DSR Theories, Conformal Group and Generalized Commutation Relation

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Abstract. In this paper the relationship of DSR theories and Conformal Group is reviewed. On the other hand, the relation between DSR Magueijo Smolin generators and generalized commutation relations is also shown.

1. Introduction

It has been claimed that the special relativity must be modified, that a Lorentz symmetry breaking could be observable in some experimental tests in the near future as in high energy cosmic ray spectra experiences [1,2]. On the other hand, more than breaking, corrections in the dispersion relation $E^2 - p^2 = m^2$ have been proposed instead [1]. Furthermore, quantum gravity models suggest that it could be desirable to review the Lorentz invariance relations and string theories consider some modifications to the very structure of the space time at high energy scales. Some data seem to invite to introduce a minimal length in physical theories, indeed, there are theories that have some fundamental quantities: the Planck longitude $l_p = \sqrt{\hbar G/c^3}$, its associated time scale $t_p = l_p/c$ and the Planck energy $E_p = \hbar/t_p$. It is supposed that beyond these thresholds, the physics should change dramatically. However, these absolute values of longitude, time or energy are not in agreement with the Lorentz transformations, this point reinforces the idea of modifying the Lorentz symmetry.

Solutions for the problem on how to modify the Lorentz boosts, have been proposed. In particular, a very interesting solution is Doubly (or Deformed, after some authors), Special relativity (DSR) theories [3,4,5]. These theories are based on a generalization of Lorentz transformations through the more broad point of view of conformal transformations, they have two observed independent scales: velocity of light and Planck length.

DSR theories are of increasing interest because they can be useful as a new tool in gravity theories, in Cosmology as an alternative to inflation [6,7], and in other fields like propagation of light [8], that is related, for instance, to cosmic microwave background radiation.

More specifically, it is conjectured that the deformations in momentum (the Magueijo Smolin formulation), can be understood as linear combination of conformal generators, but the inclusion of a new generator is needed. This new generator can be obtained from the same theory, and it completes the set of symmetries of the massless Klein Gordon equation. Then, a DSR massless particle is shown to be isomorphic to a normal Lorentz particle living in a $d+2$ space. Furthermore the inclusion of Magueijo Smolin momentum generates a modified algebra similar to the one proposed by Kempf et al. and L.N. Chang.
2. Extended Non-linear Conformal Algebra

It is possible to introduce a new generator to the usual Conformal Group: \( \tilde{K}_\mu \):

\[
\tilde{K}_\mu = 2p_\mu xp - p^2 x_\mu
\]  

(1)

The nature of this generator is totally new because it produces conformal transformations on the momentum space and the finite version of this transformations is:

\[
p'_\mu = -p_\mu - \alpha_\mu p^2 \left(1 - \frac{2}{1 - 2\alpha p + \alpha^2 p^2}\right)
\]  

(2)

This is is a symmetry of the massless Klein-Gordon equation since: \( \{ \tilde{K}, P^2 \} = 2P^2 P_\mu = 0 \).

So, after the inclusion of \( \tilde{K} \), aside the usual relations we have:

\[
\{ M_{\mu\nu}, \tilde{K}_\lambda \} = \eta_{\mu\lambda} \tilde{K}_\nu - \eta_{\nu\lambda} \tilde{K}_\mu, \{ D, \tilde{K}_\mu \} = \tilde{K}_\mu,
\]

\[
\{ \tilde{K}_\mu, P_\nu \} = 2P_\mu P_\nu - P^2 \eta_{\mu\nu},
\]

\[
\{ \tilde{K}_\mu, \tilde{K}_\nu \} = 0,
\]

\[
\{ \tilde{K}_\mu, K_\nu \} = 4(M_{\mu\nu} - D_\eta_{\mu\nu})D - 2P_\mu K_\nu + \frac{2(2DP_\mu - \tilde{K}_\mu)(2DP_\nu - \tilde{K}_\nu)}{KP}
\]

Due to the last bracket, the set of generators form a non linear algebra. Conformal Algebra is by its own non-linear due to the nonlinear (quadratic) transformations in \( x_i \) generated by the special conformal generator. Furthermore, the inclusion of \( \tilde{K} \) generates a bigger algebra that is non-linear because the brackets can not be written as linear combinations of the generators, but as linear combinations of products of generators.

The non-linear feature of the extended Conformal Algebra seems not to be important in order to obtain the DSR generators, but it is worth noting that the inclusion of \( \tilde{K} \) modifies the algebra nature.

Now, we have two separate set of symmetries. Each set form a dynamic \( SO(2,1) \) symmetry of the Klein Gordon equation: one that includes \( K_\mu \), generating special conformal transformations in the position space and the other that includes \( \tilde{K}_\mu \), generating special conformal transformations in the momentum space.

The existence of \( \tilde{K}_\mu \) could produce some consequences in the symmetry group of non relativistic particles since there is a canonical relation between a non relativistic particle in \( R^4 \) and the relativistic one in \( R^{1,1} \) [9]. This can be of some interest because Fluid Mechanics symmetries are intimately related to the non relativistic free particle symmetry [10].

Each separated group of \( SO(2,1) \) symmetries can be used to construct the DSR transformations in position or momentum space.

3. The Magueijo Smolin DSR momentum transformation

Starting with the ordinary Lorentz generators:

\[ L_{ab} = p_a \frac{\partial}{\partial p^b} - p_b \frac{\partial}{\partial p^a}, \]

(3)

the modified Lorentz boost proposed by Magueijo and Smolin [?] is:

\[ K_i = L_{0i} + l_p p_i p_\mu \frac{\partial}{\partial p_\mu}, \]

(4)

where the second term is the deformation proposed and \( l_p \) is the Planck length. It can be exponentiated as:
\[
K^i = U^{-1}(p_0)L_0U(p_0)
\]

where \( U(p_0) = \exp(l_p p_0 p_\mu \frac{\partial}{\partial p_\mu}) \), and the action of \( U(p_0) \) over \( p_\mu \) is

\[
U(p_0)p_\mu = \frac{p_\mu}{1 - l_p p_0}
\]

Following the Maguejo Smolin procedure it can be seen that, for example, boosts in the z direction are:

\[
p'_0 = \frac{\gamma(p_0 - v p_z)}{1 + l_p (\gamma - 1)p_0 - l_p \gamma v p_z}
\]

\[
p'_z = \frac{\gamma(p_z - v p_0)}{1 + l_p (\gamma - 1)p_0 - l_p \gamma v p_z}
\]

\[
p'_x = \frac{p_x}{1 + l_p (\gamma - 1)p_0 - l_p \gamma v p_z}
\]

\[
p'_y = \frac{p_y}{1 + l_p (\gamma - 1)p_0 - l_p \gamma v p_z}
\]

This transformations are identical to those obtained by Fock [11,12], but applied to momentum space. They can be retrieved replacing \( p \) by \( x \) and \( p_0 \) by \( t \), instead. The Fock like transformations can also be obtained as usual Lorentz transformations for the transformed \( p' \) from eq 6 [11,12], so we need deal just with the extra term of the deformed boosts.

The transformation (6) can be yielded as result of the action of an \( \tilde{K}_\mu = 2p_\mu (x p) - p^2 x_\mu \) generator, that produces:

\[
p'^\mu = -\frac{p^\mu - \alpha^\mu p^2}{1 - 2\alpha p + \alpha^2 p^2}
\]

the identification should be performed on the surface \( p^2 = 0 \).

So, the Maguejo Smolin transformation can be seen as a combination of

\[-M_{0\mu} + \tilde{K}_\mu,
\]

Then, we can obtain the Maguejo Smolin transformations for a particle living on the light cone through the projection of the action of a linear combination of conformal generators. But in order to achieve the identification with (6), it is necessary to transform the constrain \( \psi_1 = p^2 \approx 0 \) in an exact identity, the transformation parameter must be \( \alpha^\mu = (l_p/2,0,0,0) \) and the global minus sign must be removed.

4. DSR and generalized commutation relations

It is possible to find a relationship between noncommutativity, deformed algebra and DSR theories. It is possible to see that that deformed algebra in the way that Kempf et al. and L.N. Chang. present [13,14,15,16,17,18] , can be seen as a first order approximation of some DSR theories. Effectively, if the commutator between the position operator and the Maguejo-Smolin momentum operator \( \pi \) is calculated, we can obtain:

\[
[x_i,p_j] = i\delta_{ij}, \quad i,j = 1,\ldots n
\]

\[
\pi_j = \frac{p_j}{(1 - l_p p^2)},
\]
\[ [x_i, \pi_j] = i \frac{\delta_{ij}}{(1 - l_p \lambda^2)} + i \frac{2l_p p_i p_j}{(1 - l_p \lambda^2)^2}. \] (8)

and using the definition of \( \pi_j \):

\[ [x_i, \pi_j] = 2i \delta_{ij} l_p f + 2il_p \pi_i \pi_j \] (9)

where

\[ f = \frac{\pi^2}{\sqrt{1 + 4l_p \pi^2} - 1} \] (10)

at first order gives:

\[ [x_i, \pi_j] \approx i \delta_{ij}(1 + \ell_p \pi^2) + 2i\ell_p \pi_i \pi_j \] (11)

and that is exactly the relation proposed by Kempf et al. and L.N. Chang.

On the other hand, using a very similar treatment to the one performed by Leiva to obtain the DSR generators, J. Romero and A. Zamora [19] obtained the Snyder commutation relations. This is a new clue that all these theories have a very important underlying relationship.

5. Discussion and outlook

To conclude, let us summarize the obtained results and discuss shortly some problems that deserve a further attention.

Magueijo Smolin DSR theories can be presented as linear combination of Conformal Group generators. On the other hand the very fruitful activity about generalized commutation relations can be see as first order theories of deformed momenta presented by Magueijo Smolin.

There are some problems that deserve more attention: The problem of how to include the mass term in this formulation, this is a hard problem because introducing a mass scale the Conformal Symmetry is broken. To extend the studies to fields formulation could be a interesting task that can be solved by the experience of Yamaguchi about harmonic oscillator with minimal length.

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