Conditional G-expectation in $L^p$ and Related Itô’s Calculus

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In this paper, we define a dynamically consistent conditional G-expectation in space $L^p$, and give the related stochastic calculus of Itô’s type, especially get Itô’s formula for a general $C^{1,2}$-function.

Keywords conditional G-expectation, Itô’s integral, stopping times, Itô’s formula

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1 Introduction

The notion of G-expectation, which is a typical nonlinear expectation, was proposed by Peng [5], [6], and [7]. It can be regarded as a nonlinear generalization of Wiener probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = C([0, \infty), \mathbb{R}^d)$, $\mathcal{F} = \mathcal{B}(\Omega)$, and $P$ is a Wiener probability measure defined on $(\Omega, \mathcal{F})$. On the same canonical space $\Omega$, G-expectation is a sublinear expectation, such that the same canonical process $B_t(\omega) := \omega_t, t \geq 0$ is a G-Brownian motion, i.e., it is a continuous process with stable and independent increments. One important feature of G-expectation is its time consistency. To be precise, let $\xi$ be a random variable and $Y_t := \mathbb{E}^G_t[\xi]$ be the conditional G-expectation, then one has $\mathbb{E}^G_s[\xi] = \mathbb{E}^G_s[\mathbb{E}^G_t[\xi]]$ for any $s < t$. For this reason, the conditional G-expectation is called a G-martingale, or a martingale under G-expectation.

A well-known and fundamentally important fact in probability theory is that the linear space $L^1_P$ coincides with the $\mathbb{E}P[\cdot]$-norm completion of the space of bounded and $\mathcal{F}$-measurable functions $B_b(\Omega)$, or bounded and continuous functions $C_b(\Omega)$, or even smaller one, the space $\text{Lip}_{b,cyl}(\Omega) \subset C_b(\Omega)$ of bounded and Lipschitz cylinder functions. While in the theory of G-expectation, Denis, et al. [1] proved that the $\mathbb{E}[]$-norm completion of $\text{Lip}_{b,cyl}(\Omega)$ and $C_b(\Omega)$ are the same space $L^1_G(\Omega)$, the random variables $X = X(\omega)$ which are quasi-continuous with respect to the natural Choquet capacity $c(A) := \mathbb{E}^G[I_A], A \in \mathcal{B}(\Omega)$, but they are strict subspace of the $\mathbb{E}[][\cdot]$-norm completion of $B_b(\Omega)$. Moreover the latter one is, again, a strict subspace of $L^1$, the space of all $\mathcal{F}$-measurable random variables $X$ such that $\mathbb{E}[||X||] < \infty$.

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In Peng [5], [6], and [7], G-expectation and the related Itô’s calculus are mainly based on space $L^1_G(\Omega)$. Then Li and Peng [4] extend the Itô’s integral to space without the quasi-continuity, obtain Itô’s integral on stopping time interval, and get Itô’s formula for a general $C^{1,2}$-function, which generalizes the previous results of Peng [5], [6], and [7] and its improved version of Gao [2] and Zhang et al. [8].

But in Li and Peng [4], the conditional G-expectation has not been defined, so whether the martingale properties still hold for the stochastic integral $(\int_0^t \eta_s dB_s)_{0 \leq t \leq T}$ and the conditional G-expectation of random variables without quasi-continuous condition is open.

This paper is organized as follows. In section 2, we give the definition of conditional G-expectation of random variables in space $L^1$. In section 3, we define the related Itô’s integral in space $M^2(0,T)$. In section 4, we prove the Itô’s formula for general $C^{1,2}$ function.

2 Conditional G-expectation in $L^p$

2.1 G-Brownian motion and G-expectation

We first present some preliminaries in the theory of G-expectation and the related space of random variables. More relevant details can be found in Peng [5], [6], [7].

Let $\Omega$ be a given set and let $\mathcal{H}$ be a linear space of real functions defined on $\Omega$ such that $c \in \mathcal{H}$ for each constant $c$ and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. A sublinear expectation $\mathbb{E}$ on $\mathcal{H}$ is a functional $\mathbb{E}: \mathcal{H} \rightarrow \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$,

(a) Monotonicity: If $X \geq Y$ then $\mathbb{E}[X] \geq \mathbb{E}[Y]$.

(b) Constant preserving: $\mathbb{E}[c] = c$.

(c) Sub-additivity: $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.

(d) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \forall \lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space. In the literature, for technical convenience, $\mathcal{H}$ is taken as the space satisfying that if $X_1, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$ where $C_{l,Lip}(\mathbb{R}^n)$ denotes the linear space of (local Lipschitz) functions $\varphi$ satisfying

$$| \varphi(x) - \varphi(y) | \leq C(1 + |x|^m + |y|^m) | x - y |, \forall x, y \in \mathbb{R}^n,$$

for some $C > 0, m \in \mathbb{N}$ depending on $\varphi$. The linear space $C_{l,Lip}(\mathbb{R}^n)$ can be replaced by $L^\infty(\mathbb{R}^n), C_b(\mathbb{R}^n), C_b^k(\mathbb{R}^n), C_{unif}(\mathbb{R}^n), C_{b,Lip}(\mathbb{R}^n)$, and $L^0(\mathbb{R}^n)$. In this case $X = (X_1, \ldots, X_n)$ is called an $n$-dimensional random vector, denoted by $X \in \mathcal{H}^n$. 2
DEFINITION 2.1. In a nonlinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, a random vector $Y \in \mathcal{H}^n$ is said to be independent from another random vector $X \in \mathcal{H}^n$ under $\mathbb{E}[:]$ if for each test function $\varphi \in C_{1,Lip}(\mathbb{R}^{m+n})$ we have

$$\mathbb{E}[\varphi(X,Y)] = \mathbb{E}[\mathbb{E}[\varphi(x,Y)]_{x=X}].$$

Let $\Omega = C_0^d(\mathbb{R}^+) \times \mathbb{R}_+^d$ be the space of all $\mathbb{R}^d$-valued continuous paths $(\omega_t)_{t \in \mathbb{R}^+}$, with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i} [\max_{t \in [0,i]} |\omega^1_t - \omega^2_t|] \wedge 1.$$ 

For each fixed $T \in [0, \infty)$, we set $\Omega_T := \{\omega_{\cdot \wedge T} : \omega \in \Omega\}.$

Let

$$Lip(\Omega_T) := \{\phi(B_{t_1 \wedge T}, \cdots, B_{t_n \wedge T}) : n \in \mathbb{N}, t_1, \cdots, t_n \in [0, \infty), \phi \in C_{1,Lip}(\mathbb{R}^{d \times n})\}$$

with $B_t = \omega_t$, $t \in [0, \infty)$ for $\omega \in \Omega$, and

$$Lip(\Omega) := \bigcup_{n=1}^{\infty} Lip(\Omega_n).$$

Let $(\xi_i)_{i=1}^{\infty}$ be a sequence of $d$-dimensional random vectors on a sublinear expectation space $(\hat{\Omega}, \hat{\mathcal{H}}, \hat{\mathbb{E}})$ such that $\xi_i$ is $G$-normal distributed and $\xi_{i+1}$ is independent from $(\xi_1, \cdots, \xi_i)$ for each $i = 1, 2, \cdots$. For each $X \in Lip(\Omega)$ with

$$X = \phi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}),$$

some $\phi \in C_{1,Lip}(\mathbb{R}^{d \times n})$ and $0 = t_0 < t_1 < \cdots < t_n < \infty$, define $G$-expectation $\hat{\mathbb{E}}[:]$ as

$$\hat{\mathbb{E}}[\phi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})] := \hat{\mathbb{E}}[\phi(\sqrt{t_1 - t_0} \xi_1, \cdots, \sqrt{t_n - t_{n-1}} \xi_n)].$$

And the related conditional $G$-expectation of

$$X = \phi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})$$

under $\Omega_{t_j}$ is defined by

$$\hat{\mathbb{E}}[X|\Omega_{t_j}] := \psi(B_{t_1}, \cdots, B_{t_j} - B_{t_{j-1}}),$$

where

$$\psi(x_1, \cdots, x_j) = \hat{\mathbb{E}}[\phi(x_1, \cdots, x_j, \sqrt{t_j+1 - t_j} \xi_{j+1}, \cdots, \sqrt{t_n - t_{n-1}} \xi_n)].$$
\( \hat{E}[\cdot] \) consistently defines a sublinear expectation on \( L_{ip}(\Omega) \) and \((B_t)_{t \geq 0}\) is a G-Brownian motion.

The sublinear expectation \( \hat{E}[\cdot] : L_{ip}(\Omega) \to \mathbb{R} \) defined through the above procedure is called a G-expectation. The corresponding canonical process \((B_t)_{t \geq 0}\) on the sublinear expectation space \((\Omega, L_{ip}(\Omega), \hat{E})\) is called a G-Brownian motion.

Let \( L_{pG}(\Omega_T), p \geq 1 \), denotes the completion of \( L_{ip}(\Omega_T) := \{ \varphi(B_{t_1 \wedge T}, \cdots, B_{t_n \wedge T}) : n \in \mathbb{N}, t_1, \cdots, t_n \in [0, \infty), \varphi \in C_{1, \text{Lip}}(\mathbb{R}^{d \times n}) \} \) under the norm \( \| X \|_p := (\hat{E}[|X|^p])^{\frac{1}{p}} \). And set \( L_{ip}(\Omega) := \bigcup_{n=1}^{\infty} L_{ip}(\Omega_n) \). The definition of G-expectation and conditional G-expectation can be extended to space \( L_{pG}(\Omega) \).

2.2 Conditional G-expectation in \( L^p \)

Let \( \mathcal{M} \) be the collection of all probability measures on \((\Omega, \mathcal{B}(\Omega))\), and
\( L^0(\Omega) \): the space of all \( \mathcal{B}(\Omega) \)-measurable real functions;
\( L^0(\Omega_t) \): the space of all \( \mathcal{B}(\Omega_t) \)-measurable real functions;
\( B_b(\Omega) \): all bounded elements in \( L^0(\Omega) \); \( B_b(\Omega_t) \): all bounded elements in \( L^0(\Omega_t) \);
\( C_b(\Omega) \): all bounded and continuous elements in \( L^0(\Omega) \); \( C_b(\Omega_t) \): all bounded and continuous elements in \( L^0(\Omega_t) \).

**THEOREM 2.2.** (Denis, et al.) There exists a weakly compact subset \( \mathcal{P} \subseteq \mathcal{M} \), such that

\[
\hat{E}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi], \forall \xi \in L_{ip}^1(\Omega).
\]

\( \mathcal{P} \) is called a set that represents \( \hat{E} \).

The upper expectation of probability measure set \( \mathcal{P} \) is defined in Huber and Strassen [3]: For each \( X \in L^0(\Omega) \) such that \( E_P[X] \) exists for each \( P \in \mathcal{P} \), the upper expectation about \( \mathcal{P} \) is defined as

\[
E[X] = E^{\mathcal{P}}[X] := \sup_{P \in \mathcal{P}} E_P[X].
\]

Denote

\[
c(A) = \sup_{P \in \mathcal{P}} P(A), A \in \mathcal{B}(\Omega).
\]

Then \( c(\cdot) \) is a Choquet capacity. A set \( A \) is called polar if \( c(A) = 0 \), and a property holds “quasi-surely” (q.s.) if it holds outside a polar set. Let
\[
\mathcal{L}^p := \{ X \in L^0(\Omega) : \mathbb{E}[|X|^p] < \infty \}, 0 < p < \infty,
\]
\[
\mathcal{L}^\infty := \{ X \in L^0(\Omega) : \exists \text{ a constant } M, \text{s.t.} |X| \leq M, \text{q.s.} \},
\]
\[
\mathcal{N} := \{ X \in L^0(\Omega) : X = 0, \text{c - q.s.} \},
\]
\[
\mathcal{L}^p := \mathcal{L}^p/\mathcal{N}.
\]
Then for $0 < p < 1$, $\mathbb{L}^p$ is a complete metric space under the distance $d(X, Y) := \mathbb{E}[|X - Y|^p]$, and for $1 \leq p < \infty$, $\mathbb{L}^p$ is a Banach space under the norm $\|X\|_p := (\mathbb{E}[|X|^p])^{\frac{1}{p}}$.

Similarly, we can define space $\mathbb{L}^p(\Omega_t)$. And we denote by $\mathbb{L}^p_b(\Omega_t)$ the completion of $B_b(\Omega_t)$ and $\mathbb{L}^p_c(\Omega_t)$ the completion of $C_b(\Omega_t)$ under norm $\|\cdot\|_p = \mathbb{E}[|\cdot|^p]$, $0 \leq t \leq \infty$.

Denis, et al.\cite{1} proved that $\mathbb{L}^1_c(\Omega_t) = \mathbb{L}^1_{G}(\Omega_t) \subset \mathbb{L}^1_b(\Omega_t)$.

$\mathbb{L}^\infty := \mathbb{L}^\infty / \mathcal{N}$ is a Banach space under the norm

$$\|X\|_{\infty} := \inf\{M \geq 0 : |X| \leq M, \text{q.s.}\}.$$

**LEMMA 2.3.** If for any $p > 0$, $X \in \mathbb{L}^p$, then $c(|X| = \infty) = 0$, that is $|X| < \infty$, q.s.

Proof. Otherwise, $c(|X| = \infty) > 0$, then

$$\mathbb{E}[|X|^p] \geq \mathbb{E}[|X|^p I_{\{|X| = \infty\}}] = \infty,$$

which means $X \notin \mathbb{L}^p$.

Denote

$$\mathbb{L}_S(\Omega_{s,t}) := \{\eta = \sum_{j=1}^N I_{A_j}\eta^j, \text{ where } \{A_j\}_{j=1}^N \text{ is an } \mathcal{F}_s - \text{partition of } \Omega, \text{ and } \eta^j, j = 1, \ldots, N \text{ are } \mathcal{F}_t^* - \text{measurable}\}.$$ 

**DEFINITION 2.4.** We define a mapping, $\mathbb{E}_s[\cdot] : \mathbb{L}_S(\Omega_{s,t}) \rightarrow \mathbb{L}_S(\Omega_{s,s})$ which has the following properties,

(i) if $\xi \geq \eta$, then $\mathbb{E}_s[\xi] \geq \mathbb{E}_s[\eta]$.
(ii) $\mathbb{E}_s[\eta] = \eta$, if $\eta \in \mathbb{L}_S(\Omega_{s,s})$.
(iii) $\mathbb{E}_s[\xi] - \mathbb{E}_s[\eta] \leq \mathbb{E}_s[\xi - \eta]$.
(iv) $\mathbb{E}_s[\eta \xi] = \eta^+ \mathbb{E}_s[\xi] + \eta^- \mathbb{E}_s[-\xi], \text{if } \eta \in \mathbb{L}_S(\Omega_{s,s})$.
(v) $\mathbb{E}_s[\eta] = \mathbb{E}[\eta]$, if $\eta$ is independent from $\mathcal{F}_s$.

Then for any $\eta \in \mathbb{L}_S(\Omega_{s,t})$, we have

$$I_{A_j} \mathbb{E}_s[\eta] = \mathbb{E}_s[I_{A_j} \sum_{j=1}^N I_{A_j}\eta^j] = \mathbb{E}_s[I_{A_j}\eta^j] = I_{A_j} \mathbb{E}[\eta^j].$$

Summarizing over $j$, we have

$$\mathbb{E}_s[\eta] = \sum_{j=1}^N I_{A_j} \mathbb{E}[\eta^j].$$

So we can define the conditional $G$-expectation of $\eta$ as

$$\mathbb{E}_s[\eta] := \mathbb{E}[\eta \mid \mathcal{F}_s] = \sum_{j=1}^N I_{A_j} \mathbb{E}[\eta^j].$$
Denote
\[ \mathbb{L}_S(\Omega_{s,t}) := \{ \eta = \sum_{j=1}^{N} I_{A_j} \eta^j, r, t \}, \] where \( \{A^i_j\}_{j=1}^N \) is an \( \mathcal{F}_s \) - partition of \( \Omega \),
\[ \eta^j, r, j = 1, \ldots, N \] are \( \mathcal{F}^s_r \) - measurable , \( \eta^j, r, j = 1, \ldots, N \) are \( \mathcal{F}^r_t \) - measurable }.

**Proposition 2.5.** \( \mathbb{E}_s[\cdot] : \mathbb{L}_S(\Omega_{s,t}) \to \mathbb{L}_s(\Omega_{s,s}) \) is dynamically consistent.

Proof. For any random variable \( \eta \in \mathbb{L}_s(\Omega_{s,t}) \), we have
\[
\mathbb{E}_s[\eta] = \sum_{j=1}^{N} I_{A_j} \mathbb{E}[\eta^j, r, t] = \sum_{j=1}^{N} I_{A^i_j} \mathbb{E}[\mathbb{x}[\eta^j, r, t]_x = \eta^j] \\
= \sum_{j=1}^{N} I_{A^i_j} \mathbb{E}[\eta^j, r, t].
\]

And
\[
I_{A^i_j} \mathbb{E}_r[\eta] = \mathbb{E}_r[I_{A^i_j} \eta^j, r, t] = I_{A^i_j} \eta^j, r, \mathbb{E}[\eta^j, r, t],
\]
Summarizing over \( j \), we have
\[
\mathbb{E}_r[\eta] = \sum_{j=1}^{N} I_{A^i_j} \eta^j, s, \mathbb{E}[\eta^j, r, t].
\]

Hence
\[
\mathbb{E}_s[\mathbb{E}_r[\eta]] = \mathbb{E}_s[\sum_{j=1}^{N} I_{A^i_j} \eta^j, r, t] = \sum_{j=1}^{N} I_{A^i_j} \mathbb{E}[\eta^j, r, t] = \mathbb{E}_s[\eta].
\]

So \( \mathbb{E}_s[\cdot] \) is dynamically consistent.

By the proof of lemma 43 of Denis, et al. [1], “the collection of processes \((\theta_t)_{t \in [s,T]}\) with \( \{\theta_t = \sum I_{A_j} \theta^j : \{A_j\}_{j=1}^N \} \) is an \( \mathcal{F}_s \) partition of \( \Omega \), \( \theta^j \) is \((\mathbb{F}^s)\)-adapted} is dense in \( \mathcal{A}_s \)”. So for any indicator function \( I_{A_j} \in \mathcal{F}_j \), there exist sequence \( \zeta^i = \sum_{j=1}^{N_1} I_{A^i_j} \eta^j, i = 1, \ldots \), where for every \( i \), \( \{A^i_1\}_{j=1}^{N_1} \) is an \( \mathcal{F}_s \)-partition of \( \Omega \) and \( \eta^j, i \) are \( \mathcal{F}^s_r \)-measurable, such that \( \zeta^i \to I_{A_i}, i \to \infty \).

While for any \( \mathcal{F}_t \)-measurable random variable \( \eta \), there exists simple function sequence \( \xi^i = \sum_{j=1}^{N_2} I_{A^i_j} \eta^j, i = 1, 2, \ldots \) where \( \{A^i_1\}_{j=1}^{N_2} \) is an \( \mathcal{F}_i \)-partition of \( \Omega \), and \( \eta^j, i \) are constants, such that \( \xi^i \to \eta, i \to \infty \).

So \( \mathbb{L}_S(\Omega_{s,t}) \) is dense in \( L^0(\Omega) \), and \( \mathbb{L}_S(\Omega_{s,s}) \) is dense in \( \mathbb{L}_s(\Omega_{s,s}) \) as well as in \( L^0(\Omega) \).

Hence for any \( \eta \in \mathbb{L}^1(\Omega) \), there exists \( \eta^i \in \mathbb{L}_S(\Omega_{s,t}) \) such that \( \eta^i \to \eta, i \to \infty \). We define the conditional G-expectation of \( \eta \) as
\[
\mathbb{E}_s[\eta] = \lim_{i \to \infty} \mathbb{E}_s[\eta^i]. \quad (2.1)
\]

The conditional G-expectation \( \mathbb{E}_s[\cdot] : \mathbb{L}^1(\Omega) \to \mathbb{L}^1(\Omega) \) defined in (2.1) has the following properties.
**PROPOSITION 2.6.** For each $X, Y \in \mathbb{L}^1(\Omega)$,

(i) if $X \geq Y$, then $\mathbb{E}_s[X] \geq \mathbb{E}_s[Y]$.

(ii) $\mathbb{E}_s[\eta] = \eta$, if $\eta \in \mathbb{L}^1(\Omega_s)$.

(iii) $\mathbb{E}_s[X] - \mathbb{E}_s[Y] \leq \mathbb{E}_s[X - Y]$.

(iv) $\mathbb{E}_s[\eta X] = \eta^+\mathbb{E}_s[X] + \eta^-\mathbb{E}_s[-X]$ for each bounded $\eta \in \mathbb{L}^1(\Omega_s)$.

(v) $\mathbb{E}_s[\mathbb{E}_t[X]] = \mathbb{E}_{t \wedge s}[X]$, in particular, $\mathbb{E}[\mathbb{E}_s[X]] = \mathbb{E}[X]$.

3 \hspace{1em} **Itô’s integral in $\mathbb{M}^2(0, T)$**

For $T \in \mathbb{R}^+$, a partition $\pi_T$ of $[0, T]$ is a finite ordered subset $\pi_T = \{t_0, t_1, \ldots, t_N\}$ such that $0 = t_0 < t_1 < \ldots < t_N = T$. Let $\mu(\pi_T) := \max\{|t_{i+1} - t_i| : i = 0, 1, \ldots, N - 1\}$, and use $\pi_T^N = \{t_0^N, t_1^N, \ldots, t_N^N\}$ to denote a sequence of partitions of $[0, T]$ such that $\lim_{N \to \infty} \mu(\pi_T^N) = 0$.

Let $p \geq 1$ be fixed. We consider the following type of simple processes: for a given partition $\pi_T = \{t_0, t_1, \ldots, t_N\}$ of $[0, T]$ we set

$$\eta_k(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega)I_{(t_k, t_{k+1})}(t),$$

where $\xi_k \in \mathbb{L}^p(\Omega_{t_k})$, $k = 0, 1, 2, \ldots, N - 1$ are given. The collection of these processes is denoted by $\mathbb{M}^{p,0}(0, T)$.

**DEFINITION 3.1.** For each $p \geq 1$, we denote by $\mathbb{M}^p(0, T)$ the completion of $\mathbb{M}^{p,0}(0, T)$ under the norm

$$\| \eta \|_{\mathbb{M}^p(0, T)} := \left\{ \mathbb{E}\left[ \int_0^T |\eta|^p dt \right] \right\}^{\frac{1}{p}}.$$

It is clear that $\mathbb{M}^p(0, T) \supset \mathbb{M}^q(0, T)$ for $1 \leq p \leq q$.

**DEFINITION 3.2.** For an $\eta \in \mathbb{M}^{p,0}(0, T)$ with $\eta_k(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega)I_{(t_k, t_{k+1})}(t)$, the related Bochner integral is

$$\int_0^T \eta(\omega)dt := \sum_{k=0}^{N-1} \xi_k(\omega)(t_{k+1} - t_k).$$

For each $\eta \in \mathbb{M}^{p,0}(0, T)$, set

$$\tilde{\mathbb{E}}_T[\eta] := \frac{1}{T} \mathbb{E}\left[ \int_0^T \eta(\omega) dt \right].$$

$\tilde{\mathbb{E}}_T : \mathbb{M}^{p,0}(0, T) \to \mathbb{R}$ forms a sublinear expectation, so under the natural norm $\| \eta \|_{\mathbb{M}^p(0, T)}$, the Bochner integral can be extended from $\mathbb{M}^{p,0}(0, T)$ to $\mathbb{M}^p(0, T)$.
We now introduce the definition of Itô’s integral. For simplicity, we first introduce Itô’s integral with respect to 1-dimensional G-Brownian motion.

Let \((B_t)_{t \geq 0}\) be a 1-dimensional G-Brownian motion with \(G(\alpha) = \frac{1}{2}(\sigma^2\alpha^+ - \sigma^2\alpha^-)\), where \(0 \leq \sigma \leq \bar{\sigma} < \infty\).

**DEFINITION 3.3.** For each \(\eta \in \mathbb{M}^{2,0}(0,T)\) with \(\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) I_{[t_k, t_{k+1})}(t)\), define

\[
I(\eta) = \int_0^T \eta_t dB_t := \sum_{k=0}^{N-1} \xi_k (B_{t_{k+1}} - B_{t_k}).
\]

**LEMMA 3.4.** The mapping \(I: \mathbb{M}^{2,0}(0,T) \rightarrow \mathbb{L}^2(\Omega_T)\) is a continuous linear mapping and thus can be continuously extended to \(I: \mathbb{M}^2(0,T) \rightarrow \mathbb{L}^2(\Omega_T)\), and we have

\[
\mathbb{E} \left[ \int_0^T \eta_t dB_t \right] = 0,
\]

\[
\mathbb{E} \left[ (\int_0^T \eta_t dB_t)^2 \right] \leq \bar{\sigma}^2 \mathbb{E} \left[ \int_0^T \eta_t^2 dt \right].
\]

Proof. Notice that \(B_{t_{i+1}} - B_{t_i}\) is independent of \(F_{t_i}\), so for \(\xi_i \in \mathbb{L}^1(\Omega_{t_i})\), we have

\[
\mathbb{E}[\xi_i(B_{t_{i+1}} - B_{t_i})] = \mathbb{E}[-\xi_i(B_{t_{i+1}} - B_{t_i})] = 0,
\]

and

\[
\mathbb{E}[\xi_i^2(B_{t_{i+1}} - B_{t_i})^2 - \bar{\sigma}^2\xi_i^2(t_{i+1} - t_i)] = \mathbb{E}[\mathbb{E}_t[\xi_i^2(B_{t_{i+1}} - B_{t_i})^2 - \bar{\sigma}^2\xi_i^2(t_{i+1} - t_i)]] = 0.
\]

Hence we get (3.1) and (3.2) by the same procedure as Peng [7].

**DEFINITION 3.5.** We define, for a fixed \(\eta \in \mathbb{M}^2(0,T)\), the stochastic integral

\[
\int_0^T \eta_t dB_t := I(\eta).
\]

It is clear (3.1) and (3.2) still holds for \(\eta \in \mathbb{M}^2(0,T)\).

The Itô’s integral has the following properties,

**PROPOSITION 3.6.** Let \(\xi, \eta \in \mathbb{M}^2(0,T)\), and let \(0 \leq s \leq r \leq t \leq T\). Then we have

\[
(i) \int_s^t \eta_u dB_u = \int_s^r \eta_u dB_u + \int_r^t \eta_u dB_u, \text{q.s.},
\]

\[
(ii) \int_s^t (\alpha \eta_u + \theta_u) dB_u = \alpha \int_s^t \eta_u dB_u + \int_s^t \theta_u dB_u, \text{ if } \alpha \in \mathbb{L}^2(\Omega_s),
\]

\[
(iii) \mathbb{E}[X + \int_s^T \eta_u dB_u | \Omega_s] = \mathbb{E}[X | \Omega_s], \text{ for } X \in \mathbb{L}^1(\Omega).
\]
For the multi-dimensional case. Let $G(\cdot) : S(d) \to \mathbb{R}$ be a given monotonic and sublinear function and let $(B_t)_{t \geq 0}$ be a d-dimensional G-Brownian motion. For each fixed $a \in \mathbb{R}^d$, we use $B_t^a := \langle a, B_t \rangle$. Then $(B_t^a)_{t \geq 0}$ is a 1-dimensional $G_a$-Brownian motion with $G_a(\alpha) = \frac{1}{2}(\sigma^2_{aa^T} \alpha^+ - \sigma^2_{aa^T} \alpha^-)$, where $\sigma^2_{aa^T} = 2G(aa^T)$ and $\sigma^2_{aa^T} = -2G(-aa^T)$. Similar to 1-dimensional case, we can define Itô’s integral by

$$I(\eta) := \int_0^T \eta_t dB_t^a,$$

for $\eta \in M^2(0, T)$.

Thus $Q_{0,T} : M^{1,0}(0, T) \to L^1(\Omega_T)$ is a continuous linear mapping. Consequently, $Q_{0,T}$ can be uniquely extended to $M^{1}(0, T)$, and we have

$$E[|\int_0^T \eta_t dB_t^a|] \leq \bar{\sigma}^2 E[\int_0^T |\eta_t| dt], \quad \forall \eta \in M^{1}(0, T).$$

Proof. Notice that $E[|\xi_j|(|\langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j}) - \bar{\sigma}^2|\xi_j|(|t_{j+1} - t_j)|] = 0$. Then it is easy to check that (3.5) as well as (3.6) holds.

**PROPOSITION 3.8.** For any $\eta \in M^{2}(0, T)$, we have

$$E[(\int_0^t \eta_s dB_s)^2] = E[\int_0^T \eta_s^2 d\langle B\rangle_s].$$

Proof. For $\eta \in M^{2,0}(0, T)$, it is easy to check that (3.7) holds. We can continuously extend the above equality to the case $\eta \in M^{2}(0, T)$ and get (3.7).

Similar to Li and Peng(2011), we can prove the following proposition.

**PROPOSITION 3.9.** For any $\xi \in M^{1}(0, T)$, $\eta \in M^{2}(0, T)$, and $0 \leq t \leq T$, $\int_0^t \xi_s ds$, $\int_0^t \eta_s dB_s$, and $\int_0^t \xi_s d\langle B\rangle_s$ are well defined processes which are continuous in $t$ quasi-surely.
A stopping time $\tau$ with respect to filtration $(\mathcal{F}_t)$ is a mapping $\tau : \Omega \to [0, T]$ such that for every $t$, $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$.

**LEMMA 3.10.** For each stopping time $\tau$ and $\eta \in \mathcal{M}^p(0, T)$, we have $I_{[0,\tau)}(\eta) = I_{[0,\tau]}(\eta) \in \mathcal{M}^p(0, T)$.

Proof. For a given stopping time $\tau$, let

$$
\tau_n = \sum_{k=1}^n t_k^\tau I_{[t_{k-1}^\tau, t_k^\tau]} + T I_{[\tau, \infty]}.
$$

Then

$$
I_{[\tau_n, T]}(t) = I_{[\sum_{i=1}^n t_i^\tau \mathbb{I}_{[t_{i-1}^\tau, t_i^\tau]} + T I_{[\tau_n, T]}]}(t)
$$

$$
= \sum_{i=1}^n I_{[t_i^\tau, T]}(t) I_{[t_{i-1}^\tau, t_i^\tau]} + T I_{[\tau_n, T]}
$$

$$
= \sum_{i=1}^n \sum_{k=1}^{n-1} I_{[t_k^\tau, t_{k+1}^\tau]}(t) I_{[t_{i-1}^\tau, t_i^\tau]} + T I_{[\tau_n, T]}
$$

$$
= \sum_{k=1}^{n-1} \left( \sum_{i=1}^k I_{[t_i^\tau, t_{i+1}^\tau]}(t) \right) I_{[t_{i-1}^\tau, t_i^\tau]}(t).
$$

Since $\sum_{i=1}^k I_{[t_i^\tau, t_{i+1}^\tau]} \in L^p(\Omega_{\mathcal{E}})$, we have $I_{[\tau_n, T]} \in \mathcal{M}^{p,0}(0, T)$.

For any $\eta \in \mathcal{M}^p(0, T)$, there exists a sequence of simple processes $\eta_i^\tau = \sum_k \mathbb{I}_{[t_k^\tau, t_{k+1}^\tau]}(t)$ with $\mathcal{F}_t \in \mathcal{M}^p(\Omega_{\mathcal{E}})$ such that $\eta_i^\tau \to \eta, i \to \infty$ in $\mathcal{M}^p(0, T)$. Obviously, $I_{[\tau_n, T]} \eta_i^\tau \in \mathcal{M}^{p,0}(0, T)$.

It is easy to check that $I_{[\tau_n, T]} \eta_i^\tau \to I_{[\tau_n, T]} \eta_i, i \to \infty$ in $\mathcal{M}^p(0, T)$, which means that $I_{[\tau_n, T]} \eta_i^\tau \in \mathcal{M}^p(0, T)$.

Now we prove $I_{[\tau_n, T]} \eta_i^\tau \to I_{[\tau_n, T]} \eta$ in $\mathcal{M}^p(0, T)$. We have

$$
\mathbb{E}\left[\int_0^T |I_{[\tau_n, T]}(t) \eta_i^\tau - I_{[\tau_n, T]}(t) \eta|^p dt\right]
\leq C \mathbb{E}\left[\int_0^T |I_{[\tau_n, T]}(t) \eta_i^\tau - I_{[\tau_n, T]}(t) \eta_i|^p dt\right] + C \mathbb{E}\left[\int_0^T |I_{[\tau_n, T]}(t) \eta_i^\tau - I_{[\tau_n, T]}(t) \eta|^p dt\right]
\leq \int_0^T |I_{[\tau_n, T]}(t) | \eta_i^\tau - \eta|^p dt\right].
$$

Since $\eta_i^\tau \to \eta, i \to \infty$ in $\mathcal{M}^p(0, T)$, for any $\epsilon > 0$, there exists $I$ such that when $i \geq I$, $\mathbb{E}\left[\int_0^T |\eta_i^\tau - \eta|^p dt\right] < \frac{\epsilon}{3}$. Hence for some fixed $i \geq I$, we have

$$
C \mathbb{E}\left[\int_0^T |I_{[\tau_n, T]}(t) \eta_i^\tau - I_{[\tau_n, T]}(t) \eta_i|^p dt\right] < \frac{\epsilon}{3},
$$

$$
C \mathbb{E}\left[\int_0^T |I_{[\tau_n, T]}(t) \eta_i - I_{[\tau_n, T]}(t) \eta|^p dt\right] < \frac{\epsilon}{3}.
$$

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and
\[
\mathbb{E}\left[ \int_0^T I_{[\tau_n,T]}(t)\eta^i_t - I_{[\tau,T]}(t)\eta^i_t dt \right] \leq \sum_k |\xi^i_k|^p \mu(\pi^n_T) < \frac{\epsilon}{3}, \text{ for some fixed } i \geq I \text{ and } n \text{ large enough}.
\]

So \( I_{[\tau_n,T]} \eta \to I_{[\tau,T]} \eta \) in \( M^p(0,T) \), which means \( I_{[\tau,T]} \eta \in M^p(0,T) \), and consequently \( I_{[0,\tau_n]} \eta \to I_{[0,\tau]} \eta \) in \( M^p(0,T) \). As a special case, \( I_{[0,\tau)} \in M^p(0,T) \).

By lemma 3.10, the integral \( \int_0^t I_{[0,\tau]}(s) dB_s \) and \( \int_0^t I_{[0,\tau]}(s) \eta_s dB_s \) for \( \eta \in M^2(0,T) \) is well defined.

**Lemma 3.11.** For each stopping time \( \tau \) and \( \eta \in M^p(0,T) \), we have
\[
\int_0^{t \wedge \tau} \eta_s dB_s = \int_0^t I_{[0,\tau]}(s) \eta_s dB_s, \text{ q.s.} \tag{3.8}
\]

Proof. Let
\[
\tau_n = \sum_{k=1}^n t^n_k I_{[t^n_{k-1},t^n_k]} + T I_{[\tau>T]} = \sum_{k=1}^n I_{[0,t^n_k]}(0<T),
\]
with \( A^n_k = [t^n_{k-1} < \tau \leq t^n_k] \), and \( A^n = [\tau > T] \).

For any \( \eta \in M^p(0,T) \), we have
\[
\int_0^{\tau_n} \eta_s dB_s = \int_0^{\sum_{k=1}^n I_{[t^n_{k-1},t^n_k]}} \eta_s dB_s = \sum_{k=1}^n I_{[0,t^n_k]}(0<T) \int_0^{t^n_k} \eta_s dB_s = \int_0^t \sum_{k=1}^n I_{[0,t^n_k]}(s) I_{[0,t^n_k-1]} \eta_s dB_s.
\]
Thus we have
\[
\int_0^{\tau_n} \eta_s dB_s = \int_0^t I_{[0,\tau_n]}(s) \eta_s dB_s, \tag{3.9}
\]
and
\[
\int_0^{t \wedge \tau} \eta_s dB_s = \int_0^t I_{[\tau_n,t \wedge \tau]}(s) \eta_s dB_s \to 0,
\]
by the continuity of \( \int_0^t \eta_s dB_s \). Hence
\[
\lim_{n \to \infty} \int_0^{\tau_n} \eta_s dB_s = \int_0^{t \wedge \tau} \eta_s dB_s = \lim_{n \to \infty} \int_0^{t \wedge \tau} I_{[0,t \wedge \tau]}(s) \eta_s dB_s = \int_0^{t \wedge \tau} \eta_s dB_s. \tag{3.10}
\]

By the proof of lemma 3.8, \( I_{[0,\tau_n]} \eta \to I_{[0,\tau]} \eta, n \to \infty, \text{ in } M^p(0,T) \), so we have
\[
\int_0^t I_{[0,\tau_n]}(s) \eta_s dB_s \to \int_0^t I_{[0,\tau]}(s) \eta_s dB_s \text{ in } L^2(\Omega_t).
\]

By Denis, Hu and Peng(2011) proposition 17, there exists a subsequence \( \int_0^t I_{[0,\tau_n]}(s) \eta_s dB_s \) such that
\[
\int_0^t I_{[0,\tau_n]}(s) \eta_s dB_s \to \int_0^t I_{[0,\tau]}(s) \eta_s dB_s, \text{ q.s., as } k \to \infty. \tag{3.11}
\]

From (3.9), (3.10), and (3.11), (3.8) holds.
LEMMA 3.12. For $\eta \in \mathbb{M}^2(s, t)$ with $s < t$, we have

$$
\mathbb{E}_s[\int_s^t \eta_d^2 du] \leq (t - s)\mathbb{E}_s[\int_s^t \eta_d^2 du],
$$
\hfill (3.12)

$$
\mathbb{E}_s[\int_s^t \eta_d d\langle B \rangle_u^2] \leq \bar{\sigma}^4(t - s)\mathbb{E}_s[\int_s^t \eta_d^2 du].
$$
\hfill (3.13)

Proof. For each fixed $\omega \in \Omega$, $\eta_d(\omega)$ is a measurable function on $[s, t]$. By lemma 2.3, $\int_s^t \eta_d^2 du < \infty, q.s.$, So for fixed $\omega \in \Omega$ such that $\int_s^t \eta_d^2 (\omega) du < \infty$, we have

$$
|\int_s^t \eta_d(\omega) du|^2 \leq (\int_s^t |\eta_d(\omega)| du)^2 \leq (t - s) \int_s^t |\eta_d(\omega)|^2 du.
$$
\hfill (3.14)

Then we have

$$
\mathbb{E}_s[\int_s^t \eta_d^2 dt] \leq (t - s)\mathbb{E}_s[\int_s^t \eta_d^2 dt].
$$
\hfill (3.15)

Now we prove (3.13). For $\eta^n = \sum_{i=0}^{n-1} \eta_i^n \mathbf{1}_{[t_i, t_{i+1})}$, we have

$$
\mathbb{E}_s[\int_s^t \eta_d^2 \langle B \rangle_u^2] = \mathbb{E}_s[\sum_{i=0}^{n-1} \eta_i^n (\langle B \rangle_{t_{i+1}} - \langle B \rangle_{t_i})^2] = \mathbb{E}_s[\sum_{i,j=0}^{n-1} |\eta_{i}^{n} \eta_{j}^{n}| (\langle B \rangle_{t_{i+1}} - \langle B \rangle_{t_i})(\langle B \rangle_{t_{j+1}} - \langle B \rangle_{t_j})]
$$
\hfill (3.16)

$$\leq \bar{\sigma}^4 \mathbb{E}_s[\sum_{i,j=0}^{n-1} |\eta_{i}^{n} \eta_{j}^{n}| (t_{i+1} - t_i)(t_{j+1} - t_j)] = \bar{\sigma}^4 \mathbb{E}_s[(\int_s^t \eta_d^2 dt)^2] \leq \bar{\sigma}^4 (t - s)\mathbb{E}_s[\int_s^t \eta_d^2 dt].$$

Thus (3.13) holds for $\eta^n \in \mathbb{M}^2,0(0, T)$. We can continuously extend the above equality to the case $\eta \in \mathbb{M}^2(0, T)$ and get (3.13).

4 Itô’s Formula

LEMMA 4.1. Let $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^n)$ with $\partial_t \varphi, \partial_{x^i} \varphi, \partial^2_{x^i x^j} \varphi \in C_{b, Lip}([0, T] \times \mathbb{R}^n)$ for $\mu, \nu = 1, \ldots, n$. Let $s \in [0, T]$ be fixed and $X = (X^1, \ldots, X^n)^T$ be an $n$-dimensional process on $[s, T]$ of the form

$$
X^i_t = X^i_s + \alpha^i(t - s) + \eta^\nu(t \langle B^i_t \rangle_t - \langle B^i_t \rangle_s) + \beta^\nu(B^i_t - B^i_s),
$$
\hfill (3.17)
where for \( \nu = 1, \ldots, n, i, j = 1, \ldots, d, \alpha^i, \eta^i \in L^4(\Omega_s), \beta^{ij} \in L^8(\Omega_s) \) and \( X_s = (X_s^1, \ldots, X_s^n)^T \) is a given random vector in \( L^2(\Omega_s) \). Then for each \( t \geq s \), we have, in \( L^2(\Omega_s) \),

\[
\varphi(t, X_t) - \varphi(s, X_s) = \int_s^t [\partial_t \varphi(u, X_u) + \partial_x \varphi(u, X_u) \alpha^i] du + \int_s^t \partial_{x^i} \varphi(u, X_u) \beta^{ij} dB_u^j \\
+ \int_s^t [\partial_x \varphi(u, X_u) \eta^{ij} + \frac{1}{2} \partial_{x^i x^j} \varphi(u, X_u) \beta^{ij} \beta^{kl}] d\langle B^k, B^l \rangle_u. 
\]

(4.1)

Here we use the above repeated indices \( \mu, \nu, i \) and \( j \) imply the summation.

**Proof.** For each positive integer \( N \), we set \( \delta_N = (t - s)/N \) and take the partition

\[
\pi^N_{[s, t]} = \{t_0^N, t_1^N, \ldots, t_N^N\} = \{s, s + \delta_N, \ldots, s + N\delta_N = t\}.
\]

We have

\[
\varphi(t, X_t) - \varphi(s, X_s) = \sum_{k=0}^{N-1} [\varphi(t_{k+1}^N, X_{t_{k+1}^N}) - \varphi(t_k^N, X_{t_k^N})]
\]

\[
= \sum_{k=0}^{N-1} \{\partial_t \varphi(t_k^N, X_{t_k^N})(t_{k+1}^N - t_k^N) + \partial_x \varphi(t_k^N, X_{t_k^N})(X_{t_{k+1}^N}^\nu - X_{t_k^N}^\nu)
\]

\[
+ \frac{1}{2} [\partial_{x^i x^j} \varphi(t_k^N, X_{t_k^N})(X_{t_{k+1}^N}^\nu - X_{t_k^N}^\nu)(X_{t_{k+1}^N}^\nu - X_{t_k^N}^\nu) + \kappa_k^N]\}
\]

(4.2)

where

\[
\kappa_k^N = \partial_{x^i} \varphi(t_k^N + \theta\delta_N, X_{t_k^N} + \theta(X_{t_{k+1}^N}^\nu - X_{t_k^N}^\nu))(t_{k+1}^N - t_k^N)^2
\]

\[
+ 2\partial_{x^i x^j} \varphi(t_k^N + \theta\delta_N, X_{t_k^N} + \theta(X_{t_{k+1}^N}^\nu - X_{t_k^N}^\nu))(t_{k+1}^N - t_k^N)(X_{t_{k+1}^N}^\nu - X_{t_k^N}^\nu)
\]

\[
+ [\partial_{x^i x^j} \varphi(t_k^N + \delta_N, X_{t_k^N} + \theta(X_{t_{k+1}^N}^\nu - X_{t_k^N}^\nu)) - \partial_{x^i x^j} \varphi(t_k^N, X_{t_k^N})(X_{t_{k+1}^N}^\nu - X_{t_k^N}^\nu)]^2
\]

with \( \theta \in [0, 1] \). We have, since \( \partial_{x^i x^j} \varphi \in C_b(L^p([0, T] \times \mathbb{R}^n)) \),

\[
\mathbb{E}[\sum_{k=0}^{N-1} \kappa_k^N |^2] \leq CN[\delta_N^2 + \delta_N^3] \to 0,
\]

where \( C \) is a constant independent of \( k \).

The rest terms in the summation of the right side of (4.2) are \( \xi_t^N + \zeta_t^N \) with

\[
\xi_t^N = \sum_{k=0}^{N-1} \{\partial_t \varphi(t_k^N, X_{t_k^N})(t_{k+1}^N - t_k^N)
\]

\[
+ \partial_x \varphi(t_k^N, X_{t_k^N})[\alpha^i(t_{k+1}^N - t_k^N) + \eta^{ij}(\langle B^i, B^j \rangle_{t_{k+1}^N} - \langle B^i, B^j \rangle_{t_k^N}) + \beta^{ij}(B_{t_{k+1}^N}^i - B_{t_k^N}^i)]
\]

\[
+ \frac{1}{2} \partial_{x^i x^j} \varphi(t_k^N, X_{t_k^N})\beta^{ij}(B_{t_{k+1}^N}^i - B_{t_k^N}^i)(B_{t_{k+1}^N}^j - B_{t_k^N}^j)\}.
\]
and

\[
\zeta^N_t = \frac{1}{2} \sum_{k=0}^{N-1} \{ \partial_{x^k}^2 \varphi(t^N_k, X_{t^N_k})[\alpha^k(t^N_k - t^N_{k+1}) + \eta^k|\alpha^k(t^N_k - t^N_{k+1})| + \eta^k|B^i - B^j|_{t^N_{k+1}} - (B^i, B^j)_{t^N_k}] \\
\times [\alpha^k (t^N_{k+1} - t^N_k) + \eta|\alpha^k (t^N_{k+1} - t^N_k)| + \eta|B^i - B^j|_{t^N_{k+1}} - (B^i, B^j)_{t^N_k}] \\
+ 2[\alpha^k (t^N_{k+1} - t^N_k) + \eta|\alpha^k (t^N_{k+1} - t^N_k)| + \eta|B^i - B^j|_{t^N_{k+1}} - (B^i, B^j)_{t^N_k}]\beta^k|\alpha^k(t^N_k - t^N_{k+1})|] \}.
\]

Now we prove \( \zeta^N_t \) converges to the right side of (4.1) and \( \zeta^N_t \) converges to 0 in \( L^2(\Omega) \).

Firstly, we have the following estimates.

\[
\mathbb{E}_s[\int_{t_k}^{t} |\partial_u \varphi(u, X_u) - \sum_{k=0}^{N-1} \partial_u \varphi(t^N_k, X_{t^N_k})I_{[t^N_k,t^N_{k+1})}(u)|^2 du] = \sum_{k=0}^{N-1} \int_{t^N_k}^{t_{k+1}} \mathbb{E}_s[|\partial_u \varphi(u, X_u) - \partial_u \varphi(t^N_k, X_{t^N_k})|^2] du \\
\leq \sum_{k=0}^{N-1} \int_{t^N_k}^{t_{k+1}} C_1 \mathbb{E}_s[|u - t^N_k|^2 + |X_u - X_{t^N_k}|^2] du \\
\leq \sum_{k=0}^{N-1} \int_{t^N_k}^{t_{k+1}} \mathbb{E}_s[|C_1 + C_2(\alpha^k)^2|u - t^N_k|^2 + C_2(\eta^k)^2|B^i - B^j|_{t^N_k}|^2 + C_2(\beta^k)^2|B^i - B^j|_{t^N_k}|^2] du \\
\leq (C_1 + C_2(\alpha^k)^2)(t - s)\delta_N^2 + C_2(\eta^k)^2(t - s)\delta_N^2 + C_2(\beta^k)^2(t - s)\delta_N^2.
\]

Similarly,

\[
\mathbb{E}_s[\int_{t_k}^{t} |\partial_{x^k} \varphi(u, X_u) - \sum_{k=0}^{N-1} \partial_{x^k} \varphi(t^N_k, X_{t^N_k})I_{[t^N_k,t^N_{k+1})}(u)|^2 du] \leq (C_1 + C_2(\alpha^k)^2)(t - s)\delta_N^2 + C_2(\eta^k)^2(t - s)\delta_N^2 + C_2(\beta^k)^2(t - s)\delta_N,
\]

\[
\mathbb{E}_s[\int_s^t |\partial_{x^k} \varphi(u, X_u) - \sum_{k=0}^{N-1} \partial_{x^k} \varphi(t^N_k, X_{t^N_k})I_{[t^N_k,t^N_{k+1})}(u)|^2 d(B^i - B^j)_u] \leq (C_1 + C_2(\alpha^k)^2)(t - s)\delta_N^2 + C_2(\eta^k)^2(t - s)\delta_N^2 + C_2(\beta^k)^2(t - s)\delta_N,
\]

and

\[
\mathbb{E}_s[\int_s^t |\partial_{x^k,x^l} \varphi(u, X_u) - \sum_{k=0}^{N-1} \partial_{x^k,x^l} \varphi(t^N_k, X_{t^N_k})I_{[t^N_k,t^N_{k+1})}(u)|^2 d(B^i - B^j)_u] \leq (C_1 + C_2(\alpha^k)^2)(t - s)\delta_N^2 + C_2(\eta^k)^2(t - s)\delta_N^2 + C_2(\beta^k)^2(t - s)\delta_N.
\]
Of course, (4.3)-(4.6) implies that \( \sum_{k=0}^{N-1} \partial \varphi(t_k^N, X_{t_k^N}) I_{t_k^N t_{k+1}^N}(\cdot) \) converges to \( \partial \varphi(\cdot, X) \), \( \sum_{k=0}^{N-1} \partial_x \varphi(t_k^N, X_{t_k^N}) I_{t_k^N t_{k+1}^N}(\cdot) \) converges to \( \partial_x \varphi(\cdot, X) \), and \( \sum_{k=0}^{N-1} \beta_{x,x}^2 \varphi(t_k^N, X_{t_k^N}) I_{t_k^N t_{k+1}^N}(\cdot) \) converges to \( \partial_{x,x}^2 \varphi(\cdot, X) \) in \( M^2(0, T) \).

Then

\[
\mathbb{E} \left[ \int_s^t \left( \partial_u \varphi(u, X_u) + \partial_x \varphi(u, X_u) \alpha^\nu d\mu^\nu \right) du + \partial_x \varphi(u, X_u) \beta^{ij} d\nu^i d\nu^j \right] \\
+ \mathbb{E} \left[ \sum_{k=0}^{N-1} \partial_x \varphi(t_k^N, X_{t_k^N}) I_{t_k^N t_{k+1}^N}(u) \right] \alpha^\nu \mathbb{E} du^2 \]

\[
\leq 5 \mathbb{E} \left[ \int_s^t \left( \partial_u \varphi(u, X_u) - \sum_{k=0}^{N-1} \partial_x \varphi(t_k^N, X_{t_k^N}) I_{t_k^N t_{k+1}^N}(u) \right) \alpha^\nu du^2 \right] \\
+ \mathbb{E} \left[ \sum_{k=0}^{N-1} \partial_x \varphi(t_k^N, X_{t_k^N}) I_{t_k^N t_{k+1}^N}(u) \right] \beta^{ij} \mathbb{E} du^2 \]

\[
\leq 5(t - s) \mathbb{E} \left[ \int_s^t \left( \partial_u \varphi(u, X_u) - \sum_{k=0}^{N-1} \partial_x \varphi(t_k^N, X_{t_k^N}) I_{t_k^N t_{k+1}^N}(u) \right) du^2 \right] \\
+ \mathbb{E} \left[ \alpha^\nu \mathbb{E} \left[ \int_s^t \left( \partial_x \varphi(u, X_u) - \sum_{k=0}^{N-1} \partial_x \varphi(t_k^N, X_{t_k^N}) I_{t_k^N t_{k+1}^N}(u) \right) du^2 \right] \right] \\
+ K \mathbb{E} \left[ \beta^{ij} \mathbb{E} \left[ \int_s^t \left( \partial_x \varphi(u, X_u) - \sum_{k=0}^{N-1} \partial_x \varphi(t_k^N, X_{t_k^N}) I_{t_k^N t_{k+1}^N}(u) \right)^2 du^2 \right] \right] \\
+ K \mathbb{E} \left[ \eta^{ij} \mathbb{E} \left[ \int_s^t \left( \partial_x \varphi(u, X_u) - \sum_{k=0}^{N-1} \partial_x \varphi(t_k^N, X_{t_k^N}) I_{t_k^N t_{k+1}^N}(u) \right)^2 du^2 \right] \right] \\
+ \frac{1}{2} K \mathbb{E} \left[ (\beta^{ij})^2 \mathbb{E} \left[ \int_s^t \left( \partial_x \varphi(u, X_u) - \sum_{k=0}^{N-1} \partial_x \varphi(t_k^N, X_{t_k^N}) I_{t_k^N t_{k+1}^N}(u) \right)^2 du^2 \right] \right].
\]

By estimates (4.3)-(4.6), and \( \alpha^\nu, \eta^{ij} \in L^4(\Omega_x) \), \( \beta^{ij} \in L^8(\Omega_x) \) we can prove the right side of (4.7) converges to 0 as \( N \to \infty \).
LEMMA 4.2. We then have proved (4.8).

Proof. For $\alpha$, the boundedness of $\partial_t^\nu \varphi$ and

$$\mathbb{E}[(\langle B^1, B^1 \rangle^N_t - \langle B^1, B^1 \rangle_t^N)^2]$$

$$= \mathbb{E}[(\langle B^1, B^1 \rangle_t^N - \langle B^1, B^1 \rangle_t^N)^2]$$

$$\leq \left\{ \mathbb{E}[(\langle B^1, B^1 \rangle_t^N - \langle B^1, B^1 \rangle_t^N)^4] \right\}^{\frac{1}{2}},$$

we have

$$\mathbb{E}[\zeta_t^N] \leq NC_3 \sum_{k=0}^{N-1} \left\{ \mathbb{E}[\partial^2_{x^i,x^j} \varphi(t^N_k, X_t^N)]^2 \right\} (\alpha^\nu)^4 (t^N_k - t^N_s)^4 + (\eta^\nu)^4 (\langle B^1, B^1 \rangle_t^N - \langle B^1, B^1 \rangle_t^N)^4$$

$$+ 2(\alpha^\nu)^2 (t^N_k - t^N_s)^2 (\langle B^1, B^1 \rangle_t^N - \langle B^1, B^1 \rangle_t^N)^2$$

$$+ 2(\eta^\nu)^2 (\langle B^1, B^1 \rangle_t^N - \langle B^1, B^1 \rangle_t^N)^2$$

$$\leq NC_4 \sum_{k=0}^{N-1} \left\{ \mathbb{E}[\alpha^\nu] + \langle B^1, B^1 \rangle_t^N \right\}^{\frac{1}{2}} + 2((\alpha^\nu)^2 + |\eta^\nu|^2)^2 \delta^3_N$$

$$\rightarrow 0, \text{ in } \mathbb{L}^2(\Omega_t).$$

We then have proved (4.8).

LEMMA 4.2. Let $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^n)$ such that $\partial_t \varphi, \partial_{x^i} \varphi, \partial^2_{x^i,x^j} \varphi \in C_b, L^\infty(\mathbb{R}^n)$. Let $X = (X^1, \ldots, X^n)$ and

$$X_t = X_s + \int_s^t \alpha^\nu du + \int_s^t \eta^\nu d\langle B^1, B^1 \rangle_u + \int_s^t \beta^\nu dB^1_u,$$

where $\alpha^\nu, \eta^\nu \in \mathbb{M}^4(0, T)$ and $\beta^\nu \in \mathbb{M}^8(0, T)$. Then for each $t \geq s$, we have, in $\mathbb{L}^2(\Omega_t)$,

$$\varphi(t, X_t) - \varphi(s, X_s) = \int_0^t [\partial_t \varphi(u, X_u) + \partial_{x^i} \varphi(u, X_u) \alpha^\nu_u] du + \int_0^t \partial_{x^i} \varphi(u, X_u) \beta^\nu du + \int_0^t \partial_{x^i} \varphi(u, X_u) \eta^\nu du$$

$$+ \int_0^t \partial_{x^i} \varphi(u, X_u) \beta^\nu + \frac{1}{2} \partial^2_{x^i,x^j} \varphi(u, X_u) \beta^\nu \beta^\nu du.$$

Proof. For $\alpha^\nu, \eta^\nu \in \mathbb{M}^4(0, T)$ and $\beta^\nu \in \mathbb{M}^8(0, T)$, there exist sequences of simple processes $\alpha^{m,\nu} \xrightarrow{\mathbb{M}^4} \alpha^\nu, \eta^{m,\nu} \xrightarrow{\mathbb{M}^4} \eta^\nu, \beta^{m,\nu} \xrightarrow{\mathbb{M}^8} \beta^\nu$ as $m \to \infty$. Let

$$X_t^m = X_s + \int_s^t \alpha^{m,\nu}_u du + \int_s^t \eta^{m,\nu}_u d\langle B^1, B^1 \rangle_u + \int_s^t \beta^{m,\nu}_u dB^1_u, m = 1, 2, \ldots.$$
From lemma 4.1 we have
\[
\varphi(t, X_t^m) - \varphi(s, X_s^m) = \int_s^t [\partial_t \varphi(u, X_u^m) + \partial_{x^m} \varphi(u, X_u^m) \alpha_u^{m, \nu}] du + \int_s^t \partial_{x^m} \varphi(u, X_u^m) \beta_u^{m, \nu} dB_u^j + \int_s^t [\partial_{x^m} \varphi(u, X_u^m) \eta_u^{m, \nu} + \frac{1}{2} \partial_{x^m, x^m}^2 \varphi(u, X_u^m) \beta_u^{m, \nu} \beta_u^{m, \nu}] d\langle B^j, B^j \rangle_u.
\]
For any \( s \leq t \leq T \),
\[
\mathbb{E} \left[ \sup_{s \leq t \leq T} |X_t - X_t^m|^4 \right] \leq C \mathbb{E} \int_s^T \left[ (\alpha_u^{m, \nu} - \alpha_u^{\nu})^4 + |\eta_u^{m, \nu} - \eta_u^{\nu}|^4 + |\beta_u^{m, \nu} - \beta_u^{\nu}|^4 \right] du \to 0,
\]
where \( C \) is a constant. Then we have proved \( \partial_{x^m} \varphi(\cdot, X^m) \alpha^{m, \nu} \to \partial_{x^m} \varphi(\cdot, X) \alpha^{\nu} \) in \( \mathbb{M}^2(0, T) \).
Similarly, we can prove that in \( \mathbb{M}^2(0, T) \),
\[
\begin{align*}
\partial \varphi(\cdot, X^m) &\to \partial \varphi(\cdot, X), \\
\partial_{x^m} \varphi(\cdot, X^m) \eta^{m, \nu} &\to \partial_{x^m} \varphi(\cdot, X) \eta^{\nu}, \\
\partial_{x^m} \varphi(\cdot, X^m) \beta^{m, \nu} &\to \partial_{x^m} \varphi(\cdot, X) \beta^{\nu}, \\
\partial_{x^m, x^m} \varphi(\cdot, X^m) \beta^{m, \nu} \beta^{m, \nu} &\to \partial_{x^m, x^m} \varphi(\cdot, X) \beta^{\nu} \beta^{\nu}.
\end{align*}
\]
We then pass to limit as \( m \to \infty \) in both sides of (4.9) to get (4.8).

**LEMMA 4.3.** Let \( \varphi \in C^{1,2}([0, T] \times \mathbb{R}^n) \) such that \( \varphi, \partial_t \varphi, \partial_{x^m} \varphi, \partial_{x^m, x^m}^2 \varphi \) are bounded and uniformly continuous on \( [0, T] \times \mathbb{R}^n \). Let \( X = (X^1, \ldots, X^n) \) and
\[
X_t = X_s + \int_s^t \alpha_u^\nu du + \int_s^t \eta_u^{\nu i} d\langle B^i, B^j \rangle_u + \int_s^t \beta_u^{\nu i} dB_u^i,
\]

(4.9)
where \( \alpha', \eta^{ij} \in \mathbb{M}^4(0, T) \) and \( \beta^{ij} \in \mathbb{M}^8(0, T) \). Then for each \( t \geq s \), we have, in \( \mathbb{L}^2(\Omega_t) \),

\[
\varphi(t, X_t) - \varphi(s, X_s) = \int_0^t [\partial_t \varphi(u, X_u) + \partial_x \varphi(u, X_u) \alpha^{ij}_u] du + \int_0^t \partial_{x^i} \varphi(u, X_u) \beta^{ij}_u dB^j_u
\]

\[
+ \int_0^t [\partial_{x^i} \varphi(u, X_u) \eta^{ij}_u + \frac{1}{2} \partial^2_{x^i x^j} \varphi(u, X_u) \beta^{ij}_u \beta^{ij}_u dB^i_u, B^j_u].
\]

(4.10)

Proof. We take \( \{ \varphi_m \}_{m=1}^\infty \) such that, for each \( m \), \( \varphi_m \) and all its first order and second order derivatives are in \( C_{b,\mathbb{L}^p}([0, T] \times \mathbb{R}^n) \) and such that, as \( m \to \infty \), \( \varphi_m, \partial_t \varphi_m, \partial_x \varphi_m \), and \( \partial^2_{x^i x^j} \varphi_m \) converge respectively to \( \varphi, \partial_t \varphi, \partial_x \varphi \), and \( \partial^2_{x^i x^j} \varphi \) uniformly on \([0, T] \times \mathbb{R}^n\). We then use the Itô’s formula (4.8) to \( \varphi_m(t, X_t) \), i.e.,

\[
\varphi_m(t, X_t) - \varphi_m(s, X_s) = \int_0^t [\partial_t \varphi_m(u, X_u) + \partial_x \varphi_m(u, X_u) \alpha^{ij}_u] du + \int_0^t \partial_{x^i} \varphi_m(u, X_u) \beta^{ij}_u dB^j_u
\]

\[
+ \int_0^t [\partial_{x^i} \varphi_m(u, X_u) \eta^{ij}_u + \frac{1}{2} \partial^2_{x^i x^j} \varphi_m(u, X_u) \beta^{ij}_u \beta^{ij}_u dB^i_u, B^j_u].
\]

Passing to the limit as \( m \to \infty \), we get (4.10).

**THEOREM 4.4.** Let \( \varphi \in C^{1,2}([0, T] \times \mathbb{R}^n) \) and \( X = (X^1, \ldots, X^n) \) with

\[
X_t = X_0 + \int_0^t \alpha_u^i du + \int_0^t \eta^{ij}_u d\langle B^i, B^j \rangle_u + \int_0^t \beta^{ij}_u dB^j_u,
\]

where \( \alpha', \eta^{ij} \in \mathbb{M}^4(0, T) \) and \( \beta^{ij} \in \mathbb{M}^8(0, T) \). Then for each \( t \geq s \), we have, in \( \mathbb{L}^2(\Omega_t) \),

\[
\varphi(t, X_t) - \varphi(s, X_s) = \int_0^t [\partial_t \varphi(u, X_u) + \partial_x \varphi(u, X_u) \alpha^{ij}_u] du + \int_0^t \partial_{x^i} \varphi(u, X_u) \beta^{ij}_u dB^j_u
\]

\[
+ \int_0^t [\partial_{x^i} \varphi(u, X_u) \eta^{ij}_u + \frac{1}{2} \partial^2_{x^i x^j} \varphi(u, X_u) \beta^{ij}_u \beta^{ij}_u dB^i_u, B^j_u].
\]

(4.11)

Proof. For simplicity, we only prove the case of \( n = d = 1 \). We set

\[
\gamma_t = |X_t| + \int_0^t (|\beta_u| + |\beta_u|^3 + |\alpha_u|^4) du,
\]

and for \( k = 1, 2, \ldots, \tau_k = \inf\{t \geq 0; \gamma_t \geq k\} \). Let \( \varphi_k \in C^{1,2}([0, T] \times \mathbb{R}^n) \), such that \( \partial_t \varphi_k, \partial_x \varphi_k, \partial_{x^i x^j} \varphi_k \) are bounded and uniformly continuous and \( \varphi_k = \varphi \), for \( |x| \leq 2k \) and \( t \in [0, T] \). By lemma [3.10] \( I_{[0,\tau_k]} \alpha, I_{[0,\tau_k]} \eta, I_{[0,\tau_k]} \beta \in \mathbb{M}^2(0, T) \), so we have

\[
X_{t \wedge \tau_k} = X_0 + \int_0^{t \wedge \tau_k} \alpha_u^i du + \int_0^{t \wedge \tau_k} \eta_u^i dB^i_u + \int_0^{t \wedge \tau_k} \beta_u^i dB^i_u,
\]

Then we can apply the Itô’s formula (4.10) to \( \varphi_k(t, X_{t \wedge \tau_k}) \) to obtain

\[
\varphi_k(t, X_{t \wedge \tau_k}) - \varphi_k(s, X_s) = \int_0^t [\partial_t \varphi_k(u, X_u) + \partial_x \varphi_k(u, X_u) \alpha_u] du
\]

\[
+ \int_0^t \partial_x \varphi_k(u, X_u) \beta_u I_{[0,\tau_k]} dB^i_u
\]

\[
+ \int_0^t [\partial_x \varphi_k(u, X_u) \eta_u + \frac{1}{2} \partial^2_{x^i x^j} \varphi_k(u, X_u) \beta^2_u I_{[0,\tau_k]} dB^i_u, B^j_u].
\]
Passing to the limit as $k \to \infty$, we get (4.11).

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