A TWO WEIGHT THEOREM FOR $\alpha$-FRACTIONAL SINGULAR INTEGRALS IN HIGHER DIMENSION

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ABSTRACT. Let $\sigma$ and $\omega$ be locally finite positive Borel measures on $\mathbb{R}^n$ with no common point masses, and let $T^{\alpha}$ be a standard $\alpha$-fractional Calderón-Zygmund operator on $\mathbb{R}^n$ with $0 \leq \alpha < n$. Furthermore, assume as side conditions the $A^2_\alpha$ conditions and the $\alpha$-energy conditions. Then $T^{\alpha}$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$ if the cube testing conditions hold for $T^{\alpha}$ and its dual.

Conversely, if $0 \leq \alpha < n$ and the vector of $\alpha$-fractional Riesz transforms $R^{\alpha}_n$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$, or more generally a strongly elliptic vector transform $T^{\alpha}$ is bounded, then the $A^2_\alpha$ conditions hold. This exhibits the energy conditions as the fundamental enemy in higher dimensions. We also show that reversal of a key Energy inequality in the proof, that is typically used to derive the necessity of the energy conditions, doesn’t hold in the plane even for the infinite collection of all convolution Calderón-Zygmund operators.

The innovations in this higher dimensional setting are the Monotonicity and Energy Lemmas using special Haar functions, the equivalence of energy and functional energy modulo $A^2_\alpha$, the necessity of the $A^2_\alpha$ conditions for strongly elliptic vectors, the failure of energy reversal for all convolution Calderón-Zygmund operators in the plane, and the extension of certain one-dimensional arguments to higher dimensions in light of the differing Poisson integrals used in $A_2$ and modified energy conditions. The arguments of our indicator/interval paper with M.Lacey, along with the argument used by M. Lacey in his recent solution of the NTV conjecture for the Hilbert transform, are then adapted to higher dimensions.

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We prove a two weight inequality for standard $\alpha$-fractional Calderón-Zygmund operators $T^\alpha$ in Euclidean space $\mathbb{R}^n$, where we assume the $n$-dimensional $A_2^\alpha$ and $\alpha$-energy conditions as side conditions (in higher dimensions the Poisson kernels used in these two conditions differ, and the energy conditions can be mildly weakened). In particular, we show that for locally finite Borel measures $\sigma$ and $\omega$ in $\mathbb{R}^n$ with no common point masses, and assuming the energy condition

$$(E_\alpha) \equiv \sup_{Q \subseteq \mathbb{R}^n} \frac{1}{|Q|} \sum_{r=1}^{\infty} \left( \frac{P^\alpha (Q_r, 1_{Q \setminus Q_r} \sigma)}{|Q_r|} \right)^2 \left( \int_{Q_r} |x - E^\omega_{Q_r}|^2 d\omega \right) < \infty$$

and its dual, a strongly elliptic collection of standard $\alpha$-fractional Calderón-Zygmund operators $T^\alpha$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$ if and only if the $A_2^\alpha$ condition

$$A_2^\alpha \equiv \sup_{Q \subseteq \mathbb{R}^n} \mathcal{P}^\alpha (Q, \sigma) \frac{|Q|}{|Q|^{1-\alpha}} < \infty$$

and its dual hold, along with the following cube testing conditions:

$$\int_Q |T^\alpha (1_Q \sigma)|^2 \omega \leq \int_Q d\sigma \quad \text{and} \quad \int_Q |(T^\alpha)^* (1_Q \omega)|^2 \sigma \leq \int_Q d\omega,$$

for all cubes $Q$ in $\mathbb{R}^n$.

The recent proof by M. Lacey [Lac] of the Nazarov-Treil-Volberg conjecture for the Hilbert transform is the culmination of a large body of work on two-weighted inequalities beginning with the work of Nazarov, Treil and Volberg ([NaVo], [NTV1], [NTV2], [NTV4] and [Vol]) and continuing with that of Lacey and the authors ([LaSaUr1], [LaSaUr], [LaSaShUr] and [LaSaShUr2]), just to mention a few. See the references for further work.

In attempting to extend this result to higher dimensions, several difficulties immediately arise, most notably regarding functions of minimal bounded fluctuation, and the functional energy condition. However, Lacey’s stopping time and recursion argument avoids minimal bounded fluctuation altogether, and we prove here that the functional energy condition is implied by the energy condition and $A_2^\alpha$, leaving the energy condition as the fundamental enemy in higher dimensions.

The main innovations in this paper are:

1. (a) the necessity of the $A_2^\alpha$ conditions for the boundedness of the vector of $\alpha$-fractional Riesz transforms $\mathbf{R}^\alpha$, and more generally strongly elliptic collections,

(b) the Monotonicity and Energy Lemmas using special Haar functions,

(c) the equivalence of the functional energy condition with the energy condition modulo $A_2^\alpha$,
(d) the failure of reversing the Energy Lemma in dimension \( n = 2 \) even for
the infinite collection of all convolution Calderón-Zygmund integrals,
and finally an adaptation of the arguments from our previous work \[LaSaShUr2\],
and the clever stopping time and recursion arguments of M. Lacey \[Lac\],
in view of the differing Poisson kernels and modified energy.

By a convolution Calderón-Zygmund integral \( T \) in the plane, we mean a principal
value operator with convolution kernel \( K^\alpha \) \((x) = \frac{\Omega(|x|)}{|x|^{2-\alpha}} \) where \( 0 \leq \alpha < 2 \) and \( \Omega \) is
homogeneous of degree zero on \( \mathbb{R}^2 \). While even a weak form of the Energy Lemma
cannot be reversed for the collection of all classical Calderón-Zygmund operators
in the plane (those convolution Calderón-Zygmund integrals with \( \int_{S^1} \Omega(\theta) \, d\theta = 0 \)),
the weak form is easily reversed for the positive fractional integral operator \( I^\alpha \),
which for \( \alpha > 0 \) is an example of an \( \alpha \)-fractional singular integral. Since the
weak form of energy reversal implies the energy condition, we see that the energy
conditions are necessary for boundedness of \( I^\alpha \), but in general they remain a basic
obstacle to two weight theory in higher dimensions. However, the energy conditions
are implied by the natural generalizations of all side conditions used previously for
the Hilbert transform, including doubling conditions on the measures and more
generally the Energy Hypothesis of \[LaSaUr\].

The basic idea of the fractional generalization is that all of the decompositions
of functions are carried out independently of the fractional parameter \( \alpha \), while
the estimates of the resulting nonlinear forms depend on the \( \alpha \)-Poisson integrals
and the \( \alpha \)-energy conditions. In order to state our theorem precisely, we
need to define standard fractional singular integrals, the two different Poisson kernels, and
a modified energy condition which remains sufficient for use in the proof. These
are introduced in the following three subsections respectively.

1.1. Standard fractional singular integrals. Consider a kernel function \( K(x, y) \)
defined on \( \mathbb{R}^n \times \mathbb{R}^n \) satisfying the following standard size and smoothness estimates:

\[
|K(x, y)| \leq C|x - y|^{-n},
\]

\[
|K(x, y) - K(x', y)| \leq C \frac{|x - x'|}{|x - y|} |x - y|^{-n}, \quad \frac{|x - x'|}{|x - y|} \leq \frac{1}{2},
\]

\[
|K(x, y) - K(x, y')| \leq C \frac{|y - y'|}{|x - y|} |x - y|^{-n}, \quad \frac{|y - y'|}{|x - y|} \leq \frac{1}{2}.
\]

Remark 1. The adjective ‘standard’ is usually reserved for a more general smoothness condition involving a Dini function \( \eta(t) \) on \((0, 1)\), and in the case at hand we are restricting to \( \eta(t) = t \).

We define a standard Calderón-Zygmund operator associated with such a kernel as follows.

Definition 1. We say that \( T \) is a standard singular integral operator with kernel
\( K \) if \( T \) is a bounded linear operator on \( L^q(\mathbb{R}^n) \) for some fixed \( 1 < q < \infty \), that is

\[
\|Tf\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^q(\mathbb{R}^n)}, \quad f \in L^q(\mathbb{R}^n),
\]

if \( K(x, y) \) is defined on \( \mathbb{R}^n \times \mathbb{R}^n \) and satisfies (1.1), and if \( T \) and \( K \) are related by

\[
Tf(x) = \int K(x, y)f(y) \, dy, \quad \text{a.e.-} x \notin \text{supp } f,
\]
whenever $f \in L^q(\mathbb{R}^n)$ has compact support in $\mathbb{R}^n$. We say $K(x,y)$ is a standard singular kernel if it satisfies (1.1).

We will also consider generalized fractional integrals, including the Cauchy integral in the plane. The setup is essentially the same as above but with a fractional variant of the size and smoothness conditions (1.1) on the kernel. Here are the details. Let $0 \leq \alpha < n$. Consider a kernel function $K^\alpha(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the fractional size and smoothness conditions,

\begin{equation}
|K^\alpha(x,y)| \leq C |x-y|^{-\alpha-n},
\end{equation}

\begin{equation}
|K^\alpha(x,y) - K^\alpha(x',y)| \leq C \frac{|x-x'|}{|x-y|} |x-y|^{-\alpha-n}, \quad \frac{|x-x'|}{|x-y|} \leq \frac{1}{2},
\end{equation}

\begin{equation}
|K^\alpha(x,y) - K^\alpha(x,y')| \leq C \frac{|y-y'|}{|x-y|} |x-y|^{-\alpha-n}, \quad \frac{|y-y'|}{|x-y|} \leq \frac{1}{2}.
\end{equation}

**Example 1.** The Cauchy integral $C^1$ in the complex plane arises when $K(x,y) = \frac{1}{x-y}$, $x, y \in \mathbb{C}$. The fractional size and smoothness condition (1.4) holds with $n = 2$ and $\alpha = 1$ in this case.

Then we define a standard $\alpha$-fractional Calderón-Zygmund operator associated with such a kernel as follows.

**Definition 2.** We say that $T^\alpha$ is a standard $\alpha$-fractional integral operator with kernel $K^\alpha$ if $T^\alpha$ is a bounded linear operator from some $L^p(\mathbb{R}^n)$ to some $L^q(\mathbb{R}^n)$ for some fixed $1 < p \leq q < \infty$, that is

$$\|T^\alpha f\|_{L^q(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n),$$

if $K^\alpha(x,y)$ is defined on $\mathbb{R}^n \times \mathbb{R}^n$ and satisfies (1.4), and if $T^\alpha$ and $K^\alpha$ are related by

$$T^\alpha f(x) = \int K^\alpha(x,y)f(y)dy, \quad a.e. x \notin \text{supp } f,$$

whenever $f \in L^p(\mathbb{R}^n)$ has compact support in $\mathbb{R}^n$. We say $K^\alpha(x,y)$ is a standard $\alpha$-fractional kernel if it satisfies (1.4).

A typical example is the $\alpha$-fractional Riesz vector of operators

$$\mathbb{R}^{n,\alpha} = \{ R^\alpha_{n,\ell} : 1 \leq \ell \leq n \}, \quad 0 \leq \alpha < n.$$

The Riesz transforms $R^\alpha_{n,\ell}$ are convolution fractional singular integrals $R^\alpha_{n,\ell}f = K^\alpha_{n,\ell} * f$ with odd kernel defined by

$$K^\alpha_{n,\ell}(w) \equiv \frac{w^\ell}{|w|^{n+1-\alpha}}.$$

1.2. **Poisson integrals.** It turns out that in higher dimensions, there are two natural ‘Poisson integrals’ $P$ and $\mathcal{P}$ that arise, the usual Poisson integral $P$ that emerges in connection with energy considerations, and a much smaller ‘reproducing’ Poisson integral $\mathcal{P}$ that emerges in connection with size considerations - in dimension $n = 1$ these two Poisson integrals coincide. For any cube $Q$ and any positive Borel measure $\mu$, let

$$P(Q, \mu) = \int_{\mathbb{R}^n} \frac{|Q|^\frac{1}{n}}{(|Q|^\frac{1}{n} + |x-x_Q|)^{n+1}} d\mu(x),$$
be the usual Poisson integral of \( \mu \) at the point \((x_Q, |Q|)\) in the upper half space \( \mathbb{R}^{n+1}_+ \), and let

\[
P(Q, \mu) \equiv \int_{\mathbb{R}^n} \left( \frac{|Q|^{\frac{1}{n}}}{(|Q|^{\frac{1}{n}} + |x-x_Q|)^2} \right)^n d\mu(x).
\]

We also need the fractional analogues of the two Poisson integrals of a measure \( \mu \) on a cube \( Q \):

\[
P^\alpha(Q, \mu) \equiv \int_{\mathbb{R}^n} \frac{|Q|^{\frac{1}{n}}}{(|Q|^{\frac{1}{n}} + |x-x_Q|)^{n+1-\alpha}} d\mu(x),
\]

\[
\mathcal{P}^\alpha(Q, \mu) \equiv \int_{\mathbb{R}^n} \left( \frac{|Q|^{\frac{1}{n}}}{(|Q|^{\frac{1}{n}} + |x-x_Q|)^{2}} \right)^{n-\alpha} d\mu(x).
\]

Note that

- for \( 0 \leq \alpha < n - 1 \), \( P^\alpha \) is strictly larger than \( \mathcal{P}^\alpha \),
- for \( \alpha = n - 1 \), \( P^\alpha \) and \( \mathcal{P}^\alpha \) coincide,
- for \( n - 1 < \alpha < n \), \( P^\alpha \) is strictly smaller than \( \mathcal{P}^\alpha \).

The standard Poisson integral \( P^\alpha \) appears in the energy conditions, while the reproducing Poisson kernel \( \mathcal{P}^\alpha \) appears in the \( A^2_\alpha \) conditions.

1.3. Energy condition. It turns out that in higher dimensions, we can modify the definition of the energy condition from a direct generalization of the one-dimensional energy condition, and the resulting somewhat weaker condition suffices for use in all arguments. The point of introducing this variant of energy is that it is necessary for the two weight inequality in those situations where the Energy Lemma can be reversed.

Define \( P^\mu_I \) to be orthogonal projection onto the subspace of \( L^2(\mu) \) consisting of functions supported in \( I \) with \( \mu \)-mean value zero. In addition, define \( \tilde{P}^\mu_I \) to be orthogonal projection onto the subspace \( L^2_{H(I)}(\mu) \) of \( L^2(\mu) \) consisting of those functions \( f \in L^2(\mu) \) whose Haar support is contained in \( H(I) \equiv \{ J \in \mathcal{D}^n : \text{either } |J|^\frac{1}{n} > 2^{-r} |I|^\frac{1}{n} \text{ or } J \Subset I \} \), and where the notation \( J \Subset I \), read \( J \) is \((r, \varepsilon)-deeply embedded in \( I \), means that \( J \subset I \), \( |J|^\frac{1}{n} \leq 2^{-r} |I|^\frac{1}{n} \), and that \( J \) satisfies the ‘good’ condition relative to the cube \( I \):

\[
dist(J, \partial I) > \frac{1}{2} |J|^\frac{1}{n} |I|^\frac{1-n}{n}.
\]

Here \( r \in \mathbb{N} \) and \( 0 < \varepsilon < 1 \) are the parameters in the definition of the ‘good’ dyadic cubes below, and will be taken sufficiently large and small respectively depending on the dimension \( n \).

**Definition 3.** Let \( r \in \mathbb{N} \) and \( 0 < \varepsilon < 1 \). A dyadic cube \( J \) is \((r, \varepsilon)-good, or simply good, if for every dyadic supercube \( I \), it is the case that either \( J \) has side length at least \( 2^{-r} \) times that of \( I \), or \( J \Subset I \) is \((r, \varepsilon)-deeply embedded in \( I \).
We thus have
\[ \|P^\mu_I x\|_{L^2(\mu)}^2 \leq \|P^\mu_I x\|_{L^2(\mu)}^2 = \int_I \left( \frac{1}{|I|} \int_I x \, dx \right)^2 \, d\mu(x), \quad x = (x_1, \ldots, x_n), \]
where \( P^\mu_I x \) is the orthogonal projection of the identity function \( x : \mathbb{R}^n \to \mathbb{R}^n \) onto the vector-valued subspace of \( \oplus_{k=1}^n L^2(\mu) \) consisting of functions supported in \( I \) with \( \mu \)-mean value zero.

Recall that in dimension \( n = 1 \) for \( \alpha = 0 \), we defined the energy condition by
\[
\sum_{I \supset \cup I} |I| \omega \int (I, \mu) \alpha^2 P^\alpha (I, 1, \sigma)^2 \leq (\mathcal{E}_2)^2 |I|_{\sigma},
\]
where
\[
\mathcal{E}_2 (I, \mu) = \frac{1}{|I|} \left( \mathcal{E}_2 \| \frac{x}{|I|}^2 \right)_{L^2(\mu)}.
\]

Our extension of the energy conditions to higher dimensions will use the smaller projection \( P^\mu_I x \) in place of \( P^\mu_I x \), and as a result, it is convenient to define the soft energy of \( \mu \) on a cube \( J \) by
\[
\mathcal{E}_{soft} (I, \mu) = \frac{1}{|I|} \left( \mathcal{E}_{soft} \| \frac{x}{|I|}^2 \right)_{L^2(\mu)}.
\]
Thus \( \mathcal{E}_{soft} (I, \mu) \) involves precisely those Haar coefficients \( (x, h_{\omega, I}^\alpha) \) for which \( J \) is either close to \( I \) or deeply embedded in \( I \). In particular, \( \mathcal{E}_{soft} (I, \mu) \) involves all of the Haar coefficients \( (x, h_{\omega, I}^\alpha) \) for which \( J \) is good and contained in \( I \), plus others. Then we define the forward energy condition in dimension \( n \geq 2 \) for \( 0 < \alpha < n \) by
\[
\sum_{I \supset \cup I} |I| \omega \mathcal{E}_{soft} (I, \mu)^2 P^\alpha (I, 1, \sigma)^2 \leq (\mathcal{E}_2)^2 |I|_{\sigma}.
\]
Note that this definition of the energy condition depends on the choice of goodness parameters \( r \) and \( \varepsilon \).

1.4. Statement of the Theorem. We can now state our main theorem. Let \( Q^n \) denote the collection of all cubes in \( \mathbb{R}^n \), and denote by \( D^n \) a dyadic grid in \( \mathbb{R}^n \).

**Theorem 1.** Suppose that \( T^\alpha \) is a standard \( \alpha \)-fractional Calderón-Zygmund operator on \( \mathbb{R}^n \), and that \( \omega \) and \( \sigma \) are positive Borel measures on \( \mathbb{R}^n \) without common point masses. Set \( T^\omega _{\alpha} f = T^\alpha (f \sigma) \) for any smooth truncation of \( T^\alpha _{\omega} \). Then

(1) Suppose \( 0 \leq \alpha < n \). Then the operator \( T^\omega _{\alpha} \) is bounded from \( L^2 (\sigma) \) to \( L^2 (\omega) \), i.e.,
\[
\|T^\omega _{\alpha} f\|_{L^2(\omega)} \leq \mathfrak{M} \|f\|_{L^2(\sigma)},
\]
uniformly in smooth truncations of \( T^\alpha \), and moreover
\[
\mathfrak{M} \leq C_\alpha \sqrt{A_2^\alpha + A_2 + \varepsilon_\alpha + \varepsilon + \varepsilon},
\]
provided that
\[
\varepsilon \leq C_\alpha \left( \sqrt{A_2^\alpha + A_2 + \varepsilon_\alpha + \varepsilon + \varepsilon} \right).
\]
(a) the two dual $A^2_\alpha$ conditions hold,
\[
A^2_\alpha \equiv \sup_{Q \in \mathcal{Q}} \mathcal{P}^\alpha (Q, \sigma |Q|^{-\frac{\alpha}{2}} < \infty,
\]
\[
A^{2,\ast}_\alpha \equiv \sup_{Q \in \mathcal{Q}} \frac{|Q|}{|Q|^1} \mathcal{P}^\alpha (Q, \omega) < \infty,
\]
(b) and the two dual testing conditions hold,
\[
\| \mathcal{P}^\alpha (Q, (1_Q \sigma)) \|_{L^2(\omega)} < \infty,
\]
\[
\mathcal{P}^\alpha (Q, (1_Q \omega)) \|_{L^2(\sigma)} < \infty,
\]
(c) and the two dual energy conditions hold,
\[
(\mathcal{E}_\alpha)^2 \equiv \sup_{Q \subset \mathcal{Q}_\alpha} \frac{1}{|Q|_\sigma} \sum_{r=1}^\infty \left( \mathcal{P}^\alpha (Q_r, 1_{Q \setminus Q_r}) \right)^2 \| P_{Q_r}^\alpha x \|_{L^2(\omega)}^2 < \infty,
\]
\[
(\mathcal{E}'\alpha)^2 \equiv \sup_{Q \subset \mathcal{Q}_\alpha} \frac{1}{|Q|_\omega} \sum_{r=1}^\infty \left( \mathcal{P}^\alpha (Q_r, 1_{Q \setminus Q_r}) \right)^2 \| P_{Q_r}^\alpha x \|_{L^2(\sigma)}^2 < \infty,
\]
uniformly over all dyadic grids $\mathcal{D}_\alpha$, and where the goodness parameters $r$ and $\varepsilon$ implicit in the definition of $P$ are fixed sufficiently large and small respectively depending on dimension.

(2) Conversely, suppose $0 \leq \alpha < n$ and that $\{T^\alpha_j\}_{j=1}^J$ is a collection of Calderón-Zygmund operators with standard kernels $\{K^\alpha_j\}_{j=1}^J$. In the range $0 \leq \alpha < \frac{n}{2}$, we assume the following ellipticity condition: there is $c > 0$ such that for each unit vector $u$ there is $j$ satisfying
\[
|K^\alpha_j (x, x + tu)| \geq ct^{\alpha-n}, \quad t \in \mathbb{R}.
\]
For the range $\frac{n}{2} \leq \alpha < n$, we assume the following strong ellipticity condition: for each $m \in \{1, -1\}^n$, there is a sequence of coefficients $\{\lambda^m_j\}_{j=1}^J$ such that
\[
\sum_{j=1}^J \lambda^m_j K^\alpha_j (x, x + tu) \geq ct^{\alpha-n}, \quad t \in \mathbb{R}.
\]
holds for all unit vectors $u$ in the $n$-ant
\[
V_m = \{x \in \mathbb{R}^n : m_i x_i > 0 \text{ for } 1 \leq i \leq n\}, \quad m \in \{1, -1\}^n.
\]
Furthermore, assume that each operator $T^\alpha_j$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$,
\[
\|T^\alpha_j f\|_{L^2(\omega)} \leq \mathcal{M}_0 \|f\|_{L^2(\sigma)}.
\]
Then the fractional $A^2_\alpha$ condition holds, and moreover,
\[
\sqrt{A^2_\alpha + A^{2,\ast}_\alpha} \leq C \mathcal{M}_0.
\]

**Remark 2.** Inequality (1.7) reverses the size inequality in (1.6) in the direction of the unit vector $u$ for one of the operators $T^\alpha_j$. 

Remark 3. The collection consisting of the Hilbert transform kernel \( \frac{1}{x} \) alone is an example of a kernel satisfying both (1.6) and (1.7) for \( \alpha = 0 \), while the complex Cauchy kernel \( \frac{1}{z} \) satisfies both (1.6) and (1.7) for \( \alpha = 1 \) in the plane. In general however, the boundedness of an individual operator \( T^\alpha \) cannot imply the finiteness of either \( A_{\alpha}^2 \) or \( E_\alpha \). For a trivial example, if \( \sigma \) and \( \omega \) are supported on the \( x \)-axis in the plane, then the second Riesz transform \( R_2 \) is the zero operator from \( L^2(\sigma) \) to \( L^2(\omega) \), simply because the kernel \( K_2(x,y) \) of \( R_2 \) satisfies
\[
K_2((x_1,0),(y_1,0)) = \frac{0-0}{|x_1-y_1|} = 0.
\]

Remark 4. The collection of fractional Riesz transform kernels
\[
\left\{ c_j \frac{x_j - y_j}{|x-y|^{n+1-\alpha}} \right\}_{j=1}^n
\]
is an example of a collection satisfying both (1.6) and (1.7) for \( 0 \leq \alpha < n \).

We discuss the failure of reversing the Energy Lemma (3.4) in the final section of the paper.

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2. Necessity of the \( A_{\alpha}^2 \) conditions

First we recall the necessity of the usual \( A_2 \) condition for elliptic operators in Euclidean space \( \mathbb{R}^n \) from [LaSaUr1], where this result was proved also for \( 1 < p < \infty \).

Lemma 1. Suppose that \( \sigma \) and \( \omega \) have no point masses in common, and that \( \{K_j\}_{j=1}^J \) is a collection of standard kernels satisfying the ellipticity condition (1.6) with \( \alpha = 0 \). If there are corresponding Calderón-Zygmund operators \( T_j \) satisfying
\[
\|\chi_E T_j(f\sigma)\|_{L^2(\omega)} \leq C \|f\|_{L^2(\sigma)}, \quad E = \mathbb{R}^n \setminus \text{supp } f,
\]
for \( 1 \leq j \leq J \), then the two weight \( A_2 \) condition holds.

Now we prove the necessity of the fractional \( A_{\alpha}^2 \) condition when \( 0 \leq \alpha < n \), for the \( \alpha \)-fractional Riesz vector transform \( \mathbf{R}_\alpha \) defined by
\[
\mathbf{R}_\alpha(f\sigma)(x) = \int_{\mathbb{R}^n} R_\alpha^j(x,y) f(y) \, d\sigma(y), \quad K_\alpha^j(x,y) = \frac{x^j - y^j}{|x-y|^{n+1-\alpha}},
\]
whose kernel \( K_\alpha^j(x,y) \) satisfies (1.4) for \( 0 \leq \alpha < n \). Parts of the following argument involving (1.6) are taken from unpublished material obtained in joint work with M. Lacey.

Lemma 2. Suppose \( 0 \leq \alpha < n \). Let \( T_\alpha \) be any collection of operators with \( \alpha \)-standard fractional kernel satisfying the ellipticity condition (1.6), and in the case
\( \frac{2}{\alpha} \leq \alpha < n \), we also assume the more restrictive ellipticity condition \([1.7]\). Then for \( 0 \leq \alpha < n \) we have
\[
\mathcal{A}_2^0 \lesssim \mathcal{N}_\alpha (T^\alpha).
\]

**Remark 5.** Cancellation properties of \( T^\alpha \) play no role the proof below. Indeed the proof shows that \( \mathcal{A}_2^0 \) is dominated by the best constant \( C \) in the restricted inequality
\[
\| \chi_E T^\alpha (f \sigma) \|_{L^2(\omega)} \leq C \| f \|_{L^2(\sigma)}, \quad E = \mathbb{R}^n \setminus \text{supp} \; f.
\]

**Proof.** First we give the proof for the case when \( T^\alpha \) is the \( \alpha \)-fractional Riesz transform \( \mathbf{R}^\alpha \), whose kernel is \( K^\alpha (x, y) = \frac{x-y}{|x-y|^{n+\alpha}} \). Define the \( 2^n \) generalized \( n \)-ants \( Q_m \) for \( m \in \{-1, 1\}^n \), and their translates \( Q_m(w) \) for \( w \in \mathbb{R}^n \) by
\[
Q_m = \{(x_1, ..., x_n): m_kx_k > 0\},
Q_m(w) = \{z: z - w \in Q_m\}, \quad w \in \mathbb{R}^n.
\]

Fix \( m \in \{-1, 1\}^n \) and a cube \( I \). For \( a \in \mathbb{R}^n \) and \( r > 0 \) let
\[
\ell(I)(x) = \ell(I) + |x - \zeta_I|,
\]
\[
f_{a,r}(y) = 1_{Q_m(a) \cap B(a,r)} (y) s_I(y)^{n-\alpha},
\]
where \( \zeta_I \) is the center of the cube \( I \). Now
\[
\ell(I)|x - y| \leq \ell(I)|x - \zeta_I| + \ell(I)|\zeta_I - y| \\
\leq [\ell(I) + |x - \zeta_I|][\ell(I) + |\zeta_I - y|]
\]
implies
\[
\frac{1}{|x - y|} \geq \frac{1}{\ell(I)} s_I(x) s_I(y), \quad x, y \in \mathbb{R}^n.
\]

Now the key observation is that with \( L\zeta \equiv m \cdot \zeta \), we have
\[
L(x - y) = m \cdot (x - y) \geq |x - y|, \quad x \in Q_m(y),
\]
which yields
\[
L(K^\alpha(x, y)) = \frac{L(x - y)}{|x - y|^{\alpha+1-\alpha}} \\
\geq \frac{1}{|x - y|^{\alpha-\alpha}} \geq \ell(I)^{\alpha-n} s_I(x)^{n-\alpha} s_I(y)^{n-\alpha},
\]
provided \( x \in Q_{\tau, +}(y) \). Now we note that \( x \in Q_m(y) \) when \( x \in Q_m(a) \) and \( y \in Q_{-m}(a) \) to obtain that for \( x \in Q_m(a) \),
\[
L(T^\alpha(f_{a,r}(\sigma))(x)) = \int_{Q_m(a) \cap B(0,r)} L(x - y) s_I(y) d\sigma(y) \\
\geq \ell(I)^{\alpha-n} s_I(x)^{n-\alpha} \int_{Q_m(a) \cap B(0,r)} s_I(y)^{2\alpha-2\alpha} d\sigma(y).
\]

Applying \( |L\zeta| \leq \sqrt{n} |\zeta| \) and our assumed two weight inequality for the fractional Riesz transform, we see that for \( r > 0 \) large,
\[
\ell(I)^{2\alpha-2n} \int_{Q_m(a)} s_I(x)^{2\alpha-2\alpha} \left( \int_{Q_m(a) \cap B(0,r)} s_I(y)^{2\alpha-2\alpha} d\sigma(y) \right)^2 d\omega(x) \\
\leq \| LT(\sigma f_{a,r}) \|_{L^2(\omega)}^2 \lesssim \mathcal{N}_\alpha (\mathbf{R}^\alpha)^2 \| f_{a,r} \|_{L^2(\sigma)}^2 = \mathcal{N}_\alpha (\mathbf{R}^\alpha)^2 \int_{Q_m(a) \cap B(0,r)} s_I(y)^{2\alpha-2\alpha} d\sigma(y).
\]
Rearranging the last inequality, we obtain
\[ \ell (I)^{2n-2} \int_{Q_m(a)} s_I(x) x^{2n-2} \omega(x) \int_{Q_m(a) \cap B(0,r)} s_I(y) y^{2n-2} d\sigma(y) \lesssim \mathcal{N}_\alpha(\mathbb{R}^\alpha)^2, \]
and upon letting \( r \to \infty \),
\[ \int_{Q_m(a)} \ell (I)^{2-\alpha} \int_{Q_m(a)} \frac{\ell (I)^{2-\alpha}}{(\ell I + |x - \xi|)^{4-2\alpha}} d\omega(x) \int_{Q_m(a)} \frac{\ell (I)^{2-\alpha}}{(\ell I + |y - \xi|)^{4-2\alpha}} d\sigma(y) \lesssim \mathcal{N}_\alpha(\mathbb{R}^\alpha)^2. \]
Note that the ranges of integration above are pairs of opposing \( n \)-ants.

Fix a cube \( Q \), which without loss of generality can be taken to be centered at the origin, \( \xi_Q = 0 \). Then choose \( a = (2\ell(Q), \ell(Q)) \) and \( I = 1 \) so that we have
\[ \left( \int_{Q_m(a)} \frac{\ell (Q)^{n-\alpha}}{\ell (Q) + |x|^{2n-2\alpha}} d\omega(x) \right) \left( \ell (Q)^{\alpha-n} \int_Q d\sigma \right) \lesssim \mathcal{N}_\alpha(\mathbb{R}^\alpha)^2. \]

Now fix \( m = (1, 1, ..., 1) \) and note that there is a fixed \( N \) (independent of \( \ell(Q) \)) and a fixed collection of rotations \( \{\rho_k\}_{k=1}^N \), such that the rotates \( \rho_k Q_m(a) \), \( 1 \leq k \leq N \), of the \( n \)-ant \( Q_m(a) \) cover the complement of the ball \( B(0, 4\sqrt{\ell}(Q)) \):
\[ B(0, 4\sqrt{\ell}(Q))^c \subset \bigcup_{k=1}^N \rho_k Q_m(a). \]

Then we obtain, upon applying the same argument to these rotated pairs of \( n \)-ants,
\[ \left( \int_{B(0, 4\sqrt{\ell}(Q))} \frac{\ell (Q)^{n-\alpha}}{\ell (Q) + |x|^{2n-2\alpha}} d\omega(x) \right) \left( \ell (Q)^{\alpha-n} \int_Q d\sigma \right) \lesssim \mathcal{N}_\alpha(\mathbb{R}^\alpha)^2. \]

Now we assume for the moment the tailless \( A_2^\alpha \) condition
\[ \ell (Q')^{2(\alpha-n)} \left( \int_{Q'} d\omega \right) \left( \int_{Q'} d\sigma \right) \leq A_2^\alpha. \]
If we use this with \( Q' = 4\sqrt{\ell}Q \), together with (2.2), we obtain
\[ \left( \int \frac{\ell (Q)^{n-\alpha}}{\ell (Q) + |x|^{2n-2\alpha}} d\omega(x) \right)^{\frac{1}{2}} \left( \ell (Q)^{\alpha-n} \int_Q d\sigma \right)^{\frac{1}{2}} \lesssim \mathcal{N}_\alpha(\mathbb{R}^\alpha) \]
or
\[ \ell (Q)^{\alpha} \left( \frac{1}{|Q|} \int \frac{1}{(1 + |x-\xi_Q|/\ell(Q)^{\alpha})^{2n-2\alpha}} d\omega(x) \right)^{\frac{1}{2}} \left( \frac{1}{|Q|} \int d\sigma \right)^{\frac{1}{2}} \lesssim \mathcal{N}_\alpha(\mathbb{R}^\alpha). \]

Clearly we can reverse the roles of the measures \( \omega \) and \( \sigma \) and obtain
\[ A_2^\alpha \lesssim \mathcal{N}_\alpha(\mathbb{R}^\alpha) + A_2^\alpha \]
for the kernels \( K^\alpha \), \( 0 \leq \alpha < n \).

More generally, to obtain the case when \( T^\alpha \) is elliptic and the tailless \( A_2^\alpha \) condition holds, we note that the key estimate (2.1) above extends to the kernel \( K_j^\alpha \) of
Now we have used to prove Lemma 1 in [LaSaUr1]. Indeed, with notation as in that proof, and suppressing some of the initial work there, then

$$A = \text{product measure on } \mathbb{R}^\leq 0 \text{ and } \alpha \leq 2$$

In the range 0 to show, using that the side length of $$P$$ equals $$\alpha$$, and there is in addition sufficient separation between opposing cones, which in turn may require a larger constant than $$4\sqrt{N}$$ in the choice of $$Q'$$ above.

Finally, we turn to showing that the tailless $$A_2^\alpha$$ condition is implied by the norm inequality, i.e.

$$A_2^\alpha = \sup_{Q'} \ell(Q')^{\alpha} \left( \frac{1}{|Q'|} \int_{Q'} d\omega \right)^{\frac{1}{2}} \left( \frac{1}{|Q'|} \int_{Q'} d\sigma \right)^{\frac{1}{2}} \lesssim \mathcal{H}_{\alpha}(\mathbb{R}^\alpha);$$

i.e. $$\left( \int_{Q'} d\omega \right) \left( \int_{Q'} d\sigma \right) \lesssim \mathcal{H}_{\gamma}(\mathbb{R}^{\gamma})^2 |Q'|^{2-\frac{2\alpha}{\gamma}}.$$  

In the range 0 ≤ α < 2, we invoke the argument used to prove Lemma 1 in [LaSaUr1]. Indeed, with notation as in that proof, and suppressing some of the initial work there, then $$A_2(\omega, \sigma; Q) = |Q|_{\omega \times \sigma}$$ where $$\omega \times \sigma$$ denotes product measure on $$\mathbb{R}^n \times \mathbb{R}^n$$, and we have

$$A_2(\omega, \sigma; Q_0) = \sum_{\zeta} A_2(\omega, \sigma; Q_\zeta) + \sum_{\beta} A_2(\omega, \sigma; P_\beta).$$

Now we have

$$\sum_{\zeta} A_2(\omega, \sigma; Q_\zeta) = \sum_{\zeta} |Q_\zeta|_{\omega \times \sigma} \leq \sum_{\zeta} \mathcal{H}_{\alpha}(\mathbb{R}^\alpha)^2 |Q_\zeta|^{1-\frac{\alpha}{\gamma}},$$

and

$$\sum_{\zeta} |Q_\zeta|^{1-\frac{\alpha}{\gamma}} = \sum_{k \in \mathbb{Z}}: 2^k \leq \ell(Q_0) \sum_{\ell(Q_\zeta) = 2^k} (2^{2nk})^{1-\frac{\alpha}{\gamma}}$$

$$\approx \sum_{k \in \mathbb{Z}: 2^k \leq \ell(Q_0)} \left( \frac{2^k}{\ell(Q_0)} \right)^{-n} (2^{2nk})^{1-\frac{\alpha}{\gamma}} \text{ (Whitney)}$$

$$= \ell(Q_0)^n \sum_{k \in \mathbb{Z}: 2^k \leq \ell(Q_0)} 2^{2nk(-1+2n-\frac{\alpha}{\gamma})}$$

$$\leq C_\alpha \ell(Q_0)^n \ell(Q_0)^n \left( 1 - \frac{\alpha}{2} \right) = C_\alpha |Q_0 \times Q_0|^{2-\frac{\alpha}{\gamma}} = C_\alpha |Q_0|^{1-\frac{\alpha}{\gamma}},$$

provided 0 ≤ α < 2. Since ω and σ have no point masses in common, it is not hard to show, using that the side length of $$P_\beta = P_\beta \times P_\beta$$ is $$2^{-N}$$ and dist $$(P_\beta, D) \leq C 2^{-N},$$

that we have the following limit,

$$\sum_{\beta} A_2(\omega, \sigma; P_\beta) \rightarrow 0 \text{ as } N \rightarrow \infty.$$  

Indeed, if σ has no point masses at all, then

$$\sum_{\beta} A_2(\omega, \sigma; P_\beta) = \sum_{\beta} |P_\beta|_{\omega} |P'_{\beta}|_{\sigma}$$

$$\leq \left( \sum_{\beta} |P_\beta|_{\omega} \right) \sup_{\beta} |P'_{\beta}|_{\sigma}$$

$$\leq C |Q_0|_{\omega} \sup_{\beta} |P'_{\beta}|_{\sigma} \rightarrow 0 \text{ as } N \rightarrow \infty.$$
while if $\sigma$ contains a point mass $c\delta_x$, then

$$\sum_{\beta: \ x \in P_{\beta}^b} A_2(\omega, \sigma; P_{\beta}^b) \leq C \left( \sum_{\beta: \ x \in P_{\beta}^b} |P_{\beta}|_{\omega} \sup_{\beta: \ x \in P_{\beta}^b} |P_{\beta}|_{\sigma} \right) \to 0$$

since $\omega$ has no point mass at $x$. This continues to hold if $\sigma$ contains finitely many point masses disjoint from those of $\omega$, and a limiting argument finally applies. This completes the proof that $A_2^b \lesssim \mathcal{N}_b(\mathbb{R}^n)$ for the range $0 \leq \alpha < \frac{n}{2}$.

Now we turn to proving $A_2^b \lesssim \mathcal{N}_b(\mathbb{R}^n)$ for the range $0 < \alpha < n$, where we assume the stronger ellipticity condition $\mathbf{(L.7)}$. So fix a cube $Q = \prod_{i=1}^{n} Q_i$ where $Q_i = [a_i, b_i]$. Choose $\theta_1 \in [a_1, b_1]$ so that both

$$\left| [a_1, \theta_1] \times \prod_{i=2}^{n} Q_i \right| \omega, \quad \left| [\theta_1, b_1] \times \prod_{i=2}^{n} Q_i \right| \omega \geq \frac{1}{2} |Q|_{\omega}.$$

Now denote the two intervals $[a_1, \theta_1]$ and $[\theta_1, b_1]$ by $[a_1^*, b_1^*]$ and $[a_1^{**}, b_1^{**}]$ where the order is chosen so that

$$\left| [a_1^*, b_1^*] \times \prod_{i=2}^{n} Q_i \right| \sigma \leq \left| [a_1^{**}, b_1^{**}] \times \prod_{i=2}^{n} Q_i \right| \sigma.$$

Then we have both

$$\left| [a_1^*, b_1^*] \times \prod_{i=2}^{n} Q_i \right| \omega \geq \frac{1}{2} |Q|_{\omega},$$

$$\left| [a_1^{**}, b_1^{**}] \times \prod_{i=2}^{n} Q_i \right| \sigma \geq \frac{1}{2} |Q|_{\sigma}.$$

Now choose $\theta_2 \in [a_2, b_2]$ so that both

$$\left| [a_2^*, b_2] \times [a_2, \theta_2] \times \prod_{i=3}^{n} Q_i \right| \omega, \quad \left| [a_2^*, b_2] \times [\theta_2, b_2] \times \prod_{i=3}^{n} Q_i \right| \omega \geq \frac{1}{4} |Q|_{\omega},$$

and denote the two intervals $[a_2, \theta_2]$ and $[\theta_2, b_2]$ by $[a_2^*, b_2^*]$ and $[a_2^{**}, b_2^{**}]$ where the order is chosen so that

$$\left| [a_2^{**}, b_2^{**}] \times [a_2^*, b_2^*] \times \prod_{i=3}^{n} Q_i \right| \sigma \leq \left| [a_2^{**}, b_2^{**}] \times [a_2^{**}, b_2^{**}] \times \prod_{i=2}^{n} Q_i \right| \sigma.$$

Then we have both

$$\left| [a_1^*, b_1^*] \times [a_2^*, b_2^*] \times \prod_{i=3}^{n} Q_i \right| \omega \geq \frac{1}{4} |Q|_{\omega},$$

$$\left| [a_1^{**}, b_1^{**}] \times [a_2^{**}, b_2^{**}] \times \prod_{i=2}^{n} Q_i \right| \sigma \geq \frac{1}{4} |Q|_{\sigma}.$$
Then we choose $\theta_3 \in [a_3, b_3]$ so that both
\[
\left| a_1^* b_1^* \times [a_2^*, b_2^*] \times [a_3, \theta_3] \times \prod_{i=4}^n Q_i \right|_\omega \geq \frac{1}{8} |Q|_\omega,
\]
\[
\left| a_1^* b_1^* \times [a_2^*, b_2^*] \times [\theta_3, b_3] \times \prod_{i=4}^n Q_i \right|_\omega \geq \frac{1}{8} |Q|_\omega,
\]
and continuing in this way we end up with two rectangles,
\[
G \equiv [a_1^*, b_1^*] \times [a_2^*, b_2^*] \times \cdots [a_n^*, b_n^*],
\]
\[
H \equiv [a_1^{**}, b_1^{**}] \times [a_2^{**}, b_2^{**}] \times \cdots [a_n^{**}, b_n^{**}],
\]
that satisfy
\[
|G|_\omega = |[a_1^*, b_1^*] \times [a_2^*, b_2^*] \times \cdots [a_n^*, b_n^*]|_\omega \geq \frac{1}{2^{n}} |Q|_\omega,
\]
\[
|H|_\sigma = |[a_1^{**}, b_1^{**}] \times [a_2^{**}, b_2^{**}] \times \cdots [a_n^{**}, b_n^{**}]|_\sigma \geq \frac{1}{2^{n}} |Q|_\sigma.
\]
However, the rectangles $G$ and $H$ lie in opposing $n$-ants at the vertex $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$, and so we can apply \((1.4)\) to obtain that for $x \in G$,
\[
\sum_{j=1}^J \lambda_j^n T^\alpha_j (1_H \sigma) (x) = \left| \int_H \sum_{j=1}^J \lambda_j^n K^\alpha_j (x, y) \, d\sigma (y) \right| \geq \int_H |x-y|^{|\alpha|-n} \, d\sigma (y) \geq |Q|^{\frac{n}{2|-1}} |H|_\sigma.
\]
Then from the norm inequality we get
\[
|G|_\omega \left( |Q|^{\frac{n}{2|-1}} |H|_\sigma \right)^2 \lesssim \int_G \left| \sum_{j=1}^J \lambda_j^n T^\alpha_j (1_H \sigma) \right|^2 \, d\omega
\]
\[
\lesssim n \sum_{j=1}^J \lambda_j^n T^\alpha_j \int_H 1^2_\sigma \, d\sigma \equiv n \sum_{j=1}^J \lambda_j^n T^\alpha_j |H|_\sigma,
\]
from which we deduce that
\[
|Q|^{2 \left( \frac{3}{2} - 1 \right)} |Q|_\omega |Q|_\sigma \lesssim 2^{2n} |Q|^{2 \left( \frac{3}{2} - 1 \right)} |G|_\omega |H|_\sigma \lesssim 2^{2n} n \sum_{j=1}^J \lambda_j^n T^\alpha_j,
\]
and hence
\[
A^2_2 \lesssim 2^{2n} n \sum_{j=1}^J \lambda_j^n T^\alpha_j.
\]
This completes the proof of Lemma \([2]\) \(\Box\)

3. Monotonicity lemma and Energy lemma

The Monotonicity Lemma below will be used in proving our theorem. It will use the \textit{n-coordinate} Haar functions $\{h_{pek}^J\}_{k=1}^n$ associated with the cube $J$ defined in \((4.1)\) below, where $e_k = (1, \ldots, 1, 0, 1, \ldots, 1)$ is the multiindex in $\{0, 1\}^n$ with 0 in the $k^{th}$ position and 1 elsewhere, i.e. the complement of the unit $k^{th}$ coordinate vector.
3.1. A weighted Haar basis. We will use a specific construction of the Haar basis in \( \mathbb{R}^n \) that is adapted to a measure \( \mu \) (c.f. [Hyt]). Consider the unit cube \( Q_0 \equiv [0, 1)^n \). Define

\[
h^1 \equiv 1_{[0, \frac{1}{2})} + 1_{[\frac{1}{2}, 1]},
\]

\[
h^0 \equiv -1_{[0, \frac{1}{2})} + 1_{[\frac{1}{2}, 1]},
\]

and for every multiindex \( a \in \{0, 1\}^n \) and \( x \in Q_0 \), define

\[
H^a(x) = \prod_{k=1}^n h^{a_k}(x_k).
\]

Then \( H^a(x) \) equals \( \pm 1 \) on each of the \( 2^n \) dyadic subcubes of \( Q_0 \), and \( \int H^a(x) \, dx = 0 \) for all \( a \neq 1 \equiv (1, 1, \ldots, 1) \). Indeed, if \( a \neq 1 \) then \( \int_{Q_0} h^a(x) \, dx = \int_{Q_0} h^b(x) \, dx = 0 \) and

\[
\int H^a(x) \, dx = \prod_{k=1}^n \int h^{a_k}(x_k) \, dx_k = 0.
\]

Thus the functions \( \{H^a\}_{a \in \{0, 1\}^n \setminus \{1\}} \) are the unweighted Haar functions associated with the unit cube \( Q_0 \).

We now adapt these Haar functions to a locally finite positive Borel measure \( \mu \) on \( Q_0 \), and for convenience we assume that \( |Q_\beta|_\mu > 0 \) for the dyadic children \( \{Q_\beta\}_{\beta \in \{0, 1\}^n} \) of \( Q_0 \). Here the cube \( Q_\beta \) is the child whose vertex closest to the origin is the point \( \frac{1}{2} \beta = \left( \frac{\beta_1}{2}, \frac{\beta_2}{2}, \ldots, \frac{\beta_n}{2} \right) \). We define the weighted Haar functions \( \{H^a_\mu\}_{a \in \{0, 1\}^n \setminus \{1\}} \) by

\[
H^a_\mu(x) \equiv \frac{1}{\gamma_\mu(Q_0)} H^a(x) \sum_{\beta \in \{0, 1\}^n} \frac{1}{|Q_\beta|_\mu} 1_{Q_\beta}(x)
\]

where the constant \( \gamma_\mu(Q_0) \) is chosen so that \( \|H^a_\mu\|_{L^2(\mu)} = 1 \) for all \( a \in \{0, 1\}^n \setminus \{1\} \), i.e.

\[
\gamma_\mu(Q_0) = \sqrt{\int \sum_{\beta \in \{0, 1\}^n} \left( \frac{1}{|Q_\beta|_\mu} \right)^2 1_{Q_\beta} \, d\mu} = \sqrt{\sum_{\beta \in \{0, 1\}^n} \frac{1}{|Q_\beta|_\mu}}.
\]

Clearly, since \( H^a(x) \) equals the constant \( \mathbb{E}^x_{Q_\beta} H^a \) on \( Q_\beta \), we have

\[
\int H^a_\mu \, d\mu = \int \left( \frac{1}{\gamma_\mu(Q_0)} H^a(x) \sum_{\beta \in \{0, 1\}^n} \frac{1}{|Q_\beta|_\mu} 1_{Q_\beta}(x) \right) \, d\mu(x)
\]

\[
= \frac{1}{\gamma_\mu(Q_0)} \sum_{\beta \in \{0, 1\}^n} \left( \mathbb{E}^x_{Q_\beta} H^a \right) \frac{1}{|Q_\beta|_\mu} \int 1_{Q_\beta}(x) \, d\mu(x)
\]

\[
= \frac{1}{\gamma_\mu(Q_0)} \sum_{\beta \in \{0, 1\}^n} \mathbb{E}^x_{Q_\beta} H^a = \frac{1}{\gamma_\mu(Q_0)} \int H^a(x) \, dx = 0,
\]

for \( a \in \{0, 1\}^n \setminus \{1\} \). Thus the functions \( \{H^a_\mu\}_{a \in \{0, 1\}^n \setminus \{1\}} \) are the Haar functions associated with \( Q_0 \), and the remainder of the Haar basis

\[
\left\{ (H^a_\mu)_Q \right\}_{a \in \Gamma_n, \ Q \in \mathcal{B}}, \quad \Gamma_n \equiv \{0, 1\}^n \setminus \{1\},
\]
for $L^2(\mu)$ is obtained by appropriately translating and dilating the functions $\{H^\alpha_{\mu}\}_{\alpha \in \Gamma_n}$ to the dyadic cubes $Q$ in the grid $\mathcal{D}$. In order to match our one-dimensional notation as closely as possible, we will denote the Haar function $(H^\alpha_{\mu})_Q$ by $h^\mu_{Q,\alpha}$:

$$h^\mu_{Q,\alpha} = \frac{1}{\sqrt{\sum_{Q' \in \mathcal{C}(Q)} |a(Q')|^2 |Q'|_\mu}} \sum_{Q' \in \mathcal{C}(Q)} a(Q') \frac{1}{|Q'|_\mu} 1_{Q'},$$

where $\mathcal{C}(Q)$ denotes the children of $Q$, and where

$$a(Q') = \mathbb{E}_{Q'} H^\alpha \in \{-1, 1\},$$

if $Q' \in \mathcal{C}(Q)$ is a child of $Q$ that occupies the same relative position inside $Q$ as the child $Q_{\beta} \in \mathcal{C}(Q_0)$ does inside the unit cube $Q_0$.

### 3.2. The Monotonicity Lemma.

For $0 \leq \alpha < n$, we recall the fractional Poisson integral

$$P^\alpha(J, \mu) = \int_{\mathbb{R}^n} \frac{|J|^\frac{n}{\alpha}}{(|J|^\frac{n}{\alpha} + |y - c_J|)^{n+1-\alpha}} d\mu(y).$$

We can now state the Monotonicity Lemma, so-called because the right hand side is monotone increasing in the measure $\mu$.

**Lemma 3 (Monotonicity).** Suppose that $I, J$ and $J^*$ are cubes in $\mathbb{R}^n$ such that $J \subset J^* \subset 2J^* \subset I$, and that $\mu$ is a positive measure on $\mathbb{R}^n$ supported outside $I$. Suppose that $h^\omega_{J,\alpha}$ is a Haar function associated with $J$. Finally suppose that $T^\alpha$ is a standard fractional singular integral on $\mathbb{R}^n$ as defined in Definition 2 with $0 < \alpha < n$. Then we have the estimate

$$|\langle T^\alpha \mu, h^\omega_{J^*,\alpha} \rangle_\omega| \lesssim \frac{P^\alpha(J^*, \mu)}{|J^*|^{\frac{n}{\alpha}}} \hat{X}^\omega(J),$$

where

$$\hat{X}^\omega(J) = \sum_{\ell=1}^n \langle x^\ell - c^\ell_J, h^\omega_{J,\ell} \rangle_\omega = \sum_{\ell=1}^n \hat{x}^\ell(J, e_\ell),$$

and $c^\ell_J = (c^1_J, ..., c^n_J)$ is the center of $J$.

**Proof.** The general case follows easily from the case $J^* = J$, so we assume this restriction. The inner product $\langle x^\ell - c^\ell_J, h^\omega_{J,\ell} \rangle_\omega$ is positive since the function $(x^\ell - c^\ell_J) h^\omega_{J,\ell}(x)$ is nonnegative and supported on the cube $J$. It follows that

$$\sum_{\ell=1}^n \langle x^\ell - c^\ell_J, h^\omega_{J,\ell} \rangle_\omega = \int \left( \sum_{\ell=1}^n (x^\ell - c^\ell_J) h^\omega_{J,\ell}(x) \right) d\omega(x) = \sum_{\ell=1}^n \int |x^\ell - c^\ell_J| |h^\omega_{J,\ell}(x)| d\omega(x).$$
Now we use the smoothness estimate \((4.3)\), the assumptions that \(x \in J\) and \(y \notin 2J\), and then \((5.3)\) and the fact that \(|h_{j}^{\omega,a}| = |h_{j}^{\omega,a'}|\) for all \(a, a'\), to obtain

\[
\|\langle T^{\alpha} \mu, h_{j}^{\omega,a} \rangle_{\omega} \| = \left| \int \left\{ \int K^{\alpha} (x, y) h_{j}^{\omega,a} (x) \, d\omega (x) \right\} \, d\mu (y) \right| = \left| \int \langle K_{y}^{\alpha}, h_{j}^{\omega,a} \rangle_{\omega} \, d\mu (y) \right|
\]

\[
= \left| \int \langle K_{y}^{\alpha} (x) - K_{y}^{\alpha} (c_{j}), h_{j}^{\omega,a} \rangle_{\omega} \, d\mu (y) \right|
\]

\[
\leq \left\langle \int_{y \notin 2J} |K_{y}^{\alpha} (x) - K_{y}^{\alpha} (c_{j})| \, d\mu (y), |h_{j}^{\omega,a} (x)| \right\rangle_{\omega}
\]

\[
\leq C \left\langle \int_{y \notin 2J} \frac{|x - c_{j}|}{|y - c_{j}|^{n+1-\alpha}} \, d\mu (y), |h_{j}^{\omega,a} (x)| \right\rangle_{\omega}
\]

\[
\leq C \frac{P^{\alpha} (J, \mu)}{|J|^{\frac{n}{\alpha}}} \left\langle \sum_{\ell=1}^{n} |x^{\ell} - c_{j}^{\ell}|, |h_{j}^{\omega,a}| \right\rangle_{\omega}
\]

\[
= C \frac{P^{\alpha} (J, \mu)}{|J|^{\frac{n}{\alpha}}} \sum_{\ell=1}^{n} \langle x^{\ell} - c_{j}^{\ell}, h_{j}^{\omega,a} \rangle_{\omega} = C \frac{P^{\alpha} (J, \mu)}{|J|^{\frac{n}{\alpha}}} \hat{X}_{\omega} (J).
\]

\(\square\)

When \(n = 1\) and \(\alpha = 0\), the estimate in the Monotonicity Lemma was reversed for the Hilbert transform in \([LaSaShUr2]\), i.e.

\[
P \left( \frac{J^{*}, \mu}{|J^{*}|^{\frac{1}{n}}} \right) \hat{X}_{\omega} (J) \lesssim \langle H_{\mu}, h_{j}^{\omega,a} \rangle_{\omega},
\]

and this turned out to be the key to deriving control of the functional energy constant \(\delta\) by \(A_{2}\) and the testing conditions for the Hilbert transform.

We thank Michael Lacey for pointing out to us an example showing that inequality \((3.2)\) cannot be reversed in higher dimensions for the vector of Riesz transforms. His example is similar to the following modification of an example in the first version of this paper.

**Example 2.** (M. Lacey) The simple reverse monotonicity inequality

\[
\frac{P^{\alpha} (J, \mu)}{|J|^{\frac{n}{\alpha}}} \hat{X}_{\omega} (J) \lesssim \left( \sum_{k \in F} \left| \langle R_{k}^{\alpha} \mu, h_{j}^{\omega,a} \rangle_{\omega} \right|^{2} \right)^{\frac{1}{2}}
\]

is false. Indeed, take \(T = R_{k}\) and let \(\mu = \delta_{z_{0}}\) be a point mass located outside \(CJ\). Then the quantities \(\frac{\partial R_{k}^{\alpha}}{\partial x_{j}} (c_{j})\) are constants when integrated against \(\mu\), and thus the equation \(\sum_{\ell=1}^{n} (x^{\ell} - c_{j}^{\ell}) \frac{\partial R_{k}^{\alpha}}{\partial x_{j}} (c_{j}) = 0\) defines a hyperplane in \(\mathbb{R}^{n}\) that contains the point \(c_{j}\). The basic idea of the example is to take \(k = 1, 2, \ldots, n - 1\), consider the intersection of the \(n - 1\) hyperplanes that define a line in \(\mathbb{R}^{n}\) through \(c_{j}\), and let \(\omega\) consist of two point masses located on that line away from \(c_{j}\) but within \(J\). Then the right hand side of the monotonicity inequality is large, while the left hand side is of order \(\varepsilon\) if the two point masses on the line are balanced correctly, so the monotonicity lemma cannot be reversed by simply using the Riesz transform.

To make this example fit our hypotheses, let \(z_{0} = c_{j} + t (1, 1, \ldots, 1)\) for \(t \in \mathbb{R}\). Then
the \( n - 1 \) vectors \( v_k = \left[ \frac{\partial R}{\partial x^k} (c_j, z_0) \right]_{\ell=1}^n \) for \( k = 1, 2, \ldots, n - 1 \) are linearly independent. Thus the intersection of the hyperplanes \( \sum_{\ell=1}^n (x^\ell - c^\ell_j) \frac{\partial R}{\partial x^\ell} (c_j, z_0) = 0 \) defines a line \( L \) through \( c_j \). Let \( \omega = \sum_{m=1}^{2n+1} \delta_{x_m} \) be a sum of \( 2^n + 1 \) point masses of mass 1 each. Put two of the point masses, say \( \delta_{s_1} \) and \( \delta_{s_2} \), on the line \( L \) in one of the children very close to opposite boundaries of \( J \), and locate the remaining \( 2^n - 1 \) point masses very close to \( c_j \), at distance comparable to \( \varepsilon \). Then all the children of \( J \) are charged, the Haar functions associated to \( J \) are balanced, and the right hand side of the monotonicity inequality is large. This is due to the fact that the masses \( \delta_{s_1} \) and \( \delta_{s_2} \) contribute almost \( 2 \) while the others contribute order \( \varepsilon \), because for those point masses the vector \( \hat{X} (J) = \left[ (x^\ell - c^\ell_j, h^\omega e^\ell_j) \right]_{\ell=1}^n = \left[ x^\ell (J, e_k) \right]_{\ell=1}^n \) has all components comparable in size to \( \varepsilon \). On the other hand, the left hand side of the monotonicity inequality is of size \( \varepsilon \) for \( T = R_k \) \( 1 \leq k \leq n - 1 \), on account of the construction of the line \( L \), the contribution of point masses \( \varepsilon \)-close to \( c_j \), the balanced positioning of \( \delta_{s_1} \) and \( \delta_{s_2} \), and finally the above error estimate (\( \mathcal{E} \)) of size \( \varepsilon \).

### 3.3. The energy lemma

Suppose now we are given a cube \( J \in \mathcal{D}^\omega \), and a subset \( \mathcal{H} \) of the dyadic subgrid \( \mathcal{D}^\omega (J) \) of cubes from \( \mathcal{D}^\omega \) that are contained in \( J \). Let \( P^\omega_{\mathcal{H}} = \sum_{J \in \mathcal{H}} \Delta^\omega_J \) be the \( \omega \)-Haar projection onto \( \mathcal{H} \) and define the \( \mathcal{H} \)-energy \( E_{\mathcal{H}} (J, \omega) \) of \( \omega \) on the cube \( J \) by

\[
E_{\mathcal{H}} (J, \omega)^2 = \frac{1}{|J|} \int_j \left( E^\omega_{\mathcal{H}} (dx) \frac{P^\omega_{\mathcal{H}}(dx)}{|J|} (X - X') \right)^2 \, d\omega (x) \]

\[
= \frac{1}{|J|} \int_j \left( \frac{P^\omega_{\mathcal{H}} X}{|J|^{\frac{n}{2}}} \right)^2 \, d\omega (x) \]

\[
= \frac{1}{|J|} \sum_{J' \in \mathcal{H}} \sum_{a \in \Gamma_n} \left| \left( \frac{X}{|J|^{\frac{n}{2}}} h_{J, a}^\omega \right) \right|^2 \approx \frac{1}{|J|} \sum_{J' \in \mathcal{H}} \left| \hat{X} (J') \right|^2 .
\]

For \( \nu \) a signed measure on \( \mathbb{R}^n \), and \( \mathcal{H} \) a subset of the dyadic subgrid \( \mathcal{D}^\omega (J) \), and \( 0 \leq \alpha < n \), we define the functional

\[
\Phi^\alpha_{\mathcal{H}} (J, \nu) = \left( \frac{P^\alpha (J, \mu)}{|J|^{\frac{n}{2}}} \right)^2 \sum_{J' \in \mathcal{H}} \left| \hat{X} (J') \right|^2.
\]

**Lemma 4 (Energy Lemma).** Let \( J \) be a cube in \( \mathcal{D}^\omega \). Let \( \Psi_J \) be an \( L^2 (\omega) \) function supported in \( J \) and with \( \omega \)-integral zero. Let \( \nu \) be a signed measure supported in \( \mathbb{R}^n \setminus 2J \) and denote the Haar support of \( \Psi_J \) by \( \mathcal{H} = \text{supp} \hat{\Psi}_J \). Let \( T^\alpha \) be a standard \( \alpha \)-fractional Calderón-Zygmund operator with \( 0 \leq \alpha < n \). Then we have

\[
\left| \left( T^\alpha (\nu), \Psi_J \right) \right| \leq C \| \Psi_J \|_{L^2 (\omega)} \Phi^\alpha_{\mathcal{H}} (J, \nu)^{\frac{1}{2}} .
\]
Proof. We calculate

\[ |\langle T^\alpha \nu, \Psi_J \rangle_\omega| = \left| \int_J \int_{\mathbb{R}^n \setminus 2J} K^\alpha(x, y) \Psi_J(x) \, d\nu(y) \, d\omega(x) \right| \]

\[ = \left| \int_J \int_{\mathbb{R}^n \setminus 2J} K^\alpha(x, y) \sum_{J' \in \mathcal{H}} \sum_{a \in \Gamma_n} \langle \Psi_J, h^{\omega,a}_{J, y} \rangle \, h^{\omega,a}_{J, y} \, (x) \, d\nu(y) \, d\omega(x) \right| \]

\[ = \left| \sum_{J' \in \mathcal{H}} \int_{\mathbb{R}^n \setminus 2J} \sum_{a \in \Gamma_n} \langle K^\alpha_{J, y}, h^{\omega,a}_{J, y} \rangle \, \widetilde{\Psi}_J(J') \, d\nu(y) \right| \]

\[ = \left| \sum_{J' \in \mathcal{H}} \int_{\mathbb{R}^n \setminus 2J} \sum_{a \in \Gamma_n} \int_J \left[ K^\alpha_{J, y}(x) - K^\alpha_{J, y}(c_j) \right] h^{\omega,a}_{J, y} \, (x) \, d\omega(x) \, \widetilde{\Psi}_J(J') \, d\nu(y) \right| , \]

and so we have

\[ |\langle T^\alpha \nu, \Psi_J \rangle_\omega| \lesssim \sum_{J' \in \mathcal{H}} \frac{P^\alpha(J', |\nu|)}{|J'|^{\frac{\alpha}{n}}} \hat{X}^\omega(J') \, \widetilde{\Psi}_J(J') \]

\[ \lesssim \left( \sum_{J' \in \mathcal{H}} \left( \frac{P^\alpha(J', |\nu|)}{|J'|^{\frac{\alpha}{n}}} \right)^2 \hat{X}^\omega(J')^2 \right)^\frac{1}{2} \left( \sum_{J' \in \mathcal{H}} \left| \widetilde{\Psi}_J(J') \right|^2 \right)^\frac{1}{2} \]

\[ = \Phi^\alpha_J(J, \nu)^\frac{1}{2} \|\Psi_J\|_{L^2(\omega)}. \]

In particular we have the following dual formulation of the case when \( \mathcal{H} \) consists of all dyadic subcubes of \( J \).

**Corollary 1.** Let \( J \) be a cube in \( \mathcal{D} \). Let \( \nu \) be a signed measure supported in \( \mathbb{R}^n \setminus 2J \) and let \( T^\alpha \) be a standard \( \alpha \)-fractional Calderón-Zygmund operator with \( 0 \leq \alpha < n \). Then we have

\[ \frac{|J|}{2} E^\omega_J \| E^\omega_J \| \| T^\alpha \nu \, (x) - T^\alpha \nu \, (z) \|^2 = \int_J \| T^\alpha \nu \, (x) - E^\omega_J (T^\alpha \nu) \|^2 \, d\omega \leq C \Phi^\alpha_J (J, \nu). \]

It is useful to note the many faces of the energy functional \( \Phi^\alpha (J, \mu) \) that will arise in the sequel:

\[ \Phi^\alpha (J, \mu) = |J| \omega \mathcal{E}(J, \omega)^2 \mathcal{P}^\alpha (J, \mu)^2 \]

\[ = \left( \frac{\mathcal{P}^\alpha (J, \mu)}{|J|^{\frac{\alpha}{n}}} \right)^2 \int_J |x - E^\omega_J x|^2 \, d\omega (x) \]

\[ = \frac{\mathcal{P}^\alpha (J, \mu)}{|J|^{\frac{\alpha}{n}}} \| \mathcal{P}^\omega_J x \|_{L^2(\omega)}^2 \]

\[ = \left( \frac{\mathcal{P}^\alpha (J, \mu)}{|J|^{\frac{\alpha}{n}}} \right)^2 \sum_{J' \subset J} \sum_{a \in \Gamma} |\hat{x}(J', a)|^2 \]

\[ \approx \left( \frac{\mathcal{P}^\alpha (J, \mu)}{|J|^{\frac{\alpha}{n}}} \right)^2 \sum_{J' \subset J} \left| \hat{X}^\omega(J') \right|^2 , \]
where
\[ E(J, \omega)^2 = E_{d\omega(x)}^{\omega}(e_{d\omega(x)}^2 \frac{|x - z|}{|J|^{1/2}})^2 = 2E_{d\omega(x)}^{\omega} \frac{|x - E_{d\omega(x)}^{\omega} x|}{|J|^{1/2}}. \]

There is a similar equivalence involving the soft energy \( E_{soft}(J, \omega) \) and the projection \( P_{J}^{\omega} \).

Finally, we mention that in the plane, the Energy inequality (3.4) cannot be reversed even for the collection of all Calderón-Zygmund convolution operators with smooth odd kernel. See Lemma 9 in the final section below.

4. Equivalence of Energy and Functional Energy Modulo \( A_2^\sigma \)

We begin by adapting to higher dimensions three definitions that are relevant to functional energy.

**Definition 4.** A collection \( F \) of dyadic cubes is \( \sigma \)-Carleson if
\[ \sum_{F \in F : F \subset S} |F|_\omega \leq C_{\sigma} |S|_\sigma, \quad S \in F. \]
The constant \( C_{\sigma} \) is referred to as the Carleson norm of \( F \).

**Definition 5.** Let \( F \) be a collection of dyadic cubes. A collection of functions \( \{g_F\}_{F \in F} \) in \( L^2(\omega) \) is said to be \( F \)-adapted if for each \( F \in F \), there is a collection \( J(F) \) of cubes in \( D^\sigma \) such that
\[ J(F) \subset \{J \in D^\sigma : J \in F\} \]
and such that each of the following three conditions hold:
1. For each \( F \in F \), the Haar coefficients \( \tilde{g}_F(J,a) = (g_F,h_{\omega,a})_\omega \) of \( g_F \) are non-negative and supported on \( J(F) \), i.e.
\[ \begin{cases} \tilde{g}_F(J) \geq 0 & \text{for all } J \in J(F), \\ \tilde{g}_F(J) = 0 & \text{for all } J \notin J(F), \end{cases} \quad F \in F, \]
2. The sets \( \{J(F)\}_{F \in F} \) are pairwise disjoint,
3. There is a positive constant \( C \) such that if \( J^*(F) \) consists of the maximal cubes in \( J(F) \), then for every cube \( I \) in \( D^\sigma \), the set of pairs of cubes \( (F,J^*) \) that ‘straddle’ \( I \),
\[ B_I \equiv \{(F,J^*) : J^* \in J^*(F) \text{ and } J^* \subset I \subset F\}, \]
satisfies the overlap condition
\[ \sum_{(F,J^*) \in B_I} \mathbf{1}_{J^*} \leq C, \quad I \in D^\sigma. \]

**Definition 6.** Let \( \mathfrak{S}_r \) be the smallest constant in the ‘functional energy’ inequality below, holding for all non-negative \( h \in L^2(\sigma) \), all \( \sigma \)-Carleson collections \( F \), and all \( F \)-adapted collections \( \{g_F\}_{F \in F} \):
\begin{equation}
\sum_{F \in F} \sum_{J^* \in J^*(F)} P^\omega(J^*, h) \left| \left\langle \frac{x}{|J^*|^{1/2}}, g_F \mathbf{1}_{J^*} \right\rangle_\omega \right|^2 \leq \mathfrak{S}_r \|h\|^2_{L^2(\sigma)} \left( \sum_{F \in F} \|g_F\|^2_{L^2(\omega)} \right)^{1/2}.
\end{equation}

Now we show that the functional energy constants are equivalent to the energy constants modulo \( A_2^\sigma \). We proceed in two propositions.
Proposition 1.

\[ \mathfrak{F}_\alpha \lesssim \mathcal{E}_\alpha + \sqrt{A_2^\alpha} \quad \text{and} \quad \mathfrak{F}^*_\alpha \lesssim \mathcal{E}^*_\alpha + \sqrt{A_2^{\alpha^*}}. \]

To prove this first proposition, we fix \( \mathcal{F} \) as in (4.1) and set

\[ (4.2) \quad \mu \equiv \sum_{F \in \mathcal{F}} \sum_{J^* \in \mathcal{J}^*(F)} \left\| \mathbb{P}_{F, J^*} \frac{x}{|J^*|^\frac{1}{2}} \right\|_{L^2(\omega)}^2 \cdot \delta \left( c(J^*), |J^*|^\frac{1}{2} \right), \]

where \( \mathcal{J}^*(F) \) is defined in Definition 3 and the projections \( \mathbb{P}_{F, J^*} \) onto Haar functions are defined by

\[ \mathbb{P}_{F, J^*} = \sum_{J \subset J^*: J \in \mathcal{J}(F)} \sum_{a \in \Gamma_n} \triangle^\omega_a. \]

We can replace \( x \) by \( x - c \) for any choice of \( c \) we wish; the projection is unchanged. Here \( \delta_q \) denotes a Dirac unit mass at a point \( q \) in the upper half plane \( \mathbb{R}^2_+ \). More generally for any cube \( A \) we define

\[ \mathbb{P}_{F, A} = \sum_{J \subset A: J \in \mathcal{J}(F)} \sum_{a \in \Gamma_n} \triangle^\omega_a. \]

We prove the two-weight inequality

\[ (4.3) \quad \| \mathbb{P}^\alpha(f) \|_{L^2(\mathbb{R}^{n+1}_+, \mu)} \lesssim \| f \|_{L^2(\sigma)}, \]

for all nonnegative \( f \) in \( L^2(\sigma) \), noting that \( \mathcal{F} \) and \( f \) are not related here. Above, \( \mathbb{P}^\alpha(\cdot) \) denotes the \( \alpha \)-fractional Poisson extension to the upper half-space \( \mathbb{R}^{n+1}_+ \).

\[ \mathbb{P}^\alpha(\nu)(x, t) \equiv \int_{\mathbb{R}^n} \frac{t}{(t^2 + |x|^2)^\frac{n+1}{2}} d\nu(x), \]

so that in particular

\[ \| \mathbb{P}^\alpha(f) \|^2_{L^2(\mathbb{R}^{n+1}_+, \mu)} = \sum_{F \in \mathcal{F}} \sum_{J^* \in \mathcal{J}^*(F)} \| \mathbb{P}^\alpha(f) (c(J^*), |J^*|^\frac{1}{2}) \|^2 \left\| \mathbb{P}_{F, J^*} \frac{x}{|J^*|^\frac{1}{2}} \right\|^2_{L^2(\omega)}, \]

and so (4.3) implies (4.1) by the Cauchy-Schwarz inequality. By the two-weight inequality for the Poisson operator in [Saw3], inequality (4.3) requires checking these two inequalities

\[ (4.4) \quad \int_{\mathbb{R}^{n+1}_+} \mathbb{P}^\alpha(1_I \sigma)(x, t) d\mu(x, t) \equiv \| \mathbb{P}^\alpha(1_I \sigma) \|^2_{L^2(\tilde{I}, \mu)} \lesssim \left( A_2^\alpha + \mathcal{E}_\alpha^2 \right) \sigma(I), \]

\[ (4.5) \quad \int_{\mathbb{R}} \| \mathbb{P}^\alpha(1_I \mu) \|^2 d\sigma(x) \lesssim \left( A_2^\alpha + \mathcal{E}_\alpha \sqrt{A_2^\alpha} \right) \int_{\tilde{I}} t^2 d\mu(x, t), \]

for all dyadic cubes \( I \in \mathcal{D} \), where \( \tilde{I} = I \times [0, |I|] \) is the box over \( I \) in the upper half-space, and

\[ \mathbb{P}^\alpha(1_I \mu) \equiv \int_{\tilde{I}} \frac{t^2}{(t^2 + |x-y|^2)^\frac{n+1}{2}} d\mu(y, t). \]

It is important to note that we can choose for \( \mathcal{D} \) any fixed dyadic grid, the compensating point being that the integrations on the left sides of (4.4) and (4.5) are taken over the entire spaces \( \mathbb{R}^n_+ \) and \( \mathbb{R}^n \) respectively.
Remark 6. There is a gap in the proof of the Poisson inequality at the top of page 542 in [SaWh]. However, this gap can be fixed as in [SaWh] or [LaSaUr].

The following elementary Poisson inequalities will be used extensively.

Lemma 5. Suppose that $J, K, I$ are cubes satisfying $J \subset K \subset 2K \subset I$, and that $\mu$ is a positive measure supported in $\mathbb{R}^n \setminus I$. Then

$$\frac{P^{\alpha}(J, \mu)}{|J|^\frac{n}{2}} \lesssim \frac{P^{\alpha}(K, \mu)}{|K|^\frac{n}{2}} \lesssim \frac{P^{\alpha}(J, \mu)}{|J|^\frac{n}{2}}.$$  

Proof. We have

$$\frac{P^{\alpha}(J, \mu)}{|J|^\frac{n}{2}} = \frac{1}{|J|^\frac{n}{2}} \int \frac{|J|^\frac{n}{2}}{(|J|^\frac{n}{2} + |x - c_J|)^{n+1-\alpha}} d\mu(x),$$

where

$$|J|^\frac{n}{2} + |x - c_J| \approx |K|^\frac{n}{2} + |x - c_J|, \quad x \in \mathbb{R}^n \setminus I.$$  

4.1. The Poisson testing inequality. We choose the dyadic grid $\mathcal{D}$ in the testing conditions (4.4) and (4.5) to be the grid $\mathcal{D}^\omega$ that arises in the definition of the measure $\mu$ in (4.2). In particular all of the intervals $J^*$ lie in the good subgrid $\mathcal{D}_{\text{good}}^\omega$ of $\mathcal{D}$. Fix $I \in \mathcal{D}$. We split the integration on the left side of (4.4) into a local and global piece:

$$\int_{\mathbb{R}^n} P^{\alpha}(1, \sigma)^2 d\mu = \int_{\mathcal{I}} P^{\alpha}(1, \sigma)^2 d\mu + \int_{\mathbb{R}^n \setminus \mathcal{I}} P^{\alpha}(1, \sigma)^2 d\mu.$$  

The global piece turns out to be controlled solely by the $A_2^\sigma$ condition, so we leave that term for later, and turn now to estimating the local term.

An important consequence of the fact that $I$ and $J^*$ lie in the same grid $\mathcal{D} = \mathcal{D}^\omega$, is that $(c(J^*), |J^*|) \in \mathcal{I}$ if and only if $J^* \subset I$. Thus we have

$$\int_{\mathcal{I}} P^{\alpha}(1, \sigma)(x, t)^2 d\mu(x, t)$$

$$= \sum_{F \in \mathcal{F}} \sum_{J^* \in \mathcal{J}^*(F): J^* \subset I} \mathbb{P}^{\alpha}(1, \sigma)(c(J^*), |J^*|)^2 \left\| P_{F, J^*}^{\omega} \left( \frac{x}{|J^*|^\frac{n}{2}} \right) \right\|_{L^2(\omega)}^2$$

$$= \sum_{F \in \mathcal{F}} \sum_{J^* \in \mathcal{J}^*(F): J^* \subset I} \mathbb{P}^{\alpha}(J^*, 1, \sigma)^2 \left\| P_{F, J^*}^{\omega} \left( \frac{x}{|J^*|^\frac{n}{2}} \right) \right\|_{L^2(\omega)}^2.$$

Note that the collections $\mathcal{J}^*(F)$ are pairwise disjoint for $F \in \mathcal{F}$, and that for $J^* \in \mathcal{J}^*(F)$ we have

$$\left\| P_{F, J^*}^{\omega} \left( \frac{x}{|J^*|^\frac{n}{2}} \right) \right\|_{L^2(\omega)}^2 = \left\| P_{F, J^*}^{\omega} \left( \frac{x - E_{J^*}^{\omega} x}{|J^*|^\frac{n}{2}} \right) \right\|_{L^2(\omega)}^2 \leq E_{\text{soft}}(J^*, \omega)^2 |J^*|_\omega.$$  

(4.6)
In the first stage of the proof, we ‘create some holes’ by restricting the support of \( \sigma \) to the interval \( F \) in the sum below.

\[
\sum_{F \in \mathcal{F}} \sum_{J^* \in J^*(F) : J^* \subseteq I} P^\alpha(J^*, 1_{F \cap I_\sigma})^2 \left\| P_{F,J^*} \frac{\chi}{|J^*|^\frac{n}{2}} \right\|^2_{L^2(\omega)}
\]

\[
= \left\{ \sum_{F \in \mathcal{F} : F \subseteq I} + \sum_{F \in \mathcal{F} : F \supseteq I} \right\} \sum_{J^* \in J^*(F) : J^* \subseteq I} P^\alpha(J^*, 1_{F \cap I_\sigma})^2 \left\| P_{F,J^*} \frac{\chi}{|J^*|^\frac{n}{2}} \right\|^2_{L^2(\omega)}
\]

\[
= A + B.
\]

Then

\[
A \leq \sum_{F \in \mathcal{F}} \sum_{J^* \in J^*(F) : J^* \subseteq I} P^\alpha(J^*, 1_{F \cap I_\sigma})^2 E_{\text{soft}}(J^*, \omega) |J^*|_{\omega} \leq \sum_{F \in \mathcal{F}} E_{\alpha}^2 \sigma(F \cap I) \lesssim E_{\alpha}^2 \sigma(I),
\]

where the constant \( E_{\alpha} \) is defined in the energy condition

\[
\sup_{\cup_{i=1}^n I_i \subseteq I} \sum_i P(I_i, 1_{I_\sigma})^2 E_{\text{soft}}(I_i, \omega)^2 |I_i|_{\omega} \leq E_{\alpha}^2 |I|_{\sigma}, \quad I \in \mathcal{D}^\sigma.
\]

We also used that the stopping cubes \( \mathcal{F} \) satisfy a \( \sigma \)-Carleson measure estimate,

\[
\sum_{F \in \mathcal{F} : F \subset F_0} |F|_{\sigma} \lesssim |F_0|_{\sigma},
\]

which implies that if \( \{F_j\} \) are the maximal \( F \in \mathcal{F} \) that are contained in \( I \), then

\[
\sum_{F \in \mathcal{F}} \sigma(F \cap I) \leq \sum_j \sum_{F \subset F_j} \sigma(F) \lesssim \sum_j \sigma(F_j) \leq \sigma(I).
\]

Now let \( \mathcal{J}(I) \) consist of those \( J^* \subset I \) that lie in \( J^*(F) \) for some \( F \supset I \). For \( J^* \in \mathcal{J}(I) \) there are only two possibilities:

\[ J^* \subseteq I \text{ or } J^* \not\subseteq I. \]

If \( J^* \subseteq I \) and \( F \supset I \), then \( J^* \in \mathcal{F} \) by the definition of \( J^* \) good, and then by Property (3) in the definition of \( \mathcal{J}(F) \), Definition 5, we have the following two conclusions:

- for each \( J^* \subset I \) there are at most \( C \) cubes \( F \) with \( J^* \in \mathcal{J}(F) \) and \( F \supset I \),
- the cubes \( J^* \in \mathcal{J}(I) \) with \( J^* \in \mathcal{J}(F) \) have overlap bounded by \( C \), independent of \( I \).

As for the other case \( J^* \in \mathcal{J}(I) \) and \( J^* \not\subseteq I \), there are at most \( 2^n(r+1) \) such cubes \( J^* \), and they can be easily estimated without regard to their overlap if we let \( F_{J^*} \) be the unique interval \( F_{J^*} \supset \) with \( J^* \in J^*(F_{J^*}) \). The Poisson inequality then shows that term \( B \) satisfies
Lemma 6. Suppose \( \mathcal{G} \) is a collection of dyadic subcubes of \( I \) with bounded overlap \( C \):
\[
\sum_{G \in \mathcal{G}} 1_G \leq C.
\]

Then we can write \( \mathcal{G} = \bigcup_{\ell = 1}^C \mathcal{G}_\ell \) where each collection \( \mathcal{G}_\ell \) consists of pairwise disjoint cubes.

Proof. Let \( \mathcal{G}_1 \) consist of the maximal cubes in \( \mathcal{G} \). Then let \( \mathcal{G}_2 \) consist of the maximal cubes in \( \mathcal{G} \setminus \mathcal{G}_1 \), and continue by inductively defining \( \mathcal{G}_\ell \) to consist of the maximal cubes in \( \mathcal{G} \setminus \bigcup_{\ell' < \ell} \mathcal{G}_{\ell'} \) for \( \ell \geq 3 \). Clearly the cubes in each \( \mathcal{G}_\ell \) are pairwise disjoint, and moreover, \( \mathcal{G}_\ell = \emptyset \) for \( \ell > C \) since if there is a cube \( G \in \mathcal{G}_\ell \) then there is a unique tower of cubes \( G_k \supseteq G_{k-1} \supseteq \ldots \supseteq G_2 \supseteq G_1 \) with \( G_k \in \mathcal{G}_k \).

It remains then to show the following inequality with ‘holes’, where the support of \( \sigma \) is restricted to the complement of the interval \( F \):
\[
\left| \sum_{F \in \mathcal{F}_1} \sum_{J^* \in J^*(F)} \left( \frac{P^{\alpha} (J^*, 1_{I \setminus F})}{|J^*|^{1/2}} \right)^2 \| \mathcal{P}^\omega_{F, J^*} x \|_{L^2(\omega)}^2 \right| \lesssim E_0^2 \sigma(I),
\]
where \( \mathcal{F}_1 \) consists of those \( F \in \mathcal{F} \) with \( F \subset I \). Let \( \mathcal{M}(F) \) consist of the maximal deeply embedded good subcubes of \( F \). Using the Poisson inequalities in Lemma 5 together with the additivity of the projections \( \| \mathcal{P}^\omega_{F, J^*} x \|_{L^2(\omega)}^2 \), we obtain
\[
\left| \sum_{F \in \mathcal{F}_1} \sum_{J^* \in J^*(F)} \left( \frac{P^{\alpha} (J^*, 1_{I \setminus F})}{|J^*|^{1/2}} \right)^2 \| \mathcal{P}^\omega_{F, J^*} x \|_{L^2(\omega)}^2 \right| \lesssim \sum_{F \in \mathcal{F}_1} \sum_{J^* \in J^*(F)} \left( \frac{P^{\alpha} (J^*, 1_{I \setminus F})}{|J^*|^{1/2}} \right)^2 \left( \sum_{J^* \in J^*(F): J^* \subset J^{**}} \| \mathcal{P}^\omega_{F, J^{**}} x \|_{L^2(\omega)}^2 \right),
\]
where for \( J^{**} \in \mathcal{M}(F) \), we define \( \mathcal{P}^\omega_{F, J^{**}} \equiv \sum_{J^* \in J^*(F): J^* \subset J^{**}} \mathcal{P}^\omega_{F, J^*} \). Thus we can replace \( J^*(F) \) with \( \mathcal{M}(F) \) in the above sum, and if we revert to writing \( J^* \) in place of \( J^{**} \), it suffices to prove the following lemma.
Lemma 7. We have
\begin{equation}
(4.7) \sum_{F \in \mathcal{F}_I} \sum_{J^* \in \mathcal{M}(F)} \left( F^\alpha \frac{(J^*, 1_{|F|})}{|J^*|^{\frac{1}{2}}}; \frac{\|P^\omega_{F^*, J^*} x\|^2_{L^2(\omega)}}{|F|^{\frac{1}{2}}} \right)^2 \lesssim \mathcal{C}_{\alpha}(I).
\end{equation}

Proof. We consider the space $\ell_2^2$ of square summable sequences on the index set $F$
where
\begin{equation}
F = \{ (F, J^*) : F \in \mathcal{F}_I \text{ and } J^* \in \mathcal{M}(F) \}
\end{equation}
is the index set of pairs $(F, J^*)$ occurring in the sum in (4.7). We now take a
sequence $a = \{ a_{F, J^*} \}_{(F, J^*) \in \ell_2^2}$ with $a_{F, J^*} \geq 0$ and estimate
\begin{equation}
S = \sum_{F \in \mathcal{F}_I} \sum_{J^* \in \mathcal{M}(F)} \frac{\|P^\omega_{F^*, J^*} x\|^2_{L^2(\omega)}}{|F|^{\frac{1}{2}}} a_{F, J^*},
\end{equation}
by the Poisson inequalities in Lemma 5. We now invoke
\begin{align*}
&\sum_{F \in \mathcal{F}_I} \sum_{J^* \in \mathcal{M}(F)} \|P^\omega_{F^*, J^*} x\|^2_{L^2(\omega)} a_{F, J^*} \\
\lesssim &\left( \sum_{F \in \mathcal{F}_I} \sum_{J^* \in \mathcal{M}(F)} \|P^\omega_{F^*, J^*} x\|^2_{L^2(\omega)} \right)^{\frac{1}{2}} \times \left( \sum_{F \in \mathcal{F}_I} \sum_{J^* \in \mathcal{M}(F)} a_{F, J^*} \right)^{\frac{1}{2}} \\
\lesssim &\|P^\omega_{F^*, K^*} x\|^2_{L^2(\omega)} \|Q^\omega_{F^*, K^*} a\|_{\ell_2^2},
\end{align*}
where for $K^* \in \mathcal{M}(F')$ and $f \in L^2(\omega)$,
\begin{equation}
P^\omega_{F^*, K^*} \equiv \sum_{F \in \mathcal{F}_I} \sum_{J^* \in \mathcal{M}(F)} P^\omega_{F^*, J^*},
\end{equation}
while for $K^* \in \mathcal{M}(F')$ and $a = \{ a_{G, L^*} \}_{(G, L^*) \in \ell_2^2},$
\begin{align*}
Q^\omega_{F^*, K^*} a &\equiv \sum_{F \in \mathcal{F}_I} \sum_{J^* \in \mathcal{M}(F)} Q^\omega_{F^*, J^*} a; \\
Q^\omega_{F^*, J^*} a &\equiv \{ 1_{(F^*, J^*)} a_{G^*, L^*} \}_{(G^*, L^*) \in \ell_2^2}.
\end{align*}
Thus we can write
\[ |F| \lesssim 2^{-dk} |I|, \quad k \geq 0. \]

Thus we can write
\[
|S| \lesssim \sum_{F' \in \mathcal{F}_1} \sum_{K^* \in \mathcal{M}(F')} \frac{P^\alpha (K^*, 1_{\mathcal{F}' \setminus \mathcal{F}'^*})}{|K^*|^{\frac{1}{2}}} \left\| P_{\mathcal{F}', K^*} a \right\|_{L^2(\omega)} \left\| Q^\omega_{\mathcal{F}', K^*} a \right\|_{\ell^2_F}
\]
\[
= \sum_{k=0}^{\infty} \sum_{F' \in \mathcal{F}_1} \sum_{d(F')=k} \frac{P^\alpha (K^*, 1_{\mathcal{F}' \setminus \mathcal{F}'^*})}{|K^*|^{\frac{1}{2}}} \left\| P_{\mathcal{F}', K^*} a \right\|_{L^2(\omega)} \left\| Q^\omega_{\mathcal{F}', K^*} a \right\|_{\ell^2_F}
\]
\[
= \sum_{k=0}^{\infty} A_k,
\]
where
\[
A_k \lesssim \left( \sum_{F' \in \mathcal{F}_1} \sum_{d(F')=k} \frac{P^\alpha (K^*, 1_{\mathcal{F}' \setminus \mathcal{F}'^*})}{|K^*|^{\frac{1}{2}}} \right)^2 \left\| P_{\mathcal{F}', K^*} a \right\|_{L^2(\omega)} \left\| Q^\omega_{\mathcal{F}', K^*} a \right\|_{\ell^2_F}
\]
\[
\times \left( \sum_{F' \in \mathcal{F}_1} \sum_{d(F')=k} \left\| Q^\omega_{\mathcal{F}', K^*} a \right\|_{\ell^2_F} \right)^{\frac{1}{2}}
\]
\[
\lesssim \left( \mathcal{E}_2^2 \sum_{F'' \in \mathcal{F}_1} |F''| \right)^{\frac{1}{2}} \|a\|_{\ell^2_F}
\]
\[
\lesssim \mathcal{E}_2 (2^{-dk} |I|)^{\frac{1}{2}} \|a\|_{\ell^2_F},
\]
and we finally obtain
\[
|S| \lesssim \sum_{k=0}^{\infty} \mathcal{E}_2 (2^{-dk} |I|)^{\frac{1}{2}} \|a\|_{\ell^2_F} \lesssim \mathcal{E}_2 \sqrt{|I|} \|a\|_{\ell^2_F}.
\]

By duality of \(\ell^2_F\) we now conclude that
\[
\sum_{F \in \mathcal{F}_1} \sum_{J^* \in \mathcal{M}(F)} \left( \frac{P^\alpha (J^*, 1_{\mathcal{F}' \setminus \mathcal{F}'^*})}{|J^*|^{\frac{1}{2}}} \right)^2 \left\| P_{\mathcal{F}', J^*} a \right\|_{L^2(\omega)}^2 \lesssim \mathcal{E}_2^2 |I|,
\]
which is (4.7).

Now we turn to proving the following estimate for the global part of the first testing condition (4.4):
\[
\int_{\mathbb{R}^2 \setminus \mathcal{F}} P^\alpha (1_I)^2 d\mu \lesssim A^2 |I|.
\]
We begin by decomposing the integral on the left into four pieces:

\[
\int_{\mathbb{R}^2 \setminus \hat{I}} P^\alpha (1_I \sigma)^2 \, d\mu = \sum_{J^* : (c(J^*), |J^*|^{1/\alpha}) \in \mathbb{R}_{\alpha+1}^{*+1}} P^\alpha (1_I \sigma) \left( c(J^*), |J^*|^{1/\alpha} \right)^2 \sum_{F \sim J^*} \left\| P_{F,J^*} \frac{x}{|J^*|^{1/\alpha}} \right\|_{L^2(\omega)}^2
\]

\[
= \left\{ \begin{array}{ll}
\sum_{J^* \cap I = \emptyset} + \sum_{J^* \subset 3I \setminus I} + \sum_{J^* \supset I} + \sum_{J^* \cap I = \emptyset} \end{array} \right\} P^\alpha (1_I \sigma) \left( c(J^*), |J^*|^{1/\alpha} \right)^2 \sum_{F \sim J^*} \left\| P_{F,J^*} \frac{x}{|J^*|^{1/\alpha}} \right\|_{L^2(\omega)}^2
\]

\[= A + B + C + D.\]

We further decompose term A according to the length of $J^*$ and its distance from $I$, and then use (4.6) and $E(J^*, \omega) \leq 1$ to obtain:

\[A \lesssim \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} \sum_{J^* \subset 3F \setminus K} \left( \frac{2^{-m} |I|^{1/\alpha}}{\text{dist} (J^*, I)^{\alpha+1}} |I|_{\sigma} \right)^2 |J^*|_{\omega},\]

\[\lesssim \sum_{m=0}^{\infty} 2^{-2m} \sum_{k=1}^{\infty} \frac{|I|^{1/\alpha} |I|_{\sigma} |3k+1 I \setminus 3k I|_{\omega}}{|3k I|^{2(1 + \frac{1}{\alpha})}} |I|_{\sigma},\]

\[\lesssim \sum_{m=0}^{\infty} 2^{-2m} \sum_{k=1}^{\infty} 3^{-2k} \left\{ \frac{|3k+1 I|_{\sigma} |3k+1 I|_{\omega}}{|3k I|^{2(1 - \frac{1}{\alpha})}} \right\} |I|_{\sigma} \lesssim A_2 |I|_{\sigma}.\]

We further decompose term B according to the length of $J^*$ and now use the fractional version of the Poisson inequality (essentially in \[\text{Vol}\]),

\[
P^\alpha (J^*, 1_I \sigma)^2 \lesssim \left( \frac{|J^*|^{1/\alpha}}{|I|^{1/\alpha}} \right)^{2 - 2(n+1-\alpha)\varepsilon} P^\alpha (1_I \sigma)^2,
\]

which uses the fact that our grid $\mathcal{D}_{\text{good}}^\omega$ is a good subgrid of $\mathcal{D} = \mathcal{D}^\omega$. Indeed, we have

\[
P \left( J, \sigma \chi_{J \cap I} \right) \approx \sum_{k=0}^{\infty} 2^{-k} \frac{1}{|2^k J|} \int_{(2^k J) \cap (\hat{I} \setminus I)} \sigma,\]

and $(2^k J) \cap (\hat{I} \setminus I) \neq \emptyset$ requires

\[
\text{dist} (J, e(I)) \leq 2^k |J|^{1/\alpha}.\]

Let $k_0$ be the smallest such $k$. By our distance assumption we must then have

\[
|J|^{1/\alpha} |I|^{1-\alpha} \leq \text{dist} (J, e(I)) \leq 2^{k_0} |J|^{1/\alpha},\]

or

\[
2^{-k_0} \leq \left( \frac{|J|^{1/\alpha}}{|I|^{1/\alpha}} \right)^{1-\varepsilon}.\]
Now let $k_1$ be defined by $2^{k_1} = \frac{|I|^\frac{1}{n}}{|J|^\frac{1}{n}}$. Then assuming $k_1 > k_0$ (the case $k_1 \leq k_0$ is similar) we have

$$P^\alpha \left( J, \sigma \chi_{\hat{I}} \right) \approx \begin{cases} \sum_{k=k_0}^{k_1} \sum_{k=k_1}^{\infty} \frac{|J|^\frac{1}{n}}{|2^{k}J|^\frac{n+1-\alpha}{n}} \int_{(2^{k}J) \cap (\hat{I} \setminus J)} d\sigma \\ \geq \frac{|I|^{\frac{1}{n}}}{|2^{k_0}J|^{\frac{n+1-\alpha}{n}}} \left( \frac{1}{|I|^{\frac{n}{n}}} \int_{(2^{k_1}J) \cap (\hat{I} \setminus J)} d\sigma \right) + 2^{-k_1} P^\alpha \left( I, \sigma \chi_{\hat{I}} \right) \\ \leq \left( \frac{|J|^{\frac{1}{n}}}{|I|^{\frac{n}{n}}} \right)^{(1-\varepsilon)(n+1-\alpha)} \left( \frac{|I|^{\frac{1}{n}}}{|J|^{\frac{n}{n}}} \right)^{n-\alpha} P^\alpha \left( I, \sigma \chi_{\hat{I}} \right) + \frac{|J|^{\frac{1}{n}}}{|I|^{\frac{n}{n}}} P^\alpha \left( I, \sigma \chi_{\hat{I}} \right), \end{cases}$$

which is the inequality \[4,8\].

We then obtain

$$B \lesssim \sum_{m=0}^{\infty} \sum_{J^* \subset 3I \setminus I} (2^{-m})^{2-2(n+1-\alpha)\varepsilon} \left( \frac{|I|_\sigma}{|I|^{1-\frac{n}{n}}} \right)^2 |J^*|_\omega$$

$$\leq \sum_{m=0}^{\infty} (2^{-m})^{2-2(n+1-\alpha)\varepsilon} \left( \frac{|3I|_\sigma}{|3I|^{1-\frac{n}{n}}} \right) |I|_\sigma \lesssim A^2_2 |I|_\sigma.$$

For term $C$ we will have to group the cubes $J^*$ into blocks $B_i$, and then exploit the mutual orthogonality in the pairs $(F, J^*)$ of the projections $P^\ast_{F,J^*}$ defining $\mu$, in order to avoid overlapping estimates. We first split the sum according to whether or not $I$ intersects the triple of $J^*$:

$$C \approx \begin{cases} \sum_{J^*: |I \cap J^*| = 0} \sum_{J^*: \{J^* \subset 3J \setminus I\}} \left( \frac{|J^*|^{\frac{1}{n}}}{dist(J^*, I)^{n+1-\alpha} |I|_\sigma} \right)^2 \sum_{F \sim J^*} \left\| P^\ast_{F,J^*} \frac{X}{|J^*|^\frac{1}{n}} \right\|_{L^2(\omega)}^2 \\ \sum_{J^*: |I \cap J^*| \neq 0} \sum_{J^*: \{J^* \subset 3J \setminus I\}} \left( \frac{|J^*|^{\frac{1}{n}}}{dist(J^*, I)^{n+1-\alpha} |I|_\sigma} \right)^2 \sum_{F \sim J^*} \left\| P^\ast_{F,J^*} \frac{X}{|J^*|^\frac{1}{n}} \right\|_{L^2(\omega)}^2 \end{cases}$$

$$= C_1 + C_2.$$
\( P_{F,J}^\omega x = P_{F,J}^\omega (x - c(B_i)) \) along with the mutual orthogonality of the \( P_{F,J}^\omega \):

\[
C_1 \leq \sum_{i=1}^{\infty} \sum_{J^*: \text{all } J^* \subset B_i} \left( \frac{1}{\text{dist}(J^*, I)} \right)^{n+1-\alpha} \| P_{F,J}^\omega x \|_{L^2(\omega)}^2 \sum_{F \sim J^*} \| P_{F,J}^\omega x \|_{L^2(\omega)}^2
\]

\[
\lesssim \sum_{i=1}^{\infty} \left( \frac{1}{\text{dist}(B_i, I)} \right)^{n+1-\alpha} \| P_{F,J}^\omega x \|_{L^2(\omega)}^2
\]

\[
\lesssim \sum_{i=1}^{\infty} \left( \frac{1}{\text{dist}(B_i, I)} \right)^{n+1-\alpha} \| P_{F,J}^\omega x \|_{L^2(\omega)}^2 \leq A_2^\omega |I|_{\sigma}
\]

since \( \text{dist}(B_i, I) \approx |B_i|^{1/2} \) and

\[
\sum_{i=1}^{\infty} \left( \frac{1}{|B_i|^{2(1-\frac{\alpha}{n})}} \right) |I|_{\sigma} \lesssim A_2^\omega |I|_{\sigma}
\]

Next we turn to estimating term \( C_2 \) where the triple of \( J^* \) contains \( I \) but \( J^* \) itself does not. Note that there are at most \( 2^n \) such cubes \( J^* \) of a given side length, one in each ‘generalized octant’ relative to \( I \). So with this in mind we sum over the cubes \( J^* \) according to their lengths and use (4.4) to obtain

\[
C_2 = \sum_{m=0}^{\infty} \sum_{J^*: \text{all } J^* \subset B_i \cap \{ |J^*| = 2^m |I| \}} \left( \frac{|I|_{\sigma}}{|J^*|^{\frac{n}{n+1-\alpha}}} \right)^2 \| P_{F,J}^\omega x \|_{L^2(\omega)}^2 \sum_{F \sim J^*} \| P_{F,J}^\omega x \|_{L^2(\omega)}^2
\]

\[
\lesssim \sum_{m=0}^{\infty} \left( \frac{|I|_{\sigma}}{|2^m I|^{\frac{n}{n+1-\alpha}}} \right)^2 \| 3 \cdot 2^m I \|_{\omega} = \left( \frac{|I|_{\sigma}}{|I|^{\frac{n}{n+1-\alpha}}} \right) \sum_{m=0}^{\infty} \left( \frac{|I|_{\sigma}}{2^m |I|^{\frac{n}{2}}|I|^{2(1-\frac{\alpha}{n})}} \right) |I|_{\sigma}
\]

\[
\lesssim \left( \frac{|I|_{\sigma}}{|I|^{\frac{n}{n+1-\alpha}}} \right) \| P^\omega (I, \omega) \|_{\omega} \lesssim A_2^\omega |I|_{\sigma},
\]

since

\[
\sum_{m=0}^{\infty} \left( \frac{|I|_{\sigma}}{2^m |I|^{2(1-\frac{\alpha}{n})}} \right) = \int \sum_{m=0}^{\infty} \left( \frac{|I|_{\sigma}}{2^m |I|^{2(1-\frac{\alpha}{n})}} \right) 1_{3 \cdot 2^m I} (x) \ d\omega (x) \lesssim P^\alpha (I, \omega).
\]
Finally, we turn to term \( D \), which is handled in the same way as term \( C_2 \). The intervals \( J^* \) occurring here are included in the set of ancestors \( A_k \equiv \frac{n}{n} (k) I \) of \( I \), \( 1 \leq k < \infty \). We thus have

\[
D = \sum_{k=1}^{\infty} \mathbb{P}^\alpha \left( (1 \sigma) \left( c(A_k), |A_k| \right)^2 \right) \sum_{F \sim A_k} \left\| \mathbb{P}^\omega_{F^*, J^*} \frac{x}{|A_k|^{1/n}} \right\|_{L^2(\omega)}^2
\]

\[
\lesssim \sum_{k=1}^{\infty} \left( \frac{|I|}{|A_k|} \right)^2 \left| A_k \right|_{\omega} = \left\{ \frac{|I|}{|A_k|} \right\} \sum_{k=1}^{\infty} \frac{|I|^{1-\frac{1}{n}}}{|A_k|^{2(1-\frac{1}{n})}} \left| A_k \right|_{\omega} |I|_{\omega}
\]

\[
\lesssim \left\{ \frac{|I|}{|A_k|} \right\} \mathbb{P}^\alpha (I, \omega) \left| I \right|_{\omega} \lesssim A_2^\alpha |I|_{\omega},
\]

since

\[
\sum_{k=1}^{\infty} \frac{|I|^{1-\frac{1}{n}}}{|A_k|^{2(1-\frac{1}{n})}} |A_k|_{\omega} = \int \sum_{k=1}^{\infty} \frac{|I|^{1-\frac{1}{n}}}{|A_k|^{2(1-\frac{1}{n})}} 1_{A_k}(x) d\omega (x)
\]

\[
= \int \sum_{k=1}^{\infty} \frac{1}{|A_k|^{2(1-\frac{1}{n})}} \frac{|I|^{1-\frac{1}{n}}}{|I|^{2(1-\frac{1}{n})}} 1_{A_k}(x) d\omega (x)
\]

\[
\lesssim \int \left( \frac{|I|^{\frac{1}{n}} \left( |I| + dist(x, I) \right)^{n-\alpha}}{|I|^{\frac{1}{n}} + dist(x, I)} \right)^2 d\omega (x) = \mathbb{P}^\alpha (I, \omega).
\]

4.2. The dual Poisson testing inequality. Again we split the integration on the left side of (4.4) into local and global parts:

\[
\int_{\mathbb{R}} \left[ \mathbb{P}^{\alpha*} (t \check{1}_{I} \mu) \right]^2 \sigma = \int_{I} \left[ \mathbb{P}^{\alpha*} (t \check{1}_{I} \mu) \right]^2 \sigma + \int_{\mathbb{R} \setminus I} \left[ \mathbb{P}^{\alpha*} (t \check{1}_{I} \mu) \right]^2 \sigma.
\]

We begin with the local part. Note that the right hand side of (4.3) is

\[
(4.9) \quad \int_{I} t^2 d\mu (x, t) = \sum_{F \subseteq F, J \subseteq J^*(F)} \sum_{J^* \subseteq I} \|\mathbb{P}^\alpha_{F, J^*} z\|_{L^2(\omega)}^2,
\]

where we are using the dummy variable \( z \) to denote the argument of \( \mathbb{P}^\alpha_{F, J^*} \) so as to avoid confusion with the integration variable \( x \) in \( d\sigma (x) \). Compute

\[
(4.10) \quad \mathbb{P}^{\alpha*} (t \check{1}_{I} \mu) (x) = \sum_{F \subseteq F, J \subseteq J^*(F)} \sum_{J^* \subseteq I} \frac{\|\mathbb{P}^\alpha_{F, J^*} z\|_{L^2(\omega)}^2}{\left( |J^*|^{\frac{1}{n}} + |x - c(J^*)| \right)^{n+1-\alpha}},
\]

and then expand the square and integrate to obtain that the left hand side of (4.5) is

\[
\sum_{F \subseteq F, J \subseteq J^*(F)} \sum_{F' \subseteq F, J \subseteq J^*(F')} \int \left[ \mathbb{P}^\alpha_{F, J^*} z\right]_{L^2(\omega)}^2 \frac{\|\mathbb{P}^\alpha_{F, J^*} z\|_{L^2(\omega)}^2}{\left( |J^*|^{\frac{1}{n}} + |x - c(J^*)| \right)^{n+1-\alpha}} d\sigma (x),
\]

where we are now dropping the superscripts * from the cubes \( J^* \) and \((J^*)^*\), but not from \( J^* (F) \) and \( J^* (F') \), for clarity of display. We fix an integer \( s \), and consider
those cubes $J$ and $J'$ with $|J|^{\frac{1}{n}} = 2^{-s} |J|^{\frac{1}{n}}$. The expression to control for fixed $s$ is

$$U_s = \sum_{F \in \mathcal{F}} \sum_{J \in J^*(F)} \sum_{F' \in \mathcal{F}} \sum_{J' \in J^*(F')} \sum_{\epsilon > 0} \int_{I} \frac{\| P_{F,J}^\omega \|^2_{L^2(\omega)}}{\left( |J|^{\frac{1}{n}} + |x - c(J)| \right)^{n+1-\alpha}} \frac{\| P_{F',J'}^\omega \|^2_{L^2(\omega)}}{\left( |J'|^{\frac{1}{n}} + |x - c(J')| \right)^{n+1-\alpha}} d\sigma(x).$$

Note that $J$ uniquely fixes $F$ and $J'$ uniquely fixes $F'$.

Our first decomposition is to write

$$U_s = T_s + T_{s, close},$$

with

$$T_{s, close} = \sum_{F \in \mathcal{F}} \sum_{J \in J^*(F)} \sum_{F' \in \mathcal{F}} \sum_{J' \in J^*(F') \colon |J|^{\frac{1}{n}} = 2^{-s} |J|^{\frac{1}{n}} \atop \text{dist}(J,J') \geq 2^{(1+\epsilon)} |J|^{\frac{1}{n}}} \int_{I \setminus B(J,J')} \frac{\| P_{F,J}^\omega \|^2_{L^2(\omega)}}{\left( |J|^{\frac{1}{n}} + |x - c(J)| \right)^{n+1-\alpha}} \frac{\| P_{F',J'}^\omega \|^2_{L^2(\omega)}}{\left( |J'|^{\frac{1}{n}} + |x - c(J')| \right)^{n+1-\alpha}} d\sigma(x),$$

where $\epsilon > 0$ will be chosen later in the proof and $B(J,J') = B(c(J), \text{dist}(J,J'))$, is the ball centered at $c(J)$ with radius $\text{dist}(J,J')$. We will exploit the restriction of integration to $I \setminus B(J,J')$, together with the condition $\text{dist}(J,J') \geq 2^{(1+\epsilon)} |J|^{\frac{1}{n}}$, in establishing \ref{4.13} below, which will then give an estimate for the term $T_{s, close}$ using an argument dual to that used for the remaining term $T_s$.

We begin with an analysis of the term $T_s$ and write

$$T_s = \sum_{F \in \mathcal{F}} \sum_{J \in J^*(F)} \sum_{F' \in \mathcal{F}} \sum_{J' \in J^*(F')} \sum_{\epsilon > 0} \int_{I} \frac{\| P_{F,J}^\omega \|^2_{L^2(\omega)}}{\left( |J|^{\frac{1}{n}} + |x - c(J)| \right)^{n+1-\alpha}} \frac{\| P_{F',J'}^\omega \|^2_{L^2(\omega)}}{\left( |J'|^{\frac{1}{n}} + |x - c(J')| \right)^{n+1-\alpha}} d\sigma(x) \leq M_s \sum_{F \in \mathcal{F}} \sum_{J \in J^*(F)} \| P_{F,J}^\omega \|^2_\omega;$$

$$M_s = \sup_{F \in \mathcal{F}} \sup_{J \in J^*(F)} A_s(J);$$

$$A_s(J) = \sum_{F \in \mathcal{F}} \sum_{J' \in J^*(F')} \int_{I} S(J',J)(x) d\sigma(x);$$

$$S(J',J)(x) = \frac{1}{\left( |J|^{\frac{1}{n}} + |x - c(J)| \right)^{n+1-\alpha}} \frac{\| P_{F',J'}^\omega \|^2_{L^2(\omega)}}{\left( |J'|^{\frac{1}{n}} + |x - c(J')| \right)^{n+1-\alpha}},$$

where the additional restriction due to removal of $T_{s, close}$ from $U_s$ in \ref{4.11} results in a restricted range of integration $I \setminus B(J,J')$ in the integral in $A_s(J)$ in certain
situations, and this will be understood throughout the proof, without being explicitly indicated in the notation. This additional restriction will however be needed to establish (4.12) below.

Now fix $J$ as in the definition of $M_s(J)$, and decompose the sum over $J'$ in $A_s(J)$ by

$$A_s(J) = \sum_{J' \in \mathcal{F}} \sum_{J' \in \mathcal{F}'(J')} \int_I S(J',J)(x) \, d\sigma(x)$$

$$= \sum_{J' \subset 2J} \int_I S(J',J)(x) \, d\sigma(x) + \sum_{\ell=1}^{\infty} \sum_{J' \subset 2^{\ell+1}J \setminus 2^\ell J} \int_I S(J',J)(x) \, d\sigma(x) \equiv \sum_{\ell=0}^{\infty} A_s^{\ell}(J);$$

and then decompose the integrals over $I$ by

$$A^0_s(J) = \sum_{J' \subset 2J} \int_{I \setminus A_J} S(J',J)(x) \, d\sigma(x) + \sum_{J' \subset 2J} \int_{I \cap A_J} S(J',J)(x) \, d\sigma(x)$$

$$A^\ell_s(J) = \sum_{J' \subset 2^{\ell+1}J \setminus 2^\ell J} \int_{I \setminus 2^{\ell+2}J \setminus 2^\ell J} S(J',J)(x) \, d\sigma(x)$$

$$+ \sum_{J' \subset 2^{\ell+1}J \setminus 2^\ell J} \int_{I \setminus 2^{\ell+1}J \setminus 2^{\ell-1}J} S(J',J)(x) \, d\sigma(x)$$

$$+ \sum_{J' \subset 2^{\ell+1}J \setminus 2^\ell J} \int_{I \setminus 2^{\ell+1}J \setminus 2^{\ell-1}J} S(J',J)(x) \, d\sigma(x)$$

$$= A^\ell_{s,far}(J) + A^\ell_{s,near}(J) + A^\ell_{s,close}(J), \quad \ell \geq 1.$$ 

Note the important point that the close terms $A^\ell_{s,close}(J)$ vanish for $\ell > \varepsilon s$ because of the decomposition (4.11):

$$A^\ell_{s,close}(J) = 0, \quad \ell > \varepsilon s.$$ 

Indeed, if $J' \subset 2^{\ell+1}J \setminus 2^\ell J$, then we have

$$2^{\ell} |J|^{\frac{n}{n}} \approx \text{dist}(J, J'),$$

and if $\ell > \varepsilon s$, then

$$\text{dist}(J, J') \geq 2^{\varepsilon s} |J|^{\frac{n}{n}} = 2^{(1+\varepsilon)s} |J'|^{\frac{n}{n}}.$$ 

It now follows from the definition of $T_{s,close}$ and $T_s$ in (4.11), that in the term $T_s$, the integration is taken over the set $I \setminus B(J, J')$. But in the term $A_{s,close}^\ell(J)$ that is derived from $T_s$ we are restricted to integrating over the cube $2^{\ell-1}J$, which is contained in $B(J, J')$ by (4.13). Thus the range of integration in the term $A_{s,close}^\ell(J)$ is the empty set, and so $A_{s,close}^\ell(J) = 0$. 

Now using $\|P_{F^r,T}^\omega z\|_{L^2(\omega)}^2 \leq |J|^\frac{2}{\alpha} |J'|_{\omega}$ we have

$$A_{s,far}^0 (J) = \sum_{J' \subset 2J} \int_{I_\lambda(4J)} S_{(J',J)} (x) \, d\sigma (x)$$

$$\lesssim \sum_{J' \subset 2J} \int_{I_\lambda(4J)} \frac{|J'|_{\omega}}{|J|^{\frac{2}{\alpha}} + |x - c(J)|}^{2(n+1-\alpha)} |J|^\frac{2}{\alpha} |J'|_{\omega} \, d\sigma (x)$$

$$= 2^{-2s} \left( \sum_{J' \subset 2J} |J'|_{\omega} \right) \int_{I_\lambda(4J)} \frac{|J|^\frac{2}{\alpha}}{|J|^\frac{2}{\alpha} + |x - c(J)|}^{2(n+1-\alpha)} |J|^\frac{2}{\alpha} \, d\sigma (x),$$

which is dominated by

$$2^{-2s} |2J|_{\omega} \int_{I_\lambda(4J)} \frac{1}{|J|^\frac{2}{\alpha} + |x - c(J)|}^{2(n-\alpha)} \, d\sigma (x)$$

$$\approx 2^{-2s} \frac{|2J|_{\omega}}{|2J|^{\frac{2}{\alpha}} + |x - c(J)|} \int_{I_\lambda(4J)} \frac{|J|^\frac{2}{\alpha}}{|J|^\frac{2}{\alpha} + |x - c(J)|}^{n-\alpha} \, d\sigma (x)$$

$$\lesssim 2^{-2s} \frac{|2J|_{\omega}}{|2J|^{\frac{2}{\alpha}}} P^\alpha (2J, \sigma) \lesssim 2^{-2s} A_2^\alpha.$$ 

To estimate the near term $A_{s,near}^0 (J)$, we initially keep the energy $\|P_{F^r,T}^\omega z\|_{L^2(\omega)}^2$ and use the energy constant $\mathcal{E}_\alpha$ as follows:

$$A_{s,near}^0 (J) = \sum_{J' \subset 2J} \int_{I_\lambda(4J)} S_{(J',J)} (x) \, d\sigma (x)$$

$$\approx \sum_{J' \subset 2J} \int_{I_\lambda(4J)} \frac{1}{|J|^{\frac{2}{\alpha}} + |x - c(J)|} \frac{\|P_{F^r,T}^\omega z\|_{L^2(\omega)}^2}{|J|^{\frac{2}{\alpha}} + |x - c(J)|} \, d\sigma (x)$$

$$= \frac{1}{|J|^{\frac{2}{\alpha}} + |x - c(J)|} \sum_{J' \subset 2J} \|P_{F^r,T}^\omega z\|_{L^2(\omega)}^2 \int_{I_\lambda(4J)} \frac{1}{|J|^{\frac{2}{\alpha}} + |x - c(J)|} \, d\sigma (x)$$

$$= \frac{1}{|J|^{\frac{2}{\alpha}} + |x - c(J)|} \sum_{J' \subset 2J} \|P_{F^r,T}^\omega z\|_{L^2(\omega)}^2 \frac{P^\alpha (J', 1_{I_\lambda(4J)} \sigma)}{|J|^{\frac{2}{\alpha}}}$$

and by Cauchy-Schwarz this is dominated by

$$\lesssim \frac{1}{|J|^{\frac{2}{\alpha}} + |x - c(J)|} \left( \sum_{J' \subset 2J} \|P_{F^r,T}^\omega z\|_{L^2(\omega)}^2 \right)^{\frac{1}{2}} \left( \sum_{J' \subset 2J} \|P_{F^r,T}^\omega z\|_{L^2(\omega)}^2 \left( \frac{P^\alpha (J', 1_{I_\lambda(4J)} \sigma)}{|J|^{\frac{2}{\alpha}}} \right)^{\frac{1}{2}} \right)$$

$$\lesssim \frac{1}{|J|^{\frac{1}{2}} + |x - c(J)|} \left( \sum_{J' \subset 2J} |J'|^\frac{2}{\alpha} |J'|_{\omega} \right)^{\frac{1}{2}} \mathcal{E}_\alpha \sqrt{|I_\lambda(4J)|_{\sigma}}$$

$$\lesssim 2^{-s} |J|^\frac{2}{\alpha} \sqrt{2J|_{\omega} \mathcal{E}_\alpha \sqrt{|4J|_{\sigma}}} \lesssim 2^{-s} \mathcal{E}_\alpha \sqrt{|J|^\frac{2}{\alpha} |J|_{\omega} |J|_{\sigma}} \lesssim 2^{-s} \mathcal{E}_\alpha \sqrt{A_2^\alpha}.$$
Similarly, for $\ell \geq 1$, we can estimate the far term

$$A_{s,\text{far}}^\ell (J) = \sum_{J' \subset (2^{\ell+1}J) \setminus (2^\ell J \cup 2^{\ell+2}J)} S(J',J) \, d\sigma(x)$$

$$\lesssim \sum_{J' \subset (2^{\ell+1}J) \setminus (2^\ell J \cup 2^{\ell+2}J)} \int_{I \setminus (2^\ell J \cup 2^{\ell+2}J)} \frac{|J'|^\frac{1}{n} |\omega|}{(|J|^{\frac{1}{n}} + |x - c(J)|)^{2(n+1-\alpha)}} \, d\sigma(x)$$

$$= 2^{-2s} \left( \sum_{J' \subset (2^{\ell+1}J) \setminus (2^\ell J \cup 2^{\ell+2}J)} |J'| \right) \int_{I \setminus (2^\ell J \cup 2^{\ell+2}J)} \frac{|J|^\frac{2}{n}}{|2^\ell J|^\frac{1}{n} + |x - c(2^\ell J)|^{2(n+1-\alpha)}} \, d\sigma(x)$$

$$\approx 2^{-2s} 2^{-\frac{\ell}{2}} \left( \sum_{J' \subset (2^{\ell+1}J) \setminus (2^\ell J \cup 2^{\ell+2}J)} |J'| \right) \int_{I \setminus (2^\ell J \cup 2^{\ell+2}J)} \frac{|2^\ell J|^{\frac{2}{n}}}{|2^\ell J|^{\frac{1}{n}} + |x - c(2^\ell J)|^{2(n+1-\alpha)}} \, d\sigma(x)$$

which is at most

$$\lesssim 2^{-2s} 2^{-\frac{\ell}{2}} |2^{\ell+1}J| \omega \int_{I \setminus (2^\ell J \cup 2^{\ell+2}J)} \frac{1}{|2^\ell J|^{\frac{1}{n}} + |x - c(2^\ell J)|^{2(n+1-\alpha)}} \, d\sigma(x)$$

$$\approx 2^{-2s} 2^{-\frac{\ell}{2}} \frac{|2^{\ell+1}J| \omega}{|2^\ell J|^{\frac{1}{n}}} \int_{I \setminus (2^\ell J \cup 2^{\ell+2}J)} \left( \frac{|2^\ell J|^{\frac{1}{n}}}{|2^\ell J|^{\frac{1}{n}} + |x - c(2^\ell J)|^{2(n+1-\alpha)}} \right)^{n-\alpha} \, d\sigma(x)$$

$$\lesssim 2^{-2s} 2^{-\frac{\ell}{2}} \left\{ \frac{|2^{\ell+1}J| \omega}{|2^\ell J|^{\frac{1}{n}}} \int_{I \setminus (2^\ell J \cup 2^{\ell+2}J)} \left( \frac{|2^\ell J|^{\frac{1}{n}}}{|2^\ell J|^{\frac{1}{n}} + |x - c(2^\ell J)|^{2(n+1-\alpha)}} \right)^{n-\alpha} \, d\sigma(x) \right\} \lesssim 2^{-2s} 2^{-\frac{\ell}{2}} A_2^\sigma$$

and the near term

$$A_{s,\text{near}}^\ell (J)$$

$$= \sum_{J' \subset 2^{\ell+1}J \setminus 2^\ell J} \int_{I \setminus (2^\ell J \cup 2^{\ell+2}J)} S(J',J) \, d\sigma(x)$$

$$\approx \sum_{J' \subset 2^{\ell+1}J \setminus 2^\ell J} \int_{I \setminus (2^\ell J \cup 2^{\ell+2}J)} \frac{1}{|2^\ell J|^{\frac{1}{n}} + |x - c(2^\ell J)|^{2(n+1-\alpha)}} \, d\sigma(x)$$

$$= \frac{1}{|2^{\ell+1}J|^{\frac{1}{n}} + |x - c(2^\ell J)|^{2(n+1-\alpha)}} \sum_{J' \subset 2^{\ell+1}J \setminus 2^\ell J} \frac{||P_{\omega} J'||_{L^2(\omega)}^2}{|2^{\ell+2}J \setminus 2^{\ell+1}J|^{n+1-\alpha}} \, d\sigma(x)$$
which is dominated by
\[
\leq \frac{1}{|2^\ell - 1|^{\frac{1}{n}} (n+1-\alpha)} \sum_{J \subset 2^{\ell+1} J \cap 2^\ell J} \|P_{\omega} F^{\prime}, J^\prime Z\|_{L^2(\omega)}^2 \frac{P_\alpha (J^\prime, 1_{I \cap (2^{\ell+2} J) \sigma})}{|J^\prime|^{\frac{1}{n}}}
\]
\[
\leq \frac{1}{|2^\ell - 1|^{\frac{1}{n}} (n+1-\alpha)} \left( \sum_{J \subset 2^{\ell+1} J \cap 2^\ell J} \|P_{\omega} F^{\prime}, J^\prime Z\|_{L^2(\omega)}^2 \right)^{\frac{1}{2}}
\times \left( \sum_{J \subset 2^{\ell+1} J \cap 2^\ell J} \|P_{\omega} F^{\prime}, J^\prime Z\|_{L^2(\omega)} \left( \frac{P_\alpha (J^\prime, 1_{I \cap (2^{\ell+2} J) \sigma})}{|J^\prime|^{\frac{1}{n}}} \right)^2 \right)^{\frac{1}{2}}
\]
and which can now be estimated using \( \|P_{\omega} F^{\prime}, J^\prime Z\|_{L^2(\omega)}^2 \leq |J^\prime|^\frac{2}{n} |J^\prime|_\omega \) and the energy constant \( \mathcal{E}_\alpha \) to get
\[
A_{s,near}^\ell (J) \lesssim 2^{-s} 2^{-\frac{\ell}{4}} \frac{|2^\ell J|^{\frac{1}{n}}}{|2^{\ell - 1} J|^{\frac{1}{n}} (n+1-\alpha)} \sqrt{|2^{\ell+1} J|_\omega \mathcal{E}_\alpha \sqrt{|2^{\ell+2} J|_\sigma}}
\]
\[
\lesssim 2^{-s} 2^{-\frac{\ell}{4}} \mathcal{E}_\alpha \sqrt{\frac{|2^{\ell+2} J|_\omega}{|2^{\ell+2} J|_\sigma}} \frac{|2^{\ell+2} J|_\alpha}{|2^{\ell+2} J|^{1 - \frac{\alpha}{2}}}
\]
\[
\lesssim 2^{-s} 2^{-\frac{\ell}{4}} \mathcal{E}_\alpha \sqrt{A_2^\ell}.
\]
These estimates are summable in both \( s \) and \( \ell \).

Now we turn to the term \( S_{close}^\ell (J) \), and recall from (4.12) that \( S_{close}^\ell (J) = 0 \) if \( \ell > \varepsilon s \). So we now suppose that \( \ell \leq \varepsilon s \). We have
\[
A_{s,close}^\ell (J) = \sum_{J \subset 2^{\ell+1} J \cap 2^\ell J} \int_{I \cap (2^{\ell-1} J)} S^{\prime} (J^\prime, J) (x) \, d\sigma (x)
\]
\[
\lesssim |J|^{\frac{2}{n}} \mathcal{E}_\alpha \sqrt{\frac{|2^{\ell+2} J|_\omega}{|2^{\ell+2} J|_\sigma}} \frac{|2^{\ell+2} J|_\alpha}{|2^{\ell+2} J|^{1 - \frac{\alpha}{2}}} \int_{I \cap (2^{\ell-1} J)} \left( \frac{1}{|J|^{\frac{1}{n}} + |x - c (J)|} \right)^{n+1-\alpha} \, d\sigma (x)
\]
Now we use the inequality \( \|P_{\omega} F^{\prime}, J^\prime Z\|_{L^2(\omega)} \leq |J^\prime|^\frac{2}{n} |J^\prime|_\omega \) to get the relatively crude estimate
\[
A_{s,close}^\ell (J) \lesssim 2^{-s} |J|^{\frac{2}{n}} \frac{|2^{\ell+1} J|_\omega}{|2^{\ell+1} J|^{(n+1-\alpha)}} \int_{I \cap (2^{\ell-1} J)} \left( \frac{1}{|J|^{\frac{1}{n}} + |x - c (J)|} \right)^{n+1-\alpha} \, d\sigma (x)
\]
\[
\lesssim 2^{-s} |J|^{\frac{2}{n}} \frac{|2^{\ell+1} J|_\omega}{|2^{\ell+1} J|^{(n+1-\alpha)}} \frac{|2^{\ell-1} J|_\sigma}{|2^{\ell+1} J|^{\frac{1}{n}} |2^{\ell+1} J|^{1 - \frac{\alpha}{2}}} \lesssim 2^{-s} \frac{|2^{\ell+1} J|_\omega}{|2^{\ell+1} J|^{\frac{1}{n}} |2^{\ell+1} J|^{1 - \frac{\alpha}{2}}} 2^{\ell(n-1-\alpha)}
\]
\[
\lesssim 2^{-s} 2^{\ell(n-1-\alpha)} A_2^\ell \lesssim 2^{-s} A_2^\ell
\]
provided that \( \ell \leq \frac{\varepsilon}{n} \). But we are assuming \( \ell \leq \varepsilon s \) here and so we obtain a suitable estimate for \( S_{close}^\ell (J) \) provided we choose \( 0 < \varepsilon < \frac{1}{n} \).
Remark 7. We cannot simply sum the estimate
\[
A^\ell_{s, \text{close}} (J) \lesssim 2^{-2s} |J|^\frac{\omega}{2} |2^\ell J|^\omega \frac{1}{|2^\ell J|^\frac{\omega}{2(n+\alpha)}} P^\alpha (J, 1_{2^\ell - 1} \sigma),
\]
over all \( \ell \geq 1 \) to get
\[
\sum_\ell A^\ell_{s, \text{close}} (J) \lesssim 2^{-2s} P^\alpha (J, \sigma) \sum_\ell \frac{|J|^\frac{\omega}{2}}{|2^\ell J|^\frac{\omega}{2(n+\alpha)}} |2^\ell J|^\omega \lesssim 2^{-2s} P^\alpha (J, \omega) P^\alpha (J, \omega),
\]
since we only have control of the product \( P (J, \sigma) P (J, \omega) \) in dimension \( n = 1 \), where the two Poisson kernels \( P \) and \( \mathcal{P} \) coincide, and the two-tailed \( A_2 \) condition is known to hold.

Now we return to the term,
\[
T_{s, \text{close}} \equiv \sum_{F \in \mathcal{F}} \sum_{J \in I^* (F)} \sum_{J' \in I^* (F')} \sum_{\text{dist} (J, J') > 2^{(1+\alpha)} |J|^\frac{\omega}{2}} \int_{I \cap B (J, J')} \frac{\|P^\omega_F, z \|^2_{L^2 (\omega)}}{|J|^\frac{\omega}{2} + |x - c (J)|} \frac{\|P^\omega_{F'}, z \|^2_{L^2 (\omega)}}{|J'|^\frac{\omega}{2} + |x - c (J')|} \omega (x) d\sigma (x).
\]

It will suffice to show that \( T_{s, \text{close}} \) satisfies the estimate,
\[
T_{s, \text{close}} \lesssim 2^{-s} \mathcal{E}_\alpha \sqrt{A_2^2} \sum_{F \in \mathcal{F}} \sum_{J \in I} \|P^\omega_F, z \|^2_{L^2 (\omega)} = 2^{-s} \mathcal{E}_\alpha \sqrt{A_2^2} \int_I t^2 d\mu (x, t).
\]

We can write (suppressing some notation for clarity),
\[
T_{s, \text{close}} = \sum_{J, J'} \int_{I \cap B (J, J')} \frac{\|P^\omega_F, z \|^2_{L^2 (\omega)}}{|J|^\frac{\omega}{2} + |x - c (J)|} \frac{\|P^\omega_{F'}, z \|^2_{L^2 (\omega)}}{|J'|^\frac{\omega}{2} + |x - c (J')|} \frac{1}{\text{dist} (J, J')^n + 1 - \alpha} d\sigma (x)
\]
\[
\approx \sum_{J, J'} \|P^\omega_F, z \|^2_{L^2 (\omega)} \|P^\omega_{F'}, z \|^2_{L^2 (\omega)} \frac{1}{\text{dist} (J, J')^n + 1 - \alpha} \frac{P^\alpha (J, 1_{I \cap B (J, J')} \sigma)}{|J|^\frac{\omega}{2}}
\]
\[
\leq \sum_{J} \|P^\omega_{F'}, z \|^2_{L^2 (\omega)} \sum_{J'} \frac{1}{\text{dist} (J, J')^n + 1 - \alpha} \|P^\omega_{F', z} \|^2_{L^2 (\omega)} \frac{P^\alpha (J, 1_{I \cap B (J, J')} \sigma)}{|J|^\frac{\omega}{2}},
\]
and it remains to show that for each \( J' \),
\[
S_{s, \text{close}} (J') \equiv \sum_j \frac{\|P^\omega_{F', z} \|^2_{L^2 (\omega)}}{\text{dist} (J, J')^n + 1 - \alpha} \frac{P^\alpha (J, 1_{I \cap B (J, J')} \sigma)}{|J|^\frac{\omega}{2}} \lesssim \mathcal{E}_\alpha \sqrt{A_2^2}.
\]
We write
\[ S_{s, \text{close}} (J') \approx \sum_{k=1}^{\infty} \frac{1}{(2^k |J'|^{\frac{1}{n}})^{n+1-\alpha}} \sum_{J: \text{dist}(J, J') \approx 2^k |J'|^{\frac{1}{n}}} \left\| P^s_{F,J} \right\|^2_{L^2(\omega)} \frac{P^s (J, 1_{\mathcal{I}(J, J')})}{|J|^{\frac{1}{2}}}. \]
\[ = \sum_{k=1}^{\infty} \frac{1}{(2^k |J'|^{\frac{1}{n}})^{n+1-\alpha}} S^k_{s, \text{close}} (J'). \]

Now we apply Cauchy-Schwartz to get
\[ S^k_{s, \text{close}} (J') \leq \left( \sum_{J: \text{dist}(J, J') \approx 2^k |J'|^{\frac{1}{n}}} \left\| P^s_{F,J} \right\|^2_{L^2(\omega)} \right)^{\frac{1}{2}} \times \left( \sum_{J: \text{dist}(J, J') \approx 2^k |J'|^{\frac{1}{n}}} \left( \frac{P^s (J, 1_{\mathcal{I}(J, J')})}{|J|^{\frac{1}{2}}} \right)^2 \right)^{\frac{1}{2}} \]
\[ \leq \left( \sum_{J: \text{dist}(J, J') \approx 2^k |J'|^{\frac{1}{n}}} |J|^{\frac{2}{n}} |J|_{\omega} \right)^{\frac{1}{2}} \left( \frac{P^s (J, 1_{\mathcal{I}(J, J')})}{|J|^{\frac{1}{2}}} \right)^{\frac{1}{2}} \]
\[ \leq \varepsilon_\alpha 2^s |J'|^{\frac{2}{n}} \sqrt{2^{2k} |J|_{\omega}} \sqrt{C 2^{2k} |J|_{\sigma}} \leq \varepsilon_\alpha \sqrt{A_2^{2s} 2^k |J'|^{\frac{2}{n}} |J|^{1-\frac{2}{n}}} \]
\[ = \varepsilon_\alpha \sqrt{A_2^{2s} 2^{k(n-\alpha)} |J'|^{\frac{2}{n}(n+1-\alpha)}}, \]
provided
\[ B (c (J), \text{dist} (J, J')) = B (J, J') \subset C 2^k J'. \]

But this follows from \( \text{dist} (J, J') \approx 2^k |J'|^{\frac{1}{n}} \) and \( k \geq s \). Of course \( k \geq s \) follows from the facts that \( |J|^{\frac{1}{n}} = 2^s |J'|^{\frac{1}{n}} \) and that \( J \) must fit inside \( 2^k J' \). But we can do better from the definition of \( T_{s, \text{close}} \), namely
\[ (4.14) \quad k \geq (1 + \varepsilon) s. \]

Indeed, in the term \( T_{s, \text{close}} \) we have \( \text{dist} (J, J') \geq 2^{(1+\varepsilon)s} |J'|^{\frac{1}{n}} \), and combined with \( \text{dist} (J, J') \approx 2^k |J'|^{\frac{1}{n}} \), we obtain \((4.14)\). Then we have
\[ S_{s, \text{close}} (J') = \sum_{k \geq (1+\varepsilon)s} \frac{1}{(2^k |J'|^{\frac{1}{n}})^{n+1-\alpha}} S^k_{s, \text{close}} (J') \]
\[ \lesssim \varepsilon_\alpha \sqrt{A_2^{2s}} \sum_{k \geq (1+\varepsilon)s} \frac{1}{(2^k |J'|^{\frac{1}{n}})^{n+1-\alpha}} 2^{s-k} 2^{k(n-\alpha)} |J'|^{\frac{2}{n}(n+1-\alpha)} \]
\[ \lesssim \varepsilon_\alpha \sqrt{A_2^{2s}} \sum_{k \geq (1+\varepsilon)s} 2^{-s-k} \lesssim 2^{-\varepsilon s} \varepsilon_\alpha \sqrt{A_2^{2s}}, \]
which is summable in \( s \). This completes the proof of the estimate for the local part of the second testing condition \((4.5)\).
It remains to prove the following estimate for the global part of the second testing condition \((4.3)\):
\[
\int_{\mathbb{R} \setminus \mathcal{I}} \|P_{\omega}^\alpha(t \mathbf{1}_{\mathcal{I}} \mu)\|^2 \sigma \lesssim A_2^\alpha |\mathcal{I}|_\sigma.
\]
We decompose the integral on the left into two pieces:
\[
\int_{\mathbb{R} \setminus \mathcal{I}} \|P_{\omega}^\alpha(t \mathbf{1}_{\mathcal{I}} \mu)\|^2 \sigma = \int_{3 \mathcal{I} \setminus \mathcal{I}} \|P_{\omega}^\alpha(t \mathbf{1}_{\mathcal{I}} \mu)\|^2 \sigma + \int_{3 \mathcal{I} \setminus \mathcal{I}} \|P_{\omega}^\alpha(t \mathbf{1}_{\mathcal{I}} \mu)\|^2 \sigma = A + B.
\]
We further decompose term \(A\) in annuli and use \((4.10)\) to obtain
\[
A = \sum_{m=1}^{\infty} \int_{3^{m+1} \mathcal{I} \setminus 3^m \mathcal{I}} \|P_{\omega}^\alpha(t \mathbf{1}_{\mathcal{I}} \mu)\|^2 \sigma
\]
\[
= \sum_{m=1}^{\infty} \int_{3^{m+1} \mathcal{I} \setminus 3^m \mathcal{I}} \sum_{F \in \mathcal{F}, J^* \in \mathcal{J}^*(F), \mathcal{J}^* \subset \mathcal{I}} \frac{\|P_{\mathcal{F}, J^*}^\alpha \|_{L^2(\omega)}^2}{(|J^*| + |x - c(J^*)|)^{n+1-\alpha}} d\sigma (x)
\]
\[
\lesssim \sum_{m=1}^{\infty} \int_{3^{m+1} \mathcal{I} \setminus 3^m \mathcal{I}} \sum_{F \in \mathcal{F}, J^* \in \mathcal{J}^*(F), \mathcal{J}^* \subset \mathcal{I}} \frac{\|P_{\mathcal{F}, J^*}^\alpha \|_{L^2(\omega)}^2}{(3^m |\mathcal{I}|_\sigma)^{2(n+1-\alpha)}}
\]
\[
\lesssim \left\{ \sum_{m=1}^{\infty} 3^{-2m} \frac{3^{m+1} |\mathcal{I}|_\sigma \|P_{\mathcal{F}, J^*}^\alpha \|_{L^2(\omega)}^2}{3^{m+1} |\mathcal{I}|_\sigma^{2(1-\alpha)}} \right\} \left( \int_{\mathcal{I}} t^2 d\mu \right) \lesssim A_2^\alpha \int_{\mathcal{I}} t^2 d\mu,
\]
where of course \(A_2^\alpha \leq A_2^\alpha\).

Finally, we estimate term \(B\) by using \((4.10)\) to write
\[
B = \int_{3 \mathcal{I} \setminus \mathcal{I}} \sum_{F \in \mathcal{F}, J^* \in \mathcal{J}^*(F), \mathcal{J}^* \subset \mathcal{I}} \frac{\|P_{\mathcal{F}, J^*}^\alpha \|_{L^2(\omega)}^2}{(|J^*| + |x - c(J^*)|)^{n+1-\alpha}} d\sigma (x),
\]
and then expanding the square and calculating as in the proof of the local part given earlier. The details are similar and left to the reader.

4.3. **Control of energy by functional energy.** Now we use an easy duality argument to show that conversely, the energy condition is a consequence of the functional energy condition.

**Proposition 2.**
\[
\mathcal{E}_\alpha \lesssim \mathcal{F}_\alpha \text{ and } \mathcal{E}_\alpha^* \lesssim \mathcal{F}_\alpha^*.
\]
Proof. To prove this second proposition, we fix a subpartition \( \{ I_r \}_{r=1}^{\infty} \) of the interval \( I \) into \( D \)-dyadic subcubes \( I_r \) as in the definition of the energy constant. Let \( \mathcal{F} \equiv \{ I_r \}_{r=1}^{\infty} \cup \{ I \} \), and note that \( \mathcal{F} \) trivially satisfies the Carleson condition (4). Set

\[
\tilde{M}(I_r) \equiv \bigcup_{J^* \in M(I_r)} \{ J \in D : J \subset J^* \},
\]

\[
\tilde{M}(I) \equiv \bigcup_{r=1}^{\infty} \tilde{M}(I_r).
\]

Given a sequence \( \{ a_J \}_{J \in \tilde{M}(I)} \) of nonnegative numbers (with all but finitely many vanishing), define

\[
g_{I_r} = \sum_{J \in \tilde{M}(I_r)} a_J \sum_{k=1}^{n} h_{J_r}^k, \quad \text{for } 1 \leq r < \infty \text{ and } J^* \in M(I_r).
\]

Then \( \{ g_{I_r} \}_{r=1}^{\infty} = \{ g_F \}_{F \in \mathcal{F} \setminus \{ I \}} \) is \( \mathcal{F} \)-adapted and the functional energy inequality (4.1) with \( h = 1_I \) gives

\[
\sum_{r=1}^{\infty} \left( \frac{\mathbb{P}^\alpha (I_r, 1_I)}{|I_r|^{\frac{1}{n}}} \right) \sum_{J^* \in M(I_r)} \sum_{J \subset J^*} \hat{X}^\omega(J) a_J
\]

\[
= \sum_{F \in \mathcal{F}} \sum_{J^* \in M(F)} \mathbb{P}^\alpha(J^*, h) \left| \left( \frac{x}{|J^*|^{\frac{1}{n}}} g_F 1_{J^*} \right) \right| \omega
\]

\[
\leq \tilde{s}_\alpha \| h \|_{L^2(\sigma)} \left( \sum_{F \in \mathcal{F}} \| g_F \|_{L^2(\omega)}^2 \right)^{1/2}
\]

\[
\lesssim \tilde{s}_\alpha \sqrt{|I|_{\sigma}} \left( \sum_{J \in \tilde{M}(I)} |a_J|^2 \right)^{1/2}.
\]

Duality now yields

\[
\sum_{r=1}^{\infty} \left( \frac{\mathbb{P}^\alpha (I_r, 1_I)}{|I_r|^{\frac{1}{n}}} \right)^2 \sum_{J^* \in M(I_r)} \sum_{J \subset J^*} \hat{X}^\omega(J) a_J^2 \lesssim \tilde{s}_\alpha^2 |I|_{\sigma}.
\]

Combining this with the easy estimate

\[
\sum_{r=1}^{\infty} \left( \frac{\mathbb{P}^\alpha (I_r, 1_I)}{|I_r|^{\frac{1}{n}}} \right)^2 \sum_{J \subset I_r : |J|^{\frac{1}{n}} \geq 2^{-r} |I_r|^{\frac{1}{n}}} \hat{X}^\omega(J) \lesssim \sum_{r=1}^{\infty} \left( \frac{\mathbb{P}^\alpha (I_r, 1_I)}{|I_r|^{\frac{1}{n}}} \right)^2 |I_r|_{\omega} \lesssim A_2^2 |I|_{\sigma}
\]

gives \( E_\alpha \lesssim \tilde{s}_\alpha + \sqrt{A_2} \). \hfill \Box

5. Completion of the proof of the Main Theorem

Now we briefly describe how to apply the one-dimensional arguments from [LaSaShUr2] and [Lac] to complete the proof of Theorem 1. Details can be found in Chapters 8-10 of our expanded version of this paper [SaShUr] on the arXiv.
Very broadly speaking, the method in [LaSaShUr3] and [Lac] uses the NTV splitting of an appropriate bilinear form into Paraproduct, Neighbour and Stopping terms.

The Paraproduct term uses only cube testing.

The Neighbour term further splits into Short range, Middle range and Long range terms. The Short and Middle range terms use only the tailless $A^2_2$ condition, while the Long range term uses the one-tailed ‘reproducing’ condition $A^2_2$, and so its proof requires some modification. But these modifications are straightforward and the reader can find the details in Chapter 8 of [SaShUr].

The Stopping term is further split into a Near term and a Far term. Our paper [LaSaShUr2] or [LaSaShUr3] shows that the Far term is controlled by the functional energy condition as defined there, and it is easy to check that our definition of functional energy in this paper works just as well in generalizing the arguments to higher dimension. Note that our definition is more restrictive in that points (2) and (3) in our Definition 5 of $\mathcal{F}$-adapted are stronger requirements than those in the definition given in [LaSaShUr2], but nevertheless apply in all situations encountered. By one of the main results in this paper, the functional energy condition is in turn controlled by the energy condition and the $A^2_2$ condition. Finally, the Near term is handled by Lacey in [Lac] using a stopping time and recursion that carry over to all standard $\alpha$-fractional singular integrals in higher dimension. A detailed proof of this extension can be found in Chapter 10 of [SaShUr].

6. Failure of reverse energy

While the Monotonicity Lemma as stated earlier cannot be reversed, we now turn to the question of whether or not there exists a reversal of the dual energy inequality in Corollary [1] for certain vector transforms $T^{\alpha,\alpha}$. It turns out not to be possible for the vector of $\alpha$-fractional Riesz transforms $R^{\alpha,\alpha}$ in dimension $n \geq 2$, with an even more spectacular failure in the plane, which we present in Lemma 9 below.

6.1. The reverse energy inequality. Recall the energy $E(J, \omega)$ of $\omega$ on a cube $J$,

$$E(J, \omega)^2 = \frac{1}{|J|} \frac{1}{|\omega|} \int_J \int_J \frac{|x - z|^2}{|J|^2} \, dz \, dw(\omega) = 2 \frac{1}{|J|} \int_J \frac{x - \bar{E}_J x}{|J|^2} \, dw(\omega).$$

Define its associated coordinate energies $E^j(J, \omega)$ by

$$E^j(J, \omega)^2 = \frac{1}{|J|} \frac{1}{|\omega|} \int_J \int_J \frac{|x^j - z_j|^2}{|J|^2} \, dz \, dw(\omega), \quad j = 1, 2, \ldots, n,$$

and the rotations $E^j_R(J, \omega)$ of the coordinate energies by a rotation $R \in SO(n)$, which we refer to as partial energies,

$$E^j_R(J, \omega)^2 = \frac{1}{|J|} \frac{1}{|\omega|} \int_J \int_J \frac{|x^j_R - z^j_R|^2}{|J|^2} \, dz \, dw(\omega), \quad j = 1, 2, \ldots, n,$$

where for $R \in SO(n)$, $x_R = (x^j_R)_{j=1}^n = R(x^j)_{j=1}^n = Rx$. Set $E_R(J, \omega)^2 = E^1_R(J, \omega)^2 + \ldots + E^n_R(J, \omega)^2$. We have the following elementary computations.
Lemma 8. For $R \in SO(n)$ we have
$$E_R(J, \omega)^2 = E^1_R(J, \omega)^2 + \ldots + E^n_R(J, \omega)^2 = E(J, \omega)^2.$$ 
More generally, if $R = \{R_j\}_{j=1}^n \subset SO(n)$ is a collection of rotations such that the matrix $M_R = \begin{bmatrix} R_1 e^1 \\ \vdots \\ R_n e^1 \end{bmatrix}$ with rows $R_\ell e^1$ is nonsingular, then
$$E(J, \omega)^2 \leq \frac{1}{\epsilon_R} \sum_{\ell=1}^n E^1_{R_\ell}(J, \omega)^2,$$
where $\epsilon_R$ is the least eigenvalue of $M_R^* M_R$.

Proof. We have
$$|x^1_R - z^1_R|^2 + \ldots + |x^n_R - z^n_R|^2 = |R(x - z)|^2 = |x - z|^2 = |x^1 - z^1|^2 + \ldots + |x^n - z^n|^2,$$
so that
$$E_R(J, \omega)^2 = E^1_R(J, \omega)^2 + \ldots + E^n_R(J, \omega)^2 = E(J, \omega)^2.$$ 
More generally, if $M_{R_\ell}$ denotes the $\ell$th row of the matrix $M_R$, we have
$$\epsilon_R |x - z|^2 \leq (x - z)^{tr} M_{R_\ell}^* M_R (x - z) \leq \sum_{\ell=1}^n |R_\ell e^1 \cdot (x - z)|^2,$$
so that
$$\epsilon_R E(J, \omega)^2 = \left( \frac{1}{|J|} \right)^2 \int_J \int_J \epsilon_R |x - z|^2 d\omega(x) d\omega(z) \leq \left( \frac{1}{|J|} \right)^2 \int_J \int_J \left\{ \sum_{\ell=1}^n |R_\ell e^1 \cdot (x - z)|^2 \right\} d\omega(x) d\omega(z) \leq \sum_{\ell=1}^n E^1_{R_\ell}(J, \omega)^2.$$ 

The point of the estimate (6.1) is that it could hopefully be used to help obtain a reversal of energy for a vector transform $T^{n,\alpha} = \{T^{n,\alpha}_\ell\}_{\ell=1}^n$, where the convolution kernel $K^{n,\alpha}_\ell(w)$ of the operator $T^{n,\alpha}_\ell$ has the form
$$K^{n,\alpha}_\ell(w) = \frac{\Omega^{n,\alpha}_\ell \left( \frac{w}{|w|} \right)}{|w|^{n-\alpha}},$$
and where $\Omega^{n,\alpha}_\ell$ is smooth on the sphere $S^{n-1}$. We refer to the operator $T^{n,\alpha}_\ell$ as an $\alpha$-fractional convolution Calderón-Zygmund operator. If in addition we require that $\Omega^{n,\alpha}_\ell$ has vanishing integral on the sphere $S^{n-1}$, we refer to $T^{n,\alpha}_\ell$ as a classical $\alpha$-fractional Calderón-Zygmund operator.
However, we now dash this hope, at least for the most familiar singular operators in the plane, in a spectacular way. We show that strong reversal of energy fails for all convolution Calderón-Zygmund operators in the plane, and that when $\alpha = 1$, weak reversal of energy fails for all classical Calderón-Zygmund operators in the plane. The following definitions will make this precise.

We say that a vector $T^\alpha = \{T^\alpha_\ell\}_{\ell=1}^2$ of $\alpha$-fractional transforms in the plane satisfies a strong reversal of $\omega$-energy on a cube $J$ if there is a positive constant $C_0$ such that for all $\gamma > 2$ sufficiently large and for all positive measures $\mu$ supported outside $\gamma J$, we have the inequality

\[
 E(J, \omega)^2 P^\alpha (J, \mu)^2 \leq C_0 E_J^\omega |T^\alpha \mu(x) - T^\alpha \mu(z)|^2 .
\]

The right hand side of (6.3) is clearly dominated by $C_0 E_J^\omega |T^\alpha \mu|^2$, and so we say that $T^\alpha = \{T^\alpha_\ell\}_{\ell=1}^2$ satisfies a weak reversal of $\omega$-energy on a cube $J$ if for $\gamma$ and $\mu$ as above, we have the weaker inequality

\[
 E(J, \omega)^2 P^\alpha (J, \mu)^2 \leq C_0 E_J^\omega |T^\alpha \mu|^2 .
\]

We thank M. Lacey for pointing out the example $\mu = d\theta$ on the circle and $\omega$ is supported near the origin, which demonstrates the failure of energy reversal for the Cauchy transform.

**Lemma 9** (Failure of Reverse Energy). (1) Suppose that $J$ is a square in the plane $\mathbb{R}^2$ and $\gamma > 2$. Suppose also that $0 \leq \alpha < 2$ and that $T^\alpha = \{T^\alpha_\ell\}_{\ell=1}^2$ is a vector of $\alpha$-fractional convolution Calderón-Zygmund operators with kernels $K^\alpha_\ell (w) = \Omega_\ell (\frac{|w|}{|w|^\alpha})$. Finally suppose that $C_0 > 0$ is given. Then for $\gamma$ sufficiently large, there exists a positive measure $\mu = \mu_{\alpha, \gamma}$ on $\mathbb{R}^2$ supported outside $\gamma J$ and depending only on $\alpha$ and $\gamma$, such that for all differentiable choices of $\Omega_\ell$, the strong reversal of energy inequality (6.3) fails.

(2) Let $0 \leq \alpha < 2$ and suppose that $T^\alpha$ is a single classical Calderón-Zygmund operator with kernel $K^\alpha (w) = \Omega (\frac{|w|}{|w|^\alpha})$. Then for any square $J$ and $\gamma > 2$ sufficiently large, there is a positive measure $\mu = \mu_{\alpha, \gamma, T^\alpha}$ on $\mathbb{R}^2$ supported outside $\gamma J$ and depending only on $\alpha$, $\gamma$ and $T^\alpha$, such that the weak reversal of energy inequality (6.4) fails.

**Proof.** Let $\varepsilon > 0$. We have

\[
 T^\alpha_\ell \mu(x) = \int K^\alpha_\ell(x, y) d\mu(y) = \int \frac{\Omega_\ell (y - x)}{|y - x|^{2-\alpha}} d\mu(y) = \int \left\{ \frac{1}{|y - c_J|^{2-\alpha}} + (x - c_J) \cdot \nabla_x \left( \frac{1}{|y - x|^{\alpha}} \right) \right|_{x=c_J} \times [\Omega_\ell (y - c_J)] \right\} d\mu(y) + E_x,
\]
and so
\[ T_\alpha^\ell \mu (x) - T_\alpha^\ell \mu (z) = \int \left[ (x - z) \cdot \nabla_x \left( \frac{\Omega_\ell (y - x)}{|y - x|^{d-\alpha}} \right) \right] |x = c_j \] \[ d\mu (y) + \left[ E_\alpha^\ell,x - E_\alpha^\ell,z \right] \]
\[ \equiv \Lambda_\alpha^\ell + \left[ E_\alpha^\ell,x - E_\alpha^\ell,z \right], \]
where if \( \gamma > 2 \) is sufficiently large,
\[ |E_\alpha^\ell,x - E_\alpha^\ell,z| \leq C \frac{\bar{P}_\alpha (J, \mu)}{|J|^2} |x - z| \leq \varepsilon \frac{\bar{P}_\alpha (J, \mu)}{|J|^2} |x - z|. \]

The point of this inequality (6.5) is that it permits the replacement of the difference \( T_\alpha^\ell \mu (x) - T_\alpha^\ell \mu (z) \) in (6.3) by the linear part \( \Lambda_\alpha^\ell \) of the Taylor expansion of the kernel \( K_\alpha^\ell \).

Now we make the choice
\[ \Omega_\ell (w) = \Omega (\theta_\ell (w)); \]
\[ \theta_\ell (w) \equiv \tan^{-1} \frac{\omega^{\ell'}}{\omega^\ell}, \quad 1 \leq \ell \leq 2, \]
where \( \omega^{\ell'} \) denotes the coordinate variable other than \( \omega^\ell \), i.e. \( \ell + \ell' = 3 \), and calculate the gradient of the convolution kernel
\[ K_\alpha^\ell (w) = \frac{\Omega_\ell (w)}{|w|^{2-\alpha}} = \frac{\Omega (\theta_\ell (w))}{|w|^{2-\alpha}} = \frac{\Omega \left( \tan^{-1} \frac{\omega^{\ell'}}{\omega^\ell} \right)}{|w|^{2-\alpha}}, \]
using the temporary notation \( w = (w^\ell, w^{\ell'}) \) in which we write the \( \ell \)th variable first. We then have
\[ \nabla_w \tan^{-1} \frac{\omega^{\ell'}}{\omega^\ell} = \left( \frac{\partial}{\partial w^\ell} \tan^{-1} \frac{\omega^{\ell'}}{\omega^\ell}, \nabla_w \tan^{-1} \frac{\omega^{\ell'}}{\omega^\ell} \right) \]
\[ = \left( \frac{1}{1 + \left( \frac{\omega^{\ell'}}{\omega^\ell} \right)^2 \omega^\ell \omega^{\ell'}}, \frac{1}{1 + \left( \frac{\omega^{\ell'}}{\omega^\ell} \right)^2 \omega^\ell \omega^{\ell'}} \right) \]
\[ = \frac{1}{|w|^2} (-\omega^{\ell'}, \omega^\ell), \]
and
\[ \nabla |w|^{\alpha-2} = \left( \frac{\partial}{\partial w^1} \left( (w^1)^2 + (w^2)^2 \right)^{\frac{\alpha-2}{2}}, \frac{\partial}{\partial w^2} \left( (w^1)^2 + (w^2)^2 \right)^{\frac{\alpha-2}{2}} \right) \]
\[ = \frac{\alpha - 2}{2} \left( (w^1)^2 + (w^2)^2 \right)^{\frac{\alpha-2}{2}} - 1 w \]
\[ = (\alpha - 2) |w|^{\alpha-4} w. \]

Remark 8. In previous versions of this shortened paper, we mistakenly used the negative of the right hand side of this last equality, and with disastrous consequences.
Thus we obtain with $\theta_\ell (w) = \tan^{-1} \frac{w^\ell}{w^\ell}$,
\[
\nabla K_\ell^\alpha (w) = \nabla \left( \frac{\Omega_\ell (w)}{|w|^{1-\alpha}} \right) = \Omega_\ell (w) \nabla |w|^{\alpha-2} + |w|^{\alpha-2} \Omega_\ell (w) \nabla \theta_\ell \\
= \Omega_\ell (\theta_\ell (w)) (\alpha - 2) |w|^{\alpha-4} w + |w|^{\alpha-2} \Omega_\ell (\theta_\ell (w)) \frac{1}{|w|^2} (-w^\ell, w^\ell) \\
= \frac{(\alpha - 2) \Omega_\ell (\theta_\ell (w))}{|w|^{\alpha-\ell}} \cdot w + \Omega_\ell (\theta_\ell (w)) \cdot w^\perp,
\]
where $w \equiv (w^\ell, w^\ell)$ and $w^\perp \equiv (-w^\ell, w^\ell)$. Now we compute that the linear part $\Lambda^\alpha_\ell$ is given by
\[
\Lambda^\alpha_\ell = (x - z) \cdot \int \nabla K_\ell^\alpha (c_J) \, d\mu (y) \equiv (x - z) \cdot Z^\alpha_{\Omega_\ell} (c_J),
\]
where
\[
Z^\alpha_{\Omega_\ell} (c_J) = \int_{\mathbb{R}^2} (\alpha - 2) \Omega (\theta_\ell (y - c_J)) \frac{(y - c_J) + \Omega_\ell (\theta_\ell (y - c_J)) (y - c_J)}{|y - c_J|^{1-\alpha}} d\mu (y) \\
= \int_{S^1} \left\{ (\alpha - 2) \Omega (\theta_\ell (w)) \frac{w^\ell}{|w|} - \Omega_\ell (\theta_\ell (w)) \frac{w^\ell}{|w|} \right\} \, e^\ell \, d\Psi_\mu \\
+ \int_{S^1} \left\{ (\alpha - 2) \Omega (\theta_\ell (w)) \frac{w^\ell}{|w|} + \Omega_\ell (\theta_\ell (w)) \frac{w^\ell}{|w|} \right\} \, e^\ell \, d\Psi_\mu,
\]
and $e^\ell$ is the coordinate vector with a 1 in the $\ell$th position. We have now reverted to the usual ordering of the components of $w = (w^1, w^2)$ in which $w^1$ occurs first. Here $\Psi_\mu$ is an essentially arbitrary positive finite measure on the circle $S^1$ given formally by
\[
\frac{d\Psi_\mu}{d\theta} (\xi) = \int_0^\infty r^{\alpha-3} d\mu_\xi (r) = \int_0^\infty r^{\alpha-3} d\mu (r\xi), \quad \xi = \frac{w}{|w|} = (\cos \theta_\ell, \sin \theta_\ell) \in S^1.
\]
Now we use
\[
\tan \theta_\ell (w) = \frac{w^\ell}{w^\ell}, \\
\csc \theta_\ell (w) = \sqrt{1 + \cot^2 \theta_\ell (w)} = \sqrt{1 + \left( \frac{w^\ell}{w^\ell} \right)^2} = \frac{|w|}{w^\ell}, \\
\sin \theta_\ell (w) = \frac{w^\ell}{|w|} \text{ and } \cos \theta_\ell (w) = \frac{w^\ell}{|w|},
\]
to obtain
\[
Z^\alpha_{\Omega_\ell} (c_J) = \int_{S^1} \left\{ (\alpha - 2) \Omega (\theta_\ell (w)) \cos \theta_\ell (w) - \Omega_\ell (\theta_\ell (w)) \sin \theta_\ell (w) \right\} e^\ell \, d\Psi_\mu \\
+ \int_{S^1} \int \left\{ (\alpha - 2) \Omega (\theta_\ell (w)) \sin \theta_\ell (w) + \Omega_\ell (\theta_\ell (w)) \cos \theta_\ell (w) \right\} e^\ell \, d\Psi_\mu \\
\equiv \int_{S^1} \left\{ A_{2,\alpha} (\theta_\ell (w)) e^\ell + B_{2,\alpha} (\theta_\ell (w)) e^\ell \right\} \, d\Psi_\mu.
\]
Now we show that reversal of energy on $J$ is ‘essentially’ equivalent to linear independence of the vectors $Z_{\Omega \ell}^\alpha (c_J)$ and $Z_{\Omega 2}^\alpha (c_J)$. More precisely, assume for the moment that $\left| Z_{\Omega \ell}^\alpha \right| \geq \frac{P^\alpha (J, \mu)}{|J|^2}$ for $1 \leq \ell \leq 2$, and that there is a positive number $\eta_\alpha > 0$, depending only on $\alpha$, such that the pair of vectors $\{Z_{\Omega \ell}^\alpha (c_J)\}_{\ell=1}^2$ satisfies the matrix inequality

\begin{equation}
\begin{bmatrix}
Z_{\Omega 1}^\alpha \\
Z_{\Omega 2}^\alpha
\end{bmatrix}^* \begin{bmatrix}
Z_{\Omega 1}^\alpha \\
Z_{\Omega 2}^\alpha
\end{bmatrix} \geq \eta_\alpha I,
\end{equation}

where $A \succeq B$ means that $A - B$ is nonnegative semidefinite. From (6.3), (6.6) and Lemma 8, we would then immediately obtain inequality (6.3) if we choose $\alpha \ell$ so that $\eta_\alpha, \eta_\eta \geq \frac{P^\alpha (J, \mu)}{2}$.

Conversely, if (6.6) fails, then $Z_{\Omega \ell}^\alpha (c_J)$ is nonnegative semidefinite. From (6.5), (6.6) and Lemma 8, we would then immediately obtain inequality (6.3) if we choose $\varepsilon > 0$ sufficiently small, is at least

\begin{equation}
\begin{aligned}
\int \int |T^\alpha \mu (x) - T^\alpha \mu (z)|^2 d\omega (x) d\omega (z) \\
= \sum_{\ell=1}^2 \int \int |(x - z) \cdot Z_{\Omega \ell}^\alpha (c_J) + [E_{\Omega 1}^\alpha (x) - E_{\Omega 2}^\alpha (z)]|^2 d\omega (x) d\omega (z) \\
\geq \sum_{\ell=1}^2 \int \int \frac{P^\alpha (J, \mu)}{|J|^2} |x - z| \cdot \frac{|Z_{\Omega \ell}^\alpha (c_J) (c_J)|^2}{Z_{\Omega \ell}^\alpha (c_J) (c_J)} d\omega (x) d\omega (z) \\
- C \sum_{\ell=1}^2 \int \int \varepsilon \frac{P^\alpha (J, \mu)}{|J|^2} |x - z|^2 d\omega (x) d\omega (z),
\end{aligned}
\end{equation}

which for $\varepsilon > 0$ sufficiently small, is at least

\begin{equation}
\begin{aligned}
P^\alpha (J, \mu)^2 \left\{ \sum_{\ell=1}^2 \frac{J_\omega^2}{|J|^2} E_{\Omega \ell}^2 (J, \omega) \right\} - C\varepsilon^2 P^\alpha (J, \mu)^2 \frac{J_\omega^2}{|J|^2} E (J, \omega)^2 \\
\geq (\eta_\alpha - C\varepsilon^2) P^\alpha (J, \mu)^2 \frac{J_\omega^2}{|J|^2} E (J, \omega)^2 \geq \frac{1}{2} \eta_\alpha \frac{J_\omega^2}{|J|^2} E (J, \omega)^2 P^\alpha (J, \mu)^2.
\end{aligned}
\end{equation}

Conversely, if (6.6) fails, then $Z_{\Omega \ell}^\alpha (c_J)$ are parallel, and we can only reverse a partial energy at best, not the full energy. So it suffices to show the failure of (6.6).

Taking $\ell = 1$, setting $\theta = \theta_1$ and dropping some subscripts we obtain

$$A_\alpha (\theta) \equiv (\alpha - 2) \Omega (\theta) \cos \theta - \Omega' (\theta) \sin \theta,$$

$$B_\alpha (\theta) \equiv (\alpha - 2) \Omega (\theta) \sin \theta + \Omega' (\theta) \cos \theta.$$

Now in the case $\alpha = 1$ these coefficients are perfect derivatives,

$$A_1 (\theta) = -\Omega (\theta) \cos \theta - \Omega' (\theta) \sin \theta = -|\Omega (\theta) \sin \theta|,'$$

$$B_1 (\theta) = -\Omega (\theta) \sin \theta + \Omega' (\theta) \cos \theta = -|\Omega (\theta) \cos \theta|. $$
and so have vanishing integral on the circle. Thus with the choice \( d\Psi_\mu (\theta) = d\theta \) we have

\[
Z_\Omega (c_J) = \int_{S^1} \{ A_1 (\theta) e^1 + B_1 (\theta) e^2 \} \, d\theta = 0
\]

the zero vector, for every choice of differentiable \( \Omega \) on the circle.

In the more general case \( 0 < \alpha < 2 \), we can take \( d\Psi_\mu (\theta) = \eta_\alpha (\theta) \, d\theta \) with density \( \eta_\alpha (\theta) \) so that

\[
Z_\alpha^2 (c_J) = \int_{S^1} \{ A_\alpha (\theta) e^1 + B_\alpha (\theta) e^2 \} \eta_\alpha (\theta) \, d\theta.
\]

Now the \( e^1 \) component of \( Z_\alpha^2 (c_J) \) will vanish for all choices of differentiable \( \Omega \) provided we have, using integration by parts on the term involving \( \Omega' (\theta) \),

\[
\int_{S^1} A_\alpha (\theta) \eta_\alpha (\theta) \, d\theta = - \int_{S^1} \{ (2 - \alpha) \Omega (\theta) \cos \theta + \Omega' (\theta) \sin \theta \} \eta_\alpha (\theta) \, d\theta = 0.
\]

This will occur for all differentiable \( \Omega \) exactly when \( \eta_\alpha \) satisfies the equation

\[
0 = (1 - \alpha) \cos \theta \eta_\alpha (\theta) - \sin \theta \eta'_\alpha (\theta);
\]

\[
\frac{\eta'_\alpha (\theta)}{\eta_\alpha (\theta)} = (1 - \alpha) \frac{\cos \theta}{\sin \theta};
\]

\[
\ln |\eta_\alpha (\theta)| = (1 - \alpha) \ln |\sin \theta|;
\]

which gives

\[
\eta_\alpha (\theta) = C |\sin \theta|^{1-\alpha},
\]

a locally integrable density for all \( 0 < \alpha < 2 \).

Similarly the \( e^2 \) component of \( Z_{\Omega^2, \alpha} (c_J) \) will vanish for all choices of differentiable \( \Omega \) provided we have, using integration by parts on the term involving \( \Omega' (\theta) \),

\[
\int_{S^1} B_\alpha (\theta) \eta_\alpha (\theta) \, d\theta = - \int_{S^1} \{ (\alpha - 2) \Omega (\theta) \sin \theta + \Omega' (\theta) \cos \theta \} \eta_\alpha (\theta) \, d\theta = 0.
\]
which will occur if \( \eta_\alpha \) satisfies the equation

\[
0 = (\alpha - 1) \sin \theta \eta_\alpha (\theta) - \cos \theta \eta'_\alpha (\theta) ;
\]

\[
\frac{\eta'_\alpha (\theta)}{\eta_\alpha (\theta)} = (\alpha - 1) \frac{\sin \theta}{\cos \theta} ;
\]

\[
\ln |\eta_\alpha (\theta)| = (\alpha - 1) \ln |\cos \theta| ;
\]

which gives

\[
\eta_\alpha (\theta) = C |\cos \theta|^{\alpha - 1} ,
\]

a locally integrable density for all \( 0 < \alpha < 2 \).

These two densities are of course different when \( \alpha \neq 1 \), and so we cannot have a measure \( d\Psi_\mu \) for which the vector \( Z^\alpha_{\Omega_1} (c_J) \) vanishes for all differentiable \( \Omega \) when \( \alpha \neq 1 \).

However the vectors \( Z^\alpha_{\Omega_1} (c_J) \) and \( Z^\alpha_{\Omega_2} (c_J) \) can be parallel when \( \alpha \neq 1 \), i.e.

\[
\det \left[ Z^\alpha_{\Omega_1} (c_J), Z^\alpha_{\Omega_2} (c_J) \right] = 0.
\]

Indeed, we have

\[
\det \left[ Z^\alpha_{\Omega_1} (c_J), Z^\alpha_{\Omega_2} (c_J) \right] = \det \left[ \int_{S^1} A_\alpha (\theta_1) \eta_\alpha (\theta_1) \, d\theta_1 \int_{S^1} A_\alpha (\theta_2) \eta_\alpha (\theta_2) \, d\theta_2 \int_{S^1} B_\alpha (\theta_1) \eta_\alpha (\theta_1) \, d\theta_1 \int_{S^1} B_\alpha (\theta_2) \eta_\alpha (\theta_2) \, d\theta_2 \right] ,
\]

which is the determinant of the matrix whose two columns are

\[
\left( -\int_{S^1} \Omega (\theta_1) \{ (1 - \alpha) \cos \theta_1 \eta_\alpha (\theta_1) - \sin \theta_1 \eta'_\alpha (\theta_1) \} \, d\theta_1 \right);
\]

and

\[
\left( -\int_{S^1} \Omega (\theta_2) \{ (1 - \alpha) \cos \theta_2 \eta_\alpha (\theta_2) - \sin \theta_2 \eta'_\alpha (\theta_2) \} \, d\theta_2 \right).
\]

Now \( \theta_2 = \theta_1 + \frac{\pi}{2} \) and so

\[
\cos \theta_2 = \cos \left( \theta_1 + \frac{\pi}{2} \right) = -\sin \theta_1 ,
\]

\[
\sin \theta_2 = \sin \left( \theta_1 + \frac{\pi}{2} \right) = \cos \theta_1 ,
\]

and writing \( \theta \) for \( \theta_1 \) once again, we obtain that the above matrix is

\[
\left[ \int_{S^1} \Omega (\theta) \{ (\alpha - 1) \cos \theta \eta_\alpha (\theta) + \sin \theta \eta'_\alpha (\theta) \} \, d\theta - \int_{S^1} \Omega (\theta) \{ (\alpha - 1) \sin \theta \eta_\alpha (\theta) - \cos \theta \eta'_\alpha (\theta) \} \, d\theta \right].
\]

This matrix has determinant

\[
\left[ \int_{S^1} \Omega (\theta) \{ (\alpha - 1) \cos \theta \eta_\alpha (\theta) + \sin \theta \eta'_\alpha (\theta) \} \, d\theta \right]^2 - \left[ \int_{S^1} \Omega (\theta) \{ (\alpha - 1) \sin \theta \eta_\alpha (\theta) - \cos \theta \eta'_\alpha (\theta) \} \, d\theta \right]^2 ,
\]

which vanishes if and only if

\[
\int_{S^1} \Omega (\theta) \{ (\alpha - 1) \cos \theta \eta_\alpha (\theta) + \sin \theta \eta'_\alpha (\theta) \} \, d\theta = \pm \int_{S^1} \Omega (\theta) \{ (\alpha - 1) \sin \theta \eta_\alpha (\theta) - \cos \theta \eta'_\alpha (\theta) \} \, d\theta .
\]
Taking the plus sign in (6.7), we get that equality holds for all differentiable $\Omega$ if and only if
\[
(\alpha - 1) \cos \theta \eta_\alpha (\theta) + \sin \theta = (\alpha - 1) \sin \theta \eta_\alpha (\theta) - \cos \theta \eta'_\alpha (\theta);
\]
\[
(1 - \alpha) (\cos \theta - \sin \theta) \eta_\alpha (\theta) = (\cos \theta + \sin \theta) \eta'_\alpha (\theta);
\]
\[
\frac{\eta'_\alpha (\theta)}{\eta_\alpha (\theta)} = (1 - \alpha) \frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} = (1 - \alpha) \frac{1 - \tan \theta}{1 + \tan \theta};
\]
\[
\ln |\eta_\alpha (\theta)| = (1 - \alpha) \int \frac{1 - \tan \theta}{1 + \tan \theta} d\theta = \frac{1 - \alpha}{2} \ln (2 \sin 2\theta + 2),
\]
which gives
\[
\eta_\alpha (\theta) = C (1 + \sin 2\theta)^{\frac{1 - \alpha}{2}}.
\]
This is a locally integrable density with singularity at $\theta = -\frac{\pi}{4}$ and results in
\[
Z^{(\alpha)} (c J) = \left( \begin{array} {c} A_\alpha \\ A_\alpha \end{array} \right) = A_\alpha \left( \begin{array} {c} 1 \\ 1 \end{array} \right);
\]
\[
A_\alpha = \int_{S^2} \Omega (\theta) \left\{ (\alpha - 1) \cos \theta \eta_\alpha (\theta) + \sin \theta \eta'_\alpha (\theta) \right\} d\theta,
\]
where since $\eta_\alpha (\theta) = C (1 + \sin 2\theta)^{\frac{1 - \alpha}{2}}$ is not the special weight $\eta_\alpha (\theta) = C |\sin \theta|^{1 - \alpha}$, there is $\Omega$ such that $A_\alpha \neq 0$. However, in this case $Z_{\Omega_2^{2,\alpha}} (c J)$ is parallel to $Z_{\Omega_1^{2,\alpha}} (c J)$, and so we can only reverse the partial energy $E_{\omega}^{2,\alpha} (J, \omega)$ at best, and not the full energy $E (J, \omega)$. Similar considerations apply when taking the minus sign in (6.7) above.

So for each $\alpha$, there is a density $d\Psi_\mu (\theta) = \eta_\alpha (\theta) d\theta$ and a measure $\omega$ such that for $\gamma$ so large that $\epsilon \ll C_0$, the strong reversal of $\omega$-energy inequality (6.3) fails for all differentiable $\Omega$ with this single measure $\mu$, even without the assumption of vanishing integral $\int_{S^1} \Omega (\theta) = 0$. Indeed, we have that
\[
\int_{S^1} \left[ T^{\alpha} \mu (x) - T^{\alpha} \mu (z) \right]^2 d\omega (x) d\omega (z)
\]
\[
= \sum_{\ell = 1}^{2} \int_{S^1} \int_{S^1} \left[ (x - z) \cdot Z_{\Omega_\ell^{2,\alpha}} (c J) + \left[ E_{\ell,x}^{\alpha} - E_{\ell,z}^{\alpha} \right] \right]^2 d\omega (x) d\omega (z)
\]
\[
\leq \sum_{\ell = 1}^{2} \int_{S^1} \int_{S^1} \left\{ \frac{P^{\alpha} (J, \mu)}{|J|^{\frac{1}{2}}} (x - z) \cdot \frac{Z_{\Omega_\ell^{2,\alpha}} (c J) (c J)}{|Z_{\Omega_\ell^{2,\alpha}} (c J) (c J)|} \right\}^2 d\omega (x) d\omega (z)
\]
\[
+ C \sum_{\ell = 1}^{2} \int_{S^1} \int_{S^1} \left\{ \frac{P^{\alpha} (J, \mu)}{|J|^{\frac{1}{2}}} |x - z| \right\}^2 d\omega (x) d\omega (z)
\]
\[
\leq E_{\omega}^{1,\frac{1}{2}} (J, \omega)^2 P^{\alpha} (J, \mu)^2 + C \epsilon^2 E (J, \omega)^2 P^{\alpha} (J, \mu)^2
\]
\[
\leq \frac{1}{10} C_0 E (J, \omega)^2 P^{\alpha} (J, \mu)^2,
\]
provided we choose $\gamma$ so large that $C \epsilon^2 \leq \frac{1}{10} C_0$ and provided we choose $\omega$ so that $E_{\omega}^{1,\frac{1}{2}} (J, \omega) = 0$ but $E_{\omega}^{1,\frac{1}{2}} (J, \omega) > 0$. This completes the proof of part (1) of Lemma 9.

To show the failure of the weak reversal of energy inequality (6.4) in part (2), we exploit the assumption that $\int_{S^1} \Omega (\theta) = 0$ together with the following observation.
We have
\[
\mathbb{E}_x^L \mu^s = \int_{\mathbb{R}^2} \left\{ \frac{1}{|J|} \int_{J} \frac{\Omega(y-x)}{|y-x|^{2-\alpha}} d\omega(x) \right\} d\mu(y)
\]
\[
= \int_{\xi \in \mathbb{S}^1} \Omega(\xi) \left\{ \frac{1}{|J|} \int_{(x,y) \in J \times \mathbb{R}^2} \frac{d\omega(x) d\mu(y)}{|\xi|^{2-\alpha}} \right\} d\theta
\]
\[
= \int_{\mathbb{S}^1} \Omega(\theta) d\Phi_\mu(\theta),
\]
where
\[
\frac{d\Phi_\mu}{d\theta}(\xi) = \frac{1}{|J|} \int_{(x,y) \in J \times \mathbb{R}^2} \frac{d\omega(x) d\mu(y)}{|y-x|^{2-\alpha}}, \quad \xi = (\cos \theta, \sin \theta) \in \mathbb{S}^1.
\]

We will now apply a transformation to $\mu$ that moves its mass along rays away from $cJ$, but leaves the density $\frac{d\Phi_\mu}{d\theta}$ invariant. Given a function $\varphi : \mathbb{S}^1 \to [1, \infty)$, define the measure $\mu$ in the plane by (we ignore the issue of measurability)
\[
d\tilde{\mu}(y) \equiv \varphi \left( \frac{y-c_J}{|y-c_J|} \right)^{3-\alpha} d\mu \left( c_J + \frac{y-c_J}{\varphi(\frac{y-c_J}{|y-c_J|})} \right),
\]
so that for $\xi \in \mathbb{S}^1$,
\[
\frac{d\Psi_{\tilde{\mu}}}{d\theta}(\xi) = \int_0^\infty r^{\alpha-3} d\tilde{\mu}(r\xi) = \int_0^\infty r^{\alpha-3} \varphi(\xi)^{3-\alpha} d\mu \left( \frac{r\xi}{\varphi(\xi)} \right)
\]
\[
= \int_0^\infty s\varphi(\xi)^{\alpha-3} \varphi(\xi)^{3-\alpha} d\mu(s\xi) = \frac{d\Psi_\mu}{d\theta}(\xi).
\]

Now we compute
\[
\int_{\mathbb{S}^1} \Omega(\theta) d\Phi_\mu(\theta) = \int_{\{\Omega(\theta) > 0\}} \Omega(\theta) d\Phi_\mu(\theta) + \int_{\{\Omega(\theta) < 0\}} \Omega(\theta) d\Phi_\mu(\theta)
\]
and if this integral does not already vanish, then we may assume without loss of generality that it is negative. Pick an arc $K$ with $\gamma K$ contained in the set $P_\delta = \{\theta : \Omega(\theta) > \delta\}$ for some $\delta > 0$. We now apply a transformation of the above type to $\mu$ with
\[
\varphi(\theta) = \begin{cases} 1 & \text{if } \theta \notin P_\delta \\ M & \text{if } \theta \in P_\delta \end{cases},
\]
and where $M \geq 1$. From the definition of $\frac{d\Psi_\mu}{d\theta}$, and the change of variable
\[
y' = c_J + \frac{y-c_J}{\varphi(\frac{y-c_J}{|y-c_J|})},
\]
we have
\[
\frac{y'-c_J}{|y'-c_J|} = \frac{\varphi \left( \frac{y-c_J}{|y-c_J|} \right)}{\varphi \left( \frac{y-c_J}{|y-c_J|} \right)} = \frac{y-c_J}{|y-c_J|},
\]
and so
\[
\frac{d\Phi_{\bar{\mu}}}{d\theta}(\xi) = \frac{1}{|J|_\omega} \int_{(x,y) \in J \times \mathbb{R}^2} \frac{d\omega(x) \, d\bar{\mu}(y)}{|y-x|^{2-\alpha}} \]
\[
= \frac{1}{|J|_\omega} \int_{(x,y) \in J \times \mathbb{R}^2} \frac{\varphi(y-cJ)}{|y|^{\alpha}} \frac{d\omega(x) \, d\mu(y')}{|y-x|^{2-\alpha}} \]
\[
= \frac{1}{|J|_\omega} \int_{(x,y') \in J \times \mathbb{R}^2} \frac{\varphi(y'-cJ)}{|y'|^{\alpha}} \frac{d\omega(x) \, d\mu(y')}{|y'-x|^{2-\alpha}}.
\]

It is clear from this formula that for \( \theta \in K \), only the values of \( \frac{d\Phi_\mu}{d\theta} \) on \( \gamma K \) are modified by the transformation, and since \( \gamma K \subset P_\delta \) with \( \delta > 0 \), we conclude that for \( \theta \in K \), we have \( \lim_{M \to \infty} \frac{d\Phi_\mu}{d\theta}(\theta) = \infty \). Thus there is a choice of \( M \) such that the integral of \( \Omega(\theta) \, d\Phi_\mu(\theta) \) over \( \{ \Omega(\theta) > 0 \} \) equals \( -
\int_{\{\Omega(\theta) > 0\}} \Omega(\theta) \, d\Phi_\mu(\theta) \).

Then for the resulting measure \( \bar{\mu} \), the density \( \frac{d\Phi_\bar{\mu}}{d\theta} = \frac{d\Phi_\mu}{d\theta} \) remains unchanged, and the density \( \frac{d\Phi_\bar{\mu}}{d\theta} \) satisfies
\[
\mathbb{E}_ J T^\alpha \bar{\mu} = \int_{S^1} \Omega(\theta) \, d\Phi_{\bar{\mu}}(\theta) = 0.
\]

Then starting with \( d\Phi_\mu(\theta) = \eta_\alpha(\theta) \, d\theta \) as above, we conclude that \( T^\alpha \) and the transformed measure \( \bar{\mu} \) fail the strong reversal of energy inequality (6.3), and also that \( \mathbb{E}_ J T^\alpha \bar{\mu} = 0 \). Combining these two facts and taking \( \gamma \) sufficiently large gives the failure of the weak weak reversal of energy inequality (6.4) for \( \bar{\mu} \). Indeed, we have
\[
\int J \int J |T^\alpha \bar{\mu}(x)|^2 \, d\omega(x) \, d\omega(z) = \frac{1}{2} \int J \int J |T^\alpha \bar{\mu}(x) - T^\alpha \bar{\mu}(z)|^2 \, d\omega(x) \, d\omega(z),
\]
and we can now apply inequality (6.8) with \( \bar{\mu} \) in place of \( \mu \). This completes the proof of part (2) of Lemma [3].

\[\square\]

**Corollary 2** (of the proof). For every rotation \( R_\phi \) of the circle by angle \( \phi \), the density \( \eta_\alpha^\phi(\theta) = \eta_\alpha(\theta - \phi) = C (1 + \sin 2(\theta - \phi))^{\frac{1+\alpha}{2}} \) is such that
\[
Z_{R_\phi \Omega^\alpha_1}(cJ) = Z_{R_\phi \Omega^\alpha_2}(cJ) = AR_\phi \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

Thus every unit vector can occur as a singular direction for the energy.

**Remark 9.** The argument in the proof of Lemma [3] above can be modified to show that weak energy reversal fails for the \( \alpha \)-fractional Riesz transform vector \( R^{2,\alpha} \) in \( \mathbb{R}^2 \), for all \( 0 \leq \alpha < 2 \). The idea is to construct a measure \( \bar{\mu} \) as in the proof above with \( T^\alpha = R^{2,\alpha} \), but where
\[
\mathbb{E}_ J R^{2,\alpha}_J \bar{\mu} = \int_{S^1} \Omega(\theta) \, d\Phi_{\bar{\mu}}(\theta) = 0,
\]
for both $\ell = 1$ and $\ell = 2$. This can be achieved by choosing an appropriate pair of disjoint arcs $K_1$ and $K_2$ and using some simple algebra, and the details are left to the interested reader. In the case $\alpha = 1$, the density $\eta (\theta)$ is constant, and it is easy to find a measure $\omega$ with $E_{\omega} J R_1 \mu = 0$ for $\ell = 1, 2$ and $E (J, \omega) > 0$, and an example of this form was communicated to us by M. Lacey. There are also extensions for other operators to dimension $n \geq 3$, but due to the complexity of algebra involved, we will not pursue them here.

6.1.1. Weak reversal of energy. Finally, we point out that we do have weak reversal of energy \((6.4)\) for the usual (positive) fractional integral operator

$$I^{n,\alpha} f (x) = \int_{\mathbb{R}^n} \frac{1}{|y - x|^{n-\alpha}} f (y) \, dy,$$

i.e.

$$E (J, \omega)^2 P^{\alpha} (J, \mu)^2 \leq C_{n,\alpha,\gamma} E_{\omega}^d |I^{n,\alpha} \mu|^2.$$

When $\alpha > 0$, the operator $I^{n,\alpha}$ is an example of an $\alpha$-fractional singular integral $T^{\alpha}$ as defined above, since $I^{n,\alpha}$ maps unweighted $L^p$ to unweighted $L^q$ for $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

To see the weak reversal of energy for $I^{n,\alpha}$ we note that

$$\left( \int_{\mathbb{R}^n} \frac{|J|^{\frac{1}{p}}}{(|J|^{\frac{1}{p}} + |J|^\frac{1}{q} + |y|)^{\alpha + 1 - \alpha}} \, d\mu (y) \right)^2 \leq \left( \int_{\mathbb{R}^n} \frac{1}{(|J|^\frac{1}{p} + |y|)^{n-\alpha}} \, d\mu (y) \right)^2 \approx \left( \frac{1}{|J|} \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} \frac{1}{|y - x|^{n-\alpha}} \, d\mu (y) \right] \, d\omega (x) \right)^2 \leq \frac{1}{|J|} \sum_{\omega} E_{\omega}^d (|I^{n,\alpha} \mu|^2).$$

6.2. Necessity of the energy conditions. The purpose of this final subsection is to show that in those situations where we have weak reversal of energy for an $\alpha$-fractional vector transform $T^{n,\alpha}$ in $\mathbb{R}^n$, then the energy conditions are indeed necessary. Of course the previous subsection shows this fails for the class of operators of greatest interest, namely the $\alpha$-fractional Riesz transform vector $R^{n,\alpha}$ in $\mathbb{R}^n$, and for classical Calderón-Zygmund operators in the plane.

Standing assumption for this subsection: We assume for the remainder of this subsection that the reverse energy inequality \((9)\) holds for some finite collection $T^{n,\alpha}$ of Calderón-Zygmund operators.

We emphasize that in the standing assumption, we do not assume that the operators $T^{n,\alpha}$ are of convolution type, nor that there are any Lebesgue measure cancellation conditions on the kernel. The case when $T^{n,\alpha}$ is the fractional integral $I^{\alpha}$ provides a trivial example in which this standing assumption holds.

We first define a preliminary energy constant $E^{\text{weak}}_{\alpha}$ and its dual $E^{\text{weak}*}_{\alpha}$ in higher dimensions. Provided $0 \leq \alpha < 1$, we prove they are controlled by $A^2_2$ and the cube
testing conditions for $L^{n,\alpha}$. The proofs are a relatively straightforward application of the properties obtained earlier.

We say that a collection $\{I_r\}_{r=1}^{\infty}$ of cubes is a subpartition of a cube $I$ if $I_r \subset I$ and $I_r \cap I_{r'} = \emptyset$ for $r \neq r'$. Now let $\{I_r\}_{r=1}^{\infty}$ be a subpartition of a cube $I$. For each $I_r \in \mathcal{D}$ let $\mathcal{M}(I_r)$ consist of the maximal deeply embedded subcubes $J$ of $I_r$.

Furthermore, let $\gamma > 1$ be such that the expanded cubes $J_r^{**} \equiv \gamma J_r^*$ for $J_r^* \in \mathcal{M}(I_r)$ satisfy a bounded overlap condition

$$\sum_{J_r^{**} \in \mathcal{M}(I_r)} 1_{J_r^{**}} \lesssim \beta 1_{I_{hr}}, \quad 1 \leq r < \infty,$$

for some constant $\beta$.

**Definition 7.** Define the weak energy constant by

$$(\mathcal{E}^{\text{weak}}_{\alpha})^2 \equiv \sup \frac{1}{|I|} \sum_{r=1}^{\infty} \sum_{J_r \in \mathcal{M}(I_r)} \left( \frac{P_{\alpha} (J_r^{**}, 1_{I_r \setminus J_r^{**}})}{|J_r^{**}|^{\frac{1}{2}}} \right)^2 \|P_{J_r} x\|_{L^2(\omega)}^2,$$

where notation is as above, and the supremum is taken over

1. all dyadic grids $\mathcal{D}$,
2. all $D$-dyadic cubes $I$,
3. and where $\{I_r\}_{r=1}^{\infty}$ equals $\mathcal{M}(I)$.

There is a similar definition for the backward weak energy constant $\mathcal{E}^{\text{weak}}_{\alpha}$. Note that our decomposition of $I_r$ into the collection $\mathcal{M}(I_r)$ of maximal deeply embedded subcubes is uniquely determined by $I_r$, and the decomposition of $I$ into $\{I_r\}_{r=1}^{\infty} = \mathcal{M}(I)$ is uniquely determined by $I$. Finally, we recall that using (3.3), the reverse energy inequality (9),

$$\mathbb{E} (J, \omega) \overset{\text{P}_{\alpha} (J, \mu)}{\lesssim} C_{T_{\alpha}} T_{\alpha}^* |T^{n,\alpha} \mu(x) - T^{n,\alpha} \mu(z)|^2$$

implies the weak form (6.4):

$$(6.9) \quad \|P_{J} x\|_{L^2(\omega)}^2 \left( \frac{P_{\alpha} (J, \mu)}{|J|^{\frac{1}{2}}} \right)^2 \lesssim C_{T_{\alpha}} \int |T^{n,\alpha} \mu(x)|^2 d\omega(x),$$

which is the form needed in the following lemma.

**Lemma 10.** Let $0 \leq \alpha < 1$. We have

$$\mathcal{E}^{\text{weak}}_{\alpha} \lesssim T_{\alpha}^{n,\alpha}.$$
Proof. Using (39) with \( \mu = 1_{I \setminus J_r^* \cdot \sigma} \), we then ‘plug the hole’ in \( I \setminus J_r^* \) to obtain

\[
\sum_{r = 1}^{\infty} \sum_{J_r^* \in \mathcal{M}(I_r)} \left( \frac{P^\alpha (J_r^* \cdot 1_{I \setminus J_r^* \cdot \sigma})}{|J_r^*|^\frac{1}{n}} \right)^2 \left\| P_{J_r^*} x \right\|_{L^2(\omega)}^2
\]

\[
\lesssim \sum_{r = 1}^{\infty} \sum_{J_r^* \in \mathcal{M}(I_r)} \int_{J_r^*} |T^{n, \alpha} 1 \cdot 1_{I \setminus J_r^* \cdot \sigma}|^2 d\omega
\]

\[
\lesssim \sum_{r = 1}^{\infty} \sum_{J_r^* \in \mathcal{M}(I_r)} \int_{J_r^*} |T^{n, \alpha} 1 \cdot 1_{I \setminus J_r^* \cdot \sigma}|^2 d\omega + \sum_{r = 1}^{\infty} \sum_{J_r^* \in \mathcal{M}(I_r)} \int_{J_r^*} |T^{n, \alpha} 1 \cdot 1_{I \setminus J_r^* \cdot \sigma}|^2 d\omega
\]

\[
\lesssim \int_I |T^{n, \alpha} 1 \cdot 1_{I \setminus J_r^* \cdot \sigma}|^2 d\omega + \sum_{r = 1}^{\infty} \sum_{J_r^* \in \mathcal{M}(I_r)} \int_{J_r^*} |T^{n, \alpha} 1 \cdot 1_{I \setminus J_r^* \cdot \sigma}|^2 d\omega
\]

\[
\lesssim \mathcal{T} n, \alpha |I|_\sigma + \sum_{r = 1}^{\infty} \sum_{J_r^* \in \mathcal{M}(I_r)} \mathcal{T} n, \alpha |J_r^*|_\sigma \lesssim \mathcal{T} n, \alpha |I|_\sigma ,
\]

upon using the bounded overlap property of the squares \( \{J_r^*\} \) \( J_r^* \in \mathcal{M}(I_r) \) and \( r \geq 1 \). This completes the proof of Lemma \( \text{III} \).

6.2.1. Necessity of the energy condition. Now we define an extension to higher dimensions of the one-dimensional energy constants \( \mathcal{E}^\alpha \) and \( \mathcal{E}^{\alpha, \ast} \). This will involve the smaller projection \( \tilde{P}_K^\omega \), introduced in the introduction, that satisfies

\[
\left\| \tilde{P}_K^\omega x \right\|_{L^2(\mu)}^2 = \sum_{a \in \Gamma_n} \sum_{J \in \mathcal{H}(K)} \left| \langle \tilde{x}, \tilde{h}_{J}^{\omega, a} \rangle_\omega \right|^2 \approx \sum_{J \in \mathcal{H}(K)} \tilde{X}^{\omega}(J)^2 ,
\]

where

\[
\mathcal{H}(K) \equiv \left\{ J \in \mathcal{D} : J \subset K \text{ and either } |J|^\frac{1}{n} \geq 2^{-r} |K|^\frac{1}{n} \text{ or } J \in K \right\} .
\]

Provided \( 0 \leq \alpha < 1 \), we prove the energy constants \( \mathcal{E}^\alpha \) and \( \mathcal{E}^{\alpha, \ast} \) are controlled by \( A_3^\alpha \) and cube testing for admissible local transforms. Recall that a collection \( \{I_r\}_{r = 1}^{\infty} \) of cubes is a subpartition of a cube \( I \) if \( I_r \subset I \) and \( I_r \cap I_r' = \emptyset \) for \( r \neq r' \).

**Definition 8.** Define the energy constant

\[
\mathcal{E}^\alpha_2 \equiv \sup_{I \cdot \sigma} \frac{1}{|I|_\sigma} \sum_{r = 1}^{\infty} \left( \frac{|I|_\sigma}{|I|^\frac{1}{n}} \right)^2 \sum_{J \in \mathcal{H}(I_r)} \tilde{X}^{\omega}(J)^2 ,
\]

where the supremum is taken over

\[
(1) \text{ all dyadic grids } \mathcal{D},
(2) \text{ all } \mathcal{D} \text{-dyadic cubes } I,
(3) \text{ and all subpartitions } \{I_r\}_{r = 1}^{\infty} \text{ of the interval } I \text{ into } \mathcal{D} \text{-dyadic subcubes } I_r.
\]

There is a similar definition for the backward energy constant \( \mathcal{E}^\alpha_{-} \). Recall that the classical tailless \( A_3^\alpha \) condition,

\[
A_3^\alpha \equiv \sup_{Q} |Q|^{2(\frac{n}{n} - 1)} |Q|_\omega |Q|_\sigma < \infty ,
\]

satisfies \( A_3^\alpha \lesssim A_3^\alpha \). The basic result proved here is this.
Lemma 11. Let $0 \leq \alpha < 1$. We have the energy condition,

$$\mathcal{E}_\alpha \lesssim \mathcal{X}_{T^n, \alpha} + \sqrt{A_2^\alpha}.$$ 

Proof. We have

$$\frac{1}{|I|} \sum_{r=1}^{\infty} \left( \frac{P^\alpha(I_r, 1_{I_r})}{|I_r|^{\frac{\alpha}{2}}} \right)^2 \sum_{J \in \mathcal{H}(I_r)} \hat{X}^\omega(J)^2 \lesssim \frac{1}{|I|} \sum_{r=1}^{\infty} \left( \frac{P^\alpha(I_r, 1_{I_r})}{|I_r|^{\frac{\alpha}{2}}} \right)^2 \sum_{J \in \mathcal{H}(I_r)} \hat{X}^\omega(J)^2
\quad + \frac{1}{|I|} \sum_{r=1}^{\infty} \left( \frac{P^\alpha(I_r, 1_{I_r})}{|I_r|^{\frac{\alpha}{2}}} \right)^2 \left( \sum_{J \in \mathcal{H}(I_r)} \hat{X}^\omega(J)^2 \right)
\quad \equiv A + B,$$

where

$$B \lesssim \frac{1}{|I|} \sum_{r=1}^{\infty} \left( \frac{|I_r|}{|I_r|^{\frac{\alpha}{2}} |I_r|^\frac{\alpha}{2}} \right)^2 \left( |I_r|^\frac{\alpha}{2} |I_r|^\omega \right) \lesssim A_2^{\omega} \frac{1}{|I|} \sum_{r=1}^{\infty} |I_r| \lesssim A_2^{\alpha}.$$

Now from Lemma 10 we obtain

$$A = \frac{1}{|I|} \sum_{r=1}^{\infty} \sum_{J^* \in \mathcal{M}(I_r)} \left( \frac{P^\alpha(J^*, 1_{J^*})}{|J^*|^{\frac{\alpha}{2}}} \right)^2 \left( \sum_{J \subseteq J^*} \hat{X}^\omega(J)^2 \right)
\quad + \frac{1}{|I|} \sum_{r=1}^{\infty} \left( \frac{P^\alpha(J^*, 1_{J^*})}{|J^*|^{\frac{\alpha}{2}}} \right)^2 \left( \sum_{J \subseteq J^*} \hat{X}^\omega(J)^2 \right)
\quad \lesssim \frac{1}{|I|} \sum_{r=1}^{\infty} \sum_{J^* \in \mathcal{M}(I_r)} \left( \frac{P^\alpha(J^*, 1_{J^*})}{|J^*|^{\frac{\alpha}{2}}} \right)^2 \left( \sum_{J \subseteq J^*} \hat{X}^\omega(J)^2 \right)
\quad + \frac{1}{|I|} \sum_{r=1}^{\infty} \left( \frac{P^\alpha(J^*, 1_{J^*})}{|J^*|^{\frac{\alpha}{2}}} \right)^2 C |I_r|^\omega
\quad \lesssim \left( E_{\alpha}^{\text{weak}} \right)^2 + A_2^{\omega} \lesssim \left( \mathcal{X}_{T^n, \alpha} \right)^2 + A_2^{\omega},$$

since \( \frac{P^\alpha(J^*, 1_{J^*})}{|J^*|^{\frac{\alpha}{2}}} \approx \frac{P^\alpha(\gamma J^*, 1_{J^*})}{|\gamma J^*|^{\frac{\alpha}{2}}} \). The estimate in the middle line above requires

$$\frac{P^\alpha(I_r, 1_{I_r})}{|I_r|^{\frac{\alpha}{2}}} \lesssim \frac{P^\alpha(J^*, 1_{J^*})}{|J^*|^{\frac{\alpha}{2}}}, \quad J^* \in \mathcal{M}(I).$$

However, if $B \subset A$, then using

$$\frac{P^\alpha(K, 1_{A\setminus B})}{|K|^{\frac{\alpha}{2}}} = \int_{A \setminus B} \frac{1}{(|K|^{\frac{\alpha}{2}} + |y - c_K|^{\alpha})^{n+1-\alpha} \sigma(y)},$$
and the fact that $|y - c_K| \geq |K|^{\frac{1}{n}}$ if $K = J^* \in \mathcal{M}(B)$ or $K = B$, we conclude that

\[
\frac{P^\alpha(B, 1_{A \setminus B} \sigma)}{|B|^\frac{1}{n}} \leq \int_{A \setminus B} \frac{1}{|y - c_B|^{n+1-\alpha}} d\sigma(y) \leq \int_{A \setminus B} \frac{1}{|y - c_{J^*}|^{n+1-\alpha}} d\sigma(y) \leq \frac{P^\alpha(J^*, 1_{A \setminus B} \sigma)}{|J^*|^\frac{1}{n}},
\]

does indeed hold for $J^* \in \mathcal{M}(B)$. □

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