New $R$-matrices for small quantum groups

Simon Lentner and Daniel Nett
Algebra and Number Theory, Hamburg University,
Bundesstraße 55, D-20146 Hamburg

Abstract. In this article we construct a large family of $R$-matrices for various extensions of small quantum groups by grouplike elements. The extensions are in correspondence to lattices between root and weight lattice and admit $R$-matrices in many cases where the original small quantum group does not. The results of this work extend some results that are well-known for $u_q(sl_2)$. Especially for $A_n$ a combinatorial theorem about roots of unity was necessary, which is interesting on its own, and has been proven in a separate paper [LN14].

Contents

Introduction and Summary 2
1. Preliminaries 6
1.1. Lie Theory 6
1.2. Quantum groups 7
1.3. $R$-matrices 9
2. Ansatz for $R$ 10
2.1. Quasi-$R$-matrix and Cartan-part 10
2.2. A set of equations 11
3. The first type of equations 14
3.1. Equations of group-type 14
3.2. Results for all fundamental groups of Lie algebras 15
4. Quotient diamonds and the second type of equations 16
4.1. Quotient diamonds and equations of diamond-type 16
4.2. Cyclic fundamental group $G = \mathbb{Z}_N$ 18
4.3. Example: $B_2$ 21
5. Solutions 21
References 31
Hopf algebras with $R$-matrices, so called quasitriangular Hopf algebras, give rise to braided tensor categories, which have many interesting applications. For example (assuming an additional strong non-degeneracy condition we do not address in this article) they give rise to knot invariants and conformal field theories with an action of the modular group.

For quantum groups, Lusztig gives in [Lus93] Sec. 32 essentially an $R$-matrix, but it is not clear, if this gives rise to an $R$-matrix over the small quantum groups $u_q(\mathfrak{g})$. In fact for most considered examples Lusztig’s small quantum group does not admit an $R$-matrix. This has been resolved in two ways in literature:

- Several authors consider slightly smaller quotients, i.e. $K^e = 1$ for $e$ half the exponent in Lusztig’s definition, where one can obtain indeed an $R$-matrix for $q$ of odd order (see e.g. [Kas95] for $\mathfrak{sl}_2$). For some applications however, it is desirable that the quotient is taken precisely with Lusztig’s choice and one wishes to focus on the even case.
- For $q$ an even root of unity, some authors consider $R$-matrices up to outer automorphism ([Tan92], [Res95]), or quadratic extensions of $u_q(\mathfrak{g})$, e.g. explicitly in the case of $u_q(\mathfrak{sl}_2)$ in [FGST06] and more generally in [GW98] for $u_q(\mathfrak{sl}_n)$.

To our knowledge there were no results for $\mathfrak{g}$ other than $\mathfrak{sl}_n$.

In this article we generalize both approaches in a Lie theoretic setting and determine all those $R$-matrices that can be obtained through Lusztig’s ansatz [Lus93], Sec. 32.1. We consider extensions $u_q(\mathfrak{g}, \Lambda)$ of $u_q(\mathfrak{g})$ for each choice of a lattice $\Lambda_R \subset \Lambda \subset \Lambda_W$ between root and weight lattice, which corresponds to a choice of a complex connected Lie group associated to $\mathfrak{g}$. In particular this includes the quadratic extensions mentioned above. Depending on $\mathfrak{g},q,\Lambda$ we determine explicitly all solutions to the defining relations of an $R$-matrix and thus find many $R$-matrices for different variants of $u_q(\mathfrak{g})$. In particular we find indeed that for $q$ of even order (or divisible by 4 for multiply-laced $\mathfrak{g}$) we get $R$-matrices for (mostly extensions of) Lusztig’s original quotient. The main calculations in the present article are formulated in terms of certain sublattices of $\Lambda$. These sublattices depend heavily on the Lie algebras and the roots of unity in question.

This article is organized as follows.

In Section 1 we fix the Lie theoretic notation and prove some technical preliminaries. In particular, we introduce some sublattices of the weight lattice $\Lambda_W$ of a simple complex Lie algebra, e.g. the so-called centralizer $\text{Cent}(\Lambda_R)$ of $\Lambda_R$ in $\Lambda_W$ (in a braided sense). We then give the definition of the finite dimensional quantum groups $u_q(\mathfrak{g}, \Lambda, \Lambda')$ for lattices $\Lambda$, $\Lambda'$, where $\Lambda'$ is a suitable sublattice of $\text{Cent}(\Lambda_R)$. Choices of $\Lambda'$ correspond to the choice of a quotient, see above. We recall also the definition of an $R$-matrix.

In Section 2 we review the ansatz $R = R_0\Theta$ for $R$-matrices by Lusztig, with fixed $\Theta \in u_q(\mathfrak{g}, \Lambda)^+ \otimes u_q(\mathfrak{g}, \Lambda)^-$ and free $R_0 = \sum_{\mu,\nu \in \Lambda/\Lambda'} f(\mu, \nu) K_\mu \otimes K_\nu$. We find equations for the free parameters $f(\mu, \nu)$ that are equivalent to $R$ being an $R$-matrix and depend on the fundamental group $\pi_1 = \Lambda_W/\Lambda_R$ of $\mathfrak{g}$ and on some sublattices of $\Lambda_W$ associated to $q$. This ansatz was also used by Müller [Müll98a], [Müll98b] for determining $R$-matrices for quadratic extensions of $u_q(\mathfrak{sl}_n)$.
In Section 3 we will first consider those equations on \( f(\mu, \nu) \), that only depend on \( \pi_1 \) as a group, the so-called group-equations for the coefficients of the ansatz in Section 2. We will give all solutions of the group-equations of a group \( G \), where \( G \) is cyclic or equal to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), since these are the relevant cases for \( G = \pi_1 \) the fundamental group of the Lie algebras in interest. The case \( g = A_n \) with fundamental group \( \mathbb{Z}_{n+1} \) is particularly hard and requires an apparently new combinatorial assertion, which is proven in a separate article [LN14].

We then consider in Section 4 a certain constellation of sublattices of \( \Lambda \), which we call a diamond. For such a diamond-like constellation of subgroups we define diamond-equations and derive a necessary condition for the existence of solutions and some results again for the cyclic case.

In Section 5 we give the main result of this article in Theorem 5.1, a list of \( R \)-matrices obtained by Lusztig’s ansatz. These are obtained by solving the corresponding group- and diamond-equations, depending on the fundamental group \( \Lambda_R \subset \Lambda \subset \Lambda_W \), kernel \( \Lambda' \subset \text{Cent}(\Lambda_W) \cap \Lambda_R \) and the \( \ell \)-th root of unity \( q \). Here, \( \text{Cent}(\Lambda_W) \) denotes the lattice orthogonal to \( \Lambda_W \) mod \( \ell \), i.e. the set of \( \lambda \in \Lambda \) with \( \ell \mid (\lambda, \mu) \) for all weights \( \mu \).

We develop general results that allow us to compute all \( R \)-matrices fulfilling Lusztig’s ansatz depending on \( g, \Lambda, \Lambda' \). Under our assumption \( \Lambda' \subset \text{Cent}(\Lambda_W) \cap \Lambda_R \) (which simplifies some calculations), we find that in fact only \( \Lambda' = \Lambda^{[\ell]}_R \) allows the existence of an \( R \)-matrix.

**Theorem 5.1.** Let \( g \) be a finite-dimensional simple complex Lie algebra with root lattice \( \Lambda_R \), weight lattice \( \Lambda_W \) and fundamental group \( \pi_1 = \Lambda_W / \Lambda_R \). Let \( q \) be an \( \ell \)-th root of unity, \( \ell \in \mathbb{N}, \ell > 2 \). Then we have the following \( R \)-matrix of the form \( R = R_0 \bar{\Theta} \), with \( \Theta \) as in Theorem 2.2:

\[
R = \left( \frac{1}{|\Lambda/\Lambda'|} \sum_{(\mu, \nu) \in (\Lambda_1/\Lambda' \times \Lambda_2/\Lambda')} q^{-\langle \mu, \nu \rangle} \omega(\bar{\mu}, \bar{\nu}) K_\mu \otimes K_\nu \right) \cdot \bar{\Theta},
\]

for the quantum group \( u_q(\mathfrak{g}, \Lambda, \Lambda') \) with \( \Lambda_i \) the preimage of a certain subgroup \( H_i \subset \pi_1 \) in \( \Lambda_W \) (\( i = 1, 2 \)), a certain group-pairing \( \omega: H_1 \times H_2 \to \mathbb{C}^\times \) and \( \Lambda' = \Lambda^{[\ell]}_R \) as in Def. 1.4.

In the following table we list for all root systems the following data, depending on \( \ell \): Possible choices of \( H_1, H_2 \) (in terms of fundamental weights \( \lambda_k \)), the group-pairing \( \omega \), and the number of solutions \( # \). If the number has a superscript * one obtains \( R \)-matrices for Lusztig’s original choice of \( \Lambda' \). For \( g = D_n \), \( 2 \mid n \), with \( \pi_1 = \mathbb{Z}_2 \times \mathbb{Z}_2 \) we get the only cases \( H_1 \neq H_2 \) and denote by \( \lambda \neq \lambda' \in \{\lambda_{n-1}, \lambda_n, \lambda_{n-1} + \lambda_n\} \) arbitrary elements of order 2 in \( \pi_1 \).

| \( g \) | \( \ell \) | \# | \( H_i \cong \) | \( H_i (i=1,2) \) | \( \omega \) |
|---|---|---|---|---|---|
| \( A_{n \geq 1} \) | \( \ell \) odd | \( \mathbb{Z}_d \) | \( \langle \frac{n+1}{d} \lambda_n \rangle \) | \( \omega(\lambda_n, \lambda_n) = \xi_d^k \), if \( d \mid (n+1), 1 \leq k \leq d \) and \( \gcd(n+1, d\ell, k\ell - \frac{n+1}{d}n) = 1 \) |
| \( \ell \) even | * | | | |
| \( B_{n \geq 2} \) | \( \ell \) odd | \( \mathbb{Z}_4 \) | \( \{0\} \) | \( \omega(0,0) = 1 \) |
| \( \ell \) even | 1 \( \mathbb{Z}_2 \) | \( \langle \lambda_n \rangle \) | \( \omega(\lambda_n, \lambda_n) = (-1)^{n-1} \) |
| \( \ell \equiv 2 \mod 4 \) | 2 \( \mathbb{Z}_4 \) | \( \langle \lambda_n \rangle \) | \( \omega(\lambda_n, \lambda_n) = \pm 1 \) |
| \( \ell \equiv 0 \mod 4 \) | 2* \( \mathbb{Z}_2 \) | \( \{0\} \) | \( \omega(0,0) = 1 \), if \( n \) even |
| \( \ell \not\equiv 4 \) | 1* \( \mathbb{Z}_1 \) | \( \{0\} \) | \( \omega(0,0) = 1 \), if \( n \) even |


\[ \begin{array}{|c|c|c|c|} \hline \ell \mod 4 & \ell \neq 4 & \omega(0,0) = 1 & \omega(0,0) = 1, \text{if } n \text{ even} \\ \hline \ell \equiv 2 \mod 4 & \ell \equiv 0 \mod 4 & \omega(0,0) = 1 & \omega(0,0) = 1 \\ \hline \ell \equiv 0 \mod 4 & \ell \equiv 3 \mod 4 & \omega(0,0) = 1 & \omega(0,0) = 1 \\ \hline \ell \equiv 2 \mod 4 & \ell \equiv 0 \mod 4 & \omega(0,0) = 1 & \omega(0,0) = 1 \\ \hline \end{array} \]

**Table 1: Solutions for \( R_0 \)-matrices**
The cases $B_n, C_n, F_4$, $\ell = 4$ and $G_2$, $\ell = 3, 6$ and $\ell = 4$ respectively, can be obtained in the table for $A_1^{\infty}, D_n, D_4$, and again $A_2$ and $A_3$ respectively, as outlined in Remark 2.1.

Note, that Lusztig’s $R$-matrix for $\Lambda = \Lambda_R$ correspond to the case $H = Z_1$ and $\omega = 1$. The known quadratic extension for $\mathfrak{sl}_2$ is the case $A_1$ with $H = Z_2$ in the example below.

**Remark.** We indicate in which sense our results are *not* complete:

- Technically, one could even allow $\Lambda_R \subset \Lambda \subset \Lambda_W^\vee$, but then loose the topological interpretation as different choices of a Lie group associated to $\mathfrak{g}$.
- Our assumption $\Lambda' \subset \text{Cent}(\Lambda_W) \cap \Lambda_R$ was to simplify calculations and prove uniqueness. In general $\Lambda' \in \text{Cent}(\Lambda)$ would suffice (and could yield more solutions), but one would have to deal with possible $2$-cocycles in $H^2(\Lambda/\Lambda', \pi_1)$ in Lemma 2.5.

Then, one could ask in general whether all $R$-matrices are necessarily given by Lusztig’s ansatz and hence of our form. Note that this would also give an effective way to classify factorizable $R$-matrices.

**Example.** For $\mathfrak{g} = \mathfrak{sl}_2$ with root system $A_1$ the fundamental group is $\pi_1 = \mathbb{Z}_2$. Let $\alpha$ be the simple root, generating the root lattice $\Lambda_R$, and $\lambda = \frac{1}{2} \alpha$ the fundamental dominant weight, generating the weight lattice $\Lambda_W$. We will give the $R$-matrices for the quantum groups $u = u_q(\mathfrak{g}, \Lambda, \Lambda')$ for $\ell$-th root of unity $q$ and lattices $\Lambda_R \subset \Lambda \subset \Lambda_W$ and $\Lambda' = \Lambda_R^{[\ell]}$, which equals in the simply laced case $\ell \Lambda_R$.

The quasi $R$-matrix $\Theta$ (see Theorem 2.2) depends only on the root lattice and exist in $u^+ \otimes u^-$ with Borel parts $u^\pm$, generated by $E_\alpha, F_\alpha$. With $\ell_\alpha = \ell / \text{gcd}(\ell, 2d_\alpha) = \ell / \text{gcd}(\ell, 2)$ we have

$$\Theta = \sum_{k=0}^{\ell_\alpha - 1} (-1)^k (q - q^{-1})^k \frac{q^{-k(k-1)/2} F_\alpha^k \otimes E_\alpha^k}{[k]_q!} \quad \text{and} \quad \bar{\Theta} = \sum_{k=0}^{\ell_\alpha - 1} (q - q^{-1})^k \frac{q^{k(k-1)/2} E_\alpha^k \otimes F_\alpha^k}{[k]_q!},$$

with $q$-factorial $[k]_q!$ as defined in Definition 1.8. The $R_0$-part is given by

$$R_0 = \frac{1}{|\Lambda/\Lambda_R^{[\ell]}|} \sum_{\mu, \nu \in \Lambda/\Lambda'} q^{-\langle \mu, \nu \rangle} \omega(\bar{\mu}, \bar{\nu}) K_\mu \otimes K_\nu,$$

for $H$ and $\omega$: $H \times H \to \mathbb{C}^\times$ as in Table 2. The possible solutions depend on $\ell$. We now check the condition $\text{gcd}(2, d\ell, k\ell - 2/d) = 1$ from the theorem above ($n = 1$ and $d = 1, 2$). For odd $\ell$, we get the following solutions by Theorem 5.1

$$H = \mathbb{Z}_1, \quad \omega: \mathbb{Z}_1 \times \mathbb{Z}_1 \to \mathbb{C}^\times, \quad \omega(0, 0) = 1,$$

$$H = \mathbb{Z}_2, \quad \omega: \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{C}^\times, \quad \omega(\lambda, \lambda) = 1.$$  

For even $\ell$, the solution for $H = \mathbb{Z}_1$, i.e. for $\Lambda = \Lambda_R$, does not exist (since $2 \not| \ell$ and $2 \not| (\ell - 2)$), rather we get both possible solutions on the full support $H = \mathbb{Z}_2$:

$$H = \mathbb{Z}_2, \quad \omega: \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{C}^\times, \quad \omega(\lambda, \lambda) = \pm 1.$$
In these cases, the $R$-matrices are explicitly given by
\[
R = \left( \frac{1}{2\ell} \sum_{i,j=0}^{2\ell-1} q^{-i\lambda_j} (\pm 1)^{ij} K_{\alpha+x} \otimes K_{\beta+y} \right) \cdot \Theta \\
= \frac{1}{2\ell} \sum_{k=0}^{\ell - 1} \sum_{i,j=0}^{2\ell-1} (q - q^{-1})^k q^{k(k-1)/2} q^{-ij/2} (\pm 1)^{ij} K^i_{\lambda} E^j_{\alpha} \otimes K^j_{\lambda} F^i_{\alpha} \\
= \frac{1}{2\ell} \sum_{k=0}^{\ell - 1} \sum_{i,j=0}^{2\ell-1} (q - q^{-1})^k q^{k(k-1)/2} q^{j(j-1)/2} (\pm 1)^{ij} E^k_{\alpha} K^i_{\lambda} \otimes F^j_{\alpha} K^i_{\lambda}
\]

Acknowledgements. Partly supported by the DFG Priority Program 1388 “Representation theory”. We thank Christoph Schweigert for several helpful discussions.

1. Preliminaries

At first, we fix a convention.

Convention 1.1. In the following, $q$ is an $\ell$-th root of unity. We fix $q = \exp(\frac{2\pi i}{\ell})$ and for $a \in \mathbb{R}$ we set $q^a = \exp(\frac{2\pi ia}{\ell})$, $\ell > 2$.

1.1. Lie Theory. Let $\mathfrak{g}$ be a finite-dimensional, semisimple complex Lie algebra with simple roots $\alpha_i$, indexed by $i \in I$, and a set of positive roots $\Phi^+$. Denote the Killing form by $(\cdot, \cdot)$, normalized such that $(\alpha, \alpha) = 2$ for the short roots $\alpha$. The Cartan matrix is given by
\[
\alpha_{ij} = \langle \alpha_i, \alpha_j \rangle = 2 \frac{\langle \alpha_i, \alpha_j \rangle}{(\alpha_i, \alpha_i)}.
\]

For a root $\alpha$ we call $d_\alpha := (\alpha, \alpha)/2$ with $d_\alpha \in \{1, 2, 3\}$. Especially, $d_i := d_{\alpha_i}$ and in this notation $(\alpha_i, \alpha_j) = d_i a_{ij}$. The fundamental dominant weights $\lambda_1, \ldots, \lambda_n$ are given by the condition $2(\alpha_i, \lambda_j)/(\alpha_i, \alpha_i) = \delta_{ij}$, hence the Cartan matrix expresses the change of basis roots to weights.

Definition 1.2. The root lattice $\Lambda_R = \Lambda_R(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ is the abelian group with rank $\text{rank}(\Lambda_R) = \text{rank}(\mathfrak{g}) = |I|$, generated by the simple roots $\alpha_i$, $i \in I$. The Killing form implements an integral pairing of abelian groups, turning $\Lambda_R$ into an integral lattice.

Definition 1.3. The weight lattice $\Lambda_W = \Lambda_W(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ is the abelian group with rank $\text{rank}(\Lambda_W) = \text{rank}(\mathfrak{g})$, generated by the fundamental dominant weights $\lambda_i$, $i \in I$. It is standard fact of Lie theory (cf. [Hum72], Section 13.1) that the root lattice is contained in the weight lattice.

Definition 1.4. Let $\Lambda_R$, $\Lambda_W$ the root, resp. weight, lattice of the Lie algebra $\mathfrak{g}$ with generators $\alpha_i$, resp. $\lambda_i$, for $i \in I$.

(i) For any positive integer $\ell$, the $\ell$-lattice $\Lambda_R^{(\ell)}$, resp. $\Lambda_W^{(\ell)}$, is defined as
\[
\Lambda_R^{(\ell)} = \left\langle \frac{\ell}{\gcd(\ell, 2d_i)} \alpha_i, \alpha_i \in \Lambda_R \right\rangle \quad \text{resp.} \quad \Lambda_W^{(\ell)} = \left\langle \frac{\ell}{\gcd(\ell, 2d_i)} \lambda_i, \lambda_i \in \Lambda_W \right\rangle.
\]

Following Lusztig, we define $\ell_\alpha := \ell / \gcd(\ell, 2d_i)$, which is the order of $q^{2d_i}$, where $q$ is a primitive $\ell$-th root of unity. More generally, we define for any root $\ell_\alpha := \ell / \gcd(\ell, 2d_\alpha)$.
(ii) For any positive integer \( \ell \), the lattice \( \Lambda_R^{[\ell]} \), resp. \( \Lambda_W^{[\ell]} \), is defined as
\[
\Lambda_R^{[\ell]} = \left\{ \frac{\ell}{\gcd(\ell, d_i)} \alpha_i \mid \alpha_i \in \Lambda_R \right\} \quad \text{resp.} \quad \Lambda_W^{[\ell]} = \left\{ \frac{\ell}{\gcd(\ell, d_i)} \lambda_i \mid \lambda_i \in \Lambda_W \right\}.
\] (1.2)

**Definition 1.5.** For \( \Lambda_1, \Lambda_2 \subseteq \Lambda \) with \( \Lambda_2 \subseteq \Lambda_1 \) we define \( \Cent_{\Lambda_1}(\Lambda_2) = \{ \eta \in \Lambda_1 \mid (\eta, \lambda) \in \ell \mathbb{Z} \ \forall \lambda \in \Lambda_2 \} \). In the situation \( \Lambda_1 = \Lambda_W \) we simply write \( \Cent_{\Lambda_W}(\Lambda_2) = \Cent(\Lambda_2) \).

 Especially, \( \langle K_{\eta} \mid \eta \in \Cent(\Lambda_R) \rangle \) are the group elements in the center of the quantum group \( U_q(\mathfrak{g}, \Lambda_W) \), cf. Section 1.2.

**Lemma 1.6.** For a Lie algebra \( \mathfrak{g} \) we have \( \Cent(\Lambda_R) = \Lambda_W^{[\ell]} \). We call the elements of \( \Cent(\Lambda_R) \) central weights.

**Proof.** Let \( \lambda = \sum_{j \in I} m_j \lambda_j \in \Lambda_W \) with fundamental weights \( \lambda_i \). For a simple root \( \alpha_i \) we have \( (\alpha_i, \lambda) = (\alpha_i, \sum_{j \in I} m_j \lambda_j) = d_i m_i \). Thus, \( \lambda \) is central weight if \( \ell | d_i m_i \) for all \( i \), hence \( (\ell/\gcd(\ell, d_i)) | m_i \) for all \( i \).

The same calculation gives the following lemma.

**Lemma 1.7.** For a Lie algebra \( \mathfrak{g} \) we have \( \Cent(\Lambda_W) \cap \Lambda_R = \Lambda_R^{[\ell]} \).

1.2. Quantum groups. For a finite-dimensional complex simple Lie algebra \( \mathfrak{g} \), lattices \( \Lambda, \Lambda' \) with \( \Lambda_R \subseteq \Lambda \subseteq \Lambda_W \) and \( 2\Lambda_R^{[\ell]} \subseteq \Lambda' \subseteq \Cent(\Lambda_W) \cap \Lambda_R \), and primitive \( \ell \)-th root of unity \( q \), we aim to define the finite-dimensional quantum group \( U_q(\mathfrak{g}, \Lambda, \Lambda') \), also called small quantum group. We construct \( U_q(\mathfrak{g}, \Lambda, \Lambda') \) by using rational and integral forms of the deformed universal enveloping algebra \( U_q(\mathfrak{g}) \) for an indeterminate \( q \). In the following we collect the definitions and give the statements without proofs and follow the lines of [Len14]. The different choices of \( \Lambda \) are already in [Lus93], Sec. 2.2. We shall give a dictionary to translate Lusztig’s notation to the one used here.

**Definition 1.8.** For \( q \in \mathbb{C}^\times \) or \( q \) an indeterminate and \( n \leq k \in \mathbb{N}_0 \) we define
\[
[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! := [1]_1 [2]_q \cdots [n]_q, \quad [n]_q^{[m]} := \begin{cases} \frac{[n]_q!}{[k]_q! [n-k]_q!}, & 0 \leq k \leq n, \\ 0, & \text{else.} \end{cases}
\]

**Definition 1.9.** Let \( q \) be an indeterminate. For each abelian group \( \Lambda \) with \( \Lambda_R \subseteq \Lambda \subseteq \Lambda_W \) we define the rational form \( U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda) \) over the ring of rational functions \( k = \mathbb{Q}(q) \) as follows:

As algebra, let \( U_q^{\mathbb{Q}(q)}(\mathfrak{g}, \Lambda) \) be generated by the group ring \( k[\Lambda] \), spanned by \( K_\lambda, \lambda \in \Lambda \), and additional generators \( E_{\alpha_i}, F_{\alpha_i} \), for each simple root \( \alpha_i, i \in I \), with relations:
\[
K_\lambda E_{\alpha_i} K_\lambda^{-1} = q^{(\lambda, \alpha_i)} E_{\alpha_i}, \quad (1.3)
\]
\[
K_\lambda F_{\alpha_i} K_\lambda^{-1} = q^{-(\lambda, \alpha_i)} F_{\alpha_i}, \quad (1.4)
\]
\[
E_{\alpha_i} F_{\alpha_j} - F_{\alpha_j} E_{\alpha_i} = \delta_{ij} \frac{K_{\alpha_i} - K_{\alpha_i}^{-1}}{q_{\alpha_i} - q_{\alpha_i}^{-1}}, \quad (1.5)
\]
and Serre relations for any \( i \neq j \in I \)

\[
\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \frac{1-a_{ij}}{r} \right]_{q_i} E_{\alpha_i}^{-a_{ij}-r} E_{\alpha_j} F_{\alpha_i}^r = 0, \quad (1.6)
\]

\[
\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \frac{1-a_{ij}}{r} \right]_{\bar{q}_i} F_{\alpha_i}^{-a_{ij}-r} F_{\alpha_j} E_{\alpha_i}^r = 0, \quad (1.7)
\]

where \( \bar{q} := q^{-1} \), the quantum binomial coefficients are defined in Definition 1.8 and by definition \( q^{(\alpha_i, \alpha_j)} = (q^\theta)^{a_{ij}} = q_i^{a_{ij}} \).

As a coalgebra, let the coproduct \( \Delta \), the counit \( \varepsilon \) and the antipode \( S \) be defined on the group-Hopf algebra \( k[\Lambda] \) as usual

\[
\Delta(K_\lambda) = K_\lambda \otimes K_\lambda, \quad \varepsilon(K_\lambda) = 1, \quad S(K_\lambda) = K_{-\lambda},
\]

and on the generator \( E_{\alpha_i}, F_{\alpha_i} \), for each simple root \( \alpha_i, i \in I \) as follows

\[
\Delta(E_{\alpha_i}) = E_{\alpha_i} \otimes K_{\alpha_i} + 1 \otimes E_{\alpha_i}, \quad \Delta(F_{\alpha_i}) = F_{\alpha_i} \otimes 1 + K_{-\alpha_i}^{-1} \otimes F_{\alpha_i},
\]

\[
\varepsilon(E_{\alpha_i}) = 0, \quad \varepsilon(F_{\alpha_i}) = 0,
\]

\[
S(E_{\alpha_i}) = -E_{\alpha_i} K_{-\alpha_i}^{-1}, \quad S(F_{\alpha_i}) = -K_{\alpha_i} F_{\alpha_i}.
\]

**Theorem 1.10** (cf. [Len14], Theorem 3.3). The rational form \( U_q^{Q(q)}(g, \Lambda) \) of Definition 1.9 is a Hopf algebra over the field \( k = \mathbb{Q}(q) \).

Moreover, we have a triangular decomposition: Consider the subalgebras \( U_q^{Q(q),+} \), generated by the \( E_{\alpha_i} \), and \( U_q^{Q(q),-} \), generated by the \( F_{\alpha_i} \), and \( U_q^{Q(q),0} = k[\Lambda] \), spanned by the \( K_\lambda \). Then the multiplication in \( U_q^{Q(q)} = U_q^{Q(q)}(g, \Lambda) \) induces an isomorphism of vector spaces:

\[
U_q^{Q(q),+} \otimes U_q^{Q(q),0} \otimes U_q^{Q(q),-} \xrightarrow{\cong} U_q^{Q(q)}.
\]

**Definition 1.11.** The so-called restricted integral form \( U_q^{Z[q,q^{-1}]}(g, \Lambda) \) is generated as a \( \mathbb{Z}[q,q^{-1}] \)-algebra by \( \Lambda \) and the following elements in \( U_q^{Q(q)}(g, \Lambda)^\pm \), called divided powers:

\[
E_{\alpha}^{(r)} := \frac{E_{\alpha}}{\prod_{s=1}^{r} q_{a}^s - q_{-a}^s}, \quad F_{\alpha}^{(r)} := \frac{F_{\alpha}}{\prod_{s=1}^{r} q_{a}^s - q_{-a}^s}, \quad \text{for all } \alpha \in \Phi^+, r > 0,
\]

and by the following elements in \( U_q^{Q(q)}(g, \Lambda)^0 \):

\[
K_{\alpha_i}^{(r)} := \left[ K_{\alpha_i}; 0 \right] r := \prod_{s=1}^{r} \frac{K_{\alpha_i} q_{\alpha_i}^{-s} - K_{-\alpha_i}^{-1} q_{-\alpha_i}^{-s}}{q_{\alpha_i}^s - q_{-\alpha_i}^{-s}}, \quad i \in I.
\]

This definitions can also be found in Lusztig’s book [Lus93]. In order to translate Lusztig’s notation to the one used here, one has to match the terms in the following way...
| Lusztig’s notation | notation used here |
|-----------------|---------------------|
| Index set $I$   | simple roots $\{\alpha_i \mid i \in I\}$ |
| $X$             | root lattice $\Lambda_R$ |
| $Y$             | lattice $\Lambda_R \subset \Lambda \subset \Lambda_W$ |
| $i' \in X$     | $\alpha_i$ |
| $i \in Y$      | $\alpha_i$ coroot |
| $i \cdot j, i, j \in \mathbb{Z}[I]$ | $(\alpha_i, \alpha_j)$ |
| $\langle i, j' \rangle = 2\frac{\ell}{\mathbb{Z}}$, $i \in Y, j' \in X$ | $\langle \alpha_i, \alpha_j \rangle$ |
| $\tilde{K}_i = K_{i, i}$ | $K_{\alpha_i}$ |

We now define the \textit{restricted specialization} $U_q(\mathfrak{g}, \Lambda)$. Here, we specialize $q$ to a specific choice $q \in \mathbb{C}^\times$.

**Definition 1.12.** The infinite-dimensional Hopf algebra $U_q(\mathfrak{g}, \Lambda)$ is defined by

$$U_q(\mathfrak{g}, \Lambda) := U_q(\mathfrak{g}, \Lambda) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}_q,$$

where $\mathbb{C}_q = \mathbb{C}$ with the $\mathbb{Z}[q,q^{-1}]$-module structure defined by the specific value $q \in \mathbb{C}^\times$.

From now on, $q$ will be an primitive $\ell$-th root of unity. We choose explicitly $q = \exp(\frac{2\pi i}{\ell})$, see Convention 1.1.

**Definition 1.13.** For a finite-dimensional complex simple Lie algebra $\mathfrak{g}$ and lattices $\Lambda, \Lambda'$ with $\Lambda_R \subset \Lambda \subset \Lambda_W$ and $2\Lambda^{(\ell)}_R \subset \Lambda' \subset \text{Cent}(\Lambda_W) \cap \Lambda_R$, we define the \textit{small quantum group} $u_q(\mathfrak{g}, \Lambda, \Lambda')$ as the algebra $U_q(\mathfrak{g}, \Lambda)$ from 1.12 generated by $K_\lambda$ for $\lambda \in \Lambda$ and $E_\alpha, F_\alpha$ with $l_\alpha > 0, \alpha \in \Phi^+$ not necessarily simple, together with the relations

$$E_\alpha^{l_\alpha} = 0, \quad F_\alpha^{l_\alpha} = 0 \quad \text{and} \quad K_\lambda = 1 \text{ for } \lambda \in \Lambda'.$$

This is a finite dimensional Hopf algebra of dimension

$$|\Lambda/\Lambda'| \prod_{\alpha \in \Phi^+, \ell_\alpha > 1} \ell_\alpha^2.$$

The fact, that this gives a Hopf algebra for $\Lambda' = 2\Lambda_R^{(\ell)}$ is in Lusztig, [Lus90], Sec. 8.

### 1.3. R-matrices.

**Definition 1.14.** A Hopf algebra $H$ is called \textit{quasitriangular} if there exists an invertible element $R \in H \otimes H$ such that

$$\Delta^\text{op}(h) = R \Delta(h) R^{-1},$$

$$\Delta \otimes \text{Id}(R) = R_{13}R_{23},$$

$$\text{Id} \otimes \Delta(R) = R_{13}R_{12},$$

with $\Delta^\text{op}(h) = \tau \circ \Delta(h)$, where $\tau : H \otimes H \to H \otimes H$, $a \otimes b \mapsto b \otimes a$ and $R_{12} = R \otimes 1, \ R_{23} = 1 \otimes R, \ R_{13} = (\tau \otimes \text{Id})(R_{23}) = (\text{Id} \otimes \tau)(R_{12}) \in H^{\otimes 3}$. Such an element is called \textit{R-matrix} of $H$. 
2. Ansatz for \( R \)

2.1. Quasi-\( R \)-matrix and Cartan-part. The goal of this paper is to construct new families of \( R \)-matrices for small quantum groups and certain extensions (see Def. 1.13).

Our starting point is Lusztig’s ansatz in [Lus93], Sec. 32.1, for a universal \( R \)-matrix of \( U_q(\mathfrak{g}, \Lambda) \). This ansatz has been translated by Müller in his Dissertation [Müll98a], resp. in [Müll98b], for small quantum groups, which we will use in the following. Note, that this ansatz has been successfully generalized to general diagonal Nichols algebras in [AY13].

For a finite-dimensional, semisimple complex Lie algebra \( \mathfrak{g} \), an \( \ell \)-th root of unity \( q \) and lattices \( \Lambda, \Lambda' \) as in Section 1.2 we write \( u = u_q(\mathfrak{g}, \Lambda, \Lambda') \). Let \( \tilde{\Delta} : u \otimes u \to u \otimes u \) be the \( \mathbb{Q}(q) \)-algebra automorphism given by \( \tilde{\otimes} \). Let \( \Delta : u \to u \otimes u \) be the \( \mathbb{Q}(q) \)-algebra homomorphism given by \( \Delta(x) = \tilde{\Delta}(x) \) for all \( x \in U \). We have in general \( \Delta \neq \tilde{\Delta} \).

Assume in the following, that 
\[
\ell_i > 1 \text{ for all } i \in I, \text{ and } \ell_i > - (\alpha_i, \alpha_j) \text{ for all } i, j \text{ with } i \neq j. \tag{2.1}
\]

**Theorem 2.1 (Lem14).** For a root system \( \Phi \) of a finite-dimensional simple complex Lie algebra and \( \ell \)-th root of unity \( q \), the condition (2.1) fails only in the following cases \( (\Phi, \ell) \).

In each case, the small quantum group \( u_q(\mathfrak{g}) \) is described by a different \( \tilde{\Phi} \) fulfilling (2.1), hence the present work also provides results for these cases by consulting the results for \( \Phi \).

| \( \Phi \) | (all) | \( B_n \) | \( C_n \) | \( F_4 \) | \( G_2 \) | \( G_2 \) |
|---|---|---|---|---|---|---|
| \( \ell \) | 1, 2 | 4 | 4 | 3, 6 | 4 |
| \( \tilde{\Phi} \) | (empty) | \( A_1 \times \ldots \times A_1 \) | \( D_n \) | \( D_n \) | \( A_2 \) | \( A_3 \) |

The following theorem is essentially in [Lus93]. Note that the roles of \( E, F \) will be switched in our article to match the usual convention.

**Theorem 2.2** (cf. [Müll98a], Thm. 8.2). (a) There is a unique family of elements \( \Theta_\nu \in u_\nu^+ \otimes u_\nu^- \), \( \nu \in \Lambda_R \), such that \( \Theta_\emptyset = 1 \otimes 1 \) and \( \Theta = \sum_\nu \Theta_\nu \in u \otimes u \) satisfies \( \Delta(x)\Theta = \Theta \Delta(x) \) for all \( x \in u \).

(b) Let \( B \) be a vector space-basis of \( u_\nu^+ \), such that \( B_\nu = B \cap u_\nu^+ \) is a basis of \( u_\nu^+ \) for all \( \nu \). Here, \( u_\nu^+ \) refers to the natural \( \Lambda_R \)-grading of \( u_\nu^+ \). Let \( \{ b^* \mid b \in B_\nu \} \) be the basis of \( u_\nu^- \) dual to \( B_\nu \) under the non-degenerate bilinear form \( (\cdot, \cdot) : u^+ \otimes u^- \to \mathbb{C} \). We have
\[
\Theta_\nu = (-1)^{tr \nu} q_\nu \sum_{b \in B_\nu} b^* \otimes b^* \in u_\nu^+ \otimes u_\nu^- \tag{2.2}
\]
where \( q_\nu = \prod_i q^{\alpha_i} \), \( tr \nu = \sum_i \nu_i \) for \( \nu = \sum_i \nu_i \alpha_i \in \Lambda_R \).

**Remark 2.3.** (i) The element \( \Theta \) is called the *Quasi-\( R \)-matrix* of \( u = u_q(\mathfrak{g}, \Lambda, \Lambda') \).

(ii) Since the element \( \Theta \) is unique, the expressions \( \sum_{b \in B_\nu} b^+ \otimes b^* \) in part (b) of the theorem are independent of the actual choice of the basis \( B \).

(iii) For example, if \( \mathfrak{g} = A_1 \), i.e. there is only one simple root \( \alpha = \alpha_1 \), and \( E = E_\alpha \), \( F = F_\alpha \). Thus we have
\[
\Theta = \sum_{n=0}^{\ell_\alpha-1} (-1)^n (q - q^{-1})^n q^{-n(n-1)/2} E^n \otimes F^n.
\]
(iv) The Quasi-R-matrix $\Theta$ is invertible and it is $\Theta^{-1} = \bar{\Theta}$, i.e. the expression one gets by changing all $q$ to $\bar{q} = q^{-1}$.

**Theorem 2.4** (cf. [Mül98b], Theorem 8.11). Let $\Lambda' \subset \{\mu \in \Lambda \mid K_\mu, \text{central in } u_q(\mathfrak{g}, \Lambda)\}$ a subgroup of $\Lambda$, and $H_1, H_2$ subgroups of $\Lambda/\Lambda'$, containing $\Lambda_R/\Lambda'$. In the following, $\mu, \mu_1, \mu_2 \in H_1$ and $\nu, \nu_1, \nu_2 \in H_2$.

The element $R = R_0\bar{\Theta}$ with $R_0 = \sum_{\mu,\nu} f(\mu,\nu)K_\mu \otimes K_\nu$ is a R-matrix for $u_q(\mathfrak{g}, \Lambda, \Lambda')$, if and only if for all $\alpha \in \Lambda_R$ and $\mu, \nu$ the following holds:

\[ f(\mu + \alpha, \nu) = q^{-(\nu,\alpha)} f(\mu, \nu), \quad f(\mu, \nu + \alpha) = q^{-(\mu,\alpha)} f(\mu, \nu), \]

\[ \sum_{\mu_1, \mu_2 = \mu} f(\mu_1, \nu_1) f(\mu_2, \nu_2) = \delta_{\mu_1,\mu_2} f(\mu_1, \nu_1), \quad \sum_{\mu_1, \mu_2 = \mu} f(\mu_1, \nu_1) f(\mu_2, \nu_2) = \delta_{\nu_1,\nu_2} f(\mu, \nu_1), \]

\[ \sum_{\nu} f(\mu, \nu) = \delta_{\nu,0}, \quad \sum_{\mu} f(\mu, \nu) = \delta_{\mu,0}. \]

Condition **2.5** follows from **2.3** and **2.4** if there exists $c \in \mathbb{C}$ such that $f(\mu, 0) = f(0, \nu) = c$ for all $\mu, \nu$. There are conditions on the order of $q$: For all $\mu, \nu$ for which there exist $\bar{\mu}, \bar{\nu}$ such that $f(\mu, \bar{\nu}) \neq 0$, $f(\bar{\mu}, \nu) \neq 0$ we have

\[ q^{2i(\mu,\alpha)} = q^{2i(\nu,\alpha)} = 1. \]

If this condition is satisfied then $f$ is well-defined on the preimages of $H_1 \times H_2$ under $\Lambda \to \Lambda/\Lambda'$. (In particular, this is the case under our assumption $\Lambda' \in \text{Cent}(\Lambda_W).$)

### 2.2. A set of equations.

**Lemma 2.5.** Let $\Lambda \subset \Lambda_W$ a sublattice and $\Lambda' \subset \Lambda$. Assume in addition, $\Lambda' \subset \text{Cent}(\Lambda_W)$.

(i) Let $f : \Lambda/\Lambda' \times \Lambda/\Lambda' \to \mathbb{C}$, satisfying condition **2.3** of Theorem 2.4. Then

\[ g(\bar{\mu}, \bar{\nu}) := |\Lambda_R/\Lambda'| q^{(\mu,\nu)} f(\mu, \nu), \]

defines a function $\pi_1 \times \pi_1 \to \mathbb{C}$.

(ii) If, in addition, $f$ satisfies conditions **2.4**-**2.5**, the function $g$ in (i) satisfies the following equations:

\[ \sum_{\mu_1, \mu_2 = \mu} \delta_{(\mu_2-\mu_1\in\text{Cent}(\Lambda_R))} q^{(\mu_2-\mu_1,\bar{\nu}_1)} g(\bar{\mu}_1, \bar{\nu}_1) g(\bar{\mu}_2, \bar{\nu}_2) = \delta_{\mu_1,\mu_2} g(\bar{\mu}_1, \bar{\nu}_1), \]

\[ \sum_{\bar{\mu}_1, \bar{\mu}_2 = \bar{\mu}} \delta_{(\bar{\nu}_2-\nu_1\in\text{Cent}(\Lambda_R))} q^{(\nu_2-\nu_1,\bar{\mu}_1)} g(\bar{\mu}_1, \bar{\nu}_1) g(\bar{\mu}_2, \bar{\nu}_2) = \delta_{\nu_1,\nu_2} g(\bar{\mu}, \bar{\nu}_1), \]

\[ \sum_{\bar{\mu}} \delta_{(\bar{\mu}\in\text{Cent}(\Lambda_R))} q^{-(\mu,\bar{\mu})} g(\bar{\mu}, \bar{\nu}) = \delta_{\mu,0}, \]

\[ \sum_{\bar{\nu}} \delta_{(\nu\in\text{Cent}(\Lambda_R))} q^{-(\nu,\bar{\mu})} g(\bar{\mu}, \bar{\nu}) = \delta_{\nu,0}. \]

Here, the sums range over $\pi_1$ and expressions like $\delta_{(\mu\in\text{Cent}(\Lambda_R))}$ equals 1 if $\mu$ is a central weight and 0 otherwise.

Before we proceed with the proof we will comment on the relevance of this equations and introduce a definition. For a given Lie algebra $\mathfrak{g}$ with root lattice $\Lambda_R$ and weight lattice $\Lambda_W$ the solutions of the $g(\bar{\mu}, \bar{\nu})$-equations give solutions for an $R_0$ in the ansatz $R = R_0\bar{\Theta}$. Hence, we get possible $R$-matrices for the quantum group $u_q(\mathfrak{g}, \Lambda_W, \Lambda')$. 

We divide the equations in two types.

**Definition 2.6.** For central weights 0 we call the equations (2.7)-(2.8) *group-equations*:

\[
g(\bar{\mu}, \bar{\nu}) = \sum_{\bar{\nu}_1 + \bar{\nu}_2 = \bar{\nu}} g(\bar{\mu}, \bar{\nu}_1)g(\bar{\mu}, \bar{\nu}_2),
\]

\[
g(\bar{\mu}, \bar{\nu}) = \sum_{\bar{\mu}_1 + \bar{\mu}_2 = \bar{\mu}} g(\bar{\mu}_1, \bar{\nu})g(\bar{\mu}_2, \bar{\nu}),
\]

\[
1 = \sum_{\bar{\nu}} g(0, \bar{\nu}),
\]

\[
1 = \sum_{\bar{\mu}} g(\bar{\mu}, 0).
\]

For \(\pi_1 = \Lambda_W/\Lambda_R\) of order \(n\) this gives us \(2n^2 + 2\) group-equations.

For central weight \(0 \neq \zeta \in \text{Cent}(\Lambda_R)/\Lambda'\), we call the equations (2.7)-(2.8) *diamond-equations* (for reasons that will become transparent later):

\[
0 = \sum_{\bar{\nu}_1 + \bar{\nu}_2 = \bar{\nu}} q^{(\zeta, \bar{\nu})}g(\bar{\mu}, \bar{\nu}_1)g(\bar{\mu} + \bar{\zeta}, \bar{\nu}_2),
\]

\[
0 = \sum_{\bar{\mu}_1 + \bar{\mu}_2 = \bar{\mu}} q^{(\bar{\mu}, \zeta)}g(\bar{\mu}_1, \bar{\nu})g(\bar{\mu}_2, \bar{\nu} + \bar{\zeta}),
\]

\[
0 = \sum_{\bar{\nu}} q^{-(\bar{\nu}, \zeta)}g(\bar{\mu} + \bar{\zeta}, \bar{\nu}),
\]

\[
0 = \sum_{\bar{\mu}} q^{-(\bar{\mu}, \zeta)}g(\bar{\mu}, \bar{\nu} + \bar{\zeta}).
\]

This gives up to \((|\text{Cent}(\Lambda_R)/\Lambda'| - 1)(2n^2 + 2)\) diamond-equations.

**Proof of Lemma 2.5.**

(i) Since \(\Lambda' \in \text{Cent}(\Lambda_W)\) we have \(q^{(\Lambda_W, \Lambda')} = 1\) and terms \(q^{(\mu, \nu)}\) for \(\mu, \nu \in \Lambda/\Lambda'\) do not depend on the residue class representatives modulo \(\Lambda'\). We check that the function \(g\) is well-defined. Let \(\mu, \nu \in \Lambda\) and \(\lambda' \in \Lambda'\). Thus,

\[
g(\mu + \lambda', \nu) = |\Lambda_R/\Lambda'| |q^{(\mu+\lambda', \nu)}f(\mu + \lambda', \nu)
\]

\[
= |\Lambda_R/\Lambda'| |q^{(\mu+\lambda', \nu)}q^{-\lambda', \nu}f(\mu, \nu)
\]

\[
= |\Lambda_R/\Lambda'| |q^{(\mu, \nu)}f(\mu, \nu)
\]

\[
= g(\mu, \nu),
\]

and analogously for \(g(\mu, \nu + \lambda')\).

(ii) We consider equations (2.4). Let \(\nu_1, \nu \in \Lambda/\Lambda'\) and write \(\nu_i = \bar{\nu}_i + \alpha_i\) and \(\nu = \bar{\nu} + \alpha\) with \(\bar{\nu}_i, \bar{\nu} \in \Lambda_W/\Lambda_R\) and \(\alpha_i, \alpha \in \Lambda_R\), \(i = 1, 2\). For the sum \(\nu = \nu_1 + \nu_2\) we get \(\bar{\nu} \equiv \bar{\nu}_1 + \bar{\nu}_2\) in \(\Lambda_W/\Lambda_R\), i.e. there is a cocycle \(\sigma(\nu_1, \nu_2) \in \Lambda_R\) with \(\bar{\nu} = \bar{\nu}_1 + \bar{\nu}_2 + \sigma(\nu_1, \nu_2)\) in \(\Lambda_W\).
and \( \alpha = \alpha_1 + \alpha_2 - \sigma(\nu_1, \nu_2) \). We will write \( \sigma \) for \( \sigma(\nu_1, \nu_2) \).

\[
\sum_{\nu_1 + \nu_2 = \nu} f(\mu_1, \nu_1) f(\mu_2, \nu_2) = \sum_{\nu_1 + \nu_2 = \nu} q^{-\nu_1} (\mu_1, \nu_1) + (\mu_2, \nu_2) f(\bar{\mu}_1, \bar{\nu}_1) f(\bar{\mu}_2, \bar{\nu}_2) \\
= \sum_{\nu_1 + \nu_2 = \nu} \sum_{\alpha_1 + \alpha_2 = \alpha + \sigma} q^{-\nu_1} (\mu_1, \nu_1) + (\mu_2, \nu_2) f(\bar{\mu}_1, \bar{\nu}_1) f(\bar{\mu}_2, \bar{\nu}_2) \\
= \sum_{\nu_1 + \nu_2 = \nu} q^{-\nu_1} (\mu_1, \nu_1) + (\mu_2, \nu_2) f(\bar{\mu}_1, \bar{\nu}_1) f(\bar{\mu}_2, \bar{\nu}_2) \sum_{\alpha_1 + \alpha_2 = \alpha + \sigma} q^{-\nu_1} (\mu_1, \nu_1) - (\mu_2, \nu_2) \\
\leq 0 \quad \text{otherwise.}
\]

Firstly, we consider the second sum over the roots (\( \mu_1, \mu_2 \) are fixed).

\[
\sum_{\alpha_1 + \nu_2 = \alpha + \sigma} q^{-\nu_1} (\mu_1, \nu_1) - (\mu_2, \nu_2) = \sum_{\alpha_1 \in \Lambda_R/\Lambda'} q^{-\nu_1} (\mu_1, \nu_1) - (\mu_2, \nu_2) \\
= \sum_{\alpha_1 \in \Lambda_R/\Lambda'} q^{-\nu_1} (\mu_1, \nu_1) - (\mu_2, \nu_2) \\
= \sum_{\alpha_1 \in \Lambda_R/\Lambda'} q^{-\nu_1} (\mu_1, \nu_1)
\]

The last sum equals \( |\Lambda_R/\Lambda'| \) iff \( \ell | (\mu_2 - \mu_1, \alpha_1) \) for all \( \alpha_1 \in \Lambda_R/\Lambda' \), i.e. \( \mu_2 - \mu_1 \in \Cent(\Lambda_R) \), and 0 otherwise. Hence, with \( C = |\Lambda_R/\Lambda'| \cdot \delta_{(\mu_2 - \mu_1) \in \Cent(\Lambda_R)} \), the sum (*) simplifies to

\[
C \cdot \sum_{\nu_1 + \nu_2 = \nu} q^{-\nu_1} (\mu_1, \nu_1) + (\mu_2, \nu_2) q^{-\nu_1} (\mu_1, \nu_1) f(\bar{\mu}_1, \bar{\nu}_1) f(\bar{\mu}_2, \bar{\nu}_2) \\
= C \cdot \sum_{\nu_1 + \nu_2 = \nu} q^{-\nu_1} (\mu_1, \nu_1) + (\mu_2, \nu_2) q^{-\nu_1} (\mu_1, \nu_1) f(\bar{\mu}_1, \bar{\nu}_1) f(\bar{\mu}_2, \bar{\nu}_2) \\
= C \cdot q^{-\nu_1} (\mu_1, \nu_1) q(\bar{\mu}_1, \bar{\nu}_1) f(\bar{\mu}_1, \bar{\nu}_1) q(\bar{\mu}_2, \bar{\nu}_2) f(\bar{\mu}_2, \bar{\nu}_2),
\]

Comparing this with the right hand side of the first equation of (2.4) gives

\[
C \cdot q^{-\nu_1} (\mu_1, \nu_1) q(\bar{\mu}_1, \bar{\nu}_1) f(\bar{\mu}_1, \bar{\nu}_1) q(\bar{\mu}_2, \bar{\nu}_2) f(\bar{\mu}_2, \bar{\nu}_2) = \delta_{\mu_1, \mu_2} q^{-\nu_1} (\mu_1, \nu_1) f(\bar{\mu}_1, \bar{\nu}_1),
\]

and with the definition of \( g(\bar{\mu}, \bar{\nu}) = |\Lambda_R/\Lambda'| q(\mu, \nu) f(\mu, \nu) \) we get the following equation

\[
\sum_{\nu_1 + \nu_2 = \nu} \delta_{\mu_2 - \mu_1 \in \Cent(\Lambda_R)} q(\bar{\mu}_1, \bar{\nu}_1) g(\bar{\mu}_1, \bar{\nu}_1) g(\bar{\mu}_2, \bar{\nu}_2) = \delta_{\mu_1, \mu_2} g(\bar{\mu}_1, \bar{\nu}_1).
\]

Analogously, we get the equation of the sum \( \sum_{\nu_1 + \nu_2 = \nu} \).
We now consider the equations (2.5). Again, \( \nu = \bar{\nu} + \alpha \) as above.

\[
\sum_{\nu \in \Lambda / \Lambda'} f(\mu, \nu) = \sum_{\nu} q^{-\mu, \nu} f(\bar{\mu}, \bar{\nu}) \\
= \sum_{\bar{\nu}} q^{\bar{\mu}, \bar{\nu}} f(\bar{\mu}, \bar{\nu}) \sum_{\alpha \in \Lambda R / \Lambda'} q^{-\mu, \alpha} \\
= \sum_{\bar{\nu}} q^{-\mu, \bar{\nu}} f(\bar{\mu}, \bar{\nu}) \sum_{\alpha \in \Lambda R / \Lambda'} q^{-\mu, \alpha} \\
= \delta_{(\mu \in \text{Cent}(\Lambda R))} |\Lambda R / \Lambda'| \sum_{\bar{\nu}} q^{-\mu, \bar{\nu}} f(\bar{\mu}, \bar{\nu}) \\
= \delta_{(\mu \in \text{Cent}(\Lambda R))} \sum_{\bar{\nu}} q^{-\mu, \bar{\nu}} g(\bar{\mu}, \bar{\nu}) \\
= \delta_{R, 0}.
\]

3. THE FIRST TYPE OF EQUATIONS

3.1. Equations of group-type.

**Definition 3.1.** For an abelian group \( G \) we define a set of equations for \( |G|^2 \) variables \( g(x, y) \), \( x, y \in G \), which we call group-equations.

\[
g(x, y) = \sum_{y_1 + y_2 = y} g(x, y_1) g(x, y_2), \quad (3.1)
g(x, y) = \sum_{x_1 + x_2 = x} g(x_1, y) g(x_2, y), \quad (3.2)
1 = \sum_{y \in G} g(0, y), \quad (3.3)
1 = \sum_{x \in G} g(x, 0). \quad (3.4)
\]

Thus, there are \( 2|G|^2 + 2 \) group-equations in \( |G|^2 \) variables with values in \( \mathbb{C} \).

These equations are the equations in Lemma 2.5 and the following Definition for central weight \( \zeta = 0 \).

**Theorem 3.2.** Let \( G \) be an abelian group of order \( N \), \( H_1, H_2 \) subgroups with \( |H_1| = |H_2| = d \). Let \( \omega : H_1 \times H_2 \to \mathbb{C}^\times \) be a pairing of groups. Here, the group \( G \) is written additively and \( \mathbb{C}^\times \) multiplicatively, thus we have \( \omega(x, y)^d = 1 \) for all \( x \in H_1, y \in H_2 \). Then the function

\[
g : G \times G \to \mathbb{C}, \quad (x, y) \mapsto \frac{1}{d} \omega(x, y) \delta_{(x \in H_1)} \delta_{(y \in H_2)} \quad (3.5)
\]

is a solution of the group-equations (3.1)-(3.4) of \( G \).
Proof. Let $G, H_1, H_2$ and $\omega$ be as in the theorem. We insert the function $g$ as in (3.5) in the group-equation (3.1) of $G$. Let $x, y \in G$.

$$
\sum_{y_1 + y_2 = y} g(x, y_1)g(x, y_2) = \left(\frac{1}{d}\right)^2 \sum_{y_1 + y_2 = y} \omega(x, y_1)\omega(x, y_2)\delta_{(x \in H_1)}\delta_{(y_1 \in H_2)}\delta_{(y_2 \in H_2)}
$$

$$
= \left(\frac{1}{d}\right)^2 \sum_{y_1 + y_2 = y} \omega(x, y_1 + y_2)\delta_{(x \in H_1)}\delta_{(y_1 \in H_2)}\delta_{(y_2 \in H_2)}
$$

$$
= \left(\frac{1}{d}\right)^2 \left|H_2\right| \omega(x, y)\delta_{(x \in H_1)}\delta_{(y \in H_2)}
$$

$$
g(x, y).
$$

Analogously for the sum in (3.2). We now insert the function $g$ in (3.3):

$$
\sum_{y \in G} g(0, y) = \frac{1}{d} \sum_{y \in G} \omega(0, y)\delta_{(y \in H_2)} = \frac{1}{d} \sum_{y \in H_2} 1 = 1.
$$

Question 3.3. Are these all solutions of the group-equations for a given group $G$?

3.2. Results for all fundamental groups of Lie algebras. We now treat the cases $G = \mathbb{Z}_N$ for $N \geq 1$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, since these are the only examples of fundamental groups $\pi_1$ of root systems.

**Theorem 3.4.** In the following cases, the functions $g$ of Theorem 3.3 are the only solutions of the group-equations (3.1)-(3.4) of $G$.

(a) For $G = \mathbb{Z}_N$, the cyclic groups of order $N$. Here, we get $\sum_d N d$ different solutions.

(b) For $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. Here we get 35 different solutions.

**Proof.** (a) This is the content of [LN14], Theorem 5.6.

(b) We have checked this explicitly via MAPLE.

**Example 3.5.** Let $G = \mathbb{Z}_N$, $N \geq 1$. For any divisor $d$ of $N$ there is a unique subgroup $H = \frac{N}{d}\mathbb{Z}_N \cong \mathbb{Z}_d$ of $G$ of order $d$. By Theorem 3.2, we have, that for any pairing $\omega: H \times H \to \mathbb{C}^\times$, the function $g$ as in (3.5) is a solution of the group-equations (3.1)-(3.4). We give the solution explicitly. For $H = \langle h \rangle$, $h \in \frac{N}{d}\mathbb{Z}_N$, we get a pairing $\omega: H \times H \to \mathbb{C}^\times$ by $\omega(h, h) = \xi$ with $\xi$ a $d$-th root of unity, not necessarily primitive. Thus, the function (3.5) translates to

$$
g: G \times G, \quad (x, y) \mapsto \frac{1}{d} \xi^{\frac{x y}{N/d^2}} \delta_{\langle x \rangle} \delta_{\langle y \rangle} \delta_{\langle y \rangle}.
$$

**Example 3.6.** Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b \rangle$. For $H_1 = H_2 = G$ there are $2^4 = 16$ possible pairings, since a pairing is given by determining the values of $\omega(x, y) = \pm 1$ for $x, y \in \{a, b\}$.

In $G$, there are 3 different subgroups of order 2, hence there are 9 possible pairs $(H_1, H_2)$ of groups $H_i$ of order 2. For each pair, there are two possible choices for $\omega(x, y) = \pm 1$, $x, y$ being the generators of $H_1$, resp. $H_2$. Thus, we get 18 pairings for subgroups of order $d = 2$. For $H_1 = H_2 = \{0\}$ there is only one pairing, mapping $(0, 0)$ to 1. Thus, we have 35 pairings in total.

| # | $H_i \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ | $H_1$ | $H_2$ | $\omega$ |
|---|---|---|---|---|
| 16 | $\langle a, b \rangle$ | $\langle a, b \rangle$ | $\omega(x, y) = \pm 1$ for $x, y \in \{a, b\}$ |
| 9×2 | $\mathbb{Z}_2$ | $\{a, b, a + b\}$ | $\{a, b, a + b\}$ | $\omega(x, y) = \pm 1$ |
| 1 | $\mathbb{Z}_1$ | $\{0\}$ | $\{0\}$ | $\omega(0, 0) = 1$ |
4. Quotient diamonds and the second type of equations

4.1. Quotient diamonds and equations of diamond-type.

**Definition 4.1.** Let $G$ and $A$ be abelian groups and $B, C, D$ subgroups of $A$, such that $D = B \cap C$. We call a tuple $(G, A, B, C, D, \varphi_1, \varphi_2)$ with injective group morphisms $\varphi_1: A/B \to G^\ast = \text{Hom}(G, \mathbb{C}^\times)$ and $\varphi_2: A/C \to G$ a diamond for $G$. We will visualize the situation with the following diagram:

\[
\begin{array}{ccc}
G^\ast & \leftarrow & A \\
\downarrow & & \downarrow \\
B & \rightarrow & C \\
\downarrow & & \downarrow \\
D & \rightarrow & \emptyset
\end{array}
\]

**Definition 4.2.** Let $(G, A, B, C, D, \varphi_1, \varphi_2)$ be a diamond for $G$. For $a \in A$ and not in $B \cap C$ we define the following equations for the $|G|^2$ variables $g(x, y)$, $x, y \in G$:

\[
\begin{align*}
0 &= \sum_{y_1+y_2=y, y_1, y_2 \in G} \varphi_1(a)(y_1)g(x, y_1)g(x + \varphi_2(a), y_2), \quad (4.1) \\
0 &= \sum_{x_1+x_2=x, x_1, x_2 \in G} \varphi_1(a)(x_1)g(x_1, y)g(x_2, y + \varphi_2(a)), \quad (4.2) \\
0 &= \sum_{y \in G} (\varphi_1(a)(y))^{-1}g(\varphi_2(a), y), \quad (4.3) \\
0 &= \sum_{x \in G} (\varphi_1(a)(x))^{-1}g(x, \varphi_2(a)). \quad (4.4)
\end{align*}
\]

We call this set of equations diamond-equations for the diamond of $G$. Here, $\varphi_i(a)$ denotes the image of $a + B$, resp. $a + C$, for $a \in A$ under $\varphi_1$, resp. $\varphi_2$.

These are up to $(|A| - 1)(2|G|^2 + 2)$ equations in $|G|^2$ variables with values in $\mathbb{C}$.

We show how these equations arise in the situation of Lemma 2.5.

**Lemma 4.3.** Let $G = \pi_1$, the fundamental group of a root system $\Phi$. Assume $\Lambda'$ is a sublattice of $\Lambda_R$, contained in $\text{Cent}(\Lambda_W)$. Let $A = \text{Cent}(\Lambda_R)/\Lambda'$, $B = \text{Cent}(\Lambda_W)/\Lambda'$, $C = \text{Cent}(\Lambda_R) \cap \Lambda_R/\Lambda'$ and $D = \text{Cent}(\Lambda_W) \cap \Lambda_R/\Lambda'$. Then there exist injections $\varphi_1: A/B \to \pi_1^+$ and $\varphi_2: A/C \to \pi_1$, such that $(G, A, B, C, D, \varphi_1, \varphi_2)$ is a diamond for $G$.

\[
\begin{array}{ccc}
\text{Cent}(\Lambda_R)/\Lambda' & \leftarrow & \pi_1^+ \\
\downarrow & \swarrow & \downarrow \\
\text{Cent}(\Lambda_W)/\Lambda' & \nearrow & \text{Cent}(\Lambda_R) \cap \Lambda_R/\Lambda' \\
\downarrow & \searrow & \downarrow \\
\text{Cent}(\Lambda_W) \cap \Lambda_R/\Lambda' & \rightarrow & \pi_1
\end{array}
\]

**Proof.** Recall from Lemmas 1.6 and 1.7 that we have $\text{Cent}(\Lambda_R) = \Lambda_{\cdot}^\llbracket R \rrbracket$ and $\text{Cent}(\Lambda_W) \cap \Lambda_R = \Lambda_{\cdot}^\llbracket W \rrbracket / (\Lambda_{\cdot}^\llbracket W \rrbracket \cap \Lambda_R)$ and $\Lambda_{\cdot}^\llbracket R \rrbracket \subset \Lambda_{\cdot}^\llbracket W \rrbracket$. We have $A/C \cong \Lambda_{\cdot}^\llbracket W \rrbracket / (\Lambda_{\cdot}^\llbracket W \rrbracket \cap \Lambda_R)$ and $\Lambda_{\cdot}^\llbracket R \rrbracket \subset \Lambda_{\cdot}^\llbracket W \rrbracket$. 

...
To show the existence of an injective morphism $\varphi_2 : A/C \to \pi_1$, we define $\tilde{\varphi}_2$ on $\Lambda^{[\ell]}_W$ and calculate the kernel. By Definition 1.4 the generators of $\Lambda^{[\ell]}_W$ are $\ell_{[i]}\lambda_i$ for all $i \in I$, with $\ell_{[i]} := \ell / \gcd(\ell, d_i)$. Thus

$$\tilde{\varphi}_2 : \Lambda^{[\ell]}_W \to \pi_1, \ \ell_{[i]}\lambda_i \mapsto \ell_{[i]}\lambda_i + \Lambda_R$$

gives a group morphism. Since $\Lambda' \subset \Lambda_R = \ker \tilde{\varphi}_2$, this induces a well-defined map $\varphi_2 : A/\Lambda' \to \pi_1$. Obviously, the kernel of this map is $\Lambda^{[\ell]}_W \cap \Lambda_R$, hence the desired injection $\varphi_2 : A/C \to \pi_1$ exists and is given by taking $\lambda + (\Lambda^{[\ell]}_W \cap \Lambda_R)$ modulo $\Lambda_R$, $\lambda \in \Lambda^{[\ell]}_W$.

Now, we show the existence of $\varphi_1$. The map

$$f : \text{Cent}(\Lambda_R) \to \text{Hom}(\Lambda_W, \mathbb{C}^\times), \ \lambda \mapsto (\Lambda_W \to \mathbb{C}^\times, \ \eta \mapsto q^{(\lambda,\eta)})$$

is a group morphism. We define $g : \text{Hom}(\Lambda_W, \mathbb{C}^\times) \to \text{Hom}(\Lambda_W/\Lambda_R, \mathbb{C}^\times)$ by $g(\psi) := \psi \circ p$, where $p$ is the natural projection $\Lambda_W \to \Lambda_W/\Lambda_R$. Thus, the upper right triangle of the following diagram commutes.

$$\begin{array}{ccc}
\text{Cent}(\Lambda_R) & \xrightarrow{f} & \text{Hom}(\Lambda_W, \mathbb{C}^\times) \\
\downarrow & & \downarrow \text{id} \\
\text{Cent}(\Lambda_R)/\text{Cent}(\Lambda_W) & \xrightarrow{g \circ f} & \text{Hom}(\pi_1, \mathbb{C}^\times)
\end{array}$$

It is $\lambda \in \ker g \circ f$, iff $q^{(\lambda,\tilde{\eta})} = 1$ for all $\tilde{\eta} \in \pi_1$. Since $\lambda \in \text{Cent}(\Lambda_R)$, this is equivalent to $q^{(\lambda,\eta)} = 1$ for all $\eta \in \Lambda_W$, hence $\lambda \in \text{Cent}(\Lambda_W)$. Thus, we get $\varphi_1$ as desired, which is well defined as map from $\text{Cent}(\Lambda_R) / \Lambda'/\text{Cent}(\Lambda_W) / \Lambda'$ since $\Lambda' \subset \text{Cent}(\Lambda_W) = \ker f$.

**Lemma 4.4.** Let $(G, A, B, C, D, \varphi_1, \varphi_2)$ be a diamond as in Lemma 4.3. If $\text{Cent}(\Lambda_W) \cap \Lambda_R/\Lambda' \neq 0$, then the diamond-equations (4.1)-(4.4) have no solutions, which are also solutions of the group-equations (3.1)-(3.4). Hence under our assumptions (!), the existence of an $R$-matrix requires necessarily the choice $\Lambda' = \text{Cent}(\Lambda_W) \cap \Lambda_R$.

**Proof.** If $\text{Cent}(\Lambda_W) \cap \Lambda_R/\Lambda' \neq 0$, then there exist a root $\zeta \in \text{Cent}(\Lambda_W)$, not contained in the kernel $\Lambda'$. Thus, there are diamond-equations with $\varphi_1(\zeta) = 1$ and $\varphi_2(\zeta) = 0$, i.e. the set of equations:

$$0 = \sum_{y_1 + y_2 = y} g(x, y_1)g(x, y_2), \quad (4.5)$$
$$0 = \sum_{x_1 + x_2 = x} g(x_1, y)g(x_2, y), \quad (4.6)$$
$$0 = \sum_{y \in G} g(0, y), \quad (4.7)$$
$$0 = \sum_{x \in G} g(x, 0). \quad (4.8)$$

Since this are group-equations as in Definition 3.1 but with left-hand side equal to 0, solutions of the group-equations does not solve the diamond-equations in this situation. 


Before examining in which case a solution of the group-equations as in Theorem 3.2 is also a solution of the diamond-equations (4.3), (4.4), we show that it is sufficient to check the diamond-equations (4.3) and (4.4).

**Lemma 4.5.** Let $G$ be an abelian group of order $N$, $H_1$, $H_2$ subgroups with $|H_1| = |H_2| = d$ and $\omega : H_1 \times H_2 \to \mathbb{C}^\times$ a group-pairing, such that $g : G \times G \to \mathbb{C}$, $(x, y) \mapsto 1/d \sum \omega(x, y)\delta(x \in H_1)\delta(y \in H_2)$ is a solution of the group-equations (3.1), (3.2). Then the following holds:

If $g$ is a solution of the diamond-equations (4.1), (4.2), then $g$ solves the diamond-equations (4.3), (4.4) as well.

**Proof.** Let $g$ be a solution of the group-equations as in Theorem 3.2. Assume that $g$ solves (4.1) and (4.2). Let $\varphi_1, \varphi_2$ as in Definition 4.1 and $0 \neq \zeta \in A$ a non-trivial central weight. Then, for $x, y \in G$ we get by inserting $g$ in (4.1)

$$0 = \sum_{y_1, y_2 = y} \varphi_1(\zeta)(y_1)g(x, y_1)g(x + \varphi_2(\zeta), y_2)$$

$$= \sum_{y_1, y_2 = y} \varphi_1(\zeta)(y_1)\frac{1}{d^2} \omega(x, y_1)\omega(x + \varphi_2(\zeta), y_2)\delta(x \in H_1)\delta(y_1 \in H_2)\delta(x + \varphi_2(\zeta) \in H_1)\delta(y_2 \in H_2)$$

$$= \delta(x \in H_1)\delta(y \in H_2)\delta(\varphi_2(\zeta) \in H_1)\frac{1}{d^2} \sum_{y_1, y_2 = y} \varphi_1(\zeta)(y_1)\omega(x, y_1)\omega(x, y_2)\omega(\varphi_2(\zeta), y_2)$$

$$= \delta(x \in H_1)\delta(y \in H_2)\delta(\varphi_2(\zeta) \in H_1)\frac{1}{d^2} \omega(x, y)\sum_{y_1, y_2 = y} \varphi_1(\zeta)(y_1)\omega(\varphi_2(\zeta), y_2)$$

$$= \delta(x \in H_1)\delta(y \in H_2)\delta(\varphi_2(\zeta) \in H_1)\frac{1}{d^2} \omega(x, y)\varphi_1(\zeta)(y)\sum_{y_1, y_2 = y} \varphi_1(\zeta)(y_2)^{-1}\omega(\varphi_2(\zeta), y_2).$$

In particular, this holds for $x = y = 0$, and in this case the expression vanishes iff

$$\delta(\varphi_2(\zeta) \in H_1)\frac{1}{d^2} \sum_{y_1, y_2 = y} \varphi_1(\zeta)(y)^{-1}\omega(\varphi_2(\zeta), y) = 0,$$

which is (4.3). Analogously, it follows that if $g$ solves (4.2) it solves (4.4). \qed

**4.2. Cyclic fundamental group** $G = \mathbb{Z}_N$. In the following, $G$ will always be a fundamental group of a simple complex Lie algebra, hence either cyclic or equal to $\mathbb{Z}_2 \times \mathbb{Z}_2$ for the case $D_n$, $n$ even. In this section, we will derive some results for the cyclic case.

In Example 3.5 we have given solutions of the group-equations for $G = \mathbb{Z}_N$, i.e. for all $d \mid N$ the functions

$$g: G \times G \to \mathbb{C}, \ (x, y) \mapsto \frac{1}{d} \xi \sum_{y_1, y_2 = y} \delta(x \in H_1)\delta(y_1 \in H_2)\delta(x + \varphi_2(\zeta) \in H_1)\delta(y_2 \in H_2)$$

with $\xi$ a $d$-th root of unity, not necessarily primitive. In the following, we denote by $\xi_d$ the primitive $d$-th root of unity $\exp(2\pi i/d)$.
Lemma 4.6. Let \( l \geq 2, m, n \in \mathbb{N} \) and \( G = \mathbb{Z}_N = \langle \lambda \rangle \). We consider the following diamonds \( (G, A, B, C, D, \varphi_1, \varphi_2) \) with \( A = \mathbb{Z}_N \) and injections \( \varphi_1 \) and \( \varphi_2 \) given by

\[
\varphi_1 : A \rightarrow G^*, \ a \mapsto (\xi_N^m(\cdot))^{-1}, \quad \text{with} \quad (\xi_N^m(\cdot)) : G \rightarrow \mathbb{C}^*, \ x \mapsto \xi_N^{mx},
\]

\[
\varphi_2 : A \rightarrow G^*, \ a \mapsto i\lambda,
\]

with primitive \( N \)-th root of unity \( \xi_N, B = \ker \varphi_1 \) and \( C = \ker \varphi_2 \) and \( D = \mathbb{Z}_1 \).

Possible solutions of the group-equations (3.1)-(3.4) are given for any choice of integers \( 1 \leq k \leq d \) and \( d \mid N \) as in Example 3.5 by

\[
g : G \times G \rightarrow \mathbb{C}, \ (x, y) \mapsto \frac{1}{d} \left( \xi_d^k \right)^{\frac{xy}{N/d^2}} \delta_{\frac{N}{d}|z|} \delta_{\frac{N}{d}|y|},
\]

(4.10)

with primitive \( d \)-th root of unity \( \xi_d = \exp(2\pi i/d) \). These are solutions also to the diamond-equations (4.1)-(4.4), iff \( N \mid m, l \) or the following condition hold:

\[
\gcd(N, dl, kl - \frac{N}{d} m) = 1.
\]

(4.11)

Proof. For \( N \mid m, l \) there is no non-trivial diamond-equation, hence all solutions of the group-equations as in Example 3.5 are possible. Assume now, that not both \( N \mid m \) and \( N \mid l \). We insert the function \( g \) from (4.10) in the diamond-equations (4.1)-(4.4) and get requirements for \( d, k, l \) and \( N \). By Lemma 4.5 it is sufficient to consider only equations (4.3) and (4.4). Since for cyclic \( G \) the function \( g \) is symmetric we choose equation (4.3) for the calculation. Let \( 1 \leq z < N, a \in A \) and \( y \in G \), then

\[
\sum_{y=1}^{N} (\varphi_1(za)(y))^{-1} g(\varphi_2(za), y) = \frac{1}{d} \sum_{y=1}^{N} \xi_N^{-nym} \left( \xi_d^k \right)^{\frac{y}{N/d^2}} \delta_{\frac{N}{d}|z|} \delta_{\frac{N}{d}|y|}
\]

\[
= \frac{1}{d} \sum_{y=1}^{N} \xi_d^{-nym} \left( \xi_d^k \right)^{\frac{y}{N/d^2}} \delta_{\frac{N}{d}|z|} \delta_{\frac{N}{d}|y|},
\]

\[
= \frac{1}{d} \sum_{y=1}^{d} \left( \xi_d^{-y} \right)^{ym} \left( \xi_d^k \right)^{\frac{y}{N/d^2}} \delta_{\frac{N}{d}|z|},
\]

with the substitution \( y' = y/(N/d) \). This sum equals 0 iff \( N/d \nmid zl \) or \( d \nmid z(\text{lcm}(d, l) - m) \). This is equivalent to \( N \nmid zdl \) or \( N \notmid z(\text{lcm}(d, l) - m) \), hence \( N \nmid \gcd(zdl, z(\text{lcm}(d, l) - m)) \). Since this condition has to be fulfilled for all \( z \) we get that \( N \nmid \gcd(\text{lcm}(d, l) - (\text{lcm}(d, l) - m)) \), hence \( \gcd(N, dl, kl - (\text{lcm}(d, l) - m)) = 1 \).

We spell out the condition for explicit values \( m \) and \( l \).

Example 4.7. Let \( G = \mathbb{Z}_N = \langle \lambda \rangle, \ l \geq 2, m, n \in \mathbb{N} \) and diamond \( (G, A, B, C, D, \varphi_1, \varphi_2) \) as in Lemma 4.6. Depending on \( m, l \) we get the following criteria for solutions of the diamond-equations. Here, we give \( \varphi_1 \) and \( \varphi_2 \) shortly by the generator of its image.

(I) If \( N \mid m \) and \( N \mid l \) we have the diamond \( (\mathbb{Z}_N, \mathbb{Z}_N, \mathbb{Z}_N, \mathbb{Z}_N, \mathbb{Z}_1, 1, 0) \) and all solutions of the form \( (4.10) \) are also solutions to the diamond-equations (4.1)-(4.4). (Since \( B, C = A \), there are no no-trivial diamond-equations.)

(II) If \( N \mid m \) and \( N \notmid l \) we have the diamond \( (\mathbb{Z}_N, \mathbb{Z}_N, \mathbb{Z}_N, \mathbb{Z}_{\gcd(l, N)}, \mathbb{Z}_1, 1, \lambda) \). In this case the function \( g \) as in (4.10) is a solution to the diamond-equations (4.1)-(4.4) if \( \gcd(N, dl, kl) = 1 \).
(III) If \( \gcd(m, N) = 1 \) and \( N \nmid l \) we have the diamond \((\mathbb{Z}_N, \mathbb{Z}_N, \mathbb{Z}_1, \mathbb{Z}_{\gcd(l, N)}, \mathbb{Z}_4, \xi_N, (\lambda)).\) In this case the function \( g \) as in (4.10) is a solution to the diamond-equations (4.1)-(4.4) if

\[
\gcd(N, dl, kl - \frac{N}{\ell} m) = 1.
\]

In most cases, \( N \) is prime or equals 1, hence we consider the two special cases

1. If \( d = 1, \) (4.12) simplifies to \( \gcd(N, l, l - Nm) = 1 \), which is equivalent to \( \gcd(N, l) = 1. \)
2. If \( d = N, \) (4.12) simplifies to \( \gcd(N, lN, kl - m) = 1 \), which is equivalent to \( \gcd(N, kl - m) = 1. \)

Finally, we consider the Lie algebras with cyclic fundamental group in question and determine the values \( m \) and \( l \) according to the Lie theoretic data and thereby the corresponding diamonds.

**Example 4.8.** Let \( G = \mathbb{Z}_N \) the fundamental group of a simple complex Lie algebra \( \mathfrak{g} \), generated by the fundamental dominant weight \( \lambda_n \). Let \( \ell \in \mathbb{N}, q = \exp(2\pi i/\ell), \) \( \ell_{[n]} = \ell / \gcd(\ell, d_n), m_{[n]} := N(\lambda_n, \lambda_n) / \gcd(\ell, d_n) \) and \((G, A, B, C, D, \varphi_1, \varphi_2)\) a diamond as in Lemma 4.3, such that the corresponding diamond-equations (4.1)-(4.4) have a solution that is also a solution to the group-equations (3.1)-(3.4). Then, the diamond is

\[
(G, \mathbb{Z}_N, \mathbb{Z}_{\gcd(m_{[n]}, N)}, \mathbb{Z}_{\gcd(\ell_{[n]}, N)}, \mathbb{Z}_1, \varphi_1, \varphi_2),
\]

with injections

\[
\varphi_1: A \to G^*, \ell_{[n]}\lambda_n \mapsto (\xi_N^{m_{[n]}})^{-1}, \quad \text{with} \quad (\xi_N^{m_{[n]}})^{-1} : G \to \mathbb{C}^\times, \quad x \mapsto \xi_N^{m_{[n]}x},
\]

\[
\varphi_2: A \to G, \quad \ell_{[n]}\lambda_n \mapsto \ell_{[n]}\lambda_n,
\]

with primitive \( N \)-th root of unity \( \xi_N = \exp(2\pi i/N). \) The group \( A = \text{Cent}(\Lambda_R)/\Lambda' = \Lambda_{W}/\Lambda_R' \) is generated by \( \ell_{[n]}\lambda_n \) and \( q(\ell_{[n]}\lambda_n, \lambda_n) = (\xi_N^{N}(\lambda_n, \lambda_n) / \gcd(\ell, d_n)) \). Since the order of \( \xi_N^{m_{[n]}} \) in \( \mathbb{C}^\times \) is \( N / \gcd(m_{[n]}, N) \) and the order of \( \ell_{[n]} \) in \( \mathbb{Z}_N \) is \( N / \gcd(\ell_{[n]}, N) \), the injections \( \varphi_1, \varphi_2 \) determine the diamond (4.13).

In the following table, we give the values \( \ell_{[n]} \) and \( m_{[n]} \) for all root systems of simple Lie algebras with cyclic fundamental group.

| \( \mathfrak{g} \) | \( A_{n\geq1} \) | \( B_{n\geq2} \) | \( C_{n\geq3} \) | \( D_{n\geq5} \) | \( E_6 \) | \( E_7 \) | \( E_8 \) | \( F_4 \) | \( G_2 \) |
|-----------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| \( \pi_1 \)     | \( \mathbb{Z}_{n+1} \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \) | \( \mathbb{Z}_4 \) | \( \mathbb{Z}_4 \) | \( \mathbb{Z}_4 \) | \( \mathbb{Z}_4 \) | \( \mathbb{Z}_4 \) | \( \mathbb{Z}_4 \) |
| \( N \)         | \( n + 1 \) | \( 2 \) | \( 2 \) | \( 4 \) | \( 3 \) | \( 2 \) | \( 1 \) | \( 1 \) | \( 1 \) |
| \( d_n \)       | \( 1 \) | \( 1 \) | \( 2 \) | \( 1 \) | \( 1 \) | \( 1 \) | \( 1 \) | \( 1 \) | \( 3 \) |
| \( \ell \)       | all | all | \( 2 \nmid \ell \) | \( 2 \nmid \ell \) | all | all | all | all | \( 3 \nmid \ell \) | \( 3 \nmid \ell \) |
| \( \gcd(\ell, d_n) \) | \( 1 \) | \( 1 \) | \( 2 \) | \( 1 \) | \( 1 \) | \( 1 \) | \( 1 \) | \( 1 \) | \( 3 \) |
| \( (\lambda_n, \lambda_n) \) | \( \frac{n_{[n+1]}}{n} \) | \( \frac{n}{2} \) | \( n \) | \( \frac{n}{2} \) | \( \frac{1}{3} \) | \( \frac{1}{3} \) | \( \frac{1}{3} \) | \( 2 \) | \( 1 \) | 6 |
| \( \ell_{[n]} \) | \( \ell \) | \( \ell \) | \( \ell/2 \) | \( \ell \) | \( \ell \) | \( \ell \) | \( \ell \) | \( \ell/3 \) | \( \ell \) |
| \( m_{[n]} \) | \( n \) | \( n \) | \( 2n \) | \( n \) | \( n \) | \( 4 \) | \( 3 \) | \( 2 \) | \( 1 \) | 6 |
| cases           | (III) | (I)-(III) | (II) | (I)-(III) | (III) | (II) | (I) | (I) | (I) | (I) |

In the last row we indicate which cases in Example 4.7 apply. This will guide the proof of Theorem 5.1. Note that case (II) only appears for \( B_n, n \) even and \( \ell \) odd, and for \( C_n, n \) even and \( \ell \equiv 2 \mod 4 \) or odd \( \ell \).
4.3. Example: $B_2$. For $\mathfrak{g}$ with root system $B_2$ we have $\pi_1 = \mathbb{Z}_2$. There is one long root, $\alpha_1$, and one short root, $\alpha_2$, hence $d_1 = 2$ and $d_2 = 1$. The symmetrized Cartan matrix $\tilde{C}$ is given below. The fundamental dominant weights $\lambda_1, \lambda_2$ are given as in [Hum72], Section 13.2. Here, $\lambda_1$ is a root and $\lambda_2$ is the generator of the fundamental group $\mathbb{Z}_2$. The matrix $id_W^R$ gives the coefficients of the fundamental dominant weights in the basis $\{\alpha_1, \alpha_2\}$.

$$\tilde{C} = \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}, \quad id_W^R = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

Thus, $(\lambda_2, \lambda_2) = 1$. The lattice diamonds, depending on $\ell$, are:

(i) For odd $\ell$ we have $A = \ell \Lambda_W$ and $C = D = \ell \Lambda_R$. Since $(\lambda_2, \lambda_2) = 1$, we have $B = \ell \Lambda_W$. (Since $(\lambda_n, \lambda_n) = n/2$, in the general case $B_n$, the group $\text{Cent}(\Lambda_W)$ depends on $n$: for even $n$ we have $B = \ell \Lambda_W$, and $B = \ell \Lambda_R$ for odd $n$.)

(ii) For even $\ell$ we have $A = C = B = \ell(\frac{1}{2} \lambda_1, \lambda_2)$ and $D = \ell(\frac{1}{2} \alpha_1, \alpha_2)$. (Again, $B$ depends on $n$, hence we have $B = (\frac{1}{2} \lambda_1, \ldots, \frac{1}{2} \lambda_{n-1}, \lambda_n)$ if $n$ is even and $B = (\frac{1}{2} \lambda_1, \ldots, \frac{1}{2} \lambda_{n-1}, 2 \lambda_n)$ if $n$ is odd.)

We calculate the quotient diamonds for kernel $\Lambda' = \Lambda_R^{[\ell]}$ since by the necessary criterion of Lemma 4.3 this is the only case where possible solutions exist. We calculate Lusztig’s kernel $2\Lambda_R^{[\ell]}$ as well and compare it with $\Lambda_R^{[\ell]}$. We then determine the solutions of the corresponding diamond-equations according to Example 4.7.

(i) For odd $\ell$ it is $\Lambda_R^{[\ell]} = \ell \Lambda_R = 2 \Lambda_R^{2\ell}$, $\ell[n] = \ell$ and $m[n] = n = 2 = N$. Thus, for $\Lambda' = \Lambda_R^{[\ell]}$ the quotient diamond is given by $(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_1, 1, \lambda_2)$. By Example 4.7 (II), one has to check for which $d, k$ it is $\text{gcd}(2d, d\ell, k\ell) = 1$. This gives the 2 solutions: $(d, k) = (1, 1)$ and $(d, k) = (2, 1)$.

(ii) For $\ell \equiv 2 \pmod{4}$ it is $\Lambda_R^{[\ell]} = \ell(\frac{1}{2} \alpha_1, \ldots, \frac{1}{2} \alpha_{n-1}, \alpha_n) \neq \ell \Lambda_R = 2 \Lambda_R^{[\ell]}$, $\ell[n] = \ell$ and $m[n] = N$ as above. Here, we have $\text{gcd}(\ell, N) = N = 2$, thus for $\Lambda' = \Lambda_R^{[\ell]}$ we get the quotient diamond $(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, 1, 0)$. Thus, all 3 solutions of the group-equations are solutions to the diamond-equations as well by 4.7 (I).

(ii.b) For $\ell \equiv 0 \pmod{4}$ it is $\Lambda_R^{[\ell]} = \ell(\frac{1}{2} \alpha_1, \ldots, \frac{1}{2} \alpha_{n-1}, \alpha_n) = 2 \Lambda_R^{[\ell]}$. Thus in this case the quotient diamond as in (ii.a) is the same for Lusztig’s kernel, namely $(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, 1, 0)$ and again all 3 solutions of the group-equations are solutions to the diamond-equations as well.

$$\begin{array}{c}
1 \leftrightarrow \mathbb{Z}_2 \\
\mathbb{Z}_2 \searrow \mathbb{Z}_1 \uparrow \lambda_2 \quad \mathbb{Z}_2 \searrow \mathbb{Z}_1 \uparrow 0 \rightarrow 0
\end{array}$$

quotient diamond in case (i) 

quotient diamond in cases (ii.a), (ii.b)

5. Solutions

Theorem 5.1. Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra with root lattice $\Lambda_R$, weight lattice $\Lambda_W$ and fundamental group $\pi_1 = \Lambda_W/\Lambda_R$. Let $q$ be an $\ell$-th root of unity,
$\ell \in \mathbb{N}$, $\ell > 2$. Then we have the following $R$-matrix of the form $R = R_0\hat{\Theta}$, with $\Theta$ as in Theorem [2.2]

$$R = \left( \frac{1}{|\Lambda/\Lambda'|} \sum_{(\mu,\nu) \in (\Lambda_1/\Lambda'_1 \times \Lambda_2/\Lambda'_2)} q^{-\langle \mu,\nu \rangle} \omega(\bar{\mu}, \bar{\nu}) K_{\mu} \otimes K_{\nu} \right) \cdot \Theta,$$

for the quantum group $u_\ell(\mathfrak{g}, \Lambda, \Lambda')$ with $\Lambda_1$ the preimage of a certain subgroup $H_i \subset \pi_1$ in $\Lambda_W$ ($i = 1, 2$), a certain group-pairing $\omega: H_1 \times H_2 \to \mathbb{C}^\times$ and $\Lambda' = \Lambda'_{R_0}$ as in Def. [1.4].

In the following table we list for all root systems the following data, depending on $\ell$: Possible choices of $H_1, H_2$ (in terms of fundamental weights $\lambda_i$), the group-pairing $\omega$, and the number of solutions $\#$. If the number has a superscript $\ast$, we obtain $R$-matrices for Lusztig’s original choice of $\Lambda'$. For $\mathfrak{g} = D_n, 2 \mid n$, with $\pi_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$ we get the only cases $H_1 \neq H_2$ and denote by $\lambda \neq \lambda' \in \{\lambda_{n-1}, \lambda_n, \lambda_{n-1} + \lambda_n\}$ arbitrary elements of order 2 in $\pi_1$.

| $\mathfrak{g}$ | $\ell$ | $\#$ | $H_i \cong$ | $H_i (i=1,2)$ | $\omega$ |
|----------------|--------|-----|-------------|--------------|--------|
| $A_{n \geq 1}$ | $\ell$ odd | $Z_d$ | $\omega(\lambda, \lambda) = \xi_d^k$, if $d \mid (n + 1)$, $1 \leq k \leq d$ and $\gcd(n + 1, d\ell, k\ell - \frac{n+1}{d}n) = 1$ |
| $\pi_1 = Z_{n+1}$ | $\ell$ even | * | | | |
| $B_{n \geq 2}$ | $\ell$ odd | $Z_1$ | $\{0\}$ | $\omega(0, 0) = 1$ |
| $\pi_1 = Z_2$ | | 1 | $Z_2$ | $\langle \lambda_n \rangle$ | $\omega(\lambda_n, \lambda_n) = (1)^{n-1}$ |
| | | 1 | $Z_1$ | $\{0\}$ | $\omega(0, 0) = 1$, if $n$ even |
| | $\ell \equiv 2 \mod 4$ | 2 | $Z_2$ | $\langle \lambda_n \rangle$ | $\omega(\lambda_n, \lambda_n) = \pm 1$ |
| | | 1 | $Z_1$ | $\{0\}$ | $\omega(0, 0) = 1$, if $n$ even |
| | $\ell \not\equiv 4$ | 1* | $Z_1$ | $\{0\}$ | $\omega(0, 0) = 1$, if $n$ even |
| | $\ell \equiv 2 \mod 4$ | 1 | $Z_1$ | $\{0\}$ | $\omega(0, 0) = 1$ |
| | | 1 | $Z_2$ | $\langle \lambda_n \rangle$ | $\omega(\lambda_n, \lambda_n) = -1$ |
| | | 1 | $Z_1$ | $\{0\}$ | $\omega(0, 0) = 1$ |
| | $\ell \not\equiv 4$ | 2* | $Z_2$ | $\langle \lambda_n \rangle$ | $\omega(\lambda_n, \lambda_n) = \pm 1$ |
| | | 1* | $Z_1$ | $\{0\}$ | $\omega(0, 0) = 1$, if $n$ even |
| $C_{n \geq 3}$ | $\ell$ odd | $Z_2$ | $\langle \lambda \rangle$ | $\omega(\lambda, \lambda) = -1$ |
| $\pi_1 = Z_2$ | | 3 | $Z_2$ | $\langle \lambda \rangle$ | $\omega(\lambda, \lambda) = -1$ |
| | $\ell \equiv 2 \mod 4$ | 6 | $Z_2 \neq Z_2'$ | $\langle \lambda, \lambda' \rangle$ | $\omega(\lambda, \lambda') = 1$ |
| | $\ell \not\equiv 4$ | 1 | $Z_2 \times Z_2$ | $\langle \lambda_{n-1}, \lambda_n \rangle$ | $\omega(\lambda_{i}, \lambda_j) = 1$ |
| | | 1 | $Z_2 \times Z_2$ | $\langle \lambda_{n-1}, \lambda_n \rangle$ | $\omega(\lambda_{i}, \lambda_j) = -1$ |
| $D_{n \geq 4}$ | $\ell$ odd | $Z_2 \times Z_2$ | $\omega(\lambda_{n-1}, \lambda_{n-1}) = \pm 1$ |
| $n$ even | | 2 | $Z_2 \times Z_2$ | $\langle \lambda_{n-1}, \lambda_n \rangle$ | $\omega(\lambda_{n-1}, \lambda_{n-1}) = 1$ |
| | | | $\omega(\lambda_{n-1}, \lambda_{n-1}) = 1$ |
| | $\pi_1 = Z_2 \times Z_2$ | | 2 | $Z_2 \times Z_2$ | $\langle \lambda_{n-1}, \lambda_n \rangle$ | $\omega(\lambda_{n-1}, \lambda_{n-1}) = -1$ |
| | | | $\omega(\lambda_{n-1}, \lambda_{n-1}) = -1$ |
| | | | $\omega(\lambda_{n-1}, \lambda_{n-1}) = \mp 1$ |
| | | | $\omega(\lambda_{n-1}, \lambda_{n-1}) = \pm 1$ |
| | | | $\omega(\lambda_{n-1}, \lambda_{n-1}) = 1$ |
| | | | $\omega(\lambda_{n-1}, \lambda_{n-1}) = -1$ |
Table 2: Solutions for $R_0$-matrices

| $D_{n>5}$, $n$ odd, $\pi_1 = Z_4$ | $\ell$ even | 16 | $Z_2 \times Z_2$ | $\langle \lambda_{n-1}, \lambda_n \rangle$ | $\omega(\lambda_i, \lambda_j) \in \{\pm 1\}$ |
| $\ell$ odd | 1 | $Z_1$ | $\{0\}$ | $\omega(0, 0) = 1$ |
| | 1 | $Z_2$ | $\langle 2\lambda_n \rangle$ | $\omega(2\lambda_n, 2\lambda_n) = -1$ |
| | 2 | $Z_4$ | $\langle \lambda_n \rangle$ | $\omega(\lambda_n, \lambda_n) = \pm 1$ |
| $\ell \equiv 2 \mod 4$ | 4 | $Z_4$ | $\langle \lambda_n \rangle$ | $\omega(\lambda_n, \lambda_n) = c, c^4 = 1$ |
| $\ell \equiv 0 \mod 4$ | 4 | $Z_4$ | $\langle \lambda_n \rangle$ | $\omega(\lambda_n, \lambda_n) = c, c^4 = 1$ |

| $E_6$, $\pi_1 = Z_3$ | $\ell$ odd, $3 \nmid \ell$ | 1 | $Z_1$ | $\{0\}$ | $\omega(0, 0) = 1$ |
| | 2 | $Z_3$ | $\langle \lambda_6 \rangle$ | $\omega(\lambda_6, \lambda_6) = 1, \exp(\frac{2\pi i}{3})$ |
| $\ell$ even, $3 \nmid \ell$ | 1* | $Z_3$ | $\{0\}$ | $\omega(0, 0) = 1$ |
| | 2* | $Z_3$ | $\langle \lambda_6 \rangle$ | $\omega(\lambda_6, \lambda_6) = 1, \exp(2\frac{2\pi i}{3})$ |
| $\ell$ odd, $3 \mid \ell$ | 3 | $Z_3$ | $\langle \lambda_6 \rangle$ | $\omega(\lambda_6, \lambda_6) = c, c^4 = 1$ |
| $\ell$ even, $3 \mid \ell$ | 3* | $Z_3$ | $\langle \lambda_6 \rangle$ | $\omega(\lambda_6, \lambda_6) = c, c^4 = 1$ |

| $E_7$, $\pi_1 = Z_2$ | $\ell$ odd | 1 | $Z_1$ | $\{0\}$ | $\omega(0, 0) = 1$ |
| | 1 | $Z_2$ | $\langle \lambda_7 \rangle$ | $\omega(\lambda_7, \lambda_7) = 1$ |
| $\ell$ even | 2 | $Z_2$ | $\langle \lambda_7 \rangle$ | $\omega(\lambda_7, \lambda_7) = \pm 1$ |

| $E_8$, $\pi_1 = Z_1$ | $\ell$ odd | 1 | $Z_1$ | $\{0\}$ | $\omega(0, 0) = 1$ |
| | 1 | $Z_2$ | $\langle \lambda_7 \rangle$ | $\omega(\lambda_7, \lambda_7) = 1$ |
| $\ell$ even | 1* | $Z_1$ | $\{0\}$ | $\omega(0, 0) = 1$ |

| $F_4$, $\pi_1 = Z_1$ | $\ell$ odd | 1 | $Z_1$ | $\{0\}$ | $\omega(0, 0) = 1$ |
| | 1 | $Z_1$ | $\{0\}$ | $\omega(0, 0) = 1$ |
| $\ell \equiv 2 \mod 4$ | 1 | $Z_1$ | $\{0\}$ | $\omega(0, 0) = 1$ |
| $\ell \neq 4$ | 1* | $Z_1$ | $\{0\}$ | $\omega(0, 0) = 1$ |

| $G_2$, $\pi_1 = Z_1$ | $\ell$ odd | 1 | $Z_1$ | $\{0\}$ | $\omega(0, 0) = 1$ |
| | 1 | $Z_1$ | $\{0\}$ | $\omega(0, 0) = 1$ |
| $\ell$ even | 1* | $Z_1$ | $\{0\}$ | $\omega(0, 0) = 1$ |

Proof. We treat the root systems case by case and determine the solutions of the form

$$g: G \times G \to \mathbb{C}, \ (x, y) \mapsto \frac{1}{d} \omega(x, y) \delta_{x \in H_1} \delta_{y \in H_2}$$

with subgroups $H_1, H_2$ of $G = \pi_1$ as in Theorem 3.2.

For this, we first determine the lattices $A = \text{Cent}(\Lambda_R) = \Lambda^{[\ell]}_W$, $B = \text{Cent}(\Lambda_W)$, $C = \text{Cent}(\Lambda_R) \cap \Lambda_R$, $D = \text{Cent}(\Lambda_W) \cap \Lambda_R = \Lambda_R^{[\ell]}$, depending on $\ell$. For the Lie algebras with cyclic fundamental group (all but for root system $D_n$ with even $n$), we then determine the values $m_{[\ell]}$ and $\ell_{[\ell]}$, depending on $\ell$, $n$, and the order of $\pi_1$, and thereby the quotient diamonds and which solutions of the group equations are solutions to the corresponding diamond equations. In these cases, the $\omega$-part of the solutions to the group equations are of the form

$$\omega: H \times H \to \mathbb{C}^\times, \ (x, y) \mapsto \left(\frac{x^k}{y^d} \right)^{\frac{m_{[\ell]} x^k y^d}{\ell_{[\ell]}}}$$

for subgroup $H = \frac{N}{d} \mathbb{Z}_N$ of $\pi_1$ of order $d$. We give the solutions by pairs $(d, k)$, which we determine by applying Lemma 4.6 and Example 4.7. An overview of the possible cases gives Example 4.8.

For $D_n$ with even $n$ and fundamental group $\mathbb{Z}_2 \times \mathbb{Z}_2$ we also determine all quotient diamonds.
(depending on \( \ell \)) and check which solutions of the group equations are solution to the diamond equations in a rather case by case calculation.

(1) For \( \mathfrak{g} \) with root system \( A_n, n \geq 1 \), we have \( \pi_1 = \mathbb{Z}_{n+1} \) for all \( n \). The roots are \( \alpha_1, \ldots, \alpha_n \) and \( d_i = 2 \) for \( 1 \leq i \leq n \). The symmetrized Cartan matrix \( \tilde{C} \) is given below. The fundamental dominant weights \( \lambda_i \) are given as in [Hum72], Section 13.2, and \( \lambda_n \) is the generator of the fundamental group \( \mathbb{Z}_{n+1} \). The matrix \( \text{id}^R_W \) gives the coefficients of the fundamental dominant weights in the basis \( \{ \alpha_1, \ldots, \alpha_n \} \).

\[
\tilde{C} = \begin{pmatrix}
2 & -1 & 0 & \ldots & 0 \\
-1 & 2 & -1 & 0 & \ldots & 0 \\
0 & -1 & 2 & -1 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \ldots & -1 & 2
\end{pmatrix}
\]

\( \text{id}^R_W = a_{ij} \) with \( a_{ij} = \begin{cases} 
\frac{1}{n+1} i(n-j+1), & \text{if } i \leq j, \\
\frac{1}{n+1} j(n-i+1), & \text{if } i > j.
\end{cases} \)

The lattice diamonds, depending on \( \ell \), are:

(i) For even \( \ell \) we have \( A = \ell \Lambda_W, B = D = \ell \Lambda_R \) and \( C = \ell / \text{gcd}(n+1, \ell) \Lambda_W \).

(ii) For odd \( \ell \), the same lattices as in (i).

We calculate the quotient diamonds for kernel \( \Lambda' = \Lambda_R^{[\ell]} \) and compare it with Lusztig’s kernel \( 2 \Lambda_R^{(\ell)} \). We then determine the solutions of the corresponding diamond-equations according to Example 4.7.

(i) For odd \( \ell \) it is \( \Lambda_R^{[\ell]} = \ell \Lambda_R \neq 2 \ell \Lambda_R = 2 \Lambda_R^{(\ell)}, \ell_{[n]} = \ell \) and \( m_{[n]} = n \). Thus, the quotient diamond is given by \( (\mathbb{Z}_{n+1}, \mathbb{Z}_{n+1}, \mathbb{Z}_{\text{gcd}(\ell, n+1)}, \mathbb{Z}_{\ell}, \mathbb{Z}_{n+1}, \ell \lambda_n) \), hence we are in case (III) of Example 4.7. We get solutions \( (d, k) \) iff \( \text{gcd}(n+1, d \ell, k \ell - \frac{n-1}{d} n) = 1 \).

(ii) For odd \( \ell \) it is \( \Lambda_R^{[\ell]} = \ell \Lambda_R = 2 \Lambda_R^{(\ell)}, \ell_{[n]} = \ell \) and \( m_{[n]} = n \). Thus, the quotient diamonds and solutions are as in (i).

(2) For \( \mathfrak{g} \) with root system \( B_n, n \geq 2 \), we have \( \pi_1 = \mathbb{Z}_2 \) for all \( n \). The long roots are \( \alpha_1, \ldots, \alpha_{n-1} \) and the short root \( \alpha_n \), hence \( d_i = 2 \) for \( 1 \leq i \leq n-1 \) and \( d_n = 1 \). The symmetrized Cartan matrix \( \tilde{C} \) is given below. The fundamental dominant weights \( \lambda_i \) are given as in [Hum72], Section 13.2. Here, \( \lambda_1, \ldots, \lambda_{n-1} \) are roots and \( \lambda_n \) is the generator of the fundamental group \( \mathbb{Z}_2 \). The matrix \( \text{id}^R_W \) gives the coefficients of the fundamental dominant weights in the basis \( \{ \alpha_1, \ldots, \alpha_n \} \).

\[
\tilde{C} = \begin{pmatrix}
4 & -2 & 0 & 0 & \ldots & 0 \\
-2 & 4 & -2 & 0 & \ldots & 0 \\
0 & \ldots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & -2 & 4 & -2 \\
0 & 0 & 0 & \ldots & 0 & -2 & 2
\end{pmatrix}
\]

\[
\text{id}^R_W = \begin{pmatrix}
1 & 1 & \ldots & 1 & \frac{1}{2} \\
1 & 2 & \ldots & 2 & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
1 & 2 & 3 & \ldots & n-1 & \frac{n-1}{2} \\
1 & 2 & 3 & \ldots & n-1 & \frac{n-1}{2}
\end{pmatrix}
\]

The lattice diamonds, depending on \( \ell \), are:

(i) For odd \( \ell \) we have \( A = \ell \Lambda_W \) and \( C = D = \ell \Lambda_R \). Since \( (\lambda_n, \lambda_n) = n/2 \), the group \( \text{Cent}(\Lambda_W) \) depends on \( n \). It is \( B = \ell \Lambda_W \) for even \( n \) and \( B = \ell \Lambda_R \) for odd \( n \).
(ii) For even \( \ell \) we have \( A = C = \ell \left( \frac{1}{2} \lambda_1, \ldots, \frac{1}{2} \lambda_{n-1}, \lambda_n \right) \) and \( D = \ell \left( \frac{1}{2} \alpha_1, \ldots, \frac{1}{2} \alpha_{n-1}, \alpha_n \right) \).

Again, \( B \) depends on \( n \), and we have \( B = \ell \left( \frac{1}{2} \lambda_1, \ldots, \frac{1}{2} \lambda_{n-1}, \lambda_n \right) \) for even \( n \) and \( B = \ell \left( \frac{1}{2} \lambda_1, \ldots, \frac{1}{2} \lambda_{n-1}, 2 \lambda_n \right) \) for odd \( n \).

We calculate the quotient diamonds for kernel \( \Lambda' = \Lambda_{\ell}^{(\ell)} \) and compare it with Lusztig’s kernel \( 2 \Lambda_{\ell}^{(\ell)} \). We then determine the solutions of the corresponding diamond-equations according to Example 4.7.

(i) For odd \( \ell \) it is \( \Lambda_{\ell}^{(\ell)} = \ell \Lambda_R = 2 \ell \Lambda_{\ell}^{(\ell)} \), \( \ell \equiv \ell [n] = \ell \) and \( m_{[n]} = n \). In this case, the quotient diamond is given by either \((\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, Z_1, 1, \lambda_n)\) for even \( n \), or by \((\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, Z_1, -1, \lambda_n)\) for odd \( n \). Thus we are either in case (II), or in case (III) of Example 4.7. In the first case (even \( n \)) we get solutions by \((d, k) = (1, 1)\) and \((2, 1)\). For odd \( n \) we get solutions \((d, k) = (1, 1)\) and \((2, 2)\).

(ii.a) For \( \ell \equiv 2 \mod 4 \) it is \( \Lambda_{\ell}^{(\ell)} = \ell \left( \frac{1}{2} \alpha_1, \ldots, \frac{1}{2} \alpha_{n-1}, \alpha_n \right) \neq \ell \Lambda_R = 2 \ell \Lambda_{\ell}^{(\ell)} \), \( \ell [n] = \ell \) and \( m_{[n]} = n \). The quotient diamond is given by either \((\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, 1, 0)\) for even \( n \), or by \((\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_1, -1, 0)\) for odd \( n \). Thus we are either in case (I) or in case (III) of Example 4.7. In the first case (even \( n \)) we get all possible 3 solutions \((d, k) = (1, 1), (2, 1)\) and \((2, 1)\). For odd \( n \) we get solutions \((d, k) = (2, 1)\) and \((2, 2)\).

(ii.b) For \( \ell \equiv 0 \mod 4 \) it is \( \Lambda_{\ell}^{(\ell)} = \ell \left( \frac{1}{2} \alpha_1, \ldots, \frac{1}{2} \alpha_{n-1}, \alpha_n \right) = 2 \Lambda_{\ell}^{(\ell)} \), \( \ell [n] = \ell \) and \( m_{[n]} = n \). Thus the quotient diamonds and solutions are as in (ii).

(3) For \( g \) with root system \( \mathfrak{C}_n \), \( n \geq 3 \), we have \( \pi_1 = \mathbb{Z}_2 \) for all \( n \). The short roots are \( \alpha_1, \ldots, \alpha_{n-1} \) and the long root \( \alpha_n \), hence \( d_i = 1 \) for \( 1 \leq i \leq n-1 \) and \( d_n = 2 \). The symmetrized Cartan matrix \( \tilde{C} \) is given below. The fundamental dominant weights \( \lambda_i \) are given as in [Hum2]. Section 13.2, and \( \lambda_n \) is the generator of the fundamental group \( \mathbb{Z}_2 \). The matrix \( id_{W}^{R} \) gives the coefficients of the fundamental dominant weights in the basis \{\( \alpha_1, \ldots, \alpha_n \)\}.

\[
\begin{pmatrix}
2 & -1 & 0 & 0 & . & 0 \\
-1 & 2 & -1 & 0 & . & 0 \\
. & . & . & . & . & . \\
0 & 0 & 0 & -1 & 2 & -2 \\
0 & 0 & 0 & . & 0 & -2 & 4
\end{pmatrix}
= id_{W}^{R} =
\begin{pmatrix}
1 & 1 & . & 1 & 1 \\
1 & 2 & . & 2 & 2 \\
. & . & . & . & . \\
1 & 2 & . & n-1 & n-1 \\
\frac{1}{2} & 1 & . & \frac{n-1}{2} & \frac{n}{2}
\end{pmatrix}
\]

The lattice diamonds, depending on \( \ell \), are:

(i) For odd \( \ell \) we have \( A = B = \ell \Lambda_W \) and \( C = D = \ell \Lambda_R \).

(ii) For \( \ell \equiv 2 \mod 4 \) we have \( A = \ell \left( \alpha_1, \ldots, \alpha_{n-1}, \frac{1}{2} \lambda_n \right) \) and \( C = D = \ell \Lambda_W \). Since \( \left( \lambda_n, \lambda_n \right) = n \), \( B = \text{Cent}(\Lambda_W) \) depends on \( n \). For odd \( n \) it equals \( \ell \Lambda_W \) and for even \( n \) it is equal to \( A \).

(iii) For \( \ell \equiv 0 \mod 4 \) we have \( A = C = \ell \left( \alpha_1, \ldots, \alpha_{n-1}, \frac{1}{2} \lambda_n \right) \) and \( D = \ell \Lambda_W \). Here again, \( B = \text{Cent}(\Lambda_W) \) depends on \( n \). For odd \( n \) it equals \( \ell \Lambda_W \) and for even \( n \) it is equal to \( A \).

We calculate the quotient diamonds for kernel \( \Lambda' = \Lambda_{\ell}^{(\ell)} \) and compare it with Lusztig’s kernel \( 2 \Lambda_{\ell}^{(\ell)} \). We then determine the solutions of the corresponding diamond-equations according to Example 4.7.

(i) For odd \( \ell \) it is \( \Lambda_{\ell}^{(\ell)} = \ell \Lambda_R \neq 2 \ell \Lambda_R = 2 \Lambda_{\ell}^{(\ell)} \), \( \ell [n] = \ell \) and \( m_{[n]} = 2n \). In this case, the quotient diamond is given by \((\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, Z_1, 1, \lambda_n)\). Thus we are in case (II) of Example 4.7, hence the 2 solutions are given by \((d, k) = (1, 1)\) and \((2, 1)\).
(ii) For \( \ell \equiv 2 \mod 4 \) it is \( \Lambda_R^{(\ell)} = \ell \langle \alpha_1, \ldots, \alpha_{n-1}, \frac{1}{2} \alpha_n \rangle \neq \ell \Lambda_R = 2 \Lambda_R^{(\ell)} \), \( \ell \) is even, \( m_{[n]} = n \). The quotient diamond is given by either \((\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_1, 1, \lambda_n)\) for even \( n \), or by \((\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_1, -1, \lambda_n)\) for odd \( n \). Thus we are in either in case (II) or in case (III) of Example 4.7. In the first case (even \( n \)) we get solutions \((d,k) = (1,1)\) and \((2,1)\). For odd \( n \) we get solutions \((d,k) = (1,1)\) and \((2,2)\).

(ii) For \( \ell \equiv 0 \mod 4 \) it is \( \Lambda_R^{(\ell)} = \ell \langle \frac{1}{2} \alpha_1, \ldots, \frac{1}{2} \alpha_{n-1}, \alpha_n \rangle = 2 \Lambda_R^{(\ell)} \), \( \ell \) is even, \( m_{[n]} = n \). The quotient diamond is given by either \((\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, 1, 0)\) for even \( n \), or by \((\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_1, -1, 0)\) for odd \( n \). Thus we are in either in case (I) or in case (III) of Example 4.7. In the first case (even \( n \)) we get all 3 possible solutions \((d,k) = (1,1), (2,1)\) and \((2,2)\). For odd \( n \) we get solutions \((d,k) = (2,1)\) and \((2,2)\).

(4) For \( g \) with root system \( D_n \), \( n \geq 4 \) even, we have \( \pi_1 = \mathbb{Z}_2 \times \mathbb{Z}_2 \) for all \( n \). The roots are \( \alpha_1, \ldots, \alpha_n \) and \( d_i = 1 \) for \( 1 \leq i \leq n \). The symmetrized Cartan matrix \( \tilde{C} \) is given below. The fundamental dominant weights \( \lambda_i \) are given as in [Hum72], Section 13.2, and \( \lambda_{n-1}, \lambda_n \) are the generators of the fundamental group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( \lambda_{n-1} + \lambda_n \) is the other element of order 2. The matrix \( id_W^R \) gives the coefficients of the fundamental dominant weights in the basis \{\( \alpha_1, \ldots, \alpha_n \)\}, and since \( d_i = 1 \) for all \( i \), also the values \( (\lambda_i, \lambda_j) \) for \( 1 \leq i, j \leq n \).

\[
\tilde{C} = \begin{pmatrix}
2 & -1 & 0 & 0 & \cdots & 0 \\
-1 & 2 & -1 & 0 & \cdots & \cdots \\
& & & & & \cdots \\
& & & & & \cdots \\
& & & & & \cdots \\
0 & 0 & 0 & 0 & -1 & 2 \\
0 & 0 & 0 & 0 & -1 & 0 \\
\end{pmatrix}
\]

\[
id_W^R = \begin{pmatrix}
1 & 1 & 1 & 1 & \frac{1}{2} & \frac{1}{2} \\
1 & 1 & 2 & 2 & 1 & 1 \\
1 & 2 & 3 & 3 & \frac{3}{2} & \frac{3}{2} \\
1 & 2 & 3 & n - 2 & \frac{n-2}{2} & \frac{n-2}{2} \\
\frac{1}{2} & 1 & \frac{3}{2} & \frac{n-2}{4} & \frac{n-2}{4} & \frac{n-2}{4} \\
\frac{1}{2} & 1 & \frac{3}{2} & \frac{n-2}{4} & \frac{n-2}{4} & \frac{n-2}{4} \\
\end{pmatrix}
\]

The lattice diamonds, depending on \( \ell \), are:

(i) For odd \( \ell \) we have \( A = \ell \Lambda_W \) and \( B = C = D = \ell \Lambda_R \).

(ii) For even \( \ell \) we have \( A = C = \ell \Lambda_W \) and \( B = D = \ell \Lambda_R \).

We calculate the quotient diamonds for kernel \( \Lambda' = \Lambda_R^{(\ell)} \) and compare it with Lusztig’s kernel \( 2 \Lambda_R^{(\ell)} \). We then determine the solutions of the corresponding diamond-equations by a case by case calculation.

(i) For odd \( \ell \) it is \( \Lambda_R^{(\ell)} = \ell \Lambda_R \neq 2 \ell \Lambda_R = 2 \Lambda_R^{(\ell)} \). Thus, the quotient diamond is given by \((\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_1, \varphi_1, \varphi_2)\) with injections

\[
\varphi_1: \ell \langle \lambda_{n-1}, \lambda_n \rangle \rightarrow \pi_1, \ell \lambda_{n-1} \mapsto q^{\ell \langle \lambda_{n-1}, - \rangle}, \ell \lambda_n \mapsto q^{\ell \langle \lambda_n, - \rangle},
\]

\[
\varphi_2: \ell \langle \lambda_{n-1}, \lambda_n \rangle \rightarrow \pi_1, \ell \lambda_{n-1} \mapsto \lambda_{n-1}, \ell \lambda_n \mapsto \lambda_n.
\]

In the following, we will write \( a := \lambda_{n-1}, b := \lambda_n \) and \( c := \lambda_{n-1} + \lambda_n \) for the 3 elements of order 2 of \( \pi_1 \). Since \( (\lambda_j, \lambda_j) = n/4 \) for \( j \in \{n - 1, n\} \), and \( (\lambda_i, \lambda_j) = (n - 2)/4 \) for \( i \neq j, i,j \in \{n - 1, n\} \) we get
Consider the diamond equations (4.3) and (4.4) by Lemma 4.5. We get the following system of equations for all possible 9 solutions with \( \omega \) we get 4 symmetric solutions and 2 non-symmetric solutions, which are given by their values (\( \omega \)). Since it suffices to consider the diamond equations (4.3) and (4.4) by Lemma 4.5, we check which function \( g : G \times G \to \mathbb{C}, \ (x, y) \mapsto \frac{1}{d} \omega(x, y) \delta_{x \in H_1} \delta_{y \in H_2} \) with subgroups \( H_i \) of \( G = \mathbb{Z}_2 \times \mathbb{Z}_2 \) of order \( d \) and a pairing \( \omega \) as in Example 3.6 is a solution to these equations. We get the following system of equations for \( g \):

\[
\begin{align*}
1 &= g(0, 0) + g(a, 0) + g(b, 0) + g(c, 0) \\
1 &= g(0, 0) + g(0, a) + g(0, b) + g(0, c) \\
0 &= g(0, a) \pm g(a, a) \mp g(b, a) - g(c, a) \\
0 &= g(a, 0) \pm g(a, a) \mp g(a, b) - g(a, c) \\
0 &= g(0, b) \mp g(a, b) \pm g(b, b) - g(c, b) \\
0 &= g(b, 0) \mp g(a, b) \pm g(b, b) - g(b, c) \\
0 &= g(b, 0) \mp g(a, b) \pm g(b, b) - g(b, c) \\
0 &= g(0, c) - g(a, c) - g(b, c) + g(c, c) \\
0 &= g(c, 0) - g(c, a) - g(c, b) + g(c, c)
\end{align*}
\]

(5.1)

where the \( \pm, \mp \) possibilities depend on whether \( \ell \equiv 0 \) or \( 2 \mod 4 \). It is easy to see that the trivial solution on \( H_1 = H_2 = \mathbb{Z}_1 \) is a solution. For \( H_i \cong \mathbb{Z}_2 \) the solution has one of the following two structures. For symmetric solutions \( H_1 = H_2 = \langle \lambda \rangle \) we get \( \omega(\lambda, \lambda) = -1 \). If \( H_1 = \langle \lambda \rangle \neq \langle \lambda' \rangle = H_2 \) we get \( \omega(\lambda, \lambda') = 1 \). This gives all possible 9 solutions with \( H_i \cong \mathbb{Z}_2 \), and we check which functions on \( G = \mathbb{Z}_2 \times \mathbb{Z}_2 \) are solutions to the diamond equations. We get 4 symmetric solutions and 2 non-symmetric solutions, which are given by their values (\( \omega(x, y) \)). On generator pairs:

\[
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}, \begin{pmatrix}
-1 & -1 \\
-1 & -1
\end{pmatrix}, \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}, \begin{pmatrix}
-1 & 1 \\
1 & -1
\end{pmatrix}, \begin{pmatrix}
-1 & 1 \\
-1 & -1
\end{pmatrix}, \begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}
\]

(ii) For even \( \ell \) it is \( \Lambda_R^{[\ell]} = \ell \Lambda_R = 2 \Lambda_R^{[\ell]} \). Thus the quotient diamond is given by \( (\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_2 \times \mathbb{Z}_2, \varphi_1, 0) \) and the injection \( \varphi_2 \) is trivial. We get an analogue block of equations as [5.1], but without non-zero “shift” \( \varphi_2(x), x \in A \). We can add appropriate equations and get the \( 1 = 4g(0, 0) \), hence only pairings of \( H_1 = H_2 = \pi_1 \) are solutions. It is now easy to check, that all 16 possible pairings on \( \pi_1 \times \pi_1 \) are solutions to the diamond equations.

(5) For \( g \) with root system \( D_n \), \( n \geq 5 \) odd, we have \( \pi_1 = \mathbb{Z}_4 \) for all \( n \). The root and weight data is as for even \( n \) in [4]. The weight \( \lambda_n \) is the generator of the fundamental group \( \mathbb{Z}_2 \).

The lattice diamonds, depending on \( \ell \), are:
(i) For odd $\ell$ we have $A = \ell \Lambda_W$ and $B = C = D = \ell \Lambda_R$.
(ii) For $\ell \equiv 2 \pmod{4}$ we have $A = \ell \Lambda_W$, $C = \ell (\lambda_1, \ldots , \lambda_{n-2}, 2\lambda_{n-1}, 2\lambda_n)$ and $B = D = \ell \Lambda_R$.
(iii) For $\ell \equiv 0 \pmod{4}$ we have $A = C = \ell \Lambda_W$ and $B = D = \ell \Lambda_R$.

We calculate the quotient diamonds for kernel $\Lambda' = \Lambda_{R, \ell}^\perp$ and compare it with Lusztig’s kernel $2\Lambda_{R, \ell}$. We then determine the solutions of the corresponding diamond-equations according to Example 4.7.

(i) For odd $\ell$ it is $\Lambda_{R, \ell}^\perp = \ell \Lambda_R \neq 2\ell \Lambda_R = 2\Lambda_{R, \ell}^\perp$, $\ell_{[n]} = \ell$ and $m_{[n]} = n$. Thus, the quotient diamond is given by $(\mathbb{Z}_4, \mathbb{Z}_4, \mathbb{Z}_1, \mathbb{Z}_1, \xi_1, \lambda_n)$, hence we are in case (III) of Example 4.7. We get solutions $(d, k) = (1, 1), (2, 1), (4, 2)$ and $(4, 4)$.
(ii) For $\ell \equiv 2 \pmod{4}$ it is $\Lambda_{R, \ell}^\perp = \ell \Lambda_R = 2\Lambda_{R, \ell}^\perp$, $\ell_{[n]} = \ell$ and $m_{[n]} = n$. Thus, the quotient diamond is given by $(\mathbb{Z}_4, \mathbb{Z}_4, \mathbb{Z}_1, \mathbb{Z}_1, \xi_1, 2\lambda_n)$, hence we are in case (III) of Example 4.7. We get all 4 solutions $(d, k) = (1, 1), (4, 1), (4, 2), (4, 3)$ and $(4, 4)$ on $H = \mathbb{Z}_4$.
(iii) For $\ell \equiv 0 \pmod{4}$ it is $\Lambda_{R, \ell}^\perp = \ell \Lambda_R = 2\Lambda_{R, \ell}^\perp$, $\ell_{[n]} = \ell$ and $m_{[n]} = n$. Thus, the quotient diamond is given by $(\mathbb{Z}_4, \mathbb{Z}_4, \mathbb{Z}_1, \mathbb{Z}_1, \xi_1, 4\lambda_n)$, hence we are in case (III) of Example 4.7. We get the same 4 solutions as in (ii).

(6) For $g$ with root system $E_6$, we have $\pi_1 = \mathbb{Z}_3$. The roots are $\alpha_1, \ldots, \alpha_6$ and $d_i = 1$ for $1 \leq i \leq 6$. The symmetrized Cartan matrix $\tilde{C}$ is given below. The fundamental dominant weights $\lambda_i$ are given as in [Hum72], Section 13.2, and $\lambda_6$ is the generator of the fundamental group $\mathbb{Z}_3$. The matrix $id_{W}^R$ gives the coefficients of the fundamental dominant weights in the basis $\{\alpha_1, \ldots, \alpha_6\}$, and since $d_i = 1$ for all $i$, also the values $(\lambda_i, \lambda_j)$ for $1 \leq i, j \leq 6$.

$$\tilde{C} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 4 & 5 & 2 & 4 & 2 \\ 1 & 2 & 3 & 2 & 1 \\ 5 & 2 & 10 & 3 & 4 \\ 3 & 4 & 6 & 4 & 3 \\ 4 & 2 & 8 & 4 & 2 \\ 3 & 1 & 3 & 2 & 3 \end{pmatrix} = id_{W}^R$$

The lattice diamonds, depending on $\ell$, are:
(i) For $3 \nmid \ell$ we have $A = \ell \Lambda_W$ and $B = C = D = \ell \Lambda_R$.
(ii) For $3 \mid \ell$ we have $A = C = \ell \Lambda_W$ and $B = D = \ell \Lambda_R$.

We calculate the quotient diamonds for kernel $\Lambda' = \Lambda_{R, \ell}^\perp$ and compare it with Lusztig’s kernel $2\Lambda_{R, \ell}$. We then determine the solutions of the corresponding diamond-equations according to Example 4.7.

(i.a) For $3 \nmid \ell$ and $\ell$ odd it is $\Lambda_{R, \ell}^\perp = \ell \Lambda_R \neq 2\ell \Lambda_R = 2\Lambda_{R, \ell}^\perp$, $\ell_{[n]} = \ell$ and $m_{[n]} = 4$. Thus, the quotient diamond is given by $(\mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_1, \mathbb{Z}_1, \xi_3, \lambda_6)$, hence we are in case (III) of Example 4.7. Since $\ell \equiv 2 \pmod{3}$ we get solutions $(d, k) = (1, 1), (3, 1)$ and $(3, 3)$.

(i.b) For $3 \nmid \ell$ and $\ell$ even it is $\Lambda_{R, \ell}^\perp = \ell \Lambda_R = 2\Lambda_{R, \ell}^\perp$, $\ell_{[n]} = \ell$ and $m_{[n]} = 4$. Thus, the quotient diamond is given by $(\mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_1, \mathbb{Z}_1, \xi_3, \lambda_6)$, and we are again in case (III) of Example 4.7. Since $\ell \equiv 1 \pmod{3}$ we get solutions $(d, k) = (1, 1), (3, 2)$ and $(3, 3)$. 
(ii.a) For $3 \mid \ell$ and $\ell$ odd it is $\Lambda_R^{[\ell]} = \ell \Lambda_R \neq 2 \ell \Lambda_R = 2 \Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = 4$. Thus, the quotient diamond is given by $(\mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_1, \mathbb{Z}_1, \xi_3, 0)$, hence we are in case (III) of Example 4.7. We get all 3 solutions $(d, k) = (3, 1)$, $(3, 2)$ and $(3, 3)$ on $\mathbb{Z}_3$.

(ii.b) For $3 \mid \ell$ and $\ell$ even it is $\Lambda_R^{[\ell]} = \ell \Lambda_R = 2 \Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = 4$. Thus, the quotient diamond is given by $(\mathbb{Z}_3, \mathbb{Z}_3, \mathbb{Z}_1, \mathbb{Z}_1, \xi_3, 0)$, and we the same solutions as in (ii.a).

(7) For $g$ with root system $E_7$, we have $\pi_1 = \mathbb{Z}_2$. The roots are $\alpha_1, \ldots, \alpha_7$ and $d_i = 1$ for $1 \leq i \leq 7$. The symmetrized Cartan matrix $\tilde{C}$ is given below. The fundamental dominant weights $\lambda_i$ are given as in [Hum72], Section 13.2, and $\lambda_7$ is the generator of the fundamental group $\mathbb{Z}_2$. The matrix $id_W^{R}$ gives the coefficients of the fundamental dominant weights in the basis $\{\alpha_1, \ldots, \alpha_7\}$, and since $d_i = 1$ for all $i$, also the values $(\lambda_i, \lambda_j)$ for $1 \leq i, j \leq 7$.

\[
\tilde{C} = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
id_W^{R} = \begin{pmatrix}
2 & 2 & 3 & 4 & 3 & 2 & 1 \\
2 & 7 & 4 & 6 & 9 & 3 & 3 & 3 \\
3 & 4 & 6 & 8 & 6 & 4 & 2 \\
4 & 6 & 8 & 12 & 9 & 6 & 3 \\
3 & 9 & 6 & 9 & 15 & 5 & 5 & 5 \\
2 & 3 & 4 & 6 & 5 & 4 & 2 \\
1 & 3 & 2 & 3 & 5 & 2 & 3 & 2 \\
\end{pmatrix}
\]

The lattice diamonds, depending on $\ell$, are:

(i) For odd $\ell$ we have $A = \ell \Lambda_W$ and $B = C = D = \ell \Lambda_R$.

(ii) For even $\ell$ we have $A = C = \ell \Lambda_W$ and $B = D = \ell \Lambda_R$.

We calculate the quotient diamonds for kernel $\Lambda = \Lambda_R^{[\ell]}$ and compare it with Lusztig’s kernel $2 \Lambda_R^{(\ell)}$. We then determine the solutions of the corresponding diamond-equations according to Example 4.7.

(i) For $\ell$ odd it is $\Lambda_R^{[\ell]} = \ell \Lambda_R \neq 2 \ell \Lambda_R = 2 \Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = 3$. Thus, the quotient diamond is given by $(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_1, \xi_2, \lambda_7)$ and we are in case (III) of Example 4.7. We get solutions $(d, k) = (1, 1)$ and $(2, 2)$.

(ii) For $\ell$ even it is $\Lambda_R^{[\ell]} = \ell \Lambda_R = 2 \Lambda_R^{(\ell)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = 3$. Thus, the quotient diamond is given by $(\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_1, \xi_2, 0)$ and we are again in case (III) of Example 4.7. We get all 2 solutions $(d, k) = (2, 1)$ and $(2, 2)$ on $\mathbb{Z}_2$.

(8) For $g$ with root system $E_8$, we have $\pi_1 = \mathbb{Z}_1$. The roots are $\alpha_1, \ldots, \alpha_8$ and $d_i = 1$ for $1 \leq i \leq 8$. The symmetrized Cartan matrix $\tilde{C}$ is given below. The fundamental dominant weights $\lambda_i$ are given as in [Hum72], Section 13.2, and are roots. The matrix $id_W^{R}$ gives the coefficients of the fundamental dominant weights in the basis $\{\alpha_1, \ldots, \alpha_8\}$, and since $d_i = 1$ for all $i$, also the values $(\lambda_i, \lambda_j)$ for $1 \leq i, j \leq 8$. 

\[
\tilde{C} = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
id_W^{R} = \begin{pmatrix}
2 & 2 & 3 & 4 & 3 & 2 & 1 \\
2 & 7 & 4 & 6 & 9 & 3 & 3 & 3 \\
3 & 4 & 6 & 8 & 6 & 4 & 2 \\
4 & 6 & 8 & 12 & 9 & 6 & 3 \\
3 & 9 & 6 & 9 & 15 & 5 & 5 & 5 \\
2 & 3 & 4 & 6 & 5 & 4 & 2 \\
1 & 3 & 2 & 3 & 5 & 2 & 3 & 2 \\
\end{pmatrix}
\]
\[
\tilde{C} = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & 1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix} \quad \text{id}_W^R = \begin{pmatrix}
4 & 5 & 7 & 10 & 8 & 6 & 4 & 2 \\
5 & 8 & 10 & 15 & 12 & 9 & 6 & 3 \\
7 & 10 & 14 & 20 & 16 & 12 & 8 & 4 \\
10 & 15 & 20 & 30 & 24 & 18 & 12 & 6 \\
8 & 12 & 16 & 24 & 20 & 15 & 10 & 5 \\
6 & 9 & 12 & 18 & 15 & 12 & 8 & 4 \\
4 & 6 & 8 & 12 & 10 & 8 & 6 & 3 \\
2 & 3 & 4 & 6 & 5 & 4 & 3 & 2 \\
\end{pmatrix}
\]

The lattice diamonds, depending on \(\ell\), are:

(i) For odd \(\ell\) we have \(A = B = C = D = \ell \Lambda_W = \ell \Lambda_R\).

(ii) For even \(\ell\): same as in (i).

We calculate the quotient diamonds for kernel \(\Lambda' = \Lambda_R^\ell\) and compare it with Lusztig’s kernel \(2\Lambda_R^{(\ell)}\). We then determine the solutions of the corresponding diamond-equations according to Example 4.7.

(i) For odd \(\ell\) it is \(\Lambda_R^\ell = \ell \Lambda_R \neq 2\ell \Lambda_R = 2\Lambda_R^{(\ell)}, \ell | n = \ell\) and \(m | n = 2\). Thus, the quotient diamond is given by \((Z_1, Z_1, Z_1, Z_1, Z_1, 1, 0)\) and we are in case (1) of Example 4.7. We get the only solution \((d, k) = (1, 1)\).

(ii) For even \(\ell\) it is \(\Lambda_R^\ell = \ell \Lambda_R = 2\Lambda_R^\ell, \ell | n = \ell\) and \(m | n = 2\). We get the same diamond and solution as in (i).

(9) For \(g\) with root system \(F_4\), we have \(\pi_1 = Z_1\). The roots \(\alpha_1, \alpha_2\) are long, \(\alpha_3, \alpha_4\) are short, hence \(d_1 = d_2 = 2\) and \(d_3 = d_4 = 1\). The symmetrized Cartan matrix \(\tilde{C}\) is given below. The fundamental dominant weights \(\lambda_i\) are given as in [Hum72], Section 13.2, and are roots. The matrix \(\text{id}_W^R\) gives the coefficients of the fundamental dominant weights in the basis \(\{\alpha_1, \ldots, \alpha_4\}\).

\[
\tilde{C} = \begin{pmatrix}
4 & -2 & 0 & 0 \\
-2 & 4 & -2 & 0 \\
0 & -2 & 2 & -1 \\
0 & 0 & -1 & 2 \\
\end{pmatrix} \quad \text{id}_W^R = \begin{pmatrix}
4 & 6 & 4 & 2 \\
6 & 12 & 8 & 4 \\
4 & 8 & 6 & 3 \\
2 & 4 & 3 & 1 \\
\end{pmatrix}
\]

The lattice diamonds, depending on \(\ell\), are:

(i) For odd \(\ell\), we have \(A = B = C = D = \ell \Lambda_W = \ell \Lambda_R\).

(ii) For even \(\ell\), we have \(A = B = C = D = \ell \Lambda(W, \Lambda(\ell)) = \ell \Lambda(\ell)\).

We calculate the quotient diamonds for kernel \(\Lambda' = \Lambda_R^\ell\) and compare it with Lusztig’s kernel \(2\Lambda_R^{(\ell)}\). We then determine the solutions of the corresponding diamond-equations according to Example 4.7.

(i) For odd \(\ell\) it is \(\Lambda_R^\ell = \ell \Lambda_R \neq 2\ell \Lambda_R = 2\Lambda_R^{(\ell)}, \ell | n = \ell\) and \(m | n = 1\). Thus, the quotient diamond is given by \((Z_1, Z_1, Z_1, Z_1, Z_1, 1, 0)\) and we are in case (1) of Example 4.7. We get the only solution \((d, k) = (1, 1)\).

(ii) For \(\ell \equiv 2 \mod 4\) it is \(\Lambda_R^\ell = \ell \Lambda_R \neq 2\ell \Lambda_R = 2\Lambda_R^{(\ell)}, \ell | n = \ell\) and \(m | n = 1\). We get the same diamond and solution as in (i).

(iii) For \(\ell \equiv 0 \mod 4\) it is \(\Lambda_R^\ell = \ell \Lambda_R \neq 2\ell \Lambda_R = 2\Lambda_R^{(\ell)}, \ell | n = \ell\) and \(m | n = 1\). We get the same diamond and solution as in (i).

(10) For \(g\) with root system \(G_2\), we have \(\pi_1 = Z_1\). The root \(\alpha_1\) is short and \(\alpha_2\) is long, hence \(d_1 = 1\) and \(d_2 = 3\). The symmetrized Cartan matrix \(\tilde{C}\) is given below. The
fundamental dominant weights $\lambda_i$ are given as in [Hum72], Section 13.2, and are roots. The matrix $id_W^R$ gives the coefficients of the fundamental dominant weights in the basis $\{\alpha_1, \ldots, \alpha_2\}$.

$$\tilde{C} = \begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix} \quad id_W^R = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

The lattice diamonds, depending on $\ell$, are:

(i) For $3 \nmid \ell$, we have $A = B = C = D = \ell \Lambda_W = \ell \Lambda_R$.

(ii) For $3 | \ell$, we have $A = B = C = D = \ell(\lambda_1, \frac{1}{3}\lambda_2)$.

We calculate the quotient diamonds for kernel $\Lambda' = \Lambda_R^{|\ell|}$ and compare it with Lusztig’s kernel $2\Lambda_R^{(|\ell|)}$. We then determine the solutions of the corresponding diamond-equations according to Example 4.7

(i.a) For $3 \nmid \ell$ and $\ell$ odd it is $\Lambda_R^{(|\ell|)} = \ell \Lambda_R \neq 2\ell \Lambda_R = 2\Lambda_R^{(|\ell|)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = 6$. Thus, the quotient diamond is given by $(Z_1, Z_1, Z_1, Z_1, 1, 0)$ and we are in case (I) of Example 4.7. We get the only solution $(d, k) = (1, 1)$.

(i.b) For $3 \nmid \ell$ and $\ell$ even it is $\Lambda_R^{(|\ell|)} = \ell \Lambda_R = 2\Lambda_R^{(|\ell|)}$, $\ell_{[n]} = \ell$ and $m_{[n]} = 6$. We get the same diamond and solution as in (i.a).

(ii.a) For $3 | \ell$ and $\ell$ odd it is $\Lambda_R^{(|\ell|)} = \ell(\alpha_1, \frac{1}{3}\alpha_2) \neq 2\ell(\alpha_1, \frac{1}{3}\alpha_2) = 2\Lambda_R^{(|\ell|)}$, $\ell_{[n]} = \ell/3$ and $m_{[n]} = 2$. We get the same diamond and solution as in (i.a).

(ii.b) For $3 | \ell$ and $\ell$ even it is $\Lambda_R^{(|\ell|)} = \ell(\alpha_1, \frac{1}{3}\alpha_2) = 2\Lambda_R^{(|\ell|)}$, $\ell_{[n]} = \ell/3$ and $m_{[n]} = 2$. We get the same diamond and solution as in (i.a).

References

[AY13] I. Angiono, H. Yamane, The $R$-matrix of quantum doubles of Nichols algebras of diagonal type, Preprint, arXiv:1304.5752 2013.

[CP95] V. Chari and A. Pressley, A guide to quantum groups, Cambridge University Press, 1995.

[FGST06] B. Feigin, A. Gainutdinov, A. Semikhatov, I. Tipunin, Modular Group Representations and Fusion in Logarithmic Conformal Field Theories and in the Quantum Group Center, Commun. Math. Phys., 265, pp. 47-93, 2006.

[GW98] S. Gelaki and S. Westreich, On the Quasitriangularity of $U_q(sl_n)$, Journal of the London Mathematical Society 57.1: 105-125, 1998.

[Hum72] J. E. Humphreys, Introduction to Lie algebras and representation theory, Graduate texts in mathematics, Vol. 9, Springer, 1972.

[Kas95] C. Kassel, Quantum groups, Graduate texts in mathematics, Vol. 155, Springer, 1995.

[Len14] S. Lentner, A Frobenius homomorphism for Lusztig’s quantum groups over arbitrary roots of unity, Preprint, arXiv:1406.0865 2014.

[LN14] S. Lentner, D. Nett, A theorem on roots of unity and a combinatorial principle, arXiv:1409.5822 Preprint, 2014.

[Lus90] G. Lusztig, Quantum groups at roots of 1, Geom.Ded. 35, 89-114, 1990.

[Lus93] G. Lusztig, Introduction to quantum groups, Birkhäuser, 1993.

[Müll98a] E. Müller, Quantengruppen im Einheitswurzelfall, Dissertation, LMU, 1998.

[Müll98b] E. Müller, Some topics on Frobenius-Lusztig kernels, I, II, Journal of Algebra, 206(2), 624-681, 1998.

[Res95] N. Reshetikhin, Quasitriangularity of quantum groups at roots of 1, Communications in mathematical physics 170, no. 1, 79-99, 1995.

[Tan92] T. Tanisaki, Killing forms, Harish-Chandra homomorphisms and universal $R$-matrices for quantum algebras, Infinite Analysis, Advanced Series in Mathematical Physics 16, pp. 941-62, 1992.