Analysis Of The Girth For Regular Bi-partite
Graphs With Degree 3

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Abstract. The goal of this paper is to derive the detailed description of
the Enumeration Based Search Algorithm from the high level description
provided in [16], analyze the experimental results from our implementa-
tion of the Enumeration Based Search Algorithm for finding a regular
bi-partite graph of degree 3, and compare it with known results from the
available literature. We show that the values of $m$ for a given girth $g$
for $(m, 3)$ BTUs are within the known mathematical bounds for regular
bi-partite graphs from the available literature.

1 Introduction

The goal of this paper is to develop the detailed description of the Enumeration
Based Search Algorithm from the high level description provided in [16] and
analyze the implementation results of the Enumeration Based Search Algorithm
for finding a regular bi-partite graph of degree 3, and compare it with known
results from the available literature. $(m, r)$ BTU is our notation for a regular
bi-partite graph that has been introduced in [1]. The high level description of
the Enumeration Based Search Algorithm for searching a girth maximum $(m, r)$
BTU has been described in [16]. The theoretical background behind BTUs has
been introduced and explained in detail in [1] and [2].

2 Girth Maximization as a Extremal Graph Theory
question

We consider the problem of searching for a girth maximum $(m, r)$ BTU as a
question in Extremal Graph Theory by raising two related questions.

1. Given girth $g$ and $r \in \mathbb{N}$, what is the minimum value of $m$ such that a $(m, r)$
   BTU has girth $g$.
2. Given $m, r \in \mathbb{N}; m \gg r$, what is the maximum attainable girth for a $(m, r)$
   BTU?

2.1 Definitions

We review definitions from [1] and [2].
Definition 1. \((m,r)\) BTU

A \((m,r)\) Balanced Tanner Unit (BTU) is a regular bi-partite graph that can be represented by a \(m \times m\) square matrix with \(r\) non-zero elements in each of its rows and columns. Every \((m,r)\) BTU has a bipartite graph representation and an equivalent matrix representation.

Definition 2. Girth maximum \((m,r)\) BTU

A labeled \((m,r)\) BTU \(A\) is girth maximum if there does not exist another labeled \((m,r)\) BTU \(B\) with girth greater than that of \(A\).

Definition 3. \(\Phi(\beta_1, \beta_2, \ldots, \beta_{r-1})\) where \(\beta_i \in P_2(m)\) for \(1 \leq i \leq r-1\)

\(\Phi(\beta_1, \beta_2, \ldots, \beta_{r-1})\) refers to the family of all labeled \((m,r)\) BTUs with compatible permutations \(p_1,p_2,\ldots,p_r \in S_m; p_i \notin C(p_1,p_2,\ldots,p_{i-1})\) for \(1 < i \leq r\) that occur in the same order on a complete \(m\) symmetric permutation tree, \(x_{1,1} < x_{2,1} < \ldots < x_{r,1}\) where \(p_j = (x_{j,1}x_{j,2}\ldots x_{j,m}); 1 \leq j \leq r\), such that \(\beta_i\) is the partition between permutations \(p_{i-1}\) and \(p_i\) for all integer values of \(i\) given by \(1 < i < r\).

Definition 4. Optimal partition parameters for girth maximum \((m,r)\) BTU. 

\(\beta_1, \beta_2, \ldots, \beta_{r-1} \in P_2(b \ast k^{r-1})\) refer to optimal partitions derived in [2] such that there exists a girth maximum \((m,r)\) BTU in \(\Phi(\beta_1, \beta_2, \ldots, \beta_{r-1})\), where \(\beta_i\) refers to \(\sum_{j=1}^{r-1} b \ast k^j = b \ast k^{r-1}\) for \(1 \leq i \leq r-1\), with \(k \in \mathbb{N}\) obtained as a solution to \(b \ast k^{r-1} = m\) such that \(b \in \mathbb{N}\) is minimized. Thus, \(\beta_1, \beta_2, \ldots, \beta_{r-1} \in P_2(b \ast k^{r-1})\) are \(\sum_{j=1}^{r-2} b \ast k = b \ast k^{r-1}\), \(\sum_{j=1}^{r-3} b \ast k^2 = b \ast k^{r-1}\), \ldots, \(\sum_{j=1}^{k} b \ast k^{r-2} = b \ast k^{r-1}\), and \(\sum_{j=1}^{1} b \ast k^{r-1} = b \ast k^{r-1}\) respectively.

2.2 Search for girth maximum \((m,r)\) BTU

Search for a girth maximum \((m,r)\) BTU refers to search for an optimal labelled \((m,r)\) BTU in a family of labelled BTUs that we refer to as \(\Phi(\beta_1, \beta_2, \ldots, \beta_{r-1})\) where \(\beta_i \in P_2(m)\) for \(1 \leq i \leq r-1\).

3 Girth Maximization as a Extremal Graph Theory question

We consider the problem of searching for a girth maximum \((m,r)\) BTU as a question in Extremal Graph Theory by raising two related questions.

1. Given girth \(g\) and \(r \in \mathbb{N}\), what is the minimum value of \(m\) such that a \((m,r)\) BTU has girth \(g\)?

2. Given \(m, r \in \mathbb{N}; m \gg r\), what is the maximum attainable girth for a \((m,r)\) BTU?
4 Maximum Attainable Girth

4.1 Maximum Attainable Girth for a \((m, r)\) BTU

We denote the maximum Attainable Girth for a \((m, r)\) BTU as a function \(g_{\text{max}} : \{\mathbb{N} \cup \{0\}\}^2 \rightarrow \mathbb{N} \cup \{0\}\).

**Theorem 1.** The maximum attainable girth of a \((m, r)\) BTU satisfies the inequality \(g_{\text{max}}(m, r) < 2 \times k\) where \(k \in \mathbb{N}\) is obtained by minimizing \(b \in \mathbb{N}\) such that \(m = b \times k^{r-1}\) for \(r \geq 3\).

**Proof.** From the optimal partition result from [2] for a \((m, r)\) BTU for \(r \geq 3\), we obtain that the maximum possible length of the maximum known cycle is \(2 \times k\), where the optimal partitions are \(\beta_i\) refers to \(\sum_{j=1}^{r-1} b \times k^i = b \times k^{r-1}\) for \(1 \leq i \leq r - 1\) and \(k\) is obtained by minimizing \(b \in \mathbb{N}\) such that \(m = b \times k^{r-1}\) for \(r \geq 3\). We now need to show that \(g_{\text{max}}\) cannot equal \(2 \times k\) for \(r \geq 3\). This follows because of micro-partition cycles defined in [2] and their combinations which do not permit \(g_{\text{max}}\) to equal \(2 \times k\) for \(r \geq 3\). Hence, the result follows.

5 High Level Description Of Enumeration Based Search from [16]

5.1 Enumeration Based Search algorithm for girth maximum \((m, r)\) BTU for \(r > 3\)

We find \(b, k \in \mathbb{N}\) such that \(b\) is the smallest integer satisfying \(m = b \times k^{r-1}\); for \(i = 2; i < r; i++\) \{ 
\[ p_i = C_j; \min(b \times k^{i-1} - j, j) > b \times k^{i-2} \text{ such that } (j, b \times k^{i-1}, b \times k^{i-1} - j) \text{ are relatively prime; } \]
if \(i == 2\) 
\[ p_{i-1} = I_{bk^{i-1}}; \]
else \{ 
Rearrange the \((b \times k^{i-1}, i)\) BTU such that \(p_{i-1} = I_{bk^{i-1}};\)
Find \(q_i = S_{bk^{i-2}}\) such that it maximizes girth of \((b \times k^{i-1}, i)\) BTU is formed by \(p_1, \ldots, p_i \in S_{bk^{i-2}}; p_x = k \times q_x; 1 \leq x \leq i - 2;\)
if \(i != r - 1\)
Scale permutations \(p_y = k \times q_y; 1 \leq y \leq i;\)
\}
\}

5.2 Enumeration Based Search algorithm for a girth maximum \((m, 3)\) BTU where \(m = b \times k^2\)

We find \(b, k \in \mathbb{N}\) such that \(b\) is the smallest integer satisfying \(m = b \times k^2\); for \(i = 2; i < 3; i++\) \{ 
\[ p_i = C_j; \min(b \times k^{i-1} - j, j) > b \times k^{i-2} \text{ such that } (j, b \times k^{i-1}, b \times k^{i-1} - j) \text{ are relatively prime; } \]
if \(i == 2\) 
\[ p_{i-1} = I_{bk^{i-1}}; \]
else \{ 
Rearrange the \((b \times k^{i-1}, i)\) BTU such that \(p_{i-1} = I_{bk^{i-1}};\)
Find \(q_i = S_{bk^{i-2}}\) such that it maximizes girth of \((b \times k^{i-1}, i)\) BTU is formed by \(p_1, \ldots, p_i \in S_{bk^{i-1}}; p_x = k \times q_x; 1 \leq x \leq i - 2;\)
if \(i != r - 1\)
Scale permutations \(p_y = k \times q_y; 1 \leq y \leq i;\)
\}
\}
relatively prime;
if( \( i == 2 \) )
\( p_{i-1} = I_{b \cdot k^{i-1}} \);
else {
Rearrange the \((b \cdot k^{i-1}, i)\) BTU such that \( p_{i-1} = I_{b \cdot k^{i-1}} \);
Find \( q_1 \in S_{b \cdot k} \) such that a girth maximum \((b \cdot k^2, 3)\) BTU is formed by \( p_1, p_2, p_3 \in S_{b \cdot k^{i-1}} ; p_1 = k \cdot q_1 \);
}

5.3 Reorganizing the \((b \cdot k^{i-1}, i)\) BTU such that \( p_{i-1} = I_{b \cdot k^{i-1}} \)

Without loss of generality, we apply suitable permutations on depth and permutations labels on the \((b \cdot k^{i-1}, i)\) BTU in order to obtain \( p_{i-1} = I_{b \cdot k^{i-1}} \). Permutations on depth and permutations labels have been explained and defined in [1] and preserve isomorphism since they correspond to row permutations and column permutations on the matrix representation of the \((b \cdot k^{i-1}, i)\) BTU.

6 Detailed Description Of Enumeration Based Search for a girth maximum \((k^2, 3)\) BTU

To find permutation a \( q_1 \in S_k \) such that a girth maximum \((k^2, 3)\) BTU is formed by \{p_1, \ldots, p_3\} {
We enumerate all permutations \( q_1 \) with node at depth 1 fixed, such that partition between \( q_1 \) and \( q_2 = I_k \) is \( (k) \in P_2(k) \);
for(each enumerated permutation \( q_1 \) ) {
We scale up \( q_1 \) by \( k \) and \( p_2 = I_{k^2} \);
\( p_3 = C_j \) where \((j, k^2, k^2 - j)\) are relatively prime;
We compute the girth of this \((k^2, 3)\) BTU;
}
We choose permutation \( q_1 \) that gives us the best girth;

7 Detailed Description Of Enumeration Based Search for a girth maximum \((m, r)\) BTU where \( b, k \in \mathbb{N} \) such that \( b \) is the smallest integer satisfying \( m = b \cdot k^{r-1} \)

To find permutations \( \{q_1, \ldots, q_{i-2}\} \in S_{b \cdot k^{i-2}} \) such that a girth maximum \((b \cdot k^{i-1}, i)\) BTU is formed by \( \{p_1, \ldots, p_i\} \) {
We enumerate all permutations \( q_{i-2} \) with node at depth 1 fixed, such that partition between \( q_{i-2} \) and \( q_{i-1} = I_{b \cdot k^{i-2}} \) is \( (b \cdot k^{i-2}) \);
for(each enumerated permutation \( q_{i-2} \) ) {
We permute \( \{q_1, \ldots, q_{i-3}\} \) such that all partitions between any two permutations in the set \( \{q_1, \ldots, q_{i-2}, q_{i-1}\} \) are preserved;
We scale up \( \{q_1, \ldots, q_{i-2}\} \) by \( k \) and \( p_{i-1} = I_{b \cdot k^{i-1}} ; p_i = C_j \) where \((j, b \cdot k^{i-1}, b \cdot k^{i-1} - j)\) are relatively prime;
We compute the girth of this \((b \ast k^{i-1}, i)\) BTU;

\}

We choose permutation \(q_{i-2}\) that gives us the best girth;

\section{Algorithm to Find Permutations Of \(\{q_1, \ldots, q_{i-2}\}\)}

We permute \(\{q_1, \ldots, q_{i-2}\}\) such that all partitions between any two permutations in the set \(\{q_1, \ldots, q_{i-2}, I_{b \ast k^{i-2}}\}\) are preserved

for( \(j = 2; j < b \ast k^{i-2}; j++\) { \}

\(d = \) Label at depth \(j\) of \(q_{i-2}\):
Permutations On Depth \((d, j)\):
Permutations On Labels \((d, j)\):
We calculate the partition between permutations \(k \ast q_{i-2}\) and \(p_i\) and girth;
We accept the change to \(\{q_1, \ldots, q_{i-2}\}\) if it improves the girth;

\(q_{i-1}\) returns to \(I_{b \ast k^{i-2}}\) after each run of the loop.

\section{Experimental Results for Implementation Of Enumeration Based Search}

Girth obtained for various values of \(m\) and for \(r = 3\) has been shown Table 1.
We find that the values of \(m\) for a given value of girth \(g\) lie between the lower bound for \(m\) and improved lower bound for \(m\) from [13]. The execution time is too long for \(k > 10\) due to the algorithm being in EXPTIME.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\(k\) & \(m\) & \(r\) & \(g\) \\
\hline
5 & 25 & 3 & 8 \\
6 & 36 & 3 & 8 \\
7 & 49 & 3 & 10 \\
8 & 64 & 3 & 10 \\
9 & 81 & 3 & 10 \\
10 & 100 & 3 & 10 \\
\hline
\end{tabular}
\caption{Girth obtained for various of \(m\) and for \(r = 3\) from Implementation}
\end{table}

\section{Bound from [12]}

For \(q\) being a power of a prime \(k \geq 3\), Lazebnik in [12] describes explicit construction of a \(q\)-regular bipartite graph on \(v = 2 \ast q^k\) vertices with girth \(g \geq k + 5\).

If we consider this as a \((m, r)\) BTU, we get \(r\) a power of a prime and \(m = r^k; k \geq 3\), girth \(g \geq \log_r(m) + 5\). For \(g \geq 12\), we obtain \(\log_r(m) \geq 7\) which gives us \(m \geq r^7\) and we hence obtain \(m \geq 3^7 = 343 \ast 9 = 3087\).
11 Lower bounds from [9]

We quote the main theorem from [9], "Let \( G = (V_L, V_R, E) \) be a bi-partite graph of girth \( g = 2 * r \), with \( n_L = |V_L| \) and \( n_R = |V_R| \), the number of vertices on the left and right sides, and \( m = |E| \) the number of edges. Assume further that all vertex degrees in \( G \) are \( \geq 2 \). Then: \( n_L \geq \sum_{i=0}^{r-1} (A_L)^{\text{ceil}(i/2)}(A_R)^{\text{floor}(i/2)} \) and \( n_R \geq \sum_{i=0}^{r-1} (A_L)^{\text{ceil}(i/2)}(A_R)^{\text{floor}(i/2)} \). For a \((m, r)\) BTU with girth \( g \), we obtain \( m \geq \sum g^{2-1} (r-1)^{\text{ceil}(i/2)}(r-1)^{\text{floor}(i/2)} \).

Therefore, \( m \geq \sum_{i=0}^{g/2-1} (r-1)^{\text{ceil}(i/2)+\text{floor}(i/2)} \).

For even integers \( i \), \( \text{ceil}(i/2) + \text{floor}(i/2) = i \)

For odd integers \( i \), \( \text{ceil}(i/2) + \text{floor}(i/2) = (i + 1)/2 + (i - 1)/2 = i \)

Therefore, \( m \geq \sum_{i=0}^{g/2-1} (r-1)^i = (r-1)^{g/2-1} = \frac{(r-1)^{g/2-1}}{(r-2)} \)

Putting \( r = 3 \) and \( g = 12 \) we get \( m \geq \frac{(2)^{6}-1}{3-2} = 63 \). Putting \( r = 3 \) and \( g = 10 \) we get \( m \geq \frac{(2)^{5}-1}{3-2} = 31 \). Putting \( r = 3 \) and \( g = 8 \) we get \( m \geq \frac{(2)^{4}-1}{3-2} = 15 \).

11.2 From Main Theorem in [9]

Derived from the main theorem, From [9], \( n_L \geq \sum_{i=0}^{r-1} (A_R)^{\text{ceil}(i/2)}(A_L)^{\text{floor}(i/2)} \) and \( n_R \geq \sum_{i=0}^{r-1} (A_L)^{\text{ceil}(i/2)}(A_R)^{\text{floor}(i/2)} \). For a \((m, r)\) BTU with girth \( g \), we obtain, \( A_R = \{(r-1)^{r/(mr^r)}\}^m = r-1 \) and \( A_L = \{(r-1)^{r/(mr^r)}\}^m = r-1 \). Thus, \( m \geq \sum_{i=0}^{g/2-1} (r-1)^{\text{ceil}(i/2)}(r-1)^{\text{floor}(i/2)} \). Therefore, \( m \geq \sum_{i=0}^{g/2-1} (r-1)^{\text{ceil}(i/2)+\text{floor}(i/2)} \).

For even integers \( i \), \( \text{ceil}(i/2) + \text{floor}(i/2) = i \).

For odd integers \( i \), \( \text{ceil}(i/2) + \text{floor}(i/2) = (i + 1)/2 + (i - 1)/2 = i \).

Therefore, \( m \geq \sum_{i=0}^{g/2-1} (r-1)^i = (r-1)^{g/2-1} = \frac{(r-1)^{g/2-1}}{r-1} \)

Putting \( r = 3 \) and \( g = 12 \) we get \( m \geq \frac{(2)^{6}-1}{3-2} = 63 \).

Putting \( r = 3 \) and \( g = 10 \) we get \( m \geq \frac{(2)^{5}-1}{3-2} = 31 \).

Putting \( r = 3 \) and \( g = 8 \) we get \( m \geq \frac{(2)^{4}-1}{3-2} = 15 \).

12 Other Related Research

Irregular LDPC codes with girth 20 in [11] and Regular LDPC codes of girth at least 10 from [10].
13 Results from [15]

We quote Theorem from [15] for even values of $g$ since our current interest is only in bi-partite graphs. "For $g \geq 3$ and $\delta \geq 3$ put $n_0(g, \delta) = \frac{2 \cdot (\delta - 1)^{(g/2) - 1}}{(g - 1)}$ if $g$ is even. Then a graph $G$ with minimal degree $\delta$ and girth $g$ has at least $n_0(g, \delta)$ vertices." We use this result to compute $n_0(g, \delta)$ for $\delta = 3$ and various values of $g$ in Table 2 by simplifying the equation as $n_0(g, 3) = 2 \cdot \{(2)^{g/2} - 1\}$

Table 2. Minimum value of $n_0(g, 3)$ for different girths $g$ for $\delta = 3$ from [15]

| $g$ | $n_0(g, 3)$ |
|-----|-------------|
| 4   | 6           |
| 6   | 14          |
| 8   | 30          |
| 10  | 62          |
| 12  | 126         |
| 14  | 254         |

14 Results from [13]

We quote theorems from [13].

1. "Given $\delta \geq 3$ and $g \geq 3$, there exists a $G^n, n \leq (2 + \delta)^g$ with minimal degree of at least $\delta$ and girth of at least $g$".

2. "Lower Bound $n(g, \delta) \geq \frac{1 + \delta (\delta - 1)^{(g - 1)/2} - 1}{(g - 2)}$ if $g$ is odd. $n(g, \delta) \geq \frac{(\delta - 1)^{(g/2)} - 1}{(g - 2)}$ if $g$ is even. Equality holds for $\delta = 3$ and $g = \{3, 4, 5, 6, 7, 8\}$ and $g = 4, \delta \geq 3$".

3. "If $g$ is odd, $n(g + 1, \delta) \leq 2 \cdot n(g, \delta)$".

4. "Upper Bound $n(g, \delta) \leq \frac{2 \cdot (\delta - 1)^{(g - 1)/2} - 1}{(g - 2)}$ if $g$ is odd. $n(g, \delta) \leq \frac{4 \cdot (\delta - 1)^{(g - 2)/2} - 1}{(g - 2)}$ if $g$ is even".

5. "Let $m \geq \sum_{i=0}^{g-2} (\delta - 1)^i = \frac{(\delta - 1)^{g-1} - 1}{(\delta - 2)}$ be an integer. Then there exists a $\delta$-regular graph of order $2 \cdot m$ and girth of at least $g$".

6. "Most significant improvement of the bound for $\delta = 3$, $n(g, 3) \leq 2^{g^2 - 1}$.

15 Bound derived from [13]

We derive the following bound from [13], $\frac{(\delta - 1)^{g-1} - 1}{(\delta - 2)} \leq n(g, \delta) \leq \frac{4 \cdot (\delta - 1)^{g-2} - 1}{(\delta - 2)}$ for the minimum order $n(g, \delta)$ where $g$ is its girth and $\delta$ is its degree. By putting $\delta = 3$, we obtain a simplified form of the above equation, $(2)^{g/2} - 1 \leq n(g, 3) \leq 4 \cdot (2)^{g-2} - 1$ which could be further simplified as $2^{g/2} - 1 \leq n(g, 3) \leq 2^g - 1$. We calculate the bounds for $\delta = 3$ and the improved upper bound corresponds to $n(g, 3) \leq 2^{g-1}$ from [13] in Table 3.
Table 3. Lower Bound, Upper Bound and Improved Upper Bound for $n(g, 3)$ for different girths $g$ for $\delta = 3$ from [13]

| $g$ | Lower Bound $n(g, 3)$ | Upper Bound $n(g, 3)$ | Improved upper bound $n(g, 3)$ |
|-----|------------------------|------------------------|--------------------------------|
| 4   | 3                      | 15                     | 8                             |
| 6   | 7                      | 63                     | 32                            |
| 8   | 15                     | 255                    | 128                           |
| 10  | 31                     | 1023                   | 512                           |
| 12  | 63                     | 4095                   | 2048                          |
| 14  | 127                    | 16383                  | 8192                          |

16 Analysis for [17] and [18]

From [17], we quote the following result, "If the degree is $D \geq 3$ and girth $g = 2 \ast r + 1; r \geq 2$, a simple lower bound for number of vertices of a regular graph is given by $n_o(g, D) = 1 + \frac{D}{D-2}((D-1)^r - 1)$." For $D = 3$ we simplify the equation as follows $n_o(g, 3) = 1 + 3((2)^{g-1}/2 - 1)$. While the exponent is similar to the lower bound in [13], we cannot apply the result as the girths take odd values and do not directly apply for bi-partite graphs.

17 Analysis for [19]

We analyze the girths obtained for various size of the matrices from [19] in Table 4. However, these matrices have irregular degrees and hence a direct comparison with our obtained results might not be possible.

Table 4. Girth obtained for various size of the matrices in [19]

| Girth | Minimum $N$ |
|-------|-------------|
| 6     | 5           |
| 8     | 9           |
| 10    | 39          |
| 12    | 97          |

18 Analysis for [14]

We quote from [14], "Ramanujan graphs $X^{p,q}$ are $p + 1$ regular Cayley graphs of the group $PSL(2, \mathbb{Z}/q\mathbb{Z})$ if the Legendre symbol $\left(\frac{q}{p}\right) = 1$ and of $PGL(2, \mathbb{Z}/q\mathbb{Z})$ if the Legendre symbol $\left(\frac{q}{p}\right) = -1$. $X^{p,q}$ is bi-partite of order $n = |(X^{p,q})| = \ldots"
$q \ast (q^2 - 1)$ and a bound on the girth is given by the equation, $g(X^{p,q}) \geq 4 \log_p(q) - \log_p(4)$.

Putting $p = 2$ in order to get degree $k = p + 1 = 3$, we obtain the inequality $g \geq 4 \log_2(q) - \log_2(4)$ which can be simplified as $(g + 2)/4 \geq \log_2(q)$ in order to obtain $2^{(g+2)/4} \geq q$.

For each value of girth $g$, we calculate the minimum value of $q$ such that $q \geq 2^{(g+2)/4}$ and the Legendre symbol $(p/q) = -1$ and then calculate $n = q \ast (q^2 - 1)$ for $p = 2$ and degree $k = 3$ in Table 5.

| Table 5. Analysis for [14] |
|-----------------------------|
| Girth min $q, q \geq 2^{(g+2)/4}$, $(p/q) = -1$ | $n = q \ast (q^2 - 1)$ | Chosen $p$ | Degree $k = p + 1$ |
| 6   | 5     | 120   | 2   | 3               |
| 8   | 11    | 1320  | 2   | 3               |
| 10  | 11    | 1320  | 2   | 3               |
| 12  | 13    | 2184  | 2   | 3               |

19 Conclusion

Our implementation for the Enumeration based Search for a girth maximum $(m, r)$ BTU finds the maximum attainable girth of a $(m, r)$ BTU for $r = 3$ and various values of $m$. The values of $m$ for a given girth $g$ are within the known mathematical bounds for regular bi-partitite graphs from the available literature. When we compare our results with bounds for more general graphs, or graphs with irregular graphs, a direct comparison may not possible since it is well known that for a given $g$ and average degree, a lower number of vertices can be reached for irregular graphs.

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