A unified view of acoustic-electrostatic solitons in complex plasmas

J F McKenzie\textsuperscript{1,2} and T B Doyle\textsuperscript{2}

\textsuperscript{1} Max-Planck Institut f"ur Aeronomie, 37191 Katlenburg-Lindau, Germany
\textsuperscript{2} School of Pure and Applied Physics, University of Natal, Durban 4041, South Africa

E-mail: DOYLE@nu.ac.za

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Abstract. A fluid dynamic approach is used in a unified fully nonlinear treatment of the properties of the dust-acoustic, ion-acoustic and Langmuir-acoustic solitons. The analysis, which is carried out in the wave frame of the soliton, is based on total momentum conservation and Bernoulli-like energy equations for each of the particle species in each wave type, and yields the structure equation for the ‘heavy’ species flow speed in each case. The heavy (cold or supersonic) species is always compressed in the soliton, requiring concomitant contraints on the potential and on the flow speed of the electrons and protons in the wave. The treatment clearly elucidates the crucial role played by the heavy species sonic point in limiting the collective species Mach number, which determines the upper limit for the existence of the soliton and its amplitude, and also shows the essentially similar nature of each soliton type. An exact solution, which highlights these characteristic properties, shows that the three acoustic solitons are in fact the same mathematical entity in different physical disguises.

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1. Introduction

The properties of acoustic (electrostatic) solitons are usually deduced within a hydrodynamic framework from the first integral of Poisson’s equation, which leads to the interpretation that the soliton is analogous to a particle moving in a pseudo-potential well (see e.g. Sagdeev [1] for the case of the ion-acoustic soliton, IAW, or Rao et al [2] for the dust-acoustic soliton, DAW). In the present work we adopt a fluid dynamic approach, which emphasizes the nature of the flow of each species (i.e. whether it is subsonic or supersonic), and shows that the strength of the soliton is limited by the cooler and heavier (in the IAW and DAW cases) species, and that the cooler and heavier flow is compressed and accelerated towards its sonic point. In this unified approach total momentum conservation (equivalent to the first integral of Poisson’s equation) is cast as the structure equation for the flow speed of the cool component, which is obtained using the Bernoulli energy equations, to eliminate the speeds of the other species, and the equation of motion of the cool species, to eliminate the electric field.

The structure equation is an expression of the fact that, at any point in the wave, the change in the total particle momentum is balanced by the electric stress. Thus the centre of the wave, which is an equilibrium point at which the electric field is zero, is given by the zero of the total particle momentum function. It can be shown, in general, that this momentum function, may have two zeros, other than the initial point, in the compressive range speed of the cool species. The nearest of these to the initial point is interlaced by a maximum of the function at the charge neutral point (at which the electric stress maximizes) and by a minimum at the cool sonic point. Also, there exists a critical collective Mach number above which smooth solitons cannot be constructed because these two zeros coalesce at the cool sonic point and the flow becomes choked. It is this condition which limits the strength of the wave and provides the collective Mach number regimes in which solitons may exist.

In a special case, which nevertheless provides a very good approximation if the hot species are ‘very hot’, we find that a simplified structure equation applies to all acoustic solitons, including the ‘high frequency’ Langmuir-acoustic, as well as the IAW and DAW solitons. This fully nonlinear equation shows that in each wave type the equilibrium point (or centre of the wave) corresponds to a compression in the cool species at a cool species speed given by

\[ u_{ceq} = \frac{2}{M} - 1, \]

where \( M \) is the appropriate collective wave Mach number, which must lie in the range

\[ 1 < M < 2. \]

These results are summarized in table 1, which illustrates how the momentum balance, the length scale, the collective Mach number and amplitude are interlinked for each kind of acoustic soliton. It is interesting to note that a magnetic soliton, propagating transversely to the magnetic field in a cold plasma (Adlam and Allan [3] and Karpmann [4]), is also governed by the same structure equation as the acoustic soliton, with the Alfvén Mach number \( M_A \) replacing the acoustic collective Mach number. The compression in the field is then \( 2M_A - 1 \), which is equivalent to the fluid compression given by equation (1).

There is a considerable body of work on solitons involving perturbative reduction techniques, which lead to KdV and Schrödinger type wave equations, which may be found in Infeld and Rowlands [5] and references therein. Our approach adopts a gas dynamic viewpoint and is totally nonlinear without recourse to expansion techniques.
### Table 1. Comparison of momentum balance, length scale, collective Mach number and amplitude for each type of acoustic soliton.

| Wave type       | Momentum balance | Length scale $l$ | Collective Mach number | Amplitude         |
|-----------------|------------------|-----------------|----------------------|-------------------|
| Dust-acoustic   | ‘Cold’ heavies   | $U/\omega_{ph}$ | $1 - M_{hc} \leq 2$ | $u_h = (2/M_{hc}) - 1$ |
|                 | ‘Hot’ protons and electrons | | | |
| Ion-acoustic    | ‘Cold protons’   | $U/\omega_{pp}$ | $1 - M_{ph} \leq 2$ | $u_p = (2/M_{ph}) - 1$ |
|                 | ‘Hot’ electrons  | | | |
| Langmuir-       | ‘Cold’ electrons | $U/\omega_{pce}$ | $1 - m \leq 2$ | $u_e = (2/m) - 1$ |
| acoustic        | ‘Massive’ protons| | | |

2. General formulation for electrostatic stationary waves

We consider a plasma consisting of both hot ($h$) and cold ($c$) electron components, protons ($p$) and other heavier species ($d$) and, for mathematical convenience, formulate the problem in a stationary wave frame, in which the flow is steady (i.e. $\partial / \partial t = 0$) and all species stream along the $x$-axis with the wave speed $U$ at $x = -\infty$. The mass flux $m_i n_i u_i$ of each species is a constant, $M_i = m_i n_{io} U$, where $n_{io}$ is the unperturbed species density at $x = -\infty$ ($i = c, h, p, d$).

Conservation of total momentum flux in the direction of propagation ($x$-axis) requires

$$\frac{e_0 E^2}{2} \equiv \frac{e_0}{2} \left( \frac{d\phi}{dx} \right)^2 = U^2 \sum_i m_i n_{io} P_i(u_i),$$

where $E$ and $\phi$ are the electric field and potential, respectively. The function $P_i(u_i)$, which is the change in the particle momentum for each species, is given by

$$P_i(u_i) = u_i - 1 + \frac{1}{\gamma_i M_i^2} \left( \frac{1}{u_i^{\gamma_i}} - 1 \right),$$

where $P_i(u_i)$ and the speeds $u_i$ have been normalized to the species dynamic pressure $m_i n_{io} U^2$ and to the wave speed, respectively, and we have also assumed adiabatic flow, i.e. $p_i \propto n_i^{\gamma_i} \propto u_i^{-\gamma_i}$. The first term on the right-hand side of equation (4a) is the change in the dynamic pressure, and the second is the change in the thermal pressure $p_i$. The species Mach number $M_i$ is defined as

$$M_i^2 = U^2 / c_{io}^2, \quad c_{io}^2 = \gamma p_{io} / m_i n_{io}.$$  

The energy for each species is given by

$$\epsilon_i(u_i) = \frac{1}{2} (u_i^2 - 1) + \frac{1}{(\gamma_i - 1) M_i^2} \left[ \frac{1}{u_i^{(\gamma_i-1)}} - 1 \right] = - \frac{e\phi Z_i}{U^2 m_i},$$

so that the energy changes in each species are related through

$$m_i Z_i \epsilon_i(u_i) = m_p \epsilon_p(u_p) = - m_e \epsilon_e(u_e).$$
Figure 1. The behaviour of the energy function $\epsilon(u_i)$ as a function of $u_i$ for supersonic ($M_i > 1$) and subsonic ($M_i < 1$) initial conditions. If $M_i > 1$ positive (negative) charged particles are decelerated (accelerated) in a potential hill ($\phi > 0$), which is depicted as intersections between horizontal straight lines and the energy function curve. If $M_i < 1$, positive (negative) charged particles are accelerated (decelerated) in a potential hill ($\phi > 0$). Note that the energy function attains a minimum value where the flow speed becomes sonic.

In equation (5a) the first term on the right-hand side represents the change in kinetic energy and the second the enthalpy change.

The energy, $\epsilon_i$, and momentum, $P_i$, functions are of Bernoulli type in that they exhibit minima at the species sonic points $u_i = M_i^{-\frac{2}{\gamma_i+1}}$. The species energy equation (equation (5a)) lends itself to the simple graphical interpretation shown in figure 1, where the energy curves for subsonic ($M_i < 1$) and for supersonic ($M_i > 1$) conditions are intersected by the horizontal ‘equi-potential’ lines. This construction shows that if $M_i > 1$, positive (negative) charged particles are decelerated (accelerated) in a potential hill ($\phi > 0$), while if $M_i < 1$, positive (negative) charged particles are accelerated (decelerated) in a potential well ($\phi < 0$). For example, as we shall see subsequently for a DAW soliton, the cold ($M_d > 1$) and heavier species are compressed and, if $Z_d > 0$, the electrons ($M_e < 1$) are also compressed, while the protons ($M_p < 1$) are rarefied, thus necessitating a potential maximum (hill). The converse is true for $Z_d < 0$. The maximum compression in the heavier species (and hence the maximum potential) occurs at its sonic point, where $\epsilon_i(u_i)$ has a minimum. It is this transonic flow feature which limits the amplitude of the wave because the flow becomes choked at the sonic point.

In general the structure equation for the soliton may be written in the form

$$\frac{1}{2} \left[ \left( 1 - \frac{1}{M_j u_j^{\gamma_j+1}} \right) u_j \frac{du_j}{dx} \right]^2 = \frac{\omega_{pj}^2}{U^2} \sum_i m_j n_j \rho_i P_i(u_i) \equiv R(u_j)$$

(6)
in which the plasma frequency of the $j$th species is

$$\omega_{pj}^2 = \frac{(eZ_j)^2 n_j}{\epsilon_0 m_j}$$

and there is an associated length scale, $l_j = U/\omega_{pj}$, for the soliton. Equation (6) is a differential equation for the chosen species speed $u_j$, in which the electric field has been eliminated in favour of $u_j$ through the equation of motion. The total momentum function $R(u_j)$, which is a function only of the chosen species speed $u_j$, is derived with the aid of the energy relations (equations (5)) in order to express the other species’ speed momentum functions as functions of $u_j$.

For a ‘high frequency’ Langmuir-acoustic wave, in a two-component electron plasma, the momentum function $R(u_c)$, for the ‘cold’ electrons, takes the form

$$R(u_c) = \frac{1}{l^2}(P - \epsilon)_c + \frac{1}{l^2}(P - \epsilon)_h,$$

in which the energy relation $\epsilon_h = \epsilon_c$ is used to eliminate $u_h$ in favour of $u_c$. Similarly for an ion-acoustic wave the momentum function takes the form

$$R(u_p) = \frac{1}{l^2}(P - \epsilon)_p + \frac{1}{l^2}(P - \epsilon)_e,$$  \hspace{1cm} (9a)

when other heavier ions are included, but if they are absent we have

$$R(u_p) = \frac{1}{l^2} \left[ P_p + \frac{m_e}{m_p} P_e \right].$$  \hspace{1cm} (9b)

Finally in the dust-acoustic wave the momentum function $R(u_d)$ is given by

$$R(u_d) = \frac{1}{l^2} \left[ P_d + \frac{m_p n_{po}}{m_d n_{do}} P_p + \frac{m_e n_{eo}}{m_d n_{do}} P_e \right].$$  \hspace{1cm} (10)

It is the properties of this total particle momentum function $R(u_i)$ which determine the necessary and sufficient conditions for the existence of soliton-like solutions of the structure equation which, in the form of equation (6), shows the importance of the sonic point, $u_j = M_j^{-2/(\gamma j + 1)}$, where $du_j/dx = \infty$, and the flow becomes choked.

3. Properties of the structure equation

A prerequisite for the existence of a soliton is that, in the neighbourhood of the initial point $(u_j - 1 \equiv \delta_j \ll 1)$, the momentum function $R(u_j)$ has a double positive zero. The structure equation then approximates to

$$\frac{d\delta_j}{dx} = \pm \frac{\kappa}{l_j} \delta_j,$$  \hspace{1cm} (11a)

where, for the Langmuir-acoustic soliton,

$$\kappa^2 = \left( \frac{n_h}{n_c} \right)_o \frac{1}{(M_h^{-2} - 1)} + \frac{1}{(M_c^{-2} - 1)},$$  \hspace{1cm} (11b)

for the ion-acoustic soliton,

$$\kappa^2 = M_{ep}^{-2} - \frac{1}{(1 - M_p^{-2})},$$  \hspace{1cm} (11c)
Figure 2. The total plasma momentum function \( R(u_d) \) as a function of the heavy (dust) species speed \( u_d \) for three values of the collective Mach number \( M_{dc} \), one sub-critical, one critical and one super-critical. The sub-critical case yields an equilibrium point between the charge neutral point and the heavy species sonic point. Solitons exist if the flow is super dust-acoustic and subcritical.

and for the dust-acoustic soliton,

\[
\kappa^2 = M_{dc}^2 - \frac{1}{(1 - M_d^{-2})}. \tag{11d}
\]

If \( \kappa^2 > 0 \), the structure equation has exponential solutions corresponding to ‘evanescent’ stationary waves with dispersion equations given by equations (11). This requirement is equivalent to the collective wave Mach number being ‘supersonic’. Once this necessary condition is satisfied (which is equivalent to \( R(u) \) having a double positive zero at \( u = 1 \)) the momentum function \( R(u_j) \) may have two other simple zeros in the compressive range \( u_j < 1 \). The zero nearest to the initial point corresponds to the centre of the wave and defines its amplitude. If the collective Mach number is less than a critical value this equilibrium point is located between the charge-neutral point, where the electric stresses maximize, and the species sonic point. Thus, for example, in the case of the dust-acoustic wave, the derivative of \( R(u_d) \) is

\[
\frac{\partial R(u_d)}{\partial u_d} = \left(1 - \frac{1}{M_{dc}^2u_d^{2\gamma_e+1}}\right) \left(1 - \frac{n_{e0}u_d}{Z_d n_d u_e} + \frac{n_{p0}u_d}{Z_d n_{d0}u_p}\right). \tag{12}
\]

At the zero of the last bracket, corresponding to the charge-neutral point \( (Z_d n_d + n_p = n_e) \), \( R(u_d) \) has a positive maximum. A zero of the first bracket corresponds to the dust sonic point where \( R(u_d) \) has a negative minimum. This behaviour of \( R(u_d) \) is shown in figure 2 for three values of the collective Mach number \( M_{dc} \). At the critical value, \( M_c \), of the collective Mach number the two zeros of \( R(u_d) \) coalesce at the dust sonic point. For wave Mach numbers in excess of this critical value \( R(u_d) \) has no real zeros so that solitons cannot be constructed. The particle momentum functions for the ion-acoustic and Langmuir-acoustic waves display exactly the same behaviour, with the collective Mach number defined according to the appropriate mode.
in the dispersion equation. The nature of the flow speeds in each of the wave types will be
discussed in detail in the next section.

At this stage, however, it is possible to summarize the basic properties of each kind of
soliton. The DAW soliton is characterized by a compression in the heavier species necessitating
a potential hill (dip) if \( Z_d > 0 \) \((< 0)\), while the protons are rarefied (compressed) if \( Z_d > 0 \)
\((< 0)\), and the electrons are compressed (rarefied) if \( Z_d > 0 \) \((< 0)\). In an IAW soliton both
the protons and the electrons are compressed in a potential hill, with the electrons initially lagging
the protons up to the charge-neutral point, after which they run ahead of the protons up to the
centre of the wave (which is completed by a reflection about this point). The Langmuir-acoustic
soliton is characterized by a compression in the cold electron component, a rarefaction in the
hot electrons, and a potential dip. These features are all highlighted in the next section by the
existence of a simplified structure equation which describes all three solitons.

4. Simplified structure equation (exact solution)

The general points made above on the nature of the acoustic solitons can readily be elucidated
in certain approximations. As an example, in a dust-acoustic soliton, it can be assumed that
the pressure (and enthalpy) of the electrons and protons dominate their dynamic pressure (and
kinetic energy). This is equivalent to assuming that their flows are highly subsonic \( M_{e,p} \ll 1 \).
On the other hand the heavier species is ‘cold’ and highly supersonic \( M_h \gg 1 \) so that the energy
hodograph relations may be written as

\[
\epsilon_d(u_d) = \frac{1}{2}(u_d^2 - 1),
\]

\[
M_{id}^2 = \frac{U^2 m_d n_{io}}{\gamma_i p_{io}},
\]

in which

\[
u_i = \left[ 1 \mp (\gamma_i - 1)M_{id}^2 \epsilon(u_d)/Z_d \right]^{-1/(\gamma_i - 1)},
\]

(13a)

(13b)

(13c)

Here \( M_{id} \) is a compound Mach number, which is based on the electron (proton) pressure and
the heavy mass. With the values for the electron and proton speeds given by equation (13a) the
total particle momentum function \( R(u_d) \) now takes the simple form

\[
R(u_d) = \frac{1}{2I_d^2}(u_d^2 - 1)^2 \left[ \left( \frac{M_{dc}^2}{2} \right) (u_d + 1)^2 - 1 \right],
\]

(14a)

in which the collective dust-acoustic number \( M_{dc} \) is defined as

\[
M_{dc}^2 = \frac{(n_{po}M_{pd}^2 + n_{eo}M_{ed}^2)}{Z_d^2 n_{do}}.
\]

(14b)

Note that we have assumed \( \gamma_p = \gamma_e = 2 \) for algebraic convenience, but this makes no significant
impact on the analysis if the electrons and protons are sufficiently hot (as an expansion of equation
(13a) clearly demonstrates).

The structure equation for the DAW now assumes the transparent form (McKenzie and
Doyle [6])

\[
u_d \frac{du_d}{dx} = \pm \frac{1}{I_d^2}(u_d - 1) \left\{ \left( \frac{M_{dc}}{2} \right) (u_d + 1)^2 - 1 \right\}^{1/2},
\]

(15)
Figure 3. The regimes in Mach number speeds for dust-acoustic solitons for values of the (heavy) dust charge quantum $Z_d = \pm 2$. The regimes lie, respectively, between the solid and dashed curves.

which, if $M_{dc} > 1$, describes a compressive soliton in $u_d$ whose amplitude is given by

$$u_{deq} = \frac{2}{M_{dc}} - 1,$$

and from which we deduce that solitons exist in the range

$$1 < M_{dc} < 2.$$  \hfill (16b)

Figure 3 shows this permitted region in $(M_{pd}, M_{eh})$ space. The energy hodograph (equation (13a)), showing $(u_p, u_e)$ as functions of $u_d$, is shown in figure 4, and the corresponding characteristic spatial signature for a DAW soliton with a negatively charged heavy species is given in figure 5.

The ion-acoustic and Langmuir-acoustic solitons are equivalently described by equation (15a), but with $u_d$ replaced by $u_p$, or $u_e$, and the Mach number $M_{dc}$ by the ion-acoustic Mach number $M_{ep}$, or the Langmuir-acoustic number (or 'soliton number', $m$, which has been defined elsewhere [6] by: $m = ((n_h/n_e)_o[(1/M_h^2) - 1])^{1/2}$. In the latter case the hot electrons must be sufficiently abundant. A detailed description of the properties of ion-acoustic and Langmuir-acoustic solitons has been given by McKenzie [7, 8]. The energy hodograph for the ion-acoustic wave is shown in figure 6, which illustrates how the protons and electrons are both compressed, with the electrons initially lagging behind the protons up to the charge-neutral point, after which they run ahead of the protons up to the equilibrium point, which lies between the charge neutral point and the proton sonic point. The various properties of these waves are summarized in table 1, which also reveals the their close structural resemblance.

5. Summary

The necessary and sufficient conditions for the existence of acoustic-electrostatic solitons follow from conservation of total momentum of the system which, through the Bernoulli energy relations
Figure 4. Energy hodograph $u_{e,p}$ versus $u_d$ for a (heavy) dust charge quantum $Z_d = 2$ (see text).

Figure 5. The spatial signature for a dust-acoustic soliton with a (heavy) dust charge quantum $Z_d = -2$. The electrons are rarefied, while the dust and protons are compressed, and there is a potential minimum at the centre of the soliton.

and the appropriate equation of motion, is cast as the structure equation for the appropriate species flow speed. In a simplified case, of wide applicability, this structure equation possesses exact analytical solutions, which highlight and unify the properties of each type of soliton.

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Figure 6. The hodograph relation between the electron and proton speeds for the ion-acoustic wave, when the electron enthalpy changes balance proton kinetic energy changes. The equilibrium point (b) lies between the the charge neutral point (a) (where the line $u_e = u_p$ intersects the hodograph) and the proton sonic point (c) if the wave speed is super ion-acoustic and the Mach number is sub-critical (i.e. $M < M_c$). The point (d) is the other (inaccesible) root of $R(u_p) = 0$.

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