Constitutive relations for electromagnetic field in a form of 6 × 6 matrices derived from the geometric algebra

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Abstract

To have a closed system, the Maxwell equations should be supplemented by constitutive relations which connect the primary electromagnetic fields \((E, B)\) with the secondary ones \((D, H)\) induced in a medium. Recently [Opt. Commun. 354, 259 (2015)] the allowed shapes of the constitutive relations that follow from the relativistic Maxwell equations formulated in terms of geometric algebra were constructed by author. In this paper the obtained general relativistic relations between \((D, H)\) and \((E, B)\) fields are transformed to four 6 × 6 matrices that are universal in constructing various combinations of constitutive relations in terms of more popular Gibbs-Heaviside vectorial calculus frequently used to investigate the electromagnetic wave propagation in anisotropic, birefringent, bianisotropic, chiral etc media.

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I. INTRODUCTION

The Maxwell equations are not closed. The so-called constitutive relations (CR) are needed to describe the electromagnetic (EM) field propagation in a medium. The CR connects the pair \((E, B)\) of electric and magnetic fields with the induced ones \((D, H)\) in the medium where the wave propagates, for example in a solid or plasma. A concrete form of the CR depends on mathematics used to describe the fields. In Gibbs-Heaviside (GH) analysis [1, 2], where the fields are 3D vectors, the constitutive relations between the fields are expressed by \(3 \times 3\) matrices, for example, \(D = \varepsilon E\), or in components \(D_i = \sum_{j=1}^{3} \varepsilon_{ij} E_j\), where \(\varepsilon\) is the symmetric permittivity matrix with \(\varepsilon_{ij} = \varepsilon_{ji}\) [3]. If relativistic effects are important then one introduces the metric tensor and goes either to tensor calculus [4], which is popular among physicists, or to dyadic or GH calculus [5] which are more popular within the engineering community.

A modern and more fundamental approach to electrodynamics is based on differential p-forms [6] and Clifford geometric algebras [7–11]. The former allows to formulate the premetric electrodynamics and perform initial calculations without defining the space-time metric at all. The GA requires to specify the metric to be used but, due to 8-periodicity GA theorem and natural inclusion of lower dimensional algebras and spaces as one goes to higher dimensional spaces, even for spaces characterized by different metrics, the GA automatically connects the classical and relativistic calculations in a simple manner. For electrodynamics the two of GA’s are the most important, namely, \(Cl_{3,0}\) which is related with the Euclidean 3D space and \(Cl_{1,3}\) (sometimes \(Cl_{3,1}\)) which describes the Minkowski 4D space-time and contains \(Cl_{3,0}\) as a subalgebra. In the former, \(Cl_{3,0}\) algebra, the fields \(E\) and \(D\) are represented as vectors while \(B\) and \(H\) are represented as oriented planes, i. e. planes that are spanned by two vectors \(a\) and \(b\) and have two orientations which are encoded in the outer product, \(a \wedge b\) and \(b \wedge a = -a \wedge b\). In accordance with \(Cl_{3,0}\), the electric field vector and magnetic field bivector of a propagating EM wave in the vacuum lie in a single plane, called the polarization plane. In contrast, according to GH vector analysis [1, 3], where the magnetic field is treated as an axial vector, there arise interpretational difficulties. Since in this interpretation the electric and magnetic fields are perpendicular to EM wave propagation direction (Poynting vector) there appears ambiguity as to which of the fields (electric or magnetic) represents the true polarization of EM wave in the vacuum. Usually we identify...
the polarization vector with electric field if the EM wave falls onto dielectric. However, if EM wave penetrates into magnetic material the polarization should be ascribed to the magnetic field since now the polarization is totally controlled by the magnetic susceptibility of the medium. Thus it appears that the polarization of the EM wave in vacuum depends on material properties onto which if falls. The paradox arises due to mathematical inconsistency, since the axial vector is not a member of the vectorial space where the Maxwell equations are usually constructed. In GA interpretation the magnetic field is an oriented plane (bivector) rather than the vector. The different behavior of \( \mathbf{E} \) and \( \mathbf{B} \) fields under reflection – so difficult to explain to the students – is very simple to understand if GA is appealed to. Since in 3D Euclidean space there are three mutually perpendicular planes (which may be identified with the bivectors if the arrows pointing anticlockwise are drawn on the planes), one may find one-to-one correspondence between GH axial vectors and \( \text{Cl}_3,0 \) algebra bivectors. For this purpose it is enough the GH axial vector (magnetic field) to replace by oriented basis plane which is perpendicular to the axial vector. This explains why in classical \( \text{Cl}_3,0 \) interpretation the electric and magnetic fields lie in a single bivector plane which may identified with the polarization. In addition, the Poynting vector lies in the same oriented plane. However, in 4D space one has four basis vectors and six basis planes. As a result the concept of axial vector in relativity theory fails and one must start from the beginning with new mathematics. In GA the polarization of EM wave is an integral part of the respective algebra, thus no interpretational ambiguities with the polarization arise. From GA point of view, the commonly used GH vectorial calculus \([1, 2]\) is nothing else but a crippled quaternionic algebra. The quaternions belong to \( \text{Cl}_{0,2} \) algebra, which in its turn is the subalgebra of \( \text{Cl}_{3,0} \), while the latter in its turn is the subalgebra of the relativistic \( \text{Cl}_{1,3} \) algebra. This hierarchy and coherent mathematical notation allows to formulate all physics, including the mechanics, electrodynamics, quantum mechanics and gravitation theory, in a single mathematical picture. However the most of physicists are still unaware of this new kind of mathematics.

General forms of CR’s formulated in terms of classical \( \text{Cl}_{3,0} \) and relativistic \( \text{Cl}_{1,3} \) algebras were deduced in papers \([12, 13]\). Since the GA is not widely known to physicists the aim of this paper is to display the constitutive relations that follow from space-time structure encoded in the relativistic GA in a form of \( 6 \times 6 \) matrices which can be easily transformed to Gibbs-Heaviside form and applied to investigate properties of EM waves in various media.
Before presenting the matrices it may be useful to list some of essential properties of $Cl_{1,3}$ algebra that will help the reader to grasp how the relativity and CR’s are built in this GA.

1) $Cl_{1,3}$ is constructed from four orthogonal vectors $\gamma_i$ which define the basis vectors in the space-time. $\gamma_0$ is the time axis and $\gamma_1$, $\gamma_2$, and $\gamma_3$ are the space axes. The algebra of vectors $\gamma_i$ are isomorphic to algebra of Dirac matrices usually used in the relativistic quantum mechanics. The squares of $\gamma_i$’s satisfy $\gamma_0^2 = 1$ and $\gamma_i^2 = -1$ and thus define $(+,−,−,−)$ metric of the space-time.

2) Apart from the vectors the space-time contains more geometric objects: bivectors $\gamma_i \wedge \gamma_j \equiv \gamma_{ij}$ (six oriented planes), trivectors $\gamma_i \wedge \gamma_j \wedge \gamma_l$ (four oriented 3D volumes) and the pseudoscalar which is equal to outer products of all basis vector, $I = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3$. The EM field in GA is represented by a general bivector plane that can be decomposed into six projections, or six basis bivectors $\gamma_i \wedge \gamma_j$, $i \neq j$. The number of basis bivectors is equal to the number of EM field components in relativistic electrodynamics. Three bivectors $\sigma_i \equiv \gamma_i \wedge \gamma_0$, where $\gamma_0$ is the time coordinate, are time-like, i.e. odd with respect to spatial inversion, and are connected with the electric field. Their squares are $\sigma_i^2 = +1$. The remaining bivectors $l\sigma_1 = \gamma_3 \wedge \gamma_2$, $l\sigma_2 = \gamma_1 \wedge \gamma_3$ and $l\sigma_3 = \gamma_2 \wedge \gamma_1$ that represent the magnetic field are space-like (even with respect to spatial inversion). Their squares are negative, $(l\sigma_i)^2 = -1$.

Thus, in $Cl_{1,3}$ algebra the primary EM field $F = E + B$ called the Faraday bivector can be decomposed into six elementary bivectors (projections) that represent six oriented planes in the 4D Minkowski space-time:

$$E = E_1 \sigma_1 + E_2 \sigma_2 + E_3 \sigma_3, \quad E^2 > 0,$$

$$B = B_1 l\sigma_1 + B_2 l\sigma_2 + B_3 l\sigma_3, \quad B^2 < 0. \quad (2)$$

The real-valued coefficients before time- and space-like basis bivectors mathematically can be obtained by relativistic space-time splitting operation [9]. They represent the projections of 3D electric and magnetic fields $E = (E_1, E_2, E_3)$ and $B = (B_1, B_2, B_3)$ that are accessible to experiment.

3) The important property of electrodynamics formulated in GA terms is that there is no need for additional space-time symmetry considerations. The automorphisms or involution symmetries of GA, namely, the identity, inversion, reversion and Clifford conjugation are isomorphic to discrete Gauss-Klein group of four $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ [14], which in its turn is isomorphic to group consisting of identity operation, space $P$ and time $T$ reversals, and the combina-
tion $PT$. Thus the discrete space-time symmetry operations $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \cong \{1, P, T, PT\}$ are satisfied in GA automatically. The idea that involutions define physically important subspaces was expressed for the first time by J. Dauns in his article entitled “Metrics are Clifford algebra involutions” \[15\].

From all this follows that the multiplicative CR’s that connect primary and secondary EM fields are integrated in relativistic $Cl_{1,3}$ algebra and there is no need for additional assumptions. However, the additive CR’s such as spontaneous electric $\mathbf{P}$ and magnetic $\mathbf{M}$ polarizations are not included in the results presented below. The reader who is unfamiliar with the GA may go directly to final results, i.e., to matrices (6), (7), (13) and (15) which may be used to construct various CR’s in a more conventional GH form.

II. GENERAL FORM OF CONSTITUTIVE RELATIONS FOR EM FIELDS

We shall assume that the medium is lossless, linear, and unbounded, with instantaneous response to external fields. Then the constitutive relation between the Faraday field $\mathbf{F}$ and excitation field $\mathbf{G}$ induced in the medium is

$$
\mathbf{G} = \mathbf{D} + \mathbf{H} = \chi(\mathbf{E} + \mathbf{B}) = \chi(\mathbf{F}),
$$

(3)

where the operator $\chi$ is a linear bivector-valued function of the bivector argument. Also, we shall assume that CR between $\mathbf{F}$ and $\mathbf{G}$ is local. The Maxwell equations in $Cl_{1,3}$ written in terms of primary $\mathbf{F}$ and secondary $\mathbf{G}$ fields can be found in \[9\]. We shall adopt that the vacuum constants are normalized, $\varepsilon_0 = \mu_0 = 1$, so that Eq. (3) is dimensionless. Conversion to dimensional form was given in \[13\].

If $\mathbf{F}$ and $\mathbf{G}$ are columns that represent component of the bivectors in the order ($\sigma_1, \sigma_2, \sigma_3, I\sigma_1, I\sigma_2, I\sigma_3$) than the relation between the primary and secondary fields in (3) can be expressed through $6 \times 6$ matrix $\hat{\chi}$ with elements $\hat{\chi}_{ij,kl} = \gamma_{ij} \cdot (\hat{\chi} \gamma_{kl})$, where $ij$ and $kl$ are compound indices that run from 1 to 6, and where the dot means the inner GA product. Then the transformation (3) becomes

$$
\begin{pmatrix}
\mathbf{D} \\
\mathbf{H}
\end{pmatrix} = \begin{pmatrix}
\text{Diel.Birefr.} & \text{Fizeau} \\
\text{Fizeau} & \text{Magn.Birefr.} \\
\text{Diel.Faraday} & \text{Opt.Act.} \\
\text{Opt.Act.} & \text{Magn.Faraday}
\end{pmatrix}
\begin{pmatrix}
\mathbf{E} \\
\mathbf{B}
\end{pmatrix}.
$$

(4)
3 x 3 submatrices in (4) are named according to some characteristic physical effect they represent. The first 6 x 6 matrix is symmetric. Its diagonal blocks are related with dielectric and magnetic birefringence. The off-diagonal blocks mix electric and magnetic fields and are responsible for the Fizeau effect. The second asymmetric matrix is responsible for dielectric and magnetic Faraday effects and optical activity. The division into two, symmetric and antisymmetric matrices in (4), is related with the adjoint transformation in GA (analogue of transpose in the matrix notation under which matrix rows and columns are interchanged with an appropriate sign). Since an arbitrary adjoint transformation in GA can be written as a sum of symmetric and antisymmetric transformations [9], the two matrices in (4) correspond to this partition. The constitutive relation is an operator $\chi$ which acts on the bivector $F$ and returns new bivector $G = \chi(F)$ in the Minkowski space. In the second, antisymmetric matrix of (4), after transformation, in addition, the orientation of the new bivector $G$ is changed to opposite. One can also imagine that opposite surfaces of the resulting plane $G$ have been interchanged. This is a geometric content of the CR that follows from $Cl_{1,3}$ algebra. The described linear GA transformation can be cast into 6 x 6 matrix that describes how the primary EM field components go to secondary field components.

In the following we shall rewrite the equation (4) in terms of coefficients that correspond to transformations within or between time-like $\sigma$ and space-like $l\sigma$ triads of elementary bivectors

$$
\begin{bmatrix}
\sigma \\
l\sigma
\end{bmatrix} = 
\begin{pmatrix}
\varepsilon & \gamma^s, \gamma^a \\
\gamma^s, \gamma^a & \mu^{-1}
\end{pmatrix}_{\text{sym}} + 
\begin{pmatrix}
n & s^s, s^a \\
s^s, s^a & m
\end{pmatrix}_{\text{antisym}}
\begin{bmatrix}
\sigma \\
l\sigma
\end{bmatrix},
$$

(5)

where various effects were replaced by respective symbols they represent. In short, the individual submatrices in (5) interconnect general time-like $\sigma$ and space-like $l\sigma$ GA bivectors. If only the upper-left block of the first “sym” matrix is taken into account then we have a constitutive relation that connects time-like bivector with time-like bivector, i. e. we have a symmetrical $(\sigma, \sigma)_s$ coupling between primary and secondary fields. There are more couplings such as $(\sigma, l\sigma)_s$, $(l\sigma, \sigma)_s$, and $(l\sigma, l\sigma)_s$ that will represent different linear electromagnetic and optical effects. Similarly, the second, antisymmetric part provides a set of bivector couplings $(\sigma, \sigma)_a$, $(\sigma, l\sigma)_a$, $(l\sigma, \sigma)_a$, and $(l\sigma, l\sigma)_a$ which represent different
physical effects and which have, as mentioned, opposite orientations of secondary EM field bivectors.

III. SYMMETRIC PART

The symmetric part consists of three different $3 \times 3$ submatrices that correspond to \((\sigma, \sigma)_s\), \((l\sigma, l\sigma)_s\) and \((\sigma, l\sigma)_s = (l\sigma, \sigma)_s\) couplings.

a. \((\sigma, \sigma)_s\) and \((l\sigma, l\sigma)_s\) couplings — electrical and magnetic birefringence. The coupling \((\sigma, \sigma)_s\) describes the effect of anisotropic dielectric on light propagation, for example the birefringence of light in a quartz. The permittivity submatrix $\hat{\varepsilon}$ generated by GA transformation is symmetric, $\varepsilon_{ij} = \varepsilon_{ji}$, and consists of 6 independent scalars which connect two time-like bivectors, $E$ and $D$, as the transformation symbol $(\sigma, \sigma)_s$ implies.

Similarly, in the $(l\sigma, l\sigma)_s$ coupling the space-like bivector $B$ is transformed to other space-like bivector $H$. It corresponds to the inverse permeability matrix $\hat{\mu}^{-1}$ which is characterized by six scalars too. So the compound transformation describes the bianisotropic medium which transforms the EM field $F = E + B$ to $D + H = \chi_{\sigma,\mu^{-1}}(F)$. The GA transformation operator $\chi_{\sigma,\mu^{-1}} = \chi_{\varepsilon} + \chi_{\mu^{-1}}$ yields the matrix $\hat{\chi}_{\varepsilon,\mu^{-1}}$ with elements $\gamma_{ij} \cdot \chi_{\varepsilon,\mu^{-1}}(\gamma_{kl})$,

$$
\hat{\chi}_{\varepsilon,\mu^{-1}} = \begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & 0 & 0 & 0 \\
\varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} & 0 & 0 & 0 \\
\varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & \mu_{11} & \mu_{12} & \mu_{13}^{-1} \\
0 & 0 & 0 & \mu_{12}^{-1} & \mu_{22} & \mu_{23}^{-1} \\
0 & 0 & 0 & \mu_{13}^{-1} & \mu_{23}^{-1} & \mu_{33}
\end{bmatrix}.
$$

(6)

It is understood that this matrix acts on the column-vector $(E_1, E_2, E_3, B_1, B_2, B_3)^T$, or in short $(E, B)^T$ where T means “Transpose”. The result is the excitation column-vector $(D, H)^T$.

b. $(\sigma, l\sigma)_s$ coupling — Fresnel-Fizeau effect. This coupling transforms the space-like bivector to time-like bivector, $\sigma_i \rightarrow l\sigma_j$, or vice versa, $l\sigma_i \rightarrow \sigma_j$. Since the structure of upper-right and lower-left Fresnel-Fizeau submatrices in (5) is similar (the submatrices are antisymmetric with respect to main diagonal), the structure of both transformations, $\sigma \rightarrow l\sigma$ and $l\sigma \rightarrow \sigma$, is similar too. In its turn, the individual $3 \times 3$ submatrices may be decomposed into sum of even and odd parts, consequently the transformation $\chi_{\gamma} = \chi_{\gamma}(F)$
can be divided into sum of matrices \( \hat{\chi}_\gamma = \hat{\chi}_s^\gamma + \hat{\chi}_a^\gamma \), where the superscripts \( s \) and \( a \) indicate the symmetric and antisymmetric parts. Thus we find that the most general transformation which is allowed by GA can be rewritten in the following matrix form \[13\],

\[
\hat{\chi}_\gamma = \begin{bmatrix}
0 & 0 & 0 & \gamma_{11}^s & \gamma_{12}^a - \gamma_{12}^a & \gamma_{13}^a - \gamma_{13}^a \\
0 & 0 & 0 & \gamma_{12}^s - \gamma_{12}^s & \gamma_{22}^a & \gamma_{23}^a + \gamma_{23}^a \\
0 & 0 & 0 & \gamma_{13}^s + \gamma_{13}^a & \gamma_{23}^a - \gamma_{23}^a & \gamma_{33}^a \\
\gamma_{12}^s - \gamma_{12}^a & \gamma_{22}^s & \gamma_{23}^s - \gamma_{23}^a & 0 & 0 & 0 \\
\gamma_{13}^s + \gamma_{13}^a & \gamma_{23}^s + \gamma_{23}^a & \gamma_{33}^a & 0 & 0 & 0 \\
\gamma_{12}^s - \gamma_{12}^a & \gamma_{23}^s + \gamma_{23}^a & \gamma_{33}^a & 0 & 0 & 0 \\
\end{bmatrix}. \tag{7}
\]

The matrix (7) represents various nonreciprocal effects. All in all, it contains 9 independent scalar parameters, three of which \((\gamma_{12}^a, \gamma_{13}^a, \gamma_{23}^a)\) belong to the skew symmetric part of submatrices and can be represented by a vector.

The combined action of EM birefringence and Fizeau effects is given by sum of (6) and (7). In a simple case when the permittivity and permeability are scalars, and the coupling \((\sigma, I\sigma)_s\) has vectorial form the sum simplifies to

\[
\hat{\chi}_s = \begin{bmatrix}
\varepsilon & 0 & 0 & 0 & \gamma_{12}^a & -\gamma_{13}^a \\
0 & \varepsilon & 0 & -\gamma_{12}^a & 0 & \gamma_{23}^a \\
0 & 0 & \varepsilon & \gamma_{13}^a & -\gamma_{23}^a & 0 \\
0 & -\gamma_{12}^a & \gamma_{13}^a & \mu^{-1} & 0 & 0 \\
\gamma_{12}^a & 0 & -\gamma_{23}^a & 0 & \mu^{-1} & 0 \\
-\gamma_{13}^a & \gamma_{23}^a & 0 & 0 & 0 & \mu^{-1} \\
\end{bmatrix}. \tag{8}
\]

In the standard GH notation with the axial magnetic field vector \( \mathbf{B} \) introduced, the matrix (8) leads to the following constitutive relation between the fields

\[
\begin{align*}
\mathbf{D} &= \varepsilon \mathbf{E} - \gamma^a \times \mathbf{B}, \\
\mathbf{H} &= \gamma^a \times \mathbf{E} + \mu^{-1} \mathbf{B},
\end{align*}
\tag{9, 10}
\]

where the cross indicates standard vectorial product and \( \gamma^a = (\gamma_{23}, \gamma_{13}, \gamma_{12}) \) is the coupling vector. The same form of the constitutive relations was obtained earlier using simpler \( Cl_{3,0} \) algebra \[12\], which describes the classical electrodynamics in 3D Euclidean space and where time is a parameter rather than the space-time vector. In (9) and (10) the anisotropy
of the medium is characterized by coupling vector $\gamma^a$, which may be, for example, the velocity $v$ of a fluid as in the Fizeau experiment, or the velocity of a moving dielectric slab. However it should be noted that the matrix (7) was generated by $Cl_{1,3}$ algebra that describes the Minkowski space-time and therefore this matrix represents the relativistic constitutive relation. The matrix (7) can be applied even when the velocity of medium approaches to the light velocity. Finally, we shall remind that the total relativistic matrix $\hat{\chi}_\gamma$ is symmetric while the asymmetry appears in individual $3 \times 3$ blocks only. In conclusion, the symmetric matrices (6) and (7) represent transformation that corresponds to adjoint symmetric GA transformation. In GA terms it can be written as a sum $G = (D + H) = \chi_{\text{sym}} F = (\chi_\varepsilon + \chi_{\mu^{-1}} + \chi_\gamma) F$, where linear GA transformation operators have their respective matrix analogues $\hat{\chi}_{\text{sym}} = \hat{\chi}_\varepsilon + \hat{\chi}_{\mu^{-1}} + \hat{\chi}_\gamma$ in equations (6) and (7).

**IV. ANTISYMMETRIC PART**

The second, antisymmetric term in (5), in addition, takes into account the change of bivector direction after the transformation, figuratively speaking, due to interchange of opposite surface colors of the oriented bivector plane. This 4D space-time property of $Cl_{1,3}$ which is encoded in the automorphisms has no analogy in the classical electrodynamics [12].

c. $(\sigma, \sigma)_a$ and $(l\sigma, l\sigma)_a$ couplings — electric and magnetic Faraday effects. In the electric Faraday effect characterized by $(\sigma, \sigma)_a$ coupling, the electric field $\mathcal{E}$ induces an excitation or displacement bivector $\mathcal{D}$ of the form

$$\mathcal{D} = \chi_n(\mathcal{E}) = (E_3 n_2 - E_2 n_3) \sigma_1 +$$

$$+ (E_1 n_3 - E_3 n_1) \sigma_2 + (E_2 n_1 - E_1 n_2) \sigma_3,$$

(11)

where $n_1, n_2, n_3$ are material parameters which in the standard notation are elements of $3 \times 3$ skew symmetric matrix.

In the magnetic Faraday effect characterized by $(l\sigma, l\sigma)_a$ coupling the magnetic field bivector $\mathcal{B}$ is transformed to excitation bivector $\mathcal{H}$. Similarly, this transformation can be expressed by material bivector $m = m_1 l\sigma_1 + m_2 l\sigma_2 + m_3 l\sigma_3$, where $m_i$ are the scalars that are magnitudes of the projections of space-like bivector and define the strength and angular properties of the physical effect. If transformed to the coordinate form, this transformation
is similar to (11)
\[
\mathcal{H} = \chi_m(\mathcal{B}) = (B_2 m_3 - B_3 m_2) I \sigma_1 + \\
(B_3 m_1 - B_1 m_3) I \sigma_2 + (B_1 m_2 - B_2 m_1) I \sigma_3.
\] (12)

Both the Faraday electric and magnetic transformations can be rearranged as a single matrix [13]

\[
\hat{\chi}_{n,m} = \\
\begin{pmatrix}
0 & -n_3 & n_2 & 0 & 0 & 0 \\
n_3 & 0 & -n_1 & 0 & 0 & 0 \\
-n_2 & n_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & m_3 & -m_2 & \\
0 & 0 & 0 & -m_3 & 0 & m_1 \\
0 & 0 & 0 & m_2 & -m_1 & 0 \\
\end{pmatrix}
\] (13)

which may be cast into the standard GH vectorial form as two constitutive relations,

\[
\mathbf{D} = \mathbf{n} \times \mathbf{E}, \quad \mathbf{H} = -\mathbf{m} \times \mathbf{B}
\] (14)

where \( \mathbf{n} = (n_1, n_2, n_3) \) and \( \mathbf{m} = (m_1, m_2, m_3) \) are the material vectors.

d. \((\sigma, I\sigma)_a\) coupling — optical activity. In the optical activity, the electric field \( \mathcal{E} \) is converted to magnetic excitation \( \mathcal{H} \). Similar conversion occurs between \( \mathcal{B} \) and \( \mathcal{D} \). Expression (15) shows that the optical activity is described by two off-diagonal submatrices. As follows from GA, both submatrices should be equal but have opposite signs so that the optical activity in fact is characterized by a single 3 \times 3 submatrix with 9 parameters which can be expressed as a sum of symmetric and skew symmetric parts \( \hat{\chi}_s \) and \( \hat{\chi}_a \). In a matrix form the combined transformation \( \hat{\chi}_a = \hat{\chi}_s + \hat{\chi}_a \) has the following structure,

\[
\hat{\chi}_a = \\
\begin{pmatrix}
0 & 0 & 0 & s_{11}^s & s_{12}^s - s_3^a & s_{13}^s + s_2^a \\
0 & 0 & 0 & s_{21}^s + s_3^a & s_{22}^s & s_{23}^s - s_1^a \\
0 & 0 & 0 & s_{31}^s - s_2^a & s_{32}^s + s_1^a & s_{33}^a \\
-s_{11}^s & -s_{21}^s - s_3^a & -s_{31}^s + s_2^a & 0 & 0 & 0 \\
-s_{12}^s + s_3^a & -s_{22}^s & -s_{32}^s - s_1^a & 0 & 0 & 0 \\
-s_{13}^s - s_2^a & -s_{23}^s + s_1^a & -s_{33}^a & 0 & 0 & 0 \\
\end{pmatrix}
\] (15)

where without losing notational generality it may be assumed that \( s_{ij}^s = s_{ji}^a \).

Then, in the symmetric part \( \hat{\chi}_s \) we can select 6 independent scalar components which geometrically can be represented by material ellipsoid. In the skew symmetric part \( \hat{\chi}_a \) we
can select 3 independent components and represent them by vector \( s^a = (s_1^a, s_2^a, s_3^a) \). Thus, in the standard vectorial notation the transformation matrix (15) can be represented as

\[
D = \hat{\chi}_s^a B + s^a \times B, \quad (16)
\]

\[
H = s^a \times E - \hat{\chi}_s^a E. \quad (17)
\]

The symmetric part \( \hat{\chi}_s^a \) is related with the chirality of medium. If \( s_{ij}^a = 0 \) when \( i \neq j \) and \( s_{11}^a = s_{22}^a = s_{33}^a \) then we have a single chiral coupling constant between primary and secondary fields, the case which is frequently met in the computer modelling of chiral materials. The cross-product terms in (16) and (17) also appear in the Fizeau effect, equations (9) and (10). However, in contrast to equations (9) and (10), now we see that the coupling vector \( s^a \) in equations (16) and (17) has the same sign. This difference in signs brings about principally different physical effects.

e. Axionic part. Axionic contribution comes from the trace of the full transformation matrix (5), actually from the trace of the symmetric part of \( 6 \times 6 \) matrix. The respective CR is described by product of unit matrix and scalar constant (trace) \( \alpha \). In GA where the metric is predetermined the axionic term does not appear. It should be remarked that at present it remains unclear whether the axionic field exists at all. The experimental attempts to detect this field and respective particle (axion) up till now were unsuccessful.

To sum up, the Clifford geometric algebra provides a coherent picture of how the constitutive relations for a homogeneous EM media originate from all possible linear transformations between time-like and space-like bivectors of relativistic \( Cl_{1,3} \) algebra and automorphisms (involutions) of this algebra without any need for additional assumptions on space-time properties. In the paper the obtained set of 36 independent real coefficients generated by \( Cl_{1,3} \) algebra coincides with that found by E. J. Post from space-time symmetry consideration using the tensorial calculus. All possible transformations that follow from internal GA structure here were cast into a form of \( 6 \times 6 \) matrices (6), (7), (13) and (15) which in turn were separated into \( 3 \times 3 \) susceptibility submatrices according to physical effects they describe. The obtained constitutive \( 3 \times 3 \) matrices were found to be either symmetric (represented by ellipsoids) or skew symmetric (represented by vectors). They may be applied to investigate EM wave propagation in optics and electrodynamics.

The obtained \( 6 \times 6 \) susceptibility matrices (6), (7), (13) and (15) are rather general. Since they are related with the linear transformations between primary \( (E, B) \) and sec-
ondary \((D, H)\) fields, they may be added in various combinations to produce a multitude of physical effects in EM wave propagation. Since the geometric algebra is real the coupling coefficients may assume real values only including zero, of course. After transformation of constitutive matrices to the GH notation the EM fields may be treated as complex vectors as it is usually done when the EM wave propagation equations are written directly in this notation. However, the summation of smaller \(3 \times 3\) susceptibility submatrices which, as mentioned, appear in the frequently used GH notation, must be in accord with the symmetry and summation rules for larger \(6 \times 6\) matrices \(6, 7, 13\) and \(15\) to avoid forbidden combinations between pairs of polar \((E, B)\) and axial \((D, H)\) fields.

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