Abstract. We study tensor products of infinite dimensional representations (not corepresentations) of the SU(2) quantum group. Eigenvectors of certain self-adjoint elements are obtained, and coupling coefficients between different eigenvectors are computed. The coupling coefficients can be considered as $q$-analogs of Bessel functions. As a result we obtain several $q$-integral identities involving $q$-hypergeometric orthogonal polynomials and $q$-Bessel-type functions.

1. Introduction

Many identities for special functions have a structure coming from representation theory of Lie or quantum algebras. In e.g. [17], [20], [13] summation and integral identities are derived from tensor products of irreducible $*$-representations of the Lie algebras $su(2)$, $su(1,1)$ and quantum versions of these Lie algebras. In this paper we consider representations of the SU(2) quantum group, i.e. the quantized algebra $A_q = A_q(SU(2))$ of functions on the Lie group SU(2). In the classical $q=1$ case the representation theory of this algebra is trivial, but in the quantum case the algebra has a class of infinite dimensional irreducible $*$-representations. We consider two- and threefold tensor products of these representations, and compute coupling coefficients between eigenvectors of certain self-adjoint elements in the algebra $A_q$. This leads to several $q$-integral identities involving different classes of $q$-hypergeometric special functions. The identities we obtain can be viewed as connection formulas between orthogonal bases in Hilbert spaces of functions in two variables.

In [24] Koornwinder studies the tensor product of two infinite dimensional representations of $A_q$. Using spectral analysis of a self-adjoint element $\gamma \beta \in A_q$, the tensor product can be decomposed into irreducible representations. The element $\gamma \beta$ is a special case of the self-adjoint element $\rho_{\tau,\sigma} \in A_q$, $\tau, \sigma \in \mathbb{R}$. The latter was introduced by Koornwinder in [25], in which an explicit expression for the Haar-functional on the subalgebra generated by $\rho_{\tau,\sigma}$ is derived using Askey-Wilson polynomials. Koelink and Verding [23] obtained the Haar-functional in a different way using spectral analysis of $\rho_{\tau,\sigma}$ in an infinite dimensional representation of $A_q$. The latter approach uses eigenvectors which are given explicitly in terms of $q$-hypergeometric orthogonal polynomials. In this paper we combine the approaches from [23] and [24] to derive several $\rho_{\tau,\sigma}$-eigenvectors for tensor product representations. Having different explicit eigenvectors, it may be expected to find ‘nice’ explicit expressions for the coupling coefficients between them.

The coupling coefficients we obtain can be considered as $q$-analogs of Bessel functions. The most general ones in this paper are the $q$-Meixner functions [14]. It is shown that several known types of $q$-Bessel functions can be considered as limit cases of the $q$-Meixner functions. This leads to the scheme of $q$-Bessel functions in Figure 1. For the definitions of the functions in this scheme and their Hankel-type orthogonality relations see Sections 5 and 6. Let us remark that Jackson’s [10] well-known $q$-analogs of Bessel functions are included in this scheme: the Stieltjes-Wigert case is closely related to Jackson’s second $q$-Bessel function, and the Hahn-Exton $q$-Bessel function is Jackson’s third $q$-Bessel function. Furthermore, Jackson’s first and second $q$-Bessel functions are basically the same function, see e.g. [15, Theorem 14.1.3].

The solid arrows in the scheme correspond to limit relations between elements in the algebra $A_q$. The left side of the scheme contains coupling coefficients for three-fold tensor product representation, the right side for two-fold tensor products.

Date: May 23, 2014.
The scheme in Figure 1 can be considered as being part of an extended Askey-scheme of $q$-hypergeometric integral transforms [22], which contains the well-known Askey-scheme of $q$-hypergeometric polynomials, see [18]. Figure 1 is also closely related to the $q$-Meixner scheme of indeterminate moment problems within the Askey-scheme of $q$-hypergeometric orthogonal polynomials, see [5, p.24]. Indeed, the $q$-Meixner functions are $q$-Meixner polynomials for specific values of the spectral variable, and therefore the orthogonality measure for the $q$-Meixner functions gives a solution for the corresponding indeterminate moment problem, see [14]. The other cases in Figure 1 also correspond to indeterminate moment problems, except the Hahn-Exton $q$-Bessel functions.

The outline of the paper is as follows. In Section 2 we give preliminaries on representations of $A_q(SU(2))$. In particular we consider the tensor product of two infinite dimensional irreducible $*$-representations and corresponding eigenvectors, and we introduce useful subalgebras depending on two parameters $\tau, \sigma \in \mathbb{R}$. In Sections 3 and 4 we determine explicitly the Clebsch-Gordan coefficients corresponding to eigenvectors of $\rho_{\tau,\sigma}$. First in Section 3 we do this in case $\sigma = \infty$, and in Section 4 the general case is considered. In Section 5 we first determine eigenvectors of $\rho_{\tau,\sigma}$, in case of a two-fold tensor product representation, which are different from the eigenvectors from Section 3. Then we determine explicitly coupling coefficients between the different eigenvectors. This leads to a $q$-integral identity involving several types of $q$-hypergeometric orthogonal polynomials and functions closely related to the Al-Salam–Carlitz II polynomials. In Section 6 we first determine eigenvectors for $\rho_{\tau,\sigma}$ in a three-fold tensor product representation. We show that the coupling coefficients between different eigenvectors are given in terms of $q$-Meixner functions. This leads to more $q$-integral identities.

**Notation.** We denote the set of integers by $\mathbb{Z}$, and by $\mathbb{N}$ we denote the set of nonnegative integers. $\mathbb{T} = \{x \in \mathbb{C} \mid |x| = 1\}$, the unit circle in $\mathbb{C}$. By $\sqrt{\cdot}$ we denote the principal branch of the square root. Throughout the paper $q$ is a fixed number in $(0, 1)$. We use notations for $q$-shifted factorials, $\theta$-functions and $q$-hypergeometric functions as in [11], i.e.,

\[
(a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j), \quad (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}, \quad n \in \mathbb{Z},
\]

\[
\theta(a; q) = \frac{(a; q)_\infty}{(q/a; q)_\infty}, \quad a \notin q\mathbb{Z},
\]

\[
(a_1, a_2, \ldots, a_k; q)_n = \prod_{j=1}^{k} (a_j; q)_n, \quad \theta(a_1, a_2, \ldots, a_k; q) = \prod_{j=1}^{k} \theta(a_j; q),
\]

\[
r\varphi_s\left(\frac{a_1, a_2, \ldots, a_r}{b_1, b_2, \ldots, b_s}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n} \left((-1)^n q^{\frac{1}{2}n(n-1)}\right)^{s-r-1} \frac{z^n}{n!},
\]
and we also use the shorthand notations
\[
(ab^{\pm 1}; q)_n = (ab, a/b; q)_n, \quad \theta(ab^{\pm 1}; q) = \theta(ab, a/b; q).
\]

An identity that we frequently use, often without mentioning, is the \(\theta\)-product identity:
\[
\theta(xq^k; q) = (-x)^{-k}q^{-\frac{1}{2}k(k-1)}\theta(x), \quad k \in \mathbb{Z}.
\]

We also need the Jackson \(q\)-integral, see [[1]], which is defined by
\[
\int_0^\alpha f(x) \, dq^x = (1 - q) \sum_{k=0}^{\infty} f(\alpha q^k)q^k, \quad \int_0^\infty f(x) \, dq^x = \int_0^\infty f(x) \, dq_x - \int_0^\beta f(x) \, dq_x,
\]
for a function \(f\) such that the sums converge absolutely. Observe that for \(\alpha > 0\) and \(\beta < 0\),
\[
\int_0^\alpha f(x) \, dq^x = (1 - q) \sum_{x \in \beta q^n, \alpha q^n} f(x) |x|.
\]

2. The SU(2) Quantum Group

The SU(2) quantum group is the complex unital associative algebra \(A_q = A_q(SU(2))\) generated by \(\alpha, \beta, \gamma, \delta\), which satisfy the relations
\[
\alpha \beta = q \beta \alpha, \quad \alpha \gamma = q \gamma \alpha, \quad \beta \delta = q \delta \beta, \quad \gamma \delta = q \delta \gamma, \quad \beta \gamma = \gamma \beta, \quad \alpha \delta - q \beta \gamma = 1 = \delta \alpha - q^{-1} \beta \gamma.
\]

There is a \(\ast\)-structure defined on the generators by
\[
\alpha^\ast = \delta, \quad \beta^\ast = -q \gamma, \quad \gamma^\ast = -q^{-1} \beta, \quad \delta^\ast = \alpha.
\]

\(A_q\) is a Hopf-\(\ast\)-algebra with comultiplication \(\Delta\), which is defined on the generators by
\[
\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma, \quad \Delta(\beta) = \alpha \otimes \beta + \beta \otimes \delta, \quad \Delta(\gamma) = \gamma \otimes \alpha + \delta \otimes \gamma, \quad \Delta(\delta) = \delta \otimes \delta + \gamma \otimes \beta.
\]

We do not need the antipode in this paper. The irreducible \(\ast\)-representation of \(A_q\) are either 1-dimensional, or infinite dimensional. The infinite dimensional irreducible \(\ast\)-representations are labeled by \(\phi \in [0, 2\pi]\), and we denote a representation by \(\pi_\phi\). The representation space of \(\pi_\phi\) is \(\ell^2(\mathbb{N})\), and the generators \(\alpha, \beta, \gamma, \delta\) act on the standard orthonormal basis \(\{e_n\}_{n \in \mathbb{N}}\) of \(\ell^2(\mathbb{N})\) as raising and lowering operators:
\[
\pi_\phi(\alpha) e_n = \sqrt{1 - q^{2n}} e_{n-1}, \quad 
\pi_\phi(\beta) e_n = -e^{-i\phi} q^{n+1} e_n, \quad 
\pi_\phi(\gamma) e_n = e^{i\phi} q^n e_n, \quad 
\pi_\phi(\delta) e_n = \sqrt{1 - q^{2n+2}} e_{n+1}.
\]

Note that the actions of \(\beta\) and \(\delta\) can be found from the actions of \(\alpha\) and \(\gamma\) using the \(\ast\)-structure.
2.1. The tensor product representation. In this paper we are mainly interested in the tensor product representation

$$\mathcal{T} = \mathcal{T}_{\phi, \psi} = (\pi_\phi \otimes \pi_\psi) \Delta, \quad \phi, \psi \in [0, 2\pi),$$

of $A_q$ on $L^2(N) \otimes L^2(N)$. Koornwinder \cite{24} determined the decomposition of $\mathcal{T}$ into irreducible representations. The result is as follows.

Let $\rho$ be the direct integral representation $\rho \equiv \int_0^{2\pi} \pi_\phi \, d\phi$ acting on $L^2(0, 2\pi) \otimes L^2(N) \cong \int_0^{2\pi} L^2(N) \, d\phi$. Let $\{v_m\}_{m \in \mathbb{Z}}$ denote the standard orthonormal basis of $L^2(0, 2\pi)$, $v_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{-im\phi}$.

One checks directly the actions of the generators on standard basis elements:

$$
\begin{align*}
\rho(\alpha) v_m \otimes e_n &= \sqrt{1 - q^{2m}} v_{m-1} \otimes e_n, \\
\rho(\beta) v_m \otimes e_n &= -q^{n+1} v_{m+1} \otimes e_n, \\
\rho(\gamma) v_m \otimes e_n &= q^n v_{m-1} \otimes e_n, \\
\rho(\delta) v_m \otimes e_n &= \sqrt{1 - q^{2n+2}} v_{m+1} \otimes e_{n+1}.
\end{align*}
$$

Theorem 2.1. For $\phi, \psi \in [0, 2\pi)$ the tensor product representation $\mathcal{T}_{\phi, \psi}$ is unitarily equivalent to the direct integral representation $\rho$.

It is instructive to go through the proof of Theorem 2.1. We compute the Clebsch-Gordan coefficients corresponding to the standard basis $\{e_n\}_{n \in \mathbb{N}}$ of $L^2(N)$. Observe that by \ref{2.4} the standard basis vector $e_n$ is an eigenvector of $\pi_\phi(\gamma\gamma^*)$ for eigenvalue $q^{2n}$. We consider $\mathcal{T}(\gamma\gamma^*)$ acting on basis elements $e_n \otimes e_m$. From \ref{2.3} we obtain

$$
\Delta(\gamma\gamma^*) = -q^{-1}(\gamma\beta \otimes \alpha\delta + \gamma\alpha \otimes \alpha\beta + \delta\beta \otimes \gamma\delta + \delta\alpha \otimes \gamma\beta),
$$

and then \ref{2.4} gives

$$
\mathcal{T}(\gamma\gamma^*) e_{n_1} \otimes e_{n_2} = e^{i(\phi - \psi)} q^{n_1+n_2-1} \sqrt{(1 - q^{2n_1})(1 - q^{2n_2})} e_{n_1-1} \otimes e_{n_2-1} + [q^{2n_1} + q^{2n_2} - q^{2n_1+2n_2} (1 + q^2)] e_{n_1} \otimes e_{n_2} + e^{-i(\phi - \psi)} q^{n_1+n_2+1} \sqrt{(1 - q^{2n_1+2})(1 - q^{2n_2+2})} e_{n_1+1} \otimes e_{n_2+1}.
$$

We define

$$
\mathcal{F}_n^p = \begin{cases} 
\{e_n \otimes e_{n+p}, & p \geq 0, \\
\{e_{n-p} \otimes e_n, & p < 0,
\end{cases}
$$

and let $H_p$, $p \in \mathbb{Z}$, be the Hilbert space defined by

$$
H_p = \text{span} \{ \mathcal{F}_n^p | \ n \in \mathbb{N} \} \cong L^2(N).
$$

Then we see that $\mathcal{T}(\gamma\gamma^*)$ leaves $H_p$ invariant, and restricted to $H_p$ it acts as a Jacobi operator (or tridiagonal operator). To diagonalize $\mathcal{T}(\gamma\gamma^*)|_{H_p}$ we need the Wall polynomials, see \ref{15}, which are defined by

$$
(2.7) \quad p_n(y; a; q) = 2\varphi_1 \left( \frac{q^{-n}, 0}{aq}, \frac{q}{q} : q, qy \right).
$$

For $0 < a < q^{-1}$ these polynomials satisfy the orthogonality relations,

$$
\sum_{x \in \mathbb{N}} p_m(q^x; a; q)p_n(q^x; a; q) \frac{(aq)^x}{(q; q)_x} = \delta_{mn} \frac{(aq)^n(q; q)_n}{(aq; q)_n(aq; q)_n},
$$

and they form a basis for the corresponding weighted $\ell^2$-space. Let the function $\tilde{p}_n(q^x; a; q)$ be defined by

$$
(2.8) \quad \tilde{p}_n(q^x; a; q) = (-1)^{n+x} \sqrt{\frac{(aq)^{x-n}(aq; q)_x(aq; q)_n}{(q; q)_n(q; q)_x}} p_n(q^x; a; q),
$$
then from the orthogonality relation for the Wall polynomials and from completeness we obtain the orthogonality relations
\[
\sum_{n \in \mathbb{N}} \bar{p}_n(q^x; a; q) \bar{p}_m(q^y; a; q) = \delta_{nm}, \quad \sum_{n \in \mathbb{N}} \bar{p}_n(q^x; a; q) \bar{p}_n(q^y; a; q) = \delta_{xy},
\]
for \(0 < a < q^{-1}\). The latter, the dual orthogonality relations, actually correspond to orthogonality relations for the Al-Salam–Carlitz II polynomials, see [18]. The three-term recurrence relation for the Wall polynomials is equivalent to
\[
q^x \bar{p}_n(q^y; a; q) = q^n \sqrt{a(1 - q^n)(1 - qa^n)} \bar{p}_{n+1}(q^x; a; q) + \left[q^n(1 - aq^{n+1}) + qa^n(1 - q^n)\right] \bar{p}_n(q^y; a; q) + q^n \frac{1}{\sqrt{a(1 - q^n)(1 - qa^n)}} \bar{p}_{n-1}(q^x; a; q),
\]
with \(\bar{p}_{-1}(q^x) = 0\) and \(\bar{p}_0(q^x) = (q^{-1/2} + \sqrt{1 - q^2})/2\). We define
\[
c_{x,p,n} = \begin{cases} \bar{p}_n(q^{2x}; q^{2p}; q^2), & p \in \mathbb{N}, \\ \bar{p}_n(q^{2x}; q^{-2p}; q^2), & -p \in \mathbb{N}, \end{cases}
\]
then it follows that the vector
\[
V_{x,p} = \begin{cases} \sum_{n \in \mathbb{N}} e^{i(n+p)\phi + i\bar{z}(\phi - \psi)} c_{x,p,n} e_n \otimes e_{n+p}, & p \geq 0, \\ \sum_{n \in \mathbb{N}} e^{i(n-p)\phi - i\bar{z}(\phi - \psi)} c_{x,p,n} e_{n-p} \otimes e_n, & p \leq 0, \end{cases}
\]
is an eigenvector of \(T(\gamma^*)\) for eigenvalue \(q^{2x}\), \(x \in \mathbb{N}\). We define \(c_n = 0\) for \(n \in -\mathbb{N}_{\geq 1}\), then the eigenvector \(V_{x,p}\) can actually be defined for all \(p \in \mathbb{Z}\) by the first expression in (2.10). This is a consequence of the identity
\[
\bar{p}_n(q^x; q^{-p}; q) = \begin{cases} \bar{p}_{n-p}(q^x; q^p; q), & 0 \leq p \leq n, \\ 0, & p > n, \end{cases}
\]
which follows from the \(\varphi x\)-expression for \(\bar{p}_n\), and, for \(p \in \mathbb{N}\),
\[
(q^{-1}p; q)\sum_{\alpha \in \mathbb{N}} \frac{A_{\alpha}(q^{1+p}; q)_{\alpha}}{(q,q^{-1}; q)_{\alpha}} = (q^{1+p}; q)\sum_{\alpha \in \mathbb{N}} \frac{A_{\alpha}(q^{1-p}; q)_{\alpha}}{(q^{-1}; q)_{\alpha}} = (q^{1+p}; q)\sum_{\alpha \in \mathbb{N}} \frac{A_{\alpha}}{(q,q^{1+p}; q)_{\alpha}}.
\]
From the dual orthogonality relations for \(\bar{p}_n\) it follows that \(\{V_{x,p} \mid x \in \mathbb{N}, p \in \mathbb{Z}\}\) is an orthonormal basis for \(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})\), and we see that (2.10) is equivalent to
\[
e_{n_1} \otimes e_{n_2} = \sum_{x \in \mathbb{N}} e^{i\varphi x - i\varphi^n - i\bar{z}(\phi - \psi)} c_{x,n_2-n_1,n_1} V_{x,n_1}.
\]
Finally, we need to determine the actions of the \(A_{\beta}\)-generators on \(V_{x,p}\). Note that \(T(\delta)\) leaves \(H_p\) invariant, and from the commutation relations (2.4) we see that \(T(\gamma \gamma^*) = q^{-2}(\gamma^*)^2 \delta\), so \(T(\delta)V_{x,p}\) is an eigenvector in \(H_p\) of \(T(\gamma \gamma^*)\) for eigenvalue \(q^{2x+2}\), which implies \(T(\delta)V_{x,p} = CV_{x+1,p}\) for some \(C \in \mathbb{C}\). To determine the value of \(C\) we use \(\delta = \alpha + \beta \otimes \gamma\), and see that (2.2) and (2.3),
\[
C(V_{x+1,p}, e_0 \otimes e_p) = \langle T(\delta)V_{x,p}, e_0 \otimes e_p \rangle = \langle V_{x,p}, T(\alpha)e_0 \otimes e_p \rangle = \langle V_{x,p}, e_0 \otimes e_p \rangle = -e^{i(\phi - \psi)} q^{(p+1)}(V_{x,p}, e_0 \otimes e_p).
\]
From \(c_{x,p,0} = (-1)^x q^{x(p+1)} \sqrt{(q^{2p+2}; q)\infty/(q^2; q^2)\infty}\) it now follows that \(C = \sqrt{1 - q^{2x+2}}\). Using \(\alpha \delta = 1 + q^2 \gamma \gamma^*\), the action of \(\alpha\) is obtained from the action of \(\delta\). For the action of \(\gamma\) we note that \(T(\gamma)\) sends \(H_p\) to \(H_{-p}\), and from \(\gamma^*(\gamma^*) = (\gamma^*)^2\) it then follows that \(T(\gamma)V_{x,p}\) is an eigenvector in \(H_{-p-1}\) of \(T(\gamma \gamma^*)\) for eigenvalue \(q^{2x}\), so \(T(\gamma)V_{x,p} = DV_{x,p-1}\) for some \(D \in \mathbb{C}\). The value of \(D\)
can be computed similarly as above. De the action of $\beta$ can be obtained from the action of $\gamma$. We now obtained

$$
\mathcal{T} (\alpha) V_{x,p} = \sqrt{1 - q^{2x}} V_{x-1,p},
$$

$$
\mathcal{T} (\beta) V_{x,p} = q^{x+1} V_{x,p+1},
$$

$$
\mathcal{T} (\gamma) V_{x,p} = q^{x} V_{x,p-1},
$$

$$
\mathcal{T} (\delta) V_{x,p} = \sqrt{1 - q^{2x+2}} V_{x+1,p}.
$$

Comparing this with the actions of the generators $\alpha$, $\beta$, $\gamma$, $\delta$ in the direct integral representation of $\rho$, we see that the unitary operator $\Lambda : l^2(\mathbb{N}) \otimes l^2(\mathbb{N}) \to L^2(0, 2\pi) \otimes l^2(\mathbb{N})$ defined by

$$
\Lambda V_{x,p} = v_p \otimes e_x,
$$

intertwines $\mathcal{T}(X)$ with $\rho(X)$ for any $X \in \mathcal{A}_q$. This proves Theorem 2.1.

**Remark 2.2.** The actions of $\alpha, \beta, \gamma, \delta$ on the eigenvectors $V_{x,p}$ imply contiguous relations for the Clebsch-Gordan coefficients $c_{x,p,n}$, which in turn imply the following contiguous relations for Wall polynomials:

$$(1 - a)p_n(q^x; a/q; q) = (1 - aq^n)p_n(q^x; a; q) - a(1 - q^n)p_{n-1}(q^x; a; q),$$

$$q^x p_n(q^x; a/q; q) = q^n (1 - aq)p_n(q^x; a; q) - p_{n+1}(q^x; a; q),$$

$$(1 - q^n)p_n(q^{x-1}; a; q) = (1 - aq^{n+1})p_{n+1}(q^x; a; q) + aq^{n+1}p_n(q^x; a; q).$$

These relations can of course also be proved directly. To obtain the the first relation, we expand $p_n(q^x; a/q; q)$ in terms of $p_n(q^x; a; q)$,

$$p_n(q^x; a/q; q) = \sum_{k=0}^{n} c_k p_k(q^x; a; q).$$

The coefficient $c_n$ can be found by comparing leading coefficients. Next we multiply both sides with $p_m(q^x; a; q)w(q^x; a; q)$, $0 \leq m \leq n - 1$, and sum over $x$ from 0 to $\infty$. Using the orthogonality relation for the Wall polynomials on both sides, we find the value of $c_n$, and we find $c_k = 0$ for $k \leq n - 2$. The other contiguous relations are proved in the same way. Note that, having the contiguous relations, the actions of the generators on $V_{x,p}$ can be derived from them.

**2.2. Special elements in $\mathcal{A}_q$.** For $\tau, \sigma \in \mathbb{R}$ the element $\rho_{\tau,\sigma} \in \mathcal{A}_q$ is defined by

$$
\rho_{\tau,\sigma} = \frac{1}{2} \left( \alpha^2 + \delta^2 + q\gamma^2 + q^{-1} \beta^2 + i(q^{-\sigma} - q^\sigma)(\delta \gamma + \beta \alpha) \right. \nonumber \\left. - i(q^{-\tau} - q^\tau)(\delta \beta + q \gamma \alpha) + (q^{-\sigma} - q^\sigma)(q^{-\tau} - q^\tau) \right),
$$

and furthermore,

$$
\rho_{\tau,\infty} = \lim_{\sigma \to \infty} 2q^{\tau + \sigma - 1}\rho_{\tau,\sigma} = iq^{-\sigma}(\delta \gamma + q^{-1} \beta \alpha) + q^{-1}(1 - q^{-2\sigma})\gamma \beta,
$$

$$
\rho_{\infty,\sigma} = \lim_{\tau \to \infty} 2q^{\tau + \sigma - 1}\rho_{\tau,\sigma} = -iq^{-\sigma}(q^{-1} \delta \beta + \gamma \alpha) + q^{-1}(1 - q^{-2\sigma})\gamma \beta.
$$

Note that $\rho_{\tau,\infty} = \rho_{\tau,\sigma}$. Observe also that $\rho_{\infty,\infty} = \lim_{\tau \to \infty} \rho_{\tau,\infty} = -\gamma \tau$. The element $\rho_{\tau,\sigma}$, first introduced by Koornwinder, plays an important role in the harmonic analysis on $\mathcal{A}_q$, see e.g. [25],[19]. In this paper we are mainly interested in the spectral analysis of $\mathcal{T}(\rho_{\tau,\sigma})$. The following elements of $\mathcal{A}_q$ will be useful. For $\tau, \sigma \in \mathbb{R} \cup \{\infty\}$ we define, cf. [19] Prop. 6.5],

$$
\alpha_{\tau,\sigma} = q^{\frac{1}{2}\sigma} \alpha - iq^{\sigma - \frac{1}{2}\beta} + iq^{\tau + \frac{1}{2}\gamma} + q^{\sigma + \tau - \frac{1}{2}\delta},
$$

$$
\beta_{\tau,\sigma} = -q^{\sigma + \frac{1}{2}\alpha} - iq^{\sigma + \frac{1}{2}\beta} - iq^{\tau + \sigma + \frac{1}{2}\gamma} + q^{\sigma - \frac{1}{2}\delta},
$$

$$
\gamma_{\tau,\sigma} = -q^{\tau + \frac{1}{2}\alpha} + iq^{\tau + \sigma - \frac{1}{2}\beta} + iq^{\tau + \frac{1}{2}\gamma} + q^{-\frac{1}{2}\delta},
$$

$$
\delta_{\tau,\sigma} = q^{\tau + \sigma + \frac{1}{2}\alpha} + iq^{\tau + \frac{1}{2}\beta} - iq^{\sigma + \frac{1}{2}\gamma} + q^{-\frac{1}{2}\delta},
$$

(2.15)
where $q^\infty = 0$. We collect a few useful relations.

Adjoint:

$$
(2.16) \quad \alpha^*_\tau,\sigma = q\delta_{\tau-1,\sigma-1}, \quad \beta^*_\tau,\sigma = -\gamma_{\tau-1,\sigma+1}, \quad \gamma^*_\tau,\sigma = -\beta_{\tau+1,\sigma-1}, \quad \delta^*_\tau,\sigma = q^{-1}\alpha_{\tau+1,\sigma+1}.
$$

Commutation relations:

$$
(2.17) \begin{align*}
\beta_{\tau+1,\sigma-1}\gamma_{\tau,\sigma} &= 2q^{\tau+\sigma}\rho_{\tau,\sigma} - q^{2\sigma-1} - q^{2\tau+1}, \\
\gamma_{\tau-1,\sigma+1}\beta_{\tau,\sigma} &= 2q^{\tau+\sigma}\rho_{\tau,\sigma} - q^{2\sigma+1} - q^{2\tau-1}, \\
\alpha_{\tau+1,\sigma+1}\delta_{\tau,\sigma} &= 2q^{\tau+\sigma+1}\rho_{\tau,\sigma} + 1 + q^{2\tau+2\sigma+2}, \\
\delta_{\tau-1,\sigma-1}\alpha_{\tau,\sigma} &= 2q^{\tau+\sigma-1}\rho_{\tau,\sigma} + 1 + q^{2\tau+2\sigma-2}, \\
\alpha_{\tau,\sigma}\rho_{\tau,\sigma} &= \rho_{\tau-1,\sigma-1}\alpha_{\tau,\sigma}, \\
\beta_{\tau,\sigma}\rho_{\tau,\sigma} &= \rho_{\tau-1,\sigma+1}\beta_{\tau,\sigma}, \\
\gamma_{\tau,\sigma}\rho_{\tau,\sigma} &= \rho_{\tau+1,\sigma-1}\gamma_{\tau,\sigma}, \\
\delta_{\tau,\sigma}\rho_{\tau,\sigma} &= \rho_{\tau+1,\sigma+1}\delta_{\tau,\sigma},
\end{align*}
$$

Decompositions:

$$
(2.19) \begin{align*}
\alpha_{\tau,\sigma} &= \alpha_{\tau,\infty} + q^\tau\beta_{\tau,\infty}, \\
\beta_{\tau,\sigma} &= \beta_{\tau,\infty} - q^\tau\alpha_{\tau,\infty}, \\
\gamma_{\tau,\sigma} &= \gamma_{\tau,\infty} + q^\tau\delta_{\tau,\infty}, \\
\delta_{\tau,\sigma} &= \delta_{\tau,\infty} - q^\tau\gamma_{\tau,\infty},
\end{align*}
$$

$$
(2.20) \begin{align*}
\alpha_{\tau,\sigma} &= \alpha_{\infty,\sigma} + q^\tau\beta_{\infty,\sigma}, \\
\beta_{\tau,\sigma} &= \beta_{\infty,\sigma} + q^\tau\delta_{\infty,\sigma}, \\
\gamma_{\tau,\sigma} &= \gamma_{\infty,\sigma} - q^\tau\alpha_{\infty,\sigma}, \\
\delta_{\tau,\sigma} &= \delta_{\infty,\sigma} - q^\tau\beta_{\infty,\sigma},
\end{align*}
$$

Now $\rho_{\tau,\sigma}$ can be expressed as

$$
(2.21) \quad 2\rho_{\tau,\sigma} = q^{-\tau-\sigma-1}(\alpha_{\tau+1,\infty} - q^{\sigma+1}\beta_{\tau+1,\infty})(\delta_{\tau,\sigma} - q^\sigma\gamma_{\tau,\infty}) + q^{-\tau-\sigma-1} + q^{\tau+\sigma+1}.
$$

Coproduct: for $\lambda \in \mathbb{R} \cup \{\infty\}$

$$
(2.22) \begin{align*}
\Delta(\alpha_{\tau,\sigma}) &= \frac{1}{1 + q^{2\lambda+1}} \left( q^{-\frac{1}{2}}\alpha_{\tau,\lambda+1} \otimes \alpha_{\lambda+1,\sigma} + q^{\frac{1}{2}}\beta_{\tau,\lambda} \otimes \gamma_{\lambda,\sigma} \right), \\
\Delta(\beta_{\tau,\sigma}) &= \frac{1}{1 + q^{2\lambda+1}} \left( q^{-\frac{1}{2}}\alpha_{\tau,\lambda+1} \otimes \beta_{\lambda+1,\sigma} + q^{\frac{1}{2}}\beta_{\tau,\lambda} \otimes \delta_{\lambda,\sigma} \right), \\
\Delta(\gamma_{\tau,\sigma}) &= \frac{1}{1 + q^{2\lambda+1}} \left( q^{-\frac{1}{2}}\gamma_{\tau,\lambda+1} \otimes \alpha_{\lambda+1,\sigma} + q^{\frac{1}{2}}\delta_{\tau,\lambda} \otimes \gamma_{\lambda,\sigma} \right), \\
\Delta(\delta_{\tau,\sigma}) &= \frac{1}{1 + q^{2\lambda+1}} \left( -q^{-\frac{1}{2}}\gamma_{\tau,\lambda+1} \otimes \beta_{\lambda+1,\sigma} + q^{\frac{1}{2}}\delta_{\tau,\lambda} \otimes \delta_{\lambda,\sigma} \right).
\end{align*}
$$

This gives us

$$
(2.23) \quad 2\Delta(\rho_{\tau,\sigma} + q^{-\tau-\sigma-1} + q^{\tau+\sigma+1}) = \frac{q^{-\tau-\sigma-1}}{(1 + q^{2\lambda+1})^2} \left( q^{-\frac{1}{2}}\alpha_{\tau+1,\lambda+1} \otimes \alpha_{\lambda+1,\sigma+1} + q^{\frac{1}{2}}\beta_{\tau+1,\lambda} \otimes \delta_{\lambda+1,\sigma+1} + q^{-\frac{1}{2}}\gamma_{\tau+1,\lambda} \otimes \gamma_{\lambda+1,\sigma} + q^{\frac{1}{2}}\delta_{\tau+1,\lambda} \otimes \delta_{\lambda+1,\sigma} \\
+ [2q^{\tau+\lambda+1}\rho_{\tau,\lambda} + q^{2\lambda+2\tau+2} + 1] \otimes [2q^{\sigma+\lambda+1}\rho_{\lambda,\sigma} + q^{2\lambda+2\sigma+2} + 1] \\
+ [2q^{\alpha+\lambda+1}\rho_{\tau,\lambda+1} - q^{2\tau+1} - q^{2\tau+1}] \otimes [2q^{\lambda+\sigma+1}\rho_{\lambda+1,\sigma} - q^{2\lambda+1} - q^{2\sigma+1}] \right).
$$

Let us denote by $A_q^{\alpha,\sigma}$ the subalgebra of $A_q$ generated by

$\alpha_{\tau+n,\sigma+m}, \beta_{\tau+n,\sigma+m}, \gamma_{\tau+n,\sigma+m}, \delta_{\tau+n,\sigma+m}, \quad n, m \in \mathbb{Z},$

and subalgebras $A_q^{\alpha, \infty}, A_q^{\infty, \sigma}$ are defined similarly. Observe that $A_q \supset A_q^{\alpha, \infty} \supset A_q^{\infty, \sigma}$ and $A_q \supset A_q^{\infty, \sigma} \supset A_q^{\alpha, \sigma}$. This structure allows us to ‘build up’ representations of $A_q^{\alpha, \sigma}$ from (simpler) representations of $A_q, A_q^{\alpha, \infty}$ and $A_q^{\infty, \sigma}$. It has been shown by Stokman in [27] that the algebra $A_q^{\alpha, \sigma}$ is closely related to the SU(2) dynamical quantum group [10, 21].

3. $(\tau, \infty)$-Clebsch-Gordan coefficients

In this section we compute Clebsch-Gordan coefficients corresponding to eigenvectors of $T(\rho_{\tau,\infty})$. 
3.1. Eigenvectors of $\rho_{\tau,\infty}$. Eigenvectors of $\pi_\phi(\rho_{\tau,\infty})$ can be found in [23] [5]. We use a different normalization here. To define the eigenvectors we need the Al-Salam–Carlitz polynomials, see [11,15], defined by

\begin{equation}
U_n^{(a)}(x; q) = (-a)^n q^{\frac{1}{2} n(n-1)} \varphi_1 \left( q^{-n}, x^{-1}; q, \frac{qx}{a} \right).
\end{equation}

They satisfy the three-term recurrence relation

\begin{equation}
x U_n^{(a)}(x; q) = U_{n+1}^{(a)}(x; q) + (a + 1) q^n U_n^{(a)}(x; q) - a q^{n-1}(1 - q^n) U_{n-1}^{(a)}(x; q),
\end{equation}

and the orthogonality relation

\begin{equation}
\int_a^1 (qx, qx/a; q) \infty U_n^{(a)}(x; q) U_m^{(a)}(x; q) \, dq \, x = \delta_{mn} (-a)^n (1 - q^n)(q; q)_n(q; q)_\infty \theta(a; q) q^{\frac{1}{2} n(n-1)}
\end{equation}

for $a < 0$. Since the polynomials $U_n^{(a)}$ are orthogonal on a countable set and complete, they also satisfy the dual orthogonality relations

\begin{equation}
\sum_{n \in \mathbb{N}} U_n^{(a)}(x; q) U_n^{(a)}(y; q) q^{-\frac{1}{2} n(n-1)} (-a)^n (q; q)_n \delta_{xy} |x|(qx, qx/a; q) \infty = \delta_{xy} |x|(qx, qx/a; q) \infty \theta(a; q),
\end{equation}

provided $a < 0$ and $x, y \in a q^{\mathbb{N}} \cup q^{\mathbb{N}}$.

From the definition (2.14) of $\rho_{\tau,\infty}$ and the actions (2.14) of the $A_q$-generators we obtain

\begin{equation}
\pi_\phi(\rho_{\tau,\infty}) e_n = i q^n e \sqrt{1 - q^{2n+2}} e_{n+1} - q^{2n}(1 - q^{2\tau}) e_n - i q^n q^{n-1} \sqrt{1 - q^{2\tau}} e_{n-1}.
\end{equation}

We can diagonalize this using the Al-Salam–Carlitz polynomials. We define the function $m_{x,n}^{\tau,\infty}$ by

\begin{equation}
m_{x,n}^{\tau,\infty} = i^n \tilde{U}_n^{(-q^{2\tau})}(-x; q^2),
\end{equation}

where

\begin{equation}
\tilde{U}_n^{(a)}(x; q) = \left( q^{-\frac{1}{2} n(n-1)} |x|(qx, qx/a; q) \infty \right) (-a)^n (q; q)_n \theta(a; q).
\end{equation}

The three-term recurrence relation for the Al-Salam–Carlitz polynomials translates to

\begin{equation}
-x m_{x,n}^{\tau,\infty} = -iq^n \sqrt{1 - q^{2n+2}} m_{x,n+1}^{\tau,\infty} + (1 - q^{2\tau}) q^{2n} m_{x,n}^{\tau,\infty}(x) + iq^{n-1} \sqrt{1 - q^{2\tau}} m_{x,n-1}^{\tau,\infty}.
\end{equation}

Comparing this with (3.2) we find the following result.

**Proposition 3.1.** For $x \in -q^{2\mathbb{N}} \cup q^{2\mathbb{N}+2\mathbb{N}}$ the vector

\begin{equation}
v_x^{\tau,\infty} = \sum_{n \in \mathbb{N}} e^{i n \phi} m_{x,n}^{\tau,\infty} e_n
\end{equation}

is an eigenvector of $\pi_\phi(\rho_{\tau,\infty})$ for eigenvalue $x$. Moreover, $\{v_x^{\tau,\infty} \mid x \in -q^{2\mathbb{N}} \cup q^{2\mathbb{N}+2\mathbb{N}}\}$ is an orthonormal basis for $L^2(\mathbb{N})$.

**Proof.** It follows from the action of $\rho_{\tau,\infty}$ on the standard orthonormal basis and from (3.3) that $v_x^{\tau,\infty}$ is an eigenvector. Using the dual orthogonality relations for the Al-Salam–Carlitz polynomials we obtain orthogonality relations for the eigenvectors.

We can determine the explicit actions of $\alpha_{\tau,\infty}, \beta_{\tau,\infty}, \gamma_{\tau,\infty}, \delta_{\tau,\infty}$ on the eigenvectors $v_x^{\tau,\infty}$.

**Proposition 3.2.** For $x \in -q^{2\mathbb{N}} \cup q^{2\mathbb{N}+2\mathbb{N}}$,

\begin{align*}
\pi_\phi(\alpha_{\tau,\infty}) v_x^{\tau,\infty} &= i e^{i \phi} q^{\frac{1}{2}} \sqrt{1 + x} v_x^{\tau-1,\infty}, \\
\pi_\phi(\beta_{\tau,\infty}) v_x^{\tau,\infty} &= i e^{-i \phi} q^{\frac{1}{2}} \sqrt{1 - qx^2 - xq^{2\tau}} v_x^{\tau-1,\infty}, \\
\pi_\phi(\gamma_{\tau,\infty}) v_x^{\tau,\infty} &= i e^{i \phi} q^{\frac{1}{2}} \sqrt{1 - qx^2 - xq^{2\tau}} v_x^{\tau+1,\infty}, \\
\pi_\phi(\delta_{\tau,\infty}) v_x^{\tau,\infty} &= -i e^{-i \phi} q^{\frac{1}{2}} \sqrt{1 + xq^2} v_x^{\tau+1,\infty},
\end{align*}

where $v_{x-q^2}^{\tau,\infty} = v_{q^{2\tau}x}^{\tau,\infty} = 0$. 

Remark 3.3. Because of the shift in the parameter \( \tau \), the actions of \( \alpha, \beta, \gamma, \delta \) on the standard basis of \( \ell^2 (\mathbb{N}) \) have \('dynamic'\) raising and lowering operators.

For later references we also need an orthonormal basis on which \( \pi_\phi (\rho_{\infty, \sigma}) \) acts diagonally. This basis is

\[
v_x^{\infty, \sigma} = \sum_{n \in \mathbb{N}} e^{-i n \phi} m_{x, n}^{\infty, \sigma} e_n,
\]

where

\[
m_{x, n}^{\infty, \sigma} = i^n q^{-n \sigma} q^{-\frac{1}{2} (n-1)} \sqrt{\frac{|x| (-q^2 x, q^{2 - 2\sigma}; q^2)_\infty}{(q^2, q^2)_n (q^2, -q^{2\sigma}, -q^{2 - 2\sigma}; q^2)_\infty}} L_n (-q^2x; q^2).
\]

The vector \( v_x^{\infty, \sigma} \) is an eigenvector of \( \pi_\phi (\rho_{\infty, \sigma}) \) for eigenvalue \( x \), and

\[
\pi_\phi (\alpha_{\infty, \sigma}) v_x^{\infty, \sigma} = ie^{-i \phi} q^{\frac{1}{2}} \sqrt{1 + x} v_x^{\infty, \sigma - 1},
\]

\[
\pi_\phi (\beta_{\infty, \sigma}) v_x^{\infty, \sigma} = ie^{-i \phi} q^{\frac{1}{2}} \sqrt{1 - x q^{2 \sigma}} v_x^{\infty, \sigma + 1},
\]

\[
\pi_\phi (\gamma_{\infty, \sigma}) v_x^{\infty, \sigma} = ie^{-i \phi} q^{-\frac{1}{2}} \sqrt{1 - x q^{2 - 2\sigma}} v_x^{\infty, \sigma - 1},
\]

\[
\pi_\phi (\delta_{\infty, \sigma}) v_x^{\infty, \sigma} = -ie^{-i \phi} q^{-\frac{1}{2}} \sqrt{1 + x q^{2}} v_x^{\infty, \sigma + 1}.
\]

3.2. Clebsch-Gordan coefficients. Next we consider the action of \( \rho_{\infty, \infty} \) in the tensor product representation \( T \), see (2.5). We will need the big \( q \)-Laguerre polynomials, a subclass of the big \( q \)-Jacobi polynomials [3], [18]. The big \( q \)-Laguerre polynomials are defined by

\[
L_n (x; a, b; q) = 3 \varphi_2 \left( \frac{q^{-n}, 0, x}{aq, bq}, q \right), \quad n \in \mathbb{N}.
\]

For \( 0 < a < q^{-1} \) and \( b < 0 \) they satisfy the orthogonality relations

\[
\int_{bq}^{aq} \frac{(x/a, x/b; q)_\infty}{(x; q)_\infty} L_m (x; a, b; q) L_n (x; a, b; q) \, dx = \delta_{m n} a q (1 - q) \frac{(q; q)_\infty \theta (b/a; q)}{(aq, bq; q)_\infty} \frac{\theta (q; q)_\infty}{(aq, bq; q)_n} (-abq^2)^n q^{\frac{1}{2} n (n-1)}. \]

Note that \( a \) and \( b \) can be interchanged, so the orthogonality relations are also valid for \( a < 0 \) and \( 0 < b < q^{-1} \). From (2.11) it follows that, for \( p \in \mathbb{N} \), the polynomials \( L_n \) satisfy the identity

\[
(q^{1-p}; q)_\infty L_n (x; q^{-p}, b; q) = \begin{cases} \frac{(q^{1-p}; q)_\infty (q^{-n}, x; q)_p q^p L_n (xq^p; q^p, bq^p; q)}{(aq, bq; q)_p}, & 0 \leq p \leq n, \\ 0, & p > n, \end{cases}
\]

which will be useful later on. We define orthonormal functions related to the big \( q \)-Laguerre polynomials by

\[
\tilde{L}_n (x; a, b; q) = (-abq^2)^{-n/2} q^{-\frac{1}{2} n (n-1)} \sqrt{|x| (x/a, x/b, aq, bq; q)_\infty (aq, bq; q)_n} L_n (x; a, b; q).
\]
These functions satisfy the orthogonality relations and dual orthogonality relations
\[ \sum_{x \in \ell^2(N) \otimes \ell^2(N)} L_m(x; a, b; q) L_n(x; a, b; q) = \delta_{mn}, \]
\[ \sum_{n \in \mathbb{N}} L_m(x; a, b; q) \tilde{L}_n(y; a, b; q) = \delta_{xy}, \]
provided \( x, y \in \ell^2(N) \otimes \ell^2(N) \) in the last relation. The three-term recurrence relation for \( L_n \) gives the relations
\[ \frac{x}{bq} \tilde{L}_n(x) = [q^n(a/b + a + 1) - apq^{n+1}(1 + q)] \tilde{L}_n(x) \]
\[ - q^{n/2} \sqrt{-a/b(1 - q^{n+1})(1 - bq^{n+1})} \tilde{L}_{n+1}(x) \]
\[ - q^{n(1-1/2)} \sqrt{-a/b(1 - q^n)(1 - apq^n)} \tilde{L}_{n-1}(x). \]

We now consider the operator \( T(\rho_{\infty}) \). Taking the appropriate limits in (3.23) we have
\[ \Delta(\rho_{\infty}) = \rho_{\infty} \otimes 1 + q^{-1}(1 + q^2)\rho_{\infty} + (1 - q^2) \otimes \gamma \beta + i \beta_{\rho_{\infty}} \otimes 1 + i \gamma \beta_{\rho_{\infty}} \otimes \rho_{\infty} \]
Using Proposition 3.2 and (2.4) we see that \( T(\rho_{\infty}) \) acts ‘nicely’ on the \( \ell^2(N) \otimes \ell^2(N) \)-basis \( \{ v_\lambda^{\tau, \infty} \otimes e_n \mid y \in -q^{2n} \cup q^{2n+2}N, n \in \mathbb{N} \} \);
\[ T(\rho_{\infty}) v_\lambda^{\tau, \infty} \otimes e_n = [y(1 - q^2 - q^{n+2}) - q^n(1 - q^{2\tau})] v_\lambda^{\tau, \infty} \otimes e_n \]
\[ + ire^{-i(\psi - 2\beta)} q^{n+\tau} \sqrt{(1 - q^{n+2})(1 + q^2)}(1 - yq^{-2\tau}) v^{\tau, \infty}_{\lambda q^2} \otimes e_{n+1} \]
\[ - ire^{-i(\psi - 2\beta)} q^{n+\tau-1} \sqrt{(1 - q^n)(1 + y)(1 - q^{-2\tau})} v^{\tau, \infty}_{\lambda q^2} \otimes e_{n-1}. \]
We can identity a restriction of \( T(\rho_{\infty}) \) with the Jacobi operator corresponding to the big \( q \)-Laguerre polynomials. We define, for \( y \in -q^{2\tau} \cup q^{2\tau+2\tau} \), the Clebsch-Gordan coefficients
\[ e_x^{\tau, \infty} = (\text{sgn}(y))^n L_n(-xq^{2\tau}; y, y^{2\tau+q^2}). \]

**Proposition 3.4.** For \( x \in -q^{2n} \cup q^{2n+2}N \) and \( y \in -q^{2\tau} \cup q^{2\tau+2\tau} \) the vector
\[ V_{x,y}^{\tau, \infty} = \sum_{n \in \mathbb{N}} e^{iN(\psi - 2\beta)} c_{x,y,n}^{\tau, \infty} v^{\tau, \infty}_{\lambda q^2} \otimes e_n, \]
is an eigenvector of \( T(\rho_{\infty}) \) for eigenvalue \( x \), using the conventions \( v^{\tau, \infty}_{\lambda q^2} = e_n = 0 \) for \( n \in \mathbb{N}_1 \) and \( \lambda \in \{-1, q^{2\tau}\} \). Furthermore,
\[ \{ V_{x,y}^{\tau, \infty} \mid x \in -q^{2n} \cup q^{2n+2}N, y \in -q^{2\tau} \cup q^{2\tau+2\tau} \} \]
is an orthonormal basis for \( \ell^2(N) \otimes \ell^2(N) \), i.e.
\[ \left\langle V_{x,y}^{\tau, \infty}, V_{x,y'}^{\tau, \infty} \right\rangle_{\ell^2(N) \otimes \ell^2(N)} = \delta_{x,y}\delta_{y,y'}, \]

**Proof.** The proof is very similar to the proof of Proposition 3.1 We consider the action of \( T(\rho_{\infty}) \) on basis vectors \( v_{-n}^{\tau, \infty} \otimes e_n \). Write \( y = \lambda q^{2m} \) with \( \lambda \in \{-1, q^{2\tau}\} \), \( m \in \mathbb{N} \), and let \( p = m - n \). For \( p \geq 0 \) the result follows from comparing (3.8) with the three-term recurrence relation (3.7). For \( p < 0 \) the action (3.8) of \( T(\rho_{\infty}) \) actually corresponds to the three-term recurrence relation of \( \tilde{L}_m(x; q^{-2\tau}, -\lambda^2 q^{-2\tau}; q^2) \), up to a phase factor. In this case we obtain
\[ \sum_{m \in \mathbb{N}} (\text{sgn}(y))^m e^{iN(m)(\psi - 2\beta)} \tilde{L}_m(-\lambda xq^{-2\tau}; q^{-2p}, -\lambda^2 q^{-2\tau}; q^2) v_{\lambda q^2}^{\tau, \infty} \otimes e_{m} \]
as an eigenvector for eigenvalue \( x \). Using the conventions \( v_{\lambda q^2}^{\tau, \infty} = e_n = 0 \) for \( n \in \mathbb{N}_1 \) and identity (3.6), it follows that this eigenvector coincides with \( V_{x,y}^{\tau, \infty} \). \( \square \)
Remark 3.5. In the limit \( \tau \to \infty \) ‘half’ of the spectrum of \( \rho_{\tau,\infty} \), namely the part \( q^{2\tau+2N} \), vanishes. For \( x \in -q^{2N} \) and \( y \in -q^{2Z} \) (both corresponding to the remaining part of the spectrum), we have
\[
\lim_{\tau \to \infty} L_n(-xyq^{2-2\tau}; -y, yq^{-2\tau}; q^2) = \rho_n(-x; -y; q^2),
\]
which follows directly from the \( q \)-hypergeometric expressions of both functions. Furthermore, in this limit the orthogonality relations for the big \( q^2 \)-Laguerre polynomials \( L_n(-xyq^{2-2\tau}; -y, yq^{-2\tau}; q^2) \) go over into the orthogonality relations for the Wall polynomials \( p_n(-x; -y; q^2) \) (at least formally). In this way we see that the limit of a Clebsch-Gordan coefficient for \( \rho_{\tau,\infty} \) is a Clebsch-Gordan coefficient for \( \rho_{\infty,\infty} = -\gamma^* \).

Similarly as for \( v^\tau_{x,y} \) we can determine the actions of \( \alpha_{\tau,\infty}, \beta_{\tau,\infty}, \gamma_{\tau,\infty}, \delta_{\tau,\infty} \) on the vectors \( V^\tau_{x,y} \).

Proposition 3.6. For \( x \in -q^{2N} \cup q^{2\tau+2N} \) and \( y \in -q^{2Z} \cup q^{2\tau+2Z} \),
\[
\mathcal{T}(\alpha_{\tau,\infty}) V^\tau_{x,y} = i e^{i(\psi-\phi)} q^{\tau \frac{1}{2}} \sqrt{1+x} V^{\tau-1,\infty}_{x/q^2,y},
\]
\[
\mathcal{T}(\beta_{\tau,\infty}) V^\tau_{x,y} = i e^{i(\phi-\psi)} q^{\tau \frac{1}{2}} \sqrt{1-xyq^{2-2\tau}} V^{\tau-1,\infty}_{x/q^2,y},
\]
\[
\mathcal{T}(\gamma_{\tau,\infty}) V^\tau_{x,y} = i e^{i(\phi-\psi)} q^{\tau \frac{1}{2}} \sqrt{1-xyq^{2-2\tau}} V^{\tau+1,\infty}_{x/q^2,y},
\]
\[
\mathcal{T}(\delta_{\tau,\infty}) V^\tau_{x,y} = -i e^{i(\phi-\psi)} q^{\tau \frac{1}{2}} \sqrt{1+xyq^{2}} V^{\tau+1,\infty}_{x/q^2,y}.
\]
Proof. Let us calculate the action of \( \beta_{\tau,\infty} \). First observe that
\[
\Delta(\beta_{\tau,\infty}) = \beta_{\tau,\infty} \otimes \delta - i q^{-1} \alpha_{\tau,\infty} \otimes \beta
\]
by (2.22) and (2.15). Using Proposition 3.2 and (2.4) this gives
\[
\mathcal{T}(\beta_{\tau,\infty}) v^\tau_{x,y} \otimes e_n = c_1 v^\tau_{x,y} \otimes e_n + c_2 v^\tau_{x,y} \otimes e_n,
\]
for certain coefficients \( c_j, j = 1, 2 \). Since \( \beta_{\tau,\infty} \rho_{\tau,\infty} = \rho_{\tau-1,\infty} \beta_{\tau,\infty} \) by (2.13), we conclude that
\[
\mathcal{T}(\beta_{\tau,\infty}) V^\tau_{x,y} = C V^{\tau-1,\infty}_{x,y/q^2},
\]
where the value of \( C \) still needs to be determined. To determine this value we use
\[
\Delta(\beta^*_{\tau,\infty}) = -\Delta(\gamma_{\tau-1,\infty}) = -\gamma_{\tau-1,\infty} \otimes \alpha - i q \delta_{\tau-1,\infty} \otimes \gamma,
\]
see (2.10) and (2.17), then
\[
C(V^{\tau-1,\infty}_{x,y/q^2} \otimes e_n) = -\langle V^{\tau-1,\infty}_{x,y/q^2} \otimes e_n | \mathcal{T}(\gamma_{\tau-1,\infty}) v^\tau_{x,y/q^2} \otimes e_n \rangle = e^{i(\phi-\psi)} q^{\tau \frac{1}{2}} \sqrt{1+y} \langle V^{\tau-1,\infty}_{x,y/q^2} \otimes e_n | v^\tau_{x,y/q^2} \otimes e_n \rangle.
\]
The value of \( C \) now follows from the explicit expression for \( v^\tau_{x,y,0} \). From \( \beta_{\tau+1,\infty} \gamma_{\tau,\infty} = q \rho_{\tau,\infty} - q^{2\tau+1} \) we obtain the action of \( \gamma_{\tau,\infty} \). The actions of \( \alpha_{\tau,\infty} \) and \( \delta_{\tau,\infty} \) are obtained in a similar way. \( \square \)

The action of \( \alpha_{\tau,\infty} \) implies the following contiguous relation for big \( q \)-Laguerre polynomials:
\[
(1 - \frac{x}{bq}) L_n(x; a, bq; q) = (1 - \frac{1}{bq})(1 - aq^{n+1}) L_{n+1}(x; a, b; q) - \frac{a}{b} (1 - bq)^n L_n(x; a, b; q).
\]
The action of \( \gamma_{\tau,\infty} \) implies the same contiguous relation with \( a \) and \( b \) interchanged.

We can also find eigenvectors of \( \mathcal{T}(\rho_{\infty,n}) \). Define for \( x \in -q^{2N} \cup q^{2\sigma+2N}, y \in -q^{2Z} \cup q^{2\sigma+2Z}, \)
\( n \in \mathbb{N}, \) the Clebsch-Gordan coefficients by
\[
e^\infty_{x,n,y} = (\text{sgn}(y))^{n} L_n(-xyq^{2-2\sigma}; -y, yq^{-2\sigma}; q^2),
\]
then for the vector
\[
V^{\infty}_{x,y} = \sum_{n \in \mathbb{N}} e^{i n(\psi-2\phi)} e^\infty_{x,n,y} e_n \otimes v^\infty_{y/q^2 e_n},
\]
is an eigenvector of \( T(\rho_{\tau,\sigma}) \) for eigenvalue \( \lambda \). Furthermore,

\[
\begin{align*}
T(0_{\infty,\rho}) V_{x,y}^{\infty,\sigma} &= i e^{i(\phi-\psi)} q^{\frac{x}{2}} \sqrt{1+x} V_{x/y}^{\infty,\sigma+1}, \\
T(\beta_{\infty,\rho}) V_{x,y}^{\infty,\sigma} &= i q^{\frac{x}{2}} \sqrt{1-x} V_{x/y}^{\infty,\sigma+1}, \\
T(\gamma_{\infty,\rho}) V_{x,y}^{\infty,\sigma} &= i q^{\frac{x}{2}} \sqrt{1-x} V_{x/y}^{\infty,\sigma+1}, \\
T(\delta_{\infty,\rho}) V_{x,y}^{\infty,\sigma} &= -i e^{i(\psi-\phi)} q^{\frac{x}{2}} \sqrt{1+x} V_{x/y}^{\infty,\sigma+1}.
\end{align*}
\]  
(3.9)

4. \((\tau,\sigma)\)-Clebsch-Gordan coefficients.

We determine the Clebsch-Gordan coefficients corresponding to (generalized) eigenvectors of \( T(\rho_{\tau,\sigma}) \).

4.1. Eigenvectors of \( \rho_{\tau,\sigma} \). We first give eigenvectors of \( \pi_\phi(\rho_{\tau,\sigma}) \). These eigenvectors, with a different normalization, can be found in [23] [6]. To define the eigenvectors we need the Al-Salam–Chihara polynomials [2]. These polynomials are Askey-Wilson polynomials, see [4], [18], with two parameters equal to zero. Later on we also need the continuous dual Askey-Wilson polynomials, see [23, §18], with two parameters equal to zero.

The continuous dual \( \phi \)-Hahn polynomials are polynomials in \( x \) with different normalization, can be found in [23, §18]. From (2.11) we find the useful identity, for \( p \neq n \),

\[
\begin{align*}
&\int_{-1}^{1} p_n(x) p_m(x) w(x; a, b, c|q) \, dx + \sum_{k \in \mathbb{N}} p_n(x_\alpha^k) p_n(x_\alpha^{k+1}) w(x_\alpha^{k+1}; a, b, c|q) = \delta_{nm} (q_{n+1}; abq^n, acq^n, bcq^n; q)_\infty,
\end{align*}
\]  
(4.3)

where \( \alpha \) is any of the parameters \( a, b, c \). Assuming \( \alpha = a \) for the discrete part, the weight functions are given by

\[
\begin{align*}
w(x; a, b, c|q) &= \frac{1}{2\pi \sqrt{1-x}} (x^{\pm 2}; q)_\infty \quad x \in \mathbb{T}, \\
w(x_\alpha^k; a, b, c|q) &= \frac{(a^{-2}; q)_\infty}{(q, ab, ac, b/a, c/a; q)_\infty} (1 - a^2 q^{2k})(a^2, ab, ac; q)_k^{-k} (1 - a^2 q^{-1})(a^2, ab, ac; q)_k^{k-1} (a^2bc)^{-k}.
\end{align*}
\]

We denote by \( I = I_{a,b,c|q} \) the support of the orthogonality measure, so \( I \) consists of the interval \([-1,1]\) and a finite (possibly empty) discrete part. We define orthonormal functions by

\[
p_n(x; a, b, c|q) = \sqrt{w(x; a, b, c|q)} (q^n; abq^n, acq^n, bcq^n; q)_\infty p_n(x; a, b, c|q),
\]
then \( \{ \tilde{p}_n \}_{n \in \mathbb{N}} \) is an orthonormal basis for \( L^2(I) \), and the three-term recurrence relation becomes
\[
2\mu_x \tilde{p}_n(\mu_x) = \sqrt{(1 - q^{n+1})(1 - abq^n)(1 - acq^n)(1 - bcq^n)} \tilde{p}_{n+1}(\mu_x) \\
+ \left[ abc q^{2n-1}(1 + q) - q^n(a + b + c) \right] \tilde{p}_n(\mu_x) \\
+ \sqrt{(1 - q^n)(1 - abq^{n-1})(1 - acq^{n-1})(1 - bcq^{n-1})} \tilde{p}_{n-1}(\mu_x).
\]
(4.5)

The Al-Salam–Chihara polynomials \( q_n \) are obtained from the continuous dual \( q \)-Hahn polynomials by setting \( c = 0 \), i.e. \( q_n(\mu_x; a, b \mid q) = \tau_n(\mu_x; a, b, 0 \mid q) \). Explicit expressions for \( q_n \) are
\[
q_n(\mu_x; a, b \mid q) = a^{-n}(ab \mid q)_n \frac{\varphi_2 \left( q^{-n}, ax, a/x; ab \mid q \right)}{ab, 0} \\
= x^n(b/x \mid q)_n \frac{\varphi_1 \left( q^{-n}, ax, axq^{-n}/b \mid q \right)}{q, q}.
\]
(4.6)

The three-term recurrence relation and orthogonality relations are obtained by letting \( c \to 0 \) in (4.2) and (4.4). We define
\[
\tilde{q}_n(\mu_x; a, b \mid q) = \lim_{c \to 0} \tau_n(\mu_x; a, b, c \mid q),
\]
then \( \{ \tilde{q}_n \}_{n \in \mathbb{N}} \) is an orthonormal basis for \( L^2(Ia, b, 0 \mid q) \).

To determine eigenvectors of \( \rho_{\tau, \sigma} \) we consider the action on the vectors \( v^\tau_{y} \infty \) from Proposition 3.1. First we observe from (2.24) that
\[
2\rho_{\tau, \sigma} = (q^{1-\sigma - \tau} - q^{1+\sigma - \tau})\rho_{\tau, \infty} + q^{-\tau} \beta_{\tau+1, \infty} \delta_{\tau, \infty} - q^{-1-\tau} \alpha_{\tau+1, \infty} \gamma_{\tau, \infty}.
\]
Using Propositions 3.1 and 3.2 we obtain, for \( y \in -q^{2n} \cup q^{2n+2N} \),
\[
2\pi_\phi(\rho_{\tau, \sigma}) v^\tau_{y} \infty = e^{-2i\phi} \sqrt{(1 + yq^2)(1 - yq^{-2})} v^\tau_{y} \infty \\
+ (q^{1-\sigma - \tau} - q^{1+\sigma - \tau}) y v^\tau_{y} \infty + e^{2i\phi} \sqrt{(1 + y)(1 - yq^{-2})} v^\tau_{-q^{-2n}}.
\]
(4.7)

This operator can be matched to the Jacobi operator for the Al-Salam–Chihara polynomials. For \( y \in -q^{2n} \cup q^{2n+2N} \) we define
\[
m^\tau_{x, y} = \tilde{q}_n(\mu_x; -\lambda q^{-1-\sigma} + 1, \lambda q^{1-\sigma} \mid q),
\]
where \( y = \lambda q^{2n} \) with \( \lambda \in \{ -1, q^{2\tau} \} \). We also write \( I_{-\lambda q^{1-\sigma} - 1, \lambda q^{1-\sigma} - 0 \mid q} = I_{\lambda}^{\tau} \) for the corresponding support.

**Proposition 4.1.** The operator \( \Theta^{\tau, \sigma} : L^2(N) \to L^2(I_{\tau}^{\tau} \sigma) \otimes L^2(I_{\tau}^{\tau} \sigma) \) defined by
\[
\Theta^{\tau, \sigma} v^\tau_{y} \infty(\mu_x) = e^{-2i\phi} m^\tau_{x, y}, \quad y = \lambda q^{2n},
\]
is unitary and intertwines \( \pi_{\phi}(\rho_{\tau, \sigma}) \) with the multiplication operator \( M \) on \( L^2(I_{\tau}^{\tau} \sigma) \otimes L^2(I_{\tau}^{\tau} \sigma) \).

Here the multiplication operator \( M \) is defined by \( M f(\mu_x) = \mu_x f(\mu_x) \) almost everywhere.

**Proof.** The operator \( \Theta^{\tau, \sigma} \) is unitary, since it maps one orthonormal basis to another. We set \( y = \lambda q^{2n} \) in (1.7). Comparing (1.7) with the three-term recurrence relation (4.5) with \( a = -\lambda q^{1-\sigma - \tau}, b = \lambda q^{1-\sigma - \tau} \) and \( c = 0 \), we see that \( \Theta^{\tau, \sigma} \) intertwines \( \pi_{\phi}(\rho_{\tau, \sigma}) \) with \( M \). \( \square \)

Note that \( \pi_{\phi}(\rho_{\tau, \sigma}) \) has continuous spectrum \([-1, 1]\) with multiplicity two and (possibly empty) finite discrete spectrum
\[
\{ \mu_{q^{-1} - \sigma + 2k} \mid k \in \mathbb{N}, q^{-1} - \sigma + 2k > 1 \} \cup \{ \mu_{q^{-1} - \sigma + 2k} \mid k \in \mathbb{N}, q^{-1} - \sigma + 2k > 1 \}.
\]
Since the spectrum is (partly) continuous, we do not have eigenvectors in general. We can, however, formulate Proposition 4.1 in terms of generalized eigenvectors. Let \( v^\tau_{x} \sigma \lambda \) denote the formal sum
\[
v^\tau_{x} \sigma \lambda = \sum_{n \in \mathbb{N}} e^{-2i\phi} m^\tau_{x, \lambda q^{2n}} q^\tau_{x} \sigma \lambda q^{2n}, \quad \lambda = -1, q^{2\tau}.
\]
For \( \mu_x \in [-1, 1] \), i.e., \( x \in \mathbb{T} \), this can be considered as a generalized eigenvector of \( \pi_{\phi}(\rho_{\tau, \infty}) \) for eigenvalue \( \mu_x \). For \( \mu_x \) in the discrete spectrum this is a genuine eigenvector.
Remark 4.2. In this paper we will use generalized eigenvectors as if they were genuine eigenvectors, in particular we consider inner products of generalized eigenvectors with other vectors $v$. In general the inner products do not exist, but for specific vectors $v$ the inner products do make sense. For example,
\[
(v_{x}^{\tau,\sigma}, v_{\lambda q_{2n}}^{\tau,\infty}) = e^{-2in\theta_{\lambda}q_{2n}}(v_{x}^{\tau,\sigma}, v_{\lambda q_{2n}}^{\tau,\infty}) = \Theta^{\tau,\sigma} v_{\lambda q_{2n}}^{\tau,\infty}(\mu_{x}).
\]

Similar as in Proposition 3.2 we can determine actions of $\alpha_{\gamma_{\tau,\sigma}}$, $\beta_{\gamma_{\tau,\sigma}}$, $\gamma_{\tau,\sigma}$, $\delta_{\tau,\sigma}$ on the (generalized) eigenvectors. We do not need these actions later on, so we omit the details. The result is
\[
\pi_{\phi}(\alpha_{\tau,\sigma}) v_{x}^{\tau,\sigma} = ie^{-i\phi q^{+}\sigma + \frac{1}{2}}\sqrt{(1 + q^{1-\sigma} - q^{-1}\tau)}(1 + q^{1-\sigma} - q^{-1}\tau) v_{x}^{\tau-1,\sigma-1},
\]
\[
\pi_{\phi}(\beta_{\tau,\sigma}) v_{x}^{\tau,\sigma} = ie^{-i\phi q^{-}\sigma - \frac{1}{2}}\sqrt{(1 - q^{1-\tau} - q^{-1}\tau)}(1 - q^{1-\tau} - q^{-1}\tau) v_{x}^{\tau-1,\sigma+1},
\]
\[
\pi_{\phi}(\gamma_{\tau,\sigma}) v_{x}^{\tau,\sigma} = ie^{i\phi q^{0}}\sqrt{(1 - q^{1-\tau} - q^{-1}\tau)}(1 - q^{1-\tau} - q^{-1}\tau) v_{x}^{\tau+1,\sigma-1},
\]
\[
\pi_{\phi}(\delta_{\tau,\sigma}) v_{x}^{\tau,\sigma} = -ie^{i\phi q^{0} + \frac{1}{2}}\sqrt{(1 - q^{1-\tau} - q^{-1}\tau)}(1 - q^{1-\tau} - q^{-1}\tau) v_{x}^{\tau+1,\sigma+1}.
\]
Alternatively, we can reformulate these actions as the function $m_{x,y}^{\tau,\sigma}$ using the operator $\Theta^{\tau,\sigma}$ from Proposition 3.1 Define $\Theta = \bigoplus_{m,n \in \mathbb{Z}} \Theta^{m,\sigma,\sigma+n}$, then
\[
\pi_{\phi}(X)m_{x,y}^{\tau,\sigma} = \Theta \pi_{\phi}(X)\Theta^{x,y} \in A_{q},
\]
gives an action of $\mathcal{A}_{q}^{\tau,\sigma}$ on $\bigoplus_{m,n \in \mathbb{Z}} L^{2}(I^{1+m,\sigma+n}) \otimes L^{2}(I^{1+m,\sigma+n})$.

For the genuine eigenvectors $v_{x}^{\tau,\sigma}$, i.e. with $\mu_{x}$ in the discrete spectrum, we can rewrite the actions as 'dynamic' raising and lowering operators. Indeed, let $x_{k} = -q^{1-\gamma_{\tau,\sigma}} + 2k$ such that $v_{x}^{\tau,\sigma}$ is in the discrete spectrum of $\pi_{\phi}(\rho_{\tau,\sigma})$. We set $v_{k}^{\tau,\sigma} = v_{x_{k}}^{\tau,\sigma}$, then
\[
\pi_{\phi}(\alpha_{\tau,\sigma}) v_{k}^{\tau,\sigma} = ie^{-i\phi q^{+}\sigma + \frac{1}{2}}\sqrt{(1 - q^{2-2\sigma - 2\tau + 2k})(1 - q^{-2k})} v_{k-1,\sigma-1}^{\tau,\sigma},
\]
\[
\pi_{\phi}(\beta_{\tau,\sigma}) v_{k}^{\tau,\sigma} = ie^{-i\phi q^{-}\sigma - \frac{1}{2}}\sqrt{(1 + q^{2+2\tau + 2k})(1 + q^{2-2k})} v_{k-1,\sigma+1}^{\tau,\sigma},
\]
\[
\pi_{\phi}(\gamma_{\tau,\sigma}) v_{k}^{\tau,\sigma} = ie^{i\phi q^{0}}\sqrt{(1 + q^{2+2\tau + 2k})(1 + q^{2-2k})} v_{k+1,\sigma-1}^{\tau,\sigma},
\]
\[
\pi_{\phi}(\delta_{\tau,\sigma}) v_{k}^{\tau,\sigma} = -ie^{i\phi q^{0} + \frac{1}{2}}\sqrt{(1 - q^{2-2\tau - 2k})(1 - q^{-2k})} v_{k+1,\sigma+1}^{\tau,\sigma}.
\]

There is a similar result for the vectors $v_{x_{k}}^{\tau,\sigma}$, with $x_{k} = q^{1-\gamma_{\tau,\sigma} + 2k}$ such that $v_{x_{k}}^{\tau,\sigma}$ is in the discrete spectrum.

4.2. Clebsch-Gordan coefficients. To determine Clebsch-Gordan coefficients, we diagonalize the action of $\mathcal{T}(\rho_{\tau,\sigma})$ on a suitable basis. From letting $\lambda \to \infty$ in (2.2) we obtain
\[
\Delta(\rho_{\tau,\sigma}) = \frac{1}{2} q^{\tau-\sigma-1}(q^{1-\gamma_{\tau,\sigma}} \otimes \alpha_{\lambda,\sigma+1} + \beta_{\lambda,\sigma} + \gamma_{\lambda,\sigma+1} \delta_{\lambda,\sigma} + \gamma_{\lambda,\sigma+1} \delta_{\lambda,\sigma}) + q^{2}(1 + q^{2}) \rho_{\tau,\sigma} \otimes \rho_{\sigma,\sigma} + q^{2}(1 - q^{2}) \rho_{\tau,\sigma} \otimes 1 + q^{2}(1 - q^{2}) \otimes \rho_{\sigma,\sigma}
\]
\[
(4.8)
\]
We now let $\mathcal{T}(\rho_{\tau,\sigma})$ act on the $\ell^{2}(N) \otimes \ell^{2}(N)$-basis \{\(v_{y}^{\tau,\infty} \otimes v_{z}^{\infty,\sigma} | y \in -q^{2N} \cup q^{2\sigma + 2N}, z \in -q^{2N} \cup q^{2\sigma + 2N}\), see Propositions 3.1, 3.2 and (3.4). This gives
\[
\mathcal{T}(\rho_{\tau,\sigma}) v_{y}^{\tau,\infty} \otimes v_{z}^{\infty,\sigma} =
\]
\[
\frac{1}{2} \left[ y q^{1-\tau-\sigma} (1 + q^{2}) + y q^{1-\tau-\sigma} (1 - q^{2}) + z q^{1-\tau-\sigma} (1 - q^{2}) \right] v_{y}^{\tau,\infty} \otimes v_{z}^{\infty,\sigma}
\]
\[
+ \frac{1}{2} q^{2(\phi-\psi)} \sqrt{(1 + y)(1 - q^{2}y^{2})} (1 + z)(1 - q^{2z}) v_{y}^{\tau,\infty} \otimes v_{z}^{\infty,\sigma}
\]
\[
+ \frac{1}{2} q^{2(\phi-\psi)} \sqrt{(1 + yq^{2})(1 - q^{2}y^{2})} (1 + z)(1 - q^{2z}) v_{y}^{\tau,\infty} \otimes v_{z}^{\infty,\sigma}.
\]
We can identify this with the Jacobi operator corresponding to continuous dual $q^{2}$-Hahn polynomials. We define, for $y \in -q^{2N} \cup q^{2\tau + 2N}$ and $z \in -q^{2\sigma + 2N}$,
\[
c_{z,y,z}^{\sigma} = \rho_{y}(\mu_{z} ; -\lambda q^{1+\tau-\sigma}, \lambda q^{1-\tau-\sigma}, zq^{1+\tau-\sigma}/|\gamma|^{2}), \quad y = \lambda q^{2n}, \quad \lambda \in \{-1, q^{2}\}.
\]
For the corresponding support we write $L_{\lambda q^{1-r}+\tau,\lambda q^{1-r}-\tau}xq^{1-r}-\tau/x_{\lambda q^{1-r}} = I_{\tau}^{r}$. Let us remark that if $z$ is too large the pairwise product of the continuous dual $q$-Hahn parameters is not smaller than 1, so at first sight it seems that the orthogonality relations for the continuous dual $q$-Hahn polynomials are not valid. If this is the case we use \textbf{(3.3)} so that all conditions for orthogonality are again satisfied.

**Proposition 4.3.** Let $z \in -q^{2z} \cup q^{2s+2z}$ and let $\mathcal{H}_z$ be the subspace of $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ given by

$$\mathcal{H}_z = \mathfrak{sp} \mathfrak{su} \left\{ v_{\lambda q^{2n}}^{\infty} \otimes v_{q^{2n}}^{\infty} | n \in \mathbb{N}, \lambda \in \{-1, q^{2z}\} \right\}.$$ 

Then the operator $\mathcal{T} : \mathcal{H}_z \to L^2(I_{-1,z}^{\infty}) \oplus L^2(I_{z}^{\infty})$ defined by

$$\mathcal{T}_z(\lambda q^{2n} \otimes v_{q^{2n}}^{\infty} (\mu_z)) = e^{2\langle \mu \psi \rangle} v_{\lambda q^{2n},z}^{\infty} \sigma, \quad \lambda \in \{-1,q^{2z}\},$$

is unitary and intertwines $\mathcal{T}(\rho_{r,\sigma})|_{\mathcal{H}_z}$ with the multiplication operator $M$ on $L^2(I_{-1,z}^{\infty}) \oplus L^2(I_{z}^{\infty})$.

Similar as before, we use here the convention $\psi_k^{\infty,2k} = 0$ for $k \in \mathbb{N}_{\geq 1}$ and $\nu \in \{-1, q^{2z}\}$. Observe that $\mathcal{T}(\rho_{r,\sigma})|_{\mathcal{H}_z}$ has $[-1,1]$ as continuous spectrum, and the points in the discrete spectrum $\Sigma^{r,\sigma}_{\infty}$ are of the form $\mu_q^{1-r+s+2k}$ or $\mu_q^{1-r-s+2k}$ for some integer $k$.

**Remark 4.4.** Recall that $\rho_{r,\infty} = \lim_{\sigma \to \infty} 2q^{2r+q-1} \rho_{r,\sigma}$. We see that in the limit $\sigma \to \infty$ the continuous spectrum vanishes. Furthermore, the number of points in the discrete spectrum increases as $\sigma$ increases, and then we see that

$$\lim_{\sigma \to \infty} 2q^{2r+q-1}\Sigma^{r,\sigma}_{\infty} = -q^{2z} \cup q^{2r+2z},$$

as expected from Proposition \textbf{3.3}. Furthermore, for $z \in -q^{2z}$ and $x_k = \nu q^{1-r+s+2k}$ with $\nu \in \{-1, q^{2z}\}$, we have

$$\lim_{\sigma \to \infty} 3\varphi_2 \left( q^{-2n} - \lambda q^{1-r+s} x_k, -\lambda q^{1-r-s}/x_k : q^2, q^2 \right) = 3\varphi_2 \left( q^{-2n} - \lambda q^{2-2r}, -\lambda q^{2} : q^2, q^2 \right),$$

which corresponds to the limit from the continuous dual $q^2$-Hahn polynomials to the big $q^2$-Laguerre polynomials. To be precise, this gives the big $q^2$-Laguerre polynomials from the previous section in case $p < 0$, see the proof of Proposition \textbf{3.3}. From this we obtain

$$\lim_{\sigma \to \infty} v_{x_k,\lambda q^{2n}}^{\tau,\sigma} = c_{\nu q^2k,-\lambda \lambda n}^{\infty}. $$

In terms of generalized eigenvectors, the vectors

$$(4.9) \quad V_{x_k}^{\tau,\sigma} = \sum_{n \in \mathbb{N}} e^{2\langle \mu \psi \rangle} e_{x_k,\lambda q^{2n}}^{\tau,\sigma} v_{\lambda q^{2n}}^{\infty} \otimes v_{q^{2n}}^{\infty}, \quad \lambda = -1, q^{2r},$$

are both generalized eigenvectors of $\mathcal{T}(\rho_{r,\sigma})$ for eigenvalue $\mu_x$. Alternatively, we may write

$$\psi_y^{\infty} \otimes \psi_{x}^{\infty} = \int_{-1}^{1} e^{-2\langle \mu \psi \rangle} e_{y,\nu q^{2n}}^{\tau,\sigma} \otimes v_{\lambda q^{2n}}^{\infty} \otimes v_{q^{2n}}^{\infty} d\mu_x + \sum_{k} e^{-2\langle \mu \psi \rangle} e_{x_k,y,\nu q^{2n}}^{\tau,\sigma} \otimes v_{\lambda q^{2n}}^{\infty} \otimes v_{q^{2n}}^{\infty},$$

for $y = \lambda q^{2n}$ and $x_k$ is in the discrete spectrum. The actions of the generators of $\mathcal{A}_\sigma^{\tau}$ are:

- $\mathcal{T}(\alpha_{r,\sigma})V_{x_k}^{\tau,\sigma} = -e^{\langle \psi \psi \rangle} q^{2r-1/2 \sqrt{(1+q^{1-r-s})/\lambda q^{1-r-s}}} V_{x_k}^{\tau,\sigma}$
- $\mathcal{T}(\beta_{r,\sigma})V_{x_k}^{\tau,\sigma} = e^{\langle \psi \psi \rangle} q^{-1/2 \sqrt{(1-q^{1-r-s})/\lambda q^{1-r-s}}} V_{x_k}^{\tau,\sigma}$
- $\mathcal{T}(\gamma_{r,\sigma})V_{x_k}^{\tau,\sigma} = -e^{\langle \psi \psi \rangle} q^{1/2 \sqrt{(1-q^{1-r-s})/\lambda q^{1-r-s}}} V_{x_k}^{\tau,\sigma}$
- $\mathcal{T}(\delta_{r,\sigma})V_{x_k}^{\tau,\sigma} = -e^{\langle \psi \psi \rangle} q^{1/2 \sqrt{(1+q^{1-r-s})/\lambda q^{1-r-s}}} V_{x_k}^{\tau,\sigma}$

**Remark 4.5.** By symmetry in $\sigma$ and $\tau$, we may also define, for $y \in -q^{2z} \cup q^{2r+2z}$, the subspace $\mathcal{H}_y$ of $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ by

$$\mathcal{H}_y = \mathfrak{sp} \mathfrak{su} \left\{ v_{\lambda q^{2n}}^{\infty} \otimes v_{\xi q^{2n}}^{\infty} | n \in \mathbb{N}, \xi \in \{-1, q^{2\sigma}\} \right\}.$$
Then the operator $\tilde{T} : \tilde{H}_y \to L^2(I^{\sigma,\tau}_{-1,y}) \otimes L^2(I^{\rho,\tau}_{q^2\sigma,y})$ defined by

$$\tilde{T}_{y\xi q^2n} \otimes \tilde{v}_{\xi q^2n}^{\infty} (\mu_x) = e^{2in(\psi - \phi)}v_{x\xi q^2n,y}^{\rho,\tau}, \quad \xi \in \{-1, q^{2\sigma}\},$$

is unitary and intertwines $T(\rho,\tau)|_{\tilde{H}_y}$ with $M$. So, for $\xi = -1, q^{2\sigma}$, the vectors

$$\tilde{v}_{x,y}^{\tau,\sigma,\xi} = \sum_{n \in \mathbb{N}} e^{2in(\psi - \phi)}v_{x\xi q^2n,y}^{\tau,\sigma,\xi} \tilde{v}_{yq^2n}^{\infty} \otimes \tilde{v}_{\xi q^2n}^{\infty},$$

are both generalized eigenvectors of $T(\rho,\sigma)$ for eigenvalue $\mu_x$. These are related to the generalized eigenvectors \([4.39]\) by

\[
\begin{pmatrix}
(\tilde{v}_{x,y}^{\tau,\sigma,\xi - 1})_{x,\lambda \xi q^2k} \\
(\tilde{v}_{x,y}^{\tau,\sigma,\xi})_{x,\lambda \xi q^2k}
\end{pmatrix} =
\begin{pmatrix}
\delta_{\lambda, -1}\delta_{\xi, -1} - \delta_{\lambda,q^2\sigma}\delta_{\xi,q^2\sigma} \\
\delta_{\lambda, -1}\delta_{\xi,q^2\sigma} \delta_{\xi, q^2\sigma}
\end{pmatrix}
\begin{pmatrix}
(\tilde{v}_{x,y}^{\tau,\sigma,\xi - 1})_{x,\xi q^2k} \\
(\tilde{v}_{x,y}^{\tau,\sigma,\xi})_{x,\xi q^2k}
\end{pmatrix}.
\]

5. COUPLING COEFFICIENTS FOR TWO-FOLD TENSOR PRODUCTS

In this section we determine coupling coefficients between different (generalized) eigenvectors of $T(\rho,\sigma)$. For simplicity we take the representation labels $\phi$ and $\psi$ both equal to 0. Note that there is hardly any loss of generality in doing so, because the representation labels only occur in phase factors in eigenvectors in the previous sections. First we introduce a class of functions that we will need.

5.1. Al-Salam–Carlitz II functions. The Al-Salam–Carlitz II polynomials $P_n$ and related functions $Q_n$, see \([12]\), are defined by

\[
P_n(x) = P_n(x; c, d; q) = (-c)^n q^{-\frac{1}{2}n(n-1)} 2\varphi_0 \left(\frac{q^{-n}, cx}{-1}; q, \frac{dq^n}{c}\right), \quad n \in \mathbb{N},
\]

and

\[
Q_n(x) = Q_n(x; c, d; z_-, z_+; q) = (-d)^n q^{-\frac{1}{2}n(n+1)} \frac{(cx, q^2\sigma)_{\infty} \theta(dz_-, dz_+; q)}{(q/dz, q^{n+1}x/dz_-, dz_+; q)_{\infty}}
\times 1\varphi_1 \left(\frac{q/cx}{dz_- z_+ q^{-n}/x}; q, \frac{cz_- z_+ q^n}{x}\right), \quad n \in \mathbb{Z}.
\]

Both $P_n$ and $Q_n$ are symmetric in the parameters $c$ and $d$, and $Q_n$ is also symmetric in $z_-$ and $z_+$. For notational convenience we also define $P_n = 0$ for $n \in -\mathbb{N}_{\geq 1}$.

Let $\tau = d, z_+ > 0$ and $z_- < 0$, then the functions $P_n$ and $Q_n$ satisfy the orthogonality relations

\[
\begin{align*}
\frac{1}{1-q} \int_{\infty(z_-)}^{\infty(z_+)} P_n(x)P_m(x)w(x) \, dq \, dx &= \delta_{mn} p_n^P, \\
\frac{1}{1-q} \int_{\infty(z_-)}^{\infty(z_+)} Q_n(x)Q_m(x)w(x) \, dq \, dx &= \delta_{mn} q_n^Q, \\
\frac{1}{1-q} \int_{\infty(z_-)}^{\infty(z_+)} P_n(x)Q_m(x)w(x) \, dq \, dx &= 0,
\end{align*}
\]

where $m, n \in \mathbb{Z}$ and

\[
w(x) = w(x; a, b; q) = \frac{1}{(cx, dx; q)_{\infty}},
\]

\[
h_n^P(c, d; z_-, z_+; q) = \begin{cases} z_+(q; q)_n (cd)^n q^{-n^2} (q; q)_{\infty} \theta(z_-, dz_-, dz_+, d_+; q), & n \geq 0, \\
0, & n < 0
\end{cases}
\]

\[
h_n^Q(c, d; z_-, z_+; q) = z_+ (q; q)_{\infty} \theta(z_-, dz_+, d_+; q),
\]

\[
h_n^Q(c, d; z_-, z_+; q) = z_+(q; q)_{\infty} (cdz_+ q^{-n+1}; q)_{\infty} (q; q)_{\infty}^2 \theta(z_-, dz_+, d_+; q).
\]
Moreover, \( \{P_n\}_{n \in \mathbb{N}} \cup \{Q_n\}_{n \in \mathbb{Z}} \) is an orthogonal basis for the corresponding \( L^2 \)-space, which implies the dual orthogonality relations

\[
\sum_{n \in \mathbb{N}} P_n(x) P_n(y) \frac{1}{h_n^2} + \sum_{n \in \mathbb{Z}} Q_n(x) Q_n(y) \frac{1}{h^2} = \delta_{xy} \frac{1}{|x|w(x)},
\]

for \( x, y \in z_q \mathbb{Z} \cup z_q \mathbb{Z}^2 \).

The following \( q \)-integral evaluation formulas turn out to be useful. The proof is given in the appendix.

**Proposition 5.1.** For \( n \in \mathbb{N} \),

\[
\int_{z^-}^{z^+} P_n(x) \frac{q(x/z^- q/x/z^+; q\infty)}{(cx, dx; q\infty)} dx = (1 - q)z^+ (-d)^n q^{-n(n-1)} \frac{\theta(z^-/z^+; q)}{(cz^- q, dz^- q^2, c z^+ q, d z^+ q; q\infty)} 2\varphi_1 \left( \frac{q^{-n}, dz^+}{q^{1-n}/cz^-; q, dz^-} \right),
\]

\[
\int_{z^-}^{z^+} Q_n(x) \frac{q(x/z^- q/x/z^+; q\infty)}{(cx, dx; q\infty)} dx = (1 - q)z^+(c/|q|)n q^{-n(n-1)} \frac{\theta(z^-/z^+; q)}{(cz^- q, q^{n+1}; q\infty)} 2\varphi_1 \left( \frac{q^{-n}, q/dz^-}{cz^+ q^{-n}; q, dz^+} \right).
\]

Furthermore, for \( n \in -\mathbb{N}_{\geq 1} \),

\[
\int_{z^-}^{z^+} Q_n(x) \frac{q(x/z^- q/x/z^+; q\infty)}{(cx, dx; q\infty)} dx = 0.
\]

**A limit case.** We also need a limit case of the Al-Salam–Carlitz II functions. We let \( c \to q/z^+ \) and \( d \to 0 \). Clearly the polynomials \( P_n \) vanish in this limit. For the limit of \( Q_n(x) \) we distinguish between \( x \in z_q \mathbb{Z}^- \) and \( x \in z_q \mathbb{Z}^+ \). Using (1.1) we obtain

\[
Q_n(z_q^k) = (z^-)^{-n} \left( \frac{z^+}{z^-} \right)^k q^{k(n+k)} \frac{(c z_q^k q, dz_q^k q, dz_q^{-n-k}; q\infty)}{dz_q^{-q-n-k}; q, c z_q-q^{-n-k}},
\]

so that

\[
\lim_{c \to q/z^+} Q_n(z_q^k) = (z^-)^{-n} \left( \frac{z^+}{z^-} \right)^k q^{k(n+k)} \frac{(q^{k+1}; q\infty)}{0, q^{-z^{-1} n-k} z^+},
\]

which equals zero if \( k \in -\mathbb{N}_{\geq 1} \). Reversing the order of summation in the \( 1\varphi_1 \)-series,

\[
1\varphi_1 \left( q^{-k} ; z q^{-1-k} \right) = (z q^{-k})^k 1\varphi_1 \left( q^{-k} ; q^{1+k} \right),
\]

we can now recognize the limit of \( Q_n \) as a Stieltjes–Wigert polynomial, which is defined by

\[
S_k(x; q) = \frac{1}{(q; q)_k} 1\varphi_1 \left( q^{-k}; 0; -x q^{1+k} \right),
\]

so that

\[
\lim_{c \to q/z^+} Q_n(z_q^k) = \frac{(q; q\infty)}{(z^-)^n} S_k(-z_q^{n+1}; q).
\]

In the same way we can express the limit of \( Q_n(z_q^k) \) in terms of the functions

\[
M_{k}^{(l)}(x; q) = \frac{1}{(q; q\infty)^l} 1\varphi_1 \left( -t q^{-k}; 0; -x q^{1+k/l} / t \right),
\]

for \( x, y \in z_q \mathbb{Z} \cup z_q \mathbb{Z}^2 \).
which are closely related to Jackson’s second $q$-Bessel functions. We have
\[ Q_n(z_q^{-k}) = (z_+)^{-n} \frac{z_-}{z_+} \frac{kq^{k(n+1)}(z_q^{-k+1}/z_+; q)_\infty 1 \Psi_1 
\quad \times \frac{z_q^{-k}/z_-}{0; q, q^{1-n-k}}}{(q^{k+1}; q)_\infty}
\]
\[ = (z_-)^{-n} (z_q^{-k+1}/z_+; q)_\infty 1 \Psi_1 \frac{z_q^{-k}/z_-}{0; q, q^{1-n+k}}
\]
\[ = (z_-)^{-n} (q, z_q^{-k+1}/z_+; q)_\infty M_k^{(-z_+/z_-)} (-z_q^{-k}/z_-; q).
\]

From the orthogonality relations for $Q_n$ we obtain (formally)
\[ \sum_{k \in \mathbb{N}} S_k \left( \frac{z_q^{-m}}{z_-}; q \right) S_k \left( \frac{z_q^{-n}}{z_-}; q \right) \frac{z_q^{-k}}{(q^{k+1}; q)_\infty}
\]
\[ = \sum_{k \in \mathbb{Z}} M_k^{(-z_+/z_-)} \left( \frac{z_q^{-m}}{z_-}; q \right) M_k^{(-z_+/z_-)} \left( \frac{z_q^{-n}}{z_-}; q \right)
\]
\[ = \delta_{mn} z_+ \left( \frac{z_q^{-n}}{z_+} q^{-\frac{1}{2} n(n+1)} \theta(z_-/z_+; q),
\]

and from the dual orthogonality relations we find
\[ \sum_{n \in \mathbb{Z}} S_k \left( \frac{z_q^{-m}}{z_-}; q \right) M_l \left( \frac{z_q^{-n}}{z_-}; q \right) \frac{z_q^{-n}}{(q^{l+1}; q)_\infty}
\]
\[ = \sum_{n \in \mathbb{Z}} S_k \left( \frac{z_q^{-m}}{z_-}; q \right) S_l \left( \frac{z_q^{-n}}{z_-}; q \right) \frac{z_q^{-n}}{(q^{l+1}; q)_\infty} = \delta_{kl} q^{-k(l+1); q}_\infty \theta(z_-/z_+; q),
\]
\[ \sum_{n \in \mathbb{Z}} M_k^{(-z_+/z_-)} \left( \frac{z_q^{-m}}{z_-}; q \right) M_l^{(-z_+/z_-)} \left( \frac{z_q^{-n}}{z_-}; q \right) \frac{z_q^{-n}}{(q^{l+1}; q)_\infty}
\]
\[ = \delta_{kl} \left( \frac{z_q^{-m}}{z_-} q^{\frac{1}{2} n(n+1)} \theta(z_-/z_+; q),
\]

These relations are proved by Christiansen and Koelink in [6, Theorem 3.5].

5.2. Coupling coefficients. In Section 4.2 we diagonalized $\mathcal{T}(\rho, \sigma)$ by considering the action on basis elements $v_{\tau, \nu}^{\infty} \otimes v_{\infty}^{\infty}$. We can also diagonalize $\mathcal{T}(\rho, \sigma)$ using the actions of $\alpha_{\tau, \infty}, \beta_{\tau, \infty}, \gamma_{\tau, \infty}, \delta_{\tau, \infty}$, on the orthonormal basis $\{v_{\tau, \rho}^{\infty} \mid y \in -q^{2n} \cup q^{2r+2n}, z \in -q^{2s} \cup q^{2r+2s}\}$ of $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$, see Propositions 4.4 and 4.5. We find
\[ \mathcal{T}(\rho_{\tau, \infty}) V_{\tau, \rho}^{\infty} = \sqrt{(1 + yq^2)(1 - yq^{-2}\tau)} v_{\tau, \rho}^{\infty} z_q^{-2n} + (q^{-1 - \sigma - \tau} - q^{1 + \sigma - \tau}) y v_{\tau, \rho}^{\infty} + \sqrt{(1 + y)(1 - yq^{-2}\tau)} v_{\tau, \rho}^{\infty} z_q^{-2n}.
\]

Comparing this with Proposition 4.1 we obtain the following result.

Proposition 5.2. Define for $z \in -q^{2s} \cup q^{2r+2s}$ the subspace
\[ \mathcal{H}_z = \text{span} \left\{ V_{\tau, \rho}^{\infty} \mid n \in \mathbb{N}, \nu \in \{ -1, q^{2r} \} \right\} \subset \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}),
\]
then the unitary operator $\vartheta: \mathcal{H}_z \to L^2(\mathcal{H}_z^\perp) \oplus L^2(\mathcal{H}_z^\perp)$ defined by
\[ \vartheta V_{\tau, \rho}^{\infty} (x) = m_{x, \nu}^{\tau, \sigma},
\]
intertwines $\mathcal{T}(\rho_{\tau, \sigma})|_{\mathcal{H}_z}$ with multiplication operator $M$ on $L^2(I_{\tau, \sigma}^\perp) \oplus L^2(I_{\tau, \sigma}^\perp)$.

In terms of generalized eigenvectors, Proposition 5.2 says that
\[ U_{\tau, \nu}^{\infty} = \sum_{n \in \mathbb{N}} m_{x, \nu}^{\tau, \sigma} V_{\tau, \rho}^{\infty} z_q^{-2n}, \quad \nu = -1, q^{2T},
\]
are generalized eigenvectors of $\mathcal{T}(\rho_{\tau, \sigma})$ for eigenvalue $\mu_x$, $x \in T$. For $\mu_x$ in the discrete spectrum, these are genuine eigenvectors.
From here on we assume \( x \in T \) in this section. Since \( V^\tau,\sigma_{x,y} \) defined by (5.9) are also generalized eigenvectors of \( T(\rho,\sigma) \) for eigenvalue \( \mu_\perp \) there exists a (formal) expansion

\[
(U^\tau,\sigma_{x,y}^{-1} x, \xi, \nu, \lambda) = \sum_{y \in -q^2xzq^2 + z} S^\tau,\sigma_{x,y,z} (V^\tau,\sigma_{x,y}^{-1} x, \xi, \nu, \lambda)
\]

where the coupling coefficients \( S^\tau,\sigma_{x,y,z} \) are \( 2 \times 2 \)-matrices;

\[
S^\tau,\sigma_{x,y,z} = \left( (S^\tau,\sigma_{x,y,z})_{\nu,\lambda} \right)_{\nu,\lambda \in \{-1, q^2\}^T}.
\]

Alternatively, \( S^\tau,\sigma_{x,y,z} \) is the kernel of the unitary integral operator \( S : L^2(\Gamma^\tau,\sigma_{0,1}) \oplus L^2(\Gamma^\tau,\sigma_{1,2}) \rightarrow L^2(\Gamma^\tau,\sigma_{0,1}) \oplus L^2(\Gamma^\tau,\sigma_{1,2}) \) that satisfies \( (S \circ \vartheta f)(x) = \Upsilon f(x) \) (almost everywhere), where \( \vartheta \) and \( \Upsilon \) are the operators defined in Propositions 5.2 and 5.4. The unitarity of \( S \) is equivalent to the matrix orthogonality relations

\[
\sum_{y \in -q^2xzq^2 + z} S^\tau,\sigma_{x,y,z} (S^\tau,\sigma_{x,y,z})^* = \delta_{z_1z_2} I,
\]

\[
\sum_{z \in -q^2xzq^2 + z} (S^\tau,\sigma_{x,y,z})^* S^\tau,\sigma_{x,y,z} = \delta_{y_1y_2} I.
\]

In order to determine explicit expressions for the matrix coefficients of \( S^\tau,\sigma_{x,y,z} \) we use the following result.

**Lemma 5.3.** Let \( \xi \in \{-1, q^2\} \), \( k \in \mathbb{Z} \) and \( m, n \in \mathbb{N} \). Then for \( \lambda = \xi \),

\[
\sum_{y \in -q^2xzq^2 + z} (S^\tau,\sigma_{x,y,\xi q^2k})_{\nu,\lambda} = 0.
\]

Furthermore, \( (S^\tau,\sigma_{x,y,\xi q^2k})_{\nu,\lambda} = 0 \) for \( \lambda \neq \xi \).

**Proof.** This follows from (5.7) by taking inner products with the orthonormal basis vectors \( v_{\nu q^2m} \otimes e_m \). □

**Theorem 5.4.** Let \( x \in T \), \( y \in -q^{2z} \cup q^{2z} \), \( \lambda, \nu, \xi \in \{-1, q^2\} \), and \( z = \xi q^{2k} \) with \( k \in \mathbb{Z} \).

(i) If \( \xi \neq \lambda \), then \( S^\tau,\sigma_{x,y,z} = 0 \).

(ii) If \( \xi = \lambda \), then

\[
(S^\tau,\sigma_{x,y,z})_{\nu,\lambda} = \begin{cases} (\text{sgn}(\nu))^k \frac{|y|w(y; c; d; q^2)}{h^\nu_k(c, d; z_-, z_+; q^2)} P_k(y; c, d; q^2), & \text{if } \nu \neq \lambda, \\
(\text{sgn}(\nu))^k \frac{|y|w(y; c; d; q^2)}{h^\nu_k(c, d; z_-, z_+; q^2)} Q_k(y; c, d; z_-, z_+; q^2), & \text{if } \nu = \lambda, \end{cases}
\]

with

\[
(c, d, z_-, z_+) = \begin{cases} (-\nu q^{1-\tau - \sigma} / x, -\nu x q^{1-\tau - \sigma}, -1, q^{2\sigma}), & \text{if } \nu \neq \lambda, \\
(x q^{1-\tau - \sigma} / \nu, q^{1+\tau - \sigma} / \nu x, -1, q^{2\sigma}), & \text{if } \nu = \lambda. \end{cases}
\]

Note that \( \lambda = -q^{2\tau} / \nu \) in case \( \nu \neq \lambda \).

**Proof.** The case \( \xi \neq \lambda \) is already proved, so we assume \( \xi = \lambda \). We first show that (5.9) is satisfied. The coefficients \( (S^\tau,\sigma_{x,y,\xi q^2k})_{\nu,\lambda} \) do not depend on \( m \) and \( n \), so we may choose \( m \) and \( n \) in a convenient way. We choose \( n = m = 0 \), then for \( z = \xi q^{2k} \) with \( k \in \mathbb{N} \) the explicit identity is, after canceling
common terms,
\[
\sum_{y \in -q^{2\mathbb{L}} \cup q^{2\mathbb{Z}^{+}}} \left( \varepsilon_{y, z} \right)_{\nu, \lambda} \frac{[y]_{q}^{\frac{1}{2}}(-yq^{2}, yq^{-2x}; q^{2})_{\infty}}{(yq^{1+\tau-\sigma}x_{1+1}; /\lambda; q^{2})_{\infty}} \frac{\theta(-q^{2\tau}; q^{2})(-\lambda q^{1-\tau+\sigma}x_{1}; q^{2})_{\infty}}{\theta(-q^{2\tau+2}; q^{2})(-\lambda q^{2-2\tau}; q^{2})_{\infty}} \frac{x}{(q^{2}, -\nu q^{\sigma-\tau+1}x_{1}; q^{2})_{\infty}} \frac{(\nu q^{-1+\sigma-\tau}/x, q^{1+\tau-\sigma}/\nu x_{1}; q^{2})_{\infty}}{\frac{1}{\nu x_{1}}}
\]
\]

For \( k \in -\mathbb{N}_{\geq 1} \) the right hand side is equal to zero. Note that \((-yq^{2}, yq^{-2x}; q^{2})_{\infty} = 0 \) for \( y \not\in -q^{2\mathbb{Z}} \cup q^{2\mathbb{Z}^{+}} \), so we can write the sum as a \( q \)-integral of the form \( \int_{1}^{\infty} f(y) dy \). Expressions for \( (S_{y, z})_{\nu, \lambda} \) are now obtained from Proposition 5.1 with parameters

\[
(c, d, z, +, z+) = \begin{cases} 
(-\nu q^{1-\tau-\sigma}/x, -\nu xq^{1-\tau-\sigma}, -1, q^{2\sigma}), & \nu \neq \lambda, \\
(xq^{1+\tau-\sigma}/\nu, q^{1+\tau-\sigma}/\nu x_{1}, q^{2\sigma}), & \nu = \lambda.
\end{cases}
\]

The expressions obtained in this way can be simplified using the \( \theta \)-product identity (1.1) and

\[
(5.10) \quad \frac{q^{2\tau}}{\lambda} \theta(-q^{2\tau+2}; q^{2}) = \theta(-\lambda q^{2-2\tau}; q^{2}), \quad \frac{(q^{2}, -\lambda q^{2-2\tau})_{k}}{(\lambda q^{2-2\tau}, -\lambda q^{2}; q^{2})_{k}} = 1.
\]

Next we show \( S_{y, z} \) is indeed the kernel in a unitary operator by verifying the orthogonality relations (5.3). First we assume \( z_{i} \in -q^{2\mathbb{Z}} \) for \( i = 1, 2 \), then

\[
\sum_{y \in -q^{2\mathbb{L}} \cup q^{2\mathbb{Z}^{+}}} S_{y, z, 1}^{\tau, \sigma} (S_{y, z, 2}^{\tau, \sigma})^{*} = \sum_{y \in -q^{2\mathbb{L}} \cup q^{2\mathbb{Z}^{+}}} \left( \begin{array}{c}
(S_{y, z, 1}^{\tau, \sigma})_{-1, -1} \\
0 \\
0 \\
(S_{y, z, 2}^{\tau, \sigma})_{q^{2\tau}, -1}
\end{array} \right) \left( \begin{array}{c}
(S_{y, z, 2}^{\tau, \sigma})_{-1, -1} \\
0 \\
0 \\
(S_{y, z, 1}^{\tau, \sigma})_{q^{2\tau}, -1}
\end{array} \right) = \delta_{z_{1} z_{2}} \left( \begin{array}{c}
1 \\
0 \\
0 \\
1
\end{array} \right),
\]

where the last line follows from writing the matrix coefficients in terms of the functions \( P_{k} \) and \( Q_{k} \) and using the orthogonality relations (5.3). E.g., with \( z_{i} = -q^{2k_{i}} \),

\[
\sum_{y \in -q^{2\mathbb{L}} \cup q^{2\mathbb{Z}^{+}}} \left( S_{y, z, 1}^{\tau, \sigma} \right)_{-1, -1} (S_{y, z, 2}^{\tau, \sigma})_{q^{2\tau}, -1} = K \int_{(q^{2\tau})} Q_{k_{1}}(y; c, d; -1, q^{2\sigma}; q^{2}) P_{k_{2}}(y; c, d; q^{2}) w(y; c, d; q^{2}) d_{x} y = 0,
\]

where \( (c, d) = (-xq^{1+\tau-\sigma}, -q^{1+\tau-\sigma}/x) \) and \( K \) is independent of \( y \). The cases involving \( z_{i} \in q^{2\mathbb{Z}^{+}} \) are proved in the same way.

The dual orthogonality relations are also proved in this way, now using the dual orthogonality relations for \( P_{k} \) and \( Q_{k} \).

As a result we obtain the following \( q \)-integral evaluation formulas involving Al-Salam–Carlitz II polynomials \( P_{n} \) (5.1) and functions \( Q_{n} \) (5.2), continuous dual \( q \)-Hahn polynomials \( p_{n} \) (1.1), Al-Salam–Chihara polynomials \( q_{n} \) (1.3), big \( q \)-Laguerre polynomials \( L_{n} \) (5.5), and Al-Salam–Carlitz polynomials \( U_{n} \) (3.1).
Theorem 5.5. Let $x \in \mathbb{T}$, $s, t > 0$, $\nu \in \{-1, 1\}$, and $m, n \in \mathbb{N}$. For $k \in \mathbb{N}$,

$$
\int_{-q^{-n}}^{q^{-n}} p_n(\mu_x; \frac{1}{x}, \sqrt{qst}, \frac{1}{x}, \sqrt{qt/s}, -\nu y \sqrt{q/st}; q) U_m^{(\nu)}(-y^q; q)
\times P_k(y; -\nu x \sqrt{q/st}, -\nu^2 \sqrt{q/st}; q) \frac{(-y^{q+1} q^{1+n}, y^{q+1} q^{1+n}/s; q)_\infty}{(-\nu x^{q+1} \sqrt{q/st}; q)_\infty} dy
\times (1 - q)^{-m} \nu^{m-k} (q/st)^{k/2} (st/q)^{m/2} q^{-n} q^{m(m-n) q^{-1/4}}
\times (t q^{n-m+1}/\nu, -q^{n-m+1}/\nu; q)_\infty
\times (q^{m-n+k+1}/t; q)_\infty
\times q_{m-n+k}(\mu_x; -\nu \sqrt{q/st}; q)_\infty L_m(q^{k+1}; t q^{n-m}/\nu, -q^{n-m}/\nu; q)_\infty
$$

and for $k \in \mathbb{Z}$

$$
\int_{-q^{-n}}^{q^{-n}} p_n(\mu_x; -\nu \sqrt{q/st}, -\nu \sqrt{q/st}; q) U_m^{(\nu)}(-y^q; q)
\times Q_k(y; \frac{1}{x}, \sqrt{q/st}, -\frac{1}{x}, \sqrt{q/st}; -1, s; q) \frac{(-y^{q+1} q^{1+n}, y^{q+1} q^{1+n}/s; q)_\infty}{(-\nu^{q+1} x^{q+1} \sqrt{q/st}; q)_\infty} dy
\times (1 - q) \nu^{-m-k} (t/q s)^{k/2} (st/q)^{m/2} q^{-n} q^{m(m-n) q^{-1/4}}
\times (-\nu q^{m-n+1}, -\nu q^{m-n+1}/t; q)_\infty
\times (q^{m-n+k+1}/t; q)_\infty
\times q_{m-n+k}(\mu_x; -\nu \sqrt{q/st}; q)_\infty L_m(-\nu^2 q^{k+1}/t; -\nu q^{n-m}, -\nu q^{n-m}/t; q)_\infty
$$

Proof. This follows from writing (5.3) explicitly in terms of the corresponding special functions, substituting $(q^{2x}, q^{2t}) \to (s, t)$ and replacing $q^2$ by $q$. The first $q$-integral identity corresponds to the case $\lambda \neq \nu$; the second identity corresponds to the case $\lambda = \nu$. \qed

We substitute $(x, \lambda, \nu) \to (-q^2 \sqrt{q/st}, -1, -1)$ in the second identity of Theorem 5.5 and let $t \to 0$. This limit corresponds to $\rho_{\tau, \sigma} \to \rho_{\infty, \sigma}$ in $A_q$. This gives the following identity involving big $q$-Laguerre polynomials $L_n$ (3.35), Al-Salam–Carlitz polynomials $U_n$ (3.1), Wall polynomials (2.7), Stieltjes-Wigert polynomials $S_n$ (5.4) and related $q$-Bessel functions $M_n$ (5.3).

Corollary 5.6. For $k, m, n, x \in \mathbb{N}$ and $s > 0$,

$$
\sum_{y \in \mathbb{N}} L_n(q^{x+y+1} - s q^y; q, q^y; q) U_m^{(\nu)}(-s q^{y+n}; q) S_{x+y}(q^{k-2x}; q)
\times (q^{y+1}; q)_x (-s q^{y+1}; q)_\infty (q^y)^{-n} q^{x+y}
\sum_{y \in \mathbb{N}} L_n(\nu q^{x+y+1}/s; q^y, -q^{x+y}/s; q) U_m^{(\nu)}(q^{y+n}/q; q^y; q) M_{x+y}^{(\nu)}(q^{k-2x}; q)
\times (-q^{y+1}/s; q)_x (q^{y+1}; q)_\infty q^{-n} q^{x+y}
\times (q^{x+y+1}/s; q)_x (-q^{x+y+1}; q)_\infty q^{-n} q^{x+y}
\times (q^{x+y+1}/s; q)_x (-q^{x+y+1}; q)_\infty q^{-n} q^{x+y}
\Delta_{x-k} q^{-x(x-1)} q^{-n(n+1)/2} q^{k(k-1)/2} q^{n(m+k)+k(x-m)} \theta(-s; q)
\times (q^{x+n-1}; q)_m (q^{m-n+k+1}; q)_\infty L_m^{(\nu)}(-s q^x; q, q^x; q)_\infty M_{x+n+k}^{(\nu)}(q^x; q)_\infty
$$

Remark 5.7.

(i) Define the $T(\rho_{\infty, \sigma})$-eigenvectors $U_{x, \rho}^{\infty, \sigma} \in L^2(\mathbb{N}) \otimes L^2(\mathbb{N})$ by

$$
U_{x, \rho}^{\infty, \sigma} = \sum_{n \in \mathbb{N}} m_{x,n}^{\infty, \sigma} V_{n, p+n}, \quad p \in \mathbb{Z},
$$

where $V_{n,p}$ is the eigenvector of $T(\gamma^x)$ defined by (2.10). $U_{x, \rho}^{\infty, \sigma}$ is an eigenvector of $T(\rho_{\infty, \sigma})$ for eigenvalue $x \in -2q^{2\mathbb{N}} \cup q^{2\mathbb{N}+1}$. Corollary 5.6 shows that the coupling coefficients between $T(\rho_{\infty, \sigma})$-eigenvectors $V_{x, \rho}^{\infty, \sigma}$ and $U_{x, \rho}^{\infty, \sigma}$ can be expressed in terms of Stieltjes-Wigert polynomials $S_k$ and related $q$-Bessel functions $M_k$. 

(ii) In the \((\infty, \infty)\)-case there is no eigenvector similar to \(U^{\tau,\sigma}\) or \(U^{\infty,\sigma}\), so the limit \(s \to 0\) in Corollary 5.10 will not give any interesting identity.

Similarly as the generalized eigenvector \(5.6\), we also have the generalized eigenvector

\[
\tilde{U}^{\tau,\sigma,\nu}_{x,y} = \sum_{n \in \mathbb{N}} m^{\tau,\sigma}_{x,y}(q^{2n} + yq^{2n-1}, \nu = -1, q^{2\sigma},
\]

of \(T(\rho,\sigma)\) for eigenvalue \(\mu_x\). Here \(y \in -q^{2\mathbb{Z}} \cup q^{2\mathbb{Z}+2\mathbb{Z}}\). Essentially this is the generalized eigenvector \(U^{\tau,\sigma,\nu}_{x,y}\) with \(\tau \leftrightarrow \sigma\). We define, for \(y \in -q^{2\mathbb{Z}} \cup q^{2\mathbb{Z}+2\mathbb{Z}}, z \in -q^{2\mathbb{Z}} \cup q^{2\mathbb{Z}+2\mathbb{Z}}\), the \(2 \times 2\)-matrix valued coupling coefficients \(T^{\tau,\sigma}_{x,y,z}\) by

\[
\begin{pmatrix}
\tilde{U}^{\tau,\sigma,\nu-1}_{x,y} \\
\tilde{U}^{\tau,\sigma,\nu^2}_{x,y}
\end{pmatrix}
= \sum_{z \in -q^{2\mathbb{Z}} \cup q^{2\mathbb{Z}+2\mathbb{Z}}} T^{\tau,\sigma}_{x,y,z}
\begin{pmatrix}
U^{\tau,\sigma,\nu-1}_{x,z,\nu} \\
U^{\tau,\sigma,\nu^2}_{x,z,\nu}
\end{pmatrix}.
\]

These coupling coefficients can be expressed in terms of the the coupling coefficients \(S\) as follows. We start by expanding \(\tilde{U}^{\tau,\sigma}_{x,y}\) in terms of \(V^{\tau,\sigma}_{x,v}\), using \(5.7\) with \(\tau \leftrightarrow \sigma\);

\[
\begin{pmatrix}
\tilde{U}^{\tau,\sigma,\nu-1}_{x,y} \\
\tilde{U}^{\tau,\sigma,\nu^2}_{x,y}
\end{pmatrix}
= \sum_{k \in \mathbb{Z}} S^{\tau,\sigma}_{x,-q^{2k},y} V^{\tau,\sigma,\nu-1}_{x,q^{2k},\nu} + S^{\tau,\sigma}_{x,q^{2k}+2k,y} V^{\tau,\sigma,\nu^2}_{x,q^{2k}+2k,\nu} = \sum_{k \in \mathbb{Z}} S^{\tau,\sigma}_{x,-q^{2k},y} V^{\tau,\sigma,\nu-1}_{x,q^{2k}+2k,\nu} + S^{\tau,\sigma}_{x,q^{2k}+2k,y} V^{\tau,\sigma,\nu^2}_{x,q^{2k}+2k,\nu},
\]

For the second identity we used \(4.10\) to write \(\tilde{V}^{\tau,\sigma}_{x,v}\) in terms of \(V^{\tau,\sigma}_{x,v}\). Then we transform \(V^{\tau,\sigma}_{x,v}\) to \(U^{\tau,\sigma}_{x,z}\) using the inverse of \(5.7\),

\[
\begin{pmatrix}
V^{\tau,\sigma,\nu}_{x,v} \\
V^{\tau,\sigma,\nu^2}_{x,v}
\end{pmatrix}
= \sum_{z \in -q^{2\mathbb{Z}} \cup q^{2\mathbb{Z}+2\mathbb{Z}}} \begin{pmatrix}
(S^{\tau,\sigma}_{x,-q^{2k},z})_{-1,\nu} & (S^{\tau,\sigma}_{x,q^{2k}+2k,z})_{1,\nu} \\
(S^{\tau,\sigma}_{x,q^{2k}+2k,z})_{-1,\nu} & (S^{\tau,\sigma}_{x,-q^{2k},z})_{1,\nu}
\end{pmatrix}
\begin{pmatrix}
U^{\tau,\sigma,\nu-1}_{x,z,\nu} \\
U^{\tau,\sigma,\nu^2}_{x,z,\nu}
\end{pmatrix}.
\]

Combining then gives the following result:

\[
T^{\tau,\sigma}_{x,y,z} = \sum_{v \in -q^{2\mathbb{Z}} \cup q^{2\mathbb{Z}+2\mathbb{Z}}} \tilde{S}^{\tau,\sigma}_{x,v,y} \tilde{S}^{\tau,\sigma}_{x,v,z}
\]

where \(\tilde{S}^{\tau,\sigma}_{x,v,y} = S^{\tau,\sigma}_{x,v,y}\), and

\[
\tilde{S}^{\tau,\sigma}_{x,v,z} = \begin{pmatrix}
(S^{\tau,\sigma}_{x,-q^{2k},z})_{-1,\nu} & (S^{\tau,\sigma}_{x,q^{2k}+2k,z})_{1,\nu} \\
(S^{\tau,\sigma}_{x,q^{2k}+2k,z})_{-1,\nu} & (S^{\tau,\sigma}_{x,-q^{2k},z})_{1,\nu}
\end{pmatrix}, \quad v = \nu q^{2k}.
\]

The coupling coefficients \(T^{\tau,\sigma}_{x,y,z}\) are self-dual, i.e. \((T^{\tau,\sigma}_{x,y,z})^* = T^{\tau,\sigma}_{x,y,z}\), and they satisfy the orthogonality relations

\[
\sum_{z \in -q^{2\mathbb{Z}} \cup q^{2\mathbb{Z}+2\mathbb{Z}}} T^{\tau,\sigma}_{x,y,z} (T^{\tau,\sigma}_{x,y,z})^* = \delta_{gy} I.
\]

6. COUPLING COEFFICIENTS FOR THREE-FOLD TENSOR PRODUCTS

In this section we consider a three-fold tensor product representation of \(A_n\) and we compute the coupling coefficients between two different generalized eigenvectors of \(\rho_{\tau,\sigma}\). The coupling coefficients can be considered as \(6j\)-symbols or Racah coefficients. We start by introducing a class of functions that we need later on.
6.1. \(q\)-Meixner functions. The \(q\)-Meixner function is introduced in [14], and is defined by

\[
\phi_q(x) = \phi_q(x; a, b; q) = 2 \varphi_2 \left( -1/x, -1/\gamma ; a, b ; q, a^b \gamma x \right).
\]

It is an entire function in \(x\) and in \(\gamma\), and it is self-dual, i.e., \(\phi_q(x) = \phi_q(x)\). Note that, for \(n \in \mathbb{N}\), \(\phi_{q^n}(x)\) is a polynomial in \(x\) of degree \(n\); the \(q\)-Meixner polynomial. For parameters \(a, b, t\) satisfying \(a = b\), \(a \in \mathbb{C} \setminus \mathbb{R}\), \(t > 0\), the \(q\)-Meixner functions satisfy the orthogonality relations

\[
\int_{-1}^{\infty(t)} \phi_{\gamma_1}(x) \phi_{\gamma_2}(x) w(x) dx = \delta_{\gamma_1, \gamma_2} (1 - q)^{K_t} K_{q/abt} \gamma_1 | w(\gamma_1) |, \quad \gamma_1, \gamma_2 \in -q^N \cup \mathbb{Q}/abt,
\]

where

\[
w(x) = w(x; a, b; q) = \frac{(-qx; q)_\infty}{(-ax, -bx; q)_\infty}, \quad K_t = K_t(a, b; q) = \frac{(q; q)_\infty \theta(-t; q)}{(a, b, -at, -bt; q)_\infty}.
\]

Furthermore, \(\{ \phi_q | \gamma \in -q^N \cup \mathbb{Q}/abt \}\) is an orthogonal basis for the corresponding \(L^2\)-space. Note that the dual orthogonality relations are equivalent to (6.2).

The following \(q\)-integral involving \(q\)-Meixner functions is useful; the proof is given in Appendix A.2.

Proposition 6.1. For \(m \in \mathbb{N}\),

\[
\int_{-1}^{t} \phi_q(xq^m; aq^m, bq^m; q)(xq/t; q)_\infty \phi_q(x; a, b; q) dx =
\]

\[
(1 - q) K_{b} m (aq^m, bq^m; q) (abt \gamma q^m; q)_\infty (-qt; q)_m q^{-m} \varphi_2 ((-bt, -at; q)_\infty) \phi_q(xq^m, -bt, -at; q, \gamma q^m, -qt; q)\]

Limit cases. The \(q\)-Meixner functions can be considered as generalizations of certain \(q\)-Bessel functions. We set \(t = c/a\), \(a = q^{\alpha+1}\), \(b = \beta q^n\) with \(\alpha > -1\), \(c > 0\) and \(n \in \mathbb{N}\), then the limit \(n \to \infty\) of the \(q\)-Meixner function with \(\gamma \in \mathbb{Q}^n/abt\) gives

\[
\lim_{n \to \infty} \phi_{q^{n+1} + \beta q^n; q} = \frac{1}{q^{\alpha+1}} \frac{1}{c} = J_{\alpha, k}(-x; q),
\]

where \(J_{\alpha, k}(-x; q)\) denotes a \(q\)-Bessel function [14 (6.1)]. For \(x \in -q^N\) this is a \(q\)-Laguerre polynomial in \(q^n\). Furthermore, for \(\gamma \in -q^N\) we obtain

\[
\lim_{n \to \infty} \phi_{-q^n}(x; q^{\alpha+1} + \beta q^n; q) = 0.
\]

Taking the limit in the orthogonality relations (6.2) for the \(q\)-Meixner functions find formally

\[
\int_{-1}^{\infty(q^{-1} - \alpha)} J_{\alpha, k}(-x; q) J_{\alpha, l}(-x; q) \frac{(-qx; q)_\infty}{(-xq^{\alpha+1}; q)_\infty} dx =
\]

\[
\delta_{k, l} (1 - q) \frac{(q; q)_\infty}{(q^{\alpha+1}; q)_\infty} \frac{\theta(-q^{\alpha+2}; q)}{\theta(-q^{\alpha+1}; q)} q^{-k} \frac{1}{c} (-q^{\alpha+1}; q)_\infty, \quad k, l \in \mathbb{Z},
\]

\[
\sum_{k \in \mathbb{Z}} J_{\alpha, k}(-x; q) J_{\alpha, k}(-y; q) \frac{q^{k+1} \theta(-q^{\alpha+1}; q)}{(-q^{\alpha+1}; q)_\infty} =
\]

\[
\delta_{x, y} \frac{(q; q)_\infty}{(q^{\alpha+1}; q)_\infty} \frac{\theta(-q^{\alpha+2}; q)}{\theta(-q^{\alpha+1}; q)} |x| (-qx; q)_\infty, \quad x, y \in -q^N \cup cq^\infty - 1 - \alpha,
\]

where we used the \(\theta\)-product identity (1.1) for the right hand side of (6.2). These orthogonality relations are proved by Ciccoli, Koelink and Koornwinder in [1], so there is no need to make the limit transitions rigorous. Note that the self-duality property of the \(q\)-Meixner functions is lost in the limit.
We can take one more limit. We set \( c = q^{-\alpha} \), \( m \in \mathbb{Z} \) and let \( x \in c q^{2-\alpha} - 1 \), then

\[
\lim_{m \to \infty} J_{\alpha,k}^{-m} (-q^{m-\alpha-1}; q) = q^{(\frac{\alpha}{1+\alpha})} \frac{\varphi_1(0, q^{\alpha+1}; q, q^{\alpha+1})}{\varphi_1(0, q^{\alpha+1}; q, q^{\alpha+1})} = J_{\alpha}(q^{\alpha+1}; q),
\]

where \( J_{\alpha} \) is the Hahn-Exton q-Bessel function, or Jackson’s third q-Bessel function. For \( x \in -q^{N} \) we have

\[
\lim_{m \to \infty} J_{\alpha,k}^{-m} (-q^{n}; q) = 0.
\]

The limit of the orthogonality relations becomes

\[
\sum_{k \in \mathbb{Z}} J_{\alpha}(q^{k+m}; q^2) J_{\alpha}(q^{k+n}; q^2) \varphi^{(k+1)} = \delta_{m,n} q^{-n(\alpha+1)} \frac{(q; q^2)^2}{(q^{\alpha+1}; q^2)^2}, \quad m, n \in \mathbb{Z},
\]

which are the q-Hankel orthogonality relations proved by Koorwinder and Swartoutw in [26]. Note that these q-Bessel functions are again self-dual.

**Remark 6.2.** The Al-Salam–Carlitz II functions from [5, 1] and their orthogonality relations can also formally be obtained from the q-Meixner functions. We will not need this in this paper.

### 6.2. Coupling coefficients

We consider the threefold tensor product representations of \( \mathcal{A}_q \) on \( l^2(\mathbb{N})^{\otimes 3} \) given by

\[
\mathcal{T}^{(3)} = (\pi_0 \otimes \pi_0 \otimes \pi_0)(1 \otimes \Delta) \Delta.
\]

We first determine eigenvectors of \( \mathcal{T}^{(3)}(\rho_{r,s}) \) in the same way as in [4, 2]. From [18] we find

\[
(1 \otimes \Delta) \Delta(\rho_{r,s}) =
\frac{1}{2} q^{r^2-\sigma-1} \left( q^{\alpha} \alpha_{r+1,1} \gamma_{r,\infty} \times \Delta(\alpha_{\infty,1} + \beta_{\infty,\sigma}) + q^{\beta_{r+1,1}} \delta_{r,\infty,1} \times \Delta(\gamma_{\infty,1} + \delta_{r,\infty,1})
\right.
\]

\[
+ q^{2}(1 + q^{2}) \rho_{r,\infty} \times \Delta(\rho_{\infty,\sigma}) + q^{2}(1 - q^{2r}) \rho_{r,\infty} \times \Delta(1) + q^{2}(1 - q^{2r}) \times \Delta(1).
\]

We now let \( \mathcal{T}^{(3)}(\rho_{r,s}) \) act on the \( l^2(\mathbb{N})^{\otimes 3} \)-basis \( \{ e^{x \infty}_{x,y,z} \otimes y^{\infty}_{y,z} \mid x \in -q^{2N} \times q^{2r+2N}, y \in -q^{2N} \times q^{2r+2N}, z \in -q^{2N} \times q^{2r+2N} \} \), see Propositions 3.1, 3.2 and 3.3. This gives

\[
\mathcal{T}^{(3)}(\rho_{r,s}) e^{x \infty}_{x,y,z} \otimes y^{\infty}_{y,z} =
\frac{1}{2} \left( (1 + x^{2})(1 - x^{2r}) \right) e^{x \infty}_{x,y,z} \otimes y^{\infty}_{y,z} + q^{2}(1 + q^{2}) \left. \right.
\]

\[
\frac{1}{2} \left( (1 + x^{2})(1 - x^{2r}) \right) e^{x \infty}_{x,y,z} \otimes y^{\infty}_{y,z} + q^{2}(1 + q^{2}) \left. \right.
\]

This can be matched to the three-term recurrence relation for certain continuous dual q^2-Hahn polynomials, and then similar as in [18] we find the generalized eigenvectors

\[
P_{r,s,x,y,z} = \sum_{n \in \mathbb{N}} C_{r,s,x,y,z}^{x,y,z} e^{x \infty}_{x,y,z} \otimes y^{\infty}_{y,z}, \quad \lambda = 1, q^{2r},
\]

for eigenvalue \( \mu_{\lambda} \).

Next we determine another generalized eigenvector of \( \mathcal{T}^{(3)}(\rho_{r,s}) \) using the cocommutativity of \( \Delta \), i.e., \( (1 \otimes \Delta) \Delta = (\Delta \otimes 1) \Delta \). From [18] we find

\[
(1 \otimes \Delta) \Delta(\rho_{r,s}) =
\frac{1}{2} q^{r^2-\sigma-1} \left( q^{\alpha} \alpha_{r+1,1} \gamma_{r,\infty} \times \Delta(\alpha_{\infty,1} + \beta_{\infty,\sigma}) + q^{\beta_{r+1,1}} \delta_{r,\infty,1} \times \Delta(\gamma_{\infty,1} + \delta_{r,\infty,1})
\right.
\]

\[
+ q^{2}(1 + q^{2}) \rho_{r,\infty} \times \Delta(\rho_{\infty,\sigma}) + q^{2}(1 - q^{2r}) \Delta(\rho_{r,\infty}) \times \Delta(1) + q^{2}(1 - q^{2r}) \times \Delta(1).
\]
Now we let $\mathcal{T}^{(3)}(\rho_{\tau,\sigma})$ act on the $\ell^2(\mathbb{N})^\otimes 3$-basis vectors $V_{x,y}^{\tau,\omega} \otimes v_{z}^{\infty,\sigma}$,

$$
\mathcal{T}^{(3)}(\rho_{\tau,\sigma}) V_{x,y}^{\tau,\omega} \otimes v_{z}^{\infty,\sigma} = \frac{1}{2} \left[ x q^{1-\tau-\sigma} (1 + q^2) + z q^{1-\tau-\sigma} (1 - q^{2\sigma}) + z q^{1-\tau-\sigma} (1 - q^{2\tau}) \right] V_{x,y}^{\tau,\omega} \otimes v_{z}^{\infty,\sigma}
$$

$$
+ \frac{1}{2} \sqrt{1 + x} (1 - x q^{-2\tau}) (1 + z) (1 - q^{2\omega}) \right] V_{x/q, y/q}^{\tau,\omega} \otimes v_{z/q}^{\infty,\sigma}
$$

$$
+ \frac{1}{2} \sqrt{1 + x q^2} (1 - x q^{-2\omega}) (1 + z) (1 - q^{2\tau}) \right] V_{x q^2, y q^2}^{\tau,\omega} \otimes v_{z q^2}^{\infty,\sigma},
$$

and then we obtain the generalized eigenvector

$$
G_{u,x;y}^{\tau,\omega;\sigma;k} = \sum_{k \in \mathbb{N}} c_{u,\xi q^2k}^{x,y} V_{u,x,y}^{\tau,\omega} \otimes v_{z}^{\infty,\sigma},
$$

for eigenvalue $\mu_u$ (compare with Remark 6.3).

We define $R_{u,x;y}^{\tau,\omega;\sigma;k}$ as the $2 \times 2$-matrix-valued coupling coefficients between the generalized eigenvectors;

$$
\begin{align*}
(F_{u,x;y}^{\tau,\omega;\sigma;k}) = & \sum_{z,v} R_{u,x;y}^{\tau,\omega;\sigma;k} \left( G_{u,z}^{\tau,\omega;\sigma;k} \right), \\
R_{u,x;y}^{\tau,\omega;\sigma;k} = & \left( (R_{u,x;y}^{\tau,\omega;\sigma;k})_{\lambda,\xi} \right)_{\lambda \in \{-1, q^2\}, \xi \in \{-1, q^2\}}.
\end{align*}
$$

The coupling coefficients satisfy the orthogonality relations

$$
\sum_{z,v} (R_{u,x;y}^{\tau,\omega;\sigma;k})^* R_{u,x;y}^{\tau,\omega;\sigma;k} = \delta_{z_1, z_2} \delta_{y_1, y_2} I,
$$

$$
\sum_{z,v \in -q^{2\omega} + q^{2\tau} + q^2} R_{u,x;y}^{\tau,\omega;\sigma;k} (R_{u,x;y}^{\tau,\omega;\sigma;k})^* = \delta_{x_1, x_2} \delta_{y_1, y_2} I,
$$

where $I$ is the $2 \times 2$ identity matrix, and $A^*$ denotes the complex transpose of $A$. Furthermore, since $(x, y, z, v, \sigma, \tau) \rightarrow (z, v, x, y, \tau, \sigma)$ interchanges $F_{u,x;y}^{\tau,\omega;\sigma;k}$ with $G_{u,z}^{\tau,\omega;\sigma;k}$, we have the symmetry property

$$
(R_{u,x;y}^{\tau,\omega;\sigma;k})_{\lambda,\xi} = (R_{u,x;y}^{\tau,\omega;\sigma;k})_{\xi,\lambda}.
$$

Because of this symmetry property the two orthogonality relations (6.6) are equivalent.

We use (6.4) to determine an explicit expression for $(R_{u,x;y}^{\tau,\omega;\sigma;k})_{\lambda,\xi}$. First a preliminary result.

**Lemma 6.3.** The coupling coefficients satisfy

$$
(R_{u,x;y}^{\tau,\omega;\sigma;k})_{\lambda,\xi} = \delta_{y, \xi q^2 r} \delta_{v, \lambda q^2 r} R_{u,r,x;y}^{\tau,\omega;\lambda,\xi},
$$

where $R_{u,r,x;y}^{\tau,\omega;\lambda,\xi}$ is determined by

$$
\sum_{z,v \in -q^{2\omega} + q^{2\tau} + q^2} R_{u,z;x,y}^{\tau,\omega;\lambda,\xi} c_{u,\xi q^2 k}^{x,y} c_{z,y}^{\tau,\omega;\lambda,\xi} = \delta_{\lambda, \nu} \lambda q^{2n},
$$

for $k, m, n \in \mathbb{N}$.

**Proof.** We have

$$
(F_{u,x;y}^{\tau,\omega;\sigma;k})_{\lambda,\xi} = \delta_{y, \xi q^2 r} \delta_{v, \lambda q^2 r} c_{u,\xi q^2 k}^{x+y,\lambda q^{2n}+m},
$$

and

$$
(F_{u,z;x}^{\tau,\omega;\lambda,\xi})_{\lambda,\xi} = \delta_{\lambda, \nu} \lambda q^{2n},
$$

Now taking the inner product of (6.5) with basis vectors $c_{u,\xi q^2 k}^{x+y,\lambda q^{2n}+m} \otimes \epsilon_m \otimes \xi q^2 k$ gives the result. □
Note that by (6.8), for fixed $y, v, \xi, \lambda$, only one of the four matrix coefficients of $R_{\alpha, \beta, \gamma}^{n}$ is nonzero. We are now ready to show that the coefficients $R_{\alpha, \beta, \gamma}^{n}$ are orthonormal Meixner functions.

**Theorem 6.4.** Let $u \in \mathbb{T}$, $\lambda \in \{1, q^{2}\}$, $\xi \in \{-1, q^{2}\}$, $\tau \in \mathbb{Z}$, $x \in -q^{2\tau} \cup q^{2\tau+2\tau}$ and $z \in -q^{2\tau} \cup q^{2\tau+2\tau}$ and define $x' = \xi x q^{2\tau-2\sigma}$ and $z' = \lambda z q^{2\tau-2\tau}$. Then for $x' \in -q^{2\tau} \cup q^{2\tau+2\tau}$ and $z' \in -q^{2\tau} \cup q^{2\tau+2\tau}$ we have

$$R_{\alpha, \beta, \gamma}^{n} = \text{sgn}(\lambda \xi)\sqrt{\frac{|x' z'| w(z'; a, b; q^{2}) w(x'; a, b; q^{2})}{K_{\alpha}(a, b; q^{2}) K_{\beta}/a \kappa(a, b; q^{2})}} \phi_{x'}(z'; a, b; q^{2}),$$

with

$$(a, b, t) = (-q^{1+\sigma+\tau-2\tau} u / \lambda \xi, -q^{1+\sigma+\tau-2\tau} / u \lambda \xi, \lambda q^{2\tau-2\tau}).$$

For $x' \in -q^{2\tau} \cup q^{2\tau+2\tau}$, we have $R_{\alpha, \beta, \gamma}^{n} = 0$.

**Proof.** We need to verify that (6.9) is satisfied, and we need to verify the orthogonality relations (6.10) for $R_{\alpha, \beta, \gamma}^{n}$. The orthogonality relations follow directly from the orthogonality relations (6.2) for the $q$-Meixner functions.

Since the coupling coefficients are independent of $k$ and $m$, it is enough to show that (6.10) holds for $k = m = 0$ (so $r = n$), i.e.

$$\begin{align*}
\left(\begin{array}{c}
q^{2\tau}(q^{2}, -\xi q^{2} q^{2-2\sigma}, -\lambda q^{2+2n}, \lambda q^{2+2n-2\tau} ; q^{2})_{\infty} \\
(-\xi q^{1-\tau} u \pm 1 / q^{2}, \xi q^{1-\tau-\sigma} u \pm 1 / q^{2})_{\infty} \theta(-q^{2\tau} ; q^{2})
\end{array}\right)_{2}^{+} \\
\left(\begin{array}{c}
q^{2\tau} \left(q^{2} q^{2} q^{2-2\sigma}, -\lambda q^{2+2n}, \lambda q^{2+2n-2\tau} ; q^{2})_{\infty} \\
(-\xi q^{1-\tau} u \pm 1 / q^{2}, \xi q^{1-\tau-\sigma} u \pm 1 / q^{2})_{\infty} \theta(-q^{2\tau} ; q^{2})
\end{array}\right)_{2}^{+}
\end{align*}
\times \sum_{z \in -q^{2\tau} \cup q^{2\tau+2\tau}} R_{\alpha, \beta, \gamma}^{n} \frac{|z|^{2} (z q^{2\tau} q^{2}, -z q^{2} q^{2})_{\infty}}{(z q^{1+\tau-\sigma} u \pm 1 / q^{2}, -q^{2} q^{2})_{\infty} \theta(-q^{2\tau} ; q^{2})}
\times \frac{\left(q^{2} q^{2} q^{2-2\sigma}, -\lambda q^{2+2n}, \lambda q^{2+2n-2\tau} ; q^{2})_{\infty} \theta(-q^{2\tau} ; q^{2})}{\left(q^{1+\tau-\sigma} u \pm 1 / q^{2}, -q^{2} q^{2})_{\infty} \theta(-q^{2\tau} ; q^{2})}
\times (z q^{2\tau} q^{2}, -z q^{2} q^{2})_{\infty} \lambda q^{2-2\tau} q^{2}, q^{2})_{\infty} \lambda q^{2-2\tau} q^{2}, q^{2})_{\infty}
\end{align*}$$

We will use Proposition 6.1. First note that the sum over $z$ in (6.10) can be written as a $q$-integral of the form $f_{1}^{+} f(z) d_{q} z$. The summand can be expressed in terms of the $q$-Meixner weight function $w_{\alpha, \beta, \gamma}$ using

$$\frac{(z q^{2\tau} q^{2}, -z q^{2} q^{2})_{\infty}}{(z q^{1+\tau-\sigma} u \pm 1 / q^{2}, -q^{2} q^{2})_{\infty} \theta(-q^{2\tau} ; q^{2})} = \frac{w(\lambda q^{2\tau} q^{2}, -z q^{2} q^{2})_{\infty}}{w(\lambda q^{2\tau} q^{2}, -z q^{2} q^{2})_{\infty} \theta(-q^{2\tau} ; q^{2})}.$$
with \( t = \lambda^2 q^{-2r} \). For \( \xi = q^{2r} \) this is also true, but we need to write the \( 3\varphi_2 \)-function in (6.10) as

\[
3\varphi_2(q^{-2n}, -\lambda q^{1-r-s}u; -\lambda q^{1-r-s}/u; q^2, q^2) = -\lambda^2 q^{-2r} \varphi(q^{2n}; q^2) \\
\]

which follows from the \( 3\varphi_2 \)-transformation [11 (III.11)]. For \( \lambda = q^{2r} \) it now is a straightforward calculation using Proposition 6.1 to show that (6.10), hence also (6.9), is satisfied with the expression for \( R_{\alpha, \beta, \gamma, \delta} \) given in the theorem. For \( \lambda = -1 \) the calculation is the same after application of the substitution rule for \( q \)-integrals,

\[
\int_{-1}^{z} f(-zq^{-2r}) dq = q^{2r} \int_{-1}^{q^{-2r}} f(z) dz.
\]

**Remark 6.5.** Note that the symmetry property (6.7) for the coupling coefficient \( s \) corresponds to (4.1), big \( \lambda \)-integrals, big \( \gamma \)-integrals, big \( \tau \)-integrals, by writing it in terms of the orthogonal polynomials/functions, and from careful bookkeeping. We also use the identity

\[
\int_{-\xi u + 1}^{\xi u + 1} \frac{d\xi}{\sqrt{q^2 - \lambda u^2}} = \frac{\sqrt{q^2 - \lambda u^2}}{\lambda u}.
\]

Taking the limits \( s \rightarrow 0 \) and \( s, t \rightarrow 0 \), which corresponds to \( \rho_{r, \sigma} \rightarrow \rho_{r, \infty} \) and \( \rho_{r, \sigma} \rightarrow -\gamma \gamma^* \) in the algebra \( A_q \), we obtain the following results. The functions involved are big \( q \)-Laguerre polynomials \( L_n(q) \), big \( q \)-Meixner functions \( \phi_n(q) \), and big \( q \)-Bessel functions (6.3)

**Theorem 6.6.** Let \( u \in T, k, m, n \in \mathbb{N}, s, t > 0, x \in -q^{2n} \cup sq^{N-n}, \xi \in \{-1, s\}, \lambda \in \{-1, t\}, \) and set \( r = k - m + n, \) then

\[
\int_{-q^{-k}}^{q^{-k}} p_k(\mu; -\xi \sqrt{q^2 - \lambda u^2}) L_m(-\lambda q^{1+r}/t; -\lambda q^{n-m}, \lambda q^{n-m}/t; q) \\
\times \phi_{\xi q^{r}/s}(\lambda q^{r}/t; -\lambda q^{n-m}; \lambda q^{n-m}/t; q) \infty \frac{d\xi}{\sqrt{q^2 - \lambda u^2}} \\
= (1 - q) (-\xi u + 1; \lambda q^{n-m}, \lambda q^{n-m}/t; q) L_m(-\lambda q^{1+r}/s; -\lambda q^{n-m}, \lambda q^{n-m}/s; q).
\]

**Proof.** This follows from identity (6.9), with \( (q^{2r}, q^{2r}) \rightarrow (s, t) \) and \( q^2 \rightarrow q \), by writing it in terms of the orthogonal polynomials/functions, and from careful bookkeeping. We also use the identity

\[
\int_{-\xi u + 1}^{\xi u + 1} \frac{d\xi}{\sqrt{q^2 - \lambda u^2}} = \frac{\sqrt{q^2 - \lambda u^2}}{\lambda u}.
\]

**Corollary 6.7.** Let \( k, m, n, u \in \mathbb{N}, x \in \mathbb{Z} \) and \( t > 0 \). The following identities hold:

\[
\int_{-1}^{t} L_k(zq^{1+k+n}/t - z, z/t; q) L_m(zq^{1+k-m+n}/t; q^{n-m}, -q^{n-m}/t; q) \\
\times \phi_{\xi q^{r}/s}(\lambda q^{r}/t; -zq^{k-m-n}/t; q) \frac{d\xi}{\sqrt{q^2 - \lambda u^2}} \\
= (1 - q) t^{m-1} q^{m(m-1)-mn-m+n+k} \frac{d\xi}{\sqrt{q^2 - \lambda u^2}} \\
\times \phi_{\xi q^{r}/s}(\lambda q^{r}/t; -zq^{k-m-n}/t; q) L_m(-zq^{n-m+1}/t; q^{n-m}, -q^{n-m)/(k-m)} \\
\times L_n(-zq^{n-m}/t; -1/t; q^{n-m+1}/t; q^{n-m} - q^{n-m+k-n+1}/t; q) \infty \frac{d\xi}{\sqrt{q^2 - \lambda u^2}}.
\]
and
\[ \sum_{z \in \mathbb{N}} q^{(u-k+1)(m-k+n+u+1)} \frac{(q^m; q)_\infty}{(q; q)_\infty} J_{m-k+n+u}(q^{x+z}; q)p_k(q^u; q^z)p_n(q^{k+z}; q^{n-m}; q) = q^{x(k-m)} \frac{(q^{k+m}; q)_m}{(q^{n-m}; q)_m} J_{m-k+n+u}(q^{x+z}; q)p_k(q^u; q^z)p_n(q^{n+z}; q^{k-m}; q). \]

**Proof.** For the first identity substitute \((u, x, \xi, \lambda) \mapsto (-q^u \sqrt{q}/s, -q^z, -1, -1)\) in Theorem 6.6 and let \(s \to 0\). For the second identity we also substitute \(z \mapsto -q^z\), and let \(t \to 0\). \(\square\)

**Remark 6.8.**

(i) Corollary 6.7 shows that the \((\tau, \infty)\)-coupling coefficients, respectively the \((\infty, \infty)\)-coupling coefficients, can be expressed in terms of big \(q\)-Bessel functions, respectively Hahn-Exton \(q\)-Bessel functions.

(ii) The second identity in Corollary 6.7 is recently obtained in [8] by De Commer and Koelink in a different quantum group setting; they use the quantum linking groupoid between quantum SU(2) and quantum E(2). They consider the identity as a \(q\)-analog of Erdélyi’s identity [9] which roughly says that the Hankel transform maps a product of two Laguerre polynomials to the product of two Laguerre polynomials. The first identity in Corollary 6.7 can be considered as another \(q\)-analog of Erdélyi’s identity.

**Appendix A. \(q\)-Integral evaluations**

**A.1. Proof of Proposition 5.1.** We prove the following result. For \(n \in \mathbb{N},\)

\[
\frac{1}{1-q} \int_{z_-}^{z_+} P_n(x) \frac{(q/z, q/x; q)_\infty}{(x, dq; q)_\infty} \, dq \, x
\]

\[
= z_+ (-d)^n q^{-\frac{1}{2}n(n-1)} (q, q dq, z q^n; q)_\infty \vartheta(z_-/z_+; q) \frac{(q^{-n}, dz_+; q^n, q)}{(q^{-n}/cz_-, q^-; dz_+; q)_\infty} 2 \varphi_1 \left( \begin{array}{c} q^{-n-1}/cz_-; q, q \n \end{array} \right) \]

\[
= z_+ (-d)^n q^{-\frac{1}{2}n(n-1)} (q, q dq, z q^n; q)_\infty \vartheta(z_-/z_+; q) \frac{(q^{-n}, dz_+; q^n, q)}{(q^{-n}/cz_-, q^-; dz_+; q)_\infty} 2 \varphi_1 \left( \begin{array}{c} q^{-n-1}/cz_-; q, q \n \end{array} \right),
\]

and

\[
\frac{1}{1-q} \int_{z_-}^{z_+} Q_n(x) \frac{(q/z, q/x; q)_\infty}{(x, dq; q)_\infty} \, dq \, x
\]

\[
= z_+ (c/q)^n q^{-\frac{1}{2}n(n-1)} (q, q dq, c q^n; q)_\infty \vartheta(z_-/z_+; q) \frac{(q^{-n}, q/pcz_+; q,cz_-, q^{-n})}{(q^{-n}/cz_-, q^-; cz_+; q)_\infty} 2 \varphi_1 \left( \begin{array}{c} q^{-n}, q/pcz_+; q,cz_-, q^{-n} 
\end{array} \right) \]

\[
= z_+ (c/q)^n q^{-\frac{1}{2}n(n-1)} (q, q dq, c q^n; q)_\infty \vartheta(z_-/z_+; q) \frac{(q^{-n}, q/pcz_+; q,cz_-, q^{-n})}{(q^{-n}/cz_-, q^-; cz_+; q)_\infty} 2 \varphi_1 \left( \begin{array}{c} q^{-n}, q/pcz_+; q,cz_-, q^{-n} \n \end{array} \right).
\]

Furthermore, for \(n \in -\mathbb{N}_{\geq 1},\)

\[
\int_{z_-}^{z_+} Q_n(x) \frac{(q/z, q/x; q)_\infty}{(x, dq; q)_\infty} \, dq \, x = 0.
\]

Here \(P_n\) and \(Q_n\) are defined by (5.1) and (5.2).

The functions \(P_n\) and \(Q_n\) are essentially two instances of the same function. Indeed, we define

\[
\psi_\gamma(x; c, d) = (cq \gamma x, dx; q)_\infty \vartheta_1 \left( \begin{array}{c} q^{-n}, cq \gamma x 
\end{array} \right),
\]

and

\[
\varphi_\gamma(x) = \frac{\vartheta(cz_+, dx; q)}{\vartheta(d/c; q)} \psi_\gamma(x; c, d) + (c \leftrightarrow d),
\]

(A.4)
where \((c \leftrightarrow d)\) means that the preceding expression is repeated, but with \(c\) and \(d\) interchanged. Now \(P_n\) and \(Q_n\) are related to \(\varphi_\gamma\) by

\[
\varphi_\gamma(x) = \begin{cases} 
(cdz_+)^{-n}(q^{n+1};q)_\infty \theta(cz_+,dz_+;q) P_n(x), & \text{if } \gamma = q^n, \\
(z_-)^n(q^{n+2}/cdz_-z_+;q)_\infty Q_n(x), & \text{if } \gamma = \frac{q^{n+1}}{cdz_-z_+},
\end{cases}
\]

(\(A.5\))

see [12, \[3.4\].]

To prove identities \([A.1],[A.2]\) and \([A.3]\) we evaluate the \(q\)-integral

\[
J_\gamma(c,d) = \int_{z_-}^{z_+} \varphi_\gamma(x) \left(\frac{q^{x/z_-}q^{x/z_+};q}_\infty \right) \frac{(ex,dx;q)_\infty}{(cx,dx;q)_\infty} d_qx.
\]

We will use the evaluation formula [11, (2.10.20)]

\[
\int_A^B \frac{(qx/A,qx/B;q)_\infty}{(Cx,Dx;q)_\infty} dq = (1-q)^2 \frac{(q,ABCD;q)_\infty\theta(A/B;q)}{(AC,AD,BC,BD;q)_\infty},
\]

which is the nonterminating \(q\)-Vandermonde sum written as a \(q\)-integral. By \([A.4]\) the \(q\)-integral \(J_\gamma\) splits as

\[
J_\gamma(c,d) = \frac{\theta(cz_+,dz_+;q)}{\theta(d/c;q)} I_\gamma(c,d) + (c \leftrightarrow d),
\]

where

\[
I_\gamma(c,d) = \int_{z_-}^{z_+} \psi_\gamma(x;c,d) \left(\frac{q^{x/z_-}q^{x/z_+};q}_\infty \right) \frac{(ex,dx;q)_\infty}{(cx,dx;q)_\infty} d_qx.
\]

To evaluate \(I_\gamma\) we write \(\psi_\gamma\) as

\[
\psi_\gamma(x;c,d) = (dx,q,q,cq/d;q)_\infty \psi_1 \left(\frac{0,ex}{cq/d};q,\gamma\right), \quad |\gamma| < q^{-1},
\]

see [12, (2.3)]. We interchange the order of summation and \(q\)-integration, which is allowed by absolute convergence, then from \([A.6]\) we obtain

\[
I_\gamma(c,d) = (q,\gamma,cq/d;d)_\infty \sum_{n=0}^{\infty} \frac{(q)_n^{(q\gamma)^n}}{(q,q,q/d)_n} \int_{z_-}^{z_+} \frac{(q^{x/z_-}q^{x/z_+};q)_\infty}{(q^{\gamma}x;q)_\infty} d_qx
\]

(\(A.7\))

\[
= (1-q)^n (q,q,q,cq/d;q)_\infty \theta(z_-/z_+;q) \frac{(cz_-,cz_+;q)_\infty}{(cq/d)q,q}\psi_1 \left(\frac{cz_-,cz_+}{cq/d};q,q\right),
\]

provided \(|\gamma| < q^{-1}\). So we have

\[
J_\gamma(c,d) = K_\gamma \left(\frac{q^{cz_+}}{cz_+,d/c;\gamma}_\infty \right) \frac{\theta(dz_+\gamma;q)}{(cz_-,d/c;\gamma)_\infty} \psi_1 \left(\frac{cz_-,cz_+}{cq/d};q,q\gamma\right) + (c \leftrightarrow d), \quad |\gamma| < q^{-1},
\]

(\(A.8\))

where \(K_\gamma = (1-q)\theta(z_-/z_+;q)\). For any \(x \in z_- - z_+^2\) the function \(\gamma \mapsto \psi_\gamma(x;c,d)\) is an entire function, so \(\gamma \mapsto \psi_\gamma(x)\) is analytic on \(C \setminus \{0\}\). Therefore, by absolute convergence \(\gamma \mapsto J_\gamma(c,d)\) is also analytic on \(C \setminus \{0\}\). An analytic continuation of the right hand side of \((A.8)\) in \(\gamma\) can be obtained by applying the three-term transformation formula for \(2\psi_1\)-functions [11, (III.32)] with parameters \((A,B,C,Z) = (cz_-,dz_-z_+,q/cd/dz_+\gamma)\), which gives us

\[
J_\gamma(c,d) = K_\gamma \left(\frac{q^{dz_+}}{dz_+,dz_-;\gamma}_\infty \right) \frac{\theta(dz_-z_+\gamma;q)}{(cz_-,d/c;\gamma)_\infty} \psi_1 \left(\frac{cz_-,dz_-}{q,cz_-z_+\gamma};q,q\right), \quad |\gamma| > \frac{q}{|cz_--z_+|}.
\]

(\(A.9\))

Applying Heine's transformation [11, (III.2)] in \((A.9)\) gives

\[
J_\gamma(c,d) = K_\gamma \left(\frac{q^{dz_+}}{dz_+,dz_-;\gamma}_\infty \right) \frac{\theta(dz_-z_+\gamma;q)}{(cz_-,d/c;\gamma)_\infty} \psi_1 \left(\frac{1/\gamma,dz_-}{q,cz_-;\gamma};q,q\right), \quad \frac{|d/dz_+|}{|dz_+|} < 1.
\]

Setting \(\gamma = q^n\), we see that the \(2\psi_1\)-series terminates, so that the condition \(|q/dz_+| < 1\) is not needed. Using \([A.5]\) this gives us the second identity in \([A.1]\). The first one is obtained by interchanging \(z_-\) and \(z_+\), or, alternatively, reversing the order of summation in the \(2\psi_1\)-series and using \(c \leftrightarrow d\) symmetry.
The results involving $Q_n$ follow from applying Heine’s transformation [11 (III.1)] in (A.9);

$$J_\gamma(c,d) = K_\gamma(z_-, z_+) \frac{(cdz_++z\gamma, q/dz_+; q)_{\infty}}{(ez_-; q)_{\infty}} \phantom{|} 2\varphi_1 \left( \frac{q/dz_+, q/cdz_++z\gamma; q, dz_-}{q/dz_+; q} \right), \quad |dz_-| < 1.$$

We set $\gamma = q^{n+1}/cdz_++z$ with $n \in \mathbb{Z}$. In case $n \in \mathbb{N}$, the $2\varphi_1$-series is a finite sum, and then the condition $|dz_-| < 1$ is not needed. Using (A.5) we obtain the first expressions in (A.3).

Interchanging $z_-$ and $z_+$ gives the other expression. In case $n \in -\mathbb{N}_0$, $J_\gamma(c,d)$ vanishes, since $(cdz_++z\gamma; q)_{\infty} = 0$, which proves (A.3).

A.2. Proof of Proposition 6.1. We prove

$$\int_{-1}^{t} \varphi_1(xq^m; aq^{-m}, bq^{-m}; q) \frac{(-xq, xq/t; q)_{\infty}}{(-ax, -bx; q)_{\infty}} dx = \frac{(1 - q)t (q, abtq^{-m}; q)_{\infty} \theta(-1/t; q)}{aq^{-m}; b, -at, -bt; q)_{\infty} 2\varphi_2 \left( \frac{q^{-m}, -bt, -at}{aq^{-m}, -qt; q} \right),$$

where $\phi_\gamma$ denotes the $q$-Meixner function (6.1).

We write the $q$-Meixner function as a $2\varphi_1$-series;

$$\phi_\gamma(xq^m; aq^{-m}, bq^{-m}; q) = \frac{(-a\gamma q^{-m}; q)_{\infty}}{(aq^{-m}; q)_{\infty}} 2\varphi_1 \left( -\frac{1}{\gamma}, -bx \frac{aq^{-m}; q, a\gamma q^{-m}}{bq^{-m}; q} \right), \quad |a\gamma q^{-m}| < 1.$$

By interchanging the order of summation and $q$-integration, and using (A.6), we obtain

$$\int_{-1}^{t} \varphi_1(xq^m; aq^{-m}, bq^{-m}; q) \frac{(-xq, xq/t; q)_{\infty}}{(-ax, -bxq^n; q)_{\infty}} dx = \sum_{n=0}^{\infty} \frac{(-1/\gamma; q)_n(-a\gamma q^{-m})^n}{(q, bq^{-m}; q)_n} \int_{-1}^{t} \frac{(-xq, xq/t; q)_{\infty}}{(-ax, -bxq^n; q)_{\infty}} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1/\gamma; q)_n(-a\gamma q^{-m})^n}{(aq^{-m}q; q)_n} \frac{(1 - q)t (q, abtq^n; q)_{\infty} \theta(-1/t; q)}{(a, bq^{-m}; -at, -btq^n; q)_{\infty}}$$

$$= (1 - q) \frac{(q, -a\gamma q^{-m}; -abtq^n; q)_{\infty} \theta(-1/t; q)}{(a, aq^{-m}; b, -at, -bt; q)_{\infty}} 2\varphi_2 \left( -\frac{1}{\gamma}, -bt \frac{aq^{-m}, b, -at, -bt; q, a\gamma q^{-m}}{bq^{-m}, -abt; q, -a\gamma q^{-m}} \right).$$

We apply transformation formulas [11 (III.9)] with parameters $(A, B, C, D, E) = (-bt, b, -1/\gamma, bq^{-m}, -abt)$, and [11 (III.13)] with $(B, C, D, E) = (-bt, bq^{-m}, -abt, bq^{-m})$, then we find

$$\phi_\gamma(xq^m; aq^{-m}, bq^{-m}; q)_{\infty} 2\varphi_2 \left( \frac{q^{-m}, -bt, -abtq^{-m}}{-abt, -a\gamma q^{-m}; q} \right) = \frac{(a, abtq^{-m}; q)_{\infty} \theta(-1/t; q)}{(bt, -a\gamma q^{-m}; q)_{\infty}} 2\varphi_2 \left( \frac{q^{-m}, -bt, -abtq^{-m}}{-abt, -a\gamma q^{-m}; q} \right).$$

This proves the result for $|a\gamma q^{-m}| < 1$. By analytic continuation it holds for all $\gamma$.

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Technische Universiteit Delft, DIAM, PO Box 5031, 2600 GA Delft, the Netherlands
E-mail address: w.g.m.groenevelt@tudelft.nl