VOLUME FUNCTIONAL OF COMPACT MANIFOLDS WITH A
PRESCRIBED BOUNDARY METRIC

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Abstract. We prove that a critical metric of the volume functional on a four-dimensional compact manifold with boundary satisfying a second-order vanishing condition on the Weyl tensor must be isometric to a geodesic ball in a simply connected space form \( \mathbb{R}^4, \mathbb{H}^4 \) or \( \mathbb{S}^4 \). Moreover, we provide an integral curvature estimate involving the Yamabe constant for critical metrics of the volume functional, which allows us to get a rigidity result for such critical metrics on four-dimensional manifolds.

1. Introduction

Let \((M^n, g)\) be a connected compact Riemannian manifold with dimension \(n\) at least three. Following the terminology adopted by Corvino, Eichmair and Miao [13] as well as Miao and Tam [22], we say that \(g\) is a V-static metric if there is a smooth function \(f\) on \(M^n\) and a constant \(\kappa\) satisfying the V-static equation

\[
\mathcal{L}_g^*(f) = -(\Delta f)g + Hess\ f - fRic = \kappa g,
\]

where \(\mathcal{L}_g^*\) is the formal \(L^2\)-adjoint of the linearization of the scalar curvature operator \(\mathcal{L}_g\), which plays a fundamental role in problems related to prescribing the scalar curvature function. Here, \(Ric, \Delta\) and \(Hess\) stand, respectively, for the Ricci tensor, the Laplacian operator and the Hessian form on \(M^n\). Such a function \(f\) is called V-static potential.

In the work [13], Corvino, Eichmair and Miao proved that V-static metrics arises from the modified problem of finding stationary points for the volume functional on the space of metrics whose scalar curvature is equal to a given constant and such metrics are useful as an attempt to better understand the interplay between scalar curvature and volume. In this context, Corvino, Eichmair and Miao [13] were able to show that when the metric \(g\) does not admit non-trivial solution to Eq. (1.1), then one can achieve simultaneously a prescribed perturbation of the scalar curvature that is compactly supported in a bounded domain \(\Omega\) and a prescribed perturbation of the volume by a small deformation of the metric in \(\Omega\). It is worth to point out that a Riemannian manifold \((M^n, g)\) for which there exists a nontrivial function \(f\) satisfying (1.1) must have constant scalar curvature \(R\) (cf. Proposition 2.1 in [13] and Theorem 7 in [22]).

The case where \(\kappa \neq 0\) in (1.1) and the potential function \(f\) vanishes on the boundary \(\partial M\) was studied by Miao and Tam [22]. To be precise, Miao and Tam [22] showed that these critical metrics arise as critical points of the volume functional on...
when restricted to the class of metrics $g$ with prescribed constant scalar curvature such that $g|_{\partial M} = h$ for a prescribed Riemannian metric $h$ on the boundary. In this background, a **Miao-Tam critical metric** is a 3-tuple $(M^n, g, f)$, where $(M^n, g)$ is a compact Riemannian manifold of dimension at least three with a smooth boundary $\partial M$ and $f: M^n \to \mathbb{R}$ is a smooth function such that $f^{-1}(0) = \partial M$ satisfying the overdetermined-elliptic system

\begin{equation}
\Sigma^*_g(f) = -(\Delta f)g + \text{Hess } f - f\text{Ric} = g.
\end{equation}

For more details on such a subject, we refer the reader to [3, 4, 5, 6, 13, 22, 23] and [25].

Some explicit examples of Miao-Tam critical metrics were built on connected domain with compact closure in $\mathbb{R}^n$, $\mathbb{H}^n$ and $\mathbb{S}^n$ (cf. [22]). Moreover, some results obtained in [22] suggest that critical metrics with a prescribed boundary metric seem to be rather rigid. From this perspective, it is natural to ask whether these quoted examples are the only Miao-Tam critical metrics. In order to motivate our main results, we now briefly recall a few relevant partial answers to this question under vanishing conditions involving zero, first, or specific second order derivatives of the Weyl tensor $W$, which is defined by the following decomposition formula

\begin{equation}
R_{ijkl} = W_{ijkl} + \frac{1}{n-2} \left( R_{ik} g_{jl} + R_{jl} g_{ik} - R_{il} g_{jk} - R_{jk} g_{il} \right)
- \frac{R}{(n-1)(n-2)} \left( g_{jl} g_{ik} - g_{il} g_{jk} \right),
\end{equation}

where $R_{ijkl}$ stands for the Riemann curvature operator. Indeed, inspired by ideas developed by Kobayashi [20], Kobayashi and Obata [21], Miao and Tam [23] proved that a locally conformally flat (i.e. $W = 0$, for $n \geq 4$) simply connected, compact Miao-Tam critical metric $(M^n, g, f)$ with boundary isometric to a standard sphere $\mathbb{S}^{n-1}$ must be isometric to a geodesic ball in a simply connected space form $\mathbb{R}^n$, $\mathbb{H}^n$ or $\mathbb{S}^n$. Afterward, motivated by [3], Barros, Diôgenes and Ribeiro [5] proved that a Bach-flat simply connected, four-dimensional compact Miao-Tam critical metric with boundary isometric to a standard sphere $\mathbb{S}^3$ must be isometric to a geodesic ball in a simply connected space form $\mathbb{R}^4$, $\mathbb{H}^4$ or $\mathbb{S}^4$. The Bach-flat condition can be seen as a vanishing condition involving second and zero order terms in the Weyl tensor. Recently, Kim and Shin [19] showed that a simply connected, compact Miao-Tam critical metric with harmonic curvature and boundary isometric to a standard sphere $\mathbb{S}^3$ must be isometric to a geodesic ball in a simply connected space form $\mathbb{R}^4$, $\mathbb{H}^4$ or $\mathbb{S}^4$. The Bach-flat condition can be seen as a vanishing condition involving second and zero order terms in the Weyl tensor. Recently, Kim and Shin [19] showed that a simply connected, compact Miao-Tam critical metric with harmonic curvature and boundary isometric to a standard sphere $\mathbb{S}^3$ must be isometric to a geodesic ball in a simply connected space form $\mathbb{R}^4$, $\mathbb{H}^4$ or $\mathbb{S}^4$. But, since a Miao-Tam critical metric has constant scalar curvature, we conclude that the assumption of harmonic curvature in the Kim-Shin result can be replaced by the harmonic Weyl tensor condition (i.e. $\text{div} W = 0$). See also [3] and [23] for further related results.

Before presenting the main results, let us remark that recently Catino, Mastrolia and Monticelli [11] obtained an important classification for gradient Ricci solitons admitting a fourth-order vanishing condition on the Weyl tensor. To be precise, they showed that any $n$-dimensional ($n \geq 4$) gradient shrinking Ricci soliton with fourth-order divergence-free Weyl tensor (i.e. $\text{div}^4 W = 0$) is either Einstein or a finite quotient of $N^{n-k} \times \mathbb{R}^k$, $(k > 0)$, the product of an Einstein manifold $N^{n-k}$ with the Gaussian shrinking soliton $\mathbb{R}^k$. We highlight that $\text{div}^4 W = W_{ijkl,ikjl}$, where the indexes after the comma are the covariant derivatives.
At the same time, it is well-known that dimension four display fascinating and peculiar features, for this reason very much attention has been given to this dimension; see, for instance [7, 15, 24], for more information about this specific dimension. In this paper, motivated by [11], we shall classify four-dimensional compact Miao-Tam critical metrics under the second order divergence-free Weyl tensor condition
\[(1.4) \quad \text{div}^2 W = W_{ijkl,ik} = 0,\]
which is clearly weaker than locally conformally flat and harmonic Weyl tensor conditions considered in [23] and [19]. More precisely, we have established the following result.

**Theorem 1.** Let \((M^4, g, f)\) be a simply connected, compact Miao-Tam critical metric with second order divergence-free Weyl tensor and boundary isometric to a standard sphere \(S^3\). Then \((M^4, g)\) is isometric to a geodesic ball in a simply connected space form \(\mathbb{R}^4, \mathbb{H}^4\) or \(S^4\).

In order to proceed, it is important to remember that the Yamabe constant for Riemannian manifolds with boundary is given by
\[(1.5) \quad \mathcal{Y}(M, \partial M, [g]) = \inf_{\phi \neq 0} \frac{\int_M \left( \frac{4(n-1)}{n-2} |\nabla \phi|^2 + R\phi^2 \right) dV_g + 2 \int_{\partial M} H \phi^2 dS_g}{\left( \int_M |\phi|^{\frac{2n}{n-2}} dV_g \right)^{\frac{n-2}{n-4}}},\]
where \(H\) is the mean curvature of \(\partial M\) and \(\phi\) is a smooth positive function on \(M^n\).

For more details, we refer the reader to [14] the references therein.

In dimension 4, it is known that the Yamabe invariant alone is too weak to classify a given manifold. For this reason, it is expected additional conditions. In [10], Catino proved that any four-dimensional compact gradient shrinking Ricci soliton satisfying some \(L^2\)-pinching condition is isometric to a quotient of the round sphere; see also [18] for further related results. Here, by adapting the method outlined in [10] as well as [16], we provide an integral curvature estimate involving the Yamabe constant for four-dimensional compact Miao-Tam critical metrics. More precisely, we have the following result.

**Theorem 2.** Let \((M^4, g, f)\) be a simply connected, compact Miao-Tam critical metric with positive scalar curvature. Then we have:
\[(1.6) \quad \mathcal{Y}(M, \partial M, [g]) \Phi(M) \leq 4\sqrt{3} \left( \int_M \left( |W|^2 + |\tilde{Ric}|^2 \right) dV_g \right)^{\frac{1}{2}} \Phi(M) + 3(3\sqrt{2} - 1) \int_M |\tilde{Ric}|^2 |\nabla f|^2 dV_g,
\]
where \(\mathcal{Y}(M, \partial M, [g])\) is given by (1.5) and \(\Phi(M) = \left( \int_M f^4 |\tilde{Ric}|^4 dV_g \right)^{\frac{1}{2}}\). Moreover, if the equality occurs in (1.6), then \(M^4\) is isometric to a geodesic ball in \(S^4\).

The article is organized as follows. In Section 2, we review some basic facts and classical tensors. Moreover, we prove a couple of key lemmas which will be used in the proof of the main results. In Section 3, we prove the main results.
2. Background and Key Lemmas

In this section we shall present some preliminaries and key lemmas which will be useful in the proof of our main results. To start with, we remember that a Riemannian manifold \((M^n, g)\) is a \(V\)-static metric if there exists a smooth function \(f\) and a constant \(\kappa\) such that

\[
- (\Delta f)g + \text{Hess} f - f\text{Ric} = \kappa g.
\]

In particular, tracing (2.1) we deduce that the \(V\)-static potential \(f\) also satisfies the linear equation

\[
- (n - 1)\Delta f = Rf + kn.
\]

From this, it is easy to check that

\[
\text{Hess} f - f\text{Ric} = -Rf + \kappa n - \frac{1}{n-1} g
\]

and

\[
f\hat{\text{Ric}} = \hat{\text{Hess}} f,
\]

where \(\hat{T} = T - \frac{1}{n} \text{tr} T g\) stands for the traceless of tensor \(T\).

To fix notation we recall some special tensors in the study of curvature for a Riemannian manifold \((M^n, g)\), \(n \geq 3\). The first one is the Weyl tensor \(W\) which is defined by the following decomposition formula

\[
R_{ijkl} = W_{ijkl} + \frac{1}{n-2} \left( R_{ikg_{jl}} + R_{jlg_{ik}} - R_{ilg_{jk}} - R_{jkg_{il}} \right) - \frac{R}{(n-1)(n-2)} \left( g_{jl}g_{ik} - g_{il}g_{jk} \right),
\]

where \(R_{ijkl}\) stands for the Riemann curvature operator. The second one is the Cotton tensor \(C\) given by

\[
C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)} \left( \nabla_i R g_{jk} - \nabla_j R g_{ik} \right).
\]

Easily one verifies that these two tensors are related as follows

\[
C_{ijk} = -\frac{n-2}{n-3} \nabla_i W_{ijkl}.
\]

provided \(n \geq 4\). Moreover, it is easy to see that

\[
C_{ijk} = -C_{jik} \quad \text{and} \quad C_{ijk} + C_{jki} + C_{kij} = 0.
\]

In particular, by its skew-symmetry and Schur’s lemma, we have

\[
g^{ij} C_{ijk} g^{jk} C_{ijk} = g^{ik} C_{ijk} = g^{jk} C_{ijk} = 0.
\]

Next, the Schouten tensor \(A\) is defined by

\[
A_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij}.
\]

Combining Eqs. (2.5) and (2.10) we have the following decomposition

\[
R_{ijkl} = \frac{1}{n-2} (A \odot g)_{ijkl} + W_{ijkl},
\]
where ⊙ is the Kulkarni-Nomizu product. Further, we recall that the Bach tensor on a Riemannian manifold \((M^n, g)\), \(n \geq 4\), is defined in term of the components of the Weyl tensor \(W_{ikjl}\) as follows

\[
B_{ij} = \frac{1}{n-3} \nabla^k \nabla^l W_{ikjl} + \frac{1}{n-2} R_{kl} W_{i}^k \, j^l ,
\]

while for \(n = 3\) it is given by

\[
B_{ij} = \nabla^k C_{kij}.
\]

For more details about these tensors, we refer to \([1, 7]\). We say that \((M^n, g)\) is Bach-flat when \(B_{ij} = 0\). It is easy to check that locally conformally flat metrics as well as Einstein metrics are Bach-flat.

In order to set the stage for the proof to follow let us recall some useful results obtained in \([2, 5]\).

**Lemma 1** \((2, 5)\). Let \((M^n, g, f, \kappa)\) be a connected, smooth Riemannian manifold and \(f\) is a smooth function on \(M^n\) satisfying the \(V\)-static equation \((1.1)\). Then

\[
f(\nabla_i R_{jk} - \nabla_j R_{ik}) = R_{ijkl} \nabla_l f + \frac{R}{n-1} (\nabla_i f g_{jk} - \nabla_j f g_{ik}) - (\nabla_i R_{jk} - \nabla_j R_{ik}).
\]

In particular, it is not difficult to check that Riemannian manifolds satisfying the \(V\)-static equation must to satisfy

\[
f C_{ijk} = T_{ijk} + W_{ijkl} \nabla_l f,
\]

where the covariant 3-tensor \(T_{ijk}\), defined previously in \([5]\), is given by

\[
T_{ijk} = \frac{n-1}{n-2} (R_{ik} \nabla_j f - R_{jk} \nabla_i f) - \frac{R}{n-2} (g_{ik} \nabla_j f - g_{jk} \nabla_i f)
\]

\[+ \frac{1}{n-2} (g_{ik} R_{js} \nabla_s f - g_{jk} R_{is} \nabla_s f).\]

Before presenting the proof of main results it is necessary to prove a couple of key lemmas. To do so, it is crucial to recall a Böchner type formula for \(V\)-static metrics obtained recently in \([2]\) (cf. Theorem 2 in \([2]\)). More precisely, Baltazar and Ribeiro proved that a connected, smooth Riemannian manifold \(M^n\) and a smooth function \(f\) on \(M^n\) satisfying the \(V\)-static equation \((1.1)\) must to satisfy

\[
\frac{1}{2} \text{div}(f \nabla |\hat{Ric}|^2) = \frac{n-2}{n-1} f|C|^2 + f|\nabla \hat{Ric}|^2 + \frac{nk}{n-1} |\hat{Ric}|^2 + 2Rf |\hat{Ric}|^2 + \frac{2nf}{n-2} \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki} \]

\[+ \frac{n-2}{n-1} W_{ijkl} C_{ijk} \nabla_l f - 2f W_{ijkl} \hat{R}_{ik} \hat{R}_{jl}.\]

In the sequel, we shall use \((2, 16)\) to get the following lemma.
Lemma 2. Let \((M^n, g, f, \kappa)\) be a connected, smooth Riemannian manifold and \(f\) is a smooth function on \(M^n\) satisfying the V-static equation \((1.1)\). Then we have:

\[
\frac{1}{2} \text{div}(f^2 \nabla |\hat{\text{Ric}}|^2) = f^2 |\nabla \hat{\text{Ric}}|^2 + \frac{n f^2}{n-1} |\hat{\text{Ric}}|^2 + \frac{2 R f^2}{n-1} |\hat{\text{Ric}}|^2 + 2 f \nabla, R_{jk} R_{ik} \nabla_j f - \frac{1}{2} (f \nabla |\hat{\text{Ric}}|^2, \nabla f) + \frac{2 n f^2}{n-2} R_{ij} R_{jk} \hat{\dot{R}}_{ki}.
\]

\[(2.17)\]

Proof. To begin with, we use Eq. \((2.16)\) to achieve

\[
\frac{1}{2} \text{div}(f^2 \nabla |\hat{\text{Ric}}|^2) = \frac{1}{2} f \text{div}(f \nabla |\hat{\text{Ric}}|^2) + \frac{1}{2} (f \nabla |\hat{\text{Ric}}|^2, \nabla f)
\]

\[
= f \left[ \frac{n-2}{n-1} f |C|^2 + f^2 |\nabla \hat{\text{Ric}}|^2 + \frac{n f}{n-1} |\hat{\text{Ric}}|^2 + \frac{2 R f^2}{n-1} |\hat{\text{Ric}}|^2 + \frac{2 n f^2}{n-2} R_{ij} R_{jk} \hat{\dot{R}}_{ki} - \frac{n-2}{n-1} W_{ijkl} C_{ijk} \nabla_l f - 2 f W_{ijkl} \hat{\dot{R}}_{ik} \hat{\dot{R}}_{jl} \right] + \frac{1}{2} (f \nabla |\hat{\text{Ric}}|^2, \nabla f),
\]

so that

\[
\frac{1}{2} \text{div}(f^2 \nabla |\hat{\text{Ric}}|^2) = \frac{n-2}{n-1} f^2 |C|^2 + f^2 |\nabla \hat{\text{Ric}}|^2 + \frac{n f}{n-1} |\hat{\text{Ric}}|^2 + \frac{2 R f^2}{n-1} |\hat{\text{Ric}}|^2 + \frac{2 n f^2}{n-2} R_{ij} R_{jk} \hat{\dot{R}}_{ki} - \frac{n-2}{n-1} W_{ijkl} C_{ijk} \nabla_l f - 2 f^2 W_{ijkl} \hat{\dot{R}}_{ik} \hat{\dot{R}}_{jl} + \frac{1}{2} (f \nabla |\hat{\text{Ric}}|^2, \nabla f).
\]

\[(2.18)\]

On the other hand, by using Eq. \((2.14)\) we have

\[
W_{ijkl} C_{ijk} \nabla_l f = (f C_{ijk} - T_{ijk}) C_{ijk} = f |C|^2 - T_{ijk} C_{ijk}.
\]

Next, since \(M^n\) has constant scalar curvature we may use \((2.6)\) jointly with \((2.15)\) to infer

\[
W_{ijkl} C_{ijk} \nabla_l f = f |C|^2 - \frac{n-1}{n-2} (R_{ik} \nabla_j f - R_{jk} \nabla_i f) (\nabla_i R_{jk} - \nabla_j R_{ik})
\]

\[
= f |C|^2 - \frac{2(n-1)}{n-2} (\nabla_i R_{jk} R_{ik} \nabla_j f - \nabla_j R_{ik} R_{ik} \nabla_i f)
\]

\[
= f |C|^2 - \frac{2(n-1)}{n-2} \langle \nabla_i R_{jk} R_{ik} \nabla_j f + \frac{2(n-1)}{n-2} \nabla_j R_{ik} R_{ik} \nabla_i f.
\]

From this it follows that

\[
\frac{n-2}{n-1} f W_{ijkl} C_{ijk} \nabla_l f = \frac{n-2}{n-1} f^2 |C|^2 - 2 f \nabla, R_{jk} R_{ik} \nabla_j f + 2 f \nabla_j R_{ik} R_{ik} \nabla_j f.
\]

\[(2.19)\]
Substituting (2.19) into (2.18) we get
\[
\frac{1}{2} \text{div}(f^2 \nabla |\text{Ric}|^2) = \frac{n-2}{n-1} f^2 |C|^2 + \frac{nkf}{n-1} |\text{Ric}|^2 + \frac{2Rf^2}{n-1} |\text{Ric}|^2 \\
+ \frac{2nf^2}{n-2} \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki} - \frac{n-2}{n-1} f^2 |C|^2 + 2f \nabla_i R_{jk} R_{ik} \nabla_j f \\
- 2f \nabla_j R_{ik} R_{ik} \nabla_j f - 2f^2 W_{ijkl} \hat{R}_{ij} \hat{R}_{kl} + \frac{1}{2} (f \nabla |\text{Ric}|^2, \nabla f)
\]
which finishes the proof of the lemma. \(\square\)

To finish this section, we shall deduce an integral formula for Miao-Tam critical metrics, which plays a fundamental role in the proof of Theorem 2.

**Lemma 3.** Let \((M^n, g, f)\) be a compact, oriented, connected Miao-Tam critical metric with smooth boundary \(\partial M\). Then we have:
\[
2 \int_M f^2 W_{ijkl} \hat{R}_{ik} \hat{R}_{jl} dV_g = -2 \int_M |\text{Ric}(\nabla f)|^2 dV_g + \int_M f^2 |\nabla \text{Ric}|^2 dV_g \\
+ \frac{n}{n-1} \int_M f |\text{Ric}|^2 dV_g + \frac{2R}{n-1} \int_M f^2 |\text{Ric}|^2 dV_g \\
+ \frac{n-4}{2n} \int_M f \Delta f |\text{Ric}|^2 dV_g + \frac{1}{2} \int_M |\text{Ric}|^2 |\nabla f|^2 dV_g \\
+ \frac{4}{n-2} \int_M f^2 \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki} dV_g.
\]

**Proof.** We start integrating the expression obtained in Lemma 2 over \(M\) to achieve
\[
0 = \int_M f^2 |\nabla \text{Ric}|^2 dV_g + \frac{n}{n-1} \int_M f |\text{Ric}|^2 dV_g + \frac{2R}{n-1} \int_M f^2 |\text{Ric}|^2 dV_g \\
+ 2 \int_M f \nabla_i \hat{R}_{jk} \hat{R}_{ik} \nabla_j f dV_g - \frac{1}{2} \int_M (f \nabla |\text{Ric}|^2, \nabla f) dV_g \\
- 2 \int_M f^2 W_{ijkl} \hat{R}_{ik} \hat{R}_{jl} dV_g + \frac{2n}{n-2} \int_M f^2 \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki} dV_g.
\]

Easily one verifies that
\[
\int_M f \nabla_i \hat{R}_{jk} \hat{R}_{ik} \nabla_j f dV_g = \int_M \nabla_i (f \hat{R}_{jk} \hat{R}_{ik} \nabla_j f) dV_g - \int_M \hat{R}_{ik} \hat{R}_{ik} \nabla_i f \nabla_j f dV_g \\
- \int_M f \hat{R}_{jk} \hat{R}_{ik} \nabla_i \nabla_j f dV_g,
\]
and this combined with (2.3) yields
\[
\int_M f \nabla_i \hat{R}_{jk} \hat{R}_{ik} \nabla_j f dV_g = - \int_M |\text{Ric}(\nabla f)|^2 dV_g - \int_M f \hat{R}_{jk} \hat{R}_{ik} (f \hat{R}_{ij} + \frac{\Delta f}{n} g_{ij}) dV_g,
\]
which can be rewritten as
\[ \int_{M} f \nabla_{j} \hat{R}_{jk} \hat{R}_{ik} \nabla_{j} f dV_{g} = - \int_{M} |\hat{Ric}(\nabla f)|^{2} dV_{g} - \int_{M} f^{2} \hat{R}_{jk} \hat{R}_{ik} \hat{R}_{ij} dV_{g} \]
\[ - \frac{1}{n} \int_{M} f \Delta f |\hat{Ric}|^{2} dV_{g}. \]

(2.23)

Hence we use this data into (2.22) to infer

\[ 0 = \int_{M} f^{2} |\hat{Ric}|^{2} dV_{g} + \frac{n}{n - 1} \int_{M} f |\hat{Ric}|^{2} dV_{g} + \frac{2R}{n - 1} \int_{M} f^{2} |\hat{Ric}|^{2} dV_{g} \]
\[ - 2 \int_{M} |\hat{Ric}(\nabla f)|^{2} dV_{g} - 2 \int_{M} f^{2} \hat{R}_{jk} \hat{R}_{ik} \hat{R}_{ij} dV_{g} - \frac{2}{n} \int_{M} f \Delta f |\hat{Ric}|^{2} dV_{g} \]
\[ - \frac{1}{2} \int_{M} \langle f \nabla |\hat{Ric}|^{2}, \nabla f \rangle dV_{g} - 2 \int_{M} f^{2} W_{ijkl} \hat{R}_{ik} \hat{R}_{jl} dV_{g} \]
\[ + \frac{2n}{n - 2} \int_{M} f^{2} \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{kl} dV_{g}, \]

in other words,

\[ 0 = \int_{M} f^{2} |\nabla \hat{Ric}|^{2} dV_{g} + \frac{n}{n - 1} \int_{M} f |\hat{Ric}|^{2} dV_{g} + \frac{2R}{n - 1} \int_{M} f^{2} |\hat{Ric}|^{2} dV_{g} \]
\[ - \frac{1}{2} \int_{M} \langle f \nabla |\hat{Ric}|^{2}, \nabla f \rangle dV_{g} - 2 \int_{M} |\hat{Ric}(\nabla f)|^{2} dV_{g} - \frac{2}{n} \int_{M} f \Delta f |\hat{Ric}|^{2} dV_{g} \]
\[ - 2 \int_{M} f^{2} W_{ijkl} \hat{R}_{ik} \hat{R}_{jl} dV_{g} + \frac{4}{n - 2} \int_{M} f^{2} \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{kl} dV_{g}. \]

(2.24)

To proceed, by a direct computation, we can check that

\[ \text{div}(f|\hat{Ric}|^{2} \nabla f) = f |\hat{Ric}|^{2} \Delta f + \langle \nabla (f|\hat{Ric}|^{2}), \nabla f \rangle \]
\[ = f \Delta f |\hat{Ric}|^{2} + f \langle \nabla |\hat{Ric}|^{2}, \nabla f \rangle + |\hat{Ric}|^{2} |\nabla f|^{2}. \]

Of which we deduce

\[ 0 = \int_{M} f \Delta f |\hat{Ric}|^{2} dV_{g} + \int_{M} f \langle \nabla |\hat{Ric}|^{2}, \nabla f \rangle dV_{g} + \int_{M} |\hat{Ric}|^{2} |\nabla f|^{2} dV_{g}. \]

Thus,

\[ \int_{M} f \langle \nabla |\hat{Ric}|^{2}, \nabla f \rangle dV_{g} = - \int_{M} f \Delta f |\hat{Ric}|^{2} dV_{g} - \int_{M} |\hat{Ric}|^{2} |\nabla f|^{2} dV_{g}. \]

(2.25)
One notices that combining (2.25) with (2.24) we arrive at
\[
0 = \int_M f^2 |\nabla \hat{Ric}|^2 dV_g + \frac{n}{n-1} \int_M f |\hat{Ric}|^2 dV_g + \frac{2R}{n-1} \int_M f^2 |\hat{Ric}|^2 dV_g \\
+ \frac{1}{2} \int_M f \Delta f |\hat{Ric}|^2 dV_g + \frac{1}{2} \int_M |\hat{Ric}|^2 |\nabla f|^2 dV_g - 2 \int_M |\hat{Ric}(\nabla f)|^2 dV_g \\
- \frac{2}{n} \int_M f \Delta f |\hat{Ric}|^2 dV_g - 2 \int_M f^2 W_{ijkl} \hat{R}_{ik} \hat{R}_{jl} dV_g \\
+ \frac{4}{n-2} \int_M f^2 \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki} dV_g \\
= \int_M f^2 |\nabla \hat{Ric}|^2 dV_g + \frac{n}{n-1} \int_M f |\hat{Ric}|^2 dV_g + \frac{2R}{n-1} \int_M f^2 |\hat{Ric}|^2 dV_g \\
+ \frac{n-4}{2n} \int_M f \Delta f |\hat{Ric}|^2 dV_g + \frac{1}{2} \int_M |\hat{Ric}|^2 |\nabla f|^2 dV_g - 2 \int_M |\hat{Ric}(\nabla f)|^2 dV_g \\
- 2 \int_M f^2 W_{ijkl} \hat{R}_{ik} \hat{R}_{jl} dV_g + \frac{4}{n-2} \int_M f^2 \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ki} dV_g, 
\]
that was to be proved. \(\square\)

3. PROOF OF THE MAIN RESULTS

3.1. Proof of Theorem 1. To start with, we shall present a crucial integral formula.

**Lemma 4.** Let \((M^n, g, f)\) be a Miao-Tam critical metric. Then we have:
\[
\int_M f^2 |C|^2 dV_g + \int_M f^2 \text{div}^3 C dV_g + 2 \int_M \text{div} C(\nabla f, \nabla f) dV_g = 0.
\]

**Proof.** Taking into account that \(M^n\) has constant scalar curvature, we use (2.6) together with (2.8) to infer
\[
\int_M f^2 |C|^2 dV_g = \int_M f^2 (\nabla_i R_{jk} - \nabla_j R_{ik}) C_{ijk} dV_g \\
\quad = 2 \int_M f^2 \nabla_i R_{ijk} C_{ijk} dV_g. 
\]
(3.1)

Next, by using the Stokes’s formula and that \(f\) vanishes on the boundary we achieve at
\[
0 = \int_M \nabla_i (f^2 R_{ijk} C_{ijk}) dV_g \\
= \int_M \nabla_i f^2 R_{ijk} C_{ijk} dV_g + \int_M f^2 \nabla_i R_{ijk} C_{ijk} dV_g \\
+ \int_M f^2 R_{jk} \nabla_i C_{ijk} dV_g, 
\]
and this substituted into (3.1) gives
\[
\int_M f^2 |C|^2 dV_g = -2 \int_M \nabla_i f^2 R_{ijk} C_{ijk} dV_g - 2 \int_M f^2 R_{jk} \nabla_i C_{ijk} dV_g. 
\]
Taking into account that the Cotton tensor has trace-free in any two indices we use the fundamental equation (1.2) to obtain

\( (3.3) \quad \int_M f^2 |C|^2 dV_g = -4 \int_M \nabla_i f \nabla_j \nabla_k f C_{ijk} dV_g - 2 \int_M f \nabla_j \nabla_k f \nabla_i C_{ijk} dV_g. \)

One notices that

\[ \nabla_j \left( \nabla_i f \nabla_k f C_{ijk} \right) = \nabla_j \nabla_i f \nabla_k f C_{ijk} + \nabla_i \nabla_j \nabla_k f C_{ijk} - \text{div} C(\nabla f, \nabla f). \]

But, by using once more that the Cotton is skew-symmetric in the first two indices, it is not difficult to check, by replacing the indices, that

\[ \nabla_j \nabla_i f \nabla_k f C_{ijk} = -\nabla_j \nabla_i f \nabla_k f C_{ijk} = 0. \]

Therefore, on integrating (3.4) over \( M^0 \) we may use this data jointly with Stokes’s formula to arrive at

\( (3.5) \quad \int_M \nabla_i f \nabla_j \nabla_k f C_{ijk} dV_g = \int_M \text{div} C(\nabla f, \nabla f) dV_g + \int_{\partial M} C(\nabla f, \nu, \nabla f) dS_g, \)

where \( \nu = \pm \frac{\nabla f}{|\nabla f|} \), according to the potential function is either positive or negative on \( M \).

In the meantime, notice that

\[ \nabla_j \left( f \nabla_k f \nabla_i C_{ijk} \right) = \nabla_j f \nabla_k f \nabla_i C_{ijk} + f \nabla_j \nabla_k f \nabla_i C_{ijk} + f \nabla_k f \nabla_j \nabla_i C_{ijk} \]

\[ = \text{div} C(\nabla f, \nabla f) + f \nabla_j \nabla_k f \nabla_i C_{ijk} + \frac{1}{2} f \nabla_k f^2 \nabla_j \nabla_i C_{ijk}. \]

Upon integrating the above expression over \( M \) we use again the Stokes’s formula to deduce

\[ \int_M f \nabla_j \nabla_k f \nabla_i C_{ijk} dV_g = - \int_M \text{div} C(\nabla f, \nabla f) dV_g - \frac{1}{2} \int_M \nabla_k f^2 \nabla_j \nabla_i C_{ijk} dV_g, \]

which can be rewritten as

\[ \int_M f \nabla_j \nabla_k f \nabla_i C_{ijk} dV_g = - \int_M \text{div} C(\nabla f, \nabla f) dV_g + \frac{1}{2} \int_M f^2 \nabla_k \nabla_j \nabla_i C_{ijk} dV_g \]

\( (3.6) \quad = - \int_M \text{div} C(\nabla f, \nabla f) dV_g + \frac{1}{2} \int_M f^2 \text{div}^3 C dV_g. \)

Now, substituting (3.3) and (3.6) into (3.3) we obtain

\[ \int_M f^2 |C|^2 dV_g + 2 \int_M \text{div} C(\nabla f, \nabla f) dV_g = \int_M \nabla_k f^2 \nabla_j \nabla_i C_{ijk} dV_g, \]

that is,
This is what we wanted to prove.

Proceeding, since \( M^4 \) has second order divergence-free Weyl tensor, we may use (3.7) jointly with (2.7) to deduce that the Cotton tensor \( C \) is identically zero in \( M^4 \). This implies that the Weyl tensor is harmonic, and then it suffices to apply Theorem 10.3 in [19] to conclude that \( M^4 \) is isometric to a geodesic ball in a simply connected space form \( \mathbb{R}^4, \mathbb{H}^4 \) or \( \mathbb{S}^4 \). This finishes the proof of Theorem 1.

3.2. Proof of Theorem 2

**Proof.** We start invoking Lemma 3 jointly with the classical Kato’s inequality to obtain

\[
0 \geq \int_M f^2 |\nabla \hat{Ric}|^2 dV_g + \frac{n}{n-1} \int_M f |\hat{Ric}|^2 dV_g \\
+ \frac{2R}{n-1} \int_M f^2 |\hat{Ric}|^2 dV_g + \frac{n-4}{2n} \int_M f \Delta f |\hat{Ric}|^2 dV_g \\
+ \frac{1}{2} \int_M |\hat{Ric}|^2 |\nabla f|^2 dV_g - 2 \int_M |\hat{Ric}(\nabla f)|^2 dV_g \\
-2 \int_M f^2 W_{ijkl} \hat{R}_{ik} \hat{R}_{jl} dV_g + \frac{4}{n-2} \int_M f^2 \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ik} dV_g.
\]

(3.8)

On the other hand, by a result by Catino (cf. [10], Proposition 2.1, see also [8], Lemma 4.7), on every \( n \)-dimensional Riemannian manifold the following estimate holds

\[
| - W_{ijkl} \hat{R}_{ik} \hat{R}_{jl} + \frac{2}{n-2} \hat{R}_{ij} \hat{R}_{jk} \hat{R}_{ik} | \leq \sqrt{\frac{n-2}{2(n-1)}} \left( |W|^2 + \frac{8}{n(n-2)} |\hat{Ric}|^2 \right)^{\frac{1}{2}} |\hat{Ric}|^2.
\]

This employed into (3.8) achieves

\[
0 \geq \int_M f^2 |\nabla \hat{Ric}|^2 dV_g + \frac{n}{n-1} \int_M f |\hat{Ric}|^2 dV_g + \frac{2R}{n-1} \int_M f^2 |\hat{Ric}|^2 dV_g \\
+ \frac{n-4}{2n} \int_M f \Delta f |\hat{Ric}|^2 dV_g + \frac{1}{2} \int_M |\hat{Ric}|^2 |\nabla f|^2 dV_g - 2 \int_M |\hat{Ric}(\nabla f)|^2 dV_g \\
- \sqrt{\frac{2(n-2)}{n-1}} \int_M f^2 \left( |W|^2 + \frac{8}{n(n-2)} |\hat{Ric}|^2 \right)^{\frac{1}{2}} |\hat{Ric}|^2 dV_g.
\]

(3.9)

For what follows it is essential to remark that from (1.5) we have

\[
\frac{n-2}{4(n-1)} \mathcal{V}(M, \partial M, [g]) \left( \int_M |\phi|^\frac{2n}{n-2} dV_g \right)^{\frac{n-2}{n}} \leq \int_M |\nabla \phi|^2 dV_g + \frac{n-4}{4(n-1)} \int_M R \phi^2 dV_g + \frac{n-2}{2(n-1)} \int_{\partial M} H \phi^2 dS_g.
\]

(3.10)

where \( H \) is the mean curvature of \( \partial M \). Since \( f^{-1}(0) = \partial M \), we deduce that \( f \) does not change sign. Then, we assume that \( f \) is nonnegative. In particular, \( f > 0 \) at the
interior of \( M \). Therefore, choosing \( \phi = f|\tilde{R}c| \) in (3.10) and using that \( f \) vanishes on the boundary we obtain

\[
\int_M |\nabla (f|\tilde{R}c|)|^2 dV_g \geq \frac{n-2}{4(n-1)} \mathcal{V}(M, \partial M, [g]) \left( \int_M f^{2n-2} |\tilde{R}c|^{\frac{2n-2}{n-2}} dV_g \right)^{\frac{n-2}{n}} - \frac{(n-2)R}{4(n-1)} \int_M f^2 |\tilde{R}c|^2 dV_g - \int_M |\tilde{R}c|^2 |\nabla f|^2 dV_g.
\]

Moreover, taking into account that

\[
|\nabla (f|\tilde{R}c|)|^2 = |f \nabla |\tilde{R}c| + |\tilde{R}c| \nabla f|^2 = f^2 |\nabla |\tilde{R}c||^2 + 2f |\tilde{R}c| (\nabla |\tilde{R}c|, \nabla f) + |\tilde{R}c|^2 |\nabla f|^2 = f^2 |\nabla |\tilde{R}c||^2 + f (\langle \nabla |\tilde{R}c|^2, \nabla f \rangle + |\tilde{R}c|^2 |\nabla f|^2,
\]

we arrive at

\[
\int_M f^2 |\nabla |\tilde{R}c||^2 dV_g \geq \frac{n-2}{4(n-1)} \mathcal{V}(M, \partial M, [g]) \left( \int_M f^{2n-2} |\tilde{R}c|^{\frac{2n-2}{n-2}} dV_g \right)^{\frac{n-2}{n}} - \frac{(n-2)R}{4(n-1)} \int_M f^2 |\tilde{R}c|^2 dV_g - \int_M |\tilde{R}c|^2 |\nabla f|^2 dV_g.
\]

By using (2.24) we have

\[
\int_M f^2 |\nabla |\tilde{R}c||^2 dV_g \geq \frac{n-2}{4(n-1)} \mathcal{V}(M, \partial M, [g]) \left( \int_M f^{2n-2} |\tilde{R}c|^{\frac{2n-2}{n-2}} dV_g \right)^{\frac{n-2}{n}} - \frac{(n-2)R}{4(n-1)} \int_M f^2 |\tilde{R}c|^2 dV_g - \int_M \frac{2}{n-1} f^2 |\tilde{R}c|^2 dV_g + \int_M f \Delta f |\tilde{R}c|^2 dV_g,
\]

(3.11)

Next, comparing (3.10) with (3.11) we immediately obtain

\[
0 \geq \frac{n-2}{4(n-1)} \mathcal{V}(M, \partial M, [g]) \left( \int_M f^{2n-2} |\tilde{R}c|^{\frac{2n-2}{n-2}} dV_g \right)^{\frac{n-2}{n}} - \frac{(n-2)R}{4(n-1)} \int_M f^2 |\tilde{R}c|^2 dV_g + \int_M f \Delta f |\tilde{R}c|^2 dV_g + \frac{n}{n-1} \int_M f |\tilde{R}c|^2 dV_g + \frac{2R}{n-1} \int_M f^2 |\tilde{R}c|^2 dV_g + \frac{n-4}{2n} \int_M f \Delta f |\tilde{R}c|^2 dV_g + \int_M \frac{1}{2} |\tilde{R}c| |\nabla f|^2 dV_g - \int_M |\tilde{R}c| (\nabla f)^2 dV_g - \sqrt{\frac{2(n-2)}{n-1}} \int_M f^2 \left( |W|^2 + \frac{8}{n(n-2)} |\tilde{R}c|^2 \right)^{\frac{1}{2}} |\tilde{R}c|^2 dV_g,
\]

which can rewritten as
\[0 \geq \frac{n-2}{4(n-1)} \mathcal{V}(M, \partial M, [g]) \left( \int_M f^{\frac{2n}{n-2}} |\hat{\text{Ric}}|^{\frac{2n}{n-2}} \, dV_g \right)^{\frac{n-2}{n}} + \frac{1}{2} \int_M |\hat{\text{Ric}}|^2 |\nabla f|^2 \, dV_g \]

\[-2 \int_M |\hat{\text{Ric}}(\nabla f)|^2 \, dV_g + \frac{n}{n-1} \int_M f |\hat{\text{Ric}}|^2 \, dV_g - \frac{(n-10)R}{4(n-1)} \int_M f^2 |\hat{\text{Ric}}|^2 \, dV_g \]

\[-\sqrt{\frac{2(n-2)}{n-1}} \int_M f^2 \left( |W|^2 + \frac{8}{n(n-2)} |\hat{\text{Ric}}|^2 \right)^{\frac{n}{2}} |\hat{\text{Ric}}|^2 \, dV_g \]

+ \frac{3n-4}{2n} \int_M f \Delta f |\hat{\text{Ric}}|^2 \, dV_g,

and by Eq. (2.2) we get

\[0 \geq \frac{n-2}{4(n-1)} \mathcal{V}(M, \partial M, [g]) \left( \int_M f^{\frac{2n}{n-2}} |\hat{\text{Ric}}|^{\frac{2n}{n-2}} \, dV_g \right)^{\frac{n-2}{n}} + \frac{1}{2} \int_M |\hat{\text{Ric}}|^2 |\nabla f|^2 \, dV_g \]

\[-2 \int_M |\hat{\text{Ric}}(\nabla f)|^2 \, dV_g + \frac{n}{n-1} \int_M f |\hat{\text{Ric}}|^2 \, dV_g - \frac{(n-10)R}{4(n-1)} \int_M f^2 |\hat{\text{Ric}}|^2 \, dV_g \]

\[-\frac{(3n-4)R}{2n(n-1)} \int_M f^2 |\hat{\text{Ric}}|^2 \, dV_g - \frac{(3n-4)}{2(n-1)} \int_M f |\hat{\text{Ric}}|^2 \, dV_g \]

\[-\sqrt{\frac{2(n-2)}{n-1}} \int_M f^2 \left( |W|^2 + \frac{8}{n(n-2)} |\hat{\text{Ric}}|^2 \right)^{\frac{n}{2}} |\hat{\text{Ric}}|^2 \, dV_g.

Of which we deduce

\[0 \geq \frac{n-2}{4(n-1)} \mathcal{V}(M, \partial M, [g]) \left( \int_M f^{\frac{2n}{n-2}} |\hat{\text{Ric}}|^{\frac{2n}{n-2}} \, dV_g \right)^{\frac{n-2}{n}} + \frac{1}{2} \int_M |\hat{\text{Ric}}|^2 |\nabla f|^2 \, dV_g \]

\[-2 \int_M |\hat{\text{Ric}}(\nabla f)|^2 \, dV_g - \frac{(n-4)R}{2(n-1)} \int_M f |\hat{\text{Ric}}|^2 \, dV_g \]

\[-\frac{(n^2-4n-8)R}{4n(n-1)} \int_M f^2 |\hat{\text{Ric}}|^2 \, dV_g \]

(3.12)

\[-\sqrt{\frac{2(n-2)}{n-1}} \int_M f^2 \left( |W|^2 + \frac{8}{n(n-2)} |\hat{\text{Ric}}|^2 \right)^{\frac{n}{2}} |\hat{\text{Ric}}|^2 \, dV_g.

Next, the Hölder inequality implies that

\[0 \geq \frac{n-2}{4(n-1)} \mathcal{V}(M, \partial M, [g]) \left( \int_M f^{\frac{2n}{n-2}} |\hat{\text{Ric}}|^{\frac{2n}{n-2}} \, dV_g \right)^{\frac{n-2}{n}} \]

\[-\sqrt{\frac{2(n-2)}{n-1}} \left( \int_M \left( |W|^2 + \frac{8}{n(n-2)} |\hat{\text{Ric}}|^2 \right)^{\frac{n}{2}} \, dV_g \right)^{\frac{n}{2}} \left( \int_M f^{\frac{2n}{n-2}} |\hat{\text{Ric}}|^{\frac{2n}{n-2}} \, dV_g \right)^{\frac{n-2}{n}} \]

\[+ \frac{1}{2} \int_M |\hat{\text{Ric}}|^2 |\nabla f|^2 \, dV_g - 2 \int_M |\hat{\text{Ric}}(\nabla f)|^2 \, dV_g - \frac{(n-4)R}{2(n-1)} \int_M f |\hat{\text{Ric}}|^2 \, dV_g \]

\[-\frac{(n^2-4n-8)R}{4n(n-1)} \int_M f^2 |\hat{\text{Ric}}|^2 \, dV_g.

(3.13)
Now, we claim that
\begin{equation}
|\hat{\text{Ric}}(\nabla f)|^2 \leq \frac{(n-1)\sqrt{2n}}{2n} |\hat{\text{Ric}}|^2 |\nabla f|^2.
\end{equation}
To prove this claim, first, notice that a straightforward computation gives
\begin{equation}
|\text{Ric}(\nabla f)|^2 = -\frac{1}{8} (df \otimes df \otimes g)_{ijkl} (\text{Ric} \otimes \hat{\text{Ric}})_{ijkl}
\end{equation}
and
\begin{equation}
|(df \otimes df \otimes g)_{ijkl} (\text{Ric} \otimes \hat{\text{Ric}})_{ijkl}| \leq |df \otimes df \otimes g|^2 |\text{Ric} \otimes \hat{\text{Ric}}|^2,
\end{equation}
where $\otimes$ is the Kulkarni-Nomizu product. We further have
\begin{equation}
|df \otimes df \otimes g|^2 = 4(n-1) |\nabla f|^4
\end{equation}
and
\begin{equation}
|\text{Ric} \otimes \hat{\text{Ric}}|^2 = 8 |\text{Ric}|^4 - 8 |\hat{\text{Ric}}|^2.
\end{equation}
In particular, it is easy to check that $\text{tr} \text{Ric}^2 = |\text{Ric}|^2$ and then we immediately have $|\hat{\text{Ric}}|^2 \geq \frac{|\text{Ric}|^4}{n}$, which allows to deduce that
\begin{equation}
|\text{Ric} \otimes \hat{\text{Ric}}|^2 \leq \frac{8(n-1)}{n} |\text{Ric}|^4.
\end{equation}
Whence, it follows that
\begin{equation}
|(df \otimes df \otimes g)_{ijkl} (\text{Ric} \otimes \hat{\text{Ric}})_{ijkl}| \leq \frac{32(n-1)^2}{n} |\text{Ric}|^4 |\nabla f|^4.
\end{equation}
Thereby, we immediately achieve
\begin{equation}
|\text{Ric}(\nabla f)|^2 \leq \frac{(n-1)\sqrt{2n}}{2n} |\text{Ric}|^2 |\nabla f|^2,
\end{equation}
which settles our claim.

Substituting (3.14) into (3.13) we get
\begin{equation}
0 \geq \frac{n-2}{4(n-1)} \mathcal{V}(M, \partial M, [g]) \left( \int_M \int_M \frac{2}{n-1} |\text{Ric}|^2 \, dv_g \right)^{\frac{n-2}{2}}
\end{equation}
\begin{equation}
- \sqrt{\frac{8}{n-1}} \left( \int_M \left( |W|^2 + \frac{8}{n(n-2)} |\text{Ric}|^2 \right)^{\frac{4}{n}} \, dv_g \right)^{\frac{2}{n}} \left( \int_M \frac{2}{n-1} |\text{Ric}|^2 \, dv_g \right)^{\frac{n-2}{2}}
\end{equation}
\begin{equation}
- \frac{2(n-1)\sqrt{2n}}{2n} \int_M |\text{Ric}|^2 |\nabla f|^2 \, dv_g - \frac{(n-4)}{2(n-1)} \int_M f |\text{Ric}|^2 \, dv_g
\end{equation}
\begin{equation}
- \frac{(n^2 - 4n - 8)}{4n(n-1)} \int_M f^2 |\text{Ric}|^2 \, dv_g.
\end{equation}
\begin{equation}
(3.15)
\end{equation}
In particular, by choosing $n = 4$ in the above expression we achieve
\begin{equation}
0 \geq \left[ \frac{1}{6} \mathcal{V}(M, \partial M, [g]) - \frac{2}{\sqrt{3}} \left( \int_M \left( |W|^2 + |\text{Ric}|^2 \right) \, dv_g \right)^{\frac{4}{3}} \left( \int_M f^4 |\text{Ric}|^4 \, dv_g \right)^{\frac{1}{3}} \right]
\end{equation}
\begin{equation}
- \frac{3\sqrt{2} - 1}{2} \int_M |\text{Ric}|^2 |\nabla f|^2 \, dv_g + \frac{R}{6} \int_M f^2 |\text{Ric}|^2 \, dv_g.
\end{equation}
\begin{equation}
(3.16)
\end{equation}
which gives the stated inequality.

Proceeding, we suppose that \((1.6)\) is actually an equality. Then, it is not difficult to see from \((3.16)\) that

\[
\int_{M} f^{2} |\tilde{Ric}|^{2} dV_{g} = 0.
\]

But, we already know that, choosing appropriate coordinates (e.g. harmonic coordinates), \(f\) and \(g\) are analytic (cf. Theorem 2.8 in [12] or Proposition 2.1 in [13]). Whence, we have \(|\tilde{Ric}|^{2} = 0\) and then \(M^{4}\) is Einstein. Finally, we may invoke Theorem 1.1 in [23] to conclude that \(M^{4}\) is isometric to a geodesic ball in \(S^{4}\).

So, the proof is completed. \(\Box\)

**Acknowledgement.** The authors want to thank A. Barros, E. Barbosa and R. Batista for fruitful conversations about this subject. E. Ribeiro and H. Baltazar were partially supported by grants from CNPq/Brazil.

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