Research Article

Existence and Uniqueness of Weak Solutions for Novel Anisotropic Nonlinear Diffusion Equations Related to Image Analysis

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This paper establishes the existence and uniqueness of weak solutions for the initial-boundary value problem of anisotropic nonlinear diffusion partial differential equations related to image processing and analysis. An implicit iterative method combined with a variational approach has been applied to construct approximate solutions for this problem. Under some a priori estimates and a monotonicity condition, the existence of unique weak solutions for this problem has been proven. This work has been complemented by a consistent and stable approximation scheme showing its great significance as an image restoration technique.

1. Introduction

In the last three decades, nonlinear diffusion equations have inspired numerous research studies in various application ranges. Perona and Malik [1] were the first to introduce such equation in image processing and analysis in the following manner:

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nabla \cdot [c(|\nabla u|)\nabla u] & = 0, \text{ in } \Omega \times (0, T], \\
\langle c(|\nabla u|)\nabla u, n \rangle & = 0, \text{ on } \partial \Omega \times (0, T], \\
u(x; 0) & = u_0(x), \text{ in } \Omega,
\end{align*}
\]

where $\Omega$ is an image domain in $\mathbb{R}^2$ and $c$ is a positive decreasing function defined on $\mathbb{R}_+$.

When it comes to processing a digital image, Perona and Malik chose the above model to preserve meaningful features such as edges while reducing irrelevant information such as noise in the homogeneous area. Nevertheless, this model, known as an isotropic nonlinear diffusion equation, handles an image feature with the same amount of blurring in all its directions. For instance, this process cannot successfully eliminate noises at edges [2]. Accordingly, it might be wise to consider the orientation of essential features by using anisotropic diffusion. Weickert [2] introduced this property by defining an orientation descriptor using the structure tensor, which is convenient to identify features such as corners and T-junctions. Besides, digital images present some structural difficulties; that is, they are discrete in space and image intensity values. Accordingly, it would be of great interest to adapt the diffusion to digital images’ structure by considering vertical, horizontal, and diagonal differential operators. Due to these reasons, we modeled and developed anisotropic nonlinear diffusion equations using a novel diffusion tensor.

Various tools can be used to examine the existence of solutions for nonlinear partial differential equations (PDEs), such as variational techniques, monotonicity method, fixed-point theorems, iterative methods, and truncation techniques; for more detailed information, we refer to [3–7] and the references therein. These PDEs have been motivated by various applications such as image restoration and reconstruction (see, for example, [3, 4, 8–11]). Moreover, the
image processing of the brain allows the localization of epileptogenic foci for the patient. A noninvasive method has been examined numerically as an inverse problem in [12].

Under some challenging conditions, the existence and uniqueness of weak solutions for the Perona and Malik model have been investigated in the bounded variation space $BV(\Omega)$ [3, 13]. In some other functional frameworks, Wang and Zhou have thoroughly studied in [4] and proved the existence and uniqueness of weak solutions in the Orlicz space $L_{\log L}(\Omega)$ using a new diffusion function $c(s) = ((s + (s + 1)\log(s + 1))/(s(s + 1)))$ for all $s \geq 0$.

In this paper, we suppose that $\Omega$ is an open-bounded domain of $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$, and $T$ is a positive number. We denote

$$ z_{x_1} u := \nabla u \cdot e_1, $$

$$ z_{x_2} u := \nabla u \cdot e_2, $$

$$ z_{x_{12}} u := \nabla u \cdot \frac{e_1 + e_2}{|e_1 + e_2|}, $$

$$ z_{x_{-12}} u := \nabla u \cdot \frac{-e_1 + e_2}{|e_1 + e_2|}, $$

where $(e_1, e_2)$ is the canonical basis of $\mathbb{R}^2$. We consider the following anisotropic nonlinear parabolic initial-boundary value problem:

$$ \begin{aligned}
\frac{\partial u}{\partial t} - \nabla \cdot [D_{\nabla u} \nabla u] &= 0, \quad \text{in } \Omega \times (0, T], \\
\langle D_{\nabla u} \nabla u, n \rangle &= 0, \quad \text{on } \partial\Omega \times (0, T], \\
u(x; 0) &= u_0(x), \quad \text{in } \Omega,
\end{aligned} $$

where $D_{\nabla u}$, the diffusion tensor, is a real symmetric positive definite matrix of $\mathbb{R}^{2 \times 2}$ defined as follows:

$$ D_{\nabla u} = \begin{pmatrix}
g\left(\left|u_{x_1}\right|\right) + \frac{g\left(\left|u_{x_{11}}\right|\right) + g\left(\left|u_{x_{-11}}\right|\right)}{2} & \frac{g\left(\left|u_{x_{11}}\right|\right) - g\left(\left|u_{x_{-11}}\right|\right)}{2} \\
\frac{g\left(\left|u_{x_{11}}\right|\right) - g\left(\left|u_{x_{-11}}\right|\right)}{2} & g\left(\left|u_{x_3}\right|\right) + \frac{g\left(\left|u_{x_{12}}\right|\right) + g\left(\left|u_{x_{-12}}\right|\right)}{2}
\end{pmatrix}, $$

and $g: \mathbb{R} \rightarrow \mathbb{R}$ is a $C^1$ positive decreasing function. Then, we can define $\phi: \mathbb{R} \rightarrow \mathbb{R}$ as a $C^2$ function such that

$$ \phi(s) = \int_0^s r g(r) dr, \quad s \geq 0, $$

satisfying
\[
\phi(0) = \phi'(0) = 0, \quad \phi(s) > 0, \phi'(s) > 0, \quad \text{for } s \in \mathbb{R}^*_+, \\
\phi''(s) \geq 0, \quad s \phi''(s) \leq \phi'(s), \quad \text{for } s \in \mathbb{R}_+, \\
0 < \lim_{s \to \infty} \frac{\phi(s)}{s} \log(s) < \infty, \quad 0 < \lim_{s \to \infty} \frac{\phi'(s)}{\log(s)} < \infty, \\
\lim_{s \to 0^+} \frac{\phi'(s)}{s} > 0, \quad \lim_{s \to \infty} \frac{\phi'(s)}{s} = 0.
\]

(6)

To construct an adaptive diffusion tensor, the function \( g \) is approximated numerically by a cubic Hermite spline \([14]\) that interpolates numeric data specified at \( 0 = k_0 < k_1 < \cdots < k_m \) with \( m \in \mathbb{N}^* \):

\[
g(s) = \left\{ \begin{array}{ll}
p_{k_0}P_{1,k_0,k_1}(s) + v_{k_0}P_{2,k_0,k_1}(s) + p_{k_1}P_{1,k_1,k_2}(s) + v_{k_1}P_{2,k_1,k_2}(s), & i \in \{0,1,\ldots,m-1\}, \\
p_{k_m}P_{g_{k_m,1}}(s) + v_{k_m}P_{g_{k_m,2}}(s), & s \in [k_m,\infty[ \end{array} \right.
\]

(7)

where \( p \) and \( v \) are the coefficients used to define the position and the velocity vector at a specific point, \( k_i \) are the threshold parameters, \( \{P_{j,cd}\} \) is the family of the basis functions composed of polynomials of degree 3 used on the interval \([c,d]\) such that

\[
\begin{align*}
P_{1,cd}(s) &= \frac{(s-d)^2(2s+d-3c)}{(d-c)^3}, \\
P_{2,cd}(s) &= \frac{(s-d)^2(s-c)}{(d-c)^2}.
\end{align*}
\]

(8)

And we may consider

\[
\begin{align*}
g_{k_m,1}(s) &= \frac{k_m}{\log(k_m) + 2} \frac{2s(\log(s) + 1) - k_m \log(k_m)}{s^2}, \\
g_{k_m,2}(s) &= \frac{k_m^2}{\log(k_m) + 2} \frac{s(\log(s) + 1) - k_m \log(k_m) + 1}{s^2}.
\end{align*}
\]

(9)
From the definition of $\phi$, we can deduce

$$
\phi(s) = \begin{cases} 
C_i + \sum_{j=0}^{3} A_{k_{m-j+1}} s + 2, \\
A_{k_{m+2} s} \log(s) + A_{k_{m+1}} \log(s) + C_m,
\end{cases}
$$

where $C_i$ and $C_m$ are constants determined by the continuity of $\phi$ at each $k_i$. In this case, the values of the coefficients $A_{k_{m-j+1}}$ are determined experimentally provided that $\phi$ satisfies the above conditions on $[0, k_m]$. Besides, we may introduce some sufficient conditions on $k_m$ and $A_{k_m}$ that guarantee the properties of $\phi$ on $[k_m, \infty]$:

$$
\begin{align*}
    &k_m \geq 1, \\
    &A_{k_{m+2}} > 0, \\
    &A_{k_{m+1}} < k_m A_{k_{m+2}}, \\
    &A_{k_{m+1}} \geq -\frac{k_m \log(k_m)}{2} A_{k_{m+2}}.
\end{align*}
$$

Anisotropic diffusion model (3) allows strong directional smoothing within the areas where $|u_{x_1}|$, $|u_{x_2}|$, or $|u_{x_12}|$ is small and prevents blurring boundaries, contours, or corners that separate neighboring areas, where one or a combination of these differential operators has significant value.

Moreover, the matrix $D_{\nu u}$ has two eigenvalues $\lambda_{+, \lambda}$:

$$
\lambda_{+\lambda} = \frac{1}{2} \left( g\left(|u_{x_1}|\right) + g\left(|u_{x_2}|\right) + g\left(|u_{x_1}x_2|\right) + g\left(|u_{x_1}|\right) \right) \\
\pm \sqrt{\left(g\left(|u_{x_1}|\right) - g\left(|u_{x_2}|\right)\right)^2 + \left(g\left(|u_{x_1}|\right) - g\left(|u_{x_1}x_2|\right)\right)^2}.
$$

with $\theta_{+, \lambda}$ are the corresponding eigenvectors. We can then expand the first equation of (3) into

$$
\frac{\partial u}{\partial t} = \nabla \cdot \left[ \lambda_{+, \lambda} \theta_{+, \lambda} \nabla u \right] + \nabla \cdot \left[ \lambda_{-, \lambda} \theta_{-, \lambda} \nabla u \right].
$$

Accordingly, it is clear from the expression of $\lambda_{+, \lambda}$ that $\lambda_{+} \geq \lambda_{-} > 0$, which means that the diffusion towards $\theta_{-}$ is privileged over $\theta_{+}$. In fact, the difference

$$
LlogL^{k_m} (\Omega) = \left\{ u: \Omega \rightarrow \mathbb{R} \mid \int_{\Omega \cap \{|u| \geq k_m\}} |u| \log (|u|) dx < \infty \right\}.
$$

Next, we define weak solutions for problem (3) on $Q_T = \Omega \times (0, T]$ with $T > 0$:

**Definition 1.** A function $u: \overline{Q}_T \rightarrow \mathbb{R}$ is a weak solution for problem (3) if the following conditions are satisfied:

(i) $u \in C([0, T]; L^2 (\Omega)) \cap L^1 (0, T; W^{1,1} (\Omega))$ with $\partial_{x_i} u \in L^{k_m} (\Omega)$ for $i = 1, 2$.

(ii) For any $\varphi \in C^1 (\overline{Q}_T)$ with $\varphi (., 0) = 0$, we have

$$
-\int_{\Omega} u_0 (x) \varphi (x, 0) dx + \int_0^T \int_{\Omega} [-u \varphi_t + D_{\nu u} \nabla u \cdot \nabla \varphi] dx dt = 0.
$$

Now, we state our main theorem.

**Theorem 1.** Under assumption (14), there exists a unique weak solution for initial-boundary value problem (3).
Inspired by [4], this paper will investigate the existence and uniqueness of weak solutions for problem (3) according to the following steps:

(i) First, we approximate nonlinear evolution problem (3) by nonlinear elliptic problems using an implicit iterative method (discretization in time-variable only), and then we prove the existence of a unique weak solution for each elliptic problem adopting a variational approach. These solutions constitute approximate solutions for problem (3).

(ii) Next, we show the uniqueness of solutions for initial-boundary value problem (3) using the monotonicity of the vector field \( D \nabla u \).

(iii) Finally, passing to limits in some a priori energy estimates and using the monotonicity condition (17), we demonstrate the existence of weak solutions for problem (3).

\[ D_{\xi_1} \xi_1 - D_{\xi_0} \xi_0 = \int_0^1 \int \left[ (\xi_1 \cdot e_1) (\xi_1 - \xi_0) \cdot e_1 \right] \, dt + \int_0^1 \int \left[ (\xi_1 \cdot e_2) (\xi_1 - \xi_0) \cdot e_2 \right] \, dt \]

Since \( \phi''(s) = g(s) + sg'(s) \), then we obtain

\[ D_{\xi_1} \xi_1 - D_{\xi_0} \xi_0 = \int_0^1 \int \left[ \phi''(\xi_1 \cdot e_1) (\xi_1 - \xi_0) \cdot e_1 \right] \, dt + \int_0^1 \int \left[ \phi''(\xi_1 \cdot e_2) (\xi_1 - \xi_0) \cdot e_2 \right] \, dt \]

We conclude then

\[ (D_{\xi_1} \xi_1 - D_{\xi_0} \xi_0) \cdot (\xi_1 - \xi_0) = \int_0^1 \phi''(\xi_1 \cdot e_1) (\xi_1 - \xi_0) \cdot e_1 \, dt + \int_0^1 \phi''(\xi_1 \cdot e_2) (\xi_1 - \xi_0) \cdot e_2 \, dt \]

which completes the proof.

Lemma 3. Uniform integrability and weak convergence [15]. Assume \( \Omega \subset \mathbb{R}^2 \) is bounded, and let \( \{u_i\}_{i=1}^{\infty} \) be a sequence of functions in \( L^1(\Omega) \) satisfying

\[ \sup_i \|u_i\|_{L^1(\Omega)} < \infty. \]

2. Preliminaries

In this section, we state some useful lemmas that will be used later in the proofs.

Lemma 1. For all \( a \geq 0 \) and \( b \geq 1 \), we have \( ab \leq a \exp (a) + b \log (b) \).

Proof. If \( b \leq \exp (a) \), then \( ab \leq a \exp (a) \leq a \exp (a) + b \log (b) \).

If \( \exp (a) < b \), then \( a \log (b) < a \exp (a) + b \log (b) \).

Lemma 2. Suppose \( \phi: \mathbb{R} \rightarrow \mathbb{R} \) is a \( C^2 \) convex function. Then, for all \( \xi_0, \xi_1 \in \mathbb{R}^2 \), we have

\[ (D_{\xi_1} \xi_1 - D_{\xi_0} \xi_0) \cdot (\xi_1 - \xi_0) \geq 0. \]

Proof. For each \( t \in [0, 1] \), we put \( \xi_t = (1-t)\xi_0 + t\xi_1 \). Then, we have

\[ \lim_{i \to \infty} \sup \int_{\Omega} |u_i| \, dx = 0. \]

Then, there exist a subsequence \( \{u_{i_j}\}_{j=1}^{\infty} \) and \( \bar{u} \in L^1(\Omega) \) such that

\[ u_{i_j} \rightharpoonup \bar{u}, \quad \text{weakly in } L^1(\Omega). \]
Lemma 4. Assume $\Omega \subset \mathbb{R}^2$ is bounded, and let $\{u_i\}_{i=1}^\infty$ be a sequence of functions in $L^1(\Omega)$ such that

$$\sup \int_{\Omega \cap \{|u_i| \geq k_m\}} |u_i| \log(|u_i|) \, dx < \infty. \quad (24)$$

Then, there exist a subsequence $\{u_j\}_{j=1}^\infty$ and $\bar{u} \in L^1(\Omega)$ such that

$$u_j \rightharpoonup \bar{u}, \quad \text{weakly in } L^1(\Omega), \quad (25)$$

with $\bar{u} \in L^{\log L^{k_m}}(\Omega)$.

Proof. Given $M > 0$, we may find an $l \geq k_m$ such that $M \leq s \log(s)$ for all $s \geq l$. Consequently,

$$\int_{\Omega} |u_j| \, dx = \int_{\Omega \cap \{|u_j| < k_m\}} |u_j| \, dx + \int_{\Omega \cap \{|u_j| \geq k_m\}} |u_j| \, dx \leq k_m |\Omega| + \frac{1}{M} \int_{\Omega \cap \{|u_j| \geq k_m\}} |u_j| \log(|u_j|) \, dx, \quad (26)$$

which implies that

$$\sup \int_{\Omega} |u_j| \, dx < \infty. \quad (27)$$

On the other hand, there exists a positive constant $C$ such that

$$\left| f'(|\bar{u}|) \right|_{[k_m \leq |\bar{u}| \leq N]} \in L^\infty(\Omega) \quad \text{and passing to limits as } j \to \infty, \quad \text{we get}$$

$$\int_{\Omega} f(|\bar{u}|) \, dx \leq \liminf_{j \to \infty} \int_{\Omega \cap \{|u_j| \geq k_m\}} f(|u_j|) \, dx < \infty. \quad (33)$$

Then, passing to limits as $N \to \infty$, we deduce

$$\int_{\Omega \cap \{|u| \geq k_m\}} |\bar{u}| \log(|\bar{u}|) \, dx < \infty. \quad (34)$$

It follows then $\bar{u} \in L^{\log L^{k_m}}(\Omega)$. This finishes the proof. \qed

3. Approximate Solutions

In this section, we will discretize the time-variable interval $[0, T]$ to get approximate solutions for problem (3). We denote $h = (T/N)$ with $N \in \mathbb{N}^*$, and we designate by $u_n$ an approximate solution at time $nh$. We define gradually from $n = 1, 2, \ldots, N$ the following elliptic problems:

$$\int_{\Omega \cap \{|u_n| \geq h\}} |u_n| \, dx \leq \frac{1}{M} \int_{\Omega \cap \{|u_n| \geq h\}} |u_n| \log(|u_n|) \, dx \leq \frac{1}{M} \int_{\Omega \cap \{|u_n| \geq h\}} |u_n| \log(|u_n|) \, dx \quad (28)$$

$$\leq \frac{C}{M} = c, \quad (29)$$

which is true for all $i$ and arbitrary $\varepsilon > 0$. It follows then that

$$\lim_{i \to \infty} \sup \int_{\Omega \cap \{|u_i| \geq h\}} |u_i| \, dx = 0. \quad (30)$$

Then, from Lemma 3, there exist a subsequence $\{u_j\}_{j=1}^\infty$ of $\{u_i\}_{i=1}^\infty$ and a function $\bar{u} \in L^1(\Omega)$ such that

$$u_j \rightharpoonup \bar{u}, \quad \text{weakly in } L^1(\Omega). \quad (31)$$

It remains to prove that $\bar{u} \in L^\log L^{k_m}(\Omega)$.

We know that the function $f(s) = s \log(s)$ for $s \geq 1$ is increasing and convex, and then the function $f(|u|)$ is also convex for all $s \geq 1$. Therefore, we obtain

$$f(|\bar{u}|) \leq f\left(|u_i|\right) + f'\left(|\bar{u}|\right)\left(|u| - u_i\right). \quad (32)$$

Integrating the above inequality over $\Omega_N \cap \{|u_i| \geq k_m\}$ with $\Omega_N = \Omega \cap \{k_m \leq |u| \leq N\}$, we have

$$\int_{\Omega_N} f\left(|\bar{u}|\right) \, dx \leq \int_{\Omega_N} f\left(|u_i|\right) \, dx + f'\left(|\bar{u}|\right)\int_{\Omega_N} \left(|u| - u_i\right) \, dx \quad (33)$$

and

$$\left\{ \begin{array}{l}
\frac{u_n - u_{n-1}}{h} - \nabla \cdot [D_{\nu u} \nabla u_n] = 0, \quad \text{in } \Omega, \\
\langle D_{\nu u} \nabla u_n, n \rangle = 0, \quad \text{on } \partial \Omega.
\end{array} \right. \quad (35)$$

To solve these equations step by step, we only need to prove the existence and uniqueness of weak solutions of the following elliptic problems:

$$\int_{\Omega \cap \{|u| \geq h\}} |u| \, dx \leq \int_{\Omega \cap \{|u| \geq h\}} |u| \log(|u|) \, dx \leq \frac{1}{M} \int_{\Omega \cap \{|u| \geq h\}} |u| \log(|u|) \, dx \quad (28)$$

$$\leq \frac{C}{M} = c, \quad (29)$$

where $h > 0$ and $u_0 \in L^2(\Omega)$. We have

$$\int_{\Omega \cap \{|u| \geq h\}} |u| \, dx \leq \int_{\Omega \cap \{|u| \geq h\}} |u| \log(|u|) \, dx \leq \frac{1}{M} \int_{\Omega \cap \{|u| \geq h\}} |u| \log(|u|) \, dx \quad (28)$$

$$\leq \frac{C}{M} = c, \quad (29)$$

and when $\varphi$ is a constant function, we obtain

$$\int_{\Omega} \frac{u - u_0}{h} \varphi \, dx + \int_{\Omega} D_{\nu u} \nabla u \cdot \varphi \, dx = 0. \quad (37)$$

Definition 2. A function $u \in L^2(\Omega) \cap W^{1,1}(\Omega)$ with $\frac{1}{\nu} u \in L^{\log L^{k_m}}(\Omega)$ for $i = 1, 2$ is called a weak solution for problem (36); if for any $\varphi \in C^1(\bar{\Omega})$, we have

\[ \int_{\Omega} \ud x = \int_{\Omega} u_0 \ud x. \]  

(38)

In order to prove the existence and uniqueness of weak solutions for problem (36), we consider the variational problem

\[ U = \left\{ u \in L^2(\Omega) \cap W^{1,1}(\Omega) \mid \partial_{x_i} u \in L\log L^{k_m}(\Omega) \right\} \]

for \( i = 1, 2 \), \( \int_{\Omega} \ud x = \int_{\Omega} u_0 \ud x \),

(40)

and when \( u \in U \), the functional \( E \) is defined as

\[ E(u) = \int_{\Omega} \left[ \phi\left( |u_{x_1}| \right) + \phi\left( |u_{x_2}| \right) + \phi\left( |u_{x_{1,2}}| \right) \right] + \frac{1}{2h} \int_{\Omega} (u - u_0)^2 \ud x. \]

(41)

It is easy to prove that (36) is the Euler–Lagrange equations of the functional \( E \) [16].

**Theorem 2.** Problem (36) has a unique weak solution.

**Proof.** Since

\[ 0 \leq \inf_{u \in U} E(u) \leq E(0) = \frac{1}{2h} \int_{\Omega} u_0^2 \ud x, \]

(42)

then we can construct a minimizing sequence \( \{u_q\}_{q=1}^{\infty} \) in \( U \) such that \( E(u_q) \to E(0) + 1 \) and

\[ \lim_{q \to \infty} E(u_q) = \inf_{u \in U} E(u). \]

(43)

Besides,

\[ \int_{\Omega^\prime} \left[ \phi\left( |\partial_{x_i} u_q| \right) \log\left( |\partial_{x_i} u_q| \right) \right] \ud x \leq C \int_{\Omega^\prime} \left[ \phi\left( |\partial_{x_i} u_q| \right) \right] \ud x \leq C E(u_q) < C(E(0) + 1) \]

(46)

with \( C = (\epsilon_0 + (1/A_{k_m})) > 0 \) and \( i = 1, 2 \). It follows then that for \( i = 1, 2 \),

\[ \sup_q \int_{\Omega^\prime} \left[ \phi\left( |\partial_{x_i} u_q| \right) \right] \ud x < \infty. \]

(47)

Therefore, thanks to Lemma 4 and the weak compactness of \( L^2(\Omega) \), we can find a subsequence \( \{u_{q_j}\}_{j=1}^{\infty} \) of \( \{u_q\}_{q=1}^{\infty} \) and a function \( u_1 \in L^2(\Omega) \cap W^{1,1}(\Omega) \) such that

\[ u_{q_j} \to u_1, \quad \text{weakly in } L^2(\Omega), \]

(48)

and for \( i = 1, 2 \),

\[ \partial_{x_i} u_{q_j} \to \partial_{x_i} u_1, \quad \text{weakly in } L^1(\Omega), \]

\[ \partial_{x_i} u_1 \in L\log L^{k_m}(\Omega). \]

(49)

Therefore, we have

\[ \int_{\Omega} u_1 \ud x = \lim_{j \to \infty} \int_{\Omega} u_{q_j} \ud x = \int_{\Omega} u_0 \ud x, \]

(50)

and following the reasoning in the proof of Lemma 4, it is easy to show that for any \( a \in \{x_1, x_2, x_{12}, x_{-12}\} \) and for a fixed \( \epsilon > 0 \), there exists \( l \geq k_m \) such that

\[ \min\left\{ E(u) \mid u \in U \right\}, \]

(39)
we can prove that

\[
\int_{\Omega \cap \{|u| \leq l\}} \phi(\partial_u u_1) \, dx \leq \liminf_{j \to \infty} \int_{\Omega \cap \{|u_{\eta j}| \leq l\}} \phi(\partial_u u_{\eta j}) \, dx.
\]

Similarly, since \( \phi \) is increasing and convex in \([0, l] \), then we can prove that

\[
\int_{\Omega \cap \{|u| < l\}} \phi(\partial_u u_1) \, dx \leq \liminf_{j \to \infty} \int_{\Omega \cap \{|u_{\eta j}| < l\}} \phi(\partial_u u_{\eta j}) \, dx.
\]

Therefore, we obtain from (52) and (53) that

\[
\int_\Omega \phi(\partial_u u_1) \, dx = \int_{\Omega \cap \{|u| < l\}} \phi(\partial_u u_1) \, dx + \int_{\Omega \cap \{|u| \geq l\}} \phi(\partial_u u_1) \, dx
\leq (\epsilon + A_{k, 2}) \left( \epsilon + \frac{1}{A_{k, 2}} \right) \liminf_{j \to \infty} \int_\Omega \phi(\partial_u u_{\eta j}) \, dx.
\]

Thus, by letting \( \epsilon \to 0 \), we get

\[
\int_\Omega \phi(\partial_u u_1) \, dx \leq \liminf_{j \to \infty} \int_\Omega \phi(\partial_u u_{\eta j}) \, dx,
\]
for any \( a \in \{x_1, x_2, x_{12}, x_{-12}\} \). It follows then that

\[
E(u_1) \leq \liminf_{j \to \infty} E(u_{\eta j}) = \inf_{u \in U} E(u),
\]
which signifies that \( u_1 \in U \) is a minimizer of the energy functional \( E(u) \), i.e.,

\[
E(u_1) = \inf_{u \in U} E(u).
\]

Furthermore, for all \( \phi \in C^1(\overline{\Omega}) \) and \( t \in \mathbb{R} \), we have \( u_1 + t(\phi - \phi_1) \in U \) with \( \phi_1 = (1/|\Omega|) \int_\Omega \phi \, dx \). Then, \( \rho(0) \leq \rho(t) \) where

\[
\rho(t) = E(u_1 + t(\phi - \phi_1)).
\]
Hence, we have \( \rho'(0) = 0 \), which means

\[
\int_\Omega \frac{u_1 - u_0}{h} (\phi - \phi_1) \, dx + \int_\Omega D_{\nabla u_1} \nabla \phi \, dx = 0.
\]

Thanks to Lemma 2, we deduce that

\[
\int_\Omega \frac{(\bar{u} - u)^2}{h} \, dx + \int_\Omega \left[ D_{\nabla \bar{u}} \nabla \bar{u} - D_{\nabla u_1} \nabla u_1 \right] : (\nabla \bar{u} - \nabla u_1) \, dx = 0.
\]

Because of (50), we get

\[
\int_\Omega \frac{u_1 - u_0}{h} \phi \, dx + \int_\Omega D_{\nabla u_1} \nabla u_1 \cdot \nabla \phi \, dx = 0.
\]

We conclude then that \( u_1 \) is a weak solution for problem (36).

Now, assume that there is another weak solution \( \bar{u} \) of (36). Then, for every \( \phi \in C^1(\overline{\Omega}) \), we have

\[
\int_\Omega \frac{\bar{u} - u_0}{h} \phi \, dx + \int_\Omega D_{\nabla \bar{u}} \nabla \bar{u} \cdot \nabla \phi \, dx = 0,
\]
which leads to

\[
\int_\Omega \frac{\bar{u} - u}{h} \phi \, dx + \int_\Omega \left[ D_{\nabla \bar{u}} \nabla \bar{u} - D_{\nabla u_1} \nabla u_1 \right] : (\nabla \bar{u} - \nabla u_1) \, dx = 0.
\]

Then, if we choose \( \phi = \bar{u} - u_1 \) as a test function in (62), we get

\[
\int_\Omega \frac{(\bar{u} - u_1)^2}{h} \, dx = 0.
\]
Therefore, \( \tilde{u} = u_1 \) a.e. in \( \Omega \).

In conclusion, we have shown that there exists a unique weak solution \( u_n \in U \) satisfying (35) for every \( n \in \{1, 2, \ldots, N\} \). Consequently, we define an approximate solution \( u_h \) for problem (3) as

\[
u_h(x, t) = \begin{cases} 
u_0(x), & t = 0, \\ u_i(x), & t \in (0, h], \\ \ldots, & \ldots, \\ u_j(x), & t \in ((j - 1)h, jh], \\ \ldots, & \ldots, \\ u_N(x), & t \in ((N - 1)h, T], \\ \end{cases}
\]

(65)

for every \( h = (T/N) \).

\[ \square \]

4. Existence and Uniqueness of Weak Solutions

Proof. of Theorem 1. In the beginning, we establish the uniqueness of solutions for problem (3). For this purpose, we suppose there exist two weak solutions \( u \) and \( v \) for problem (3). Then, we obtain the following:

\[
\begin{align*}
& \frac{\partial (u - v)}{\partial t} - \nabla \cdot [\mathbf{D}_{\nabla u} \nabla u - \mathbf{D}_{\nabla v} \nabla v] = 0, & \text{in } Q_T, \\
& \langle \mathbf{D}_{\nabla u} \nabla u - \mathbf{D}_{\nabla v} \nabla v, n \rangle = 0, & \text{on } \partial \Omega \times (0, T], \\
& (u - v)(x; 0) = 0, & \text{in } \Omega.
\end{align*}
\]

(66)

By multiplying the first equation of the above problem by \((u - v)\) and integrating over \( \Omega \) and \([0, T]\), we get

\[
\frac{1}{2} \int_{\Omega} (u - v)^2(t) \, dx + \int_0^T \int_{\Omega} \mathbf{D}_{\nabla u} \nabla u \cdot \nabla (u - v) \, dx \, dt = 0,
\]

(67)

for every \( t \in (0, T] \). Since the second term of the above equation is nonnegative (thanks to Lemma 2), it follows then \( u = v \) a.e. in \( Q_T \).

Let us now find our weak solution for problem (3). We intend to send \( h \) to zero and show that a subsequence of our solutions \( u_h \) of the approximate problems (35) converges to a weak solution for problem (3). To this end, we need to find some a priori estimates.

It follows from (35) that for every \( \varphi \in C^1(\Omega) \),

\[
\int_{\Omega} \frac{u_n - u_{n-1}}{h} \varphi \, dx + \int_{\Omega} \mathbf{D}_{\nabla u} \nabla u_n \cdot \nabla \varphi \, dx = 0.
\]

(68)

Then, by taking \( u_n \) as a test function in (68) and using \( u_{n-1} \leq \frac{((u_n^2 + u_{n-1}^2))}{2} \), we get

\[
\frac{1}{2} \int_{\Omega} u_n^2 \, dx + h \int_{\Omega} \mathbf{D}_{\nabla u} \nabla u_n \cdot \nabla u_n \, dx \leq \frac{1}{2} \int_{\Omega} u_{n-1}^2 \, dx.
\]

(69)

For each \( t \in (0, T) \), we can find \( j \in \{1, \ldots, N\} \) such that \( t \in ((j - 1)h, jh] \). Then, by adding all the inequalities (69) from \( n = 1 \) to \( n = j \), we get

\[
\text{sup}_{0 \leq t \leq T} \int_{\Omega} u_h^2(x, t) \, dx + \int_0^T \int_{\Omega} \mathbf{D}_{\nabla u} \nabla u_h \cdot \nabla u_h \, dx \, dt \leq 2 \int_{\Omega} u_0^2 \, dx.
\]

(73)

Recalling that \( 0 \leq \phi(s) \leq s \phi'(s) \) for all \( s \geq 0 \), then we can derive the following:

\[
\mathbf{D}_{\nabla u} \nabla u_h \cdot \nabla u_h = |\nabla u_h \cdot e_1| \phi'(|\nabla u_h \cdot e_1|) + |\nabla u_h \cdot e_2| \phi'(|\nabla u_h \cdot e_2|) + |\nabla u_h \cdot e_{12}| \phi'(|\nabla u_h \cdot e_{12}|) \geq \phi(|\nabla u_h \cdot e_1|) + \phi(|\nabla u_h \cdot e_2|) + \phi(|\nabla u_h \cdot e_{12}|).
\]

(74)
Besides, as in (46), for $|\partial_{x_i} u_h|, |\partial_{x_j} u_h| \geq k_m$, we may find a positive constant $C$ such that

$$|\partial_{x_i} u_h| \log \left( |\partial_{x_i} u_h| \right) + |\partial_{x_j} u_h| \log \left( |\partial_{x_j} u_h| \right) \leq C \left( \phi \left( |\nabla u_h \cdot e_1| \right) + \phi \left( |\nabla u_h \cdot e_2| \right) \right) \leq CD \nabla u_h \cdot \nabla u_h.$$  

(75)

Thus, we conclude

$$\begin{align*}
\sup_{0 \leq t \leq T} \int_{\Omega} u_h^2(x,t) dx &< \infty, \\
\int_0^T \int_{\Omega} \left[ |\partial_{x_i} u_h| \log \left( |\partial_{x_i} u_h| \right) \right] dx dt &< \infty, \\
\int_0^T \int_{\Omega} \left[ |\partial_{x_j} u_h| \log \left( |\partial_{x_j} u_h| \right) \right] dx dt &< \infty.
\end{align*}$$  

(76)

By Lemma 4, we can find a subsequence of $\{u_{h_k}\}$ (for simplicity, we also denote it by $u_k$) such that [17]

$$u_{h_k} \rightharpoonup u, \quad \text{weakly } * \text{ in } L^{\infty}(0,T; L^2(\Omega)),
$$

$$u_k \rightarrow u, \quad \text{weakly } L^1(0,T; W^{1,1}(\Omega)).$$  

(77)

with

$$\begin{align*}
\sup_{0 \leq t \leq T} \int_{\Omega} u^2(x,t) dx &< \infty, \\
\int_0^T \int_{\Omega} \left[ |\partial_{x_i} u| \log \left( |\partial_{x_i} u| \right) \right] dx dt &< \infty, \\
\int_0^T \int_{\Omega} \left[ |\partial_{x_j} u| \log \left( |\partial_{x_j} u| \right) \right] dx dt &< \infty.
\end{align*}$$  

(78)

So, it remains to prove that $u$ is just a weak solution for problem (3). Let us now denote $\xi_h = D_{\nabla u_h} \nabla u_h$. We will show that $\xi_h$ is bounded in $[L^2(Q_T)]^2$, so we may find a subsequence of $\xi_h$ that converges weakly in $[L^2(Q_T)]^2$ to a particular vector-valued function. Then, we will prove that this vector-valued function is equal almost everywhere to $D_{\nabla u_h} \nabla u_h$ through monotonicity condition (17).

From the expression of $D_{\nabla u_h}$, we can derive the following:

$$\begin{align*}
|\xi_h| &= \left| \frac{\partial_{x_i} u_h}{\partial_{x_i} u_h} \right| \phi \left( \left| \frac{\partial_{x_i} u_h}{\partial_{x_i} u_h} \right| e_1 \right) + \left| \frac{\partial_{x_j} u_h}{\partial_{x_j} u_h} \right| \phi \left( \left| \frac{\partial_{x_j} u_h}{\partial_{x_j} u_h} \right| e_2 \right) \\
&+ \left| \frac{\partial_{x_i} u_h}{\partial_{x_i} u_h} \right| \phi \left( \left| \frac{\partial_{x_i} u_h}{\partial_{x_i} u_h} \right| e_{12} \right) + \left| \frac{\partial_{x_j} u_h}{\partial_{x_j} u_h} \right| \phi \left( \left| \frac{\partial_{x_j} u_h}{\partial_{x_j} u_h} \right| e_{12} \right) \\
&\leq 4 \phi \left( \left| \frac{\partial_{x_i} u_h}{\partial_{x_i} u_h} \right| + \left| \frac{\partial_{x_j} u_h}{\partial_{x_j} u_h} \right| \right).
\end{align*}$$  

(79)

Given $\epsilon_1, \epsilon_2 > 0$, we may find $l_1 = l_2 = k_m$ such that

$$\phi'(s) \leq M \log(s),$$

$$s \leq \epsilon_2 s \log(s),$$  

(80)

for all $s \geq k_m$ with $M = (\epsilon_1 + A_{\log^2(\cdot)}).$ Thus, we can distinguish two cases:

(i) If $|\partial_{x_i} u_h| + |\partial_{x_j} u_h| < k_m$ then $|\xi h|^2 \leq (4 \phi'(k_m))^2$.

(ii) If $|\partial_{x_i} u_h| + |\partial_{x_j} u_h| \geq k_m$ then

Then, $\{\xi_h\}$ is bounded in $[L^2(Q_T)]^2$, which means that we can find a subsequence of $\{\xi_h\}$ (denote it also by $\{\xi_h\}$) and a function $\xi \in [L^2(Q_T)]^2$ such that

$$\xi_h \rightharpoonup \xi, \quad \text{weakly in } [L^2(Q_T)]^2.$$  

(82)

Since $s \rightarrow s \exp(s) (s \geq 0)$ is increasing and convex, then as in the proof of Lemma 4, we deduce that

$$\int_0^T \int_{\Omega} \left[ |\partial_{x_i} u_h| + |\partial_{x_j} u_h| \right] |\xi| \exp \left( \frac{|\xi|}{4M} \right) dx dt \leq$$

$$\liminf_{h \rightarrow 0} \int_0^T \int_{\Omega} \left[ |\partial_{x_i} u_h| + |\partial_{x_j} u_h| \right] |\xi_h| \exp \left( \frac{|\xi_h|}{4M} \right) dx dt < \infty.$$  

(83)

Then, by using Lemma 1, we get
\[
\int_0^T \int_\Omega |\xi \cdot \nabla u| dx \, dt \leq \int_0^T \int_\Omega |\xi||\nabla u| dx \, dt \\
\leq \int_0^T \int_\Omega |\xi| (|\partial_{x_1} u| + |\partial_{x_2} u|) dx \, dt \\
\leq k_m \int_0^T \int_{\Omega \cap \{|\partial_{x_1} u| + |\partial_{x_2} u| < k_m\}} |\xi| dx dt \\
+ \int_0^T \int_{\Omega \cap \{|\partial_{x_1} u| + |\partial_{x_2} u| \geq k_m\}} |\xi| \exp\left(\frac{|\xi|}{4M}\right) dx dt \\
+ 4M (1 + \varepsilon_2) \left[\int_0^T \int_{\Omega \cap \{|\partial_{x_1} u| \geq k_m\}} |\partial_{x_1} u| \log\left(1 + |\partial_{x_1} u|\right) dx dt \right] < \infty
\] (84)

It follows then \(\xi \cdot \nabla u \in L^1(Q_T)\). Next, we will show that
\(\xi = D_{\nabla u} \nabla u\) a.e. in \(Q_T\).

For each \(\phi \in C^1(Q_T)\) with \(\phi(., T) = 0\), we take \(\phi(x, nh)\) as a test function in (35):
\[
\int_\Omega \frac{u_n(x) - u_{n-1}(x)}{h} \phi(x, nh) dx + \int_\Omega D_{\nabla u_n} \nabla \phi(x, nh) dx = 0,
\] (85)
with \(n \in \{1, 2, \ldots, N\}\). By summing \(n\) from 1 to \(N\), we obtain
\[
\sum_{n=0}^{N-1} \int_\Omega \phi(x, nh) \frac{\phi(x, (n+1)h) - \phi(x, nh)}{h} dx = -\sum_{n=0}^{N-1} \int_\Omega \frac{\phi(x, t)}{h} \phi_t(x, t) dx dt
\]
\[
= -\frac{1}{h} \int_0^T \int_\Omega \phi_0(x, 0) dx + \frac{N}{h} \int_0^T \int_\Omega \phi(x, nh) dx dt.
\] (87)

Therefore,
\[
-\int_\Omega \phi_0(x) \phi(x, 0) dx - \int_0^T \int_\Omega \phi_t(x, t) \phi_t(x, t) dx dt + \int_0^T \int_\Omega D_{\nabla u_n} \nabla \phi(x, nh) \cdot \nabla \phi dx dt
+ \sum_{n=1}^N \int_\Omega \frac{\phi(x, nh)}{h} \frac{\phi(x, (n+1)h) - \phi(x, nh)}{h} dx dt = 0.
\] (88)

Letting \(h\) tend to zero, we get
\[
\int_\Omega \phi_0(x) \phi(x, 0) dx + \int_0^T \int_\Omega \phi_t(x, t) dx dt = \int_0^T \int_\Omega \xi \cdot \nabla \phi dx dt.
\] (89)
On the other hand, we let \( \nu \in L^1(Q_T) \) with
\[
\int_0^T \int_{\Omega} \left| \frac{\partial}{\partial t} v \right|^2 dx \, dt < \infty, \tag{90}
\]
for \( i = 1, 2 \). We sum up inequalities (69):
\[
\frac{1}{2} \int_{\Omega} u_h^2(T) \, dx + \int_0^T \int_{\Omega} D_{v_nh} \nabla u_h \cdot \nabla u_h dx \, dt \leq \frac{1}{2} \int_{\Omega} u_0^2 \, dx. \tag{91}
\]
We have from Lemma 2 that
\[
\int_0^T \int_{\Omega} \left( D_{v_nh} \nabla u_h - D_{v_nh} \nabla \nu \right) \cdot (\nabla u_h - \nabla \nu) dx \, dt \geq 0. \tag{92}
\]
Then, we obtain
\[
\frac{1}{2} \int_{\Omega} u_h^2(T) \, dx + \int_0^T \int_{\Omega} D_{v_nh} \nabla u_h \cdot \nabla \nu dx \, dt + \int_0^T \int_{\Omega} D_{v_nh} \nabla \nu \cdot \nabla u_h dx \, dt
\]
\[
- \int_0^T \int_{\Omega} D_{v_nh} \nabla \nu \cdot \nabla \nu dx \, dt \leq \frac{1}{2} \int_{\Omega} u_0^2 \, dx. \tag{93}
\]
Letting \( h \to 0 \) and noting that
\[
\int_{\Omega} u^2(T) \, dx \leq \liminf_{h \to 0} \int_{\Omega} u_h^2(T) \, dx, \tag{94}
\]
we obtain
\[
\frac{1}{2} \int_{\Omega} u^2(T) \, dx + \int_0^T \int_{\Omega} \xi \cdot \nabla \nu dx \, dt + \int_0^T \int_{\Omega} D_{\nu} \nabla \nu \cdot \nabla u dx \, dt
\]
\[
- \int_0^T \int_{\Omega} D_{\nu} \nabla \nu \cdot \nabla \nu dx \, dt \leq \frac{1}{2} \int_{\Omega} u_0^2 \, dx. \tag{95}
\]
By using \( \varphi = u \) in (89), we get
\[
\frac{1}{2} \int_{\Omega} u^2(T) \, dx + \frac{1}{2} \int_{\Omega} u_0^2 \, dx = \int_0^T \int_{\Omega} \xi \cdot \nabla \nu dx \, dt. \tag{96}
\]
Combining (95) with (96), we have
\[
\int_0^T \int_{\Omega} \left( \frac{\xi}{D_{\nu}} \nabla v \cdot (\nabla v - \nabla \nu) dx \, dt \leq - \int_{\Omega} \frac{u^2}{T} dx. \tag{97}
\]
Now, setting \( v = u + \lambda \omega \) for any \( \lambda > 0 \), \( \omega \in W^{1,2}(Q_T) \), we derive from the above inequality that
\[
\int_0^T \int_{\Omega} \left( \frac{\xi}{D_{\nu}} \nabla (u + \lambda \omega) \cdot \nabla \omega dx \, dt \leq 0. \tag{98}
\]
By letting \( \lambda \to 0 \) and using Lebesgue’s dominated convergence theorem, we obtain
\[
\int_0^T \int_{\Omega} \left( \frac{\xi}{D_{\nu}} \nabla u \cdot \psi dx \, dt = 0, \tag{99}
\]
for every \( \psi \in [L^2(\Omega)]^2 \). It follows then
\[
\xi = D_{\nu} \nabla u, \quad \text{a.e. in} \ Q_T. \tag{100}
\]
\[
\begin{aligned}
&\frac{\partial u_h}{\partial t} - \nabla \cdot \left[ D_{v_h} \nabla v_h - D_{v_u} \nabla u \right] = 0, \quad \text{in } \Omega \times (0, T], \\
&\langle D_{v_h} \nabla v_h - D_{v_u} \nabla u, n \rangle = 0, \quad \text{on } \partial \Omega \times (0, T], \\
w_h(x; 0) = u(x, h) - u_0(x), \quad \text{in } \Omega. \\
\end{aligned}
\]

For each \( t_0 \in [0, T] \), we may choose \( u_h \) as a test function in the first equation for problem (104) over \([0, t_0]:\)

\[
\int_\Omega |u(x, t_0 + h) - u(x, t_0)|^2 dx \leq \int_\Omega |u(x, h) - u_0(x)|^2 dx. 
\]

Because of Lemma 2, we deduce

\[
\int_\Omega \left| u(x, t_0 + h) - u(x, t_0) \right|^2 dx \leq \int_\Omega \left| u(x, h) - u_0(x) \right|^2 dx. 
\]

Now, in order to prove that \( u \in C([0, T], L^2(\Omega)) \), we need to prove

\[
\limsup_{n \to 0} \int_\Omega \left| u(x, h) - u_0(x) \right|^2 dx = 0. 
\]

We suppose that (107) is not true. Then, there exist a positive number \( \delta \) and a sequence \( \{h_i\} \) with \( h_i \to 0 \) as \( i \to \infty \) such that

\[
\lim_{h \to 0} \int_\Omega \left| u(x, h) - u_0(x) \right|^2 dx \geq \delta. 
\]

From estimate (72), we have

\[
\int_\Omega |u(x, h_i)|^2 dx \leq \int_\Omega |u_0(x)|^2 dx. 
\]

Then, from (108), we get

\[
\liminf_{h \to 0} \int_\Omega \left| u_0(x) \right|^2 dx - \int_\Omega u_0(x) u(x, h_i) dx \geq \frac{\delta}{2}. 
\]

From (109), we conclude that \( \{u(x, h_i)\} \) is a bounded sequence in \( L^2(\Omega) \). Then, we may find a subsequence (denote it also by \( \{u(x, h_i)\} \)) such that there exists a function \( \bar{u}_0 \in L^2(\Omega) \) such that

\[
\text{Since } u \in C(0, T; H^{-1}(\Omega)), \text{ it follows that } u(x, h_i) \rightharpoonup u_0, \quad \text{weakly in } H^{-1}(\Omega). 
\]

Therefore, we must have \( \bar{u}_0 = u_0 \), and since \( u \in C(0, T; H^{-1}(\Omega)) \), it follows that

\[
\text{Therefore, we conclude that (107) is true and } u \in C([0, T], L^2(\Omega)). \text{ This completes the proof of Theorem 1.} \]

\[\square\]

5. Numerical Implementation and Experimental Results

5.1. Consistent and Stable Symmetric Finite Difference Approximation. In this section, we provide a consistent and stable discretization scheme using symmetric finite difference approximation: at time \( t_n = n \delta_t, n \geq 0 \), and the mesh points \( x_i = i \delta, \ y_j = j \delta (0 \leq i \leq N + 1 \text{ and } 0 \leq j \leq M + 1) \), and we denote by \( u_{i,j} \) the finite difference approximation of \( u(x_i, y_j; t_n) \). The time-space derivatives are discretized in the following manner:
\[ u_{x_i}(x_j, y_j; t_n) = \frac{u(x_{i+1/2}, y_j; t_n) - u(x_{i-1/2}, y_j; t_n)}{\delta} + \mathcal{O}(\delta^2), \]
\[ u_{y_j}(x_j, y_j; t_n) = \frac{u(x_i, y_{j+1/2}; t_n) - u(x_i, y_{j-1/2}; t_n)}{\delta} + \mathcal{O}(\delta^2), \]
\[ u_{x_{i+1/2}}(x_j, y_j; t_n) = \frac{u(x_{i+1/2}, y_j; t_n) - u(x_{i-1/2}, y_j; t_n)}{\sqrt{2\delta}} + \mathcal{O}(\delta^2), \]
\[ u_{x_{i-1/2}}(x_j, y_j; t_n) = \frac{u(x_{i-1/2}, y_j; t_n) - u(x_{i+1/2}, y_j; t_n)}{\sqrt{2\delta}} + \mathcal{O}(\delta^2), \]
\[ u_t(x_i, y_j; t_{n+1}) = \frac{u(x_i, y_j; t_{n+1}) - u(x_i, y_j; t_n)}{\delta_t} + \mathcal{O}(\delta_t). \]

By assume \( \delta = 1 \) and denote

\[
\begin{aligned}
g^n_{N_{i,j}} &= g\left(\left|\Delta_N u^n_{i,j}\right|\right), \\
g^n_{E_{i,j}} &= g\left(\left|\Delta_E u^n_{i,j}\right|\right), \\
g^n_{S_{i,j}} &= g\left(\left|\Delta_S u^n_{i,j}\right|\right), \\
g^n_{W_{i,j}} &= g\left(\left|\Delta_W u^n_{i,j}\right|\right), \\
g^n_{NE_{i,j}} &= g\left(\left|\Delta_{NE} u^n_{i,j}\right|\right), \\
g^n_{SE_{i,j}} &= g\left(\left|\Delta_{SE} u^n_{i,j}\right|\right), \\
g^n_{SW_{i,j}} &= g\left(\left|\Delta_{SW} u^n_{i,j}\right|\right), \\
g^n_{NW_{i,j}} &= g\left(\left|\Delta_{NW} u^n_{i,j}\right|\right),
\end{aligned}
\]

Then, we may approximate problem (3) using the above scheme to obtain the following nonlinear diffusion filter:

\[
\begin{aligned}
u_{i,j}^{n+1} &= u_{i,j}^n + \delta_t \left[ g_N \Delta_N u^n + g_E \Delta_E u^n + g_S \Delta_S u^n + g_W \Delta_W u^n + \frac{g_{NE} \Delta_{NE} u^n + g_{SE} \Delta_{SE} u^n + g_{SW} \Delta_{SW} u^n + g_{NW} \Delta_{NW} u^n}{2} \right]_{i,j},
\end{aligned}
\]
for $1 \leq i \leq N$, $1 \leq j \leq 1 \leq i \leq M$, and $n \geq 0$, with the initial condition $u_{i,j}^0$ and the discrete Neumann boundary condition:

$$
\begin{align*}
&u_{0,j}^n = u_{1,j}^n, u_{N+1,j}^n = u_{N,j}^n, \quad \text{for } 1 \leq j \leq M, \\
&u_{i,0}^n = u_{i,1}^n, u_{i,M+1}^n = u_{i,M}^n, \quad \text{for } 1 \leq i \leq N, \\
&u_{0,0}^n = u_{1,1}^n, u_{N+1,0}^n = u_{N,1}^n, \\
&u_{0,M+1}^n = u_{1,M}^n, u_{N+1,M+1}^n = u_{N,M}^n.
\end{align*}
$$

A unique sequence $(u^n)_{n \in \mathbb{N}}$ is produced when using filter (116) on a particular initial image $u^0$ [2]. Besides, due to the continuity of the function $g$, the sequence $u^n$ depends continuously on $u^0$ for every finite $n$. Furthermore, equation (116) satisfies the following maximum-minimum principle, which describes a stability condition for the discrete scheme.

**Theorem 3.** Discrete extremum principle [1, 2].

For an iteration step satisfying

$$0 < \delta_i < \frac{1}{6g(0)},$$

scheme (116) satisfies

$$\min_{i,j} u_{i,j}^0 \leq u_{i,j}^n \leq \max_{i,j} u_{i,j}^0,$$

for all $1 \leq i \leq N$, $1 \leq j \leq M$, and $n \in \mathbb{N}$.

5.2. Experimental Results. This section will show the performance of proposed diffusion filter (116) in the image denoising process, under the boundary and initial conditions (117) while respecting the requirements concerning $\phi$ (Section 1), and $\delta_i$ (118). We will use the Peak Signal-to-Noise Ratio (PSNR that is a positive value) [18] and the Structural SIMilarity Index (SSIM that lies in $(0,1)$) [19] to evaluate the quality of the restored images. The best results for the denoising process are equivalent to the higher value of these metrics.

For comparative purposes, we will examine the proposed diffusion function with another one that has the same properties using the same filter (116). Therefore, we will use the following diffusion functions:

(i) The proposed diffusion function ($m = 1$ for instance):

$$\begin{align*}
g(s) &= \left\{ \begin{array}{ll}
p_0 P_{1,0k}(s) + v_k P_{2,0k}(s) + p_k P_{1,\infty}(s) + v_k P_{2,\infty}(s), & s \in [0,k], \\
p_k g_{k,1}(s) + v_k g_{k,2}(s), & s \in [k,\infty].
\end{array} \right.
\end{align*}$$

(ii) The Wang and Zhou diffusion function (WZ) [4]:

$$g(s) = \frac{1}{s+1} + \frac{\log(s+1)}{s}.$$  

Additionally, we will consider real test images Figure 1 and evaluate our model's performance on these images, which will be corrupted with different levels of Gaussian white noises with zero mean and variance $\sigma^2$.

Table 1 shows the quantitative results on real images, corrupted with various Gaussian noises, filtered by discrete model (116) using proposed diffusion function (120) and the one proposed by WZ (121). These results are obtained using the optimal parameters determined experimentally, as in Table 2 for each diffusion function.

It can be seen from Table 1 and Figure 2 that the proposed model shows remarkable results against the WZ model. From a visual comparison, Figure 2 shows that the restored images using the proposed diffusion function have considerable noise removal and preserve the image essential features better than the restored images by the WZ diffusion function. Besides, compared with the WZ diffusion function,
Table 1: PSNR and SSIM values of the images in Figure 1 affected by different values of Gaussian noise $\sigma^2$ and their corresponding iteration number for both functions.

| $\sigma^2$ | PSNR | SSIM | PSNR | SSIM | Iter | PSNR | SSIM | Iter |
|------------|------|------|------|------|------|------|------|------|
| Patient30  |      |      |      |      |      |      |      |      |
| 0.005      | 23.4311 | 0.3426 | 33.3757 | 0.9114 | 30 | **33.8333** | **0.9372** | 13 |
| 0.010      | 20.6839 | 0.2432 | 31.3367 | 0.8766 | 41 | **31.5737** | **0.9145** | 21 |
| 0.015      | 19.1190 | 0.1960 | 30.1565 | 0.8521 | 48 | **30.2008** | **0.8980** | 24 |
| 0.020      | 17.9763 | 0.1660 | 29.0905 | 0.8298 | 56 | **29.0007** | **0.8825** | 26 |
| 0.100      | 12.0756 | 0.0956 | **22.6821** | 0.6535 | 117 | **21.8953** | **0.7389** | 56 |
| Patient50  |      |      |      |      |      |      |      |      |
| 0.005      | 23.6686 | 0.4424 | 31.1185 | 0.8708 | 24 | **31.2934** | **0.8769** | 11 |
| 0.010      | 20.7561 | 0.3278 | 29.0893 | 0.8186 | 35 | **29.1527** | **0.8258** | 19 |
| 0.015      | 19.1313 | 0.2707 | 27.9137 | 0.7820 | 41 | **27.8981** | **0.7920** | 26 |
| 0.020      | 17.9838 | 0.2334 | 27.0011 | 0.7508 | 47 | **26.9714** | **0.7625** | 27 |
| 0.100      | 11.9985 | 0.0911 | **21.4469** | 0.5585 | 93 | **20.9674** | **0.6237** | 62 |
| Patient55  |      |      |      |      |      |      |      |      |
| 0.005      | 24.0179 | 0.3867 | 31.3190 | 0.9021 | 26 | **31.4668** | **0.9258** | 19 |
| 0.010      | 21.2303 | 0.2892 | 29.1310 | 0.8600 | 36 | **29.0997** | **0.8887** | 35 |
| 0.015      | 19.5990 | 0.2403 | **27.6640** | 0.7938 | 44 | **27.5635** | **0.8305** | 27 |
| 0.020      | 18.4292 | 0.2096 | 26.7717 | 0.7938 | 49 | **26.5468** | **0.8305** | 42 |
| 0.100      | 12.2234 | 0.0882 | **20.6948** | 0.5642 | 98 | **20.1755** | **0.6237** | 99 |

Table 2: The best possible parameters for different diffusion functions.

| $\sigma^2$ | $\delta_1$ | $\delta_k$ | $k$ | $p_0$ | $p_k$ | $v_0$ | $v_k$ |
|------------|-------------|------------|-----|-------|-------|-------|-------|
| Patient30  |            |            |     |       |       |       |       |
| 0.005      | 0.08331    | 0.14701    | 4.61411 | 1.13191 | 0.66151 | -0.00011 | -0.10441 |
| 0.010      | 0.08331    | 0.14991    | 5.00191 | 1.10891 | 0.45921 | -0.00011 | -0.04351 |
| 0.015      | 0.08331    | 0.15051    | 5.20221 | 1.10601 | 0.45671 | -0.00021 | -0.03851 |
| 0.020      | 0.08331    | 0.14701    | 5.86411 | 1.13281 | 0.46991 | -0.00021 | -0.03961 |
| 0.100      | 0.08331    | 0.15001    | 5.08081 | 1.10111 | 0.45791 | -0.00091 | -0.03951 |
| Patient50  |            |            |     |       |       |       |       |
| 0.005      | 0.08331    | 0.14941    | 3.18011 | 1.09241 | 0.56271 | -0.00021 | -0.01151 |
| 0.010      | 0.08321    | 0.16231    | 1.89931 | 1.00171 | 0.56991 | -0.00001 | -0.00201 |
| 0.015      | 0.08331    | 0.16591    | 2.06951 | 0.99011 | 0.56981 | -0.00011 | -0.11471 |
| 0.020      | 0.08331    | 0.16101    | 1.79891 | 1.00001 | 0.56981 | -0.00011 | -0.01051 |
| 0.100      | 0.08331    | 0.15301    | 3.87001 | 0.98701 | 0.55511 | -0.00031 | -0.13411 |
| Patient55  |            |            |     |       |       |       |       |
| 0.005      | 0.08321    | 0.14681    | 3.50081 | 1.12891 | 0.58971 | -0.00021 | -0.16571 |
| 0.010      | 0.08331    | 0.14701    | 3.31431 | 1.12621 | 0.58281 | -0.00131 | -0.17311 |
| 0.015      | 0.08331    | 0.15141    | 3.09991 | 1.09021 | 0.58881 | -0.00401 | -0.18991 |
| 0.020      | 0.08331    | 0.14881    | 3.39691 | 1.08721 | 0.53891 | -0.00011 | -0.15801 |
| 0.100      | 0.08331    | 0.14991    | 2.70091 | 1.08901 | 0.56861 | -0.00021 | -0.20951 |

Figure 2: Continued.
the results from Table 1 prove that the suggested approach has higher values in SSIM, whereas the WZ model shows significant results in PSNR while $\sigma^2$-value increases.

6. Conclusion

This paper principally investigates the class of anisotropic diffusion partial differential equations related to image processing and analysis. The existence and uniqueness of weak solutions for this problem have been proven under sufficient conditions satisfied by $\phi$. A consistent and stable numerical approximation has been applied, and a discrete nonlinear filter has been tested and revealed its efficiency in the image restoration field.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] P. Perona and J. Malik, “Scale-space and edge detection using anisotropic diffusion,” IEEE Transactions on Pattern Analysis and Machine Intelligence, vol. 12, no. 7, pp. 629–639, 1990.
[2] J. Weickert, Anisotropic Diffusion in Image Processing, Springer Vieweg Verlag, Wiesbaden, Germany, 1998.
[3] G. Aubert and P. Kornprobst, Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations, Springer, New York, NY, USA, 2006.
[4] L. Wang and S. Zhou, “Existence and uniqueness of weak solutions for a nonlinear parabolic equation related to image analysis,” Journal of Partial Differential Equations, vol. 19, no. 2, pp. 97–112, 2006.
[5] L. C. Evans, Partial Differential Equations, American Mathematical Society, Providence, RI, USA, 2010.
[6] N. Biranvand and A. Salari, ”Energy estimate for impulsive fractional advection dispersion equations in anomalous diffusions,” Journal of Nonlinear Functional Analysis, vol. 2018, pp. 1–17, 2018.
[7] K. P. P. Candito and U. Guarnotta, “Two solutions for a parametric singular $p$-laplacian problem,” Journal of Nonlinear and Variational Analysis, vol. 4, pp. 455–468, 2020.
[8] Z. Feng and Z. Yin, “On weak solutions for a class of nonlinear parabolic equations related to image analysis,” Nonlinear Analysis: Theory, Methods & Applications, vol. 71, no. 7-8, pp. 2506–2517, 2009.
[9] P. Chen, ”Existence and uniqueness of weak solutions for a class of nonlinear parabolic equations,” Electronic Research Announcements in Mathematical Sciences, vol. 24, pp. 38–52, 2017.
[10] S. Li and P. Li, ”Weak solutions for a class of generalised image restoration models,” International Journal of Dynamical Systems and Differential Equations, vol. 8, no. 3, p. 190, 2018.
[11] T. Humphries, M. Loreto, B. Halter, W. O’Keeffe, and L. Ramírez, ”Comparison of regularized and superiorized methods for tomographic image reconstruction,” Journal of Applied and Numerical Optimization, vol. 2, pp. 77–99, 2020.
[12] M. Jourhanme, Méthodes numériques de résolution d’un problème d’électro-encéphalographie, Ph.D. thesis, University of Rennes 1, Rennes, France, 1993.
[13] G. Aubert and L. Vese, “A variational method in image recovery,” SIAM Journal on Numerical Analysis, vol. 34, no. 5, pp. 1948–1979, 1997.
[14] J. Stoer and R. Bulirsch, Interpolation, pp. 37–144, Springer, New York, NY, USA, 2002.
[15] L. C. Evans and R. F. Gariepy, Measure Theory and Fine Properties of Functions, Taylor & Francis Group, New York, NY, USA, 2015.
[16] A. Tiarimti Alaoui and M. Jourhmane, "Existence and uniqueness of weak solutions for a new class of convex optimization problems related to image analysis," 2020.

[17] L. C. Evans, Weak Convergence Methods for Nonlinear Partial Differential Equations, American Mathematical Society, Providence, RI, USA, 1990.

[18] R. C. Gonzalez and R. E. Woods, Digital Image Processing, Prentice-Hall, Upper Saddle River, NJ, USA, 3rd edition, 2006.

[19] Z. Wang, A. C. Bovik, H. R. Sheikh, and E. P. Simoncelli, "Image quality assessment: from error visibility to structural similarity," IEEE Transactions on Image Processing, vol. 13, no. 4, pp. 600–612, 2004.

[20] F. Gaillard, "Normal brain mri," 2016.

[21] F. Gaillard, "Normal brain mri (tle protocol)," 2015.

[22] F. Gaillard, "Normal Mri Brain including MR venogram," 2017.