CRITICAL POINTS FOR SURFACE MAPS AND THE BENEDICKS-CARLESON THEOREM

HIROKI TAKAHASI

ABSTRACT. We give an alternative proof of the Benedicks-Carleson theorem on the existence of strange attractors in Hénon-like maps in the plane. To bypass a huge inductive argument, we introduce an induction-free explicit definition of dynamically critical points. The argument is sufficiently general and in particular applies to the case of non-invertible maps as well. It naturally raises the question of an intrinsic characterization of dynamically critical points for dissipative surface maps.

CONTENTS

1. Introduction 3
  1.1. Statement of the result 4
  1.2. Overview of the paper 5
2. Basic estimates and constructions 7
  2.1. Curvature and distortion 7
  2.2. Hyperbolicity and regularity 8
  2.3. Admissible curves 8
  2.4. Mostly contracting directions 9
  2.5. Long stable leaves 9
  2.6. Precritical points 10
  2.7. Creation of new precritical points 10
  2.8. Strong regularity and good precritical points 11
  2.9. Admissible position 11
  2.10. Derivative recovery 11
  2.11. Critical points 12
  2.12. Hyperbolic times 12
3. The dynamics 13
  3.1. Exponential growth condition 13
  3.2. Capture argument 13
  3.3. Controlled orbits 16
  3.4. Proof of Theorem A 16
4. A model problem 18
  4.1. The model 18
  4.2. Critical points 18
  4.3. Smooth continuations 19
  4.4. Structure of the simplest bad parameter sets 19
  4.5. Issues to be addressed 21

Date: February 1, 2008.
1. Introduction

Strange attractors are of fundamental importance in the study of dynamical systems. While they are quite often observed numerically, a theoretical study of them still remains a challenge. The first existence theorem was obtained by Benedicks and Carleson [2], on the Hénon family \((x, y) \rightarrow (1 - ax^2 + y, bx)\) for a positive measure set of parameters close to \((2, 0)\). Mora and Viana [10], Díaz, Rocha, and Viana [8] pushed their argument further and proved the existence of strange attractors in very general bifurcation mechanisms, such as homoclinic tangencies or critical saddle-node cycles. See also Wang and Young [19] for a more geometric treatment which yields advanced properties of the attractor.

A breakthrough in this direction had taken place before in the context of the quadratic family \(f_a: x \rightarrow 1 - ax^2\). With a careful control of the recurrence of the critical point \(x = 0\), Jakobson [9] constructed a positive measure set of parameters such that the corresponding maps admit absolutely continuous invariant probability measures. See Collet and Eckmann [7], Benedicks and Carleson [1] taking other approaches.

[2] is a very creative extension of their previous argument in one-dimension [1]. Since the Hénon map is a diffeomorphism, there is no critical point in the usual sense. However, they remarkably invented dynamically critical points for certain Hénon maps, which allowed them to develop a parameter selection argument with some partial resemblance to the one-dimensional case.

In [2] [10] [19], the construction of critical points relies on a huge inductive scheme. To recover the assumption of the induction, parameter selections are made with a careful control of the recurrence of critical points constructed at early stages. As such, the assumption of the induction has to incorporate both phase space dynamics and structures in parameter space relative to the old critical points, and necessarily becomes complicated.

The aim of the present paper is to improve this point by providing an alternative proof of the Benedicks-Carleson theorem. A key ingredient is an induction-free explicit definition of critical points. A strong dissipation and an exponential growth of derivatives along the orbits of critical points together imply the existence of strange attractors with positive Lyapunov exponent (Theorem A). The set of parameters satisfying this growth condition is shown to have positive Lebesgue measure (Theorem B). The definition of critical points is a purely analytic one and makes sense for any smooth dissipative surface maps. It is interesting to ask whether it has any intrinsic meaning. A similar question is addressed and some results are given in [11].

Our argument is sufficiently general and in particular applies to the case of non-invertible maps such that the unstable manifold intersects itself. While no explicit result has been known in this case (see the next paragraph), non-invertible Hénon-like families with singularities naturally appear: e.g. in homoclinic bifurcations of surface maps; in connection with certain reaction-diffusion equations.

A crucial fact used in [2] [10] [19] is that tangent directions of two nearby horizontal pieces of the unstable manifold are nearby as well, for them to avoid intersecting each other. A new difficulty in the non-invertible case is the obvious failure of this property. Meanwhile, the same difficulty appears in dimension higher than two, and Viana [18] dealt with this by taking the closeness of tangent directions as an
independent assumption. Although far from straightforward, this implies that one can deal with the non-invertible case in two-dimension by adapting his argument. See also Remark 2.7.3.

The present paper lays a foundation of further developments, e.g. the basin problem for the case of non-invertible maps with fold singularities. It is a question on the coincidence of the asymptotic distribution of Lebesgue almost every point in the basin of attraction. Based on the present paper we shall give a positive solution to this problem \[14\]. A positive solution to the same problem for invertible case was initially given by Benedicks and Viana \[3\], and then by Wang and Young \[19\], under certain regularity condition on the Jacobian of the map. While this condition has been removed in \[13\], the absence of singularities remains crucial.

1.1. Statement of the result. An Hénon-like family is a continuous two parameter family of not necessarily invertible maps $H_{a,b} : [-2, 2]^2 \to \mathbb{R}^2$ of the form

\[
H_{a,b} : (x, y) \mapsto \left( 1 - ax^2 + bu(a, b, x, y), bvu(a, b, x, y) \right),
\]

where $(a, b)$ is close to $(2, 0)$, and $u, v$ are $C^4$ with respect to $a, x, y$. We assume

\[
\partial_x v(2, 0, 0, 0) \neq 0.
\]

Let $Q$ denote the hyperbolic fixed point which is near $(-1, 0)$. For $b > 0$ small, two straight lines $[-2, 2] \times \{\pm 1/10\}$ cut two curves $S_1$ and $S_2$ in the stable set of $Q$, such that $Q \in S_1$ and $H(S_2) \subset S_1$. Define $D = D_{a,b}$ to be the closed region surrounded by these two lines and two curves. Clearly, $P \in \text{Int} D$ holds. It is easy to see that there exists a closed set $\Omega \subset \mathbb{R}^2$ such that $H_{a,b}(D) \subset D$ for $(a, b) \in \Omega$, and for any open neighborhood $U$ of $(2, 0)$, $\Omega \cap U$ contains an open set. We only consider parameters contained in $\Omega$.

Let $P$ denote the hyperbolic fixed point which is not $Q$. Regardless of whether $H$ is invertible or not, the unstable manifold $W^u(P)$ is obtained as an immersed real line. To bypass its possible self-intersections, define

\[
T_z W^u(P) = \{ v \in T_z \mathbb{R}^2 : \text{there exists a segment in } W^u(P) \text{ which is tangent to } v \}.
\]

The result splits into two theorems. The first one gives a sufficient condition for the existence of strange attractors, in the form of exponential growth condition $(EG)_n$. It is a condition on the growth of orbits of critical points of order $n$. We need to wait until Section 3 to correctly define this.

**Theorem A.** For an Hénon-like family $(H_{a,b})$ there exists $N > 0$ such that if $(a, b)$, $b > 0$ is sufficiently close to $(2, 0)$ and $H = H_{a,b}$ satisfies $(EG)_n$ for all $n \geq N$, then:

(a) there exists a countable set $C \subset W^u(P)$ near $(0, 0)$ such that:

\[
\begin{align*}
(a-i) \quad & \|D^H (H(\zeta)) (0)\| \geq e^\frac{99}{100} \log^2 2^n \text{ for every } \zeta \in C \text{ and } n \geq 1; \\
(a-ii) \quad & \text{for every } \zeta \in C \text{ there exists a unique (up to sign) unit vector } e \in T_{H(\zeta)} W^u(P) \text{ such that } \|D^H (H(\zeta)) e\| \leq (Kb)^n \text{ for every } n \geq 1, \text{ where } K > 0 \text{ is a uniform constant}; \\
(a-iii) \quad & \text{for all } z \in W^u(P) \setminus \bigcup_{n=-\infty}^{\infty} H^n(C) \text{ and } v \in T_z W^u(P),
\end{align*}
\]

\[
\limsup_{n \to +\infty} \frac{1}{n} \log \|D^H (z) v\| \geq \frac{\log 2}{3};
\]
(b) For any periodic point \( p \in [-2, 2]^2 \),
\[
\limsup_{n \to +\infty} \frac{1}{n} \log \|DH^n(p)\| \geq \frac{\log 2}{3}.
\]

The following theorem states that the condition in Theorem A is not empty from a measure theoretical point of view. These two theorems together imply the Benedicks-Carleson theorem.

**Theorem B.** For an Hénon-like family \((H_{a,b})\) and \( b > 0 \) small, there exists a positive measure set \( \Omega_b \) of \( a \)-values near 2 such that \( H = H_{a,b} \) satisfies \((EG)_n\) for all \( n \geq N \) whenever \( a \in \Omega_b \).

Several remarks are in order on the scope of the theorems. The present setting may be considerably extended along the line that is now well-understood. In the definition of the Hénon-like family, one may replace the quadratic family by the so-called transversal family of uni/multimodal maps and keep the conclusion the same. While only the two dimensional case is treated here, the argument may be extended to higher dimensions with additional geometric considerations, as in [18] [20]. We have suppressed these possible extensions for simplicity.

For \( \text{cl}(W^u(P)) \) to deserve the name of attractor, its basin of attraction should have nonempty interior. This is known to be the case when the map is invertible: see [12] Appendix 3. Otherwise the same argument does not hold due to the existence of self-intersections of the unstable manifold. Meanwhile, Benedicks personally communicated to us that he has an argument which holds even if the map is non-invertible.

One can derive some known properties of the attractor under the same assumption on critical points as in Theorem A. For example, developing a large deviation argument in phase space, one can prove that Lebesgue almost every point in \( W^u(P) \) has a dense forward orbit in \( \text{cl}(W^u(P)) \). Adapting [5] [6] to our setting (and perhaps under an weaker condition on critical points), one can prove the existence of physical measures with nice statistical properties.

1.2. **Overview of the paper.** The rest of this paper consists of seven sections and one appendix. Section 2 provides basic estimates and constructions which will be frequently used later. Some are new and some are old, already appearing in [2] [10] [19] in one form or another. Building on some of them we define (pre) critical points (Sect. 2.6, Sect. 2.11). Intuitively, they are points of tangencies between stable and unstable manifolds having regular backward orbits.

One important problem is the analysis of the growth of orbits starting from neighborhoods of critical points. Assuming **strong regularity condition** on critical orbits and **admissible position** (Sect. 2.8), we prove that an exponential growth of derivatives prevails (Proposition 2.10.2). At this point, a precise distortion estimate in Lemma 2.1.2 is crucial in order to faithfully copy the growth of the critical orbit.

In Section 3 we introduce the exponential growth condition \((EG)_n\) on the orbit of critical points of order \( n \). This condition is sufficient to develop a capture argument which systematically assigns suitable critical points (binding points) to every free return. As a by-product we conclude a proof of Theorem A.
Sections from 4 to 7 deal with parameter issues. The goal is the construction of the parameter set in Theorem B. Parameters which satisfy \((EG)_{n-1}\) but not \((EG)_n\) are discarded at step \(n\). We begin by introducing in Section 4 a toy model of the Hénon map, according to [2] Section 3, in order to describe how our parameter exclusion argument unfolds in that much simpler context. This will help grasp the meanings of constructions and arguments that follow later.

The condition \((EG)_n\) is not well-adapted to our inductive scheme. Thus we introduce in Section 6.2 a stronger condition, called \((RR)_n\). Parameters have to satisfy this condition to be selected.

We pay attention to the complement of good parameter sets. This idea has been borrowed from the work of Tsujii [15] [16] in one-dimension. He proved that parameters discarded at step \(n\) are contained in a finite union of well-structured sets the measures of which are quantified through the sum of essential return depths. We show that essentially the same thing prevails in two-dimension. In doing this, two issues intrinsic to two-dimension need to be considered and remedies are made accordingly, as explained in the next two paragraphs.

Critical points disappear when parameters are varied. Hence we are obliged to work with quasi critical points (Sect. 5.1) rather than critical points itself. Proposition 5.3.1 asserts the existence of smooth continuations of quasi critical points in a sufficiently large interval. This sets the stage for considering the dynamics of critical curves, in Section 7. Under the assumption of \((RR)_{n-1}\), we manage to recover three consequences which are known to hold in one-dimension: good distortion and curvature estimates (Proposition 7.2.2); a large amount of expansion in parameter space at essential returns (Proposition 7.3.1); existence of binding points for critical curves (Proposition 7.4.1).

There are uncountably many critical points. Nevertheless, the total number of combinatorially equivalent classes of critical points needed to be considered at step \(n\) is finite and not too large. Here, we regard two distinct critical points as combinatorially equivalent, if their backward and forward orbits are characterized by the same set of discrete data, called sample points (Sect. 5.3), essential return times (Sect. 6.1), maximal hyperbolic times (Sect. 7.2), essential return depths (Sect. 7.3). Each equivalence class makes multiple holes in good parameter sets. A crucial point is to show that these holes constitute a set which is well-structured in the above sense (Lemma 8.2.2, Lemma 8.2.4, Lemma 8.2.5). It then follows that the measure of parameters discarded at step \(n\) is smaller than the total number of equivalence classes multiplied by some exponentially small number in \(n\). Consequently, a positive measure set is left over (Proposition 8.1.2).

Fairly long and computational proofs are postponed to Appendix to ensure an easy access of readers to the heart of the argument.

I am grateful to Masato Tsujii for having brought this problem to my attention. I have to say his notes [17] is very important for the existence of this paper. Most of this work has been done while I was at Instituto de Matemática Pura e Aplicada, Rio de Janeiro, Brazil. Above all, I am grateful to Vítor Araújo, Samuel Senti,

\[\text{8.4.1}\]

1 The orbits of two combinatorially equivalent critical points may get apart, namely, they are not analytically equivalent in general. This is why each equivalence class makes multiple holes. See Sect. 5.4.
2. Basic estimates and constructions

This section is devoted to basic estimates and constructions which will be frequently used later. To begin with, we introduce absolute constants which are definitely fixed throughout the argument. They are

\[ \Delta \gtrsim 2 \log 2, \sigma = 100, \ell \lesssim 1/2, \lambda \gtrsim \log 2. \]

In particular, the norms of all the partial derivatives of \((a, z) \rightarrow H_a(z)\) are bounded by \(e^\Delta\). Other constants entirely determined by the family \((H_{a,b})\) are mostly denoted by \(K\). Keep in mind that the values of \(K\) are different in different places. We reserve \(K_0, K_1\) for special uses as follows:
- \(K_0\) concerns hyperbolic behaviors away from the critical region (Lemma 2.2.1);
- \(K_1\) determines the angle of vertical cones in which the mostly contracting directions reside (Lemma 2.4.4).

We introduce system constants which are allowed to change, provided that a finite number of relations are satisfied. They are \(\alpha, M, \delta, \theta, b\), chosen in this order. We have \(\alpha, \delta, \theta, b \ll 1\) and \(M, \beta \gg 1\). A smaller \(\Omega\) is needed as \(\beta\) gets bigger.

We use the following notation:
\[ A_i = H^i(A) \text{ for a set } A \subset D \text{ and } i \geq 0. \]

A sequence of nonzero tangent vectors \(\{v_i(z_i)\}_{i=0}^n\) such that \(v_i(z_i) = DH^i(z_0)v_0(z_0)\) is called a vector orbit of \(H\).

2.1. Curvature and distortion.

Lemma 2.1.1. Let \(v = \{v_i(z_i)\}_{i=0}^n\) be a vector orbit, and \(\gamma_0 \subset D\) a \(C^2\) curve which is tangent to \(v_0(z_0)\). Let \(\kappa_j(z_j)\) denote the curvature of \(\gamma_j\) at \(z_j\). Then for \(1 \leq j \leq n\),

\[ \kappa_j(z_j) \leq (Kb)^j \frac{||v_0||^3}{||v_j||^3} \kappa_0(z_0) + \sum_{\ell=1}^j (Kb)^j \frac{||v_{j-\ell}||^3}{||v_j||^3}. \]

We say \(v\) is \(\kappa\)-expanding, or simply expanding, if there exists \(\kappa \geq b^{1/4}\) such that

\[ ||v_i|| \geq \kappa^i ||v_0|| \text{ for every } 1 \leq i \leq n. \]

We say \(v\) is \(\kappa\)-expanding, or simply expanding, if there exists \(\kappa \geq b^{1/4}\) such that

\[ ||v_i|| \geq \kappa^i ||v_0|| \text{ for every } 1 \leq i \leq n. \]

Choose a large integer \(M > 0\) such that \(ne^{-\alpha \sigma n} \leq 1/2\) for every \(n \geq M\). For a \(C^1\) curve \(\gamma_0\) and \(z_0 \in \gamma_0\), let \(t_{\gamma_0}(z_0)\) denote the unit vector tangent at \(z_0\) to \(\gamma_0\).
Corollary 2.2.2. Let \( n \geq M \) and suppose that \( v = \{v_i(z_i)\}_{i=0}^n \) is expanding. Let \( \gamma_0 \subset D \) be a \( C^2 \) curve which is tangent to \( v_0(z_0) \), length\( (\gamma_0) \leq \Xi(v) \), and curvature \( \leq 1 \) everywhere. For every \( 1 \leq i \leq n \) and all \( y_0 \in \gamma_0 \),
\[
\text{Log} \left| \frac{DH^t(z_0)v_t(z_0)}{DH^t(y_0)v_t(y_0)} \right| \leq \frac{1}{2},
\]

See Appendix for the proof.

2.2. Hyperbolicity and regularity. The following lemma ensures certain amount of hyperbolicity outside of the critical region \( C_\delta = (-\delta, \delta) \times [-1/10, 1/10] \).

Lemma 2.2.1. There exists \( K_0 \leq 1 \) such that for all \( \lambda \leq \log 2, \alpha, \delta > 0 \), the following holds for \( H = H_{a,b} \) with \((a,b)\) close to \((2,0)\) and \( \lambda = \lambda - \alpha > 0 \): let \( \{v_i(z_i)\}_{i=0}^n \), \( n \geq 1 \) be a vector orbit of \( H \) such that slope\( (v_0) \leq K_0b \).

(a) If \( z_0, z_1, \cdots, z_n \not\in C_\delta \), then slope\( (v_i) \leq K_0b \) and \( \|v_j\| \geq K_0e^{\lambda(n-i)}\|v_i\| \) for \( 0 \leq i \leq n \);
(b) If moreover \( |z_n| \geq 2|z_0| \), then \( \|v_n\| \geq K_0e^{\lambda n}\|v_0\| \);
(c) If \( n \geq 2 \) and \( \|v_n\| \geq e^{-2K_0\delta}\|v_i\| \) for \( i = n-1, n-2 \), then slope\( (v_n) \leq K_0b \).

Proof. We only give a proof of (c) because the rest is well-known. Suppose that \( z_{n-1} \not\in C_{\delta^2} \). Then \( \|DH^2(z_{n-2})\| \leq K_0\delta^2 \), and thus \( \|v_n\| \leq K\delta^2\|v_{n-2}\| \). This yields a contradiction. Hence \( z_{n-2} \not\in C_{\delta^2} \) holds. By the same reasoning we obtain \( z_{n-1} \not\in C_{\delta^2} \).

Suppose that slope\( (v_{n-2}) \geq \delta^{-1} \). Then we have \( \|v_{n-1}\| \leq K\delta^{-1}\|v_{n-2}\| \), and thus \( \|v_n\| \leq K\delta^{-2}\|v_{n-2}\| \). This yields a contradiction. Hence slope\( (v_{n-2}) \leq \delta^{-1} \) holds.

Then we have slope\( (v_{n-1}) \leq K_0K\delta^{-3}b \) and slope\( (v_n) \leq K_0K\delta^{-3}b^2 \leq K_0b \).

A vector orbit \( \{v_i(z_i)\}_{i=0}^n \) is called \( r \)-regular \( (r > 0) \) if
\[
\|v_n\| \geq K_0r\delta\|v_i\| \quad \text{for} \quad 0 \leq i \leq m.
\]
It is easy to see that the following holds.

Corollary 2.2.2. Let \( r \geq e^{-2} \), \( n \geq 2 \), and suppose that \( \{v_i(z_i)\}_{i=0}^n \) is an \( r \)-regular vector orbit of \( H \) as in Lemma 2.2.1. Then \( \{v_i(z_i)\}_{i=0}^m \) is \( r \)-regular, where \( m = \min\{i \geq n : z_i \in C_\delta\} \).

2.3. Admissible curves. A \( C^2 \) curve \( \gamma_0 \) is called admissible if:

(A1) slope\( (t_{\gamma_0}(z_0)) \leq K_0b \) for all \( z_0 \in \gamma_0 \);
(A2) the curvature is \( \leq 1 \) everywhere on \( \gamma_0 \).

Lemma 2.3.1. Let \( n \geq M \). Suppose that a vector orbit \( v = \{v_i(z_i)\}_{i=0}^n \) is \( \kappa \)-expanding and \( e^{-4} \)-regular. Let \( \gamma_0 \) be a \( C^2 \) curve which is tangent to \( v_0(z_0) \), length\( (\gamma_0) = \Xi(v) \), curvature \( \leq 1 \) everywhere. Then \( \gamma_n \) is an admissible curve and
\[
\text{length}(\gamma_n) \geq e^{-3\Delta n}\kappa^{3n}.
\]

Proof. By Lemma 2.1.2 we have
\[
\text{length}(\gamma_n) \geq e^{-\alpha\sigma n^{-1/2}}\|v_n\| \cdot \Xi(v) \geq e^{-2\sigma n}\|v_n\| \cdot \min_{0 \leq i \leq n} \|v_i\| \cdot \min_{1 \leq j \leq n} \|v_j\|^2.
\]

Using \( \kappa^i\|v_0\| \leq \|v_i\| \leq e^{\Delta i}\|v_0\| \) for \( 0 \leq i \leq n \) we obtain the lower estimate of the length. (A1) follows from (c) in Lemma 2.2.1 (A2) follows from Lemma 2.1.1 and the regularity of \( v \).
2.4. Mostly contracting directions. Let $M$ be a $2 \times 2$ matrix. Denote by $e(M)$ the unit vector (up to sign) such that $\|Me(M)\| \leq \|Mu\|$ holds for any unit vector $u$. We call $e(M)$, when it exists, the mostly contracting direction of $M$. We analogously define the unit vector $f(M)$ which is mostly expanded by $M$. Clearly $Me(M) \perp Mf(M)$, and moreover $e(M) \perp f(M)$ holds.\footnote{Proof: consider the dual $M^*$. Then $e(M^*)$, $f(M^*)$ is well-defined and $M^*e(M^*) \perp M^*f(M^*)$. Since $Me(M) \in \ker f(M^*)$ and $Mf(M) \in \ker e(M^*)$ we have $M^*Me(M) \in \ker M^*f(M^*)$ and $M^*Mf(M) \in \ker M^*e(M^*)$. This implies $e(M) \perp f(M)$.}

For a sequence of matrices $M_1, M_2 \cdots$, we use $M^{(i)}$ to denote the matrix product $M_1 \cdots M_2 M_1$, and $e_i$ to denote the mostly contracting direction of $M^{(i)}$. We assume $|\det M_i| \leq K_0 b$ and $\|M_i\| \leq e^A$. We quote some results in \cite{19} without proofs.

Lemma 2.4.1. (\cite{19} Lemma 2.1) Let $i \geq 2$, and suppose that $\|M^{(i)}\| \geq \kappa_i$ and $\|M^{(i-1)}\| \geq \kappa_i^2$ for some $\kappa \geq b^{1/4}$. Then $e_i$ and $e_{i-1}$ are well-defined, and satisfy

$$\|e_i \times e_{i-1}\| \leq \left( \frac{Kb}{\kappa^2} \right)^{i-1}.$$  

Corollary 2.4.2. (\cite{19} Corollary 2.1) If $\|M^{(i)}\| \geq \kappa_i$ for $1 \leq i \leq n$, then

(a) $\|e_n - e_1\| \leq \kappa^{-1} K b$;

(b) $\|M^{(i)} e_n\| \leq (K b)^i$ holds for $1 \leq i \leq n$.

Next we consider parametrized matrices $M_i(s_1, s_2, s_3)$ such that $\|\partial M_i(s_1, s_2, s_3)\| \leq e^A$ and $|\det M_i(s_1, s_2, s_3)| \leq e^A$, where $\partial$ denotes any first order partial derivatives.

Corollary 2.4.3. (\cite{19} Corollary 2.2) Suppose that $\|M^{(i)}(s_1, s_2, s_3)\| \geq \kappa_i$ for $1 \leq i \leq n$. Then for $2 \leq i \leq n$,

$$|\partial (e_i \times e_{i-1})| \leq \left( \frac{Kb}{\kappa^3} \right)^{i-1}.$$  

For $z \in D$ and $n \geq 1$, define $e_n(z) = e(DH^n(z))$ when it makes sense.

Lemma 2.4.4. There exists $K_1$ such that if $z = (x, y) \notin C_d$ then $e_1(z)$ is well-defined and

$$\text{slope}(e_1(z)) \geq K_1^{-1} |x|^2 b^{-1} \text{ and } \|\partial e_1(z)\| \leq K_1 |x|^{-2}.$$  

If moreover $\|DH^i(z)\| \geq \kappa_i$ for $1 \leq i \leq n$ then the same estimates hold for $e_n$.

See Appendix for the proof.

2.5. Long stable leaves. A long stable leaf of order $k$ is an integral curve of $e_k$ having the form

$$\Gamma = \{(x(y), y) \in D: |y| \leq 1/10, |\dot{x}(y)| \leq K_1 \delta^{-2} b, |\dot{y}(y)| \leq K_1 \delta^{-2}.$$  

For a long stable leaf $\Gamma$ and $r > 0$, define a strip

$$\Gamma(r) = \{(x, y) \in D: |x - x(y)| \leq r\}.$$  

The following proposition asserts the existence of long stable leaves around expanding orbits. While similar constructions have already appeared in \cite{2} \cite{10} \cite{19}, we work with the distortion estimate in Lemma 2.4.1 rather than the so-called matrix perturbation Lemma (\cite{2} Lemma 5.5). This yields a better estimate on the width.
of the strip which plays a crucial role in Proposition 2.10.2. See Appendix for the proof.

**Proposition 2.5.1.** Let \( n \geq M, z_0 \notin C_d \), and define \( w = \{w_i(z_i)\}_{i=0}^n \) by \( w_i = DH^i(z_0) (\frac{1}{\beta}) \). If \( w \) is expanding, then for \( 1 \leq k \leq n - 1 \):

(a) the maximal integral curve \( \Gamma^{(k)} \) of \( e_k \) through \( z_0 \) is a long stable leaf;

(b) For all \( z_i^{(k)}(\Pi_0^{\max(M,k+1)} w \) and \( 1 \leq i \leq k + 1 \),

\[
\log \left| \frac{DH^i(z_0) (\frac{1}{\beta})}{DH^i(z_0^{(k)}) (\frac{1}{\beta})} \right| \leq 1.
\]

In particular, \( e_1, e_2, \ldots, e_{k+1} \) are well-defined on \( \Gamma^{(k)}(\Pi_0^{\max(M,k+1)} w \).

(c) If \( z_0 \in H(C_d) \), then the curvature of the stable leaves are \( \leq 2K_1 \).

2.6. Precritical points. Suppose that \( \gamma_0 \) is an admissible curve in \( C_d \). We say \( \zeta_0 \in \gamma_0 \) is a precritical point of order \( n \) on \( \gamma_0 \), if:

(P1) \( \|DH^i(\zeta_0)\| \geq e^{-1} \) for every \( 1 \leq i \leq n \);

(P2) \( e_\alpha(\zeta_1) \) is tangent to \( DH(\zeta_0) t_{\gamma_0}(\zeta_0) \).

**Remark 2.6.1.** By Lemma 2.4.1 and Lemma 2.4.4, we have

\[
\text{slope}(DH_{\gamma_0}(\zeta_0)) \geq K_1^{-1}b^{-1}.
\]

This implies that all precritical points are contained in a small neighborhood of the origin, for sufficiently small \( b \).

**Remark 2.6.2.** Every admissible curve admits no more than two precritical points of the same order. This follows from (c) in Proposition 2.5.1 and the fact that two distinct long stable leaves do not intersect each other, according to the uniqueness of solutions in ordinary differential equations.

2.7. Creation of new precritical points. The following two lemmas the proofs of which are given in Appendix are used to create new precritical points around the existing ones. For related discussions, see: [2] p.113, Lemma 6.1; [10] sect.7A, 7B; [19] Lemma 2.10, 2.11. Our proof is a slight adaptation of the them. Here, all admissible curves are assumed to be parametrized by arc length.

**Lemma 2.7.1.** Let \( \gamma_0 \) be an admissible curve in \( C_d \), where \( \gamma_0(0) = \zeta_0 \) is a precritical point of order \( m \). Let \( \varepsilon \in [Kb, e^{-10\beta}] \), and suppose that \( \gamma_0(s) \) is defined for \( s \in [-\varepsilon^{m/2}, \varepsilon^{m/2}] \). Suppose that there exists \( j \in [\beta^{-1}m, \beta m] \) such that \( \|DH^i(\zeta_1)\| \geq 1 \) holds for every \( 1 \leq i \leq j \). Then there exists a precritical point \( \tilde{\zeta}_0 \) of order \( j \) on \( \gamma_0 \) such that \( |\zeta_0 - \tilde{\zeta}_0| \leq \varepsilon^{m/2} \).

**Lemma 2.7.2.** There exists an integer \( m_0 \) depending only on (\( H_{a,b} \)) such that the following holds: let \( \gamma \) and \( \tilde{\gamma} \) be two admissible curves in \( C_d \) such that:

(i) \( \tilde{\gamma}(s) \) are defined for \( s \in [-\varepsilon^{m/2}, \varepsilon^{m/2}] \), where \( \varepsilon \in (0, e^{-10\Delta}] \);

(ii) \( \gamma(0) \) is a precritical point of order \( m \) and \( \|DH^i(\gamma(0))\| \geq e \) for \( 1 \leq i \leq m \);

(iii) the \( x \)-coordinates of \( \gamma(0) \) and \( \tilde{\gamma}(0) \) coincide;

(iv) \( |\gamma(0) - \tilde{\gamma}(0)| \leq \min(Kb, e^m) \) and angle(\( \gamma(0), \tilde{\gamma}(0) \) \( \leq \varepsilon^{m} \).

Then there exists \( s_0 \in [-\varepsilon^{m/2}, \varepsilon^{m/2}] \) such that \( \tilde{\gamma}(s_0) \) is a precritical point of order \( m \).
Remark 2.7.3. In [2] [10] [19], γ and  are assumed to be disjoint, which is crucial. The smallness of angle(γ(0), ̇) automatically follows from this, for them to avoid intersecting each other. In the present context, we need to allow γ to intersect , and thus the smallness of the angle needs to be taken as an independent assumption as in (iv).

2.8. Strong regularity and good precritical points. Let ζ0 be a precritical point of order n ≥ M on an admissible curve γ0. A vector orbit w = {wi(ζ0)j}j=0 defined by w0 = DHi(ζ1)j(1) is called a forward vector orbit of ζ0. We say w is strongly regular if:

(S1) ||wj|| ≥ e(λ−α)(j−i)−ασi||wi|| 0 ≤ ∀i ≤ ∀j ≤ βn;

(S2) for every k ∈ [0, βn] there exists χ(k) ∈ [(1 − ασ)k, k] such that Π0χ(k)w is 1-regular.

We say ζ0 is good if w is strongly regular.

Remark 2.8.1. (S1) is not sufficient for our purpose because it does not care the slope of tangent vectors. (S2) and (c) in Lemma 2.2.1 imply slope(vχ(k)) ≤ K0b.

Remark 2.8.2. By Remark 2.6.1 and f2(0) = −1 = f(−1), it follows that for an arbitrarily large integer N, one may assume that all precritical points of order ≤ N are good, shrinking Ω close to (2, 0) if necessary.

2.9. Admissible position. Suppose that ζ0 is a good precritical point of order n ≥ M on an admissible curve γ0. A nonzero vector v0(z0) is in admissible position relative to ζ0 if:

(AP1) v0(z0) is tangent to γ0;

(AP2) ||wβn|| ≤ |ζ0 − z0| ≤ (L−1Ξ(w))1/2, where L = |f2′(0)| = 4.

We say v0(z0) is in critical position relative to ζ0 if

(CP) |ζ0 − z0| ≤ ||wβn||1/2.

We say v0(z0) is related to ζ0 if it is either in critical position or in admissible position relative to ζ0. The definition of admissible position makes sense by the next

Lemma 2.9.1. For the above w, we have

Ξ(w) · ||wβn||2−2ℓ ≥ e(1−2ℓ)λβn/2.

Proof. Fix i ∈ [0, βn]. Using (S1) we have ||wj|| ≥ e−ασβn||wi|| for i ≤ j ≤ βn. This implies Θ(w, i)||wβn|| ≥ e−3ασβn, and thus Θ(w, i)||wβn||2−2ℓ ≥ e(1−2ℓ)λβn/2. Since i ∈ [0, βn] is arbitrary we obtain the desired inequality.

2.10. Derivative recovery. Define

\[ p = \left[ \frac{(1 - ℓ)\beta Δn}{-\log \sqrt{b}} \right] + 1, \]

and

\[ q = \chi(\beta n), \]

where [·] is the Gauss symbol. We call p the folding period, and q the binding period.
Remark 2.10.1. The binding period is the time of duration in which the orbit of the point in admissible position shadows the critical orbit in a sufficiently regular way. During this period one can compare the growth of these two orbits in light of Lemma 2.12. The folding period is a moment at which the corresponding two vectors become sufficiently parallel to each other.

Proposition 2.10.2. Suppose that a nonzero vector $v_0(z_0)$ is in admissible position relative to a good precritical point $\zeta_0$ of order $n \geq M$. Then:

(a) $\|v_i\| \leq \|v_0\| e^{-\beta i} \leq \|v_0\| e^{-\beta i} \leq \|v_0\| e^{-\beta i} \leq \|v_0\| e^{-\beta i}$;  
(b) $L|\zeta_0 - z_0|^{1+\alpha} \|v_0\| \leq \|v_p\| \leq L|\zeta_0 - z_0|^{1-\alpha} \|v_0\|$, where $\alpha$ is a constant which can be made arbitrarily small by choosing small $b$;
(c) $\|v_{q+1}\| \geq \|v_0\| e^{\frac{3\alpha}{1+\alpha}} (q+1)$;
(d) $\log |\zeta_0 - z_0|^{-\frac{3}{2}-\frac{3}{2}} \leq q \leq \log |\zeta_0 - z_0|^{-\frac{3}{2}}$;
(e) $\|v_0\| |\zeta_0 - z_0|^{-1+3(1-2\ell)} \leq \|v_{q+1}\| \leq \|v_0\| |\zeta_0 - z_0|^{-1-\alpha + \frac{3\alpha}{1+\alpha}}$;
(f) $|\zeta_i - z_i| \leq e^{-\alpha q/2} \leq \|v_{q+1}\| \leq \|v_0\| |\zeta_0 - z_0|^{-1-\alpha + \frac{3\alpha}{1+\alpha}}$;
(g) $\|v_{q+1}\| \geq e^{-1} K_0 \delta \|v_i\| \geq e^{-1} K_0 \delta \|v_i\| \geq 0 \leq \forall i \leq q + 1$;
(h) $\frac{\|v_j\|}{\|v_i\|} \geq \left( \frac{\|v_p\|}{\|v_0\|} \right)^{1+\frac{3\alpha}{1+\alpha}} \geq \left( \frac{\|v_p\|}{\|v_0\|} \right)^{1+\frac{3\alpha}{1+\alpha}} \geq 0 \leq \forall i \leq j \leq q + 1$.

See Appendix for the proof.

2.11. Critical points. Put

$$N = -\Delta^{-1} \log \delta.$$  

We say a precritical point $\zeta_0$ of order $n \geq N$ on an admissible curve $\gamma$ is a critical point of order $n$, if:

(C1) $\|DH_i(H(\zeta_0))\| \geq 1$ for every $1 \leq i \leq n$;
(C2) there exists an $e^{-2}$-regular and $e^{-10\Delta}$-expanding orbit $\{w_i(\zeta_i)\}_{i=-n}^0 \subset D$ such that $\zeta_n \notin \mathcal{C}_\delta$ and $w_0(z_0)$ is tangent to $\gamma_0$ at $\zeta_0$.

Remark 2.11.1. (C1) is slightly stronger than (P1).

Remark 2.11.2. Critical points of order $n$ are contained in $D_n$. In other words, critical points of higher order dig deeper inside.

Remark 2.11.3. Considering uncountably many critical points is not essential. Instead, one may request that critical points are contained in $W^u(P)$. However, the proof gets slightly more complicated.

Remark 2.11.4. (C2) implies that the long stable leaf of order $n$ through $\zeta_n$ is well-defined. It intersects the boundary of $D$, and thus $\zeta_0$ is approximated by $\partial D_n$. This fact strongly suggests that our argument is based on the following informal principle as in [19]: use $\partial D_n (n = 0, 1, \cdots)$ as guidewires to control everything.

2.12. Hyperbolic times. Let $v = \{v_i(z_i)\}_{i=0}^m$ be a vector orbit. An integer $h \in [0, m]$ is called a hyperbolic time if:

(H1) $z_{m-h} \notin \mathcal{C}_\delta$;
(H2) $\Pi_{m-h}^m v$ is $e^{-10\Delta}$-expanding.
The next lemma asserts the existence of plenty of hyperbolic times in regular orbits which are nicely distributed. See [2] Lemma 6.6, [10] Lemma 9.1, [19] Claim 5.1 for related discussions. See Appendix for the proof.

**Lemma 2.12.1.** Let $m \geq N$ and suppose that $v = \{v_i(z_i)\}_{i=0}^m$ is $e^{-3}$-regular. Then there exists a sequence of hyperbolic times $h_1 < h_2 < \cdots < h_s$ such that:

(a) $\Pi_{n-h_i}^n v$ is $e^{-9\Delta}$-expanding;
(b) $h_{i+1}/16 \leq h_i \leq h_{i+1}/4$ for $1 \leq i \leq s - 1$;
(c) $[m/2] - 1 \leq h_s$.

Let $\zeta_0$ be a critical point of order $n$, and suppose that $h_1 < h_2 < \cdots < h_s$ is a sequence of hyperbolic times which is obtained by applying Lemma 2.12.1 to the backward orbit of $\zeta_0$. We define a sequence of hyperbolic times associated to $\zeta_0$ by \{h_1 < h_2 < \cdots < h_s \leq n\}. In other words, we add $n$ to the sequence unless $h_s = n$. This yields no contradiction because of (C2).

3. The dynamics

In this section we introduce the condition $(EG)_n$ in Theorem A. Assuming this we develop an argument to find a suitable precritical points to which Proposition 2.10.2 applies. Consequently we obtain a proof of Theorem A.

3.1. Exponential growth condition. Let $n \geq N$. We say $H$ satisfies $(EG)_n$ if any critical point of order $\leq n$ on any admissible curve is good.

3.2. Capture argument. The following proposition asserts that one can associate suitable critical points (binding points) to all $e^{-1}$-regular orbits which fall inside $C_\delta$.

**Proposition 3.2.1.** Suppose that $H$ satisfies $(EG)_n$ for some $n \geq N$. Let $\{v_i(z_i)\}_{i=0}^m$ be a $e^{-1}$-regular vector orbit of $H$ such that $m \geq N$ and $z_m \in C_\delta$. Let $\{h_i\}_{i=1}^s$ denote the sequence of hyperbolic times associated with $\{v_i(z_i)\}_{i=0}^m$ given by Lemma 2.12.1. Let $i_0$ denote the largest integer such that $h_{i_0} \leq n$. Then one of the following occurs:

(a) there exists a good precritical point of order $\leq h_{i_0}$ relative to which $v_m(z_m)$ is in admissible position;
(b) there exists a good critical point of order $h_{i_0}$ relative to which $v_m(z_m)$ is in critical position.

In the case of (a) $\{v_i(z_i)\}_{i=0}^{m+q+1}$ is $e^{-1}$-regular, where $q$ is the binding period.

**Remark 3.2.2.** It is important that no relation between $m$ and $n$ is assumed. In particular $m$ is allowed to be larger than $n$.

**Proof of Proposition 3.2.1.** We fix some notation. For a nonzero vector $v(z)$ and $r > 0$, let $\gamma(v(z), r)$ denote the straight line of length $r$ which is centered at $z$ and tangent to $v(z)$. Put $\rho = e^{-50\Delta}$. Put $\gamma^{(i)} = H^{h_i}(\gamma(v_{m-h_i}, \rho^{h_i}))$ for $1 \leq i \leq s$. Since $h_i$ is a hyperbolic time we have $\rho^{h_i} \leq \Xi(\{v_j\}_{j=m-h_i})$. By Lemma 2.3.1 $\gamma^{(i)}$ is an admissible curve with length $\geq \rho^{2h_i}$. In particular it makes sense to speak about the existence of precritical points on $\gamma^{(i)}$.

**Lemma 3.2.3.** Let $i \leq i_0 - 1$, and suppose that there exists a good critical point of order $h_i$ on $\gamma^{(i)}$ relative to which $v_m(z_m)$ is in critical position. Then one of the following occurs:
(a) there exists a good precritical point of order \( \in [h_i + 1, h_i + 1] \) on \( \gamma^{(i+1)} \) relative to which \( v_m(z_m) \) is in admissible position;
(b) there exists a good critical point of order \( h_{i+1} \) on \( \gamma^{(i+1)} \) relative to which \( v_m(z_m) \) is in critical position.

Proof. Let \( \zeta^{(h_i,i)}_0 \) denote the good critical point of order \( h_i \) on \( \gamma^{(i)} \) relative to which \( v_m \) is in critical position. Take \( z \in \gamma^{(i+1)} \) whose \( x \)-coordinate coincides with that of \( \zeta^{(h_i,i)}_0 \). Such \( z \) uniquely exists because of the lower bound on the length of \( \gamma^{(i+1)} \) and the assumption that \( v_m(z_m) \) is in critical position relative to \( \zeta^{(h_i,i)}_0 \). Let \( w = \{ w_i \}_{i=0}^{\beta h_i} \) denote the forward vector orbit of \( \zeta^{(h_i,i)}_0 \).

Sublemma 3.2.4. We have:

- \(|\zeta^{(h_i,i)}_0 - \tilde{z}| \leq K ||w_{\beta h_i}||^{2\ell-2};
- \( \angle(t_{\gamma^{(i)}} \zeta^{(h_i,i)}_0, t_{\gamma^{(i+1)}}(\tilde{z})) \leq K ||w_{\beta h_i}||^{2\ell-2}.\)

Proof. Parametrize \( \gamma^{(i)} \) and \( \gamma^{(i+1)} \) by arc length so that \( \gamma^{(i)}(0) = z_m = \gamma^{(i+1)}(0) \) and the \( x \)-components of the derivatives have the same sign. Then

\[
|\gamma^{(i)}(s) - \gamma^{(i+1)}(s)| \leq K \int_0^s \|\dot{\gamma}^{(i)}(t) - \dot{\gamma}^{(i+1)}(t)\| dt.
\]

Since \( \gamma^{(i)} \) and \( \gamma^{(i+1)} \) are admissible curves which are tangent to \( v_m(z_m) \), we have \( \dot{\gamma}^{(i)}(0) = \dot{\gamma}^{(i+1)}(0) \) and \( \|\dot{\gamma}^{(i)}(0)\|, \|\dot{\gamma}^{(i+1)}(0)\| \leq 1. \) Thus

\[
\int_0^s \|\dot{\gamma}^{(i)}(t) - \dot{\gamma}^{(i+1)}(t)\| dt \leq K \int_0^s t dt \leq Ks^2.
\]

This implies (a). (b) follows from the bound on the curvatures of \( \gamma^{(i)} \) and \( \gamma^{(i+1)}. \)

Since \( \beta > 1, \gamma^{(i)} \) (resp. \( \gamma^{(i+1)} \)) contains a curve of length \( > \|w_{\beta h_i}\|^{\ell-1} \) centered at \( \zeta^{(h_i,i)}_0 \) (resp. \( \tilde{z} \)). By Sublemma 3.2.4 and Lemma 2.7.2 there exists a precritical point of order \( h_i \) on \( \gamma^{(i+1)} \), called \( \zeta^{(h_i,i+1)}_0 \), such that \( |z - \zeta^{(h_i,i+1)}_0| \leq K \|w_{\beta h_i}\|^{\ell-1}. \)

Sublemma 3.2.5. For every \( 1 \leq k \leq \beta h_i, \)

\[
\left| \log \frac{\|DH^k(\zeta^{(h_i,i+1)}_0)(\frac{1}{0})\|}{\|DH^k(\zeta^{(h_i,i)}_0)(\frac{1}{0})\|} \right| \leq 1.
\]

Proof. (a) in Sublemma 3.2.4 gives \( |\zeta^{(h_i,i)}_0 - \zeta^{(h_i,i+1)}_0| \leq K \|w_{\beta h_i}\|^{\ell-1}. \) Using Lemma 8.11.1 and Lemma 2.9.1 we obtain \( \zeta^{(h_i,i+1)}_0 \in \Gamma(\beta h_i - 1)(\Xi(w)). \) Hence the inequality follows.

For every \( k \in [h_i + 1, h_i + 1], \) Lemma 2.7.1 yields a precritical point of order \( k \) on \( \gamma^{(i+1)} \), called \( \zeta^{(k,i+1)}_0 \). In fact, \( \zeta^{(h_i,i+1)}_0 \) is a good critical point of order \( h_{i+1} \), because of \( (EG)_n, h_{i+1} \leq n \), and the fact that there exists a \( e^{-2} \)-regular backward orbit of length \( h_{i+1} \), by Lemma 2.1.2. Hence all \( \zeta^{(k,i+1)}_0 \) is a good precritical point for every \( h_i + 1 \leq k \leq h_{i+1} - 1. \)
Sublemma 3.2.6. Suppose that $\zeta_0, \zeta'_0$ are good precritical points of order $m$ and $m+1$ on an admissible curve $\gamma_0$ such that $|\zeta_0 - \zeta'_0| \leq (Kb)^{m/2}$. Let $w = \{w_i(\zeta_{i+1})\}_{i=0}^{\beta m}$, $w' = \{v'_i(\zeta'_{i+1})\}_{i=0}^{\beta(m+1)}$ denote the respective forward vector orbits. Then

$$\Xi(w') \cdot \|w_{\beta m}\|^{2-2\ell} \geq e^{(1-2\ell)\lambda \beta m/2}. $$

Proof. Using $|\log \|w'_{\beta m}\| - \log \|w_{\beta m}\|| \leq 1$ by (b) in Proposition 2.5.1 and the strong regularity of $w'$, for every $0 \leq i \leq \beta m$ we have

$$\|w'_i\| \leq e^{\|w_{\beta m}\|/\|w'_{\beta m}\|} \|w'_{\beta m}\| \leq e^{\beta \|w_{\beta m}\|/\|w'_{\beta m}\|}. $$

Meanwhile, for $\beta m \leq i \leq \beta(m + 1)$ we have $\|w'_i\| \leq e^{\beta \Lambda} \|w'_{\beta m}\|$, and thus $\|w'_i\| \leq e^{\beta \Lambda + 1} \|w'_{\beta m}\|$. Using these and $\|w'_i\| \geq e^{-\alpha \beta(m+1)} \|w'_{\beta m}\|$ for $0 \leq i \leq j \leq \beta(m + 1)$ we obtain $\Xi(w') \geq e^{-4 \alpha \beta m} \|w_{\beta m}\|^{-1}$. This implies the desired inequality. \qed

Sublemma 3.2.6 implies that $v_m(z_m)$ is related to $\zeta_0^{(h_i, i+1)}$. Suppose that $v_m(z_m)$ is in critical position relative to $\zeta_0^{(h_i, i+1)}$. In this case, it follows from Sublemma 3.2.6 that $v_m(z_m)$ is related to $\zeta_0^{(h_i+1, i+1)}$. If $v_m(z_m)$ is in admissible position relative to $\zeta_0^{(h_i+1, i+1)}$, then it is done. Otherwise, we again use Sublemma 3.2.6 and repeat the same argument. Eventually, only two possibilities are left: there exists $k \in [h_i + 1, h_{i+1}]$ such that $v_m(z_m)$ is in admissible position relative to $\zeta_0^{(k, i+1)}$, or else $v_m(z_m)$ is in critical position relative to $\zeta_0^{(h_i+1, i+1)}$. This completes the proof of Lemma 3.2.3. \qed

We now complete the proof of Proposition 3.2.1. We firstly consider the case $z_m \notin \mathcal{C}_{\delta^0}$. Choose a large integer $R$ which do not depend on $\delta$, and consider $H = H_{a,b}$ such that $(a, b)$ is close enough to $(2,0)$ so that all precritical points of order $\leq R$ are good. Take a straight segment $\gamma_0$ which is tangent at $z_m$ to $v_m$ and intersects both $\{\delta\} \times \mathbb{R}$ and $\{-\delta\} \times \mathbb{R}$. Clearly, $\gamma_0$ is an admissible curve, and there exists a good precritical point of order $M$ on $\gamma_0$ to which $v_m(z_m)$ is related. Since all precritical points of order $\leq R$ are good, we can successively apply Lemma 2.7.2 to create good precritical points of higher order on $\gamma_0$.

We claim that there exists a precritical point of order $\leq R$ on $\gamma_0$ relative to which $v_m(z_m)$ is in admissible position. Indeed, Sublemma 3.2.6 implies that if $v_m(z_m)$ is in critical position relative to a precritical point $\zeta_0$ of order $j < R$ on $\gamma$, then $v_m(z_m)$ is related to the precritical point of order $j + 1$ on $\gamma_0$. This leaves out only two possibilities: either there exists a precritical point of order $\leq R$ on $\gamma_0$ relative to which $v_m(z_m)$ is in admissible position, or else $v_m(z_m)$ is in critical position relative to the precritical point of order $R$ on $\gamma_0$. However, the second possibility is eliminated by the fact that all precritical points are contained in $\mathcal{C}_{\delta^0}$, and $R$ can be made arbitrarily large after $\delta$ is fixed. Hence the claim follows.

It is left to consider the case $z_m \in \mathcal{C}_{\delta^0}$. Since $\text{length}(\gamma^{(1)}) \geq \rho^N$, the admissible curve $\gamma^{(1)}$ intersects both $\{\delta^{10}\} \times \mathbb{R}$ and $\{-\delta^{10}\} \times \mathbb{R}$. Hence there exists a good precritical point of order $N$ on $\gamma^{(1)}$ to which $v_m(z_m)$ is related. If $v_m(z_m)$ is related to it then it is done. If not, we appeal to Lemma 3.2.3. This finishes the proof of the first half of the assertion of the proposition. That $\psi'$ is $e^{-1}$-regular follows from $\|v_{m+q+1}\| \geq \|v_m\|$ and (g) in Proposition 2.10.2. \qed
3.3. Controlled orbits. Suppose that $H$ satisfies $(EG)_n$. Consider a vector orbit $\mathbf{v} = \{v_i(z_i)\}_{i=0}^m$. We say an integer $i \in [0, m]$ is a return time if $z_i \in C_\delta$ holds. We say $\mathbf{v}$ is controlled up to time $m$, if $z_0 \in H(C_\delta)$ and $\text{slope}(v_0) \leq K_0b$, and no return takes place up to time $m$, or else there exists a sequence of return times $m_0 < m_1 \cdots < m_t \leq m$ such that:

- (CO1) there is no return time before $m_0$;
- (CO2) for every $0 \leq s \leq t$ there exists a binding point of order $\leq \min(m_s, n)$ relative to which $\text{v}_{m_s}(z_{m_s})$ is in admissible position;
- (CO3) $m_{s+1} = \min\{i : i \geq m_s + q_s + 1, z_i \in C_\delta\}$ for $0 \leq s \leq t - 1$, where $q_s$ is the corresponding binding period;
- (CO4) $m_t \leq m \leq m_t + q_t + 1$, or $m > m_t + q_t + 1$ and no return takes place from $m_t + q_t + 1$ to $m - 1$.

We call $i$ bound if $i \in [m_s + 1, m_s + q_s]$ for some $s \in [0, t]$. We call $i$ free if it is not bound.

**Lemma 3.3.1.** If $\mathbf{v} = \{v_i\}_{i=0}^m$ is controlled, then for every free iterate $0 \leq i \leq m$,

$$\|v_i\| \geq K_0\delta e^{\lambda i/3}\|v_0\|.$$ 

**Proof.** By Lemma 2.2.1 for every $i \leq m_0$ we have $\|v_i\| \geq K_0\delta e^{\lambda i}\|v_0\|$. Since $m_0 \geq N$, we have $\|v_{m_0}\| \geq e^{\lambda i/3}\|v_0\|$. We claim that $\|v_i\| \geq e^{\lambda i/3}\|v_0\|$ holds for every $i \in \cup_{s=0}^{q_s+1}(m_s + q_s + 1)$. Indeed, by (c) in Proposition 2.10.2, the inequality holds for $i = m_0 + q_0 + 1$. If it holds for some $i = m_s + q_s + 1$, then (b) in Lemma 2.2.1 and (c) in Proposition 2.10.2 together yield the inequality for $i = m_{s+1} + q_{s+1} + 1$.

We complete the proof of the lemma. Using $\text{slope}(v_{m_s+q_s+1}) \leq K_0b$ and Lemma 2.2.1 for $m_s + q_s + 1 \leq i \leq m_{s+1}$ we have

$$\frac{\|v_i\|}{\|v_0\|} \geq \frac{\|v_i\|}{\|v_{m_s+q_s+1}\|} \frac{\|v_{m_s+q_s+1}\|}{\|v_0\|} \geq K_0\delta e^{\lambda(i-m_s-q_s-1)} e^{\lambda/3(m_s+q_s+1)} \geq K_0\delta e^{\lambda i/3}.$$

By the same reasoning we have $\|v_i\| \geq K_0\delta e^{\lambda i/3}\|v_0\|$ for every free iterate in between $m_t$ and $m$. □

3.4. Proof of Theorem A. We are in position to prove Theorem A. We fix $\alpha, M, \beta, \delta$, once and for all. For small $b > 0$, let $\Omega(0)\}$ denote a small a-interval such that $\{(a, b) : a \in \Omega(0)\} \subset \Omega$. We moreover assume that $\{(a, b) : a \in \Omega(0)\}$ is close enough to (2,0) so that all the previous estimates and arguments hold. In what follows we only consider $H = H_{a, b}$ such that $a \in \Omega(0)$.

Suppose that $H$ satisfies $(EG)_n$ for every $n \geq N$. For $z_0 \in W^u(P)$, take an integer $k_0 \geq 0$ such that the set of preimages $H^{-k_0}(z_0)$ intersects $W^u(P)$. Pick one point from $H^{-k_0}(z_0) \cap W^u_{\text{loc}}(P)$ and denote it by $z_{-k_0}$. Then $z_i = H^{i+k_0}(z_{-k_0})$ is uniquely determined for $i \leq -k_0$. For an arbitrary $j \leq \min\{-k_0, N\}$, define a vector orbit $\{v_i(z_i)\}_{i=j}^{k_0}$ by $v_i = DH^{i+k_0}t_{w_{\text{loc}}}(P)(z_{-k_0})$. Since $P$ is a hyperbolic fixed point, we have $\|v_i\| \geq \|v_i(z_i)\|$ for $j \leq i \leq -k_0$. Let $m_0 = \min\{i : H^i(z_{-k_0}) \in C_\delta\}$. By Lemma 2.2.1 and $\text{slope}(v_{k_0}) \leq K_0b$, we have $\|v_{m_0-k_0}\| \geq K_0\delta\|v_i\|$ for $j \leq i \leq m_0 - k_0$. Since $m_0 - k_0 - j \geq -j \geq N$, the necessary conditions are satisfied for the capture argument in Proposition 3.2.1 to work. For an arbitrary $j$ we apply the capture argument and end up with one of the following: obtain a good precritical point relative to which $v_{m}(z_{m})$ is in admissible position; not so, namely, $v_{m}(z_{m})$ is in critical position.
relative to all the precritical points assigned by the capture argument. In the first case, we run the system further. When the next free return takes place we apply the capture argument again. This is feasible by the last assertion in Proposition 3.2.1 and Corollary 2.2.2. By the same reasoning the two possibilities are left.

By now it is clear how to define $C$. Define $C$ to be the set of all $z_0 \in W^u(P)$ such that there exists a controlled vector orbit $\{v_i(z_i)\}_{i=-j}^0$ such that: (i) $z_{-j}$ is near $P$ and $v_{-j}$ is tangent to $W^u_{ loc}(P)$; (ii) $z_0$ is a free return; (iii) $v_0(z_0)$ is in critical position relative to any critical point which is assigned by the capture argument. Let us see $C$ satisfies the desired properties.

First of all, by Lemma 2.3.1 and the fact that $W^u_{ loc}(P)$ is an admissible curve, any $z_0 \in C$ is contained in the interior of some admissible curve, say $\gamma$, which is contained in $W^u(P)$. For now let us suppose that there is no self intersection of $W^u(P)$. Lemma 2.7.2 and the definition of $C$ implies the existence of a sequence of infinitely many good precritical points of arbitrarily high order on $\gamma$, converging on $z_0$. This implies $C \cap \gamma = \{z_0\}$. Let us see why this is so. Suppose that $z_0' \in C \cap \gamma$. Then, by the same reasoning, there exists a sequence of infinitely many precritical points of arbitrarily high order on $\gamma$ which converges on $z_0'$. Since $\gamma$ is an admissible curve, there exists no more than two distinct critical points on $\gamma$ of the same order. This implies that the two sequences must converge on the same point. Hence $z_0' = z_0$, and the claim follows. Let us now suppose that there is a self intersection of $W^u(P)$. In this case, the above argument is slightly incomplete because there may exist two distinct critical points on two distinct admissible curves which intersect each other. To deal with this, consider an immersion $\iota: \mathbb{R} \to W^u(P)$. Then the above argument shows that $\iota^{-1}(\gamma \cap C)$ contains exactly one point. Consequently $C$ is a countable set in this case as well.

For $z_0 \in C$, let $y_n$ denote the good precritical point of order $n$ which belongs to the sequence converging on $z_0$. Since the speed of this convergence is exponential which does not depend on $z_0$, (a-i) follows. Let $\Gamma^{(n)}$ denote the long stable leaf of order $n$ through $H(y_n)$. It follows from the proof of Proposition 2.5.1 that $\{\Gamma^{(n)}\}_{n=1}^\infty$ forms a Cauchy sequence in the $C^2$ topology. Let $\Gamma^{(\infty)}$ denote its $C^2$ limit. Since $\Gamma^{(n)}$ is tangent to $H(\gamma)$ at $H(y_n)$ and $H(y_n) \to z_1$, $\Gamma^{(\infty)}$ is tangent at $z_1$ to $H(\gamma)$. This yields (a-ii). (a-iii) follows from the definition of $C$ and Lemma 3.3.1.

It is left to prove (b). Since the Lyapunov exponents of all periodic points of $f_2$ are $\log 2$, we may assume that the largest Lyapunov exponents of all periodic points of $H$ with period $\leq N$ are $\geq \log 2/3$. For a periodic orbit $O$ with period $p \geq N$, there exists a sub-orbit of length $N$ which stays outside of $C_8$. Along this orbit we construct an $e^{-1}$-regular vector orbit of length $N$ and then apply the capture argument. If the vector orbit is always in admissible position as we run the system, then the largest Lyapunov exponent of $O$ is $\geq \log 2/3$, by Lemma 3.3.1. Otherwise, there exists a vector orbit of length $\geq \sqrt{\beta}N$ which shadows the orbit of the critical point. In particular it is $e^{-1}$-regular and grows exponentially fast in norm. If $\sqrt{\beta}N \geq p$, then clearly the largest Lyapunov exponent of $O$ is $\geq \lambda - \alpha$. If $\sqrt{\beta}N \leq p$, then we apply the capture argument to this longer vector orbit and repeat the same argument. Since $p$ is finite, this argument stops sooner or later. Consequently, the largest Lyapunov exponents of all periodic points are $\geq \log 2/3$. $\square$
4. A MODEL PROBLEM

The aim of this section is to help the readers grasp the meanings of constructions and arguments in later sections. As we said in the introduction, our parameter exclusion argument is an extension of that of Tsujii [15] [16] in one-dimension: rather than good parameter sets we pay attention to its complement. A new difficulty arises at this point: at each step of induction we already need to deal with infinitely many critical points of arbitrarily high order. According to [2] we introduce a toy model of the Hénon map having infinitely many critical points and describe how our parameter exclusion argument unfolds in this context. Our argument is totally different from that in [2] Section 3 which deals with only a finite number of critical points at each step of induction.

4.1. The model. Let \( b \in (0, 1) \), and let \( M = [-1, 1] \times K(b) \) where \( K(b) \) is the mid-\((1 − b)\) Cantor set in \([0, 1]\). Let \( a(\kappa) = a + \phi(\kappa) \) where \( a \) is a parameter and \( \phi \) is some small Lipschitz function. Consider the map \( f: M → M \) given by

\[
f(x, \kappa) = (1 - a(\kappa)x^2, \kappa_1), \quad \begin{cases} 
\kappa_1 = b\kappa/2 & \text{if } x < 0 \\
\kappa_1 = 1 - b\kappa/2 & \text{if } x \geq 0.
\end{cases}
\]

The map \( f \) sends every half horizontal \([-1, 0) \times \{\kappa\} \) and \([0, 1) \times \{\kappa\} \) inside some horizontal \([-1, 1] \times \{\kappa_1\} \) in a quadratic fashion, with a critical point \( x = 0 \) in the usual sense. Each horizontal line plays the role of the unstable manifold in Hénon-like maps. The parameter \( b \) controls the amount of contraction: let \((x_0, \kappa_0), (\tilde{x}_0, \tilde{\kappa}_0) \in M\), write \( f^i(x_0, \kappa_0) = (x_i, \kappa_i) \) for \( i \geq 0 \). If \( x_i, \tilde{x}_i < 0 \) or \( x_i, \tilde{x}_i \geq 0 \) for \( 0 \leq i \leq n - 1 \), then \(|\kappa_n - \tilde{\kappa}_n| \leq (b/2)^n(\kappa_0 - \tilde{\kappa}_0)\).

4.2. Critical points. Clearly, adapted definitions of critical points\(^\dagger\) of order \( n \), and the condition \((EG)_n\) make sense in this model. We say \( \zeta_0 \in \{0\} \times K(b) \) is a critical point of order \( n \) if there exists a sequence \( \{\zeta_i\}_{i=-n}^0 \) such that:

\[(CT1) \ f(\zeta_i) = \zeta_{i+1} \text{ for } -n \leq i \leq -1 \text{ and } \zeta_{-n} \notin C; \]

\[(CT2) \ \{df^i(\zeta_{-n})\}_{i=0}^n \text{ is } e^{-10} \text{-expanding and } e^{-2} \text{-regular.} \]

We say \( \zeta_0 \) is a *quasi critical point* of order \( n \) if \((CT1)\) holds and \((CT2)\) with \( e^{-10}, e^{-2} \) replaced by \( e^{-11}, e^{-3} \) holds.

The following theorem follows as a much easier version of Theorem A:

**Theorem C.** There exists \( N > 0 \) such that if \( \phi \) is small, \((a, b)\) is close to \((2, 0)\), and \( f_a \) satisfies \((EG)_n\) for every \( n \geq N \), then:

\[
\begin{align*}
\text{(a)} & \quad \text{there exists a countable set } C \subset \{0\} \times K(b) \text{ such that } \|DH^n(H(\zeta))\| \geq e^{\frac{99}{100} \log 2 \cdot n} \\
& \quad \text{for all } \zeta \in C \text{ and } n \geq 1; \\text{(b)} & \quad \text{for all } z \in M - \bigcup_{n \geq 0} H^n(\{0\} \times K(b)), \\
& \quad \limsup_{n \to +\infty} \frac{1}{n} \log \|DH^n(z)v\| \geq \frac{\log 2}{3};
\end{align*}
\]

\(^\dagger\)It might look strange to purposely define critical points in this way, for it is obvious where the usual critical points are. Keep in mind that this definition is just to fulfill the aim of this section.
(c) For any periodic point \( p \in M \),
\[
\limsup_{n \to +\infty} \frac{1}{n} \log \|DH^n(p)\| \geq \frac{\log 2}{3}.
\]

The following theorem follows as a much easier version of Theorem B:

**Theorem D.** For \( b > 0 \) small, there exists a positive measure set \( \Omega_b \) of \( a \)-values near 2 such that \( H = H_{a,b} \) satisfies \((EG)_n\) for all \( n \geq N \) whenever \( a \in \Omega_b \).

For the rest of this section we sketch a proof of Theorem D. Some terminologies and lemmas are used prior to their introductions. We have tried to make explicit which part of the paper should be referred on such occasions.

### 4.3. Smooth continuations.

**Lemma 4.3.1.** Let \( \zeta_0 \) be a critical point of order \( \xi \geq n \) of \( f_{a,*} \), with \( h \geq n \) the \( n \)-maximal hyperbolic time \((\ref{5.3})\). There exists a constant function \( a \in [a_* - e^{-\lambda_0h/17}, a_* + e^{-\lambda_0h/17}] \to \zeta_0(a) \in \{0\} \times K(b) \) such that:

(a) \( \zeta_0(a) \) is a quasi critical point of order \( h \) of \( H_a \);
(b) let \( \zeta_{-h}(a) = (x(a), \kappa(a)) \) and \( \zeta_{-h} = (x, \kappa) \). Then \( \kappa(a) = 1 \) and \( |x - x(a)| \leq e^{-100\Delta n} \).

(c) \( |\zeta_0 - \zeta_0(a_0)| \leq (b/2)^h \).

**Proof.** For \( z = (x, \kappa) \in M \), define \( |z| = |x| \).

**Sublemma 4.3.2.** We have \( |\zeta_{-h+i}| \geq e^{-11\Delta i} \) for \( -h \leq i \leq -1 \).

**Proof.** Suppose that \( |\zeta_{-h+i}| \leq e^{-11\Delta i} \) for some \( i \in [-h, -1] \). Since \( |Df| \leq e^a \) we have \( |Df^{i+1}(\zeta_{-h})| \leq e^{a} |\zeta_{-h+i}| \leq e^{-10\Delta i} \). This yields a contradiction to the fact that \( h \) is a hyperbolic time. \( \square \)

Take \( z_0 \in [-1, 1] \times \{1\} \) whose \( x \)-coordinate coincides with that of \( \zeta_{-h} \). A recursive use of Sublemma 4.3.2 implies that the orbits of \( z_0 \) and \( \zeta_{-h} \) share the same kneading sequences up to time \( h - 1 \) and in particular \( |z_i| \geq |\zeta_{-h+i}| - (b/2)i \). This implies \( f_{a}^{i} y \notin \{0\} \times K(b) \) for \( 0 \leq i \leq h - 1 \) if \( a \in [a_* - e^{-\lambda_0h/17}, a_* + e^{-\lambda_0h/17}] \) and \( y \in [\zeta_{-h} - e^{-100\Delta n} \zeta_{-h} + e^{-100\Delta n}] \times \{1\} \). Take a horizontal segment \( \gamma_0 \) of length \( e^{-100\Delta n} \) centered at \( z_0 \). We claim that \( f_{a}^{i}\gamma_0 \) intersects \( \{0\} \times K(b) \). This follows from \( |f_{a} z_0 - f_{a} z_0| \leq e^{-\beta h/18} \) and length\( f_{a}^{i}\gamma_0 \), by Lemma 2.1.2. Define \( \zeta_0(a) := f_{a}^{i}\gamma_0 \cap \{0\} \times K(b) \) and \( \zeta_{i}(a) := f_{a}^{-1}\zeta_{i+1}(a) \). By construction, (a) (b) hold. Hence (c) follows from the fact that the orbits of \( \zeta_{-h}(a_0) \) and \( \zeta_{-h} \) share the same kneading sequences up to time \( h - 1 \). \( \square \)

### 4.4. Structure of the simplest bad parameter sets. 

The set of parameters discarded at step \( n \) is contained in a finite union of well-structured sets the measures of which are easy to estimate. We restrict ourselves to the simplest case and explain the structure of this set.

We now subdivide \([-1, 1] \times \{1\} \) into \( e^{100\Delta n} \) segments of equal length. Let \( S(n) \) denote the set of mid points of these intervals. Clearly we have card\( (S(n)) \leq e^{100\Delta n} \). Fix \( z \in S(n) \) and three integers \( d \geq \max\{\alpha \beta n/100, -\log \delta\} \), \( \nu \in [0, \beta n] \), \( m \in [\beta(n - 1) + 1, \beta n] \). Define \( \Omega^{(n)}(\nu, d, z, n, m) \) to be the set of all \( a \in RR_{n-1} \) such that:
(BT1) there exists a critical point \( \zeta_0 \) of order \( \xi \geq n \) such that \( m - 1 \) is the largest integer up to which the forward orbit \( w = \{ w_i(\zeta_{i+1}) \}_{i=0}^{\beta} \) of \( \zeta_0 \) is reluctantly recurrent (Sect. 6.2);

(BT2) the forward orbit of \( \zeta_0 \) makes essential returns exactly at \( \nu \) up to time \( \beta n \) with \( d = d(\nu) \) (Sect. 6.1);

(BT3) \( n \) is a hyperbolic time of the backward orbit of \( \zeta_0 \) is \( h \) and the corresponding smooth continuation of order \( n \): \( a \to \zeta_0(a) \) as in Lemma 4.3.1 satisfies \( |\zeta_{-n}(a) - z| \leq e^{-100\Delta n} \).

Let \( a \in \Omega^{(n)}(\cdot) \). We say a critical point \( \zeta_0 \) of \( H_a \) of order \( \geq n \) is responsible for \( a \) if \( \zeta_0 \) satisfies (BT1) (BT2) (BT3). The following two lemmas are crucial in dealing with infinitely many critical points. For the definition of intervals \( J(\cdot, \cdot) \), see Sect. 7.2

**Lemma 4.4.1.** Let \( a_* \in \Omega^{(n)}(\cdot) \), and let \( \zeta_0 \) denote a critical point which is responsible for \( a_* \). The set \( J(a_*, \zeta_0, \nu, 0) - J(a_*, \zeta_0, \nu, d) \) does not intersect \( \Omega^{(n)}(\cdot) \).

*Proof.* Consider the continuation \( b \in J(a_*, \zeta_0, \nu, 0) \to \zeta_0(b) \). Let \( a \in J(a_*, \zeta_0, \nu, 0) - J(a_*, \zeta_0, \nu, d) \) and suppose that \( a \in \Omega^{(n)}(\cdot) \). Let \( \tilde{\zeta}_0 \) denote any critical point which is responsible for \( a \). Consider the corresponding continuation \( \tilde{\zeta}_0(\cdot) \). We claim that \( \zeta_0(a) = \tilde{\zeta}_0(a) \) holds. Indeed, this follows from \( a \in J(a_*, \zeta_0, \nu, 0) \cap J(a, \tilde{\zeta}_0, \nu, 0) \) and (BT3). Meanwhile, by Lemma 7.4.1 and the assumption on \( a \), \( \zeta_{\nu+1}(a) \) is in admissible position. Moreover Proposition 7.3.1 implies \( |\zeta_{\nu+1}(a) - \zeta_{\nu+1}(a)| \geq |\zeta_{\nu+1}|^{1-\alpha n/2} \).

This implies \( |\zeta_{\nu+1}(a)| \geq e^{-d/\nu}. \) By (c) in Lemma 4.3.1 we have \( |\zeta_{\nu+1} - \tilde{\zeta}_{\nu+1}(a)| \leq (b/2)^{e^{\Delta \nu}} \leq (b/3)^n \). Consequently we obtain

\[
|\tilde{\zeta}_{\nu+1}| \geq |\zeta_{\nu+1}(a) - |\tilde{\zeta}_{\nu+1} - \zeta_{\nu+1}(a)| > e^{-d/2}.
\]

This yields a contradiction to the assumption that \( \tilde{\zeta}_0 \) is responsible for \( a \) because (BT2) is not satisfied.

**Lemma 4.4.2.** Let \( a, \tilde{a} \in \Omega^{(n)}(\cdot) \). Suppose that \( \zeta_0, \tilde{\zeta}_0 \) are critical points which are respectively responsible for \( a \) and \( \tilde{a} \). Assume that:

(i) \( J(a, \zeta_0, \nu, 0) \cap J(\tilde{a}, \tilde{\zeta}_0, \nu, 0) \neq \emptyset \);

(ii) \( \tilde{a} \notin J(a, \zeta_0, \nu, 0) \).

Then we have \( J(a, \zeta_0, \nu, 0) \subset J(\tilde{a}, \tilde{\zeta}_0, \nu, 0) \).

*Proof.* Consider two continuations \( b \in J(a, \zeta_0, \nu, 0) \to \zeta_0(b) \) and \( b \in J(\tilde{a}, \tilde{\zeta}_0, \nu, 0) \to \tilde{\zeta}_0(b) \). By Corollary 7.2.2 and Proposition 7.3.1 there exist \( c \in J(a, \zeta_0, \nu, d) \) and \( \tilde{c} \in J(\tilde{a}, \tilde{\zeta}_0, \nu, d) \) such that \( \zeta_{\nu+1}(c), \tilde{\zeta}_{\nu+1}(\tilde{c}) \in \{0\} \times K(b) \). We claim that \( c = \tilde{c} \).

Indeed, the construction of smooth continuation readily implies \( \zeta_0(c) = \tilde{\zeta}_0(c) \) for \( c \in J(a, \zeta_0, \nu, 0) \cap J(\tilde{a}, \tilde{\zeta}_0, \nu, 0) \). Thus \( J_{\nu+1}(a, \zeta_0, \nu, 0) \cup J_{\nu+1}(\tilde{a}, \tilde{\zeta}_0, \nu, 0) \) is an admissible curve, which intersects \( \{0\} \times K(b) \) exactly at one point. Hence the claim follows. This claim and (ii) together imply that one of the connected components of \( J(a, \zeta_0, \nu, 0) - J(a, \zeta_0, \nu, d) \) is contained in \( J(\tilde{a}, \tilde{\zeta}_0, \nu, d) \), and thus \( J(a, \zeta_0, \nu, 0) \subset J(\tilde{a}, \tilde{\zeta}_0, \nu, 0) \).
In view of [8.4] it is not difficult to see from Proposition 4.4.1 and Lemma 4.4.2 that there exists a finite set of parameters \( \{a_1, \cdots, a_\ell_1, \cdots, a_\ell_2 \} \subset \Omega^{(n)}(\cdot) \) such that

\[
\Omega^{(n)}(\cdot) \subset \bigcup_{j_1=1}^{\ell_1} J(a_{j_1}, \nu_j, 0),
\]

where the intervals which take part in the union are two by two disjoint. By now it is easy to estimate the measure of \( \Omega^{(n)}(\cdot) \).

4.5. Issues to be addressed. The above case is the simplest one and does not exhaust all the possibilities. In reality we need to consider the effect of multiple essential returns made by a single critical point. Needless to say, this is an issue already present in one-dimension. We need to consider a deeper structure in parameter space as in Lemma 8.2.4 accordingly.

As a reminder we point out some important differences between the model and the Hénon-like case.

(a) Smooth continuations of critical points are not constant functions in general. To show that their parameter dependence is small, a very sharp derivative estimate based on the Hadamard lemma is necessary (Proposition 5.3.1).

(b) Critical points probably do not lie on any vertical straight line. Therefore, to yield a contradiction to (B2) (a counterpart of (BT2), see §8.1), it is necessary to show that the dependence of return depths on binding points is sufficiently small, as in Sublemma 8.2.3.

5. Parameter dependence of critical points

In this section we deal with parameter dependence of critical points. We introduce quasi critical points and prove that they continue to exist in a sufficiently large parameter interval with small derivatives.

5.1. Quasi critical points. A precritical point \( \zeta_0 \) of order \( n \geq N \) on an admissible curve \( \gamma_0 \) is a primary quasi critical point if:

(PQ) there exists an \( e^{-3} \)-regular and \( e^{-11\Delta} \)-expanding orbit \( \{w_i(\zeta_i)\}_{i=-n}^0 \) such that \( \zeta_{-n} \notin \mathcal{C}_\delta \) and \( w_0(\zeta_0) \in T_{\zeta_0} \gamma_0 \).

We say \( \zeta_0 \) is a secondary quasi critical point if:

(SQ) there exists an \( e^{-12\Delta} \)-expanding vector orbit \( \{w_i(\zeta_i)\}_{i=-n}^0 \) such that \( \zeta_{-n} \notin \mathcal{C}_\delta \) and \( w_0(\zeta_0) \in T_{\zeta_0} \gamma_0 \).

The following lemma asserts that near critical points there exists a stack of primary quasi critical points of lower order. Notice that the assumption is slightly stronger than (PQ).

Lemma 5.1.1. Let \( \zeta^{(j)}_0 \) be a primary quasi critical point of order \( h_j \) on \( \gamma_0 \), with \( \{h_i\}_{i=1}^j \) the associated sequence of hyperbolic times. Assume that the backward orbit \( \{w_i\}_{i=-h_j}^0 \) is \( e^{-11.5\Delta} \)-expanding and \( e^{-2.5} \)-regular. For every \( 1 \leq i \leq j \) there exists a primary quasi critical point \( \hat{\zeta}^{(i)}_0 \) of order \( h_i \) on an admissible curve \( \gamma^{(i)} := H^{(h_i) \gamma}(w_{-h_i}, \rho^{h_i}) \) such that

\[
|\hat{\zeta}^{(i)}_0 - \zeta^{(j)}_0| \leq \sum_{k=i}^{j} (Kb)^{h_k/3}.
\]
Proof. Clearly, the assertion with \( i = j \) holds, because \( \gamma_0 \) and \( \gamma^{(j)} \) are tangent at \( \zeta_0^{(j)} \). Let \( i \in [1, j - 1] \), and suppose that there exists a primary quasi critical point \( \zeta_0^{(i+1)} \) of order \( h_{i+1} \) on \( \gamma^{(i+1)} \) with \( |\zeta_0^{(i+1)} - \zeta_0^{(j)}| \leq \sum_{k=i+1}^{j} (Kb)^{hk/3} \). Then the lower bound on the length of \( \gamma^{(i+1)} \) implies that \( \zeta_0^{(i+1)} \) is located around the middle of \( \gamma^{(i+1)} \). This permits us to use Lemma 2.7.2 to yield a precritical point of order \( h_i \) on \( \gamma^{(i)} \), called \( \tilde{\gamma}^{(h_i,i+1)} \), such that \( |\zeta_0^{(i+1)} - \zeta_0^{(h_i,i+1)}| \leq (Kb)^{h_{i+1}/2} \). Let \( z_0 \in \gamma^{(i)} \) denote the point whose \( x \)-coordinate coincides with that of \( \zeta_0^{(h_i,i+1)} \). Such \( z_0 \) uniquely exists because \( \text{length}(\gamma^{(i)}) \gg |\zeta_0^{(i+1)} - \zeta_0^{(h_i,i+1)}| \) holds.

Claim 5.1.2. We have:

(a) \( |\zeta_0^{(h_i,i+1)} - z_0| \leq (Kb)^{h_i/2} \);
(b) \( \text{angle}(\hat{t}_{\gamma^{(i)}}(\zeta_0^{(h_i,i+1)}), t_{\gamma^{(i)}}(z_0)) \leq (Kb)^{h_i/2} \).

Proof. Since \( h_i \) is a hyperbolic time we have \( |z_{-h_i} - \zeta_0^{(h_i,i+1)}| \leq e|z_0 - \zeta_0^{(h_i,i+1)}||w_{-h_i}| \).

Since \( \gamma^{(i+1)} \) and \( \gamma^{(i)} \) are admissible curves which are tangent to \( w_0 \), we have \( |z_0 - \zeta_0^{(h_i,i+1)}| \leq |\zeta_0^{(h_i,i+1)} - \zeta_0^{(h_i,i+1)}| \). Using the assumption of the induction,

\[
|z_{-h_i} - \zeta_0^{(h_i,i+1)}| \leq e^{10\Delta h_i} \left( (Kb)^{h_{i+1}/2} + \sum_{k=i+1}^{j} (Kb)^{hk/3} \right) \leq (Kb)^{h_i/4}.
\]

Thus the long stable leaf \( \Gamma^{(h_i)} \) of order \( h_i \) through \( \zeta_0^{(h_i,i+1)} \) is well-defined. In view of the proof of Proposition 5.2.1, the desired inequality follows if \( \Gamma^{(h_i)} \) intersects \( \gamma(w_{-h_i}, \rho^{h_i}) \). This follows from Sublemma 5.2.3 and the fact that \( \gamma(w_{-h_i}, \rho^{h_i}) \) is a straight segment.

By Claim 5.1.2 and Lemma 2.7.2 there exists a precritical point \( \tilde{\gamma}_0^{(i)} \) of order \( h_i \) on \( \gamma^{(i)} \) such that \( |\zeta_0^{(i)} - z_0| \leq (Kb)^{h_i/2} \). Consequently,

\[
|\zeta_0^{(i)} - \zeta_0^{(j)}| \leq |\zeta_0^{(i)} - z_0| + |z_0 - \zeta_0^{(h_i,i+1)}| + |\zeta_0^{(h_i,i+1)} - \zeta_0^{(i+1)}| + |\zeta_0^{(i+1)} - \zeta_0^{(j)}| \leq 2(Kb)^{h_i/2} + (Kb)^{h_{i+1}/2} + \sum_{k=i+1}^{j} (Kb)^{hk/3} \leq 3(Kb)^{h_i/2} + \sum_{k=i+1}^{j} (Kb)^{hk/3} \leq \sum_{k=i}^{j} (Kb)^{hk/3}.
\]

This restores the assumption of the induction and completes the proof.

5.2. Smooth continuations. For \( a_* \in \Omega^{(0)} \) and \( h > 0 \), define

\[
\tilde{J}(a_*, h) = [a_* - e^{-\lambda h\beta/17}, a_* + e^{-\lambda h\beta/17}] \cap \Omega^{(0)}.
\]

Proposition 5.2.1. Let \( a_* \in \Omega^{(0)} \), and suppose that \( \zeta_0 \) is a good primary quasi critical point of order \( h \) of \( H_{a_*} \). There exists a \( C^3 \) map \( a \in \tilde{J}(a_*, h) \rightarrow \zeta_0(a) \) such that:

(a) \( \zeta_0(a) \) is a secondary quasi critical point of \( H_a \) of order \( h \);
Claim 5.2.3. \( \angle(w_h, w_{-h}) \geq e^{-12\Delta h} \).

Proof. Put \( \psi = \angle(e_h, w_{-h}) \). Split \( w_{-h} = \|w_{-h}\|(\cos\psi \cdot e_h + \sin\psi \cdot f_h) \). Then

\[
e^{-20\Delta h} \leq \|w_{-h}\|^{-2} \leq (K^b)^2 \cos^2\psi + e^{2\Delta h} \sin^2\psi \leq (K^b)^2 + e^{2\Delta h} \sin^2\psi.
\]

Taking the both sides of the inequality and rearranging gives the inequality. \( \square \)

The argument is not affected even if we assume that \( w_{-h} \) is a unit vector, and we do so. Split \( w_{-h} = \xi_{e_h} + \eta_{f_h} \). By Claim 5.2.3 we have \( |\eta| \geq e^{-10\Delta h} \) and thus \( \|w_{-h}\| \approx \|DH_i^{\ell}f_h\| \) for \( i \geq h/10 \). For \( \ell \in [1, 9h/10] \), by Lemma 2.5.1 we obtain

\[
\frac{\|v_{h-\ell}\|}{\|v_h\|} \leq e \frac{\|DH^{h-\ell}f_h\|}{\|DH^hf_h\|} \leq e \cdot \|w_{-\ell}\| \leq K_0^{-1} \delta^{-1} e^A.
\]
For $t \in [9h/10, h]$ we have

$$\frac{\|v_{h-t}\|}{\|v_h\|} = \frac{\|v_{h-t}\|}{\|v_0\|} \leq e^{\Delta(h-t)} e^{12\Delta h} \leq e^{13\Delta t}. \tag{7}$$

Substituting (6), (7) into the sum we obtain the bound on the curvature. (6) with $t = 1, 2$ and (c) in Lemma 2.2.1 yields that the slopes of tangent directions of $H^h_a \hat{\gamma}$ are $\leq K_0 b$. Consequently $H^h_a \hat{\gamma}$ is an admissible curve. \hfill $\square$

In the same spirit as the beginning of the proof of Proposition 2.5.1 we have

$$\text{angle}(v_h, w_0) \leq (Kb)^{-1} \sum_{i=0}^{h} \frac{\|v_i\|}{\|v_h\|} \|w_{i-h}\|.$$  

To bound the sum, we use (6), (7) and $\|w_{i-h}\| \leq K_0^{-1} e^{3\delta - 1} \|w_0\|$. This yields $\text{angle}(v_h, w_0) \leq (Kb)^{1/2}$. Take a straight segment $\gamma_0$ of length $2h$ which is centered at $\zeta_{-h}$ and tangent to $w_{-h}$. Then $\gamma_h$ is an admissible curve of length $\geq 2h$ by Lemma 2.3.1. Applying Lemma 2.7.2 to the pair of admissible curves $\gamma_h, H^h_a \hat{\gamma}$, we conclude the existence of a precritical point $\zeta_0$ of order $h$ on $H^h_a \hat{\gamma}$. Since the distortion estimate in Lemma 2.1.2 holds on $\hat{\gamma}$, $\zeta_0$ has an $e^{-11.5\Delta}$-expanding backward orbit of length $h$, which in addition is linked to $\bar{\zeta}$, by construction. Hence $\zeta_0$ is a secondary quasi critical point of order $h$.

Put $z_i(a) = H^h_a z_0$.

**Claim 5.2.4.** For all $a \in \hat{J}(a, h)$ there exists a unique $\zeta(a) \in \bar{\gamma}$ such that:

(a) the $x$-coordinate of $H^h_a \zeta(a)$ coincides with that of $z_h(a)_a$.

(b) $|z_h(a) - H^h_a \zeta(a)| \leq |z_h(a_*) - z_h(a)| \leq e^{-\lambda h/18}$.

(c) $\text{angle}(DH^h_a(\zeta(a)) (\frac{1}{h}), v_{h}(a_*)) \leq e^{-\lambda h/18}$

Proof. Since $\|z_i(a)\| \leq he^h_h$ we have $|z_h(a_*) - z_h(a)| \leq he^{-\Delta h} |a_* - a| \leq e^{-\lambda h/18}$, and thus length($H^h_a \hat{\gamma}$) $\geq |z_h(a_*) - z_h(a)|$. This, and the fact that $H^h_a \hat{\gamma}$ and $H^h_a \tilde{\gamma}$ are admissible curves together imply the unique existence of $\zeta(a) \in \tilde{\gamma}_0$ with (a). The fact that $H^h_a \tilde{\gamma}$ is an admissible curve and the ”Pythagoras theorem” yield (b). (b) implies angle($v_h(a), DH^h_a(\zeta(a)) (\frac{1}{h})$) $\leq e^{-\lambda h/18}$, and using (1) we have (c). \hfill $\square$

Put $\tilde{\gamma}_h(a) = H^h_a (\tilde{\gamma})$, and parametrize $\tilde{\gamma}_h(a)$ so that $\tilde{\gamma}_h(a)(0) = H^h_a (\zeta(a))$ holds. By Lemma 5.2.2 and (b) in Claim 5.2.4, $\tilde{\gamma}_h(a)(s)$ is well-defined for $s \in [-e^{-\lambda h}, e^{-\lambda h}]$. This and (b) (c) in Claim 5.2.3 permits us to apply Lemma 2.7.2 to conclude that there exists $s \in [-e^{-\lambda h}, e^{-\lambda h}]$ such that $\tilde{\gamma}_h(a)(s)$ is a precritical point of order $h$ of $H_a$. For the rest of the argument we appeal to the following lemma the proof of which is given in Appendix:

**Lemma 5.2.5.** Let $\gamma$ be an admissible curve in $C_\delta$, where $\gamma(0) = \zeta_0$ is a precritical point of order $m$ of $H_a$. Assume that $\varepsilon \ll 1$, and $\gamma(s)$ is defined for $s \in [-\varepsilon^m/2], \varepsilon^m/2]$. Then for all $a \in [a_* - \varepsilon^m, a_* + \varepsilon^m]$ there exists $\tilde{s}(a) \in [-\varepsilon^m/2, \varepsilon^m/2]$ such that $\tilde{\gamma}(\tilde{s}(a))$ is a precritical point of order $m$ of $H_a$.

According to Lemma 5.2.5 there exists a precritical point of order $h$ of $H_a$ on $H^h_a \tilde{\gamma}$. By construction and (5), it is a secondary quasi critical point of order $h$. This finishes the proof of (a) (b) (c). (d) follows from Lemma 2.7.2.
Lemma 5.1.1 has a smooth continuation of $\zeta_0(a)$. Parametrize $\tilde{\gamma}$ by arc length and let $s(a)$ be the one such that $\zeta_0(a) = H_a^{h}(\tilde{\gamma}(s(a)))$. We estimate the derivatives of $s(a)$. For $(s, a) \in \tilde{\gamma} \times \hat{J}$, define

$$v(s, a) = \frac{DH_a^{h+1}(\tilde{\gamma}(s)) \left(\frac{1}{0}\right)}{\|DH_a^{h+1}(\tilde{\gamma}(s)) \left(\frac{1}{0}\right)\|} \quad \text{and} \quad w(s, a) = e_h(a)(H_a^{h+1}(\tilde{\gamma}(s))).$$

Notice that $v(s(a), a) - w(s(a), a) \equiv 0$. Let $\kappa$ denote the curvature of $H_a^{h+1}\tilde{\gamma}$ at $\tilde{\gamma}_{h+1}(a)$. It is easy to see that $\kappa = \mathcal{O}(b^{-2})$. Let $\{w_i(a)\}_{i=-h}^{0}$ denote the backward vector orbit of $\zeta_0(a)$. Using [2], for small variance $\|s\|$ we have $\|v(s+ds, a) - v(s, a)\| \geq Kbkd\|w_h(a)\|^{-1}$. Taking limit $ds \to 0$ we have $\|\partial_s v(s, a)\| \geq Kb^{-1}\|w_h(a)\|^{-1}$. On the other hand, by Lemma 2.3.4 we have $\|\partial_s w\| \leq K\|w_h(a)\|^{-1}$. Hence we obtain

$$\|\partial_s v(s, a) - \partial_s w(s, a)\| \geq K\|w_h(a)\|^{-1} \geq Ke^{-15\Delta h}.$$ 

In particular, one of the component of the difference is $\geq Ke^{-20\Delta h}$. By the implicit function theorem we obtain

$$(8) \quad |\dot{s}(a)|, |\ddot{s}(a)|, |\dddot{s}(a)| \leq Ke^{70\Delta h}.$$ 

Put $Z_i(a) = H_a^{i}(\tilde{\gamma}(s(a)))$. Then we have $Z_h(a) = \zeta_0(a)$. Using $Z_i(a) = \mathcal{H}(a, Z_{i-1}(a))$ we have $\dot{Z}_i = \partial_s \mathcal{H}(a, Z_{i-1}) + DH_a(Z_{i-1})\dot{Z}_{i-1}$. Using this for $\ell$-times ($\ell \leq i$),

$$(9) \quad \dot{Z}_i = DH_a^\ell(Z_{i-\ell})\dot{Z}_{i-\ell} + \sum_{s=0}^{\ell-1} DH_a^s(Z_{i-s})\partial_a \mathcal{H}(a, Z_{i-s-1}).$$

Substituting $\ell = i$, and then $i = h$, and using (8) we obtain

$$(10) \quad \|\dot{Z}_h\| \leq he^{\Delta h} + e^{\Delta h}\|\dddot{s}(a)\| \leq e^{100\Delta h}.$$ 

To estimate $\|\dot{Z}_h\|$ we differentiate (9) and use the second order derivative estimate in (8). The estimate of $\|\dot{Z}_h\|$ is analogous. The details are left as exercises. This completes the proof of Proposition 5.2.1.

The following corollary is actually contained in Proposition 5.2.1.

**Corollary 5.2.6.** Let $a_* \in \Omega^{(0)}$, and suppose that $\zeta_0$ is a good primary quasi critical point of order $h$ of $H_{a_*}$. There exists a secondary quasi critical point $\zeta_0(a_*)$ of $H_{a_*}$ such that $|\zeta_0 - \zeta_0(a_*)| \leq (Kb)^{h/2}$.

5.3. Derivative estimates of smooth continuations. The bound on the derivatives in (e) in Proposition 5.2.1 is too coarse to be adapted to our argument. To rectify this we derive much finer derivative estimates.

**Proposition 5.3.1.** Let $\zeta_0$ be a critical point of $H_{a_*}$ of order $\xi$, with $\{h_j\}_{j=1}^{s}$ the sequence of hyperbolic times associated with its backward orbit $w = \{w_i(\zeta_j)\}_{i=-\xi}^{0}$. For every $1 \leq j \leq s$ the primary quasi critical point $\zeta_0(a_j)$ of order $h_j$ given by Lemma 5.1.1 has a smooth continuation $a \in \hat{J}(a_*, h_j) \to \zeta_0^{(j)}(a)$ such that:

(a) $\|\dot{\zeta}_0^{(j)}(a)\|, \|\ddot{\zeta}_0^{(j)}(a)\| \leq \delta$;
(b) if the forward vector orbit of $\zeta_0$ is strongly regular up to time $m \in [M, \beta \xi]$, then for every $1 \leq i \leq \min\{m, \beta h_j\}$,
\[
\left| \log \frac{\|DH_{a_i}^s(\zeta_i) \left( \frac{1}{2i} \right) \|}{\|DH_{a_i}^s(\zeta_i^{(j)}(a_*)) \left( \frac{1}{2i} \right) \|} \right| \leq 1.
\]

We call the map $a \to \zeta^{(j)}(a)$ a smooth continuation of order $h_j$ of $\zeta_0$.

**Proof.** By Lemma 5.1.1 we have $|\zeta_0 - \tilde{\zeta}^{(j)}(a)| \leq \sum_{k=1}^s (Kb)^{h_k/3}$. Applying Proposition 5.2.1 to $\tilde{\zeta}^{(j)}$ we obtain a smooth continuation $\zeta^{(j)}(a)$ on $\hat{J}(a_*, h_j)$. By Lemma 5.1.1 and (d) in Proposition 5.2.1 we have $|\zeta_0 - \zeta^{(j)}| \leq |\zeta_0 - \tilde{\zeta}^{(j)}(a)| + |\tilde{\zeta}^{(j)}(a) - \zeta^{(j)}| \leq (Kb)^{h_j/4}$. This and (b) in Proposition 2.5.1 together imply (b).

Before entering the proof of (a) we sketch the argument. The idea is to apply the next lemma to $\zeta^{(i+1)}_0(a) - \zeta^{(i)}_0(a)$ for $1 \leq i < j - 1$:

**Lemma 5.3.2.** (Hadamard) Let $g \in C^2[0, L]$ be such that $|g| \leq M_0$ and $|g''| < M_2$. If $4M_0 < L^2$ then $|g'| \leq \sqrt{M_0(1 + M_2)}$.

To apply this lemma, a strong bound on the distance $|\zeta^{(i+1)}_0(a) - \zeta^{(i)}_0(a)|$ is needed. However, the construction of smooth continuations does not imply any correlation between $\zeta^{(i+1)}_0(a)$ and $\zeta^{(i)}_0(a)$. In order to bound the distance we consider another expression of smooth continuations in light of Corollary 5.2.6. To this end we need some definitions.

Let $h \geq N$. Cut the segment $\mathcal{I}$ into $e^{100\Delta h}$ subsegments of equal length. The mid points of them are called $n$-sample points, or simply sample points. Let $S(h)$ denote the set of all $h$-sample points. Clearly we have
\[
\text{Card}(S(h)) = e^{100\Delta h}.
\]

We say a vector orbit $w = \{w_i(z_i)\}_{i=-h}^0$ is linked to $\tilde{z} \in S(h)$ if:

- **(L1)** the long stable leaf $\Gamma$ of order $h$ through $z_h$ is well-defined;
- **(L2)** $|\mathcal{I} \cap \Gamma - \tilde{z}| \leq e^{-100\Delta h}$.

For $1 \leq j \leq s$, let $z^{(j)} \in S(h_j)$ denote an $h_j$-sample point to which $\Pi^0_{-h_j} w$ is linked. We construct a $C^3$ parameter family of primary quasi critical points $\{\tilde{\zeta}^{(i)}_{0,a}\}_{a \in J(a_*, h_j)}$ of order $h_j$ whose backward orbits share the same set of hyperbolic times and sample points. Then, applying Lemma 5.1.1 to $\tilde{\zeta}^{(j)}_{0,a}$ we obtain for $1 \leq i \leq j$ a primary quasi critical point $\tilde{\zeta}^{(i)}_{0,a}$ of order $h_i$ of $H_a$. By Corollary 5.2.6 we obtain an associated secondary quasi critical point $\zeta^{(i)}_{0,a}$ of order $h_i$. By construction it follows that $\zeta^{(i)}_{0,a}$ is linked to $z^{(i)}$. The construction of continuations and the fact that any admissible curve admits at most one precritical point of the same order (Remark 2.6.2) imply $\zeta^{(i)}_{0,a} = \zeta^{(i)}_0(a)$. Thus it is enough to consider $|\zeta^{(i+1)}_{0,a} - \zeta^{(i)}_{0,a}|$, which can be bound by Lemma 5.1.1 and Corollary 5.2.6.

The following lemma allows us to interpolate between two critical points for different parameters.

**Lemma 5.3.3.** Let $\zeta_0$ be a primary quasi critical point of $H_{a_*}$, with $\{h_i\}_{i=1}^s$ the sequence of hyperbolic times associated with the backward orbit $w = \{w_i(\zeta_i)\}_{i=-\xi}^0$. 

Let $a \in \mathcal{J}(a_s, h_i)$ and suppose that $\zeta_0$ is a primary quasi critical point of $H_a$, with $\hat{w} = \{\hat{w}_i(\zeta_i)\}_{i=-\xi}^0$ the backward orbit. Suppose that:

(i) $h_i$ is a hyperbolic time of $\hat{w}$;
(ii) let $\Gamma$ (resp. $\tilde{\Gamma}$) denote the long stable leaf of order $h_i$ through $\zeta_{-h_i}$ (resp. $\tilde{\zeta}_{-h_i}$). Then $|\Gamma \cap I - \tilde{\Gamma} \cap I| \leq e^{-10^5h_i}$.

Then:

(a) $h_1, h_2, \ldots, h_{i-1}$ are hyperbolic times of $\hat{w}$;
(b) For $1 \leq j < i$, $\Pi_{-h_j}^0 w$ and $\Pi_{-h_j}^0 \hat{w}$ are linked to the same $h_j$-sample point.

We now start the proof of the proposition. Let $\hat{\zeta}_{0,i}$ denote the primary quasi critical point of order $h_j$ which is constructed from $\zeta_0$ by Lemma 5.3.3. Let $w(a) = \{w_i(a)\}_{i=-h_j}^0$ denote its backward vector orbit. It can be read out from the proof of Proposition 5.2.1 that $\{H_{a_j}^h \gamma(w_{-h_j}(a_s), \rho_{h_j})\}_{a \in \mathcal{J}(a_s, h_j)}$ is a family of sufficiently long admissible curves close to one another. Mimicking the proof of Proposition 5.2.1 and using Lemma 5.2.5, we construct a primary quasi critical point $\hat{\zeta}_{0,a}$ of order $h_j$ of $H_a$ on $H_{a_j}^h \gamma(w_{-h_j}(a_s), \rho_{h_j})$ such that the assumptions (i) (ii) in Lemma 5.3.3 are satisfied with respect to $\{\hat{\zeta}_{0,a}\}_{a \in \mathcal{J}(a_s, h_j)}$. Let $w(a)$ denote the backward orbit of $\hat{\zeta}_{0,a}$. We apply Lemma 5.3.3 to $\{\hat{\zeta}_{0,a}\}_{a \in \mathcal{J}(a_s, h_j)}$ and get that, $h_1, h_2, \ldots, h_{j-1}$ are hyperbolic times of $w(a)$ as well and $\Pi_{-h_j}^0 w(a)$ are linked to $z^{(i)}$.

By construction, $w(a)$ is $e^{-11.5\Delta}$-expanding and $e^{-2.5}$-regular. This allows us to apply Lemma 5.1.1 to $\hat{\zeta}_{0,a}$ to yield a primary quasi critical point $\hat{\zeta}_{0,a}$ of order $h_i$ which is linked to $z^{(i)}$. Meanwhile, Corollary 5.2.6 asserts that near each $\hat{\zeta}_{0,a}$ there exists an associated secondary quasi critical point $\zeta_{0,a}^{(i)}$ which is linked to $z^{(i)}$. We now recall that there exists a smooth continuation $a \in \mathcal{J}(a_s, h_i) \rightarrow \zeta_{0,a}^{(i)}(a)$. Since $\mathcal{J}(a_s, h_i) \supset \mathcal{J}(a_s, h_j)$, $\zeta_{0,a}^{(i)}(a)$ is well-defined. The construction of $\zeta_{0,a}^{(i)}$, $\zeta_{0,a}^{(i)}(\cdot)$, and Remark 2.0.2 together imply $\zeta_{0,a}^{(i)} = \zeta_{0,a}^{(i)}(a)$. Using this, Lemma 5.1.1 and Corollary 5.2.6

$$\|\zeta_{0,a}^{(i+1)}(a) - \zeta_{0,a}^{(i)}(a)\| \leq \|\zeta_{0,a}^{(i+1)} - \hat{\zeta}_{0,a}^{(i+1)}\| + \|\hat{\zeta}_{0,a}^{(i+1)} - \hat{\zeta}_{0,a}^{(i)}\| + \|\hat{\zeta}_{0,a}^{(i)} - \hat{\zeta}_{0,a}^{(i)}\|$$

$$\leq 4(Kb)^{h_i}.$$ 

The second order derivative estimate in (e) in Proposition 5.2.1 permits us to apply Lemma 5.3.2 to yield $\|\zeta_{0,a}^{(i+1)}(a) - \zeta_{0,a}^{(i)}(a)\| \leq (Kb)^{h_i}$. Meanwhile we clearly have $\|\zeta_{0,a}^{(i+1)}(a)\| \leq \delta$, because $b$ is chosen to be small after $\delta$. Consequently,

$$\|\hat{\zeta}_{0,a}^{(i)}(a)\| \leq \|\hat{\zeta}_{0,a}^{(i)}(a)\| + \sum_{i=1}^{j-1} \|\hat{\zeta}_{0,a}^{(i+1)}(a) - \zeta_{0,a}^{(i)}(a)\| \leq Kb + \delta/2 \leq \delta.$$ 

The second order derivative estimate is done in the same way. We use Lemma 5.3.2 with respect to $\zeta_{0,a}^{(i+1)}(a) - \zeta_{0,a}^{(i)}(a)$ together with the third order derivative estimate in (e) in Proposition 5.2.1. This completes the proof of Proposition 5.3.1. \qed
6. Inductive assumption

The assumption \((EG)_n\) in itself is not well-adapted to our parameter exclusion argument. For this we need a more sophisticated assumption of inductive nature, called reluctant recurrence condition \((RR)_n\). We show that \((RR)_n\) implies \((EG)_{n+1}\).

6.1. Essential returns. Suppose that \(H\) satisfies \((EG)_n\) for some \(n \geq N\). Suppose that \(w = \{w_i\}_{i=0}^m\) makes a free return at \(m_i \leq m\). If \(w_{m_i}\) is in admissible position relative to some critica point, define
\[
d(m_i) = -\log \frac{\|w_{m_i+p_i}\|}{\|w_{m_i}\|},
\]
where \(p_i\) is the folding period. If \(w_{m_i}\) is in critical position relative to any critical point, define
\[
d(m_i) = \alpha m_i.
\]
Let \(0 < m_i < m_{i+1} < \cdots < m_j \leq m\) denote consecutive free returns of \(w\). We say \(m_j\) is subject to \(m_i\) if
\[
\sum_{i+1 \leq k \leq j} d(m_k) \leq 10d(m_i).
\]
A free return \(\nu\) is called essential if it is the first return time, or else it is not subject to any previous free return. We say \(w\) is reluctantly recurrent up to time \(m\) if
\[
\sum_{\nu \leq j: \text{essential}} d(\nu) \leq \frac{\alpha j}{100} \text{ for every } 0 \leq j \leq m.
\]

6.2. Reluctant recurrence condition. Suppose that \(H\) satisfies \((EG)_n\) for some \(n \geq N\). We say \(H\) satisfies \((RR)_n\) if the forward orbit of every critical point is controlled and reluctantly recurrent up to time \(\min(\beta(n+1), \beta\xi) - 1\), where \(\xi\) is the order of the critical point. To simplify formalism, we say \(H_{a,b}\) satisfies \((RR)_{N-1}\) if \(a \in \Omega(0)\).

Remark 6.2.1. An inductive nature lurks behind the definition of \((RR)_n\), on the relation between the order of binding points and that of controlled critical points. No contradiction arises at this point because of the following two facts: forward orbits of critical points of order \(N\) are obviously controlled and reluctantly recurrent; to control forward orbits of critical points at most up to time \(\beta(n+1)\), only those critical points of order \(\leq \alpha(n+1)/100\) are used. This follows from (13).

Proposition 6.2.2. Suppose that \(H\) satisfies \((EG)_n\), and \(\zeta_0\) is a critical point of order \(m\). If the forward orbit of \(\zeta_0\) is reluctantly recurrent up to time \(k \leq \beta m - 1\), then it is strongly regular up to time \(k+1\). In particular, if \(H\) satisfies \((RR)_n\) then \((EG)_{n+1}\) holds.

See Appendix for the proof.
7. Dynamics of critical curves

Suppose that \( a \in J \to \zeta_0(a) \) is a smooth continuation defined on an interval \( J \subset \Omega(0) \). Define \( \zeta_i(a) = H_{i}^{*}(\zeta_0(a)) \) for \( a \in J \) and \( i \geq 0 \). The aim of this section is to study the behavior of critical curves \( J_i := \{ \zeta_{i+1}(a) : a \in J \} \) under the assumption \((RR)_{n-1}\).

7.1. Distortion with respect to parameterized curves. Let \((RR)_{n-1}\) denote the set of \( a \in \Omega(0) \) such that \( H_{a,b} \) satisfies \((RR)_{n-1}\). Let \( a_* \in (RR)_{n-1} \), and suppose that a vector orbit \( w = \{w_i(z_i)\}_{i=0}^{m} \) of \( H_{a_*} \) is reluctantly recurrent up to time \( m - 1 \). Define

\[
\Phi(w) = e^{-10\Delta} \cdot \left[ \sum_{0 \leq i \leq m-1} \Theta(w,i)^{-1} \right]^{-1}.
\]

Put \( \alpha_0 = \frac{\alpha \lambda \sigma}{200 \Delta} \), and define

\[
J(a_*, w, d) = [a_* - e^{-\alpha_0 d/2} \Phi(w), a_* + e^{-\alpha_0 d/2} \Phi(w)] \cap \Omega(0).
\]

**Proposition 7.1.1.** Let \( c_0 : J(a_*, w, 0) \to \mathbb{R}^2 \) be a \( C^2 \) map such that:

(i) \( c_0(a_*) = z_0 \) and \( z_0 \in H_{a_*}(C_0) \);

(ii) \( \| \dot{c}_0(a) \| \leq K \delta \). \( \| \dot{c}_0(a) \| \leq K \delta \).

Then for every free iterate \( 1 \leq i \leq m \) of \( w \),

(a) \( c_i(J(a_*, w, 0)) \) is an admissible curve;

(b) for all \( a \in J(a_*, w, 0) \),

(b-i) \( \log \| \dot{c}_i(a) \| \leq 1 + 10 \sum_{0 \leq k \leq i-1} \left[ \Phi(w) \Theta(w,k)^{-1} + \|w_k\|^{-\frac{1}{2}} \right] \); (b-ii) \( \log \| DH^i_a(c_0(a)) (0) \| \leq 1 + 10 \sum_{0 \leq k \leq i-1} \left[ \Phi(w) \Theta(w,k)^{-1} + \|w_k\|^{-\frac{1}{2}} \right] \); (b-iii) \( \| \dot{c}_{i-k}(a) \| \leq (K \delta)^{-k} \| \dot{c}_i(a) \|^2 \) \( 0 \leq \forall k \leq i \).

See Appendix for the proof.

7.2. Distortion with respect to smooth continuations. We fix some assumptions and notation for the rest of this section. Let \( a_* \in (RR)_{n-1} \), and suppose that \( \zeta_0 \) is a critical point of order \( \xi \geq n \) of \( H_{a_*} \). Let \( m - 1 \leq \beta n - 1 \) denote the largest integer up to which the forward vector orbit \( w = \{w_i(z_i)\}_{i=0}^{\beta \xi - 1} \) of \( \zeta_0 \) is reluctantly recurrent. Recall that \((RR)_{n-1}\) implies \( m - 1 \geq \beta (n - 1) - 1 \). Put \( \alpha_0 = \frac{\alpha \lambda \sigma}{200 \Delta} \), and for \( \nu \leq m \) define

\[
J(a_*, \zeta_0, \nu, d) = J(a_*, \Pi_0^{\nu} w, d).
\]

Let \( \zeta_0 \) be a critical point of order \( \xi \geq n \), with \( \{h_i\}_{i=1}^{n} \) the associated sequence of hyperbolic times. We say \( h_i \) is the \( n \)-maximal hyperbolic time if \( h_{i-1} < n \leq h_i \). Since the sequence of hyperbolic times is strictly monotone, the \( n \)-maximal hyperbolic time is uniquely determined.

**Lemma 7.2.1.** Let \( h_i \) denote the \( n \)-maximal hyperbolic time of a critical point \( \zeta_0 \) of order \( \xi \geq n \). Then \( n \leq h_i < 16n \).
Proof. By (b) in Lemma 7.2.1 and the maximality we have $h_i \leq 16h_{i-1} < 16n$. □

We now apply Proposition 7.2.2 to critical curves and obtain the following

**Corollary 7.2.2.** Let $a_s \in RR_{n-1}$, and suppose that $\zeta_0$ is a critical point of $H_{a_s}$ of order $\xi \geq n$, with $\{h_j\}_{j=1}^n$ the associated sequence of hyperbolic times. Let $h_{j_0}$ denote the $n$-maximal hyperbolic time. For every $1 \leq j \leq j_0$, let $\zeta_0^{(j)}(a)$ denote the smooth continuation of order $h_j$ defined on $J(a_s, h_j)$. For every free iterate $\nu \in [\beta h_j/16, \beta h_j]$, $\nu \leq m$ we have:

(a) $J(a_s, \zeta_0, \nu, 0) \subset J(a_s, h_j)$;
(b) $J_\nu = \{\zeta_0^{(j)}(a) \in J(a_s, \zeta_0, \nu, 0)\}$ is an admissible curve;
(c) for all $a \in J(a_s, \zeta_0, \nu, 0)$,

$$\left| \log \frac{\|\zeta_0^{(j)}(a)\|}{\|\zeta_0^{(j)}(a)\|} \right| \leq 20.$$

*Proof.* Let us recall from the proof of Proposition 6.2.2 that $\chi(\cdot)$ is a free iterate. Thus $\Phi(\Pi_0^\nu w) \leq \Theta(\Pi_0^\nu w, \chi(\nu - 1))$ holds. By (S1) and the assumption on $\nu$ we have

$$\Theta(\Pi_0^\nu w, \chi(\nu - 1)) \leq \|w_{\chi(\nu - 1)}\|^{-1} \leq e^{-\lambda \beta h_j/17}.$$

This implies (a). Put $c_i(a) = \zeta_0^{(j)}(a)$. Then $c_0$ clearly satisfies the assumption (i) in Proposition 7.2.2 (ii) is also satisfied by virtue of Proposition 5.3.1. Thus (b) follows. Since the number in the right hand side of (b-i) is $\leq 20$, we obtain (c). □

**Remark 7.2.3.** It is worth to call attention to subtleties behind the proof of the proposition. In the first place, it involves a double induction with respect to $n$ and $i$. When considering the case for general $n$, it is necessary that binding structures for $w$ are available uniformly on $J(a_s, \zeta_0, \nu, 0)$. To be more precise, let $k < n$ denote the order of a binding point $\zeta_0$ at a free return $i \in [0, m]$ of $w$. We need that the secondary quasi critical point of order $k$ associated with $\zeta_0$ has a smooth continuation on $J(a_s, \zeta_0, \nu, 0)$ whose forward orbits obey a uniform distortion estimate in the form of (b-ii) in Proposition 7.1.1. This follows if $\Phi(w) \leq \Phi(\tilde{w})$, where $\tilde{w}$ is the forward orbit of $\zeta_0$. Let us see this. The condition ($RR)_k$ implies $\chi(\beta k) \leq \alpha i$, and hence $\beta k \leq \alpha i \leq \alpha n \ll n$, and in particular

$$\Phi(w) \leq \|w_n\|^{-1} \leq e^{-2\alpha \sigma \beta k - \Delta \beta k} \leq \frac{1}{\beta k} \min_{1 \leq i \leq \beta k} \Theta(\tilde{w}, i) \leq \Phi(\tilde{w}).$$

The following lemma is a slight adaptation of [19] Proposition 6.1 to our context. This is used for the proof of Proposition 7.1.1 as well as for later arguments with $c_i(a_s) = \zeta_0^{(j)}(a_s)$. We have the warranty for omitting the proof because it is almost the same as theirs in which Lemma 8.16.1 plays a crucial role.

**Lemma 7.2.4.** There exists $D_1, D_2 > 0$ such that for every $0 \leq i \leq \nu$,

$$D_1 \leq \frac{\|\dot{c}_i(a_s)\|}{\|w_i\|} \leq D_2.$$
7.3. Expansion at essential returns. Let $0 < \nu_1 < \nu_2 < \cdots < \nu_t \leq \beta n$ denote
the maximal sequence of essential returns up to time $\beta n$. For $i \in [0, t]$, let $s(i) \in [1, s]$ denote
the smallest integer such that $\nu_i \leq \beta h_{s(i)}$ holds. By definition we have $h_{s(i)} \leq h_{j_0}$.

**Proposition 7.3.1.** The secondary quasi critical point $\zeta_0^{(s(i))}$ has a smooth continuation
on $J(a_s, \zeta_0, \nu_i, 0)$. Moreover, for all $a \in J(a_s, \zeta_0, \nu_i, 0) - J(a_s, \zeta_0, \nu_i, d(\nu_i))$,
\[
|\zeta_0^{(s(i))}(a_s) - \zeta_0^{(s(i))}(a)| \geq |\tilde{\zeta}_0 - \zeta_{\nu_i+1}|^{1-\alpha_0/2},
\]
where $\tilde{\zeta}_0$ is a critical point relative to which $w_{\nu_i}$ is in admissible or in critical position.

**Proof.** For now we prove the first half of the assertion. By Proposition 7.2.2 it is enough to prove
$\beta h_{s(i)}/16 \leq \nu_i \leq \beta h_{s(i)}$. The right hand side is obvious by definition. Regarding the left hand side, since $\nu_i \geq \nu_1 > \beta h_1$ we have $s(i) \geq 2$. Thus $\beta h_{s(i)}/16 \leq \beta h_{s(i)-1} < \nu_i$ holds, by Lemma 2.12.1.

**Lemma 7.3.2.** We have
\[
\Phi(\Pi_0^a w) \cdot \|w_{\nu_i}\| \geq |\tilde{\zeta}_0 - \zeta_{\nu_i+1}|^{1-\alpha_0}.
\]

The second half of the assertion is an immediate consequence of this lemma the proof of which is given in Appendix. To see this, recall that $\nu_i$ is an essential return and hence it is free. Thus $\mathcal{J}_{\nu_i}$ is an admissible curve. By Corollary 7.2.2 and Lemma 7.2.4
\[
|\zeta_0^{(s(i))}(a_s) - \zeta_0^{(s(i))}(a)| \geq e^{-3}\|w_{\nu_i}\|\|a_s - a\| \geq e^{-3}\|w_{\nu_i}\|\Phi(\Pi_0^a w) e^{-\alpha_0 d(\nu_i)}.
\]
Therefore, Lemma 7.3.2 yields the desired inequality.

7.4. Binding points for critical curves. The following lemma asserts that one can find binding points for all critical values at any essential return.

**Lemma 7.4.1.** Suppose that $\nu_i$ is an essential return and $w_{\nu_i}(\zeta_{\nu_i+1})$ is in admissible or in critical position relative to a critical point $\zeta_0$. For all $a \in J(a_s, \zeta_0, \nu_i, 0) - J(a_s, \zeta_0, \nu_i, d(\nu_i))$ such that $\zeta_0^{(s(i))}(a) \in \mathcal{C}_3$, there exists a precritical point $\zeta_0(a)$ of $H_a$ relative to which $(\zeta_0^{(s(i))}(a), \zeta_{\nu_i+1}(a))$ is in admissible position. Moreover we have
\[
(14) \quad -\log |\zeta_0(a) - \zeta_0^{(s(i))}(a)| \leq (1 - \alpha_0)d(\nu_i).
\]

**Proof.** Let $\{k_j\}_{j=1}^t$ denote the sequence of hyperbolic times associated with the backward orbit of $\tilde{\zeta}_0$. Let $a \in J(a_s, k_j) \rightarrow \zeta_0^{(j)}(a)$ denote the smooth continuation of order $k_j$. Since $k_j \leq \nu_i$, we have $J(a_s, \zeta_0, \nu_i, 0) \subset J(a_s, k_j) \subset J(a_s, \zeta_0, \nu_i, d(\nu_i))$. Proposition 7.3.1 permits us to apply Lemma 2.7.2 to create a precritical point $\zeta_0^{[k_j]}(a)$ of $H_a$ of order $k_j$ near $\zeta_0^{(0)}(a_s)$ on $J_{\nu_i}$.

We apply Lemma 2.7.1 to construct a sequence of precritical points of lower order. There are two cases: $\zeta_0^{[k_{j-1}]}(a), \ldots, \zeta_0^{[\beta^{-1}k_j]}(a)$ are created on $J_{\nu_i}$, or else there exists some $\ell \in [\beta^{-1}k_j + 1, k_j]$ such that $\zeta_0^{[\ell]}$ is so close to the boundary of $J_{\nu_i}$ that there is no room for $\zeta_0^{[\ell-1]}$ to be created. In the second case, we stop further construction. In the first case, take $s'$ to be the smallest integer such that $h_{s'} \geq \beta^{-1}t$, and apply Lemma 2.7.2 with respect to $\zeta_0^{(s')} (a)$ to create a precritical point of order $h_{s'}$ on $J_{\nu_i}$. 

Since any admissible curve admits only one precritical point of the same order, $\mathcal{O}_0^{[s]}$ coincides with the one which was constructed at the previous step. We repeat the same construction using $\tilde{\mathcal{O}}_0^{(s)}(a)$ instead of $\tilde{\mathcal{O}}_0^{(t)}(a)$. Put $Z_{\nu}(a) = \mathcal{O}_0^{(s(i))}(a)$.

**Sublemma 7.4.2.** Suppose that $Z_{\nu} \in \mathcal{C}_3$. If $\mathcal{O}_0^{[k_t-1]}, \mathcal{O}_0^{[k_t-2]}, \ldots, \mathcal{O}_0^{[k_t]}$ are created as above and $|\mathcal{O}_0^{[\ell]} - \mathcal{O}_0^{[\ell]}| \geq 1/3 \cdot \text{length}(J_{\nu} \cap \mathcal{C}_3)$, then $(Z_{\nu}, \mathcal{O}_0^{[\ell]} \mathcal{C}_3)$ is related to $\mathcal{O}_0^{[\ell]}$.

**Proof.** By Lemma 2.7.1 we have $|\mathcal{O}_0^{[\ell]} - \mathcal{O}_0^{[\ell]}| \leq (Kb)^{\ell}$. Thus the assumption implies

$$k \leq \frac{\log(1/3 \cdot \text{length}(J_{\nu} \cap \mathcal{C}_3))}{\log(Kb)} =: c.$$

Suppose that $(Z_{\nu}, \mathcal{O}_0^{[\ell]})$ is not related to $\mathcal{O}_0^{[\ell]}$. Then we have $|Z_{\nu} - \mathcal{O}_0^{[\ell]}| \geq e^{-c \Delta_{3} \beta} \geq K \cdot (\text{length}(J_{\nu} \cap \mathcal{C}_3))^{1/2}$. This yields a contradiction because $Z_{\nu}, \mathcal{O}_0^{[\ell]} \in J_{\nu} \cap \mathcal{C}_3$ and $\text{length}(J_{\nu} \cap \mathcal{C}_3) < 1$. \hfill \Box

Let $k_0 < k_t$ denote the largest integer such that $\mathcal{O}_0^{[k_0]}$ is well-defined and $(Z_{\nu}, \mathcal{O}_0^{[k_0]})$ is related to $\mathcal{O}_0^{[k_0]}$. We claim that $k_0$ exists. To see this it is enough to show that there exists a precritical point to which $(Z_{\nu}, \mathcal{O}_0^{[k_0]})$ is related. This is indeed the case when the sequence of all precritical points are contained in the $1/3 \cdot \text{length}(J_{\nu})$-neighborhood of $\mathcal{O}_0^{[k_0]}$. Otherwise, we appeal to Sublemma 7.4.2.

Suppose that $(Z_{\nu}, \mathcal{O}_0^{[k_0]})$ is in critical position relative to $\mathcal{O}_0^{[k_0]}$. Then it is related to $\mathcal{O}_0^{[k_0+1]}$, by Sublemma 3.2.6. By the maximality of of $k_0$ we have $k_0 = k_t - 1$. On the other hand, by Proposition 7.3.1 $(Z_{\nu}, \mathcal{O}_0^{[k_t]})$ is not related to $\mathcal{O}_0^{[k_t]}$. This yields a contradiction. Therefore, $(Z_{\nu}, \mathcal{O}_0^{[k_0]})$ is in admissible position relative to $\mathcal{O}_0^{[k_0]}$. \hfill \Box

8. **Proof of Theorem B**

In this section we prove that the set of $a \in \Omega^{(0)}$ such that $H_a$ satisfies $(EG)_n$ for all $n \geq N$ has positive Lebesgue measure.

8.1. **Definition of bad parameter sets.** Let $n \geq N$. We define a subset of $\Omega^{(0)}$ which contains $RR_{n-1} - RR_n$. Fix four integers $r \leq -\Delta \beta n / \log \delta$, $R \geq \alpha \beta n / 100$, $h \in [n, 16n]$, $m \in [\beta(n-1) + 1, \beta n]$. Define $\mathcal{N}_r$ to be the set of all strictly monotone sequences of integers $n = \{n_i\}_{i=1}^{r}$ in $[0, \beta n]$. Define $\mathcal{D}_R$ to be the set of all sequences of integers $d = \{d_i\}_{i=1}^{r}$ such that

$$-\log \delta \leq d_i \text{ and } \sum_{i=1}^{r} d_i = R.$$

Let $\mathcal{P}$ denote the set of sequences $p = \{(h_i, z^{(i)})\}_{i=1}^{s}$ such that:

- **(P1)** $h_i$ are nonzero positive integers;
- **(P2)** $h_a = h$;
- **(P3)** $h_{i+1}/16 \leq h_i \leq h_{i+1}/4$ for $1 \leq i \leq s - 1$;
- **(P4)** $z^{(i)} \in S(h_i)$.

Define $\Omega^{(0)}(n, d, p, m)$ to be the set of all $a \in RR_{n-1}$ such that:
(B1) there exists a critical point \( \zeta_0 \) of order \( \xi \geq n \) such that \( m - 1 \) is the largest integer up to which the forward orbit \( w = \{w_i(\zeta_{i+1})\} \) of \( \zeta_0 \) is reluctantly recurrent; 
(B2) the forward orbit of \( \zeta_0 \) makes essential returns exactly at \( \nu_1 < \nu_2 < \cdots < \nu_r \leq \beta n \) up to time \( \beta n \). For every \( 1 \leq i \leq r \), \( d_i = d(\nu_i) \); 
(B3) the \( n \)-maximal hyperbolic time of the backward orbit is \( h \); 
(B4) \( \Pi_{-h}^0 w \) is linked to \( z^{(i)} \in S(h_i) \) for every \( 1 \leq i \leq s \).

Define

\[
\Omega^{(n)} = \bigcup_{R,r,h} \bigcup_{n,d,p,m} \Omega^{(n)}(n,d,p,m),
\]

where the unions run over all possible combinations of the subscripts. The following lemma is more or less automatic from the above definition.

**Lemma 8.1.1.** For every \( n \geq N \) we have \( RR_{n-1} - RR_n \subset \Omega^{(n)} \).

**Proof.** Suppose that \( a \in RR_{n-1} - RR_n \). By definition, there exists a critical point \( \zeta_0 \) of \( H_a \) of order \( \xi \geq n \) whose forward orbit \( w = \{w_i(\zeta_{i+1})\} \) is not reluctantly recurrent up to time \( \beta n - 1 \). Let \( h \) denote the \( n \)-maximal hyperbolic time, and take a sequence \( p \) of pair of hyperbolic times and sample points. Then (B3) is satisfied by Lemma 2.12.1. Let \( m - 1 \) denote the largest integer up to which \( w \) is reluctantly recurrent. By \( (RR)_{n-1} \) we have \( \beta(n - 1) \leq m - 1 \). Clearly, \( m \) is an essential free return. Let \( n = \{\nu_1 < \nu_2 < \cdots < \nu_r \leq m\} \) denote all the essential returns up to time \( m \), with \( d = \{d_i\}_{i=1}^r \) the corresponding sequence of essential return depths. By Sublemma 8.15.1 two consecutive essential returns are separated by at least \( \Delta^{-1} \log \delta^{-1} \) iterates. Hence \( r \leq \Delta \beta n / \log \delta^{-1} \) holds. Since \( w \) is not reluctantly recurrent up to time \( m \), we have

\[
R := \sum_{i=1}^r d_i \geq \frac{\alpha m}{100}.
\]

Hence we obtain \( a \in \Omega^{(n)}(n,d,p,m) \). \( \square \)

Let \( |\cdot| \) denote the one-dimensional Lebesgue measure.

**Proposition 8.1.2.** For every \( n \geq N \),

\[
|\Omega^{(n)}| \leq |\Omega^{(0)}| \cdot e^{-\alpha_0 \alpha \beta n / 4}.
\]

As a corollary we obtain

\[
\left| \bigcup_{n \geq N} \Omega^{(n)} \right| < |\Omega^{(0)}| \sum_{n \geq N} e^{-\alpha_0 \alpha \beta n / 4} < |\Omega^{(0)}|,
\]

where the last inequality follows from the fact that large \( \beta \) is chosen after \( \alpha \) is fixed. Hence, the set \( \bigcap_{n \geq N} RR_n \) contains a positive measure subset. By Proposition 6.2.2 this implies Theorem B.

A proof of Proposition 8.1.2 needs some preliminary considerations and thus we postpone it to the end of this section.
8.2. Structure in parameter space. Let \( a \in \Omega^{(n)}(\cdot) \). We say a critical point \( \zeta_0 \) of \( H_a \) of order \( \geq n \) is responsible for \( a \) if \( \zeta_0 \) satisfies (B1) (B2) (B3) (B4).

**Lemma 8.2.1.** Let \( a, \tilde{a} \in \Omega^{(n)}(\cdot) \). Suppose that \( \zeta_0, \tilde{\zeta}_0 \) are critical points which are responsible for \( a \) and \( \tilde{a} \) respectively. Let \( \zeta_0^{(i)}(\cdot), \tilde{\zeta}_0^{(i)}(\cdot) \) denote the smooth continuations of order \( h_i \leq h \) of \( \zeta_0 \) and \( \tilde{\zeta}_0 \). If \( J(a, \zeta_0, \nu, 0) \cap J(\tilde{a}, \tilde{\zeta}_0, \nu, 0) \neq \emptyset \) holds for some \( \nu \in [\beta h_i/16, \beta h_i] \), then \( \zeta_0^{(i)}(b) = \tilde{\zeta}_0^{(i)}(b) \) holds for all \( b \in J(a, \zeta_0, \nu, 0) \cap J(\tilde{a}, \tilde{\zeta}_0, \nu, 0) \).

**Proof.** Keeping (B3) in mind, recall the construction of smooth continuations in Section 5 and use the fact that one admissible curve does not admit more than two precritical points of the same order (Remark 2.6.2). \( \square \)

**Lemma 8.2.2.** Let \( a_* \in \Omega^{(n)}(\cdot) \), and let \( \zeta_0 \) denote a critical point which is responsible for \( a_* \). For every \( i \in [1, r] \), the set \( J(a_*, \zeta_0, \nu_i, 0) - J(a_*, \zeta_0, \nu_i, d_i) \) does not intersect \( \Omega^{(n)}(\cdot) \).

**Proof.** Consider the smooth continuation \( b \in J(a_*, \zeta_0, \nu_i, 0) \rightarrow \zeta_0^{(s(i))}(b) \) of order \( h_s(i) \) of \( \zeta_0 \) given by Proposition 5.2.1. Let \( a \in J(a_*, \zeta_0, \nu_i, 0) - J(a_*, \zeta_0, \nu_i, d_i) \) and suppose that \( a \in \Omega^{(n)}(\cdot) \). Let \( \tilde{\zeta}_0 \) denote any critical point which is responsible for \( a \). Consider the smooth continuation \( \zeta_0^{(s(i))}(\cdot) \) of order \( h_s(i) \) of \( \tilde{\zeta}_0 \). By \( a \in J(a_*, \zeta_0, \nu_i, 0) \cap J(a_*, \zeta_0, \nu_i, 0) \) and Lemma 8.2.1 we have \( \zeta_0^{(s(i))}(a) = \zeta_0^{(s(i))}(a) \). Meanwhile, by Lemma 7.4.1 and the assumption on \( a \), \( \zeta_0^{(s(i))}(a) \) is in admissible position. By Lemma 5.1.1 and Corollary 5.2.6 \( \zeta_0^{(s(i))}(a) \) is in admissible position as well.

**Sublemma 8.2.3.** Suppose that \( v_0(z_0) \) is in admissible position relative to two critical points \( \zeta_0 \) and \( \tilde{\zeta}_0 \). Then

\[ -\log |\zeta_0 - z_0| \leq -(1 + \alpha_0) \log |\tilde{\zeta}_0 - z_0| \]

**Proof.** Let \( n \) and \( \tilde{n} \) denote the orders of \( \zeta_0 \) and \( \tilde{\zeta}_0 \) respectively. Suppose that \( \tilde{n} \in [\Delta^{-1}\lambda n, \Delta\lambda^{-1}n] \). Split \( \xi e_n + \eta f_n = DH(z)v(z) = \xi e_n + \tilde{\eta} f_{\tilde{n}} \). Since angle(\( e_n, e_{\tilde{n}} \) \( \leq (\text{Kb})^\min(n, n, \tilde{n}) \leq (\text{Kb})^{\Delta^{-1}\lambda n} \)), we have \( |\tilde{\eta} - \eta| \leq (\text{Kb})^{\Delta^{-1}\lambda n} \), and thus \( |\eta| \approx |\tilde{\eta}| \). By Lemma 8.1.1 this implies the desired inequality.

It is left to prove \( \tilde{n} \in [\Delta^{-1}\lambda n, \Delta\lambda^{-1}n] \). Suppose that \( \tilde{\zeta}_0 \) is closer to \( z_0 \) than \( \zeta_0 \). Then (AP2) implies \( \tilde{n} \geq \Delta^{-1}\lambda n \). By the same reasoning, we have \( \tilde{n} \leq \Delta\lambda^{-1}n \) when \( \zeta_0 \) is closer to \( z_0 \) than \( \tilde{\zeta}_0 \). \( \square \)

By Sublemma 8.2.3 and Proposition 7.3.1 the essential return depth \( d(\nu_i) \) of the forward orbit of \( \zeta_0 \) at time \( \nu_i \) is strictly smaller than \( d_i \). Thus (B2) does not hold. This yields a contradiction to the assumption that \( \tilde{\zeta}_0 \) is responsible for \( a \). \( \square \)

**Lemma 8.2.4.** Let \( a, \tilde{a} \in \Omega^{(n)}(\cdot) \). Suppose that \( \zeta_0, \tilde{\zeta}_0 \) are critical points which are responsible for \( a \) and \( \tilde{a} \) respectively. Let \( \nu_i, \nu_j \in n \) and suppose that \( \nu_i < \nu_j \). If \( J(a, \zeta_0, \nu_i, d_i) \cap J(\tilde{a}, \tilde{\zeta}_0, \nu_j, d) \neq \emptyset \) holds for some \( d \geq -\log \delta \), then \( J(\tilde{a}, \tilde{\zeta}_0, \nu_j, 0) \subset J(a, \zeta_0, \nu_i, d_i - \alpha_0^{-1}) \).

**Proof.** By Proposition 7.2.2, the critical curve \( \{\zeta_0^{(s(i))}(b) : b \in J(a, \zeta_0, \nu_i, d_i)\} \) is an admissible curve. By Lemma 7.3.2 there exists \( \tilde{a} \subset J(a, \zeta_0, \nu_i, d_i) \) such that \( \zeta_0^{(s(i))}(\tilde{a}) \) is a critical point of order \( \nu_i \) of \( H_{\tilde{a}} \). We claim that \( \tilde{a} \notin J(\tilde{a}, \tilde{\zeta}_0, \nu_j, 0) \) holds. This
implies that one of the connected components of \( J(\tilde{a}, \tilde{\zeta}_i, \nu_j, 0) - J(\tilde{a}, \tilde{\zeta}_0, \nu_j, d) \) is contained in \( J(a, \nu_i, d_i) \). This implies
\[
2^{-1}(1 - e^{-\alpha_0 d^2/2})|J(\tilde{a}, \tilde{\zeta}_0, \nu_j, 0)| \leq |J(a, \nu_i, d_i)|.
\]
Using \( d \geq -\log \delta \) and the fact that \( \delta \) is chosen after \( \alpha_0 \), we obtain the inclusion.

It is left to prove the claim. Suppose that \( \hat{a} \in J(\tilde{a}, \tilde{\zeta}_0, \nu_j, 0) \). Consider the smooth continuation \( \tilde{c}(s(i)) \) of the secondary quasi critical point associated with \( \tilde{\zeta}_0 \). By Lemma 8.2.1, we have \( \tilde{c}(s(i)) = \tilde{c}(s(i)) \), and thus \( \tilde{c}(s(i)) \) is a critical point of order \( s(i) \). This yields a contradiction to the fact that points on the critical curve is in admissible position relative to some critical point, which was already proved in the proof of Lemma 8.16.7.

**Lemma 8.2.5.** Let \( a, \tilde{a} \in \Omega^{(\nu)}(\cdot) \). Suppose that \( \zeta_0, \tilde{\zeta}_0 \) are critical points which are respectively responsible for \( a \) and \( \tilde{a} \). Assume that:

1. \( J(a, \zeta_0, \nu_i, 0) \cap J(\tilde{a}, \tilde{\zeta}_0, \nu_i, 0) \neq \emptyset \);
2. \( \tilde{a} \notin J(a, \zeta_0, \nu_i, 0) \).

Then we have \( J(a, \zeta_0, \nu_i, 0) \subset J(\tilde{a}, \tilde{\zeta}_0, \nu_i, 0) \).

**Proof.** Consider the two smooth continuations of order \( h_{s(i)}, b \in J(a, \zeta_0, \nu_i, 0) \rightarrow \zeta_0(b) \) and \( \tilde{b} \in J(\tilde{a}, \tilde{\zeta}_0, \nu_i, 0) \rightarrow \tilde{\zeta}_0(b) \). By Proposition 7.2.2 and 7.3.1, there exist \( c \in J(a, \zeta_0, \nu_i, d_i) \) and \( \tilde{c} \in J(\tilde{a}, \tilde{\zeta}_0, \nu_i, d_i) \) such that \( \zeta_{\nu_i+1}(c) \) and \( \tilde{\zeta}_{\nu_i+1}(\tilde{c}) \) are precritical points of order \( \nu_i \). Suppose that \( c \neq \tilde{c} \). By Lemma 8.2.1, \( \zeta_0(c) = \zeta_0(\tilde{c}) \) holds for all \( e \in J(a, \zeta_0, \nu_i, 0) \cap J(\tilde{a}, \tilde{\zeta}_0, \nu_i, 0) \). Thus, it follows that \( J_{\nu_i+1}(a, \zeta_0, \nu_i, 0) \cup J_{\nu_i+1}(\tilde{a}, \tilde{\zeta}_0, \nu_i, 0) \) is an admissible curve which admits two distinct precritical points of the same order. This is a contradiction. Hence \( c = \tilde{c} \) holds. This implies that one of the connected component of \( J(a, \zeta_0, \nu_i, 0) - J(a, \zeta_0, \nu_i, d_i) \) is contained in \( J(\tilde{a}, \tilde{\zeta}_0, \nu_i, d_i) \), and thus \( J(a, \zeta_0, \nu_i, 0) \subset J(\tilde{a}, \tilde{\zeta}_0, \nu_i, 0) \).

### 8.3. Total number of combinations.

**Lemma 8.3.1.** There exists \( \tau(\delta) > 0 \) such that \( \tau(\delta) \rightarrow 0 \) as \( \delta \rightarrow 0 \) and
\[
\text{card}(\mathcal{N}_r) \leq e^{\tau(\delta)3n} \quad \text{and} \quad \text{card}(\mathcal{D}_R) \leq e^{\tau(\delta)R}.
\]

**Proof.** The cardinality of \( \mathcal{D}_R \) is smaller than the total number of combinations of dividing \( R \) objects into \( r \) groups. Hence we have \( \text{card}(\mathcal{D}_R) \leq (\binom{R+r}{r}) \).

**Sublemma 8.3.2.** For any \( c > 0 \), there exists \( s_0 > 0 \) such that
\[
\binom{n+s}{s} \leq e^{cn}
\]
holds for all positive integers \( n, s \) such that \( s \leq s_0 n \).

**Proof.** Choose \( s_0 > 0 \) such that \( s_0 \leq c/3 \), \( s_0^{-s_0} \leq e^{c/3} \), and \( (1 + s_0)^{s_0} \leq e^{c/3} \). The Stirling formula for factorials \( k! \in [1 + 1/4k] \sqrt{2\pi kk^k} e^{-k} \) gives
\[
\binom{n+s}{s} = \frac{(n+s)!}{n!s!} \leq \frac{(n+s)^{n+s}}{n^n s^s} \leq \left( \frac{n+s}{n} \right)^{n} \left( \frac{n+s}{s} \right)^{s}.
\]
Regarding the first term,
\[
\left(\frac{n+s}{n}\right)^n = \left(1 + \frac{s}{n}\right)^n = e^{n \log \left(1 + \frac{s}{n}\right)} \leq e^s \leq e^{\frac{cn}{10}}.
\]

Regarding the second term,
\[
\left(\frac{n+s}{s}\right)^s = \left[\left(\frac{s}{n(1 + s/n)}\right)^{-s/n}\right]^n \leq \left[\left(\frac{s}{n}\right)^{-s/n}\left(1 + \frac{s}{n}\right)^{s/n}\right]^n \leq e^{2cn/3}.
\]

Sublemma 8.3.2 and \(r \leq R/\log \delta^{-1}\) yields the desired inequality. The same argument applies to \(\mathcal{N}_r\) because \(\text{card}(\mathcal{N}_r) \leq (\beta^n_r) \leq (\beta^{n+r}_r)\) and \(r \leq \Delta/\beta_n/\log \delta^{-1}\).

**Lemma 8.3.3.** We have \(\text{card}(\mathcal{P}) \leq e^{500\Delta n}\).

**Proof.** By (P2) and (P3) we have \(h_i \leq 4^{i-s}h\). Using (P4) we have
\[
\text{card}(\mathcal{P}) \leq \sum_{s=1}^{n} \left[\left(\frac{n}{s}\right) \cdot \exp \left(100\Delta \sum_{i=1}^{s} 4^{i-s}h\right)\right].
\]
Using \(\sum_{s=0}^{n} \binom{n}{s} = 2^n\) we obtain
\[
\text{card}(\mathcal{P}) \leq \left[\sum_{s=1}^{n} \binom{n}{s}\right] \cdot \left[\sum_{s=1}^{n} \exp \left(100\Delta \sum_{i=1}^{s} 4^{i-s}h\right)\right]
\leq 2^n \sum_{s=1}^{n} e^{400\Delta h/3}
\leq e^{500\Delta n}.
\]

\[\Box\]

8.4. **Proofs of Proposition 8.1.2.** For \(a \in \Omega^{(n)}(\cdot)\) and \(d \geq 0\), denote by \(J(a, \nu, d)\) any parameter interval of the form \(J(a, \zeta, \nu, d)\), where \(\zeta\) is a critical point which is responsible for \(a\).

We consider the following operation. Choose some \(a_1 \in \Omega^{(n)}(\cdot)\). If \(\Omega^{(n)}(\cdot) \subset J(a_1, \nu_1, 0)\), then stop the operation. If not, choose \(a_2 \in \Omega^{(n)}(\cdot) - J(a_1, \nu_1, 0)\) and ask whether \(\Omega^{(n)}(\cdot) \subset J(a_1, \nu_1, 0) \cup J(a_2, \nu_1, 0)\) or not. If so, then stop the operation. If not, choose \(a_3 \in \Omega^{(n)}(\cdot) - J(a_1, \nu_1, 0) - J(a_2, \nu_1, 0)\) and ask whether \(\Omega^{(n)}(\cdot) \subset J(a_1, \nu_1, 0) \cup J(a_2, \nu_1, 0) \cup J(a_3, \nu_1, 0)\) or not. Repeat this. Since the length of intervals of the form \(J(a, \nu_1, 0)\) are bounded from below, this operation stops sooner or later and we end up with a finite set of parameters \(S_1 = \{a_1, \ldots, a_{\ell_1}\} \subset \Omega^{(n)}(\cdot)\) such that
\[
(15) \quad \Omega^{(n)}(\cdot) \subset \bigcup_{j_1=1}^{\ell_1} J(a_{j_1}, \nu_1, 0).
\]
By Lemma 8.2.5, any two of the intervals in the union does not intersect each other, unless one is contained in the other. Hence, \(\Omega^{(n)}(\cdot)\) is contained in the union of two by two disjoint intervals which is maximal with respect inclusion among unions with the same property. Without loss of generality we may assume that the intervals in (15) are two by two disjoint, and we do so for simplicity.

We extend this operation in the following way. Let \(i \geq 1\), and denote by \(\mathbf{j}(i) = (j_1, j_2, \ldots, j_i)\) the multi index. Suppose that we are given a finite set of parameters
S_i = \{a_{j(i)}\} \subset \Omega^{(n)}(\cdot) such that \Omega^{(n)}(\cdot) \subset \bigcup_{a_{j(i)} \in S_i} J(a_{j(i)}, \nu_i, 0). For each a_{j(i)} \in S_i, applying the above operation to J(a_{j(i)}, \nu_i, 0) \cap \Omega^{(n)}(\cdot) in the place of \Omega^{(n)}(\cdot), we define a finite set of parameters S_{j(i)} = \{a_{j(i)}, a_{j(i)}_2, \ldots, a_{j(i)}_\ell_{i+1}\} \subset \Omega^{(n)}(\cdot) such that

\[ J(a_{j(i)}, \nu_i, 0) \cap \Omega^{(n)}(\cdot) \subset \bigcup_{j=i+1}^{\ell_{i+1}} J(a_{j(i)}, j_{i+1}, \nu_{i+1}, 0). \]

Define S_{i+1} = \bigcup_{j(i)} S_{j(i)}. Then \Omega^{(n)}(\cdot) \subset \bigcup_{a_{j(i+1)} \in S_{i+1}} J(a_{j(i+1)}, \nu_{i+1}, 0) holds. For the same reason as before, we may assume that the intervals in the union are two by two disjoint. We repeat this construction up to i = r.

**Claim 8.4.1.**

\[ \sum_{j=1}^{\ell_i} |J(a_{j1}, \nu_1, d_i)| \leq |\Omega^{(0)}| \cdot e^{-a_0d_i/2}. \]

**Proof.** It holds that

\[ \sum_{j=1}^{\ell_i} |J(a_{j1}, \nu_1, d_i)| \leq e^{-a_0d_i} \sum_{j=1}^{\ell_i} |J(a_{j1}, \nu_1, 0)|. \]

Since \{J(a_{j1}, \nu_1, 0)\}_{j=1}^{\ell_i} are two by two disjoint intervals which are contained in \Omega^{(0)}, we get the claim. \[ \square \]

**Claim 8.4.2.** For every 1 \leq i \leq r - 1 and a_{j(i)} \in S_i,

\[ \sum_{j=i+1}^{\ell_{i+1}} |J(a_{j(i), j_{i+1}}, \nu_{i+1}, d_{i+1})| \leq e^{-a_0d_{i+1}/2} \cdot |J(a_{j(i)}, \nu_i, d_i)|. \]

**Proof.** It holds that

\[ \sum_{j=i+1}^{\ell_{i+1}} |J(a_{j(i), j_{i+1}}, \nu_{i+1}, d_{i+1})| \leq e^{-a_0d_{i+1}} \sum_{j=i+1}^{\ell_{i+1}} |J(a_{j(i), j_{i+1}}, \nu_{i+1}, \alpha_0^{-1})|. \]

Since the intervals \{J(a_{j(i), j_{i+1}}, \nu_{i+1}, \alpha_0^{-1})\}_{j=i+1}^{\ell_{i+1}} are two-by-two disjoint, it is enough to show that they are contained in J(a_{j(i)}, \nu_i, d_i - \alpha_0^{-1}). This follows from Lemma 8.2.4 and J(a_{j(i)}, \nu_i, d_i) \cap J(a_{j(i)}, \nu_i, d_i) \neq \emptyset for every j_{i+1}, by construction. \[ \square \]

We are now in position to estimate the measure of \Omega^{(n)}(\cdot). Lemma 8.2.2 gives \Omega^{(n)}(\cdot) \subset \bigcup_{j(r)} J(a_{j(r)}, \nu_r, d_r), and thus

\[ |\Omega^{(n)}(\cdot)| \leq \sum_{j(r)} |J(a_{j(r)}, \nu_r, d_r)| = \sum_{j(r-1)} \sum_{j_{r-1}}^{\ell} |J(a_{j(r-1), j_{r}}, \nu_r, d_r)|. \]

Notice the nested nature of the expression of the right hand side: \ell_r depends on j(r - 1). Using Lemma 8.4.2

\[ \sum_{j(r)} |J(a_{j(r)}, \nu_r, d_r)| \leq e^{-a_0d_r} \sum_{j(r-1)} |J(a_{j(r-1)}, \nu_{r-1}, d_{r-1})|. \]

Using this recursively we obtain

(16) \[ |\Omega^{(n)}(\cdot)| \leq \sum_{j(r)} |J(a_{j(r)}, \nu_r, d_r)| \leq |\Omega^{(0)}| e^{-a_0R/2}. \]
We now estimate the measure of \( \Omega^{(n)} \). By definition we have

\[
|\Omega^{(n)}| \leq \sum_{R,r,h} \sum_{n,d,p,m} |\Omega^{(n)}(n,d,p,m)|.
\]

Using (16),

\[
|\Omega^{(n)}| \leq |\Omega^{(0)}| \sum_{R,r,h} \text{card}(\mathcal{N}_r \times \mathcal{D}_R \times \mathcal{P}) \cdot \beta \cdot e^{-\alpha_0 R/2}.
\]

Using Lemma 8.3.1 and Lemma 8.3.3,

\[
|\Omega^{(n)}| \leq |\Omega^{(0)}| \cdot \beta \cdot \sum_{R,r,h} e^{-\alpha_0 R/2 + \tau(\delta)R + \tau(\delta)\beta n + 500\Delta n} \leq |\Omega^{(0)}| \cdot \beta \cdot \sum_{R,r,h} e^{-\alpha_0 R/3}.
\]

Using \( h \leq 16n \) and \( r \leq R \),

\[
|\Omega^{(n)}| \leq |\Omega^{(0)}| \cdot 16\beta n \cdot \sum_{R} e^{-\alpha_0 R/4}.
\]

Using \( R \geq \alpha \beta n/100 \) we obtain

\[
|\Omega^{(n)}| \leq |\Omega^{(0)}| \cdot 16\beta n \cdot e^{-\alpha_0 \alpha \beta n/400} \leq |\Omega^{(0)}| e^{-\alpha_0 \alpha \beta n/401}.
\]

This finishes the proof of Proposition 8.1.2, and hence that of Theorem B. \( \square \)

APPENDIX: COMPUTATIONAL PROOFS

8.5. Proof of Lemma 2.1.1. Parametrize \( \gamma_0 \) by \( s \in [0,1] \), and suppose that \( z_0 = \gamma_0(s_0) \). Let \( \gamma_i(s) = H^i(\gamma_0(s)) \) for \( i \geq 0 \). Let

\[
DH = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

and

\[
D^2H(\gamma_{i-1}(s)) = \begin{pmatrix} \langle \nabla A, \dot{\gamma}_{i-1}(s) \rangle & \langle \nabla B, \dot{\gamma}_{i-1}(s) \rangle \\ \langle \nabla C, \dot{\gamma}_{i-1}(s) \rangle & \langle \nabla D, \dot{\gamma}_{i-1}(s) \rangle \end{pmatrix}.
\]

It is easy to see that \( \|\dot{\gamma}_{i}(s_0)\|^3 \cdot \kappa_i(z_i) \leq I + II \), where

\[
I = Kb \cdot \|\dot{\gamma}_{i-1}(s_0)\|^3 \kappa_{i-1}(z_{i-1})
\]

and

\[
II = \|DH(\gamma_{i-1}(s_0))\dot{\gamma}_{i-1}(s_0) \times D^2H(\gamma_{i-1}(s_0))\dot{\gamma}_{i-1}(s_0)\|.
\]

The vector product in \( II \) is degree three homogeneous in \( \|\dot{\gamma}_{i-1}(s_0)\| \). Moreover, since the \( C^1 \)-norms of \( B, C, D \) are bounded by \( Kb \), the second components of the two vectors in the product have a factor \( b \). Therefore

\[
\kappa_i(z_i) \leq \frac{\|v_{i-1}\|^3}{\|v_i\|^3} (Kb + Kb \cdot \kappa_{i-1}(z_{i-1})).
\]

A recursive use of this inequality gives the desired one. \( \square \)
8.6. Proof of Lemma 2.1.2. Let $\kappa_i$ denote the maximum of the curvature of $\gamma_i$. Then

$$\text{length}(\gamma_0) \leq \Xi(\mathbf{v})\Theta(\mathbf{v}, 0) - 1\Theta(\mathbf{v}, 0) \leq \Xi(\mathbf{v})\Theta(\mathbf{v}, 0) - 1 \|v_1\|^2 / \|v_0\|^2 \leq e^{\Delta - \alpha\sigma_n} / \|v_1\|.$$ 

Since $n \geq M$ it is enough to prove the following by induction on $i \in [0, n - 1]$: 

$$\text{(17)} \quad (1 + \kappa_i) \cdot \text{length}(\gamma_i) \leq e^{2\Delta - \alpha\sigma_n} \|v_{i+1}\| / \|v_i\|,$n

$$\text{(18)} \quad \left| \log \frac{\|DH_i^{i+1}(z_0)t_{\gamma_i}(z_0)\|}{\|DH_i^{i+1}(y_0)t_{\gamma_i}(y_0)\|} \right| \leq (i + 1)e^{2\Delta - \alpha\sigma_n/2} \forall y_0 \in \gamma_0.$$ 

Notice that (17) for $i = 0$ follows from the above inequality.

$(17) \Rightarrow (18)$. Let $0 \leq j \leq i$ and $y_0 \in \gamma_0$. Put $u_j = DH_i(y_0)t_{\gamma_i}(y_0)$. Using (17),

$$\left| \frac{\|v_{j+1}\|}{\|v_j\|} - \frac{\|u_{j+1}\|}{\|u_j\|} \right| \leq e^\Delta (1 + \kappa_j) \text{length}(\gamma_j) \leq e^{3\Delta - \alpha\sigma_n} \|v_{j+1}\| / \|v_j\|,$n

and thus

$$\frac{\|u_{j+1}\|}{\|u_j\|} \geq \left| \frac{\|v_{j+1}\|}{\|v_j\|} - \frac{\|u_{j+1}\|}{\|u_j\|} \right| \geq (1 - e^{3\Delta - \alpha\sigma_n}) \|v_{j+1}\| / \|v_j\|.$$ 

Taking logs,

$$\left| \log \frac{\|v_{j+1}\|}{\|v_j\|} - \log \frac{\|u_{j+1}\|}{\|u_j\|} \right| \leq e^{3\Delta - \alpha\sigma_n/2}.$$ 

Using this for every $0 \leq j \leq i$ we obtain (18).

$(18) \Rightarrow (17)$ with $i = i + 1$. Using (18),

$$\text{length}(\gamma_{i+1}) \leq e \cdot \frac{\|v_{i+1}\|}{\|v_0\|} \text{length}(\gamma_0) \leq e \cdot \Xi(\mathbf{v}) \frac{\|v_{i+1}\|}{\|v_0\|}.$$ 

Using Lemma 2.1.1 and $\kappa_0 \leq 1$,

$$(1 + \kappa_{i+1}) \cdot \text{length}(\gamma_{i+1}) \leq \Xi(\mathbf{v}) (I + II + III),$$

where

$$I = e \frac{\|v_{i+1}\|}{\|v_0\|},$$

$$II = e^4(Kb)^{i+1} \frac{\|v_0\|^2}{\|v_{i+1}\|^2},$$

$$III = e^4 \frac{\|v_{i+1}\|}{\|v_0\|} \sum_{j=1}^{i+1} (Kb)^j \frac{\|v_{i+1-j}\|^3}{\|v_{i+1}\|^3}.$$ 

By the definition of $\Theta(\mathbf{v}, i + 1)$,

$$\text{(19)} \quad \Theta(\mathbf{v}, i + 1)^{-1} \Theta(\mathbf{v}, i + 1) \leq \Theta(\mathbf{v}, i + 1)^{-1} \frac{\|v_0\|}{\|v_{i+1}\|} \frac{\|v_{i+2}\|^2}{\|v_{i+1}\|^2},$$

and therefore

$$I \leq e^{\Delta} \Theta(\mathbf{v}, i + 1)^{-1} \frac{\|v_{i+2}\|}{\|v_{i+1}\|}.$$
Using (19) and the expansivity of $\mathbf{v}$,
\[
II \leq e^4(Kb)^{i+1} \Theta(\mathbf{v}, i + 1)^{-1} \frac{\|v_0\|^3}{\|v_{i+1}\|^3} \left(\frac{\|v_{i+2}\|}{\|v_{i+1}\|}\right)^2
\]
\[
\leq (Kb)^{i+1}b^{-\frac{3(i+1)}{4}} e^{4+\Delta} \Theta(\mathbf{v}, i + 1)^{-1} \frac{\|v_{i+2}\|}{\|v_{i+1}\|}
\]
\[
\leq \Theta(\mathbf{v}, i + 1)^{-1}\frac{\|v_{i+2}\|}{\|v_{i+1}\|}.
\]

Regarding $\text{III}$, for every $0 \leq k \leq n$ we have
\[
\frac{\|v_{i+1}\| \|v_{i+1-j}\|^3}{\|v_0\| \|v_{i+1}\|^3} = \Theta(\mathbf{v}, k)^{-1} \Theta(\mathbf{v}, k) \frac{\|v_{i+1}\| \|v_{i+1-j}\|^3}{\|v_0\| \|v_{i+1}\|^3}
\]
\[
= \Theta(\mathbf{v}, k)^{-1} \min_{k \leq \ell \leq n} \frac{\|v_{i+1}\| \|v_{\ell}\|^2 \|v_{i+1-j}\|^3}{\|v_k\| \|v_k\|^2 \|v_{i+1}\|^3}.
\]

Substituting $k = i + 1 - j \leq n - 1$ into the right hand side and then using $\min_{i+1-j \leq \ell \leq n} \|v_\ell\|^2 \leq \|v_{i+1}\| \|v_{i+2}\|$, we have
\[
\frac{\|v_{i+1}\| \|v_{i+1-j}\|^3}{\|v_0\| \|v_{i+1}\|^3} \leq \Theta(\mathbf{v}, i + 1 - j)^{-1}\frac{\|v_{i+2}\|}{\|v_{i+1}\|}.
\]

Consequently,
\[
\text{III} \leq \frac{\|v_{i+2}\|}{\|v_{i+1}\|} \sum_{j=1}^{i+1} (Kb)^j \cdot \Theta(\mathbf{v}, i + 1 - j)^{-1}.
\]

Altogether these three inequalities and the definition of $\Xi(\mathbf{v})$ yield (17) with $i + 1$ in the place of $i$. \qed

8.7. Proof of Lemma 2.4.4. The well-definedness follows from $|\det DH(z)| \leq Kb$ and $\|DH(z)\| \geq 2\delta \gg \sqrt{Kb}/\pi$. Let $\|DH e_1\| = \lambda$, and
\[
DH(z) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]

The Lagrange method of undetermined coefficients gives
\[
e_1 = \rho^{-1}(B^2 + D^2 - \lambda^2, -(AB + CD)),
\]
where $\rho > 0$ is the normalizing constant. We have $\lambda \leq Kb\|DH(z)\|^{-1} \leq 2K|x|^{-1}$, and (1) implies that $B, C, D$ are $O(b)$, and $|A| \leq K|x|$. Altogether these imply the lower estimate of the slope of $e_1(z)$.

Using the fact that $\lambda$ is the smaller eigenvalue of $DH(z)^*DH(z)$ we have
\[
\lambda = \frac{I - \sqrt{T^2 - 4II}}{2},
\]
where $I = A^2 + B^2 + C^2 + D^2$, $II = A^2D^2 + B^2C^2 - 2ABCD$. Since all the partial derivatives of $B, C, D$ are $O(b)$ and $\|\partial A\| \leq K$, we have $\rho \geq Kb|x|$. This yields $I, \|\partial I\| \leq K|x| \leq \sqrt{T^2 - 4II}$, $|\partial II| = O(b)$, and in particular $\|\partial \rho\| \leq Kb$ and $\|\partial \lambda\| \leq K|x|$. Putting altogether these we obtain the upper estimate of $\|\partial e_1(z)\|$. The rest of the assertion follows from Corollary 2.4.2 and Corollary 2.4.3. \qed
8.8. **Proof of Proposition 2.5.1.** We prove (a) (b) by induction. (c) is not an inductive issue and can easily be read out from the argument and Lemma 2.4.4.

It is easy to see by perturbation and Lemma 2.4.4 that (a) holds for \(k = 1\). (b) for \(k = 1\) holds because \(\|DH^i(z_0) (\frac{1}{i}) - DH^i(z_0') (\frac{1}{i})\| \ll \kappa^i\) for \(z_0' \in \Gamma(1)(\Pi_0^{\max(M,2)} w)\) and \(i = 1, 2\).

**Sublemma 8.8.1.** Let \(2 \leq j \leq n - 1\), and assume (a) (b) for \(1 \leq k \leq j - 1\). Then \(\Gamma^{(j)}\) is a long stable leaf and satisfies \(\Gamma^{(j)} \subset \Gamma^{(j - 1)}(\Pi_0^{\max(M, J)} w)\).

**Proof.** Parametrize \(\Gamma^{(j)}\) and \(\Gamma^{(j - 1)}\) by arc length and assume that \(z_0 = \Gamma^{(j)}(0) = \Gamma^{(j - 1)}(0)\). Suppose that \(\Gamma^{(j)}(s)\) is well-defined for \(s \in [0, s_0]\). For any such \(s\), using Lemma 2.4.1 and Lemma 2.4.4

\[
\|e_j(\Gamma^{(j)}(s)) - e_{j - 1}(\Gamma^{(j - 1)}(s))\| \leq \|e_j(\Gamma^{(j)}(s)) - e_{j - 1}(\Gamma^{(j)}(s))\| + \|e_{j - 1}(\Gamma^{(j)}(s)) - e_{j - 1}(\Gamma^{(j - 1)}(s))\|
\]

\[
\leq \left(\frac{Kb}{\kappa^2}\right)^{j - 1} + K_1 \delta^{-2} |\Gamma^{(j)}(s) - \Gamma^{(j - 1)}(s)|.
\]

Therefore

\[
|\Gamma^{(j)}(s) - \Gamma^{(j - 1)}(s)| = \left|\int_0^s \dot{\Gamma}^{(j)}(s) - \dot{\Gamma}^{(j - 1)}(s) ds\right|
\]

\[
\leq \int_0^s \|e_j(\Gamma^{(j)}(s)) - e_{j - 1}(\Gamma^{(j - 1)}(s))\| ds
\]

\[
\leq \left(\frac{Kb}{\kappa^2}\right)^{j - 1} s + K_1 \delta^{-2} \int_0^s |\Gamma^{(j)}(s) - \Gamma^{(j - 1)}(s)| ds
\]

\[
\leq K_1 \delta^{-2} s + \left(\frac{Kb}{\kappa^2}\right)^{j - 1} s
\]

\[
\vdots
\]

\[
\leq \frac{(K_1 \delta^{-2} s)^m}{m!} + \frac{(Kb)}{\kappa^2} \sum_{k = 1}^{m} \frac{(K_1 \delta^{-2} s)^k}{k!}.
\]

The third inequality follows from substituting \(|\Gamma^{(j)}(s) - \Gamma^{(j - 1)}(s)| \leq 1\) into the inside of the integral. Similarly, the \(m\)-th inequality \((m \geq 4)\) follows from substituting the \(m - 1\)-th one into the same place. Substituting \(s = s_0\) and passing \(m \to +\infty\) we obtain \(|\Gamma^{(j)}(s_0) - \Gamma^{(j - 1)}(s_0)| \leq b^{\frac{2}{j - 1}}\). In other words, \(\Gamma^{(j)}(s_0)\) hits neither the left nor the right side boundary of \(\Gamma^{(j - 1)}(\Pi_0^{\max(M, J)} w)\) on which \(e_j\) is well-defined by (b). This implies that \(\Gamma^{(j)}(s)\) is defined for all \(s \in [-1/10, 1/10]\). Lemma 2.4.4 implies that \(\Gamma^{(j)}\) is indeed a long stable leaf with the desired derivative estimate. The inclusion is obvious from the argument. \(\square\)

**Sublemma 8.8.2.** Under the same assumption as in Sublemma 8.8.1, let \(z_0' \in \Gamma^{(j)}\) and define \(w'_i = DH^i(z_0') (\frac{1}{i})\). For \(2 \leq i \leq j\),

\[
|\log \frac{\|w'_{i + 1}\|}{\|w'_i\|} - \log \frac{\|w'_{i + 1}\|}{\|w'_i\|} | \leq b^{\frac{1}{j - i}}.
\]
Proof. Put $A = DH(z_{i-1})$, $A' = DH(z'_{i-1})$. Then

$$\angle(w_i, w'_i) = \frac{\|w_i \times w'_i\|}{\|w_i\| \cdot \|w'_i\|}$$

$$= \frac{\|A'w_{i-1} \times A'w'_{i-1} + (A - A')w_{i-1} \times A'w'_{i-1}\|}{\|w_i\| \cdot \|w'_i\|}$$

$$\leq \frac{\|w_{i-1}\| \cdot \|w'_{i-1}\| (|\det A'| \cdot \angle(w_{i-1}, w'_{i-1}) + Kb|z_{i-1} - z'_{i-1}|)}{\|w_i\| \cdot \|w'_i\|}$$

$$\leq \frac{\|w_{i-1}\| \cdot \|w'_i\| (Kb \cdot \angle(w_{i-1}, w'_{i-1}) + (Kb)^i - 1)}{\|w_i\| \cdot \|w'_i\|}.$$

Using this recursively and then $|\log \|w_i\| - \log \|w'_i\|| \leq 1$, which follows from (b) for $k = j - 1$ and Sublemma 8.8.1, we have

$$\angle(w_i, w'_i) \leq (Kb)^{i-1} \sum_{t=0}^i \frac{\|w_t\| \cdot \|w'_t\|}{\|w_i\| \cdot \|w'_i\|} \leq b^{i-1}.$$

Therefore

$$\left\| \frac{w_{i+1}}{\|w_i\|} - \frac{w'_{i+1}}{\|w'_i\|} \right\| \leq \left\| DH(z_i) \right\| \left\| \frac{w_i}{\|w_i\|} - \frac{w'_i}{\|w'_i\|} \right\| + \left\| DH(z_i) - DH(z'_i) \right\| \left\| \frac{w'_i}{\|w'_i\|} \right\| \leq b^{i-1}.$$

Using $\|w_{i+1}\|/\|w_i\|^{-1} \geq e^{-\Delta k^i} \gg b^{i-1}$ for $2 \leq i \leq j$, we obtain the desired inequality.\[\square\]

For an arbitrary $z''_0 \in \Gamma^{(j)}(\Pi_{0}^{\max\{M,j+1\}}w)$, take $z'_0 \in \Gamma^{(j)}$ whose $y$-coordinate coincides with that of $z''_0$. Then $|z'_0 - z''_0| \leq \Xi(\Pi_{0}^{\max\{M,j+1\}}w)$, and thus by Lemma 2.1.2 we have $|\log \|w'_i\| - \log \|w''_i\|| \leq 1/2$ for $1 \leq i \leq j + 1$, where $w''_i = DH(z''_0)$ ($\odot$).

As we have already proved in the beginning, $|\log \|w_i\| - \log \|w'_i\|| \leq 1/4$ holds for $i = 1, 2$. Combining this with Sublemma 8.8.2 we have $|\log \|w_i\| - \log \|w''_i\|| \leq 1/2$ for $1 \leq i \leq j + 1$. Consequently we obtain $|\log \|w_i\| - \log \|w'_i\|| \leq 1/2$ for $1 \leq i \leq j + 1$. Hence (b) holds for $k = j \leq n - 1$. This restores the assumption of the induction and completes the proof.\[\square\]

8.9. Proof of Lemma 2.7.1. Let $\Gamma^{(j-1)}$ denote the long stable leaf of order $j - 1$ through $\zeta_1$. Let $\mathbf{w}$ denote the forward vector orbit of $\zeta_0$. Using the fact that $\mathbf{w}$ is expanding and the upper bound on the length of $\gamma_0$, it is easy to see that $\gamma_1 \subset \Gamma^{(j-1)}(\Pi_0^2\mathbf{w})$ holds. Hence it makes sense for $z_0 \in \gamma_0$ to consider the expressions

$$DH(z_0)t_{\gamma_0}(z_0) = \tilde{\xi}e_j(z_1) + \tilde{\eta}f_j(z_1) \quad \text{and} \quad DH(z_0)t_{\gamma_0}(z_0) = \xi e_m(z_1) + \eta f_m(z_1).$$

Put $\psi(z_0) = \angle(e_m(z_1), e_j(z_1))$. We clearly have $\tilde{\eta} = \eta \cos \psi \pm \xi \sin \psi$, the sign being chosen as the case may be. By Lemma 2.4.1 and Lemma 2.4.4

$$\psi(z_0) \leq \angle(e_m(z_1), e_j(z_1)) \leq K|z_0 - z_0| + (Kb)^m.$$

In particular we have $\psi(z_0) \ll 1$. Suppose that $z_0$ is the endpoint of $\gamma_0$. Then the assumption implies $\psi(z_0) \leq K|z_0 - z_0|$. Lemma 8.11.1 implies $|\eta(z_0)| = |z_0 - z_0|$, $|\xi(z_0)| \leq 2K_1b$, and $\eta(z_0)\eta(z') < 0$, where $z'$ is the other endpoint of $\gamma_0$. Without loss of generality we may assume $\eta(z_0) > 0$. Then $\eta(z_0) \geq |z_0 - z_0| (1/2 - 2K_1b) > 0$,
and \( \tilde{\eta}(z') < 0 \) on the other hand. By the intermediate value theorem there exists \( \tilde{\zeta} \in \gamma_0 \) such that \( \tilde{\eta}(\tilde{\zeta}_0) = 0 \). In other words \( \tilde{\zeta}_0 \) is a critical point of order \( j \).

\[ \square \]

8.10. Proof of Lemma 2.7.2.

Sublemma 8.10.1. We have:

(a) \( \text{slope}(DH\tilde{\gamma}(0)) \geq K b^{-1} \),

(b) \( \text{angle}(DH\tilde{\gamma}(0), DH\tilde{\gamma}(0)) \leq K b^{-1}(|\gamma(0) - \tilde{\gamma}(0)| + \|\gamma(0) - \tilde{\gamma}(0)\|) \).

Proof. Let us see that (b) follows from (a). (a) implies \( \text{angle}(DH\tilde{\gamma}(0), DH\tilde{\gamma}(0)) \ll 1 \), and thus

\[
\text{angle}(DH\tilde{\gamma}(0), DH\tilde{\gamma}(0)) \leq \frac{\|DH\tilde{\gamma}(0) - DH\tilde{\gamma}(0)\|}{\min(\|DH\tilde{\gamma}(0)\|, \|DH\tilde{\gamma}(0)\|)}.
\]

The denominator is \( \geq K b \), by (1) (2) and the fact that the slopes of \( \gamma(0) \) and \( \tilde{\gamma}(0) \) are \( \leq K b \). Hence (b) follows.

Put \( DH\tilde{\gamma}(0) = (\xi, \eta), \) \( DH\tilde{\gamma}(0) = (\xi, \eta) \). We show \( |\xi| \leq 2K_1 \delta^{-b} |\eta| \) which is equivalent to (a). Put \( \gamma(0) = \rho \cdot (1, \theta) \) and \( \tilde{\gamma}(0) = \tilde{\rho} \cdot (1, \tilde{\theta}) \), where \( \rho, \tilde{\rho} \approx 1 \) are the normalizing constants. By (2) and the fact that \( |\theta|, |\tilde{\theta}| \leq K_0 b \ll 1, |\eta|, |\tilde{\eta}| \) have the order \( b \). Thus

\[
|\xi/\eta| \leq (K b)^{-1} |\xi| \leq (K b)^{-1} |\xi|
\]

\[
+ K^{-1}(|\partial_x u(\gamma(0)) - \partial_x u(\tilde{\gamma}(0))| + |\theta| |\partial_y u(\gamma(0)) - \partial_y u(\tilde{\gamma}(0))|)
\]

\[
+ K^{-1} |\theta - \tilde{\theta}| |\partial_y u(\gamma(0))|).
\]

Using \( |\gamma(0) - \tilde{\gamma}(0)| \leq K b \) in the assumption (iv) of Lemma 2.7.2 and \( |\theta - \tilde{\theta}| \leq K b \),

\[
|\xi/\eta| \leq (K b)^{-1} |\xi| + K b \leq K|\xi|/|\eta| + K b \leq K_0 b + K b \leq 2K_1 b,
\]

where the third inequality uses the fact that \( \gamma(0) \) is a precritical point. \( \square \)

By the same reasoning as in the proof of Lemma 2.7.1 it is easy to see that \( e_m \) is well-defined on a neighborhood of \( \tilde{\gamma}_1 \). Hence it makes sense for \( z_0 \in \tilde{\gamma} \) to consider the expressions

\[
DHt\tilde{\gamma}(z_0) = \xi t\tilde{\gamma}_1(\tilde{z}_1) + \eta t\tilde{\gamma}_1(\tilde{z}_1)\frac{1}{\lambda} \quad \text{and} \quad DHt\tilde{\gamma}(z_0) = \xi e_m(z_1) + \tilde{\eta} f_m(z_1).
\]

Then \( \tilde{\eta} = \eta \cos\psi \pm \xi \sin\psi \) holds, where \( \psi = \text{angle}(DH\tilde{\gamma}(0), e_m(z_1)) \). By (a) in Sublemma 8.10.1 and Lemma 8.11.1, we have \( \eta = L|\gamma(0) - z_0| \) and \( |\xi| \leq K_1 b \).

Suppose that \( z_0 \) is one of the endpoints of \( \tilde{\gamma} \). Using the fact that \( DH\tilde{\gamma}(0) \) is collinear to \( e_m(H(\gamma(0))) \), (b) in Sublemma 8.10.1 and then (i) (iv) we have

\[
|\gamma(0) - z_0| \leq |\text{angle}(DH\tilde{\gamma}(0), DH\tilde{\gamma}(0)) + \text{angle}(e_m(H(\gamma(0))), e_m(z_1))|
\]

\[
\leq (K b^{-1} \varepsilon_m^2 + 1)|\gamma(0) - z_0|.
\]

For the same reason as in the proof of Lemma 2.7.1 we may assume \( \tilde{\eta}(z_0) > 0 \). Then

\[
\tilde{\eta}(z_0) \geq L|\gamma(0) - z_0| \cos\psi - Kb \sin\psi \geq |\gamma(0) - z_0| \left(1 - Kb - K \varepsilon^m/2\right) > 0,
\]

where the last inequality follows from the assumption on \( m, \varepsilon \). On the other hand we have \( \tilde{\eta}(z') < 0 \), where \( z' \) is the other endpoint of \( \tilde{\gamma} \). By the intermediate value
that $\Gamma \subset C_0$, and $\zeta(0) = \zeta_0$. Using the Taylor expansion around $\zeta_0$, we obtain a function $\gamma(t)$ such that $\gamma(0) = \zeta_0$ and $\gamma'(0) = z$. Then we have

\[ DH(z)(t_{\gamma_0}(z), t_{\gamma_1}(z)) = (t_{\gamma_1}(\zeta_1), t_{\gamma_1}(\zeta_1))T_1DH(z)T_0^{-1}. \]

The number $\xi(z)$ corresponds to the $(2,1)$-entry of $T_1DH(z)T_0^{-1}$, and hence (a) follows. The number $\eta(z)$ corresponds to the $(1,1)$-entry of the same matrix. A direct computation using $b \ll \delta$ gives

\[ (1 - \theta/2)|f''(0)| \leq \left| \frac{d\eta(z)}{dx} \right| \leq (1 + \theta/2)|f''(0)|. \]

Using the Taylor expansion around $\zeta_0 = (x_0, y_0)$ and $\eta(\zeta_0) = 0$,

\[ (1 - \theta)L|x_0 - x| \leq |\eta(z)| \leq (1 + \theta)L|x_0 - x|, \]

for small $\delta$ and $a$ close to 2. This implies (b) because $|x_0 - x| \approx |\zeta_0 - z|$ holds. \hfill $\Box$

**Claim 8.11.2.** Let $\Gamma^{(b)}$ denote the long stable leaf of order $q - 1$ through $\zeta_1$. Then $z_1 \in \Gamma^{(b)}(\Xi(w))$.

**Proof.** Suppose that $\zeta_1 - z_1 = (\xi, \eta)$. Let $z'$ (resp. $z''$) denote the unique point in $\Gamma^{(n)}$ (resp. $\Gamma^{(b)}$) whose $y$-coordinate coincides with that of $z_1$. Then $\zeta_1 - z' = (\xi', \eta)$ and $\zeta_1 - z'' = (\xi'', \eta)$ hold for some $\xi', \xi''$. Parametrize $\Gamma^{(n)}$ by arc length and assume that $\Gamma^{(n)}(0) = \zeta_1$. Define $\varphi(s) = \text{angle}(e_n(\Gamma^{(n)}(s)), e_n(\zeta_1))$. Then we have $\varphi(0) = 0$ and $|\varphi'(s)| \leq K$. Thus

\[ |\xi'| \leq K \int_0^{\eta}|\varphi(s)|ds \leq K \int_0^{\eta}sds \leq K\eta^2. \]

By Lemma 8.11.1, we have $\eta^2 \leq K_1b|\xi|$, and thus $|\xi'| \leq KK_1b|\xi|$. Hence $|\xi - \xi'| \leq |\xi| + |\xi'| \leq 2|\xi|$, and by Lemma 8.11.1 again,

\[ |\xi| \leq (1 + 2\theta)\int L|\zeta_0 - z|dz, \]
This implies the claim. □

Split \( v_1(z_1) = \xi e_{\beta n}(z_1) + \eta f_{\beta n}(z_1) \). We estimate \( |\eta| \). By Lemma 2.4.4,

\[
\angle(e_{\beta n}(\xi_1), e_{\beta n}(z_1)) \leq \|D e_{\beta n}\| |\xi_1 - z_1| \leq K K_1 \delta|\xi_0 - z_0|.
\]

By Lemma 2.4.1 and the left hand side of (2.9),

\[
\angle(e_{\beta n}(\xi_1), e_{\beta n}(z_1)) \leq (Kb)^n \leq |\xi_0 - z_0|^2.
\]

Thus \( \angle(e_{\beta n}(\xi_1), e_{\beta n}(z_1)) \leq K \delta|\xi_0 - z_0| \), and this implies

(20) \[
|\eta| \simeq L|\xi_0 - z_0||v_0|.
\]

We prove (a). Using (20), for every \( 0 \leq i \leq p \) we have

\[
\|D H^i \eta f_{\beta n}(z_1)\| \leq \|D H^i(z_1)\||\eta| \leq e^{-\alpha \beta n}|v_0|.
\]

Using \( 2p \leq \alpha \sigma n \),

\[
\|D H^i \eta f_{\beta n}(z_1)\| \leq e^{-2p \beta n} \|v_0\| \leq e^{-2\beta i} \|v_0\|.
\]

This and \( \|D H^i \xi e_{\beta n}(z_1)\| \leq (Kb)^i \|v_0\| \) yield (a).

We prove (b). For every \( 0 \leq i \leq \beta n \),

\[
\|D H^i f_{\beta n}(z_1)\| \geq e^{-1} \|w_i\| \geq e^{(\lambda - \alpha)i - 1}.
\]

Since \( z_0 \) is in admissible position, (20) implies

\[
|\eta| \geq \|w_{\beta n}\|^{\ell - 1} \|v_0\| \geq e^{(\ell - 1) \Delta \beta n} \|v_0\|.
\]

Using the definition of \( p \), for every \( p \leq i \leq \beta n \) we have

\[
\frac{\|D H^i \xi e_{\beta n}(z_1)\|}{\|D H^i \eta f_{\beta n}(z_1)\|} \leq \frac{(Kb)^i}{e^{-1} e^{(\ell - 1) \Delta \beta n} e^{(\lambda - \alpha)i}} \leq b^{i/2} \leq \theta.
\]

This implies

(21) \[
(1 - \theta) \|D H^i \eta f_{\beta n}(z_1)\| \leq \|v_{i + 1}\| \leq (1 + \theta) \|D H^i \eta f_{\beta n}(z_1)\|.
\]

Take small \( \tilde{\alpha} > 0 \) such that \( \Delta p/n - \alpha \tilde{\alpha} \beta \sigma < 0 \) holds. Then

\[
\frac{\|v_p\|}{\|v_0\|} \leq (1 + \theta) L \cdot |z_0 - z_0| \|D H^p f_{\beta n}(z_1)\|
\]

\[
\leq L |z_0 - z_0|^{1 - \tilde{\alpha}} e^{\Delta p - \alpha \tilde{\alpha} \beta n}
\]

\[
\leq |z_0 - z_0|^{1 - \tilde{\alpha}}.
\]

This yields the upper estimate in (b). On the other hand, \( p \geq 1 \) and \( \|D H^p f_{\beta n}(z_1)\| \geq e^{-1} \|w_p\| \geq e^{\lambda - \alpha - 1} \) gives

\[
\frac{\|v_p\|}{\|v_0\|} \geq L e^{\lambda - \alpha - 1} |z_0 - z_0| \geq L |z_0 - z_0|^{1 + \tilde{\alpha}}.
\]
We prove (c). Using (21) for $p - 1 \leq i \leq \beta n$ we have

$$\log \frac{\|v_{j+1}\|}{\|v_{i+1}\|} - \log \frac{\|w_j\|}{\|w_i\|} \leq 1. \tag{22}$$

Therefore

$$\|v_{\beta n+1}\| \geq \|DH^{\beta n} \eta f_{\beta n}(z_1)\| \geq e^{-1} \|w_{\beta n}\| \|v_0\| \geq e^{(\lambda - \alpha)\beta n} \|v_0\|$$

and

$$\frac{\|v_{q+1}\|}{\|v_0\|} = \frac{\|v_{\beta n+1}\|}{\|v_{\beta n}\|} \|v_{q+1}\| \geq e^{(\lambda - \alpha)\beta n - \Delta \alpha \beta n} \geq e^{\frac{\log 2}{q+1}(q+1)} \|v_0\|. \tag{23}$$

We prove (d). Let $\tau_0$ denote the straight segment whose endpoints are $z_1$ and $z''$. Integrating (20) and using $\eta^2 \leq K_1\delta^{-1}b$, we have $\text{length}(\tau_0) \simeq |\zeta_0 - z_0|^2$. Using (22),

$$\text{length}(\tau_{\beta n}) \leq e \cdot |\zeta_0 - z_0|^2 \|w_{\beta n}\| \leq e \cdot \text{\Xi}(w) \|w_{\beta n}\| \leq e^{1 - \alpha \sigma \beta n}. \tag{24}$$

On the other hand,

$$\text{length}(\tau_{\beta n}) \geq e^{-1} |\zeta_0 - z_0|^2 \|w_{\beta n}\| \geq |\zeta_0 - z_0|^2 e^{-1 + (\lambda - \alpha)\beta n}. \tag{25}$$

These two inequalities together imply the upper estimate of $\beta n$ in terms of $|\zeta_0 - z_0|$, and hence that of $q$. On the other hand,

$$e^{-1} \|w_{\beta n}\|^{2\ell - 1} \leq e^{-1} |\zeta_0 - z_0|^2 \|w_{\beta n}\| \leq \text{length}(\tau_{\beta n}) \leq e \cdot |\zeta_0 - z_0|^2 \|w_{\beta n}\|, \tag{26}$$

and thus

$$|\zeta_0 - z_0|^2 \geq e^{-2} \|w_{\beta n}\|^{2\ell - 2} \geq e^{-\Delta(2 - \ell)\beta n - 4}. \tag{27}$$

Taking logs and rearranging we obtain the lower estimate of $\beta n$ and hence that of $q$ in the desired form, because $q \geq (1 - \alpha \sigma)\beta n$.

We prove (e). By Lemma 21.2 for every $0 \leq i \leq \beta n$ we have

$$e^{-1} \|w_i\| \leq \frac{\text{length}(\tau_i)}{\text{length}(\tau_0)} \leq e \|w_i\|. \tag{28}$$

Hence

$$\text{length}(\tau_{\beta n}) \geq e^{-1} \|w_{\beta n}\| |\zeta_0 - z_0|^2 \geq e^{-3} \|w_{\beta n}\|^{2\ell - 1} \geq e^{-(1 - 2\ell)\lambda \beta n}. \tag{29}$$

Rearranging this and using the upper estimate of $\beta n$ in the proof of (d),

$$\|DH^{\beta n} f_{\beta n}(z_1)\| \geq e^{-1} \|w_{\beta n}\| \geq |\zeta_0 - z_0|^{-2} e^{-(1 - 2\ell)\lambda \beta n} \geq |\zeta_0 - z_0|^{2 - 3(1 - 2\ell)}. \tag{30}$$

Hence

$$\frac{\|v_{q+1}\|}{\|v_0\|} \geq e^{-\Delta \alpha \sigma \beta n} |\zeta_0 - z_0|^{-1 + 3(1 - 2\ell)} \geq |\zeta_0 - z_0|^{-2 + 4(1 - 2\ell)}. \tag{31}$$

On the other hand, using (23) for $i = p$ and $\beta n$,

$$\frac{\|v_{p+1}\|}{\|v_p\|} \leq e |\zeta_0 - z_0|^{1 - \alpha} \frac{\|w_{\beta n}\|}{\|w_{p-1}\|} \leq e^2 |\zeta_0 - z_0|^{1 - \alpha} \frac{\text{length}(\tau_{\beta n})}{\text{length}(\tau_{p-1})}. \tag{32}$$

Using (d) we have $\text{length}(\tau_{\beta n}) \leq e^{-\alpha \sigma \beta n} \leq |\zeta_0 - z_0|^{\frac{3 - 2\ell}{\lambda \beta n}}$. Meanwhile we have $\text{length}(\tau_{p-1}) \geq \text{length}(\tau_0) \geq |\zeta_0 - z_0|^2$. Substituting these into the right hand side we obtain the upper estimate of $\|v_{\beta n+1}\|$, and hence that of $\|v_{q+1}\|$. We prove (f). We clearly have $|z_i - z''_{i-1}| \leq \|w_i\| \cdot \text{\Xi}(w) \leq e^{-\alpha \sigma q}$. Since $z''_0 \in \Gamma^{(q-1)}$ we have $|\zeta_i - z''_{i-1}| \leq |\zeta_1 - z''_0|$ for $1 \leq i \leq q$. Moreover, by Lemma 8.11.1 we have...
\[ |\zeta_i - z_0''| \leq K_1 \delta^{-1} b |\zeta_0 - z_0|^2 \leq e^{-\alpha q}. \] Hence we obtain \[ |\zeta_i - z_i''| \leq |\zeta_i - z_i''' + |z_i - z_i'''| | \leq 2e^{-\alpha q} \leq e^{-\alpha q}/2. \]

We prove (g). Using (a) (c), for every \( 0 \leq i \leq p \) we have
\[
\frac{\|v_{q+1}^i\|}{\|v_i\|} \geq \frac{\|v_{q+1}^i\|}{\|v_0\|} \geq e^{-\frac{\log 2}{3} (q+1)} \geq e^{-1} K_0 \delta.
\]
Using (b) in Proposition 2.5.1, for every \( p + 1 \leq i \leq q \) we have
\[
\frac{\|v_{q+1}^i\|}{\|v_i\|} \geq e^{-1} \cdot \frac{\|v_q^i\|}{\|v_{i-1}\|} \geq e^{-1} K_0 \delta.
\]

We prove (h). There are three cases: \( i \leq j \leq p \); \( i \leq p \leq j \); \( p \leq i \leq j \). In the first case, using \( \|v_j\| \geq e^{-\Delta p} \|v_p\| \) and (a) (b),
\[
\frac{\|v_j\|}{\|v_i\|} \geq \frac{\|v_p\|}{\|v_i\|} \cdot \frac{\|v_j\|}{\|v_p\|} \geq e^{-\Delta p} \geq \left( \frac{\|v_p\|}{\|v_0\|} \right)^{1+\Delta} \geq \left( \frac{\|v_p\|}{\|v_0\|} \right)^{1+3\alpha \sigma \lambda (1+\Delta)}.
\]
The remaining cases have similar proofs. Using (b) in Proposition 2.5.1 for all \( p \leq i \leq j \leq q \) we have
\[
\frac{\|v_j\|}{\|v_i\|} \geq e^{-2} \frac{\|v_j\|}{\|v_i\|} \geq e^{-\alpha q} \geq e^{-\alpha q}.
\]
Substituting (b) (d) into this we obtain
\[
\frac{\|v_j\|}{\|v_i\|} \geq \left( \frac{\|v_p\|}{\|v_0\|} \right)^{3\alpha \sigma \lambda (1+\Delta)} \geq \left( \frac{\|v_p\|}{\|v_0\|} \right)^{1+3\alpha \sigma \lambda (1+\Delta)}.
\]
This finishes the proof in the last case. In the second case, the above inequality with \( i = p \) and \( \|v_i\| \leq \|v_0\| \) in (a) yields the desired one. \( \square \)

8.12. Proof of Lemma 2.12.1.

Sublemma 8.12.1. [cf. [19], Claim 5.1] For every \( i \in [N, m] \), there exists a hyperbolic time \( i' \in [[i/2], i] \).

Proof. Consider the graph, denoted by \( G \), of the function \( k \rightarrow \log \|v_k\| \) defined on \([m - i, m]\). Let \( L \) be the infinite line through \((m, \log \|v_m\|)\) with slope \( \Delta \). Clearly, all points of \( G \) lies above \( L \). Let \( P \) be the point of intersection between \( L \) and the line \( \{x = m - [i/2]\} \). Let \( L \) be pivoted at \( P \) and rotate it clockwise until it hits \( G \). With \( L \) in its final position, \( G \) still lies above \( L \). Define an integer \( i'' \) so that \((m - i'', \log \|v_{m-i''}\|)\) belongs to the set of the first hit. We clearly have \( i'' \in [[i/2], i] \).

Since \( \|v_m\| \geq K_0 \delta e^{-3} \|v_{m-i''}\| \) and \( i \geq N \), the slope of \( L \) in its final position is bigger than
\[
-\Delta + \log \|v_m\| - \log \|v_{m-i''}\| \geq \frac{-\Delta + \log \|v_{m-i''}\|}{[i/2]} \geq -\Delta + 2i^{-1} \log(K_0 e^{-3} \delta) \geq -4\Delta.
\]
This implies that \( \Pi^n_{m-i''} v \) is \( e^{-4\Delta} \)- expanding. Define \( i' = i'' - 1 \) if \( z_{m-i''} \in C_\delta \), and \( i' = i'' \) otherwise. Then \( z_{m-i'} \notin C_\delta \) and \( i' \in [[i/2], i] \) hold. Moreover, for every \( 1 \leq j \leq i' \) we have
\[
\|v_{m-i'j}\| = \|v_{m-i''j+1}\| \geq e^{-4\Delta(j+1)} \|v_{m-i''j}\| \geq e^{-4\Delta(j-5\delta)} \|v_{m-i''j}\| \geq e^{-9\Delta j} \|v_{m-i''j}\|,
\]
where the second inequality follows from \( \|v_{m-i''j}\| \geq e^{-\Delta} \|v_{m-i''j}\| \). Hence \( i' \) is a hyperbolic time. \( \square \)
We now complete the proof of the lemma. Define \( \{ \tilde{h}_i \}_{i=1}^s \) to be the strictly monotone increasing sequence of hyperbolic times which is maximal with respect to inclusion as a subset of \( \{ i \}^m_{i=N} \). Suppose that \( \tilde{h}_{i+1} = j' \) for some \( j \in [N, m] \). If \( j \leq 2N \), then \( \tilde{h}_{i+1} \leq 2N \) holds, by Sublemma 8.12.1. On the other hand we have \( \tilde{h}_i \geq N/2 \), and therefore \( \tilde{h}_{i+1} \leq 4\tilde{h}_i \). Suppose that \( j > 2N \). Then

\[
\tilde{h}_{i+1} > \lfloor j/2 \rfloor > \lfloor j/2 \rfloor - 1 > (\lfloor j/2 \rfloor - 1)'.
\]

Since \( \tilde{h}_i \) and \( \tilde{h}_{i+1} \) are two consecutive hyperbolic times, we have

\[
\tilde{h}_i \geq (\lfloor j/2 \rfloor - 1)' \geq (\lfloor j/2 \rfloor - 1)/2.
\]

This and \( \tilde{h}_{i+1} \leq j \) yields \( \tilde{h}_{i+1} \leq 4\tilde{h}_i \).

We define a subsequence \( \mathcal{I} \) of \( \{ \tilde{h}_i \}_{i=1}^s \) as follows. Define \( \tilde{h}_z \in \mathcal{I} \). Suppose that \( \tilde{h}_i \in \mathcal{I} \) and \( i \geq 2 \). Let \( \tilde{h}_{k(i)} \) denote the smallest hyperbolic time such that \( \geq \tilde{h}_i/4 \). Such \( k(i) \) always exists by \( i \geq 2 \) and \( \tilde{h}_{i-1} \geq \tilde{h}_i/4 \). We define \( \tilde{h}_{i-1}, \tilde{h}_{i-2}, \ldots, \tilde{h}_{k(i)} \notin \mathcal{I} \). If \( k(i) = 1 \), then we stop the construction. If \( k(i) \geq 2 \), then we define \( \tilde{h}_{k(i)} \notin \mathcal{I} \). If \( k(i) - 1 = 1 \) then we stop the construction. If \( k(i) - 1 \geq 2 \), then we repeat the same selection procedure. Let \( \mathcal{I} = \{ \tilde{h}_i \}_{i=1}^s \). Then we have \( \tilde{h}_i = \tilde{h}_z \geq [m/2] - 1 \) and \( h_i \leq \tilde{h}_{i+1}/4 \). Suppose that \( h_{i+1} = h_j \) for some \( j \). Then we have \( \tilde{h}_{k(j)} \geq h_j/4 \) and \( \tilde{h}_i = \tilde{h}_{k(j)} \). Thus \( h_i \geq \tilde{h}_{k(j)}/4 \), and consequently \( h_i \geq h_{i+1}/16 \) follows. \( \square \)

8.13. Proof of Lemma 5.7.5 Let \( w = \{ w_i \}_{i=0}^{\beta m} \) denote the forward vector orbit of \( \zeta_0 \). Let \( \Gamma^{(m)} \) denote the forward vector orbit of \( \zeta_0 \) through \( H_{a, \zeta} \). Then we have \( H_{a, \gamma} \subset \Gamma^{(m)}(\Xi(\Pi^m_0 w)) \), because \( \varepsilon \ll 1 \) implies \( |H_{a, \zeta} - H_{a, \zeta}| \ll \Xi(\Pi^m_0 w) \) and \( \text{diam}(H_{a, \gamma}) \ll \Xi(\Pi^m_0 w) \). Hence, for \( 1 \leq i \leq m \), the constrictive field under the iteration of \( D H_{a, \zeta} \), denoted by \( e_i(a, \zeta) \), is well-defined on a neighborhood of \( H_{a, \zeta} \). Define \( w(a) = \{ w_i(a) \}_{i=m}^m \) by \( w_i(a) = D H_{a, \zeta}^i (H_{a, \zeta})(0) \) and \( w'(a) = \{ w'_i(a) \}_{i=m}^m \) by \( w'_i(a) = D H_{a, \zeta}^i (H_{a, \zeta})(0) \). The same type of estimate as in 5 applies and we have \( \log \| w_i(a) \| - \| w_i'(a) \| \leq 1 \) for \( 1 \leq i \leq m \). In particular, \( w(a) \) is expanding up to time \( m \). On the other hand, by \( |H_{a, \zeta} - H_{a, \zeta}| \ll \Xi(w(a)) \) and \( b \) in Proposition 2.5.1 we have \( \log \| w_i(a) \| - \| w'_i(a) \| \leq 1 \) for \( 1 \leq i \leq m \). Hence \( w'(a) \) is expanding up to time \( m \). By a reasoning similar to that of Lemma 2.7.1 with parameter dependence in mind. For \( z \in \gamma \), split

\[
DH_{a, \zeta} = \xi e_n(a, \zeta) + \eta f_n(a, \zeta)
\]

and

\[
DH_{a, \zeta} = \xi_0 e_n(a, \zeta) + \eta f_n(a, \zeta).
\]

By Lemma 8.11.1 we have \( \eta = |\zeta - \zeta| \) and \( |\zeta| \leq K_1b \). Put

\[
\psi = \angle (e_m(a, \zeta), e_m(a, \zeta)).
\]

Comparing the coefficients of the both sides of the identity \( DH_{a, \zeta} = DH_{a, \zeta} + (DH_{a, \zeta} - DH_{a, \zeta}) \), we have \( \tilde{\eta} = \eta \cos \psi \pm \xi \sin \psi + R \), where \( |R| \leq |DH_{a, \zeta} - DH_{a, \zeta}| \leq
Suppose that \( z \) is one of the two endpoints of \( \gamma \). Then \( \psi \leq K e^\Delta |\zeta - z| \) holds. Without loss of generality we may assume \( \eta(z) > 0 \). Then
\[
\tilde{\eta}(z) \geq |\zeta - z|/(1 - 2Kb) - |R| > 0.
\]
In the same way we have \( \tilde{\eta}(z') < 0 \), where \( z' \) is the other endpoint of \( \gamma \). By the intermediate value theorem, there exists \( s(a) \in [-e^{m/2}, e^{m/2}] \) such that \( \tilde{\eta}(\hat{\gamma}(s(a))) = 0 \). In other words, \( H_a \hat{\gamma}(s(a)) \) is a critical point of \( H_a \) of order \( m \).

8.14. Proof of Lemma 5.3.3. We prove (a). Take \( z \in \tilde{\Gamma} \) whose \( y \)-coordinate coincides with that of \( \zeta_{-h_i} \). By the assumption (ii) and the fact that \( e_{h_i} \) is Lipschitz, we have
\[
|z - \zeta_{-h_i}| \leq Ke^{-10^{4}\Delta h_i}.
\]
Thus for \( -h_i < j \leq 0 \),
\[
(24) \quad |\zeta_j - \tilde{\zeta}_j| \leq e^{-10^{4}\Delta h_i + \Delta(j + h_i)} + (Kb)^{j + h_i} \leq e^{-10^{4}\Delta h_i} + Kb.
\]
This implies \( \tilde{\zeta}_j \notin \mathcal{C}_{2\delta} \) for \( j = -h_{i-1}, \ldots, -h_1 \), and hence (H1) follows.

Let \( j < i \). We prove (H2), that is, \( \Pi^0_{-h_j} \tilde{w} \) is \( e^{-10\Delta} \)-expanding. In view of the computation in the proof of Proposition 2.5.1 we have
\[
\angle(w_{-h_j}, \tilde{w}_{-h_j}) \leq \frac{||w_{-h_{i-1}}|| ||\tilde{w}_{-h_{i-1}}||}{||w_{-h_j}|| ||\tilde{w}_{-h_j}||} \times (Kb \cdot \angle(w_{-h_{i-1}}, \tilde{w}_{-h_{i-1}}) + K|\zeta_{-h_{i-1}} - \tilde{\zeta}_{-h_{i-1}}| + K|a_* - a|).
\]
Using this recursively and then \( |\zeta_{-h_{i-1}} - \tilde{\zeta}_{-h_{i-1}}| \leq (Kb)^{h_{i-1}} \), \( |a - a_*| \leq e^{-\beta \lambda h_i/17} \),
\[
\angle(w_{-h_j}, \tilde{w}_{-h_j}) \leq (Kb)^{h_{i-1}} ||w_{-h_j}|| ||\tilde{w}_{-h_j}|| \sum_{k = -h_i}^{\frac{-h_j}{52\Delta h_i}} ||w_k|| ||\tilde{w}_k||.
\]
To bound the right hand side we begin by estimating \( h_i - h_j \) from below. Using Lemma 2.12.1 we have \( h_j \leq h_i/4 \), and thus \( h_i - h_j \geq 3h_i/4 \). Using this and the assumption (i),
\[
||w_k|| ||\tilde{w}_k|| \leq e^{20\Delta h_i} e^{2\Delta(k + h_i)} \leq e^{52\Delta h_i}.
\]
Consequently we obtain \( \angle(w_{-h_j}, \tilde{w}_{-h_j}) \leq e^{-50\Delta h_i} \). This implies for \( 1 \leq k \leq h_j \),
\[
||w_{-h_j+k}|| ||\tilde{w}_{-h_j+k}|| \leq Ke^{2k} |\zeta_{-h_j} - \tilde{\zeta}_{-h_j}| + e^{\Delta k} e^{-50\Delta h_j} \ll ||w_{-h_j+k}||.
\]
This implies that \( \Pi^0_{-h_j} \tilde{w} \) is \( e^{-10\Delta} \)-expanding.

We prove (b). By the uniformly Lipschitz property of the mostly contracting directions, it is enough to prove \( |\zeta_{-h_j} - \tilde{\zeta}_{-h_j}| \leq e^{-99\Delta h_j} \). This follows from substituting \( k = -h_j \) into the left hand side of (24) and using \( h_i - h_j \geq 3h_i/4 \). \( \square \)
8.15. Proof of Proposition 6.2.2. We firstly prove (S1).

Case I: no free return takes place in \((i, j)\) and \(i\) is free. \((S1)\) clearly holds if \(K_0e^{αj}δ ≤ 1\), because \(ζ_0\) is a critical point and thus no return takes place up to time \(j\). If \(K_0e^{αj}δ ≥ 1\), Lemma 2.2.1 and \(σ ≥ 1\) gives

\[ \|w_j\| ≥ K_0δe^{λ(j-i)}\|w_i\| = K_0δe^{(λ-α)(j-i)}e^{-αi}\|w_i\| ≥ e^{(λ-α)(j-i)-ασi}\|w_i\|. \]

Case II: some free returns take place in \((i, j)\) and both \(i, j\) are free. Let \(i < m_{i_0} < m_{i_0+1} \cdots < m_{j_0} < j\) denote the maximal sequence of free returns. Then

\[ \frac{\|w_j\|}{\|w_i\|} = \frac{\|w_{j_0}\|}{\|w_{m_{i_0}+q_{i_0}+1}\|} \cdot \prod_{i=i_0}^{j_0-1} \frac{\|w_{m_{i}+q_{i}+1}\|}{\|w_{m_{i}}\|} \cdot \prod_{i=i_0}^{j_0} \frac{\|w_{m_{i}+q_{i}}\|}{\|w_{m_{i}}\|}. \]

Using \(\|w_{m_{i}+q_{i}}\| ≥ \|w_{m_{i}}\|\) for \(i_0 ≤ i ≤ j_0\) and Lemma 2.2.1 with respect to the first and last fractions,

\[ \frac{\|w_j\|}{\|w_i\|} ≥ K_0^{j_0-i_0+1}δ \exp \left[ λ \left( j - i - \sum_{i=i_0}^{j_0} q_i \right) \right]. \]

Since \(ζ_0\) is a critical point and some return takes place before \(j\), we have \(K_0δe^{αj/10} ≥ 1\). Thus

\[ \frac{\|w_j\|}{\|w_i\|} ≥ K_0^{j_0-i_0} \exp \left[ λ \left( j - i - \sum_{i=i_0}^{j_0} q_i \right) - αj/10 \right]. \]

To bound the sum of the binding periods we argue as follows. Using (b) (d) in Proposition 2.10.2

\[ \sum_{i=i_0}^{j_0} q_i ≤ \frac{3}{λ(1- α)} \sum_{i=i_0}^{j_0} - \log \frac{\|w_{m_{i}+p_{i}}\|}{\|w_{m_{i}}\|}. \]

Since each \(m_{i}\) is an essential return unless subject to some previous essential return,

\[ \sum_{i=i_0}^{j_0} q_i ≤ \frac{3}{λ(1- α)} × 11 \times \sum_{m_{i} < j \text{ essential}} - \log \frac{\|v_{m_{i}+p_{i}}\|}{\|v_{m_{i}}\|} ≤ \frac{αj}{2}, \]

where the last inequality follows from (13). To bound \(K_0^{j_0-i_0}\) we use the next sub-lemma and obtain \(j_0 - i_0 ≤ \frac{Δ(j-i)}{λ\log δ}\), a proof of which is left as an exercise: consider a perturbation from \(H_{2,0}\).

Sublemma 8.15.1. max\(\{i ∈ \mathbb{N}: H^i(C_δ) \cap C_δ = ∅\}\) ≥ \(-Δ^{-1}\log δ\).

Substituting these two inequalities into the above one we have

\[ \|w_j\| ≥ e^{(λ-α)(λ+1)/2j-λi} ≥ e^{(λ-α)(λ+1)/4(j-i)}\|w_i\| ≥ e^{(λ-α)(j-i)-ασi}\|w_i\|. \]
Case III: some free returns take place in \((i, j)\), \(i\) is free, \(j\) is bound. Let \(m_{j_0}\) denote the free return such that \(m_{j_0} < j \leq m_{j_0} + q_{j_0} + 1\). Then

\[
\frac{\|w_j\|}{\|w_i\|} = \frac{\|w_j\|}{\|w_{m_{j_0} + q_{j_0} + 1}\|} \cdot \frac{\|w_{m_{j_0} + q_{j_0} + 1}\|}{\|w_i\|}.
\]

Regarding the first term, we have

\[
\|w_j\| \geq e^{-\Delta(m_{j_0} + q_{j_0} + 1 - j)}\|w_{m_{j_0} + q_{j_0} + 1}\| \geq e^{-\Delta q_{j_0}}\|w_{m_{j_0} + q_{j_0} + 1}\| \geq e^{-\Delta \alpha j/10}\|w_{m_{j_0} + q_{j_0} + 1}\|.
\]

Using this and applying (25) to the second term, we obtain

\[
\|w_j\| \geq e^{(\lambda - \alpha (\lambda + 1)/4)(j - i) - \alpha \Delta j/10}\|w_i\| \geq e^{(\lambda - \alpha)(j - i) - \alpha \sigma_i}\|w_i\|.
\]

Case IV: some free returns take place in \((i, j)\), \(i\) is bound, \(j\) is free. Let \(m_{i_0}\) denote the free return such that \(m_{i_0} < i \leq m_{i_0} + q_{i_0} + 1\). Suppose that \(i \leq m_{i_0} + p_{i_0}\). By (a) in Proposition 2.10.2 we have

\[
(26) \quad \frac{\|w_{i_0}\|}{\|w_i\|} = \frac{\|w_{j_0}\|}{\|w_{m_{i_0}}\|} \cdot \frac{\|w_{m_{i_0}}\|}{\|w_i\|} \geq \frac{\|w_{j_0}\|}{\|w_{m_{i_0}}\|}.
\]

Since \(m_{i_0}\) and \(j\) are free, (25) applies to the right hand side. Since \(m_{i_0} < i\), we obtain the desired inequality. Suppose that \(i > m_{i_0} + p_{i_0}\). (21) implies

\[
\|w_i\| \leq (1 + \theta)L|\tilde{\zeta}_0 - \zeta_{m_{i_0} + 1}| e^{\Delta(i - m_{i_0})\|w_{m_{i_0}}\|},
\]

where \(\tilde{\zeta}_0\) is a critical point relative to which \(w_{m_{i_0}}\) is in admissible position. Since \(i - m_{i_0} \leq q_{i_0} \leq \alpha m_{i_0}/10\) and \(|\tilde{\zeta}_0 - \zeta_{m_{i_0} + 1}| \leq \delta\) we have \(\|w_i\| \leq \sqrt{\delta} e^{\Delta m_{i_0}\|w_{m_{i_0}}\|}\). Using this and (25),

\[
\|w_{j_0}\| = \|w_{m_{i_0}}\| \|w_{j_0}\| \geq e^{(\lambda - \alpha (\lambda + 1)/4)(j - m_{i_0})} e^{-\Delta m_{i_0}/10} \geq e^{(\lambda - \alpha)(j - i) - \alpha \sigma_i}.
\]

8.15.2. Case V: both \(i\) and \(j\) are bound. Suppose that \(i\) and \(j\) are bound to different free returns. In this case, there exists a free return \(m_{i_0}\) such that \(i < m_{i_0} < j\). Using the estimates in III and IV we have

\[
\|w_{j_0}\| = \|w_{m_{i_0}}\| \|w_{j_0}\| \geq e^{(\lambda - \alpha (\lambda + 1)/4)(j - i) - \alpha \Delta(i + m_{i_0})/10} \geq e^{(\lambda - \alpha)(j - i) - \alpha \sigma_i}.
\]

Suppose that \(i\) and \(j\) are bound to the same free return \(m_{i_0}\). Let \(\tilde{\zeta}_0\) denote the critical point of order \(k\) relative to which \(w_{m_{i_0}}\) is in admissible position. Let \(\tilde{w} = \{\tilde{w}_i\}_{i=0}^{\beta k}\) denote the forward vector orbit of \(\tilde{\zeta}_0\). By \((EG)_a\), \(\tilde{w}\) is strongly regular. Three cases need to be considered separately:

(i) \(m_{i_0} + p_{i_0} \leq i < j\). Using (22) we have

\[
\|w_{j_0}\| \geq e^{-\|\tilde{w}_{j-m_{i_0}} - 1\|\|w_{i-m_{i_0}} - 1\|} \geq e^{-2} e^{(\lambda - \alpha)(j - i) - \alpha \sigma(i - m_{i_0} - 1)} \geq e^{(\lambda - \alpha)(j - i) - \alpha \sigma i}.
\]
Combining these two inequalities we obtain the desired one.

The definition of the folding period gives \( i \leq \chi j \) for every \( 0 \leq j \).

This and \((ii)\) returns in \( [1 − \alpha \sigma] \). Meanwhile, any binding period is \( j \geq −\alpha \sigma j \).

Rearranging this and using \( \Delta \), we have \( \| w_{m_0} \| = 2 \| \tilde{w}_j \| \| w_{m_0} \| \geq e^{−\alpha \sigma j} \| w_{m_0} \| \). This and \( \| w_i \| \leq \| v_{m_0} \| \) yield the desired inequality.

\((iii)\) \( m_0 \leq i < j < m_0 + p_0 \). Using the estimate in \((ii)\) and \( p_0 \ll \alpha m_0 \) we have \( \| w_{m_0 + p_0} \| \geq e^{\alpha m_0} \| w_j \| \leq e^{\alpha \sigma} \| w_j \| \). The definition of the folding period gives \( \| w_{m_0 + p_0} \| \leq e^{\Delta (m_0 + p_0)} \| w_j \| \leq e^{\Delta m_0} \| w_j \| \leq e^{\alpha \sigma} \| w_j \| \).

Combining these two inequalities we obtain the desired one.

It is left to define the function \( \chi(\cdot) \) in \((S2)\). To this end we introduce the following terminology: \( j \in [0, m_0 + 1) \) is isolated if \((1)\) it is free, and \((2)\) there is no return before \( j \), or else \( j \geq j' + q - \lambda^{-1} e \log (K_0 \delta) \) holds for the last free return \( j' \) before \( j \) with the binding period \( q \). Define \( \chi(j) \) to be the largest integer in \([0, j)\) which is isolated.

Let us see \( \chi(\cdot) \) indeed satisfies the desired properties. They are clearly satisfied when there is no return before \( j \), by Lemma \((2.2.1)\) and \( \chi(j) = j \) in this case. Suppose that \( j' \) is the last free return before \( \chi(j) \). Since there is no return in between \( j' + q \) and \( \chi(j) \), and by Lemma \((2.2.1)\) we have \( \| w_{\chi(j)} \| \geq K_0 \| w_i \| \) for every \( j' + q + 1 \leq i \leq \chi(j) \). On the other hand, by Proposition \((2.10.2)\) we have \( \| w_{j' + q + 1} \| \geq e^{-1} K_0 \| w_i \| \) for every \( 0 \leq i \leq j' + q + 1 \), and therefore \( \| w_{\chi(j)} \| \| w_{j' + q + 1} \| \| w_i \| \geq K_0 \| w_{j' + q + 1} \| \| w_{\chi(j)} \| \| w_i \| \geq K_0 \delta e^{\lambda (\chi(j) - j' - q)} e^{-1} K_0 \delta \geq K_0 \delta \).

It is left to prove \( \chi(j) \in [(1 - \alpha \sigma) j, j] \). If \( j \) is isolated then it is done because \( \chi(j) = j \) by definition. Suppose the contrary, and let \( \psi(j) \) denote the last free return which takes place before \( j \). We derive a contradiction assuming that there exists \( k \geq 1 \) such that \( \psi(j), \psi^k(j) = \psi \cdots \psi(j) (k \text{-composite}) \) are not isolated and \( \psi^k(j) \leq (1 - \alpha \sigma) j \). By the definition of isolated iterates, two consecutive free returns in \([(1 - \alpha \sigma) j, j]\) are close to each other. More precisely, one free return takes place right after \( -\lambda^{-1} \log (K_0 \delta) \) iterates of the end of the binding period of another at the latest. Meanwhile, any binding period is \( \geq -\lambda^{-1} \log (K_0 \delta) \), by Lemma \((2.10.2)\). This implies that the proportion of total bound iterates in \([j - \alpha \sigma, j]\) is bigger than a certain uniform constant which only depends on \( \Delta \) and \( \lambda \). On the other hand, the total number of bound iterates in \([(1 - \alpha \sigma) j, j]\) is clearly smaller than the sum of the binding periods of free returns which take place before \( j \), which is \( \leq \alpha j \) as was already proved. These two estimates yield a contradiction. This completes the proof of Proposition \((6.2.2)\).
8.16. Proof of Proposition 7.1.1. Before entering the proof we need a very useful inequality which is an adaptation of [19] Lemma 6.2 to our context.

Lemma 8.16.1. Suppose that $H$ satisfies $(RR)_{n-1}$, and that $\{w_j(z_j)\}_{j=0}^{i}$ is reluctantly recurrent up to time $i - 1$. Then for every $0 \leq s \leq i$,

$$\|DH^{i-s}(z_0)\| \leq Ke^{-\lambda s/2}\|w_i\|.$$  

Proof. Let $q_t$ denote the binding period of a free return $t \leq i$, and define $I_t = [t - q_t, t + q_t]$. These intervals are not necessarily two by two disjoint and it does not matter.

Claim 8.16.2. For every $s \notin \cup I_t$ and $j \in [1, i - s]$,

$$\|w_{s+j}\| \geq e^{-2\Delta j}\|w_s\|.$$  

Proof. Fix $s$, and then fix $j$. Let $r$ be the last free return between $s$ and $s+j$. If no such $r$ exists, then the inequality follows because $s$ is free. Let $j' \geq j$ be the smallest integer such that $z_{s+j'}$ is free. Notice that $j'$ may be bigger than $i$ and it does not matter. Using the fact that $s$ is free,

$$\|w_{s+j}\| \geq e^{-\Delta(j'-j)}\|w_{s+j'}\| \geq e^{-\Delta(j'-j)}|t_0 - z_r|\|w_s\| \geq e^{-\Delta(j'-j)}e^{-\lambda r/3}\|w_s\|,$$

where $t_0$ is the binding point for $z_r$. Since $r$ is the last free return, $s + j' \leq r + q_r$ holds, and thus $j' \leq j + q_r$. Since $s < r - q_r < r \leq s + j$, we have $q_r \leq j$. This yields the desired inequality. □

Suppose that $s \notin \cup I_t$. Then $e_k(z_s)$ is well-defined for $1 \leq k \leq i - s$. Since $s$ is free, slope($w_s$) $\leq K_0$. Hence we obtain

$$\|DH^{i-s}(z_s)\| \leq K\|w_i\|/\|w_s\| \leq Ke^{-\lambda s}\|w_i\|,$$

where the last inequality follows from the strong regularity of $w$.

Suppose that $s \in \cup I_t$. Let $r$ denote the last return such that $s \in I_r$. Since $w$ is reluctantly recurrent, we have $q_r \leq 10\alpha s$. If $i \in I_r$, then

$$\|DH^{i-s}(z_s)\| \leq e^{\Delta q_r} \leq e^{10\alpha s} \leq e^{-\lambda s/2}\|w_i\|.$$  

Suppose that $i \notin I_r$. Suppose that $s \geq (1 - 10\alpha)i$. Then we have

$$\|DH^{i-s}(z_s)\| \leq e^{\Delta i} \leq e^{\Delta i} \leq e^{-\lambda s/2}\|w_i\|.$$  

It is left to consider the case $s < (1 - 10\alpha)i$. We consider the following operation. Put $s_1 = r_0 + 10q_{r_0}$. Ask whether $s_1 \notin \cup I_t$ or not. If so, then stop the operation. If not, then let $r_1$ denote the last return such that $s_1 \in I_{r_1}$. Put $s_2 = r_1 + 10q_{r_1}$, and ask whether $s_2 \notin \cup I_t$ or not. If so, then stop the operation. If not, then let $r_2 \leq i$ denote the last return such that $s_2 \in I_{r_2}$. Put $s_3 = r_2 + 10q_{r_2}$. Repeat this. This operation defines an increasing sequence of integers. Denote by $\{s_i\}_{i=0}^{\ell}$ such a sequence which is maximal with respect to inclusion as a set. Suppose that $s_{\ell} \in \cup I_t$. This implies $s_{\ell} \geq i$. By construction, $s_{i+1} - s_i \leq 2q_r$. This implies

$$\sum_{i=0}^{\ell} q_{r_i} \geq s_{\ell} - s_0 \geq i - s_0 \geq 10\alpha i.$$
On the other hand, since \( w \) is reluctantly recurrent, \( \sum_{i=0}^{\ell} q_i \leq \alpha s_\ell \leq \alpha i \) holds. This yields a contradiction. Consequently, \( s_\ell \notin I_\ell \) holds. Then

\[
\|DH^{-s}(z_s)\| \leq \|DH^{-s}(z_{s_i})\| \prod_{i=0}^{\ell-1} \|DH^{s_{i+1}-s_i}(z_{s_i})\|
\]

\[
\leq Ke^{-\lambda s/2}e^{-\lambda s/2}\|w_i\|e^{\alpha s_\ell}
\]

\[
\leq Ke^{-\lambda s/2}\|w_i\|.
\]

This finishes the proof of Lemma 8.16.1 \( \square \)

We now start the proof of Proposition 7.1.1 by estimating \( |\dot{c}_1(a)| \) for \( a \in J(a_*, w, 0) \). Let \( c_i(a) = (x_i(a), y_i(a)) \). Then we have \( |\dot{x}_1(a)| = |x_0(a)|^2 + 2ax_0(a)\dot{x}_0(a) + O(b) \) and \( |\dot{y}_1(a)| = O(b) \). Using \( x_0(a) \approx 1 \) and \( |\dot{x}_0(a)| \leq \|\dot{c}_0(a)\| \leq K\delta \) in (a),

\[
(1 - \delta)|x_0(a)|^2 \leq |\dot{c}_1(a)| \leq (1 + \delta)|x_0(a)|^2.
\]

Using \( \|\dot{c}_0(a)\| \leq K\delta \) we have \( \|\dot{c}_1(a)\| \leq K\delta \). Hence the curvature of \( J_1 := c_1(J(a_*, w, 0)) \) is \( \leq 1 \) everywhere. Meanwhile it is easy to see that slope(\( \dot{c}_1(a) \)) \( \leq K\delta \) in (a). Consequently, \( J_1 \) is an admissible curve. Using (27),

\[
\log \frac{\|\dot{c}_1(a_*)\|}{\|\dot{c}_1(a)\|} \leq \frac{2|x_0(a_*) - x_0(a)|}{|x_0(a)|} \leq |c_0(a_*) - c_0(a)| \leq K\delta|a_* - a| \leq 1,
\]

and thus (b-i) holds for \( i = 1 \). By a similar reasoning we obtain (b-ii) for \( i = 1 \). (b-iii) for \( i = 1 \) clearly holds. This completes the proof for \( i = 1 \).

Let \( i \in [1, m-1] \) be a free iterate. If \( i \) is a return, then let \( q \) denote the corresponding binding period. Otherwise, let \( q = 0 \). We prove the assertion for \( i = i + q + 1 \), assuming that they hold for \( i \).

We prove (b-i). Define

\[
D(a, i) = \log \frac{\|\dot{c}_{i+1}(a)\|}{\|\dot{c}_i(a)\|} - \log \frac{\|w_{i+1}\|}{\|w_i\|}.
\]

If \( \dot{c}_{i+1}(a) = 0 \) (as it really never does), we define \( D(a, i) = +\infty \). By the chain rule and the assumption of the induction, it is enough to prove \( 2D(a, i) \leq \Phi(w) \cdot \Theta(w, i)^{-1} + \|w_i\|^{-\frac{1}{2}} \) for all \( a \in J(a_*, w, 0) \). To show this, split \( D(a, i) \leq A + B \), where

\[
A = \log \frac{\|DH^{q+1}(c_i(a))\dot{c}_i(a)\|}{\|\dot{c}_i(a)\|} - \log \frac{\|w_{i+1}\|}{\|w_i\|},
\]

\[
B = \log \frac{\|DH^{q+1}(c_i(a))\dot{c}_i(a)\|}{\|\dot{c}_i(a)\|} - \log \frac{\|\dot{c}_{i+1}(a)\|}{\|\dot{c}_i(a)\|}.
\]

It is enough to prove the following

**Lemma 8.16.3.** We have:

(a) \( A \leq (1 - e^{-\lambda})^{-1}(\Phi(w) \cdot \Theta(w, i)^{-1} + \|w_i\|^{-\frac{1}{2}}) \);

(b) \( B \leq \|w_i\|^{-\frac{1}{2}} \).

**Proof.** We claim that (a) follows from

\[
(28) \quad A \leq \frac{\|w_{i+q+1}\|}{\|w_i\|} \left[ \Phi(w)\Theta(w, i)^{-1} + \|w_i\|^{-\frac{1}{2}} \right],
\]

...
where
\[ A := \left| \frac{\|DH_a^{q+1}(c_i(a))\dot{c}_i(a)\|}{\|\dot{c}_i(a)\|} - \frac{\|w_{i+q+1}\|}{\|w_i\|} \right|. \]

Indeed, by the definition of \( \Phi(w) \) and (S1), the number in the biggest parenthesis in (28) is \( \leq e^{-\lambda i} + e^{-10 \Delta} \leq e^{-\lambda} < 1 \), we have
\[ \|DH_a^{q+1}(c_i(a))\dot{c}_i(a)\| \geq (1 - e^{-\lambda})\|w_{i+q+1}\|. \]

Taking logs we obtain (a).

We prove (28). Split \( A \leq I + II + III + IV + V + VI \), where
\[
I = \left| \frac{\|DH_a^{q+1}(c_i(a))\dot{c}_i(a)\|}{\|\dot{c}_i(a)\|} - \frac{\|DH_a^{q+1}(c_i(a))\dot{c}_i(a)\|}{\|\dot{c}_i(a)\|} \right|,
\]
\[ II = 2 \cdot \|DH_a^{q+1}(c_i(a))\| \left| \frac{\dot{c}_i(a)}{\|\dot{c}_i(a)\|} - \frac{\dot{c}_i(a)}{\|\dot{c}_i(a)\|} \right|,
\]
\[ III = \|DH_a^{q+1}(c_i(a))\| \left| \frac{\dot{c}_i(a)}{\|\dot{c}_i(a)\|} - \frac{\dot{c}_i(a)}{\|\dot{c}_i(a)\|} \right|,
\]
\[ IV = 2 \cdot \|DH_a^{q+1}(c_i(a))\| \left| \frac{\dot{c}_i(a)}{\|\dot{c}_i(a)\|} - \frac{w_i}{\|w_i\|} \right|,
\]
\[ V = \|DH_a^{q+1}(c_i(a))\| \left| \frac{\dot{c}_i(a)}{\|\dot{c}_i(a)\|} - \frac{w_i}{\|w_i\|} \right|,
\]
\[ VI = \|DH_a^{q+1}(c_i(a)) - DH_a^{q+1}(c_i(a))\|.
\]

Suppose that \( q = 0 \). Using (b-i),
\[ I, II, III, VI \leq K|c_i(a) - c_i(a)| \leq e|a - a| \|\dot{c}_i(a)\|.
\]

Using Lemma 7.2.4 and (b) in Proposition 5.3.1
\[ I, II, III, VI \leq \Phi(w)\Theta(w, i)\Theta(w, i)^{-1}\|w_i\| \leq \Phi(w)\Theta(w, i)\Theta(w, i)^{-1}\|w_{i+q+1}\|. \]

Sublemma 8.16.4. (19) Lemma 6.3 If \( \dot{c}_i(a) \neq 0 \), then
\[ \angle(\dot{c}_i(a), w_i(a)) \leq \frac{\|w_0(a)\|}{\|w_1(a)\|} \left( \sum_{s=1}^{i} \frac{\|w_s(a)\|}{\|w_1(a)\|} b_j - s + \frac{\|w_0(a)\|}{\|w_1(a)\|} b_j \right). \]

Using Sublemma 8.16.4 we have IV, V \leq K\|w_i\|^{-1}. Hence we obtain (28).

Suppose that \( q \neq 0 \). Let \( \zeta_0 \) denote a binding point of order \( \xi \) at the free return \( i \) and \( \tilde{\omega} = \{\tilde{w}_i\}_{i=0}^{i_\Delta} \) the corresponding forward vector orbit. Let \( p \) denote the folding period. By Remark 7.2.3, there exists a smooth continuation \( a \in J(a_\xi, w, 0) \rightarrow \zeta_0(a) \) such that the corresponding forward vector orbits \( \tilde{\omega}(a) \) obey (b-ii).

The rest of the argument needs three sublemmas.

Sublemma 8.16.5. Let \( a, b \in J(a_\xi, w, 0) \). The tangent vector \( (c_i(a), \dot{c}_i(a)) \) is in admissible position relative to \( \zeta_0(b) \). In particular, \( H_{bc_i}(a) \subseteq \Gamma^{(\beta_\xi-1)}(\tilde{\omega}(b)) \) holds.
Proof. Using Lemma 7.2.3,
\[ |c_i(a_*) - c_i(a)| \leq \|w_i\|\Phi(w) \leq \|w_i\|\Theta(w, i) \]
\[ \leq \left( \frac{\|u_i\|}{\|w_{i+p}\|} \right)^2 L^2 |\tilde{\zeta}_0 - \zeta_{i+1}|^{2(1-\alpha)} \ll |\tilde{\zeta}_0 - \zeta_{i+1}|. \]
This and the fact that \( J \) is an admissible curve together imply that \((c_i(a), \tilde{c}_i(a))\) is in admissible position relative to \( \zeta_0 \), provided that \((c_i(a_*), \tilde{c}_i(a_*))\) is in admissible position relative to \( \tilde{\zeta}_0 \). This is indeed the case by Sublemma 8.16.4. On the other hand, by Proposition 5.2.1 and (a) in Proposition 5.3.1,
\[ |\tilde{\zeta}_0 - \zeta_0(b)| \leq |\tilde{\zeta}_0 - \tilde{\zeta}_0(a_*)| + |\tilde{\zeta}_0(a_*) - \zeta_0(b)| \ll |\tilde{\zeta}_0 - \zeta_{i+1}|. \]
Hence the first assertion follows. The last assertion follows from this. \( \Box \)

Sublemma 8.16.6. For every \( 1 \leq k \leq q \), \( a \in J(a_*, w_0) \), \( z \in \Gamma(q^{-1})(\omega(a)) \),
\[ \| \partial(DH^k_a(z)) \| \leq K e^{2\alpha k} \| \tilde{w}_k \|^2, \]
where \( \partial \) denotes any partial derivative of the first order.

Proof. For \( 1 \leq s \leq k \) we have
\[ \| D^{s-1}H_a^k(z) \| \leq e \| \tilde{w}_{s-1}(a) \| \leq e^{-\lambda \alpha(\chi(k) - s+1)} e^{\alpha \sigma \chi(k) + 1} \| \tilde{w}_s(a) \|. \]
Since \( \chi(k) \) is free, \( \| \tilde{w}_k(a) \| \leq \| \tilde{w}_s(a) \| \), and thus \( \| \tilde{w}_k(a) \| \leq e \| \tilde{w}_k(a) \| \leq e^{1+\alpha \sigma} \| \tilde{w}_k \|. \) Using this and Lemma 8.16.1,
\[ \| \partial(DH^k_a(z)) \| \leq K \sum_{s=1}^k \| D^{s-1}H^s_a(z) \| \| DH^s_a(z) \| \leq e^{2\alpha \sigma} \| \tilde{w}_k \|^2. \]
This finishes the proof. \( \Box \)

Sublemma 8.16.7. For all \( a \in J(a_*, w_0) \) we have
\[ I \leq \frac{\|w_{i+q+1}\|}{\|w_i\|} \left( \frac{\|w_i\|}{\|w_{i+p}\|} \right)^2 |c_i(a_*) - c_i(a)| \]

Proof. By Sublemma 8.16.5 we have \( H_{a_*}(c_i(a)) \in \Gamma^{(q^{-1})}((\tilde{w}(a_*))) \), and hence the contractive directions \( e_i \) \( (i = 1, \cdots, q) \) under the iterations of \( H_{a_*} \) are well-defined at \( H_{a_*}(c_i(a)) \). Split
\[ \frac{DH_{a_*}(c_i(a_*))c_i(a_*)}{\|c_i(a_*)\|} = \xi e_q(c_i(a_*)) + \eta f_q(c_i(a_*)) \]
and
\[ \frac{DH_{a_*}(c_i(a))c_i(a)}{\|c_i(a)\|} = \tilde{\xi} e_q(H_{a_*}c_i(a)) + \tilde{\eta} f_q(H_{a_*}c_i(a)). \]
Then \( I \leq A + B + C + D \), where
\[ A = |\xi - \tilde{\xi}| \| DH_{a_*}^q e_q(H_{a_*}c_i(a)) \|, \]
\[ B = |\eta - \tilde{\eta}| \| DH_{a_*}^q f_q(c_i(a_*)) \|, \]
\[ C = |\xi| \| DH_{a_*}^q e_q(c_i(a_*)) - DH_{a_*}^q e_q(H_{a_*}c_i(a)) \|, \]
\[ D = |\eta| \| DH_{a_*}^q f_q(c_i(a_*)) - DH_{a_*}^q f_q(H_{a_*}c_i(a)) \|. \]
We estimate $A$, $B$, $C$, $D$ one by one. It can be read out from the proof of Lemma \[8.11.1\] that the Lipschitz continuity of the first order derivatives of $H$ and the fact that $\mathcal{J}_i$ is an admissible curve together imply

$$A \leq \|\xi - \hat{\xi}\| \leq K|c_i(a_* - \xi(a)|.$$

Applying the capture argument, we can find an admissible curve $\gamma$ which contains $Z_i(a_*)$ and a critical point in its boundary. Applying the argument in the proof of Lemma \[8.11.1\] to $\gamma \cup \mathcal{J}_i$, we have $|\eta - \hat{\eta}| \leq K|c_i(a_*) - \xi(a)|$, and thus

$$B \leq K|c_i(a_*) - \xi(a)|\|w_\xi\|.$$

Let $z \in \Gamma^{(\beta - 1)}(w)$. By the chain rule and Lemma \[8.16.1\],

$$\|D(DH_{a_*}^q(z)) \cdot e_q(z)\| \leq e^\Delta \sum_{s=1}^q \|DH_{a_*}^{q-s}(z_s)\| \|DH_{a_*}^{s-1}(z)c_q(z)\| \leq \|w_\xi\|.$$

we have

$$\|DH_{a_*}^q(z) \cdot De_q(z)\| = \|DH_{a_*}^q(z)f_q(z)\| \leq K\|w_\xi\|.$$

Using these and the mean value theorem,

$$C \leq K|c_i(a_*) - \xi(a)|\|w_\xi\|.$$

(b) in Proposition \[2.10.2\] implies

$$\|\tilde{w}_\xi\| \leq K|\tilde{\eta}_0 - \zeta_{i+1}^{-1}\|w_{i+q+1}\|/\|w_i\|.$$

Using this and Sublemma \[8.16.6\] for $k = q$, and then \[31\] and (b) (e) in Proposition \[2.10.2\]

$$\|\partial(DH_{a_*}^q(z))\| \leq |\tilde{\eta}_0 - \zeta_{i+1}^{-1}|\|w_{i+q+1}\|/\|w_i\| \left(\|w_i\|/\|w_{i+p}\|\right)^{3/2}.$$

By the mean value theorem and $|\eta| = |\tilde{\eta}_0 - \zeta_{i+1}|$,

$$D \leq |\eta|\|D(DH_{a_*}^q f_q(\cdot))\| |c_{i+1}(a_*) - H_{a_*} c_i(a)|
\leq |\eta| \left(\|D(DH_{a_*}^q(\cdot))\| + \|DH_{a_*}^q e_q(\cdot)\|\right) e^\Delta |c_i(a_*) - \xi(a)|
\leq \|w_{i+q+1}\|/\|w_i\| \left(\|w_i\|/\|w_{i+p}\|\right)^2 |c_i(a_*) - \xi(a)|\|w_{i+p}\|^2/c_i(a_* - \xi(a)|.$$

Consequently we obtain the desired upper estimate of $I$. \[\square\]

Back to the proof of \[28\] for $q \neq 0$, Sublemma \[8.16.7\] gives

$$I \leq \|w_{i+q+1}\|/\|w_i\| \left(\|w_i\|^2/\|w_{i+p}\|^2\right) |w_i|\Phi(w)\Theta(w, i)\Theta(w, i)^{-1} \leq \|w_{i+q+1}\|/\|w_i\|\Phi(w)\Theta(w, i)^{-1}.$$

Regarding $II$ and $III$, we have $\|DH_{a_*}^{q+1}(c_i(a))\| \leq \|DH_{a_*}^{q}(c_{i+1}(a))\| \leq \|\tilde{w}_\xi\|$, by Sublemma \[8.16.5\]. This yields

$$II, III \leq \left(\|w_i\|^2/\|w_{i+p}\|^2\right)^{1+\Delta} \|w_{i+q+1}\|/\|w_i\|\Phi(w) \leq \|w_{i+q+1}\|/\|w_i\|\Phi(w)\Theta(w, i)^{-1}.$$
Using Sublemma \ref{Lemma9.16.4},

\[ IV, V \leq \frac{\|w_{i+q+1}\|}{\|w_i\|} \left( \frac{\|w_i\|}{\|w_{i+p}\|} \right)^{1+\alpha} \leq \frac{\|w_{i+q+1}\|}{\|w_i\|} \left( \frac{\|w_0\|}{\|w_i\|} \right) \frac{1}{2}. \]

Now it is left to consider VI. Fix \( a \), and consider the matrix valued function \( \varphi: b \to DH_a^{q+1}(c_i(a)) \). Denote by \( D_b \) the \( b \)-derivative. The chain rule gives

\[ \|D_b \varphi(b)\| = \|D_b(DH_a^q(H_b(c_i(a))) \cdot DH_b(c_i(a)))\| \leq K \|D_b(DH_a^q(H_b(c_i(a))))\| + e^\Delta \|DH_b^q(H_b(c_i(a)))\|. \]

Let \( z \in \Gamma(\tilde{w}(b)) \). Using Sublemma \ref{Lemma9.16.6},

\[ \|D_b(DH_b^q(z))\| \leq \frac{\|w_{i+q+1}\|}{\|w_i\|} \frac{\|w_i\|^3}{\|w_{i+p}\|^3}. \]

By the mean value theorem,

\[ VI \leq \frac{\|w_{i+q+1}\|}{\|w_i\|} \frac{\|w_i\|^3}{\|w_{i+p}\|^3} \Phi(w)\Theta(w,i)\Theta(w,i)^{-1} \leq \frac{\|w_{i+q+1}\|}{\|w_i\|} \Phi(w)\Theta(w,i)^{-1}, \]

where the last inequality follows from \( \|w_0\| \leq \|w_{i+p}\| \). Consequently, (28) follows when \( q \neq 0 \) as well. This completes the proof of (a).

We prove (b). In view of (2) we have

\[ \|\hat{c}_{i+q+1}(a) - DH_a^{q+1}(c_i(a))\| \leq e^{\Delta q}. \]

Dividing both sides by \( \|\hat{c}_i(a)\| \) and then using \( q \leq \alpha i \) and (S1),

\[ \frac{\|\hat{c}_{i+q+1}(a)\|}{\|\hat{c}_i(a)\|} - \frac{\|DH_a^{q+1}(c_i(a))\|}{\|\hat{c}_i(a)\|} \leq e^{\Delta q} \|w_i\|^{-1} \leq \|w_i\|^{-1/2}. \]

This and (29) together imply

\[ \frac{\|\hat{c}_{i+q+1}(a)\|}{\|\hat{c}_i(a)\|} \geq (1 - e^{-\lambda}) \frac{\|w_{i+q+1}\|}{\|w_i\|} - \|w_i\|^{-1/2} \geq \frac{1}{2} \frac{\|w_{i+q+1}\|}{\|w_i\|}. \]

Taking logs and rearranging gives

\[ B \leq \frac{2\|w_i\|^{3/2}}{\|w_{i+q+1}\|} \leq \|w_i\|^{-\frac{1}{2}}, \]

where the last inequality follows from \( \|w_i\| \leq \|w_{i+q+1}\| \). This completes the proof Lemma \ref{Lemma9.16.3} and hence that of (b-i). \( \square \)

A proof of (b-ii) for \( i = i + q + 1 \) goes analogously, with

\[ \tilde{D}(a,i) = \left| \frac{\|w_{i+q+1}(a)\|}{\|w_i(a)\|} - \frac{\|w_{i+q+1}\|}{\|w_i\|} \right| \]

in the place of \( D(a,i) \). We have

\[ \tilde{D}(a,i) \leq \left| \frac{\|DH_a^{q+1}(c_i(a))w_i(a)\|}{\|w_i(a)\|} - \frac{\|w_{i+q+1}\|}{\|w_0\|} \right| + VI, \]

and the first term can be estimated similarly to the case of \( I \).
We now prove (b-iii) for $i = i + q + 1$. Let $1 \leq k \leq i$. By (b-i) and Lemma 7.2.4 we have $\|\hat{c}_{i+q+1}(a)\| \geq K_0\delta\|\hat{c}_i(a)\|$, regardless of whether $q = 0$ or not. From this and the inductive assumption we have

$$\|\hat{c}_{i+q+1-(k+q+1)}\| \leq (K_0\delta)^{-3k}\|\hat{c}_i\|^3 \leq (K_0\delta)^{-3(k+1)}\|\hat{c}_{i+q+1}\|^3 \leq (K_0\delta)^{-3(k+q+1)}\|\hat{c}_{i+q+1}\|^3.$$ 

Hence it is enough to prove $\|\hat{c}_j(a)\| \leq (K_0\delta)^{-3(i+q+1-j)}\|\hat{c}_{i+q+1}(a)\|$ for $i+1 \leq j \leq i + q + 1$. Let $k \in [1, q + 1]$. We compute $\hat{c}_{i+k}$ in view of (9) and split $\|\hat{c}_{i+k}\|/\|\hat{c}_{i+q+1}\|^3 \leq A + B + C + D$, where

$$A = \|\hat{c}_{i+q+1}\|^{-3}\|DH^k_a(c_i)\|,$$

$$B = \|\hat{c}_{i+q+1}\|^{-3}\sum_{s=0}^{k-1} DH^s_a(c_{i+k-s}) \left(\partial^2_a H + \partial_a(\partial_a H)\hat{c}_{i+k-s-1}\right),$$

$$C = \|\hat{c}_{i+q+1}\|^{-3}\|\partial_a(DH^k_a(c_i))\|,$$

$$D = \|\hat{c}_{i+q+1}\|^{-3}\sum_{s=0}^{k-1} \partial_a(DH^s_a(c_{i+k-s}))\|\partial_a H\|$$

where all the partial derivatives of $H$ inside the two sums are taken at $(a, c_{i+k-s-1})$.

Using the previous inequality and the strong regularity of $\hat{w}$ gives

$$\|DH^k_a^{-1}(c_{i+1})\| \leq K\left\|\frac{\hat{w}_{i+q+1}}{\hat{w}_i}\right\| \leq \left(\|\hat{w}_i\|^{-1/3}\right)^{1+1/3} \leq \left(\frac{\|\hat{w}_i\|^{1+\alpha}}{\|\hat{w}_{i+q+1}\|^{1+\alpha}}\right)^{1+1/3}.$$ 

Using this and $\|\hat{c}_i\| \leq \|\hat{c}_i\|^3$, which is part of the assumption of the induction,

$$A \leq \frac{\|DH^k_a^{-1}(c_{i+1})\|}{\|\hat{c}_{i+q+1}\|^3} \leq \frac{\|\hat{c}_i\|^3\|DH^k_a^{-1}(c_{i+1})\|}{\|\hat{c}_{i+q+1}\|^3} \leq \frac{\|\hat{c}_i\|^3}{\|\hat{c}_{i+q+1}\|^3} \left(\frac{\|\hat{w}_i\|^{1+\alpha}}{\|\hat{w}_{i+q+1}\|^{1+\alpha}}\right)^{1+1/3} \leq 1/4.$$ 

**Claim 8.16.8.** If $\ell \in [1, q + 1]$, then $\|\hat{c}_{i+\ell}\| \leq \|\hat{w}_{i}\|^{1+\ell}$. 

**Proof.** If $q = 0$ then $\ell = 1$, and we have $\|\hat{c}_{i+1}\| \leq 4\delta\|\hat{c}_i\| \leq \|\hat{w}_{i}\|^{1+\ell}$. If $q \neq 0$ then

$$\|\hat{c}_{i+\ell}\| \leq \|\hat{c}_{i+\ell} - DH^\ell_a^{i+\ell}(c_{i+\ell})\| + \|DH^\ell_a^{i+\ell}(c_{i+\ell})\|.$$ 

By (322), the first term is $\leq e^{\Delta\sigma(i+\ell)}$. To estimate the second term, we use the fact that $\chi(i+\ell)$ is a free iterate before $i$, (b-i), and Lemma 7.2.4. Then $\|\hat{c}_{i+\ell}\| \leq e^{\Delta\sigma(i+\ell)}(1 + \|\hat{c}_{i+\ell}(c_{i+\ell})\|) \leq \|\hat{w}_{i}(c_{i+\ell})\|^{1+\ell} \leq e^{\Delta\sigma} \|\hat{w}_i\| \leq \|\hat{w}_i\|^{1+\frac{1}{3}}$. \qed

Using Claim **8.16.8**

$$B \leq e^{\Delta q} \frac{\|\hat{c}_i\|^2}{\|\hat{c}_{i+q+1}\|^3} \leq e^{\Delta q} \frac{1}{\|\hat{c}_{i+q+1}\|^3} \leq \frac{1}{4}.$$
We estimate $C$. By the chain rule,
\[ \| \partial_a(DH^k_a(c_i(a))) \| \leq K \| DH^{k-1}_a(c_{i+1}) \| + K \| \partial_a(DH^{k-1}_a(c_{i+1})) \|. \]
Using Claim 8.16.6 and $\| \dot{c}_{i+1} \| \leq K \| \dot{c}_i \|,$
\[ \| \partial_a(DH^k_a(c_i(a))) \| \leq \frac{\| \tilde{w}_k \|}{\| \tilde{w}_0 \|} + e^{\alpha q} \| \tilde{w}_k \| \| \dot{c}_i \| \leq e^{-(\lambda-\alpha)(q-k)+\alpha q(1+e^{\alpha q})} \frac{\| \tilde{w}_k \|^2}{\| \tilde{w}_0 \|^2}. \]
Using (31) and $q \leq \alpha i$ we obtain
\[ \| \partial_a(DH^k_a(c_i(a))) \| \leq e^{3\alpha i} \| \dot{c}_i \| \frac{\| w_{i+q+1} \|^2}{\| w_i \|^2} \leq \| w_{i+q+1} \|^2, \]
and therefore $C \leq 1/4$.

We estimate $D$. Using the chain rule and Claim 8.16.8
\[ \| \partial_a DH^k_a(c_{i+k-s}) \| \leq K se^{\Delta s} \| c_{i+k-s} \| \leq e^{2\Delta s} \| w_i \|^{1/3}. \]
This yields
\[ D \leq e^{\Delta q} \frac{\| w_i \|^{1+1/3}}{\| w_{i+q+1} \|^{3/3}} \leq \frac{1}{4}. \]
Altogether these yield (b-iii) for $i = i + q + 1$.

It is left to restore the assumption of the induction, that is, to prove that $J_{i+q+1}$ is an admissible curve. For an arbitrary $i$ and $a \in J(a_s, w_0)$, let $\kappa_i(a)$ denote the curvature of $J_i$ at $c_i(a)$. Split $\kappa_{i+1}(a) \leq \kappa_{i+1}^{(1)}(a) + \kappa_{i+1}^{(2)}(a)$, where
\[ \kappa_{i+1}^{(1)} = \frac{\| DH_a(c_i(a)) \dot{c}_i(a) \times \dot{c}_i(a) \|}{\| \dot{c}_{i+1}(a) \|^3}, \]
\[ \kappa_{i+1}^{(2)} = \frac{\| \dot{c}_{i+1}(a) \|}{\| \dot{c}_{i+1}(a) \|^3}. \]

**Sublemma 8.16.9.** For every $i \geq 0$,
\[ \kappa_{i+1}^{(1)} \leq Kb \cdot \frac{\| \dot{c}_i \|^3}{\| \dot{c}_{i+1} \|^3} (\kappa_{i+1}^{(1)} + \kappa_{i+1}^{(2)} + 1). \]

**Proof.** Split $\kappa_{i+1}^{(1)} \leq I + II + III$, where
\[ I = \| \dot{c}_{i+1} \|^{-3} \| DH_a(c_i) \dot{c}_i \times \partial_a^2 H \|, \]
\[ II = \| \dot{c}_{i+1} \|^{-3} \| DH_a(c_i) \dot{c}_i \times \partial_a (\partial_a H) \cdot \dot{c}_i \|, \]
\[ III = \| \dot{c}_{i+1} \|^{-3} \| DH_a(c_i) \dot{c}_i \times DH_a(c_i) \dot{c}_i \|, \]
where all the partial derivatives are taken at $(a, c_i(a))$. Since $H$ is a small perturbation of $(x, y) \to (1 - ax^2, 0)$, the $C^0$ norm of $\partial_a^2 H(a, c_i(a))$ is close to zero. In particular we have
\[ I \leq Kb \frac{\| \dot{c}_i \|}{\| \dot{c}_{i+1} \|^3}. \]
Clearly, $\| \partial_a(\partial_a H(a, c_i)) \| \leq K \| \dot{c}_i \|$ holds, and thus the numerator of $II$ is free three homogeneous in $\| \dot{c}_i(a) \|$. Moreover, it is easy to see that the second components
of the two vectors involved in the product is smaller than \( Kb \) in norm. Hence we obtain

\[
II \leq Kb \left\| \frac{\dot{c}_i}{\| \dot{c}_{i+1} \|} \right\|^3.
\]

Meanwhile it is easy to see that

\[
III \leq Kb \left( \left\| \frac{\dot{c}_i}{\| \dot{c}_{i+1} \|} \right\|^3 \left( \kappa_i^{(1)} + \kappa_i^{(2)} \right) \right).
\]

Putting these three inequalities together we obtain the desired one. \( \square \)

A recursive use of this inequality in Sublemma 8.16.9 yields

\[
\kappa_{i+q+1}^{(1)} \leq (Kb)^{i+q} \frac{\| \dot{c}_i \|^3}{\| \dot{c}_{i+q+1} \|^3} \kappa_0^{(1)} + \sum_{\ell=0}^{i+q} (Kb)_{\ell+1} \frac{\| \dot{c}_{i+q+1-\ell} \|^3}{\| \dot{c}_{i+q+1} \|^3} (\kappa_{i+q+1-\ell}^{(2)} + 1).
\]

Using (b-iii) for \( i = i + q + 1 \),

\[
\frac{\| \dot{c}_{i+q+1-\ell} \|^3}{\| \dot{c}_{i+q+1} \|^3} \kappa_{i+q+1-\ell}^{(2)} \leq (K_0^\delta)^{-\ell}.
\]

Lemma 7.2.4 gives

\[
\frac{\| \dot{c}_0 \|}{\| \dot{c}_{i+q+1} \|} \leq e^{\frac{\| w_0 \|}{\| w_{i+q+1} \|}} \leq e^{2K_0^{-1}\delta^{-1}},
\]

regardless of whether \( q = 0 \) or not. Substituting these into the above inequality we obtain \( \kappa_{i+q+1}^{(1)} \leq 1 \). Hence we obtain \( \kappa_{i+q+1} \leq 1 \). Regarding the slope, recall that \( q \) is a free iterate of \( w(a) \) for all \( a \in J(a_*, w, 0) \). Thus \( \text{slope}(w_q(a)) \leq K_0 b \) holds. This and Lemma 8.16.4 together yield \( \text{slope}(\dot{c}_{i+q+1}(a)) \leq K_0 b \). Hence \( J_{i+q+1} \) is an admissible curve. This completes the proof of Proposition 7.1.1. \( \square \)

8.17. Proof of Lemma 7.3.2 Put \( \nu_i = \nu \). Let \( 0 < m_0 < m_1 < \cdots < m_t < \nu \) denote the set of all free returns which take place before \( \nu \). Let \( p_s, q_s \) \((0 \leq s \leq t)\) denote the corresponding folding and binding periods.

Sublemma 8.17.1. For every \( 0 \leq s \leq t \) and \( m_s \leq i \leq m_s + q_s + 1 \),

\[
\min_{i \leq j \leq \nu} \frac{\| w_j \|}{\| w_i \|} \geq \min_{s \leq u \leq t} e^{-3d(m_u)}.
\]

Proof. There are three cases: \( j \leq m_s + q_s + 1 \); \( j > m_s + q_s + 1 \) and \( j \) is free; \( j > m_s + q_s + 1 \) and \( j \) is bound. In the first case, the desired inequality immediately follows from (h) in Proposition 2.10.2. In the second case, split

\[
\frac{\| w_j \|}{\| w_i \|} = \frac{\| w_j \|}{\| w_{m_s+q_s+1} \|} \frac{\| w_{m_s+q_s+1} \|}{\| w_i \|}.
\]

The first term is \( \geq K_0\delta \), because \( m_s + q_s + 1 \) and \( j \) are free. Applying (h) in Proposition 2.10.2 to the second term,

\[
\frac{\| w_j \|}{\| w_i \|} \geq K_0\delta e^{-4d(m_s)} \geq e^{-3d(m_s)}.
\]
In the last case, there exists \( u \in [s + 1, t] \) such that \( j \in [m_u + 1, m_u + q_u + 1] \). Split
\[
\|w_j\| = \frac{\|w_{m_u}\|}{\|w_i\|} \|w_{m_u}\|.
\]
Using (h) again and \( \|w_{m_u}\| \geq K_0 e^{-1}\delta \|w_i\| \),
\[
\frac{\|w_j\|}{\|w_i\|} \geq K_0 e^{-1}\delta e^{-(1 + \frac{\alpha}{1 + \alpha})d(m_u)} \geq e^{-3d(m_u)}.
\]

**Sublemma 8.17.2.** For every \( 0 \leq s \leq t \),
\[
\sum_{i=m_{s-1}+q_{s-1}+1}^{m_s} \Theta(\Pi'_0 w, i) \leq \frac{\|w_{m_s}\|}{1 - e^{-\lambda}} \max_{s \leq u \leq t} e^{6d(m_u)}.
\]

**Proof.** Let \( j \in [i, \nu] \). Suppose that \( j \geq m_s \). By Sublemma 8.17.1,
\[
(34) \quad \frac{\|w_j\|}{\|w_i\|} = \frac{\|w_{m_u}\|}{\|w_i\|} \frac{\|w_{m_u}\|}{\|w_{m_u}\|} \geq \frac{\|w_{m_u}\|}{\|w_{m_u}\|} \geq \min_{s \leq u \leq t} e^{-3d(m_u)}.
\]
Suppose that \( j < m_s \). Then \( \|w_j\| \geq K_0 \delta \|w_i\| \) holds because \( i \) is free and no return takes place until \( j \). Hence the inequality in (34) holds in this case as well. Substituting (34) and \( \|w_i\| \leq \|w_{m_u}\| e^{-\lambda(m_u - i)} \) into the definition of \( \Theta(\Pi'_0 w, i) \),
\[
\Theta(\Pi'_0 w, i) \leq \|w_{m_u}\| e^{-\lambda(m_u - i)} \max_{s \leq u \leq t} e^{6d(m_u)}.
\]
Summing up this for every \( i \in [m_{s-1} + q_{s-1} + q, m_s] \) yields the desired inequality. \( \square \)

**Sublemma 8.17.3.** We have
\[
\sum_{i=\mu(t)+q(t)+1}^{\nu-1} \|w_{\nu'}\|^{-1} \cdot \Theta(\Pi'_0 w, i) \leq -\lambda^{-1} \log(K_0 \delta) \cdot \delta^{\frac{\alpha \lambda \delta}{100 \lambda}}.
\]

**Proof.** Put \( s_0 = -2\lambda^{-1} \log(K_0 \delta) \gg 1 \). Since no return takes place from \( i \) to \( \nu \),
\[
\|w_{\nu'}\| \cdot \Theta(\Pi'_0 w, i) = \min_{i \leq j \leq \nu} \frac{\|w_{\nu'}\|}{\|w_i\|} \left( \frac{\|w_j\|}{\|w_i\|} \right)^2 \geq (K_0 \delta)^2 e^{\lambda(\nu - i)} \geq e^{\lambda(\nu - i - s_0)},
\]
and thus
\[
(35) \quad \sum_{m_t + q_t + 1 \leq i \leq \nu - 1} \|w_{\nu'}\|^{-1} \cdot \Theta(\Pi'_0 w, i) \leq \sum_{i=0}^{\infty} e^{-\lambda i} = \frac{1}{1 - e^{-\lambda}}.
\]
Suppose that \( i \geq \nu - s_0 \). Let \( j \in [i, \nu] \) denote an integer such that
\[
\Theta(\Pi'_0 w, i) = \frac{\|w_0\|}{\|w_i\|} \frac{\|w_j\|}{\|w_i\|} \frac{\|w_j\|}{\|w_i\|}.
\]
Let \( x_i \) denote the x-coordinate of \( z_i \), and put \( x_{j_0} = \min_{i \leq j \leq j-1} |x_k| \). Using (b) in Lemma 2.2.1 successively we have \( |x_{j_0}| |w_i| \leq \|w_j\| \), and thus
\[
\|w_{\nu'}\| \cdot \Theta(\Pi'_0 w, i) \geq |x_{j_0}|^2 \frac{\|w_{\nu'}\|}{\|w_{j_0}\|}.
\]
Suppose that $|x_{j_0}| \geq \delta^{1/100}$. Then $\|w_\nu\| \cdot \Theta(\Pi'_0 w, i) \geq \delta^{1/50}$. Suppose that $\delta \leq |x_{j_0}| \leq \delta^{1/100}$. In this case, although $j_0$ is not a return time, we can consider a binding period $q$ initiated at $j_0$, and it is easy to show that the same estimates as in Proposition 2.10.2 holds. In particular,

**Claim 8.17.4.** $|x_{j_0}|\|w_{j_0+q+1}\| \geq \|w_{j_0}\|$.  

**Proof.** It is easy to see that the lower estimate of (d) in Proposition 2.10.2 remains valid even if we replace $\Delta$ by some constant $c_0 \approx \log 2$, because the argument only involves the first order derivative of the map. Then we have $q \geq -(3 \log 2/c_0) \log |x_{j_0}|$. Since $w_{j_0+1} \leq K_0 b$ and the orbit keeps staying close to $(-1,0)$ until the end of the binding period, there exists $c_1 \approx \log 2$ such that $\|w_{j_0+q+1}\| \geq e^{c_1 \eta}\|w_{j_0+1}\|$. We have

$$\frac{\|w_{j_0+q+1}\|}{\|w_{j_0}\|}|x_{j_0}| \geq |x_{j_0}|^2 e^{c_1 \eta} \geq |x_{j_0}|^{-1/2}.$$  

This yields the claim. \hfill \Box

By (f) in Proposition 2.10.2 and the fact that $f_{2}^{2}(0) = -1 = f_{2}(-1)$, we have $|x_{j_0+q+2} + 1| \leq \delta^{\frac{m_0}{1000}}$. Since $x_\nu \in (-\delta, \delta)$, we have $\nu - j_0 - q - 1 \geq -\frac{\alpha}{10000} \log \delta$. This yields $|x_{j_0}|\|w_\nu\| \geq \delta^{1-\frac{\alpha}{10000}}\|w_{j_0+q+1}\|$. Putting all these together we have

$$\|w_{\nu}\| \cdot \Theta(\Pi'_0 w, i) \geq |x_{j_0}|^2 \frac{\|w_{\nu}\|}{\|w_{j_0+q+1}\|} \frac{\|w_{j_0+q+1}\|}{\|w_{j_0}\|} \geq \delta^{1-2\alpha_0}.$$

Therefore

$$\sum_{\substack{m_0+q+1 \leq i \leq \nu-1 \\
\nu - s \geq s_0}} \|w_\nu\|^{-1} \Theta(\Pi'_0 w, i)^{-1} \leq s_0 \delta^{2\alpha_0-1}.$$  

This and (35) yield the desired inequality because $s_0 \delta^{2\alpha_0-1} \to +\infty$ as $\delta \to 0$. \hfill \Box

We are now in position to conclude a proof of the lemma. It is enough to show that for every $0 \leq s \leq t$,

$$\|w_\nu\| \cdot \left[ \sum_{i=m_0+q_0+1}^{m_0+q_1+1} \Theta(\Pi'_0 w, i)^{-1} \right]^{-1} \geq |\tilde{\zeta}_0 - \zeta_{\nu+1}|^{3/5} \delta^{-(t-s)/1000}.$$  

Indeed, taking reciprocals of both sides and summing up for all $0 \leq s \leq t$ we obtain

$$\|w_\nu\|^{-1} \Phi(\Pi'_0 w)^{-1} = \sum_{1 \leq i \leq \nu-1} \|w_\nu\|^{-1} \Theta(\Pi'_0 w, i)^{-1}$$

$$= \sum_{s=0}^{t} \left( \sum_{i=m_0+q_0+1}^{m_0+q_1+1} \right) + \sum_{i=m_t+q_{t+1}}^{\nu-1}$$

$$\leq -\log \delta \cdot \delta^{2\alpha_0-1} + |\tilde{\zeta}_0 - \zeta_{\nu+1}|^{-3/5} \sum_{s=0}^{t} \delta^{(t-s)/1000}$$

$$\leq |\tilde{\zeta}_0 - \zeta_{\nu+1}|^{\alpha_0-1}.$$  

Taking the reciprocals of both sides we obtain the desired inequality.
It is left to prove (36). Using this and Sublemma 8.17.2

\[ \| w_\nu \| \cdot \left[ \sum_{i=m_{s-1}+q_{s-1}+1}^{m_s} \Theta(\Pi_0' w_i)^{-1} \right]^{-1} \geq \| w_\nu \| e^{-6d(m_s)}. \]

Suppose that \( t = s \). Since \( \nu \) is an essential return, \( d(\nu) \geq 10d(m_t) \) holds. Meanwhile, by the definition of \( d(\cdot) \) we have \( -\log |\tilde{\zeta}_0 - \zeta_{\nu+1}| \geq 11d(m_t) \), regardless of whether \( \zeta_{\nu+1} \) is in admissible or in critical position. Substituting this into (37) we obtain (36).

Suppose that \( 0 \leq s \leq t - 1 \). On the first term of the right hand side of (37),

\[ \| w_\nu \| \cdot \left[ \sum_{i=m_{s-1}+q_{s-1}+1}^{m_s} \Theta(\Pi_0' w_i)^{-1} \right]^{-1} \geq \prod_{s \leq u \leq t} \| w_{m_u+q_u+1} \| e^{-3d(m_u)/5}. \]

Since \( \nu \) is an essential return, for every \( 0 \leq s \leq t - 1 \),

\[ 10d(m_s) \leq d(\nu) + \sum_{s+1 \leq u \leq t} d(m_u). \]

Therefore

\[ \| w_\nu \| \cdot \left[ \sum_{i=m_{s-1}+q_{s-1}+1}^{m_s} \Theta(\Pi_0' w_i)^{-1} \right]^{-1} \geq e^{-3d(\nu)/5} \prod_{s+1 \leq u \leq t} \| w_{m_u+q_u+1} \| e^{-3d(m_u)/5}. \]

By Lemma 2.10.2

\[ \| w_{m_u+q_u+1} \| e^{-3d(m_u)/5} \geq e^{-d(m_u)/100} \geq \delta^{-1/100}. \]

Sustituting this into the right hand side we obtain (36). This completes the proof of Lemma 7.3.2 and hence that of Proposition 7.3.1.

**References**

[1] M. Benedicks and L. Carleson - On iterations of \( 1 - ax^2 \) on \((-1, 1)\). Ann. of Math. (2) 122 (1985), no. 1, 1–25.

[2] M. Benedicks and L. Carleson - The dynamics of the Hénon map. Ann of Math. (2) 133, 73–169, (1991)

[3] M. Benedicks and M. Viana - Solution of the basin problem for Hénon-like attractors. Invent math. 143, 375–434, (2001)

[4] M. Benedicks and M. Viana - Random perturbations and statistical properties of Hénon-like maps. Ann. Inst. H. Poincare Anal. Non Lineaire 23 (2006), no. 5, 713–752.

[5] M. Benedicks and L-S. Young - Sinai-Bowen-Ruelle measures for certain Hénon maps. Invent Math. 112, 541–576, (1993)

[6] M. Benedicks and L-S. Young - Markov extensions and decay of correlations for certain Hénon maps. Asterisque No. 261 (2000), xi, 13–56.

[7] P. Collet and J. P. Eckmann - Positive Lyapunov exponents and absolute continuity for maps of the interval. Ergod. Th. & Dynam. Sys. (1983), 3, 13-46

[8] L. Díaz, J. Rocha, and M. Viana - Strange attractors in saddle-node cycles: prevalence and globality. Invent. Math. 125 (1996), no. 1, 37–74.
[9] M. Jakobson - Absolutely continuous invariant measures for one-parameter families of one-dimensional maps. *Comm. Math. Phys.* **81** (1981), 39–88.

[10] L. Mora and M. Viana - Abundance of strange attractors. *Acta Math.* **171** 1–71. (1993)

[11] E. Pujals and F. R. Hertz - Critical points for surface diffeomorphisms. preprint IMPA.

[12] J. Palis and F. Takens - *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations. Fractal dimensions and infinitely many attractors*. Cambridge Studies in Advanced Mathematics, **35**. Cambridge University Press. (1993)

[13] H. Takahasi - On the basin problem for Hénon-like attractors, *Journal of Math Kyoto Univ.* **46** no.2 (2006) 303-348

[14] H. Takahasi - Critical points for surface maps and the basin problem for Hénon-like attractors, in preparation.

[15] M. Tsujii - A proof of Benedicks-Carleson-Jakobson Theorem. *Tokyo J. Math.* 16 (1993), no. 2, 295–310.

[16] M. Tsujii Positive Lyapunov exponents in families of one-dimensional dynamical systems. *Invent. Math.* **111** (1993), no. 1, 113–137

[17] M. Tsujii - Strange attractors of the Hénon map, in Japanese.

[18] M. Viana - Strange attractors in higher dimensions. *Bol. Soc. Brasil. Mat.* **24** (1993), no. 1, 13–62.

[19] Q. Wang and L-S. Young - Strange attractors with one direction of instability. *Comm. Math. Phys.* **218** (2001), 1–97.

[20] Q. Wang and L-S. Young - Toward a theory of rank one attractors, to appear in *Ann. of Math.*

**Estrada Dona Castorina 110 22460-320 IMPA Rio de Janeiro Brazil**

**E-mail address:** takahash@impa.br