The Word Problem for the Singular Braid Monoid

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Abstract

We give a solution to the word problem for the singular braid monoid \( SB_n \). The complexity of the algorithm is quadratic in the product of the word length and the number of the singular generators in the word. Furthermore we algebraically reprove a result of Fenn, Keyman and Rourke that the monoid embeds into a group and we compute the cohomological dimension of this group.

1 Introduction

Back in the 20th Emil Artin introduced the braid group [Art25]. He gave a presentation and already showed how to solve the word problem for this group.

When the theory of Vassiliev knot invariants started in the early 90ths, it became also interesting - both from the point of view of mathematics as of physics - to look at singular braids (see e.g. [Bir93, Bae92, Hut98, FRZ96, FKR96, Ver98]), where transversal self-intersections are allowed. These singular braids form a monoid.

While for the word problem in the braid group many different solutions are known ([Gar69, ECH+92, BKL, FGR+98]), for the singular braid monoid such an algorithm was not known and it seems very difficult to extend one of the solutions of the word problem in the braid group to the singular braid monoid.

The aim of this paper is to give an algorithm solving the word problem in the singular braid monoid and - as Artin did for the braid group - to give informations on the algebraical structure of the singular braid monoid. This will be done by using traditional algebraic tools, such as properties of HNN-extensions of groups.

We proceed as follows. As proved by Fenn, Keyman and Rourke [FKR96] the singular braid monoid \( SB_n \) embeds into a group \( SG_n \). Since the proof given there involved some geometrical arguments which do not seem to have generalizations for much more general settings, for example for other “singular Artin groups”, in the course of this text we will give a group theoretical proof of it.

This embedding theorem allows us to use the tools of classical group theory for the solution of the word problem. We will work out the structure of a certain subgroup of finite index in \( SG_n \) as an iterated HNN-extension - with some nice properties - of a subgroup of the braid

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group. Britton’s lemma together with the known solution to the word problem for the braid group now allows us to give a solution to the word problem for the singular braid monoid.

Along the path of our proof we can give some information about the group $SG^n$. For example we will compute its cohomological dimension to be $n - 1 + \lfloor n/2 \rfloor$. For the easiest case $SG^3$ we will construct a $K(SG^3, 1)$–space and compute the homology of the group.

In Section 9 we will give some technical but necessary proofs. The trustful reader can skip this section.

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## 2 The singular braid monoid

The theory of Vassiliev invariants made it interesting to investigate knotted objects having a finite number of transversal self-intersections. As such a generalization of the braid group $B^n$ we get the singular braid monoid $SB^n$ generated by the elementary singular braids $\sigma_1, \ldots, \sigma_{n-1}$ and $\tau_1, \ldots, \tau_{n-1}$ depicted in Figure 1.

![Figure 1: $\sigma_i, \sigma_i^{-1}$ and $\tau_i$](image)

Joan Birman [Bir93] and independently John Baez [Bae92] gave a presentation for the singular braid monoid. The generators $\tau_j, j > 1$, can be expressed in terms of $\sigma_1, \ldots, \sigma_{n-1}$ and $\tau_1$, and one can show that the monoid presentation is equivalent to the following presentation:

**Proposition 2.1 ([DG98])** The monoid $SB^n$ is generated by the elements

$\sigma_i^{\pm 1}, i = 1, \ldots, n - 1, \text{ and } \tau_1$

satisfying the following relations:

$$\sigma_i \sigma_i^{-1} = 1 \text{ for all } i$$

(1)

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ and } \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } j > i + 1$$

(2)

$$\sigma_2 \sigma_2^2 \sigma_2 \tau_1 = \tau_1 \sigma_2^2 \sigma_2$$

(3)

$$\sigma_i \tau_1 = \tau_1 \sigma_i \text{ for } i \neq 2$$

(4)

$$\sigma_2 \sigma_3 \sigma_1 \sigma_2 \tau_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \tau_1 = \tau_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \tau_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \text{ for } n > 3.$$  

(5)

It would be very arduous to work with a semigroup so in the course of this article we will heavily make use of the following embedding theorem:

**Theorem 2.2 ([FKR96])** The singular braid monoid $SB^n$ embeds into a group that will be denoted by $SG^n$. That means $SG^n$ is the group that we get by regarding the presentation of $SB^n$ as a group presentation.

Since the original proof of this theorem involved geometrical considerations and since the theorem lies on our road, in Section 8 we will give an algebraic proof of it.

For later use we will need the following theorem:
Theorem 2.3 ([FRZ96]) For a braid $\beta \in B_n$ the following are equivalent:

(i) $\sigma_j \beta = \beta \sigma_k$

(ii) $\sigma^r_j \beta = \beta \sigma^r_k$ for some nonzero integer $r$

(iii) $\tau_j \beta = \beta \tau_k$

(iv) $\tau^r_j \beta = \beta \tau^r_k$ for some positive integer $r$.

Actually, we only need a lemma that is proved in [FRZ96] as an application. Let $\eta$ be the homomorphisms

$$\eta: SB_n \longrightarrow \mathbb{Z}B_n$$

of the singular braid monoid into the integral group ring of the braid group, that is induced by the map: $\tau_i \mapsto \sigma_i - \sigma_i^{-1}$ and $\sigma_i \mapsto \sigma_i$.

We have:

Lemma 2.4 ([FRZ96]) The homomorphism $\eta$ is injective for the subset $SB_n^{(2)}$ of $SB_n$ of all singular braids having at most two singularities.

Remark The more general conjecture of Joan Birman [Bir93], that the homomorphism $\eta$ is injective is still open. For a discussion of this problem see [FRZ96], [JJ], [Zhu97] and [DG98].

3 HNN-extensions of groups

Our main tool is the concept of HNN-extensions of groups (see e.g. [LS77] or [MKS76]). Let $H = \langle S \mid \text{rel. } H \rangle$ be a group with a set of generators $S$ and relations rel. $H$ and $U$ and $V$ two isomorphic subgroups of $H$ together with an isomorphism $\Phi$.

The HNN-extension $G$ of $H$ relative to $U$ and $V$ is

$$G \cong \langle H, t \mid \text{rel. } H, tut^{-1} = \Phi(u), u \in U \rangle.$$ 

The element $t$ is called stable letter and $H$ is the base group. In our cases $\Phi$ is always the identity, so from now on we only consider such HNN-extensions. By the classical result of Higman, Neumann and Neumann the group $H$ is embedded into $G$, that means the subgroup of $G$ that is generated by the elements of $H$ is isomorphic to $H$.

Central for our solution to the word problem for the singular braid monoid is the following beautiful result of Britton, often quoted as Britton’s lemma:

Lemma 3.1 ([Bri63]) Let $H = \langle S \mid \text{rel. } H \rangle$ be a presentation of the group $H$ with a set of generators $S$ and relations rel. $H$ in these generators.

Furthermore let $G$ be an HNN-extension of $H$ of the following form:

$$G = \langle S, t \mid \text{rel. } H, t^{-1}ut = u, u \in U \rangle$$

for some subgroup $U \subset H$.

Let $w$ be a word in the generators of $G$ which involves $t$. If $w = 1$ in $G$ then $w$ contains a subword $t^{-1}ut$ or $utu^{-1}$ where $u$ is a word in $S$, and $u$, regarded as an element of the group $H$, belongs to the subgroup $U$. 

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We will apply Britton’s Lemma to a subgroup of finite index in the singular braid group $SG_n$ to show that this subgroup and hence $SG_n$ itself has a solvable word problem.

In general, if the base group $H$ has a solvable word problem it does not mean that an $HNN$-extension $G$ as in Britton’s Lemma has to have one. In addition there must be a test whether a given element of $H$ is in $U$ or not, or equivalently whether it commutes with $t$ or not.

In our proof $t$ is always a singular generator and when $u$ lies in the braid group $B_n$ the test whether $u$ commutes with $t$ is established by the solution to the word problem in $B_n$ and Lemma 2.4.

4 The subgroups $SGD_{n,i}$ of $SG_n$

In general a good reference for all used facts about braid groups is [Bir74]. Our notation in the following is a modification of the notation in [Cho48]. Especially we think of $B_n-1$ as the subgroup of $B_n$ generated by $\{\sigma_2, \ldots, \sigma_{n-1}\}$ rather than of the one generated by $\{\sigma_1, \ldots, \sigma_{n-2}\}$.

**Definition 4.1** Let $SGD_{n,i}$ be the preimage of

$$\Sigma_{n-i} = Sym(\{i+1, \ldots, n\}) \subset \Sigma_n = Sym(\{1, \ldots, n\})$$

of the natural homomorphism

$$SG_n \rightarrow \Sigma_n$$

and let $D_{n,i}$ (resp. $SD_{n,i}$) be the corresponding subgroup of $B_n$ (resp. the submonoid of $SB_n$). Especially we have $SGD_{n,0} = SG_n$. We will call the kernel $SGP_n$ of the homomorphism in (6) the pure singular braid group. So $SGP_n := SGD_{n,n-1}$.

**Lemma 4.2** The underlying geometry gives us an embedding

$$\phi_{n,i} : SG_{n-i} \rightarrow SGD_{n,i}$$

induced by the map

$$\sigma_j \mapsto \sigma_{j+i}, \quad \tau_j \mapsto \tau_{j+i}.$$  

The same holds for the embedding $SB_{n-i}$ into $SD_{n,i}$ and $B_{n-i}$ into $D_{n,i}$.

**Lemma 4.3** A system of Schreier right cosets of $SGD_{n,j-1}$ modulo $SGD_{n,j}$ is given by

$$M_{i,j} := \sigma_i \sigma_{i+1} \cdots \sigma_j, \quad j = i, \ldots, n-1$$

and the identity.

We get as generators for $SGD_{n,i}$:

$$a_{k,j} := \sigma_k \sigma_{k+1} \cdots \sigma_j^{-1} \sigma_{j-1}^{-1} \cdots \sigma_k^{-1}, \quad 1 \leq k \leq j \leq n-1, \quad k \leq i$$

$$X_{k,j} := \sigma_j \sigma_{j-1} \cdots \sigma_k \tau_k^{-1} \sigma_{k+1}^{-1} \cdots \sigma_{j-1}^{-1} \sigma_j^{-1}, \quad i \geq j \geq k \geq 1$$

$$\tau_{i+1} \quad \text{and} \quad \sigma_{i+1}, \ldots, \sigma_{n-1}.$$
4.1 A presentation for the subgroup $SGD_{n,1}$

Essentially for our considerations will be to work out a group presentation for the subgroup $SGD_{n,1}$ of $SG_n$. A presentation for the corresponding subgroup of $B_n$ was given by Chow:

**Theorem 4.4 (Chow, [Cho48])** The subgroup $D_{n,1}$ of $B_n$ is generated by the elements $a_1, \ldots, a_{1,n-1}$ and $\sigma_2, \ldots, \sigma_{n-1}$ subject to the relations

(i) The relations of $B_{n-1}$ generated by $\sigma_2, \ldots, \sigma_{n-1}$ hold

(ii) \[
\begin{align*}
\sigma_i a_{1,k} \sigma_i^{-1} &= a_{1,k} \\ &\text{for } k \neq i, i-1 \\
\sigma_i a_{1,i} \sigma_i^{-1} &= a_{1,i-1} \\
\sigma_i a_{1,i-1} \sigma_i^{-1} &= a_{1,i-1}^{-1} a_{1,i} a_{1,i-1}.
\end{align*}
\]

Furthermore the subgroup of $D_{n,1}$ generated by $a_{1,1}, \ldots, a_{1,n-1}$ is a free subgroup of rank $n-1$ and lies normal in $D_{n,1}$.

**Remark**

This presentation may be simplified according to the philosophy of our paper. With the setting $a := a_{1,1} = \sigma_1^2$ one can get:

**Corollary 4.5** The subgroup $D_{n,1}$ of $B_n$ is generated by the elements $a$ and $\sigma_2, \ldots, \sigma_{n-1}$ subject to the usual braid relations and the relations:

\[
\begin{align*}
\sigma_i a &= a \sigma_i \quad \text{for } i \geq 3, \\
\sigma_2 a \sigma_2 &= a \sigma_2 a \sigma_2.
\end{align*}
\]

Corollary 4.5 shows that $D_{n,1}$ again is an Artin group. It recently gained some new interest when tom Dieck [tD94] studied representations of it. Because of its own geometrical meaning it is also called cylinder braid group $ZB_{n-1}$.

**Theorem 4.6** $SGD_{n,1}$ is generated by the elements $\sigma_2, \ldots, \sigma_{n-1}$ and

\[
a_{1,j} = \sigma_1 \cdots \sigma_{j-1} \sigma_j^2 \sigma_{j-1}^{-1} \cdots \sigma_1^{-1}
\]
as well as by the singular elements

\[
\begin{align*}
X_{1,1} &= \sigma_1 \tau_1 \\
\tau_2 &= \sigma_1 \sigma_2 \tau_1 \sigma_2 \sigma_1^{-1}.
\end{align*}
\]

$SGD_{n,1}$ is defined by the following relations:

(i) The relations of $D_{n,1} \subset B_n$

(ii) The relations involving singular generators coming from the embedding

\[
\phi_{n,1} : SG_{n-1} \to SGD_{n,1}:
\]

\[
\begin{align*}
\sigma_i \tau_2 &= \tau_2 \sigma_i \quad \text{for } i \geq 2; i \neq 3 \\
\sigma_3 \sigma_2^2 \sigma_3 \tau_2 &= \tau_2 \sigma_3 \sigma_2^2 \sigma_3 \\
\sigma_3 \sigma_4 \sigma_2 \sigma_3 \tau_2 \sigma_3 \sigma_4 \sigma_2 \sigma_3 \tau_2 &= \tau_2 \sigma_3 \sigma_4 \sigma_2 \sigma_3 \tau_2 \sigma_3 \sigma_4 \sigma_2 \sigma_3
\end{align*}
\]
(iii) $a_1\tau_2 = \tau_2 a_1, \quad \text{for } i \geq 3$

(iv) $a_1a_1\tau_2 = \tau_2 a_1a_1$

(v) $\sigma_i X_{1,1} = X_{1,1}\sigma_i \quad \text{for } i \geq 3$

(vi) $a_1X_{1,1} = X_{1,1}a_1$

(vii) $X_{1,1}\sigma_2a_1\sigma_2 = \sigma_2a_1\sigma_2 X_{1,1}$

(viii) $X_{1,1}\sigma_2\sigma_3a_1a_1\sigma_3\sigma_2 = \sigma_2\sigma_3a_1a_1\sigma_3\sigma_2 X_{1,1}$

(ix) $X_{1,1}\sigma_2\sigma_3\tau_3^{-1}\sigma_2^{−1} = \sigma_2\sigma_3\tau_3^{-1}\tau_2^{−1} X_{1,1}.$

4.2 The HNN-group-structure of the pure singular group $SG_P n$

**Proposition 4.7** There is a presentation for the subgroup $SGD_{n,i}$ of $SG_P n$ in terms of the generators $a_{k,j}$ and $X_{k,j}$ as in Lemma 4.3 so that the relations are either of the following forms:

(i) relators coming from the subgroup $D_{n,i} \subset B_n$. 

(ii) The relations coming from the embedding $\Phi_{n,i} : SG_{n-i} \rightarrow SGD_{n,i}.$

(iii) $X_{k,l}w = w X_{k,l}$, where $k \leq l \leq i$ and $w$ is an element of the pure braid group $P_n$, written in terms of the generators of $D_{n,i}$.

(iv) $X_{k,l}\sigma_j = \sigma_j X_{k,l}$ for some $j > i + 1$ and $k \leq l \leq i$.

(v) $X_{k,l}\tau_i+1 = \tau_{i+1} X_{k,l}$, $k \leq l \leq i$.

(vi) $\tau_{i+1} w = w \tau_{i+1}$ where $w$ is an element of the pure braid group $P_n$.

(vii) $X_{l,k}\sigma_i+1\sigma_{i+2}\tau_{i+1}\sigma_{i+2}\sigma_{i+1}^{−1} = \sigma_{i+1}\sigma_{i+2}\tau_{i+1}\sigma_{i+2}\sigma_{i+1}^{−1} X_{l,k}$

for some $l \leq k \leq i$.

(viii) $X_{k,l} w X_{r,s} w^{-1} = w X_{r,s} w^{-1} X_{k,l}$ for some $k < r \leq i$, where $w$ is a word in $D_{n,i}$.

Clearly if we consider the subgroup $SGP_n := SGD_{n,n−1}$ relations involving $\tau_i$ and $\sigma_i$ no longer occur. Hence from our proposition it immediately follows:

**Theorem 4.8** Let $X$ be the collection \{ $X_{i,j}, 1 \leq i \leq j \leq n−1$ \} of the generators of $SGP_n$ involving singularities, and let $A$ be the collection \{ $a_{i,j}, 1 \leq i \leq j$ \} of non-singular generators.

For each choice of $X_{i,j} \in X$ is $SGP_n$ isomorphic to an HNN-extension of the subgroup $H_{i,j}$ of $SGP_n$ that is generated by all $x \in X - \{X_{i,j}\}$ and all $a \in A$:

$$SGP_n = \langle H_{i,j}, X_{i,j} | \text{rel.} H_{i,j}, X_{i,j} U_{i,j} = U_{i,j} X_{i,j} \rangle$$

for some subgroups $U_{i,j}$ in $H_{i,j}$.

Hence, the group $SGP_n$ is an iterated HNN-extension of the group $P_n$. This gives us the first Betti number:
**Corollary 4.9** The first homology group with integer coefficients is:

\[ H_1(SGP_n, \mathbb{Z}) \cong \mathbb{Z}^{n(n-1)}. \]

**Proof** The first homology group for the pure braid group \( P_n \) is well-known to be free abelian of rank \( n(n-1)/2 \). It follows e.g. immediately from the fact that the short exact sequence

\[ \{0\} \rightarrow F_{n-1} \rightarrow P_n \rightarrow P_{n-1} \rightarrow \{0\} \]

splits (see e.g. [Bir74]). Here \( F_{n-1} \) is the free subgroup of rank \( n-1 \) in \( P_n \) generated by \( \{a_{1,1}, \ldots, a_{1,n-1}\} \) (for notations confer Lemma 4.3) and the homomorphism \( P_n \rightarrow P_{n-1} \) is given by pulling out the first strand of a pure braid. Therefore - by induction - the first homology group of the group \( P_n \) is free abelian of rank \( (n(n-1))/2 \).

If \( G \) is a group which abelianization is free abelian of rank \( k \) then the abelianization of an HNN-extension has rank \( k + 1 \). Since the group \( SGP_n \) is an iterated HNN-extension by Theorem 4.8 and since the cardinality of the set \( X \) of stable letters is of size \( (n(n-1))/2 \) we get the desired result. \( \square \)

We will need an additional lemma to Proposition 4.7 which follows easily from geometrical considerations:

**Lemma 4.10** A relation

\[ X_{i,j}w_1X_{k,l}w_2 = w_1X_{k,l}w_2X_{i,j} \]

with \( w_1, w_2 \in P_n \) cannot occur in \( SGP_n \) if \( i = k \) or \( i = l + 1 \) or \( j + 1 = k \) or \( j = l \).

As a corollary to Theorem 4.8 and Lemma 4.10 we get a presentation for the following factor group:

**Corollary 4.11** Let \( N \) be the subgroup of \( SGP_n \) normally generated by \( P_n \). Then \( SGP_n/N \) has the presentation

\[ \langle X \mid X_{i,j}X_{k,l} = X_{k,l}X_{i,j} \text{ for } i \neq k, l + 1, \text{ and } j \neq l, k - 1 \rangle. \]

### 4.3 Example: The pure singular braid group on three strands

**Example 4.12** The group \( SGP_3 \) is generated by the elements \( a_{1,1}, a_{1,2}, a_{2,2} \) as well as \( X_{1,1}, X_{1,2} \) and \( X_{2,2} \).

The relations are:

\[
\begin{align*}
    a_{2,2}a_{1,2}a_{2,2}^{-1} &= a_{1,1}^{-1}a_{1,2}a_{1,1} \\
    a_{2,2}a_{1,2}a_{1,1}a_{2,2}^{-1} &= a_{1,2}a_{1,1} \\
    X_{2,2}a_{2,2} &= a_{2,2}X_{2,2} \\
    X_{2,2}a_{1,2}a_{1,1} &= a_{1,2}a_{1,1}X_{2,2} \\
    X_{1,1}a_{1,1} &= a_{1,1}X_{1,1} \\
    X_{1,1}a_{1,2}a_{1,1}a_{2,2} &= a_{1,2}a_{1,1}a_{2,2}X_{1,1} \\
    X_{1,2}a_{1,2}^{-1}a_{1,2}a_{1,1} &= a_{1,1}^{-1}a_{1,2}a_{1,1}X_{1,2} \\
    X_{1,2}a_{2,2}a_{1,1} &= a_{2,2}a_{1,1}X_{1,2}
\end{align*}
\]
5 A solution for the word problem in $SB_n$

To give a solution to the word problem in $SB_n$ we will proceed as follows: We know that $SB_n$ embeds into a group $SG_n$. Especially we know that two words $w_1$ and $w_2$ are in $SB_n$, where $w_1$, $w_2$ and $s$ are in $SB_n$, are equivalent if and only if $w_1$ and $w_2$ are equivalent in $SB_n$.

We note that - for our purposes - it is sufficient to solve the word problem for any two words $w_1$ and $w_2$ in $SGP_n$ with positive exponents for each singular generator.

Again let $X$ be the set of singular generators $\{X_{i,j}\}$ of $SGP_n$.

**Theorem 5.1** Let $w_1 = \alpha_1 Y_1 \alpha_2 \cdots \alpha_m Y_m$ and $w_2 = Z_1 \beta_1 \cdots Z_r \beta_r$ be two words in $SP_n \subset SGP_n$ with $Y_j, Z_k \in X$ and $\alpha_j, \beta_k \in P_n$.

Then $w_1 = w_2$ if and only if the following hold:

There is a $j$ such that

\[ Z_j = Y_m \text{ and } Z_i \neq Y_m \text{ for } i > j \] (15)

\[ Z_{r-l} \beta_{r-l} \cdots \beta_{r-1} \beta_r Y_m \beta_{r-1}^{-1} \cdots \beta_{r-1}^{-1} Z_{r-1} = \beta_{r-l} \cdots \beta_{r-1} \beta_r Y_m \beta_{r-1}^{-1} \cdots \beta_{r-1}^{-1} Z_{r-1} \] (16)

for all $r - l > j$

\[ \beta_j \cdots \beta_r Y_m = Y_m \beta_j \cdots \beta_r \] (17)

\[ \alpha_1 Y_1 \alpha_2 \cdots Y_{m-1} \alpha_m = Z_1 \beta_1 \cdots \beta_{j-1} \beta_j \cdots Z_r \beta_r. \] (18)

This gives us a solution to the word problem.

**Proof** By our Theorem 4.8 we know that we can regard $SGP_n$ as HNN-extension with stable letter $Y_m$. Thus, by Britton’s lemma, if $w_1 = \alpha_1 Y_1 \alpha_2 \cdots \alpha_m Y_m = w_2 = Z_1 \beta_1 \cdots Z_r \beta_r$ then there must be a $j$ satisfying (15) and

\[ Y_m \beta_j Z_{j+1} \cdots Z_r \beta_r = \beta_j Z_{j+1} \cdots Z_r \beta_r Y_m \] (19)

and therefore (18).

The converse is also true.

If we now consider $Z_r$ as the stable letter with the same argument - Britton's Lemma and Theorem 4.8 - we see that Equation (19) is equivalent to

\[ Z_r \beta_r Y_m \beta_r^{-1} = \beta_r Y_m \beta_r^{-1} Z_r \]

\[ Y_m \beta_j Z_{j+1} \cdots \beta_{r-1} \beta_r = \beta_j Z_{j+1} \cdots \beta_{r-1} \beta_r Y_m. \]

Hence we will end up with Equations (16) and (17).

Since the word problem is solvable for the braid group the Equations (16) and (17) are testable by Lemma 2.4.

Equation (15) is easy to test and the test for Equation (18) is given by induction on the number of singular generators in a word. $\square$

**Remark** It is not hard to see that along the same line one can actually get a solution for the word problem in the whole group $SG_n$. 


6 The complexity of the algorithm

We know by the approach of Birman, Ko and Lee [BKL] that the complexity for the word problem in $B_n$ is in $O(|w|^2 n)$, where $|w|$ is the word length in terms of the generators $\sigma_j$ of $B_n$.

To avoid messy details and computations in the sequel we are only interested in the complexity for the word problem for a fixed number $n$ of strands. Our aim is to give the complexity for the word problem for the singular braid group in terms of $|w|$ the total word length and $|w|_s$, the number of singular generators in a word.

The pure braid group is generated by $a_{k,j} := \sigma_k \sigma_{k-1} \cdots \sigma_j^2 \sigma_{j+1} \cdots \sigma_k^{-1}$. Since we fixed the number of strands, the complexity for the word problem in this group is also $O(|w|^2)$ where now $|w|$ is the word length in terms of the new generators $a_{k,j}$.

Let $w_1$ and $w_2$ be two given words in the singular braid monoid $SB_n$. We regard $w_1$ and $w_2$ also as elements of the group $SG_n$.

The factor group $SG_n/SGP_n$ is isomorphic to the symmetric group on $n$ elements. Hence, to compute the right coset class of $w_1$ and $w_2$ modulo the subgroup $SGP_n$ of $SG_n$ is clearly linear in the word length of $w_1$ or $w_2$. If they are in different classes, we are done. Otherwise we work with $w_1 M_{i,j}^{-1}$ and $w_2 M_{i,j}^{-1}$ instead, where $M_{i,j}$ is the representative of the right coset class of $w_1$ and $w_2$.

Rewriting the new $w_1$ and $w_2$ in terms of the generators $X_{i,j}$ and $a_{i,j}$ of $SGP_n$ does not change the word length or the number of singular generators. So we can assume without loss of generality that $w_1$ and $w_2$ are already in $SGP_n$ and are given as products of the generators $X_{i,j}$ and $a_{i,j}$.

Now let for two fixed words $w_1$ and $w_2$ in $SGP_n$ involving only positive exponents of the singular generators the values $|w|$ (resp. $|w|_s$) be the maximum of $|w_1|$ and $|w_2|$ (resp. the maximum of $|w_1|_s$ and $|w_2|_s$).

A short look at (15) - (18) gives the following:

(i) To check (15) is linear in $O(|w|_s)$.

(ii) To check (16) is in $O(|w|_s |w|^2)$ since the word problem for $B_n$ is in $O(|w|^2)$.

(iii) To check (17) is in $O(|w|^2)$.

By (18) we know that we have to check (15) - (17) at most $|w|_s$-times.

Hence we have proved:

**Theorem 6.1** The complexity of the word problem in $SB_n$ with the above definitions is in $O(|w|^2 |w|^2)$.

7 The group $SG_n$ is torsion-free

There are many proofs for the well-known fact that the braid groups $B_n$ are torsion-free. Until most recently, however, none of them could be considered as being elementary.

Now Dehornoy’s ordering of the braid group and especially the interpretation of it, given in [FGR+98], yields an easy way to show this result.

The proof of the torsion-freeness of the group $SG_n$ was announced in [FKR96] but as far as we know was never proved. The proof, however, follows directly from our approach:

**Theorem 7.1 (Fenn, Keyman, Rourke)** The group $SG_n$ is torsion-free.
Proof First we note that by the structure theorems for HNN-groups (e.g. [MKS76], [LS77]) torsion must lie in the base group. Thus, by Theorem 1.3, torsion in the normal subgroup $SGP_n$ of $SG_n$ must lie in $P_n$ which is torsion free as a subgroup of $B_n$. So $SGP_n$ is torsion free.

The subgroup $N$ of $SG_n$ normally generated by $\tau_1\sigma_1^{-1}$ is a subgroup of $SGP_n$ and therefore also torsion-free.

As it easily follows from Proposition 2.1 the group $SG_n/N$ is isomorphic to $B_n$. Since $B_n$ is torsion-free a torsion element must lie in $N$ and we are done. □

7.1 The cohomological dimension of $SG_n$

A good reference for almost all facts that we use about the cohomological dimensions $cd(G)$ of a group $G$ is [Bro94] or [Ser71]. Especially we use Serre’s Theorem that the cohomological dimension of a torsion-free group is equal to the one of each of the subgroups of finite index. Furthermore the cohomological dimension of a subgroup must be less or equal to the cohomological dimension of the group. Since the cohomological dimension of a free abelian group of rank $n$ is $n$ this means that the following lemma gives us a lower bound. The first part - for the braid group - is of course very well known.

Lemma 7.2 The pure braid group $P_n$ contains a free abelian subgroup of rank $n - 1$. The pure singular braid group contains a free abelian subgroup of rank $n - 1 + \lfloor n/2 \rfloor$.

Proof The images of $a_{k,j} = \sigma_k\sigma_{k-1} \cdots \sigma_j^{-1}\sigma_{j-1} \cdots \sigma_k^{-1}$, $n - 1 \geq j \geq k \geq 1$, form a basis for the commutator factor group of $P_n$, which is free abelian of rank $n(n - 1)/2$.

Correspondingly the images of the $a_{i,j}$ and of $X_{k,j} = \sigma_j\sigma_{j-1} \cdots \sigma_k\sigma_{k+1} \cdots \sigma_j^{-1}$ generate by Lemma 1.3 commutator factor group of $SGP_n$, which is free abelian of rank $n(n - 1)$.

By a result of Chow [Cho48] the center of the braid group $B_n$ is infinite cyclic and is generated by $c_n := (a_{1,n-1} \cdots a_{1,1})(a_{2,n-1} \cdots a_{2,2}) \cdots (a_{n-1,n-1}).$ This was generalized by Fenn, Rourke and Zhu [FRZ90] to the singular braid monoid $SB_n$. For $n \geq 3$ the center is also infinite cyclic and generated by $c_n$.

For $n = 2$ the group $B_2$ is infinite cyclic and $SG_2$ is free abelian of rank 2. For $n = 3$ the element $a_{2,2}$ and the center $c_3$ form a free abelian subgroup of rank 2 in $P_3$ and $a_{2,2}, c_3$ and $X_{2,2}$ form a free abelian subgroup of rank 3 in $SGP_3$.

Now for $n > 3$ the elements $c_n$ and $a_{n-1,n-1}$ in $P_n$ both commute with each other and with $P_{n-2} = \{a_{k,j}, k,j \leq n - 3\} \subset P_n$ and are independent in the commutator factor group. Therefore, by induction, $P_n$ contains a free abelian subgroup of rank $n - 1$.

Correspondingly, $X_{n-1,n-1}, a_{n-1,n-2}$ and $c_n$ commute with each other and $SGP_{n-2}$. The claim follows for $SGP_n$. □

Again for the braid group itself the following theorem is well-known. (See e.g. [Vas92] for an account to results of Arnold and Fuchs.) We only include a proof for completeness.

Theorem 7.3 The group $B_n$ has cohomological dimension $n - 1$. The group $SG_n$ has cohomological dimension $n - 1 + \lfloor n/2 \rfloor$.

Proof The braid group $B_n$ is torsion-free and since $SG_n$ is torsion-free by Theorem 7.1 it is enough by Serre’s Theorem to prove the theorem for a subgroup of finite index in $SG_n$ and in $B_n$. We choose $SGP_n$ and $P_n$ for this purpose.
First we give the argument for $B_n$. By Lemma 7.2 we already know that $n - 1$ is a lower bound for the cohomological dimension $cd(B_n)$.

Now for a group $G$ and a normal subgroup $H$ in $G$ the relation $cd(G) \leq cd(H) + cd(G/H)$ holds. The kernel of the natural map $P_n \to P_{n-1}$ is free of rank $n - 1$. Therefore its cohomological dimension is 1. Furthermore $P_2$ is infinite cyclic and therefore also of cohomological dimension 1. The result follows by induction.

For $SGP_n$ the situation is more complicated. The subgroup $SGP_n$ has by Theorem 4.8 the structure of an iterated HNN-extension of the subgroup $P_n$ in $B_n$.

We know (see [Bie76]) that for an HNN-extension $G = \langle H, t \mid rel. H, tUt^{-1} = U \rangle$ of a group $H$ with subgroup $U$ the relation

$$
cd(G) \leq \max(cd(H), cd(U) + 1)
$$

holds.

In the following we change the notation for reasons of simplifications. We define $Y_{i,j} := X_{i,j+1}$ if $i < j$ and $Y_{i,j} := X_{j,i+1}$ if $j < i$. This means $Y_{i,j}$ is a singular pure braid so that string $i$ intersects string $j$ once.

For an index set $I \subset \{1, \ldots, n\}$ let $X_I$ be the set of all singular generators $Y_{i,j}$ with $i, j \in I$.

We will show that the subgroup $H_I$ of $SGP_n$ generated by $P_n$ and $X_I$ has cohomological dimension less or equal to $\lfloor |I|/2 \rfloor + n - 1$.

If $|I| = 2$ then there is just one singular generator, say $Y_{j,k}$ in $X_I$ and $H_I$ is HNN-extension of $P_n$ so by (20) we have $cd(H_I) \leq cd(P_n) + 1 = n$.

If $|I| = 3$ then there are three singular generators in $X_I$, say $Y_{j,k}, Y_{k,l}$ and $Y_{j,l}$. By Lemma 4.10 we know that each of these three generators cannot commute with a word that includes one of the others. Therefore by our structure theorem $H_I$ is an HNN-extension

$$\langle SGP_n, Y_{j,k}, Y_{k,l}, Y_{j,l} \mid rel. SGP_n, Y_{j,k}U_{j,k} = U_{j,k}Y_{j,k}, Y_{k,l}U_{k,l} = U_{k,l}Y_{k,l}, Y_{j,l}U_{j,l} = U_{j,l}Y_{j,l} \rangle$$

with three subgroups $U_{j,k}, U_{k,l}$ and $U_{j,l}$ of $P_n$. Therefore all three subgroups have cohomological dimension less or equal to $cd(P_n)$ and thus $cd(H_I) \leq cd(P_n) + 1 = n$.

For $I = I' \cup \{j\}$ for some $j$ we know that $Y_{j,i}, i \in I'$, cannot commute by Lemma 4.10 with a word that includes one singular generator $Y_{k,i}$ for some $k \in I'$. So the subgroup of $H_{I'}$ with which $Y_{j,i}$ commutes is by our structure theorem actually a subgroup of $H_{I'-\{i\}}$.

Since by induction $cd(H_{I'-\{i\}}) \leq n - 1 + |I'|-1$ we know that the HNN-extension

$$\langle Y_{j,i}, H_{I'} \mid rel. H_{I'}, Y_{j,i}U_{j,i} = U_{j,i}Y_{j,i} \rangle$$

must have cohomological dimension less or equal than

$$\max(cd(H_{I'}), cd(H_{I'-\{i\}}) + 1) = n - 1 + \lceil |I'|/2 \rceil
$$

Now if we successively add all other $Y_{j,k}$, $k \in I'$, to this group then by the same arguments we still have this upper bound for it, since $Y_{i,k}, i \in I'$, only commutes with a subgroup of $H_{I'-\{k\}}$. □

Example 7.4 We will show how to use the HNN-structure of

$$SG_3 \cong \langle B_3, \tau_1 \mid rel. B_3, \tau_1\sigma_1 = \sigma_1\tau_1, \tau_1(\sigma_2\sigma_1^2\sigma_2) = (\sigma_2\sigma_1^2\sigma_2)\tau_1 \rangle$$

to compute the homology of this group.
Let $K$ be the trefoil knot embedded in $S^3$. Let $U$ be a tubular neighborhood of $K$. By $C = S^3 \setminus U$ we denote the closure of the complement of this tubular neighborhood. Obviously $\partial C = \partial U$ is homeomorphic to the torus $T^2$.

It is well known that the space $C$ is a $K(B_3,1)$-space. In fact, the fundamental group of $C$ is isomorphic to $B_3$ and since $C$ is the closure of the complement of the tubular neighborhood of a knot, the higher homotopy groups are trivial (see e.g. [BZ85]).

Moreover, the embedding of $\partial C$ into $C$ induces an injection $i$ from $\pi_1(\partial C, \ast) \cong \mathbb{Z} \oplus \mathbb{Z}$ into $B_3$. The image of $i$ is generated by $\sigma_1$ and $\sigma_2 \sigma_1^2 \sigma_2$ (see figure below).

![Figure 2: $\omega = \sigma_2 \sigma_1^2 \sigma_2$](image)

This allows us to construct a $K(SG_3, 1)$-space in the following way: Consider the space $E = S^1 \times S^1 \times I$. Its boundary consists of two solid tori $T_0 = S^1 \times S^1 \times \{0\}$ and $T_1 = S^1 \times S^1 \times \{1\}$. Take a function $f : T \to \partial C$ which sends the longitude of $T$ to $\sigma_2 \sigma_1^2 \sigma_2$ and the meridian of $T$ to $\sigma_1$. Attach both tori, $T_0$ and $T_1$, to $C$ using $f$ as an attaching map, in order to obtain a space $X = E \bigsqcup f f C$. Note that, due to the special structure of our attaching map, the image $\tilde{E}$ of $E$ in $X$ is homeomorphic to $S^1 \times S^1 \times S^1$.

The space $X$ is a $K(SG_3,1)$-space. Indeed, it may be easily seen that $\pi_1(X, \ast) \cong SG_3$ using the theorem of Seifert-van Kampen. The fact that $\pi_n(X, \ast) = 0$ for $n > 1$ follows from a general argument using covering spaces. In fact, attaching several copies of the universal covering space of $E$ to the universal covering space of $C$ in an appropriate way, yields a covering space of $X$ which is homotopy equivalent to a wedge $\bigvee_{i \in I} S^1$ of circles. Since $\bigvee_{i \in I} S^1$ is an Eilenberg-MacLane-space, so is $X$.

Hence, we can compute the homology of $SG_3$ by calculating the homology of the space $X$.

Since $X$ is three-dimensional, we immediately get $H_n(X) = H_n(SG_3) = 0$ for $n \geq 4$, as it follows from Theorem 7.3. Moreover, $H_0(SG_3) \cong \mathbb{Z}$ and $H_1(SG_3) \cong \mathbb{Z} \oplus \mathbb{Z}$ since the
abelianization of $SG_3$ is free abelian of rank 2.

It remains to compute $H_2(X)$ and $H_3(X)$. This will be done by using the Mayer-Vietoris-Sequence

$$\ldots \to H_3(\partial C) \to H_3(C) \oplus H_3(\hat{E}) \to H_3(X) \to H_2(\partial C) \to H_2(C) \oplus H_2(\hat{E}) \to H_2(X) \to H_1(\partial C) \to \ldots$$

which arises from the natural decomposition $X = C \cup \hat{E}$.

It is easy to see, that the map from $H_2(\partial C)$ to $H_2(C) \oplus H_2(\hat{E})$ is injective. Therefore the map from $H_3(X)$ to $H_2(\partial C)$ must be the trivial map. Since $H_3(\partial C) = H_3(T) = 0$ and $H_3(C) = 0$ this implies that $H_3(X) \cong H_3(SG_3)$ is isomorphic to $H_3(\hat{E}) \cong H_3(S^1 \times S^1 \times S^1) \cong \mathbb{Z}$.

We are left with the case of $H_2(X)$. As in the last case, we see that the map from $H_1(\partial C)$ to $H_1(C) \oplus H_1(\hat{E})$ is injective. Thus the map from $H_2(X)$ to $H_1(\partial C)$ is the trivial map. Since $H_2(C) = 0$ this implies, that $H_2(X)$ is obtained as a quotient from $H_2(\hat{E})$. In fact, a close examination of the map from $H_2(\partial C)$ to $H_2(\hat{E})$ shows that $H_2(X) \cong H_2(SG_3)$ must be isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

Remark Results concerning the homology of the infinite singular braid group $SG_\infty$ may be found in [Ver98].

8 The singular braid monoid embeds in a group

In [FKR96] Theorem 2.2 is proved with the help of geometrical considerations. For further generalizations it might be useful to have an algebraic proof of it at hand.

On the other hand, other embedding theorems like Ore's theorem - that was for example used in in Garside's solution for the word and conjugacy problem for the braid groups [Gar63] when showing that the semigroup of positive braids embeds into the braid groups - do not seem to be applicable.

Using our tools, however, it is quite easy to give an algebraic proof of the theorem:

Theorem 8.1 ([FKR96]) The singular braid monoid $SB_n$ embeds into a group $SG_n$.

Proof Let $w_1$ and $w_2$ be two different elements of $SB_n$ that have equal images in $SG_n$, also denoted by $w_1$ and $w_2$. For an element $w$ in $SB_n$ the image in the symmetric group $\Sigma_n$ under the natural map $SB_n \to \Sigma_n$ is the same as under the map $SB_n \to SG_n \to \Sigma_n$.

Hence, by multiplying both elements with the same element in $B_n$ we can assume that $w_1$ and $w_2$ map to the subgroup $SGP_n$ of $SG_n$. Therefore we can regard them as given as words in the generators $a_{i,j} := \sigma_i \sigma_{j+1} \cdots \sigma_i^2 \sigma_{i-1}^{-1} \cdots \sigma_j^{-1}$ and $X_{i,j} := \sigma_j \sigma_{j-1} \cdots \sigma_i \sigma_i^{-1} \cdots \sigma_j$, where the $X_{i,j}$ only occur with positive exponents.

We assume that $w_1$ and $w_2$ are minimal examples with these properties, that means the sum of the number of singular generators in $w_1$ and $w_2$ is minimal.

We know that $B_n$ embeds into both $SB_n$ and $SG_n$ and therefore there must be at least one singular generator in $w_1$ or $w_2$.

Now we apply Theorem 5.1 to the two words $w_1$ and $w_2$ that are equal in $SGP_n$. We have in mind that we used the fact there - which was already induced by the embedding theorem - that for $w_1, w_2, s \in SB_n$ we have $w_1 s = w_2 s \iff w_1 = w_2$. This cannot cause trouble here since we assume $w_1$ and $w_2$ to be minimal examples.
Now we know by Theorem 5.1 that \( w_1 \) and \( w_2 \) are either already equal in \( SB_n \) or there are two different words \( v_1 \) and \( v_2 \) in \( SB_n \) that both maps to the same element in \( SGP_n \) and the sum of the numbers of singular generators in \( v_1 \) and \( v_2 \) is less than the sum of the numbers of singular generators in \( w_1 \) and \( w_2 \).

Therefore the theorem follows. \( \Box \)

9 Proofs of Theorem 4.6 and Theorem 4.7

Proof of Theorem 4.6

First we note that the Relations (vi) and (vii) hold in \( SGD_{n,1} \).

Furthermore we already have in \( D_{n,1} \):

\[
\begin{align*}
    a_{1,2}a_{3}a_{1,1}a_{2} &= \sigma_{3}a_{1,3}\sigma_{2}a_{1,2} \\
    a_{1,1}(\sigma_{2}\sigma_{3}a_{1,3}\sigma_{2}a_{1,2}) &= (\sigma_{2}\sigma_{3}^{2}a_{1,3}\sigma_{2}a_{1,2})a_{1,1} \\
    a_{1,3} &= \sigma_{3}^{-1}a_{1,1}\sigma_{2}\sigma_{3}.
\end{align*}
\]

By Reidemeister-Schreier we know that we get all relations in \( SGD_{n,1} \) by applying the rewriting process on all \( M_{i}R_{i}M_{i}^{-1} \), where \( M_{i} \) runs through the Schreier system of right cosets of \( SGD_{n,1} \) in \( SG_{n} \) (see Lemma 4.3) and \( R_{i} \) runs through all relations of the presentation of \( SG_{n} \).

Since we already know a presentation for the subgroup \( D_{n,1} \) of \( B_{n} \) and since the Schreier cosets are in \( B_{n} \) we only have to look at the relations involving singular generators:

(i) Relations coming from \( \sigma_{i}\tau_{1} = \tau_{1}\sigma_{i} \), for \( i \neq 2 \): These are precisely the relations in (ii), (iii), (iv) and (v).

Especially from \( \sigma_{2}a_{1,3} = a_{1,3}\tau_{2} \) and (23) it follows that

\[
a_{1,1}(\sigma_{2}\sigma_{3}\tau_{2}^{-1}\sigma_{3}^{-1}) = (\sigma_{2}\sigma_{3}\tau_{2}^{-1}\sigma_{3}^{-1})a_{1,1}
\]

(ii) Relations coming from \( \sigma_{2}\sigma_{3}^{2}\sigma_{2}\tau_{1} = \tau_{1}\sigma_{2}\sigma_{3}^{2}\sigma_{2} \)

\[
\begin{align*}
    M_{j}, j \geq 3 & \quad \sigma_{3}\sigma_{2}^{2}\sigma_{3}\tau_{2} = \tau_{2}\sigma_{3}\sigma_{2}^{2}\sigma_{3} \\
    M_{2} & \quad a_{1,2}a_{1,1}a_{2} = \tau_{2}a_{1,2}a_{1,1} \\
    M_{1} & \quad \sigma_{2}^{2}a_{1,2}X_{1,1} = X_{1,1}\sigma_{2}a_{1,1}\sigma_{2} \\
    \iff & \quad \sigma_{2}a_{1,1}\sigma_{2}X_{1,1} = X_{1,1}\sigma_{2}a_{1,1}\sigma_{2}^{-1} \\
    M_{0} & \quad \sigma_{2}a_{1,1}\sigma_{2}X_{1,1}^{-1} = X_{1,1}\sigma_{2}^{-1}\sigma_{2}a_{1,2}
\end{align*}
\]

The relation corresponding to \( M_{0} \) follows directly from the relation corresponding to \( M_{1} \) and (v).

(iii) Relations coming from \( \sigma_{2}\sigma_{3}\sigma_{1}\sigma_{2}\tau_{1}\sigma_{2}\sigma_{3}\sigma_{1}\sigma_{2} = \tau_{1}\sigma_{2}\sigma_{3}\sigma_{1}\sigma_{2}\tau_{1}\sigma_{2}\sigma_{3}\sigma_{1}\sigma_{2} \)

\[
\begin{align*}
    M_{j}, j \geq 4 & \quad \sigma_{3}\sigma_{2}\sigma_{3}\tau_{2}\sigma_{3}\sigma_{4}\sigma_{2}\sigma_{3} = \tau_{2}\sigma_{3}\sigma_{4}\sigma_{2}\sigma_{3}\tau_{2}\sigma_{3}\sigma_{4}\sigma_{2}\sigma_{3} \\
    M_{3} & \quad \sigma_{3}a_{1,3}\sigma_{2}a_{1,2}X_{1,1}\sigma_{2}\sigma_{3} = \tau_{2}\sigma_{3}a_{1,3}\sigma_{2}a_{1,2}X_{1,1}\sigma_{2}\sigma_{3} \\
    M_{2} & \quad a_{1,2}\sigma_{3}a_{1,1}\sigma_{2}X_{1,1}^{-1} = \tau_{2}a_{1,2}\sigma_{3}a_{1,1}\sigma_{2}X_{1,1}^{-1}\sigma_{2}\sigma_{3} \\
    M_{1} & \quad \sigma_{2}\sigma_{3}a_{1,3}\sigma_{2}a_{1,2}X_{1,1} = X_{1,1}\sigma_{2}\sigma_{3}a_{1,2}a_{1,1}\sigma_{2}\sigma_{3} \\
    M_{0} & \quad \sigma_{2}\sigma_{3}a_{1,2}\sigma_{3}a_{1,1}\sigma_{2}X_{1,1}^{-1} = X_{1,1}\sigma_{2}\sigma_{3}a_{1,2}a_{1,1}\sigma_{2}\sigma_{3}.
\end{align*}
\]
Now it is easy to see that the equations in (26) - with the first relation as an exception - follow from (viii), (24), (31), (21) and (23). □

**Proof of Theorem 4.7**

By Lemma 4.3 a system of Schreier right cosets of $SGD_{n,i}$ modulo $SGD_{n,i-1}$ is given by $M_{i,j} = \sigma_1 \cdots \sigma_j$, $j = i, \ldots, n-1$ and the identity. Hence, by application of the Reidemeister-Schreier process we know that all relations for $SGD_{n,i}$ can be obtained by rewriting $M_{i,j}R_{i-1,k}M_{i,j}^{-1}$ in terms of the generators of $SGD_{n,i}$, where $j = i, \ldots, n-1$ and $R_{i-1,k}$ runs through all relations of $SGD_{n,i-1}$.

More precisely: If $R_{i-1,k} = v_1v_2 \cdots v_r$ is a relator in $SGD_{n,i-1}$ then

$$\prod_{t=1}^{r} (M_{i,j}v_tM_{i,j}^{-1})(M_{i,j}v_{t+1}M_{i,j}^{-1}) \cdots (M_{i,j}v_rM_{i,j}^{-1})$$

is a relator in $SGD_{n,i}$ and all necessary relators in $SGD_{n,i}$ are of this form.

By Theorem 4.4 the claim is true for $SGD_{n,1}$. Now assume it is true for $SGD_{n,i-1}$. We will show that all $M_{i,j}R_{i-1,k}M_{i,j}^{-1}$ have the form that we claimed.

First we will look at the terms in (23) coming from the singular generators in $SGD_{n,i-1}$. Since the $X_{k,l}$, $1 \leq k \leq l \leq i-1$, are already in the normal subgroup $SGP_n$ of $SG_n$ it follows that $M_{i,j}X_{k,l} = X_{k,l}$ for each right coset $M_{i,j}$.

Moreover:

$$M_{i,j}X_{k,l}M_{i,j}^{-1} = X_{k,l} \quad \text{for} \quad i > l + 1 \geq k + 1$$

$$M_{i,j}X_{k,i}M_{i,j}^{-1} = X_{k,i}$$

Especially this means:

(i) A relation $X_{k,l}w = wX_{k,l}$ where $w$ is a word in the pure braid group $P_n$ (in terms of the generators of $D_{n,i-1}$) yields a relation $X_{k,m}w = \tilde{w}X_{k,m}$ for some $m$ and $\tilde{w}$ a word in the pure braid group $P_n$ (in terms of the generators of $D_{n,i}$).

(ii) A relation $X_{k,l}\sigma_j = \sigma_jX_{k,l}$, $j \geq i$, leads to a relation either of the form $X_{k,m}\sigma_q = \sigma_qX_{k,m}$ for some $m$ and $q > i$ or $X_{k,m}a_{i,q} = a_{i,q}X_{k,m}$ for some $m$ and $q$.

(iii) A relation of the form (31) yields relations of the form (30) and form (viii).

(iv) A relation $\tau_i w = w \tau_i$, $w \in P_n$ yields relations of either of the following types:

We know that $\overline{M_{i,j}}\overline{\tau_i} = M_{i,j}$ for $j > i$, $\overline{M_{i,i}}\overline{\tau_i} = id$ and $\overline{\tau_i} = M_{i,i}$.

Therefore we have for $j > i$:

$$\prod_{t=1}^{r} (M_{i,j}^{\tau_i}\overline{M_{i,j}}^{-1})(M_{i,j}^{\tau_i}w\overline{M_{i,j}}^{-1}) \quad \text{for some} \quad \tilde{w} \in P_n.$$
Finally,

\[(\tau_i M_{i,i}^{-1})(M_{i,i} w M_{i,i}^{-1}) = w(\tau_i M_{i,i}^{-1})\]
\[\iff X_{i,i} a_{i,i}^{-1} w = w X_{i,i} a_{i,i}^{-1}\]
\[\iff X_{i,i} w a_{i,i}^{-1} = w X_{i,i} a_{i,i}^{-1}\]
\[\iff X_{i,i} w = w X_{i,i}\]

(v) The rewriting process for the relation

\[M_{i,j} X_{l,k} \sigma_i \sigma_{i+1} \tau_i \sigma_{i+1}^{-1} a_{i,i}^{-1} M_{i,j}' = M_{i,j} \sigma_i \sigma_{i+1} \tau_i \sigma_{i+1}^{-1} X_{l,k} M_{i,j}'\]

for a suitable \(j'\) and for \(l \leq k \leq i - 1\) yields for \(j > i + 1:\)

\[X_{l,k'} \sigma_i \sigma_{i+1} \tau_i \sigma_{i+1}^{-1} a_{i,i}^{-1} \sigma_{i+1}^{-1} = \sigma_{i+1} \sigma_i \sigma_{i+1} \tau_i \sigma_{i+1}^{-1} X_{l,k'}\]

for some \(k' \leq i\).

Furthermore we have for \(j = i + 1\), since \(\sigma_i \sigma_{i+1} a_{i,i} = a_{i,i} \sigma_{i+1}:\)

\[X_{l,k} \sigma_i \sigma_{i+1} X_{i,i} \sigma_{i+1}^{-1} a_{i,i}^{-1} M_{i,j}' = X_{i,i} \sigma_{i+1}^{-1} M_{i,j}'\]
\[\iff X_{l,k} \sigma_i \sigma_{i+1} X_{i,i} \sigma_{i+1}^{-1} a_{i,i}^{-1} X_{l,k'} = a_{i,i} \sigma_{i+1}^{-1} M_{i,j}'\]

for some \(k' \leq i\).

For \(j = i\) we get:

\[X_{l,k} a_{i,i} \sigma_i \sigma_{i+1} X_{i,i} \sigma_{i+1}^{-1} a_{i,i}^{-1} \sigma_{i+1}^{-1} = a_{i,i} \sigma_{i+1}^{-1} X_{i,i} \sigma_{i+1}^{-1} a_{i,i}^{-1} \sigma_{i+1}^{-1} X_{l,k'}\]

This is covered by Relation (33) and an additional relation that we can add:

\[X_{l,k} a_{i,i} \sigma_i \sigma_{i+1} a_{i,i}^{-1} \sigma_{i+1}^{-1} a_{i,i}^{-1} = a_{i,i} \sigma_{i+1}^{-1} a_{i,i}^{-1} \sigma_{i+1}^{-1} a_{i,i}^{-1} X_{l,k'}\]

(37)

The additional relation is of the form (33).

Finally for the identity as the right coset we get:

\[X_{l,k} \tau_{i+1} = \tau_{i+1} X_{l,k}\]

(38)

(vi) A relation \(X_{k,l} w X_{r,s} w^{-1} = w X_{r,s} w^{-1} X_{k,l}\) becomes:

\[(M_{i,j} X_{k,l} M_{i,j}^{-1})(M_{i,j} w M_{i,j}^{-1})(M_{i,j} X_{r,s} M_{i,j}^{-1})(M_{i,j} w^{-1} M_{i,j}^{-1}) =\]
\[(M_{i,j} w^{-1} M_{i,j}^{-1})(M_{i,j} X_{r,s} M_{i,j}^{-1})(M_{i,j} w M_{i,j}^{-1})(M_{i,j} X_{k,l} M_{i,j}^{-1})\]
\[\iff X_{k,l} \tilde{w} X_{r,s} \tilde{w}^{-1} = \tilde{w} X_{r,s} \tilde{w}^{-1} X_{k,l}\]

for some \(\tilde{j}, \tilde{l}, \tilde{s}\) and a word \(\tilde{w}\) in \(D_{n,i}\).
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