Retracts of vertex sets of trees
and the almost stability theorem

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Abstract

Let $G$ be a group, let $T$ be an (oriented) $G$-tree with finite edge stabilizers, and let $VT$ denote the vertex set of $T$. We show that, for each $G$-retract $V'$ of the $G$-set $VT$, there exists a $G$-tree whose edge stabilizers are finite and whose vertex set is $V'$. This fact leads to various new consequences of the almost stability theorem.

We also give an example of a group $G$, a $G$-tree $T$ and a $G$-retract $V'$ of $VT$ such that no $G$-tree has vertex set $V'$.

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1 Outline

Throughout the article, let $G$ be a group, and let $\mathbb{N}$ denote the set of finite cardinals, $\{0, 1, 2, \ldots \}$. All our $G$-actions will be on the left.

The following extends Definitions II.1.1 of [3] (where $A$ is assumed to have trivial $G$-action).

1.1 Definition. Let $E$ and $A$ be $G$-sets.

Let $(E, A)$ denote the set of all functions from $E$ to $A$. An element $v$ of $(E, A)$ has the form $v: E \rightarrow A$, $e \mapsto v(e)$. There is a natural $G$-action on $(E, A)$ such that $(gv)(e) := g(v(g^{-1}e))$ for all $v \in (E, A)$, $g \in G$, $e \in E$.

Two elements $v$ and $w$ of $(E, A)$ are said to be almost equal if the set

$$\{ e \in E \mid v(e) \neq w(e) \}$$

is finite. Almost equality is an equivalence relation; the equivalence classes are called the almost equality classes in $(E, A)$.

A subset $V$ of $(E, A)$ is said to be $G$-stable if $V$ is closed under the $G$-action. In general, a $G$-stable subset is the same as a $G$-subset.

In this article, we wish to strengthen the following result.

1.2 The almost stability theorem [3, Theorem III.8.5]. If $E$ is a $G$-set with finite stabilizers, and $A$ is a nonempty set with trivial $G$-action, and $V$ is a $G$-stable almost equality class in the $G$-set $(E, A)$, then there exists a $G$-tree with finite edge stabilizers and vertex set $V$.

In the light of Bass-Serre theory, the almost stability theorem can be thought of as a broad generalization of Stallings’ ends theorem.

Let us now recall the notion of a $G$-retract of a $G$-set. The following alters Definition III.1.1 of [3] slightly.
1.3 Definition. A $G$-retract $U$ of a $G$-set $V$ is a $G$-subset of $V$ with the property that, for each $w \in V - U$, there exists $u \in U$ such that $G_w \leq G_u$, or, equivalently, with the property that there exists a $G$-map, called a $G$-retraction, from $V$ to $U$ which is the identity on $U$. 

Chapter IV of [3] collects together a wide variety of consequences of the almost stability theorem [2]. In some of these applications, the conclusions assert that certain naturally arising $G$-sets are $G$-retracts of vertex sets of $G$-trees with finite edge stabilizers. This leads to the question of whether or not the class of vertex sets of $G$-trees with finite edge stabilizers is closed under taking $G$-retracts. We are now able to answer this in the affirmative; in Section 2 below, we prove that any $G$-retract of the vertex set of a $G$-tree with finite edge stabilizers is itself the vertex set of a $G$-tree with finite edge stabilizers.

In Section 3 we record the resulting generalizations of the almost stability theorem and the applications which are affected. In the most classic example, if $G$ has cohomological dimension one, and $\omega ZG$ is the augmentation ideal of the group ring $ZG$, one can deduce that $G$ acts freely on a tree whose vertex set is the $G$-set $1 + \omega ZG$, and, hence, $G$ is a free group; this is a slightly more detailed version of a theorem of Stallings and Swan.

In Section 4 we record an even more general form of the almost stability theorem in which the $G$-action on $A$ need not be trivial.

In Section 5 we construct a group $G$ and a $G$-retract of a vertex set of a $G$-tree (with infinite edge stabilizers) that is not itself the vertex set of a $G$-tree.

2 Operations on trees

Throughout this section we will be working with the following.

2.1 Hypotheses. Let $T = (T,V,E,\iota,\tau)$ be a $G$-tree, as in [3] Definition I.2.3.

We write $VT = V$ and $ET = E$, and we view the underlying $G$-set of $T$ as the disjoint union of $T$ and $E$, written $T = V \cup E$. Here $\iota: E \to V$ is the initial vertex map and $\tau: E \to V$ is the terminal vertex map.

We first consider a simple form of retraction, which amplifies Definitions III.7.1 of [3]. Recall that a vertex $v$ of a tree is called a sink if every edge of the tree is oriented towards $v$.

2.2 The compressing lemma. Suppose that Hypotheses 2.1 hold.

Let $E'$ be a $G$-subset of $E$ such that each component of the subforest $T - E'$ of $T$ has a (unique) sink. Let $V'$ denote the set of sinks of the components of $T - E'$.

Let $i: E' \to E$ denote the inclusion map, and let $\phi: V \to V'$ denote the $G$-retraction which assigns, to each $v \in V$, the sink of the component of $T - E'$ containing $v$.

Then the $G$-graph $T' = (T',V',E',\phi \circ \iota \circ i, \phi \circ \tau \circ i)$ is a $G$-tree.

Let $E'' = E - E'$ and let $V'' = V - V'$. Then $T - E'$ is the $G$-subforest of $T$ with vertex set $V$ and edge set $E''$. For each $v \in V$, $\phi(v)$ is reached in $T$ by starting at $v$ and travelling as far as possible along edges in $E''$ respecting the orientation. The initial vertex map $\iota: E \to V$ induces a bijective map $E'' \to V''$.

We say that $T'$ is obtained from $T$ by compressing the closures of the elements of $E''$ to their terminal vertices or by compressing the components of $T - E'$ to their sinks.

In applications, we usually first $G$-equivariantly reorient $T$ and then, in the resulting tree, compress a $G$-set of closed edges to their terminal vertices; we then call the combined procedure a $G$-equivariant compressing operation.
Proof of Lemma 2.2. The map \( \phi \) induces a surjective \( G \)-map \( T \to T' \) in which the fibres are the components of \( T - E' \). It follows that \( T' \) is a \( G \)-tree. \( \square \)

We now recall the sliding operation of Rips-Sela [3] p. 59 as generalized by Forester [7] Section 3.6; see also the Type 1 operation of [3] p. 146]. We find it convenient to express the result and the proof in the notation of [3].

2.3 The sliding lemma. Suppose that Hypotheses 2.1 hold.

Let \( e \) and \( f \) be elements of \( E \).

Suppose that \( \tau e = \tau f \), \( G_e \leq G_f \), and \( G f \cap Ge = \emptyset \).

Let \( \tau' : E \to V \) denote the map given by

\[
e' \mapsto \tau'(e') := \begin{cases} \tau(e') & \text{if } e' \in E - Ge, \\ \tau(gf) & \text{if } e' = ge \text{ for some } g \in G, \end{cases}
\]

for all \( e' \in E \).

Then the \( G \)-graph \( T' = (T', V, E, \iota, \tau') \) is a \( G \)-tree.

Here, we say that \( T' \) is obtained from \( T \) by \( G \)-equivariantly sliding \( \tau e \) along \( f \) from \( \iota f \) to \( \tau f \).

In applications, we usually first \( G \)-equivariantly reorient \( Ge \), or \( G f \), or both, or neither, and then, in the resulting tree, \( G \)-equivariantly slide \( \tau e \) along \( f \) from \( \iota f \) to \( \tau f \), and then reorient back again. We then call the combined procedure a \( G \)-equivariant sliding operation.

Proof of Lemma 2.4. It is clear that \( T' \) is a \( G \)-graph.

Let \( X \) be the \( G \)-graph obtained from \( T \) by deleting the two edge orbits \( Ge \cup Gf \), and then inserting one new vertex orbit \( Gv \) and three new edge orbits \( Ge' \cup Gf_1 \cup Gf_2 \), with \( G_v = G_e, G_e = G_{f_1} = G_{f_2} = G_f \), and setting

\[
\iota(e') = \iota(e), \quad \iota(f_1) = \iota(f) = \tau(e), \quad \iota(f_2) = \tau(e) = \tau(f_1) = v, \quad \tau(f_2) = \tau(f).
\]

Thus we are \( G \)-equivariantly subdividing \( f \) into \( f_1 \) and \( f_2 \) by adding \( v \), and then sliding \( \tau e \) along \( f_1 \) from \( \iota f_1 \) to \( \tau f_1 = v \).

Then \( T \) is recovered from \( X \) by \( G \)-equivariantly compressing the closure of \( f_1 \) to \( \iota(f_1) \), and renaming \( f_2 \) as \( f, e' \) as \( e \). Thus \( X \) maps onto \( T \) with fibres which are trees. It follows that \( X \) is a tree; see [3] Proposition III.3.3.

Also \( T' \) is recovered from \( X \) by \( G \)-equivariantly compressing the closure of \( f_2 \) to \( \tau(f_2) \), and renaming \( f_1 \) as \( f, e' \) as \( e \). By Lemma 2.2 \( T' \) is a tree. \( \square \)

3 Filtrations

Throughout this section we will be working with the following.

3.1 Hypotheses. Let \( T = (T, V, E, \iota, \tau) \) be a \( G \)-tree, let \( U \) be a \( G \)-retract of the \( G \)-set \( V \), and let \( W = V - U \). \( \square \)

3.2 Conventions. We shall use interval notation for ordinals; for example, if \( \kappa \) is an ordinal, then \([0, \kappa)\) denotes the set of all ordinals \( \alpha \) such that \( \alpha < \kappa \).

If we have an ordinal \( \kappa \) and a specified map from a set \( X \) to \([0, \kappa)\), then we will understand that the following notation applies. Denoting the image of each \( x \in X \) by \( \height(x) \in [0, \kappa) \), we write, for each \( \alpha \in [0, \kappa) \) and each \( \beta \in [0, \kappa) \),

\[
X[\alpha] := \{ x \in X \mid \height(x) = \alpha \} \quad \text{and} \quad X[0, \beta) := \{ x \in X \mid \height(x) < \beta \}.
\]
3.3 Definitions. Suppose that Hypotheses 3.1 hold.
Let $P(T)$ denote the set of paths in $T$, as in Definitions I.2.3 of \cite{3}. Thus, for each $p \in P(T)$, we have the initial vertex of $p$, denoted $i_p$, the terminal vertex of $p$, denoted $\tau p$, the set of edges which occur in $p$, denoted $E(p) \subseteq E$, the length of $p$, denoted $\text{length}(p) \in \mathbb{N}$, and the $G$-stabilizer of $p$, denoted $G_p \leq G$.

Let $\kappa$ be an ordinal and let
\[
(3.3.1) \quad T \to [0, \kappa), \quad x \mapsto \text{height}(x)
\]
be a map. Since $T$ is nonempty, $\kappa$ must be nonzero. As a set, $T = V \cup E$. Thus, for each $\alpha \in [0, \kappa)$, we have $T[\alpha]$, $E[\alpha]$ and $V[\alpha]$, and, for each $\beta \in [0, \kappa)$, we have $T[0, \beta)$, $E[0, \beta)$ and $V[0, \beta)$.

For each $w \in W$, we then define
\[
P_T(w) := \{ p \in P(T) \mid i_p = w, G_p = G_w, \text{height}(\tau p) < \text{height}(w), \text{height}(E(p)) \subseteq \{\text{height}(w), \text{height}(w) + 1\} \}.
\]

We say that (3.3.1) is a $U$-filtration of $T$ if all of the following hold:
\begin{align*}
(3.3.2) \quad & \text{for each } \beta \in [0, \kappa), T[0, \beta) \text{ is a } G\text{-subforest of } T; \\
(3.3.3) \quad & T[0] = U; \\
(3.3.4) \quad & \text{for each } \alpha \in [1, \kappa), T[\alpha] \text{ is a } G\text{-finite } G\text{-subset of } T; \text{ and,} \\
(3.3.5) \quad & \text{for each } w \in W, P_T(w) \text{ is nonempty.}
\end{align*}

3.4 Lemma. If Hypotheses 3.1 hold, then there exists a $U$-filtration of $T$.

Proof. We shall recursively construct a family $(E[\alpha] \mid \alpha \in [0, \kappa))$ of $G$-subsets of $E$, for some nonzero ordinal $\kappa$.
We take $E[0] = \emptyset$.

Suppose that $\gamma$ is a nonzero ordinal, and that we have a family $(E[\alpha] \mid \alpha \in [0, \gamma))$ of $G$-subsets of $E$.

For each $\beta \in [0, \gamma)$, we define
\[
E[0, \beta) := \bigcup_{\alpha \in [0, \beta)} E[\alpha] \quad \text{and} \quad V[0, \beta) := \begin{cases} 
\emptyset & \text{if } \beta = 0, \\
U \cup (E[0, \beta)) \cup \tau(E[0, \beta)) & \text{if } \beta > 0.
\end{cases}
\]

For each $\alpha \in [0, \gamma)$, we define $V[\alpha] := V[0, \alpha + 1) - V[0, \alpha)$. Thus
\[
V[0, \beta) = \bigcup_{\alpha \in [0, \beta)} V[\alpha].
\]

If $E[0, \gamma) = E$, we take $\kappa = \gamma$ and the construction terminates.
Now suppose that $E[0, \gamma) \subset E$. We shall explain how to choose $E[\gamma]$.
If $\gamma$ is a limit ordinal or $1$, we take $E[\gamma]$ to be an arbitrary single $G$-orbit in $E - E[0, \gamma)$.
If $\gamma$ is a successor ordinal greater than $1$ then there is a unique $\alpha \in [1, \gamma)$ such that $\gamma = \alpha + 1$, and we want to construct $E[\alpha + 1]$. Notice that $V[0, \alpha]$ is a $G$-retract of $V$ because $V[0, \alpha)$ contains $U$. Thus we can $G$-equivariantly specify, for each $w \in V[\alpha]$, a $T$-geodesic $p = p(w)$ from $w$ to an element $v = v(w) \in V[0, \alpha)$ fixed by $G_w$. Since $G_w$ fixes both ends of $p$, $G_w$ fixes $p$. Hence we may assume that $v$ is the first, and hence only, vertex of $p$ that lies in $V[0, \alpha)$. Clearly $G_p$ fixes $w$. Thus $G_w = G_p$. Let $P_{n+1}$ denote the set of edges which occur in the $p(w)$, as $w$
ranges over $V[\alpha]$. Then $P_{\alpha+1} \subseteq E - E[0, \alpha]$, since each element of $E[0, \alpha]$ has both vertices in $V[0, \alpha]$. If $P_{\alpha+1} \subseteq E[\alpha]$, we choose $E[\alpha+1]$ to be an arbitrary single $G$-orbit in $E - E[0, \alpha + 1]$. If $P_{\alpha+1} \not\subseteq E[\alpha]$, we take $E[\alpha + 1] = P_{\alpha+1} - E[\alpha]$. This completes the description of the recursive construction.

We now verify that we have a $U$-filtration of $T$.

It can be seen that, for each ordinal $\gamma$ such that $(E[\alpha] \mid \alpha \in [0, \gamma))$ is defined, the $E[\alpha]$, $\alpha \in [1, \gamma)$, are pairwise disjoint, nonempty, $G$-subsets of $E$. Hence the cardinal of $\gamma$ is at most one more than the cardinal of $E$. Therefore the construction terminates at some stage. This implies that there exists a nonzero ordinal $\kappa$ such that $E[0, \kappa) = E$. Also $V[0, \kappa) = V$, and $(V[\alpha] \mid \alpha \in [0, \kappa))$ gives a partition of $V$. Thus we have an implicit map $T \to [0, \kappa)$ and we denote it by $x \mapsto \text{height}(x)$.

Clearly (3.3.2), (3.3.3) and (3.3.4) hold. If $\alpha \in [1, \kappa)$ and $E[\alpha]$ is $G$-finite, then either $E[0, \alpha + 1] = E$ or $V[\alpha]$, $P_{\alpha+1}$ and $E[\alpha + 1]$ are $G$-finite. It follows, by transfinite induction, that $E[\alpha]$ and $V[\alpha]$ are $G$-finite for all $\alpha \in [1, \kappa)$. Thus (3.3.4) holds.

\section{The main result}

Let us introduce a technical concept which generalizes that of a finite subgroup.

\subsection{Definitions.}

A subgroup $H$ of $G$ is said to be $G$-conjugate incomparable if, for each $g \in G$, $H^g \not\subseteq H$ (if and) only if $H^g = H$. This clearly holds if $H$ is finite.

We say that a $G$-set $X$ has $G$-conjugate-incomparable stabilizers if, for each $x \in X$, the $G$-stabilizer $G_x$ is a $G$-conjugate-incomparable subgroup, that is, for each $g \in G$, $G_x \not\subseteq G_{gx}$ (if and) only if $G_x = G_{gx}$.

Throughout this section we will be working with the following.

\subsection{Hypotheses.}

Let $T = (T, V, E, \iota, \tau)$ be a $G$-tree, let $U$ be a $G$-retract of the $G$-set $V$, and let $W = V - U$.

Suppose that the $G$-set $W$ has $G$-conjugate-incomparable stabilizers.

Let $\kappa$ be an ordinal and let

\[(4.2.1) \quad \text{height} : V \cup E \to [0, \kappa), \quad x \mapsto \text{height}(x),\]

be a $U$-filtration of $T$.

\subsection{Definitions.}

Suppose that Hypotheses 4.2 hold.

Let $w \in W$. Define $d_T(w) := \min\{\text{length}(p) \mid p \in P_T(w)\}$. Then $d_T(w)$ is a positive integer and

\[(4.3.1) \quad d_T(gw) = d_T(w) \text{ for all } g \in G.\]

For $v_0, v_1$ in $V$, we say that $v_1$ is lower than $v_0$ if one of the following holds:

\[(4.3.2) \quad \text{height}(v_0) > \text{height}(v_1);\]
\[(4.3.3) \quad \text{height}(v_0) = \text{height}(v_1) > 0 \text{ and } G_{v_0} < G_{v_1}; \text{ or},\]
\[(4.3.4) \quad \text{height}(v_0) = \text{height}(v_1) > 0 \text{ and } G_{v_0} = G_{v_1} \text{ and } d_T(v_0) > d_T(v_1).\]

An edge $e$ of $T$ is said to be problematic if it joins vertices $v_0, v_1$ such that $\text{height}(e) = \text{height}(v_1) = \text{height}(v_0) + 1$. Notice that $\text{height}(e)$ is a successor ordinal and that $v_0$ is lower than $v_1$.

For each $v_0 \in W$, there exists a path

\[(4.3.5) \quad v_0, e_1^1, v_1, e_2^2, v_2, \ldots, e_d^d, v_d \text{ in } P_T(v_0) \text{ such that } d = d_T(v_0).\]
Here $\text{height}(v_1) \leq \text{height}(v_0) + 1$. We say that $v_0$ is a problematic vertex of $T$ if there exists a path as in (4.3.1) such that $\text{height}(v_1) = \text{height}(v_0) + 1$. In this event $\text{height}(e_1) = \text{height}(v_1)$ and $e_1$ is a problematic edge of $T$.

4.4 Lemma. If Hypotheses 4.2 hold, then applying some transfinite sequence of $G$-equivariant sliding operations to $T$ yields a $G$-tree $T' = (T', V, E, \iota', \tau')$ such that (4.2.1) is also a $U$-filtration of $T'$ and $T'$ has no problematic vertices.

Proof. We shall construct a family of trees

$$(T_\beta = (T_\beta, V, E, \iota, \tau_\beta) \mid \beta \in [0, \kappa])$$

such that, for each $\beta \in [0, \kappa]$, (4.2.1) is a $U$-filtration of $T_\beta$, and $T_\beta$ has no problematic vertices in $V[0, \beta]$.

We take $T_0 = T$.

For each successor ordinal $\beta = \alpha + 1 \in [0, \kappa)$, $T_{\alpha+1}$ will be obtained from $T_\alpha$ by altering, if necessary, $\iota_\alpha$ and $\tau_\alpha$ on $E[\alpha+1]$, as described below.

For each limit ordinal $\beta \in [0, \kappa]$, we let $\iota_\beta$ be given on $E[\alpha]$ by $\iota_\alpha$, for each $\alpha \in [0, \beta)$, and similarly for $\tau_\beta$.

Suppose then that $\beta = \alpha + 1 \in [0, \kappa)$, that we have a tree $T_\alpha = (T_\alpha, V, E, \iota_\alpha, \tau_\alpha)$, and that (4.2.1) is a $U$-filtration of $T_\alpha$, and that $T_\alpha$ has no problematic vertices in $V[0, \alpha]$. We now describe a crucial problem-reducing procedure that can be applied in the case where there exists some $v_0 \in V[\alpha]$ which is a problematic vertex of $T_\alpha$.

Let $d = d_{T_\alpha}(v_0)$. Thus, there exists a path

$$v_0, e_1^1, v_1, e_2^2, v_2, \ldots, e_d^d, v_d$$

in $P_{T_\alpha}(v_0)$ such that $v_1 \in V[\alpha+1]$. Hence, $e_1 \in E[\alpha+1]$. Without loss of generality, let us assume that $e_1 = -1$.

There exists a least $i \in [2, d]$ such that $v_i \in V[0, \alpha+1)$. Then

$$\{v_1, \ldots, v_{i-1}\} \subseteq V[\alpha+1] \quad \text{and, hence,} \quad \{e_1, \ldots, e_i\} \subseteq E[\alpha+1].$$

We claim that $Ge_1 \cap \bigcup_{j=2}^i Ge_j = \emptyset$. Suppose this fails. Then $e_1 \in \bigcup_{j=2}^i Ge_j$. Here, $v_0 \in \bigcup_{j=1}^i Gv_j$. Since $v_0 \in V[\alpha]$ and $\bigcup_{j=1}^{i-1} Gv_j \subseteq V[\alpha+1]$ we see that $v_0 \in Gv_1$. Hence $v_i \in V[\alpha]$ and, by Lemma 4.3, $d_{T_\alpha}(v_i) = d_{T_\alpha}(v_0) = d$. But $Gv_0 = Gp \subseteq Gv_i$. Since $Gv_0$ is a $G$-conjugate-incomparable subgroup, $Gv_0 = Gv_i$. It follows that

$$v_i, e_{i+1}^i, v_{i+1}, \ldots, e_d^d, v_d$$

lies in $P_{T_\alpha}(v_i)$. Hence $d_{T_\alpha}(v_i) \leq d - i$, which is a contradiction. This proves the claim.

By Lemma 2.8, we can $G$-equivariantly slide $\iota e_1$ along $e_2^2$ from $v_1$ to $v_2$, and then $G$-equivariantly slide $\iota e_1$ along $e_3^3$ from $v_2$ to $v_3$, and so on, up to $v_i$. We then get a new $G$-tree $T_{\alpha,1} = (T_{\alpha,1}, V, E, \iota_{\alpha,1}, \tau_{\alpha,1})$ by $G$-equivariantly sliding $\iota e_1$ along our path from $v_1$ to $v_i$.

Let $e'_1$ denote $e_1$ viewed as an edge of $T_{\alpha,1}$. Wherever $v_1, e_1, v_0$ occurs in a path in $T_{\alpha,1}$, it can be replaced with the sequence

$$v_1, e_2^2, v_2, \ldots, v_{i-1}, e_i^i, v_i, e_1', v_0$$

to obtain a path in $T_{\alpha,1}$. It is important to note that all the edges involved here lie in $E[\alpha+1]$. In terms of the free groupoid on $E[\alpha+1]$, $e_1 = e_2^2 e_3^3 \cdots e_i^i e_1'$, and we are performing the change-of-basis which replaces $e_1$ with $e_1'$. 

It is easy to see that Hypotheses 4.2 hold and Hypertheses 4.5 Lemma. Thus \( E^{[\alpha + 1]} \) is a \( G \)-filtration of \( T_{\alpha,1} \). Notice that \( T_{\alpha,1} \), like \( T_{\alpha} \), has no problematic vertices in \( V[0, \alpha] \). We have reduced the number of \( G \)-orbits of problematic edges in \( E^{[\alpha + 1]} \).

This completes the description of a problem-reducing procedure.

Since \( E^{[\alpha + 1]} \) is \( G \)-finite by Hypotheses 4.3, on repeating problem-reducing procedures as often as possible, we find some \( m \in \mathbb{N} \), and a sequence

\[ T_\alpha = T_{\alpha,0}, T_{\alpha,1}, \ldots, T_{\alpha,m}, \]

such that \( T_{\alpha,m} \) has no problematic vertices in \( V[0, \alpha] \cup V[\alpha] = V[0, \alpha + 1] \). We define \( T_{\alpha+1} = (T_{\alpha+1}, V, E, \tau_{\alpha+1}, \tau_{\alpha+1}) \) to be \( T_{\alpha,m} \). Notice that \( \tau_{\alpha+1} \) agrees with \( \tau_\alpha \) on \( E - E^{[\alpha + 1]} \), and similarly for \( \tau_{\alpha+1} \).

Continuing this procedure transfinitley, we arrive at a tree \( T_\alpha \) which has no problematic vertices.

4.5 Lemma. If Hypotheses 4.2 hold and \( T \) has no problematic vertices, then applying some \( G \)-equivariant compressing operation on \( T \) yields a \( G \)-tree with vertex set \( U \).

Proof. We claim that any sequence in \( V \) is finite if each term is lower than all its predecessors.

Let \( \alpha \in [0, \kappa) \).

If \( v_0, v_1 \) are elements of the same \( G \)-orbit of \( V[\alpha] \), then \( v_1 \) is not lower than \( v_0 \), that is, Hypotheses 4.3.2, 4.3.3 all fail; this follows from 4.3.1 and the fact that \( V[\alpha] \) has \( G \)-conjugate-incomparable stabilizers.

Thus, if \( n \in \mathbb{N} \) and \( v_1, v_2, \ldots, v_n \) is a sequence in \( V[\alpha] \) such that each term is lower than all its predecessors, then \( G v_1, G v_2, \ldots, G v_n \) are pairwise disjoint, and \( n \) is at most the number of \( G \)-orbits in \( V[\alpha] \). It follows that any sequence in \( V[\alpha] \) is finite if each term is lower than all its predecessors. The claim now follows.

Let us \( G \)-equivariantly reorient \( T \) so that, for each edge \( e, \iota e \) is not lower than \( \tau e \).

Let \( v_0 \in W \). Let us \( G \)-equivariantly choose a path

\[ v_0, e_1^{e_1}, v_1, e_2^{e_2}, v_2, \ldots, e_d^{e_d}, v_d \]

in \( P_T(v_0) \) such that \( d = d_T(v_0) \). Then we call \( e_1 \) the distinguished edge associated to \( v_0 \), and \( v_1 \) the distinguished neighbour of \( v_0 \).

Let \( E'' \) denote the set of distinguished edges chosen in this way.

Let us consider the above path for \( v_0 \). From Definitions 4.3 we see that, since \( T \) has no problematic vertices, \( \text{height}(v_0) \geq \text{height}(v_1) \). We claim that \( v_1 \) is lower than \( v_0 \). The claim is clear if \( \text{height}(v_0) > \text{height}(v_1) \) (in which case, \( d = 1 \)), and we may assume that \( \text{height}(v_0) = \text{height}(v_1) \) (\( > 0 \)). Again, the claim is clear if \( G v_0 < G v_1 \), and we may assume that \( G v_0 = G v_1 \). Here \( G v_1 \) fixes \( p \), and the path

\[ v_1, e_2^{e_2}, v_2, \ldots, e_d^{e_d}, v_d \]

shows that \( d_T(v_1) \leq d - 1 < d = d_T(v_0) \), and the claim is proved. Hence \( e_1 = 1 \).

Thus \( \iota \) induces a bijection \( E'' \to W \).

Moreover, in travelling along the distinguished edge \( e_1 \) respecting the orientation, from \( v_0 \) to its distinguished neighbour \( v_1 \), we move to a lower vertex.

Thus, starting at any element \( v \) of \( V \), after travelling a finite number of steps along distinguished edges respecting the orientation, we arrive at a vertex, denoted \( \phi(v) \), with no distinguished neighbours, that is, \( \phi(v) \in U \).

By Lemma 4.2, compressing the closures of the distinguished edges to their terminal vertices gives a \( G \)-tree with vertex set \( U \) and edge set \( E - E'' \).
We now come to our main result. In Section 7, we will see that the $G$-conjugate-incomparability hypotheses cannot be omitted.

4.6 Theorem. Let $T$ be a $G$-tree, and let $U$ be a $G$-retract of the $G$-set $VT$. Suppose that the $G$-set $ET$ has $G$-conjugate-incomparable stabilizers, or, more generally, that the $G$-set $VT - U$ has $G$-conjugate-incomparable stabilizers.

Then applying to $T$ some transfinite sequence of $G$-equivariant sliding operations followed by some $G$-equivariant compressing operation yields a $G$-tree $T'$ such that $VT' = U$.

Here $ET'$ is a $G$-subset of $ET$, and there exists a $G$-set isomorphism $ET' \cong VT - VT' = VT - U$.

Proof. For each $w \in VT - U$, there exists $u \in U$ such that $G_w \leq G_u$. If $e$ denotes the first edge in the $T$-geodesic from $w$ to $u$, then $G_e = G_w$. Thus, if $E$ has $G$-conjugate-incomparable stabilizers, then the same holds for $VT - U$.

By Lemma 3.4, we may assume that Hypotheses 4.2 hold. By Lemma 4.4, we may assume that $T$ itself has no problematic vertices. Applying Lemma 4.5, we obtain the result; the final assertion follows from Lemma 2.2. □

We record the special case of Theorem 4.6 that is of interest to us.

4.7 The retraction lemma. Let $T$ be a $G$-tree whose edge stabilizers are finite, and let $U$ be any $G$-retract of the $G$-set $VT$. Then there exists a $G$-tree whose edge stabilizers are finite and whose vertex set is the $G$-set $U$. □

5 The almost stability theorem and applications

We now combine the almost stability theorem 1.2 and the retraction lemma 4.7.

5.1 Theorem. Let $E$ and $A$ be $G$-sets such that $E$ has finite stabilizers and $A$ has trivial $G$-action. If $V$ is a $G$-retract of a $G$-stable almost equality class in $(E, A)$, then there exists a $G$-tree whose edge stabilizers are finite and whose vertex set is the $G$-set $V$.

Proof. By the almost stability theorem 1.2 there exists a $G$-tree whose edge stabilizers are finite and whose vertex set is the given $G$-stable almost equality class in $(E, A)$. By the retraction lemma 4.7, there exists a $G$-tree whose edge stabilizers are finite and whose vertex set is $V$. □

We now recall Definitions IV.2.1 and IV.2.2 of [3].

5.2 Definitions. Let $M$ be a $G$-module, that is, an additive abelian group which is also a $G$-set such that $G$ acts as group automorphisms on $M$. Thus a $G$-module is simply a left module over the integral group ring $\mathbb{Z}G$.

If $d: G \to M$ is a derivation, that is, a map such that $d(xy) = d(x) + xd(y)$ for all $x, y \in G$, then $M_d$ denotes the set $M$ endowed with the $G$-action

$$G \times M \to M, \quad (g, m) \mapsto g \cdot m := gm + d(g)$$

for all $g \in G$ and all $m \in M$.

It is straightforward to show that $M_d$ is a $G$-set. This construction has made other appearances in the literature; see [1, Remarque 4.4].

We say that $M$ is an induced $G$-module if there exists an abelian group $A$ such that $M$ is isomorphic, as $G$-module, to $AG := \mathbb{Z}G \otimes_{\mathbb{Z}} A$.

We say that $M$ is a $G$-projective $G$-module if $M$ is isomorphic, as $G$-module, to a direct summand of an induced $G$-module. □
5.3 Example. If $R$ is any ring and $P$ is a projective left $RG$-module, then there exists a free left $R$-module $F$ such that $P$ is isomorphic, as $RG$-module, to an $RG$-summand of

$$RG \otimes_R F = ZG \otimes_Z R \otimes_R F = ZG \otimes_Z F = FG.$$ 

Hence $P$ is $G$-projective.

The following generalizes Theorem IV.2.5 and Corollary IV.2.8 of [3].

5.4 Theorem. If $P$ is a $G$-projective $G$-module, and $d: G \to P$ is a derivation, then there exists a $G$-tree whose edge stabilizers are finite and whose vertex set is the $G$-set $P_d$.

**Proof.** There exists an abelian group $A$ such that $P$ is isomorphic to a $G$-summand of $AG$. We view $P$ as a $G$-submodule of $AG$. There exists an additive $G$-retraction $\pi: AG \to P$.

We view $AG$ as the almost equality class of $(G, A)$ which contains the zero map. Thus $AG$ is a $G$-submodule of $(G, A)$, and we have a derivation

$$d : G \to P \subseteq AG \subseteq (G, A).$$

By a classic result of Hochschild’s, there exists $v \in (G, A)$ such that, for all $g \in G$, $d(g) = gv - v$. For example, we can take $v: x \mapsto -(d(x))(x)$, for all $x \in G$. See the proof of Proposition IV.2.3 in [3].

Let $U = v + P$ and $V = v + AG$. Then $U \subseteq V \subseteq (G, A)$, and $V$ is the almost equality class which contains $v$. Also, $U$ and $V$ are $G$-stable, since, for each $g \in G$, $gv = v + d(g) \in v + P \subseteq v + AG$. The map

$$V \to U, \quad v + m \mapsto v + \pi(m), \text{ for all } m \in AG,$$

is a $G$-retraction, since, for all $m \in AG$,

$$g(v + m) = v + gm + d(g) \mapsto v + \pi(gm + d(g)) = v + g\pi(m) + d(g) = g(v + \pi(m)).$$

By Theorem 5.1 there exists a $G$-tree whose edge stabilizers are finite and whose vertex set is the $G$-set $U$.

The bijective map $P \to U$, $p \mapsto v + p$, is an isomorphism of $G$-sets $P_d \cong U$. Now the result follows.

5.5 Remark. Notice that, in Theorem 5.4 the stabilizer of a vertex $p \in P_d$ is precisely the kernel of the derivation

$$d + \text{ad } p: G \to P, \quad g \mapsto d(g) + gp - p = (g - 1)(v + p).$$

The following generalizes Corollary IV.2.10 of [3] and is used in the proof of Lemma 5.16 of [3].

5.6 Corollary. Let $M$ be a $G$-module, let $P$ be a $G$-projective $G$-submodule of $M$, and let $v$ be an element of $M$. If the subset $v + P$ of $M$ is $G$-stable, then there exists a $G$-tree whose edge stabilizers are finite and whose vertex set is the $G$-set $v + P$.

**Proof.** The inner derivation $\text{ad } v: G \to M$ restricts to a derivation $d: G \to P$, $g \mapsto gv - v \in P \subseteq M$, for all $g \in G$. The bijective map $P \to v + P$, $p \mapsto v + p$, is then an isomorphism of $G$-sets $P_d \cong v + P$. Now the result follows from Theorem 5.4. □
5.7 Example. Let $R$ be a nonzero associative ring, and let $\omega RG$ be the augmentation ideal of the group ring $RG$.

Notice that, in the (left) $G$-set $RG$, both the coset $1 + \omega RG$ and $RG - \{0\}$ are $G$-stable, and that the $G$-set $RG - \{0\}$ has finite stabilizers.

If $\omega RG$ is projective as left $RG$-module, then, by Corollary 5.6, there exists a $G$-tree $T$ with $VT = 1 + \omega RG \subseteq RG - \{0\}$; hence $T$ has finite stabilizers. This sheds some light on the main step in the characterization of groups of cohomological dimension at most one over $R$. See, for example, [3, Theorem IV.3.13].

6 A more general form

We next want to generalize Theorem 5.1.

The following is similar to Lemma 2.2 of [4], and the proof is straightforward.

6.1 Lemma. Let $E$ and $A$ be $G$-sets such that, for each $e \in E$, $G_e$ acts trivially on $A$.

Let $\tilde{A}$ denote the $G$-set with the same underlying set as $A$ but with trivial $G$-action.

Let $E_0$ be a $G$-transversal in $E$.

For each $\phi \in (E, A)$, let $\tilde{\phi} \in (E, \tilde{A})$ be defined by $\tilde{\phi}(ge) = g^{-1} \cdot \phi(ge)$ for all $(g, e) \in G \times E_0$, where $\cdot$ denotes the $G$ action on $A$.

For each $\psi \in (E, A)$, let $\tilde{\psi} \in (E, A)$ be defined by $\tilde{\psi}(ge) = g \cdot \psi(ge)$ for all $(g, e) \in G \times E_0$.

Then

$$(E, A) \rightarrow (E, \tilde{A}), \quad \phi \mapsto \tilde{\phi}, \quad \text{and} \quad (E, \tilde{A}) \rightarrow (E, A), \quad \psi \mapsto \tilde{\psi},$$

are mutually inverse isomorphisms of $G$-sets which preserve almost equality between functions.

Combined, Lemma 6.1 and Theorem 5.1 give the most general form that we know of the almost stability theorem.

6.2 Theorem. Let $E$ and $A$ be $G$-sets such that, for each $e \in E$, $G_e$ is finite and acts trivially on $A$. If $V$ is a $G$-retract of a $G$-stable almost equality class in $(E, A)$, then there exists a $G$-tree whose edge stabilizers are finite and whose vertex set is the $G$-set $V$.

For each $e \in E$, if $G_e$ is trivial, then $G_e$ is finite and acts trivially on $A$. It was this case that was useful in [4].

7 An example

In this section, we shall give an example of a group $G$ and a retract of a vertex set of a $G$-tree that is not the vertex set of any $G$-tree.

We shall use two technical lemmas. Recall that, for $x, y \in G$, $x^y$ denotes $y^{-1}xy$.

7.1 Lemma. Let $G = \langle x, y \mid \rangle$, let $n \in \mathbb{N}$, and let $g \in G$.

(i) If $x^n y^n x^{-n} \in (x^2, y^2)^g$, then $n \neq 0$ and $g \in \langle x^2, y^2 \rangle$.

(ii) If $x^n y^n x^{-n} \in (x^4, xyx, y^4)^g$, then $n \neq 1$ and $g \in \langle x^4, xyx, y^4 \rangle$.
Proof. Let $T = X(G, \{x, y\})$, the Cayley graph of $G$ with respect to \{x, y\}, as in [3 Definitions I.2.1]. Each (oriented) edge of $T$ is labelled $x$ or $y$.

Let $H \leq G$, and let $w = x^2y^{-2}x^2 \in G$. Let $X := H \setminus T$, let $Y := \langle w \rangle \setminus T$, and let $Z := G \setminus T$.

The pullback of the two natural maps $X \to Z$, $Y \to Z$ provides detailed information about all nontrivial subgroups of $G$ of the form $\langle w \rangle \cap H^g$; see [2 p. 380]. However, this pullback can be rather cumbersome and we do not require detailed information. For our purposes, special considerations will suffice, as follows.

Define $g^{-1}X := (H^g) \setminus T$.

There is a graph isomorphism $X \simeq g^{-1}X$, $Hx \leftrightarrow H^g g^{-1}x$.

The fundamental group of $X$ with basepoint $H1$, $\pi(X, H1)$, is naturally isomorphic to $H$, with the elements of $H$ being read off closed paths based at $H1$.

Similarly, $H^g$ is naturally isomorphic to $\pi(g^{-1}X, H^g1)$, and this in turn is naturally isomorphic to $\pi(X, Hg)$ via the graph isomorphism $g^{-1}X \simeq X$.

Suppose that $w$ lies in $H^g$. Then $w$ can be read off a closed path in $X$ based at $Hg$. Since $w$ is a cyclically reduced word, the closed path is cyclically reduced. The smallest subgraph of $X$ which contains all the cyclically reduced closed paths in $X$ is called the core of $X$, denoted $\text{core}(X)$. It follows that the vertex $Hg$ lies in $\text{core}(X)$, and that we can start at $Hg$, read $w$ and stay inside $\text{core}(X)$.

(i) Suppose that $H = \langle x^2, y^2 \rangle$.

Here $\text{core}(X)$ has vertex set $\{H1, Hx, Hy\}$ and labelled-edge set

$$\{(H1, x, Hx), (Hx, x, Hx^2), (H1, y, Hy), (Hy, y, Hx^2)\}$$

with $Hx^2 = Hy^2 = H1$.

We note that $Hxy$ and $Hyx$ are outside $\text{core}(X)$.

Since $(Hy)x = Hyx$ does not lie in $\text{core}(X)$, we see that $Hg \neq Hy$. Hence, $Hg \in \{H1, Hx\}$.

Notice that $(H1)(xy) = Hxy$ and $(Hx)(yx) = Hyx$. These lie outside $\text{core}(X)$. Thus $n \neq 0$. Hence, $x^2 \in H$.

Notice that $(Hx)(x^2y) = Hxy$ lies outside $\text{core}(X)$. Thus $Hg \neq Hx$. Hence, $Hg = H1$, that is, $g \in H$.

This proves (i).

(ii). Suppose that $H = \langle x^3, xyx, y^4 \rangle$.

Here $\text{core}(X)$ has vertex set

$$\{H1\} \cup \{Hx^i, Hy^i \mid 1 \leq i \leq 3\}$$

and labelled-edge set

$$\{(Hx^i, x, Hx^{i+1}), (Hy^i, y, Hy^{i+1}) \mid 0 \leq i \leq 3\} \cup \{(Hx, y, Hxy)\},$$

with $Hx^4 = Hy^4 = H1$ and $Hxy = Hx^3$.

We note that $Hxy^2 = Hx^2y^2, Hyx, Hy^2x$ and $Hx^3y$, all lie outside $\text{core}(X)$.

For any $j$ with $1 \leq j \leq 3$, $(Hy^j)(x) = Hy^jx$ lies outside $\text{core}(X)$. It follows that $Hg \neq Hy^j$. Hence $Hg = Hx^i$ for some $i$ with $0 \leq i \leq 3$.

Notice that $(Hx)(xy) = Hx^2y, (Hx^2)(xy) = Hx^3y$, and $(Hx^3)(xy) = Hxy$. These all lie outside $\text{core}(X)$. Thus, if $n = 0$, then $Hg = H1$.

Notice that $(H1)(x^2y) = Hx^2y, (Hx)(x^2y) = Hx^3y, (Hx^2)(x^2y^2x) = Hy^2x$, and $(Hx^3)(x^2y^2) = Hxy^2$. These all lie outside $\text{core}(X)$. Thus $n \neq 1$.

Now suppose that $n \geq 2$. Thus $x^{2n} = (x^4)^{2n-2} \in H$.

Notice that $(Hx)(x^2n^2y^2) = Hxy^2, (Hx^2)(x^2n^2y) = Hx^2y, and (Hx^3)(x^2n^2y) = Hx^3y$. These all lie outside $\text{core}(X)$. Thus $Hg = H1$.

This proves (ii).
It is straightforward to prove the following.

7.2 Lemma. Let \( G = (x, y, t \mid x^4t = x^8, y^4t = y^8, x^2y^2x^2 = x^4y^4x^4) \) and let \( n \in \mathbb{N} \).

(i) If \( n \neq 1 \), then \((xy)x)^n = x^{2n}y^{2n}x^{2n}\) in \( G \).

(ii) \((xy)x)^{n+2} = (x^4)^{2n}(y^4)^{2n}(x^4)^{2n}\) in \( G \). \( \square \)

Throughout the remainder of the section we work with the following example.

7.3 Hypotheses. Let \( G = (x, y, t \mid x^4t = x^8, y^4t = y^8, x^2y^2x^2 = x^4y^4x^4) \).

Let \( T = (T, V, E, \iota, \tau) \) be the \( G \)-graph given by the following data, where \( \lor \) denotes the disjoint union:

\[
\begin{align*}
V &= Gu \lor Gw, \quad G_u = \langle x, y \rangle, \quad G_w = \langle x^4, y^4 \rangle, \\
E &= Ge \lor Gf, \quad G_e = \langle x^4, xyx, y^4 \rangle, \quad G_f = \langle x^4, y^4 \rangle, \\
\iota(e) &= u, \quad \tau(e) = t^2w, \quad \iota(f) = w, \quad \tau(f) = tw.
\end{align*}
\]

Using Lemma 7.2 we see that the following hold:

\[
\begin{align*}
G_e &\leq G_u, \quad G_{1-2e} = G_{e}^{t^2} = \langle x^{16}, x^4y^4x^4, y^{16} \rangle \leq G_w, \\
G_f &\leq G_w, \quad G_{t^{-1}f} = G_{f}^{t} = \langle x^8, y^8 \rangle \leq G_w.
\end{align*}
\]

Thus \( T \) is a well-defined \( G \)-graph.

Let \( U = Gu \).

Let \( H = \langle x, y \rangle \leq G \).

For any subset \( S \) of \( T \), we let \( S^{xyz} \) denote \( \{s \in S \mid (xyx)s = s \} \). \( \square \)

Since \( G_w \leq G_u \), it is clear that \( U \) is a \( G \)-retract of \( V \). We shall see that \( T \) is a \( G \)-tree, and that no \( G \)-tree has vertex set \( U \).

7.4 Lemma. If Hypotheses 7.3 hold, then the \( G \)-graph \( T \) is a tree, and \( H \) is freely generated by \( \langle x, y \rangle \).

Proof. Let us momentarily forget Hypotheses 7.3.

Let \( Y = (Y, \overline{V}, \overline{E}, \overline{\iota}, \overline{\tau}) \) be the graph given as follows.

\[
\overline{V} = \{\pi, \overline{\pi}\}, \quad \overline{E} = \{\pi, \overline{f}\}, \quad \overline{\iota}(\overline{f}) = \overline{\pi}, \quad \overline{\tau}(\overline{f}) = \overline{\pi}(\overline{f}) = \overline{\tau}(\overline{f}) = \overline{\pi}.
\]

Let \( Y_0 := (Y_0, \overline{V}, \{\pi\}, \overline{\iota}, \overline{\tau}) \) be the unique maximal subtree of \( Y \).

Using the notation of Definitions I.3.1 of [3], let \( (G(-), Y) \) be the graph of groups given by the following data.

\[
\begin{align*}
G(\pi) &= \langle x, y \rangle, \quad G(\overline{\pi}) = \langle x', y' \rangle, \quad G(\pi) = \langle x^4, xyx, y^4 \rangle, \quad G(\overline{f}) = \langle x', y' \rangle, \\
(x^4)^{\overline{\iota}} &= x'^4, \quad (xyx)^{\overline{\iota}} = x'y'x', \quad (y^4)^{\overline{\iota}} = y'^4, \quad (x')^{\overline{\tau}} = x'^2, \quad (y')^{\overline{\tau}} = y'^2.
\end{align*}
\]

Recall that, in the notation of Definitions I.3.1 of [3], \((-)^{\overline{\tau}} \) denotes the edge-group monomorphism associated to \( \overline{\tau} \).

Let \( G := \pi(G(-), Y, Y_0) \), as in Definitions I.3.4 of [3]. Writing \( t \) for the element of \( G \) that realizes the monomorphism \( \overline{\tau}: G(\overline{f}) \to G(\overline{\pi}) \), we have

\[
G = \langle x, y, x', y', t \mid x^4t = x'^4, xyx = x'y'x', y^4 = y'^4, x'^t = x'^2, y'^t = y'^2 \rangle.
\]

Then \( \langle x, y \rangle = G(\pi) \leq G \) by Corollary I.7.5 of [3].
Now \( x'^2 = x'^2t = x'^4 = x^4 \). Thus \( x' = x^{4t^{-2}} \). Similarly, \( y' = y^{4t^{-2}} \). Hence we can write

\[
G = \langle x, y, t \mid x^4 = x'^{16t^{-2}}, \ xyx = x'^{4t^{-2}}y'^{4t^{-2}}x^{-1}, \ y^4 = y'^{16t^{-2}} \rangle,
\]

\[
= \langle x, y, t \mid x^4t = x'^{8}, \ xyx^2x^{-2} = x'^4y'^{-4}x^4, \ y^4 = y'^{8} \rangle.
\]

Let \( T = (T, V, E, \iota, \tau) \) be \( T(G(-), Y, y_0) \), as in Definitions I.3.4 of \( [3] \). Thus

\[
V = G\pi \lor G\nu, \quad G\pi = \langle x, y \rangle, \quad G\nu = \langle x', y' \rangle = \langle x^4, y^4 \rangle^{t^{-2}},
\]

\[
E = G\nu \lor G\overline{f}, \quad G\nu = \langle x^4, xyx, y^4 \rangle, \quad G\overline{f} = \langle x', y' \rangle = \langle x^4, y^4 \rangle^{t^{-2}},
\]

\[
\iota(\nu) = \pi, \quad \tau(\nu) = \nu, \quad \iota(\overline{f}) = \nu, \quad \tau(\overline{f}) = W.
\]

By Bass-Serre Theory, \( T \) is a \( G \)-tree; see \( [3] \) Theorem I.7.6].

Let \( u := \nu, \ w := t^{-2}w, \ e := \pi, \ f := t^{-2}f \).

Then \( w = u, \ w = t^2w, \ i = w, \ t^f = tw \).

Thus the above \( G \) and \( T \) agree with the \( G \) and \( T \) of Hypotheses \( \text{T3} \) and the result is proved.

7.5 Lemma. Let \( n \in \mathbb{N} \). If Hypotheses \( \text{T3} \) hold, then the following also hold.

(i) \( (tn^2G_w e)_{xyx} = \{tn^e\} \) if \( n \neq 1 \).

(ii) \( (tn^2G_w t^{-2}e)_{xyx} = \begin{cases} \{tn^e\} & \text{if } n \neq 1, \\ \emptyset & \text{if } n = 1. \end{cases} \)

(iii) \( (tn^2G_w t^{-1}f)_{xyx} = \begin{cases} \{tn^{+1}f\} & \text{if } n \neq 0, \\ \emptyset & \text{if } n = 0. \end{cases} \)

(iv) \( (tn^2G_w f)_{xyx} = \{tn^2f\} \).

Proof. (i). Let \( g \in G_w = \langle x, y \rangle \).

Suppose that \( n \neq 1 \) and that \( (xyx)tn^g = tn^ge \). Then \( (xyx)^{tn}g \in G_e \). By Lemma \( \text{T2} \text{[3]} \)

\[
(x^2y^2x^2)g \in G_e = \langle x^4, xyx, y^4 \rangle.
\]

By Lemma \( \text{T4} \text{[3]} \), \( g \in \langle x^4, xyx, y^4 \rangle = G_e \). Hence \( t^ng = t^n e \). It is now easy to see that (i) holds.

(ii). Let \( g \in G_w = \langle x^4, y^4 \rangle \).

Suppose that \( (xyx)tn^+g t^{-2}e = t^{n+2}gt^{-2}e \). Then \( (xyx)^{tn+2}gt^{-2} \in G_e \). By Lemma \( \text{T2} \text{[3]} \),

\[
\langle (x^4)^{tn} (y^4)^{2n} (x^4)^{2n} \rangle \in G_e^2 = \langle x^4, xyx, y^4 \rangle^2 = \langle x^4, x^4, y^4, x^4 \rangle.
\]

By Lemma \( \text{T1} \text{[3]} \), \( n \neq 1 \) and \( g \in \langle x^4, x^4, x^4 \rangle = G_e^2 \). Hence \( t^{n+2}gt^{-2}e = t^n e \).

It is now clear that (ii) holds.

(iii). Let \( g \in G_w = \langle x^4, y^4 \rangle \).

Suppose that \( (xyx)tn^+g t^{-1}f = t^{n+2}gt^{-1}f \). Then \( (xyx)^{tn^+2}gt^{-1} \in G_f \). By Lemma \( \text{T2} \text{[3]} \),

\[
\langle (x^4)^{2n} (y^4)^{2n} (x^4)^{2n} \rangle \in G_f^t = \langle x^4, y^4 \rangle = \langle x^8, y^8 \rangle.
\]
By Lemma 7.3, \( n \neq 0 \) and \( g \in \langle x^8, y^8 \rangle = G_f \). Hence \( t^ngt^{-1}f = t^{n-1}f \). It is now clear that (iii) holds.

(iv) By Lemma 7.2, \( (xyzt)^{n+2} \in \langle x^4, y^4 \rangle = G_f = G_w \). \( \Box \)

7.6 Lemma. If Hypotheses 7.3 hold, then

\[ V^{xyz} = \{ t^n u \mid n \in \mathbb{N} - \{1\} \} \cup \{ t^{n+2} w \mid n \in \mathbb{N} \}. \]

Proof. Let \( n \in \mathbb{N} \).

From [8 Definitions I.3.4], we obtain the following.

\[ \iota^{-1}(t^n u) = t^n G_u e, \quad \tau^{-1}(t^n u) = \varnothing, \]
\[ \iota^{-1}(t^{n+2} w) = t^{n+2} G_w f, \quad \tau^{-1}(t^{n+2} w) = t^{n+2} G_w t^{-2} e \cup t^{n+2} G_w t^{-1} f. \]

By Lemma 7.5(ii), (iii) and (iv), the edges of \( T^{xyz} \) incident to \( t^2 w \) are \( e \) and \( t^2 f \), the edges of \( T^{xyz} \) incident to \( t^3 w \) are \( t^2 f \) and \( t^3 f \), and, for \( n \geq 2 \), the edges of \( T^{xyz} \) incident to \( t^{n+2} w \) are \( t^n e \), \( t^{n+1} f \) and \( t^{n+2} f \).

Hence, in \( T^{xyz} \), the neighbours of \( t^2 w \) are \( u \) and \( t^3 w \), the neighbours of \( t^3 w \) are \( t^2 w \) and \( t^4 w \), and, for \( n \geq 2 \), the neighbours of \( t^{n+2} w \) are \( t^n u \), \( t^{n+1} w \) and \( t^{n+3} w \).

By Lemma 7.6, and the fact that \( t^{n+1}w \) is \( t^n e \), and hence the unique neighbour of \( t^n u \) in \( T^{xyz} \) is \( t^{n+2} w \).

The result now follows. \( \Box \)

We now have the desired example.

7.7 Theorem. There exists a group \( G \) and a \( G \)-set \( U \) such that \( U \) is a \( G \)-retract of the vertex set of some \( G \)-tree but \( U \) is not the vertex set of any \( G \)-tree.

Proof. We assume that Hypotheses 7.3 hold.

By Lemma 7.3, \( U \) is a \( G \)-retract of the vertex set of some \( G \)-tree.

Suppose that there exists a \( G \)-tree \( T' \) with \( V T' = U = Gu \). We will derive a contradiction.

Temporarily returning to the tree \( T \), we let \( L \) denote the subtree of \( T \) with vertex set \( \{ t \} w \) and edge set \( \{ t \} f \). Then \( L \) is homeomorphic to \( \mathbb{R} \) and \( t \) acts on \( L \) by translation. In particular, \( \{ t \} \) acts freely on \( V T \). Hence, \( \{ t \} \) acts freely on \( V T' \subseteq V T \).

As in [8 Proposition I.4.11], there exists a subtree \( L' \) of \( T' \) homeomorphic to \( \mathbb{R} \) on which \( t \) acts by translation.

Let \( v' \) denote the vertex of \( L' \) closest to \( u \) in \( T' \). It is well known, and easy to prove, that the \( T' \)-geodesics from \( u \) to \( t^2 u \), denoted \( T'[u, t^2 u] \), is the concatenation of the four \( T' \)-geodesics \( T'[u, v'], T'[v', t^2 v'], T'[t^2 v', t^2 u], \) and \( T'[t^2 v', t^2 u] \).

By Lemma 7.3 and the fact that \( \{ t \} \) acts freely on \( V T' \),

\[(7.7.1) \quad V T^{xyz} = (Gu)^{xyz} = \{ t^n u \mid n \in \mathbb{N} - \{1\} \} \cup \{ t^n u \mid n \in \mathbb{N} \} - \{ tu \}. \]

By (7.7.1), or by direct calculation, \( xyzt \) fixes \( u \), moves \( tu \), and fixes \( t^2 u \). Thus, \( xyzt \) fixes \( T'[u, t^2 u] \), and, hence, \( xyzt \) fixes \( v' \), fixes \( t^2 v' \), and fixes \( t^2 v' \).

In particular, \( tu \neq v' \), hence \( u \neq v' \), that is, \( u \notin L' \).

Since \( xyzt \) fixes \( v' \), we see, by (7.7.1), that \( v' = t^n u \) for some \( n \in \mathbb{N} - \{1\} \). Hence \( u = t^{-n} v' \in t^{-n} L' = L' \). This is a contradiction. \( \Box \)

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