Charm-quark fragmentation
with an effective coupling constant

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ABSTRACT: We use a recently proposed non-perturbative model, based on an effective strong coupling constant, to study $c$-flavoured hadron production in $e^+e^-$ annihilation. Charm-quark production is described in the framework of perturbative fragmentation functions, with NLO coefficient functions, NLL DGLAP evolution and NNLL large-$x$ resummation. We model hadronization effects by means of the effective coupling constant in the NNLO approximation and compare our results with experimental data taken at the $Z^0$ pole and at the $\Upsilon(4S)$ resonance. We find that, within the experimental and theoretical errors, our model is able to give a reasonable description of $D^*$-meson spectra from ALEPH and BELLE for $x < 0.85$, while discrepancies are present when comparing with $D$ data from BELLE and CLEO and $D^*$ spectra from CLEO. Moreover, the fairly large theoretical uncertainties call for a full NNLO/NNLL analysis.

KEYWORDS: QCD, Heavy Quark Physics, NLO Computations
1. Introduction

The hadronization of partons into hadrons, for the time being, cannot be calculated from first principles, but it is usually described in terms of phenomenological models, such as the Kartvelishvili [1] or Peterson [2] non-perturbative fragmentation functions, containing few parameters which need to be tuned to experimental data. It was recently proposed [3, 4], however, a non-perturbative model, based on the work in Refs. [5, 6], including power corrections via an effective strong coupling constant, which does not exhibit the Landau pole any longer and includes absorptive effects due to gluon branching. The interesting feature of such a model is that it does not contain any extra free parameter to be fitted to the data, besides the ones entering in the parton-level calculation. In [3], such a model was used in the framework of $B$-meson decays and it was found good agreement with the data on the photon spectrum and on the hadron-mass distribution in radiative and semileptonic decays, respectively. In [4] the effective coupling was employed in the framework of bottom-quark fragmentation and, within the theoretical uncertainties, a reasonable fit of LEP and SLD data on $B$-hadron spectra was obtained in both $x$ and moment spaces.

Although the results in [3, 4] are encouraging, it is nonetheless mandatory to consider more data and observables to validate the effective-coupling model. In this paper, we consider charm-quark fragmentation in $e^+e^-$ processes and investigate how our non-perturbative model fares against $D$-meson data from LEP and $B$-factories. In fact, charm production involves pretty different scales with respect to $b$-quark fragmentation, and therefore the comparison with $D$-hadron spectra should help to shed light on our model. Considering charm production at the $Z^0$ pole and at the $\Upsilon(4S)$ resonance, furthermore, is also
interesting to understand how our model behaves when the process hard scale changes. In the following, perturbative charm production will be described in the framework of perturbative fragmentation functions [7], as in [4], and the effective coupling constant will be our only source of non-perturbative power corrections.

The plan of the present paper is the following. In section 2 we shall review the main points of the parton-level computation, based on the perturbative fragmentation formalism, and including large-\(x\) resummation in both coefficient function and initial condition of the perturbative fragmentation function. In section 3 we shall discuss the effective coupling constant and the inclusion of non-perturbative corrections to charm-quark fragmentation. In section 4 we shall compare the results with charmed-meson spectra from LEP and \(B\)-factories. We shall finally summarize our main results in section 5.

2. Charm-quark production

In this section we shall discuss our calculation for charm-quark production. As it is essentially the same computation which was carried out in [4] for \(b\)-quark fragmentation, we shall just point out its main issues and refer to [4] for further details.

2.1 Perturbative fragmentation functions

We consider \(c\bar{c}\)-pair production in \(e^+e^-\) annihilation at next-to-leading order (NLO) in the strong coupling constant \(\alpha_S\):

\[
e^+e^+ \to P(Q) \to c(p_c)\bar{c}(p_{\bar{c}}) (g(p_g))
\]  

where \(P(Q)\), in the following, will be a \(Z^0\) boson or a \(\Upsilon(4S)\) resonance produced at a \(B\)-factory, and define the charm-quark energy fraction:

\[
x = \frac{2p_c \cdot Q}{Q^2}. \tag{2.2}
\]

The perturbative fragmentation approach [7], up to power corrections, factorizes the energy distribution of a heavy quark, the charm quark in our case, as the convolution of a coefficient function, associated with the emission off a massless parton, and a perturbative fragmentation function, expressing the transition of the light parton into a heavy quark. This way, the \(c\)-quark spectrum reads:

\[
\frac{1}{\sigma} \frac{d\sigma}{dx}(x, Q, m_c) = \sum_i \int_x^1 \frac{dz}{z} \left[ \frac{1}{\sigma} \frac{d\hat{\sigma}_i}{dz}(z, Q, \mu_R, \mu_F) \right]_{\overline{\text{MS}}} D_i^{\overline{\text{MS}}}(\frac{x}{z}, \mu_F, m_c)
\]

+ \(O((m_c/Q)^p)\). \quad \tag{2.3}

In Eq. (2.3), \(p \geq 1\), \(d\hat{\sigma}_i/dz\) is the differential cross section for the production of a massless parton \(i\) after subtracting the collinear singularity in the \(\overline{\text{MS}}\) factorization scheme; \(\mu_R\) and \(\mu_F\) are the renormalization and factorization scales; \(\sigma\) is the NLO \(e^+e^- \to q\bar{q}(g)\) cross section. Hereafter, we shall neglect charm production via \(g \to c\bar{c}\) splitting. In fact, we can anticipate that, when comparing with data, secondary \(c\bar{c}\) production will be subtracted by
the sample which we shall consider. This implies that \( i = c \) in Eq. (2.3) and \( D_{c}^{\overline{\text{MS}}} \) is the perturbative fragmentation function expressing the fragmentation of a massless \( c \) into a massive \( c \). The NLO \( \overline{\text{MS}} \) coefficient function for \( e^{+}e^{-} \rightarrow q\bar{q} \) processes can be found in [8].

The perturbative fragmentation function follows the DGLAP evolution equations [10]; its value at a any scale \( \mu_{F} \) can be obtained once an initial condition at \( \mu_{0F} \) is given. In [9] the initial condition \( D_{c}^{\text{ini}}(x, \mu_{0F}, m_{c}) \) was calculated in the NLO approximation and its process-independence was established on more general grounds in [11]. It is given by:

\[
D_{c}^{\text{ini}}(x, \alpha_{S}(\mu_{0F}^{2}), \mu_{0F}^{2}, m_{c}^{2}) = \delta(1-x) + \frac{\alpha_{S}(\mu_{0F}^{2})C_{F}}{2\pi} \left[ \frac{1}{1-x} \left( \ln \frac{\mu_{0F}^{2}}{m_{c}^{2}} - 2\ln(1-x) - 1 \right) \right]_{+}.
\]

(2.4)

As discussed in [4], solving the DGLAP equations for an evolution from \( \mu_{0F} \) to \( \mu_{F} \), with a NLO kernel, allows one to resum leading (LL) \( \alpha_{S}^{2}\ln^{n}(\mu_{F}^{2}/\mu_{0F}^{2}) \) and next-to-leading (NLL) \( \alpha_{S}^{3}\ln^{n-1}(\mu_{F}^{2}/\mu_{0F}^{2}) \) logarithms. Setting \( \mu_{0F} \simeq m_{c} \) and \( \mu_{F} \simeq Q \), one resums the large \( \ln(Q^{2}/m_{c}^{2}) \) appearing in the massive NLO spectrum [7]. The resummation of such mass logarithms is usually called collinear resummation. For the sake of working in the same perturbative framework as in [4], in the following we shall consider NLO coefficient functions and initial condition, along with NLL DGLAP evolution. However, one could go beyond such a level of accuracy and include NNLO corrections to the coefficient function [12], initial condition [13] and to the non-singlet splitting functions [14] entering in the kernel of the DGLAP equations.

2.2 Large-\( x \) resummation

Both coefficient function [7] and initial condition (2.4) contain terms, \( \sim 1/(1-x)_{+} \) and \( \sim \ln(1-x)/(1-x)_{+} \), which are enhanced when \( x \) approaches 1, which corresponds to soft- or collinear-gluon radiation. One needs to resum such contributions to all orders to improve the perturbative prediction (threshold resummation). As in [4], we shall implement threshold resummation, which is process-dependent in the coefficient function and process-independent in the initial condition [11], in the next-to-next-to-leading logarithmic (NNLL) approximation, following the general method of [15, 16]. Large-\( x \) resummation is typically performed in Mellin moment-space, where the Mellin transform of the differential cross section reads:

\[
\sigma_{N} = \int_{0}^{1} dx \ x^{N-1} \frac{d\sigma}{d\sigma}.
\]

(2.5)

In \( N \)-space, the enhanced contributions \( \sim \alpha_{S}/(1-x)_{+} \) and \( \sim \alpha_{S}[\ln(1-x)/(1-x)]_{+} \) correspond to single \( \sim \alpha_{S}\ln N \) and double \( \sim \alpha_{S}\ln^{2} N \) logarithms of the Mellin variable \( N \). The resummed coefficient function is given by the following generalized exponential function [11]:

\[
\Delta_{N}^{(C)}[\alpha_{S}(\mu_{R}^{2}), \mu_{R}^{2}, \mu_{F}^{2}, Q^{2}] = \exp \left\{ G_{N}^{(C)}[\alpha_{S}(\mu_{R}^{2}), \mu_{R}^{2}, \mu_{F}^{2}, Q^{2}] \right\},
\]

(2.6)

where

\[
G_{N}^{(C)}[\alpha_{S}(\mu_{R}^{2}), \mu_{R}^{2}, \mu_{F}^{2}, Q^{2}] = \int_{0}^{1} dz \ z^{N-1} \left\{ \int_{\mu_{F}^{2}}^{Q^{2}(1-z)} \frac{dk^{2}}{k^{2}} A[\alpha_{S}(k^{2})] + B[\alpha_{S}(Q^{2}(1-z))] \right\}.
\]

(2.7)
The exponent $G^{(C)}_N[\alpha_S(\mu^2_R), \mu^2_R, \mu^2_F, Q^2]$ resums the large logarithms of the Mellin variable; in the NNLL approximation, one keeps in the exponent terms $\sim \alpha_S^3 \ln^{n+1} N$ (LL), $\sim \alpha_S^3 \ln^n N$ (NLL) and $\sim \alpha_S^3 \ln^{n-1} N$ (NNLL). As in [16], the integration variables are $z = 1-x, x$ being the gluon energy fraction, and $k^2 = (p_c+p_g)^2(1-z)$. In soft approximation, $z \simeq x$; for small-angle radiation $k^2 \simeq k_1^2$, the gluon transverse momentum with respect to the c.

Function $A(\alpha_S)$ resums soft and collinear radiation, while $B(\alpha_S)$ includes all-order collinear and hard emissions. They can be expanded as a series in $\alpha_S$ as:

$$A(\alpha_S) = \sum_{n=1}^{\infty} \left( \frac{\alpha_S}{\pi} \right)^n A^{(n)}; \quad (2.8)$$

$$B(\alpha_S) = \sum_{n=1}^{\infty} \left( \frac{\alpha_S}{\pi} \right)^n B^{(n)}. \quad (2.9)$$

In the NLL approximation, one needs to include the first two coefficients of $A(\alpha_S)$ and the first of $B(\alpha_S)$; to NNLL accuracy, $A^{(3)}$ and $B^{(2)}$ are also needed. The coefficients $A^{(1)}$, $A^{(2)}$ and $B^{(2)}$ can be found in [16]; more recent is the calculation of the NNLL contributions $A^{(3)}$ [17] and $B^{(2)}$ [18].

Likewise, the threshold-resummed initial condition reads [11]:

$$\Delta^{(D)}_N[\alpha_S(\mu^2_R), \mu^2_R, \mu^2_F, m^2_c] = \exp \left\{ G^{(D)}_N[\alpha_S(\mu^2_R), \mu^2_R, \mu^2_F, m^2_c] \right\}, \quad (2.10)$$

where

$$G^{(D)}_N[\alpha_S(\mu^2_R), \mu^2_R, \mu^2_F, m^2_c] = \int_0^1 dz \left[ \frac{z^{N-1} - 1}{1-z} \right] A[\alpha_S(k^2)] + D[\alpha_S(m^2_c(1-z)^2)]. \quad (2.11)$$

with $k^2$ and $z$ defined as in (2.7). To NNLL accuracy, we need $A^{(1)}$, $A^{(2)}$ and $A^{(3)}$ and the first two coefficients of

$$D(\alpha_S) = \sum_{n=1}^{\infty} \left( \frac{\alpha_S}{\pi} \right)^n D^{(n)}, \quad (2.12)$$

namely $D^{(1)}$ and $D^{(2)}$. Function $D(\alpha_S)$, called $H(\alpha_S)$ in [11], is characteristic of the fragmentation of heavy quarks and resums soft and large-angle radiation. Its $O(\alpha_S)$ coefficient can be found in [11], while $D^{(2)}$ can be read from the formulas in [13]. In any case, all relevant NNLL threshold-resummation coefficients are reported in [4].

In the phenomenological analysis of [4], the inclusion of NNLL effects, and especially the contribution $\sim \alpha_S^3 A^{(3)}$ to function $A(\alpha_S)$, turned out to be necessary to reproduce the $b$-fragmentation data. In fact, as we shall point out in the next section, when using the effective coupling constant we need to redefine the threshold-resummation coefficients from the third order on, in such a way that $A^{(3)}$ gets enhanced. The inclusion of NNLL terms in the resummed exponents shifted the $B$-hadron spectrum towards lower $x$ values and played a crucial role to obtain a reasonable description of LEP and SLD data (see Fig. 4 in Ref. [4]).
As in [3, 4], the Mellin transforms of our resummed expressions will be performed exactly and not according to the step-function approximation, which was instead employed in the resumptions of [11, 16]. In fact, as we shall discuss in section 3, we will model non-perturbative effects to charm fragmentation by means of an effective coupling constant and it was found in [22] that the step-function approximation would suppress most power corrections included in the physical observables via the analytic coupling. In any case, as thoroughly detailed in [4], the issue of the power corrections which are transferred to the cross section by the effective coupling, and whether it is a better approximation performing the Mellin transforms in an exact or approximated way is currently an open issue and we cannot draw any firm conclusion. A careful analysis, along the lines of [23], will be anyway very welcome to clarify this point. For the time being, the exactness of the Mellin transforms should be seen as part of our non-perturbative model.

As in [4] the resummed results are matched to the exact NLO coefficient function and initial condition. A difference with respect to the approach followed in Ref. [4] is that we implement the so-called ln \( R \)-matching [14, 20], corresponding to matching the logarithms of resummed and NLO expressions. We briefly review this matching strategy and how it compares with the standard method implemented in [4]. Referring, e.g., to the coefficient function, matched to the exact NLO one, it can be written (see, e.g., Eq. (4.2) of Ref. [4]) as:

\[
C_N^{\text{res}}[\alpha_S(\mu_R^2), \mu_R^2, \mu_F^2, Q^2] = K^{(C)}[\alpha_S(\mu_R^2), \mu_R^2, \mu_F^2, Q^2] \Delta_N^{(C)}[\alpha_S(\mu_R^2), \mu_R^2, \mu_F^2, Q^2] + d_N^{(C)}[\alpha_S(\mu_R^2), \mu_R^2, \mu_F^2, Q^2].
\]  

(2.13)

In (2.13), \( \Delta_N^{(C)} \) is the resummed coefficient function, given in Eq. (2.6),

\[
K^{(C)}[\alpha_S(\mu_R^2), \mu_R^2, \mu_F^2, Q^2] = 1 + \alpha_S(\mu_R^2)Q(\mu_F^2, Q^2)
\]  

(2.14)

is a hard factor, including the constant terms which are present in the NLO coefficient function but are not resummed in \( \Delta_N^{(C)} \),

\[
d_N^{(C)}[\alpha_S(\mu_R^2), \mu_R^2, \mu_F^2, Q^2] = \alpha_S(\mu_R^2)Y(\mu_F^2, Q^2)
\]  

(2.15)

is a remainder function, collecting the left-over NLO terms, suppressed at large \( N \). The explicit expression for functions \( Q(\mu_F^2, Q^2) \) and \( Y(\mu_F^2, Q^2) \) can be read from the formulas in [4]. A similar expression holds for the resummed initial condition matched to the NLO result (see, e.g., Eq. (5.17) in Ref. [4]).

According to the ln \( R \)-matching, functions \( K^{(C)} \) and \( d^{(C)} \) are to be replaced by exponential functions of their \( \mathcal{O}(\alpha_S) \) terms and Eq. (2.13) should read:

\[
C_N^{\text{res}}[\alpha_S(\mu_R^2), \mu_R^2, \mu_F^2, Q^2] = \exp[\alpha_S(\mu_R^2)Q(\mu_F^2, Q^2)] \times \Delta_N^{(C)}[\alpha_S(\mu_R^2), \mu_R^2, \mu_F^2, Q^2] \times \exp[\alpha_S(\mu_R^2)Y(\mu_F^2, Q^2)].
\]  

(2.16)

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\(^1\)In [11, 20], the longitudinal-momentum integration is done after performing the replacement \( x^N \to 1 - \Theta(1 - z - \frac{\alpha_S}{N}) \), which is a correct approximation to NLL accuracy. Beyond NLL, it can be generalized following the prescription presented in [21].
From Eq. (2.16), one can easily check that the logarithms of our NLO and resummed functions are matched. Furthermore, Eq. (2.16) differs from (2.13) only by terms of \( O(\alpha_s^2) \) or higher, but it is smoother at small and large values of \( N(x) \), thanks to the exponential functions in Eq. (2.16). It was in fact pointed out in [4] that, since the remainder function contains terms \( \sim \ln x \) and \( \sim \ln (1-x) \), the physical differential cross sections exhibit oscillating behaviour near \( x \approx 0 \) and \( x \approx 1 \). Exponentiating the \( O(\alpha_S) \) contributions to the remainder function should therefore improve the prediction for small and large values of \( x \). The \( \ln R \)-matching prescription will be adopted in the following even for the resummed initial condition of the perturbative fragmentation function.

The \( c \)-quark spectrum will finally read in \( N \)-space as follows:

\[
\sigma_N^c \left[ \alpha_S(\mu_0^2 R), \alpha_S(\mu_F^2 R), \mu_0^2 R, \mu_F^2 R, \mu_0^2 F, \mu_F^2 F, m_c^2, Q^2 \right] = C_{N}^{\text{res}} \left[ \alpha_S(\mu_0^2 R), \alpha_S(\mu_F^2 R), \mu_0^2 R, \mu_F^2 F, m_c^2, Q^2 \right] \\
\times E_N \left[ \alpha_S(\mu_0^2 R), \alpha_S(\mu_F^2 R) \right] \\
\times D_{N}^{\text{ini, res}} \left[ \alpha_S(\mu_0^2 R), \mu_0^2 R, \mu_0^2 F, m_c^2 \right].
\]

In Eq. (2.17), \( E_N \left[ \alpha_S(\mu_0^2 R), \alpha_S(\mu_F^2 R) \right] \) is the DGLAP evolution operator, whose explicit expression can be found, e.g., in [7], for an evolution between the scales \( \mu_0 \) and \( \mu_F \).

### 3. Effective coupling constant

We shall include non-perturbative corrections to charm fragmentation using, as in [4], a model, based on an extension of Refs. [5, 6], which includes power corrections via an effective strong coupling constant, and does not introduce any further parameter to be tuned to experimental data. We review below the main points of our model.

As discussed in Ref. [24], in resummed calculations the momentum-independent coupling constant is replaced by the following integral over the discontinuity of the gluon propagator:

\[
\alpha_S \to i \frac{1}{2\pi} \int_0^{k^2} ds \ \text{Disc}_s \frac{\alpha_S(-s)}{s},
\]

where \( k^2 \) is the gluon transverse momentum relative to the emitter, defined, e.g., as in Eq. (2.7). In Eq. (3.1), the discontinuity is given by:

\[
\text{Disc}_s F(s) = \lim_{\epsilon \to 0^+} [F(s + i\epsilon) - F(s - i\epsilon)].
\]

At LO, e.g., \( \alpha_S(-s) \) reads:

\[
\alpha_{S, \text{LO}}(-s) = \frac{1}{\beta_0 \ln(|s|/\Lambda^2) - i\pi \Theta(s)},
\]

where \( \beta_0 = (33 - 2n_f)/(12\pi) \) is the first-order term of the QCD \( \beta \)-function, \( n_f \) is number of active flavours, and \( \Lambda \) is the QCD scale, e.g., in the MS renormalization scheme.

The integral (3.1) is usually carried out neglecting the imaginary part, \( \sim i\pi \), in the denominator of \( \alpha_S(-s) \), i.e. assuming

\[
\ln \frac{|s|}{\Lambda^2} \gg \pi
\]
in Eq. (3.3). The approximation (3.4) allows one to avoid the Landau pole, so that the integral (3.1) turns out to be roughly equal to the strong coupling constant evaluated at the upper integration limit:

\[ \frac{i}{2\pi} \int_0^{k^2} ds \, \text{Disc}_s \frac{\alpha_S(-s)}{s} \simeq \alpha_S(k^2). \]  

(3.5)

In fact, resummed formulas typically use the transverse momentum \( k^2 \) as the scale of the strong coupling constant [16].

As in [4], we shall follow a different approach and avoid the Landau pole by using in Eq. (3.1) a regularized coupling constant \( \bar{\alpha}_S \), defined as follows [5]:

\[ \bar{\alpha}_S(k^2) = \frac{1}{2\pi i} \int_0^{\infty} \frac{ds}{s + k^2} \text{Disc}_s \alpha_S(-s). \]  

(3.6)

Inserting in (3.6) the LO expression (3.3) and performing the integration, we obtain:

\[ \bar{\alpha}_{S,\text{LO}}(k^2) = \frac{1}{\beta_0} \left[ \frac{1}{\ln(k^2/\Lambda^2)} - \frac{\Lambda^2}{k^2 - \Lambda^2} \right]. \]  

(3.7)

If we compare Eq. (3.7) with the LO standard coupling, i.e.

\[ \alpha_{S,\text{LO}}(k^2) = \frac{1}{\beta_0 \ln(k^2/\Lambda^2)}, \]  

(3.8)

we learn that in Eq. (3.7) a power-suppressed term, relevant at small \( k^2 \), has subtracted off the Landau pole \( k^2 = \Lambda^2 \), which is instead present in (3.8). At large \( k^2 \), \( \bar{\alpha}_S(k^2) \) is nonetheless still roughly equal to \( \alpha_S(k^2) \). Such results can be generalized to higher accuracy levels, using the two- and three-loop beta function, as done in [4].

Our effective coupling constant \( \tilde{\alpha}_S(k^2) \) will be defined as in Eq. (3.1), but using the analytic coupling (3.6) in the integrand function:

\[ \tilde{\alpha}_S(k^2) = \frac{i}{2\pi} \int_0^{k^2} ds \, \text{Disc}_s \frac{\bar{\alpha}_S(-s)}{s}. \]  

(3.9)

Using the LO result (3.7), we can perform the integral (3.9) and obtain our LO effective coupling constant:

\[ \tilde{\alpha}_{S,\text{LO}}(k^2) = \frac{1}{\beta_0} \left\{ \frac{1}{2} - \frac{1}{\pi} \arctan \left[ \frac{\ln(k^2/\Lambda^2)}{\pi} \right] \right\}. \]  

(3.10)

The NLO and NNLO expressions of \( \tilde{\alpha}_S(k^2) \) can be found in [4]. It is straightforward to show that Eq. (3.10), as well as its higher-order generalizations in [4], are free from the Landau pole and include power-suppressed effects at small momenta. Also, as discussed in [22], Eq. (3.9) accounts for absorptive effects due to gluon branching.

In principle, both analytic coupling constants (3.6) and (3.9) are possible candidates to model non-perturbative corrections \(^2\). However, as debated in [4], it is only (3.9) which

\(^2\)In the literature [4, 5], one usually refers to \( \bar{\alpha}_S(k^2) \) and \( \tilde{\alpha}_S(k^2) \) as space- and time-like coupling constants, respectively.
gives an acceptable description of $b$-fragmentation data and we shall therefore stick to \( \hat{\alpha}_S(k^2) \) to model power corrections to charm fragmentation as well.

The relation between effective and standard coupling constant for \( \ln(k^2/\Lambda^2) \gg \pi \) reads:

\[
\hat{\alpha}_S(k^2) = \alpha_S(k^2) - \frac{(\pi \beta_0)^2}{3} \alpha_S^3(k^2) + O(\alpha_S^4).
\]  

(3.11)

From Eq. (3.11) we learn that at high energy the difference between \( \hat{\alpha}_S(k^2) \) and \( \alpha_S(k^2) \) starts from \( O(\alpha_S^3) \). Moreover, Eq. (3.11) dictates that, when employing the effective coupling constant, we will have to redefine the soft-resummation coefficients from order \( \alpha_S^3 \) on. As anticipated in subsection 2.2, the coefficient \( A^{(3)} \) of the \( O(\alpha_S^3) \) term of function \( A(\alpha_S) \), entering in Eqs. (2.7) and (2.11), will get enhanced according to:

\[
A^{(3)} \rightarrow \tilde{A}^{(3)} = A^{(3)} + \frac{(\pi \beta_0)^2}{3}A^{(1)}.
\]  

(3.12)

The other assumptions contained in our model are also detailed in Ref. [4] and we do not report them here for the sake of brevity. We just point out that, when dealing with higher orders of \( \hat{\alpha}_S \), we shall adopt the so-called ‘power-expansion’ choice, which implies that we shall evaluate the powers \( \hat{\alpha}_S^n(k^2) \) after computing the integral over the discontinuity:

\[
\hat{\alpha}_S^n(k^2) = \left[ \frac{i}{2\pi} \int_0^{k^2} ds \ \text{Disc}_s \ \frac{\hat{\alpha}_S(-s)}{s} \right]^n.
\]  

(3.13)

On the contrary, the original proposal in [3] consisted in calculating the discontinuity of \( \hat{\alpha}_S^n(-s) \) before integrating over \( s \) (‘non power-expansion’ choice). As discussed in [4], the non-power expansion prescription would yield a rather poor description of \( b \)-fragmentation data.

The purpose of the present paper is indeed to push our model to lower energies and compare its predictions with data on \( c \)-flavoured hadron production. We shall use \( \hat{\alpha}_S(k^2) \), evaluated to three-loop accuracy, everywhere in our calculation, i.e. in both coefficient function and perturbative fragmentation function. Hereafter, the effective coupling constant (3.9) will be our only source of non-perturbative corrections and we shall not introduce any further non-perturbative fragmentation function.

It was pointed out in [1] that power-correction effects in the initial condition of the perturbative fragmentation function are more relevant than in the coefficient function. The typical scales at which the coupling constant is evaluated are, in fact, \( C = Q\sqrt{1-x} \) in the coefficient function and \( S = m_c(1-x) \) in the initial condition. \( C \) and \( S \) are the integration limits in the resummed exponents as well as the arguments of \( \alpha_S \) in functions \( B[\alpha_S(C^2)] \) and \( D[\alpha_S(S^2)] \), appearing in the large-\( x \) resummation expressions (2.7) and (2.11). If we calculate \( C \) and \( S \) for \( Q = m_Z \) and \( x = 0.5 \), where, as will be shown in the next section, the \( D \)-meson spectrum in \( e^+e^- \) annihilation is roughly peaked, we shall get \( C \simeq 46 \text{ GeV}, \ S \simeq 0.9 \text{ GeV}, \ \hat{\alpha}_S(C^2) \simeq 0.13 \) and \( \hat{\alpha}_S(S^2) \simeq 0.35 \). Therefore, non-perturbative corrections are more important in the initial condition, depending on \( S \), than in the coefficient function. Comparing now the values of \( \hat{\alpha}_S \) at the charm- and bottom-mass scales, we find that \( \hat{\alpha}_S(m_c^2) \simeq 0.3 \) is appreciably higher than \( \hat{\alpha}_S(m_b^2) \simeq 0.2 \). However, it is
interesting to notice that the scales $S$ and $C$, and hence $\tilde{\alpha}_S(S^2)$ and $\tilde{\alpha}_S(C^2)$, are roughly the same for bottom and charm production if evaluated at the maxima of the respective spectra at LEP, i.e. $x=0.5$ for $D$- and $x=0.8$ for $B$-hadron energy distributions.

Before closing this section, we would like to stress that, in its current formulation, our parameter-free model works in the same fashion for $B$ as well $D$ mesons, up to the replacement $m_b \to m_c$. Also, our model does not distinguish among baryons and mesons, spin-1 and spin-0, charged and neutral hadrons. It was therefore argued in [4] that possible extensions of our model may consist in including a correcting term, so that

$$\tilde{\alpha}_S(k^2) \to \tilde{\alpha}_S(k^2) + \delta\tilde{\alpha}_S(k^2),$$

where $\tilde{\alpha}_S(k^2)$ is still the effective coupling discussed above, and $\delta\tilde{\alpha}_S(k^2)$ may depend, e.g., on whether we have baryons or mesons, $B$’s or $D$’s, and so on. The analysis which we shall undertake herafter should therefore be helpful to establish, for the time being, whether the contribution $\delta\tilde{\alpha}_S(k^2)$ is mandatory or not.

### 4. Results

In this section we compare our results with experimental data on $c$-flavoured hadron production in $e^+e^-$ annihilation. Hadronization effects will be accounted for by employing the analytic coupling constant \((3.9)\) at NNLO. Whenever we use $\tilde{\alpha}_S(k^2)$ instead of the standard $\alpha_S(k^2)$, the charm-quark energy fraction will be replaced by its hadron-level counterpart:

$$x_D = \frac{2p_D \cdot Q}{Q^2},$$

with $p_D$ being the momentum of a $D$-hadron. The $D$ spectrum in moment space will be written in a form analogous to Eq.\((2.17)\), up to $\alpha_S \to \tilde{\alpha}_S$:

$$\sigma^D_N(\mu_R^2, \mu_F^2, \mu_{0R}^2, \mu_{0F}^2, \mu_c^2, m_c^2, Q^2) = C^\text{res}_N[\tilde{\alpha}_S(\mu_R^2), \tilde{\alpha}_S(\mu_F^2)] \times E_N[\tilde{\alpha}_S(\mu_{0R}^2), \tilde{\alpha}_S(\mu_{0F}^2)] \times D^\text{ini, res}_N[\tilde{\alpha}_S(\mu_{0R}^2), \mu_{0R}^2, \mu_{0F}^2, m_c^2].$$

The $x$-space result is then recovered by performing an inverse Mellin transform:

$$\sigma^D(x_D; \mu_R^2, \mu_F^2, \mu_{0R}^2, \mu_{0F}^2, \mu_c^2, m_c^2, Q^2) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dN}{2\pi i} x_D^{-\gamma} \sigma^D_N(\mu_R^2, \mu_{0R}^2, \mu_{0F}^2, \mu_c^2, m_c^2, Q^2),$$

where $\gamma$ is a positive constant. As discussed in [4], since the effective $\tilde{\alpha}_S(k^2)$ does not exhibit the Landau pole any longer, we do not need any prescription, such as the well-known minimal prescription \([25]\), to avoid the Landau pole in the integration \((4.3)\). The integral will be performed in a numerical way, along the lines of [4]; it was checked that the results are stable when varying the integration contour, i.e. the constant $\gamma$.

As in Ref. \([26]\), we shall consider LEP data from the ALEPH Collaboration \([27]\), taken at the $Z^0$ pole, and data from the experiments CLEO \([28]\) and BELLE \([29]\), at the $\Upsilon(4S)$ resonance. We shall investigate neutral as well as charged $D$- and $D^*$-meson production;
in fact, we just pointed out that our model does not distinguish the hadron electric charge or spin.

As discussed in [26], electromagnetic initial-state radiation (ISR) effects can modify the shape of charmed-meson spectra. Such effects are important especially at B-factories, where the emission of photons from the $e^+e^-$ pair, whose rate is $\sim \alpha \ln(Q^2/m_c^2)$, $m_c$ being the electron mass, may significantly decrease the energy in the centre-of-mass system. The CLEO and BELLE data did not account for such effects, which were instead implemented in the analysis [23]. In the following, we shall compare with data corrected for ISR effects: a discussion on the impact of such contributions on $D$-spectra in $x$- and $N$-spaces can be found in [26]. Such effects were also implemented to correct the ALEPH data, but it was understood that at the $Z^0$ pole they are quite negligible.

The non-perturbative model based on the effective coupling constant (3.9) does not have any free parameter to tune to the data which we shall consider. We shall nonetheless vary the parameters entering in the perturbative calculation in such a way to give an estimate of the theoretical uncertainty on our prediction. We change each quantity separately, keeping the others to their default values, in such a way to deal with a reasonable number of runs.

Following [4], the default values of our perturbative parameters will be $\mu_R = \mu_F = Q$ and $\mu_{0R} = \mu_{0F} = m_c$, where $\mu_R$ and $\mu_F$ are the renormalization and factorization scales in the coefficient function, and $\mu_{0R}$ and $\mu_{0F}$ in the initial condition of the perturbative fragmentation function. The hard scale will be $Q = m_Z$ or $m_{T(4S)}$ at LEP or B-factories, with $m_Z = 91.19$ GeV and $m_{T(4S)} = 10.58$ GeV. We shall vary $\mu_R$ and $\mu_F$ between $Q/2$ and $2Q$, $\mu_{0R}$ and $\mu_{0F}$ between $m_c/2$ and $2m_c$. As in [4], we shall let $\alpha_S(m_Z^2)$ run in the range $0.117 < \alpha_S(m_Z^2) < 0.121$, using $\alpha_S(m_Z^2) = 0.119$ as our default value. The corresponding variation range of the effective coupling constant is $0.115 < \tilde{\alpha}_S(m_Z^2) < 0.119$.

For the purpose of $m_c$, as thoroughly discussed in [4], using the pole or the $\overline{\text{MS}}$ heavy-quark mass definition in the initial condition, is equivalent for calculations relying on the NLO/NLL approximation. However, in the NNLL large-$x$ resummation of the initial condition, and in particular in the definition of the coefficient $D^{(2)}$ in Eq. (2.11), we are employing results of the NNLO computation in [13], which uses the heavy-quark pole mass. Hence, we should use the charm pole mass as well. Nonetheless, as pointed out in [4], when we use the effective coupling constant to describe hadronization corrections, it is not uniquely determined whether $m_c$ should be the quark or the hadron mass. As done for the purpose of the bottom-quark mass, we shall adopt a conservative choice and vary $m_c$ in the range $1.5$ GeV $< m_c < 2.1$ GeV, that includes the current estimations for the charm pole mass as well as $D$-hadron masses [30]. Our default value will be $m_c=1.8$ GeV.

In the DGLAP evolution operator $E_N[\tilde{\alpha}_S(\mu_{0P}^2)\tilde{\alpha}_S(\mu_F^2)]$ we shall use $n_f = 4$ and $n_f = 5$ as the number of active flavours below and above the bottom-quark mass threshold, respectively. The $b$-quark mass will be varied in the range $4.7$ GeV $< m_b < 5.3$ GeV, as in [4], with $m_b = 5$ GeV being our default value. Elsewhere, $n_f$ will be consistently chosen according to the energy scale we are dealing with.
4.1 Comparison with ALEPH data

We shall first consider ALEPH data on $D^*+\overline{\text{c}}\overline{\text{c}}$ production. As detailed in [27], such mesons can be in general produced from a $Z^0 \rightarrow \overline{\text{c}}\overline{\text{c}}$ decay, from the decay of a primary $b$-flavoured hadron produced in $Z^0 \rightarrow b\overline{b}$, from gluon splitting to $\overline{\text{c}}\overline{\text{c}}$ or $b\overline{b}$ pairs, which subsequently hadronize or decay into a $D^*+$. The ALEPH Collaboration was able to subtract the $Z^0 \rightarrow b\overline{b}$ and gluon-splitting contributions and publish data on $D^*$ spectra just from the $\overline{\text{c}}\overline{\text{c}}$ source. In the following, we shall compare the predictions of our model with such a subsample, which will allow us to neglect secondary charm production in the perturbative calculation discussed in section 2. In Fig. 1 we present the spectrum given by our model, along with the $D^*$ ALEPH data, and investigate the dependence on the factorization scales $\mu_F$ and $\mu_0F$ (Fig. 1(a)), and on the choice of $\alpha_S(m^2_Z)$ and $m_c$ (Fig. 1(b)). For the sake of comparison, both data and theoretical predictions are normalized to unity. As already observed in [4], the dependence on $\mu_0F$, the scale entering in the initial condition of the perturbative fragmentation function, is fairly large, while the impact of the choice of $\mu_F$ is pretty small. In particular, setting a lower value of $\mu_0F$, e.g. $\mu_0F = m_c/2$, tends to deplete the small-$x_D$ region of the spectrum and enhances the event fraction around the peak. Also, the peak is slightly shifted to higher $x_D$ if we choose $\mu_0F = m_c/2$. The prediction obtained for $\mu_0F = 2m_c$ reproduces quite well the low-$x_D$ data, while discrepancies are still present in the middle-high range.

The dependence on $\alpha_S(m^2_Z)$ and $m_c$ is also quite relevant, as can be learned from Fig. 1(b). In particular, it is interesting to notice that a low value of $m_c$, i.e. $m_c = 1.5$ GeV, consistent with the quark mass rather than the $D^*$-meson mass, gives a pretty good description of the peak, but it fares rather poorly with respect to the data at $x_D > 0.7$. On the contrary, a high value of $m_c$, such as 2.1 GeV, significantly moves the peak towards large $x_D$ and worsens the overall comparison. As for the effect of the variation of $\alpha_S(m^2_Z)$, we find that it shifts the position of the peak: the lower $\alpha_S(m^2_Z)$, the higher the value of $x_D$ at which the $D$ spectrum is peaked. The dependence on the renormalization scales $\mu_R$ and $\mu_0R$ is very little, and we do not present the corresponding plots for the sake of brevity. We also varied $m_b$, the bottom-quark mass entering in the matching condition of the DGLAP evolution operator, but found out that it has a negligible impact on the energy distribution.

Overall, we can say that our model gives an acceptable description of the raise at low and average values of $x_D$, while discrepancies are present around the peak, unless one sets a relatively low value for $m_c$, and at very large $x_D$. Our curves tend to be harder than the data and approach zero at large $x_D$ more rapidly. Although the comparison at very large $x_D$ is not completely satisfactory, using the $\ln R$-matching prescription, discussed in subsection 2.2., has nonetheless improved the spectrum near the endpoint $x_D = 1$, as it is smoother and not oscillating any longer. We checked that if we had used the standard matching between NLO and resummed expressions as in [4], the charmed-meson distributions would have become negative for $x_D \gtrsim 0.9$. In any case, we are aware that our model, based on an extrapolation of perturbation theory, up to the replacement of the coupling constant $\alpha_S(k^2) \rightarrow \tilde{\alpha}_S(k^2)$, cannot be completely reliable at very large $x_D$. One
can roughly estimate \[ x_{D,\text{max}} \simeq 1 - \Lambda/m_c \simeq 0.85 \] the maximum value of \( x_D \) at which our model, or any model, such as the non-perturbative fragmentation functions \[1, 2\], based on simple parametrizations of power corrections, can be trusted.

It is therefore safe to discard few points at very large \( x_D \) and limit our analysis to \( x_D \leq 0.85 \) when evaluating the \( \chi^2 \) from the comparison with the data. Even in this range, using our default values for the parameters in the parton-level computation, we are not able to acceptably reproduce the data, as we obtain \( \chi^2/\text{dof} = 56.47/17 \). A better description of the data is nonetheless obtained if, e.g., we keep all quantities to their default values, but set \( \mu_0F = 2m_c \) (\( \chi^2/\text{dof} = 27.18/17 \)) or \( \alpha_S(m_Z^2) = 0.121 \) (\( \chi^2/\text{dof} = 30.52/17 \)). Setting \( m_c = 1.5 \) GeV, we find \( \chi^2/\text{dof} = 32.29/17 \). As we are not fitting any non-perturbative parameter to the data, such values of \( \chi^2 \) are perfectly acceptable. Also, they are of similar magnitude to those obtained in \[4\] for the comparison with \( B \)-hadron energy distributions at the \( Z^0 \) pole.

The overall impact of the inclusion of non-perturbative corrections at LEP energies via our model can be learned from Fig. 2, where we compare our best-fit prediction, i.e. the one obtained with \( \mu_0F = 2m_c \) and the other quantities set to their default values, with the purely perturbative results of Ref. \[11\], where the authors used the standard coupling constant and resummed NLL soft and collinear contributions to the coefficient function and perturbative fragmentation function. We also present in Fig. 2 the ALEPH \( D^{*+} \) data. The role played by power corrections is clearly remarkable throughout all \( x \)-spectrum, and is essential to obtain an acceptable description of the data. In fact, the parton-level calculation of \[11\] needs to be convoluted with a non-perturbative fragmentation function to reproduce the data.

Before closing this subsection, we remind that the possible reasons determining the fairly large theoretical uncertainties were already listed and detailed in \[4\]. In particular, we have resummed large-\( x \) contributions to the coefficient function and initial condition in
the NNLL approximation, but we still have matched the resummation to the NLO exact results, thus generating a mismatch between the NNLL terms in the resummed exponents \( \sim \alpha_s^2 \ln N, \) etc.) and the remainder functions. We believe that the uncertainties should be milder if we used the exact NNLO results \([12, 13]\). Moreover, lower theoretical errors should be expected if we also employed NNLL DGLAP evolution equations, using NNLO non-singlet splitting functions \([14]\). With respect to the analysis on \( B \)-hadron production, the effect of the choice of scales and masses on charmed-meson spectra is even larger: in fact, the dependence on such quantities is typically logarithmic, hence larger once they vary around smaller values, e.g. around \( m_c \) rather than \( m_b \). It is however interesting to notice that, unlike the comparison with the \( B \)-hadron data, where setting \( \mu_0 = m_b/2 \) gave the best description of the data \([3]\), the charm-fragmentation data seem to prefer a quite high value of \( \mu_0 \), since \( \mu_0 = 2m_c \) yields the best fit. We believe that a full NNLO/NNLL analysis should clarify this issue as well.

\[ \text{Figure 2: Our best-fit result for } D \text{-meson production at LEP (} j = D, \text{ solid), compared with the perturbative parton-level calculation of } [11] \text{ (} j = c, \text{ dashes) and the ALEPH } D^*+ \text{ data.} \]

4.2 Comparison with CLEO and BELLE data

We would like to compare the predictions of our model with data from the experiments CLEO \([28]\) and BELLE \([29]\), collected at the \( \Upsilon(4S) \) resonance. Such a comparison is quite interesting, since the value of the hard scale is much smaller than at LEP, and in the analysis \([26]\) it was found out that the same non-perturbative function, albeit provided with three tunable parameters, was not able to describe at the same time both LEP and \( B \)-factory \( D \)-meson data. Our case is clearly different, as our non-perturbative model is not tunable to data, but it will be nonetheless cumbersome to investigate how our predictions fare with respect to the different data sets and estimate the theoretical uncertainty.
In Fig. 3 we present the comparison with CLEO and BELLE data on $D^0$ production; both data sets are separately normalized to 1, for the sake of a consistent comparison with the theory curves, whose first moment reads $\sigma_{N=1}^{(D)} = 1$. We vary renormalization and factorization scales, $m_c$ and $\alpha_S(m_Z^2)$ along the lines of our comparison with ALEPH. Fig. 3 (a) exhibits the dependence on $\mu_F$ and $\mu_0F$; Fig. 3 (b) the one on $\alpha_S(m_Z^2)$ and $m_c$. We do not present the effect of changing $\mu_R$, $\mu_0R$ and $m_b$, since it is very little, as already found at the $Z^0$ pole.

Unlike the comparison with the ALEPH data, where, though within the experimental and theoretical uncertainties, we succeeded in getting a reasonable fit of the data, our prediction lies quite far from the CLEO and BELLE $D^0$ data and there is no choice of parameters and scales, within our ranges, which can accommodate the experimental data. In fact, such data exhibit very small errors and, even if we limit our analysis to $x_D < 0.85$, as we did before, we still obtain quite large $\chi^2$, typically $\chi^2/\text{dof} \gtrsim \mathcal{O}(10)$. It is nonetheless interesting to notice that the best comparison is obtained for $m_c = 1.5$ GeV; in this case, one is able at least to reproduce the rise of the spectrum up to $x_D \simeq 0.6$, but still incapable of describing the peak and the large-$x_D$ tail. As pointed out when comparing with ALEPH, a full NNLO/NNLL analysis is mandatory to reduce the theoretical error and should shed light on the dependence on the quark (meson) mass as well. Ref. [26] also presented $D^+$ data from CLEO and BELLE; the comparison with our predictions is however qualitatively similar to the one presented in Fig. 3 and we do not show it for brevity.

![Figure 3](image-url)

**Figure 3:** As in Fig. 1 but comparing with $D^0$ data from CLEO and BELLE experiments. In (a) we vary the factorization scales $\mu_F$ (solid) and $\mu_0F$ (dashes); in (b) $\alpha_S(m_Z^2)$ (solid) and $m_c$ (dashes).

We instead present in Fig. 4 the comparison of the predictions yielded by our model with CLEO and BELLE data on $D^{*0}$-meson production. Unlike the $D^0$ data just considered, the $D^{*0}$ spectrum measured by BELLE exhibits larger errors, so that we are able to reproduce the data at $x_D < 0.85$ with quite small $\chi^2$ values. With our default parametrization, we obtain $\chi/\text{dof} = 45.23/36$, while an even lower value, $\chi/\text{dof} = 32.10/36$, is obtained if we set $\mu_0F = 2m_c$, the same value of $\mu_0F$ yielding the best fit to the ALEPH $D^{*+}$ spec-
trum. The comparison with the CLEO $D^{*0}$ data, which instead are still affected by pretty small errors, is quite unsatisfactory and the $\chi^2$ values are pretty large.

Figure 4: As in Fig. 3, but comparing with the $D^{*0}$ data from BELLE and CLEO. The scales are varied as in Figs. 1 and 3.

Figure 5: The solid line is our best-fit prediction for $D$-hadron production at the $\Upsilon(4S)$ resonance ($j = D$); the dashed line is the purely perturbative $c$-quark spectrum ($j = c$) yielded by the computation in [11]. Also presented are the BELLE data on $D^{*0}$ production.

As done for the purpose of the comparison with the ALEPH $D^*$ data, we present in Fig. 5 the comparison with our best prediction for the $B$-factory data, i.e. the one obtained for $\mu_{QF} = 2m_c$, along with the BELLE $D^{*0}$ spectrum and the NLO/NLL perturbative prediction from Ref. [11]. We note that the parton-level result is sharply peaked at large $x$, even more than in Fig. 2. In fact, the smaller phase space available at the $\Upsilon(4S)$ resonance with respect to the $Z^0$ pole enhances the probability of producing $c\bar{c}$ pairs near
the threshold $x = 1$. Overall, the impact of non-perturbative corrections in the coupling constant at the $\Upsilon(4S)$ resonance looks even more important than at LEP energies.

5. Conclusions

We studied charm-quark fragmentation in $e^+e^-$ annihilation and used a recently proposed model, based on an effective strong coupling constant, as the only source of non-perturbative effects. Such a model was already employed in \cite{3,4} and gave a reasonable description of $b$-quark fragmentation in $e^+e^-$ annihilation and some $B$-meson decay data. We described charm-quark perturbative production following the perturbative fragmentation approach, with NLO coefficient function and initial condition of the perturbative fragmentation function, NLL DGLAP evolution and NNLL large-$x$ resummation. Resummed expressions were matched to the exact NLO ones using the so-called $\ln R$-prescription, which turned out to significantly improve the spectrum near the $x = 1$ endpoint. The effective coupling was implemented in the NNLO approximation, as in Refs. \cite{3,4}.

We compared the predictions of our model with data from ALEPH, BELLE and CLEO, corrected for initial-state photon-radiation effects as in \cite{26}. Throughout our analysis, we limited ourselves to the range $x < 0.85$ and, since our non-perturbative model has no tunable parameter, we varied the quantities in the perturbative calculation within typical ranges, according to the values quoted in \cite{30}.

We found that our model is able to acceptably describe, within the theoretical and experimental errors, the $D^{*+}$ spectrum from ALEPH and the $D^{*0}$ one from BELLE. In particular, the best fits to the data are obtained, within our chosen ranges, if we set the factorization scale entering in the initial condition to $\mu_0_F = 2m_c$. A value of $m_c$ consistent with the charm pole mass, rather than the $D$-meson mass, improves the comparison at small $x_D$ and around the peak. Significant discrepancies were instead found when comparing with other experimental spectra, such as $D^0$ data from BELLE and CLEO or $D^{*0}$ data from CLEO. The experimental distributions, in fact, exhibit very small errors and we did not manage to determine a combination of our perturbative parameters giving an acceptable value of $\chi^2$/dof. We just noticed that setting $m_c = 1.5$ GeV gives a good description of the data for $x_D < 0.6$, but some disagreement is still present for larger values of $x_D$.

In summary, we can say that our results confirm that, as found in \cite{26}, one is not able to fit all ALEPH and $B$-factory data on $c$-flavoured hadron production using the same power-correction model, i.e. the parameter-free analytic coupling constant in our case and a non-perturbative fragmentation function with three tunable parameters in \cite{26}. It is however remarkable that, although within fairly large theoretical and experimental errors and after discarding few data points at very large $x_D$, our model succeeded in reproducing ALEPH as well as BELLE data on $D^*$-meson production. The discrepancy of the prediction yielded by our model with respect to the very precise $D^0$ data from CLEO and BELLE clearly deserves further investigation. The results in this paper, along the ones reported in \cite{4}, seem to indicate that the model works quite well for heavy-quark fragmentation at the $Z^0$ pole, while problems show up once the hard scale is lowered.
It was argued in [26] that missing power corrections to the coefficient function, mostly relevant at large $x_D$ ($N$), may explain why the same non-perturbative fragmentation function is not able to fit both LEP and $B$-factory data. Our case is slightly different since, besides discrepancies at large $x_D$, some disagreement with the data at the $\Upsilon(4S)$ resonance was also found around the peak and at smaller values of $x_D$. In any case, we found out that playing around the input parameters in the perturbative calculation, and in particular setting a relatively low value for $m_c$, we managed to improve at least the comparison at small $x_D$ and around the peak. Therefore, before drawing any firm conclusion on the origin of the discrepancy with CLEO and BELLE data, we should first perform a complete NNLO/NNLL analysis, using the calculations in Refs. [12, 13, 14] and decrease the theoretical error on our results.

A similar remark holds for the purpose of Eq. (3.14) and whether the correcting term $\tilde{\delta}_S(k^2)$ should be added to our effective coupling constant to discriminate among $b$- and $c$-flavoured hadrons, spin-0 and spin-1, charged and neutral charmed mesons. Apparently, a correcting term might be necessary to accommodate at least the $D^0$ data from $B$-factories. However, before speculating about its functional form, we believe that we still need a NNLO/NNLL investigation to reduce the theoretical uncertainty and deal with a more stable theoretical prediction. In fact, without a NNLO/NNLL analysis, function $\delta S(k^2)$ may largely depend on the values chosen for the perturbative parameters and it may considerably vary according to whether, e.g., one sets $m_c = 1.5$ or $2.1$ GeV, $\mu_0 = m_c/2$ or $2m_c$, and so on. Furthermore, the change (5.12) in the $O(\alpha_S^3)$ large-$x$ resummation coefficient $A^{(3)}$ has been implemented in the threshold NNLL expressions, but not yet in the splitting functions, whose NNLO corrections do contain a contribution $\sim A^{(3)}$ [14]. Including such a term in the splitting functions, along with the redefinition $A^{(3)} \rightarrow \tilde{A}^{(3)}$, may shift the $x_D$ spectrum and possibly improve the comparison with the $B$-factory data.

Moreover, we plan to use our model to describe $D$- and $B$-hadron production at the Tevatron accelerator, along the lines of Refs. [31, 32], and extend the results to LHC energies. Also, we can use the NLO perturbative calculations in [33, 34], along with the the effective coupling constant, to predict bottomed-hadron spectra in top ($t \rightarrow bW$) or Higgs ($H \rightarrow b\bar{b}$) decays at the Tevatron and LHC. Finally, the $c$-fragmentation results here presented can be compared with the ones yielded by Monte Carlo generators, extending the analysis carried out in [35], where parton showers and resummed calculations were used to describe $B$-hadron production in $e^+e^-$ annihilation, top and Higgs decays. For such a comparison to be consistent, however, even HERWIG [36] and PYTHIA [37] will have to be tuned to the same LEP and $B$-factory data used throughout this paper. It will also be very interesting to implement the effective coupling constant to replace, e.g., the cluster model [38] which simulates the hadronization in HERWIG and investigate how the Monte Carlo results fare with respect to the experimental data on $D$- and $B$-hadron production. This is in progress as well.

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