Universal security for randomness expansion

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Abstract
We show that any spatially separated multi-part quantum device demonstrating nonlocality can be used in an untrusted-device protocol for randomness expansion with unconditional quantum security. A consequence is that the noise tolerance for secure randomness expansion only needs to be small enough that it rules out deterministic behavior of the device. This greatly reduces the requirement on implementation precision. For example, for the CHSH game, the noise can be 10.3%, compared with 1.5% in the previous bound. We also show that similar results hold with nonlocality replaced by the broader concept of contextuality, and the spatial separation requirement replaced by the broader compatibility requirement. This is the first full quantum security proof for contextuality-based randomness expansion.

For both nonlocality and contextuality, we have identified the minimum device requirement. Our results imply in particular the equivalence of quantum security with classical security for the protocols considered. Our main technical contribution is a strong Schatten-norm uncertainty principle which applies to arbitrary pairs of noncommuting binary measurements.
1 Motivation

Randomness is indispensable for modern day information processing. It captures the essence of secrecy. This is because a message being secretive means precisely that it is random to the adversary. It also drives randomized algorithms (such as physics simulation), besides many other applications. However most practical random number generators (RNG) are heuristics without theoretical guarantees. There are known vulnerabilities in the methods currently in use [9].

More recently, RNGs based on quantum measurements have emerged in the market. While a (close to) perfect implementation of certain measurements can theoretically guarantee randomness, current technology is still far from reaching that precision. This raises a serious question: would the implementation imperfections open the door to adversary attacks? An additional concern is, even if in the future when the implementation technology is satisfactory, could there be “backdoors” in the generator inserted by a malicious party? It is difficult for the user, as a classical being, to directly verify the inner-working of the quantum device.

Those considerations motivated the study of untrusted-device quantum protocols, which are deterministic procedures interacting with (necessarily) multiple “untrusted” quantum devices. The user makes no prior assumptions about inner-workings of the devices. In particular, the devices may be entangled among themselves, or even with the external adversary. This protocol includes a certification procedure which decides whether the outputs should be “accepted” or “rejected.” Ideally, two types of errors should be minimized. The “completeness error” is the chance of rejecting an honest implementation (that is, a correct implementation with a possible limited amount of noise, or device deficiency), and the “soundness error” is the chance of accepting when the generated output is not uniformly random.

An untrusted-device protocol necessarily needs a classical input X to begin with that is not fully known to the adversary-device system. In this paper we assume that X is a small uniformly random seed, and our goal is to expand it into a much longer output which is also (nearly) uniformly random. This is randomness expansion (or “seeded” extraction in the terminology of [3]).

In his Ph.D. thesis [4], Colbeck formulated the problem of randomness expansion and proposed protocols based on quantum non-local games. New protocols and security analyses followed. Several authors proved classical security only [17, 8, 18, 5]. Vazirani and Vidick [21] was the first to prove full quantum security. Their protocol is also exponentially expanding using just two non-communicating devices. In [16], the present authors developed a different approach for the security analysis, and proved quantum security together with several new desirable properties including robustness (i.e., the honest implementation being imperfect), cryptographic security, and unit size quantum memory requirement for each device.

In [16] and in the current paper, we work with the “spot-checking protocol” developed in [21] and [5]. Informally, the protocol proceeds as follows: an n-player nonlocal game G is chosen, and a specific n-letter input string (a₁, . . . , aₙ) from the game is selected. We suppose the existence of an untrusted n-part quantum device D. At each round of the protocol, the user choses a bit g ∈ {0, 1} according to a biased (1 − q, q) distribution (with q > 0 small). If g = 1 (“game round”), she plays the game with D; if g = 0 (“generation round”) she merely gives the input string a = (a₁, . . . , aₙ) to D. At the end of the protocol, the total number of wins during game rounds is computed, and if it is above a certain threshold (“acceptance threshold”), the user accepts the results and applies a randomness extractor to the outputs of D to produce the final outputs of the protocol. (See figure 3 in section 7 for a more formal description.)

In the current work, we ask the following:
What is the minimum requirement for a device to guarantee quantum security in an untrusted-device randomness expansion protocol?

Our goal is to identify the essential features that guarantee full security. This leads to several more specific questions.

What is the broadest class of devices that can be used securely? The analysis in [16] allows $G$ to be any binary XOR game that has the strong self-testing property [15]. There are plenty of binary games that are not strong self-tests, and far more non-local games that are not binary. Furthermore, non-local games, which are based on Bell inequalities, are a proper subset of contextuality games, which are based on Kochen-Specker inequalities. There have been proposals and experiments for randomness expansion using contextuality without non-local games [10, 1, 20, 6]. No full quantum-security proof for those contextuality-based protocols is known.

Is quantum entanglement necessary? All protocols proved to be quantum-secure require at least a linear (in the output length) amount of entanglement [21, 16]. Yet the optimal quantum strategy for a contextuality game does not necessarily require entanglement. For example, the KCBS inequality [12] can be maximumly violated by an unentangled qutrit [12, 14]. Thus understanding what contextual (but not non-local) game can be used securely will shed light on the role of entanglement.

What is the largest amount of noise tolerable? Here “noise” refers to device deficiency, i.e., the gap between the device’s probability of winning the game and the optimal probability of winning the game. The answer to this question is important for the implementation. The analysis in [16] requires that the noise be a sufficiently small constant. For example, for the well-known CHSH game, the level of noise with quantum-security guarantee implied by [16] is $\sim 1.5\%$, which is still challenging for experimental implementation and is far smaller than the full classical-quantum gap, which is $\cos^2 \frac{\pi}{8} - \frac{3}{4} \approx 10.3\%$.

Are there protocols that are classically secure but not quantum-secure? If only classical security (i.e., security against an adversary who does not have quantum memory) is required, then the noise tolerance and class of games are already well understood [5]. This raises the question of whether there could be protocols that are classically secure but not quantum secure. Indeed, there are classical-quantum states $(A, E)$ such that $A$ and $E$ are highly correlated, but to a “classical” adversary (i.e., one who is forced to make a measurement on $E$ before using it to eavesdrop on $A$) the two systems appear almost independent (see, e.g., [7]). Could such systems occur as outputs in randomness expansion?

2 Our contributions

The result of this paper answers each of the questions above. We use the notion of a contextuality game, which is a generalization of nonlocal games broad enough to encompass all Kochen-Specker inequalities. For any contextuality game $G$, and chosen input $a$, denote by $w_G^*$ the optimal quantum winning probability. Let $w^*_a$ denote the optimal winning probability among all quantum strategies that produce deterministic output on input $a$. Refer to $\delta^*_a := w^*_G - w^*_a$ as the quantum-deterministic gap of $G$ on $a$. We define Protocol K, an analogue of Protocol R for contextuality games (see figure 2). We prove the following (see Theorem 6.4).

**Theorem 2.1** (Main Theorem; Informal). Let $(G, a)$ be a contextuality game with selected input. Let $u$ (the acceptance threshold) be a real number between $w^*_a$ and $w^*_G$. Then, when Protocol K is executed for $N$ rounds (with $G, a, u$ as parameters), it produces at least $f(u)N$ quantum-proof extractable bits, where

$$f(u) = 2(\log e)(u - w^*_a)^2.$$  (2.1)
The same result also holds for Protocol R and nonlocal games (see Theorem 7.1).

The crucial aspect of this theorem is that the function $f$ is nonzero over the whole interval $(w^a_G, w^*_G)$. Therefore quantum security is achieved whenever the acceptance threshold $u$ lies in this interval. Of course, any acceptance threshold less than $w^a_G$ cannot guarantee security, since the device could give deterministic outputs during all generation rounds. So the range of security thresholds $(w^a_G, w^*_G)$ cannot be made larger. One can show that any super-classical device for a game $G'$ can be used for playing a restricted game $G$ with a positive quantum-deterministic gap on some input. Thus being super-classical is the minimum device requirement.

Answers to the other questions also follow. The largest alloweable noise tolerance is the quantum-deterministic gap $\delta_G$, and the class of contextuality games that are usable are precisely those for which $\delta_G > 0$. Classical security is equivalent to quantum security for spot-checking protocols. (The number of quantum-proof extractable bits is at least linearly related to the number of classically-proof bits.) Entanglement is not necessary for randomness expansion, provided that contextuality can be used as a basis for security.

Comparison with Miller-Shi [16]. Our security analysis in this work falls into the paradigm of our earlier work [16]. In this work, we have introduced new ingredients that allow us to obtain generalizations of the results in [16]. The main improvements in the current work are (1) that we work with arbitrary nonlocal or contextual games (whereas [16] was restricted to binary XOR games) and (2) that we enlarge (maximally) the amount of noise permitted in Protocol R.

We note that in the context of binary XOR games, Theorem 2.1 is complementary (neither stronger nor weaker) to Corollary I.3 from [16]. The rate curve (2.1) in Theorem 2.1 is nonzero over a larger interval, but the rate curve in Corollary I.3 approaches a rate of 1 as the acceptance threshold approaches $w^*_G$ (which is not true of (2.1)).

The strong robust self-testing property for a nonlocal game $G$ asserts that any near-optimal strategy for $G$ must be close to a certain unique optimal strategy. This property was used crucially in the proofs of [16]. One interesting consequence of the current paper is that this property is not necessary: games that do not satisfy self-testing can still be used for randomness expansion.

Outline and proof techniques. We summarize the new ingredients in this paper. The main technical contribution of this paper is a new universal uncertainty principle for the Schatten norm $\|\cdot\|_{1+\epsilon}$. Once introduced into the framework of [16] (in place of the old uncertainty principle, Theorem E.2), the new principle implies the strong security claims above.

Let $H$ be a quantum system in state $\tau$, and let $\{\tau_0, \tau_1\}$ and $\{\tau_+, \tau_-\}$ be states of $H$ arising from anticommuting measurements on $H$. Suppose for simplicity that $\|\tau\|_{1+\epsilon} = 1$. Then, we prove the following.

$$\|\begin{bmatrix} \tau_0 & \tau_1 \\ \tau_1^* & \tau_0 \end{bmatrix}\|_{1+\epsilon} \leq 1 - \frac{\epsilon}{2} (1 - 2 \|\tau_-\|_{1+\epsilon})^2 + O(\epsilon^2).$$  (2.2)

(See the proof of Theorem 4.2) The critical aspect of this inequality is that the function on the right hand side (which determines the rate curve (2.1) is bounded below 1 as long as $\|\tau_-\|_{1+\epsilon}$ is bounded away from 1/2. The basis for this assertion is the uniform convexity of the Schatten norm [2]. Specifically, we exploit the uniform convexity of the function

$$g(t) = \|\begin{bmatrix} \tau_0 & tX \\ tX^* & \tau_1 \end{bmatrix}\|_{1+\epsilon}$$  (2.3)
where $\tau = \begin{bmatrix} \tau_0 & X \\ X^* & \tau_1 \end{bmatrix}$, and use the fact that $\|X\|_{1+\epsilon}$ is an approximate upper bound for the quantity $1 - 2\|\tau_--\|_{1+\epsilon}$.

Having proved (2.2), the next step is to generalize the class of measurements that can be used. In [16] we focused just on measurements that are partially trusted (i.e., partially anti-commuting), but this too can be extended. A quantity that is used in other uncertainty principles (e.g. [13]) to measure the non-commutativity of a pair of POVMs $\{A_0, A_1\}, \{A_2, A_3\}$ is the following:

$$d = \max_{i \in \{0,1\}} \max_{j \in \{2,3\}} \left\| \sqrt{A_i} \sqrt{A_j} \right\|^2$$

(2.4)

The use of this term is the crucial step for closing the quantum-classical gap. We prove a version of (2.2) which incorporates $d$ (Theorem 4.4).

We state a new protocol (Protocol U) which phrases randomness expansion with minimal assumptions: we need only a device $D$ which has one of two measurement settings at each round ($\{A_0^{(n)}, A_1^{(n)}\}$ or $\{A_2^{(n)}, A_3^{(n)}\}$) such that the commutativity parameters (2.4) have a uniform upper bound. The uncertainty principle (2.2) implies security for Protocol U, which specializes to provide the proof of security for Protocol K. (See Theorem 6.4)

Our proof (like that of [16]) suggests a deep relationship between quantum security and the geometry of the Schatten norm. This is an avenue that would be good for further exploration.

3 Preliminaries

For any Hermitian operator $X$ on a finite-dimensional Hilbert space $V$, let us say that an enlargement of $X$ is an embedding $i: V \to V'$ of $V$ into a larger finite-dimensional Hilbert space together with a Hermitian operator $X'$ on $V'$ satisfying $X = i^* X' i$.

The acronym POVM stands for positive operator-valued measure, and denotes a collection of positive semidefinite operators $M_i$ on a Hilbert space $V$ satisfying $\sum_i M_i = 1$.

We use the symbols 0, 1, +, – (in appropriate context) to denote respectively the vectors

$$(1, 0), (0, 1), \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) \in \mathbb{C}^2.$$  

The key mathematical concept in our proofs of randomness is the $(1 + \epsilon)$-Schatten norm. For any linear operator $Z$ and any $\epsilon \in (0, 1]$, the $(1 + \epsilon)$-Schatten norm is given by

$$\|Z\|_{1+\epsilon} = \text{Tr}( (Z^* Z)^{\frac{1+\epsilon}{2}} )^{\frac{1}{1+\epsilon}}.$$  

(3.1)

If $Z$ is positive semidefinite, then $\|Z\|_{1+\epsilon} = \text{Tr}(Z^{1+\epsilon})^{\frac{1}{1+\epsilon}}$. Note that if $Z$ is positive semidefinite and $\|Z\|_{1+\epsilon} \leq 1$, then

$$\|Z\|_{1+\epsilon} - O(\epsilon) \leq \text{Tr}(Z^{1+\epsilon})^{\frac{1}{1+\epsilon}} \leq \|Z\|_{1+\epsilon}.$$  

(3.2)

This norm has properties closely related to those of the 1-norm. Although the $(1 + \epsilon)$-norm does not approximate the 1-norm in the strictest sense (since $\|X\|_{1+\epsilon} / \|X\|_1$ can be arbitrarily small) it has many of the same properties modulo terms that vanish as $\epsilon \to 0$. If $X$ and $Y$ are positive semidefinite operators satisfying $\|X\|_{1+\epsilon}, \|Y\|_{1+\epsilon} \leq 1$, then

$$\|X\|_{1+\epsilon} + \|Y\|_{1+\epsilon} - O(\epsilon) \leq \|X + Y\|_{1+\epsilon} \leq \|X\|_{1+\epsilon} + \|Y\|_{1+\epsilon},$$  

(3.3)

$$\text{Tr}(X^{1+\epsilon}) + \text{Tr}(Y^{1+\epsilon}) \leq \text{Tr}[(X + Y)^{1+\epsilon}] \leq \text{Tr}(X^{1+\epsilon}) + \text{Tr}(Y^{1+\epsilon}) + O(\epsilon).$$  

(3.4)
Additionally, if $Z$ is an operator on $\mathbb{C}^2 \otimes \mathbb{C}^m$, satisfying $\|Z\|_{1+\epsilon} \leq 1$, then we have the following (see Proposition 1 in [19]).

$$\|\text{Tr}_{\mathbb{C}^2}(Z)\|_{1+\epsilon} \leq 1 + O(\epsilon).$$

(3.5)

**Device models.**

**Definition 1.** A binary device $D$ with commutativity parameter $\ell$ (with $\ell \in [\frac{1}{2}, 1]$) consists of the following:

1. A quantum system $Q$ and a density operator $\Phi: Q \to Q$.
2. For every positive integer $n$, a pair of binary POVMs on $Q$, $\{A_0^{(n)}, A_1^{(n)}\}$ and $\{A_2^{(n)}, A_3^{(n)}\}$, satisfying

$$\max_{j \in \{0,1\}} \left\| \sqrt{A_i^{(n)}} \sqrt{A_j^{(n)}} \right\|^2 \leq \ell, \text{ for all } n.$$ 

The device $D$ begins with the quantum system $Q$ in state $\Phi$. At the $n$th use of the device, it accepts a single bit $x_n$ as input. If $x_n = 0$, then $D$ applies the nondestructive measurement $Q \to Q \otimes \mathbb{C}^2$ given by

$$X \mapsto \sqrt{A_0^{(1)}}X\sqrt{A_0^{(1)}} \otimes |0\rangle \langle 0| + \sqrt{A_1^{(1)}}X\sqrt{A_1^{(1)}} \otimes |1\rangle \langle 1|$$

(3.6)

and outputs the resulting bit $y_n$. If $x_n = 1$, then $D$ applies the same measurement with $A_0$ replaced by $A_2$ and $A_1$ replaced by $A_3$.

Note that the measurements $A_i^{(n)}$ could be such that they encode the bits $x_n$ and $y_n$ into the state of $Q$; thus this device model allows memory.

**Definition 2.** A contextual measurement device $D$ with $m$ settings consists of the following.

1. A quantum system $Q$ and a density operator $\Phi: Q \to Q$.
2. A set of contexts $S \subseteq \text{Pow}\{1, 2, \ldots, m\}$ which contains all the singleton sets $\{k\}$, $k \in \{1, 2, \ldots, m\}$, and is closed under taking subsets: $B \subseteq A \in S \Rightarrow B \in S$.
3. For every positive integer $n$, a sequence of binary POVMs

$$\{A_1^{(n)}, I - A_1^{(n)}\}, \{A_2^{(n)}, I - A_2^{(n)}\}, \ldots, \{A_m^{(n)}, I - A_m^{(n)}\}$$

(3.7)

satisfying the condition that for any nonempty $T \in S$, the operators $\{A_i^{(n)} | i \in T\}$ are simultaneously diagonalizable.

On the $n$th round, the contextual measurement device accepts a context $T = \{a_1, \ldots, a_k\} \in S$ as input, performs the nondestructive measurements

$$X \mapsto \sqrt{A_{a_i}^{(n)}}X\sqrt{A_{a_i}^{(n)}} \otimes |0\rangle \langle 0| + \sqrt{A_{a_i}^{(n)}}X\sqrt{A_{a_i}^{(n)}} \otimes |1\rangle \langle 1|$$

(3.8)

for each $i \in T$ and outputs the results as a $k$-tuple of bits $(b_1, \ldots, b_k)$.

We will also use the notion of a multi-part quantum device, with a definition similar to that of Definition 4 in [16].

**Definition 3.** A quantum device $D$ with $r$ components and alphabet size $b$ consists of the following.
1. Quantum systems $Q_1, \ldots, Q_r$ whose initial state is given by a density operator
   \[
   \Phi: (Q_1 \otimes \ldots \otimes Q_r) \rightarrow (Q_1 \otimes \ldots \otimes Q_r). \tag{3.9}
   \]

2. For any $n > 0, i \in \{1, 2, \ldots, r\}$, and $j \in \{1, 2, \ldots, b\}$, a POVM
   \[
   \{ M^{(n)}_{ijk} \}_{k \in \{1, 2, \ldots, b\}} \tag{3.10}
   \]
on $Q_i$.

3. A sequence of unitary operators $(U_1, U_2, U_3, \ldots)$ on $Q_1 \otimes \ldots \otimes Q_r$.

   Such a device acts as follows (on the $n$th iteration). First, inputs (from the set $\{1, 2, \ldots, b\}$)
   are received and the components perform corresponding measurements: if the $i$th component
   receives input $j$, then it performs the POVM $\{ M^{(n)}_{ijk} \}$ on $Q_i$ and outputs the result $k \in \{1, 2, \ldots, b\}$.
   Then, the unitary operation $U_i$ is applied to the entire system $Q_1 \otimes \ldots \otimes Q_r$. (This model permits
   communication between the components in between rounds, including sharing previous inputs
   and outputs.) The process then repeats using the operators $\{ M^{(n+1)}_{ijk} \}$ and $U_{n+1}.$

**Simulation.** We use the term “simulation” in the same sense as in Section B.3 of [16]. Specifically,
a procedure $P_1$ “simulates” another procedure $P_2$ if, for any purifying system $E_1$ for the devices
used in $P_1$ and any purifying system $E_2$ for the devices used in $P_2$, the joint state $O_1E_1$ of the
outputs of $P_1$ together with $E_1$ is isomorphic to the state of $O_2E_2$.

# Uncertainty Principles

**Anti-commuting measurements.** The starting point for the results in this section is the next
proposition, which follows from Theorem 1 in [2].

**Proposition 4.1.** For any $\varepsilon \in (0, 1)$, and any linear operators $W$ and $Z$ such that $\| W \|_{1+\varepsilon} = \| Z \|_{1+\varepsilon} = 1,$
\[
\left\| \frac{W + Z}{2} \right\|_{1+\varepsilon} \leq 1 - \frac{\varepsilon}{8} \left\| W - Z \right\|_{1+\varepsilon}^2. \tag{4.1}
\]

**Theorem 4.2.** Let $H$ be a finite-dimensional Hilbert space, and let $\varepsilon \in (0, 1]$. Let $R: \mathbb{C}^2 \otimes H \rightarrow \mathbb{C}^2 \otimes H$
be a positive semidefinite operator such that the operator $\rho := \text{Tr}_{\mathbb{C}^2}(R)$ has $\| \rho \|_{1+\varepsilon} = 1.$ For any $v \in \mathbb{C}^2,$
let $\rho_v = \text{Tr}_{\mathbb{C}^2}[(vv^* \otimes I)R].$ Then,
\[
\left\| \begin{bmatrix} \rho_0 & 0 \\ 0 & \rho_1 \end{bmatrix} \right\|_{1+\varepsilon} \leq 1 - \frac{\varepsilon}{2} (1 - 2 \| \rho_- \|_{1+\varepsilon}^2) + O(\varepsilon^2). \tag{4.2}
\]

**Proof.** We will first dualize the statement of the result. Purify $R$ by taking an additional Hilbert
space $K$ and a vector $r \in \mathbb{C}^2 \otimes H \otimes K$ such that $R = \text{Tr}_K(rr^*).$ Let $\tau = \text{Tr}_H(rr^*)$ and define
$\tau_v = \text{Tr}_{\mathbb{C}^2}[(vv^* \otimes I_K)\tau]$ for any $v \in \mathbb{C}^2$. The operators $\tau, \tau_0, \tau_1, \tau_-$ have the same eigenvalues
respectively as $\rho, \rho_0, \rho_1, \rho_-$. In particular, $\| \tau \|_{1+\varepsilon} = 1$. To prove (4.2), we need only to show that
the same relation holds for the operators $\tau_v$.

Write $\tau$ as
\[
\tau = \begin{bmatrix} \tau_0 & X \\ X^* & \tau_1 \end{bmatrix}. \tag{4.3}
\]
Applying Proposition 4.1 to the operators \( W = \tau \) and \( Z = (\sigma_z \otimes I)\tau(\sigma_z \otimes I) \), we have the following:

\[
\left\| \begin{bmatrix} \tau_0 & 0 \\ X & \tau_1 \end{bmatrix} \right\|_{1+\epsilon} = \frac{1}{2} \left\| \begin{bmatrix} \tau_0 & X \\ X^* & \tau_1 \end{bmatrix} + \begin{bmatrix} -\tau_0 & -X \\ -X^* & \tau_1 \end{bmatrix} \right\|_{1+\epsilon}
\]

\[
\leq 1 - \frac{\epsilon}{8} \left\| \begin{bmatrix} \tau_0 & X \\ X^* & \tau_1 \end{bmatrix} - \begin{bmatrix} \tau_0 & -X \\ -X^* & \tau_1 \end{bmatrix} \right\|^2_{1+\epsilon}
\]

\[
= 1 - \frac{\epsilon}{8} \left\| \begin{bmatrix} 0 & 2X \\ 2X^* & 0 \end{bmatrix} \right\|^2_{1+\epsilon}
\]

\[
\leq 1 - \frac{\epsilon}{8} \|2X + 2X^*\|_{1+\epsilon}^2 + O(\epsilon^2)
\]

\[
= 1 - \frac{\epsilon}{8} \|2(\tau_+ - \tau_-)\|_{1+\epsilon}^2 + O(\epsilon^2)
\]

\[
= 1 - \frac{\epsilon}{2} \|\tau - 2\tau_-\|_{1+\epsilon}^2 + O(\epsilon^2)
\]

\[
\leq 1 - \frac{\epsilon}{2} \left(1 - 2 \|\tau_-\|_{1+\epsilon}^2\right)^2 + O(\epsilon^2),
\]

as desired.

The generalizations of Theorem 4.2 that follow use techniques from known uncertainty principles (summarized in [22]; see especially [13]). We borrow a term (see the left side of equation (4.11) below) that measures the noncommutativity of two POVMs.

**Theorem 4.3.** Let \( \{A_0, A_1\} \) and \( \{A_2, A_3\} \) be two binary POVMs on a finite-dimensional Hilbert space \( V \), satisfying

\[
\max_{i \in \{0, 1\}} \|\sqrt{A_i} \sqrt{A_j}\|^2 = \frac{1}{2}.
\]

Let \( R: V \otimes H \to V \otimes H \) be a positive semidefinite operator such that the operator \( \rho := \text{Tr}_V(R) \) has \( \|\rho\|_{1+\epsilon} = 1 \). Let \( \rho_i = \text{Tr}_V((A_i \otimes I_H)R) \). Then,

\[
\left\| \begin{bmatrix} \rho_0 & 0 \\ \rho_1 \end{bmatrix} \right\|_{1+\epsilon} \leq 1 - \frac{\epsilon}{2} \left(1 - 2 \|\rho_3\|_{1+\epsilon}^2\right)^2 + O(\epsilon^2).
\]

**Proof of Theorem 4.3** To begin with, we can consider a Hilbert space of the form \( V \oplus V_1 \), with \( \dim V_1 = \dim V \), and choose a projective enlargement \( \{\overline{A}_0, \overline{A}_1\} \) on \( V \oplus V_1 \). Then,

\[
\left\| \sqrt{\overline{A}_0} \sqrt{A}_2 \right\|^2 = \left\| \sqrt{A_2} \sqrt{A}_0 \right\|^2
\]

\[
= \left\| \sqrt{A_2} \sqrt{A}_0 \right\|^2
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= \left\| \sqrt{A_2} \sqrt{A}_0 \right\|^2
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= \left\| \sqrt{A_0} \sqrt{A}_2 \right\|^2.
\]

Similar simplifications show that the quantity on the left side of (4.11) remains the same when \( \{A_0, A_1\} \) is replaced by \( \{\overline{A}_0, \overline{A}_1\} \). Therefore, we will simply assume at the outset that \( \{A_0, A_1\} \) is projective.
Taking an appropriate choice of basis, we may assume that $A_0 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ and $A_1 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$.

Note that both $A_0 A_2 A_0$ and $A_0 (I - A_2) A_0$ have operator norm $\leq \frac{1}{2}$, which is possible if and only if $A_0 A_2 A_0 = A_0 / 2$. Generalizing this reasoning, we can put $A_2$ and $A_3$ in the form

$$A_2 = \begin{bmatrix} I/2 & Y \\ Y^* & I/2 \end{bmatrix}, \quad A_3 = I - A_2.$$ \hspace{1cm} (4.17)

Note that we must have $\|Y\| \leq \frac{1}{2}$. The following projective measurement is an enlargement of \{A_2, A_3\}.

$$\mathcal{A}_2 = \begin{bmatrix} I/2 & Y/2 & -\sqrt{I/4 - Y^*Y} & Y^* \\ -\sqrt{I/4 - Y^*Y} & I/2 & Y^* & \sqrt{I/4 - Y^*Y} \\ Y & \sqrt{I/4 - Y^*Y} & I/2 & I/2 \end{bmatrix}, \quad \mathcal{A}_3 = I - A_2.$$

This measurement is equivalent under further basis change to

$$\{ \begin{bmatrix} I/2 & I/2 \\ I/2 & I/2 \end{bmatrix}, \begin{bmatrix} I/2 & -I/2 \\ -I/2 & I/2 \end{bmatrix} \}$$

which reduces the proof to the case of Theorem 4.2.

**General measurements.**

**Theorem 4.4.** Let $\epsilon \in (0, 1]$. Let \{A_0, A_1\} and \{A_2, A_3\} be two binary POVMs on a finite-dimensional Hilbert space $V$, and let

$$c = 2 \left( \max_{i \in \{0, 1\}} \|\sqrt{A_i} \sqrt{A_j}\|^2 \right) - 1. \hspace{1cm} (4.19)$$

Let $R, \rho, \rho_i$ be as in Theorem 4.3 (with $\|\rho\|_{1+\epsilon} = 1$). Then,

$$\left\| \begin{bmatrix} \rho_0 \\ \rho_1 \end{bmatrix} \right\|_{1+\epsilon} \leq 1 - \frac{c}{2} \left( \max \{ (1 - 2 \|\rho_3\|_{1+\epsilon}) - c, 0 \} \right)^2 + O(\epsilon^2). \hspace{1cm} (4.20)$$

**Proof of Theorem 4.4.** As in the proof of Theorem 4.3 we may assume that $A_0 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$. Let $M = A_2 - A_3$, and write this operator as

$$M = \begin{bmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{bmatrix}.$$

Note that $\|M_{00}\|, \|M_{11}\| \leq c$ (from the definition of $c$) and $\|M_{01}\|, \|M_{10}\| \leq 1$. We therefore have
the following.

\[
1 - 2 \| \rho_3 \|_{1+\epsilon} = \| \rho - 2\rho_3 \|_{1+\epsilon} + O(\epsilon) \tag{4.22}
\]

\[
= \| \text{Tr}_V (R(\rho_1 \otimes \mathbb{I}_V)) \|_{1+\epsilon} + O(\epsilon) \tag{4.23}
\]

\[
\leq \left\| \text{Tr}_V \left( R \left( \begin{bmatrix} M_{00} & 0 \\ 0 & M_{11} \end{bmatrix} \otimes \mathbb{I}_V \right) \right) \right\|_{1+\epsilon} + O(\epsilon) \tag{4.24}
\]

\[
+ \left\| \text{Tr}_V \left( R \left( \begin{bmatrix} 0 & M_{01} \\ M_{10} & 0 \end{bmatrix} \otimes \mathbb{I}_V \right) \right) \right\|_{1+\epsilon} + O(\epsilon) \tag{4.25}
\]

\[
\leq c + \left\| \text{Tr}_V \left( R \left( \begin{bmatrix} 0 & M_{01} \\ M_{10} & 0 \end{bmatrix} \otimes \mathbb{I}_V \right) \right) \right\|_{1+\epsilon} + O(\epsilon). \tag{4.26}
\]

Let \( T \) be the second summand in (4.26). Applying Theorem 4.3 to the measurements \( \{A_0, A_1\} \) and \( \left\{ \begin{bmatrix} \mathbb{I}/2 & M_{01}/2 \\ M_{10}/2 & \mathbb{I}/2 \end{bmatrix}, \begin{bmatrix} \mathbb{I}/2 & -M_{01}/2 \\ -M_{10}/2 & \mathbb{I}/2 \end{bmatrix} \right\} \) yields

\[
\left\| \begin{bmatrix} \rho_0 \\ \rho_1 \end{bmatrix} \right\|_{1+\epsilon} \leq 1 - \frac{\epsilon}{2} \tau^2 + O(\epsilon^2). \tag{4.28}
\]

Combining this bound with (4.26) yields the desired result.

\[ \square \]

## 5 A Universal Protocol

The central object of this section is Protocol U (see Figure 1) which is an abstraction of the spot-checking protocol (developed in [5] and [21]).

In order to use Theorem 4.4, we will first restate it in the following alternate form which is more compatible with [16]. (Compare with Theorem E.2 in [16].)

**Corollary 5.1.** For any \( \ell \in \left[ \frac{1}{2}, 1 \right] \), there exists a continuous function \( \Pi_\ell : (0, 1] \times [0, 1] \) such that the following holds.

1. Let \( D \) be a binary device with commutativity parameter \( \ell \), let \( E \) be a purifying system for \( D \), and let \( \rho_0, \rho_1, \rho_2, \rho_3 \) denote the (subnormalized) states of \( E \) corresponding to the input-output combinations 00, 01, 10, 11 for \( D \). Let

\[
\delta = \text{Tr}(\rho_3^{1+\epsilon})/\text{Tr}(\rho_1^{1+\epsilon}). \tag{5.2}
\]

Then,

\[
\frac{\text{Tr}(\rho_0^{1+\epsilon} + \rho_1^{1+\epsilon})}{\text{Tr}(\rho_1^{1+\epsilon})} \leq \left( \frac{1}{2} \right)^{\epsilon \Pi_\ell(\ell, \delta)}. \tag{5.3}
\]

2. The function \( \pi_\ell(y) = \lim_{x \to y} \Pi_\ell(x, y) \) satisfies

\[
\pi_\ell(y) = \begin{cases} 2(\log e)(1 - \ell - y)^2 & \text{if } y \in [0, 1 - \ell] \\ 0 & \text{if } y \in (1 - \ell, \ell) \\ 2(\log e)(y - \ell)^2 & \text{if } y \in [\ell, 1]. \end{cases} \tag{5.4}
\]
Arguments:
- \( N \): A positive integer (the output length).
- \( \ell \): A real number from the interval \([\frac{1}{2}, 1]\). (The commutativity parameter.)
- \( \eta \): A real number from the interval \((0, \frac{1}{2})\). (The error tolerance.)
- \( q \): A real number from the interval \((0, 1]\). (The test probability.)
- \( D \): A binary device with commutativity parameter \( \ell \).

Protocol \( U \):
1. Bits \( g_1, g_2, \ldots, g_N \) are chosen at independently at random according to a \((1 - q, q)\) distribution, and given (in order) to device \( D \) to produce output bits \( o_1, o_2, \ldots, o_N \).
2. If
   \[
   \sum_{k=1}^{N} g_i o_i > \eta q N, \tag{5.1}
   \]
   then the protocol aborts. Otherwise, it succeeds, and the output is the bit sequence \( o_1, \ldots, o_N \).

Figure 1: Abstract Protocol for Randomness Expansion

The following theorem now holds by repeating the reasoning from Sections H and I in [16]. Let \( G = (g_1, \ldots, g_N) \) and \( O = (o_1, \ldots, o_N) \) denote the input and output registers for Protocol \( U \). If \( E \) is a purifying system for the device \( D \), let \( \Gamma_{EGO} \) denote the final state of \( E, G, \) and \( O \), and let \( \Gamma^s_{EGO} \) denote the subnormalized state corresponding to the “success” event.

Theorem 5.2. Let \( \ell \in [\frac{1}{2}, 1], \eta \in (0, \frac{1}{2}) \), and \( \delta > 0 \) be real numbers. Then, there exist positive reals \( b \) and \( q_0 \) such that the following holds. If Protocol \( U \) is executed with the parameters \( N, \ell, \eta, q, D \), with \( q \leq q_0 \), and \( E \) denotes a purifying system for \( D \), then
\[
H^\varepsilon_{\min} (\Gamma^s_{EGO} | EG)) \geq N \cdot (\pi_\ell(\eta) - \delta), \tag{5.5}
\]
where \( \varepsilon = \sqrt{2} \cdot 2^{-bq N} \) and \( \pi_\ell \) denotes the function from Corollary [5.7].

6 Kochen-Specker Inequalities

Randomness expansion from Kochen-Specker Inequalities has previously been explored in [10, 11, 20, 6]. In this section we give a full proof of security for such expansion. We begin with a formalism which is similar to that of other papers on Kochen-Specker inequalities [12, 11, 11].

Definition 4. A contextuality game \( G \) with \( m \) measurement settings is a is a multilinear polynomial \( f(X_1, \ldots, X_m) = \sum_{T \subseteq \{1,2,\ldots,m\}} f_T \prod_{i \in T} X_i \) satisfying \( \sum_{T} |f_T| = 1 \).

Such a polynomial encodes rules for a game as follows. Let \( D \) be a contextual measurement device whose set of contexts contains \( \text{Supp } f \) (that is, contains every element \( T \subseteq \{1,2,\ldots,m\} \) for which \( f_T \neq 0 \)). To play the game, choose a subset \( T \) at random under the probability distribution \( \{|f_T|\} \), and give \( T \) as input to the device \( D \). Let \( (b_1, \ldots, b_k) \) be the output bits. The score of the
game is then given by

$$f_T \frac{f_T}{|f_T|} (−1)^{b_1 \oplus b_2 \oplus ... \oplus b_k}. \quad (6.1)$$

**Remark 1.** In Definition 4 we have restricted the scoring functions to be XOR functions and also made assumptions on the probability distribution used to choose the inputs. A more general definition would allow for an arbitrary probability distribution \(\{p_T : T \in \{1, \ldots, m\}\}\), and allow the score for each context \(T\) to be given by arbitrary functions \(\{g^T: \{0,1\}^T \rightarrow \mathbb{R}\}\) which assigns to each possible outcome a real number. But in fact such a scoring rule can be rewritten in the form of Definition 4 (modulo linear scaling). Let

\[
\mathcal{F}(X_1, \ldots, X_n) = \sum_T p_T \sum_{b_1, \ldots, b_T \in \{0,1\}} g^T(b_1, \ldots, b_T) \left(\frac{1 + (-1)^{b_1} X_1}{2}\right) \cdots \left(\frac{1 + (-1)^{b_T} X_T}{2}\right).
\]

Let \(c\) be the sum of the absolute values of the coefficients of the polynomial \(\mathcal{F}\), and let \(f = \mathcal{F}/c\). Then, for any compatible contextual measurement device \(D\), the expected score awarded to \(D\) by \((\{p_T\}, \{g^T\})\) is \(c\) times the expected score awarded by \(f\).

A protocol for randomness expansion from contextuality games is given in Figure 2.

**Arguments:**
- \(N\): A positive integer (the **output length**).
- \(m\): A positive integer (the **number of measurement settings**).
- \(\eta\): A real number from the interval \((0, \frac{1}{2})\). (The **error tolerance**.)
- \(q\): A real number from the interval \((0, 1)\). (The **test probability**.)
- \(f\): A contextuality game.
- \(D\): A contextual measurement device \(D\) whose set of contexts contains \(\text{Supp} f\).

**Protocol K:**
1. Choose a bit \(g \in \{0,1\}\) according to the distribution \((1-q, q)\).
2. If \(g = 0\) ("generation round"), then give the context \(\{1\}\) to \(D\) and record the output bit \(o\).
3. If \(g = 1\) ("test round"), then play game \(f\) with \(D\). If the score is \(+1\), then let \(o = 0\). If the score is \(-1\), then let \(o = 1\).
4. Repeat to obtain a sequence of bits \(g_1, \ldots, g_N\) and \(o_1, \ldots, o_N\).
5. If

\[
\sum_{k=1}^{N} g_k o_k < \eta q N, \quad (6.2)
\]

then the protocol **aborts**. Otherwise, it **succeeds**, and the output is the bit sequence \(o_1, \ldots, o_N\).

**Figure 2:** Randomness Expansion from Kochen-Specker Inequalities (Protocol K)

One convenient aspect of this polynomial formulation is that it is easy to express the supremum of possible expected scores for the game that can be achieved by a contextual measurement.
device. If the measurements used by \( D \) (which we may assume to be projective) are \( \{P_i, \mathbb{I} - P_i\}_{i \in \{1, \ldots, m\}} \), and the initial state of \( D \) is \( \Phi \), then the expected score is

\[
\text{Tr}(f(Y_1, \ldots, Y_n)\Phi)
\]

where \( Y_i = 2P_i - \mathbb{I} \). Therefore the optimal upper bound on possible expected scores is

\[
q_f := \sup_{n \in \mathbb{N}, Y_i : C^n \to C^n, Y_i^2 = \mathbb{I}, \{Y_i|i \in T\} \text{ commute } \forall T \in \text{Supp } f} \left\| f(Y_1, \ldots, Y_m) \right\|.
\]

Another important quantity is the largest possible score that can be achieved by a noncontextual, deterministic device. This quantity is given by

\[
\mathcal{C}_f := \max_{a_i \in \{-1, 1\}} |f(a_1, \ldots, a_m)|.
\]

If a device achieves a score above \( \mathcal{C}_f \), then some of its outputs must be random. For the purposes of randomness expansion it is more useful to have a guarantee that a particular output is random (an observation made in [1]). Therefore we will use the following quantity:

\[
q'_f := \sup_{n \in \mathbb{N}, Y_i : C^n \to C^n, Y_i^2 = \mathbb{I}, \{Y_i|i \in T\} \text{ commute } \forall T \in \text{Supp } f} \left\| \left( \frac{I + (-1)^dY_1}{2} \right) f(Y_1, \ldots, Y_m) \left( \frac{I + (-1)^dY_1}{2} \right) \right\|.
\]

This is the optimal upper bound on the expected score that can be achieved by a device \( D \) given that its output on input \( \{1\} \) is deterministic.

Also let

\[
w_G = (q_G + 1)/2 \quad \text{and} \quad w'_G = (q'_G + 1)/2.
\]

These are the corresponding bounds on “winning probabilities” (i.e., probabilities of obtaining a score of \( (+1) \)).

The KCBS game. An example of a contextuality game is the KCBS game from [12] (which was used in [6] for randomness expansion). We express this game as

\[
g(X_1, X_2, X_3, X_4, X_5) = -\frac{1}{5}X_1X_2 - \frac{1}{5}X_2X_3 - \frac{1}{5}X_3X_4 - \frac{1}{5}X_4X_5 - \frac{1}{5}X_5X_1.
\]

**Proposition 6.1.** Let \( g \) be given by (6.8). Then, \( q'_g = 0.6 \).

**Proof.** It is clear that a score of \( 3/5 \) can be achieved by a deterministic device (say, by a device which outputs 0, 1, 0, 1, 0 on inputs 1, 2, 3, 4, 5, respectively). Suppose, for the sake of contradiction, that there is a device \( D \) compatible with the KCBS game which outputs a score above \( 3/5 \), and which gives a deterministic output on input 1. Let \( B_1, B_2, B_3, B_4, B_5 \) be contextual random variables which represent the outputs of \( D \) on inputs 1, 2, 3, 4, 5, and let \( Z_i = 1 - 2B_i \). It is easy to see that

\[
\langle Z_iZ_j \rangle \geq -1 + |\langle Z_i \rangle + \langle Z_j \rangle|
\]
for any \(i,j \in \{1,2,3,4,5\}\) with \(j = (i + 1) \mod 5\). (Here we are using \(\langle \cdot \rangle\) to denote expectation.) Therefore,
\[
-3 > \langle Z_1 Z_2 \rangle + \langle Z_2 Z_3 \rangle + \langle Z_3 Z_4 \rangle + \langle Z_4 Z_5 \rangle + \langle Z_5 Z_1 \rangle \\
\geq -5 + |\langle Z_1 \rangle| - |\langle Z_2 \rangle| - |\langle Z_3 \rangle| - |\langle Z_4 \rangle| - |\langle Z_5 \rangle| |
\geq -5 + |\langle Z_1 \rangle| - |\langle Z_2 \rangle| - |\langle Z_3 \rangle| - |\langle Z_4 \rangle| - |\langle Z_5 \rangle| - |\langle Z_5 \rangle| - |\langle Z_3 \rangle|
\]

Using the inequality \(|a - b| + |b - c| \geq |a - c|\), this bound implies
\[
-3 > -5 + |\langle Z_1 \rangle| - |\langle Z_1 \rangle|.
\]

Given that \(\langle Z_1 \rangle = \pm 1\) by assumption, this is a contradiction. \(\blacksquare\)

As proved in [12], the value of \(q_g\) is at least 0.788. Therefore there is a gap between \(q_g\) and \(q'_g\), which, as we will see momentarily, enables randomness expansion.

**Universal security.** We are now prepared to prove security for Protocol K. All that is needed, in fact, is to write the quantity \(w'_f\) defined above (see (6.6)) in a form that is more compatible with Corollary 5.1.

**Lemma 6.2.** Let \(N\) be a Hermitian operator on \(\mathbb{C}^n\) satisfying \(\|N\| \leq 1\). Then,
\[
\|N\| = 2 \max \{\| (I + N)/2 \|, \| (I - N)/2 \| \} - 1.
\]

**Proof.** The quantities \(2 \| (I \pm N)/2 \| - 1\) compute the absolute values of the highest eigenvalue and the lowest eigenvalue of \(N\). The result is immediate. \(\blacksquare\)

As a consequence of the lemma and the identity \(\| PNP \| = \| P\sqrt{N} \|^2\), we have the following alternative formula for \(w'_f\):
\[
w'_f := \sup_{n \in \mathbb{N}: Y \in \mathcal{C}^n: Y_i^2 = I, \\{Y_i\}_{i \in T} \text{ commute} \forall T \in \text{Supp } f} \left\| \sqrt{\frac{I + (-1)^d Y_1}{2}} \sqrt{\frac{I + (-1)^e f(Y_1, \ldots, Y_m)}{2}} \right\|^2.
\]

**Proposition 6.3.** For any contextual XOR game \(f\) and any compatible contextual measurement device \(D\), there exists a binary device \(D'\) with commutativity parameter \(\leq w'_f\) such that Protocol K with parameters \(N, m, \eta, q, f, D\) is simulated by Protocol U with parameters \(n, \ell, \eta, q, D'\).

**Proof.** Let \(D'\) have the same initial state as \(D\), and let the measurements for \(D'\) be defined in terms of the measurements \(A_i^{(n)}\) used by \(D\) via the expressions
\[
\left\{ \frac{I + Y_1^{(n)}}{2}, \frac{I - Y_1^{(n)}}{2} \right\}, \left\{ \frac{I + f(Y_1^{(n)}, \ldots, Y_m^{(n)})}{2}, \frac{I - f(Y_1^{(n)}, \ldots, Y_m^{(n)})}{2} \right\}
\]
where \(Y_i^{(n)} = 2A_i^{(n)} - I\). Then the desired simulation holds, and by definition \(w'_f\) is an upper bound for the commutativity parameter of \(D'\). \(\blacksquare\)

We therefore have the following by Theorem 5.2.
Theorem 6.4. Let $f$ be a contextuality game with $m$ measurement settings, and let $\eta \in (0, \frac{1}{2})$, and $\delta > 0$ be real numbers. Then, there exist positive reals $b$ and $q_0$ such that the following holds. If Protocol $K$ is executed with the parameters $N, m, \eta, q, f, D$, with $q \leq q_0$, and $E$ denotes a purifying system for $D$, then

$$H_{\min}^c (\Gamma_{EGO}^e \mid EG) \geq N \cdot (\pi_{w_f'}(\eta) - \delta), \quad (6.14)$$

where $\epsilon = \sqrt{2} \cdot 2^{-bqN}$ and $\pi_{w_f'}$ denotes the function from Corollary 5.1.

Note that by definition, $\pi_{w_f'}(\eta)$ is nonzero for any $\eta > w_f'$.

7 Nonlocal Games

Finally, we note that the reasoning in section 6 carries over easily to the context of nonlocal games. Let $n, b$ be positive integers, and define a binary measurement simulation procedure with parameters $n, b$ to be a triple $(p, F, V)$, where $p: \{1, 2, \ldots, b\}^n \to \mathbb{R}$ is a probability distribution, $F: \{1, 2, \ldots, b\}^2 \to \{0, 1\}$, (7.1)

if a function and $V: \{1, 2, \ldots, b\}^n \to \{0, 1\}$ is another function. This first two elements $p, F$ prescribe a nonlocal game with $n$ players as follows: a tuple $x \in \{1, 2, \ldots, b\}^n$ is chosen randomly according to $p$ and the terms $x_1, \ldots, x_n$ are given as input respectively to the components $D_1, \ldots, D_n$ of an $n$-part quantum device. The outputs $y = (y_1, \ldots, y_n) \in \{1, 2, \ldots, b\}^n$ (which are also assumed to be in the alphabet $\{1, 2, \ldots, b\}$) are received and then the function $F$ is applied to $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ to obtain the outcome of the game. (If $F = 0$ the game is won, and if $F = 1$, the game is lost.)

In Figure 3, we have written a general version of Protocol R (randomness expansion for nonlocal games) which uses the concept of a binary measurement simulation procedure.

Let $w$ denote the supremum of the winning probabilities for quantum strategies for the game $(p, F)$. Let $w'$ denote the same supremum taken just over quantum strategies that give strictly deterministic outputs when the input $(1, 1, \ldots, 1)$ is given and the function $V$ is applied (as in a generation round in Protocol R). Repeating the reasoning from section 6 shows that Protocol R simulates Protocol U with commutativity parameter $\ell = w'$. Therefore we have the following.

Theorem 7.1. Let $\eta \in (0, \frac{1}{2})$, and $\delta > 0$ be real numbers. Then, there exist positive reals $c$ and $q_0$ such that the following holds. If Protocol $R$ is executed with parameters $N, \eta, q, X, D$, with $q \leq q_0$, and $E$ denotes a purifying system for $D$, then

$$H_{\min}^c (\Gamma_{EGO}^e \mid EG) \geq N \cdot (\pi_{w_f'}(\eta) - \delta), \quad (7.3)$$

where $\epsilon = \sqrt{2} \cdot 2^{-cqN}$ and $\pi_{w_f'}$ denotes the function from Corollary 5.1.

8 Acknowledgements

Many thanks to Dong-Ling Deng and Kihwan Kim for sharing with us their work on randomness expansion, and to Patrick Ion for introducing us to the literature on Kochen-Specker inequalities.
Arguments:

$N$ : A positive integer (the output length).
$\eta$ : A real number from the interval $(0, \frac{1}{2})$. (The error tolerance.)
$q$ : A real number from the interval $(0, 1)$. (The test probability.)
$X$ : A binary measurement simulation procedure $X = (p, F, V)$ with parameters $n, b$.
$D$ : An $n$-part quantum device (with input and output alphabets $\{1, 2, \ldots, b\}$).

Protocol R:

1. Choose a bit $g \in \{0, 1\}$ according to the distribution $(1 - q, q)$.
2. If $g = 1$ ("test round"), then play game $p, F$ with $D$, and let $o$ be the outcome of the game.
3. If $g = 0$ ("generation round"), then give the $n$ inputs $(1, 1, \ldots, 1)$ to $D$, let $(r_1, \ldots, r_n) \in \{1, 2, \ldots, b\}^n$ denote the resulting outputs, and let $o = V(r_1, \ldots, r_n)$.
4. Repeat to obtain a sequence of bits $g_1, \ldots, g_N$ and $o_1, \ldots, o_N$.
5. If
   \[ \sum_{k=1}^{N} g_k o_k > \eta q N, \]
   then the protocol aborts. Otherwise, it succeeds, and the output is the bit sequence $o_1, \ldots, o_N$.

Figure 3: Randomness Expansion for Nonlocal Games (Protocol R)

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