Additive lattice kirigami

Toen Castle, Daniel M. Sussman, Michael Tanis, Randall D. Kamien*

Kirigami uses bending, folding, cutting, and pasting to create complex three-dimensional (3D) structures from a flat sheet. In the case of lattice kirigami, this cutting and rejoining introduces defects into an underlying 2D lattice in the form of points of nonzero Gaussian curvature. A set of simple rules was previously used to generate a wide variety of stepped structures; we now pare back these rules to their minimum. This allows us to describe a set of techniques that unify a wide variety of cut-and-paste actions under the rubric of lattice kirigami, including adding new material and rejoining material across arbitrary cuts in the sheet. We also explore the use of more complex lattices and the different structures that consequently arise. Regardless of the choice of lattice, creating complex structures may require multiple overlapping kirigami cuts, where subsequent cuts are not performed on a locally flat lattice. Our additive kirigami method describes such cuts, providing a simple methodology and a set of techniques to build a huge variety of complex 3D shapes.

INTRODUCTION

Adding folds to thin sheets provides a natural mechanism to manipulate both their shape and elastic properties (1, 2). However, achieving complex target shapes with only bends or folds is computationally expensive and is an essential prerequisite to creating functional devices. The difficulty in controlling shape and the techniques devised to overcome it provide the basis of the art and science of origami. Structures such as Buckminster Fuller–style geodesic domes fundamentally owe their strength and curvature to the judicious introduction of lattice defects into the flat tiling system defined by otherwise hexagonal arrangements of struts (3). In contrast, the model sheets in origami design are intrinsically flat, and no amount of folding or bending can induce a sheet of paper to bulge outward in a smooth way. A bulging form can be achieved with crinkles though, as in the unavoidable wrinkles that appear when a round shape is gift-wrapped. Many origami techniques are artful variants of this wrinkling phenomenon, using curves and folds to create specific crinkles that introduce the desired mean (extrinsic) and apparent Gaussian (intrinsic) curvature (4, 5).

In recent papers, we used a related method, known as kirigami, which supplements the folds of origami with moves that can cut and rejoin the sheet (6–10). We used these kirigami elements to create collections of actual Gaussian curvature dipoles on a sheet (6). We restricted ourselves to kirigami performed on a lattice (6, 8), focusing for simplicity on cuts that respected both an underlying hexagonal lattice and the dual triangular lattice, allowing us to use the physics of dislocations and disclinations for guidance. In adopting this simplification, a number of essential constraints were described, and a second batch of simplifying assumptions was also introduced, along with the design rules they implied. These restrictions provided a natural way of exploring the design paradigm without being swamped by the possibilities allowed by arbitrary cuts. In this setting, kirigami introduces lattice defects into otherwise hexagonal arrangements of fixed-length struts. We showed how to combine the curvature dipoles generated by these lattice defects with fold lines to form simple shapes and algorithmically demonstrated how to combine large arrays of these kirigami elements to make arbitrary stepped surfaces of hexagonal or triangular units (8). Although the design capacity of origami is already remarkable, moving to this algorithmic lattice style of kirigami allows for similarly complicated shapes to be formed while greatly reducing the complexity of the design process and the amount of wasted material.

This paper first explores the breadth of structures possible when some of the nonessential simplifying assumptions are discarded. We maintain the first half of our first rule, “We assume that our sheet cannot shear or stretch...,” while leaving the consideration of the second part, “and can only be bent and cut along straight lines” for a subsequent paper. We relax our second rule, no longer restricting ourselves to kirigami elements that simultaneously maintain both the lattice and its dual, instead preserving only one of the lattices. Finally, we abandon our third rule, allowing cuts and folds that lead to nonvertical sidewalls.

These changes allow a more general paradigm of lattice kirigami and naturally extend our earlier ideas. Previously, our method of introducing intrinsic curvature into the sheet involved excising connected regions of material and then resealing the boundary of the excision. The removal of wedges reduces the amount of angular material around the tip of the wedge and adds angular material to the base. In the new freer regime considered here, a wider variety of wedge shapes and joining mechanisms are considered, including wedges with angles above π and spatial relations between the wedges that require extra material to be inserted rather than excised. For this reason, we call these methods “additive lattice kirigami.” These generalizations provide a huge variety of new kirigami motifs when composed together, some of which neither add nor remove material. They also provide a natural description of the cell division and rearrangement ideas that initiated our studies on kirigami (11–13).

A particular goal of kirigami design is to distribute Gaussian curvature arbitrarily on a surface while simultaneously using folds to impart mean curvature and control the embedding. Because we limit ourselves to kirigami on a lattice, this goal becomes the arbitrary redistribution of Gaussian curvature onto lattice locations. In the remainder of this paper, we consider different approaches to achieving this goal. We first discuss varying cut types to relax our previous restriction of imparting angular deficits and excesses at lattice points of the hexagonal lattice to multiples of π/3. We then consider a much...
broader generalization of the original kirigami cuts that permits a versatile distribution of Gaussian curvature across the lattice. By going beyond our previous work (6, 8) in this manner, we can explicitly consider constructing kirigami surfaces that are no longer, for instance, the graphs of single-valued height functions. Finally, we vary the lattice types and explore more complex kirigami transformations.

RESULTS

Changing the fold and cut angles

There are natural generalizations of our earlier kirigami cutting patterns that make the kirigami structures more versatile while remaining in the kirigami rubric. The kirigami element in Fig. 1A has five degrees of freedom corresponding to changes in the excised irregular hexagon: the climb ($\mathbf{\ell}_c$) and glide ($\mathbf{\ell}_g$) of the displacement $\mathbf{\ell}$ of the dislocation relative to the Burgers vector $\mathbf{b}_1 = -\mathbf{b}_2$, the length of the Burgers vector $||\mathbf{b}||$, and the angles of each wedge. One choice of parameters sets the climb and glide to 0, resulting in a diamond-shaped excision that, when rejoined, will generate a sheet with one point of angular excess centered between two points of angle defect (Fig. 2A). This excised shape can be further modified by changing the wedge angles. These angles were previously presumed to be less than $\pi$ to be compatible with simple stepped structures (because a large excised angle creates a large angle deficit and an incompatibly sharp cone).

Our kirigami method is based on a lattice; thus, Gaussian curvature cannot be created arbitrarily but rather occurs in (tuples of) point dipoles (6, 8). The amount of curvature is directly proportional to the angle excess or defect at a point, so the ability to vary these angles provides increased versatility. However, on a hexagonal lattice, using the restrictions and the terminology of our original paper (6), the angles are limited to multiples of $\pi/3$, such as the example in Fig. 1B. Specifically, a 5-7 dislocation pair has an angle difference of $\pi/3$, while a 2-4 or a 4-8 pair has an angle difference of $2\pi/3$. All kirigami surface rearrangements are composed of combinations of these dislocation pairs, sometimes with common points, so the lattice points have an angle excess or defect that is a multiple of $2\pi/3$, whereas the lattice points have multiples of $\pi/3$. To achieve other gradations of angles and thus curvature, there are two options: releasing the constraint that maintains the dual lattice (with constant intrinsic edge lengths) or using a different lattice.

As shown in Fig. 1B, it is possible to have nonvertical sidewalls while preserving both the lattice and its dual. However, the full constraint still requires us to maintain highly symmetric cuts and joins. If we consider a vertex that serves as the top of a cutout wedge, the total angle must be less than $2\pi$. Because the wedges must have an angle of $\pi/3$ or $2\pi/3$, the resulting structures are limited and it can be shown that only an equilateral triangular plateau is possible.

Ignoring the maintenance of the dual lattice allows more possibilities, with angle defects and excesses free to vary away from multiples of $\pi/3$. In practice, it might be appropriate to ignore the dual lattice because only the lattice itself is important and the dual is not, such as in the case of uniform-length rods being assembled together to form the frame of a geodesic dome. Alternatively, neither lattice may be physically relevant, with the lattice being used only for its useful “bookkeeping” role. Varying angles of wedges on the hexagonal/triangular lattice are shown in Fig. 3. As described in the caption, in some of the cases the edges are bisected by the excised wedge, but not at a right angle, so that they are then rejoined at a slight angle. This operation respects the edge lengths but not the lattice itself. Typically, these new options identify different basis points across the excised wedge. Other examples are also included in Fig. 3 for comprehensiveness, demonstrating motifs that satisfy the original criteria but have not been previously illustrated. These include excised wedges whose cuts run from lattice points to dual lattice points and wedges oriented at an angle to the lattice.

Generalizing kirigami: Area-preserving kirigami

We now seek out kirigami cut configurations that preserve the sheet material under the cutting action. In the simplest approach, we try to avoid removing material; thus, we must first avoid losing material to a dislocation, which we can achieve by setting the climb to 0 (Fig. 2A). Now, the only material lost is in the excised wedges themselves, and the amount of material in these wedges depends on their angles. In the previous discussion, when relaxing the constraint that angles need to be a multiple of $\pi/3$, the angles were still presumed to be less than $\pi$ to be compatible with simple stepped structures (note that a large excised angle creates a large angle deficit and an incompatibly sharp cone). We now allow these angles to increase to $\pi$, shrinking the excised diamond
Fig. 2. General moves. (A to D) Simple kirigami cuts with zero climb and glide and their effects on the lattice. The cuts are drawn partially open for emphasis. Large angle deficits and excesses result in severe lattice defects. Each motif, composed of two curvature dipoles and with negative poles coinciding, has net zero curvature. Points with an angle defect are marked with a star. (A) An excised diamond results in two pentagons and an octagon. (B) Setting angles $\alpha_1 = \alpha_2 = \pi$ is equivalent to just cutting a straight slit and pinching it closed sideways. Closing a slit by pinching the middle creates a large angle excess $(2\pi)$ in the center of the scar. (C) Angle $\alpha_1$ or $\alpha_2$ can be greater than $\pi$ (in this example, $\alpha_1 = 2\pi/3$ and $\alpha_2 = 4\pi/3$). (D) The angle $\alpha_2 = 5\pi/3$ causes a severe lattice defect. (E to H) Kirigami slits with nonzero glide on a hexagonal lattice. Each slit is of length 3, running along lines between lattice locations of the dual lattice and hence forming defects on the dual lattice. Each pinch point, which identifies with no other point under the kirigami action, is indicated by a star. The large gray numbers indicate the change in the number of sides of the hexagons and hence the angle defect or excess. The small numbers label edges for clarity. Compared with zero-glide configurations, the curvature is diluted, reducing the magnitude of the lattice defects. (E to H) A series of zigzag motifs with varying angles on the lattice (left), zoomed in (center), and the cut-and-rejoined final configuration (right). Motif (G), which minimally changes the magnitude of the induced Gaussian curvature, is modeled in paper in (H). (I to K) Composition of kirigami motifs. (I) A repeated series of simple zig cuts (bottom) is equivalent to a longer zigzagging cut (top), both of which form a pair of $\pm 5\pi$ dislocations on a hexagonal lattice or more generally a pair of $\pm 1$ dislocations. These dislocation pairs are necessarily at the angle specified by this geometric construction as other angles require the addition or removal of material. The arrows show the relative movement of the hexagons to each other, as the wedges are removed and reinserted. (J) A slit pattern can be “closed up” by pinching the points marked with stars, forming points and shifting the Gaussian curvature to the tips of the slits. (K) Combined with the natural folds along the lattice (between the centers and edges of the cuts), the slit pattern becomes a lattice of tetrahedra.

Castle et al. Sci. Adv. 2016; 2: e1601258 23 September 2016
perhaps unsurprising in that both plastic surgery and our kirigami techniques are based on cutting, rearranging, and rejoining a two-dimensional surface. Z-plasty allows skin to be moved where it is needed without being detached from the blood supply that keeps it alive. The elastic properties of the skin mean that the points of nonzero Gaussian curvature created this way are spread out by the stretching of the skin, outside the unstretchable and unshearable regime we consider. Just as the “v” cuts can be composed together into zigzagging chains and more complex structures, so too is z-plasty extended to a variety of techniques such as “v-y plasty,” “v-m plasty,” “double z plasty,” and so on (19).

These line-cut motifs can be composed together, just as the simpler kirigami motifs in our previous papers were (6, 8). In the case of the “zigzagging” or z cut, adjacent zigzags can be overlaid, as shown in Fig. 2. These combined zigzags create a pair of 5-7 dislocations (or, equivalently, a pair of Gaussian curvature dipoles) displaced from each other by the total number of “zigs” in the zigzag. This extended motif provides an area-preserving means of first creating two dipoles in a flat sheet and then displacing them from each other. The direction of the displacement is explained below in the context of negative climb.

The motifs can also be combined in noncontiguous ways, such as the regular array of slits in Fig. 2 (J and K). Once the sheet is perturbed away from the undeformed-lattice limit, and with none of our earlier constraints that made defects interact locally, it is necessary to think of these z cuts as ±1 Gaussian curvature defects independent of the original coordination of the site—many such cuts in the same location of the lattice can change the Gaussian curvature of the lattice point regardless of whether that lattice point was previously a sixfold coordinated site.

**Generalizing kirigami: Additive kirigami**

We originally considered rejoining edges after excising material (6) ($\ell_c > 0$), and above, we discussed an area-preserving scheme of rejoining edges after cutting straight slits (with $\ell_c = 0$). Thus, a natural extension is to set $\ell_c$ to a negative number. Although physically incongruous in the context of cutting paper, this is merely the formalization of inserting extra material between excised wedges.

One natural occurrence to motivate this consideration is the growth process in layers of hexagonally arranged cells, following the work of Rivier et al. (13). Figure 4A shows a diagram of cell splitting at a set orientation within a crab cuticle to form a ridge or furrow, together with several kirigami interpretations (Fig. 4B to D) of the final arrangement of cells. The first interpretation introduces two dislocations which face away from each other. This removal of material to simulate a “growth process” is inelegant because it requires dislocations comparable to the system size, removing material all the way to the boundary of the sheet. A much more natural scenario is to insert material instead of removing it (Fig. 4C). In this simple case, the normal kirigami process—excising a pair of joined wedges, bringing the sides together to make a slit, and then rejoining across the slit—is reversed. Instead, a straight cut is made and “pulled open” so that a pair of joined wedges can be inserted into the gap. This “additive kirigami” is thus a reversal of the “subtractive” kirigami that we previously introduced (6, 8). The final example in Fig. 4D shows the most formally useful viewpoint: The climb is negative, so a strip of extra material is inserted, which is partially canceled at its ends by the overlapping wedges removed at each end of the dislocation. This shows a particularly simple context in which both our original kirigami approach and this newer one can be used to target a particular distribution of curvature on the lattice.

---

**Fig. 3. A compendium of possible kirigami cutting angles on the hexagonal/triangular lattice.** The gray arrows indicate Burgers vectors for different vector pairs. a and b show various cutting lines from one basis point of the lattice A to another, separated by length. The blue cuts of a are shorter, and the green cuts of b are longer. Cuts bisect the lattice edges, but not at right angles, producing angle defects not generally equal to multiples of $\pi/3$. c and d show excision motifs compatible with vertical sidewalls. c shows wedges based on the dual lattice $\Lambda$ both aligned and misaligned with the lattice directions. d shows a wedge whose endpoints include both A and $\Lambda$.
Combining the original subtractive kirigami paradigm with negative values of climb provides a natural language to discuss quite complex cutting patterns. For example, consider again the zigzag cut in Fig. 2G. The Burgers vectors of the “zig” and the “zag” define a parallelogram of material, which must be inserted to join the dislocations. By a happy geometric coincidence, this area exactly cancels with the two triangles that form the dislocations, allowing the rearrangement in Fig. 2G to occur without addition or subtraction of material. For this to be a material-neutral action, the climb must have exactly the right negative value to compensate for the formation of dislocations. The amount of glide is immaterial to the net material requirements, although a geometric realization of the kirigami action, which does not disconnect the sheet, may only be possible for certain (small) glide values.

Generalizing kirigami: Complex cuts
As the above results suggest, a versatile mechanism to manipulate the Gaussian curvature of an initially flat sheet (in such a way that the total Gaussian curvature still sums to 0) is to cut an appropriately shaped slit or hole and then pinch the gap closed again in a different sense by choosing new points to contain the angle defect. In this section, we show that any cut or hole located appropriately on the lattice and joined up in this fashion is equivalent to a composition of the above-discussed kirigami actions, with the caveat that in any composition of kirigami actions, all but the first will have to navigate the lattice defects induced by the preceding kirigami elements. Earlier, we avoided these interactions. We introduced large collections of dislocation pairs in forming our pluripotent algorithmic kirigami sheet (8), but we kept the dislocations small in extent and disjoint, in the sense that no kirigami cut ran between the curvature poles of another dislocation. In a more complex curvature landscape, locally isolated disclinations destroy the Euclidean idea of parallelism and thus raise questions about how antiparallel Burgers vectors can cancel. These questions are overcome by replacing the requirement of parallelism with the existence of a path along which the Burgers vector can be parallel transported to cancel with the corresponding dislocation.

With this in mind, we demonstrate that arbitrary cuts can be viewed as compositions of simpler kirigami elements iteratively: We start with the simplest case of a slit on a hexagonal lattice and then extend it to cover all cases. Consider a slit on the lattice, perhaps with bends and branch points, and identify the locations that will be pinched together (in the manner shown in the figures above) as the surface is rejoined across the slit. These pinch points need to be located at appropriate cyclic lattice distances from each other around the slit, to respect the lattice while closing up. Among other stronger requirements on this spacing, the total lattice length of the bounding cycle must be even. One first selects any of the pinch points and performs a material-neutral kirigami action analogous to those in Fig. 2 (B to G) centered at that location to identify the two adjacent edges. This will achieve part of the desired boundary identification at the cost of unintentionally identifying the edges on the other side of the slit, forming a “spur” branching off the slit. However, this spur can be moved along the slit with a further kirigami action: One of the motifs in Fig. 2 (B to G) is again inserted into the lattice, with one end of the wedges inside the spur and the other in the slit. Each insertion of this kind moves the spur one edge length along the cut, with edge (de-)identifications forming a new spur as it removes the old one. This movement is repeated until the spur is brought to coincide with one of the other designated pinch points. This entire “pinch and slide” process is repeated until the desired edge identifications have been achieved as illustrated in Fig. 5.

In this way, a connected set of straight-line cuts—whether the combined cut is straight, bent, or branched—can be closed up in a different manner (that is, have its edges reidentified differently) using only the generalized kirigami cuts discussed above to shuffle the curvature around to the desired points. If the initial cut involves material excision instead of just slit formation, the same result holds: Standard kirigami actions remove this void material, reducing it to a branched slit, and then the above procedure is followed. The void material can be shrunk in any convenient way, repeatedly using a kirigami motif similar to the one in Fig. 2A to collapse its boundaries together, as shown in Fig. 5A. As stated above, the accidentally formed spurs that may be created from this process can be moved along the cut to a desired location; thus, their formation is not a problem. One potential problem, though, occurs when the hole is of an odd-sized perimeter, such as the triangle in Fig. 5B [recall that the total cut must be of even length; thus, there must be a second hole along the cut (also of odd size) that should be compensated for], so that the kirigami cuts, which identify edges pairwise, cannot reduce the hole to a branching slit. This can be overcome by creating a superlattice with halved lattice dimensions; thus, every edge previously of length 1 is now of length 2 (or, equivalently, only allowing even length–perimeter cuts). This new hole now has even length and can be removed as before. This may be difficult to achieve in a real material, which perhaps already has a small or otherwise irreducible scale. We note, though, that the use of the superlattice is only a temporary measure (all disclinations are migrated back to the original lattice) introduced to simplify the constructive algorithm above, so only the intermediate states of the kirigami action require the use of the superlattice.
Fig. 5. Collapsing an excision to become a puckered slit. (A) Shrinking an excision to identify adjacent edges using subtractive kirigami as indicated by the stars and Burgers vector edges. (B) Void spaces with odd perimeters (lattice locations in black) can be removed by temporarily halving the lattice dimensions (new locations have white centers). The small spurs thus generated are later united with other small spurs, and the half-dimension lattice can be omitted. (C to H) An example of a complex kirigami cut being the composition of simpler kirigami actions. The sequence from (C) to (H) shown by double arrows indicates the pinch-and-slide algorithm described in the text, whereas the single arrow sequence indicates that the same result can be achieved by the composition of subtractive and additive kirigami. (C) The bent slit is pinched closed at two antipodal points around its perimeter, marked with orange and blue stars. The Burgers vectors required to close the dislocations of just the two edges around each star are marked; they are misaligned and do not cancel. (D) A strip of triangles is inserted using additive kirigami, as illustrated in Fig. 4. (E) With this new configuration, the Burgers vectors of the two pinch points and their neighboring edges do cancel, using traditional subtractive kirigami with wedge angles $x$ and positive climb and glide. (F) Starting with the “pinch” part of the algorithmic procedure, the pinching action in Fig. 2C is applied around the blue star along the part of the slit marked in green. (G) As expected, this forms an unwanted spur that is “slid” along the cut by the same method. In this example, a second pinch at the orange star would have exactly the same effect. (H) Final lattice rearrangement. (I) Excising a hexagon from a hexagonal lattice and replacing it with a twist to create a ring of 5 and 7 curvature sites. In this case, most +1 and −1 sites cancel, leaving just two 5–7 pairs arranged in a stepped configuration with nonvertical sidewalls. Two sides of the hexagon are marked with orange stars and lines to convey the twisted geometry.

The above construction is presented to conceptually unify the new methods with the original lattice kirigami techniques, but an actual desired kirigami transformation can often have much simpler and more elegant alternatives. For example, the transformation of the bent slit in Fig. 5C can be simply achieved by first adding a line of material (with positive ±1 dislocations) and then taking advantage of the new disclination points to remove that material with a subtractive dislocation pair (Fig. 5, D, E, and H). This example was chosen for its simplicity; thus, the alternate method is no easier than the algorithm above (Fig. 5, F to H). However, similar methods can be applied to more complex (and larger) kirigami rearrangements to require fewer actions.

We have shown that any arbitrary material removal is equivalent to a series of generalized kirigami cuts, but we also know that the extension to negative climb allows material insertion. Recall that material addition is the reversal of the standard material removal, as shown in Fig. 4C. Thus, if a cut is made and material with an equal perimeter is incorporated into the gap, the result could have been created by additive kirigami. Further, these two events can occur sequentially: First, a patch is chopped out (and the sheet is sealed up), and then a patch is inserted along the scar line of the previous cut. Quite generally shaped patches can be used as long as they respect the lattice spacing; thus, it is simplest to choose a patch whose perimeter is composed of segments between (dual) lattice locations along the (dual) lattice directions. The patch dimensions do not need to match except for the perimeter, but it may be appropriate for design reasons to excise a patch and reinsert it with an offset. Figure 5I shows a simple example of a hexagon being excised and then replaced with a twist. In general, such a hexagon twist action produces 12 disclination points, corresponding to the inner and outer corners of the hexagon; however, in this case, the dimensions of the irregular hexagon and the amount of twist have been specifically chosen so that eight of these points cancel, leaving a simple structure with just four disclinations (two 5–7 curvature dipoles).

The construction in the proof above provides an algorithm to replicate a hole or slit cutting/patch insertion/sheet rejoining with generalized kirigami—that is, kirigami extended to include negative climb. Thus, the word “kirigami” can be meaningfully extended to describe any such process of cutting, insertion, or removal of material; re-identification of the perimeter; and rejoining.

Crafting structure from complex cuts

The versatility and power of this new kirigami technique lie in the combination and juxtaposition of multiple kirigami elements. Examples of an array of mixed pyramid types and an array of octahedra are shown in Fig. 6. Interestingly, it is possible to maintain the same cutting pattern
that creates a planar array of octahedra, but by varying the fold types (mountain to valley and vice versa) one can cause individual octahedra to protrude downward from the base layer instead of upward. This forms both a protruding nubbin on the bottom as well as a hole on the top, allowing multiple stacked layers to interlock. The geometry of this locking mechanism is tight enough that disengagement of the layers requires partial unfolding. In addition, the entire structure is composed of equilateral triangles (inherited from the flat equilateral triangles folded from the flat sheet in Fig. 6E via the octahedral faces) and so is extremely rigid once assembled.

As an aside, we note that the positive-climb dislocations of subtractive kirigami correspond to the removal of material. However, for practical applications where only the “top” surface is exposed and the dislocation is not seamlessly rejoined but rather just brought together with fasteners, it may be useful to keep the dislocation material attached for use as fastening tabs. A minimal-cutting technique to make a zero-glidel/positive-climb dislocation is shown in Fig. 7A, with a central fold and two slits. This same technique is easily generalized to a nonzero glide/zero climb, where the dislocation is replaced with a single cut and where the two excised wedges each have a fold down their center (see Fig. 7B). A greater challenge is posed by a dislocation with both (positive) climb and glide; the diagonal nature of the dislocation means that a central fold in the style of Fig. 7A identifies points together that are not aligned along the Burgers vector. A solution to this is to insert a number of slits (as shown in Fig. 7C) together with folds that are perpendicular to the Burgers vector. The dislocation is broken down into a series of parallelograms, each inhabited by a rectangle and two triangles. As the two sides of the dislocation are brought together, the rectangles twist and are overlaid by the triangles, which fold on top of them. In the initial and final states, all panels are flat; however, the twist of the rectangle is accompanied by a bend across its diagonal during the transition. If there is a positive-energy cost for bending away from the folds, then the structure is bistable with a snap-through transition between the two configurations. Such a bistable configuration is not possible with a straight pure climb dislocation, but the dislocation can be broken down into parts with alternating positive...
and negative glide, which together total zero glide. In this way, a pure climb configuration can be built from zigzagging sections of mixed climb and glide.

The slitted structure in Fig. 7 (C to G) is not only a convenient modification of our original subtractive kirigami paradigm (which, by a happy accident, has a bistable conformation) but also an example of a composition of complex kirigami cuts as described above. Each slit in Fig. 7C is an example of area-preserving kirigami of the form of Fig. 2B, where a slit is cut and the center of each side is pinched to reconnect the sheet “sideways” across the slit. Away from the ends of the dislocation, the points with π angular deficit become the folded π/2 corners of the rectangle, whereas the endpoints of the slits now have an angular excess of 2π and are the common points of neighboring rectangular tabs. At the end slits, the outer pinch point defines the folded π/2 corner of the triangular tabs of the end wedges, while the inner pinch point becomes the corner of a rectangular tab as before. It is only the folding of the end triangles that now defines the 5-7 disclinations; different fold angles and different lattice locations could terminate the dislocation with, for example, 4-8 or 2-4 dislocation pairs.

Changing the lattice

Only the hexagonal/triangular and square lattices have been considered previously (6, 8) and thus far in this paper, but extensions to other lattices are both possible and productive. We first consider Bravais lattices without bases. These are lattices where each vertex is translationally equivalent to every other vertex; hence, these lattices are naturally limited in their complexity. There are five Bravais lattices in translationally equivalent to every other vertex; hence, these lattices are Bravais lattices without bases. These are lattices where each vertex is also equivalent to every other vertex.

The hexagonal/triangular dual lattices can sustain disclinations of multiples of π/3 in a number of directions, whereas the square lattice has the benefit of familiarity despite being limited to disclinations of multiples of π/2. The hexagonal lattice, dual to the triangular lattice, is itself a triangular Bravais lattice with a basis (Fig. 1A). It is the presence of this basis that allows the smaller disclination angle π/3 not (isotropically) available to other basis-free lattices.

In contrast with these basis-free lattices, lattices with a basis can be arbitrarily complex. There can be any number of extra vertices inside the unit cell, arranged arbitrarily to allow for a huge variety of designs. As the basis becomes larger, the geometric significance of the lattice...
On a lattice, with a basis of 4 vertices and 10 edges. Rather than being concerned with the lattice and dual lattice as in our previous work, the relevant feature for kirigami design purposes is the tiling and the Voronoi dual of the tiling. The dual is a packing of irregular pentagons meeting three and four to a vertex. This irregularity makes the dual tiling an unappealing target for kirigami design because of the difficulty of getting edges to meet up across excisions. However, cuts can be made along the edges of the 4.3.4.3.3 tiling in a protean way: The tiling has a symmetry that they can be put under the rubric of combining “subtractive” lattice kirigami with negative-climb elements. Together, these techniques greatly expand the ways in which kirigami motifs can be used to distribute Gaussian curvature at will on the different lattice sites.

We note that just as multiple kirigami cuts can be combined together into a sheet, so too can distinct kirigami sheets be combined together into a single interlocking structure. This idea has been explored without cuts in modular origami designs (21). For static origami structures, the individual pieces are typically lodged firmly in place, while in dynamic contexts, the shapes are glued together to maintain cohesion throughout the movement. Kirigami offers a more flexible paradigm, where adjacent pieces can be geometrically locked together or intertwined. There are two straightforward avenues by which such modular kirigami can be achieved. The first takes advantage of forming kirigami sheets of periodically repeating geometric shapes, such as

![Fig. 8. Details of the 4.3.4.3.3 tiling.](image-url)

(A) A 4.3.4.3.3 tiling with both primitive vectors of the centered rectangular lattice shown. (B to D) A square array of square antiprisms formed by excising every second square from the 4.3.4.3.3 tiling. (B) Top view. (C) Bottom view. (D) The loose arrangement of antiprisms in the plane allows the sheet to bend a certain angle out of the plane, until the octahedra touch. (E) A cutting pattern to make a layer of cuboctahedra. The cuts impose chirality on the tiling. (F) A layer of cuboctahedra is rigid, especially if touching squares are stuck together, because there are no out-of-plane deformations (to linear order). The chiral cutting pattern once again becomes a chiral scar pattern on the achiral layer. (G) The pinch motif shown in Figs. 2E and 7B applied along the slit allows the triangles to fold on themselves, reducing the cuboctahedron exterior to its component squares, that is, a cube.
the array of octahedra in Fig. 6F. These polyhedra can have interstitial polyhedral gaps (tetrahedra, in the case of the octahedral array), allowing a second array of complementary polyhedral shapes to interlock with the first. With an appropriate design, the sheets may be “zipped” together row by row, folding the sheets back on themselves to expose the sites into which the complementary shapes dock. The second approach uses the construction process of the kirigami directly, interlocking unfinished sheets, that is, those in which the kirigami cuts have not yet been rejoined. This adds considerable degrees of freedom to the structure. This manner of polyhedral interdigitation is illustrated in Fig. 6 (G and H), and additional structures using these two methods of attachment are shown in Fig. 9.

In analogy with modular origami tubes (22), it is also possible to roll kirigami tubes up into cylinders with specifically modulated surfaces and then to arrange the cylinders into rod packings. Just as is the case with sheets, the surfaces of the rods can be designed to interlock with other rods either before or after the cut edges are rejoined. Figure 9 shows the rolling of sheets into tubes, sample tube geometries, and a junction of three tubes mid-assembly with unconnected edges. In general, modular flat-foldable structures have a host of advantages over traditionally engineered structures, including the simplicity with which they can be transported, deployed, and extended, and kirigami will enable simple implementation of such designs.

We anticipate that these modular ideas, together with the general relaxation of our lattice kirigami rules, will be useful in assembling arbitrary (albeit not pluripotent) structures out of flat sheets, lifting our previous restriction of being able to recreate surfaces that are graphs of functions with sufficiently small slopes (8). The price to be paid is the loss of an algorithmic approach to inverse design, in which a given target shape can be readily mapped to a precise sequence of area-preserving or additive kirigami moves. This is a direction of current research, which will particularly require addressing the problem of “parallel transport” mentioned above to systematically address the placement of defects in a (locally) nonflat background.

**MATERIALS AND METHODS**

**Experimental design**

The kirigami designs were cut from sheets of paper or cardboard to build the models shown. Folding was done by hand along lines marked with standard origami fold indications, and preparatory scoring was performed along folds in the thicker media. Cutting and scoring were done by hand or by using a Graphtec CE6000 Series Cutting Plotter.

**REFERENCES AND NOTES**

1. T. A. Witten, Stress focusing in elastic sheets. Rev. Mod. Phys. 79, 643–675 (2007).
2. J. L. Silverberg, A. A. Evans, L. McLeod, R. C. Hayward, T. Hull, C. D. Santangelo, I. Cohen, Using origami design principles to fold reprogrammable mechanical metamaterials. Science 345, 647–650 (2014).
3. H. Kenner, Geodesic Math and How to Use It (University of California Press, Berkeley, 2003).
4. T. Tachi, 3D origami design based on tucking molecule, in Origami4 (CRC Press, Boca Raton, 2009), pp. 259–272.
5. T. Tachi, Freeform variations of origami. J. Geom. Graph 14, 203–215 (2010).
6. T. Castle, Y. Cho, X. Gong, E. Jung, D. M. Sussman, S. Yang, R. D. Kaimen, Making the cut: Lattice kirigami rules. Phys. Rev. Lett. 113, 245502 (2014).
7. B. F. Grosso, E. J. Mele, Bending rules in graphene kirigami. Phys. Rev. Lett. 115, 195501 (2015).
8. D. M. Sussman, Y. Cho, T. Castle, X. Gong, E. Jung, S. Yang, Algorithmic lattice kirigami: A route to pluripotent materials. Proc. Natl. Acad. Sci. U.S.A. 112, 7449–7453 (2015).
9. B. G. Chen, B. Liu, A. A. Evans, J. Paulose, I. Cohen, V. Vitelli, C. D. Santangelo, Topological mechanics of origami and kirigami. Phys. Rev. Lett. 116, 135501 (2016).
10. L. H. Dudte, E. Vouga, T. Tachi, L. Mahadevan, Programming curvature using origami tessellations. Nat. Mater. 15, 583–588 (2016).
11. J.-F. Sadoc, N. Rivier, J. Charvolin, Phyllotaxis: A non-conventional crystalline solution to packing efficiency in situations with radial symmetry. Acta Crystallogr. 68, 470–483 (2012).
12. J. F. Sadoc, J. Charvolin, N. Rivier, Phyllotaxis on surfaces of constant Gaussian curvature. J. Phys. A Math. Theor. 46, 295202 (2013).
13. N. Rivier, M. F. Miri, C. Oguey, Plasticity and topological defects in cellular structures: Extra matter, folds and crab moulting. Colloids Surf. A 263, 39–45 (2005).
14. M. K. Blees, A. W. Barnard, P. A. Rose, S. P. Roberts, K. L. McGill, P. Y. Huang, A. R. Ruyack, J. W. Kevek, B. Kobrin, D. A. Muller, P. L. McEuen, Graphene kirigami. Nature 524, 204–207 (2014).
15. T. C. Shyu, P. F. Damasceno, P. M. Dodd, A. Lamoureux, L. Xu, M. Shlian, M. Shtein, S. C. Glotzer, N. A. Kotov, A kirigami approach to engineering elasticity in nanocomposites through patterned defects. Nat. Mater. 14, 785–789 (2015).
16. A. Lamoureux, K. Lee, M. Shlian, S. R. Forrest, M. Shtein, Dynamic kirigami structures for integrated solar tracking. Nat. Commun. 6, 8092 (2015).
17. A. Limberg, S. Wolfe, The Planning of Local Plastic Operations on the Body Surface: Theory and Practice (Collamore Press, Lexington, MA, 1984).
18. A. D. McGregor, I. A. McGregor, Fundamental Techniques of Plastic Surgery (Churchill Livingstone, London, ed. 10, 2000).
19. Local flaps: Z, VY and combinations, www.eatonhand.com/flip/webflaps.htm [accessed 16 May 2016].
20. C. Kittel, Introduction to Solid State Physics (John Wiley and Sons, Hoboken, NJ, ed. 7, 1996).
Additive lattice kirigami
Toen Castle, Daniel M. Sussman, Michael Tanis and Randall D. Kamien

Sci Adv 2 (9), e1601258.
DOI: 10.1126/sciadv.1601258

ARTICLE TOOLS  http://advances.sciencemag.org/content/2/9/e1601258

REFERENCES  This article cites 16 articles, 4 of which you can access for free
http://advances.sciencemag.org/content/2/9/e1601258#BIBL

PERMISSIONS  http://www.sciencemag.org/help/reprints-and-permissions

Use of this article is subject to the Terms of Service