SPINORIAL REPRESENTATIONS OF SYMMETRIC GROUPS

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Abstract. A real representation \( \pi \) of a finite group may be regarded as a homomorphism to an orthogonal group \( O(V) \). For symmetric groups \( S_n \), alternating groups \( A_n \), and products \( S_n \times S_{n'} \) of symmetric groups, we give criteria for whether \( \pi \) lifts to the double cover \( \text{Pin}(V) \) of \( O(V) \), in terms of character values. From these criteria we compute the second Stiefel-Whitney classes of these representations.

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1. Introduction

A real representation \( \pi \) of a finite group \( G \) can be viewed as a group homomorphism from \( G \) to the orthogonal group \( O(V) \) of a Euclidean space \( V \). Recall the double cover \( \rho : \text{Pin}(V) \rightarrow O(V) \). We say that \( \pi \) is spinorial, provided it lifts to \( \text{Pin}(V) \), meaning there is a homomorphism \( \hat{\pi} : G \rightarrow \text{Pin}(V) \) so that \( \rho \circ \hat{\pi} = \pi \).

When the image of \( \pi \) lands in \( SO(V) \), the representation is spinorial precisely when its second Stiefel-Whitney class \( w_2(\pi) \) vanishes. Equivalently, when the associated vector bundle over the classifying space \( BG \) has a spin structure. (See Section 2.6 of [Ben98], [GKT89], and Theorem II.1.7 in [LM16].) Determining spinoriality of Galois representations also has applications in number theory: see [Ser84], [Del76], and [PR95].

In this paper we give lifting criteria for representations of the symmetric groups \( S_n \), the alternating groups \( A_n \), and a product \( S_n \times S_{n'} \) of two symmetric groups. Write \( s_i \in S_n \) for the transposition \( (i, i + 1) \), in cycle notation. A key result of this paper is the following:

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Theorem 1.1. Let \( n \geq 4 \).

1. A representation \( \pi \) of \( S_n \) is spinorial iff \( \chi_{\pi}(1) \equiv \chi_{\pi}(s_1s_3) \mod 8 \) and \( \chi_{\pi}(1) - \chi_{\pi}(s_1) \) is congruent to 0 or 6 mod 8.
2. A representation \( \pi \) of \( A_n \) is spinorial iff \( \chi_{\pi}(1) \equiv \chi_{\pi}(s_1s_3) \mod 8 \).

Combining this with the main result of [GPS] on character values, one deduces that as \( n \to \infty \), “100%” irreducible representations of \( S_n \) are spinorial. (See Corollary 3.6.)

Next, we leverage this result to compute the second Stiefel-Whitney classes for (real) representations \( \pi \) of \( S_n \):

\[
2w_2(\pi) = \left[ \frac{\chi_{\pi}(1) - \chi_{\pi}(s_1)}{4} \right] e_{\text{cup}} + \frac{\chi_{\pi}(1) - \chi_{\pi}(s_1s_3)}{4} w_2(\pi_n),
\]

where \( \pi_n \) is the standard representation of \( S_n \), and \( e_{\text{cup}} \in H^2(G,\mathbb{Z}/2\mathbb{Z}) \) is a certain cup product.

This formula allows us to compute the second Stiefel-Whitney classes of representations of \( S_n \times S_{n'} \) through Künneth theory, and therefore to identify spinorial representations of this product. To state the result, let \( \Pi = \pi \boxtimes \pi' \) be the external tensor product of representations \( \pi \) of \( S_n \) and \( \pi' \) of \( S_{n'} \). Let \( g = \frac{1}{2}(\chi_{\pi}(1) - \chi_{\pi}(s_1)) \), the multiplicity of \(-1\) as an eigenvalue of \( \pi(s_1) \) and similarly write \( g' \) for the corresponding quantity for \( \pi' \).

Theorem 1.2. The representation \( \Pi \) of \( S_n \times S_{n'} \) is spinorial iff the restrictions of \( \Pi \) to \( S_n \times \{1\} \) and \( \{1\} \times S_{n'} \) are spinorial, and

\[
\text{deg} \Pi + 1)gg' \equiv 0 \mod 2.
\]

We now describe the layout of this paper. Section 2 reviews the group \( \text{Pin}(V) \) and other conventions. The \( S_n \) case of Theorem 1.1 is proven in Section 3 by means of defining relations for the \( s_i \). Additionally we note corollaries of Theorem 1.1 primarily the aforementioned “100%” result, a connection with skew Young tableau numbers, and the important case of permutation modules, meaning the induction of the trivial character from a Young subgroup of \( S_n \). In particular we demonstrate that the regular representation of \( S_n \) is spinorial for \( n \geq 4 \).

Representations of the alternating groups are treated in Section 4 again via generators and relations. The main result is the \( A_n \) case of Theorem 1.1. We enumerate the spinorial irreducible representations of \( A_n \) in Theorem 4.2. Data for spinoriality of irreducible representations of \( S_n \) and \( A_n \) for small \( n \) is presented in Tables 1 and 2 of Section 5.

In Section 6 we review the axioms of Stiefel-Whitney classes of real representations, and then deduce the Stiefel-Whitney class of a real representation of \( S_n \). In Section 7 we apply Künneth theory to this formula to compute Stiefel-Whitney classes for real representations of \( S_n \times S_{n'} \). From this it is straightforward to deduce Theorem 1.2.

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2. Notation and Preliminaries

2.1. Representations. All representations are on finite-dimensional vector spaces, which are always real, except in Section 6 where they may be specified as complex. For a representation \((\pi, V)\) of a group \(G\), write \(\det \pi\) for the composition \(\det \circ \pi\); it is a linear character of \(G\). Also write \(\chi_x\) for the character of \(\pi\). If \(H \leq G\) is a subgroup, write \(\pi|_H\) for the restriction of \(\pi\) to \(H\). A real representation \(\pi : G \to \text{GL}(V)\) can be conjugated to have image in \(\text{O}(V)\), so we will assume that this is the case. When \(\det \pi\) is trivial, it maps to \(\text{SO}(V)\), and the spinoriality question is whether it lifts to the double cover \(\text{Spin}(V)\) (which we review in the next section).

Let \(\text{sgn} : S_n \to \{\pm 1\}\) be the usual sign character. For \(G = S_n\), we say that \(\pi\) is chiral provided \(\det \pi = \text{sgn}\) and \(\pi\) is achiral provided \(\det \pi = 1\). Write \(\pi_n : S_n \to \text{GL}_n(\mathbb{R})\) for the standard representation of \(S_n\) by permutation matrices.

2.2. The Pin Group. We essentially review [BtD95, Chapter 1] for defining the groups \(\text{Spin}(V)\) and \(\text{Pin}(V)\), where \(V\) is a Euclidean (i.e., a normed finite-dimensional real vector) space. The Clifford algebra \(C(V)\) is the quotient of the tensor algebra \(T(V)\) by the two-sided ideal generated by the set

\[\{v \otimes v + |v|^2 : v \in V\}\]

Write \(C(V)^\times\) for its group of units.

We identify \(V\) as a subspace of \(C(V)\) through the natural injection \(i : V \to C(V)\).

Write \(\alpha\) for the unique involution of the \(\mathbb{R}\)-algebra \(C(V)\) with the property that \(\alpha(x) = -x\) for \(x \in V\). One has

\[C(V) = C(V)^0 \oplus C(V)^1,\]

where \(C(V)^0\) is the 1-eigenspace of \(\alpha\) and \(C(V)^1\) is the \(-1\)-eigenspace.

Write \(t\) for the unique anti-involution of \(C(V)\) with \(t(x) = x\). For \(x \in C(V)\), define \(\overline{x} = t(\alpha(x))\); it is again an algebra anti-involution. Define

\[N : C(V) \to C(V)\]

by \(N(x) = x\overline{x}\). Put

\[\Gamma_V = \{x \in C(V)^\times \mid \alpha(x)Vx^{-1} = V\}.$

Let \(\rho : \Gamma_V \to \text{GL}(V)\) be the homomorphism given by \(v \mapsto \alpha(x)vx^{-1}\). We will repeatedly use the fact that if \(v\) is a unit vector, then \(\rho(v)\) is the reflection determined by \(v\). Write \(\text{Pin}(V)\) for the kernel of the restriction of \(N\) to \(\Gamma_V\). The restriction of \(\rho\) to \(\text{Pin}(V)\) is a double cover of \(\text{O}(V)\) with kernel \(\{\pm 1\}\). The preimage of \(\text{SO}(V)\) under \(\rho\) is denoted \(\text{Spin}(V)\). Alternately, \(\text{Spin}(V) = \text{Pin}(V) \cap C(V)^0\).

3. Symmetric Groups

3.1. Lifting Criteria. Let \(n \geq 2\). The group \(S_n\) is generated by the transpositions \(s_i = (i, i + 1)\) for \(1 \leq i \leq n - 1\), with the following relations:

\begin{align*}
(1) \quad & s_i^2 = 1, \quad 1 \leq i \leq n - 1, \\
(2) \quad & s_is_k = s_ks_i, \quad \text{when} \quad |i - k| > 1, \\
(3) \quad & (s_is_{i+1})^3 = 1, \quad 1 \leq i \leq n - 2.
\end{align*}

Therefore, defining a homomorphism from \(S_n\) to a group \(G\) is equivalent to choosing elements \(x_1, \ldots, x_{n-1} \in G\) satisfying the same relations. Let us call the relation \(x_i^2 = 1\) the “first lifting condition”, the relation \(x_ix_k = x_kx_i\) the “second
lifting condition”, and \((x_i x_{i+1})^3 = 1\) the “third lifting condition”. Note that this second condition is vacuous for \(n < 4\).

Let \(\pi : S_n \to O(V)\) be a representation of degree \(d\). For each \(\pi(s_i) \in O(V)\) there are \(\pm c_i \in \text{Pin}(V)\) with \(p(\pm c_i) = \pi(s_i)\), and the question is whether we may choose these signs so that the \(x_i = \pm c_i\) satisfy these lifting conditions.

Let \(g_\pi = \chi_\pi(1) - \chi_\pi(s_1)\), as in [APS17]. This is the multiplicity of the eigenvalue \(-1\) of \(\pi(s_1)\), and the eigenvalue 1 occurs with multiplicity \(d - g_\pi\). Put \(c_i = u_1 \cdots u_{g_\pi} \in \text{Pin}(V)\), where \(u_1, \ldots, u_{g_\pi}\) is an orthonormal basis of the \(-1\)-eigenspace of \(\pi(s_1)\). Since \(\pi(s_i)\) is the product of the reflections in each \(u_j\), the elements \(c_i\) and \(-c_i\) are the lifts of \(\pi(s_i)\). One computes that

\[
\left(\frac{c_i^2}{2}\right) = \left(-c_i^2\right) = \left(-1\right)^{\frac{1}{2}g_\pi(g_\pi+1)},
\]

and therefore the first lifting condition is satisfied iff \(g_\pi\) is congruent to 0 or 3 modulo 4. It does not matter for this whether we choose \(c_i\) or \(-c_i\).

Consider the sequence \((c_1 c_2)^3, (c_2 c_3)^3, \ldots \in \text{Pin}(V)\). Since each \((\pi(s_i) \pi(s_{i+1}))^3 = 1\), this must be a sequence of \(\pm 1\)'s. For the third lifting condition these must each be 1. Thus \(c_1\) may take either sign, but then the signs for \(c_2, c_3, \ldots\) are determined. Moreover this does not affect the first lifting condition. Thus:

**Proposition 3.1.** The first and third lifting conditions hold iff \(g_\pi \equiv 0 \text{ or } 3 \mod 4\).

Now let \(|i-k| > 1\), and suppose as above that \(c_i^2 = 1 = c_k^2 = 1\). Then the second lifting condition holds iff \((c_i c_k)^2 = 1\). By conjugating we may assume that \(i = 1\) and \(k = 3\). So put \(h_\pi = \frac{\chi_\pi(1) - \chi_\pi(s_1 s_3)}{2}\); as above the condition is equivalent to \(h_\pi \equiv 0, 3 \mod 4\). However:

**Lemma 3.2.** The integer \(h_\pi\) is even.

**Proof.** Let \(\zeta_4\) be a 4-cycle in \(S_n\). Then \(\zeta_4^3\) is conjugate to \(s_1 s_3\). Let \(m\) be the multiplicity of \(i = \sqrt{-1}\) as an eigenvalue of \(\pi(\zeta_4)\). Then \(h_\pi\), the multiplicity of \(-1\) as an eigenvalue of \(\pi(s_1 s_3)\), is \(2m\).

Therefore:

**Proposition 3.3.** The second lifting condition holds iff \(h_\pi\) is a multiple of 4.

\(\square\)

We summarize the above as the following:

**Theorem 3.4.** Let \(n \geq 4\), and \(\pi\) a representation of \(S_n\). The following are equivalent:

1. The representation \(\pi\) is spinorial.
2. \(g_\pi \equiv 0 \text{ or } 3 \mod 4\), and \(h_\pi \equiv 0 \mod 4\).
3. \(\chi_\pi(1) - \chi_\pi(s_1) \equiv 0 \text{ or } 6 \mod 8\), and \(\chi_\pi(1) \equiv \chi_\pi(s_1 s_3) \mod 8\).

When \(\pi\) is spinorial it has two lifts.

When \(\pi\) is spinorial, note that \(g_\pi \equiv 0\) iff \(\pi\) is achiral, and \(g_\pi \equiv 3\) iff \(\pi\) is chiral.

**Proof.** The equivalence should be clear; the two lifts correspond to the choice of sign for \(c_1\).

\(\square\)

**Remark:** The two lifts correspond to the two members of \(H^1(S_n, \mathbb{Z}/2\mathbb{Z})\); see Theorem II.1.7 of [LM16].
Corollary 3.5. A representation \( \pi \) of \( S_n \) is spinorial iff its restrictions to the cyclic subgroups \( \langle s_1 \rangle \) and \( \langle s_1 s_3 \rangle \) are both spinorial.

Proof. Indeed, the \( g_\pi \) condition corresponds to the subgroup \( \langle s_1 \rangle \), and the \( h_\pi \) condition corresponds to \( \langle s_1 s_3 \rangle \). \( \Box \)

The irreducible representations of \( S_n \) are the Specht modules \((\sigma_\lambda, V_\lambda)\), indexed by partitions of \( n \). (See for instance [JK09].) Write \( f_\lambda = f_{\pi_\lambda} \), and similarly for \( g_\lambda, h_\lambda \). Write \( p(n) \) for the number of partitions of \( n \).

Corollary 3.6. We have \( \lim_{n \to \infty} \frac{\# \{ \lambda \vdash n \mid \sigma_\lambda \text{ is achiral and spinorial} \} }{p(n)} = 1 \).

In other words, as \( n \to \infty \), 100% of irreducible representations of \( S_n \) are achiral and spinorial.

Proof. According to [GPS], as \( n \to \infty \), 100% of partitions \( \lambda \) of \( n \) have
\[ \chi_\lambda(1) \equiv \chi_\lambda(s_1) \equiv \chi_\lambda(s_1 s_3) \equiv 0 \pmod{8}. \]
The conclusion then follows from Theorem 1.1. \( \Box \)

3.2. Connection with Skew Young Tableaux. Let \( \mu, \lambda \) be partitions for which the Young diagram of \( \lambda \) contains that of \( \mu \). The notion of standard Young tableaux generalizes to “skew diagrams” \( \lambda/\mu \). Following Section 7.10 in [Sta99], write \( f_{\lambda/\mu} \) for the number of SYT on \( \lambda/\mu \). (If the Young diagram of \( \mu \) is not contained in that of \( \lambda \), put \( f_{\lambda/\mu} = 0 \).)

Proposition 3.7. We have
\[ (1) \ g_\lambda = f_{\lambda/(1,1)} \quad \text{and} \quad (2) \ h_\lambda = 2 \cdot (f_{\lambda/(3,1)} + f_{\lambda/(2,2)}). \]

Proof. Let \( \mu + k \) for some \( k \leq n \), and let \( \mu \) be the partition of \( n \) defined by adding \( (n-k) \) 1’s, i.e. \( \mu = \mu + \underbrace{\underbrace{1 + \cdots + 1}}_{(n-k) \text{ times}} \). Write \( w_\mu \in S_n \) be a permutation with cycle type \( \mu \). According to [Sta99], Exercise 7.62, we have
\[ \chi_\lambda(w_\mu) = \sum_{\nu \vdash k} \chi_\nu(w_\mu) \cdot f_{\lambda/\nu}. \]

Taking \( \mu = (2) \) gives
\[ \chi_\lambda(s_1) = f_{\lambda/(2)} - f_{\lambda/(1,1)}, \]
and taking \( \mu = (1,1) \) gives
\[ \chi_\lambda(1) = f_{\lambda/(2)} + f_{\lambda/(1,1)}, \]
so that \( g_\lambda = f_{\lambda/(1,1)} \).

Similarly, taking \( \mu = (2,2) \) and using the character table for \( S_4 \), we compute
\[ (3.1) \ \chi_\lambda(s_1 s_3) = f_{\lambda/(4)} - f_{\lambda/(3,1)} + 2f_{\lambda/(2,2)} - f_{\lambda/(2,1,1)} + f_{\lambda/(1,1,1,1)}. \]

Taking \( \mu = (1,1,1,1) \) gives
\[ (3.2) \ \chi_\lambda(1) = 3f_{\lambda/(4)} + 3f_{\lambda/(3,1)} + 2f_{\lambda/(2,2)} + 3f_{\lambda/(2,1,1)} + f_{\lambda/(1,1,1,1,1)}. \]

Combining (3.1) and (3.2) gives the formula for \( h_\lambda \). \( \Box \)
3.3. **Permutation Representations.** Another important class of representations of $S_n$ are the permutation representations, which are also indexed by partitions of $n$. Let $\lambda = (\lambda_1, \ldots, \lambda_{\ell}) \vdash n$, and consider the set $\mathcal{P}_\lambda$ of ordered partitions of $\{1, 2, \ldots, n\}$ with shape $\lambda$. Thus a member of $\mathcal{P}_\lambda$ is an $\ell$-tuple $(X_1, \ldots, X_\ell)$ of disjoint sets with each $|X_i| = \lambda_i$ and $\bigcup_{i=1}^\ell X_i = \{1, \ldots, n\}$. Note that $\mathcal{P}_\lambda$ has cardinality

$$\left(\begin{array}{c} n \\ \lambda_1, \ldots, \lambda_\ell \end{array}\right).$$

(3.3)

The group $S_n$ acts on $\mathcal{P}_\lambda$ in the obvious way, and we obtain the permutation representation $\mathbb{R}[\mathcal{P}_\lambda]$. This representation space is given by formal linear combinations of elements of $\mathcal{P}_\lambda$, so its degree is given by $(3.3)$.

For example, if $\lambda = (1, \ldots, 1) \vdash n$, then $\mathbb{R}[\mathcal{P}_\lambda]$ is the regular representation of $S_n$. If $\lambda = (n-1, 1)$, then $\mathbb{R}[\mathcal{P}_\lambda]$ is the standard representation $\pi_n$ of $S_n$ on $\mathbb{R}^n$. Note that $S_n$ acts transitively on $\mathcal{P}_\lambda$ with a stabilizer equal to the “Young subgroup” $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_{\ell}}$, so we can also view $\mathbb{R}[\mathcal{P}_\lambda]$ as the induction from $S_\lambda$ to $S_n$ of the trivial representation.

The characters of the $\mathbb{R}[\mathcal{P}_\lambda]$, though typically reducible, form an important basis of the representation ring of $S_n$. See, for example, Section 2.2 of [JK09].

Recall that, for a representation $\pi$ coming from a group action, the character value $\chi_\pi(g)$ is the number of fixed points of $g$. Write $\Theta_\lambda$ for the character $\chi_{\mathbb{R}[\mathcal{P}_\lambda]}$.

From this fixed-point principle we compute

$$\Theta_\lambda(s_1) = \sum_{|\lambda_i| \geq 2} \left(\begin{array}{c} n-2 \\ \lambda_i, \ldots, \lambda_i - 2, \ldots, \lambda_\ell \end{array}\right),$$

since a partition in $\mathcal{P}_\lambda$ is fixed by $s_1$ iff 1 and 2 lie in the same part $X_i$ for some $i$. Similarly, $\Theta_\lambda(s_1 s_3)$ equals

$$\sum_{1 \leq i < j \leq \ell \atop |\lambda_i| \geq 2, |\lambda_j| \geq 2} \left(\begin{array}{c} n-4 \\ \lambda_i, \ldots, \lambda_i - 2, \ldots, \lambda_j - 2, \ldots, \lambda_\ell \end{array}\right) + \sum_{|\lambda_i| \geq 4} \left(\begin{array}{c} n-4 \\ \lambda_i, \ldots, \lambda_i - 4, \ldots, \lambda_\ell \end{array}\right).$$

This is because a partition in $\mathcal{P}_\lambda$ is fixed by $s_1 s_3$ iff either the elements 1, 2, 3, 4 all lie in the same part $X_i$, or 1, 2 lie in some $X_i$ and 3, 4 lie some other part $X_j$.

These character values may be used to compute $g_{\mathbb{R}[\mathcal{P}_\lambda]}$ and $h_{\mathbb{R}[\mathcal{P}_\lambda]}$. (Compare Lemma 17 in [APS17].) For instance if $\lambda = (1, \ldots, 1) \vdash n$, then $\Theta_\lambda(s_1) = \Theta_\lambda(s_1 s_3) = 0$, so $g_{\mathbb{R}[\mathcal{P}_\lambda]} = h_{\mathbb{R}[\mathcal{P}_\lambda]} = \frac{n^2}{2}$. Thus the regular representation of $S_n$ is achiral and spinorial.

The standard representation $\pi_n$ corresponds to $\lambda = (n-1, 1)$. For this $\lambda$ we have $\Theta_\lambda(s_1) = n - 2$, so $g_{\mathbb{R}[\mathcal{P}_\lambda]} = 1$ and it follows that $\pi_n$ is aspinorial.

For easy reference, we collect here results about common representations of $S_n$:

**Proposition 3.8.** For $n \geq 2$ the standard representation $\pi_n$ is achiral and aspinorial, and the sign representation is achiral and aspinorial. For $n \geq 4$, the regular representation of $S_n$ is achiral and spinorial.

4. **Alternating Groups**

Now we turn to the alternating group $A_n$, for $n \geq 4$. 

4.1. Spinoriality Criterion. The group $A_n$ is generated by the permutations
\[ u_i = s_1 s_{i+1}, \quad (i = 1, 2, \ldots, n - 2) \]
with relations:
\[ u_i^3 = u_i^2 = (u_{j-1} u_j)^3 = 1, \quad (2 \leq j \leq n - 2), \]
\[ (u_i u_j)^2 = 1, \quad (1 \leq i < j - 1, j \leq n - 2). \]

(See for instance [CM80].)

Note that $u_1$ is a 3-cycle and the other $u_i$ are $(2, 2)$-cycles.

For a real representation $(\pi, V)$ of $A_n$ again put $h_\pi = \frac{\chi_\pi(1) - \chi_\pi(s_1 s_3)}{2}$. Since this is the multiplicity of the eigenvalue $-1$ of $\pi(s_1 s_3)$, which has determinant 1, the integer $h_\pi$ is necessarily even.

**Theorem 4.1.** A real representation $(\pi, V)$ of $A_n$ is spinorial if and only if $h_\pi$ is a multiple of 4. In this case there is a unique lift.

**Proof.** As in Section 3.1 we may choose $c_i$ with $\rho(c_i) = \pi(u_i)$ satisfying the same relations as the $u_i$. Let $c_1$ be a lift of $\pi(u_1)$. Since $\rho(c_1)^3 = \pi(u_1)^3 = 1$, we have $c_1^3 = \pm 1$. This determines the sign of $c_1$.

The $u_j$ and $u_i u_j$ as above, for $j > 1$, are all conjugate to $u_2$ in $A_n$. Therefore all the conditions $c_j^2 = 1$ and $(c_i c_j)^2 = 1$ are equivalent to the condition $c_2^2 = 1$. As before, this is equivalent to $h_\pi$ being congruent to 0 or 3 mod 4, but since $h_\pi$ is even, it must be a multiple of 4.

Finally, there is a unique choice of signs normalizing $c_2, \ldots, c_{n-2}$ so that
\[ (c_1 c_2)^3 = (c_2 c_3)^3 = \ldots = 1. \]

**Example:** If $\rho$ is the regular representation of $A_n$ (on the group algebra $\mathbb{R}[A_n]$), then $h_\rho = \frac{n!}{4}$, so $\rho$ is spinorial iff $n \neq 4, 5$.

**Example:** For the standard representation $\pi_n$ of $S_n$, $h_{\pi_n} = 2$, so the restriction of $\pi_n$ to $A_n$ is spinorial.

4.2. Real Irreducible Representations. Let us review the relationship between real and complex irreducible representations of a finite group $G$, following [BID95]. If $(\pi, V)$ is a complex representation of a group $G$, write $(\pi_\mathbb{R}, V_\mathbb{R})$ for the realization of $\pi$, meaning that we simply forget the complex structure on $V$ and regard it as a real representation. If moreover $(\pi, V)$ is an orthogonal complex representation, meaning that it admits a $G$-invariant symmetric nondegenerate bilinear form, then there is a unique real representation $(\pi_0, V_0)$, up to isomorphism, so that $\pi \cong \pi_0 \otimes_{\mathbb{R}} \mathbb{C}$.

It is not hard to see that $\pi_0$ is self-dual iff $\pi$ is self-dual, and that an orthogonal $\pi$ is spinorial, i.e., lifts to Spin$(V)$, iff $\pi_0$ is spinorial, i.e., lifts to Spin$(V_0)$.

Every real irreducible representation $\sigma$ of $G$ is either of the form
\begin{align*}
(1) & \quad \sigma = \pi_0, \text{ for an orthogonal irreducible complex representation } \pi \text{ of } G, \\
(2) & \quad \sigma = \pi_\mathbb{R}, \text{ for an irreducible complex representation } \pi \text{ of } G \text{ which is not orthogonal.}
\end{align*}

In the case of $G = S_n$, all complex representations are orthogonal.
4.3. Real Irreducible Representations of $A_n$. For a partition $\lambda$, write $\lambda'$ for its conjugate partition. Furthermore write $\epsilon_\lambda = 1$ when the number of cells in the Young diagram of $\lambda$ above the diagonal is even, and $\epsilon_\lambda = -1$ when this number is odd.

For example, let $\lambda = (4, 3, 2, 1)$. Then $\lambda = \lambda'$, and there are 4 cells above the diagonal, shaded in the Young diagram below, so $\epsilon_\lambda = 1$.

```
   0 0 0 1
   0 0 1 1
   0 1 1 2
```

Let $\sigma_\lambda$ be the (real) Specht module corresponding to $\lambda$, as before. Write $\pi_\lambda = \sigma_\lambda \otimes \mathbb{C}$ for its complexification, i.e., the complex Specht module corresponding to $\lambda$.

If $\lambda \neq \lambda'$, then $\pi_\lambda$ restricts irreducibly to $A_n$. When $\lambda = \lambda'$, the restriction of $\pi_\lambda$ to $A_n$ decomposes into a direct sum of two nonisomorphic representations $\pi_\lambda^+$ and $\pi_\lambda^-$. Either of $\pi_\lambda^\pm$ is the twist of the other by $\sigma_\lambda(w)$ for any odd permutation $w$.

The set of $\pi_\lambda$ with $\lambda \neq \lambda'$, together with the $\pi_\lambda^\pm$ for $\lambda = \lambda'$, is a complete set of irreducible complex representations of $A_n$.

For $\lambda = \lambda'$ we have

$$
\chi_\lambda^+(s_1s_3) = \chi_\lambda^-(s_1s_3) = \frac{1}{2} \chi_\lambda(s_1s_3).
$$

If moreover $\epsilon_\lambda = 1$, then the representations $\pi_\lambda^+$ and $\pi_\lambda^-$ are orthogonal. We may then define real irreducible representations of $A_n$ by $\sigma_\lambda|A_n \subseteq \pi_\lambda^+$ and $\sigma_\lambda|A_n \subseteq \pi_\lambda^-$. Since

$$(\pi_\lambda^+)_R \oplus (\pi_\lambda^-)_R \cong (\sigma_\lambda|A_n \otimes \mathbb{C})_R \cong \sigma_\lambda|A_n,$$

we have isomorphisms of real $A_n$-representations:

$$(\pi_\lambda^+)_R \cong (\pi_\lambda^-)_R \cong \sigma_\lambda|A_n.$$
whether the restriction of $\sigma_\lambda$ to $A_n$ is spinorial, by Theorem 4.2. Table 2 below lists for self-conjugate $\lambda$ with $\epsilon_\lambda = 1$, whether the constituents $\sigma_\lambda^+$ and $\sigma_\lambda^-$ are spinorial, following Theorem 4.2. This is done for all such $\lambda$ with $3 \leq |\lambda| \leq 15$.

Table 1. Spinoriality/Chirality of $\sigma_\lambda$ with $2 \leq |\lambda| \leq 6$

| $\lambda$ | Chirality of $\sigma_\lambda$ | Spinoriality of $\sigma_\lambda$ | Spinoriality of $\sigma_\lambda|_{A_n}$ |
|----------|-----------------------------|-------------------------------|-------------------------------------|
| $|\lambda| = 2$ |                                 |                               |                                     |
| (2)      | achiral                     | spinorial                     | spinorial                           |
| (1,1)    | chiral                      | spinorial                     | spinorial                           |
| $|\lambda| = 3$ |                                 |                               |                                     |
| (3)      | achiral                     | spinorial                     | spinorial                           |
| (2,1)    | chiral                      | aspinorial                    | spinorial                           |
| (1,1)    | chiral                      | aspinorial                    | spinorial                           |
| $|\lambda| = 4$ |                                 |                               |                                     |
| (4)      | achiral                     | spinorial                     | spinorial                           |
| (3,1)    | chiral                      | aspinorial                    | aspinorial                          |
| (2,2)    | chiral                      | aspinorial                    | spinorial                           |
| (2,1^2)  | achiral                     | aspinorial                    | aspinorial                          |
| (1^4)    | chiral                      | aspinorial                    | spinorial                           |
| $|\lambda| = 5$ |                                 |                               |                                     |
| (5)      | achiral                     | spinorial                     | spinorial                           |
| (4,1)    | chiral                      | aspinorial                    | aspinorial                          |
| (3,2)    | achiral                     | aspinorial                    | aspinorial                          |
| (3,1^2)  | chiral                      | spinorial                     | spinorial                           |
| (2^2,1)  | chiral                      | aspinorial                    | aspinorial                          |
| (2,1^3)  | chiral                      | aspinorial                    | aspinorial                          |
| (1^5)    | chiral                      | aspinorial                    | spinorial                           |
| $|\lambda| = 6$ |                                 |                               |                                     |
| (6)      | achiral                     | spinorial                     | spinorial                           |
| (5,1)    | chiral                      | aspinorial                    | aspinorial                          |
| (4,2)    | chiral                      | aspinorial                    | spinorial                           |
| (4,1^2)  | achiral                     | aspinorial                    | aspinorial                          |
| (3^4)    | achiral                     | aspinorial                    | aspinorial                          |
| (3,2,1)  | achiral                     | spinorial                     | spinorial                           |
| (3,1^4)  | achiral                     | aspinorial                    | aspinorial                          |
| (2^5)    | chiral                      | aspinorial                    | aspinorial                          |
| (2^2,1^4)| achiral                     | aspinorial                    | spinorial                           |
| (2,1^4)  | achiral                     | aspinorial                    | aspinorial                          |
| (1^6)    | chiral                      | aspinorial                    | spinorial                           |
Table 2. Spinoriality of $\sigma_\lambda^+ \times \chi$ with $\lambda = \lambda', \epsilon_\lambda = 1$, and $3 \leq |\lambda| \leq 15$

| $\lambda$    | $|\lambda|$ | $\sigma_\lambda^+$ |
|-------------|-------------|------------------|
| (3, 1, 1)   | 5           | aspinorial        |
| (3, 2, 1)   | 6           | spinorial         |
| (5, 1^4)    | 9           | spinorial         |
| (5, 2, 1^4) | 10          | spinorial         |
| (4, 3, 2, 1)| 10          | spinorial         |
| (4, 3, 3, 1)| 11          | aspinorial        |
| (7, 1^6)    | 13          | spinorial         |
| (7, 2, 1^4) | 14          | spinorial         |
| (6, 3^4, 1^4)| 15         | spinorial         |
| (5, 4, 3, 2, 1)| 15     | spinorial         |
| (4^3, 3)    | 15          | spinorial         |

6. Stiefel-Whitney Classes

6.1. Basic Properties. Let $G$ be a finite group and $\pi$ a real representation of $G$. Stiefel-Whitney classes $w_i(\pi)$ are defined for $0 \leq i \leq \deg \pi$ as members of the cohomology groups $H^i(G) = H^i(G, \mathbb{Z}/2\mathbb{Z})$. Here $\mathbb{Z}/2\mathbb{Z}$ is trivial as a $G$-module. One considers the total Stiefel-Whitney class in the $\mathbb{Z}/2\mathbb{Z}$-cohomology ring:

$$w(\pi) = w_0(\pi) + w_1(\pi) + \cdots + w_d(\pi) \in H^*(G) = \bigoplus_{i=0}^{\infty} H^i(G),$$

where $d = \deg \pi$.

According to, for example [GKT89], these characteristic classes satisfy the following properties:

1. $w_0(\pi) = 1$.
2. $w_1(\pi) = \det \pi$, regarded as a linear character in $H^1(G) \cong \text{Hom}(G, \{\pm 1\})$.
3. If $\pi'$ is another real representation, then $w(\pi \oplus \pi') = w(\pi) \cup w(\pi')$.
4. If $f : G' \to G$ is a group homomorphism, then $w(\pi \circ f) = f^*(w(\pi))$, where $f^*$ is the induced map on cohomology.
5. Suppose $\det \pi = 1$. Then $w_2(\pi) = 0$ iff $\pi$ is spinorial.

Note in particular that $w(\chi) = 1 + \chi$, if $\chi$ is a linear character of $G$.

The last property generalizes as follows:

**Proposition 6.1.** A real representation $\pi$ is spinorial iff $w_2(\pi) = w_1(\pi) \cup w_1(\pi)$.

We will deduce this proposition from the following lemma.

**Lemma 6.2.** Let $\pi' = \pi \oplus \det \pi$. Then $\pi$ is spinorial iff $\pi'$ is spinorial.

**Proof.** Let $V'$ be the representation space of $\pi'$; say $V' = V \oplus \mathbb{R}v'$ for some unit vector $v'$ perpendicular to $V$. Write $\iota : C(V) \to C(V')$ for the canonical injection. Note that if $x \in C(V)^0$, then $\iota(x)v' = v' \cdot \iota(x)$, and if $x \in C(V)^1$, then $\iota(x)v' = -v' \cdot \iota(x)$. Write $\rho' : \text{Pin}(V') \to O(V')$ for the usual double cover.

Define $\varphi : O(V) \to \text{SO}(V')$ by

$$\varphi(g) = g \oplus \det(g),$$

where $\det(g)$ is the determinant of $g$. Then $\varphi(g) = g \oplus \det(g)$ for all $g \in O(V)$. Hence $w_2(\pi) = w_1(\pi) \cup w_1(\pi)$.

We will deduce this proposition from the following lemma.
so that $\pi' = \varphi \circ \pi$. Write $\Phi_V < SO(V')$ for the image of $\varphi$. The essential problem is to construct a lift of $\varphi \circ \rho$. Since $\operatorname{Spin}(V) = \operatorname{Pin}(V) \cap C(V)^0$, the map $\tilde{\varphi} : \operatorname{Pin}(V) \to \operatorname{Spin}(V')$ defined by

$$
\tilde{\varphi}(x) = \begin{cases} 
\iota(x), & \text{if } x \in \operatorname{Spin}(V) \\
\iota(x)v', & \text{if } x \notin \operatorname{Spin}(V)
\end{cases}
$$

is a group homomorphism. Note that

$$(6.1)\quad \rho' \circ \tilde{\varphi} = \varphi \circ \rho.
$$

Let us see that $\tilde{\varphi}$ is injective; suppose $\tilde{\varphi}(x_1) = \tilde{\varphi}(x_2)$. Clearly if $x_1$ and $x_2$ are both in $\operatorname{Spin}(V)$, or both not in $\operatorname{Spin}(V)$, then $x_1 = x_2$. Suppose $x_1 \in \operatorname{Spin}(V)$ but $x_2 \notin \operatorname{Spin}(V)$. Then $v' = \iota(x_2^{-1}x_1)$ and in particular $v' \in \iota(C(V))$. But this is impossible, say by Corollary 6.7 in Chapter I of [BtD95]. We conclude that $\tilde{\varphi}$ is injective.

Write $\tilde{\Phi}_V < \operatorname{Spin}(V')$ for the image of $\tilde{\varphi}$; then $\tilde{\varphi}$ is an isomorphism from $\operatorname{Pin}(V)$ onto $\tilde{\Phi}_V$ and

$$(\rho')^{-1}\Phi_V = \tilde{\Phi}_V.
$$

If $\hat{\pi}$ is a lift of $\pi$, then $\tilde{\varphi} \circ \hat{\pi}$ is a lift of $\pi'$. Conversely, suppose $\hat{\pi}'$ is a lift of $\pi'$. Then its image lies in $\tilde{\Phi}_V$, and therefore $\hat{\pi}' = \tilde{\varphi} \circ \hat{\pi}$ for some homomorphism $\hat{\pi} : G \to \operatorname{Pin}(V)$. Since

$$
\rho' \circ \hat{\pi}' = \varphi \circ \pi,
$$

it follows that

$$
\varphi \circ \rho \circ \hat{\pi} = \varphi \circ \pi.
$$

Thus $\hat{\pi}$ is a lift of $\pi$.

\[\Box\]

Proof: (of Proposition 6.1) Note that $\det \pi' = 1$, so $\pi$ is spinorial iff $w_2(\pi') = 0$. But

$$
w(\pi') = w(\pi) \cup w(\det \pi) = (1 + \det \pi + w_2(\pi) + \cdots) \cup (1 + \det \pi) = 1 + w_2(\pi) + w_1(\pi) \cup w_1(\pi) + \cdots,
$$

whence the theorem. \[\Box\]

6.2. The group of order 2. Let $C$ be a cyclic group of order 2, and write ‘$\operatorname{sgn}$’ for its nontrivial linear character. Then $H^2(C) \cong \mathbb{Z}/2\mathbb{Z}$; the nonzero element is ‘$\operatorname{sgn} \cup \operatorname{sgn}$’. Let $\pi$ be the sum of $m$ copies of the trivial representation with $n$ copies of $\operatorname{sgn}$. Then

$$
w(\pi) = w(\operatorname{sgn}) \cup \cdots \cup w(\operatorname{sgn}) = 1 + n \cdot \operatorname{sgn} + \binom{n}{2} \cdot \operatorname{sgn} \cup \operatorname{sgn} + \cdots.
$$

In particular, $w_2(\pi) = \binom{n}{2} \cdot \operatorname{sgn} \cup \operatorname{sgn}$. By Proposition 6.1, $\pi$ is spinorial iff $n^2 \equiv \binom{n}{2} \mod 2$; equivalently, $n \equiv 0$ or 3 mod 4.
6.3. Calculation for $S_n$. Write

$$e_{\cup} = w_1(\text{sgn}) \cup w_1(\text{sgn}) = w_2(\text{sgn} \oplus \text{sgn}) \in H^2(S_n).$$

Again write $\pi_n$ for the standard representation of $S_n$ on $\mathbb{R}^n$. From [Ser84, Section 1.5] we know that $e_{\cup}$ and $w_2(\pi_n)$ comprise a basis for the $\mathbb{Z}/2\mathbb{Z}$-vector space $H^2(S_n)$.

**Proposition 6.3.** The map

$$\Phi : H^2(S_n) \to H^2(\langle s_1 \rangle) \oplus H^2(\langle s_1 s_3 \rangle),$$

given by the two restrictions, is an isomorphism for $n \geq 4$.

Thus the second $\mathbb{Z}/2\mathbb{Z}$-cohomology of $S_n$ is “detected” by these cyclic subgroups; compare Corollary 3.5 above and Theorem VI.1.2 in [AM04].

**Proof.** Since $\Phi$ is a linear map between 2-dimensional $\mathbb{Z}/2\mathbb{Z}$-vector spaces, it suffices to prove that its rank is 2. Let $b_1$ be the generator of $H^2(\langle s_1 \rangle)$, and $b_2$ be the generator of $H^2(\langle s_1 s_3 \rangle)$.

The restriction of $\pi_n$ to $\langle s_1 \rangle$ decomposes into a trivial $(n-1)$-dimensional representation plus one copy of $\text{sgn}$. The restriction to $\langle s_1 s_3 \rangle$ contains two copies of $\text{sgn}$. Therefore $\Phi(w_2(\pi_n)) = (0, b_2)$. Similarly $\Phi(w_2(\text{sgn} \oplus \text{sgn})) = \Phi(e_{\cup}) = (b_1, 0)$.

Thus $\Phi$ has rank 2, as required.

**Theorem 6.4.** For $\pi$ a real representation of $S_n$, with $n \geq 4$, we have

$$w_2(\pi) = \left[\frac{g_\pi}{2}\right] e_{\cup} + \frac{h_\pi}{2} w_2(\pi_n)$$

$$= \left[\chi_V(1) - \chi_V(s_1)\right] e_{\cup} + \frac{\chi_V(1) - \chi_V(s_1 s_3)}{4} w_2(\pi_n).$$

Here $[\cdot]$ denotes the greatest integer function.

**Proof.** Suppose first that $\pi$ is achiral. Since $e_{\cup}$ and $w_2(\pi_n)$ form a basis of $H^2(S_n)$ we must have

$$w_2(\pi) = c_1 e_{\cup} + c_2 w_2(\pi_n),$$

for some $c_1, c_2 \in \mathbb{Z}/2\mathbb{Z}$. Thus $\Phi(w_2(\pi)) = c_1 b_1 + c_2 b_2$. By the Stiefel-Whitney class properties [4] and [5].

$$c_1 = 0 \iff \pi|_{\langle s_1 \rangle} \text{ is spinorial} \iff 4 | g_\pi$$

and

$$c_2 = 0 \iff \pi|_{\langle s_1 s_3 \rangle} \text{ is spinorial} \iff 4 | h_\pi.$$ 

Thus $c_1 \equiv \frac{g_\pi}{2} \mod 2$ and $c_2 \equiv \frac{h_\pi}{2} \mod 2$.

If $\pi$ is chiral, then $\pi' = \pi \oplus \text{sgn}$ is achiral. From the identity $w_2(\pi) = w_2(\pi') + e_{\cup}$, we deduce that

$$w_2(\pi) = \frac{g_\pi - 1}{2} e_{\cup} + \frac{h_\pi}{2} w_2(\pi_n).$$

□
Thus $\pi$ is an element of $7.1$. machinery of Stiefel-Whitney classes.

Remark: Since the groups $H^2(A_n, \mathbb{Z}/2\mathbb{Z})$ have order 1 or 2, computing the Stiefel-Whitney class of a real representation of $A_n$ is equivalent to determining its spinoriality, which we have already done.

7. Products

Spinoriality for representations of $S_n \times S_n'$ can also be determined by means of generators and relations. (See Theorem 5.4.1 in [Gan19].) However we will instead obtain a satisfactory criterion by simply feeding our calculation of $w_2(\pi)$ into the machinery of Stiefel-Whitney classes.

7.1. External tensor products. Let $G, G'$ be finite groups, let $(\pi, V)$ be a real representation of $G$, and let $(\pi', V')$ be a real representation of $G'$. Write $\pi \boxtimes \pi'$ for the external tensor product representation of $G \times G'$ on $V \otimes V'$. One computes

$$\det(\pi \boxtimes \pi') = \det(\pi)^\deg \pi' \cdot \det(\pi')^{\deg \pi},$$

and hence

$$w_1(\pi \boxtimes \pi') = \deg \pi' \cdot w_1(\pi) + \deg \pi \cdot w_1(\pi'),$$

which is an element of

$$H^1(G \times G') \cong H^1(G) \oplus H^1(G').$$

The famous “splitting principle” (e.g., proceeding as in Problem 7-C of [MS16]) similarly gives

$$w_2(\pi \boxtimes \pi') = \deg \pi' \cdot w_2(\pi) + \left(\frac{\deg \pi'}{2}\right) w_1(\pi) \cup w_1(\pi) + (\deg \pi \deg \pi' - 1)w_1(\pi) \otimes w_1(\pi')$$

$$+ \left(\frac{\deg \pi}{2}\right) w_1(\pi') \cup w_1(\pi') + \deg \pi \cdot w_2(\pi'),$$

as an element of

$$H^2(G \times G') \cong H^2(G) \oplus (H^1(G) \otimes H^1(G')) \oplus H^2(G').$$

Finally, $w_2(\pi \boxtimes \pi') + w_1(\pi \boxtimes \pi') \cup w_1(\pi \boxtimes \pi')$ comes out to be

$$\deg \pi' \cdot w_2(\pi) + \left(\frac{\deg \pi' + 1}{2}\right) w_1(\pi) \cup w_1(\pi) + (\deg \pi \deg \pi' + 1)w_1(\pi) \otimes w_1(\pi')$$

$$+ \left(\frac{\dim \pi + 1}{2}\right) w_1(\pi') \cup w_1(\pi') + \deg \pi \cdot w_2(\pi').$$

Thus $\pi \boxtimes \pi'$ is spinorial (by Proposition 5.1) iff all of the following vanish:

1. $\deg \pi' \cdot w_2(\pi) + \left(\frac{\deg \pi' + 1}{2}\right) w_1(\pi) \cup w_1(\pi)$,
2. $(\deg \pi \deg \pi' + 1)w_1(\pi) \otimes w_1(\pi')$, and
3. $\left(\frac{\deg \pi + 1}{2}\right) w_1(\pi') \cup w_1(\pi') + \deg \pi \cdot w_2(\pi').$
7.2. **Products of Symmetric Groups.** We now prove Theorem 1.2. Let $\pi, \pi'$ be representations of $S_n$ and $S'_{n'}$. Write $f = f_\pi$, $f' = f_{\pi'}$ and similarly for $g, h, g'$ and $h'$. Let $\Pi = \pi \boxtimes \pi'$; all representations of $S_n \times S'_{n'}$ are sums of such representations.

**Proof.** By Proposition 6.1, $\Pi$ is spinorial iff $w_2(\Pi) = w_1(\Pi) \cup w_1(\Pi)$. From Theorem 6.4 and (1)-(3) of Section 7.1 we deduce that $\Pi$ is spinorial iff all of the following are even:

1. $f' \cdot \frac{h}{2}$,
2. $f' \left[ \frac{g}{2} \right] + (\frac{f' + 1}{2}) g$,
3. $(f f' + 1) g g'$,
4. $f \cdot \frac{h'}{2}$, and
5. $f \left[ \frac{g'}{2} \right] + (\frac{f + 1}{2}) g'$.

Note that if $\Pi$ is spinorial, then its restriction to $S_n \times \{1\}$, which amounts to $f'$ copies of $\pi$, is spinorial. From before, this implies that $f' \cdot \frac{h}{2}$ is even, and $f' g$ is congruent to 0 or 3 mod 4. One can verify this $f'g$ condition is equivalent to (2) being even. Thus (1),(2),(4), and (5) above are all even iff the restrictions of $\Pi$ to $S_n \times \{1\}$ and $\{1\} \times S'_{n'}$ are spinorial. Theorem 1.2 follows from this, since $ff' = \deg \Pi$. 

**References**

[AM04] Alejandro Adem and R. James Milgram. *Cohomology of finite groups*, volume 309 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, second edition, 2004.

[APS17] Arvind Ayyer, Amritanshu Prasad, and Steven Spallone. Representations of symmetric groups with non-trivial determinant. *J. Combin. Theory Ser. A*, 150:208–232, 2017.

[Ben98] D. J. Benson. *Representations and cohomology. II*, volume 31 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, second edition, 1998.

[BtD95] Theodor Bröcker and Tammo tom Dieck. *Representations of compact Lie groups*, volume 98 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1995.

[CM80] H. S. M. Coxeter and W. O. J. Moser. *Generators and relations for discrete groups*, volume 14 of Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas]. Springer-Verlag, Berlin-New York, fourth edition, 1980.

[Del76] Pierre Deligne. Les constantes locales de l’équation fonctionelle de la fonction $L$ d’Artin d’une représentation orthogonale. *Invent. Math.*, 35:299–316, 1976.

[Gan19] Jyotirmoy Ganguly. *Spinorial Representations of Symmetric and Alternating Groups*. PhD thesis, Indian Institute of Science Education and Research, Pune, 2019.

[GKT89] J. Guimarães, C. Kahn, and C. Thomas. *Stiefel-Whitney classes of real representations of finite groups*. *J. Algebra*, 126(2):327–347, 1989.

[GPS] Jyotirmoy Ganguly, Amritanshu Prasad, and Steven Spallone. On the divisibility of character values of the symmetric group. *arXiv preprint https://arxiv.org/abs/1904.12130*.

[JK09] Gordon James and Adalbert Kerber. *The Representation Theory of the Symmetric Group*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2009.

[LM16] H Blaine Lawson and Marie-Louise Michelsohn. *Spin geometry (pms-38)*, volume 38. Princeton university press, 2016.

[MS16] John Milnor and James D Stasheff. *Characteristic Classes (AM-76)*, volume 76. Princeton university press, 2016.
[PR95] Dipendra Prasad and Dinakar Ramakrishnan. Lifting orthogonal representations to spin groups and local root numbers. Proc. Indian Acad. Sci. Math. Sci., 105(3):259–267, 1995.

[Ser84] Jean-Pierre Serre. L’invariant de Witt de la forme $\text{Tr}(x^2)$. Comment. Math. Helv., 59(4):651–676, 1984.

[Sta99] Richard P. Stanley. Enumerative combinatorics. Vol. 2, volume 62 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

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