(Non)decoupling of the Higgs triplet effects

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Abstract

We consider the electroweak theory with an additional Higgs triplet at one loop using the hybrid renormalization scheme based on \( \alpha_{\text{EM}} \), \( G_F \) and \( M_Z \) as input observables. We show that in this scheme loop corrections can be naturally split into the Standard Model part and corrections due to “new physics”. The latter, however do not decouple in the limit of infinite triplet mass parameter, if the triplet trilinear coupling to SM Higgs doublets grows along with the the triplet mass. In electroweak observables computed at one loop this effect can be attributed to radiative generation in this limit of a nonvanishing vacuum expectation value of the triplet. We also point out that whenever tree level expressions for the electroweak observables depend on vacuum expectation values of scalar fields other than the Standard Model Higgs doublet, tadpole contribution to the “oblique” parameter \( T \) should in principle be included. In the Appendix the origin of nondecoupling is discussed on the basis of symmetry principles in a simple scalar field theory.
1 Introduction

Extensions of the Standard Model (SM) constructed with the aim of solving the hierarchy problem are often based on gauge groups larger than $SU(2) \times U(1)$ and often lead to the presence in the low energy limit of Higgs fields in representations other than the doublet one. For example, string inspired models usually predict an extra $U(1)$ gauge group factor resulting in an additional massive neutral vector boson. Some of the Little Higgs models predict also the existence of charged massive vector bosons and/or of a triplet of Higgs scalar fields. As a result in these models the custodial symmetry is explicitly broken and the $\rho$ parameter deviates from unity already at the tree level.

Precision electroweak data severely constrain such models. Some of the published constraints were derived by computing relevant electroweak observables at the tree level only [1]. Others result from computing corrections due to new physics to the parameters $S$, $T$ and $U$ either at the tree level only [2] or by adding also one-loop corrections due to new physics [3,4]. The last two approaches implicitly assume that the SM and new physics corrections can be separated from each other.

However recently doubts have been expressed about the validity of such an approach [5–7]. It has been argued that if $\rho \neq 1$ at the tree level, the whole structure of radiative corrections is changed [5,6] and the constraints on such models can only be worked out by computing observables in the complete model using a dedicated renormalization scheme: Electroweak sectors of such models depend, apart from the two $SU(2) \times U(1)$ gauge couplings $g_2$, $g_y$ and the Higgs doublet vacuum expectation value $v_H$, also on additional free parameters (the gauge coupling(s) related to the extra $U(1)$ group, or the vacuum expectation value $v_\phi$ of the Higgs triplet) and - as has been argued in [5–7] - more than three input observables must be chosen in order to define the renormalization scheme. A simple $SU(2) \times U(1)$ model with a Higgs doublet and a Higgs triplet has been recently analysed in detail in ref. [7] using such a scheme (and fitted to the data in ref. [8]) based on $\alpha_{EM}$, $G_F$, $M_Z$ and $\sin^2 \theta$ as input observables. It has been found that the $W$ mass depends on the top mass only logarithmically (in contrast to the quadratic dependence in the SM) and that contributions of the heavy Higgs scalars to electroweak observables do not decouple. Similar effects were also reported in [9] for the Littlest Higgs model [10] also containing an $SU(2)$ triplet and earlier in [6] for the case of $U(1)$ extensions of the SM.

These results seemed surprising, especially in the case of $U(1)$ extensions of the SM: on the basis of the Appelquist-Carrazzone decoupling theorem one would expect that effects of a heavy $Z'$ should be negligible and the SM structure of the loop corrections to electroweak observables should be only delicately perturbed. In order to clarify the issue we have reconsidered in [11] the $U(1)$ extension of the SM using a different renormalization scheme. We have pointed out that in fact the use of more input observables is not necessary and argued that it is precisely the use of low energy observables to fix the additional parameters of the extended electroweak sector which is responsible for the lack of the Appelquist-Carrazzone
decoupling of heavy sector effects in electroweak observables. The renormalization scheme used in [11] is a hybrid combination of the scheme based on three input observables $\alpha_{\text{EM}}, G_F$ and $M_W$ (for convenience) with the $\overline{\text{MS}}$ scheme. Its advantage is that it can be applied uniformly to both, the SM and its extension. This enables direct comparison of the extended model predictions for various observables (other than the input ones) with the predictions of the SM and assessment whether they become identical in the limit of infinitely heavy additional particles predicted by the extended model (i.e. whether there is decoupling or not). We have analysed the model with the extra $U(1)$ symmetry at one loop using this scheme and showed that the Appelquist-Carrazzone decoupling of a heavy $Z'$ effects is manifest and that the radiative corrections naturally split into the SM part plus corrections due to $Z'$ which are suppressed by its mass. In particular, $M_W^2$ depends on the top quark mass quadratically as in the SM. For the $U(1)$ extension of the SM the proposed renormalization scheme justifies, therefore, combining the electroweak observables computed in the SM with higher orders corrections included with only the tree level corrections due to the extended gauge structure.

In this paper we apply our renormalization scheme to the model with an additional triplet analysed in [7,8] and earlier in [12]. Our motivation is to check whether the reported results are not due to the renormalization scheme adopted in [7]. In our scheme we determine the two gauge couplings $g_2, g_y$ and the expectation value (VEV) $v_H$ of the doublet in terms of $\alpha_{\text{EM}}, G_F$ and $M_Z$ (as is customary in the SM) and treat the triplet VEV $v_\phi$ as the running $\overline{\text{MS}}$ parameter (at some fixed but arbitrary renormalization scale $Q$). The electroweak observables computed at one loop are independent of the chosen renormalization scale $Q$ due to the renormalization group running of the tree level $v_\phi$ provided the tadpole contributions to the vector boson self energies are properly included. In this scheme the loop corrections are consistently organized in such a way that for $v_\phi \rightarrow 0$ the SM part can be separated from the new physics corrections. In particular, $M_W^2$ and $\rho$ parameters depend quadratically on the top quark mass as in the SM.

Our renormalization scheme allows us to clarify the issue of (non)decoupling. As we show, at the tree level effects of the triplet in electroweak observables can be made arbitrarily small by decreasing the triplet VEV $v_\phi$ in the limit in which the triplet mass parameter $m_\phi^2$ becomes large. In this limit the additional neutral $K^0$ and charged $H^\pm$ scalars become heavy. At one loop, however, the decoupling of the triplet effects can be spoiled (as found in [7]). This happens if the dimensionful coupling of two Higgs doublets with the triplet is kept of the same order as $m_\phi$. In our approach the violation of the decoupling in electroweak observables is at one loop due to the tadpole contributions and can be attributed to a nonvanishing effective triplet VEV: while all additional one-particle irreducible contributions to electroweak observables vanish in the limit $m_\phi^2 \rightarrow \infty$, the tadpole contributions do not: for generic values of the other parameters a nonzero triplet VEV is generated by radiative corrections in this limit and adds to the (vanishing in the $m_\phi^2 \rightarrow \infty$ limit) tree level VEV $v_\phi$. Therefore, in this limit the electroweak observables are modified as if there was a nonzero triplet VEV in the tree level contributions. Of course, one
can always assume that the values of the other parameters are such that for actual value of $m_\phi^2$ tadpole contributions vanish but this requires a severe fine tuning.\footnote{Beyond one loop canceling tadpoles by tuning parameters is also possible but there are also other contributions of heavy particles which do not decouple (e.g. to the vertex $H^+ W^- Z^0$ mentioned at the end of Sec. 5) and it is not likely that simultaneous canceling of all heavy particle contributions to all electroweak observables is possible.} The tadpole contributions to the triplet VEV vanish in the limit $m_\phi^2 \rightarrow \infty$ only if the the triplet dimensionfull trilinear coupling is kept of order of the electroweak scale (this is radiatively stable).

We also elucidate the lack of quadratic dependence of $M_W^2$ on the top quark mass $m_t$ reported in [6–9]. It is simply related to the fact that the $W$ mass depends on $m_t$ only logarithmically also in the SM, if $\alpha_{\text{EM}}, G_F$ and $\sin^2 \theta_{\ell}^{\text{eff}}$ are used as the input observables instead of $\alpha_{\text{EM}}, G_F$ and $M_Z$. (In the SM quadratically on $m_t$ depends then $M_Z^2$.) If the decoupling of the triplet effects does hold (the trilinear triplet coupling does not grow with $m_\phi$) the calculation of $M_W^2$ in the renormalization scheme based on $\alpha_{\text{EM}}, G_F, M_Z$ and $\sin^2 \theta_{\ell}^{\text{eff}}$ can be reinterpreted as the the calculation of $M_W^2$ in the SM in the renormalization scheme based on $\alpha_{\text{EM}}, G_F$ and $\sin^2 \theta_{\ell}^{\text{eff}}$.

The plan is as follows. In Section 2 we present the model at the tree level. In particular we identify the limits in which the triplet VEV vanishes. In section 3 we define our renormalization scheme and demonstrate its working in section 4 by computing the low energy $\rho$ defined in terms of the neutral and charged current low energy processes. We discuss here the (non)decoupling of the triplet effects. In section 5 we discuss the calculation of the $W$ mass and compare it with the calculation in the renormalization scheme of ref [7]. Finally, in the last section we discuss the role of tadpoles in the Peskin-Takeuchi $S, T$ and $U$ parameters [18] and comment on the problem generated by the presence of $SU(2)$ Higgs triplets in Grand Unified Theories. Since the nondecoupling of Higgs triplet effects may seem counter-intuitive, and has important consequences for model building, in Appendix B we investigate it from another point of view in the model with Higgs fields only. Appendix A contains necessary formulae.

2 The model

As in [4, 7] we consider for simplicity the SM model extended with a $Y = 0$ Higgs (weak hypercharge) triplet $\Phi = 1/2 \tau^a \phi^a$. We would like to investigate whether the triplet can decouple, that is whether there is a limit in which the model predictions for all “low energy” observables (i.e. observables which can be defined also in the SM) are the same as in the SM.

Assuming that both, the doublet and the triplet acquire vacuum expectation values (VEVs) $v_H$ and $v_\phi$, respectively, the tree level expressions for gauge boson masses read

$$M_Z^2 = \frac{1}{4} (g_2^2 + g_y^2) v_H^2,$$

\footnote{Beyond one loop canceling tadpoles by tuning parameters is also possible but there are also other contributions of heavy particles which do not decouple (e.g. to the vertex $H^+ W^- Z^0$ mentioned at the end of Sec. 5) and it is not likely that simultaneous canceling of all heavy particle contributions to all electroweak observables is possible.}
\[ M_W^2 = \frac{1}{4} g_2^2 v_H^2 + g_2^2 v_\phi^2. \]  

(1)

The gauge boson couplings to fermions are as in the SM (similarly as the Yukawa terms). The weak mixing angle is also defined by \( s^2 = \frac{g_1^2}{(g_2^2 + g_3^2)^2} \). The three customary SM input observables are given by

\[ M_Z^2 = \frac{1}{s^2 c^2} v_H^2, \]
\[ \alpha_{\text{EM}} = \frac{e^2}{4\pi}, \]
\[ \sqrt{2} G_F = \frac{e^2}{4s^2 M_W^2} = \frac{1}{v_H^2 + 4v_\phi^2}. \]  

(2)

so that

\[ e^2 = 4\pi \alpha_{\text{EM}} \equiv e_0^2, \]
\[ v_H^2 = \frac{1}{\sqrt{2} G_F} - 4v_\phi^2, \]
\[ s^2 = \frac{1}{2} \left( 1 - \sqrt{\frac{4\pi \alpha_{\text{EM}}}{\sqrt{2} G_F M_Z^2}} \left( 1 - 4\sqrt{2} G_F v_\phi^2 \right) \right) \equiv s_0^2. \]  

(3)

In terms of \( M_Z^2, \alpha_{\text{EM}} \) and \( G_F \) the \( W \) boson mass is then given by

\[ M_W^2 = \frac{\pi \alpha_{\text{EM}}}{\sqrt{2} G_F s_0^2} \equiv (M_W^2)_0. \]  

(4)

and all measurable \( \rho \) parameters that can be defined [11] are equal:

\[ \rho_{\text{low}} = \rho = \rho_{\text{Zf}} = \frac{1}{1 - 4\sqrt{2} G_F v_\phi^2}. \]  

(5)

It is therefore clear that the first condition for decoupling, i.e. for decoupling at the tree level, of the triplet is vanishing (in the limit) of its tree level VEV \( v_\phi \).

The tree level VEVs \( v_\phi \) and \( v_H \)

\[ \langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_H \end{pmatrix}, \quad \langle \phi^0 \rangle = v_\phi \]  

(6)

are determined by the Higgs potential whose most general form is

\[ V = m_H^2 H^\dagger H + \frac{\lambda_H}{4} (H^\dagger H)^2 + m_\phi^2 \text{tr} (\Phi^2) + \frac{\lambda_\phi}{4} (\text{tr}\Phi^2)^2 \\
+ \kappa H^\dagger H \text{tr} (\Phi^2) + \mu H^\dagger \Phi H \]  

(7)

where \( \text{tr}(\Phi^2) = \frac{1}{2} (\phi^0 \phi^0 + 2\phi^+ \phi^-) \). The minimization conditions determining \( v_\phi \) and \( v_H \) read

\[ v_H \left( m_H^2 + \frac{\lambda_H}{4} v_H^2 + \frac{\kappa}{2} v_\phi^2 - \frac{1}{2} \mu v_\phi \right) = 0 \]
\[ m_\phi^2 v_\phi + \frac{\lambda_\phi}{4} v_\phi^2 + \frac{\kappa}{2} v_H^2 v_\phi - \frac{1}{4} \mu v_H^2 = 0 \]  

(8)

There are two limits in which \( v_\phi \to 0 \):
• For $\mu = 0$ and for $m_{\phi}^2$ and $\kappa$ positive ($\lambda_{\phi}$ must be positive for stability anyway) the solution to the second condition is clearly $v_{\phi} = 0$ (and evidently $v_{\phi} \neq 0$ for $\mu \neq 0$ if $v_H$ is nonzero). It is therefore easy to see that $v_{\phi} \propto \mu$ for $\mu \to 0$, independently of the magnitude of $m_{\phi}^2$.

• For positive $m_{\phi}^2 \to \infty$ and $\lambda_{\phi}$, $\kappa$ fixed the first term in the second equation of (8) cannot be canceled by the other terms unless $v_{\phi} \to 0$. In the limit $v_{\phi} \sim \frac{1}{2} \mu (v_H^2/m_{\phi}^2)$.

Thus, in both these limits the effects of the triplet in low energy electroweak observables vanish at the tree level.

To investigate whether the triplet decouples also with quantum corrections taken into account we first determine the masses of the physical scalars. They are given by

$$V_{\text{quadr}} = \left( \mu v_{\phi} + \frac{v_H^2 \mu}{4v_{\phi}} \right) H^+ H^- + \frac{1}{2} \left( h^0, \phi^0 \right) \begin{pmatrix} \frac{1}{2} \lambda_{H} v_{H}^2 & (\kappa v_{\phi} - \frac{1}{2} \mu) v_{H} & \frac{1}{2} \lambda_{\phi} v_{\phi}^2 + \frac{v_{H}^2 \mu}{4v_{\phi}} \\ (\kappa v_{\phi} - \frac{1}{2} \mu) v_{H} & \frac{1}{2} \lambda_{\phi} v_{\phi}^2 + \frac{v_{H}^2 \mu}{4v_{\phi}} & \phi^0 \end{pmatrix} \left( h^0, \phi^0 \right)$$  \hspace{1cm} (9)

The physical charged scalar $H^+$ and $G_W^+$ which becomes the longitudinal component of the massive $W^\pm$ are given by the following combinations of the charged doublet ($G^+$) and triplet ($\phi^+$) components

$$G_W^+ = c_{\delta} G^+ - s_{\delta} \phi^+ \hspace{1cm} G^+ = c_{\delta} G_W^+ + s_{\delta} H^+,$$

$$H^+ = s_{\delta} G^+ + c_{\delta} \phi^+ \hspace{1cm} \phi^+ = -s_{\delta} G_W^+ + c_{\delta} H^+,$$  \hspace{1cm} (10)

where

$$s_{\delta} = 2 \frac{v_{\phi}}{v_H} \frac{1}{\sqrt{1 + 4 v_{\phi}^2 / v_H^2}}, \hspace{1cm} c_{\delta} = \frac{1}{\sqrt{1 + 4 v_{\phi}^2 / v_H^2}}.$$  \hspace{1cm} (11)

The neutral mass eigenstates $H^0$ and $K^0$ are given by

$$\begin{pmatrix} h^0 \\ \phi^0 \end{pmatrix} = \begin{pmatrix} c_{\gamma} & -s_{\gamma} \\ s_{\gamma} & c_{\gamma} \end{pmatrix} \begin{pmatrix} H^0 \\ K^0 \end{pmatrix}.$$  \hspace{1cm} (12)

Useful relations between $c_{\gamma}$, $s_{\gamma}$ and $H^0$ and $K^0$ masses are given in Appendix A.

Here we note only that for $v_{\phi} \to 0$ the mixing between $G^+$ and $\phi^+$ as well as the mixing between $h^0$ and $\phi^0$ vanishes: For $v_{\phi} \to 0$ one has

$$s_{\delta} \approx s_{\gamma} \approx \frac{\mu v_H}{2 m_{\phi}^2} \to 0.$$  \hspace{1cm} (13)

From the formulae given above it is clear that for $\mu \to 0$ the masses of $H^+$ and of $K^0$ approach $M_{H^+} = M_{K^0} \sim m_{\phi}$. Hence although $v_{\phi} \to 0$ in this limit, loop contributions of $H^+$ and $K^0$ to electroweak observables will not be suppressed if $m_{\phi}$
is not large, and decoupling cannot hold. In contrast, for $m^2_\phi \to \infty$, $M_{H^+} = M_{K^0} \to m_\phi$ and decoupling can be expected. However, as we will show, it holds only in the limit $m^2_\phi \to \infty$, $\mu$ fixed. In the limit $\mu \propto m_\phi \to \infty$ it is broken by quantum corrections and unless the other parameters are tuned appropriately the effects of $H^+$ and $K^0$ cannot be neglected.

3 Renormalization scheme

Beyond the tree level the model can be most easily renormalized using the minimal subtraction (at some fixed renormalization scale), so that its predictions for observables are given as (finite) functions of the renormalized running Lagrangian parameters. It is customary to invert the appropriate number of these relations and to express all running parameters in terms of a chosen set of input observables (the number of the input observables must be then equal to the number of renormalized parameters), so that other observables are predicted in terms of the chosen ones. This step is however not necessary. In principle it is perfectly possible to fit the renormalized parameters to the data directly. It is also possible, as proposed in [11], to invert only a smaller set of relations and to express predictions of the model in terms of some smaller number (smaller than the number of renormalized parameters) of input observables and a complementary number of renormalized parameters.

Following this logic, if the SM is naturally embedded in an extended model, the same three “low energy” input observables can be chosen for the SM and its extension and used to eliminate the same three combinations of the renormalized parameters (e.g. $g_y$, $g_2$ and $v_H$) in both models keeping additional parameters of the extended model as renormalized free parameters. This enables direct comparison of the extended model predictions for other observables with the predictions of the SM because the structure of radiative corrections in the extended model is such that the SM contributions can be clearly separated. The decoupling of heavy extra particles cannot be then superficially spoiled and becomes easy to assess: it holds if in the limit of infinitely heavy additional particles (not present in the SM) predictions of the extended model for observables (other than the input ones) become identical with the predictions of the SM. In fact this is almost the unique way of checking the decoupling. A possible modification would be to determine additional parameters of the extended model using observables which do not exist in the SM (“high energy observables”) such as physical masses and couplings of the additional particles. This step is however an unnecessary technical complication.

Renormalization schemes which use different numbers of “low energy” input observables for the SM and for its extension do not allow for checking decoupling directly. Using more measured “low energy” input observables to fix other parameters of the extended model usually spoils natural correlations (existing in the extended theory) of the values of these parameters with masses of the heavy particles which are necessary for the decoupling to hold. Such schemes can, therefore, spoil decoupling superficially. This was demonstrated in [11] in the case of a $U(1)$ extension in
which fixing the $Z'$ gauge coupling in terms of an additional low energy observable (e.g. $\sin^2 \theta_W^{\text{eff}}$) disables in fact taking $M_{Z'}$ to infinity, if the running coupling constant is not to become nonperturbatively large.

In order to avoid such complications we shall analyse the triplet extension of the SM using for the input observables $\alpha_{\text{EM}}, G_F, M_Z$ as in the SM. The model has 4 free parameters: the gauge couplings $g_Y, g_2$ and the VEVs $v_H$ and $v_\phi$. Beyond the tree level they are interpreted as the renormalized running parameters. For given $\hat{v}_\phi$ we express $\hat{\alpha}, \hat{s}$ and $\hat{v}_H$ in terms of $\alpha_{\text{EM}}, G_F, M_Z$.

Explicit expressions for $\Delta_G$ are given in Appendix A. The last term in the expression for $\alpha_{\text{EM}}$ can be computed along the lines given in [11, 14].

Solving for $\hat{\alpha}, \hat{s}$ and $\hat{v}_H$ with one loop accuracy we find

\begin{align}
\hat{\alpha} & = \alpha_{\text{EM}} \left( 1 - \tilde{\Pi}_\gamma(0) + \frac{\alpha_{\text{EM}}}{\pi} \ln \left( \frac{\hat{M}_W^2(0)}{Q^2} \right) \right) \\
\hat{v}_H^2 & = \frac{1}{\sqrt{2} G_F} (1 + \Delta_G) - 4 \hat{v}_\phi^2 \\
\hat{s}^2 \hat{c}^2 & = \frac{\pi \alpha_{\text{EM}}}{\sqrt{2} G_F M_Z^2} \left( 1 - \tilde{\Pi}_\gamma(0) + \frac{\alpha_{\text{EM}}}{\pi} \ln \left( \frac{\hat{M}_W^2(0)}{Q^2} \right) + \frac{\tilde{\Pi}_{ZZ}(M_Z^2)}{M_Z^2} \right) \left[ 1 - 4 \sqrt{2} G_F \hat{v}_\phi^2 + \Delta_G \right]
\end{align}

where $(M_W^2)_{(0)}$ is defined in (4). The last relation gives for $\hat{s}^2$

\begin{equation}
\hat{s}^2 = \frac{1}{2} \left( 1 - \frac{4 \pi \alpha_{\text{EM}}}{\sqrt{2} G_F M_Z^2} \left( 1 - 4 \sqrt{2} G_F \hat{v}_\phi^2 + \Delta \right) \right)
\end{equation}
\[ \Delta = \Delta_G - \left(1 - 4\sqrt{2}G_F\hat{v}_\phi^2\right) \left(\hat{\Pi}_\gamma(0) - \frac{\alpha_{EM}}{\pi} \ln \left(\frac{(M_W^2(0))^2}{Q^2}\right) - \frac{\hat{\Pi}_{ZZ}(M_Z^2)}{M_Z^2}\right) \]  
\[ (19) \]

so that to one loop accuracy
\[ s^2 = s^2_0 + \frac{s^2_0 c^2_0}{c^2_0 - s^2_0} \left[ \frac{\Delta_G}{1 - 4\sqrt{2}G_F\hat{v}_\phi^2} - \hat{\Pi}_\gamma(0) + \frac{\alpha_{EM}}{\pi} \ln \left(\frac{M_W^2}{Q^2}\right) \right] \]  
\[ (20) \]

where \( s^2_0 \) is defined in (3). The formulae (17) and (20) form the basis of our renormalization scheme: they allow to express \( \hat{\alpha}, \hat{\nu}_H^2, \hat{s}_2 \) and \( \hat{c}_2 \) in formulae for electroweak observables in terms of input observables to one loop accuracy.

4 \( \rho_{\text{low}} \) in the electroweak model with \( Y = 0 \) triplet

In order to check decoupling of the triplet we compute at one loop the parameter \( \rho_{\text{low}} \) defined in terms of the ratio of the neutral and charged current terms in the low energy effective Lagrangian. It is given by
\[ \rho_{\text{low}} = \frac{a - b}{\sqrt{2}G_F}, \]  
\[ (21) \]

where \( a \) and \( b \) are the coefficients in the neutral current low energy effective Lagrangian written in the form
\[ \mathcal{L}_{\text{eff}} = [\bar{\psi}_e \gamma^\lambda (a \mathbf{P}_L + b \mathbf{P}_R) \psi_e] [\bar{\psi}_{\nu_\mu} \gamma^\lambda \mathbf{P}_L \psi_{\nu_\mu}] \]  
\[ (22) \]

In terms of the running Lagrangian parameters we get
\[ \sqrt{2}G_F \rho_{\text{low}} = \frac{\hat{e}^2}{4 \hat{s}^2 \hat{c}^2 M_Z^2} \left\{ 1 - \frac{\hat{\Pi}_{ZZ}(0)}{M_Z^2} - \frac{\hat{e}^2 \hat{c}^2}{4\pi^2 \hat{s}^2} \ln \left(\frac{\hat{M}_W^2}{Q^2}\right) \right. \]
\[ + \frac{\hat{e}^2}{64\pi^2 \hat{s}^2 \hat{c}^2} \left( 3 - 12 \hat{s}^2 + 24 \hat{s}^4 + 16 \hat{c}^4 \frac{\hat{M}_Z^2}{M_W^2(0)} \right) \]  
\[ \right\} \]  
\[ (23) \]

where the last two terms are the contributions of the vertex and box diagram corrections, respectively. These are computed as in the SM except that one cannot use the relation \( \hat{M}_W^2 = \hat{c}^2 \hat{M}_Z^2 \). To one loop accuracy, using the relations (17) and (11) we have therefore
\[ \sqrt{2}G_F \rho_{\text{low}} = \frac{\sqrt{2}G_F}{1 - 4\sqrt{2}G_F\hat{v}_\phi^2} \left\{ 1 - \frac{\Delta_G}{1 - 4\sqrt{2}G_F\hat{v}_\phi^2} - \frac{\hat{e}^2_0 \hat{c}^2_0}{4\pi^2 \hat{s}^2_0} \ln \left(\frac{M_W^2(0)^2}{Q^2}\right) \right. \]
\[ + \frac{\hat{e}^2_0}{64\pi^2 \hat{s}^2_0 \hat{c}^2_0} \left( 3 - 12 \hat{s}^2_0 + 24 \hat{s}^4_0 + 16 \hat{c}^4_0 \frac{M_Z^2}{M_W^2(0)} \right) \]  
\[ \right\} \]  
\[ (24) \]
One has to remember that the $Z^0$ and $W$ self energies (in $\Delta_G$) include tadpole contributions. Note also that in agreement with (4) to one loop

$$
\frac{1}{1 - 4\sqrt{2}G_F\tilde{v}_\phi^2 (M^2_{\tilde{v}})_{(0)}} \frac{\tilde{\Pi}_{WW}(0)}{M^2_z} - \frac{\tilde{\Pi}_{ZZ}(0)}{M^2_z} = \frac{\tilde{\Pi}_{WW}(0)}{c^2(0)M^2_z} - \frac{\tilde{\Pi}_{ZZ}(0)}{M^2_z} + \ldots \quad (25)
$$

We can now examine different contributions to $\rho$. Since the coupling of fermions to gauge bosons are as in the SM for the one particle irreducible contribution of top and bottom to $\rho_{\text{low}}$ we get

$$
\rho_{\text{low}} = \frac{1}{1 - 4\sqrt{2}G_F\tilde{v}_\phi^2} \left( 1 + \frac{\sqrt{2}G_F}{1 - 4\sqrt{2}G_F\tilde{v}_\phi^2} \frac{N_c}{16\pi^2} g(m_t, m_b) + \ldots \right) \quad (26)
$$

where $g(m_t, m_b) \approx m_t^2$ is defined in [Appendix A]. This is finite as in the SM. Moreover, it is clear that in the limit $m_\phi^2 \to \infty$, when $\tilde{v}_\phi \to 0$, the expression (26) reduces to the well known SM expression. Fermions contribute to $\rho$ also through the tadpoles, but this second contribution vanishes as $\tilde{v}_\phi \to 0$.

The full expressions for $\tilde{\Pi}_{WW}(q^2)$ and $\tilde{\Pi}_{ZZ}(q^2)$ are given in [Appendix A]. It is easy to see that the limit $m_\phi^2 \to \infty$ ($s_\gamma, s_\delta \to 0$) gauge boson contributions and most of the 1PI contributions of scalars approach the SM limit and one is left with the following dangerous, because they cancel out, so that the 1PI contributions of extra scalars to $\tilde{\Pi}_{WW}(q^2)$ and $\tilde{\Pi}_{ZZ}(q^2)$ are as in the SM for the one particle irreducible contribution of top

The complete 1PI bosonic contribution to $\rho_{\text{low}}$ is not independent of the renormalization scale $Q$. Another explicit dependence on $Q$ is introduced by tadpole contributions to $\tilde{\Pi}_{WW}$ and $\tilde{\Pi}_{ZZ}$. Due to our renormalization scheme (in which $\tilde{v}_\phi$ is not traded for an observable), the contribution of tadpoles to $\rho_{\text{low}}$ (and other observables) does not cancel out but is found to be

$$
\rho_{\text{low}} = \frac{1}{1 - 4\sqrt{2}G_F\tilde{v}_\phi^2 \left( 1 - 8\frac{\sqrt{2}G_F\tilde{v}_\phi}{1 - 4\sqrt{2}G_F\tilde{v}_\phi^2} \left( s_\gamma \frac{T_H}{M^2_{H_0}} + c_\gamma \frac{T_K}{M^2_{K^0}} \right) + \ldots \right)} \quad (28)
$$

\text{Note that for } m_\phi^2 \to \infty \ A(0, M_{K^0, M_{H^+}}) \to 0.
where $\langle H^0 \rangle_{\text{loop}} = -i T_H$, $\langle K^0 \rangle_{\text{loop}} = -i T_K$. For example fermions give

$$T_H = -2\sqrt{2}c_g \sum_f N_i^{(f)} \hat{Y}_f \hat{m}_f a(\hat{m}_f)$$

$$T_K = +2\sqrt{2}s_g \sum_f N_i^{(f)} \hat{Y}_f \hat{m}_f a(\hat{m}_f)$$

where $\hat{Y}_f$ are the Yukawa couplings. Explicit dependence of tadpoles on $Q$ is necessary to render $\rho_{\text{low}}$ renormalization scale independent [11, 15]. This is because the tree level expression for $\rho_{\text{low}}$ depends on the running parameter $v_\phi$ which also changes with the renormalization scale. Therefore to one loop accuracy

$$\rho_{\text{low}} = \frac{1}{1 - 4\sqrt{2}G_F v_\phi^2(Q)} \{1 + \ldots\}$$

$$= \frac{1}{1 - 4\sqrt{2}G_F v_\phi^2(Q^')} \left\{1 + 2\frac{\sqrt{2}G_F}{1 - 4\sqrt{2}G_F v_\phi^2(\hat{q}_\phi)} \ln \frac{Q^2}{Q'^2} + \ldots\right\} (29)$$

where $v_\phi^2 \equiv Qd\hat{v}_\phi^2/dQ$. Using the relations [A.2] it is for example easy to see that the terms $\sim \hat{Y}^4_f$ in the renormalization group equation [A.10], when inserted in [29], for $v_\phi^2$ properly change $Q$ into $Q'$ in the fermionic tadpoles. Similarly, $Q$ dependence of the gauge boson (and ghost) tadpoles

$$s_\gamma \frac{T_H^{(W,Z)}}{M_{H^0}^2} + c_\gamma \frac{T_K^{(W,Z)}}{M_{K^0}^2} = \frac{1}{16\pi^2 \text{DET}} \left[3\frac{\hat{c}_H^2}{2} \hat{v}_H^2 \tilde{\phi}^2 \hat{M}_W^2 \left(\ln \frac{\hat{M}_W^2}{Q^2} - \frac{1}{3}\right) - \frac{3}{2} \frac{\hat{c}_Z^2}{2} \hat{v}_H^2 \left(\hat{\kappa} \hat{v}_\phi - \frac{1}{2} \hat{\mu}\right) \hat{M}_W^2 \left(\ln \frac{\hat{M}_W^2}{Q^2} - \frac{1}{3}\right) - \frac{3}{4} \frac{\hat{c}_Z^2}{2} \hat{v}_H^2 \left(\hat{\kappa} \hat{v}_\phi - \frac{1}{2} \hat{\mu}\right) \hat{M}_Z^2 \left(\ln \frac{\hat{M}_Z^2}{Q^2} - \frac{1}{3}\right)\right] (30)$$

where DET is given by [A.3] combines with the terms $\propto \hat{g}_2^2, \hat{g}_3^2, \hat{g}_2 \hat{g}_3, \hat{g}_4$ in [A.10]. Contribution of $G^0$ (the $Z^0$ Goldstone) to the combination of tadpoles in (28) vanishes whereas the renormalization scale dependence of the $G^+_W, G^+_H$ (W± Goldstones) contribution

$$s_\gamma \frac{T_H^{(G^\pm)}}{M_{H^0}^2} + c_\gamma \frac{T_K^{(G^\pm)}}{M_{K^0}^2} = \frac{1}{16\pi^2 \hat{v}_H^2 + 4\hat{v}_\phi^2} \frac{4\hat{v}_\phi}{a(\hat{M}_W)} (31)$$

adds to the remaining $Q$ dependence of the bosonic 1PI contributions to $\rho_{\text{low}}$ and together they can be seen to properly match the term $\propto \hat{g}_2^2$ in [A.10].

The $H^\pm$ contribution to the tadpole combination in (28) is

$$s_\gamma \frac{T_H^{(H^\pm)}}{M_{H^0}^2} + c_\gamma \frac{T_K^{(H^\pm)}}{M_{K^0}^2} = \frac{1}{16\pi^2 \text{DET}(1 + 4\hat{v}_\phi^2/\hat{v}_H^2)}$$

$$\times \left[2\hat{\lambda}_H \hat{\mu} \hat{v}_\phi^2 + \frac{1}{4} \hat{\lambda}_H \hat{\lambda}_H \hat{v}_H^2 \hat{v}_\phi - 2\hat{\kappa} \hat{\mu} \hat{v}_\phi^2 - \hat{\kappa}^2 \hat{v}_H^2 \hat{v}_\phi - \frac{3}{2} \hat{\kappa} \hat{\mu} \hat{v}_\phi^2 + \frac{1}{2} \hat{\kappa} \hat{v}_H^2 \hat{\mu} \hat{v}_\phi\right] a(\hat{M}_{H^+}) (32)$$
whereas the neutral scalars contribution to the tadpole combination in \((28)\) reads

\[
\frac{s_\gamma T_H^{(H^0,K^0)}(H^0,K^0)}{M_H^2} + c_\gamma \frac{T_K^{(H^0,K^0)}(H^0,K^0)}{M_K^2} = \frac{1}{\text{DET}} \left[ -\frac{3}{4} \lambda_H \hat{v}_H^2 \left( \hat{v}_\phi - \frac{1}{2} \hat{\mu} \right) \left( c_\gamma^2 a(\hat{M}_H^0) + s_\gamma^2 a(\hat{M}_K^0) \right) \\
+ \frac{3}{8} \lambda_H \hat{v}_H^2 \hat{v}_\phi \left( s_\gamma^2 a(\hat{M}_H^0) + c_\gamma^2 a(\hat{M}_K^0) \right) \\
+ \frac{1}{4} \lambda_H \hat{v}_H^3 \left( a(\hat{M}_H^0) - a(\hat{M}_K^0) \right) 2 s_\gamma c_\gamma \\
- \frac{1}{2} \hat{v}_H \left( \hat{v}_\phi - \frac{1}{2} \hat{\mu} \right) \left( s_\gamma^2 a(\hat{M}_H^0) + c_\gamma^2 a(\hat{M}_K^0) \right) \\
+ \frac{1}{2} \lambda_H \hat{v}_H^2 \left( \hat{v}_\phi - \frac{1}{2} \hat{\mu} \right) \left( c_\gamma^2 a(\hat{M}_H^0) + s_\gamma^2 a(\hat{M}_K^0) \right) \\
- \frac{1}{2} \hat{v}_H \left( \hat{v}_\phi - \frac{1}{2} \hat{\mu} \right)^2 \left( a(\hat{M}_H^0) - a(\hat{M}_K^0) \right) 2 s_\gamma c_\gamma \right]
\]

In the limit \(m_\phi \to \infty, \mu \propto m_\phi\) the \(K^0\) and \(H^+\) tadpole contributions do not vanish in general. Their contribution to \(\rho_{\text{low}}\) is in this limit given by \((28)\) with

\[
\hat{v}_\phi \left( s_\gamma \frac{T_H^{(H^+,K^0)}}{M_H^2} + c_\gamma \frac{T_K^{(H^+,K^0)}}{M_K^2} \right) = \frac{1}{16\pi^2} \frac{3 \hat{\mu} \hat{v}_\phi (\frac{1}{2} \hat{v}_H^2 + \hat{v}_\phi) \left( -1 + \ln \frac{\hat{m}_\phi^2}{Q^2} \right)}{2 M_H^2}
\]

Hence, unless one assumes that for the particular renormalization scale \(Q\) chosen for calculations \(\ln(\hat{m}_\phi^2(Q)/Q^2) \approx 1\) or that \(\hat{v}(Q)\hat{v}_H^2(Q) + 2 \hat{\mu}(Q) \hat{v}_\phi(Q) \approx 0\) (these relations would be, of course, modified in higher orders), there is no decoupling in this limit. Our approach allows to understand this peculiarity (observed in [7]) as due to the behaviour of the triplet VEV. In the above limit the tree level VEV \(\hat{v}_\phi\) vanishes but the one loop corrections to the true VEV of the triplet do not. This is because once the \(SU(2)\) symmetry is broken (by the VEV of the doublet), the triplet is no longer protected from acquiring a nonzero VEV radiatively. Thus, although the tree level triplet VEV, as well as all 1PI one loop contributions to \(\rho_{\text{low}}\), vanish for \(\hat{m}_\phi \to \infty, \hat{\mu} \propto \hat{m}_\phi\), there is a nonzero correction to the SM result which can be accounted for by simply replacing \(v_\phi\) by one loop correction to it in the tree level term in \(\rho_{\text{low}}\). Of course, if the dimensionful coupling \(\hat{\mu}\) is kept fixed, the \(K^0\) and \(H^+\) contribution to tadpoles vanish in the limit \(\hat{m}_\phi \to \infty\) and one recovers the SM result. In order to better understand the origin of nondecoupling we analyze it in Appendix B in the simplified model with Higgs fields only and point out its connections with the custodial symmetry breaking.

## 5 The W boson mass

In this section we compare the one loop expression for the \(W\) boson mass in our scheme and in the scheme of ref. [7]. This will allow us to elucidate the question of its dependence on the top quark mass.
In our scheme the $W$ boson mass is at one loop given by

$$M_W^2 = \frac{\hat{e}^2}{4s^2}(\hat{v}_H^2 + 4\hat{\nu}_0^2) + \hat{\Pi}_{WW}(M_W^2)$$  \hspace{1cm} (34)$$

Using the one loop expressions for $\hat{e}^2$, $\hat{s}^2$ and $\hat{v}_H^2$ (17), (18) and (19) this takes the form

$$M_W^2 = \frac{\pi\alpha_{EM}}{\sqrt{2}G_F s^2(0)} \left\{ 1 - \hat{\Pi}_Z(0) + \frac{\alpha_{EM}}{\pi} \ln \left( \frac{M_W^2(0)}{Q^2} \right) + \Delta_G + \frac{\hat{\Pi}_{WW}(M_W^2(0))}{M_W^2(0)} \right. \right.$$

$$- \frac{c^2(0)}{c^2(0) - s^2(0)} \left[ \frac{\hat{\Pi}_{ZZ}(M_Z^2)}{M_Z^2} - \frac{\hat{\Pi}_{WW}(0)}{c^2(0) M_Z^2} - \hat{\Pi}_Z(0) + \frac{\alpha_{EM}}{\pi} \ln \left( \frac{M_W^2(0)}{Q^2} \right) \right.$$

$$+ \frac{1}{1 - 4\sqrt{2} G_F \hat{v}_\phi^2} \left( B_{WZ} + B_{WZ^0} + 2\hat{\Lambda} + \hat{\Sigma}_{EL} + \hat{\Sigma}_{\nu L} \right) \} \right)$$

The non-tadpole fermion contribution to this formula has formally exactly the same form as in the SM. Hence, it is finite and renormalization scale independent. Expressed in terms of the input observables $\alpha_{EM}$, $G_F$ and $M_Z$ it differs, however, from the corresponding contribution in the SM in that $s^2(0)$ and $c^2(0)$ given by (3) depend on $\hat{v}_\phi$. Still, from the expressions for $\hat{\Pi}_{WW}$ and $\hat{\Pi}_{ZZ}$ collected in the Appendix A it is clear that $M_W^2$ depends on $m_t$ quadratically. In the limit $\hat{m}_\phi^2 \to \infty$, in which $\hat{v}_\phi$ vanishes, $s^2(0)$ and $c^2(0)$ approach their SM values and the coefficient of $m_t^2$ on the right-hand side of this formula becomes as in the SM.

The renormalization scale dependence of this expression can be checked as in the case of $\rho_{low}$ using the formula

$$\frac{1}{s^2(0)(Q)} \approx \frac{1}{s^2(0)(Q')} \left\{ 1 + \frac{c^2(0)}{c^2(0) - s^2(0)} \frac{4\sqrt{2} G_F}{1 - 4\sqrt{2} G_F \hat{v}_\phi^2} \hat{v}_\phi^2 \ln \frac{Q}{Q'} \right\}$$

Similarly as for $\rho_{low}$, it can be checked that in the limit $\hat{m}_\phi \to \infty$ the 1PI contributions of extra scalars decouple from $M_W$, but the tadpoles do not cancel out and generically there is no decoupling if the dimensionful coupling $\hat{\mu}$ in (7) grows along with $\hat{m}_\phi$. Compared to the SM, the one-loop prediction of the triplet model for $M_W$ is in this limit modified only by the radiatively generated nonzero VEV of the triplet.

Of course, if $\hat{\mu}$ is kept fixed all extra contributions to $M_W$ disappear in the limit $\hat{m}_\phi \to \infty$. This limit is useful to elucidate the relation of the results obtained in [7] for $M_W$ to the SM prediction. The renormalization scheme of [7] is based on four input observables: $\alpha_{EM}$, $G_F$, $M_Z$ and $\sin^2\theta_{eff}$ (the last quantity is defined by the coupling of on-shell $Z^0$ to on-shell charged lepton-antilepton pair). The same scheme have been also used earlier in [12, 17].

At one loop the basic formulae in the scheme of refs. [7, 12, 17] read

$$M_Z^2 = \frac{1}{4} \frac{\hat{e}^2}{s^2(0)^2} \hat{v}_H^2 + \hat{\Pi}_{ZZ}(M_Z^2)$$
\[ \alpha_{\text{EM}} = \frac{e^2}{4\pi} \left( 1 + \tilde{\Pi}_\gamma(0) - \frac{\tilde{\alpha}}{\pi} \ln \frac{M_W^2}{Q^2} \right) \] (37)

\[ \sqrt{2}G_F = \frac{\hat{e}^2}{4\hat{s}\hat{v}M_W^2} = \frac{1}{\hat{s}^2 + 4\hat{v}^2} (1 + \Delta_G) \]

\[ \sin^2 \theta^\text{eff}_\ell = \hat{s}^2 (1 + \Delta_{s^2}) \]

with \( \Delta_G \) given in (10) and

\[ \Delta_{s^2} = (1 - 2\hat{s}^2) \left( \hat{\Sigma}_{VR} - \hat{\Sigma}_{VL} \right) - \frac{\hat{c}}{\hat{s}} \hat{\Pi}_Z \gamma(M_Z^2) M_Z^2 \left\{ 1 + \hat{\Pi}_\gamma(0) - \alpha_{\text{EM}} \frac{\pi}{\alpha_{\text{EM}}} \ln \frac{(M_W^2(0))}{Q^2} \right\} \]

\[ - \left( 1 - \sin^2 \theta^\text{eff}_\ell \frac{\cos^2 \theta^\text{eff}_\ell}{\alpha_{\text{EM}}} M_Z^2 \frac{\hat{\Pi}_Z \gamma(M_Z^2)}{M_Z^2} \right) \Delta_{s^2} - \frac{\hat{c}}{\hat{s}} \hat{F}_R - \frac{2\hat{s}\hat{c}}{\hat{e}} \left( \hat{F}_L - \hat{F}_R \right) \] (38)

where \( \hat{F}_L, \hat{F}_R \) are the 1PI corrections to the \( Z^0\ell\ell \) vertex and \( \hat{\Sigma}_{VL}, \hat{\Sigma}_{VR} \) are the vector parts of the charged lepton self energies. Solving the relations (37) to one loop accuracy gives

\[ \hat{\alpha} = \alpha_{\text{EM}} \left( 1 - \hat{\Pi}_\gamma(0) + \alpha_{\text{EM}} \frac{\pi}{\alpha_{\text{EM}}} \ln \frac{(M_W^2(0))}{Q^2} \right) \]

\[ \hat{s}^2 = \sin^2 \theta^\text{eff}_\ell (1 - \Delta_{s^2}) \]

\[ \hat{v}_H^2 = \frac{\sin^2 \theta^\text{eff}_\ell \cos^2 \theta^\text{eff}_\ell}{\pi \alpha_{\text{EM}}} M_Z^2 \left\{ 1 + \hat{\Pi}_\gamma(0) - \alpha_{\text{EM}} \frac{\pi}{\alpha_{\text{EM}}} \ln \frac{(M_W^2(0))}{Q^2} \right\} \]

\[ - \left( 1 - \sin^2 \theta^\text{eff}_\ell \frac{\cos^2 \theta^\text{eff}_\ell}{\alpha_{\text{EM}}} \right) \Delta_{s^2} - \frac{\hat{c}}{\hat{s}} \hat{\Pi}_Z \gamma(M_Z^2) \frac{\hat{G}}{M_Z^2} \] (39)

\[ 4\hat{v}_\phi^2 = \frac{1}{\sqrt{2}G_F} (1 + \Delta_G) - \frac{\sin^2 \theta^\text{eff}_\ell \cos^2 \theta^\text{eff}_\ell}{\pi \alpha_{\text{EM}}} M_Z^2 \left\{ 1 + \hat{\Pi}_\gamma(0) \right\} \]

\[ - \alpha_{\text{EM}} \frac{\pi}{\alpha_{\text{EM}}} \ln \frac{(M_W^2(0))}{Q^2} - \left( 1 - \sin^2 \theta^\text{eff}_\ell \frac{\cos^2 \theta^\text{eff}_\ell}{\alpha_{\text{EM}}} \right) \Delta_{s^2} - \frac{\hat{c}}{\hat{s}} \hat{\Pi}_Z \gamma(M_Z^2) \frac{\hat{G}}{M_Z^2} \] where now \( (M_W^2(0)) = \pi \alpha_{\text{EM}}/\sqrt{2}G_F \sin^2 \theta^\text{eff}_\ell \). In this scheme the one-loop formula for \( M_W^2 \) reads

\[ M_W^2 = \frac{\pi \alpha_{\text{EM}}}{\sqrt{2}G_F \sin^2 \theta^\text{eff}_\ell} \left\{ 1 - \hat{\Pi}_\gamma(0) + \alpha_{\text{EM}} \frac{\pi}{\alpha_{\text{EM}}} \ln \frac{M_W^2}{Q^2} + \Delta_{s^2} + \Delta_G + \frac{\hat{\Pi}_{WW}(M_W^2)}{(M_W^2(0))} \right\} \] (40)

It has formally the same form as the formula obtained in the SM with \( \alpha_{\text{EM}}, G_F \) and \( \sin^2 \theta^\text{eff}_\ell \) (instead of \( M_Z \)) taken for the input observables. It is also clear that tadpole contribution cancels out between \( \hat{\Pi}_{WW}(M_W^2)/(M_W^2(0)) \) and \( \Delta_G \). The dependence on the top quark mass is only logarithmic: quadratic dependence cancels out between \( \hat{\Pi}_{WW}(M_W^2)/(M_W^2(0)) \) and \( -\hat{\Pi}_{WW}(0)/(M_W^2(0)) \) in \( \Delta_G \). The difference with the SM is of course that in \( \Pi_{WW}, \Delta_{s^2}, \Delta_G \) and in \( \hat{\Pi}_\gamma(0) \) there are contributions of the extended scalar sector. They would disappear (cancel out) in the limit \( s\gamma, s\phi \to 0, \bar{M}_{K^0}, \bar{M}_{H^+} \to \hat{m}_\phi \) and one would then get the expression for \( M_W \) which is identical to the SM result in the scheme based on \( \alpha_{\text{EM}}, G_F \) and \( \sin^2 \theta^\text{eff}_\ell \), in which \( M_W^2 \) also depends on \( m_t \) only logarithmically. However, in the renormalization scheme based
on $M_Z$, $\alpha_{\text{EM}}$, $G_F$ and $\sin^2 \theta^\text{eff}_\ell$ as input observables this limit cannot be freely taken: while in terms of renormalized $\overline{\text{MS}}$ parameters the limit $\hat{m}_\phi \to \infty$, $\hat{M}_K^0, \hat{M}_{H^+} \to \hat{m}_\phi$ formally entails $s_\gamma, s_\delta \to 2\hat{v}_\phi/\hat{v}_H$, in the discussed scheme $\hat{v}_\phi$ and $\hat{v}_H$ are in fact determined by the equations (39) and their correlation with the magnitude of $\hat{m}_\phi$ is lost. In other words, in this scheme, whether $\hat{v}_\phi \to 0$ and whether this limit corresponds to the limits $\hat{M}_K^0, \hat{M}_{H^+} \to \hat{m}_\phi$ and $s_\gamma, s_\delta \to 0$ is dictated by the data and the SM contributions to observables and not by theoretical considerations (as is possible in our scheme).

One can wonder however, how decoupling of the triplet degrees of freedom in the limit, $\hat{m}_\phi \to \infty$, $\hat{\mu}$ fixed could manifest itself in the scheme based on four input observables. In particular one can wonder how the celebrated SM quadratic dependence of the $M_W \leftrightarrow M_Z$ interrelation on $m_t$ would be recovered? To discuss this it is easier to imagine for a while that the four experimental input data $\alpha_{\text{EM}}$, $G_F$, $M_Z$ and $\sin^2 \theta^\text{eff}_\ell$ can be varied freely and that they are such that (to one loop accuracy)

$$\frac{\sin^2 \theta^\text{eff}_\ell \cos^2 \theta^\text{eff}_\ell}{\pi \alpha_{\text{EM}}} M_Z^2 = \frac{1}{\sqrt{2} G_F}$$

so that that the equation for $4\hat{v}_\phi$ in (39) can be satisfied by $\hat{v}_\phi = 0$ in the decoupling limit $\hat{m}_\phi \to \infty$, $\hat{\mu}$ fixed (all additional contributions of the heavy particles cancel out or vanish in the right hand side of the equation for $4\hat{v}_\phi$). Then this equation relates the input observable $M_Z^2$ to the three other input observables $\alpha_{\text{EM}}, G_F$ and $\sin^2 \theta^\text{eff}_\ell$. It is then obvious that if the model is to fit the data in the decoupling limit, the measured $M_Z^2$ which allows for $\hat{v}_\phi \approx 0$ must change quadratically with the change of the input value of $m_t$. For $\hat{v}_\phi = 0$ the last equation in (39) is (to one loop accuracy) equivalent to

$$M_Z^2 = \frac{\pi \alpha_{\text{EM}}}{\sqrt{2} G_F \sin^2 \theta^\text{eff}_\ell \cos^2 \theta^\text{eff}_\ell} \left\{ 1 - \tilde{\Pi}_\gamma(0) + \frac{\alpha_{\text{EM}}}{\pi} \ln \frac{(M_W^2(0))}{Q^2} \right.\right.$$

$$\left. + \left( 1 - \frac{\sin^2 \theta^\text{eff}_\ell}{\cos^2 \theta^\text{eff}_\ell} \right) \Delta s^2 + \Delta G + \frac{\tilde{\Pi}_Z(M_Z^2)}{M_Z^2} \right\}$$

which is precisely the formula for $M_Z^2$ in the SM renormalized with $\alpha_{\text{EM}}, G_F$ and $\sin^2 \theta$ as the input observables which depends on $m_t$ quadratically.

Let us also note that if $\alpha_{\text{EM}}, G_F$ and $\sin^2 \theta$ were taken for the input observables in the triplet model instead of $\alpha_{\text{EM}}, G_F$ and $M_Z$, the tadpole contributions to the resulting one loop expression for $M_W$ in the triplet model would cancel out and the formula (39) would reduce to the SM expression in the limit $\hat{m}_\phi \to \infty$, even for $\hat{\mu} \propto \hat{m}_\phi$. This can be easily understood: the only dangerous effect (which spoils decoupling e.g. in $\rho_{\text{low}}$) are those which can be interpreted as corrections to the tree level VEV $\hat{v}_\phi$. Since the tree level expression for $M_W$ in the triplet model with $\alpha_{\text{EM}}, G_F$ and $\sin^2 \theta$ as the input observables is independent of $\hat{v}_\phi$, the one loop tadpole contributions must cancel out. In this scheme tadpoles, which for $\mu \propto m_\phi$ would
not decouple, would enter $M_Z$ for which the tree level expression would depend on $\hat{v}_\phi$.

Finally is also worthwhile discussing calculation of $\rho_{\text{low}}$, $M_W^2$ and other electroweak observables in a renormalization scheme using as the input observables $\alpha_{\text{EM}}$, $G_F$, $M_Z$ and in addition one "high energy" observable, which is absent in the SM. For example for the additional observable one could take the on shell value of the formfactor $\Lambda$ proportional to $g_{\mu\nu}$ in the $H^+ Z^0 W^+$ coupling to the $Z^0 W^+$ pair. Following the general procedure one would then express $\hat{\alpha}$, $\hat{s}^2$, $\hat{v}_H$ and $\hat{v}_\phi$ in terms of $\alpha_{\text{EM}}$, $G_F$, $M_Z$ and $\Lambda$. At the tree level one would then have

$$
\begin{align*}
\hat{\alpha} &= \alpha_{\text{EM}} \\
\hat{s}^2 &= \frac{1}{2} \left\{ \frac{1}{1 + \frac{1}{4\sqrt{2}G_F M_Z^4}} \left[ 1 + \frac{\Lambda^2}{4\sqrt{2}G_F M_Z^2} \right]^2 \right. \\
\hat{v}_H^2 &= \frac{1}{\sqrt{2}G_F} \left[ 1 - \frac{\Lambda^2}{4\pi \alpha_{\text{EM}} M_Z^2 \hat{s}^2} \right] \\
\hat{v}_\phi^2 &= \frac{1}{4\sqrt{2}G_F \sqrt{4\pi \alpha_{\text{EM}}} M_Z^2 \hat{s}^2}
\end{align*}
$$

(43)

The decoupling limit would then correspond to taking $\Lambda \to 0$ (this is possible, as $\Lambda$ is not fixed by experiment yet).

Formulae necessary to express $\hat{\alpha}$, $\hat{s}^2$, $\hat{v}_H$ and $\hat{v}_\phi$ to one loop can be obtained by substituting in (43)

$$
\begin{align*}
\alpha_{\text{EM}} &\to \alpha_{\text{EM}} \left( 1 - \Pi_{\gamma}(0) + \frac{\alpha_{\text{EM}}}{\pi} \ln \left( \frac{M_W^2(0)}{Q^2} \right) \right) \\
\frac{1}{\sqrt{2}G_F} &\to \frac{1}{\sqrt{2}G_F} (1 + \Delta_G) \\
M_Z^2 &\to M_Z^2 \left( 1 - \frac{\Pi_{ZZ}(M_Z^2)}{M_Z^2} \right) \\
\Lambda &\to \Lambda - \delta\Lambda
\end{align*}
$$

(where $\delta\Lambda$ is a one loop correction to the on shell $H^+ Z^0 W^+$ vertex) and expanding the resulting expressions appropriately.

Since in such a scheme the tree level expression for electroweak observables do not depend on $\hat{v}_\phi$, the tadpole contributions must cancel out (just as they do in the SM). Nondecoupling of the Higgs triplet effects would then manifest itself through the corrections $\delta\Lambda$. Indeed, the $g_{\mu\nu}$ formfactor of the $H^+ Z^0 W^+$ coupling receives, among others, a contribution from the one loop diagram with $H^+ G_W^+ H^0$ coupling and $G_W^+$, $H^0$ and $Z^0$ circulating in the loop. Since the particles in the loop are light, the loop integral is not suppressed by any heavy mass factor. Moreover, as is easy to check, the $H^+ G_W^+ H^0$ coupling is proportional to $\mu$ and in fact grows in the limit $\mu \sim m_\phi \to \infty$. In electroweak observables the correction $\delta\Lambda$ is always multiplied by $\Lambda$. Now, for $\mu \propto m_\phi$ and fixed values of the dimensionless Higgs potential
couplings, the limit \( M_{K^0} \sim M_{H^+} \rightarrow m_\phi \rightarrow \infty \) requires that \( \Lambda \) vanishes only as \( 1/\mu \). Hence, \( \propto \Lambda \delta \Lambda \) contribution to electroweak observables does not disappear and the decoupling is violated. One should stress however, that introducing an additional “high energy” observable like \( \Lambda \) makes the analysis of the decoupling much more complicated not only from the point of view of practical calculations but also conceptually.

6 \( S, T, U \) parameters and other issues

Our calculation carries also an important message for the calculations of the \( S, T \) and \( U \) parameters introduced in [18] and widely used to constrain extensions of the SM. (Application of these parameters to the model considered in this paper can be found in [4]. Applications of the parameters \( S, T, U \) to models with triplets have been also considered in [20]) Using these parameters implicitly assumes working in the renormalization scheme defined in section 3 with \( \alpha_{\text{EM}}, G_F \) and \( M_Z \) used as the only input observables and treating other parameters of the tested model as renormalized running parameters.

At one loop the expressions for these parameters in the triplet model read [4]

\[
\begin{align*}
\alpha_{\text{EM}}T &= \frac{\hat{\Pi}_{WW}^{\text{new}}(0)}{(c^2 M_W^2)_{(0)\text{SM}}} - \frac{\hat{\Pi}_{ZZ}^{\text{new}}(0)}{M_Z^2} + 4\sqrt{2}G_F \hat{v}_\phi^2 \\
\alpha_{\text{EM}}S &= 4(s^2 c^2)_{(0)\text{SM}} \left\{ \hat{\Pi}_{ZZ}^{\text{new}} - \left( \frac{c^2 - s^2}{c s} \right)_{(0)\text{SM}} \hat{\Pi}_{Z\gamma}^{\text{new}} - \hat{\Pi}_{\gamma}^{(0)\text{new}} \right\} \\
\alpha_{\text{EM}}U &= 4s^2_{(0)\text{SM}} \left\{ \hat{\Pi}_{WW}^{\text{new}} - c^2_{(0)\text{SM}} \hat{\Pi}_{ZZ}^{\text{new}} - 2(sc)_{(0)\text{SM}} \hat{\Pi}_{Z\gamma}^{\text{new}} - s^2_{(0)\text{SM}} \hat{\Pi}_{\gamma}^{(0)\text{new}} \right\}
\end{align*}
\]

where \( \hat{\Pi}(g^2) = \hat{\Pi}(0) + g^2 \hat{\Pi}' \). Those corrections to electroweak observables that can be interpreted as corrections to the gauge boson propagators can be expressed in terms of these parameters. For example, in terms of \( S, T \) and \( U \) the corrections to the \( W \) boson mass due to the triplet extension of the SM read

\[
\delta M_W^2 = (M_W^2)_{(0)\text{SM}} \left\{ \left( \frac{c^2}{c^2 - s^2} \right)_{(0)\text{SM}} \alpha_{\text{EM}}T + \frac{1}{4s^2_{(0)\text{SM}}} \alpha_{\text{EM}}U - \frac{1}{2} \left( \frac{1}{c^2 - s^2} \right)_{(0)\text{SM}} \alpha_{\text{EM}}S \right\}
\]

This agrees with the full one loop corrections to \( M_W \) (35) if one sets \( \hat{v}_\phi^2 = 0 \) in the one loop part of (35), neglects the (“nonoblique”) box and vertex corrections and expands \( s^2_{(0)} \) in the prefactor of (35) to first order in \( \hat{v}_\phi^2 \) (this contribution is accounted by the term \( 4\sqrt{2}G_F \hat{v}_\phi^2 \) in \( T \) in (44)). It is therefore clear that the expression for \( T \) should also include the tadpole contribution. Neglecting tadpoles in \( T \) is equivalent to the (tacit) assumption that the the model parameters are taken at the renormalization scale \( Q \) for which tadpole contribution to \( T \) happens to vanish and that it is just at this scale \( Q \) that \( \hat{v}_\phi \) in the tree level term in \( T \) is small.

In general however, in all SM extensions, in which the tree level expressions for electroweak observables depend on VEVs of additional Higgs bosons tadpoles must
be included in $T$. The minimal supersymmetric extension (the MSSM) is special here because because the tree level masses of the gauge bosons depend on the same combination of the corresponding two VEVs.

Finally let us notice that Grand Unified Theories (GUTs) generically give rise to $SU(2)$ triplets which are assumed to have mass parameters $m_\phi \sim M_{\text{GUT}}$. Since in GUTs the analog of the parameter $\mu$ is also typically of the same order (it arises from a GUT gauge symmetry breaking VEV) nonsupersymmetric GUTs generically suffer from the problem of nondecoupling of $SU(2)$ triplets (the problem of justifying vanishing of their effects adds to the standard hierarchy problem of such models).

The problem does not arise in supersymmetric GUTs because there fermion contributions to tadpoles cancel against bosonic ones in the limit of exact supersymmetry. Therefore, in realistic models tadpoles are suppressed by $m_{\text{soft}}/M_{\text{GUT}}$ where $m_{\text{soft}} \sim \mathcal{O}(1 \text{ TeV})$ is a typical soft supersymmetry breaking scale.

## 7 Discussion

We have applied the renormalization scheme based on three input observables $\alpha_{\text{EM}}$, $G_F$ and $M_Z$ to the extension of the standard model with a Higgs field transforming as an $SU(2)$ triplet. As we have argued, such a scheme allows for straightforward investigation of the question of Appelquist-Carrazzone decoupling of additional heavy particles. We have explicitly shown that in the model with an $Y = 0$ triplet the decoupling does not hold if the dimensionful trilinear coupling grows along with the triplet mass parameter. Our approach allowed us to attribute this effect to a nonzero triplet VEV generated by radiative corrections for nonzero VEV of the Higgs doublet. We have also checked that similar nondecoupling of heavy particle effects is present in models with $Y = \pm 1$ triplets which arise in Littlest Higgs models [10] or in models aiming at protecting primordial baryon asymmetry [19]. At one loop the effects of the heavy triplet in electroweak observables can be negligible only for severely tuned parameters of the model. It appears, however, unlikely that they can be eliminated in this way from all electroweak observables in higher orders.

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Appendix A Useful formulae

Relations for scalar masses. From (9) and (12) the following useful relations can be derived:

\[
c^2_{\gamma} M^2_{H^0} + s^2_{\gamma} M^2_{K^0} = \frac{1}{2} \lambda_H v_H^2 \\
s^2_{\gamma} M^2_{H^0} + c^2_{\gamma} M^2_{K^0} = \frac{1}{2} \lambda_\phi v_\phi^2 + \frac{v_H^2 \mu}{4 v_\phi} \\
c_{\gamma} s_{\gamma} (M^2_{H^0} - M^2_{K^0}) = v_H (\kappa v_\phi - \frac{1}{2} \mu)
\]

and

\[
\frac{s^2_{\gamma}}{M^2_{H^0}} + \frac{c^2_{\gamma}}{M^2_{K^0}} = \frac{1}{2} \lambda_H v_H^2 \text{DET} \\
\frac{c^2_{\gamma}}{M^2_{H^0}} + \frac{s^2_{\gamma}}{M^2_{K^0}} = \frac{1}{2} \lambda_\phi v_\phi^2 + \frac{v_H^2 \mu}{4 v_\phi} \\
- s_{\gamma} c_{\gamma} \left( \frac{1}{M^2_{H^0}} - \frac{1}{M^2_{K^0}} \right) = -v_H (\kappa v_\phi - \frac{1}{2} \mu) \text{DET}
\]

Where

\[
\text{DET} \equiv M^2_{H^0} M^2_{K^0} = \frac{1}{4} v_H^2 v_\phi^2 \left[ \lambda_H \lambda_\phi + \frac{\lambda_H v_H^2 \mu}{v_\phi^2} - 4 \kappa^2 + 4 \frac{\mu^2}{v_\phi^2} - \frac{\lambda_H v_H^2 \mu}{v_\phi^2} \right]
\]

Box and vertex correction contributions to $\Delta_G$ (10). They can be calculated as in the SM (except that the relation $\hat{M}_W^2 = \hat{c}^2 \hat{M}_Z^2$ cannot be used). One then finds

\[
B_{\text{boxes}} + 2\tilde{A} + \tilde{S}_{\nu L} + \tilde{S}_{e L} = \frac{\hat{c}}{4\pi s^2} \left\{ -4 \eta_{\text{div}} - 4 \ln \frac{\hat{M}_Z^2}{Q^2} + 6 \right. \\
+ \left. \left( \frac{1}{2} - \frac{5}{2} s^2 + \frac{7 - 14 s^2 + 10 s^4}{4 s^2} \right) \ln \frac{\hat{M}_W^2}{\hat{M}_Z^2 - \hat{M}_W^2} \right\} \text{ (A.4)}
\]

For $\hat{M}_W = \hat{c}^2 \hat{M}_Z^2$ this reduces to the SM result.

Gauge boson self energies. For $\Pi_{Z\gamma}(q^2)$ in units $\hat{c}^2/16\pi^2 \hat{s} \hat{c}$ (in the Feynman gauge) we get

\[
\frac{1}{2} \sum_q N_c^{(f)} |Q_f| (1 - 4 |Q_f| s^2) \left[ 4 \hat{A}(q^2, \hat{m}_f, \hat{m}_f) + q^2 b_0(q^2, \hat{m}_f, \hat{m}_f) \right] \\
- \hat{c}^2 \left[ 8 \hat{A}(q^2, \hat{M}_W, \hat{M}_W) + (4 q^2 + 2 \hat{M}_W^2) b_0(q^2, \hat{M}_W, \hat{M}_W) - \frac{2}{3} q^2 \right] \\
- \frac{1}{2} \hat{c}^2 v_H^2 \left( 1 - \frac{\hat{c}^2 4 v_H^2}{\hat{s}^2 v_H^2} \right) b_0(q^2, \hat{M}_W, \hat{M}_W) \\
- 2 \left( (\hat{c}^2 - \hat{s}^2) c_\delta^2 + 2 \hat{c}^2 \hat{s}_\delta^2 \right) \hat{A}(q^2, \hat{M}_W, \hat{M}_W) \\
- 2 \left( (\hat{c}^2 - \hat{s}^2) \hat{s}_\delta^2 + 2 \hat{c}^2 \hat{c}_\delta^2 \right) \hat{A}(q^2, \hat{M}_H^+, \hat{M}_H^+)
\]
For the 1PI contribution to $\Pi_{ZZ}(q^2)$ the one loop formula (in units $\bar{\epsilon}^2/16\pi^2s^2\bar{\epsilon}^2$) is

\[
\frac{1}{2} \sum_f N_{cf} a'_f \left[ 2\tilde{A}(q^2, \tilde{m}_f, \tilde{m}_f) + \left( \frac{q^2}{2} - \tilde{m}_f^2 \right) b_0(q^2, \tilde{m}_f, \tilde{m}_f) \right] \\
+ \frac{1}{2} \sum_f N_{cf} a'_f m_f^2 b_0(q^2, \tilde{m}_f, \tilde{m}_f) + 3 \left[ \tilde{A}(q^2, 0, 0) + \frac{q^2}{4} b_0(q^2, 0, 0) \right] \\
- \bar{\epsilon}^4 \left[ 8\tilde{A}(q^2, \tilde{M}_W, \tilde{M}_W) + (4q^2 + 4\tilde{M}_W^2) b_0(q^2, \tilde{M}_W, \tilde{M}_W) - \frac{2}{3} q^2 \right] \\
- \left[ (\bar{e}^2 - s^2) c_\delta^2 + 2\bar{e}^2 \bar{s}^2 \right]^2 \tilde{A}(q^2, \tilde{M}_W, \tilde{M}_W) \\
- \left[ (\bar{e}^2 - s^2)s_\delta^2 + 2\bar{e}^2 c_\delta^2 \right]^2 \tilde{A}(q^2, \tilde{M}_H^+, \tilde{M}_H^+) \\
- 2c_\delta^2 \bar{s}^2 \tilde{A}(q^2, \tilde{M}_W, \tilde{M}_H^+) \\
- c_\gamma^2 \tilde{A}(q^2, \tilde{M}_Z, \tilde{M}_H^o) \\
- s_\gamma^2 \tilde{A}(q^2, \tilde{M}_Z, \tilde{M}_K^o) \\
+ 2\bar{e}^2 \bar{c}^2 \left( \frac{1}{2} v_H \hat{s} c_\delta - c_\delta \bar{c} \hat{s} \right)^2 b_0(q^2, \tilde{M}_W, \tilde{M}_W) \\
+ 2\bar{e}^2 \bar{c}^2 \left( \frac{1}{2} v_H \hat{s} \bar{s} c_\delta + c_\delta \bar{c} \bar{s} \right)^2 b_0(q^2, \tilde{M}_W, \tilde{M}_H^+) \\
+ \tilde{M}_Z^2 c_\gamma^2 b_0(q^2, \tilde{M}_Z, \tilde{M}_H^o) \\
+ \tilde{M}_Z^2 s_\gamma^2 b_0(q^2, \tilde{M}_Z, \tilde{M}_K^o) \\
\]  
\[(A.5)\]

where the sums in the two first terms extend to all fermions except neutrinos (which are accounted for by the third term), $a'_f = 1 - 4|Q_f|\hat{s}^2 + 8|Q_f|^2\bar{s}^4$, $a'_f = -4|Q_f|\bar{s}^2 + 8|Q_f|^2\hat{s}^4$.

Finally, in units $\bar{\epsilon}^2/16\pi^2s^2$, for the 1PI part of $\Pi_{WW}(q^2)$ we have

\[
\frac{1}{2} \sum_{k=1}^3 \left[ 4\tilde{A}(q^2, \tilde{m}_{e_k}, 0) + (q^2 - \tilde{m}_{e_k}^2) b_0(q^2, \tilde{m}_{e_k}, 0) \right] \\
+ \frac{3}{2} \sum_{k,l=1}^3 |V_{CKM}^{kl}|^2 \left[ 4\tilde{A}(q^2, \tilde{m}_{u_k}, \tilde{m}_{u_l}) + (q^2 - \tilde{m}_{u_k}^2 - \tilde{m}_{u_l}^2) b_0(q^2, \tilde{m}_{u_k}, \tilde{m}_{u_l}) \right] \\
- s^2 \left[ 8\tilde{A}(q^2, \tilde{M}_W, 0) + (4q^2 + \tilde{M}_W^2) b_0(q^2, \tilde{M}_W, 0) - \frac{2}{3} q^2 \right] \\
- \bar{\epsilon}^2 \left[ 8\tilde{A}(q^2, \tilde{M}_W, \tilde{M}_Z) + (4q^2 + \tilde{M}_W^2 + \tilde{M}_Z^2) b_0(q^2, \tilde{M}_W, \tilde{M}_Z) - \frac{2}{3} q^2 \right] \\
- 4 \left( s_\gamma^2 + \frac{1}{2} c_\gamma c_\delta \right)^2 \tilde{A}(q^2, \tilde{M}_W, \tilde{M}_H^o) \\
- 4 \left( -s_\gamma^2 + \frac{1}{2} c_\gamma c_\delta \right)^2 \tilde{A}(q^2, \tilde{M}_H^+, \tilde{M}_H^o) \\
- 4 \left( c_\gamma \delta - \frac{1}{2} s_\gamma c_\delta \right)^2 \tilde{A}(q^2, \tilde{M}_W, \tilde{M}_K^o) \\
\]
\[ -4 \left( -c_\gamma c_\delta - \frac{1}{2} s_\gamma s_\delta \right)^2 \tilde{A}(q^2, \tilde{M}_{H^+}, \tilde{M}_{K^0}) \]
\[ - c_\delta^2 \tilde{A}(q^2, \tilde{M}_W, \tilde{M}_Z) \]
\[ - s_\delta^2 \tilde{A}(q^2, \tilde{M}_{H^+}, \tilde{M}_Z) \]
\[ + \varepsilon^2 \left( \frac{1}{2} v_H c_\delta + v_\phi s_\delta \right)^2 b_0(q^2, \tilde{M}_W, 0) \]
\[ + \varepsilon^2 \left( \frac{1}{2} v_H \frac{\hat{s}}{c} c_\delta - v_\phi \frac{\hat{c}}{s} s_\delta \right)^2 b_0(q^2, \tilde{M}_W, \tilde{M}_Z) \]
\[ + \varepsilon^2 \left( \frac{1}{2} v_H \frac{\hat{s}}{c} s_\delta + v_\phi \frac{\hat{c}}{s} c_\delta \right)^2 b_0(q^2, \tilde{M}_{H^+}, \tilde{M}_Z) \]
\[ + \frac{\varepsilon^2}{s^2} \left( \frac{1}{2} v_H c_\gamma + 2 v_\phi s_\gamma \right)^2 b_0(q^2, \tilde{M}_W, \tilde{M}_{H^0}) \]
\[ + \frac{\varepsilon^2}{s^2} \left( - \frac{1}{2} v_H s_\gamma + 2 v_\phi c_\gamma \right)^2 b_0(q^2, \tilde{M}_W, \tilde{M}_{K^0}) \]

**Trilinear couplings of $H^0$ and $K^0$ relevant for tadpoles.** The couplings are written as e.g. $\mathcal{L} \supset - \lambda^{0}_{HGG} H^0 G^0 G^0 - \lambda^{\pm}_{HGG} H^0 G^+_W G^-_W$ etc. and the factors $\lambda$ read

\[
\lambda^0_{HGG} = \frac{1}{2} \left[ \frac{3}{2} \lambda_H v_H c_\gamma + \left( \kappa v_\phi - \frac{1}{2} \mu \right) c_\gamma \right]
\]
\[ \lambda^0_{KGG} = \frac{1}{2} \left[ - \frac{3}{2} \lambda_H v_H s_\gamma + \left( \kappa v_\phi - \frac{1}{2} \mu \right) s_\gamma \right] \]
\[ \lambda^0_{HHH} = \frac{1}{2} \left[ \frac{1}{2} \frac{3}{2} \lambda_H v_H c_\gamma^3 + \frac{1}{2} \lambda_\phi v_\phi s_\gamma^3 + \kappa v_H c_\gamma s_\gamma^2 + \left( \kappa v_\phi - \frac{1}{2} \mu \right) c_\gamma^2 s_\gamma \right] \]
\[ \lambda^0_{KHH} = \frac{1}{2} \left[ \frac{1}{2} \lambda_H v_H^2 c_\gamma^2 s_\gamma + \frac{3}{2} \lambda_\phi v_\phi c_\gamma^2 s_\gamma - \kappa v_H (s_\gamma^3 - 2 c_\gamma^2 s_\gamma) + \left( \kappa v_\phi - \frac{1}{2} \mu \right) (c_\gamma^3 - 2 c_\gamma s_\gamma^2) \right] \]
\[ \lambda^0_{HHK} = \frac{1}{2} \left[ - \frac{1}{2} \lambda_H v_H s_\gamma^3 + \frac{3}{2} \lambda_\phi v_\phi c_\gamma^2 s_\gamma + \kappa v_H (c_\gamma^3 - 2 c_\gamma s_\gamma^2) + \left( \kappa v_\phi - \frac{1}{2} \mu \right) (c_\gamma^3 - 2 c_\gamma s_\gamma^2) \right] \]
\[ \lambda^0_{KHK} = \frac{1}{2} \left[ - \frac{1}{2} \lambda_H v_H^2 s_\gamma^2 c_\gamma + \frac{3}{2} \lambda_\phi v_\phi c_\gamma^2 s_\gamma - \kappa v_H c_\gamma^2 s_\gamma + \left( \kappa v_\phi - \frac{1}{2} \mu \right) c_\gamma^2 s_\gamma \right] \]
\[ \lambda^0_{HHH} = \frac{1}{2} \left[ \frac{1}{2} \lambda_H v_H c_\gamma^3 + \frac{1}{2} \lambda_\phi v_\phi c_\gamma^2 s_\gamma + \kappa v_H c_\gamma s_\gamma^2 + \left( \kappa v_\phi + \frac{1}{2} \mu \right) c_\gamma c_\delta - \mu c_\gamma s_\delta c_\delta \right] \]
\[ \lambda^0_{KGG} = \frac{1}{2} \left[ - \frac{1}{2} \lambda_H v_H s_\gamma c_\delta + \frac{1}{2} \lambda_\phi v_\phi c_\gamma s_\delta - \kappa v_H s_\gamma s_\delta^2 + \left( \kappa v_\phi + \frac{1}{2} \mu \right) c_\gamma c_\delta^2 + \mu s_\gamma s_\delta c_\delta \right] \]
\[ \lambda^0_{HHH} = \frac{1}{2} \left[ \frac{1}{2} \lambda_H v_H c_\gamma^3 + \frac{1}{2} \lambda_\phi v_\phi c_\gamma^2 s_\delta + \kappa v_H c_\gamma s_\delta^2 + \left( \kappa v_\phi + \frac{1}{2} \mu \right) c_\gamma c_\delta^2 + \mu c_\gamma s_\delta c_\delta \right] \]
\[ \lambda^0_{KHH} = \frac{1}{2} \left[ - \frac{1}{2} \lambda_H v_H s_\gamma c_\delta - \frac{1}{2} \lambda_\phi v_\phi c_\gamma c_\delta + \kappa v_\phi c_\gamma + v_H c_\gamma c_\delta s_\delta + \frac{1}{2} \mu (c_\gamma^2 - s_\delta^2) c_\gamma \right] \]
\[ \lambda^0_{HGH} = \frac{1}{2} \left[ \frac{1}{2} \lambda_H v_H c_\gamma c_\delta s_\delta - \frac{1}{2} \lambda_\phi v_\phi c_\gamma c_\delta c_\delta + \kappa c_\gamma c_\delta + v_H c_\gamma c_\delta s_\delta + \frac{1}{2} \mu (c_\gamma^2 - s_\delta^2) c_\gamma \right] \]
Renormalization group equations. For the $Y = 0$ triplet model the renormalization group equations for the potential parameters are given in [16] and read

\[ Q \frac{dm_H^2}{dQ} = \left[ - \left( \frac{9}{2} g_y^2 + \frac{3}{2} g_y^2 - 2 \sum_f Y_f^2 - 3 \lambda_H \right) m_H^2 + 3 \kappa m_H^2 + \frac{3}{2} \mu^2 \right] \]

\[ Q \frac{dm_2^2}{dQ} = \left[ - \left( 12 g_y^2 - \frac{5}{2} \lambda_\phi \right) m_2^2 + 4 \kappa m_H^2 + \mu^2 \right] \quad (A.8) \]

\[ Q \frac{d\lambda_H}{dQ} = \left[ \frac{9}{2} g_y^4 + 3 g_y^2 g_\gamma^2 + \frac{3}{2} g_\gamma^4 - \left( 9 g_y^2 + 3 g_y^2 - 4 \sum_f Y_f^2 \right) \lambda_H + 6 \lambda_H^2 + 6 \kappa^2 - 8 \sum_f Y_f^4 \right] \]

\[ Q \frac{d\lambda_\phi}{dQ} = \left[ 48 g_y^4 - 24 g_y^2 \lambda_\phi + \frac{11}{2} \lambda_\phi^2 + 8 \kappa^2 \right] \]

\[ Q \frac{dk}{dQ} = \left[ 6 g_y^4 - \left( \frac{33}{2} g_y^2 + \frac{3}{2} g_\gamma^2 - 2 \sum_f Y_f^2 \right) k - 3 \lambda_H k + \frac{5}{2} \lambda_\phi k + 4 \kappa^2 \right] \]

\[ Q \frac{d\mu}{dQ} = \mu \left( - \frac{21}{2} g_y^2 - \frac{3}{2} g_\gamma^2 + 2 \sum_f Y_f^2 + \lambda_H + 4 \kappa \right) \quad (A.9) \]

The renormalization group equation for $v_\phi$ can be derived combining the equations (A.8) with (A.9). After some algebra one gets

\[ \frac{1}{\lambda_H} [\ldots] \dot{v}_\phi^2 = -16 \sum_f N_c^{(f)} Y_f^2 (kv_\phi - \frac{1}{2} \mu) \frac{v_H^2}{\lambda_H v_\phi} \]

\[ + 3 (g_y^2 + g_\gamma^2)^2 (kv_\phi - \frac{1}{2} \mu) \frac{v_H^2}{\lambda_H v_\phi} \]

\[ + 6 g_y^4 (kv_\phi - \frac{1}{2} \mu) \frac{v_H^2 + 4 v_\phi^2}{\lambda_H v_\phi} \]

\[ - 12 g_y^2 (v_H^2 + 4 v_\phi^2) \]

\[ + 12 g_y^2 v_\phi^2 \frac{1}{\lambda_H} [\ldots] \]

\[ - \frac{5}{2} \lambda_\phi \mu \frac{v_H^2}{\lambda_H} - 3 \lambda_\phi v_\phi^2 + 4 \lambda_H v_H^2 \]

\[ - 8 \mu^2 - 8 \kappa^2 v_\phi^2 - 2 \lambda_H \mu v_\phi^2 \]

\[ + 4 \kappa \mu v_H^2 + 6 \kappa^2 \mu v_\phi^2 - 24 \frac{\kappa^2 \mu v_\phi}{\lambda_H} \]

\[ + 4 \lambda_\phi \frac{v_H^2}{\lambda_H} v_\phi^2 + 16 \kappa^3 v_\phi^2 \frac{v_H^2}{\lambda_H} + 20 \kappa \mu \frac{v_\phi}{\lambda_H} \]

\[ - 2 \lambda_\phi \kappa \mu \frac{v_H^2}{\lambda_H} - 3 \kappa \mu \frac{v_H^2}{\lambda_H v_\phi^2} - 6 \frac{\mu}{\lambda_H} \frac{v_\phi}{v_\phi} \]

(A.10)
where we have denoted by [...] the expression
\[ [...] \equiv \lambda_H \lambda_\phi + 4 \frac{\kappa \mu}{v_\phi} - 4 \kappa^2 - \frac{\mu^2}{v_\phi^2} + \frac{\lambda_H \mu v_H^2}{v_\phi^2} \] (A.11)
proportional to DET in (A.3): DET = \( \frac{1}{4} v_H^2 v_\phi^2 \ldots \).

Some loop functions. For completeness we recall the definitions:

\[ 16\pi^2 a(m) = m^2 \left( \eta_{\text{div}} - 1 + \ln \frac{m^2}{\mu^2} \right) \] (A.12)

\[ 16\pi^2 b_0(q^2, m_1, m_2) = \eta_{\text{div}} + \int_0^1 dx \ln \frac{q^2 x (x - 1) + x m_1^2 + (1 - x) m_2^2}{\mu^2} \] (A.13)

\[ \tilde{A}(q^2, m_1, m_2) = \frac{1}{6} a(m_1) - \frac{1}{6} a(m_2) + \frac{1}{6} (m_1^2 + m_2^2 - \frac{q^2}{2}) b_0(q^2, m_1, m_2) \]
\[ + \frac{m_1^2 - m_2^2}{12 q^2} \left[ a(m_1) - a(m_2) - (m_1^2 - m_2^2) b_0(q^2, m_1, m_2) \right] \]
\[ - \frac{1}{16\pi^2} \frac{1}{6} (m_1^2 + m_2^2 - \frac{q^2}{3}) \] (A.14)

\( \tilde{A}(0, m_1, m_2) \) is finite and reads

\[ 16\pi^2 \tilde{A}(0, m_1, m_2) = -\frac{1}{8} \left[ m_1^2 + m_2^2 - \frac{2 m_1^2 m_2^2}{m_1^2 - m_2^2} \log \frac{m_1^2}{m_2^2} \right] \equiv -\frac{1}{8} g(m_1, m_2) \] (A.15)

**Appendix B  Nondecoupling of the Higgs triplet**

Since the nondecoupling of the Higgs triplet effects may have important consequences for model building we elucidate it here from another point of view in a simple model with two fields only, the doublet \( H \) and the triplet \( \phi \). The most general interaction potential is given by (7). It is instructive to look at this problem from the symmetry point of view. The original triplet model (7) has the global \( SU(2) \times U(1) \) symmetry. VEVs of \( H \) and \( \phi \) break this symmetry down to \( U(1) \) (identified in the electroweak model with the electromagnetic symmetry). After spontaneous symmetry breaking by VEVs (6) the fields \( G^0 \) and \( G^\pm_W \) are massless to all orders and their scattering amplitudes have properties specific for Goldstone bosons (in particular they vanish at the threshold). Decoupling of the triplet would mean that in the limit \( m_\phi^2 \to \infty \) all scattering amplitudes of these Goldstone bosons and \( H^0 \) can be reproduced by starting from the Lagrangian containing only the doublet \( H \):

\[ L_{\text{eff}} = z_H \partial_\mu H^\dagger \partial^\mu H + z_H m_\text{eff}^2 H^\dagger H + z_H^2 \frac{\lambda_{\text{eff}}}{4} (H^\dagger H)^2, \] (B.1)
where $H = (G^+_W, (v + H^0 + iG^0)/\sqrt{2})^T$. The factors $z_H = 1 + \delta z_H$, $m_{\text{eff}}^2 = m_H^2 + \delta m_H^2$ and $\lambda_{\text{eff}} = \lambda_H + \delta \lambda_H$ would be then determined order by order in perturbation calculus. The model (B.1) is known to possess the $SU(2)_L \times SU(2)_R$ symmetry which the $H$ VEV breaks down to the so-called custodial $SU(2)_V$ symmetry. The $G^0$ and $G^\pm_W$ amplitudes calculated in the model (B.1) satisfy therefore $SU(2)_V$ relations which in principle do not follow from the original theory: in the potential (4) the larger $SU(2)_L \times SU(2)_R$ symmetry is explicitly broken by the $\mu$ term. Hence, one can expect that when $\mu$ grows along with $m_\phi$ the effects of explicit $SU(2)_L \times SU(2)_R$ breaking do not disappear. If it is indeed the case, then no effective renormalizable Lagrangian for fields $G^0, G^\pm_W$ and $H^0$ can reproduce the amplitudes obtained from (7). This is because in the original theory (7) $G^0, G^\pm_W$ are true Goldstone bosons and such an effective Lagrangian would have to ensure their masslessness to all orders. This is only possible if they are Goldstone bosons also at the effective Lagrangian level. (B.1) is however the only renormalizable Lagrangian that ensures masslessness of $G^0, G^\pm_W$ but it leads to exact $SU(2)_V$ symmetry. This is in contrast to what one could expect from looking at Feynman diagrams in the symmetric phase of the original theory with $H$ and $\phi$: one could think that even for $\mu \sim m_\phi$ the decoupling should hold with the tree level matching condition (corrected successively in higher loops)

$$\lambda_{\text{eff}} = \lambda_H - \frac{\mu^2}{2m_\phi^2}. \quad (B.2)$$

This is indeed how the decoupling works at the tree level even in the broken phase because for $m_\phi \to \infty$ the tree level VEV $v_\phi$ vanishes irrespectively of the behaviour of $\mu$.

To show that for $\mu \sim m_\phi$ the Lagrangian (B.1) indeed cannot reproduce amplitudes of the original theory it is sufficient to point out only one contribution which does not follow from (B.1). To this end we consider one loop tadpole corrections to the $H^0G^0G^0$ and $H^0G^+G^-$ couplings shown in figure 4. If the decoupling holds, such contributions, although 1-particle reducible, should be (up to terms suppressed as $m_\phi \to \infty$) reproduced by the 1-PI $H^0G^0G^0$ and $H^0G^+G^-$ vertices calculated in the effective theory (B.1). The contribution of the diagram 4 reads

$$-i2\lambda^0_{KG^0G^0} \frac{1}{M_{K^0}^2 - q^2} \left\{2\lambda^0_{KH^0} \frac{T_H}{M_{H^0}^2} + 2\lambda^0_{KH^K} \frac{T_K}{M_{K^0}^2} \right\}$$

$$= -i2\lambda^0_{KG^0G^0} \frac{1}{M_{K^0}^2 - q^2} \left\{ + \left(\kappa v_\phi - \frac{1}{2}\mu \right) \left(c_\gamma^2 - s_\gamma^2\right) \left(\frac{c_\gamma}{M_{H^0}^2} T_H - \frac{s_\gamma}{M_{H^0}^2} T_K\right) + \ldots \right\} \quad (B.3)$$

where we have displayed only the most important term. By using the relations (A.2) it is easy to show that the leading terms in the combination of tadpoles appearing in (B.3) comes from the contribution of $K^0$ and $H^+$

$$\frac{c_\gamma}{M_{H^0}^2} T_H - \frac{s_\gamma}{M_{H^0}^2} T_K = \frac{1}{M_{H^0}^2 M_{K^0}^2} v_H \left(\kappa m_\phi^2 + \frac{1}{2}\mu^2\right) \left[\frac{1}{2}a(M_{K^0}) + a(M_{H^+}^2)\right] \quad (B.4)$$
Since $a(M_{K^0}) \sim a(M_{H^\pm}^2) \sim m_\phi^2$ this is of order $m_\phi^2$. Therefore, expanding (B.3) in powers of $q^2/M_{K^0}^2$ we get for $\mu \sim m_\phi$ among the unsuppressed terms the contribution

$$-i 2\lambda_{KGG}^0 \frac{q^2}{M_{H^\pm}^4} \left( -\frac{1}{2}\mu \right) \frac{1}{M_{H^0}^2 M_{K^0}^2} v_H \left( \kappa m_\phi^2 + \frac{1}{2} \mu^2 \right) \left[ \frac{1}{2} a(M_{K^0}) + a(M_{H^\pm}^2) \right]$$  \(\text{(B.5)}\)

Since $2\lambda_{KGG}^0 = (-\frac{1}{2}\mu) + \ldots$ we see that if $\mu \sim m_\phi$ there is a contribution to the vertex $H^0G^0G^0$ which is unsuppressed. It is also a matter of simple analysis to see that nontadpole contributions (as well as diagrams with tadpoles attached directly to the $H^0G^0G^0$ vertex via the $H^0H^0G^0G^0$ and $H^0K^0G^0G^0$ quartic couplings) cannot give such a contribution. Clearly, this contribution cannot be reproduced by the renormalizable Lagrangian (B.1). Moreover, since the coupling $\lambda_{KGG}^\pm$ has the leading term proportional to $\mu$ with opposite sign compared to $2\lambda_{KGG}^0$, it is clear that the similar contribution to the $H^0G^+_{W^+}G^-_{W^-}$ vertex is different, thus breaking the custodial $SU(2)_V$ symmetry. Another contribution to the $H^0G^+_{W^+}G^-_{W^-}$ vertex comes from the diagrams shown in figure 1b and c. For the leading terms they give

$$-i \left( \frac{1}{2}\mu \right) \left[ \frac{p_1^2}{M_{H^\pm}^4} + \frac{p_2^2}{M_{H^\pm}^4} \right] \left( -\frac{1}{2}\mu \right) \frac{v_H}{M_{H^0}^2 M_{K^0}^2} \left( \kappa m_\phi^2 + \frac{1}{2} \mu^2 \right) \left[ \frac{1}{2} a(M_{K^0}) + a(M_{H^\pm}^2) \right]$$  \(\text{(B.6)}\)

which again cannot be reproduced by the renormalizable Lagrangian.

Thus, we have shown that indeed, for $\mu \sim m_\phi$ in the original theory there are corrections to the $H^0G^0G^0$ and $H^0G^+_{W^+}G^-_{W^-}$ vertices which would require nonrenormalizable terms

$$\Delta L_{\text{eff}} \propto \frac{v_H}{M_{H^0}^2} \left[ \frac{1}{2} G^0G^0(\partial^2 H^0) - G^+_{W^+}G^-_{W^-}(\partial^2 H^0) + (\partial^2 G^+_{W^+})G^0_{W^-}H^0 + G^+_{W^-}(\partial^2 G^-_{W^-})H^0 \right]$$  \(\text{(B.7)}\)

and which break the custodial $SU(2)_V$ symmetry.
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