q-fractional functional integro-differential equation with q-integral condition

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Abstract

In this paper, we study the existence and the uniqueness of the solution of the non-local problems of q-fractional functional integro-differential equation with non-local q-integral condition. The continuous dependence of the solution will be studied. Two examples will be given to illustrate results.

Keywords: Functional equations, existence of solutions, continuous dependence, q-calculus.

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1. Introduction

q-calculus, This area of research has several applications, see [2, 10, 11] and references therein. There are several developments and applications of the q-calculus in mathematical physics, the theory of relativity and special functions [1, 4]. In several papers [13, 15], integro-differential equation with infinite-point boundary conditions have been studied. In [5–7] El-Sayed et al. introduced and studied integro-differential equation with infinite-point boundary conditions.

In this paper, we are concerned with q-fractional functional integro-differential equation

\[
\frac{dx}{dt} = f(t, x(t), I_q^s g(t, x(t))), \quad \text{a.e. } t \in (0, 1],
\]

with the non-local q-integral

\[
\int_0^1 x(s) ds = x_0.
\]

The existence and the uniqueness of solution, under certain conditions, will be proved. The continuous dependence of the solution on x_0, will be studied.

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2. q-calculus

First, we write the basic definitions of the q-fractional integral and q-derivatives, for more details.

- Let a real parameter $q \in (0, 1)$, we define a q-real number $[a]_q$ by
  \[ [a]_q = \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}. \]

- A q-analog of the Pochhammer symbol (q-shifted factorial) is defined by
  \[ (a; q)_n = \begin{cases} 1, & n = 0, \\ \prod_{i=1}^{n-1} (1 - a(q^{-1})^i), & n \in \mathbb{N}. \end{cases} \]

- The q-analog of the power $(a - b)^n$ is given by
  \[ (a - b)^n = \begin{cases} 1, & n = 0, \\ \prod_{i=1}^{n-1} (a - bq^i), & n \in \mathbb{N}, \quad a, b \in \mathbb{R}. \end{cases} \]

- Also,
  \[ (a - b)^n = a^n (b/a; q)_n, \quad a \neq 0. \]
  Notice that, $\lim_{n \to \infty} (a; q)_n$ exists and we will denote it by $(a; q)_\infty$.

- More generally, for $\lambda \in \mathbb{R}$, $aq^\lambda \neq q^{-n}(n \in \mathbb{N})$, we define
  \[ (a; q)_{\lambda} = \frac{(a; q)_{\infty}}{(aq^\lambda; q)_{\infty}} \quad \text{and} \quad (a - b)_{\lambda} = a^\lambda \frac{(b/a; q)_{\infty}}{(q^\lambda b/a; q)_{\infty}}. \]

- The q-gamma function is defined by
  \[ \Gamma_q(t) = \frac{G(q^t)}{(1 - q)(1 - q^t)G(q)}, \quad t \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}, \]
  where $G(q^t) = \frac{1}{(q^t; q)_{\infty}}$. Or, equivalently, $\Gamma_q(t) = \frac{1}{1 - q^{t-1}}$, and satisfies $\Gamma_q(t + 1) = [t]_q \Gamma_q(t)$.

**Definition 2.1 ([2]).** Let $f$ be a function defined on $[0, 1]$. The fractional q-integral of the Riemann-Liouville type of order $\delta \geq 0$ is given by

\[ (I_q^\delta f)(t) = \begin{cases} f(t), & \delta = 0, \\ \frac{1}{\Gamma_q(\delta)} \int_0^t (t - qs)^{\delta - 1} f(s) d_q s = (1 - q)^{\delta} t^\delta \sum_{i=0}^{\infty} q^{i(\delta - 1)} \frac{(q^\delta; q)_i}{(q; q)_i} f(q^\delta t), & \delta > 0. \end{cases} \]

**Lemma 2.2 ([16]).** For $\delta > 0$, using q-integration by parts, we have

\[ (I_q^\delta 1)(t) = t^{\delta(\delta + 1)}. \]
The equivalence of (1.1)-(1.2) and integral equation is given in the following lemma.

**Lemma 2.3.** If the solution of the non-local problem (1.1)-(1.2) exists, then the functional q-fractional integro-differential equation (1.1)-(1.2) and the functional q-integral equation

\[
x(t) = \frac{1}{\int_0^1 dq s} [x_0 - \int_0^1 \int_0^s f(\theta, x(\theta), I_q^\delta g(\theta, x(\theta)))d\theta dq s] + \int_0^t f(s, x(s), I_q^\delta g(s, x(s)))ds.
\]  

(2.1)

are equivalent.

**Proof.** Let \( x \) be a solution of the non-local problem (1.1)-(1.2), integrating both sides of (1.1) we get

\[
x(t) = x(0) + \int_0^t f(s, x(s), I_q^\delta g(s, x(s)))ds.
\]  

(2.2)

Using the non-local q-condition (1.2), we get

\[
\int_0^1 x(s) dq s = \int_0^1 x(0) dq s + \int_0^1 \int_0^s f(\theta, x(\theta), I_q^\delta g(\theta, x(\theta)))d\theta dq s,
\]

then

\[
x(0) = \frac{1}{\int_0^1 dq s} [x_0 - \int_0^1 \int_0^s f(\theta, x(\theta), I_q^\delta g(\theta, x(\theta)))d\theta dq s].
\]  

(2.3)

Using (2.2) and (2.3), we obtain (2.1). To complete the proof, suppose that \( x \) satisfies equation (2.1), differentiating (2.1) we obtain

\[
\frac{dx}{dt} = \frac{d}{dt} \left\{ \frac{1}{\int_0^1 dq s} [x_0 - \int_0^1 \int_0^s f(\theta, x(\theta), I_q^\delta g(\theta, x(\theta)))d\theta dq s] \right\} \int_0^t f(s, x(s), I_q^\delta g(s, x(s)))ds
\]

\[
= 0 + \frac{d}{dt} \int_0^t f(s, x(s), I_q^\delta g(s, x(s)))ds = f(t, x(t), I_q^\delta g(t, x(t)))
\]

and

\[
\int_0^1 x(\tau) dq \tau = \frac{1}{\int_0^1 dq s} [x_0 - \int_0^1 \int_0^s f(\theta, x(\theta), I_q^\delta g(\theta, x(\theta)))d\theta dq s] \int_0^1 dq \tau + \int_0^1 \int_0^\theta \int_0^1 dq dq s \int_0^1 f(s, x(s), I_q^\delta g(s, x(s)))ds dq \tau.
\]

Then

\[
\int_0^1 x(\tau) dq \tau = x_0.
\]

\[\square\]

3. Existence of solution

In the following theorem, using Schauder fixed point theorem [8], we establish existence of at least one solution of (1.1)-(1.2).

**Theorem 3.1.** Let \( f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R} \) and \( g : [0, 1] \times \mathbb{R} \to \mathbb{R} \) satisfies Carathéodory condition, if there exist functions \( c_{1,2} \in L^1[0, 1] \) and positive constants \( b_{1,2} > 0 \) such that

\[
|f(t, x, y)| \leq c_1(t) + b_1|x| + b_1|y|, \quad |g(t, x)| \leq c_2(t) + b_2|x|,
\]

and

\[
\sup_{t \in [0, 1]} \int_0^t c_1(s)ds \leq M_1, \quad \sup_{t \in [0, 1]} \int_0^t I_q^\delta c_2(s)ds \leq M_2, \quad (2b_1 + \frac{2b_1b_2}{\Gamma(\delta + 1)}) \in [0, 1),
\]

then the non-local problem (1.1)-(1.2) has at least one solution.
Proof. Define the operator $A$ associated with the integral equation (2.1) by

$$Ax(t) = \frac{1}{\int_0^1 d_4s} \left[ x_0 - \int_0^t \int_0^s f(\theta, x(\theta), I_4^s g(\theta, x(\theta))) d\theta d_4s \right] + \int_0^t f(s, x(s), I_4^s g(s, x(s))) ds.$$ 

Let $Q_r = \{ x(t) \in \mathbb{R} : \|x\| \leq r \}$, where $r = \frac{|x_0| + 2M_1 + 2b_1M_2}{2b_1b_2 \left( 1 - \frac{2b_1 + (b_1 + 1)f_{\theta}^s}{\delta + 1} \right)}$. Then, for $x \in Q_r$, we have

$$|Ax(t)| \leq \frac{1}{\int_0^1 d_4s} \left[ |x_0| + \int_0^t \int_0^s |f(\theta, x(\theta), I_4^s g(\theta, x(\theta)))| d\theta d_4s \right] + \int_0^t |f(s, x(s), I_4^s g(s, x(s)))| ds$$

$$\leq \frac{1}{\int_0^1 d_4s} \left[ |x_0| + \int_0^t \int_0^s (c_1(\theta) + b_1|x(\theta)| + b_1 I_4^s g(\theta, x(\theta))) d\theta d_4s \right]$$

$$+ \int_0^t (c_1(s) + b_1|x(s)| + b_1 I_4^s g(s, x(s))) ds$$

$$\leq \frac{1}{\int_0^1 d_4s} \left[ |x_0| + \int_0^t \left( M_1 + b_1 r + b_1 \int_0^s I_q^s c_2(\theta) + b_2|x(\theta)| d\theta d_s \right) \right]$$

$$+ M_1 + b_1 r + b_1 \int_0^t I_q^s c_2(s) + b_2|x(s)| ds$$

$$\leq \frac{1}{\int_0^1 d_4s} \left[ |x_0| + \int_0^t \left( M_1 + b_1 r + b_1 M_2 + b_1 b_2 r \int_0^s \frac{\theta^\delta}{\Gamma_q(\delta + 1)} d\theta \right) d_4s \right]$$

$$+ M_1 + b_1 r + b_1 M_2 + b_1 b_2 r \int_0^t \frac{s^\delta}{\Gamma_q(\delta + 1)} ds$$

$$= \frac{|x_0|}{\int_0^1 d_4s} + 2M_1 + 2b_1 r + 2b_1 M_2 + \frac{2b_1 b_2 r}{(\delta + 1)\Gamma_q(\delta + 1)} = r.$$ 

This proves that $A : Q_r \rightarrow Q_r$ and the class of functions $\{Ax\}$ is uniformly bounded in $Q_r$. Now, let $t_1, t_2 \in (0, 1)$ such that $|t_2 - t_1| < \delta$, then

$$|Ax(t_2) - Ax(t_1)| = \left| \int_{t_1}^{t_2} f(s, x(s), I_4^s g(s, x(s))) ds - \int_{t_1}^{t_1} f(s, x(s), I_4^s g(s, x(s))) ds \right|$$

$$\leq \int_{t_1}^{t_2} |f(s, x(s), I_4^s g(s, x(s)))| ds$$

$$\leq \int_{t_1}^{t_2} (c_1(s) + b_1|x(s)| + b_1 I_4^s g(s, x(s))) ds$$

$$\leq \int_{t_1}^{t_2} c_1(s) ds + (t_2 - t_1)b_1 r + b_1 \int_{t_1}^{t_2} I_q^s c_2(s) ds + b_1 b_2 r \int_{t_1}^{t_2} \frac{s^\delta}{\Gamma_q(\delta + 1)} ds.$$ 

This means that the class of functions $\{Ax\}$ is equi-continuous in $Q_r$. Let $x_n \in Q_r$, $x_n \rightarrow x(n \rightarrow \infty)$, then from continuity of the functions $f$ and $g$, we obtain $f(t, x_n(t), y_n(t)) \rightarrow f(t, x(t), y(t))$ and $g(t, x_n(t)) \rightarrow g(t, x(t))$ as $n \rightarrow \infty$. Also

$$\lim_{n \rightarrow \infty} Ax_n(t) = \lim_{n \rightarrow \infty} \left[ \frac{1}{\int_0^1 d_4s} \left[ x_0 - \int_0^1 \int_0^s f(\theta, x_n(\theta), I_4^s g(\theta, x(\theta))) d\theta d_4s \right] \right.$$

$$+ \left. \int_0^t f(s, x_n(s), I_4^s g(s, x_n(s))) ds \right].$$

(3.1)
Using (1.1)-(1.2) and Lebesgue dominated convergence Theorem [12], from (3.1) we obtain

\[
\lim_{n \to \infty} A x_n(t) = \left[ \frac{1}{\int_0^t dq_s} \left[ x_0 - \int_0^t \int_0^s f(\theta, x_n(\theta), I_q^\delta g(\theta, x(\theta))) \, dq_s \, d\theta d\theta \right] \right. \\
+ \left. \int_0^t \lim_{n \to \infty} \left[ f(s, x_n(s), I_q^\delta g(s, x_n(s))) \right] ds \right] = A x(t).
\]

Then \(A x_n \to A x\) as \(n \to \infty\). This means that the operator \(A\) is continuous. Then by Schauder fixed point Theorem [8] there exist at least one solution \(x \in C[0,1]\) of the functional equation (1.1)-(1.2).

\(\square\)

4. Uniqueness of the solution

In the following theorem, we establish existence of exactly one solution of (1.1)-(1.2).

**Theorem 4.1.** Let \(f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}\) is measurable in \(t\) for any \(x,y \in \mathbb{R}\) and continuous in \(x, y\) for all \(t \in [0,1]\) and \(g : [0,1] \times \mathbb{R} \to \mathbb{R}\) is measurable in \(t\) for any \(x \in \mathbb{R}\) and continuous in \(x\) for all \(t \in [0,1]\). If there exists \(b_{1,2} > 0\) with

\[
|f(t, x, y) - f(t, u, v)| \leq b_1|x - u| + b_1|y - v| \quad \text{and} \quad |g(t, x) - g(t, u)| \leq b_2|x - u|,
\]

then the solution of the non-local problem (1.1)-(1.2) is unique.

**Proof.** Let \(x, y\) be two the solution of (1.1)-(1.2), then

\[
|x(t) - y(t)| = \left| \frac{1}{\int_0^t dq_s} \left[ x_0 - \int_0^t \int_0^s f(\theta, x(\theta), I_q^\delta g(\theta, x(\theta))) \, dq_s \, d\theta d\theta \right] \right.
\\
- \left. \frac{1}{\int_0^t dq_s} \left[ x_0 - \int_0^t \int_0^s f(\theta, y(\theta), I_q^\delta g(\theta, y(\theta))) \, dq_s \, d\theta d\theta \right] \right| ds
\\
- \left. \int_0^t |f(s, x(s), I_q^\delta g(s, x(s))) - f(s, y(s), I_q^\delta g(s, y(s)))| ds \right| ds
\\
\leq \left. \int_0^t \left[ |f(s, x(s), I_q^\delta g(s, x(s))) - f(s, y(s), I_q^\delta g(s, y(s)))| \right] ds \right| ds
\\
\leq 2b_1 |x - y| + \frac{2b_1 b_2}{(\delta + 1) \Gamma_q(\delta + 1)} \|x - y\| = (2b_1 + \frac{2b_1 b_2}{(\delta + 1) \Gamma_q(\delta + 1)}) \|x - y\|.
\]

Hence

\[
(1 - 2b_1 + \frac{2b_1 b_2}{(\delta + 1) \Gamma_q(\delta + 1)}) \|x - y\| \leq 0.
\]

since \((2b_1 + \frac{2b_1 b_2}{(\delta + 1) \Gamma_q(\delta + 1)}) < 1\), then \(x = y\) and the solution of the non-local problem (1.1)-(1.2) is unique. \(\square\)
5. Continuous dependence

In the following, we give the definition of solutions continuously depend on initial data.

**Definition 5.1.** The solution $x \in C[0, 1]$ of the non-local problem (1.1)-(1.2) depends continuously on $x_0$, if

$$\forall \epsilon > 0, \quad \exists \delta_1(\epsilon) \quad \text{s.t.} \quad |x_0 - x_0^*| < \delta_1 \Rightarrow ||x - x^*|| < \epsilon,$$

where $x^*$ is the solution of the non-local problem

$$\frac{dx^*}{dt} = f(t, x^*(t), I_q^s g(t, x^*(t))), \quad \text{a.e.} \quad t \in (0, 1), \quad (5.1)$$

with the non-local $q$-condition

$$\int_0^1 x^*(s)ds = x_0^*. \quad (5.2)$$

**Theorem 5.2.** Let the assumptions of Theorem 4.1 be satisfied, then the solution of the non-local problem (1.1)-(1.2) depends continuously on $x_0$.

**Proof.** Let $x, x^*$ be two solutions of the non-local problem (1.1)-(1.2) and (5.1)-(5.2), respectively. Then

$$|x(t) - x^*(t)| = \frac{1}{\int_0^1 dq_s} \left[ |x_0 - \int_0^1 f(\theta, x(\theta), I_q^s g(\theta, x(\theta))) d\theta dq_s| + \int_0^t f(s, x(s), I_q^s g(s, x(s))) ds \right.

- \left. \frac{1}{\int_0^1 dq_s} \left[ |x_0 - \int_0^1 f(\theta, x^*(\theta), I_q^s g(\theta, x^*(\theta))) d\theta dq_s| - \int_0^t f(s, x^*(s), I_q^s g(s, x^*(s))) ds \right] \right]

\leq \frac{|x_0 - x_0^*|}{\int_0^1 dq_s} + \frac{1}{\int_0^1 dq_s} \int_0^1 \left[ |f(\theta, x(\theta), I_q^s g(\theta, x(\theta))) - f(\theta, y(\theta), I_q^s g(\theta, y(\theta)))| d\theta dq_s \right.

+ \left. \int_0^t |f(s, x(s), I_q^s g(s, x(s))) - f(s, y(s), I_q^s g(s, y(s)))| ds \right]

\leq \frac{|x_0 - x_0^*|}{\int_0^1 dq_s} + \frac{1}{\int_0^1 dq_s} \int_0^1 \left[ |b_1||x - y|| + b_1 I_q^s g(\theta, x(\theta))

- g(\theta, y(\theta))| d\theta dq_s + \int_0^t |(b_1||x - y|| + b_1 I_q^s g(s, x(s)) - g(s, y(s)))| ds \right]

\leq \frac{\delta_1}{\int_0^1 dq_s} + 2b_1||x - y|| + \frac{2b_1b_2}{(\delta + 1)I_q^s (\delta + 1)}||x - y|| = \frac{\delta_1}{\int_0^1 dq_s} + (2b_1 + \frac{2b_1b_2}{(\delta + 1)I_q^s (\delta + 1)})||x - y||.

Hence

$$||x - x^*|| \leq \frac{\delta_1}{\int_0^1 dq_s [1 - (2b_1 + \frac{2b_1b_2}{(\delta + 1)I_q^s (\delta + 1)})]} = \epsilon.$$

This means that the solution of the non-local problem (1.1)-(1.2) depends continuously on $x_0$.

6. Examples

In this section we offer some examples to illustrate our results

**Example 6.1.** Consider the following nonlinear integro-differential equation

$$\frac{dx}{dt} = t^3 e^{-t} + \frac{\ln(1 + x(t))}{3 + t^2} + I_{0.9}^1 \left( \frac{1}{9}(3t^3) + t^5 \cos x(t) + e^{-t} x(t) \right), \quad \text{a.e.} \quad t \in (0, 1), \quad (6.1)$$
with q-condition
\[ \int_0^1 x(s) \, ds = x_0. \] (6.2)

Set
\[ f(t, x(t), I_q^t g(t, x(t))) = t^3 e^{-t} + \ln\left(1 + x(t)\right) + \frac{1}{3} t^5 (\cos(3t + 3) + t^5 \cos x(t) + e^{-t} x(t)). \]

Then
\[ |f(t, x(t), I_q^t g(t, x(t)))| \leq t^3 e^{-t} + \frac{1}{3} |x| + \frac{1}{3} t^5 \frac{1}{3} |(\cos(3t + 3) + t^5 \cos x(t) + e^{-t} x(t))|, \]
and also
\[ |g(t, x(t))| \leq \frac{1}{3} |\cos(3t + 3)| + \frac{1}{3} |x(t)|. \]

It is clear that the assumptions of Theorem 3.1 are satisfied with
\[ c_1(t) = t^3 e^{-t} \in L^1[0, 1], \quad c_2(t) = \frac{1}{2} |\cos(3t + 3)| \in L^1[0, 1], \]
\[ b_1 = \frac{1}{3}, \quad b_2 = \frac{1}{3}, \quad 2b_1 + \frac{2b_1 b_2}{(\delta + 1) \Gamma_q(\delta + 1)} = \frac{2}{3} + \frac{4}{9} \frac{\Gamma_{1.5}(\frac{7}{5})}{\Gamma_{1.5}(\frac{3}{5})} = 0.8390712688 < 1, \]
by applying to Theorem 3.1, the given non-local problem (6.1)-(6.2) has a continuous solution.

**Example 6.2.** Consider the following nonlinear integro-differential equation
\[ \frac{dx}{dt} = t^3 + t + 1 + \frac{x(t)}{\sqrt{t+3}} + \frac{1}{4} I_q^t (\sin^2(3t + 3) + \frac{tx(t)}{2^t (1 + x(t))}), \quad \text{a.e.} \quad t \in (0, 1), \] (6.3)
with q-condition
\[ \int_0^1 x(s) \, ds = x_0. \] (6.4)

Set
\[ f(t, x(t), I_q^t g(t, x(t))) = t^3 + t + 1 + \frac{x(t)}{\sqrt{2t+4}} + \frac{1}{4} I_q^t (\sin^2(3t + 3) + \frac{tx(t)}{2^t (1 + x(t))}). \]

Then
\[ |f(t, x(t), I_q^t g(t, x(t)))| \leq t^3 + t + 1 + \frac{1}{3} |x| + \frac{1}{3} I_q^t \frac{3}{4} |\sin^2(3t + 3) + \frac{tx(t)}{2^t (1 + x(t))}|, \]
and also
\[ |g(s, x(s))| \leq \frac{3}{4} |\sin^2(3s + 3)| + \frac{3}{8} |x(s)|. \]

It is clear that the assumptions of Theorem 3.1 are satisfied with
\[ c_1(t) = t^3 + t + 1 \in L^1[0, 1], \quad c_2(t) = \frac{3}{4} |\sin^2(3s + 3)| \in L^1[0, 1], \]
\[ b_1 = \frac{1}{3}, \quad b_2 = \frac{3}{8}, \quad 2b_1 + \frac{2b_1 b_2}{(\delta + 1) \Gamma_q(\delta + 1)} = \frac{2}{3} + \frac{1}{9} \frac{\Gamma_{1.5}(\frac{7}{5})}{\Gamma_{1.5}(\frac{3}{5})} = 0.8986471417 < 1, \]
by applying to Theorem 3.1, the given non-local problem (6.3)-(6.4) has a continuous solution.
7. Conclusion

In this work, the existence of continuous solution using Schauder fixed point Theorem, its uniqueness, and the continuous dependence of the q-fractional functional integro-differential equation on an initial data have been studied. Some examples are introduced to illustrate the benefits of our results. In the future, generalization of q-fractional functional differential equation in time scale can be examined.

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References

[1] T. Abdeljawad, D. Baleanu, Caputo q-fractional initial value problems and a q-analogue mittag-leffler function, Commun. Nonlinear Sci. Numer. Simul., 16 (2011), 4682–4688.
[2] R. P. Agarwal, Certain fractional q-integrals and q-derivatives, Proc. Cambridge Philos. Soc., 66 (1969), 365–370.
[3] A. Boutiara, M. Benbachir, M. K. A. Kaabar, F. Martinez, M. E. Samei, M. Kaplan, Explicit iteration and unbounded solutions for fractional q-difference equations with boundary conditions on an infinite interval, J. Inequal. Appl., 2022 (2022), 27 pages.
[4] R. I. Butt, T. Abdeljawad, M. A. Alqudah, M. ur Rehman, Ulam stability of caputo q-fractional delay difference equation: q-fractional Gronwall inequality approach, J. Inequal. Appl., 2019 (2019), 13 pages.
[5] A. M. A. El-Sayed, R. G. Ahmed, Existence of Solutions for a Functional Integro-Differential Equation with Infinite Point and Integral Conditions, Int. J. Appl. Comput. Math., 5 (2019), 15 pages.
[6] A. M. A. El-Sayed, R. G. Ahmed, Solvability of a coupled system of functional integro-differential equations with infinite point and Riemann-Stieltjes integral conditions, Appl. Math. Comput., 370 (2020), 18 pages.
[7] A. El-Sayed, R. Gamal, Infinite point and Riemann-Stieltjes integral conditions for an integro-differential equation, Nonlinear Anal. Model. Control, 24 (2019), 733–754.
[8] K. Goebel, W. A. Kirk, Topics in metric fixed point theory, Cambridge University Press, Cambridge, (1990).
[9] T. S. Hassan, R. G. Ahmed, A. M. A. El-Sayed, R. A. El-Nabulsi, O. Moaaz, M. B. Mesmouli, Solvability of a State-Dependence Functional Integro-Differential Inclusion with Delay non-local Condition, Mathematics, 10 (2022), 11 pages.
[10] M. Jleli, M. Mursaleen, B. Samet, Q-integral equations of fractional orders, Electron. J. Differential Equations, 2016 (2016), 14 pages.
[11] V. Kac, P. Cheung, Quantum Calculus, Springer-Verlag, New York, (2001).
[12] A. N. Kolomogorov, S. V. Fomin, Introductory real analysis, Dover Publications, New York, (1975).
[13] X. Q. Liu, L. S. Liu, Y. H. Wu, Existence of positive solutions for a singular nonlinear fractional differential equation with integral boundary conditions involving fractional derivatives, Bound. Value Probl., 2018 (2018), 21 pages.
[14] M. M. Matar, M. I. Abbas, J. Alzabut, M. K. A. Kaabar, S. Etemad, S. Rezapour, Investigation of the p-Laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives, Adv. Difference Equ., 2021 (2021), 18 pages.
[15] D. D. Min, L. S. Liu, Y. H. Wu, Uniqueness of positive solutions for the singular fractional differential equations involving integral boundary value conditions, Bound. Value Probl., 2018 (2018), 18 pages.
[16] P. M. Rajković, S. D. Marinković, M. S. Stanković, Fractional integrals and derivatives in q-calculus, Appl. Anal. Discrete Math., 1 (2017), 311–323.
[17] S. Rezapour, A. Imran, A. Hussain, F. Martinez, S. Etemad, M. K. A. Kaabar, Condensing functions and approximate endpoint criterion for the existence analysis of quantum integro-difference FBVPs, Symmetry, 13 (2021), 15 pages.
[18] M. E. Samei, R. Ghaifari, S. W. Yao, M. K. A. Kaabar, F. Martinez, M. Inc, Existence of Solutions for a Singular Fractional q-Differential Equations under Riemann-Liouville Integral Boundary Condition, Symmetry, 13 (2021), 12 pages.
[19] M. E. Samei, L. Karimi, M. K. A. Kaabar, To investigate a class of multi-singular pointwise defined fractional q-integro-differential equation with applications, AIMS Math., 7 (2022), 7781–7816.