Loop groups and quantum fields

Alan L. Carey

Department of Pure Mathematics, University of Adelaide, Adelaide, South Australia 5005, Australia.
acarey@maths.adelaide.edu.au

and

Edwin Langmann

Theoretical Physics, Royal Institute of Technology, S-10044 Sweden.
langmann@theophys.kth.se

Autumn 2001

Abstract

This article surveys the application of the representation theory of loop groups to simple models in quantum field theory and to certain integrable systems. The common thread in the discussion is the construction of quantum fields using vertex operators. These examples include the construction and solution of the Luttinger model and other 1+1 dimensional interacting quantum field theories, the construction of anyon field operators on the circle, the ‘2nd quantization’ of the Calogero-Sutherland model using anyons and the geometric construction of quantum fields on Riemann surfaces. We describe some new results on the elliptic Calogero-Sutherland model.

1 Introduction

The examples we discuss in this review support the viewpoint (cf. [PS]) that much of 1+1 dimensional quantum field theory is the representation theory of infinite dimensional groups. This is well understood at the Lie algebra level for conformal quantum field theory where the representation theory of the Virasoro algebra and Kac-Moody algebras is central. We take a somewhat different point of view in this article and one that is more closely related to the ideas generated by the representation theory of loop groups (see e.g. [CR, F, K, KRi, PS]) and the analysis of superselection sectors [BMT]. This view is closely related to string theory and the analysis of integrable systems. We spend some time surveying a number of older developments and describe in detail two newer applications. The first of these is the Calogero-Sutherland model and its construction using anyon fields, while the second reviews the properties of Fermion fields in Riemann surfaces which is related to the study of the Landau-Lifshitz equation. All of our examples have in common that they can be made mathematically precise using an approach which is particularly simple (even though we will only touch on those technical issues which are not essential to the results).

Loop groups Map(S^1, G) are infinite dimensional Lie groups of smooth maps from the circle to some finite dimensional Lie group G such as SU(n) or U(n). We will mostly restrict
ourselves to the simplest case $\text{Map}(S^1, U(1))$. We will review some of the important results from the projective representation theory of this group and show how these results are applied to models in quantum field theory. Especially interesting are applications which yield new information, unattainable in a precise mathematical form in any other way. The most interesting use of these methods is in constructing the solutions of integrable systems. To make the later discussion accessible we begin in Part A with the construction and solution of the *Luttinger model*, a simple model for interacting fermions in 1+1 dimensions which, in a rather delicate limit, reduces to the massless Thirring model. We will then discuss modifications of the method which leads to the Luttinger model and show how other examples can be obtained with a particular emphasis on fields with braid group statistics.

Brief descriptions of these other models are contained in Subsection 3.2 and in Part B, Subsection 6.1 with references to the literature. From the viewpoint of conformal field theory these are mostly genus zero examples. In Part B we discuss the geometric viewpoint on free fermions at non-zero temperature and find that they provide a genus 1 example i.e. fields on a torus. This leads into a discussion of fermion field theories on higher genus Riemann surfaces. We find that associated to these are representations of ‘generalised’ loop groups. (These apply, in the spirit of the other applications discussed in this review, to the construction of the soliton solutions of the Landau-Lifshitz equation which we briefly describe).

The latter portion of Part A is a review of our recent paper [CL], with some extensions, on constructing quantum fields that are neither bosons nor fermions but satisfy non-trivial algebraic ‘exchange relations’. Crucial to this is the existence of a *boson-anyon correspondence* which shows that the fields satisfying braid statistics (the so-called anyon field operators) can be obtained as a limit of operators representing certain special elements of the loop group. This result allows explicit computation of all anyon correlation functions.

To be specific, *anyons on the circle* are quantum field operators obeying exchange relations

$$\varphi^\nu(x)\varphi^{\nu'}(y) = e^{i\pi\nu\nu'\text{sgn}(x-y)}\varphi^{\nu'}(y)\varphi^\nu(x) \quad \forall x, y \in S^1, x \neq y.$$  

(1)

For $\nu$ and $(\nu')^2$ both even (odd) integers these fields are bosonic (fermionic), but we are interested in anyons where $\nu$ and $\nu'$ can be any real numbers. The basic idea of how to generalize the boson-fermion correspondence to anyons is quite old, see e.g. [K], provided one is not too concerned with the exact mathematical meaning of the construction. However, there are delicate technical points related to the distributional nature of quantum fields. We will show how the theory of loop groups allows us to handle these difficulties quite elegantly. Our method can construct anyon field operators satisfying Eq. (1) only if $\nu$ and $\nu'$ are integer multiples of some fixed $\nu_0 \in \mathbb{R}$. These anyon field operators can be used to solve the *Calogero-Sutherland model* [Su] which is defined by the Hamiltonian

$$H_{N,\nu^2} = -\sum_{k=1}^N \frac{\partial^2}{\partial x_k^2} + \sum_{1 \leq k < \ell \leq N} 2\nu^2(\nu^2 - 1)V(x_k - x_\ell)$$

(2)

with $-L/2 \leq x_j \leq L/2$ coordinates on a circle of length $L$, $\nu > 0$, $N = 2, 3, \ldots$, and

$$V(r) = -\frac{\partial^2}{\partial r^2} \log \sin(\frac{\pi r}{L})$$

(3)

i.e. $V(r) = (\frac{\pi}{L})^2 \sin^{-2}(\frac{\pi r}{L})$. This is an integrable quantum mechanical model of $N$ interacting particles moving on a circle. This model has received considerable attention recently, especially in the context of the quantum Hall effect and conformal field theory, see e.g. [AMOS1, AMOS2, I, MS].
The idea of [CL] is to generalize the construction of the wedge representation of the $W_{1+\infty}$-algebra (see e.g. [KRd]), which uses fermions, to anyons. As a motivation of this, we discuss in some detail how, in the fermion case, a ‘generating function’ for the operators representing the Abelian subalgebra of the $W_{1+\infty}$-algebra can be obtained as a simple application of loop group theory. These operators $W^{s+1}$, $s = 0, 1, 2 \ldots$ generalize the fermion charge ($s = 0$) and the free fermion Hamiltonian ($s = 1$) in a natural manner. They are all local and quadratic in the fermion fields, and we obtain alternative representations which are local and in powers of boson fields. This corresponds to a generalization of the Sugawara constructions (see e.g. [GO]), and we refer to these as generalized Kronig identities (which is the special case $s = 1$). Remarkably a further generalisation to anyons with an arbitrary statistics parameter $\nu$ is possible and which is straightforward only for $s = 0, 1$. The first non-trivial case is $W^3$ which can be regarded as a second quantization of the Calogero-Sutherland Hamiltonian with the coupling constant determined by the statistics parameter $\nu$. The anyon-analog of $W^3$ obeys exchange relations with products of anyon field operators which may be exploited to construct iteratively, eigenfunctions of the Calogero-Sutherland model and thus recover the solution of this model [CL] found originally in [Su].

We end Part A with an outline of a further generalization of this construction in which anyons at a finite temperature $1/\beta$ are constructed and used to find a second quantization of the elliptic generalization of the Calogero-Sutherland model in which the interaction potential $V(r)$ is equal to the Weierstrass elliptic function $\wp(r)$ with periods $L/2$ and $i\beta L/(2\pi)$ [L3].

1.1 Summary

This review starts in Section 2 with the projective representation theory of loop groups based on the quasi-free representations of fermion field algebras. This is the so-called wedge representation of the loop group. In this Section we also review the boson-fermion correspondence. In Section 3 we outline how these results are used to treat the Luttinger model. Our discussion of the $W_{1+\infty}$ algebra is contained in Section 4. Section 5 explains the results on the Calogero-Sutherland model and anyons. Part B begins with an overview of other two dimensional quantum field theories which can be constructed using representations of loop groups (Section 6). In Section 7 we revisit the theory of free fermions at non-zero temperature and show how these fields may be interpreted as living on a torus. (The exposition follows [CH1]. The result is, however, folklore.) This motivates a discussion of fermions on Riemann surfaces in Section 8 synthesising examples and ideas from [S2] [CEH] [CH1] [CHM] [CHMS] [CHa] but mainly following [CH2]. The main point of our exposition is to sketch how the geometry of a Riemann surface with boundary determines a representation of an associated infinite dimensional group of $U(1)$-valued functions on the boundary. We then focus on the construction of vertex operators on Riemann surfaces by generalising the construction of Part A.

The discussion of the quantum field theory applications is based on many papers [CEH]–[CW], [GL1]–[GLR], [L1]–[L3]–[LS] with emphasis on [GLR] [CL] [CH1] [CH2]. Most of Section 4 and Subsection 5.2.2 are new. A portion of Part A appeared in [LC] and a recent pedagogical introduction to some of the material described here is given in [L4].

In the present review we restrict ourselves to quasi-free second quantization of fermions and two dimensional quantum field theory models. We only mention in passing that bosons can be treated in a similar manner, see e.g. [Ru, L1], and that there is a super-version of quasi-free second quantization in which bosons and fermions are treated simultaneously and on the same footing using $Z_2$-graded algebraic structures [GL1, GL2]. We also mention that there is an interesting relation between quasi-free second quantization and Connes’ noncommutative
geometry [Co] (a recent textbook is [GVF]) and higher dimensional quantum gauge theories. Some of these developments were reviewed in [L2] (see also [GVF]).

**PART A: LOOP GROUPS, FERMIONS AND PHYSICS**

2 Loop groups and quantized fermions

In this section we review some mathematical results on loop groups and quasi-free representations of fermion field algebras which will play a central role in the following. The material is standard, see e.g. [Ar, K, KRi, Mi, PS]. We follow mainly the discussions in [CR] and [S1].

2.1 Notation

Throughout this part, $x \in [-L/2, L/2]$ is a coordinate on a circle of length $L$ which we denote as $S^1_L$. Let $\mathcal{G} = \text{Map}(S^1_L; U(1))$ be the set of smooth maps $S^1_L \to U(1)$. We note that each loop $\varphi \in \mathcal{G}$ can be written as

$$\varphi(x) = e^{if(x)}, \quad f(x) = w \frac{2\pi}{L} x + \alpha(x)$$

where $w = [(f(L/2) - f(-L/2))/2\pi]$ is an integer called the *winding number*, and $\alpha$ is a smooth map $S^1_L \to \mathbb{R}$. We will find it convenient to decompose such maps into positive-, negative- and zero Fourier components,

$$\alpha(x) = \alpha^+(x) + \alpha^-(x) + \tilde{\alpha}$$

$$\alpha^\pm(x) = \frac{1}{L} \sum_{\pm p > 0} \hat{\alpha}(p) e^{ipx}, \quad \tilde{\alpha} = \frac{1}{L} \hat{\alpha}(0)$$

where we use the following conventions for Fourier transformation of loops,

$$\hat{\alpha}(p) = \int_{-L/2}^{L/2} dx \alpha(x) e^{-ipx} \quad p \in \Lambda^*$$

where

$$\Lambda^* := \left\{ p = \frac{2\pi}{L} n \left| n \in \mathbb{Z} \right. \right\}.$$

2.2 Loop group of maps $S^1_L \to U(1)$

We note that $\mathcal{G}$ is a Lie group under point-wise multiplication, $(\varphi_1 \cdot \varphi_2)(x) = \varphi_1(x) \varphi_2(x)$. It is known that $\mathcal{G}$ has an interesting central extension $\hat{\mathcal{G}} = U(1) \times \mathcal{G}$ with the group multiplication

$$(\gamma_1, \varphi_1) \cdot (\gamma_2, \varphi_2) := (\gamma_1 \gamma_2 \sigma(\varphi_1, \varphi_2), \varphi_1 \cdot \varphi_2)$$

$$(\gamma_i \in U(1), \; \varphi_i \in \mathcal{G})$$

where

$$\sigma(e^{if_1}, e^{if_2}) = e^{-iS(f_1, f_2)/2},$$
\[ S(f_1, f_2) = \frac{1}{4\pi} \left[ f_1 \left( \frac{L}{2} \right) f_2 \left( -\frac{L}{2} \right) - f_1 \left( -\frac{L}{2} \right) f_2 \left( \frac{L}{2} \right) \right] + \frac{1}{4\pi} \int_{-L/2}^{L/2} dx \left( \frac{df_1(x)}{dx} f_2(x) - f_1(x) \frac{df_2(x)}{dx} \right) \]  

is a two cocycle of the group \( G \): it satisfies

\[ \sigma(\varphi_1, \varphi_2) \sigma(\varphi_1 \cdot \varphi_2, \varphi_3) = \sigma(\varphi_1, \varphi_2 \cdot \varphi_3) \sigma(\varphi_2, \varphi_3) \]

which is equivalent the associativity of the group product defined in Eq. \( \Theta \).

Below we will describe in some detail the construction of the so-called wedge-representation of \( \hat{G} \), \( (\gamma, \varphi) \rightarrow \gamma \Gamma(\varphi) \), on the fermion Fock space \( F \) over \( L^2(S^1_L) \). The \( \Gamma(\varphi) \) are unitary operators satisfying

\[ \Gamma(\varphi_1) \Gamma(\varphi_2) = \sigma(\varphi_2, \varphi_2) \Gamma(\varphi_1 \cdot \varphi_2) \]  

(8)

and

\[ \Gamma(\varphi)^* = \Gamma(\varphi^*) \]  

(9)

(for simplicity in notation, we denote the Hilbert space adjoint and complex conjugation by the same symbol \( * \)). Moreover, there is a vector \( \Omega \in \mathcal{F} \) such that for all \( f \) of the form Eq. \( \Pi \),

\[ \langle \Omega, \Gamma(e^{iL})\Omega \rangle = \delta_{w,0}e^{-iS(\alpha^-\alpha^+)} \]  

(10)

where \( \langle \cdot, \cdot \rangle \) is the inner product in \( \mathcal{F} \). Note that

\[ iS(\alpha^-, \alpha^+) = \sum_{p>0} \frac{p}{2\pi L} \hat{\alpha}(-p) \hat{\alpha}(p) \]

is positive definite.

We now describe how this representation \( \Gamma \) of \( \hat{G} \) is constructed.

### 2.3 Quasi-free second quantization of fermions

#### 2.3.1 Fermion field algebras

Let \( \mathcal{H} \) be a separable Hilbert space. The fermion field algebra \( \mathcal{A} \) over \( \mathcal{H} \) is then defined as the \( C^* \)-algebra generated by elements \( a^*(f) \) and \( a(f) = a^*(f)^* \) such that \( f \rightarrow a^*(f) \) is linear,

\[ ||a^*(f)||^2 = \langle f, f \rangle_{\mathcal{H}}, \]

and the canonical anticommutation relations (CAR) hold,

\[ a(f) a(g) + a(g) a(f) = 0, \quad a(f) a(g)^* + a(g)^* a(f) = \langle f, g \rangle_{\mathcal{H}} I \]  

(11)

(here and in the following, \( I \) denotes the identity operator). The Fermion Fock space \( \mathcal{F} \) over \( \mathcal{H} \) is the Hilbert space obtained by completing the exterior algebra \( \wedge \mathcal{H} \) over \( \mathcal{H} \) in the obvious Hilbert space topology. We define an action of \( a(g)^* \) by

\[ a(g)^* g_1 \wedge g_2 \wedge \ldots \wedge g_n = g \wedge g_1 \wedge g_2 \wedge \ldots \wedge g_n \]

for \( g_j \) in \( \mathcal{H} \). Then \( a(g) \) may be identified with the Hilbert space adjoint of \( a(g)^* \) and it is easy to see that the anti-commutation relations \( \Pi \) hold. In this way one obtains the so-called Fock-Cook representation of the fermion field algebra \( \mathcal{A} \).

**Remark:** In applications to models in physics \( \mathcal{H} \) is taken as the Hilbert space of 1-particle states. For example for statistical mechanical models of fermions with spin on a finite lattice \( \Lambda \), the 1-particle states are \( \mathbb{C}^2 \)-valued function on \( \Lambda \), i.e. \( \mathcal{H} \cong \mathbb{C}^{2|M|} \) is actually finite.
dimensional. In this case the algebra $\mathcal{A}$ and the Fock space $\mathcal{F}$ are also finite dimensional\footnote{One can check that $\dim_{\mathbb{C}}(\mathcal{F}) = 2^{\dim_{\mathbb{C}}(\mathcal{H})}$ if $\dim_{\mathbb{C}}(\mathcal{H}) < \infty$.} all the irreducible representations are $\mathcal{A}$ are unitarily equivalent, and all one ever needs is the Fock-Cook representation described above. The situation becomes mathematically more interesting if $\mathcal{H}$ is infinite dimensional. This is the situation for quantum field models on a continuous manifold $M$ where the appropriate 1-particle space $\mathcal{H}$ is typically a space of square integrable functions on $M$.

### 2.3.2 Quasi-free representations I. Irreducible case

Let $P_-$ be a projection operator on $\mathcal{H}$ (i.e. $P_-^2 = P_-$) and let $P_+ = 1 - P_-$. Then there is a representation $\pi_{P_-}$ of $\mathcal{A}$ on the fermion Fock space $\mathcal{F}$ over $\mathcal{H}$ which is determined by the following conditions,

$$\psi(P_+ f)\Omega = 0 = \psi^*(P_- f)\Omega \quad \forall f \in \mathcal{H}$$

where we write $\psi(f) = \pi_{P_-}(a(f))$; $\Omega$ is the cyclic (or vacuum) vector in the $\mathcal{F}$. One can prove that the representations $\pi_{P_-}$ are irreducible [BR].

**Remark:** In applications, $P_-$ usually is determined by a self-adjoint operator $D$ on $\mathcal{H}$ which represents the 1-particle Hamiltonian (i.e. energy operator) of a specific model and typically is some Dirac operator with a spectrum which is unbounded from above and below. In this situation $P_-$ is taken as the spectral projection of $D$ corresponding to the interval $(-\infty, 0)$ i.e. $P_- = \theta(-D)$, where $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$. Then the state $\Omega$ above corresponds to the so-called filled Dirac sea and is the ground state (i.e. state of least energy) of the many particle Hamitlonian corresponding to $D$, as explained in more detail below.

### 2.3.3 Second quantization of 1-particle operators

Let $g_1$ the set of all bounded operators $X$ on $\mathcal{H}$ such that $P_\pm XP_\mp$ is Hilbert-Schmidt. $g_1$ is a Lie algebra, and for each $X \in g_1$ there is operator $d\Gamma(X)$ acting on $\mathcal{F}$ such that [CR]

$$[d\Gamma(X), \psi^*(f)] = \psi^*(Xf),$$

$$d\Gamma(X) = d\Gamma(X^*)^*,$$

and

$$<\Omega, d\Gamma(X)\Omega>_{\mathcal{F}} = 0.$$ (15)

The construction of these operators requires a regularization — physicists refer to it as ‘normal ordering’ — and due to this, $X \rightarrow d\Gamma(X)$ is not a representation but rather a *projective representation* of $g_1$: One has relations

$$[d\Gamma(X), d\Gamma(Y)] = d\Gamma([X,Y]) + i\hat{S}(X,Y)I$$

where

$$i\hat{S}(X,Y) = \text{Trace}_{\mathcal{H}}(P_-XP_+YP_- - P_-YP_+XP_-)$$ (17)

is a *non-trivial two cocycle* of the Lie algebra $g_1$ [CR, L]. In the physics literature $i\hat{S}$ is known as the Schwinger term. For $X \in g_1$ one also has

$$P_+XP_- = 0 \Rightarrow d\Gamma(X)\Omega = 0$$ (18)
which is called *highest weight condition*.

We note that even for bounded $X$, the operators $d\Gamma(X)$ are unbounded, but one can easily construct a common dense invariant domain on which all the relations above are well-defined [CR, GL2]. Moreover, the construction of $d\Gamma(X)$ can be naturally extended to certain algebras of *unbounded* operators $X$ on $\mathcal{H}$ which have a common dense invariant domain of definition [GL2]. All relations given above naturally extend to this larger Lie algebra of operators on $\mathcal{H}$. In the following the same symbol $g_1$ will be used also for such Lie algebras of unbounded operators on $\mathcal{H}$.

Let $G_1$ be the set of all unitary operators $U$ on $\mathcal{H}$ with $P_\pm U P_\mp$ Hilbert-Schmidt. This is a Lie group with a Lie algebra containing the self-adjoint operators in $g_1$. It has a projective representation $U \to \Gamma(U)$ on $\mathcal{F}$ such that

$$\Gamma(U)\psi(f)\Gamma(U)^{-1} = \psi(Uf).$$

We say that $\Gamma(U)$ *implements* $U$. ‘Projective’ here means that relations

$$\Gamma(U)\Gamma(V) = \hat{\sigma}(U,V)\Gamma(UV)$$

hold with $\hat{\sigma}$ a non-trivial phase factor. An explicit formula for this cocycle $\hat{\sigma}$ was derived in [L1].

**Remark:** In a specific model, 1-particle observables are given by self-adjoint $X$ on $\mathcal{H}$ and $d\Gamma(X)$ (if it exists) represents the corresponding many particle observable. For example, if $D$ is the 1-particle Hamiltonian then $d\Gamma(D)$ is the many particle Hamiltonian. One can show that in the quasi-free representation $\pi_{P_-}$ with $P_- = \theta(-D)$, $d\Gamma(H)$ is always positive. This is the reason why the quasi-free representations are needed. One essential physical requirement in every quantum model is the existence of a *ground state* i.e. state of lowest energy. If the 1-particle Hamiltonian is not bounded from below then there is no groundstate, neither in the 1-particle Hilbert space nor in the Fock-Cook representation, and therefore they both have to be rejected. On the other hand, a representation in which the many particle Hamiltonian is bounded from below allows for a ground state.

### 2.3.4 Quasi-free representations II. Reducible case

A more general class of representations is obtained by replacing $P_-$ by a self-adjoint operator $A$ on $\mathcal{H}$ with $0 < A < 1$. These representations, denoted $\pi_A$, are constructed as follows. We let $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$ and form the fermion algebra over $\mathcal{K}$, denoted $\mathcal{A}(\mathcal{K})$. Define a projection on $\mathcal{K}$ by

$$P(A) = \begin{pmatrix} A & A^{1/2}(1-A)^{1/2} \\ A^{1/2}(1-A)^{1/2} & 1-A \end{pmatrix}$$

Then the representation $\pi_A$ is by definition the restriction of the representation $\pi_{P(A)}$ of $\mathcal{A}(\mathcal{K})$ to the subalgebra $\mathcal{A}(\mathcal{H} \oplus (0))$.

**Remark:** These representations can be used to describe quantum field theory models at *finite temperature*: If $D$ is the 1-particle Hamiltonian then

$$A = \frac{1}{e^{\beta D} + 1}$$

(21)

gives rise to the representation at temperature $T = 1/\beta > 0$. In the zero temperature limit one recovers an irreducible representation.
We now are ready to describe the relation between loop groups and fermion quantization. As underlying Hilbert space for the fermions we take $H = L^2(S^1_L) \cong \ell^2(\Lambda_0^*)$ where

$$\Lambda_0^* = \left\{ k = \frac{2\pi}{L}(n + \frac{1}{2}) \mid n \in \mathbb{Z} \right\}.$$ 

These are identified via the Fourier transform,

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-L/2}^{L/2} dx f(x) e^{-ikx}$$

for $k \in \Lambda_0^*$. An orthogonal basis of $L^2(S^1_L)$ is provided by the functions

$$e_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}, \quad k \in \Lambda_0^*,$$

and then we have $f = \frac{2\pi}{L} \sum_k \hat{f}(k)e_k$. The spectral projection $P_-$ we use is defined as

$$(P_- f)(k) = \hat{f}(k)$$

for $k < 0$ and $= 0$ otherwise.

**Remark:** Note that $P_- = \theta(-D)$ where $D$ is the self-adjoint operator given by $De_k = ke_k$ for all $k \in \Lambda_0^*$. Of course $D$ is a self-adjoint extension of $-i\partial_x$, the (chiral) Dirac operator on the circle $S^1_L$.

Each smooth function $\alpha \in \text{Map}(S^1_L, \mathbb{C})$ naturally defines a bounded operator on $L^2(S^1_L)$ which we denote by the same symbol, $(\alpha f)(x) = \alpha(x)f(x)$ for all $f \in L^2(S^1_L)$. A central result in the theory of loop groups is that all these operators $\alpha$ are in $g_1$, and $[\text{CR}]

$$\hat{S}(\alpha_1, \alpha_2) = \frac{1}{4\pi} \int_{-L/2}^{L/2} dx \left( \frac{d\alpha_1(x)}{dx} \alpha_2(x) - \alpha_1(x) \frac{d\alpha_2(x)}{dx} \right).$$

Moreover,

$$d\Gamma(\alpha^-)\Omega = d\Gamma(\alpha^+)^*\Omega = 0$$

follows from Eqs. $[18]$ and $[14]$. Especially, all $\varphi \in \mathcal{G}$ are in $G_1$, and $U \rightarrow \Gamma(U)$ is precisely the wedge-representation of $\mathcal{G}$ discussed above. The choice of phase of $\Gamma(\varphi)$ is important to obtain the explicit form for $\sigma$ given in Eq. $[8]$. To fix the phase completely we need $R = \Gamma(\varphi_1)$ corresponding to $\varphi_1(x) = e^{2\pi i x/L}$ (for an explicit construction of $\Gamma(\varphi_1)$ see e.g. $[Ru]$). This unitary operator obeys

$$R^{-w}d\Gamma(\alpha^+)R^w = d\Gamma(\alpha^+), \quad R^{-w}QR^w = Q + wI$$

for all integer $w$. Here we introduced the operator

$$Q := d\Gamma(I)$$

which can be interpreted as the charge operator. Writing general loops as in Eqs. $[4]$, $[5]$ we now can define

$$\Gamma(e^{i\lambda}) := e^{i\lambda Q/2}R^w e^{i\lambda Q/2}e^{i\lambda d\Gamma(\alpha^+ + \alpha^-)}.$$
Then a straightforward computation gives
\[ S(f_1, f_2) = (w_1 \alpha_2 - \bar{\alpha}_1 w_2) + \tilde{S}(\alpha_1, \alpha_2) \] (29)
identical with Eq. (I). A similar computation implies Eq. (10).

We find it convenient to introduce normal ordering for implementers of loops
\[ \Gamma(e^{if}) = e^{-iS(\alpha^-, \alpha^+)/2} \times \Gamma(e^{if}) \times \] (30)
with the numerical factor chosen such that
\[ \langle \Omega, \times \Gamma(e^{if}) \times \Omega \rangle = 1 \quad \text{if} \quad w = 0 \] [cf. Eq. (10)]. Note that
\[ e^{id} \Gamma(\alpha^\pm) \] are not operators, however, but have to be interpreted as sesquilinear forms. This definition naturally extends to products of implementers,
\[ \times \Gamma(e^{if_1}) \Gamma(e^{if_2}) \cdots \Gamma(e^{if_N}) \times := \times \Gamma(e^{if_1} e^{if_2} \cdots e^{if_N}) \times \] (32)
and operators of the form
\[ \times d\Gamma(\alpha_1) \cdots d\Gamma(\alpha_m) \Gamma(e^{if}) \times := \left. \frac{\partial^n}{\partial \alpha_1 \cdots \partial \alpha_m} \times e^{ia_1 d\Gamma(\alpha_1) \cdots e^{ia_m d\Gamma(\alpha_m)} \Gamma(e^{if}) \times} \right|_{a_j = 0}. \] (33)

Note that operators between normal ordering symbols commute. We also note the following relations
\[ \times \Gamma(e^{if_1}) \times \times \Gamma(e^{if_2}) \times = e^{-i\tilde{S}(f_1, f_2)} \times \Gamma(e^{if_1}) \Gamma(e^{if_1}) \times \] (34)
with
\[ \tilde{S}(f_1, f_2) = w_1 \alpha_2 - \bar{\alpha}_1 w_2 + 2S(\alpha_1^-, \alpha_2^+) = -S(f_2, f_1)^* \] (35)
which will be useful in the following.

### 2.4.1 Bosons from fermions

We define \( \epsilon_p(x) = e^{-ipx} \) for \( p \in \Lambda^* \) and set
\[ \hat{\rho}(p) = d\Gamma(\epsilon_p). \] (36)
Then \( \hat{\rho}(-p) = \hat{\rho}(p)^* \), \( \hat{\rho}(0) = Q \), and the equations given above imply
\[ [\hat{\rho}(p), \hat{\rho}(p')] = \frac{p}{2\pi} \delta_{-p, p'} L \] (37)
and
\[ \hat{\rho}(p) \Omega = 0 \quad p \geq 0. \] (38)

The \( \hat{\rho}(p) \) can be naturally interpreted as boson field operators.

\(^2\)to show this one can use \( e^{a_1} e^{a_2} = e^{[a_1, a_2]/2} e^{a_1 + a_2} \) for \( a_j = id\Gamma(\alpha_j) \), and \( R^n e^{irQ} R^{-n} = e^{-inr} e^{irQ} \) for real \( r \) and integer \( n \).
2.4.2 Fermions from bosons

In Ref. [S1] a so-called ‘blip’ function was introduced which equals, up to the sign,

\[
\frac{e^{i(x-y)2\pi/L} - \lambda}{1 - \lambda e^{i(x-y)2\pi/L}}, \quad 0 < \lambda < 1.
\]

This is the exponential of a smoothed out step function: Writing it as \( e^{if_{y,\varepsilon}} \) with \( \lambda = e^{-2\varepsilon/L} \) one gets

\[
f_{y,\varepsilon}(x) = \frac{2\pi}{L}(x - y) + \alpha^+_{y,\varepsilon}(x) + \alpha^-_{y,\varepsilon}(x)
\]

with

\[
\alpha^\pm_{y,\varepsilon}(x) = \pm i \log(1 - e^{2\pi i(x-y)\pm\varepsilon}/L) = \pm i \sum_{n=1}^{\infty} \frac{1}{n} e^{\pm 2\pi i n(x-y)/L} e^{-2\pi \varepsilon n/L}.
\]

Note that the winding number of \( f_{y,\varepsilon} \) equals 1. Since \( f_{y,\varepsilon}(x) \) for \( \varepsilon \downarrow 0 \) converges to \( i\pi \text{sgn}(x-y) \) we will also use the following suggestive notation,

\[
\text{sgn}_\varepsilon(x-y) := \frac{1}{\pi} f_{y,\varepsilon}(x).
\]

Later we will also need the function \( \delta_{y,\varepsilon} = \partial_y f_{y,\varepsilon}/2\pi \) i.e.

\[
\delta_{y,\varepsilon}(x) = \frac{1}{L} + \delta^+_{y,\varepsilon}(x) + \delta^-_{y,\varepsilon}(x)
\]

with

\[
\delta^\pm_{y,\varepsilon}(x) = \frac{1}{L} \sum_{n>0} e^{\pm 2\pi i n(x-y)/L} e^{-2\pi \varepsilon n/L}.
\]

This smoothed out \( \delta \)-function will play an important role in Sections 4 and 5.2.

These functions have the following important properties.\(^3\)

\[
S(\alpha^-_{y,\varepsilon}, \alpha^+_{y',\varepsilon'}) = \alpha^+_{y',\varepsilon+\varepsilon'}(y)
\]

\[
S(f_{y,\varepsilon}, f_{y',\varepsilon'}) = \pi \text{sgn}_{\varepsilon+\varepsilon'}(y - y')
\]

\[
S(\delta^\pm_{y,\varepsilon}, \alpha^\mp_{y',\varepsilon'}) = -\delta^\pm_{y',\varepsilon+\varepsilon'}(y)
\]

Note that for \( \varepsilon > 0 \) the operators

\[
\varphi^\pm_\varepsilon(y) := \frac{1}{\pi} \Gamma(e^{i\varepsilon f_{y,\varepsilon}}) \varphi^\pm_\varepsilon(y)
\]

are well-defined, and from Eqs. \( 44 \) and \( 8 \) we conclude

\[
\varphi^\nu_\varepsilon(y) \varphi^\nu_\varepsilon'(y') = e^{i\pi \text{sgn}_{\varepsilon+\varepsilon'}(y-y')} \varphi^\nu_\varepsilon(y') \varphi^\nu_\varepsilon(y)
\]

for \( \nu, \nu' = \pm 1 \). In the limit \( \varepsilon, \varepsilon' \downarrow 0 \) these formally become anticommutator relations. This suggests that \( \varphi^\pm_\varepsilon(x) \) in the limit \( \varepsilon \downarrow 0 \) should be proportional to fermion fields. Indeed one can prove the

**Theorem:** For all \( f \in L^2(S^1_L) \) such that \( \hat{f}(p) \) has a compact support the following identity holds,

\[
\psi^*(f) = \lim_{\varepsilon \downarrow 0} \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} dx f(x) \varphi^1_\varepsilon(y)
\]

\(^3\)The proof is a straightforward calculation
in the sense of strong convergence on a dense domain.

This is the central result of what is usually called the boson-fermion correspondence; see e.g. [CHu, Fr, PS].

We now sketch a proof of this result to introduce some techniques which we will generalize later ([CR] gives a different proof). Note that there are subtleties which require careful specification of the domains on which the identities hold however we will ignore those for brevity. The idea is to first show by explicit computations that the

$$
\varphi^1(f) = \lim_{\varepsilon \downarrow 0} \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} dx \; f(x) \varphi^1_\varepsilon(y), \quad \varphi^{-1}(f) = \lim_{\varepsilon \downarrow 0} \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} dx \; \overline{f(x) \varphi^{-1}_\varepsilon(y)} = \varphi^1(f)^* \tag{48}
$$

obey the same CAR as the operators $\psi^{(s)}(f)$. For that we use Eqs. (30), (32), and (44) which imply

$$
\varphi^\nu_\varepsilon(x) \varphi^{\nu'}_\varepsilon(y) = b_\varepsilon(r)^{\nu \nu'} \varphi^\nu_\varepsilon(x) \varphi^{\nu'}_\varepsilon(y) \tag{49}
$$

where $b_\varepsilon(r) = -2ie^{-\pi \varepsilon/L} \sin \frac{\pi}{L}(r + i\varepsilon), \; r = x - y$, and $\varepsilon = \varepsilon + \varepsilon'$. Thus

$$
\varphi^1_\varepsilon(x) \varphi^1_\varepsilon(y) + \varphi^1_\varepsilon(y) \varphi^1_\varepsilon(x) = -2ie^{-\pi \varepsilon/L} \left[ \sin \frac{\pi}{L}(r + i\varepsilon) - \sin \frac{\pi}{L}(r - i\varepsilon) \right] \varphi^1_\varepsilon(x) \varphi^1_\varepsilon(y) \tag{49}
$$

where the r.h.s. obviously becomes zero after smearing with appropriate test functions and sending $\varepsilon, \varepsilon'$ to zero. This proves $\{ \varphi^1(f), \varphi^1(g) \} = 0$. Similarly,

$$
\varphi^{-1}_\varepsilon(x) \varphi^1_\varepsilon(y) + \varphi^1_\varepsilon(y) \varphi^{-1}_\varepsilon(x) = [\cdots] \varphi^{-1}_\varepsilon(x) \varphi^1_\varepsilon(y) \tag{50}
$$

where

$$
[\cdots] = \left[ \frac{e^{i\pi r/L}}{1 - e^{i\pi r/L} e^{-2\pi i\varepsilon/L}} + \frac{e^{-i\pi r/L}}{1 - e^{-i\pi r/L} e^{-2\pi i\varepsilon/L}} \right] = L\delta_{y,\varepsilon}(x)
$$

with the smoothed out $\delta$-function introduced above. Since $\varphi^{-1}_\varepsilon(x) \varphi^1_\varepsilon(y)$ becomes the identity operator for $x = y$ and $\varepsilon = \varepsilon'$, a simple argument implies $\{ \varphi^1(f), \varphi^{-1}(g) \} = (f, g)I$, and this completes the proof of the CAR. Next one shows

$$
\varphi^{-1}(P_+ f)\Omega = 0 = \varphi^1(P_- f)\Omega . \tag{50}
$$

To prove this we use Eqs. (15), (31), (25) and (26) which imply

$$
\varphi^\pm_\varepsilon(x) \Omega = e^{-i\pi x/L} e^{\mp i\Gamma(\alpha^+_{x,\varepsilon})} R^\pm \Omega .
$$

Now

$$
d\Gamma(\alpha^+_{x,\varepsilon}) = \frac{2\pi i}{L} \sum_{p > 0} \frac{1}{p} e^{-ipx - |p|\varepsilon} \hat{\rho}(-p)
$$

$(p \in \Lambda^*; \; \text{cf. Eqs. (39)–(40))}$, i.e. it is a sum of terms with negative Fourier coefficients only. By expanding the second exponential on the r.h.s. of this equation one can therefore only generate terms with negative Fourier coefficients, and thus $\int_{-L/2}^{L/2} dx \; e^{ikx} \varphi^\pm_\varepsilon(x) = 0$ for all negative $k \in \Lambda^*_0$. This proves Eq. (50).

These arguments show that the operators $\varphi^\pm_\varepsilon(f)$ also provide the same quasi-free representation of the CAR on $\mathcal{F}$ as the $\psi^{(s)}(f)$, and with the same vacuum vector $\Omega$. 

To complete the proof one can check by explicit computation that the operators \( \varphi^\pm_1(f) \) have the same commutator resp. exchange relations with the operators \( \hat{\rho}(p) \) resp. \( R \) as the \( \psi^{(s)}(f) \) and use the following

**Lemma:** [CHOU] The vectors

\[
\prod_{n=1}^{\infty} \hat{\rho}(-\frac{2\pi}{L} n)^{m_n} R^\nu \Omega
\]

with \( m_n \in \mathbb{N}_0 \) and \( \nu \in \mathbb{Z} \) such that \( \sum_{n=0}^{\infty} m_n < \infty \), are dense in \( \mathcal{F} \).

### 3 Quantum field theory in 1+1 dimensions

In this Section we discuss quantum field theory models of interacting fermions on one dimensional space. To be specific we concentrate on the Luttinger model [ML], a simple model for a one dimensional metal. Our first purpose is to illustrate how the mathematical results summarized above are used to construct and solve 1+1 dimensional quantum field theory models. Our second purpose is to give a physical motivation for various operators which we construct and study in the next Section.

#### 3.1 The Luttinger model

We start with a physical motivation for this model. We consider spinless fermions in a one dimensional metal (wire) of length \( L \) which can be characterized by a band relation \( E(p) = E(-p) \) describing the energy as a function of the (pseudo-) momentum \( p \). If the band is filled up to the chemical potential \( \mu \), the Fermi surface consists of two points \( p = \pm p_F \) where \( E(p) - \mu \) vanishes. Physically one expects that the states close to the Fermi surface are the most important ones. For those one can Taylor expand the band relations about the Fermi surface, and one gets two branches,

\[
E(\pm p_F \pm (k - p_F)) - \mu = \pm v_F (k - p_F) + \frac{1}{2} m^{-1} (k - p_F)^2 + \ldots
\]

where \( v_F \) (Fermi velocity) is the slope and \( m^{-1} \) (inverse mass) the curvature of the band at the Fermi surface. With that we obtain a multi particle Hamiltonian \( H_0 = v_F (W_2^2 + W_2^2) + \frac{1}{2} m^{-1} (W_3^2 + W_3^2) + \ldots \) where

\[
W^{s+1}_\pm = \int_{-L/2}^{L/2} dx \psi^*_\pm(x)(\pm \hat{p})^s \psi_\pm(x), \quad \hat{p} = -i \frac{d}{dx}
\]

for \( s = 1, 2 \) with \( \psi_\pm \) the fermion field operators describing the excitations of the two branches. The model \( H_0 \) describes non-interacting fermions and thus trivially is soluble. However, if one only takes into account the linear term in the Taylor expansion Eq. (52), the model remains soluble even in presence of an interaction.

The Luttinger model thus is formally defined by the Hamiltonian \( H = H_0 + H' \) where\(^{[3]}\)

\[
H_0 = \int_{-L/2}^{L/2} dx \, \psi^*(x) \sigma_3 \hat{p} \psi(x), \quad \psi^* = (\psi_+, \psi_-)
\]

\(^{[3]}\)we set \( v_F = 1 \)
\[(\sigma_3)_{\sigma\sigma'} = \sigma\delta_{\sigma,\sigma'}\] is the free part, and
\[H' = \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy \rho_+(x)v(x-y)\rho_-(y), \quad \rho_\pm(x) = \psi_\pm^*(x)\psi_\pm(x)\] (55)
the interaction (the interaction potential \(v\) will be further specified below). It is worth noting that \(H_0\) equals a Hamiltonian of free relativistic fermions in 1+1 dimensions. A crucial point in the correct treatment of the model is the construction of the fermion fields \(\psi_\pm\): if one would use ‘naive’ fermions with a ‘vacuum’ \(\Omega_{\text{unphys.}}\) such that \(\psi_\pm\Omega_{\text{unphys.}} = 0\), the Hamiltonians \(H_0\) and \(H\) would not be bounded from below. Since there is no groundstate then, this model would be unphysical. The physical idea for solving this problem is the ‘filling of the Dirac sea’. The theory of quasi-free representations of CAR algebras described is a general formalism which allows one to construct physical representations of the fermion fields for non-interacting relativistic fermion models. It turns out that the quasi-free representation in which \(H_0\) is positive is also the one in which \(H\) exists and also is bounded from below. This is also the case for other 1+1 dimensional models mentioned further below. It is this precisely this property which makes these 1+1 dimensional models simpler than corresponding models in higher dimensions.

We now describe how to construct the physical representation for the free Hamiltonian \(H_0\). The 1-particle Hilbert space is \(L^2(S^1)\otimes \mathbb{C}^2\), and writing functions in this space as \(f = (f_+, f_-)\) we define \((\hat{P}_- f)_\pm(\mp k) = \hat{f}_\pm(\mp k)\) for all \(k > 0\) and \(= 0\) otherwise \((k \in \Lambda^*_0)\). Then \(\pi_{\hat{P}_-}\) is the physical representation of the CAR algebra \(\mathcal{A}\) over \(L^2(S^1)\otimes \mathbb{C}^2\): The operator \(H_0 = d\Gamma(-i\sigma_3 d/dx)\) is self-adjoint and positive. Moreover, \(\hat{\rho}_\pm(p) = d\Gamma(\frac{1}{2}(1\pm \sigma_3)\epsilon_p), \epsilon_p(x) = e^{-ipx}\), can be identified with the Fourier modes of the fermion currents \(\rho_\pm(x)\). Thus
\[H' = \frac{2\pi}{L} \sum_{p \in \Lambda^*} \hat{\rho}_+(p)\dot{\psi}(p)\hat{\rho}_-(-p)\] (56)
which can be shown to be such that \(H = H_0 + H'\) is self-adjoint and bounded from below if and only if the following condition holds \([\text{ML}]\),
\[|\dot{\psi}(p)| < 1, \quad \sum_{p \in \Lambda^*} |p||\dot{\psi}(p)|^2 < \infty\] (57)
where \(\dot{\psi}(p) = \frac{1}{\pi\tau} \int_{-L/2}^{L/2} dx \, v(x)e^{-ipx}\) are the Fourier modes of the interaction potential. To complete the construction of the model, one can specify a common dense invariant domains of definition for all operators \(H_0, H', \hat{\rho}_\pm(p)\) etc., see e.g. \([\text{GL2}]\).

We note that the results described in Subsection 2.6ff immediately apply to the fermions \(\psi_+\). It is clear that there are similar formulas for the \(\psi_-\)-fermions. Especially, due to the non-trivial representation \(\pi_{\hat{P}_-}\), the commutators of the fermion currents are not zero, but equal to Schwinger terms. This allows the interpretation of the fermion currents as boson fields, as discussed. The appearance of this Schwinger term in the commutator relations of the fermion currents is an example of an anomaly. It has drastic consequences for the physical properties of the model.

The important relation which allows a solution of the Luttinger model is the so-called Kronig identity\(^5\)
\[W^2_\pm = \frac{\pi}{L} \sum_{p \in \Lambda^*} \hat{\rho}_\pm(p)\hat{\rho}_\pm(-p)\] (58)
\(^5\)we sketch a proof of this relation below
Physically this means that the free fermion Hamiltonian equals a free boson Hamiltonian. Since $H'$ is also quadratic in the boson fields, the Luttinger Hamiltonian $H$ equals a free boson Hamiltonian which is diagonalized by a unitary operator $U$ which can be constructed explicitly [ML, HSU]. Then the ground state of the Luttinger model is found as $U \Omega$. Moreover, one can also compute all Green function of the model explicitly. This is due to the boson-fermion correspondence which allows one to write the fermions $\psi_{\pm}(x)$ as a limit of exponential of boson fields. This means it is possible to compute the ‘interacting fermion fields’ $\Psi(t, x) := U(t)^* \psi_{\pm}(x) U(t), U(t) = e^{-itH} U$, explicitly. The computation of Green functions reduces then to normal ordering of products of implementers using Eqs. (34), (35) [HSU].

The construction and solution for the Luttinger models described here was for zero temperature. A similar construction and solution of the Luttinger model at finite temperature was given in [CHa].

3.2 Other models

In the limit where space becomes infinite, $L \to \infty$, and the interaction local, i.e.

$$\hat{v}(p) = g \quad \text{independent of } p, \quad |g| < 1,$$

the Luttinger model reduces to the massless Thirring model [T]. This latter limit is non-trivial and quite instructive: for the potential Eq. (59) the condition (57) fails, and the operator $U$ does not exist. To construct this limit, one needs an additional multiplicative regularization. Due to this, the interacting fields $\Psi(t, x)$ for the massless Thirring model are not fermions but more singular (this is nicely explained in [W], e.g.). To see in detail how the interacting fields turn from fermions to these more singular operators, one can construct the Thirring model as a limit $\ell \to 0$ of the Luttinger models with potentials $\hat{v}_\ell(p) = g(g^2 + (1 - g^2)e^{\ell|p|} - 1)/2$ [GLR] (for an alternative approach see [CRW]).

Other interacting quantum field theory models which can be constructed and solved by similar methods include the Schwinger model [Ma], i.e. 1+1 dimensional quantum electrodynamics with massless fermions, the Luttinger-Schwinger model, i.e. the gauged Luttinger model [GLR], and diagonal QCD$_{1+1}$ [CW]. A similar construction of QCD$_{1+1}$, i.e. the non-abelian version of the Schwinger model, was given in [LS].

4 $W_{1+\infty}$-algebra: Generalizing the Kronig identity

We now discuss an interesting mathematical application of the formalism in Section 2 to the so-called $W_{1+\infty}$-algebra (see e.g. [KRd]).

As motivation, we recall from the last Section that one can interpret the operators $H_0 = v_F W^2 + \frac{1}{2} m^{-1} W^3 + \ldots, W^s = W_+^s$ Eq. (53), as a (part of a) fermion Hamiltonian. In this section we show that the $W^s$ are examples of operators which represent elements in the algebra $W_{1+\infty}$, and moreover that the Kronig identity Eq. (58) for $W^2$ is only ‘the tip of an iceberg’. There is a beautiful generalization of the Kronig identity to the full $W_{1+\infty}$-algebra.

As in Section 2 we only consider one branch $\psi = \psi_+$ of fermions here.

---

6 this specific form of the ‘regularized’ local interacting potential results in simple explicit formulas for the interacting fields

7 As in Section 2 we only consider one branch $\psi = \psi_+$ of fermions here.
4.1 Definition of \( W_{1+\infty} \)

The \( W_{1+\infty} \)-algebra is a central extension of a Lie algebra \( w_{\infty} \) defined as follows. Consider the differential operators

\[
w_p^s := e^{-i p x / 2} (-i \partial_x)^{s-1} e^{-i p x / 2}
\]

for \( p \in \Lambda^* \) and \( s \in \mathbb{N} \). It is easy to see that these operators generate a Lie algebra with the Lie bracket given by the commutator. To write the commutator relations for these operators without a lengthy derivation it is convenient to proceed less formally and introduce the ‘generating function’

\[
w_p(a) = \sum_{s=1}^{\infty} \left( -ia \right)^{s-1} \frac{1}{(s-1)!} w_p^s, \quad p \in \Lambda^*, \tag{61}\]

i.e. \( w_p(a) = e^{-i p x / 2} e^{-a \partial_x} e^{-i p x / 2} \), is to be understood in the sense of formal power series in \( a \).

We then compute

\[
w_p(a)w_q(b) = e^{-i p x / 2} e^{-a \partial_x} e^{-i q x / 2} e^{-b \partial_x} e^{-i q x / 2} = e^{-i p x / 2} e^{-i q(x-a)/2} e^{-a \partial_x} e^{-b \partial_x} e^{-i p(x+b)/2} = e^{i(qa-pb)/2} w_{p+q}(a + b)
\]

and thus obtain

\[
[w_p(a), w_q(b)] = (e^{i(qa-pb)/2} - e^{-i(qa-pb)/2}) w_{p+q}(a + b). \tag{62}
\]

The Lie algebra \( w_{\infty} \) is defined by Eqs. (61) and (62) (these relations do not really depend on the ‘generating function’ argument we used to write them down). Similarly, the \( W_{1+\infty} \)-algebra is generated by elements \( W_p^s \) collected in a ‘generating function’

\[
W_p(a) = \sum_{s=1}^{\infty} \left( -ia \right)^{s-1} \frac{1}{(s-1)!} W_p^s, \quad p \in \Lambda^*, \tag{63}\]

together with a central element \( c \),

\[
[W_p(a), c] = 0, \tag{64}\]

and the relations

\[
[W_p(a), W_q(b)] = (e^{i(qa-pb)/2} - e^{-i(qa-pb)/2}) W_{p+q}(a + b) + c \delta_{p,-q} \frac{\sin(\frac{p}{2}(a + b))}{\sin(\frac{\pi}{L}(a + b))}. \tag{65}
\]

Remark: One can check by direct calculation that the bracket defined in Eq. (65) obeys the Jacobi identity. We will of course give a representation of this Lie algebra next which will make it clear in what sense we interpret the generators of this algebra as operators. We also note that Eqs. (65) and (63) imply

\[
[W_p^{1}, W_q^{1}] = (p - q)W_{p+q}^{1} + \delta_{p,-q} \frac{L}{2\pi} \frac{c}{12} p \left( p^2 - \left( \frac{2\pi}{L} \right)^2 \right), \tag{66}\]

which shows that \( W_p^{1} \) and \( c \) generate the Virasoro algebra \( \text{Vir} \).\footnote{To see that these are indeed the usual defining relations of \( \text{Vir} \), set \( L_p = W_p^{1} \) and \( L = 2\pi \) so that \( \Lambda^* = \mathbb{Z} \).} \( \text{Vir} \) is a Lie subalgebra of \( W_{1+\infty} \).
4.2 Fermion representation of $W_{1+\infty}$

We can naturally identify the differential operators in Eq. (60) with operators on the Hilbert space $L^2(S^1_L)$ defined as

$$w_p^s e_k = \left(k - \frac{p}{2}\right)^{s-1} e_{k-p} \quad \forall k \in \Lambda_0^*$$

for $e_k$ given by Eq. (23). From our general results in Section 2.3 we thus expect that the operators $d\Gamma(w_p^s)$ should give a representation of a central extension of $w_\infty$. Indeed one can prove the

**Theorem:** The operators

$$d\Gamma(w_p^s)$$

with $w_p^s$ as in Eq. (67), and $c \equiv I$, give a unitary highest weight representation of $W_{1+\infty}$, i.e. the relations in Eqs. (64)–(63) and in addition, using the notation $d\Gamma(w_p^s) \equiv W_p^s$, we have

$$(W_p^s)^* = W_{-p}^s \quad \forall p$$

and

$$W_p^s \Omega = 0 \quad \forall p \geq 0$$

hold true for all $s \in \mathbb{N}$ on some common, dense, invariant domain.

The use of the notation $W_p^s$ to denote the generators of $W_{1+\infty}$ in this particular representation will not cause any confusion as no other representations are introduced here. To prove this theorem one only needs to show that all $w_p^s \in g_1$ so that the general results in Section 2 apply. In particular, the relations in Eq. (65) follow from Eqs. (16)–(17) where the central term is obtained from

$$iS(w_p(a), w_q(b)) = \sum_{k \in \Lambda_0^*} <e_k, (P_- w_p(a) P_+ w_q(b) P_--P_- w_q(b) P_+ w_p(a) P_-) e_k>$$

by a straightforward computation (use $w_p(a) e_k = e^{-ia(k - \frac{p}{2})} e_{k-p}$, $P_\pm e_k = \theta(\pm k) e_k$ and $<e_k, e_{k'} > = \delta_{k,k'}$). Moreover, Eq. (69) follows from Eq. (14) and $(w_p^s)^* = w_{-p}^s$ (the latter can be easily checked using the definition Eq. (67)), and Eq. (70) follows from Eq. (18).

We also note

$$[W_p^s, \psi^*(k)] = \left(k - \frac{p}{2}\right)^{s-1} \psi^*(k-p)$$

which follows from Eqs. (16) and (67).

4.3 Boson representation of $W_{1+\infty}$

We recall the Kronig identities which played a central role for solving the Luttinger model, $W_0^2 = \frac{\pi}{2} \sum_{p \in \Lambda^*} \hat{\rho}(p) \hat{\rho}(-p) \hat{x}$. It is well-known that this identity has a generalization to the Virasoro algebra i.e. all operators $W_p^2$ (this is the Sugawara construction; see e.g. [GO]). We now ask: Is there a generalization of the Kronig identity to the full $W_{1+\infty}$-algebra?

The desired result is summarized in the following

**Theorem:** Let

$$W_\epsilon(y; a) = N(a) \left(\hat{\varphi}_\epsilon(y + \frac{\alpha}{2})\varphi_\epsilon^{-1}(y - \frac{\alpha}{2}) - I\right),$$

(72)
with the normalization constant
\[ N(a) = \frac{i}{2L \sin(\frac{\pi}{2}a)}. \tag{73} \]

Then the operators defined by the following equation
\[ W_p(a) := \lim_{\varepsilon \to 0} \int_{-L/2}^{L/2} dy \, e^{-ipy} \mathcal{W}_\varepsilon(y; a) = \sum_{s=1}^{\infty} \frac{(-ia)^{s-1}}{(s-1)!} W_p \tag{74} \]
equal the operators in Eq. (68):
\[ W_s^\alpha = W_p^s \text{ for all } s \in \mathbb{N} \text{ and } p \in \Lambda^*. \]

To see that this theorem allows us to compute formulas for all the operators \( W_p^s \) in terms of the boson operators \( \hat{\rho}(p) \) we write
\[ [\ldots] = d\Gamma(f_{y+\frac{\varepsilon}{2},\varepsilon}) - d\Gamma(f_{y-\frac{\varepsilon}{2},\varepsilon}) = -2\pi \left[ a\rho_\varepsilon(y) + \frac{a^3}{24} \partial_y^2 \rho_\varepsilon(y) + \ldots \right] \]
where \( \partial_y = \partial/\partial y \), and
\[ \rho_\varepsilon(y) = d\Gamma(\delta_{y,\varepsilon}) = \frac{1}{L} \sum_{p \in \Lambda} \hat{\rho}(p) e^{ipy} e^{-|p|\varepsilon} \tag{75} \]
is the regularized fermion current in position space. Inserting this in the l.h.s. of Eq. (74), expanding in powers of \( a \) and comparing with the r.h.s. of Eq. (74) one obtains
\[ W^1_p = \int_{-L/2}^{L/2} dy \, e^{-ipy} \hat{\rho}(y) \hat{\rho}(y) \mid_{\varepsilon \to 0} = \hat{\rho}(p) \]
\[ W^2_p = \pi \int_{-L/2}^{L/2} dy \, e^{-ipy} \hat{\rho}(y) \hat{\rho}(y) \mid_{\varepsilon \to 0} = \frac{1}{2} \left( \frac{2\pi}{L} \right) \sum_{q \in \Lambda^*} \hat{\rho}(q) \hat{\rho}(p-q) \]
\[ W^3_p = \frac{4\pi^2}{3} \int_{-L/2}^{L/2} dy \, e^{-ipy} \left( \hat{\rho}(y)^3 - \frac{1}{4L^2} \hat{\rho}(y)^2 \right) \mid_{\varepsilon \to 0} = \frac{1}{3} \left( \frac{2\pi}{L} \right)^2 \sum_{q_1,q_2 \in \Lambda^*} \hat{\rho}(q_1) \hat{\rho}(q_2) \hat{\rho}(p-q_1-q_2) + \ldots \]
\[ \vdots \]
\[ W^{s+1}_p = \frac{1}{s+1} \left( \frac{2\pi}{L} \right)^s \sum_{q_1,\ldots,q_s \in \Lambda^*} \hat{\rho}(q_1) \cdots \hat{\rho}(q_s) \hat{\rho}(p-q_1-\cdots-q_s) + \ldots \]
where ‘+...’ refers to those terms involving fewer \( \hat{\rho} \)’s.

We now sketch how this theorem can be proved by using the results summarized in Section 2. We recall Eq. (17) which shows that \( L^{-1/2} \varphi_\varepsilon^\dagger(y) = L^{-1/2} x^0 \Gamma(e^{i f_{y,\varepsilon}}) \hat{\rho}(y) \) equals a regularized fermion operator \( \psi^\dagger(y) \). Using that the argument is simple: we compute the commutator of \( \mathcal{W}_\varepsilon(y; a) = N(a) \hat{\Gamma}(e^{i f_{y,\varepsilon}}) \hat{\varphi}_\varepsilon(x) \) with \( \varphi_\varepsilon^\dagger(x) \) using Eqs. (34), (35) and (44). We obtain
\[ [\mathcal{W}_\varepsilon(y; a), \varphi_\varepsilon^\dagger(x)] = (\cdots) \hat{\Gamma}(e^{i f_{y,\varepsilon} + f_{y,\varepsilon}^2)} \hat{\varphi}_\varepsilon(x) \]
with
\[ (\cdots) = N(a) \left[ \frac{\sin \frac{\pi}{2}(y - \frac{a}{2} - x + i\varepsilon)}{\sin \frac{\pi}{2}(y - \frac{a}{2} - x + i\varepsilon)} - c.c. \right] = \frac{i}{2L} \left[ \cot \frac{\pi}{2}(y - \frac{a}{2} - x + i\varepsilon) - c.c. \right] \]
\footnote{At this point the reason for our normalization constant \( N^1(a) \) becomes obvious.}
Recalling Eq. (74) and (71) we thus see that

\[ \pm \frac{i}{2L} \cot \frac{\pi}{L}(y - \frac{a}{2} - x \pm i\bar{\varepsilon}) = \frac{1}{2L} + \delta_{\bar{x},\bar{\varepsilon}}(y - \frac{a}{2}), \tag{77} \]

which implies \((\cdots) = \delta_{\bar{x},\bar{\varepsilon}}(y - \frac{a}{2})\). We conclude that

\[ \lim_{\varepsilon \downarrow 0} \int_{-L/2}^{L/2} dy \, e^{-ipy}[\mathcal{W}_{\varepsilon}(y; a), \varphi_{\varepsilon}^{1}(x)] = e^{-ip(x + \frac{a}{2})} \Gamma(e^{if_{x+a,\varepsilon}})^{\varepsilon} \]

equivalent with

\[ [\mathcal{W}(a), \varphi_{\varepsilon}^{1}(x)] = e^{-ip(x + \frac{a}{2})} \varphi_{\varepsilon}^{1}(x + a) \]

Using now

\[ \hat{\psi}^{*}(k) = \lim_{\varepsilon \downarrow 0} \frac{1}{L} \int_{-L/2}^{L/2} dx \, e^{ikx} \varphi_{\varepsilon}^{1}(y) \]

which follows from Eq. (47) we conclude that

\[ [\mathcal{W}_{p}(a), \hat{\psi}^{*}(k)] = e^{-ia(k - \frac{p}{2})}\hat{\psi}^{*}(k - p). \]

Recalling Eq. (74) and (71) we thus see that \([\mathcal{W}^{s}_{p}, \hat{\psi}^{*}(k)] = [W^{s}_{p}, \psi^{*}(k)]\) always. Moreover, by definition of \(\mathcal{\bar{z}} \cdots \mathcal{\bar{z}}\) we also get \(\mathcal{W}(a)p^{\Omega} = 0\) for all \(p \geq 0\), i.e. \(\mathcal{W}^{s+1}_{p}\Omega = 0\) for all \(p \geq 0\). It is also easy to check that \((\mathcal{W}^{s}_{p})^{*} = \mathcal{W}^{s}_{-p}\), and the theorem therefore follows by applying the following Lemma to \(A = \mathcal{W}^{s}_{p} - \mathcal{W}^{s}_{-p}\) for \(p \geq 0\).

**Lemma:** [CR] For linear operators \(A\) on \(\mathcal{F}\), \([A, \hat{\psi}^{*}(k)] = 0\) for all \(k \in \Lambda^{s}_{0}\), and \(A\Omega = 0\) imply \(A = 0\).

## 5 Anyons and the Calogero-Sutherland model

### 5.1 Boson-anyon correspondence

In this section we discuss how to generalize the boson-fermion correspondence to anyons.

#### 5.1.1 Construction of anyon field operators

To construct anyons we have to extend the relations Eq. (46) to any non-integer \(\nu\nu'\). The naive idea would be to define \(\varphi_{\nu}^{\varepsilon}(y) = \mathcal{\bar{z}} \Gamma(e^{i\nu f_{y,\varepsilon}})^{\varepsilon}\) for arbitrary \(\nu\), and these objects would then (formally) obey the desired relations. However, since the functions \(e^{i\nu f_{y,\varepsilon}(x)}\) are not periodic if \(\nu\) is not an integer, the operator \(\Gamma(e^{i\nu f_{y,\varepsilon}})\) does not exist in the Fock representation in general. This technical difficulty indicates that anyon field operators are delicate objects whose consistent construction requires some care.

To circumvent this problem, we note that \(S(f_{1}, f_{2})\) Eq. (29) is invariant under changes \(\alpha_{i} \rightarrow \alpha_{i}\lambda\) and \(w_{i} \rightarrow w_{i}/\lambda\) with an arbitrary scaling parameter \(\lambda\). We use this to construct a function \(\tilde{f}_{y,\varepsilon}(x)\) which has the following properties,

\[ \begin{align*}
(i) & \quad e^{i\nu f_{y,\varepsilon}(x)} \text{ is periodic for all } \nu, \\
(ii) & \quad S(\tilde{f}_{y,\varepsilon}, \tilde{f}_{y,\varepsilon}) = S(f_{y,\varepsilon}, f_{y,\varepsilon}).
\end{align*} \tag{78} \]

\(^{10}\)This is easily seen by expanding the l.h.s as a Taylor series in \(e^{\pm i(y-x)2\pi/L}e^{-c2\pi/L}\).
Since the functions $\nu \tilde{f}_{y,\varepsilon}(x)$ have winding numbers different from zero, the first requirement can only be fulfilled for $\nu$ values which are an integer multiple of some fixed number $\nu_0 > 0$. Then
\[ \tilde{f}_{y,\varepsilon}(x) = \frac{2\pi}{L\nu_0} x - \frac{2\pi\nu}{L} y + \alpha_{y,\varepsilon}^+(x) + \alpha_{y,\varepsilon}^-(x) \] (79)
has the desired properties. Thus the operators
\[ \varphi_{\nu_0}^{\varepsilon}(y) := \sqrt{\Gamma(e^{i\nu \tilde{f}_{y,\varepsilon}})} \overline{\varphi_{\nu_0}^{-\varepsilon}(y)} \] (80)
are well-defined for $\varepsilon > 0$, and they obey the exchange relations Eq. (46) but now for all $\nu, \nu'$ which are integer multiples of $\nu_0$. Thus the theory of loop groups provides a simple and rigorous construction of regularized free anyon field operators $\varphi^{\nu}(x)$.

5.1.2 Anyon correlation functions

We now can easily compute all anyon correlations functions: Eqs. (30), (32), and (44) imply
\[ \langle \Omega, \varphi^{\nu_1}(y_1) \cdots \varphi^{\nu_N}(y_N) \Omega \rangle = \delta_{\nu_1 + \ldots + \nu_N, 0} J_{\nu_1, \ldots, \nu_N}(y_1, \ldots, y_N). \] (84)

5.2 Second quantized Calogero-Sutherland Hamiltonian

We recall the operators $W_{s+1}^{p} \equiv W_{p=0}^{s+1}$ which (formally) obey (see Eq. (71))
\[ [W_{s+1}^{p}, \psi^{*}(x)] = i^{s} \frac{\partial^{s}}{\partial x^{s}} \psi^{*}(x) \] (85)
where $\psi^{*}(x) = \varphi^{1}(x)$ are the fermion operators. We now try to find generalizations of these operators to the case of anyons. Without loss of generality, we assume in the following $\nu = \nu_0 > 0$.

We first generalize the operators $W_{s+1}^{p} \equiv W_{p=0}^{s+1}$ defined in Eqs. (72)–(74) to general $\nu$.
\[ W^{\nu}(a) := \lim_{\varepsilon \downarrow 0} \int_{-L/2}^{L/2} dy W^{\nu}_{\varepsilon}(y; a) = \sum_{s=1}^{\infty} \frac{(-ia)^{s-1}}{(s-1)!} W_{s}^{\nu, a} \] (86)
with
\[ W^{\nu}_{\varepsilon}(y; a) := N^{\nu}(a) \left( e^{i\nu \tilde{f}_{y+a, \varepsilon}} \tilde{f}_{y, \varepsilon} \right)^{\varepsilon} - I \] (87)
and
\[ N^{\nu}(a) = \frac{i}{2L\nu^{2} \cos^{2}(\frac{x}{2}a) \tan(\frac{x}{2}a)}. \] (88)
Similarly as in the Proof of Theorem 4.3 we compute

\[ [W^\nu_\xi(y; a), \varphi^\nu_\xi(x)] = (\cdots) \times \Gamma(e^{i\nu[f_{\xi,\xi} + f_{\xi+a,\xi'} - f_{\xi,\xi'}]}) \times \]

with

\[ (\cdots) := N^\nu(a) \left[ \left( \frac{\sin \frac{\pi}{L} (y + a + x + i \bar{\varepsilon})}{\sin \frac{\pi}{L} (y + x + i \bar{\varepsilon})} \right)^{\nu^2} - c.c. \right] \]

\[ = N^\nu(a) \cos^{\nu^2} \left( \frac{x}{L} a \right) (1 + \tanh(\frac{x}{L} a) \cot \frac{\pi}{L} (y - x + i \bar{\varepsilon}))^{\nu^2} + c.c. \]

and \( \bar{\varepsilon} = \varepsilon + \varepsilon' \). Expanding this in powers of \( a \) and using \( \cot^2(z) = -1 - \frac{d}{dz} \cot(z) \) and Eq. (77) we obtain

\[ (\cdots) = \delta_{x,\bar{\varepsilon}}(y) - \frac{1}{2} (\nu^2 - 1) a \partial_y \delta_{x,\bar{\varepsilon}}(y) + O(a^2). \]

Thus

\[ [W^\nu(a), \varphi^\nu_\xi(x)] = \varphi^\nu_\xi(x + a) + i\pi \nu (\nu^2 - 1) a \times [\tilde{\rho}_{\xi}(x + a) - \tilde{\rho}_{\xi}(x)] \varphi^\nu_\xi(x + a) \times + O(a^3). \]

Comparing now equal powers of \( a \) on both sides of Eq. (89) we see that the generalization of Eq. (85) to anyons holds true only for \( s = 0, 1 \),

\[ [W^{\nu,s+1}, \varphi^\nu_\xi(x)] = \nu^{1-s} \frac{\partial^s}{\partial x^s} \varphi^\nu_\xi(x) \quad s = 0, 1 \]

but for \( s > 2 \) we get correction terms, e.g.

\[ [W^{\nu,3}, \varphi^\nu_\xi(x)] = \frac{i^2}{\nu} \frac{\partial^2}{\partial x^2} \varphi^\nu_\xi(x) + 2\pi i (\nu^2 - 1) \times [\tilde{\rho}_{\xi}(x)'] \varphi^\nu_\xi(x) \times \]

where

\[ \tilde{\rho}_{\xi}(y) = -\frac{1}{2\pi} d\Gamma(\partial_y f_{\xi,\xi}) = \rho_{\xi}(y) + \frac{\nu - 1}{L} Q. \]

Here and in the following we only consider the first non-trivial case \( s = 2 \). We now need to cancel the second term in Eq. (92). This can be partly done by an operator

\[ C \propto i \lim_{\varepsilon \to 0} \int_{-L/2}^{L/2} dy \times \rho^+_{y,\xi} \partial_y \rho^-_{y,\xi} \]

where \( \rho^\pm_{y,\xi} = d\Gamma(\delta^\pm_{y,\xi}) \). By explicit computation similar to the one above one can prove that

\[ C \varphi^\nu_\xi(x) + \varphi^\nu_\xi(x)C = 2\pi i \times [\tilde{\rho}_{2\varepsilon}(x)'] \varphi^\nu_\xi(x) \times + 2 \times C \varphi^\nu_\xi(x) \times . \]

The first term can be used to cancel the second term in Eq. (92). The second term seems somewhat strange, however, it disappears when applying this equation to vectors of the form \( R^w\Omega, w \) an arbitrary integer (in contrast to the first term!). We thus see that the operator

\[ \mathcal{H}^{\nu,3} = \nu W^{\nu,3} + (1 - \nu^2) C \]

obeys the relation

\[ [\mathcal{H}^{\nu,3}, \varphi^\nu_\xi(x)] R^w\Omega \simeq i^2 \frac{\partial^2}{\partial x^2} \varphi^\nu_\xi(x) R^w\Omega \]
where ‘≃’ means ‘equal up to a regular term which vanished for ε ↓ 0’. This seems to be the best we can do to generalize the relation Eq. (85) for \( s = 2 \) to the anyon case. However, to fully appreciate this operator \( H^{\nu,3} \), one has to extend the computation above to a product of multiple anyon operators \([CL]\). One thus obtains the following

**Theorem:** There exists an operator \( H^{\nu,3} \) which obeys the following relations,

\[
[H^{\nu,3}, \varphi_\varepsilon^\nu(y_1) \cdots \varphi_\varepsilon^\nu(y_N)] R^w \Omega \simeq H^{\varepsilon}_N, \nu^2 \cdot \varphi_\varepsilon^\nu(y_1) \cdots \varphi_\varepsilon^\nu(y_N) R^w \Omega
\]  

(96)

for all integer \( w \), where

\[
H^{\varepsilon}_N, \nu^2 = -\sum_{k=1}^N \frac{\partial^2}{\partial y_k^2} + \sum_{1 \leq k < \ell \leq N} 2\nu^2 (\nu^2 - 1) V_\varepsilon(y_k - y_\ell)
\]

(97)

with

\[
V_\varepsilon(r) = -\frac{\partial^2}{\partial r^2} \log b_\varepsilon(r)
\]

(98)

is a regularized version of the Calogero-Sutherland Hamiltonian defined in Eq. (2).

### 5.2.1 Constructing eigenfunctions for the Calogero-Sutherland Hamiltonian

We now sketch how the theorem in the previous section can be used to find eigenfunctions of the Calogero-Sutherland Hamiltonian: Suppose we found a common eigenvector \( \eta \) of the operators \( H^{\nu,3} \) and \( Q \),

\[
H^{\nu,3} \eta = E \eta, \quad Q \eta = N \eta.
\]

(99)

Then the theorem in Section 5.2 and the relation

\[
H^{\nu,3} \Omega = 0,
\]

(100)

imply that

\[
F_\eta(x_1, \ldots, x_N) = \lim_{\varepsilon \downarrow 0} \langle \eta, \varphi_\varepsilon^\nu(x_1) \cdots \varphi_\varepsilon^\nu(x_N) \Omega \rangle,
\]

(101)

is an eigenfunction of the Calogero-Sutherland Hamiltonian with the eigenvalue \( E \). This follows immediately if we sandwich Eq. (96) between \( \eta \) and \( \Omega \), use Eq. (100) and take the limit \( \varepsilon \downarrow 0 \).

To find such vectors \( \eta \) one again can use Eqs. (96) and (100). The idea is to consider the Fourier modes of the anyon field operators,

\[
\hat{\varphi}_\varepsilon^\nu(p) = \lim_{\varepsilon \downarrow 0} \int_{-L/2}^{L/2} \ dx \ e^{i\pi \nu Qy/L} \varphi_\varepsilon^\nu(x) e^{i\pi \nu Qy/L} e^{ipx}
\]

(102)

where \( p \in \Lambda^* \). Note that the anyon field operators are not periodic, and we have to remove the non-periodic factors before Fourier transformation. Using the theorem in Section 5.2 one then can show that linear combinations of vectors

\[
\eta = \hat{\varphi}_\varepsilon^\nu(p_1) \cdots \hat{\varphi}_\varepsilon^\nu(p_m) R^{N-m} \Omega, \quad p_k - p_{k+1} \geq 0, \quad 0 \leq m \leq N
\]

obey Eq. (96) which eigenvalues \( E \) one can easily compute. The simplest case is \( \eta = R^N \Omega \) where we obtain a eigenstate which is essentially the known groundstate of the Calogero-Sutherland model \([Su]\) (up to a phase factor which corresponds to a non-zero center-of-mass ‘motion’). In general, we obtain all the eigenfunctions of the Calogero-Sutherland model \([CL]\) which were found originally by Sutherland \([Su]\).
5.2.2 Finite temperature anyons and the elliptic Calogero-Sutherland model

Recently the construction described above was generalized to the elliptic Calogero-Sutherland model i.e. the Hamiltonians in Eq. (2) with an interaction potential

\[
V(r) = -\frac{\partial^2}{\partial r^2} \log \left( \sin \left( \frac{\pi r}{L} \right) \prod_{n=1}^{\infty} [1 - 2q^{2n} \cos(\frac{2\pi r}{L}) + q^{4n}] \right)
\] (103)

(0 ≤ q < 1) which is equal, up to an additive constant, to the Weierstrass elliptic function \( \wp(r) \) with periods \( L/2 \) and \( i \log(1/q) \). The idea was to consider finite temperature anyons i.e. construct the anyons as described above but using a quasi-free reducible fermion representation corresponding to a finite temperature \( 1/\beta \), as described in Section 2.3.4. The motivation came partly from the geometric interpretation of this fermion representation which we describe in Section 7.

Introducing a non-zero temperature quasi-free state only changes the normal ordering prescription i.e. all formulas in Section 5.1.2 remain true except Eq. (83) which is changed to

\[
b_\varepsilon(r) = -2ie^{-\pi\varepsilon/L} \sin \left( \frac{\pi r}{L} + i\varepsilon \right) \prod_{n=1}^{\infty} [1 - 2q^{2n} e^{-2\pi\varepsilon/L} \cos(\frac{2\pi r}{L}) + q^{4n} e^{-4\pi\varepsilon/L}] \] (104)

where \( q = \exp(-\beta L/(2\pi)) \). With that modification, the theorem in Section 5.2 remains true, and this gives a second quantization of the elliptic Calogero-Sutherland system. To further generalize the results in [CL] there is one complication: for \( \beta < \infty \), the operator \( H_N^{\nu,\varepsilon} \) no longer obeys Eq. (100), and thus the argument leading to a solution algorithm seems to fail. However, one can prove that the weaker condition

\[
\langle \Omega, [H_N^{\nu,\varepsilon}, \varphi^\nu_\varepsilon(x_N)^* \cdots \varphi^\nu_\varepsilon(x_1)^* \varphi^\nu_\varepsilon(x_1) \cdots \varphi^\nu_\varepsilon(x_N)] \rangle = 0
\] (105)

still holds, and this is enough to obtain a solution algorithm: the theorem in Section 5.2 and this relation imply

\[
H_N^{\nu,\varepsilon}(x) F_{N,\nu^2}^{\varepsilon,\nu}(y, x) \simeq H_N^{\nu^2}(y) F_{N,\nu^2}^{\varepsilon,\nu}(y, x)
\] (106)

where \( F_{N,\nu^2}^{\varepsilon,\nu}(y, x) = \langle \Omega, \varphi^\nu_\varepsilon(x_N)^* \cdots \varphi^\nu_\varepsilon(x_1)^* \varphi^\nu_\varepsilon(x_1) \cdots \varphi^\nu_\varepsilon(x_N) \rangle \) and the Hamiltonians on the two sides act on different arguments \( x \) and \( y \), as indicated. From Eqs. (84) and (82) we obtain

\[
F_{N,\nu^2}^{\varepsilon,\nu}(y, x) = \frac{\prod_{1 \leq j < j' \leq N} b_{2\varepsilon}(y_j - y_{j'}) \nu^2 \prod_{1 \leq k < k' \leq N} b_{2\varepsilon}(x_k - x_{k'}) \nu^2}{\prod_{j,k=1}^{N} b_{\varepsilon,\nu}(y_j - x_k) \nu^2}.
\] (107)

This remarkable identity Eq. (106) together with Eq. (107) can be used to construct eigenfunctions of the elliptic Calogero-Moser Hamiltonian as linear combinations of the anyon correlation functions

\[
\lim_{\varepsilon \downarrow 0} \langle \Omega, \varphi^\nu_\varepsilon(p_N)^* \cdots \varphi^\nu_\varepsilon(p_1)^* \varphi^\nu_\varepsilon(x_1) \cdots \varphi^\nu_\varepsilon(x_N) \rangle
\]

which can be computed from \( F_{N,\nu^2}^{\varepsilon,\nu}(y, x) \) by Fourier transformation in the variables \( y \) and taking the limits \( \varepsilon, \varepsilon' \downarrow 0 \). This provides a generalization of the solution of the Calogero-Sutherland model [Su] to the elliptic case [L3].
PART B: LOOP GROUPS, 1+1 DIMENSIONAL QFT AND RIEMANN SURFACES

6 Overview

This part presents two main themes. First we give a summary of some of the literature on various applications of loop group representation theory in quantum field theory. This list is not exhaustive and is confined to examples in which we have had some involvement. Second we extend the elementary discussion of the examples of the previous section. This extension consists of two related discussions. First the K.M.S. state (or finite temperature state) on the fermion algebra over $L^2(S^1)$ for the free Dirac Hamiltonian is shown to be interpretable geometrically as describing fermions on a torus. This then leads into a discussion of fermions on higher genus Riemann surfaces. This latter exposition is more mathematically sophisticated and assumes some knowledge of the geometry of Riemann surfaces. The idea here is to sketch how one constructs quantum fields and vertex operators on Riemann surfaces from representations of loop groups.

6.1 A guide to various examples

We introduce some examples of quasifree representations of the fermion field algebra following on from our exposition in 2.3.2 and 2.3.4.

**Notation**

(i) We let $P_-$ denote the projection on $L^2(\mathbb{R}, \mathbb{C}^N)$ (resp. $L^2(S^1, \mathbb{C}^N)$) onto functions which are boundary values of functions holomorphic in the lower half plane in $\mathbb{C}$ (resp. exterior of the unit disc).

(ii) Let $A(\beta)$ denote the operator on $L^2(S^1, \mathbb{C}^N)$ (resp. $L^2(\mathbb{R}, \mathbb{C}^N)$) which is given by multiplication by the function

$$k \to e^{-\beta k}/(1 + e^{-\beta k}), \quad k \in \mathbb{Z} \ (\text{resp.} k \in \mathbb{R}) \quad (\beta \geq 0)$$

on the Fourier transform.

(iii) Let $A(m)$ denote the operator on $L^2(\mathbb{R}, \mathbb{C}^N)$ given by multiplication on the Fourier transform by the function

$$p \to (1 - p/(p^2 + m^2)^{1/2})/2, \quad (m \geq 0).$$

These operators arise respectively as follows.

(i) The operator $P_-$ is the spectral projection of the massless Dirac Hamiltonian corresponding to the negative part of the spectrum. Then the resulting representation of $A$ is the usual ‘infinite wedge representation’ or equivalently, that obtained by ‘filling the Dirac sea’.

(ii). The operator $A(\beta)$ defines a K.M.S. state (or temperature state at inverse temperature $\beta$) on the Fermion algebra $A$ for the one parameter group of automorphisms generated by the massless Dirac operator (see 2.3.4).

(iii). The operator $P_{A(m)}$ is the spectral projection of the massive Dirac Hamiltonian corresponding to the interval $(-\infty, -m]$.

In this exposition we cannot provide details of all of the applications of loop groups to quantum field theory. The following is a brief guide to a number of papers which deal with models in 1 + 1-dimensional space-time.

- The standard free field construction of the basic representation of the affine Lie algebra $A^{(1)}_{N-1}$ can be obtained by taking the underlying Hilbert space to be $L^2(S^1, \mathbb{C}^N)$ and the
fermion representation to be \( \pi_{P^-} \). The construction is a simple generalization of the previous discussion of the wedge representation of the loop group of \( U(1) \) to the construction of the wedge representation of the loop group of \( U(N) \). The affine Lie algebra \( \mathcal{A}^{(1)}_{N-1} \) arises as the Lie algebra of this central extension of the loop group of \( U(N) \) which is acting on the fermion Fock space for \( \pi_{P^-} \). There is also an analogous construction of projective representations of the loop groups of \( SU(N) \) and \( SO(N) \) (see [CR]).

- The Cayley transform from the circle to the real line may be used to realise the preceding affine Lie algebra representation as the infinitesimal version of a projective representation of the group \( \text{Map}(\mathbb{R}, U(N)) \) of smooth maps from the real line into \( U(N) \). This representation acts on the fermion Fock space over \( L^2(\mathbb{R}, \mathbb{C}^N) \) [CR].

- Temperature or KMS states on \( \mathcal{A}^{(1)}_{N-1} \) may be obtained by realizing this affine algebra in the representation space of free fermions at inverse temperature \( \beta \). Here the underlying Hilbert space is \( L^2(S^1, \mathbb{C}^N) \) and the fermion representation is \( \pi_{A(\beta)} \). The corresponding state on the fermion algebra \( \mathcal{A} \) over \( L^2(S^1, \mathbb{C}^N) \) is a K.M.S. state. The cyclic vector (or vacuum) for the resulting representation of the C*-algebra generated by the operators

\[
\{ \rho_{A(\beta)}(\varphi) : \varphi \in \text{Map}(S^1, U(N)) \}
\]

defines a K.M.S. state on this algebra. Thus, in this way, we obtain temperature states on loop groups. Generalising from Subsection 2.6 we can use blip functions to reconstruct the fermions in the non-zero temperature (K.M.S.) representation \( \pi_{A(\beta)} \). When \( N = 1 \) the correlation functions of the approximate fermion operators give theta function identities [CHa]. Interestingly this example can be re-interpreted geometrically as quantum field theory on the torus [CHI], a fact which we will explain in more detail in the next section.

- Starting with the underlying Hilbert space \( L^2(S^1, \mathbb{C}) \oplus L^2(S^1, \mathbb{C}) \) and the fermion representation \( \pi_{PA(\beta)} \), we can use boson algebra automorphisms to ‘twist’ the vertex operators so as to obtain the fields of the non-zero temperature Luttinger model [CHa].

- By using massive fermions over \( L^2(\mathbb{R}, \mathbb{C}^N) \), that is, the representation \( \pi_{A(m)} \) of the fermion algebra over \( L^2(\mathbb{R}, \mathbb{C}^N) \) one may obtain a type III\(_1\) factor representation of the group \( \text{Map}_0(\mathbb{R}, U(N)) \) consisting of smooth maps \( \varphi : \mathbb{R} \to U(N) \) with \( \varphi(0) = 1 \). This may be used to construct sine-Gordon fields at the critical value of the coupling constant where the theory is free [CR].

Note that in this last example if we put \( m = 0 \) we can construct the massless Thirring model [CRW]. A less complicated version of this construction is what we used in Subsection 3.1 for the Luttinger model. Finally we remark on a basic limitation of this approach. In all the examples there is somewhere in the background a free quantum field theory. A situation which we would like to understand and for which the methods of this paper do not apply is the massive Thirring model or equivalently, the sine-Gordon model for general coupling constant.

### 7 Free fields on the torus

In this section we provide a re-interpretation of the quasifree representation of the fermions at inverse temperature \( \beta \) as a theory of free fermions on the torus. This is the simplest case of the more general theory of free fields on Riemann surfaces. We regard the torus as constructed from two annuli by joining along the boundary. Equivalently (at least topologically) this is the same as joining two cylinders to form the torus.
We let $R$ be the annulus (in the complex plane) $\{w : e^{-\beta/2} \leq |w| \leq 1\}$ and $\bar{R}$ be the annulus $\{w : 1 \leq |w| \leq e^{-\beta/2}\}$ where $\beta > 0$. We may now form the torus $\Sigma = R \sqcup \partial R \sqcup \bar{R}$ by first joining $R$ and $\bar{R}$ along their common boundary and then identifying the remaining boundary circles. Notice that $\Sigma$ then possesses an anticonformal involution: $w^\pi = 1/\bar{w}$ (an example of the so-called Schottky involution).

With respect to the canonical homology basis

$$A = \{e^{\beta/2+i\theta}, 0 \leq \theta \leq 2\pi\}, \quad B = \{r \in \mathbb{R} : e^{-\beta/2} < r < e^{\beta/2}\},$$

where $w = re^{i\theta}$ are polar coordinates, we may think of the torus as the complex plane modulo the lattice $2\pi \mathbb{Z} + i\beta(\mathbb{Z} + 1/2)$.

Introduce the four spin structures (square roots of the cotangent bundle) $L^\alpha$ where $\alpha = (0,0), (1/2,0), (0,1/2), (1/2,1/2)$ denotes the so-called theta characteristics which summarise the behaviour of their sections as we shift through the periods of the $A$ and $B$ cycles. These sections may be defined as functions on the complex plane with well defined equivariance properties under the action of the translations defining the lattice. Specifically these are: periodic under both generators, periodic under one and antiperiodic under the other and antiperiodic under both respectively. So for example for $\alpha = (0,1/2)$ this means the sections regarded as functions on the complex plane satisfy

$$f(w + 2\pi) = f(w), \quad f(w + i\beta) = -f(w).$$

The Hilbert spaces for the fermionic algebras live on the boundary of $R$ (or equivalently $\bar{R}$)

We form pre-Hilbert spaces of smooth sections of each $L^\alpha$ restricted to the boundary with $L^2$ norms defined as follows. Let $K = (L^\alpha)^2$ denote the cotangent bundle with transition functions $z_{\gamma \delta}$ then $|K|$ is the bundle with transition functions $|z|_{\gamma \delta}$ so that if $f$ a section of the square root $L^\alpha$ of $K$ then $|f|^2$ is a section of $|K|$, that is, a measure on the boundary circles. So it makes sense to integrate $|f|^2$ on the two boundary circles to define the norm:

$$||f||^2 = \int_{\partial R} |f|^2$$

We denote the completions by $H^\alpha = H^\alpha(\partial R)$. Equivalently we could work with $\bar{R}$ and the space $H^\alpha(\partial \bar{R})$. The appropriate representation of the fermion algebra $A(H^\alpha)$ is, in the case of the first three spin structures, given by the projection $P^\alpha$ onto the subspace of $H^\alpha$ which is the closure of the subspace obtained by restriction to the boundary of the holomorphic sections. (For the last spin structure there is a problem with this definition but we will not resolve it here.)

To see that this is an interesting thing to do we need to give these projections explicitly. In each case they are given by the Szego kernel which in Fay’s notation $\sigma_\alpha(\bar{x}, y)$.

We will write this down explicitly in the case where it reproduces the KMS states on the loop group [CHa].

Explicitly, on restriction to the circle $\{w : |w| = 1\}$ in the boundary of $R$ we have, in polar co-ordinates, the Szego kernel as the function on $S^1 \times S^1$ given by

$$\sigma_{(0,0)}(e^{-i\xi}, e^{i\varphi}) = \theta_3(\varphi - \xi) \theta_1'(0)[\theta_3(0) \theta_1(\varphi - \xi)]^{-1} \sqrt{dw \sqrt{dz}}$$

where $\theta_3$ and $\theta_1$ are the classical theta functions\footnote{With $q = e^{-\beta/2}$,}

$$\theta_1(\xi) = 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin((n + 1/2)\xi), \quad \theta_3(\xi) = 1 + 2 \sum_{n=1}^{\infty} q^n \cos(n\xi)$$

With $w = e^{i\varphi}$, $z = e^{i\xi}$ and the factor
\[ \sqrt{dw}\sqrt{dz} \] is a notation for a section of \( L^{1/2,1/2} \). These sections of \( L^{1/2,1/2} \) are 1/2-forms in each variable, the result of the fact that the definition of the Szego kernel involves the so-called prime form which is constructed from a quotient of the theta function in the denominator by sections of \( L^{1/2,1/2} \). Note that our notation for the theta functions is classical and differs from that of Fay.

On the other hand in [CHA] there is a novel proof using quantum field theory of the (well known) identity:

\[ \Sigma_n e^{-\beta(n+1/2)}(1 + e^{-\beta(n+1/2)})^{-1} e^{in(\varphi - \xi)} = \theta_3(\varphi - \xi) \theta_1'(0) [\theta_3(0) \theta_1(\varphi - \xi)]^{-1} e^{i(\xi - \varphi)/2}. \]

This may be interpreted as giving the Fourier expansion of the Szego kernel. Using this identity let us compute the effect of applying the projection defined by integration against \( \Sigma \) to the other boundary circle of \( R \). We consider an element \( (f \sqrt{dz}, 0) \in H^{1/2,0}(\partial R) \) where \( f \in L^2(S^1) \) and integrate to define an extension of this section of \( L^{1/2,0} \) to a holomorphic section on the interior of \( R \):

\[ \int \sigma_{(0,0)}(e^{-\xi}, w) f(\xi) \sqrt{dz} \equiv g(w). \]

Now we restrict this section \( g(w) \) to \( \{ w : |w| = 1 \} \) and to the other boundary circle respectively to give a pair of sections \( (g_1 \sqrt{dw}, g_2 \sqrt{dw}) \) where \( g_j \) are in \( L^2(S^1) \).

After some calculation using

\[ f(\xi) = (2\pi)^{-1} \Sigma_n \hat{f}_n e^{in\xi}. \]

the end result is

\[ g_1(\varphi) \sqrt{dw} = \Sigma_n e^{-\beta(n+1/2)}(1 + e^{-\beta(n+1/2)})^{-1} e^{in\varphi} \hat{f}_n \sqrt{dw} \]
\[ g_2(\varphi) \sqrt{dw} = \Sigma_n e^{-\beta(n+1/2)/2}(1 + e^{-\beta(n+1/2)})^{-1} e^{in\varphi} \hat{f}_n \sqrt{dw}. \]

The significance of this formula is that it represents the action on \( (f, 0) \in L^2(S^1) \oplus L^2(S^1) \) of the projection operator \( P(A(\beta)) \) of Subsection 2.3.4 with \( A(\beta) \) given by (19) where \( D \) in that formula is the operator of multiplication on the \( n^{th} \) Fourier coefficient by \( n + 1/2 \). That is

\[ P(A(\beta)) \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}. \]

As we noted in 2.3.4, \( A(\beta) \) defines the quasifree KMS state on the \( C^* \)-algebra \( A(L^2(S^1)) \) for each \( f \in L^2(S^1) \). This situation can be described succinctly by saying that the effect of considering quantum field theory on a genus one Schottky double is to consider temperature states on the appropriate fermion algebra.

8 Free fields on Riemann surfaces

8.1 Overview

Loop groups are intimately related to conformal field theory. The viewpoint of Segal [S2] gives an axiomatic framework for conformal field theories. Explicit examples have been
constructed in [CH2] generalising the construction of the previous section. Roughly speaking the idea is that if one has a set of oriented circles one may ‘interpolate’ between them using Riemann surfaces. With each boundary circle we can associate a Hilbert space identified with $L^2(S^1, \mathbb{C}^N)$ by choosing a local complex co-ordinate which parametrises a neighbourhood of each boundary circle. The direct sum over all boundary circles of these $L^2$ spaces is the boundary Hilbert space. By restricting the holomorphic sections of certain rank $N$ bundles over the Riemann surface to the boundary one obtains a subspace of the boundary Hilbert space. The orthogonal projection onto this subspace may be used to define a quasifree representation $\pi$ of the fermion algebra built over the boundary Hilbert space. Smooth functions from the boundary into the structure group of the bundle on the Riemann surface form an analogue of the loop group (cf the torus example). There is a representation of this group of functions in the Hilbert space of the quasifree fermion algebra representation $\pi$.

We will now explain this construction in more precise terms for the case $N = 1$ (which is in a sense sufficient see [CH2]). Thus we will be considering groups of maps into the group $U(1)$ which has been the main focus throughout this article. Consider a Riemann surface with boundary a smooth oriented 1-manifold $S$ (which may be thought of concretely as a disjoint union of circles). A spin structure on a Riemann surface is a real line bundle $\lambda$ such that the tensor product with itself $\lambda \otimes \lambda$ is the cotangent bundle whose sections are one forms on the Riemann surface. Restricting the spin structure to $S$ we get a real space $K_\mathbb{R} = S(S, \lambda)$ of smooth sections of $\lambda$. This has a canonical quadratic form which pairs sections $\alpha_1$ and $\alpha_2$ to give

$$\langle \alpha_1, \alpha_2 \rangle = \int_{\partial \Sigma_1} \alpha_1 \otimes \alpha_2.$$  \hspace{1cm} (108)

(Note that $\alpha_1 \otimes \alpha_2$ is a one form and hence can be integrated along $S$.) Using this bilinear form we can construct the complex Clifford $*$-algebra $C(K_\mathbb{R})$ over $K_\mathbb{R}$ which is the associative algebra generated by the identity $I$ and the elements of $K_\mathbb{R}$ subject to the relations

$$\alpha_1 \alpha_2 + \alpha_2 \alpha_1 = (\alpha_1, \alpha_2)I.$$  

It is well known that this Clifford algebra has a unique irreducible $*$-representation with ‘positive energy’ (i.e. the generator of rotations has positive spectrum) for any parametrisation of $S$. Independence of parametrisation follows because diffeomorphisms are implemented by unitaries in the positive energy representation of the Clifford algebra thus giving an equivalence of representations defined by different parametrisations. The infinitesimal version of this action of the diffeomorphism group of the circle is well known: it is just a representation of the Virasoro algebra.

Now suppose that $S$ is the boundary of a Riemann surface $\Sigma_1$. Let $L_1$ be a line bundle on $\Sigma_1$ whose restriction to $S$ is $\lambda \otimes \mathbb{C}$. Then the Hilbert space $\mathcal{H}$ on which the Clifford algebra representation acts is given by the completion of $C(K_\mathbb{R})/\mathcal{J}$ where $\mathcal{J}$ is the left ideal in the Clifford algebra generated by sections of $\lambda$ which extend over $\Sigma_1$ to holomorphic sections of $L_1$. We let the space of such sections be denoted by $K_1$. (Another way to think about this representation of the Clifford algebra is that it is the Fock representation corresponding to the projection from $K_\mathbb{R}$ to $K_1$ and we will describe it more explicitly later.)

Given $\Sigma_1$ with boundary $S$ there are many ways to ‘cap’ the boundary circles to give a Riemann surface without boundary. Thinking of the example of the last subsection let us take $\Sigma_1 = R$, the annulus. Then we could glue on another annulus to form the torus or we could think of $R$ as a cylinder and cap the boundary with two discs to form a sphere. More generally let us instead start with a Riemann surface without boundary $\Sigma$ and a decomposition $\Sigma = \Sigma_1 \cup \Sigma_2 \cdots \cup \Sigma_n$...
\[ \Sigma_1 \cup \Sigma_2 \text{ into two submanifolds which intersect in their common smooth boundary, } \partial \Sigma_1 = S = \partial \Sigma_2, \text{ and a line bundle } L \text{ over } \Sigma \text{ such that } L|_{\Sigma_j} = L_j. \] The decomposition of \( \Sigma \) as \( \Sigma_1 \cup \Sigma_2 \) naturally defines a decomposition of \( \mathcal{K} \) as a direct sum of subspaces \( \mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \) which are the restrictions to the common boundary of holomorphic sections of \( L_j \). Each of these is isotropic with respect to the bilinear form in Eq. (108) above and this data in turn defines a Fock representation of the Clifford algebra on the exterior algebra over \( \mathcal{K}_1 \). This space is isomorphic to the irreducible *-representation space \( \mathcal{H} \).

There is a geometric way to understand this isomorphism whenever there is an anti-holomorphic diffeomorphism identifying \( \Sigma_1 \) and \( \Sigma_2 \) (we say in this case that \( \Sigma \) is the Schottky double of \( \Sigma_1 \) and the diffeomorphism is called the Schottky involution.) Provided the line bundle \( L \) is compatible with the Schottky involution then the involution may be used to define a complex conjugation on \( \mathcal{K}_R = \mathcal{K} \) which gives it a natural Hilbert space inner product when combined with (108). Moreover in the case of the Schottky double it is natural to use Araki’s self-dual CAR formalism, \( [Ar] \) in which the complex conjugation on \( \mathcal{K} \), is extended to a conjugate linear involution on \( C(\mathcal{K}) \) (see the next section) and this then enables us to connect up to the fermion algebra description of earlier sections.

In this context one has some natural generalisations of the loop group. Take a complex line bundle \( L \) on \( \Sigma \) compatible with the Schottky involution such that the tensor product \( L \otimes L \) is the complexification of the cotangent bundle over \( S \). Then (108) is a pairing between \( \mathcal{K}_1 \) and \( \mathcal{K}_2 = \overline{\mathcal{K}_1} \) and each is an isotropic subspace of \( \mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \). Now introduce the group \( \text{Map}(\partial \Sigma, U(1)) \) of real analytic \( U(1) \) valued maps on the boundary \( \partial \Sigma_1 \) (that is they are the restriction to \( S \) of a \( \mathbb{C}^* \) valued function analytic in a neighbourhood of \( S \) in \( \Sigma \)). This group acts on \( \mathcal{K} \) by multiplication and so defines a group of automorphisms of the Clifford algebra \( C(\mathcal{K}) \). We will see in the next section that these automorphisms are in fact implementable so that a central extension of \( \text{Map}(\partial \Sigma, U(1)) \) has a representation \( \Gamma \) on \( \mathcal{H} \) as in earlier sections.

(Note: at the Lie algebra level one obtains a representation of a Heisenberg algebra thus generalising \([JLK]\)). The main interest in conformal field theory is in the properties of the representation of \( \text{Map}(\partial \Sigma, U(1)) \) (or more generally in the groups of smooth compact Lie group valued functions on \( \partial \Sigma_1 \)). When \( \Sigma \) is a Schottky double (using standard tools of representation theory together with results of Segal \([SI]\) and Carey, Ruijsenaars and Palmer \([CR\ [CP]\]) this representation is cyclic (in fact irreducible), with cyclic vector \( \Omega \) say, and one may explicitly compute the ‘matrix elements’

\[ \langle \Omega, \Gamma(g)\Omega \rangle, \quad g \in \text{Map}(\partial \Sigma, U(1)). \]

The resulting formulae imply those involving the tau-functions of \([KNTY]\) at least in certain cases.

In the geometric setting Schottky doubles are rather special. To consider more general cases one has to work in the complex Clifford algebra formalism not with the fermion algebra as there is no natural involution. This departs somewhat from the main theme of this review, however, in the next subsection we will give a brief outline of how this theory develops. It has an interesting application to the Landau-Lifshitz equation to which we return at the end of the paper. From a more conventional conformal field theory viewpoint one may describe what we do in the next subsection as providing a geometric interpretation of \([KNTY]\). We do not however attempt to derive explicit formulae for correlation functions. Further examples may be found in \([CHM]\) and \([CHMS]\) while much of the original physics literature can be traced from the work in \([ABNMV\ [ANMV\ [C\ DJKM\ [E\ N\ RI\ [R2]\].

The point of particular interest in the context of this review is the existence of a generalisation of Segal’s vertex operators \([SI]\) (for genus zero) to surfaces of arbitrary genus.
8.2 Fermions

We now make the discussion of the overview much more explicit but at the expense of having to assume considerable familiarity with the theory of Riemann surfaces. A standard reference is for example [GH]. Our first task is to show how splitting the Riemann surface into two submanifolds leads to a polarisation of our underlying space $K$. Thus as before $L$ is a line bundle over a Riemann surface $\Sigma$. As a preliminary to considering a decomposition of $\Sigma$, we suppose that the surface has an open covering by two sets $U_1$ and $U_2$. Writing $S(\Sigma, \mathcal{O}(L))$ for the global sections of the sheaf $\mathcal{O}(L)$ of germs of holomorphic sections of $L$ and $H^1(\Sigma, \mathcal{O}(L))$ for the first cohomology group with coefficients in the sheaf, the Mayer-Vietoris sequence can be written as

$$0 \to S(\Sigma, \mathcal{O}(L)) \to S(U_1, \mathcal{O}(L)) \oplus S(U_2, \mathcal{O}(L)) \to S(U_1 \cap U_2, \mathcal{O}(L)) \to H^1(\Sigma, \mathcal{O}(L)) \to 0.$$ 

In the case of fermions we choose $L$ to be an even spin structure (a square root of the cotangent bundle) for which $S(\Sigma, \mathcal{O}(L))$ vanishes (as happens generically, [Pa]). By Serre duality $H^1(\Sigma, \mathcal{O}(L))$ then also vanishes and the sequence reduces to

$$0 \to S(U_1, \mathcal{O}(L)) \oplus S(U_2, \mathcal{O}(L)) \to S(U_1 \cap U_2, \mathcal{O}(L)) \to 0,$$

from which we deduce that there is a decomposition

$$S(U_1 \cap U_2, \mathcal{O}(L)) = S(U_1, \mathcal{O}(L)) \oplus S(U_2, \mathcal{O}(L)).$$

Now we return to the situation where $\Sigma_1$ and $\Sigma_2$ are closed submanifolds of $\Sigma$ which intersect in their common smooth boundary

$$\Sigma_1 \cap \Sigma_2 = \partial \Sigma_1 = \partial \Sigma_2.$$

For $j = 1$ and $2$ we choose a sequence of neighbourhoods $U_j$ which shrink down to $\Sigma_j$, so that $S(U_1 \cap U_2, \mathcal{O}(L))$ increases to $\mathcal{K} = S(\Sigma_1 \cap \Sigma_2, \mathcal{O}(L)) = S(\partial \Sigma_1, \mathcal{O}(L))$. The spaces $S(U_j, \mathcal{O}(L))$ then increase to give spaces $\mathcal{K}_j$ such that

$$\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$$

establishing the existence of the splitting as required.

Since $L$ is a spin bundle the tensor product of sections $\alpha_j \in S(U_j, \mathcal{O}(L))$ gives a section of the cotangent bundle $K$. Choosing an orientation of $\partial \Sigma_1$ we may integrate $\alpha_1 \otimes \alpha_2$ round the boundary to get the natural symmetric non-degenerate bilinear form on $\mathcal{K}$ given by (108). If both sections $\alpha_1$ and $\alpha_2$ have holomorphic extensions to $U_1$ (or $U_2$) then their product also extends and by Cauchy’s theorem the integral defining $(\alpha_1, \alpha_2)$ vanishes. From this we deduce that $S(U_j, \mathcal{O}(L))$ and its limit $\mathcal{K}_j$ are isotropic, for $j = 1$ or $2$. It is easy to see that (108) is a non-degenerate bilinear form on $\mathcal{K}$ and therefore defines a pairing of the subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$.

Any decomposition of an inner product space into isotropic subspaces

$$\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2,$$

gives rise to a natural representation $\Psi_{21}$ of the Clifford algebra of $\mathcal{K}$ on the exterior algebra $\wedge \mathcal{K}_1$. Elements $\alpha$ of $\mathcal{K}_1$ act by exterior multiplication,

$$\Psi_{21}(\alpha) : \alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_r \mapsto \alpha \wedge \alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_r,$$
whilst elements of $\mathcal{K}_2$ act by inner multiplication,

$$
\Psi_{21}(\alpha) : \alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_r \mapsto \sum_{k=1}^{r} (-1)^{k-1}(\alpha, \alpha_k) \wedge \alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_{k-1} \wedge \alpha_{k+1} \ldots \wedge \alpha_r.
$$

(The pairing of the isotropic subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$ extends to their exterior algebras and the inner multiplication action of $\mathcal{K}_2$ is just the transpose of exterior multiplication on $\wedge \mathcal{K}_2$.)

These conditions determine $\Psi_{21}$ and ensure the usual relations

$$
\Psi_{21}(\beta)\Psi_{21}(\alpha) + \Psi_{21}(\alpha)\Psi_{21}(\beta) = (\beta, \alpha),
$$

for all $\beta$ and $\alpha$ in $\mathcal{K}_1 \oplus \mathcal{K}_2$.

For $j = 1$ or 2, there is a cyclic vector $\Omega_j = 1 \oplus 0 \oplus 0 \ldots \in \wedge \mathcal{K}_j$, called the vacuum vector. With respect to the pairing of $\wedge \mathcal{K}_1$ and $\wedge \mathcal{K}_2$, $\Psi_{21}$ and $\Psi_{12}$ are dual representations of the Clifford algebra.

We summarise the discussion above.

**Proposition I:** Associated to every decomposition of a Riemann surface $\Sigma$ as the union of submanifolds $\Sigma_1$ and $\Sigma_2$ with common boundary and a generic even spin structure $L$ over $\Sigma$, we have the following data

1. A non-degenerate bilinear form (108) on the real analytic sections $\mathcal{K}$ of $L$ restricted to $\Sigma_1 \cap \Sigma_2$.
2. A Fock representation of the Clifford algebra over $\mathcal{K}$ defined by (108) on the exterior algebra over the space of sections of $L$ restricted to either $\Sigma_1$ or $\Sigma_2$. These representations are dual to each other.

We now introduce the (not necessarily orthogonal) projection $P_{kj}$ onto $\mathcal{K}_j$ along $\mathcal{K}_k$. Since $\mathcal{K}_1$ and $\mathcal{K}_2$ are isotropic we have

$$
(\beta, P_{21}\alpha) = (P_{12}\beta, P_{21}\alpha) = (P_{12}\beta, \alpha),
$$

so that $P_{21}$ and $P_{12}$ are transpose maps with respect to the bilinear form. It follows from the definition of $\Psi_{21}$ that

$$
(\Omega_2, \Psi_{21}(\beta)\Psi_{21}(\alpha)\Omega_1) = (\beta, P_{21}\alpha).
$$

Given another decomposition $\mathcal{K} = \mathcal{K}_3 \oplus \mathcal{K}_2$, there is a natural map $T^2_{13}$ from $\wedge \mathcal{K}_3$ to $\wedge \mathcal{K}_1$ which maps $\Omega_3$ to $\Omega_1$ and intertwines the Clifford algebra representations $\Psi_{23}$ and $\Psi_{21}$, which is defined by

$$
T^2_{13}\Psi_{23}(\alpha)\Omega_3 = \Psi_{21}(\alpha)\Omega_1.
$$

This is well-defined since $\Psi_{23}(\alpha)\Omega_3$ vanishes if and only if $\alpha$ is in the ideal generated by $\mathcal{K}_2$ and then $\Psi_{21}(\alpha)\Omega_1$ vanishes too.

The normal arena for quantum field theory is a Hilbert space, which, by the Riesz representation theorem, means that there is an antilinear identification of the space and its dual. Such an antilinear map arises naturally from the geometry if one takes $\Sigma$ to be a Schottky double (cf [JKL, CH]) with its natural antiholomorphic involution taking $z \in \Sigma_1$ to the corresponding point $\bar{z}$ in $\Sigma_2$ (thus fixing each point of the boundary). Thus, as a real manifold, $\Sigma_2$ is an oppositely oriented copy of $\Sigma_1$. For more on Schottky doubles, see [Fe, H].

**Proposition II:** Let $\Sigma$ be a Schottky double.

1. The Schottky involution induces maps of forms and $\frac{1}{2}$-forms, written for brevity as $\alpha(z) \mapsto$
The image is an antiholomorphic $\frac{1}{2}$-form, so that its complex conjugate is holomorphic.

(ii) Defining $\alpha(\bar{z}) = \bar{\alpha}(z)$, we obtain an antilinear map, $\sim$ with

$$(\bar{\alpha}, \bar{\beta}) = \int_{\partial\Sigma_1} \overline{\alpha(z)}\beta(z) = (\alpha, \beta),$$

(i.e. $\sim$ is antiorthogonal).

(iii) The map in (ii) satisfies $(\bar{\alpha}, \alpha) = \int_{\partial\Sigma_1} |\alpha|^2$, and hence $(\alpha, \beta) = (\bar{\alpha}, \beta)$ defines an inner product on $\mathcal{K}$.

(iv) There is a natural isomorphism of the Clifford algebra over $\mathcal{K}$ with the fermion algebra over $\mathcal{K}_1$ (regarded as a pre-Hilbert space in the inner product in (iii)) given by

$$a(\alpha)^* = \Psi_{21}(\alpha) \quad \alpha \in \mathcal{K}_1$$

where we use the notation of 2.3.1 for the fermions.

We use $[\mathcal{F}_1]$ for the notation on theta functions employed in the next result which generalises the discussion for the torus.

**Lemma.** [CHM] The projection $P_1$ onto the first component in $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ is given by an integral operator. Its kernel is the Szeg"o kernel, $\Lambda$, which can be written explicitly in terms of the theta function $\theta[e]$ associated to the same even half-period $e$ which specifies the choice of spin bundle $L$, and the Schottky-Klein prime form $E$, which is a $-\frac{1}{2}$-form in each of its arguments:

$$\Lambda(x, y) = \frac{\theta[e](y - x)}{2\pi i \theta[e](0) E(y, x)}.$$  

(Actually this formula makes it clear that $\Lambda$ can be defined for any surface, $\Sigma$ whether or not it is a Schottky double. For the proof we refer to [CH2]).

## 8.3 Equivalence of representations

There is a special case of the preceding situation for which more detailed information is available. Henceforth we assume, following Segal [S2], that the boundary of $\Sigma_1$ consists of parametrised circles (we make this assumption precise in our next result).

**Lemma.** Assume there are coordinate charts containing each boundary circle such that in terms of a local coordinate $z$, $|z| = 1$ is the boundary circle. Then the Hilbert space representations of the CAR defined by different Riemann surfaces $\Sigma_1$ and spin bundles $L$ which have the same boundary $\partial\Sigma_1$ and restriction $L|_{\partial\Sigma_1}$ which we constructed in in Proposition I in Subsection 8.2 are all equivalent.

This is proved in [CH2] using the method of [PS], Section 8.11. It is enough for our discussion here to understand the polarisation. The spin bundle can be trivialised in such a way that its sections can be identified either with functions on the circle or with functions multiplied by $z^{1/2}$. Thus square-integrable sections are either identified with $L^2(S^1)$ or with $z^{1/2}L^2(S^1)$. Just as there is a standard polarisation of $L^2(S^1)$ into the two Hardy spaces $H_+$ and $H_-$, so $z^{1/2}L^2(S^1)$ can be polarised into $z^{1/2}H_+$ and $z^{1/2}H_-$. Then the representation defined on $\mathcal{F}_1$ using the decomposition into holomorphic sections on $\Sigma_1$ and its reflection
\(\Sigma_4 = \varphi(\Sigma_1)\) is equivalent to that defined by using the appropriate Hardy space decomposition of the sections of \(L_{|\partial \Sigma_1}\), and so all such representations are equivalent.

**Remark:** Having established this equivalence with the standard representation one knows (see [PS]) that the existence of the equivalence does not depend on the precise choice of holomorphic local coordinate as the group \(\text{Diff}(S^1)\) acts in the Hilbert space of this standard representation (enabling us to change parametrisation).

In the physics literature on conformal field theory it is not usual to assume that the Riemann surface is a Schottky double and in fact fermion correlation functions are written down for many examples. A case of particular interest is the class of representations defined by the Krichever map (see [PS] for a discussion of the latter). To understand what these correlation functions mean we need to extend the discussion of the previous subsection.

To this end let us now compare the theory obtained by capping \(\Sigma_1\) by its Schottky dual \(\Sigma_2\) with that obtained when one caps it with another space \(\Sigma_-\) to give a closed surface. To do this we need to suppose that \(\Sigma_-\) is the Schottky dual of \(\Sigma_+\). We now have three different ways of decomposing \(K\):

\[
K = K_1 \oplus K_2 = K_1 \oplus K_- = K_+ \oplus K_-.
\]

The first and third of these define Fock representations \(\Psi_+\) and \(\Psi_1\) which, by the lemma in Subsection 8.3, are intertwined by some unitary operator \(U\). Denoting transpose with respect to our bilinear form by \(\top\) and using our earlier definitions we have

\[
(\Omega_2, \Psi_1(\alpha)U\Omega_+) = (T_{2-}^1 \Omega_-, \Psi_{21}(\alpha)U\Omega_+) = (\Omega_-, (T_{2-}^1)\top \Psi_{12}(\alpha)\top U\Omega_+) = (\Omega_-, \Psi_{1-}(\alpha)\top (T_{2-}^1)\top UT_{+1}^2\Omega_1).
\]

Now, from the earlier equivalences, \((T_{2-}^1)\top UT_{+1}^2\) intertwines \(\Psi_{-1} = \Psi_{1-}^\top\) with itself and so, by Schur’s Lemma, is a multiple, \(k\), of the identity, giving

\[
(\Omega_2, \Psi_1(\alpha)U\Omega_+) = k(\Omega_-, \Psi_{-1}(\alpha)\Omega_1).
\]

Applying the same argument to products in the Clifford algebra now gives

\[
\langle \Omega_1, \Psi_1(\alpha)\Psi_1(\beta)U\Omega_+ \rangle = (\Omega_2, \Psi_1(\alpha)\Psi_1(\beta)U\Omega_+) = k(\Omega_-, \Psi_{-1}(\alpha)\Psi_{-1}(\beta)\Omega_1) = k(\alpha, P_{-1}\beta).
\]

Thus correlation functions involving two different Fock cyclic vectors, \(\Omega_1\) and \(\Omega_+\), can also be computed purely in terms of the geometrical projection involving the surface obtained by capping \(\Sigma_1\) with the Schottky dual of \(\Sigma_+\).

This provides an interpretation of the correlation functions involved in the Krichever construction where one uses for \(\Sigma_-\) a union of discs. In some papers these correlation functions are misleadingly written as inner products involving the same cyclic vector (i.e. \(\Omega_1\) is identified with \(\Omega_+\)). Under the construction we have given here this may only be done in the Schottky double case. This is what distinguishes the latter from other possibilities: it is only in the Schottky case that there is a geometrically defined inner product on \(K\) in terms of which the correlation functions are positive definite and hence one can obtain a quantum field theory that satisfies the Wightman axioms.
The boson-fermion correspondence

The discussion in the previous subsection has a simple consequence.

**Corollary:** The representation defined by any Riemann surface $\Sigma_1$ is equivalent to that obtained simply by capping the $p+1$ circles which make up $\partial\Sigma_1$ by discs, that is, it is equivalent to a tensor product of $p + 1$ standard fermion representations for a single circle.

In the remaining sections we shall take $\Sigma$ to be the Schottky double formed from $\Sigma_1$ and with parametrised boundary circles. The representation of the Clifford algebra over the space $K$ we denote by $\Psi$ for short. (Note that although in this case $\Psi$ can be regarded as a representation of the fermion algebra we will persist with this Clifford notation.) The group $\text{Map}(\partial\Sigma, U(1)) \equiv \text{Map}(\partial\Sigma_1, U(1))$ of smooth functions from $\partial\Sigma_1$ to the complex numbers of modulus 1 acts unitarily by pointwise multiplication on $K = L^2(\partial\Sigma_1)$. This group also acts as automorphisms of the Clifford algebra: for $\xi \in \text{Map}(\partial\Sigma, U(1))$ and $\alpha \in K$ we have

$$\xi : \Psi(\alpha) \mapsto \Psi(\xi \cdot \alpha).$$

We will refer to $\text{Map}(\partial\Sigma, U(1))$ somewhat loosely as the ‘bosons’ even though strictly speaking it is the Lie algebra of this group which can be given the structure of a Heisenberg algebra and hence may be regarded as representing bosons.

As $\text{Map}(\partial\Sigma, U(1))$ is the product of groups of smooth maps on each connected component of the boundary and the representation $\Psi$ is equivalent to the standard representation obtained by capping the boundary by discs, this automorphism is implemented by an irreducible projective representation $\Gamma$ with 2-cocycle $\sigma$, that is

$$\Gamma(\xi_1)\Gamma(\xi_2) = \sigma(\xi_1, \xi_2)\Gamma(\xi_1\xi_2).$$

Choose a base point $c_j$ on the $j$-th boundary circle for $j = 0, 1, 2, \ldots, p$. Let $\xi \in \text{Map}(\partial\Sigma, U(1))$ and $\xi = e^{if}$. Define $\Delta_j f$ to be the change in the value of a function $f$ after one circuit of that boundary circle. The Lie algebra of $\text{Map}(\partial\Sigma, U(1))$ consists of those $f$ with $\Delta_j f = 0$ for all $j = 0, 1, 2, \ldots, p$.

Following Segal [S1] we note that for each $\xi \in \text{Map}(\partial\Sigma, U(1))$ there is a choice of unitary $\Gamma(\xi)$ such that the cocycle has the form

$$\sigma(e^{if_1}, e^{if_2}) = e^{-is(f_2,f_1)/4\pi} = \exp \left( -\frac{i}{4\pi} \left( \int_{\partial\Sigma^+} f_2 df_1 + \sum_{j=1}^{p} f_2(c_j) \Delta_j f_1 \right) \right).$$

Although we have not chosen to do so here this cocycle may be derived by a purely geometric argument as in [CHM]. On the Lie algebra of $\text{Map}(\partial\Sigma, U(1))$ the bilinear form $s$ in the expression for $\sigma$ is symplectic, equipping this Lie algebra with the structure of an infinite dimensional Heisenberg algebra.

The representation $\Gamma$ has a number of interesting properties for a discussion of which we refer the reader to [CH2]. We restrict the exposition here to describing the boson-fermion correspondence on $\Sigma$.

For each boundary component it follows from the corollary at the beginning of this subsection that the fermion representation can be recovered from the boson operators (this is the boson-fermion correspondence of Subsection 2.6). Explicitly we choose an annular neighbourhood of a boundary circle and a local coordinate $z$ such that the boundary is $|z| = 1$. 
As we showed in Section 2 following [PS, CHu, CR] the fermions can be reconstructed using the special loops or 'blips' \( \gamma_a \in \text{Map}(\partial \Sigma, U(1)) \) where \( |a| < 1 \) and

\[
\gamma_a(z) = \left( \frac{\tau}{a} \right)^{\frac{1}{2}} \left( \frac{z - a}{2a - 1} \right).
\]

In fact, it is known that for \( \alpha \) concentrated on the annular neighbourhood of that boundary circle

\[
\Psi(\alpha) = \lim_{r \to 1} \int_{|a| = r} \alpha(a) \left( (1 - |a|^2)^{-\frac{3}{2}} \Gamma(\gamma_a) \right) \sqrt{\frac{da}{a}}.
\]

where the the limit means strong convergence on a dense domain (see [CR] for details). Note that since \( \alpha \) is a half-form the factor \( \sqrt{\frac{da}{a}} \) turns this into a form which can then be integrated. We shall now show that these local blips can also be interpreted on the Riemann surface in the sense that they may be extended holomorphically to the whole of \( \Sigma_1 \) so that the implementers of the blips are regularised vertex operators on the Riemann surface.

First let us consider the renormalisation factor \( (1 - |a|^2)^{-\frac{3}{2}} \). In terms of the local coordinate the prime form can be expressed as

\[
E(x, y) = \frac{y - x}{\sqrt{dxdy}} + O((x - y)^3),
\]

([Fa] Cor 2.5). The local expression for the involution is \( \tilde{a} = \frac{1}{a} \), so that we have

\[
E(\tilde{a}, a)^{-1} = \frac{\sqrt{d(1/a)da}}{a - a} = \frac{\sqrt{-da}da}{(1 - |a|^2)},
\]

whose square root is almost what we want.

Now this is not quite well defined in general, and one should rather use the expression

\[
\frac{i\theta(e - \tilde{a} + a)}{\theta(e)E(\tilde{a}, a)},
\]

for \( e \) a half period fixed by the Schottky involution and having prescribed behaviour on certain cycles in \( \Sigma_1 \). This period is chosen to have the same limiting behaviour near the boundary and is, as Fay shows ([Fa] Cor 6.15 et seq), a positive section of the bundle \( |K| \otimes (2\text{Re}(e)) \) where \( \text{Re}(e) \) denotes the line bundle associated with the real part of \( e \). Considering for the moment the case when \( e = 0 \), we see that we may take a positive square root as a section of \( |K|^\frac{1}{2} \). Since the boundary is oriented the restrictions of \( |K| \) and \( K \) there can be naturally identified. The general case of non-zero \( e \) can be handled by multiplying through by an appropriate \( \exp(-\sum e_k \int \omega_k^0) \) to convert it to the previous case. One then ends up with a square root which is a section of \( |K|^\frac{1}{2} \otimes e \) restricted to an annular region containing the boundary circle. It converges to the half form \( (id\theta)^{\frac{1}{2}} = (dw/w)^{\frac{1}{2}} \) on the circle. It follows that in the annular region \( \alpha(a)E(\tilde{a}, a)^{-\frac{1}{2}} \) is a one-form and we have:

\[
\Psi(\alpha) = \lim_{\lambda \to 1} \int_{|a| = \lambda} \alpha(a)E(\tilde{a}, a)^{-\frac{1}{2}} \Gamma(\gamma_a).
\]

To interpret the blip \( \gamma_a \) we recall that there is a family of distinguished meromorphic functions on \( \Sigma_1 \) whose values on \( \partial \Sigma_1 \) have modulus 1 and with the minimal number of
zeroes, Theorem 6.6. Tailoring the result to our needs, for \( a \in \Sigma_1 \) and \( s \) a suitable even half period we take

\[
\epsilon_a(z) = \frac{\theta(z - \tilde{a} - s) E(z, a)}{\theta(z - a - s) E(z, \tilde{a})} \exp \left( \frac{1}{2} \sum_{j=1}^{p} \mu_j \int_{a}^{\tilde{a}} \omega_j \right),
\]

where \( 1 + \mu_j \) agrees modulo 2 with the winding number of \( \epsilon_a \) round the \( j \)-th boundary component \( \partial_j \Sigma_1 \). Since \( \epsilon_a \) has modulus 1 on the boundary circles it represents an element of \( \text{Map}(\partial \Sigma, U(1)) \). To see that near the boundary circle it behaves in the correct fashion we record the following fact.

**Lemma:** In sufficiently small annular neighbourhoods of the boundary circles the function \( \gamma_a^{-1} \epsilon_a \) is defined and converges pointwise as \( a \to w \) on the unit circle to the constant function 1.

One way to interpret this lemma is that as \( a \to w \) on the unit circle the meromorphic function \( \epsilon_a \) on the Riemann surface approaches a (singular) distribution. Recalling Segal’s blip construction of vertex operators described in Section 2 (see [PS]) this suggests how to construct regularised ‘vertex operators’ on the Riemann surface which give a precise analytic meaning to the boson-fermion correspondence. With some extra work one now proves [CH2]:

**Theorem:** For \( \Phi \) in a dense domain of the Fock space

\[
\Psi(\alpha) \Phi = \lim_{\lambda \to 1} \int_{|a|=\lambda} \alpha(a) \left( E(\tilde{a}, a)^{-\frac{1}{2}} \Gamma(\epsilon_a) \right) \Phi.
\]

### 8.5 The Landau-Lifshitz equation

In the previous subsection we emphasised the situation when the splitting of the Riemann surface is into the two halves of a Schottky double. When the decomposition of the Riemann surface is not symmetrical then there are additional complexities. These are illustrated very clearly in the application of the general ideas of the preceding subsections to the completely integrable non-linear Landau-Lifshitz (LL) equation in [CHMS]. The interest in this example stems from the fact that the spectral curve \( \Sigma \) which arises from the Lax form of the LL equations is an elliptic curve. This means that there is no immediate generalisation of the methods of solving integrable systems which are applicable when the spectral curve is the Riemann sphere. For example, a key ingredient in the Riemann sphere case is that a generic \( \text{SL}_2(\mathbb{C}) \)-valued loop on the unit circle has a Birkhoff factorisation as a product of loops, one holomorphic inside the unit disc, the other holomorphic outside the disc (see [PS] for a discussion of Birkhoff factorisation). There is, however, no such factorisation in general for a disc in \( \Sigma \).

The approach to the study of the LL equation in [CHMS] was partly modelled on the study of the KdV equation in [SW]. The first step is to find the appropriate group of functions on the elliptic curve to play the role that the loop group of \( \text{SL}_2(\mathbb{C}) \) does for integrable systems such as KdV with spectral curve the Riemann sphere. The group was constructed by first decomposing the elliptic curve (or torus) into two submanifolds. The first submanifold, \( \Sigma_1 \), is a union of four disjoint discs and the second is the closure of the complement of \( \Sigma_1 \). If we regard the torus as the complex plane modulo a lattice \( 2\pi \mathbb{Z} + \tau \mathbb{Z} \) as in Section 7 then
the four discs are centred on the points $0, \pi, \tau/2, \tau/2 + \pi$. The boundary of $\Sigma_1$ consists of four circles and the group of interest in this case consists of the real analytic maps of these four circles into $SL_2(\mathbb{C})$ which are equivariant under the discrete symmetries of the torus generated by translation through the half periods $\pi, \tau/2$ of the underlying lattice in $\mathbb{C}$. The discs are clearly permuted by this symmetry group. The imposition of equivariance under the discrete symmetry quite remarkably turns out to be exactly what is needed for the existence of an appropriate analogue of Birkhoff factorisation. This factorisation then enables one to construct the soliton solutions of LL by a method that generalises that in [SW](see [CHMS]).

References

[ABNMV] Alvarez-Gaumé L., Bost J.B., Moore G., Nelson P., Vafa C.: Bosonization on higher genus Riemann surfaces, *Commun. Math. Phys.* **112** 503 (1987)

[AMOS1] Awata H., Matsuo Y., Odake S., Shiraishi J.: Excited states of Calogero-Sutherland model and singular vectors of the $W(N)$ algebra, *Nucl. Phys.* **B449**, 347 (1995)

[AMOS2] Awata H., Matsuo Y., Odake S., Shiraishi J.: Collective field theory, Calogero-Sutherland model and generalized matrix models, *Phys. Lett.* **B347** 49 (1995)

[AMV] Alvarez-Gaumé L., Moore G., Vafa C.: Theta functions, modular invariance, and strings, *Commun. Math. Phys.* **106**, 1 (1986)

[ANMV] Alvarez-Gaumé, L, Nelson, P, Moore, G, Vafa, C: Bosonization in arbitrary genus *Phys. Lett.* **B178** 41 (1986)

[Ar] Araki H: Bogoliubov automorphisms and Fock representations of canonical anticommutation relations. In *Operator algebras and mathematical physics (Iowa City, Iowa, 1985)*, 23–141, Contemp. Math. **62**, Amer. Math. Soc. , Providence, RI, 1987

[BMT] Buchholz D., Mack G., Todorov I.: Localized automorphisms of the $U(1)$ current algebra on the circle: an instructive example. In *The algebraic theory of superselection sectors (Palermo, 1989)*, 356–378, World Sci. Publishing, River Edge, NJ, 1990

[BR] Bratteli O., Robinson D.W.: *Operator Algebras and Quantum Statistical Mechanics. 2*. Second edition, Springer, Berlin, 1997

[C] Cardy J.L: Conformal invariance. In *Phase transitions and critical phenomena, Vol. 11*, 55–126, Academic Press, London, 1987

[CEH] Carey A.L, Eastwood M.G, Hannabuss K.C: Riemann surfaces, Clifford algebras and infinite dimensional groups, *Commun. Math. Phys.* **130**, 217 (1990)

[CH1] Carey A.L, Hannabuss K.C: Some simple examples of conformal field theories, *Int. J. Mod. Phys.* **B4**, 1059 (1990)

[CH2] Carey A.L, Hannabuss K.C: Infinite dimensional groups and Riemann surface field theories *Commun. Math. Phys.* **176** 321 (1996)
[CHM] Carey A.L, Hannabuss K.C, Murray M.K: Free fermions on Riemann surfaces and spectral curves of the chiral Potts model, in Topological and geometrical methods in field theory (Turku, Finland, 1991), 48-63, Singapore, World Scientific 1992

[CHMS] Carey A.L, Hannabuss K.C, Mason L.J, Singer M.A: The Landau-Lifshitz equation, elliptic curves and the Ward transform, Commun. Math. Phys. 154, 25 (1993)

[CHOU] Carey A.L., Hurst C.A., O’Brien D.M.: Fermion currents in 1+1 dimensions, J. Math. Phys. 24 2212 (1983); see also Uhlenbrock D.A., Commun. Math. Phys. 4, 64 (1967)

[CHa] Carey A.L., Hannabuss K.C.: Temperature states on the loop groups, theta functions and the Luttinger model, J. Func. Anal. 75, 128 (1987)

[CHu] Carey A.L., Hurst C.A.: A note on the boson-fermion correspondence and infinite dimensional groups, Commun. Math. Phys. 98, 435 (1985)

[CL] Carey A.L., Langmann E.: Loop groups, anyons, and the Calogero-Sutherland model, Commun. Math. Phys. 201, 1 (1999)

[CP] Carey A.L, Palmer J: Infinite complex spin groups, J. Func. Anal. 83, 1 (1989)

[CR] Carey A.L., Ruijsenaars S.N.M.: On fermion gauge groups, current algebras and Kac-Moody algebras, Acta Appl. Mat. 10, 1 (1987)

[CRW] Carey A.L., Ruijsenaars S.N.M., Wright J.D.: The massless Thirring model: positivity of Klaiber’s n-point functions, Commun. Math. Phys. 99, 347 (1985)

[CW] Carey A.L., Wright J.D.: Hilbert space representations of the gauge groups of some two dimensional field theories, Rev. Math. Phys. 5, 551 (1993)

[Co] Connes A.: Noncommutative Geometry, Academic Press, San Diego, CA, 1994

[DJKM] Date E., Jimbo M., Kashiwara M., Miwa T.: Transformation groups for soliton equations. In Nonlinear integrable systems—classical theory and quantum theory (Kyoto, 1981), 39–119, World Sci. Publishing, Singapore, 1983

[E] Eguchi T.: Chiral bosonization on a Riemann surface. In Conformal field theory, anomalies and superstrings (Singapore, 1987), 372–390, World Sci. Publishing, Singapore, 1988

[Fa] Fay J.D: Theta functions on Riemann surfaces, Springer Lecture Notes in Mathematics 352, Springer, Berlin, 1973

[F] Frenkel I.B.: Two constructions of affine Lie algebra representations and boson-fermion correspondence in quantum field theory, J. Funct. Anal. 44, 259 (1981)

[GH] Griffiths P., Harris J.: Principles of algebraic geometry, Wiley-Intersci., New York, 1978

[GL1] Grosse H., Langmann E.: On super current algebras and super Schwinger terms’, Lett. Math. Phys. 21, 69 (1991)

[GL2] Grosse H., Langmann E.: A super-version of quasi-free second quantization. I. Charged particles, J. Math. Phys. 33, 1032 (1992)
[GLR] Grosse H., Langmann E., Raschhofer E.: On the Luttinger–Schwinger model, *Ann. Phys. (N.Y.)* **253**, 310 (1997)

[GO] Goddard P., Olive D.: Kac-Moody and Virasoro algebras in relation to quantum physics, *Int. J. Mod. Phys* A**1**, 303 (1986); see also Bardakci K., Halpern M. B., *Phys. Rev. D* **3**, 2493 (1971)

[GVF] Gracia-Bondía M., Várilly J.C., Figueroa H.: *Elements of noncommutative geometry*, Birkhäuser Boston, Boston, MA, 2001

[H] Hejhal D.A: Theta functions, kernel functions and Abelian integrals, *Mem. Amer. Math. Soc.* **129**, 1972

[HSU] Heidenreich R., Seiler R., Uhlenbrock D.A.: The Luttinger model, *J. Statist. Phys.* **22**, 27 (1980)

[I] Iso S.: Anyon basis in $c = 1$ conformal field theory, *Nucl. Phys.* B**443** [FS], 581 (1995)

[JKL] Jaffe A, Klimek S, Lesniewski A.: Representations of the Heisenberg algebra on a Riemann surface, *Commun. Math. Phys.* **126**, 421 (1989)

[KNTY] Kawamoto N., Namikawa Y., Tsuchiya A., Yamada Y.: Geometric realization of conformal field theory on Riemann surfaces, *Commun. Math. Phys.* **116**, 247 (1988)

[K] Kac V.G.: *Infinite dimensional Lie algebras*, Third edition, Cambridge Univ. Press, Cambridge, 1990

[KRd] Kac V.G., Radul A.: Quasifinite highest weight modules over the Lie algebra of differential operators on the circle, *Commun. Math. Phys.* **157** 429 (1993); see also Bilal A., *Phys. Lett.* B**227** 406 (1989); Bakas I., *Phys. Lett.* B**228** 57 (1989)

[KRi] Kac V.G., Raina A.K.: *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras*, World Sci. Publishing, Teaneck, NJ, 1987

[K] Klaiber B.: The Thirring model, In *Quantum theory and statistical physics* Vol. XA, p141, Barut A.O., Brittin W.E. (eds.), Lectures in Theoretical Physics, New York: Gordon & Breach, 1967; see also Hagen C.H., *Nuovo Cim.* B**51B**, 169 (1967)

[L1] Langmann E.: Cocycles for boson and fermion Bogoliubov transformations, *J. Math. Phys.* **35**, 96 (1994)

[L2] Langmann E.: Quantum gauge theories and noncommutative geometry, *Acta Phys. Pol. B* **27**, 2477 (1996) [hep-th/9608003]

[L3] Langmann E.: Anyons and the elliptic Calogero-Sutherland model, *Lett. Math. Phys.* (in print) [math-ph/0007036]; Second quantization of the elliptic Calogero-Sutherland model, [math-ph/0102005]; and work in progress

[L4] Langmann E.: Quantum Theory of Fermion Systems: Topics between Physics and Mathematics. In *Geometric methods for quantum field theory (Villa de Leyva, Colombia, 1999)*, Paycha S., Reyes A., Campo H.O. (eds.), World Scientific, to appear.
[LC] Langmann E., Carey A.L.: Loop groups, Luttinger model, Anyons, and Sutherland systems, Proc. of International Workshop “Mathematical Physics” in Kiev, Ukraine (May 1997), Ukrainian J. Phys. 6-7 vol. 43, 817 (1998)

[LS] Langmann E., Semenoff G.W.: QCD(1+1) with massless quarks and gauge covariant Sugawara construction, Phys. Lett. B341, 195 (1994)

[L] Lundberg L.-E.: Quasi-free “second quantization”, Commun. Math. Phys. 50, 103 (1976)

[Ma] Manton N. S.: The Schwinger model and its axial anomaly, Ann. Phys. (N.Y.) 159, 220 (1985); see also Schwinger J., Phys. Rev. 128, 2425 (1962)

[MS] Marotta V., Sciarrino A.: From vertex operators to Calogero-Sutherland models. Nucl. Phys. B476, 351 (1996)

[ML] Mattis D.C., Lieb E.H.: Exact solution of a many-fermion system and its associated boson field, J. Math. Phys. 6, 304 (1965); see also Luttinger J. M., J. Math. Phys. 4, 1154 (1963)

[Mi] Mickelsson J.: Current Algebras and Groups, Plenum Monographs in Nonlinear Physics, Plenum, New York., 1989

[N] Namikawa Y: A conformal field theory on Riemann surfaces realized as quantized moduli theory on Riemann surfaces. In Theta functions—Bowdoin 1987, Part 1 (Brunswick, ME, 1987), 413–443, Proc. Sympos. Pure Math., Part 1, Amer. Math. Soc., Providence, RI, 1989

[PS] Pressley A., Segal G.: Loop Groups, Oxford Math. Monographs, Oxford, 1986

[R1] Raina, A.K.: Fay’s trisecant identity and Wick’s theorem: an algebraic geometry viewpoint, Exposition. Math. 8, 227 (1990)

[R2] Raina, A.K.: An algebraic geometry view of currents in a model quantum field theory on a curve, C. R. Acad. Sci. Paris Sér. I Math. 318, 851 (1994)

[Ru] Ruijsenaars S.N.M.: On Bogoliubov transformations for systems of relativistic charged particles, J. Math. Phys. 18, 517 (1977)

[S1] Segal G.B: Unitary representations of some infinite-dimensional groups, Commun. Math. Phys. 80, 301 (1981)

[S2] Segal G.B.: The definition of conformal field theory, (draft of paper)

[Su] Sutherland B.: Exact results for a quantum many body problem in one-dimension, Phys. Rev. A4 2019 (1971) and A5, 1372 (1972); see also Calogero F., J. Math. Phys. 10, 2197 and 2197 (1969) and 12, 419 (1971)

[SW] Segal G.B. and Wilson G.: Loop groups and equations of KdV type, Publ. Math. IHES 61, 5 (1989)

[T] Thirring W.: A soluble relativistic field theory?, Ann. Phys. (N.Y.) 3, 91 (1958)

[W] Wilson K.G.: Operator product expansions and anomalous dimensions in the Thirring model, Phys. Rev. D2, 1473 (1970)