Decomposability of local determinantal representations of hypersurfaces

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Abstract. Let ℳ be a matrix whose entries are power series in several variables and determinant det(ℳ) does not vanish identically. The equation det(ℳ) = 0 defines a hypersurface singularity and the (co)-kernel of ℳ is a maximally Cohen-Macaulay module over the local ring of this singularity.

Suppose the determinant det(ℳ) is reducible, i.e. the hypersurface is locally reducible. A natural question is whether the matrix is equivalent to a block-diagonal or at least to an upper-block-triangular. (Or whether the corresponding module is decomposable or at least is an extension.) We give various necessary and sufficient criteria.

Two classes of such matrices of functions appear naturally in the study of decomposability: those with many generators (e.g. maximally generated or Ulrich maximal) and those that descend from birational modifications of the hypersurface by pushforwards (i.e. correspond to modules over bigger rings). Their properties are studied.

Contents

1. Introduction 1
2. Preliminaries and background 7
3. Decomposability of maximally generated determinantal representations 16
4. Saturated determinantal representations 25
5. Some applications 27
References 28

1. Introduction

Let k be an algebraically closed, normed, complete field of characteristic zero, e.g. the complex numbers. Let kn be the affine space of dimension n over k, let (kn, 0) be a small neighborhood of the origin. Denote the corresponding ring of regular functions by O(kn, 0). This means:

- rational functions that are regular at the origin: k[x1, ..., xn](m),
- or locally converging power series, k{x1, ..., xn},
- or formal power series, k[[x1, ..., xn]].

In this paper by a curve/hypersurface we always mean the germ at the singular point, which is assumed to be the origin 0 ∈ (kn, 0). The hypersurfaces are considered with their multiplicities, not just as zero sets of functions. We denote the zero/identity matrices by 0, respectively I. Denote by ℳ♭ the adjoint matrix of ℳ, so ℳ♭♭ = det(ℳ)I.
1.1. Setup. Let $\mathcal{M}$ be a $d \times d$ matrix whose entries are functions in $\mathcal{O}_{(k^n,0)}$. We always assume $f = \det(\mathcal{M}) \neq 0$ and $d > 1$, and the matrix vanishes at the origin, $\mathcal{M}|_0 = 0$. So the matrix defines the (algebraic, analytic or formal) hypersurface $(X, 0) = \{ \det(\mathcal{M}) = 0 \} \subset (k^n, 0)$, of dimension $(n - 1)$. This hypersurface is mostly singular, can be reducible/non-reduced.

Such matrices of functions occur in various fields. In algebraic geometry they are called local determinantal representations of the hypersurface $(X, 0)$. They correspond also to some elements of the local class group $\text{Cl}(X, 0)$. In commutative algebra they appear as matrix factorizations or the syzygy matrices in the resolutions of modules over hypersurface singularities. They appear as local homomorphisms of vector bundles (and degeneracy loci), as maps $(k^n, 0) \to \text{Mat}(d \times d)$, as matrix families in operator theory, as transfer functions in control theory, in semi-definite programming etc. For a short mixture of results cf. §1.5.

With the applications in mind, the determinantal representations are studied up to the local equivalence $\mathcal{M} \sim A\mathcal{M}B$ for $A, B \in \text{GL}(d, \mathcal{O}_{(k^n,0)})$, i.e. $A, B$ are invertible at the origin. This equivalence preserves the embedded hypersurface pointwise. Any matrix is locally equivalent to a block-diagonal, $\mathcal{M} \sim I \oplus \mathcal{M}'$, where $\mathcal{M}'|_0 = 0$, property 2.12. Hence we mostly assume that $\mathcal{M}$ vanishes at the origin.

Determinantal representations are well studied in simple cases, e.g. when the hypersurface singularity is of one of $A, D, E$ types, or for locally irreducible plane curve singularities. For more complicated singularities the determinantal representations are not so well understood. In particular the following decomposability question has not been addressed.

**Problem** Suppose $\det(\mathcal{M})$ is reducible, i.e. the hypersurface $\{ \det \mathcal{M} = 0 \}$ is locally decomposable. When is $\mathcal{M}$ decomposable, i.e. $\mathcal{M} \sim \mathcal{M}_1 \oplus \mathcal{M}_2$? When is $\mathcal{M}$ an extension, i.e. equivalent to an upper-block-triangular matrix?

In more modern language this question reads: when is the (co)kernel module decomposable or extension?

The main goal of this work is to treat this question and provide necessary and/or sufficient decomposability criteria. Our analysis shows two particular types of matrix functions (or modules) with nice properties: those of “maximal size” and those that ”descend from a modification of singularity”. They admit especially strong criteria. On the other hand, they happen to be particularly important in applications, [L.K.M.V.-book]. Any such decomposability criterion is useful as it reduces a ”matrix problem” to a simpler blocks.

1.2. Maximally generated determinantal representations, i.e. matrices of maximal size. Restrict $\mathcal{M}$ to the hypersurface $(X, 0)$, i.e. consider $\mathcal{M}$ as a matrix with entries in the quotient ring $\mathcal{O}_{(X,0)} = \mathcal{O}_{(k^n,0)}/(\det \mathcal{M})$. Two determinantal representations are equivalent over $\mathcal{O}_{(k^n,0)}$ iff they are equivalent over $\mathcal{O}_{(X,0)}$, (proposition 2.13).

Let $\text{mult}(X, pt)$ be the multiplicity of the hypersurface at the point, i.t. the order of vanishing of $\det \mathcal{M}$. At each point $\text{corank} \mathcal{M}|_{pt} \leq \text{mult}(X, pt)$, (property 2.12). This motivates the following

**Definition 1.1.** The representation is called maximally generated at the point $0 \in (X, 0) \subset (k^n, 0)$ if $\text{corank} \mathcal{M}|_0 = \text{mult}(X, 0)$. The representation is called maximally generated near the point $0 \in X \subset k^n$ if it is maximally generated in some neighborhood of $0 \in k^n$.

For example, any determinantal representation of $X$ is maximally generated at any smooth point of $X$. Determinantal representations that are maximally generated at the origin $0 \in k^n$ correspond to Ulrich-maximal Cohen-Macaulay modules, [Ulrich1984]. For an
isolated hypersurface singularity (e.g. reduced plane curves) being maximally generated at the point and maximally generated near the point is the same.

Sometimes we specify the neighborhood where $M$ is maximally generated. For example it can be the set of all the smooth points of $X$ (or the smooth points of the reduced locus $X_{\text{red}}$), so the neighborhood is punctured. Or the set of all the points where the multiplicity of $X$ is bounded by some number.

The notions of maximally generated at the point and maximally generated in the punctured neighborhood of the point are essentially different. For example, if $(X,0)$ is an isolated singularity and $M$ any of its determinantal representations then the block-diagonal matrix $\oplus^r M$ is a determinantal representation of $(rX,0)$, maximally generated on the punctured neighborhood of the origin. On the other hand, if $M$ is a maximally generated determinantal representation of $(X,0)$, then by inserting into the matrix $\oplus^r M$, above the diagonal blocks, some generic polynomials, vanishing at the origin, we get a presentation which is maximally generated at the origin but not in the punctured neighborhood of the origin.

1.3. Saturated determinantal representations. Let $(X',0) \xrightarrow{\nu} (X,0)$ be a finite modification. Here $(X',0)$ is a multi-germ and $\nu$ is a proper, surjective, finite, birational morphism of pure dimensional schemes that is an isomorphism outside the singular locus of $(X,0)$. For example, it could be (in the trivial case) the identity isomorphism: $(X,0) \xrightarrow{\sim} (X,0)$. Or, in the 'maximal case', the normalization $(\tilde{X},0) \rightarrow (X,0)$. For the details cf. §2.2.2.

Associated to this morphism is the relative adjoint ideal $\text{Adj}_{X'/X} \subset \mathcal{O}_{k^n,0}$, defined by

$$\text{Adj}_{X'/X} := \{ g \in \mathcal{O}_{k^n,0} : \nu^*(g)\mathcal{O}_{(X',0)} \subset \nu^{-1}\mathcal{O}_{(X,0)} \},$$

Note that $X$ can be non-reduced here. The restriction of $\text{Adj}_{X'/X}$ to $(X,0)$ defines the relative conductor ideal $I^d_{X'/X} = \text{Ann}_{\mathcal{O}_{(X,0)}} \mathcal{O}_{(X',0)} \mathcal{O}_{(X,0)} \subset \mathcal{O}_{(X,0)}$. For more detail cf. §2.2.3.

Definition 1.2. The determinantal representation $M$ is called $X'/X$-saturated if every element of the adjoint matrix $M'$ belongs to the adjoint ideal $\text{Adj}_{X'/X}$.

(This definition is easy to check in particular cases. In 1.4.3 we give an equivalent but more conceptual definition.) Note that the properties of being maximally generated or $X'/X$ saturated are invariant with respect to the local equivalence.

1.4. Contents of the paper. The matrices of functions appear in various fields, our results are relevant for broad audience. Hence we describe briefly several approaches in §1.5 and throughout the paper we recall some known facts. We provide many (counter-)examples. In §5 we give some applications of the decomposability to the study of determinantal representations of particular hypersurface singularities.

Section 2 contains preliminaries and background. In §2.1 we discuss various notions of locality, i.e. the dependence on the base rings: $k[[m]]$, $k\{\}$ or $k[[\cdot]]$. In §2.2 we discuss curve and hypersurface singularities and their finite modifications $(X',0) \rightarrow (X,0)$. In §2.2.3 we discuss the corresponding conductor and adjoint ideals, $I^d_{X'/X}$ and $\text{Adj}_{X'/X}$. In §2.4 we introduce the (co)kernel of a determinantal representation, $E$, which is a maximal Cohen-Macaulay module of rank 1 over $(X,0)$.

1.4.1. Decomposability of modules with many generators. An arbitrary determinantal representation cannot be brought to an upper-block-triangular form, even in the case of plane
curve singularity that is an ordinary multiple point, i.e. the union of smooth pairwise non-tangent branches. However, modules with many generators tend to be decomposable or extensions.

Proposition 3.5. Suppose \( E_{(X,0)} \) is minimally generated by \( d(E) \) elements and its restrictions \( E_i := E|_{(X_i,0)} \mod \text{Torsion} \) are generated by \( d(E_i) \) elements. Suppose the restriction \( \text{tr}(E)_i \) (of Auslander’s transpose) is generated by \( d(\text{tr}(E))_i \) elements. Then \( E \) is an extension iff \( d(E) = d(E_1) + d(\text{tr}(E)_2) \) or \( d(E) = d(E_2) + d(\text{tr}(E)_1) \).

In other words, suppose \( M_{d \times d} \) vanishes at the origin. Consider the \( \mathcal{O}_{(X,0)} \) module spanned by the columns of \( \mathcal{M}'_{|(X,0)} \). Suppose this module is minimally generated by \( d(E_i) \) columns. Suppose the module of the rows of \( \mathcal{M}'_{|(X,0)} \) is generated by \( d(\text{tr}(E))_i \) rows. Then \( M \) is locally equivalent to an upper-block-triangular form iff \( d = d(E_1) + d(\text{tr}(E)_2) \) or \( d = d(E_2) + d(\text{tr}(E)_1) \).

In the case of curves we have a much stronger criterion:

Theorem 3.7 Let \( (C,0) = (C_1,0) \cup (C_2,0) \subset (k^2,0) \), with \( (C_i,0) \) possibly further reducible, non-reduced but with no common components. Let \( \mathcal{M} \) be a determinantal representation of \( (C,0) \).

1. \( \mathcal{M} \sim \begin{pmatrix} \mathcal{M}_1 & * \\ 0 & \mathcal{M}_2 \end{pmatrix} \) iff \( \mathcal{M} \sim \begin{pmatrix} \mathcal{M}_2 & * \\ 0 & \mathcal{M}_1 \end{pmatrix} \), where \( \mathcal{M}_i, \mathcal{M}_\tilde{i} \) are some determinantal representations of \( (C_i,0) \).

2. If \( \mathcal{M} \) is maximally generated at the origin then it is equivalent to an upper-block-triangular matrix, i.e. the corresponding module is an extension.

In general this extension of modules is non-trivial, the matrix is indecomposable. However the decomposability holds if the components of the curve are not tangent.

Theorem 3.9. Let \( (C,0) = (C_1,0) \cup (C_2,0) \subset (k^2,0) \) where \( (C_i,0) \) can be further reducible or non-reduced but have no common tangents. If \( \mathcal{M} \) is a determinantal representation of this curve that is maximally generated at the origin, then it is decomposable: \( \mathcal{M} \sim \mathcal{M}_1 \oplus \mathcal{M}_2 \).

The last two results reduce the classification of local maximally generated determinantal representations of plane curve singularities (i.e. families of matrices depending on two parameters) essentially to multiple branches, i.e. \( (rC,0) \) with \( (C,0) \) locally irreducible. In this case we have:

Theorem 3.10 Let \( (rC,0) \subset (k^2,0) \), where \( (C,0) \) is a locally irreducible, reduced plane curve.

1. Let \( \mathcal{M} \) be a determinantal representation of \( (rC,0) \) maximally generated at the origin. Then \( \mathcal{M} \) is equivalent to an upper-block-triangular matrix, the blocks on the diagonal are determinantal representations of \( (C,0) \).

2. Let \( \mathcal{M} \) be a determinantal representation maximally generated on the punctured neighborhood of the origin. Then \( \mathcal{M} \) is totally decomposable: \( \mathcal{M} = \oplus \mathcal{M}_i \) where \( \mathcal{M}_i \) is a determinantal representation of \( (C,0) \).

In higher dimensional case we give an analog of theorem 3.9. Let \( (X,0) = (X_1,0) \cup (X_2,0) \subset (k^n,0) \).

Theorem 3.11. 1. If the intersection \( (X_1,0) \cap (X_2,0) \) is reduced, i.e. the components are reduced and generically transverse, then any determinantal representation that is maximally generated on the smooth points of \( (X_1,0) \cap (X_2,0) \) is decomposable.

2. More generally, if the projectivized tangent cones, \( \mathbb{P}T_{(X_1,0)}, \mathbb{P}T_{(X_2,0)} \subset \mathbb{P}(k^n) \), intersect
transversally then any determinantal representation of \((X, 0)\) that is maximally generated near the origin is decomposable.

As we show by numerous examples the assumptions of these criteria are almost necessary, so these sufficient criteria are in some sense the best possible.

1.4.2. Relation to matrix factorizations. Recall that a matrix factorization of a function \(f \in \mathcal{O}(k^n, 0)\) is the matrix identity \(AB = f I\). (We always assume \(A|_0 = B|_0 = 0\), for more detail see §1.5.3.) If \(f\) is irreducible then \(A\) is a determinantal representation of some power of \(f\). Which determinantal representations arise in this way? An immediate corollary of our approach is:

Corollary 3.3. Let \(\mathcal{M}\) be a determinantal representation of \(\prod f_\alpha^p\). It can be augmented to a matrix factorization of \(\prod f_\alpha\) (i.e. there exists \(B\) with \(MB = \prod f_\alpha I\)) iff \(\mathcal{M}\) is maximally generated at smooth points of the reduced hypersurface \(\{\prod f_\alpha = 0\} \subset (k^n, 0)\).

1.4.3. Saturated modules. In §4 we study \(X'/X\)-saturated determinantal representations. For a finite modification \((X', 0) \xrightarrow{\nu} (X, 0)\) the kernel \(E\) of \(\mathcal{M}\) can be pulled back to \(\nu^*E/Torsion\), a module on \((X', 0)\). Usually pulling back adds many new elements, as the initial kernel \(E\) is not a module over the bigger ring \(\mathcal{O}(X', 0)\). This gives a reformulation of definition 1.2:

Proposition 4.3. The determinantal representation \(\mathcal{M}\) is \(X'/X\)-saturated iff the corresponding kernel module \(E\) is \(X'/X\)-saturated, i.e. the canonical embedding \(E \hookrightarrow \nu_*(\nu^*E/Torsion)\) is an isomorphism. The kernel \(E\) of \(\mathcal{M}\) is \(X'/X\)-saturated iff the kernel \(\text{tr}(E)\) of \(\mathcal{M}^T\) is \(X'/X\)-saturated.

This proposition is helpful in proving various decomposability criteria. For example:

Theorem 4.4. A determinantal representation of \((X_1, 0) \cup (X_2, 0)\) is decomposable iff it is \((X_1, 0) \bigcup (X_2, 0) \bigcup (X_2, 0)\) saturated.

We emphasize that it is very simple to check saturatedness, by using definition 1.2, i.e. by checking the entries of \(\mathcal{M}^\vee\).

Then we get an immediate:

Corollary. Suppose \((X, 0)\) is reduced and its normalization \((\tilde{X}, 0) \xrightarrow{\nu} (X, 0)\) is a smooth variety. (For example, the normalization of a reduced curve is always smooth.) There exists unique \(\tilde{X}/X\)-saturated module (or \(\tilde{X}/X\)-saturated determinantal representation): \(\nu_*(\mathcal{O}(\tilde{X}, 0))\). It is maximally generated.

1.5. A brief introduction and overview. We recall here the local aspects of determinantal representations only, for some references on the global aspects cf.[Kerner-Vinnikov2009].

1.5.1. A view from singularities. The modern study started probably from the seminal paper [Arnold1971] and was mentioned in [Arnold-problems, 1975-26, pg. 23]. Many works studied the miniversal deformations of a constant matrix for various equivalences (i.e. to write a normal form for a linear family of matrices), cf. e.g. [Tannenbaum81, Chapter5], [Khabbaz-Stengle1970] or [Lancaster-Rodman2005]. For the deformation theory from commutative algebra point of view cf. [Ile2004].

Recently various singularity invariants of such "matrices of functions" have been established: [Bruce-Tari2004], [Bruce-Goryunov-Zakalyukin2002], [Goryunov-Zakalyukin2003], [Goryunov-Mond2005].

1.5.2. The case of one variable is elementary, e.g. [Gantmacher-book, chapter VI]: any square, non-degenerate matrix with formal entries is locally equivalent to the unique diagonal
matrix: $\oplus x^{d_l} \mathbb{I}_{r_l \times r_l}$ for $d_1 < \cdots < d_n$ and \{r_l\} are some integers. In more modern language: any $k[[x]]$ module is the direct sum of cyclic modules.

1.5.3. Matrix factorizations and maximally Cohen-Macaulay modules. For an introduction to the case of more variables see [Yoshino-book] and [Leuschke-Wiegand-book].

Let $\mathcal{M}$ be a local determinantal representation of $f \in \mathcal{O}_{(k^n,0)}$. Let $E$ be its kernel spanned by the columns of $\mathcal{M}$ as a module over $\mathcal{O}_{(X,0)} := \mathcal{O}_{(k^n,0)}/(f)$. Then $E$ has a period two resolution by free $\mathcal{O}_{(X,0)}$ modules:

$$\ldots \xrightarrow{\mathcal{M}^\vee} \mathcal{O}^{\oplus d}_{(X,0)} \xrightarrow{\mathcal{M}} \mathcal{O}^{\oplus d}_{(X,0)} \xrightarrow{\mathcal{M}^\vee} \mathcal{O}^{\oplus d}_{(X,0)} \rightarrow E \rightarrow 0$$

One can show that $\text{depth}(E) = n - 1 = \text{dim} \mathcal{O}_{(X,0)}$, hence $E$ is a maximally Cohen-Macaulay (MCM) module. Maximally Cohen-Macaulay modules over reduced curves are just the torsion-free modules.

Vice-versa [Eisenbud1980]: any maximally Cohen Macaulay (MCM) module $E$ over the hypersurface ring $\mathcal{O}_{(X,0)}$ as above has a resolution of period two:

$$\ldots \xrightarrow{\mathcal{M}_1} \mathcal{O}^{\oplus d}_{(X,0)} \xrightarrow{\mathcal{M}_2} \mathcal{O}^{\oplus d}_{(X,0)} \xrightarrow{\mathcal{M}_1} \mathcal{O}^{\oplus d}_{(X,0)} \rightarrow E \rightarrow 0$$

corresponding to the matrix factorization: $f \mathbb{I} = M_1 M_2$. Note that here the dimensions of $\{M_i\}$ are not fixed. So, in general $\mathcal{M}_i$ are determinantal representations of $\prod f_i^{p_i}$, for $f_{\text{reduced}} = \prod f_i$.

Suppose $f$ is homogeneous, of degree $d$. Then, by [Backelin-Herzog-Sanders1988], $f$ admits a matrix factorization in linear matrices: $f \mathbb{I} = A_1 \ldots A_d$, i.e. all the entries of $\{A_i\}$ are homogeneous linear forms.

For an MCM module $E$ over $\mathcal{O}_{(X,0)}$ the minimal number of generators of $E$ is not bigger than $\text{multiplicity}(X,0) \times \text{rank}(E)$, [Ulrich1984, §3]. Modules for which the equality occurs are called Ulrich’s modules. In our case, with $\text{rank}(E) = 1$, they are precisely the maximally generated determinantal representations. For an arbitrary algebraic hypersurface Ulrich modules, of high rank, exist [Backelin-Herzog1989, Theorem 1]. Hence, for any $\{f = 0\} \subset (k^n,0)$ its multiples $\{f^p = 0\}$, for $p$ high enough, have maximally generated determinantal representations.

Let $E$ be an MCM-module (of any rank) on $(X,0) = \{f = 0\}$. Its resolution provides a matrix factorization of $f$. The syzygy of $E$ gives a determinantal representation of a "multiple" of $f$, i.e. a hypersurface $(Y,0)$ that contains $(X,0)$ and whose reduced locus coincides with the reduced locus of $(X,0)$. Thus $E$ can be considered as a rank one module on $(Y,0)$. So there exists a natural embedding of the theory of MCM-modules on hypersurfaces into the theory of rank one MCM-modules on (non-reduced) hypersurfaces.

MCM modules over a given hypersurface singularity, i.e. matrices of formal series with the given determinant, have been classified in some particular cases. A hypersurface singularity is called of finite/tame CM-representation type if it has a finite/countable number of indecomposable MCM’s, up to isomorphism.

- A series of papers resulted in [Buchweitz-Greuel-Schreyer1987]: a hypersurface singularity is of finite CM-representation type iff it is the ring of a simple (ADE) singularity.
- The MCM’s of rank 1 over locally irreducible plane curve singularities were thoroughly studied in [Greuel-Pfister-1993].
- The MCM modules over the surface $\sum_{i=1}^3 x_i^3$ and matrix factorizations were classified in [Laza-Pfister-Popescu2002]. The MCM modules of rank 2 over the hypersurface $\sum_{i=1}^4 x_i^3$ were classified in [Baciu-Ene-Pfister-Popescu2005].
• The MCM modules over the ring $k[[x, y]]/(x^n)$ were classified in [Ene-Popescu2008].
• The MCM modules over the ring $k[[x, y]]/(xy^2)$ were classified in [Buchweitz-Greuel-Schreyer1987].
• The MCM modules over surface singularities were studied in [Burban-Drozd2008], in particular the modules over $k[[x, y, z]]/xy$ and $k[[x, y, z]]/x^2y - z^2$ were classified. See also [Burban-Drozd2010].
• The MCM modules over Thom-Sebastiani rings, i.e. $k[[x_1, \ldots, x_k, y_1, \ldots, y_n]]/(f(x) \oplus g(y))$, were studied in [Herzog-Popescu1997]. In particular, the modules over $k[[x_1, \ldots, x_n, y]]/(f(x) + y^3)$ were related to those over $k[[x_1, \ldots, x_n]]/(f(x))$. Some MCM modules over $k[[x, y]]/(x^n + y^3)$ were classified.
• The possible rank of an MCM module without free summand, on a reduced hypersurface is bounded from below. In particular, the conjecture in [Buchweitz-Greuel-Schreyer1987] reads:

$$rank(E) \geq 2^{\left\lfloor \dim \text{Sing}(X, 0) - 2 \right\rfloor}$$

1.5.4. Some applications.
1.5.4.1. Semi-definite programming is probably the most important new development in optimization in the last two decades. The general goal is to study the linear matrix inequalities, i.e. the positive-definiteness of $\sum x_i M_i$, where $\{M_i\}$ are real symmetric matrices. The literature on the subject is vast, for the general introduction cf. [S.I.G.-book].

A related object is the subsets of $\mathbb{R}^n$, presentable by linear matrix inequalities, called spectrahedra. This notion was introduced and studied in [Goldman-Ramana1995], for the recent advances cf. [Helton-Vinnikov2007].

1.5.4.2. The famous Lax conjecture relates the hyperbolic polynomials and self-adjoint positive definite determinantal representations, [Lax-1958]. The initial form, for homogeneous polynomials in three variables has been proved in [Lewis-Parrilo-Ramana2005] and [Helton-Vinnikov2007]. The general case is still open.

1.5.4.3. The spectral analysis of pairs of commuting non-selfadjoint or non-unitary operators is essentially based on the determinantal representations of plane algebraic curves cf. [L.K.M.V.-book].

2. Preliminaries and background

2.1. On the base rings. When studying determinantal representations several rings appear naturally:
- the ring of rational functions that are regular at the origin (i.e. the localization at the the origin of the polynomial ring $k[x_1, \ldots, x_n]_{(m)}$),
- the ring of locally analytic functions $k\{x_1, \ldots, x_n\}$
- the ring of formal power series $k[[x_1, \ldots, x_n]]$.

We always denote the maximal ideal by $m$.

The ring $k\{x_1, \ldots, x_n\}$ comes inevitably in the local considerations. For example, an algebraic hypersurface singularity can be irreducible over $k[x_1, \ldots, x_n]_{(m)}$ but reducible over $k\{x_1, \ldots, x_n\}$. The (ir)reducibility over $k\{x_1, \ldots, x_n\}$ and $k[[x_1, \ldots, x_n]]$ coincide, i.e. if $(X, 0)$ is a locally analytic germ and $(X, 0) = \cup_i (X_i, 0)$ is its decomposition into irreducible formal germs, then all $(X_i, 0)$ are locally analytic. This follows e.g. from Artin and Pfister-Popescu approximation theorems.
Even if one restricts to locally converging power series, in some inductive arguments the formal power series might appear, fortunately just as an intermediate step. The final result (the determinantal representation and the matrices of equivalence) can always be chosen locally analytic due to the approximation theorems:

**Theorem 2.1.** [Artin1968][GLS-book1, pg.32] \(\text{Let } x, y \text{ be the multi-variables and } f_1, \ldots, f_k \in k\{x, y\} \text{ the locally analytic series. Suppose there exist formal power series } \hat{Y}_1(x), \ldots, \hat{Y}_l(x) \in k[[x]] \text{ solving the equations, i.e.}:
\)
\[
(5) \quad f_i(x, \hat{Y}_1(x), \ldots, \hat{Y}_l(x)) \equiv 0, \quad i = 1..k
\]
Then there exists a locally analytic solution \(Y_1(x), \ldots, Y_l(x) \in k\{x\}:
\)
\[
(6) \quad f_i(x, Y_1(x), \ldots, Y_l(x)) \equiv 0, \quad i = 1..k
\]

**Theorem 2.2.** [Pfister-Popescu1975] \(\text{Let } F_1 = 0, F_2 = 0, \ldots F_k = 0 \text{ be a system of polynomial equations over a complete local ring } (R, m). \text{ The system has a solution in } R \text{ iff it has a solution in } R/m^N \text{ for any } N.
\)

An immediate application of these theorems is:

**Corollary 2.3.** Let \(M\) be a matrix with entries in \(k\{x_1, \ldots, x_n\}\). Suppose for each \(N\) the matrix, considered as matrix over \(k\{x_1, \ldots, x_n\}/m^N\), is equivalent to a block diagonal \(\text{(} \begin{smallmatrix} * & \emptyset \\ \emptyset & \ast \end{smallmatrix} \text{)}\) (or an upper-block-triangular \(\text{(} \begin{smallmatrix} * & \ast \\ \emptyset & \ast \end{smallmatrix} \text{)})\). Then \(M\) is equivalent to a block-diagonal (or an upper-block-triangular) matrix over \(k\{x_1, \ldots, x_n\}\).

Indeed, we have here the system of locally analytic equations, corresponding to the zero blocks of \(AMB = \text{(} \begin{smallmatrix} * & \emptyset \\ \emptyset & \ast \end{smallmatrix} \text{)}\) or \(AMB = \text{(} \begin{smallmatrix} * & \ast \\ \emptyset & \ast \end{smallmatrix} \text{)}. And these equations have a solution over \(k\{x_1, \ldots, x_n\}/m^N\) for any \(N\).

Over the ring of locally analytic or formal power series the Krull-Schmidt theorem holds: the decomposition of \(E\) into irreducible components (i.e. of \(M\) into indecomposable blocks) is unique up to permutation. For the situation over the rings of rational functions see [Wiegand2001] or [Leuschke-Wiegand-book].

Comparing the determinantal representations over various base rings we have two natural and well studied questions, cf. [Eisenbud-book], [Yoshino-book].

- (injectivity) Let \(M_1, M_2\) be determinantal representations with rational/locally converging entries. Suppose they are formally equivalent. Are they rationally/locally converging equivalent? This is indeed true, as the completion is a faithful functor.

- (surjectivity) Which formal determinantal representations of an analytic/algebraic hypersurface are equivalent to determinantal representations with analytic/rational entries?

By the classical theorem [Elkik1973, Théorème 3] any formal determinantal representation of an analytic hypersurface with isolated singularity is equivalent to an analytic determinantal representation. For affine rings, i.e. the ring of rational functions, this property usually fails.

Any formal determinantal representation of a zero dimensional hypersurface singularity, i.e. \(f(x) \in k[[x]]\), is equivalent to an algebraic (cf. the beginning of §1.5.2). For the determinantal representations of plane curve singularities one has:
Determinantal representations

Property 2.4. 1. If \( \det(M) \) defines an algebraic plane curve singularity that is locally irreducible (over \( k[[x,y]] \)), then \( M \) is equivalent to a matrix of polynomials.

2. More generally, if \( \det(M) \) defines an algebraic plane curve singularity all of whose irreducible components (over \( k[[x,y]] \)) are algebraic, then \( M \) is equivalent to a matrix of polynomials.

This is proved e.g. in [Frankild-Sather-Wagstaff-Wiegand2008, Proposition 3.3]. A down-to-earth proof is in [Belitskii-Kerner2010, Appendix A]

More generally, for any formal determinantal representation \( M \) of an algebraic hypersurface, there exists another, \( N \), such that \( M \oplus N \) is equivalent to a determinantal representation with polynomial entries, see [Frankild-Sather-Wagstaff-Wiegand2008, Corollary 3.5].

2.2. Hypersurface singularities.

2.2.1. Some local decompositions. For local considerations we always assume the (singular) point to be at the origin and the ring is either \( k\{x_1,\ldots,x_n\} \) or \( k[[x_1,\ldots,x_n]] \).

Associated to any germ \((X,0) = \{f = 0\}\) is the decomposition \((X,0) = \cup (p_iX_i,0) = \{\prod_i f_i^{p_i} = 0\}\). Here each \((X_i,0)\) is reduced and locally irreducible. The tangent cone \( T_{(X,0)} \) is formed as the limit of all the tangent planes at smooth points. Let \( f = f_p + f_{p+1} + \ldots \) be the Taylor expansion, then the tangent cone is \( \{f_p = 0\} \subset (k^n,0) \). For curves the tangent cone is the collection of tangent lines, each with the corresponding multiplicity.

The tangent cone is in general reducible, associated to it is the tangential decomposition: \((X,0) = \cup (X_\alpha,0)\). Here \( \alpha \) runs over all the (set-theoretical) components of the tangent cone, each \((X_\alpha,0)\) can be further reducible, non-reduced.

The simplest invariant of the hypersurface singularity \( \{f_p + f_{p+1} + \cdots = 0\} \) is the multiplicity \( p = \text{mult}(X,0) \), for the tangential components denote \( p_\alpha = \text{mult}(X_\alpha,0) \).

2.2.2. The normalization and intermediate modifications. A singularity is normal if its local ring is a domain integrally closed in its field of fractions, [Eisenbud-book, pg.118]. A reduced hypersurface singularity is normal iff its singular locus is of codimension at least two. For example, a reduced curve is normal if it is smooth and a reducible hypersurface is not normal.

The normalization of a (non-normal) germ is a (unique) finite proper birational morphism \((\tilde{X},0) \rightarrow (X,0)\), with \((\tilde{X},0)\) pure dimensional and normal. Note that \((\tilde{X},0)\) is usually a multi-germ, as the normalization separates the components. For brevity we write \((\tilde{X},0)\) instead of \( \prod_i(\tilde{X}_i,0_i) \). In the non-reduced case, \( X = \cup (p_iX_i,0) \), the normalization is: \( \prod_i(p_i\tilde{X}_i,0) \).

Algebraically, if \( \mathcal{O}_{(X,0)} \) is the local ring of \((X,0)\) then the normalization is induced by the inclusion \( \mathcal{O}_{(X,0)} \subset \mathcal{O}_{(\tilde{X},0)} \), where \( \mathcal{O}_{(\tilde{X},0)} \) is the integral closure.

As any reduced normal curve is smooth, the normalization of a reduced curve singularity is its resolution \((\tilde{C},0) \rightarrow (C,0)\). Usually the normalization \((\tilde{X},0) \rightarrow (X,0)\) can be (nontrivially) factorized: \((\tilde{X},0) \rightarrow (X',0) \rightarrow (X,0)\). Here both maps are finite surjective bi-rational morphisms. Usually this factorization can be done in many distinct ways. All the possible intermediate steps form an oriented graph, usually not a tree. The initial vertex of this graph is the full normalization, the final is the original hypersurface. Algebraically, the intermediate steps correspond to
extensions of the local rings:

\[ (7) \quad \mathcal{O}(x,0) \overset{\nu_{X/Y}}{\to} \mathcal{O}(x',0) \overset{\nu_{X'/Y'}}{\to} \mathcal{O}(\tilde{x},0) \]

**Example 2.5.** The curve singularity of type \( A_n \): \( y^2 = x^{n+1} \).

- \( n = 2l \). In this case the curve is a branch, i.e. is locally irreducible. The normalization is given by \( x = t^2 \), \( y = t^{2l+1} \) or by the extension of the local ring: \( k\{t^2, t^{2l+1}\} \subset k\{t\} \). All the intermediate modifications correspond to the intermediate rings:

\[ (8) \quad k\{t^2, t^{2l+1}\} \subset k\{t^2, t^{2l-1}\} \subset k\{t^2, t^{2l-3}\} \subset \ldots \subset k\{t^2, t^{3}\} \subset k\{t\} \]

Or geometrically we have the surjections of plane curves singularities of types:

\[ (9) \quad A_0 \to A_2 \to \ldots \to A_{2l-2} \to A_{2l} \]

- \( n = 2l-1 \). In this case the curve has two branches. The normalization is given by \( x = t_1 + t_2 \), \( y = t_1^l - t_2^l \) or by the extension of the local ring: \( k\{t_1 + t_2, t_1^l - t_2^l\}/(t_1t_2) \subset k\{t_1\} \times k\{t_2\} \). All the intermediate modifications correspond to the intermediate rings:

\[ (10) \quad k\{t_1 + t_2, t_1^l - t_2^l\}/(t_1t_2) \subset k\{t_1 + t_2, t_1^{l-1} - t_2^{l-1}\}/(t_1t_2) \subset \ldots \subset k\{t_1, t_2\}/(t_1t_2) \subset k\{t_1\} \times k\{t_2\} \]

Or geometrically:

\[ (11) \quad A_0 \sqcup A_0 \to A_1 \to A_3 \to \ldots \to A_{2l-1} \]

**Example 2.6.** Consider the germ of the type \( x^p = y^p \). This is the ordinary multiple point, i.e. the intersection of \( p \) smooth pairwise non-tangent branches. Here the tangent cone consists of the lines \( \{x = w y\} \), for \( w^p = 1 \). The tangential decomposition coincides with the branch decomposition. The normalization separates all the branches and is defined by \( (x = t_i, y = w_i t_i) \), here \( w_i \) are all the \( p \)'th roots of unity. This corresponds to the embedding \( k\{x, y\}/x^p = y^p \subset \prod_i k\{t_i\} \).

The graph of modifications for \( p = 3 \) is:

\[ (12) \quad A_0 \sqcup A_0 \sqcup A_0 \to A_0 \sqcup A_1 \to \text{Spec}\left( k\{t_1, t_2, t_3\}/t_it_j, \ i \neq j \right) \to \{x^3 = y^3\} \]

Here \( k\{t_1, t_2, t_3\}/(t_it_j, \ i \neq j) \) is the local ring of the curve singularity formed by three pairwise non-tangent smooth branches. Its embedding dimension is 3, i.e. this is a non-planar singularity.

The term \( A_0 \sqcup A_1 \) corresponds to the separation of one branch from the remaining two. (By permutation this can be done in three ways.) The term \( A_0 \sqcup A_0 \sqcup A_0 \) corresponds to the total separation of branches, i.e. the normalization.

Note that in general most modifications lead to non-planar and even non-Gorenstein singularities.

**Example 2.7.** A particular kind of modification is the separation of all the locally irreducible components: \( \bigsqcup (X_i, 0_i) \to (\bigcup X_i, 0) \). It is isomorphism when restricted to each particular component. For the rings: \( \mathcal{O}(\bigcup X_i, 0) \overset{\iota}{\to} \mathcal{O}(X_i, 0) = \prod \mathcal{O}(X_i, 0_i) \). If \( E \) is a module over \( \mathcal{O}(\bigcup X_i, 0) \), then it is lifted to the collection of modules \( \{ E \otimes \mathcal{O}(X_i, 0) \} / \text{Torsion} \), defined by the diagonal embedding \( 1_{(X,0)} \to \oplus 1_{(X_i,0)} \).
2.2.3. Adjoint and conductor ideals. Let \((X, 0) \subset (k^n, 0)\) be a (possibly non-reduced) hypersurface singularity and \((X', 0) \xrightarrow{\nu} (X, 0)\) a finite modification as above, e.g. the normalization. As \((X, 0)\) is usually reducible, its modification is usually a multi-germ, \((X', 0) = \bigsqcup_{i} (X'_i, 0_i)\), with the morphisms \((X'_i, 0) \xrightarrow{\nu_i} (X, 0)\).

**Definition 2.8.** The relative conductor ideal is:

\[
\mathcal{O}(X, 0) \supset I_{X'/X}^{cd} := \text{Ann}_{\mathcal{O}(X, 0)}(\mathcal{O}(X', 0)/\mathcal{O}(X, 0)) = \{g \mid \forall i : \nu_i^*(g) \mathcal{O}(X'_i, 0) \subset \nu_i^{-1}(\mathcal{O}(X, 0))\}
\]

Note that by its definition \(I_{X'/X}^{cd}\) is the maximal ideal both in \(\mathcal{O}(X, 0)\) that is also an ideal in \(\mathcal{O}(X', 0)\).

Consider the ideals in \(\mathcal{O}(k^n, 0)\) whose restriction to \((X, 0)\) is contained in \(I_{X'/X}^{cd}\). Call the maximal among them: the relative adjoint ideal \(\text{Adj}_{X'/X} \subset \mathcal{O}(k^n, 0)\). So, \(\text{Adj}_{X'/X} |_{(X, 0)} = I_{X'/X}^{cd}\). For various properties of the adjoint/conductor ideals for normalization in the reduced case cf. [Serre-book, §IV.11], [Fulton-2002] [GLS-book1, I.3.4, pg 214] and

**Example 2.9.** Continue example 2.5. Let the local ring of \((C', 0)\) be \(k\{t^2, t^{2l+1}\}\), i.e. the modification is \(A_{2l'} \to A_{2l}\). Then

\[
I_{C'/C}^{cd} = \langle t^{2l-2l'}, t^{2l+1} \rangle \subset \mathcal{O}(C, 0) = k\{t^2, t^{2l+1}\}, \quad \text{Adj}_{C'/C} = \langle x^{l-l'}, y \rangle \subset \mathcal{O}(k^2, 0)
\]

Similarly, for the modification \(A_{2l'-1} \to A_{2l-1}\) one has:

\[
I_{C'/C}^{cd} = \langle t_1^{l-l'} + t_2^{l-l'}, t_1^{l-l'} - t_2^{l-l'} \rangle \subset \mathcal{O}(C, 0) = k\{t_1 + t_2, t_1^{l-l'}, t_2^{l-l'} \}, \quad \text{Adj}_{C'/C} = \langle x^{l-l'}, y \rangle \subset \mathcal{O}(k^2, 0)
\]

**Proposition 2.10.** 1. For the modifications \((X''', 0) \to (X', 0) \to (X, 0)\) one has: \(I_{X'''/X}^{cd} \subset I_{X'/X}^{cd} \cap I_{X'/X'''}^{cd} \subset I_{X'''/X}^{cd} \subset \mathcal{O}(X, 0)\).

2. Let \((X, 0) = (\bigcup X_i, 0)\) where \((X_i, 0)\) can be further reducible but with no common components. Let \(f\) and \(\{f_i\}\) be the defining equations of \((X, 0)\) and \(\{(X_i, 0)\}_i\). Then

\[
\text{Adj}_{\bigcup_{i} (X_i, 0)/(\bigcup X_i, 0)} = \langle \frac{f}{f_1}, \ldots, \frac{f}{f_k} \rangle
\]

3. More generally, if \((X', 0) = \bigsqcup_{i} (X'_i, 0_i) \xrightarrow{\nu_i} (X, 0) = (\bigcup X_i, 0)\) is a finite modification then

\[
I_{X'/X}^{cd} = \{ \sum_{i} f_i g_i, \quad g_i \in I_{X'_i/X_i}^{cd} \}
\]

**Proof.** 1. The product of any element of \(I_{X'''/X}^{cd}\) with an element of \(I_{X'/X}^{cd}\) lies in \(\mathcal{O}(X, 0)\), hence one can speak about the inclusion. The statement is immediate.

2. Suppose \(g \in \text{Adj}_{\bigcup_{i} (X_i, 0)/(\bigcup X_i, 0)}\), so that \(\nu^*(g)1_i \in \nu_i^{-1}(\mathcal{O}(X_i, 0))\), for \(1_i \in \mathcal{O}(X_i, 0)\). As \(\nu^*(g)1_i |_{(X_i, 0)} = 0\) for \(j \neq i\) we get that \(\nu^*(g)1_i\) is divisible by \(\frac{f_i}{f_j}\). So \(g \in \langle \frac{f_i}{f_j} \rangle\). By going over all the components one get \(g \in \langle \frac{f_1}{f_k}, \ldots, \frac{f_l}{f_k} \rangle\). So, \(\text{Adj}_{\bigcup_{i} (X_i, 0)/(\bigcup X_i, 0)} \subset \langle \frac{f_1}{f_k}, \ldots, \frac{f_l}{f_k} \rangle\). The converse inclusion is immediate.

3. The \(\supset\) inclusion. Let \(g = \sum_{i} \frac{f_i}{f_k} g_i\), with \(\{g_i \in I_{X'_i/X_i}^{cd}\}_i\). Let \(h = \sum h_i \in \mathcal{O}_{\bigcup (X'_i, 0_i)}\). Then

\[
h g = \sum_{i} \frac{f_i}{f_k} g_i h_i, \text{ as } h_j g_i = 0 \text{ for } j \neq i.
\]

Moreover \(\{g_i h_i \in \mathcal{O}(X, 0)\}_i\), hence \(gh \in \mathcal{O}(X, 0)\).

The \(\subset\) inclusion. Let \(g \in I_{X'/X}^{cd}\), so by the second part: \(g = \sum \frac{f_i}{f_k} g_i\). Then \(\frac{f_i}{f_k} g_i \mathcal{O}(X'_i, 0) \subset \mathcal{O}(X, 0)\). But for any \(h_j \in \mathcal{O}(X'_j, 0)\), with \(j \neq i\): \(\frac{f_i}{f_k} g_i h_j |_{(X'_j, 0)} = 0\). Therefore \(\frac{f_i}{f_k} g_i h_i \in \frac{f_i}{f_k} \mathcal{O}(X, 0)\), because \(\mathcal{O}(k^n, 0)\) is UFD. Hence \(g_i h_i \in \mathcal{O}(X, 0)\), i.e. \(g_i \in I_{X'_i/X_i}^{cd}\).
Remark 2.11. • In general neither \( I_{C'/C}^{cd} \subseteq \mathcal{O}_{(C,0)} \) nor even \( \nu^* I_{C'/C}^{cd} \subseteq \mathcal{O}_{(C',0)} \) are principal ideals. For example for the modification
\[
(C',0) = \text{Spec}(k\{t^3,t^5,t^7\}) \xrightarrow{\nu} (C,0) = \text{Spec}(k\{t^3,t^7\}) \quad \nu^* I_{C'/C}^{cd} = \langle t^7, t^9, .. \rangle \subset k\{t^3,t^5,t^7\}
\]

• In general \( I_{C'/C}^{cd} \supsetneq I_{C'/C}^{cd} \). For example,
\[
\mathcal{O}_{(C,0)} = k\{t^3,t^7\} \supsetneq \mathcal{O}_{(C',0)} = k\{t^3,t^5,t^7\} \subset \mathcal{O}_{(C'',0)} = k\{t^3,t^4,t^5\},
\]
\[
I_{C'/C}^{cd} = \langle t^7, t^9, .. \rangle, \quad I_{C'/C'}^{cd} = \langle t^3, t^5, .. \rangle, \quad I_{C'/C}^{cd} = \langle t^9, t^{10}, .. \rangle
\]

2.3. The matrix and its adjoint. We work with (square) matrices, their sub-blocks and particular entries. Sometimes to avoid confusion we emphasize the dimensionality, e.g. \( M_{d \times d} \). Then \( M_{i \times i} \) denotes an \( i \times i \) block in \( M_{d \times d} \) and \( \det(M_{i \times i}) \) the corresponding minor. By \( M_{ij} \) we mean a particular entry.

Note that the adjoint of \( \begin{pmatrix} M_1 & M_2 \\ 0 & M_2 \end{pmatrix} \) is \( \begin{pmatrix} \det(M_2)M_1' & -M_1'M_3M_2' \\ 0 & \det(M_1)M_2' \end{pmatrix} \).

When working with matrices of functions, several natural notions arise:
• \( \text{deg}_x(M) \) = the maximal degree of \( x_i \) in the entries of \( M \). This is infinity unless all the entries of \( M \) are polynomials in \( x_i \). Similarly for \( \text{deg}(M) \), the total degree.
• \( \text{ord}_{x_i}(M) \) = the minimal degree of \( x_i \) appearing in \( M \). If an entry of \( M \) does not depend on \( x_1 \), the order is zero, if \( A = 0 \) then \( \text{ord}_{x_1}(A) = \infty \). Similarly \( \text{ord}(M) \) and \( \text{ord}_x(M_{ij}) \) for a particular entry. So, e.g. \( \text{ord}(M) \geq 1 \) iff \( M|_0 = 0 \).
• \( \text{jet}_k(M) \) is obtained from \( M \) by truncation of all the monomials with total degree higher than \( k \).

2.3.1. Reduction to a minimal form. Let \( M \in \text{Mat}(d \times d, \mathcal{O}_{(k^n,0)}) \), without the assumption \( M|_0 = 0 \). Let the multiplicity of the hypersurface germ \( \{ \det(M) = 0 \} \subset (k^n,0) \) be \( \text{mult}(X,0) \geq 1 \).

Property 2.12. 1. Locally \( M_{d \times d} \) is equivalent to \( \text{Id}_{(d-p) \times (d-p)} \begin{pmatrix} 0 \\ M_{p \times p} \end{pmatrix} \) with \( M_{p \times p}|_{(0,0)} = 0 \) and \( 1 \leq p \leq \text{mult}(X,0) \).

2. The stable equivalence (i.e. \( \text{Id} \oplus M_1 \sim \text{Id} \oplus M_2 \)) implies the ordinary local equivalence (\( M_1 \sim M_2 \)).

This can be proved just by row and column operations of linear algebra. From the algebraic point of view the first statement is the reduction to a minimal presentation of the module. The second is the uniqueness of such a reduction. Both are proved e.g. in [Yoshino-book, pg. 58].

The first statement is proved for the symmetric case in [Piontkowski2006, lemma 1.7]. In the first statement both bounds are sharp, regardless of \( d, p \) and \( \text{mult}(X,0) \).

2.3.2. Equivalence over \( (k^n,0) \) vs equivalence over \( (X,0) \).

Proposition 2.13. Two determinantal representations are equivalent over \( (k^n,0) \) iff they are equivalent over \( (X,0) \).

Proof. The direct statement is trivial. For the converse statement, let \( M_1 \equiv AM_2B \mod \det(M_1) \). One can assume both \( M_1 \) and \( M_2 \) vanish at the origin. Then \( M_1 = AM_2B + M_1M_2'Q \), for some matrix \( Q \) with entries in \( \mathcal{O}_{(k^n,0)} \). Hence \( M_1 = AM_2B(\text{Id} - M_2'Q)^{-1} \). Here \( (\text{Id} - M_2'Q) \) is invertible as \( M_2'Q|_0 = 0 \). \( \blacksquare \)

Note that here \( (X,0) \) is considered with its multiplicities, not just as a set. The equivalence
$\mathcal{M}_1 \sim \mathcal{M}_2$ over $(X_{\text{red}}, 0)$ does not imply that over $(X, 0)$. For example, both $\begin{pmatrix} x & 0 \\ 0 & x^3 \end{pmatrix}$ and $\begin{pmatrix} x^2 & 0 \\ 0 & x^2 \end{pmatrix}$ restrict to zero matrix over $\{x = 0\}$. But they are not equivalent when restricted to $\{x^4 = 0\}$.

2.3.3. The corank of the matrix. Let $\mathcal{M}$ be a determinantal representation of $(X, 0) \subset (k^n, 0)$, let $\mathcal{M}^\vee$ be the adjoint matrix of $\mathcal{M}$, so $\mathcal{M} \mathcal{M}^\vee = \det(\mathcal{M}) I_{d \times d}$.

The corank of $\mathcal{M}_{d \times d}$ at the point $pt \in k^n$ is the maximal number $p$ such that the determinant of any $(d - p + 1) \times (d - p + 1)$ minor of $\mathcal{M}$ vanishes at $pt$. The matrix $\mathcal{M}$ is non-degenerate on $(k^n, 0) \setminus (X, 0)$ and the corank at the point $pt \in X$ satisfies:

$$1 \leq \text{corank}(\mathcal{M}|_{pt}) \leq \text{mult}(X, pt)$$

(To see this, note that $\mathcal{M}$ is equivalent to $I \oplus \mathcal{N}$, where $\mathcal{N}|_0 = \emptyset$ and the corank of $\mathcal{M}$ equals the size of $\mathcal{N}$.) Hence any determinantal representation of a smooth hypersurface is maximally generated, cf. definition 1.1. For a reduced hypersurface the adjoint matrix $\mathcal{M}^\vee$ is not zero at smooth points of $X$. As $\mathcal{M}^\vee|_X \times \mathcal{M}|_X = \emptyset$ the rank of $\mathcal{M}^\vee$ at any smooth point of $X$ is 1. If $\text{corank}(\mathcal{M}|_{pt}) \geq 2$ then $\mathcal{M}^\vee|_{pt} = \emptyset$. Note that $\mathcal{M}^\vee = (\det \mathcal{M})^{d-2} \mathcal{M}$ and $\det \mathcal{M}^\vee = (\det \mathcal{M})^{d-1}$.

We have an immediate

**Proposition 2.14.** If the representation is maximally generated at the origin, i.e. $\text{rank}(\mathcal{M}_{d \times d}|_0) = d - \text{mult}(X, 0)$ then for the minimal form $\mathcal{M}_{p \times p}$: $p = \text{mult}(X, 0)$ and $\det(\text{jet}_1 \mathcal{M}_{p \times p}) \neq 0$ and $\det(\text{jet}_{p-1} \mathcal{M}^\vee_{p \times p}) \neq 0$.

2.3.4. Fitting ideals.

**Definition-Proposition 2.15.** The fitting ideal $I_k(\mathcal{M}) \subset O_(k^n, 0)$, is generated by all the $k \times k$ minors of $\mathcal{M}$. It is invariant under the local equivalence.

**Proof.** First consider the case $k = 1$, i.e. the ideal $I_1(\mathcal{M})$ is generated by the entries of $\mathcal{M}$. Then immediately: $I_1(\mathcal{A}M\mathcal{B}) \subset I_1(\mathcal{M})$. As $A, B$ are locally invertible the opposite inclusion holds too.

For arbitrary $k$ note that the wedge $\wedge^k \mathcal{M}$ is the collection of all the $k \times k$ minors, hence continue as for $k = 1$. ■

**Remark 2.16.** A trivial observation about the fitting ideals. Suppose $\mathcal{M}_{p \times p}$ is locally decomposable as $\begin{pmatrix} \mathcal{M}_{p_1 \times p_1} & 0 \\ 0 & \mathcal{M}_{p_2 \times p_2} \end{pmatrix}$. Then $I_1(\mathcal{M})$ is generated by at most $p^2 - 2p_1p_2$ elements.

Similarly, if $\mathcal{M}$ can be locally brought to an upper-block-triangular form then $I_1(\mathcal{M})$ is generated by at most $p^2 - p_1p_2$ elements.

2.4. Kernel modules. Given a local determinantal representation with $\mathcal{M}|_0 = \emptyset$, define the kernel module over $O_{(k^n, 0)}$ as follows. Let $E \subset O_{(k^n, 0)}^{\geq d}$ be the collection of all the kernel vectors, i.e. $v$ such that $\mathcal{M}v$ is divisible by $\det(\mathcal{M})$. 
Lemma 2.17. 1. $E$ is a module over $\mathcal{O}_{(k^n,0)}$, minimally generated by the columns of $\mathcal{M}$. It is supported on $(X,0)$.

2. Its restriction to the hypersurface (i.e. $E \otimes \mathcal{O}_{(X,0)}$) is a torsion-free module.

3. For the reduced hypersurface the module $E \otimes \mathcal{O}_{(X,0)}$ is free iff $\mathcal{M}$ is a $1 \times 1$ matrix.

Proof. 1. (This statement is also proved in [Yoshino-book, pg.56].) Let $E'$ be the $\mathcal{O}_{(k^n,0)}$ module generated by the columns of $\mathcal{M}$. Then $E' \subset E$. Let $v \in E$, so $Mv = det(M) \begin{pmatrix} a_1 \\ \vdots \\ a_p \end{pmatrix}$.

Let $v_1\.v_p$ be the columns of $\mathcal{M}$, then $M(v - \sum a_i v_i) = 0 \in \mathcal{O}_{(k^n,0)}$. As $\mathcal{M}$ is non-degenerate on $(k^n,0)$ we get $v \in E'$. Hence $E' = E$. By linear independence on $(k^n,0)$, the columns of $\mathcal{M}$ form a minimal set of generators.

2. The module $E_{(X,0)}$ is torsion-free as a submodule of the free module $\mathcal{O}_{(k^n,0)} \oplus \mathcal{O}_{(k^n,0)}$.

3. Suppose $E$ is free and $v_1\.v_p$ are the columns of $\mathcal{M}$, i.e. a minimal set of generators: $E \approx \oplus \mathcal{O}_{(X,0)} v_i$. Then $\mathcal{M} v_i |_{(X,0)} = 0$ implies the linear relations among $\{v_i\}$, contradicting the freeness of $E$, except in the case $\mathcal{M}_{1 \times 1}$.

By its definition the kernel module has a natural basis $\{v_1\.v_d\} = \text{the columns of } \mathcal{M}$. The embedded kernel with its basis determines the determinantal representation:

Property 2.18. 1. Let $\mathcal{M}_1, \mathcal{M}_2 \in \text{Mat}(d \times d, \mathcal{O}_{(k^n,0)})$ be two local determinantal representation of the same hypersurface and $E_1, E_2$ be the corresponding kernel modules. Then $\mathcal{M}_1 = \mathcal{M}_2 \quad \text{or} \quad \mathcal{M}_1 = A \mathcal{M}_2 \quad \text{or} \quad \mathcal{M}_1 = A \mathcal{M}_2 B$ (for $A, B$ locally invertible) iff $(E_1, \{v_1\.v_d\}) = (E_2, \{v_1\.v_d\}) \subset \mathcal{O}_{(k^n,0)} \oplus \mathcal{O}_{(k^n,0)}$ or $E_1 = E_2 \subset \mathcal{O}_{(k^n,0)}$ or $E_1 \approx E_2$.

2. In particular, if two kernel modules of the same hypersurface are abstractly isomorphic then their isomorphism is induced by a unique ambient automorphism of $\mathcal{O}_{(k^n,0)}^{\oplus d}$.

3. In particular: $\mathcal{M}$ is decomposable (or locally equivalent to an upper-block-triangular form) iff $E$ is a direct sum (or an extension).

Here in the first statement we mean the coincidence of the natural bases/the coincidence of the embedded modules/the abstract isomorphism of modules.

Proof. 1,2. The direction $\Rightarrow$ in all the statements is immediate. The converse follows from the uniqueness of minimal free resolution [Eisenbud-book].

As the kernel is spanned by the columns of $\mathcal{M}$ the statement is straightforward, except possibly for the last part: if $E_1 \approx E_2$ then $\mathcal{M}_1 = A \mathcal{M}_2 B$.

Let $\phi : E_1 \xrightarrow{\sim} E_2$ be an abstract isomorphism of modules, i.e. an $\mathcal{O}_{(X,0)}$-linear map. The module $E_1$ has a minimal free resolution. The isomorphism $\phi$ provides an additional minimal free resolution:

$$
0 \rightarrow E_1 \rightarrow \mathcal{O}_{(X,0)}^{\oplus d} \xrightarrow{\mathcal{M}_1} \mathcal{O}_{(X,0)}^{\oplus d} \rightarrow \cdots \\
\phi \downarrow \quad \quad \quad \psi \downarrow \\
0 \rightarrow E_2 \rightarrow \mathcal{O}_{(X,0)}^{\oplus d} \xrightarrow{\mathcal{M}_2} \mathcal{O}_{(X,0)}^{\oplus d} \rightarrow \cdots
$$

(19)

By the uniqueness of minimal free resolution, [Eisenbud-book, §20], we get the existence of $\psi \in Aut(\mathcal{O}_{(X,0)}^{\oplus d})$.

3. Suppose $E = E_1 \oplus E_2$, let $F_2 \xrightarrow{\mathcal{M}} F_1 \rightarrow E \rightarrow 0$ be the minimal resolution. Let
Proposition 2.20. Let \( F^{(i)}_2 \xrightarrow{\mathcal{M}} F^{(i)}_1 \to E_i \to 0 \) be the minimal resolutions of \( E_1, E_2 \). Consider their direct sum:

\[
F^{(1)}_2 \oplus F^{(2)}_2 \xrightarrow{\mathcal{M}_1 \oplus \mathcal{M}_2} F^{(1)}_1 \oplus F^{(2)}_1 \to E_1 \oplus E_2 = E \to 0
\]

This resolution of \( E \) is minimal. Indeed, by the decomposability assumption the number of generators of \( E \) is the sum of those of \( E_1, E_2 \), hence \( \text{rank}(F_1^{(i)}) = \text{rank}(F_2^{(i)}) + \text{rank}(F_1^{(i)}) \). Similarly, any linear relation between the generators of \( E \) (i.e. a syzygy) is the sum of relations for \( E_1 \) and \( E_2 \). Hence \( \text{rank}(F_2) = \text{rank}(F_2^{(2)}) + \text{rank}(F_2^{(1)}) \).

Finally, by the uniqueness of the minimal resolution we get that the two proposed resolutions of \( E \) are isomorphic, hence the statement.

Similarly for the extension of modules. ■

2.4.1. Auslander transpose of \( E \). In addition to the kernel of \( \mathcal{M} \), spanned by the columns of \( \mathcal{M}' \), one sometimes considers the left kernel: \( \text{tr}(E) = \text{Ker}(\mathcal{M}^T) \), called Auslander's transpose. It is spanned by the rows of \( \mathcal{M}' \). The propositions above hold for \( \text{tr}(E) \) with obvious alterations. The modules \( E, \text{tr}(E) \) are non-isomorphic in general. However (by the last proposition) \( E \) is decomposable or an extension iff \( \text{tr}(E) \) is. In addition, the minimal numbers of generators of \( E \) and \( \text{tr}(E) \) coincide.

2.5. Pulback of modules: liftings and restrictions. Suppose a map of germs \( (Y,0) \xrightarrow{i} (X,0) \) is given. In our context this will be either a finite modification \( (X',0) \to (X,0) \), as in the introduction, or the embedding \( (X_1,0) \hookrightarrow (X_1 \cup X_2,0) \), here \( (X_i,0) \subset (k^n,0) \) are hypersurfaces, possibly reducible/non-reduced, but with no common components.

2.5.1. Two ways to pullback. The ordinary pull-back of a module is \( E \otimes_{\mathcal{O}(Y,0)} \mathcal{O}(X,0) \), usually it has torsion. We always consider the torsion-free part: \( E \otimes_{\mathcal{O}(Y,0)}/\text{Torsion} \).

Example 2.19.
- Let \( (C,0) = \{x^p = y^q\} \subset (k^2,0) \) with \( (p,q) = 1 \) and \( q > p \). Consider the maximal ideal: \( \mathfrak{m} = \langle x,y \rangle \mathcal{O}_{(C,0)} \). Then the normalization \( \nu : (\tilde{C},0) \to (C,0) \) is \( t \to (x = t^q, y = t^p) \) and \( \nu^*(\mathfrak{m}) \) contains torsion. For example \( \nu^*(x) - t^{q-p} \nu^*(y) \) is annihilated by \( t^p = \nu^*(x) \in k\{t\} \).
  
  If we quotient by the torsion we get a free module: \( \nu^*(\mathfrak{m})/\text{Torsion} \approx \mathcal{O}_{(\tilde{C},0)}/(\nu^* y) \).

- Let \( E \) be the kernel module of \( \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \), where \( f_1, f_2 \in k\{x_1,..,x_n\} \) are mutually prime. Then

\[
E \otimes_{\mathcal{O}(k^n,0)/(f_1 f_2)} \mathcal{O}(k^n,0)/(f_1) = \mathcal{O}(k^n,0)/(f_1, s_1, s_2)/(f_2 s_2)
\]

while

\[
(E \otimes_{\mathcal{O}(k^n,0)/(f_1 f_2)} \mathcal{O}(k^n,0)/(f_1)) / \text{Torsion} = \mathcal{O}(k^n,0)/(f_1, s_1)
\]

In our case the kernel module is naturally embedded: \( E_{(X,0)} \xrightarrow{i} \mathcal{O}_{(X,0)}^{\oplus d} \). Hence the map \( \mathcal{O}_{(X,0)}^{\oplus d} \xrightarrow{j^*} \mathcal{O}_{(Y,0)} \) provides another version of pullback: \( j^*(i(E)) \subset \mathcal{O}_{(Y,0)}^{\oplus d} \). Here \( j^*(i(E)) \) is generated by the columns of \( \mathcal{M}'|_{(Y,0)} \). The two pullbacks are compatible:

**Proposition 2.20.** The \( \mathcal{O}_{(Y,0)} \) modules \( j^*(i(E)) \) and \( E \otimes_{\mathcal{O}(X,0)} \mathcal{O}_{(Y,0)}/\text{Torsion} \) are isomorphic.
Proof. Let $E_{(X,0)}$ be generated by $\{s_k\}_k$. Any element of $j^*(i(E))$ is presentable as $\sum a_k j^*(s_k)$, where $a_k \in \mathcal{O}_{(Y,0)}$. Thus a natural map is:

$$\phi : j^*(i(E)) \ni \sum a_k j^*(s_k) \to \left[ \sum a_k \otimes s_k \right] \in E \otimes_{\mathcal{O}_{(X,0)}} \mathcal{O}_{(Y,0)}/Torsion$$

This map is well defined. Indeed, if $\sum a_k j^*(s_k) = \sum b_k j^*(s_k)$ then

$$\phi\left(\sum a_k j^*(s_k)\right) - \phi\left(\sum b_k j^*(s_k)\right) = \left[ \sum (a_k - b_k) \otimes s_k \right] = 0 \in E \otimes_{\mathcal{O}_{(X,0)}} \mathcal{O}_{(Y,0)}/Torsion$$

The map is linear and surjective by construction. Injectivity: if $\sum a_k \otimes s_k = 0 \in E \otimes_{\mathcal{O}_{(X,0)}} \mathcal{O}_{(Y,0)}/Torsion$, then there exists a non-zero divisor $g \in \mathcal{O}_{(Y,0)}$ such that $g(\sum a_k \otimes s_k) = 0 \in E \otimes_{\mathcal{O}_{(X,0)}} \mathcal{O}_{(Y,0)} \subset \mathcal{O}^{ad}_{(Y,0)}$. But then, as $g$ is a non-zero divisor, $\sum a_k \otimes s_k = 0 \in E \otimes_{\mathcal{O}_{(X,0)}} \mathcal{O}_{(Y,0)}$. Hence the statement. 

2.5.2. Restriction to a component does not preserve Cohen-Macaulayness. Suppose the hypersurface is locally reducible: $(X, 0) = \cup (X_i, 0)$. Consider the restriction $E_i = E|_{(X_i,0)}/Torsion$, spanned by the column of $\mathcal{M}^\vee|_{(X_i,0)}$. By construction $E_i$ is torsion-free, in particular if $(X, 0)$ is a curve, i.e. $n = 2$, then $E_i$ is Cohen-Macaulay.

In higher dimensions $E_i$ is not necessarily Cohen-Macaulay. For example

$$\mathcal{M} = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}, \quad \mathcal{M}^\vee = \begin{pmatrix} z & -y \\ 0 & x \end{pmatrix}, \quad E = \mathcal{O}_{(X,0)}(s_1, s_2)/(xs_1, ys_1 + zs_2)$$

Here $E|_{x=0}$ is torsion free and isomorphic to the maximal ideal $\mathfrak{m} \subset k\{x, y\}$. So it is a non-free module over a regular ring, hence cannot be Cohen-Macaulay. On the other hand $E|_{x=0}/Torsion$ is free of rank 1.

In particular, in higher dimensions the minimal number of generators of $E_i$ can differ from that of $tr(E)|_{(X_i,0)}/Torsion$.

3. Decomposability of maximally generated determinantal representations

Suppose the hypersurface is locally reducible $(X, 0) = (X_1, 0) \cup (X_2, 0)$, where $(X_i, 0)$ can be further reducible, non-reduced but without common components. Let $E$ and $E_i = E|_{(X_i,0)}/Torsion$ be the kernels of determinantal representations. In this section we show that modules with large number of generators (e.g. maximally generated) tend to be extensions or even decomposable.

3.1. Preparations.

Lemma 3.1. Let $\mathcal{M}$ be an arbitrary square matrix with entries in $\mathcal{O}_{(k^n,0)}$.
1. Let $I \subset \mathcal{O}_{(k^n,0)}$ be a radical ideal that is a complete intersection. Suppose for any $i \times i$ minor of $\mathcal{M}$ one has: $\det(\mathcal{M}_{i\times i}) \in I^l$. Then for any $(i+1) \times (i+1)$ minor: $\det(\mathcal{M}_{(i+1)\times (i+1)}) \in I^{l+1}$.
2. In particular, suppose for any $i \times i$ minor $\mathcal{M}_{i\times i}$ the determinant is divisible by $g^l$ and $g$ has no multiple factors. Then, for any $(i+1) \times (i+1)$ minor $\mathcal{M}_{(i+1)\times (i+1)}$, the determinant is divisible by $g^{l+1}$.
3. Consider the hypersurface germ $\{ \prod_{i=1} f_i^{p_i} = 0 \} \subset (k^n,0)$ for $\{ f_i \}$ reduced. Let $\mathcal{M}$ be its determinantal representation, $\mathcal{M}|_0 = \emptyset$. Suppose it is maximally generated on the locus $\cap_{j \in J} p_j X_j$ for $J \subseteq \{ 1, \ldots, r \}$. Then all the entries of $\mathcal{M}^\vee$ belong to the power of the radical of the ideal generated by $\{ f_j \}_{j \in J}$: $\left( \text{Rad}(\{ f_j \}_{j \in J}) \right)^{\sum_{j \in J} p_j - 1}$. 
4. In particular, $\mathcal{M}$ is maximally generated at the smooth points of the reduced locus $X_{\text{red}}$ iff all the entries of $\mathcal{M}^\vee$ are divisible by $\prod f_i^{p_i-1}$.

Proof. 1. Let $\mathcal{M}_{(i+1)\times(i+1)}$ be any minor, let $\mathcal{M}_{(i+1)\times(i+1)}^\vee$ be its adjoint matrix. By the assumption, every element of this adjoint matrix lies in $I^i$. Hence

$$\left(\det \mathcal{M}_{(i+1)\times(i+1)}\right)^i = \det \left(\mathcal{M}_{(i+1)\times(i+1)}^\vee\right) \in I^{(i+1)}$$

Let $I = \langle g_1, \ldots, g_k \rangle$ be a minimal set of generators. Consider the projection $\mathcal{O}_{(k^n,0)} \to \mathcal{O}_{(k^n,0)}/\langle g_2, \ldots, g_k \rangle$. Let $\langle g_1 \rangle \subset \mathcal{O}_{(k^n,0)}/\langle g_2, \ldots, g_k \rangle$ be the image of $I$ under this projection.

So, the image of $\left(\det \mathcal{M}_{(i+1)\times(i+1)}\right)^i$ lies in $\langle g_1^{(i+1)} \rangle$. As $I$ is a complete intersection and $g_1$ is not a zero divisor one has:

$$\left(\frac{\det \mathcal{M}_{(i+1)\times(i+1)}}{g_1^i}\right)^i \in \langle g_1^i \rangle \subset \mathcal{O}_{(k^n,0)}/\langle g_2, \ldots, g_k \rangle$$

As $g_1$ has no multiple factors one gets: the image of $\det \mathcal{M}_{(i+1)\times(i+1)}$ in $\mathcal{O}_{(k^n,0)}/\langle g_2, \ldots, g_k \rangle$ is divisible by $g_1^{i+1}$.

Finally note that the same holds for any generator of $I$. For example, for any $k$-linear combination of the generators. Hence the statement.

2. This is just the case of principal ideal, $I = \langle g \rangle$, for $g$ without multiple factors.

3. Let $pt \in \bigcap_{j \in J} X_j$. By the assumption we have: $\text{corank}(\mathcal{M}|_{pt}) \geq \sum_{j \in J} p_j$. So the determinant of any $(d - \sum_{j \in J} p_j + 1) \times (d - \sum_{j \in J} p_j + 1)$ minor of $\mathcal{M}$ belongs to the radical of the ideal generated by $\{f_j\}_{j \in J}$. By the first statement we get: any entry of $\mathcal{M}^\vee$ belongs to $\left(\text{Rad}(\langle f_j \rangle_{j \in J})\right)_{\sum_{j \in J} p_j - 1}$.

4. By the assumption, for any smooth point $pt \in \{f_i = 0\} \setminus \{\prod_{j \neq i} f_j = 0\}$ we have: $\text{corank}(\mathcal{M}|_{pt}) = p_i$. So, any $(d - p_i + 1) \times (d - p_i + 1)$ minor of $\mathcal{M}$ is divisible by $f_i$ near $pt$. By the second statement we get: any entry of $\mathcal{M}^\vee$ is divisible by $f_i^{p_i-1}$ near $pt$. Taking the closure we get the divisibility everywhere. Going over all the $\{f_i\}_i$ we get the direct statement.

For the converse statement, let $pt$ be a smooth point of the reduced locus. Can assume it is the origin. Rectify the hypersurface $\{\prod f_i^{p_i} = 0\}$ locally near this point, so the corresponding local ring is $\mathcal{O}_{(k^n,0)}/x_1^{p_1}$. Restrict to the line $x_2 = 0 = \ldots = x_n$. Then, analyzing the zero-dimensional case, one gets: the corank of $\mathcal{M}$ at this point is $p_a$. Hence $\mathcal{M}$ is maximally generated at the smooth points of the reduced locus.

Remark 3.2. The conditions on the ideal in the proposition are relevant.

- If the ideal is not a complete intersections the statement does not hold. For example, let $\mathcal{M}_{3 \times 3}$ be a matrix of indeterminates, let $I_2(\mathcal{M})$ be the ideal generated by all the $2 \times 2$ minors. One can check that $I_2(\mathcal{M})$ is radical. Hence any $2 \times 2$ minor belongs to $I_2(\mathcal{M})$ but certainly $\det(\mathcal{M}) \not\in I_2(\mathcal{M})^2$.
- In the third statement it is important to take maximally generated near the point. For
example, $\mathcal{M} = \begin{pmatrix} y & x \\ 0 & y \end{pmatrix}$ is maximally generated at the origin but not near the origin. And not all the entries of $\mathcal{M}^\vee$ are divisible by $y$.

Now we prove the statement of §1.4.2

**Corollary 3.3.** Let $\mathcal{M}$ be a determinantal representation of $\prod f_\alpha^{p_\alpha}$. It can be augmented to a matrix factorization of $\prod f_\alpha$, (i.e. there exists $B$ such that $\mathcal{M}B = \prod f_\alpha I$) iff $\mathcal{M}$ is maximally generated at the smooth points of the reduced locus $\{ \prod f_\alpha = 0 \}$.

**Proof.** $\Leftarrow$ If $\mathcal{M}$ is maximally generated at smooth points of the reduced locus then by theorem 3.1 the adjoint matrix $\mathcal{M}^\vee$ is divisible by $\prod f_\alpha^{p_\alpha-1}$. Hence

$$\mathcal{M} \mathcal{M}^\vee \prod f_\alpha^{p_\alpha-1} = \prod f_\alpha I$$

$\Rightarrow$ Suppose $\mathcal{M}B = \prod f_\alpha I$, for some matrix $B$. Then $B = \prod f_\alpha^{-1} \mathcal{M}^\vee$, i.e. $\mathcal{M}^\vee$ is divisible by $\prod f_\alpha^{p_\alpha-1}$. Now, by proposition 3.1, $\mathcal{M}$ is maximally generated at the smooth points of the reduced locus. □

**Theorem 3.4.** Let $(X, 0) = \{ f_1, f_2 = 0 \} \subset (k^n, 0)$, where $f_i$ can be further reducible, non-reduced, but are relatively prime. A determinantal representation $\mathcal{M}$ of $(X, 0)$ decomposes as $\mathcal{M}_1 \oplus \mathcal{M}_2$ iff every element of $\mathcal{M}^\vee$ belongs to the ideal $\langle f_1, f_2 \rangle \subset \mathcal{O}_{(k^n, 0)}$.

This proof uses only linear algebra. A more conceptual proof is in §4.

**Proof.** $\Rightarrow$ is obvious.

$\Leftarrow$ By the assumption $\mathcal{M}^\vee = f_2 \mathcal{M}_1^\vee + f_1 \mathcal{M}_2^\vee$, where $\mathcal{M}_i$ are some (square) matrices with elements in $\mathcal{O}_{(k^n, 0)}$. Multiply the equality by $\mathcal{M}$, then one has:

$$f_1 f_2 I = \mathcal{M} \mathcal{M}^\vee = f_2 \mathcal{M} \mathcal{M}_1^\vee + f_1 \mathcal{M} \mathcal{M}_2^\vee$$

As $f_1, f_2$ are relatively prime, $\mathcal{M} \mathcal{M}_i^\vee$ is divisible by $f_i$. Therefore one can define the matrices $\{ A_i \}$, $\{ B_i \}$ by $f_i A_i := \mathcal{M} \mathcal{M}_i^\vee$ and $f_i B_i := \mathcal{M}_i^\vee \mathcal{M}$. By definition: $A_1 + A_2 = I$ and $B_1 + B_2 = I$. We prove that in fact $A_1 \oplus A_2 = I$ and $B_1 \oplus B_2 = I$. The key ingredient is the identity:

$$\mathcal{M}_j^\vee f_i A_i = \mathcal{M}_j^\vee f_i \mathcal{M}_i^\vee = f_j B_j \mathcal{M}_j^\vee$$

Let $m_1, m_2$ be the multiplicities of $f_1, f_2$ at the origin. It follows that $\mathcal{M}_j^\vee A_i$ is divisible by $f_j$ and thus $jet_{m_j-1}(\mathcal{M}_j^\vee A_i) = 0$ for $i \neq j$. Hence, due to the orders of $\mathcal{M}_j^\vee \mathcal{M}$ we get: $jet_{m_j}(\mathcal{M}_j^\vee A_i) = jet_{m_j} (f_j A_j A_i) = 0$, implying:

$$jet_0(A_j) jet_0(A_i) = jet_0(A_j A_i) \text{ for } i \neq j, \quad \text{and } \sum jet_0(A_i) = I \Rightarrow I = \oplus jet_0(A_i)$$

The equivalence $\mathcal{M} \rightarrow U \mathcal{M} X$ results in: $A_i \rightarrow U A_i U^{-1}$ and $B_j \rightarrow X B_j X^{-1}$. So, by the conjugation by (constant) matrices can assume the block form:

$$jet_0(A_1) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad jet_0(A_2) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

Apply further conjugation to remove the terms of $A_i$ in the columns of the i'th block to get:

$$A_1 = \begin{pmatrix} I & * \\ 0 & * \end{pmatrix}, \quad A_2 = \begin{pmatrix} * & 0 \\ * & I \end{pmatrix}$$
Finally, use $A_1 + A_2 = I$ to obtain $A_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$.

Do the same procedure for $B_i$’s, this keeps $A_i$’s intact. Now use the original definition, to write: $M^\vee_i = \frac{d_i}{f_i} M^\vee A_i$ and $M^\vee = \frac{d_i}{f_i} B_i M^\vee$. This gives:

\[(34) \quad M^\vee = f_2 M^\vee_1 \oplus f_1 M^\vee_2\]

\[
\]

3.2. Modules with many generators tend to be extensions. Let $M_{d\times d}$ be a local determinantal representation of a reducible hypersurface, $\{\det(M) = 0\} = (X_1 \cup X_2, 0)$. Here $(X_1, 0) = \{f_i = 0\}$ can be further reducible, non-reduced, but with no common components. As always, we assume $\mathcal{M}|_0 = \emptyset$, i.e. the kernel $E$ is minimally generated by $d$ elements.

Suppose the restriction $E_i = E|_{(X_i, 0)}/\text{Torsion}$, (cf.§2.5), is minimally generated by $d(E_i)$ elements. In other words, the maximal number of the columns of $M^\vee|_{(X_i, 0)}$, none of which belongs to the $\mathcal{O}_{(X_i, 0)}$ span of the others, is $d(E_i)$. Similarly, let $d(tr(E)_i)$ be the minimal number of generators for the restrictions of the left kernels $tr(E)_i$, see §2.4.1.

**Proposition 3.5.** 0. $d(E_i) > 0$ and $\max(d(E_1), d(E_2)) \leq d \leq d(E_1) + d(E_2)$. Similarly for $d(tr(E)_i)$.

1. $M^\vee \sim \begin{pmatrix} f_i A_1 & * \\ 0 & f_2 A_2 \end{pmatrix}$, where $0$ is a $(d - d(\text{tr}(E)_2)) \times (d - d(E_1))$ block of zeros and $A_i$ some matrices with values in $\mathcal{O}_{(X, 0)}$. Similarly, $M^\vee \sim \begin{pmatrix} f_2 \tilde{A}_2 & * \\ 0 & f_1 \tilde{A}_1 \end{pmatrix}$, where $0$ is a $(d - d(\text{tr}(E)_1)) \times (d - d(E_2))$ block of zeros.

2. In particular, $E$ is an extension (i.e. $M$ is equivalent to an upper block triangular) iff $d = d(\text{tr}(E)_1) + d(E_2)$ or $d = d(E_1) + d(\text{tr}(E)_2)$.

**Proof. 0, 1.** The inequalities $0 \leq d(E_i) \leq d$ are obvious. Consider $M^\vee|_{(X_i, 0)}$. By the assumption the module of the columns of $M^\vee|_{(X_i, 0)}$ is generated by $d_i$ elements, hence one can assume that the first $(d - d(E_1))$ columns of $M^\vee$ are divisible by $f_1$. Similarly, the module of the rows of $M^\vee|_{(X_2, 0)}$ is generated by $d(E_2)$ elements. Hence one can assume that the last $(d - d(E_2))$ rows of $M^\vee$ are divisible by $f_2$. As $f_1, f_2$ are relatively prime, one has:

\[(35) \quad M^\vee \sim \begin{pmatrix} f_i A_1 & * \\ f_1 f_2 (...) & f_2 A_2 \end{pmatrix}\]

Hence $M^\vee|_{(X, 0)} \sim \begin{pmatrix} f_i A_1 & * \\ 0 & f_2 A_2 \end{pmatrix}$. Now, by proposition 2.13, we have $M^\vee \sim \begin{pmatrix} f_2 \tilde{A}_2 & * \\ 0 & f_1 \tilde{A}_1 \end{pmatrix}$ over $(k^n, 0)$.

Similarly one obtains $M^\vee \sim \begin{pmatrix} f_i A_1 & * \\ 0 & f_2 A_2 \end{pmatrix}$.

Finally, if $d_1 + d_2 < d$ then from the presentation above one gets: $\det(M^\vee) \equiv 0$ on $(k^n, 0)$.

2. The direction $\Rightarrow$ is obvious. For the converse statement, by the first part we can assume $M^\vee = \begin{pmatrix} f_i (...) & * \\ 0 & f_2 (...) \end{pmatrix}$, where $0$ is a $(d - d(\text{tr}(E)_2)) \times (d - d(E_1))$ block of zeros. Hence if $d = d(E_1) + d(\text{tr}(E)_2)$ then $M = \begin{pmatrix} M_1 & * \\ 0 & M_2 \end{pmatrix}$ with $\det(M_i) = f_i$. ■
Remark 3.6. • In general a matrix is not equivalent to an upper-block-triangular in two ways. Namely, \( \mathcal{M} \sim \begin{pmatrix} \mathcal{M}_1 & * \\ 0 & \mathcal{M}_2 \end{pmatrix} \), with \( \mathcal{M}_i \) a determinantal representation of \((X_i, 0)\), does not imply \( \mathcal{M} \sim \begin{pmatrix} \tilde{\mathcal{M}}_2 & * \\ 0 & \tilde{\mathcal{M}}_1 \end{pmatrix} \), with \( \tilde{\mathcal{M}}_i \) a determinantal representation of \((X_i, 0)\). For example \( \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \) is not equivalent to \( \begin{pmatrix} z & * \\ 0 & x \end{pmatrix} \). For curves however this property does exist, as is shown in theorem 3.7.

• The assumption on the number of generators is essential. Consider the matrix

\[
\mathcal{M} = \begin{pmatrix} x^p - y^p & x^p y^{p-1} \\ x^p + y^p & x^p \end{pmatrix}, \quad p > 2
\]

This determinantal representation of an ordinary multiple point is not locally equivalent to an upper triangular form. Note that \( I_1(\mathcal{M}) \) is minimally generated by 4 elements, apply remark 2.16.

• In general maximally generated determinantal representations are indecomposable. Consider

\[
\mathcal{M} = \begin{pmatrix} y + x^q & x^q \\ 0 & y - x^q \end{pmatrix}, \quad 0 < q < l
\]

To see that \( \mathcal{M} \) is not locally decomposable note that \( I_1(\mathcal{M}) = \langle y, x^q \rangle \). If \( \mathcal{M} \sim \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \) then \( f_i \) are the equations of branches of the curve \( y^2 = x^q \) and \( x^q \notin \langle f_1, f_2 \rangle \).

• Even worse, being maximally generated at the origin does not imply equivalence to an upper-block-triangular. For example,

\[
\mathcal{M} = \begin{pmatrix} x & -y & 0 \\ z & y & 0 \\ 0 & z & x \end{pmatrix}
\]

is a determinantal representation of \( \{xyz = 0\} \subset (k^3, 0) \). The representation is maximally generated at the origin but is not equivalent to an upper-block-triangular. If it were, the corank of \( \mathcal{M} \) would be at least 2 on one of the intersections \( x = 0 = y \) or \( x = 0 = z \) or \( y = 0 = z \). But \( \text{corank}(\mathcal{M}) = 1 \) on all the intersections.

3.3. The case of curves. For \( n = 2 \) various strong criteria are possible.

3.3.1. The criterion for being an extension.

Theorem 3.7. Let \((C, 0) = (C_1, 0) \cup (C_2, 0) \subset (k^2, 0)\), with \((C_i, 0)\) possibly further reducible, non-reduced but with no common components. Let \( \mathcal{M} \) be a determinantal representation of \((C, 0)\).

1. \( \mathcal{M} \sim \begin{pmatrix} \mathcal{M}_1 & * \\ 0 & \mathcal{M}_2 \end{pmatrix} \) iff \( \mathcal{M} \sim \begin{pmatrix} \tilde{\mathcal{M}}_2 & * \\ 0 & \tilde{\mathcal{M}}_1 \end{pmatrix} \), where \( \mathcal{M}_i, \tilde{\mathcal{M}}_i \) are some determinantal representations of \((C_i, 0)\).

2. If \( \mathcal{M} \) is maximally generated at the origin then it is equivalent to an upper-block-triangular matrix.

Proof. 1. Let \( E_i \) and \( \text{tr}(E)_i \) be the restrictions of \( \text{Ker}(\mathcal{M}) \) and \( \text{Ker}(\mathcal{M}^T) \) to the local components, cf. §2.5. Recall from §2.4.1 that for curves both \( E_i \) and \( \text{tr}(E)_i \) are Cohen-Macaulay, i.e. torsion-free, hence their minimal number of generators coincide.
Suppose \( M \sim \begin{pmatrix} M_1 & \ast \\ \ast & M_2 \end{pmatrix} \), so \( M^\vee \sim \begin{pmatrix} f_2M^\vee_1 & \ast \\ \ast & f_1M^\vee_1 \end{pmatrix} \). Here \( M_i \) is a \( p_i \times p_i \) matrix. We get: \( tr(E)_1 \) is minimally generated by \( p_1 \) elements. By the remark above: \( E_1 \) is minimally generated by \( p_1 \) elements. Hence, the span of the columns of \( M^\vee|_{(X_1,0)} \) is generated by \( f_1 \). And this form can be achieved by operations on columns only.

Similarly, \( E_2 \) is generated by \( p_2 \) elements, hence the same for \( tr(E)_2 \). Thus, after some row operations, one can assume that the last \( p_2 \) rows on \( M^\vee \) are divisible by \( f_2 \).

Therefore \( M_{(C,0)} \) has a \( p_2 \times p_1 \) block of zeros, hence by property 2.13 the matrix \( M \) has a block of zeros too.

2. Again, as the restrictions \( E_i \) are Cohen-Macaulay, they are generated by at most \( mult(C_i,0) \) elements. Hence, the conditions of theorem 3.5 are satisfied: \( d = mult(C,0) = d_1 + d_2 \) and the module is an extension. 

Remark 3.8. In the first statement of the theorem, the matrices \( M_i \) are in general not equivalent to \( M_i \) or to \( M_i^T \). For example, consider a determinantal representation of \( y(y^2 - x^{k+l}) \):

\[
\begin{pmatrix}
y & 0 & x \\
0 & y & x_l \\
0 & x^k & y
\end{pmatrix} \sim \begin{pmatrix}
y & 0 & x \\
-x^{k-1}y & y & 0 \\
0 & 0 & x^k
\end{pmatrix} \sim \begin{pmatrix}
y & 0 & x \\
0 & y & 0 \\
x^{k+1} & y & x^k
\end{pmatrix} \sim \begin{pmatrix}
y & 0 & x \\
0 & x^{k+1} & 0 \\
x^k & 0 & y
\end{pmatrix}
\]

And \( \begin{pmatrix}
y & x^k \\
x & y
\end{pmatrix} \sim \begin{pmatrix}
y & x^{k+1} \\
x & y
\end{pmatrix} \).

3.3.2. Decomposability in the non-tangent case. If the components of the curve are non-tangent then the determinantal representations tend to be decomposable. Recall the tangential decomposition \( (C,0) = U \cup \alpha(C_\alpha,0) \) from §2.2.1. Let \( mult(C,0) = p \) and \( mult(C_\alpha,0) = p_\alpha \). As always we assume \( M_{(0)} = 0 \).

Theorem 3.9. Let \( M_{p \times p} \) be a determinantal representation of the plane curve \( (C,0) \), maximally generated at the origin. Corresponding to the tangential decomposition of \( (C,0) \), the matrix \( M \) is locally equivalent to a block diagonal: \( M_{p \times p} \sim \bigoplus \alpha M_{p_\alpha \times p_\alpha} \). Here \( \{ M_{p_\alpha \times p_\alpha} \} \) are maximally generated determinantal representations of \( \{(C_\alpha,0)\} \).

Proof. The theorem states that there exists a solution to the problem:

\[
(\mathbb{I} + A) M (\mathbb{I} + B) = \begin{pmatrix} M_{m_1 \times m_1} & 0 & \ldots & 0 \\
0 & \ldots & \ldots & 0 \\
0 & \ldots & M_{m_k \times m_k} \end{pmatrix}, \quad A\mid_{(0,0)} = B\mid_{(0,0)}, \quad \det(M_{m_\alpha \times m_\alpha}) = f_\alpha
\]

for the unknowns \( A, B, \{ M_{m_\alpha \times m_\alpha} \}_\alpha \). Using Artin’s and Pfister-Popescu theorems, §1.5.1, it is enough to prove that the solution exists in \( k[x,y]/\mathfrak{m}^N \) for any \( N \).

By the assumption \( M \) vanishes at the origin, while the property 2.12 gives: \( \det(jet_1 M) \neq 0 \).

Step 1. By \( GL(p,k) \times GL(p,k) \) bring \( jet_1(M) \) to the Jordan form. For that, let \( jet_1(M) = xP + yQ \) with \( P, Q \) constant matrices. Assume that the curve is not tangent to coordinate axes. Hence \( P \) and \( Q \) are of full rank. By \( GL(p,k) \times GL(p,k) \) bring \( P \) to \( \mathbb{I} \). The remaining transformation of \( GL(p,k) \times GL(p,k) \) preserving \( P = \mathbb{I} \) is the conjugation: \( M \to UMU^{-1} \).
Hence $Q$ can be assumed in the Jordan form.

**Step 2.** The matrix $\mathcal{M}$ is naturally subdivided into the blocks $B_{ij}$, which are $p_i \times p_j$ rectangles (corresponding to the fixed eigenvalues of $jet_1(\mathcal{M})$). We should remove the off-diagonal blocks, $B_{ij}$ for $i \neq j$. We do this by induction, at the $N$th step removing all the terms whose order is $\leq N$.

Let $N = \min_{i \neq j}(\text{ord} M_{ij})$ for $(ij)$ not in a diagonal block (thus $N > 1$). Consider $jet_N(\mathcal{M})$, i.e. truncate all the monomials whose total degree is bigger than $N$. Suppose the block $B_{12} \subset jet_N(\mathcal{M})$ is non-zero, i.e. there is an entry of order $N$.

As $l_1, l_2$ are linearly independent, by a linear change of coordinates in $(k^2, 0)$ can assume $l_1 = x$, $l_2 = y$. Decompose: $B_{12} = xT + yR$, where $T, R$ are $p_1 \times p_2$ matrices, with $\text{ord}(T) \geq N - 1$ and $\text{ord}(R) \geq N - 1$. From the last row of $B_{12}$ subtract the rows

$$jet_N M_{p_1+1,*}, jet_N M_{p_1+2,*}, \ldots, jet_N M_{p_1+p_2,*}$$

of $jet_N(\mathcal{M})$ multiplied by $R_{p_11}, R_{p_12}..R_{p_1p_2}$. By the assumptions this does not change $jet_N(\mathcal{M})$ outside the block $B_{12}$. After this procedure every entry of the last row of $B_{12}$ is divisible by $x$. Thus subtract from the columns of $B_{12}$ the column $jet_N M_{*p_1}$ multiplied by the appropriate factors.

Now the last row of $B_{12}$ consists of zeros, while $jet_N(\mathcal{M})$ is unchanged outside $B_{12}$. Do the same procedure for the row $jet_N M_{p_1-1,*}$ of $B_{12}$ (using the rows $jet_N M_{p_1+1,*}, jet_N M_{p_1+2,*}, \ldots, jet_N M_{p_1+p_2,*}$ and the column $jet_N M_{*p_1-1}$). And so on.

**Step 3.** After the last step one has the refined matrix $jet_N(\mathcal{M}')$ which coincides with $jet_N(\mathcal{M})$ outside the block $B_{12}$ and has zeros inside this block. Do the same thing for all other (off-diagonal) blocks. Then one has a block diagonal matrix $jet_N(\mathcal{M}')$.

Now repeat all the computation starting from non-truncated version $\mathcal{M}$. This results in the increase of $N$. Continue by induction. Thus, for each $N$ can bring $\mathcal{M}$ to such a form that the $jet_N(\mathcal{M})$ is block diagonal. Then by the initial remark the statement follows.

3.3.3. **The case of multiple curve.** The results above reduce the decomposability questions to determinantal representations of a multiple curve $(rC, 0) \subset (k^n, 0)$, where $(C, 0)$ is locally irreducible and reduced.

**Theorem 3.10.** Let $(rC, 0) \subset (k^2, 0)$, where $(C, 0)$ is a locally irreducible, reduced plane curve.

1. Let $\mathcal{M}$ be its determinantal representation maximally generated at the origin. Then $\mathcal{M}$ is equivalent to an upper-block-triangular matrix, the blocks on the diagonal are determinantal representations of $(C, 0)$.

2. Let $\mathcal{M}$ be a determinantal representation maximally generated on the punctured neighborhood of the origin. Then $\mathcal{M}$ is totally decomposable: $\mathcal{M} = \bigoplus M_i$ where $M_i$ is a determinantal representation of $(C, 0)$.

**Proof.** Let $(\tilde{C}, 0) \rightarrow (C, 0) = \{f = 0\}$ be the normalization of the reduced curve. It defines valuation on $\mathcal{O}_{(C, 0)}$ by $\text{val}(g) := \text{ord} u^*(g)$, for $g \in \mathcal{O}_{(C, 0)}$. In the non-reduced case the valuation on $\mathcal{O}_{(rC, 0)}$ is defined by the pair:

$$\text{val}(g) := (\text{ord}_f g, \text{val}(g/\text{ord}_fg))$$
Here \( \text{ord}_f(g) \) is the maximal \( k \) such that \( g \) is divisible by \( f^k \). In other words: \( g \in f^kO_{(r,C,0)} \) but \( g \not\in f^{k+1}O_{(r,C,0)} \).

The natural order on pairs for this valuation is defined by
\[
(a_1,a_2) < (b_1,b_2) \text{ if } a_1 < b_1 \text{ or } \left( \begin{array}{c} a_1 = b_1 \\ a_2 < b_2 \end{array} \right)
\]

Let \( p = \text{mult}(C,0) \), so \( M \) is a \( pr \times pr \) matrix.

1. Compare the valuations of the entries of \( M^\vee \). After a permutation of rows and columns we can assume that \( M^\vee_{1,1} \) has the minimal valuation in \( M^\vee \) and in the first row and column the valuations are increasing:
\[
\text{val}(M^\vee_{1,1}) < \text{val}(M^\vee_{1,2}) < \ldots < \text{val}(M^\vee_{1,pr}) \quad \text{val}(M^\vee_{1,1}) < \text{val}(M^\vee_{2,1}) < \ldots < \text{val}(M^\vee_{pr,1})
\]

Note that \( M^\vee_{1,1} \not\in O_{(r,C,0)} \). Note that \( O_{(C,0)} \) is generated as a \( O_{(r,C,0)} \) module by \( p \) elements. Therefore we can assume (possibly after a subtraction of columns) that the elements \( M^\vee_{1,j} \) are divisible by \( f^{\lfloor \frac{1}{p} \rfloor} \). Similarly, after some row subtraction we can assume that the elements \( M^\vee_{j,1} \) are divisible by \( f^{\lfloor \frac{1}{p} \rfloor} \).

Finally, recall that \( \text{rank}(M^\vee_{|r,C,0)}) \leq 1 \), i.e. any two rows or columns are proportional. Hence, in the chosen basis the matrix is:
\[
M^\vee = \begin{pmatrix}
* & * & * & * \\
* & * & 0 & \\
.. & .. & .. & . \\
* & 0 & .. & 0
\end{pmatrix}
\]

i.e. is equivalent to an upper-block-triangular.

2. By Proposition 3.1 the adjoint matrix \( M^\vee \) is divisible by \( f^{r-1} \). Let \( N^\vee_{p \times pr} \) be the submatrix of \( \tilde{M}^\vee \) formed by lower \( p \) rows. Consider the module over \( O_{(r,C,0)} \) spanned by the columns of \( N^\vee \). This module is generated by \( p \) elements. This can be seen, for example, by checking the valuation of the columns, by \( \tilde{C} \rightarrow (C,0) \).

Hence the matrix \( \tilde{M}^\vee \) is equivalent to the upper-block-triangular matrix, with the zero block \( \emptyset_{p \times (r-1)p} \). Hence \( \hat{M}^\vee \) is equivalent to the upper-block-triangular matrix. Assume \( M^\vee \) in this form. Now consider the submatrix of \( \tilde{M}^\vee \) formed by the last \( p \) columns. By the argument as above one gets: \( M^\vee \) is equivalent to a block diagonal, with blocks \( p \times p \) and \( (r-1)p \times (r-1)p \).

Continue in the same way to get the statement. \( \Box \)

3.4. Higher dimensional case. In some cases we have decomposability according to the tangential decomposition of a reduced hypersurface.

**Theorem 3.11.** Let \( n \geq 3 \) and \( (X,0) = (X_1,0) \cup (X_2,0) \).

1. If the intersection \( (X_1,0) \cap (X_2,0) \) is reduced, i.e. the components are reduced and generically transverse, then any determinantal representation that is maximally generated on the smooth points of \( (X_1,0) \cap (X_2,0) \) is decomposable.
2. More generally, if the projectivized tangent cones, \( \mathbb{P}T_{(X_1,0)}, \mathbb{P}T_{(X_2,0)} \subset \mathbb{P}^n(k) \), intersect transversally then any determinantal representation of \( (X,0) \) that is maximally generated near the origin is decomposable.

**Proof.** 1. By part 3 of proposition 3.1 every entry of \( M^\vee \) belongs to \( \langle f_1, f_2 \rangle \subset O_{(k^n,0)} \). Then the decomposability follows by proposition 3.4.
2. As $\mathcal{M}$ is maximally generated near the origin, for any point $pt \in X_1 \cap X_2$ the order of vanishing of any element of the adjoint matrix satisfies: $ord_{pt}\mathcal{M}_{ij}^\vee \geq mult(X, pt) − 1$. We claim that this implies $\mathcal{M}^\vee_{ij} \in \langle f_1, f_2 \rangle \subset \mathcal{O}_{(k^n, 0)}$, hence as above $\mathcal{M}$ is decomposable (by proposition 3.4).

So, we should prove the following statement: given $f_1, f_2, h \in \mathcal{O}_{(k^n, 0)}$, such that the projectivized tangent cones $\mathbb{P}T_{(f_1 = 0)}$, $\mathbb{P}T_{(f_2 = 0)} \subset \mathbb{P}(k^n)$ intersect transversely and for any point $pt \in (k^n, 0)$: $ord_{pt}(h) \geq ord_{pt}(f_1) + ord_{pt}(f_2) - 1$. Then $h \in \langle f_1, f_2 \rangle \subset \mathcal{O}_{(k^n, 0)}$.

Let $Z = \{f_1 = 0 = f_2\}$, then at any point of $Z$ the order of $h$ is at least one, i.e. $h$ vanishes on $Z$, i.e. $h \in \text{Rad}(f_1, f_2) \subset \mathcal{O}_{(k^n, 0)}$. We should prove that $h$ belongs to the ideal $\langle f_1, f_2 \rangle$ itself. The proof is by induction on $n$.

Suppose $n = 2$, i.e. $\{f_1 = 0\}$ are curve singularities whose tangent cones intersect at the origin only. Let $H^0(\mathcal{O}_{(k^2, 0)}(d))$ be the vector space of all the homogeneous polynomials in two variables of degree $d$. Let $H^0(\mathcal{O}_{(k^2, 0)}(-f_1)(d)) \subset H^0(\mathcal{O}_{(k^2, 0)}(d))$ be the subspace of all the polynomials divisible by $f_1$. Then we have the exact sequence:

\[
0 \to H^0(\mathcal{O}_{(k^2, 0)}(-f_1 - f_2)(d)) \to H^0(\mathcal{O}_{(k^2, 0)}(-f_1)(d)) \oplus H^0(\mathcal{O}_{(k^2, 0)}(-f_2)(d)) \to H^0(\mathcal{O}_{(k^2, 0)}(d))
\]

By the assumption $ord_0 h \geq ord_0(f_1) + ord_0(f_2) - 1$. Hence it is enough to show that the map $H^0(\mathcal{O}_{(k^2, 0)}(-f_1)(d)) \oplus H^0(\mathcal{O}_{(k^2, 0)}(-f_2)(d)) \to H^0(\mathcal{O}_{(k^2, 0)}(d))$ is surjective for $d \geq ord_0(f_1) + ord_0(f_2) - 1$. This is checked by computing the dimensions:

\[
\begin{align*}
dim H^0(\mathcal{O}_{(k^2, 0)}(d)) &= d + 1, \\
dim H^0(\mathcal{O}_{(k^2, 0)}(-f_1)(d)) &= d - d_1 + 1, \\
dim H^0(\mathcal{O}_{(k^2, 0)}(-f_2)(d)) &= d - d_2 + 1
\end{align*}
\]

here $d_1 = ord(f_1)$. Hence for $n = 2$ we get: $h \in \langle f_1, f_2 \rangle \subset \mathcal{O}_{(k^n, 0)}$.

Suppose the statement has been proven for the case of $(n - 1)$ variables. Let $pt \in Z = X_1 \cap X_2$, let $(k^{n-1}, pt) \subset (k^n, pt)$ be a hyperplane transversal to the tangent cones $T_{(X_1, pt)}$, $T_{(X_2, pt)}$. Then the restrictions $f_1|_{(k^{n-1}, pt)}$, $f_2|_{(k^{n-1}, pt)}$, $h|_{(k^{n-1}, pt)}$ satisfy the assumptions of the statement. Hence by the induction assumption: $h = a_1 f_1 + a_2 f_2 + lh'$, where $a_1, a_2, h'$ are some regular functions, while $l$ is the locally defining equation of the hyperplane $(k^{n-1}, pt)$. Note that $h'$ itself satisfies the assumptions of the statement on the punctured neighborhood of $pt \in Z$. As the vanishing order does not increase under small deformations we get that $h'$ satisfies the assumptions of the statement on the whole neighborhood of $pt \in Z$. Then reiterating procedure we get $h = a'_1 f_1 + a'_2 f_2 + l^2 h''$ etc. As $h$ is locally analytic, this process stops after a finite number of steps, giving $h \in \langle f_1, f_2 \rangle \subset \mathcal{O}_{(k^n, 0)}$. ■

Remark 3.12. Note that for curves (theorem 3.9) we ask for maximally generated at the origin, while in higher dimensions (theorem 3.11) we ask for maximally generated on an open set near the origin. This is essential. For example $\mathcal{M} = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ is maximally generated at the origin. And the hypersurface $\{\det(\mathcal{M}) = xz = 0\}$ consists of two transverse hyperplanes. But the determinantal representation is not maximally generated near the singular point and is indecomposable. A similar example is in remark 3.6.

The theorem implies an immediate

Corollary 3.13. Let $(X, 0) = \cup_i (X_i, 0)$ be the reduced union of pairwise non-tangent smooth hypersurfaces, e.g. an arrangement of hyperplanes. Then $(X, 0)$ has the unique determinantal representation maximally generated on the neighborhood of $0 \in k^n$: the diagonal matrix.

3.5. Limits of kernel fibres. One often imposes the following conditions of linear independence. Let $(X, 0) = \cup (X_i, 0)$ be reduced, $E_X \subset X \times k^d$ and $E_i = E|_{(X_i, 0)}/\text{Torsion}$. Let
Consider the topological closure of the embedded line bundle: \( \overline{E_{Y_i}} \subset X_i \times k^d \). Denote its fibre at the origin by \( \overline{E_{Y_i}|_0} \).

**Proposition 3.14.** Let \( (X,0) = \bigcup (X_i,0) \subset (k^n,0) \) be a collection of reduced, smooth hypersurfaces. The determinantal representation \( M_{p \times p} \) is completely decomposable iff the fibres \( \{ \overline{E_{Y_i}|_0} \} \) are one dimensional vector subspaces of \( k^d \) that are linearly independent: \( \text{Span}(\bigcup \overline{E_{Y_i}|_0}) = \oplus \overline{E_{Y_i}|_0} \).

**Proof.** \( \Rightarrow \) is obvious.

\( \Leftarrow \) By the assumption \( M|_0 = 0 \) hence the corank of \( M|_0 \) equals the number of (smooth) components, i.e. the multiplicity of the singularity. By continuity of the fibres (embedded vector spaces) this happens also at the neighboring points. Hence \( M \) is maximally generated near the origin.

Now, fix \( \overline{E_{Y_1}|_0} = (1,0,...,0) \in k^d \) and “rectify” the fibres locally. Namely, after a \( GL(k^d) \) transformation one can assume: \( \overline{E_{Y_1}} = (1,0,...,0) \) over some neighborhood of the origin, while \( \overline{E_{Y_i > 1}} \subset \{ z_1 = 0 \} \subset k^d \) near the origin. Hence, in this basis

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & * & * & * \\
\vdots & \vdots & \vdots & \vdots \\
0 & * & * & *
\end{pmatrix}
\]

(48)

Repeat for other components. □

**Remark 3.15.** It is not clear whether the conditions can be weakened.

- The fibre at the origin, \( \overline{E_{Y_i}|_0} \), can be not a one-dimensional vector space, cf. the last example in remark 3.6.
- The smoothness of the components in the statement is important. For example, consider \( M = \begin{pmatrix} x^a & y^{d+1} \\ y^c & x^b y \end{pmatrix} \), a determinantal representation of \( y(x^{a+b} - y^{c+d}) \) for \( (a+b, c+d) = 1 \). Assume also \( c > 1 \) and \( d > 0 \). This determinantal representation is not equivalent to an upper-triangular. Otherwise one would have \( I_1(M) \ni y \).

On the other hand the limits of the kernel sections are linearly independent. \( \begin{pmatrix} x^b y & -y^{d+1} \\ -y^c & x^a \end{pmatrix} \). So, on \( y = 0 \) the kernel is generated by \( \begin{pmatrix} 0 \\ x^a \end{pmatrix} \), whose limit is \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). On \( x^{a+b} = y^{c+d} \) both columns of \( M^\vee \) are non-zero, but linearly dependent. So, for \( a > d + 1 \) or \( c - 1 > b \) their (normalized) limit at the origin is \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).

- It is important to ask for the common linear independence of the fibres, not just the pairwise linear dependence. Recall the first example in remark 3.6. There the branches are smooth and the limits of any two fibres are independent. But altogether they are not linearly independent.

## 4. Saturated determinantal representations

Let \( (X',0) \xrightarrow{\nu} (X,0) \) be a finite modification (cf. introduction). Given a torsion-free module \( E_{(X,0)} \), let \( \nu^* E_{(X,0)}/\text{Torsion} \) be its torsion free pull-back (§2.5). Then \( E \) is naturally embedded into the pushforward \( E \subset \nu_*(\nu^* E/\text{Torsion}) \).
Definition 4.1. The module $E_{(X,0)}$ is called $X'/X$ saturated if this embedding is an isomorphism of modules: $E \sim \nu_* (\nu^* E/\text{Torsion})$

Example 4.2. Consider the torsion-free modules of rank 1 over the $A_n$ singularity $g^2 = x^{n+1}$, continuing examples 2.5 and 2.9. Every such module can be embedded as $\mathcal{O}_{(C,0)} \subset E \subset \mathcal{O}_{(C',0)}$.

- $n = 2k$. The torsion-free modules of rank 1 are $\{E_i := \mathcal{O}_{(C,0)}(1, t^{2i+1}) \}_{0 \leq i < k}$. Let $\mathcal{O}_{(C',0)} = k\{t^2, t^{2i+1}\}$ and $(C',0) \hookrightarrow (C,0)$ the corresponding modification. Then $\nu^*(E_i)/\text{Torsion}$ is a free $\mathcal{O}_{(C',0)}$ module. In fact $E_i = \nu_* \mathcal{O}_{(C',0)}$. The corresponding determinantal representation is $\begin{pmatrix} y & x^{k-1} \\ x^{i+k+1} & y \end{pmatrix}$.

- $n = 2k - 1$. The torsion-free modules of rank 1 are $\{E_i := \mathcal{O}_{(C,0)}(1, t_1^i - t_2^i) \}_{0 \leq i < k}$. Let $\mathcal{O}_{(C',0)} = k\{t_1 + t_2, t_1^i - t_2^i\}$ and $(C',0) \hookrightarrow (C,0)$ the corresponding modification. Then $E_i = \nu_* \mathcal{O}_{(C',0)}$. The corresponding determinantal representation is $\begin{pmatrix} y + x^k & x^{k-1} \\ 0 & y - x^k \end{pmatrix}$.

Note that in both cases the ideal generated by the entries of $\mathcal{M}^\nu$ is precisely the relative adjoint ideal $\text{Adj}_{C'/C}$.

Proposition 4.3. 0. Every torsion-free module over $(X,0)$ is $X/X$ saturated.
1. For any torsion-free module $E_{(X,0)}$ there exists the unique maximal finite modification such that $E$ is $X'/X$-saturated. Namely, if $E$ is also $X'//X$ saturated then the modification $(X',0) \rightarrow (X,0)$ factorizes as $(X',0) \rightarrow (X'',0) \rightarrow (X,0)$.
2. Let $E$ be the kernel of a determinantal representation $\mathcal{M}$ of $(X,0)$. Then $E$ is $X'/X$-saturated iff $\mathcal{M}$ is $X'/X$-saturated, in the sense of definition 1.2.
3. $E$ is $X'/X$-saturated iff $\text{tr}(E)$ is $X'/X$-saturated.

Proof. 0. Trivial.

1. Suppose $E$ is both $X_1/X$ and $X_2/X$ saturated for the extensions of local rings: $\mathcal{O}_{(X,0)} \subset \mathcal{O}_{(X_1,0)}, \mathcal{O}_{(X,0)} \subset \mathcal{O}_{(X_2,0)}$, here $\tilde{X}$ is the normalization. Let $R \subset \mathcal{O}_{(\tilde{X},0)}$ be the subring generated by $\mathcal{O}_{(X_1,0)}, \mathcal{O}_{(X_2,0)}$. Geometrically we have the diagram on the right.

Spec$(R) \rightarrow (X_2,0)$
\[ \downarrow \]
\[(X_1,0) \rightarrow (X,0)\]

Now, by construction, $E$ is Spec$(R)/X$ saturated. By taking such unions of the local rings (and staying inside $\mathcal{O}_{(\tilde{X},0)}$) the unique maximal modification is constructed.

2. $\Rightarrow$ If $E$ is $X'/X$-saturated then it is an $\mathcal{O}_{(X',0)}$ module. Recall that $E$ is spanned by the columns of $\mathcal{M}^\nu$. Hence for any entry of $\mathcal{M}^\nu$: $\mathcal{O}_{(X',0)} \mathcal{M}^\nu_{ij} \subset \mathcal{O}_{(X,0)}$, i.e. $\mathcal{M}^\nu_{ij} \subset \text{Adj}_{X'/X}$.

\[ \Leftarrow \] If all the entries of $\mathcal{M}^\nu$ belong to $\text{Adj}_{X'/X}$ then $\nu_* \nu^*(E)/\text{Torsion}$ is generated (as an $\mathcal{O}_{(X,0)}$ module) by some columns with entries in $\mathcal{O}_{(X,0)}$. Let $s \in \nu_* \nu^*(E)/\text{Torsion}$, then $\mathcal{M}s = 0 \in \mathcal{O}_{(X,0)}$. But then, by definition, $s \in E$. Hence the statement.

3. Note that $\mathcal{M}$ is $X'/X$-saturated iff $\mathcal{M}^\nu$ is. The notion of being $X'/X$ saturated suits perfectly for the decomposition criterion.

Theorem 4.4. Let $(X,0) = (X_1,0) \cup (X_2,0) \subset (k^n,0)$ where $(X_i,0) = \{f_i = 0\}$ can be further reducible, non-reduced, but without common components. Let $X' = (X_1,0) \coprod (X_2,0) \rightarrow (X,0)$ be the finite modification that separates the components. Let $E$ be the kernel of a determinantal representation $\mathcal{M}$ of $(X,0)$. The following are equivalent.

1. $E$ is $X'/X$ saturated.

1'. $\mathcal{M}$ is $X'/X$ saturated, i.e. every element of $\mathcal{M}^\nu$ belongs to the ideal $\langle f_1, f_2 \rangle \subset \mathcal{O}_{(k^n,0)}$. 

2. $E = E_1 \oplus E_2$, where $E_i = E^{(X_i,0)}/\text{Torsion}$.

2'. $\mathcal{M} \sim \mathcal{M}_1 \oplus \mathcal{M}_2$, where $\mathcal{M}_i$ is a determinantal representation of $(X_i, 0)$.

In particular, if $(X, 0) = \{ f = 0 \}$, with $f = \prod f_i$, and $\mathcal{M}^\nu = \sum \frac{f_i}{f} \mathcal{M}^\nu_i$ then the determinantal representation is completely decomposable: $\mathcal{M} \sim \oplus \mathcal{M}_i$.

Note that (for the statement 1') the relative adjoint ideal of $\text{Adj}(X', 0)/(X, 0)$ was computed in Proposition 2.10.

**Proof.** The equivalence of 1 and 1' is proven in Proposition 4.3. The equivalence of 2 and 2' is proven in Proposition 2.18. The implications 2'⇒1' and 2⇒1 are obvious.

The implication 1⇒2. Note that for $(X', 0) \rightarrow (X, 0)$ the pullback decomposes: $\nu^*(E)/\text{Torsion} = E_1 \oplus E_2$. Here $E_i$ is supported on $(X_i, 0)$. As $E$ is $X'/X$ saturated one has: $E = \nu_*(E_1 \oplus E_2) \approx E_1 \oplus E_2$.

This proves the theorem. Note that in proposition 3.4 we give also a direct proof of 1'⇒2', purely in terms of linear algebra. ■

5. Some applications

We restrict here to the case of curves, for higher dimensions cf. §3.4. Let $\mathcal{M}$ be a maximally generated dr of the plane curve $(C, 0)$. Then $\mathcal{M}$ is decomposable according to the tangential decomposition (Theorem 3.9). We study its blocks, each of them is a maximally generated determinantal representation of a curve singularities whose tangent cone has just one line.

**Corollary 5.1.** Let $(C, 0) = \cup(p_\alpha C_\alpha)$, where each $C_\alpha$ is smooth and $T_{(C,0)} = \{ x_1^p = 0 \}$. Let $\mathcal{M}$ be a determinantal representation of $(X, 0)$, maximally generated at the origin.

Then $\mathcal{M}$ is equivalent to

$$
\begin{pmatrix}
(f_1 & \beta_1 x_2^{n_1} & h_{13}(x_2) & \ldots & \ldots & h_{1n}(x_2) \\
0 & f_2 & \beta_2 x_2^{n_2} & h_{24}(x_2) & \ldots & h_{2n}(x_2) \\
\ldots & \ldots & \ldots & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & 0 & f_p
\end{pmatrix}
$$

with $\beta_i \in \{0, 1\}$ and either $h_{ij}(x_2) \equiv 0$ or $h_{ij}(x_2)$ a polynomial in $x_2$ such that $\text{ord}_{x_2}(h_{ij}) \geq 1$ and $\text{deg}(h_{ij}) < \min(l_i, l_j)$.

**Proof.** 1. The matrix is equivalent to an upper triangular by theorems 3.7 and 3.10. Then by columns subtraction one can kill all the $x_i$ dependent terms in the entries above the diagonal.

2. Consider the diagonal $\{(i, i+1)\}$. Represent each nonzero element $\mathcal{M}_{i,i+1}(x_2)$ as $x_2^{n_i} \tilde{\mathcal{M}}_{i,i+1}$, where $\tilde{\mathcal{M}}_{i,i+1}\mid_{(0,0)} \neq 0$, i.e. is locally invertible. If $n_i \geq \min(l_i, l_{i+1})$ then by adding the $i$th column to the column $(i + 1)$ and subtracting the row $(i + 1)$ from the row $i$ the $x_2$-order can be increased. Continue this process inductively, thus killing this entry. Hence, if for some element $\mathcal{M}_{i,i+1}$ the $x_2$-order is at least $l_i$ or $l_{i+1}$ the element can be just set to zero.

The remaining non-zero elements $x^{n_i} \tilde{\mathcal{M}}_{i,i+1}$ are set to $x^{n_i}$ by the conjugation $\mathcal{M} \rightarrow U^{-1} \mathcal{M} U$.
with

\[ U = \begin{pmatrix} 
\prod_{i \geq 1} \tilde{M}_{i,i+1} & 0 & \ldots & 0 \\
0 & \prod_{i \geq 2} \tilde{M}_{i,i+1} & \ldots & 0 \\
0 & \ldots & \ldots & 0 \\
\tilde{M}_{k-1,k} 
\end{pmatrix} \]

Regarding the remaining entries \( h_{ij}(x) \) with \( j - i \geq 2 \), bring them to the needed form diagonal-by-diagonal. This is done again by the standard procedure: add \( y + x^l_i \), subtract \( y + x^l_j \) etc. ■

Example 5.2. • Any maximally generated determinantal representation of \( y(y+x^{l_1})(y-x^{l_2}) \) is equivalent to either:

\[ \begin{pmatrix} 
y + x^{l_1} & x^{n_1} & h(x) 
y & y & x^{n_2} 
y & 0 & y - x^{l_2} 
\end{pmatrix}, \quad n_i < l_i, \quad 1 \leq \text{ord}(h(x)) < \min(n_1,n_2) \text{ or } h(x) \equiv 0 \]

or

\[ \begin{pmatrix} 
y + x^{l_1} & 0 & x^{n_1} 
y & y & x^{n_2} 
y & 0 & y - x^{l_2} 
\end{pmatrix}, \quad \begin{pmatrix} 
y + x^{l_1} & x^{n_1} & x^{n_2} 
y & y & 0 
y & 0 & y - x^{l_2} 
\end{pmatrix}, \quad \begin{pmatrix} 
y + x^{l_1} & 0 & x^{n_1} 
y & 0 & y 
y & 0 & y - x^{l_2} 
\end{pmatrix} \]

• For \( (C,0) = \{y(y^2 - x^{2l+1}) = 0\} \) any maximally generated determinantal representation is equivalent to

\[ \begin{pmatrix} 
y & p_1(x) & p_2(x) 
y & x^{2l+1-m} 
y & 0 
\end{pmatrix}, \quad p_1(0) = 0 = p_2(0), \quad \text{deg}(p_1(x)) < m, \quad \text{deg}(p_2(x)) < 2l + 1 - m \]

Here the pair of polynomials \((p_1(x), p_2(x))\) is determined up to scaling. Hence the space of maximally generated representations is parameterized by \( H^0(O_{\mathbb{P}^1}(m-2)) \times H^0(O_{\mathbb{P}^1}(2l - m))/\sim \), where \( H^0(O_{\mathbb{P}^1}(j)) \) is the space of homogeneous polynomials in two variables of degree \( j \) and the equivalence relation is the scaling.

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