A NOTE ON A THEOREM OF HEATH-BROWN AND SKOROBOGATOV

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Abstract. We generalise a result of Heath-Brown and Skorobogatov [5] to show that a certain class of varieties over a number field \( k \) satisfies Weak Approximation and the Hasse Principle, provided there is no Brauer-Manin obstruction.

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1. Introduction

Let \( k \) be a number field with \( [k : \mathbb{Q}] = m \), and ring of integers \( \mathfrak{o} \). Let \( K \) be a finite extension of \( k \) with \( [K : k] = n \), and let \( \tau_1, \ldots, \tau_n \) be a \( k \)-basis of \( K \). For \( x \in k^n \), we let \( N(x) = N_{K/k}(x^{(1)}\tau_1 + \cdots + x^{(n)}\tau_n) \) be a norm form of \( K/k \). The subject of this note is the affine variety \( X \), defined by the Diophantine equation

\[
P(t) = N(x),
\]

where \( P(t) \) is a polynomial with coefficients in \( k \). Let \( \overline{k} \) be an algebraic closure of \( k \). If \( P(t) \) has exactly two solutions in \( k \), and no other roots in \( \overline{k} \), then we can immediately change variables to obtain the equation

\[
t^{a_0}(1 - t)^{a_1} = \alpha N(x), \tag{1.1}
\]

where \( \alpha \in k^* \) and \( a_0, a_1 \) are positive integers. The culmination of [3] and [5] is the following theorem, under the additional assumption that \( k = \mathbb{Q} \):

**Theorem 1.** The Brauer-Manin obstruction is the only obstruction to the Hasse Principle and Weak Approximation on any smooth projective model of the open subset of the variety (1.1), given by \( P(t) \neq 0 \).

There was only a modest link missing to show this theorem for general \( k \), which is straightforward by present standards, and is our aim here. The key step of [3] and [5] is a descent argument, which reduces the problem to showing the validity of the Hasse principle and weak approximation on the smooth affine quasi-projective variety \( Y \subset \mathbb{P}^{2n} \) defined by
for given \( a, b \in \mathfrak{o} \). In [5] this was achieved by finding an asymptotic lower bound for the number of suitably constrained integer solutions to (1.2) in a large box. The principle tool was the Hardy-Littlewood circle method for \( k = \mathbb{Q} \). We shall use a more general version of the circle method here to handle arbitrary number fields.

In [3], the Brauer group of the variety \( X \) was calculated for some special cases, to identify some situations where the Brauer–Manin obstruction is empty. For example if \( a_0 \) and \( a_1 \) are coprime, and \( K/k \) does not contain any non trivial cyclic extension of \( k \), then \( \text{Br}(k) = \text{Br}(X) \), and so the Hasse principle and weak approximation both hold. On the other hand, it is known that there can be obstructions to weak approximation if \( K \) is a cyclic extension of \( k \). For an example due to Coray, see [4, §9].

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2. Notation

Let \( \mathfrak{o} \) be the ring of integers of \( k \). Without loss of generality, suppose that \( \tau_1, \ldots, \tau_n \) is a \( \mathfrak{o} \)-basis of \( K \). Let \( \mathfrak{n} \) be an integral ideal of \( \mathfrak{o} \), with \( \mathbb{Z} \)-basis \( \omega_1, \ldots, \omega_m \). Let \( \sigma_1, \ldots, \sigma_{n_1} \) be the distinct real embeddings of \( k \), and let \( \sigma_{n_1+1}, \ldots, \sigma_{n_1+n_2} \) be the distinct complex embeddings, such that \( \sigma_{n_1+i} \) is conjugate to \( \sigma_{n_1+n_2+i} \). Put \( k_i \) to be the completion of \( k \) with respect to the embedding \( \sigma_i \), for \( i = 1, \ldots, n_1 + n_2 \).

Define \( V \) to be the commutative \( \mathbb{R} \)-algebra \( \bigoplus_{i=1}^{n_1+n_2} k_i \cong k \otimes \mathbb{Q} \mathbb{R} \). For an element \( x \in V \), we write \( \pi_i(x) \) for its projection onto the \( i \)th summand, \( (so x = \bigoplus \pi_i(x)) \). There is a canonical embedding of \( k \) into \( V \) given by \( \alpha \to \bigoplus \sigma_i(\alpha) \). We identify \( k \) with its image in \( V \). Under this image, \( \mathfrak{n} \) forms a lattice in \( V \), and \( \omega_1, \ldots, \omega_n \) form a real basis for \( V \). We define trace and norm maps on \( V \) as

\[
\text{Tr}(\alpha) = \sum_{i=1}^{n_1} \pi_i(\alpha) + 2 \sum_{i=n_1+1}^{n_1+n_2} \mathfrak{R}(\pi_i(\alpha)),
\]

\[
\text{Nm}(\alpha) = \prod_{i=1}^{n_1} \pi_i(\alpha) \prod_{i=n_1+1}^{n_1+n_2} |\pi_i(\alpha)|^2,
\]
respectively. We also define a distance function $|\cdot|$ on $V$,

$$|x| = |x_1\omega_1 + \cdots + x_m\omega_m| = \max_i |x_i|.$$  

This extends to $V^s$, for $s \in \mathbb{N}$: if $x = (x^{(1)}, \ldots, x^{(s)}) \in V^s$, then

$$|x| = \max_j |x^{(j)}|.$$  

We note that there will be some constant $c$, dependent only on $k$ and our choice of basis $\omega_1, \ldots, \omega_m$, such that

$$|\pi_i(x)| \leq c|x| \quad (2.1)$$

for all $x \in V$ and $1 \leq i \leq m$ (since each $\pi_i$ is linear, this is clear). Also for any $v, w \in V$, we have

$$|vw| \ll |v||w|, \quad \text{Nm}(v) \ll |v|^m \quad \text{and} \quad |v^{-1}| \ll \frac{|v|^{m-1}}{\text{Nm}(v)} \quad (2.2)$$

For any point $v \in V^s$, let $B(v)$ be the box

$$B(v) = \{ x \in V^s : |x - v| < \rho \}, \quad (2.3)$$

where $\rho$ is a fixed real number $0 < \rho < 1$. For a set $A \subset V^s$, and positive real number $P$, we define $PA$ to be the set $\{ x \in V^s : P^{-1}x \in A \}$.  

### 3. Statement of the Main Lemma

Consider the smooth quasi-projective variety $Y'$ given by the equation (1.2) together with the inequalities $x \neq 0$, $y \neq 0$, $z \neq 0$, $N(x) \neq 0$, $N(y) \neq 0$. It is sufficient to prove weak approximation on $Y'$, since weak approximation is a birational invariant on smooth varieties.

We assume equation (1.2) has a solution in $k_{\nu}$ for all places $\nu$ of $k$. Suppose we are given a finite set of places $S$ and a set of local solutions $(x_{\nu}, y_{\nu}, z_{\nu}) \in Y'(k)$ for each $\nu \in S$. For any fixed $\eta > 0$, our task is to find a $k$-point $(x, y, z) \in Y'(k)$ such that

$$|x^{(i)} - x^{(i)}_{\nu}|_\nu < \eta, \quad |y^{(i)} - y^{(i)}_{\nu}|_\nu < \eta, \quad |z - z_{\nu}|_\nu < \eta$$

for all $1 \leq i \leq n$, and $\nu \in S$, where $| \cdot |_\nu$ denotes the valuation on $k_{\nu}$. Without loss of generality, we can assume that $S$ contains all the infinite places.

For the finite places, we note that by the Chinese Remainder Theorem, finding a rational point which is $p$-adically close to some set of $p$-adic points, is equivalent to finding an integral point which is restricted to some congruence class modulo some integral ideal. In our case, we shall let the ideal be $n$ as in the notation section. So we
are given \((x_n, y_n, z_n) \in \mathfrak{a}^{2n+1}\) which is a non-singular solution of (1.2) modulo \(n\).

Our task is now to find a solution \((x, y, z) \in \mathfrak{a}^{2n+1}\) with

\[
|x^{(i)} - Px^{(i)}|_\nu < P\eta, \quad |y^{(i)} - Py^{(i)}|_\nu < P\eta, \quad |z - Pz|_\nu < P\eta
\]

for each infinite place \(\nu\), and

\[
x^{(i)} \equiv x_n^{(i)}, \quad y^{(i)} \equiv y_n^{(i)}, \quad z \equiv z_n \mod n.
\]

Our main lemma is then the following:

Lemma 2. Suppose that for each prime \(p\) there is a non-singular solution to (1.2) satisfying (3.2) in \(p\)-adic integers. Then (1.2) has a solution in \(\mathfrak{a}^{2n+1}\) satisfying (3.1) and (3.2), provided \(P\) is sufficiently large.

This will be enough to prove weak approximation on the variety \(Y\), and will thus establish Theorem 1 for general \(k\).

4. The Circle Method

We set

\[
S_1(\alpha) = \sum_x e(\text{Tr}(aaN(x))),
\]

\[
S_2(\alpha) = \sum_y e(\text{Tr}(abN(y))),
\]

\[
S_3(\alpha) = \sum_z e(\text{Tr}(az^n)),
\]

with all sums running over modulo classes defined by (3.2), and inside the dilated boxes \(P\mathfrak{B}_1 \subset V^n, P\mathfrak{B}_2 \subset V^n, P\mathfrak{B}_3 \subset V\) respectively, where

\[
\mathfrak{B}_1 = \mathfrak{B} \left( \bigoplus_{i=1}^{n_1+n_2} x_{\nu_i} \right), \quad \mathfrak{B}_2 = \mathfrak{B} \left( \bigoplus_{i=1}^{n_1+n_2} y_{\nu_i} \right), \quad \mathfrak{B}_3 = \mathfrak{B} \left( \bigoplus_{i=1}^{n_1+n_2} z_{\nu_i} \right),
\]

\(\nu_i\) being the place corresponding to the embedding \(\sigma_i\).

Also, we let \(\mathfrak{B}' \subset V^{2n+1}\) be the product \(\mathfrak{B}' = \mathfrak{B}_1 \times \mathfrak{B}_2 \times \mathfrak{B}_3\). From the observation that the constant \(c\) in (2.1) exists, we see that to satisfy (3.1), it will be sufficient that \((x, y, z) \in P\mathfrak{B}'\), where \(\rho = \rho(\eta)\) has been chosen appropriately small in the definition (2.3). Furthermore, by choosing \(\rho\) sufficiently small, we can guarantee that \((x, y, z) \in Y'(K)\).

We define \(\mathcal{I}\) as:

\[
\mathcal{I} := \{\alpha = \alpha_1\omega_1 + \cdots + \alpha_m\omega_m \in V : 0 \leq \alpha_i \leq 1\}.
\]
Let $\mathcal{N}(P)$ be the number of points $(x, y, z) \in \mathfrak{o}^{2n+1} \cap P\mathfrak{B}'$ which are a solution to (1.2), and such that the conditions (3.2) are satisfied. We have

$$\mathcal{N}(P) = \int_I S_1(\alpha)S_2(\alpha)S_3(-\alpha)\,d\alpha.$$ 

For any $\gamma \in k$, define the denominator ideal of $\gamma$ as

$$a_\gamma = \{ \kappa \in o : \kappa \gamma \in n \}.$$ 

We also set

$$M_\gamma(\theta) = \{ x \in I : |x - \gamma| \leq P^{-n+m(n-1)\theta} \},$$

for some $\theta > 0$ to be fixed later, and define a special subset of $I$,

$$M = M(\theta) = \bigcup_{\gamma \in k} \mathcal{M}_\gamma,$$

which we shall call the ‘major arcs’. We define the ‘minor arcs’ as the compliment of the major arcs, $M(\theta) = I \setminus M(\theta)$.

Finally we shall state once and for all that implied constants in any $\ll, \gg$, or $O(\cdot)$ quantifiers, are dependent only on $k, K, n$ with fixed choice of basis, and $\mathfrak{B}$.

### 4.1. The Minor Arcs.

First we shall get suitable estimates for $S_1(\alpha)$, and $S_2(\alpha)$. Note that $N$ is a norm form on $K/\mathbb{Q}$ with $\mathbb{Z}$-basis $\{ \omega_i\tau_j \}$. So the argument of [2, Lemma 1] holds here (in fact we have extra restrictions on our variables but this does not affect the argument). This results in the estimate

$$\int_I |S_j(\alpha)|^2\,d\alpha \ll P^{mn+\varepsilon}$$

for $j = 1, 2$, and any $\varepsilon > 0$.

Now we want to get a bound on $|S_3(\alpha)|$ for $\alpha$ on the minor arcs.

**Lemma 3.** Let $\varepsilon > 0$ and suppose $0 < \Delta < 1$. Either:

(i) $|S_3(\alpha)| \ll P^{m-\Delta/2n-1+\varepsilon}$, or

(ii) there exists $0 \neq \mu \in n, \lambda \in n$ such that

$$|\mu| \ll P^{(n-1)\Delta} \text{ and } |\mu \alpha - \lambda| < P^{-n+(n-1)\Delta}.$$ 

**Proof.** Consider the sum

$$S'_3(\alpha) = \sum_z e(\text{Tr}(\alpha(z + z_n)^n)).$$
where $z$ now runs over the set $\mathbb{n} \cap P\mathbb{B}_3$. By comparing the domains of summation, we see that

\[ S_3(\alpha) = S'_3(\alpha) + O(P^{m-1}), \]

and thus if assumption (i) fails, then it also fails with $S_3(\alpha)$ replaced by $S'_3(\alpha)$. Put $f(z) = \sum_{i=1}^n \text{Tr}(\omega_i(z + z_n)^n)\omega_i$. Then $f$ is of the type defined by [1, Eq 2.6]. Furthermore, in the notation of [1],

\[ S'_3(\alpha) = \sum_{z \in P\mathbb{B}_3} e[\alpha \cdot f(z)], \]

so our result is given by [1, Lemma 3]. Note that this lemma was for exponential sums over $\mathfrak{o}$ rather than general $\mathfrak{n}$, but it is trivial to generalise to this setting. □

Under the assumption that $\alpha$ satisfies (ii), we have (using (2.2))

\[
\left| \alpha - \frac{\lambda}{\mu} \right| \ll |\mu|^{-1}||\mu\alpha - \lambda| \\
\ll |\mu|^{m-1}P^{-n+(n-1)\Delta} \\
\ll P^{-n+m(n-1)\Delta}.
\]

If we put $\gamma = \frac{\lambda}{\mu}$, we see that $\langle \mu \rangle \subset a_\gamma$, and so

\[ \text{Nm}(a_\gamma) \leq \text{Nm}(\langle \mu \rangle) \ll P^{m(n-1)\Delta}. \]

Hence $\alpha \in M(\Delta)$. So we deduce

\[ |S_3(\alpha)| \ll P^{m-\Delta/2^{n-1}+\varepsilon}, \quad (4.2) \]

for all $\alpha \in \mathfrak{m}(\Delta)$.

Combining this with (4.1) and using Cauchy’s inequality we obtain:

**Lemma 4.**

\[ \int_{\mathfrak{m}(\Delta)} S_1(\alpha)S_2(\alpha)S_3(-\alpha)d\alpha \ll P^{(n+1)m-\delta} \]

for some $\delta = \delta(\Delta) > 0$.

4.2. **The Major Arcs.** For $k = (k^{(1)}, \ldots, k^{(2n+1)}) \in \mathbb{n}^{2n+1}$, we define the function

\[
F(k^{(1)}, \ldots, k^{(2n+1)}) = aN(k^{(1)} + x^{(1)}_n, \ldots, k^{(n)} + x^{(n)}_n) \\
+ bN(k^{(n+1)} + y^{(1)}_n, \ldots, k^{(2n)} + y^{(n)}_n) - (k^{(2n+1)} + z_n)^n.
\]

Note that the assumption of Lemma 2 is equivalent to the assumption that $F(k) = 0$ has a non-singular solution in $\mathbb{n}_p$ for every prime $p$. 
Put
\[ S_\gamma = \text{Nm}(a_\gamma)^{-2n+1} \sum_{k \mod na_\gamma} e(\text{Tr}(\gamma F(k))), \]
the sum being over \( k \in n^{2n+1} \). We then define
\[ \mathcal{S}(\Delta) = \sum'_{\text{Nm}(a_\gamma) \leq P^\Delta} S_\gamma, \]
where the dash indicates that only one \( \gamma \) should be taken from each equivalence class modulo \( n \). We call this the singular series. Finally, put
\[ I(\Delta) = \int_{|\beta| < P^\Delta} \int_{\mathbb{B}'(\Delta)} e(\text{Tr}(\beta F(k))) d\beta d\kappa. \]
This is the singular integral.

**Lemma 5.** For \( \Delta \) sufficiently small,
\[ \int_{\mathfrak{M}(\Delta)} S_1(\alpha)S_2(\alpha)S_3(-\alpha) d\alpha = \mathcal{S}(\Delta)I(\Delta) P^{(n+1)m} + O(P^{(n+1)m-\delta}), \]
for some \( \delta = \delta(\Delta) > 0 \).

*Proof.* This follows from [6, Lemma 7]. \[\square\]

Combining this lemma with Lemma 4, we see
\[ \mathcal{N}(P) = \int_{\mathfrak{M}(\Delta)} S(\alpha) d\alpha + \int_{\mathfrak{m}(\Delta)} S(\alpha) d\alpha \]
\[ = \mathcal{S}(\Delta)I(\Delta) P^{(n+1)m} + O(P^{(n+1)m-\delta}). \]
So all that remains to show is that under the assumption of Lemma 2 \( \mathcal{S}(\Delta) \) and \( I(\Delta) \) have strictly positive limits as \( P \to \infty \).

**Lemma 6.** For our box \( \mathfrak{B}' \) chosen as before, \( I(\Delta) \to I_0 \), a constant as \( P \to \infty \). Furthermore \( I_0 > 0 \).

*Proof.* We define the polynomial
\[ F^*(x) = F(x_1^{(1)} \omega_1 + \cdots + x_m^{(1)} \omega_m, \ldots, x_1^{(s)} \omega_1 + \cdots + x_m^{(s)} \omega_m), \]
considered as a real polynomial in the \( sm \) variables \( \{x_1^{(1)}, \ldots, x_m^{(s)}\} \). In the definition of \( I \), we can just as easily think of the inner integral being over \( \mathbb{R}^{mn} \) with \( F \) replaced by \( F^* \), and the outer integral as being over the real variables \( \beta_1, \ldots, \beta_m \), where \( \beta = \beta_1 \omega_1 + \cdots + \beta_m \omega_m \). Then this lemma is routine, and indeed an argument analogous to the one used in [5] can be used. The key point is that the box is centred at a nonsingular point in \( V^n \) (note that a non-singular solution to \( F \) in \( V^n \) corresponds to a non-singular solution to \( F^* \) in \( \mathbb{R}^{mn} \)). \[\square\]
Lemma 7. We have

(i) \( \mathcal{S}(\infty) \) exists,
(ii) \( \mathcal{S}(\Delta) - \mathcal{S}(\infty) \ll P^{-\zeta}, \) for some positive \( \zeta = \zeta(\Delta), \) and
(iii) \( \mathcal{S}(\infty) > 0. \)

We follow the arguments of [5]. Consider the sum

\[ T_1(\gamma) = \sum_{k_1 \mod a_\gamma} e(\text{Tr}(\gamma F_1(k_1))), \]

where \( F_1(k_1) = aN(k_1 + x_{n}^{(1)}, \ldots, k_n + x_{n}^{(n)}). \) Define \( T_2 \) analogously, and set

\[ T_3(\gamma) = \sum_{k \mod a_\gamma} e(\text{Tr}(\gamma(k + z_n)^n)). \]

Then clearly \( S_\gamma = Nm(a_\gamma)^{-2n+1}T_1(\gamma)T_2(\gamma)T_3(\gamma). \) We will consider the dyadic range:

\[ \mathcal{S}_R = \sum'_{R/2 < Nm(a_\gamma) \leq R} Nm(a_\gamma)^{-2n+1}T_1(\gamma)T_2(\gamma)T_3(\gamma). \]

If we repeat the argument of Lemmas 5 and 6 with \( |S_1(\alpha)|^2 \) in place of \( S_1(\alpha)S_2(\alpha)S_3(-\alpha), \) we find that

\[ \sum'_{Nm(a_\gamma) \leq P^\Delta} \int_{\mathfrak{M}_{\gamma}(P^\Delta)} |S_1(\alpha)|^2 d\alpha = \Sigma_1 J_1 + O(P^{mn-\delta}), \]

for some \( \delta = \delta(\Delta) > 0, \) and where

\[ \Sigma_1 = \sum'_{Nm(a_\gamma) \leq P^\Delta} Nm(a_\gamma)^{-2n}|T_1(\gamma)|^2, \]

and

\[ J_1 \sim CP^{mn} \]

for some positive constant \( C. \) But

\[ \sum'_{Nm(a_\gamma) \leq P^\Delta} \int_{\mathfrak{M}_{\gamma}(P^\Delta)} |S_1(\alpha)|^2 d\alpha \leq \int_{I} |S_1(\alpha)|^2 d\alpha \ll P^{mn+\varepsilon} \]

by (4.1). Note that the estimate holds for any \( P \geq 1, \) and \( \varepsilon > 0. \) So if we choose \( P \) such that \( P^\Delta = R, \) and put \( \varpi = \varepsilon/\Delta, \) we see that

\[ \sum'_{Nm(a_\gamma) \leq R} Nm(a_\gamma)^{-2n}|T_1(\gamma)|^2 \ll R^{\varpi}, \]

for any \( R \geq 1, \) and \( \varpi > 0. \) Similarly we have
\[ \sum_{Nm(a,\gamma) \leq R} Nm(a,\gamma)^{-2n} |T_2(\gamma)|^2 \ll R^\omega, \]

and so

\[ \sum_{R/2 < Nm(a,\gamma) \leq R} Nm(a,\gamma)^{-2n} |T_1(\gamma)T_2(\gamma)| \ll R^\omega \quad (4.3) \]

by Cauchy’s inequality.

Now we bound \( T_3(\gamma) \). Let \( N = Nm(a,\gamma) \), and note that

\[ |N^{-1}T_3(\gamma)| = |N^{-m} \sum_{z \mod (N)} e(\text{Tr}(\gamma(z + z_n)^n))| \]

\[ = |N^{-m} \sum_{z \in \mathfrak{z} \cap N}\mathfrak{z} e(\text{Tr}(\gamma(z + z_n)^n)), \]

where \( \mathfrak{z} = \{ x \in V : 0 \leq x_i < 1 \ \forall \ i \} \). So now we can use Lemma 3, replacing \( S_3 \) with the exponential sum on the last line, taking \( P = N \), and \( \Delta < 1/m(n-1) \). If alternative (i), holds we have

\[ |N^{-1}T_3(\gamma)| \ll N^{-m}N^{m-\Delta/2n-1+\delta} = N^{-\Delta/2n-1+\delta} \quad (4.4) \]

for any \( \delta > 0 \). On the other hand, alternative (ii) implies the existence of some \( \mu, \lambda \in \mathfrak{n} \) where

\[ 0 < |\mu| \ll N^{(n-1)\Delta}, \]

and

\[ |\mu\gamma - \lambda| \ll N^{-(n+1)\Delta}. \]

Note that \( \mathfrak{a}_\gamma(\mu\gamma - \lambda) \subseteq \mathfrak{n} \), so that if \( \mu\gamma - \lambda = \sum \theta_i\omega_i \),

then \( \theta_iN \in \mathbb{Z} \) for all \( i \). But \( |\theta_iN| < N^{(n-1)(\Delta-1)} < 1 \), and so \( \theta_i = 0 \) for all \( i \). It follows that \( \mu \in \mathfrak{a}_\gamma \), so that \( N|Nm(\mu)\). But

\[ Nm(\mu) \ll |\mu|^m \ll N^{m(n-1)\Delta} \ll N^{1-\varepsilon}, \]

for some positive \( \varepsilon \), and since \( Nm(\mu) \neq 0 \), this is a contradiction if \( N \) is large enough. Therefore (4.4) holds, and combining this with (4.3), we arrive at the estimate

\[ \mathcal{S}_R \ll R^{\omega-\Delta/2n-1+\delta}. \]

Since \( \omega, \delta \) were arbitrary, (i) and (ii) of our lemma follow immediately. The proof of (iii) is routine. For any prime ideal \( \mathfrak{p} \), we define

\[ \mu(\mathfrak{p}) = \sum_{j=1}^\infty \sum_{a,\gamma=\mathfrak{p}j} S_\gamma, \]
and then
\[ S(\infty) = \prod_p \mu(p). \]

Standard arguments show that the assumption that \( F \) has a non-singular solution in each \( n_p \) implies that each \( \mu(p) > 0 \), and that the product is strictly positive. This completes the proof.

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