Conditions for the stability of a Cournot duopoly model with tax evasion and time delay

Raúl Villafuerte-Segura¹  Eduardo Alvarado-Santos¹
Benjamín A. Itzá-Ortiz² *

¹ Centro de Investigación en Computación y Sistemas, Universidad Autónoma del Estado de Hidalgo, Pachuca, Hidalgo, MÉXICO
² Centro de Investigación en Matemáticas, Universidad Autónoma del Estado de Hidalgo, Pachuca, Hidalgo, MÉXICO

February 18th, 2019

Abstract

We study a Cournot duopoly model with tax evasion and time delay. We prove that if the marginal production costs of both competing firms are equal then the equilibrium point is asymptotically stable and independent of time delay. As consequence, our model can not have bifurcations if the delay, as a parameter, is varied. It further imply that less tax evasion and higher public revenue can be achieved either by increasing the effectiveness of audits or by adjusting the penalties for tax evasion.

Introduction

Since its introduction in 1838, the Cournot model [3] has provided abundant cases of study for both basic and applied research. As time-delay has become an inherent property often needed when modeling natural phenomena, including economic dynamics, the importance of discussing a Cournot model with time-delay was recognized decades ago [8]. Recently, it has become an active research area [4][6][7][10][11]. In [5], a Cournot oligopoly model with tax evasion was introduced, and subsequently studied, in the duopoly setting, by other authors [1][9][12], where in addition, the introduction of a time delay was made. The time delay in the Cournot duopoly model is justified just by considering that a firm in the duopoly enters the market first followed by the second firm some time later. Besides the two classical variables in the model representing the production of each firm in the duopoly, the modeling of tax evasion has introduced two new variables representing the declared revenue upon which tax due is calculated. As consequence,
opportunities to formulate and study examples of stabilities, bifurcations and instabilities has been created.

The aim of this paper is to provide conditions for the stability of the equilibrium point of a Cournot duopoly model with tax evasion and time delay. Furthermore, we verify that the stability is independent of the delay. Somewhat similar results where obtained in [7, Theorem 4] for a particular duopoly Cournot model with time delay but without the tax evasion modelling. It turns out that our conditions, besides including some standard mathematical economics conditions, such as strict monotonicity and strict convexity of the cost and penalty functions, it requires both firms in the duopoly to have same marginal production costs and same second derivatives of the cost functions at the equilibrium point. We emphasize that proofs of our results do not require explicit formulas of the functions involved, just its mathematical economics conditions. In addition, a technical condition on the second partial derivatives of the profit functions will be required. It has been observed [5, Page 723] that this technical condition means that tax evasion alone is likely to be responsible for a reduction of public funds.

We divide this work in four sections. In Section 1, we present the model and prove some basic results which will be used in subsequent sections. The main results of the paper are stated and proved in Section 2. In Section 3 we provide some examples of cost functions and inverse demand functions which satisfy the conditions of stability presented in the paper. Finally, conclusions are stated in Section 4.

1 The model

In this section we will define and explain the Cournot model studied in this paper and will also derive some elementary results needed for later. This Cournot model was originally introduced in [5]. We begin by letting $x_1$ and $x_2$ denote the quantities produced by each firm in the duopoly, while $z_1$ and $z_2$ will denote the amount of money each firm declares as revenue to the tax authority. The inverse demand function $p(x)$, where $x = x_1 + x_2$, is a positive, strictly decreasing, twice differentiable function, that is, $p'(x) < 0$, and such that

$$p'(x_1 + x_2) + x_i p''(x_1 + x_2) < 0. \tag{1}$$

The cost function $C_i(x_i)$ for each firm will be a positive twice differentiable function with positive first derivative and nonnegative second derivative, that is, it is an strictly increasing and convex function. A penalty $F(x_i p(x_1 + x_2) - z_i)$ is imposed on the $i$-th firm which is caught evading taxes; regarded as a function, we assume $F$ to be nonnegative and twice differentiable, having positive first and second derivatives and such that $F(0) = 0$. The tax rate and the detection probabilities are denoted by $0 < \sigma < 1$ and $0 < q_i < 1$, respectively.

For $i = 1, 2$, we define the profit function $P_i$ as the sum of two functions. The first addend is

$$(1 - q_i) \left( x_i p(x_1 + x_2) - C_i(x_i) - \sigma z_i \right),$$

which represent the probability $1-q_i$ of not being caught evading tax times the result of subtracting to the income $x_i p(x_1 + x_2)$ the cost $C_i(x_i)$ and the tax rate paid on the declared income $z_i \leq$
Let
\(x_i p(x_1 + x_2)\). The second addend in the profit function \(P_i\) is

\[ q_i \left( (1 - \sigma_x) x_i p(x_1 + x_2) - C_i(x_i) - F(x_i p(x_1 + x_2) - z_i) \right), \]

which represent the probability \(q_i\) of being caught evading tax times the result of subtracting to the income \(x_i p(x_1 + x_2)\) of the \(i\)th firm, the cost \(C_i(x_i)\), the tax rate due \(\sigma_x p(x_1 + x_2)\) and a penalty \(F(x_i p(x_1 + x_2) - z_i)\) on the evaded amount. Hence, we may rewrite the profit function as

\[ P_i = P_i(x_1, x_2, z_i) = (1 - q_i \sigma_x) x_i p(x_1 + x_2) - (1 - q_i) \sigma z_i - C_i(x_i) - q_i F(x_i p(x_1 + x_2) - z_i). \]

The following computations will prove useful throughout the paper.

**Lemma 1.1.** Let \(1 \leq i, j \leq 2\) with \(i \neq j\). The second derivatives of the profit function \(P_i\) defined in (2) are given by the formulas

\[
0 > \frac{\partial^2 P_i}{\partial x_j \partial z_i} = q_i x_i p'(x_1 + x_2) F''(x_i p(x_1 + x_2) - z_i),
\]

\[
0 > \frac{\partial^2 P_i}{\partial z_i^2} = -q_i F''(x_i p(x_1 + x_2) - z_i),
\]

\[
\frac{\partial^2 P_i}{\partial x_i^2} = \left( 1 - q_i (\sigma + F'(x_i p(x_1 + x_2) - z_i)) \right) (2p'(x_1 + x_2) + x_i p''(x_1 + x_2)) - C_i''(x_i)
- q_i (p(x_1 + x_2) + x_i p'(x_1 + x_2))^2 F''(x_i p(x_1 + x_2) - z_i),
\]

\[
\frac{\partial^2 P_i}{\partial x_j \partial x_i} = \left( 1 - q_i (\sigma + F'(x_i p(x_1 + x_2) - z_i)) \right) (p'(x_1 + x_2) + x_i p''(x_1 + x_2))
- q_i x_i p'(x_1 + x_2) (p(x_1 + x_2) + x_i p'(x_1 + x_2)) F''(x_i p(x_1 + x_2) - z_i),
\]

\[
\frac{\partial^2 P_i}{\partial z_i \partial x_i} = q_i (p(x_1 + x_2) + x_i p'(x_1 + x_2)) F''(x_i p(x_1 + x_2) - z_i).
\]

**Proof.** The proof is a straightforward computation, we omit details. The two inequalities follow from the properties of the functions involved.

**Proposition 1.2.** The profit functions \(P_i\), as defined in (2), has a maximum at the points \(x_i = x_i^*, z_i = z_i^*\), for \(i = 1, 2\), which satisfy the system of four equations

\[
(1 - \sigma_x) p(x_1 + x_2) + x_i p'(x_1 + x_2) - C_i'(x_i) = 0,
\]

\[
\sigma - q_i \left( \sigma + F'(x_i p(x_1 + x_2) - z_i) \right) = 0.
\]

**Proof.** For \(i = 1, 2\), the critical points satisfy the following two equations

\[
\frac{\partial P_i}{\partial x_i} = \left( 1 - q_i \sigma - q_i F'(x_i p(x_1 + x_2) - z_i) \right) (p(x_1 + x_2) + x_i p'(x_1 + x_2)) - C_i'(x_i) = 0,
\]

\[
\frac{\partial P_i}{\partial z_i} = -(1 - q_i) \sigma + q_i F'(x_i p(x_1 + x_2) - z_i) = 0.
\]
From the second equation we obtain \( \sigma = q_i \sigma + q_i F'(x_i p(x_1 + x_2) - z_i) \) and substituting this \( \sigma \) in the first equation we obtain the desired equations (3). It only remains to check that the solution of this system of equations is a maximum. For this purpose, using Lemma 1.1 we verify that \( \frac{\partial^2 P_i}{\partial z_i^2} < 0 \).

Moreover, we have

\[
\frac{\partial^2 P_i}{\partial z_i^2} \frac{\partial^2 P_i}{\partial x_i^2} - \left( \frac{\partial^2 P_i}{\partial x_i \partial z_i} \right)^2 = -q_i F''(x_i p(x_1 + x_2) - z_i) \left( (1 - \sigma) \left( 2p'(x_1 + x_2) + x_i p''(x_1 + x_2) \right) \right)
\]

is positive since using condition (1) and the fact that \( F \) and \( C \) are convex, the factor \( -q_i F'' \) is negative and other factor in the parenthesis is also negative. This completes the proof. \( \square \)

It is possible to rewrite the system of equations (3) in Proposition 1.2 as

\[
\begin{align*}
(1 - \sigma) \frac{d}{dx_i} (x_i p(x_1 + x_2)) &= C'_i(x_i) \\
(1 - \sigma) \frac{d}{dx_i} \left( F(x_i p(x_i + x_1) - z_i) \right) &= \frac{\sigma (1 - q_i)}{q_i} C'_i(x_i).
\end{align*}
\]

The first equation above states that, at the equilibrium point, the marginal cost and the marginal profit, after tax, coincide in each firm, as was expected from the theory of Cournot models. The second equation relates the marginal penalty on the declared amount with the marginal cost.

Combining Lemma 1.1, Proposition 1.2, assumption (1) and the monotonicity and convexity of the functions involved, we obtain the following.

**Corollary 1.3.** At the points \( x_i = x_i^*, z_i = z_i^* \) where the profit function \( P_i \), as defined in (2), reaches a maximum, we have

\[
0 < \frac{C'_i(x_i)}{1 - \sigma} = p(x_1 + x_2) + x_i p'(x_1 + x_2)
\]

Furthermore,

\[
0 < \frac{\partial^2 P_i}{\partial z_i \partial x_i} = \frac{q_i}{1 - \sigma} F''(x_i p(x_1 + x_2) - z_i) C'_i(x_i),
\]

\[
0 > \frac{\partial^2 P_i}{\partial x_i^2} = (1 - \sigma) \left( 2p'(x_1 + x_2) + x_i p''(x_1 + x_2) \right) - C''_i(x_i) - q_i \left( p(x_1 + x_2) + x_i p'(x_1 + x_2) \right)^2 F''(x_i p(x_1 + x_2) - z_i),
\]

\[
\frac{\partial^2 P_i}{\partial x_j \partial x_i} = (1 - \sigma) \left( p'(x_1 + x_2) + x_i p''(x_1 + x_2) \right) - q_i x_i p'(x_1 + x_2) \left( p(x_1 + x_2) + x_i p'(x_1 + x_2) \right) F''(x_i p(x_1 + x_2) - z_i).
\]
For the delay Cournot duopoly model, we will assume the first firm enters the market first, so that its profit function remains unchanged, that is
\[ P_1 (x_1, x_2, z_1, z_2, x_{1\tau}, x_{2\tau}, z_{1\tau}, z_{2\tau}) = P_1 (x_1, x_2, z_1, z_2). \]
For the case of the second firm, its profit function is modified to reflect it enters the market after a delay \( \tau > 0 \), so that
\[
P_2 (x_1, x_2, z_1, z_2, x_{1\tau}, x_{2\tau}, z_{1\tau}, z_{2\tau}) = (1 - q_2) \left( x_2(t) p (x_{1\tau}(t) + x_2(t)) - C_2(x_2(t)) - \sigma z_2(t) \right) + q_2 \left( (1 - \sigma) x_2(t) p (x_{1\tau}(t) + x_2(t)) - C_2(x_2(t)) - F(x_2(t) p (x_{1\tau}(t) + x_2(t)) - z_2(t)) \right).
\]
Let \( \bar{x}(t) = (x_1(t), x_2(t), z_1(t), z_2(t)) \) and \( \bar{x}_\tau(t) = (x_{1\tau}(t), x_{2\tau}(t), z_{1\tau}(t), z_{2\tau}(t)) \) where \( x_{i\tau}(t) = x_i(t - \tau) \) and \( z_{i\tau}(t) = z_i(t - \tau) \). Denote constants \( k_1, k_2, k_3, k_4 > 0 \). To create our dynamical system, we assume
\[
\frac{dx_i}{dt} = k_i \frac{\partial P_i}{\partial x_i} \quad \text{and} \quad \frac{dz_i}{dt} = k_{i+2} \frac{\partial P_i}{\partial z_i}.
\]
The delay Cournot duopoly with tax evasion is then
\[
\frac{d\bar{x}}{dt} = \left( k_1 \frac{\partial P_1}{\partial x_1}(\bar{x}, \bar{x}_\tau), k_2 \frac{\partial P_2}{\partial x_2}(\bar{x}, \bar{x}_\tau), k_3 \frac{\partial P_1}{\partial z_1}(\bar{x}, \bar{x}_\tau), k_4 \frac{\partial P_2}{\partial z_2}(\bar{x}, \bar{x}_\tau) \right).
\]
Notice that the fixed points of this dynamical system (5) is precisely the equilibrium points computed in Proposition 1.2.

**Proposition 1.4.** The quasi-polynomial associated to the linearization of the delay Cournot duopoly at its fixed point is given by the formula
\[
Q(\lambda) = p_1 (\lambda) p_2 (\lambda) - e^{-\lambda \tau} g_1(\lambda) g_2(\lambda),
\]
where
\[
p_i(\lambda) = \lambda^3 - \left( k_i \frac{\partial^2 P_i}{\partial x_i^2} + k_{i+2} \frac{\partial^2 P_i}{\partial z_i^2} \right) \lambda + k_i \frac{\partial^2 P_i}{\partial x_i^2} k_{i+2} \frac{\partial^2 P_i}{\partial z_i^2} - k_i \frac{\partial^2 P_i}{\partial z_i \partial x_i} k_{i+2} \frac{\partial^2 P_i}{\partial x_i \partial z_i},
\]
\[
g_1(\lambda) = k_1 \frac{\partial^2 P_1}{\partial x_1 \partial x_1}(\lambda - k_1 \frac{\partial^2 P_1}{\partial x_2 \partial x_1} k_3 \frac{\partial^2 P_1}{\partial z_1^2}) + k_1 \frac{\partial^2 P_1}{\partial z_1 \partial x_1} k_3 \frac{\partial^2 P_1}{\partial x_2 \partial z_1},
\]
and
\[
g_2(\lambda) = k_2 \frac{\partial^2 P_2}{\partial x_2 \partial x_2}(\lambda - k_2 \frac{\partial^2 P_2}{\partial x_1 \partial x_2} k_4 \frac{\partial^2 P_2}{\partial z_2^2}) + k_2 \frac{\partial^2 P_2}{\partial z_2 \partial x_2} k_4 \frac{\partial^2 P_2}{\partial x_1 \partial z_2}.
\]

**Proof.** The linearized system is of the form
\[
\frac{d\bar{x}}{dt} = A\bar{x} + B\bar{x}_\tau,
\]
Proposition 1.5. The characteristic polynomial

\[ Q(\lambda) = p_1(\lambda)p_2(\lambda) - g_1(\lambda)g_2(\lambda) = \lambda^4 + \alpha_3 \lambda^3 + \alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0, \]

corresponding to the linearization of the Cournot duopoly model (3) without delay is asymptotically stable if \( \alpha_1 > 0 \) and \( \alpha_1 \alpha_2 \alpha_3 > \alpha_1^2 + \alpha_2^2 \alpha_0 \).

Proof. We first observe that the inequalities \( \alpha_0 > 0, \alpha_1 > 0 \) and \( \alpha_1 \alpha_2 \alpha_3 > \alpha_1^2 + \alpha_2^2 \alpha_0 \) imply \( \alpha_2 \alpha_3 > \alpha_1 + \frac{\alpha_2^2 \alpha_0}{\alpha_1} > \alpha_1 \). Therefore, the inequalities \( \alpha_0 > 0, \alpha_1 > 0, \alpha_3 > 0 \) and \( \alpha_1 \alpha_2 \alpha_3 > \alpha_1^2 + \alpha_2^2 \alpha_0 \) imply \( \frac{1}{\alpha_3} > 0, \frac{\alpha_2^2}{\alpha_2 \alpha_3 - \alpha_1} > 0, \frac{(\alpha_2 \alpha_3 - \alpha_1)^2}{\alpha_3 (\alpha_1 \alpha_2 \alpha_3 - \alpha_1^2 - \alpha_2^2 \alpha_0)} > 0 \) and \( \alpha_0 > 0 \), which are the Routh-Hurwitz conditions for the desired result. Thus, in order to establish our proposition, we only need to verify the two inequalities \( \alpha_0 > 0 \) and \( \alpha_3 > 0 \).

To prove \( \alpha_3 > 0 \), we notice \( \alpha_3 \) is nothing but the addition of the linear coefficient of \( p_1(\lambda) \) plus the linear coefficient of \( p_2(\lambda) \), that is,

\[ \alpha_3 = -\sum_{i=1}^{2} \left( k_i \frac{\partial^2 P_i}{\partial x_i^2} + k_{i+2} \frac{\partial^2 P_i}{\partial z_i^2} \right) > 0, \]

which is then positive since \( k_j > 0 \) and, by Lemma 1.4, \( \frac{\partial^2 P_i}{\partial x_i^2} < 0 \) and \( \frac{\partial^2 P_i}{\partial z_i^2} < 0 \).

Finally, an application of Proposition 1.4 and Lemma 1.1 allow us to compute
\[
\frac{\alpha_0}{(k_1k_2k_3k_4)} = \prod_{i=1}^{2} \left( \frac{\partial^2 P_i}{\partial x_i^2} \frac{\partial^2 P_i}{\partial z_i^2} - \left( \frac{\partial^2 P_i}{\partial z_i \partial x_i} \right)^2 \right) - \prod_{i=1}^{2} \left( \frac{\partial^2 P_i}{\partial x_i \partial x_i} \frac{\partial^2 P_i}{\partial z_i \partial z_i} - \frac{\partial^2 P_i}{\partial z_i \partial x_i} \frac{\partial^2 P_i}{\partial x_i \partial z_i} \right)
\]

\[
\geq (1 - \sigma)^2 \left( \prod_{i=1}^{2} q_i F''(x_i p(x_1 + x_2) - z_i) \right)
\]

\[
\left( \prod_{i=1}^{2} (2p'(x_1 + x_2) + x_i p''(x_1 + x_2)) - \prod_{i=1}^{2} (p'(x_1 + x_2) + x_i p''(x_1 + x_2)) \right)
\]

\[
= (1 - \sigma)^2 \left( \prod_{i=1}^{2} q_i F''(x_i p(x_1 + x_2) - z_i) \right)
\]

\[
p'(x_1 + x_2) \left( p'(x_1 + x_2) + \sum_{i=1}^{2} (p'(x_1 + x_2) + x_i p''(x_1 + x_2)) \right)
\]

\[
> 0.
\]

\[
\square
\]

2. Same marginal production costs

In this section we prove the main result of this paper. We will give conditions for the stability of the Cournot duopoly model with tax evasion and in fact, we will also show that the stability won’t be affected by any delay.

As mentioned in the introduction, we first assume the marginal production costs and the second derivatives of the cost functions on the equilibrium point are equal, that is,

\[
C'_1(x_1^*) = C'_2(x_2^*) \quad \text{and} \quad C''_1(x_1^*) = C''_2(x_2^*). \tag{8}
\]

The following proposition is a slight generalization of an observation in [5, Page 717].

**Proposition 2.1.** In the Cournot duopoly model defined in (2), \( C'_1(x_1^*) = C'_2(x_2^*) \) if and only if \( x_1^* = x_2^* \). Furthermore, \( q_1 = q_2 \) if and only if \( z_1^* = z_2^* \).

**Proof.** Subtracting the first equation in Proposition 1.2 for \( i = 1 \) to \( i = 2 \), we obtain

\[
(x_1^* - x_2^*)p'(x_1^* + x_2^*) = 0.
\]

Since \( p \) is strictly decreasing we obtain \( x^* = x_1^* = x_2^* \). Conversely, if \( x^* = x_1^* = x_2^* \) then using again the first equation in Proposition 1.2 we obtain \( C_1(x_1^*) = p(2x^*) + x^*p'(2x^*) = C_2(x_2^*) \) and so \( C_1(x_1^*) = C_2(x_2^*) \) For the equality of the \( z_i \)'s, assuming \( q_1 = q = q_2 \) we now use the second equation in Proposition 1.2 to obtain

\[
F'(x^*p(2x^*) - z_1^*) = \frac{(1 - q_1)\sigma}{q_1} = \frac{(1 - q)\sigma}{q} = \frac{(1 - q_2)\sigma}{q_2} = F'(x^*p(2x^*) - z_2^*)
\]
and using that $F'$ is strictly increasing we conclude $z_1^* = z_2^*$. Conversely, if $z^* = z_1^* = z_2^*$ using again the second equation in Proposition 1.2

$$\left(1 - q_1\right)\sigma = F'(x^* p(2x^*) - z^*) = \frac{(1 - q_2)\sigma}{q_2},$$

which implies $q_1 = q_2$, as wanted.

In addition to the hypothesis on the marginal costs, we are going to assume the the probabilities $q = q_1 = q_2$ of being caught evading taxes are the same for both firms. We remark that this assumption $q_1 = q_2$ was also made in the original introduction of the Cournot duopoly model with tax evasion [5] and also in subsequent works, e.g [12]. In addition, we will assume $k_1 = k_3$ and $k_2 = k_4$, that is, both firms have the same strategies for adapting their productions and income declared for tax purposes. Finally, we impose one technical condition on the second partial derivatives of the profit function:

$$-\frac{\partial^2 P_1}{\partial x_1^2} > \left| \frac{\partial^2 P_1}{\partial x_2 \partial x_1} \right|,$$

where the inequality is on the evaluation at the equilibrium point. For easy reference, we establish the following.

**Definition 2.2.** We define the Cournot duopoly model with equal marginal costs, as the delay Cournot duopoly model with tax evasion [5] such that conditions (8) and (9) together with the assumptions $q = q_1 = q_2$ and $(k_1, k_3) = (k_2, k_4)$.

**Proposition 2.3.** In the Cournot duopoly model with equal marginal costs given in Definition 2.2, the inequality $p(x_1 + x_2) + 2x_1 p'(x_1 + x_2) \geq 0$ implies the technical condition (9).

**Proof.** It will suffice to prove that $-\frac{\partial^2 P_1}{\partial x_1^2} \pm \frac{\partial^2 P_1}{\partial x_2 \partial x_1} > 0$. Using Corollary 1.3 we compute

$$\begin{align*}
-\frac{\partial^2 P_1}{\partial x_1^2} + \frac{\partial^2 P_1}{\partial x_2 \partial x_1} &= C_1''(x_1) - (1 - \sigma)p'(x_1 + x_2) \\
&\quad + q_1 p(x_1 + x_2) (p(x_1 + x_2) + x_1 p'(x_1 + x_2)) F''(x_1 p(x_1 + x_2) - z_i) \\
&> 0.
\end{align*}$$

and using the hypothesis we finally obtain

$$\begin{align*}
-\frac{\partial^2 P_1}{\partial x_1^2} - \frac{\partial^2 P_1}{\partial x_2 \partial x_1} &= C_1''(x_1) - (1 - \sigma) \left( p'(x_1 + x_2) + 2 \left( p'(x_1 + x_2) + x_1 p''(x_1 + x_2) \right) \right) \\
&\quad + \frac{q_1}{1 - \sigma} \left( p(x_1 + x_2) + 2x_1 p'(x_1 + x_2) \right) F''(x_1 p(x_1 + x_2) - z_i) C_1'(x_1) \\
&> 0.
\end{align*}$$

\[\square\]
One may wonder if the technical condition (9), stated for the first firm in the duopoly, is valid for the second firm. That this is the case, under the conditions already described, is a consequence of the following.

**Proposition 2.4.** The second derivatives of the profit functions (2) corresponding to the Cournot duopoly model with equal marginal costs given in Definition 2.2 coincide at the equilibrium point. Thus, the quasipolynomial corresponding to the linearization of the form (7) reduces to

\[ Q(\lambda) = p^2(\lambda) - e^{-r_{\lambda}}g^2(\lambda), \]

where \( p(\lambda) \) and \( g(\lambda) \) are second and first degree polynomials, respectively.

**Proof.** We combine Lemma 1.1, Corollary 1.3 and Proposition 8 together with the assumptions of the Cournot duopoly model with same marginal costs to obtain the desired result. \( \square \)

The following theorem formalizes the claim that the conditions for the Cournot duopoly model with equal marginal costs given in Definition 2.2 are sufficient for the equilibrium point to be asymptotically stable. In particular, they must satisfy the equivalent conditions for the model presented in [12, Proposition 5].

**Theorem 2.5.** The equilibrium point of the Cournot duopoly model with equal marginal costs given in Definition 2.2 is asymptotically stable for \( \tau = 0 \).

**Proof.** Using Proposition 2.4, we may assume \( p(\lambda) = \lambda^2 + m_1\lambda + m_0 \) and \( g(\lambda) = c_1\lambda + c_0 \), so that the characteristic polynomial corresponding to the linearization of the Cournot duopoly model with equal marginal costs and without delay, is

\[
Q(\lambda) = (\lambda^2 + m_1\lambda + m_0)^2 - (c_1\lambda + c_0)^2 \\
= \lambda^4 + 2m_1\lambda^3 + (2m_0 + m_1^2 - c_0^2)\lambda^2 + (2m_0m_1 - 2c_0c_1)\lambda + m_0^2 - c_0^2 \\
= \lambda^2 + \alpha_3\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0.
\]

According to Proposition 1.5, we only need to verify that \( \alpha_1 > 0 \) and \( \alpha_1\alpha_2\alpha_3 > \alpha_1^2 + \alpha_3^2\alpha_0 \). First, we combine the formulas in Lemma 1.1 and Proposition 1.4 to obtain

\[
m_0 = -c_0 - q_1k_1k_3p'(x_1 + x_2)F''(x_1p(x_1 + x_2) - z_1) + q_1k_1k_3C''(x_1)F''(x_1p(x_1 + x_2) - z_1) \\
= -c_0 + r,
\]

where \( r > 0 \) and

\[
-c_0 = k_1 \frac{\partial^2 P_1}{\partial x_2 \partial x_1}k_3 \frac{\partial^2 P_1}{\partial z_1^2} - k_1 \frac{\partial^2 P_1}{\partial z_1 \partial x_1}k_3 \frac{\partial^2 P_1}{\partial x_2 \partial z_1} \\
= -q_1k_1k_3(1 - \sigma)(p(x_1 + x_2) + x_1p'(x_1 + x_2))F''(x_1p(x_1 + x_2) - z_1) \\
> 0.
\]
On the other hand, we have

\[ m_1 = -k_1 \frac{\partial^2 P_1}{\partial x_1^2} - k_2 \frac{\partial^2 P_1}{\partial z_1^2} \]
\[ > -k_1 \frac{\partial^2 P_1}{\partial x_1^2} \]
\[ = -c_1 + q_1 k_1 (p(x_1 + x_2) + x_1 p'(x_1 + x_2))p(x_1 + x_2)F''(x_1 p(x_1 + x_2) - z_1) \]
\[ - (1 - \sigma) k_1 p'(x_1 + x_2) + k_1 C_1''(x_1) \]
\[ > -c_1. \]

Thus

\[ m_0 m_1 = (-c_0 + r) m_1 > (-c_0) m_1 > (-c_1)(-c_0) = c_0 c_1. \]

Hence

\[ \alpha_1 = 2(m_0 m_1 - c_0 c_1) > 0. \]

Finally, the technical condition (9) gives us

\[ m_1 = -k_1 \frac{\partial^2 P_1}{\partial x_1^2} - k_2 \frac{\partial^2 P_1}{\partial z_1^2} \]
\[ > k_1 \left| \frac{\partial^2 P_1}{\partial x_2 \partial x_1} \right| \]
\[ = |c_1|. \]

so that \( m_1^2 - c_1^2 > 0. \) Thus, we compute

\[ \alpha_3 \alpha_2 \alpha_1 = (2m_1)(2m_0 + m_1^2 - c_1^2)(2m_0 m_1 - 2c_0 c_1) \]
\[ = (2m_1)(2m_0)(2m_0 m_1 - 2c_0 c_1) + (2m_1)(m_1^2 - c_1^2)(2m_0 m_1 - 2c_0 c_1) \]
\[ = 4m_0^2 m_1^2 - 8m_0 m_1 c_0 c_1 + 4c_0^2 c_1^2 + 4m_0^2 m_1^2 - 4c_0^2 c_1^2 + (2m_1)(m_1^2 - c_1^2)(2m_0 m_1 - 2c_0 c_1) \]
\[ = \alpha_1^2 + 4m_1^2(m_2^2 - c_0^2) + 4m_1^2 c_0^2 - 4c_0^2 c_1^2 + (2m_1)(m_1^2 - c_1^2)(2m_0 m_1 - 2c_0 c_1) \]
\[ = \alpha_1^2 + \alpha_3^2 c_0 + (m_1^2 - c_1^2)(4c_0^2 + \alpha_3(m_1^2 - c_1^2)\alpha_1) \]
\[ > \alpha_1^2 + \alpha_3^2 c_0, \]

as was to be proved. \( \square \)

**Proposition 2.6.** The quasipolinomial (4) corresponding to the linealization (7) of the Cournot duopoly model with equal marginal costs given in Definition 2.2 does not have imaginary roots.

**Proof.** By contradiction, suppose \( \lambda = i \omega, \omega \neq 0, \) is an imaginary root of \( Q(\lambda) = p^2(\lambda) - e^{i\omega \tau} g^2(\lambda). \) Since \( p(\lambda) \) and \( g(\lambda) \) are polynomials with real coefficients, the imaginary roots of the quasipolynomial \( Q(\lambda) \) occur in conjugate pairs. Therefore, we may assume without loss of generality that \( \omega > 0. \) We first observe that

\[ |p(i\omega)| = |g(i\omega)|. \tag{10} \]
On the other hand, by the technical condition (9)

\[
\frac{1}{w} \text{Im}(p(iw)) = -k_1 \frac{\partial^2 P_1}{\partial x_1^2} - k_2 \frac{\partial^2 P_1}{\partial z_1^2} > k_1 \frac{\partial^2 P_1}{\partial x_1^2} \\
> k_1 \left| \frac{\partial^2 P_i}{\partial x_j \partial x_i} \right| \\
= \frac{1}{w} |\text{Im}(g(i\omega))|.
\]

Furthermore

\[
\text{Re}(p(iw)) = k_1 \frac{\partial^2 P_1}{\partial x_1^2} k_2 \frac{\partial^2 P_1}{\partial z_1^2} - k_1 \frac{\partial^2 P_i}{\partial x_i \partial x_i} k_2 \frac{\partial^2 P_i}{\partial x_i \partial z_i} - w^2 \\
> k_1 \frac{\partial^2 P_1}{\partial x_1^2} k_3 \frac{\partial^2 P_1}{\partial z_1^2} - k_1 \frac{\partial^2 P_i}{\partial x_i \partial z_i} k_3 \frac{\partial^2 P_i}{\partial x_i \partial x_i} \\
> k_1 \frac{\partial^2 P_1}{\partial x_2 \partial x_1} k_3 \frac{\partial^2 P_1}{\partial z_1^2} - k_1 \frac{\partial^2 P_i}{\partial x_1 \partial z_1} k_3 \frac{\partial^2 P_i}{\partial x_2 \partial z_1} \\
= -c_0 \\
= |\text{Re}(g(i\omega))|.
\]

The inequalities above give us

\[
|p(i\omega)|^2 = \text{Re}(p(iw))^2 + \text{Im}(p(iw))^2 > |\text{Re}(g(i\omega))|^2 + |\text{Im}(g(i\omega))|^2 = |g(i\omega)|^2.
\]

Hence $|p(i\omega)| > |g(i\omega)|$, a contradiction of (10). This completes the proof.

We have proved in Theorem 2.5 that the equilibrium point of the Cournot duopoly model with equal marginal costs is asymptotically stable for $\tau = 0$, and in the previous Proposition 2.6 we showed that the quasipolynomial corresponding to its linealization does not have imaginary roots. Therefore, there are no roots of the quasipolynomial which cross the imaginary axis as the value of the delay $\tau$ increases. As consequence, see e.g. [2], we have obtained the following theorem, the main result of our paper.

**Theorem 2.7.** The equilibrium point of the Cournot duopoly model with equal marginal costs given in Definition 2.2 is asymptotically stable and independent of the delay.

**Remark 2.8.** As a direct consequence of the above theorem, we can assure that the Cournot duopoly model with equal marginal costs does not have bifurcations under parametric variations of the delay $\tau \geq 0$. 

11
3 Examples

In this section we will provide some examples which satisfy the conditions for the Cournot duopoly model with equal marginal costs given in Definition 2.2. As motivation, let us observe that in case the proposed conditions are satisfied then the system of equation (3) can be rewritten as

\[ \begin{align*}
    p(2x^*) + x^* p'(2x^*) &= \frac{C'(x^*)}{1 - \sigma} \\
    F'(x^*p(2x^*) - z^*) &= \frac{\sigma(1 - q)}{q}
\end{align*} \]  

(11) \hspace{1cm} (12)

If \( G = F' \) then \( G \) is strictly increasing by definition of \( F \), so that solving for \( z^* \) in Equation (12) we obtain

\[ z^* = x^*p(2x^*) - G^{-1}\left(\frac{\sigma(1 - q)}{q}\right). \]

Thus, the amount \( z^* \) declared as revenue by the firms in the duopoly will be closer to their actual revenue \( x^*p(2x^*) \) either when the effectiveness of audits is increased, that is, the value of \( q \) representing the probability of being caught evading taxes increases, or by adjusting the penalties for tax evasion, that is, introducing a penalty function such that the value of \( G^{-1} \) at \( \frac{\sigma(1 - q)}{q} \) is as low as possible. In addition, Theorem 2.7 assures that the equilibrium point of our model is asymptotically stable and independent of time delay.

We will next provide a family of cost functions which satisfy condition (8) and two examples of inverse demand functions which allow the profit functions to satisfy the desired technical condition (9). Thus, for our examples, the penalty function \( F \) does not affect the stability of the system.

3.1 Examples of cost functions

Let us assume the cost function, for \( i = 1, 2 \), is given by the formula \( C_i(x_i) = f_i + dx_i + cx_i^2 \), where \( f_i \geq 0, d > 0 \) and \( c \geq 0 \) are constants. We claim that this functions satisfy condition (8). Indeed, subtracting the first equation of the system (3) for \( i = 2 \) from \( i = 1 \) we obtain

\[ (1 - \sigma)(x_1^* - x_2^*)p'(x_1^* + x_2^*) = 2c(x_1^* - x_2^*). \]

By contradiction, if \( x_1^* \neq x_2^* \) then the above equation implies \( p'(x_1^* + x_2^*) = \frac{2c}{1 - \sigma} \geq 0 \), contradicting \( p'(x) < 0 \). An application of Proposition 2.1 completes the proof of the claim.

3.2 Examples of inverse demand functions

We now provide examples of inverse demand functions \( p(x) \) which satisfy the inequality \( p(2x^*) + 2x^*p'(2x^*) \geq 0 \) which, according to Proposition 2.3 implies the desired technical condition (9). We claim that the functions \( p_1(x) = 1/x \) and \( p_2(x) = a - bx \) with \( a, b > 0 \), both do. For the first function, we compute

\[ p_1(2x^*) + 2x^*p'_1(2x^*) = \frac{1}{2x^*} - \frac{2x^*}{4(x^*)^2} = 0, \]
as desired. Finally, for the second function \( p_2(x) \), we solve Equation (11) to obtain \( x^* = \frac{a}{3b} - \frac{C'(x^*)}{3b(1-\sigma)}. \) Therefore, using the cost function from previous subsection, we get \( x^* = \frac{a(1-\sigma)-d}{3b(1-\sigma)+2c}. \) Since the inequality \( p_2(2x^*) + 2x^*p'_2(2x^*) \geq 0 \) is equivalent to the inequality \( a - 4bx^* \geq 0 \), we substitute the value of \( x^* \) and obtain
\[
2ac + 4bd \geq ab(1 - \sigma),
\]
which is the desired condition for stability.

4 Conclusions

In this paper a stability analysis of a Cournot duopoly model with tax evasion under parametric variations of the delay is presented. As consequence of this analysis, we give conditions for the equilibrium point of the above model to be delay-independent and asymptotically stable. In particular, there are no bifurcations under parametric variations of the time-delay, in other words, under the given conditions, the equilibrium point of the duopoly can not be made unstable by a late insertion of the second firm of the duopoly in the market. Evidently, our conditions for stability are not necessary, as examples in the literature show [12 Section 5]; however, our conditions for stability are surprisingly simple and apply for a variety of classical functions found in the literature, as was exhibited in previous section.

Under the proposed assumptions, we are able to suggest that either by increasing the effectiveness of audits or by adapting the penalties for tax evasion it may result in the rise of the tax revenue, more precisely, in the declared amount of revenue to the tax authority, which is asymptotically stable, thus inhibiting tax evasion and increasing public revenue.

References

[1] Bundau O., Neamtu M. and Opris D. (2007). Rent seeking games with tax evasion. \texttt{arXiv:0706.0664v1 [math.DS]}. Preprint.

[2] Cooke K. L. and van den Driessche P.(1986). On zeroes of some transcendental equations. Funkcial. Ekvac. vol. 29, no. 1, 77–90.

[3] Cournot A. (1838). Recherches sur les principes mathématiques de la théorie de richesses. Paris. Chez L. Hachette.

[4] Elsadany A. A. and Matouk A. E. (2014). Dynamic Cournot duopoly game with delay. Journal of Complex Systems, vol. 2014, Article ID 384843.

[5] Goerke L., and Runkel M. (2011). Tax Evasion and Competition. Scottish Journal of Political Economy, Vol. 58, Issue 5, pp. 711-736.

[6] Gori L., Guerrini L. and Sodini M. (2015). A continuous time Cournot duopoly with delays. Chaos, solitons & fractal, vol. 79, 166-177.
[7] Guerrini L., Matsumoto A. and Szidarovszky F. (2018). Delay Cournot duopoly models revisited. Chaos 28, 093113; https://doi.org/10.1063/1.5020903

[8] Howroyd T. D. and Russel A. (1984). Cournot oligopoly models with time delays. J. Math. Econ., vol. 13, 97–108.

[9] Itzá-Ortiz B. A. and Mera-Lorenzo Y. (2012). Modelos de duopolio de Cournot con evasión de impuestos. Miscelánea Matemática, vol. 55, 79–97.

[10] Matsumoto A. and Szidarovszky F. (2013) Discrete and continuous dynamics in nonlinear monopolies. Economic Institute of Chuo University. Discussion Paper No. 210. Available in http://ir.c.chuo-u.ac.jp/repository/search/binary/p/5515/s/3353/

[11] Matsumoto A., Szidarovszky F. and Yoshida, H. (2009) Dynamics in Delay Cournot Duopoly. Economic Institute of Chuo University. Discussion Paper No. 114. Available in http://www2.tamacc.chuo-u.ac.jp/keizaiken/discussno114.pdf

[12] Neamtu M. (2010) Deterministic and stochastic Cournot duopoly games with tax evasion. WSEAS Transactions on Mathematics. vol. 9, 618–627.