Generalizations of the relationship between quasi-hereditary algebras and directed bocses

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Abstract

Koenig, Külshammer and Ovsienko showed that Morita equivalence classes of quasi-hereditary algebras are in one-to-one correspondence with equivalence classes of the module categories over directed bocses. In this article, we extend this result to ∆-filtered algebras and ∆-filtered algebras.

1 Introduction

Quasi-hereditary algebras were introduced by Cline, Parshall and Scott to study the highest weight categories in Lie theory [CPS]. So far, many results have been obtained for quasi-hereditary algebras. For example, quasi-hereditary algebras have finite global dimensions, and every algebra is isomorphic to the endomorphism ring of a projective module over a quasi-hereditary algebra. On the other hand, bocses theory was introduced in the context of Drozd’s tame and wild dichotomy theorem and Crawley-Boevey applied it to analyze the module categories over tame algebras [C-B]. The module categories over bocses behave differently from those over algebras. Koenig, Külshammer and Ovsienko connected these theories by giving equivalences between the categories of modules over directed bocses and those of ∆-filtered modules over quasi-hereditary algebras.

In this article, we would like to extend their results to ∆-filtered algebras and ∆-filtered algebras. To do so, from Sections 2 to 5, we review several facts. In Section 2, we define quasi-hereditary, ∆-filtered (or standardly stratified algebras), and ∆-filtered algebras. In Section 3, we define bocses and the categories of modules over them. In Section 4, we introduce A∞-categories and, according to [Kel], show that every Ext-algebra has a structure of an A∞-category. In Section 5, we define twisted modules whose category gives an equivalence between F(∆) and mod B for the category F(∆) of ∆-filtered modules over a ∆-filtered algebra and a one-cyclic directed bocs B. Now we recall the result of [KKO] (we call this KKO theory), as our main interest of this article, which describes the relation between the quasi-hereditary algebras and directed bocses. Their main result is as follows.

Theorem 1.1 ([KKO] Theorem 1.1, Corollary 1.3, [BKK] Theorem 3.13). We have a bijection

\[ \{ \text{Morita equivalence classes of quasi-hereditary algebras} \} \leftrightarrow \{ \text{Equivalence classes of the module categories over directed bocses} \}. \]
Let a quasi-hereditary algebra $A$ and a directed bocs $B = (B, W)$ correspond via the above bijection. Then the right Burt-Butler algebra $R_B$ of $B$ is Morita equivalent to $A$. Moreover, $R_B$ has a homological exact Borel subalgebra $B$.

Section 6 gives a generalization of Theorem 1.1 to $\Delta$-filtered algebras and our theorem is as follows.

**Theorem 1.2** (Theorem 6.15). We have a bijection

\[
\{\text{Morita equivalence classes of } \Delta\text{-filtered algebras}\} \leftrightarrow \{\text{Equivalence classes of the module categories over one-cyclic directed bocses}\}.
\]

Let a $\Delta$-filtered algebra $A$ and a one-cyclic directed bocs $B = (B, W)$ correspond via the above bijection. Then the right Burt-Butler algebra $R_B$ of $B$ is Morita equivalent to $A$. Moreover, $R_B$ has a homological exact Borel subalgebra $B$.

When we generalize Theorem 1.1 in Theorem 6.15, we face three problems. The first problem concerns with the dimension of the Ext-algebra. In [KKO], the directed bocs is constructed by using the Ext-algebra of standard modules over a quasi-hereditary algebra. But in general, the Ext-algebra of properly standard modules over a $\Delta$-filtered algebra is not finite dimensional. To avoid infinite dimensional algebras, we will use a finite dimensional subspace of the Ext-algebra. In Subsection 6.1, we show that the method for construction of bocses by using the subspaces is the generalization of the one used in [KKO]. The second problem is on the dimension of $B$ of the bocs $B = (B, W)$ induced from a $\Delta$-filtered algebra. Since the Gabriel quiver of $B$ has loops but no cycles of length more than 1, it suffices to show that each $e_iB e_i$ is finite dimensional, which is of course equivalent to the fact that $B$ is so. And they are argued in Subsection 6.1. The third problem occurs in the discussion related to [DR] Theorem 2. It shows the relationship between Morita equivalence classes of quasi-hereditary algebras and the categories $\mathcal{F}(\Delta)$ of $\Delta$-filtered modules over quasi-hereditary algebras and it is proved by using the fact that an abelian category with some objects $\Theta$ realizes a quasi-hereditary algebra as an endomorphism algebra of Ext-projective objects of $\mathcal{F}(\Theta)$. But we can not similarly show the results on $\Delta$-filtered algebras in place of those for quasi-hereditary algebras, because in general, properly standard modules over $\Delta$-filtered algebras has self-extensions. Hence we can not construct Ext-projective objects of $\mathcal{F}(\Delta)$ by a way similar to that in [DR]. Thus, we use [ADL] Theorem 2.3, which claimed that for any module category over a finite dimensional algebra $C$, there exists a $\Delta$-filtered algebra $A$ such that $\mathcal{F}(\Delta_C) \simeq \mathcal{F}(\Delta_A)$. And this is discussed in Subsection 6.2.

Finally, we remark that a generalization of Theorem 1.1 to $\Delta$-filtered algebras is obtained by arguments which are almost the same as those given in [KKO]. The precise statement is as follows.

**Theorem 1.3** (see [BPS] Theorem 12.9). We have a bijection

\[
\{\text{Morita equivalence classes of } \Delta\text{-filtered algebras}\} \leftrightarrow \{\text{Equivalence classes of the module categories over weakly directed bocses}\}.
\]

Let a $\Delta$-filtered algebra $A$ and a weakly directed bocs $B = (B, W)$ correspond via the above bijection. Then the right Burt-Butler algebra $R_B$ of $B$ is Morita equivalent to $A$. Moreover, $R_B$ has a homological exact Borel subalgebra $B$. 

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Note that \( \Delta \)-filtered algebras and \( \Sigma \)-filtered algebras are quite natural generalizations of quasi-hereditary algebras, and it is concluded that our main results in this article (Theorem \[6.15\] above) and \[BPS\] theorem 12.9 generalize Theorem 1.1 to those algebras.

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2 Notations and definitions

Throughout this article, let \( K \) be an algebraically closed field and \( A \) a finite dimensional \( K \)-algebra with \( n \) simple modules (up to isomorphisms). The category of a finitely generated left \( A \)-modules will be denoted by \( \text{mod}_A \) and call its objects just \( A \)-modules. We denote simple \( A \)-modules by \( S_A(i) \) for \( 1 \leq i \leq n \) and corresponding projective indecomposable \( A \)-modules by \( P_A(i) \). But when there is not much danger of confusion, we also write \( S(i) \), \( P(i) \) for \( S_A(i) \), \( P_A(i) \), respectively. We write \( D \) to mean the standard \( K \)-dual \( \text{Hom}_K(-, K) \). For an \( A \)-module \( X \), we write the Jordan-H"older multiplicity of \( S(i) \) in \( X \) by \([X : S(i)]\).

We assume that all categories are \( K \)-categories. In this article, let \( \mathcal{L} \) be the trivial category over \( \{1, \ldots, n\} \). Let \( \mathcal{C} \) be a category and \( X \) and \( Y \) objects of \( \mathcal{C} \). We often write \( \mathcal{C}(X, Y) \) to mean \( \text{Hom}_\mathcal{C}(X, Y) \). The dual category \( \mathcal{D} \mathcal{C} \) of \( \mathcal{C} \) is defined as follows. Its objects coincide with those of \( \mathcal{C} \) and \( \mathcal{D} \mathcal{C}(X, Y) := \mathcal{D}(\mathcal{C}(X, Y)) \) where the right hand side is given by the \( K \)-dual as a vector space. A category \( \mathcal{C} \) is called \( \mathbb{Z} \)-graded if every space of morphisms \( \mathcal{C}(X, Y) \) have a decomposition \( \mathcal{C}(X, Y) = \bigoplus_{k \in \mathbb{Z}} \mathcal{C}(X, Y)_k \), and for \( f \in \mathcal{C}(X, Y)_k \) and \( g \in \mathcal{C}(Y, Z)_l \), their composition \( gf \) is in \( \mathcal{C}(X, Z)_{k+l} \). For an element \( f \in \mathcal{C}(X, Y)_k \), the label \( k \in \mathbb{Z} \) is called degree of \( f \) and we write by \([f]\) the degree of \( f \). Let \( A_1, \ldots, A_m \) be \( \mathbb{Z} \)-graded categories over \( \{1, \ldots, n\} \). Then their tensor product \( A_m \otimes \mathcal{L} \cdots \otimes \mathcal{L} A_1 \) over \( \mathcal{L} \) is the category whose objects are also \( 1, \ldots, n \), and

\[
(A_m \otimes \mathcal{L} \cdots \otimes \mathcal{L} A_1)((i_0, i_m)_k := \bigoplus_{1 \leq i_1 \leq \cdots \leq i_m \leq n, k=k_1+\cdots+k_m} A_m(i_{m-1}, i_m)_k \otimes \cdots \otimes \mathcal{K} A_1(i_0, i_1)_{k_m}.
\]

In particular, when \( A = A_1 = \cdots = A_m \), we write \( A \otimes^m = A_m \otimes \mathcal{L} \cdots \otimes \mathcal{L} A_1 \). Moreover, define the tensor category \( \mathcal{C}[A] \) of \( A \) over \( \mathcal{L} \) as follows. Its objects are also \( 1, \ldots, n \), and the \( K \)-vector space of morphisms is defined by \( \mathcal{C}[A](i, j) = \bigoplus_{r \geq 0} A^{\otimes r}(i, j) \) with \( A^{\otimes 0} = \mathcal{L} \).

Let

\[
X = ( \cdots \longrightarrow X_{i+1} \overset{\partial^X_i}{\longrightarrow} X_i \longrightarrow \cdots, \quad Y = ( \cdots \longrightarrow Y_{i+1} \overset{\partial^Y_i}{\longrightarrow} Y_i \longrightarrow \cdots )
\]

be complexes of \( A \)-modules. For any \( k \in \mathbb{Z} \), denote by \( Y[k] \) the complex \( Y \) shifted by \( k \) degrees to the left: \( Y[k]_i = Y_{i-k} \) and \( \partial^Y[k] = (-1)^k \partial^Y \). Then we write \( \text{Hom}_A(X, Y[k]) \) as the set of collections \((f^{(t)} : X_i \to Y_{i-k})_{i \in \mathbb{Z}}\) of \( A \)-homomorphisms i.e.

\[
\cdots \longrightarrow X_{i+1} \overset{\partial^X_i}{\longrightarrow} X_i \longrightarrow \cdots \quad \Downarrow f^{(i+1)} \quad \Downarrow f^{(i)}
\]

\[
\cdots \longrightarrow Y_{i-k+1} \overset{(-1)^k \partial^Y_{i-k+1}}{\longrightarrow} Y_{i-k} \longrightarrow \cdots
\]
Moreover, if such \( f = (f^{(l)}) \) is compatible with the differentials of \( X \) and \( Y \), that is \( f^{(l)} \partial^{X}_{l} = (-1)^{k} \partial^{Y}_{l} k f^{(l+1)} \) for any \( l \in \mathbb{Z} \), call this \( f \) a chain map. And if a map \( f \in \text{Hom}_{A}(X,Y[k]) \) is not a chain map, it is called a non-chain map.

Now, we will recall the definitions of some classes of algebras and important modules over them.

**Definition 2.1.** (1) For each \( i \in \{1, \ldots, n\} \), the \( A \)-module \( \Delta_{A}(i) \), or just \( \Delta(i) \), called the standard module, is defined by the maximal factor module of \( P(i) \) having only composition factors \( S(j) \) with \( j \leq i \). Moreover the maximal factor module \( \Delta(i) \) of \( \Delta(i) \) such that \( [\Delta(i) : S(j)] = 1 \) is called the proper standard module. Let \( \Delta = \{\Delta(1), \ldots, \Delta(n)\} \) and \( \Delta = \{\Delta(1), \ldots, \Delta(n)\} \), and we call them a standard system and a properly standard system, respectively.

(2) We say that a module \( X \) has a \( \Delta \)-filtration, or \( X \) is \( \Delta \)-filtered, if there is a submodule sequence \( 0 = X_{m} \subset \cdots \subset X_{1} \subset X_{0} = X \) such that for each \( 1 \leq k \leq m \), \( X_{k-1}/X_{k} \cong \Delta(j) \) for some \( j \in \{1, \ldots, n\} \). Write \( F(\Delta) \) to mean the full subcategory of \text{mod} \( A \) whose objects are modules with \( \Delta \)-filtrations. Similarly we define \( \Delta \)-filtered modules and the category \( F(\Delta) \).

(3) A pair of an algebra and a total order \((A, \leq)\), or just \( A \), is called a \( \Delta \)-filtered algebra (resp. a \( \Delta \)-filtered algebra) provided that every \( P(i) \) has a \( \{\Delta(i), \ldots, \Delta(n)\} \)-filtration (resp. a \( \{\Delta(i), \ldots, \Delta(n)\} \)-filtration).

(4) If \((A, \leq)\) is a \( \Delta \)-filtered algebra and \( \Delta = \Delta \), then it is called a quasi-hereditary algebra.

**Remark 2.2.** (1) A \( \Delta \)-filtered algebra is also called a standardly stratified algebras.

(2) The standard modules and properly standard modules are also defined by the following:

\[
\Delta(i) = \sum_{j \geq i} \text{Im} \varphi, \quad \Delta(i) = \sum_{j \geq i} \text{Im} \varphi.
\]

Hereafter, let \( e_{1}, \ldots, e_{n} \) be pairwise orthogonal basic primitive idempotents of an algebra \( B \) and write \( B_{i} \) by \( P_{B}(i) \).

**Definition 2.3.** (1) Let \( A \) be a \( \Delta \)-filtered algebra. A subalgebra \( B \) of \( A \) is called an exact Borel subalgebra if the following conditions hold.

- \( B \) is directed, i.e., \( \text{rad}_{B}(P_{B}(i), P_{B}(j)) = 0 \) for \( i \leq j \).
- \( A \odot_{B} - : \text{mod} B \rightarrow \text{mod} A \) is exact.
- \( \Delta_{A}(i) \cong A \odot_{B} S_{B}(i) \).

(2) Let \( A \) be a \( \Delta \)-filtered algebra. A subalgebra \( B \) of \( A \) is called a proper Borel subalgebra if the following conditions hold.

- \( B \) is one-cyclic directed, i.e., \( \text{rad}_{B}(P_{B}(i), P_{B}(j)) = 0 \) for \( i < j \).
- \( A \odot_{B} - : \text{mod} B \rightarrow \text{mod} A \) is exact.
- \( \Delta_{A}(i) \cong A \odot_{B} S_{B}(i) \).

(3) A subalgebra \( B \) of \( A \) is homological if for any \( B \)-modules \( X, Y \), natural maps

\[
\text{Ext}^{k}_{B}(X, Y) \rightarrow \text{Ext}^{k}_{A}(A \odot_{B} X, A \odot_{B} Y)
\]

are epimorphisms for \( k \geq 1 \) and isomorphisms for \( k \geq 2 \).
3 Bocses

The bocs theory was introduced by Drozd’s tame and wild dichotomy theorem, and Crawley-Boevey studied bocses in [C-B].

**Definition 3.1.** A bocs is $B = (B, W, \varepsilon, \mu)$, or just $(B, W)$, consisting of a basic $K$-algebra $B$ and a $B$-bimodule $W$ which has a $B$-coalgebra structure, that is, there exist a $B$-bilinear counit $\varepsilon : W \to B$ and a $B$-bilinear comultiplication $\mu : W \ni w \mapsto \sum w_1 \otimes w_2 \in W \otimes_B W$ (using sigma notation), and the following diagrams are commutative:

\[
\begin{array}{c}
W \xrightarrow{\mu} W \otimes W \\
\mu \downarrow \downarrow \mu \\
W \otimes W \xrightarrow{\mu \otimes \id_W} W \otimes W \otimes W
\end{array}
\quad \begin{array}{c}
B \otimes W \xrightarrow{\varepsilon \otimes \id_W} W \otimes W \xrightarrow{\id_W \otimes \varepsilon} W \otimes B \\
\mu \downarrow \downarrow \mu \\
W \xrightarrow{\varepsilon \otimes \id_W} W \otimes B
\end{array}
\]

with isomorphisms $l_W : B \otimes W \ni b \otimes w \mapsto bw \in W$ and $r_W : W \otimes B \ni w \otimes b \mapsto wb \in W$.

We will always assume that the algebra $B$ and $B$-bimodule $W$ of a bocs are finite dimensional and the counit $\varepsilon$ is surjective.

**Definition 3.2.**

1. A bocs $B = (B, W)$ is said to have a projective kernel if $W := \ker \varepsilon$ is a projective $B$-bimodule.
2. A bocs $B = (B, W)$ with a projective kernel is called
   \[
   \begin{cases}
   \text{directed} & \text{if } \text{rad}_B(P_B(i), P_B(j)) = 0 \text{ for } i \leq j \text{ and } W \cong \bigoplus_{i \geq j} (Be_i \otimes_K e_j B)^{d_{ij}}, \\
   \text{weakly directed} & \text{if } \text{rad}_B(P_B(i), P_B(j)) = 0 \text{ for } i \leq j \text{ and } W \cong \bigoplus_{i \leq j} (Be_i \otimes_K e_j B)^{d_{ij}}, \\
   \text{one-cyclic directed} & \text{if } \text{rad}_B(P_B(i), P_B(j)) = 0 \text{ for } i < j \text{ and } W \cong \bigoplus_{i > j} (Be_i \otimes_K e_j B)^{d_{ij}},
   \end{cases}
   \]
   for some $d_{ij} \geq 0$.

**Definition 3.3.** The category $\text{mod} B$ of finite dimensional modules over a bocs $B = (B, W)$ is defined as follows:

- **objects** finite dimensional left $B$-modules
- **morphisms** for $B$-modules $X$ and $Y$, define $\text{Hom}_B(X, Y) = \text{Hom}_B(W \otimes_B X, Y)$

composition for $B$-modules $X, Y, Z$ and morphisms $f \in \text{Hom}_B(X, Y), g \in \text{Hom}_B(Y, Z)$, the composition $gf$ of $f$ and $g$ is given by:

\[
W \otimes X \xrightarrow{\mu \otimes \id_X} W \otimes W \otimes X \xrightarrow{\id_W \otimes f} W \otimes Y \xrightarrow{g} Z.
\]

unites the unite morphism $\id_X^B \in \text{End}_B(X)$ of a $B$-module $X$ is given by the composition of the following maps:

\[
W \otimes X \xrightarrow{\varepsilon \otimes \id_X} B \otimes X \xrightarrow{l_X} X.
\]
Remark 3.4. Since a standard adjunction gives the canonical isomorphism
\[ \Hom_B(W \otimes_B X, Y) \cong \Hom_{B \otimes B^{\mathit{op}}}(W, \Hom_K(X, Y)), \]
we may also define \( \Hom_{\mathcal{B}}(X, Y) \) by \( \Hom_{B \otimes B^{\mathit{op}}}(W, \Hom_K(X, Y)) \). In this case, the composition and unites in \( \text{mod} \mathcal{B} \) are as follows.

**composition** for \( f \in \Hom_{\mathcal{B}}(X, Y) \) and \( g \in \Hom_{\mathcal{B}}(Y, Z) \), the composition \( gf \) is given by:
\[
W \xrightarrow{\mu} W \otimes_B W \xrightarrow{g \otimes f} \Hom_K(Y, Z) \otimes_B \Hom_K(X, Y) \xrightarrow{\circ} \Hom_K(X, Z),
\]
where \( \circ \) is the usual composition of maps.

**unites** the unite morphism \( \text{id}_B^X \in \text{End}_B(X) \) is given by the composition of the following maps:
\[
W \xrightarrow{\varepsilon} B \xrightarrow{\gamma_X} \text{End}_K(X),
\]
where \( \gamma_X(1_B) = \text{id}_X \).

Definition 3.5 ([BB]). Let \( \mathcal{B} = (B, W) \) be a bocs. The right Burt-Butler algebra \( R = R_{\mathcal{B}} \) of the bocs \( \mathcal{B} \) is defined by \( \text{End}_B(B)^{\mathit{op}} \) and whose multiplication is the composition of morphisms in \( \text{mod} \mathcal{B} \) with \( 1_R = \text{id}_B^B \).

Similarly we can also define the left Burt-Butler algebra \( L_B = \text{End}_B^{\mathit{op}}(B) \) of \( \mathcal{B} \). Now we give some facts for right Burt-Butler algebras which are necessary for our argument.

Lemma 3.6 ([BB]). Let \( \mathcal{B} \) be a bocs with projective kernel and \( R \) the right Burt-Butler algebra of \( \mathcal{B} \). Then the following conditions hold.

1. There is an equivalence between categories
\[ R \otimes_B - : \text{mod} \mathcal{B} \rightarrow \text{Ind}(R, B), \]
where \( \text{Ind}(B, R) \) is the full subcategory of \( \text{mod} R \) consisting of the modules \( R \otimes_B X \) for \( B \)-modules \( X \).

2. There are surjective maps
\[ \Ext^k_B(X, Y) \rightarrow \Ext^k_R(R \otimes X, R \otimes Y) \] for \( k \geq 1 \),
and they are bijective for \( k \geq 2 \).

3. The algebra \( R \) is a projective \( B \)-module.

4 \( A_\infty \)-categories

In this section, we will recall the definition of \( A_\infty \)-categories. As an important example, Ext-algebras have an \( A_\infty \)-structure, see Example 4.4, and we will use this fact in Subsection 6.1 to construct bocses from \( \Delta \)-filtered algebras.
Definition 4.1. A $\mathbb{Z}$-graded $K$-category $\mathcal{A}$ is called an $A_\infty$-category if there are $K$-bilinear functors $m_r : \mathcal{A}^\otimes r \to \mathcal{A}$ for $r \geq 1$ such that each functor $m_r$ of degree $2 - r$ satisfies the condition

$$
\sum_{k = r + 1 + u, r,u \geq 0} (-1)^{r + tu} m_{r+1+u}(id^\otimes r \otimes m_1 \otimes id^\otimes u) = 0, \text{ for any } k \geq 1.
$$

We often write the $A_\infty$-category by a pair $(\mathcal{A}, m)$.

Lemma 4.2 ([Kel] (3.6) Lemma). Let $\mathcal{A}$ be a $\mathbb{Z}$-graded category and $m_r : \mathcal{A}^\otimes r \to \mathcal{A}$ for $r \geq 1$ graded functors of degree $2 - r$. We define the suspension $s\mathcal{A}$ of $\mathcal{A}$ given by $(s\mathcal{A})_k = \mathcal{A}_{k+1}$ and consider the family of graded functors $b_r : s\mathcal{A}^\otimes r \to s\mathcal{A}$ of degree 1 which make the following diagram commutative

$$
\begin{array}{ccc}
(s\mathcal{A})^\otimes r & \xrightarrow{b_r} & s\mathcal{A} \\
\downarrow s & & \downarrow s \\
\mathcal{A}^\otimes r & \xrightarrow{m_r} & \mathcal{A}
\end{array}
$$

where $s : \mathcal{A} \to s\mathcal{A}$ is the canonical functor of degree $-1$. Then the following are equivalent

1. $(\mathcal{A}, m)$ is an $A_\infty$-category.
2. For any $k \geq 1$, we have
   $$
   \sum_{k = r + 1 + u, r,u \geq 0} b_{r+1+u}(id^\otimes r \otimes b_1 \otimes id^\otimes u) = 0.
   $$
3. A differential coalgebra $\overline{T}s\mathcal{A} = \oplus_{r \geq 1}(s\mathcal{A})^\otimes r$ with the comultiplication
   $$
   \mu : \overline{T}s\mathcal{A} \to \overline{T}s\mathcal{A} \otimes \overline{T}s\mathcal{A} ; a_r \otimes \ldots \otimes a_1 \mapsto \sum_{t = 1}^{r-1} (a_r \otimes \ldots \otimes a_{t+1}) \otimes (a_t \otimes \ldots \otimes a_1),
   $$
   gives $b : \overline{T}s\mathcal{A} \to \overline{T}s\mathcal{A}$ of degree 1 such that $b^2 = 0$ and $\mu b = (1 \otimes b + b \otimes 1)\mu$.

In particular, the equivalence between (2) and (3) is given by defining the composition of $b$ that maps $(s\mathcal{A})^\otimes k$ to $(s\mathcal{A})^\otimes l$ as $\sum_{k = r + 1 + u, r,u \geq 0} b_{r+1+u}(id^\otimes r \otimes b_1 \otimes id^\otimes u)$, where $l = r + 1 + u$.

Proposition 4.3 ([Kel] (3.1), [KKO] Theorem 4.3). Let $(\mathcal{A}, \lambda)$ be an differential graded category. Then the homology category $H^*(\mathcal{A})$ of $\mathcal{A}$, where the homology is taken with respect to $\lambda_1$, is also an $A_\infty$-category $(H^*(\mathcal{A}), m)$ with $m_1 = 0$ and $m_2$ being induced from $\lambda_2$.

Example 4.4 ([KKO] (4.3)). Let $X$ be an $A$-module and $\mathbb{P}_X$ a projective resolution of $X$ with differential $\partial_X$. Let $\mathcal{A}$ be a category with one object $X$ and $\mathcal{A}(X, X) = \text{End}_A(\mathbb{P}_X)$ the endomorphism ring of $\mathbb{P}_X$ as a complex of $A$-modules. Then the $K$-linear map $\lambda_1 : \mathcal{A} \to \mathcal{A}$ with $f \mapsto \partial_X f - (-1)^k f\partial_X$ for homogeneous $f$ of degree $k$, the natural composition $\lambda_2 : \mathcal{A}^\otimes 2 \to \mathcal{A}$, and $\lambda_r = 0$ for $r \geq 3$ make $(\mathcal{A}, \lambda)$ a differential graded category. Moreover the morphism space $H^*(\mathcal{A})(X, X)$ of the homology category $H^*(\mathcal{A})$ of $\mathcal{A}$ with respect to $\lambda_1$ is just the Ext-algebra $\text{Ext}^*_A(X, X) = \oplus_{k \geq 0} \text{Ext}^k_A(X, X)$ of $X$ and is an $A_\infty$-category with $m_1 = 0$ and $m_2$ being the Yoneda product.
5 Twisted modules

In this section, we define twisted modules in order to prepare for the proof of Theorem 6.12. For detail, refer to [Kel] or [KKO].

Let \((\mathcal{A}, m)\) be an \(A_\infty\)-category. First, we define the category \(\text{add}\, \mathcal{A}\). Its objects are \(\mathcal{L}\)-modules, i.e., functors \(X : \mathcal{L} \to \text{mod} \, K\), and for \(\mathcal{L}\)-modules \(X\) and \(Y\), we have

\[
\text{add}\, \mathcal{A}(X, Y)_k := \bigoplus_{1 \leq i, j \leq n} \text{Hom}_K(X(i), Y(j)) \otimes \mathcal{A}(i, j)_k.
\]

Moreover we define the graded multiplications \(b^g_r\) of \(\text{add}\, \mathcal{A}\) as follows:

\[
b^g_r((f_r \otimes sa_r) \otimes \cdots \otimes (f_1 \otimes sa_1)) := -f_r \circ \cdots \circ f_1 \otimes b_r(sa_r \otimes \cdots \otimes sa_1).
\]

Then \(\text{add}\, \mathcal{A}\) satisfies the condition (2) in Lemma 4.2 which was shown in [KKO] Lemma 5.1, and hence \(\text{add}\, \mathcal{A}\) has an \(A_\infty\)-structure.

Next we define twisted modules. A \textbf{pretwisted module} \((X, \delta)\) is a pair of an \(\mathcal{L}\)-module \(X\) and \(\delta = \sum_r(f_r \otimes a_r) \in \text{add}\, \mathcal{A}(X, X)_1\). A \textbf{pretwisted submodule} \((X', \delta')\) of \((X, \delta)\) is defined by the following: A family of subspaces \(X'(i) \subset X(i)\) such that the restriction \(f_r|_{X'(i)}\) of \(f_r : X(i) \to X(j)\) to \(X'(i)\) maps into \(X'(j)\). Moreover \(\delta'\) is given by \(\sum_r(f_r|_{X'} \otimes a_r)\). There is an obvious way to define the notion of a \textbf{pretwisted factor module} \((X/X', \delta/\delta')\). Finally, a pretwisted module \((X, \delta)\) is called a \textbf{twisted module} if the following conditions hold:

1. There is a chain of pretwisted submodules

\[
(0, 0) = (X_N, \delta_N) \subset \cdots \subset (X_1, \delta_1) \subset (X_0, \delta_0) = (X, \delta)
\]

of \((X, \delta)\) such that for each \(1 \leq i \leq N\), \((X_{i-1}, \delta_{i-1})/(X_i, \delta_i) = (X_{i-1}/X_i, 0)\).

2. \[\sum_{r \geq 1} (-1)^{(r-1)/2}m_r(\delta, \ldots, \delta) = 0.\]

The first condition is called \textbf{triangularity} and the second one is called the \textbf{Maurer-Cartan equation}. Moreover for any morphism \(X \to X'\) in \(\text{add}\, \mathcal{A}\), we define a morphism of twisted modules \((X, \delta) \to (X', \delta')\), and we will denote \(\text{twmod}\, \mathcal{A}\) the category of twisted modules. Further we give the graded multiplications \(b^w_r\) of \(\text{twmod}\, \mathcal{A}\) by

\[
b^w_r(st_r \otimes \cdots \otimes st_1) := \sum_{i_0, \ldots, i_r \geq 0} b^g_{i_0 + \cdots + i_r + r}(s_0 \otimes s_{i_0 + 1} \otimes \cdots \otimes st_1 \otimes s_0 \otimes \cdots \otimes s_0^r).
\]

Then \(\text{twmod}\, \mathcal{A}\) also satisfies the condition (2) in Lemma 4.2 see [KKO] Lemma 5.3. Thus \(\text{twmod}\, \mathcal{A}\) is an \(A_\infty\)-category.

In order to prove Theorem 6.12 we mention the next lemma without the proof.

\textbf{Lemma 5.1} ([Kel] (7.7), [KKO] Theorem 5.4). \textit{For an \(n\)-tuple of \(A\)-modules \(X = (X_1, \ldots, X_n)\), we have \(\mathcal{F}(X) \simeq H^0(\text{twmod}\, \mathcal{A}(X, X))\).}

Next we consider the category \(\text{conv}\, \mathcal{A}\) whose objects are \(\mathcal{L}\)-modules and

\[
\text{conv}\, \mathcal{A}(X, Y)_k := \bigoplus_{1 \leq i, j \leq n} \text{Hom}_K(\mathbb{D}\mathcal{A}(i, j)_k, \text{Hom}_K(X(i), Y(j))).
\]
For Lemma 5.2 the next lemma and proposition hold. Let \( L = (X, \delta) \) be a pretwisted module over \( A \) and \( \chi \) be a \( \mathbb{Z} \)-graded \( (sF, \phi) \) with \( \text{sgn}(\chi) = 0 \). Then \( H \) is a \( s\)-twisted functor on \( \mathcal{F}_A \) with \( \text{sgn}(\chi) = 0 \), and \( d_r \) satisfies the condition (2) in Lemma 4.2.

The next lemma and proposition hold. Next, we construct a functor \( b_r^c \) for \( s(\mathcal{C}A) \) satisfying (2) in Lemma 4.2.

The functor \( b_r^c : s(\mathcal{C}A)^\otimes r \to s(\mathcal{C}A) \) is defined by \( sF_r \otimes \cdots \otimes sF_1 \to v_r[sF_r \otimes \cdots \otimes sF_1]d_r \).

Then \( b_r^c \) satisfies the condition (2) in Lemma 4.2 for \( s\mathcal{A} \), and we have \( Mb_r^c = b_r^cM^r \) (KKO Lemma 6.1). Finally, we consider objects corresponding to twisted modules. Let \( Q = \mathbb{D}s\mathcal{A} \). Then the next lemma and proposition hold.

**Lemma 5.2 (KKO) Lemma 6.2.** There is a one-to-one correspondence between \( \mathcal{L}[Q_0] \)-modules \( X_\delta \) and pairs \( (X, M(\delta)) \) where \( (X, \delta) \) are pretwisted modules over \( A \). This relation is given by \( X_\delta = \bigoplus_{r \geq 1} v_r \circ (\mathbb{D}(\delta))^{\otimes r} \) as an \( \mathcal{L}[Q_0] \)-module.

**Proposition 5.3 (KKO) Proposition 6.3.** Let \( (X, \delta) \) be a pretwisted module over \( A \) and \( X_\delta \) an \( \mathcal{L}[Q_0] \)-module given in Lemma 5.2. Put \( \delta = \bigoplus_{r \geq 1} (f_r \otimes a_r) \). Then the following hold:

1. For \( \chi \in Q_0(i, j), x \in X \), we have \( X_\delta(\chi)(x) = \bigoplus_{r \geq 1} \chi(sa_r)f_r(x) \).
2. \( \delta = 0 \) if and only if \( X_\delta \) is semi-simple.
3. \( (X', \delta') \) is a pretwisted submodule of \( (X, \delta) \) if and only if \( X_{\delta'} \) is a submodule of \( X_\delta \).
4. If \( (X', \delta') \subset (X, \delta) \), then \( X_{\delta/\delta'} = X_{\delta}/X_{\delta'} \).
5. \( (X, \delta) \) satisfies the triangularity condition if and only if for some \( r \geq 1 \), \( v_r \circ (\mathbb{D}(\delta))^{\otimes r} : (Q_0)^{\otimes r} \to \text{End}_A(X) \) is the zero map.
6. \( (X, \delta) \) satisfies Maurer-Cartan equation if and only if \( X_\delta \circ d = 0 \) with \( d = (d_r)_{r \geq 1} \) if and only if \( X_\delta \) is a \( \mathbb{B} = (\mathcal{L}[Q_0]/(\mathcal{L}[Q_0] \cap \text{Im} d)) \)-module, where the functors \( d_r : \mathbb{D}s\mathcal{A} \to \mathbb{D}s\mathcal{A}^{\otimes r} \) are defined by

\[
d_r(\chi)(sa_r \otimes \cdots \otimes sa_1) = \chi(b_r(sa_r \otimes \cdots \otimes sa_1)) = \chi(sm_r(a_r \otimes \cdots \otimes a_1)).
\]

This proposition will be used in Theorem 6.12 to show the equivalence between \( \text{mod } \mathcal{B} \) and \( H^0(\text{twmod } H^*(A)) \).
6 $\Delta$-filtered algebras versus one-cyclic directed bocses

In this section, let $A$ be a $\Delta$-filtered algebra, unless otherwise noted. We will show the relationship between $\Delta$-filtered algebras and one-cyclic directed bocses. To do this, we imitate the arguments in [KKO]. In the case of a $\Delta$-filtered algebra, we must face three problems. They come from the $\Delta$-filtered algebras and one-cyclic directed bocses. To do this, we imitate the arguments by Example 4.4. Further, $\Delta$ is not finite dimensional, in general. This is the first problem. $\Delta$ cannot be. It is the second problem. So in order to avoid infinite $\Delta$-filtered algebra, unless otherwise noted. We will show the relationship $\Delta$-filtered algebra and $\Delta$. Following, in [KKO], they used the Ext-algebra $\text{Ext}^s_A(\Delta, \Delta) = \bigoplus_{k \geq 0} \bigoplus_{1 \leq i, j \leq n} \text{Ext}^k_A(\Delta(i), \Delta(j))$ of $\Delta$ and it is finite dimensional for a quasi-hereditary algebra $A$. But in the case for $A$ being $\Delta$-filtered algebra, $\text{Ext}^s_A(\Delta, \Delta)$ is not finite dimensional, in general. This is the first problem. Moreover in [KKO], the differential $b$ of the tensor algebra $T(s \text{Ext}^s_A(\Delta, \Delta))$ can be controlled by finitely many elements since we have $b_r = 0$ for $k > n$. Obviously, for a $\Delta$-filtered algebra $A$, the algebra $T(s \text{Ext}^s_A(\Delta, \Delta))$ cannot be. It is the second problem. In order to avoid infinite dimensional algebras, we generalize a method of constructing bocses. Let $\mathbb{P}_i$ be a projective resolution of $\Delta(i)$ with differential $\partial$ for any $1 \leq i \leq n$ and $A$ the $\mathbb{Z}$-graded category with objects $1, \ldots, n$ and $A^k(i, j) = \text{Hom}_A(\mathbb{P}_i, \mathbb{P}_j[k])$. Then $A$ is a differential graded category with the multiplication being compositions of morphisms and differential $d$ defined by $d(f) = \partial \circ f - (-1)^k f \circ \partial$ for $f \in A^k$. Now let $Z^*(A)$, $B^*(A)$, and $H^*(A)$ be the cocycles, coboundaries, and cohomology of $A$, respectively. Then $Z^*(A)$ is the set of chain morphisms in $A$ and $B^*(A)$ is that of null-homotopic morphisms i.e. morphisms homotopic to zero morphisms. And we can identify $H^*(A)$ with $\text{Ext}^s_A(\Delta, \Delta)$ by Example 4.4. Further $H^*(A)$ is an $A_\infty$-category with multiplications $m_r$ by Example 4.4 again.

At first, we recall that the method of construction of directed bocses from quasi-hereditary algebras in [KKO]. We will write $sH^k(A)$ to mean $(sH^s(A))^k = H^{k+1}(A)$.

Step 1 Let $A$ be a quasi-hereditary and $H^*(A) = \text{Ext}^s_A(\Delta, \Delta)$.

Step 2 Consider the dual maps $d_k : \mathcal{Q} \to \mathcal{Q}^{\otimes r}$ of $b_r$ given in Lemma 4.2 where $\mathcal{Q}_k(i, j) = D((sH^k(A))(i, j))$.

Step 3 Let $\mathcal{L}[\mathcal{Q}]$ be the tensor category of $\mathcal{Q}$ over $\mathcal{L}$. Then $\mathcal{L}[\mathcal{Q}]$ is a differential graded category with differential $d$.

Step 4 Let $U = \mathcal{L}[\mathcal{Q}]/I$, where the ideal $I$ of $\mathcal{L}[\mathcal{Q}]$ is generated by $\mathcal{Q}_{-1}$ and $d(\mathcal{Q}_{-1})$. Since $I$ is a differential ideal with respect to $d$, the factor $U$ is also a differential graded category. Moreover, $U$ is freely generated over $B = \mathcal{L}[\mathcal{Q}_{0}]/(\mathcal{L}[\mathcal{Q}_{0}] \cap I)$ by $\mathcal{Q}_{1}$.

Step 5 Put $W = U_1/d(B)$ and take the natural epimorphism $\pi : U_1 \to W$. Consider the two homomorphisms $\mu : W \to W \otimes W$ and $\varepsilon : W \to B$ such that the following diagrams.
commute, respectively,

\[
\begin{array}{ccc}
U_1 & \xrightarrow{d} & U_1 \otimes U_1 \\
\pi & & \pi \otimes \pi \\
W & \xrightarrow{\mu} & W \otimes W
\end{array}
\quad
\begin{array}{ccc}
U_1 & \cong & (\bigoplus_i B \omega_i B) \oplus \overline{U} \\
\pi & & \pi \\
W & \xrightarrow{\varepsilon} & B
\end{array}
\]

where \( \omega_i \in Q_1(i, i) \) are elements corresponding to \( \text{id}_{\Delta(i)} \), and \( \varepsilon \) maps \( \omega_i \) to \( e_i \) and \( \overline{U} \) to zero. Then \((B, W, \varepsilon, \mu)\) is a directed bocs.

The first problem lies in Steps 1 and 2. We notice that for a \( \Delta \)-filtered algebra \( A \), the Ext-algebra \( H^*(A) = \text{Ext}^*_A(\Delta, \Delta) \) is infinite dimensional in general. It is difficult to take the dual of \( sH^*(A) \) with respect to \( K \)-bases. In order to argue Step 2 for finite dimensional spaces, we consider a subspace of of \( sH^*(A) \).

**Proposition 6.1.** 1. Let \( b_r : sH^*(A)^{\otimes r} \to sH^*(A) \) be a graded maps of degree 1 induced from the \( A_{\infty} \)-category \( H^*(A) \) by Lemma 4.2. Consider the graded linear maps \( b'_r : (sH \leq 1(A))^{\otimes r} \to sH \leq 1(A) \) of degree 1 defined by

\[
b'_r(sa_r, \ldots, sa_1) = \begin{cases} b_r(sa_r, \ldots, sa_1) & \text{for } \sum_{t=1}^{r'} |s_{a_t}| \leq 0, \\ 0 & \text{for } \sum_{t=1}^{r'} |s_{a_t}| \geq 1. \end{cases}
\]

Then for any \( k \geq 1 \), we have

\[
\sum_{k=r+t+u \atop r, u \geq 0, t \geq 1} b'_{r+1+t}(\text{id}^{\otimes r} \otimes b'_t \otimes \text{id}^{\otimes u})i^{\otimes k} = 0,
\]

where \( i : sH^{\leq 0}(A) \to sH^{\leq 1}(A) \) is the canonical injection.

2. Take the dual statement of the above and let \( Q = sH^*(A) \). Then we get graded maps \( d'_r : Q_{\geq 1} \to Q_{\geq 2}^{\otimes r} \) of degree 1 satisfying

\[
\sum_{k=r+t+u \atop r, u \geq 0, t \geq 1} p^{\otimes k}(\text{id}^{\otimes r} \otimes d'_t \otimes \text{id}^{\otimes u})d'_{r+1+u} = 0,
\]

for \( k \geq 1 \), where \( p : Q_{\geq 1} \to Q_{\geq 0} \) is the canonical surjection.

3. Consider the factor category \( T(Q_{\geq 0})/d'(Q_{\geq 1}) \). Then it is a differential graded category with differential \( d' \) induced from the maps \( d'_r \).

**Proof.** The claims 1. and 2. can be checked by routines. So we prove 3. Let \( d' : T(Q_{\geq 1}) \to T(Q_{\geq 1}) \) be the graded map whose component mapping \( Q_{\geq 1}^{\otimes 1} \) to \( Q_{\geq 1}^{\otimes k} \) is defined by

\[
\sum_{k=r+t+u \atop r, u \geq 0, t \geq 1} (\text{id}^{\otimes r} \otimes d'_t \otimes \text{id}^{\otimes u}),
\]
where \( l = r + 1 + u \). Then we have \( p \circ d'^2 = 0 \) by 2. Moreover \( d' \) satisfies \( d'm' = m'(1 \otimes d' + d' \otimes 1) \) for the natural multiplication

\[
m' : T(Q_{\geq -1}) \otimes T(Q_{\geq -1}) \to T(Q_{\geq -1}); \; (q_r \otimes \cdots \otimes q_{i+1}) \cdot (q_i \otimes \cdots \otimes q_1) \mapsto q_r \otimes \cdots \otimes q_1
\]

of \( T(Q_{\geq -1}) \). Since the ideal \( I' \) of \( T(Q_{\geq -1}) \) generated by \( Q_{-1} \) and \( d'(Q_{-1}) \) is closed under \( d' \), the factor \( T(Q_{\geq -1})/I' \cong T(Q_{\geq 0})/d'(Q_{-1}) \) is a differential graded category.

Hence the algebra \( U \) in Step 4 can be replaced with such the differential graded algebra \( T(Q_{\geq 0})/d'(Q_{-1}) \) with differential \( d' \) above.

Now we consider the construction of a bocs \( B = (B, W) \) from a \( \Delta \)-filtered algebra \( A \) as follows.

Step 1 Let \( A \) be a \( \Delta \)-filtered algebra and \( H^*(A) = \Ext^*(\Delta, \Delta) \).

Step 2 Consider the maps \( d'_r : Q_{\geq -1} \to Q_{\geq -1}^\otimes r \) given in Proposition 6.11 where \( Q_k(i, j) = D((sH^k(A))(i, j)) \).

Step 3 Let \( L[Q_{\geq -1}] \) be the tensor category of \( Q_{\geq -1} \) over \( L \).

Step 4 Put \( U = L[Q_{\geq -1}]/I' \), where the ideal \( I' \) of \( L[Q_{\geq -1}] \) is generated by \( Q_{-1} \) and \( d'(Q_{-1}) \). Then the factor \( U \) is a differential graded category by Proposition 6.11 Moreover, \( U \) is freely generated over \( B = L[Q_0]/d'(Q_{-1}) \) by \( Q_1 \).

Step 5 Put \( W = U_1/d'(B) \) and take the natural epimorphism \( \pi : U_1 \to W \). Consider the two homomorphisms \( \mu : W \to W \otimes W \) and \( \varepsilon : W \to B \) such that the following diagrams commute, respectively,

\[
\begin{array}{ccc}
U_1 & \xrightarrow{d} & U_1 \\
\pi & \downarrow & \pi \\
W & \xrightarrow{\mu} & W \otimes W \\
\end{array}
\quad
\begin{array}{ccc}
U_1 & \xrightarrow{\cong} & \bigoplus_i B \omega_i B \\
\pi & \downarrow & \varepsilon \\
W & \xrightarrow{\varepsilon} & B \\
\end{array}
\]

where \( \omega_i \in Q_1(i, i) \) are elements corresponding to \( \text{id}_{\Delta(i)} \), and \( \varepsilon \) maps \( \omega_i \) to \( e_i \) and \( \sum \) to zero.

Then \( B = (B, W, \varepsilon, \mu) \) is a bocs with projective kernel, see [KKO] Lemmas 7.5 to 7.7.

We will show that this \( B = (B, W) \) is a one-cyclic direct bocs with \( B \) being finite dimensional. But when \( A \) is \( \Delta \)-filtered algebra, [KKO] did not guarantee that \( B \) is finite dimensional in Step 4. And this is the second problem for the generalization of Theorem 1.1. Remark that \( B \) is finite dimensional if so are \( e_i B e_i \) for all \( 1 \leq i \leq n \), because \( B \) is one-cyclic directed by the construction (see Lemma 6.11 below). In order to show that \( e_i B e_i \) are finite dimensional, we prove that \( \End_A(\Delta(i)) \) and \( e_i B e_i \) are Morita equivalent in Theorem 6.10 below. To show this, we use the following remark and theorem.

Remark 6.2 ([LPWZ], [M]). We have a decomposition

\[
\mathcal{A} = Z^*(\mathcal{A}) \oplus L^*(\mathcal{A}) = H^*(\mathcal{A}) \oplus B^*(\mathcal{A}) \oplus L^*(\mathcal{A})
\]

of the graded differential category \( \mathcal{A} \) for a subspace \( L^*(\mathcal{A}) \) of \( \mathcal{A} \) and identify \( L^*(\mathcal{A}) \) with \( \mathcal{A}/Z^*(\mathcal{A}) \). Consider the graded map \( G : \mathcal{A} \to \mathcal{A} \) of degree \(-1\) satisfying \( G|_{L^*(\mathcal{A})} = 0 \) and \( G|_{B^*(\mathcal{A})} = 0 \).
(d|_{k-1,(A)})^{-1}. Define a sequence of linear maps $\lambda_r$ of degree $2 - k$ as follows. There is no map $\lambda_1$ but we define the composition $G\lambda_1$ by $-\text{id}_A$. The map $\lambda_2$ is the same as the multiplication of $A$. And for $r \geq 3$, we inductively define $\lambda_r$ by

$$
\lambda_r = \sum_{t=1}^{r-1} (-1)^{t+(r-t)(\sum_{j=1}^t a_j) + 1} \lambda_2(G\lambda_{r-t}(a_r, \ldots, a_{r-t}), G\lambda_t(a_t, \ldots, a_1))
$$

for $a_1, \ldots, a_r \in A$. Let $p : A \to H^*(A)$ and $i : H^*(A) \to A$ be the canonical projection and injection, respectively. Then $H^*(A)$ is an $A_\infty$-algebra with multiplications $m_r = p\lambda_r i^{\otimes r} : H^*(A)^{\otimes r} \to H^*(A)$.

**Theorem 6.3 ([LPWZ] Theorem A).** Let $E = \bigoplus_{k \geq 0} E_k$ be a graded algebra with finite dimensional spaces $E_k$ for any $k \geq 0$. Let $m = \bigoplus_{k \geq 1} E_k$ and $Q = m/m^2$. Let $R = \bigoplus_{k \geq 2} R_k$ be a minimal graded space of relations of $E$, with $R_k$ chosen so that

$$
R_k \subset \bigoplus_{1 \leq i \leq k-1} Q_i \otimes E_{k-i} \subset \left( \bigoplus_{r \geq 2} Q^{\otimes r} \right)_k.
$$

For each $t \geq 2$ and $k \geq 2$, let $i_k : R_k \to (\bigoplus_{r \geq 2} Q^{\otimes r})_k$ be the inclusion map and let $i^t_k$ be the composite

$$
R_k \xrightarrow{i_k} \left( \bigoplus_{r \geq 2} Q^{\otimes r} \right)_k \xrightarrow{} (Q^{\otimes t})_k.
$$

Then there is a choice of $A_\infty$-category $(H^*, m)$ for $H^* = \text{Ext}^*(S_E, S_E)$ so that in any degree $-k$, the multiplication $m_1$ of $H^*$ restricted to $(H^1)^{\otimes t}_k$ is equal to the map

$$
D(i^t_k) : ( (H^1)^{\otimes t} )_{-k} = D((Q^{\otimes t})_k) \longrightarrow D(R_k) \subset H^{2t}_k.
$$

For simplicity we forget the grading of $E$, i.e. assume that $E$ is just a finite dimensional algebra and $m = \text{rad } A$. Then this theorem implies that the relations of $E$ induces the $A_\infty$-multiplication on $H^1$. Of course, the algebra generated by $D(H^1)$ with relations $D(m_t(H^1)^{\otimes t})$ for $t \geq 2$ is Morita equivalent to $E$.

**Lemma 6.4.** Let $A$ be a $K$-algebra and

$$
P_M : \cdots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \quad \text{and} \quad P_N : \cdots \longrightarrow P_2' \xrightarrow{\partial'_2} P_1' \xrightarrow{\partial'_1} P_0'
$$

be projective resolutions of some $A$-modules $M$ and $N$. Assume that $k \geq 1$ and that a chain morphism $f = (f^{(i)})_{i \in \mathbb{Z}} : P_M \to P_N[k]$ satisfies

$$
\cdots \longrightarrow P_{k+2} \xrightarrow{\partial_{k+2}} P_{k+1} \xrightarrow{\partial_{k+1}} P_k \xrightarrow{f^{(k+2)}} P_{k+1} \xrightarrow{f^{(k+1)}} P_k \xrightarrow{\partial_k} P_0
$$

$$
\cdots \longrightarrow P'_2 \xrightarrow{(-1)^k \partial'_2} P'_1 \xrightarrow{(-1)^k \partial'_1} P'_0 \xrightarrow{f^{(k)}} P'_0
$$

satisfies $f^{(k)} = \partial' u^{(k)}$ for some $A$-homomorphism $u^{(k)} : P_k \to P'_1$. Then the morphism $f$ is null-homotopic. In particular, a chain morphism $f$ is uniquely determined by $f^{(k)}$, up to homotopy.
Proof. We will show that for each \( l \geq k \), there are \( A \)-homomorphisms \( u^{(l)} : P_l \to P_{l-k+1}^{l} \) and \( u^{(l-1)} : P_{l-1} \to P_{l-k}^{l} \) such that \( f^{(l)} = \partial_{l-k+1}^{l} u^{(l)} - (-1)^{k-1} u^{(l-1)} \partial_{l} \). We proceed by induction on \( l \). If \( l = k \), then we have \( u^{(k)} : P_k \to P_{k-1}^{k} \) such that \( f^{(k)} = \partial_{k} u^{(k)} \) by our assumption. Assume that \( l > k \) and that there are \( u^{(l-1)} : P_{l-1} \to P_{l-k}^{l} \) and \( u^{(l-2)} : P_{l-2} \to P_{l-k-1}^{l} \) such that \( f^{(l-1)} = \partial_{l-k}^{l} u^{(l-1)} - (-1)^{k-1} u^{(l-2)} \partial_{l-1} \). Then we have equalities

\[
\partial_{l-k}^{l} f^{(l)} = (-1)^{k} f^{(l-1)} \partial_{k} \\
= (-1)^{k} (\partial_{l-k}^{l} u^{(l-1)} - (-1)^{k-1} u^{(l-2)} \partial_{l-1}) \partial_{l} \\
= (-1)^{k} \partial_{l-k}^{l} u^{(l-1)} \partial_{l}.
\]

Hence \( \text{Im}(f^{(l)} - (-1)^{k} u^{(l-1)} \partial_{l}) \subset \text{Ker} \partial_{l-k}^{l} = \text{Im} \partial_{l-k+1}^{l} \). Since \( P_{l} \) is projective, there exists \( u^{(l)} : P_{l} \to P_{l-k+1}^{l} \) such that \( f^{(l)} - (-1)^{k} u^{(l-1)} \partial_{l} = \partial_{l-k+1}^{l} u^{(l)}. \) Thus we conclude that \( f^{(l)} = \partial_{l-k+1}^{l} u^{(l)} - (-1)^{k-1} u^{(l-2)} \partial_{l-1} \partial_{l} \).

\( \square \)

Fix \( 1 \leq i \leq n \). Define \( E \) as the opposite algebra of \( \text{End}_{A}(\Delta(i)) \) and let \( \mathcal{A}_{i} \) be a full subcategory of \( \mathcal{A} \) whose object is \( i \). The following lemmas show that we can choose bases of \( H^{k}(\mathcal{A}_{i}) \), \( B^{k}(\mathcal{A}_{i}) \) and \( L^{k-1}(\mathcal{A}_{i}) \) just by concentrating on their \( k \)-th components.

Lemma 6.5. A chain morphism \( g \in Z^{k}(\mathcal{A}_{i}) \) is in \( B^{k}(\mathcal{A}_{i}) \) if and only if \( g^{(k)} \in \text{rad}(P_{k}, P_{0}) \).

Proof. By Remark 222 we have \( \text{Im} \partial_{k} \equiv \sum \text{Im} \varphi \) for \( j \geq i \) and \( \varphi \in \text{rad}(P_{i}, P_{j}) \). So if \( g^{(k)} \in \text{rad}(P_{k}, P_{0}) \), there exists \( u^{(k)} : P_{k} \to P_{1} \) such that \( g^{(k)} = \partial_{1} u^{(k)} \). Apply the previous lemma, we conclude that \( g \) is null-homotopic. On the other hand, suppose \( g^{(k)} \notin \text{rad}(P_{k}, P_{0}) \). Since differentials \( \partial_{l} \) of \( P_{l} \) are in \( \text{rad}(P_{l}, P_{l-1}) \) for any \( l \geq 0 \), compositions \( \partial_{1} u^{(k)} \) and \( u^{(k-1)} \partial_{k} \), and hence their sum, must be in \( \text{rad}(P_{l}, P_{l-1}) \) where \( u^{(k-1)} : P_{k-1} \to P_{0} \) and \( u^{(k)} : P_{k} \to P_{1} \). Hence \( g^{(k)} \neq \partial_{1} u^{(k)} - (-1)^{k-1} u^{(k-1)} \partial_{k} \) for any \( u^{(k-1)} \) and \( u^{(k)} \), and this implies that \( g \) is not null-homotopic.

\( \square \)

Lemma 6.6. 1. For a projective resolution

\[
P_{1} : \cdots \xrightarrow{\partial_{3}} P_{2} \xrightarrow{\partial_{2}} P_{1} \xrightarrow{\partial_{1}} P_{0}
\]

of \( \Sigma(i) \), we write \( P_{k} = P(i)^{\oplus c_{k}} \oplus P_{k}^{\Sigma} \) for \( k \geq 1 \), where \( c_{k} \geq 0 \) and \( P_{k}^{\Sigma} \) does not include \( P(i) \) as direct summands. Then the chain morphisms \( f_{l} : \Sigma_{l} \to P_{l}[k] \) with \( f_{l}^{(k)} = [\pi_{l}, 0] : P(i)^{\oplus c_{k}} \oplus P_{k}^{\Sigma} \to P_{0} \) form a basis of \( \text{Ext}^{k}(\Sigma(i), \Sigma(i)) \) for \( 1 \leq l \leq c_{k} \). Here, \( \pi_{l} : P(i)^{\oplus c_{k}} \to P(i) \) are the canonical projections on \( l \)-th components.

2. We can choose, as a basis of \( L^{k-1}(\mathcal{A}_{i}) \), non-chain morphisms \( u_{l} \in L^{k-1}(\mathcal{A}_{i}) \) for \( u_{l}^{(k-1)} = 0 \) and a subset \( \{ u_{l}^{(k)} \} \) of a basis of \( \text{Hom}_{A}(P_{k}, P_{1}) \) satisfying \( \partial_{1} u_{l}^{(k)} \neq 0 \).

Proof. For the claim 1. If \( c_{k} = 0 \), then any morphisms \( f \in Z^{k}(\mathcal{A}_{i}) \) satisfies \( f^{(k)} \in \text{rad}(P_{k}, P_{0}) \). Hence \( f \notin H^{k}(\mathcal{A}_{i}) \) by Lemma 6.5 and so \( H^{k}(\mathcal{A}_{i}) = 0 \). We may assume \( c_{k} \geq 1 \). Let \( f_{l} : \Sigma_{l} \to P_{l}[k] \) be in \( Z^{k}(\mathcal{A}_{i}) \) satisfying \( f_{l}^{(k)} = [\pi_{l}, 0] : P(i)^{\oplus c_{k}} \oplus P_{k}^{\Sigma} \to P_{0} \) for some \( 1 \leq l \leq c_{k} \). Then \( f_{l}^{(k)} \) is not in \( \text{rad}(P_{k}, P_{0}) \). Hence by Lemma 6.5 the chain morphism \( f_{l} \) is not in \( B^{k}(\mathcal{A}_{i}) \). Thus, it is a non-zero element of \( H^{k}(\mathcal{A}_{i}) = \text{Ext}^{k}(\Sigma(i), \Sigma(i)) \). Moreover, it immediately turns out \( f_{1}, \ldots, f_{c_{k}} \) form a basis of \( H^{k}(\mathcal{A}_{i}) = \text{Ext}^{k}(\Sigma(i), \Sigma(i)) \). For the claim 2. We first show that any \( g \in B^{k}(\mathcal{A}_{i}) \) can be written by \( d(u) \).
for some \( u \in L^{k-1}(A_i) \) such that \( u^{(k-1)} = 0 \). By Lemma 6.3 again, \( g^{(k)} \in \text{rad}_A(P_k, P_0) \), and hence \( \text{Im} \, g^{(k)} \subseteq \text{Im} \, \partial_1 \), which is from the definition of \( \Delta(i) \), see Remark 2.2. By the projectivity of \( P_k \), there exits \( u^{(k)} : P_k \to P_1 \) such that \( g^{(k)} = \partial_1 u^{(k)} \). Hence we can take \( u \in L^k(A_i) \) as \( u^{(k-1)} = 0 \). Moreover if \( \partial_1 u^{(k)} = 0 \), we have \( d(u)^{(k)} = 0 \). So any non-zero \( u \in L^k(A_i) \) satisfies \( \partial_1 u^{(k)} \neq 0 \).

**Proposition 6.7.** For \( a_1, \ldots, a_r \in A^1 \), we have \( \lambda_r(a_{r-1}, \ldots, a_1) \in Z^2(A_i) \). Moreover, let \( a_1, \ldots, a_r \in H^1(A_i) \). Then \( \lambda_r(a_{r-1}, \ldots, a_1) \in H^2(A_i) \) if and only if \( (a_r \circ G \lambda_{r-1}(a_{r-1}, \ldots, a_1))^{(2)} \) is surjective if and only if for

\[
(G \lambda_{r-1}(a_{r-1}, \ldots, a_1))^{(2)} = \sum h_j \in \text{Hom}_A(P_2, P(i))^{\oplus c_1} \oplus \text{Hom}_A(P_2, P_1),
\]

there exists \( h_j \in \text{Hom}_A(P_2, P(i)) \) is surjective.

**Proof.** By the definition of \( \lambda_r \), in Remark 6.2 we have

\[
d(\lambda_r(a_r, \ldots, a_1)) = \sum_{t=1}^{r-1} (-1)^{t(r-t)+1} G \lambda_{r-t}(a_k, \ldots, a_{t+1}) \circ G \lambda_t(a_t, \ldots, a_1)
\]

\[
+(-1)^{|G \lambda_{r-t}(a_{r-1}, a_{t+1})|} \sum_{t=1}^{r-1} (-1)^{t(r-t)+1} G \lambda_{r-t}(a_r, \ldots, a_{t+1}) \circ d G \lambda_t(a_t, \ldots, a_1).
\]

On the first term, we have

\[
\sum_{t=1}^{r-1} (-1)^{t(r-t)+1} G \lambda_{r-t}(a_r, \ldots, a_{t+1}) \circ G \lambda_t(a_t, \ldots, a_1)
\]

\[
= \sum_{t=1}^{r-1} (-1)^{t(r-t)+1} \lambda_t(a_r, \ldots, a_{r-t+1}) \circ G \lambda_{r-t}(a_{r-t}, \ldots, a_1)
\]

\[
= \sum_{t=1 \atop u=1}^{r-1} (-1)^{\sigma} G \lambda_u(a_r, \ldots, a_{r-u+1}) \circ G \lambda_{t-u}(a_{r-u}, \ldots, a_{r-t+1}) \circ G \lambda_{r-t}(a_{r-t}, \ldots, a_1),
\]

where \( \sigma = t(r-t) + u(t-u) \). Since \( |G \lambda_{r-t}(a_r, \ldots, a_{t+1})| = 1 \), the second term equals to

\[
- \sum_{t=1}^{r-1} (-1)^{t(r-t)+1} G \lambda_{r-t}(a_r, \ldots, a_{t+1}) \circ \lambda_t(a_t, \ldots, a_1)
\]

\[
= - \sum_{t=1 \atop u=1}^{r-1} (-1)^{\sigma} G \lambda_{r-t}(a_r, \ldots, a_{t+1}) \circ G \lambda_{t-u}(a_t, \ldots, a_{u+1}) \circ G \lambda_u(a_u, \ldots, a_1).
\]

Since \( \sigma = u(r-t) + (t-u)(r-t) + u(t-u) \), the signs of each term in the two polynomials coincide. Hence \( d(\lambda_r(a_r, \ldots, a_1)) = 0 \), that is, \( \lambda_r(a_r, \ldots, a_1) \in Z^2(A_i) \).

Next, assume \( a_1, \ldots, a_r \in H^1(A_i) \) and let \( f = \lambda_r(a_r, \ldots, a_1) \). If \( f^{(2)} \) is surjective, so are at least one of morphisms

\[
(G \lambda_{r-t}(a_r, \ldots, a_{t+1}) \circ G \lambda_t(a_t, \ldots, a_1))^{(2)} = (G \lambda_{r-t}(a_r, \ldots, a_{t+1}))^{(1)} \circ (G \lambda_t(a_t, \ldots, a_1))^{(2)}.
\]

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Since \(G\lambda_{r-t}(a_r, \ldots, a_{t+1})\) are in \(L^1(A_i)\) for \(1 \leq t \leq r-2\), we have \(G\lambda_{r-t}(a_r, \ldots, a_{t+1})^{(1)} = 0\). Hence
\[
\left(\sum_{t=1}^{r-1} \left(-1\right)^{(r-t)+1} G\lambda_{r-t}(a_r, \ldots, a_{t+1}) \circ G\lambda_t(a_t, \ldots, a_1)\right)^{(2)} = \left((-1)^{r} a_r \circ G\lambda_{r-1}(a_{r-1}, \ldots, a_1)\right)^{(2)}.
\]
This shows that \(f^{(2)}\) is surjective if and only if \((a_r \circ G\lambda_{r-1}(a_{r-1}, \ldots, a_1))^{(2)}\) is so. Moreover since \(a_r^{(1)}\) is a surjection onto \(P(i)\), the morphism \((G\lambda_{r-1}(a_{r-1}, \ldots, a_1))^{(2)}\) has a summand \(h_j \in \text{Hom}_A(P_2, P(i)) \subset \text{Hom}_A(P_2, P(i))^{\oplus c_1} \oplus \text{Hom}_A(P_2, P_1)\), which is surjective.

Now, let \(n_j = [A : P(j)]\) and \(\{e_j^A\}_{1 \leq j \leq n}\) be a set of pairwise orthogonal idempotents of \(A\) with \(Ae_j^A = P(j)\). Consider the factor algebra \(A' = A/AeA\) for \(e = \sum_{j=i}^{n} e_j^A\). Then \(\Delta(i)\) and \(\Delta(i)\) are also \(A'\)-modules because \(e\Delta(i) = 0\) and \(e\Delta(i) = 0\). We will write \(\bigoplus_{k \geq 0} \text{Hom}_{A'}(P'_i, P'_i[k])\) by \(A'_i\), where
\[
P'_i : \cdots \rightarrow P'_2 \rightarrow P'_1 \rightarrow P'_0
\]
is a projective resolution of \(\Delta(i)\) in mod \(A'\). Of course, \(A'_i\) is a differential category whose differential \(d' = d|_{A'_i}\) is the restriction of that of \(A_i\). Moreover there are multiplications \(m'_k\) such that \((H(A'_i), m')\) is an \(A_{\infty}\)-category by Example 4.4. Further the maps \(\lambda'_i : A''_i^{\otimes r} \rightarrow A'_i\) is defined by a way similar to one in Remark 1.2. Now numbers \(c_k\) coincide with the numbers of copies of \(P_A(i)\) in \(P'_k\), because the numbers \(c_k\) independent of \(P_A(i+1), \ldots, P_A(n)\), or \(\Delta(i+1), \ldots, \Delta(n)\). Moreover the previous lemmas can be argued by ignoring all \(P'_k\), hence by replacing \(A\) with \(A'\). We get the following fact.

**Proposition 6.8.** There is an isomorphism \(H(A_i) \cong H(A'_i)\) as \(\mathbb{Z}\)-graded categories. Moreover for \(a_1, \ldots, a_{r} \in H^1(A_i) \cong H^1(A'_i)\), \(\lambda_i(a_r, \ldots, a_1) \in H^2(A_i)\) if and only if \(\lambda'_i(a_r, \ldots, a_1) \in H^2(A'_i)\).

**Proof.** We notice that the \(\mathbb{Z}\)-graded categories \(H(A_i)\) and \(H(A'_i)\) are isomorphic. Indeed, Lemma 6.6 showed us that a \(K\)-basis of \(H^k(A_i)\) are formed by chain morphisms \(f : P_i \rightarrow P_i[k]\) with the \(k\)-th component \(f^{(k)} = [\pi_i, 0]\). Moreover the \(k\)-th component \(P'_k\) of the resolution \(P'_i\) is isomorphic to \(P_A(i)^{\oplus c_k}\). Assign \(f\) to \(f' : P'_i \rightarrow P'_i[k]\) with \(f'_k = \pi'_i : P_A(i)^{\oplus c_k} \rightarrow P_A(i)\) the canonical projection, then we get an isomorphism from \(H(A_i)\) to \(H(A'_i)\) as \(\mathbb{Z}\)-graded categories.

Next we show that they have also the relationship on their \(A_{\infty}\)-multiplications when we restrict them to \(H^1(A_i)\) and \(H^1(A'_i)\). By Lemmas 5.5 and 5.6 we may fix bases of \(H^k(A_i)\), \(B^k(A_i)\) and \(L^k(A_i)\) as follows.

- **\(H^k(A_i)\):** chain morphisms \(f_i\)’s for \(1 \leq l \leq c_k\) with \(f_i^{(k)} = [\pi_i, 0]\)
- **\(B^k(A_i)\):** chain morphisms \(g_i\)’s for a basis \(\{g_i^{(k)}\}_l\) of \(\text{rad}_A(P_k, P_0)\).
- **\(L^k(A_i)\):** non-chain morphisms \(u_i\)’s for \(u_i^{(k)} = 0\) and a subset \(\{u_i^{(k+1)}\}_l\) of a basis of \(\text{Hom}_A(P_{k+1}, P_1)\) satisfying \(\partial_i u_i^{(k+1)} \neq 0\).

By Proposition 6.7 \(\lambda_i(a_r, \ldots, a_1) \in H^2(A_i)\) if and only if for \((G\lambda_{r-1}(a_{r-1}, \ldots, a_1))^{(2)} = \sum h_j \in \text{Hom}_A(P_2, P(i))^{\oplus c_1} \oplus \text{Hom}_A(P_2, P_1)\), there exists \(h_j \in \text{Hom}_A(P_2, P(i))\) is surjective. And the latter condition depends on only automorphisms between \(P(i)\). Thus the multiplications of \(H^1(A_i)\) is calculated by that of \(\text{Aut}_A(P_A(i))\), and the arguments above can be done similarly for \(A'_i\). Moreover we have isomorphisms \(\text{Aut}_A(P_A(i)) \cong \text{Aut}_{A'}(P_A'(i))\) as algebras. Hence there is also a one-to-one correspondence between the multiplications in \(H^1(A_i)\) and \(H^1(A'_i)\).
Proposition 6.9. There is an isomorphism $H(A') \cong \text{Ext}^*_E(S_E, S_E)$ as $A_\infty$-categories.

Proof. We have $\text{Hom}_{A'}(\Delta(i), \Delta(i)) \cong S_E$ and $\text{Hom}_{A'}(\Delta(i), \Delta(i)) \cong P_E$ as $E$-modules. Moreover since $\Delta(i) \cong P_A(i)$ is a projective $A'$-module, the functor $\text{Hom}_{A'}(\Delta(i), -) : \text{mod} A' \to \text{mod} E$ is exact. Hence, the complex

$$
P_E = \text{Hom}_{A'}(\Delta(i), P'_i) : \cdots \to P_E^{b_{c_2}} \to P_E^{b_{c_1}} \to P_E^{b_{c_0}}$$

is a projective resolution of $S_E$. So there is an isomorphism between $\bigoplus_{k \geq 1} \text{Hom}_E(P_E, P_E[k])$ and $A'_i = \bigoplus_{k \geq 1} \text{Hom}_{A'}(P'_i, P'_i[k])$ as differential graded categories. This implies that $\text{Ext}^*_E(S_E, S_E) \cong H(A'_i)$ as $A_\infty$-categories.

These propositions imply that $e_i B e_i$ is finite dimensional as follows.

Theorem 6.10. Let $E = \text{End}_A(\Delta(i))$ and $B_A = (B, W)$ be the bocs induced from $A$. Then $E$ and $e_i B e_i$ are Morita equivalent. In particular, $e_i B e_i$ is finite dimensional.

Proof. By the construction of $B$, the algebra $e_i B e_i$ is generated by a $K$-basis of $Q_0(i, i) = Ds(H^1(A_i))$ and has relations induced from $Q_{-1}(i, i) = Ds(H^2(A_i))$. Similarly, Theorem 6.8 implies that a $K$-basis of $D \text{Ext}^1_E(S_E, S_E)$ generates $E$ and that a $K$-basis of $D \text{Ext}^2_E(S_E, S_E)$ constitutes relations of $E$. Next, identify morphisms in $H^1(A_i)$ and $\text{Ext}^1_E(S_E, S_E)$. Then the previous propositions show that the multiplications of morphisms in $H^1(A_i)$ is included in $H^2(A_i)$ if and only if that is in $\text{Ext}^2_E(S_E, S_E)$. Since these multiplications as $A_\infty$-categories are induced from the multiplications of the algebras $e_i B e_i$ and $E$, these algebras are Morita equivalent.

The following lemma shows that the bocs $B_A = (B, W)$ is one-cyclic directed. And this bocs is what we want i.e. it satisfies that $F(\Delta_A) \simeq \text{mod} B_A$, which is proved in Theorem 6.12 below.

Lemma 6.11. Let $A$ be a $\Delta$-filtered algebra. Then the bocs $B_A$ given above is one-cyclic directed.

Proof. The conditions

$$\text{End}_A(\Delta(i)) \cong K, \text{Hom}_A(\Delta(i), \Delta(j)) = 0 \text{ and } \text{Ext}^1_A(\Delta(i), \Delta(j)) = 0 \text{ for } i > j$$

implies that

$$W \cong \bigoplus_{i > j} (B e_i \otimes e_j B)^{d_{ij}} \text{ and } \text{Ext}^1_B(S(i), S(j)) = 0 \text{ for } i > j.$$ 

The second condition shows $\text{rad}_B(P_B(j), P_B(i))/\text{rad}_B^2(P_B(j), P_B(i)) = 0$ for $j < i$, and hence $\text{rad}_B(P_B(j), P_B(i)) = 0$ for $i < j$. 

Theorem 6.12. Let $A$ be a $\Delta$-filtered algebra. Then the one-cyclic directed bocs $B_A = (B, W)$ constructed above satisfies $\text{mod} B_A \simeq F(\Delta_A)$.

Proof. Let $H^*(A)$ be an $A_\infty$-category over $\{1, 2, \ldots, n\}$ such that $H^k(A)(i, j) = \text{Ext}^k_A(\Delta, \Delta)$. We already claimed $F(\Delta) \simeq H^0(\text{twmod} H^*(A))$ in Lemma 6.1. So it suffices to give an equivalence $F : H^0(\text{twmod} H^*(A)) \to \text{mod} B_A$ for the bocs $B_A$ given above. For objects, we confirmed a one-to-one correspondence between $(X, \delta)$ and $X_\delta$ in Proposition 6.3. For morphisms, $F$ assigns $f \in H^0(\text{twmod} H^*(A))$ to $M(\ast f) \in \bigoplus_{i \leq j \leq n} \text{Hom}_K(Q(i, j), \text{Hom}_K(X(i, Y(j))))$ such that
\[ M(sf)(\varphi_k) = 0 \text{ for } \varphi_k \in Q_1 \cap d(B). \] Although the later is not a morphism of \( \text{mod} \mathcal{B}_A \), we can identify them by the following isomorphisms as vector spaces

\[
\text{Hom}_B(X,Y) = \text{Hom}_B \otimes B^{\text{op}}(( \bigoplus_k B\varphi_k B)/d(B), \text{Hom}_K(X,Y))
\]

\[ \cong \{ \Phi \in \text{Hom}_B \otimes B^{\text{op}}(( \bigoplus_k B\varphi_k B, \text{Hom}_K(X,Y)) \mid \Phi(d(B)) = 0 \}
\]

\[ \cong \{ g \in \bigoplus_{1 \leq i,j \leq n} \text{Hom}_K(Q_1(i,j), \text{Hom}_K(X(i), Y(j))) \mid \Phi_g(d(B)) = 0 \}, \]

where \( \varphi_k \) are generators of \( Q_1 \) over \( K \) and \( \Phi_g \) is given by \( \Phi_g(b\varphi_kb') = bg(\varphi_kb') \). Then \( F \) is an equivalence. We omit to check that here since the calculation is the same to that in [KKO]. \qed

Thus we give a new construction of one-cyclic directed bocses \( \mathcal{B}_A \) from \( \overline{\Xi} \)-filtered algebras \( A \) with \( \mathcal{F}(\overline{\Xi}_A) \cong \text{mod} \mathcal{B}_A \), which is generalization of the method in [KKO]. Indeed, if \( A \) is quasi-hereditary, the bocs constructed by our way is the same to that by using the methods in [KKO].

### 6.2 \( \overline{\Xi} \)-filtered algebras from one-cyclic directed bocses

In this subsection, we construct an algebra \( R \) from a one-cyclic directed bocs \( B \) with \( \mathcal{F}(\Delta_R) \cong \text{mod} \mathcal{B} \). Since Burt and Butler’s theory are confirmed for bocses with projective kernels, we can apply them to one-cyclic directed bocses. By immitating the arguments in [KKO], we will show that the right Burt-Butler algebra of a one-cyclic directed bocs is a \( \overline{\Xi} \)-filtered algebra.

**Theorem 6.13.** Let \( B = (B, W) \) be a one-cyclic directed bocs. Then its right Burt-Butler algebra \( R \) of \( B \) is a \( \overline{\Xi} \)-filtered algebra with a homological proper Borel subalgebra \( B \) such that \( \mathcal{F}(\overline{\Xi}_R) \cong \text{mod} \mathcal{B} \).

**Proof.** Remark that the algebra \( R \) is basic since so is \( B \). Put \( \overline{\Xi}_R(i) = R \otimes S_B(i) \). Recall that \( \text{Ind}(B, R) \) is a full subcategory of \( \text{mod} R \) whose objects are of the form \( R \otimes X \) for \( B \)-modules \( X \). By the definition of \( \overline{\Xi}_R \) in this proof, we have an equivalence

\[ T : \text{Ind}(B, R) \to \mathcal{F}(\overline{\Xi}_R) \text{ with } T(R \otimes S_B(i)) \cong \overline{\Xi}_R(i). \]

Indeed, obviously this functor is dense and \( \text{Ind}(B, R) \) and \( \mathcal{F}(\overline{\Xi}_R) \) are full subcategories of \( \text{mod} R \). On the other hand, we also have an equivalence

\[ R \otimes_B - : \text{mod} B \to \text{Ind}(B, R) \]

by Lemma 3.6 [11]. Hence we obtain an equivalence \( T \circ (R \otimes_B -) : \text{mod} B \to \mathcal{F}(\overline{\Xi}_R) \). Next we show that \( R \) is a \( \overline{\Xi} \)-filtered algebra. Now \( \overline{\Xi}_R(i) \) is a factor module of \( P_R(i) \) because there is a surjection \( P_R(i) \to \overline{\Xi}_R(i) \) induced from the canonical epimorphism \( P_B(i) \to S_B(i) \). By the equivalence
$T \circ (R \otimes B -)$, we have an isomorphism as $K$-vector spaces

$$
\text{Hom}_R(P_R(i), \overline{\Sigma_R}(j)) = \text{Hom}_R(R \otimes_B P_B(i), R \otimes_B S_B(j)) \\
\cong \text{Hom}_B(P_B(i), S_B(j)) \\
\cong \text{Hom}_B(W \otimes_B P_B(i), S_B(j)) \\
\cong \text{Hom}_B((B \oplus W) \otimes_B P_B(i), S_B(j))
$$

$$
\cong \left( \text{Hom}_B(B \otimes_B P_B(i), S_B(j)) \right) \oplus \left( \text{Hom}_B(\bigoplus_{k > l} (B e_k \otimes_K e_l B)^{d_{kl}} \otimes_B P_B(i), S_B(j)) \right)
$$

$$
\cong \left( \text{Hom}_B(P_B(i), S_B(j)) \right) \oplus \left( \text{Hom}_B(\bigoplus_{j > l} (P_B(j) \otimes_K e_l B e_i)^{d_{jl}}, S_B(j)) \right)
$$

$$
\cong \left( \text{Hom}_B(P_B(i), S_B(j)) \right) \oplus \left( \text{Hom}_B(\bigoplus_{j > l} P_B(j)^{d_{jl} \dim e_l B e_i}, S_B(j)) \right).
$$

As a result, we have

$$
\dim \text{Hom}_R(P_R(i), \overline{\Sigma_R}(j)) = \begin{cases} 
1 + \sum_{j > l} d_{jl} \dim e_l B e_i & (i \leq j) \\
0 & (i > j),
\end{cases}
$$

and hence $\overline{\Sigma_R}(j)$ has the composition factor only of the form $S(i)$ with $i \leq j$ and $[\overline{\Sigma_R}(j) : S_R(j)] = 1$.

Finally, by using Lemma 3.6 [2], we have $\dim \text{Ext}^1_R(\overline{\Sigma_R}(i), \overline{\Sigma_R}(j)) \leq \dim \text{Ext}^1_B(S_B(i), S_B(j)) = 0$ for $i > j$. On the other hand, $R \cong R \otimes B$ is filtered by $\overline{\Sigma_R}$. The above conditions show that $\overline{\Sigma_R}$ is the properly standard system over $R$. Hence the algebra $R$ is $\overline{\Sigma}$-filtered. Moreover, the functor $R \otimes_B -$ is exact, by Lemma 3.6 [4]. Thus $B$ is a homological proper Borel subalgebra of $R$.

6.3 Relation between $\overline{\Sigma}$-filtered algebras and $\mathcal{F}(\overline{\Sigma})$

In this subsection, we show that $\overline{\Sigma}$-filtered algebras are uniquely determined by $\mathcal{F}(\overline{\Sigma})$, up to Morita equivalence. But we cannot prove this proposition similarly to DR Theorem 2. This is because a set $\{\overline{\Sigma}_1, \ldots, \overline{\Sigma}_n\}$ of objects, satisfying the homological properties of a properly standard system, of an abelian $K$-category $\mathcal{C}$ may have self-extensions. It may require that we take extensions of objects of $\mathcal{F}(\overline{\Sigma}_i)$ by $\overline{\Sigma}_i$, repeatedly when constructing Ext-projective object $P_{\overline{\Sigma}_i}(i)$. In order for such a situation not to occur, $\mathcal{C}$ will be assumed to be the category mod $C$ for some finite dimensional algebra $C$. Fortunately, we just need a one-to-one correspondence between Morita equivalence classes of $\overline{\Sigma}$-filtered algebras and equivalence classes of categories of modules with $\overline{\Sigma}$-filtrations. So we give the next theorem given in ADL Theorem 2.3.

**Theorem 6.14 (ADL) Theorem 2.3.** Let $C$ be a finite dimensional algebra. Then there exists a $\overline{\Sigma}$-filtered algebra $A$, unique up to Morita equivalence, such that the category $\mathcal{F}(\overline{\Sigma}_C)$ and $\mathcal{F}(\overline{\Sigma}_A)$ are equivalent. In particular, $\overline{\Sigma}$-filtered algebras $A$, and $C$ are Morita equivalent if and only if there exists an equivalence $F : \mathcal{F}(\overline{\Sigma}_A) \to \mathcal{F}(\overline{\Sigma}_C)$.
6.4 Main result

As the final subsection of this article, we sum up the previous three subsections and get the following theorem giving a generalization of Theorem 1.1.

**Theorem 6.15.** We have a bijection

\[
\{\text{Morita equivalence classes of } \Delta\text{-filtered algebras}\} \\
\uparrow \\
\{\text{Equivalence classes of the module categories over one-cyclic directed bocses}\}.
\]

Let a \(\Delta\)-filtered algebra \(A\) and a one-cyclic directed bocs \(B = (B, W)\) correspond via the above bijection. Then the right Burt-Butler algebra \(R_B\) of \(B\) is Morita equivalent to \(A\). Moreover, \(R_B\) has a homological proper Borel subalgebra \(B\).

**Proof.** The second half condition is shown in Theorem 6.13. On the first half condition, let \(A\) be a \(\Delta\)-filtered algebra. Then we can construct a one-cyclic directed bocs \(B = B_A\) given in Subsection 6.1. Moreover, the right Burt-Butler algebra \(R\) of \(B\) is a \(\Delta\)-filtered algebra by the discussion in Subsection 6.2. In these cases, there are equivalences

\[
\mathcal{F}(\Delta_A) \cong \text{mod} B \cong \mathcal{F}(\Delta_R)
\]

by Theorems 6.12 and 6.13. Finally, Theorem 6.14 guarantees \(A\) and \(R\) are Morita equivalent.

The following theorem is the result for \(\Delta\)-filtered algebras. Actually, almost the same arguments in [KKO] give its proof. And we remark that in [BPS] Theorem 12.9, Bautista, Pérez and Salméron gave the result stronger than this.

**Theorem 6.16.** We have a bijection

\[
\{\text{Morita equivalence classes of } \Delta\text{-filtered algebras}\} \\
\uparrow \\
\{\text{Equivalence classes of the module categories over weakly directed bocses}\}.
\]

Let a \(\Delta\)-filtered algebra \(A\) and a weakly directed bocs \(B = (B, W)\) correspond via the above bijection. Then the right Burt-Butler algebra \(R_B\) of \(B\) is Morita equivalent to \(A\). Moreover, \(R_B\) has a homological exact Borel subalgebra \(B\).

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