Realization of fractional-order capacitor based on passive symmetric network

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HIGHLIGHTS
• A new realization of the fractional capacitor using passive symmetric networks is proposed.
• General analysis of this network regardless of the internal impedances composition is introduced.
• Three scenarios based on RC circuit or integer Cole-Impedance circuit or both are utilized.
• The network size is optimized using Minimax and least mth optimization techniques.
• Monte Carlo simulations and experimental results are provided with applications.

ABSTRACT
In this paper, a new realization of the fractional capacitor (FC) using passive symmetric networks is proposed. A general analysis of the symmetric network that is independent of the internal impedance composition is introduced. Three different internal impedances are utilized in the network to realize the required response of the FC. These three cases are based on either a series RC circuit, integer Cole-Impedance circuit, or both. The network size and the values of the passive elements are optimized using the minimax and least mth optimization techniques. The proposed realizations are compared with well-known realizations achieving a reasonable performance with a phase error of approximately 2°. Since the target of this emulator circuit is the use of off-the-shelf components, Monte Carlo simulations with 5% tolerance in the utilized elements are presented. In addition, experimental measurements of the proposed capacitors are performed, therein showing comparable results with the simulations. The proposed realizations can be used to emulate the FC for experimental verifications of new fractional-order circuits and systems. The functionality of the proposed realizations is verified using two oscillator examples: a fractional-order Wien oscillator and a relaxation oscillator.

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Introduction

Fractional-order circuits and systems have attracted the attention of researchers worldwide due to the nature of the fractional behaviour, which can model many natural phenomena [1].
Fractional-order modelling considers the effects of the history and is thus practical and more suitable for modelling, analysing, and synthesizing electrical, chemical, and biological systems [2–10]. In addition, fractional-order modelling adds extra degrees of freedom in controlling the frequency behaviour, which makes it superior to traditional integer-order models and able to describe the behaviour of complex systems and materials [11]. Recently, fractional calculus has been applied extensively to electrical circuits. Many theorems and generalized fundamentals, such as stability theorems, filters, fractional-order oscillators and charging circuits, have been introduced using fractional-order circuits [12–21].

The first logical definitions for fractional calculus were introduced by Liouville, Riemann and Grünwald in 1834, 1847 and 1867, respectively [22]. However, the idea of fractional calculus, as an extension of calculus, was proposed much earlier by L’Hopital and Leibniz in 1695. The Laplace transform of the derivative of a function, \( f(t) \), in the fractional domain is \( L \{ \alpha D^\alpha f(t) \} = s^\alpha F(s) \) for zero initial conditions. Based on this definition, the general electrical element is defined as \( Z(s) = ks^\alpha \), which is called a constant phase element, CPE, where the phase, \( \theta \), is \( \tan^{-1}(\frac{\alpha}{\tilde{\beta}}) \), a constant and function of the fractional order \( \alpha \). When \( \alpha = 0, -1 \) and 1, this element is known in the circuit community as resistor, capacitor and inductor, respectively. This element is either capacitive for \( \alpha < 0 \) or inductive for \( \alpha > 0 \). In addition, the CPE is referred to as a fractional capacitor (FC) for \( -1 < \alpha < 0 \). In practice, the frequency-dependent losses in the capacitor and the inductor elements are modelled as a CPE, as previously proved [23–24].

Moreover, fractional theory was extended to include memristive elements [25]. Due to the importance of the fractional behaviour, there have been many attempts to realize a solid-state constant-phase element as a two-terminal device. Solid-state CPEs are realized using different composites and materials, for example, electrochemical materials and a composition of resistive and capacitive film layers [26–29]. All these attempts remain in the research phase and have yet to become commercially available. Thus, researchers tend to synthesis circuits that mimic the frequency behaviour of fractional elements for a certain band of frequencies. The realization of fractional emulation circuits is divided into two main categories:

(a) Passive realizations based on specific types of RC ladder structures such as that shown in Fig. 1 [30–33]. These passive realizations are based on an approximation of the fractional integral/differential operator \( s^\alpha \) as an integer-order transfer function. For example, the Oustaloup approximation provides a rational finite-order transfer function that can be realized using well-known transfer function realization techniques such as that of Causer and Foster [34]. Another way to realize the FC was introduced by Valsa [35], where the poles and zeros are arranged to have the order required to simplify the FC realization. However, these techniques require a wide range of resistor and capacitor values.

(b) Active realizations based on operational amplifiers (opamps) or current feedback opamps (CFOAs) with some passive components [36–39].

In addition, a summary and comparison between the active realizations of CPEs are introduced showing the complexity, performance and working frequency range [40]. Moreover, there are many recent publications that try to realize the fractional order element with minimum area using different ways based on transistor levels [41–42], using a single active element [43].

In this paper, we investigate a new passive realization technique for CPE and FC based on a passive symmetric network. Three RC circuits are used in a symmetric network for approximating the fractional behaviour in the range [100Hz – 10kHz]. This frequency range is chosen as an arbitrary example to verify the proposed circuits and expressions; any frequency range can be used and optimized over. The wider the frequency range is, the higher the number of stages. The minimax optimization technique [44] is used to fit the circuit network magnitude and phase response to the CPE. The advantage of the proposed realization is that the spread of the element values is much less than other realizations such as Valsa, Foster etc.

This paper is organized as follows: Section 2 introduces the mathematical analysis for the proposed symmetric network. Then, the formulation of the optimization technique is introduced and applied for three proposed circuits in Section 3. A comparison among these circuits and well-known realizations is introduced, in addition to Monte Carlo simulations and experimental results. Section 4 discusses the application of the proposed circuits in sinusoidal and relaxation oscillators to check the functionality of the proposed circuits. Finally, the conclusion and future work are given.

Proposed symmetric network analysis

Previous passive realizations are based on using different resistors and capacitors. In this paper, we investigate replicating the same impedance in the network to obtain the fractional behaviour. Fig. 1(c) shows the circuit diagram of the proposed symmetric network. We use basic circuit network theory and the proposed approach to analyse fractional-order \( 2 \times n \) RLC networks [45] and obtained the equivalent impedance for the proposed network shown in Fig. 1(c). In Fig. 1(d), by applying Kirchhoff’s current law at nodes \( c \) and \( d \), the equation of the currents can be written as

\[
l_k - l_{k-1} = -q_k - q_{k-1} = l_k,
\]

and according to Kirchhoff’s voltage law, the voltage equations of the \( k \)th and \((k-1)\) loops can be expressed as

\[
z_1l_{k-1} + z_2l_k - z_1l_{k-1} = 0,
\]

\[
z_1l_{k-1} + z_2l_k - z_{k-1}l_k - z_1l_{k-1} = 0,
\]

respectively. Subtracting Eq. (2b) from Eq. (2a) and then substituting by Eq. (1),

\[
z_0(2l_k - l_{k-1} - l_{k+1}) + 2z_1l_k = 0,
\]

which can be rewritten as

\[
l_{k+1} = 2(1 + \hat{\lambda})l_k - l_{k-1},
\]

where \( \hat{\lambda} = \frac{\tilde{\beta}}{z_2} \). Eq. (4) can be written as

\[
l_{k+1} = (p + q)l_k - pl_k,
\]

where

\[
p = q = 2(1 + \hat{\lambda}),
\]

and the combination of both of them can be used to approximate the fractional order capacitor.

By solving Eq. (6), the values of \( p \) and \( q \) can be written as

\[
p = 1 + \hat{\lambda} + \sqrt{2\hat{\lambda} + \hat{\lambda}^2}, \quad \text{and} \quad q = 1 + \hat{\lambda} - \sqrt{2\hat{\lambda} + \hat{\lambda}^2},
\]

respectively. Eq. (5) represents the recursive current relation between the network nodes. Thus, we can obtain

\[
l_{k+1} = p^{-1}(l_2 - pl_1),
\]

\[
l_{k+1} = q^{-1}(l_2 - ql_1).
\]
By subtracting Eq. (8b) from Eq. (8a), the solution $I_k$ based on $I_1, I_2$ is

$$I_k = \frac{1}{p - q}\{p^{k-1}(I_2 - qI_1) - q^{k-1}(I_2 - pI_1)\},$$

where $k = 3, 4, 5, \ldots$. From the first loop, the relation between $I_2$ and $I_1$ is given by

$$I_2 = (1 + 2\lambda)I_1 - 2\lambda I,$$

and from Eq. (6), $2\lambda = p + q - 2$, $I_2$ is

$$I_2 = (p + q - 1)I_1 - (p + q - 2)I.$$

From Fig. 1(d), the current at point $a$ is given by

$$I = \sum_{i=1}^{n+1} I_i.$$

By substituting Eqs. (9) and (11) into Eq. (12),

$$I_1 = I\left(1 - \frac{p^n - q^m}{p^{n+1} - q^{m+1}}\right).$$

The voltage equation between the two points $a$ and $b$ for the network shown in Fig. 1(c) is given by

$$V_{ab} = I_2z_0.$$

Hence, the network equivalent impedance between $a$ and $b$ is given as follows:
where $Z_{ab}$ is the input impedance of the network. Now, the parameters of the network required to behave similarly to an FC or CPE must be found. In the next section, the optimization formulated to find the optimal network parameters is introduced.

**Optimal realization for fractional-order capacitor**

**Optimization problem formulation**

The equivalent impedance of fractional capacitor with order $x$ is given by

$$Z_{eq}(\omega) = \frac{1}{C_x(j\omega)^x},$$

(16)

and has constant phase depend on the value of $x$ and equals ($-\frac{2\pi}{x}$). Now, it is required to find the circuit values which give a close response to the required fractional-order capacitor. Thus, an optimization problem is constructed to fit the fractional-order capacitor behavior to design each network shown in Fig. 1(c). Each one of these problems can be written as a Minimax problem function of choosing $z_0$ and $z_1$. The objective function of the Minimax optimization problem is constructed between the phase of the proposed network and the required FC phase, $-2\pi/x$ [39]. Also, it is constructed to find the elements values for $z_0$ and $z_1$ and the number of network, $n$, over the required frequency range for instance 100 Hz to 10 kHz. The error function between the phase of proposed network and fractional order capacitor can be expressed as

$$\phi(n, z_0, z_1) = \arg(Z_{ab}) + \frac{2\pi}{2}.$$  (17a)

The largest elemental error $\phi_i = \phi(n, z_0, z_1)|_{j...j}$ should be minimized and, therefore, the $L_n$ norm of error function should be used. The $L_n$ of the error function Eq. (17a) is numerically equal to $\max(|\phi_m(n, z_0, z_1)|)$ then the minimization of the $L_n$ norm can be given by:

$$\min \left[ \max(|\phi_m(n, z_0, z_1)|) \right]$$

$$n, z_0, z_1, \quad m = 1, 2, \ldots, N$$

(17b)

where $\phi_m(n, z_0, z_1) = \arg(Z_{ab}) + \frac{2\pi}{2}f_{j...j}$ and $N$ the number of points in frequency range [100 Hz to 10 kHz]. To use the least $m^\text{th}$ optimization function, the objective function in Eq. (17b) should be rewritten in form of one minimize function as follows:

$$\min[F(n, z_0, z_1)],$$

(18a)

$$F(n, z_0, z_1) = \left\{ \frac{1}{N} \left[ \left| \phi_m(n, z_0, z_1) \right| \right]^m \right\}^{\frac{1}{m}},$$

(18b)

where $m$ is a positive integer number. The impedances $z_0$ and $z_1$, can be chosen to be any integer-order resistive network. In this work, three approximations for the fractional order impedances are investigated; the first one is series-connected RC circuit shown in Fig. 1(e). The second circuit model is due to replacement RC series by the first-order Cole-Impedance model connected in Fig. 1(e). The other one is circuit model of combined between series-connected RC and the first-order Cole-Impedance model connected.

The proposed approach to evaluate the values for elements circuits is summarized in the following steps:

**Step 0:** Define the required order, $x$, and the phase error $\varepsilon$.

**Step 1:** Set the network size to one ($n = 1$).

**Step 2:** Solve the optimization problem (18) by any optimization package software.

**Step 3:** Evaluate the maximum value of absolute error in phase response $|\phi_m|_{\text{max}}$.

**Step 4:** If $|\phi_m| \leq \varepsilon$ end; otherwise increment $n$ and go to step 2.

Simply, this algorithm can be seen as a search algorithm which searches for the values of the network that best fit the required fractional response under two conditions: the phase error should be less than $\varepsilon$ and the minimum number of networks, $n$.

**Series RC-based network realization**

Assume that $z_0$ and $z_1$ are series-connected RC, as shown in Fig. 1(e). The impedance equations are

$$z_0 = R_0 + \frac{1}{C_{00}}, \quad z_1 = R_1 + \frac{1}{C_{11}},$$

(19a)

where

$$R_0 = \frac{\mu R_0}{C_x}, \quad C_0 = \frac{C_x C_1}{\mu}, \quad R_1 = \frac{\mu R_1}{C_x}, \quad C_1 = \frac{C_x}{\mu},$$

(19b)

and $\mu$ is a constant parameter used to control the network magnitude response. It is important to highlight that the argument of $Z_{ab}$ in Eq. (15) is independent of $\mu$ and $C_x$ because the equivalent impedance $Z_{eq}$ is a function of $\lambda$, which is the ratio between $z_1$ and $z_0$. Thus, the optimization problem (18) can be written as

$$\min [F(n, R_1, R_2, C_1)],$$

subject to $n \in N, R_1, R_0 \geq 0$ and $C_0, C_1 > 0$, where $F(n, R_1, R_0, C_0, C_1)$ is given by

$$F(n, R_1, R_0, C_0, C_1) = \left\{ \frac{1}{N} \left[ \left| \phi_m(n, R_1, R_0, C_0, C_1) \right| \right]^m \right\}^{\frac{1}{m}},$$

(20a)

The optimization package in Mathematica is used to solve this optimization problem. To find the global minimum of the optimization problem (20) subject to $R_1, R_0 \geq 0$ and $C_0, C_1 > 0$, the NMinimize Function in Mathematica is used. The optimized values for the circuit elements for different values of the fractional order $x$ are summarized in Table 1. Note that the proposed problem in Eq. (20) is based only on the phase response of the fractional-order capacitor. The value of $\mu$ is used to control the network magnitude response to fit the capacitor magnitude response. To find the value of $\mu$, a problem based on fitting between $|Z_{eq}(\omega)|$ and $|Z_{ab}|$ is established and can be solved by the “FindFit” function in Mathematica. The values of $\mu$ for different values of the fractional order $x$ are summarized in Table 1.

**Cole-Impedance based network realization**

Assume $z_0$ and $z_1$ are Cole-impedance connected in Fig. 1(e). Then, the impedance equations are as follows:

$$z_0 = R_{00} + \frac{R_{00}}{sC_0R_{00} + 1}, \quad z_1 = R_{11} + \frac{R_{11}}{sC_{11}R_{11} + 1},$$

(21a)
Table 1
The optimized values for series RC, first-order cole-impedance and RC-Cole-impedance connected networks.

|       | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $R_0$ | 1.48 | 2.55 | 4.37 | 8.98 | 17.9 | 35.9 | 71.9 | 143 | 285 | 570 |
| $C_0$ | 1.05 | 1.24 | 1.69 | 2.42 | 3.87 | 7.87 | 15.7 | 31.5 | 63.0 | 126.0 |
| $n$   | 0.98 | 1.48 | 2.55 | 4.37 | 8.98 | 17.9 | 35.9 | 71.9 | 143 | 285 |

The maximum values of the absolute error in the phase for the proposed network using the parameter optimal values for the optimization problem in Eq. (18) can be written as

$$F(n, R_0, R_1, C_0, C_1) = \sum_{m=1}^{N} [\phi_m(n, R_1, R_0, R_0, C_0, C_1)]^2.$$

In this case, assume that $x_0$ or $x_1$ is series RC connected and that the other remaining impedance is Cole-impedance connected. Then, the impedance equations are as follows:

$$z_0 = R_0 + \frac{R_0}{sC_0R_0 + 1}, z_1 = R_1 + \frac{1}{sC_1},$$

or

$$z_0 = R_0 + \frac{1}{sC_0}, z_1 = R_1 + \frac{R_1}{sC_1R_1 + 1}.$$

Similarly, we form the optimization problem as in the previous two cases and use the “NMinimize” and “FindFit” functions in Mathematica. Tables 1 and 2 show two optimal values for $n, R_1, R_0, R_0, C_0, C_1$ and the control parameter $\mu$ for different maximum absolute errors between $[1.2^\alpha, 3.7^\alpha]$ and $[1.2^\alpha, 2^\alpha]$ for different values of the fractional order $\alpha$ in the range frequency of 100 Hz to 10 kHz.

Simulation results and comparison

The discussion and the comparison between the different models can be summarized in the following points:

- The maximum values of the absolute error in the phase for the three models under different values of $\alpha$ are tabulated in Table 3. From this table, the errors in the phase

Table 2
The optimized values for the RC-Cole-impedance model connected network with Max. Abs. Error between $[1.2^\alpha, 2^\alpha]$.

|       | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\mu$ | 8.98 | 1.48 | 2.55 | 4.37 | 8.98 | 17.9 | 35.9 | 71.9 | 143 | 285 |
| $R_0$ | 0.98 | 1.48 | 2.55 | 4.37 | 8.98 | 17.9 | 35.9 | 71.9 | 143 | 285 |
| $R_1$ | 1.05 | 1.24 | 1.69 | 2.42 | 3.87 | 7.87 | 15.7 | 31.5 | 63.0 | 126.0 |
| $n$   | 0.98 | 1.48 | 2.55 | 4.37 | 8.98 | 17.9 | 35.9 | 71.9 | 143 | 285 |
for the RC series model and Cole-impedance connected model are between $[2.9: 5.9]$ and $[1.2: 1.9]$, respectively. Although the error in the Cole-impedance model is less than $2^\circ$, the number of networks $n$ is larger than that of the RC series model for all values of $\alpha$. For example, when the fractional order $\alpha = 0.9$, network numbers of $n = 1$ and $n = 5$ achieve errors of $2.9^\circ$ and $1.2^\circ$ for the RC-series model and Cole-impedance model, respectively.

Table 3
The maximum values of the absolute error in the phase response of the three proposed models.

| $\alpha$ | $n$ | RC series model | Cole-impedance model | RC-Cole-impedance model |
|---------|-----|-----------------|----------------------|--------------------------|
|         |     | Max. abs. error (rad) | Max. abs. error (rad) | Max. abs. error (rad) |
| 0.9     | 1   | 0.05206         | 0.02143              | 0.02177                  |
| 0.8     | 2   | 0.08814         | 0.03288              | 0.0368                  |
| 0.7     | 5   | 0.0809         | 0.03439              | 0.05403                  |
| 0.6     | 7   | 0.08458         | 0.029322             | 0.0446                  |
| 0.5     | 18  | 0.08586        | 0.03122              | 0.03568                  |
| 0.4     | 10  | 0.0999         | 0.0318               | 0.0644                  |
| 0.3     | 3   | 0.103          | 0.03056              | 0.0608                  |

Fig. 2. The phase responses and errors for the proposed circuits for (a, b) $\alpha = 0.9$, (c, d) $\alpha = 0.8$ (e, f) $\alpha = 0.5$, and (g, h) comparison with other well-known techniques.
The element values for the three proposed circuits for $C_\alpha = 10^{-6} \text{F/s}^{1-\alpha}$.

| Circuit | Cole-impedance model | RC model |
|---------|-----------------------|----------|
| $\alpha$ | 0.9 | 0.5 | 0.3 | 0.9 | 0.5 | 0.3 |
| $R_{0_l}(\Omega)$ | 6.355 | 121947.088 | 94190.616 | 8.3858 | 44515.21 | 112059.288 |
| $R_{0_h}(\Omega)$ | 37084.053 | 128946.76 | 116306.61 | – | – | – |
| $C_{0_l}(\text{F})$ | $3.61 \times 10^{-7}$ | $9.99 \times 10^{-9}$ | $1.29 \times 10^{-8}$ | $3.679 \times 10^{-7}$ | $5.379 \times 10^{-8}$ | $2.786 \times 10^{-8}$ |
| $R_{1_l}(\Omega)$ | 6494.929 | 4169.646 | 1428.3297 | 0 | 0 | 0 |
| $R_{1_h}(\Omega)$ | 16572.333 | 15361079.6 | $22.9 \times 10^6$ | – | – | – |
| $C_{1_l}(\text{F})$ | $3.85 \times 10^{-7}$ | $3.67 \times 10^{-6}$ | $1.99 \times 10^{-5}$ | $7.66 \times 10^{-7}$ | $6.628 \times 10^{-8}$ | $2.346 \times 10^{-9}$ |
| $n$ | 5 | 25 | 7 | 1 | 18 | 3 |

| RC-Cole-impedance model | $\alpha = 0.8$ |
|-------------------------|----------------|
| $R_{0_l}(\Omega)$ | 39.984 |
| $R_{0_h}(\Omega)$ | 65126.96 |
| $C_{0_l}(\text{F})$ | $0.130 \times 10^{-6}$ |
| $R_{1_l}(\Omega)$ | 1132.88 |
| $C_{1_l}(\text{F})$ | $349.6 \times 10^{-6}$ |

- There are 8 elements in the RC-series model and 17 elements in the RC-Cole-impedance model with fractional order $\alpha = 0.9$. However, for lower fractional orders, for example, $\alpha = 0.8$ or $0.7$, the number of elements and the phase error for the RC-series model are larger than those of the RC-Cole-impedance model. There are 10 and 17 elements in the RC-Cole-impedance model and 14 and 32 elements in the RC-series model for $\alpha = 0.8$ and 0.7, respectively.

- The phase response in the two proposed models when $\alpha = 0.9$ and $\alpha = 0.5$ is shown in Fig. 2(a) and (e), respectively. It is clear from Fig. 2(a) and (e) that the phase response for the two proposed circuits is near the phase of the fractional-order capacitor in the frequency range of 100 Hz to 10 kHz.

- From the absolute errors shown in Fig. 2(b) and (d), the phase response of the Cole-impedance model is better matched with the fractional-order capacitor. However, the number of networks ($n = 25$) in the Cole-impedance model is large compared to the RC-series model ($n = 18$). In some cases, the absolute error of the RC-Cole-impedance model is smaller than that of the other proposed models even though the network size ($n$) is equal to or less than the RC-Cole-impedance model. For example, when $\alpha = 0.8$, the network size is $\{1, 4\}$, $2$ and $5$ with error $\{3.3^9, 2.1^9\}$, $5^9$ and $1.9^9$ in the RC-Cole-impedance model, RC model, and Cole-impedance model, respectively. The number of elements in the RC-Cole-impedance model is less than that of the Cole-impedance model even though the error and network size are almost equal.

- Fig. 2(c) and (d) show the absolute errors and phase responses for the RC-Cole-impedance model for $\alpha = 0.8$. These figures clearly show that the phase response for the two cases ($n = 1$ and $n = 4$) is near the phase of the fractional-order capacitor in the frequency range of 100 Hz to 10 kHz. The magnitude responses of the three proposed circuits are studied for $C_\alpha = 10^{-6} \text{F/s}^{1-\alpha}$. The proposed circuit elements are summarized in Table 4. These values are calculated from Table 1 and using Eqs. (19b) and (21b).

- Fig. 2(g) shows the absolute errors of the 6th- and 11th-order approximations of $s^{\alpha}$ by the El-Khazali approximation and Oustaloup’s approximation $[46-47]$, respectively. It is illustrated from this figure that Oustaloup’s approximation is good approximation at low frequencies. However, the proposed approximate responses are approximately $\pm 25\%$ in the frequency range design from 100 Hz to 10 kHz. In addition, Fig. 2(h) shows the absolute error in the phase when $\alpha = 0.8$ for the Foster II, Valsa $[34,35]$ and RC-Cole-impedance models ($n = 1$). This figure shows that the Valsa model is better than the RC-Cole-impedance model. However, the Valsa model is an asymmetric circuit, where the values of the elements are not equal, and the proposed model is a symmetric model with reasonable phase. The symmetry property is one of the advantages of the proposed models compared to other models, and it may facilitate the future manufacture of fractional-order capacitors.

- Fig. 3 shows the magnitude response of the two proposed circuits when $\alpha = 0.9$ and 0.3. These figures clearly show that the relative error for the Cole-impedance model is smaller than the RC model error for each case. The circuit elements used in the design are summarized in Tables 4 when $\alpha = 0.3$, 0.8 and 0.9 for the different proposed models.

- Fig. 3(c) and (d) show the errors and magnitude response of RC-Cole-impedance model when $\alpha = 0.8$. These Figures show that the response for the two cases of the RC-Cole-impedance model exactly matches the response of the fractional-order capacitor.

- Fig. 3(g) shows the number of RC networks required to realize a fractional-order capacitor with order $\alpha$ with absolute error less than 0.09. Clearly, the maximum number of RC networks is needed for $\alpha = 0.5$ and decreases with increasing or decreasing the order since the device becomes more capacitive or resistive towards 1 or zero, respectively.

### The Monte Carlo analysis and experimental results

The behaviour of the proposed models for different values of $\alpha$ and $\mu$ was studied using Monte Carlo analysis. For $\alpha = 0.8$ and $n = 4$ in the RC-Cole-impedance model, the phase and magnitude responses with 5% tolerance in the resistors and capacitors are shown in Fig. 4, in addition to the variability curves of $\alpha$ and $C_\alpha$. Table 5 shows the effects of applying a 5% tolerance to the resistors and capacitors of the proposed models. The Monte Carlo analysis is performed over 1000 runs. The mean and standard deviation of the designed element parameters ($\alpha, C_\alpha$) are found as follows: for $\alpha = 0.9$ realized using the RC-series model, the mean and standard deviation are $(0.9026, 0.982 \times 10^{-6})$ and $(0.0019, 0.468 \times 10^{-7})$, respectively; for the $\alpha = 0.3$ element realized using the Cole-impedance model, the mean and standard deviation are $(0.3022, 0.982 \times 10^{-6})$ and $(0.004, 0.437 \times 10^{-7})$, respectively; and for the $\alpha = 0.8$ element realized using the RC-Cole-impedance model, the mean and standard deviation are $(0.8063, 0.99 \times 10^{-6})$ and $(0.0023, 0.548 \times 10^{-7})$ for $n = 1$ and $(0.8027, 1.002 \times 10^{-6})$ and $(0.0045, 0.37 \times 10^{-7})$ for $n = 4$, respectively.
Two fractional-order capacitors of different order are realized using the RC model and the RC-Cole-impedance model. The EC-Lab control software and SP-150 BioLogic instrument are used for the characterization. Fig. 4(e) and (f) show the characterizations of the proposed capacitor elements of the RC model and RC-Cole-impedance model of fractional order 0.9 and 0.8, respectively. In the case of $\alpha = 0.9$, the exact phase is $-82, -72$ degrees, and the error is $\pm 3.5$ degrees for the two cases.

**Applications**

To validate the proposed approximation models, two applications are investigated: the Wien fractional-order oscillator presented in Radwan et al. [48] and the fractional-order relaxation oscillator presented in Nishio [49] with their circuit simulations.

**Application (1): fractional-order Wien oscillator**

The following system is describing the fractional-order Wien oscillator shown in Fig. 5(a):

$$\begin{bmatrix}
D^\alpha V_{c1} \\
D^\alpha V_{c2}
\end{bmatrix} = \begin{bmatrix}
\frac{A}{K_{L1}} - \frac{1}{K_{C1}} \\
\frac{1}{K_{C2}} - \frac{1}{K_{L2}}
\end{bmatrix} \begin{bmatrix}
V_{c1} \\
V_{c2}
\end{bmatrix},$$

where $A = 1 + \frac{R_2}{R_1}$. The linear fractional-order system (24) can admit sinusoidal oscillations if and only if there exists a value of $\omega$ that satisfies simultaneously the two equations [48].
\[
\omega^{\alpha+\beta} \cos \left( \frac{(\alpha + \beta)\pi}{2} \right) + \frac{1}{R_1C_2} \omega^\alpha \cos \left( \frac{\beta \pi}{2} \right) - \left( \frac{A - 1}{R_2C_1} \right) \omega^\beta \cos \left( \frac{\pi}{2} \right) + \frac{1}{C_1C_2R_1R_2} = 0. \tag{25a}
\]

\[
\omega^{\alpha+\beta} \sin \left( \frac{(\alpha + \beta)\pi}{2} \right) + \frac{1}{R_1C_2} \omega^\alpha \sin \left( \frac{\beta \pi}{2} \right) - \left( \frac{A - 1}{R_2C_1} \right) \omega^\beta \sin \left( \frac{\beta \pi}{2} \right) = 0. \tag{25b}
\]

The gain \( A \) and the oscillation frequency \( \omega \) do not have closed-form formulas and need to be solved numerically. The Wien oscillator with the proposed fractional-order capacitors is simulated using LTspice. To design the fractional-order Wien oscillator from Eqs. (25a) and (25b), assume the values of \( A, C_1, C_2, \alpha \) and \( \beta \) and solve Eqs. (25a) and (25b) at the required frequency of oscillation to obtain the values of \( R_1 \) and \( R_2 \).

As a special case, when \( \alpha = \beta \), the gain and frequency of oscillation are derived in [50] and are given by

\[
\omega = \left( \frac{1}{R_1R_2C_1C_2} \right)^{1/2\alpha}, \tag{26a}
\]

\[
A = 1 + \frac{R_2}{R_1} + \frac{C_1}{C_2} + 2 \sqrt{\frac{R_2C_1}{R_1C_2} \cos \frac{\beta \pi}{2}}. \tag{26b}
\]

For a 1kHz oscillation, the values of \( R \) and \( C \) satisfying Eqs. (25a) and (25b) are given in Table 6 with different values of \( \alpha \) and \( \beta \). The oscillator is simulated by LTspice using the TL1001 op amp with the discrete elements listed in Table 6. The simulation results are shown in Fig. 5 for different cases, which perform efficiently with the proposed capacitors. Fig. 5(e) shows the Fast Fourier Transform of the time-domain signal for the C-I realization with order 0.9. The total harmonic distortion of this oscillator is approximately 0.114.

**Application (2): Fractional-order relaxation oscillator**

The circuit shown in Fig. 6(a) represents a free-running multivibrator with a FC \( C_2 \). For \( \gamma < 1 \), the oscillation period, \( T \), and time
constant, $\tau$, are related by the following closed-form expression [49]:
\[
\frac{1 - B}{1 + B} = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{1}{2})^k}{(k^2 + 1)}
\]
(27)
where $B = \frac{\beta}{R_3 \cdot R_2}$ and $\tau = RC$. The oscillation period, $T$, has a closed-form solution at $\gamma = 1$ only. Thus, to find the time constant $\tau$ required to obtain a certain oscillation period, this equation needs to be solved numerically. To test the RC-Cole-impedance capacitor model (with $n = 1$) in this oscillator, we chose the oscillation frequency to be 1kHz and $\alpha = 0.8$. The values of $R$ and $C$, chosen to satisfy (27), are $R = 1k\Omega$, $C_1 = 10^{-6}$, $R_1 = 1k\Omega$ and $R_2 = 2.6k\Omega$. The oscillator is simulated by LTspice using a TL1001 opamp. Fig. 6(b) shows the time-domain response of the oscillator. In addition, the FFT of the time-domain voltage is shown in Fig. 6(c).
Conclusions

In this work, the proposed network-based FC realization is analyzed, and its equivalent impedance is deduced. Three symmetric circuits used to approximate fractional-order capacitors are proposed. The values of the proposed circuit elements are summarized in Tables 1 and 2 for certain capacitor orders; one can use these tables for capacitor designs in the design range of 100Hz to 10kHz. A simple approach is proposed based on the minimax technique and least mth optimization function to validate the magnitude and phase response of the FC. The proposed FC realizations were tested on the fractional-order Wien oscillator and relaxation oscillator using LTspice. The simulation responses were reasonable and acceptable.

As future work, the proposed fractional-order realizations will be experimentally tested and measured. Furthermore, the designed

![Circuit diagram]

**Fig. 5.** (a) Circuit realization for fractional-order Wien oscillator responses for different realizations. (b, c) R-C model, (d) C-I model, and (e) FFT of the transient response of C-I case shown in (d).

| $\alpha$ | $\beta$ | $C_1$ (μF/s$^{1-\alpha}$) | $C_2$ (μF/s$^{1-\beta}$) | $R_1$ (Ω) | $R_2$ (Ω) | $R_3$ (kΩ) | $R_4$ (kΩ) |
|---------|---------|---------------------------|---------------------------|-----------|-----------|-----------|-----------|
| 0.9     | 1       | 1                         | 1                         | 214.05    | 312.5     | 15        | 29.4 – 30.4 |
| 0.5     | 1       | 1                         | 1                         | 1279      | 2379      | 15        | 30.13     |
| 0.5     | 0.5     | 1                         | 1                         | 24,371    | 6530      | 15        | 30.9      |

**Table 6**

The values of R and C for Wien oscillator for 1 kHz oscillation.
elements will be designed over wider frequency ranges. The designed FCs will be utilized to facilitate fractional-order applications such as filters and control.

Conflict of interest

The authors have declared no conflict of interest.

Compliance with Ethics Requirements

This article does not contain any studies with human or animal subjects.

Acknowledgments

The authors would like to thank the Science and Technology Development Fund (STDF, Egypt) for funding the project # 25977 and the Nile University for facilitating all procedures required to complete this study.

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