Numerical solution of special 2D Fredholm-Volterra integral equations using barycentric Gegenbauer interpolation collocation method

Hongyan Liu and Jin Huang*

School of Mathematical Sciences, University of Electronic Science and Technology, Chengdu Sichuan 611731, China
*Corresponding author: huangjin12345@163.com

Abstract. In this paper, an efficient collocation method based on two dimensional barycentric Gegenbauer interpolation is used to solve a kind of special two dimensional Fredholm-Volterra integral equations (2D-FVIEs). The explicit barycentric weights for the Gegenbauer-Gauss nodes not only reduce the complicated calculation but also preserve the numerical stability. The combination of the barycentric interpolation and the Legendre-Gauss quadrature formula transforms the 2D-FVIEs into a system of discrete algebraic equations whose solution is a set of the nodal function values. The numerical experiments provide the maximum absolute error and the convergent order to assess the accuracy and the convergence of the method.

1. Introduction

This paper we're inspired by the high-order and stable barycentric Gegenbauer pseudospectral method in [1] and devoted to solving the following two dimensional Fredholm-Volterra integral equation

\[ u(x, y) + \int_0^1 k_1(x, t)u(t, y)dt + \int_0^y k_2(y, s)u(x, s)ds = f(x, y), \quad (x, y) \in D = [0,1]^2, \]  

where \(k_1(x, t), k_2(y, s)\) and \(f(x, y)\) are known functions in \(D\), and \(u(x, y)\) is unknown function. The numerical solution of the special 2D-FVIEs is in fact significant in many practical applications, for example, the contact problems in [2, 3]. Recently, there are many efficient numerical schemes are used to solve these equations, for example, the block pulse function operational matrix method in [4, 5].

The rest arrangement of this paper includes, in Section 2, we review the Gegenbauer interpolation, barycentric Gegenbauer interpolation and two dimensional barycentric Gegenbauer interpolation. In Section 3, the two dimensional barycentric Gegenbauer interpolation collocation method together with the Legendre-Gauss quadrature formula is employed to numerically solve the special 2D-FVIEs. In Section 4, numerical examples are tested to testify that the method has high accuracy and high order.
\[(G_m^{(a)}, G_n^{(a)})_{L_w^{(a)}[-1,1]}^2 = (G_m^{(a)}, G_n^{(a)})_{w^{(a)}}^2 = \int_{-1}^{1} G_m^{(a)}(x)G_n^{(a)}(x)w^{(a)}(x)\,dx = \gamma_m^{(a)}\delta_{mn}, \quad (2)\]

where

\[
\gamma_m^{(a)} = (G_m^{(a)}, G_m^{(a)})_{L_w^{(a)}[-1,1]}^2 = \left\| G_m^{(a)} \right\|^2_{w^{(a)}} = \frac{2^{1-2a}m!^2(m+2a)}{m!(m+a)!^2}, \quad (3)
\]

is the nonzero normalization factor, and \(\delta_{mn}\) is the Kronecker Delta function.

### 2.1. Gegenbauer interpolation

The Gegenbauer interpolation of any real function \(u \in L_w^{(a)}[-1,1]\) may be expanded as

\[
u(x) = \sum_{j=0}^{\infty} \hat{u}_j G_j^{(a)}(x), \quad \hat{u}_j = (f_j^{(a)})^{-1} (u, G_j^{(a)}(x))_{w^{(a)}}, \quad (4)
\]

where \(\{\hat{u}_j\}_{j=0}^{\infty}\) are the orthogonal polynomial expansion coefficients. A natural truncated form of the Gegenbauer interpolation of \(u \in L_w^{(a)}[-1,1]\) is defined to be

\[
p_m^{(a)} u(x) = \sum_{j=0}^{m} \hat{u}_j G_j^{(a)}(x), \quad \hat{u}_j = (f_j^{(a)})^{-1} \sum_{i=0}^{m} w_{om,i}^{(a)} u(x_{m,i}) G_j^{(a)}(x_{m,i}), \quad (5)
\]

where the expansion coefficients \(\{\hat{u}_j\}_{j=0}^{m}\) are exactly evaluated using the orthogonal property, the Gegenbauer-Gauss quadrature formula with the Gegenbauer-Gauss quadrature nodes and weights \(\{(x_{m,i}, w_{om,i}^{(a)})\}_{i=0}^{m}\), and the interpolation conditions \(p_m^{(a)} u(x_{m,i}) = u(x_{m,i})\). Then the Lagrange form of nodal Gegenbauer interpolation of \(u\) at the Gegenbauer-Gauss nodes is

\[
p_m^{(a)} u(x) = \sum_{i=0}^{m} u_{m,i}^{(a)} L_{m,i}^{(a)}(x) = w_{om,i}^{(a)} \sum_{j=0}^{m} f_j^{(a)} (x_{m,i}) G_j^{(a)}(x_{m,i}), \quad (6)
\]

where \(u_{m,i}^{(a)} = u(x_{m,i})\) and \(L_{m,i}^{(a)}(x)\) are the Lagrange interpolation basis functions.

As is known, the evaluations of the modal interpolation (5) and the nodal interpolation (6) both require \(O(m^2)\) operations. In addition, the interpolation (6) is numerically unstable for dense partition, and a new computation of each \(\{L_{m,i}^{(a)}(x)\}_{i=0}^{m+1}\) is required for adding a new data pair \((x_{m+1,i}^{(a)}, u_{m+1,i}^{(a)})\).

### 2.2. Barycentric Gegenbauer interpolation

Barycentric interpolation is an efficient variant of some polynomial and rational interpolations, which has recently attracted widespread attentions [1, 6–8]. And the corresponding stabilities at the set of Gauss points are analysed in [9–13]. The barycentric Gegenbauer interpolation of a real function \(\nu(x), x \in [-1,1]\) at Gegenbauer-Gauss nodes \(\{x_{m,i}^{(a)}\}\) is defined by

\[
p_{B,m}^{(a)} u(x) = \sum_{i=0}^{m} G_{B,m,i}^{(a)}(x) u_{m,i}^{(a)}, \quad G_{B,m,i}^{(a)}(x) = \frac{\omega_{m,i}^{(a)}}{x - x_{m,i}^{(a)}} \left( \sum_{j=0}^{m} \frac{\omega_{m,j}^{(a)}}{x - x_{m,j}^{(a)}} \right)^{-1}, \quad (7)
\]

where \(u_{m,i}^{(a)} = u(x_{m,i}^{(a)}), i = 0, 1, ..., m\), \(G_{B,m,i}^{(a)}(x)\) are the barycentric Gegenbauer interpolation basis functions, and \(\omega_{m,i}^{(a)}\) are the barycentric weights. The explicit barycentric weights for the Gegenbauer-Gauss nodes are provided in [14] are as follows

\[
\omega_{m,j}^{(a)} = (-1)^i \left[ 1 - \left( x_{m,i}^{(a)} \right)^2 \right]^{1/2}, \quad (8)
\]

which may be still unstable. Later, the numerically stable barycentric weights \(\omega_{m,i}^{(a)}\) at Gegenbauer-Gauss nodes provided in [1] are as follows

\[
\omega_{m,i}^{(a)} = (-1)^i \sin \left( \cos^{-1} x_{m,i}^{(a)} \right) \left( w_{om,i}^{(a)} \right)^{1/2}. \quad (9)
\]
The barycentric Gegenbauer interpolation only need $O(m)$ operations for known barycentric weights.

2.3. Two dimensional barycentric Gegenbauer interpolation
Let $\{G_{B,m,j}^{(a)}(x)\}_{i=0}^{m}$ and $\{G_{B,n,j}^{(a)}(y)\}_{j=0}^{n}$ be two sets of barycentric Gegenbauer basis functions of $P_{B,m}$ and $P_{B,n}$, respectively. Then, the 2D barycentric Gegenbauer basis functions $\tilde{G}_{B,mn,j}^{(a)}(x,y)$ of $P_{B,mn}$ is constructed by using tensor product of $G_{B,m,j}^{(a)}(x)$ and $G_{B,n,j}^{(a)}(y)$, more precisely,

$$P_{B,mn}^{(a)} = \text{span}\{G_{B,m,j}^{(a)}(x)G_{B,n,j}^{(a)}(y) : i = 0, 1, \ldots, m, j = 0, 1, \ldots, n\}. \quad (10)$$

Then 2D barycentric Gegenbauer interpolation of an arbitrary smooth function $u(x,y)$ on a rectangle domain $(x,y) \in [-1,1]^2$ associated with tensor product nodes $\{(x_{0,j}, y_{0,j})\}$ is denoted by

$$P_{B,mn}^{(a)} u(x,y) = \sum_{(i,j) = (0,0)}^{(m,n)} G_{B,mn,ij}^{(a)}(x,y)u_{mn,ij}^{(a)} = \sum_{(i,j) = (0,0)}^{(m,n)} G_{B,m}^{(a)}(x)G_{B,n}^{(a)}(y)u_{mn,ij}^{(a)}, \quad (11)$$

where $u_{mn,ij}^{(a)} = u(x_{i}, y_{j})$.

3. Numerical Algorithm
In this section, we solve the special 2D-FVIEs (1) using the 2D barycentric Gegenbauer interpolation collocation method. Without loss of generality, we first convert $D = [0,1]^2$ into $\tilde{D} = [-1,1]^2$ by the variable transformations $x = (1 + \tilde{x})/2$ and $y = (1 + \tilde{y})/2, (\tilde{x}, \tilde{y}) \in [-1,1]^2$. Under the notations

$$\tilde{u}(\tilde{x}, \tilde{y}) = u(((1 + \tilde{x})/2, (1 + \tilde{y})/2), \tilde{f}(\tilde{x}, \tilde{y}) = f(((1 + \tilde{x})/2, (1 + \tilde{y})/2).$$

The equation (1) can be rewritten as the following form,

$$\tilde{u}(\tilde{x}, \tilde{y}) + \int_{0}^{1} \tilde{k}_1((1 + \tilde{x})/2, \tilde{t}) u(\tilde{t}, (1 + \tilde{y})/2) d\tilde{t} + \int_{0}^{1} \tilde{k}_2((1 + \tilde{y})/2, \tilde{s}) u((1 + \tilde{x})/2, \tilde{s}) d\tilde{s} = \tilde{f}(\tilde{x}, \tilde{y}). \quad (12)$$

Further, we set $t = (1 + \tilde{t})/2, s = (1 + \tilde{s})/2, (\tilde{t}, \tilde{s}) \in [-1,1] \times [-1,1]$, and give the notations

$$\tilde{k}_1(\tilde{x}, \tilde{t}) = \frac{1}{2} k_1((1 + \tilde{x})/2, (1 + \tilde{t})/2), \tilde{k}_2(\tilde{y}, \tilde{s}) = \frac{1}{2} k_2((1 + \tilde{y})/2, (1 + \tilde{s})/2).$$

The domain $D = [0,1]^2$ in (1) becomes the domain $\tilde{D} = [-1,1]^2$ in the equation below,

$$\tilde{u}(\tilde{x}, \tilde{y}) + \int_{-1}^{1} \tilde{k}_1(\tilde{x}, \tilde{t}) \tilde{u}(\tilde{t}, \tilde{y}) d\tilde{t} + \int_{-1}^{1} \tilde{k}_2(\tilde{y}, \tilde{s}) \tilde{u}(\tilde{x}, \tilde{s}) d\tilde{s} = \tilde{f}(\tilde{x}, \tilde{y}), (\tilde{x}, \tilde{y}) \in \tilde{D}. \quad (13)$$

Next a collocation method to the equation (13) at the Gegenbauer-Gauss nodes $\{(\tilde{x}_{m,k}, \tilde{y}_{n,i})\}$ is

$$\tilde{u}_{mn,kl}^{(a)} + \int_{-1}^{1} \tilde{k}_1(\tilde{x}_{m,k}, \tilde{t}) \tilde{u}(\tilde{t}, \tilde{y}_{n,i}) d\tilde{t} + \int_{-1}^{1} \tilde{k}_2(\tilde{y}_{n,i}, \tilde{s}) \tilde{u}(\tilde{x}_{m,k}, \tilde{s}) d\tilde{s} = \tilde{f}_{mn,kl}^{(a)}, \quad (14)$$

where $\tilde{u}_{mn,kl}^{(a)} = \tilde{u}(\tilde{x}_{m,k}, \tilde{y}_{n,i})$, $\tilde{f}_{mn,kl}^{(a)} = \tilde{f}(\tilde{x}_{m,k}, \tilde{y}_{n,i})$. For a further approximation, we use the linear transformation $\tilde{z}(\tilde{x}_{m,k}, \tilde{y}_{n,i}) = (\tilde{y}_{n,i} + 1)\tilde{y}/2 + (\tilde{y}_{n,i} - 1)\tilde{y}/2, \tilde{y} \in [-1,1]$, and have

$$\tilde{u}_{mn,kl}^{(a)} + \int_{-1}^{1} \tilde{k}_1(\tilde{x}_{m,k}, \tilde{t}) \tilde{u}(\tilde{t}, \tilde{y}_{n,i}) d\tilde{t} + \int_{-1}^{1} \tilde{k}_2(\tilde{y}_{n,i}, \tilde{s}) \tilde{u}(\tilde{x}_{m,k}, \tilde{s}) d\tilde{s} = \tilde{f}_{mn,kl}^{(a)}.$$ 

With $\tilde{c}_{n,i}^{(a)} = (\tilde{y}_{n,i} + 1)/2$. Now we approximate the integral terms by the Legendre-Gauss quadrature formulas with the quadrature nodes and weights denoted by $\{\tilde{x}_{p}, \tilde{w}_{p}\}_{p=1}^{m}$ and $\tilde{y}_{q}, \tilde{w}_{q} \}_{q=1}^{n}$, that is,
\[ \tilde{u}_{mn,kl}^{(a)} + \sum_{p=1}^{\eta} \tilde{c}_{p}^{(a)} \tilde{k}_{1}(\tilde{x}_{m,k}^{(a)}, \tilde{y}_{n,l}^{(a)}) \tilde{u}(\tilde{x}_{m,k}, \tilde{y}_{n,l}) \]
\[ + \sum_{q=1}^{\eta} \tilde{c}_{q}^{(a)} \tilde{k}_{2}(\tilde{y}_{n,l}^{(a)}, \tilde{z}(\tilde{y}_{n,l}^{(a)}, \tilde{\eta}_{q})) \tilde{u}(\tilde{x}_{m,k}, \tilde{z}(\tilde{y}_{n,l}^{(a)}, \tilde{\eta}_{q})) = \tilde{f}_{mn,kl}^{(a)}. \]  
(16)  

Finally, we use the two dimensional interpolation to find \( \tilde{u}_{mn}(x, y) \) such that
\[ \tilde{u}_{mn,kl}^{(a)} + \sum_{(i,j)=(0,0)}^{(m,n)} \{ \sum_{p=1}^{\eta} \tilde{c}_{p}^{(a)} \tilde{k}_{1}(\tilde{x}_{m,k}^{(a)}, \tilde{y}_{n,l}^{(a)}) G_{B, mn, ij}(\tilde{x}_{m,k}, \tilde{y}_{n,l}) \}
\[ + \sum_{q=1}^{\eta} \tilde{c}_{q}^{(a)} \tilde{k}_{2}(\tilde{y}_{n,l}^{(a)}, \tilde{z}(\tilde{y}_{n,l}^{(a)}, \tilde{\eta}_{q})) G_{B, mn, ij}(\tilde{x}_{m,k}, \tilde{z}(\tilde{y}_{n,l}^{(a)}, \tilde{\eta}_{q})) \} \tilde{u}_{mn, ij}^{(a)} = \tilde{f}_{mn,kl}^{(a)}. \]  
(17)

The approximate nodal function values \( \tilde{u}_{mn, ij}^{(a)} \) at the Gegenbauer-Gauss nodes \( (x_{m,k}, y_{n,l}) \) are gained by solving the linear algebraic equation systems (17). Then the approximate solution of equation (1) is
\[ u_{mn}(x, y) = \sum_{(i,j)=(0,0)}^{(m,n)} G_{B, mn, ij}(x, y) \tilde{u}_{mn, ij}^{(a)}. \]  
(18)

4. Numerical Experiments  
This section two special 2D-FVIEs are demonstrated to test the numerical accuracy and convergent order of the present method based on the algorithm in Section 3. The absolute error is defined by
\[ e_{ij} = \left| u(x_{m,i}, y_{n,j}) - u_{mn}(x_{m,i}, y_{n,j}) \right|, \quad i = 0, 1, ..., m, \quad j = 0, 1, ..., n, \]  
(19)
where \( u(x, y) \) and \( u_{mn}(x, y) \) are respectively the exact solution and the numerical solution. Further, the maximum error and the convergent order are respectively defined by
\[ e_{mn}^{\max} = \max_{i,j} \{ e_{ij} \}, \quad \text{Order} = \log_{2} \left( \frac{e_{mn}^{\max}}{e_{mn}^{2^{m,n}}} \right). \]  
(20)

Example 1. Consider the special two dimensional Fredholm-Volterra integral equation
\[ u(x, y) + \int_{0}^{1} (5x + 2t^2 - 5) u(t, y) \, dt + \int_{0}^{y} (-4y^2 + ys) u(x, s) \, ds = f(x, y), \]  
(21)
with \( 0 \leq x, y \leq 1 \), and \( f(x, y) \) is determined by the exact solution \( u(x, y) = x y + 3 \).

| Table 1. Numerical results based on the barycentric Gegenbauer interpolation, Example 1. |
| --- | --- | --- | --- | --- | --- |
| \( \alpha \) | \( m = n = 2^2 \) | \( m = n = 2^3 \) | \( m = n = 2^4 \) | \( m = n = 2^5 \) | \( m = n = 2^6 \) | \( m = n = 2^7 \) |
| 0 | \( e_{mn}^{\max} \) | 1.1012e-01 | 4.7962e-02 | 1.9028e-02 | 1.7108e-02 | 7.4499e-03 |
| Order | -- | 1.1991 | 1.3338 | 0.1535 | 1.1994 |
| 0.5 | \( e_{mn}^{\max} \) | 3.1770e-02 | 1.0556e-02 | 3.1747e-03 | 2.3565e-03 | 7.3579e-04 |
| Order | -- | 1.5896 | 1.7334 | 0.4300 | 1.6793 |
| 1 | \( e_{mn}^{\max} \) | 1.1911e-02 | 3.1956e-03 | 8.4887e-04 | 4.4277e-04 | 1.0022e-04 |
| Order | -- | 1.8981 | 1.9125 | 0.9390 | 2.1434 |
| 2 | \( e_{mn}^{\max} \) | 2.7701e-03 | 5.7213e-04 | 1.1327e-04 | 2.3374e-05 | 3.0821e-06 |
| Order | -- | 2.2755 | 2.3366 | 2.2768 | 2.9229 |
| 4 | \( e_{mn}^{\max} \) | 1.1989e-02 | 5.9946e-04 | 1.8097e-05 | 4.3721e-07 | 2.0228e-08 |
| Order | -- | 4.3219 | 5.0498 | 5.3713 | 4.4339 |

The equation (21) is considered to be solved by the two dimensional barycentric Gegenbauer interpolation collocation method for the \( 3 \times 3 \) Legendre-Gauss quadrature formula. Numerical results in Table 1 indicate that the maximum nodal error and the convergent order are dependent on the nodal numbers and the parameter \( \alpha \). The error distributions by the present method are near \( e^{-07} \) for \( m = n = 64 \) in Figure 1(a) while by the numerical method based on the two dimensional block pulse functions in [4] are near \( e^{-02} \) for \( m = n = 64 \). From Figure 1(b), one can see that the errors are still decreasing.
for denser partition. Table 1, Figure 1(a) and Figure 1(b) demonstrate that the scheme has a high order numerical approximation for a suitable parameter $\alpha$.

![Figure 1(a) Nodal error distributions based on the two dimensional barycentric Gegenbauer interpolation collocation method with $m = n = 64$, $\alpha = 4$ for the fixed $3 \times 3$ Gauss-Legendre quadrature formula.](image1)

![Figure 1(b) Nodal error distributions based on the two dimensional barycentric Gegenbauer interpolation collocation method with $m = n = 128$, $\alpha = 4$ for the fixed $3 \times 3$ Gauss-Legendre quadrature formula.](image2)

**Example 2.** Consider the special two dimensional Fredholm-Volterra integral equation

$$u(x, y) + \int_0^1 \frac{t}{x+z} u(t, y) dt + \int_0^\beta \exp(y + s)u(x, s)ds = f(x, y), \quad (22)$$

with $0 \leq x, y \leq 1$, and $f(x, y)$ is determined by the exact solution $u(x, y) = \exp(x - y)$.

| $\alpha$ | $m = n = 2^4$ | $m = n = 2^5$ | $m = n = 2^6$ | $m = n = 2^7$ |
|---|---|---|---|---|
| 0  | $e_{\text{max}}^{m,n}$ | 2.5748e-02 | 6.1790e-02 | 5.2927e-02 | 2.5473e-02 |
|   | Order | -- | 0.1696 | 0.2234 | 1.0550 |
| 0.5 | $e_{\text{max}}^{m,n}$ | 4.8583e-02 | 2.4577e-02 | 1.3345e-02 | 7.8410e-03 | 2.4976e-03 |
|   | Order | -- | 0.9831 | 0.8810 | 0.7672 | 1.6505 |
| 1.1 | $e_{\text{max}}^{m,n}$ | 7.8076e-02 | 2.2259e-02 | 5.8772e-03 | 1.5375e-03 | 3.9499e-04 |
|   | Order | -- | 1.8105 | 1.9212 | 1.9345 | 1.9607 |

The equation (22) is considered to be solved by the 2D barycentric Gegenbauer interpolation collocation method for the $5 \times 5$ Legendre-Gauss quadrature formula. Numerical results in Table 2 indicate that the maximum nodal error and the convergent order depend on the nodal numbers and the parameter $\alpha$. The error distributions by the present method are near e-03 for $m = n = 64$ in Figure 2(a) while by the two dimensional block pulse function operational matrix method of [4] are near e-02 for $m = n = 64$. From Figure 2(b), one can see that the errors are still decreasing for denser partition. Table 2, Figure 2(a) and Figure 2(b) show that the scheme has a high order numerical approximation for a suitable parameter $\alpha$.

### 5. Conclusion

In this paper, we approximate the solution of the special two dimensional Fredholm-Volterra integral equations (2D-FVIEs) by the two dimensional barycentric Gegenbauer interpolation. The special 2D-FVIEs are transformed into the corresponding system of discrete algebraic equations by combining the two dimensional barycentric Gegenbauer interpolation collocation method with the double Legendre-Gauss quadrature formula. The solution of the discrete equations is a set of the nodal function values at the Gegenbauer-Gauss nodes. Numerical results illustrate that the scheme is high order and efficient.
for a suitable choice of the parameter $\alpha$. Additionally, a comparison with the numerical method based on the two dimensional block pulse functions show that the current method has a better performance.

![Figure 2(a) Nodal error distributions based on the two dimensional barycentric Gegenbauer interpolation collocation method with $m = n = 64, \alpha = 1.1$ for the fixed 5 $\times$ 5 Gauss-Legendre quadrature formula.](image1)

![Figure 2(b) Nodal error distributions based on the two dimensional barycentric Gegenbauer interpolation collocation method with $m = n = 128, \alpha = 1.1$ for the fixed 5 $\times$ 5 Gauss-Legendre quadrature formula.](image2)

**Acknowledgements**

The authors are very grateful to the team members and the reviewers for their valuable advices. This job was supported by the National Natural Science Foundation of China (Grant no. 11801062).

**References**

[1] Elgindy K 2017 High-order, stable, and efficient pseudospectral method using barycentric Gegenbauer quadratures. *Appl. Numer. Math.* 113: 1-25.

[2] Kovalenko E 1984 The solution of contact problems of creep theory for combined ageing foundations. *J. Appl. Math. Mech.* 48(6): 739-745.

[3] Abdou M 2003 On asymptotic methods for Fredholm-Volterra integral equation of the second kind in contact problems. *J. Comput. Appl. Math.* 154: 431-446.

[4] Babolian E, Maleknejad K, Mordad M and Rahimi B 2011 A numerical method for solving Fredholm-Volterra integral equations in two-dimensional spaces using block pulse functions and an operational matrix. *J. Comput. Appl. Math.* 235: 3965-3971.

[5] Xie J, Huang Q and Zhao F 2017 Numerical solution of nonlinear Volterra-Fredholm-Hammerstein integral equations in two-dimensional spaces based on block pulse functions. *J. Comput. Appl. Math.* 317: 565-572.

[6] Berrut J and Trefethen L 2004 Barycentric Lagrange interpolation. *SIAM Rev.* 46(3): 501-517.

[7] Floater M and Hormann K 2007 Barycentric rational interpolation with no poles and high rates of approximation. *Numer. Math.* 107(2): 315-331.

[8] Sadiq B and Viswanath D 2014 Barycentric Hermite interpolation. *SIAM J. Sci. Comput.* 35(3): A1254-A1270.

[9] Higham N 2004 The numerical stability of barycentric Lagrange interpolation. *IMA J. Numer. Anal.* 24(4): 547-556.

[10] Mascarenhas W and Camargo A 2014 On the backward stability of the second barycentric formula for interpolation. *Dolomites. Res. Notes. Approx.* 7(1): 1-12.

[11] Mascarenhas W 2014 The stability of barycentric interpolation at the Chebyshev points of the second kind. *Numer. Math.* 128(2): 265-300.

[12] Camargo A 2016 On the numerical stability of Floater-Hormanns rational interpolant. *Numer. Algorithms* 72(1): 131-152.

[13] Lawrence P and Corless R 2012 Numerical stability of barycentric Hermite root-finding.
International Workshop on Symbolic-Numeric Computation pp 147-148.

[14] Wang H, Huybrechs D and Vandewalle S 2014 Explicit barycentric weights for polynomial interpolation in the roots or extrema of classical orthogonal polynomials. *Math. Comput.* 83(290): 2893-2914.