Behaviour of the Centrifugal Force 
and of Ellipticity for a Slowly Rotating Fluid 
Configuration with Different Equations of State

Anshu Gupta, Sai Iyer and A. R. Prasanna 
Physical Research Laboratory 
Ahmedabad 380 009, India

Abstract

We have evaluated the centrifugal force acting on a fluid element and the ellipticity 
of the fluid configuration, which is slowly rotating, using the Hartle-Thorne solution 
for different equations of state. The centrifugal force shows a maximum in every case, 
whereas the reversal in sign could be seen in only one case, and the system becomes 
unstable in other cases. The ellipticity as calculated from the usual definition shows 
maxima, whereas the definition obtained from the equilibration of the inertial forces, 
shows a negative behaviour, indicating that the system is prolate and not oblate. 
This prolate shape of the configuration is similar to the one earlier found by Pfister 
and Braun for a rotating shell of matter, using the correct centrifugal force expression 
for the interior. The location of the centrifugal maxima gets farther away from the 
Schwarzschild radius ($R_s$) as the equation of state gets softer.

Introduction

One of the most important aspects in the discussion of the dynamics of a rotating 
neutron star is the equation of state of matter contained within. There have been
a large number of studies in this regard starting from the pioneering paper of Oppenheimer and Volkoff \[1\] in 1939 and later by Pandharipande (1971) \[2\] and several others over the last twenty five years. In spite of all the efforts put in, it is still enigmatic as to the correct nature of the equation of state and its transitions within, as it involves the actual short range interactions about which there exists no exact theory as yet. However, there have been several different equations of state, considering the nuclear interactions at various levels and potentials, and it is important to see the effects that they may have in the actual structure and dynamics of a neutron star.

It is indeed well known that a rotating body breaks spherical symmetry and the equilibrium configuration would be a Maclaurin spheroid or a Jacobi ellipsoid due to ellipticity arising from the anisotropy of the equilibrating forces. Our interest is to consider the dynamics of the interior of ultra compact bodies in the realm of general relativity as the gravitational potential is quite large. Though the original formulation of general relativity does not express the equilibrium in terms of forces, a recent 3+1 formulation allows one to introduce the language of forces without affecting any of the relativistic effects. On the other hand, through the language of forces one gets some new results which were not apparent earlier. One of the interesting new features is the realisation that the centrifugal force acting on a fluid element of the configuration has a maximum value and that it also reverses its direction at a value of \( r \), outside the event horizon \[3\], \[4\]. Further, it has also been seen that the ellipticity of the spheroidal configuration attains a maximum \[1\], a result which was earlier shown by Miller et al. \[5\]. As the change in ellipticity would have to do with the shape of the compact object, it is indeed interesting to check how the ellipticity and centrifugal force change with different equations of state. As shown earlier in
the case of homogeneous distribution, the centrifugal force reverses for a configuration with $R/R_s \approx 1.45$, and the ellipticity for the same configuration is maximum at $R/R_s \approx 2.75$.

In this, we extend this study to the case of an inhomogeneous density distribution with four different equations of state as given below:

(A) Pandharipande (hyperonic matter) [6],

(B) Wiringa, Fiks and Fabrocini’s beta-stable model: UV14+UVII [7],

(C) Walecka’s model corresponding to pure neutron matter [8], and

(D) Sahu, Basu and Datta’s model based on the chiral sigma model [9].

Model (A) is Pandharipande’s (hyperonic matter) EOS which studies the behaviour of dense matter using a many body theory based upon the variational approach. Hyperons are considered one of the baryonic constituents of the neutron star’s interior. Model (B) considers the main constituents of neutron star matter to be neutrons, protons, electrons and muons. This is one of the three models given by Wiringa, Fiks and Fabrocini which includes three-nucleon interactions using a non-relativistic approach based on the variational method. Walecka’s model (C), which characterizes the effective interaction by the meson parameters (masses and the coupling constants), is for pure neutron matter and emphasises the scalar and vector meson exchange interactions by considering the relativistic approach. Model (D) is a field theoretical EOS for neutron rich matter in beta equilibrium based on the chiral sigma model.
Model A is the softest equation of state among the considered EOS, whereas the most stiff EOS is model D. B and C are intermediate EOS of which C is the stiffer of the two.

Using the ACL formalism of 3+1 conformal slicing of the space time

\[ ds^2 = dl^2 + g_{00}(dt + 2\omega_\alpha dx^\alpha)^2, \]  

with \( dl^2 \) representing the positive definite metric of the absolute 3-space \( \bar{g}_{\mu\nu}dx^\mu dx^\nu \) (where Greek indices take values from 1 to 3 and \( \bar{\cdot} \) denotes the quantities in absolute 3-space), the four force acting on a fluid element of a slowly rotating perfect fluid distribution, represented by the spacetime metric

\[ ds^2 = -e^{2\nu}dt^2 + e^{2\psi}d\phi^2 + e^{2\mu_1}dr^2 + e^{2\mu_2}d\theta^2, \]

in a locally non rotating frame (LNRF), is given by

\[ f_0 = \Phi^{-1}(\rho + p)\bar{U}^\mu \partial_\mu U_0 + h_0^\mu p_{,\mu} \]  

\[ f_\alpha = \Phi^{-1}(\rho + p) \left[ \bar{U}^\mu \nabla_\mu \bar{U}_\alpha + \frac{M_0^2}{2\Phi} \partial_\alpha \Phi \right] + h_\alpha^\mu p_{,\mu} \]  

Here \( \Phi \) represents the gravitational potential \(-g_{00}\), \( p \) the pressure, \( \rho \) the energy density and \( U^\mu \) the velocity four vector of a fluid element. An overhead tilde represents the quantity in the projected 3-space of the optical reference geometry. The terms \( \bar{U}^\mu \nabla_\mu \bar{U}_\alpha \) and \( (M_0^2/2\Phi)\partial_\alpha \Phi \) represent the centrifugal \( (F_{cf}) \) and gravitational acceleration \( (F_g) \) respectively and have the form

\[ F_{cf} = e^{2\psi + 2\nu}(\Omega - \omega)^2(\psi' - \nu') \left[ e^{2\nu} - e^{2\psi}(\Omega - \omega)^2 \right]^{-1} \]  

\[ F_g = e^{2\nu}\nu', \]

for metric (2).
The Newtonian force balance equation

\[ g_{\text{equator}} - a\Omega^2 = g_{\text{pole}}(1 - e^2)^{1/2} \]  

may be expressed in general as

\[ F_{ge} - F_{cf} = F_{gp}(1 - e^2)^{1/2}, \]  

with \( F_{ge} \) and \( F_{gp} \) representing the accelerations at equator and pole respectively.

In the limit of slow rotation, the ellipticity is given by

\[ \epsilon = \frac{1}{2} e^2, \]  

where \( e \) is the eccentricity of the spheroid. Using equations (7) and (8), ellipticity is seen to be

\[ \epsilon = \frac{1}{2} \left( 1 - \left[ \frac{F_{cf} - F_{ge}}{F_{gp}} \right]^2 \right). \]  

For the Hartle-Thorne [10] metric

\[ F_{cf} = r^2\omega^2(1/r - \nu_0/2), \]  

\[ F_{ge} = \frac{1}{2} e^\nu_0 \nu_0'(1 + 2h_0 - h_2 + 2h_0' - h_2'), \]  

\[ F_{gp} = \frac{1}{2} e^\nu_0 \nu_0'(1 + 2h_0 + 2h_2 + 2h_0' + 2h_2'), \]

yielding for the ellipticity function:

\[ \epsilon = 3(h_2 + h_2'/\nu_0) + \frac{r^2\omega^2}{e^\nu_0'} (2/r\nu_0' - 1). \]
Results and Discussions:

Table 1 shows the location of extrema for the centrifugal force as well as for the ellipticity and their respective values for both the Hartle-Thorne definition ($\bar{\epsilon}_{H-T}$) and our definition ($\bar{\epsilon}$). Figures (1)-(3) give the plots of centrifugal force $\bar{F}_{cf}$ and the two ellipticities (expressed in dimensionless units $J^2/M^4$ and $J^2/M^5$ respectively) as a function of $R/R_s$, $R_s$ being the Schwarzschild radius.

As is seen, the centrifugal force keeps increasing as the configuration size gets smaller and then attains a maximum somewhere between $R/R_s = 2.1$ and 2.3, for different equations of state. However, unlike in the case of a homogeneous spheroid where the centrifugal force reverses sign at $R = 1.45 R_s$, with these different equations of state the reversal is seen only in the case of Wiringa et al. model (B), at $R \simeq 1.454 R_s$. In other cases the equilibrium configuration becomes unstable before reaching the value $R = 1.5 R_s$, which in fact is the radius of the orbit of the particle for which the centrifugal force is zero in the Schwarzschild space time.

Considering the nature of ellipticity it appears that the behaviour is different for the inhomogeneous distribution than the case of homogeneous distribution. As seen from Fig. 2, the ellipticity ($\bar{\epsilon}$) keep reducing as the configuration gets smaller, becomes zero (meaning that the shape becomes spherical) and further on gets to a negative value. However, the negative value attains a minimum and then again the ellipticity starts increasing.

The fact that the ellipticity starts decreasing and further becomes negative, could be
due to the reason that the force balance equation used in defining the ellipticity is perhaps true only for a homogeneous distribution of the fluid, whereas we have in the above varying density configurations. However, a point that needs to be checked carefully is that when Pfister and Braun [11] used the correct centrifugal force expression for obtaining the solution for the interior of a mass shell, they found that a proper boundary fit of the exterior and the interior solutions for the shell, was possible only if the configuration is prolate rather than oblate. Here, in our approach also we start from the equilibration of the forces within the framework of general relativity and get prolate configuration for distributions with inhomogeneous density.

It is also worth noting that for the same configurations, when ellipticity is defined in terms of the radii of the object with constant surface density embedded in a 3-dimensional flat space, a la Hartle-Thorne[10], one gets the ellipticity maximum as was in the case of a purely homogeneous configuration. In this case, the location of the maxima changes with the equation of state, shifting inwards as the equation of state gets stiffer. As the equation of state of any configuration describes the pressure-density relation, the equilibrium of the configuration in a sense depicts the balancing of the various forces like gravity, material binding and the centrifugal force. As the equations of state gets stiffer the intra-nucleonic forces, which are effectively repulsive at very short range become larger, thus requiring the configuration to get more compact before similar behaviour of ellipticity extrema is attained. However, a matter of concern is the result concerning the difference in behaviour of the ellipticity function in the two different treatments, for the inhomogeneous distribution while the conventional Hartle-Thorne way of defining it via embedding in a 3-flat geometry shows the function to be positive, defining it through the balancing of inertial forces a la Maclaurin and Newton, shows it to be negative. It is important to look deeper
into this question to ascertain whether the treatment of defining inertial forces for a fluid configuration has to be different from the approach used for a single particle dynamics, particularly for inhomogeneous distributions.
References

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Figure Captions

Fig. 1 Centrifugal force $\tilde{F}_{cf}$ in units of $(J^2/M^5)$ for decreasing values of radius $R$ in terms of Schwarzschild radius $R_s$ for various equations of state. EOS A is the softest and EOS D represents the stiffest among the four considered equations of state.

Fig. 2 Ellipticity $\bar{\epsilon}$ derived in optical reference geometry, in units of $J^2/M^4$.

Fig. 3 Ellipticity $\bar{\epsilon}_{H-T}$ as defined by Hartle and Thorne, in units of $J^2/M^4$. 

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Table Captions

Table 1 Location of extrema for the centrifugal force ($\tilde{F}_{cf}$) and ellipticity ($\tilde{\epsilon}$) (column 7, 5) and their values (column 6, 4) for the equations of state (Models A, B, C, D) considered in this paper as well as for the homogeneous distribution. Column 8 gives the location of reversal in $\tilde{\epsilon}$. The radius is expressed in terms of Schwarzschild radius $R_s(= 2M)$, whereas the ellipticity and the centrifugal force are expressed in the dimensionless units of $J^2/M^5$ and $J^2/M^4$, respectively.
Table 1

| EOS     | $\tilde{\epsilon}_{H-T}$ | $R_{\tilde{\epsilon}_{H-T}}$ | $\tilde{\epsilon}$ | $R_{\tilde{\epsilon}}$ | $\tilde{F}_{cf}$ | $R_{\tilde{F}_{cf}}$ | $R_{\tilde{\epsilon}_{rev.}}$ |
|---------|---------------------------|-------------------------------|---------------------|-------------------------|---------------|----------------------|--------------------------|
| A       | 0.953                     | 3.278                         | -0.257              | 1.603                   | 0.0335        | 2.234                | 2.297                    |
| B       | 0.849                     | 2.948                         | -0.403              | 1.610                   | 0.0271        | 2.204                | 3.301                    |
| C       | 0.832                     | 2.857                         | -0.395              | 1.769                   | 0.0261        | 2.180                | 3.631                    |
| D       | 0.813                     | 2.625                         | -0.388              | 2.014                   | 0.0252        | 2.112                | 4.343                    |
| Homo.   | 0.761                     | 2.3                           | 1.207               | 2.75                    | 0.0157        | 2.1                  |                          |
Fig. 3

\[\overline{\epsilon_{H-T}} \text{ vs. } \frac{R}{R_s}\]

- EOS A
- EOS B
- EOS C
- EOS D