CR-manifolds of dimension 5: A Lie algebra approach

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Abstract. We study real-analytic Levi degenerate hypersurfaces $M$ in complex manifolds of dimension 3, for which the CR-automorphism group $\text{Aut}(M)$ is a real Lie group acting transitively on $M$. We provide large classes of examples for such $M$, compute the corresponding groups $\text{Aut}(M)$ and determine the maximal subsets of $M$ that cannot be separated by global continuous CR-functions. It turns out that all our examples, although partly arising in different contexts, are locally CR-equivalent to the tube $\mathcal{T} = \mathcal{C} \times i\mathbb{R}^3 \subset \mathbb{C}^3$ over the future light cone $\mathcal{C} := \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = x_3^2, x_3 > 0\}$ in 3-dimensional space-time.

1. Introduction

The notion of a Cauchy-Riemann-manifold (CR-manifold for short) generalizes that of a complex manifold. While two complex manifolds of the same dimension are always locally equivalent, this is no longer true for CR-manifolds. A basic invariant for a CR-manifold is the so-called Levi form. In case this form is nondegenerate, CHERN and MOSER give in their celebrated paper [5] further invariants which characterize up to equivalence the local CR-structure of real hypersurfaces in $\mathbb{C}^{n+1}$. For hypersurfaces with degenerate Levi form the corresponding program seems to be much harder to overcome. Of course, in case the connected real-analytic (locally closed) hypersurface $M \subset \mathbb{C}^{n+1}$ has a Levi nondegenerate point, all points in a dense open subset of $M$ are of this type. Therefore, of a special interest are those hypersurfaces which have everywhere degenerate Levi form but are not Levi flat. Clearly, the smallest $n$ for which hypersurfaces $M \subset \mathbb{C}^{n+1}$ with this property can occur is $n = 2$, that is, where $M$ has real dimension 5. In fact, the tube $\mathcal{T} := \mathcal{C} \times i\mathbb{R}^3 \subset \mathbb{C}^3$ over the future light cone $\mathcal{C} := \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = x_3^2, x_3 > 0\}$ in 3-dimensional space-time is a well known example of this type. This CR-manifold, which is the starting point for our studies, is even homogeneous in the sense that a Lie group of CR-automorphisms acts transitively. In a way, $\mathcal{T}$ may be considered as a model surface [7] for locally homogeneous Levi degenerate surfaces, similar to the sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ for spherical surfaces in the Levi nondegenerate case, compare e.g. [4] for the classification of homogeneous surfaces of this type and [1] for general homogeneous CR-manifolds.

In this paper we examine several naturally occurring CR-manifolds of dimension 5 with degenerate Levi-form. Most (but not all) of these are locally homogeneous and 2-nondegenerate. Surprisingly, it turns out that all of the latter are locally equivalent to the tube $\mathcal{T}$ over the light cone. From the global point of view, however, there are large classes of pairwise CR-nonequivalent manifolds which all are locally CR-equivalent to the tube $\mathcal{T}$. The classification of all homogeneous CR-manifolds of this type can essentially be reduced to the study of homogeneous domains in the real projective space $\mathbb{P}_3(\mathbb{R})$.

The paper is organized as follows:

In Section 2 we fix notation and recall some basic facts concerning the geometry of CR-manifolds.

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In Section 3 we recall some (more or less known) properties of the model example \( \mathcal{F} \) and focus on an explicit description of the Lie algebra of all infinitesimal CR-automorphisms.

In Section 4 we investigate hypersurface germs at \( 0 \in \mathbb{C}^3 \) which correspond to the partial normal form \( (A,\ii,2) \) in [6] (the only one out of 8 for holomorphically nondegenerate hypersurfaces to which uniformly Levi degenerate hypersurfaces can belong). Besides the fact that a partial normal form presentation of a hypersurface germ \( (M,0) \) is not an invariant in the strict sense (since it still depends on the choice of an adapted coordinate system and a further group of local CR-automorphisms), in general, it is rather difficult to deduce from the information encoded in the partial normal form only whether \( (M,0) \) is locally homogeneous or not. In Proposition 4.9 we give for certain germs a criterion for local homogeneity. We then introduce a family \( (\mathcal{M}_t)_{t \in \mathbb{R}} \) of germs belonging to the partial normal form \( (A,\ii,2) \) and compute explicitly the Lie algebras \( \mathfrak{hol}(\mathcal{M}_t,0) \) of of germs of infinitesimal CR-transformations at \( 0 \). It turns out that \( (\mathcal{M}_t)_{t \in \mathbb{R}} \) varies from \( \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{su}(2) \) \((t < 0)\) over a non semi-simple Lie algebra \( (t = 0) \) to \( \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}) \) \((t > 0)\) and jumps for \( t = 1 \) to the 10-dimensional Lie algebra \( \mathfrak{so}(2,3) \). As main result of the section we show that actually \( \mathcal{M}_1 \) is locally CR-equivalent to the tube \( \mathcal{F} \) and provide an explicit local CR-isomorphism. Moreover, the family \( (\mathcal{M}_t)_{t \in \mathbb{R}} \) helps to clarify several results which can be found in the literature, compare e.g. the concluding paragraph in Section 4.

In Section 5 we study global properties of CR-manifolds that are locally CR-equivalent to the tube \( \mathcal{F} \). Our first result states that on the universal covering of \( \mathcal{F} \), which is infinitely sheeted, every global continuous CR-function is the pull-back of a function defined on \( \mathcal{F} \). We then introduce a CR-deformation family \( (\mathcal{Z}_t)_{t > 0} \) of \( \mathcal{F} \) with interesting properties: The \( \mathcal{Z}_t \) are pairwise CR-nonequivalent CR-manifolds locally CR-equivalent to \( \mathcal{F} \) and \( \mathcal{Z}_1 = \mathcal{F} \). On the other hand, all \( \mathcal{Z}_t \) are diffeomorphic to \( \mathcal{F} \) as real manifolds. What concerns the separation properties by global CR-functions, the \( \mathcal{Z}_t \) behave differently for various \( t \): For instance, if \( t = p/q \) with \( p,q \in \mathbb{N} \) relatively prime, there are precisely \( p \) points in \( \mathcal{Z}_t \) that cannot be separated from a given point in \( \mathcal{Z}_t \) by global CR-functions. Further, for \( t \) irrational, there is a closed hypersurface \( \mathcal{N}_t \subset \mathcal{Z}_t \) such that every continuous CR-function on \( \mathcal{Z}_t \) is real-analytic outside \( \mathcal{N}_t \), while there do exist continuous CR-functions on \( \mathcal{Z}_t \) which are not globally real-analytic.

In Section 6 we consider a hypersurface \( \mathcal{R} \subset \mathbb{C}^3 \) which, in a certain sense, is universal for all homogeneous CR-manifolds that are locally CR-equivalent to the tube \( \mathcal{F} \). This hypersurface occurs as the smooth boundary part of the Lie ball in \( \mathbb{C}^3 \) (biholomorphic image of Siegel’s upper half-plane in the space of symmetric 2\(\times\)2-matrices via a Cayley transformation). One result is that every simply-connected homogeneous CR-manifold locally CR-equivalent to \( \mathcal{F} \) acts transitively. What concerns some natural group actions, we observe that the nonclosed orbits in the complex manifold \( \text{SL}(2,\mathbb{C}) \) under the action of the group \( \text{SL}(2,\mathbb{R}) \times \text{SL}(2,\mathbb{R}) \) given by \( z \mapsto gz\overline{h}^{-1} \) or the nonclosed orbits in the same complex manifold, but under a different action, namely of \( \text{SL}(2,\mathbb{C}) \cong \text{Spin}(1,3) \) acting by \( z \mapsto g\overline{z}f \), are all CR-equivalent to certain homogeneous domains in \( \mathcal{R} \).

We close by indicating in Section 7 how some results from Section 5 can be generalized to higher dimensions.

2. Preliminaries

In this paper we use essentially the same conventions and notation as in [11, section 2]: Let \( Z \) be a complex manifold \( Z \) and \( \pi : TZ \to Z \) its tangent bundle. Then \( TZ \) also has the structure of a complex manifold with \( \pi \) being a holomorphic submersion. In particular, every tangent space \( T_aZ, a \in Z, \) is a complex vector space with \( \text{dim}_\mathbb{C} T_aZ = n \) if \( Z \) has complex
dimension $n$ at $a$. By $\mathfrak{hol}(Z)$ we denote the complex Lie algebra of all holomorphic vector fields on $Z$, that is, of all holomorphic sections $\xi : Z \to TM$ in the tangent bundle over $Z$. For every $a \in Z$ we denote the corresponding tangent vector in $T_a Z$ with $\xi_a$ (and not with $\xi(a)$).

A (locally closed) real submanifold $M \subset Z$ is called a CR-manifold if the complex dimension of the holomorphic tangent space $H_a M := T_a M \cap iT_a M$ is a locally constant function of $a \in M$ (here every tangent space $T_a M$ is considered as an $\mathbb{R}$-linear subspace of $T_a Z$). A smooth function $f : M \to \mathbb{C}$ is called CR if for every $a \in M$ the restriction of its differential to $H_a M$ is complex linear. More generally, a smooth mapping $\varphi : M \to M'$ between CR-manifolds is CR, if the differential $d_a \varphi : T_a M \to T_{\varphi(a)} M'$ maps $H_a M$ to $H_{\varphi(a)} M'$ and is complex linear thereon. The notion of CR-function and CR-mapping can also be generalized to the nonsmooth case; then the conditions have to hold in the distribution sense, see [2] as a general reference for CR-manifolds.

In the following we only consider connected real-analytic CR-submanifolds $M$ of $Z$. With $\mathfrak{hol}(M)$ we denote the real Lie algebra of all vector fields $\eta : M \to TM \subset TZ$ on $M$ with the following property: To every $a \in M$ there is an open neighbourhood $U$ of $a$ with respect to $Z$ and a holomorphic vector field $\xi \in \mathfrak{hol}(U)$ with $\xi_z = \eta_z$ for all $z \in U \cap M$. Clearly, in case the CR-submanifold $M$ is generic in $Z$ (that is, satisfies $T_a Z = T_a M + iT_a M$ for every $a \in M$), the local holomorphic extension $\xi$ above can always be chosen in such a way that $U$ is an open neighbourhood of all of $M$ in $Z$. The elements of $\mathfrak{hol}(M)$ are also called infinitesimal CR-transformations on $M$, the reason being that the corresponding local flow on $M$ consists of real-analytic CR-transformations. The vector field $\eta \in \mathfrak{hol}(M)$ is called complete on $M$ if the local flow actually consists of a one-parameter family $(g_t)$, $t \in \mathbb{R}$, of global transformations. Then we write $\exp(\eta) := g_1$ and get a mapping $\exp : \text{aut}(M) \to \text{Aut}(M)$, where $\text{aut}(M) \subset \mathfrak{hol}(M)$ is the subset of all complete vector fields and $\text{Aut}(M)$ is the group of all real-analytic CR-diffeomorphisms of $M$. In general, $\text{aut}(M) \subset \mathfrak{hol}(M)$ is not closed under addition nor under taking brackets. But in case there is a Lie subalgebra $g \subset \mathfrak{hol}(M)$ of finite dimension containing $\text{aut}(M)$, then a result of [14] implies: $\text{aut}(M)$ itself is a Lie algebra of finite dimension and $\text{Aut}(M)$ has a unique Lie group structure such that $\exp : \text{aut}(M) \to \text{Aut}(M)$ is locally bianalytic in a neighbourhood of $0 \in \text{aut}(M)$.

For every $a \in M$ denote by $\mathfrak{hol}(M, a)$ the space of all germs at $a$ of infinitesimal CR-transformations defined in arbitrary open neighbourhoods of $a$. Then $\mathfrak{hol}(M, a)$ is a real Lie algebra, and $\text{aut}(M, a) := \{ \xi \in \mathfrak{hol}(M, a) : \xi_a = 0 \}$ is a Lie subalgebra of finite codimension. We call the CR-manifold $M$ locally homogeneous if the evaluation map $\mathfrak{hol}(M, a) \to T_a M$, $\xi \mapsto \xi_a$, is surjective for every $a \in M$. Local homogeneity implies that to every pair $a, b$ of points in $M$ there exist open neighbourhoods $U, V$ of $a, b$ in $M$ together with a real-analytic CR-diffeomorphism $U \to V$. We call $M$ homogeneous if there exists a connected Lie group $G$ together with a group homomorphism $\Phi : G \to \text{Aut}(M)$ such that the mapping $G \times M \to M$, $(g, x) \mapsto \Phi(g)x$, is real-analytic and $G$ acts transitively on $M$ via $\Phi$. Clearly, ‘homogeneous’ implies ‘locally homogeneous’.

The CR-manifold $M \subset Z$ is called holomorphically nondegenerate if for every domain $U \subset Z$ and every $\xi \in \mathfrak{hol}(U)$ with $\xi_x, i\xi_x \in T_x M$ for all $x \in U \cap M$ necessarily $\xi_x = 0$ holds for all $x \in U \cap M$. In case $M$ is a real-analytic hypersurface in $Z$, holomorphic nondegeneracy is equivalent to $\dim \mathfrak{hol}(M, a) < \infty$ for all $a \in M$, see e.g. [2] p. 367 for this and related results. In this note (except in the final section 7) the complex manifold $Z$ always has dimension 3 (either $\mathbb{C}^3$ or a nonsingular quadric in $\mathbb{P}_4(\mathbb{C})$) and $M$ is a real hypersurface that is 2-nondegenerate, a property that implies ‘holomorphically nondegenerate’.

**Convention for notating vector fields.** In this paper we do not need the complexified tangent bundle $TM \otimes \mathbb{R} \mathbb{C}$ of $M$. All vector fields occurring here correspond to ‘real vector fields’ elsewhere. In particular, if $E$ is a complex vector space of finite dimension and $U \subset E$ is an open subset
then the vector fields $\xi \in \mathfrak{hol}(U)$ correspond to holomorphic mappings $f : U \to E$, and the correspondence is given in terms of the canonical trivialization $TU \cong U \times E$ by identifying the mapping $f$ with the vector field $\xi = (\operatorname{id}_U, f)$. To have a short notation we also write

$$\xi = f(z) \frac{\partial}{\partial z}.$$ 

As an example, if $E$ is the space of all complex $n \times m$-matrices and $c$ is an $m \times n$-matrix, then $zc\partial/\partial z$ denotes the quadratic vector field on $E$ corresponding to the holomorphic mapping $E \to E, z \mapsto zc$. As soon as the vector field $\xi = f(z) \frac{\partial}{\partial z}$ is considered as differential operator, special caution is necessary: $\xi$ applied to the smooth function $h$ on $U$ is $\xi h = f(z) \frac{\partial}{\partial z}h + \overline{f}(z) \frac{\partial}{\partial \overline{z}} h$. We therefore stress again that we write

$$\xi(z) = f_1(z) \frac{\partial}{\partial z_1} + f_2(z) \frac{\partial}{\partial z_2} + \ldots + f_n(z) \frac{\partial}{\partial z_n},$$ 

where $f = (f_1, \ldots, f_n) : U \to \mathbb{C}^n$ is holomorphic.

3. The tube over the light cone

In this section we introduce the light cone tube $\mathcal{T}$, a real hypersurface of $\mathbb{C}^3$ that is everywhere Levi degenerate but has finite dimensional Lie algebra $\mathfrak{hol}(\mathcal{T}, a)$ at every point $a \in \mathcal{T}$. From [11] we recall the explicit form of this Lie algebra and present a root decomposition in terms of vector fields. This will enable us in the next section to show that a certain local equation defines a CR-manifold locally equivalent to $\mathcal{T}$.

A convenient model for 3-dimensional space time is the linear subspace $V \subset \mathbb{R}^{2\times 2}$ of all real symmetric $2 \times 2$-matrices with the (normalized) trace as time coordinate. There

$$\Omega := \{ v \in V : v \text{ positive definite} \}$$

is the **future cone** and its smooth boundary part is the **future light cone**

$$\mathcal{C} := \{ v \in V : \det(v) = 0, \operatorname{tr}(v) > 0 \} = \left\{ \begin{pmatrix} t + x_1 & x_2 \\ x_2 & t - x_1 \end{pmatrix} \in V : t^2 = x_1^2 + x_2^2, \ t > 0 \right\}. $$

It is obvious that there exists a 2-dimensional group of linear transformations on $\mathbb{R}^3$ acting simply transitive on $\mathcal{C}$.

The main object of our interest is the tube

$$\mathcal{T} = \mathcal{C} \oplus iV = \{ z \in V \oplus iV : \det(z + \overline{z}) = 0, \operatorname{Re} \operatorname{tr}(z) > 0 \}$$

over $\mathcal{C}$, where we identify the complexification $V \oplus iV$ in the obvious way with the space $E$ of all symmetric complex $2 \times 2$-matrices. $\mathcal{T}$ is the smooth boundary part of the **tube domain** $\mathcal{H} := \Omega \oplus iV$ in $E$ (Siegel’s upper half plane up to the factor $i$) and is a locally closed real-analytic hypersurface of $E$ with everywhere degenerate Levi form. Actually it is well known that $\mathcal{T}$ is everywhere 2-nondegenerate as CR-manifold.
The CR-automorphisms. The 7-dimensional Lie group of affine transformations on $E$

$$\{ z \mapsto g z' + i v : g \in \text{GL}(2, \mathbb{R}), v \in V \}$$

acts transitively on $\mathcal{T}$ and $\mathcal{H}$, where $g'$ denotes the transpose of $g$. It is known [11] that $\text{Aut}(\mathcal{T})$ is the group of all transformations (3.2) while $\text{Aut}(\mathcal{H})$ is the 10-dimensional group of all biholomorphic transformations

$$z \mapsto (a z - i b)(i c z + d)^{-1},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the real symplectic subgroup $\text{Sp}(2, \mathbb{R}) \subset \text{SL}(4, \mathbb{R})$ with $a, b, c, d \in \mathbb{R}^{2 \times 2}$, see [12] p. 351. Differentiating the action of $\text{Sp}(2, \mathbb{R})$ gives

$$\mathfrak{g} := \text{aut}(\mathcal{H}) = \{ (b + c z + z c' + z d z) \partial/\partial z : b, d \in \mathbb{C} \} \cong \mathfrak{sp}(2, \mathbb{R}) \cong \mathfrak{so}(2, 3)$$

with convention (2.1) in effect. All vector fields in $\text{aut}(\mathcal{H})$ are polynomial of degree $\leq 2$ on $E$, in particular,

$$\text{aut}(\mathcal{H}) \subset \mathfrak{hol}(\mathcal{T}) \subset \mathfrak{hol}(\mathcal{T}, a)$$

in a canonical way for all $a \in \mathcal{T}$. One of the main results of [11] states (even for higher dimensional examples of this type) that all three Lie algebras above coincide, see Proposition 4.3 in [11].

The vector fields in $\mathfrak{g}$ corresponding to $c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in (3.4) are

$$\zeta_1 := 2 z_0 \partial/\partial z_0 + z_1 \partial/\partial z_1 \quad \text{and} \quad \zeta_2 := z_1 \partial/\partial z_1 + 2 z_2 \partial/\partial z_2,$$

when expressed in the coordinates

$$(z_0, z_1, z_2) \mapsto \begin{pmatrix} z_0 & z_1 \\ z_1 & z_2 \end{pmatrix}$$

on $E$. These give the following decomposition

$$\mathfrak{g} = \bigoplus_{\nu \in \mathbb{Z}^2} \mathfrak{g}^\nu \quad \text{with} \quad [\mathfrak{g}^\mu, \mathfrak{g}^\nu] \subset \mathfrak{g}^{\mu + \nu},$$

where $\mathfrak{g}^\nu := \{ \xi \in \mathfrak{g} : [\zeta_j, \xi] = \nu_j \xi \quad \text{for} \quad j = 1, 2 \}$ for $\nu = (\nu_1, \nu_2)$ and, in particular, $\mathfrak{g}^0 = \mathbb{R} \zeta_1 \oplus \mathbb{R} \zeta_2$. For every root (i.e. $\nu \neq 0$ and $\mathfrak{g}^\nu \neq 0$) the corresponding root space $\mathfrak{g}^\nu$ has real dimension 1, and the set of all roots is visualized by the eight vectors in Figure 1, the root system of the complex simple Lie algebra $\mathfrak{so}(5, \mathbb{C}) \cong \mathfrak{sp}(2, \mathbb{C})$.

![Figure 1](image1.png)

![Figure 2](image2.png)
An explicit choice of root vectors $\xi^\nu \in \mathfrak{g}^\nu$ is as follows:

\[(3.7)\]
\[
\begin{align*}
\xi^{0,2} &= i z_1^2 \partial/\partial z_0 + i z_1 z_2 \partial/\partial z_1 + i z_2^2 \partial/\partial z_2 \\
\xi^{-1,1} &= 2z_1 \partial/\partial z_0 + z_2 \partial/\partial z_1 \\
\xi^{-2,0} &= i \partial/\partial z_0 \\
\xi^{-1,-1} &= i \partial/\partial z_1 \\
\xi^{1,1} &= 2i z_0 z_1 \partial/\partial z_0 + i(z_1^2 + z_0 z_2) \partial/\partial z_1 + 2i z_1 z_2 \partial/\partial z_2 \\
\xi^{2,0} &= i z_0^2 \partial/\partial z_0 + i z_0 z_1 \partial/\partial z_1 + i z_1^2 \partial/\partial z_2 \\
\xi^{1,-1} &= -z_0 \partial/\partial z_1 - 2z_1 \partial/\partial z_2 \\
\xi^{0,-2} &= i \partial/\partial z_2.
\end{align*}
\]

**Remarks.** The choice of the vector fields (3.5) gives the Cartan subalgebra $\mathfrak{g}^0 = \mathbb{R}\zeta_1 \oplus \mathbb{R}\zeta_2$ of $\mathfrak{g}$. Unlike the complex situation there may exist several nonconjugate Cartan subalgebras in a real semi-simple Lie algebra. In our particular situation $\mathfrak{g} \cong \mathfrak{so}(2,3)$ there are precisely four conjugacy classes of Cartan subalgebras.

Note that all vector fields in $\mathfrak{g}^0$ vanish simultaneously in a unique point of the nonsmooth boundary part $iV$ of $\mathcal{H}$, namely in the origin $0 \in E$. In Section 4 we will consider another Cartan subalgebra of $\mathfrak{g}$ that vanishes in a point of the smooth boundary part $\mathcal{T}$ of $\mathcal{H}$, and in Section 6 even a third one is considered that vanishes in a point of $\mathcal{H}$, compare (6.2).

It is not a coincidence that the Cartan subalgebras above vanish at some point of $E$. To put it in a broader perspective, $E$ admits a natural embedding in a compact complex flag manifold $Z$ together with a global action of $\mathfrak{Sp}(2,\mathbb{R})$ on $Z$ that extends the action (3.3). By Borel’s fixed point theorem every torus $H := \exp(\mathfrak{h})$, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, has a fixed point in $Z$.

The only nonclosed orbits in $E$ of the group

\[H := \{ z \mapsto g z g' + i v : g \in \mathrm{SL}(2,\mathbb{R}), v \in V \}\]

are $\mathcal{T}$ and $-\mathcal{T}$. Furthermore, the real function $\det(z + \overline{z})$ is constant on $H$-orbits and its sign determines the signature of the Levi form for the orbits.

### 4. A local realization

In this section we introduce a family $(\mathcal{M}_t)_{t \in \mathbb{R}}$ of local CR-submanifolds in $\mathbb{C}^3$ that may be considered as local deformations of the tube $\mathcal{T}$, are pairwise CR-inequivalent and have semi-simple Lie algebras $\mathfrak{hol}(\mathcal{M}_t,0)$, $t \neq 0$, of dimension $\geq 6$. In particular, with $\mathcal{M}_1$ we get a normalized equation for $\mathcal{T}$ in the sense of [7].

In the following we consider in $\mathbb{C}^3$ with coordinates $(w,z_1,z_2)$ hypersurfaces $M$ of the following type. $M$ is given near $0 \in \mathbb{C}^3$ by an equation

\[(4.1)\]
\[w + \bar{w} = 2z_1 \bar{z}_1 + (z_1^2 \bar{z}_2 + z_2^2 \bar{z}_1) + q(\operatorname{Im}(w), z_1, \bar{z}_1, z_2, \bar{z}_2),\]

where $q$ is a convergent real power series whose nonzero monomial terms either have degree $\geq 4$ or have degree 3 and then contain $\operatorname{Im}(w)$. Then (4.1) is just the partial normal form (A.ii,2) in [6]. In particular, $M$ is 2-nondegenerate at 0 and there the Levi kernel is the $z_2$-coordinate axis. Further, $\dim \mathfrak{hol}(M,0) < \infty$ holds.

Consider on $\mathbb{C}^3$ the group of all linear transformations $(w,z_1,z_2) \mapsto (s^2 w, s t z_1, t^2 z_2)$ with $s \in e^{i\mathbb{R}}$ and $t \in e^{i\mathbb{R}}$. Clearly, this group is the exponential of the Lie algebra spanned in $\mathfrak{aut}(\mathbb{C}^3)$ by the two (real) holomorphic vector fields $\zeta_1$ and $i \zeta_2$ where, using convention (2.1), $\zeta_1$ and $\zeta_2$ are given in the local coordinates by

\[(4.2)\]
\[\zeta_1 := 2w \partial/\partial w + z_1 \partial/\partial z_1 \quad \text{and} \quad \zeta_2 := z_1 \partial/\partial z_1 + 2z_2 \partial/\partial z_2.\]
The coordinate expression of these vector fields coincides with those in (3.5) (if the variable $w$ is renamed $z_0$). The same will occur in Section 6.

From now on we assume for the rest of the section that $\mathfrak{g} := \mathfrak{ho}(M,0)$ contains the vector fields $\zeta_1$ and $i\zeta_2$. For the sake of clarity let us emphasize that under this assumption the vector field $\zeta_2$ cannot be contained in $\mathfrak{g}$ since $M$ is holomorphically nondegenerate.

Our first goal is to show that under the assumption $\zeta_1, i\zeta_2 \in \mathfrak{g}$ there is a unique equation of the form (4.1) such that $M$ is locally homogeneous at 0 $\in M$. It will turn out that in this case $M$ is locally CR-equivalent to the tube $\mathcal{F}$ over the future light cone in $\mathbb{C}^3$ as considered in Section 2.

We need some preparation: Let $\mathfrak{p}$ be the complex Lie algebra of all polynomial holomorphic vector fields on $\mathbb{C}^3$ with coordinates $w, z_1, z_2$. The natural adjoint action of $\zeta_1$ and $i\zeta_2$ induces a grading of $\mathfrak{p}$: Consider the lattice
\begin{equation}
\Lambda := \{ n + im \in \mathbb{Z} + i\mathbb{Z} : n + m \in 2\mathbb{Z} \}
\end{equation}
in $\mathbb{C}$ and denote by $\mathfrak{p}^\lambda$ the $\lambda$-eigenspace of $\mathfrak{ad}(\zeta_1 + i\zeta_2)$ in $\mathfrak{p}$. Note that every monomial vector field is contained in some $\mathfrak{p}^\lambda$: For all $m, n, l \in \mathbb{N}$ and $k = 0, 1, 2$ the vector field $w^m z_1^n z_2^i \partial/\partial z_k$ (with $\partial/\partial z_0 := \partial/\partial w$ to simplify notation) is in $\mathfrak{p}^\lambda$ for $\lambda = (2m + n + k - 2) + i(n + 2l - k)$. In particular, every $\mathfrak{p}^\lambda$ has finite dimension and
\begin{equation}
\mathfrak{p} = \bigoplus_{\lambda \in \Lambda} \mathfrak{p}^\lambda,
\quad [\mathfrak{p}^\lambda, \mathfrak{p}^\mu] \subset \mathfrak{p}^{\lambda + \mu}.
\end{equation}
A small check shows that $\mathfrak{p}^\lambda = 0$ if $\min(\text{Re}\lambda, \text{Im}\lambda, \text{Re}\lambda + \text{Im}\lambda) < -2$. For instance, for $\lambda \in \{-2, -1 - i, -2i\}$ the spaces $\mathfrak{p}^\lambda = \mathbb{C}^\xi^\lambda$ are 1-dimensional with generators
\begin{equation}
\xi^{-2} := i\partial/\partial w, \quad \xi^{-1-i} := \partial/\partial z_1, \quad \xi^{-2i} := \partial/\partial z_2.
\end{equation}
In Figure 2 the nonzero complex dimensions of $\mathfrak{p}^\lambda$ are listed for all $\lambda = m + in$ with $m \leq 6$ and $n \leq 4$.

We now explain how $\mathfrak{g}$ is related to the decomposition $\mathfrak{p} = \bigoplus \mathfrak{p}^\lambda$. A priori, the finite dimensional Lie algebra $\mathfrak{g}$ is contained in $\mathfrak{ho}(U)$ for some open neighbourhood $U$ of 0 $\in \mathbb{C}^3$. The sum $\mathfrak{l} := \mathfrak{g} + i\mathfrak{g}$ in $\mathfrak{ho}(U)$ actually is a direct sum of real subspaces. Since the complex Lie algebra $\mathfrak{l}$ contains the Euler vector field $(\zeta_1 + i\zeta_2)/2$ necessarily $\mathfrak{l}$ is contained in $\mathfrak{p}$. Both $\mathfrak{l}$ and $\mathfrak{g}$ are invariant under $\mathfrak{ad}(\zeta_1 + i\zeta_2)$. This gives immediately the decomposition
\begin{equation}
\mathfrak{l} = \bigoplus_{\lambda \in \Lambda} \mathfrak{l}^\lambda, \quad \mathfrak{l}^\lambda := \mathfrak{l} \cap \mathfrak{p}^\lambda.
\end{equation}
Let $\xi \mapsto \overline{\xi}$ be the conjugation of $\mathfrak{l}$ with respect to the real form $\mathfrak{g}$. Then
\begin{equation}
\overline{\mathfrak{l}^\lambda} = \mathfrak{l}^{-\bar{\lambda}}
\end{equation}
for all $\lambda$ and consequently $\mathfrak{g} \cap \mathfrak{l}^\lambda = 0$ if $\lambda \notin \mathbb{R}$. Furthermore, $\mathfrak{g}$ admits the generalized eigenspace decomposition
\begin{equation}
\mathfrak{g} = \bigoplus_{\text{Im}\lambda \geq 0} \mathfrak{g}^{[\lambda]}, \quad \mathfrak{g}^{[\lambda]} := \mathfrak{g} \cap (\mathfrak{l}^\lambda + \overline{\mathfrak{l}^\lambda}).
\end{equation}
Next we claim that

\[(4.8) \quad g^{[0]} = \mathbb{R}\zeta_1 \oplus \mathbb{R}i\zeta_2 \]

holds. Indeed, assume to the contrary that there exists \( \xi \in g^{[0]} \) with \( \xi \notin (\mathbb{R}\zeta_1 + \mathbb{R}i\zeta_2) \). Then \( \xi \in \mathfrak{F}^0 \) and, after subtracting a suitable linear combination of \( \zeta_1, i\zeta_2 \), we may assume \( \xi = \alpha w \partial/\partial w + \beta z_2 \partial/\partial z_2 \) for certain \( \alpha, \beta \in \mathbb{C} \). Applying \( \xi \) to the defining equation (4.1) yields on \( M \) for \( r := \text{Re}(\alpha) \) and \( s := \text{Im}(\alpha) \) the identity

\[
2r(2z_1\bar{z}_1 + z_1^2\bar{z}_2 + \bar{z}_1^2 z_2) - 2s \text{Im}(w) = \beta z_1^2 \bar{z}_2 + \bar{\beta} z_1^2 \bar{z}_2 + \cdots
\]

up to terms of degree \( \geq 4 \) or of degree 3 containing \( \text{Im}(w) \) (the convention (2.1) has to be observed). This forces \( \alpha = \beta = 0 \) and, in particular, \( \xi \in (\mathbb{R}\zeta_1 + \mathbb{R}i\zeta_2) \). \( \square \)
Our first result now states

4.9 Proposition. Suppose that the CR-manifold $M$ is given locally by the equation (4.1) and that $\mathfrak{g} = h\mathfrak{o}(M, 0)$ contains the vector fields $\zeta_1, i\zeta_2$. Then the following conditions are equivalent.

(i) $M$ is locally homogeneous at 0.
(ii) $\mathfrak{g}$ contains the vector fields
$$\eta := z_2 \partial/\partial w + (1 - z_2) \partial/\partial z_2$$
and $\chi := z_1^2 \partial/\partial w - z_1 z_2 \partial/\partial z_1 + (1 - z_2^2) \partial/\partial z_2$.
(iii) The term $q$ in the defining equation (4.1) is
$$q = (z_1 \bar{z}_1 + z_2^2 \bar{z}_2 + \bar{z}_1^2 z_2) \cdot \sum_{j=1}^{\infty} (z_2 \bar{z}_2)^j,$$
that is, the defining equation reads $w + w = (z_1 \bar{z}_1 + z_2^2 \bar{z}_2 + \bar{z}_1^2 z_2)(1 - z_2 \bar{z}_2)^{-1}$.

Proof. (i) $\implies$ (ii) Suppose that $M$ is locally homogeneous at 0. Then $\{\xi_0 : \xi \in \{\} = T_0 M + iT_0 M \cong \mathbb{C}^3$. Since $t^\lambda$ vanishes at 0 for every $\lambda \notin \{-2, -1 - i, -2i\}$ we get $\dim t^\lambda = 1$ for all $\lambda \in \{\}$ where we have used (4.6). Since every $\mathfrak{g}^{[\lambda]}$ is invariant under $\text{ad}(i\zeta_2)$ we have $\{\xi_0 : \xi \in \mathfrak{g}^{[\lambda]}\} = \{\xi_0 : \xi \in t^\lambda\}$ if $\text{Im}(\lambda) < 0$. Therefore there exists a unique $\xi \in \mathfrak{H}^{-1+i}$ with $\eta := \xi^{-1+i} + \xi \in \mathfrak{g}$, that is, $\eta = \partial/\partial z_1 + \alpha z_1 \partial/\partial w + \beta z_2 \partial/\partial z_2 \in \mathfrak{g}$ for certain $\alpha, \beta \in \mathbb{C}$. Applying $\eta$ to the equation (4.1) and comparing homogeneous terms immediately gives $\alpha = 2$ and $\beta = -1$. In the same way we get a vector field $\chi := \partial/\partial z_2 + z_1^2 \partial/\partial w - \delta z_1 z_2 \partial/\partial z_1 - \varepsilon z_2^2 \partial/\partial z_2 \in \mathfrak{g}^{[2]}$ for suitable $\delta, \varepsilon \in \mathbb{C}$. But then $[\mathfrak{g}^{-1+i}, \mathfrak{g}^{[2]}) \subset \mathfrak{g}^{-1+i}$ and
$$[\eta, \chi] = (2z_2 + 2(\delta - 1)z_1 z_2) \partial/\partial w + (1 - \delta z_2 + (\delta - \varepsilon)z_2^2) \partial/\partial z_1 \in \mathfrak{g}^{-1+i},$$
imply $[\eta, \chi] = \eta$, i.e. $\delta = \varepsilon = 1$.

(ii) $\implies$ (i) In case $\eta \in \mathfrak{g}^{-1+i}$, $\chi \in \mathfrak{g}^{[2]}$ the linear subspace $\mathfrak{g}^{-1+i} \oplus \mathfrak{g}^{[2]} \subset \mathfrak{g}$ spans the holomorphic tangent space $H_0(M) \subset T_0 M$. The vector field $[\eta, [i\zeta_2, \eta]] \in \mathfrak{g}^{-2}$ does not vanish and thus spans $T_0 M/H_0 M$, that is, $\mathfrak{g}$ spans the full tangent space $T_0 M$.

(iii) $\implies$ (ii) This is easily checked.

(ii) $\implies$ (iii) The $\mathbb{R}$-linear span of $\xi^{-2}$, $\eta$, $\chi$, $[i\zeta_2, \eta]$, $[i\zeta_2, \chi]$, $i\zeta_2$ is a Lie subalgebra $\mathfrak{a} \subset \mathfrak{g}$ of dimension 6 which also spans the full tangent space of $M$ at 0. Therefore $M$ is the local integral manifold of $\mathfrak{a}$ near 0 $\in \mathbb{C}^3$. Denote by $M$ the hypersurface defined near 0 $\in \mathbb{C}^3$ by the equation (4.1), where $q$ is replaced by $\widetilde{q} = (2z_1 \bar{z}_1 + z_2^2 \bar{z}_2 + \bar{z}_1^2 z_2) \cdot \sum_{j=1}^{\infty} (z_2 \bar{z}_2)^j$. By a routine check it is verified that also $\mathfrak{a} \subset h\mathfrak{o}(M, 0)$ holds, that is, also $M$ is the local integral manifold of $\mathfrak{a}$ near 0 $\in \mathbb{C}^3$. This implies that the germs of $M$ and $\bar{M}$ at 0 coincide and hence that $q = \widetilde{q}$ as power series.

\[ \square \]

A family of CR-manifolds. In the following we fix $t \in \mathbb{R}$ and consider the CR-hypersurface $\mathcal{M}_t$ given in $\mathbb{C}^3$ near the origin by the equation

$$w + \bar{w} = (2z_1 \bar{z}_1 + z_2^2 \bar{z}_2 + \bar{z}_1^2 z_2)(1 - t z_2 \bar{z}_2)^{-1}. \tag{4.10}$$

Clearly, this is a special case of (4.1), and it is easily checked that the vector fields $\zeta_1, i\zeta_2$ are contained in $\mathfrak{g} := \mathfrak{g}_t := h\mathfrak{o}(\mathcal{M}_t, 0)$. The manifold $\mathcal{M}_t$ for $t = 0$ has already been studied in [8] and for $t = 1$ in [9], where also the vector fields $\eta, \chi$ from (4.9) occur.

Our next goal is to describe how the Lie algebra $\mathfrak{g}$ depends on the parameter $t$. Put $\Psi := \{\pm 2, \pm 2i\}$ and define for each $\lambda \in \Psi \subset \Lambda$ the vector fields $\xi^\lambda \in \mathfrak{H}^\lambda$ as follows:

$$\xi^{2i} := z_1^2 \partial/\partial w - t z_1 z_2 \partial/\partial z_1 - t z_2^2 \partial/\partial z_2,$$
$$\xi^{2} := it w z_1 \partial/\partial w + it w z_1 \partial/\partial z_1 - iz_2^2 \partial/\partial z_2$$
and $\xi^{-2i}, \xi^{-2}$ as in (4.5). Furthermore put

$$\eta^\lambda := \begin{cases} \xi^\lambda + \xi^{\overline{\lambda}} & \text{Im}(\lambda) \geq 0 \\ i\xi^\lambda - i\xi^{\overline{\lambda}} & \text{otherwise} \end{cases}$$

(4.12)
and notice that
\[ \eta^\lambda \in \mathfrak{g}, \quad [\zeta_1, \eta^\lambda] = \text{Re}(\lambda) \eta^\lambda \quad \text{and} \quad [i\zeta_2, \eta^\lambda] = \text{Im}(\lambda) \eta^\lambda \]
hold for every \( \lambda \in \Psi \). By applying vector fields to equation (4.10) it is not difficult to see that the Lie algebra
\[
\mathfrak{h} := (\mathbb{R}\eta^{-2} \oplus \mathfrak{C} \zeta_1 \oplus \mathbb{R}\eta^2) \oplus (\mathbb{R}\eta^{-2i} \oplus \mathbb{R}i\zeta_2 \oplus \mathbb{R}\eta^{2i})
\]
(4.13)
is contained in \( \mathfrak{g} \). In case \( t \neq 1 \) the CR-manifold \( \mathcal{M} \) cannot be locally homogeneous at 0 since then the vector field \( \chi \) from (4.9) is not contained in \( \mathfrak{g}^{[2]} = \mathbb{R}\eta^{2i} \oplus \mathbb{R}\eta^{-2i} \), compare Proposition 4.9. In case \( t = 0 \) the right hand side of (4.10) reduces to a cubic polynomial and, with some computation, it can be seen that \( \mathfrak{g} = \mathfrak{h} \) in this case, compare also case B2 in \([8, p. 94]\). In all other cases \( \mathfrak{h} \) is semi-simple. We shall use this in the following.

**Explicit determination of \( \mathfrak{hol}(\mathcal{M}_t, 0) \).** For fixed \( t \in \mathbb{R} \) and \( \mathfrak{l} = \mathfrak{g} \oplus i \mathfrak{g} \) with \( \mathfrak{g} = \mathfrak{hol}(\mathcal{M}_t, 0) \) put
\[ \Phi := \{ \lambda \in \Lambda : \lambda \neq 0 \quad \text{and} \quad \mathfrak{t}^\lambda \neq 0 \} \].
Then \( \Psi \subset \Phi \) and \( \Phi \) is invariant under \( \lambda \mapsto \overline{\lambda} \). For every \( k \in \mathbb{Z} \) denote by \( d_k \) the dimension of the eigenspace \( \{ \xi \in \mathfrak{l} : [\zeta_1, \xi] = k\xi \} = \bigoplus_{\text{Re} \lambda = k} \mathfrak{t}^\lambda \). In case \( t \neq 0 \) the representation theory of \( \mathfrak{sl}(2, \mathbb{C}) \cong \mathbb{C}\eta^{-2} \oplus \mathbb{C}\zeta_1 \oplus \mathbb{C}\eta^2 \) implies \( d_k = d_{-k} \) for all \( k \), and thus \( d_1 \in \{ 0, 2 \} \), \( d_2 = 1 \) and \( d_k = 0 \) for \( k > 2 \). As a consequence, \( \Phi \) is also invariant under \( \lambda \mapsto \overline{\lambda} \) and \( \mathfrak{t}^\lambda \) has dimension 1 for every \( \lambda \in \Phi \). Because of \( (-1 - i) \notin \Phi \) in case \( t \neq 1 \) we therefore have \( \Phi = \Psi \) in this situation. This proves:

**4.14 Proposition.** In case \( t \neq 1 \) the Lie algebra \( \mathfrak{hol}(\mathcal{M}_t, 0) \) coincides with \( \mathfrak{h} \) and, in particular, has dimension 6.

It remains to consider the case \( t = 1 \). Applying \( \eta \) to equation (4.10) implies \( \eta \in \mathfrak{g} = \mathfrak{hol}(\mathcal{M}_1, 0) \). In particular \( (-1 - i) \in \Phi \) and hence \( \Phi := \{ \pm 2, \pm 1 \pm i, \pm 2i \} \), that is, \( \Phi \) can also be visualized by Figure 1, while \( \Psi \subset \Phi \) corresponds to the subset of long arrows. Define for every \( \lambda \in \Phi \) the vectors \( \xi^\lambda \in \mathfrak{P}^\lambda \) by (4.5) and
\[
\begin{align*}
\xi^{1+i} &= 2z_1 \partial/\partial w - z_2 \partial/\partial z_1 \\
\xi^{1-i} &= 2z_1 \partial/\partial z_2 - w \partial/\partial z_1 \\
\xi^{2i} &= 2wz_1 \partial/\partial w + (z_1^2 - wz_2) \partial/\partial z_1 + 2z_1 z_2 \partial/\partial z_2 \\
\xi^2 &= iw^2 \partial/\partial w + iwz_1 \partial/\partial z_1 - iz_1^2 \partial/\partial z_2 .
\end{align*}
\]
(4.15)
Notice that after replacing \( z_2 \) by \(-z_2\) all vector fields in (4.15) and (4.5) become complex multiples of those in (3.7).

Now define \( \eta^\lambda \) by (4.12) for every \( \lambda \in \Phi \). Differentiating the defining equation for \( M_1 \) along \( \eta^\lambda \) gives \( \eta^\lambda \in \mathfrak{g} \) and hence \( \mathfrak{t}^\lambda = \mathbb{C}\xi^\lambda \) for all \( \lambda \in \Phi \). For all \( \lambda, \mu \in \Phi \) with \( \lambda + \mu \neq 0 \) the identity \( [\mathfrak{t}^\lambda, \mathfrak{t}^\mu] = \mathbb{C}\xi^\lambda \) can be verified, that is, \( \mathfrak{t} \) is isomorphic to \( \mathfrak{so}(5, \mathbb{C}) \). As real form of \( \mathfrak{t} \) therefore \( \mathfrak{g} \) is isomorphic to \( \mathfrak{so}(5), \mathfrak{so}(1, 4) \) or \( \mathfrak{so}(2, 3) \). The isotropy subalgebra \( \mathfrak{i}_0 \) of \( \mathfrak{t} \) at the origin of \( \mathbb{C}^3 \) is maximal parabolic. Therefore, \( \mathcal{M}_1 \) occurs as piece of an R-orbit in a 3-dimensional quadric \( Z \subset \mathbb{P}_4(\mathbb{C}) \) or in \( \mathbb{P}_3(\mathbb{C}) \), where R is a real form of \( \mathbb{SO}(5, \mathbb{C}) \cong \mathbb{Sp}(2, \mathbb{C}) \). Since the only 2-nondegenerate hypersurfaces orbits occur in case of \( \mathbb{SO}(2, 3) \) acting on \( Z \), the Lie algebra \( \mathfrak{g} \) can only be isomorphic to \( \mathfrak{so}(2, 3) \). Since the tube \( \mathcal{T} \) over the future light cone actually can be realized as an open piece of an \( \mathbb{SO}(2, 3) \)-orbit in \( Z \), we get (a direkt proof is given by formula (4.18)): 
4.16 Proposition. The hypersurface $\mathcal{M}_1$ defined near $0 \in \mathbb{C}^3$ by the equation

\[ w + \overline{w} = (2z_1 \overline{z}_1 + z_1^2 \overline{z}_2 + \overline{z}_2^2 z_2)(1 - z_2 \overline{z}_2)^{-1} \tag{4.17} \]

is locally CR-isomorphic to the tube $\mathcal{T}$ over the future light cone.

The local CR-equivalence between $\mathcal{M}_1$ and $\mathcal{T}$ can be made explicit. For this, a transformation in $SO(5, \mathbb{C})$ has to be found that the corresponding $SO(2,3)$-orbits in the quadric $Z$ maps to each other. Let $U := \{(w, z_1, z_2) \in \mathbb{C}^3 : |z_2| < 1\}$ and consider $\mathcal{M}_1$ as closed CR-submanifold of $U$ via the equation (4.17). Then

\[ (w, z_1, z_2) \mapsto \frac{1}{1 + z_2} \left( \frac{w + wz_2 + z_1^2 \sqrt{2z_1}}{\sqrt{2z_1}} , \frac{\sqrt{2z_1}}{1 - z_2} \right) \tag{4.18} \]

defines a biholomorphic mapping $\varphi$ from $U$ to an open subset of $E$ with $\varphi(\mathcal{M}_1) \subset \mathcal{T}$. The inverse $\varphi^{-1}$ is given by

\[ \left( \begin{array}{c} x \\ y \\ t \end{array} \right) \mapsto \frac{1}{1 + t} \left( x + xt - y^2, \sqrt{2y}, 1 - t \right). \]

More generally, we consider $\mathcal{M} := \mathcal{M}_t$ as a closed CR-hypersurface of $U_t := \{(w, z_1, z_2) \in \mathbb{C}^3 : tz_2 \overline{z}_2 < 1\}$ and $\mathfrak{g} = \mathfrak{hol}(\mathcal{M}, 0)$ as a Lie algebra of holomorphic vector fields on $\mathbb{C}^3$. For fixed $t \neq 1$

\[ N := \{(w, z_1, 0) \in \mathbb{C}^3 : w = z_1 \overline{z}_1\} \]

is a transversal slice in $\mathcal{M}$ to the infinitesimal action of $\mathfrak{g}$ on $\mathcal{M}$. An elementary computation gives

\[ \begin{pmatrix} 2 & 2z_1 \\ 2z_1 & 2tz_1 \overline{z}_1 \end{pmatrix} \]

for the Levi matrix, and hence $4(t - 1)z_1 \overline{z}_1$ for the Levi determinant at the point $(w, z_1, 0) \in N$. In particular, the infinitesimal $\mathfrak{g}$-orbit of the origin, that is $\{(w, 0, z_2) \in \mathbb{C}^3 : w \in i\mathbb{R}, tz_2 \overline{z}_2 < 1\}$, coincides with the set of all Levi degenerate points of $\mathcal{M}$. Furthermore, the Levi form at every point of $\mathcal{M} \setminus N$ is definite if $t > 1$ and indefinite otherwise.

The family $\mathcal{M}_t$ yields counter examples to several statements found in the literature: For instance, contrary to Theorem 4 in [7], for every $\mathcal{M}_t$ the vector field $\xi^2 = itw^2 \partial \partial w + itwz_1 \partial \partial z_1 - iz_1^2 \partial \partial z_2 \in \mathfrak{hol}(\mathcal{M}_t, 0)$ has vanishing 1-jet at 0. Furthermore, in [9] it is claimed that “$\mathcal{M}_1$ is a new example of a uniformly Levi-degenerate CR-manifold apart from $\mathcal{T}$,” which cannot be true due to Proposition 4.16. Finally, translated to our notation, it is claimed in [8] p. 194 for the model surface $\mathcal{M}_0$ that “$\dim \mathfrak{hol}(\mathcal{M}_0, 0) \leq \dim \mathfrak{hol}(\mathcal{M}_0, 0) = 6$ holds for every hypersurface $\mathcal{M}$ given by an equation of the form (4.1)”. But for $\mathcal{M}_1$ this dimension is 10.

5. Global properties

Let again $V, E, \mathbb{C}, \mathcal{T}, \mathcal{H}$ have the same meaning as in section 2. The tube manifold $\mathcal{T}$ is a homogeneous 2-nondegenerate CR-manifold of dimension 5. In the following we want to present further CR manifolds of this type that are locally but not globally CR-equivalent to $\mathcal{T}$. 
Let us denote by \( \widetilde{T} \) the universal covering of \( T \) in the following and by \( \mu : \widetilde{T} \to T \) the corresponding covering map. For convenience we identify \( \widetilde{T} \) with \( \mathbb{R}^2 \times V \) and \( \mu \) with the mapping

\[
(r, \varphi, v) \mapsto e^r \left( \begin{array}{cc}
1 + \cos \varphi & \sin \varphi \\
\sin \varphi & 1 - \cos \varphi
\end{array} \right) + iv, \quad r, \varphi \in \mathbb{R}, v \in V.
\]

(5.1)

Clearly, the CR-structure on \( \widetilde{T} = \mathbb{R}^2 \times V \) is uniquely determined by the condition that \( \mu \) is a local CR-isomorphism.

Our first result states in particular that every continuous CR-function on \( \widetilde{T} \) is constant on \( \mu \)-fibers. For this denote by \( \hat{\mathcal{F}} := \mathcal{F} \cup \mathcal{H} \) the convex hull of \( \mathcal{F} \) in \( E \). Also, call for every open subset \( U \subset \hat{\mathcal{F}} \) a continuous mapping \( f : U \to \mathbb{C}^n \), \( n \in \mathbb{N} \), holomorphic if its restriction to \( U \cap \mathcal{H} \) is holomorphic in the usual sense.

**5.2 Proposition.** To every continuous CR-function \( f \) on \( \widetilde{T} \) there exists a unique holomorphic function \( h \) on \( \hat{\mathcal{F}} \) with \( f = h \circ \mu \).

**Proof.** There exists a connected Lie group \( G \) (for instance, the group of all transformations (3.2) with \( \det(g) > 0 \) acting continuously on \( \hat{\mathcal{F}} \) by biholomorphic transformations (in the extended sense defined above) such that \( \mathcal{F} \) and \( \mathcal{H} \) are \( G \)-orbits. We may assume that \( G \) is simply connected, otherwise replace \( G \) by its universal covering group. This guarantees that the action of \( G \) on \( \mathcal{F} \) lifts to a transitive action of \( G \) on \( \mathcal{F} \). The sheaf \( \mathcal{O} \) over \( \hat{\mathcal{F}} \) of germs of local holomorphic functions (also in the extended sense) is a Hausdorff space for which the canonical projection \( \pi : \mathcal{O} \to \hat{\mathcal{F}} \) is a local homeomorphism. Now fix an arbitrary point \( a \in \mathcal{F} \) and choose an open neighbourhood \( V \) of \( a \) in \( \mathcal{F} \). For \( V \) small enough there exists a continuous section \( \sigma : V \to \hat{\mathcal{F}} \), i.e. \( \mu \circ \sigma = \text{id}_V \). Since \( \mathcal{F} \) is the smooth boundary part of the Stein domain \( \mathcal{H} \), the Levi cone at \( a \in \mathcal{F} \), (which is non zero) points into the direction of \( \mathcal{H} \). Consequently, there exists a connected open neighbourhood \( U \) of \( a \) with respect to \( \hat{\mathcal{F}} \) such that every continuous CR-function on \( V \) has a holomorphic extension to \( U \), compare [3] p. 256 or [2] p. 205. Fix now an arbitrary continuous CR-function \( f \) on \( \mathcal{F} \). For every \( g \in G \) there exists a unique holomorphic function \( f_g \) on \( U \) which coincides with \( f \circ g \circ \sigma \) on \( U \cap V \). Denote by \( \mathcal{F}_g \subset \mathcal{O} \) the subset of all germs induced by the holomorphic function \( f_g \circ g^{-1} \), defined on \( g(U) \). Clearly, every \( \mathcal{F}_g \) is connected. Observe that for sufficiently close \( g_1, g_2 \in G \), the intersection \( \mathcal{F}_{g_1} \cap \mathcal{F}_{g_2} \) is not empty. Since \( G \) is connected, there is a unique connected component \( \mathcal{F} \) of \( \mathcal{O} \) containing all \( \mathcal{F}_g \), \( g \in G \).

We claim that \( \pi : \mathcal{F} \to \hat{\mathcal{F}} \) is a covering map. Since \( \pi \) is a local homeomorphism it is sufficient to show that every continuous curve \( \gamma : [0, 1] \to \hat{\mathcal{F}} \) has a lifting to a continuous curve \( \tau : [0, 1] \to \mathcal{F} \) with \( \pi \circ \tau = \gamma \). Without lost of generality we may consider only curves \( \gamma \) in \( \mathcal{H} \). The claim follows from the existence of a continuous curve \( t \mapsto g_t \) in \( G \) with \( \gamma(t) = g_t(\gamma(0)) \) for all \( t \in [0, 1] \). Since \( \hat{\mathcal{F}} \) is simply connected we get that \( \pi : \mathcal{F} \to \hat{\mathcal{F}} \) is a bijection. Therefore \( \mathcal{F} \) is a continuous section in \( \mathcal{O} \) over \( \hat{\mathcal{F}} \) and determines the required holomorphic function \( h \) on \( \hat{\mathcal{F}} \).

**5.3 Corollary.** To every \( \tilde{g} \in \text{Aut}(\widetilde{T}) \) there exists a unique \( g \in \text{Aut}(\mathcal{F}) \) with \( \mu \circ \tilde{g} = g \circ \mu \). The mapping \( \text{Aut}(\widetilde{T}) \to \text{Aut}(\mathcal{F}) \), \( \tilde{g} \mapsto g \), realizes \( \text{Aut}(\widetilde{T}) \) as universal covering group of \( \text{Aut}(\mathcal{F}) \). In particular, \( \text{Aut}(\widetilde{T}) \) is a simply connected Lie group of dimension \( 7 \) with two connected components, acting transitively on \( \mathcal{F} \).

**Proof.** Denote by \( \iota : \mathcal{F} \to E \) the canonical injection. By Proposition 5.2 the mapping \( \iota \circ \mu \circ \tilde{g} \) is constant on \( \mu \)-fibers and hence factors over \( \mu \). The group \( \text{Aut}(\mathcal{F}) \) is given by all affine transformations (3.2) and hence there is a canonical isomorphism of Lie groups

\[
\text{Aut}(\mathcal{F}) \cong \text{GL}(E) \times V,
\]
where \( GL(\mathcal{C}) := \{ g \in GL(V) : g(\mathcal{C}) = \mathcal{C} \} \cong (GL(2, \mathbb{R})/\{ \pm e \}) \),
e \in GL(2, \mathbb{R}) \) is the unit matrix and the semi-direct product refers to the canonical injection \( \rho : GL(\mathcal{C}) \to GL(V) \). In particular, \( \text{Aut}(\mathcal{F}) \cong GL(\mathcal{C}) \times V \) with \( \mathcal{G} \cong GL(\mathcal{C}) \cong GL(2, \mathbb{R}) \) the universal covering group of \( GL(\mathcal{C}) \).

Proof. \( \square \)

A CR-deformation family for \( \mathcal{F} \). For every \( \psi \in \mathbb{R} \) and \( \mu_{\psi} := \begin{pmatrix} \cos \psi/2 & \sin \psi/2 \\ -\sin \psi/2 & \cos \psi/2 \end{pmatrix} \) define \( \lambda_{\psi} \in GL(\mathcal{C}) \) by \( \lambda_{\psi} v = \mu_{\psi} v \mu_{\psi}' \). For every \( z = (s, \psi, w) \in \mathbb{R}^2 \times V \) then

\[
\theta_z(r, \varphi, v) := (r + s, \varphi + \psi, e^s \lambda_{\psi} v + w)
\]
defines an affine transformation \( \theta_z \in \text{Aut}(\mathcal{F}) \) and \( \Theta := \{ \theta_z : z \in \mathbb{R}^2 \times V \} \) is a Lie group acting freely and transitively on \( \mathcal{F} \). Clearly, \( \Theta \) has many discrete subgroups \( \Gamma \), each of which gives a CR-manifold \( \mathcal{F}/T \) locally CR-equivalent to \( \mathcal{F} \). Here we restrict our attention to the following one-parameter family \( (\Gamma_t)_{t>0} \) of discrete subgroups: For every real \( t > 0 \) put \( \gamma_t := \theta(0, 2\pi t, 0) \), \( \Gamma_t := \{ \gamma_{nt} : n \in \mathbb{Z} \} \) and \( \mathcal{F}_t := \mathcal{F}/\Gamma_t \). Then every \( \mathcal{F}_t \) is isomorphic to \( \mathcal{F} \) as real-analytic manifold while \( \mathcal{F}_t \) is equivalent to \( \mathcal{F} \) as CR-manifold. We will see later that actually the \( \mathcal{F}_t \) are pairwise nonequivalent as CR-manifolds, compare Proposition 5.10. The family \( (\mathcal{F}_t)_{t>0} \) may be considered as a CR-deformation family of \( \mathcal{F} \cong \mathcal{F}_1 \): For \( \mathbb{R}^+ := \{ t \in \mathbb{R} : t > 0 \} \) consider the CR-manifold \( \mathcal{F} \times \mathbb{R}^+ \), on which \( \mathbb{Z} \) acts freely by \( (z, t) \mapsto (\gamma_{nt} z, t), n \in \mathbb{Z} \). Then \( (\mathcal{F} \times \mathbb{R}^+)/\mathbb{Z} \) is a real-analytic CR-manifold, and the canonical projection \( \pi : (\mathcal{F} \times \mathbb{R}^+)/\mathbb{Z} \to \mathbb{R}^+ \) is a CR-mapping whose fibers give the family \( (\mathcal{F}_t)_{t>0} \).

5.5 Proposition. For every \( t > 0 \) and every \( a \in \mathcal{F}_t \) the cardinality \( \sigma_t(a) \) of the set

\[
\Sigma_t(a) := \{ z \in \mathcal{F}_t : f(z) = f(a) \text{ for every continuous CR-function } f \text{ on } \mathcal{F}_t \}
\]
is given by

\[
\sigma_t(a) = \begin{cases} p & t = p/q \text{ for relatively prime integers } p, q > 0 \\ \infty & \text{in all other cases.} \end{cases}
\]

Proof. The group \( \Gamma := \Gamma_1/(\Gamma_1 \cap \Gamma_t) \) acts freely on \( \mathcal{F}_t \), and by Proposition 5.2 every continuous CR-function on \( \mathcal{F}_t \) is constant on \( \Gamma \)-orbits. In case \( t \) irrational \( \Gamma \cong \mathbb{Z} \) implies \( \sigma_t(a) = \infty \). Therefore we may assume in the following that \( t = p/q \) with relatively prime integers \( p, q > 0 \). Since then \( \Gamma \) has order \( p \) we get \( \sigma_t(a) \geq p \). On the other hand, the CR-manifold \( \mathcal{F}_{1/q} \) is separable with respect to real-analytic CR-functions. Indeed, the orbits of the finite subgroup \( \Lambda := \{ \lambda_{2\pi s} : s \in q^{-1}\mathbb{Z} \} \subset GL(V) \subset GL(E) \) are separated by the \( \Lambda \)-invariant holomorphic polynomials on \( E \). Since we have a CR-covering map \( \mathcal{F}_{p/q} \to \mathcal{F}_{1/q} \) of degree \( p \) this implies \( \sigma_t(a) \leq p \).

\( \square \)

It is not difficult to see that the centralizer of \( \gamma_t \) in \( \text{Aut}(\mathcal{F}) \) acts transitively on \( \mathcal{F} \) if and only if \( t \in \mathbb{N} \). As a consequence, \( \mathcal{F}_t \) is homogeneous as CR-manifold if and only if \( t \) is an integer. In any case, the centralizer of \( \gamma_t \) contains the 1-parameter subgroup \( \{ \gamma_s : s \in \mathbb{R} \} \) of \( \text{Aut}(\mathcal{F}) \), which therefore also acts on \( \mathcal{F} \).

5.7 Proposition. For every irrational \( t > 0 \) the following properties hold:

(i) For every \( a \in \mathcal{F}_t \), the set \( \Sigma_t(a) \) defined in (5.6) is the circle \( \Sigma_t(a) = \{ \gamma_s(a) : 0 \leq s < t \} \).

(ii) There exists a closed hypersurface \( \mathcal{N}_t \subset \mathcal{F}_t \) such that every continuous CR-function on \( \mathcal{F}_t \) is real-analytic on the complement \( \mathcal{F}_t \setminus \mathcal{N}_t \).

(iii) There exists a continuous CR-function on \( \mathcal{F}_t \) which is not real-analytic.
Lemma 5.8 separates the circle $\{ \gamma_s(a) : s \in \mathbb{R} \}$ we get $\Sigma_t(a) \subset \{ \gamma_s(a) : s \in \mathbb{R} \}$. This implies that $f \circ \mu_t$ is $\gamma_s$-invariant for every $s \in \mathbb{R}$ and hence that $h$ is invariant under $\text{SO}(2)$ acting on $E$ as in (3.2). Now the proposition follows from the next lemmata 5.8 and 5.9, which may be of independent interest. The inclusion $\{ \gamma_s(a) : s \in \mathbb{R} \} \subset \Sigma_t(a)$ follows from the fact that the mapping $\varphi$ defined in Lemma 5.8 separates the $\text{SO}(2)$-orbits in $\mathcal{T}$.

5.8 Lemma. Consider in $\mathcal{T}$ the hypersurface

$\mathcal{N} := \{ c + i(rc + se) : c \in \mathbb{C} \text{ and } r, s \in \mathbb{R}, \text{ e the unit matrix},$

and denote by $\mathcal{M} := \mathcal{T} \setminus \mathcal{N}$ its complement. Let $\text{tr}$ be the normalized trace on $E$ (i.e. $\text{tr}(e) = 1$) and define the $\text{SO}(2)$-invariant holomorphic mapping $\varphi : E \to \mathbb{C}^2$ by $\varphi(z) := (\text{tr}(z), \text{tr}(z^2) - \text{tr}(z)^2)$. Then

$\varphi(\mathcal{M}) = \{(x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2 : x_1 > 0, x_2 < x_1 - (2x_1)^{-2}y_2^2\}$

$\varphi(\mathcal{N}) = \{(x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2 : x_1 > 0, x_2 = x_1 - (2x_1)^{-2}y_2^2\},$

and to every continuous $\text{SO}(2)$-invariant CR-function $h$ on $\mathcal{T}$ there exists a unique continuous function $f$ on $\varphi(\mathcal{T})$ with $h = f \circ \varphi$ such that the restriction of $f$ to the interior $\varphi(\mathcal{M})$ is holomorphic. In particular, every continuous $\text{SO}(2)$-invariant CR-function on $\mathcal{T}$ is real-analytic on the dense domain $\mathcal{M} \subset \mathcal{T}$.

Proof. The expressions for $\varphi(\mathcal{M})$ and $\varphi(\mathcal{N})$ are checked by direct computation. Every continuous $\text{SO}(2)$-invariant function $h$ on $\mathcal{T}$ is of the form $h = f \circ \varphi$ with $f$ a continuous function on $\varphi(\mathcal{T})$. In case $h$ is CR in addition, $h$ admits an extension to an $\text{SO}(2)$-invariant holomorphic function on $\mathcal{M}$. This implies that $f$ is holomorphic on $\varphi(\mathcal{M}) = \varphi(\mathcal{N})$.

For the proof of 5.7.iii it would be enough to find directly a continuous function $g$ on $\varphi(\mathcal{T})$ such that the pull-back $f := g \circ \varphi$ is CR but not real-analytic on $\mathcal{T}$. Instead, we use invariant integration for the explicit determination of such an $f$.

5.9 Lemma. There exists a continuous $\text{SO}(2)$-invariant function on $\mathcal{T}$ that is not real-analytic.

Proof. Let $s$ be the unique continuous branch of the square root on $H := \{ s \in \mathbb{C} : \text{Re}(s) \geq 0 \}$ with $s = 1$. Then $\lambda(z) = z \text{defines a complex linear form } \lambda : E \to \mathbb{C}$ with $\lambda(\mathcal{H}) \subset H$ for $\hat{\mathcal{T}} := \mathcal{T} \cup \mathcal{H}$. The function $\varphi(z) := s(\lambda(z))$ is continuous on $\hat{\mathcal{T}}$ and holomorphic on $\mathcal{H}$. For easier computation let us introduce on $E$ the coordinates $(w, z_1, z_2) \mapsto (w - z_1 z_2, w + z_1) ;$ and then $\lambda(w, z_1, z_2) = w - z_1$. For every $t \in \mathbb{R}$ define $\gamma_t \in \text{GL}(E)$ by

$(w, z_1, z_2) \mapsto (w, z_1 \cos t + z_2 \sin t, -z_1 \sin t + z_2 \cos t).$

Then

$h(z) := \int_0^{2\pi} \varphi(\gamma_t z) \, dt$

defines a continuous $\text{SO}(2)$-invariant function $h$ on $\hat{\mathcal{T}}$ that is holomorphic on $\mathcal{H}$. As local uniform limit of holomorphic functions therefore the restriction $f := h|_\mathcal{H}$ is a continuous $\text{SO}(2)$-invariant CR-function on $\mathcal{T}$. Define on the open unit disk $\Delta \subset \mathbb{C}$ the holomorphic function $g$ by

$g(s) := h(1, s, 0) = \int_0^{2\pi} \sqrt{1 - s \cos t} \, dt = \sum_{k=0}^{\infty} \binom{1/2}{k} c_k s^k, \quad c_k := \int_0^{2\pi} (\cos t)^k \, dt,$
and denote by $R \geq 1$ the radius of convergence of the power series expansion. From the recursion $c_0 = 2\pi$, $c_1 = 0$ and $c_k = \frac{k+1}{k}c_{k-2}$ for $k \geq 2$ we get $R = 1$. Because of $(1, s, 0) \in \mathcal{T}$ for $|s| = 1$ the function $g$ has a continuous extension to the closure $\overline{\Delta}$.

Now assume that $f$ is real-analytic on $\mathcal{T}$. Then $g$ would have a holomorphic extension to an open neighbourhood of $\overline{\Delta}$ in $\mathbb{C}$. This contradicts $R = 1$, that is, $f$ cannot be real-analytic on $\mathcal{T}$.

\[ \square \]

5.10 Proposition. \textbf{The manifolds $\mathcal{T}_s, \mathcal{T}_t$ are globally CR-equivalent if and only if $s = t$.}

\textbf{Proof.} Assume that $\mathcal{T}_s, \mathcal{T}_t$ are globally CR-equivalent. Then there exists a transformation $g \in \text{Aut}(\mathcal{T})$ with $g \Gamma_s = \Gamma_t g$, that is, $g \gamma_s = \gamma_t g$ with $r = \pm t$. In case $s \in \mathbb{N}$ the CR-manifold $\mathcal{T}_s$ is homogeneous. But then $\sigma_t(a) = \sigma_s(b)$ for all $a \in \mathcal{T}_s$ and $b \in \mathcal{T}_s$ implies $s = t$ as a consequence of Proposition 5.5. We therefore may assume $s, t \notin \mathbb{N}$. Inspecting the homomorphic image of the equation $g \gamma_s = \gamma_t g$ in $\text{Aut}(\mathcal{T})$ shows $g \in \Theta$ and thus $s = t$, see (5.4) for the definition of the group $\Theta$.

\[ \square \]

6. The bounded realization

Let again $E = \{z \in \mathbb{C}^{2 \times 2} : z' = z\}$ be the linear space of all complex symmetric $2 \times 2$-matrices with $e = 1_2 \in E$ the unit matrix. Put

\[ \mathcal{D} := \{z \in E : e-z\overline{z} \text{ positive definite}\} \]

(6.1) \[ \mathcal{B} := \{z \in E : \det(e-z\overline{z}) = 0, \text{tr}(e-z\overline{z}) > 0\} = \{z \in \partial \mathcal{D} : z\overline{z} \neq e\} \]

\[ \mathcal{S} := \{z \in E : z \text{ unitary}\} \cong U(2)/O(2) \cong (S^1 \times S^2)/\mathbb{Z}_2, \]

where the action of $\mathbb{Z}_2$ is generated by the antipodal map in each of the two spheres $S_1, S^2$. Then $\mathcal{D}$ is the open unit ball in $E$ with respect to the operator norm and is also called a \textbf{Lie ball}. The boundary $\partial \mathcal{D}$ decomposes into its smooth part $\mathcal{B}$ and the Shilov boundary $\mathcal{S}$, which is a totally real submanifold of $E$.

The CR-manifold $\mathcal{R}$ is fibered in its \textbf{holomorphic arc components}, which all are affinely equivalent to the open unit disk $\Delta \subset \mathbb{C}$, compare [16]. By definition, $A \subset \mathcal{R}$ is a holomorphic arc component of $\mathcal{R}$, if it is minimal with respect to the property: $A \neq \emptyset$, and $f(\Delta) \subset A$ for every holomorphic map $f : \Delta \to E$ with $f(\Delta) \subset \mathcal{R}$ and $f(\Delta) \cap A \neq \emptyset$. For instance, the set $A$ of all diagonal matrices $z \in E$ with $z_{11} = 1 > |z_{22}|$ is such an arc component, and every other is of the form $uAu'$ with $u \in U(2)$. This implies that the space of all arc components in $\mathcal{R}$ can be identified with $U(2)/\{(O(1) \times U(1)) \cong \mathbb{P}_3(\mathbb{R})$. Since every $g \in \text{Aut}(\mathcal{R})$ respects holomorphic arc components there is an $\text{Aut}(\mathcal{R})$-equivariant fiber bundle $\Xi : \mathcal{R} \to \mathbb{P}_3(\mathbb{R})$ with fibers the holomorphic arc components of $\mathcal{R}$.

It is well known that the \textbf{Cayley transformation} $z \mapsto (z-e)(z+e)^{-1}$ defines a biholomorphic map $\gamma : \mathcal{H} \to \mathcal{D}$ and also gives a CR-isomorphism $\mathcal{I} \cong \{z \in \mathcal{R} : \det(e-z) \neq 0\}$. In fact, $E$ can be considered as Zariski-open subset of a nonsingular quadric $Z \subset \mathbb{P}_4(\mathbb{C})$ in such a way that $\gamma$ extends to a biholomorphic automorphism of $Z$, and then $\gamma^2(z) = -z^{-1}$ for all invertible $z \in E$. Also, there exists a unique antiholomorphic involution $\tau : Z \to Z$ with $\text{Fix}(\tau) = \mathcal{I}$, and then

\[ G := \{g \in \text{Aut}(Z) : g(\mathcal{D}) = \mathcal{D}\} = \{g \in \text{Aut}(Z) : g(\mathcal{R}) = \mathcal{R}\} \]

\[ = \{g \in \text{Aut}(Z) : \tau g \tau = g\}^0 \cong \text{SO}(2,3)^0 \]

acts transitively on $\mathcal{D}, \mathcal{R}, \mathcal{S}$. Via these actions the groups $G$, $\text{Aut}(\mathcal{D})$ and $\text{Aut}(\mathcal{R})$ are canonically isomorphic and hence are identified in the following. We also identify the Lie algebra

\[ \mathfrak{g} := \text{aut}(\mathcal{D}) = \{(a+cz + zc' - z\overline{z}) \partial / \partial z : a \in E, c \in u(2)\} \]
of \( G \) with \( \text{aut}(\mathcal{R}) = \mathfrak{hol}(\mathcal{R}) \), where \( \mathfrak{u}(2) \subseteq \mathfrak{gl}(2, \mathbb{C}) \) is the Lie subalgebra of all skew hermitian matrices.

The complexification \( I = \mathfrak{g} \oplus i \mathfrak{g} \) has the decomposition
\[
I = \bigoplus_{\nu \in \mathbb{Z}^2} I^\nu \quad \text{with} \quad I^\nu := \{\xi \in I : [\zeta_j, \xi] = \nu_j \xi \quad \text{for} \quad j = 1, 2\}
\]
with \( \zeta_j \in I \) defined by (3.5) and (3.7) again being a choice of root vectors \( \xi^\nu \in I^\nu \). But unlike to (3.6) the real form \( \mathfrak{g} \) is embedded in a different way here:

\[
\mathfrak{g} \cap (I^\nu + I^{-\nu}) = \begin{cases} \mathbb{R}(\xi^\nu + \xi^{-\nu}) \oplus \mathbb{R}(i\xi^\nu - i\xi^{-\nu}) & \nu \neq 0 \\ \mathbb{R}i\zeta_1 \oplus \mathbb{R}i\zeta_2 & \text{otherwise} \end{cases}
\]

\( \mathcal{R} \) has the remarkable property that every CR-equivalence between domains \( \mathcal{U}, \mathcal{V} \) in \( \mathcal{R} \) extends to a CR-automorphism of \( \mathcal{R} \), compare [11]. In particular, \( \text{Aut}(\mathcal{U}) \cong \{g \in G : g(\mathcal{U}) = \mathcal{U}\} \) and the groups \( \text{Aut}(\mathcal{U}), \text{Aut}(\mathcal{V}) \) are conjugate in \( \text{Aut}(\mathcal{R}) \) for any pair of CR-equivalent domains \( \mathcal{U}, \mathcal{V} \subseteq \mathcal{R} \). Via the Cayley transformation \( \mathcal{R} \) contains a copy of the CR-manifold \( \mathcal{F} \) and hence may be considered as a CR-extension of \( \mathcal{F} \). We may ask whether \( \mathcal{R} \) is maximal with respect to CR-extensions, that is, whether \( \mathcal{R} \) can be a proper domain in some other connected CR-manifold. A partial answer is given by the following result.

6.3 Proposition. Let \( M \) be a connected, locally homogeneous CR-manifold and let \( D \subset M \) be a domain that is CR-isomorphic to a covering of \( \mathcal{R} \). Then \( D = M \).

Proof. Assume to the contrary \( D \neq M \) and fix a boundary point \( c \) of \( D \) in \( M \). Since \( M \) is locally homogeneous there exists a connected open neighbourhood \( U \subset M \) of \( c \) and a vector field \( \eta \in \mathfrak{hol}(U) \) having an integral curve \( \gamma : [0, 1] \to U \) with \( \gamma(0) = c \) and \( a := \gamma(1) \in D \). Since \( D \) is CR-equivalent to a covering of \( \mathcal{R} \) there exists a vector field \( \xi \in \mathfrak{hol}(D) \) having the same germ as \( \eta \) at \( a \). Since \( \xi \) is complete on \( D \) there exists an integral curve \( g : \mathbb{R} \to D \) of \( \xi \) with \( g(0) = a \). But then, for every \( s \in [0, 1] \), we have \( g_{-s} = \gamma_{1-s} \in D \). In particular, \( c = g_{-1} \in D \) gives a contradiction.

6.4 Proposition. Let \( M \) be a homogeneous, connected, simply connected real-analytic CR-manifold which is locally CR-isomorphic to \( \mathcal{R} \). Then \( M \) is CR-equivalent to the universal covering \( \tilde{\mathcal{U}} \) of a homogeneous domain \( \mathcal{U} \subset \mathcal{R} \).

Proof. For every \( a \in M \) let \( \mathcal{F}_a \) be the set of all germs at \( a \) of CR-equivalences \( U \to \mathcal{V} \), where \( U \) is a an open neighbourhood of \( a \in M \) and \( \mathcal{V} \) is open in \( \mathcal{R} \). Consider the disjoint union \( \mathcal{F} \) of all \( \mathcal{F}_a \), \( a \in M \), in the usual way as sheaf over \( M \) with sheaf projection \( \mu : \mathcal{F} \to M \) and fix a connected component \( N \) of \( \mathcal{F} \). By assumption there is a connected Lie group \( H \) of CR-automorphisms of \( M \) acting transitively on \( M \). This implies that \( \mu : N \to \mathcal{R} \) is a covering map and hence bijective. The evaluation map \( N \to \mathcal{R} \) defines a CR-covering map onto the image \( \mathcal{U} \subset \mathcal{R} \). Since this map is \( H \)-equivariant, \( \mathcal{U} \) is a homogeneous domain in \( \mathcal{R} \).

Proposition 6.4 reduces the classification problem for CR-manifolds locally CR-isomorphic to \( \mathcal{R} \) to the study of homogeneous domains in \( \mathcal{R} \). But these are in 1-1-correspondence to \( G \)-homogeneous domains in \( \mathbb{P}_3(\mathbb{R}) \). More precisely, every homogeneous domain \( \mathcal{U} \subset \mathcal{R} \) is the full \( \mathfrak{E} \)-pre-image of \( U := \mathfrak{E}(\mathcal{U}) \). Indeed, fix an arbitrary holomorphic arc component \( A \) of \( \mathcal{R} \) with \( A \cap \mathcal{U} \neq \emptyset \). Then the group \( H := \{g \in \text{Aut}(\mathcal{R}) : g(A) = A\} \) has the orbit \( A \cap \mathcal{U} \) that is open in \( A \cong \Delta \). Therefore \( H \) acts transitively on \( A \) implying the claim. As a consequence, \( \mathcal{U} \to U \) is a fiber bundle with contractible fiber, implying that the homogeneous domain \( \mathcal{U} \subset \mathcal{R} \) and its \( \mathfrak{E} \)-image \( U \subset \mathbb{P}_3(\mathbb{R}) \) always have the same homotopy type. In particular, \( \mathcal{R} \) has fundamental group \( \mathbb{Z}_2 \).
For certain homogeneous domains in $\mathcal{R}$ the same argument as in the proof of Proposition 5.2 yields the following holomorphic extension property.

**6.5 Proposition.** Let $\mu : \tilde{\mathcal{U}} \to \mathcal{U}$ be the universal covering of a homogeneous domain $\mathcal{U} \subset \mathcal{R}$ and assume that $\text{Aut}(\mathcal{U}) \subset \text{Aut}(\mathcal{R})$ acts transitively on $\mathcal{D}$. Then to every continuous CR-function $f$ on $\tilde{\mathcal{U}}$ there exists a unique continuous function $h$ on $\mathcal{U} \cup \mathcal{D}$ with the following two properties.

(i) $f = h \circ \mu$,
(ii) $h$ is holomorphic on $\mathcal{D}$.

To get further examples of homogeneous domains in $\mathcal{R}$ write

$$G_Y := \{g \in G : g(Y) = Y\} \quad \text{and} \quad g_Y := \{\xi \in g : \exp(\mathbb{R} \xi) \subset G_Y\}$$

for every subset $Y \subset Z$. Then every subgroup $G_a, a \in \mathcal{R}$, has a unique open orbit in $\mathcal{R}$. In case $a = -e$ this orbit is just the image $\gamma(\mathcal{T}) \subset \mathcal{R}$ of the tube manifold $\mathcal{T}$. Also, for every holomorphic arc component $A$ of $\mathcal{R}$ the group $G_A$ has a unique open orbit in $\mathcal{R}$. Even the intersection $G_A \cap G_a$ has a unique open orbit in $\mathcal{R}$, provided $a \in \mathcal{R}$ is in the closure of $A$ with respect to $\partial \mathcal{D}$. All these groups act transitively on $\mathcal{D}$ and thus give homogeneous domains in $\mathcal{R}$ with the holomorphic extension property of Proposition 6.5. But there also exist homogeneous domains $\mathcal{U} \subset \mathcal{R}$ such that $\text{Aut}(\mathcal{U})$ is not transitive on $\mathcal{D}$.

**6.6 Example.** Let $F \subset E$ be the $\mathbb{C}$-linear subspace of all diagonal matrices. Then the intersection $F \cap \mathcal{D}$ is biholomorphically equivalent to the bidisk $\Delta^2 \subset \mathbb{C}^2$ and $G_{F \cap \mathcal{D}}$ has Lie algebra

$$g_{F \cap \mathcal{D}} := \{(a + cz + zc' - z\bar{\sigma}z)\partial/\partial z : a \in F, c \in F \cap \mathfrak{u}(2)\} \cong \mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R}).$$

The corresponding open orbit $\mathcal{U} \subset \mathcal{R}$ has the following equivalent description, which may be of interest for itself: The group $H := \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ acts on the complex manifold $X := \text{SL}(2, \mathbb{C})$ by $z \mapsto gzh^{-1}$. For every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in X$ put $\delta(z) := \det(z + \bar{\sigma}) - 2 = 2\text{Re}(a\bar{\sigma} - b\bar{\sigma})$. Then $\delta$ is an $H$-invariant function on $X$ with $\delta(X) = \mathbb{R}$. It is easily checked that multiplication by $j := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ induces a biholomorphic automorphism of $X$ with $\delta(jz) = -\delta(z)$. The critical set of $\delta$ is $\{z \in X : \bar{\sigma} = \pm z\} = \text{SL}(2, \mathbb{R}) \cup j \cdot \text{SL}(2, \mathbb{R})$. In particular, $\delta$ has critical values $\pm 2$.

For every $\varepsilon \in \{\pm i, 0\}$ consider the orbit $M_\varepsilon := H \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$. The inversion $z \mapsto z^{-1}$ maps $M_\varepsilon$ to $M_{-\varepsilon}$ and $\overline{M_\varepsilon} \cap \overline{M_{-\varepsilon}} = M_0 = \text{SL}(2, \mathbb{R})$. It can be shown that $M_\varepsilon$ is CR-equivalent to $\mathcal{U} \subset \mathcal{R}$. Furthermore, $M_\varepsilon$ is diffeomorphic to $\mathcal{C} \times \text{SL}(2, \mathbb{R})$ and hence has fundamental group $\mathbb{Z}^2$ (recall that the tube manifold $\mathcal{T}$ is diffeomorphic to $\mathcal{C} \times \mathbb{V}$).

**6.7 Example.** The Lie group $\text{SU}(2)/\{\pm e\} \cong \text{SO}(3)$ acts on $E$ by $z \mapsto g zg'$. Choose an $\mathbb{R}$-linear subspace $W$ of $E$ that is invariant under this action, e.g. $W = \{z \in E : z_{11} - \overline{z}_{22} = z_{12} + \overline{z}_{12} = 0\}$. Denote by $\tau : E \to E$ the unique conjugate linear involution fixing every point of $W$. Then $\tau(\mathcal{D}) = \mathcal{D}$ and the intersection $W \cap \mathcal{D}$ is a euclidian ball in $W$. The group $G_{W \cap \mathcal{D}} = \{g \in G : \tau g \tau = g\}$ has Lie algebra

$$g_{W \cap \mathcal{D}} := \{(a + cz + zc' - z\bar{\sigma}z)\partial/\partial z : a \in W, c \in \mathfrak{su}(2)\} \cong \mathfrak{sl}(2, \mathbb{C}).$$

Again, the corresponding open orbit $\mathcal{U} \subset \mathcal{R}$ has an equivalent realization: Let $X := \text{SL}(2, \mathbb{C})$ and consider $H := \text{SL}(2, \mathbb{C})$ as real Lie group. Then $H$ acts on $X$ by $z \mapsto g zg^*$, where $g^*$ is the conjugate transpose of $g$. Clearly, $z \mapsto -z$ commutes with the action of $H$ and hence permutes $H$-orbits. The function $\delta(z) := \det(z + \bar{\sigma}z) - 2 = a\bar{\sigma} + d\bar{\sigma} - b\bar{\sigma} - c\bar{\sigma}$ on $X$ is $H$-invariant with critical values $\pm 2$ and critical set $\{z \in X : \bar{\sigma} = \pm z\}$. This set is the union of three totally real
orbits, more precisely:

**Critical value** 2: There are the two critical orbits \( N := \{ h \in X : h^* = h > 0 \} \) and \( -N = \{ h \in X : h^* = h < 0 \} \). A diffeomorphism \( \{ h = h^* \in \mathbb{C}^{2 \times 2} : \text{tr}(h) = 0 \} \to N \) is given by the exponential mapping. By elementary calculus it is seen that \( \delta \) attains a local maximum at every point of \( \pm N \). As a consequence, \( N \) is the only \( H \)-orbit in \( X \) having \( N \) in its closure (and the same with \( -N \)).

**Critical value** \(-2\): The only critical orbit is \( N := \{ ih : h = h^* \in \mathbb{C}^{2 \times 2}, \det(h) = -1 \} \). The hypersurface orbit \( M := H \left( \frac{1}{i} \begin{pmatrix} 1 & i \\ i & 0 \end{pmatrix} \right) \) has \( N \) in its closure. It can be shown that \( M \) is CR-equivalent to \( \mathcal{U} \).

7. Some final remarks.

The idea of proof in Propositions 5.2 and 6.5 can also be applied in higher dimensions: For \( n \geq 3 \) fixed let \( V \subset \mathbb{R}^{n \times n} \) be the subspace of all symmetric \( n \times n \)-matrices and \( E := V \oplus iV \subset \mathbb{C}^{n \times n} \) its complexification. For all \( p, q \geq 0 \) with \( k := n - (p + q) \geq 0 \) denote by \( \mathcal{C}_{p,q} \subset V \) the cone of all real symmetric \( n \times n \)-matrices of type \((p, q)\), that is, having \( p \) positive and \( q \) negative eigenvalues. The group \( GL(n, \mathbb{R}) \) acts on \( V \) by \( x \mapsto gxg^t \), and the cones \( \mathcal{C}_{p,q} \) are the corresponding orbits. Because of \( \mathcal{C}_{q,p} = -\mathcal{C}_{p,q} \) we may restrict our attention to the special case \( p \geq q \). The cone \( \Omega := \mathcal{C}_{n,0} \) is convex open, and \( \mathcal{H} := \Omega \oplus IV \subset E \) is a symmetric tube domain (Siegel’s upper half plane of rank \( n \) up to the factor \( i \)). The group \( \text{Aut}(\mathcal{H}) \) consists of all biholomorphic transformations \((3.3)\) with \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) in the symplectic subgroup \( Sp(n, \mathbb{R}) \subset SL(2n, \mathbb{R}) \), compare [12] p. 345.

In analogy to our setting in Section 3 with \( n = 2 \) we denote for every \( p, q \) by \( \mathcal{T}_{p,q} := \mathcal{C}_{p,q} \oplus iV \) the tube manifold over the cone \( \mathcal{C}_{p,q} \). Then \( \mathcal{T}_{p,q} \) is open precisely if \( k = 0 \) and is closed if \( k = n \). In all other cases \( \mathcal{T}_{p,q} \) is a 2-nondegenerate CR-manifold and \( \text{Aut}(\mathcal{T}_{p,q}) \) consists of all transformations \((3.2)\) (with \( GL(2, \mathbb{R}) \) replaced by \( GL(n, \mathbb{R}) \), compare [10]). In [10] also the following holomorphic extension property has been shown: In case \( pq \neq 0 \) every continuous CR-function on \( \mathcal{T}_{p,q} \) has a holomorphic extension to all of \( E \). In case \( p > 0 \) every continuous CR-function on \( \mathcal{T}_{p,0} \) has a holomorphic extension to \( \mathcal{H} \) that is continuous up to \( \mathcal{T}_{p,0} \subset \mathcal{H} \) in a certain sense.

Note that every \( \mathcal{T}_{p,q} \) is homogeneous under \( GL(n, \mathbb{R})^0 \times V \). Applying the decomposition theorem of Mostow one deduces that \( \mathcal{T}_{p,q} \) is a bundle with contractible fibers over the homogeneous space \( SO(n)/S(O(p) \times O(q) \times O(k)) \). Explicit computation shows that \( SO(n) \) is a maximal compact subgroup of \( GL(n, \mathbb{R})^0 \) and that for the diagonal matrix \( a := \begin{pmatrix} 1_p & -1_q & 0_k \end{pmatrix} \in \mathcal{T}_{p,q} \), the subgroup \( S(O(p) \times O(q) \times O(k)) \subset SO(n) \) is maximal compact in the isotropy subgroup \( \{ g \in GL(n, \mathbb{R}^n)^0 : gag' = a \} \). As a consequence, employing the long exact homotopy sequence, we determine the fundamental group of \( \mathcal{T}_{p,q} \) for \( n \geq 3 \) and \( 0 < p + q < n \) as

\[
\pi_1(\mathcal{T}_{p,q}) = \begin{cases} 
Q_8 & n = 3, \quad p = q = 1 \\
\mathbb{Z}_2 \oplus \mathbb{Z}_2 & n > 3, \quad p > 0, \quad q > 0 \\
\mathbb{Z}_2 & \text{otherwise}
\end{cases}
\]

where \( Q_8 \) is the quaternion group of order 8.

For the universal covering \( \tilde{\mathcal{T}}_{p,q} \to \mathcal{T}_{p,q} \) of \( \mathcal{T}_{p,q} \) we get the following higher dimensional analog of Proposition 5.2.

7.1 Proposition. In case \( 0 < p + q < n \) every continuous CR-function on \( \tilde{\mathcal{T}}_{p,q} \) is constant on \( \mu \)-fibers.

Proof. Let \( H \) be the group of all transformations \( z \mapsto g z g' + i v \) on \( E \) with \( g \in GL(n, \mathbb{R}) \) and \( v \in V \). To begin with assume first that \( p-q > 0 \) and fix a continuous CR-function \( f \) on \( \tilde{\mathcal{T}}_{p,q} \).
Let $U \subset E$ be the smallest open $H$-invariant subset containing $\mathcal{T}_{p,q}$, that is, $U$ is the union of all $\mathcal{T}_{p',q'}$ with $p' > p$ and $q' > q$. Denote by $\mathcal{O}$ the sheaf over $U$ of all germs of holomorphic functions. Since for every $a \in \mathcal{T}_{p,q}$ the Levi cone at $a$ spans the full normal space to $\mathcal{T}_{p,q}$ in $E$, for every $x \in \mathcal{T}_{p,q}$ the function $f$ induces a germ $f_x \in \mathcal{O}_x$. Denote by $\mathcal{F}$ the connected component of $\mathcal{O}$ containing all $f_x$ with $x \in \mathcal{T}_{p,q}$. As in the proof of Proposition 5.2 it follows that $\mathcal{F}$ is a covering of $U$. As a consequence of the following Lemma 7.2 every loop in $\mathcal{T}_{p,q}$ is zero-homotopic in $U$. Therefore $\{f_x : x \in \mathcal{T}_{p,q}\}$ is a trivial covering of $\mathcal{T}_{p,q}$, that is, $f = g \circ \mu$ for some CR-function on $\mathcal{T}_{p,q}$.

It remains to consider the case $q = 0$. But then $\mathcal{T}_{p,0} \cup \mathcal{H}$ is simply connected and the claim follows in the same way as in the proof of Proposition 5.2 (for every open subset $V$ of $\mathcal{T}_{p,0} \cup \mathcal{H}$ a function $g$ on $V$ should be called holomorphic if its restriction to $V \cap \mathcal{H}$ is holomorphic in the usual sense and if for every $a \in V \cap \mathcal{T}_{p,q}$ there is a neighbourhood $W$ of $a$ in $\mathcal{T}_{p,q}$ and a wedge $\Gamma$ with edge $W$ such that the restriction $g|_{\Gamma}$ extends continuously to $W$).

**7.2 Lemma.** For every $p, q$ with $p + q < n$ the canonical injection $\mathcal{C}_{p,q} \hookrightarrow \mathcal{C}_{p+1,q} \cup \mathcal{C}_{p,q} \cup \mathcal{C}_{p,q+1}$ induces the trivial homomorphism between the corresponding fundamental groups.

**Proof.** Consider the 1-parameter subgroup of $\text{SO}(n)$ given by

$$g_t := \begin{pmatrix} \cos t & -\sin t & 1_{n-2} \\ \sin t & \cos t & 0 \\ 0 & 0 & 1_{n-2} \end{pmatrix}$$

for all $t \in \mathbb{R}$. Furthermore fix diagonal matrices $a, b \in \mathcal{C}_{p,q}$ with $a_{11} = 1 = -b_{11}$ and $a_{22} = b_{22} = 0$. Then the loops $\gamma, \sigma : [0, \pi) \to \mathcal{C}_{p,q}$ defined by $\gamma(t) := g_ag_t$ and $\sigma(t) := g_bg_t$ generate the group $\pi_1(\mathcal{C}_{p,q})$. For every $s \in \mathbb{R}$ let $a(s), b(s)$ be the matrices obtained from $a, b$ by putting $a(s)_{22} = b(s)_{22} = s$ and leaving all other entries unchanged. Then $a(0) = a$ and $a(s) \in \mathcal{C}_{p+1,q}$ for all $s > 0$. But then the loops $\gamma_s : [0, \pi) \to \mathcal{C}_{p,q} \cup \mathcal{C}_{p+1,q}$, $s \geq 0$, defined by $\gamma_s(t) := g_ag_s$ give a homotopy from $\gamma$ to the constant loop $\gamma_1$. In the same way the loops $\sigma_s : [0, \pi) \to \mathcal{C}_{p,q} \cup \mathcal{C}_{p,q+1}$, $s \leq 0$, defined by $\sigma_s(t) := g-bg_s$ give a homotopy from $\sigma$ to the constant loop $\sigma_{-1}$. \hfill \Box

It is well known that the tube domain $\mathcal{H}$ has a canonical realization as bounded symmetric domain

$$\mathcal{D} := \{z \in E : 1_n - z\mathbb{I} \text{ positive definite}\}$$

given by the Cayley transformation $\gamma : z \mapsto (z - 1_n)(z + 1_n)^{-1}$. The boundary $\partial \mathcal{D}$ is the union of the CR-submanifolds of $E$

$$\mathcal{R}_{p,0} := \{z \in E : (1_n - z\mathbb{I}) \in \mathcal{C}_{p,0}\}, \quad 0 \leq p < n.$$ 

The group $\text{Aut}(\mathcal{D})$ acts transitively on each $\mathcal{R}_{p,0}$ and can be identified with $\text{Aut}(\mathcal{R}_{p,0})$ this way for every $p > 0$, see [11]. The Cayley transformation $\gamma$ gives a CR-isomorphism from $\mathcal{T}_{p,0}$ to the dense domain $\{z \in \mathcal{R}_{p,0} : \det(1_n - z\mathbb{I}) \neq 0\}$ in $\mathcal{R}_{p,0}$. Furthermore, the manifold $\mathcal{R}_{p,0}$ is an $\text{Aut}(\mathcal{H})$-equivariant fiber bundle over $\text{U}(n)/(\text{O}(n-p) \times \text{U}(p))$ with fibers the holomorphic arc components. In particular, $\pi_1(\mathcal{R}_{p,0}) = \mathbb{Z}_2$ for $p < n$. As in the proof of Proposition 5.2 it is shown for every $p > 0$ that every continuous CR-function on the universal covering $\mu : \mathcal{R}_{p,0} \rightarrow \mathcal{R}_{p,0}$ is constant on $\mu$-fibers.

Every $\mathcal{T}_{p,0}$ has the following local description in terms of normalized coordinates generalizing (4.17): For $k := n - p$ put

$$W := \{w \in \mathbb{C}^{k \times k} : w' = w\}, \quad E_1 := \mathbb{C}^{p \times k}, \quad E_2 := \{z_2 \in \mathbb{C}^{p \times p} : z_2' = z_2\}$$

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and denote by $\mathcal{M}$ in $U := \{(w, z_1, z_2) \in W \times E_1 \times E_2 : 1_p - z_2\overline{z}_2 > 0\}$ the CR-submanifold given by the matrix equation

$$w + \overline{w} = x + \overline{x} \quad \text{with} \quad x := \overline{z}_1(1_p - z_2\overline{z}_2)^{-1}(z_1 + z_2\overline{z}_1).$$

Then

$$w, z_1, z_2 \mapsto \left(\frac{w + z_1' y z_1}{\sqrt{2} y z_1}, \frac{\sqrt{2} z_1' y}{(1_p - z_2)y}\right) \quad \text{with} \quad y := (1_p + z_2)^{-1}$$

defines a biholomorphic mapping $\varphi$ from $U$ to a domain in $E$ such that $\varphi(\mathcal{M})$ is open in $\mathcal{T}_{p,0}$.

Due to the coincidence $\mathfrak{sp}(2, \mathbb{R}) \cong \mathfrak{so}(2, 3)$ between symplectic and orthogonal groups in low dimensions there are two ways to generalize the tube $\mathcal{T}$ over the light cone $\mathcal{C} \subset \mathbb{R}^3$ to higher dimensions: The first possibility in terms of symmetric $n \times n$-matrices and the symplectic group has been explicitly described above. The corresponding tube manifolds $\mathcal{T}_{p,q}$, $0 < p + q < n$, are all not simply connected and hence provide nontrivial examples for Proposition 7.1. Let us close with a description of the second way associated with orthogonal groups: For fixed $n \geq 2$ let

$$\langle z | w \rangle := z_1 w_1 + \ldots + z_n w_n$$

be the standard symmetric bilinear form and $z \mapsto \overline{z} = (\overline{z}_1, \ldots, \overline{z}_n)$ the standard conjugation on $\mathbb{C}^n$. Denote by

$$\mathcal{C} := \{(t, x) \in \mathbb{R}^2 \times \mathbb{R}^n : t_1 t_2 = \langle x | x \rangle, \ t_1 + t_2 > 0\}$$

the future light cone in $(n + 2)$-dimensional space time and by $\mathcal{T} := \mathcal{C} \oplus i\mathbb{R}^{n+2}$ the corresponding tube manifold over $\mathcal{C}$. Then $\mathcal{T}$ is a simply connected, 2-nondegenerate homogeneous CR-manifold with $\mathfrak{so}(\mathcal{T}, a) \cong \mathfrak{so}(2, n+2)$ for all $a \in \mathcal{T}$, see [11] for the last statement. Then the hypersurface $\mathcal{M}$, given in $U := \{(w, z_1, z_2) \in \mathbb{C} \times \mathbb{C}^n \times \mathbb{C} : |z_2| < 1\}$ by the equation

$$w + \overline{w} = (2\langle z_1 | \overline{z}_1 \rangle + \langle z_1 | z_1 \rangle\overline{z}_2 + \langle \overline{z}_1 | \overline{z}_1 \rangle z_2)(1 - z_2\overline{z}_2)^{-1},$$

is CR-equivalent to a domain in $\mathcal{T}$. As in (4.17) an explicit equivalence is obtained by

$$(w, z_1, z_2) \mapsto (1 + z_2)^{-1}(w + wz_2 + \langle z_1 | z_1 \rangle, 1 - z_2, \sqrt{2} z_1).$$

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