Generalized fractional integral inequalities for exponentially \((s, m)\)-convex functions

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Abstract
In this paper we have derived the fractional integral inequalities by defining exponentially \((s, m)\)-convex functions. These inequalities provide upper bounds, boundedness, continuity, and Hadamard type inequality for fractional integrals containing an extended Mittag-Leffler function. The results about fractional integral operators for \(s\)-convex, \(m\)-convex, \((s, m)\)-convex, exponentially convex, exponentially \(s\)-convex, and convex functions are direct consequences of presented results.

Keywords: Convex function; \((s, m)\)-convex function; Mittag-Leffler function; Fractional integral operators; Boundedness

1 Introduction
Convex functions are very useful in mathematical analysis due to their fascinating properties and convenient characterizations.

**Definition 1** A function \(f : I \rightarrow \mathbb{R}\) is said to be convex function if the following inequality holds:

\[
f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)
\]  

(1.1)

for all \(a, b \in I\) and \(t \in [0, 1]\). If inequality (1.1) holds in reverse order, then the function \(f\) is called concave function.

A graphical interpretation of a convex function \(f\) over an interval \([a, b]\) provides at a glance the following well-known Hadamard inequality:

\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.
\]  

(1.2)

This inequality has been studied extensively, and a lot of its versions have been published by defining new functions obtained from inequality (1.1). Next we define some of these definitions.
Definition 2 ([10]) Let $s \in [0,1]$. A function $f : [0, \infty) \to \mathbb{R}$ is said to be $s$-convex function in the second sense if

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$$

holds for all $a, b \in [0, \infty)$ and $t \in [0,1]$.

In [22], Toader gave the following definition of $m$-convex function.

Definition 3 A function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be $m$-convex if

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

holds, where $m \in [0,1]$, $x, y \in [0, b]$, and $t \in [0,1]$.

In [4], Awan et al. gave the following definition of exponentially convex function.

Definition 4 A function $f : K \to \mathbb{R}$, where $K$ is an interval, is said to be an exponentially convex function if

$$f(ta + (1-t)b) \leq t^\alpha a + (1-t)^\alpha b$$

holds for all $a, b \in K$, $t \in [0,1]$, and $\alpha \in \mathbb{R}$. If the inequality in (1.3) is reversed, then $f$ is called exponentially concave.

In [12], Mehreen and Anwar gave the following definition of exponentially $s$-convex function.

Definition 5 ([12]) Let $s \in (0,1]$ and $K \subseteq [0, \infty)$ be an interval. A function $f : K \to \mathbb{R}$ is said to be exponentially $s$-convex in the second sense if

$$f(ta + (1-t)b) \leq t^\alpha a + (1-t)^\alpha b$$

holds for all $a, b \in K$, $t \in [0,1]$, and $\alpha \in \mathbb{R}$. If the inequality in (1.4) is reversed, then $f$ is called exponentially $s$-concave function.

In [1], Anastassiou gave the following definition of $(s,m)$-convex function.

Definition 6 ([1]) A function $f : [0, b] \to \mathbb{R}$ is said to be an $(s,m)$-convex function, where $(s,m) \in [0,1]^2$ and $b > 0$, if for every $x, y \in [0, b]$ and $t \in [0,1]$ we have

$$f(ta + m(1-t)b) \leq tf(a) + m(1-t^t)f(b).$$

The aim of this paper is to define a further generalization named exponentially $(s,m)$-convex function (Definition 9) and explore the bounds of generalized fractional integral.
operators containing Mittag-Leffler functions in their kernels. The Mittag-Leffler function $E_\sigma(t)$ was introduced by Gosta [13] in 1903:

$$E_\sigma(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\sigma n + 1)},$$

where $t, \sigma \in \mathbb{C}, \Re(\sigma) > 0$ and $\Gamma(\cdot)$ is the gamma function.

The Mittag-Leffler function is a direct generalization of the exponential function to which it reduces for $\sigma = 1$. In the solution of fractional integral equations and fractional differential equations, the Mittag-Leffler function arises naturally. Due to its importance, the Mittag-Leffler function has been further generalized and extended by many researchers, we refer the reader to [3, 9, 19, 20]. Recently in [2], Andrić et al. introduced a generalized Mittag-Leffler function defined as follows.

**Definition 7** Let $\mu, \sigma, l, \gamma, c \in \mathbb{C}, \Re(\mu), \Re(\sigma), \Re(l) > 0, \Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$, and $0 < k \leq \delta + \Re(\mu)$. Then the extended generalized Mittag-Leffler function is defined by

$$E_{\mu,\sigma,l}^{\gamma,\delta,k,c}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_nk}{\Gamma(\mu n + \sigma)} \frac{t^n}{(l)_n},$$

where $\beta_p$ is the generalized beta function defined as follows:

$$\beta_p(x,y) = \int_{0}^{1} t^{x-1}(1-t)^{y-1} e^{-pt} dt$$

and $(c)_nk$ is the Pochhammer symbol defined by $(c)_nk = \frac{\Gamma(c+nk)}{\Gamma(c)}$.

**Remark 1** The function given in (1.5) is a generalization of the following Mittag-Leffler functions:

(i) If $p = 0$ in (1.5), then it reduces to the Salim–Faraj function defined in [19].

(ii) If $l = \delta = 1$ in (1.5), then it reduces to the function defined by Rahman et al. in [15].

(iii) If $p = 0$ and $l = \delta = 1$ in (1.5), then it reduces to the Shukla–Prajapat function defined in [20], see also [21].

(iv) If $p = 0$ and $l = \delta = k = 1$ in (1.5), then it reduces to the Prabhakar function defined in [14].

Derivative property of the generalized Mittag-Leffler function is given in following lemma.

**Lemma 1** ([2]) If $m \in \mathbb{N}, \omega, \mu, \sigma, l, \gamma, c \in \mathbb{C}, \Re(\mu), \Re(\sigma), \Re(l) > 0, \Re(c) > \Re(\gamma) > 0$ with $p \geq 0, \delta > 0$ and $0 < k < \delta + \Re(\mu)$, then

$$\left( \frac{d}{dt} \right)^m \left[ t^{\sigma-m-1} E_{\mu,\sigma,l}^{\gamma,\delta,k,c}(\omega t^\mu;p) \right] = t^{\sigma-m-1} E_{\mu,\sigma-m,l}^{\gamma,\delta,k,c}(\omega t^\mu;p), \quad \Re(\sigma) > m.$$ (1.6)

Fractional integral operators are very useful in advancement of mathematical inequalities. Many researchers have established fractional integral inequalities due to different kinds of fractional and conformable integral operators, see [1, 2, 5, 6, 8, 11, 16–18, 23].
The Mittag-Leffler function is used to define generalized fractional integral operators. The left-sided and right-sided fractional integral operators containing Mittag-Leffler function are defined as follows.

**Definition 8** ([2]) Let \( \omega, \mu, \sigma, l, k, c, s, \gamma, m \in \mathbb{C} \), \( \Re(\mu), \Re(\sigma), \Re(l) > 0 \), \( \Re(c) > \Re(\gamma) > 0 \) with \( p \geq 0 \), \( \delta > 0 \) and \( 0 < k \leq \delta + \Re(\mu) \). Let \( f \in L_1[a, b] \) and \( x \in [a, b] \). Then the generalized fractional integral operators containing Mittag-Leffler function are defined by

\[
(\epsilon_{\mu, \sigma, l, a}^{\gamma, k, c} f)(x; p) = \int_{x}^{b} (x - t)^{\mu - 1} E_{\mu, \sigma, l}^{\gamma, k, c} (\omega(t - x)^{\mu}; p) f(t) \, dt,
\]

(1.7)

\[
(\epsilon_{\mu, \sigma, b}^{\gamma, k, c} f)(x; p) = \int_{a}^{x} (t - x)^{\mu - 1} E_{\mu, \sigma, b}^{\gamma, k, c} (\omega(t - x)^{\mu}; p) f(t) \, dt,
\]

(1.8)

where \( E_{\mu, \sigma, l}^{\gamma, k, c} (\cdot) \) is the Mittag-Leffler function given in (1.5).

**Remark 2** Integral operators given in (1.7) and (1.8) are the generalization of the following fractional integral operators containing Mittag-Leffler function:

(i) If we take \( p = 0 \), it reduces to the fractional integral operators defined by Salim and Faraj in [19].

(ii) If we take \( l = \delta = 1 \), it reduces to the fractional integral operators defined by Rahman et al. in [15].

(iii) If we take \( p = 0 \) and \( l = \delta = 1 \), it reduces to the fractional integral operators defined by Srivastava and Tomovski in [21].

(iv) If we take \( p = 0 \) and \( l = \delta = k = 1 \), it reduces to the fractional integral operators defined by Prabhakar in [14].

(v) If we take \( p = \omega = 0 \), it reduces to the right-sided and left-sided Riemann–Liouville fractional integrals.

In [8], Farid et al. proved that

\[
(\epsilon_{\mu, \sigma, l, a}^{\gamma, k, c} 1)(x; p) = (x - a)^{\mu} E_{\mu, \sigma, l}^{\gamma, k, c} (w(x - a)^{\mu}; p)
\]

(1.9)

and

\[
(\epsilon_{\mu, \sigma, b}^{\gamma, k, c} 1)(x; p) = (b - x)^{\mu} E_{\mu, \sigma, b}^{\gamma, k, c} (w(b - x)^{\mu}; p).
\]

(1.10)

We will follow the upcoming notations in the main results:

\[
D_{a, a}^{\gamma, k, c} (x; p) = (\epsilon_{\mu, \sigma, l, a}^{\gamma, k, c} 1)(x; p),
\]

(1.11)

\[
D_{a, b}^{\gamma, k, c} (x; p) = (\epsilon_{\mu, \sigma, l, b}^{\gamma, k, c} 1)(x; p).
\]

(1.12)

In the upcoming section we define a new definition named exponentially \((s, m)\)-convex function which generalizes convex, \(s\)-convex, \(m\)-convex, exponentially convex, and exponentially \(s\)-convex functions. Further this definition is used to establish the upper bounds of left-sided and right-sided generalized fractional integral operators (1.7) and (1.8). The upper bounds provide the continuity of these operators. A modulus inequality is obtained for differentiable functions which in absolute value are exponentially \((s, m)\)-convex. Furthermore a fractional version of the Hadamard inequality is proved.
2 Main results

Definition 9 Let $s \in [0, 1]$ and $K \subseteq [0, \infty)$ be an interval. A function $f : K \to \mathbb{R}$ is said to be exponentially $(s, m)$-convex function in the second sense if

$$f(ta + m(1 - t)b) \leq t^s \frac{f(a)}{e^{sa}} + m(1 - t)^s \frac{f(b)}{e^{mb}}$$

holds for all $a, b \in K$, $m \in [0, 1]$, and $\alpha \in \mathbb{R}$.

Remark 3

(i) For $m = 1$, one can get an exponentially $s$-convex function.

(ii) For $\alpha = 0$, one can get an $(s, m)$-convex function.

(iii) For $\alpha = 0$, $m = 1$, one can get an $s$-convex function in the second sense.

(iv) For $\alpha = 0$, $s = 1$, $m = 1$, one can get a convex function.

Theorem 1 Let $f : K \subseteq [0, \infty) \to \mathbb{R}$ be a real-valued function. If $f$ is positive and exponentially $(s, m)$-convex, then for $a, b \in K, a < b$, and $\sigma, \tau \geq 1$, the following fractional integral inequality for generalized integral operators (1.7) and (1.8) holds:

$$\left( \epsilon_{\mu, \sigma, \lambda, \alpha, \tau}^\gamma f \right)(x; p) + \left( \epsilon_{\mu, \tau, \lambda, \alpha, \sigma}^\gamma f \right)(x; p) \leq \left( \frac{f(a)}{e^{sa}} + \frac{mf\left( \frac{x}{m} \right)}{e^{mb}} \right) \frac{(x - a)D_{\alpha, \sigma}^{\tau}(x; p)}{s + 1} + \left( \frac{f(b)}{e^{mb}} + \frac{mf\left( \frac{x}{m} \right)}{e^{mb}} \right) \frac{(b - x)D_{\alpha, \tau}^{\sigma}(x; p)}{s + 1}, \quad x \in [a, b] \alpha, \beta \in \mathbb{R}. \quad (2.1)$$

Proof Let $x \in [a, b]$. Then, for $t \in [a, x]$ and $\sigma \geq 1$, one can have the following inequality:

$$(x - t)^{\sigma - 1}E_{\mu, \sigma, \lambda}^{\gamma, \delta, \kappa, \epsilon} \omega(x - t)^{\mu}; p \leq (x - a)^{\sigma - 1}E_{\mu, \sigma, \lambda}^{\gamma, \delta, \kappa, \epsilon} \omega(x - a)^{\mu}; p. \quad (2.2)$$

As $f$ is exponentially $(s, m)$-convex, therefore one can obtain

$$f(t) \leq \left( \frac{x - t}{x - a} \right)^{\tau} \frac{f(a)}{e^{sa}} + m \left( \frac{t - a}{x - a} \right)^{\tau} \frac{f\left( \frac{x}{m} \right)}{e^{mb}}, \quad \alpha \in \mathbb{R}. \quad (2.3)$$

By multiplying (2.2) and (2.3) and then integrating over $[a, x]$, we get

$$\int_a^x (x - t)^{\sigma - 1}E_{\mu, \sigma, \lambda}^{\gamma, \delta, \kappa, \epsilon} \omega(x - t)^{\mu}; p)f(t) \ dt$$

$$\leq \frac{(x - a)^{\sigma - 1}E_{\mu, \sigma, \lambda}^{\gamma, \delta, \kappa, \epsilon} \omega(x - a)^{\mu}; p}{(x - a)^{\sigma}} \times \left( \frac{f(a)}{e^{sa}} \int_a^x (x - t)^{\tau} dt + \frac{mf\left( \frac{x}{m} \right)}{e^{mb}} \int_a^x (t - a)^{\tau} dt \right),$$

that is, the left integral operator satisfies the following inequality:

$$\left( \epsilon_{\mu, \sigma, \lambda, \alpha, \tau}^\gamma f \right)(x; p) \leq \frac{(x - a)D_{\alpha, \sigma}^{\tau}(x; p)}{s + 1} \left( \frac{f(a)}{e^{sa}} + m \frac{f\left( \frac{x}{m} \right)}{e^{mb}} \right). \quad (2.4)$$
On the other hand, for $t \in (x, b]$ and $\tau \geq 1$, one can have the following inequality:

$$(t - x)^{-1} E^{\gamma, h, k, c}_{\mu, \tau, l} (\omega(t - x)\mu; p) \leq (b - x)^{-1} E^{\gamma, h, k, c}_{\mu, \tau, l} (\omega(b - x)\mu; p).$$

Again from exponential $(s, m)$-convexity of $f$, we have

$$f(t) \leq \left( \frac{t - x}{b - x} \right)^{s} f(b) \frac{t}{e^{\beta b}} + m \left( \frac{b - t}{b - x} \right)^{s} f\left( \frac{b}{e^{\beta x}} \right), \quad \beta \in \mathbb{R}.$$  \hspace{2cm} (2.6)

By multiplying (2.5) and (2.6) and then integrating over $[x, b]$, we get

$$\int_{x}^{b} (t - x)^{-1} E^{\gamma, h, k, c}_{\mu, \tau, l} (\omega(t - x)\mu; p) f(t) \, dt \leq \frac{(b - x)^{-1} E^{\gamma, h, k, c}_{\mu, \tau, l} (\omega(b - x)\mu; p)}{b - x},$$

$$\times \left( \frac{f(b)}{e^{\beta b}} \int_{x}^{b} (t - x)^{s} \, dt + \frac{mf\left( \frac{b}{e^{\beta x}} \right)}{e^{\beta x}} \int_{x}^{b} (b - t)^{s} \, dt \right),$$

that is, the right integral operator satisfies the following inequality:

$$(\epsilon_{\mu, \tau, l_{0}, [a, b]}^{\gamma, h, k, c} f)(x; p) \leq \frac{(b - x) D_{\tau - 1, b}^{-} (x; p) f(b) + mf\left( \frac{b}{e^{\beta x}} \right)}{s + 1} \left( \frac{\frac{b}{e^{\beta x}}}{e^{\beta x}} \right).$$ \hspace{2cm} (2.7)

By adding (2.4) and (2.7), the required inequality (2.1) can be obtained. 

The following special cases are considered.

**Corollary 1** If we set $\sigma = \tau$ in (2.1), then the following inequality is obtained:

$$(\epsilon_{\mu, \sigma, l_{0}, [a, b]}^{\gamma, h, k, c} f)(x; p) + (\epsilon_{\mu, \sigma, l_{0}, [a, b]}^{\gamma, h, k, c} f)(x; p) \leq \left( \frac{f(a)}{e^{\alpha a}} + \frac{mf\left( \frac{a}{e^{\alpha a}} \right)}{e^{\alpha a}} \right) \frac{(x - a) D_{\tau - 1, a}^{+} (x; p)}{s + 1}$$

$$+ \left( \frac{f(b)}{e^{\beta b}} + \frac{mf\left( \frac{b}{e^{\beta x}} \right)}{e^{\beta b}} \right) \frac{(b - x) D_{\tau - 1, b}^{-} (x; p)}{s + 1}, \quad x \in [a, b].$$ \hspace{2cm} (2.8)

**Corollary 2** Along with the assumption of Theorem 1, if $f \in L_{\infty} [a, b]$, then the following inequality is obtained:

$$(\epsilon_{\mu, \sigma, l_{0}, [a, b]}^{\gamma, h, k, c} f)(x; p) + (\epsilon_{\mu, \sigma, l_{0}, [a, b]}^{\gamma, h, k, c} f)(x; p) \leq \frac{\|f\|_{\infty} (1 + \frac{m}{e^{\alpha a}})}{s + 1} \frac{(x - a) D_{\tau - 1, a}^{+} (x; p)}{s + 1}$$

$$+ \frac{1}{e^{\beta b}} \frac{(b - x) D_{\tau - 1, b}^{-} (x; p)}{s + 1}.$$ \hspace{2cm} (2.9)

**Corollary 3** For $\sigma = \tau$ in (2.9), we get the following result:

$$(\epsilon_{\mu, \sigma, l_{0}, [a, b]}^{\gamma, h, k, c} f)(x; p) + (\epsilon_{\mu, \sigma, l_{0}, [a, b]}^{\gamma, h, k, c} f)(x; p)$$
\[ \|f\|_\infty \leq \frac{\|f\|_\infty}{s + 1} \left( \left( \frac{1}{e^a} + \frac{m}{e^c} \right)(x - a)D_{\alpha-1,a^*}(x;p) + \left( \frac{1}{e^{gb}} + \frac{m}{e^{c^*}} \right)(b - x)D_{\beta-1,b^*}(x;p) \right). \]  

(2.10)

Corollary 4 For \( s = 1 \) in (2.9), we get the following result:

\[ \left( e^{\gamma \lambda \delta \kappa \ell_{\mu, \sigma, \tau\omega, \cdot}^f}(x;p) + e^{\gamma \lambda \delta \kappa \ell_{\mu, \sigma, \tau\omega, \cdot}^f}(x;p) \right) \leq \frac{2\|f\|_\infty}{s + 1} \left( \left( \frac{1}{e^a} + \frac{m}{e^c} \right)(x - a)D_{\alpha-1,a^*}(x;p) + \left( \frac{1}{e^{gb}} + \frac{m}{e^{c^*}} \right)(b - x)D_{\beta-1,b^*}(x;p) \right). \]  

(2.11)

Theorem 2 With the assumptions of Theorem 1, if \( f \in L_\infty(a, b) \), then operators defined in (1.7) and (1.8) are continuous.

Proof If \( f \in L_\infty(a, b) \), then from (2.4) we have

\[ \left| \left( e^{\gamma \lambda \delta \kappa \ell_{\mu, \sigma, \tau\omega, \cdot}^f}(x;p) \right) \right| \leq \frac{2\|f\|_\infty(x - a)D_{\alpha-1,a^*}(x;p)}{s + 1} \left( \left( \frac{1}{e^a} + \frac{m}{e^c} \right) \right), \]

(2.12)

that is, \( \left| \left( e^{\gamma \lambda \delta \kappa \ell_{\mu, \sigma, \tau\omega, \cdot}^f}(x;p) \right) \right| \leq M\|f\|_\infty \), where \( M = \frac{2(b - a)D_{\alpha-1,a^*}(b;p)}{s + 1} \left( \left( \frac{1}{e^a} + \frac{m}{e^c} \right) \right) \). Therefore \( \left( e^{\gamma \lambda \delta \kappa \ell_{\mu, \sigma, \tau\omega, \cdot}^f}(x;p) \right) \) is bounded, also it is easy to see that it is linear, hence this is a continuous operator. On the other hand, from (2.7) one can obtain

\[ \left| \left( e^{\gamma \lambda \delta \kappa \ell_{\mu, \sigma, \tau\omega, \cdot}^f}(x;p) \right) \right| \leq K\|f\|_\infty \]

where \( K = \frac{2(b - a)D_{\alpha-1,a^*}(b;p)}{s + 1} \left( \left( \frac{1}{e^a} + \frac{m}{e^c} \right) \right) \). Therefore \( \left( e^{\gamma \lambda \delta \kappa \ell_{\mu, \sigma, \tau\omega, \cdot}^f}(x;p) \right) \) is bounded, also it is linear, hence continuous.

The next result provides the boundedness of a sum of left and right integrals at an arbitrary point for functions whose derivatives in absolute values are exponentially \((s, m)\)-convex.

Theorem 3 Let \( f : K \subseteq [0, \infty) \rightarrow \mathbb{R} \) be a real-valued function. If \( f \) is differentiable and \( |f'| \) is exponentially \((s, m)\)-convex, then for \( a, b \in K, a < b \), and \( \sigma \geq 1 \), the following fractional integral inequality for generalized integral operators (1.7) and (1.8) holds:

\[ \left| \left( e^{\gamma \lambda \delta \kappa \ell_{\mu, \sigma, \tau\omega, \cdot}^f}(x;p) + e^{\gamma \lambda \delta \kappa \ell_{\mu, \sigma, \tau\omega, \cdot}^f}(x;p) \right) \right| - \left( D_{\alpha-1,a^*}(x;p)f(a) + D_{\beta-1,b^*}(x;p)f(b) \right) \]

\[ \leq \left( \left| f'(a) \right| + \left| \frac{m|f'(x)|}{e^a} \right| \right) \left( \left( \frac{1}{e^a} + \frac{m}{e^c} \right) \right) (x - a)D_{\alpha-1,a^*}(x;p) \]

\[ + \left( \left| f'(b) \right| + \left| \frac{m|f'(x)|}{e^b} \right| \right) \left( \left( \frac{1}{e^b} + \frac{m}{e^{c^*}} \right) \right) (b - x)D_{\beta-1,b^*}(x;p), \quad x \in [a, b], \alpha, \beta \in \mathbb{R}. \]  

(2.13)
Proof. Let \( x \in [a, b] \) and \( t \in [a, x] \), by using exponential \((s, m)\)-convexity of \(|f'|\), we have

\[
|f'(t)| \leq \left( \frac{x - t}{x - a} \right)^s \frac{|f'(a)|}{e^{a\alpha}} + m \left( \frac{x - a}{x - a} \right)^s \frac{|f'(\frac{x}{m})|}{e^{\frac{a\alpha}{m}}}. \tag{2.14}
\]

From (2.14), one can have

\[
f'(t) \leq \left( \frac{x - t}{x - a} \right)^s \frac{|f'(a)|}{e^{a\alpha}} + m \left( \frac{x - a}{x - a} \right)^s \frac{|f'(\frac{x}{m})|}{e^{\frac{a\alpha}{m}}}. \tag{2.15}
\]

The product of (2.2) and (2.15) gives the following inequality:

\[
(x - t)^{\sigma - 1} E_{\mu, \sigma, l}^{\gamma, \delta, k, \epsilon} (\sigma(x - t)^\mu; p)|f'(t)| dt
\]

\[
\leq (x - a)^{\sigma - 1} E_{\mu, \sigma, l}^{\gamma, \delta, k, \epsilon} (\sigma(x - a)^\mu; p) \left( \frac{|f'(a)|}{e^{a\alpha}} (x - t)^s + \frac{m|f'(\frac{x}{m})|}{e^{\frac{a\alpha}{m}}} (t - a)^s \right). \tag{2.16}
\]

After integrating the above inequality over \([a, x]\), we get

\[
\int_a^x (x - t)^{\sigma - 1} E_{\mu, \sigma, l}^{\gamma, \delta, k, \epsilon} (\sigma(x - t)^\mu; p)|f'(t)| dt
\]

\[
\leq (x - a)^{\sigma - 1} E_{\mu, \sigma, l}^{\gamma, \delta, k, \epsilon} (\sigma(x - a)^\mu; p)
\times \left( \frac{|f'(a)|}{e^{a\alpha}} \int_a^x (x - t)^s dt + \frac{m|f'(\frac{x}{m})|}{e^{\frac{a\alpha}{m}}} \int_a^x (t - a)^s dt \right)
\]

\[
= \frac{(x - a)^{\sigma} E_{\mu, \sigma, l}^{\gamma, \delta, k, \epsilon} (\sigma(x - a)^\mu; p)}{s + 1} \left( \frac{|f'(a)|}{e^{a\alpha}} + \frac{m|f'(\frac{x}{m})|}{e^{\frac{a\alpha}{m}}} \right). \tag{2.17}
\]

The left-hand side of (2.17) is calculated as follows:

\[
\int_a^x (x - t)^{\sigma - 1} E_{\mu, \sigma, l}^{\gamma, \delta, k, \epsilon} (\sigma(x - t)^\mu; p)|f'(t)| dt. \tag{2.18}
\]

Put \( x - t = z \), that is, \( t = x - z \), also using the derivative property (1.6) of Mittag-Leffler function, we have

\[
\int_0^{x - a} z^{\sigma - 1} E_{\mu, \sigma, l}^{\gamma, \delta, k, \epsilon} (\sigma z^\mu; p)|f'(x - z)| dz
\]

\[
= (x - a)^{\sigma - 1} E_{\mu, \sigma, l}^{\gamma, \delta, k, \epsilon} (\sigma(x - a)^\mu; p)f(a) - \int_0^{x - a} z^{\sigma - 2} E_{\mu, \sigma, l}^{\gamma, \delta, k, \epsilon} (\sigma z^\mu; p)f(x - z) dz.
\]

Now putting \( x - z = t \) in the second term of the right-hand side of the above equation and then using (1.7), we get

\[
\int_0^{x - a} z^{\sigma - 1} E_{\mu, \sigma, l}^{\gamma, \delta, k, \epsilon} (\sigma z^\mu; p)f(x - z) dz
\]

\[
= (x - a)^{\sigma - 1} E_{\mu, \sigma, l}^{\gamma, \delta, k, \epsilon} (\sigma(x - a)^\mu; p)f(a) - \left( E_{\mu, \sigma + 1, l, \omega, \alpha}^{\gamma, \delta, k, \epsilon}(x; p) \right)(x; p).
\]

Therefore (2.17) takes the following form:

\[
\left( D_{\sigma - \alpha, \gamma}(x; p) \right)f(a) - \left( E_{\mu, \sigma + 1, l, \omega, \alpha}^{\gamma, \delta, k, \epsilon}(x; p) \right)(x; p)
\]
\[
\leq \frac{(x-a)D_{\sigma^{-1},\alpha^*}(x;p)}{s+1}\left(\frac{|f'(a)|}{e^{sa}} + \frac{m|f'(\frac{x}{m})|}{e^{\frac{s}{m}}}\right). \tag{2.19}
\]

Also from (2.14) one can have
\[
f'(t) \geq -\left(\frac{x-t}{x-a}\right)^{\frac{s}{a}}|f'(a)| e^{sa} + m \left(\frac{t-a}{x-a}\right)^{\frac{s}{a}}|f'(\frac{x}{m})| e^{\frac{s}{m}}. \tag{2.20}
\]

Following the same procedure as we did for (2.15), one can obtain
\[
\left(e^{y, d, k, c}_{\mu, \sigma + 1, l, \alpha^*} f\right)(x;p) - D_{\sigma^{-1}, \alpha^*}(x;p)f(a)
\leq \frac{(x-a)D_{\sigma^{-1}, \alpha^*}(x;p)}{s+1}\left(\frac{|f'(a)|}{e^{sa}} + \frac{m|f'(\frac{x}{m})|}{e^{\frac{s}{m}}}\right). \tag{2.21}
\]

From (2.19) and (2.21), we get
\[
\left|\left(e^{y, d, k, c}_{\mu, \sigma + 1, l, \alpha^*} f\right)(x;p) - D_{\sigma^{-1}, \alpha^*}(x;p)f(a)\right|
\leq \frac{(x-a)D_{\sigma^{-1}, \alpha^*}(x;p)}{s+1}\left(\frac{|f'(a)|}{e^{sa}} + \frac{m|f'(\frac{x}{m})|}{e^{\frac{s}{m}}}\right). \tag{2.22}
\]

Now we let \(x \in [a, b]\) and \(t \in (x, b)\). Then, by exponential \((s, m)\)-convexity of \(|f'|\), we have
\[
|f'(t)| \leq \left(\frac{t-x}{b-x}\right)^{\frac{s}{b}}|f'(b)| e^{sb} + m \left(\frac{b-t}{b-x}\right)^{\frac{s}{b}}|f'(\frac{x}{m})| e^{\frac{s}{m}}, \quad \beta \in \mathbb{R}. \tag{2.23}
\]

On the same lines as we have done for (2.2), (2.15), and (2.20), one can get from (2.5) and (2.23) the following inequality:
\[
\left|\left(e^{y, d, k, c}_{\mu, \sigma + 1, l, \alpha^*} f\right)(x;p) - D_{\sigma^{-1}, \beta^*}(x;p)f(b)\right|
\leq \frac{(b-x)D_{\sigma^{-1}, \beta^*}(x;p)}{s+1}\left(\frac{|f'(b)|}{e^{sb}} + \frac{m|f'(\frac{x}{m})|}{e^{\frac{s}{m}}}\right). \tag{2.24}
\]

From inequalities (2.22) and (2.24) via the triangular inequality, (2.13) can be obtained. \(\square\)

**Corollary 5** If we put \(\sigma = \tau\) in (2.13), then the following inequality is obtained:
\[
\left|\left(e^{y, d, k, c}_{\mu, \sigma + 1, l, \alpha^*} f\right)(x;p) + \left(e^{y, d, k, c}_{\mu, \sigma + 1, l, \alpha^*} f\right)(x;p)
- \left(D_{\sigma^{-1}, \alpha^*}(x;p)f(a) + D_{\sigma^{-1}, \beta^*}(x;p)f(b)\right)\right|
\leq \left(\frac{|f'(a)|}{e^{sa}} + \frac{m|f'(\frac{x}{m})|}{e^{\frac{s}{m}}}\right)\frac{(x-a)D_{\sigma^{-1}, \alpha^*}(x;p)}{s+1}
+ \left(\frac{|f'(b)|}{e^{sb}} + \frac{m|f'(\frac{x}{m})|}{e^{\frac{s}{m}}}\right)\frac{(b-x)D_{\sigma^{-1}, \beta^*}(x;p)}{s+1}, \quad x \in [a, b], \alpha, \beta \in \mathbb{R}. \tag{2.25}
\]

**Definition 10** Let \(f : [a, b] \to \mathbb{R}\) be a function, we will say that \(f\) is exponentially \(m\)-symmetric about \(\frac{a+b}{2}\) if
\[
f(x) = \frac{f(\frac{a+b-x}{2})}{e^{\frac{a+b-x}{2}}}, \quad x \in \mathbb{R}. \tag{2.26}
\]
It is required to give the following lemma which will be helpful to produce Hadamard type estimations for the generalized fractional integral operators.

**Lemma 2** Let $f : K \subseteq [0, \infty) \to \mathbb{R}, a, b \in K, a < b$, be an exponentially $(s, m)$-convex function. If $f$ is exponentially $m$-symmetric about $\frac{a+b}{2}$, then the following inequality holds:

$$f \left( \frac{a + b}{2} \right) \leq \left( 1 + \frac{m}{2} \right) \frac{f(x)}{2^m e^{\alpha x}}, \quad \alpha \in \mathbb{R}. \quad (2.27)$$

**Proof** Since $f$ is exponentially $(s, m)$-convex, so

$$f \left( \frac{a + b}{2} \right) \leq f(at + (1-t)b) + \frac{mf \left( \frac{a(1-t)+bt}{m} \right)}{2^m e^{\alpha t}}, \quad t \in [0, 1]. \quad (2.28)$$

Let $x = at + (1-t)b$, where $x \in [a, b]$. Then we have $a + b - x = bt + (1-t)a$, and we get

$$f \left( \frac{a + b}{2} \right) \leq f(x) + \frac{mf \left( \frac{a+b-x}{m} \right)}{2^m e^{\alpha x}}. \quad (2.29)$$

Now, using that $f$ is exponentially $m$-symmetric, we will get (2.27). \qed

**Theorem 4** Let $f : K \subseteq [0, \infty) \to \mathbb{R}, a, b \in K, a < b$, be a real-valued function. If $f$ is positive, exponentially $(s, m)$-convex and exponentially $m$-symmetric about $\frac{a+b}{2}$, then for $\sigma, \tau > 0$, the following fractional integral inequality for generalized integral operators (1.7) and (1.8) holds:

$$\frac{2^\gamma h(\alpha)}{1 + m^x} f \left( \frac{a + b}{2} \right) \left[ D_{\tau,1,b}^\alpha (a;p) + D_{\sigma,1,a}^\beta (b;p) \right]$$

$$\leq \left( \epsilon^{\gamma, \delta, k, c}_{\mu, \nu, \tau, \omega, a, b} f \right)(a;p) + \left( \epsilon^{\gamma, \delta, k, c}_{\mu, \nu, \tau, \omega, a, b} f \right)(b;p)$$

$$\leq \left[ D_{\tau,1,b}^\alpha (a;p) + D_{\sigma,1,a}^\beta (b;p) \right] \frac{(b-a)^2 s + 1}{e^m} \left( \frac{f \left( \frac{a}{m} \right)}{e^{\frac{a}{m}}} + \frac{f \left( \frac{b}{m} \right)}{e^{\frac{b}{m}}} \right), \quad \alpha, \beta \in \mathbb{R}. \quad (2.30)$$

where $h(\alpha) = e^{\alpha b}$ for $\alpha < 0$ and $h(\alpha) = e^{\alpha a}$ for $\alpha \geq 0$.

**Proof** For $x \in [a, b]$, we have

$$(x-a)^\gamma E_{\mu, \nu, \tau, \omega, a}^{\gamma, \delta, k, c} (\omega(x-a)^\nu; p) \leq (b-a)^\gamma E_{\mu, \nu, \tau, \omega, a}^{\gamma, \delta, k, c} (\omega(b-a)^\nu; p), \quad \tau > 0. \quad (2.31)$$

As $f$ is exponentially $(s, m)$-convex, so for $x \in [a, b]$, we have

$$f(x) \leq \left( \frac{x-a}{b-a} \right)^\tau f(b) e^{\alpha b} + m \left( \frac{b-x}{b-a} \right)^\tau f(b) e^{\alpha b}, \quad \alpha \in \mathbb{R}. \quad (2.32)$$

By multiplying (2.31) and (2.32) and then integrating over $[a, b]$, we get

$$\int_a^b (x-a)^\gamma E_{\mu, \nu, \tau, \omega, a}^{\gamma, \delta, k, c} (\omega(x-a)^\nu; p) f(x) dx$$

$$\leq (b-a)^{-\gamma} E_{\mu, \nu, \tau, \omega, a}^{\gamma, \delta, k, c} (\omega(b-a)^\nu; p) \left( \frac{f(b)}{e^{\alpha b}} \int_a^b (x-a)^\gamma dx + \frac{mf \left( \frac{b}{m} \right)}{e^{\frac{b}{m}}} \int_a^b (b-x)^\gamma dx \right),$$
from which we have

\[
(e_{\mu,\tau+1,0,a,b,f}(x;p) \leq \frac{(b-a)^{\tau+1} E_{\mu,\tau+1,0,a,b}(\omega(b-a)^{\mu};p)}{\frac{f(b)}{e^{ab}} + \frac{mf(\frac{a}{m})}{e^{\frac{m}{m}}}}, \tag{2.33}
\]

\[
(e_{\mu,\tau+1,0,a,b,f}(x;p) \leq \frac{(b-a)^{\tau} D_{\tau-1,b^*}(a;p)}{\frac{f(b)}{e^{ab}} + \frac{mf(\frac{a}{m})}{e^{\frac{m}{m}}}}), \tag{2.34}
\]

On the other hand, for \(x \in [a, b]\), we have

\[
(b-x)^{\tau} E_{\mu,\tau,0}(\omega(b-a)^{\mu};p) \leq (b-a)^{\tau} E_{\mu,\tau,0}(\omega(b-a)^{\mu};p), \quad \alpha > 0. \tag{2.35}
\]

By multiplying (2.32) and (2.35) and then integrating over \([a, b]\), we get

\[
\int_a^b (b-x)^{\tau} E_{\mu,\tau,0}(\omega(b-a)^{\mu};p) \frac{f(x)}{e^{ab}} dx \leq (b-a) \frac{f(b)}{e^{ab}} \int_a^b (x-a)^{\tau} dx + \frac{mf(\frac{a}{m})}{e^{\frac{m}{m}}} \int_a^b (b-x)^{\tau} dx,
\]

from which we have

\[
(e_{\mu,\tau+1,0,a,b,f}(b;p) \leq \frac{(b-a)^{\tau+1} E_{\mu,\tau+1,0,a}(\omega(b-a)^{\mu};p)}{\frac{f(b)}{e^{ab}} + \frac{mf(\frac{a}{m})}{e^{\frac{m}{m}}}}, \tag{2.36}
\]

\[
(e_{\mu,\tau+1,0,a,b,f}(b;p) \leq \frac{(b-a)^{\tau} D_{\tau-1,b^*}(b;p)}{\frac{f(b)}{e^{ab}} + \frac{mf(\frac{a}{m})}{e^{\frac{m}{m}}}}). \tag{2.37}
\]

Adding (2.34) and (2.37), we get

\[
(e_{\mu,\tau+1,0,a,b,f}(a;p) + (e_{\mu,\tau+1,0,a,b,f}(b;p) \leq \frac{(b-a)^{\tau}}{\frac{f(b)}{e^{ab}} + \frac{mf(\frac{a}{m})}{e^{\frac{m}{m}}}} D_{\tau-1,b^*}(a;p) + D_{\tau-1,b^*}(b;p) \frac{(b-a)^{\tau}}{\frac{f(b)}{e^{ab}} + \frac{mf(\frac{a}{m})}{e^{\frac{m}{m}}}}). \tag{2.38}
\]

Multiplying (2.27) with \((x-a)^{\tau} E_{\mu,\tau,0}(\omega(x-a)^{\mu};p)\) and integrating over \([a, b]\), we get

\[
f \left( \frac{a+b}{2} \right) \int_a^b (x-a)^{\tau} E_{\mu,\tau,0}(\omega(x-a)^{\mu};p) dx \leq \frac{m+1}{2^{\tau}} \int_a^b (x-a)^{\tau} E_{\mu,\tau,0}(\omega(x-a)^{\mu};p) \frac{f(x)}{e^{\alpha x}} dx. \tag{2.39}
\]

By using (1.8) and (1.12), we get

\[
f \left( \frac{a+b}{2} \right) D_{\tau+1,b^*}(a;p) \leq \frac{m+1}{2^{\tau}} h(a) (e_{\mu,\tau+1,1,a,b,f}(a;p)). \tag{2.40}
\]

Multiplying (2.27) with \((b-x)^{\tau} E_{\mu,\tau,0}(\omega(b-x)^{\mu};p)\) and integrating over \([a, b]\), also using (1.7) and (1.11), we get

\[
f \left( \frac{a+b}{2} \right) D_{\tau+1,b^*}(b;p) \leq \frac{m+1}{2^{\tau+1}} h(a) (e_{\mu,\tau+1,1,a,b,f}(b;p)). \tag{2.41}
\]
By adding (2.40) and (2.41), we get

\[
\frac{2!}{1 + m^2} h(\alpha) \left( \frac{a + b}{2} \right) \left[ D_{\tau+1,b^-}(a;p) + D_{\sigma+1,a^+}(b;p) \right] \\
\leq \left( \epsilon_{\mu+1,1,\alpha,\mu,l,\rho}(f)(a;p) \right) + \left( \epsilon_{\mu+1,1,\alpha,\mu,l,\rho^*}(f)(b;p) \right)
\]

(2.42)

By combining (2.38) and (2.42), inequality (2.30) can be obtained.

**Corollary 6** If we put \( \sigma = \tau \) in (2.30), then the following inequality is obtained:

\[
\frac{2!}{1 + m^2} e^{\alpha x} \left( \frac{a + b}{2} \right) \left[ D_{\sigma+1,b^-}(a;p) + D_{\sigma+1,a^+}(b;p) \right] \\
\leq \left( \epsilon_{\mu+1,1,\alpha,\mu,l,\rho}(f)(a;p) \right) + \left( \epsilon_{\mu+1,1,\alpha,\mu,l,\rho^*}(f)(b;p) \right)
\]

(2.43)

### 3 Concluding remarks

This paper has investigated generalized fractional integral inequalities which provide the bounds of fractional integral operators containing Mittag-Leffler functions in their kernels. By setting different values of parameters involved in the Mittag-Leffler function, the results for various known fractional operators can be obtained. For example, by setting \( p = 0 \), fractional integral inequalities for fractional operators defined by Salim and Faraj in [19] can be obtained; by setting \( l = \delta = 1 \), fractional integral inequalities for fractional operators defined by Rahman et al. in [15] can be deduced, by setting \( p = 0 \) and \( l = \delta = 1 \), fractional integral inequalities for fractional operators defined by Shukla and Prajapati in [20] (see also [21]) can be deduced, by setting \( p = 0 \) and \( l = \delta = k = 1 \), fractional integral inequalities for fractional operators defined by Prabhakar in [14] can be deduced, by setting \( p = \omega = 0 \) fractional integral inequalities for Riemann–Liouville fractional integrals can be deduced. Also all the results of this paper hold for \( s \)-convex, \( m \)-convex, exponentially convex, exponentially \( s \)-convex, and convex functions. In particular results for \((s,m)\)-convex functions, which are proved in [7], can be obtained.

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