M-REGULAR DECOMPOSITIONS FOR
PUSHFORWARDS OF PLURICANONICAL BUNDLES
OF PAIRS TO ABELIAN VARIETIES

ZHI JIANG

ABSTRACT. We extend the so called Chen-Jiang decompositions
for pushforwards of pluricanonical bundles to abelian varieties
to the setting of klt pairs. We also provide a geometric application
of this decomposition.

1. INTRODUCTION

Recall the following result, which is very useful in the study of irregular varieties.

Theorem 1.1. Let $X$ be a smooth projective variety and $f : X \to A$ be a morphism from $X$ to an abelian variety. Then

(1) for any $i \geq 0$,
\[ R^i f_* \omega_X \cong \bigoplus_{p_B : A \to B} \bigoplus_k p_B^* \mathcal{F}_{B,k} \otimes Q_{B,k}, \]

where $p_B$ are surjective morphisms between abelian varieties, $\mathcal{F}_{B,k}$ are M-regular sheaves on $B$, and $Q_{B,k}$ are torsion line bundles;

(2) there exists a quotient of abelian variety $p_0 : A \to B_0$ such that for any $i \geq 2$,
\[ f_* \omega_X^i = \bigoplus_k p_0^* \mathcal{F}_k \otimes Q_k, \]

where $\mathcal{F}_k$ are M-regular sheaves on $B_0$ and $Q_k$ are torsion line bundles.

Remark 1.2. (1) The first statement was first proved in [CJ] when $f$ is generically finite onto its image and the general form is proved in [PPS] and the decomposition is referred as Chen-Jiang decomposition. Indeed, the formula is proved in more general
Kähler setting in [PPS]. We will also adapt this terminology in this note.

(2) The second statement was proved in [LPS].

Both statements are quite powerful and it is then natural to ask if we can extend these decomposition theorems to pairs. The main result of this note is to extend both statements to the setting of klt pairs.

**Theorem 1.3.** Let $(X, \Delta)$ be a klt pair and let $f : X \to A$ be a primitive morphism from $X$ to an abelian variety $A$.

1. Suppose that a Weil divisor $D \sim \mathbb{Q} K_X + \Delta$. Then $R^j f_* \mathcal{O}_X(D)$ admits Chen-Jiang decomposition for each $j \geq 0$.

2. Suppose that a Cartier divisor $D \sim \mathbb{Q} m(K_X + \Delta)$ for some $m \geq 2$. Then there exists a quotient between abelian varieties with connected fibers $p : A \to B$, IT$^0$ sheaves $\mathcal{F}_i$ on $B$, and $Q_i \in \text{Pic}^0(A)$ torsion line bundles, for $1 \leq i \leq N$; such that

$$f_* \mathcal{O}_X(D) = \bigoplus_{1 \leq i \leq N} p^* \mathcal{F}_i \otimes Q_i.$$  

**Remark 1.4.** After this note was finished, the author was informed by Fanjun Meng that he proved essentially the same result by some different method. The author likes to thank him by sharing the preprint [M].

In the last section, we give a geometric application of this theorem. Let $S$ be a klt surface of general type and let $D$ be a $\mathbb{Q}$-Cartier Weil divisor on $S$ which is $\mathbb{Q}$-linear equivalent to $K_S$. When $S$ is smooth, by Hirzebruch-Riemann-Roch, we know that $\chi(S, \mathcal{O}_S(D)) > 0$. However, when $S$ is singular, the conclusion does not hold (see Example 4.2). When the klt surface $S$ is moreover of maximal Albanese dimension, we can completely describe the structure of $(S, D \sim \mathbb{Q} K_S)$ such that $\chi(S, \mathcal{O}_S(D)) \leq 0$ (see Theorem 4.3).

2. **Preliminaries**

In [PS], Popa and Schnell proved that the pushforward of pluricanonical bundles of log canonical pairs are GV.

**Theorem 2.1.** Let $(X, \Delta)$ be a log canonical pair and $f : X \to A$ be a morphism to an abelian variety. Suppose that $k(K_X + \Delta)$ is Cartier, then $f_* \mathcal{O}_X(k(K_X + D))$ is GV.

Moreover, Shibata [Sh] studied the cohomological support loci of pluricanonical bundles of log canonical pairs.
**Theorem 2.2.** Let \((X, \Delta)\) be a log canonical pair and let \(f : X \to A\) be a morphism from \(X\) to an abelian variety \(A\).

- Assume that \(D\) is a Cartier divisor such that \(D \sim_{\mathbb{Q}} m(K_X + \Delta)\) for some integer \(m \geq 2\), then
  \[V^j_D(D, f) := \{ P \in \text{Pic}^0(A) \mid h^0(X, \mathcal{O}_X(D) \otimes f^*P) \geq j \}\]
  is a finite union of torsion translates of abelian subvarieties of \(\text{Pic}^0(A)\);
- If \((X, \Delta)\) is moreover log smooth, then for any Cartier divisor \(D \sim_{\mathbb{Q}} K_X + \Delta\), then
  \[V^j_D(D, f) := \{ P \in \text{Pic}^0(A) \mid h^i(X, \mathcal{O}_X(D) \otimes f^*P) \geq j \}\]
  is a finite union of torsion translates of abelian subvarieties of \(\text{Pic}^0(A)\).

These two results strongly imply that the pushforward of pluricanonical bundles of pairs should also admit Chen-Jiang decompositions.

The following result is also very crucial (see [WJ, Corollary 2.13]).

**Theorem 2.3.** Let \(f : X \to Y\) be a surjective morphism between smooth projective varieties and let \(\Delta\) be a \(\mathbb{Q}\)-effective divisor on \(X\) with SNC support and \([\Delta] = 0\). Assume for \(m > 1\) such that \(m\Delta\) is Cartier, \(c_1(f_*\mathcal{O}_X(mK_{X/Y} + m\Delta)) = 0 \in H^2(Y, \mathbb{R})\), then \(f_*\mathcal{O}_X(mK_{X/Y} + m\Delta)\) is a vector bundle with an Hermitian flat metric.

We have the vanishing theorem for klt pairs (see [K1, Corollary 10.15 and Corollary 10.16]).

**Theorem 2.4.** Let \((X, \Delta)\) be a klt pair and let \(f : X \to Y\) be a surjective morphism between projective varieties. Assume that \(M\) is a Cartier divisor on \(X\) such that \(M \equiv K_X + \Delta + N\) with \(N\) an \(\mathbb{Q}\)-divisor.

Then,

1. if \(N\) is \(f\)-nef and big, \(H^i(Y, R^j f_*\mathcal{O}_X(M)) = 0\) for all \(i > 0\) and \(j \geq 0\);
2. if \(N \equiv f^*N_Y\) for some \(\mathbb{Q}\)-Cartier divisor \(N_Y\) on \(Y\), then \(R^i f_*\mathcal{O}_X(M)\) is torsion free for \(i \geq 0\);
3. if \(f\) is birational, \(L \equiv N + \Delta\), and \(N\) is an \(f\)-nef \(\mathbb{Q}\)-divisor, then \(R^i f_* (K_X \otimes L) = 0\) for \(i > 0\).

This result was partially generalized by Fujino ([F, Theorem 2.48]).

**Theorem 2.5.** Let \((X, \Delta)\) be a klt pair and let \(L\) be a \(\mathbb{Q}\)-Cartier Weil divisor on \(X\). Assume that \(L - (K_X + B)\) is nef and big over \(V\), where \(\pi : X \to V\) is a proper morphism. Then \(R^q \pi_* \mathcal{O}_X(L) = 0\) for any \(q > 0\).
3. The proof of the main result

**Proposition 3.1.** Let \((X, \Delta)\) be a klt pair and let \(f : X \to A\) be a morphism from \(X\) to an abelian variety \(A\). Suppose that a Weil divisor \(D \sim_{\mathbb{Q}} K_X + \Delta\). Then \(R^j f_* \mathcal{O}_X(D)\) admits Chen-Jiang decomposition for each \(j \geq 0\).

**Proof.** Take a log resolution \(\mu : X' \to X\) such that \(K_{X'} + \Delta' + E_1 = \mu^*(K_X + \Delta) + E_2\), where \(\Delta'\) is the strict transforms of \(\Delta\), \(E_1\) and \(E_2\) are effective \(\mu\)-exceptional divisors without common components, and \(\Delta' + E_1 + E_2\) has SNC components. Note that \([\Delta' + E_1] = 0\) since \((X, \Delta)\) is klt. Let \(E_2' := [E_2] - E_2\). We then have

\[
K_{X'} + \Delta' + E_1 + E_2' = \mu^*(K_X + \Delta) + [E_2].
\]

Note that \((X', \Delta' + E_1 + E_2')\) is still klt. Let \(D' := \mu^*D + [E_2]\). Then \(D' \sim_{\mathbb{Q}} K_{X'} + \Delta' + E_1 + E_2'\). Since \(\mu\) is birational, by Theorem 2.5, we have

\[
R^j f_* \mathcal{O}_{X'}(D') = \mu_* \mathcal{O}_{X'}(D') = \mathcal{O}_X(D).
\]

Hence \(R^j f_* \mathcal{O}_X(D) = R^j (f \circ \mu)_* \mathcal{O}_{X'}(D')\). Hence we may simply assume that \(X\) is a smooth projective variety and \(\Delta\) has SNC support and \([\Delta] = 0\). Let \(C = D - K_X\). Take the minimal positive integer \(N\) such that \(N\Delta\) is an effective Cartier divisor and \(NC \sim N\Delta\). Take the resolution of singularities of the corresponding \(N\)-cyclic cover, we get a generically finite morphism \(\pi : Y \to X\) such that

\[
\pi_* \omega_Y = \omega_X \otimes \left( \bigoplus_{k=0}^{N-1} \mathcal{O}_X(kC - [k\Delta]) \right).
\]

In particular, for \(k = 1\), we see that \(\mathcal{O}_X(K_X + C) = \mathcal{O}_X(D)\) is a direct summand of \(\pi_* \omega_X\). Thus \(R^i (f \circ \pi)_* \omega_Y\) has a direct summand \(R^i f_* \mathcal{O}_X(D)\). Since \(R^i (f \circ \pi)_* \omega_Y\) admits Chen-Jiang decomposition, we conclude that \(f_* \mathcal{O}_X(D)\) also admits a Chen-Jiang decomposition by [LPS] Porposition 4.6.

\[\square\]

**Proposition 3.2.** Let \((X, \Delta)\) be a klt pair and let \(f : X \to A\) be a primitive morphism from \(X\) to an abelian variety \(A\). Suppose that a Cartier divisor \(D \sim_{\mathbb{Q}} m(K_X + \Delta)\) for some \(m \geq 2\). Then there exists a quotient between abelian varieties with connected fibers \(p : A \to B\), \(IT^0\) sheaves \(\mathcal{F}_i\) on \(B\), and \(Q_i \in \text{Pic}^0(A)\) torsion line bundles, for \(1 \leq i \leq N\); such that

\[
f_* \mathcal{O}_X(D) = \bigoplus_{1 \leq i \leq N} p^* \mathcal{F}_i \otimes Q_i.
\]
Proof. Let \( g : X \rightarrow Y \) be the Iitaka fibration of \((X, D)\). After birational modifications, we may assume that \( g \) is a morphism and let \( F \) be a general fiber of \( g \). Then \((F, \Delta|_F)\) is also a klt pair and \( \kappa(F, D|_F) = \kappa(F, (K_F + \Delta|_F)) = 0 \). Hence \( f(F) \) is a translate of a fixed abelian subvariety \( K \) of \( A \) (see for instance \([WY, \text{Theorem A}]\)). Let \( p : A \rightarrow B = A/K \) be the quotient. We have a commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & A \\
\downarrow g & & \downarrow p \\
Y & \xrightarrow{h} & B.
\end{array}
\]

By Theorem \(2.2\), there exists \( Q_i, 1 \leq i \leq N \) torsion line bundles on \( A \) such that \( V_0^0(f^*\mathcal{O}_X(D)) = \bigcup_{1 \leq i \leq N} (-Q_i + B_i) \), where each \( B_i \) is an abelian subvariety of \( \text{Pic}^0(A) \). It is clear from the construction of \( B \) that each \( B_i \) is contained in a translate of \( \text{Pic}^0(B) \).

On the other hand, for any torsion line bundle \( P \in V_0^0(f^*\mathcal{O}_X(D)) \), \( p^*(f^*\mathcal{O}_X(D) \otimes P) \) is non-trivial. We claim that \( F_P := p^*(f^*\mathcal{O}_X(D) \otimes \mathcal{J}(||(m - 1)(K_X + \Delta)||) \otimes P) \) is IT\(^0\) on \( B \). Indeed, by Theorem \(2.4\), we know that \( F'_P := p^*(f^*\mathcal{O}_X(D) \otimes \mathcal{J}(||(m - 1)(K_X + \Delta)||) \otimes P) \) is IT\(^0\) on \( B \). Moreover, for any \( Q \in \text{Pic}^0(B) \), from the property of multiplier ideal sheaves (see \([L]\)) and the structure of each \( V_0^0(f^*\mathcal{O}_X(D)) \) (see Theorem \(2.2\)), we have

\[
H^0(B, F'_P \otimes Q) = H^0(B, F_P \otimes Q).
\]

We also know by Theorem \(2.1\) that \( F_P \) is GV. Then it is easy to see that the quotient sheaf \( F_P/F'_P \) is also GV and \( V_0^0(F_P/F'_P) \) is empty. Thus \( F_P = F'_P \) is IT\(^0\).

Thus \( V_0^0(f^*\mathcal{O}_X(D)) \) is a disjoint union of torsion translates of \( \text{Pic}^0(B) \) and we may write \( V_0^0(f^*\mathcal{O}_X(D)) = \bigcup_{1 \leq i \leq N} (-Q_i + \text{Pic}^0(B)) \) for some torsion line bundles \( Q_i \in \text{Pic}^0(A) \).

Let \( \mathcal{F}_i := p^*(f^*\mathcal{O}_X(D) \otimes Q_i^{-1}) \) be the corresponding IT\(^0\) sheaf on \( B \). We know consider the natural map

\[
\varphi : \mathcal{F} := \bigoplus_{1 \leq i \leq N} p^*\mathcal{F}_i \otimes Q_i \rightarrow f^*\mathcal{O}_X(D).
\]

From the construction, we know that for any \( P \in \text{Pic}^0(A) \),

\[
H^0(A, \mathcal{F} \otimes P) \xrightarrow{\varphi} H^0(A, f^*\mathcal{O}_X(D) \otimes P)
\]

is an isomorphism.
Moreover, we claim that \( \varphi \) is injective. First, we show that
\[
f_*\mathcal{O}_X(D) |_{A_y} \simeq \bigoplus_{1 \leq i \leq N} Q_i^{a_i} |_{A_y},
\]
for \( y \in B \) a general point of \((p \circ f)(X)\). Indeed, by base change we know that \( f_*\mathcal{O}_X(D) |_{A_y} \) is isomorphic to \( f_{y*}\mathcal{O}_{X_y}(D_y) \), where \( X_y \) is the corresponding over \( y \). Since \((X_y, \Delta|_{X_y})\) is klt and \( D|_{X_y} \sim_{\mathbb{Q}} m(K_{X_y} + \Delta|_{X_y}) \), \( f_{y*}\mathcal{O}_{X_y}(D_y) \) is a GV-sheaf. On the other hand, we know from Generic vanishing theory that the natural map \( \text{Pic}^0_y \) corresponding over \( c \) for \( y \) \( \mathcal{V} \) bundles. We then conclude from the structure of \( a \) there exists \( F \)
\[
\text{rk} \mathcal{F} = a_i \text{ and } \varphi \text{ is an injective map.}
\]
Let \( \mathcal{D} := f_*\mathcal{O}_X(D)/\mathcal{F} \). We have the short exact sequence
\[
0 \to \mathcal{F} \to f_*\mathcal{O}_X(D) \to \mathcal{D} \to 0.
\]
Since both \( \mathcal{F} \) and \( f_*\mathcal{O}_X(D) \) are GV sheaves, so is \( \mathcal{D} \). Moreover, \( V^0(\mathcal{D}) \subset V^0(\mathcal{F}) \cup V^0(f_*\mathcal{O}_X(D)) = V^0(\mathcal{F}) = \bigcup_{1 \leq i \leq N} (-Q_i + \text{Pic}^0(B)) \). For all \( Q \in V^0(\mathcal{F}) \), we know that \( p_* (\mathcal{F} \otimes Q) = p_* (f_*\mathcal{O}_X(D) \otimes Q) \). Hence we have the exact sequence
\[
0 \to f_* (\mathcal{D} \otimes Q) \to R^1p_*(\mathcal{F} \otimes Q) \to R^1p_*(f_*\mathcal{O}_X(D) \otimes Q).
\]
We may assume that \( |Q| = [-Q_1] + [Q_2] \in -Q_1 + \text{Pic}^0(B) \), where \( Q_2 \in \text{Pic}^0(B) \), then \( R^1p_*(\mathcal{F} \otimes Q) = (\mathcal{F}_1 \otimes Q_2)^{\oplus h^1(\mathcal{O}_K)} \) is torsion free on its support. Since the map
\[
R^1p_*(\mathcal{F} \otimes Q) \to R^1p_*(f_*\mathcal{O}_X(D) \otimes Q)
\]
is isomorphism on the generic point of their support, we conclude that this map is injective. Hence \( f_* (\mathcal{D} \otimes Q) = 0 \) for all \( Q \in V^0(\mathcal{F}) \). Thus \( V^0(\mathcal{D}) = \emptyset \) and hence \( \mathcal{D} = 0 \). \( \square \)

**Question 3.3.** Does Chen-Jiang decompositions hold for (pluri)-canonical bundles of log canonical pairs?

4. Geometric applications

**Proposition 4.1.** Let \((X, \Delta)\) be a klt pair. Assume that a big \( \mathbb{Q} \)-Cartier Weil divisor \( D \sim_{\mathbb{Q}} K_X + \Delta \) and there exists a morphism \( f : \)
$X \rightarrow A$ which is generically finite onto its image. Then $V^0(f_*, \mathcal{O}_X(D))$ generates $\text{Pic}^0(A)$.

**Proof.** This is a direct generalization of Chen and Hacon’s theorem in the smooth case. The same argument works here. Assume that $V^0(f_*, \mathcal{O}_X(D))$ generates an abelian subvariety $\hat{B}$ of $\text{Pic}^0(A)$. Considering the morphisms:

$$
\begin{array}{c}
X \\
\downarrow \quad \downarrow \quad \downarrow \\
A \\
\downarrow \quad \downarrow \\
B,
\end{array}
$$

where $p$ is the dual morphism of the inclusion $\hat{B} \hookrightarrow \hat{A}$ and $g = p \circ f$.

Let $X_y$ be a connected component of a general fiber of $g$ and let $K$ be the kernel of $p$ and let $f_y : X_y \rightarrow K$ be the corresponding morphism. Then by Proposition 3.1 and base change, $f_y_* \mathcal{O}_{X_y}(D|_{X_y})$ is a direct sum of torsion line bundles. After étale cover of $\pi$, we may assume that $f_y$ is primitive and hence $f_y_* \mathcal{O}_{X_y}(D|_{X_y}) = \mathcal{O}_K$ and hence $f$ is birational. Since $H^0(X_y, \mathcal{O}_{X_y}(D|_{X_y})) \simeq \text{Hom}(\mathcal{O}_{X_y}(D|_{X_y}), \omega_{X_y})^\vee$ and $D|_{X_y} \sim_K K_{X_y} + \Delta|_{X_y}$, we conclude that $D|_{X_y} \sim_K K_{X_y}$. Note that $D$ is big on $X$ and the family of $X_y$ covers $X$, hence $D|_{X_y}$ is a big $\mathbb{Q}$-Cartier divisor. We get a contradiction. $\square$

Let $S$ be a klt surface. Assume that a $\mathbb{Q}$-Cartier Weil divisor $D \sim \mathbb{Q} K_S$ and there exists a morphism $f : S \rightarrow A$ from $S$ to an abelian variety $A$ which is generically finite onto its image. By the main theorem, we know that $\chi(S, \mathcal{O}_S(D)) = h^0(S, \mathcal{O}_S(D) \otimes P) \geq 0$ for $P \in \text{Pic}^0(A)$ general. A natural question is the following: assume moreover that $D$ is big, is it possible that $\chi(S, \mathcal{O}_S(D)) = 0$?

If $S$ is a smooth surface of general type, we know by Riemann-Roch that $\chi(S, \omega_S) > 0$.

**Example 4.2.** Let $C_i \rightarrow E_i$ be bi-elliptic curves of genus $\geq 2$ with the natural involution $\tau_i$, for $i = 1, 2$. Let $S := (C_1 \times C_2)/\tau_1 \times \tau_2$. Then $f : S \rightarrow E_1 \times E_2$ is a degree $2$ morphism. Note that $S$ has quotient and hence klt singularities. It is easy to see that $\chi(S, \omega_S) = \chi(C_1 \times C_2, \omega_{C_1 \times C_2}) > 0$. Let $\pi : C_1 \times C_2 \rightarrow S$ be the quotient morphism. Then $\pi_* \omega_{C_1 \times C_2} = \omega_S \oplus \mathcal{O}_S(D)$, where $D \sim \mathbb{Q} K_S$ since $\pi$ is quasi-étale. Hence $\chi(S, \mathcal{O}_S(D)) = 0$.

**Theorem 4.3.** Let $S$ be a klt surface. Assume that there exists $f : S \rightarrow A$ a primitive morphism which generically finite onto its image and a big $\mathbb{Q}$-Cartier Weil divisor $D \sim \mathbb{Q} K_S$ such that $\chi(S, \mathcal{O}_S(D)) = 0$. 


Then \( f \) is surjective and is of degree 2. Moreover, after an étale cover, \( f \) is birationally equivalent to the morphism \( f \) as in Example 4.2.

**Proof.** By Proposition 3.1, \( f_*\mathcal{O}_S(D) \) admits Chen-Jiang decomposition. After étale covers, we may assume that all torsion line bundles in the decomposition formula are trivial line bundles. Moreover, if \( \mathcal{O}_A \) is a direct summand of \( f_*\mathcal{O}_S(D) \). Then \( f \) is surjective and \( H^2(A, f_*\mathcal{O}_S(D)) \neq 0 \). Then \( H^2(S, \mathcal{O}_S(D)) \neq 0 \). Since klt singularities are Cohen-Macaulay, we have \( H^0(S, \omega_S(-D)) \neq 0 \) by Serre duality. Then \( D \sim K_S \) and \( \chi(S, \omega_S) = 0 \). Let \( S' \to S \) be a resolution of singularities. Then \( \chi(S', \omega_{S'}) = 0 \). This implies that \( S' \) is not of general type and hence fibered by elliptic curves or is birational to an abelian surface. In both case it is easy to see that \( K_S \) cannot be big, which is a contradiction.

Hence

\[
f_*\mathcal{O}_S(D) = \bigoplus_{1 \leq i \leq k} p_i^*\mathcal{F}_i,
\]

where \( p_i : A \to E_i \) are quotients to elliptic curves and \( \mathcal{F}_i \) are ample vector bundles on \( E_i \). By Proposition 4.1 we know that \( k \geq 2 \). We claim that \( \text{rk}\mathcal{F}_i = 1 \). Indeed, \( \mathcal{F}_i = R^1p_i_*(f_*\mathcal{O}_S(D)) \). By Lemma 4.4 we know that the induced morphism \( p_i \circ f : S \to E_i \) is an algebraic fiber space. As \( D \sim Q K_S \), \( E_i = R^1p_i_*(f_*\mathcal{O}_S(D)) \neq 0 \) implies that \( D|_F \sim K_F \), where \( F \) is a general fiber of \( p_i \circ f \) and \( \text{rk}\mathcal{F}_i = 1 \).

Take the smallest integer \( N \) such that \( N(D - K_S) \sim 0 \). We now take this isomorphism to construct the corresponding cyclic quasi-étale cover (see for instance [KM, Definition 2.52]) \( \pi : \tilde{S} \to S \) such that \( \pi^*D \sim K_{\tilde{S}} \). Write

\[
\pi_*\mathcal{O}_{\tilde{S}} = \mathcal{O}_S \bigoplus_{1 \leq i \leq N-1} \mathcal{O}_S(V_i),
\]

where \( V_i = i(K_S - D) \sim Q 0 \) are \( \mathbb{Q} \)-Cartier Weil divisors on \( S \) for \( 1 \leq i \leq N-1 \). Since \( \pi \) is quasi-étale, \( \tilde{S} \) also has klt singularities (see [KM, Proposition 5.20]). Let

\[
g : \tilde{S} \to S \to A
\]

be the composition of morphisms.

We now consider Chen-Jiang decomposition for \( g_*\omega_{\tilde{S}} \). Let \( \tilde{S} \to C_i \xrightarrow{\pi_i} E_i \) be the Stein factorization of the morphism \( p_i \circ g : \tilde{S} \to E_i \). We claim that \( \pi_i \) is also a cyclic cover of degree \( N \). Indeed, since \( \pi \) is a cyclic cover, we may write

\[
(2) \quad \pi_*\mathcal{O}_{\tilde{S}}(K_{\tilde{S}}) = \mathcal{O}_S(K_S) \bigoplus_{1 \leq i \leq N-1} \mathcal{O}_S(K_S - V_i).
\]
Note that since $K_S|_F \sim D|_F$, $V_i|_F \sim 0$. Thus $R^1p_i^*\mathcal{O}_S(K_S - V_i) \neq 0$ is of rank 1 on $E_i$. Moreover, $\omega_{C_i} = R^1g_i^*\omega_S$ (see [K1] Theorem 10.19 and [K2]). We then conclude that

$$\pi_i^*\omega_{C_i} = \omega_{E_i} \bigoplus_{1 \leq i \leq N-1} R^1p_i^*f_*\mathcal{O}_S(K_S - V_i).$$

Hence $\deg \pi_i = N$ and $S$ is indeed birational to the main component of $S \times_{E_i} C_i$. Thus $\pi_i$ is a cyclic cover of degree $N$ between smooth projective curves. Let $\mathcal{V}_i := \bigoplus_{1 \leq i \leq N-1} R^1p_i^*\mathcal{O}_S(K_S - V_i)$ be the direct sum of $N - 1$ line bundles on $E_i$. We then write the Chen-Jiang decomposition of $g_*\mathcal{O}_S(K_S)$ as follows:

$$g_*\mathcal{O}_S(K_S) = \mathcal{O}_A \bigoplus_{1 \leq i \leq k} p_i^*\mathcal{V}_i \oplus \mathcal{M}.$$  

(3)

A priori, $\mathcal{M}$ may not be M-regular. Since $\text{rank } g_*\mathcal{O}_S(K_S) = Nk$, $\text{rank } \mathcal{M} = k - 1$.

Moreover, since $\mathcal{F}_i = R^1p_i^*(f_*\mathcal{O}_S(D)) = R^1p_i^*(f_*\mathcal{O}_S(K_S - V_i))$ is an ample line bundle of $E_i$, $C_i$ is a curve of genus $\geq 2$. For any $1 \leq i \neq j \leq k$, $g$ factors as $\tilde{S} \rightarrow (C_i \times C_j) \times (E_i \times E_j)A$. Hence $p_i^*\mathcal{F}_i \otimes p_j^*\mathcal{F}_j$ is also a direct summand of $g_*\mathcal{O}_S(K_S)$. Hence we can consider the morphisms whose composition is the identity:

$$\text{Id}: p_1^*\mathcal{F}_1 \otimes p_2^*\mathcal{F}_2 \rightarrow g_*\mathcal{O}_S(K_S) \rightarrow p_1^*\mathcal{F}_1 \otimes p_2^*\mathcal{F}_2.$$  

By [3], it is clear that $p_1^*\mathcal{F}_1 \otimes p_2^*\mathcal{F}_2$ is a direct summand of $\mathcal{M}$. Similarly, for any $2 \leq i \leq k$, $p_i^*\mathcal{F}_1 \otimes p_i^*\mathcal{F}_i$ is a direct summand of $\mathcal{M}$. Since there is no nonzero morphisms between $p_i^*\mathcal{F}_1 \otimes p_i^*\mathcal{F}_i$ and $p_i^*\mathcal{F}_1 \otimes p_j^*\mathcal{F}_j$ for $i \neq j$. We conclude that

$$\mathcal{M} \simeq \bigoplus_{2 \leq i \leq k} p_i^*\mathcal{F}_1 \otimes p_i^*\mathcal{F}_i.$$  

(4)

Apply the same argument, we also have the isomorphism

$$\mathcal{M} \simeq \bigoplus_{i \neq 2} p_2^*\mathcal{F}_2 \otimes p_i^*\mathcal{F}_i.$$  

(5)

Combining (4) and (5), we have

$$\bigoplus_{2 \leq i \leq k} p_i^*\mathcal{F}_1 \otimes p_i^*\mathcal{F}_i \simeq \bigoplus_{i \neq 2} p_2^*\mathcal{F}_2 \otimes p_i^*\mathcal{F}_i.$$  

Restrict this isomorphism on a general fiber of $A \rightarrow E_1$, we see that this isomorphism can hold only when $k = 2$. Hence $f$ is of degree 2 and

$$g_*\mathcal{O}_S(K_S) = \mathcal{O}_A \bigoplus_{1 \leq i \leq 2} p_i^*\mathcal{V}_i \bigoplus (p_1^*\mathcal{F}_1 \otimes p_2^*\mathcal{F}_2).$$
Since $g$ factors as $\tilde{S} \to (C_1 \times C_2) \times (E_1 \times E_2) A$, we conclude that $2N \geq N^2$. Hence $N = 2$ and $g$ is birational to the natural morphism

$$(C_1 \times C_2) \times (E_1 \times E_2) A \to A,$$

where both $C_i \to E_i$ are double covers. Hence after étale base change, $f$ is birational equivalent to the morphism in Example (4.2). □

**Lemma 4.4.** Let $S$ be as in Theorem 4.3. Then a smooth model of $S$ is of general type. Moreover, there does not exist a surjective morphism from $S$ to a smooth projective curve of genus $\geq 2$.

**Proof.** Let $\rho : S' \to S$ be a desingularization. Assume that $S'$ is not of general type, then by [Ka], $f \circ \rho : S' \to A$ is either birational or there exists a morphism $g : S' \to C$ to a smooth projective curve of genus $\geq 2$, a quotient between abelian varieties $A \to B$ with connected fiber of dimension 1, and a generically finite morphism $C \to B$ onto its image such that the induced morphism $S' \to C \times_B A$ is birational.

In the first case, $f : S \to A$ is birational. Then $f_*\mathcal{O}_S(D)$ is of rank 1, which is obviously a contradiction to Proposition 4.1.

In the second case, since $(S, \Delta)$ is klt, there exits a natural morphism $h : S \to C$. Since for any $P \in \text{Pic}^0(A)$, $h_*(\mathcal{O}_S(D) \otimes f^*P) \otimes \omega_C^{-1}$ is nef, we conclude that $H^0(S, \mathcal{O}_S(D) \otimes P) \neq 0$, which is again a contradiction. □

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Shanghai Center for Mathematical Sciences, Xingjiangwan Campus, Fudan University, Shanghai 200438, P. R. China
E-mail address: zhijiang@fudan.edu.cn