DEFINABLE SETS IN ORDERED STRUCTURES

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1. Introduction. We introduce the notion of an O-minimal theory of ordered structures, such a theory being one such that the definable subsets of its models are particularly simple. The theory of real closed fields will be an example. For $T$ an O-minimal theory we prove that over every subset $A$ of a model there is a prime model, which is unique up to $A$-isomorphism. We also prove in our model-theoretic context results on the structure of semialgebraic sets. Our work was directly stimulated by the paper of van den Dries [4].

2. Definitions and examples. $L$ will be a finitary first order language which contains, among other things, a symbol $<$. We shall be concerned with infinite $L$-structures $M$ in which $<$ denotes a linear ordering of $M$. For example if $L$ has symbols $<, +, 0$, then an ordered group is just an $L$-structure $G$ which satisfies the axioms for ordered groups. A definable subset of $M^n$ is a subset $X \subset M^n$ of the form $\{a \in M^n : M \models \varphi(a, m)\}$ for some $L$-formula $\varphi(x, y)$ and $m \in M^r$, $r < \omega$. So the definable sets are those which are obtained from the sets defined with parameters from the basic relations and functions on $M$, by closing under finite unions, finite intersections, complementation and projection. An interval of $M$ is something of the form $(a, b), [a, b], (a, b]$ or $[a, b)$, where $a, b \in M$ (or $a = -\infty$, or $b = +\infty$). (Such an interval is clearly definable.)

Definition 1. (i) $M$ is O-minimal if every definable set $X \subset M$ is a finite union of rational intervals of $M$.

(ii) A complete $L$-theory $T$ is O-minimal if every model of $T$ is O-minimal.

Note that in Definition 1 no condition is placed on the definable subsets of $M^n$. An important consequence of the definition is that an O-minimal structure is definably complete; namely every definable subset of $M$ which is bounded above has a supremum in $M$ (and similarly for infimums).

A consequence of the Tarski-Seidenberg theory [3] (i.e. quantifier elimination), is that any real closed field is O-minimal. In fact if $K$ is a real closed field then the definable subsets of $K^n$, $n < \omega$, are precisely the semialgebraic sets over $K$. The following is proved, essentially using the "definable completeness" of O-minimal structures:

Theorem 2. (i) An ordered group $G$ is O-minimal just if $G$ is abelian and divisible.

(ii) An ordered unitary ring $R$ is O-minimal just if $R$ is a real closed field.