BOUNDARY $C^{2,\alpha}$ REGULARITY FOR THE OBLIQUE BOUNDARY VALUE PROBLEM OF MONGE-AMPÈRE EQUATIONS

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Abstract. We study the good shape property of boundary sections of convex solutions of the oblique boundary value problem for Monge-Ampère equations
\[
\det D^2 u = f(x) \text{ in } \Omega, \quad D^2 u = \phi(x) \text{ on } \partial \Omega.
\]
In the two-dimensional case, we prove the global $C^{2,\alpha}$ estimate for the solution. When the dimension $n \geq 3$, we show that this estimate still holds if the solution is bounded from above by a quadratic function in the tangent direction. We also obtain an existence result for the convex solution of Monge-Ampère equations with Robin oblique boundary conditions.

Key Words: oblique boundary value problem, Monge-Ampère equation, $C^{2,\alpha}$ regularity

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Running head: Regularity for the oblique derivative problem of Monge-Ampère equation

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1. Introduction

In this paper, all solutions to the Monge-Ampère equations are convex functions. We always assume that \( n \geq 2, \alpha \in (0, 1) \) is a constant, \( \Omega \) is a bounded convex domain in \( \mathbb{R}^n \), and \( 0 < \lambda \leq f(x) \leq \Lambda \) on \( \Omega \), and \( \nu \) is the inward normal vector field on \( \partial \Omega \) if \( \Omega \in C^1 \). Lions, Trudinger and Urbas [14] studied the Neumann problem of the Monge-Ampère equation \[
\det D^2 u = f(x) \text{ in } \Omega, \quad D_{\nu} u = \phi(x) \text{ on } \partial \Omega.
\]
Under the assumptions of \( f, \phi \) and \( \partial \Omega \) are sufficiently smooth, and \( \Omega \) is uniformly convex, they proved the existence and uniqueness of smooth solutions in the space \( C^{3,\alpha}(\bar{\Omega}) \) \((0 < \alpha < 1)\). Subsequently, Urbas [21] and Wang [25] investigated the corresponding oblique derivative problem \[
(1.1) \quad \det D^2 u = f(x) \text{ in } \Omega, \quad D_{\beta} u = \phi(x) \text{ on } \partial \Omega.
\]
In two-dimensional case, when \( \Omega \) is \( C^{2,1} \) and uniformly convex \( f \in C^{1,1}(\bar{\Omega}), \phi \in C^{1,1}(\partial \Omega) \), and \( \beta \in C^{2,1}(\partial \Omega; \mathbb{R}^n) \) is a small \( C^1 \) perturbation of the inner vector field \( \nu \). Urbas [22] proved that the solution belongs to \( C^{2,\alpha}(\bar{\Omega}) \), and this was extended to higher dimensions in [24] under the further assumption that \( \Omega \) is uniformly convex and \( C^{3,1} \) smooth. The method of [14] can also be applied to the oblique derivative problem for other fully nonlinear equations, such as the oblique derivative problem for Hessian equation studied by Ma and Qiu [16], and for the augmented Monge-Ampère equations studied by Jiang and Trudinger [13].

We study the pointwise Schauder regularity at the boundary of the Aleksandrov solution to (1.1), which means we investigate the \( C^{2,\alpha} \) regularity of the solution when the positive coefficient \( f \) is only in \( C^{\alpha}(\bar{\Omega}) \). The interior Schauder regularity of strictly convex solutions is obtained by Caffarelli [2], and a simplified proof can be found in Jian and Wang [12]. The boundary Schauder regularity for the Dirichlet problem is given by Trudinger and Wang in [19], and a pointwise Schauder estimate is provided by Savin in [18]. The global estimate for the second boundary value problem of the Monge-Ampère equation in optimal transportation can be found in works by Delanoé [8], Urbas [23] and Caffarelli [6]. Recently, Chen, Liu, and Wang [6] made substantial improvements to the global \( C^{2,\alpha} \) regularity result for the natural boundary value problem in optimal transport in [6].

When \( n \geq 3 \), the boundary Schauder regularity of (1.1) is not naturally established, as shown in [25] or Example 9.1 in this paper. Following the study of the Dirichlet problem in [18], we additionally assume that \( u \) has quadratic growth at the given boundary point \( x_0 \). That is, there exists a constant \( C_0 > 0 \) and a subdifferential \( p_{x_0} \in \partial u(x_0) \) such that \[
(1.2) \quad u(x) - u(x_0) - p_{x_0} \cdot (x - x_0) \leq C_0 |x - x_0|^2, \quad \forall x \in \partial \Omega.
\]
An example that satisfies the quadratic growth condition is when \( \partial \Omega \) is \( C^{1,1} \) and \( x_0 \) is the maximum value point of \( u \).

Based on the convexity of the solution \( u \), we interpret the equation \( D_{\beta} u = \phi \) in (1.1) in terms of the Dini derivative (see (4.1)).
Theorem 1.1. Let \( n = 2 \) and \( u \in C(\Omega) \) be a solution of (1.1). Suppose that \( \partial \Omega \in C^{1,\alpha} \), \( f \in C^\alpha(\Omega) \), \( \phi \in C^{1,\alpha}(\Omega) \), \( \beta \in C^{1,\alpha}(\partial \Omega; \mathbb{R}^n) \) is an oblique vector field. Then, \( u \in C^{2,\alpha}(\Omega) \).

Theorem 1.2. Let \( n \geq 3 \) and \( u \in C(\Omega) \) be a solution of (1.1). Suppose that \( \partial \Omega \in C^{1,\alpha} \), \( f \in C^\alpha(\Omega) \), \( \phi \in C^{1,\alpha}(\Omega) \), \( \beta \in C^{1,\alpha}(\partial \Omega; \mathbb{R}^n) \) is an oblique vector field. If (1.2) holds, then \( u \in C^{2,\alpha}(x_0) \).

Remark 1.3. If we replace the assumption (1.2) in Theorem 1.2 with

\[ u(x) - u(x_0) - p_{x_0} \cdot (x - x_0) \leq C_0 |x - x_0|^2 + \varepsilon, \quad \forall x \in \partial \Omega \]

where \( \varepsilon > 0 \) is a sufficiently small constant depending on some universal constants, the conclusion still holds. In particular, the solution is actually a \( C^{2,\alpha} \) function around \( x_0 \).

Obviously, Theorem 1.1 can be seen as the extension of the Schauder regularity for Neumann problems of Poisson equations in two dimensions. When \( n \geq 3 \), Theorem 1.2 actually implies that if \( u \) is \( C^{1,1} \) in all tangential directions, then \( u \in C^{2,\alpha}(\Omega) \). We will prove in Section 3 that the quadratic growth condition (1.2) can ensure the strict convexity, and the strict convexity of the solution, especially in the tangential direction, which is crucial for our boundary Schauder regularity. As shown in Example 9.3, we can always construct non-strictly convex solution of the oblique derivative problem, which are \( C^{1,1-\varepsilon} \) along the tangent direction.

Our main tools are normalization and perturbation methods, with key points being the pre-compactness of the oblique derivative problem under normalization, the pre-compactness of the normalized solution, and the compactness of the corresponding oblique boundary values under convergence.

For the normalized oblique derivative problem (1.1), our approach is to study the section section

\[ S_{\xi,\nabla u(x_0)}(x_0) \]

of \( u \) with height \( h \), based point \( x_0 \in \partial \Omega \) and a specific subgradient \( \xi \in \partial u(x_0) \), defined by

\[ S_{\xi,\nabla u(x_0)}(x_0) := \{ x \in \Omega \mid u(x) < u(x_0) + \xi \cdot (x - x_0) + h \}, \]

and its oblique boundary \( G_{h}^{\xi}(x_0) := S_{h}^{\xi}(x_0) \cap \partial \Omega \). In Section 5, we choose a special \( \xi = \nabla u(x_0) \) and write \( S_{\xi,\nabla u(x_0)}(x_0) \) as \( S_{\xi}^{\xi}(x_0) \) or \( S_{\xi}(x_0) \). We will prove that \( S_{\xi}^{\xi}(x_0) \) shrinks to \( \{ x_0 \} \), and there exists a positive constant \( c \) such that for every small \( h \), \( S_{\xi}^{\xi}(x_0) \) satisfies

\[ ch \leq |S_{\xi}^{\xi}(x_0)| \leq c^{-1}h \]

and

\[ cP^{x_0}S_{\xi}^{\xi}(x_0) + (1 - c)P^{x_0}x_0 \subset P^{x_0}G_{h}^{\xi}(x_0) \cap P^{x_0}(2x_0 - G_{h}^{\xi}(x_0)), \]

where \( P^{x_0} = P^{x_0,\beta(x_0)} \) denotes the projection mapping along the direction \( \beta(x_0) \) to the tangent plane \( \partial \Omega \) of at \( x_0 \). We refer to this as the good shape property of \( S_{\xi}^{\xi}(x_0) \), which is invariant under convergence and any linear diagonal transformation \( D \) that keeps \( \beta \) and the tangent plane invariant. As a result, the normalized oblique derivative problems are pre-compactness.

\(^1\)\( \beta \) is oblique (point inward) at \( x_0 \) if \( \{ x_0 + t\beta(x_0) \mid t > 0 \} \cap \Omega \neq \emptyset \).
In the case \( n = 2 \), the good shape properties holds at any boundary point, the engulfing property of sections is naturally induced, and it can be shown that the solution is always \( C^{1,\alpha} \) up to the boundary. However, when \( n \geq 3 \), there are two obstacles, stemming from the lack of compactness of solutions and the corresponding oblique boundary values.

To ensure the compactness of the normalized solution, we prove that the solution with zero oblique boundary values is uniformly strictly convex near the given boundary point. In Section 6, we will use this fact and the good shape property to construct a universal strictly convex modulus for solutions of normalized oblique derivative problems, which is equivalent to a pointwise uniform \( C^{1,\alpha} \) estimate at the given boundary point \( x_0 \).

In Section 7, we study the compactness of oblique boundary values. By combining the interior equation and the convexity of the domain, we investigate the upper semicontinuous (convex) viscosity subsolutions of (1.1). Several theorems on existence and compactness are presented here.

In Section 8, we will study the blow-up limit and prove that the normalized oblique boundary \( \tilde{G}_h \) tends to flat, so that the blow-up limit corresponds to solutions to the Neumann problem for the Monge-Amp`ere equation in the half-space under some specific restrictions.

**Theorem 1.4 ([11] Theorem 1.1).** Let \( u \in C(\mathbb{R}^n_+) \) be a convex solution of
\[
\det D^2u = 1 \text{ in } \mathbb{R}^n_+, \quad D_nu = ax_1 \text{ on } \partial \mathbb{R}^n_+.
\]
If \( n = 2 \), or if \( n \geq 3 \) and \( a = 0 \), or \( n \geq 3 \) and \( u \) satisfies \( \lim_{z \in \mathbb{R}^{n-2}, z \to \infty} \frac{u(0,z,0)}{|z|^2} < \infty \), then \( u \) is a quadratic polynomial.

Then a standard perturbation methods as in [12] gives the \( C^{2,\alpha} \) estimate. In Section 9, we will list some examples that illustrate the obstacles that may arise in oblique derivative problems.

### 2. Preliminaries

In this paper, a point in \( \mathbb{R}^n \) is written as
\[
x = (x_1, \ldots, x_{n-1}, x_n) = (x', x_n),
\]
and if \( n \geq 3 \), it will also be written as \( x = (x_1, x'', x_n) \). Define \( \mathcal{P}x = x' \), the projection mapping along the \( x_n \)-axis to the hyperplane \( \mathbb{R}^{n-1} := \{x_n = 0\} \), and set
\[
\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n | x_n > 0\}.
\]
Denote by \( I \) [or \( I', I'' \)] the identity matrix of size \( n \) [or \( n-1, n-2 \)] and \( B_r(x) \) [or \( B'_r(x'), B''_r(x'') \)] the ball of radius \( r \) centered at \( x \) [or \( x', x'' \)] in \( \mathbb{R}^n \) [or \( \mathbb{R}^{n-1}, \mathbb{R}^{n-2} \)].

The term “universal constant” refers to a positive constant that depends only on \( \text{diam}(\Omega) \) and \( \|\partial\Omega\|_{C^{1,\alpha}}, \alpha, \beta, \Lambda, \|\phi\|_{Lip}, \) the oblique constant \( \eta \) of \( \beta \), \( \|\beta\|_{Lip} \), the quadratic constant \( C_0 \) whenever they are involved, but not on the specific functions \( u \) and \( v \). We denote these universal constants
by $c$ if small or $C$ if large. Moreover, we denote $C_q$ as a positive quantity that depends only on the universal constants and the parameters $q$. Given two non-negative quantities $b_1$ and $b_2$, we will use the notation

$$ b_1 \lesssim b_2 $$

(2.1)

to indicate that $b_1 \leq C b_2$ for a universal constant $C$. We also denote

$$ b_1 \approx b_2 $$

if $b_1 \lesssim b_2$ and $b_2 \lesssim b_1$. We will write $A \leq B$ for matrices $A$ and $B$ when $B-A$ is positive semidefinite. Similarly, we use $A \lesssim B$ for two positive semidefinite matrices $A$ and $B$ to indicate that $A \leq C B$, and use $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. The symbol $\gtrsim$ is understood in a similar way. If the constant $C$ is replaced by $C_q$, then we use the symbols $\lesssim_q$, $\gtrsim_q$ and $\approx_q$ instead.

In this paper, we need some preliminaries on convex solutions to Monge-Ampère equations. Readers can refer to related chapters in [1, 2, 3, 9, 10, 20].

For $E, F \subset \mathbb{R}^n$ and $\kappa \in \mathbb{R}$ we define

$$ E + F := \{x + y| x \in E, y \in F\}, \quad \kappa E = \{\kappa x| x \in E\}, $$

and we write $-E$ for $(-1)E$, $E - F$ for $E + (-F)$ and $E + x$ for $E + \{x\}$, where $x \in \mathbb{R}^n$. The Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$ is denoted by $|E|$. For $\kappa > 0$, a bounded convex set $E \subset \mathbb{R}^n$ is said to be $\kappa$-balanced about a point $x$ if

$$ t(x - E) \subset E - x \text{ for all } t \in [0, \kappa]. $$

If $\kappa$ can be taken to be universal, then we say that $E$ is balanced about $x$.

**Lemma 2.1** (John’s Lemma). Suppose $S \subset \mathbb{R}^n$ is a bounded convex set and $\mathring{S} \neq \emptyset$. Then there is an ellipsoid $E$ (called the John ellipsoid, the unique ellipsoid of the maximal volume contained in $S$) such that

$$ E \subset S \subset C(E - x_E) + x_E, $$

where $x_E$ is the mass center of $E$, and $C$ depends only on $n$.

**Lemma 2.2** ([11, Lemma 2.4]). Given convex set $E \subset \mathbb{R}^n$ and point $x \in \mathbb{R}^n$. Suppose that the line $x + te_n : t \in \mathbb{R}$ intersects $E$ at the points $y$ and $z$. Then, we have

$$ |y_n - z_n| \cdot \text{Vol}_{n-1} \mathcal{P}E \lesssim |E|. $$

Moreover, if $G$ is $\kappa$-balanced about $G$, then we have the inverse inequality

$$ |y_n - z_n| \cdot \text{Vol}_{n-1} \mathcal{P}E \gtrsim \kappa |E|. $$

The subdifferential of a convex function $u$ at $x_0 \in \Omega$ is

$$ \partial u(x_0) = \{p \in \mathbb{R}^n| u(x) \geq u(x_0) + p \cdot (x - x_0) \text{ for all } x \in U\}.$$
Each element \( p \in \partial u(x_0) \) is called a subgradient of \( u \) at \( x_0 \), and the function \( u(x_0) + p \cdot (x - x_0) \) is a supporting hyperplane function of \( u \) at \( x_0 \). We denote \( \partial u(E) := \bigcup_{x \in E} \partial u(x) \) for \( E \subset \subset \Omega \), which is measurable if \( E \) is open or closed, and hence is measurable if \( E \) is Borel. The Monge-Ampère measure \( M_u \) is defined by

\[
M_u(E) = |\partial u(E)| \quad \text{for each Borel set } E \subset \subset U,
\]

we say \( u \) is a (generalized/Aleksandrov) solution to \( \det D^2 u = f \) if \( M_u = f dx \). We will use the following comparison principle for generalized solutions to Monge-Ampère equations, which can be found in, e.g., [9] or [10].

**Lemma 2.3.** Suppose \( \Omega \) is a bounded convex set, and \( u \in C(\overline{\Omega}) \) and \( v \in C(\overline{\Omega}) \) are convex functions. If \( \det D^2 u \leq \det D^2 v \) in \( \Omega \), \( u \geq v \) on \( \partial \Omega \), then \( u \geq v \) in \( \Omega \).

As a corollary,

**Lemma 2.4.** Suppose \( \Omega \) is a bounded convex set, \( G_1 \) and \( G_2 \) are closed sets on \( \partial \Omega \) with no interior intersection and satisfy \( G_1 \cup G_2 = \partial \Omega \), and \( \beta \) is oblique on \( \partial \Omega \setminus G_1 \). Let \( u \in C(\overline{\Omega}) \) and \( v \in C(\overline{\Omega}) \) be two convex functions that satisfy

\[
\begin{aligned}
\det D^2 v &\leq \det D^2 u \quad \text{in } \Omega, \\
u < v &\quad \text{on } G_1, \\
D_\beta v &< D_\beta u \quad \text{on } G_2.
\end{aligned}
\]

Then, \( u < v \) in \( \overline{\Omega} \).

The following lemmas are well known for convex functions and are corollaries of the Aleksandrov-Bakelman-Pucci maximum principle, see [10, Theorem 1.4.2].

**Lemma 2.5** (Aleksandrov’s Maximum Principle). Suppose \( u \in C(\overline{\Omega}) \) is convex, \( u = 0 \) on \( \partial \Omega \). There exists \( C > 0 \), depending only on the dimension \( n \), such that

\[
u(x) \geq -C \left( (\text{diam}(\Omega))^n - 1 \text{ dist } (x, \partial \Omega) M_u(\Omega) \right)^{\frac{1}{n}}.
\]

By comparison with an appropriate quadratic function, we can obtain from Lemma 2.3 that

**Lemma 2.6.** Let \( u \) be a convex function on \( \Omega \) satisfying \( \det D^2 u > \lambda \) in \( \Omega \), then \( |\Omega| \leq C(\lambda) \|u\|_{L^\infty(\Omega)}^\frac{2}{n} \). In particular, we always have \( |S_{h,\rho}^u(x_0)| \lesssim h^\frac{2}{n} \).

### 3. A Qualitative Strict Convexity Lemma

In this section, we introduce the qualitative strict convexity lemma for solutions of Monge-Ampère equations. We first review some lemmas about the strict convexity of convex functions, which can be found in [11, 3].
Lemma 3.1. Assuming $n=2$, $\Omega \subset \mathbb{R}^2$ is a convex domain, $u \in C(\Omega)$ is convex, and
\[ \lambda \leq \det D^2u \leq \Lambda \text{ in } \Omega. \]

Then $u$ is strictly convex and $C^1$ in $\Omega$.

Lemma 3.2. Suppose that $n \geq 3$, $\Omega \subset \mathbb{R}^n$ is a convex domain, $u \in C(\bar{\Omega})$ is convex function, and
\[ \lambda \leq \det D^2u \leq \Lambda \text{ in } \Omega, \quad u = \varphi \text{ on } \partial \Omega. \]

If $\varphi \in C^{1,\alpha}(\partial \Omega)$ for $\alpha > 1 - \frac{2}{n}$, then $u$ is strictly convex and $C^1$ in $\Omega$.

Lemma 3.3. Suppose that $u \in C(\Omega)$ is convex and satisfies $0 < \lambda \leq \det D^2u \leq \Lambda$. For $x \in \Omega$ and $p \in \partial u(x)$, we define $\Sigma := \{ y | u(y) = u(x) + p \cdot (y - x) \}$. Then, either $\Sigma = \{ x \}$ or $\Sigma$ has no extreme point inside $\Omega$.

As is well-known, the $C^{1,\alpha}$ and $C^{2,\alpha}$ interior estimates of strictly convex solutions to the Monge-Ampère equation were established by Caffarelli [1, 2, 3]. Also see the related chapters in [9, 10].

In this section, we always assume that $0 \in \partial \Omega$ and
\[ \Omega \cap B_c(0) := \{ (x', x_n) | x_n > g(x'), x' \in B'_c(0) \}, \]

such that the part of the boundary $\partial \Omega$ near 0 is described by a locally Lipschitz function $g$ (or $C^{1,\alpha}_{loc}$ if $\partial \Omega \in C^{1,\alpha}$) as
\[ G := \{ (x', x_n) | x_n = g(x'), x' \in B'_c(0) \}. \]

Since $\Omega$ is convex, $g$ is convex and satisfies
\[ 0 \leq g(x') \leq C|x'|, \quad \forall x' \in B'_c(0). \]

We use $Gx' = G(x', 0)$ to denote the point $(x', g(x'))$, where We will use the following notation,
\[ Gx = \begin{cases} (x', x_n) & \text{if } x_n \geq g(x'), \\ (x', g(x')) & \text{if } x_n \leq g(x'). \end{cases} \]

Lemma 3.4. Let $u \in C(\bar{\Omega})$ be a convex function, $u \geq 0$ and satisfies
\[ 0 < \lambda \leq \det D^2u \leq \Lambda \text{ in } \Omega. \]

Then there exists a module $\sigma_1$ such that if $\|u\|_{L^\infty(\Omega \cap B_c)} \leq K$ and
\[ \frac{\text{Vol}_{n-1}(P \hat{S}_h(0) \cap (-P \hat{S}_h(0)) \cap B'_1(0))}{\hat{h}^{n-1}} \geq K^{-1}, \quad \forall h \geq \sigma_1, \]

where $\hat{S}_h(0) := \{ x \in \Omega | u(x) \leq h \}$, then $u(0) \geq \sigma_1(K^{-1})$.

\[ ^2 \text{A point is an extreme point of a convex set } E \text{ if it does not lie in any open line segment joining two points of } E. \]
\[ ^3 \text{In this paper, a module } \sigma_i(t) \text{ represents a non-negative, strictly increasing, and continuous function that depends on the parameter } t \text{ and satisfies } \sigma_i(0) = 0. \]
Proof. Assume that \( u(0) \leq \sigma_1 \) (to be determined) is very small. For any \( h \geq C\sigma_1 \), let \( a_h = \sup \{ t \mid t e_n \in \hat{S}_h(0) \} \). Note that \( u(he_n) \leq hu(e_n) + (1 - h)u(0) \leq Kh + \sigma_1 \), so we have \( a_h \geq ch \), where \( c \) depends on \( K \). On the other hand, by Lemma 2.2 and Lemma 2.6 we have

\[ a_h \text{Vol}_{n-1}(\mathcal{P}\hat{S}_h(0) \cap (-\mathcal{P}\hat{S}_h(0))) \leq a_h \text{Vol}_{n-1}(\mathcal{P}\hat{S}_h(0)) \lesssim |\hat{S}_h(0)| \lesssim h^{\frac{2}{n}}, \]

which together with (3.5) implies that \( a_h \leq Ch \). Therefore, \( ch \leq a_h \leq Ch \). For each \( t \geq C\sigma_1 \), we now consider the non-increasing function

\[ (3.6) \quad b(t) := \frac{u(t e_n)}{t} \approx 1. \]

We claim that there exists \( \delta \in (0,1) \) depending on \( K \) such that

\[ (3.7) \quad b(t/2) < (1 - \delta)b(t), \quad \forall t \geq \delta^{-1}\sigma_1. \]

In contrast to (3.7), suppose there exists a positive constant \( t_0 \geq C\delta^{-1}\sigma_1 \) such that \( b(t_0/2) \geq (1 - \delta)b(t_0) \). For simplicity, we can assume that

\[ b(t_0) = 1 \quad \text{and} \quad b(t_0/2) \geq 1 - \delta. \]

Let

\[ E := \{(x', x_n) \in \hat{S}_{t_0}(0) \mid x' \in \mathcal{P}\hat{S}_{t_0}(0) \cap (-\mathcal{P}\hat{S}_{t_0}(0))\}. \]

Then \( \mathcal{P}E \) is balanced about 0, by Lemma 2.2 we have

\[ (3.8) \quad 0 \leq \sup \{ x \mid x = (x', x_n) \in E \} < \frac{t_0}{4}. \]

Now, let \( v(x) = u(x) - x_n \) and take the points \( Y_1 = t_0e_n \) and \( Y_2 = \frac{t_0e_n}{2} \). We have,

\[ -\delta t_0 \leq v(Y_1) \leq 0, \quad -\delta t_0 \leq v(Y_2) \leq 0, \quad \text{and} \quad v \leq u \leq \delta t_0 \quad \text{in} \quad E. \]

Consider the cone domains

\[ \Gamma_1^+ := \{ Y_1 + s(z - Y_1) \mid s \in (0,1), z \in \mathcal{P}E \}, \quad \text{and} \quad \Gamma_2^- := \{ Y_2 - s(z - Y_2) \mid s \in (1/2,2), z \in \mathcal{P}E \}. \]

By convexity, we have

\[ -C\delta t_0 \leq v \leq C\delta t_0 \quad \text{in} \quad \Gamma_1^+ \cap \Gamma_2^-. \]

Then Lemma 2.6 implies \( |\Gamma_1^+ \cap \Gamma_2^-| \lesssim (\delta t_0)^{\frac{2}{n}} \), which contradicts

\[ |\Gamma_1^+ \cap \Gamma_2^-| \geq c_{t_0} \text{Vol}_{n-1}(\mathcal{P}E) \geq cK^{-1}\delta^{-1}(\delta t_0)^{\frac{2}{n}} \]

provided \( \delta \) is small. Thus, we have proved (3.7). By iteration, (3.7) implies

\[ b(t) \lesssim t^{\left|\log_2(1 - \delta)\right|} \quad \text{for} \quad t \geq \delta^{-1}\sigma_1, \]

which contradict (3.6) when \( \sigma_1 \) is small.

□
Lemma 3.5. Let $u \in C(\Omega)$ be a convex function, $u \geq 0$, and $0 < \lambda \leq \det D^2u \leq \Lambda$ in $\Omega$. Assume that $u \leq 1$ in $\Omega$ and satisfies

\begin{equation}
(3.9) \quad u(x) \leq |x''|^2 \quad \text{on } G \cap \{x_1 = 0\},
\end{equation}

then there exists a modulus $\sigma_2$ such that $u(x) \geq \sigma_2(|x_1| + |x_n|)$ in $\Omega \cap B_r(0)$.

Proof. Fix $\rho > 0$ small and take a point $y = (y_1, y', y_n) \in \Omega$ satisfying $|y_1| + y_n = \rho$. Our goal is to get a lower bound $u(y)$ in terms of $\rho$. We can simplify the problem by using the reflection transformation with respect to the $e_1$ axis and the sliding transformation $Ax := (x_1, x'' - \frac{x_1 + x_n}{\rho} y'', x_n)$. Therefore, without loss of generality, we can assume that $y = y_1 e_1 + y_n e_n$. Then, we apply a rotational transformation $B$ on the plane spanned by $e_1$ and $e_n$, which transforms the vectors $y_1 e_1 + y_n e_n$ and $y_n e_1 - y_1 e_n$ into $\rho e_1$ and $\rho e_n$, respectively. After that, we consider a translation transformation $T$ that moves the point $\frac{1}{\rho} e_1$ to the origin. Let $v(x) = u(B^{-1} T^{-1} x)$ and

$$
\Omega_v := TB \{\{ty + (1-t)z| z \in \Omega \cap \{x_1 = 0\} : t \in (0,1)\},
\end{equation}

then

$$
0 \leq v \lesssim 1 \quad \text{and} \quad \det D^2v \approx_\rho 1 \quad \text{in } \Omega_v.
$$

By locally writing

$$
G_v := TB \{\{ty + (1-t)z| z \in G \cap \{x_1 = 0\} : t \in (0,1)\} = \{(x', x_n)| x_n = g_v(x')\},
\end{equation}

we can see that $g_v$ is convex in $B'_\rho(0)$ and satisfies $0 \leq g_v(x') \lesssim_\rho |x'|$. Combining convexity and the assumption (3.9), we have

$$
v(x) \lesssim \rho^{-1}|x''|^2 + u(y) \quad \text{on } G_v \cap B^+_{\rho \lambda}(0),
$$

which implies that $\tilde{S}_h := \{x \in \Omega_v| v(x) \leq h\}$ satisfies

$$
\frac{\text{Vol}_{n-1}(P'' \tilde{S}_h \cap (-P'' \tilde{S}_h) \cap E)}{h} \lesssim_\rho 1 \quad \text{for } h \geq 2u(y).
$$

Then, a modified version of Lemma 3.3 gives a lower bound for $v(0)$, which in turn gives a lower bound for $u(y)$.

\square

4. Global Lipschitz Regularity of solutions

If $\partial \Omega \in C^1$, $\beta$ is oblique (point inward) on $\partial \Omega$ means that $\beta \cdot \nu \geq \eta > 0$ for some $\eta$, where $\nu$ is the inward normal vector field. If $\Omega$ is only a bounded convex domain, we say that $\beta$ is oblique (point inward) at $x_0$ provided $\{x_0 + t\beta(x_0)| t > 0\} \cap \Omega \neq \emptyset$. Since $\beta$ is continuous, by compactness we can assume the existence of constant $\eta$, called the oblique constant of $\beta$, depending on $\|\beta\|_{C^\alpha}, \|\partial \Omega\|_{\text{Lip}}$, such that

$$
\eta \leq \|\beta\| \leq \eta^{-1}, \quad \text{and} \quad \{x_0 + t(\beta(x_0) + B_\eta(0))| t \in (0, \eta)\} \subset \Omega.
$$

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Combining the convexity of $\Omega$ and the oblique property $\beta$, the set
\[
\{y + t\beta(y) \mid y \in \partial \Omega, t > 0\}
\]
covers a neighborhood of $\partial \Omega$.

Let $u$ be a solution of (1.1). Locally, after translation and rotation, we assume that the boundary point $x_0 = 0$, and
\[
\Omega \subset \{x_n \geq 0\} \quad \text{satisfies (3.1)-(3.3)}.
\]
Therefore, we always consider $\phi$ and $\beta$ as functions depending only on the variable $x'$, and use $\phi(x') = \phi(Gx')$ and $\beta(x') = \beta(Gx')$ to denote them, respectively.

The Dini derivative of $u$ along any direction $\gamma$ at $x$, where $\gamma$ is oblique if $x \in \partial \Omega$, is denoted by
\[
D_\gamma u(x) := D_\gamma^+ u(x) = \lim_{t \to 0^+} \sup_{p \in \partial u(x + t\gamma(x))} p \cdot \gamma \in [-\infty, +\infty).
\]
If $\gamma$ is continuous, then $D_\gamma u$ is an upper semicontinuous function in $\Omega$. If $u$ is Lipschitz near $x_0$, which is always true for $x_0 \in \Omega$, then we have
\[
D_\gamma u(x) = \sup_{p \in \partial u(x)} p \cdot \gamma = \lim_{t \to 0^+} D_\gamma u(x + t\gamma(x)).
\]

We assume $\beta(0) = e_n$ by considering the function $u(\mathcal{B}y)$. Here, the sliding transformation $\mathcal{B}x = \beta_n x + \sum_{i=1}^{n-1} \beta_i x_i e_i$ is defined by the expression $\beta(0) = (\beta', \beta_n)$. By proving that $u \in \text{Lip}(\bar{\Omega})$, we locally replace the oblique equation $D_\beta u = \phi$ with the Dini derivative $D_\gamma u$. We let $\gamma = e_n$ and choose $\nabla u(x)$ such that (4.2) holds, where such a definition is not unique and depends on the oblique vector $\gamma$ near $x_0$. Additionally, we use $\nabla u(x)$ to denote any element in $\partial u(x)$ for any $x \in \Omega$. The choice of $\nabla u(x)$ does not affect the proof. Finally, by subtracting the support function $u(0) + \nabla u(0) \cdot x$, we can assume that
\[
u(0) = 0, \quad \nabla u(0) = 0, \quad u \geq 0 \quad \text{in} \quad \Omega, \quad \beta(0) = e_n, \quad \phi(0) = 0.
\]
For simplicity, we will always write $S^w_{u,\nabla u(0)}(0)$ and $G^w_{h,\nabla u(0)}(0)$ as $S_h$ and $G_h$, respectively.

To provide a Lipschitz bound, we now demonstrate how to control the gradient in the remaining directions from the oblique direction.

**Lemma 4.1.** For any convex domain $\Omega$ satisfying (3.1)-(3.3), assume that $u \in C \left(\Omega \cap B^+_e(0)\right)$ is convex, for any $x \in \Omega \cap \bar{B}^+_e(0)$ and $p \in \partial u(x)$, we have
\[
|p'| \lesssim \frac{2 \|u\|_{L^\infty(\Omega \cap B^+_2(0))} + |p_n| (x_n + |x'|)}{|x'|}
\]
and
\[
|p'| \lesssim \inf_{r \geq |x'|} \left( \frac{2 \|u\|_{L^\infty(\Omega \cap B^+_2(0))}}{r} - p_n \right).
\]
Proof. We may assume that $p' \neq 0$. By choosing the point $y \in \partial \Omega$ such that $y' = x' + \frac{p'}{|p'|}x'$, we have $y_n \lesssim |y'| \lesssim |x'|$ and

$$|p'| \cdot |x'| + |p_n| (x_n + |x'|) \lesssim |p'| \cdot |x'| + p_n(y_n - x_n) = p \cdot (y - x) \leq u(y) - u(x) \leq \|u\|_{L^\infty(\partial \Omega \cap B_{2|x|}(0))}.$$ 

For any $r > |x'|$, by choosing a point $z \in \Omega$ such that $|p'| \cdot (z - x) = r(c|p'|, |p'|)$, we have

$$r(p_n + c|p'|) = p \cdot (z - x) \leq u(z) - u(x) \leq \|u\|_{L^\infty(\Omega \cap B_{2|x|}(0))}.$$

Lemma 4.2. Let $u$ be a solution to (1.1). Suppose $x_0 \in \partial \Omega$, then there exists $p \in \partial u(x_0)$ satisfying (4.2), for $\gamma = \beta$, denoted by $\nabla u(x_0)$, such that

$$|\nabla u(x_0)| \lesssim \omega_u(\Omega) + \|\min \{\phi, 0\}\|_{L^\infty(\partial \Omega)},$$

where $\omega_u(\Omega)$ denotes the oscillation of $u$ on $\Omega$. Therefore,

$$\|u\|_{L^\infty(\Omega)} \leq \omega_u(\Omega) + \|\min \{\phi, 0\}\|_{L^\infty(\partial \Omega)}.$$

Proof. Applying Lemma 4.1 we obtain that for each $x_t := x_0 + t\beta$ is bounded by $C \left(\omega_u(\Omega) + \|\min \{\phi, 0\}\|_{L^\infty(\partial \Omega)}\right)$, and as $t \to 0^+$, they subconverge to some support function $\ell_0$ of $u$ at $x_0$. It can be verified that the gradient $p := \nabla \ell_0$ satisfies (4.2).

The proof is completed by applying Lemma 4.1 and noting that $\{y + t\beta(y) | y \in \partial \Omega, t \geq 0\}$ covers a neighborhood of the $\partial \Omega$ and $u$ is convex.

Theorem 4.3. Let $u$ be a solution to (1.1). Suppose $\phi \in C(\partial \Omega)$, then

$$\omega_u(\Omega) \lesssim \text{diam}(\Omega) \|\min \{\phi, 0\}\|_{L^\infty(\partial \Omega)}.$$

Hence, $\|u\|_{L^\infty(\Omega)} \leq (1 + \text{diam}(\Omega)) \|\min \{\phi, 0\}\|_{L^\infty(\partial \Omega)}$.

Proof. We can assume that $u$ attains its maximum at the boundary point $0$, $\beta(0) = e_n$ and $\Omega$ satisfies (3.1)-(3.3). By writing $\nabla u(0) = \phi(0)e_n + be$ with unit vector $e \in \mathbb{R}^{n-1}$ and $b \geq 0$, we have

$$bt + \phi(0)g(te_n) = \nabla u(0) \cdot G(te) \leq u(G(te)) - u(0) \leq 0, \quad \forall t > 0 \text{ small}.$$ 

Therefore, $b \lesssim |\min \{\phi(0), 0\}| \leq \|\min \{\phi, 0\}\|_{L^\infty(\partial \Omega)}$ and then $|\nabla u(0)| \lesssim \|\min \{\phi, 0\}\|_{L^\infty(\partial \Omega)}$, which implies that

$$\inf_{\Omega} u \geq \inf_{\Omega} (u(0) + \nabla u(0) \cdot x) \geq \sup_{\Omega} u - Cdiam(\Omega) \|\min \{\phi, 0\}\|_{L^\infty(\partial \Omega)}.$$ 

Lemma 4.4. Let $u$ be a solution to (1.1). Suppose $\phi \in C(\partial \Omega)$, $0 \in \partial \Omega$, $\beta(0) = e_n$, $\Omega$ satisfies (3.1)-(3.3), and (4.3) holds. Then

$$|D_n u(\mathcal{G}x') - \phi(x')| \lesssim \|u\|_{L^p(\Omega)} |\beta(x') - e_n| \lesssim |x'| \quad \text{on} \quad \partial \Omega \cap B_{\epsilon}(0).$$
If we also assume that \( u(x) \lesssim |x|^{1+\alpha_0} \) on \( \partial\Omega \) for \( \alpha_0 \in [0, 1] \), then we can improve (4.4) to:

\[
(4.5) \quad |D_n u(\mathcal{G}x') - \phi(x')| \lesssim |x|^{1+\alpha_0} \text{ on } \partial\Omega \cap B_c(0).
\]

Proof. Given a point \( x = \mathcal{G}x' \in \partial\Omega \cap B_c(0) \), based on (4.2) and the definition of \( \nabla u(x) \), we obtain that

\[
D_n u(\mathcal{G}x') = \sup_{p \in \partial u(x)} p \cdot e_n \geq \nabla u(x) \cdot e_n \geq \nabla u(x) \cdot \beta(x') - C|\beta(x') - e_n| \cdot |\nabla u(x)| \\
\geq \phi(x') - C \|u\|_{Lip(\overline{\Omega})} |x'|.
\]

Since \( \beta \in Lip(\partial\Omega) \) and \( \beta(0) = e_n \), for every fixed \( x = \mathcal{G}x' \in \partial\Omega \cap B_c(0) \) and \( t > 0 \) small, we can write \( x + te_n = x_s + s\beta(x_s') \), where \( x_s \in \partial\Omega \) and \( s > 0 \) satisfying

\[
s \approx t, \quad |\beta(x_s') - e_n| \leq \frac{1}{2} |x_s'| \quad \text{and} \quad |x_s'| \leq 2|x'|.
\]

Then, by abuse the notation \( \nabla \), we have

\[
D_n u(x + te_n) \leq D_{\beta(x_s')} u(x_s + s\beta(x_s')) + C|\beta(x_s') - e_n| \cdot |\nabla u(x_s + s\beta(x_s'))|.
\]

Now consider the family of functions \( \{\phi_t\}_{t \geq 0} \) defined by \( \phi_t(y') := D_{\beta(y')} u(\mathcal{G}y' + t\beta(y')) : B_0(0) \to R \). \( \phi_t \) are all upper semicontinuous and converge decreasingly to the continuous function \( \phi \) as \( t \to 0^+ \). Therefore, this convergence is uniform and there exists a modulus \( \sigma_3(t) \) (depending on \( u \)) with \( \sigma_3(0) = 0 \), such that

\[
(4.7) \quad \phi_t(y') \leq \phi(y') + \sigma_3(t).
\]

Hence,

\[
D_n u(\mathcal{G}x') = \lim_{t \to 0^+} D_n u(x + te_n) \\
\leq \lim_{t \to 0^+} \left[ D_{\beta(x_s')} u(x_s + s\beta(x_s')) + C|\beta(x_s') - e_n| \cdot |\nabla u(x_s + s\beta(x_s'))| \right] \\
\leq \lim_{t \to 0^+} \left( \phi(x_s') + \sigma_3(Cs) + C|\beta(x_s') - e_n| \cdot |\nabla u(x_s + s\beta(x_s'))| \right) \\
\leq \phi(x') + C|\beta(x') - e_n| \cdot \|u\|_{Lip(\overline{\Omega})} \\
\leq \phi(x') + C|x'| \cdot \|u\|_{Lip(\overline{\Omega})},
\]

and (4.4) follows.

Finally, if \( u(x) \lesssim |x|^{1+\alpha_0} \) on \( \partial\Omega \), note that \( \beta \) is still oblique, (4.4) and a modified version of Lemma 4.1 imply that \( |\nabla u(x)| \lesssim |x|^{\alpha_0} \) on \( \partial\Omega \), and the same proof yields (4.5). \( \square \)

5. Good Shape Property

Let \( u \) be a solution to (1.1), and assume that \( \partial\Omega \in C^{1,\alpha}, f \in C^0(\overline{\Omega}) \) with \( 0 < \lambda \leq f(x) \leq \Lambda \), \( \phi \in C^{1,\alpha}(\overline{\Omega}) \), \( \beta \in C^{1,\alpha}(\partial\Omega; \mathbb{R}^n) \) is an oblique vector field. As in Section 3, we always assume that

\[
x_0 = 0, \quad \text{and } \Omega \subset \{x_n \geq 0\} \text{ satisfies } 3.1 - 3.3.
\]
and
\[ u(0) = 0, \quad \nabla u(0) = 0, \quad u \geq 0 \text{ in } \Omega, \quad \beta(0) = e_n, \quad \phi(0) = 0. \]
Furthermore, when \( n \geq 3 \), we assume that (1.2), i.e.
\[ 0 \leq u(x) \lesssim |x|^2, \quad \forall x \in \partial \Omega, \]
and then \( |\nabla u(x)| \lesssim |x'| \) on \( \partial \Omega \).

Lemma 5.1. For all small \( h > 0 \) and \( x \in \mathbb{R}^n \), suppose the line \( \ell(t) := x + t e_n, \ t \in \mathbb{R} \) intersects \( S_h(0) \) at the points \( y \) and \( z \), then
\[ |y_n - z_n| \cdot \text{diam } S_h(x_0) \lesssim \frac{|S_h(0)|}{h^{n/2}} \lesssim h. \]

Proof. From Lemma 2.6, we know that \( |S_h(0)| \lesssim h^{n/2} \), so the lemma holds naturally for \( n = 2 \). For \( n \geq 3 \), we notice that \( h^{1/2} B_c(0) \subset P S_h(0) \) and so
\[ |y_n - z_n| \cdot \text{diam } S_h(x_0) \cdot h^{n/2} \lesssim |S_h(0)| \lesssim h^{n/2}. \]

By Lemma 5.3 we now have

Lemma 5.2. There exists a module \( \sigma : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( \lim_{h \to 0} \sigma(h) = 0 \) such that
\[ \text{diam } S_h(x_0) \leq \sigma(h). \]

Starting from this Section, unless otherwise stated, we always consider the mixed problem:
\[ (5.2) \quad \det D^2 u = f \text{ in } \Omega, \quad D_n u = \phi^0 \text{ on } G := \partial \Omega \cap B_c(0), \]
where \( \phi^0(x') := D_n u(G x') \) satisfies \(|\phi^0| \leq C|x'| \) around 0. The purpose of this section to show the good shape property around 0.

Theorem 5.3. For all small \( h > 0 \), we have
\[ (5.3) \quad c P S_h(0) \subset P G_h(0) \subset P S_h(0), \]
\[ (5.4) \quad c h^{n/2} \leq |S_h(0)| \leq C h^{n/2}, \]
\[ (5.5) \quad P G_h(0) \subset -C P G_h(0). \]

Remark 4.5. By noting that
\[ u(x) - u(y) - \nabla u(y) \cdot (x - y) \lesssim C|x - y|^2 + \varepsilon_0, \quad \forall x \in \partial \Omega \]
holds at \( y \in \partial \Omega \) sufficiently close to 0 for small \( \varepsilon_0 > 0 \). We can also show the good shape properties at \( y \in \partial \Omega \) sufficiently close to 0 for \( h \geq \sigma_4(\varepsilon_0) \). Then the boundary section \( S_h(y) \) looks like the cylindrical domain
\[ \{ z + t \beta(y) | y + t \beta(y) \in S_h(y) \text{ and } z \in P^u G_h(y) \} \]
with the projection \( P^u G_h(y) \) being balanced about \( y \), where \( P^u \) is the projection along the \( \beta(y) \) direction to the tangent plane of \( \partial \Omega \) at \( y \).
The right side of (5.4) is the result of Lemma 2.6. The remaining parts will be proven by Lemmas 5.5, 5.6 and 5.7.

**Lemma 5.5.** For all small $h > 0$, we have

\begin{equation}
\mathcal{P}S_h(0) \subset \mathcal{P}S_h(0) \subset (1 + C\|\phi^0\|_{L^\infty}) \mathcal{P}G_h(0) \subset C\mathcal{P}G_h(0).
\end{equation}

**Proof.** For any fixed unit vector $e$ such that $e \perp e_n$, we let

\[ K := \sup \{ s \mid s \in \mathcal{P}S_h(0) \} / \sup \{ s \mid t \in \mathcal{P}G_h(0), \forall t \in (0, s) \}. \]

To simplify, let us assume that $e = e_1$ and that the two maxima in the above equation are achieved by points $y = (y_1, 0, y_n) \in \overline{S_h(0)}$ and $z = (z_1, 0, z_n) \in \overline{G_h(0)}$ respectively, where $y_1 = \sup \{ s \mid s \in \mathcal{P}S_h(0) \}$ and $z_1 = \sup \{ s \mid s \in \mathcal{P}G_h(0), \forall t \in (0, s) \}$. Then, $u(z) = u(y) = h$, $y_1 = Kz_1$, and $y_n \geq 0$ is small. Additionally, $z$ lies below the line connecting $y$ and $0$, and we let

\[ H := \frac{y_n}{K} - z_n \leq \frac{y_n}{K} \lesssim K^{-1}. \]

Lemma 5.1 implies that $Hy_1 \lesssim h$, and hence $KHz_1 \lesssim h$. Since $y_1 = Kz_1$, we see that

\[ z + He_n = K^{-1}y \in \{ sy, s \in (0, 1) \} \cap \{ z + te_n, t \in [0, \infty) \}. \]

The convexity implies that

\[ u(K^{-1}y) \leq K^{-1}u(y) + (1 - K^{-1})u(0) = K^{-1}h, \]

while (4.4) implies that

\[ u(z + He_n) \geq u(z) - \|\phi^0\|_{L^\infty} Hz_1 \geq (1 - CK^{-1}\|\phi^0\|_{L^\infty})h. \]

Combining these two inequalities, we obtain that $K \leq 1 + C\|\phi^0\|_{L^\infty}$. \hfill $\square$

Now, for any point $x \in \Omega$ near 0 and constant $\kappa \in [0, 1]$, we can take the point

\begin{equation}
y_{\kappa, x} = \mathcal{G}(c\mathcal{P}(\kappa x)) \in \partial \Omega,
\end{equation}

so that $y_{\kappa, x}' = c\kappa x'$ and

\[ u(y_{\kappa, x}) \leq u(\kappa x) \leq \kappa u(x) + (1 - t)u(0) \leq \kappa u(x). \]

**Lemma 5.6.** For all small $h > 0$, we have

\[ |S_h(0)| \gtrsim h^{\nabla}. \]

**Proof.** Let $h$ be small. In contrast to our Lemma, we now assume that $\epsilon := |S_h(0)| / h^{\nabla}$ is very small. Let $m_h$ denote the mass center of $S_h := S_h(0)$, then $z := y_{\epsilon, m_h} \in S_h$. Note that $0 \in \mathcal{P}S_h$, so $\mathcal{P}S_h$ is balanced about $z'$. By John’s Lemma 2.2, we can find a transformation $D' = \text{diag} \{ d_1, \cdots, d_{n-1} \}$ in some suitable orthogonal frame such that

\begin{equation}
\mathcal{P}S_h - z' \subset D'B_{\kappa}(0) \quad \text{and} \quad \det D' \approx \text{Vol}_{n-1} \mathcal{P}S_h.
\end{equation}
Now, let \( d_n := \sup \{ t | z + t e_n \in S_h(0) \} \) and \( D = \text{diag} \{ D', d_n \} \), then
\[
(5.9) \quad \det D \approx |S_h|,
\]
Let \( Ax := x - z - \ell(x')e_n \), where \( \ell \) is the support function of \( G \) at \( z \), then \( D^{-1}AS_1 \subset B_{G}(0) \).
Consider the function
\[
w(x) = \frac{u(A^{-1}Dx)}{h}.
\]
Then \( w(0) = \frac{d_n}{h} \leq \frac{1}{2} \), and by Lemma 5.1, we have on \( D^{-1}A(\partial S_h \setminus G_h) \) that
\[
|D_n w(x)| = \left| \frac{d_n \phi^0(D'x')}{h} \right| \lesssim \frac{d_n |D'x'|}{h} \lesssim d_n \text{diam} (PS_h) \frac{|x'|}{h} \lesssim \frac{|S_h| |x'|}{h} \lesssim \epsilon |x'|.
\]
Therefore,
\[
\begin{cases}
\det D^2 w = \frac{(\det D)^2 f(A^{-1}Dx)}{h^2} \approx \epsilon^2 & \text{in } D^{-1}A S_h, \\
|D_n w| \lesssim \epsilon & \text{on } D^{-1}A G_h, \\
w = 1 & \text{on } D^{-1}A (\partial S_h \setminus G_h).
\end{cases}
\]
We now claim that for large enough \( K > 0 \), the convex function
\[
v(y) = w(0) + \sum_{i=1}^{n} \frac{y_i^2}{8nK^2} + \frac{y_n}{4K}
\]
satisfies
\[
\begin{cases}
\det D^2 w < \det D^2 v & \text{in } D^{-1}A S_h, \\
D_n w < D_n v & \text{on } D^{-1}A G_h, \\
v < w = 1 & \text{on } D^{-1}A (\partial S_h \setminus G_h).
\end{cases}
\]
Since the oblique assumption \( e_n \cdot \nu > 0 \) still holds, according to Lemma 2.1, we have \( v < w \), which contradicts the fact that \( v(0) = w(0) \). This completes the proof.

By direct computation, we can verify our claim as follows:
- in \( D^{-1}A S_h \), \( \det D^2 v \gtrsim 1 > C \epsilon^2 \approx \det D^2 w \);
- on \( D^{-1}A G_h \), \( D_n v \gtrsim 1 > C \epsilon \gtrsim D_n w \);
- on \( D^{-1}A (\partial S_h \setminus G_h) \), \( v < w(0) + \frac{1}{2} \lesssim 1 = w \).

\[\square\]

**Lemma 5.7.** For all small \( h > 0 \), we have
\[
(5.10) \quad PG_h(0) \subset -C PG_h(0).
\]
Therefore, \( PG_h(0) \) is balanced about 0.

**Proof.** As in the proof of Lemma 5.6, we choose the point \( m_h \) and set \( z = y_{1 \cdots m_h} \). We also choose an appropriate orthogonal so that the diagonal transformation \( D' = \text{diag} \{ d_1, \cdots, d_{n-1} \} \) satisfies Equation 58, and let \( D = \text{diag} \{ D', d_n \} \). For simplicity, we assume that \( d := d_1 \geq d_2 \geq \cdots \geq d_{n-1} \gtrsim h^{\frac{1}{2}} \).
Considering the function
\[
\tilde{u}(x) = \frac{u(Dx)}{h} \quad \text{and} \quad \tilde{\phi}^0(x') = D_nu(Dx) = \frac{d_nD_n\phi^0(D'x')}{h},
\]
we write $\tilde{S}_1 := \tilde{S}_1^n = D^{-1}S_h(0)$ and $\tilde{G}_1 := \tilde{G}_1^n = D^{-1}G_h(0)$.

Let $K$ be a large constant, and let $\delta = \frac{1}{K^{1/4}}$ and $\kappa = K^{-\frac{1}{8}}$ be small. If

$$\mathcal{P}G_h(0) \not\subset -\delta^{-1}\mathcal{P}G_h(0).$$

From [5.3] (Lemma [5.5]), we have

$$\text{dist}(0, \partial \mathcal{P} \tilde{S}_1) \approx \text{dist}(0, \partial \mathcal{P} \tilde{G}_1) \leq \delta.$$ 

This implies that $d \gtrsim \delta^{-1}h^{\frac{1}{3}}$. Therefore, we have on $\tilde{G}_1$ that

$$|\tilde{\vartheta}(x')| \lesssim \frac{d_n|D'x'|}{h} \lesssim \begin{cases} \frac{d_n|d_1|}{h} \lesssim \delta & \text{if } n = 2, \\ \frac{d_n|d_1| + d_n d_2}{h} \lesssim \frac{h^\frac{1}{2}}{d_2} |x_1| + \frac{h^\frac{1}{2}}{d_2} \lesssim \frac{h^\frac{1}{2} |x_1|}{d_2} + \delta & \text{if } n \geq 3. \end{cases}$$

Furthermore, when $n \geq 3$, we will describe the boundary values more carefully. Take a point $q = \tilde{G}q'$ such that $\text{dist}(0, q') = \text{dist}(0, \partial \mathcal{P} \tilde{G}_1)$, and without loss of generality, assume that $q_1 \geq 0$.

We consider two cases: $d_2 \leq K h^\frac{1}{2}$ or $d_2 \geq K h^\frac{1}{2}$.

**Case 1:** $d_2 \leq K h^\frac{1}{2}$. Then we have

$$B_{cK^{-1}}(0) \subset \mathcal{P} \tilde{S}_1 \subset B_{c}(0).$$

By convexity, we also have $\sup \{ t | t e_1 \in \mathcal{P} \tilde{G}_1 \} \lesssim K \delta \lesssim K^{-2}$, so (5.11) becomes

$$|\tilde{\vartheta}(x')| \lesssim K^{-1}$$

provided $x_1 \geq -\frac{1}{K}$.

**Case 2:** $d_2 \geq K h^\frac{1}{2}$. Then (5.11) implies

$$|\tilde{\vartheta}(x')| \lesssim K^{-1}.$$ 

In this case, we consider a new orthogonal coordinate system with origin at zero and $e_1 = q' / |q'|$ as the axis, while keeping the normal and tangent planes invariant. We still denote this coordinate system as $(e_1, e_2, \cdots, e_n)$.

Regardless of the case and dimension, we always define

$$E = \Omega \cap \left\{ x_1 \geq -\frac{1}{K} \right\} = \Omega \cap \left\{ -\frac{1}{K} \leq x_1 \leq \frac{C}{K^2} \right\}$$

and let $m_E$ be the mass center of $E$. We take $z_E = y_{c, m_E}$ and $A_{E}x := x - z_E - \ell_E(x')e_n$, where $\ell_E$ is the support function of $\tilde{G}$ at $z_E$, and consider the function

$$w(x) := \frac{\tilde{u} (A_{E}^{-1} x)}{h}.$$ 

We have $w(0) \leq \kappa$, and the domain $F = A_{E}E$ satisfies

$$F \subset [-CK^{-1}, CK^{-1}] \times B_{cK^{-1}}(0) \times [0, CK^{-1}].$$

Let $a = \frac{1}{K} - z_E \cdot e_1$. Since $A_{E} \left\{ x_1 \geq -K^{-1} \right\} = \left\{ x_1 \geq -a \right\}$, we can write $\partial F = \partial_1 F \cup \partial_2 F \cup \partial_3 F$, where

$$\partial_1 F = \partial F \cap A_{E} \tilde{G}_1(0) \cap \{ y_1 > -a \}, \quad \partial_2 F = \partial F \cap \{ y_1 = -a \}, \quad \partial_3 F = \partial F \setminus (\partial_1 F \cup \partial_2 F).$$
Since $|z_E \cdot e_1| \leq C \kappa K^{-1}$, we have $a \approx K^{-1}$. Noting that $-y \cdot p \leq w(0) - w(x) \leq w(0) \leq \kappa$ holds for any $x$ on $\partial_2 F$ any any $p \in \partial w(x)$, we have the following.

\[
\begin{aligned}
&\text{det} D^2 w \approx c \quad \text{in } F, \\
&D_n w \lesssim K^{-1} \quad \text{on } \partial_1 F, \\
&D_{-y} w \lesssim \kappa \quad \text{on } \partial_2 F, \\
&w = 1 \quad \text{on } \partial_3 F.
\end{aligned}
\]

Let

\[
Q(x) = \frac{1}{4n} \left( \frac{x^2}{2} + 2x \right), \quad x \in \mathbb{R}
\]

and consider the convex function

\[
v(y) = w(0) + \left[ Q \left( \frac{y_1}{a} \right) + \kappa \left( Q \left( \frac{\kappa y_n}{C} \right) + \sum_{i=2}^{n-1} Q \left( \frac{y_i}{C} \right) \right) \right].
\]

We claim that

\[
\begin{aligned}
&\text{det} D^2 w < \text{det} D^2 v \quad \text{in } F, \\
&D_n w < D_n v \quad \text{on } \partial_1 F, \\
&D_{-y} w < D_{-y} v \quad \text{on } \partial_2 F, \\
v < w \quad \text{on } \partial_3 F.
\end{aligned}
\]

Then, we can apply the comparison principle to obtain $v < w$, which contradicts the fact that $v(0) = w(0)$. This completes the proof.

By direct computation, we can verify (5.13) as follows:

- in $F$, $\text{det} D^2 v \gtrsim \kappa^{n+1} a^{-2} \gtrsim K^2 \kappa^{n+1} \gtrsim K > \text{det} D^2 w$;
- on $\partial_1 F$, $v \leq v(0) + \frac{3n}{4n} = w(0) + \frac{3}{4} \leq \kappa + \frac{3}{4} < 1 = w$;
- on $\partial_2 F$, $D_{-y} v \geq D_{-y} y \left( \frac{y_n}{C} \right) - C \kappa \geq c - C \kappa > \kappa \geq D_{-y} w(0)$;
- on $\partial_1 F$, $D_n v \gtrsim \kappa^2 > CK^{-1} \gtrsim D_n w$.

6. Uniformly Strict Convexity for Normalization Family

Following Section 5 we now introduce the (Sliding) Normalization Family related to the oblique derivative problem. We can assume that $\nabla' \phi(x') = ae_1$. Fix a small $h$ and $y_h \in PS_h(0)$ such that

\[
|y_h \cdot e_1| = \sup \{ x \cdot e_1 | x \in PS_h(0) \}.
\]

Consider the sliding transformation

\[
A_h x = x + \sum_{i=2}^{n-1} \frac{y_h \cdot e_1}{y_h \cdot e_1} x_i e_i.
\]

\[\text{A short segment along the vector } -y \text{ at point } y \in \partial_1 F \cap \partial_2 F \text{ lies in } \bar{F}, \text{ and a modified version of Lemma 2.4 still implies } v < w.\]
Using the good shape lemma (Theorem 5.3), we can find a new orthogonal coordinates in which the directions of \( e_1 \) and \( e_n \) remain unchanged, and
\[
\mathcal{D}_hB_{c(n)}(0) \subset \mathcal{A}_h^{-1}S_h(0) \subset \mathcal{D}_hB_{c(n)}(0)
\]
holds for some diagonal matrix \( \mathcal{D}_h := \text{diag}(d_1(h), \cdots, d_n(h)) \), where we further require
\[
\Pi_{i=1}^n d_i(h) = \det \mathcal{D}_h = |S_h|.
\]
Note that the sliding transformation \( \mathcal{A}_h \) does not change the tangent plane \( \mathbb{R}^{n-1} \). Assuming that \( d_2 \geq d_3 \geq \cdots \geq d_{n-1} \geq ch^{\frac{n}{2}} \), we also have
\[
d_1d_2d_n \leq h^{\frac{n}{2}} \text{ when } n \geq 3.
\]

**Definition 6.1.** Let \( \mathcal{T}_h = \mathcal{A}_h \circ \mathcal{D}_h = \text{diag}(\mathcal{T}'_h, d_n(h)) \), we define the sliding normalization \((\tilde{u}, \tilde{\Omega})\) of \((u, \Omega)\) as follows
\[
\tilde{u}_h(x) := \frac{u(\mathcal{T}_h x)}{h}, \quad x \in \tilde{\Omega}_h := \mathcal{T}_h^{-1}\Omega.
\]
For simplicity, we omit the subscript \( h \) when there is no confusion. \( \tilde{u} \) is a solution to
\[
\begin{cases}
\det D^2\tilde{u} = \tilde{f} & \text{in } \tilde{S}_1, \\
D_n\tilde{u} = \tilde{\phi}^0 & \text{on } \tilde{G}_1, \\
\tilde{u} = 1 & \text{on } \partial\tilde{S}_1 \setminus \tilde{G}_1,
\end{cases}
\]
where
\[
\tilde{f}(x) = \frac{(\det D)^2}{h^n} f(\mathcal{T}_h x), \quad \tilde{\phi}^0(x') = \frac{d_n\phi^0(\mathcal{T}'_h x')}{h}.
\]

We now define the Neumann coefficient
\[
a_h = \frac{d_1(h)d_n(h)}{h}a, \quad \text{then } |a_h| \lesssim 1, \quad \text{and if } n \geq 3, \quad |a_h| \lesssim \frac{h\frac{1}{2}}{d_2} \lesssim 1.
\]
We will see that there exists \( a_0 > 0 \) such that
\[
|\tilde{\phi}_h(x')^0 - a_h x_1| \lesssim |d_n(h)|^\alpha \text{ on } \tilde{G}_1.
\]
Also, the quadratic growth assumption gives,
\[
|a_h|\tilde{u}_h(G(0, x'')) \lesssim \sum_{i=2}^{n-1} d_i(h)^2 x_i^2
\]
In this section, we discuss the uniformly strict convexity of the solutions of normalization families, which will be used in the next section to compensate for the lack of compactness.

**Theorem 6.2** (Uniformly Strict Convexity). There exists small positive constant \( \delta_0 \) such that
\[
\begin{align*}
(1 + \delta_0)\delta_0 S_h(0) \cap \Omega & \subset S_{\delta_0h}(0), \\
(1 + \delta_0)S_{\delta_0h}(0) \cap \Omega & \subset S_h(0), \\
S_{\delta_0h}(x) & \subset cS_h(0) \text{ if } x \in S_{\delta_0h}(0).
\end{align*}
\]
By iteration, \((6.4)\) and \((6.5)\) imply for \(m = 1, 2, \cdots\),
\[
\delta_0^{m(\log(1+\delta_0)+1)} S_h(0) \subset S_{\delta_0^{m-1}}(0) \subset \delta_0^{m(\log(1-\delta_0))} S_h(0).
\]

Thus,
\[
c_{\delta_0} |x - x_0|^\alpha \leq u(x) - u(x_0) - \nabla u(x_0) \cdot (x - x_0) \leq C_{\delta_0} |x - x_0|^1 \alpha_0 \quad \text{for} \quad x \in S_{t,p}(0),
\]
with \(\alpha_0 = \inf \left\{ \log(1 + \delta_0), \frac{\log(1-\delta_0)}{\log(1-\delta_0)} \right\} \in (0, 1)\). Recalling Lemma 4.1, it will imply that
\[
(6.7) \quad |\nabla \tilde{u}(x)| \lesssim |x|^\alpha_0 \quad \text{in} \quad S_1(0).
\]

We refer to \((6.4)\) as the upper uniformly strict convexity lemma and \((6.5)\) as the lower strict convexity lemma. By \((6.7)\), the engulf property \((6.6)\) is a direct consequence of \((6.4)\) and \((6.5)\).

**Proof of Theorem 6.2 in the case of \(n = 2\)**. We can apply Lemma 6.2 to the normalization \((\tilde{u}_h, \tilde{S}_h)\) to obtain that \(\text{diam} \tilde{S}_h(0) \leq \sigma(t)\), which yields \((6.5)\). Then, \((6.4)\) follows from \((6.5)\) and the good shape lemma \((6.3)\). \(\square\)

**Remark 6.3.** When \(n = 2\), recall \((6.7)\), or when \(n \geq 3\), recall \((4.3)\), we always have \(|\phi_0(x') - ax_1| \lesssim |x'|^{1+\alpha_0}\) on \(G\), and thus \((6.2)\) is given by direct calculation as follows:
\[
|\phi_0(x') - ax_1| \lesssim \frac{|d_1| \cdot |d_n|^{1+\alpha_0}}{h} \lesssim |d_n|^{\alpha_0} \lesssim |\sigma(h)|^{\alpha_0} \quad \text{on} \quad \tilde{G}_1.
\]

**Remark 6.4.** Let \(\Theta\) be any local estimate or property that is related to \((\tilde{u}, \tilde{\Omega})\), and assume that \(\Theta\) is invariant under the normalization transformation and linear transformations that preserve the tangent plane. From \((6.6)\), it follows that if \(\Theta\) holds locally (near 0), then \(\Theta\) holds globally in \(B_{\sigma}(0)\). In particular, we can use Theorem 6.2 to obtain Lipschitz estimates for \(\tilde{u}\) and \(\tilde{g}\) on \(B_{\sigma}(0)\) and \(B'_{\sigma}(0)\).

The remaining part of this section is mainly devoted to the proof of Theorem 6.2 in the case of \(n \geq 3\). This is divided into two cases: the first case where \(|a_h|\) is small and the result will hold naturally, and the second case where \(|a_h|\) is not small and we need to use the quadratic growth assumption \((5.1)\) and the qualitative strict convexity lemma \((5.5)\).

### 6.1. The Zero Neumann Boundary Value Problem

We first consider the case where \(|a_h| = 0\), which leads us to study the following degenerate problem. In this section, we consider the reflection \(R(x', x_n) = (x', -x_n)\), and the reflection image of a set \(S\) is denoted by \(RS := \{(x, -x_n) | x \in S\}\)

**Theorem 6.5.** Assume that \(E\) is a bounded convex domain symmetric about the plane \(\mathbb{R}^{n-1} := \{x_n = 0\}\), and let \(f\) satisfies \(0 < \lambda \leq \hat{f} \leq \Lambda\) and \(\hat{f} \circ R = \hat{f}\) in \(E\). Let
\[
S := \{x \in \hat{E} | x_n \geq \hat{g}(x') \geq 0, x \in \hat{PE}\}
\]
be a closed convex subset satisfying \(\mathcal{P}S = \mathcal{P}E\), where \(\hat{g}\) is nonnegative. Assume that \(v\) is a convex solution to
\[
(6.8) \quad \det D^2 v = \hat{f} \chi_{S \cup RS} \quad \text{in} \ E, \ v = 0 \quad \text{on} \ \partial E,
\]
where $\chi$ is the characteristic function. Then, $D_n v = 0$ on $\hat{G} := \partial S \setminus \partial E$, $v$ satisfies the uniformly strict convexity Theorem 6.2 on $\hat{G}$, and $v \in C^{1,\alpha}_\text{loc}(E)$ for small constant $\alpha = \alpha_0(n, \lambda, \Lambda)$.

Proof.

$$\hat{G} x = \begin{cases} (x', |x_n|) & x \in S \cup RS, \\ (x', \hat{g}(x')) & x \in E \setminus (S \cup RS). \end{cases}$$

Without loss of generality, we assume that $|S| \neq 0$. By convexity, we have $u \circ \hat{G} \geq v$. For any points $y, z \in E$, the point $\hat{G}(\frac{y + z}{2})$ is contained in the simplex generated by the vertices $\hat{G}y, \hat{G}z, R\hat{G}y$ and $R\hat{G}z$. Thus,

$$\hat{G} \left( \frac{y + z}{2} \right) = \kappa \left( \hat{G}y + \hat{G}z \right) + (1 - \kappa) \left( \frac{R\hat{G}y + R\hat{G}z}{2} \right)$$

holds for some $\kappa \in [0, 1]$. Therefore, $v \left( \hat{G} \left( \frac{y + z}{2} \right) \right) \leq \frac{v(\hat{G}y) + v(\hat{G}z)}{2}$, and hence $u \circ \hat{G}$ is a convex function satisfying

$$\det D^2 u \circ \hat{G} \geq \hat{f}_S \circ \partial RS = \det D^2 v \text{ in } E, u \circ \hat{G} = 0 = v \text{ on } \partial E.$$

By the comparison principle, we have $v \geq u \circ \hat{G}$, so $u = u \circ \hat{G}$. In particular, we have $D_n v = 0$ on $E \setminus S$, which means that all the information of $v$ is contained in $\hat{S}$.

For any fixed point $y \in \hat{G}$, we now choose $p \in \partial v(y)$ such that $p \cdot e_n \geq 0$ and let $S_h^u(y) := S_{h,p}^u(y)$. We claim that if $S_h^u(y) \cap S$ is strictly contained in $E$, then $S_h^u(y)$ is of good shape at $y$. More precisely, there exist positive constants $c$ and $C$ depending only on $n, \lambda$ and $\Lambda$ such that

$$\mathcal{P} \left( S_h^u(y) \cap \hat{G} \right) = \mathcal{P} \left( S_h^u(y) \cap S \right) = \mathcal{P} S_h^u(y),$$

$$c h^\frac{2}{n} \leq |S_h^u(y) \cap S| \leq C h^\frac{2}{n},$$

$$\mathcal{P} G_h^u(y) \subset -\mathcal{C} \mathcal{P} G_h^u(y),$$

where $G_h^u(y) = S_h^u(y) \cap \hat{G}$. We obtain (6.9) from the fact $v = v \circ \hat{G}$, while the right-hand side of (6.10) comes from Lemma 2.6. Now let $z'$ be the mass center of $\mathcal{P} S_h(y)$, let $z = (z', 0)$, nd consider the sliding transformation $A_z x = (x', x_n - \ell_z(x'))$, where $\ell_z$ is any support function of $\hat{G}$ at $z$. Let $S_z = A_z (S_h(y) \cap S) \cup R A_z (S_h(y) \cap S)$, $E_z := A_z S_h(y) \cup R A_z S_h(y)$, and consider the function

$$v_z(x) = v \left( A_z^{-1}(x', |x_n|) \right) - v(y) - p' \cdot (x - y) - h, \quad x \in E_z.$$}

We can normalize $v_z$ and $E_z$ such that $B_{\epsilon}(z) \subset E_z \subset B_C(z)$ and $\| \det D^2 v_z \|_{L^\infty} \approx 1$. Noting that $\mathcal{P} S_z = \mathcal{P} E_z$ and $f_{\chi_{E_z}}$ satisfies the double measure property at $z$ (see [3] or Section 3.1 in [10] for the definition), we have

$$-v_z(y) \geq -v_z(z) \approx 1.$$}

Then, the Aleksandrov’s Maximum Principle implies that $E_z$ is balanced about $z$. This gives (6.10) and (6.11).

By following the standard proof of Theorem 1 in [1], we can see that the good shape lemma will prohibit the existence of lines on the graph of $v$ with endpoints on $\hat{G} \cup R\hat{G}$, thereby giving the corresponding uniform strict convexity and $C^{1,\alpha}$ regularity of the solution.
6.2. Proof of Theorem 6.2

Proof. By the good shape lemma, we have the decomposition
\[ c \left( \mathcal{P} \tilde{G}_1(0) + (\tilde{S}_1(0) \cap \{t \geq 0\}) \right) \subset \tilde{S}_1(0) \subset C \left( \mathcal{P} \tilde{G}_1(0) + (\tilde{S}_1(0) \cap \{t \geq 0\}) \right). \]

Using a similar iterative method, the proof of Theorem 6.2 is equivalent to showing separate proofs along the normal direction \( \tilde{S}_1(0) \cap \{t \geq 0\} \) and the tangent direction \( \mathcal{P} \tilde{G}_1(0) \).

We first prove (6.4) in the normal direction. Let \( k = \frac{1}{2} < 1 \), it is enough to prove
\[ \delta \tilde{S}_1(0) \cap \{t \geq 0\} \subset k \tilde{S}_1(0) \cap \{t \geq 0\} \text{ for a small } \delta > 0. \]

Otherwise, we have
\[ \tilde{u}(t) \geq kt \text{ for } t \geq \delta. \]

According to Lemma 3.4 and good shape Lemma (Theorem 5.3),
\[ \text{Vol}_{n-1} \left( \mathcal{P} \tilde{S}_1(0) \cap \mathcal{P}(-\tilde{S}_1(0)) \cap B'(0) \right) \geq ck \text{ for } t \geq c\delta. \]

Since \( \|\tilde{u}\|_{L^\infty} \leq 1 \), Lemma 3.4 implies \( \delta \geq \sigma_1(k) \), which is a contradiction.

Let \( \kappa < \sigma_1(k) \) be a small positive constant, and let \( \alpha_0 > 0 \) be a constant that is smaller than the values given by Theorem 6.5, taking
\[ K = \kappa^{-n-1}, \delta = K^{-\frac{2(n+7)(1+\alpha_0)}{n}} \text{ and } \delta_0 = \delta^{2n+6}. \]

To prove (6.4) in the tangent direction and (6.5), we assume that \( h \) is small enough such that \( \text{diam } S_{h_0}(0) \leq \delta^4 \) and consider two cases.

Case 1. There exists \( s \in [\delta^2 h, h] \) such that \( d_2(s) \geq K s^{\frac{1}{2}} \). For simplicity, let us assume that \( s = h \) and \( d_2(h) \geq Kh^{\frac{1}{2}} \). In this case, (6.2) implies \( \tilde{v} \leq -C \sigma(h)^\alpha \lesssim K^{-\frac{1}{2}} \) on \( B'(h) \). Thus, we can use Lemma 5.5 for \( \tilde{u} \) to obtain
\[ \mathcal{P} \tilde{S}_1(0) \subset (1 + CK^{-1}) \mathcal{P} \tilde{G}_1(0). \]

In Theorem 6.5 we take \( \tilde{f} = \tilde{f}, S = \tilde{S}_1 \). Let \( E \) be the convex hull of \( S \cup RS \), and let \( v \) be the solution to
\[ \det D^2 v = \tilde{f} \chi_{S \cup RS} \text{ in } E, \quad v = 0 \text{ on } \partial E. \]

Combined with the Aleksandrov’s Maximum Principle, we have
\[ v(x) \geq -C(n, \Lambda) \text{ dist}(x, \partial E)^{\frac{1}{2}} \geq -CK^{-\frac{1}{2}} \geq -\kappa, \quad \forall x \in \partial \tilde{S}_1 \setminus \partial \tilde{G}_1. \]

By comparing with the functions \( w^\pm(x) := \frac{1 + v(x)}{1 + \kappa} \pm 2\kappa(2C - x_n) \), we obtain that
\[ \|\tilde{u} - 1 - v\|_{L^\infty} \lesssim \kappa. \]
Since \( \hat{u}(0) = 0 \), we have
\[
\begin{align*}
\min \{ \hat{u}(Gx'), \hat{u}(G(-x')) \} & \leq \min \{ v(Gx'), v(G(-x')) \} - 1 \\
& \leq \min \{ v(Gx'), v(G(-x')) \} - v(0) + C\kappa \\
& \leq C|x'|^{1+\alpha_0} + C\kappa.
\end{align*}
\]
and
\[
\begin{align*}
\max \{ \hat{u}(Gx'), \hat{u}(G(-x')) \} & \geq \max \{ v(Gx'), v(G(-x')) \} \\
& \geq \max \{ v(Gx'), v(G(-x')) \} - v(0) - C\kappa \\
& \geq c|x'|^{1+\alpha_0} - C\kappa.
\end{align*}
\]
Recalling (6.11), we obtain
\[
c|x'|^{1+\alpha_0} - C\kappa \leq \hat{u}(Gx') \leq C|x'|^{1+\alpha_0} + C\kappa,
\]
which proves both (6.1) and (6.3) in the tangent direction.

Next, we prove (6.5) in the normal direction. That is
\[
\tilde{S}_{\delta_0}(0) \cap \{te_n | t \geq 0 \} \subset \{1 - \delta_0\} \tilde{S}_1(0).
\]
Suppose for contradiction that there exists a point \( y \in \tilde{S}_{\delta_0}(0) \setminus \{1 - \delta_0\} \tilde{S}_1(0) \) such that \( y \cdot e_n > 1 - \delta_0 \) and \( u(y) \leq \delta_0 \). Let \( z = (1 - \delta_0)e_n \). By convexity, we have \( \hat{u}(z) \leq \delta_0 \).

Without loss of generality, assume that \( \hat{u}(e_n) = 1 \). By noting that
\[
\hat{u}(x) \geq c|x'|^{1+\alpha_0} - C\kappa \geq c|x_n|^{1+\alpha_0} - C\kappa \geq 4\delta_0 \left( x_n - \frac{1}{2} \right) \text{ on } \partial \tilde{S}_1 \cap \left\{ x_n \geq \frac{1}{2} \right\},
\]
we see that \( F := \left\{ x \in \tilde{S}_1 | \hat{u}(x) \leq 3\delta_0 \left( x_n - \frac{1}{2} \right) \right\} \) satisfies \( F \subset \subset \tilde{S}_1 \). Consider the convex function \( w(x) = \hat{u}(x) - 3\delta_0 \left( x_n - \frac{1}{2} \right) \), we have \( w(z) \approx -\delta_0 \approx \inf_F w \). Since \( \lambda \leq \det D^2w \leq \Lambda \) in \( F \) and \( w = 0 \) on \( \partial F \), \( F \) should be balanced about \( z \). However, this is impossible since \( z = (1 - \delta_0)e_n \), but \( \frac{\delta}{8} e_n \in F \) and \( e_n \notin F \), which can be inferred from \( w \left( \frac{\delta}{8} e_n \right) < 0 \) and \( w(e_n) > 0 \).

**Case 2.** We have \( d_2(s) \leq KS_{\delta}^{\frac{1}{2}} \) for all \( s \in [\delta^2, h] \). Note that \( d_2(s) \geq \cdots \geq d_{n-1}(s) \geq cs^{\frac{1}{2}} \), which implies that
\[
(6.12) \quad cs^\frac{1}{2} l \leq DKs^{\frac{1}{2}} l, \quad \forall s \in [\delta^2, h].
\]
Hence, we have
\[
(6.13) \quad C \sum_{i=2}^{n-1} K^{-2}x_i^2 \leq \hat{u}(G(0,x'')) \leq C \sum_{i=2}^{n-1} K^2x_i^2 \text{ for } |x''| \geq CK\delta.
\]
Since (6.4) holds along the normal direction, given \( t \geq \delta^2 \), we still have
\[
\hat{u}(te_n) \lessapprox t^{1+\alpha_0}, \quad \text{i.e.,} \quad t^{1+\alpha_0} d_n(h) \lessapprox d_n(th).
\]
(6.12) also shows
\[
\Pi_{i=2}^{n-1} d_i(th) \geq K^{\frac{n-2}{2}} t^{\frac{\alpha_0}{2}} \cdot \Pi_{i=2}^{n-1} d_i(h).
\]
Recalling (6.4), we have
\[
d_1(th) \lessapprox K^{\frac{n-2}{2}} t^{\frac{\alpha_0}{2}} d_1(h).
\]
This means that
\begin{equation}
\mathcal{P}\hat{S}_t \subset \mathcal{P}\hat{S}_1 \cap \left\{ |x_1| \leq CK^{\frac{n-2}{2}} t^{\frac{2n}{n+2}} \right\}.
\end{equation}

Now, choose a point \( y_t \in \mathcal{P}\hat{S}_t \) such that \( y_t \cdot e_1 \geq \frac{d_t(t)}{d_t(h)} \), then we have
\begin{equation}
t \lesssim y_t \cdot e_1 \lesssim K^{\frac{n-2}{2}} t^{\frac{2n}{n+2}}.
\end{equation}

First, we make the following observation. Given points \( P \in \mathcal{P}\hat{S}_t \) and \( Q \notin \mathcal{P}\hat{S}_t \), the balancing property of \( \mathcal{P}\hat{S}_t \) implies that \( \pm cP \in \mathcal{P}\hat{S}_t \) and \( \pm CQ \notin \mathcal{P}\hat{S}_t \). From (6.13), we obtain
\begin{equation}
ct^{\frac{1}{2}} B''_1(0) \subset \mathcal{P}\hat{S}_t \cap \mathbb{R}^{n-2} \subset C t^{\frac{1}{2}} B''_1(0).
\end{equation}

Consider the cones
\[
\Gamma^+_1(P) := \left\{ \pm cP + s(x \pm cP) \mid s \geq 0, \; x \in \pm t^{\frac{1}{2}} B''_1(0) \right\}
\]
and
\[
\Gamma^+_2(Q) := \mathbb{R}^{n-1} \setminus \left\{ \pm CQ - s(x \mp CQ) \mid s \geq 0, \; x \in \pm t^{\frac{1}{2}} B''_1(0) \right\}.
\]

Let \( \Gamma_1(P) = \Gamma^+_1(P) \cap \Gamma^-_1(P) \) and \( \Gamma_2(Q) = \Gamma^+_2(Q) \cap \Gamma^-_2(Q) \).

By considering rays starting from \( \pm cP \in \mathcal{P}\hat{S}_t \) passing through the ball \( C t^{\frac{1}{2}} B''_1(0) \), we notice that \( \mathcal{P}\hat{S}_t \cap \{ \pm x_n \geq 0 \} \) is contained in the cone \( \Gamma^+_1(P) \), hence \( \mathcal{P}\hat{S}_t \subset \Gamma_1(P) \). Similarly, by considering rays starting from \( \pm CQ \notin \mathcal{P}\hat{S}_t \) passing through the ball \( C t^{\frac{1}{2}} B''_1(0) \), we also have \( \mathcal{P}\hat{S}_t \subset \Gamma_2(Q) \). We now divide the proof into 3 steps.

**Step 1.** We prove (6.15) in the tangent direction. Let \( \bar{t} = t^{\frac{2n}{2n+2}} \), where \( t \) is very small such that \( CK^{\frac{n-2}{2}} t^{\frac{2n}{n+2}} \leq \frac{1}{2} \). From (6.14), we have
\begin{equation}
\mathcal{P}\hat{S}_t(0) \subset \Gamma_1(y_t) \subset \left\{ |x_1| \geq c(\bar{t}/K)^{\frac{1}{2}}(|x''| - CK^{\frac{1}{2}} \bar{t}^{\frac{1}{2}}) \right\}.
\end{equation}

Since \( \bar{t} \geq t \), (6.14) and (6.16) imply
\[
\mathcal{P}\hat{S}_t(0) \subset \mathcal{P}\hat{S}_1(0) \cap \mathcal{P}\hat{S}_t(0) \subset \left\{ |x_1| \geq c(\bar{t}/K)^{\frac{1}{2}}(|x''| - CK^{\frac{1}{2}} \bar{t}^{\frac{1}{2}}) \right\} \cap \left\{ |x_1| \lesssim K^{\frac{n-2}{2}} t^{\frac{2n}{n+2}} \right\}
\]
\[
\subset \left\{ |x''| \lesssim K^{\frac{1}{2}} \bar{t}^{\frac{1}{2}} + K^{\frac{n-2}{2}} \bar{t}^{\frac{2n}{n+2}} \right\} \cap \left\{ |x_1| \lesssim K^{\frac{n-2}{2}} t^{\frac{2n}{n+2}} \right\}
\]
\[
\subset \left\{ |x''| \lesssim K^{\frac{n-2}{2}} \bar{t}^{\frac{1}{2}} \right\} \cap \left\{ |x_1| \lesssim K^{\frac{n-2}{2}} \bar{t}^{\frac{1}{2}} \right\}
\]
\[
\subset \left\{ |x''| \lesssim K^{\frac{n-2}{2}} \bar{t}^{\frac{1}{2}} \right\} \cap \left\{ |x_1| \lesssim K^{\frac{n-2}{2}} \bar{t}^{\frac{1}{2}} \right\}
\]
\[
\subset CK^{\frac{n-2}{2}} \bar{t}^{\frac{1}{2}} \mathcal{P}\hat{S}_1 = CK^{\frac{n-2}{2}} t^{\frac{2n}{n+2}} \mathcal{P}\hat{S}_1,
\]
the proof is complete.

**Step 2.** The proof of (6.15) in the normal direction is the same as in Case 1, where we only used (6.15) in the tangent direction.
Step 3. Finally, we prove the upper strict convexity lemma (6.14) in the tangent direction. By iteration, (6.16) implies
\[ \tilde{u}(x) \geq c|x|^{\frac{1 + \alpha_0}{\alpha_0}} \text{ on } \tilde{S}_t. \]
Using similar discussions as in (6.13) and (6.14), we have for any \( t \geq \delta^2 \) that
\[ d_n(th) \leq ct^{\alpha_0} d_n(h), \quad d_1(th) \geq CK^2 t^{-\frac{1}{1+\alpha_0}}. \]
In particular, we can find \( y_t \in \mathcal{P}\tilde{S}_t(0) \) such that
\[ y_t \cdot c \geq \frac{d_1(th)}{d_1(h)} \geq cK^2 t^{-\frac{1}{1+\alpha_0}}. \]
Letting \( b_1(t) = \sup \left\{ \mu | \mu e_1 \in \mathcal{P}\tilde{S}_t(0) \right\} \), we assert that
\[ b_1(\delta_0) \geq K\delta_0. \]
This would imply \( \tilde{u}(te_1) \leq K^{-1}t \) for \( t \leq \delta_0 \), and combining with (6.13), we complete the proof.

Suppose (6.18) is not true, since \( \frac{b_1(t)}{t} \) monotonically decreases and has a positive lower bound, we have \( ct \leq b_1(t) \leq Kt \) for all \( t \geq \delta_0 \), implying
\[ \mathcal{P}\tilde{S}_t(0) \subset \Gamma_2(Ke_1) \subset \{ |x_1| \leq CK^2 t^{\frac{2}{3}} |x''| + CKt \}. \]
Recalling (6.10), we obtain
\[ \mathcal{P}\tilde{S}_t(0) \subset \left\{ c(t/K)^{\frac{2}{3}} |x''| - Ct \leq |x_1| \leq CK^2 t^{\frac{2}{3}} |x''| + CKt \right\}. \]
Let \( N = CK^2, r = \frac{1}{2N^2}, t = \delta^2 \) and \( s = rt \geq \delta^2 \). For any point \( y \in \mathcal{P}\tilde{S}_y \subset \mathcal{P}\tilde{S}_t \), we have
\[ c(t/K)^{\frac{2}{3}} |y''| - Ct \leq |y_1| \leq CKK^2 s^{\frac{2}{3}} |y''| + CKs \leq Ns^{\frac{2}{3}} |y''| + CKs, \]
which implies
\[ |y''| \leq CK \frac{t + s}{(t/K)^{\frac{2}{3}} - Ns^{\frac{2}{3}}} = CKK^2 t^{\frac{2}{3}} \frac{1 + r}{1 - N(Kr)^{\frac{2}{3}}} \leq Nt^{\frac{2}{3}} \frac{1 + r}{1 - N^2 r^{\frac{2}{3}}}, \]
and thus
\[ |y_1| \leq Ns^{\frac{2}{3}} |y''| + CKs \leq N(rt)^{\frac{2}{3}} Nt^{\frac{2}{3}} \frac{1 + r}{1 - N^2 r^{\frac{2}{3}}} + Nrt \leq Nt \left( r + Nr^{\frac{2}{3}} \frac{1 + r}{1 - N^2 r^{\frac{2}{3}}} \right). \]
Now taking \( y = y_s \) and revisiting (6.17), we get
\[ cK^2 t^{\frac{2}{3}} (rt)^{\frac{1}{1+\alpha_0}} \leq y_s \cdot e_1 \leq Nt \left( r + Nr^{\frac{2}{3}} \frac{1 + r}{1 - N^2 r^{\frac{2}{3}}} \right). \]
Therefore,
\[ \delta^2 = t \geq \left[ NK^\frac{2}{3} r^{-\frac{1}{1+\alpha_0}} \left( r + Nr^{\frac{2}{3}} \frac{1 + r}{1 - N^2 r^{\frac{2}{3}}} \right) \right]^{-\frac{1+\alpha_0}{\alpha_0}} \]
\[ \geq \left[ NK^\frac{2}{3} N^{-\frac{1}{1+\alpha_0}} N^{-1} \right]^{-\frac{1+\alpha_0}{\alpha_0}} \]
\[ \geq \left[ K^{-\frac{2}{3}} K^{-\frac{2}{3}} \right]^{-\frac{1+\alpha_0}{\alpha_0}} \]
\[ \geq \delta, \]
which is impossible.
7. Viscosity Subsolutions and the Compactness of Oblique Boundary Values.

Equation (6.1) has a well-behaved form and Theorem 6.2 provides compactness of solutions. However, $n \geq 3$, due to the lack of a uniform $C^1$ modulus, we cannot guarantee the compactness of the corresponding oblique boundary values. Therefore, we employ the viscosity subsolution.

The assumptions in this section are independent of the assumptions in our main theorem. We always assume that $f$ is bounded and positive. We will use $USC(E)$ (or $LSC(E)$) to denote the family of all upper (lower) semicontinuous functions on the set $E$.

Recalling the oblique derivative problem

\[
\det D^2 u = f(x) \quad \text{in } \Omega, \quad D_\beta u = \phi(x) \quad \text{on } \partial \Omega.
\]

Caffarelli [10] established the equivalence between the definition of a generalized solution and a viscosity solution of $\det D^2 u = f$ when $f$ is continuous. Assuming that $\det D^2 u = f$, we say that $u \in USC(\bar{\Omega})$ is a subsolution to $D_\beta u = \phi$ on $\partial \Omega$, if for any $x_0 \in \partial \Omega$ and any convex $v \in C^1(\bar{\Omega})$ such that $u - v$ has a local maximum at $x_0$, we have

\[
D_\beta v(x_0) \geq \phi(x_0).
\]

We define supersolutions in a similar way. If $u \in C(\bar{\Omega})$, then the definition of viscosity subsolution to $D_\beta u \geq \phi$ is weaker than the definition in terms of Dini derivatives. On the other hand, we have

**Lemma 7.1.** Let $u \in L^\infty(\bar{\Omega}) \cap USC(\bar{\Omega})$ be a subsolution of (7.1). Assume that $\beta \in Lip(\partial \Omega)$ and $\phi \in LSC(\partial \Omega)$. Then, we have $D_\beta u \geq \phi$ on $\partial \Omega$ in the Dini sense and $u \in Lip(\bar{\Omega})$.

**Proof.** We can assume by contradiction that $D_\beta u(x_0) < \phi(x_0)$ at $x_0 \in \partial \Omega$. Without loss of generality, we let $x_0 = 0$, $\Omega$ satisfies (3.1)-(3.3), $u(0) = 0$, and $\beta(0) = e_n$. Let $b = \phi(0) \in \mathbb{R}$, and there exist small positive constants $\tau$ and $\epsilon$ such that $D_n u(\tau e_n) \leq b - \epsilon$. Therefore, we have

\[
(7.2) \quad u(te_n) \leq u(0) + (b - \epsilon)t \quad \text{for } t \in [0, \tau].
\]

Let $\eta$ be the oblique constant, $\kappa = \frac{1}{8(1 + \|\beta\|_{Lip})}$, and $r = r(\epsilon, \tau, u) \leq \min \{\epsilon, \eta_0 \eta_1, \frac{1}{8} \kappa \eta^3 \epsilon \tau\}$ sufficiently small such that,

\[
\phi(x) \geq b - \frac{\epsilon}{2} \quad \text{on } B_{2\eta^{-1}r}(0) \cap \partial \Omega, \quad u(x) \leq \frac{1}{8} \kappa \eta^3 \epsilon \tau \quad \text{in } B_{2\eta^{-1}r}(0) \cap \Omega,
\]

and

\[
u(x) \leq u(\tau e_n) + \frac{1}{8} \kappa \eta^3 \epsilon \tau \quad \text{in } B_{2r}(\tau e_n) \cap \Omega.
\]

Consider the cylindrical domain

\[
\Gamma = (B'_\eta(0) \times [0, \tau]) \cap \Omega
\]
and the convex function
\[ v(x) = \frac{\kappa \eta \epsilon}{r^2} |x'|^2 + (b - \epsilon)x_n. \]

Let \( G_1 = \partial \Omega \setminus \partial \bar{\Omega} \) and \( G_2 = \partial \Omega \cap \partial \Omega \). It is clear that \( G_2 \subset (B'_r(0) \times [0, \tau]) \cap \partial \Omega \) and \( \partial G_1 \subset B_{2\eta^{-1}r}(0) \setminus B_r(0) \). By calculation, we have
\[ v > u \text{ on } \partial G_1 \cup B'_r(\tau e_n). \]

Note that \( v \) is linear along the \( e_n \) direction and \( u \) is a convex function, so we have \( v > u \) on \( G_1 \). For points \( z \in G_2 \), we also have
\[ D_{\beta(z)} v(z) \leq b - \epsilon + \left( \frac{2 \kappa \eta \epsilon}{r} + 1 \right) |\beta(z) - \beta(0)| \]
\[ \leq \phi(z) - \frac{1}{2} \epsilon + \left( \frac{2 \kappa \eta \epsilon}{r} + 1 \right) \| \beta \|_{\text{Lip}} r < \phi(z) = D_{\beta(z)} u(z). \]

In summary,
\[
\begin{aligned}
&\det D^2 v = 0 \leq \det D^2 u & \text{in } \Gamma, \\
u < v & \text{ on } G_1, \\
D_{\beta} v < D_{\beta} u & \text{ on } G_2.
\end{aligned}
\]

By the comparison principle, we find that \( u < v \) in \( \Gamma \), which contradicts \( v(0) = 0 = u(0) \). Therefore, we have proved that \( D_{\beta} u \geq \phi \) on \( \partial \Omega \).

Using the same discussion as in the proof of Lemma 7.2, we have
\[
|\nabla u(x_0 + t\beta(x_0))| \lesssim (\omega_u(\Omega) + \| \min \{ \phi, 0 \} \|_{L^\infty(\partial \Omega)}), \quad \forall x_0 \in \partial \Omega \text{ and } t > 0 \text{ small}.
\]

Now, we consider the Lipschitz extension of \( u, u^*(x) = \overline{\min}_{y \to x} u(y) \). Since, \( u \in USC(\Omega) \) is convex, \( u^* \leq u \) on \( \bar{\Omega} \) and \( u = u^* \) in 0. If \( u^* < u \) at some boundary point \( x_0 \), say \( x_0 = 0 \), we still have (7.2), and the same discussion shows that this is impossible. Therefore, we have \( u = u^* \in Lip(\Omega) \).

Let \( \Omega \) satisfies (5.1)-(5.3), we consider the following mixed problem,
\[
\begin{aligned}
&\det D^2 u = f \text{ in } \Omega, \\
&D_{\alpha} u = \phi \text{ on } G := \partial \Omega \cap B_\epsilon(0).
\end{aligned}
\]

For our purposes, we make some additional assumptions to give the following existence and compactness results, one of which is a qualitative strict convexity assumption on \( u \), where for any fixed \( \alpha_0 > 0 \), we assume that for \( \kappa_0 = \kappa_0(\alpha_0) > 0 \) that
\[
\kappa_0 |x|^{1+\alpha_0} - \kappa_0 \leq u(x) \leq C|x|^{1+\alpha_0} + \kappa_0.
\]

**Lemma 7.2.** Suppose \( u^- \in USC(\bar{\Omega}) \) is a subsolution and \( u^+ \in LSC(\bar{\Omega}) \) is a supersolution to (5.2), with \( u^- \leq u^+ \) and both satisfying (7.4). If \( \phi = \phi(x') \in C(B'_r(0)) \) is concave, then there exists a convex function \( u \in Lip(\bar{\Omega} \cap B_{\epsilon}(0)) \) such that
\[ u^- \leq u \leq u^+ \text{ on } \bar{\Omega} \cap B_{\epsilon}(0) \]
and
\[ \det D^2 u = f \text{ in } \Omega, \quad D_{\alpha} u = \phi \text{ on } G := \partial \Omega \cap B_\rho(0), \]
where \( \rho > 0 \) is a small universal constant.
Proof. Consider the non-empty set
\[ V := \{ v \in USC(\Omega) \mid v \text{ is a subsolution to problem (5.2) and } u^- \leq v \leq u^+ \} \,.
\]
Then the function
\[ u(x) = \sup_{v \in V} v(x) \in Lip (\overline{\Omega} \cap B_c(0)) \]
is locally bounded and convex in \( \Omega \), and the classical interior discussion shows that \( \det D^2 u = f \) in \( \Omega \), as seen in [7 Section 9]. By Lemma 4.2 and Lemma 7.1, functions in \( V \) are uniformly Lipschitz on \( \Omega \cap B_c(0) \), and for any point \( x_0 \in \partial \Omega \cap B_c(0) \), we can find a sequence \( \{v_k\} \subset V \) such that \( v_k(x_0) \to u(x_0) \), then the supporting planes of \( v_k \) at \( x_0 \) subconverge to the supporting plane of \( u \) at \( x_0 \). Thus, we have \( D_n u \geq \phi \) on \( \partial \Omega \cap B_c(0) \).

It remains to be shown that \( D_n u \leq \phi \) on \( B_r(0) \cap G \). We assume by contradiction that \( D_n u(x_0) \geq \phi(x_0) + 3\epsilon \) for some \( x_0 \in G \cap B_r(0) \) and \( \epsilon > 0 \). Let \( E := \{ u(x) = u(x_0) + \nabla u(x_0) \cdot (x - x_0) \} \cap \overline{\Omega}. \)

From (7.2), we have \( E \subset B_c(0) \). Since \( \det D^2 u \approx 1 \), \( E \) cannot have any interior extreme points in \( \Omega \). Therefore,
\[ \mathcal{P}E = \mathcal{P}(E \cap G), \quad \text{and } \partial E \cap (\partial \mathcal{P}E \times \mathbb{R}) = \partial E \cap G. \]
Recall that \( D_n u \in USC(\overline{\Omega} \cap B_c(0)) \), \( D_n u \geq \phi \) on \( G \), and \( \phi \) is concave. Thus, we can choose \( y \in E \cap G \) such that \( y' \) is an exposed point of \( \mathcal{P}E \) and satisfies \( D_n u(y) \geq \phi(y) + 3\epsilon \). Without loss of generality, we assume
\[ y = 0, \quad \nabla u(0) = 0, \quad \mathcal{P}E \subset \{ x_1 \leq 0 \} \quad \text{and} \quad \mathcal{P}E \cap \{ x_1 = 0 \} = \{ 0 \}. \]

According to the definition of supersolutions, we have \( u(0) < u^+(0) \), observing that \( u^+ - u \in LSC \), we can choose positive \( \tau \) and \( r \), where \( r << \epsilon \), such that
\[ u^+ - u \geq \tau > 0 \text{ in } B_r(0) \cap \Omega \]
and
\[ \phi(x) \leq \phi(0) + \epsilon \leq -2\epsilon \text{ in } B_r(0) \cap \partial \Omega. \]

Let \( S_h(0) := S_{h,\nabla u(0)}^u(0) \), choose a small constant \( h > 0 \) (to be determined later), and let
\[ t_h := \sup \{ t \mid t e_n \in S_h(0) \} = \frac{h}{\varepsilon_h} \]
where \( \varepsilon_h \to 0 \) as \( h \to 0 \). By Lemma 2.0 we have \( |S_h(0)| \leq Ch^\frac{2}{n} \). So Lemma 2.2 implies that
\[ \text{Vol}_{n-1} \mathcal{P}S_h(0) \lesssim \frac{|S_h(0)|}{t_h} \lesssim \varepsilon_h^{\frac{n-2}{2}}. \]

By John’s Lemma 2.1 there exists point \( x_h \in \mathcal{P}S_h(0) \) and an affine transformation \( T' \) on \( \mathbb{R}^{n-1} \) such that
\[ \mathcal{P}S_h(0) - x_h \subset \{ x' \in \mathbb{R}^{n-1} \mid |T' x'|^2 \leq c \} \quad \text{and} \quad \text{Vol}_{n-1} (\mathcal{P}S_h(0)) \det T' \approx 1. \]

---

7 An exposed point of a convex set \( E \) is a point \( x \in E \) at which some continuous linear functional attains its strict maximum over \( E \), the set of exposed points is a non-empty subset of the set of extreme points.
Note that $u \in \text{Lip}(\Omega)$ and $u(0) = 0$ ensure that $cB_0(0) \subset \mathcal{P}S_h(0)$ and $h \|T'\| \leq c$, and together with $0 \in \mathcal{P}S_h(0)$, we obtain $\mathcal{P}S_h(0) \subset \{x' \in \mathbb{R}^{n-1} | |T'|^2 \leq 1\}$ for small $h > 0$.

Let

$$v = u + \epsilon x_n, \quad E_h = (\{x' \in \mathbb{R}^{n-1} | |T'|^2 \leq 1\} \times [0, \epsilon^{-1}h]) \cap \bar{\Omega},$$

we have

$$(7.5) \hat{S}_h^0 := \{x \in \Omega | v(x) < h\} \subset \{u \leq h\} \cap \{x_n \leq \epsilon^{-1}h\} \subset E_h.$$  

Let $a_1(h) := \sup \{t | \mathcal{G}(te_1) \in \mathcal{S}_h(0)\}$ and consider the set

$$F_h = \hat{S}_h^0 \cap \{-16a_1(h) \leq x_1 \leq a_1(h)\}.$$  

Since $\lim_{h \to 0} (S_h(0) \cap \{-16a_1(h) \leq x_1 \leq a_1(h)\}) = E \cap \{x_n = 0\} = \{0\}$ in the Hausdorff sense, and $\mathcal{P}E \subset \{x_1 \leq 0\}$, we have $\lim_{h \to 0} F_h \to \{0\}$. We now choose $h > 0$ small such that $F_h \subset B_r(0)$. Let

$$Q^h(x) = P^h(x) + \frac{hx_1}{8a_1(h)},$$

where

$$(7.6) P^h(x) = \frac{h}{4} + \frac{h}{2n} \left[|T'x'|^2 + \left(\frac{x_n}{\epsilon^{-1}h}\right)^2\right] - \epsilon x_n.$$  

Then,

$$Q^h(x) \leq P^h(x) + \frac{h}{8} < h = u \quad \text{on} \quad \{\partial E_h \setminus G\} \cap \{x_1 \geq -8a_1(h)\}$$

and thus

$$Q^h(x) \leq P^h(x) - 2h < -h < u \quad \text{on} \quad \{\partial F_h \setminus G\} \cap \{x_1 = -8a_1(h)\}.$$  

Noting that $\det D^2Q^h = \det D^2P^h \geq C\epsilon^2 \det T'^2h^{n-2} \geq \frac{C\epsilon^2}{\epsilon_h^2} \gg 2\Lambda$, we obtain

$$\begin{cases}
\det D^2Q^h > 2\Lambda & \text{in } F_h, \\
D_nQ^h > \phi + \frac{\epsilon}{2} & \text{on } \partial F_h \cap G, \\
Q^h(x) < u + \frac{\epsilon}{8} & \text{on } \partial F_h \setminus G.
\end{cases}$$

In addition, since the function $w = \max \{Q^h, u\} \chi_{F_h} + u \chi_{\bar{F}_h}$ satisfies $w \leq u^+$, so we have $w \in V$. However, $w(0) = Q^h(0) > u(0)$, which contradicts the definition of $u$.

\[\square\]

Similarly, we have compactness results for the following mixed problems.

**Lemma 7.3.** Let $u_k, \ k = 1, 2, \cdots$, satisfy

$$\det D^2u_k = f_k \quad \text{in } \Omega_k \cap B_r(0), \quad D_nu_k = \phi_k \quad \text{on } \partial \Omega_k \cap B_r(0),$$

where $\Omega_k = \{x | x_n \geq g_k(x'), \ x' \in B'_{r}(0)\} \text{ satisfies (3.1) - (3.3), } \lambda \leq f_k \leq \Lambda, \text{ and } \phi_k \in L^\infty(B_r(0))$ converges uniformly to a concave function $\phi \in C(B_r(0))$. Suppose all $u_k$ satisfy (7.3). Then $(u_k, \Omega_k)$ subconverges to some $(u, \Omega)$, and $u$ satisfies

$$\lambda \leq \det D^2u \leq \Lambda \quad \text{in } \Omega \cap B_r(0), \quad D_nu = \phi \quad \text{on } \partial \Omega \cap B_r(0).$$
Proof. By Theorem 4.3, $u_k$ is locally uniformly Lipschitz, hence $u_k$ subconverges to the subsolution $u$ of some mixed problem

$$\det D^2 u = f, \quad \lambda \leq f \leq \Lambda \text{ in } \Omega,$$

and

$$D_{\beta} u \geq \phi \text{ on } \partial \Omega.$$ 

The same proof as in Lemma 7.2 can also verify that this is an upper solution. □

Furthermore, we state the following existence and compactness theorems, although they will not be used in the proof.

**Theorem 7.4.** Assuming that $f$ is bounded and positive, and $\beta \in \text{Lip}(\partial \Omega; \mathbb{R}^n)$ is oblique, and $\phi(x, r) \in C(\Omega \times R)$ with $D_v \phi(x, r) \geq b > 0$ for some constant $b$. Suppose $n = 2$ or $\Omega$ is a strictly convex domain, then the Robin problem

$$(7.7) \quad \det D^2 u = f(x) \text{ in } \Omega, \quad D_{\beta} u = \phi(x, u) \text{ on } \partial \Omega.$$ 

has a solution $u \in \text{Lip}(\bar{\Omega})$.

**Proof of Theorem 7.4.** By choosing appropriate positive constants $K_1$ and $K_2$, we have $u^+(x) = K_1$ as a supersolution, and $u^-(x) = -K_2 + \Lambda|x - y|^2$ as a subsolution of problem (7.1). By the same proof as in Lemma 7.2 we only need to show that the function $u(x) = \sup_{v \in V} v(x)$ satisfies $D_{\beta} u = \phi(x, u)u$, where $V := \{v \in \text{USC}(\bar{\Omega}) \mid v \text{ is a subsolution to problem (7.7), } u^- \leq v \leq u^+\}$.

Without loss of generality, we assume by contradiction that $0 \in \partial \Omega$, $\Omega$ satisfies (3.1)-(3.3), $\beta(0) = e_n$, $u(0) = 0$, $\nabla u(0) = 0$, $D_n u(0) \geq \phi(0, 0) + 3\epsilon$. Then, we can proceed with the same discussion and notation as in the proof of the theorem, the only difference being that the proof is simpler in this case. Here, we only point out where the differences lie. We choose $\delta_h^k$ as in (7.6) and $P_h$ as in (7.6). If $n = 2$, then by Lemma Lemma 3.3 and the fact $\nu \geq u$, we have $\delta_h^k(0) \subset S_h(0) \rightarrow \{0\}$ as $h \rightarrow 0$. If $\Omega$ is strict convex, by (7.6) we have $\delta_h^k(0) \subset \{x_n \leq \epsilon^{-1}h\} \rightarrow \{0\}$ as $h \rightarrow 0$. By writing $T'x' = \sum_{i=1}^{n-1} a_i^{-1} x_i e_i$ in suitable orthogonal coordinates, we obtain

$$\left| D_{x'} \left( \frac{h|T'x'|^2}{2n} \right) \right| \lesssim \sum_{i=1}^{n-1} h a_i^{-2} x_i \lesssim \sum_{i=1}^{n-1} h a_i^{-1} \lesssim 1,$$

and then

$$D_{\beta} P_h \geq \beta'(x') \cdot D_{x'} \left( \frac{h|T'x'|^2}{2n} \right) - C|\beta_n(x') - \beta_n(0)| - \epsilon > -2\epsilon \text{ on } \partial E_h \cap G.$$ 

Therefore, we still have

$$\begin{cases}
\det D^2 P_h > 2\Lambda & \text{in } E_h, \\
D_{\beta} P_h > \phi & \text{on } \partial E_h \cap G, \\
P_h(x) < u & \text{on } \partial E_h \setminus G.
\end{cases}$$

In addition, since the function $\hat{w} = \max \left\{ P_h, u \right\} \chi_{E_h} + u \chi_{E_h^c}$ satisfies $\hat{w} \leq u^+$, so we have $\hat{w} \in V$. However, $\hat{w}(0) = P_h(0) > u(0)$, which contradicts the definition of $u$. □
Similarly, we state the following compactness results without proof.

**Theorem 7.5.** Let \( u_k, k = 1, 2, \ldots \), satisfy
\[
\det D^2 u_k = f_k \text{ in } \Omega_k, \quad D_{\beta_k} u_k = \phi_k \text{ on } \partial \Omega_k,
\]
where \( \Omega_k = \{ x | x_n \geq g_k(x'), \ x' \in B'(0) \} \) satisfies (3.1)-(3.3), \( \lambda \leq f_k \leq \Lambda, \ \beta_k \) is uniformly oblique on \( \partial \Omega_k \) for \( k \), and \( \phi_k \in L^\infty(B_c(0)) \) converges uniformly to a concave function \( \phi \in C(B_c(0)) \).

Suppose \( n = 2 \) or \( \partial \Omega \) is strictly convex. Then, up to a constant, \( u_k \) subconverges to a solution \( u \) of
\[
\det D^2 u = f \text{ in } \Omega, \quad D_{\beta} u = \phi \text{ on } \partial \Omega.
\]

8. **Stationary Lemma and \( C^{2,\alpha} \) Regularity**

Following Section 6, by the virtue of (6.1)-(6.2), Theorem 6.2 and Lemma 7.3 imply that the (Sliding) normalization of \( u \) and the corresponding oblique derivative problem are pre-compact. And as \( h \to 0 \), we can ensure convergence in arbitrarily large domain. Suppose \( \hat{u} \) is one of the limits, and assume for simplicity that
\[
\begin{cases}
\det D^2 \hat{u} = 1 \text{ in } \hat{S}_1, \\
D_n \hat{u} = ax_1 \text{ on } \hat{G}_1, \\
\hat{u} = 1 \text{ on } \partial \hat{S}_1 \setminus \hat{G}_1.
\end{cases}
\]

**Lemma 8.1.** We have \( \hat{u}(te_n) \lesssim t^2 \).

**Proof.** If \( n = 2 \), If \( \hat{u} \in C^1_{loc}(\hat{\Omega} \cup \hat{G}_1) \) and strictly convex in \( \hat{\Omega} \), which is always true when \( n = 2 \), we have that \( \hat{u} \) is smooth in \( \Omega \). Therefore, the function \( \zeta = D_n \hat{u} - ax_1 \) is continuous in \( \hat{\Omega} \cup \hat{G}_1 \) and satisfies
\[
\begin{cases}
\hat{U}^{ij} D_{ij} \zeta = 0 \text{ in } \hat{S}_1(0), \\
\zeta \lesssim 1 \text{ on } \partial \hat{S}_1(0), \\
\zeta = 0 \text{ on } \hat{G}_1(0),
\end{cases}
\]
where \( \hat{U}^{ij} \) is the cofactor matrix of \( D^2 \hat{u} \). The Lemma is completed by using
\[
w^+(x) = C_1 \left[ \hat{u} - \frac{n}{2} x_n D_n \hat{u} + C_2 n x_n \right]
\]
as an upper barrier, where \( C_1 \) and \( C_2 \) are sufficiently large.

If \( n \geq 3 \), for any fixed \( \epsilon > 0 \) small, we now approximate \( \hat{S}_1(0) \) from the inside using a smooth convex domain \( E \), and take smooth functions \( f_\epsilon \) and \( w_\epsilon \) such that \( D_n f_\epsilon = 0 \) and
\[
\| f_\epsilon - 1 \|_{L^1(E)} + \| w_\epsilon - \hat{u} \|_{L^\infty(\partial E)} \text{ is arbitrary small.}
\]

By solving the Dirichlet problem
\[
\det D^2 u_\epsilon = f_\epsilon \text{ in } E, \quad u_\epsilon = w_\epsilon \text{ on } \partial E,
\]
we can always find a convex function such that
\[
\| u_\epsilon - u \|_{L^\infty(E)} \leq \epsilon
\]
We further assume that \( S^\epsilon := \hat{S}_1(0) + \epsilon \hat{e}_n \subset E \) and let \( G^\epsilon := \hat{G}_1(0) + \epsilon \hat{e}_n \).
By convexity, we have $D_n u_\epsilon \lesssim 1$ in $S^\epsilon$. Note that there exists $\sigma_3$ (depending on $\tilde{u}$) with $\sigma_3(0) = 0$ such that

$$0 \leq D_n \tilde{u} \left( \tilde{G} x' + te_n \right) - D_n \tilde{u}(\tilde{G} x') \leq \sigma_3(t).$$

Therefore, for any point $q \in S^\epsilon$, we have by convexity that

$$D_n u_\epsilon(p) \leq \frac{\tilde{u} \left( p + \epsilon^\frac{1}{2} e_n \right) - \tilde{u}(p) + 2\epsilon}{\epsilon^\frac{1}{2}} \leq \ell(p') + \sigma_3(2\epsilon^\frac{1}{2}) + \epsilon^\frac{1}{2},$$

and

$$D_n u_\epsilon(p) \geq \frac{\tilde{u} \left( p - \epsilon^\frac{1}{2} e_n \right) - \tilde{u}(p) - 2\epsilon}{\epsilon^\frac{1}{2}} \geq \ell(p') - \sigma_3(\epsilon^\frac{1}{2}) - \epsilon^\frac{1}{2}.$$  

And Lemma 8.1 implies that locally $\|\nabla u_\epsilon\| \leq C$. Thus, $\zeta_\epsilon = D_n u_\epsilon - \ell(x)$ is a bounded solution of

$$\begin{cases}
U_{ij}^\ell D_{ij} \zeta_\epsilon = 0 & \text{in } S^\epsilon, \\
\zeta_\epsilon \leq C & \text{on } \partial S^\epsilon, \\
\zeta_\epsilon \leq \sigma(2\epsilon^\frac{1}{2}) + \epsilon^\frac{1}{2} & \text{on } G^\epsilon.
\end{cases}$$

(8.1)

According to Theorem 6.2, there exists a universal small constant $\varrho > 0$ such that

$$\bar{S}_2^\epsilon(0) \subset \frac{1}{8\varrho} \bar{S}_\varrho(0),$$

holds for for each normalized solution $\tilde{u}$. This property remains invariant under uniform convergence. Therefore, it also holds for $\tilde{u}$. Let

$$\tilde{u}_\epsilon(x) = u_\epsilon(x) - u_\epsilon(y_\epsilon) - \nabla \tilde{u}(y_\epsilon) \cdot (x - y_\epsilon),$$

where $y_\epsilon = \epsilon^\frac{1}{2} e_n$, we have

$$\tilde{u}_\epsilon \geq c > 0 \text{ on } \partial S^\epsilon \setminus G^\epsilon.$$  

In fact, we notice that the function $\ell(x) = u_\epsilon(y_\epsilon) + \nabla \tilde{u}(y_\epsilon) \cdot (x - y_\epsilon) - \varrho - 2\epsilon$ satisfies

$$\ell(0) \geq -\varrho - c_1 \epsilon^\frac{1}{2}, \quad \ell \leq 0 \text{ on } \bar{S}_\varrho(0).$$

Looking back at (8.2), we find that

$$\ell + \varrho \leq \frac{1}{4} \text{ on } \partial S_2^\epsilon \setminus \tilde{G}_2^\epsilon.$$  

This means that $u - \ell - \varrho \geq \frac{1}{4}$ on $\partial S_2^\epsilon \setminus \tilde{G}_2^\epsilon$, and hence

$$\tilde{u}_\epsilon = u_\epsilon - \ell - \varrho - 2\epsilon \geq \frac{1}{8} \text{ on } \partial S^\epsilon \setminus G^\epsilon.$$  

The Lemma is completed by using

$$w^+_\epsilon := C_1 \left[ \tilde{u}_\epsilon - \frac{n}{2} (x_n - \epsilon^\frac{1}{2}) D_n u_\epsilon + nC(x_n - \epsilon^\frac{1}{2}) + \sigma(2\epsilon^\frac{1}{2}) + \epsilon^\frac{1}{2} \right],$$

as supersolution for the equation of $D_n u_\epsilon$ and then letting taking $\epsilon \to 0$. \hfill \Box

By Lemma 8.1 there exists $\delta_1 > 0$ and module $\sigma^*$ such that when $\omega_f(\Omega) + \omega_{D\varrho}(\Omega) + h \leq \delta_1$,

$$\tilde{u}(te_n) \leq Ct^2 + \sigma^*(\delta_1) \text{ for } h \leq h_0.$$
This will ensure that the Neumann boundary $\tilde{G}$ converges locally uniformly to the plane $\mathbb{R}^{n-1}$, and by verifying the assumptions in Liouville Theorem, the limit $\tilde{u}$ must be quadratic, giving us

**Lemma 8.2.** For any small constant $\varepsilon_0 > 0$, we can find constant $\delta_1 > 0$ such that if $\omega_f(\Omega) + \omega_D(\Omega) + h \leq \delta_1$, then there exists a quadratic function $P_h$ satisfying

$$P_h(0) = \nabla^T P_h(0) = 0, \quad \det D^2 P_h = f(0),$$

and

$$\|D_n P_h - D_n \tilde{u}\|_{L^\infty(S_1)} + \|\tilde{u} - P_h\|_{L^\infty(S_1)} \leq \varepsilon_0.$$

**Proof of Lemma 8.2** Recalling [33], by iteration, this means that for any fixed $\varepsilon > 0$, we have

$$d_n(h) \geq c(h)^{1+\varepsilon},$$

and thus we obtain $T'_n \leq C(h)^{\frac{1+\varepsilon}{2}} T'$. By taking $\varepsilon = \frac{k}{2}$, we find that

$$|\bar{g}_h(x')| \leq \frac{1}{d_n(h)}|T'_n x'|^{1+\varepsilon} \lesssim h^\frac{k}{2} |x'|^{\varepsilon} \to 0, \quad \text{as } h \to 0^+,$$

so the Neumann boundary $\tilde{G}_h$ converges locally uniformly to the plane $\mathbb{R}^{n-1}$.

By recalling [52] and [63], and letting $\delta_1 \to 0$ (so that $h \to 0$), for any sequence of normalized solutions, we can assume that one of the following cases holds for a subsequence: either $a_h \to 0$, thus $D_n \tilde{u} \to 0$; or $|a_h| \gtrsim K^{-1}$ for some constant $K > 0$ depending on the sequence, and we obtain $\tilde{u}(0, x'', 0) \lesssim K|x''|^2$. By applying [14] the limit $\tilde{u}$ must be quadratic.

Note that the sections with height 1 has already been normalized, so the limit functions are pre-compact. Therefore, for any $\varepsilon > 0$, when $\delta_1$ is small, there exists a quadratic function $P_h$ satisfying

$$P_h(0) = \nabla^T P_h(0) = 0, \quad \det D^2 P_h = f(0),$$

and

$$\|D_n P_h - D_n \tilde{u}\|_{L^\infty(S_1)} + \|\tilde{u} - P_h\|_{L^\infty(S_1)} \leq \varepsilon^2.$$

By convexity, we have

$$D_n \tilde{u}(x', \tilde{g}(x')) \geq D_n P_h(x', \tilde{g}(x')) - \varepsilon^2$$

and

$$D_n P_h(x) - C \varepsilon \leq D_n \tilde{u}(x', \tilde{g}(x')) \leq D_n P_h(x) + C \varepsilon \text{ if } x_n \geq \tilde{g}(x') + \varepsilon,$$

which means that

$$\|D_n P_h - D_n \tilde{u}\|_{L^\infty(S_1)} \lesssim \varepsilon,$$

and then the proof is completed by choosing $\varepsilon = c\varepsilon_0$. □

Next, we use the standard perturbation method to prove that if $\tilde{u}_h$ is approximated by a quadratic function in $\tilde{S}_1^\alpha$, and the known data $\tilde{f}$ and $\tilde{\phi}^0$ are smaller perturbations of constant and linear functions, then $\tilde{u}_{\mu,h}$ has a better approximation in $\tilde{S}_1^{\alpha,h}$, where $\mu$ might be very small, ultimately giving Theorem 1.2.
Let $\mathcal{F}_a$ denote the set of quadratic functions given by $P(x) := Q(x) + \ell(x')$, where $\ell$ is a linear function and the quadratic term $Q(x) = \sum_{i,j=1}^n a_{ij}x_ix_j$ satisfies $D_n Q(x',0) = ax_1$ and $\det D^2 Q = 1$. Given $Q + \ell \in \mathcal{F}_a$, let $(\gamma_2, \cdots, \gamma_{n-1}) = (D_{12}Q, \cdots, D_{11n-1}Q)(D_{2n}^2Q)^{-1}$. By considering the sliding transformation $Bx = x - 2 \sum_{i=2}^{n-1} \gamma_i x_i \epsilon_i$ and then a rotation transformation in $x''$, we can always find a new coordinate system in which $Q$ takes the form of

$$Q_{a,\kappa}(x) := \frac{1}{2} \left[ \sqrt{\kappa^{-2} + a_0^2(x_1^2 + x_n^2)} + 2ax_1x_n + \kappa^{-1} \sum_{i=2}^{n-1} x_i^2 \right],$$

where $a \in \mathbb{R}$ and $\kappa > 0$. Clearly, $\det D^2 Q_{a,\kappa}(x) \equiv 1$ and $D_n Q_{a,\kappa}(x',0) = ax_1$.

In Theorem 8.3, Lemma 8.4 and Lemma 8.5 below, we will abuse the notation $u$, which refer to different functions in each lemma.

**Theorem 8.3.** Let $u \in C(\overline{S_1})$ be a solution to

$$\begin{cases}
\det D^2 u = 1 & \text{in } S_1, \\
D_n u = ax_1 & \text{on } G_1, \\
u = 1 & \text{on } \partial S_1 \setminus G_1.
\end{cases}
$$

Suppose $u(0) = 0$, $\nabla u(0) = 0$, $u \geq 0$, $|a| \leq C$, $B_r^+(0) \cap \overline{S_1} \subset S_1 \subset B_C(0)$, and the defining function $g$ of $G_1$ satisfies $\|g\|_{\text{Lip}(B_r^+(0))} \leq C$. Then there exists a universal constant $\epsilon_0 > 0$ such that if

$$\|g\|_{C^{1,\alpha}(B_r^+(0))} + \|u - P\|_{L^\infty(S_1)} + \|D_n u - D_n P\|_{L^\infty(S_1)} \leq \epsilon_0$$

holds for some $P(x) = \ell(x') + Q(x) \in \mathcal{F}_a$ with $B_{c/4}(0) \subset \{P \leq 1\}$, then we have

$$\|u\|_{C^{2,\alpha}(B_r^+(0) \cap \overline{S_1/2})} \leq C.$$

Theorem 8.3 relies on the following lemma.

**Lemma 8.4.** Let $u \in C(\overline{S_1})$ be a solution to (8.4). Suppose $P(x) = \ell(x') + Q(x) \in \mathcal{F}_a$ satisfies $B_{c/4}(0) \subset \{P \leq 1\}$ and

$$\|g\|_{C^{1,\alpha}(B_r^+(0))} + \|u - P\|_{L^\infty(S_1)} + \|D_n u - D_n P\|_{L^\infty(S_1)} \leq \epsilon,$$

then

$$|D\ell| \leq C \epsilon^{\frac{1}{3}}.
$$

For any small constant $\mu > 0$, there exists $\epsilon_0 > 0$, depending on $n$, $|a|$, $\mu$ such that if $\epsilon \leq \epsilon_0$, then we can find a better approximation $P^0 = \ell^0 + Q^0 \in \mathcal{F}_a$ of $u$ so that

$$|u - P^0| \leq C \epsilon \mu^\frac{2}{3}, \quad |D_n u - D_n Q^0| \leq C \epsilon \mu \text{ in } S_{\mu}^0,$$

and

$$\|D^2 Q^0 - D^2 Q\| \leq 2 \epsilon.$$

The proof of Lemma 8.4 is the same as [11, Lemma 6.2], we put it in the Appendix A.
Proof of Theorem 8.3. Let $h_k = h^k$ for $k = 1, 2, \ldots$. By using Lemma 8.4, we will prove the following by induction on $k$. For simplicity, we omit all subscripts $u_k$ for domains and boundaries.

Claim. For $k \geq 0$, there exists a sequence of constants $\kappa_k \in (c, C)$ and transformations

$$\mathcal{M}_k = \text{diag} \{ \mathcal{M}'_{k,1}, \mathcal{M}_{k,nn} \}, \quad \det \mathcal{M}_k = 1,$$

such that at height $h_k$, the normalization solution $(u_k, \Omega_k)$ of $(u, \Omega)$ (see Definition 6.1) given by

$$u_k(x) := \frac{u(T_k h_k^\omega x)}{h_k^\omega}, \quad \text{for } x \in \Omega_k := T_k^{-1}(\Omega) \text{ with } T_k = \Pi_{i=0}^k \mathcal{M}_k$$

satisfies

$$\epsilon_k := \|g_k\|_{C^{\alpha}(B^c_0(0))} + \|u_k - P_k\|_{L^\infty(S_1)} + \|D_n u_k - D_n Q_{a,\kappa_k}\|_{L^\infty(S_1)} \leq \epsilon_0$$

for some $P_k = \ell_k + Q_{a,\kappa_k} \in F_a$, where $g_k$ is the defining function of the oblique boundary of $u_k$, and

$$Q_{a,\kappa}(x) := \frac{1}{2} \left[ \sqrt{\kappa^2 + a^2(x_1^2 + x_n^2) + 2ax_1x_n + \kappa^{-1} \sum_{i=2}^{n-1} x_i^2} \right].$$

Moreover,

$$\|\mathcal{M}_k - I\| + |\kappa_k - \kappa_{k-1}| \leq C \epsilon_{k-1} \text{ and } |D\ell_k| \leq C \epsilon_{k-1}.$$

The statement for $k = 0$ is trivial by taking $\mathcal{M} = I$, we also assume for simplicity that $\kappa_0 = 1$. Assuming the statement holds at $k$, we will prove the statement at $k + 1$ by applying Theorem 8.4 to $u_k$. We will estimate $\|T_k\|$ and $\epsilon_k$. Assuming $\epsilon_0$ and $\mu$ are sufficiently small, once we prove the statement, we will have

$$\epsilon_i \leq h_i^\omega \quad \text{for } i = 0, 1, \ldots, k$$

Then we have $\|\mathcal{M}_i - I\| \leq 2\epsilon_i \leq 2h_i^\omega$ for $i = 0, 1, \ldots, k$, and

$$\|T_k - I\| \leq \sum_{i=0}^k \|\mathcal{M}_i - I\| \leq 4 \sum_{i=0}^k h_i^\omega \leq \frac{1}{2} \text{ and then } \frac{1}{2} \leq \kappa_k \leq 2.$$

We then apply Lemma 8.4 to $g_k$ and obtain a better approximation $\tilde{P}_{k+1} = \tilde{\ell}^+ \tilde{Q} \in F_a$. By (8.8), we can find $\kappa_{k+1} > 0$ and positive definite matrix $\mathcal{M}_{k+1} = \text{diag} \{ \mathcal{M}'_{k+1,1}, \mathcal{M}_{k+1,nn} \}$ such that

$$D^2 \tilde{Q} = \mathcal{M}_{k+1} D^2 Q_{a,\kappa_{k+1}} \mathcal{M}'_{k+1,1}$$

and

$$|\kappa_{k+1} - \kappa_k| + \|\mathcal{M}_{k+1} - I\| \leq C \epsilon_k.$$

Let $\ell_{k+1} = \tilde{\ell} \circ \mathcal{M}_{k+1}$. By calculation, we still have

$$\|T_{k+1} - I\| \leq \frac{1}{2}, \quad \frac{1}{2} \leq \kappa_{k+1} \leq 2$$

and

$$\|g_{k+1}\|_{C^{\alpha}(B^c_0(0))} \leq C h_k^\omega \| T_{k+1} \|^\omega \leq C h_k^\omega \| T_k \|^\omega \| \mathcal{M}_{k+1} \| \leq h_{k+1}^\omega.$$
Therefore, we obtain (8.9) and the claim is complete.

Here, since we also obtained universal bounds for $\kappa_0$ and $\|T_k\|$, the same discussion allows us to refine the estimate (8.9) to $\epsilon_k \leq C\alpha h_k^2$. Now, we can let $\kappa_0$ and $T_k$ converge geometrically to $\kappa_\infty$ and $T_\infty$, respectively, i.e.,

$$|\kappa_k - \kappa_\infty| \leq C_\alpha h_k^2 \text{ and } \|T_\infty T_k^{-1} - I\| \leq C_\alpha h_k^2.$$ 

Replacing each $T_k$ with $T_\infty$, we see that

$$|u(x) - h_k^2 \ell_k(T_\infty^{-1}x) - Q_{\alpha, \kappa_\infty}(T_\infty^{-1}x)| \leq C_\alpha h_k^{1+\frac{1}{\alpha}} \text{ in } B_{\rho_k}(0).$$

Recalling that $|D\ell_k| \leq C_\alpha \epsilon_k^{-1}$, we conclude that

$$|u(x) - Q_{\alpha, \kappa_\infty}(T_\infty^{-1}x)| \leq C_\alpha |x|^{2+\alpha}.$$ 

Thus, we obtain the pointwise $C^{2,\alpha}$ module of $u$ at 0.

Note that the above discussion is valid for every point $x_0 \in G_{h_k}(0)$. Therefore, we find that $u$ is $C^{2,\alpha}$ on the boundary $G_{h_k}(0)$. Assuming that $t > 0$ is sufficiently small, for any given point $z \in \partial \Omega$ and point $y = z + tv(z) \in \Omega$ close to the origin. Let $Q_z$ denote the approximate quadratic polynomial of $u$ at $z$. At this point, the section $S_{t, \ell_k}(y)$ is contained in $\Omega$ and satisfies

$$B_{c_t}(y) \subset S_{t, \ell_k}(y) \subset B_{c_{\ell_k}}(y).$$

Note that

$$\det D^2 u(x) = 1 \quad \text{in } S_{t, \ell_k}(y),$$
$$|u - Q_z(x)| \leq C|t|^{2+\alpha} \quad \text{on } \partial S_{t, \ell_k}(y).$$

The classical interior $C^{2,\alpha}$ theory for Monge-Ampère equations then gives $\|u\|_{C^{2,\alpha}(B_{c_t}(y))} \leq C$, hence $u \in C^{2,\alpha}(\Omega \cap B_\rho(0))$ for some small universal $\rho$.

\textbf{Lemma 8.5.} Let $\epsilon > 0$, suppose $u \in \overline{C(S_1)}$ satisfies

$$\begin{cases}
|\det D^2 u - 1| \leq \delta \epsilon & \text{in } S_1 \\
|D_n u - ax| \leq \delta \epsilon & \text{in } G_1 \\
u = 1 & \text{on } \partial S_1 \setminus G_1
\end{cases}$$

Assume that for some $P(x) = \ell(x') + Q(x) \in \mathcal{F}_a$ with $B_{c/4}(0) \subset \{P \leq 1\}$, we have

$$\|g\|_{C^{1,\alpha}(B_{c/4}(0))} + \|u - P\|_{L^\infty(S_1)} \leq \epsilon,$$

then $|D\ell| \leq C\epsilon^{\frac{1}{\alpha}}$. For any small constant $\mu > 0$, there exists small positive $\delta_0$ and $\epsilon_0$, depending on $n, |a|, \mu$ such that whenever $\epsilon \leq \epsilon_0$ and $\delta \leq \delta_0$, there is a better approximation $P^0 = t^0 + Q^0 \in \mathcal{F}_a$ of $u$ such that

$$|u - P^0| \leq C\epsilon \mu^{\frac{1}{\alpha}} \text{ in } S^Q_{\mu^2} \text{ and } \|D^2 Q^0 - D^2 Q\| \leq 2\epsilon.$$ 

\textbf{Proof.} By applying Lemma 7.2, we obtain a solution $w$ of

$$\begin{cases}
|\det D^2 w = 1 & \text{in } S^t_1(0), \\
D_n w = ax & \text{in } G^t_\rho(0), \\
u^- \leq w \leq u^+ & \text{in } S^t_1(0),
\end{cases}$$

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In fact, for $C$ sufficiently large, we can take
\[ u^- := (1 + C\delta\epsilon)(u + C^2\delta\epsilon x_n) - 2C^3\delta\epsilon \]
and
\[ u^+ := (1 - C\delta\epsilon)(u - C^2\delta\epsilon x_n) + 2C^3\delta\epsilon \]
are the subsolution and supersolution, respectively.

We can see that
\[ \|u - w\|_{L^\infty} \leq C\delta\epsilon, \]
and then
\[ \|g\|_{C^{1,\alpha}(B'_c(0))} + \|w - P\|_{L^\infty(S_{1/4})} \leq 2\epsilon. \]
According to Theorem 8.3, $w$ is uniformly $C^{2,\alpha}$ up to the Neumann boundary. The $C^1$ estimate for the Neumann problem of the uniformly elliptic equation implies
\[ \|g\|_{C^{1,\alpha}(B'_c(0))} + \|w - P\|_{L^\infty(S_{1/4})} \leq C\epsilon. \]
Thus, by applying Lemma 8.4, we obtain a better approximation of $w$, and therefore, a better approximation of $u$, provided that $\delta \leq \delta_0$ is sufficiently small.

**Proof of Theorem 1.2.** By Lemma 8.2, we may assume that $x_0 = 0$ and start with a suitable normalization of $((\tilde{u}, \tilde{S}_1))$ at $h = h_0$, where $h_0$ is small enough. Then, $\tilde{u}$ solves
\[
\begin{align*}
\det D^2\tilde{u} &= \tilde{f} &\text{in } \tilde{S}_1, \\
D_n\tilde{u} &= \tilde{\varphi} &\text{on } \tilde{G}_1, \\
\tilde{u} &= 1 &\text{on } \partial\tilde{S}_1 \setminus \tilde{G}_1,
\end{align*}
\]
where
\[
\tilde{f}(x) = \frac{(\det D)^2}{h^n} f(Thx), \quad \tilde{\varphi}(x') = \frac{d_n \varphi(T_h x')}{h}.
\]
Here, we may assume that $\tilde{u}(0) = 0$, $\nabla \tilde{u}(0) = 0$, $\tilde{u} \geq 0$, $B_\epsilon^+(0) \cap \overline{\tilde{S}_1} \subset \tilde{S}_1 \subset B_C(0)$ and let the defining function $\tilde{g}$ of $G_1$ satisfies $\|\tilde{g}\|_{Lip (B'_c(0))} \leq C$. We also assume that
\[
\tilde{f}_{h_0}(0) = 1, \quad D\tilde{\varphi}_{h_0}(0) = ae_1, \quad |a| \leq C,
\]
and there exists $P(x) = \ell(x') + Q(x) \in \mathcal{F}_a$ with $B_{c/4}(0) \subset \{P \leq 1\}$ satisfying
\[ \|\tilde{u} - P\|_{L^\infty(\tilde{S}_1)} \leq \epsilon_0. \]

The proof is then the same as in Theorem 8.3. By applying Lemma 8.5, we can still use the induction argument to prove that $h_k = h_0\mu^k$, $\epsilon_k := \epsilon_0\mu^k$ (and then for $\epsilon_k := C\epsilon_0\mu^k$) the following.

**Claim.** For $k \geq 0$, there exists a sequence of constants $\kappa_k \in (c, C)$ and transformations
\[
\mathcal{M}_k = \text{diag} \{\mathcal{M}_{k1}, \mathcal{M}_{k,n}\}, \quad \text{det } \mathcal{M}_k = 1,
\]
such that at height $h_k$, the normalization solution $(u_k, \Omega_k)$ of $(u, \Omega)$ given by
\[ u_k(x) := \frac{u(T_k h_k^x)}{h_k} \text{ for } x \in \Omega_k := \mathcal{T}_k^{-1}(\Omega) \]
with $T_k = T_{h_0}^k \Pi_x \mathcal{M}_k$. 37
satisfies
\[
\left\{ \begin{array}{l}
|\det D^2 u_k - 1| \leq \delta \epsilon_k & \text{in } S_1 \\
|D_n u_k - ax_1| \leq \delta \epsilon_k & \text{on } G_1 \\
u_k = 1 & \text{on } \partial S_1 \setminus G_1
\end{array} \right.
\]

And we have
\[
\|g_k\|_{C^{1,\alpha}(B_{\rho}(0))} + \|u_k - p_k\|_{L^\infty(S_1)} + \|D_n u_k - D_n Q_{a,\kappa_k}\|_{L^\infty(S_1)} \leq \epsilon_k \leq \epsilon_0
\]
for some \( p_k = \ell_k + Q_{a,\kappa_k} \in F_a \), where \( g_k \) is the defining function of the oblique boundary of \( u_k \), and
\[
Q_{a,\kappa}(x) := \frac{1}{2} \left[ \sqrt{\kappa n - 2 + a^2(x_1^2 + x_n^2)} + 2ax_n + \kappa^{-1} \sum_{i=2}^{n-1} x_i^2 \right].
\]
Moreover,
\[
\|\mathcal{M}_k - I\| + |\kappa_k - \kappa_{k-1}| \leq C\epsilon_{k-1} \text{ and } |D\ell_k| \leq C\epsilon_{k-1}^{1/2}.
\]

And the same proof gives the \( C^{2,\alpha} \) estimates of the solution as that of Theorem 8.3. When \( n = 2 \), the \( C^{2,\alpha} \) regularity is holds at every boundary point, and for points far away from the boundary, the maximum height of their sections inside the domain has a positive lower bound, giving a global \( C^{2,\alpha} \) estimate.

9. Examples

In this section, we will give some examples of the oblique derivative problem by introducing convex functions for which \( \det D^2 u \approx 1 \) in \( B_{\rho}(0) \), to illustrate the obstacles for the boundary Schauder regularity in the oblique derivative problem for Monge-Ampère equations when the dimension \( n \geq 3 \).

Example 9.1. Consider classical Pogorelov’s function
\[
u(x) = (1 + x_n^2)|x'|^{2-\frac{n}{2}} \text{ in } B_{\rho}(0), \quad n \geq 3.
\]
Here, \( u \) is only \( C^{1,1-2/n} \cap C^\infty(B_{\rho}(0) \setminus \{|x'| = 0\}) \), the oblique boundary value \( D_{n} u \) is \( C^{1,1-2/n} \) on boundary \( \partial B_{\rho}(0) \). Moreover, we can take a smooth oblique vector field \( \beta \) such that \( \beta(x', x_n) = (-x'x_n, (1 - \frac{1}{n})(1 + x_n^2)) \) around the points \( \{\pm pe_n\} \) such that \( u \) solves (1.1) with both \( \Omega := B_{\rho}(0) \), \( f \), \( \beta \), \( \phi \) being smooth.

Example 9.2. Consider the merely Lipschitz function appearing in Caffarelli
\[
u(x) = |x'| + (1 + x_n^2)|x'|^{n/2} \text{ in } B_{\rho}(0), \quad n \geq 3.
\]
Here, the Neumann boundary value \( D_{n} u \) is Lipschitz on boundary \( \partial B_{\rho}(0) \), and \( \det D^2 u \) is as smooth as \( \left(1 + \frac{n-2}{2} |x'|^{n/2-1} (1 + x_n^2)\right)^{n-2} \). And we can always take an appropriate smooth oblique vector field \( \beta \) such that \( \phi = D_{\beta} u \) is as smooth as \( |x'|^{n/2} \).
Finally, we recall a singular homogeneous function from our recent paper [11, Example 4.3 and Remark 4.4]. Suppose \( n \geq 3 \), for any small \( \delta > 0 \) and \( a, b \in (1, \infty) \) satisfying
\[
\frac{1}{a} = \frac{1}{2} + \frac{\delta}{n - 2} \quad \text{and} \quad b = \frac{1}{1 - \delta},
\]
we let
\[
W_{a,b}(x_1, x'') = \left\{ \begin{array}{ll}
|x''|^a + |x''|^a - \frac{2}{b} |x_1|^2, & \text{if } |x''|^a \geq |x_1|^b, \\
2k + a - ab |x_1|^b + \frac{ab - a}{b} |x_1|^b - \frac{2k}{a} |x''|^2, & \text{if } |x_1|^b \geq |x''|^a,
\end{array} \right.
\]
and
\[
W(x_1, x'', x_n) = (1 + x_n^2)W_{a,b}(x').
\]

Let \( \rho = \rho(\delta) > 0 \) be small. Then, the function \( W \) is convex in \( \{|x_n| \leq \rho\} \) and satisfies
\[
c(\delta) \leq \det D^2W \leq C(\delta), \quad D_nW(x', 0) = 0.
\]
Moreover, given a constant \( R > 0 \), the solution \( W_R^+ \) to the Dirichlet problem
\[
\begin{cases}
\det D^2W^+ = c(\delta) & \text{in } B'_R(0) \times [-\rho, \rho], \\
W^+ = W & \text{on } \partial(B'_R(0) \times [-\rho, \rho])
\end{cases}
\]
satisfies
\[
W(x', 0) \leq W^+(x', t) \leq (1 + \rho^2)W(x', 0)
\]
Additionally, we will verify this in Appendix B the following Example 9.3, which shows that the assumption \( (1.2) \) in Theorem 1.2 is optimal in some sense for local problems.

**Example 9.3.** The function \( u(y_1, y'', y_n) = W^+(y_n, y'', y_1) \) satisfies \( u \in C^{1, a-1}_{loc}(B'_R(0)) \), and is a solution to
\[
\det D^2u = c \text{ in } B'^+_R(0), \quad D_nu = 0 \text{ on } B'_R(0).
\]
But \( u \notin C^{1, \delta + \epsilon} \) for any \( \epsilon > 0 \).

**APPENDIX A. PROOF OF LEMMA 8.4**

Proof. The estimate of \( |D\ell| \) is obtained based on convexity, which we omit here. Contrary to our statement \( (5.7) \), there exists a sequence of solutions \( u_k \) to
\[
\begin{cases}
\det D^2u_k = 1 & \text{in } S'^{u_k}_1, \\
D_nu_k = a_kx_1 & \text{on } G'^{u_k}_1, \\
u_k = 1 & \text{on } \partial S^{u_k}_1 \setminus G^{u_k}_1,
\end{cases}
\]
with good approximations \( P_k = \ell_k + Q_k \) satisfying
\[
\epsilon_k := \|g_k\|_{C^{1, a}(B'_R(0))} + \|u_k - P_k\|_{L^\infty(S'^{u_k}_1)} + \|D_nu_k - D_nQ_k\|_{L^\infty(S'^{u_k}_1)} \to 0,
\]
but \( (8.7) \) fails for \( u_k \) and all polynomials in \( \mathcal{F}_{a_k} \).
According to the assumption \(B_{\epsilon^4}(0) \subset \{P_k \leq 1\}\), we always have \(cT \leq D^2Q_k \leq CT\). By considering a subsequence and using affine transformations, without loss of generality, we may assume that for all \(k\),

\[
Q_k(x) := \frac{1}{2} \left[ \sqrt{\kappa_k^{-2} + a_k^2 (x_1^2 + x_n^2)} + 2a_kx_1x_n + \kappa_k^{-1} \sum_{i=2}^{n-1} x_i^2 \right].
\]

We assume that \(\lim_{k \to \infty} \kappa_k = 1\) and \(\lim_{k \to \infty} a_k = a\), and let

\[
Q_a(x) := \frac{1}{2} \left[ \sqrt{1 + a^2 (x_1^2 + x_n^2)} + 2ax_1x_n + \sum_{i=2}^{n-1} x_i^2 \right].
\]

In the following discussion, we omit all subscripts \(u_k\) for the domains and boundaries, and we consider the functions

\[
w_k = \frac{u_k - \ell_k - Q_k}{\epsilon_k}.
\]

**Step 1.** As \(k \to \infty\), \(w_k\) converges locally uniformly to a function \(w\) satisfying

\[
\mathcal{L}[w] := \sqrt{1 + a^2(w_{11} + w_{nn})} + aw_{1n} + \sum_{i=2}^{n-1} w_{ii} = 0 \text{ in } B_1^+(0).
\]

We consider convergence on the compact subset

\[
E = \left\{ x \in \mathbb{R}^n | x_n \geq C\epsilon_k^2 \right\} \cap S_1(0) \cap S_{1/2}^Q(y).
\]

For every point \(y \in E\) satisfying \(\text{dist}(y, \partial G_1) > C\epsilon_k^2\), we choose a constant \(\rho = c\text{dist}(y, \partial G_1)\) such that the ball \(B_{C\rho}(y) \subset E\). Then we have

\[
B_{C\rho}(y) \subset S_{C\rho^2}^Q(y) \subset B_{C\rho}(y).
\]

Recall that \(|u_k - Q| \leq \epsilon_k\). Since \(\epsilon_k \leq c\rho^2\), we have \(B_{2c\rho}(y) \subset S_{C\rho^2}^Q(y) \subset B_{C\rho}(y)\). The standard estimate of the Monge-Ampère equation implies that

\[
\|u_k\|_{C^m(B_{c\rho}(y))} \leq C_m\rho^{2-m} \text{ and } cT \leq D^2u_k \leq CT \text{ in } B_{c\rho}(y) \text{ for } m \geq 0.
\]

Note that

\[
(1.1) \quad 0 = \det D^2u_k - \det D^2Q_k = Tr(AD^2(u_k - Q_k)) = \epsilon_k Tr(AD^2w_k),
\]

where

\[
A := [A_{ij}]_{n \times n} = \int_0^1 \text{cof}((1 - t)D^2Q_k + tD^2u_k) \text{d}t
\]

and \(\text{cof} \mathcal{M}\) denotes the cofactor matrix of a matrix \(\mathcal{M}\). The operator \(\mathcal{L}_A[\cdot] := A_{ij}D_{ij}(\cdot)\) is uniform elliptic with smooth coefficients. From \(1.1\), we obtain

\[
(1.2) \quad \|w_k\|_{C^m(B_{c\rho}(y))} \leq C\rho^{-m},
\]

which implies

\[
|D^2u_k - D^2Q_k| \leq C\epsilon_k\rho^{-2} \text{ and } |A_{ij} - D_{ij}Q_k| \leq C\epsilon_k\rho^{-2}.
\]

Letting \(\epsilon_k \to 0\), we see that \((w_k, D_n w)\) converges locally uniformly in \(B_{C\rho}(y)\) to \((w, D_n w)\) with \(\mathcal{L}w = 0\).
Step 2. By giving an uniform control of \((w_k, D_n w_k)\) near \(\{x_n = 0\}\), we have that
\[
(1.3) \quad Lw = 0 \text{ in } B^+_x(0), \quad w(0) = 0, \quad D_n w = 0 \text{ on } \{x_n = 0\}.
\]

We claim that
\[
(1.4) \quad |D_n w_k(x)| \lesssim x_n + \epsilon_k \text{ in } S^+_x(0),
\]
and
\[
(1.5) \quad \omega_{w_k}(B_r(x)) \lesssim \max \left\{ r^\frac{1}{4}, \epsilon_k^\frac{1}{2} \right\} \text{ for } x \in G^+_x.
\]

Fixing \(k\), since we can construct a smooth approximation of \(u_k\) using the solution of Dirichlet problems, as we did in Lemma 8.1, without breaking the structure of the problem, we may assume that \(w_k\) is smooth. Then, \(\zeta = D_n w_k\) satisfies
\[
\begin{cases}
U^{ij}_k D_{ij} \zeta = 0 & \text{in } S_1(0), \\
|\zeta| \leq C & \text{in } S_1(0), \\
|\zeta| = 0 & \text{on } G_1(0).
\end{cases}
\]
where \(U^{ij}_k\) denote the cofactor matrix of \(D^2 u_k\). Let \(C_1\) and \(C_2\) be sufficiently large, the function
\[
v(x) = C_1 \left[ u_k(x) - \frac{n}{2} x_n D_n u_k + C_2 (x_n + \epsilon_k) \right].
\]
satisfies
\[
\begin{cases}
U^{ij}_k D_{ij} v = 0 & \text{in } S_1(0), \\
v \geq C & \text{on } \partial S_1(0) \setminus G_1(0), \\
v \geq 0 & \text{on } G_1(0).
\end{cases}
\]
Hence, \(v\) and \(-v\) are the upper and lower barriers of \(w_k\) at 0, respectively. Therefore,
\[
|D_n w_k(te_n)| \lesssim t + \epsilon_k.
\]
By considering the function \(u_{k,x_0} = [u_k(x) - u_k(x_0) - \nabla u_k(x_0) \cdot (x - x_0)]\) and transformation \(A x = (x - x_0) - D^t g_k(x_0) \cdot x e_n\), a similar discussion applies to every point on \(G^+_x\). Therefore, we have
\[
|D_n w_k(x)| \lesssim |A x \cdot e_n| + \epsilon_k \lesssim x_n + \epsilon_k,
\]
and (1.4) is proved.

From (1.4) that, we conclude that
\[
|w_k(x', g_k(x')) - w_k(x', x_n)| \lesssim x_n^2 + \epsilon_k x_n.
\]
To prove (1.5), we estimate \(|w_k(p) - w_k(q)|\) for points \(p = (y_1, g_k(y_1))\) and \(q = (q_1, g_k(q_1))\) on \(G^+_x(0)\). Let
\[
r = |p' - q'| \text{ and } \rho = C \max \left\{ r^\frac{1}{4}, \epsilon_k^\frac{1}{2} \right\}
\]
and take points \(y = (p', \rho)\) and \(z = (q', \rho)\). Since \(\rho \geq C \epsilon_k^\frac{1}{2}, \) (1.2) implies that
\[
|w_k(y) - w_k(z)| \lesssim r \rho^{-1},
\]
and then
\[ |w_k(p) - w_k(q)| \lesssim \rho^2 + \epsilon_k \rho + r \rho^{-1} \lesssim \max\left\{ t^2, \epsilon_k^2 \right\}. \]

This gives (1.5).

**Step 3.** Now, \( w \) is a continuous viscosity solution of (1.3), from (1.4) and the standard theory for elliptic partial differential equations, we have \( w \in C^1_{loc}. \) Note that \( w(0) = D_n w(0) = D' D_n w(0) = 0. \) There exists a quadratic function

\[ R(x) = \sum_{1 \leq i,j \leq n-1} a_{ij} x_i x_j + \frac{a_{nn}}{2} x_n^2 + \sum_{i=1}^{n-1} b_i x_i, \]

such that
\[ |w(x) - R(x)| \lesssim |x|^3 \text{ and } |D_n w(x) - a_{nn} x_n| \lesssim |x|^2, \]

for which \( a_{ij}, a_{nn}, b_i \) are bounded and satisfy \( \sqrt{1 + a_j^2 (a_{11} + a_{nn}) + \sum_{i=2}^{n-1} a_{ii}} = 0. \) As \( \epsilon_k \to 0, \) recalling that \( w \) is the limit of \( w_k \) in Step 2, we have
\[ |w_k(x) - R(x)| \leq \sigma + C |x|^3 \text{ and } |D_n w_k(x) - a_{nn} x_n| \leq \sigma + C |x|^2, \]

where \( \sigma = \sigma(\epsilon_k) \to 0. \) Suppose
\[ |a_k - a| + |\kappa_k - 1| + \sigma^2 + \epsilon_k^4 \leq \mu, \]

then
\( (1.6) \)
\[ |u_k - (Q_k + \epsilon_k R)| \leq C \epsilon_k \mu^3 \text{ in } S_{\mu}^{Q_k}. \]

and
\( (1.7) \)
\[ |D_n u_k - (D_n Q_k + \epsilon_k D_n R)| \leq C \epsilon_k \mu \text{ in } S_{\mu}^{Q_k}. \]

Fixed \( k, \) observing that \( \det D^2 u_k = \det (D^2 Q_k + \epsilon_k D^2 w_k) = 1, \) we see that the function \( k(t) = \det (D^2 Q_k + \epsilon_k D^2 R + t \Sigma) \) satisfies \( |k(0) - 1| \leq \epsilon_k \left( |u_k - a| + |\kappa_k - 1| \right) + O(\epsilon_k^2) \leq \epsilon_k \mu \) and \( k'(t) \approx 1. \) The equation \( k(t) = 1 \) has a solution \( t_0 \) satisfying
\( (1.8) \)
\[ |t_0| \lesssim \epsilon_k \mu. \]

Take
\[ P_0(x) = Q_k(x) + \epsilon_k R(x) + \frac{t_0}{2} |x|^2, \]

we have \( \det D^2 P_0 = 1, \) \( D_n P_0 = a x_n \) on \( \{ x_n = 0 \}. \) Thus, \( P_0 \in F_a. \) Combining (1.6), (1.7) and (1.8), we obtain
\[ |u_k(x) - P_0(x)| \leq C t_0 \mu + C \epsilon_k \mu^3 \leq C \epsilon_k \mu^3 \text{ in } S_{\mu}, \]

which together with the result from Step 1 implies
\[ |u_k(x) - P_0(x)| \leq C \epsilon_k \mu^3 + C \delta \epsilon \leq C \epsilon_k \mu^3 \text{ in } S_{\mu}. \]

This contradicts the definition of \( u_k \) if we replace \( \mu \) with \( \mu/2 \) and note that \( S_{\mu/2}(0) \cup S_{\mu/2}(0) \subset S_{\mu}. \) Therefore, we have proved (8.7), and (8.8) follows from calculation. \( \square \)
Appendix B. Verification of Example 9.3

Similar to Lemma 4.1 we can prove

**Lemma B.1.** Write \( x \in \mathbb{R}^n \) as \( x = (x_1, \ldots, x_k) \), \( x_j \in \mathbb{R}^a_j \), \( \sum_{i=1}^k a_i = n \). Let \( u \in C(B_r(0)) \) be a convex function, satisfying

\[
    u(x) \leq u(0) + \sum_{i=1}^k \sigma_i(|x^i|) \text{ in } B_r(0).
\]

Then

\[
    |\nabla_{x_j} u(0)| \leq C \inf_{0 \leq t \leq cr} \frac{2\sigma(t)}{t}, \quad i = 1, 2, \ldots, k.
\]

In particular, locally bounded convex functions are locally Lipschitz.

Before verifying Example 9.3, we introduce some properties of the singular function \( W_{a,b}(x') \) on the plane \( \mathbb{R}^{n-1} \). Denote \( v(x') = W_{a,b}(x') \), \( \bar{v}(x') = (1 + \rho^2)W_{a,b}(x') \).

Given \( t > 0 \), consider the sections

\[
    F_t = S_t^v(0), \quad \tilde{F}_t = S_t^{\bar{v}}(0)
\]

and the diagonal transformation

\[
    D_t = \text{diag} \left\{ t^{1/b}, t^{1/a}I'' \right\}.
\]

For simplicity, we only consider the case \( t = 1 \) in the following lemmas since \( v \) is a homogeneous function and these lemmas are invariant under the normalization

\[
    \tilde{v}(x) := \frac{v(D_t x)}{t} = v(x).
\]

**Lemma B.2.** Given a linear function \( L(x') \) satisfying \( L(x') \leq \bar{v} \). Denote the section \( S_L = \{ x' \mid v \leq L(x') \} \). There exists universal constants \( c, C \) such that if \( S_L \cap \partial F_t \neq \emptyset \), then \( S_L \subset \tilde{F}_{Ct} \setminus F_{ct} \).

**Proof.** Suppose that \( p \in S_L \cap \partial F_1 \neq \emptyset \). Then \( |p| \leq C \), the upper barrier relation \( L(x') \leq \bar{v} \) and \( L(x') \geq 0 \) implies that \( \|DL\| = \|DL(p)\| \leq C \) (see Lemma 4.1). While the function \( v \) is super-linearity at infinite, thus \( S_L \subset B_{Ct}(0) \subset \tilde{F}_{Ct} \), for some universal constant \( C_1 \). This also implies the opposite relation when \( c_1 C_1 \) is small enough. Otherwise, if \( S_L \cap F_{c_1} \neq \emptyset \), then \( S_L \subset \tilde{F}_{c_1 C_1} \subset F_{C_1} \), which contradicts the assumption \( S_L \cap \partial F_t \neq \emptyset \). \( \square \)

**Lemma B.3.** Given points \( p' \) and denote \( t = v(p') \). There exists universal constants \( c, C \) such that if \( s \in [0, c] \) then

\[
    cs^{\frac{1}{2}}D_tB_{s}^t(0) \subset S_{st}^{\bar{v}}(p') - p' \subset Cs^{\frac{1}{2}}D_tB_{s}^t(0)
\]

and thus

\[
    \text{Vol}_{n-1} S_{st}^{\bar{v}}(p') \geq cs^{\frac{n-1}{2}}t^{\frac{n}{2}}.
\]
Lemma B.4. Suppose that function $L(x')$ is a linear function such that $L(x') \leq \hat{v}$. There exists universal constants $C$ such that if the section $S_L = \{x' | v \leq L(x')\}$ satisfies $S_L \cap \partial F_t \neq \emptyset$, then 
\[
\|t^{-1}D_t \cdot DL\| \leq C, \quad \text{and}
\]
\[
\hat{v}(x') - L(x') \leq C(v(x' - q') + t), \quad \forall q \in \tilde{F}_c t \setminus F_c t. 
\]

Proof. Suppose that $t = 1$. Then $\|DL\| \leq C$ and $p' \in B'_C(0)$. Note that $v$ is a homogeneous function which is super-linearity at infinite. Thus, for $C_1$ large enough,
\[
C|x'| \leq C_1 v(x' - q') + C_1,
\]
and
\[
\hat{v}(x') \leq C_1 v(x' - q') + C_1.
\]

These two inequalities imply our lemma. \qed

Now we are in the position to verify Example 9.3 which will be completed by three Steps.

Step 1. Denote $w(x) = W^+_R(x)$ and $E = B'_R(0) \times (-\rho, \rho)$, we claim that
\[
w \in C^1(E) \cap C^2_{\text{loc}}(E \setminus \{te_n\}),
\]
and
\[
D_1 w(0, x'', x_n) = 0 \quad \text{and} \quad D_n w(x_1, x'', 0) = 0.
\]

Consider the set $\Gamma$ formed by the intersection of the image of $w$ and the lines $l$. Let $x_0$ be a endpoint of $l$. The classical interior strict convexity lemma states that the endpoints of $\Gamma$ is not inside $E$, therefore $x_0$ is on the boundary. Assume by way of contradiction that $x_0' \neq 0$, then $w$ is $C^2$ at point $x_0$ for some $\epsilon < \frac{x_0'}{\rho}$, the Pogorelov strict convexity lemma implies that $x_0$ cannot be the extreme point $\Gamma$. Therefore, $\Gamma$ is contained in the axis $e_n$. Thus, $w \in C^2(E \setminus \{|x'| = 0\})$. Note that [B.1] and [9.1] imply $w$ is pointwise $C^{1,b-1}$ on axis $e_n$, the claim is completed.

Step 2. Given point $q = (q', q_n) \in E \cap B'_R(0)$. Suppose that $q' \neq 0$. Denote $t = v(q')$. Let $S^w_{h_q}(q) = \{x \in E | w(x) \leq w(q) + \nabla w(q) \cdot (x - q) + h_q\}$ be the maximum section with base point $p$, and $L(x) = w(x) \leq w(q) + \nabla w(q) \cdot (x - q) + h_q$. We aim to prove that $h_q \approx t$, where $t = v(q') \approx w(q)$.

We claim $h_q \leq Ct$ by proving that
\[
P S^w_{h_q}(q) \subset \tilde{F}_c t \setminus F_c t.
\]
For each $r \in [-\rho, \rho]$, consider the function
\[
v_r(x') = w(x', r) \quad \text{and} \quad L_r(x') = L(x', r),
\]
and the sections
\[ H_r = \{ x' \mid v_r(x') \leq L_r(x) \}, \]
\[ G_r = \{ x' \mid v(x') \leq L_r(x) \}. \]

By (9.1)
\[ H_r \subset G_r. \]

Since \( q \) is near axis \( e_n \), the maximum section touches \( \partial E \) at point \( p \in \{ |x_n| = \rho \} \). For simplicity, we assume that \( q_n \geq 0 \), then
\[ D_n w(q) \geq D_n w(q', 0) = 0. \]

Therefore, for any constants \( r_2 \leq r_1 \leq \rho \),
\[ L_{r_2}(x') \leq L_{r_1}(x') \leq \bar{v}(x'), \]
and
\[ G_{r_2} \subset G_{r_1}. \]

Thus, we can assume that \( p_n = \rho \). Then
\[ q' \subset G_{q_n} \subset G_{p_n}. \]

Recall Lemma [372] we obtain
\[ G_r \subset \bar{F}_{ct} \setminus F_{ct}. \]

This is (2.2).

Next, we show that \( h_q \geq ct \). Let \( s \geq 0 \). By Lemma [2.0],
\[ \left| S_{h_q + st}^w(q) \right| \leq C(h_q + st)^{\frac{a}{2}}. \]
Recall (2.2),
\[ u(p) \geq ct. \]

Note that \( L_\rho \) is the support function of \( \bar{v} \) at point \( p \). Lemma [2.2] and Lemma [313] mean
\[ \left| S_{h_q + st}^w(q) \right| \geq cp \text{Vol}_{n-1} \{ S_{h_q + st}^w(q) \cap \{ x_n = \rho \} \} \geq cps^{\frac{n+1}{a} - 1} t^{\frac{n}{2}}. \]
Thus,
\[ C(h_q + st)^{\frac{a}{2}} \geq cps^{\frac{n+1}{a} - 1} t^{\frac{n}{2}}. \]
i.e.
\[ \frac{h_q}{t} \geq \sup_s \left\{ c(p)s^{\frac{n+1}{a}} - s \right\} \geq c. \]

**Step 3.** We prove that \( w \) satisfies
\[ (2.3) \]
\[ w(x) - w(q) - \nabla w(x) \cdot (x - q) \leq C(|x_1|^b + |x''|^a + x_n^2) \]
for any \( q \in B_{\frac{\rho}{2}}(0) \).

The case that \( q \) is on axis \( e_n \) follows from (9.1). We then assume that \( q' \neq 0. \)
If \( x \in S_{\bar{h}_q}^w(q) \). Note that \( S_{\bar{h}_q}^w(q) \) is interior section, the first equation of problem (2.2) means that
\[
\left| S_{\bar{h}_q}^w(q) \right| \approx h_q^w \approx t^w,
\]
and
\[
\left| \mathcal{F} \cap [-\rho, \rho] \right| \leq Ct^w.
\]
Note that \( S_{\bar{h}_q}^w(q) \) is balanced about \( q \). Therefore, according to Lemma 2.2,
\[
S_{s_0 t}^{w}(q) - q \approx D_t B_s^1(0) \times [-\rho, \rho].
\]
for \( s_0 = \frac{h_q}{t} \approx c. \) Then the classical interior \( C^{1,1} \) regularity results implies if \( s \in [0, s_0] \), then
\[
(2.4) \quad cs_0^w D_t B_s^1(0) \times [-cs_0^w, cs_0^w] \subset S_{s t}^{w}(q) - q \subset Cs_0^w D_t B_s^1(0) \times [-Cs_0^w, Cs_0^w].
\]
And (2.4) implies that (2.3).

If \( x \notin S_{\bar{h}_q}^w(q) \). Then
\[
v(x' - q') \geq ct \text{ outside } S_{\bar{h}_q}^w(q).
\]
Recall (2.2), we have \( \| D_n L \| \leq Ct \). By (2.1),
\[
w(x) - w(q) - \nabla w(x) \cdot (x - q) \leq \bar{v}(x') - L_{q_n}(x') + Ct
\]
\[
\leq C(v(x' - q') + t)
\]
\[
\leq Cv(x' - q').
\]
This is (2.3).

In particular, by virtue of Lemma \( 2.1 \) and (2.3), we see that \( w \) is \( C^{1,1-a} \) on the plane \( x_1 = 0 \), thus we finish the proof.

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