Remarks about Static Back-Reaction on Black Hole Spacetimes

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Abstract

Recently, it has been claimed that the back reaction of vacuum polarization on a black hole spacetime naturally regularizes infinities in the black hole entropy. We examine the back reaction calculation and find no such short-distance cut-off, in contradiction with these recent claims. Moreover, the intuitive expectation that the perturbative calculation breaks down near the event horizon is confirmed. The new surface gravity diverges and the metric is degenerate at the stretched horizon.

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It has recently been claimed \cite{1} that the back-reaction on a black hole spacetime due to vacuum polarization provides a cut-off needed to regularize the entropy. The 1-loop quantum correction to the entropy from a scalar field near the event horizon diverges, as shown by ‘t Hooft \cite{2}. The divergence results since an infinite number of states can contribute at the horizon. One might conjecture that if a particle could never actually reach the event horizon, then only a finite number of states would contribute to the entropy and the infinities would be regularized. In Ref. \cite{1}, Lousto claims to have found such a brick wall by computing the back reaction due to vacuum polarization on the spacetime. In the corrected metric, he argues, the acceleration required to keep a particle at rest vanishes at some distance outside the stretched horizon and at that point a particle can remain in stable equilibrium. Therefore a natural barrier, and hence a short-distance cut-off, arises. We critically examine this claim and arrive at contrary conclusions.

We find that the back-reaction-corrected metric has a new event horizon, stretched by an amount proportional to the Planck length outside the unperturbed horizon. Unlike in Ref. \cite{1}, there is no barrier to reaching the horizon. The acceleration required to keep a particle stationary never vanishes. In fact, the acceleration diverges at the stretched horizon. The new surface gravity diverges as well. Not surprisingly, we also find that our perturbative calculation fails before the horizon can be reached. In short, this perturbative back-reaction calculation is unable to shed light on the physics near the event horizon.

Following the method employed by Lousto in Ref. \cite{1}, we consider the back reaction due to a conformally invariant scalar field in the Boulware vacuum \cite{4}. The one-loop renormalised energy momentum tensor in the spacetime of a Schwarzschild black hole is \cite{5,6} (G=c=1, \hbar\neq 1)

\[
< B|T^\nu_\mu|B >^{ren} = \frac{\alpha M^4}{r^6}\left\{ \frac{(2 - 3M/r)^2}{(1 - 2M/r)^2}(-\delta^\nu_\mu + 4\delta^0_\mu \delta^\nu_0) + 6(3\delta^0_\mu \delta^\nu_0 + \delta^1_\mu \delta^\nu_1) \right\},
\]

where

\[
\alpha = \frac{1}{1440\pi^2} \left( \frac{M_{Pl}}{M} \right)^2.
\]

The expression in Eqn. (1) differs from that calculated in the Hartle-Hawking thermal state \cite{7} $|H\rangle$ (often misleadingly referred to as a vacuum state) by a purely thermal contribution from the bath of Hawking radiation,

\[
< H|T^\nu_\mu|H >^{ren} - < B|T^\nu_\mu|B >^{ren} = \frac{\sigma T^4_{loc}}{3} \left( \delta^\nu_\mu - 4\delta^0_\mu \delta^\nu_0 \right),
\]

where \sigma is the Stephan-Boltzman constant, $T_{loc} = T_H(1 - 2M/r)^{-1/2}$ is the temperature of the black hole as measured by a static observer at radial position $r$ and $T_H$ is the usual Hawking temperature. The Boulware vacuum and the Hartle-Hawking thermal state are both time-reversal invariant and thus suffer no particle production. Consequently, we are only considering the back-reaction due to vacuum polarization.

A more realistic calculation of back-reaction effects would require us to use the Unruh vacuum \cite{8} which allows for particle production and fluxes of energy-momentum due to
Hawking radiation. However, we shall confine our current analysis to back-reaction in the Boulware vacuum in order to critically examine the results in Ref. [1].

Choosing the Schwarzschild gauge, the metric for a static spherically symmetric space-time is

\[
ds^2 = -a(r)dt^2 + b(r)^{-1}dr^2 + r^2d\Omega^2, \quad d\Omega^2 = d\vartheta^2 + \sin^2\vartheta d\phi^2.
\]

(4)

Einstein’s equations yield [3]

\[
b(r) = 1 - 2\frac{M_t}{r} + \frac{1}{r}\int^r_\infty \tilde{r}^2 < B|T^t_t|B >^\text{ren} \tilde{r}^2 d\tilde{r},
\]

(5)

and

\[
a(r) = b(r) \exp \left\{ \int^r_\infty < B|T^r_r - T^t_t|B >^\text{ren} \frac{\tilde{r}}{b(\tilde{r})} d\tilde{r} \right\}.
\]

(6)

The quantity \( M_t \) represents the total mass due to the vacuum energy and the black hole as measured by asymptotic observers. In order to parallel the treatment in Ref. [1] we shall assume \( M_t = M \) where \( M \) is the unperturbed black hole mass. Still, it is unclear that this assumption can be justified. Perhaps it could be argued that since no matter has been added or removed, the mass should not change. However, when using the Hartle-Hawking thermal state, it is certainly incorrect to equate \( M_t \) and \( M \) since radiation has been added to the system [9].

Inserting Eq. (4) into Eq. (5), with \( M_t = M \), gives

\[
b(r) = 1 - 2\frac{M}{r} + \frac{1}{r}\int^r_\infty \tilde{r}^2 < B|T^t_t|B >^\text{ren} \tilde{r}^2 d\tilde{r},
\]

(7)

where we differ from Lousto’s expression for \( b(r) \) in Ref [1] in the third and sixth terms. It is the sign difference in the sixth term which accounts for the majority of the differences between our work and that in Ref [1]. We find that \( b(r) \) vanishes at \( b(r_h) = 0 \) where

\[
r_h = M \left( 2 + \frac{\sqrt{3\alpha}}{4} - \frac{3\alpha}{32} \ln \left( \frac{3\alpha}{64} \right) + \frac{9\alpha}{64} + \ldots \right).
\]

(8)

The leading correction to the position of the event horizon, \( M\sqrt{\alpha} \), is proportional to \( M_{\text{Pl}} \), so the stretched horizon at \( r_h \) is roughly a Planck length from the unperturbed horizon.

In the neighbourhood of \( r_h \) we find \( b(r) \) approaches zero linearly,

\[
b(r_h + x) = \frac{x}{M} \left( 1 + \frac{\sqrt{3\alpha}}{8} \ln \left( \frac{3\alpha}{64} \right) - \frac{\sqrt{3\alpha}}{8} + \ldots \right) + \mathcal{O}\left( \frac{x^2}{M^2} \right).
\]

(9)

The above equation exhibits the important feature that the limit \( \alpha \to 0 \) is non-analytic in the neighborhood of \( r_h \) since

\[
\lim_{\alpha \to 0} b(r_h + x) = \frac{x}{M} + \mathcal{O}\left( \frac{x^2}{M^2} \right),
\]

(10)
while for the uncorrected metric with $\alpha = 0$ we have $r_h = 2M$ and
\[ b_{\alpha=0}(2M) = \frac{x}{2M}. \tag{11} \]

This is a clear sign that our perturbation theory breaks down at $r_h$ as our results are no longer analytic (perturbative) in $\alpha$. This can easily be understood by considering the energy momentum tensor near the horizon and noticing that it diverges as $r$ approaches $2M$ due to the redshift factor to the fourth power. This factor of $(1 - 2M/r)^{-2}$ is proportional to $1/\alpha$ at the stretched horizon $r_h$, and has the effect of canceling the overall small expansion parameter $\alpha$. The vacuum energy is no longer a small perturbation near the horizon and our approximation scheme breaks down.

Continuing our analysis, despite our misgivings, we proceed to calculate the metric function $a(r)$. The integral in Eqn. (6) is difficult to evaluate in closed form and we were only able to obtain explicit expressions for $a(r)$ when $\alpha << 1$ and $r >> r_h$. The restriction $r >> r_h$ occurs because the integrand is singular at $r_h$. With these restrictions we find
\[
a(r) = 1 - 2\frac{M}{r} + \frac{\alpha}{8\left(1 - 2\frac{M}{r}\right)} \left[ 20\frac{M^5}{r^5} - 32\frac{M^4}{r^4} - 14\frac{M^3}{r^3} + 24\frac{M^2}{r^2} - 6\frac{M}{r} \right. \nn\left. - 18\frac{M^2}{r^2} \ln\left(1 - 2\frac{M}{r}\right) + 15\frac{M}{r} \ln\left(1 - 2\frac{M}{r}\right) - 3 \ln\left(1 - 2\frac{M}{r}\right) \right] + \mathcal{O}(\alpha^2), \tag{12} \]

In order to evaluate the integral in the neighborhood of $r_h$ we employed the method of steepest descents and found that $a(r)$ near $r = r_h$ is given by
\[
a(r_h + x) = C \left(\frac{x}{M}\right)^{(1/3 - \sqrt{3} \alpha \ln(3\alpha/64)/12 + ...)} \tag{13} \]

The constant $C$ is an uninteresting numerical factor. We see that $a(r)$ does vanish at $r = r_h$, showing that $r_h$ is indeed the position of the new event horizon.

To complement the above analytic expressions for $a(r)$ in the regions $r >> r_h$ and $r \approx r_h$ we display a numerical evaluation of $a(r)$ in Fig. 1. for the choice $\alpha = 0.1$. This value of $\alpha$ was chosen to facilitate comparison with Lousto’s plot of $a(r)$ in Fig. 1. of Ref. [1] where by contrast $a(r)$ fails to pass through zero but rather reaches a minimum before asymptoting to $+\infty$ as $r$ approaches $2M$.

Considering the equations of motion for a radially directed test particle with unit energy at spatial infinity, we find the coordinate velocity is given by
\[
\left(\frac{dr}{dt}\right)^2 = ab(1 - a). \tag{14} \]

According to Ref. [1], Fig. 1., a “brick wall” is encountered just outside of $r = 2M$ at the point where $a(r) = 1$ and the coordinate velocity vanishes. An analogous “brick wall” is also said to exist for strings, membranes and scalar fields, thus providing an ultraviolet cut-off for entropy calculations. From our analytic expressions (12), (13), and the numerical evaluation displayed in our Fig.1. we see $a(r) \leq 1$ and no “brick wall” is encountered.
FIG. 1. The dashed line shows $a(r)$ in the region between $r = r_h$ and $r = 5M$ for the choice $\alpha = 0.1$, while the solid line shows $a(r)$ for the unperturbed Schwarzschild metric.

It is also stated in Ref. [1] that the coordinate acceleration, $\ddot{r} = \frac{d^2r}{dt^2}$, vanishes at the “brick wall”, allowing particles to be placed in stable equilibrium ($\dot{r} = \ddot{r} = 0$) just outside the horizon. We disagree with that assertion. In order to remain stationary against the pull of the black hole, an observer must be accelerated by amount $A^\mu = \nabla^\mu \ln V$, where $V$ is the norm of the timelike Killing vector $\xi^\mu = (a(r), 0, 0, 0)$. The magnitude of the acceleration can be written in terms of the metric components as

$$A = (A^\mu A_\mu)^{1/2} = \frac{a'b^{1/2}}{2a}. \quad (15)$$

We see that neither the proper acceleration $A$, nor the coordinate acceleration $\ddot{r} = aA$ vanish outside of $r = r_h$. Symbolically, using expressions (5) and (6), the proper acceleration is

$$A = \frac{1}{2b^{1/2}} \left[ b' + \langle B|T^r_r - T^i_i|B \rangle \right]. \quad (16)$$

This quantity diverges on the horizon since $b$ approaches zero while the term in brackets is finite and non-vanishing. Explicitly, we find $A(r_h + x) \propto x^{-1/2}$ and $\ddot{r}(r_h + x) \propto x^{-1/6}$ to leading order in $\alpha$. The acceleration not only does not vanish, but actually diverges near $r = 2M$.

To complete our analysis we shall calculate the surface gravity and the metric determinant in our back-reaction corrected spacetime. The breakdown of our calculation at the stretched horizon will be apparent in these quantities.

The surface gravity is the force which would need to be exerted at infinity in order to keep an observer stationary at the event horizon:

$$\kappa \equiv \lim_{r \to r_h} \left[ V(A^\mu A_\mu)^{1/2} \right] = \frac{a'}{2} \left( \frac{b}{a} \right)^{1/2} \bigg|_{r=r_h}. \quad (17)$$

The surface gravity is then
\[ \kappa = \left( b' + < B | T_r^r - T_t^t | B > r \right) |_{r=r_h} \times \frac{1}{2} \exp \left[ \frac{1}{2} \int_{\infty}^{r} < B | T_r^r - T_t^t | B > \frac{r}{b(r)} d\tilde{r} \right], \] (18)

which diverges at \( r_h \) since the exponential is singular at \( r_h \). Explicitly, we find that \( \kappa \) diverges as \( x^{-1/3} \) as \( x \) tends to zero. In addition, a naive application of our results indicates that the metric is degenerate at the stretched horizon since

\[
\sqrt{-g} (r_h + x) = r^2 \sin \theta \exp \left\{ \int_{\infty}^{r_h+x} < B | T_r^r - T_t^t | B > r_{en} \frac{r}{2b(r)} d\tilde{r} \right\}
\]

\[
\propto \left( \frac{x}{M} \right)^{(1/3 + \sqrt{3} \alpha \ln(3\alpha/64)}24+ \ldots) \] . (19)

Both the degeneracy of the corrected metric and the infinite surface gravity at the stretched horizon are a result of our perturbative calculation failing in that region and tell us nothing about the true situation near the unperturbed event horizon.

The breakdown in our calculation might be taken as support for the viewpoint [10] that quantum gravity effects need to be included to obtain sensible results near the event horizon. A more conservative conclusion would be that the complete, time-dependent back-reaction calculation should be performed. The time-dependent approach could properly include Hawking radiation as well as have a sensible perturbative treatment. For instance, in the Unruh vacuum, \(< T_{\mu\nu} >\) is finite since the divergence in the static vacuum polarization is precisely canceled by the contribution from the outgoing flux of Hawking radiation [11].

What we can say for certain is that we found no evidence that vacuum polarization gives rise to a short-distance cut-off.

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