Singularly Perturbed Solutions for a Class of Thermoelastic Weakly Coupled Problems

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Abstract. Based on the basic equation of Green Lindsay (G-L) theory, the thermoelastic weak coupling problem under the basic equation is discussed, that is, two thermal relaxation parameters are added to the constitutive equation, the influence of the coupling term on the temperature field and elastic field is considered, and the asymptotic solution of the governing equation is constructed. Firstly, in order to obtain the asymptotic solution, the singularly perturbed expansion method is used. Then, combined with the corresponding boundary conditions, the partial differential equation method is used to solve the external solution and the boundary layer correction term. Secondly, in the case of weak coupling, the uniformly efficient estimation of the remainder of the asymptotic solution is obtained by using Gronwall inequality, so as to obtain the uniformly efficient of the formal asymptotic solution. Finally, the first term of the asymptotic solution is numerically analyzed by using the singularly perturbed numerical method. The present work will be conducive to the analysis of thermoelastic processes and numerical simulation of different materials in the case of weak coupling.

Keywords. Thermoelastic coupling, singular perturbation, numerical solution, uniform valid estimate.

1. Introduction

In thermoelastic theory, with the emergence of ultrashort laser pulses, non Fourier law has a significant effect on the generation of heat flow, and then on the elastic field, and no longer meets the conditions of classical thermoelastic theory. Based on the green Naghdi (G-N) theory of type III, Othman et al. [1] studied the effect of laser pulse on the heat load of three-dimensional generalized micropolar thermoelastic homogeneous isotropic medium, solved the dimensionless equation of the problem by using normal analysis technology. Rajneesh et al.[2] studied the homogeneous isotope micropolar porous thermoelastic circular plate by using the eigenvalue method in the thermoelastic three-phase lag theory caused by thermomechanical source. The expressions of each component were obtained in the transformation domain by using Laplace transform and Hankel transform. Paul et al. [3] studied the thick plate with diffusion of two-dimensional magnet thermoelastic problem, the analytical solution was deduced in Laplace Fourier transform domain, and obtained the numerical inversion of stress, displacement and chemical potential in space-time domain. Shakeriaski et al. [4] studied the nonlinear thermoelastic response of elastic media heated by short laser pulses, established the nonlinear coupled L-S-type generalized thermoelastic equation by using the finite strain theory (FST), and solved the time-dependent nonlinear equations by using Newmark's numerical time integration technique and an updated finite element algorithm.

In recent years, many scholars have modified and extended the classical thermoelastic coupling theory by adding thermal relaxation parameters in combination with practical problems. Tang et al. [5]
obtained the combined form of the basic equations of L-S theory, extended L-S theory and G-L theory. On this basis, the thermoelastic coupling problem of semi infinite medium surface after instantaneous heating is considered, and the equations of motion and energy are solved by numerical inverse Laplace transform. El-Maghraby et al. [6] obtained the state space equation of one-dimensional generalized thermoelastic problem with a thermal relaxation parameter based on L-S theory. Youssef et al. [7, 8] established a new generalized thermoelastic theory with two thermal relaxation parameters, obtained the uniqueness theorem of the two temperature generalized linear thermoelastic equation of a homogeneous isotropic object, and used this theory to solve the thermoelastic interaction problem of a cylindrical cavity subjected to thermal shock on the boundary surface and a moving heat source with constant velocity in an elastic infinite medium. Kumar et al. [9-11] studied the thermoelastic interaction under the dual temperature G-L theory, deeply analyzed the plane harmonic in the dual temperature thermoelastic medium with two relaxation parameters, obtained the dispersion relation solution of its propagating in the medium. Shivay et al. [12] established the temperature rate dependent two temperature thermoelasticity theory which based on the generalized thermodynamic law, obtained a new and more general two temperature relationship, and proved the uniqueness of the solution of the general mixed initial boundary value problem for anisotropic media under that theory. Jangid [13] considered homogeneous and anisotropic materials under the background of this model, based on the generalized thermodynamic law, a modified trdt theory with a new two temperature relationship is established, and an alternative equation for the mixed boundary value initial value problem is proposed. Using this equation, the convolution variational principle is derived, and the reciprocal principle of this theory is derived. Zhang et al. [14] proposed a thermal mechanical coupling propagation model based on non Fourier heat conduction, established a dimensionless three-dimensional generalized thermal mechanical coupling equation of equal diameter two spherical particles and the corresponding initial boundary value conditions.

The exact analytical solution of ultrasonic wave excited by pulsed laser in materials is very difficult, because this physical process includes not only transient thermal diffusion, but also transient elastic wave excitation and propagation in finite space. The singular perturbation method is mainly used to solve the problem with small parameters in the high-order term. When the thermal relaxation parameter is added to the thermoelastic coupling problem, the singular perturbation method can be used to analyze this kind of problem. Therefore, it is very necessary to study it from the perspective of theoretical research and practical application.

In this paper we describe the temperature field generated by pulsed laser acting on the material surface by using the non Fourier heat conduction law, and the thermoelastic coupling model is established in conjunction with the motion equation in G-L theory. It is considered that the coupling coefficient has a positive linear relationship with the time of heat flow relaxation. Using the singularly perturbed asymptotic expansion method, the internal and external solutions of the equation are constructed. The solution of the coupled equation is expanded into parabolic equation and hyperbolic equation. The formal asymptotic solutions are solved one by one through the corresponding boundary conditions. The uniform validity of the remainder of the asymptotic solution is obtained by Gronwall inequality and $L^2$ estimation, so as to obtain the distribution of temperature field and elastic field in the bounded region. Then the numerical analysis method is used to approximate the asymptotic solution equation, and the effectiveness of the numerical method is verified by a numerical example.

2. Governing Equations
In this paper, we consider a class of the Green-Lindsay models with weak coupling, so as to study the properties of temperature field and elastic field of thermoelastic plane wave from uniform isotropic heat conduction in finite solid space. According to reference [5], the basic governing equations of the homogeneous isotropic thermal conductive elastic medium model (when the density and thermal conductivity remain unchanged) are as follows (in general Cartesian coordinates system):

The strain-displacement relation:
\[ e_{i,j} = \frac{1}{2}(u_{i,j} + u_{j,i}) \] (1)

The constitutive relations:

\[ \sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} - \gamma(T - T_0 + \tau)\delta_{ij} \] (2)

The equation of motion:

\[ \sigma_{ij} = \rho \ddot{u} \] (3)

Cattaneo-Vernotte (C-V) non-Fourier heat flow equation:

\[ q_i + \tau_i \dot{q}_i = -k\dot{T}_i \] (4)

Combining equation (2) and (3), we get the movement displacement equation is as follows:

\[ \rho \ddot{u} = (\lambda + \mu u_{i,j} + \mu u_{j,i} - (3\lambda + 2\mu)\alpha \Phi \biggl( \frac{\ddot{u} + \dddot{T}}{\rho} \biggr) \] (5)

The energy balance equation which is derived from Cattaneo-Vernotte (C-V) non-Fourier heat flow equation:

\[ k \dot{T}_i = \rho C \dot{\tau}_i - \dot{T}_0 \biggl( \tau_0 + \gamma \dot{\tau}_0 \biggr) \] (6)

In the above equations, \( u_i \) are the displacement components, \( e_{ij} \) are the components of the elastic strain tensor, \( \sigma_{ij} \) are the components of the elastic stress tensor, \( \lambda \) and \( \mu \) are Lamé’s constants, \( \alpha \) is the coefficient of linear thermal expansion, \( \gamma = (3\lambda + 2\mu)\alpha \) is the parameter of thermoelastic coupling, \( \rho \) is the constant mass density, \( C \) is the specific heat at constant strain, \( q_i \) are the components of the heat flux vector, \( k \) is the thermal conductivity of the materia, \( \tau_0 \) and \( \tau_1 \) are the thermal relaxation parameters.

3. Theoretical Analysis

One dimensional temperature and deformation fields are considered, for simplifications we will use the following non-dimensional variables:

\[ x' = \left( \frac{\lambda + 2\mu}{\rho} \right) \frac{k}{\rho \gamma \kappa} x, \quad U' = U' = \left( \frac{\lambda + 2\mu}{\rho} \right) \frac{\rho C E}{(3\lambda + \mu)\alpha T_0 \kappa} \] (7)

Equations (5)-(6) in the non-dimensional forms (after suppressing the primes) reduce to:

\[ U' - U = \dot{U} + \varepsilon_i \dot{U}, \quad \dot{\theta} = \dot{\theta} + \varepsilon_\theta \dot{U} + \delta U, \quad \sum = U' - (f + \varepsilon_\theta \phi) \]

where \( \varepsilon_i = \left( \frac{\lambda + 2\mu}{\rho} \right) \frac{\rho C E}{k} \tau_i \), \( \delta = \left( \frac{3\lambda + 2\mu}{\rho} \right) \frac{\alpha \gamma}{(\lambda + 2\mu)\rho C E} \), we order \( \sigma_0 = \varepsilon_1 = \varepsilon_\theta = \gamma \), then considering an initial values and boundary conditions, thus, we can rewrite the equation (7) as:
\begin{equation}
U_{xx} = U_{\theta} \theta + \varepsilon \theta_{x0}
\end{equation}
\begin{equation}
\theta_{xx} = \theta + \varepsilon \theta_{x0} + \beta \varepsilon U_{xx}
\end{equation}
\begin{equation}
\theta(x,t)|_{t=0} = \varphi_{0}(x)
\end{equation}
\begin{equation}
\theta(x,t)|_{x=0} = \varphi_{x0}(x)
\end{equation}
\begin{equation}
\theta(x,t)|_{x=L} = \varphi_{Lx}(x)
\end{equation}
\begin{equation}
U(x,t)|_{t=0} = \psi_{0}(x)
\end{equation}
\begin{equation}
U(x,t)|_{x=0} = \psi_{x0}(x)
\end{equation}
\begin{equation}
U(x,t)|_{x=L} = \psi_{Lx}(x)
\end{equation}

where \( x \in [0, L], t \in (0, T) \). We make the following assumptions:

\[ [H_1] \quad (x,t) \in \Omega, \Omega = \bar{\Omega} \times (0, T), \Omega = (0, L), \text{L and T are finite values} \]

\[ [H_2] \quad \varphi_1(x), \varphi_2(x), \psi_1(x), \psi_2(x) \text{are some known arbitrary functions} \]

\[ [H_3] \quad \varphi_3(t), \varphi_4(t), \psi_3(t), \psi_4(t) \text{are any known function of at least first order differentiable} \]

By canonical expansion of the solution of equation (7), we obtain

\begin{equation}
U(x,t) = \sum_{i=0}^{N} \varepsilon^{i} U_i(x,t) \quad \theta(x,t) = \sum_{i=0}^{N} \varepsilon^{i} \theta_i(x,t)
\end{equation}

Comparing the same power of \( \varepsilon \), we have:

\begin{equation}
U_{0 xx} = U_{\theta} \theta \quad \text{0 xx} = \theta \quad \theta_{0 xx} = \theta \quad \theta_{0 x0} = \varphi \quad \theta_{0 Lx} = \varphi \quad \theta_{x0} = \varphi \quad \theta_{Lx} = \varphi 
\end{equation}

By separating variables to solve the equation (10), we can obtain:

\begin{equation}
\theta_{i}(x,t) = V(x,t) + \sum_{n=1}^{\infty} A_{n} \sin \frac{n \pi x}{L} + B_{n} \cos \frac{n \pi x}{L}
\end{equation}

By separating variables to solve the equation (11), we can obtain:

\begin{equation}
U_{0} = W(x,t) + \sum_{n=1}^{\infty} \left( B_{n} \cos \frac{n \pi x}{L} + C_{n} \sin \frac{n \pi x}{L} \right) \sin \frac{n \pi x}{L} + \sum_{n=1}^{\infty} \frac{1}{\pi} \int_{0}^{\tau} \tilde{W}_{n}(\tau) \sin \frac{n \pi (\tau-t)}{L} \, d\tau \sin \frac{n \pi x}{L}
\end{equation}
where

\[ V(x,t) = \frac{\varphi_1(t) - \varphi_2(t)}{L} + \varphi_3(t) \quad A_n = \frac{2}{L} \int_0^L (\varphi_1(x) - V(x,0)) \sin \frac{n \pi x}{L} \, dx \]

\[ \bar{V}(x,t) = V(x,t) = \sum_{n=1}^{\infty} \bar{V}_n(t) \sin \frac{n \pi x}{L} \]

\[ W(x,t) = \frac{\psi_1(t) - \psi_2(t)}{L} x + \psi_3(t) \quad B_n = \frac{2}{L} \int_0^L (\psi_1(x) - W(x,0)) \sin \frac{n \pi x}{L} \, dx \quad C_n = \frac{2}{L} \int_0^1 (\psi_2(x) - W(x,0)) \sin \frac{n \pi x}{L} \, dx \]

\[ \bar{W}(x,t) = W_n(t) + \theta_{0\alpha} = \sum_{n=1}^{\infty} \bar{W}_n(t) \sin \frac{n \pi x}{L} \]

Now give the composite expansion of the solution:

\[ U^* = \sum_{i=0}^{N} \varepsilon^i U_i(x,t) + \sum_{i=0}^{N-1} \varepsilon^{i+1} P_j(x,t)e^{-\varepsilon^j} \quad \theta^* = \sum_{i=0}^{N} \varepsilon^i \theta_i(x,t) + \sum_{i=0}^{N-1} \varepsilon^{i+1} Q_j(x,t)e^{-\varepsilon^j} \quad (14) \]

Combining initial values and boundary conditions, we can be concluded that \( P_0(x,t) = P_1(x,t) = P_2(x,t) = 0 \), then, comparing the same power of \( \varepsilon \), we have:

\[ -Q_{0,xx} = Q_{xx}, \quad Q_0(x,t)|_{t=0} = 0 \quad \theta(x,t)|_{t=0} = \varphi_2(x) \quad (15) \]

\[ Q_{j+1,xx} = -Q_{j+1,tt} + \beta P_{j+1} + \beta P_{j+1} \quad \theta_{j+1}(x,t)|_{t=0} = 0 \]

\[ Q_{j+1}(x,t)|_{t=L} = 0 \quad Q_{j+1}(x,t)|_{t=0} = 0 \quad (16) \]

\[ P_{j+3} = P_{j+1,x} \alpha P_{j+1} + \beta P_{j+1,t} \quad (17) \]

By separating variables, we can obtain

\[ Q_0(x,t) = \sum_{n=1}^{\infty} D_n e^{n^2 \pi^2 \varepsilon^j} \sin \frac{n \pi x}{L} \sin \frac{n \pi x}{L} \quad (18) \]

where \( D_n = \frac{2}{L} \int_0^L (\varphi_2(x) - \varphi_2(x)) \sin \frac{n \pi x}{L} \, dx \), according to reference [15], on the bound domain, the existence and uniqueness of equation (11) and equation (16) can be obtained.

4. Remainder Estimation

Lemma 1. The remainder of solution of equation (8)

\[ U(x,t,\varepsilon) = \sum_{i=0}^{N} \varepsilon^i U_i(x,t) + \sum_{i=0}^{N-1} \varepsilon^{i+1} P_j(x,t)e^{-\varepsilon^j} + \varepsilon^N R \]

\[ \theta(x,t,\varepsilon) = \sum_{i=0}^{N} \varepsilon^i \theta_i(x,t) + \sum_{i=0}^{N-1} \varepsilon^{i+1} Q_j(x,t)e^{-\varepsilon^j} + \varepsilon^N S \quad (19) \]

Satisfies
Then, the formal asymptotic solution of equation (8) is uniformly valid. 

**Proof.** Replace equation (19) with equation (8), we have

\[
\begin{align*}
R_\alpha &= R_\alpha^0 + \varepsilon S_\alpha + \varepsilon S_\beta + F_1, \\
S_\alpha &= \varepsilon S_\alpha^0 + S_\beta + \beta \varepsilon S_\alpha + F_2,
\end{align*}
\]

\[\begin{align*}
S(x,t)|_{t=0} &= \varphi_1(x), \\
S(x,t)|_{x=L} &= \varphi_3(t), \\
R(x,t)|_{x=0} &= \psi_1(x), \\
R(x,t)|_{x=L} &= \psi_3(t).
\end{align*}\]

where

\[F_1 = \theta_{N,\mu} + Q_{N-1,\mu} e^{-\varepsilon t} \quad F_2 = \theta_{N,\mu} + Q_{N-1,\mu} e^{-\varepsilon t} + \beta U_{N,\mu} + \beta P_{N-1,\mu} e^{-\varepsilon t}\]

Similar to reference [16], equation (21) multiplies $R$ on both sides of the first equation, multiplies $S$ on both sides of the second equation, then integrate on $\Omega_I$ and use the corresponding boundary conditions to obtain

\[
\begin{align*}
\frac{1}{2} \int_0^L R_\alpha^2 dx + \frac{1}{2} \int_0^L S_\alpha^2 dx + \int_0^L S_\alpha R_\alpha dx + \varepsilon \int_0^L S_\alpha R_\alpha dx dt + \int_0^L S_\beta R_\alpha dx dt = 0 \\
\frac{1}{2} \int_0^L S_\alpha^2 dx + \frac{1}{2} \varepsilon \int_0^L S_\alpha^2 dxdt + \frac{1}{2} \varepsilon \int_0^L S_\alpha R_\alpha dx dt + \frac{1}{2} \varepsilon \int_0^L F_1 S_\alpha dx dt = 0
\end{align*}\]

According to equation (22), we have:

\[
\frac{\beta}{2} \int_0^L R_\alpha^2 dx + \frac{\beta}{2} \int_0^L S_\alpha^2 dx + \frac{1}{2} \int_0^L S_\alpha dx + \frac{1}{2} \varepsilon \int_0^L S_\alpha dx + \frac{1}{2} \varepsilon \int_0^L S_\beta dx dt \\
\leq \frac{\beta}{2} \int_0^L S_\alpha^2 dx + \beta \int_0^L R_\alpha^2 dx dt + \frac{1}{2} \varepsilon \int_0^L F_1 S_\alpha dx dt \quad \text{(23)}
\]

Now we considering two cases of $\beta$

(i) When $0 < \beta < 1$, we have

\[
\frac{1}{2} \int_0^L \beta F_1^2 + F_2^2 dx dt = M_i
\]

\[
\frac{1}{2} \int_0^L (S_\alpha^2 + \beta R_\alpha^2) dx \leq M_i + \beta \int_0^L (S_\alpha^2 + R_\alpha^2) dx dt \leq M_i + \int_0^L (S_\alpha^2 + \beta R_\alpha^2) dx dt
\]

According to the Gronwall inequality, we have

\[
\int_0^L (S_\alpha^2 + \beta R_\alpha^2) dx \leq 2M_i e^{2t}
\]

Thus
\[ \int_0^L \int_0^L (S_i^2 + \beta R_i^2) \, dx \, dt \leq M_1 e^{2T} \leq M_1 M_2 \]  

(26)

where \( M_2 = e^{2T} - 1 \).

(i) When \( 1 \leq \beta < e^{-1} \), we have

\[ \frac{1}{2} \int_0^L (S_i^2 + \beta R_i^2) \, dx \leq M_1 + \beta \int_0^L (S_i^2 + R_i^2) \, dx \, dt \leq M_1 + \beta \int_0^L (S_i^2 + \beta R_i^2) \, dx \, dt \]  

(27)

Similarly, it can be obtained

\[ \int_0^L \int_0^L (S_i^2 + \beta R_i^2) \, dx \, dt \leq \int_0^L 2M_1 e^{2\beta T} \, dt \leq \beta M_1 M_3 \]  

(28)

where \( M_3 = e^{2\beta T} - 1 \).

From equation (23), we can obtain

\[ \int_0^L \int_0^L S_i^2 \, dx \, dt \leq 2M_1 (M_4 + 1) \quad \int_0^L \int_0^L R_i^2 \, dx \, dt \leq \frac{2TM_1}{\beta} (M_4 + 2) \]  

(29)

where \( M_4 = M_2 \) or \( M_4 = \beta M_3 \), because

\[ \int_0^L \int_0^L (R_i^2) \, dx \, dt = 2\int_0^L \int_0^L R_i \, dx \, dt \leq \int_0^L \int_0^L R_i \, dx \, dt + \int_0^L \int_0^L R_i \, dx \, dt \]  

(30)

Combining equation (26), equation (28) and equation (30), according to Gronwall inequality, we have:

\[ \int_0^L \int_0^L R_i^2 \, dx \, dt \leq M_5 \quad \int_0^L \int_0^L S_i^2 \, dx \, dt \leq M_6 \]  

(31)

where

\[ M_5 = \frac{M_1 M_4}{\beta} (e^T - 1), M_6 = 2M_1 (M_4 + 1) (e^T - 1) \]

\[ M = 3 \max \{ M_1 M_4, \frac{M_1 M_4}{\beta}, 2M_1 (M_4 + 1), \frac{2TM_1 (M_4 + 2)}{\beta}, M_5, M_6 \} \]

According to reference [16], we obtain that the remainder terms \( R \) and \( S \) uniformly establish the estimation equation (20) on the regional \( \Omega_T \). Therefore, \( R \) and \( S \) are bounded, that is, equation (14) is uniformly valid.

5. Numerical Approximation

Define grid on area \( [0, L] \times [0, T] \):

\[ x_i = i h \quad i = 0, 1, 2, \ldots, m \quad m = L / h \]

\[ t_j = j \tau \quad j = 0, 1, 2, \ldots, n \quad n = T / \tau \]

Then, we shall use further the following notations:
According to references [17] and [18], we order \( r_0 = \tau^2 h^{-2} \), \( r_1 = \tau^2 h^{-1} \), \( r_2 = \tau h^{-2} \), for the heat conduction equation, the Grank Nicolson scheme is adopted, and the second-order part of the non-homogeneous hyperbolic equation adopts the central difference scheme. Equation (10) is transformed into:

\[
\frac{r_2}{2} \theta_0^{i+1} + (1 + r) \frac{r_2}{2} \theta_i^{j-1} + \frac{r}{r_2} \theta_i^{j-1} + \frac{r}{r_2} \theta_i^{j-1} + \frac{r}{r_2} \theta_i^{j-1} = \frac{r}{r_2} \theta_i^{j-1} + \frac{r}{r_2} \theta_i^{j-1} + \frac{r}{r_2} \theta_i^{j-1}
\]

\[
\theta_0^{i+1} = \theta_0^i + \varphi(x_i)
\]

\[
\theta_0^{i+1} = \varphi(t_j)
\]

\[
\theta_0^{i+1} = \varphi(t_j)
\]

\[
U_{0, i}^{j+1} = r U_{i, 0}^{j+1} + 2(1 - r) U_{i, 0}^{j+1} + \varphi(x_i)
\]

\[
U_{0, i}^{j+1} = \varphi(x_i)
\]

\[
U_{0, i}^{j+1} = \varphi(t_j)
\]

\[
U_{0, i}^{j+1} = \psi(t_j)
\]

Using the equation (32), the equation (15) is transformed into:

\[
Q_{0, i}^{j+1} = -r Q_{i, 0}^{j+1} + (1 - r) Q_{2, i}^{j+1} + \varphi(x_i)
\]

\[
Q_{0, i}^{j+1} = \theta_0^i - \varphi_2(x_i)
\]

\[
Q_{0, i}^{j+1} = 0
\]

\[
Q_{0, i}^{j+1} = 0
\]

According to references [17] and [18], the truncation errors of equations (33), equations (34) and equations (35) are \( O(h^2 + \tau^2) \), \( O(h^2 + \tau^2) \), \( O(h^2 + \tau^2) \).

6. Numerical Example

We order \( L = \pi, T = 1, \beta_1 = \left[ -1 + \left( \frac{23}{108} \right)^2 \right]^i, \beta_2 = \left[ -1 - \left( \frac{23}{108} \right)^2 \right]^i \), the given equation and definite solution conditions are as follows:

\[
p_{xx} = p_i
\]

\[
p(x, t) |_{x=0} = e^{\beta_1 x}
\]

\[
p(x, t) |_{x=L} = e^{\beta_2 x}
\]

\[
p(x, t) |_{x=0} = e^{\beta_1 x + \beta_2 x}
\]
The exact solution of equation (36), equation (37) and equation (38) are

\[ p = e^{(\beta x + \beta t)} \quad q = e^{(\beta x + \beta t)} \quad r = e^\beta \sin x \]

For equations (36) and (37), \( h = 0.01 \times \pi, \tau = 0.01 \), the numerical error diagram is as follows:

![Figure 1](image1.png)  
*Figure 1 Error surface graph of \( p \).*  

![Figure 2](image2.png)  
*Figure 2 Error surface graph of \( q \).*  

For equation (38), we select \( h = 0.1 \times \pi, \tau = 0.01 \), then the numerical error diagram is as follows:

![Figure 3](image3.png)  
*Figure 3 Error surface graph of \( r \).*  

**7. Conclusions**

By adjusting the coupling coefficient and small parameter scale coefficient, the method can be applied to G-L thermoelastic coupling problems with different materials and corresponding different coupling coefficients. The original G-L thermoelastic coupling model is reduced by the singular perturbation method, and the asymptotic solutions of singularly perturbed hyperbolic and parabolic equations in bounded domain are obtained by using the matched asymptotic expansion method. The \( L^2 \) estimate of the remainder of the asymptotic solution is given. By numerical method, the numerical solution of formal asymptotic solution can be obtained in turn. Thus, the specific forms of temperature field and stress field in bounded domain are described.

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