Abstract—This paper presents an improved active disturbance rejection control scheme (IFO-ADRC) with an improved fractional-order extended state observer (IFO-ESO). The structural information of the system is utilized in IFO-ESO rather than buried as in the typical fractional-order extended state observer (FO-ESO) and help significantly improve the performance of IFO-ESO and closed-loop system. Compared with the integer-order active disturbance rejection controller (IO-ADRC), the auxiliary tracking controller of IFO-ADRC has a simpler form and fewer parameters need to be tuned. Frequency-domain analysis shows that IFO-ESO has better performance over the larger frequency band than FO-ESO, and time-domain simulation shows that IFO-ADRC has better transient performance and is more robust against the parameter variations than traditional fractional-order active disturbance rejection controller (FO-ADRC) and IO-ADRC. The IFO-ADRC is applied to permanent magnet synchronous motor (PMSM) servo control system and demonstrates its capability in the real-world application.

Index Terms—active disturbance rejection control, fractional-order active disturbance rejection control, fractional-order extended state observer, robustness.

I. INTRODUCTION

Fractional calculus has been applied to different fields in recent years [1], [2], [3], [4]. With a deep understanding of the system, there is a growing need for fractional-order control and modelling [5], [6]. Some real-world phenomena demonstrate fractional-order characteristics [7], [8]. Permanent magnet synchronous motor [8], gas-turbine [9], and heating–furnace [10] can be identified as fractional-order models. On the other hand, the fractional-order controller has the potential to achieve better and more robust control performance than the integer-order controller [11]. Many fractional-order controllers have been proposed, including fractional-order sliding mode controller [12], [13], fractional-order intelligent PID controller [14], fractional-order PID controller [15], and so on.

The active disturbance rejection control (ADRC) was proposed in [16] as an alternative paradigm for control system design. The core idea of ADRC is to improve the robustness of the system using extended state observer (ESO). ESO, an important part ADRC, can estimate the total disturbances, including internal disturbance caused by system uncertainties and external disturbance. Gao [17] proposed the linear ADRC to broaden the application range of ADRC by linearizing ESO (IO-ESO) and feedback control law. Applying IO-ESO, a m-order integer-order system can be approximately converted into m unit gain integer-order integrators in series. According to the design idea of the linear IO-ADRC, a linear feedback control law of m parameters need to be designed to realize the stable closed-loop system. Furthermore, Gao [18] proposed an ADRC structure involving the fractional-order tracking differentiator, the fractional-order PID controller and the fractional-order extended state observer for nonlinear fractional-order systems. The stability region of the fractional-order system can be larger than that of the integer-order system in the complex plane [19]. FO-ADRC can provide a possibility to realize closed-loop stability with a simpler controller. In [1], Chen et al. applied FO-ESO to approximately convert a typical second-order system into a cascaded fractional-order integrator and the stable closed-loop system can be realized using merely a simple proportional controller. For a fractional-order system, it is natural to use a fractional-order controller to achieve closed-loop stability [20], [21]. A FO-ADRC based on FO-ESO was proposed by Li [22] to approximately convert the fractional-order system to a cascaded fractional-order integrator. The IO-ADRC was proved by Li [23] to estimate the disturbances in fractional-order systems, which considers the fractional-order dynamics as a common disturbance.

An integer-order system can be approximately converted to a fractional-order system (a cascaded fractional-order integrator) by using FO-ESO and then an auxiliary tracking controller is designed to meet the actual needs [1]. However, the satisfactory approximation can only be made in the low-frequency band, making it difficult to ensure the system’s robustness when the control system bandwidth is large. This paper is along the research line of proposing a new type of fractional-order ESO structure and then fractional-order ADRC. The main contributions of the paper can be summarized as follows. First, the new type of ESO, called IFO-ESO, will utilize structural information in the total disturbance given in FO-ESO. IFO-ESO will be used to compensate for the system uncertainties and external disturbance. The compensated system is approximately converted into a fractional-order integrator. It will be shown that the approximation performance is improved by IFO-ESO over FO-ESO, making the system more robust to system uncertainties and external

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An Improved Fractional-Order Active Disturbance Rejection Control: Performance Analysis and Experiment Verification

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disturbances. The theoretical analysis is verified by the control experiment on the PMSM servo speed control system. Second, compared with IO-ADRC, the auxiliary tracking controller of IFO-ADRC has a simpler form and fewer parameters need to be tuned. Thirdly, the stability criteria of the IFO-ESO and the IFO-ADRC closed-loop system are given when the disturbance is assumed to be bounded. Unlike the stability analysis of the ESO and the closed-loop system in [1] and [22], the stability criteria of the high-order ESO and high-order ADRC closed-loop system in this paper are given.

This paper is organized as follows. The structures of IFO-ESO and IFO-ADRC are proposed in Section II. The BIBO stability criteria of the IFO-ESO and the closed-loop system are given in Section III. The performance analysis of the IFO-ESO in the frequency-domain is shown in Section IV. Section V presents the time-domain simulation results, followed by the experimental results on PMSM servo system in VI. The paper is concluded in Section VII.

Notations. \(I_n\) is an \(n \times n\) identity matrix and \(0_{m \times n}\) is an zero vector matrix with size specified by the subscript.

II. AN IMPROVED STRUCTURE OF ACTIVE DISTURBANCE REJECTION CONTROLS

In this paper, we consider an integer-order linear system as follows:

\[
G(s) = \frac{Y(s)}{U(s)} = \frac{b}{s^{m} + \sum_{i=1}^{m-1} a_i s^i + a_0}
\]

where \(s\) is Laplace operator, \(a_i, a_0\) and \(b\) are real numbers, \(m\) and \(i\) are positive integers with \(m\) representing the maximum order of the system. The differential equation form of system (1) with the external disturbance, denoted by \(d\), is

\[
y^{(m)} = - \sum_{i=1}^{m-1} a_i y^{(i)} - a_0 y + b u + d
\]

Caputo derivative is adopted as the fractional-order derivative method in this paper and described as follows,

\[
f^{(\gamma)}(t) = \frac{C}{\Gamma(m_0 - \gamma)} \int_{0}^{t} f^{(m_0)}(\tau) (t - \tau)^{\gamma - m_0 + 1} d\tau
\]

where \(m_0\) is an integer satisfying \(m_0 - 1 < \gamma < m_0\) and \(\Gamma(\bullet)\) is Euler's gamma function. From the adopted definition, one can see that \(D^\gamma f(t)(d\gamma) = C D^{\gamma + d} f(t)\) where \(d\) is a positive number [24].

Let \(\gamma\) be a fractional number satisfying \(m - 1 < \gamma < m\), \(n\) be a positive integer number satisfying \(\gamma < n < \frac{\gamma}{\gamma + 1}\), and \(\gamma = \frac{m-1}{n}\) be a fractional number. As a result, \(n \gamma < m < (n+1)\gamma\). Define the quantity \(q(y^{(m)}, y^{(m)}, t) = y^{(m\gamma)} - y^{(m)}\). Equation (2) can be rewritten as follows,

\[
y^{(m\gamma)} = y^{(m\gamma)} - y^{(m)} - \sum_{i=1}^{m-1} a_i y^{(i)} - a_0 y + b u + d
\]

\[
y^{(m\gamma)} = y^{(m\gamma)} - y^{(m)} - \sum_{i=1}^{m-1} a_i y^{(i)} - a_0 y + (b - b_0) u + b_0 u + d
\]

\[
y^{(m\gamma)} = f^{ifo}(y^{(1)}, y^{(2)}, \ldots, y^{(m-1)}, y^{(m)}, y, u, t) + b_0 u
\]

Note that \(f^{ifo} = - \sum_{i=1}^{m-1} a_i y^{(i)} - a_0 y + (b - b_0) u\) can be regarded as the total disturbance where the term \(- \sum_{i=1}^{m-1} a_i y^{(i)} - a_0 y + (b - b_0) u\) is the internal disturbance due to uncertain parameters and \(d\) is the external disturbance. The aim of this paper is to design \(u\) such that the system output tracks a sufficiently smooth reference trajectory \(r\) and the ultimate tracking error stays in the neighborhood of the origin, i.e., \(\lim_{t \to \infty} \|y(t) - r(t)\| < \epsilon\), when the reference signal and its derivatives, i.e., \(r, \dot{r}, \ddot{r}, \ldots, r^{(m-1)}, r^{(m\gamma - m+1)}, \) are bounded.

As inspired by FO-ADRC in [1], we propose the tracking controller as follows

\[
u = \frac{u_0 - \hat{q} - \hat{f}^{ifo}}{b_0}
\]

where \(\hat{q}\) and \(\hat{f}^{ifo}\) are estimators for signals \(q\) and \(f^{ifo},\) respectively, and \(u_0\) is the auxiliary tracking controller to be specified later. If signals \(q\) and \(f^{ifo}\) are approximately estimated by \(\hat{q}\) and \(\hat{f}^{ifo}\), the closed-loop system composed of (4) and (5) can be approximately converted into a fractional-order integrator as follows

\[
y^{(m\gamma)} = u_0 + (f^{ifo} - \hat{f}^{ifo}) + q - \hat{q} \approx u_0.
\]

As will be explained in Remark 2.2, a big advantage of IFO-ADRC and FO-ADRC is that a simpler auxiliary controller can be designed and easily tuned.

Remark 2.1: For FO-ADRC in [1], equation (2) is written as

\[
y^{(m\gamma)} = f^{ifo}(y^{(1)}, y^{(2)}, \ldots, y^{(m-1)}, y^{(m)}, y^{(m)}, y, u, t) + b_0 u
\]

where the term \(f^{ifo} = y^{(m\gamma)} - y^{(m)} - \sum_{i=1}^{m-1} a_i y^{(i)} - a_0 y + (b - b_0) u\). In comparison to FO-ADRC, IFO-ADRC will separate the structurally certain term \(q\) from \(f^{ifo}\). It is explicitly included in (4) and estimated in a new type of ESO introduced later. As will be demonstrated in Section IV and V, the structural certainty of the term \(q\) can help significantly improve the performance of ESO and the closed-loop system.

Let \(x_1 = y, x_2 = y^{(\gamma)}, \ldots, x_n = y^{((n-1)\gamma)}, x_{n+1} = f^{ifo}, h^{ifo} = f^{ifo}(y^{(1)}, y^{(2)}, \ldots, y^{(m-1)}, y, u, t)\) where
$x_1, x_2, \ldots, x_n$ represent system states and $x_{n+1}$ is an extended state, the state-space representation of (4) is given as follows:

$$
\begin{align*}
\dot{x}(\gamma) &= Ax + Bu + Eh_{iJo} + Fq \\
y &= Cx 
\end{align*}
$$

where

$$
\begin{align*}
x &= \begin{bmatrix} x_1, x_2, \ldots, x_n, x_{n+1} \end{bmatrix}^T, \\
A &= \begin{bmatrix} 0_{n \times 1} & I_n \\
0 & 0_{1 \times n} \end{bmatrix} \\
B &= \begin{bmatrix} 0, 0, \ldots, b_0, 0 \end{bmatrix}^T, \\
C &= \begin{bmatrix} 1, 0, \ldots, 0, 0 \end{bmatrix}, \\
E &= \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T, \\
F &= \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T. 
\end{align*}
$$

(9)

Then, IFO-ESO is designed to estimate $x_1, x_2, \ldots, x_n, x_{n+1}$ as follows:

$$
z(\gamma) = Az + Bu + Fq + L(y - \hat{y}) \\
\hat{y} = Cz
$$

(10)

where

$$
z = \begin{bmatrix} z_1, z_2, \ldots, z_n, z_{n+1} \end{bmatrix}^T \\
L = \begin{bmatrix} \beta_1, \beta_2, \ldots, \beta_n, \beta_{n+1} \end{bmatrix}^T \\
\hat{q} = (z_1^{(\gamma)} - z_n^{(m-n\gamma+1)})
$$

(11)

Note that $L$ are extended state observer gains. $z_1, z_2, \ldots, z_n, z_{n+1}$ are the estimation of the state $x_1, x_2, \ldots, x_n, x_{n+1}$, respectively and $b_0$ is the nominal value of $b$. Let $f_{iJo}$ in (5) be $f_{iJo} = z_{n+1}$.

With IFO-ESO, the tasking can be fulfilled with the auxiliary tracking controller $u_0$ in (5), which is designed as follows:

$$
u_0 = k_p(r - z_1) + k_{d_1}(\dot{r} - \dot{z}_1) + \ldots \\
+ k_{dm-2}(r^{(m-2)} - z_1^{(m-2)}) + r^{(n\gamma)}
$$

(12)

where $k_p, k_{d_1}, \ldots, k_{dm-2}$ are the parameters of the feedback control law. The closed-loop system composed of (2), (5) and (12), is called IFO-ADRC system.

**Remark 2.2:** For IO-ADRC in [17], the (2) is written as

$$
y^{(m)} = f_{iSo}(y^{(1)}, y^{(2)}, \ldots, y^{(m-1)}, y, u, t) + b_0u
$$

where $f_{iSo} = - \sum_{i=1}^{m-1} a_i y^{(i)} - a_0 y + (b - b_0)u$. Any part of controller similar to (5) where the auxiliary tracking controller $u_0$ is designed as

$$
u_0 = k_p(r - z_1) + k_{d_1}(\dot{r} - \dot{z}_1) + \ldots \\
+ k_{dm-1}(r^{(m-1)} - z_1^{(m-1)}) + r^{(m)}
$$

(13)

Note that the number of parameters for the auxiliary tracking controller of IFO-ADRC is less than that of IO-ADRC. In other words, we can achieve the trajectory tracking for an integer-order plant with a simpler auxiliary controller $u_0$ process, which simplifies the parameter tuning. As will be demonstrated in Section V, a P auxiliary tracking controller can be adopted to achieve the trajectory tracking of a second-order integer-order plant, and a PD auxiliary tracking controller can be adopted for a third-order integer-order plant. Section V will show the system transient performance of IFO-ADRC is better than that of IO-ADRC.

### III. Stabilty Analysis of IFO-ADRC

In this section, the stability criteria for the ESO and the IFO-ADRC system are provided. Let the observer error be

$$
e_i = x_i - z_i, i = 1, \ldots, n + 1.
$$

(14)

From (8) and (10), the equation of the extended state observer error can be written as

$$
e_i(\gamma) = Ae - Le_1 + F(e_n^{(\gamma)} - e_n^{(m-n\gamma+1)})
$$

(15)

where $e = [e_1, e_2, \ldots, e_n, e_{n+1}]^T$. The characteristic matrix of (15) is [25]:

$$
\lambda(s) = \begin{bmatrix} s^\nu + \beta_1 & -1 & 0 & \cdots & 0 \\
\beta_2 & s^\nu & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\beta_n & 0 & 0 & \cdots & s^\nu -1 \\
\beta_{n+1} & 0 & 0 & \cdots & 0 & s^\nu \end{bmatrix}
$$

(16)

From (16), the characteristic polynomial of the system (15) can be obtained:

$$
\lambda(s) = s^{\nu + \gamma}(s^{n\gamma} + \sum_{i=1}^{n-1} \beta_i s^{((n-i)\gamma)}) + \beta_n s^\gamma + \beta_{n+1}
$$

(17)

where $\nu = m - n\gamma$ and $0 < \nu < \gamma$. Then, Theorem 3.1 will present the bounded-input bounded-output (BIBO) stability of the error system (15), when $h_{iJo}$ is bounded. If the boundedness of the external disturbance $d$ rather than $h_{iJo}$ is assumed, Theorem 3.2 will give the condition of BIBO stability of the closed-loop system in terms of roots of a polynomial. In this case, the tracking error converges into a neighborhood of the origin. A special case of Theorem 3.2 when $m = n = 2$ is elaborated in Proposition 3.1, showing that when the observer gain is selected sufficiently large, the system is BIBO. The proofs of both theorems and the proposition are given in the Appendix.

**Theorem 3.1:** Consider the error dynamics of IFO-ESO (15). Let $\omega_i > 0$ and $\beta_i = C^{(n+1)}\omega_i$ for $i = 1, 2, \ldots, n+1$. If $h_{iJo}$ is bounded, then the IFO-ESO is BIBO stable, regarding $h_{iJo}$ as the input and $e_1$ as the output.

**Theorem 3.2:** Consider the IFO-ADRC closed-loop system composed of (2), (5) and (12). Let $p_1, p_2, q_1$, and $q_3$ positive prime such that $\nu = \frac{p_1}{q_1}$ and $\gamma = \frac{p_2}{q_2}$. Define a polynomial

$$
P(w) = (w^{n+p_2q_1} + \sum_{i=0}^{m-1} a_i w^{i+p_2q_2})(k_p + \sum_{i=1}^{m-2} k_{d_i} w^{i+p_2q_2} + \sum_{i=1}^{m-2} k_{d_{i}} w^{i+p_2q_2} + \sum_{i=1}^{m-2} k_{d_{i}} w^{i+p_2q_2}) + \beta_{n+1} + \beta_n w^{(n-p_2q_1)}
$$

(18)

If $b = b_0, k_p, k_{d_1}, \ldots, k_{d_{m-2}}$ and $\beta_1, \ldots, \beta_{n+1}$ are selected such that all the roots of (18) are located in $|\arg(w_i)| > \frac{\pi}{2q_2q_3}$, then the IFO-ADRC closed-loop system is BIBO stable.
Moreover, the tracking error $r(t) - y(t)$ converges to a small neighborhood of the origin as $t \to \infty$.

Proposition 3.1: Consider the IFO-ADRC closed-loop system composed of (2), (5) and (12) with $m = n = 2$. Suppose the plant (2) is stable or marginally stable, i.e., $a_1 \geq 0$ and $a_0 \geq 0$. Let $\beta_i = C^i_{\alpha+1}\omega^i_y$ for $i = 1, 2, 3, k_p > 0$. Then, there always exists a constant $\omega > 0$, such that the closed-loop system is BIBO stable. Moreover, the tracking error $r(t) - y(t)$ converges to a small neighborhood of the origin as $t \to \infty$.

IV. Performance Analysis of IFO-ESO in Frequency-domain

In this section, we will compare the performance of the IFO-ESO proposed in Section II with the FO-ESO proposed in [1]. The FO-ESO is as follows

$$z^{(\gamma)} = Az + Bu + L(y - \hat{y})$$
$$\hat{y} = Cz$$

(19)

where $z$, $A$, $B$, $C$ and $L$ are given in (11). Similar to the analysis in Section II, the controller $u$ utilizing the estimation of the FO-ESO in [1] is

$$u = \frac{u_0 - \hat{f}_{fo}}{b_0}$$

(20)

which is aimed to approximately convert the perturbed system into a pure cascaded integrator shown as follows

$$y^{(\gamma)} = u_0 + (f_{fo} - \hat{f}_{fo}) \approx u_0$$

(21)

Note that the role of the ESO in the framework of ADRC including IO-ESO, FO-ESO and IFO-ESO is to estimate the uncertain dynamics and external disturbances to improve the robustness of the system. If the IFO-ESO can perfectly estimate $\hat{q}$ and $\hat{f}_{fo}$, the IFO-ADRC in (5) and the FO-ADRC in (20) can convert the original system into a cascaded fractional-order integrator $1/s^{\gamma}$ (looking from $u_0$ to $y$). Therefore, we are motivated to use the model difference between $Y(s)/U_0(s)$ and $1/s^{\gamma}$ to assess the performance of the two ESOs. We adopt mean square error between $Y(s)/U_0(s)$ and $1/s^{\gamma}$ to evaluate how different the two models are. The mean square error (MSE) of two linear model is defined as

$$e_\omega(\omega) = |\Delta_\omega(\omega)|^2$$

(22)

where

$$\Delta_\omega(\omega) = 1 - (j\omega)^{\gamma}Y(j\omega)/U_0(j\omega)$$

(23)

The MSE (22) was used in [26] for the model identification where the problem is re-casted into an optimal problem of minimizing the model difference between the identified and ideal model in terms of the MSE. Therefore, the MSE can be used to evaluate model difference in the frequency-domain.

As in [1], for simplicity, we consider the second-order system as follow

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s(s + a_0)},$$

(24)

where the external disturbance and system uncertainty are not considered. In this case, the FO-ESO in (19) is simplified with

$$z = [z_1, z_2, z_3]^T, \quad A = \begin{bmatrix} 0 & I_3 \\ 0 & 0 \end{bmatrix}$$
$$B = [0, b_0, 0]^T, \quad C = [1, 0, 0].$$

(25)

Similarly, the IFO-ESO is

$$z^{(\gamma)} = Az + Bu + E\hat{\varphi} + L(y - \hat{y}),$$
$$\hat{\varphi} = (z^{(\gamma)} - z_2^{(2-\gamma)}),$$
$$\hat{y} = Cz,$$

(26)

where $E = [0, 1, 0]^T$.

Next, let us calculate $Y(s)/U_0(s)$ for FA-DRC and IFO-ADRC. For the fair comparison, we choose the same group of the observer gains $\beta_1 = 3\omega_\gamma, \beta_2 = 3\omega_\gamma^2, \beta_3 = \omega_\gamma^3$ for IFO-ESO and FO-ESO. For the IFO-ESO, all the parameters meet the conditions of Theorem 3.1, and thus the dynamics of the estimation error for IFO-ESO is asymptotically stable (because the disturbance and uncertainties do not exist). Conducting the Laplace transform on the both sides of (26) gives

$$Z_1(s) = \frac{(3\omega_\gamma s^2 + 3\omega_\gamma^2 s^{2\gamma} + \omega_\gamma^3)Y(s)}{s^{2+\gamma} + 3s^{2\gamma} + 5s^{2\gamma} + 3s^{2\gamma} + \omega_\gamma^3}$$
$$+ b_0sU_0(s) + s^{2+\gamma} + 3s^{2\gamma} + 3s^{2\gamma} + \omega_\gamma^3$$
$$Z_2(s) = \frac{(3\omega_\gamma s^2 + 3\omega_\gamma^2 s^{2\gamma} + \omega_\gamma^3)Y(s)}{s^{2+\gamma} + 3s^{2\gamma} + 3s^{2\gamma} + 3s^{2\gamma} + \omega_\gamma^3}$$
$$+ b_0sU_0(s) + s^{2+\gamma} + 3s^{2\gamma} + 3s^{2\gamma} + \omega_\gamma^3$$
$$Z_3(s) = \frac{(3\omega_\gamma s^2 + 3\omega_\gamma^2 s^{2\gamma} + 3s^{2\gamma} + \omega_\gamma^3)Y(s)}{s^{2+\gamma} + 3s^{2\gamma} + 3s^{2\gamma} + \omega_\gamma^3}$$
$$+ b_0sU_0(s) - s^{2+\gamma} + 3s^{2\gamma} + 3s^{2\gamma} + \omega_\gamma^3$$
$$Q(s) = (s^{2\gamma} - s^{2\gamma} - 2\gamma)Z_2(s).$$

(27)

where $Z_1(s), Z_2(s), Y(s), U(s)$ and $Q(s)$ are the Laplace transforms of signals $z_1, z_2, y, u$, and the quantity $q$, respectively. Conducting the Laplace transform on the both sides of (5) and substituting the result and (27) into (24) obtain

$$P_{fo}(s) = \frac{Y(s)}{U_0(s)} = \frac{N_1}{D_1}$$

(28)

where

$$N_1 = b_0(s^{2\gamma} + 3s^{2\gamma} + 3s^{2\gamma} + \omega_\gamma^3)$$
$$D_1 = b_0\omega_\gamma^3s^{2\gamma}(3s^{2\gamma} + 3s^{2\gamma} + \omega_\gamma^3)$$
$$+ a_0b_0s^{2\gamma}(s^{2\gamma} + 3s^{2\gamma} + 3s^{2\gamma} + \omega_\gamma^3)$$
$$+ b_0s^{2\gamma}(s^{2\gamma} + 3s^{2\gamma} + 3s^{2\gamma} + \omega_\gamma^3)$$

(29)

Similarly, the Laplace transform of the both side of FO-ESO...
It is clearly shown in Fig. 1 that the amplitude and the performance of FO-ESO and IFO-ESO with observer gains

\[
Z_1(s) = \frac{(3\omega_0 s^2+ 3 \omega_0^2 s^2 + \omega_0^3)}{b_0 s^3 + 3 s^2 \omega_0 + 3 \gamma \omega_0^2 + \omega_0^3} Y(s)
\]

\[
+ \frac{3 s^2 \omega_0 + 3 \gamma \omega_0^2 + \omega_0^3}{b_0 s^2 + 3 s \omega_0 + \omega_0^3} Y(s)
\]

\[
Z_2(s) = \frac{3 s^2 + 3 s \omega_0 + 3 \gamma \omega_0^2 + \omega_0^3}{b_0 s^2 + 3 s \omega_0 + \omega_0^3} Y(s)
\]

\[
- \frac{3 s^2 + 3 s \omega_0 + 3 \gamma \omega_0^2 + \omega_0^3}{b_0 s^2 + 3 s \omega_0 + \omega_0^3} Y(s)
\]

Conducting the Laplace transform on both sides of (20) and substituting the result and (30) into (24) obtain

\[
P_{fo}(s) = \frac{Y(s)}{U_0(s)} = \frac{N_2}{D_2}
\]

where

\[
N_2 = b(s^2 + \omega_0^2)^3
\]

\[
D_2 = b(s^2 + \omega_0^2)^3
\]

The model difference between the fractional-order integrator (1/s^2) and \( P_o(P_{fo} \text{ or } P_{fo}) \) at \( \omega \) can be expressed as:

\[
\Delta_o = 1 - (j\omega)^2 P_o(j\omega)
\]

Then, the mean-square error between the fractional-order integrator and \( P_{ifo} \) can be expressed as follows:

\[
\Delta_{ifo} = \frac{N_4}{D_4}
\]

where

\[
N_4 = a_0 j w ((j\omega)^2 + 3 \omega_o (j\omega)^2 + 3 \omega_o^2)
\]

\[
D_4 = a_0 j w ((j\omega)^2 + 3 \omega_o (j\omega)^2 + 3 \omega_o^2) + (j\omega)^2 ((j\omega)^2 + \omega_o^2)
\]

From (22) and (31), the mean-square error between the fractional-order integrator and \( P_{fo} \) can be expressed as follows:

\[
\Delta_{fo} = \frac{N_4}{D_4}
\]

where

\[
N_4 = \omega_o^3 j w ((j\omega)^2 + \omega_o^2) + (a_0 j w + (j\omega)^2) \omega_o (j\omega)^2 + 3 \omega_o^2
\]

\[
D_4 = \omega_o^3 j w ((j\omega)^2 + (a_0 j w + (j\omega)^2) \omega_o (j\omega)^2 + 3 \omega_o^2)
\]

Now, let us use the Bode diagram to study the performance of FO-ESO and IFO-ESO with observer gains \( L = [3\omega_0, 3 \omega_0^2, \omega_0^3] \). Moreover, the system parameters and the observer gain parameters are adapted from [1], i.e., \( a_0 = 26.08 \), \( b_0 = 38.635 \), \( \omega_o = 700 \) and \( \gamma = 0.75 \). The Bode diagram of \( P_{ifo} \) in (28) and \( P_{fo} \) in (31) are illustrated in Fig. 1. It is clearly shown in Fig. 1 that the amplitude and the phase diagram of \( P_{ifo} \) are close to that of 1/s^1.5 within a larger frequency band than that of \( P_{fo} \). In particular, the approximation of \( P_{fo} \) to 1/s^1.5 gets worse in high-frequency band. Fig. 2 shows the mean-square error \( e_{fo} \) and \( e_{ifo} \) with respect to frequency \( \omega \). The result coincides with Fig. 1 that \( P_{fo} \) better approximates the integrator 1/s^1.5 than \( P_{fo} \) does. In other words, the IFO-ESO has better performance in terms of disturbance estimation than the FO-ESO. Therefore, the closed-loop system resulting from the IFO-ESO is more robust than that from the FO-ESO.

Fig. 3 shows the curves of the mean-square error \( e_{fo} \) and \( e_{ifo} \) with different model parameter \( a_0 \) when \( \omega_o = 2000 \) and \( \gamma = 0.75 \). Fig. 4 shows the curves of the mean-square error \( e_{fo} \) and \( e_{ifo} \) with different observer gains \( \omega_o \) when \( a_0 = 10 \) and \( \gamma = 0.75 \). Fig. 5 demonstrates the variation of the mean-square error with different order \( \gamma \) when \( a_0 = 10 \) and \( \omega_o = 2000 \). As shown in Fig. 3, Fig. 4, and Fig. 5, the mean-square error \( e_{ifo} \) is less prone to the variation of system parameter \( a_0 \), controller parameters \( \omega_o \), and the order \( \gamma \) than the mean-square error \( e_{fo} \) is.

Remark 4.1: Note that if the performance of ESO is robust to the variation of the closed-loop system parameters \( a_0 \), \( \omega_o \), and \( \gamma \), the auxiliary tracking controller design of \( \tau_0 \) based on the compensated system (6) or (21) can achieve better performance when these parameters vary. In other words, the IFO-ADRC closed-loop system is more robust with respect to the variations of the plant \( G(s) \) parameters and ESO.
AN IMPROVED FRACTIONAL-ORDER ACTIVE DISTURBANCE REJECTION CONTROL: PERFORMANCE ANALYSIS AND EXPERIMENT VERIFICATION

V. TIME-DOMAIN SIMULATION AND COMPARISON

In this section, we will show the performance of the IFO-ADRC in the time-domain using MATLAB/Simulink and compare it with IO-ADRC and FO-ADRC. The plant used for the simulation is (24) which is adopted from the example of [1]. The structures of IFO-ADRC, FO-ADRC and IO-ADRC are similar and presented in Fig. 6, 7 and 8, respectively. An advantage of IFO-ADRC and FO-ADRC is that a simpler auxiliary controller $u_0$ can be used. Note that the auxiliary controller $u_0$ for IFO-ADRC and FO-ADRC is a P controller, while PD controllers must be used in IO-ADRC to ensure the stability. The controller in IFO-ADRC and FO-ADRC is

$$C_p(s) = K_{fp},$$

where $K_{fp}$ is the parameter of P controller and the PD controller in IO-ADRC is

$$C_{pd}(s) = K_{ip}(1 + K_{id}s),$$

where $K_{ip}$ and $K_{id}$ are the PD parameters. The IO-ESO used in IO-ADRC is given [17],

$$\dot{z} = Az + Bu + \mathcal{L}(y - \hat{y})$$

$$\hat{y} = Cz$$

Refer to (25) for the matrices $z$, $A$, $B$, and $C$. The controller $u$ in IO-ADRC is given

$$u = \frac{u_0 - \hat{f}_{io}}{b_0}$$

The observer gains $\mathcal{L} = [\beta_1, \beta_2, \beta_3]^T = [3\omega_0, 3\omega_0^2, \omega_0^3]^T$ and the order $\gamma$ are adopted from the reference paper [1], i.e.,

$a_o = 26.08$, $b_0 = 383.635$, $b = b_0$, $\omega_o = 700$ rad/s, and
Fig. 8. Structure of the IO-ADRC

\[ \gamma = 0.75. \] The fractional-order operator \( s^\gamma \) is discretized by the impulse response invariant method [27] where the discrete frequency for IFO-ESO and FO-ESO is 8000 Hz and the discrete order of the fractional-order operators is 7. Let the P controller parameter be \( K_{fp} = 356 \). Note that the observer parameters satisfy conditions of Theorem 3.2 and Theorem 1 in [1], the IFO-ADRC and FO-ADRC closed-loop systems are BIBO stable.

For the fair comparison, based on the open-loop transfer function, the PD controller used in IO-ADRC parameters are chosen to ensure that the gain crossover frequency \( \omega_c^* \) and the phase margin \( \phi_m \) of the IO-ADRC are the same as that of the IFO-ADRC [11], [11]. For IFO-ADRC, the open-loop transfer function can be described as

\[ G_{ifo}(s) = C_p(s) \frac{Z_1(s)}{U_0(s)} = K_{fp} P_{ifo}(s) \frac{Z_1(s)}{Y(s)} \quad (42) \]

where

\[ \frac{Z_1(s)}{Y(s)} = \frac{s^2 + 3\omega_0s^2 + a_0s^2 + 3\omega_0^2s^2 + \omega_0^3}{s^2 + 3\omega_0s^2 + 3\omega_0^2s^2 + \omega_0^3} \quad (43) \]

The gain crossover frequency \( \omega_c^* = 42 \) rad/s and the phase margin \( \phi_m = 39.9^\circ \) can be calculated from

\[ \angle G_{ifo}(j\omega_c^*) = -\pi + \phi_m \]

\[ |G_{ifo}(j\omega_c^*)| = 1 \quad (44) \]

For IO-ADRC, the open-loop transfer function is

\[ G_{io}(s) = C_{pd}(s) P_{io}(s) \frac{Z_1(s)}{Y(s)} \quad (45) \]

where

\[ P_{io}(s) = \frac{Y(s)}{U_0(s)} = \frac{(s + \omega_0)^3}{s(a_0 + s)(s + \omega_0)^3} \quad (46) \]

\[ \frac{Z_1(s)}{Y(s)} = \frac{s^2 + 3\omega_0s^2 + a_0s^2 + 3\omega_0^2s^2 + \omega_0^3}{s^2 + 3\omega_0s^2 + 3\omega_0^2s^2 + \omega_0^3} \]

Then, the PD controller for IO-ADRC can be designed as

\[ C_{pd}(s) = 1559.83(1 + 0.0199s) \quad (47) \]

to ensure that the gain crossover frequency \( \omega_c^* = 42 \) rad/s and the phase margin \( \phi_m = 39.9^\circ \).

The open-loop Bode diagram of the IO-ADRC and IFO-ADRC systems is illustrated in Fig. 9 showing that the two systems have the same gain crossover frequency \( \omega_c^* = 42 \) rad/s and phase margin \( \phi_m = 39.9^\circ \). From the open-loop Bode diagram of the IO-ADRC system, the IO-ADRC closed-loop system is stable.

The step responses of the IO-ADRC, FO-ADRC, and IFO-ADRC systems are shown in Fig. 10. It is shown in Fig. 10 that the IFO-ADRC system has better dynamic response performance than the FO-ADRC and IO-ADRC systems. The IFO-ADRC system has smaller overshoot than the FO-ADRC system.

Now, let us consider the system performance variation against controller parameters. Let us multiple the P controller parameter \( K_{fp} \) and PD controller parameter \( K_{ip} \) by \( K = 0.8 \) and \( K = 1.2 \). The nominal value is used when \( K = 1 \). Fig. 11, Fig. 12 and Fig. 13 are the step responses of three closed-loop systems when different controller parameters are imposed. As shown in Fig. 12 and Fig. 13, the IFO-ADRC system are robust to controller gain variations.

When \( K = 0.8, K = 1.0, \) or \( K = 1.2 \) are set respectively, the maximum speed of the step response are denoted as \( M_k \). The overshoot fluctuation is calculated as

\[ \text{overshoot fluctuation} = \max\{M_{0.8}, M_{1.0}, M_{1.2}\} - \min\{M_{0.8}, M_{1.0}, M_{1.2}\} \quad (48) \]

The step responses of three closed-loop systems for different \( K \) are given in TABLE I. Note that the overshoots of IFO-ADRC system and the IO-ADRC system are similar and smaller than that of the FO-ADRC system. The settling time...
VI. Experiments: PMSM Speed Servo Control

In this section, the control performance of IFO-ADRC, FO-ADRC, and IO-ADRC are compared in a real-world application. Experiments are carried out on the PMSM speed servo control system. Fig. 14 is the block diagram of the PMSM given in dq-axis frame. The diagram encircled by the blue and red dotted lines are the electromagnetic and the mechanical part of the PMSM, respectively.

The armature current of DC motor can be represented by q-axis component of the current $i_q$ [8]. The q-axis voltage equation of PMSM is described as follows

$$u_q - E = u_q - C_e n = R_s i_q + L_q \frac{di_q}{dt}$$  \hspace{1cm} (49)

where $u_q$ is the q-axis voltage, $E$ is the back electromotive force, $C_e$ is the electromotive force coefficient, $n$ is the actual speed of motor rotor, $R_s$ is motor phase armature resistance, $i_q$ is q-axis current and $L_q$ is q-axis inductance. Taking $u_q$ as the input voltage and $i_q$ as the output current, the transfer functions of the electromagnetic part is

$$G_e(s) = \frac{I_q(s)}{U_q(s)} = \frac{1/L_q}{s + R_s/L_q}$$  \hspace{1cm} (50)

where $I_q(s)$ and $U_q(s)$ are the Laplace transforms of $i_q$ and $u_q$, respectively. Note that $E$ is regarded as a constant disturbance input of the q-axis current loop and is therefore not shown in the transfer function above [28].

The motion equation of PMSM can be described as follows

$$T - T_L = C_m (i_q - i_L) = \frac{GD^2}{375} \frac{dn}{dt} + K_e B n$$  \hspace{1cm} (51)
where $T$ is the electromagnetic moment, $T_L$ is the load moment, $C_m$ is the torque coefficient, $i_L$ is the equivalent current of load torque, $GD^2$ is flywheel inertia, $K_v$ is a speed conversion factor, $B$ is coefficient of viscous friction, and $n$ is the actual speed of the motor rotor. Taking $i_q$ as input current and $n$ as output speed, the transfer functions of the mechanical part is

$$G_m = \frac{N(s)}{I_q(s)} = \frac{375C_m}{GD^2s + K_vB}$$

(52)

where $N(s)$ is the Laplace transforms of $n$, Note that the load moment is not considered in this paper, i.e., $i_L = 0$, $K_m$ is determined by the actual hardware.

The block circled by the green dash-dotted line in Fig. 15(a) is the block diagram of the current loop $G_i(s)$ of the PMSM speed servo system. In Fig. 15(a), $i_{qr}$ and $i_{qr}$ are the per unit of the actual voltage and the actual current, respectively, $i_q$ is the reference input of the current loop, $K_0$ and $K_2$ are the voltage and current conversion factors, respectively. The PI controller in current loop is designed to ensure that $G_i(s) = 1$ in its operating frequency band of the speed loop. Fig. 15(b) is the block diagram of the speed loop of the PMSM speed servo system using IFO-ADRC. In Fig. 15(b), $K_1$ is the speed conversion factor, $T_i$ is the speed feedback filter coefficient, and $n_r$ is the per unit of the actual speed. Since $G_i(s) = 1$, thus the plant of the speed loop is

$$G(s) = \frac{N_r(s)}{I_q(s)} = \frac{375C_mK_1}{T_iGD^2s^2 + (K_vBT_i + GD^2)s + K_vB}$$

(53)

where $N_r(s)$ and $I_q(s)$ are Laplace transforms of $n_r$ and $i_q$, respectively.

The closed-loop controller is implemented on digital signal processor (DSP) illustrated in Fig. 16. The PMSM is 60ST-M00630C and MOSFET is adopted as the gate driver. The specification of the PMSM is shown in Table II. The closed-loop controller is implemented on digital signal processor (DSP) illustrated in Fig. 16. The PMSM is 60ST-M00630C and MOSFET is adopted as the gate driver. The specification of the PMSM is shown in Table II. $K_0 = 20.3$ is determined by the actual hardware, $K_1 = 1/1200$, $K_2 = 1/9.9$, $T_i = 1/100$ are configured by software, and $K_v = 30/\pi$ is the speed conversion factor from rad/s to r/min. The speed sampling period was set as 0.125 ms, and the current loop sampling period is set as 0.0625 ms. The motor speed waveform is collected by DSP Emulator and CCS software.

### Table II: Specification of the PMSM

| Parameters | Unit | Value |
|------------|------|-------|
| $R_s$      | $\Omega$ | 0.3   |
| $L_d$      | $\text{mH}$ | 1.377 |
| $GD^2$     | $\text{Kgm}^2$ | 0.00147 |
| $B$        | $\text{Ns/rad}$ | 0.05880 |
| $C_m$      | $\text{Nm/A}$ | 0.112 |

When the specification of the PMSM in Table II is used, the plant (53) of the PMSM becomes

$$G(s) = \frac{2380.9}{s^2 + 138.1s + 3819.7}$$

(54)

We use the same controllers in Section V for IFO-ADRC, FO-ADRC and IO-ADRC. Let the observer gain be $L = [\beta_1, \beta_2, \beta_3]^T = [3\omega_0, 3\omega_0^2, \omega_0]^T$, $\omega_0 = 700$ rad/s, and $\gamma = 0.75$. The operator $s^\gamma$ is discretized by the impulse response invariant method where the discrete frequency is 8000 Hz and the discrete order of the fractional-order operators is 5. The parameter of the P controller (see (38)) used in the IFO-ADRC and the FO-ADRC is $K_{fp} = 750$, while the PD controller used in the IO-ADRC design method in Section V is designed as

$$C_{pd}(s) = 2314.69(1 + 0.0185s)$$

(55)

For the IFO-ADRC closed-loop system, all parameters meet the conditions of Theorem 3.2, thus the closed-loop of IFO-ADRC is BIBO stable. Also, the IO-ADRC and FO-ADRC closed-loop systems are ensured to be BIBO stable.

Let us multiple the P controller parameter $K_{fp}$ and PD controller parameter $K_{fp}$ by $K = 0.6$ and $K = 1.4$, while the nominal value is used when $K = 1$. Fig. 18 and Fig. 19 are simulation results of step responses with different controller parameters for the FO-ADRC and IFO-ADRC systems, respectively. As shown in Fig. 18 and Fig. 19, the IFO-ADRC system is robust to the controller parameter variations. Fig. 20 is step responses of the IO-ADRC, FO-ADRC and IFO-ADRC systems for the experiment setup. It is illustrated that the step response of the IFO-ADRC system has smaller overshoot, oscillation magnitude and shorter settling time than the FO-ADRC and IO-ADRC system. Fig. 22 and Fig. 23 are experiment results of step responses of three
different systems when the controller parameter varies. Similar to simulation result, the IFO-ADRC system is the most robust to controller parameter variations. Table III summarizes results of the step responses for three different systems, showing that the IFO-ADRC system has smaller overshoot, less settling time, and overshoot fluctuation than the IO-ADRC and IFO-ADRC system.

**TABLE III**
COMPARISON OF THE RESPONSES WITH THREE CONTROL SYSTEMS (EXPERIMENT)

| Controller | Overshoot(%) (K=1.0) | Settling time(s) (K=1.0) | Overshoot fluctuation(%) (K=0.6,1,1.4) |
|------------|-----------------------|--------------------------|-----------------------------------------|
| IO-ADRC    | 25.30                 | 0.191                    | 4.883                                   |
| FO-ADRC    | 28.03                 | 0.233                    | 15.035                                  |
| IFO-ADRC   | 21.40                 | 0.185                    | 2.584                                   |

**VII. CONCLUSION**

An improved active disturbance rejection control scheme with IFO-ESO is proposed in this paper. IFO-ESO can ap-
proximately convert an integer-order system into a cascaded fractional-order integrator for which a simpler feedback law can be designed. The approximation to a cascaded fractional-order integrator by IFO-ESO behaves well across a larger frequency band than FO-ESO, ensuring the closed-loop system is robust to controller gain, ESO parameters, and plant parameters variations. The frequency-domain analysis and PMSM speed servo control experiments verify that the proposed IFO-ADRC achieves better performance than FO-ADRC and IO-ADRC.

\[ R = -\sum_{i=1}^{n-1} a_i r_{i+1}^{(\gamma)} - a_0 r_1^{(\gamma)} + d \] (62)

where \( \sigma = \frac{\gamma}{\mu+1}, \) \( 0 < \sigma < 1 \). According to Kharitonov-Based Method [29], the two boundary polynomials are:

\[ \begin{align*}
1\lambda(w) &= w(1 + \sum_{i=1}^{n-1} \beta_i) + \beta_n + \beta_{n+1} \\
2\lambda(w) &= w(u^n + \sum_{i=1}^{n-1} \beta_i u^{(n-i)}) + \beta_n w + \beta_{n+1}
\end{align*} \] (57)

Substitute \( \beta_i = C_{n+i}^{\gamma_0} \), \( i = 1, 2, \cdots, n+1 \) into \( 2\lambda(w) \) gives \( 2\lambda(w) = (w + \omega_0)^{n+1} \). Then, the roots of the two boundary polynomials are:

\[ \begin{align*}
1\lambda(w) &: w_1 = -\frac{-m_0 w^n + \omega_0^{n+1}}{1 + \sum_{i=1}^{n} C_{n+i}^{\gamma_0}} \\
2\lambda(w) &: w_i = -\omega_0, i = 1, 2, 3, \cdots, n + 1
\end{align*} \] (58)

Since all the roots of the boundary polynomials are located in \( |\arg(w_i)| > \frac{n}{n+1} \), all the roots of (56) are located in \( |\arg(w_i)| > \frac{n}{2(n+1)} \) [30], i.e., \( \lambda(s) \) is Hurwitz, system (15) is BIBO stable, regarding \( h_{ifo} \) as input and \( e_1 \) as output.

**Proof of Theorem 3.2:**

Let \( (r_1, r_2, r_3, \cdots, r_m, r_m+1) = (r, r^{(\gamma)}, r^{(2\gamma)}, \cdots, r^{(m\gamma)}, r^{(m\gamma+m+1)}) \). From (6) and (15), one has

\[ y^{(\gamma)} = k_p(r_1 - z_1) + k_d(r_1' - \dot{z}_1) + \cdots + k_d(m-2) \]

\[ r_{m+2}^{(m-2)} - z_{m+2}^{(m-2)} + f_{ifo} - \tilde{f}_{ifo} + q - \tilde{q} \] (59)

It follows from (14) that

\[ y^{(\gamma)} - r_1^{(\gamma)} = k_p(r_1 - x_1 + e_1) + k_d(r_1' - \dot{x}_1 + \dot{e}_1) + \cdots + k_d(m-2)(r_1^{(m-2)} - x_1^{(m-2)} + e_1^{(m-2)}) + e_{n+1} + e_{n}^{(\gamma)} - e_{n}^{(x)} \] (60)

Let \( q_1 = r_1 - x_1, q_i = \dot{q}_{i-1}, i = 2, 3, \cdots, v - 1 \), (60) can be written as

\[ \dot{q}_1 = q_2, \dot{q}_2 = q_3, \cdots, \dot{q}_{m-1} = q_m \]

\[ q_m^{(n\gamma-m+1)} = q_{m+1} = -(k_p q_1 + k_d q_2 + \cdots + k_d(m-2) q_{m-1}) - (k_p e_1 + k_d \dot{e}_1 + \cdots + k_d(m-2) e_1^{(m-2)}) + e_{n+1} + e_{n}^{(\gamma)} - e_{n}^{(x)} \] (61)

The system (15) can be written:

\[ e_1^{(\gamma)} = -\beta_1 e_1 + e_2 \]

\[ e_2^{(\gamma)} = -\beta_2 e_1 + e_3 \]

\[ \vdots \]

\[ d(\gamma) e_n \]

\[ \frac{d(\gamma) e_{n+1}}{d(\gamma) t} = -\beta_n e_n + e_{n+1} \]

\[ e_{n+1}^{(\gamma)} = -\beta_{n+1} e_1 + h_{ifo} \]

\[ h_{ifo} = a_0 q_1^{(\gamma)} + a_1 q_2^{(\gamma)} + a_2 q_3^{(\gamma)} + \cdots + a_{m-1} q_m^{(\gamma)} + R \]

**APPENDIX**

**Proof of Theorem 3.1:** Let \( w = s^{\nu+\gamma} \), and (17) can be written:

\[ \lambda(w) = w(w^n + \sum_{i=1}^{n-1} \beta_i w^{(n-i)\sigma}) + \beta_n w^{\sigma} + \beta_{n+1} \] (56)
According to (61), it follows that,
\[ q_m^{(n\gamma-m+1)} = q_{m+1} = -(k_p q_1 + k_d q_2 + \cdots + k_{dm-2} q_{m-1}) \]
\[ - (k_p e_1 + k_d e_2 + \cdots + k_{dm-2} e_{1(m-2)}) \]
\[ - e_n^{(\gamma)} - \beta e_1 \]  

(63)

Note \( w_1 = k_p e_1 + k_d e_2 + \cdots + k_{dm-2} e_{1(m-2)} + e_n^{(\gamma)} + \beta e_1, w_2 = q_0 q_1^{(\gamma)} + a_1 q_2^{(\gamma)} + a_2 q_3^{(\gamma)} + \cdots + a_{m-1} q_{m-1}^{(\gamma)}. \)

Combining (63) and (62) gives the block diagram of the closed-loop system in Fig. 24. From (63), it gives

Because \( R \) is bounded, we can treat \( R \) as the disturbance and calculate the transfer function of the closed-loop system. Further calculation gives

\[ G_o(s) = -\frac{Q_1(s)}{E(s)} = G_m(s)G_n(s) \]  

(67)

\[ H_o(s) = \frac{E_1(s)}{Q_1(s)} = H_m(s)H_n(s) \]  

(68)

Combining (67) and (68), the transfer function of the closed-loop system \( P_o(s) \) can be given

\[ P_o(s) = \frac{Q_1(s)}{R_o(s)} = \frac{G_o(s)}{1 + G_o(s)H_o(s)} \]  

(69)

where \( R_o(s) \) is the Laplace transforms of \( r_o \) (see the Fig. 24). The characteristic polynomial of the closed-loop system is

\[ P(s) = (s^r \sum_{i=0}^{m-1} a_i s^i)(k_p + \sum_{i=1}^{m-2} k_d s^i + s^{n\gamma} + \sum_{i=1}^n \beta_i s^{(n-i)\gamma}) \]
\[ + (s^{2\gamma} + k_p + \sum_{i=1}^{m-2} k_d s^i)(s^{n\gamma} + \sum_{i=1}^n \beta_i s^{(n-i)\gamma}) \]
\[ + \beta_n s^{n\gamma} + \beta_{n+1} \]  

(70)

Finding prime number \( p_1, p_2, q_1, \) and \( q_1 \) such that \( \nu = \frac{p_1}{q_1}, \gamma = \frac{p_2}{q_2}, (18) \) is satisfied. Since \( R \) is bounded, the closed-loop system is BIBO stable [29].

**Proof of Proposition 3.2:** When \( m = n = 2 \), the characteristic polynomial \( P(s) \) of the closed-loop system can be written

\[ P(s) = s^r(a_0 + a_1 s)(k_p + s^{2\gamma} + \beta_1 s^3 + \beta_2) \]
\[ + (s^{2\gamma} + k_p)(s^{n\gamma} + \beta_1 s^3 + \beta_2) \]  

(71)

(72)

According to Khariotov-Based Method, when \( \beta_i = C_{i+1}^n \omega_i \) for \( i = 1, 2, 3 \), the two boundary polynomial can be written

\[ 1P(s) = A_0 s^2 + A_1 s + A_2 \]
\[ 2P(s) = B_0 s^2 + B_1 s^4 + B_2 s^3 + B_3 s^2 + B_4 s + B_5 \]  

(73)

where

\[ A_0 = 1 + k_p + 3(1 + k_p) \omega_3 \]
\[ A_1 = a_1 + k_p + 3(1 + k_p) \omega_3 \]
\[ A_2 = a_0(1 + k_p + 3(1 + k_p) \omega_3 + 3(1 + k_p) \omega_3^2) \]
\[ + (1 + k_p) \omega_3^3 \]
\[ B_0 = 1, B_1 = a_1 + 3(1 + k_p) \omega_3, B_2 = a_0 + k_p + 3a_1 \omega_3 + 3(1 + k_p) \omega_3^2 \]
\[ + (1 + k_p) \omega_3^3 \]
\[ B_3 = a_1 k_p + 3a_0 \omega_3 + 3k_p \omega_3 + 3a_2 \omega_3^2 + \omega_3^3 \]
\[ B_4 = a_0 k_p + 3a_0 \omega_3 + 3k_p \omega_3^2 + 3a_3 \omega_3^2 + k_p \omega_3^3 \]  

(74)

Firstly, we consider \( 1P(s) = 0 \). When \( a_1 \geq 0, a_0 \geq 0, k_p > 0, \) and \( \omega_3 > 0, \) from (74), we have \( A_0 > 0, A_1 > 0, \) and \( A_2 > 0. \) According to Routh-Hurwitz criterion, if \( a_1 \geq 0, a_0 \geq 0, k_p > 0, \) and \( \omega_3 > 0, \) all the roots of \( 1P(s) = 0 \) are located in left plane, i.e., when \( a_1 \geq 0 \) and \( a_0 \geq 0, \) there exists a sufficiently large \( \omega_3 > 0, \) such that all the root of \( 1P(s) = 0 \) are located in left plane.

Secondly, we consider \( 2P(s) = 0. \) According to Routh-Hurwitz criterion, \( 2P(s) = 0 \) can be written as the Routh
| $s^5$ | $B_5$ | $B_4$ | $B_3$ |
|-------|-------|-------|-------|
| $s^4$ | $B_3$ | $B_2$ | $B_1$ |
| $s^3$ | $C_2$ | $C_1$ | $C_0$ |
| $s^2$ | $C_1$ | $C_0$ | $B_0$ |
| $s^1$ | $C_0$ | $B_0$ | $B_0$ |
| $s^0$ | $C_0$ | $B_0$ | $B_0$ |

In TABLE IV, $C_1$, $C_2$, $C_3$, and $C_4$ are

$$C_1 = \frac{N_5}{D_5}, \quad C_2 = \frac{N_6}{D_6}, \quad C_3 = \frac{N_7}{D_7}, \quad C_4 = k_p\omega_n^3$$  \hspace{1cm} (75)$$

where

$$N_5 = a_0 a_1 + 3a_1^2\omega_n + 9a_1\omega_n^2 + 8\omega_n^3$$
$$D_5 = a_1 + 3a_2\omega_n$$
$$N_6 = 3(a_0 a_0 + a_1^2 k_p)\omega_n + (9a_0 a_1^2 + 15a_2^2 k_p)\omega_n^2 + a_1(10a_0 + 9a_1^2 + 18k_p)\omega_n^3 + 30a_1^2\omega_n^4 + 33a_1\omega_n^5 + 8\omega_n^6 - 3a_0 a_0 k_p - 9a_0 k_p - 3a_0\omega_n^4$$
$$D_6 = a_0 a_1 + 3a_1^2\omega_n + 9a_1\omega_n^2 + 8\omega_n^3$$
$$N_7 = 5(a_0^3 a_1 k_p + a_0 a_1^3 k_p) + (9a_0 a_1^2 k_p + 15a_0^2 a_1 k_p + a_1(10a_0 + 9a_1^2 + 18k_p)\omega_n^3 + 30a_1^2\omega_n^4 + 33a_1\omega_n^5 + 8\omega_n^6$$
$$D_7 = 3(a_1^2 k_p + \omega_n^2)$$

When $a_0 \geq 0$, $a_1 \geq 0$, and $\omega_n$ is sufficiently large, from (76), we have $B_0 > 0$, $B_1 > 0$, $C_1 > 0$, $C_2 > 0$, $C_3 \geq 0$, and $C_4 > 0$. According to Routh-Hurwitz criterion, if $B_0 > 0$, $B_1 > 0$, $C_1 > 0$, $C_2 > 0$, $C_3 \geq 0$, and $C_4 > 0$, then all the roots of $P(s) = 0$ are located in the left plane.

In summary, choosing $b_i = C_n + \omega_n i$ for $i = 1, 2, 3$, when $m = n = 2$, all the roots of a system with constant $\omega_n > 0$, such that the closed-loop is BIBO stable.

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