SCHWARTZ FUNCTIONS, HADAMARD PRODUCTS, AND
THE DIXMIER-MALLIAVIN THEOREM

DEVADATTA G. HEGDE

Abstract. In this paper we show that functions of the form $\prod_{n \geq 1} \frac{1}{1 + \frac{a_n^2}{x^2}}$
where $a_n > 0$ and $\sum_{n \geq 1} \frac{1}{a_n^2} < \infty$ are in the Schwartz space of the real
line, answering a question raised by Casselman in [Cas]. As a consequence we obtain substantial simplifications in the proofs of Dixmier
and Malliavin of their theorem that every test function on a Lie group
is a finite linear combination of convolutions of two test functions, and
an analogue of this for Fréchet space Lie group representations.

Contents

1. Introduction 1
2. The Main theorem 3
3. The Dixmier-Malliavin Theorem 6
   3.1. A special case of the Dixmier-Malliavin theorem: $\mathcal{S}(\mathbb{R})$ 6
   3.2. Some technical results 8
   3.3. Proof of the Dixmier-Malliavin theorems 11

References 13

1. INTRODUCTION

In the context of trace formulas, for example the Selberg trace formula
for compact quotients, the result of Dixmier and Malliavin below can play
an important role, as follows. Consider the right regular action $\pi$ of $G$ on
$L^2(\Gamma \backslash G)$ where $\Gamma$ is a discrete subgroup of a semi-simple real Lie group $G$
and the quotient $\Gamma \backslash G$ is compact. The trace formula expresses the trace of
the operator $\pi(\phi)$ for a test function $\phi$ in two different ways. It essential
to know that the operator $\pi(\phi)$ is of trace class before taking the trace (see
[Art05], section 1).

In the above case the action of $\phi$ on $L^2(\Gamma \backslash G)$ is given by a continuous
kernel $K(x, y) = \sum_{\gamma \in \Gamma} \phi(x^{-1} \gamma y)$ that is square integrable i.e, $K \in L^2(\Gamma \backslash G \times
\Gamma \backslash G)$. Square integrability follows from the compactness of $\Gamma \backslash G$. Therefore
the operator $\pi(\phi)$ is Hilbert-Schmidt. An operator on a Hilbert space is trace

Date: May 4, 2021.
class if it is a finite linear combination of compositions of pairs of Hilbert-Schmidt operators. One way to prove that \( \pi(\phi) \) is trace class is by using the theorem of Dixmier and Malliavin that that every test function on a Lie group is a finite linear combination of convolutions of two test functions and that the convolution \( f * g \) of two test functions on \( G \) is a test function which acts by \( \pi(f) \circ \pi(g) \) on the representation space. This was proved in the seminal paper of Dixmier and Malliavin [DM78]. Their result is:

**Theorem 1.**

**(a)** Every test function \( \phi \) on a real Lie group is a finite sum of convolutions \( \phi_i \ast \psi_i \) where \( \phi_i, \psi_i \) are test functions.

**(b)** Let \((\pi, V)\) be a continuous Fréchet space representation of a real Lie group \( G \) and let \( V^\infty \) be the Fréchet space of smooth vectors in \( V \). Then every smooth vector can be written as a finite linear combination of vectors of the form \( \pi(f)v \) where \( f \) is a test function and \( v \in V^\infty \).

In the compact quotient case the action of \( \pi(f) \) is Hilbert-Schmidt for \( f \) merely continuous. Thus a result of Cartier in [Car76], which shows that any test function can be written as a finite linear sum of \( m \)-times differentiable functions for all \( m \geq 0 \), is sufficient to show that \( \pi(f) \) is trace class. But the result of Dixmier and Malliavin, in this generality, is an essential technical result for many problems in the archimedean local aspects in the theory of automorphic forms. See [CP04] for an application in the determination of poles of Rankin-Selberg \( L \)-functions for \( GL_n \).

The technical heart of Dixmier and Malliavin’s proof is that entire functions of the form

\[
f(x) = \prod_{n \geq 1} \left( 1 + \frac{x^2}{a_n^2} \right)
\]

for \( a_n > 0 \) and \( \frac{a_{n+1}}{a_n} \geq 2 \) for all \( n \geq 1 \) have the property that \( f(x) \in S \), the Schwartz space of the real line. This allows them to construct a sufficiently large family of sequences \( \{c_n\}_{n \geq 1} \) such that for each such sequence there exists \( \phi \in S \) satisfying

\[
\sum_{j=1}^{n} c_j \frac{d^{2j} \phi}{dx^{2j}} \to \delta \quad \text{as} \quad n \to \infty
\]

in \( S' \), the dual of \( S \). From this a strong factorization theorem that every function in \( S \) is a convolution \( f * g \) for \( f, g \in S \) can be easily deduced (see theorem 7). This result is the key for all further generalizations.

In his exposition of the Dixmier-Malliavin theorem, Casselman [Cas] noted that the function

\[
\frac{x}{\sinh x} = \prod_{n \geq 1} \left( 1 + \frac{x^2}{\pi^2 n^2} \right)
\]

is in the Schwartz space and hence entire functions of the form \( f(x) = \prod_{n \geq 1} \left( 1 + \frac{x^2}{a_n^2} \right) \) might have the property that \( \frac{1}{f} \in S \) under very general circumstances and asked for a good criterion on sequences \( a_n \) so that \( \frac{1}{f} \in S \).

The main result (theorem 2) of this paper is a positive answer to the above question of Casselman showing that if \( f(x) = \prod_{n \geq 1} \left( 1 + \frac{x^2}{a_n^2} \right) \) converges to
an entire function then $\frac{1}{f} \in \mathcal{S}$, by an application of Fourier transform methods. This simplifies the technical aspects of the Dixmier-Malliavin’s proof considerably. For the convenience of the reader we give an amplified version of the Dixmier-Malliavin proof in section 3 incorporating the simplifications due to theorem 2 of this paper. The proof itself has a natural interpretation in terms of Bornological spaces and may enable a conceptual proof that Godement-Jacquet zeta integrals give the standard $L$-function attached to a cuspidal form on $GL_n$ (see [Dor20]).

Acknowledgements. I would like to thank professor Bill Casselman and professor Paul Garrett for their careful reading of the manuscript and helpful remarks. I would like to thank professor Abdelmalek Abdesselam for bringing Miyazaki’s work [Miy60] to my attention.

// Comments from the readers are welcome.

2. The Main theorem

In the following we use the standard notation: $z = x + iy$ is a complex number, $\mathcal{S} = \mathcal{S}(\mathbb{R})$ denotes the Schwartz space of rapidly decreasing smooth functions on the real line $\mathbb{R}$ whose derivatives are also rapidly decreasing, and $\hat{f}$ denotes the Fourier transform of a function or a tempered distribution $f$.

The main result of this paper is:

**Theorem 2.** Let $\{a_n\}_{n \geq 1}$ be a sequence of positive real numbers such that the following conditions are true.

(a) The sequence $a_n$ monotonically increases to infinity. That is, $a_{n+1} \geq a_n$ for all $n \geq 0$ and $a_n \to \infty$.

(b) The infinite product $f(z) = \prod \left(1 + \frac{x^2}{a_n^2}\right)$ converges uniformly on compact subsets of $\mathbb{C}$.

Then, $f|_{\mathbb{R}} = f(x)$ is in the Schwartz space of the real line.

**Remark 3.** Any sequence $a_n \geq n$ would satisfy the hypothesis of the theorem, but we do not assume this.

For the proof we need a few elementary results from distribution theory and Fourier transform. The following Fourier transform is well known (Poisson kernel) and can be computed by contour integration: for $\xi \in \mathbb{R}$ and $a > 0$

\[
\int_{\mathbb{R}} e^{-2\pi i \xi x} \frac{dx}{x^2 + a^2} = \frac{\pi}{a} e^{-2\pi a | \xi |}
\]

(2.1)

The following proposition about the distributional derivatives of non-smooth functions is standard (see theorem 3.1.3 in Hörmander [Hör90]).

**Proposition 4.** Let $f$ be a continuously differentiable function on $\mathbb{R}\setminus\{0\}$ such that $f_{0^+} = \lim_{x \to 0^+} f(x)$ and $f_{0^-} = \lim_{x \to 0^-} f(x)$ exist. Let $v(x) = \frac{df}{dx}$ for $x \neq 0$. Then the distributional derivative is $f' = v + (f_{0^+} - f_{0^-})\delta$ where $\delta$ is the Dirac delta distribution at the origin. In particular, if $f$ is continuous at 0 then $f' = v$. 
Lemma 5. Let \( f(\xi) \) be a rapidly decreasing smooth function. The convolution \( e^{-2\pi a|\xi|} * f \) is a smooth function and its derivative is \( \frac{d}{dx}(e^{-2\pi a|\xi|} * f)(x) = (\psi * f)(x) \), where

\[
\psi(\xi) = -\text{sgn}(\xi)2\pi a e^{-2\pi a|\xi|} = \begin{cases} -2\pi a e^{-2\pi a\xi} & \text{for } \xi > 0 \\ 2\pi a e^{2\pi a\xi} & \text{for } \xi < 0 \end{cases}
\]

is the distributional derivative of \( e^{-2\pi a|\xi|} \).

Proof. This is an elementary computation. We have

\[
(e^{-2\pi a|\xi|} * f)(x) = \int_{\mathbb{R}} e^{-2\pi a|x-\xi|} f(\xi) d\xi = \int_{-\infty}^{x} e^{2\pi a(\xi-x)} f(\xi) d\xi + \int_{x}^{\infty} e^{2\pi a(x-\xi)} f(\xi) d\xi
\]

so \( e^{-2\pi a|\xi|} * f \) is a smooth function. Therefore,

\[
\frac{d}{dx}(e^{-2\pi a|\xi|} * f)(x) = \left( f(x) - 2\pi a e^{-2\pi ax} \int_{-\infty}^{x} e^{2\pi ax} f(\xi) d\xi \right) + \left( -f(x) + 2\pi a e^{2\pi ax} \int_{x}^{\infty} e^{-2\pi ax} f(\xi) d\xi \right)
\]

\[
= \int_{-\infty}^{x} (-2\pi a) e^{2\pi a(\xi-x)} f(\xi) d\xi + \int_{x}^{\infty} (2\pi a) e^{2\pi a(x-\xi)} f(\xi) d\xi = (\psi * f)(x)
\]

That the distributional derivative of \( e^{-2\pi a|\xi|} \) is \( \psi \) follows from proposition Proposition 4.

Lemma 6. Let \( f \) and \( g \) be rapidly decreasing functions on \( \mathbb{R} \) which are continuous except possibly at 0 where there is at most a jump discontinuity. Then the convolution \( f * g \) is also a rapidly decreasing function.

Proof. We have

\[
|f * g(x)| \leq \int_{\mathbb{R}} |f(x-y)g(y)| dy
\]

\[
\leq \int_{|x-y| < \frac{|x|}{2}} |f(x-y)g(y)| dy + \int_{|x-y| \geq \frac{|x|}{2}} |f(x-y)g(y)| dy
\]

\[
\leq \left( \sup_{y \in \mathbb{R}}|f(y)| \right) \int_{|y| > \frac{|x|}{2}} |g(y)| dy + \left( \sup_{y \in \mathbb{R}}|g(y)| \right) \int_{|x-y| \geq \frac{|x|}{2}} |f(x-y)| dy
\]

and the result follows since \( \int_{|y| > \frac{|x|}{2}} |g(y)| dy \) and \( \int_{|x-y| \geq \frac{|x|}{2}} |f(x-y)| dy \) are rapidly decreasing at infinity. \( \square \)
Proof. (of the theorem) The function \( \phi(x) = \prod_{n \geq 1} \left( 1 + \frac{x^2}{a_{n}^2} \right) \) grows faster than any polynomial and hence \( f(x) = \frac{1}{\phi(x)} \) is rapidly decreasing. We estimate

\[
|1 + \frac{x^2}{a^2}| \geq \left| 1 + \frac{(x + iy)^2}{a_{n}^2} \right| = \frac{1}{a^2} \left| (x^2 + a^2 - y^2) + i2xy \right| \geq \left| 1 + \frac{x^2 - y^2}{a^2} \right|
\]

Thus,

\[
|f(x + iy)| = \left| \prod_{n \geq 1} \left( 1 + \frac{(x+iy)^2}{a_{n}^2} \right) \right| \leq \prod_{n \geq 1} \left( 1 + \frac{(x^2 - y^2)}{a_{n}^2} \right)
\]

The above estimate shows that \( f(x + iy) \) is rapidly decreasing for any fixed \( y \) and we move the contour of integration to a line \( y = \epsilon \) where there are no poles of \( f \) in the strip \( 0 < y \leq \epsilon \). Thus,

\[
\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i \xi x} dx = e^{2\pi \xi} \int_{\mathbb{R}} f(x + i\epsilon)e^{-2\pi i \xi x} dx
\]

and

\[
|\hat{f}(\xi)| \leq Ce^{2\pi \xi} \quad (\text{for some } C > 0)
\]

so \( \hat{f} \) is rapidly decreasing at \( -\infty \). Choosing an \( \epsilon < 0 \), we get that \( \hat{f} \) is rapidly decreasing at \( +\infty \).

To prove that \( f \in \mathcal{S} \) we show that \( \hat{f} \in \mathcal{S} \) by proving that all the derivatives of \( \hat{f} \) are also rapidly decreasing. Since \( f \) is rapidly decreasing \( \hat{f} \) is a smooth function. We write \( a = a_1 \) and

\[
f = \frac{1}{\left( 1 + \frac{x^2}{a^2} \right)} \cdot f_1 \quad \text{where} \quad f_1 = \frac{1}{\prod_{n \geq 2} \left( 1 + \frac{x^2}{a_{n}^2} \right)}.
\]

Using the Fourier transform of \( \frac{1}{x^2 + a^2} \) we get

\[
\hat{f}(\xi) = \left( \frac{a^2}{x^2 + a^2} \right) * \hat{f}_1 = \pi a \left( e^{-2\pi a |\xi|} \right) * \hat{f}_1
\]

By using lemma 5 we get

\[
\frac{df}{d\xi} = \pi a \left( \psi * \hat{f}_1 \right), \quad \psi(\xi) = -\text{sgn}(\xi)2\pi ae^{-2\pi a |\xi|}
\]

and from lemma 6 that it is a rapidly decreasing function. It follows that \( \frac{df}{d\xi} \) is also rapidly decreasing since it is an infinite product of the same form as \( f \). Further,

\[
\frac{d^n f}{d\xi^n} = \pi a \left( \psi * \frac{d^{n-1} \hat{f}_1}{d\xi^{n-1}} \right)
\]

So by induction on \( n \), using lemma 6 we conclude that \( \frac{d^n f}{d\xi^n} \) is rapidly decreasing for any \( n \). \( \square \)
3. The Dixmier-Malliavin Theorem

In this section we amplify Dixmier and Malliavin’s proof of their theorems mentioned in the introduction simplifying some technical lemmas by using theorem 2.

First we need a technical result (corollary 11) about Schwartz functions on the real line, using which we illustrate the proof mechanism of the general case by proving the strong factorization property of Schwartz functions on the real line (see the proof of theorem 7).

In subsection 3.2 we use this technical result about Schwartz functions to deduce a similar result about test functions (lemma 14) which gives a weak factorization. Then we collect an amusing trick (lemma 16), due to Dixmier and Malliavin, about convolution of compactly supported measures which reduces the Lie group case to the case of test functions on the real line.

3.1. A special case of the Dixmier-Malliavin theorem: \( \mathcal{S}(\mathbb{R}) \). In this subsection we prove

**Theorem 7.** Every function \( \phi \in \mathcal{S} \) has a factorization \( \phi = f \ast g \) where \( f, g \in \mathcal{S} \).

This result was originally proved for \( \mathbb{R}^n \) by Miyazaki [Miy60] using different methods. As in Dixmier-Malliavin [DM78] we need a sufficiently large family of sequences \( \{ c_n \}_{n \geq 1} \) such that for each such sequence there exists a \( \phi \in \mathcal{S} \) such that \( \sum_{n \geq j \geq 1} c_j \frac{d^j \phi}{d x^j} \rightarrow \delta \) in \( \mathcal{S}' \). By using the Fourier transform this can be reduced to a problem about entire functions answered by using the main theorem 2 and the Hadamard factorization theorem (see Chapter 5 of [SS03], for example). We begin by collecting a few elementary results about entire functions.

**Lemma 8.** The function \( f(z) = \sum_{n \geq 0} \frac{z^n}{(n)!} \) satisfies \( |f(z)| \leq C e^{c|z|^{1/2}} \) for some real numbers \( c \) and \( C \).

**Proof.** Consider the function \( g(z) = \sum_{n \geq 0} \frac{z^n}{(n)!} \). Since

\[
\sum_{n \geq 0} a_n b_n \leq \left( \sum_{n \geq 0} a_n \right) \left( \sum_{n \geq 0} b_n \right)
\]

for \( a_n, b_n > 0 \), we have

\[
|g(z)| \leq \sum_{n \geq 0} \frac{|z|^n}{(n)!} \leq \left( \sum_{n \geq 0} \frac{|z|^{1/2}}{n!} \right) \left( \sum_{n \geq 0} \frac{|z|^{1/2}}{n!} \right) \leq e^{2|z|^{1/2}}
\]

Similarly,

\[
|f(z)| \leq |f(|z|)| \leq g(|z|^{1/2}) \cdot g(|z|^{1/2}) \leq e^{4|z|^{1/2}}
\]

\( \Box \)
Corollary 9. If $\frac{1}{(n!)^4} \geq c_n > 0$ then the function $f(z) = \sum_{n \geq 0} c_n z^{2n}$ is an infinite product

$$f(z) = \prod_{n \geq 1} \left(1 + \frac{z^2}{a_n^2} \right)$$

for some $a_n > 0$.

Proof. Note that $f$ is an entire function, and all the zeros of $f$ are off the real line and occur in conjugate pairs. The existence of such an infinite product follows from Hadamard factorization since $|f(z)| \leq e^{4|z|^2}$ has order of growth $\leq \frac{1}{2}$. □

The following is the crucial property of functions in $S$ used in the proof of the Dixmier-Malliavin theorem.

Corollary 10. (Dixmier-Malliavin) Given a sequence of bounds $B_n > 0$, there exists a sequence of constants $0 < b_n < B_n$ and a function $\psi \in S$ such that

$$F_{\psi,n} = \sum_{j=0}^{n} (-1)^j b_j \psi^{(2j)} \to \delta \quad \text{(in $S'$)}$$

as $n \to \infty$.

Proof. Since Fourier transform is a topological isomorphism on $S'$ it is enough to show that there exists a $\psi \in S$ such that

$$\hat{F}_{\psi,n}(\xi) = \left( \sum_{0 \leq j \leq n} (2\pi)^j b_j \xi^{2j} \right) \hat{\psi}(\xi) \to 1$$

in $S'$. By theorem [2] we pick $\psi \in S$ such that

$$\hat{\psi}(\xi) = \frac{1}{\sum_{j \geq 0} (2\pi)^j b_j \xi^{2j}} = \frac{1}{\prod_{n \geq 1} \left(1 + \frac{\xi^2}{b_n^2} \right)}$$

with $b_j < \frac{1}{(2\pi)^j}$. The result now follows since

$$\frac{\sum_{0 \leq j \leq n} (2\pi)^j b_j \xi^{2j}}{\sum_{j=0}^{\infty} (2\pi)^j b_j \xi^{2j}} \to 1$$

uniformly on compact subsets of $\mathbb{R}$ as $n \to \infty$ and hence also as tempered distributions. □

By using the corollary [10] with bounds $B_j$ above, we choose a $\psi \in S$ such that

$$\sum_{0 \leq j \leq n} (-1)^j b_j \psi^{(2j)} \to \delta \quad \text{(in $S'$, with $0 < b_j < B_j$)}$$

On one hand,

$$\phi \ast \left( \sum_{j \leq n} (-1)^j b_j \psi^{(2j)} \right) \to \phi \quad \text{(in $S'$)}$$
On the other hand,
\[
\phi \ast \left( \sum_{j \leq n} (-1)^j b_j \psi^{(2j)} \right) = \psi \ast \left( \sum_{j \leq n} (-1)^j b_j \phi^{(2j)} \right)
\]

We choose bounds \(B_j\) to make \(\sum_{j \leq n} (-1)^j b_j \phi^{(2j)}\) converge in \(\mathcal{S}\). Schwartz space \(\mathcal{S}\) a Fréchet space given by semi-norms
\[
|f|_{m,n} = \sup_{x \in \mathbb{R}} |x^m f^{(n)}(x)|
\]

To show that a sequence \(\varphi_n \in \mathcal{S}\) converges in \(\mathcal{S}\) it is enough to show that the sequence is Cauchy for each of the norms \(| \cdot |_{m,n}\). That is,
\[
\sup_{x \in \mathbb{R}} |x^m (\varphi^{(n)}_i - \varphi^{(n)}_j)| \to 0 \quad \text{as} \ i, j \to \infty
\]

This is follows if the bounds \(B_j > 0\) satisfy
\[
\sum_{j \geq 0} B_j \sup_{x \in \mathbb{R}} |x^m |\phi^{(2j+n)}| < \infty \quad \text{for all} \ n, m
\]

which we obtain by the following lemma [11].

**Lemma 11.** Given a sequence of positive numbers \(b_{mn} (0 \leq m, n \in \mathbb{Z})\) we can find a sequence \(a_j (0 \leq j \in \mathbb{Z})\) such that \(\sum_{j \geq 0} a_j b_{jn} < \infty\) for all \(n\).

**Proof.** This is a diagonal argument. First choose a decreasing sequence of positive numbers \(A_1^1, A_2^1, \ldots\) such that \(\sum_{j \geq 0} A_1^1 b_{j1} < \infty\). Leaving \(A_1^1\) fixed shrink \(\{A_1^1\}_{j \geq 2}\) if necessary to obtain \(\{A_1^2\}_{j \geq 2}\) so that \(\sum_{j \geq 0} A_1^2 b_{j2} < \infty\). Leaving \(A_1^1, A_2^1\) fixed shrink \(\{A_2^1\}_{j \geq 2}\) if necessary to obtain \(\{A_2^2\}_{j \geq 3}\) so that \(\sum_{j \geq 0} A_2^2 b_{j3} < \infty\). Continuing similarly we obtain the desired sequence \(a_j = A_j^j\). \(\square\)

Let \(\Phi = \sum_{j \geq 0} (-1)^j b_j \phi^{(2j)}\) then \(\phi = \Phi \ast \psi\) where the last equality is in \(\mathcal{S}'\). Since both sides are Schwartz functions we have equality in \(\mathcal{S}\). This proves theorem [7].

**Remark 12.** This strong factorization result is specific to Schwartz space on the real line and analogous Schwartz spaces used in the theory of automorphic forms need not have strong factorization. Strong factorization provably fails for test functions on \(\mathbb{R}^n\) for \(n \geq 3\) [RST78].

3.2. **Some technical results.** Let \(\mathcal{D}\) denote the space of test functions, smooth functions on \(\mathbb{R}\) with compact support. We now prove the test function analogue of corollary [10] above about Schwartz functions. Now we add a further hypothesis that the sequences satisfy \(a_n \geq n\). Then we have a uniform bound on their derivatives.

**Claim 13.** We may find a sequence \(c_j\) satisfying \(c_j > \sup_{x \in \mathbb{R}} |\phi^{(j)}(x)|\) independent of all \(\phi\) constructed such that
\[ \hat{\phi}(\xi) = \frac{1}{\sum_{n \geq j \geq 0} (2\pi)^j b_j \xi^{2j}} = \frac{1}{\prod_{n \geq 1} \left(1 + \frac{\xi^2}{a_n^2}\right)} \]

with bounds \( 0 < b_n < \frac{1}{(2\pi)^j (m^2)} \) and \( a_n \geq n \).

**Proof.** Since \( a_n \geq n \) it follows that
\[
\prod_{n \geq 1} \left(1 + \frac{\xi^2}{a_n^2}\right) \geq \prod_{m \leq j+2} \left(1 + \frac{\xi^2}{m^2}\right)
\]

Thus we have an estimate on \( \hat{\phi} \) independent of \( \phi \)
\[
|\hat{\phi}(\xi)| \leq \frac{1}{\prod_{1 \leq m \leq j+2} \left(1 + \frac{\xi^2}{m^2}\right)}
\]

By Sobolev embedding theorem,
\[
\sup_{x \in \mathbb{R}} |\phi^{(j)}(x)| \leq C \int_{\mathbb{R}} (1 + \xi^2)^{j+1} \hat{\phi}(\xi) d\xi
\]

for a constant \( C \) independent of \( \phi \). The result follows by the estimate above on \( \hat{\phi} \). \( \square \)

**Lemma 14.** Given a sequence of bounds \( B_n > 0 \), there exists a sequence of constants \( 0 < b_n < B_n \) and a function \( f \in \mathcal{D} \) such that
\[
\sum_{i=1}^{n} (-1)^i b_j f^{(2j)} \to \delta + h \quad (\text{in } \mathcal{D}')
\]

as \( n \to \infty \) for some \( h \in \mathcal{D} \). The support of the test functions \( f \) and \( h \) can be made arbitrarily small.

**Proof.** Let \( \omega \) be a test function symmetric about the origin, supported on \([-2, 2]\) and identically 1 on \([-1, 1]\). Let \( \phi_\lambda \in \mathcal{S} \) be as in the similar lemma above about Schwartz functions with bounds \( \lambda = \{\lambda_j\}_{j \geq 0} \) to be decided later. The function \( f = \omega \cdot \phi_\lambda \) is supported on \([-2, 2]\) and \( \phi_\lambda = f \) on \([-1, 1]\). By claim[13] we can choose \( c_n = \sup_{x \in \mathbb{R}} |f^{(n)}(x)| \) independent of the sequence \( \lambda \). We pick the bounds \( b_j \) for \( \phi_\lambda = \phi \), namely \( b_j < \min\{\lambda_j, \frac{1}{j^2 c_{2j}}, \ldots, \frac{1}{j^2 c_{2j+j}}\} \) satisfying the the same assumptions as \( \lambda \). Now
\[
\sum_{j \leq n} (-1)^j b_j f^{(2j)} \to \delta + h \quad (\text{as } n \to \infty)
\]

for some test function \( h \) supported in \([-2, 2]\). For \(|x| \geq 2\) the entire sequence is zero, for \(|x| \leq 1\) then \( f = \phi \). On \(|x| \geq 1\),
\[
|b_j f^{(2n+j)}(x)| \leq b_j c_{2n+j} \leq \frac{1}{j^2}
\]
for $j$ sufficiently large so that $\sum_{j\leq m} (-1)^j b_j f^{(2j)}$ converges to a smooth function on $|x| > 1$ as $m \to \infty$. By scaling one may make the support of $f$ and $h$ arbitrarily small. \hfill \Box

Remark 15. Using the method in the proof of theorem \cite{RST78} any test function $\phi$ on the real line can be written as $\phi = \Phi \ast f - h \ast \phi$. Again, strong factorization provably fails for test functions on $\mathbb{R}^n$ for $n \geq 3$ \cite{RST78}.

The Dixmier-Malliavin theorem for Lie groups is proved by a clever use of the theorem in the case of the real line. This method requires a result about convolution of certain compactly supported measures on Lie groups, proven below.

Let $\{x_1, \ldots, x_n\}$ be a basis of $\mathfrak{g}$, the Lie algebra of a real Lie group $G$, and let $\Theta : \mathfrak{g} \to G$ be the map given by $t_1 x_1 + \ldots + t_n x_n \mapsto \exp(t_1 x_1) \ldots \exp(t_n x_n)$. The map $\Theta$ is a local diffeomorphism at $0 \in \mathfrak{g}$. Let $V \subset G$ be an open neighborhood of the identity where $\Theta^{-1}$ exists and let $\Theta^{-1} : V \to U \subset \mathfrak{g} \cong \mathbb{R}^n$ be $\Theta^{-1}(g) = (\theta_1(g), \ldots, \theta_n(g))$. For $x \in \mathfrak{g}$ and $f \in \mathcal{D}(\mathbb{R})$ we define a measure $\mu_{x,f}$ on $G$ by invoking the Riesz-Markov-Kakutani theorem as follows.

$$\mu_{x,f}(F) = \int_{\mathbb{R}} F(e^{tx}) f(t) dt \quad \text{(for all } F \in C^0(G))$$

For any test function $\nu$ on $G$ and $y \in G$ let $R_y \nu$ denote the right regular action of $y$ on $\nu$.

Lemma 16. Let $f_1, \ldots, f_n \in \mathcal{D}(\mathbb{R})$ be such that the support of $\mu_{x_1, f_1}$ is contained in an open neighborhood $W$ of $1 \in G$ satisfying $W^n = W \cdots W \subset V$. Then the distribution $\mu_{x_1, f_1} \ast \cdots \ast \mu_{x_1, f_n}$ is integration against a test function on $G$.

Proof. We denote $\mu_{x_1, f_1}$ by $\mu_i$ for clarity. The convolution of measures $\mu_{x_1, f_1} \ast \cdots \ast \mu_{x_1, f_n}$ as a distribution is given by (see the remark below)

$$(\mu_{x_1, f_1} \ast \cdots \ast \mu_{x_1, f_n})(\phi) = \int_{G^n} \phi(y_1 \ldots y_n) d\mu_1(y_1) \ldots d\mu_n(y_n)$$

$$= \int_{G^{n-1}} \left( \int_G (R_{y_2} \circ \cdots \circ R_{y_n} \phi)(y_1) d\mu_1(y_1) \right) d\mu_2 \ldots d\mu_n$$

$$= \int_{G^{n-1}} \left( \int_{\mathbb{R}} (R_{y_2} \circ \cdots \circ R_{y_n} \phi)(e^{tx_1}) f_1(t_1) dt_1 \right) d\mu_2 \ldots d\mu_n$$

$$= \int_{G^{n-2}} \left( \int_{\mathbb{R}^2} (R_{y_3} \circ \cdots \circ R_{y_n} \phi)(e^{tx_1} x_2) f_1(t_1) f_1(t_1) dt_1 dt_2 \right) d\mu_3 \ldots d\mu_n$$

$$\cdots \cdots \cdots$$

$$= \int_{\mathbb{R}^n} \phi(e^{tx_1} \ldots e^{tx_n}) f_1(t_1) \ldots f_1(t_n) dt_1 \ldots dt_n$$

$$= \int_{V \subset G} \phi(g)(f \circ \theta_1)(g) \ldots (f \circ \theta_n)(g) \cdot J(g) dg$$

$$= \int_G \phi(g)(f \circ \theta_1)(g) \ldots (f \circ \theta_n)(g) \cdot J(g) dg$$
where $dg$ is the Haar measure on $G$ and $J$ is the Jacobian arising due to change of measure. Thus, the distribution $\mu_{x_1,f_1} \ast \cdots \ast \mu_{x_n,f_n}$ is given by $f \circ \theta_1(g) \cdots f \circ \theta_n(g) \cdot J(g)$ which is a test function on $G$.

Remark 17. The expression used above for the convolution can be seen for convolution of two test functions $f, g$ by looking at the integral

$$(f \ast g)(\phi) = \int_G \int_G f(xy^{-1})g(y)\phi(x)dydx$$

which is a test function on $G$.

and for $n$ functions induction gives the desired expression. For compactly supported distributions $\lambda_1, \ldots, \lambda_n$ the assertion is $(\lambda_1 \ast \cdots \ast \lambda_n)(\phi) = (\lambda_1 \otimes \cdots \otimes \lambda_n)(\phi \circ M^n)$ where $M^n : G^n \to G$ given by $(x_1, \ldots, x_n) \mapsto x_1 \cdots x_n$, the tensor product is in the sense of the Schwartz kernel theorem, and $\lambda_1 \otimes \cdots \otimes \lambda_n$ is a distribution on $G^n$. For measures this will give the expression used above.

3.3. Proof of the Dixmier-Malliavin theorems. Let $\mathcal{D}(G)$ be the space of smooth functions with compact support on a real Lie group $G$.

Theorem 18. (Dixmier-Malliavin) Every element $\phi \in \mathcal{D}(G)$ is a finite sum of convolutions $\phi_i \ast \psi_i$ where $\phi_i, \psi_i \in \mathcal{D}(G)$

Proof. We begin with an observation. Let $\{\gamma_1, \ldots, \gamma_m\}$ be a basis for the Lie algebra $\mathfrak{g}$ of $G$ such that

$$(t_1, \ldots, t_m) \mapsto \exp(t_1\gamma_1) \cdots \exp(t_m\gamma_m)$$

is a diffeomorphism of $(-1, 1)^m$ to a neighborhood of $1 \in G$. Let $\{\alpha_j\}$ be a basis of the universal enveloping algebra $U\mathfrak{g}$, and put

$$M_{\ell n} = \sup_{g \in G} |\alpha_{\ell} x_1^{2n} \phi(g)| \quad (\text{where } \alpha_{\ell} x_1^{2n} \in U\mathfrak{g})$$

By a diagonal argument above we can choose $b_j$ positive such that $\sum_j b_j M_{j \ell} < +\infty$ for all $\ell$. This will show that various summations below will converge in the space of test functions.

Given $g \in \mathcal{D}(\mathbb{R})$ and $x \in \mathfrak{g}$ define a measure $\mu = \mu(x, g)$ on $G$ by

$$\int_G Fd\mu = \int_{\mathbb{R}} F(e^{tx})g(t)dt \quad \text{for } F \in C_c^0(G)$$

By lemma 14 with bounds $b_j$ we can find $f, h \in \mathcal{D}$ such that

$$\sum_{0 \leq j \leq n} (-1)^j b_j f^{(2j)} \to \delta + h$$

as $n \to \infty$ in $\mathcal{D}'$. For $x \in \mathfrak{g}$,

$$\left( \sum_{0 \leq j \leq n} (-1)^j b_j x^{(2j)} \right) : \mu(x, f) \to \delta + \mu(x, h)$$
Remark Theorem follows.

$\square$

usual colimit topology are never

w

ions of the Baire category theorem. Thus, Dixmier-Malliavin theorem for

union of closed sets with empty interior and hence would violate the conclu-

Theorem 19. (Dixmier-Malliavin theorem for representations) Let

$π \in D(\mathbb{R})$, on one hand

$$\left( \left( \sum (-1)^j b_j x^{(2j)} \right) \cdot \mu(x, f) \right) * \varphi \rightarrow \varphi + \mu(x, h) * \varphi$$

On the other hand,

$$\left( \left( \sum (-1)^j b_j x^{(2j)} \right) \cdot \mu(x, f) \right) * \varphi \rightarrow \mu(x, f) * Φ$$

where $Φ = \sum_{j \geq 0} (-1)^j b_j \varphi^{(2j)}$. Thus,

$$\varphi = -μ(x, h) * \varphi + μ(x, f) * Φ$$

If $\{x_1, \ldots, x_n\}$ is a basis for $g$ then by applying the above to $\varphi$ and $Φ$ on the

right $(n-1)$-times we can write $\varphi$ as a linear combination of terms of the

form $μ(x_1, f_1) * \ldots * μ(x_n, f_n) * α$ where $f_i \in D(\mathbb{R})$, $α \in D(\mathbb{R})$ and $μ(x_i, f_i)$

are compactly supported measures with support contained in $\{exp(\alpha x_i) : t \in \mathbb{R}\}$. We may further make sure that support of $f_i$ and hence $μ(x_i, f_i)$ are

arbitrarily small so that computations can be done in local coordinates. By

lemma 16 above about convolution of certain compactly supported measures,

$μ(x_1, f_1) * \ldots * μ(x_n, f_n)$ is a test function. \qed

Theorem 19. (Dixmier-Malliavin theorem for representations) Let $(π, V)$

be a smooth Fréchet space representation of a real Lie group $G$ and let $V^∞$

be the Fréchet space of smooth vectors in $V$. Then every smooth vector can

be written as a finite linear combination of vectors of the form $π(f)v$ where

$f \in D(\mathbb{R})$ and $v \in V^∞$.

Proof. Let $|.|_ℓ$ be a countable semi-norms on $V$ corresponding to the topology

on $V$. Let $\{x_j\}$ be a basis for $Ug$, the universal enveloping algebra of $g$ and

set

$$M_{n,k,ℓ} = |π(X_k x^{2n})v|_ℓ$$

By a diagonal argument similar to the one above we may choose $b_n$ such

that $\sum b_n M_{n,k,ℓ} < ∞$ for all $k, ℓ$. By lemma 13 with bounds $b_j$ we can find

$f, h \in D$ such that $\sum_{0 \leq j \leq n} (-1)^j b_j f^{(2j)} \rightarrow δ + h$ as $n \rightarrow ∞$ in $D^\prime$. We have

$$π(μx,f) * \left( \sum_{0 \leq j \leq n} (-1)^j b_n π(x^{2n}) \right) \cdot v \rightarrow v + π(μx,h)v$$

in $V$. By the above considerations $\sum_{0 \leq j \leq n} (-1)^j b_n π(x^{2n})v$ has a limit $η$ in

$V$ and then $π(μx,f)η = v + π(μx,h)v$. As in the previous theorem $v$ is a linear combination of vectors of the form $π(μx_1, f_1) * \ldots * μ(x_n, f_n)w$ for some

$w \in V$. Again by using lemma 16 about convolution of compactly supported measures the theorem follows. \qed

Remark 20. The space of test functions on a non-compact manifold with the

usual colimit topology are never Fréchet spaces since they are a countable

union of closed sets with empty interior and hence would violate the conclu-

sions of the Baire category theorem. Thus, Dixmier-Malliavin theorem for
representations does not directly imply the result for test functions. Smooth vectors in non-Fréchet space representations are more subtle. For example, for the regular representation of $\mathbb{R}$ on $\mathcal{D}$ every vector is a smooth vector since distributions are infinitely differentiable. Setting up a common notation to treat both the cases in a single theorem does not appear to be worth the clarity lost in the process.

References

[Miy60] Kenichi Miyazaki. “Distinguished elements in a space of distributions”. In: *J. Sci. Hiroshima Univ. Ser. A* 24 (1960), pp. 527–533. issn: 0386-3018.

[Car76] Pierre Cartier. “Vecteurs différentiables dans les représentations unitaires des groupes de Lie”. In: (1976), 20–34. Lecture Notes in Math., Vol. 514.

[DM78] Jacques Dixmier and Paul Malliavin. “Factorisations de fonctions et de vecteurs indéfiniment différentiables”. In: *Bull. Sci. Math.* (2) 102.4 (1978), pp. 307–330.

[RST78] L. A. Rubel, W. A. Squires, and B. A. Taylor. “Irreducibility of certain entire functions with applications to harmonic analysis”. In: *Ann. of Math. (2)* 108.3 (1978), pp. 553–567. issn: 0003-486X. doi: 10.2307/1971188. url: https://doi-org.ezp1.lib.umn.edu/10.2307/1971188

[Hör90] Lars Hörmander. *The analysis of linear partial differential operators. I.* Second. Vol. 256. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Distribution theory and Fourier analysis. Springer-Verlag, Berlin, 1990, pp. xii+440.

[SS03] Elias M. Stein and Rami Shakarchi. *Complex analysis*. Vol. 2. Princeton Lectures in Analysis. Princeton University Press, Princeton, NJ, 2003, pp. xviii+379.

[CP04] James W. Cogdell and Ilya I. Piatetski-Shapiro. “Remarks on Rankin-Selberg convolutions”. In: *Contributions to automorphic forms, geometry, and number theory*. Johns Hopkins Univ. Press, Baltimore, MD, 2004, pp. 255–278.

[Art05] James Arthur. “An introduction to the trace formula”. In: *Harmonic analysis, the trace formula, and Shimura varieties*. Vol. 4. Clay Math. Proc. Amer. Math. Soc., Providence, RI, 2005, pp. 1–263.

[Dor20] Gal Dor. *The Dixmier-Malliavin Theorem and Bornological Vector Spaces*. 2020. arXiv: 2001.05694 [math.RT].

[Cas] William Casselman. *The theorem of Dixmier and Malliavin*. https://www.math.ubc.ca/~cas. Accessed: 2020-11-10.

Email address: hegde039@umn.edu

Current address: Department of Mathematics, University of Minnesota, Minneapolis, MN - 55414.