Entanglement Classification via Operator Size: a Monoid Isomorphism

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Abstract

We develop a general framework for the entanglement classification in discrete systems, lattice gauge field theories and continuous systems. Given a quantum state, we define the dimension spectrum which is the dimensions of subspaces generated by $k$-local operators acting on the state and characterize the entanglement resource of the state. With the spectrum as coefficients, we define the entanglement polynomials which induce a homomorphism from states to polynomials. By taking quotient over the kernel of the homomorphism, we obtain an isomorphism from entanglement classes to polynomials, which classifies entanglement effectively. It implies that we can characterize and find the building blocks of entanglement by entanglement polynomials factorization. It’s also proven that an operator inducing automorphisms on all of the subspaces of $k$-local operators keeps the entanglement polynomials invariant. SLOCC and permutation are examples of such operators. We also construct a series of states called stochastic renormalized states to compute entanglement polynomials effectively by computing their ranks.

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1 Introduction

Entanglement is a nonlocal quantum correlation. A quantum state is entangled if it cannot be factorized. Entanglement is a key concept for understanding quantum gravity [1] and topological order [2], such as 1) the Ryu-Takayanagi formula [3] which relates the entanglement entropy on the boundary and the extremal surface area in the bulk in AdS/CFT correspondence [4]; and 2) ER=EPR [5] which relates the entanglement and the spacetime connectivity. So it’s important to classify entanglement, which hopefully can characterize the holographic spacetimes and quantum phases.

Entanglement classification is a long-standing open problem. Various methods were proposed [6, 7, 8, 9, 10, 11, 12, 13]. In this work, we develop a general framework for the entanglement classification in discrete systems, lattice gauge field theories and continuous systems.

The key concept is the dimension spectrum which characterizes the entanglement resource of a quantum state. Inspired by the state-dependent operator size [14], it’s defined as the dimensions of subspaces generated by $k$-local operators acting on the state. Roughly speaking, $k$-local operators are linear combinations of the tensor products of $k$ local operators. We then construct a polynomial with its dimension spectrum as coefficients. We call it the entanglement polynomial. It induces a homomorphism $f$ from states to polynomials. With the tensor product as the multiplication, quantum states form a monoid $S$. Polynomials with polynomial multiplication also form a monoid $M$. Then the first isomorphism theorem for monoids states that the quotient of $S$ over the kernel of $f$ is isomorphic to $M$,

\[ S/\text{Ker}(f) \cong M. \] (1)
We will derive this relation in Section 3.2. Each element in $S/\text{Ker}(f)$ is an entanglement class. The isomorphism keeps the product structure of states, so product states are mapped to reducible polynomials in $M$. In this way, by polynomial factorization, we can characterize and find the building blocks of entanglement.

Another reason that entanglement polynomials can classify entanglement is their symmetries. A large class of operators acting on a state keep its entanglement polynomial invariant. Two basic examples are stochastic local operations and classical communications (SLOCC) and permutations, which are key concepts in entanglement classification. For four or more qubits, SLOCC itself cannot classify entanglement effectively, i.e. there are infinitely many SLOCC classes \cite{6}. However, as we will see, the entanglement spectrum is an integer partition of the Hilbert space dimension, which means it can always classify entanglement into a finite number of classes only if the Hilbert space is finite dimensional. Since the kernel of the monoid homomorphism induces a congruence relation on the Hilbert space, there isn’t any overlap among these classes.

To compute entanglement polynomials effectively, we construct a series of states whose ranks are the dimension spectrum. We call them the stochastic renormalized states, because they are effective states under the random noise of given size. Then we apply our method to qubits and show that it matches with the SLOCC classification for three qubits.

This work is organized as follows. In Section 2, we briefly review the definition of state-dependent operator size and define the dimension spectrum. In Section 3, we define entanglement polynomials, states monoid $S$ and polynomial monoid $M$. And we show that the homomorphism $f$ induced by entanglement polynomials is an isomorphism from $S/\text{Ker}(f)$ to $M$. In Section 4, we prove a sufficient condition for entanglement polynomial symmetries and show that SLOCC and permutations satisfy this condition. In Section 5, we define the stochastic renormalized states whose ranks are the dimension spectrum and apply this method to qubits. In Section 6, we summarize the main results in the work. In Appendix A, we prove several lemmas which will be used in Section 5.

2 Dimension spectrum

The dimension spectrum is closely related to state-dependent operator size. In this section, we briefly review the relevant definitions and define the dimension spectrum.

Operator size is a measure of local degrees of freedom on which the operator has nontrivial action \cite{15, 16}. In \cite{14}, I generalized it to the state-dependent case. The state-dependent operator size (SDOS) measures how many local d.o.f. are really changed by an operator acting on a given state. Take two qubits and Pauli operators as an example. Consider the operator $O = \sigma^x \otimes \sigma^z$ and the state $|00\rangle$. The size of $O$ is 2, because it has nontrivial action on two qubits. On the other hand, since $O|00\rangle = |10\rangle$, only one d.o.f. is changed. The state-dependent operator size should be equal to 1.
Let's define the state-dependent operator size precisely. Given a discrete system, operators supported on a subsystem $B$ can be described by a local algebra, $A_B$, which is the operator algebra of all of the operators acting on $B$. An operator algebra is a linear space of operators which is closed under operator multiplication and hermitian conjugation, and includes the identity. Generally, we can define a measure for $B$, denoted by $|B|$. Since we are considering discrete systems, we can suppose $|B| \in \mathbb{N}$ without loss of generality. When we hope to emphasize that a local algebra is a linear space, we denote $A_B$ as $A^B$. Then we define a linear space $A^k$ as

$$A^k \equiv \sum_{|B|=k} A_B,$$

(2)

where $\sum$ means that $A^k$ is spanned by all the vectors in $A_B$'s. We set $A^0 \equiv \{I\}$. If $O \in A^k$, then $O$ is called a $k$-local operator. Given a state $|\psi\rangle$, we can construct the subspace

$$\mathcal{H}_k \equiv \{O|\psi\rangle \mid O \in A^k\}.$$

(3)

$\mathcal{H}_k$ can be decomposed as

$$\mathcal{H}_k = \Delta \mathcal{H}_k \oplus \mathcal{H}_{k-1}, \quad \Delta \mathcal{H}_k \perp \mathcal{H}_{k-1}, \quad 0 \leq k \leq N_\psi.$$

(4)

$N_\psi$ is the minimal value of $k$ such that $\mathcal{H}_{k+1} = \mathcal{H}_k$. We set $\mathcal{H}_{-1} \equiv \emptyset$. The meaning of $\Delta \mathcal{H}_k$ is that we only need $k$-local operators acting on $|\psi\rangle$ to generate $\Delta \mathcal{H}_k$ which cannot be generated by $(k-1)$-local operators, and each state in $\Delta \mathcal{H}_k$ can be perfectly distinguished from states that can be generated by $(k-1)$-local operators acting on $|\psi\rangle$. Then we can say that there are $k$ local d.o.f. that are changed when $|\psi\rangle$ evolves to a state in $\Delta \mathcal{H}_k$. For this reason, we call $\Delta \mathcal{H}_k$ the $k$-local subspace. We have the orthogonalization decomposition

$$\mathcal{H}_k = \bigoplus_{l=0}^{k} \Delta \mathcal{H}_l.$$

(5)

For a general operator $O$, we have

$$O|\psi\rangle = \sum_{k=0}^{N_\psi} \Delta P_k O|\psi\rangle,$$

(6)

where $\Delta P_k$ is the projector corresponding to $\Delta \mathcal{H}_k$. Then we define the state-dependent operator size as the average number of local d.o.f. that are changed by $O$ acting on $|\psi\rangle$,

$$S(O, |\psi\rangle) = \sum_{k=1}^{N_\psi} k \langle \psi | O^\dagger \Delta P_k O | \psi \rangle,$$

(7)

where the normalization $\langle \psi | O^\dagger O | \psi \rangle = 1$ is assumed. Note that if $O|\psi\rangle = O'|\psi\rangle$, we have

$$S(O, |\psi\rangle) = S(O', |\psi\rangle).$$

(8)
So SDOS can be regarded as a function of the initial state $|\psi\rangle$ and the final state. In this work, the key concept is the **dimension spectrum** defined as the dimensions of $\Delta H_k, (0 \leq k \leq N_{\psi}),$

$$d_k \equiv \dim(\Delta H_k). \quad (9)$$

From now on, we will focus on the dimension spectrum. The SDOS, which motivates us to define the dimension spectrum, is not directly relevant to the following sections though.

### 3 A monoid isomorphism

In this section, I will illustrate a monoid isomorphism from entanglement classes to polynomials. It can classify the entanglement in general systems.

#### 3.1 Entanglement polynomials

We first define an equivalence relation among states as follows. $|\psi\rangle$ is equivalent to $|\phi\rangle$ if there exist a system whose Hilbert space $\mathcal{N}$ only contains a single state such that

$$|\phi\rangle = |\psi\rangle \otimes |0\rangle, \quad |0\rangle \in \mathcal{N}. \quad (10)$$

In other words, $|0\rangle$ here is a state of a qudit with $d = 1$. We denote it as $|\psi\rangle \sim |\phi\rangle$. In fact, it’s a congruence relation with respect to tensor product, which means

$$|\psi_1\rangle \sim |\phi_1\rangle, \quad |\psi_2\rangle \sim |\phi_2\rangle \Rightarrow |\psi_1\rangle \otimes |\psi_2\rangle \sim |\phi_1\rangle \otimes |\phi_2\rangle. \quad (11)$$

For a given state $|\psi\rangle$, we denote its congruence class as $\{ |\psi\rangle \}$. Then we can define tensor product among congruence classes as

$$\{ |\psi\rangle \} \otimes \{ |\phi\rangle \} \equiv \{ |\psi\rangle \otimes |\phi\rangle \}. \quad (12)$$

These congruence classes with the tensor product as the monoid multiplication form a monoid $S$. A monoid is a group without the requirement that each element has an inverse. In $S$, the identity is $\{ |0\rangle \}$. We can check the associativity

$$(\{ |\alpha\rangle \} \otimes \{ |\beta\rangle \} \otimes \{ |\gamma\rangle \}) = \{ |\alpha\rangle \otimes |\beta\rangle \} \otimes \{ |\gamma\rangle \} = \{ |\alpha\rangle \} \otimes \{ |\beta\rangle \otimes |\gamma\rangle \} = \{ |\alpha\rangle \} \otimes (\{ |\beta\rangle \} \otimes \{ |\gamma\rangle \}) \quad (13)$$

So $S$ is indeed a monoid. On the other hand, polynomials with the polynomial multiplication as the monoid multiplication also form a monoid $M$. The identity is just $1$. We will see that there is a homomorphism from $S$ to $M$. Given a state $|\psi\rangle$, we can define a polynomial as

$$f[|\psi\rangle] = \sum_{k=0}^{N_{\psi}} d_k x^k, \quad (14)$$
where \(\{d_k\}\) is the dimension spectrum of \(|\psi\rangle\). It’s obvious that two states in the same congruence class share the same dimension spectrum, because there is no nontrivial operators acting on qudits with \(d = 1\). So Eq. (13) defines a map from \(S\) to \(M\). We call it the entanglement polynomial of the state \(|\psi\rangle\). Then we come to one of the main results in this work.

**Theorem 3.1.** Entanglement polynomials induce a homomorphism from state monoid \(S\) to polynomial monoid \(M\), i.e.

\[
f[|\psi\rangle \otimes |\phi\rangle] = f[|\psi\rangle] f[|\phi\rangle].
\]

**Proof.** Given a state \(|\psi\rangle\), Eq. (13) defines a series of subspaces denoted by \(\mathcal{H}_k(|\psi\rangle)\). Then we have

\[
\mathcal{H}_k(|\psi\rangle \otimes |\phi\rangle) = \sum_{l+m=k} \mathcal{H}_l(|\psi\rangle) \otimes \mathcal{H}_m(|\phi\rangle),
\]

where \(0 \leq l \leq N_{\psi}, 0 \leq m \leq N_{\phi}\). This restriction also holds for the following \(l\)’s and \(m\)’s. Using Eq. (5) on the r.h.s. in the above equation, we have

\[
\mathcal{H}_k(|\psi\rangle \otimes |\phi\rangle) = \oplus_{p=0}^{k} \oplus_{l+m=p} \Delta \mathcal{H}_l(|\psi\rangle) \otimes \Delta \mathcal{H}_m(|\phi\rangle).
\]

By definition Eq. (4), this implies

\[
\Delta \mathcal{H}_k(|\psi\rangle \otimes |\phi\rangle) = \oplus_{l+m=k} \Delta \mathcal{H}_l(|\psi\rangle) \otimes \Delta \mathcal{H}_m(|\phi\rangle).
\]

Denote \(\text{dim} (\Delta \mathcal{H}_k(|\psi\rangle))\) as \(d_k(|\psi\rangle)\), we get the following relation.

\[
d_k(|\psi\rangle \otimes |\phi\rangle) = \text{dim} (\Delta \mathcal{H}_k(|\psi\rangle \otimes |\phi\rangle))
= \sum_{l+m=k} \text{dim} (\Delta \mathcal{H}_l(|\psi\rangle)) \text{dim} (\Delta \mathcal{H}_m(|\phi\rangle))
= \sum_{l+m=k} d_l(|\psi\rangle) d_m(|\phi\rangle).
\]

Using this relation, we have

\[
f[|\psi\rangle] f[|\phi\rangle] = \left( \sum_{l=0}^{N_{\psi}} d_l(|\psi\rangle) x^l \right) \left( \sum_{m=0}^{N_{\phi}} d_m(|\phi\rangle) x^m \right)
= \sum_{k=0}^{N_{\psi}+N_{\phi}} \left( x^k \sum_{l+m=k} d_l(|\psi\rangle) d_m(|\phi\rangle) \right)
= \sum_{k=0}^{N_{\psi}+N_{\phi}} d_k(|\psi\rangle \otimes |\phi\rangle) x^k.
\]

Since \(N_{\psi}\) is the minimal value of \(k\) such that \(\mathcal{H}_{k+1}(|\psi\rangle) = \mathcal{H}_k(|\psi\rangle)\), we have

\[
N_{|\psi\rangle \otimes |\phi\rangle} = N_{\psi} + N_{\phi}. 
\]
This implies

\[ f(|\psi\rangle) f(|\phi\rangle) = \sum_{k=0}^{N_\psi + N_\phi} d_k(|\psi\rangle \otimes |\phi\rangle)x^k \]

\[ = \sum_{k=0}^{N_{|\psi\rangle \otimes |\phi\rangle}} d_k(|\psi\rangle \otimes |\phi\rangle)x^k \]

\[ = f(|\psi\rangle \otimes |\phi\rangle). \quad (22) \]

Theorem 3.1 implies that product states correspond to reducible polynomials. In this way, we can find building blocks of entanglement by factorization. We will illustrate this with a few examples in Section 5.

### 3.2 The isomorphism

It’s well known that a homomorphism \( f \) naturally induces a congruence relation \( R \) by

\[ |\psi\rangle R |\phi\rangle, \text{ iff } f(|\psi\rangle) = f(|\phi\rangle). \quad (23) \]

We denote the congruence class of \( |\psi\rangle \) as \( \{ |\psi\rangle \}_R \). By definition in Section 3.1, \( \{ |\psi\rangle \}_R \subseteq \{ |\psi\rangle \}_R \), and \( R \) is also a congruence relation in the state monoid \( S \)

\[ \{ |\psi\rangle \}_R \{ |\phi\rangle \}_R, \text{ iff } f(|\psi\rangle) = f(|\phi\rangle). \quad (24) \]

In this way, entanglement polynomials induce an isomorphism from congruence classes \( \{ |\psi\rangle \}_R \) to polynomials. More precisely, consider the kernel of \( f \)

\[ \text{Ker}(f) = \{(|\psi\rangle, |\phi\rangle) \mid (|\psi\rangle, |\phi\rangle) \in S \times S, f(|\psi\rangle) = f(|\phi\rangle)\}, \quad (25) \]

and the quotient monoid \( S/\text{Ker}(f) \) which is the monoid composed of congruence classes \( \{ |\psi\rangle \}_R \) with the monoid multiplication \( \otimes \)

\[ \{ |\psi\rangle \}_R \otimes \{ |\phi\rangle \}_R \equiv \{ |\psi\rangle \otimes |\phi\rangle \}_R. \quad (26) \]

Then we have the entanglement classification theorem.

**Theorem 3.2.** The entanglement polynomials induce an isomorphism from the quotient monoid \( S/\text{Ker}(f) \) to polynomial monoid \( M \)

\[ S/\text{Ker}(f) \cong M. \quad (27) \]

As we will see, each element in \( S/\text{Ker}(f) \) is an entanglement class.
3.3 Mixed states

Theorem 3.2 can be further generalized for mixed states. To do so, we need to define the dimension spectrum for a mixed state. Before that, we first define the pure state dimension spectrum relative to a subsystem $B$. Given a subsystem $B$ and a state $|\psi\rangle$, similar to Eq. (2), (3) and (4), we define

$$A_B^k \equiv \sum_{|B'|=k} A_{B'}, \ B' \subseteq B.$$  \hfill (28)

$$H_B^k \equiv \{O|\psi\rangle \mid O \in A_B^k\}.$$  \hfill (29)

$$H_B^k = \Delta H_{k-1}^B \oplus H_{k-1}^B, \ \Delta H_k^B \perp H_{k-1}^B, \ 0 \leq k \leq N_B^\psi.$$  \hfill (30)

Then we have the orthogonalization decomposition like Eq. (5)

$$H_k^B = \bigoplus_{l=0}^k \Delta H_l^B.$$  \hfill (31)

The dimension spectrum relative to $B$ is defined as

$$d_k^B \equiv \dim(\Delta H_k^B).$$  \hfill (32)

And the entanglement polynomial relative to $B$ is

$$f_B[|\psi\rangle] \equiv \sum_{k=0}^{N_B^\psi} d_k^B x^k.$$  \hfill (33)

Arguments in the proof of Theorem 3.1 also hold for $\{d_k^B\}$, so we have

$$f_B[|\psi\rangle \otimes |\phi\rangle] = f_B[|\psi\rangle] f_B[|\phi\rangle].$$  \hfill (34)

Now we turn to mixed states. Given a mixed state $\rho \in \text{End}(\mathcal{H})$, we can define its purification $|\sqrt{\rho}\rangle$ on the double copy of the original Hilbert space $\mathcal{H} \otimes \mathcal{H}_c$, $\mathcal{H}_c \cong \mathcal{H}$, which satisfies

$$\text{tr}_{\mathcal{H}_c}(|\sqrt{\rho}\rangle\langle\sqrt{\rho}|) = \rho.$$  \hfill (35)

We notice that the dimension spectrum of $|\sqrt{\rho}\rangle$ relative to subsystem $\mathcal{H}$ is independent of the purification redundancy, that is

$$d_k^H(|\sqrt{\rho}\rangle) = d_k^H(U_{\mathcal{H}_c}|\sqrt{\rho}\rangle),$$  \hfill (36)

where $U_{\mathcal{H}_c}$ is an arbitrary unitary operator acting on $\mathcal{H}_c$. So we can define the dimension spectrum of $\rho$ as

$$d_k(\rho) \equiv d_k^H(|\sqrt{\rho}\rangle)$$  \hfill (37)
which only depends on \( \rho \) itself. Then we define the entanglement polynomials for mixed states

\[
f[\rho] \equiv \sum_{k=0}^{N_\rho} d(\rho)_k x^k,
\]
(38)

Following Eq. (34), we have the homomorphism from mixed states to entanglement polynomials

\[
f[\rho \otimes \sigma] = f[\rho] f[\sigma].
\]
(39)

### 3.4 Nontrivial center

To analyze entanglement, we usually divide the system into several parts, which corresponds to the factorization of the Hilbert space. Given a system whose Hilbert space is \( \mathcal{H} \), consider a subsystem \( B \) and its complement \( \overline{B} \) and denote their Hilbert spaces as \( \mathcal{H}_B \) and \( \mathcal{H}_{\overline{B}} \) respectively. The following factorization is usually assumed.

\[
\mathcal{H} = \mathcal{H}_B \otimes \mathcal{H}_{\overline{B}}.
\]
(40)

If it is the case, Theorem 3.1 can be used to factorize the entanglement. However, if the local algebra \( A_B \) has a nontrivial center \( Z_B \), a simple factorization like Eq. (40) generally doesn’t exist, which occurs in gauge field theories [17]. \( Z_B \) is defined as

\[
Z_B \equiv \{ O \mid [O, O'] = 0, \forall O' \in A_B \}.
\]
(41)

Another useful concept is the commutant \( A'_B \) of a local algebra \( A_B \), which is defined as

\[
A'_B \equiv \{ O' \mid O' \in A, [O, O'] = 0, \forall O \in A_B \}.
\]
(42)

\( A \) is the algebra of the whole system. By definition and due to the locality, we have \( A_B \subseteq A'_B \). In many cases, the Haag duality holds [18],

\[
A_{\overline{B}} = A'_B.
\]
(43)

We will use it in the proof of Lemma A.4.

\( Z_B \) can be expressed as

\[
Z_B = A_B \cap A'_B.
\]
(44)

\( Z_B \) is nontrivial if it contains elements other than the multiples of the identity. In this case, operators in \( A_B \) cannot be factorized as

\[
O = O_B \otimes I_{\overline{B}}.
\]
(45)

where \( O_B \in \text{End}(\mathcal{H}_B) \), \( I_{\overline{B}} \in \text{End}(\mathcal{H}_{\overline{B}}) \), and \( I_{\overline{B}} \) is the identity. In general, given a local algebra \( A_B \), the Hilbert space \( \mathcal{H} \) can be decomposed as

\[
\mathcal{H} = \bigoplus_\alpha \mathcal{H}_{B_{\alpha}} \otimes \mathcal{H}_{\overline{B}_{\alpha}},
\]
(46)
such that operators $O \in \mathcal{A}_B$ and $O' \in \mathcal{A}'_B$ take the following form

$$O = \bigoplus_{\alpha} O_{B_{\alpha}} \otimes I_{\mathcal{H}_{\alpha}},$$

(47a)

$$O' = \bigoplus_{\alpha} I_{B_{\alpha}} \otimes O'_{B_{\alpha}},$$

(47b)

where $O_{B_{\alpha}} \in \text{End}(\mathcal{H}_{B_{\alpha}})$, $O'_{B_{\alpha}} \in \text{End}(\mathcal{H}_{B_{\alpha}})$. For more details of these results in von Neumann algebra, I recommend to refer to appendix A in [19].

In the presence of nontrivial center, the monoid multiplication among states is no longer tensor product anymore, so Theorem 3.1 need to be generalized. A mixed state $\rho \in \mathcal{A}_B \otimes \mathcal{B}$ can be purified on the algebra $\mathcal{A}_B \otimes \mathcal{B}_c$ of a double copy system $B \cup B_c$, $B_c \cong B$. Similar to the discussion in the last subsection, we can define the dimension spectrum of $\rho$ as the dimension spectrum of its purification relative to $B$ without purification redundancy. Then the entanglement polynomials are defined as in Eq. (38). Now suppose a system is divided into two parts $B$ and $B'$, given two states $\rho_B \in \mathcal{A}_B$ and $\rho_B' \in \mathcal{A}_B'$, we have the homomorphism

$$f[\rho_B \rho_B'] = f[\rho_B] f[\rho_B'].$$

(48)

To prove Eq. (48) and for later convenience, we define the action of the operator space $A^k$ on the Hilbert space $\mathcal{H}$,

$$A^k \mathcal{H} = \text{Span}\{O|\psi\rangle | \mathcal{O} \in A^k, |\psi\rangle \in \mathcal{H}\},$$

(49)

and the $\Delta_k$ operator

$$A^k \mathcal{H} = \Delta_k(A^k \mathcal{H}) \oplus A^{k-1} \mathcal{H}, \quad \Delta_k(A^k \mathcal{H}) \perp A^{k-1} \mathcal{H}.$$  

(50)

We first prove Eq. (48) with the assumption that $\rho_B \rho_B'$ is a pure state $|\psi\rangle\langle\psi| \in \text{End}(\mathcal{H})$, then the proof in general the case can be obtained easily. Denote the Hilbert space spanned by $|\psi\rangle$ as $\mathcal{H}^\psi$. Similar to the proof of Theorem 3.1 consider $A^k \mathcal{H}^\psi$

$$A^k \mathcal{H}^\psi = \sum_{l+m=k} A_{B^l}^m A_{B'}^l \mathcal{H}^\psi$$

$$= \sum_{l+m=k} A_{B^l}^m \bigoplus_{p=0}^{l} \Delta_p(A_{B^l}^m \mathcal{H}^\psi)$$

$$= \sum_{l+m=k} \bigoplus_{p=0}^{l} A_{B^l}^m \Delta_p(A_{B^l}^m \mathcal{H}^\psi)$$

$$= \sum_{l+m=k} \bigoplus_{p=0}^{l} \bigoplus_{q=0}^{m} \Delta_q(A_{B^l}^m \Delta_p(A_{B^l}^m \mathcal{H}^\psi)).$$

(51)

Let $\Delta_k$ act on both sides of the above equations

$$\Delta_k(A^k \mathcal{H}^\psi) = \bigoplus_{l+m=k} \Delta_m(A_{B^l}^m \Delta_l(A_{B^l}^m \mathcal{H}^\psi)).$$

(52)
Then compute the dimension

\[
\dim[\Delta_k(A^k\mathcal{H}_\psi)] = \sum_{l+m=k} \dim[\Delta_l(A_B^m\Delta_l(A_B^l\mathcal{H}_\psi))]
= \sum_{l+m=k} \dim[\Delta_l(A_B^l\mathcal{H}_\psi)]\dim[\Delta_m(A_B^m\mathcal{H}_\psi)].
\]  

(53)

The last equality holds since \(\rho\) is a product state \(\rho_B\rho_B\). Remind that these are coefficients of entanglement polynomials

\[
f[\rho_B]f[\rho_B] = \left(\sum_{l=0}^{N_{\rho_B}} d_l(\rho_B)x^l\right)\left(\sum_{m=0}^{N_{\rho_B}} d_m(\rho_B)x^m\right)
= \sum_{k} x^k \sum_{l+m=k} d_l(\rho_B)d_m(\rho_B)
= \sum_{k} d_k(\rho)x^k
= \sum_{k} d_k(\rho)x^k
= f[\rho].
\]  

(54)

In the second equality, Eq. (53) is used. In the forth equality, we used \(N_{\rho} = N_{\rho_B} + N_{\rho_B}\). So we obtain the homomorphism if \(\rho\) is pure. If \(\rho\) is mixed, we just need to purify it in a double copy system and replace the dimension spectra above by dimension spectra relative to the original system. In this way, the above arguments still hold, so Eq. (53) is proved.

### 3.5 Continuous systems

Until now, we always assume that the system is discrete. In this subsection, we generalize Theorem 3.1 for continuous systems. In this case, we cannot assume the measure of a subregion \(B\) satisfies \(|B|\in\mathbb{N}\), while it should be \(|B|\in\mathbb{R}^+\). We first define an \(\epsilon\)-dependent operator \(\Delta^{(\epsilon)}\) by

\[
A^k\mathcal{H} = \Delta^{(\epsilon)}_k(A^k\mathcal{H}) \oplus A^{k-\epsilon}\mathcal{H}, \quad \Delta^{(\epsilon)}_k(A^k\mathcal{H}) \perp A^{k-\epsilon}\mathcal{H}.
\]  

(55)

To make the dimension spectrum finite, we define it for a state \(|\psi\rangle\in\mathcal{H}\) as

\[
d_k(|\psi\rangle) \equiv \frac{1}{\epsilon|\mathcal{H}|} \dim[\Delta^{(\epsilon)}_k(A^k\mathcal{H}_\psi)].
\]  

(56)

To make it independent of \(\epsilon\) and consistent with Eq. (53), we set \(\epsilon = |\mathcal{H}|^{-1}\) such that

\[
d_k(|\psi\rangle) = \dim[\Delta^{(|\mathcal{H}|^{-1})}_k(A^k\mathcal{H}_\psi)].
\]  

(57)
Since $|\mathcal{H}|$ goes to infinity for continuous systems, we have

$$
d_k(|\psi\rangle) = \lim_{\epsilon \to 0} \frac{1}{\epsilon |\mathcal{H}|} \dim[A_k^\epsilon (A^k_{\mathcal{H}} \psi)]$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon |\mathcal{H}|} (\dim[A^k_{\mathcal{H}} \psi] - \dim[A^{k-\epsilon}_{\mathcal{H}} \psi])$$

$$= |\mathcal{H}|^{-1} \partial_k \dim[A^k_{\mathcal{H}} \psi].$$

(58)

The entanglement polynomials are defined as

$$f[|\psi\rangle] \equiv \int_0^V d_k(|\psi\rangle) x^k dk.$$  

(59)

$V$ is the measure of the whole system. As we can see, it’s more appropriate to call it entanglement function rather than polynomial. Set $x = e^{-s}$, using Eq. (58), then the entanglement function can be expressed as

$$f[|\psi\rangle] = |\mathcal{H}|^{-1} \int_0^V e^{-ks} \partial_k \dim[A^k_{\mathcal{H}} \psi] dk$$

$$= |\mathcal{H}|^{-1} (|\mathcal{H}| e^{-Vs} - 1 + s \int_0^V e^{-ks} \dim[A^k_{\mathcal{H}} \psi] dk)$$

$$= e^{-Vs} + |\mathcal{H}|^{-1} s \int_0^V e^{-ks} \dim[A^k_{\mathcal{H}} \psi] dk.$$  

(60)

So the entanglement function is the Laplace transform of the dimension spectrum up to some factors. Now we turn to the homomorphism. Given a state $|\psi\rangle = |\psi_B\rangle \otimes |\psi_B\rangle$ where $|\psi_B\rangle \in \mathcal{H}_B$ and $|\psi_B\rangle \in \mathcal{H}_{\mathcal{B}}$, we replace $\Delta$ in Eq. (61) by $\Delta^{(c)}$. then we get

$$A^k_{\mathcal{H}} \psi = \sum_{l+m=k} A^m_B A^l_B \mathcal{H} \psi$$

$$= \sum_{l+m=k} A^m_B \bigoplus_{p=0}^{l/\epsilon} \Delta_p^{(c)} (A^p_B \mathcal{H} \psi)$$

$$= \sum_{l+m=k} \bigoplus_{p=0}^{l/\epsilon} A^m_B \Delta_p^{(c)} (A^p_B \mathcal{H} \psi)$$

$$= \sum_{l+m=k} \bigoplus_{p=0}^{l/\epsilon} \bigoplus_{m/\epsilon} \Delta_p^{(c)} (A^m_B \Delta_p^{(c)} (A^p_B \mathcal{H} \psi)).$$

(61)

Let $\Delta_k^{(c)}$ act on both sides of the above equation, we have

$$\Delta_k^{(c)} (A^k_{\mathcal{H}} \psi) = \bigoplus_{l+m=k} \Delta_k^{(c)} (A^m_B \Delta_k^{(c)} (A^l_B \mathcal{H} \psi)).$$

(62)
Similar to Eq. (53), their dimensions are given by
\[
\dim(\Delta^{(c)}_k(A^k|H^\psi\rangle)) = \sum_{l+m=k} \dim[\Delta^{(c)}_l(A^l|H^\psi\rangle)\dim[\Delta^{(c)}_m(A^m|H^\psi\rangle)].
\] (63)

Using Eq. (56), we get
\[
d_k(|\psi\rangle) = \lim_{\epsilon \to 0} \sum_{l+m=k} \epsilon d_l(|\psi_B\rangle) d_m(|\psi_{\overline{\psi}}\rangle)
= \int_0^k d_l(|\psi_B\rangle) d_{k-l}(|\psi_{\overline{\psi}}\rangle) dl.
\] (64)

Then the homomorphism holds,
\[
f[|\psi_B\rangle] f[|\psi_{\overline{\psi}}\rangle] = \left( \int_0^{V_B} d_l(|\psi_B\rangle) x^l dl \right) \left( \int_0^{V_{\overline{\psi}}} d_m(|\psi_{\overline{\psi}}\rangle) x^m dm \right)
= \int_0^{V_B+V_{\overline{\psi}}} x^k \int_0^k d_l(|\psi_B\rangle) d_{k-l}(|\psi_{\overline{\psi}}\rangle) dl dk
= \int_0^{V} x^k d_k(|\psi_B\rangle \otimes |\psi_{\overline{\psi}}\rangle) dk
= f[|\psi_B\rangle \otimes |\psi_{\overline{\psi}}\rangle].
\] (65)

It should be noticed that Eq. (64) only holds without the presence of nontrivial center in the local algebra \(A_B\), otherwise the Hilbert space cannot be factorized so that \(|\mathcal{H}| = |\mathcal{H}_B| |\mathcal{H}_{\overline{\psi}}|\) doesn’t hold. We leave this more general case for future works.

4 Symmetries

4.1 Automorphisms

The power of Theorem 3.1 as a tool to classify entanglement comes from symmetries of entanglement polynomials. Besides calculating dimension spectrum directly, symmetries of entanglement polynomials can determine whether two states are in the same congruence class. By symmetries, I mean an operator \(U\) satisfying
\[
f[U|\psi\rangle] = f[|\psi\rangle].
\] (66)

Unlike group homomorphisms, the kernel of a monoid homomorphism cannot be determined by the preimage of the identity. So \(U\) generally depends on the specific state \(|\psi\rangle\). However, we can still find some state-independent symmetries. Define the adjoint action of \(U\)
\[
ad_U \mathcal{O} \equiv U^{-1}OU,
\] (67)

and its action on the operator space
\[
ad_U A^k = U^{-1} A^k U \equiv \{ad_U \mathcal{O} | \mathcal{O} \in A^k\}.
\] (68)

Then we come to another important result in this work.
Theorem 4.1. If \( \forall k, \, \text{ad}_U \) induces an automorphism on \( A_k \), then \( U \) keeps the dimension spectra and entanglement polynomials invariant, i.e.
\[
\text{ad}_U A^k = A^k, \, \forall k \Rightarrow d_k(U|\psi\rangle) = d_k(|\psi\rangle), \, f[U|\psi\rangle] = f[|\psi\rangle], \, \forall |\psi\rangle. \tag{69}
\]

Proof. Note that
\[
\dim[A^k H^U] = \dim[A^k U H^\psi] \\
= \dim[U U^{-1} A^k U H^\psi] \\
= \dim[U^{-1} A^k U H^\psi] \\
= \dim[A^k H^\psi]. \tag{70}
\]

In the first equality, we defined the operator action on the Hilbert space
\[ U H \equiv \{ U|\psi\rangle \mid |\psi\rangle \in H \}. \tag{71} \]

In the third equality, we used the fact that an invertible operator induces an automorphism on the vector space. By definition, we have
\[
d_k(U|\psi\rangle) = \dim[\Delta(A^k H^U \psi)] \\
= \dim[A^k H^U \psi] - \dim[A^k H^U \psi] \\
= \dim[A^k H^\psi] - \dim[A^k H^\psi] \\
= \dim[\Delta(A^k H^\psi)] \\
= d_k(|\psi\rangle). \tag{72}
\]

Thus
\[ f[U|\psi\rangle] = f[|\psi\rangle]. \tag{73} \]

Similar to Theorem 3.1, Theorem 4.1 can be generalized for mixed states, in the presence of nontrivial center and for continuous systems without any difficulties.

4.2 SLOCC and permutations

As an example, we will show that stochastic local operations and classical communications (SLOCC) together with permutations are symmetries of the entanglement polynomials. Suppose the Hilbert space \( \mathcal{H} \) can be factorized
\[ \mathcal{H} = \bigotimes_{i=1}^n \mathcal{H}_i. \tag{74} \]

SLOCC can be represented by local invertible operators \[ \mathcal{L} \]
\[ \mathcal{L} = \bigotimes_{i=1}^n L_i, \quad L_i \in \text{Aut}(\mathcal{H}_i). \tag{75} \]
By Eq. (2), operators in $A^k$ can be expressed as

$$\mathcal{O} = \sum_{1 \leq i_1 < \ldots < i_k \leq n} c_{i_1 \ldots i_k} \bigotimes_{p=1}^{k} \mathcal{O}_{i_p}, \quad \mathcal{O}_{i_p} \in \text{End}(\mathcal{H}_{i_p}).$$  \hspace{1cm} (76)

c_{i_1 \ldots i_k} \in \mathbb{C}$. Then we have

$$L^{-1} \mathcal{O} L = \sum_{1 \leq i_1 < \ldots < i_k \leq n} c_{i_1 \ldots i_k} \bigotimes_{p=1}^{k} L_{i_p}^{-1} \mathcal{O}_{i_p} L_{i_p} \Rightarrow \text{ad}_L \in \text{End}(A^k).$$  \hspace{1cm} (77)

Since $L$ is invertible, ad$_L$ induces an automorphism on $A^k$. Thus, Theorem 4.1 implies that SLOCC keep the entanglement polynomials invariant.

Then we suppose that $H_i \cong H$, $\forall i$, and define the swap operator $X_i$

$$X_i |a\rangle_i |b\rangle_{i+1} = |b\rangle_i |a\rangle_{i+1},$$  \hspace{1cm} (78)

for all $|a\rangle_i, |b\rangle_i \in H_i$ and $|a\rangle_{i+1}, |b\rangle_{i+1} \in H_{i+1}$. Then the action of ad$_{X_i}$ on $\mathcal{O}$ is to swap $\mathcal{O}_i$ and $\mathcal{O}_{i+1}$, such that $X_i^{-1} \mathcal{O} X_i \in A^k$. Since $X_i$ is invertible, ad$_{X_i}$ induces an automorphism. So swap operators are symmetries of entanglement polynomials. Since each permutation in Sym$_n$ is the product of a series of swap operators, it also keeps entanglement polynomials invariant. So two states in the same SLOCC class or Sym$_n$ class must belong to the same congruence class $\{|\psi\rangle\}_R$, which motivates us to call $\{|\psi\rangle\}_R$ the entanglement class.

It’s well known that there is an infinite number of SLOCC classes for four or more qubits, which means SLOCC itself is not an effective way to classify multipartite entanglement. However, notice that the dimension spectrum is an integer partition of the Hilbert space dimension, which means it can always classify states into a finite number of entanglement classes only if the Hilbert space is finite dimensional.

5 Stochastic renormalized states

5.1 Ranks

It’s difficult to calculate entanglement polynomials, especially for continuous systems. In this section, we construct a series of states such that the dimension spectrum is given by their ranks. Before that, we define the reduced states w.r.t. a local algebra $A_B$. Given a subsystem $B$ and a state $\rho$ of the whole system, the reduced state $\rho_B \in A_B$ is defined by

$$\text{tr}(\mathcal{O} \rho_B) = \text{tr}(\mathcal{O} \rho), \quad \forall \mathcal{O} \in A_B.$$  \hspace{1cm} (79)

Then we define the stochastic renormalized states as

$$\rho_k = \frac{\sum_{|B|=k} |B\rangle \langle B| \rho_B}{\text{tr}(\sum_{|B|=k} |B\rangle \langle B| \rho_B)},$$  \hspace{1cm} (80)
To illustrate the physical meaning of it, suppose the system is in a state $\rho$, and we try to measure the operator $O$ in a noisy laboratory. The effect of the noise is to disturb our observation such that we cannot access some part of the system. The noise is stochastic, so the position of the lost part $B$ is random. At each observation, the expectation value of $O$ is

$$\langle O \rangle_{\text{noise}} = \text{tr}(O \rho_B).$$  \hspace{1cm} (81)

After many experiments, the average of the expectation value is

$$\langle O \rangle_{\text{noise}} = \frac{\sum_B \text{tr}(O \rho_B)}{\text{tr}(\sum_B \rho_B)}.$$  \hspace{1cm} (82)

If the size of the noise is always equal to $k$, then we have

$$\langle O \rangle_{\text{noise}} = \text{tr}(O \rho_k).$$  \hspace{1cm} (83)

So stochastic renormalized states are effective states under the random noise of given size. Now we come to the main tool to calculate the entanglement polynomials.

**Theorem 5.1.** Given a pure state $\rho = |\psi\rangle\langle\psi|$, we have

$$d_k(|\psi\rangle) = \Delta_k \text{Rank}[\rho_k].$$  \hspace{1cm} (84)

$\Delta_k$ is the difference operator satisfying $\Delta_k a(k) = a(k) - a(k - 1)$. For a state $\rho \in \text{End}(\mathcal{H})$ of a continuous system, Eq. (84) just becomes

$$d_k(|\psi\rangle) = |\mathcal{H}|^{-1} \partial_k \text{Rank}[\rho_k].$$  \hspace{1cm} (85)

**Proof.** The derivation is as follows.

$$d_k(|\psi\rangle) = \dim[\Delta_k (A^k \mathcal{H}^\psi)]$$
$$= \Delta_k \dim[A^k \mathcal{H}^\psi]$$
$$= \Delta_k \text{Rank}[\sum_{|B|=k} \sum_{O \in A_B} O \rho \rho^\dagger]$$
$$= \Delta_k \text{Rank}[\sum_{|B|=k} \rho_B]$$
$$= \Delta_k \text{Rank}[\rho_k].$$  \hspace{1cm} (86)

The first equality is the definition of the dimension spectrum. The third equality follows Lemma [A.1]. In the forth equality, we used Lemma [A.3]. Replace $\Delta_k$ by $|\mathcal{H}|^{-1} \partial_k$, we can get the derivation of Eq. (85).  \hspace{1cm} \sqrt

The rank of a state is not a quantity that physicists usually calculate. So we introduce two tricks to address it. The first one is the replica trick.

$$\text{Rank}[\rho_k] = \lim_{n \to 0} \text{tr}(\rho_k^n).$$  \hspace{1cm} (87)
However, the analytic continuation is usually difficult. The second trick is to perturb $\rho_k$ by an infinitesimal scalar operator $\epsilon I$ and calculate its determinant. Then the rank can be read out from the exponent of $\epsilon$ in the leading order, that is

$$\det(\rho_k + \epsilon I) = O(\epsilon^{\text{Rank}[\rho_k]}).$$

(88)

5.2 Qubits

To illustrate how entanglement polynomials classify entanglement, we take qubits as a simple example. As is shown in Section 4.2, SLOCC and permutations keep the entanglement polynomials invariant, so we just need to perform calculations for representative states in SLOCC+$\text{Sym}_n$ classes. Up to three qubits, they are $|0\rangle$, $|\text{Bell}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$, $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, and tensor products of them, i.e. $|00\rangle$, $|000\rangle$, $|0|\text{Bell}\rangle$. Using Eq. (84) and Eq. (88), entanglement polynomials can be calculated directly. The results are listed in Table 1 and Table 2.

Remind that entanglement polynomials induce the isomorphism Eq. (27)

$$S/\text{Ker}(f) \cong M.$$

In this case, $M$ is generated by these irreducible polynomials, such as $1+x$, $1+3x$, $1+6x+x^2$ and $1+7x$ etc. We call the entanglement classes corresponding to these irreducible polynomials the prime states, because they are just like the prime numbers in $\mathbb{N}$. So $S/\text{Ker}(f)$ is generated by prime states, such as $\{0\}_R$, $\{|\text{Bell}\rangle\}_R$, $\{|W\rangle\}_R$ and $\{|\text{GHZ}\rangle\}_R$ etc. To find prime states, we just need to factorize polynomials. Notice that reducible polynomials over $\mathbb{Z}$ can be irreducible in $M$. For four and more qubits, there is an infinite number of
SLOCC classes, but Theorem 3.2 can still classify them into a finite number of entanglement classes.

In this way, we can characterize and find the building blocks of entanglement by computing entanglement polynomials and factorization.

6 Conclusions

In this work, we define the dimension spectrum which is the dimension of subspaces generated by $k$-local operators acting on a given state. With the dimension spectrum as coefficients, we define the entanglement polynomials which induce an monoid isomorphism from states to polynomials. This isomorphism transforms the tensor product among states to polynomial product, which implies the equivalence between entanglement characterization and polynomial factorization. If an operator induces automorphisms on all the subspaces of $k$-local operators, then it keeps the entanglement invariant. SLOCC and permutations satisfy this condition, but they are only part of the symmetries. The stochastic renormalized states are defined such that we can compute their ranks to obtain the entanglement polynomials.

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A Lemmas

In this section, we prove several lemmas used in the derivation of Eq. (84).

Lemma A.1. Given a state $|\psi\rangle$ and a set of operators $\{O_i \mid i = 1, \ldots, n\}$, we have

$$\dim\{O|\psi\rangle\} = \text{Rank}\left[\sum_{i=1}^{n} O_i|\psi\rangle\langle\psi|O_i^\dagger\right],$$

(90)

where $\{O|\psi\rangle\} \equiv \text{Span}\{O_i|\psi\rangle \mid i = 1, \ldots, n\}$.

Proof. $O_i|\psi\rangle\langle\psi|O_i^\dagger$ is a positive semi-definite operator, so

$$\langle\phi|O_i|\psi\rangle\langle\psi|O_i^\dagger|\phi\rangle = 0 \iff |\phi\rangle \in \text{Ker}(O_i|\psi\rangle\langle\psi|O_i^\dagger).$$

(91)

The sum of positive semi-definite operators is still positive semi-definite, so

$$\sum_{i=1}^{n} \langle\phi|O_i|\psi\rangle\langle\psi|O_i^\dagger|\phi\rangle = 0 \iff |\phi\rangle \in \text{Ker}\left(\sum_{i=1}^{n} O_i|\psi\rangle\langle\psi|O_i^\dagger\right),$$

(92)

and

$$\sum_{i=1}^{n} \langle\phi|O_i|\psi\rangle\langle\psi|O_i^\dagger|\phi\rangle = 0 \iff \langle\phi|O_i|\psi\rangle\langle\psi|O_i^\dagger|\phi\rangle = 0, \forall i.$$ (93)

Thus

$$\text{Ker}\left(\sum_{i=1}^{n} O_i|\psi\rangle\langle\psi|O_i^\dagger\right) = \bigcap_{i=1}^{n} \text{Ker}(O_i|\psi\rangle\langle\psi|O_i^\dagger) = \{O|\psi\rangle\}^\perp.$$ (94)

$\{O|\psi\rangle\}^\perp$ is the orthogonal complement of $\{O|\psi\rangle\}$. Compute the dimensions of both sides

$$\dim[\text{Ker}\left(\sum_{i=1}^{n} O_i|\psi\rangle\langle\psi|O_i^\dagger\right)] = |\mathcal{H}| - \dim[\{O|\psi\rangle\}]^\perp$$

(95)

Since

$$\dim[\text{Ker}\left(\sum_{i=1}^{n} O_i|\psi\rangle\langle\psi|O_i^\dagger\right)] = |\mathcal{H}| - \text{Rank}\left[\sum_{i=1}^{n} O_i|\psi\rangle\langle\psi|O_i^\dagger\right],$$

(96)

we get

$$\dim[\{O|\psi\rangle\}] = \text{Rank}\left[\sum_{i=1}^{n} O_i|\psi\rangle\langle\psi|O_i^\dagger\right].$$

\[\Box\]

Lemma A.2. Given a local algebra $A_B$, $\sum_{O \in A_B} O^\dagger O$ is in its center $Z_B$. If $Z_B$ is trivial, we have

$$\sum_{O \in A_B} O^\dagger O = |\mathcal{H}|^{-1} I,$$

(97)

where $\mathcal{H}$ is the Hilbert space and $I$ is the identity.
Proof. Suppose $U$ is a unitary operator in $A_B$ and define $O_U \equiv O U$. Use $O = O_U U^\dagger$
\[
\sum_{O \in A_B} O^\dagger O U = \sum_{O \in A_B} (O_U U^\dagger)^\dagger O_U
= U \sum_{O \in A_B} O_U^\dagger O_U
= U \sum_{O \in A_B} O^\dagger O.
\] (98)

Since any operator in $A_B$ can be expressed as a linear combination of four unitary operators, we have
\[
\forall O' \in A_B, \quad [O', \sum_{O \in A_B} O^\dagger O] = 0.
\] (99)

So $\sum_{O \in A_B} O^\dagger O$ is in the center $Z_B$. If $Z_B$ is trivial, we have
\[
\sum_{O \in A_B} O^\dagger O \propto I.
\] (100)

Taking trace on both sides, we get
\[
\frac{\sum_{O \in A_B} O^\dagger O}{\text{tr}(\sum_{O \in A_B} O^\dagger O)} = |H|^{-1} I.
\] (101)

Remind that for a local algebra $A_B$ there is a Hilbert space decomposition Eq. (46)
\[\mathcal{H} = \bigoplus_{\alpha} \mathcal{H}_{B,\alpha} \otimes \mathcal{H}_{\overline{B},\alpha},\]
such that operators $O \in A_B$ take the following form
\[
O = \bigoplus_{\alpha} O_{B,\alpha} \otimes I_{\overline{B},\alpha},
\] (102)
where $O_{B,\alpha} \in \text{End}(\mathcal{H}_{B,\alpha})$, $O_{\overline{B},\alpha} \in \text{End}(\mathcal{H}_{\overline{B},\alpha})$. Then we have
\[
\sum_{O \in A_B} O^\dagger O = \bigoplus_{\alpha} \sum_{O_{B,\alpha} \in \text{End}(\mathcal{H}_{B,\alpha})} O_{B,\alpha}^\dagger O_{B,\alpha} \otimes I_{\overline{B},\alpha}.
\] (103)

According to Lemma [A.2]
\[
\sum_{O_{B,\alpha} \in \text{End}(\mathcal{H}_{B,\alpha})} O_{B,\alpha}^\dagger O_{B,\alpha} = \text{tr}(\sum_{O_{B,\alpha} \in \text{End}(\mathcal{H}_{B,\alpha})} O_{B,\alpha}^\dagger O_{B,\alpha}) |\mathcal{H}_{B,\alpha}|^{-1} I_{B,\alpha}.
\] (104)
Thus
\[ \sum_{O \in A} O^\dagger O = \bigoplus_{\alpha} \text{tr} \left( \sum_{O_{B_{\alpha}} \in \text{End}(H_{B_{\alpha}})} O_{B_{\alpha}}^\dagger O_{B_{\alpha}} \right) |H_{B_{\alpha}}|^{-1} I_{\alpha}. \]  

It can be seen that \( \sum_{O \in A} O^\dagger O \) is equal to which element in \( Z_B \) depends on the way they are normalized in the sum. If we restrict
\[ \text{tr} \left( \sum_{O_{B_{\alpha}} \in \text{End}(H_{B_{\alpha}})} O_{B_{\alpha}}^\dagger O_{B_{\alpha}} \right) = |H_{B_{\alpha}}|, \forall \alpha \]
in the summation, then we have
\[ \sum_{O \in A} O^\dagger O = I, \]
even in the presence of nontrivial center. As a generalization of Lemma A.2 we summarize it as follows.

**Lemma A.3.** Given a local algebra \( A_B \), with the normalization Eq. (106) in the sum, Eq. (107) holds.

Then the last lemma is

**Lemma A.4.** Given a state \( \rho \) and a local algebra \( A_B \), if the Haag duality Eq. (43) holds, then we have
\[ \sum_{O \in A} O \rho O^\dagger = \rho_{\overline{T}}, \]
where the normalization Eq. (106) is assumed in the summation.

**Proof.** Suppose \( O' \) is an operator in \( A_{\overline{T}} \), then we have
\[ \text{tr} \left( O' \sum_{O \in A} O \rho O^\dagger \right) = \text{tr} \left( \sum_{O \in A} O O' \rho O^\dagger \right) \]
\[ = \text{tr} \left( \sum_{O \in A} O^\dagger O O' \rho \right) \]
\[ = \text{tr} \left( O^\dagger O O' \rho \right). \]
In the second equality, we used the cyclic property of the trace. In the third equality, we used the Lemma A.3.

Suppose \( U \) is a unitary operator in \( A_B \) and define \( O_U \equiv U O \). Using \( O = U^\dagger O_U \), we get
\[ U \sum_{O \in A} O \rho O^\dagger = \sum_{O_U \in A} O_U \rho (U^\dagger O_U)^\dagger \]
\[ = \sum_{O_U \in A} O_U \rho O_U^\dagger U \]
\[ = \sum_{O \in A} O \rho O^\dagger U. \]
Since any operator in \( A_B \) can be expressed as a linear combination of four unitary operators, we have
\[
\forall O' \in A_B, \quad [O', \sum_{O \in A_B} O^\dagger \rho O] = 0. \tag{111}
\]
Thus \( \sum_{O \in A_B} O^\dagger \rho O \) is an element of \( A_B \)'s commutant \( A_B' \). According to Haag duality, it's also in \( A_B' \). So
\[
\sum_{O \in A_B} O \rho O^\dagger = \rho_{B'}
\]
\[\square\]