Uniform synchronous criticality of diversely random complex networks

Xiang Li*
Department of Automation
Shanghai Jiao Tong University
Shanghai, 200030, P.R.China

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Abstract
We investigate collective synchronous behaviors in random complex networks of limit-cycle oscillators with the non-identical asymmetric coupling scheme, and find a uniform coupling criticality of collective synchronization which is independent of complexity of network topologies. Numerically simulations on categories of random complex networks have verified this conclusion.

Keywords: Collective synchronization, asymmetry coupling, random graph, scale-free
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1 Introduction
Traditionally, a network with complex topology is mathematically described by a random graph from the ER model proposed by Erdös and Rényi [1], which has ever dominated scientists’ thinking about complex networks for almost 40 years. Till recently, the discoveries of small-world networks [2] and scale-free networks [3] pioneered a revolution in the theoretic study of complex networks, and, surprisingly, many real-life large-scale complex networking systems are in the categories of small-world and(or) scale-free networks, ranging from biological to engineering systems, and even to social and economic systems [4]-[8].

It has been argued that network topologies significantly affect the emergent network dynamics [9], and the small-world phenomena and scale-free features of complex networks have led to a fascinating
set of common problems concerning how the complexity of network topology facilitates and constrains synchronous behaviors of networks [10].

Collective synchronization in a large population of oscillators having natural different frequencies is one of the focal points in the synchronization literature, which is a typical phenomenon in the fields of biology, physics, and engineering [11]. After the first mathematical description by Wiener [12] and the later fruitful study by Winfree [13], Kuramoto refined this connection between nonlinear dynamics and statical physics, and formalized the solution to a network of globally coupled limit-cycle oscillators [14], answering the situation why the oscillators are completely de-synchronized until the coupling strength overcomes a criticality $C_{syn}$.

The recent discoveries of topological complexity accelerate the understanding of this cooperative phenomenon in the frame of random complex networks [15]-[17]. Especially, Hong et al. reported their synchronization observations of a larger criticality of coupling strength on small-world networks than that of globally coupled networks [15]. And, the latest investigation stated the absence of critical coupling strength in frequency synchronization of a swarm of oscillators connected as a scale-free network having a power-law exponent $2 < \gamma \leq 3$ [16].

All these fruits indicate that the topology of complex networks does play an important role in collective synchronization, and different categories of networks may hold significantly different critical coupling strengths for the occurrence of synchrony in mutually coupled oscillators. However, it should be pointed out that most of these issues hold an assumption that every pair of connected nodes are coupled together with the identical coupling strength. In practice, it is more general that pairs of nodes are connected with non-identical and asymmetric couplings. Therefore, it is natural to ask the question that whether the collective synchronous behaviors of complex networks with non-identical and asymmetric coupling schemes still depend on the network topologies? In this paper, we try to explore this question, and the answer is that there does exist a uniform criticality of coupling strength to synchronize diversely random networks of limit-cycle oscillators.

The whole paper is organized as follows. Section II describes the phase evolving model of a complex network of limit-cycle oscillators, whose couplings are non-identical and asymmetric as specified in this work. A uniform critical coupling strength for collective synchrony among oscillators is analytically arrived at the same section, whose validity is verified for categories of random complex networks by numerical simulations in Section III. Section IV finally concludes this investigation.

2 Model description and main result

We consider a network of $N$ coupled limit-cycle oscillators whose phases $\theta_i, i = 1, 2, \cdots, N$, evolve as

$$\frac{d\theta_i}{dt} = \omega_i + \sum_{j=1}^{N} C_{ij} a_{ij} \sin(\theta_j - \theta_i),$$

where $C_{ij}$ is the coupling strength between node (oscillator) $i$ and node (oscillator) $j$, and $a_{ij}$ is 1(or 0) if node $i$ is connected (or disconnected) with node $j$. Frequencies $\omega_i, i = 1, 2, \cdots, N$, are randomly distributed following the given frequency distribution $g(\omega)$, which is assumed that $g(\omega) = g(-\omega)$. 
Define the non-identical asymmetric coupling scheme

\[ C_{ij} = C_j = \frac{C}{k_i}, \quad i, j = 1, \cdots, N, \quad (2) \]

where \( C \) is a positive constant, and \( k_i \) is the degree of node \( i \), which fits the given degree distribution \( P(k) \) of a network. Therefore, we have

\[ \frac{d\theta_i}{dt} = \omega_i + \frac{C}{k_i} \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i). \quad (3) \]

If for every node \( i \), its degree \( k_i = N, i = 1, 2, \cdots, N \), model 5 equals the classic Kuramoto model for globally coupled networks in [14, 9, 11].

Define the order parameter \((r, \Psi)\) as [16]:

\[ r e^{i\Psi} = \int d\omega \int dk \int d\theta g(\omega) P(k) k \rho(k, \omega; t, \theta) e^{i\theta}, \quad (4) \]

where \( \rho(k, \omega; t, \theta) \) is the density of oscillators with phase \( \theta \) at time \( t \) with the given frequency \( \omega \) and degree \( k \), which satisfies the normalization as

\[ \int_{0}^{2\pi} \rho(k, \omega; t, \theta) d\theta = 1. \quad (5) \]

Assume \( \nu \) to be the continuum limit of the right-hand side (r.h.s.) of Eq. (3), and each randomly selected edge is connected to the oscillator having degree \( k \), frequency \( \omega \), and phase \( \theta \) with probability

\[ \frac{k P(k) g(\omega) \rho(k, \omega; t, \theta)}{\int dk P(k) k}. \]

Therefore, determined by the continuity equation

\[ \frac{\partial \rho}{\partial t} = - \frac{\partial (\nu \rho)}{\partial \theta}, \quad (6) \]

the density \( \rho(k, \omega; t, \theta) \) evolves as

\[ \frac{\partial \rho(k, \omega; t, \theta)}{\partial t} = - \frac{\partial}{\partial \theta} \left[ \rho(k, \omega; t, \theta) \left( \omega + \frac{C}{k} \int dk' \int d\omega' g(\omega') P(k') k' \rho(k', \omega'; t, \theta') \sin(\theta - \theta') \right) \right]. \quad (7) \]

Substituting Eq. (4) into Eq. (7) yields

\[ \frac{\partial \rho(k, \omega; t, \theta)}{\partial t} = - \frac{\partial}{\partial \theta} \left\{ \rho(k, \omega; t, \theta) [\omega + C \sin(\Psi - \theta)] \right\}, \quad (8) \]

whose solution independent of time is

\[ \frac{\partial}{\partial \theta} \left\{ \rho(k, \omega; \theta) [\omega + C \sin(\Psi - \theta)] \right\} = 0. \quad (9) \]

where \( \rho(k, \omega; \theta) \) could be assumed to be

\[ \rho(k, \omega; \theta) = \begin{cases} \delta \left( \theta - \arcsin \left( \frac{\omega}{\sqrt{1 - \frac{C^2}{\omega^2}}} \right) \right) & \text{if } \left| \frac{\omega}{\sqrt{C^2}} \right| \leq 1 \\ \frac{D(k, \omega)}{\left| \omega - C \sin(\Psi) \right|} & \text{otherwise}. \end{cases} \quad (10) \]

Here, we assume \( \Psi = 0 \) without loss of generality, and \( D(k, \omega) \) is the normalized factor. Recall \( g(-\omega) = g(\omega) \), and substitute Eq. (10) into Eq. (4). We therefore arrive at

\[ r = \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega g(\omega) k P(k) e^{i \arcsin \left( \frac{\omega}{\sqrt{1 - \frac{C^2}{\omega^2}}} \right)} \quad (11) \]

\[ = \int_{-\infty}^{\infty} d\omega g(\omega) \sqrt{1 - \left( \frac{\omega}{\sqrt{C^2}} \right)^2} \quad (12) \]

\[ = C \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta g(C \sin \theta) d\theta. \quad (13) \]
From this self-consistency equation we finally come to
\[ C_{\text{syn}} = \frac{2}{\pi g(0)}, \]  
(14)
which is surprisingly the same as the critical coupling strength of globally coupled networks investigated by Kuramoto [14, 11].

The main point of criticality (14) is that, with the non-identical and asymmetric coupling scheme (2), different networks may exhibit the same transition of collective synchronous behaviors regardless of the significant difference of their degree distributions, i.e., the complexity of network topology. We will verify this point through extensive simulations.

3 Numerical simulations

Define the average order parameter as
\[ r_{av} = \left\langle \frac{\sum_{i=1}^{N} k_i e^{i\theta_i}}{\sum_{i=1}^{N} k_i} \right\rangle, \]  
(15)
where \( \langle \cdots \rangle \) and \([\cdots]\) denote the averages over different realizations of intrinsic networks and over different realizations of intrinsic frequencies, respectively. In all the simulations, \( r_{av} \) is averaged over 10 groups of networks satisfying the same degree distribution \( P(k) \), and each network has 5 sets of frequencies with the distribution \( g(\omega) \). We further specify
\[ g(\omega) = \begin{cases} 0.5, & \text{if } -1 < \omega < 1 \\ 0, & \text{otherwise} \end{cases}, \]  
(16)
Therefore, \( C_{\text{syn}} \approx 1.273 \) by Eq. (14) and Eq. (16).

We first numerically simulate the case of scale-free networks of the BA model [18] whose degree distribution is in the power-law form of \( P(k) \propto k^{-3} \). Figures 1-2 show the simulating outcomes on scale-free networks of BA model with \( m = m_0 = 3 \), and \( N = 256, 512, 1024, 2048 \), respectively. It can be clearly observed that even having the finite-size effect for \( N = 256, 512, 1024, 2048 \), respectively, the average order parameter \( r_{av} \) shows a rapid increase after the critical coupling strength \( C_{\text{syn}} = \frac{2}{\pi g(0)}, \) indicating that when \( C_{\text{syn}} \leq C \), the scale-free network of oscillators begin to cluster with synchrony. In Fig. 2, \( r_{av} N^{0.25} \) shows a similar dependence on the coupling strength \( C \), which rapidly increased when \( C_{\text{syn}} \leq C \).

From this simulation example of the same scale-invariant power-law degree distribution \( P(k) \propto k^{-3} \) with different network scale \( N \), we clearly conclude that the nonzero constant criticality \( C_{\text{syn}} \) is independent of the finite size of a network, showing a difference from the investigated case of scale-free identical-coupled oscillators in [10].

To investigate the independence of \( C_{\text{syn}} \) on different degree distributions \( P(k) \), we fix the scale \( N = 2048 \), and the average degree \( \langle k \rangle = 6 \). Therefore, all networks have the same number of nodes and edges with different connectivity patterns in the next simulation studies.

Two main categories of degree distributions of random complex networks are in the forms of \( P_{\text{power}}(k) \propto k^{-\gamma} \) and \( P_{\text{exp}}(k) \propto e^{-k} \). Therefore, we select the famous ER model [1] to generate networks having an
Figure 1: The average order parameter $r_{av}$ vs the coupling strength $C$ for scale-free networks of the BA model with $N = 256, 512, 1024, 2048$, respectively. All networks are started from $m = m_0 = 3$.

Figure 2: The average $r_{av}N^{0.25}$ vs the coupling strength $C$ for scale-free networks of the BA model with $N = 256, 512, 1024, 2048$, respectively. All networks are started from $m = m_0 = 3$. 
Figure 3: The average order parameter $r_{av}$ vs the coupling strength $C$ for networks generated by the ER model (dashed line with circle markers), the Local-World model (dotted line with square markers), the BA model (solid line with diamond markers), and the GKK model having power-law exponent $\gamma = 6$ (dash-dot line with right triangle markers). All networks have the same scale $N = 2048$ and the same average degree $\langle k \rangle = 6$.

exponential degree distribution $P_{exp}(k) \propto e^{-k}$, and nominate the BA model and the GKK model [20] to generate networks having the scale-invariant power-law degree distribution $P_{power}(k) \propto k^{-\gamma}$ with $\gamma = 3, 6$, respectively. The proposed local-world evolving network [19] owns a transition between the exponential degree distribution and the power-law degree distribution, therefore it is adopted as the final prototype for the coming simulations with the parameters $M = 10$, $m = m_0 = 3$.

The simulating results of Figs. 3-4 are not out of expectation of criticality [14]. There is a common critical coupling strength of the value $C_{syn} = \frac{2}{g(0)\pi}$ in these four categories of networked limit-cycle oscillators, and both the average order parameter $r_{av}$ and the average $r_{av}N^{0.25}$ increased sharply when $C_{syn} \leq C$. In other words, the significant difference among complex network topologies does not show effect on the collective synchrony of the non-identically asymmetrically coupled limit-cycle oscillators [8]. However, as concluded from [15, 17, 18], there is no such a same critical coupling strength of collective synchrony in different random complex networks of identically symmetrically coupled limit-cycle oscillators.
Figure 4: The average $r_{av}N^{0.25}$ vs the coupling strength $C$ for networks generated by the ER model (dashed line with circle markers), the Local-World model (dotted line with square markers), the BA model (solid line with diamond markers), and the GKK model having power-law exponent $\gamma = 6$ (dash-dot line with right triangle markers). All networks have the same scale $N = 2048$ and the same average degree $\langle k \rangle = 6$.

4 Conclusion

To summarize, we have investigated the collective synchronous behaviors of complex networks of oscillators having non-identical asymmetric coupling strengths, which show a uniform synchrony criticality regardless of the complexity of networking topologies. One of the direct extensions is that for scale-free networks, when synchronization is not preferred in some practical situations [21], the proposed asymmetric coupling scheme (2) could be applied to obtain a nonzero synchrony criticality which is independent of the network topologies. More important to note is the investigation in this paper indicates that, with the embedment of complicated coupling schemes, the interactions previously observed between the network collective behaviors and network complex topologies are still a small tip of an huge iceberg, most of which are left for further explorations.

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