The effect of inclined soil layers on surface vibration from underground railways using a semi-analytical approach

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Abstract. Ground vibration due to underground railways is a significant source of disturbance for people living or working near the subways. The numerical models used to predict vibration levels have inherent uncertainty which must be understood to give confidence in the predictions. A semi-analytical approach is developed herein to investigate the effect of soil layering on the surface vibration of a halfspace where both soil properties and layer inclination angles are varied. The study suggests that both material properties and inclination angle of the layers have significant effect (± 10dB) on the surface vibration response.

1. Introduction

Underground railways are a proven method for efficiently transporting large numbers of people in densely populated areas: the per-capita power consumption is low and the majority of the infrastructure is located below ground helping to eliminate congestion in urban centers. Unfortunately, ground-borne vibration from these underground railways is a major source of disturbance for individuals either working or living near subway tunnels. Vibration can cause malfunctioning of sensitive equipment [1], damage buildings [2], or impair human comfort and activity [3]. Studies show that inhabitants of urban areas who are subjected to air, road and rail traffic noise report high levels of annoyance and sleep disturbance which negatively impacts quality of life [4–8]. Griefahn et al. [8] reports higher disturbance levels associated with rail noise than from aircraft or ground traffic. This could be attributed to the multiple ways people experience railway induced vibrations: air-borne noise, vibratory motion of the floors, and re-radiated noise both in the room and from household objects such as windows or glassware.

The problem frequency range is between 15Hz to 200Hz [9, 10]: higher frequencies are quickly attenuated by the soil. Low-frequency vibrations arise from the quasi-static load of the train moving along the track [11], general wheel and rail unevenness, and periodic changes in rail-support stiffness when sleepers are present[12].

Acceptable sinusoidal vibration levels for various living and working areas are listed in BS 6472:1992 [13] and depend on many factors such as time of day and building usage. BS ISO 14837 Part 1 [14] provides guidelines on the essential considerations associated with developing prediction models and shows in outline the stages to be observed for new or modified
rail systems. Future parts of BS ISO 14837 are meant to quantify acceptable vibration levels
from underground railways, but are unavailable at this time.

As a result of more stringent ground-vibration standards, the need for quick and accurate
models which can predict the levels of vibration arising from rail traffic is increasing. If predicted
levels of vibration are high, changes can be implemented early in the design process rather than
using costly retrofits once the project is complete. The models can be used to perform parametric
studies to determine which parameters have the greatest effect on ground vibration (e.g. rail
pads, continuous slab, tunnel geometry, etc.) so that vibration mitigation measures can be
efficiently and economically employed.

Finite element and boundary element methods are common modeling schemes for
investigating ground-borne vibration [15–17]. Although FEM/BEM permit accurate modeling
of complex geometrical regions (e.g. square tunnels, piled foundations, etc.) these methods
suffer from two major drawbacks: input accuracy and speed. The dynamic properties of soils
are difficult to measure and can vary significantly over the area of interest [18] thus it is difficult
to determine the correct input parameters for the FEM models. It is therefore beneficial to
perform parametric studies to cover a range of possible soil parameters; however, common FEM
models of underground railways require tens of hours to simulate the response due to a single
loading frequency, making in-depth parametric studies intractable.

A more economical approach to simulating ground-vibration due to underground railways
involves semi-analytical methods. Researchers at the Universities of Cambridge and Nottingham
have developed the PiP software [19] to provide a means by which designers can quickly evaluate
design changes and optimize the track, tunnel and foundation to minimize the levels of vibration
transmitted to surface. The benefit of the PiP model is its speed, calculating a frequency sweep
of 200 Hz in less than a minute, which allows designers to perform hundreds of parametric
studies in the time required for a similar FEM/BEM simulation to reach a single solution. The
results from PiP have been compared with those of an alternative coupled FEM-BEM method
that accounts for a tunnel in a multi-layered half-space; the results were in good agreement for
typical soil/tunnel parameters [20].

Although the current PiP model is a useful tool for investigating underground railway
vibration problems, the uncertainty associated with some of the modeling assumptions is still
not well understood. For instance the semi-analytical approach employed by PiP assumes the
soil layering is horizontal. It was previously found by Jones and Hunt [15] that if the interface
between soil layers is inclined at small angles (< 10°) there is a significant effect on the predicted
surface vibration (±10dB); however, the BEM model used in the previous study required more
than 12 hours per computation making it an unacceptable method for implementation into
PiP. The work presented herein details a new semi-analytical formulation which can account for
inclined soil layers while maintaining fast computational speed.

2. Ground Vibration
Lord Rayleigh [21] found that disturbances within a homogeneous, isotropic halfspace propagate
through the medium as compressive waves (P-waves) and shear waves (S-waves) and along the
surface as Rayleigh waves. The three waves travel at speeds \(c_P\), \(c_S\) and \(c_R\) respectively. The
compression and shear wave speeds can be written as:

\[
c_P = \sqrt{\frac{2G(1-\nu)}{\rho(1-2\nu)}} \quad \text{and} \quad c_S = \sqrt{\frac{G}{\rho}}
\]

where \(G\), \(\nu\) and \(\rho\) are the shear modulus, Poisson’s ratio and density of the medium respectively.
The Rayleigh wave speed, \(c_R\), cannot be expressed explicitly in terms of \(G\), \(\nu\) and \(\rho\) but has
been shown by Lamb [22] to travel slightly slower than \(c_S\).
Figure 1. Wave propagation diagrams for different layering schemes

The effect these waves have on surface vibration is best illustrated through example. Consider the situation depicted in Figure 1(a) of a rigid cylinder subjected to an in-plane, impulsive line load. Surface disturbances develop as the P-wave and S-wave reach the surface, followed by a Rayleigh wave as shown (ignoring reflections). Since wave energy is inversely proportional to the distance traveled\[22], the area directly above the tunnel experiences the largest vibration amplitudes (position $B$). Houses $A$ and $C$ experience the same levels of vibration since they are spaced symmetrically over the tunnel. Peak particle velocity (PPV) shall be used herein to compare vibration levels at various points on the surface.

Consider a layer of softer material added to the surface of the halfspace (Fig 1(b)); Equation 1 states that the wave speeds must reduce as the wavefronts pass through the interface layer, resulting in wave fronts similar to those depicted in Figure 1(b) (ignoring reflections). The house at $B$ still experiences the greatest vibration while levels at $A$ and $C$ are again equivalent due to symmetry.

A final consideration is the inclination of the soil layers. It is common for soil layers to be inclined at angles between 0° and 10° to the surface, as shown in Figure 1(c). Jones and Hunt\[15] found that the reflection and refraction of body waves by the inclined interface caused
3. Inclined layers using a semi-analytical approach

The soil layering of interest is presented in Figure 2, containing various soil types and interface inclination angles. To adequately represent this section a “semi-analytical element” approach is employed (programmed in Matlab). The inclined interface is modeled in step-wise increments using hyperelements while semi-infinite elements are included at the extremes. Both element types utilize the analytical solution for wave propagating horizontally while assuming vertical displacements vary linearly through the thickness of the element, as detailed below.

3.1. Semi-infinite elements

The Navier equation governing motion for homogeneous, isotropic, linear elastic bodies, is given by Graff [23] as

\[ G \nabla^2 \{u\} + (\lambda + G) \nabla \cdot \{u\} + \rho \{b\} = \rho \ddot{\{u\}} \]  \hspace{1cm} (2)

where \{u\} is the displacement vector in the x, y, and z-directions, G and \lambda are Lame’s constants of the solid, \rho is the density of the solid, and \{b\} is the body load vector. Waas[24] shows that solving Equation 2 for plane-strain in the frequency-wavenumber domain results in a transcendental eigenvalue problem which is difficult to solve even for simple geometries. Waas suggested treating the layered region as a continuum in the horizontal direction but to discretize in the vertical direction, resulting in the following equation

\[ (A k^2 + iBk + C) \{v\} = 0 \]  \hspace{1cm} (3a)

where

\[ C = G - \omega^2 M \]  \hspace{1cm} (3b)

The vector \{v\} contains the modal coordinates in the x and z directions for the \(n\) layers at a given wavenumber, \(k\). The \(2n \times 2n\) matrices A, B and C consist of the contributions from the \(n\) individual layers and are assembled using standard FEA stiffness matrix addition; the submatrices used to construct Equation 3a are given in the Appendix where the subscript “\(j\)” refers the the \(j^{th}\) layer.
For any given frequency, Equation 3a has a non-trivial solution \( \{v\} \) iff
\[
|A k^2 + iBk + C| = 0
\] (4)
This results in a quadratic-eigenvalue problem in \( k \), where the \( k \)'s are the wavenumbers for the layered region. The solution to Equation 4 consists of \( 4n \) eigenvalues: a set of \( 2n \) values with negative imaginary parts (\( \tilde{K} \)) and \( 2n \) values with positive imaginary parts (\( \tilde{K} \)) where it can be shown \( \tilde{K} = -K \). The set of eigenvalues with negative imaginary parts correspond to waves traveling to the right while those with positive imaginary parts correspond to waves traveling to the left. The associated eigenvectors (i.e. modeshapes) are designated by \( \{X\} \) and \( \{\tilde{X}\} \), respectively.

It can be shown that for a semi-infinite region, open to the right
\[
\{P\} = R\{U\}
\] (5)
where
\[
R = iAXKX^{-1} + D
\] (6)
\( R \) is equivalent to the stiffness of the layered region in plane-strain, \( \{P\} \) is the external forcing vector, \( \{U\} \) is the displacement vector in the frequency domain; the definition of \( D \) is given in the Appendix. For a full derivation refer to Waas[24].

The TLM was developed for a soil layer resting on rigid bedrock, however Andrade[25] later extended this method to include a “halfspace” element allowing for simulation of a soil layer resting on a halfspace. The derivation outlined above remains valid with the addition of a single element in the last rows and columns of the stiffness matrix, given in the Appendix with the subscript “HS”.

3.2. Hyperelements
Kausel and Roesset[26] extended the thin-layer method to allow elements of finite length, labeled hyperelements, using the same semi-analytical formulation. This is accomplished by accounting for the waves traveling in both directions through the layer due to nodal loading at both edges of the element.

The method outlined above is first employed to determine the \( A, D, X \) and \( K \) matrices. The stiffness matrix for the hyperelement in plane-strain is then determined as
\[
S = \begin{bmatrix} R & -\tilde{R}J \\ -\tilde{R}J & \tilde{R} \end{bmatrix}^{-1} \begin{bmatrix} I & \tilde{J} \\ J & I \end{bmatrix}
\] (7a)
where
\[
J = XEX^{-1} \quad \tilde{J} = TJT \quad \tilde{R} = TRT \quad E = \text{diag}[e^{-ik_jL}]
\] (7b)
where \( L \) being the distance between lateral boundaries of the hyperelement and \( T \) is a transform between eigenvectors for left and right traveling waves (see Appendix). For a more detailed discussion see Kausel and Roesset[26].

This procedure is repeated for each column of hyperelements used to mesh the geometry of interest (see Fig. 2); the total stiffness matrix \( S_{\text{total}} \) is assembled from these elementary stiffnesses using standard finite element methods. Thus the nodal displacements due to external loading can be found using
\[
\{P\}_{\text{external}} = S_{\text{total}}\{U\}
\] (8)
where again \( \{P\}_{\text{external}} \) is the external forcing vector and \( \{U\} \) is the nodal displacements in the frequency domain.
Table 1. Boundary element model parameters

(a) Soil Parameters

| Property          | Layer 1          | Layer 2          |
|-------------------|------------------|------------------|
| Poisson’s Ratio (ν) | 0.44             | 0.44             |
| Elastic Modulus (G) | ϵE × 550(1 + 0.1i) MPa | 550(1 + 0.1i) MPa |
| Density (ρ)       | ρ × 2000 kg/m³   | 2000 kg/m³      |

(b) Parametric Variables

| Variable | Value |
|----------|-------|
| ϵE       | 4, 1, 7 |
| ϵρ       | 4, 1, 7 |
| φ        | 0, 1, 2, 5° |

Figure 3. Schematic depiction of boundary element model

4. Modeling

A two-dimensional plane-strain model, almost identical in material and geometry to that used in Jones and Hunt[15], is used to represent the layered halfspace. A schematic of the model is shown in Figure 3 with soil parameters listed in Table 1(a); these parameters are appropriate for soil types found near underground railway tunnels in the London area [27]. Two significant differences exist between the models. First, the TLM soil material includes soil damping, where no damping was included in the BEM model (see Table 1(a) where a damping coefficient of 0.1 is included in the elastic modulus). This slight damping helps stabilize the solution of the eigenvalue problems. Second, the hole in the soil representing the tunnel is not included in the TLM model; instead 16 points spaced evenly at the tunnel radius are used to apply the loading. It is assumed the presence of this extra material will have negligible effect on the results (see Section 5 for a discussion of this assumption).

Table 1(b) lists the scaling factors ϵE and ϵρ used for the parametric study. These result in wave speeds in Layer 1 which are factors of ±2, or equal to that of Layer 2, as given by Equation 1. Layers with wave speeds that differ by a factor of two are common (ie. London Clay overlaid by a gravel layer). Table 1(b) also lists the layer inclination angles investigated. As depicted in Figure 3, the inclination pivot point is located over the center of the tunnel at a depth of 4 meters. The vertical discretization into thin-layers is adapted as loading frequency increases to ensure at least four elements per shear wavelength. The incline is broken into ten steps, which results in a step height of approximately 35cm for a 2° inclination over 100 meters.

The disturbance is modeled as an impulsive 1cm vertical displacement of a rigid tunnel.

5. Results

A few representative results from the current TLM simulation are compared to the respective BEM model results[15] in Figure 4. Figure 4(a) presents the surface peak-particle velocity (PPV) for a 4m layer resting on a halfspace with a horizontal interface layer. The results compare well in magnitude and general trend, including the peak PPV directly over the tunnel and trailing off as distance from center increases. Two discrepancies are evident, however. First, the TLM results are slightly higher at the center (∼ 3dB). This is likely due to the lack of “hole” in the soil representing the tunnel (see Section 4). The continuous material transfers the wave energy from the bottom of the tunnel more efficiently to the surface; it can propagate directly upward
rather than having to travel around the soil void. Therefore, neglecting the tunnel in the model appears to significantly affect the results and must be rectified in future models. Secondly, the TLM results are lower at large distances from center. This is attributed to material damping in the TLM soil which gradually reduces the wave energy reaching the surface.

Figure 4(b) presents the surface PPV insertion gain for a layer with an inclined interface resting on a halfspace; the upper layer has an elastic modulus one quarter that of the halfspace. Insertion gain is calculated by subtracting the surface PPV for the inclined case from that of the horizontal case with equivalent layer properties. Again, the results agree well in magnitude to the BEM results. These results suggest that even small differences in layer properties and inclination angles can have a significant effect on the surface PPV (±10dB).

The BEM model required approximately 13 hours to compute the response of the region for 100 frequency steps using 0.2 meter elements. This corresponds to approximately 8 minutes per frequency. The TLM model required only 18 minutes to compute the response for 150 frequency steps, corresponding to approximately 7 seconds per frequency. Although this is approximately 150 times faster than the BEM model, it does not meet the required standard for incorporation into the PiP model (~2 seconds per frequency). The bulk of the computation time is for evaluating Equation 8 using the standard Matlab matrix division operator ($x = A\backslash b$). It is thought a custom solver suited to the banded nature of the TLM could potentially reduce the processing time to meet the PiP specifications.

6. Conclusions and Future Work
The thin-layer method, a semi-analytical formulation, has been used to simulate ground-borne vibration from underground railways using both semi-infinite elements and hyperelements. The model can account for varying material properties, loading parameters and soil-layer inclination angles. The results are comparable to a previous boundary-element model and can be computed approximately 150 times faster. Future work to include the tunnel stiffness and to increase the computational speed of the model are currently in progress.
Appendix

\[ [A]_j = \frac{h_j}{6} \begin{bmatrix} 2(2G_j + \lambda_j) & 0 & (2G_j + \lambda_j) & 0 \\ 2G_j & 0 & 2(2G_j + \lambda_j) & 0 \\ 0 & 2G_j & 0 & (2G_j + \lambda_j) \\ 0 & 0 & 0 & 2G_j \end{bmatrix} \quad [G]_j = \frac{1}{\delta_j} \begin{bmatrix} G_j & 0 & 0 & -G_j \\ (2G_j + \lambda_j) & 0 & 0 & -G_j \\ 0 & G_j & 0 & -G_j \\ 0 & 0 & -G_j & 0 \end{bmatrix} \]

\[ [D]_j = \frac{1}{2} \begin{bmatrix} 0 & \lambda_j & 0 & -\lambda_j \\ G_j & 0 & -G_j & 0 \\ 0 & \lambda_j & 0 & -\lambda_j \\ 0 & 0 & -G_j & 0 \end{bmatrix} \quad [B]_j = \frac{1}{2} \begin{bmatrix} -G_j - \lambda_j & 0 & (G_j + \lambda_j) & 0 \\ 0 & -G_j - \lambda_j & 0 & (G_j + \lambda_j) \\ (G_j + \lambda_j) & 0 & -G_j - \lambda_j & 0 \\ -(G_j + \lambda_j) & 0 & -(G_j + \lambda_j) & 0 \end{bmatrix} \]

\[ [M]_j = \frac{h^2}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \quad \mathbf{T} = \begin{bmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & 1 & \cdots \end{bmatrix} \]

\[ [A]_{HS} = \frac{iGC_s}{2\omega^2} \begin{bmatrix} (\alpha - 2)\alpha^2 & 0 & 0 \\ 0 & 0 & (1 - 2\alpha) \\ (1 - 2\alpha) & 0 & 0 \end{bmatrix} \quad [B]_{HS} = G \frac{(1 - 2\alpha)}{\alpha} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad [C]_{HS} = i\omega \rho \begin{bmatrix} C_s & 0 \\ 0 & C_p \end{bmatrix} \]

\[ [D]_{HS} = G \frac{(1 - 2\alpha)}{\alpha} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \alpha = \frac{C_s}{C_p} \]

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