On Turán-good graphs

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Abstract

For graphs $H$ and $F$, the generalized Turán number $\text{ex}(n, H, F)$ is the largest number of copies of $H$ in an $F$-free graph on $n$ vertices. We say that $H$ is $F$-Turán-good if $\text{ex}(n, H, F)$ is the number of copies in the $(\chi(F) - 1)$-partite Turán graph, provided $n$ is large enough.

We present a general theorem in case $F$ has an edge whose deletion decreases the chromatic number. In particular, this determines $\text{ex}(n, P_k, C_2^\ell + 1)$ and $\text{ex}(n, C_2^k, C_2^\ell + 1)$ exactly, if $n$ is large enough. We also study the case when $F$ has a vertex whose deletion decreases the chromatic number.

1 Introduction

A basic result in extremal Combinatorics is Turán’s theorem [18]. It states that a $K_k$-free graph on $n$ vertices cannot have more edges than the Turán graph $T_{k-1}(n)$, which is the complete $(k - 1)$-partite graph where each partite class has cardinality $\lfloor n/(k - 1) \rfloor$ or $\lceil n/(k - 1) \rceil$. In general, Turán theory deals with the function $\text{ex}(n, F)$, which is the largest number of edges in $n$-vertex $F$-free graphs. Let $\mathcal{N}(H, G)$ denote the number of copies of $H$ in $G$. Generalized Turán theory deals with $\text{ex}(n, H, F) := \max\{\mathcal{N}(H, G) : G \text{ is an } n\text{-vertex } F\text{-free graph}\}$, i.e. the largest number of copies of $H$ in $F$-free graphs on $n$ vertices.

After several sporadic results (see e.g. [2] [11] [13] [15] [12] [19]), the systematic study of this problem was initiated by Alon and Shikhelman [1]. Since then, this problem has attracted several researchers, see e.g. [3] [4] [5] [7] [8] [10] [14] [16].

However, there are not many exact results in this area (by exact result we mean that for given $H$ and $F$, we know the value of $\text{ex}(n, H, F)$ for every $n$ large enough). Most of the exact results are when the Turán graph contains the most copies of $H$. Győri, Pach and Simonovits [13] examined for what graphs $H$ do we have that $\text{ex}(n, H, K_{k+1}) = \mathcal{N}(H, T_k(n))$. Gerbner and Palmer [9] extended these investigations for arbitrary $k$-chromatic graphs. Following them, given a graph $F$ with $\chi(F) = k$, we say that $H$ is $F$-Turán-good if $\text{ex}(n, H, F) = $
\[ \mathcal{N}(H, T_{k-1}(n)) \] and \( H \) does not contain \( F \). If \( F = K_k \), we use the briefer term \( k \)-Turán good.

Let us state the main result of Győri, Pach and Simonovits \[13\] using this term.

**Theorem 1.1** (Győri, Pach and Simonovits \[13\]). Let \( r \geq 3 \) and let \( H \) be a \((k-1)\)-partite graph with \( m > k-1 \) vertices, containing \( \lfloor m/(k-1) \rfloor \) vertex disjoint copies of \( K_{k-1} \). Suppose further that for any two vertices \( u \) and \( v \) in the same connected component of \( H \), there is a sequence \( A_1, \ldots, A_s \) of \((k-1)\)-cliques in \( H \) such that \( u \in A_1, v \in A_s \), and for any \( i < s \), \( A_i \) and \( A_{i+1} \) share \( k-2 \) vertices. Then \( H \) is \( k \)-Turán-good. Moreover, if \( n \) is large enough, the Turán graph is the only \( K_k \)-free graph with \( ex(n, H, K_k) \) copies of \( H \).

Gerbner and Palmer \[9\] obtained a theorem of a similar flavor.

**Theorem 1.2** (Gerbner, Palmer \[9\]). Let \( H \) be a \( k \)-Turán-good graph. Let \( H' \) be any graph constructed from \( H \) in the following way. Choose a complete subgraph of \( H \) with vertex set \( X \), add a vertex-disjoint copy of \( K_{k-1} \) to \( H \) and join the vertices in \( X \) to the vertices of \( K_{k-1} \) by edges arbitrarily. Then \( H' \) is \( k \)-Turán-good.

Neither of the above two theorems imply the other. The main difference is that vertices in the additional clique can be connected to anything in Theorem 1.1, but only to vertices of another clique in Theorem 1.2. The trade-off is the necessity of the strong connection property of the cliques in Theorem 1.1. Let us remark that for \( k = 3 \), the assumptions of Theorem 1.1 are nothing else but that \( H \) is bipartite and has a matching of size \( \lfloor |V(H)|/2 \rfloor \); the property of the sequence of 2-cliques reduces to the property that every connected component is connected.

Observe that on their own, both theorems use \( K_{k-1} \) as building blocks, and can be used only for graphs mostly covered by vertex-disjoint copies of \( K_{k-1} \). Therefore, another difference is that in Theorem 1.2 we can start with an arbitrary \( k \)-Turán-good graph, and add cliques afterwards. Here we prove such a strengthening for Theorem 1.1.

**Proposition 1.3.** Let \( H \) be a \( k \)-Turán-good graph with a unique proper \((k-1)\)-coloring. Let \( H' \) consist of \( H \) and a copy \( K \) of \( K_{k-1} \) with vertices \( v_1, \ldots, v_{k-1} \), with additional edges between \( V(H) \) and \( V(K) \) such that for every \( i \leq k-1 \), there is a copy of \( K_{k-1} \) in \( H' \) containing \( v_i \), but not containing any \( v_j \) for \( j > i \). If \( H' \) has chromatic number \( k-1 \), then \( H' \) is \( k \)-Turán-good.

Let us show an example where this proposition is stronger than the above theorems. We will start with a slightly unbalanced complete bipartite graph. Ma and Qiu \[16\] showed that \( K_{s,t} \) with \( s \leq t \) is 3-Turán-good if and only if \( t < s + 1/2 + \sqrt{2s + 1}/4 \). Proposition 1.3 implies that if the vertices of a connected bipartite graph \( H \) can be vertex-disjointly covered by one such \( K_{s,t} \) and a matching, then \( H \) is 3-Turán-good.

Let us turn our attention to \( F \)-Turán-good graphs where \( F \) is not a clique. We show a weak version of the above results for this case. We say that an edge of a graph \( G \) is a color-critical edge if deleting it from \( G \) decreases its chromatic number. An \( m \)-chromatic graph \( F \) with a color-critical edge often behaves similarly to \( K_m \) in extremal problems. In particular, Simonovits \[17\] showed that for \( n \) large enough, the Turán graph \( T_{m-1}(n) \) contains the most...
number of edges, and it was extended by Ma and Qiu [16], who showed that $T_{m-1}(n)$ also contains the most number of $K_r$ for $r < m$. Gerbner [6] proved a stability version.

**Lemma 1.4** (Gerbner [6]). Let $F$ be a $k$-chromatic graph with a color-critical edge and $r < k$. If $G$ is an $n$-vertex $F$-free graph with chromatic number more than $k - 1$, then $ex(n, K_r, F) - N(G, K_r) = \Omega(n^{r-1})$.

Using this, we can extend the above theorems from $K_k$ to certain graphs with color-critical edges. The main idea is that if an $F$-free graph does not have too many copies of $K_k$, then those create only a negligible amount of copies of $H$.

**Theorem 1.5.** Let $F$ be a $k$-chromatic graph with a color-critical edge such that $ex(n, K_k, F) = o(n^{k-1})$, and $H$ be a graph that is both $F$-Turán-good and $k$-Turán-good. Let $H'$ be any graph constructed from $H$ in the following way. Choose a complete subgraph of $H$ with vertex set $X$, add a vertex-disjoint copy of $K_{k-1}$ to $H$ and join the vertices in $X$ to the vertices of $K_{k-1}$ by edges arbitrarily. Then $H'$ is $F$-Turán-good. Moreover, if $G$ is an $n$-vertex $F$-free graph with chromatic number more than $m - 1$, then $ex(n, H', F) - N(H', G) = \Omega(n^{|V(H')|-1})$.

**Proposition 1.6.** Let $F$ be a $k$-chromatic graph with a color-critical edge such that $ex(n, K_k, F) = o(n^{k-1})$. Let $H$ be a graph that is both $F$-Turán-good and $k$-Turán-good with a unique proper $(k - 1)$-coloring. Let $H'$ consist of $H$ and a copy $K$ of $K_{k-1}$ with vertices $v_1, \ldots, v_{k-1}$, with additional edges between $V(H)$ and $V(K)$ such that for every $i \leq k - 1$, there is a copy of $K_{k-1}$ in $H'$ containing $v_i$, but not containing $v_j$ for $j > i$. If $H'$ has chromatic number $k - 1$, then $H'$ is $F$-Turán-good. Moreover, if $G$ is an $n$-vertex $F$-free graph with chromatic number more than $m - 1$, then $ex(n, H', F) - N(H', G) = \Omega(n^{|V(H')|-1})$.

Let us note the extra assumption $ex(n, K_k, F) = o(n^{k-1})$. For $k = 3$, an example is $F = C_{2\ell+1}$, as $ex(n, K_3, C_{2\ell+1}) = O(n^{\ell+1/\ell})$ due to Győri and Li [12]. Another example is the book $B_3$, which consists of an edge $uv$ and $t$ other vertices, that are adjacent to both $u$ and $v$. Alon and Shikhelman [1] showed $ex(n, K_3, B_t) = o(n^2)$.

Gerbner and Palmer [9] conjectured that paths $P_m$ and even cycles $C_{2m}$ are $C_{2\ell+1}$-Turán-good for any $m$ and $\ell$. They proved this conjecture for $P_4$ and $\ell = 2$. They also showed that if $P_{2m}$ is $C_{2\ell+1}$-Turán-good, then $C_{2m}$ is $C_{2\ell+1}$-Turán-good too. Thus we can fully resolve their conjecture using Theorem 1.5 or Proposition 1.6.

**Corollary 1.7.** For any positive integers $m$ and $\ell$, $P_m$ and $C_{2m}$ are $C_{2\ell+1}$-Turán-good.

Gerbner and Palmer [9] also showed that $P_4$ is $B_2$-Turán-good. We can generalize it as follows.

**Corollary 1.8.** For any positive integers $m$ and $t$, $P_m$ is $B_t$-Turán-good.

Gerbner and Palmer [9] showed an example where a graph $H$ is $F$-Turán-good, and $F$ does not have a color-critical edge ($H = C_4$ and $F$ is the 2-fan, two triangles sharing a vertex). Observe that one can add additional edges to the Turán graph without violating
the $F$-free property in this case. However, those edges cannot create additional copies of $H$. A natural question is for what graphs $F$ can we find an $H$ such that $H$ is $F$-Turán-good?

We say that a vertex $v$ of a graph $G$ is color-critical if by deleting $v$ from $G$ we obtain a graph with smaller chromatic number.

**Theorem 1.9.** There exists an $F$-Turán-good graph if and only if $F$ has a color-critical vertex.

We prove Proposition 1.3, Theorem 1.5 and Proposition 1.6 in Section 2, and we prove Theorem 1.9 in Section 3. We finish the paper with some concluding remarks in Section 4.

## 2 Graphs with a color-critical edge

We start with the proof of Proposition 1.3 that we restate here for convenience.

**Proposition.** Let $H$ be a $k$-Turán-good graph with a unique proper $(k - 1)$-coloring. Let $H'$ consist of $H$ and a copy $K$ of $K_{k - 1}$ with vertices $v_1, \ldots, v_{k - 1}$, with additional edges between $V(H)$ and $V(K)$ such that for every $i < k - 1$, there is a copy of $K_{k - 1}$ in $H'$ containing $v_i$, but not containing $v_j$ for $j > i$. If $H'$ has chromatic number $k - 1$, then $H'$ is $k$-Turán-good.

**Proof.** Let $G$ be a $K_k$-free graph on $n$ vertices. We will count the copies of $H'$ in $G$ the following way. First we pick a copy $K$ of $K_{k - 1}$, then a vertex-disjoint copy $H'$ of $H$. Then we pick an actual embedding of $H$ into $H'$, and afterwards an actual embedding of $K$ into $K'$ such that the images of the remaining edges of $H'$ are present in $G$. We claim that $T_{k - 1}(n)$ gives the maximum for each of the above four factors, finishing the proof.

We have $N(K_{k - 1}, G) \leq N(H, T_{k - 1}(n))$ by a theorem of Zykov [19] (this particular case also follows from Theorem 1.4), thus the number of ways to pick a copy $K$ of $K_{k - 1}$ in $G$ is the largest if $G$ is the Turán graph. Then there are at most $\text{ex}(n - k + 1, H, K_k)$ ways to pick a vertex-disjoint copy of $H$, which is at most $T_{k - 1}(n - k + 1)$ for $n$ large enough, as $H$ is $k$-Turán-good. Observe that we have equality here in case $G = T_{k - 1}(n)$, as removing a maximal clique from the Turán graph gives a smaller Turán graph. The number of ways $H$ can be embedded into $H'$ is the number of isomorphisms of $H$ and does not depend on $G$.

After $H$ is embedded, we claim that there is at most one way to finish the embedding. We pick the images of the vertices $v_i$ from $K$ one by one, in the order of their indices. For each $v_i$, it is contained in a copy $K''$ of $K_{k - 1}$ that is already embedded. The other $k - 2$ vertices of $K''$ are already embedded, and their images have at most one common neighbor in $K'$, as $G$ is $K_k$-free. This means we only have one vertex that can be picked as $v_i$.

Finally, we show that in the Turán graph, there is a way to finish the embedding. The other $k - 2$ vertices of $K''$ that are already embedded must belong to different partite classes of the Turán graph, thus $v_i$ must be from the remaining partite class. We have to show that this way the $v_i$’s are mapped to different vertices. This is where we use the unique coloring property of $H$. Let $j$ be a color class in this unique coloring, then all the vertices of color $j$ are mapped to the same partite class $A_j$ of the Turán graph. If $v_i$ is of color $j$ in $H'$, then the other $k - 2$ vertices of $K''$ are not of color $j$, thus they are not in $A_j$, hence $v_i$ is mapped to
the vertex of $K'$ that belong to $A_j$. As the vertices of $K$ belong to different color classes in $H$, they are mapped to different partite classes of the Turán graph, finishing the proof. ■

Let us continue with the proof of Theorem 1.5 that we restate here for convenience.

Theorem. Let $F$ be a $k$-chromatic graph with a color-critical edge such that $\text{ex}(n, K_k, F) = o(n^{k-1})$, and $H$ be a graph that is both $F$-Turán-good and $k$-Turán-good. Let $H'$ be any graph constructed from $H$ in the following way. Choose a complete subgraph of $H$ with vertex set $X$, add a vertex-disjoint copy of $K_{k-1}$ to $H$ and join the vertices in $X$ to the vertices of $K_{k-1}$ by edges arbitrarily. Then $H'$ is $F$-Turán-good. Moreover, if $G$ is an $n$-vertex $F$-free graph with chromatic number more than $m - 1$, then $\text{ex}(n, H', F) - N(H', G) = \Omega(n^{\Omega(V(H')-1)})$.

Proof. By a result of Ma and Qiu [16], if $n$ is large enough, then the maximum number of copies of $K_{k-1}$ in an $F$-free graph is achieved by the Turán graph $T_{k-1}(n)$. Since $H$ is $F$-Turán-good, the Turán graph $T_{k-1}(n-k+1)$ has the maximum number of copies of $H$ among $F$-free graphs on $n-k+1$ vertices. We will show that $T_{k-1}(n)$ has the maximum number of copies of $H'$.

Let $G$ be an $F$-free graph on $n$ vertices with the maximum number of copies of $H'$. If $G$ has chromatic number at most $k-1$, then $G$ is $K_k$-free, thus we are done by Theorem 1.2. If $G$ has chromatic number more than $k-1$, then by Lemma 1.4 we have $\mathcal{N}(K_{k-1}, T_{k-1}(n)) - \mathcal{N}(K_{k-1}, G) = \Omega(n^{k-2})$.

We follow the proof of Theorem 1.2 from [9]. We take a complete subgraph $Y$ of $G$, disjoint from $K$, and consider the bipartite subgraph $G'$ of $G$ consisting of the edges between $K$ and $Y$. It was shown in [9] that if $G$ is $K_k$-free, then a matching covering $Y$ is missing from $G'$. It is easy to see that if such a matching is not missing, then not only there is a $K_k$ in $G$, but there is a $K_k$ with vertices in $Y \cup V(K)$. For sake of completeness, we repeat the argument here. By Hall’s theorem, there is a subset $Y' \subset Y$ such that all the vertices of $Y'$ are connected to all but less than $Y'$ vertices of $K$. Then $Y'$ and those vertices form a clique of size at least $k$.

Let us count first the copies of $H'$ such that there is no $K_k$ in $G$ on the vertex sets of them. For those copies there is a matching missing from $G'$. Observe that in the Turán graph between a clique of size $k-1$ and a clique of size $|Y|$, only a matching covering the smaller clique is missing. This implies that after picking a copy of $H$, there are at least as many ways to connect the appropriate sub clique of it to $K$ in the Turán graph, as in $G$.

The number of such copies of $H'$ is at most the product of the number of copies of $K_{k-1}$, the number of copies of $H$ on the remaining $n-k+1$ vertices and the number of ways to join the vertices of $K_{k-1}$ and $H$, all divided by the number of times a copy of $H'$ was counted. The first quantity is less than $\mathcal{N}(K_{k-1}, T_{k-1}(n))$ by $\Omega(n^{k-2})$. The second quantity is maximized by the Turán graph and is $O(n^{\Omega(V(H))})$, the third quantity is also maximized by the Turán graph, while the last quantity depends only on $H'$. This implies that the number of such copies of $H'$ is $\mathcal{N}(H', T_{k-1}(n)) - \Omega(n^{\Omega(V(H')-1)})$.

Let us continue with the copies of $H'$ that contain a vertex set of $K_k$ in $G$. As $G$ is $F$-free, there are $o(n^{k-1})$ copies of $K_k$ in $G$, thus $o(n^{\Omega(V(H')-1)})$ copies of $H'$. Adding up the two bounds finishes the proof. ■
Let us continue with Proposition 1.6 that we restate here for convenience. We only give a sketch of the proof, as it can be easily obtained by combining the above two proofs. We assume familiarity with those proofs.

**Proposition.** Let $F$ be a $k$-chromatic graph with a color-critical edge such that $\text{ex}(n, K_k, F) = o(n^{k-1})$. Let $H$ be an $F$-Turán-good graph with a unique proper $(k-1)$-coloring. Let $H'$ consist of $H$ and a copy $K$ of $K_{k-1}$ with vertices $v_1, \ldots, v_{k-1}$, with additional edges between $V(H)$ and $V(K)$ such that for every $i \leq k-1$, there is a copy of $K_{k-1}$ in $H'$ containing $v_i$, but not containing $v_j$ for $j > i$. If $H'$ has chromatic number $k-1$, then $H'$ is $F$-Turán-good. Moreover, if $G$ is an $n$-vertex $F$-free graph with chromatic number more than $m - 1$, then $\text{ex}(n, H', F) - N(H', G) = \Omega(n^{(H')^{-1}})$.

**Sketch of proof.** Let $G$ be an $F$-free graph on $n$ vertices. If $G$ has chromatic number $k - 1$, then it is $K_k$-free and we are done. If $G$ has chromatic number at least $k$, then by Lemma 1.4 $G$ has less copies of $K_{k-1}$ than the Turán graph by $\Omega(n^{k-2})$.

First we count copies of $H'$ in $G$ such that there are no $k$ vertices in that copy that induce a clique in $G$. For them, we can follow the proof of Proposition 1.3 with one exception: the number of $(k-1)$-cliques is less by $\Omega(n^{k-2})$, which implies that the number of copies of $H$ is less by $\Omega(n^{(H')^{-1}})$.

Then we count the other copies of $H'$: as there are $o(n^{k-1})$ copies of $K_k$ in $G$, we have that there are $o(n^{(H')^{-1}})$ such copies of $H$. Adding up the two bounds finishes the proof.

3. **Graphs with a color-critical vertex**

We will use progressive induction. This is a version of induction that can be used to prove combinatorial statements that hold only for $n$ large enough. It was introduced by Simonovits [17]. It was used for a generalized Turán problem in [5]. The statement in [17] is very general, here we state a version adapted for generalized Turán problems.

For a graph $G$ and a subgraph $G'$, we denote by $N_I(H, G, G')$ the number of copies of $H$ in $G$ that contain at least one vertex from $G'$. Given $H$ and $F$, we say that $G$ is an extremal graph if $\text{ex}(n, H, F) = N(H, G)$.

**Lemma 3.1.** Let $F$ and $H$ be graphs and $G_n$ be an $F$-free graph for every $n$. Let $\mathcal{G}_n$ be a family of $n$-vertex $F$-free graphs for every $n$, such that if $G_1, G_2 \in \mathcal{G}_n$, then $N(H, G_1) = N(H, G_2)$. Assume that there is an $n_0$ such that for every $n \geq n_0$, for every extremal graph $G$ on $n$ vertices, there is a subgraph $H'$ of $G$ with $|V(H')| \leq n/2$, such that $H'$ is also the subgraph of some $G_n \in \mathcal{G}_n$ and we have the following: $N_I(H, G, H') \leq N_I(H, G_n, H')$, with equality only if $G \in \mathcal{G}_n$.

Then for $n$ large enough, $\text{ex}(n, H, F) = N(H, G_n)$ for some $G_n \in \mathcal{G}_n$. Moreover, every extremal graph belongs to $\mathcal{G}_n$.

We omit the proof of this specialized version. It follows from the original version [17] in a straightforward way, but it is also easy to see why it holds without knowing the original proof. For small $n$ it is possible that some graph has more copies of $H$ than any $G_n \in \mathcal{G}_n$. 


However, this means a surplus of constant many copies of $H$, and then this surplus starts decreasing when $n \geq n_0$, and eventually vanishes.

We will also use the following result.

**Theorem 3.2** (Gerbner and Palmer [8]). Let $H$ be a graph and $F$ be a graph with chromatic number $k$, then $\text{ex}(n, H, F) \leq \text{ex}(n, H, K_k) + o(n^{\nu(H)})$.

The harder part of Theorem 1.9 follows from the next theorem.

**Theorem 3.3.** Let $H$ be a complete $k$-partite graph $K_{b_1,\ldots,b}$ and $F$ be a complete $(k+1)$-partite graph $K_{1,a,\ldots,a}$ such that $b > 2a - 2$. Then $H$ is $F$-Turán-good. Moreover, every extremal graph contains $T_k(n)$.

**Proof.** Let $\mathcal{G}_n$ be the family of $F$-free graphs containing $T_k(n)$, and observe that they each contain the same number of copies of $H$. Indeed, fix a $G_n \in \mathcal{G}_n$ and a copy of $T_k(n)$ in it, and let us call the edges not in that copy extra edges. Then a vertex $v$ can be incident to at most $a - 1$ extra edges. If a copy of $H$ contains an extra edge between $u$ and $v$, that means each other vertex of that $H$ is adjacent to at least one of $u$ and $v$, thus $H$ contains at most $2a - 4$ other vertices from that partite set of $T_k(n)$ where $u$ and $v$ belong. Therefore, if a copy of $H$ in $G_n$ contains a set $U$ of more than $2a - 2$ vertices from a partite set of $T_k(n)$, then there are no extra edges inside $U$, thus no edges at all inside $U$. Thus $U$ is a subset of a partite set of $H$, in particular $|U| \leq b$. Thus the only way to choose $kb$ vertices from the $k$ partite sets of $T_k(n)$ is to choose $b$ from each, and they have to form the partite sets of $H$, thus $H$ does not use any extra edge.

Let $G$ be an $F$-free graph on $n$ vertices with $\mathcal{N}(H, G) = \text{ex}(n, H, F)$ and assume indirectly that progressive induction (Lemma 3.1) cannot be applied to finish the proof, i.e. for any subgraph $H'$ of $G$ with $|V(H')| \leq n/2$, we do not have that $\mathcal{N}_1(H, G, H') \leq \mathcal{N}_1(H, G, T_k(n))$ with equality only if $G \in \mathcal{G}_n$. Let $x$ denote the minimum number of copies of $H$ that a vertex in the Turán graph $T_k(n)$ is contained in.

**Claim 3.4.** Every vertex $v$ is contained in at least $x$ copies of $H$ in $G$.

**Proof.** Otherwise let $H'$ be the graph containing only $v$, and we can apply Lemma 3.1 to finish the proof, a contradiction. 

Observe that if the degree of $v$ is $o(n)$, then we are in this situation: every copy of $H$ that contains $v$ also contains at least one vertex in its neighborhood, thus we can count the copies of $H$ containing $v$ by picking a neighbor of $v$ ($o(n)$ ways), then picking $|V(H)| - 2$ other vertices ($O(n^{\nu(H)} - 2)$ ways).

Therefore, we can assume that every vertex has linear degree. We will try to apply Lemma 3.1 with $H'$ being the complete $k$-partite graph $K_{c,\ldots,c}$ with $c = 2a - 1$. If $H'$ is not a subgraph of $G$, then $\mathcal{N}(H, G) = 0$. We pick a copy of $H'$ with partite sets $A_1, \ldots, A_k$, and with a slight abuse of notation we denote that copy by $H'$ and its vertex set by $U'$, and we also denote a copy of $H'$ in $T_k(n)$ by $H'$. 

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To apply Lemma 3.1 in this case, we need to show that \( N_t(H, G, H') \leq N_t(H, T_k(n), H') \), with equality only if \( G \in G_n \). Assume that \( N_t(H, G, H') \geq N_t(H, T_k(n), H') \). Observe that the number of copies of \( H \) containing \( t \) vertices from \( U' \) is \( \Theta(n^{|V(H)|-t}) \) in \( T_k(n) \), and \( O(n^{|V(H)|-t}) \) in \( G \). Therefore, we will focus on the main term \( n^{|V(H)|-1} \), thus on the copies of \( H \) that contain exactly one vertex from \( U' \). Let \( H_0 \) denote the complete \( k \)-partite graph \( K_{b-1,b,b,...,b} \).

Let \( G' \) be the subgraph of \( G \) induced on the vertices not in \( U' \). Observe that if a vertex in \( G' \) is connected to each \( A_i \) by at least \( a \) vertices, then they form a copy of \( F \), a contradiction. Thus for each vertex \( v \) of \( G' \), there is at least one \( A_i \) such that \( v \) is connected to at most \( a - 1 \) vertices of \( A_i \). Note that \( v \) can be connected to each vertex of some other \( A_j \). We say that a copy of \( H_0 \) in \( G' \) is **nice** if all the vertices of the \( k - 1 \) larger parts of this copy are connected to all the \( c \) vertices in some \( A_j \).

**Claim 3.5. All but \( o(n^{|V(H_0)|}) \) copies of \( H_0 \) in \( G' \) are nice.**

**Proof.** Let us consider a copy of \( H \) that contains exactly one vertex from \( U' \). It means it contains a copy of the complete \( k \)-partite graph \( H_0 := K_{b-1,b,b,...,b} \) in \( G' \). By Theorem 3.2, \( G' \) contains at most \( (1+o(1))ex(n-kc, H_0, K_{k+1}) \) copies of \( H_0 \). Theorem 1.1 shows that \( T_k(m) \) is \((k+1)\)-Turán-good for every \( m \), thus we have that \( ex(n-kc, H_0, K_{k+1}) = N(H_0, T_k(n-kc)) \).

It is easy to see that in the Turán graph, every copy of \( H_0 \) that avoids the selected copy of \( H' \) can be extended to a copy of \( H \) with one of \( c \) vertices of that copy of \( H' \) (those in the same partite class of the Turán graph). We claim that in \( G' \), every copy of \( H_0 \) can be extended to a copy of \( H \) with at most \( c \) of the vertices of \( H' \). Indeed, let \( U'' \) be the set of vertices in \( U' \) that are connected to every vertex in the \((k-1)\) larger partite sets of \( H_0 \). If \( U'' \) intersects some partite sets in at least \( a \) vertices, and has another vertex \( v \) in another partite set, then \( v \), those \( a \) vertices, and \( a \) vertices from each of the the \( k-1 \) larger partite sets of \( H_0 \) form a copy of \( F \), a contradiction. If \(|U''| > c \), then this is the case. Moreover, if \(|U''| = c \), then we have the same situation, unless \( U'' \) is a partite set of \( H' \).

The main term of \( N_t(H, G, H') \) is at most \((1+o(1))N(H_0, T_k(n-kc))\) times \( c \), minus the number of those copies of \( H_0 \) in \( G' \) that are not connected to the \( c \) vertices of one part of \( H' \) (as they should be counted at most \( c-1 \) times). If the last term is \( \Omega(n^{|V(H_0)|}) \), then the main term is smaller than the main term of \( N_t(H, T_k(n), H') \), a contradiction finishing the proof of the claim.

Let us return to the proof of the theorem. Consider a copy of \( H \) in \( G' \). We say that a vertex \( v \) of it is **replaceable** (with respect to that copy) if deleting \( v \) we obtain a nice copy of \( H_0 \), i.e. all the vertices of the \( k-1 \) larger parts of that copy of \( H_0 \) are connected to the \( c \) vertices of one part of \( H' \). If that part is \( A_i \), we say that \( v \) is replaceable by \( A_i \).

Consider an arbitrary vertex \( v \) in \( G' \). We know that \( v \) is in \( \Omega(n^{|V(H)|-1}) \) copies of \( H \) in \( G' \), and \( \Omega(n^{|V(H)|-2}) \) of those share a vertex with \( H' \). This implies that there are \( \Omega(n^{|V(H)|-1}) \) copies of \( H_0 \) in \( G' \) that can be extended to a copy of \( H \) with \( v \). \( \Omega(n^{|V(H)|-1}) \) of those copies of \( H_0 \) can also be extended to a copy of \( H \) with any one vertex from \( A_i \) for some \( i \leq k \) by Claim 3.3, i.e. \( v \) is replaceable with respect to \( \Omega(n^{|V(H)|-1}) \) copies of \( H \). Let \( B_i \) denote the set of vertices in \( G' \) such that there are \( \Omega(n^{|V(H)|-1}) \) copies of \( H_0 \) in \( G' \) that can be extended
to a copy of $H$ with $v$ or with any vertex of $A_i$, i.e. $v$ is replaceable by $A_i$ with respect to $\Omega(n^{V(H)^{-1}})$ copies of $H$.

Observe that every $v \in B_i$ is connected to less than $a$ vertices of $A_i$. Indeed, let us consider a copy of $H_0$ that $v$ extends to a copy of $H$. If $v$ is connected to $a$ vertices of $A_i$, then we can take $a$ vertices from each of the $k - 1$ larger partite classes of $H_0$, $a$ neighbors of $v$ from $A_i$ and $v$ to obtain a copy of $F$.

**Claim 3.6.** All but $o(n)$ vertices in $B_i$ are connected to all the vertices in every $A_j$ with $j \neq i$.

**Proof.** Consider a copy of $H$ in $G'$ containing a vertex $v \in B_i$. If a vertex of it is replaceable by $A_j$ with respect to $H$, then $v$ is connected to all the vertices in $A_j$. Otherwise every copy of $H_0$ obtained by removing a vertex of $H$ not in $B_i$ must belong to the $o(n^{V(H_0)^{}})$ exceptional copies in Claim 3.5. Observe that only vertices in one part of $H$ can belong to $B_i$, as vertices in other parts are connected to $c \geq a$ vertices of $A_i$.

For each vertex $v$, $\Theta(n^{V(H)^{-1}})$ copies of $H$ contain $v$, thus we obtain $\Omega(n^{V(H)^{-2}})$ copies of $H_0$ this way, as a copy of $H_0$ might be obtained $O(n)$ ways. If we have $\Omega(n)$ exceptional vertices, they would belong to $\Omega(n^{V(H)^{-1}}) = \Omega(n^{V(H_0)^{}})$ not nice copies of $H_0$, a contradiction with Claim 3.3. 

**Claim 3.7.** For every $i$, every vertex of $B_i$ is connected to less than $a$ vertices of $B_i$.

**Proof.** Recall that we started with picking an arbitrary $H'$. We obtained that $n - o(n)$ vertices of $G$ must be connected to every vertex of $k - 1$ partite classes of that $H'$, let $Q$ be their set. Consider an arbitrary $u \in Q$, that belongs to, say $B_1$. Then we obtain another copy of $H'$ from the original one if we delete a vertex of $A_1$ and add $u$ instead. Let us denote this copy by $H''$. Applying the same for $H''$, we obtain that $n - o(n)$ vertices of $G$ are connected to every vertex of $k - 1$ partite classes of $H''$. In particular, if $v \in B_j$ is connected to every vertex of $k - 1$ partite classes of $H''$, the missing partite class has to be $A_j$ (which is a partite class of $H''$). Therefore, $v$ is connected to the first partite class of the new copy of $H'$, in particular to $u$. Thus every $u \in Q \cap B_1$ is connected to all but $o(n)$ vertices in every $B_j$ for $j > 1$.

If a vertex in $B_j \cap Q$ is connected to $a$ vertices in $B_j \cap Q$, then these $a + 1$ vertices with $a$ vertices from classes of $H'$ form a copy of $F$, a contradiction. This implies that for every $j$, $|B_j \cap Q| = n/k + o(n)$. Indeed, the copies of $H$ inside $Q$ are all formed by taking a partite class from every $B_j$ by the above observation, thus their number is at most $y := \prod_{j=1}^{k} (|B_j \cap Q|) \leq N(H, T_k(n))$. It is easy to see that if the sizes of the sets $B_j$ are less balanced, then $y$ decreases by $\Omega(n^{V(H)^{}})$. On the other hand, the number of copies of $H$ containing a vertex outside $Q$ is $o(n^{V(H)^{}})$, thus the total number of copies of $H$ in $G$ is less than $N(H, T_k(n))$, a contradiction.

This also shows that for every copy of $H_0$ inside $Q$, its partite sets are contained in distinct $B_j$’s. Indeed, we can essentially repeat the argument from the first paragraph of the proof of this theorem. Again, we call an edge $uv$ an extra edge if $u, v \in B_j \cap Q$. If a copy of $H_0$ contains the extra edge $uv$, that means each other vertex of that $H_0$ is adjacent to at
least one of \( u \) and \( v \), thus \( H_0 \) contains at most \( 2a - 4 \) other vertices from \( B_j \cap Q \). Therefore, if a copy of \( H_0 \) inside \( Q \) contains a set \( U \) of more than \( 2a - 2 \) vertices from \( B_j \), then there are no extra edges inside \( U \), thus no edges at all inside \( U \). Thus \( U \) is a subset of a partite set of \( H \), in particular \( |U| \leq b \). Therefore, the only way to choose \( kb - 1 \) vertices from the sets \( B_j \cap Q \) is to choose \( b - 1 \) vertices from one of them and \( b \) vertices from the other sets, and they have to form the partite sets of \( H \).

Consider a vertex \( v \in B_i \setminus Q \). Assume first that for some \( j \neq i \), \( v \) is connected to at most \( \alpha n \) vertices of \( B_j \) for some \( \alpha < 1/k \). Consider the copies of \( H \) containing \( v \). There are \( o(n^{|V(H)| - 1}) \) copies of \( H \) containing \( v \) that also contain another vertex outside \( Q \). Consider those copies of \( H \) that have all the vertices in \( Q \) (except for \( v \)), i.e. a copy of \( H_0 \) inside \( Q \) that forms a copy of \( H \) with \( v \). Then one of the partite classes of that \( H_0 \) is inside \( B_j \), thus the vertices are chosen from the \( \alpha n \) neighbors of \( v \) in \( B_j \). This shows that \( v \) is in less than \( x \) copies of \( H \) altogether, a contradiction with Claim 3.4.

Assume now that a vertex \( v \in B_i \) is connected to \( a \) vertices of \( B_j \). Then for \( j \neq i \), \( v \) has \( n/k - o(n) \) neighbors in \( B_j \). We will build a copy of \( F \). The one-element partite class is \( v \). Its \( a \) neighbors in \( B_i \) form another partite class. Then we go through the sets \( B_j \cap Q \) \( (j \neq i) \) one by one. We always have that the \( a + 1 \) vertices we picked from \( B_i \) have \( n/k - o(n) \) neighbors in \( B_j \cap Q \), and the already picked other vertices are connected to all of those, except for \( o(n) \). Therefore, we can always pick \( a \) vertices from \( B_j \cap Q \) that are connected to all the vertices picked earlier. This way we obtain a copy of \( F \), a contradiction. ■

The above claim implies that in a copy of \( H \), vertices of \( B_i \) cannot belong to two different partite classes. Therefore, every \( B_i \) contains a partite class. Let \( G'' \) be the graph we obtain by deleting the edges inside \( B_i \) for every \( i \). Then \( \mathcal{N}(H, G) = \mathcal{N}(H, G'') \leq \mathcal{N}(H, T_k(n)) \), where the inequality follows from the facts that \( G'' \) is \( K_{k+1} \)-free and \( H \) is \((k + 1)\)-Turán-good. ■

Now we can prove Theorem 1.9, which we restate here for convenience.

**Theorem.** There exists an \( F \)-Turán-good graph if and only if \( F \) has a color-critical vertex.

**Proof.** Assume first that \( F \) does not have a color-critical vertex and let \( k = \chi(F) \). Let \( T'_{k-1}(n) \) be obtained from \( T_{k-1}(n) \) by taking a vertex \( v \) from a largest partite set of \( T_{k-1}(n) \), and connect it to every other vertex. Then deleting \( v \) from \( T'_{k-1}(n) \) we obtain a \((k - 1)\)-partite graph. As deleting any vertex from \( F \) we obtain a \( k \)-chromatic graph, \( T'_{k-1}(n) \) is \( F \)-free.

We claim that for any graph \( H \), for \( n \) large enough, either \( \mathcal{N}(H, T'_{k-1}(n)) > \mathcal{N}(H, T_{k-1}(n)) \) or \( \mathcal{N}(H, T_{k-1}(n)) = \mathcal{N}(H, T'_{k-1}(n)) = 0 \). Indeed, if there is an \( H \) in \( T_{k-1}(n) \), then there is one avoiding \( v \), but using a vertex \( w \) from the same partite set, and a vertex \( v \) from another partite set. Then we can replace \( w \) with \( v \) to obtain a copy of \( H \) that is in \( T'_{k-1}(n) \), but not in \( T_{k-1}(n) \).

If \( \mathcal{N}(H, T_{k-1}(n)) = 0 \), then the Turán graph may be extremal if \( H \) contains \( F \), but then \( H \) is not \( F \)-Turán-good. If \( \mathcal{N}(H, T'_{k-1}(n)) > \mathcal{N}(H, T_{k-1}(n)) \), then the Turán graph is not extremal, finishing the proof.

Assume now that \( F \) has a color-critical vertex. Then \( F \) is a subgraph of a complete \( k \)-partite graph \( K_{1, a, ..., a} \), thus Theorem 3.3 finishes the proof. ■
4 Concluding remarks

- We showed that if a $k$-chromatic graph $F$ has a color-critical edge and $\text{ex}(n, K_k, F) = o(n^{k-1})$, then several $k$-Turán-good graphs are also $F$-Turán-good. The proof deals only with those copies of $K_k$ that are in the vertex set of a copy of $H$. This suggests to study a local variant of generalized Turán problems. We say that a subgraph $G'$ of $G$ is $F$-free with respect to $G$ if there is no copy of $F$ induced on $V(G')$. How many subgraph of an $n$-vertex graph $G$ can be isomorphic to $H$ and be $F$-free with respect to $G$ at the same time?

If $F = K_k$ and the largest number as an answer to the above question is obtained when $G = T_{k-1}(n)$, then it is immediate that $H$ is $k$-Turán-good. The argument used in the proof of Theorem 1.3 shows that for any other $F$ with a color-critical edge and $\text{ex}(n, K_k, F) = o(n^{k-1})$, we have that $H$ is also $F$-Turán-good.

- A theme of this paper is to extend some results on $k$-Turán-good graphs to $F$-Turán-good graphs when $F$ is a $k$-chromatic graph with a color critical edge. This motivates the question: is every $k$-Turán-good graph also $F$-Turán-good?

- Maybe even more is true than what is suggested in the previous paragraph. Let us call a graph $H$ weakly $F$-Turán-good if the number of copies of $H$ is maximized by complete multipartite graph among $F$-free graphs on $n$ vertices, provided $n$ is large enough. Is every weakly $K_k$-Turán-good graph also weakly $F$-Turán-good?

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