Confinement on $R^3 \times S^1$: continuum and lattice

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(Dated: 03/06/14)

There has been substantial progress in understanding confinement in a class of four-dimensional SU(N) gauge theories using semiclassical methods. These models have one or more compact directions, and much of the analysis is based on the physics of finite-temperature gauge theories. The topology $R^3 \times S^1$ has been most often studied, using a small compactification circumference $L$ such that the running coupling $g^2(L)$ is small. The gauge action is modified by a double-trace Polyakov loop deformation term, or by the addition of periodic adjoint fermions. The additional terms act to preserve $Z(N)$ symmetry and thus confinement. An area law for Wilson loops is induced by a monopole condensate. In the continuum, the string tension can be computed analytically from topological effects. Lattice models display similar behavior, but the theoretical analysis of topological effects is based on Abelian gauge theory duality rather than on semiclassical arguments. In both cases the key step is reducing the low-energy symmetry group from $SU(N)$ to the maximal Abelian subgroup $U(1)^{N-1}$ while maintaining $Z(N)$ symmetry.

PACS numbers: 12.38.Aw, 11.15.Ha, 11.15.Kc, 11.10Wx
Keywords: quark confinement, monopoles, duality

I. INTRODUCTION: GAUGE THEORIES

One of the fundamental aspects of QCD is quark confinement: the force between widely separated quark-antiquark pairs is a constant $\sigma$, known as the string tension. This constant force implies a potential energy between a quark-antiquark pair that grows as $\sigma r$ for large distances $r$. The numerical value of $\sigma$, determined from phenomenology, is approximately $0.18 \text{ GeV}^2 \approx 0.9 \text{ GeV}/\text{fm}$. Renormalization group arguments tell us that the string tension depends on the coupling constant of QCD in a non-perturbative way: it cannot be calculated from perturbation theory. It is often convenient theoretically to view QCD in a simplified way, in a form without dynamical quarks; this is often referred to as pure $SU(3)$ gauge theory. This is a theory with no adjustable dimensionless parameters. As a consequence of dimensional transmutation, the dimensionless gauge coupling $g^2$ can be replaced by a parameter $\Lambda$ with dimensions of energy. Observables become pure numbers times an appropriate power of $\Lambda$. The pure gauge theory is thus a theory with no adjustable dimensionless parameters, making it difficult to carry out analytical approximations. Finite-temperature gauge theories offer a dimensionless parameter $T/\Lambda$ which can be used, for example, to change QCD from confining behavior at low temperatures, with a non-zero string tension, to deconfined behavior at high temperatures with zero string tension. Thus temperature allows us to probe the physics of confinement. Recent work has shown the existence of a new class of gauge theory models which provide an analytic understanding of confinement in a class of four-dimensional gauge theories. All of these models have one or more compact directions, and the most developed case is the geometry $R^3 \times S^1$, which is the geometry of Euclidean gauge theories at finite temperature when the circumference $L$ of $S^1$ is identified with the inverse temperature $\beta = 1/T$. Unlike conventional finite-temperature gauge theories, this new class can be put into a confined phase when $L \ll \Lambda$ and $g^2(L) \ll 1$ [1]. Euclidean monopoles, the constituents of finite-temperature instantons, are essential to a semiclassical calculation of the string tension in this region, explicitly revealing the nonperturbative nature of the string tension. Moreover, this small-$L$ phase is smoothly connected to the conventional, large-$L$ confining phase [2].

In the next three sections, the basic features of these confining models are developed for continuum quantum field theory. The simplest case of the gauge group $SU(2)$ is used throughout as an example. More details, as well as an introduction to some related topics, can be found in [3]. The next section introduces some basic ideas about confinement and center symmetry in finite-temperature gauge theories. Section III show how the gauge action can be modified to maintain confinement at small $L$. Section IV examines the nonperturbative content of these models in the continuum and shows how topological effects produce a nonzero string tension. Section V explores the nonperturbative content of these models on the lattice, showing a close relation between the continuum and the lattice. A final section summarizes. The notations used throughout are as follows: All field theories are taken to be in

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Euclidean space unless noted otherwise. Lower-case Greek indices are used for space-time and the metric in Euclidean space is $g_{\mu\nu} = g^{\mu\nu} = \delta_{\mu\nu}$. Roman indices in the range $j \cdots n$ generally denote "spatial" directions on $R^3 \times S^1$, i.e., the three directions orthogonal to the compact direction. Roman indices in the range $a \cdots d$ generally label group generators, while capital letters are used to denote group representations: $F, \text{Adj}, S, A, R$. $S_k$ is the $k$-dimensional surface of a $(k + 1)$-dimensional hypersphere, so $S^1$ is the unit circle. $T_k$ is the $k$-dimensional hypertorus, so $T^1$ is also $S^1$.

II. CONFINEMENT AND $Z(N)$ SYMMETRY

Gauge theories with $T \neq 0$ ("finite temperature") have a rich phase structure which has been extensively explored using a combination of analytic methods and lattice simulations. Non-Abelian gauge theories have global symmetries and associated order parameters which are analogous to magnetization in spin systems, and much of the modern formalism of critical phenomena is directly applicable. The symmetries and order parameters associated with quark confinement and chiral symmetry breaking are of particular interest as principal determinants of gauge theory phase structure.

If one or more directions in space-time are compact, the string tension may be measured using the Polyakov loop $P$, also known as the Wilson line. The Polyakov loop is essentially a Wilson loop that uses a compact direction in space-time to close the curve using a topologically non-trivial path in space time, as shown in Figure 1. The partition function is given by $Z = \text{Tr} \left[ e^{-\beta H} \right]$, with the circumference of $S^1$ given by the inverse temperature $\beta = 1/T$. In this case, we write

$$P (\vec{x}) = \mathcal{P} \exp \left[ i \int_0^{\beta} dx_4 A_4 (x) \right]$$

where $\mathcal{P}$ indicates path-ordering of the integral. The Polyakov loop one-point function $\langle TRP (\vec{x}) \rangle$ can be interpreted as a Boltzmann factor $\exp (-\beta F_R)$, where $F_R$ is the free energy required to add a static particle in the representation $R$ to the system. Of course, $\langle TRP (\vec{x}) \rangle = 0$ implies that $F_R = \infty$, which is thus a fundamental criterion determining whether particles in the representation $R$ are confined. If $\langle TRP (\vec{x}) \rangle = 0$, the string tension associated with $R$ may be determined from a two-point function

$$\langle TRP (\vec{x}) TRP^\dagger (\vec{y}) \rangle \sim e^{-\beta \sigma_R |\vec{x} - \vec{y}|}$$

as shown in Figure 2. Note that the introduction of a compact direction breaks the four-dimensional symmetry of the theory, and the string tension measured by Polyakov loops is not the same as the string tension measured by Wilson loops lying in non-compact planes. In the case of finite temperature, it is natural to use the terminology electric and magnetic string tension, respectively. In the limit where the compactification radius becomes large, i.e. $\beta \to \infty$, the two string tensions must coincide.

One of the most important concepts in our understanding of confinement is the role of center symmetry. The center of a Lie group is the set of all elements that commute with every other element. For $SU(N)$, this is $Z(N)$. Although the $Z(N)$ symmetry of $SU(N)$ gauge theories can be understood from the continuum theory, it is easier to understand from a lattice point of view. A lattice gauge theory associates link variable $U_{\mu} (x)$ with each lattice site $x$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{polyakov_loop.png}
\caption{The Polyakov loop is associated with the worldline of a heavy particle.}
\end{figure}
and direction $\mu$. The link variable is considered to be the path-ordered exponential of the gauge field from $x$ to $x + \bar{\mu}$: $U_\mu(x) = \exp[iaA_\mu(x)]$. Consider a center symmetry transformation on all the links in a given direction on a fixed hyperplane perpendicular to the direction. The standard example from $SU(N)$ gauge theories at finite temperature is $U_\mu(x,t) \rightarrow zU_\mu(x,t)$ for all $x$ and fixed $t$, with $z \in Z(N)$. Because lattice actions such as the Wilson action consist of sums of small Wilson loops, they are invariant under this global symmetry. However, the Polyakov loop transforms as $P(x) \rightarrow zP(x)$, and more generally

$$TRP\langle x \rangle \rightarrow z^{kn}TRP\langle x \rangle$$

(3)

where $k_R$ is an integer in the set $\{0, 1, ..., N - 1\}$ and is known as the $N$-ality of the representation $R$. If $k_R \neq 0$, then unbroken global $Z(N)$ symmetry implies $\langle TRP\langle x \rangle \rangle = 0$. Thus global $Z(N)$ symmetry characterizes the confining phase of an $SU(N)$ gauge theory. For pure gauge theories at non-zero temperature, the deconfinement phase transition is associated with the loss of $Z(N)$ symmetry at the critical point $T_d$. Below that point $\langle TRP\langle x \rangle \rangle = 0$ but above $T_d$, $\langle TRP\langle x \rangle \rangle \neq 0$. Notice that the case of zero $N$-ality representations is special within this framework: there is no requirement from $Z(N)$ symmetry that these representations are confined. This includes the adjoint representation, the representation of the gauge particles. However, lattice simulation indicate that $\langle TRP\langle x \rangle \rangle$ is very small for these representations in the confined phase. Although screening by gauge particles must dominate at large distances, these zero $N$-ality representations have well-defined string tensions at intermediate distances scales, e.g., on the order of a few fermi for $SU(3)$, behaving in a manner very similar to representations with non-zero $N$-ality

In the confining phase of a gauge theory, $Z(N)$ symmetry is unbroken, and all representations with non-zero $N$-ality are confined. In the deconfined phase, $Z(N)$ symmetry is completely lost, and particles are no longer confined, independent of their representation. For $N \geq 4$, additional phases are possible where $Z(N)$ is broken down to a non-trivial subgroup. In the case of $Z(4)$, there can be breaking of center symmetry down to $Z(2)$. In this partially confined phase, states consisting of two fundamental representation fermions are not confined, but single fermions are. In the case of $SU(N)$, $Z(N)$ can break to $Z(k)$, where $k$ is any divisor of $N$. States with 1 to $k - 1$ fermions are confined, but states with $k$ fermions are not. It is often convenient to include the confined and deconfined phases as $k = N$ and $k = 1$, respectively. Such partially confined phases been found in gauge theories on $R^4 \times S^1$, using both lattice simulations and perturbation theory. It should be noted that not all gauge groups have non-trivial centers. The gauge group $G(2)$ provides an interesting example of a gauge theory without a center that has received significant attention

The effective potential for the Polyakov loop should describe both the confined and deconfined phases if perturbative and nonperturbative effects are included. Perturbation theory is a reliable indicator of broken center symmetry and thus deconfinement at high temperature, because the running coupling constant $g(T)$ is small if $T \gg \Lambda$. The one-loop effective potential for a pure gauge theory in the background of a static Polyakov loop $P$ can be easily evaluated in a gauge where the background field $A_4$ is time-independent and diagonal. It is easy to see that $V^{1\text{st}}_{\text{eff}}(P)$ is given by

$$V^{1\text{st}}_{\text{eff}}(P) = 2T Tr A \sum_{n \in Z} \int \frac{d^3k}{(2\pi)^3} \log \left[(2\pi n T - A_4)^2 + \vec{k}^2\right]$$

(4)

where the factor of 2 represents the two helicity states of each mode. Note that there is no classical contribution to $V^{1\text{st}}_{\text{eff}}$. Discarding the zero-point energy term, we obtain the one-loop finite-temperature effective potential for gauge
where

\[ V_{eff}^{(1)} (P) = 2 T Tr_A \int \frac{d^3k}{(2\pi)^3} \log \left[ 1 - P \exp \left( - |\hat{k}| / T \right) \right] \] (5)

which is the free energy density of the gauge bosons in the background \( P \). The logarithm in this expression for \( V_{eff}^{(1)} (P) \) can be expanded, leading to an interpretation of \( V_{eff}^{(1)} (P) \) as a sum of contributions from gluon worldlines wrapping around the compact direction an arbitrary number of times. Explicitly, we have the expression

\[ V_{eff}^{(1)} (P) = - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^8} Tr_A P^n. \] (6)

From this form, it is easy to see that \( V_{eff}^{(1)} (P) \) is minimized when all the moments \( Tr_A P^n \) are maximized. This occurs when \( P \in Z(N) \), which gives \( Tr_A P^n = N^2 - 1 \). This indicates that the one-loop gluon effective potential favors the deconfined phase. The pressure \( p \) is the negative of the free energy density at the minimum, so

\[ p(T) = 2 \left( N^2 - 1 \right) \frac{\pi^2 T^4}{90}, \] (7)

which is exactly \( p \) for a blackbody with \( 2 \left( N^2 - 1 \right) \) degrees of freedom.

In the gauge where \( A_4 \) is diagonal and time-independent, we can parametrize \( A_4 \) in the fundamental representation of \( SU(N) \) as a diagonal, traceless \( N \times N \) matrix

\[ (A_4)_{jk} = T \theta_j \delta_{jk} \] (8)

so that

\[ P_{jk} = e^{i\theta_j} \delta_{jk} \] (9)

with \( \sum_{j=1}^{N} \theta_j = 0 \). Using the decomposition \( F \otimes \bar{F} = 1 \oplus \text{Adj} \) and the corresponding decomposition \( Tr_F P Tr_F P^+ = 1 + Tr_{A4} P \), one realizes that the \( N^2 \) eigenvalues of the product representation \( F \otimes \bar{F} \) have the form \( \exp \left[ i\Delta \theta_{jk} \right] \), where we define \( \Delta \theta_{jk} = \theta_j - \theta_k \), giving

\[ V_{eff}^{(1)} (P) = - \frac{2}{\pi^2} \sum_{j,k=1}^{N} \left( 1 - \frac{1}{N} \delta_{jk} \right) \sum_{n=1}^{\infty} \frac{1}{n^4} \exp \left[ i n \Delta \theta_{jk} \right]. \] (10)

The infinite sum over \( n \) may be carried out explicitly in terms of the fourth Bernoulli polynomial. For our purposes, a convenient explicit form is

\[ V_{eff}^{(1)} (P) = - T^4 \sum_{j,k=1}^{N} \left( 1 - \frac{1}{N} \delta_{jk} \right) \left[ \frac{\pi^2}{45} - \frac{1}{24\pi^2} |\Delta \theta_{jk}|_{2\pi}^2 \left( |\Delta \theta_{jk}|_{2\pi} - 2\pi \right)^2 \right] \] (11)

where \( |\Delta \theta_{jk}|_{2\pi} \) lies in the interval between 0 and \( 2\pi \) [11].

Confinement at low temperatures is nonperturbative in nature. In many systems, broken symmetry phases are found at low temperatures and symmetry is restored at high temperatures. The phase structure of gauge theories as a function of temperature is unusual because the broken-symmetry phase is the high-temperature phase. A lattice construction of the effective action for Polyakov loops, valid for strong-coupling, is instructive [13–16]. The spatial link variables may be integrated out exactly if spatial plaquette interactions are neglected. Each spatial link variable then appears only in two adjacent temporal plaquettes, and may be integrated out exactly using the same techniques that are used in the Migdal-Kadanoff real-space renormalization group [13,17]. The resulting effective action has the form

\[ S_{eff} = - \sum_{(jk)} K \left[ Tr_F P_j Tr_F P_k^+ + Tr_F P_k Tr_F P_j \right] \] (12)

where \( K \) is a function of the lattice gauge coupling \( g^2 \) and the extent of the lattice in the Euclidean time direction \( n_t \), which is related to the temperature by \( n_t a = 1/T \). In the strong-coupling limit of the underlying gauge theory, the explicit form for \( K \) is \( K \simeq \left( 1/g^2 N \right)^{n_t} \) to leading order. In the weak-coupling limit, a Migdal-Kadanoff bond-moving
argument gives $K \simeq 2N/g^2 n_t$. This effective action represents a $Z(N)$-invariant nearest-neighbor interaction of a spin system where the Polyakov loops are the spins. It depends only on gauge-invariant quantities. Standard expansion techniques show that the $Z(N)$ symmetry is unbroken for small $K$, and broken for $K$ large. This model explains why the high-temperature phase of gauge theories is the symmetry-breaking phase: the relation between $K$ and the underlying gauge theory parameters is such that $K$ is small at low temperatures, and large at high temperatures, exactly the reverse of a classical spin system where the coupling is proportional to $T^{-1}$. For small values of $n_t$, the deconfinement transition can be easily extracted, but the phase transition is in the strong-coupling region and far from the continuum limit. A systematic treatment of strong-coupling corrections has recently been shown to yield values for the critical lattice couplings $\beta_c \approx 2N/g^2$ for $SU(2)$ and $SU(3)$ that are within a few percent of simulation results for $4 \leq N_t \leq 16$.[15]. However, strong-coupling expansions typically have a finite radius of convergence, and are inadequate to describe the complete phase diagram.

III. PRESERVING $Z(N)$ SYMMETRY ON $R^3 \times S^1$

In the last few years, it has proven possible to construct four-dimensional gauge theories for which confinement may be reliably demonstrated using semiclassical methods valid for weak coupling[11, 2]. These models combine $Z(N)$ symmetry, the effective potential for $P$, instantons, and monopoles into a satisfying picture of confinement for a special class of models. All of the models in this class have one or more small compact directions. Models with an $R^3 \times S^1$ topology have been most investigated, and discussion here will focus on this class. The circumference $L$ of $S^1$ is taken to be small, i.e., $L \ll \Lambda^{-1}$, so that $g(L) \ll 1$ and perturbation theory and semiclassical arguments are reliable. Because the $R^3 \times S^1$ topology is natural at finite temperature, it is often useful to identify $L$ with $\beta = 1/T$ although that is not always possible. There are two distinct aspects to the behavior of these models. First, the action is modified in such a way that that center symmetry is maintained for small $L$. Second, nonperturbative effects associated with finite-temperature instantons and Euclidean-space monopoles are used to establish that the string tension is nonzero. Thus we obtain a realization of a long-held scenario for quark confinement, based on ideas originally proposed by Mandelstam[13, 20] and ’t Hooft[21, 22].

There are two closely related approaches to maintaining $Z(N)$ symmetry for small $L$. The first approach deforms the pure gauge theory by adding additional terms involving the Polyakov loop to the gauge action[2, 23, 24]. The general form for such a deformation is

$$S \rightarrow S + \beta \int d^3x \sum_{k=1}^{\infty} a_k Tr A P(\vec{x}, x_4)^k$$

(13)

where the value of $x_4$ is arbitrary and can be taken to be zero. Such terms are often referred to as double-trace deformations; see Fig. 3. If the coefficients $a_k$ are sufficiently large, they will counteract the effects of the one-loop effective potential, and $Z(N)$ symmetry will hold for small $L$. Only the first $[N/2]$ terms are necessary to ensure confinement. It is easy to prove that for a classical Polyakov loop $P$, the conditions $Tr F P^k = 0$ with $1 \leq k \leq [N/2]$ determine the unique set of Polyakov loop eigenvalues that constitute a confining solution, i.e., one for which $Tr R P = 0$ for all representations with $k_R \neq 0$[9]. The effective potential associated with $S$ is given approximately by

$$V_{eff}(P, \beta) = -\frac{2}{\pi^2 \beta^4} \sum_{n=1}^{\infty} \frac{Tr A P^n}{n^4} + \sum_{k=1}^{[N/2]} a_k Tr A P^k.$$

(14)

The explicit solution that minimizes the effective potential in the confined phase is simple: up to a factor necessary to ensure $\text{det } P = 1$, the eigenvalues of $P$ are given by the set of $N$’th roots of unity, which are permuted by a global $Z(N)$ symmetry transformation. A rich phase structure can emerge from the minimization of $V_{eff}$ for intermediate values of the coefficients $a_k$. For $N \geq 3$, the effective potential predicts that one or more phases may separate the deconfined phase from the confined phase. In the case of $SU(3)$, a single new phase is predicted, and has been observed in lattice simulations[2]. For larger values of $N$, there is a rich set of possible phases, including some where $Z(N)$ breaks down to a proper subgroup $Z(p)$. In such phases, particles in the fundamental representation are confined, but bound states of $p$-particles are not[23].

Lattice simulations of $SU(3)$ and $SU(4)$ agree for small $L$ with the theoretical predictions based on effective potential arguments[2]. The phase diagram of $SU(3)$ as a function of $T = L^{-1}$ and $a_1$ has three phases: the confined phase, the deconfined phase, and a new phase, the skewed phase. In general, the three phases of the eigenvalues of the Polyakov loop may be taken to be the set $\{\theta_1, \theta_2, \theta_3\}$ where $\theta_1 + \theta_2 + \theta_3 = 0$. For all three phases, it is possible to use $Z(3)$ symmetry to make $Tr F P$ real, and reduce the phases to the set $\{0, \theta, -\theta\}$ such that $Tr F P = 1 + 2 \cos \theta$. 
The deconfined phase is represented by $\theta = 0$, the confined phase is given by $\theta = 2\pi/3$, and the skewed phase by $\theta = \pi$. An important result obtained from the lattice simulation of $SU(3)$ is that the small-$L$ confining region, where semiclassical methods yield confinement, are smoothly connected to the conventional large-$L$ confining region. In the case of $SU(4)$, a sufficiently large value of $a_1$ leads to a partially-confining phase where $Z(4)$ is spontaneously broken to $Z(2)$. Particles with $k = 1$ are confined in this phase, i.e., $\langle Tr_F P (\vec{x}) \rangle = 0$, but particles with $k = 2$ are not, as indicated by $\langle Tr_F P^2 (\vec{x}) \rangle \neq 0$. As in the case of $SU(3)$, $Tr_F P$ can be made real. Perturbation theory then predicts a deconfined phase where the phases of the eigenvalues of $P$ are $\{0, 0, 0, 0\}$, a confined phase where they are $\{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$, and a partially confined phase where the phases are $\{\pi/2, \pi/2, -\pi/2, -\pi/2\}$.

Double-trace deformations may be applied to lattice gauge theories as easily as to continuum models. A simple double-trace deformation may be applied to the spin-model reduction of a finite-temperature lattice gauge theory, resulting in an action of the form

$$ S_{\text{eff}} = -K \sum_{\langle jk \rangle} \left[ Tr_F P_j Tr_F P_k^\dagger + Tr_F P_k Tr_F P_j^\dagger \right] + A_2 \sum_j Tr_F P_j Tr_F P_j^\dagger \tag{15} $$

where $A_2$ is the lattice analog of $a_2$ in the continuum. In this form, it is clear that the deformation term acts directly to oppose the tendency of the nearest-neighbor interaction to break $Z(N)$ symmetry.

Another approach to preserving $Z(N)$ symmetry for small $L$ uses fermions in the adjoint representation with periodic boundary conditions in the compact direction [1]. In this case, it would be somewhat misleading to use $\beta$ as a synonym for $L$, because the transfer matrix for evolution in the compact direction is not positive-definite. Periodic boundary conditions in the compact direction imply that the generating function of the ensemble, i.e., the partition function, is given by

$$ Z = Tr \left[ (-1)^F e^{-LH} \right] \tag{16} $$

where $F$ is the fermion number and $H$ is the Hamiltonian in the compact direction. This graded ensemble, familiar from supersymmetry, can be obtained from an ensemble $Tr \left[ \exp (\beta \mu F - \beta H) \right]$ with chemical potential $\mu$ by the replacement $\beta \mu \to i\pi$. This system can be viewed as a gauge theory with periodic boundary conditions in one compact spatial direction of length $L = \beta$, and the transfer matrix in the time direction is positive-definite.

The use of periodic boundary conditions for the adjoint fermions dramatically changes their contribution to the Polyakov loop effective potential. In perturbation theory, the replacement $\beta \mu \to i\pi$ shifts the Matsubara frequencies from $\beta \omega_n = (2n+1)\pi$ to $\beta \omega_n = 2n\pi$. The one loop effective potential is now essentially that of a bosonic field, but with an overall negative sign due to fermi statistics [25]. The sum of the effective potential for the fermions plus that
of the gauge bosons gives

\[ V_{\text{eff}}(P, \beta, m, N_f) = \frac{1}{\pi^2 \beta^4} \sum_{n=1}^{\infty} \frac{Tr_A P^n}{n^2} \left[ 2N_f \beta^2 m^2 K_2 (n \beta m) - \frac{2}{n^2} \right] \]  

(17)

where \( N_f \) is the number of adjoint Dirac fermions and \( m \) is their mass. Note that the first term in brackets, due to the fermions, is positive for every value of \( n \), while the second term, due to the gauge bosons, is negative.

The largest contribution to the effective potential at high temperatures is typically from the \( n = 1 \) term, which can be written simply as

\[ \frac{1}{\pi^2 \beta^4} \left[ 2N_f \beta^2 m^2 K_2 (\beta m) - 2 \right] \left[ |Tr_F P|^2 - 1 \right] \]  

(18)

where the overall sign depends only on \( N_f \) and \( \beta m \). If \( N_f \geq 1 \) and \( \beta m \) is sufficiently small, this term will favor \( Tr_F P = 0 \). On the other hand, if \( \beta m \) is sufficiently large, a value of \( P \) from the center, \( Z(N) \), is preferred. Note that an \( \mathcal{N} = 1 \) super Yang-Mills theory would correspond to \( N_f = 1/2 \) and \( m = 0 \), giving a vanishing perturbative contribution for all \( n \) [26, 27]. In that case, non-perturbative effects lead to a confining effective potential for all values of \( \beta \). In the case of \( N_f \geq 1 \), each term in the effective potential will change sign in succession as \( m \) is lowered towards zero. For larger values of \( N \), this leads to a cascade of phases separating the confined and deconfined phases [7]. Numerical investigation shows that the confined phase is obtained if \( N \beta m \lesssim 4.00398 \) [3]. As \( m \) increases, it becomes favorable that \( Tr_F P^n \neq 0 \) for successive values of \( n \). If \( N \) is even, the first phase after the confined phase will be a phase with \( Z(N/2) \) symmetry. As \( m \) increases, the last phase before reaching the deconfined phase will have \( Z(2) \) symmetry, in which \( k = 1 \) states are confined, but all states with higher \( k \) are not. Lattice simulations of \( SU(3) \) with periodic adjoint fermions are completely consistent with the picture [28] predicted by the effective potential, with a skewed phase separating the confined phase and deconfined phase. For \( N \geq 3 \), there are generally phases intermediate between the confined and deconfined phases which are not of the partially-confined type. Careful numerical analysis is necessary on a case-by-case basis to determine the phase structure for each value of \( N \) [7].

The case of \( SU(2) \) is particularly simple, because only two phases are known: the confined phase and the deconfined. For \( SU(2) \), the double-trace deformation term added to the action can be taken to be

\[ S \rightarrow S - \beta \int d^3x H_A Tr_A P (\vec{x}, A) \]  

(19)

where we have now written \( a_1 \) to be \(-\beta H_A\). If the coefficient \( H_A \) is sufficiently negative, the deformation will counteract the effects of the one-loop effective potential, and \( Z(N) \) symmetry will hold for small \( L \). The schematic form of the phase diagram in the \( T - H_A \) plane for an \( SU(2) \) gauge theory with this deformation is shown in Figure 4. Positive values of \( H_A \) favor \( Z(2) \) symmetry-breaking, and the critical temperature will decrease as \( H_A \) increases. In the limit \( H_A \rightarrow \infty \), the Polyakov loops will only take on values in \( Z(2) \); this is therefore appropriately described as an Ising limit. On the other hand, negative values of \( H_A \) favor \( Tr_F P = 0 \). This leads to a rise in the critical temperature. For the specific deformation considered here, the critical line switches to first-order behavior at a tricritical point. This behavior is familiar in \( Z(2) \) models [29]. For sufficiently negative \( H_A \), we reach the semiclassical region where the running coupling \( g(T) \) is small and classical methods may be applied reliably. Also shown in the figure is the rough equivalence of a double-trace deformation with periodic adjoint fermions. Although adjoint fermions couple in a more complicated way to the Polyakov loop than the double-trace deformation, the two approaches are very similar when the adjoint fermion mass \( m \) is very large, and the pure gauge theory is obtained in the limit \( M \rightarrow \infty \). Positive \( H_A \) corresponds to conventional anti-periodic boundary conditions while negative \( H_A \) corresponds to periodic boundary conditions. However, one cannot obtain the limits \( H_A \rightarrow \pm \infty \) by taking \( m = 0 \); this would require taking the number of adjoint flavors \( N_f \) to infinity, a limit incompatible with asymptotic freedom. For fixed \( N_f \), the \( m = 0 \) corresponds to a finite value of \( H_A \), as shown on the right-hand axis.

### IV. MONOPOLES, INSTANTONS AND CONFINEMENT ON \( R^3 \times S^1 \)

The non-perturbative dynamics of confining gauge theories on \( R^3 \times S^1 \) are based on Polyakov’s analysis of the Georgi-Glashow model in three dimensions [30]. This is an \( SU(2) \) gauge model coupled to an adjoint Higgs scalar. The four-dimensional Georgi-Glashow model is the standard example of a gauge theory with classical monopole solutions when the Higgs expectation value is non-zero. These monopoles make a non-perturbative contribution to the partition function \( Z \). In three dimensions, these monopoles are instantons. Polyakov showed that a gas of such three-dimensional monopoles gives rise to non-perturbative confinement in three dimensions, even though the theory
appears to be in a Higgs phase perturbatively. The models we are considering thus differs by the addition of a fourth compact dimension and a change to the action designed to maintain \( Z(N) \) symmetry.

Because \( L \) is small in the \( R^3 \times S^1 \) models we consider, the three-dimensional effective theory describing the behavior of Wilson loops in the non-compact directions will have many features in common with the three-dimensional theory discussed by Polyakov. In the four-dimensional theory, monopole solutions with short worldline trajectories in the compact direction exist, and behave as three-dimensional instantons in the effective theory; see Figure 4. In models on \( R^3 \times S^1 \), the role of the three-dimensional scalar field is played by the fourth component of the gauge field \( A_4 \). In a gauge where the Polyakov loop is diagonal and independent of \( x_4 \), \( P \) has a vacuum expected value induced by the perturbative effective potential. However, there is another way to understand the presence of monopoles in this phase, based on studies of instantons in pure gauge theories at finite temperature and the properties of the KvBLL caloron solution \([31–33]\). If the Polyakov loop has a non-trivial expectation value, finite-temperature instantons in \( SU(N) \) may be decomposed into \( N \) monopoles, and the locations of the monopoles become parameters of the moduli space of the instanton. In the case of \( SU(2) \), an instanton may be decomposed into a conventional BPS monopole and a so-called KK (Kaluza-Klein) monopole. The presence of the KK monopole solution differentiates the case of a gauge field at finite temperature from the case of an adjoint scalar breaking \( SU(N) \) to \( U(1)^{N-1} \), in which case there are \( N-1 \) fundamental monopoles. We will consider in detail the simplest case of \( N = 2 \).

The BPS monopole is found using the using the standard arguments \([34, 35]\). The Euclidean Lagrangian \( \mathcal{L} \) can be taken to be

\[
\mathcal{L} = \frac{1}{4} (F_{\mu\nu})^2 + V_{\text{eff}}(P) \tag{20}
\]

where \( V_{\text{eff}} \) includes both the one-loop gluonic effective potential and the additional term that prevents \( Z(N) \) symmetry breaking. This can also be written as

\[
\mathcal{L} = \frac{1}{2} (D_j A_4)^2 + \frac{1}{2} (B_j)^2 + V_{\text{eff}}(P). \tag{21}
\]

We can associate with \( \mathcal{L} \) an energy defined by

\[
E = \int d^3x \left[ \frac{1}{2} (B_j)^2 + \frac{1}{2} (D_j A_4)^2 + V_{\text{eff}}(P) \right] \tag{22}
\]
as well as an action $S = LE$. We will concern ourselves for now with the solutions in the BPS limit, in which the effective potential $V_{\text{eff}}$ is neglected, but the boundary condition on $P$ at infinity imposed by the potential is retained. We can write the energy as

$$E = \int d^3x \left[ \frac{1}{2} (B_j \pm D_j A_4)^2 \mp B_j D_j A_4 \right].$$

This expression is a sum of squares plus a term which can be converted to a surface integral, giving rise to the BPS inequality

$$E \geq \pm \int dS_j B_j A_4.$$  

The BPS inequality is saturated if the equality $B_j = \mp D_j A_4$ holds. For the case of a single monopole at the origin, we require the fields at spatial infinity to behave as

$$\lim_{r \to \infty} A_4^a = \frac{w x^a}{r},$$

$$\lim_{r \to \infty} A_i^a = e^{a ij} \frac{x^j}{gr^2}.$$  

Note that $w$ is related to the eigenvalues of $P$ at large distances by $w = 2\theta/gL$. Note that $A_4^a$ has the usual hedgehog form. $A_i^a$ is chosen such that covariant terms vanish at infinity: $(D_i A_4)^a = 0$. With the ’t Hooft-Polyakov ansatz, the general expressions for the fields become

$$A_4^a = w(h(r)) \frac{x^a}{r},$$

$$A_i^a = a(r) e^{a ij} \frac{x^j}{gr^2}.$$  

where we define $w > 0$ and require $h(\infty) = 1$ or $-1$, and $a(\infty) = 1$ to obtain the correct asymptotic behavior. We must also have $h = a = 0$ at $r = 0$ to have well-defined functions at the origin. We identify a magnetic flux

$$\Phi = \pm \int dS_j B_j^a \frac{x^a}{r} = \mp \frac{4\pi}{g}$$  

where the + sign corresponds to the case $h(\infty) = 1$ and – corresponds to $h(\infty) = -1$. The energy of the BPS monopole can be written as

$$E_{\text{BPS}} = \pm \Phi w = \frac{4\pi w}{g}.$$  

Figure 5: Short monopole worldline on $R^3 \times S^1$. 

\[ \beta = 1/T \]
In addition to the BPS monopole, there is another, topologically distinct monopole which occurs at finite temperature when \( A_4 \) is treated as a Higgs field [26]. Starting from a static monopole solution where \(|A_4| = w\) at spatial infinity, we apply a special gauge transformation

\[
U_{\text{special}} = \exp\left[\frac{i\pi x_4}{L}\tau^3\right]
\]

(29)

where \( \tau^i \) is the Pauli matrix. \( U_{\text{special}} \) transforms \( A_\mu \) in such a way that the value of \( A_4 \) at spatial infinity is shifted: \( w \to w - 2\pi/gL \). If we instead start from a static monopole solution such that \( A_4 = 2\pi/gL - w \) at spatial infinity, then the action of \( U_{\text{special}} \) gives a monopole solution with \( A_4 = -w \) at spatial infinity. A final constant gauge transformation \( U_{\text{const}} = \exp\left[i\pi\tau^2/2\right] \) yields a new monopole solution with \( A_4 = w \) at spatial infinity. The distinction between the BPS solution, which is independent of \( x_4 \), and the KK solution is made clear by consideration of the topological charge. The action of \( U_{\text{special}} \) followed by \( U_{\text{const}} \) increases the topological charge by 1 and changes the sign of the monopole charge. Thus the KK solution is topologically distinct from the BPS solution because it carries instanton number 1. The BPS antimonopole has magnetic charge opposite to the BPS monopole, and hence the same as that of the KK monopole. The KK monopole has the same magnetic charge as the BPS monopole, but carries instanton number −1. This is all completely consistent with the KvBLL decomposition of instantons in the pure gauge theory with non-trivial Polyakov loop behavior, where \( SU(2) \) instantons can be decomposed into a BPS monopole and a KK monopole. Our picture of the confined phase is one where instantons and anti-instantons have “melted” into their constituent monopoles and anti-monopoles, which effectively forms a three-dimensional gas of magnetic monopoles. In the BPS limit, both the magnetic and scalar interactions are long-ranged; this behavior appears prominently, for example, in the construction of \( N \)-monopole solutions in the BPS limit.

The BPS solution has action

\[
S_{\text{BPS}} = \frac{4\pi w L}{g} = \frac{8\pi \theta}{g^2}.
\]

(30)

For the KK solution, we have instead

\[
S_{\text{KK}} = \frac{4\pi (2\pi - gLw)}{g^2} = \frac{4\pi (2\pi - 2\theta)}{g^2}.
\]

(31)

The sum \( S_{\text{BPS}} + S_{\text{KK}} \) is exactly \( 8\pi^2/g^2 \), the action of an instanton. For \( \theta = \pi/2 \), the \( Z(2) \)-symmetric value for \( SU(2) \), \( S_{\text{BPS}} = S_{\text{KK}} \). This extends to \( SU(N) \), where the action of a monopole of any type is \( 8\pi^2/g^2 N \) in the confining phase.

Although we used the BPS construction to exhibit the existence and some properties of the monopole solutions of our system, we must move away from the BPS limit to ensure that magnetic interaction dominate at large distances, \( i.e. \), that the three-dimensional scalar interactions associated with \( A_4 \) are not long-ranged. This behavior is natural in the confined phase, where the characteristic scale of the Debye (electric) screening mass associated with \( A_4 \) is large, on the order of \( g/L \). It is well known that the BPS bound for the monopole mass holds as an equality only when the scalar potential is taken to zero. Numerical studies [36] have shown that the monopole action is given in general for \( SU(2) \) as

\[
LE_{\text{BPS}}C(\epsilon)
\]

(32)

where \( C \) a function of the quartic term in the potential that varies from \( C = 1 \) in the BPS limit to a maximum value \( C(\infty) = 1.787 \). Thus corrections to the BPS result for the monopole mass and action due to the potential terms are less than a factor of two. We will henceforth use the exact results for the actions in the BPS limit, neglecting corrections from \( V_{\text{eff}} \) for the sake of simplicity of notation.

The \( SU(2) \) construction of BPS and KK monopoles extends to \( SU(N) \) in the standard way, via the embedding of \( SU(2) \) subgroups in \( SU(N) \). There are \( N - 1 \) BPS monopoles and 1 KK monopole inside an instanton. In the confined phase, each of the \( N \) monopoles has action \( 8\pi^2/g^2 N \). It has long been thought that instanton effects must be suppressed in the large-\( N \) limit, because instanton effects would vanish as \( \exp(-cN) \) in the limit \( N \to \infty \) with \( \lambda \equiv g^2 N \) fixed [37]. In contrast, we see that the effects of monopole constituents of instantons are not suppressed by the large-\( N \) limit.

In order to understand the effects of monopoles play in the confined phase, we must analyze their interactions. We begin with a discussion of quantum fluctuations around the monopole solutions. The contribution to the partition function of a single BPS monopole at finite temperature was considered by Zarembo [38]. The measure factor \( d\mu^a \) associated with the collective coordinates (moduli) of the monopole solution, including the Jacobians from the zero modes is given by [27]

\[
\int d\mu^a = \mu^4 \int \frac{d^4 x}{(2\pi)^{3/2}} J_x \int_0^{2\pi} \frac{d\phi}{(2\pi)^1/2} J_\phi
\]

(33)
\[ \beta = \frac{1}{T} \]

Figure 6: Spatial Wilson loop in \( R^3 \times S^1 \) geometry.

where \( x \) is the position and \( \phi \) the \( U(1) \) phase of the monopole and \( \mu \) is a Pauli-Villars regulator. The label \( a \) denotes the type of monopole, \( a = \{ BPS, KK, \text{BPS}, K K \} \). The Jacobians are

\[ J_x = S_3^{3/2}, \quad J_\phi = NLS_a^{1/2}. \tag{34} \]

Each of the four zero modes contributes a factor of \( \mu \). In the BPS limit, each monopole carries an overall factor

\[ Z_a = c \mu^7/2 \left( NL \right)^{1/2} S_3^a \exp \left[ -S_a + O(1) \right] \int d^3x \]

\[ = \xi_a \exp \left[ -S_a \right] \int d^3x \tag{35} \]

in its contribution to \( Z \). The factor \( \xi_a \) is \( c \mu^7/2 \left( NL \right)^{1/2} S_3^a \) where \( c \) is a numerical constant and the factor of \( d^3x \) represents the integration over the location of the monopole. From the construction of the KK monopole, we see that we have \( \xi_{KK}(\theta) = \xi_{BPS}(\pi - \theta) \).

The renormalization of the functional determinant arising from quantum fluctuations around the monopole solution is particularly simple in the confined phase, as first observed by Davies et al. in the corresponding supersymmetric model \[26\]. The dependence on the Pauli-Villars regulator is removed, as usual, by coupling constant renormalization. The relation at one loop of the bare coupling and the regulator mass \( \mu \) to a renormalization-group invariant scale \( \Lambda \) is

\[ \Lambda^{b_0} = \mu^{b_0} e^{-8\pi^2/g^2 N} \]

where \( b_0 \) is the first coefficient of the \( \beta \) function divided by \( N \):

\[ b_0 = \frac{11}{3} - \frac{4}{3} \frac{n_f C(R_f)}{N} - \frac{1}{6} \frac{n_b C(R_b)}{N} \tag{37} \]

where \( n_f \) is the number of flavors of Dirac fermions in a representation \( R_f \), \( n_b \) is the number of flavors of real scalars in a representation \( R_b \), and \( C(R) \) is obtained from \( Tr_R \left( T^a T^b \right) = C(R) \delta^{ab} \). For the case of a pure gauge theory with a deformation, there are four collective coordinates and this gives a factor of \( \mu^4 \). The functional integral over gauge degrees of freedom gives rise to a factor \( \det' \left[-D^2\right]^{-1} \propto \mu^{-1/3} \) and the action contributes a factor \( \exp \left( -8\pi^2/g^2 N \right) \) in the confined phase. Thus the contribution of a single monopole to the partition function gives a factor

\[ \mu^{4-\frac{1}{3}} e^{-8\pi^2/g^2 N} = \mu^{11/3} e^{-8\pi^2/g^2 N} = \Lambda^{11/3}. \tag{38} \]

Thus detailed calculation tells us that \( \xi_a e^{-8\pi^2/g^2 N} \propto L^{-3} (\Lambda L)^{11/3} \). Note that the replacement of renormalization-dependent quantities with renormalization-independent quantities depends crucially on the coefficient of \( 1/g^2 \) in the action.

The interaction of the monopoles is essentially the one described by Polyakov in his original treatment of the Georgi-Glashow model in three dimensions \[30\], generalized slightly to include both the BPS and KK monopoles. Let
us consider, say, a BPS-type monopole and KK-type monopole located at \( \vec{x}_1 \) and \( \vec{x}_2 \) in the non-compact directions, with static worldlines in the compact direction. The interaction energy due to magnetic charge of such a pair is

\[
E_{BPS-KK} = -\left(\frac{4\pi}{g}\right)^2 \frac{1}{4\pi |\vec{x}_1 - \vec{x}_2|}
\]

and the associated action is approximately \( S_{BPS} + S_{KK} + LE_{BPS-KK} \). As discussed above, this will be larger than the value obtained from the Bogomolny bound, but of the same order of magnitude. There is an elegant way to capture the dynamics of the monopole plasma, using an Abelian scalar field \( \sigma \) dual to the magnetic field. Assuming that the Abelian magnetic gauge field is three-dimensional for small \( L \), we may write

\[
L \int d^3x \frac{1}{2} B_k^2 = \int d^3x \frac{g^2}{32\pi^2L} (\partial_k \sigma)^2
\]

where the normalization of \( \sigma \) is chosen to simplify the form of the interaction terms. The three-dimensional effective action is given by

\[
L_{eff} = \frac{g^2}{32\pi^2L} (\partial_j \sigma)^2 - \sum_a \xi_a e^{-S_a+iq_a\sigma}
\]

where the sum is over the set \( \{BPS, KK, \overline{BPS}, \overline{KK}\} \). Each species of monopole has its own magnetic charge sign \( q_a = \pm \) as well as its own action \( S_a \). The coefficients \( \xi_a \) represent the functional determinant associated with each kind of monopole, but the combination \( \xi_a \exp(-S_a) \) may be usefully regarded as a monopole activity in terms of the statistical mechanics of a gas of magnetic charges. The generating functional

\[
Z_\sigma = \int [d\sigma] \exp \left[ -\int d^3x L_{eff} \right]
\]

is precisely equivalent to the generating function of the monopole gas. This equivalence may be proved by expanding \( Z_\sigma \) in a power series in the \( \xi_a \)'s, and doing the functional integral over \( \sigma \) for each term of the expansion.

The magnetic monopole plasma leads to confinement in three dimensions. For our effective three-dimensional theory, any Wilson loop in a hyperplane of fixed \( x_4 \), for example a Wilson loop in the \( x_1 - x_2 \) plane, as shown in Figure 6, will show an area law. The original procedure of Polyakov [30] may be used to calculate the string tension, where the presence of a large planar Wilson loop causes the dual field \( \sigma \) to have a discontinuity on the surface associated with the loop and a half-kink profile on both sides. However, an alternative procedure is simpler in which the discontinuity in the gauge field strength induced by the Wilson loop is moved to infinity so that the string tension is obtained from the kink solution connecting the two vacua of the dual field \( \sigma \) [24].

In the confined phase, the action and functional determinant factors for all four types of monopoles are the same, so we denote them by \( S_M \) and \( \xi_M \). The potential term in the confined phase then reduces to

\[
-\sum_a \xi_a e^{-S_a+iq_a\sigma} \rightarrow 4\xi_M e^{-S_M} [1 - \cos(\sigma)]
\]

which has minima at \( \sigma = 0 \) and \( \sigma = 2\pi \); we have added a constant for convenience such that the potential is non-negative everywhere and zero at the minima. A one-dimensional soliton solution \( \sigma_a(z) \) connects the two vacua, and the string tension \( \sigma_{3d} \) for Wilson loops in the three non-compact directions is given by

\[
\sigma_{3d} = \int_{-\infty}^{+\infty} dz L_{eff} (\sigma_z(z))
\]

which can be calculated via a Bogomolny inequality to be

\[
\sigma_{3d} = \frac{4g}{\pi} \sqrt{\frac{\xi_M}{L} e^{-S_M}}.
\]

This result depends on \( L \) and is valid only in the region \( AL \ll 1 \); nevertheless, this is a concrete realization of confinement in a four-dimensional field theory via non-Abelian monopoles.

It is possible to apply the same methods used for gauge theories on \( R^4 \times S^1 \) to other geometries, such as \( S^1 \times S^3 \) or \( R^2 \times T^2 \). Meyers and Hollowood have performed a detailed study of \( SU(N) \) gauge theories on \( S^1 \times S^3 \) with
periodic adjoint fermions \[39\]. In this geometry, \( R^3 \) is replaced by \( S^3 \), so there are two length scales introduced by the geometry, the radius of the three-sphere \( R = R_{S^3} \) and \( L = R_{S^1} \). We require \( \min [R_{S^1}, R_{S^3}] \ll \Lambda \) so that we are in the weak-coupling region. The projection onto gauge-invariant states, manifested as integration over the eigenvalues of the Polyakov loop, ensures non-trivial behavior. Because the spatial volume is finite, there is no actual phase transition for finite \( N \), only a crossover as \( R/L \) is varied. However, the large-\( N \) limit does give a phase transition whose behavior is closely approximated even for moderate values of \( N \). These techniques can also be applied to the study of gauge theories at finite temperature and density on \( S^3 \times S^1 \) [40–42]. Another interesting geometry is \( R^2 \times T^2 \), where \( SU(N) \) gauge theories are in the universality class of \( Z(N) \times Z(N) \) spin models because there are two compact directions [43–45].

V. MONOPOLES AND INSTANTONS ON THE LATTICE

The nonperturbative physics of confinement detailed in the previous section is highly dependent on continuum methods that do not directly extend to lattice models. It is illuminating to see how similar results may be obtained from lattice models using rather different methods. The key is the use of highly-developed techniques for Abelian lattice duality. Before considering a four-dimensional \( SU(2) \) lattice gauge theory, we use the two-dimensional \( O(3) \) model deformed to an \( XY \) model to illustrate how lattice theories handle non-Abelian models deformed to be Abelian. The continuum Euclidean action of the \( O(3) \) model is given by

\[
S = \int d^2 x \frac{1}{2g^2} (\nabla \vec{\sigma})^2
\]  

(46)

where the field are constrained to

\[
\vec{\sigma}^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 1
\]  

(47)

Like QCD, this is an asymptotically free theory that has instantons [46]. As with finite temperature QCD, instantons can be decomposed into constituents [47]. In the case of the \( O(3) \) model, these constituents are \( XY \)-model vortices [48]. In Figure 7 a classical instanton solution is shown with the arrows denoting the components in the \( \sigma_1 - \sigma_2 \) plane, and the colors denoting the value of \( \sigma_3 \). The embedding of the vortex-antivortex solution within the instanton is obvious, and \( \sigma_3 \) is near \( \pm 1 \) precisely at the vortex cores.

The \( O(3) \) model can be deformed into an \( XY \) model by the addition of a mass term for \( \sigma_3 \) [48–50]:

\[
S \rightarrow S + \int d^2 x \frac{1}{2} \hbar \sigma_3^2
\]  

(48)

in a manner similar to finite-temperature QCD. The mass term breaks the classical conformal invariance of the model and makes it effectively Abelian at large distances. It is physically obvious that as \( \hbar \) increases, the deformed \( O(3) \) model will become more and more like an \( XY \) model, and the constituent vortices inside instantons should be identified with the Kosterlitz-Thouless vortices of the \( XY \) model. To make these identifications precise, we consider a lattice form of the deformed \( O(3) \) model.

The lattice action is given by

\[
S = -\sum_{x,\mu} K a_{a}(x) a_{a}(x + \mu) + \sum_{x} \frac{1}{2} \hbar \sigma_3^2(x)
\]  

(49)

where \( x \) is now a lattice site and \( \mu \) one of two lattice directions; the lattice parameter \( K \) corresponds to \( 1/g^2 \) in the continuum. We parametrize \( \vec{\sigma} \) as

\[
\sigma = \left( \sqrt{1 - \sigma_3^2} \cos \theta, \sqrt{1 - \sigma_3^2} \sin \theta, \sigma_3 \right).
\]  

(50)

We can decompose the action as

\[
S = -\sum_{x,\mu} K_{eff}(x, \mu) \cos \left[ \theta(x) - \theta(x + \mu) \right] + S_3
\]  

(51)
where

\[ S_3 = -\sum_{x,\mu} K \sigma_3(x) \sigma_3(x + \mu) + \sum_{x} \frac{1}{2} \hbar \sigma_3^2(x) \]  

(52)

depends only on \( \sigma_3 \).

At this point, we can follow the well-known arguments of Jose et al. \[51\] to obtain a form for the model that explicitly includes vortex effects. This is done by a series of transformations on the Abelian sector of the model. We write the partition function as

\[
Z = \hat{\mathbb{Z}}^2 \int [d\sigma] e^{-S} = \int_{-1}^{+1} [d\sigma_3(x)]  e^{-S_3} \int_{S^1} [d\theta] \prod_{x,\mu} e^{K_{eff}(x,\mu) \cos(\nabla_\mu \theta(x))} 
\]

(53)

where \( \nabla_\mu \theta(x) \equiv \theta(x+\mu) - \theta(x) \). For each link, we expand the interaction in a character expansion, which is a Fourier series:

\[
Z = \int_{-1}^{+1} [d\sigma_3(x)] e^{S_3} \int_{S^1} [d\theta] \prod_{x,\mu} \sum_{n_{\mu}(x) \in \mathbb{Z}} I_{n_{\mu}(x)}(K_{eff}(x,\mu)) e^{in_{\mu}(x) \nabla_\mu \theta(x)} 
\]

(54)

where \( I_n \) is a modified Bessel function. This step introduces integer variables \( n_{\mu}(x) \) on every link. We now make use of the asymptotic form of \( I_n \) for \( K_{eff} \gg 1 \), using what is called the Villain approximation, obtaining

\[
Z = \int_{-1}^{+1} [d\sigma_3(x)] e^{S_3} \int_{S^1} [d\theta] \prod_{x,\mu} \sum_{n_{\mu}(x) \in \mathbb{Z}} \frac{1}{\sqrt{2\pi K_{eff}(x,\mu)}} e^{K_{eff}(x,\mu) - n_{\mu}(x)^2/2K_{eff}(x,\mu)} e^{in_{\mu}(x) \nabla_\mu \theta(x)} 
\]

(55)
Although this step appears here as an approximation, it is really a small deformation of the action that does not change the critical properties of the model. It is now easy to integrate over the $\theta$ variables, which leads to the constraint $\nabla_\mu n_\mu (x) = 0$. This in turns allows us to write $n_\mu (x) = \epsilon_{\mu \nu} \nabla_\nu m(X)$ where $m(X)$ is an integer-valued field on the dual lattice site $X$ which is displaced from $x$ by half a lattice spacing in each direction. The partition function is now

$$Z = \int_{-1}^{+1} [d\sigma_3(x)] e^{-S_3} \sum_{\{m(X)\} \in Z} e^{-\sum_{X,\nu} (\nabla_\nu m(X))^2 / 2K_{eff}(x,\mu)}$$

where

$$S'_3 = S_3 - \sum_{x,\mu} \left[ K_{eff} (x,\mu) - \frac{1}{2} \log \left( 2\pi K_{eff} (x,\mu) \right) \right]$$

The final step is to introduce a new field $\phi(x) \in R$ using a periodic $\delta$-function, effectively performing a Poisson resummation:

$$Z = \int_{-1}^{+1} [d\sigma_3(x)] e^{S'_3} \int_R [d\phi(X)] e^{-\sum_{X,\nu} (\nabla_\nu \phi(X))^2 / 2K_{eff}(x,\mu)} \sum_{\{m(X)\} \in Z} e^{2\pi i m(X) \phi(\bar{x})}$$

We see from this form of the partition function that vortices are explicitly present in the functional integral, induced by the source $m(X)$ on the dual lattice. For each configuration $\{m(X)\}$, the integral over $\phi$ and $\sigma_3$ must be carried out. This can be done using standard perturbative methods. Each dual lattice site $X$ where $m(X) \neq 0$, will be the site of a vortex of charge $m(X)$. In a dilute gas approximation, we can see that the size of the vortex core will in general be set by the scale-setting parameter $h$, which determines the region around $X$ where $\sigma_3$ is significantly different from zero. The contribution of the vortex core to the total weight of a given configuration $\{m(X)\}$ can be captured in a vortex activity $y$, which represents the Boltzmann weight of the classical vortex solution times a functional determinant factor, just as in the continuum. It is clear that in the limit where $h$ is very large, $\sigma_3$ will be essentially zero everywhere, and we recover the XY model with $K_{eff} \simeq K$ and a vortex core size on the order of the lattice spacing. Note that the $Z(2)$ symmetry under $\sigma_3 \rightarrow -\sigma_3$ means that for each vortex winding number $m$, there are two types of vortices depending on the behavior of $\sigma_3$ in the core, as in the continuum. For $h > 0$, the large-distance behavior is that of an XY model, giving a continuous path between the O(3) model and the vortex Coulomb gas phase of the XY model. If we keep only the $m = 1$ contributions, we have essentially a lattice sine-Gordon model

$$Z = \int_R [d\phi(X)] \exp \left\{ -\sum_{X,\mu} \frac{1}{2K_{eff}(\bar{x})} (\nabla_\mu \phi(X))^2 + \sum_X 4y \cos (2\pi \phi(X)) \right\}$$

where $K_{eff}$ is the value of $K_{eff}$ away from the vortex cores. All of the physics associated with the shortshort-ranged $\sigma_3$ field is contained in $K_{eff}$ and $y$.

The (3+1)-dimensional SU(2) gauge theory at high temperatures can be treated in much the same way as the two-dimensional O(3) model. It is convenient to work in Polyakov gauge, where $A_4$ is diagonal and time-independent so that the Polyakov loop is given by $P = \exp \left( iA_4/T \right) = \exp \left( i\theta_0 \right)$. Working at high temperature ensures that the running coupling constant is small. A sufficiently strong deformation term will make the expected value of the timelike link variable $U_4 = \exp \left( iA_4 \right)$ significantly different from one. This in turn will give large masses to the off-diagonal parts of the $U_3$ fields. The off-diagonal fields will be important only inside monopole cores where $A_4$ is small. Outside monopole cores, the model is effectively Abelian.

A simplified approach is to take the deformation term to be very strong and assume that all the fields are independent of $x_4$. We initially take the the timelike links $U_3 (\bar{x},t)$ to be diagonal and independent of $t$:

$$U_0 (\bar{x}) = \cos \left( \theta_0 (\bar{x}) \right) + i\sigma_3 \sin \left( \theta_0 (\bar{x}) \right)$$

A strong deformation term forces $\langle Tr_F P \rangle = 0$ with an expected value for $\langle \theta_0 \rangle$, given by $N_t \langle \theta_0 \rangle = \pi/2$. As in the O(3) case, we can define the (dimensionally-reduced) spatial gauge fields as

$$U_j (\bar{x}) = \sqrt{1 - (U_j^1 (\bar{x}))^2 - (U_j^2 (\bar{x}))^2} \left[ \cos \left( \theta_j (\bar{x}) \right) + i\sigma_3 \sin \left( \theta_j (\bar{x}) \right) \right] + i\sigma_1 \cdot U_j^1 (\bar{x}) + i\sigma_2 \cdot U_j^2 (\bar{x})$$
The expectation value $\langle U_0 \rangle$ makes the $U^1_j$ and $U^2_j$ fields massive, and they do not contribute to the large-distance behavior. This leaves us with an effective three-dimensional $U(1)$ gauge theory. The dual of a a three-dimensional Abelian gauge theory is an Abelian spin system, in this case again yielding a lattice sine-Gordon model as in the continuum [53].

The above simplified approach, based on the early application of dimensional reduction, is in fact too simple. As in the $O(3)$ model, where there were two types of vortices and two types of antivortices distinguished by their behavior in the vortex core, there are four Euclidean monopole solutions, not two [26, 31, 33]. The BPS-type monopole and anti-monopole solutions can be constructed as conventional time-independent monopole solutions, and are thus included in the simplified approach. On the other hand, the KK-type solutions are constructed from the BPS solutions using an $x_4$-dependent, non-periodic gauge transformation that changes the instanton charge of a field configuration [26]. Thus, a proper treatment of both types of monopoles is necessary. After accounting carefully for both types of solutions, the dual form of the partition function in the confined phase, reduced to three dimensions, is

$$Z = \int_R [d\sigma(X)] \exp \left[ - \sum_{X, \mu} \frac{g^2}{8 N_c} \left( \nabla_\mu \sigma(X) \right)^2 + \sum_X 4y \cos(2\pi \sigma(X)) \right]$$

which has the same form as the corresponding continuum result, where the effective action has the form

$$S_{eff} = \int d^3x \left[ \frac{g^2(T) T}{32\pi^2} (\partial_j \sigma)^2 - 4y \cos(\sigma) \right].$$

The two results are equivalent after indentifying $T^{-1}$ with $N_c$ and rescaling the $\sigma$ field. However, there are some significant difference between the continuum approach and the lattice approach. In the continuum, the renormalization group structure is more apparent. In the lattice calculation, no Bogomolny bound is available for the monopole action, and the correct approach to the continuum limit must emerge in the limit where the lattice spacing is taken to zero. On the other hand, the lattice approach manifestly includes all possible combinations of monopoles as contributions to the partition function. In contrast, in the continuum calculation combinations like a $BPS$ and $\overline{BPS}$ monopole pair, which are not topologically stable, are not naturally included as an instanton effect and must be included by hand. Moreover, such a continuum configuration is unstable to annihilation of the $BPS - \overline{BPS}$ pair, an effect that is naturally avoided on the lattice.

VI. CONCLUSIONS

The reduction of non-Abelian models to Abelian effective models is a powerful technique that can be justified under certain conditions, i.e., a small compactification radius and a suitable modification of the action. Under those conditions, a very appealing picture of confinement emerges that has a clear role for monopoles, instantons, the maximal Abelian subgroup and center symmetry. Nevertheless, there remain many unanswered questions. Chief among them is the connection between the small-$L$, center-symmetric model where confinement can be demonstrated and the corresponding large-$L$ confining behavior that we wish to understand. There are also several major technical issues in the treatment of instantons and monopoles. Center symmetry is crucial in the recovery of the correct renormalization group behavior for the confining instanton-monopole plasma. This raises serious questions about the interpretation of monopole contributions when center symmetry is broken. Center symmetry is slightly broken in the low-temperature phase of QCD with dynamical quarks, where the Polyakov loop expected value is small but nonzero. Another important issue is the treatment of field configurations that have topological content but are not topologically stable. This problem, present since the discovery of instantons, is crucial here. Recent work on resurgence theory [53, 57] is promising approach to understanding the correct treatment of non-perturbative contributions in quantum field theory.

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