Information Theoretical Approach to Control of Quantum Mechanical Systems

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Fundamental limits on the controllability of quantum mechanical systems are discussed in the light of quantum information theory. It is shown that the amount of entropy-reduction that can be extracted from a quantum system by feedback controller is upper bounded by a sum of the decrease of entropy achievable in open-loop control and the mutual information between the quantum system and the controller. This upper bound sets a fundamental limit on the performance of any quantum controllers whose designs are based on the possibilities to attain low entropy states. An application of this approach pertaining to quantum error correction is also discussed.

I. INTRODUCTION

Ever since the discovery of quantum mechanics, the problem of controlling quantum systems has been an important experimental issue [1, 2]. A variety of techniques are available for controlling quantum systems, and a detailed theory of quantum control has been developed [3, 4, 5, 6]. Methods of geometric and coherent control are particularly powerful for controlling the coherent dynamics of quantum systems [2, 7].

The rapid development of quantum information technology suggests that quantum control theory might profitably be reexamined from the perspective of quantum information theory [8, 9]. In this paper we address explicitly the role of quantum information and entropy in quantum control processes. Specifically, based on classical theories [10, 11, 12, 13, 14], we prove several limiting results relating to the ability of a control device to reduce the von Neumann entropy of an arbitrary quantum system \( \rho^Q \) in the cases where (i) a controller independently acts to the state of the system (open-loop control) and (ii) the control action is influenced by some information gathered from the system (feedback control).

When a quantum system \( Q \) initially prepared in a pure state \( \rho_0 \) interacts with an environment represented by the density operator \( \rho^E \), the system \( Q \) and environment evolve according to the joint unitary evolution operator \( U_{QE} \). Then the density operator for the system \( Q \) and environment is

\[
\rho = U_{QE} (\rho_0 \otimes \rho^E) U_{QE}^\dagger.
\]  

(1)

After performing a partial trace over environment variables, the marginal density matrix of the system \( Q \) is represented by a completely positive and trace preserving map \( \mathcal{E} \), which takes the form

\[
\rho^Q = \mathcal{E} (\rho_0) = \sum_i E_i \rho_0 E_i^\dagger,
\]  

(2)

where the Kraus operators \( E_i \)'s satisfy the trace preserving property, i.e., \( \sum_i E_i^\dagger E_i = I \). This equation is known as operator-sum representation of the quantum operation \( \mathcal{E} \). Unitary evolution of the quantum system is a special case in which there is only one non-zero term in the operator sum. On the other hand, if there are two or more terms, the pure initial state becomes a mixed state. Therefore, the von Neumann entropy of the system \( Q \) increases, i.e., \( S(Q) = S(\rho^Q) > S(\rho_0) \), because of the interaction with environment. In this paper, we define the purpose of quantum control as a reduction of the entropy of the system \( Q \), e.g., quantum Maxwell demon [15, 16], quantum bang-bang control [17, 18] and quantum error correcting code [19, 20]. In the following, we shall show the information-theoretic analysis of open-loop and closed-loop (feedback) control, and give the fundamental limits on the control of quantum mechanical systems from the viewpoint of quantum information theory.

II. QUANTUM OPEN-LOOP CONTROL

Here, we present information-theoretic analysis of the quantum open-loop control. First, we shall look at a joint unitary evolution (a control unitary operation) of the quantum system \( Q \) and controller \( C \). Let the quantum system \( Q \) and the controller \( C \) be disentangled before the control unitary operation. We also assume that the states of system \( Q \) and \( C \) are respectively given by eq. (2) and

\[
\rho^C = \sum_i p_i |i\rangle_C \langle i|.
\]  

(3)

Here \( |i\rangle_C \) is an orthonormal basis of system \( C \) and \( \sum_i p_i = 1 \). Therefore, the state of the joint system...
Then, the marginal density operator of $Q$ is given by

$$\rho^{QC} = \rho^Q \otimes \rho^C = \sum_{i,j} p_i \rho^Q_j \otimes |i\rangle_C \langle i|, \quad (4)$$

where $\rho^Q_j \equiv E_j \rho_E^j$. In order to reduce the entropy of the system $Q$, a control unitary transformation $U_{open}$ is applied to joint system $QC$. Then, the system $QC$ undergoes the evolution:

$$\rho^{QC} \rightarrow U_{open} \rho^{QC} U_{open}^\dagger. \quad (5)$$

In the following we shall consider two types of control unitary operation, i.e., global unitary operation (Fig. 1) and LOCC (local quantum operation and classical communication) (see Fig. 2). In the former case, the entropy of the total system becomes

$$S(Q,C) = S(Q_{out},C_{out}) \leq S(Q_{out}) + S(C_{out}), \quad (6)$$

where we have used the subadditivity of the entropy. From this inequality, we finally obtain the entropy reduction as

$$\Delta S_Q^{open} = S(Q) - S(Q_{out}) \leq S(C_{out}) - S(C), \quad (7)$$

with equality if and only if $\rho^Q_{out} = \rho^Q \otimes \rho^{C_{out}}$. Therefore, the entropy reduction is upper bounded by the maximum amount of the entropy increase of $C$.

On the other hand, in the case of LOCC strategy (Fig. 2), the control unitary operation is given by

$$U_{open} = \sum_i U_i \otimes |i\rangle_C \langle i|. \quad (8)$$

Therefore, the state after the open loop control becomes

$$\rho^{Q_{out}C_{out}} = \sum_{i,j} p_i U_i \rho^Q_j U_i^\dagger \otimes |i\rangle_C \langle i|. \quad (9)$$

Then, the marginal density operator of $Q_{out}$ is given by

$$\rho^{Q_{out}} = \text{Tr}_C \rho^{C_{out}Q_{out}} = \sum_i p_i U_i \rho^Q U_i^\dagger. \quad (10)$$

Now using the concavity of the von Neumann entropy $S(\sum_i p_i \rho_i) \geq \sum_i p_i S(\rho_i)$, we see that

$$S(Q_{out}) = S\left(\sum_i p_i U_i \rho^Q U_i^\dagger\right) \geq \sum_i p_i S\left(U_i \rho^Q U_i^\dagger\right) = S(Q). \quad (11)$$

Therefore for open-loop control using the LOCC strategy, we finally obtain

$$\Delta S_Q^{open} \leq 0. \quad (12)$$

This means that we can never reduce the entropy of system $Q$ in contrast with the case of the global unitary operation strategy.

### III. QUANTUM FEEDBACK CONTROL

Next we shall consider the quantum feedback control. In this case, the controller $C$ performs measurements on the system $Q$ and feeds back the results of these measurements by applying operations that are the functions of the measurement results. Although both the system $Q$ and the controller $C$ are quantum mechanical in principle, the feedback operations we consider here involve feeding back classical information. The feedback quantum information via fully coherent quantum feedback was recently discussed by Lloyd and co-workers [7, 21] and experimentally demonstrated [22].

To analyze quantum feedback control, we need to consider quantum measurement processes. For simplicity, we consider a positive operator-valued measure measurement in which the entropy of the system $Q$ does not decrease, e.g., the conventional von Neumann measurement.

As in the case of preceding section, we shall investigate two types of control strategies (Figs. 3 and 4). A basic quantum feedback control using a global control unitary operation is presented in Fig. 3. Then the entropy of $C_{out}$ is calculated as

$$S(C_{out}) = S(Q_{out}, C_{out}) - S(Q_{out}) + I(Q_{out} : C_{out})$$

$$= S(Q', C') - S(Q_{out}) + I(Q_{out} : C_{out})$$

$$\leq S(Q) - S(Q_{out}) + S(C') - I(Q' : C') + I(Q_{out} : C_{out}),$$

where $I(A : B) = S(A) + S(B) - S(A, B)$ is the quantum mutual information of systems $A$ and $B$ [6]. Therefore, the entropy reduction for quantum feedback using the
global unitary operation is given by

\[
\Delta S_{Q}^{\text{feedback}} = S(Q) - S(Q_{\text{out}}) \\
\leq S(Q', C') - S(Q_{\text{out}}) + I(Q_{\text{out}} : C_{\text{out}}) \\
= S(C_{\text{out}}) - S(C') - I(Q_{\text{out}} : C_{\text{out}}) \\
+ I(Q' : C') \\
\leq \max_{U} \Delta S^{\text{open}}_{Q} + I(Q' : C').
\]  

(14)

Here \(\max_{U} \Delta S^{\text{open}}_{Q}\) is the maximum entropy reduction attained by restricting the control model to open-loop system. The equality holds if and only if \(\rho^{Q} = \rho^{Q'}\) and \(S(C_{\text{out}}) - S(C') - I(Q_{\text{out}} : C_{\text{out}}) = \max_{U} \Delta S^{\text{open}}_{Q}\). Therefore, the maximum improvement that closed-loop can give over open-loop control is limited by the quantum mutual information obtained by the controller \(C_{\text{close}}\).

Next we shall consider quantum feedback control using the LOCC (see Fig. 4), and show that the entropy reduction is upper bounded by the quantum mutual information between \(Q'\) and \(C'\), i.e., \(\Delta S_{Q}^{\text{feedback}} \leq I(Q' : C')\).

In this strategy, one performs a measurement (on the state \(\rho^{Q}\)) described by positive operators \(\{P_{i}\}\), and feeds back the results by applying a unitary transformation \(U_{i}\) when the \(i\)th outcome is found. Then the state change of the subsystem \(Q\) can be written as

\[
\rho^{Q} \rightarrow \rho^{Q'} = \sum_{i} P_{i} \rho^{Q} P_{i}^{\dagger}
\]

\[
\rightarrow \rho^{Q_{\text{out}}} = \sum_{i} U_{i} P_{i} \rho^{Q} P_{i}^{\dagger} U_{i}^{\dagger} \equiv C(\rho^{Q}).
\]

(15)

(16)

From the inequality of the entropy exchange \(S_{C}(\rho, E)\) for a quantum operation \(E\),

\[
S(E(\rho)) - S(\rho) + S_{C}(\rho, E) \geq 0,
\]

(17)

it follow that

\[
S(Q_{\text{out}}) - S(Q) + S_{C}(\rho^{Q}, C) \geq 0.
\]

(18)

Thus we have inequality for the entropy reduction,

\[
\Delta S_{Q}^{\text{feedback}} = S(Q) - S(Q_{\text{out}}) \leq S_{C}(\rho^{Q}, C).
\]

(19)

The entropy exchange is not greater than the Shannon entropy for the probabilities \(r_{i} = \text{Tr}(U_{i} P_{i} \rho^{Q} P_{i}^{\dagger} U_{i}^{\dagger})\) [22]. Thus,

\[
S_{C}(\rho^{Q}, C) \leq H(r_{i}),
\]

(20)

where equality holds if and only if the operator \(U_{i} P_{i}\) are a canonical decomposition of \(C\) with respect to \(\rho^{Q}\) [23]. Therefore we obtain

\[
\Delta S_{Q}^{\text{feedback}} \leq H(r_{i}) = - \sum_{i} r_{i} \log r_{i}.
\]

(21)

We can also show the equality \(H(r_{i}) = I(Q' : C')\) [24]. That is, in the case of the quantum feedback using LOCC, the entropy reduction is given by

\[
\Delta S_{Q}^{\text{feedback}} \leq I(Q' : C') = H(r_{i}).
\]

(22)

This implies that the maximum amount of entropy reduction is exactly equal to the quantum mutual information between subsystems \(Q'\) and \(C'\), i.e., \(I(Q' : C')\).

The quantum mutual information \(I(A : B)\) is related to correlation between subsystems \(A\) and \(B\) [9]. If a joint system \(A\) and \(B\) is a product state, then \(I(A : B) = 0\). However, \(I(A : B) > 0\) if the subsystems \(A\) and \(B\) are (classically or quantum mechanically) correlated. In the case of the quantum feedback shown in Figs. 3 and 4, the quantum measurement germinates not quantum but classical correlation between \(Q\) and \(C\). Therefore, we can conclude that the classical correlation between \(Q'\) and \(C'\) can increase the amount of the entropy reduction in compared with the case of the quantum open loop control.

Finally, we shall show the information theoretical analysis of quantum error correction [3] 22 as an example of the quantum feedback control using LOCC (Fig. 4). The quantum error correction can be thought of as a type of refrigeration process, capable of keeping a quantum system \(Q\) at a constant entropy, despite the influence of noise processes which tend to increase the entropy of system \(Q\). General error correction procedure is described by following three steps ((a) error (b) syndrome measurement and (c) error correction) 20 [24],

\[
\rho_{0} = |\psi\rangle_{Q} \langle \psi | \otimes | 0\rangle_{M} | 0\rangle
\]

\[
\overset{(a)}{\rightarrow} \rho = \sum_{i} p_{i} |e_{i}\rangle_{Q} \langle e_{i} | \otimes | 0\rangle_{M} | 0\rangle
\]

\[
\overset{(b)}{\rightarrow} \rho' = \sum_{i} p_{i} |e_{i}\rangle_{Q} \langle e_{i} | \otimes | i\rangle_{M} | i\rangle
\]

\[
\overset{(c)}{\rightarrow} \rho_{\text{out}} = |\psi\rangle_{Q} \langle \psi | \otimes \sum_{i} p_{i} |i\rangle_{M} | i\rangle
\]

(23)
Therefore, the entropy reduction is given by
\[ \rho_I = \sum_i p_i e_i |\psi_i\rangle \langle \psi_i|, \]
where \( |\psi_i\rangle \) is the (initial) encoded quantum state, \( e_i \) is the unitary error operator which acts on the state of system \( Q \) only and \( |i\rangle \) is the orthonormal basis of a measurement apparatus which acts as the controller \( C \). In the language of the control theory, steps (b) and (c) correspond to the feedback and the control operation process, respectively. In the following, we shall show an entropic analysis of the quantum error correction and show that equality of Eq. (22) holds. For simplicity, we restrict our analysis to the case of non-degenerate quantum code. From Eq. (23), we obtain the reduced density operator of the quantum system \( Q \) as
\[ \rho_Q = \text{Tr}_M \rho = \sum_i p_i e_i |\psi_i\rangle \langle \psi_i|, \]
\[ \rho_{Q \text{ out}} = \text{Tr}_M \rho_{\text{out}} = |\psi\rangle \langle \psi|. \]
Therefore, the entropy reduction is given by
\[ \Delta S_Q^{\text{EC}} = S(Q) - S(Q_{\text{out}}) = H(p_i) + \sum_i p_i S(\rho_i^Q) = H(p_i), \]
where \( \rho_i^Q \equiv e_i |\psi_i\rangle \langle \psi_i| \). On the other hand, quantum mutual information before step (c) is \( I(Q' : C') = H(p_i) \).

Therefore we have
\[ \Delta S_Q^{\text{EC}} = H(p_i) = I(Q' : C'), \]
which is the desired result, i.e., equality of Eq. (22) holds. This means that the quantum error correction is an optimal quantum feedback control system from the viewpoint of the information theory.

IV. SUMMARY

In summary, we have analyzed the quantum control system by use of the quantum information theory and showed information theoretical limits of control of quantum mechanical systems. By applying our approach to an entropic analysis of quantum error correcting code, we have shown that we can regard the quantum error correction procedure as an (information theoretically) optimal quantum feedback control system. Our result will also help in understanding quantum bang-bang control and quantum Maxwell demon.

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[24] In order to obtain the equation \( H(r_i) = I(Q' : C') \), we notice that the joint state after measurement is given by
\( \rho_Q^{C'} = \sum_i P_i^Q P_i^{C'} \otimes |i\rangle \langle i| \), where
\( P_i = P_i^Q P_i^{C'} \). Therefore the entropy of the system \( Q' C' \) is given by
\( S(Q' C') = S(\sum_i r_i P_i Q C |i\rangle \langle i|) = H(r_i) + \sum_i r_i S(P_i) = H(c) \), where in the second equality we have used the joint entropy theorem. The marginal density operators of subsystems \( Q' \) and \( C' \) are respectively given by
\( \rho_Q' = \sum_i r_i P_i |i\rangle \langle i| \) and
\( \rho_C' = \sum_i r_i P_i |c\rangle \langle c| \), so we have
\( S(Q') = S(C') = H(c) \).
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