A Renormalisation group for TCSA

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Abstract

We discuss the errors introduced by level truncation in the study of boundary renormalisation group flows by the Truncated Conformal Space Approach. We show that the TCSA results can have the qualitative form of a sequence of RG flows between different conformal boundary conditions. In the case of a perturbation by the field $\phi$, we propose a renormalisation group equation for the coupling constant which predicts a fixed point at a finite value of the TCSA coupling constant and we compare the predictions with data obtained using TBA equations.

1 Perturbed boundary conformal field theory

The Truncated Conformal Space Approach (TCSA) is a tool to study finite size effects or RG flows in perturbed conformal field theory [1, 2]. Here we consider boundary RG flows where conformal invariance of a system with a boundary is broken by a coupling to a boundary field.

$$\delta S = \lambda \int \phi(x) dx.$$ (1)

These are easier to study than bulk flows in many ways – for a unitary theory, the UV and IR fixed points must be conformal boundary conditions which are well understood, and the boundary entropy $g$ must decrease along the flow [3, 4].

An example of a problem is to study the space of flows in the tri-critical Ising model. The tri-critical Ising model is a unitary conformal field theory with central charge 7/10 and contains 6 representations of the Virasoro algebra and correspondingly 6 bulk primary fields. The

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conformal weights and labelling of the Virasoro representations are

| label  | (11) | (21) | (31) | (12) | (13) | (22) |
|--------|------|------|------|------|------|------|
| weight | 0    | 7/10 | 3/2  | 1/10 | 3/5  | 3/50 |

For this model, there are thus 6 fundamental conformal boundary conditions corresponding to the 6 primary fields [5]. It is the continuum limit of an RSOS lattice model with heights taking integer values 1 to 4, or a spin model where the spins take values −, 0 and +. One can realise the boundary conditions in terms of restrictions on the values that the spins on the edge can take. Furthermore, each conformal boundary condition supports boundary fields organised into representations of the Virasoro algebra, the representations given by the fusion rules. For the 6 fundamental boundary conditions the values of \( g \) and the conformal families of boundary fields are

| spins  | (−) | (0) | (+) | (−0) | (0+) | (−0+) |
|--------|-----|-----|-----|------|------|-------|
| labels | (11) | (21) | (31) | (12) | (13) | (22)  |
| \( g \) | 0.5127 | 0.725 | 0.5127 | 0.8296 | 0.8296 | 1.173 |
| boundary fields | (11) | (11), (31) | (11) | (11), (13) | (11), (13) | (11), (13), (12), (31) |

The condition for a boundary field to be relevant (i.e. to generate a boundary flow) is that its weight be less than 1 and in this model these are just the fields labelled (12) and (13) together with the field (11) of weight zero. The boundary RG flows of the tricritical Ising model have been well studied and the global picture in figure 1 first proposed by Affleck [6]. This can be checked using TCSA, and many of the flows can be checked by other methods as well, in particular the flows marked * agree with perturbation theory [7] and the integrable flows generated by the perturbation \( \phi_{(13)} \) have been studied using TBA equations derived from a lattice approach [8].

![Diagram](image)

Figure 1: The space of boundary flows in the tricritical Ising model
2 The Truncated Conformal Space Approach

A strip with the perturbation (1) along a boundary is equivalent to a system with a perturbed Hamiltonian

\[ H = H_0 + \lambda \phi(0) . \]  

(2)

If the strip has conformal boundary conditions (a) and (b) on its two edges then the Hilbert space on which the Hamiltonian acts is given by the fusion rule

\[ \mathcal{H} = \bigoplus_c N_{abc} \mathcal{H}_c . \]  

(3)

If we take (a) to correspond to the identity operator then the Hilbert space is a single representation of the Virasoro algebra.

\[ \mathcal{H} = \mathcal{H}_b \]

It is convenient to map the strip to the upper half plane and consider the operator

\[ R \pi H = (L_0 - \frac{c}{24}) + \lambda R \pi^{1-h} \phi(1) . \]  

(4)

The matrix elements of this operator can be calculated exactly. This gives an infinite matrix, and the idea of TCSA is to truncate the Hilbert space to states with energy less than or equal to \((N + h)\). One can then diagonalise the resulting matrices and investigate the spectrum and other properties of the truncated system.

If we perturb the (b) boundary so that in the IR it flows to a new boundary (b') then the spectrum of the Hamiltonian will interpolate that of the UV and IR boundaries. Since

\[ H_0 = \left( \frac{\pi}{R} \right) (L_0 - \frac{c}{24}) , \]

(5)

then in the case that the Hilbert space consists of a single representation of the Virasoro algebra, in all but the vacuum representation the normalised energy gaps

\[ \Delta_i = \frac{E_i - E_0}{E_1 - E_0} \]

will be integers at the UV and IR fixed points with multiplicities that are given by the characters of the two representations (b) and (b'). In the vacuum representation since the first excited state has \(L_0\) eigenvalue 2, the normalised gaps will be half-integers. For the tri-critical Ising model these multiplicities are given in table I.

As a concrete example, consider the flows away from the boundary condition (13) generated by the field \(\phi_{(13)}\). We can expect that the boundary condition flows for one sign of the coupling to the character (31) and for the other to (21) In figure 2 we show the normalised energy gaps \(\Delta_i\) for the perturbation of the theory on a strip with boundary conditions (11) and (13) by the field \(\phi_{13}\) on the (13) boundary. For zero coupling the multiplicities are those of the representation (13). For positive coupling they reorganise themselves approximately into the multiplicities of the (31) representation and for negative coupling into those of the (21) representation. in agreement with perturbative and TBA calculations. Furthermore the accuracy with which this reorganisation occurs increase with increasing truncation level — here from 82 states with \(N = 10\) to 410 states with \(N = 16\).
Table 1: Multiplicities of low lying states in the Virasoro representations entering the tri-critical Ising model

|   | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10  | ... |
|---|----|----|----|----|----|----|----|----|----|----|------|------|
| $\Delta_1$ (12) | 1  | 1  | 1  | 2  | 3  | 4  | 6  | 8  | 11 | 14 | 19  | ... |
| (13) | 1  | 1  | 2  | 2  | 4  | 5  | 7  | 9  | 13 | 16 | 22  | ... |
| (21) | 1  | 1  | 1  | 2  | 3  | 4  | 6  | 8  | 10 | 14 | 18  | ... |
| (31) | 1  | 1  | 2  | 2  | 3  | 4  | 6  | 7  | 10 | 12 | 16  | ... |
| (22) | 1  | 1  | 2  | 3  | 4  | 6  | 8  | 11 | 15 | 20 | 26  | ... |
| $2\Delta_1$ (11) | 1  | 0  | 1  | 1  | 2  | 2  | 4  | 4  | 7  | 8  | 12  | ... |

These graphs look very nice, but in fact they are deceptive. Firstly the 1st excited state has been scaled to gap 1, and secondly the range shown has been chosen carefully. Extending the range of the graph for larger positive and negative values shows that TCSA has a rather unexpected behaviour. In figure 3 we plot the normalised gaps for positive and negative coupling on a logarithmic scale. We see that the fixed points we identified earlier are not at infinite coupling but at finite values of the coupling constant and that apparently well organised behaviour continues beyond the “fixed point”. In the case of the negative direction there appears to be a sequence of values of the coupling constant at which the spectrum organises itself into the (21), (12) and (11) representations in turn. These are exactly the sequence of flows one would expect if the perturbing field $\phi_{13}$ in the UV transformed into the field $\phi_{31}$ in the IR so that one passed along the normal flow from (31) through the IR fixed point (21) and then proceeded in the reverse direction towards the (12) point and then again away from (12) towards the (11) boundary condition. In other words this is the same
Figure 3: The normalised gaps for the perturbation of the strip with boundary conditions (11) and (13) by the field $\phi_{(13)}$ with positive and negative coupling on a logarithmic scale. The positions of the approximate fixed points (31), (21), (12) and (11) are indicated by vertical lines as joining two standard flows together by identifying their mutual IR fixed point:

$$(11) \leftarrow (12) \rightarrow (21) \leftarrow (13) \rightarrow (31)$$

or the whole sequence of flows along the bottom of figure 1. The extension of this pattern in the positive direction can be seen in higher models, e.g. $M_{6,7}$ shown in figure 4.

Figure 4: The normalised gaps for the perturbation of the strip with boundary conditions (11) and (13) in the model $M_{6,7}$ by the field $\phi_{(13)}$ with negative and positive couplings each on a logarithmic scale at truncation level 16. The positions of the approximate fixed points are indicated by vertical lines.
In figure 4, the following sequence of flows can be seen, all starting from the (13) boundary condition:

\[(11) \leftarrow (12) \rightarrow (21) \leftarrow (13) \rightarrow (31) \leftarrow (14) \rightarrow \]

These are again exactly the flows one would expect if the perturbing field in the UV is \( \phi_{13} \) and in the IR is \( \phi_{31} \) (when this exists).

These sequences of flows are surprising as half of the individual flows appear to violate the \( g \)-theorem, namely the flows \((21) \rightarrow (12)\) for negative \( \lambda \) and \((31) \rightarrow (14)\) for positive \( \lambda \). It is worth noting that the same sequence of flows was seen in [9] in an apparently unrelated context. The question is whether these flows can be understood given the numerical nature of the TCSA and whether they affect the large \( N \) limit of TCSA.

In the limit \( N \to \infty \), the TCSA scheme is meant to approach the TBA or NLIE picture in which the beta function for a single perturbation is linear and the only fixed points are at infinity. To find the behaviour of TCSA as \( N \) varies we shall adapt the standard method used in the field theory investigations.

### 3 The TCSA renormalisation group equation

In this section we consider a perturbation by a single field \( \phi \) where the only relevant field appearing in the OPE of \( \phi(x)\phi(y) \) is \( \phi \) itself. This is the situation for the \( \phi_{13} \) flows presented in figures 2–4.

TCSA can be thought of as standard perturbation by a field which is projected onto states of level less than or equal to \( N \). If this projector is \( P_N \) then we consider the perturbation by \( \lambda_N \phi_N \) where \( \lambda_N \) is the effective TCSA coupling for truncation level \( N \) and \( \phi_N = P_N \phi P_N \).

We can find how \( \lambda_N \) varies with \( N \) by requiring the partition function be invariant. Rather than consider the partition function itself directly, we can consider the operator

\[ \mathcal{P} e^{-\lambda_N \int \phi_N(x) dx} = \mathcal{P} e^{-\lambda_{N+1} \int \phi_{N+1}(x) dx} , \]

whose expectation value is the partition function on the strip. Mapping this to the upper half plane, expanding this out to second order, and stripping off an integral we get (with \( y = 1 - h \))

\[
\lambda_N \phi_N(1) - \lambda_N^2 \left( \frac{R}{\pi} \right)^y \int_0^1 \phi_N(1) \phi_N(u) \frac{du}{uy} \\
= \lambda_{N+1} \phi_{N+1}(1) - \lambda_{N+1}^2 \left( \frac{R}{\pi} \right)^y \int_0^1 \phi_{N+1}(1) \phi_{N+1}(u) \frac{du}{uy} .
\]

We can take the matrix element of this expression between \( \langle \phi|\cdots|0 \rangle \) to find to second order that

\[
\lambda_{N+1} - \lambda_N = \left( \frac{R}{\pi} \right)^y \lambda_N^2 \int_0^1 \langle \phi|\phi(1)[P_{N+1} - P_N]\phi(u)|0 \rangle \frac{du}{uy} .
\]

The integrand can be identified as the coefficient of \( u^{N+1} \) in the expansion of the three point function

\[
\langle \phi|\phi(1)\phi(u)|0 \rangle = C(1 - u)^{-h} .
\]
This coefficient is
\( C \frac{\Gamma(h + N + 1)}{\Gamma(h)\Gamma(N + 2)} u^{N+1} . \) \( (11) \)
Performing the integral and taking the large \( N \) limit we find
\[ N \frac{d\lambda}{dN} = \left( \frac{R}{N\pi} \right)^y \frac{C}{\gamma} \lambda^2 , \]
where \( \gamma = \Gamma(h) \). We can solve this exactly to find \( \lambda(N) \) in terms of \( \lambda_\infty \):
\[ \lambda(N) = \frac{\lambda_\infty}{1 + \left( \frac{R}{N\pi} \right)^y \frac{C}{\gamma} \lambda_\infty} \]
(13)
We see that

1. As \( N \to \infty, \lambda \to \lambda_\infty \)
2. As \( \lambda_\infty \to \pm \infty \), \( \lambda_N \to \pm \infty \),
   \[ \lambda_N = \frac{1}{C} \left( \frac{R}{\pi} \right)^y , \]
   for \( \lambda \) positive
   \[ \lambda_N \to \pm \infty \] for \( \lambda \) negative.

Assuming that \( \lambda C > 0 \), we see that the IR fixed point at \( \lambda_\infty = \infty \) is brought in to a finite value. This value tends to \( \infty \) as \( N \) increases, so the finite value of the fixed point is indeed an artefact of truncation which would go away with increasing \( N \). Since this is only a first order perturbation theory calculation we cannot expect it to provide any information about other fixed points, for example even the closest fixed point for \( \lambda C < 0 \) is not seen by this calculation.

One thing to note is that if we consider
\[ \mu = \left( \frac{R}{N\pi} \right)^y \lambda , \]
(14)
then the RG equation becomes
\[ -N \frac{d\mu}{dN} = y\mu - \frac{C}{\gamma} \mu^2 . \]
(15)
In the limit \( h \) tends to one this reproduces the standard beta function of [7] with an effective UV cutoff \( a = R/(N\pi) \).
\[ -N \frac{d\mu}{dN} = y\mu - C \mu^2 . \]
(16)
We can also see directly from here the fixed point at \( \mu = \frac{R}{\pi} \) that
\[ \lambda_N(R/\pi)^y = \frac{y\gamma}{C} N^y . \]
(17)

4 A test of the RG equations

We can test the RG equations by calculating the spectrum at finite \( N \) with the RG improved value of \( \lambda \). Without RG correction the spectra will be \( N \) dependent; if we use the RG corrected
value of the coupling constant then the spectra for different values of $N$ should agree much more closely. This is what we see in figure 5 in the case of the boundary condition (12) perturbed by $\phi_{(13)}$ in the model $M_{4,5}$. The system is flowing from the boundary condition (12) in the UV on the left to the boundary condition (11) in the IR on the right. The (blue) dashed lines are at truncation level $N = 5$ with 12 states and the (red) solid lines are at truncation level $N = 16$ with 362 states. The points are the gaps calculated using the excited state TBA equations. On the left the TCSA data is uncorrected whereas on the right it is corrected using the RG equation (13). The agreement between different levels is improved by including the RG correction and the agreement with the TBA data is vastly improved by the RG correction.

Figure 5: Low-lying normalised energy gaps of the Hamiltonian in the tricritical Ising model on a strip with boundary conditions (11) and (12) with the latter perturbed by $\phi_{(13)}$ plotted against the logarithm of the coupling constant. On the left the TCSA data is uncorrected whereas on the right it is corrected using the RG equation (13). See text for details.

\footnote{This is a different example to that presented in the talk which was the boundary condition (14) perturbed by the field $\phi_{(13)}$ in $M_{6,7}$. The two examples show very similar properties – it has been altered to allow inclusion of excited state TBA data which shows clearly the great improvement in agreement resulting from the RG correction.}
5 Summary and Outlook

The fact that the TCSA method suffers from strong corrections for finite $N$ has been known for a long time — the corrections were highlighted in [1] and a scaling form proposed in [10]. The fact that this can bring fixed points in to finite values of the coupling constant was also known to people working in the field for a long time but was regarded as an unfortunate effect that could be removed by increasing $N$. The detailed examination presented here was motivated by the need to obtain a quantitative comparison between TCSA and the TBA results of [8]. The detailed comparison will appear later [11]. We appear to have made a first step to understanding the finite $N$-corrections to TCSA.

A similar situation occurs for the Ising model but the first correction is at third order for symmetry reasons. The standard TCSA truncation leads to a similar pattern of “fixed points” at finite values of the TCSA coupling constant with subsequent “reversed” flows as shown in figure 6. However in this case there are strong indications that the higher terms beta function depends strongly on the type of truncation - G. Zs. Tóth has constructed a different truncation which is exactly solvable but doesn’t exhibit the second fixed point seen in “naive” level truncation [12, 13].

There is however another important point which remains to be understood quantitatively and which has also been concealed in the results shown here. That is the need for an overall rescaling of the Hamiltonian, or an effective change in the strip width. In each of figures 1 to 5 it is the normalised energy gaps which have been plotted. As a final plot, in figure 7 we show an example of unnormalised energy gaps. A quantitative understanding of this rescaling still remains elusive.

![Figure 6: The normalised gaps $\Delta'_i = 2(E_i - E_0)/(E_2 - E_0)$ of the Hamiltonian in the Ising model on a strip with boundary conditions (11) and (12) perturbed by the boundary field $\phi_{(13)}$ in both positive and negative directions on a logarithmic scale - the positive on the right and the negative on the left.](image-url)
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