SERRE FIBRATIONS IN THE MORITA CATEGORY OF TOPOLOGICAL GROUPOIDS

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Abstract. In this paper, we generalize the notion of Serre fibration to the Morita category of topological groupoids and derive the associated long exact sequence of homotopy groups. We use this results for calculation of homotopy groups of various groupoids, such as the foliation groupoid of a Riemannian foliation.

1. Introduction

Topological groupoids are objects that we can use to represent singular geometric structures. For example, the leaf space of a foliation may not contain much information about the foliation, but it is well represented by the associated holonomy groupoid [12]. By reducing this groupoid to a complete transversal of the foliation, we obtain a different groupoid which represents the foliation as well as the holonomy groupoid and is Morita equivalent to the holonomy groupoid. Morita equivalence is a relation between topological groupoids, generated by the functors between topological groupoids which are called weak equivalences. Such functors are, in particular, equivalences of categories and induce homeomorphisms between the spaces of orbits of the topological groupoids [11, 12, 14]. There is the Morita category of topological groupoids in which two topological groupoids are isomorphic precisely if they are Morita equivalent. The morphisms in this category can be represented by principal bundles.

It is essential to have a way of distinguishing topological groupoids that are not Morita equivalent. There are some interesting classes of topological groupoids that are closed under Morita equivalence, such as proper groupoids and foliation groupoids [12]. A classical way to construct invariants of topological groupoids is based on the classifying space associated to a topological groupoid, which is determined uniquely up to homotopy equivalence. There are basically two standard constructions of this space, the Milnor’s infinite join construction and geometric realization of the nerve of topological groupoid [1, 3, 4, 5, 10, 11, 17, 19]. One can show that a weak equivalence between topological groupoids induces a weak homotopy equivalence between the associated classifying spaces. We can therefore study the homotopy invariants of the classifying space as the Morita invariants of the topological groupoid. With this approach, some very interesting results can be obtained. For example, the Haefliger theorem tells us that classifying space of the holonomy groupoid of a foliation on a manifold $M$ is homotopy equivalent to $M$ if and only if the holonomy cover of every leaf of the foliation is contractible [5]. In general, however, the classifying space of a topological groupoid is a complicated space and can be difficult to understand and use. Furthermore, it can be deprived of some additional geometric structure which is present on the level of groupoids, such as the smooth structure of the holonomy groupoid. An alternative approach is to define Morita invariants of topological groupoids in a manner that does not directly

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rly on the classifying space. For example, one can describe the fundamental group
of an étale groupoid $\mathcal{G}$ in terms of $\mathcal{G}$-paths [16] or the fundamental group of an
orbifold $Q$ in terms of deck transformations of the universal cover of $Q$ [15] (for
higher homotopy groups of orbifolds, see also [2]).

In this paper we study the homotopy groups of topological groupoids and
their description which is intrinsic in the Morita category and does not involve
the classifying space. To achieve this, we first generalize the Morita category of
topological groupoids to the Morita category of pairs of topological groupoids. In this
category, the $n$-th homotopy group $\pi_n(\mathcal{G}, a_0)$ of a topological groupoid $\mathcal{G}$
with base point $a_0$ can be described as the set of homotopy classes of Morita maps from
the pair $(I^n, \partial I^n)$ to the pointed topological groupoid $(\mathcal{G}, a_0)$, with natural multiplication
induced by concatenation [16]. We show that this group is isomorphic to the
$n$-th homotopy group of the associated classifying space. With our definition, it
is straightforward that $\pi_n$ is a functor, defined on the Morita category of topological
groupoids and therefore a Morita invariant.

One of our objectives is to develop some methods for efficient calculation of
homotopy groups of topological groupoids. We show that for every Morita map $P$
between pointed topological groupoids $(\mathcal{H}, b_0)$ and $(\mathcal{G}, a_0)$ there is a natural long
exact sequence of groups

$$
\ldots \to \Sigma_n(P) \to \pi_n(\mathcal{H}, b_0) \xrightarrow{\tau_n(P)} \pi_n(\mathcal{G}, a_0) \to \Sigma_{n-1}(P) \to \ldots
$$

for some groups $\Sigma_n(P)$ depending on $P$ which we describe explicitly in terms of
principal bundles. Furthermore, we generalize the notion of Serre fibration to the
Morita category of topological groupoids (we emphasize that this definition is in-
trinsic in the Morita bicategory of topological groupoids). We show that for a
Morita map $P$ that is a Serre fibration, one can identify the groups $\Sigma_n(P)$ with
the homotopy groups of some topological groupoid, which is naturally associated
to the Morita map $P$ and can be viewed as some sort of a fiber of $P$.

In the rest of the paper we apply these results to many concrete examples. We
consider a class of so called Serre groupoids (which have the property that their
source map is a Serre fibration in the classical sense) and show that the calculation
of homotopy groups of such groupoids is especially simple. Examples of Serre
groupoids are action groupoids and holonomy groupoids of some foliations.

It will turn out that there are many functors between topological groupoids that
are Serre fibrations as Morita maps. For instance, the functor from the holonomy
groupoid of the foliation of transverse principal bundle to the holonomy groupoid of
the foliation of the base space is a Serre fibration. This turns out to be useful in the
calculation of homotopy groups of the holonomy groupoid of Riemannian foliation
on a compact manifold. Another important example of a Serre fibration is the
effect functor $\text{Eff}$ from a proper étale groupoid $\mathcal{G}$ to the associated proper effective
étale groupoid $\text{Eff}(\mathcal{G})$. We will prove that the functor $\text{Eff}$ induces an isomorphism
on the $n$-th homotopy group for $n \geq 3$, whereas the relation between the first and
the second homotopy groups of $\mathcal{G}$ and $\text{Eff}(\mathcal{G})$ is expressed in terms of an exact
sequence.

2. Topological Groupoids and Principal Bundles

In this section we first recall some definitions and facts of the theory of topo-
logical groupoids and principal bundles. We introduce the Morita category of pairs
of topological groupoids and give the definition of homotopy groups of topological
groupoids in terms of Morita maps.

Throughout this notes, we will say that a (continuous) map $f : X \to Y$ between
topological spaces has local sections if for any point $y \in Y$ there exists a map
$\sigma : V \to X$, defined on an open neighbourhood $V$ of $y$ in $Y$, such that $f \circ \sigma = \text{id}_V$.

It is important to note that any map with local sections is a quotient map. More generally, a map of topological pairs $f : (X, A) \to (Y, B)$ has local sections if for any point $y \in Y$ there exists a map of topological pairs $\sigma : (V, V \cap B) \to (X, A)$, defined on an open neighbourhood $V$ of $y$ in $Y$, such that $f \circ \sigma = \text{id}_V$.

**Topological groupoid.** A groupoid $\mathcal{G}$ is a small category such that every arrow in $\mathcal{G}$ is invertible. A topological groupoid is a groupoid $\mathcal{G}$ such that its space of arrows $\mathcal{G}_1$ and its space of objects $\mathcal{G}_0$ are both topological spaces and all of the structure maps of $\mathcal{G}$ are continuous. Note that, in particular, the source map $s : \mathcal{G}_1 \to \mathcal{G}_0$ and the target map $t : \mathcal{G}_1 \to \mathcal{G}_0$ both have local sections - in fact, they both have a global section $\text{uni} : \mathcal{G}_0 \to \mathcal{G}_1$, which maps an object $x$ of $\mathcal{G}$ to the unit arrow $1_x$ at $x$.

A topological groupoid $\mathcal{G}$ is often denoted also by $(\mathcal{G}_1 \rightrightarrows \mathcal{G}_0)$.

First examples of topological groupoids are topological spaces: any topological space $X$ can be viewed as a (unit) topological groupoid $(X \rightrightarrows X)$. Any topological group $G$ is a topological groupoid with only one object, thus $(G \rightrightarrows *)$. The product of topological groupoids $\mathcal{G}$ and $\mathcal{H}$ is the topological groupoid $((\mathcal{G}_1 \times \mathcal{H}_1) \rightrightarrows (\mathcal{G}_0 \times \mathcal{H}_0))$.

**Groupoid action.** Let $\mathcal{G}$ be a topological groupoid and $X$ a topological space. A right $\mathcal{G}$-action on $X$ along a map $\epsilon : X \to \mathcal{G}_0$ is a map $\mu : X \times_{\mathcal{G}_0} \mathcal{G}_1 \to X$ (we write $\mu(x, g) = xg$), defined on the pullback $X \times_{\mathcal{G}_0} \mathcal{G}_1 = \{(x, g) \epsilon(x) = t(g)\}$, such that

(i) $\epsilon(xg) = s(g)$,
(ii) $x1_{\epsilon(x)} = x$, and
(iii) $(xg)g' = x(rg')$ for any $x \in X$, $g, g' \in \mathcal{G}_1$ with $\epsilon(x) = t(g)$ and $\epsilon(g) = t(g')$.

The space of orbits of such an action is denoted by $X/\mathcal{G}$. For such an action $\mu$ we have the associated translation groupoid $X \times \mathcal{G} = (X \times_{\mathcal{G}_0} \mathcal{G}_1 \rightrightarrows X)$, for which the source map is $\mu$ and the target map is the first projection. The multiplication $X \times \mathcal{G}$ is determined by $(x, g)(x', g') = (x, gg')$.

Analogously, one defines a left action of a topological groupoid $\mathcal{H}$ on $X$ and the associated translation groupoid $\mathcal{H} \times X$.

**Principal bundle.** Let $\mathcal{G}$ and $\mathcal{H}$ be topological groupoids. A $\mathcal{G}$-principal bundle over $\mathcal{H}$ is a space $P$, equipped with a left $\mathcal{H}$-action along a map $\pi : P \to \mathcal{H}_0$ and a right $\mathcal{G}$-action along a map $\epsilon : P \to \mathcal{G}_0$ such that

(i) the map $\pi : P \to \mathcal{H}_0$ has local sections,
(ii) for every $p \in P$ and $g \in \mathcal{G}_1$ with $\epsilon(p) = t(g)$ we have $\pi(pg) = \pi(p)$,
(iii) for every $p \in P$ and $h \in \mathcal{H}_1$ with $s(h) = \pi(p)$ we have $\epsilon(\pi(p)) = \epsilon(p)$,
(iv) the actions of $\mathcal{H}$ and $\mathcal{G}$ on $P$ commute with each other, i.e. $h(pg) = (hp)g$ for every $h \in \mathcal{H}_1$, $p \in P$ and $g \in \mathcal{G}_1$ with $s(h) = \pi(p)$ and $\epsilon(p) = t(g)$, and
(v) the map $\mu : P \times_{\mathcal{G}_0} \mathcal{G}_1 \to P \times_{\mathcal{H}_0} P$, $(x, g) \mapsto (x, xg)$, is a homeomorphism.

Any principal $\mathcal{G}$-bundle over $\mathcal{H}$ induces a map $\vartheta : P \times_{\mathcal{H}_0} P \to \mathcal{G}_1$, called the translation map, uniquely determined with the property that $p' = p \vartheta(p, p')$. Also note that the space $\mathcal{H}_0$ is homeomorphic to the orbit space $P/\mathcal{G}$, because the map $\pi$ has local sections and is therefore quotient map.

We will say that a principal $\mathcal{G}$-bundle $P$ over $\mathcal{H}$ is numerable, if the map $\pi : P \to \mathcal{H}_0$ has sections over a numerable covering of $\mathcal{H}_0$. Recall from [3][8] that a covering of a topological space is numerable if it is an open covering with subordinated locally finite partition of unity.

Let $P$ and $P'$ be principal $\mathcal{G}$-bundles over $\mathcal{H}$. A map $\varphi : P \to P'$ is a morphism of principal $\mathcal{G}$-bundles over $\mathcal{H}$ if $\epsilon = \epsilon' \circ \varphi$, $\pi = \pi' \circ \varphi$ and $\varphi$ is $\mathcal{H}$-$\mathcal{G}$-equivariant, namely $\varphi(hpg) = h\varphi(p)g$ for every $h \in \mathcal{H}_1$, $p \in P$ and $g \in \mathcal{G}_1$ with $s(h) = \pi(p)$ and
subgroupoid \( A \). A groupoids, called a marked topological groupoid. If \( P, P' \) principal \(( G, \pi )\) over \( \mathcal{H} \) and a principal \( \mathcal{H}' \)-bundle over \( \mathcal{H} \), there is a well defined tensor product \( Q \otimes P = Q \otimes_{\mathcal{H}} P \) which is a principal \( \mathcal{G} \)-bundle over \( \mathcal{H}' \). The space \( Q \otimes P \) is the space of orbits of the diagonal \( \mathcal{H}' \)-action on the fibered product \( Q \times_{\mathcal{H}_0} P \). The Morita category

\[ \text{GPD} \]

of topological groupoids is the category with topological groupoids as objects and isomorphism classes of principal bundles as morphisms, and with composition induced by the tensor product. (Considering the principal bundles, rather than their isomorphism classes, as 1-morphisms, and morphisms of principal bundles as 2-morphisms, one obtains the Morita bicategory of topological groupoids.) The category of topological spaces can be regarded as a full subcategory of the Morita category \( \text{GPD} \) in the obvious way.

**Pairs of topological groupoids.** A pair of topological groupoids is a pair \( ( \mathcal{G}, \mathcal{G}' ) \) such that \( \mathcal{G} \) is a topological groupoid and \( \mathcal{G}' \) a subgroupoid of \( \mathcal{G} \). Let \( ( \mathcal{H}, \mathcal{H}' ) \) and \( ( \mathcal{G}, \mathcal{G}' ) \) be pairs of topological groupoids. A principal \( ( \mathcal{G}, \mathcal{G}' ) \)-bundle over \( ( \mathcal{H}, \mathcal{H}' ) \) is a pair of topological spaces \( ( P, P' ) \) such that \( P \) is a principal \( \mathcal{G} \)-bundle over \( \mathcal{H} \), \( P' \) is a principal \( \mathcal{G}' \)-bundle over \( \mathcal{H}' \) with respect to the structure given by the restriction of the structure of \( P \) to \( P' \subset P \) and the map of topological pairs \( \pi : ( P, P' ) \to ( \mathcal{H}_0, \mathcal{H}'_0 ) \) has local sections.

We will say that the principal \( ( \mathcal{G}, \mathcal{G}' ) \)-bundle \( ( P, P' ) \) over \( ( \mathcal{H}, \mathcal{H}' ) \) is numerable if the map \( \pi : ( P, P' ) \to ( \mathcal{H}_0, \mathcal{H}'_0 ) \) has sections (as maps of topological pairs) over a numerable covering.

A morphism \( \phi \) from a principal \( ( \mathcal{G}, \mathcal{G}' ) \)-bundle \( ( P, P' ) \) over \( ( \mathcal{H}, \mathcal{H}' ) \) to a principal \( ( \mathcal{G}, \mathcal{G}' ) \)-bundle \( ( R, R' ) \) over \( ( \mathcal{H}, \mathcal{H}' ) \) is a morphism \( \phi : P \to R \) of principal \( \mathcal{G} \)-bundles over \( \mathcal{H} \) such that \( \phi(P') \subset R' \).

Let \( ( P, P' ) \) be a principal \( ( \mathcal{G}, \mathcal{G}' ) \)-bundle over \( ( \mathcal{H}, \mathcal{H}' ) \) and \( ( Q, Q' ) \) a principal \( ( \mathcal{H}, \mathcal{H}' ) \)-bundle over \( ( \mathcal{H}, \mathcal{H}' ) \). Then the tensor product \( ( Q, Q' ) \otimes ( P, P' ) = ( Q \otimes P, Q' \otimes P' ) \) is a principal \( ( \mathcal{G}, \mathcal{G}' ) \)-bundle over \( ( \mathcal{H}, \mathcal{H}' ) \) (note that we can naturally identify \( Q' \otimes P' \) with a subset of \( Q \otimes P \)).

The Morita category

\[ \text{GPD}^2 \]

of pairs of topological groupoids is the category with pairs of topological groupoids as objects and isomorphism classes of principal bundles as morphisms, and with composition induced by the tensor product. The category of pairs of topological spaces can be regarded as a full subcategory of the Morita category \( \text{GPD}^2 \).

**Pointed and marked topological groupoids.** Let \( \mathcal{G} \) be a topological groupoid and \( A \) a subset of \( \mathcal{G}_0 \). We can view the set \( A \) as a unit subgroupoid of \( \mathcal{G} \), namely the subgroupoid \( A = \text{uni}(A) \subset \mathcal{G} \). In this view, the pair \( ( \mathcal{G}, A ) \) is a pair of topological groupoids, called a marked topological groupoid. If \( A \) is a singleton set \( \{ a_0 \} \), then the marked topological groupoid \( ( \mathcal{G}, \{ a_0 \} ) \) is denoted simply by \( ( \mathcal{G}, a_0 ) \) and called a pointed topological groupoid. If \( ( P, P' ) \) is a principal \( ( \mathcal{G}, A ) \)-bundle over \( ( \mathcal{H}, \mathcal{H}' ) \), then we will rather see \( P' \) as a section \( \sigma \) of \( \pi : P \to \mathcal{H}_0 \) over \( \mathcal{H}'_0 \), and write \( ( P, P' ) = ( P, \sigma ) \).

The full subcategory of \( \text{GPD}^2 \), consisting of all marked topological groupoids, is the Morita category of marked topological groupoids and denoted by

\[ \text{GPD}_\square \].
The full subcategory of $\text{GPD}_\circ$, consisting of all pointed topological groupoids, is the Morita category of pointed topological groupoids and denoted by 

$\text{GPD}_\circ$.

The category of pairs of topological spaces is a full subcategory of the Morita category $\text{GPD}_\circ$, while the category of pointed topological spaces is a full subcategory of the Morita category $\text{GPD}_\circ$.

**Morita equivalence.** A functor $\phi : (\mathcal{H}, b_0) \to (\mathcal{G}, a_0)$ between pointed groupoids gives an associated principal $(\mathcal{G}, a_0)$-bundle $((\phi), (b_0, 1_{a_0}))$ over $(\mathcal{H}, b_0)$, given as pullback $$(\phi) = \mathcal{H}_0 \times_{\mathcal{G}_0} \mathcal{G}_1 = \{(y, g) | \phi(y) = t(g)\}.$$ The space $\langle \phi \rangle$ has a right $\mathcal{G}$-action along the map $s \circ \text{pr}_2 : (\phi) \to \mathcal{G}_0$ and a left $\mathcal{H}$-action along the map $\text{pr}_1 : (\phi) \to \mathcal{H}_0$ given in the obvious way. The functor $\phi$ is a weak equivalence if the map $s \circ \text{pr}_2 : (\phi) \to \mathcal{G}_0$ is a map with local sections and the diagram

$$
\begin{array}{ccc}
\mathcal{H}_0 \times \mathcal{H}_0 & \xrightarrow{\phi \times \phi} & \mathcal{G}_0 \times \mathcal{G}_0 \\
(s,t) & \downarrow & (s,t) \\
\mathcal{H}_0 & \xrightarrow{\phi} & \mathcal{G}_0
\end{array}
$$

is a pullback of topological spaces. The functor $\phi$ is a weak equivalence if and only if the isomorphism class of the bundle $\langle \phi \rangle$ is invertible in $\text{GPD}_\circ$ [10]. In this case, the principal $(\mathcal{G}, a_0)$-bundle $((\phi), (b_0, 1_{a_0}))$ over $(\mathcal{H}, b_0)$ is an isomorphism in the Morita category of pointed topological groupoids $\text{GPD}_\circ$.

In general, an isomorphism class of a principal $(\mathcal{G}, a_0)$-bundle over $(\mathcal{H}, b_0)$ is called Morita equivalence if it is an isomorphism in $\text{GPD}_\circ$.

**Homotopy.** We say that two principal $\mathcal{G}$-bundles $P$ and $Q$ over $\mathcal{H}$ are homotopic, if there exists a principal $\mathcal{G}$-bundle $E$ over $\mathcal{H} \times I$ such that $P$ is isomorphic to the restriction $E|_{\mathcal{H} \times \{0\}}$ and $Q$ is isomorphic to the restriction $E|_{\mathcal{H} \times \{1\}}$. (The restriction $E|_{\mathcal{H} \times \{0\}}$ of $E$ to the subgroupoid $\mathcal{H} \times \{0\}$ of $\mathcal{H} \times I$, naturally isomorphic to $\mathcal{H}$, is defined in the obvious way.)

More generally, two principal $(\mathcal{G}, \mathcal{G}')$-bundles $(P, P')$ and $(Q, Q')$ over $(\mathcal{H}, \mathcal{H}')$ are homotopic, if there exists a principal $(\mathcal{G}, \mathcal{G}')$-bundle $(E, E')$ over $(\mathcal{H} \times I, \mathcal{H}' \times I)$, such that $(P, P')$ is isomorphic to the restriction $(E|_{\mathcal{H} \times \{0\}}, E'|_{\mathcal{H}' \times \{0\}})$ and $(Q, Q')$ is isomorphic to the restriction $(E|_{\mathcal{H} \times \{1\}}, E'|_{\mathcal{H}' \times \{1\}})$.

**Proposition 2.1.** Homotopy is an equivalence relation on the set of principal $(\mathcal{G}, \mathcal{G}')$-bundles over $(\mathcal{H}, \mathcal{H}')$.

**Proof.** To prove that the relation is transitive, let $(E, E')$ be a principal $(\mathcal{G}, \mathcal{G}')$-bundle over $(\mathcal{H} \times I, \mathcal{H}' \times I)$, $(F, F')$ a principal $(\mathcal{G}, \mathcal{G}')$-bundle over $(\mathcal{H} \times I, \mathcal{H}' \times I)$ and $(P, P')$ a principal $(\mathcal{G}, \mathcal{G}')$-bundle over $(\mathcal{H}, \mathcal{H}')$ which is isomorphic to both $(E|_{\mathcal{H} \times \{1\}}, E'|_{\mathcal{H}' \times \{1\}})$ and $(F|_{\mathcal{H} \times \{0\}}, F'|_{\mathcal{H}' \times \{0\}})$. We cannot directly glue the bundles $(E, E')$ and $(F, F')$ together along the isomorphism between the restrictions $(E|_{\mathcal{H} \times \{1\}}, E'|_{\mathcal{H}' \times \{1\}})$ and $(F|_{\mathcal{H} \times \{0\}}, F'|_{\mathcal{H}' \times \{0\}})$ because the resulting bundle may not have local sections (although for many special classes of topological groupoids, this can in fact not happen). Therefore we proceed in the following way. We use the principal $(\mathcal{G}, \mathcal{G}')$-bundle $(P \times I, P' \times I)$ over $(\mathcal{H} \times I, \mathcal{H}' \times I)$, glue this bundle with $(E, E')$ along the isomorphism between $(E|_{\mathcal{H} \times \{1\}}, E'|_{\mathcal{H}' \times \{1\}})$ and $(P \times \{0\}, P' \times \{0\})$ and with $(F, F')$ along the isomorphism between $(F|_{\mathcal{H} \times \{0\}}, F'|_{\mathcal{H}' \times \{0\}})$ and $(P \times \{1\}, P' \times \{1\})$. After reparametrization, we obtain a principal $(\mathcal{G}, \mathcal{G}')$-bundle.
also have the induced injective map \( i \) diagram:

\[
\begin{array}{c}
\vdots
\end{array}
\]

\( B \) of maps from bundle \((E, G)\) of topological groupoids. Again, using Milnor’s infinite join construction, we get a

\( \cdot = (B, A) \)-bundle over a space \( B \).

\( G \)

\( \pi \) It is known (see [3, 4, 8]) that the quotient projection \( \pi: E\mathcal{G} \to B\mathcal{G} \) is the universal

\( \cdot \) bundles over \( B \) and \( B\mathcal{G}' \), thus the numerable principal \( \mathcal{G} \)-bundles over a space \( B \) are in bijective correspondence (via pullback) with homotopy classes of maps from \( B \) to \( B\mathcal{G} \).

We can generalize the notion of the universal numerable principal bundle to pairs of topological groupoids. Again, using Milnor’s infinite join construction, we get a bundle \( (E\mathcal{G}, E\mathcal{G}') \to (B\mathcal{G}, B\mathcal{G}') \). (The inclusion \( i: \mathcal{G}' \to \mathcal{G} \) induces an inclusion \( i: E\mathcal{G}' \to E\mathcal{G} \), so we can view \( E\mathcal{G}' \) as a subspace of \( E\mathcal{G} \) in a natural way.) We also have the induced injective map \( i: B\mathcal{G}' \to B\mathcal{G} \). Let us inspect the following diagram:

\[
\begin{array}{ccc}
E\mathcal{G}' & \to & \pi \end{array}
\]

\( \pi \) The upper map is a homeomorphism and both vertical maps are quotient maps (as they have local sections). Therefore, the lower map \( i(B\mathcal{G}') \to B\mathcal{G}' \) is continuous. This shows that \( i: B\mathcal{G}' \to B\mathcal{G} \) is an embedding, so we can view \( B\mathcal{G}' \) as a subspace of \( B\mathcal{G} \).

Using basically the same arguments as in [8] p.57] we see that the homotopy classes of numerable principal \((\mathcal{G}, \mathcal{G}')\)-bundles over topological pair \((B, A)\) are in bijective correspondence (via pullback) with the homotopy classes of maps \((B, A) \to (B\mathcal{G}, B\mathcal{G}')\).

All of the above can be in an obvious way generalized to \( n \)-tuples \((\mathcal{G}, \mathcal{G}', \mathcal{G}'', \ldots)\) of topological groupoids.

\textbf{Homotopy groups}. Let \((\mathcal{G}, a_0)\) be a pointed topological groupoid and let \( n \in \mathbb{N} \).

We define \( \pi_n(\mathcal{G}, a_0) \) to be the set of homotopy classes of principal \((\mathcal{G}, a_0)\)-bundles over \((I^n, \partial I^n)\).

\[
\pi_n(\mathcal{G}, a_0) = \text{[GPD]}((I^n, \partial I^n), (\mathcal{G}, a_0))
\]
For \( n \geq 1 \), the set \( \pi_n(\mathcal{G}, a_0) \) is in fact a group with respect to the multiplication given by the concatenation. Indeed, if \((P, \sigma)\) and \((P', \sigma')\) are two principal \((\mathcal{G}, a_0)\)-bundles over \((I^n, \partial I^n)\), there is an uniquely determined isomorphism \( h \) between the restrictions \( P|_{\{1\} \times I^{n-1}} \) and \( P'|_{\{0\} \times I^{n-1}} \) which respects the sections \( \sigma \) and \( \sigma' \). We glue the bundles \( P \) and \( P' \) along the isomorphism \( h \) to obtain a principal \((\mathcal{G}, a_0)\)-bundle over \((I^n, \partial I^n)\), after reparametrization of the base space. With this concatenation operation the set \( \pi_n(\mathcal{G}, a_0) \) becomes a group, called the \( n \)-th homotopy group of the pointed topological groupoid \((\mathcal{G}, a_0)\). Sometimes (for example, when the base point is clear from the context) we write simply \( \pi_n(\mathcal{G}, a_0) = \pi_n(\mathcal{G}) \).

Homotopy groups of pointed topological groupoids are generalizations of the classical homotopy groups of topological spaces, since clearly we have \( \pi_n(X, x_0) = \pi_n(X \rightrightarrows X, x_0) \). In the same way as for homotopy groups of topological spaces, one can see that the homotopy groups of pointed topological groupoids \( \pi_n(\mathcal{G}, a_0) \) are abelian for \( n \geq 2 \).

Any principal \((\mathcal{G}, a_0)\)-bundle \((P, p_0)\) over \((\mathcal{H}, b_0)\) induces a homomorphism (via composition) \( \pi_n(P, p_0) : \pi_n(\mathcal{H}, b_0) \to \pi_n(\mathcal{G}, a_0) \). Furthermore, this defines functors
\[
\begin{align*}
\pi_0 & : \text{GPD}_0 \to \text{Sets}_0 \\
\pi_1 & : \text{GPD}_0 \to \text{Grp} \\
\pi_n & : \text{GPD}_0 \to \text{Ab}
\end{align*}
\]
for \( n \geq 2 \), where \( \text{Sets}_0 \), \( \text{Grp} \) and \( \text{Ab} \) stand for the category of pointed sets, groups and abelian groups respectively. By definition, all this functors are Morita invariants. By the abuse of notation, we often write \( \pi_n(P, p_0) = \pi_n(P) \).

One can easily check that the following theorem holds:

**Theorem 2.3.** Let \((P, p_0)\) and \((Q, q_0)\) be homotopic principal \((\mathcal{G}, a_0)\)-bundles over \((\mathcal{H}, b_0)\). Then \((P, p_0)\) and \((Q, q_0)\) induce the same homomorphism from \( \pi_n(\mathcal{H}, b_0) \) to \( \pi_n(\mathcal{G}, a_0) \).

**Example 2.4.** (1) Let \( \text{Pair}(X) = (X \times X \to X) \) be the pair groupoid over a pointed topological space \((X, x_0)\), in which the source and the target maps are the projections. This groupoid is weakly equivalent to the space with only one point \( x_0 \), which means that \( \pi_n(\text{Pair}(M)) = 0 \) for any \( n \).

(2) Let \( p : N \to M \) be a surjective submersion and \( N \times_M N \rightrightarrows N \) the associated groupoid. Then the natural functor \( \Phi \) from \( N \times_M N \rightrightarrows N \to M \rightrightarrows M \) is a Morita equivalence, which follows from [12] p.128 and the fact that the map \( \langle \Phi \rangle \to M \) has local sections. This gives us isomorphisms
\[
\pi_n(N \times_M N \rightrightarrows N, n_0) \cong \pi_n(M \rightrightarrows M, p(n_0)) \cong \pi_n(M, p(n_0))
\]
for any \( n \in \mathbb{N} \).

**Base point change.** Let \( \mathcal{G} \) be a topological groupoid and suppose that \( a_0, a_1 \in \mathcal{G}_0 \) are connected by a path in \( \mathcal{G}_0 \). Then, using identical argument as in the case of homotopy groups of topological spaces, we get an isomorphism between \( \pi_n(\mathcal{G}, a_0) \) and \( \pi_n(\mathcal{G}, a_1) \).

Now suppose that there is an arrow \( g \in \mathcal{G}_1 \) from \( a_1 \in \mathcal{G}_0 \) to \( a_0 \in \mathcal{G}_0 \). Then the principal \((\mathcal{G}, a_0)\)-bundle \((\mathcal{G}_1, g)\) over \((\mathcal{G}, a_1)\) (with \( \pi = t : \mathcal{G}_1 \to \mathcal{G}_0 \) and \( s = s : \mathcal{G}_1 \to \mathcal{G}_0 \)) is a Morita equivalence, so it induces an isomorphism between \( \pi_n(G, a_0) \) and \( \pi_n(G, a_1) \).

It follows that if the topological groupoid \( \mathcal{G} \) is \( \mathcal{G} \)-connected (i.e. for every \( a_0, a_1 \in \mathcal{G}_0 \) there is a \( \mathcal{G} \)-path from \( a_0 \) to \( a_1 \), see [16] p.28), then the groups \( \pi_n(\mathcal{G}, a_0) \) and \( \pi_n(\mathcal{G}, a_1) \) are isomorphic.
Homotopy groups of the classifying space. From the classification of principal bundles we see that the homotopy groups $\pi_n(\mathcal{G}, a_0)$ of a pointed topological groupoid $(\mathcal{G}, a_0)$ are isomorphic to the homotopy groups of the classifying space $BG$.

Totally numerable principal bundles We recall from [3] the definition of a halo. Let $X$ be a topological space. A halo over a subset $B \subset X$ is a subset $U \subset X$ such that $B \subset U$ and there is a function $\tau : X \to [0, 1]$ with $\tau|_B = 1$ and $\text{supp}(\tau) \subset U$.

Let $(\mathcal{G}, A)$ and $(\mathcal{H}, B)$ be marked topological groupoids and $(P, \sigma)$ a numerable principal $(\mathcal{G}, A)$-bundle over $(\mathcal{H}, B)$. The bundle $(P, \sigma)$ is totally numerable if the section $\sigma : B \to P$ can be extended to a halo around $B$.

We observe that any principal $(\mathcal{G}, a_0)$-bundle $(P, \sigma)$ over $(I^n, \partial I^n)$ is homotopic to a totally numerable principal $(\mathcal{G}, a_0)$-bundle over $(I^n, \partial I^n)$. Indeed, to see this, choose a relative homotopy $H_t : (I^n \times 1, \partial I^n \times 1) \to (I^n, \partial I^n)$ from the identity to a map which retracts an open neighbourhood of $\partial I^n$ in $I^n$ to $\partial I^n$. When we take the pullback of $(P, \sigma)$ along the map $H$, we get a homotopy between the bundle $(P, \sigma)$ and a totally numerable principal $(\mathcal{G}, a_0)$-bundle over $(I^n, \partial I^n)$.

We can in fact generalize the above statement about totally numerable bundles. Let $P$ be a principal $\mathcal{G}$-bundle over $I^n$ and $\sigma : \partial I^n \to P$ a section over $\partial I^n$ such that $\epsilon(\sigma(b)) = a_0$ for every $b \in \partial I^n$. The pair $(P, \sigma)$ may not be a principal $(\mathcal{G}, a_0)$-bundle over $(I^n, \partial I^n)$ according to our definition, because we the section $\sigma$ may not have local extensions. However, by using the same argument and homotopy $H$ as above, we can see that such a bundle $P$ with a section $\sigma$ is homotopic, via a homotopy with a suitable section, to a totally numerable principal $(\mathcal{G}, a_0)$-bundle over $(I^n, \partial I^n)$. Similarly, any homotopy between principal $\mathcal{G}$-bundles over $I^n$ with sections over $\partial I^n$, equipped with a suitable section over $\partial I^n \times 1$, can be transformed into a homotopy that is itself a numerable principal $(\mathcal{G}, a_0)$-bundle over $(I^n \times 1, \partial I^n \times 1)$. It follows that the elements of the group $\pi_n(\mathcal{G}, a_0)$ can be viewed as the homotopy classes of principal $\mathcal{G}$-bundles $P$ over $I^n$, equipped with a section $\sigma : \partial I^n \to P$ with $\epsilon(\sigma(b)) = a_0$ for any $b \in \partial I^n$.

3. Serre fibrations

In Section 3 we described some basic properties of homotopy groups of pointed topological groupoids. In this section, we first show that any Morita map $P$ between pointed topological groupoids induces a long exact sequence that links the homotopy groups of the pointed topological groupoids and certain groups $\Sigma_n(P)$, which we describe explicitly and play the role of homotopy groups of “the homotopy fiber” of the Morita map $P$. Furthermore, we define what it means for a Morita map from groupoid $\mathcal{H}$ to $\mathcal{G}$ to be a Serre fibration. We show that if a Morita map $P$ between pointed topological groupoids $(\mathcal{H}, b_0)$ and $(\mathcal{G}, a_0)$ is a Serre fibration, then the groups $\Sigma_n(P)$ can be identified as the homotopy groups of a pointed topological groupoid, namely the fiber of $P$.

We know that for an ordinary map between topological spaces, one has the homotopy fiber of that map. Using this homotopy fiber, one can show that every map between topological spaces fits into a long exact sequence of homotopy groups (see [3, p.407]).

We will now give a similar construction for principal $(\mathcal{G}, a_0)$-bundle $(P, p_0)$ over $(\mathcal{H}, b_0)$, where $(\mathcal{G}, a_0)$ and $(\mathcal{H}, b_0)$ are given pointed topological groupoids. Write $L^{n+1} \subset \partial I^{n+1}$ for the face of $I^{n+1}$ determined by the equation $t_{n+1} = 1$, where $t_{n+1}$ denotes the last coordinate on $I^{n+1} = [0, 1]^{n+1} \subset \mathbb{R}^{n+1}$. Denote by $J^{n+1}$ the union of all the remaining faces of $I^{n+1}$, this $J^{n+1}$ is the closure of $\partial I^{n+1} \setminus L^{n+1}$ in $I^{n+1}$.
Let $S_n(P)$ be the set of all triples $(\alpha, \beta, h)$, where $\alpha$ is a principal $(\mathcal{X}, b_0)$-bundle over $(I^n, \partial I^n)$, $\beta$ is a principal $(\mathcal{Y}, a_0)$-bundle over $(I^{n+1}, \partial I^{n+1})$ and $h$ is an isomorphism from $\alpha \otimes P$ to $\beta|_{L^{n+1}}$. Note that here we identified $L^{n+1}$ with $I^n$ in the canonical way, and that the sections are implicit in the definition: for instance, the bundle $\alpha$ is actually a bundle $(\alpha, p)$, where $p$ is a section over $\partial I^n$ and $h$ is an isomorphism that preserves the sections.

An isomorphism between triples $(\alpha, \beta, h), (\alpha', \beta', h') \in S_n(P)$ is a pair of isomorphisms $q: \alpha \to \alpha', Q: \beta \to \beta'$ such that the diagram

$$
\begin{array}{c}
\mathit{\alpha} \otimes P \\
\downarrow h \\
\mathit{\beta}|_{L^{n+1}} \\
\downarrow \mathit{Q}|_{L^{n+1}} \\
\mathit{\alpha'} \otimes P
\end{array}
$$

commutes. We say that the triples $(\alpha, \beta, h)$ and $(\alpha', \beta', h')$ are homotopic if there is a triple $(A, B, H)$, where $A$ is a principal $(\mathcal{X}, b_0)$-bundle over $(I^n \times I, \partial I^n \times I)$, $B$ is principal $(\mathcal{Y}, a_0)$-bundle over $(I^{n+1} \times I, \partial I^{n+1} \times I)$ and $H$ is an isomorphism from $A \otimes P$ to $B|_{L^{n+1}}$ such that

$$(A|_{(I^n \times \{0\}, \partial I^n \times \{0\})}, B|_{(I^{n+1} \times \{0\}, \partial I^{n+1} \times \{0\})}, H|_{(I^n \times \{0\}, \partial I^n \times \{0\})})$$

is isomorphic to $(\alpha, \beta, h)$ and

$$(A|_{(I^n \times \{1\}, \partial I^n \times \{1\})}, B|_{(I^{n+1} \times \{1\}, \partial I^{n+1} \times \{1\})}, H|_{(I^n \times \{1\}, \partial I^n \times \{1\})})$$

is isomorphic to $(\alpha', \beta', h')$. Using similar arguments as in the proof of Proposition 24, we see that this gives us an equivalence relation on the set $S_n(P)$. We denote the set of homotopy classes of triples in $S_n(P)$ by $\Sigma_n(P)$.

Concatenation of triples is defined in the same manner as for the homotopy classes of principal $(\mathcal{X}, a_0)$-bundles over $(I^n, \partial I^n)$ and induces a group structure on $\Sigma_n(P)$.

Notice that, similarly to the case of homotopy groups, every triple $(\alpha, \beta, h)$ in $S_n(P)$ is homotopic to a numerable triple, that is, to a triple $(\alpha', \beta', h')$ such that $\alpha'$ and $\beta'$ are homotopy equal principal bundles. Furthermore, as in the case of homotopy groups, we can safely ignore the condition on local extendability of sections over $\partial I^n$, respectively $J^{n+1}$.

**Theorem 3.1.** Let $(P, p_0)$ be a principal $(\mathcal{X}, a_0)$-bundle over $(\mathcal{X}, b_0)$. Then there is a natural long exact sequence

$$\ldots \to \Sigma_n(P) \to \pi_n(\mathcal{X}, b_0) \xrightarrow{\pi_n} \pi_n(\mathcal{Y}, a_0) \to \Sigma_{n-1}(P) \to \pi_{n-1}(\mathcal{X}, b_0) \to \ldots$$

**Proof.** Let $Y_n(P)$ denote the set of triples $(\alpha, \beta, h)$, where $\alpha$ is principal $(\mathcal{X}, b_0)$-bundle over $(I^n, \partial I^n), \beta$ is principal $(\mathcal{Y}, a_0)$-bundle over $(I^{n+1} \times I, \partial I^{n+1} \times I)$ and $h$ is an isomorphism $h: \alpha \otimes P \to \beta|_{L^{n+1}}$. As in the case of $S_n(P)$, we have homotopies of such triples. We denote the set of homotopy classes of triples in $Y_n(P)$ by $\Upsilon_n(P)$.

Concatenation of triples in $Y_n(P)$ induces a group structure on $\Upsilon_n(P)$.

Let us first check that there is an isomorphism $\varphi: \Upsilon_n(P) \to \pi_n(\mathcal{X}, b_0)$ given by $\varphi(\alpha, \beta, h) = \alpha$. Indeed, for the inverse we take $\varphi^{-1}(\alpha) = (\alpha, (\alpha \otimes P) \times I, id)$, where $(\alpha \otimes P) \times I$ denotes the pullback of $\alpha \otimes P$ along the projection $(I^n \times I, \partial I^n \times I) \to (I^n, \partial I^n)$. Both maps are well defined (on the homotopy classes of triples) and the composition $\varphi \circ \varphi^{-1}$ is clearly the identity. We have to check the surjectivity of $\varphi^{-1} \circ \varphi$ is also the identity. To see this, we need to show that the triples $(\alpha, \beta, h)$ and $(\alpha, \alpha \otimes P \times I, id)$ are homotopic in $Y_n(P)$. Indeed, first we have the isomorphism $(id, h \times id)$ from $(\alpha, \alpha \otimes P \times I, id)$ to $(\alpha, \beta|_{L^{n+1}} \times I, h)$, and from here we have the homotopy to $(\alpha, \beta, h)$ of the form $(\alpha \times P, B, h \times I)$, where $B$ is the pullback of
\(\beta\) along the map \(I^n \times I \to I^n\), \((t_1, \ldots, t_{n-1}, t, t_n) \mapsto (t_1, \ldots, t_{n-1}, (1-t)t_n + t)\), and \(h \times I\) denotes the isomorphism induced by \(h\) on the corresponding pullback.

Now we have to check that the sequence
\[
\cdots \to \Sigma_n(P) \to \Upsilon_n(P) \to \pi_n(\mathcal{F}, a_0) \to \Sigma_{n-1}(P) \to \Upsilon_{n-1}(P) \to \cdots
\]
is exact. The map \(\Sigma_n(P) \to \Upsilon_n(P)\) is induced by the inclusion \(S_n(P) \to Y_n(P)\) (which restricts the implicit sections). The map \(\Upsilon_n(P) \to \pi_n(\mathcal{F}, a_0)\) maps (the homotopy class of) \((\alpha, \beta, h)\) to \(\beta|_{I^n \times \{0\}}\). The map \(\pi_n(\mathcal{F}, a_0) \to \Sigma_{n-1}(P)\) maps \(\beta\) to \(\Delta(\beta) = (b_0, \beta, \iota)\), where \(\iota\) is the uniquely determined isomorphism of bundles \(b_0\) (a bundle with global section) and \(\beta|_{I^n}\) that maps the global section of \(b_0\) to the global section of \(\beta|_{I^n}\).

(i) Exactness at \(\Upsilon_n(P)\): The composition \(\Sigma_n(P) \to \Upsilon_n(P) \to \pi_n(\mathcal{F}, a_0)\) is zero, since the bundle \(\beta|_{I^n \times \{0\}}\) is trivial \((\mathcal{F}, a_0)\)-bundle. On the other hand, if the image of the triple \((\alpha, \beta, h)\), which equals \(\beta|_{I^n \times \{0\}}\), is homotopic to the trivial \((\mathcal{F}, a_0)\)-bundle, then we just concatenate this homotopy with \(\beta\) to obtain a triple in \(\Sigma_n(P)\) which maps to \((\alpha, \beta, h)\).

(ii) Exactness at \(\pi_n(\mathcal{F}, a_0)\): The composition \(\Upsilon_n(P) \to \pi_n(\mathcal{F}, a_0) \to \Sigma_{n-1}(P)\) is zero because \((\alpha, \beta, h) \in \Upsilon_n(P)\) can be viewed, after deformation of the base space, as a homotopy from \(\Delta(\beta)|_{I^n \times \{0\}}\) to trivial triple in \(S_{n-1}(P)\). Furthermore, if \(\beta\) represents an element in \(\pi_n(\mathcal{F}, a_0)\) such that \(\Delta(\beta)\) is homotopic to the trivial triple in \(S_{n-1}(P)\), then this homotopy can be viewed, after deformation of the base space, as an element of \(Y_n(P)\) which maps to the homotopy class of \(\beta\) in \(\pi_n(\mathcal{F}, a_0)\).

(iii) Exactness at \(\Sigma_{n-1}(P)\): To see that the composition \(\pi_n(\mathcal{F}, a_0) \to \Sigma_{n-1}(P) \to \Upsilon_{n-1}(P)\) is trivial, observe that the triple \(\Delta(\beta)\) is homotopic to the trivial triple in \(Y_{n-1}(P)\) precisely because the implicit section is restricted. On the other hand, if \((\alpha, \beta, h)\) is a triple in \(S_{n-1}(P)\) which is homotopic to the trivial triple in \(Y_{n-1}(P)\), then homotopy, after deformation of the base space, represents an element \(\pi_n(G, a_0)\) which maps to the homotopy class of \((\alpha, \beta, h)\). \[\square\]

**Definition 3.2.** Let \(\mathcal{H}\) and \(\mathcal{G}\) be topological groupoids and let \(P\) be a principal \(\mathcal{G}\)-bundle over \(\mathcal{H}\). The bundle \(P\) is a Serre fibration if for every triple \((\alpha, \beta, h)\), where \(\alpha\) is a principal \(\mathcal{H}\)-bundle over \(I^n\), \(\beta\) is a principal \(\mathcal{G}\)-bundle over \(I^{n+1}\) and \(h\) is an isomorphism from \(\alpha \otimes P\) to \(\beta|_{I^n \times \{0\}}\), there is a triple \((A, B, H)\) with \(A\) a principal \(\mathcal{H}\)-bundle over \(I^{n+1}\), \(B\) a principal \(\mathcal{G}\)-bundle over \(I^{n+1}\) and \(H\) an isomorphism from \(A\otimes P\) to \(B\) of principal \(\mathcal{G}\)-bundles over \(I^{n+1}\), such that the triples \((\alpha, \beta, h)\) and \((A|_{I^n \times \{0\}}, B, H|_{I^n \times \{0\}})\) are isomorphic.

**Remark 3.3.** The notion of the isomorphism between triples \((\alpha, \beta, h)\), as above is obvious, similar to the one used in the definition of homotopy between the triples in \(S_n(P)\) - the only difference is that here the topological groupoids and principal bundles are not assumed to be pointed. Clearly, the notion of a Serre fibration is a well defined property of a Morita map between topological groupoid, although its definition is essentially intrinsic in the Morita bicategory of topological groupoids. We see that the notion of Serre fibration in GPD is a generalization of the notion of Serre fibration in the category of topological spaces, as the above definition can be presented in the diagram

\[
\begin{array}{ccc}
I^n & \xrightarrow{\alpha} & \mathcal{H} \\
\downarrow & & \downarrow \\
I^{n+1} & \xrightarrow{\beta} & \mathcal{G}
\end{array}
\]

in the Morita bicategory of topological groupoids. Note that we obtain an equivalent definition of a Serre fibration if we replace \(I^n = I^n \times \{0\}\) with \(J^{n+1}\), or by \(K^{n+1} = (I^n \times \{1\}) \cup (\partial I^n \times I) \subset I^{n+1}\), etc.
Proposition 3.4. Let $P$ be a principal $\mathcal{G}$-bundle over $\mathcal{H}$ and $Q$ a principal $\mathcal{H}$-bundle over $\mathcal{K}$.

(i) If $P$ is a Morita equivalence, then it is a Serre fibration.

(ii) If $P$ and $Q$ are both Serre fibrations, then $Q \otimes P$ is also a Serre fibration.

Proof. It is straightforward to check both assertions. To check (i), one uses the fact that if $P$ is a Morita equivalence, then the inverse of $P$ in the Morita category can be represented by the principal $\mathcal{H}$-bundle $P^{-1}$ over $\mathcal{G}$, which equals $P$ as the topological spaces, but has actions transposed. In this way, there are in fact natural isomorphisms $P \otimes P^{-1} \cong \mathcal{H}$ and $P^{-1} \otimes P \cong \mathcal{G}$, which are to be used in the argument. □

Proposition 3.5. Let $\phi : \mathcal{H} \to \mathcal{G}$ be a functor between topological groupoids such that $\phi : \mathcal{H}_0 \to \mathcal{G}_0$ is a Serre fibration and $(\phi, s) : \mathcal{H}_1 \to \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{H}_0$ is a surjective Serre fibration. Then the associated principal $\mathcal{G}$-bundle $(\phi) \overline{\mathcal{H}}$ over $\mathcal{H}$ is a Serre fibration.

Before we give the proof, let us first recall the notion of a $\mathcal{G}$-cocycle of a principal $\mathcal{G}$-bundle $P$ over a space $B$. The bundle $P$ has sections $\{\sigma_i\}_{i \in \Lambda}$ over an open covering $\{U_i\}_{i \in \Lambda}$ of $B$. Denote $g_{ij}(b) = \theta(\sigma_i(b), \sigma_j(b))$ for $b \in U_i \cap U_j$, where $\theta$ is the translation function of the bundle $P$. Write $f_i = \phi \circ \sigma_i : U_i \to \mathcal{G}_0$. Note that $s(g_{ij}(b)) = f_j(b) \circ t(g_{ij}(b)) = f_i(b)$ for any $b \in U_i \cap U_j$. Furthermore, $g_{ii}(b) = 1_{f_i(b)}$ for $b \in U_i$ and $g_{ij}(b)g_{jk}(b) = g_{ik}(b)$ for $b \in U_i \cap U_j \cap U_k$. We say that a family of functions $\{f_i, g_{ij}\}$ satisfying the above conditions is a $\mathcal{G}$-cocycle on $B$. Any $\mathcal{G}$-cocycle on $B$ on the other hand determines a principal $\mathcal{G}$-bundle over $B$ [10].

Proof of Proposition 3.5. The maps from Proposition 3.4 fit into the diagram:

It follows that $\phi : \mathcal{H}_1 \to \mathcal{G}_1$ is a surjective Serre fibration, because the maps $\phi : \mathcal{H}_0 \to \mathcal{G}_0$ and $(\phi, s) : \mathcal{H}_1 \to \mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{H}_0$ are both surjective Serre fibrations.

We have to check the Serre fibration property for the principal bundle $(\phi)$. Let $(\alpha, \beta, h)$ be a triple, where $\alpha$ is a principal $\mathcal{H}$-bundle over $I^n$, $\beta$ is a principal $\mathcal{G}$-bundle over $I^{n+1}$ and $h$ is an isomorphism from $\alpha \otimes (\phi)$ to $\beta|_{I^n \times \{0\}}$. We represent both bundles $\alpha$ and $\beta$ with cocycles, by dissecting the cube $I^n$ into a family of small cubes. More precisely, we choose a large natural number $N$ and a small positive number $\epsilon$, take $C_i = \left(\frac{1}{N} - \epsilon, \frac{i}{N} + \epsilon\right) \cap I$ for $i = 1, \ldots, N$ and

$$C_\mu = C_{\mu_1} \times C_{\mu_2} \times \cdots \times C_{\mu_n} \subset I^n$$

for any multi-index $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, $\mu_1, \mu_2, \ldots, \mu_n = 1, \ldots, n$. Then $\{C_\mu\}$ is a finite open cover of $I^n$. If we replace the intervals $C_i$ in this construction with slightly smaller closed intervals $D_i = \left[\frac{i}{N} - \frac{\epsilon}{2}, \frac{i}{N} + \frac{\epsilon}{2}\right] \cap I$, we obtain a finite closed cover $\{D_\mu\}$ of $I^n$. We do this analogously in the dimension $n + 1$, obtaining a finite open cover $C_\mu'$ and a finite closed cover $D_\mu'$ of $I^{n+1}$. For $N$ large enough, we can represent $\alpha$ and $\beta$ with cocycles $\{f_\mu, h_{\mu\nu}\}$ and $\{F_\mu', g_{\mu'\nu'}\}$ on open covers $\{C_\mu\}$ and $\{C_\mu'\}$ respectively. We can restrict these cocycles to the closed covers $\{D_\mu\}$ and $\{D_\mu'\}$ respectively $\{D_\mu'\}$. Obtaining so called “closed” cocycles which equally well represent the principal bundles (we will use the same notation for this restrictions). The principal bundle $\alpha \otimes (\phi)$ is then given by the cocycle $\{\phi \circ f_\mu, \phi \circ h_{\mu\nu}\}$. The isomorphism $h$ from $\alpha \otimes (\phi)$ to $\beta|_{I^n \times \{0\}}$ is, in terms of the cocycles, given by a family of functions $r_\mu : C_\mu \to \mathcal{G}_1$. 
Now we will lift the closed \( \mathcal{G} \)-cocycle \( \{ F_\mu, g_{\nu'\nu} \} \) to a \( \mathcal{H} \)-cocycle along \( \phi \). First observe that we can lift the functions \( r_\mu : D_\mu \rightarrow \mathcal{G} \) along \( \phi \) to functions \( \tilde{r}_\mu : D_\mu \rightarrow \mathcal{H} \) because \( (\phi, s) : \mathcal{H} \rightarrow \mathcal{G} \times \mathcal{G} \mathcal{H}_0 \) is a surjective Serre fibration. Since \( \phi : \mathcal{H}_0 \rightarrow \mathcal{G}_0 \) is a Serre fibration, we can lift \( F_{(1,\ldots,1)} \) along \( \phi \) to \( \tilde{F}_{(1,\ldots,1)} : D_{(1,\ldots,1)} \rightarrow \mathcal{H}_0 \) such that \( t \circ \tilde{r}_{(1,\ldots,1)} = \tilde{F}_{(1,\ldots,1)} \big|_{D_{(1,\ldots,1)}} \). From the elements of the cocycle already lifted we calculate the initial lift of the element \( g_{(1,\ldots,1)}(1,\ldots,1) \), and because \( (\phi, s) : \mathcal{H}_1 \rightarrow \mathcal{G}_1 \times \mathcal{G}_0 \mathcal{H}_0 \) is a Serre fibration we can lift the entire function \( g_{(1,\ldots,1)}(1,\ldots,1) \) to a function \( \tilde{g}_{(1,\ldots,1)} : D'_{(1,\ldots,1)} \cap D_{(1,\ldots,1)} \rightarrow \mathcal{H}_1 \). Now the elements of the cocycle already lifted determine the initial lift of the functions \( F_{(1,\ldots,1)} \), which can be lifted because \( \phi : \mathcal{H}_0 \rightarrow \mathcal{G}_0 \) is a Serre fibration. Proceeding in this way, we lift the entire \( \mathcal{G} \)-cocycle \( \{ F_\mu, g_{\nu'\nu} \} \), and obtain a closed \( \mathcal{H} \)-cocycle. Restricting this cocycle to the interiors of their domains, we obtain an open \( \mathcal{H} \)-cocycle representing the desired principal \( \mathcal{H} \)-bundle over \( I^{n+1} \).

Let \((P, p_0)\) be a principal \((\mathcal{H}, b_0)\)-bundle over \((\mathcal{G}, a_0)\). We see that the \( \mathcal{H} \)-action on \( P \) restricts to \( \epsilon^{-1}(a_0) \). Therefore, we get the translation groupoid

\[ \mathcal{H} \ltimes \epsilon^{-1}(a_0) = (\mathcal{H} \times \mathcal{G}_0 \epsilon^{-1}(a_0) \cong \epsilon^{-1}(a_0)), \]

which we call the fiber of \( P \) over \( a_0 \). The groupoid \( \mathcal{H} \ltimes \epsilon^{-1}(a_0) \) is also pointed with \( p_0 \in \epsilon^{-1}(a_0) \).

**Lemma 3.6.** Let \((X, \sigma)\) be principal \((\mathcal{H}, b_0)\)-bundle over \((P^0, \partial P^0)\) such that the principal \((\mathcal{G}, a_0)\)-bundle \((X \times P, \sigma \times p_0)\) over \((P^0, \partial P^0)\) is trivial. Then there exists a right \( \mathcal{H} \ltimes \epsilon^{-1}(a_0) \)-action on \( X \) such that \((X, \sigma)\) is a principal \((\mathcal{H} \ltimes \epsilon^{-1}(a_0), p_0)\)-bundle over \((P^0, \partial P^0)\).

**Proof.** Since the \((\mathcal{G}, a_0)\)-bundle \((X \times P, \sigma \times p_0)\) is trivial, it has a global section \( \tilde{\sigma} \) that extends \( \sigma \otimes p_0 \) maps to \( a_0 \) via \( X \times P \to \mathcal{G}_0 \). For any \( x \in X \) we have \( \tilde{\sigma}(\pi(x)) = x \otimes a(x) \) for an uniquely determined \( a(x) \in \epsilon^{-1}(a_0) \), this gives us a map \( \alpha : X \to \epsilon^{-1}(a_0) \). Note that \( \alpha(\sigma(h)) = p_0 \) for any \( b \in \partial P^0 \).

The right action of \( \mathcal{H} \ltimes \epsilon^{-1}(a_0) \) on \( X \) along \( \alpha \) is given by \( x(h, p) = xh \) for any \( x \in X \) and \((h, p) \in \mathcal{H} \times \mathcal{G}_0 \epsilon^{-1}(a_0) \) with \( a = hp \) (note that, by applying \( \pi \), that this equation implies \( \epsilon(x) = l(h) \)). One can check that \((X, \sigma)\) is a principal \((\mathcal{H} \ltimes \epsilon^{-1}(a_0), p_0)\)-bundle. Indeed, if \( x(h, p) = x(h', p') \), then \( x = x' \) and hence \( h = h' \), while \( a(x) = hp = h'p' \) implies \( p = p' \). This means that the \( \mathcal{H} \ltimes \epsilon^{-1}(a_0) \)-action on \( X \) is free. To see that the action is transitive along the fibers of \( \pi : X \to \partial P^0 \), let \( x, x' \in X \) with \( \pi(x) = \pi(x') \). We can choose \( h \in \mathcal{H}_1 \) such that \( xh = x' \), and derive \( x' = x(h, h^{-1}a(x)) \).

**Theorem 3.7.** Let \((P, p_0)\) be a principal \((\mathcal{H}, b_0)\)-bundle over \((\mathcal{G}, a_0)\) such that \( P \) is a Serre fibration. Then there exist a natural exact long sequence

\[ \ldots \to \pi_n(\mathcal{H} \ltimes \epsilon^{-1}(a_0)) \xrightarrow{\pi_n(\mathcal{H})} \pi_n(\mathcal{G}) \xrightarrow{\pi_n(P)} \pi_{n-1}(\mathcal{H} \ltimes \epsilon^{-1}(a_0)) \to \ldots \]

**Proof.** By Theorem 3.6 it is sufficient to prove that the groups \( \Sigma_n(P) \) are isomorphic to the groups \( \pi_n(\mathcal{H} \ltimes \epsilon^{-1}(a_0), p_0) \).

First, we define a map \( \psi : \Sigma_n(P) \to \pi_n(\mathcal{H} \ltimes \epsilon^{-1}(a_0), p_0) \), as follows: for a triple \((\alpha, \beta, h) \in S_n(P)\), we extend \( \alpha \) to \( K^{n+1} \) with a trivial \((\mathcal{H}, b_0)\)-bundle over \((\partial P^0 \times I, \partial P^0 \times I)\) and extend the isomorphism \( h \) to \( K^{n+1} \) so that it becomes an isomorphism of bundles over \((K^{n+1}, \partial P^0 \times I)\). Then the Serre fibration property of \( P \) gives us a triple \((A, B, H)\), and we use Lemma 3.6 on \( A_{|P^0 \times \{0\}} \) to get a principal \((\mathcal{H} \ltimes \epsilon^{-1}(a_0), p_0)\)-bundle \( \psi(\alpha, \beta, h) \) over \((\partial P^0, \partial P^0)\). Using again the Serre fibration property of \( P \), one can see that this map is well defined on \( \Sigma_n(P) \), i.e. depends only on the homotopy class of the triple.
Remark 4.2. That are Morita equivalent to Serre groupoids.

Proof. Proposition 4.3. We have the pullback diagram

\[
\begin{array}{ccc}
P|_{U_\lambda} & \rightarrow & \mathcal{G}_1 \\
\pi|_{U_\lambda} & \downarrow & \uparrow t \\
U_\lambda & \rightarrow & \mathcal{G}_0.
\end{array}
\]

The map \(\pi|_{U_\lambda}\) is a Serre fibration, because it is a pullback of the Serre fibration \(t\). Thus \(\pi : P \rightarrow \mathcal{H}_0\) is a Serre fibration locally over an open covering of \(\mathcal{H}_0\), which yields that it is itself a Serre fibration.

Example 3.8. Let \(\phi : (\mathcal{H}, h_0) \rightarrow (\mathcal{G}, a_0)\) be a functor between pointed topological groupoids such that the associated principal \(\mathcal{G}\)-bundle \(\langle \phi \rangle\) over \(\mathcal{H}\) is a Serre fibration. Then there exists a natural long exact sequence as in Theorem \([4, 7]\) in which the fiber \(\mathcal{H} \times \epsilon^{-1}(a_0)\) of \(\langle \phi \rangle\) equals the translation groupoid \(\mathcal{H} \times (\mathcal{H}_0 \times \mathcal{G}_0) \langle \phi \rangle\).

4. Serre Groupoids

In this section we introduce a special class of topological groupoids called Serre groupoids. We show that the calculation of homotopy groups of Serre groupoids is particularly simple. Examples will show that there are many topological groupoids that are Morita equivalent to Serre groupoids.

Definition 4.1. A Serre groupoid is a topological groupoid \(\mathcal{G}\) for which the source map \(s : \mathcal{G}_1 \rightarrow \mathcal{G}_0\) is a Serre fibration.

Remark 4.2. If \(\mathcal{G}\) is a Serre groupoid, then the target map \(t : \mathcal{G}_1 \rightarrow \mathcal{G}_0\) is also a Serre fibration.

Proposition 4.3. Let \(\mathcal{G}\) be a Serre groupoid and \(P\) a principal \(\mathcal{G}\)-bundle over \(\mathcal{H}\). Then the map \(\pi : P \rightarrow \mathcal{H}_0\) is a Serre fibration.

Proof. The bundle \(\pi : P \rightarrow \mathcal{H}_0\) has local sections over an open covering \(\{U_\lambda\}_{\lambda \in \Lambda}\).

For any \(\lambda\) we have the pullback diagram

\[
\begin{array}{ccc}
P|_{U_\lambda} & \rightarrow & \mathcal{G}_1 \\
\pi|_{U_\lambda} & \downarrow & \uparrow t \\
U_\lambda & \rightarrow & \mathcal{G}_0.
\end{array}
\]

The map \(\pi|_{U_\lambda}\) is a Serre fibration, because it is a pullback of the Serre fibration \(t\). Thus \(\pi : P \rightarrow \mathcal{H}_0\) is a Serre fibration locally over an open covering of \(\mathcal{H}_0\), which yields that it is itself a Serre fibration.

Proposition 4.4. Let \(\phi : \mathcal{H} \rightarrow \mathcal{G}\) be a continuous functor between Serre groupoids which is a Serre fibration on objects. Then the principal bundle \(\langle \phi \rangle\) associated to \(\phi\) is a Serre fibration.
Proof. Let \( \alpha \) be a principal \( \mathcal{H} \)-bundle over \( I^n \), \( \beta \) a principal \( \mathcal{G} \)-bundle over \( I^n \times I \) and \( h : \alpha \otimes \langle \phi \rangle \to \beta |_{I^n \times \{0\}} \) an isomorphisms. Both bundles \( \alpha \) and \( \beta \) have global sections, since their projections are Serre fibrations by Proposition 4.3. This means that we can view \( \alpha \) as the pullback along a map \( \alpha' : I^n \to P \) and \( \beta \) as a pullback along a map \( \beta' : I^n \times I \to \mathcal{G}_0 \). Furthermore, there exists a natural isomorphisms of functors \( \phi_0 \circ \alpha' : I^n \to \mathcal{G} \) and \( \beta' |_{I^n \times \{0\}} \to \mathcal{G} \), given by a map \( w : I^n \to \mathcal{G}_1 \). Since the target map \( \mathcal{G}_1 \to \mathcal{G}_0 \) is a Serre fibration, we can extend the map \( w \) to \( W : I^{n+1} \to \mathcal{G}_1 \) such that \( t \circ W = \beta' \). Now \( \beta'' = s \circ W \) also represents the bundle \( \beta \), and \( \phi_0 \circ \alpha' = \beta'' |_{I^n \times \{0\}} \). Finally, since \( \phi_0 \) is a Serre fibration, we can extend \( \alpha' \) to a map \( A : I^{n+1} \to \mathcal{H}_0 \) such that \( \phi_0 \circ A = \beta'' \).

Example 4.5. Let \( \phi : H \to G \) be a continuous homomorphism between topological groups. The topological groupoids \( (H \rightrightarrows \ast) \) and \( (G \rightrightarrows \ast) \) representing \( H \) and \( G \) are clearly Serre groupoids, and \( \phi \) is a functor between these two groupoids which is a Serre fibration (and in fact the identity) on objects. By Proposition 4.4 it follows that the associated principal bundle \( \langle \phi \rangle \), the total space of which equals \( G \), is a Serre fibration. Theorem 4.7 then gives us a long exact sequence

\[
\ldots \to \pi_n(H \times G) \xrightarrow{\pi_n(pr_1)} \pi_n(H \rightrightarrows \ast) \xrightarrow{\pi_n(\phi)} \pi_n(G \rightrightarrows \ast) \to \pi_{n-1}(H \times G) \to \ldots
\]

Proposition 4.6. Let \( \mathcal{H} \) and \( \mathcal{G} \) be Serre groupoids and \( P \) a principal \( \mathcal{G} \)-bundle over \( \mathcal{H} \) such that the map \( \epsilon : P \to \mathcal{G}_0 \) is a Serre fibration. Then \( P \) is a Serre fibration.

Proof. Recall that the projection \( pr_1 : \mathcal{H} \times P \times \mathcal{G} \to \mathcal{H} \) is a weak equivalence[13] and that \( (pr_1) \otimes P \cong (pr_3) \) Proposition 4.4 implies that \( (pr_3) \) is a Serre fibration. Now it follows from Proposition 5.3 that \( P \) is a Serre fibration as well.

Example 4.7. Let \( \mathcal{G} \) be a Serre groupoid, acting on a space \( X \). Then the associated translation groupoid is also a Serre groupoid.

The next theorem gives a method for calculating the homotopy groups of a Serre groupoid.

Theorem 4.8. Let \( (\mathcal{G}, a_0) \) be a pointed Serre groupoid. Then there is a natural long exact sequence

\[
\ldots \to \pi_n(s^{-1}(a_0), 1_{a_0}) \xrightarrow{\pi_n(s_0)} \pi_n(\mathcal{G}, a_0) \to \pi_n(\mathcal{G}, a_0) \to \pi_{n-1}(s^{-1}(a_0), 1_{a_0}) \to \ldots
\]

Proof. We have the obvious functor from the space \( \mathcal{G}_0 \) to the groupoid \( \mathcal{G}_1 \). The principal \( \mathcal{G} \)-bundle over \( \mathcal{G}_0 \) associated to this functor is identity on objects, hence it is a Serre fibration by Proposition 4.4. We can therefore apply Theorem 5.7.

We will now use our results to calculate homotopy groups of some topological groupoids.

Example 4.9. (1) A unit groupoid \( X \rightrightarrows X \) over topological space \( X \) is a Serre groupoid, with \( s^{-1}(x_0) = \{1_{x_0}\} \). The long exact sequence of Theorem 4.8 gives us the already mentioned isomorphisms \( \pi_n(X \rightrightarrows X) \cong \pi_n(X) \).

(2) Let \( \text{Pair}(X) \) be a pair groupoid over pointed space \( X \). It is also a Serre groupoid, because the source map is simply a projection. In particular we have \( s^{-1}(x_0) = X \).

(3) A topological group \( G \) is a topological groupoid \( (G \rightrightarrows \ast) \) with one object, and it is also a Serre groupoid. The long exact sequence in this case gives us the known result that

\[
\pi_n(G \rightrightarrows \ast) \cong \pi_{n-1}(G) \cong \pi_n(BG),
\]

where \( BG \) is a classifying space of group \( G \).
(4) Let \( \mathcal{G} \) be a Serre groupoid and \( (X, x_0) \) a pointed right \( \mathcal{G} \)-space. Then the groupoid \( X \times \mathcal{G} \) is a pointed Serre groupoid, and we get a long exact sequence
\[
\ldots \to \pi_n(s^{-1}(e(x_0))) \to \pi_n(X) \to \pi_n(X \times \mathcal{G}) \to \pi_{n-1}(s^{-1}(e(x_0))) \to \ldots
\]
In case \( \mathcal{G} \) is also étale and \( X \) is connected, this sequence reduces to the short exact sequence
\[
0 \to \pi_1(X) \to \pi_1(X \times \mathcal{G}) \to s^{-1}(e(x_0)) \to 0
\]
and isomorphisms
\[
\pi_n(X) \to \pi_n(X \times \mathcal{G})
\]
for \( n \geq 2 \).

(5) Let \( X \) be a semilocally simply connected connected pointed topological space and let \( \Pi_1(X) \) be the fundamental groupoid over \( X \). One can check that \( \Pi_1(X) \) is Serre groupoid. We have \( s^{-1}(x_0) = \tilde{X} \), the universal covering space over \( X \). We get the long exact sequence
\[
\ldots \to \pi_n(\tilde{X}) \to \pi_n(X) \to \pi_n(\Pi_1(X)) \to \pi_{n-1}(\tilde{X}) \to \ldots
\]
Because \( \pi_1(\tilde{X}) = 0 \) and \( \pi_n(\tilde{X}) \to \pi_n(X) \) are isomorphisms for \( n \geq 2 \), we have
\[
\pi_1(\Pi_1(X)) = \pi_1(X)
\]
and the other homotopy groups are zero. We can get the same result in a different way as well, because \( \Pi_1(X) \) is transitive groupoid, thus Morita equivalent to the groupoid \( (\pi_1(X) \Rightarrow \ast) \) representing the discrete group \( \pi_1(X) \).

(6) Let \( \mathcal{F} \) be the standard foliation of the open Möbius band. The associated holonomy groupoid is weakly equivalent to the translation groupoid \( \mathbb{Z}_2 \ltimes (-1, 1) \) \cite[137]{12}, therefore
\[
\pi_1(\text{Hol(Möb, } \mathcal{F})) \cong \mathbb{Z}_2
\]
and the other homotopy groups are zero.

(7) Let \( \mathcal{F} \) be the Kronecker foliation of the torus \( T^2 \). The associated holonomy groupoid is weakly equivalent to translation groupoid \( \mathbb{Z} \ltimes S^1 \) \cite[137]{12}, and we get a short exact sequence
\[
0 \to \mathbb{Z} \to \pi_1(\text{Hol}(T^2, } \mathcal{F})) \to \mathbb{Z} \to 0.
\]
The other homotopy groups are zero.

(8) Let \( (M, \mathcal{F}) \) be a foliated manifold such that every leaf is compact with finite holonomy. Then by \cite[141]{12} the source map of the associated holonomy groupoid \( \text{Hol}(M, \mathcal{F}) \) is a proper map and in fact a fiber bundle \( \mathcal{F} \) \cite[200]{9}, thus a Serre fibration. So for any \( a_0 \in L \subseteq M \), where \( L \) is a leaf of the foliation \( \mathcal{F} \), we get the long exact sequence
\[
\ldots \to \pi_n(\tilde{L}) \to \pi_n(M) \to \pi_n(\text{Hol}(M, } \mathcal{F})) \to \pi_{n-1}(\tilde{L}) \to \ldots,
\]
where \( \tilde{L} \) is the holonomy covering space of \( L \).

(9) Let \( \mathcal{G} \) be the proper étale groupoid associated to an orbifold \( Q \) of dimension \( n \). Then from \cite[144]{12} and \cite[143]{12} it follows that \( \mathcal{G} \) is Morita equivalent to the translation groupoid \( U(n) \ltimes UF(Q) \cong UF(Q) \), where \( UF(Q) \) is the unitary frame bundle associated to the orbifold \( Q \) and \( U(n) \) the unitary group. This translation groupoid is a Serre groupoid, so we get the associated long exact sequence of homotopy groups
\[
\ldots \to \pi_n(U(n)) \to \pi_n(UF(Q)) \to \pi_n(\mathcal{G}) \to \pi_{n-1}(U(n)) \to \ldots.
\]
5. Riemannian foliations

In this section we apply the theory that we have developed so far to the holonomy groupoid of transversely complete and, more generally, Riemannian foliations. For definition and properties of such foliations, see e.g. [12, 13, 15].

First let us recall the definition of a transverse principal bundle [12, p.98]. Let $G$ be a Lie group, $(M,\mathcal{F})$ a foliated manifold and $\pi : E \to M$ a (smooth) principal $G$-bundle with a foliation $\tilde{\mathcal{F}}$ on the total space $E$ such that

(i) $\tilde{\mathcal{F}}$ is preserved by the action of $G$, and

(ii) the projection $\pi : E \to M$ maps each leaf $\tilde{L}$ of $\tilde{\mathcal{F}}$ onto a leaf $L = \pi(\tilde{L})$ of $\mathcal{F}$ such that the restriction $\pi|_{\tilde{L}} : \tilde{L} \to L$ is the holonomy covering of $L$.

Then we say that $(E,\tilde{\mathcal{F}})$ is a transverse principal $G$-bundle over $(M,\mathcal{F})$.

For such a transverse principal $G$-bundle $(E,\tilde{\mathcal{F}})$ there is well defined natural projection functor

$$\text{Hol}(E,\tilde{\mathcal{F}}) \to \text{Hol}(M,\mathcal{F}).$$

If $\gamma$ is a path inside a leaf $L$ of $\mathcal{F}$, then, given some initial lift of the starting point, there is a canonical lift $\tilde{\gamma}$ of $\gamma$ along $\pi$ which lies inside a leaf of $\tilde{\mathcal{F}}$. This lifting property is well defined on the holonomy classes of paths. Indeed, suppose that $\gamma : (S^1,1) \to (M,m_0)$ is a loop in $L$ through with trivial holonomy, choose $n_0 \in \pi^{-1}(m_0)$ and write $\tilde{L}$ for the leaf of $\tilde{\mathcal{F}}$ with $n_0 \in \tilde{L}$. The canonical lift $\tilde{\gamma}$ of $\gamma$ to $\tilde{L}$ is then again a loop. We have to see that $\tilde{\gamma}$ has trivial holonomy.

Since $\gamma$ has trivial holonomy, there exists a small transversal section $T$ of $(M,\mathcal{F})$ and a map

$$\Gamma : S^1 \times T \to (M,\mathcal{F})$$

such that $\Gamma|_{S^1 \times \{m_0\}} = \gamma$, $\Gamma(S^1 \times \{m\})$ lies in a leaf of $\mathcal{F}$ for any $m \in T$ and $\Gamma(\{z\} \times T)$ is a transversal section of $(M,\mathcal{F})$ for any $z \in S^1$. We may choose $T$ so small that there exists a submanifold $\tilde{T}$ of $E$ such that $e_0 \in \tilde{T}$, $\pi : \tilde{T} \to T$ is a diffeomorphism and the tangent space of $E$ at any point $e \in \tilde{T}$ is a direct sum of the tangent space of $\tilde{T}$ at $e$, the tangent space of the fiber of $\pi$ through $e$ and the tangent space of the leaf of $\tilde{\mathcal{F}}$ through $e$. By applying the unique lifting property to the loops $\Gamma(S^1 \times \{m\})$, we obtain a lift $\tilde{\Gamma}$ of $\Gamma$ along $\pi$. Now define a map

$$\Psi : S^1 \times T \times G \to E \text{ by}$$

$$\Psi(z,m,g) = \tilde{\Gamma}(z,m)g.$$ 

This implies that $\tilde{\gamma} = \Psi|_{S^1 \times \{m_0\} \times \{1\}}$ has trivial holonomy, because $\Psi(S^1 \times \{m\} \times \{g\})$ lies in a leaf of $\tilde{\mathcal{F}}$ for any $m \in T$ and any $g \in G$, while $\Psi(\{z\} \times T \times G$ is a transversal section of $(E,\tilde{\mathcal{F}})$.

**Proposition 5.1.** Let $\pi : (E,\tilde{\mathcal{F}}) \to (M,\mathcal{F})$ be a transverse principal bundle over a foliated manifold $(M,\mathcal{F})$. The projection functor $\text{Hol}(E,\tilde{\mathcal{F}}) \to \text{Hol}(M,\mathcal{F})$ is a Serre fibration both on objects and on arrows, and the principal bundle associated to this functor is also a Serre fibration.

**Proof.** The projection functor $\phi : \text{Hol}(E,\tilde{\mathcal{F}}) \to \text{Hol}(M,\mathcal{F})$ equals the Serre fibration $\pi : E \to M$ on objects. The last part of the statement is a consequence of Proposition 5.3 as the map $(\phi,s) : \text{Hol}(E,\tilde{\mathcal{F}})_1 \to \text{Hol}(M,\mathcal{F})_1 \times_M E$ is a Serre fibration because of the unique lifting property of holonomy classes of paths discussed above. Since the projection $\text{Hol}(M,\mathcal{F})_1 \times_M E \to \text{Hol}(M,\mathcal{F})_1$ is also a Serre fibration, it follows that $\phi : \text{Hol}(E,\tilde{\mathcal{F}})_1 \to \text{Hol}(M,\mathcal{F})_1$ is a Serre fibration as well. \qed

**Theorem 5.2.** The holonomy groupoid $\text{Hol}(M,\mathcal{F})$ of any transversely complete foliation $(M,\mathcal{F})$ is a Serre groupoid.
Proof. In fact, we will prove that the source map \( s : \text{Hol}(M, \mathcal{F}) \rightarrow M \) is locally trivial fibration. We know \( [12] \) p.90 that all the leaves of foliation \((M, \mathcal{F})\) are diffeomorphic and have trivial holonomy. Let \( L \) be a leaf of \( \mathcal{F} \) and \( x_0 \in L \). We can choose complete projectable vector fields \( Y_1, \ldots, Y_q \) on \((M, \mathcal{F})\) such that the span of \((Y_1)_{x_0}, \ldots, (Y_q)_{x_0}\) is complementary to the tangent space of \( \mathcal{F} \) at \( x_0 \). The flows of these vector fields combine to a map \( T : L \times \mathbb{R}^q \rightarrow M \),

\[
T(x, (t_1, \ldots, t_q)) = (e^{t_1 Y_1} \circ \cdots \circ e^{t_q Y_q})(x),
\]

which has the property that any \( T_t = T(\cdot, t) \) is a diffeomorphism between \( L \) and another leaf of \( \mathcal{F} \). We can choose a small open ball \( B \subset \mathbb{R}^q \) centered at 0 and an open contractible neighbourhood \( U \) of \( x_0 \) in \( L \) such that \( T|_{U \times B} \) is a smooth open embedding and in fact a foliation chart. Write \( W = T(U \times B) \). Observe that \( T(L \times B) \) is an open neighbourhood of \( L \) in \( M \) which equals the saturation of \( W \).

For any \( y \in W \) denote by \((x_y, t_y)\) for the element in \( U \times B \) with \( T(x_y, t_y) = y \).

We need to show that there exists a diffeomorphism \( \phi : W \times s^{-1}(x_0) \rightarrow s^{-1}(W) \) over \( W \). To construct \( \phi \), let \( y \in W \) and let \( \gamma \) be a path in \( L \) starting at \( x_0 \). Then \( T_{y_0} \circ \gamma \) is a path inside the leaf through \( y \), and we define \( \phi(y, \gamma) \) to be the holonomy class of the concatenation of \( T_{y_0} \circ \gamma \) with a path inside \( T(U \times \{y_0\}) \) starting at \( y \). The result, of course, depends only on the holonomy class of \( \gamma \) and gives us a well defined smooth map \( \phi \).

To see that \( \phi \) is a diffeomorphism, note that \( \phi^{-1} \) can be described in a similar way: For any path \( \tau \) representing an element in \( s^{-1}(W) \) starting at a point \( y \in W \), we have the path \( (T_{y_0})^{-1} \circ \tau \) in \( L \) starting at a point in \( U \). Then \( \phi^{-1}(\tau) \) is given by the pair \((y, \gamma')\), where \( \gamma' \) is the holonomy class of the concatenation of \( \phi^{-1}(\tau) \) with a path in \( U \) starting at \( x_0 \).

\[\text{Theorem 5.3.} \quad \text{For any transversely complete foliation } \mathcal{F} \text{ on a connected manifold } M \text{ with leaf } L \text{ we have a long exact sequence} \]

\[
\ldots \rightarrow \pi_n(L) \rightarrow \pi_n(M) \rightarrow \pi_n(\text{Hol}(M, \mathcal{F})) \rightarrow \pi_{n-1}(L) \rightarrow \ldots
\]

\[\text{Proof.} \quad \text{This follows from Theorem 5.2 Theorem 1.8 and the fact that the source fibers of } \text{Hol}(M, \mathcal{F}) \text{ are diffeomorphic to } L.\]

\[\text{Let } \mathcal{F} \text{ be a Riemannian foliation of codimension } q \text{ on compact connected manifold } M. \text{ Then } [12][15] \text{ the canonical lifted foliation } \widetilde{F} \text{ to the associated transverse orthogonal frame bundle } \text{ OF}(M, \mathcal{F}) \text{ is transversely parallelizable. Also we have a well defined functor } \Phi : \text{Hol}(\text{OF}(M, \mathcal{F}), \widetilde{F}) \rightarrow \text{Hol}(M, \mathcal{F}) \text{ because } (\text{OF}(M, \mathcal{F}), \widetilde{F}) \rightarrow (M, \mathcal{F}) \text{ is a transverse principal } O(q)\text{-bundle.} \]

\[\text{Theorem 5.4.} \quad \text{Let } \mathcal{F} \text{ be a Riemannian foliation on compact connected manifold } M \text{ with base point } m_0. \text{ Then the holonomy groupoid } \text{Hol}(M, \mathcal{F}) \text{ is a Serre groupoid in we have a natural long exact sequence} \]

\[
\ldots \rightarrow \pi_n(\widetilde{L}) \rightarrow \pi_n(M) \rightarrow \pi_n(\text{Hol}(M, \mathcal{F})) \rightarrow \pi_{n-1}(\widetilde{L}) \rightarrow \ldots,
\]

where \( \widetilde{L} \) is the holonomy cover of the leaf \( L \) of \( \mathcal{F} \) through \( m_0 \).

\[\text{Proof.} \quad \text{In the commutative diagram} \]

\[
\begin{array}{ccc}
\text{Hol}(\text{OF}(M, \mathcal{F}), \widetilde{F})_1 & \phi & \text{Hol}(M, \mathcal{F})_1 \\
\downarrow s & & \downarrow s \\
\text{OF}(M) & \phi & M,
\end{array}
\]

the left vertical map and both horizontal maps are Serre fibrations (Theorem 5.2 and Proposition 5.1). From this, follows that the right \( s \) is also a Serre fibration and we get the associated long exact sequence. \[\square\]
We can now apply Proposition 5.1 to the case of Riemannian foliation \((M,\mathcal{F})\). We get the following theorem.

**Theorem 5.5.** Let \((M,\mathcal{F})\) be a Riemannian foliation of codimension \(q\) on a compact connected manifold \(M\). Then we have a natural long exact sequence of homotopy groups

\[
\cdots \to \pi_n(O(q)) \to \pi_n(\text{Hol}(OF(M,\mathcal{F}),\tilde{\mathcal{F}})) \to \pi_n(\text{Hol}(M,\mathcal{F})) \to \pi_{n-1}(O(q)) \to \cdots,
\]

where \((OF(M,\mathcal{F}),\tilde{\mathcal{F}})\) is the associated lifted foliation on the transverse orthogonal frame bundle and \(O(q)\) is the unitary group of order \(q\) viewed as a topological space.

**Proof.** The long exact sequence is obtained by Theorem 3.7 and Proposition 5.1

We only need to identify the fiber of the principal bundle associated to the functor \(\text{Hol}(OF(M,\mathcal{F}),\tilde{\mathcal{F}}) \to \text{Hol}(M,\mathcal{F})\), which is

\[
\mathcal{K} = \text{Hol}(OF(M,\mathcal{F}),\tilde{\mathcal{F}}) \ltimes (OF(M,\mathcal{F}) \times_M s^{-1}(m_0)),
\]

to be Morita equivalent to the topological space \(O(q)\). Here \(s^{-1}(m_0)\) denotes the source fiber of the groupoid \(\text{Hol}(M,\mathcal{F})\) over a chosen base point \(m_0\) of \(M\). If we choose a base point \(c_0\) in the fiber of the projection \(\pi : OF(M,\mathcal{F}) \to M\) over \(m_0\), we obtain a canonical embedding \(\iota : O(q) \to (OF(M,\mathcal{F}) \times_M s^{-1}(m_0))\), \(U \mapsto (c_0U, 1_{m_0})\). Now one can easily check that the induced groupoid \(\iota^*(\mathcal{K})\) is Morita equivalent to \(\mathcal{K}\) \cite{13} and equal to the unit groupoid \((O(q) \cong O(q))\). \(\square\)

6. **Effect of an étale groupoid**

It is natural to represent an orbifold with a proper étale groupoid \cite{11}. This groupoid may not be effective, and even if it is, it is often desirable to represent it as the effect of another, simpler étale groupoid. In this section, we study the connection between the homotopy groups of a proper étale groupoid and the homotopy groups of its associated effect groupoid.

Let \(\mathcal{G}\) be an étale Lie groupoid. We have the functor \(\text{Eff} : \mathcal{G} \to \text{Eff}(\mathcal{G})\) between étale groupoids \(\mathcal{G}\) and \(\text{Eff}(\mathcal{G})\) \cite{12}. This functor is the identity on objects and surjective local diffeomorphism on arrows. If the groupoid \(\mathcal{G}\) is proper, then \(\text{Eff}(\mathcal{G})\) is also proper and the map \(\text{Eff} : \mathcal{G}_1 \to \text{Eff}(\mathcal{G})_1\) is proper. Because both \(\mathcal{G}_1\) and \(\text{Eff}(\mathcal{G})_1\) are Hausdorff manifolds, this implies that the map \(\text{Eff} : \mathcal{G}_1 \to \text{Eff}(\mathcal{G})_1\) is in fact a covering projection (see e.g. \cite{7}).

**Proposition 6.1.** Let \(\mathcal{G}\) be a proper étale Lie groupoid. Then the principal bundle associated to the functor \(\text{Eff} : \mathcal{G} \to \text{Eff}(\mathcal{G})\) is a Serre fibration.

**Proof.** We have already seen that \(\text{Eff}\) is a covering projection on arrows, while it is the identity on objects. The proposition now follows from Proposition 3.5. \(\square\)

For a proper étale Lie groupoid \(\mathcal{G}\), let us denote by \(\mathcal{G}_0^n\) the group of ineffective arrows in the isotropy group of \(\mathcal{G}_1\) at a point \(a_0 \in \mathcal{G}_0\), namely all such arrows \(g \in \mathcal{G}_1\) from \(a_0\) to \(a_0\) such that \(\text{Eff}(g) = 1_{a_0}\).

**Theorem 6.2.** Let \((\mathcal{G},a_0)\) be a pointed proper étale Lie groupoid. Then the Eff functor induces isomorphisms \(\pi_n(\mathcal{G},a_0) \cong \pi_n(\text{Eff}(\mathcal{G}),a_0)\), for \(n = 0, 3, 4, 5, \ldots\), and there is an exact sequence

\[
0 \to \pi_1(\mathcal{G}) \to \pi_2(\text{Eff}(\mathcal{G})) \to \mathcal{G}_0^1 \to \pi_1(\mathcal{G}) \to \pi_1(\text{Eff}(\mathcal{G})) \to 0.
\]

**Proof.** We have already seen that the principal bundle associated to the functor \(\text{Eff}\) is a Serre fibration, so there is the associated long exact sequence as in Theorem 3.7. The fiber appearing in this exact sequence is the groupoid \(\mathcal{G} \ltimes \sigma^{-1}(a_0)\), where
σ is the source map of the groupoid $\mathcal{G}$. One can check that this groupoid is transitive and therefore weakly equivalent to a group, which in this case exactly the discrete group $\mathcal{G}_0$. It follows that $\pi_1(\mathcal{G}_0 \rightrightarrows \ast) = \mathcal{G}_0$, while $\pi_n(\mathcal{G}_0 \rightrightarrows \ast) = 0$ for $n \neq 1$.

\begin{proof}
Example 6.3. Let $(M, m_0)$ be a pointed manifold and let $\Gamma$ be a discrete group acting smoothly on $M$ in such a way that the translation groupoid $\Gamma \rightrightarrows M$ is proper. This groupoid is a Serre groupoid, so its effect $\text{Eff}(\Gamma \rightrightarrows M)$ is also a Serre groupoid. Using Theorem 4.8 for both groupoids, we see that $\pi_2(\Gamma \rightrightarrows M) \cong \pi_2(M)$ and $\pi_2(\text{Eff}(\Gamma \rightrightarrows M)) \cong \pi_2(M)$; furthermore, the induced isomorphism $\pi_2(\Gamma \rightrightarrows M) \cong \pi_2(\text{Eff})$. From Theorem 6.2 we get isomorphisms

$$\pi_n(\Gamma \rightrightarrows M) \cong \pi_n(\text{Eff}(\Gamma \rightrightarrows M)),$$

for $n \neq 1$, and a short exact sequence

$$0 \rightarrow \Gamma^0_{m_0} \rightarrow \pi_1(\Gamma \rightrightarrows M) \rightarrow \pi_1(\text{Eff}(\Gamma \rightrightarrows M)) \rightarrow 0,$$

where $\Gamma^0_{m_0}$ is the subgroup of $\Gamma$ containing all the elements $g \in \Gamma$ which act on $M$ by a diffeomorphism with trivial germ at $m_0$.

References

[1] Raoul Bott. Lectures on algebraic and differential topology. Lecture notes in mathematics, 279(2):1–95, 1972.
[2] Weimin Chen. On a notion of maps between orbifolds ii: homotopy and cw-complex. Communications in Contemporary Mathematics, 8(6):763–821, 2006.
[3] Albrecht Dold. Partitions of unity in the theory of fibrations. Annals of Mathematics, 78(2):223–255, 1963.
[4] Andre Haefliger. Manifolds - Amsterdam 1970. Lecture notes in mathematics, 197(2):133–164, 1970.
[5] Andre Haefliger. Structure transverse des feuilletages. Asterisque, 116:70–97, 1984.
[6] Allen Hatcher. Algebraic Topology. Cambridge University Press, 2001.
[7] Andre Henriques and David S. Metzler. Presentations of noneffective orbifolds. Trans. Amer. Math. Soc., 356:2481–2499, 2004.
[8] Dale Husemoller. Fibre Bundles. Springer–Verlag, 1994.
[9] Peter W. Michor. Topics in Differential Geometry. AMS, 2008.
[10] I. Moerdijk. Classifying Spaces and Classifying Topoi. Springer–Verlag, 1995.
[11] I. Moerdijk. Orbifolds as groupoids: an introduction. In Orbifolds in mathematics and physics (Madison, WI, 2001), volume 310 of Contemp. Math., pages 205–222. Amer. Math. Soc., Providence, RI, 2002.
[12] I. Moerdijk and J. Mrčun. Introduction to Foliations and Lie Groupoids. Cambridge University Press, 2003.
[13] I. Moerdijk and J. Mrčun. Lie groupoids, sheaves and cohomology. In Poisson geometry, deformation quantisation and group representations, volume 323 of London Math. Soc. Lecture Note Ser., pages 145–272. Cambridge Univ. Press, Cambridge, 2005.
[14] I. Moerdijk and J. Mrčun. On the developability of Lie subalgebroids.. Adv. Math., 210:1-21, 2007.
[15] P. Molino. Riemannian foliations, volume 73 of Progress in Mathematics. Birkhäuser Boston Inc., Boston, MA, 1988. Translated from the French by Grant Cairns, With appendices by Cairns, Y. Carrière, É. Ghys, E. Salem and V. Sergiescu.
[16] Janez Mrčun. Stability and Invariants of Hilsum-Skandalis Maps. PhD, 1996.
[17] Bruce L. Reinhart. Differential geometry of foliations. Springer–Verlag, 1983.
[18] William P. Thurston. Three-Dimensional Geometry and Topology. University of Minnesota, 1990.
[19] Takashi Tsuboi. Classifying spaces for groupoid structures. In Foliations, geometry, and topology, volume 498 of Contemp. Math., pages 67-81. Amer. Math. Soc., Providence, RI, 2009.

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