A Quadratic Curvature Lagrangian of Pawłowski and Rączka: A Finger Exercise with MathTensor

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Abstract

Recently Pawłowski and Rączka (P&R) proposed a unified model for the fundamental interactions which does not contain a physical Higgs field. The gravitational field equation of their model is rederived under heavy use of the computer algebra system Mathematica and its package MathTensor.

1 Introduction

The computer algebra system Mathematica, together with its package MathTensor, is very useful for executing differential geometric calculations on a four-dimensional Riemannian spacetime, as was shown by Soleng [1], for example. Here we want to demonstrate the effectiveness of these tools by picking an example from the current literature which is of some fundamental importance.

The gravitational sector of a unified model for the fundamental interactions, proposed by P&R [2], is determined by the Lagrangian density

\[ L_{\text{geom}} = \sqrt{-g} \left( -\frac{\beta}{6} (1 + \lambda R_{\Phi^\dagger \Phi} - \lambda (\Phi^\dagger \Phi)^2 - \rho C_{ijkl} C^{ijkl} \right), \]

with \( \Phi \) as Higgs field and \( R \) (curvature scalar) and \( C_{ijkl} \) (conformal Weyl curvature tensor) as gravitational fields depending on the metric tensor \( g_{ij} \) of the Riemannian spacetime, \( g := \det(g_{ij}) \). The coupling constants \( \beta, \lambda, \) and \( \rho \) are dimensionless because of the conformal invariance of the model.

The total Lagrangian density of P&R,

\[ \mathcal{L} = L_{\text{geom}} + L_{\text{field}}, \]

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depends additionally on the field part, which subsumes the contributions from all nongeometrical pieces. The explicit form of $L_{\text{field}}$ is of no relevance to us\footnote{From the Higgs-type Lagrangian, two pieces feature in the geometrical Lagrangian, namely the first two terms in (3), whereas the rest of it is attributed here to the field Lagrangian in (4).} since it is absorbed in the definition of the ‘material’ energy–momentum tensor

$$T^{ij} := \frac{2}{\sqrt{-g}} \frac{\delta L_{\text{field}}}{\delta g_{ij}}. \quad (3)$$

Hamilton’s principle yields the gravitational field equation

$$\frac{\delta L_{\text{geom}}}{\delta g_{ij}} + \frac{\delta L_{\text{field}}}{\delta g_{ij}} = 0, \quad (4)$$

or

$$-\frac{1}{\sqrt{-g}} \frac{\delta L_{\text{geom}}}{\delta g_{ij}} = \frac{1}{2} T^{ij}. \quad (5)$$

The total Lagrangian (2) is conformally invariant. Therefore it is possible, within the P&R model, to fix a conformal gauge for the scalar Higgs-type field $\Phi$ according to

$$\Phi^* \Phi = 2 = \text{constant}. \quad (6)$$

Our goal will be the computation of the left-hand side of (5) by means of the computer algebra tools mentioned above. Thereby we want to check the corresponding results of P&R \cite{2}.

## 2 Riemann Tensor and its Irreducible Pieces

Before we can commence with our calculations, we have to get hold of the definition of the Weyl curvature tensor $C_{ijkl}$. It is known that the curvature tensor of a four-dimensional Riemannian space has three irreducible pieces: the Weyl tensor $C_{ijkl}$, the tracefree Ricci tensor $R_{ij}$, and the curvature scalar $R$.

We take the conventions of Misner et al. \cite{3} and define the curvature tensor ($\Gamma^i_{jk} =$ Christoffel symbols)

$$R^i_{jkl} := \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{mk} \Gamma^m_{jl} - \Gamma^i_{ml} \Gamma^m_{jk}, \quad (7)$$

the Ricci tensor and the curvature scalar

$$R_{ij} := R^k_{ikj} \quad \text{and} \quad R := g^{ij} R_{ij}, \quad (8)$$

respectively, and, eventually, the tracefree Ricci tensor

$$JR_{ij} := R_{ij} - \frac{1}{4} g_{ij} R. \quad (9)$$

The metric has signature ($- + ++$).

The Riemann tensor (20 independent components) decomposes into three irreducible pieces, $20 = 10 \oplus 9 \oplus 1$, or

$$R_{ijkl} = (1) R_{ijkl} + (2) R_{ijkl} + (3) R_{ijkl}, \quad (10)$$
with the definitions

\[ (2) \ R_{ij}^{\ kl} := 2 R_{[i}^{ \ [k} \delta^{l]}_{j]} \quad \text{and} \quad (3) \ R_{ij}^{\ kl} := \frac{1}{6} R \delta^{[i}_l \delta^{j]}_{k]. \]  

(11)

The first irreducible piece is traceless and has to be identified with the Weyl piece: \( C_{ijkl} := (1) R_{ijkl}. \) If resolved with respect to \( C_{ijkl}. \) Eq. (11) can be read as the defining equation for the Weyl tensor.

Let us use MathTensor in order to perform the irreducible decomposition defined in (10) and (11). MathTensor recognizes \( \text{TraceFreeRicciR}. \) Therefore the pieces \( (a) R_{ijkl}, \) denoted by \( R1, R2, \) and \( R3, \) can be computed by an almost verbatim translation of (10) and (11) into MathTensor:

\[
\text{Dimension=4; Rcsign=1; (*Default*)}
\]
\[
\text{DefUnique[TraceFreeRicciR[la_,lb_],0,PairQ[la,lb]]}
\]
\[
R2[li_,lj_,lk_,ll_]=2 \text{Antisymmetrize[Antisymmetrize[}
\text{TraceFreeRicciR[lk,li] Metricg[lj,ll],[li,lj]],[lk,ll]]}
\]
\[
R3[li_,lj_,lk_,ll_]=\text{Expand[Antisymmetrize[}
\text{1/6 ScalarR Metricg[lk,li] Metricg[lj,ll],[li,lj]]}
\]
\[
R1[li_,lj_,lk_,ll_]=\text{RiemannR[li,lj,lk,li]}
\]
\[
- R2[li,lj,lk,li] - R3[li,lj,lk,li]
\]

We shall verify this decomposition with MathTensor. We first use the predefined tensor \( \text{WeylC} \) in order to check if the definitions are correct.

Furthermore we want to make sure that our decomposition is really irreducible. For this purpose we compute the traces of \( (a) R_{ijkl}, \) and find – as well as the usual symmetries of the curvature tensor – that \( C_{ijkl} \) is traceless, the Ricci tensor of \( (2) R_{ijkl} \) is \( R_{ij}, \) and \( (3) R_{ijkl} \) has neither a traceless piece nor a traceless Ricci piece, rather only the curvature scalar with the correct factor 1. The input reads:

\[
\text{TraceFreeRicciR[li,lj]/.TraceFreeRicciToRicciRule}
\]
\[
\text{diff1=}\text{CanAll[(WeylC[li,lj,lk,li]-R1[li,lj,lk,li])/.WeylToRiemannRule]}
\]
\[
\text{diff2=}\text{CanAll[Tsimplify[CanAll[diff1/.TraceFreeRicciToRicciRule]]]}
\]
\[
\text{TraceR2=}\text{Expand[Metricg[ua,ub] R2[la,li,lb,lj]]}
\]
\[
\text{TraceR3=}\text{Expand[R3[la,li,lb,lj] Metricg[ua,ub]]}
\]
\[
\text{TraceR1=}\text{Tsimplify[Expand[Metricg[ua,ub] R1[la,li,lb,lj]/.TraceFreeRicciToRicciRule]]}
\]

3
3 The Topological Euler Density

Let us come back to (1). Its last term is proportional to

\[ C^2 := \sqrt{-g} C_{ijkl} C^{ijkl}. \] (12)

Densities will be denoted by script letters. The computation of the Bach tensor \[ B^{ij} := \frac{1}{\sqrt{-g}} \frac{\delta C^2}{\delta g_{ij}} \] (13)

(see also [4]) can be simplified, if one splits off a divergence term from (12). Such a term is the topological Euler density \[ E := \frac{1}{128 \pi^2} \sqrt{-g} \varepsilon^{abcd} R^{ij}_{\ ab} R^{kl}_{\ cd} \varepsilon_{ijkl}. \] (14)

We can show that (14) represents a divergence:

\[ E = \partial_i \left( \frac{1}{128 \pi^2} D^i \right) \] (15)

with

\[ D^i := \sqrt{-g} \varepsilon^{ijkl} \varepsilon_{ab} \Gamma^a_{\ cd} \left( \frac{1}{2} R^b_{\ dkl} + \frac{1}{3} \Gamma^b_{\ mk} \Gamma^m_{\ dl} \right). \] (16)

Furthermore, also by means of MathTensor, the explicit form of (14) turns out to be

\[ E := \sqrt{-g} \left( R_{ijkl} R^{ijkl} - 4 R_{ij} R^{ij} + R^2 \right). \] (17)

The corresponding input reads:

\[
\text{Eulerd1=} -\frac{1}{128 \pi^2} \text{Epsilon}[u\_a,u\_b,u\_c,u\_d] \text{Epsilon}[l\_i,l\_j,l\_k,l\_l] \\
\quad \text{RiemannR}[u\_i,u\_j,la,lb] \text{RiemannR}[uk,ul,lc,ld] \\
\text{Eulerd2=} \text{CanAll}[\text{Expand}[\text{Eulerd1}/.\text{EpsilonProductTensorRule}]]
\]

We substitute into (12) the definition of the Weyl tensor and eliminate the emerging curvature square piece by means of (17). Then we find the simplified formula

\[ C^2 = \tilde{C}^2 + \partial_i D^i, \] (18)

with

\[ \tilde{C}^2 := 2 \left( R^{ij} R_{ij} - \frac{R^2}{3} \right) = 2 \left( \mathcal{R}^{ij} \mathcal{R}_{ij} - \frac{R^2}{12} \right), \] (19)

which has also been cross-checked by means of MathTensor.
In order to compute the variation of the remaining piece $\tilde{C}^2$ of the Lagrangian, we follow closely the scheme that was demonstrated in the lecture by Soleng [1]. After initialization, we use \texttt{Variation} in order to compute $\delta \tilde{C}^2$, followed by a series of partial integrations, rules, and simplifications. The last step actually computes the variational derivative:

\begin{verbatim}
varC1=Variation[Sqrt[-Detg]*checkCsquare,Metricg]
varC2=PIntegrate[varC1,Metricg]
varC3=PIntegrate[varC2,Metricg]
varC4=Canonicalize[Absorbg[ApplyRules[varC3,RiemannRules]]]
Bach1[ui_,uj_]=Tsimplify[VariationalDerivative[Expand[varC4/Sqrt[-Detg]],Metricg,li,lj]]

MetricgFlag=True
\end{verbatim}

The outcome of this variation reads (a semicolon denotes the covariant derivative)

\begin{align}
\text{Bach1} \equiv B_{ij} &= \frac{2}{3} R_{;ij} - 2 R_{;ij;k}^{;k} + \frac{1}{3} g_{ij} R_{;k;k} \\
&\quad - \frac{1}{3} R^2 g_{ij} + \frac{4}{3} R R_{ij} + g_{ij} R_{kl} R^{kl} - 4 R_{ikjl} R^{ikjl}, \quad (20)
\end{align}

\begin{align}
B_{ij} = B_{ji} \quad \text{and} \quad g^{ij} B_{ij} = B_{kk} = 0. \quad (21)
\end{align}

The result (20) differs slightly from that of P&R.2 In order to compute the Bach tensor (20), we could have used (12) rather than (19). In this case (20) would pick up two additional terms that compensate each other, as is explicitly shown in [3]. We feel, however, that the present detour, via $\tilde{C}^2$, pays off in conceptual and computational simplicity.

2Their result reads, see [7, Eq. (7.2)] and [2, Eq. (5.2)]:

\begin{align}
&-\frac{2}{3} R_{;ij} + 2 R_{;ij;k}^{;k} - \frac{2}{3} g_{ij} R_{;k;k} \\
&- \frac{1}{3} R^2 g_{ij} + \frac{4}{3} R R_{ij} + g_{ij} R_{kl} R^{kl} - 4 R_{ikjl} R^{ikjl}.
\end{align}

The third term of the first line carries an incorrect factor two. Therefore their Bach tensor is no longer traceless, as is required by conformal invariance. Up to a (conventional?) sign, their first line is identical to that of (20), whereas the second lines coincide.
5 The Bach Tensor Streamlined

The Bach tensor takes on a more transparent form if we express the curvature pieces in [20] exclusively in terms of the irreducible pieces.

By default, MathTensor does not recognize that the trace of $C_{ijkl}$ vanishes over arbitrary indices. Therefore we explicitly define rules to take care of this fact. After this preparatory step, we can directly reformulate the Bach tensor. Thus we put in:

```math
DefUnique[WeylC[la_,lb_,lc_,ld_],0,PairQ[la,lb] ||
   PairQ[la,lc] || PairQ[la,ld] || PairQ[lb,lc] ||
   PairQ[lb,ld] || PairQ[lc,ld]]
Bach2[li_,lj_]=Tsimplify[CanAll[Expand[Bach1[li,lj]/.
   RiemannToWeylRule/.RicciToTraceFreeRicciRule]]]
```

This computation yields

$$
B_{ij} = \frac{2}{3} R_{ij} - \frac{1}{6} R_{ik} g_{jk} - 2 R_{ij} R_{jk}^{;k} + \frac{2}{3} R R_{ij} + 4 R_{ki} R_{j}^{;k} - 6 g_{ij} R_{kl} R^{kl} - 4 R^{kl} C_{ijkl} \quad (22)
$$

If we introduce

$$
\gamma_{ij}^{kl} := \delta_{i}^{(k} \delta_{j}^{l)} - \frac{1}{4} g_{ij} g^{kl} \quad (23)
$$

where

$$
\gamma_{ij}^{kl} = \gamma_{ij}^{kl} \equiv 0 \quad \text{and} \quad g^{ij} \gamma_{ij}^{kl} \equiv 0 \quad (24)
$$

then the tracelessness $B_{ij} = 0$ and the symmetry $B_{ij} = B_{ji}$ of the Bach tensor become manifest:

$$
B_{ij} = \frac{2}{3} \gamma_{ij}^{kl} R_{kj}^{;l} - 2 R_{ij} R_{jk}^{;k} + \frac{2}{3} R R_{ij} + 4 \gamma_{ij}^{kl} R_{mk} R^{ml} - 4 R^{kl} C_{ijkl} \quad (25)
$$

The trick for the corresponding computation is again the use of a series of rules that expresses the Bach tensor in terms of $\gamma_{ij}^{kl}$ (denoted by $\Gamma$):

```math
DefineTensor[Gam,{{2,1,3,4},1,{}},1,\{1,2,4,3\},1]
DefUnique[Gam[li_,lj_,lk_,ll_],0,PairQ[li,lj]]
RuleUnique[GamRule1,Metricg[li_,lj_]
   TraceFreeRicciR[lk_,ll_] TraceFreeRicciR[lm_,ln_],
   (- 4 Gam[li,lj,um,uo]
   + 4 Symmetrize[Metricg[li,um] Metricg[lj,uo],\{um,uo\}]]
```

6
6 Gravitational Field Equation of the P&R Model

The gravitational field equation \((\ref{eq:6})\) can now be made explicit by substituting \((\ref{eq:1}), (\ref{eq:6}), \) and \((\ref{eq:13})\) into it:
\[
\frac{(1 + \beta)v^2}{12} \frac{\delta(\sqrt{-g} R)}{\delta g_{ij}} + \frac{\lambda v^4}{4} \frac{\delta\sqrt{-g}}{\delta g_{ij}} + \rho B^{ij} = \frac{1}{2} T^{ij}. \tag{26}
\]

We compute (by MathTensor) the variations \((G^{ij} = \text{Einstein tensor})\)
\[
\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g} R)}{\delta g_{ij}} = -G^{ij} := -R^{ij} + \frac{1}{2} R g^{ij} \tag{27}
\]
and
\[
\frac{1}{\sqrt{-g}} \frac{\delta\sqrt{-g}}{\delta g_{ij}} = \frac{1}{2} g^{ij}. \tag{28}
\]

By inserting these relations into \((\ref{eq:26})\), we find the (corrected) P&R field equation (see \([7, \text{Eq. 7.2}])\) and \([2, \text{Eq. 5.2})\])
\[
-\frac{(1 + \beta)v^2}{12} G^{ij} + \frac{\lambda v^4}{8} g^{ij} + \rho B^{ij} = \frac{1}{2} T^{ij}. \tag{29}
\]
Provided \(\beta \neq -1\) and \(v^2 \neq 0\), we can put \((\ref{eq:29})\) in a more conventional form,
\[
G_{ij} - \frac{3\lambda v^2}{2(1 + \beta)} g_{ij} - \frac{12\rho}{(1 + \beta)v^2} B_{ij} = -\frac{6}{(1 + \beta)v^2} T_{ij}, \tag{30}
\]
or, after some (computer) algebra \((T := T^k_k, \mathcal{T}_{ij} := T_{ij} - T g_{ij}/4)\):
\[
R_{ij} + \frac{3\lambda v^2}{2(1 + \beta)} g_{ij} - \frac{12\rho}{(1 + \beta)v^2} B_{ij} = -\frac{6}{(1 + \beta)v^2} \left(T_{ij} - \frac{1}{2} T g_{ij}\right). \tag{31}
\]

We can decompose this equation into its two irreducible pieces, the tracefree and the trace piece:
\[
\begin{align*}
R_{ij} - \frac{12\rho}{(1 + \beta)v^2} B_{ij} &= -\frac{6}{(1 + \beta)v^2} \mathcal{T}_{ij}, \tag{32}
R + \frac{6\lambda v^2}{(1 + \beta)} &= \frac{6}{(1 + \beta)v^2} T.
\end{align*}
\]
These two pieces (32) of the (corrected) gravitational field equation (29) or (31) of P&R, together with the explicit form of the Bach tensor (25), represent the general result of our considerations. Note that in the trace piece of (32) there occur only second derivatives of the metric, in contrast to the fourth order derivatives featuring in the Bach tensor of the tracefree piece. Incidentally, models leading to somewhat similar gravitational field equations have been discussed since the early 1920s by many people (see [3, 5] and [10], and the literature cited therein).

In vacuo we have

\[
\begin{align*}
R_{ij} &= \frac{12 \rho}{(1 + \beta) v^2} B_{ij}, \\
R &= -\frac{6 \lambda v^2}{1 + \beta} =: 4 \Lambda_{\text{cosm}}.
\end{align*}
\] (33)

A glance at (25) shows that we can find a special solution of the vacuum equation (33) by using the Einstein vacuum equation with a suitable cosmological constant as an ansatz:

\[
\begin{align*}
R_{ij} &= \Lambda_{\text{cosm}} g_{ij} \\
R &= 4 \Lambda_{\text{cosm}}.
\end{align*}
\] (34)

or, alternatively,

\[
\begin{align*}
R_{ij} &= 0, \\
R &= 4 \Lambda_{\text{cosm}}.
\end{align*}
\] (35)

Then the Bach tensor (25) vanishes, $B_{ij} = 0$, and (33) is fulfilled.

We have collected the MathTensor code that verifies the results of this section:

DefineTensor[B,{{1,2},1}]
DefineTensor[T,{{1,2},1}]
DefineTensor[TrFrT,{{1,2},1}]
DefUnique[B[l1_,l2_],0,PairQ[l1,l2]]
DefUnique[T[l1_,l2_],T,PairQ[l1,l2]]
lagHE=Sqrt[-Detg] ScalarR
varHE1=Variation[lagHE,Metricg]
varHE2=PIntegrate[varHE1,Metricg]
HE[ui_,uj_]=Expand[1/Sqrt[-Detg] VariationalDerivative[
  varHE2,Metricg,li,lj]]
lagconst=Sqrt[-Detg]
varconst1=Variation[lagconst,Metricg]
const[ui_,uj_]=Expand[1/Sqrt[-Detg] VariationalDerivative[
  varconst1,Metricg,li,lj]]
MetricgFlag=True
FieldEq1[li_,lj_]=(1/(12 (1+beta) v^2) HE[li,lj]) +
  (1/4 lambda v^4 const[li,lj]) +
  (rho B[li,lj]) - (1/2 T[li,lj])
FieldEq2[li_,lj_]=Simplify[Collect[Expand[-12/(1+beta) v^2]
  *FieldEq1[li,lj]],{RicciR[li,lj],ScalarR}]

8
TraceFieldEq2 = Simplify[Expand[Metricg[ui, uj] FieldEq2[li, lj]]]

RuleUnique[TRule, TrFrT[li, lj] + 1/4 T Metricg[li, lj]]

FieldEq3[li, lj] = Tsimplify[Expand[FieldEq2[li, lj] - 1/2 TraceFieldEq2 Metricg[li, lj]]]

TraceFieldEq3 = Expand[Metricg[ui, uj] FieldEq3[li, lj]]

TraceFreeFE[li, lj] = Expand[FieldEq3[li, lj] - 1/4 Metricg[li, lj] TraceFieldEq3 /. RicciToTraceFreeRicciRule /. TRule]

7 Discussion

Models very similar in their gravitational sectors to that by P&R have been developed, amongst others, by Gregorash & Papini [11, 12] and in [13, 14], the latter one, though, in a metric-affine spacetime with additional conformal invariance [14, Sect. 6]. Since conformal invariance is accommodated much more naturally in a spacetime with a Weyl piece, we believe that these post-Riemannian model should be reconsidered in the light of the more recent developments.

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