MODULI OF VECTOR-BUNDLES ON SURFACES.

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0. Introduction.

In the 1980's Donaldson proved some spectacular new results on classification of \(C^\infty\) four-manifolds by studying anti-self-dual (ASD) connections on an \(SU(2)\)-bundle. If the four-manifold underlies a complex projective surface, the set of ASD connections modulo gauge transformations is identified with a moduli space of slope-stable vector-bundles on the surface. Donaldson [D, §V] proved that for rank two these moduli spaces are generically smooth of the expected dimension (see Section (1) for precise definitions), provided the expected dimension is large enough; this implies that the polynomial invariants of a projective surface are not zero. In this paper we will present algebro-geometric results which were inspired by Donaldson’s theory; there will be no discussion of relations with Gauge theory.

First of all we will sketch our proof [O2] of a theorem proved also by Gieseker-Li [GL1,GL2].

(0.1) Theorem (Gieseker-Li, O'Grady). Let \(S\) be an irreducible smooth complex projective surface, and \(H\) an ample divisor on \(S\). There exists \(\Delta(r)\) such that the moduli of \(H\)-semistable (in the sense of Gieseker-Maruyama) rank-\(r\) torsion-free sheaves on \(S\), with Chern classes \(c_1, c_2 \in H^*(S; \mathbb{Z})\), is reduced of the expected dimension

\[
2rc_2 - (r - 1)c_1^2 - (r^2 - 1)\chi(\mathcal{O}_S) + h^1(\mathcal{O}_S),
\]

provided \(\Delta := c_2 - ((r - 1)/2r)c_1^2 > \Delta(r)\). Furthermore (for \(\Delta \gg 0\)) the open subset parametrizing \(H\)-slope-stable vector-bundles is dense, and the moduli space is irreducible.

The above statement requires a few comments. If \(r = 1\) the moduli space \(\mathcal{M}\) is isomorphic to the product of \(\text{Pic}^{c_1}(S)\) and the Hilbert scheme parametrizing length-\(c_2\) zero-dimensional subschemes of \(S\): as is well-known this Hilbert scheme is always smooth, irreducible and of the expected dimension, hence so is \(\mathcal{M}\). We are really concerned with the case \(r \geq 2\): from now on we will always assume that the rank is at least two, unless we specify otherwise. We deal with Gieseker-Maruyama semistable torsion-free sheaves, rather than with slope-stable vector-bundles, because of a theorem of Gieseker and Maruyama [G1,Ma]: The moduli space of semistable torsion-free sheaves (containing the moduli space of slope-stable vector-bundles as an open subscheme) is projective. Regarding the hypothesis

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that $\Delta$ is large: The moduli space is empty if $\Delta < 0$, by Bogomolov’s Inequality, and on the other hand it is non-empty if $\Delta \gg 0$ [Ma,LQ,G2]. For “low” non-negative values of $\Delta$ there are many examples [G2,O2] of moduli spaces which are not of the expected dimension (or which are reducible [Me]): this is a typical phenomenon occurring for surfaces of Kodaira dimension at least one. At the other extreme of the Kodaira-Enriques classification, if say $S$ is the projective plane, then Theorem (0.1) has been known for a long time in a stronger form [Ma,DL]. More generally, if $S$ is not of general type the moduli space can be somewhat analyzed [Ba,H,ES,Mk1,Mk2,F1] because its structure reflects the special properties of $S$ given by the Enriques-Kodaira classification. If instead $S$ is of general type, very little is known about moduli of vector-bundles; Theorem (0.1) is one of the few general results. After sketching a proof of this theorem we will discuss holomorphic two-forms on the moduli space (of sheaves with fixed determinant). There is a natural map, first studied by Mukai, associating to a holomorphic two-form $\omega$ on $S$ a holomorphic two-form $\omega_\xi$ on the moduli space. If the rank is two and some other hypotheses are satisfied, then $\omega_\xi$ is non-degenerate at the generic point [Mk1,O1]. As noticed by Tyurin [T] the non-degeneracy of $\omega_\xi$ implies that the image of the map

$$
\text{moduli space } \to \, CH_0(S) \\
\left[ E \right] \, \mapsto \, c_2(\xi)
$$

has "dimension" equal to that of the moduli space. Finally we will discuss the Kodaira dimension of the moduli space. We will sketch J. Li’s proof [L2] that if $S$ is of general type then the moduli space is of general type, if the rank is two and certain other hypotheses are satisfied. The results on two-forms and the Kodaira dimension had been proved when Theorem (0.1) was known in rank two only. We observe that since (0.1) holds in arbitrary rank, analogous results on the non-degeneracy of $\omega_\xi$, and on the Kodaira dimension of the moduli space, are valid if a certain conjecture (2.4) regarding vector-bundles on curves is true. We will verify this conjecture for arbitrary rank and a special choice of degree (Proposition (2.5)).

**Notation.** All schemes are defined over $\mathbb{C}$. We let $S$ be a smooth irreducible projective surface, and $K$ be its canonical divisor class. We let $H$ be an ample divisor on $S$.

Let $X$ be a projective variety of dimension $n$, and $D$ be an ample divisor on $X$: for a torsion-free sheaf $F$ on $X$ one sets

$$
slope \text{ of } F = \mu(F) := \frac{e_1(F) \cdot D^{n-1}}{\text{rk}(F)}, \quad p_F(n) := \frac{\chi(F \otimes \mathcal{O}_X(nD))}{\text{rk}(F)}.
$$

The sheaf $F$ is $D$-slope-semistable (respectively $D$-semistable) if

$$
\mu(E) \leq \mu(F) \quad (p_E(n) \leq p_F(n) \text{ for } n \gg 0),
$$

for all (non-zero) subsheaves $E \subset F$; if strict inequality holds whenever $\text{rk}(E) < \text{rk}(F)$ then $F$ is $D$-slope-stable (respectively $D$-stable). One easily checks the implications:

$$
D \text{ - slope-stable } \implies D \text{ - stable } \implies D \text{ - semistable } \implies D \text{ - slope-semistable}.
$$
Now let's specialize to the case $X = S$. For a torsion-free sheaf $F$ on $S$ the discriminant is

$$\Delta_F := c_2(F) - \frac{\text{rk}(F) - 1}{2\text{rk}(F)} c_1(F)^2.$$ 

We label moduli spaces of sheaves on $S$ with triples of sheaf data

$$\xi = (\text{rk}(\xi), \det(\xi), c_2(\xi)) \in \mathbb{N} \times \text{Pic}(S) \times H^4(S; \mathbb{Z}),$$

and we set

$$\mathcal{M}_\xi(S, H) := \{H\text{-s.s. tors.-free sheaf } F \text{ on } S$$

with $\text{rk}(F) = \text{rk}(\xi)$, $\det F \cong \det(\xi)$, $c_2(F) = c_2(\xi)\}/\text{S-equivalence}.$

To define S-equivalence one considers a Jordan-Hölder (JH) filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_n = F,$$

i.e. such that $p_{F_i} = p_F$ and $F_i/F_{i-1}$ is stable for $i = 1, \ldots, n$. The associated graded sheaf $Gr_{JH}(F) := \bigoplus_{i=1}^n F_i/F_{i-1}$ is unique up to isomorphism (although a JH filtration is not unique): two semistable sheaves $F$, $F'$ are $S$-equivalent if $Gr_{JH}(F) \cong Gr_{JH}(F')$. Thus $\mathcal{M}_\xi(S, H)$ contains an open subscheme $\mathcal{M}_\xi^s(S, H)$ parametrizing isomorphism classes of stable sheaves. By a theorem of Gieseker and Maruyama [G1, Ma], $\mathcal{M}_\xi(S, H)$ is projective. We indicate by $[F]$ the point of $\mathcal{M}_\xi(S, H)$ corresponding to a semistable sheaf $F$. We set

$$c_1(\xi) := c_1(\det(\xi)) \quad \Delta_\xi := c_2(\xi) - \frac{\text{rk}(\xi) - 1}{2\text{rk}(\xi)} c_1(\xi)^2.$$ 

Notice that we fix the determinant of sheaves, not just $c_1 \in H^2$ as in Theorem (0.1).

(0.2) Remark. How does the moduli space vary when we change the polarization $H$? This problem is studied in various papers (for example [Q, MW]). We will not discuss the known results, except for the following general fact. Let $H_1$, $H_2$ be ample divisors on $S$, and fix the rank of the sheaves: if $\Delta_\xi$ is sufficiently large the moduli spaces $\mathcal{M}_\xi(S, H_1)$, $\mathcal{M}_\xi(S, H_2)$ are birational. Thus for many purposes we can fix the polarization $H$, and this is what we will always do. To simplify notation we write $\mathcal{M}_\xi$ instead of $\mathcal{M}_\xi(S, H)$.

A family of sheaves on $X$ parametrized by $B$ consists of a sheaf on $X \times B$, flat over $B$. We say $\mathcal{M}_\xi$ is a fine moduli space if $\mathcal{M}_\xi^s = \mathcal{M}_\xi$ (i.e. semistability implies stability), and furthermore there exists a tautological family sheaves $F$ on $S$ parametrized by $\mathcal{M}_\xi$, i.e. such that $F|_{S \times [F]} \cong F$. We state below a simple condition ensuring that $\mathcal{M}_\xi$ is a fine moduli space: the verification that semistability implies stability is left to the reader, the existence of a tautological sheaf follows from [Ma (6.11), Mk2 (A.7)].

(0.3) Criterion. Assume that for $[F] \in \mathcal{M}_\xi$

$$\gcd \{ \text{rk}(F), c_1(F) \cdot H, \chi(F) \} = 1.$$ 

Then $\mathcal{M}_\xi$ is a fine moduli space.
1. Outline of the proof of Theorem (0.1).

The moduli space $\mathcal{M}$ appearing in Theorem (0.1) parametrizes sheaves with fixed rank $r$, $c_1 \in H^1(S; \mathbb{Z})$, and $c_2 \in H^4(S; \mathbb{Z})$. Let $\xi$ be a set of sheaf data with $\text{rk}(\xi) = r$, $c_1(\xi) = c_1$, $c_2(\xi) = c_2$. Since $\mathcal{M}$ is a locally-trivial fibration over $\text{Pic}^0(S)$, with fiber isomorphic to $\mathcal{M}_\xi$, Theorem (0.1) is equivalent to the analogous statement obtained replacing $\mathcal{M}$ by $\mathcal{M}_\xi$. (Of course the expected dimension of $\mathcal{M}_\xi$ is obtained subtracting $h^1(\mathcal{O}_S)$ from the expected dimension of $\mathcal{M}$.) We will outline the proof of the statement for $\mathcal{M}_\xi$: hence from now on we will only deal with $\mathcal{M}_\xi$, the moduli space with fixed determinant.

**Deformation theory and twisted endomorphisms.** References for deformation theory are [A,F2,Mk1,ST]. Let $[F] \in \mathcal{M}_\xi^{st}$, i.e. $F$ is stable. The germ of $\mathcal{M}_\xi$ at $F$ is isomorphic to $\text{Def}^0(F)$, the universal deformation space of $F$ “with fixed determinant” (i.e. it classifies deformations of $F$ which do not change the isomorphism class of $\text{det} F$). To describe $\text{Def}^0(F)$ we need the traceless Ext-groups. If $L$ is a line-bundle on $S$ we set

$$\text{Ext}^q(F, F \otimes L)^0 := \ker \left( \text{Ext}^q(F, F \otimes L) \xrightarrow{\text{Tr}} H^q(L) \right).$$

The trace $\text{Tr}$ is defined in [DL]; if $F$ is locally-free then

$$\text{Ext}^q(F, F \otimes L)^0 = H^q(\text{End}_0(F) \otimes L),$$

where $\text{End}_0(F)$ is the sheaf of traceless endomorphisms of $F$. We set

$$h^q(F, F \otimes L)^0 := \dim \text{Ext}^q(F, F \otimes L)^0.$$

The tangent space to $\text{Def}^0(F)$ is canonically identified with $\text{Ext}^1(F, F)^0$. There is a Kuranishi map

$$\text{Ext}^1(F, F)^0 \supset U \xrightarrow{\Phi} \text{Ext}^2(F, F)^0,$$

defined on an open neighborhood $U$ of the origin, such that $\text{Def}^0(F)$ is the germ at the origin of $\Phi^{-1}(0)$. Thus

$$\dim [F] \mathcal{M}_\xi \geq \dim \text{Ext}^1(F, F)^0 - \dim \text{Ext}^2(F, F)^0 = \chi(F, F)^0 = 2 \text{rk}(\xi) \Delta_\xi - (\text{rk}(\xi)^2 - 1) \chi(\mathcal{O}_S) =: \exp \dim (\mathcal{M}_\xi).$$

In fact the first equality holds because since $F$ is stable $\text{Hom}(F, F)^0 = 0$, and the second equality is just Riemann-Roch. The obstruction space $\text{Ext}^2(F, F)^0$ is Serre dual to $\text{Hom}(F, F \otimes K)^0$, hence we have the following.

**Criterion.** Assume the locus of $[F] \in \mathcal{M}_\xi^{st}$ such that

$$h^0(F, F \otimes K)^0 > 0$$

has dimension strictly smaller than the expected dimension of $\mathcal{M}_\xi$. Then $\mathcal{M}_\xi^{st}$ is a reduced local complete intersection scheme of dimension the expected one.

For $L$ a line-bundle on $S$, let

$$W^L := \{[F] \in \mathcal{M}_\xi^{st} | h^0(F, F \otimes L)^0 > 0\}.$$
(1.1) Theorem [O2]. There exist numbers \( \lambda'_0(\text{rk}(\xi), S, H, L), \lambda_1(\text{rk}(\xi), S, H) \) and \( \lambda_2(\text{rk}(\xi)) \), with \( \lambda_2(\text{rk}(\xi)) < 2 \text{rk}(\xi) \), such that

\[
\dim W^L_\xi \leq \lambda_2 \Delta_\xi + \lambda_1 \sqrt{\Delta_\xi} + \lambda'_0.
\]

Indeed the theorem implies \( \dim W^K_\xi \) is strictly less than the expected dimension, if \( \Delta_\xi \) is large enough: by the previous criterion \( M^s_\xi \) is reduced of the expected dimension. To deal with \( (M_\xi - M^s_\xi) \), i.e. strictly semistable sheaves, one needs some dimension counts: this is a technical point. For simplicity we will usually ignore strictly semistable sheaves: as a first approximation the reader may assume \( M_\xi \) is a fine moduli space (see (0.3)). Similarly the statement in Theorem (0.1) that the locus parametrizing slope-stable vector-bundles is dense follows from Theorem (1.1) together with a result of Jun Li [L1, Appendix].

Remark. The coefficients in the above theorem can be computed explicitly: they depend on \( (S, H, L) \) only via intersection numbers, in particular they are constant for families of polarized surfaces. In [O2] there are some explicit lower bounds for \( \Delta_\xi \) ensuring \( M_\xi \) is reduced of the expected dimension. Donaldson [D,F2,Z] proved Theorem (1.1) for rank two: his coefficient of \( \Delta_\xi \) is 3, which is better than our \( \lambda_2(2) = 23/6 \), but the other coefficients are not explicit. We will see later (see (2.6)) how to use Theorem (1.1) with choices of \( L \) different from \( K_S \).

In this section we will sketch a proof of Theorem (1.1) and we will give the argument for proving (asymptotic) irreducibility.

The boundary. If \( X \subset M_\xi \), the boundary \( \partial X \) consists of the subset of points parametrizing singular (i.e. not locally-free) sheaves. Our approach to the proof of Theorem (1.1) is to show that any closed subset of \( M_\xi \) of relatively small codimension has non-empty boundary. More precisely we prove the following result.

(1.2) Theorem. There exists \( \lambda_0(\text{rk}(\xi), S, H) \) such that if \( X \) is a closed irreducible subset of \( M_\xi \) with

\[
\dim X > \lambda_2 \Delta_\xi + \lambda_1 \sqrt{\Delta_\xi} + \lambda'_0,
\]

then \( \partial X \) is non-empty. (Here \( \lambda_2, \lambda_1 \) are as in Theorem (1.1).)

We will illustrate the implication Theorem(1.2) \( \implies \) Theorem(1.1) by proving the following.

(1.3) Proposition. Assume Theorem (1.2) holds. Let \( r \geq 2 \) be an integer and \( D \) be a divisor on \( S \). Suppose the following: if a torsion-free sheaf \( F \) with \( \text{rk}(F) = r \) and \( \det F \cong O_S(D) \) is semistable then it is slope-stable (e.g. if \( D \cdot H \) and \( r \) are coprime). If \( L \) is a line-bundle on \( S \) then

\[
\dim W^L_\xi < \text{exp. dim. } (M_\xi) = 2r \Delta_\xi - (r^2 - 1)\chi(O_S)
\]

for all sheaf data \( \xi \) such that \( \text{rk}(\xi) = r, \det(\xi) \cong O_S(D), \) and \( \Delta_\xi >> 0. \)

Before proving the above proposition we need some preliminaries on double-duals. Let \( F \) be a torsion-free sheaf on \( S \). Since \( \dim S = 2 \) and \( S \) is smooth the double-dual \( F^{**} \) is locally-free [OSS], since \( F \) is torsion-free the natural map \( F \to F^{**} \) is an injection. Thus we get a canonical exact sequence

\[
0 \to F \to F^{**} \to Q(F) \to 0.
\]
The length $\ell(Q(F))$ is finite. We have

\[(1-4) \quad \text{rk}(F^{**}) = \text{rk}(F), \quad \det(F^{**}) = \det(F), \quad c_2(F^{**}) = c_2(F) - \ell(Q(F)).\]

In particular $F$ is slope-stable if and only if so is $F^{**}$. Now let $\xi$ be a set of sheaf data as in the statement of Proposition (1.3). Let $X \subset M_\xi$ be a closed irreducible subset. If $[F] \in \partial X$ then by our hypothesis $F^{**}$ is slope-stable. Thus $[F^{**}] \in M_{\xi'}$, where $\xi'$ is determined by (1.4). The double-duals $F^{**}$, for $[F]$ varying in $\partial X$, are not parametrized by a single moduli space: in general $c_2(F^{**})$ will vary with $[F]$. However $\partial X$ is stratified by the double-dual strata: if $[F]$ varies in a single stratum then $[F^{**}]$ varies (algebraically) in a single moduli space (each stratum is locally closed). Let $Y \subset \partial X$ be an irreducible component of the open stratum: we set

$$Y^{**} := \{[F^{**}] | [F] \in Y\}, \quad \ell := c_2(\xi) - c_2(\xi').$$

We will need an inequality between the dimensions of $X$ and $Y^{**}$. First of all, considering short locally-free resolutions of sheaves parametrized by $X$, one gets that

$$\text{cod}(\partial X, X) \leq r - 1.$$ 

Secondly, a sheaf parametrized by $Y$ is determined by the isomorphism class of its double-dual, i.e. a point $[E] \in Y^{**}$, plus the choice of a quotient $E \to Q$, where $\ell(Q) = \ell$. A theorem of Jun Li [L1, Appendix] asserts that the generic such quotient is isomorphic to $\bigoplus_{i=1}^\ell \mathbb{C} P_i$. Putting together these facts one obtains the following.

**Lemma.** Keeping notation as above,

\[(1-5) \quad \dim Y^{**} = \dim X - 2r\ell + (r - 1)(\ell - 1) + \epsilon,\]

where $\epsilon \geq 0$. If $\epsilon = 0$ then $\partial X$ contains the isomorphism class of all sheaves $F$ fitting into an exact sequence

$$0 \to F \to E \xrightarrow{\phi} \bigoplus_{i=1}^\ell \mathbb{C} P_i \to 0,$$

where the point $[E] \in Y^{**}$, the points $P_i \in S$, and the surjection $\phi$ are chosen arbitrarily.

The reader should notice that $\text{exp.dim.}(M_{\xi'}) = (\text{exp.dim.}(M_\xi) - 2r\ell)$. The proof of the proposition will go roughly as follows. Starting from $X_0 := W^L_\xi$ we will repeatedly apply Theorem (1.2) and construct as above $Y_0 \subset \partial X_0$, $X_1 := Y_0^{**}$, $Y_1 \subset \partial X_1$, and so on. We will show that in most cases the quantity $\epsilon$ of Inequality (1.5) is strictly positive. This progressively ”inflates” the dimension of $X_i$, until it becomes too big, giving a contradiction. We still have to introduce a key ingredient in this argument, namely an a priori bound on the amount by which the actual dimension of a moduli space can exceed the expected dimension. This follows from a bound for the number of sections of semistable sheaves, obtained by Simpson [S, Cor. (1.7)]. For the purposes of this proof we only need to know that there exists $e_L(r, S, H)$ such that

\[(1-6) \quad h^0(F, F \otimes L)^0 \leq e_L(r, S, H)\]
for all slope-semistable sheaves \( F \) with \( \text{rk}(F) = r \); the point is that \( \varepsilon_L \) is independent of the discriminant \( \Delta_F \). Under the hypotheses of (1.3) we have \( M_{\xi} = M_{\xi}^{st} \), hence deformation theory gives

(1-7) \hspace{1cm} \dim M_{\xi} \leq \exp \dim(M_{\xi}) + e_K.$

Proof of Proposition (1.3). Let \( \Delta_0 \) be so large that

\[
\exp \dim(M_{\xi}) > \lambda_2(r)\Delta_0 + \Delta_0(r, S, H) + \lambda_0(r, S, H)
\]

for all \( \xi \) with \( \Delta_0 \geq \Delta_0 \) (here \( \lambda_2, \lambda_1, \lambda_0 \) are as in Theorem (1.2)). By Theorem (1.2) we have the following.

(1.8). Assume \( \Delta_0 \geq \Delta_0 \). If \( X \) is a closed irreducible subset of \( M_{\xi} \), with \( \dim X \geq \exp \dim(M_{\xi}) \), then \( \partial X \neq \emptyset \).

Now assume

(1-9) \hspace{1cm} \Delta_0 > \Delta_0 + \varepsilon_L + e_K.$

Let’s show that \( \dim W_{\xi}^L < \exp \dim(M_{\xi}) \). Suppose the contrary, and let \( X_0 \subset W_{\xi}^L \) be an irreducible component with \( \dim X_0 \geq \exp \dim(M_{\xi}) \). By (1.9) and (1.8) \( \partial X_0 \neq \emptyset \). Let \( X_0 \subset \partial X_0 \) be an irreducible component of the open double-dual stratum, and set \( X_1 := Y_0^{**} \). If \( \partial X_1 \neq \emptyset \) (\( X_1 \) is the closure of \( X_1 \) in the appropriate moduli space) we repeat the process, i.e. we consider \( Y_1 \subset \partial X_1, X_2 := Y_1^{**}, \) and continue until we reach \( X_n \) such that \( \partial X_n = \emptyset \). By Formula (1.5) we have

\[
\dim X_{i+1} = \dim X_i - 2r\ell_i + (r - 1)(\ell_i - 1) + \epsilon_i,
\]

with the obvious notation. Let \( M_{\xi_n} \) be the moduli space to which \( X_n \) belongs. The formula above gives that

\[
(*) \hspace{1cm} \dim X_n = \dim X_0 - 2r\sum_{i=0}^{n-1} \ell_i + (r - 1)\left(\sum_{i=0}^{n-1} (\ell_i - 1)\right) + \sum_{i=0}^{n-1} \epsilon_i \geq \exp \dim(M_{\xi_n}).
\]

In fact the sum of the first two terms equals \( \exp \dim(M_{\xi_n}) \), and the remaining terms are non-negative. Since we are assuming \( \partial X_n = \emptyset \), we conclude by (1.8) that \( \Delta_0 < \Delta_0 \). Since \( \Delta_0 - \Delta_0 = \sum_{i=0}^{n-1} \ell_i \),

\[
\Delta_0 - \Delta_0 \leq \sum_{i=0}^{n-1} \ell_i.
\]

Manipulating the second term of (\( * \)), and applying the above inequality we get

(1-10) \hspace{1cm} \dim X_n \geq \exp \dim(M_{\xi_n}) + \Delta_0 - \Delta_0 + \sum_{i=0}^{n-1} (\ell_i - 1).

Now comes the key observation.
Claim. Let $h_L(X_i) := \min\{h^0(F, F \otimes L)^0 | [F] \in X_i\}$. Then:

1. $0 < h_L(X_i) \leq h_L(X_{i+1})$ for $i = 0, \ldots, n - 1$.
2. If $\epsilon_i = 0$ then $h_L(X_i) < h_L(X_{i+1})$.

Proof of the claim. To prove Item (1) it suffices to show that $h_L(X_i) < h_L(X_{i+1})$ for all $i$, because $h_L(X_0) > 0$ by definition. If $F$ is a torsion-free sheaf on $S$ there is a canonical injection

$$
\rho: \text{Hom}(F, F \otimes L) \hookrightarrow \text{Hom}(F^{**}, F^{**} \otimes L)
$$

which commutes with the trace, hence it defines also an injection of the traceless Hom groups. As is easily seen this implies (1): Indeed let $[F] \in Y_i \subset \partial X_i$ be a generic point; by upper-semicontinuity $h^0(F, F \otimes L)^0 \geq h_L(X_i)$, hence $h^0(F^{**}, F^{**} \otimes L)^0 \geq h_L(X_i)$. Since $F^{**}$ is a generic point of $X_{i+1}$ we have $h_L(X_{i+1}) = h^0(F^{**}, F^{**} \otimes L)^0$; we have proved Item (1). Now let’s prove Item (2). The hypothesis together with Equation (1.5) implies that $\partial X$ contains all sheaves $F$ fitting into an exact sequence

$$
0 \to F \to E \xrightarrow{\phi} \bigoplus_{j=1}^{\ell_i} C_{P_j} \to 0,
$$

where $[E] \in X_{i+1}$. Clearly $E = F^{**}$, thus the map $\rho$ realizes $\text{Hom}(F, F \otimes L)^0$ as a subgroup of $\text{Hom}(E, E \otimes L)^0$; an element $f \in \text{Hom}(E, E \otimes L)^0$ belongs to the image of $\rho$ if and only if

$$(\star) \quad f(\text{Ker} \phi_j) \subset \text{Ker} \phi_j \otimes L, \text{ for } j = 1, \ldots, \ell_i,$$

where $\phi_j$ is the restriction of $\phi$ to the fiber over $P_j$. Now let $[E] \in X_{i+1}$ be generic: by upper-semicontinuity $h^0(E, E \otimes L)^0 = h_L(X_{i+1})$, and the latter is non-zero by Item (1). Let $f \in \text{Hom}(E, E \otimes L)^0$ be non-zero: since $f$ is not a scalar endomorphism at the generic point, we can choose $\phi$ (in fact the generic $\phi$ will do) so that $(\star)$ does not hold, i.e. $\rho$ is not surjective. Hence for generic $[F] \in Y_i \subset \partial X_i$ we have $h^0(F, F \otimes L) < h_L(X_{i+1})$. By upper-semicontinuity $h_L(X_i) < h_L(X_{i+1})$. □

Let’s conclude the proof of Proposition (1.3). Since $h_L(X_i) \leq \epsilon_i$ for all $i$ by Simpson’s bound (1.6), the claim implies that $\sum_{i=0}^{n-1}(\epsilon_i - 1) \geq -(\epsilon_L - 1)$. By (1.10) we conclude that

$$\dim X_n \geq \exp.\dim \mathcal{M}_{\xi_n} + \Delta_\xi - \Delta_0 - \epsilon_L + 1.$$

Since $\Delta_\xi$ satisfies (1.9) this inequality contradicts (1.7). □

We will sketch a proof of Theorem (1.2). First we need to discuss determinant bundles on the moduli space.

**Determinant bundles.** References for this section are [LP,L1,FM (5.3.2)]. Assume $\mathcal{M}_\xi$ is fine, thus there is a tautological sheaf $\mathcal{F}$ on $S \times \mathcal{M}_\xi$. Let $C \subset S$ be a smooth irreducible curve. Choose a vector-bundle $A$ on $C$ with the property that

$$
\chi (F|_C \otimes A) = 0 \quad \text{for all } [F] \in \mathcal{M}_\xi.
$$

(11)
The restriction $F|_{C \times \mathcal{M}_\xi}$ is flat over $\mathcal{M}_\xi$, hence by the theory of determinant line-bundles [KM] it makes sense to set

$$L(F, C, A) := \det Rq_! (F \otimes p^* A)^{-1},$$

where $p, q$ are the projections of $C \times \mathcal{M}_\xi$ to $C$ and $\mathcal{M}_\xi$ respectively. Since $A$ satisfies (1.11) the determinant line-bundle is independent of the choice of a tautological sheaf. There is a natural section of $L(F, C, A)$ whose zero-locus is supported on the subset parametrizing sheaves $F$ such that $h^0(F|_C \otimes A) > 0$ (of course it might be that this section vanishes identically on $\mathcal{M}_\xi$). Applying the Grothendieck-Riemann-Roch Theorem (for Chow groups), one gets the equality

$$c_1(L(F, C, A)) = \text{rk}(A) \pi_* \left( \left( c_2(F) - \frac{\text{rk}(F) - 1}{2 \text{rk}(F)} c_1(F)^2 \right) \cdot C \right),$$

where $\pi : S \times \mathcal{M}_\xi \to \mathcal{M}_\xi$ is the projection (use Equation (1.11)). The above formula shows that the isomorphism class of $L(F, C, A)$ only depends on the linear equivalence class $[C]$. Furthermore, since the right-hand side of (1.12) is linear in $[C]$, we can define $L(F, [C], A)$ for an arbitrary divisor class $[C]$. To get rid of the dependence from $\text{rk}(A)$ we set

$$L([C]) := \frac{1}{\text{rk}(A)} L(F, [C], A).$$

Thus we get a well-defined map $L : \text{Pic}(S) \to \text{Pic}(\mathcal{M}_\xi) \otimes \mathbb{Q}$. We set $L(n) := L([nH])$; as is easily verified $L(n)$ is a line-bundle for all $n$ divisible by $\text{rk}(F)$. For simplicity we have assumed that $\mathcal{M}_\xi$ is fine, but in fact the map $L$ can be defined without this assumption [L1,LP]: the domain of $L$ will be a certain subspace of $\text{Pic}(S)$ which always includes $\mathbb{Z}[H]$. Historically Donaldson [D] was the first to study the determinant line-bundle: his goal was to prove that the polynomial invariants of algebraic surfaces are non zero. The following theorem gives an important property of $L(n)$ [LP,L1].

**Theorem (Le Potier - J. Li).** Let $n$ be sufficiently large and divisible by $\text{rk}(\xi)$ (in particular $L(n)$ is line-bundle). Then the complete linear system $|L(n)|$ is base-point free, and it defines an embedding of the subset of $\mathcal{M}_\xi$ parametrizing $\mu$-stable locally-free sheaves.

We will use the following.

**Corollary.** Let $X \subset \mathcal{M}_\xi$ be a closed irreducible subset. If the generic point of $X$ parametrizes a $\mu$-stable locally-free sheaf then

$$c_1(L(n))^{\text{dim} X} \cdot X > 0.$$

The rational line-bundle $L(n)$ is related to the theta-divisor on the moduli space of vector-bundles on $C \in [nH]$, as follows. Let $\mathcal{A}_C \subset \mathcal{M}_\xi$ be the subset parametrizing sheaves whose restriction to $C$ is locally-free and stable; restriction defines a morphism

$$\rho : \mathcal{A}_C \to \mathcal{U}(C; \text{rk}(\xi), \text{det}(\xi)|_C)$$

to the moduli space of semistable vector-bundles on $C$ (with fixed determinant). If $\Theta$ is the theta-divisor on $\mathcal{U}(C; \text{rk}(\xi), \text{det}(\xi)|_C)$, then

$$\rho^* \Theta \sim \lambda c_1(L(n)),$$

where $\lambda$ is a positive integer.
The proof of Theorem (1.2). For simplicity we assume $\mathcal{M}_\xi$ is a fine moduli space. To lighten notation we set $r = \text{rk}(\xi)$ and $\Delta = \Delta_\xi$. The proof is by contradiction. So let’s assume $X \subset \mathcal{M}_\xi$ is an irreducible closed subset with $\partial X = \emptyset$. If $C \subset S$ is an irreducible smooth curve we set

$$X_C := \{ [F] \in X | F|_C \text{ is not stable} \}.$$ 

A key observation is that under certain hypotheses $X_C$ is non-empty.

(1.16) Proposition. Keep notation and hypotheses as above (in particular $\partial X = \emptyset$). Suppose $n$ is a positive integer such that

$$\frac{r^2 - 1}{2} H^2 n^2 + \frac{r^2 - 1}{2} K \cdot Hn < \dim X. \quad (1.17)$$

If $C \in |nH|$ is a smooth curve, then $X_C$ is non-empty, and moreover

$$\dim X_C \geq \dim X - \frac{r^2}{8} H^2 n^2 - \frac{r^2}{8} K \cdot Hn - \frac{r^2}{4}. \quad (1.18)$$

Proof. Assume that $X_C = \emptyset$. Then associating to $[F] \in X$ the $S$-equivalence class of $F|_C$ we get a well-defined morphism

$$\rho : X \to \mathcal{M}(C; \xi),$$

where $\mathcal{M}(C; \xi)$ is the moduli space of rank-$r$ semistable vector-bundles on $C$ with determinant $\text{det}(\xi)|_C$. Since the left-hand side of (1.17) equals $\dim \mathcal{M}(C; \xi)$, we have

$$(\rho^* \Theta)^{\dim X} = 0,$$

where $\Theta$ is the theta-divisor. By Equation (1.15) and Corollary (1.14) we conclude that the generic point (hence all points) of $X$ parametrizes a sheaf which is not slope-stable. This contradicts our assumption that $X_C = \emptyset$: in fact it follows directly from the definition of slope-stability that if $F|_C$ is stable (where $C \in |nH|$), then $F$ is slope-stable. This proves $X_C \neq \emptyset$. Once we know $X_C \neq \emptyset$, Inequality (1.18) follows from a straightforward dimension count. \qed

Now assume we are in the situation of Proposition (1.16). Choose $[F] \in X_C$, and let

$$0 \to L_0 \to F|_C \xrightarrow{\varphi} Q_0 \to 0 \quad (1.19)$$

be a destabilizing sequence for $F|_C$ (with $Q_0$ locally-free). Let $E$ be the locally-free sheaf on $S$ defined by the following exact sequence (an elementary modification)

$$0 \to E \to F \xrightarrow{\varphi} \iota_* Q_0 \to 0,$$

where $\iota : C \hookrightarrow S$ is the inclusion. Restricting to $C$ the above sequence we get an exact sequence

$$0 \to Q_0 \otimes \mathcal{O}_C(-C) \to E|_C \xrightarrow{f_0} L_0 \to 0.$$
Let $Y_F := \text{Quot}(E|_C; \mathcal{L}_0)$ be the Quot-scheme parametrizing quotients of $E|_C$ with Hilbert polynomial equal to that of $\mathcal{L}_0$. For $y \in Y_F$ we let $\mathcal{G}_y$ be the torsion-free sheaf on $S$ defined by the elementary modification

$$0 \to \mathcal{G}_y \to E \xrightarrow{f_y} \mathcal{I}_* \mathcal{L}_y \to 0,$$

where $f_y$ is given by the quotient of $E|_C$ parametrized by $y$. The sheaves $\mathcal{G}_y$ fit into a family parametrized by $Y_F$. One easily verifies that:

1. There is a natural isomorphism $\mathcal{G}_0 \otimes \mathcal{O}_S(C) \cong F$.
2. The sheaf $\mathcal{G}_y$ is singular if and only if so is $\mathcal{L}_y$ (i.e. if $\mathcal{L}_y$ has torsion).

Let’s assume for the moment that $\mathcal{G}_y$ is stable for all $y \in Y_F$. Then, setting $\mathcal{F}_y := \mathcal{G}_y \otimes \mathcal{O}_S(C)$, the family $\{\mathcal{F}_y\}$ defines a classifying morphism

$$\varphi: Y_F \to \mathcal{M}_\xi,$$

and by Item (1) we have $\varphi(0) = [F] \in X$. We will arrive at a contradiction if we show that there exists $y \in \varphi^{-1}X$ such that $\mathcal{L}_y$ is singular; indeed this implies $\mathcal{G}_y$ is singular by Item (1), hence $\mathcal{F}_y$ is also singular, and thus $\varphi(y) \in \partial X$, contradicting the assumption $\partial X = \emptyset$. The following elementary result is proved [O2].

**Lemma.** Let $\Sigma \subset Y_F$ be a closed irreducible subset with $\dim \Sigma > \frac{r^2}{4}$. There exists $y \in \Sigma$ such that $\mathcal{L}_y$ is singular.

To apply the lemma we notice that

$$\dim \varphi^{-1}X \geq \dim_0 Y_F + \dim X - \dim T[F]\mathcal{M}_\xi.$$

For the dimension of the Quot-scheme $Y_F$ we have

\[
\dim_0 Y_F \geq \chi(Hom(Q_0 \otimes \mathcal{O}_C(-C), \mathcal{L}_0)) \\
= \text{rk}(\mathcal{L}_0) \text{rk}(Q_0) (\mu(\mathcal{L}_0) - \mu(Q_0) + C^2 + 1 - g(C)) \\
\geq \text{rk}(\mathcal{L}_0) \text{rk}(Q_0) \left( \frac{1}{2}C^2 - \frac{1}{2}C \cdot K \right) \\
\geq \frac{1}{2}(r-1) \left( H^2n^2 - K \cdot Hn \right).
\]

(The second inequality holds because (1.19) is a destabilizing sequence.) Feeding the inequality for $\dim Y_F$ together with (1.7) into (1.21), and applying Lemma (1.20) we get the following.

**Lemma.** Assume $\dim X$ satisfies (1.17). Assume also that $\mathcal{L}_y$ is stable for all $y \in Y_F$. If

$$\dim X > 2r\Delta - (r^2 - 1)\chi(\mathcal{O}_S) + \epsilon_K - \frac{1}{2}(r - 1) \left( H^2n^2 - K \cdot Hn \right),$$

then there exists $y \in \varphi^{-1}X$ with $\mathcal{L}_y$ singular, and hence $\partial X \neq \emptyset$.

To deal with the condition that $\mathcal{L}_y$ be stable for all $y \in Y_F$ we want to choose $[F] \in X_C$ which is "very stable", i.e. such that for all subsheaves $E \subset F$ with $\text{rk}(E) < \text{rk}(F)$,

$$\mu(E) < \mu(F) - C \cdot H = \mu(F) - H^2n.$$
Carrying out some dimension counts and using (1.18) one shows it suffices that
\[(1-24) \dim X - \frac{r^2}{8} H^2 n^2 - \frac{r^2}{8} K \cdot H n - \frac{r^2}{4} > (2r - 1) \Delta + (2r - 1)(r - 1)^2 H^2 n^2 + O(n).\]
(For this we must assume \(|H|\) is base-point free.) If \(r = 2\) a weaker inequality is required [O2]. At this point we have all the elements needed to prove Theorem (1.2). If we can find \(n\) such that Inequalities (1.17)-(1.23)-(1.24) hold, then the argument sketched above shows that \(\partial X \neq \emptyset\). It is an easy exercise to determine a lower bound on \(\dim X\) guaranteeing such \(n\) exists. The reader can check that the coefficient of \(\Delta\) can be taken to be
\[\lambda_2(r) = 2r - \frac{4(r - 1)}{16r^3 - 39r^2 + 36r - 12}.\]
(If \(r = 2\) one can improve the estimates and get \(23/6\) rather than \(31/8\).)
The lower bound on \(\Delta_\xi\) ensuring that \(\mathcal{M}_\xi\) is reduced of the expected dimension can be computed explicitly. This has been carried out in [O2] for \(\text{rk}(\xi) = 2\), when \(K\) is ample and \(H = K\). The lower bound is of the form \((\text{cost.})K^2\). One can ask for sharp bounds:

**Question.** Assume \(S\) is minimal of general type. Is \(\mathcal{M}_\xi\) reduced of the expected dimension when
\[\Delta_\xi > \text{rk}(\xi)(p_g + 1),\]
for polarizations sufficiently close to \(K\)?

Notice that we must restrict the choice of polarization \(H\) or else the answer is certainly negative (see [O3 (5b.24)]): sufficiently close means that for a sheaf with Chern classes defined by \(\xi\) slope-stability (instability) for \(H\) and \(K\) coincide.

**Irreducibility.** We give the argument of Gieseker and Li [GL1] which proves that \(\mathcal{M}_\xi\) is irreducible for large enough \(\Delta_\xi\); we will make some simplifying assumptions (as in Proposition (1.3)) in order to avoid some minor technical problems.

**Theorem.** Let \(r \geq 2\) be an integer, and \(D\) be a divisor on \(S\). Suppose that every rank-\(r\) torsion-free semistable sheaf \(F\) on \(S\) with \(\text{det} F \cong \mathcal{O}_S(D)\) is actually slope-stable (e.g. if \(D \cdot H\) and \(r\) are coprime). There exists \(\Delta_1\) such that if \(\xi\) is a set of sheaf data with
\[\text{rk}(\xi) = r, \quad \text{det}(\xi) \cong \mathcal{O}_S(D), \quad \Delta_\xi > \Delta_1,\]
then \(\mathcal{M}_\xi\) is irreducible.

This section is devoted to proving Theorem (1.25). We will always assume that \(\xi\) is a set of sheaf data satisfying the hypotheses of the theorem.

Let \(\xi_0\) be a set of sheaf data (with \(\text{rk}(\xi_0) = r, \text{det}(\xi_0) = \mathcal{O}_S(D)\)), and let \(X_1, \ldots, X_n\) be the irreducible components of \(\mathcal{M}_{\xi_0}\). For \(\ell\) a positive integer, and \(i = 1, \ldots, n\), we let \(Y_\ell^i\) be the locus of moduli (in the appropriate moduli space) of sheaves \(F\) fitting into an exact sequence
\[0 \to F \to E \to \bigoplus_{j=1}^{\ell} \mathbb{C} P_j \to 0,\]
where \([E] \in X_i\) is an arbitrary point with \(E\) locally-free, and the \(P_j\)’s are pairwise distinct.
Lemma. Keep notation as above. There exists $\Delta_{\xi_0}$ such that the following holds. If $\Delta_{\xi} > \Delta_{\xi_0}$, and $\ell := c_2(\xi) - c_2(\xi_0)$, then any irreducible component of $\mathcal{M}_\xi$ contains one (at least) of the $Y^\ell_i$. Furthermore $\mathcal{M}_\xi$ is smooth at the generic point of each of the $Y^\ell_i$.

Sketch of proof. Let $\Delta_0$ be as in (1.8): hence if $\Delta_{\xi} \geq \Delta_0$ all irreducible components of $\mathcal{M}_\xi$ have non-empty boundary. Increasing $\Delta_0$ if necessary, we can assume by Proposition (1.3) that moduli spaces $\mathcal{M}_\xi$ with $\Delta_{\xi} \geq \Delta_0$ are reduced of the expected dimension. A simple application of Inequality (1.5) will show that if $\Delta_{\xi_0}$ is sufficiently larger than $\Delta_0$ the following holds. Assume $\Delta_{\xi} > \Delta_{\xi_0}$, and let $Y$ be any irreducible component of $\mathcal{M}_\xi$. Then there exists an irreducible component $Y'$ of $\mathcal{M}_\xi'$, where

$$\text{rk}(\xi') = r, \quad \det(\xi') = \mathcal{O}_S(D), \quad c_2(\xi') = c_2(\xi) - 1,$$

such that $Y$ contains the moduli point of any sheaf $F$ fitting into an exact sequence

$$0 \to F \to E \to \mathbb{C}_p \to 0,$$

where $[E]$ is an arbitrary point of $Y'$ with $E$ locally-free. Applying this same result to $\mathcal{M}_{\xi'}$ and the irreducible component $Y'$, and so on all the way down to $\Delta_{\xi_0}$, one gets the first statement of the lemma. The second statement holds because $\Delta_{\xi_0} \geq \Delta_0$, and hence the generic point $[E]$ of any irreducible component of $\mathcal{M}_{\xi_0}$ has vanishing obstruction space (i.e. $H^0(\text{End}_0(E) \otimes K) = 0$), and hence so does any sheaf whose double-dual is isomorphic to $E$. \qed

Fix $\xi_0$ as in the above lemma; then for $i = 1, \ldots, n$ there is only one irreducible component of $\mathcal{M}_\xi$ containing $Y^\ell_i$, and since each component contains at least one $Y^\ell_i$,

$$\#\text{irr.comp.}(\mathcal{M}_\xi) \leq \#\text{irr.comp.}(\mathcal{M}_{\xi_0}).$$

We will prove Theorem (1.25) by showing that if $\ell \gg 0$ then the $Y^\ell_i$ all belong to the same irreducible component. Choose $[E_i] \in X_i$, for $i = 1, \ldots, n$, with $E_i$ locally-free and with vanishing obstruction space. Thus if $[F_i] \in Y^\ell_i$ lies over $[E_i]$, i.e. $F_i^{**} \cong E_i$, the moduli space $\mathcal{M}_\xi$ is smooth at $[F_i]$, in particular the unique irreducible component containing all of $Y^\ell_i$ must contain any irreducible subset through $[F_i]$. We will construct (for $\ell \gg 0$) an irreducible subset $W \subset \mathcal{M}_\xi$ containing $[F_1], \ldots, [F_n]$; thus $\mathcal{M}_\xi$ must be irreducible. The subset $W$ is defined as follows. Let $n$ be an integer such that $E_i \otimes \mathcal{O}_S(n)$ is generated by global sections, for $i = 1, \ldots, n$. Choosing $(r - 1)$ generic sections of $E_i \otimes \mathcal{O}_S(n)$ we see that $E_i$ fits into an exact sequence

$$0 \to \mathcal{O}_S(-nH)^{(r-1)} \to E_i \to I_{Z_i}(D + (r - 1)nH) \to 0,$$

where $Z_i$ is some zero-dimensional subscheme of $S$. Choosing appropriately the surjection

$$E_i \to \bigoplus_{j=1}^\ell \mathbb{C}P^j,$$

whose kernel is $F_i$, we see that $Y_i$ contains the moduli point of a sheaf $F_i$ fitting into an exact sequence

$$0 \to \mathcal{O}_S(-nH)^{(r-1)} \to F_i \to I_{Z_i}(D + (r - 1)nH) \to 0,$$
where $\tilde{Z}_i = Z_i \cup \{P^i_1, \ldots, P^i_\ell\}$. Thus $F_i$ corresponds to a non-zero class in
\[
\text{Ext}^1 \left( I_{\tilde{Z}_i}(D + (r-1)nH), \mathcal{O}_S(-nH)^{(r-1)} \right).
\]
If $\ell$ is large enough and the points $P^i_1, \ldots, P^i_\ell$ are generic,
\[
(\dagger) \quad \dim \text{Ext}^1 \left( I_{\tilde{Z}_i}(D + (r-1)nH), \mathcal{O}_S(-nH)^{(r-1)} \right) = -\chi \left( I_{\tilde{Z}_i}(D + (r-1)nH), \mathcal{O}_S(-nH)^{(r-1)} \right).
\]
Let $d = \ell(\tilde{Z}_1) = \ldots = \ell(\tilde{Z}_n)$, and let $U \subset S^{[d]}$ be the open subset of the Hilbert scheme parametrizing length-$d$ subschemes $Z$ of $S$ such that $(\dagger)$ holds with $\tilde{Z}_i$ replaced by $Z$. We define $W \subset \mathcal{M}_\xi$ to be the subset parametrizing sheaves $\mathcal{F}_{\xi}$ which fit into an exact sequence
\[
0 \to \mathcal{O}_S(-nH)^{(r-1)} \to F \to I_{\tilde{Z}}(D + (r-1)nH) \to 0,
\]
for some $\tilde{Z} \in U$. By construction $[F_i] \in W$ for $i = 1, \ldots, n$. Since $W$ is an open subset of a bundle of projective spaces over $U$, it is irreducible. This finishes the proof of Theorem (1.25).

Remark. Notice that all the steps of the above proof can easily be made effective, except for the choice of $n$ such that $E_i \otimes \mathcal{O}_S(nH)$ is generated by global sections. In [O2] there are some effective results for complete intersections.

2. Two-forms on the moduli space.

Let $B$ be a smooth variety, and $\mathcal{F}$ be a family of torsion-free sheaves on $S$ parametrized by $B$. For $b \in B$ we set $S_b := S \times \{b\}$ and $\mathcal{F}_b := \mathcal{F}|_{S_b}$. We assume the isomorphism class of $\det \mathcal{F}_b$ is independent of $b$. Given a two-form $\omega \in \Gamma(\Omega^2_B)$ we will define a two-form $\omega_{\mathcal{F}} \in \Gamma(\Omega^2_B)$. First recall [Mm1] that given a codimension-two cycle $Z \in Z^2(S \times B)$ transverse to the projection $q : S \times B \to B$ (i.e. $Z = \sum_i n_i Z_i$ where each $Z_i$ is a subvariety intersecting the generic $S_b$ in a finite set of points) we can associate to it a two-form $\omega_Z$ on $B$. Explicitly, let $q_i : Z_i \to B$ be the restriction of $q$, and $p : S \times B \to S$ be the projection, then
\[
(2.1) \quad \omega_Z := \sum_i n_i q_{i,*} (p^* \omega|_{Z_i}).
\]
Some care must be taken in defining the push-forward at points $b \in B$ over which $q_i$ is not étale: we can circumvent this problem by considering the universal case, i.e. $B = S^{[d]}$, the Hilbert scheme parametrizing length-$d$ subschemes of $S$, and $Z$ is the cycle of the tautological subscheme of $S \times S^{[d]}$. One verifies [Be2, Prop. (5)] that there exists $\omega^{[d]} \in \Gamma(\Omega^2_{S^{[d]}})$ which restricted to the open subset parametrizing reduced subschemes equals the push-forward of $p^* \omega|_Z$. Letting $\varphi_i : B \to S^{[d_i]}$ be the rational map induced by $Z_i$, we can define the terms appearing in (2.1) by setting
\[
q_{i,*} (p^* \omega|_{Z_i}) := \varphi_{i,*} \omega^{[d_i]}.
\]
Mumford [Mm1] proved that if \( Z' \in Z^2(S \times B) \) is a cycle such that \( Z' \cdot S_b \sim Z \cdot S_b \) (\( \sim \) denotes rational equivalence) for all \( b \in B \) then \( \omega_{Z'} = \omega_Z \). In particular we get a well-defined two-form \( \omega_F \) on \( B \) if we set 
\[
\omega_F := \omega_Z, \quad Z \in Z^2(S \times B) \text{ a representative of } c_2(F) \in A^2(S \times B).
\]

If \( L \) is a line-bundle on \( B \) and \( F' := F \otimes q^*L \), then \( \omega_{F'} = \omega_F \). This allows us to define a two-form \( \omega_\xi \) on the locus \( M_\xi^0 \subset M_\xi \) of smooth (for the reduced structure) stable points. More explicitly: if \( M_\xi \) is a fine moduli space we set \( \omega_\xi := \omega_F \), where \( F \) is any tautological family of sheaves on \( S \) parametrized by \( M_\xi \) (the two-form is independent of the choice of \( F \)), if \( M_\xi \) is not a fine moduli space one can use a quasi-tautological family [Mk2, p. 407] parametrized by \( M_\xi^0 \), or resort to a patching argument. In this section we will deal with the following question: Let \( [F] \in M_\xi \) be a generic point, and view \( \omega_\xi([F]) \) as a (skew-symmetric) linear map 
\[
\omega_\xi([F]): T_{[F]}M_\xi \to \Omega_{[F]}M_\xi,
\]
what is the corank of \( \omega_\xi([F]) \)? In particular, when does there exist an open dense subset of \( M_\xi \) over which \( \omega_\xi \) is a symplectic form? Before giving a (partial) answer, we must open a digression.

(2.2) Definition. Let \( C \) be a smooth irreducible projective curve, and \( \theta \) be a theta-characteristic on \( C \). We set 
\[
\lambda_C(\theta, r, d) := h^0(\text{End}_0(V) \otimes \theta),
\]
where \( V \) is the generic stable rank-\( r \) vector-bundle on \( C \) with \( \deg V = d \). (If \( C \) has genus zero then \( \lambda_C \) is not defined, if \( C \) has genus one \( \lambda_C \) is only defined for \( r, d \) coprime.)

A result of Mumford determines the parity of \( \lambda_C \).

(2.3) Proposition [Mm2]. Let \( \theta \) be a theta-characteristic on \( C \), and \( V \) be a vector-bundle on \( C \). Then 
\[
h^0(\text{End}_0(V) \otimes \theta) \equiv (\text{rk}(V) - 1) \cdot (h^0(\theta) + \deg V) \pmod{2}.
\]

Proof. By Mumford [Mm2] the quantity \( h^0(\text{End}_0(V) \otimes \theta) \) is constant modulo two when \( V \) varies in a connected (flat) family. Since any two vector-bundles on \( C \) with the same rank and degree belong to a connected family, it suffices to check the equation for a direct sum of line-bundles; the computation is left to the reader. \( \square \)

In particular we get 
\[
\lambda_C(\theta, r, d) \equiv (r - 1) \cdot (h^0(\theta) + d) \pmod{2}.
\]

(2.4) Conjecture. Let \( C \) be a smooth irreducible projective curve of genus at least one, and \( \theta \) be a theta-characteristic on \( C \). Then
\[
\lambda_C(\theta, r, d) = \begin{cases} 0 & \text{if } (r - 1) \cdot (h^0(\theta) + d) \equiv 0 \pmod{2}, \\ 1 & \text{if } (r - 1) \cdot (h^0(\theta) + d) \equiv 1 \pmod{2}. \end{cases}
\]

In genus one the conjecture is easily settled, but for bigger genus we do not know the answer in general. When the rank is two (2.4) has been proved: in fact there exists a very quick proof [L2], a "Prym variety" proof [Be1], and a computational one [O1]. Unfortunately we have not succeeded in generalizing any of these proofs to higher rank. A different approach, explained at the end of this section, gives the following.
Proposition. Keep notation as above. Let $C$ be a smooth irreducible projective curve of genus at least two, and let $\theta$ be a theta-characteristic on $C$. Then

$$\lambda_C(\theta, r, h^0(\theta)) = 0.$$ 

Now let’s go back to moduli of vector-bundles on surfaces. We will prove (see [Mk1,O1]) the following

Theorem. Given a polarized surface $(S, H)$ there exists $\Delta(r)$ such that the following holds. Let $\omega$ be a holomorphic two-form on $S$ whose zero-locus $C$ is either empty or a smooth irreducible curve of genus at least one. Let $\xi$ be a set of sheaf data with $\Delta_\xi > \Delta(\text{rk}(\xi))$ and, in case $C$ has genus one, assume also that $\text{rk}(\xi)$, $(c_1(\xi) \cdot KS)$ are coprime. Then the corank of $\omega_\xi$ at the generic point of $M_\xi$ equals

$$\lambda_C(KS|C, \text{rk}(\xi), c_1(\xi) \cdot KS).$$

(By convention we set $\lambda_C = 0$ if $C$ is empty.)

Remark. The lower bound $\Delta(r)$ of the above theorem is not less than the quantity $\Delta(r)$ of Theorem (0.1), and hence $\dim M_\xi = 2r\Delta_\xi - (r^2 - 1)\chi(O_S)$ (here $r := \text{rk}(\xi)$).

A computation shows that

$$2r\Delta_\xi - (r^2 - 1)\chi(O_S) \equiv (r - 1) \cdot (h^0(KS|C) + c_1(\xi) \cdot KS) \pmod 2.$$ 

Hence if (2.4) holds, Theorem (2.6) gives that $\omega_\xi$ is generically symplectic if $\dim M_\xi$ is even, and ”almost symplectic” if $\dim M_\xi$ is odd.

Since (2.4) is true if the rank is two, or if $d = h^0(\theta)$ by Proposition (2.5), we get the following corollary (see [Mk1,O1] for the rank-two case) of Theorem (2.6).

Corollary. Let hypotheses be as in the previous corollary. The image of the map

$$M_\xi \to A^2(S) \quad [F] \to c_2(F)$$

has dimension equal to $\dim M_\xi$.

Example. Let $\pi: S \to \mathbb{P}^2$ be a double cover of $\mathbb{P}^2$ branched over a smooth curve of degree $8n$. A set of sheaf data $\xi$ with $\det(\xi) = \pi^*(O_{\mathbb{P}^2}(n))$ satisfies the hypotheses of Corollary (2.7) (for $\Delta_\xi \gg 0$), hence $\omega_\xi$ is generically non-degenerate.
Proof of Theorem (2.6). We maintain the notation of the introduction to this section. The first step of the proof consists in identifying \( \omega_F \) (up to multiples) with a certain two-form \( \hat{\omega}_F \) introduced by Mukai and Tyurin [Mk1,T]. Let \( b \in B \), and let

\[
\kappa: T_b(B) \to \text{Ext}^1(F_b, F_b)
\]

be the Kodaira-Spencer map of the family \( F \). We define \( \hat{\omega}_F \) at \( b \) by setting

\[
\hat{\omega}_F(v \wedge w) := \int_S \text{Tr} (\kappa(v) \cup \kappa(w)) \wedge \omega.
\]

Here "\( \cup \)" denotes Yoneda pairing, and we are viewing \( \text{Tr} (\kappa(v) \cup \kappa(w)) \) as a \((0,2)\)-form via Dolbeault’s isomorphism. If \( F_b \) is locally-free then \( \text{Ext}^1(F_b, F_b) \cong H^{0,1}(\text{End} F_b) \), and \( \text{Tr}(\cdot) \) is obtained composing the \((0,1)\)-valued endomorphisms \( \kappa(v), \kappa(w) \), and taking the trace. A local computation shows that the trace is skew-symmetric in this case. For skew-symmetry when \( F_b \) is not locally-free see [M,O1].

Proposition. Let notation be as above. Assume the isomorphism class of \( \det F_b \) is independent of \( b \in B \). Then

\[
(2.9) \quad \omega_F = \left( \frac{i}{2\pi} \right)^2 \hat{\omega}_F.
\]

Sketch of the proof. First one verifies the following:

1. Suppose the isomorphism class of \( \det F_b \) is independent of \( b \in B \). If \( L \) is line-bundle on \( S \) and \( F' = F \otimes p^* L \), then

\[
\omega_{F'} = \omega_F \quad \hat{\omega}_{F'} = \hat{\omega}_F.
\]

2. Let

\[
0 \to E \to F \to G \to 0
\]

be an exact sequence, where \( E, F, G \) are families of torsion-free sheaves on \( S \) with \( \det E_b, \det F_b, \det G_b \) constant up to isomorphism. Then

\[
\omega_F = \omega_E + \omega_G \quad \hat{\omega}_F = \hat{\omega}_E + \hat{\omega}_G.
\]

Now let’s proceed with the proof of (2.9). Replacing \( B \) by an open dense subset we can assume there is an exact sequence

\[
0 \to C_{S \times B}^{(r-1)} \to F \otimes p^* O_S(nH) \to I_Z \to 0,
\]

where \( Z \) is a family of zero-dimensional subschemes of \( S \) parametrized by \( B \), \( I_Z \) is its ideal sheaf, and \( r \) is the rank of \( F \). By Items (1)-(2) above it suffices to prove (2.9) for \( F = I_Z \). For this it is enough to consider the universal case: \( B = S^{[d]} \) and \( Z \) the tautological subscheme of \( S \times S^{[d]} \). There exists a short locally-free resolution

\[
0 \to F^1 \to F^0 \to I_Z \to 0,
\]
where the isomorphism class of $\det F^\vee$ is independent of $x \in S^{[d]}$. By Item (2) it suffices to prove (2.9) for $F^\vee$. In the de Rham cohomology of $S^{[d]}$ we have

$$[\omega_{F^\vee}] = q_*[c_2(F^\vee) \wedge p_S^*\omega],$$

where $q: S \times S^{[d]} \to S^{[d]}$ is projection (here $c_2(F^\vee) \in H^4(S \times S^{[d]}))$. On the other hand, by Chern-Weyl theory one gets [O1]

$$\left(\frac{i}{2\pi}\right)^2 [\omega_{F^\vee}] = q_*[c_2(F^\vee) \wedge p_S^*\omega].$$

Hence the two sides of (2.9) are cohomologous; since they are holomorphic and since $S^{[d]}$ is projective we conclude that they must be equal. \(\square\)

Now we can prove Theorem (2.6). By Theorem (1.1) there exists $\Delta(r)$ such that if $\Delta_x > \Delta(r)$ (where $\rk(\xi) = r$) then

$$\dim W^{2K}_x < \exp\dim\mathcal{M}_x. \tag{2-10}$$

Furthermore we can assume the generic point on every irreducible component of $\mathcal{M}_x$ represents a stable locally-free sheaf [O1]. Let $[F] \in \mathcal{M}_x$ be a generic point; thus $F$ is locally-free, stable, and by (2.10) the moduli space is smooth of the expected dimension at $[F]$ (since $H^0(K)$ has a section, $W^{2K}_x \subset W^{2K}_x$). We have $T_{[F],\mathcal{M}_x} \cong H^1(\End_0(F))$ (see Section (1)), and by (2.9)

$$\omega_x(v \wedge w) = \left(\frac{i}{2\pi}\right)^2 \int_S (v \wedge (w \cdot \omega)).$$

By Serre duality the bilinear map

$$H^1(\End_0(F)) \times H^1(\End_0(F) \otimes K) \xrightarrow{\alpha, \beta} H^2(K) \cong \mathbb{C} \xrightarrow{\Tr} \Tr(\alpha \cup \beta)$$

is a perfect pairing, hence it suffices to show that for generic $[F] \in \mathcal{M}_x$ the map

$$H^1(\End_0(F)) \xrightarrow{\omega} H^1(\End_0(F) \otimes K)$$

has corank $\lambda_C(K_S|_C, \rk(\xi), c_1(\xi) \cdot K_S)$. This certainly holds if $C = \emptyset$, hence we can assume $C \neq \emptyset$. Consider the exact sequence

$$H^0(\End_0(F) \otimes K) \to H^0(\End_0(F) \otimes K|_C) \to H^1(\End_0(F)) \xrightarrow{\omega} H^1(\End_0(F) \otimes K).$$

Since $[F]$ is generic and since (2.10) holds, we have $h^0(\End_0(F) \otimes K) = 0$. Thus we must show that

$$h^0(\End_0(F) \otimes K|_C) = \lambda_C(K_S|_C, \rk(\xi), c_1(\xi) \cdot K_S). \tag{2-11}$$

Hence it suffices to prove that if $[F] \in \mathcal{M}_x$ is generic then $F|_C$ is the generic stable vector bundle (of rank $\rk(\xi)$ and determinant $\det(\xi)|_C$). So let $[E] \in \mathcal{M}_x$ with $E$ locally-free, stable and $[E] \notin W^{2K}_x$; we claim the map $\rho: \Def^0(F) \to \Def^0(F|_C)$ defined by restriction is surjective. In fact both the domain and codomain are smooth, and the differential $d\rho$ fits into the exact sequence

$$H^1(\End_0(E)|_C) \xrightarrow{d\rho} H^1(\End_0(E)) \to H^2(\End_0(E) \otimes [-K]) \cong H^0(\End_0(E) \otimes [2K])^* = 0.$$
Proof of (2.5). We let \( C \) be a smooth irreducible projective curve of genus at least two. We will examine vector-bundles \( E \) obtained as extensions

\[
0 \to V^* \to E \to \mathcal{O}_C \to 0.
\]

(2.13) Lemma. Keep notation as above. Assume that:

1. \( h^0(\text{End}_0(V) \otimes \theta) = 0 \).
2. \( 0 \leq \deg V \leq h^0(\theta) \).
3. \( V \) is generic among vector-bundles of the same degree and rank.
4. The extension class \( \eta \in H^1(V^*) \) of (2.12) is generic.

Then there is a natural identification

\[
H^0(\text{End}_0(E) \otimes \theta) \cong \ker \left( H^0(\theta) \xrightarrow{\partial} H^1(V^* \otimes \theta) \right),
\]

where \( \partial \) is the coboundary map associated to the sequence obtained tensoring (2.12) by \( \theta \):

\[
0 \to V^* \otimes \theta \to E \otimes \theta \to \mathcal{O}_C(\theta) \to 0.
\]

Proof. Scalar endomorphisms give an injection \( H^0(\theta) \hookrightarrow H^0(\text{End}(E) \otimes \theta) \), and there is a splitting

\[
H^0(\text{End}(E) \otimes \theta) = H^0(\theta) \oplus H^0(\text{End}_0(E) \otimes \theta).
\]

Thus it suffices to give an identification

\[
H^0(\text{End}(E) \otimes \theta) / H^0(\theta) \cong \ker \left( H^0(\theta) \xrightarrow{\partial} H^1(V^* \otimes \theta) \right).
\]

Let \( \varphi \in H^0(\text{End}(E) \otimes \theta) \). First we prove

\[
\varphi(V^*) \subseteq V^* \otimes \theta.
\]

For this it suffices to show that \( \text{Hom}(V^*, V^* \otimes \theta) \hookrightarrow \text{Hom}(V^*, E \otimes \theta) \) is an isomorphism. Tensoring (2.14) by \( V \) we get a coboundary map

\[
H^0(V \otimes \theta) \xrightarrow{\partial_V} H^1(V \otimes V^* \otimes \theta).
\]

We must show \( \partial_V \) is injective. Consider the trace map

\[
H^1(V^* \otimes V \otimes \theta) \xrightarrow{\Tr} H^1(\theta).
\]

We will prove that \( \Tr \circ \partial_V \) is injective. By Serre duality we can view \( \Tr \circ \partial_V \) as a map

\[
\Tr \circ \partial_V : H^0(V \otimes \theta) \to H^0(\theta)^*.
\]

Explicitly, since the extension class \( \eta \) of (2.12) is an element of \( H^0(V \otimes K_C)^* \) (by Serre duality), we have

\[
\langle \Tr \circ \partial_V(\alpha), \beta \rangle = \langle \eta, \alpha \otimes \beta \rangle, \quad \alpha \in H^0(V \otimes \theta), \quad \beta \in H^0(\theta).
\]
Because the map
\[ H^0(\theta) \longrightarrow H^0(V \otimes K_C) \]
\[ \beta \mapsto \alpha \otimes \beta \]
is injective for any non-zero \( \alpha \in H^0(V \otimes \theta) \), there is a well-defined map
\[ \mathbb{P} := \mathbb{P} \left( H^0(V \otimes \theta) \right) \xrightarrow{\Phi} \text{Gr} \left( h^0(\theta), h^0(V \otimes K_C) \right), \]
\[ \{ \alpha \otimes \beta | \beta \in H^0(\theta) \} \]
where \( \text{Gr}(m, n) \) is the Grassmannian of \( m \)-dimensional vector subspaces of \( \mathbb{C}^n \). Let
\[ \Lambda := \bigcup_{[\alpha] \in \mathbb{P}} \{ \eta \in \mathbb{P} \left( H^0(V \otimes K_C)^* \right) \text{ vanishes on } \Phi([\alpha]) \}. \]
(Notice that \( H^0(\mathcal{O}_C \otimes \mathcal{E}) \neq 0 \) because \( \deg V \geq 0 \) by Item (2), and because \( C \) has genus at least two.) We must show that
(\( \bullet \))
\[ \Lambda \neq \mathbb{P} \left( H^0(V \otimes K_C)^* \right). \]
A dimension count gives
\[ \dim \Lambda \leq h^0(V \otimes \theta) - 1 + h^0(V \otimes K_C) - h^0(\theta) - 1 \]
(\( \dagger \))
\[ = \dim \mathbb{P} \left( H^0(V \otimes K_C)^* \right) - (h^0(\theta) - h^0(V \otimes \theta) + 1). \]
By our hypotheses \( \deg V \geq 0 \) and \( V \) is generic. This implies that
\[ h^0(V^* \otimes \theta) = 0, \]
and thus by Serre duality \( h^1(V \otimes \theta) = 0. \) Hence
\[ h^0(V \otimes \theta) = \chi(V \otimes \theta) = \deg V. \]
By (\( \dagger \)) we conclude that if \( \deg V \leq h^0(\theta) \) then (\( \bullet \)) holds. Thus for \( \eta \) generic the map \( \partial_V \) is injective. (Of course \( \deg V \leq h^0(\theta) \) is also necessary for \( \partial_V \) to be injective.) This proves (\( * \)). Now we can finish the proof of the lemma. By (\( * \)) and Item (1) the restriction of \( \varphi \) to \( V^* \) is equal to scalar multiplication by a certain section \( \sigma \in H^0(\theta) \); thus
\[ (\varphi - \sigma)(V^*) = 0, \]
or in other words
\[ H^0(\text{End}(\mathcal{E}) \otimes \theta)/H^0(\theta) \cong \text{Hom}(\mathcal{O}_C; \mathcal{E} \otimes \theta) = H^0(\mathcal{E} \otimes \theta). \]
Writing out the long exact cohomology sequence associated to (2.14), the lemma follows from (2.15). \( \square \)

Let’s prove Proposition (2.5) by induction on the rank \( r \). The case \( r = 1 \) is trivial. Let’s prove the inductive step. We assume that \( V \) is the generic stable rank-\( r \) vector-bundle with \( \deg V = h^0(\theta) \), and that (2.5) holds for \( V \). Consider the generic extension (2.12). If we show that \( h^0(\text{End}_0(\mathcal{E}) \otimes \theta) = 0 \) then we are done, because \( \text{rk}(\mathcal{E}^*) = (r + 1) \) and \( \deg \mathcal{E}^* = h^0(\theta) \). By Lemma (2.13) it suffices to prove that
\[ H^0(\theta) \xrightarrow{\partial} H^1(V^* \otimes \theta) \]
is injective. This coboundary is the transpose of the map \( \text{Tr} \circ \partial_V \) appearing in the proof of Lemma (2.13). We have proved \( \text{Tr} \circ \partial_V \) is injective, and thus \( \partial \) is injective if and only if \( h^0(\theta) = h^1(V^* \otimes \theta) \). By Serre duality \( h^1(V^* \otimes \theta) = h^0(V \otimes \theta) \), hence the result follows from (2.16).
3. Kodaira dimension of the moduli space.

The main result is due to Jun Li [L2]: moduli spaces of rank-two vector-bundles on a surface of general type are often of general type. We will prove Jun Li’s theorem for fine moduli spaces, with some additional hypotheses. The proof in general is more difficult, the main problem being the analysis of singularities coming from strictly semistable sheaves [L2]. Jun Li’s theorem, for fine moduli spaces, extends to higher rank if Conjecture (2.4) is true. In particular by Proposition (2.5) many higher-rank moduli spaces are of general type; we will give some examples. We will also quickly mention some results concerning moduli spaces on surfaces not of general type. In the proofs we will usually choose a particularly nice polarization (essentially a multiple of $K$): this is not a significant restriction because of Remark (0.2).

Throughout this section we assume $M_\xi$ is a fine moduli space; we let $F$ be a tautological sheaf on $S \times M_\xi$. To simplify notation we set $r := \text{rk}(\xi)$.

The canonical line-bundle. Let $M_\xi^{sm}$ be the subscheme of $M_\xi$ parametrizing stable sheaves $F$ with vanishing obstruction space, i.e. such that

$$\text{Ext}^2(F,F)^0 = 0.$$ 

By deformation theory $M_\xi^{sm}$ is smooth.

(3.1) Lemma. Keep notation as above. Then modulo torsion

$$K_{M_\xi^{sm}} \cong L(rK_S).$$

Proof. Let $\pi: S \times M_\xi^{sm} \to M_\xi^{sm}$ be the projection, and let $F^{sm}$ be the restriction of $F$ to $S \times M_\xi^{sm}$. Define $\text{Ext}^p_F(F^{sm},F^{sm})^0$ as the sheaf on $M_\xi^{sm}$ fitting into the exact sequence

$$0 \to \text{Ext}^p_F(F^{sm},F^{sm})^0 \to \text{Ext}^p_F(F^{sm},F^{sm}) \xrightarrow{\text{Tr}} R^p\pi_*\mathcal{O} \to 0.$$ 

Since $\text{Ext}^p_F(F^{sm},F^{sm})$ is a vector-bundle with fiber $\text{Ext}^p(F,F)$ over $[F] \in M_\xi^{sm}$, the fiber of $\text{Ext}^p_F(F^{sm},F^{sm})^0$ over $[F]$ is canonically isomorphic to $\text{Ext}^p(F,F)^0$. Thus by deformation theory

$$T_{M_\xi^{sm}} \cong \text{Ext}^1_F(F^{sm},F^{sm})^0.$$ 

Exact sequence (3.2) for $p = 1$ gives

$$c_1 \left( \text{Ext}^1_F(F^{sm},F^{sm})^0 \right) = c_1 \left( \text{Ext}^1_F(F^{sm},F^{sm}) \right).$$ 

On the other hand, by definition of $M_\xi^{sm}$ we have $\text{Ext}^p_F(F^{sm},F^{sm})^0 = 0$ for $p = 0, 2$, and hence (3.2) gives

$$c_1 \left( \text{Ext}^p_F(F^{sm},F^{sm}) \right) = 0 \quad p = 0, 2.$$ 

From the above equalities we get that in the Chow group $A^1(M_\xi^{sm})_\mathbb{Q}$

$$c_1(K_{M_\xi^{sm}}) = -c_1 \left( \text{Ext}^1_F(F^{sm},F^{sm})^0 \right) = \sum_{p=0}^2 (-1)^p c_1 \left( \text{Ext}^p_F(F^{sm},F^{sm}) \right).$$
The right-hand side can be computed by applying Grothendieck-Riemann-Roch: setting
\[ ch(F^{sm})^*: = \sum (-1)^n ch_n(F^{sm}), \]
we have
\[ \sum_{p=0}^{2} (-1)^p c_1(Ext^p_\pi(F^{sm}, F^{sm})) = \pi_* [ch(F^{sm})^* ch(F^{sm}) Td(S)]_3 \]
\[ = \pi_* \left[ \left( c_2(F^{sm}) - \frac{r-1}{2r} c_1(F^{sm})^2 \right) \cdot rK_S \right]. \]
The lemma follows from the above formula together with (3.3) and (1.12). □

**Surfaces of Kodaira dimension at most one.** First assume \( S \) is a Del Pezzo surface, and let \( H = -K \). Since \( K \cdot H < 0 \) the obstruction space
\[ \text{Ext}^2(F,F)^0 \cong (\text{Hom}(F,F \otimes K)^0)^* \]
vaneses for every \([F] \in \mathcal{M}_\xi\), hence \( \mathcal{M}_\xi \) is smooth (of the expected dimension). Thus Lemma (3.1), together with Theorem (1.13), implies that \( \kappa(\mathcal{M}_\xi) = -\infty \). In fact more is known [ES]: if \( S = \mathbb{P}^2 \) the moduli space is often rational. More in general, it is natural to expect that if \( S \) is (birationally) ruled, the moduli space is uniruled (for \( \Delta_\xi > 0 \)). Hoppe and Spindler [HS] treat the case of rank two.

If \( S \) is a \( K3 \) surface, the moduli space is smooth, hence by Lemma (3.1) we conclude that \( \kappa(\mathcal{M}_\xi) = 0 \). In fact more is true: if \( \omega \) is a non-zero two-form on \( S \), the two-form \( \omega_\xi \) is everywhere non-degenerate [Mk1], thus \( \mathcal{M}_\xi \) is holomorphically symplectic.

Finally let’s consider the case of a minimal surface of Kodaira dimension one. Let
\[ f: S \to B \]
be the elliptic fibration, and for \( b \in B \) let \( C_b := f^*(b) \). We assume that the set of sheaf data satisfies:
\[ \text{rk}(\xi) \text{ and } c_1(\xi) \cdot C_b \text{ are coprime.} \]
It is convenient to choose the polarization \( H \) to be very close to \( C_b \) in the Néron-Severi group \( NS(S) \) (how close will depend on \( \text{rk}(\xi) \) and \( \Delta_\xi \)). Such a polarization is called suitable [F1].

**Lemma.** Let notation and hypotheses be as above. Then there are no strictly \( H \)-slope-semistable torsion-free sheaves on \( S \). Furthermore, a torsion-free sheaf \( F \) on \( S \) is \( H \)-slope-stable if and only if \( F|_{C_b} \) is stable for the generic elliptic fiber \( C_b \).

Thus \( \mathcal{M}_{\text{st}} = \mathcal{M}_\xi \); furthermore \( \mathcal{M}_\xi \) is a fine moduli space (apply Remark (A.7) of [Mk2]). One also verifies that \( \mathcal{M}_\xi \) is smooth [F3], hence \( \kappa(\mathcal{M}_\xi) \) equals the dimension of
\[ X := \text{Proj} \left( \bigoplus_{n=0}^{\infty} H^0(K^{\otimes n}_{\mathcal{M}_\xi}) \right). \]
We expect that \( X \) and the canonical map \( \mathcal{M}_\xi \to X \) are described as follows, but we have not checked the details. A computation shows that \( \dim \mathcal{M}_\xi \) is even, so set
dim $\mathcal{M}_\xi = 2n$. If $[F] \in \mathcal{M}_\xi$, then by Lemma (3.4) the restriction to the generic elliptic fiber $C_b$ is stable, but there are $n$ fibers $C_{b_1}, \ldots, C_{b_n}$ such that $F|_{C_{b_i}}$ is not stable (or not locally-free). Thus we get a morphism

$$
\mathcal{M}_\xi \xrightarrow{\Phi} B^{(n)} \quad [F] \mapsto b_1 + \cdots + b_n.
$$

Then the canonical model $X$ is identified with $B^{(n)}$, and the canonical map is identified with $\Phi$. Thus

$$
\kappa(\mathcal{M}_\xi) \dim(\mathcal{M}_\xi) = \frac{1}{2} = \kappa(S) \dim(S).
$$

**Surfaces of general type.** We will prove the following result.

(3.5) **Theorem (Jun Li [L2]).** Let $S$ be a surface with ample canonical bundle, and let $H$ be a rational multiple of $K_S$. Let $\xi$ be a set of sheaf data on $S$ such that:

1. The moduli space $\mathcal{M}_\xi$ is fine.
2. The codimension of $W^r_{K_\xi}$ in $\mathcal{M}_\xi$ is at least two.
3. There exists $\omega \in \Gamma(\Omega^2_S)$ such that $\omega_\xi$ is generically non-degenerate.

Then $\mathcal{M}_\xi$ is of general type.

Combining the above theorem with (1.1), (2.6) and (2.7) one gets the following corollaries.

**Corollary.** Let hypotheses be as above, except that we replace Item (3) by:

4. $c_1(\xi) \cdot C \equiv h^0(K_S|_C) \pmod{\text{rk}(\xi)}$.

Then $\mathcal{M}_\xi$ is of general type.

(3.6) **An example.** Let $\pi: S \to \mathbb{P}^2$ be a double cover branched over a smooth curve of degree $8n$, and let $H := \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$. Let $\xi$ be a set of sheaf data such that $\det(\xi) = nH$, $\Delta_\xi > \Delta(\text{rk}(\xi))$, and

$$
\gcd\{\text{rk}(\xi), 2n, n^2 + n - c_2(\xi)\} = 1.
$$

(This last condition ensures that $\mathcal{M}_\xi$ is a fine moduli space [Ma,Mk2].) Then the hypotheses of the corollary are satisfied, hence $\mathcal{M}_\xi$ is of general type.

Let’s prove Theorem (3.5). By Items (1)-(2), together with deformation theory, the moduli space is a local complete intersection, hence its dualizing sheaf is a line-bundle; we denote it by $K_{\mathcal{M}_\xi}$. By Item (2) $\mathcal{M}_\xi$ is smooth in codimension one, hence Lemma (3.1) gives that

$$
K_{\mathcal{M}_\xi} \cong \mathcal{L}(rK) \quad \text{up to torsion.}
$$
Since $K$ is a positive multiple of $H$ the (fractional) line-bundle $L(rK)$ is big, by Theorem (1.13). Hence there exists a positive $c$ such that for $n$ large enough and sufficiently divisible

$$\Gamma(K_{\tilde{\mathcal{M}_\xi}}^\otimes n) = cn^d + O(n^{d-1}),$$

where $d := \dim \mathcal{M}_\xi$. This is not sufficient to conclude that $\mathcal{M}_\xi$ is of general type, because of the presence of singularities. Let $\rho: \tilde{\mathcal{M}}_\xi \rightarrow \mathcal{M}_\xi$ be a desingularization. We have

$$\rho^* K_{\mathcal{M}_\xi} = K_{\tilde{\mathcal{M}}_\xi} \left( \sum_i a_i E_i \right),$$

for some $a_i \in \mathbb{Z}$, where $E_i$ are the exceptional divisors of $\rho$. Let $a$ be a non-negative number such that $a \geq a_i$ for all $i$. By the above equation we have

$$\rho^* \Gamma \left( K_{\tilde{\mathcal{M}}_\xi}^\otimes n \right) \subset \Gamma \left( K_{\tilde{\mathcal{M}}_\xi}^\otimes n (aE) \right),$$

where $E := \sum_i E_i$. Now we will use the two-form $\omega_\xi$ to produce a non-zero section of $K_{\tilde{\mathcal{M}}_\xi}^\otimes (E)$. Let $\tilde{F}$ be the pull-back to $S \times \tilde{\mathcal{M}}_\xi$ of the tautological family over $S \times \mathcal{M}_\xi$; then

$$\sigma := \wedge^{d/2} \omega_{\tilde{F}} \in \Gamma \left( K_{\tilde{\mathcal{M}}_\xi} \right)$$

is non-zero because $\omega_\xi$ is generically non-degenerate, by Item (3). Since $\dim \rho(E_i) < \dim E_i$, the Kodaira-Spencer map of $\tilde{F}$ has a non-trivial kernel at the generic point of $E_i$, hence by Equation (2.9) the two-form $\omega_{\tilde{F}}$ is degenerate along $E_i$; thus

$$\sigma \in \Gamma \left( K_{\tilde{\mathcal{M}}_\xi}^\otimes (E) \right)$$

At this point we are done: letting $N := n(a + 1)$, we have an injection

$$\rho^* \Gamma \left( K_{\tilde{\mathcal{M}}_\xi}^\otimes n \right) \hookrightarrow \Gamma \left( K_{\tilde{\mathcal{M}}_\xi}^\otimes N \right),$$

By (3.7) we conclude that $\tilde{\mathcal{M}}_\xi$ is of general type, hence $\mathcal{M}_\xi$ is of general type.

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