Invariants of the $1/r^2$ Supersymmetric t-J Models

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In this work, we have studied the invariants of motion of two $SU(N)$ supersymmetric t-J models of $1/r^2$ hopping and exchange in one dimension. The first model is defined on a lattice of equal spaced sites, and the second on a non-equal spacing lattice. Using the “exchange operator formalism”, we are able to construct all the invariants for the models, by mapping the systems to mixtures of fermions and bosons. This identification shows that the supersymmetric t-J model on the chain with equal-spaced sites also belongs to Shastry-Sutherland’s “Super-Lax-Pair” family.

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Since the independent works by Haldane and Shastry, there have been renewed interests in exactly solvable models of long range interaction [1–12,14–17]. Of these systems, the 1D supersymmetric t-J model of $1/r^2$ exchange and hopping has been studied intensively [6–8,15]. The system is identified as a free Luttinger liquid [6–8], and the asymptotic correlation functions have been calculated through finite size scaling technique. The excitation spectrum of the system may be obtained with the help of the asymptotic Bethe-ansatz. In particular, for the $SU(2)$ case, the asymptotic Bethe-ansatz spectrum was explicitly shown to be exact, and the correct thermodynamics was given when the spinon rotation was properly taken into account [8]. In general, exact solvability implies existence of infinite number of constants of motion. For the long range t-J models, the complete construction of invariants of motion has remained unknown. In this work, applying the “exchange operator formalism” to a mixture of fermions and bosons, we are able to systematically provide all the invariants for the $SU(N)$ systems.

Let us first consider the $1/r^2$ supersymmetric t-J model on a one dimensional lattice of equal spaced sites. The Hamiltonian for the one-dimensional t-J model is given by

$$H = P_G \left[ - \sum_{1 \leq i \neq j \leq L} \sum_{\sigma=1}^{N} t_{ij} (c_{i \sigma}^\dagger c_{j \sigma}) + \sum_{1 \leq i \neq j \leq L} J_{ij} [P_{ij} - (1 - n_i)(1 - n_j)] \right] P_G,$$

where we take the hopping matrix and the spin exchange interaction to be $t_{ij}/2 = J_{ij} = 1/d^2(i - j)$, and $d(n) = (L/\pi) \sin(n\pi/L)$ is the chord distance, with $L$ the size of the lattice. The operator $c_{i \sigma}^\dagger$ is the fermionic operator to create an electron with spin component $\sigma$ at site $i$, $c_{i \sigma}$ is the corresponding fermionic annihilation operator. Their anti-commutation relations are given by $\{c_{i \sigma}, c_{j \sigma}^\dagger\}_+ = \delta_{ij} \delta_{\sigma, \sigma'}$, $\{c_{i \sigma}, c_{j \sigma'}\}_+ = 0$, $\{c_{i \sigma}, c_{j \sigma}^\dagger\}_+ = 0$. We assume that the spin component $\sigma$ takes values from 1 to $N$. The Gutzwiller projection operator $P_G$ projects out all the double or multiple occupancies, $P_G = \prod_{i=1}^{L} P_G(i)$, and $P_G(i) = \delta_{0, n_i} + \delta_{1, n_i}$, with $n_i = \sum_{\sigma=1}^{N} c_{i \sigma}^\dagger c_{i \sigma}$. The operator $P_{ij} = \sum_{\sigma=1}^{N} \sum_{\sigma'=1}^{N} c_{i \sigma}^\dagger c_{i \sigma} c_{j \sigma'}^\dagger c_{j \sigma'}$ exchanges the spins of the electrons at sites $i$ and $j$, if both sites are occupied. $n_i$ and $n_j$ are the electron number operators at sites $i$ and $j$.

Now, on the lattice, we may introduce two new fields, the $f$ and $b$ fields. For the new
fields, we have \( \{ f_{i\sigma}, f_{j\sigma'} \}_+ = 0, \{ f_{i\sigma}, f_{j\sigma'} \}_+ = \delta_{ij}\delta_{\sigma\sigma'}, [b_i, b_j] = 0, [b_i, b_j^\dagger] = \delta_{ij} \). The \( b \) field always commute with the \( f \) field. The size of the Hilbert space at each site is \( \infty \) in this case. However, let us project out the zero occupancy and all the double or multiple occupancy, and we work in the subspace where there is only one particle at each site. This new subspace can be shown to be equivalent to the subspace defined by the \( c \) field with no double or multiple occupancies. In particular, we may represent the fermionic electron operators \( c_{i\sigma}^\dagger \) and \( c_{i\sigma} \) in the following way:

\[
P_G(i)c_{i\sigma}^\dagger P_G(i) = \delta_{1,n_b^i+n_f^j} b_i^\dagger \delta_{1,n_b^i+n_f^j}
\]

\[
P_G(i)c_{i\sigma} P_G(i) = \delta_{1,n_b^i+n_f^j} b_i \delta_{1,n_b^i+n_f^j},
\]

(2)

where \( n_b^i + n_f^j = b_i^\dagger b_i + \sum_{\sigma=1}^{N_e} f_{i\sigma} f_{i\sigma}^\dagger \). In terms of the \( f \) and \( b \) fields, a state vector can be written as

\[
|\phi\rangle = \sum_{\sigma_1,\sigma_2,\cdots,\sigma_{N_e}} \sum_{\{x\},\{y\}} \phi(x_1\sigma_1, x_2\sigma_2, \ldots, x_{N_e}\sigma_{N_e}|y_1, y_2, \cdots, y_Q) \times \]

\[
\times f_{x_1\sigma_1}^\dagger f_{x_2\sigma_2}^\dagger \cdots f_{x_{N_e}\sigma_{N_e}}^\dagger b_{y_1}^\dagger b_{y_2}^\dagger \cdots b_{y_Q}^\dagger |0\rangle,
\]

(3)

where \( N_e \) is the number of \( f \) fermions on the lattice, \( Q \) is the number of \( b \) bosons, and we require that \( x_i \neq x_j \neq y_k \neq y_l \), and that the \( f \) fermion positions \( \{x\} \) and the \( b \) boson positions \( \{y\} \) span the whole chain. Obviously, \( N_e \) is also the number of the electrons, and \( Q \) is also the number of holes on the lattice. The amplitude \( \phi \) is anti-symmetric when exchanging \( (x_i\sigma_i) \) and \( (x_j\sigma_j) \), and symmetric in the boson coordinates \( \{y\} = (y_1, y_2, \cdots, y_Q) \). The Hamiltonian of the supersymmetric t-J model, using the mapping Eq. (4) in a straightforward way, can be written in terms of the fermionic \( f \) field and the bosonic \( b \) field.

With the above mapping, we may write the eigen-energy equation of the supersymmetric t-J model in first quantized form. Define the “exchange operator” \( M_{ij} \) as

\[
M_{ij} F(q_1, q_2, \ldots, q_i, \ldots, q_j, \ldots, q_L) = F(q_1, q_2, \ldots, q_j, \ldots, q_i, \ldots, q_L),
\]

where the function \( F \) is an arbitrary function of some position variables \( (q_1, q_2, \ldots, q_L) \), i.e., the operator \( M_{ij} \) exchanges the positions \( q_i, q_j \) of the particles \( i \) and \( j \). In terms of such exchange operators, the eigen-energy equation of the t-J model takes the form as follows
amplitude of the state vector of Eq. (3).

Using the so-called “exchange operator formalism”, they have been able to explicitly construct all the invariants of motion for the \( SU(2) \) spin chains. Let us briefly review their results. Say, \( M_{ij} \) is the exchange operator that interchanges \( q_i \) and \( q_j \), the positions of the particle \( i \) and particle \( j \), when we operate \( M_{ij} \) on a wavefunction \( F(q_1, q_2, \cdots, q_L) \). In terms of this operator, they have been able to construct an infinite sets of quantities \( I_n \) that commute among themselves,

\[
[I_n, I_m] = 0,
\]

where \( I_n = \sum_{i=1}^{L} \pi_i^n \), with \( \pi_i = \sum_{j \neq i} (z_j/z_{ij}) M_{ij} \), \( z_i = e^{2\pi i q_i/L} \), \( z_{ij} = z_i - z_j \), and \( n, m = 0, 1, 2, \cdots, \infty \). It was found that all these quantities commute with the Hamiltonian \( H = \sum_{1 \leq i \neq j \leq L} |z_i - z_j|^{-2} M_{ij} \) as long as the particles occupy the whole chain. For a system of identical bosons on the chain, the wavefunction is totally symmetric when we simultaneously interchange spins and positions of two particles. The effect of the exchange operator \( M_{ij} \) is just equivalent to the effect of the spin exchange operator alone. Using this method, they have successfully constructed all the invariants for the \( SU(N) \) Haldane-Shastry model.

We would like to stress that, in the language of the exchange operators \( M_{ij} \), the commu...
tation results proved by Fowler and Minahan hold as operator identities. The central issue is that the forms of wavefunctions of many particle systems, as well as the statistics of the particles or the types of the particles, do \textit{not} matter in order for the commutators to hold, as long as the particles occupy the whole chain. We may then apply the “exchange operator formalism” to the wavefunctions of mixtures of fermions and bosons. Therefore, from the eigen-equation Eq. (4), we thus conclude that in the first quantization, all the invariants of the t-J model are the same $I_n$’s as constructed by Fowler and Minahan, which can be written in terms of the exchange operators $M_{ij}$’s.

With the permutation properties of the amplitude $\phi$ for the mixture of bosons and fermions, it is straightforward to write all the invariants of motion of the t-J model in the second quantization form using the $I_n$’s. For instance, the exchange operation between the $f$ fermion positions is equivalent to the spin exchange operation (minus sign involved), the exchange operation between $b$ boson positions is equivalent to the hole-hole interaction term, and exchange operation between $f$ fermion and $b$ boson positions is equivalent to the electron hopping. Such procedure to reduce an $I_n$ to a second quantized form is quite simple, and we do not write all the details. Thus we provide a systematical way to construct all the invariants of motion for the $1/r^2$ supersymmetric t-J model, either in first quantized or in second quantized forms.

Recently, Shastry and Sutherland have studied the interesting relation between supersymmetry and integrability, through the so-called “Super-Lax-Pair” approach [4,5]. For this equal-spacing chain, using the mapping Eq. (2), we have been able to write the t-J model Hamiltonian Eq. (1) in terms of the exchange operators as Eq. (4). This identification shows that the “Super-Lax-Pair” results obtained by Shastry and Sutherland may apply to this t-J model [4].

Besides the above integrable t-J model on equally-spaced sites, let us consider another supersymmetric t-J model of $1/r^2$ hopping and exchange on a chain with sites not equally spaced. The positions of the sites $x_1, x_2, \ldots, x_L$ are determined by the equation
\[ x_i = \sum_{1 \leq j \neq i \leq L} \frac{2}{(x_i - x_j)^3}. \] (6)

This equation has appeared before in a paper discussing a long range spin chain of Haldane-Shastry type. Doping this spin chain, we are led to the following supersymmetric t-J model

\[ H = P_G \left[ -\sum_{1 \leq i \neq j \leq L} \sum_{\sigma=1}^{N} t_{ij} (c_\sigma^i a_\sigma^j) + \sum_{1 \leq i \neq j \leq L} J_{ij} [P_{ij} - (1 - n_i)(1 - n_j)] \right] P_G, \] (7)

where the hopping matrix and the anti-ferromagnetic exchange interaction are given by

\[ t_{ij}/2 = J_{ij} = \frac{1}{(x_i - x_j)^2}, \]

each site is occupied at most by one electron.

In the half filling case \( N_e = L \), this system reduces to the spin chain that has been studied before, which is completely solvable and similar exchange operator formalism has been developed \([10,11]\). Let us just write down the results obtained by Polychronakos:

\[ [I_n, I_m] = 0, [I_n, H] = 0, \]
\[ I_n = \sum_{i=1}^{L} h_i, h_i = a_i^\dagger a_i, \]
\[ a_i = \pi_i^\dagger + iq_i, \]
\[ \pi_i = \sum_{j(\neq i)} i(q_i - q_j)^{-1} M_{ij}, H = \sum_{i,j} (q_i - q_j)^{-2} M_{ij}, \]
\[ n, m = 0, 1, 2, \cdots, \infty. \]

Here, all the particles are put on the chain where the sites are positioned as determined by the Eq. (6).

We may relate the operation of exchanging particle positions to the operation to exchange particle spins, by assuming that we have identical bosons again, for which the wavefunctions are totally symmetric when we exchange the spins and positions of two particles simultaneously. With this assumption, from \( I_n \)'s, we thus can derive all the invariants of motion written in terms of the spin exchange operators alone.

Again, all commutation results written in terms of the exchange operators \( M_{ij} \) obtained by Polychronakos hold as operator identities, as long as the particles occupy the whole chain of the sites positioned in the special way. The forms of the wavefunctions do not matter at all.

Thus the commutation results may apply to wavefunctions of particles of arbitrary statistics or wavefunctions of mixtures of particles of different statistics on the chain. Mapping our new supersymmetric t-J model in terms of the \( b \) and \( f \) fields, we may also write the eigen-energy equation in first quantized form. In terms of the exchange operators between the positions of the bosons and fermions, the Hamiltonian takes the form

\[ H = -\sum_{1 \leq i \neq j \leq L} (q_i - q_j)^{-2} M_{ij}. \] (8)
Applying the formalism to this t-J model, in a similar way we may obtain all the invariants, either in first quantized or in the second quantized form, which commute among themselves and with the Hamiltonian. Thus this supersymmetric t-J model is also completely integrable.

In conclusion, we have studied the invariants of motion of two $SU(N)$ supersymmetric t-J models of long range hopping and exchange. The first system is on the chain of equal spaced sites, and the other on a chain of non-equal spaced sites. Mapping the corresponding t-J model Hamiltonians to those written in terms of mixed fermionic and bosonic fields, then applying the “exchange operator formalism”, we for the first time are able to construct all the invariants of the original Hamiltonian systematically.

Finally, we wish to point out that, for the second t-J model, we will also have a metal-insulator phase transition at half-filling. Away from half filling, we would also expect to have decoupled spin and charge excitations near the ground state. The system would be a Luttinger liquid. The study of the physical properties of the t-J model, such as its full excitation spectrum and finite temperature properties, is reported in our forthcoming paper. We also have Jastrow product ground state and excited state wavefunctions [13], as in the case for the model on the chain with equally spaced sites. It would also be very interesting to find out possible Shastry-Sutherland type “Super-Lax-Pair” for this new supersymmetric t-J model on non-equal spacing chain. We also hope to return to the issue of constructing invariants of the non-supersymmetric t-J models of the $1/r^2$ hopping and exchange in future.

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