Wilson Fermions on a Transverse Lattice

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I. INTRODUCTION

There are both phenomenological as well as theoretical reasons to study light-front (LF) quantization: degrees of freedom that are close to the relevant degrees of freedom in many high energy scattering experiments and the simplified vacuum structure. For recent reviews on these topics see Refs. [1-3].

One approach that seems particularly promising towards performing numerical calculations of hadron bound states in QCD_{3+1} is the transverse (⊥) lattice [4].

For our purposes, the most crucial difference between covariant field theories and LF field theories is the fact that LF field theories allow for a chirally invariant mass terms

\[ S = S_{\text{naive}} + \delta S = S_{\text{naive}} + a^2 \sum_{n_x,n_y} \int dx^+ dx^- \delta L_{n_x,n_y}, \]

where \( \delta L_{n_x,n_y} \) is the lattice approximation to \(-\alpha \bar{\psi} \gamma^+ \gamma^\perp \partial^2 \psi\). One might thus hope that chiral fermions can be formulated on a transverse lattice without the notorious species doubling problem [5]. The purpose of this paper is to give an explicit example which shows that this hope is premature. In fact, it will be shown that, after renormalization, one recovers the standard Wilson term in the interaction.

In order to keep the complexity of the equations at a minimum, the lattice regularization is first explained on a ⊥ lattice in 2+1 dimensions. This has the advantage that one does not need to specify the direction in which one does not need to specify the direction in which the transverse derivative acts. After extending the formulation to 3+1 dimensions, we proceed to apply Wilson fermions to a simple ⊥ lattice model for mesons.

II. FERMIONS ON A TRANSVERSE LATTICE

A. Naive Fermions

We start by rewriting the Lagrangian for a free fermion (2+1 dimensions, 4-spinors for the fermions) in terms of the LF-projections \( \phi \equiv \bar{x} \gamma^p \psi \) and \( \chi \equiv \gamma^\perp \gamma^\perp \psi \)

\[ \mathcal{L} = \bar{\psi} (i \beta - m) \psi \]
\[ = \sqrt{2} \phi^\dagger i \partial_+ \phi^\dagger - \phi^\dagger (i \alpha_x \partial_x + \beta m) \phi - \chi^\dagger (i \alpha_x \partial_x + \beta m) \phi, \]

where \( \alpha_x = \gamma^0 \gamma_x \) and \( \beta = \gamma^0 \). Upon discretizing the transverse (\( x \to n \)) coordinate and replacing transverse derivatives by finite differences in Eq. (2.1) one thus finds the discretized action for (naive) free fermions on a transverse lattice

\[ S_{\text{naive}} = a \sum_n \int dx^+ dx^- \mathcal{L}_n, \]

where (the dependence on the continuous longitudinal coordinates \( x^\perp \) is suppressed for notational convenience)

\[ \mathcal{L}_n = \sqrt{2} \phi^\dagger_n i \partial_+ \phi_n + \sqrt{2} \chi^\dagger_n i \partial_- \chi_n - \phi^\dagger_n \beta m \phi_n - \chi^\dagger_n i \alpha_x \frac{\chi_{n+1} - \chi_{n-1}}{2a} - \chi^\dagger_n i \alpha_x \frac{\phi_{n+1} - \phi_{n-1}}{2a}. \]

In fact, proper renormalization requires them as counter-terms.
Since Eq. \(2.3\) does not involve a “time” \((x^+)\) derivative of \(\chi\), it is convenient to use the constraint equation
\[
\sqrt{2i} \partial_- \chi_n = \beta m \phi_n + i \alpha_x \frac{\phi_{n+1} - \phi_{n-1}}{2a}
\]
(2.4)
to eliminate \(\chi\) prior to quantization, yielding the effective action for the dynamical \((\phi)\) degrees of freedom
\[
S^\phi_{\text{naive}} = a \sum_n \int dx^+ dx^- \mathcal{L}^\phi_n,
\]
(2.5)
where
\[
\mathcal{L}^\phi_n = \sqrt{2} \phi_n^\dagger i \partial_+ \phi_n - \phi_n^\dagger \frac{m^2}{i \sqrt{2} \partial_-} \phi_n - \frac{\phi_n^\dagger - \phi_{n-1}^\dagger}{2a} \frac{1}{i \sqrt{2} \partial_-} \frac{\phi_{n+1} - \phi_{n-1}}{2a}
\]
(2.6)
A simple plane wave ansatz shows that the above naive Lagrangian results in a dispersion relation which exhibits species doubling
\[
k^- = \frac{m^2 + \sin^2(k_x a)}{2k^+}
\]
(2.7)
where \(-\frac{\pi}{a} < k_x < \frac{\pi}{a}\). Thus, even for \(a \to 0\), modes with \(k_x \approx \pi/a\) have finite energy. Of course, this has nothing to do with the LF approach chosen here, but is just the familiar phenomenon of species doubling for Dirac particles (first order derivatives!) on a lattice.

The species doubling problem on a \(\perp\) lattice has been dealt with in Ref. \([a]\) using a generalization of the familiar Kogut-Susskind fermions \([b]\). However, in this paper, we will try different approaches, which are based on generalizations of Wilson fermions \([c]\).

### B. Ordinary Wilson Fermions

The most familiar “patch” to deal with species doubling is to start from the naive action, plus a “Wilson r-term” of the form \([d]\)
\[
\delta \mathcal{L}_n = \frac{r}{a} \bar{\psi}_n [\psi_{n+1} - 2 \psi_n + \psi_{n-1}]
\]
(2.8)
\[
= \frac{r}{a} \left\{ \phi_n^\dagger \beta [\chi_{n+1} - 2 \chi_n + \chi_{n-1}] + \chi_n^\dagger \beta [\phi_{n+1} - 2 \phi_n + \phi_{n-1}] \right\}
\]
where \(r = \mathcal{O}(1)\). The well known problem with the Wilson r-term is that it has to have the Dirac structure of a mass term in order to eliminate “doublers” both for particles and anti-particles and therefore chiral symmetry is explicitly broken.

Including this Wilson r-term \((2.8)\), the constraint equation for \(\chi\) reads
\[
\sqrt{2i} \partial_- \chi_n = \beta m \phi_n + i \alpha_x \frac{\phi_{n+1} - \phi_{n-1}}{2a} - \frac{\beta}{a} [\phi_{n+1} - 2 \phi_n + \phi_{n-1}],
\]
(2.9)
yielding for the effective Lagrangian of the dynamical degrees of freedom \((\phi)\)
\[
\delta \mathcal{L}^\phi_n = \frac{2mr}{a} \phi_n^\dagger \frac{1}{i \sqrt{2} \partial_-} \frac{[\phi_{n+1} - 2 \phi_n + \phi_{n-1}]}{a} - \frac{[\phi_{n+1} - 2 \phi_n + \phi_{n-1}]}{a} r^2 \frac{\phi_n^\dagger - \phi_{n-1}^\dagger}{i \sqrt{2} \partial_-} \frac{\phi_{n+1} - 2 \phi_n + \phi_{n-1}}{a}.
\]
(2.10)
The dispersion relation, with the r-term included, reads
\[
k^- = \frac{m + 2a}{a} (1 - \cos k_x a) + \frac{\sin^2 k_x a}{2k^+}
\]
(2.11)
and is, by construction, free of species doubling at \(k_x \to \pm \pi/a\).

### C. Gauging Ordinary Wilson Fermions

An action that reduces to the QCD coupling of fermions to transverse gluons in the naive continuum limit is obtained by introducing link fields \(U_n\), defined on the links that connect the site \(n\) with \(n + 1\), using the rule
\[
\bar{\psi}_n \psi_{n+1} \longrightarrow \bar{\psi}_n U_n \psi_{n+1} + \bar{\psi}_{n+1} \psi_n U_n^\dagger \psi_n.
\]
(2.12)
To keep the notation simple, we suppressed the dependence of \(U_n\) on \(x^-\). The coupling to the longitudinal gluon components is a separate issue and does not affect the main conclusions in this section. We will thus omit couplings to \(A^\pm\) here. Obviously, the resulting gauge invariance includes only gauge transformations that do not depend on \(x^\pm\). However, but if one wants to one can easily remedy this by introducing couplings to \(A^\pm\) which is what we will do in the model calculations in Section \([c]\).

One thus finds for the action including link fields
\[
\mathcal{L}_n = \sqrt{2} \phi_n^\dagger i \partial_+ \phi_n + \sqrt{2} \chi_n^\dagger i \partial_- \chi_n - \phi_n^\dagger \beta m \chi_n - \chi_n^\dagger \beta m \phi_n
\]
\[
- \phi_n^\dagger i \alpha_x \frac{U_n \chi_{n+1} - U_{n-1} \chi_{n-1}}{2a}
\]
\[
- \chi_n^\dagger i \alpha_x \frac{U_n^\dagger \phi_{n+1} - U_{n-1}^\dagger \phi_{n-1}}{2a}
\]
(2.13)
and
\[
\delta \mathcal{L}_n = \frac{r}{a} \bar{\psi}_n \left\{ U_n \psi_{n+1} - 2 \psi_n + U_{n-1}^\dagger \psi_{n-1} \right\}
\]
(2.14)
\[
= \frac{r}{a} \left\{ \phi_n^\dagger \beta \left[ U_n \chi_{n+1} - 2 \chi_n + U_{n-1}^\dagger \chi_{n-1} \right] + \chi_n^\dagger \beta \left[ U_n^\dagger \phi_{n+1} - 2 \phi_n + U_{n-1}^\dagger \phi_{n-1} \right] \right\}.
\]
After eliminating $\chi_n$ one thus finds
\[ \mathcal{L}_n^0 = \mathcal{L}_n^{\phi,0} + \mathcal{L}_n^{\phi,1} + \mathcal{L}_n^{\phi,2}, \] (2.15)
where
\[ \mathcal{L}_n^{\phi,0} = \sqrt{2}\phi_n^0 i\partial_+ \phi_n^0 - \phi_n^0 \frac{M^2}{\sqrt{2}a} \phi_n^0, \] (2.16)
with $M = m + 2r/a$, is the term containing no link fields and
\[ \mathcal{L}_n^{\phi,1} = \phi_n^1 \frac{M}{\sqrt{2}a} \left\{ \frac{U_n \phi_{n+1} + U_n^\dagger \phi_{n-1}}{a} \right\} + \beta i\alpha_x \left[ \frac{U_n \phi_{n+1} - U_n^\dagger \phi_{n-1}}{2a} \right] + h.c. \] (2.17)
is the term linear in the link fields. Once one renormalizes, the term quadratic in the link fields $[\mathcal{L}_n^{\phi,2}]$ is determined by the other two terms \[3,12\] and thus does not need to be specified for the point we are trying to make in this paper.

D. A Modified r-Term

In the context of LF quantization, it is natural not to write down the usual covariant Wilson $r$-term, but instead to modify only the kinetic mass term in the Lagrangian after $\chi$ has been eliminated, i.e. instead of Eq. (2.18) we introduce a modified $r$-term, which is formulated directly in terms of the dynamical degrees of freedom ($\phi_n$)
\[ \delta \mathcal{L}_n^0 = \phi_n^1 \frac{ar}{\sqrt{2}a} \frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{a^2}, \] (2.18)
which is the discretized version of $ar\bar{\psi} \frac{\gamma^+}{2\sqrt{2}a} \partial_x^2 \psi$. This modified $r$-term yields a dispersion relation
\[ k^- = m^2 + \frac{2m}{a} (1 - \cos k_x a) + \frac{\sin^2 k_x a}{a^2}, \] (2.19)
which is very similar to Eq. (2.11), but nevertheless does not exhibit species doubling. What makes this “modified $r$-term” especially attractive is that it is chirally invariant (on the LF — see Appendix). This feature can be best seen by re-expressing Eq. (2.18) in four component notation, yielding
\[ \delta \mathcal{L}_n^0 = ar\bar{\psi} \frac{\gamma^+}{2i\partial_-} \phi_{n+1} - 2\phi_n + \phi_{n-1}. \] (2.20)

At first, this seems to contradict the Nielsen-Ninomiya theorem \[13\], which states (under very general assumptions) that there is no chirally invariant way to solve the species doubling problem. However, the additional term [Eq. (2.18)] is non-local due to the inverse longitudinal derivative and thus the Nielsen-Ninomiya theorem does not apply.

In order to render Eq. (2.18) gauge invariant, we replace
\[ \phi_n^1 \frac{1}{i\sqrt{2}a} \phi_{n+1} \rightarrow \frac{1}{2} \phi_n^1 \left[ \frac{U_n}{i\sqrt{2}a} + \frac{1}{i\sqrt{2}a} U_n \right] \phi_{n+1} \]
\[ \phi_n^1 \frac{1}{i\sqrt{2}a} \phi_{n-1} \rightarrow \frac{1}{2} \phi_n^1 \left[ \frac{U_n}{i\sqrt{2}a} + \frac{1}{i\sqrt{2}a} U_n \right] \phi_{n-1} \]
(2.21)
and similarly for other non-local terms. Although the gauge invariant extension is not unique, Eq. (2.21) represents the minimal extension of Eq. (2.18) that is gauge invariant \[3\] hermitian and invariant under transverse parity (n → −n). Using this rule, we find the following terms contributing to the action based on this modified r-term
\[ \delta \mathcal{L}_n^0 = - \left( m^2 + \frac{2m}{a} \right) \phi_n^1 \frac{1}{i\sqrt{2}a} \phi_n, \] (2.22)
for the term containing no link fields as well as
\[ \delta \mathcal{L}_n^1 = \phi_n^1 \frac{1}{i\sqrt{2}a} \left\{ \frac{U_n \phi_{n+1} + U_n^\dagger \phi_{n-1}}{2a} \right\} \] (2.23)
for the term linear in the link field. As in the previous section, we do not exhibit the term quadratic in the link field since it follows uniquely in the renormalization procedure.

A comparison with the results from the previous section shows immediately that the naive r-term as well as the modified r-term yield, up to a redefinition of parameters, identical expressions for $\mathcal{L}_n^{\phi,0}$, $\mathcal{L}_n^{\phi,1}$ and $\delta \mathcal{L}_n^0$, $\delta \mathcal{L}_n^1$. Since these are just bare parameters, this means that up to and including linear terms in the link fields, the

2In fact, one can always start from an un-renormalized LF-Hamiltonian without “Compton terms” (i.e. the terms bilinear in both fermion and boson field) and then generate the Compton terms as counter terms.

3In the LF approach, after eliminating constrained degrees of freedom, already the canonical Lagrangian/Hamiltonian is non-local in the longitudinal direction.

4More precisely, invariant under $x^\pm$ independent gauge transformations. For the general case the same comments as in Section II apply.
standard and the modified Wilson terms yield the same interaction terms. Since the terms quadratic in the link fields are determined in the renormalization procedure, this means that the standard and modified Wilson formulations on a transverse lattice are identical.

This is the central result of this paper, since it shows that once we introduce link fields and once we determine the “Compton-terms” in the LF Hamiltonian in the renormalization procedure, also the terms quadratic in the link fields are the same for the naive Wilson r-term and the “modified” Wilson r-term. Even though our initial ansatz [Eq. (2.18)] looked very promising, all that has been achieved is that we reproduced the same interaction terms as the standard Wilson ansatz. Compared to the standard Wilson ansatz, nothing has been gained by trying to remove species doubling using the (LF-) chirally even kinetic mass term.

E. From 2+1 to 3+1 Dimensions

The main work in this paper so far was to show that the fermion terms up to and including linear in the link fields are the same for the standard and the modified Wilson formulation. Since terms nonlocal in both transverse directions (x and y) at the same time (x and y) involve 2 link fields, these do not occur when one focuses on terms up to and including linear. Therefore, the 3+1 dimensional analysis turned out to closely parallel the 2+1 dimensional analysis. Since no new physics insight was gained — in fact, the essential steps became more mysterious because of the more lengthy equations — the details of the 3+1 dimensional analysis were omitted in this work. The final result, i.e., equivalence between standard Wilson and modified Wilson formulation, was the same as in 2+1 dimensions, since coupling constant coherence between three point and four point couplings of bosons to fermions holds in both cases.

III. THE FEMTOWORM MODEL FOR MESONS

The coupling of quarks to the link-fields via the canonical terms leads to helicity flip ($\gamma$-structure: $\alpha_\perp$), while the r-term does not flip the helicity of the quarks ($\gamma$-structure: $\beta$). In order to investigate the ramifications of these basic features for the interactions of quarks, let us consider mesons in the approximation that there is at most one link field separating a quark and an anti-quark. In this approximation, the quark and anti-quark are always on the same site or on neighboring sites and mesons propagate by one-boson exchange. For example, suppose that the quark and the anti-quark start out on the same site and that the quark hops to a neighboring site. Gauge invariance requires that the quark emits at the same time a link field quantum on the link that is now connecting the quark and anti-quark. In the next step the anti-quark follows behind by absorbing the link field quantum. Propagation over more than one link proceeds by repeating this process. The resulting sequence resembles the contracting and stretching motion of an inchworm. The similarity in the propagation mechanism and the fact that the typical length scale of the mesons is not one inch but one Fermi, motivated the name for the model.

The kinetic part and self-interactions of the link fields have been introduced and discussed in Refs. [14] and will not be discussed here since the focus of this paper is on the quarks here. For details, the reader is encouraged to consult these References.

As far as the longitudinal dynamics is concerned, introducing quarks leads to both a kinetic term

$$L_{\text{kin}}^{\text{quarks}} = -\sum_{\bar{n}_\perp} \phi_{\bar{n}_\perp}^\dagger \frac{m^2}{\sqrt{2} \, \phi_{\bar{n}_\perp}} \phi_{\bar{n}_\perp}$$

(3.1)

as well as a contribution from the quarks to the vector current

$$J_{\bar{n}_\perp, i, j}^+ = J_{\bar{n}_\perp, i, j}^+ (U) + \sqrt{2} \phi_{\bar{n}_\perp}^\dagger \phi_{\bar{n}_\perp}$$

(3.2)

where $i, j$ are color indices and where $J_{\bar{n}_\perp, i, j}^+ (U)$ is the vector current from the link field which has been introduced in Ref. [3]. The main difference between the longitudinal gauge field coupling of the quarks and the link-fields is that the quarks couple only to the longitudinal gauge field at one site, while the link fields couple to the longitudinal gauge field at the two sites that are adjacent to the link. In general, i.e. without truncation of the Fock space, the physical space of gauge invariant states thus consists of closed loops of link fields plus open strings of link fields with quarks and anti-quarks at the end. For studying mesons in the large $N_C$ limit, we can focus our attention on quarks and anti-quarks connected by a string of link fields. The main approximation in this section will be to limit the length of this string to one lattice spacing. At first this seems to be a rather drastic approximation. However, if one considers that the lattice spacing in contemporary lattice calculations is on the order of 0.4 fm [4,14], limiting the length of the string to one lattice unit seems to be a reasonable approximation for a first study of meson spectra on a $\perp$ lattice.

Since the coupling of the quark to the longitudinal gauge field is local in the $\perp$ direction, the $\perp$ dynamics of the quarks is solely described by the hopping term. Using the result from Section III one finds the following for the Dirac part of the hopping terms: (Overall factors and the obligatory link-field operators are suppressed for notational convenience! It is implicitly understood that hopping is always accompanied by the emission or absorption of a link field quantum on the link connecting the initial and final site of the hopping process)

- naive term:
  hopping in x-direction:
\[ L_{x}^{\text{naive}} = - \sum \phi_{n_{\perp}}^{\dagger} \frac{i \beta \alpha_{x}}{i \partial_{x}} (\phi_{n_{\perp} + } - \phi_{n_{\perp} - }) \quad (3.3) \]

\[
= \left( \frac{1}{p_{f_{\perp}}} - \frac{1}{p_{i_{\perp}}} \right) \sum \phi_{n_{\perp}}^{\dagger} \left[ b_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) b_{i_{\perp}}(n_{\perp} + \vec{r}_{x}, p_{i_{\perp}}^{+}) - b_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) b_{i_{\perp}}(n_{\perp} - \vec{r}_{x}, p_{i_{\perp}}^{+}) - b_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) b_{i_{\perp}}(n_{\perp} - \vec{r}_{y}, p_{i_{\perp}}^{+}) + b_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) b_{i_{\perp}}(n_{\perp} + \vec{r}_{y}, p_{i_{\perp}}^{+}) - d_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) d_{i_{\perp}}(n_{\perp} + \vec{r}_{x}, p_{i_{\perp}}^{+}) + d_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) d_{i_{\perp}}(n_{\perp} - \vec{r}_{x}, p_{i_{\perp}}^{+}) + d_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) d_{i_{\perp}}(n_{\perp} - \vec{r}_{y}, p_{i_{\perp}}^{+}) - d_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) d_{i_{\perp}}(n_{\perp} + \vec{r}_{y}, p_{i_{\perp}}^{+}) \right]
\]

hopping in y-direction:

\[ L_{y}^{\text{naive}} = - \sum \phi_{n_{\perp}}^{\dagger} \frac{i \beta \alpha_{y}}{i \partial_{y}} (\phi_{n_{\perp} + } - \phi_{n_{\perp} - }) \quad (3.4) \]

\[
= \left( \frac{1}{p_{f_{\perp}}} - \frac{1}{p_{i_{\perp}}} \right) \sum \phi_{n_{\perp}}^{\dagger} \left[ b_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) b_{i_{\perp}}(n_{\perp} + \vec{r}_{y}, p_{i_{\perp}}^{+}) + b_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) b_{i_{\perp}}(n_{\perp} - \vec{r}_{y}, p_{i_{\perp}}^{+}) - b_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) b_{i_{\perp}}(n_{\perp} + \vec{r}_{x}, p_{i_{\perp}}^{+}) - d_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) d_{i_{\perp}}(n_{\perp} - \vec{r}_{y}, p_{i_{\perp}}^{+}) + d_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) d_{i_{\perp}}(n_{\perp} + \vec{r}_{y}, p_{i_{\perp}}^{+}) + d_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) d_{i_{\perp}}(n_{\perp} + \vec{r}_{x}, p_{i_{\perp}}^{+}) - d_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) d_{i_{\perp}}(n_{\perp} - \vec{r}_{x}, p_{i_{\perp}}^{+}) \right]
\]

**r-term:**

\[ L_{r} = \sum \phi_{n_{\perp}}^{\dagger} \frac{1}{i \partial_{z}} \left( \phi_{n_{\perp} + } + \phi_{n_{\perp} - } + \phi_{n_{\perp} + } + \phi_{n_{\perp} - } \right) \quad (3.5) \]

\[
= \left( \frac{1}{p_{f_{\perp}}} + \frac{1}{p_{i_{\perp}}} \right) \sum \phi_{n_{\perp}}^{\dagger} \left[ b_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) b_{i_{\perp}}(n_{\perp} + \vec{r}_{x}, p_{i_{\perp}}^{+}) + b_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) b_{i_{\perp}}(n_{\perp} - \vec{r}_{x}, p_{i_{\perp}}^{+}) + d_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) d_{i_{\perp}}(n_{\perp} - \vec{r}_{y}, p_{i_{\perp}}^{+}) + b_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) b_{i_{\perp}}(n_{\perp} + \vec{r}_{y}, p_{i_{\perp}}^{+}) + d_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) d_{i_{\perp}}(n_{\perp} + \vec{r}_{y}, p_{i_{\perp}}^{+}) + d_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) d_{i_{\perp}}(n_{\perp} - \vec{r}_{x}, p_{i_{\perp}}^{+}) + b_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) b_{i_{\perp}}(n_{\perp} + \vec{r}_{x}, p_{i_{\perp}}^{+}) + d_{i_{\perp}}^{\dagger}(n_{\perp}, p_{i_{\perp}}^{+}) d_{i_{\perp}}(n_{\perp} - \vec{r}_{x}, p_{i_{\perp}}^{+}) \right]
\]

Note that in the naive term, hopping in the the positive x or y direction \((n \rightarrow n + 1)\) yields the opposite sign from hopping in the negative direction \((n \rightarrow n - 1)\). In the r-term, the sign is the same for hopping in all directions. This observation will become important when we discuss interference between various contributions.

**A. Solutions with \( \vec{P}_{\perp}^{\text{total}} = 0 \).**

Because of manifest (discrete) translational invariance in the \( \perp \) direction, the total transverse momentum \( \vec{P}_{\perp}^{\text{total}} \) is conserved. As we will see below, the case \( \vec{P}_{\perp}^{\text{total}} = 0 \) yields particularly simple bound state equations and in the following we will focus our attention on this case.

The first simplification is a non-interference between naive hopping and r-term hopping: In general, one cannot distinguish between a link quantum emitted by the naive term and one that was emitted from the r-term. Thus a link quantum emitted from the naive term when the quark hops by one lattice unit may be absorbed by the r-term when the anti-quark follows behind and vice versa. Since the naive term flips the spin, but the r-term does not, this interference results in a non-conservation of the meson spin (which is of course a lattice artifact). However, this interference term has opposite signs for hopping into opposite directions. Therefore, for \( \vec{P}_{\perp}^{\text{total}} = 0 \), there is an exact cancellation between interference terms when the meson hops in opposite directions and there is not interference between link quanta emitted from the naive term and those emitted from the r-term.

Note that, since the naive term flips quark helicity and the r-term does not, the above result has an important consequence for the spin structure of the \( q\bar{q} \) (2 particle) Fock component: Because of the non-interference mentioned above, the only conceivable mixing of 2-particle Fock components could be between states where \( q \) and \( \bar{q} \) have both spins up or both spins down, i.e. “double helicity flip”. However, since the spin matrix element for hopping in the x-direction with double helicity flip is \((-i)(+i) = 1 \ (3.3)\), while the same matrix element for y-hopping is \((+1)(-1) = -1 \ (3.4)\), both terms cancel for a plane wave with \( \vec{P}_{\perp} = 0 \) and there is no double helicity flip hopping. As a result, the total spin in the 2-particle Fock component is conserved.

From a more general point of view, the above result about \( S_{z} \) conservation in the 2-particle Fock component emerges since the transverse lattice is invariant under rotations by \( \pi/2 \) around the z-axis, plane wave solutions with \( \vec{P}_{\perp}^{\text{total}} = 0 \) must be eigenstates under rotations around the z-axis by \( \pi/2 \) (with eigenvalues \( \pm 1 \) and \( \pm i \), i.e. in a sense \( J_{z} \) is conserved modulo 4). Since there is no orbital or gluon field angular momentum when the \( q\bar{q} \) pair is on the same site this means that \( S_{z} \), which can assume only the values 0 and \( \pm 1 \), must be conserved.

Another important observation concerns the sign of the spin splittings between mesons. For \( P_{z} = 0 \) one finds that x-hopping is attractive in the pseudoscalar channel (spin
wave function: $\uparrow\downarrow - \downarrow\uparrow$, since for example $b(\uparrow\rightarrow\downarrow) = -i$ and $d(\downarrow\rightarrow\uparrow) = -i$, i.e. together with another $(-)$ from the spin wave function and another sign from the energy denominator the total sign is $(-)$, i.e. attractive. The same result holds for $y$-hopping ($k_P = 0$), since for example $b(\uparrow\rightarrow\downarrow) = 1$ and $d(\downarrow\rightarrow\uparrow) = -1$. Likewise one can show that naive hopping yields a repulsive interaction for the vector channel (spin wave function: $\uparrow\downarrow + \downarrow\uparrow$), i.e. despite all approximations, the Weingarten inequality \cite{16} is satisfied for our model.

B. Pure $qU\bar{q}$ States

In the previous section we have seen that (discrete) rotational invariance yields a drastic simplification in the spin structure for $\bar{F}_{\perp}^{\text{total}} = 0$ solutions to our model. Again for $\bar{F}_{\perp}^{\text{total}} = 0$, we will demonstrate in this section that there are some particularly simple solutions which have a vanishing 2-particle Fock component. Mesons which mix with $q\bar{q}$ states at the same site must necessarily have total helicity $\sum h = 0, \pm 1$. However, even in the one link approximation, it is very easy to construct states which cannot mix with a $q\bar{q}$ state at the same site. The generic form of such states is

$$|h = 2\rangle = \sum_{\vec{n}_\perp} \left[ b^\dagger(\vec{n}_\perp + \vec{r}_x)d^\dagger(\vec{n}_\perp) + ib^\dagger(\vec{n}_\perp + \vec{r}_y)d^\dagger(\vec{n}_\perp) - b^\dagger(\vec{n}_\perp - \vec{r}_x)d^\dagger(\vec{n}_\perp) - ib^\dagger(\vec{n}_\perp - \vec{r}_y)d^\dagger(\vec{n}_\perp) \right]|0\rangle$$

and similarly for $h = -2$ with the longitudinal wave function ($k^\dagger$-dependence) factorized and with the link-field implicitly included. The angular dependence in Eq. \cite{3.6} resembles that of a state with $L_z = +1$, but strictly speaking on a lattice there is of course no rotational symmetry and hence no angular momentum. Nevertheless, a state with orbital structure as in Eq. \cite{3.6} cannot mix with states where a $q\bar{q}$ pair is at the same transverse site: first we note that the matrix element for hopping due to the $r$-term is independent of the direction and thus there is destructive interference when adding up the phases from the state for the quark to hop from different directions to the site of the anti-quark $(1 + i - 1 - i = 0)$. The phase of the matrix elements of the naive term for a quark with positive helicity to hop to the site of the anti-quark are $-i, 1, i, -1$ for hopping from the $+x, +y, -x, -y$ direction respectively. Together with the phase of the wave function, one thus obtains destructive interference for the amplitude due to the $r$-term as well

$$\langle \bar{q}q|H_q-\text{hop}|h = 2\rangle \propto (-i)(+1) + (+1)(+i) + (+i)(-1) + (-1)(-i) = 0.$$ (3.7)

Similarly, one can verify that the total amplitude for anti-quark hopping is zero for the $h = 2$ states \cite{3.8}. Note that it was crucial for the cancellation argument that the helicity of both $q$ and $\bar{q}$ is positive.

There are more states that do not mix with $\bar{q}q$ states for $\bar{F}_{\perp}^{\text{total}} = 0$. For example, any state with $L_z = 2$, regardless of spin, i.e. any state of the form

$$|L_z = 2\rangle = \sum_{\vec{n}_\perp} \left[ b^\dagger(\vec{n}_\perp + \vec{r}_x)d^\dagger(\vec{n}_\perp) - b^\dagger(\vec{n}_\perp + \vec{r}_y)d^\dagger(\vec{n}_\perp) + b^\dagger(\vec{n}_\perp - \vec{r}_x)d^\dagger(\vec{n}_\perp) - b^\dagger(\vec{n}_\perp - \vec{r}_y)d^\dagger(\vec{n}_\perp) \right]|0\rangle$$

(3.8)

has a vanishing matrix element with states where the $\bar{q}q$ state is on the same site. To see this, let us first consider $r$-term hopping. Since there are no phases for hopping in different directions from this term in the Hamiltonian, the only phases arise from the state and the add up to zero $1 - 1 + 1 - 1 = 0$. This argument applies regardless of spin. The matrix elements of the naive hopping term between Eq. \cite{3.8} and a pure $\bar{q}q$ state vanish, since the naive hopping term in the Hamiltonian has always opposite signs for hopping in opposite directions, the wave-function yields the same sign for separations in opposite directions. The total matrix element of the naive hopping term is thus zero due to cancellation from opposite directions. The pure $\bar{q}Gq$ states are summarized in Table I.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\textbf{helicity} & \textbf{$L_z$} & \textbf{S} & \textbf{S$_z$} & \textbf{C} & \textbf{P} \\
\hline
2 & 1 & 1 & 1 & + & + \\
2 & 2 & 0 & 0 & - & - \\
1 & 2 & 1 & -1 & + & - \\
\hline
\end{tabular}
\caption{Quantum numbers of states that cannot mix with $\bar{q}q$ states. The C and P quantum numbers contain all signs from intrinsic parities, spin and transverse symmetry but do not include signs from the longitudinal orbital wave function.}
\end{table}

\footnote{We define total helicity through the eigenvalues of the discrete rotation operator for rotations by $\pi/2$ around the z-axis. Since the resulting definition is unique only up to four times an integer, we break the ambiguity by always selecting the solution with $|h| \leq 2$.}
The total helicity of these states is obtained from the sum of the discrete orbital angular momentum and the spin (which is conserved modulo 4). The parity and C-parity quantum numbers given in Table I include all intrinsic parities as well as signs from the spin and transverse wave functions. For a state with no nodes in the longitudinal wave function, the P and C parities given in the table are thus immediately equal to the P and C parity of the corresponding state. However, for more complicated longitudinal wave functions, the P and C parities in the table need to be multiplied by the signs from the longitudinal wave function.

The longitudinal dynamics of all the states in Table I are described by the same relatively simple wave equation, which reflects the fact that (note \( N_C \to \infty \)) the quark and anti-quark interact only with the link field via the 1+1 dimensional gauge interaction

\[
M^2\psi(x, y) = \left[ \frac{m_q^2 - G^2}{x} + \frac{m_{\bar{q}}^2 - G^2}{y} + \frac{m_{\bar{q}}^2}{1 - x - y} \right] \psi(x, y)
- G^2 \int_0^{1-y} dz \frac{1 - x - y + (1 - z - y)}{(x - z)^2(1 - x - y)(1 - z - y)} \psi(z, y)
- G^2 \int_0^{1-z} dz \frac{1 - y - z + (1 - z - y)}{(y - z)^2(1 - y - z)(1 - z - y)} \psi(z, y)
\]  

(3.9)

where \( x, y \) are the longitudinal momentum fractions of the quark and anti-quark respectively and where \( G^2 = \frac{g^2N_c}{2\pi} \). The integrals in Eq. (3.9) are principal value integrals.

Note that the link fields should not be directly interpreted as gluons and therefore the mesons in Table I, i.e. the solutions to Eq. (3.9), are not necessarily hybrid or exotic mesons even if they cannot mix with pure \( \bar{q}q \) states. The physical reason why these states cannot mix with \( \bar{q}q \) states on the transverse lattice is that conservation laws (e.g. angular momentum) do not allow that \( q \) and \( \bar{q} \) sit on the same transverse coordinate, and gauge invariance on the \( \perp \) lattice requires the presence of link fields. However, hybrid mesons correspond to those solutions to Eq. (3.9) where the link field is in an excited state.

C. Numerical Calculations

In the numerical calculations, we solved the coupled Fock space equations, including \( \bar{q}q \) and \( \bar{q}Uq \) states projected onto vanishing total transverse momentum \( p_{\perp total} = 0 \).

Furthermore, after making this plane wave ansatz, one can reduce the equations of motion to equations that resemble the coupled Fock space equations in collinear QCD (with truncation to 3 particles) except that the link fields not only have vector couplings to the quarks but also scalar couplings from the \( r \)-term. Since collinear QCD models have been studied extensively in the literature, we omit these details here and refer the interested reader to Refs. [5,13].

In the Hamiltonian, we dropped the seagull interactions involving instantaneous fermion exchange! As has been explained in Ref. [13,23], gluon loops connected to gluon emission and absorption vertices are important for dynamical vertex mass generation. If one works with a truncated Fock space, such vertex corrections are included in an asymmetric way: for example, if one limits the Fock space to \( \bar{q}q \) and \( \bar{q}Uq \) then vertex corrections to boson emission can only be generated by the Hamiltonian in the “in” state but not in the “out” state because the “out” state contains already the emitted boson (and conversely for absorption). As a result, in and out states are treated asymmetrically. In order to understand the consequences of such an asymmetric treatment, consider the momentum dependence of the bare emission vertex (helicity flip)

\[
T_{\text{emission}} \propto \left( \frac{m_q}{p_i^+} - \frac{m_{\bar{q}}}{p_f^+} \right) \frac{1}{\sqrt{k_U^+}}.
\]

(3.10)

Loop corrections in the initial state connected to the vertex by instantaneous interactions renormalize only the mass term which gets divided by \( p_i^+ \) but not the mass term divided by \( p_f^+ \) (and vice versa for loop corrections in the final state) [20]. However, this is exactly what happens to an emission vertex when the Hamiltonian is diagonalized in a Fock space truncated basis. There are several possible remedies one might think of, such as perturbatively including loop corrections which are suppressed by the Fock space truncation. [23] However, in this paper we chose the most simple solution and dropped instantaneous fermion interactions altogether, which removes loop corrections connected to vertices by instantaneous interactions completely. Since we are aware that those interactions are crucial for dynamical vertex mass generation [19,20], we compensate for this omission by using constituent quark masses (300MeV) for the vertex mass and not current masses. Note that since there is also an unspecified vertex coupling constant multiplying the link filed emission terms, the precise numerical value of the vertex mass is not crucial. However, it is important to note that we used a value that does not vanish in the chiral limit. Furthermore, it is important that, for simplicity, we do not allow this coupling to run [20].

Furthermore, in the \( \bar{q}Uq \) Fock component, we use a constituent mass for the kinetic mass of the quarks as well. In the \( \bar{q}q \) component of the Fock space, where one has to add an infinite counter-term for the one loop self

\[6\text{Note that DLCQ does not completely solve this problem, even if all Fock components allowed by the discretization are included, because of the loss of boost invariance in subgraphs.}\]
energy, we use the finite part of the kinetic mass as a fit parameter to fit the $\pi$ and $\rho$ masses.

In general, the self interactions of the link field are described by a complicated effective potential \cite{4}. However, in the one link approximation only the quadratic term (the mass term) contributes, which we take to be about 300 MeV as well, which lies in the range of values suggested in Ref. \cite{4}. In the spirit of an effective link field interaction (obtained in principle by “smearing” and integrating out short range fluctuations) it would be consistent to also introduce effective interactions for the quark fields. For example, an effective interaction induced by short range fluctuations could be a spin dependent “contact interaction” similar to Gross-Neveu interactions. It is also conceivable that one could introduce a spin dependent contact term such that it partially cancels the violations of rotational invariance. However, in this first study case, we wanted to explore how well one can reproduce the light meson spectrum with canonical terms only and we considered canonical quark terms only.

The most important scale in $\perp$ lattice calculations is set by the dimensionful coupling of the quark and link fields to the instantaneous Coulomb interaction acting on each site. If one suppresses higher Fock components then this coupling directly translates into the longitudinal string tension $(\sigma = \frac{1}{2}G^2)$, which we fix at $\sigma = 1$ GeV/fm. We have also tried smaller values of the string tension, but then we were forced to use a very large coupling for the hopping term in order to generate the experimentally observed $\pi - \rho$ splitting and the spectrum became numerically unstable. This is because, for a given quark mass, smaller string tensions lead to wave functions which vanish faster at the kinematic end-points and hence lead to smaller contributions from the one boson exchange interaction which is also singular at these end-points.

In our numerical calculations we applied a sharp cutoff to the matrix element for hopping (i.e. link field emission and absorption) processes if the momentum of the final state quark is smaller than a fraction $\varepsilon$ of the momentum of the initial state quark. We varied this cutoff between $\varepsilon = 0.2$ and $\varepsilon = 0.1$ and verified that the spectra are stable with respect to variations of this momentum fraction cutoff.

As long as the $r$-term has a coupling which is about 1/4 of the coupling for the naive hopping term or smaller, the low energy spectrum for $P_\perp = 0$ is rather insensitive to its precise value. In fact, for such values, the main effect of the $r$-term is to raise the energy of mesons consisting of “doubler quarks” from the edge of the Brillouin zone. When the $r$-term was taken to be larger (about 1/2 relative to the naive hopping term or more) there was also a strong effect on the low energy spectrum for $P_\perp = 0$ and we were no longer able to simultaneously fit the $\pi$ and $\rho$ meson masses. We thus performed the numerical calculations where the value of the $r$-term coefficient is 1/4 of the naive term coefficient.

Given all constraints listed above, we fitted the only remaining free parameters — the finite part of the kinetic mass counter-term and the vertex mass times the coupling of quarks to $A_\perp$ — to the $\pi$ mass as well as the $\rho$ mass. Since rotational invariance and parity are not manifest symmetries, it is necessary to be more specific here about what we mean by the $\pi$ and the $\rho$ mesons. The $\pi$-meson is taken to be the lightest meson with $C = +$ and helicity $h = 0$. Since the $\rho$ meson has spin 1, it can have both helicity $h = 0$ as well as $h = \pm 1$. Since our approach as well as model approximations break rotational invariance, we do not find these states to be degenerate: the lightest meson with $h = 1$ and $C = -$ and the lightest meson with $h = 0$ and $C = -$ are not exactly degenerate. In the fitting procedure, we chose to fit the center of mass of these two solutions to the experimentally observed $\rho$ meson mass $m_\rho = 770$ MeV.

The resulting meson spectrum is shown in Fig. 1 for mesons with positive C-parity and in Fig. 2 for mesons with negative C-parity.
FIG. 1. Comparison of the calculated meson spectrum for helicities $h = 0, 1, 2$ with the experimental meson spectrum for states with positive C-parity. Those states where an obvious assignment for the parity quantum number seemed possible are labelled with a '+' or '-', while states with ambiguous parity quantum number are labels with a '?'. Degenerate states are displaced slightly in the vertical direction in order to make the degeneracy visible.

Unlike C-parity, the usual parity is not a manifest symmetry in the LF framework. For states with a large 2-particle component, we used the symmetry of the 2-particle component to assign a parity quantum number: for example, let us consider the vector current, which can obviously only couple to negative parity states. The vacuum to meson matrix elements of the good component $\bar{\psi}\gamma^+\psi$ of the vector current operator is proportional to $\int_0^1 dx \phi_n(x)$, where $\phi_n(x)$ is the 2-particle component of the wave function. Obviously, this matrix element can be non-vanishing only if the 2-particle component of the wave function is even under $x \leftrightarrow (1-x)$. Similar observations can be made for positive C-parity operators. We therefore assign a negative parity to states with an even wave function in the 2-particle component of the Fock space, and a positive parity to states with an odd wave function in the 2-particle component. Note that this assignment is also consistent with simple quark model considerations (i.e. free particle parity).

FIG. 2. Same as Fig. 1, but for negative C-parity.

For states with very small or vanishing 2-particle component, this procedure becomes dubious or useless. In those cases we determine the parity of a state from the symmetry of the 3-particle wave function: First there is transverse parity (which is a manifest symmetry both in LF quantization and on the $\perp$ lattice). Secondly, for the “longitudinal parity” we combine the intrinsic parity of a $q\bar{q}$ pair ($-\frac{1}{2}$) with the orbital parity. The orbital parity is determined using the free (no interactions) parity operator. Note that instead of applying the free parity operator, one can simply count nodal lines: in addition to transverse and intrinsic parity, there is a ($-\frac{1}{2}$) if the 3-particle wave function is odd under $q\bar{q}$ exchange and there is another ($-\frac{1}{2}$) if there is an odd number of nodes in the wavefunction describing the motion of the link field relative to the “center of mass” of the $q\bar{q}$ pair. For example, a state with positive transverse parity and where the longitudinal $\bar{q}Uq$ wave function has no node at all would be assigned $P = -$. Of course, counting of nodes

$^7$Note that there is no intrinsic parity for the link field.
is sometimes ambiguous, but in those cases using the free parity operator is also not very conclusive.

However, while the above parity assignments procedure seemed reasonable to us, it is not completely unambiguous and we thus plotted states with positive and negative parities in the same figure.

There are several observations one can make from Figs. 1 and 2. First of all, because of the violations of rotational invariance, states that one would expect to be degenerate (such as the h=0 and h=1 states of the ρ meson) are not degenerate. However, considering the crudeness of the model, the violations of rotational invariance are actually moderate. The second observation one can make is that even though our model properly describes the gross features of the spectrum, there are clearly states “missing” in our model and the energies of excited states typically lie too high. The missing states can be easily understood by noting that the truncation of the Fock space to 3 particles suppresses degrees of freedom associated with higher Fock components. The result that excited states lie systematically too high also has to do with the fact that a lot of physics has been left out by our truncation of the Fock space. For example, restricting the maximum separation between quark and anti-quark to one link, has a stronger effect on excited states since those tend to be larger (if one allows them to spread out).

Encouraged by the numerical spectrum, we proceeded to calculate other hadronic observables — particularly those where the use of LF field theory is advantageous: LF wave functions and parton distribution functions. LF wave functions or distribution functions for the π and the two helicity states for the ρ are shown in Fig. 3.

FIG. 3. LF wave functions $\phi_2(x_q)$ in the 2 particle component of the Fock space versus the momentum fraction carried by the quark $x_q$ for the π, and the $h = 0, 1$ states of the ρ meson.

What one observes in Fig. 3 is that all of them are rather flat with a maximum in the middle, which reflects the strong binding of these states. Thus despite the low scale (recall our large transverse lattice spacing $a \approx 0.4 fm$) the pion wave function $\phi_\pi(x)$ has a shape which already resembles the asymptotic form $\phi_\pi(x) \propto x(1-x)$, but it is much more flat. A closer look at the wave functions shows that the ρ meson wave functions are slightly more peaked in the middle than $\phi_\pi$, which reflects the weaker binding of the ρ meson compared to the π.

In Figs. 4 and 5 we show the longitudinal wave functions of the π and the ρ in the 3-particle Fock components. Fig. 4 shows the component of the 3-particle Fock component where the quark or the anti-quark helicity has flipped compared to the valence component (generated by naive hopping!), while Fig. 5 shows the component where the $q\bar{q}$ helicity wave function is the same as in the valence state.

---

8These are defined to be the longitudinal momentum space LF wave function (where momenta are measured in units of the total momentum) in the $q\bar{q}$ Fock component with zero transverse separation, i.e in our calculation the valence wave function.
In Section III B we demonstrated that there are states in the spectrum which do not mix with the $\bar{q}q$ component of the Fock space. In Fig. 6 we show the (longitudinal) LF wave functions for the four lightest solutions to Eq. (3.9). The lightest solution to Eq. (3.9), displayed in the upper left of Fig. 6 has no nodes, i.e. we interpret this solution as a $\bar{q}q$ state with orbital angular momentum around the z-axis (see the discussion in Section III B) but with the glue in its ground state. The mass of this state is about 1.63 GeV, i.e. much higher than experimentally observed mesons with $J = 2$, which we attribute to the one link approximation. The second excited solution to the $\bar{q}Uq$ equation (lower left in Fig. 6) has a node in the wave function for the link field relative to the $\bar{q}q$ pair, i.e. this state should be interpreted as a hybrid excitation. Its mass is about 2.27 GeV, but again we expect significant corrections from higher Fock components which we have not included. Finally, the first and third excited state (upper and lower right in Fig. 6 respectively) correspond to states where there are nodes in the $\bar{q}q$ relative wave function, but not relative to the link field, i.e. we interpret these solutions again as non-hybrid excited states. Their masses are 2.01 GeV and 2.32 GeV respectively.

Quark distribution functions ("structure functions") for the $\pi$ and $\rho$ mesons are shown in Fig. 7. For the $\rho$ with helicity $h = 1$, the polarized quark distribution function is also displayed. Note that, since the $\rho$ meson has spin 1, even its unpolarized quark distribution function does not need to be the same for the $h=0$ and $h=1$ states, i.e. the fact that our numerical results for the unpolarized distribution functions of the $\rho$ states with $h = 0$ and $h = 1$ should not necessarily be interpreted as a manifestation of rotational invariance violations! We should emphasize that the quark distribution functions should not vanish at the origin, because the interaction coupling the 2 particle and 3 particle Fock components is singular enough as the fermion momentum goes to zero to overcome the suppression due to the rise of the kinetic energy in the
3 particle component when the momentum of one of the particles vanishes. However, in the numerical results this singularity is suppressed due to our cutoff on the momentum fractions in a hopping (i.e. link field emission) process. Varying this cutoff \( \varepsilon \) hardly changes the structure functions except in the extreme vicinity of \( x = 0 \), where the “gap” fills up as \( \varepsilon \) is lowered. The structure functions displayed are for a value of \( \varepsilon = 0.1 \). The model yields for the momentum fraction carried by the quarks plus anti-quarks in the \( \pi \) about 75.4%, while the numbers for the \( \rho \) are slightly higher with \( \langle x_q \rangle_{\rho(h=0)} \approx 77.4\% \) and \( \langle x_q \rangle_{\rho(h=1)} \approx 76.2\% \), which again reflects the stronger binding in the \( \pi \) (resulting in a “smaller” object, where less volume is filled with gluon fields). In deep inelastic scattering experiments, the structure function of the \( \pi \) is usually determined at rather large values of \( Q^2 \). However, since the per-}

the quarks which are comparable to the experimentally measured value of about 50%. However, since the perturbative limit of the \( \perp \) lattice is still poorly understood [4], we will not attempt to evolve our numerical results to higher values of \( Q^2 \).

In the polarized quark distribution function for the \( \rho \), one can notice a sign change for small \( x \). The reason for this sign change is the following. At large \( x \) the valence quarks dominate, which have the same helicity as the parent \( \rho \). At small \( x \), the distribution function is dominated by quarks (and anti-quarks) which have just emitted a link field and link filed emission via the naive term leads to helicity flip for the quark (or anti-quark). Since the coupling of the naive hopping term is taken to be larger than the coupling of the \( r \)-term, helicity flip dominates at small \( x \) over non-flip and thus the polarized parton distribution changes sign. In order to interpret this result, we emphasize again that the parton distributions calculated should be interpreted at a very low momentum scale. Perturbative evolution would clearly cover up this sign reversal. However, it is interesting to note that, based on the above results, the spin fraction carried by the quarks spin in the \( \rho \) meson turns out to be about 52.9%. Because of the large \( N_C \) approximation employed in this model, this number should be interpreted as the non-singlet spin fraction carried by the quarks. Unfortunately, no experimental data on the structure function of the \( \rho \) meson is available, but a number which has a similar physical interpretation would be the ratio between \( g_A \) in the nucleon and its naive quark model value, which turns out to be \( \frac{2}{3} g_A \approx 0.77 \). We believe that the relatively small fraction of spin carried by the quark spins is an artifact of the constant (i.e. non-running) dynamical vertex mass which we have used for simplicity and we expect that a more refined treatment leads to less negative polarization of the quarks at small \( x \).

FIG. 7. Unpolarized quark distribution functions for the \( \pi \), the \( \rho \) with helicity \( h = 0 \) and the \( \rho \) with \( h = 1 \). For the \( \rho \) with \( h = 1 \) the polarized quark distribution is also shown.

In the polarized quark distribution function for the \( \rho \), one can notice a sign change for small \( x \). The reason for this sign change is the following. At large \( x \) the valence quarks dominate, which have the same helicity as the parent \( \rho \). At small \( x \), the distribution function is dominated by quarks (and anti-quarks) which have just emitted a link field and link field emission via the naive term leads to helicity flip for the quark (or anti-quark). Since the coupling of the naive hopping term is taken to be larger than the coupling of the \( r \)-term, helicity flip dominates at small \( x \) over non-flip and thus the polarized parton distribution changes sign. In order to interpret this result, we emphasize again that the parton distributions calculated should be interpreted at a very low momentum scale. Perturbative evolution would clearly cover up this sign reversal. However, it is interesting to note that, based on the above results, the spin fraction carried by the quarks spin in the \( \rho \) meson turns out to be about 52.9%. Because of the large \( N_C \) approximation employed in this model, this number should be interpreted as the non-singlet spin fraction carried by the quarks. Unfortunately, no experimental data on the structure function of the \( \rho \) meson is available, but a number which has a similar physical interpretation would be the ratio between \( g_A \) in the nucleon and its naive quark model value, which turns out to be \( \frac{2}{3} g_A \approx 0.77 \). We believe that the relatively small fraction of spin carried by the quark spins is an artifact of the constant (i.e. non-running) dynamical vertex mass which we have used for simplicity and we expect that a more refined treatment leads to less negative polarization of the quarks at small \( x \).

IV. SUMMARY

One of the main messages of this paper is that even though it is possible to write down a mass term that is invariant under chiral transformations (on the LF), this does not help to solve the species doubling problem in a way that is chirally invariant in the usual sense.\footnote{Once again, since LF Hamiltonians are nonlocal in the longitudinal direction, the Nielsen-Ninomiya theorem [3] does not apply directly.}

We used the kinetic mass term, which is chirally invariant in the LF approach, to remove species doubling. However the relevant 2 boson – 2 fermion coupling constants are not independent, but rather must be chosen so that they cancel divergences arising from iterating the three point interactions. Once one uses this result, the LF Hamiltonian based on standard Wilson fermions and the LF Hamiltonian based on a Wilsonian prescription (applied to the kinetic mass only) become identical.

As a first application for the transverse lattice formulation of Wilson fermions, we studied the meson spectrum in a simple model where we truncated the Fock space to \( q\bar{q} \) states (\( q \) and \( \bar{q} \) on the same site) and \( q\bar{Uq} \) states (\( q \) and \( \bar{q} \) on adjacent sites) In this approximation, we are able to separate the transverse motion, and, for \( P^\perp_{\text{total}} = 0 \), we derive an effective collinear equation for the model which can be easily solved numerically. After fitting the \( \pi \) and \( \rho \) masses as well as the longitudinal string tension,
we obtain a qualitatively reasonable meson spectrum, although excitation energies tend to be systematically too high. For the lowest hybrid meson state we obtain a mass of 2.27 GeV, which we believe is too high due to our Fock space truncation. The distribution amplitudes for the $\pi$ and $\rho$ mesons are both peaked at $x = 0.5$ and vanishing at the end-points, with the $\rho$ meson distribution function being slightly more peaked. The momentum fraction carried by the quarks (and anti-quarks) in the $\pi$ meson is about 75% with its value for the $\rho$ meson being slightly higher. For the $\rho$ meson with helicity $h = 1$ we find that the spin fraction due to the spins of the quarks is about 53%.

Considering the simplicity of the model calculations, the numerical results on hadron observables are very promising and encouraging. However, because of the simplistic nature of our first study calculations, there are many improvements possible. First of all, because of the large lattice spacing employed in current $\perp$ lattice calculations, one might also need to consider introducing “improvement” terms for quark fields. Secondly, especially for studying excited mesons, it would be very useful to include states with more than one link field in the Fock expansion, thus allowing the mesons to “expand” in the transverse direction. In connection with including higher Fock components, one also needs to address the question of dynamical vertex mass generation and the running of the effective vertex mass in more detail in order to properly incorporate the physics of dynamical chiral symmetry breaking into the effective LF Hamiltonian for QCD on a $\perp$ lattice.

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APPENDIX A: CHIRAL SYMMETRY ON THE LF

Chiral symmetry on the LF and chiral symmetry in the covariant approach are not exactly the same (here we closely follow Ref. [3]). In LF field theory, it is convenient to decompose $\psi = \psi_+ + \psi_-$, where $\psi_\pm = \frac{1}{2} \gamma^0 \gamma^\pm \psi$ and to eliminate $\psi_-$ using a constraint equation

$$
\psi_- = \frac{1}{i \partial^+} \left[ \bar{\alpha}_\perp \left( i \bar{\sigma}_\perp g A_\perp \right) + \gamma^0 m \right] \psi_+,
$$

where $i \bar{D}_\perp \equiv i \bar{\sigma}_\perp + g A_\perp$. Once $\psi_-$ has been eliminated, chiral transformations are applied to $\psi_+$ only! The constraint equation (A1) is in general inconsistent with the usual chiral transformation, since

$$
\gamma_5 \frac{1}{i \partial^+} \left[ \bar{\alpha}_\perp \left( i \bar{\sigma}_\perp + \gamma^0 m \right) \psi_+ \right] \neq \frac{1}{i \partial^+} \left[ \bar{\alpha}_\perp \left( i \bar{\sigma}_\perp + \gamma^0 m \right) \gamma_5 \psi_+ \right]
$$

(A2)

This explains how it is possible that the standard Wilson $r$-term, which breaks the usual chiral symmetry, does not break LF chiral symmetry.

It is interesting to note that only for $m = 0$ do chiral transformations on the LF and those defined covariantly agree with one another. In fact, it turns out that the generator for chiral transformations on the LF is nothing but the helicity of the fermions. This makes sense since for $m = 0$ helicity and chirality become the same (as do chiral transformations defined covariantly and on the LF) and both become conserved. Given a LF Hamiltonian, this fact makes it very easy to verify whether it is chirally invariant: if quark helicity is conserved by each interaction then the Hamiltonian is chirally invariant.

Manifest (LF-) chiral symmetry implies that the helicity of each quark is conserved. This fact gives rise to peculiar degeneracies: for example, mesons with $S_z = 0$ and $C = \pm 1$ (the $\pi$ and the $S_z = 0$ polarization states of the $\rho$) are exactly degenerate — a result which is phenomenologically not very desirable. Simple estimates show it is conceivable that a finite $\pi - \rho$ splitting remains in the chiral limit due to the singular end-point behavior of wave functions and matrix elements of the relevant interactions. However, this point requires further clarification.

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