A note on large $N$ thermal free energy in supersymmetric Chern-Simons vector models

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ABSTRACT: We compute the exact effective action for $\mathcal{N} = 3 \, U(N)_k$ and $\mathcal{N} = 4, 6 \, U(N)_k \times U(N')_{-k}$ Chern-Simons theories with minimal matter content in the 't Hooft vector model limit under which $N$ and $k$ go to infinity holding $N/k, N'$ fixed. We also extend this calculation to $\mathcal{N} = 4, 6$ mass deformed case. We show that those large $N$ effective actions except mass-deformed $\mathcal{N} = 6$ case precisely reduce to that of $\mathcal{N} = 2 \, U(N)_k$ Chern-Simons theory with one fundamental chiral field up to overall multiple factor. By using this result we argue the thermal free energy and self-duality of the $\mathcal{N} = 3, 4, 6$ Chern-Simons theories including the $\mathcal{N} = 4$ mass term reduce to those of the $\mathcal{N} = 2$ case under the limit.

KEYWORDS: Supersymmetric gauge theory, Duality in Gauge Field Theories, Chern-Simons Theories, $1/N$ Expansion

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1 Introduction

Recently there has been a big progress in study of three dimensional vector models with Chern-Simons gauge interaction (Chern-Simons vector models). A key discovery which triggered this progress is that this class of quantum field theories turned out to conserve an infinite number of higher spin currents in the ’t Hooft limit and be exactly solvable under the limit with the light-cone gauge [1, 2].

It was shown by a general argument that in a three dimensional conformal field theory keeping the “almost” conserved higher spin currents the form of the three point functions of higher spin currents is determined up to two undetermined parameters [3]. (See also [4, 5].) In a conformal Chern-Simons vector model, these two parameters basically correspond to rank of gauge group and Chern-Simons level, and the precise parameter mapping was determined by explicit computation of several three point correlators for bosonic and fermionic Chern-Simons vector models [6, 7]. These results strongly suggested that under the ’t Hooft limit these conformally symmetric Chern-Simons vector models enjoy level-rank duality known in pure Chern-Simons theories. (This suggestion was earlier made in [1].) This
non-supersymmetric duality is reminiscent of bosonization in two dimensions because the duality transformation exchanges the conserved currents of bosonic Chern-Simons vector model and those of fermionic one.

On the other hand, it is also possible that effective actions and thermal free energies in Chern-Simons vector models are computed exactly in the ‘t Hooft limit with the light-cone gauge [1, 8, 9] including chemical potential and holonomy [10]. (Related works are [11, 12]). It was shown that thermal free energies of Chern-Simons vector models also exhibit the non-supersymmetric duality mentioned above and supersymmetric duality known as Giveon-Kutasov (or Seiberg-like) duality [13, 14] by incorporating holonomy distribution obeying fermionic statistics in a high temperature limit [15]. This feature of holonomy was observed earlier in the study of Chern-Simons theory on the space of two torus by using the canonical formalism [16]. The origin of this peculiar holonomy distribution at a high temperature was clarified from the standpoint of path integral formalism in the study of Chern-Simons vector models on two sphere with thermal time [17]. It turned out that due to the fermionic holonomy distribution Chern-Simons vector models enjoy a novel thermal phase structure consistent with the duality transformation [17, 18].

The three dimensional duality can be extended to the most general renormalizable Chern-Simons vector models with one fundamental scalar and fermion under the ‘t Hooft limit [19]. This generalization enabled one to connect the known and unknown dualities by taking certain massless limits or scaling limits. As a result, strong evidence was provided for the expectation that \( N = 1, 2 \) Giveon-Kutasov duality and non-supersymmetric duality in the Chern-Simons vector models can be connected by renormalization group flow.

In this paper we explore the duality structure in Chern-Simons matter theories with higher supersymmetry by adding more fields. Especially to achieve \( N \geq 4 \) supersymmetry it is required to consider a non-simple gauge group such as \( U(N)_k \times U(N')_{-k} \) [20, 21]. We study a Chern-Simons system with such a gauge group by taking the limit defined by \( N, k \to \infty \) with \( N/k, N' \) fixed to reduce a system to a vector model.

The rest of this paper is organized as follows. In section 2, we compute the exact effective action of \( N = 3 \ U(N)_k \) and \( N = 4, 6 \ U(N)_k \times U(N')_{-k} \) Chern-Simons theories including \( N = 4, 6 \) mass terms by taking the ‘t Hooft vector model limit. In section 3, using the result obtained in section 2, we discuss the thermal free energy and self-duality of the supersymmetric Chern-Simons matter theories.\(^1\) Section 4 is devoted to summary and discussion. In appendix, supersymmetric Chern-Simons matter actions are written in our convention.

\[ 2 \text{ Exact large } N \text{ effective action} \]

\[ 2.1 \text{ Fundamental matter fields} \]

In this preliminary section we study \( U(N)_k \) Chern-Simons theory coupling to \( M \) fundamental scalar and fermionic fields in the ‘t Hooft limit, in which \( N \) and \( k \) go to infinity.

\(^1\)When we say one theory is self-dual, we mean that the dual theory thereof is in the same theory space with different parameters.
with \( \lambda = N/k \) fixed. We denote \( M \) fundamental scalar fields and fermionic ones by \( q^A, \psi_A \) respectively, where \( A = 1, 2, \cdots, M \). The case \( M = 1 \) was studied in detail in [8]. We are interested in a situation where the theory has \( U(M) \) flavor symmetry, which we assume in what follows. The main purpose of this section is to demonstrate how to generalize the previous result to \( M \) copies of the matter fields reviewing the technique employed in the previous study.

For this purpose let us start by a generic action

\[
S = \int d^3 x (\kappa L_{cs}[A] + L_m). \tag{2.1}
\]

Here \( \kappa \) is related to the Chern-Simons level \( k \) by \( \kappa = \frac{k}{4\pi} \) and

\[
L_{cs}[A] = \text{Tr} \left[ i\varepsilon^{\mu\nu\rho} \left( A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right) \right],
\]

\[
L_m = D_\mu q^\dagger_A D^\mu q^A + \psi^\dagger_A \gamma^\mu D_\mu \psi_A + V_m,
\]

where \( D_\mu \) is the covariant derivative acting on the fields in a way that

\[
D_\mu q^A = \partial_\mu q^A - i A_\mu q^A, \quad D_\mu q^\dagger_A = \partial_\mu q^\dagger_A + i q^\dagger_A A_\mu,
\]

\[
D_\mu \psi_A = \partial_\mu \psi_A - i A_\mu \psi_A, \quad D_\mu \psi^\dagger_A = \partial_\mu \psi^\dagger_A + i \psi^\dagger_A A_\mu.
\]

\( V_m \) represents a gauge-invariant potential in this system given by a function of bilinears of the elementary fields \( q^A, \psi_B \) in a flavor-singlet way. We suppress contraction of fundamental gauge indices for notational simplification. A specific example is \( N = 3 \), whose action in our notation is given in A.3.

Firstly we separate the gauge field into \( U(1) \) part and \( SU(N) \) one. The Chern-Simons coupling of \( U(1) \) gauge field is given by \( Nk \), which means the gauge propagator of \( U(1) \) gauge field has extra \( 1/N \) factor compared to that of \( SU(N) \) part. Therefore the contribution of \( U(1) \) part of the gauge field is sub-leading in the large \( N \) limit.

So let us focus on the case when the gauge group is \( SU(N) \). In order to determine the exact effective action we fix the gauge degrees of freedom by the (Euclidean) light-cone gauge [1]. This gauge fixing gets rid of the cubic interaction of gauge field, which enables us to integrate it out. From the equation of motion for \( A^+_\mu \) we obtain

\[
2\kappa \partial_- A^3_\mu = \text{Tr} \left[ i \left( q^A \partial_- q^\dagger_A - \partial_- q^A q^\dagger_A - \psi_A \gamma_- \psi^\dagger A \right) T^a \right], \tag{2.5}
\]

where \( T^a \) is a generator of \( SU(N) \) gauge group. A solution in the Fourier space is given by\(^2\)

\[
A^3_\mu(q) = \frac{1}{2\kappa i q_-} \int \frac{d^3 r}{(2\pi)^3} \text{Tr} \left[ \left( (2r + q)_- q^A (r + q) q^\dagger_A (-r) + i \psi_A (q + r) \gamma_- \psi^\dagger A (-r) \right) T^a \right]. \tag{2.6}
\]

\(^2\)In this solution we neglect zero mode (holonomy) of this gauge field. Here we suppress this for notational simplification. The holonomy can be taken into account by shifting the momentum in the propagator [10].
Plugging the solution into the action (2.1), we find \[ S = \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \left( p^A(q) - q^A(p) + \psi^A(p) i \gamma^\mu p_\mu \psi_A(p) \right) + S_m \\
+ N \int \frac{d^3P}{(2\pi)^3} \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} C_1(P, q_1, q_2) \chi^A_A(P, q_1) \chi^B_B(-P, q_2) \\
+ N \int \frac{d^3P_1}{(2\pi)^3} \frac{d^3P_2}{(2\pi)^3} \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \frac{d^3q_3}{(2\pi)^3} C_2(P_1, P_2, q_1, q_2, q_3) \\
\times \chi^A_A(P_1, q_1) \chi^B_B(P_2, q_2) \chi^C_C(-P_1 - P_2, q_3) \\
+ N \int \frac{d^3P}{(2\pi)^3} \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \frac{8\pi i N}{k(q_1 - q_2)} \xi^A_A(P, q_1) \xi^B_B(-P, q_2) + \cdots, \quad (2.7) \\
\]
where \[ \chi^A_A(P, q) = \frac{1}{N} q^A \left( \frac{P}{2} - q \right) q^B \left( \frac{P}{2} + q \right), \quad (2.8) \]
\[ \xi^A_A(P, q) = \frac{1}{2N} \psi^A \left( \frac{P}{2} - q \right) \psi_A \left( \frac{P}{2} + q \right), \quad (2.9) \]
\[ \xi^B_B(P, q) = \frac{1}{2N} \psi^A \left( \frac{P}{2} - q \right) \gamma^\mu \psi_A \left( \frac{P}{2} + q \right), \quad (2.10) \]
\[ C_1(P, q_1, q_2) = \frac{2\pi i N}{k} \frac{(-P + q_1 + q_2) (P + q_1 + q_2)}{(q_1 - q_2)}, \quad (2.11) \]
\[ C_2(P_1, P_2, q_1, q_2, q_3) = \frac{4\pi^2 N^2}{k^2} \frac{(-P_1 + P_2 + 2q_1 + 2q_2) (P_1 + P_2 + 2q_1 + 2q_2 + 2q_3)}{(P_1 + P_2 + 2q_1 - 2q_2) (P_1 - 2q_2 + 2q_3)}. \quad (2.12) \]
\[ S_m = \int d^3x V_m = S_m(\chi^A_A, \xi^B_B, \xi^A_A P, \xi^B_B P, \eta_{AB}, \bar{\eta}^{AB}) \quad (2.13) \]
where \[ \eta_{AB}(P, q) = \frac{1}{N} q^A \left( \frac{P}{2} - q \right) \psi_B \left( \frac{P}{2} + q \right), \quad \bar{\eta}^{AB}(P, q) = \frac{1}{N} \psi^A \left( \frac{P}{2} - q \right) q^B \left( \frac{P}{2} + q \right). \quad (2.14) \]
The ellipsis in (2.7) represents $1/N$ correction terms and those which contain $\eta_{AB}, \bar{\eta}^{AB}$.

The next step is to introduce auxiliary bilocal fields so that the interaction terms disappear in the action. We can add the following terms without changing the dynamics

\[ \Delta S = -N \int \frac{d^3P}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \left( \Sigma^A_B(P, q) (\alpha^A_A(-P, q) - \chi^A_A(-P, q)) \\
+ 2\Pi^A_B(P, q) (\beta^B_B(-P, q) - \xi^B_B(-P, q)) + 2\Pi^A_B(P, q) (\beta^B_B(-P, q) - \xi^B_B(-P, q)) \\
+ \Gamma^{AB}(P, q) (\gamma_{AB}(-P, q) - \eta_{AB}(-P, q) + \Gamma_{AB}(P, q) (\gamma^{AB}(-P, q) - \bar{\eta}^{AB}(-P, q)) \\
- S_{int}(\chi^A_A, \xi^B_B, \xi^A_A P, \xi^B_B P, \eta_{AB}, \bar{\eta}^{AB}) + S_{int}(\alpha^A_A, \beta^B_B, \beta^A_A, \gamma_{AB}, \bar{\gamma}^{AB}) \right), \quad (2.15) \]
where $S_{\text{int}}(\chi^B_A, \xi^B_A I, \xi^B_A \bar{I}, \eta_{AB}, \bar{q}q^A)$ denotes all interaction terms in (2.7). Adding this into (2.7) gives

\[ S + \Delta S = \int \frac{d^3 P}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \left( q^A_B \left( \frac{P}{2} - q \right), \psi^B_A \left( \frac{P}{2} - q \right) \right) \cdot \left( q^A_B + q \right) \]

\[ + S_m(\alpha^B_A, \beta^B_A I, \beta^B_A \bar{I}, \gamma_{AB}, \bar{q}q^A) \]

\[ + N \int \frac{d^3 P}{(2\pi)^3} \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_2}{(2\pi)^3} C_1(P, q_1, q_2) \alpha^B_A(P, q_1) \alpha^B_A(-P, q_2) \]

\[ + N \int \frac{d^3 P_1}{(2\pi)^3} \frac{d^3 P_2}{(2\pi)^3} \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_2}{(2\pi)^3} \frac{d^3 q_3}{(2\pi)^3} C_2(P_1, P_2, q_1, q_2, q_3) \]

\[ \times \alpha^B_A(P_1, q_1) \alpha^B_C(P_2, q_2) \alpha^C_A(-P_1 - P_2, q_3) \]

\[ + N \int \frac{d^3 P}{(2\pi)^3} \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_2}{(2\pi)^3} \frac{8\pi i N}{k(q_1 - q_2)} \beta^B_A(P, q_1) \beta^B_A(-P, q_2) \]

\[ + \Gamma^{AB}(P, q) \gamma_{AB}(-P, q) + \Gamma_{AB}(P, q) \bar{\gamma}^{AB}(-P, q) \]}

\[ + \cdots, \] (2.16)

where

\[ Q = \left( \begin{array}{ccc}
q^2 \delta^3(P) \delta^A_B + \Sigma^A_B(P, q) & \Gamma_{AB}(P, q) \\
\Gamma^{AB}(P, q) & i\gamma^A \bar{q} q^A \delta^A_B + \Pi^A_B(P, q)
\end{array} \right), \] (2.17)

and the ellipsis contains $1/N$ sub-leading and $\gamma_{AB}, \bar{\gamma}^{AB}$ terms. Since this is quadratic in terms of the elementary fields $q^A, \psi_A$, they are integrated out by gaussian integration, which results in

\[ S_{\text{eff}} = \text{STr} \log Q + S_m(\alpha^B_A, \beta^B_A I, \beta^B_A \bar{I}, \gamma_{AB}, \bar{q}q^A) \]

\[ + N \int \frac{d^3 P}{(2\pi)^3} \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_2}{(2\pi)^3} C_1(P, q_1, q_2) \alpha^B_A(P, q_1) \alpha^B_A(-P, q_2) \]

\[ + N \int \frac{d^3 P_1}{(2\pi)^3} \frac{d^3 P_2}{(2\pi)^3} \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_2}{(2\pi)^3} \frac{d^3 q_3}{(2\pi)^3} C_2(P_1, P_2, q_1, q_2, q_3) \]

\[ \times \alpha^B_A(P_1, q_1) \alpha^B_C(P_2, q_2) \alpha^C_A(-P_1 - P_2, q_3) \]

\[ + N \int \frac{d^3 P}{(2\pi)^3} \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_2}{(2\pi)^3} \frac{8\pi i N}{k(q_1 - q_2)} \beta^B_A(P, q_1) \beta^B_A(-P, q_2) \]

\[ + \Gamma^{AB}(P, q) \gamma_{AB}(-P, q) + \Gamma_{AB}(P, q) \bar{\gamma}^{AB}(-P, q) \]}

\[ + \cdots. \] (2.18)
Our interest is in the leading behavior of the large $N$ limit. For this purpose we shall focus on evaluating this on the saddle point. A natural ansatz for saddle point equations is such that solutions satisfy the translational, rotational invariance and covariance with respect to flavor indices.

\[
\langle \alpha^A_B(P,q) \rangle = (2\pi)^3 \delta^3(P) \delta^A_B \alpha(q), \\
\langle \beta^A_B(P,q) \rangle = (2\pi)^3 \delta^3(P) \delta^A_B \beta(q), \\
\langle \gamma_{AB}(P,q) \rangle = \langle \bar{\gamma}^{AB}(P,q) \rangle = 0, \\
\langle \Sigma^A_B(P,q) \rangle = (2\pi)^3 \delta^3(P) \delta^A_B \Sigma(q), \\
\langle \Pi^A_B(P,q) \rangle = (2\pi)^3 \delta^3(P) \delta^A_B \Pi(q), \\
\langle \Gamma_{AB}(P,q) \rangle = \langle \bar{\Gamma}^{AB}(P,q) \rangle = 0.
\] (2.19)

Under this assumption the above effective action in the leading of large $N$ is simplified to be

\[
S_{\text{eff}} = N MV \left[ \int \frac{d^3q}{(2\pi)^3} \left( \log(q^2 + \Sigma(q)) - \text{tr} \log(i\gamma^\mu q_\mu + \Pi(q)) \right) \right. \\
+ \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} C_1(q_1, q_2) \alpha(q_1) \alpha(q_2) \\
+ \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \frac{d^3q_3}{(2\pi)^3} C_2(q_1, q_2, q_3) \alpha(q_1) \alpha(q_2) \alpha(q_3) \\
+ \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \frac{8\pi iN}{k(q_1 - q_2)} \beta_-(q_1) \beta_1(q_2) \\
- \left. \int \frac{d^3q}{(2\pi)^3} \left( \Sigma(q) \alpha(q) + 2\Pi^f(q) \beta_f(q) + 2\Pi^-(q) \beta_-(q) \right) \right] + S_m(0, 0, 0), \\
\] (2.20)

where $V = (2\pi)^3 \delta^3(P = 0)$ and

\[
C_1(q_1, q_2) = \frac{1}{2}(C_1(0, q_1, q_2) + C_1(0, q_2, q_1)), \\
C_2(q_1, q_2, q_3) = \frac{1}{3!}(C_2(0, 0, q_1, q_2, q_3) + \text{permutation}).
\] (2.21) (2.22)

Note that the terms in the bracket are the same as those obtained in the same procedure from Chern-Simons theory with one fundamental boson and fermion ($M = 1$).

To proceed further we need to specify a potential form of matter fields. We shall do case study by using $\mathcal{N} = 3$ Chern-Simons theory in the next subsection.

### 2.1.1 $\mathcal{N} = 3 \, U(N)_k$ case

In this subsection we apply the result obtained in the previous section to $\mathcal{N} = 3 \, U(N)_k$ Chern-Simons theory with minimal matter content. The matter content is two fundamental complex scalar fields $q^A$ and fermionic fields $\psi_A$, where $A = 1, 2$. 
\( \mathcal{N} = 3 \ U(N)_k \) Chern-Simons Lagrangian is given by (A.11). The potential of the matter fields reads from (A.11) as follows.

\[
V_m^{\mathcal{N}=3} = \frac{1}{\kappa}(\psi^A \psi^A)q^B + \frac{1}{\kappa}(\psi^A q^A)q^B - \frac{1}{2\kappa}(\psi^A q^B)(\bar{\psi}^A) + \frac{1}{2\kappa}\varepsilon_{AB}\varepsilon_{CD}(\psi^A q^B)(\psi^A q^D) + \frac{1}{2\kappa}\varepsilon_{AB}\varepsilon_{CD}(q^A \psi^B)(q^C \psi^D) \]

We again contract gauge indices by bracket notation. For example, \((\bar{q}^A \psi_B) = \bar{q}^A m \psi_B^m\), where \(m\) is a gauge index of the fundamental representation. Therefore \(S_m\) in (2.13) is given by

\[
S_m^{\mathcal{N}=3}(\chi^A, \xi^A, \xi_B^A, \xi_{AB}, \eta_A^B) = N \int \frac{d^3p}{(2\pi)^3} \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \left[ \frac{2}{\kappa} \xi^A_{B1}(P, q_1) \chi^B_A(-P, q_2) \right.
\]

\[
+ \frac{N}{\kappa} \eta^B(P, q_1) \eta^A_{AB}(-P, q_2) - \frac{N}{\kappa} \eta^B(P, q_1) \eta^A_{AB}(-P, q_2)
\]

\[
+ \frac{N}{2\kappa} \varepsilon_{AB} \varepsilon_{CD} \eta^A_{AB}(P, q_1) \eta^C_{CD}(-P, q_2) + \frac{N}{2\kappa} \varepsilon_{AB} \varepsilon_{CD} \eta^A_{AB}(P, q_1) \eta^C_{CD}(-P, q_2)
\]

\[
+ N \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \frac{d^3q_3}{(2\pi)^3} \frac{N^2}{12} \chi^B_A(P, q_1) \chi^A_C(P, q_2) \chi^C_B(-P_1 - P_2, q_3)
\]

\[
- \frac{1}{12} \chi^B_A(P, q_1) \chi^C_B(P, q_2) \chi^A_C(-P_1 - P_2, q_3)
\]  

Under the assumption (2.19) this is simplified as follows.

\[
S_m^{\mathcal{N}=3}(\alpha^A_B, \beta^A_B, \beta^A_B, \alpha^A_B, 0, 0) = 2N \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \frac{d^3q_3}{(2\pi)^3} \frac{N^2}{12} \alpha(q_1) \alpha(q_2) \alpha(q_3)
\]  

One will soon notice that this is twice that of the \( N = 2 \ U(N) \) Chern-Simons theory with minimal matter content:

\[
S_m^{\mathcal{N}=3}(\alpha^A_B, \beta^A_B, \beta^A_B, \alpha^A_B, 0, 0) = 2S_m^{\mathcal{N}=2}(\alpha, \beta_1, \beta_2, 0, 0).
\]  

To show this, let us read off the matter potential in \( N = 2 \) case from (A.10).

\[
V_m^{\mathcal{N}=2} = \frac{1}{2\kappa}(\bar{\psi}^A \psi^A) + \frac{1}{2\kappa}(\bar{\psi}^A \psi^A) + \left( \frac{1}{2\kappa} \right)^2 (\bar{\psi}^A \psi^A)
\]  

In the same way, we can compute \( S_m\)

\[
S_m^{\mathcal{N}=2}(\chi, \xi, \xi_I, \eta, \bar{\eta}) = N \int \frac{d^3p}{(2\pi)^3} \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \left[ \frac{2}{\kappa} \xi_I(P, q_1) \chi(-P, q_2) + \frac{N}{2\kappa} \bar{\eta}(P, q_1) \eta(-P, q_2)
\]

\[
+ N \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \frac{d^3q_3}{(2\pi)^3} \frac{N^2}{12} \chi(P_1, q_1) \chi(P_2, q_2) \chi(-P_1 - P_2, q_3)
\]  

and under the assumption (2.19),
\[ S^N_{m=2}(\alpha, \beta_-, \beta_I, 0, 0) = NV \left[ \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \frac{2N}{\kappa} \beta_I(q_1) \alpha(q_2) + \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \frac{d^3q_3}{(2\pi)^3} \frac{N^2}{(2\pi)^3} \alpha(q_1) \alpha(q_2) \alpha(q_3) \right], \tag{2.29} \]
which proves the relation (2.26).

By taking account of (2.20), the total large \( N \) effective action in the minimal \( \mathcal{N} = 3 \) Chern-Simons theory is exactly twice as that of the minimal \( \mathcal{N} = 2 \) Chern-Simons theory in the ‘t Hooft limit.

One might wonder why the large \( N \) effective action is insensitive to the difference between the \( \mathcal{N} = 2 \) Chern-Simons theory and \( \mathcal{N} = 3 \) one. To understand this, let us consider \( \mathcal{N} = 2 \) Chern-Simons theory with one pair of chiral/anti-chiral matter fields \( (Q, \bar{Q}) \) perturbed by a superpotential of the form \( W_0 = a(\bar{Q} T^3 Q)^2 \), where \( a \) is a small positive number. It was shown in [22] that this \( \mathcal{N} = 2 \) Chern-Simons matter theory with the superpotential flows to the same \( \mathcal{N} = 2 \) one in the infra-red (IR) except that the superpotential is given by \( W = a_{IR}(\bar{Q} T^3 Q)^2 \), where \( a_{IR} \) is a fixed number of order \( 1/\kappa \), and \( \mathcal{N} = 2 \) supersymmetry is enhanced to \( \mathcal{N} = 3 \) in the IR so that the IR theory becomes the same as the \( \mathcal{N} = 3 \) Chern-Simons theory considered above. On the other hand, in the large \( N \) limit, large \( N \) factorization occurs so that the leading contribution of the superpotential is given by \( \langle W \rangle = a_{IR}(\bar{Q} Q)^2 \), which vanishes on the SU(2) symmetric vacuum (2.19).\(^3\) This is the reason why the large \( N \) effective action cannot see the difference between the \( \mathcal{N} = 3 \) Chern-Simons theory and the \( \mathcal{N} = 2 \) one with the same matter content.

### 2.2 Bi-fundamental matter fields

In this section we study \( U(N)_k \times U(N')_{-k} \) Chern-Simons theory coupling to \( M \) bi-fundamental matter fields by taking \( N \) and \( k \) to infinity and holding \( \lambda = N/k \) and \( N' \) fixed. We denote \( M \) bi-fundamental scalar fields and fermions by \( q^A, \psi_A \) respectively, where \( A = 1, 2, \cdots, M \). A generic form of the action of this class of Chern-Simons theories is given by
\[ S = \int d^3x (\kappa (\mathcal{L}_{cs}[A] - \mathcal{L}_{cs}[A']) + \mathcal{L}_m), \tag{2.30} \]
where \( \mathcal{L}_m \) is given by
\[ \mathcal{L}_m = \text{Tr} [D_\mu q_A^\dagger D^\mu q^A + \psi_A^\dagger \gamma^\mu D_\mu \psi_A] + V_m. \tag{2.31} \]
Here the covariant derivative acts on the fields in a way that
\[
\begin{align*}
D_\mu q^A &= \partial_\mu q^A - i A_\mu q^A + i q^A A'_\mu, \\
D_\mu q_A^\dagger &= \partial_\mu q_A^\dagger - i A'_\mu q_A^\dagger + i q_A^\dagger A_\mu, \\
D_\mu \psi_A &= \partial_\mu \psi_A - i A_\mu \psi_A + i \psi_A A'_\mu, \\
D_\mu \psi_A^\dagger &= \partial_\mu \psi_A^\dagger - i A'_\mu \psi_A^\dagger + i \psi_A^\dagger A_\mu. \tag{2.32}
\end{align*}
\]
\( V_m \) represents a gauge-invariant potential of the matter fields in this system. Specific examples are \( \mathcal{N} = 4 \) and \( 6 \) Chern-Simons-matter theories, whose actions in our notation are given in A.4 and A.5.

\(^3\)We used \( \sum_{m=1}^{N}(T^a)^m_p (T^b)^p_m = \delta^a_b \delta^m_p \) to rewrite the form of superpotential.
Firstly we separate the U(1) gauge fields.

\[ A_\mu \rightarrow b_\mu + A_\mu, \quad A'_\mu \rightarrow b'_\mu + A'_\mu, \]  
(2.33)

where \( b_\mu, b'_\mu \) are the trace part and \( A_\mu, A'_\mu \) are the traceless part. After this replacement \( A_\mu, A'_\mu \) always represent SU(\( N \)) SU(\( N' \)) gauge fields. Plugging this into the matter action gives

\[ \mathcal{L}_m \rightarrow \text{Tr}[(b_\mu)^2 q_A^A + ib_\mu (q_A^A D^\mu q_A^A - q_A^A A_A^{A'\mu} \gamma^\mu \psi_A)] + \mathcal{L}_m \]  
(2.34)

where \( b_\mu := b_\mu - b'_\mu \) and \( \mathcal{L}_m \) in the right-hand side is the same as (2.31) except \( A_\mu, A'_\mu \) are now SU(\( N \)) SU(\( N' \)) gauge fields. \( D^\mu \) is the covariant derivative of the SU(\( N \)) \times SU(\( N' \)) gauge group. Only a relative combination of U(1) \times U(1) gauge fields, \( b_\mu \), couples to the matter fields.

Let us turn to Chern-Simons term and also separate the U(1) part of the gauge fields from the Chern-Simons term. Substituting (2.33) into the Chern-Simons term we obtain

\[ \kappa(\mathcal{L}_{cs}[A] - \mathcal{L}_{cs}[A']) \rightarrow i\kappa \varepsilon^{\mu\nu\rho} \frac{N(N - N')}{(\sqrt{N} + \sqrt{N'})^2} \left( b^\mu_\rho \partial_\nu b^\rho_\nu + \frac{2(\sqrt{N'}N + N')}{N - N'} b^\mu_\rho \partial_\nu b^\rho_\nu \right) \]

\[ + \kappa(\mathcal{L}_{cs}[A] - \mathcal{L}_{cs}[A']), \]  
(2.35)

where we define

\[ b^\mu_\rho = b_\mu + \frac{\sqrt{N'}}{\sqrt{N}} b'_\mu \]  
(2.36)

so that the term \( \varepsilon^{\mu\nu\rho} b^\rho_\nu \partial_\nu b^\mu_\nu \) cancels. Since this \( b^\mu_\rho \) does not couple to the matter fields, one can integrate it out by solving equation of motion. The equation of motion is

\[ \varepsilon^{\mu\nu\rho}(2 \partial_\nu b^\rho_\nu + \frac{2(\sqrt{N'}N + N')}{N - N'} \partial_\nu b^\rho_\nu) = 0. \]  
(2.37)

We can solve this by

\[ b^\rho_\rho = - \frac{(\sqrt{N'}N + N')}{N - N'} b^\rho_\rho. \]  
(2.38)

Plugging back this into (2.35) gives

\[ \kappa(\mathcal{L}_{cs}[A] - \mathcal{L}_{cs}[A']) \rightarrow -i\tilde{\kappa} \varepsilon^{\mu\nu\rho} b^\rho_\mu \partial_\nu b^\mu_\nu + \kappa(\mathcal{L}_{cs}[A] - \mathcal{L}_{cs}[A']), \]  
(2.39)

where \( \tilde{\kappa} \) is defined by

\[ \tilde{\kappa} = \kappa \frac{N N'}{N - N'}. \]  
(2.40)

By collecting all the terms the whole action (2.30) becomes

\[ S \rightarrow \int d^3x \left( -i\tilde{\kappa} \varepsilon^{\mu\nu\rho} b^\rho_\mu \partial_\nu b^\mu_\nu + \text{Tr}[(b_\mu)^2 q_A^A + ib_\mu (q_A^A D^\mu q_A^A - q_A^A A_A^{A'\mu} \gamma^\mu \psi_A)] \]

\[ + \kappa(\mathcal{L}_{cs}[A] - \mathcal{L}_{cs}[A']) + \mathcal{L}_m \right). \]  
(2.41)
In summary, the first line, which contains $b_{-}^{-}$, is coming from $U(1) \times U(1)$ part and the second one is $SU(N) \times SU(N)$ part.

Now let us fix the gauge degrees of freedom by the light-cone gauge for $A_{\mu}, A'_{\mu}, b_{-}^{-}$ and integrate them out as done in the previous section. While the equation of motion for $A_{+}$ has the same form as (2.5), those for the gauge fields $b_{-}^{-}, A'$ are

$$-2\kappa \partial_{-} b_{-}^{-} = \text{Tr}_{N'} \left[ i \left( q_{A}^{\dagger} \partial_{-} q_{A} - \partial_{-} q_{A}^{\dagger} q_{A} - \psi^{1A} \gamma_{-} \psi_{A} \right) \right],$$

$$-2\kappa \partial_{-} A'_{3} = \text{Tr}_{N'} \left[ i \left( q_{A}^{\dagger} \partial_{-} q_{A} - \partial_{-} q_{A}^{\dagger} q_{A} - \psi^{1A} \gamma_{-} \psi_{A} \right) T^{a'} \right],$$

where $T^{a'}$ is a generator of $SU(N')$ gauge group and $\text{Tr}_{N'}$ is trace for $N' \times N'$ matrix. In the Fourier space they become

$$2\kappa i q_{-} b_{-}^{-} (q) = N \int \frac{d^{3}r}{(2\pi)^{3}} \text{Tr}_{N'} \left[ \left( (2r + q)_{-} \chi_{\alpha}^{A} \left( q, r + \frac{q}{2} \right) + 2i \xi_{-}^{A} \left( q, r + \frac{q}{2} \right) \right) \right],$$

$$2\kappa i q_{-} A'_{3} (q) = N \int \frac{d^{3}r}{(2\pi)^{3}} \text{Tr}_{N'} \left[ \left( (2r + q)_{-} \chi_{\alpha}^{A} \left( q, r + \frac{q}{2} \right) + 2i \xi_{-}^{A} \left( q, r + \frac{q}{2} \right) \right) T^{a'} \right],$$

where we used the notation $\chi_{B}^{A}, \xi_{-}^{A}$ analogous to (2.8), (2.10), which are now $N'$ by $N'$ matrices. Solving these and substituting back into (2.41) we find the analog of (2.7), which now also contains the contribution from the gauge fields $b_{-}^{-}, A'$.

Then we introduce auxiliary fields to eliminate all the interactions by adding the terms, which has the same form as (2.15) except the auxiliary fields are now $N' \times N'$ matrices and suitable contractions for $SU(N')$ indices. By this manipulation what we shall do is essentially to exchange $\chi, \xi$ into $\alpha, \beta$. For example the constraint equations of $b_{-}^{-}, A'_{3}$ become

$$2\kappa i q_{-} b_{-}^{-} (q) = N \int \frac{d^{3}r}{(2\pi)^{3}} \text{Tr}_{N'} \left[ \left( (2r + q)_{-} \alpha_{-}^{A} \left( q, r + \frac{q}{2} \right) + 2i \beta_{-}^{A} \left( q, r + \frac{q}{2} \right) \right) \right],$$

$$2\kappa i q_{-} A'_{3} (q) = N \int \frac{d^{3}r}{(2\pi)^{3}} \text{Tr}_{N'} \left[ \left( (2r + q)_{-} \alpha_{-}^{A} \left( q, r + \frac{q}{2} \right) + 2i \beta_{-}^{A} \left( q, r + \frac{q}{2} \right) \right) T^{a'} \right].$$

After this treatment we can integrate out the elementary fields $q^{A}, \psi_{A}$ and obtain the analog of (2.18), which contains the contribution from $b_{-}^{-}, A'$.

Then we evaluate the action at saddle points to study the leading expression in the large $N$ limit. We assume the same ansatz for saddle points such as translational, rotational invariance and covariance of flavor indices. In addition to these we also naturally expect saddle points to satisfy covariance of $SU(N')$ fundamental indices.

$$\langle \alpha_{B}^{A}(P, q) \rangle = (2\pi)^{3} \delta^{3}(P) \delta_{B}^{A} 1_{N'} \alpha(q), \quad \langle \beta_{B}^{A}(P, q) \rangle = (2\pi)^{3} \delta^{3}(P) \delta_{B}^{A} 1_{N'} \beta(q),$$

$$\langle \Sigma_{B}^{A}(P, q) \rangle = (2\pi)^{3} \delta^{3}(P) \delta_{B}^{A} 1_{N'} \Sigma(q), \quad \langle \Pi_{B}^{A}(P, q) \rangle = (2\pi)^{3} \delta^{3}(P) \delta_{B}^{A} 1_{N'} \Pi(q),$$

where $1_{N'}$ is the $N' \times N'$ unit matrix. In other words, we treat $SU(N')$ gauge symmetry as flavor symmetry under the ’t Hooft vector model limit. We set to zero for the fermionic fields. Under this ansatz consistent solutions for $b_{-}^{-}, A'_{3}$ in (2.45) become trivial. This is because under the ansatz (2.46) the right-hand side in (2.45) becomes proportional to
\( \delta^3(q) \) but the left-hand side \( q_- \), which requires \( b_3^- , A_3^q \) to vanish. Note that \( A_3^q = 0 \) is also required from gauge index contraction of the right-hand side since \( \text{Tr}_{N'} \mathcal{S}^q = 0 \). Accordingly the right-hand side in (2.45) also has to vanish so that it is required to satisfy

\[
\int \frac{d^3 r}{(2\pi)^3} (2r \alpha(r) + 2i \beta_-(r)) = 0. \tag{2.47}
\]

This has to be checked after solving saddle point equations for \( \alpha, \beta \), but we can check now because we already know that the solutions of \( \alpha, \beta \) are given by exact propagators of scalar and fermion respectively of the following form \[8\]

\[
\alpha(r) = \frac{1}{r^2 + c_{B,0}^2}, \quad \beta_-(r) = \frac{ir_-}{r^2 + c_{F,0}^2} \tag{2.48}
\]

where \( c_{B,0}, c_{F,0} \) are pole masses of scalar and fermion respectively. Plugging (2.48) into (2.47) one can see that the left-hand side vanishes by performing the angular integral. As a result \( U(N') \) sector does not contribute at all under the limit. In other words, \( U(N') \) gauge factor is so weakly gauged as to decouple from the leading contribution under the ’t Hooft vector model limit. Thus the result of the effective action is essentially the same as that of the case with one gauge group (2.20).

\[
S_{\text{eff}} = NN' MV \left[ \int \frac{d^3 q}{(2\pi)^3} \left( \log(q^2 + \Sigma(q)) - \text{tr} \log(i\gamma^\mu q_\mu + \Pi(q)) \right) \right.
\]

\[
+ \int \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_2}{(2\pi)^3} C_1(q_1, q_2) \alpha(q_1) \alpha(q_2)
\]

\[
+ \int \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_2}{(2\pi)^3} \frac{d^3 q_3}{(2\pi)^3} C_2(q_1, q_2, q_3) \alpha(q_1) \alpha(q_2) \alpha(q_3)
\]

\[
+ \int \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_2}{(2\pi)^3} \frac{d^3 q_3}{(2\pi)^3} \frac{d^3 q_4}{(2\pi)^3} \left( \frac{8\pi iN}{k(q_1 - q_2)^2} - \beta_-(q_1) \beta_-(q_2) \right)
\]

\[
- \int \frac{d^3 q}{(2\pi)^3} \left( \Sigma(q) \alpha(q) + 2\Pi'(q) \beta_1(q) + 2\Pi^-(q) \beta_-(q) \right)
\]

\[
) \left] + S_m(\alpha \delta_{A,0}^B, \beta_1 \delta_{A,0}^B, \beta_\delta_{A,0}^B, 0, 0). \tag{2.49}\right.
\]

Let us apply this result to \( \mathcal{N} = 4 \) Chern-Simons-matter theory with \( U(N)_k \times U(N')_{-k} \) case and \( \mathcal{N} = 6 \) with \( U(N)_k \times U(N')_{-k} \) (ABD) case.

### 2.2.1 \( \mathcal{N} = 4 \) \( U(N)_k \times U(N')_{-k} \) case

In this subsection we apply the result (2.49) to \( \mathcal{N} = 4 \) \( U(N)_k \times U(N')_{-k} \) Chern-Simons theory, whose matter content is two bi-fundamental complex scalar fields \( q^A \) and fermionic fields \( \psi_A \), where \( A = 1, 2 \). This Chern-Simons-matter Lagrangian is given by (A.17). The potential of the matter fields is

\[
V_{m=4}^{\mathcal{N}=4} = \text{Tr} \left[ \frac{1}{2k} \left( q_B^B q^B \psi_A^+ \psi_A - q_B^B q^B \psi_A^+ \psi_A - \varepsilon_{AC} \varepsilon_{BD} \psi_A^d \psi_A^+ \psi_A^+ \right) + \varepsilon_{AC} \varepsilon_{BD} \psi_A q_B^B q^B \psi_A^+ c_{qD} \right.
\]

\[
+ \frac{1}{k^2} \left( \frac{3}{2} q_A^A q_B^A q_C^C q^B - \frac{5}{12} q_A^A q_B^B q_C^C \right) \right]. \tag{2.50}\]
Thus $S_m$ in $\mathcal{N} = 4$ case is

$$\begin{align*}
S_m^{\mathcal{N}=4}(\chi^A_B, \xi^A_B, \xi^A_B, \eta_{AB}, \bar{\eta}^{AB}) &= N \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \frac{N^2}{(2\pi)^3} \frac{2}{\kappa} \text{Tr} \left[ 2\bar{\xi}^A_B(P,q_1)\chi^A_B(-P,q_2) - \eta_{BA}(P,q_1)\bar{\eta}^{AB}(-P,q_2) 
- \bar{\varepsilon}_{AB}\bar{\varepsilon}_{CD}(P,q_1)\eta_{CD}(-P,q_2) + \varepsilon^{AB}\varepsilon_{CD}\eta_{AB}(P,q_1)\eta_{CD}(-P,q_2) \right] \\
&+ N \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \frac{d^3q_3}{(2\pi)^3} \frac{N^2}{(2\pi)^3} \frac{2}{\kappa} \text{Tr} \left[ 3^2\chi^B_A(P_1,q_1)\chi^B_A(P_2,q_2)\chi^C_B(-P_1 - P_2,q_3) 
- 2^3 \chi^B_A(P_1,q_1)\chi^B_A(P_2,q_2)\chi^B_A(-P_1 - P_2,q_3) 
- 5^2 \chi^A_B(P_1,q_1)\chi^C_B(P_2,q_2)\chi^C_B(-P_1 - P_2,q_3) 
+ \chi^A_B(P_1,q_1)\chi^B_A(P_2,q_2)\chi^B_A(-P_1 - P_2,q_3) \right].
\end{align*}$$

Under the assumption (2.46) this reduces

$$\begin{align*}
S_m^{\mathcal{N}=4}(\alpha\delta^B_A, \beta_I\delta^B_A, \beta_-\delta^B_A, 0, 0) &= 2N'V \left[ \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \frac{2}{\kappa} \beta_I(q_1)\alpha(q_2) + \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \frac{d^3q_3}{(2\pi)^3} \frac{N^2}{(2\pi)^3} \frac{2}{\kappa} \alpha(q_1)\alpha(q_2)\alpha(q_3) \right],
\end{align*}$$

which is $2N'$ times as that of the $\mathcal{N} = 2$ U(1)$_k$ Chern-Simons theory with minimal matter content:

$$\begin{align*}
S_m^{\mathcal{N}=4}(\alpha\delta^B_A, \beta_I\delta^B_A, \beta_-\delta^B_A, 0, 0) &= 2N'S_m^{\mathcal{N}=2}(\alpha, \beta_I, \beta_-, 0, 0).
\end{align*}$$

By taking (2.49) into account, the total large $\mathcal{N}$ effective action in the $\mathcal{N} = 4$ Chern-Simons theory with minimal matter content is exactly $2N'$ as that of the minimal $\mathcal{N} = 2$ Chern-Simons theory in the 't Hooft limit.

### 2.2.2 Mass-deformed $\mathcal{N} = 4$ case

In this subsection we investigate large $\mathcal{N}$ exact effective action of the previous $\mathcal{N} = 4$ Chern-Simons matter theory deforming the theory by a mass term keeping $\mathcal{N} = 4$ supersymmetry as well as SO(4) R-symmetry [23]. The $\mathcal{N} = 4$ mass term is given by (A.24)

$$\begin{align*}
L_{\text{mass}}^{\mathcal{N}=4} &= \text{Tr} \left[ \mu \psi\dagger A \psi_A + \mu^2 q_A^+ q_A + \frac{\mu}{\kappa} (q_A^+ q_A)^2 - q_A^+ B q_B^+ q_A^+ A \right].
\end{align*}$$

where $\mu$ is a mass parameter. Since the mass term does not break the global symmetry, that of the vacuum is unchanged and thus the ansatz (2.46) holds. Under the ansatz this $\mathcal{N} = 4$ mass term becomes

$$\begin{align*}
S_{\text{mass}}^{\mathcal{N}=4}(\alpha, \beta_I, \beta_-) &= 2N'V \left[ \int \frac{d^3q_1}{(2\pi)^3} \left( 2\mu^2 \beta_I(q_1) + \mu^2 \alpha(q_1) \right) + \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \frac{d^3q_3}{(2\pi)^3} \frac{N^2}{(2\pi)^3} \frac{2}{\kappa} \alpha(q_1)\alpha(q_2) \right],
\end{align*}$$

(2.55)
which is completely the same as the term obtained from the $\mathcal{N} = 2$ mass term (A.8) with $w = 1$ reduced under the assumption (2.46) with the over all multiplicative factor $2N'$:

$$S_{\text{mass}}^{N=4}(\alpha, \beta I, \beta_-) = 2N'S_{\text{mass}}^{N=2}(\alpha, \beta I, \beta_-).$$  \hspace{1cm} (2.56)

As a result the relation of the exact effective actions for $\mathcal{N} = 2$ and $\mathcal{N} = 4$ given by (2.53) is unchanged under the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ mass deformations. Again, $\mathcal{N} = 4$ effective action reduces to that of $\mathcal{N} = 2$ with an appropriate factor including the mass terms keeping the same amount of supersymmetry.

### 2.2.3 $\mathcal{N} = 6$ $U(N)_k \times U(N')_{-k}$ (ABJ) case

In this subsection we apply the result (2.49) to ABJ theory, whose matter content is four bi-fundamental complex scalar fields $Y^A$ and fermionic fields $\Psi_A$, where $A = 1, 2, 3, 4$. This theory possesses SU(4) R-symmetry and U(1)$_b$ global symmetry. The Lagrangian of ABJ theory is given by (A.27). The potential of the matter fields is

$$V_m^{\mathcal{N}=6} = \text{Tr} \left[ \frac{1}{2\kappa} (Y^A Y^B \Psi_B - Y^A Y^B \Psi_B + 2Y^A Y^B \Psi_A \Psi_B - 2Y^A Y^B \Psi^A \Psi_B) - \varepsilon_{ABCD} Y^A Y^B Y^C Y^D + \varepsilon_{ABCD} Y^A Y^B Y^C Y_D \right] + \frac{1}{12\kappa^2} \left( -Y^A Y^B Y^C Y^D - Y^A Y^B Y^C Y_D - Y^A Y^B Y^C Y_D - 4Y^A Y^B Y^C Y_D + 6Y^A Y^B Y^C Y_D \right).$$

(2.57)

$s_m$ in $\mathcal{N} = 6$ case is

$$S_m^{\mathcal{N}=6}(\chi^A_B, \xi^A_B, \xi^A_B I, \eta_{AB}, \bar{\eta}^{AB}) = \int d^3P \int d^3q_1 \int d^3q_2 \int d^3q_3 \int d^3q 2N' \text{Tr}_{N'} \left[ 2\chi^A_B(P, q_1)\xi^B_A(-P, q_2) - \eta_{AB}(P, q_1)\bar{\eta}^{BA}(-P, q_2) + 2\eta_{BA}(P, q_1)\bar{\eta}^{A}(P, q_2) - 4\chi^A_B(P, q_1)\xi^B_A(-P, q_2) - \varepsilon_{ABCD} \eta_{AB}(P, q_1)\eta_{DA}(-P, q_2) + N' \text{Tr}_{N'} \left[ -\chi^A_B(P, q_1)\chi^B_A(P, q_2)\chi^C_A(P, q_3) - \chi^A_B(P, q_1)\chi^B_A(P, q_2)\chi^C_A(P, q_3) - \chi^A_B(P, q_1)\chi^B_A(P, q_2)\chi^C_A(P, q_3) + 6\chi^A_B(P, q_1)\chi^B_A(P, q_2)\chi^C_A(P, q_3) \right] \right].$$

Under the assumption (2.46) this reduces

$$S_m^{\mathcal{N}=6}(\alpha \delta_A^B, \beta \delta_A^B, \beta_- \delta_A^B, 0, 0) = 4N'N'V \left[ \int d^3q_1 d^3q_2 2N' \beta_I(q_1)\alpha(q_2) + \int d^3q_1 d^3q_2 d^3q_3 N' \kappa \beta_I(q_1)\alpha(q_2) \alpha(q_3) \right]$$

(2.59)

which is $4N'$ times as that of the $\mathcal{N} = 2$ $U(N)_k$ Chern-Simons theory with minimal matter content:

$$S_m^{\mathcal{N}=6}(\alpha \delta_A^B, \beta \delta_A^B, \beta_- \delta_A^B, 0, 0) = 4N' \left[ \int d^3q_1 d^3q_2 d^3q_3 N' \kappa \beta_I(q_1)\alpha(q_2) \alpha(q_3) \right].$$

(2.60)
By taking (2.49) into account, the total large $N$ effective action in ABJ theory is exactly $4N'$ times that of the minimal $N = 2$ Chern-Simons theory in the ’t Hooft limit.

The reason why the large $N$ effective action of ABJ theory has reduced to that of the $N = 2$ one will be the same as in the $N = 3$ case discussed in section 2.1.1. The ABJ(M) action can be constructed by using $N = 2$ superfield formulation [24]. In the notation of [24], the superpotential of the $N = 6$ theory is of the form $W^{N=6} \sim \varepsilon_{ABC} B_{D} \text{Tr}(Z^{A} W_{B} Z^{C} W_{D})$ up to some overall factor, where $Z^{A}, W_{B}$ ($A, B = 1, 2$) are bi-fundamental, anti-bi-fundamental chiral superfields, respectively. Under the ’t Hooft vector model limit, the contribution of the superpotential to effective action is given by $\langle W^{N=6} \rangle \sim \varepsilon_{ABC} B_{D} \text{Tr}(Z^{A} W_{B} Z^{C} W_{D})$ due to the large $N$ factorization, where $Z^{A} W_{B}$ is an $N' \times N'$ matrix. However this contribution vanishes under the SU(4) symmetric vacuum configuration (2.46), which will explain the reduction observed above.

2.2.4 Mass-deformed ABJ case

In this subsection we study large $N$ exact effective action in ABJ model deformed by a mass term keeping $N = 6$ supersymmetry [25]. The $N = 6$ mass term is given by (A.35)

$$\mathcal{L}_{mass}^{N=6} = \text{Tr} \left[ \mu \psi^{\dagger A} M_{A}^{B} \psi_{B} + \mu^{\beta} Y_{A}^{B} M_{A}^{B} Y_{B}^{C} + \frac{\mu}{\kappa} (Y_{A}^{B} M_{A}^{B} Y_{C}^{D} Y_{D}^{C} - Y_{A}^{B} M_{A}^{B} Y_{B}^{C} Y_{C}^{D}) \right]$$

where $\mu$ is a mass parameter and $M_{A}^{B} = \text{diag}(1, 1, -1, -1)$. This mass term breaks the SU(4) × U(1)$_{b}$ global symmetry to SU(2) × SU(2) × U(1) × Z$_{2}$ and thus changes the vacuum structure. A plausible ansatz respecting the global symmetry may be the following.

$$\langle \alpha_{A}^{P}(q, P) \rangle = (2\pi)^{3} \delta^{3}(P) 1_{N'}(\delta_{A}^{i} \alpha_{i}(q) + M_{A}^{i} \alpha_{i}(q))$$

$$\langle \beta_{A}^{P}(q, P) \rangle = (2\pi)^{3} \delta^{3}(P) 1_{N'}(\delta_{A}^{i} \beta_{i}(q) + M_{A}^{i} \beta_{i}(q))$$

$$\langle \Sigma_{A}^{P}(q, P) \rangle = (2\pi)^{3} \delta^{3}(P) 1_{N'}(\delta_{A}^{i} \Sigma_{i}(q) + M_{A}^{i} \Sigma_{i}(q))$$

$$\langle \Pi_{A}^{P}(q, P) \rangle = (2\pi)^{3} \delta^{3}(P) 1_{N'}(\delta_{A}^{i} \Pi_{i}(q) + M_{A}^{i} \Pi_{i}(q))$$

where $\alpha_{i}, \beta_{i}, \Sigma_{i}, \Pi_{i}$ ($i = 1, 2$) are determined by saddle point equations. Under this ansatz the effective action corresponding to (2.49) becomes

$$S_{eff}^{N=6} = NN'V \int \frac{d^{3}q}{(2\pi)^{3}} 2 \left( \log(q^{2} + \Sigma_{1}(q) + \Sigma_{2}(q)) + \log(q^{2} + \Sigma_{1}(q) - \Sigma_{2}(q)) \right.$$

$$- \text{tr} \log(iz^{\mu}q_{\mu} + \Pi_{1}(q) + \Pi_{2}(q)) - \text{tr} \log(iz^{\mu}q_{\mu} + \Pi_{1}(q) - \Pi_{2}(q)) \left.$$

$$+ \int \frac{d^{3}q_{1}}{(2\pi)^{3}} \frac{d^{3}q_{2}}{(2\pi)^{3}} C_{1}(q_{1}, q_{2}) \left( \alpha_{1}(q_{1})\alpha_{1}(q_{2}) + \alpha_{2}(q_{1})\alpha_{2}(q_{2}) \right) \right.$$  

$$+ \int \frac{d^{3}q_{1}}{(2\pi)^{3}} \frac{d^{3}q_{2}}{(2\pi)^{3}} \frac{d^{3}q_{3}}{(2\pi)^{3}} 4C_{2}(q_{1}, q_{2}, q_{3}) \left( \alpha_{1}(q_{1})\alpha_{1}(q_{2})\alpha_{1}(q_{3}) + 3\alpha_{1}(q_{1})\alpha_{2}(q_{2})\alpha_{2}(q_{3}) \right)$$

$$+ \int \frac{d^{3}q_{1}}{(2\pi)^{3}} \frac{d^{3}q_{2}}{(2\pi)^{3}} \frac{d^{3}q_{3}}{(2\pi)^{3}} 4 \times \frac{8\pi iN}{k(q_{1} - q_{2})} \left( \beta_{1}(q_{1})\beta_{1}(q_{2}) + \beta_{2}(q_{1})\beta_{2}(q_{2}) \right)$$

$$- \int \frac{d^{3}q}{(2\pi)^{3}} 4 \left( \Sigma_{1}(q)\alpha_{1}(q) + \Sigma_{2}(q)\alpha_{2}(q) + 2\Pi_{1}(q)\beta_{1}(q) + 2\Pi_{2}(q)\beta_{2}(q) \right.$$  

$$+ 2\Pi_{1}(q)\beta_{1}(q) + 2\Pi_{2}(q)\beta_{2}(q)) \right] + S_{m}^{N=6}$$

(2.66)
where $S_{m}$ is

$$S_{m}^{\mathcal{N}=6} = NM'N \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \left[ 4(2\mu \beta_2 I(q_1) + \mu^2 \alpha_2(q_1)) + \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} S_{\Sigma}^{\mu} \alpha_1(q_1) \alpha_2(q_2) \right] + \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} 4^2 \times \frac{N}{2\kappa} \left( \alpha_1(q_1) \beta_1 I(q_2) - \alpha_2(q_1) \beta_2 I(q_2) \right) + \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \frac{d^3q_3}{(2\pi)^3} N^2 \left( 12 \alpha_1(q_1) \alpha_2(q_2) \alpha_1(q_3) + 36 \alpha_1(q_1) \alpha_2(q_2) \alpha_2(q_3) \right) \right],$$

(2.67)

which includes the mass term. Let us rewrite this in terms of the following variables

$$\Sigma^{(\pm)} = \Sigma_1 \pm \Sigma_2, \quad \Pi^{(\pm)} = \Pi_1 \pm \Pi_2, \quad \alpha^{(\pm)} = \alpha_1 \pm \alpha_2, \quad \beta^{(\pm)} = \beta_1 \pm \beta_2,$$

(2.68)

which can simplify (2.66). A simple form of the large $N$ effective action for massive ABJ case is the following.

$$S_{\text{eff}}^{\mathcal{N}=6} = 2N' \left( S_{\text{eff}}^{\mathcal{N}=2}(\alpha^{(+)}, \beta_1^{(+)}, \beta_1^{(-)}, 0, 0)|_{\Sigma^{(+)}, \Pi^{(+)}} + S_{\text{eff}}^{\mathcal{N}=2}(\alpha^{(-)}, \beta_1^{(-)}, \beta_1^{(-)}, 0, 0)|_{\Sigma^{(-)}, \Pi^{(-)}} + S_{\text{mass}}^{\mathcal{N}=2}(\alpha^{(+)}, \beta_1^{(+)}), \beta_1^{(+)} - S_{\text{mass}}^{\mathcal{N}=2}(\alpha^{(-)}, \beta_1^{(-)}, \beta_1^{(-)}) \right) - NV \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \frac{2N}{\kappa} \left( \alpha^{(+)}(q_1) - \alpha^{(-)}(q_2) \right) \left( \beta_1^{(+)}(q_1) - \beta_1^{(-)}(q_2) \right) \right).$$

(2.69)

We observe a splitting of the mass term in the effective action due to the fact that the $\mathcal{N} = 6$ mass term breaks $SU(4)$ R-symmetry to two $SU(2)$s. Determining the saddle point equations and solving them is beyond the scope of this paper. As a trivial check we can see that in the case with $\mu = 0$ this effective action reduces to that of massless ABJ, because we have a solution $\alpha^{(+)} = \alpha^{(-)}$, $\beta^{(+)} = \beta^{(-)}$.

3 Comments on thermal free energy and duality

In the previous section we have obtained the large $N$ exact effective actions for $\mathcal{N} = 3, 4, 6$ Chern-Simons matter theories. Once one obtains large $N$ exact effective actions one can compute exact large $N$ thermal free energies at an arbitrary temperature by performing Wick rotation for time direction and compactifying the Euclidean time in a circle whose circumference is the inverse temperature. Due to appearance of circle one has to care about boundary conditions and holonomy. We set boundary conditions for this circle such that the scalar fields satisfy periodic one and the fermionic fields do anti-periodic one to study thermal canonical ensemble of the system. According to the boundary conditions, we exchange the integration of the momentum for the time direction into the summation over the discrete Fourier modes satisfying suitable boundary conditions. The holonomy is zero mode of gauge field on the circle and it can be taken into account by implementing a constant shift by holonomy for the thermal-time component of momentum appearing in
the propagators [10]. We normalize a thermal free energy in such a way that it vanishes at zero temperature.

For holonomy configuration determined by minimizing the free energy, a crucial argument was made in [15] that each eigenvalue of holonomy matrix obeys the fermionic statistics in the high temperature limit so that the holonomy configuration does not cramp but spread around the origin with the width $2\pi \lambda$ and height $\frac{1}{2\pi \lambda}$ in the ’t Hooft large $N$ limit for $U(N)$ level $k$ Chern-Simons theory with one fundamental boson, fermion or both. This can be confirmed not only from the canonical formalism but also from the path integral formalism [17]. Taking account of this holonomy effect one can see three dimensional duality of this class of the theories at a high temperature of order $\sqrt{N}$.

Let us consider the holonomy distribution in the situations of this paper. First let us consider $U(N)$ level $k$ Chern-Simons theory with any finite number of fundamental fields. Under the ’t Hooft limit holding the number of matter fields fixed the holonomy distribution clearly becomes the same as that in one fundamental flavor case. This implies, from the calculation in the previous section, the large $N$ free energy of $U(N)_k \mathcal{N} = 3$ Chern-Simons theory with one pair of fundamental and anti-fundamental chiral fields (quark and anti-quark) precisely reduces to twice of that of $U(N)_k \mathcal{N} = 2$ Chern-Simons theory with one chiral fundamental multiplet. Since this $\mathcal{N} = 2$ Chern-Simons theory is self-dual under the exchange of $\lambda$ and $\lambda - \text{sgn}(\lambda)$ [14, 15], this result suggests the minimal $\mathcal{N} = 3$ Chern-Simons theory is also self-dual under the same transformation of $\lambda$.

One may discuss this self-duality of $\mathcal{N} = 3$ in the following way. For this purpose we first consider $\mathcal{N} = 2$ $U(N)_k$ Chern-Simons theory with $N_F$ quark flavors $(Q^i, \tilde{Q}^j)$ with no superpotential. We call this electric theory for convenience. The dual of this theory, which we will call magnetic theory, is known as $\mathcal{N} = 2$ $U(N_F + |k| - N)_k$ Chern-Simons theory with $N_F$ dual quark flavors denoted by $(q^i, \tilde{q}^j)$ and gauge-singlet fields $M^j_i$ with superpotential $\tilde{W}_0 = \tilde{q}^i M^j_i q^i$. These two theories are considered to be equivalent in the infra-red fixed point [13]. Now consider the case with $N_F = 1$. Let us add a (marginally) relevant double trace chiral term in the superpotential $\Delta \tilde{W} = (\tilde{Q}Q)^2$ in the electric theory and flow it to the $\mathcal{N} = 3$ Chern-Simons theory [22] as discussed in section 2.1.1. What is the corresponding deformation in the magnetic side? The answer is to add the superpotential of the form $\Delta \tilde{W} = M^2$, since $M$ corresponds to the mesonic field in the electric side [13]. Clearly this gives the mass term for the field $M$, which decouples in the IR. Integrating $M$ out gives a double trace chiral term in the superpotential of the magnetic theory. Therefore the resulting IR theory of the magnetic side also achieves $\mathcal{N} = 3$ supersymmetry by using the argument of [13], which will account for the self-duality of the minimal $\mathcal{N} = 3$ theory.

Next we consider the holonomy distribution for $U(N)_k \times U(N')_{-k}$ Chern-Simons theory with any finite number of (bi-)fundamental fields. One has to take care of holonomy not only for $U(N)$ but also $U(N')$ in general $N$. But under the ’t Hooft large $N$ limit keeping $N'$ and number of (bi-)fundamental fields fixed the contribution of holonomy for $U(N')_{-k}$ reduces to trivial one and that for $U(N)_k$ becomes the same as that for Chern-Simons theory with one fundamental flavor in the leading of large $N$ limit. Therefore, as happened in $\mathcal{N} = 3$ case, the free energy of $\mathcal{N} = 4$ Chern-Simons theory (including the $\mathcal{N} = 4$ mass term) and that of ABJ theory reduce to those of $\mathcal{N} = 2$ Chern-Simons theory with
one chiral multiplet (including the $\mathcal{N} = 2$ mass term) up to overall integral factor. This suggests self-duality of $\mathcal{N} = 4$ theory including the $\mathcal{N} = 4$ mass term and ABJ theory. The self-duality of ABJ theory was already discussed in the original paper [26]. Their claim is $\mathcal{N} = 6$ theories with gauge group $U(N)_{k} \times U(N')_{-k}$ and $U(N')_{k} \times U(2N' + |k| - N)_{-k}$ are equivalent. Under the 't Hooft large $N$ limit with other parameters fixed this claim tells us that the physical quantities become the same under exchange of $\lambda$ with $\lambda - \text{sgn}(\lambda)$, which is the same self-duality transformation as that of $\mathcal{N} = 2$ case. Our result gives strong evidence for this conjecture in a non-supersymmetric situation by confirming match of the large $N$ thermal free energy under the duality transformation.

One may presumably perform analogous discussion of the self-duality of $\mathcal{N} = 4$ Chern-Simons theory including the case of finite $N$, but we leave further discussion in future.

The same holonomy distribution is also the case to mass-deformed ABJ theory under the limit. From the calculation in the previous section we observed the large $N$ effective action does not precisely reduce to that of $\mathcal{N} = 2$ with one chiral field, so neither does the large $N$ thermal free energy. Therefore it is not obvious to see self-duality of ABJ model with $\mathcal{N} = 6$ mass term from our calculation. It is intriguing to explore this more by using not only the large $N$ thermal free energy but also other tools such as three-sphere partition function. We leave detailed analysis to future work.

4 Discussion

In this paper we have computed the effective actions and thermal free energies for $\mathcal{N} = 3$ $U(N)_{k}$ and $\mathcal{N} = 4, 6$ $U(N)_{k} \times U(N')_{-k}$ Chern-Simons theories with minimal matter content including the $\mathcal{N} = 4, 6$ mass term exactly in the 't Hooft large $N$ limit with the other parameters fixed. Under this limit all of them have reduced to the effective action or thermal free energy for $\mathcal{N} = 2$ with one chiral multiplet with the overall factor $MN'$, where $M$ is the number of a chiral or anti-chiral field ($N' = 1$ for $\mathcal{N} = 3$ case), except the mass-deformed ABJ case. We have demonstrated that the self-duality of $\mathcal{N} = 3, 4, 6$ Chern-Simons theories (including the $\mathcal{N} = 4$ mass term) reduces to that of $\mathcal{N} = 2$ with one chiral field (including the $\mathcal{N} = 2$ mass term).

In section 2.2 we have shown that there is no leading contribution of the $U(N')$ gauge fields under the 't Hooft vector model limit. As a result we observed that the resulting thermal free energy showed expected duality in Chern-Simons matter theories in the limit. This result also supports the prescription given in [22] to deal with gauge fields in the large number of flavor limit in study of thermal free energy in Chern-Simons matter theories. However, at a finite Chern-Simons level $k$ there will be non-trivial contribution of $U(N')$ gauge fields. Especially the contribution of $U(1)$ part of the gauge fields will be important to see the relation between a Chern-Simons matter theory and the dual M-theory because the dual scalar field obtained by dualizing the $U(1)$ gauge field represents M-circle of the dual M-theory with the radius of order $1/k$.

There is a straight-forward generalization of the results of this paper by including chemical potential as done in [10, 19]. Under the duality transformation chemical potential for scalar fields exchanges with that for fermionic fields. But physics by including chemical
potential is not so simple because it possibly gives rise to condensation of bosonic fields known as Bose-Einstein condensation and Fermi surface of fermionic fields, which is perhaps unstable by something like the Cooper instability [27]. It was observed that the duality works in the region where both bosonic and fermionic theories are in the uncondensed phase but the duality becomes unclear in the condensed phase [19]. It is interesting to explore the duality structure beyond the uncondensed phase.

A technical but important issue is to calculate the next sub-leading correction of these theories by concurrently taking large $M$ or large $N'$ limit keeping $M/N$ or $N'/N$ fixed. (Some perturbative calculation was done in [28].) Especially the large $N'$ limit is worthwhile to study properties beyond the vector model limit of this class of Chern-Simons matter theories. Under the large $N'$ limit one has to take care of not only non-planar diagrams but also the sub-leading correction of holonomy distribution for $U(N')$ as well as $U(N)$. It is quite non-trivial to check whether the three dimensional duality holds up to the next leading of large $N$ limit. Since the Chern-Simons system reduces to $U(N')$ matrix model under the limit, the Vandermonde measure factor will play an important role to determine the correct holonomy distribution.\footnote{The author thanks S. P. Wadia for pointing this out.} It is interesting to study how the $1/N$ corrected holonomy distribution and the behavior thereof under the duality transformation is modified from that found in [17].

It is of interest to explore mass-deformed Chern-Simons vector models as in [19, 29]. In particular, a mass-deformed theory is free from infra-red divergence so that one can safely consider a scattering matrix. It is interesting to determine S-matrix of elementary particles perturbatively and exactly in the 't Hooft large $N$ limit as done in correlation functions of conserved currents [3–7]. (See [30] for a recent computation of supersymmetric correlation functions.)

It is also interesting to study a gravity theory dual to Chern-Simons vector models in the context of $AdS_4/CFT_3$ correspondence, which is conjectured as a parity violating Vasiliev theory [31] on $AdS_4$ background with suitable boundary conditions [1, 32]. (The original proposal was done in [33]. Related studies are, for example, [34–38]. See also [39–43] for reviews and recent computations of higher spin theories.) According to [32], a higher spin gravity theory dual to an $U(N)_k \times U(N')_{-k}$ bi-fundamental Chern-Simons theories such as ABJ theory is constructed from higher spin fields with $U(N')$ gauge indices. Therefore one can expand the bulk theory by a new bulk 't Hooft coupling $N'/N$ by taking the large $N', N$ limit with their ratio fixed. This indicates new confinement/deconfinement transition for higher spin fields with respect to the $U(N')$ gauge interaction. The field theory analysis by using a toy model in [32] suggested that the $U(N')$ gauge deconfinement happens at temperature of order one while the Hawking-page one occurs at temperature of order $\sqrt{N'/N}$. This implies that as $N'$ goes to $N$, the higher spin fields become heavier so that the $U(N')$ confinement/deconfinement phase transition point and the Hawking-Page one for higher spin fields coalesce into the Hawking-Page one for non-higher spin gravity theory on a certain $AdS_4$ background.\footnote{For example, the non-higher spin gravity theory dual to ABJ theory is type IIA supergravity theory on $AdS_4 \times CP^3$ with a certain B-field background.} To realize the bulk picture proposed in [32], it is...
important to clarify the confining mechanism of $U(N')$ gauge symmetry in the bulk, which may be different from that in usual QCD. This is simply because the bulk ’t Hooft coupling $N'/N$ will not get renormalized. As a result the dimensional transmutation which is often expected to happen in four dimensional Yang-Mills theories may not happen here. It is of interest to study how to compute the dynamical scale in the bulk $U(N')$ gauge theory and obtain the phase diagram thereof.

We hope this note will become useful to address these issues in the future.

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A Supersymmetric Chern-Simons-matter action

In this section we present $\mathcal{N} = 1, 2, 3, 4, 6$ supersymmetric Chern-Simons-matter action of the minimal matter content with the mass term preserving the same amount of supersymmetry in our convention.

A.1 $\mathcal{N} = 1$

$\mathcal{N} = 1$ $U(N)_k$ Chern-Simons-matter action with one chiral multiplet $(q, \psi)$ in the fundamental representation of the gauge group is given in [8]. The action is given by

$$S^{\mathcal{N}=1} = \int d^3x \left[ i\kappa \varepsilon^{\mu\nu\rho} \text{Tr} \left( A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right) + \hat{D}_\mu \tilde{q} D^\mu q + \tilde{\psi} D \psi ight. + (\tilde{q}q) W''_{qq} + (\tilde{\psi} \psi) \left( -W'_{\tilde{q}q} + \frac{1}{2\kappa} (\tilde{q}q) \right) - (\tilde{q}\psi)(\tilde{\psi}q) W''_{\tilde{q}q} 

+ \left. \left( -\frac{1}{2} W''_{\tilde{q}q} - \frac{1}{4\kappa} \right) \left( (\tilde{\psi}q)(\tilde{q}q) + (\tilde{q}\psi)(\tilde{q}\psi) \right) \right],$$

(A.1)

where $D = \gamma^\mu D_\mu$, $W_{qq}$ is a superpotential and $W'_x = \frac{dW_x}{dx}$. $\kappa$ is related to the Chern-Simons level $k$ by $\kappa = \frac{k}{4\pi}$. The covariant derivative acts as (2.4). The contraction of gauge indices is understood by our bracket notation. For example, $(\tilde{q}\psi) = \tilde{q}_m \psi^m$, where $m$ is the fundamental gauge index. Supersymmetry transformation rule is

$$\delta_\epsilon q = -\sqrt{2} \epsilon \psi, \quad \delta_\epsilon \tilde{q} = -\sqrt{2} \epsilon \tilde{\psi},$$

(A.2)

$$\delta_\epsilon \psi_\alpha = \sqrt{2} (\epsilon_\beta D^\beta_\alpha q - \epsilon_\alpha q W'(\tilde{q}q)), \quad \delta_\epsilon \tilde{\psi}_\alpha = \sqrt{2} (\epsilon_\beta D^\beta_\alpha \tilde{q} - \epsilon_\alpha W'(\tilde{q}q)\tilde{q}),$$

(A.3)

$$\delta_\epsilon A_\mu = \frac{i}{\sqrt{2}\kappa} (\epsilon \gamma_\mu \tilde{q}q - \tilde{q}\epsilon \gamma_\mu \psi).$$

(A.4)

Hereafter we shall suppress the spinor indices $\alpha, \beta$. 


Superconformal action can be obtained by restricting the superpotential to be quadratic.

\[ W(\bar{q}q) = -\frac{w}{4\kappa}(\bar{q}q)^2 \]  
(A.6)

where \( w \) is a real number. By putting \( W'_{\bar{q}q} = -\frac{w}{2\kappa} \bar{q}q, W''_{\bar{q}q} = -\frac{w}{2\kappa} \) above, we obtain the superconformal \( \mathcal{N} = 1 \) action and supersymmetry transformation.

On the other hand, the \( \mathcal{N} = 1 \) mass term can be obtained by adding a linear term in the superpotential.

\[ W_{\text{mass}}(\bar{q}q) = -\mu \bar{q}q, \]  
(A.7)

which is of the following form in the action:

\[ S_{\text{mass}}^\mathcal{N}=1 = \int d^3x \left[ \mu^2 \bar{q}q + \mu \bar{\psi}\psi + \frac{w\mu}{\kappa}(\bar{q}q)^2 \right]. \]  
(A.8)

Accordingly one has to add the following term in the fermionic supersymmetry transformation

\[ \delta'_{\epsilon}\psi = \sqrt{2}\mu \epsilon q, \quad \delta'_{\epsilon}\psi^\dagger = \sqrt{2}\mu \epsilon \bar{q}. \]  
(A.9)

A.2 \( \mathcal{N} = 2 \)

\( \mathcal{N} = 2 \) superconformal Chern-Simons-matter theory with one chiral multiplet was studied in [22]. The action with the gauge group \( \text{U}(N) \) turns out to be obtained from \( \mathcal{N} = 1 \) superconformal action with the superpotential (A.6) by setting \( w = 1 \). For convenience we write down the explicit form of the action for \( \text{U}(N) \) case.

\[ S^\mathcal{N}=2 = \int d^3x \left[ i\kappa \varepsilon^{\mu\nu\rho} \text{Tr} \left( A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right) + D_\mu \bar{q}D^\mu q + \bar{\psi} D\psi \right. \]
\[ + \frac{1}{2\kappa}(\bar{q}q)(\bar{q}q) + \frac{1}{\kappa}(\bar{\psi}\psi)(\bar{q}q) + \left( \frac{1}{2\kappa} \right)^2 (\bar{q}q)^3 \].  
(A.10)

When the gauge group is \( \text{U}(N) \), it is possible to add a mass term keeping \( \mathcal{N} = 2 \) supersymmetry, which is of the form (A.8) with \( w = 1 \).\(^6\) This is because in \( \text{U}(N) \) case one can turn on an FI D-term, which generates a mass term by integrating out auxiliary adjoint fields.

A.3 \( \mathcal{N} = 3 \)

Let us consider \( \mathcal{N} = 3 \) \( \text{U}(N)_k \) Chern-Simons-matter theory with minimal matter content, which is one fundamental hyper-multiplet. We denote two complex scalar by \( q^A \) and its

\(^6\)In a case of \( \text{SU}(N) \) gauge group there is no mass term preserving \( \mathcal{N} = 2 \) supersymmetry due to neither FI D-term nor gauge invariant superpotential.
super-partners by $\psi_A$ in the hyper-multiplet, where $A = 1, 2$. The action is given by

$$S^{N=3} = \int d^3 x \left[ ik \varepsilon^{\mu \nu \rho} \mathrm{Tr} \left( A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right) + D_\mu q_A^\dagger D^\mu q_A^\dagger + \psi^A \mathcal{D} \psi_A \right.$$ 

$$+ \frac{1}{\kappa} (\psi^A \psi_B)(q_A^\dagger q_B^\dagger) + \frac{1}{\kappa} (\psi^A q_B^\dagger)(q_A^\dagger \psi_B) - \frac{1}{2\kappa} (\psi^A q_B^\dagger)(q_A^\dagger \psi_B) \right.$$ 

$$+ \frac{1}{2\kappa} \varepsilon_{AB} \varepsilon_{CD} (\psi^A q_B^\dagger)(\psi^C q_D^\dagger) + \frac{1}{2\kappa} \varepsilon_{AB} \varepsilon_{CD} (q_A^\dagger \psi_B)(q_C^\dagger \psi_D) \right.$$ 

$$+ \frac{1}{\kappa^2} \left( \frac{1}{3} (q_A^\dagger q_B^\dagger)(q_C^\dagger q_A^\dagger)(q_B^\dagger q_C^\dagger) - \frac{1}{12} (q_A^\dagger q_B^\dagger)(q_B^\dagger q_C^\dagger)(q_C^\dagger q_A^\dagger) \right) \right], \quad (A.11)$$

where $\varepsilon_{12} = \varepsilon^{21} = 1$ and the covariant derivative acts as (2.4). Notice that the action has manifestly SU(2) R-symmetry, which accounts for $N = 3$ supersymmetry. The supersymmetry variation rule is given by

$$\delta_\omega q_A^\dagger = -\sqrt{2} \omega B \psi_B, \quad \delta_\omega A_\mu = -\sqrt{2} \omega B A_B, \quad (A.12)$$

$$\delta_\omega \psi_A = \sqrt{2} \left( \omega_{CA} \mathcal{D} q_C^\dagger + \frac{1}{\kappa} \omega_{CB} \left( q_B^\dagger q_A^\dagger q_C^\dagger - \frac{1}{2} \varepsilon_{BA} q_D^\dagger q_D^\dagger \right) \right), \quad (A.13)$$

$$\delta_\omega \psi^A = \sqrt{2} \left( \omega_{CA} \mathcal{D} q_C^\dagger + \frac{1}{\kappa} \omega_{CB} \left( q_C^\dagger q_A^\dagger q_B^\dagger - \frac{1}{2} \varepsilon_{BD} q_D^\dagger q_D^\dagger \right) \right), \quad (A.14)$$

$$\delta_\omega A_\mu = \frac{i}{\sqrt{2}\kappa} \omega_{BA} \gamma^\mu (q_A^\dagger \psi_B^\dagger + \varepsilon_{BC} \varepsilon_{DA} \psi_C q_D^\dagger), \quad (A.15)$$

where a supersymmetry parameter $\omega^{AB}$ is in the symmetric representation in SU(2) R-symmetry: $\varepsilon_{AB} \omega^{AB} = 0$. We also use the following notation.

$$\omega_{AB} := \varepsilon_{AC} \omega^{CD} \varepsilon_{DB} = (\omega^{AB})^*, \quad (A.16)$$

### A.4 $\mathcal{N} = 4$

$\mathcal{N} = 4$ Chern-Simons-matter theory with minimal matter content [21] is given by a $U(N)_k \times U(N')_{-k}$ Chern-Simons theory with one bi-fundamental hyper-multiplet denoted by $q^A$ for two complex scalar and $\psi_A$ for their super-partners, where $A = 1, 2$. The action is given by

$$S^{N=4} = \int d^4 x \left[ i k \varepsilon^{\mu \nu \rho} \mathrm{Tr} \left( A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right) \right.$$ 

$$+ D_\mu q_A^\dagger D^\mu q_A^\dagger + \psi^A \mathcal{D} \psi_A \right.$$ 

$$+ \frac{1}{2\kappa} \left( q_B^\dagger q_B^\dagger q_A^\dagger q_A^\dagger \psi_A - q_B^\dagger q_A^\dagger q_A^\dagger q_B^\dagger \psi_A - \varepsilon_{AC} \varepsilon_{BD} q_A^\dagger q_A^\dagger q_B^\dagger q_B^\dagger + \varepsilon_{AC} \varepsilon_{BD} q_B^\dagger q_B^\dagger q_A^\dagger q_A^\dagger \right.$$ 

$$+ \frac{1}{\kappa^2} \left( \frac{3}{5} q_A^\dagger q_B^\dagger q_B^\dagger q_A^\dagger q_A^\dagger q_B^\dagger q_C^\dagger q_C^\dagger + \frac{1}{5} q_A^\dagger q_B^\dagger q_B^\dagger q_C^\dagger q_C^\dagger \right) \right], \quad (A.17)$$

The covariant derivative acts on the fields by (2.32). This action has SU(2) $\times$ SU(2) R-symmetry, which explains $\mathcal{N} = 4$ supersymmetry. The supersymmetric transformation
rule is given by

\[
\delta_q^A = -\sqrt{2} \epsilon^{AB} \psi_B, \quad \delta_{\psi}^A = -\sqrt{2} \psi^1 B \epsilon_{AB}, \tag{A.18}
\]

\[
\delta_{\psi}^A = \sqrt{2} \left( \epsilon_{CA} \varphi q^C + \frac{1}{2\kappa} \epsilon_{CA} (q^C q^1_D q^1_B - q^1_D q^1_C q^1_B) \right), \tag{A.19}
\]

\[
\delta_{\psi}^A = \sqrt{2} \left( \epsilon_{CA} \varphi q^C + \frac{1}{2\kappa} \epsilon_{CA} (q^1_D q^1_C q^1_B - q^1_D q^1_C q^1_B) \right), \tag{A.20}
\]

\[
\delta_{\mu} A = \frac{i}{\sqrt{2}\kappa} \epsilon_{AB} \gamma^\mu (q^A \psi_B + \epsilon_{BC} \epsilon_{DA} \psi_C q^D), \tag{A.21}
\]

\[
\delta_{\mu} A' = \frac{i}{\sqrt{2}\kappa} \epsilon_{AB} \gamma^\mu (\psi^1 B q^A + \epsilon_{BC} \epsilon_{DA} q^1_B q^1_C), \tag{A.22}
\]

where \( \epsilon^{AB} \) is a supersymmetry parameter with two independent SU(2) indices and

\[
\epsilon_{AB} := \epsilon_{AC} \epsilon_{DB} (\epsilon^{AB})^*. \tag{A.23}
\]

A mass term preserving not only \( N = 4 \) but also SO(4)\(_R\) symmetry was constructed in [23]. In our notation, it is given by

\[
\mathcal{L}_{mass}^{N=4} = \text{Tr} \left[ \mu \psi^1 A \psi_A + \mu^2 q^1_A q^A + \frac{\mu}{\kappa} (q^1_A q^1_B q^1_B q^1_B - q^1_B q^1_B q^1_A) \right]. \tag{A.24}
\]

Accordingly we add the following variation in the fermionic supersymmetry variation.

\[
\delta_\epsilon ' \psi_A = \sqrt{2} \mu \epsilon_{BA} q^B, \quad \delta_\epsilon ' \psi^1 A = \sqrt{2} \mu \epsilon_{BA} q^1_B. \tag{A.25}
\]

### A.5 \( \mathcal{N} = 6 \)

We consider \( \mathcal{N} = 6 \) \( U(N)_k \times U(N')_{-k} \) Chern-Simons theory with four complex bi-fundamental scalars denoted by \( Y^A \) and its super-partner \( \Psi_A \), where \( A = 1, 2, 3, 4 \) [26, 44].

\footnote{In the terminology of superfield, the matter content of \( \mathcal{N} = 6 \) theory is one bi-fundamental hyper-multiplet \( (q^A, \psi_A) \) and anti-bi-fundamental (twisted) hyper-multiplet \( (\psi^1 A, \psi^1 A) \). The relation between these fields and \( (Y^A, \Psi_A) \) is given by}

\[
Y^A = (q^A, q^1_A), \quad Y_A^1 = (q_A^1, q^A) \]

\[
\psi^1 A = (\epsilon_{AB} \psi_B, \psi^1 B \epsilon_B A), \quad \psi_A = (\psi^1 B \epsilon_B A, \epsilon_{AB} \psi_B), \]

\[
\xi_{AB} = \begin{pmatrix} 0 & \epsilon_{AC} \epsilon_{CB} \\ \epsilon_{BC} \epsilon_{CA} & 0 \end{pmatrix} \tag{A.26}
\]

where we use the notation \( \mathbf{A}, \mathbf{B} \) representing SU(4) indices only here.
The action is given by
\[
S^{N=6} = \int d^3x \text{Tr} \left[ i \kappa \xi^\mu \rho \left( A_\mu \partial_\rho A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right) - \left( A'_\mu \partial_\rho A'_\rho - \frac{2i}{3} A'_\mu A'_\nu A'_\rho \right) \right] + D_\mu Y^A B^ \mu Y^A + \Psi^A \mathcal{D} \Psi_A \\
+ \frac{1}{\kappa} (Y^A B^A Y^B B^B - Y^A Y^B Y^B Y^A) + 2Y^A Y^B Y^B Y^A - 2Y^B Y^B Y^A Y^A - 2Y^A Y^B Y^B Y^A + 2Y^B Y^B Y^A Y^A - 2Y^A Y^B Y^B Y^A + 2Y^B Y^B Y^A Y^A \\
- \frac{1}{\kappa^2} \left( -4Y^A Y^B Y^B Y^A Y^A Y^A - 4Y^A Y^B Y^B Y^A Y^A Y^A - 4Y^A Y^B Y^B Y^A Y^A Y^A + 6Y^A Y^B Y^B Y^A Y^A Y^A \right). \tag{A.27}
\]

Here $\varepsilon^{1234} = \varepsilon_{1234} = 1$ and the covariant derivative acts on the fields by \((2.32)\). Note that SU(4) R-symmetry is explicitly seen and thus $N = 6$ supersymmetry. The explicit supersymmetry variation rule is
\[
\delta_\xi Y^A = -\sqrt{2} \xi^{AB} Y^B, \quad \delta_\xi Y^A = -\sqrt{2} \xi^{AB} Y^B, \tag{A.28}
\]
\[
\delta_\xi \Psi_A = \sqrt{2} \left( \xi_{BA} \gamma_\mu D_\mu Y^B + \frac{1}{2} \xi_{CB} Q_B Y^C \right), \tag{A.29}
\]
\[
\delta_\xi \Psi^A = \sqrt{2} \left( \xi^{BA} \gamma_\mu D_\mu Y^B + \frac{1}{2} \xi^{CB} (Q_B Y^C) \right), \tag{A.30}
\]
\[
\delta_\xi A_\mu = \frac{i}{\sqrt{2} \kappa} \left( Y^B \Psi^A \gamma_\mu \xi_{AB} - \xi^{AB} \gamma_\mu \Psi_B Y^A \right), \tag{A.31}
\]
\[
\delta_\xi A'_\mu = \frac{i}{\sqrt{2} \kappa} \left( \Psi^A \gamma_\mu \xi_{AB} Y^B - Y^{AC}_A \gamma_\mu \Psi_B \right), \tag{A.32}
\]
where a supersymmetry parameter $\xi^{AB}$ is in the anti-symmetric representation in SU(4) R-symmetry: $\xi^{AB} = -\xi^{BA}$. We also use the following notation:
\[
\xi_{AB} = -\frac{1}{2} \varepsilon_{ABCD} \xi^{CD} = (\xi^{AB})^* \tag{A.33}
\]
and
\[
Q^{AC} = Y^A Y^B Y^C + \delta^A Y^{[B} Y^{C]} Y^{D]} - \left( Y^B Y^C Y^A + \delta^B Y^{[A} Y^C Y^{D]} \right) \tag{A.34}
\]
where the bracket means the normalized anti-symmetrization: $X^{[AB]} = \frac{1}{2} (X^{AB} - X^{BA})$.

It is known that one can tern on a mass term keeping $N = 6$ supersymmetry [25]. The mass term is given by
\[
\mathcal{L}^N_{\text{mass}} = \text{Tr} \left[ \mu \psi^1 A_B^A Y^A_B + \mu^2 Y^A_B A_B^A Y^B + \frac{\mu}{\kappa} M_A^B (Y^A_B Y^B_C Y^C_C - Y^A_B Y^B_C Y^C_C) \right] \tag{A.35}
\]
where $M_A^B = \text{diag}(1, 1, -1, -1)$. The supersymmetry transformation is corrected so that one has to add the following variation in the fermionic supersymmetry transformation rule.
\[
\delta_\xi \psi_A = \sqrt{2} \mu \xi_{CB} M_C^A Y^C_C, \quad \delta_\xi \psi^A = \sqrt{2} \mu \xi^{CB} M_B^A Y^C_C. \tag{A.36}
\]
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