SCHATTEN PROPERTIES OF TOEPLITZ OPERATORS
ON THE PALEY-WIENER SPACE

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Abstract. We collect several old and new descriptions of Schatten class
Toeplitz operators on the Paley-Wiener space and answer a question on dis-
crete Hilbert transform commutators posed by Richard Rochberg.

1. Introduction

Given a bounded function $\varphi$ on the real line, $\mathbb{R}$, consider the Toeplitz operator $T_\varphi$
on the classical Paley-Wiener space $\text{PW}_a$,$$
T_\varphi: f \mapsto P_a(\varphi f), \quad f \in \text{PW}_a.
$$
The space $\text{PW}_a$ could be regarded as the subspace in $L^2(\mathbb{R})$ of functions with
Fourier spectrum in the interval $[-a,a]$, symbol $P_a$ above denotes the orthogonal
projection in $L^2(\mathbb{R})$ to $\text{PW}_a$. Basic theory of Toeplitz operators on $\text{PW}_a$ can be
found in paper [9] by R. Rochberg.

We are interested in description of Schatten class Toeplitz operators on $\text{PW}_a$ in
terms of their standard symbols. By the standard symbol of an operator in (1) we
mean the entire function $\varphi_{st} = F^{-1}\chi_{2a}F\varphi$, where $F$ denotes the Fourier transform
on the Schwartz space of tempered distributions, and $\chi_{2a}$ is the indicator function
of the interval $(-2a,2a)$. As we will see, a Toeplitz operator $T_\varphi$ on $\text{PW}_a$ belongs
to the Schatten class $S^p$, $0 < p < \infty$, if and only if $e^{2iax}\varphi_{st}$ belongs to a discrete
oscillation Besov space introduced in 1987 by R. Rochberg [9]. Its definition we
now recall.

For a measure $\mu$ on $\mathbb{R}$ and a function $f \in L^1_{\text{loc}}(\mu)$, the oscillation of order $n$ of $f$
on an interval $I \subset \mathbb{R}$ with respect to $\mu$ is defined by
$$
\text{osc}(f,I,\mu,n) = \inf_{P_n} \frac{1}{\mu(I)} \int_I |f(x) - P_n(x)| d\mu(x),
$$
where the infimum is taken over all polynomials $P_n$ of degree at most $n$. If $\mu(I) = 0$,
we put $\text{osc}_I(f,I,\mu,n) = 0$. Define the family $I_a$ of closed intervals
$$
I_{a,j,k} = \left[ \frac{2k \pi}{a}, \frac{2(k+1) \pi}{a} \right], \quad j, k \in \mathbb{Z}, \quad j \geq 0.
$$
Note that endpoints of intervals in $I_a$ belong to the lattice $\mathbb{Z}_a = \left\{ \frac{2\pi}{a}k, \quad k \in \mathbb{Z} \right\}$. Let $p$ be a positive real number, and let $[\frac{1}{p}]$ be the integer part of $\frac{1}{p}$. The discrete

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Section 4.3 in author’s paper [1].

Operators has been recently studied by P. Lopatto and R. Rochberg [3], see also Sin wavelet characterizations and interpolation theory of discrete Besov spaces. Theorem 2 for the case $S$ belongs to the trace class $B$ oscillation Besov space $I$ supported on subintervals for $\mu$ where $\sum_{x \in Z_a} \delta_x$ is the normalized counting measure on $Z_a$.

Our main result is the following theorem.

**Theorem 1.** Let $a$, $p$ be positive real numbers, let $\varphi$ be a bounded function on $\mathbb{R}$, and let $\varphi_{st}$ be the standard symbol of the Toeplitz operator $T_\varphi$ on $PW_a$. Then we have $T_\varphi \in S^p(PW_a)$ if and only if $e^{2i\alpha x} \varphi_{st} \in B_p(4a, \text{osc})$. Moreover, $\|T_\varphi\|_{S^p}$ is comparable to $\|e^{2i\alpha x} \varphi_{st}\|_{B_p(4a, \text{osc})}$ with constants depending only on $p$.

Theorem 1 complements a classical description of Toeplitz operators in class $S^p(PW_a)$ given by R. Rochberg [9] for $1 \leq p < \infty$ and extended by V. Peller [5] to the whole range $0 < p < \infty$. To formulate the result, consider a system $\{\nu_j\}_{j \in \mathbb{Z}}$ of infinitely smooth functions on $\mathbb{R}$ such that

$$\text{supp } \nu_j \subset [2^{j-1}, 2^j], \quad 0 \leq \nu_j \leq 1, \quad \nu_{j-1}(x) = \nu_j(x/2), \quad \sum \nu_j = 1 \text{ on } (0, 1].$$

Define $\nu_j(x) = \nu_{-j}(1-x)$ for real $x \geq \frac{1}{2}$ and integer $j \geq 1$, put $\nu_0 = 1 - \sum_{j \neq 0} \nu_j$ for $j = 0$. Finally, let $\nu_{a,j}(x) = \nu_j((x+a)/2a)$ for all $x \in [-a,a]$ and $j \in \mathbb{Z}$. Observe that system $\{\nu_{a,j}\}$ provides a resolution of unity on the interval $[-a,a]$ by functions supported on subintervals $I_j$ whose lengths are comparable to the distance from $I_j$ to the endpoints of $[-a,a]$. Rochberg-Peller theorem says that $T_\varphi$ is in $S^p(PW_a)$ for $0 < p < \infty$ if and only if

$$a \sum_{j \in \mathbb{Z}} 2^{-|j|} \| F^{-1}(\nu_{a,j} \cdot \varphi_{st})\|_{L^p(\mathbb{R})}^p < \infty,$$

with control of the norms. R. Rochberg gives yet another characterization of Toeplitz operators in class $S^p(PW_a)$, $1 \leq p < \infty$, in terms of a reproducing kernel decomposition of their standard symbols, see Theorem 5.3 in [9]. Both the statement and the proof of his result for $p = 1$ contain errors that we correct in Section 3.

As a consequence of Theorem 1 we obtain the following result.

**Theorem 2.** Let $a > 0$. The discrete Hilbert transform commutator

$$C_\psi : f \mapsto \frac{1}{\pi} \int_{Z_a} \frac{\psi(x) - \psi(t)}{x-t} f(t) \, d\mu_a(t), \quad f \in L^2(\mu_a),$$

belongs to the trace class $S^1(L^2(\mu_a))$ if and only if $\psi \in B_1(a, \text{osc}) \cap L^\infty(Z_a)$.

This answers the question posed by R. Rochberg in 1987. See Section 5 for a summary of results on discrete Hilbert transform commutators and an analogue of Theorem 2 for the case $0 < p < 1$.

We would like to mention papers [11], [12] by R. Torres for readers interested in wavelet characterizations and interpolation theory of discrete Besov spaces. The problem of membership in Schatten classes $S^p$ for general truncated Toeplitz operators has been recently studied by P. Lopatto and R. Rochberg [13], see also Section 4.3 in author’s paper [1].
2. Proof of Theorem 1 for $1 < p < \infty$

Theorem 1 for $1 < p < \infty$ follows from known results. Let $B_p(R) = \mathbb{B}^{1/p}_p(R)$ be the standard homogeneous Besov space on the real line $R$, see, e.g., Chapter 3 in [1] for definition and basic properties. Given a Toeplitz operator $T_\varphi$ on $PW_a$ with symbol $\varphi \in L^\infty(R)$, we denote

$$\varphi^-_{st} = F^{-1} \chi_{(-2a, 0)} F \varphi, \quad \varphi^+_{st} = F^{-1} \chi_{(0, 2a)} F \varphi,$$

where $\chi_S$ is the indicator function of a set $S$. As usual, $F$ stands for the Fourier transform on the Schwartz space of tempered distributions. The following result is a combination of Theorem 5.1 and its Corollary in [2].

**Theorem** (R. Rochberg). Let $1 < p < \infty$ and let $a > 0$. Then a Toeplitz operator $T_\varphi$ on $PW_a$ belongs to $S_p(PW_a)$ if and only if $\| e^{2iax} \varphi^-_{st} \|_{B_p(R)} + \| e^{-2iax} \varphi^+_{st} \|_{B_p(R)}$ is finite, in which case $\| T_\varphi \|_{S_p}$ is comparable to $\| e^{2iax} \varphi^-_{st} \|_{B_p(R)} + \| e^{-2iax} \varphi^+_{st} \|_{B_p(R)}$ with constants depending only on $p$.

Denote by $\mathcal{E}_a$ the set of tempered distributions whose Fourier transforms are supported on the interval $[-a, a]$. Next result is Theorem 1 in [2].

**Theorem** (R. Torres). Let $1 < p < \infty$ and let $f$ be a function in $\mathcal{E}_a \cap B_p(R)$ for some $a > 0$. Then its restriction to $Z_{2a}$ belongs to $B_p(2a, osc)$ and $\| f \|_{B_p(2a, osc)}$ is comparable to $\| f \|_{B_p(R)}$ with constants depending only on $p$. Moreover, every sequence in $B_p(a, osc)$ is the restriction to $Z_a$ of a unique function (modulo polynomials) in $\mathcal{E}_a \cap B_p(R)$.

**Proof of Theorem 1** ($1 < p < \infty$). Let $\varphi$ be a bounded function of $R$ and let $\varphi_{st} = F^{-1} \chi_{(-2a, 2a)} F \varphi$ be the standard symbol of the Toeplitz operator $T_\varphi \in S^p(PW_a)$. Then functions $e^{2iax} \varphi^-_{st}, e^{-2iax} \varphi^+_{st}$ belong to $\mathcal{E}_{2a} \cap B_p(R)$ by R. Rochberg’s theorem above. From theorem by R. Torres we see that $e^{2iax} \varphi^-_{st} \in B_p(4a, osc)$ and $e^{-2iax} \varphi^+_{st} \in B_p(4a, osc)$ with control of the norms. Now observe that $e^{2iax} = 1$ and $e^{2iax} \varphi^-_{st} + e^{-2iax} \varphi^+_{st}$ on $Z_{4a}$, hence $e^{2iax} \varphi_{st} \in B_p(4a, osc)$.

Conversely, assume that the restriction of $e^{2iax} \varphi_{st}$ to $Z_{4a}$ is in $B_p(4a, osc)$. Using theorem by R. Torres, find a function $f \in \mathcal{E}_{2a} \cap B_p(R)$ such that its restriction to $Z_{4a}$ agrees with $e^{2iax} \varphi_{st}$. Put $f^+ = F^{-1} \chi_{(-2a, 0)} F f$ and $f^- = F^{-1} \chi_{(0, 2a)} F f$. Observe that $\hat{\varphi} = e^{-2iax} f^+ + e^{2iax} f^-$ is an entire function of exponential type at most $2a$ coinciding with $\varphi_{st}$ on $Z_{4a}$. Since $\varphi_{st}$, $\hat{\varphi}$ are the first order distributions supported on the finite interval $[-2a, 2a]$, we have $|\hat{\varphi}(x)| + |\varphi(x)| \leq c + |x|$ for all $x \in R$ and a constant $c > 0$. It follows that the entire function $\frac{e^{2iax} \varphi^-}{x}$ of exponential type at most $2a$ is bounded on $R$ and vanishes on $Z_{4a} \setminus \{0\}$, hence $\hat{\varphi} - \varphi_{st} = p \sin(2ax)$ for a polynomial $p$ of degree at most 1. Therefore, we have $T_\varphi = T_{\hat{\varphi}} - T_{\varphi_{st}}$ on $PW_a$, see Section 2.D in [2]. Since $f^\pm \in B_p(R)$, we can use R. Rochberg’s theorem and conclude that $T_\varphi \in S^p(PW_a)$ with control of the norms: $\| T_\varphi \|_{S^p}$ is controllable by $\| e^{2iax} \varphi^- \|_{B_p(R)} + \| e^{-2iax} \varphi^+ \|_{B_p(R)} \leq c_p \| f \|_{B_p(R)} \leq c_p \| e^{2iax} \varphi_{st} \|_{B_p(4a, osc)}$. \hfill \Box

3. Reproducing kernel decomposition of standard symbols

In this section we show that the standard symbol of a Toeplitz operator on $PW_a$ from class $S_p$ can be represented as a linear combination of normalized reproducing kernels of $PW_{2a}$ with coefficients $c_k$ such that $\sum |c_k|^p < \infty$. We consider only the case $0 < p \leq 1$. Proposition 8.1 below is a corrected version of
Let $\psi$ be a bounded function on the real line $\mathbb{R}$. Consider the standard Hardy space $H^2$ in the upper half-plane $\mathbb{C}^+ = \{ \lambda \in \mathbb{C} : \Im \lambda > 0 \}$ of the complex plane $\mathbb{C}$. Denote by $H^2_{\mathbb{C}}$ the anti-analytic subspace $\{ f \in L^2(\mathbb{R}) : \hat{f} \in H^2 \}$ of $L^2(\mathbb{R})$. Recall that the classical Hankel operator $H_\psi : H^2 \to H^2$ is defined by

$$H_\psi : f \mapsto P_-(\psi f), \quad f \in H^2,$$

where $P_-$ denotes the orthogonal projection from $L^2(\mathbb{R})$ to $H^2_{\mathbb{C}}$. The operator $H_\psi$ is completely determined by its standard anti-analytic symbol $\psi_{st} = F^{-1} \chi_{(-\infty,0)} F \psi$. The latter means that $H_\psi f = H_{\psi_{st}} f$ for all $f \in H^2$ such that $\sup_{x \in \mathbb{R}} |xf(x)| < \infty$. Take a positive number $\varepsilon > 0$ and define the sets $U^+_\varepsilon, U^-_\varepsilon$ by

$$U^+_\varepsilon = \{ \lambda \in \mathbb{C} : \lambda = (1 + \varepsilon)^m (\varepsilon x \pm i) ; \ x, m \in \mathbb{Z} \}.$$

For $\lambda \in \mathbb{C}^+$ let us denote by $k_\lambda$ the reproducing kernel of $H^2$ at $\lambda$, $k_\lambda = -\frac{1}{2\pi} \frac{1}{\pi z - \lambda}$.

**Theorem** (R. Rochberg [9]). There exists a number $\varepsilon > 0$ such that $H_\psi \in S^p(H^2)$ if and only if $\psi_{st} = \sum_{\lambda \in U^+_\varepsilon} c_\lambda \frac{\chi}{|\lambda|^{1+p}}$, where $\sum |c_\lambda|^p$ is finite and the infimum of $\sum |c_\lambda|^p$ over all possible representations of $\psi_{st}$ in this form is comparable to $\|H_\psi\|_{S^p}$ with constants depending only on $p \in (0, \infty)$.

Remark that for $p \in (0,1]$ the series defining $\psi_{st}$ in the theorem above converges absolutely to a bounded function on $\mathbb{R}$, while for $p > 1$ the convergence holds only in the Besov space $B_p(\mathbb{R})$ (one need to extract constant terms from every summand to get the convergent series, see discussion in [8]). In order to prove an analogous result for Toeplitz operators on the Paley-Wiener space, let us consider the sets

$$U^\pm_{a,\varepsilon} = \{ \lambda \in U^\pm_\varepsilon : |\Im \lambda| > \varepsilon \eta \}, \quad \Lambda_{a,\varepsilon} = U^-_{a,\varepsilon} \cup \mathbb{Z}_{a,\varepsilon} \cup U^+_{a,\varepsilon}.$$

Here $\mathbb{Z}_{a,\varepsilon} = \{ \frac{\eta}{\varepsilon} k, \ k \in \mathbb{Z} \}$. Next, for $a > 0$ and $\lambda \in \mathbb{C}$, denote by $\rho_{a,\lambda}$ the reproducing kernel of the space $PW_a$ at the point $\lambda$. Recall that

$$\rho_{a,\lambda} : z \mapsto \frac{1}{\pi} \frac{1}{\sin(\pi a)(z - \lambda)}, \quad z \in \mathbb{C}.$$

We are going to prove the following proposition.

**Proposition 3.1.** Let $a > 0$ and let $\varphi \in L^\infty(\mathbb{R})$. There exist $\varepsilon > 0, \eta > 1$ such that $T_{\varphi} \in S^p(PW_a)$ if and only if $\varphi_{st} = \sum_{\lambda \in \Lambda_{a,\varepsilon}} c_\lambda \frac{\rho_{a,\lambda}}{|\lambda|^{1+p}}$, where $\sum |c_\lambda|^p$ is finite and the infimum of $\sum |c_\lambda|^p$ over all possible representations of $\varphi_{st}$ in this form is comparable to $\|T_{\varphi}\|_{S^p}$ with constants depending only on $p \in (0,1]$.

We will show how to reduce Proposition 3.1 to the above theorem for Hankel operators using a splitting of the standard symbol into three pieces: analytic, anti-analytic and a piece with “small” Fourier support.

The following two results for $0 < p \leq 1$ are consequences of Lemma 1 and Lemma 2 from [5]. The range $1 \leq p < \infty$ has been treated earlier in [9], see also Section 2 in [10].
Lemma 3.1. Let $a > 0$ and let $\varphi \in L^\infty(\mathbb{R})$. There exist bounded functions $\varphi_\ell$, $\varphi_c$, and $\varphi_r$ such that $T_\varphi = T_{\varphi_\ell} + T_{\varphi_c} + T_{\varphi_r}$ on $\mathcal{P}_a$, and we have $\|T_{\varphi}||_{S^p} \leq c_p\|T_{\varphi_\ell}||_{S^p}$ for every $s = \ell, c, r$ for $T_{\varphi} \in S^p(\mathcal{P}_a)$. Here $c_p$ is a constant depending only on $p$.

Lemma 3.2. Let $a > 0$ and let $\varphi \in L^\infty(\mathbb{R})$ be such that $\mathrm{supp} \hat{\varphi} \subset [-a, a]$. Then we have $T_{\varphi} \in S^p(\mathcal{P}_a)$ if and only if $\varphi \in L^p(\mathbb{R})$, in which case $\|\varphi||_{L^p(\mathbb{R})}$ is comparable to $\|T_{\varphi}||_{S^p}$ with constants depending only on $p$.

Proof of Proposition 3.1. Let $\varphi \in L^\infty(\mathbb{R})$ and let $\varphi_{st} = \mathcal{F}^{-1}\chi_{(-2a, 2a)}\mathcal{F}\varphi$ be the standard symbol of the operator $T_{\varphi}$ on $\mathcal{P}_a$. Then $T_{\varphi} = T_{\varphi_{st}}$, see Section 2.D in [6]. Suppose that $\varphi_{st} = \sum_{\lambda \in \Lambda_{\alpha, \varepsilon}} c_\lambda \overline{\rho_{2a, \lambda}}$ for some $\varepsilon > 0$, $\eta > 0$, and some coefficients $c_\lambda$ such that $\sum_{\lambda \in \Lambda_{\alpha, \varepsilon}} |c_\lambda|^p < \infty$. It follows from the estimate

$$\frac{|\rho_{2a, \lambda}(z)|}{\|\rho_{a, \lambda}\|^2} \leq c_\lambda |1 + \varepsilon|, \quad z \in \mathbb{C}, \quad \lambda \in \mathbb{C},$$

that this series converges absolutely to an entire function of exponential type at most $2a$ bounded on the real line $\mathbb{R}$. By triangle inequality (see, e.g., Theorem A1.1 in [6]), we have

$$\|T_{\varphi_{st}}||_{S^p} = \|T_{\varphi_{st}}||_{S^p} \leq \left( \sum_{\lambda \in \Lambda_{\alpha, \varepsilon}} |c_\lambda|^p \right) \sup_{\lambda \in \mathbb{C}} \|T_{\varphi_{st}}||_{S^p},$$

where we denoted $\varphi_{st} = \sum_{\lambda \in \Lambda_{\alpha, \varepsilon}} c_\lambda \overline{\rho_{2a, \lambda}}$. Take $\lambda \in \mathbb{C}$. For every $f, g \in \mathcal{P}_a$ we have

$$(T_{\rho_{2a, \lambda}} f, g) = (f \overline{\rho_{2a, \lambda}} g, \rho_{2a, \lambda}) = f(\lambda) \cdot g(\overline{\lambda}) = \langle f, \rho_{a, \lambda} \rangle \langle \rho_{a, \lambda}, g \rangle.$$

It follows that the operator $T_{\varphi_{st}}$ has rank one and $\|T_{\varphi_{st}}||_{S^p} = 1$. Hence $T_{\varphi}$ belongs to $S^p(\mathcal{P}_a)$ and $\|T_{\varphi}||_{S^p} \leq \sum_{\lambda \in \mathbb{C}} \|c_\lambda\|^p$.

Now let $\varphi$ be a bounded function on $\mathbb{R}$ such that $T_{\varphi} \in S^p(\mathcal{P}_a)$. We want to show that the standard symbol $\varphi_{st} = \mathcal{F}^{-1}\chi_{(-2a, 2a)}\mathcal{F}\varphi$ of $T_{\varphi}$ can be represented in the form

$$\varphi_{st} = \sum_{\lambda \in \Lambda_{\alpha, \varepsilon}} c_\lambda \frac{\rho_{2a, \lambda}}{\|\rho_{a, \lambda}\|^2}$$

for some positive numbers $\varepsilon$, $\eta$ depending only on $p$ and a sequence $\{c_\lambda\}$ such that $\sum_{\lambda \in \Lambda_{\alpha, \varepsilon}} |c_\lambda|^p$ is comparable to $\|T_{\varphi}||_{S^p}$. By Lemma 3.1 it suffices to consider separately the following three cases: (1) $\mathrm{supp} \hat{\varphi} \subset (-\infty, 0)$; (2) $\mathrm{supp} \hat{\varphi} \subset [-a, a]$; (3) $\mathrm{supp} \hat{\varphi} \subset [0, +\infty)$. Let us treat the third case first. Denote by $M_{-i\alpha x}$ the operator of multiplication by $e^{-i\alpha x}$ on $L^2(\mathbb{R})$. Since $\mathrm{supp} \hat{\varphi} \subset [0, +\infty)$, we have

$$H_{e^{-2iax}} = M_{-2iax} T_{\varphi} P a M_{-2iax},$$

where $H_{e^{-2iax}} : H^2 \rightarrow H^2$ is the Hankel operator with symbol $\psi = e^{-2iax}\varphi$. In particular, we have $\|H_{\psi}||_{S^p} \leq \|T_{\varphi}||_{S^p}$. By Rochberg’s Theorem above, the anti-analytic function $\psi_{st} = \mathcal{F}^{-1}\chi_{(-\infty, 0)}\mathcal{F}e^{-2iax}\varphi$ admits the following representation:

$$\psi_{st} = \sum_{\lambda \in \Lambda_{\alpha, \varepsilon}} c_\lambda \frac{k_\lambda}{\|k_\lambda\|^2},$$

where $k_\lambda \in \mathbb{C}$ and $\|k_\lambda\| \neq 0$. By Lemma 3.2 and the above representation, we have

$$\|\psi_{st}||_{S^p} \leq \|T_{\varphi}||_{S^p}.$$
where $\sum_{\lambda \in \mathcal{L}_a^+} |c_\lambda|^p$ is comparable to $\|H_\psi\|^p_{\mathcal{S}^p}$, and $\varepsilon > 0$ does not depend on $\psi$. This gives us decomposition for $\varphi_{st}$:

$$
\varphi_{st} = e^{2iax} \psi_{st} = \sum_{\lambda \in \mathcal{L}_a^+} c_\lambda \frac{e^{2iax_k\lambda}}{||k_{\lambda}||^2} = \sum_{\lambda \in \mathcal{L}_a^+} c_\lambda \frac{p_{2\lambda}(e^{2iax_k\lambda})}{||k_{\lambda}||^2},
$$

where $p_{2\lambda}$ denotes the orthogonal projection in $L^2(\mathbb{R})$ to $\mathcal{P}W_{2\lambda}$. It is easy to see that $p_{2\lambda}(e^{2iax_k\lambda}) = e^{2i\alpha \lambda} \rho_{2\lambda, \lambda}$ and $||\rho_{a, \lambda}||^2 \leq 2e^{2a} \|\lambda\|^2_{L^2(\mathbb{R})}$, hence

$$
\varphi_{st} = \sum_{\lambda \in \mathcal{L}_a^+} c_\lambda \beta_{\lambda} \frac{\rho_{2\lambda, \lambda}}{||\rho_{a, \lambda}||^2},
$$

for some complex numbers $\beta_{\lambda}$ such that $\sup_{\lambda} |\beta_{\lambda}| \leq 2$. Next, in the case where $\supp \varphi \subset (-\infty, 0]$ we can consider the adjoint operator $T^*_\varphi = T^*_{\varphi_{st}}$ with the standard symbol $\varphi^*_{st} : z \mapsto \varphi_{st}(\bar{z})$ and conclude that in this situation

$$
\varphi_{st} = \sum_{\lambda \in \mathcal{L}_a^+} c_\lambda \beta_{\lambda} \frac{\rho_{2\lambda, \lambda}}{||\rho_{a, \lambda}||^2}.
$$

Now let $\supp \varphi \subset [-a, a]$. By Lemma 3.2 we have $\varphi \in L^p(\mathbb{R})$. In particular, $\varphi \in \mathcal{P}W_{2a}$ and Plancherel-Polya theorem [7] yields the following decomposition:

$$
\varphi = \varphi_{st} = \frac{\pi}{2a} \sum_{\lambda \in \mathbb{Z}_{2a}} f(\lambda) \rho_{2\lambda, \lambda}, \quad \sum_{\lambda \in \mathbb{Z}_{2a}} |f(\lambda)|^p \leq c_p a^p \|\varphi\|_{L^p(\mathbb{R})},
$$

where the constant $c_p$ depends only on $p$. Put $\Lambda_\varepsilon = \mathcal{L}_a^+ \cup \mathbb{Z}_{2a} \cup \mathcal{L}_a^-$. To summarize, we have proved that for every bounded function $\varphi$ on $\mathbb{R}$ such that $T_\varphi \in \mathcal{S}^p(\mathcal{P}W_a)$ there are coefficients $c_\lambda, \lambda \in \Lambda_\varepsilon$, such that

$$
\varphi_{st} = \sum_{\lambda \in \Lambda_\varepsilon} c_\lambda \frac{\rho_{2\lambda, \lambda}}{||\rho_{a, \lambda}||^2}, \quad \sum_{\lambda \in \Lambda_\varepsilon} |c_\lambda|^p \leq c_p \|T_\varphi\|^p_{\mathcal{S}^p}. \quad (2)
$$

It remains to show that the set $\Lambda_\varepsilon$ and coefficients $c_\lambda$ in this decomposition could be replaced by the set $\Lambda_{a, \varepsilon}$ and some new coefficients $c_\lambda$ satisfying the second estimate in $[2]$. To this end, for every point $\lambda \in \Lambda_\varepsilon$ denote by $\zeta_\lambda$ the nearest point to $\lambda$ in $\Lambda_{a, \varepsilon} \subset \Lambda_\varepsilon$, where $\eta = 2^k$ and $k \in \mathbb{Z}$ is a positive integer number that will be specified later. Consider the function

$$
\hat{\varphi}^{(1)} = \sum_{\lambda \in \Lambda_\varepsilon} c_\lambda \frac{\rho_{2\lambda, \zeta_\lambda}}{||\rho_{a, \zeta_\lambda}||^2}, \quad \sum_{\lambda \in \Lambda_\varepsilon} |c_\lambda|^p \leq c_p \|T_\varphi\|^p_{\mathcal{S}^p}, \quad (3)
$$

Note that $\varphi^{(1)}$ has the required representation and $\sum |c_\lambda^{(1)}|^p \leq \sum |c_\lambda|^p$. Moreover, we have $\|T_\varphi - T_{\varphi^{(1)}}\|^p_{\mathcal{S}^p} \leq \sum_{\lambda \in \Lambda_{a, \varepsilon}} |c_\lambda|^p \cdot ||T_{\varphi, \lambda} - T_{\varphi^{(1)}, \lambda}||_{\mathcal{S}^p}^p$. On the other hand, the quasi-norm in $\mathcal{S}^p$ of the rank two operator

$$
T_{\varphi, \lambda} - T_{\varphi^{(1)}, \lambda} = \frac{\rho_{a, \lambda}}{||\rho_{a, \lambda}||} \otimes \frac{\rho_{a, \lambda}}{||\rho_{a, \lambda}||} - \frac{\rho_{a, \zeta_\lambda}}{||\rho_{a, \zeta_\lambda}||} \otimes \frac{\rho_{a, \zeta_\lambda}}{||\rho_{a, \zeta_\lambda}||}
$$

does not exceed

$$
2^{\frac{1}{p}} \left\| \frac{\rho_{a, \lambda}}{||\rho_{a, \lambda}||} - \frac{\rho_{a, \lambda}}{||\rho_{a, \lambda}||} \right\|_{L^2(\mathbb{R})} \leq 2^{\frac{1}{2} + \frac{1}{p}} \left( 1 - \frac{\text{Re} \rho_{a, \zeta_\lambda}(\lambda)}{||\rho_{a, \zeta_\lambda}|| \cdot ||\rho_{a, \lambda}||} \right)^{\frac{1}{2}}.
$$

Since $|\zeta_\lambda - \lambda| \leq \frac{\eta}{2^k}$ for all $\lambda$ by construction, one can choose a large number $\eta = 2^k$ so that $\|T_\varphi - T_{\varphi^{(1)}}\|^p_{\mathcal{S}^p} \leq \frac{1}{2} \|T_\varphi\|^p_{\mathcal{S}^p}$. Clearly, this choice of $\eta$ does not depend on $\varphi$. 

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Taking the spectrum in the interval $I = 0 < \lambda < \infty$.

We have

$$\lambda \| PW \| \| \phi \| \| \mu \| \| \rho \| \| K \| \| M \| \| N \| \| O \| \| P \| \| Q \| \| R \| \| S \| \| T \| \| U \| \| V \| \| W \| \| X \| \| Y \| \| Z \| $$

such that $\| T_x - T_{\varphi(x)} - \ldots - T_{\varphi(n)} \|_{S_r}^p \leq \frac{1}{2^r} \| T_e \|_{S_r}^p$ and $\sum_{n \geq 1} | c_{\lambda(n)} |^p \leq c_p \| T_e \|_{S_r}^p$. Since $S_r(PW_a)$ is a complete quasi-normed space and a Toeplitz operator on $PW_a$ is zero if and only if its standard symbol is zero (see Section 2.D in [9]), this gives us the required decomposition of $\varphi$ with coefficients $c_{\lambda} = \sum_{n \geq 1} c_{\lambda(n)}^2$, $\lambda \in \Lambda_{\eta \alpha \varepsilon}$.

4. Interpolation of discrete Besov sequences

Denote by $PW_{[0,a]}$ the Paley-Wiener space of functions in $L^2(\mathbb{R})$ with Fourier spectrum in the interval $[0,a]$. Recall that the reproducing kernel $k_{a,\lambda}$ of the space $PW_{[0,a]}$ at a point $\lambda \in \mathbb{C}_+$ has the form

$$k_{a,\lambda}(z) = \frac{1}{2\pi i} \frac{1 - e^{\alpha(a - \lambda)}}{z - \lambda}, \quad z \in \mathbb{C}.$$ 

Denote by $C_0(\mathbb{Z}_a)$ the set of functions on $\mathbb{Z}_a$ tending to zero at infinity. Our aim in this section is to prove the following proposition.

**Proposition 4.1.** Let $0 < p \leq 1$, let $\Lambda$ be the set $\Lambda_{\eta \alpha \varepsilon}$ from Proposition [1.3] and let $F = \sum_{\lambda \in \Lambda} c_{\lambda} \frac{k_{a,\lambda}^{||}}{\| k_{a,\lambda} \|}$ for some $c_{\lambda} \in \mathbb{C}$ such that $\sum_{\lambda \in \Lambda} | c_{\lambda} |^p < \infty$. Then the restriction of $F$ to $\mathbb{Z}_a$ belongs to $B_p(a, osc) \cap C_0(\mathbb{Z}_a)$. Conversely, for every function $f \in B_p(a, osc)$ there exists the unique function $F$ as above and a polynomial $q$ of degree at most $\left\lfloor \frac{1}{p} \right\rfloor$ such that $f = q + F$ on $\mathbb{Z}_a$. Moreover, the infimum of $\sum_{\lambda \in \Lambda} | c_{\lambda} |^p$ over all possible representations of $F = \sum_{\lambda \in \Lambda} c_{\lambda} \frac{k_{a,\lambda}^{||}}{\| k_{a,\lambda} \|}$ in this form is comparable to $\| f \|_{B_p(a, osc)}^p$ with constants depending only on $p$.

The proof of Proposition 4.1 is based on the following lemma.

**Lemma 4.1.** We have $\| k_{a,\lambda} \|_{B_p(a, osc)} \leq c_p \| k_{a,\lambda} \|^2$ for every $a > 0$, $0 < p \leq 1$, and $\lambda \in \mathbb{C}$, where the constant $c_p$ depends only on $p$.

**Proof.** At first, consider the points $\lambda$ in the support of $\mu_a$. For $\lambda \in \mathbb{Z}_a$ we have

$$k_{a,\lambda}(x) = \begin{cases} \| k_{a,\lambda} \|^2, & x = \lambda; \\ 0, & x \in \text{supp } \mu_a \setminus \{ \lambda \}. \end{cases}$$

Taking $P_I = 0$ for all intervals $I \in I_a$ in the definition of $\text{osc}(k_{a,\lambda}, I, \mu_a, [\frac{1}{p}])$, we obtain the estimate

$$\| k_{a,\lambda} \|_{B_p(a, osc)}^p \leq \sum_{I \in I_a} \left( \frac{1}{\mu_a(I)} \int_I | k_{a,\lambda}(x) | \, d\mu_a(x) \right)^p = \| k_{a,\lambda} \|^{2p} \mu_a(\{ \lambda \}) \sum_{I \in I_a} \frac{\chi(I)(\lambda)}{\mu_a(I)^p} \leq c_p \| k_{a,\lambda} \|^{2p}.$$
Now let λ be an arbitrary point in \( \mathbb{C} \setminus \text{supp } \mu_a \). Then \( k_{a, \lambda}(x) = \frac{1}{2\pi i} \frac{1-e^{-i\lambda x}}{x-\lambda} \) for all \( x \in \text{supp } \mu_a \). Thus, we need to estimate an oscillation of the function \( x \mapsto \frac{1}{x-\lambda} \) on the lattice \( \mathbb{Z}_a \). Divide collection \( \mathcal{I}_a \) from Section 1 into two parts:

\[ \mathcal{I}_{a,1} = \{ I \in \mathcal{I}_a : I = I_{a,j,k}, \Re \lambda \notin I_{a,j,k-1} \cup I_{a,j,k} \cup I_{a,j,k+1} \}, \quad \mathcal{I}_{a,2} = \mathcal{I}_a \setminus \mathcal{I}_{a,1}. \]

For an interval \( I \in \mathcal{I}_{a,1} \) with center \( x_c \), define the polynomial \( P_I \) of degree \( \left\lceil \frac{1}{p} \right\rceil \) by

\[
\frac{1}{x-\lambda} - P_I(x) = \frac{(x-x_c)^{\left\lceil \frac{1}{p} \right\rceil + 1}}{(x-\lambda)(\lambda-x_c)^{\left\lceil \frac{1}{p} \right\rceil + 1}}.
\]

Using this polynomial, we can estimate

\[
\text{osc} \left( \frac{1}{x-\lambda}, I, \mu_a, \left\lceil \frac{1}{p} \right\rceil \right) \leq \sup_{x \in I} \left| \frac{(x-x_c)^{\left\lceil \frac{1}{p} \right\rceil + 1}}{(x-\lambda)(\lambda-x_c)^{\left\lceil \frac{1}{p} \right\rceil + 1}} \right| \leq \frac{|I|^{\left\lceil \frac{1}{p} \right\rceil + 1}}{\text{dist}(\lambda, I)^{\left\lceil \frac{1}{p} \right\rceil + 2}},
\]

where \( |I| \) denotes the length of \( I \). Since \( I \in \mathcal{I}_{a,1} \), we have \( \text{dist}(\lambda, I) \geq |I| \), hence

\[
\sum_{I \in \mathcal{I}_{a,1}} \text{osc} \left( \frac{1}{x-\lambda}, I, \mu_a, \left\lceil \frac{1}{p} \right\rceil \right)^p \leq \sum_{I \in \mathcal{I}_{a,1}} \frac{1}{|I|^p} \leq c_p \cdot a^p.
\]

We also will need a more accurate estimate for the left hand side of the inequality above in the case where \( |\text{Im } \lambda| \) is large. For every \( j \geq 0 \), let \( \mathcal{I}_{a,1}^j \) be the set of intervals \( I_{a,j,k}, k \in \mathbb{Z} \), belonging to the family \( \mathcal{I}_{a,1} \). We have

\[
\sum_{I \in \mathcal{I}_{a,1}^j} \left( \frac{|I|^{\left\lceil \frac{1}{p} \right\rceil + 1}}{\text{dist}(\lambda, I)^{\left\lceil \frac{1}{p} \right\rceil + 2}} \right)^p \leq \sum_{I \in \mathcal{I}_{a,1}^j} \left( \frac{|I|^{\left\lceil \frac{1}{p} \right\rceil + 1}}{(|\text{Im } \lambda|^2 + \text{dist}(\Re \lambda, I)^2)^{\left\lceil \frac{1}{p} \right\rceil + 2}/2} \right)^p \leq c_p \left( \frac{a}{2^j} \right)^p \sum_{m \geq 1} \left( \frac{a}{2^j} \right)^2 \frac{1}{|\text{Im } \lambda|^2 + m^2} \frac{1}{|\text{Im } \lambda|^p + p} \lesssim c_p \left( \frac{a}{2^j} \right)^p \frac{1}{|\text{Im } \lambda|^p - 2p},
\]

where \( \gamma_j = \max \left( 1, \frac{1}{2} |\text{Im } \lambda| \right) \). Indeed, the last inequality follows from elementary estimates

\[
\sum_{m=1}^{\infty} m^{-1-2p} < \infty, \quad \int_1^{\infty} \frac{dx}{(r^2 + x^2)^s} \lesssim c_s r^{1-2s},
\]

where \( r > 0 \), and the constant \( c_s \) depends on \( s > 1/2 \). Put \( N_\lambda = \lceil \log_2(a |\text{Im } \lambda|) \rceil \) if \( a |\text{Im } \lambda| \geq 2 \) and \( N_\lambda = 0 \) otherwise. Note that \( \tilde{p} = -1 + \left\lceil \frac{1}{p} \right\rceil p + p \) is a positive number. It follows

\[
\sum_{I \in \mathcal{I}_{a,1}} \text{osc} \left( \frac{1}{x-\lambda}, I, \mu_a, \left\lceil \frac{1}{p} \right\rceil \right)^p \lesssim c_p \sum_{j=0}^{N_\lambda} \left( \frac{a}{2^j} \right)^p \gamma_j \lesssim c_p a^{-\tilde{p}} |\text{Im } \lambda|^{-\tilde{p}} \sum_{j=0}^{N_\lambda} 2^{\tilde{p}j} + c_p \sum_{j=N_\lambda}^{\infty} \frac{a^p}{2^{pj}} \lesssim c_p |\text{Im } \lambda|^p.
\]
Combining the last estimate with (3), we get
\[
\sum_{I \in \mathcal{I}_{a,2}} \operatorname{osc} \left( \frac{1}{\lambda - x}, I, \mu_a, \left[ \frac{1}{p} \right] \right)^p \leq c_p \min \left( a^p, \frac{1}{|\operatorname{Im} \lambda|^p} \right).
\]

Now consider the family \( \mathcal{I}_{a,2} = \mathcal{I}_{a,21} \cup \mathcal{I}_{a,22} \),
\[ \mathcal{I}_{a,21} = \{ I \in \mathcal{I}_{a,2} : |I| \leq |\operatorname{Im} \lambda| \}, \quad \mathcal{I}_{a,22} = \{ I \in \mathcal{I}_{a,2} : |I| > |\operatorname{Im} \lambda| \}. \]

For an interval \( I \in \mathcal{I}_{a,21} \) we use the polynomial \( P_I \) defined by (3). Then formula (4) implies
\[
\sum_{I \in \mathcal{I}_{a,21}} \operatorname{osc} \left( \frac{1}{\lambda - x}, I, \mu_a, \left[ \frac{1}{p} \right] \right)^p \leq \sum_{I \in \mathcal{I}_{a,21}} \left( \frac{|I|^{1/2} + 1}{|\operatorname{Im} \lambda|^{1/2} + 2} \right)^p \leq \frac{c_p}{|\operatorname{Im} \lambda|^p}.
\]

Note that if \( |\operatorname{Im} \lambda| < \frac{2\pi}{a} \), the set \( \mathcal{I}_{a,21} \) is empty. This shows that we can write
\[
\sum_{I \in \mathcal{I}_{a,21}} \operatorname{osc} \left( \frac{1}{\lambda - x}, I, \mu_a, \left[ \frac{1}{p} \right] \right)^p \leq c_p \min \left( a^p, \frac{1}{|\operatorname{Im} \lambda|^p} \right).
\]

For \( I \in \mathcal{I}_{a,22} \) we put \( P_I = 0 \). Denote by \( x_0 \) the nearest point to \( \lambda \) in \( \operatorname{supp} \mu_a \), and set \( I' = I \setminus \{ x \in \mathbb{R} : |x - \operatorname{Re} \lambda| < \pi/a \} \). We have
\[
\frac{1}{\mu_a(I)} \int_{I} \frac{1}{|x - \lambda|} \, d\mu_a(x) \leq \frac{\mu_a(\{ x_0 \})}{\mu_a(I)|x_0 - \lambda|} + \frac{1}{\mu_a(I)} \int_{I'} \frac{dx}{|x - \lambda|} \leq \frac{c}{a|I||x_0 - \lambda|} + \frac{c}{|I|} \int_{\pi a - 1}^{1} \frac{dx}{\sqrt{x^2 + |\operatorname{Im} \lambda|^2}} \leq \frac{c}{a|I||x_0 - \lambda|} + \frac{c}{|I|} \min \left( \log \frac{|I|}{\pi}, \log \left( \frac{|I|}{|\operatorname{Im} \lambda|} + 1 \right) \right).
\]

Using estimates
\[
\sum_{I \in \mathcal{I}_{a,2}} \frac{1}{|I|^p} \leq c_p a^p, \quad \sum_{I \in \mathcal{I}_{a,2}} \left( \frac{\log |I|}{|I|} \right)^p \leq c_p a^p, \quad \sum_{I \in \mathcal{I}_{a,2}} \left( \frac{1}{|I|} \log \frac{|I|}{|\operatorname{Im} \lambda|} \right)^p \leq \frac{c_p}{|\operatorname{Im} \lambda|^p},
\]
we see that
\[
\sum_{I \in \mathcal{I}_{a,2}} \operatorname{osc} \left( \frac{c_p}{\lambda - x}, I, \mu_a, \left[ \frac{1}{p} \right] \right)^p \leq \frac{c_p}{|x_0 - \lambda|^p} + c_p \min \left( a^p, \frac{1}{|\operatorname{Im} \lambda|^p} \right).
\]

Eventually, we obtain
\[
\left\| \frac{1}{x - \lambda} \right\|_{\mathcal{B}^p(a, \operatorname{osc})}^p \leq \frac{c_p}{|x_0 - \lambda|^p} + c_p \min \left( a^p, \frac{1}{|\operatorname{Im} \lambda|^p} \right).
\]

It follows that
\[
\|k_{a,\lambda}\|_{\mathcal{B}^p(a, \operatorname{osc})}^p \leq c_p (1 + e^{-a|\operatorname{Im} \lambda|}) \min \left( a^p, \frac{1}{|\operatorname{Im} \lambda|^p} \right) + c_p \left| \frac{1 - e^{-ia\lambda}}{x_0 - \lambda} \right| \leq c_p \|k_{a,\lambda}\|_{\mathcal{B}^p}^p,
\]

which is the desired estimate.

Let \( \mathcal{C}_0(\mathbb{R}) \) denote the set of all continuous functions on \( \mathbb{R} \) tending to zero at infinity. For completeness, we include the proof of the following known lemma.
Lemma 4.2. Let $0 < p \leq 1$, $a > 0$. For every function $f \in \mathcal{B}_p(\text{osc, } a)$ there exists a function $F \in \mathcal{B}_p(\mathbb{R})$ such that $F = f$ on $\mathbb{Z}_a$, and

$$\|F\|_{\mathcal{B}_p(\mathbb{R})} \leq c_p \|f\|_{\mathcal{B}_p(\text{osc, } a)},$$

where the constant $c_p$ depends only on $p$.

Proof. For $k \in \mathbb{Z}$ put $I_k = \left[\frac{2\pi a}{p} (\lfloor k \rfloor + 1), \frac{2\pi a}{p} (\lfloor k \rfloor + 2)\right]$. Interiors of intervals $I_k$ are disjoint and every set $I_k \cap \mathbb{Z}_a$ contains $\left[\frac{1}{p}\right] + 1$ points. On every $I_k$ define the polynomial $P_k$ of degree at most $\left[\frac{1}{p}\right]$ such that $P_k(x) = f(x)$ for all $x \in I_k \cap \mathbb{Z}_a$. Next, set $F(x) = P_k(x)$ for $x \in I_k$. We claim that the function $F$ is in $\mathcal{B}_p(\mathbb{R})$.

To check this, let us take an interval $J$ such that $F(J) = f(J)$ for all $x \in I_k \cap \mathbb{Z}_a$. It follows that $J = J_{j,k} = I_k \cup \ldots \cup I_{k+N}$ for some $\ell \in \mathbb{Z}$ and $N \geq 1$. Consider the polynomial $P_f$ of degree at most $\left[\frac{1}{p}\right]$ such that

$$\text{osc} (f, J, \mu_a, \left[\frac{1}{p}\right]) = \frac{1}{\mu_a(J)} \int_J |f(x) - P_f(x)| \, d\mu_a(x).$$

We have

$$\frac{1}{|J|} \int_J |F(x) - P_f(x)| \, dx = \frac{1}{|J|} \sum_{s=0}^N \int_{I_{s+k}} |P_{s+k}(x) - P_f(x)| \, dx \leq \frac{c_p}{|J|} \sum_{s=0}^N \int_{I_{s+k}} |P_{s+k}(x) - P_f(x)| \, d\mu_a(x) \leq c_p \text{osc} (f, J, \mu_a, \left[\frac{1}{p}\right]),$$

where we used the fact that

$$\int_{I_{s+k}} |P(x)| \, dx \leq c_p \int_{I_{s+k}} |P(x)| \, d\mu_a(x)$$

for every interval $I_{s+k}$ and every polynomial $P$ of degree at most $\left[\frac{1}{p}\right]$. It follows that

$$\|F\|_{\mathcal{B}_p(\mathbb{R}, \text{osc})} \leq c_p \sum_{j,k} \text{osc} (f, J, \mu_a, \left[\frac{1}{p}\right]) \leq c_p \|f\|_{\mathcal{B}_p(\text{osc, } a)},$$

and hence $F$ belongs to the space $\mathcal{B}_{1/p}(\mathbb{R}, dx, \text{osc}) = \mathcal{B}_p(\mathbb{R})$, as required. \hfill $\Box$

Proof of Proposition 4.1. Consider a function $F$ of the form

$$F = \sum_{\lambda \in \Lambda} c_\lambda \frac{k_{a,\lambda}}{\|k_{a,\lambda}\|^2}, \quad \sum_{\lambda \in \Lambda} |c_\lambda|^p < \infty.$$

Since $0 < p \leq 1$ and $|k_{a,\lambda}(x)| \leq c \|k_{a,\lambda}\|^2$ for every $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$, the series above converges absolutely to a function from $\mathcal{C}_0(\mathbb{R})$ by the Lebesgue dominated convergence theorem. By Lemma 4.2, the restriction of $F$ to $\mathbb{Z}_a$ (to be denoted by $f$) is in $\mathcal{B}_p(a, \text{osc})$ and $\|f\|_{\mathcal{B}_p(a, \text{osc})} \leq c_p \sum_{\lambda \in \Lambda} |c_\lambda|^p$ for a constant $c_p$ depending only on $p$. 
Conversely, take \( f \in \mathbb{B}_p(a, \text{osc}) \) and find a function \( \tilde{F} \in \mathbb{B}_p(\mathbb{R}) \) such that \( \tilde{F} = f \) on \( \mathbb{Z}_a \), see Lemma 4.2. Applying Theorem 2.10 from [3] to analytic and anti-analytic parts of \( \tilde{F} \), we obtain the representation

\[
\tilde{F} = q - \frac{1}{2\pi i} \sum_{\lambda \in \mathcal{U}_e} \tilde{c}_\lambda \frac{\text{Im} \lambda}{x - \lambda}, \quad x \in \mathbb{R},
\]

where the coefficients \( \tilde{c}_k \in \mathbb{C} \) are such that \( |\tilde{c}_\lambda|^p \leq \rho_p \tilde{F}^p_\mathbb{B}_p(\mathbb{R}) \), and \( q \) is a polynomial of degree at most \( \left\lfloor \frac{p}{2} \right\rfloor \). Now consider the function

\[
F = \sum_{\lambda \in \mathcal{U}_e} c_\lambda \frac{k_{\lambda, a}}{\|k_{\lambda, a}\|^2}, \quad c_\lambda = \frac{\text{Im} \lambda \cdot \|k_{\lambda, a}\|^2}{1 - e^{-ia\lambda}}.
\]

Observe that \( |c_\lambda| \leq |\tilde{c}_\lambda| \) for all \( \lambda \in \mathcal{U}_e \) and \( f = q + F \) on \( \mathbb{Z}_a \). We need to replace the set \( \mathcal{U}_e \) above to the set \( \Lambda_{\eta_0, \varepsilon} \) from Proposition 3.1. Since \( k_{\lambda, a} = e^{-ia\lambda} e^{-\frac{ia\lambda}{2}} \rho_{\lambda, a} \), we have \( \|k_{\lambda, a}\|^2 = e^{-\frac{ia\lambda}{2}} \|\rho_{\lambda, a}\|^2 \) and

\[
e^{-\frac{ia\lambda}{2}} F = \sum_{\lambda \in \mathcal{U}_e} c_\lambda e^{-\frac{ia\lambda}{2}} \frac{\rho_{\lambda/2, a, \varepsilon}}{\|k_{\lambda, a}\|^2} = \sum_{\lambda \in \mathcal{U}_e} c_\lambda e^{-\frac{ia\lambda}{2}} \frac{\rho_{\lambda/2, a, \varepsilon}}{\|k_{\lambda, a}\|^2}.
\]

From the beginning of the proof of Proposition 3.1 we see that the Toeplitz operator on \( \mathbb{P}^{a/4} \) with symbol \( e^{-\frac{ia\lambda}{2}} F \) belongs to the class \( \mathcal{S}^p(\mathbb{P}^{a/4}) \). It follows that

\[
e^{-\frac{ia\lambda}{2}} F = \sum_{\lambda \in \Lambda_{\eta_0, \varepsilon}} d_\lambda \frac{\rho_{\lambda/2, a, \varepsilon}}{\|k_{\lambda/2, a, \varepsilon}\|^2}, \quad \sum_{\lambda \in \Lambda_{\eta_0, \varepsilon}} \|d_\lambda\|^p \leq \rho_p \sum_{\lambda \in \mathcal{U}_e} |c_\lambda|^p.
\]

This yields the required representation for \( F \),

\[
F = \sum_{\lambda \in \Lambda_{\eta_0, \varepsilon}} c_\lambda \frac{k_{\lambda, a}}{\|k_{\lambda, a}\|^2}, \quad \sum_{\lambda \in \Lambda_{\eta_0, \varepsilon}} |c_\lambda|^p \leq \rho_p \|f\|_\mathbb{B}_p(a, \text{osc}),
\]

with some new coefficients \( c_\lambda \). Since \( \sum_{\lambda} |c_\lambda| < \infty \), the function \( G = e^{-\frac{ia\lambda}{2}} F \) is an entire function of exponential type at most \( a/2 \) such that \( \lim_{x \to \pm \infty} G(x) = 0 \). In particular, it is uniquely determined by values on \( \mathbb{Z}_a \). This proves uniqueness in Proposition 4.1.

\[\square\]

5. Proof of Theorem 1 for \( 0 < p \leq 1 \)

**Proof of Theorem 1** \((0 < p \leq 1)\). Let \( \varphi \in L^\infty(\mathbb{R}) \) be a function on \( \mathbb{R} \) such that the operator \( T_\varphi \) is in \( \mathcal{S}^p(\mathbb{P} a) \), and let \( \varphi_{st} = F^{-1} \chi(\cdot - 2a, 2a) F \varphi \) be the standard symbol of \( T_\varphi \). By Proposition 3.1 and Proposition 4.1 we have \( e^{2iax} \varphi_{st} \in \mathbb{B}_p(4a, \text{osc}) \) and moreover, \( \|e^{2iax} \varphi_{st}\|_{\mathbb{B}_p(4a, \text{osc})} \leq \rho_p \|T_\varphi\|_{\mathcal{S}^p} \) for a constant \( \rho_p \) depending only on \( p \).

Conversely, assume that the restriction of the function \( e^{2iax} \varphi_{st} \) to \( \mathbb{Z}_{4a} \) belongs to the space \( \mathbb{B}_p(4a, \text{osc}) \). By Proposition 4.1 there exists a function \( F \) and a polynomial \( q \) of degree at most \( \left\lfloor \frac{1}{p} \right\rfloor \) such that \( q + F = e^{2iax} \varphi_{st} \) on \( \mathbb{Z}_{4a} \) and

\[
F = \sum_{\lambda \in \Lambda_{\eta_0, \varepsilon}} c_\lambda \frac{k_{\lambda, a}}{\|k_{\lambda, a}\|^2} e^{2iax} \sum_{\lambda \in \Lambda_{\eta_0, \varepsilon}} c_\lambda e^{-2ia \text{Re} \lambda} \frac{\rho_{2a, a, \varepsilon}}{\|k_{\lambda, a}\|^2} \quad (6)
\]

for some \( c_\lambda \in \mathbb{C} \) such that \( \sum_{\lambda} |c_\lambda|^p \leq \rho_p \|e^{2iax} \varphi_{st}\|_{\mathbb{B}_p(4a, \text{osc})} \). We claim that \( T_{\tilde{\varphi}} = T_\varphi \) on \( \mathbb{P} a \), where \( \tilde{\varphi} = e^{-2iax} (q + F) \). Indeed, the entire function \( z \mapsto \tilde{\varphi} - \varphi_{st} \) has exponential type at most \( 2a \), vanishes on \( \mathbb{Z}_{4a} \), and satisfies a polynomial estimate.
on $\mathbb{R}$. Hence $\hat{\varphi} - \varphi_{st} = \hat{q}\sin(2az)$ for all $z \in \mathbb{C}$ and a polynomial $\hat{q}$. Thus, we have $T_{\varphi} = T_{\varphi_{st}} = T_{\hat{\varphi}}$. It remains to use formula (6) and Proposition 3.1. The theorem is proved. 

6. Discrete Hilbert transform commutators. Proof of Theorem [2]

Recall that $\mu_a = \frac{2\pi}{a} \sum_{x \in \mathbb{Z}_a} \delta_x$ is the scalar multiple of the counting measure on the lattice $\mathbb{Z}_a = \{\frac{2\pi}{a} k, k \in \mathbb{Z}\}$. The discrete Hilbert transform $H_{\mu_a}$ on $L^2(\mu_a)$ is defined by

$$H_{\mu_a} : f \mapsto \frac{1}{\pi} \int_{\mathbb{Z}_a} \frac{f(t)}{x-t} \, d\mu_a(t),$$

and its commutator $C_\psi = M_\psi H_{\mu_a} - H_{\mu_a} M_\psi$ with the multiplication operator $M_\psi : f \mapsto \psi f$ on $L^2(\mu_a)$ by

$$C_\psi : f \mapsto \frac{1}{\pi} \int_{\mathbb{Z}_a} \frac{\psi(x) - \psi(t)}{x-t} f(t) \, d\mu_a(t), \quad x \in \text{supp} \mu_a.$$

It is well-known that the operator $H_{\mu_a}$ admits the bounded extension from the dense subset $\mathcal{G}$ of $L^2(\mu_a)$ of finitely supported bounded functions to the whole space $L^2(\mu_a)$. A possible way to define the operator $C_\psi$ on $L^2(\mu_a)$ for any symbol $\psi$ on $\mathbb{Z}_a$ is to consider its bilinear form on elements from the dense subset $\mathcal{G} \times \mathcal{G}$ of $L^2(\mu_a) \times L^2(\mu_a)$. We will also deal with the operators $\tilde{C}_\psi : L^2(\nu_\mathbb{Z}) \to L^2(\nu_\mathbb{Z})$ defined by

$$\tilde{C}_\psi : f \mapsto \frac{1}{\pi} \int_{\mathbb{Z}_a} \frac{\psi(x) - \psi(t)}{x-t} f(t) \, d\nu_\mathbb{Z}(t), \quad x \in \text{supp} \nu_\mathbb{Z},$$

where the measure $\nu_\mathbb{Z} = \frac{2\pi}{a} \sum_{x \in \mathbb{Z}_2} \delta_x + \mathfrak{m} \mathbb{Z}_2$ is supported on the lattice $\frac{2\pi}{a} + \mathbb{Z}_2$. It can be shown that for $1 \leq p < \infty$ the operator $C_\psi : L^2(\mu_a) \to L^2(\mu_a)$ is in $S^p$ if and only if the operator $\tilde{C}_\psi : L^2(\nu_\mathbb{Z}) \to L^2(\nu_\mathbb{Z})$ is in $S^p$. As we will see, for $0 < p \leq 1$ we may have $C_\psi \notin S^p(L^2(\mu_a))$ for a function $\psi$ on $\mathbb{Z}_a$ such that the operator $\tilde{C}_\psi : L^2(\mu_a) \to L^2(\mu_a)$ is in $S^p$.

The discrete Hilbert transform commutators were investigated in details in paper [9]. In particular, it was proved in [9] that $C_\psi$ is bounded on $L^2(\mu_a)$ if and only if its symbol $\psi$ belongs to the discrete BMO($\mathbb{Z}_a$) space of functions $f$ on $\mathbb{Z}_a$ such that $\sup_{I \in \mathcal{I}_a} \text{osc}(f, I, \mu_a, 0) < \infty$, where $\mathcal{I}_a = \{I_{a,j,k}, j, k \in \mathbb{Z}, j \geq 0\}$ is the collection of intervals defined in Section 11. Another result from [9] says that $C_\psi$ is compact on $L^2(\mu_a)$ if and only if $\psi \in \text{CMO}(\mathbb{Z}_a)$, that is, $\lim_{k \to +\infty} \text{osc}(\psi, I_{a,j,k}, \mu_a, 0) = 0$ for every $j \geq 0$ and $\lim_{j \to +\infty} \text{osc}(\psi, J_j, \mu_a, 0) = 0$ for any sequence of intervals $J_j \subset \mathbb{R}$ of length $j$ with common center. Finally, the operator $C_\psi$ belongs to $S^p(L^2(\mu_a))$ for $1 < p < \infty$ if and only if $\psi \in \mathbb{B}_p(a, \text{osc})$, moreover, we have $C_\psi \in S^1(L^2(\mu_a))$ for every $\psi \in \mathbb{B}_1(a, \text{osc})$. See Theorem 6.2 in [9] and Theorem 4 in [12] for the proof of these results. It was an open question stated in Section 7 of [9] whether $C_\psi \in S^p(L^2(\mu_a))$ is equivalent to $\psi \in \mathbb{B}_p(a, \text{osc})$ for all positive $p$ (in particular, for $p = 1$). Theorem [2] gives the affirmative answer to this question for $p = 1$. On the other hand, for $0 < p < 1$ we show that there exists symbols $\psi \in \mathbb{B}_p(a, \text{osc})$ such that $C_\psi \notin S^p(L^2(\mu_a))$. In fact, the following modification of Theorem [2] holds true.
Theorem 2'. Let $0 < p < 1$. The operator $\tilde{C}_\psi : L^2(\mu_{\bar{\psi}}) \to L^2(\nu_{\bar{\psi}})$ belongs to the class $S^p$ if and only if $\psi \in B_p(a, \text{osc}) \cap \mathcal{L}^\infty(\mathbb{Z}_a)$. Moreover, the quasi-norms $\|\tilde{C}_\psi\|_{S^p}$ and $\|\psi\|_{B_p(a, \text{osc})}$ are comparable with constants depending only on $p$.

For the proof we need a result on unitary equivalence of discrete Hilbert transform commutators to some truncated Hankel operators. Given a positive number $a > 0$, we denote by $\text{PW}[{-a,0}]$ the Paley-Wiener space of functions in $L^2(\mathbb{R})$ with Fourier spectrum in the interval $[-a,0]$. Define the truncated Hankel operator $\Gamma_{\psi} : \text{PW}[0,a] \to \text{PW}[{-a,0}]$ with symbol $\psi \in L^\infty(\mathbb{R})$ by

$$\Gamma_{\psi} : f \mapsto P_{[-a,0]}(\psi f), \quad f \in \text{PW}[0,a],$$

where $P_{[-a,0]}$ stands for the projection in $L^2(\mathbb{R})$ to the subspace $\text{PW}[{-a,0}]$. It is easy to see that $\Gamma_{\psi}$ is completely determined by its standard symbol $\psi_{st,2a} = \mathcal{F}^{-1}_{\chi_{(-2a,0)}}(\psi)$, that is, $\Gamma_{\psi} f = \Gamma_{\psi_{st,2a}} f$ for all functions $f \in \text{PW}[0,a]$ such that $\sup_{x \in \mathbb{R}} |x f(x)| < \infty$. Clearly, such functions form a dense subset in $\text{PW}[0,a]$.

It is known that the embedding operator $V_{\mu_a} : \text{PW}[0,a] \to L^2(\mu_a)$ taking a function $f \in \text{PW}[0,a]$ into its restriction to $\mathbb{Z}_a$ is unitary. The same is true for the embedding operator $\tilde{V}_{\nu_a} : \text{PW}[{-a,0}] \to L^2(\nu_a)$. A general version of the following result is Lemma 4.2 of [1].

**Lemma 6.1.** Let $a > 0$, $0 < p \leq 1$, and let $\psi \in L^\infty(\mathbb{Z}_{2a})$. Then there exists an entire function $\Psi$ such that $\Psi = \psi$ on $\mathbb{Z}_{2a}$, $|F(x)| \leq c \log(e + |x|)$ for all $x \in \mathbb{R}$, and the Fourier spectrum of $F$ is contained in the interval $[-2a,0]$. Moreover, we have

$$\tilde{V}_{\nu_a} \Gamma_{\psi} V_{\mu_a}^{-1} = -i \tilde{C}_\psi.$$  

for the operators $\Gamma_{\psi} : \text{PW}[0,a] \to \text{PW}[{-a,0}]$ and $\tilde{C}_\psi : L^2(\mu_a) \to L^2(\nu_a)$.

**Proof.** Existence of such a function $\Psi$ follows from a general theory of entire functions, see, e.g., Theorem 1 in Section 21.1 of [2] and Problem 1 after its proof. In order to prove formula (7), take a pair of functions $f \in L^2(\mu_a)$, $g \in L^2(\nu_a)$ with finite support. Consider the functions $F, G$ in $\text{PW}[0,a]$ such that $F = V_{\mu_a}^{-1} f$, $G = \tilde{V}_{\nu_a}^{-1} g$. It is easy to see that $\int_{\mathbb{R}} |\Psi F G| dx < \infty$ and hence the bilinear form of $\Gamma_{\psi}$ is correctly defined on functions $F, G$. We have

$$(\tilde{V}_{\nu_a} \Gamma_{\psi} V_{\mu_a}^{-1} f, g)_{L^2(\mathbb{R})} = (\Gamma_{\psi} F, G)_{L^2(\mathbb{R})} = (FG, \tilde{\Psi})_{L^2(\mathbb{R})} = (V_{\mu_{2a}} F G, V_{\mu_{2a}} \tilde{\Psi})_{L^2(\mu_{2a})} = \frac{1}{2}(F g, \tilde{\psi})_{L^2(\nu_a)} + \frac{1}{2}(f G, \tilde{\psi})_{L^2(\mu_a)}.$$  

For every point $x \in \mathbb{R}$ we have

$$F(x) = (V_{\mu_a} F, V_{\mu_a} k_x, \mu_a)_{L^2(\mu_a)} = \frac{2}{\pi i} \int_{\mathbb{R}} \frac{f(t)}{t-x} \mu_a(t), \quad x \in \frac{\pi}{a} + \mathbb{Z}_a.$$  

Analogously, $G(t) = \frac{2}{\pi i} \int_{\mathbb{R}} \frac{g(x)}{x-t} \nu_a(x)$ for all $t \in \mathbb{Z}_a$. Using these formulas, we get

$$\tilde{V}_{\nu_a} \Gamma_{\psi} V_{\mu_a}^{-1} f, g)_{L^2(\mathbb{R})} = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{\psi(x) - \psi(t)}{x-t} f(t) g(x) d\mu_a(t) d\nu_a(x)$$  

$$= -i (\tilde{C}_\psi f, g)_{L^2(\nu_a)}.$$  

The lemma follows. \qed


Theorem 1. Let \( q \) be the operator \( \Psi : L^2(\mu_\omega) \rightarrow L^2(\nu_\omega) \) belongs to the class \( S^p \). Consider the sequence of points \( x_k = \frac{2^n}{a} k, k \in \mathbb{Z} \). Since \( 0 < p \leq 1 \), we have
\[
\sum_{k \in \mathbb{Z}} |\psi(x_{2k}) - \psi(x_{2k+1})| = a \sum_{k \in \mathbb{Z}} |(\hat{C}\Psi \delta_{x_{2k}}, \delta_{x_{2k+1}})_{L^2(\nu_\omega)}| < \infty.
\]

Hence, the function \( \psi \) is bounded on \( \mathbb{Z}_a \). Using Lemma 6.1, we can find an entire function \( \Psi \) such that \( \Psi = \psi \) on \( \mathbb{Z}_a \), \( |\Psi(x)| \leq c \log(e + |x|) \) for all \( x \in \mathbb{R} \), the Fourier spectrum of \( \Psi \) is contained in \([-a, 0]\), and relation (7) holds for the operators \( \Gamma_\Psi : \text{PW}_{[0,2]} \rightarrow \text{PW}_{[-2,0]} \) and \( \tilde{C}_\Psi : L^2(\mu_\omega) \rightarrow L^2(\nu_\omega) \). In particular, we have \( \Gamma_\Psi \in S^p \). Denote by \( M \) the multiplication operator on \( L^2(\mathbb{R}) \) by the function \( e^{t\omega x} \).

Let \( T_{e^{t\omega x}} \Psi \) be the Toeplitz operator on \( \text{PW}_{\mathbb{Z}_a} \) with standard symbol \( e^{t\omega x} \Psi \). Observe that
\[
T_{e^{t\omega x}} \Psi f = M \Gamma_\Psi M f , \tag{8}
\]
for every function function \( f \in \text{PW}_{\mathbb{Z}_a} \) such that \( \sup_{x \in \mathbb{R}} |xf(x)| < \infty \). Since \( M \) maps unitarily \( \text{PW}_{\mathbb{Z}_a} \) onto \( \text{PW}_{[0,2]} \) and \( \text{PW}_{[-2,0]} \) onto \( \text{PW}_{\mathbb{Z}_a} \), the operator \( T_{e^{t\omega x}} \Psi \) belongs to \( S^p(\text{PW}_{\mathbb{Z}_a}) \). In particular, there exists a function \( \varphi \in L^\infty(\mathbb{R}) \) such that
\[
T_{\varphi} = T_{e^{t\omega x}} \Psi \text{ and } \varphi_{st} = e^{t\omega x} \Psi + c_1 e^{-i\omega x} + c_2 e^{i\omega x} \text{ for some constants } c_1, c_2.
\]
Since \( e^{t\omega x} \varphi_{st} \) coincides with \( \psi + c_1 + c_2 \) on \( \mathbb{Z}_a \), we have \( \psi \in B_p(a,osc) \) by Theorem 4.

Moreover, the quasi-norm \( \|\tilde{C}_\varphi\|_{S^p} \) is comparable to \( \|\psi\|_{B_p(a,osc)} \) with constants depending only on \( p \in (0,1) \).

Conversely, suppose that \( \psi \in B_p(a,osc) \cap L^\infty(\mathbb{Z}_a) \). Using Lemma 6.1 again, we find an entire function \( \Psi \) such that \( \Psi = \psi \) on \( \mathbb{Z}_a \), \( |\Psi(x)| \leq c \log(e + |x|) \) for all \( x \in \mathbb{R} \), the Fourier spectrum of \( \Psi \) is contained in \([-a, 0]\), and relation (7) holds for the operators \( \Gamma_\Psi : \text{PW}_{[0,2]} \rightarrow \text{PW}_{[-2,0]} \) and \( \tilde{C}_\Psi : L^2(\mu_\omega) \rightarrow L^2(\nu_\omega) \). Since \( \psi \in L^\infty(\mathbb{Z}_a) \), the operators \( C_\psi \) and \( \Gamma_\Psi \) are bounded. Let \( \Psi_{st,a} \) be the standard symbol of the operator \( \Gamma_\Psi \). Note that \( \Psi_{st,a}(x) = \Psi(x) + q(x) \) for all \( x \in \mathbb{Z}_a \) and a polynomial \( q \) of degree at most one. In particular, we have \( \Psi_{st,a} \in B_p(a,osc) \).

By Theorem 4 the operator \( T_{e^{t\omega x}} \Psi_{st,a} \) on \( \text{PW}_{\mathbb{Z}_a} \) is in \( S^p \), hence \( \Gamma_\Psi \in S^p \) by formula (8).

It follows that the operator \( \tilde{C}_\Psi \) is in \( S^p \) as well, and, moreover, we have the estimate
\[
\|\tilde{C}_\Psi\|_{S^p} = \|\Gamma_\Psi\|_{S^p} = \left\| T_{e^{t\omega x}} \Psi_{st,a} \right\|_{S^p} \leq c_p \|\Psi_{st,a}\|_{B_p(a,osc)} = c_p \|\psi\|_{B_p(a,osc)},
\]
for a constant \( c_p \) depending only on \( p \). The theorem is proved.

Proof of Theorem 2. Let \( \psi \) be a function on the lattice \( \mathbb{Z}_a \) such that we have \( C_\psi \in S^1(L^2(\mu_\omega)) \). Then the operator \( \tilde{C}_\psi : L^2(\mu_\omega) \rightarrow L^2(\nu_\omega) \) is of trace class as well and \( \|\psi\|_{B_1(a,osc)} \leq c_1 \|\tilde{C}_\psi\|_{S^1(L^2(\mu_\omega))} \leq c_1 \|C_\psi\|_{S^1(L^2(\mu_\omega))} \) by Theorem 2.

Conversely, suppose that \( \psi \in B_1(a,osc) \cap L^\infty(\mathbb{Z}_a) \). By Lemma 4.2, we can find a function \( \Psi \in B_1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) such that \( \Psi = \psi \) on \( \mathbb{Z}_a \) and \( \|\Psi\|_{B_1(\mathbb{R})} \leq c_1 \|\psi\|_{B_1(\mathbb{R}),osc} \).

Denote \( \psi_\lambda : t \mapsto \frac{|\lambda t|^2}{|\lambda|^2} \) for \( \lambda \in \mathbb{C} \). Let us apply Theorem 2.10 in [8] to analytic and anti-analytic parts of \( \Psi \): find numbers \( c, c_\lambda \) such that \( \sum_{\lambda \in \mathcal{U}_\mu} |c_\lambda| \leq c_1 \|\Psi\|_{B_1(\mathbb{R})} \) and
\[
\psi(x) = \Psi(x) = c + \sum_{\lambda \in \mathcal{U}_\mu} c_\lambda \psi_\lambda(x), \quad x \in \mathbb{Z}_a.
\]
We claim that for every \( \lambda \in \mathcal{U}_\varepsilon \), the commutator \( C_{\psi, \lambda} \) belongs to the trace class and \( \|C_{\psi, \lambda}\|_{S^1} \leq c_1(1 + a) \) for a constant \( c_1 \) do not depending on \( \lambda \). Clearly, this will yield the desired estimate \( \|C_\psi\|_{S^1} \leq c_1(1 + a)\|\psi\|_{\mathcal{B}_1(\text{osc})} \). We have

\[
\frac{\psi_\lambda(x) - \psi_\lambda(t)}{x - t} = -\frac{|\text{Im}\lambda|^2}{(x - \lambda)^2(t - \lambda)} - \frac{|\text{Im}\lambda|^2}{(x - \lambda)(t - \lambda)^2}.
\]

Denote by \( K_{\psi, \lambda} \) the integral operator on \( L^2(\mu_a) \) with kernel \( \frac{\psi_\lambda(x) - \psi_\lambda(t)}{x - t} f(t) \):

\[
(K_{\psi, \lambda} f)(x) = \int_{\mathbb{Z}_\alpha} \frac{\psi_\lambda(x) - \psi_\lambda(t)}{x - t} f(t) \, dt = (C_{\psi, \lambda} f)(x) + \frac{2|\text{Im}\lambda|^2}{(x - \lambda)^2} f(x). \tag{9}
\]

Observe that the operator \( K_{\psi, \lambda} \) has rank 2 and

\[
\|K_{\psi, \lambda}\|_{S^p} \leq 2|\text{Im}\lambda|^2 \left\| \frac{1}{(x - \lambda)^2} \right\|_{L^2(\mu_a)} \left\| \frac{1}{x - \lambda} \right\|_{L^2(\mu_a)}.
\]

In the case where \( \text{dist}(\lambda, \mathbb{Z}_\alpha) \geq \frac{\pi}{2a} \), the last expression could be estimated from above by

\[
c_1 \left( \int_{\mathbb{R}} \frac{|\text{Im}\lambda| \, dt}{t^2 + |\text{Im}\lambda|^2} \int_{\mathbb{R}} \frac{|\text{Im}\lambda|^3 \, dt}{(t^2 + |\text{Im}\lambda|^2)^2} \right)^{\frac{1}{2}} = c_1 \left( \int_{\mathbb{R}} \frac{dt}{t^2 + 1} \int_{\mathbb{R}} \frac{dt}{(t^2 + 1)^2} \right)^{\frac{1}{2}}.
\]

Moreover, the singular numbers of the multiplication operator \( f \mapsto \frac{|\text{Im}\lambda|^2}{(x - \lambda)^3} f \) are precisely \( \frac{|\text{Im}\lambda|^2}{|x - \lambda|^3} \), \( x \in \mathbb{Z}_\alpha \), hence its norm in \( S^1(L^2(\mu_a)) \) does not exceed

\[
\sum_{x \in \mathbb{Z}_\alpha} \frac{|\text{Im}\lambda|^2}{|x - \lambda|^3} \leq \sum_{x \in \mathbb{Z}_\alpha} \frac{|\text{Im}\lambda|^2}{(x^2 + |\text{Im}\lambda|^2)^{\frac{3}{2}}} \leq c_1 a
\]

for a universal constant \( c_1 \). This tells us that \( \|C_{\psi, \lambda}\|_{S^p} \leq c_1(1 + a) \) for all \( \lambda \in \mathcal{U}_\varepsilon \) such that \( \text{dist}(\lambda, \mathbb{Z}_\alpha) \geq \frac{\pi}{2a} \). Now consider the case where \( \text{dist}(\lambda, \mathbb{Z}_\alpha) \leq \frac{\pi}{2a} \). Let \( x_\lambda \) be the nearest point to \( \lambda \) in the lattice \( \mathbb{Z}_\alpha \). The function \( \psi_\lambda \) belongs to \( L^1(\mu_a) \) and

\[
\sum_{x \in \mathbb{Z}_\alpha} |\psi_\lambda(x)| \leq |\psi_\lambda(x_\lambda)| + 2|\text{Im}\lambda|^2 \sum_{k=1}^{\infty} \frac{1}{(\frac{x_\lambda}{a} k - \frac{\pi}{2a})^2},
\]

\[
\leq \left| \frac{|\text{Im}\lambda| \lambda - x_\lambda} {\lambda - x_\lambda} \right|^2 + 2 \left( \frac{a |\text{Im}\lambda|}{2\pi} \right)^2 \sum_{k=1}^{\infty} \frac{1}{(k - \frac{1}{2})^2} \leq c_1,
\]

where the right hand side does not depend on \( \lambda \). It follows that the operator \( M_{\psi, \lambda} \) lies in \( S^1(L^2(\mu_a)) \) and \( \|M_{\psi, \lambda}\|_{S^1} \leq c_1 \). We also have

\[
\|C_{\psi, \lambda}\|_{S^p} = \|H_{\mu_a} M_{\psi, \lambda} - M_{\psi, \lambda} H_{\mu_a}\|_{S^1} \leq c_1,
\]

for another constant \( c_1 \), because the discrete Hilbert transform \( H_{\mu_a} \) is bounded on \( L^2(\mu_a) \). This completes the proof. \( \square \)

Remark that the second part of the proof of Theorem 2 is almost literal repetition of the corresponding part of the proof of Theorem 6.2 in [9]. However, the original argument in [9] has a gap: it does not involve the estimate of the \( S^1 \)-norm of the multiplication operator \( f \mapsto \frac{|\text{Im}\lambda|^2}{(x - \lambda)^3} f \) from formula (9). This technical place turns out to be crucial in the case \( 0 < p < 1 \). More precisely, we have the following result.

**Proposition 6.1.** Let \( 0 < p < 1 \) and let \( \alpha > 0 \). There exists a function \( \psi \in \mathcal{B}_p(\mathbb{Z}_\alpha) \) such that \( C_\psi \notin S^p(L^2(\mu_a)) \).
Proof. Suppose that \( C_\psi \in S^p(L^2(\mu_a)) \) for every \( \psi \in B_p(a, \text{osc}) \). Then it is easy to see from the closed graph theorem that there exists a constant \( c_{p,a} \) such that \( \| C_\psi \|_{S^p} \leq c_{p,a} \| \psi \|_{B_p(a, \text{osc})} \) for all \( \psi \in B_p(a, \text{osc}) \). Take \( \lambda \in \mathbb{C}^+ \) such that \( \text{Im} \lambda \geq \frac{2\pi}{a} \) and consider the function \( \psi_\lambda : t \mapsto \frac{\text{Im} \lambda}{(x-\lambda)(t-\lambda)} \). Analogously to (9), we have \( K_\psi = C_\psi + M_\lambda \), where \( K_\psi \) is the integral operator with kernel
\[
\psi_\lambda(x) - \psi_\lambda(t) = \frac{\text{Im} \lambda}{x-t},
\]
and \( M_\lambda : f \mapsto \frac{\text{Im} \lambda}{(x-\lambda)^2}f \) is the multiplication operator on \( L^2(\mu_a) \) by \( \frac{\text{Im} \lambda}{(x-\lambda)^2} \). Observe that \( K_\psi \) is the rank-one operator whose norm does not exceed
\[
\text{Im} \lambda \cdot \left\| \frac{1}{x-\lambda} \right\|^2_{L^2(\mu_a)} \leq c_p \int \frac{\text{Im} \lambda \, dt}{t^2 + (\text{Im} \lambda)^2} = c_p \int \frac{dt}{t^2 + 1}.
\]
It follows from our assumption and Lemma 4.1 that \( \| M_\lambda \|_{S^p} \leq c_{p,a} \) for all \( \lambda \in \mathbb{C} \) with \( \text{Im} \lambda \geq \frac{2\pi}{a} \) and a universal constant \( c_p \). On the other hand, we have
\[
\| M_\lambda \|_{S^p} = \sum_{x \in Z_a} \left( \frac{\text{Im} \lambda}{x-\lambda} \right)^p \geq a c_p \int \frac{(\text{Im} \lambda)^p \, dx}{(x^2 + (\text{Im} \lambda)^2)^p} = a c_p (\text{Im} \lambda)^{1-p} \int \frac{dt}{t^2 + 1}.
\]
Since the right hand side is unbounded in \( \lambda \), we get the contradiction. \( \square \)

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