Remarks on the K41 scaling law in turbulent fluids

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Abstract

A definition of K41 scaling law for suitable families of measures is given and investigated. First, a number of necessary conditions are proved. They imply the absence of scaling laws for 2D stochastic Navier-Stokes equations and for the stochastic Stokes (linear) problem in any dimension, while they imply a lower bound on the mean vortex stretching in 3D. Second, for 3D stochastic Navier-Stokes equations necessary and sufficient conditions for K41 are proved, translating the problem into bounds for energy and enstrophy of high and low modes respectively. The validity of such conditions in 3D remains open. Finally, a stochastic vortex model with such properties is presented.

1 Introduction

In very rough terms, the scaling law devised by Kolmogorov and Obukhov for turbulent 3D fluids (usually referred as K41, see [12] and a detailed discussion in [9]), says that $S_2(r) \sim \epsilon^{2/3} r^{2/3}$ where $S_2(r)$ is the second order structure function and $\epsilon$ is the mean energy dissipation rate. Moreover, it is specified that this law is valid at very high Reynolds numbers and for distances $r$ in a certain range between the integral range and the Kolmogorov dissipation scale, having the order $\eta = \nu^{3/4} \epsilon^{-1/4}$. Although the numerical essence of these claims may be clear, their precise mathematical interpretation is not necessarily unique and could change a little bit depending on new discoveries.

The purpose of this note is to give one possible precise mathematical formulation of this scaling law and to discuss it from a number of viewpoints. We immediately stress that we cannot prove its validity for the 3D Navier-Stokes equations, but nevertheless we obtain a number of insights that seem worth to be known.

Some of our considerations are true for quite general families of probability measures; others will be specific to the stochastic Navier-Stokes equations on the torus $[0,1]^d$, $d = 2, 3$,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p = \nu \triangle u + \sum_{\alpha} h_\alpha(x) \dot{\beta}_\alpha(t)$$

(1.1)
with $\text{div} \, u = 0$ and periodic boundary conditions, with suitable vector fields $h_\alpha(x)$ and independent Brownian motions $\beta_\alpha(t)$ (the torus instead of a more realistic framework has been chosen for mathematical simplicity). We consider this equation in the limit $\nu \to 0$. Since the force does not vanish as $\nu \to 0$, this is a singular limit problem much like the boundary layer one, and so may be considered as a prototype of high Reynolds number singular limit problem, with some mathematical simplification due to the advantages produced by stochastic analysis. It should be noted that another possible and interesting approach to the zero-viscosity limit is the one adopted in [13] (for the $2d$ case), where the amplitude of the forcing noise is proportional to the square-root of the viscosity.

In the following we shall use the parameter $\nu^{-1}$ in place of the Reynolds number; this simplification is justified in our model since the force and the domain are given, so the Reynolds number goes to infinity if and only if $\nu \to 0$.

We shall denote by $H$ the natural space of finite energy velocity fields on the torus and we shall introduce a space $P$ of probability measures on $H$ having certain symmetries and regularities (precise definitions are given in the next section). On the fields $\varphi^{(i)}(x)$ we shall assume conditions such that there exists at least one stationary probability measure $\mu \in P$ associated to (1.1) (stationary measures will be defined in the next section). We use the notation

$$\mu[f(u)] := \int_H f(u) d\mu(u)$$

whenever the integral is well defined.

For every $\mu \in P$ we introduce the second order structure function

$$S_2^\mu(r) = \mu\left[\|u(r \cdot e) - u(0)\|^2\right]$$

for some coordinate unitary vector $e$, with $r > 0$ (the results proved below extend to the so called longitudinal structure function; we consider (1.2) to fix the ideas). The measures of $P$ are supported on continuous vector fields, so the pointwise operations in (1.2) are meaningful. Moreover, the symmetries in $P$ imply that $S_2^\mu(r)$ is independent of the coordinate unitary vector $e$ (in addition most of the estimates proved in the sequel extend to every unitary vector $e$).

We are going to define K41 scaling law for a set $\mathcal{M} \subset P \times \mathbb{R}_+$. The reason is that equation (1.1) may have (a priori) more than one stationary measure for any given $\nu$ and in certain claims it seems easier to consider a set of measures for a given $\nu$. Given $\nu > 0$ we use the notation $\mathcal{M}_\nu$ for the set section

$$\{\mu \in P : (\mu, \nu) \in \mathcal{M}\}.$$

Given $(\mu, \nu) \in P \times \mathbb{R}_+$, we define the mean energy dissipation rate as

$$\epsilon = \epsilon(\mu, \nu) := \nu \cdot \mu \left[\int_{[0,1]^d} \|Du(x)\|^2 dx\right].$$

**Remark 1.1** If $\mu$ is a stationary measure of (1.1) and a mean energy equality (coming from Itô formula) can be rigorously proved, one can show that $\epsilon$ does not depend on $(\mu, \nu)$. 
Given \((\mu, \nu) \in \mathcal{P} \times \mathbb{R}_+\), we also define the quantity
\[
\eta = \eta(\mu, \nu) := \nu^{3/4} \epsilon(\mu, \nu)^{-1/4}.
\]

**Remark 1.2** In case of equations (1.1), \(\eta\) is a length scale: \(\nu\) has dimension \( [L]^{2}[T]^{-1} \), \(\epsilon\) has dimension \( [L]^{2}[T]^{-3} \), so \(\eta\) has dimension \( [L] \). The only combination of \(\nu\) and \(\epsilon\) in powers, having dimension \([L]\), is the \(\eta\) above. This is the simplest reason to choose \(\eta\) as a length scale involved in K41 theory. More refined arguments may be found in [9] and related references.

Let us come to the definition of K41 scaling law chosen in this work. Here and in the sequel, when we talk about a set \(\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+\), we tacitly assume that
\[
\mathcal{M}_\nu \neq \phi \text{ for all sufficiently small } \nu > 0,
\]
since otherwise several definitions and statements would be just empty.

**Definition 1.3** We say that a scaling law of K41 type holds true for a set \(\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+\) if there exist \(\nu_0 > 0\), \(C > c > 0\), \(C_0 > 0\), and a monotone function \(R_0 : (0, \nu_0] \to \mathbb{R}_+\) with \(R_0(\nu) > C_0\) and \(\lim_{\nu \to 0} R_0(\nu) = +\infty\), such that the bound
\[
c \cdot r^{2/3} \leq S_\mu^1(r) \leq C \cdot r^{2/3}
\]
holds for every pair \((\mu, \nu) \in \mathcal{M}\) and every \(r\) such that \(\nu \in (0, \nu_0]\) and
\[
C_0 \cdot \eta(\mu, \nu) < r < \eta(\mu, \nu) \cdot R_0(\nu).
\]

**Remark 1.4** For simplicity we could have asked the scaling property for \(C_0 \cdot \eta(\mu, \nu) < r < r_0\) for a constant \(r_0\) (a measure of the integral scale). However, such a formulation could be too restrictive. On the other hand, it is necessary that the range of \(r\)’s increases to infinity (relative to \(\eta\)) as \(\nu \to 0\), otherwise the property becomes trivial, see remark 2.3.

This is the mathematical formulation of K41 theory that we analyse in this note. Here is a list of facts we can prove around it. In summary, they have the structure of certain necessary conditions for K41, and certain almost equivalent conditions.

- We introduce a measure \(\theta\) of the length scale where dissipation takes place, defined as
\[
\theta^2 = \frac{\mu \left[ \int_{[0,1]^d} \|Du(x)\|^2 \, dx \right]}{\mu \left[ \int_{[0,1]^d} \|D^2u(x)\|^2 \, dx \right]} \quad (1.3)
\]
and prove a claim of the following form (theorem 2.2): if a scaling law holds on a range \(C_0 \cdot \eta < r < \eta \cdot R_0(\nu)\), then
\[
\theta \leq C \eta.
\]

In other words, the scale at which dissipation dominates cannot overlap with the range over which a fractal scaling law holds.
• Since $\theta$ is constant (with respect to $\nu$) for both the 2D stochastic Navier-Stokes equations and the Stokes (linear) equations, we can rule out K41 scaling law for such systems. For the theory of 2D stochastic Navier-Stokes equations this seems a remarkable fact. Moreover, these facts tell us that, in case definition 1.3 holds true in the 3D case, it is strictly due to 3D nonlinear effects. This is further emphasised by the following result.

• For the 3D stochastic Navier-Stokes equations we prove that, if K41 holds, then the mean vortex stretching

$$\mu \left[ \int T |\langle S_u \text{curl} u, \text{curl} u \rangle|^2 dx \right]$$

(where we set $S_u = \frac{1}{2}(Du + Du^T)$) must be very large, essentially at least as large as $\nu^{-3/2}$. See Corollaries 2.19 and 2.21. Vortex stretching is thus a basic mechanism in K41 theory.

• We apply a known scaling transformation (see [1]) and introduce an auxiliary family of stochastic Navier-Stokes equations with modified domain and viscosity. Then we introduce a condition on this family of equations, called Condition A, and prove it is equivalent to the scaling law of K41 type. This is conceptually interesting since Condition A is of rather qualitative nature, while its consequence (the scaling law of K41 type) is more quantitative. More specifically, the behaviour $r^{2/3}$, and also the exponent $3/4$ in the definition of $\eta$, arise from the stochastic Navier-Stokes equations themselves, through a scaling transformation, when certain bounds (without special exponents) are fulfilled for the auxiliary family.

• We give other necessary and sufficient conditions for K41, starting from Condition A. In plain words, they state that the large structures of $u$ have bounded mean square gradients (or bounded mean enstrophy), while the small structures have bounded mean energy.

• As a mild support to the belief that all these Conditions could be true in dimension $d = 3$, we finally exhibit a random field that satisfies them, and was constructed independently from this purpose in [7] as a model of turbulent fluid inspired by the vortex structures usually observed in numerical simulations.

The remainder of the article is structured as follows. The remainder of the present section introduces the notations that will be used throughout this work. In Section 2 we draw several conclusions from our formulation of the K41 scaling law that allow us to give stringent necessary conditions for it to hold. These conditions are sufficient to rule out any non-trivial scaling law in the 2D case. We proceed in Section 3 to find a condition that turns out to be equivalent to K41. This condition is then shown in Section 4 to hold for a random eddy model introduced in [7].
1.1 Notations about functions spaces

Let $\mathcal{T}$ be the torus $[0, 1]^d$, $d = 2, 3$, $L^2(\mathcal{T})$ be the space of vector fields $u : \mathcal{T} \to \mathbb{R}^d$ with $L^2(\mathcal{T})$-components, $\mathbb{H}^0(\mathcal{T})$ be the analogous Sobolev spaces, $C(\mathcal{T})$ be the analogous space of continuous fields.

Let $H$ be the space of all fields $u \in L^2(\mathcal{T})$ such that $\text{div} \ u = 0$ and $\int_{\mathcal{T}} u(x)dx = 0$ (zero mean) and the trace of $u \cdot n$ on the boundary is periodic (where $n$ is the outer normal, see [15], Ch.I, Thm 1.2). Let $V$ be the space of divergence free, zero mean, periodic elements of $\mathbb{H}^1(\mathcal{T})$ and $D(A)$ be the space of divergence free, zero mean, periodic elements of $\mathbb{H}^2(\mathcal{T})$. Finally, let $\mathcal{D}$ be the space of infinitely differentiable divergence free, zero mean, periodic fields on $\mathcal{T}$. The spaces $V$, $D(A)$ and $\mathcal{D}$ are dense and compactly embedded in $H$. Let $A : D(A) \subset H \to H$ be the (Stokes) operator $Au = -\Delta u$ (componentwise).

Sometimes we shall also need the same framework for the torus $[0, L]^d$, $d = 2, 3$, with any $L > 0$. We set $\mathcal{T}_L = [0, L]^d$, $H_L$ equal to the set of all fields $u \in L^2(\mathcal{T}_L)$ such that $\text{div} \ u = 0$ and $u \cdot n$ on the boundary is periodic, $V_L$, $D(A_L)$ and $A_L : D(A_L) \subset H_L \to H_L$ the analogous of $V$, $D(A)$ and $A$. Notice only that we define the inner product as

$$|u|_{H_L}^2 = \frac{1}{L^d} \int_{\mathcal{T}_L} |u(x)|^2 \, dx.$$  

(So that, roughly speaking, $|u|_{H_L}^2 \sim |u(0)|^2$ for homogeneous fields.)

1.2 The class $\mathcal{P}$ of probability measures

Let $\mathcal{P}_0$ be the family of all probability measures $\mu$ on $H$ (equipped with the Borel $\sigma$-algebra) such that $\mu(D(A)) = 1$ ($D(A)$ is a Borel set in $H$). Since $\mathbb{H}^2(\mathcal{T}) \subset C(\mathcal{T})$ by Sobolev embedding theorem, the elements of $D(A)$ are continuous (have a continuous element in their equivalence class). Consequently, given $x_0 \in \mathcal{T}$, the mapping $u \mapsto u(x_0)$ is well defined on $D(A)$, with values in $\mathbb{R}^d$. In particular, any expression of the form

$$\mu[f(u(x_1), \ldots, u(x_n))]$$

is well defined for given $x_1, \ldots, x_n \in \mathcal{T}$, given $\mu \in \mathcal{P}_0$, and suitable $f : \mathbb{R}^{nd} \to \mathbb{R}$ (for instance measurable non negative). It follows that $S^H_2(r)$ is well defined (possibly infinite) for every $\mu \in \mathcal{P}_0$.

The same argument does not apply to $Du(x_0)$ and $D^2u(x_0)$, at least in $d = 3$. This is why we use lengthy expressions like

$$\mu \left[ \int_{\mathcal{T}} \|Du(x)\|^2 \, dx \right], \quad \mu \left[ \int_{\mathcal{T}} \|D^2u(x)\|^2 \, dx \right]$$

which are meaningful (possibly infinite) for every $\mu \in \mathcal{P}_0$.

We denote by $\mathcal{P}$ the class of all $\mu \in \mathcal{P}_0$ such that

$$\mu \left[ \int_{\mathcal{T}} \|Du(x)\|^2 \, dx \right] < \infty$$
and, for every $a \in T$ and every rotation $R$ that transforms the set of coordinate axes in itself,

$$
\mu[f(u(-a))] = \mu[f(u)], \quad \mu[f(u(Ra))] = \mu[f(Ru(a))]
$$

(1.4)

for all continuous bounded $f : H \to \mathbb{R}$. In plain words, we impose space homogeneity and a discrete form of isotropy (compatible with the symmetries of the torus). In the following we will refer to this symmetry as partial or discrete isotropy.

Discrete isotropy is imposed for two reasons. On one hand, $S^\mu_\nu(r) < \infty$ for every $r > 0$ and $\mu \in \mathcal{P}$, by Lemma 2.1 below.

Finally, notice that $S^\mu_\nu(r) < \infty$ for every $r > 0$ and $\mu \in \mathcal{P}$, by Lemma 2.1 below.

2 Necessary conditions for K41

2.1 General results

The results of this subsection apply to suitable families of probability measures, without any use of the Navier-Stokes equations. They will be applied to the stochastic Navier-Stokes equations in the next subsection.

Given a measure $\mu \in \mathcal{P}$, $\mu \neq \delta_0$, we introduce the number $\theta = \theta(\mu)$ defined by the identity (1.3), letting $\theta = 0$ when $\mu \left[ \int_T \|D^2u(x)\|^2 \, dx \right] = \infty$. If $\mu = \delta_0$, numerator and denominator vanish and we arbitrarily define $\theta = 1$. We have $\theta \leq C$ where the constant is universal and depends only on the Poincaré constant of the torus. By the definitions, we have

$$
\theta(\mu)^2 = \frac{\epsilon(\mu, \nu)}{\nu \cdot \mu \left[ \int_T \|D^2u(x)\|^2 \, dx \right]}
$$

for every pair $(\mu, \nu) \in \mathcal{P} \times \mathbb{R}_+^\times$.

When, as in our application, the elements $u \in H$ have the meaning of velocity fields, by dimensional analysis we see that $\theta$ has the dimension of a length. We interpret it as an estimate of the length scale where dissipation is more relevant. Indeed, very roughly, from

$$
\frac{\int_T \|D^2u(x)\|^2 \, dx}{\int_T \|Du(x)\|^2 \, dx} \sim \frac{\sum |k|^2 \left( |k|^2 |\hat{u}(k)|^2 \right)}{\sum |k|^2 |\hat{u}(k)|^2}
$$
we see that $\theta(\mu)^{-2}$ has the meaning of typical square wave length of dissipation (looking at $|k|^2|\hat{u}(k)|^2$ as a sort of distribution in wave space of the dissipation).

**Lemma 2.1** For every $\mu \in \mathcal{P}$ such that $\theta(\mu) > 0$ we have

$$
\frac{1}{4d} \cdot r^2 \leq \frac{S_{\mu}^2(r)}{\mu \int_T \|Du(x)\|^2 dx} \leq r^2
$$

(2.1)

for every $r \in (0, \theta(\mu)/4d]$. The upper bound is true for every $r > 0$ even if $\theta(\mu) = 0$.

**Proof.** Since we want to use Taylor formula for elements of $D(A)$, we use the mollification described in Appendix 1. We denote by $\mu_\varepsilon$ the mollifications of $\mu$.

We prove in Appendix 1 that, for given $r$ and $\mu$,

$$
\lim_{\varepsilon \to 0} \mu_\varepsilon \left[ \|Du(0)\|^2 \right] = \mu \left[ \int_T \|Du(x)\|^2 dx \right]
$$

$$
\lim_{\varepsilon \to 0} \mu_\varepsilon \left[ \|D^2u(0)\|^2 \right] = \mu \left[ \int_T \|D^2u(x)\|^2 dx \right]
$$

$$
\lim_{\varepsilon \to 0} \mu_\varepsilon \left[ \|u(re) - u(0)\|^2 \right] = \mu \left[ \|u(re) - u(0)\|^2 \right]
$$

By space homogeneity of $\mu_\varepsilon$

$$
\mu_\varepsilon \left[ \|u(re) - u(0)\|^2 \right] \leq r^2 \int_0^1 \mu_\varepsilon \left[ \|Du(\sigma e)\|^2 \right] d\sigma
$$

$$
= r^2 \mu_\varepsilon \left[ \|Du(0)\|^2 \right]
$$

and thus, by the previous convergence results,

$$
\mu \left[ \|u(re) - u(0)\|^2 \right] \leq r^2 \mu \left[ \int_T \|Du(x)\|^2 dx \right].
$$

This implies the right-hand inequality of (2.1) for every $r > 0$.

On the other hand, for smooth vector fields we have

$$
u(re) - u(0) = Du(0)re + \frac{r^2}{2} \int_0^1 D^2u(\sigma e)(e, e) d\sigma
$$

and thus

$$
\mu_\varepsilon \left[ \|Du(0)re\|^2 \right] \leq 2\mu_\varepsilon \left[ \|u(re) - u(0)\|^2 \right]
$$

$$
+ 2\mu_\varepsilon \left[ \|r^2 \int_0^1 D^2u(\sigma e)(e, e) d\sigma\|^2 \right].
$$
Again from space homogeneity of $\mu_\varepsilon$,
\[ \mu_\varepsilon \left[ \left\| D^2 u(0) \right\|^2 \right] \leq r^4 \mu_\varepsilon \left[ \left\| D^2 u(0) \right\|^2 \right] \]
and from lemma A.3 of Appendix 1
\[ \mu_\varepsilon \left[ \left\| Du(0) e \right\|^2 \right] = \frac{1}{d} \mu_\varepsilon \left[ \left\| Du(0) \right\|^2 \right]. \]
Therefore
\[ \mu_\varepsilon \left[ \left\| u(re) - u(0) \right\|^2 \right] \geq \frac{r^2}{2d} \mu_\varepsilon \left[ \left\| Du(0) \right\|^2 \right] - r^4 \mu_\varepsilon \left[ \left\| D^2 u(0) \right\|^2 \right]. \]
We thus have in the limit
\[ S_2(r) \geq \frac{r^2}{2d} \mu \left[ \int_T \left\| Du(x) \right\|^2 dx \right] - r^4 \mu \left[ \int_T \left\| D^2 u(x) \right\|^2 dx \right] \]
and therefore, by definition of $\theta(\mu)$,
\[ S_2(r) \geq \left( \frac{1}{2d} - \frac{r^2}{\theta(\mu)} \right) \mu \left[ \int_T \left\| Du(x) \right\|^2 dx \right] \cdot r^2. \]
This implies the left-hand inequality of (2.1) for $r \in (0, \frac{\theta(\mu)}{4d}]$. The proof is complete.

**Theorem 2.2** Let $M \subset \mathcal{P} \times \mathbb{R}_+$ be a set with the following scaling property: there is a function $\tilde{\eta} : M \to \mathbb{R}_+$ (the length scale of the scaling property), a decreasing function $R_0 : [0, \infty) \to \mathbb{R}_+$, with $\lim_{\nu \to 0} R_0(\nu) = +\infty$, a scaling exponent $\alpha \in (0, 2)$ and constants $C_2 \geq C_1 > 0$, $C_3 > 0$, $\nu_0 > 0$, such that
\[ R_0(\nu) > C_3 \tilde{\eta}(\mu, \nu) \]
for every $\nu \in (0, \nu_0)$ and every $\mu \in M_\nu$. Let $\theta(\mu)$ be the dissipation length scale defined above.
Then the two length scales $\theta(\mu)$ and $\tilde{\eta}(\mu, \nu)$ are related by the property
\[ \lim sup_{\nu \to 0} \left( \sup_{\mu \in M_\nu} \frac{\theta(\mu)}{\tilde{\eta}(\mu, \nu)} \right) < \infty. \]

**Proof.** It is intuitively rather clear that (2.1) is in contradiction with (2.2) if the ranges of $r$ where the two properties hold overlap, so we need the bound (2.3). The proof below confirm this intuition by ruling out the possibility that the factor $\mu \left[ \int_T \left\| Du(x) \right\|^2 dx \right]$ may produce a compensation.
We argue by contradiction and assume that there exists a sequence \((\mu_n, \nu_n) \in \mathcal{M}\), with \(\nu_n \to 0\), such that
\[
\lim_{n \to \infty} \frac{\theta(\mu_n)}{\eta(\mu_n, \nu_n)} = +\infty. \tag{2.4}
\]
Notice that, in such a case, \(\theta(\mu_n)\) must be positive, so lemma 2.1 applies. Let us consider two sequences \(r'_n\) and \(r''_n\) defined as follows:
\[
r'_n = C_3 \eta(\mu_n, \nu_n), \quad r''_n = r'_n a_n
\]
with
\[
\lim_{n \to \infty} a_n = +\infty, \quad r''_n \leq \eta(\mu_n, \nu_n) R_0(\nu_n), \quad r''_n \leq \frac{\theta(\mu_n)}{4d}
\]
where we ask that the last two inequalities are satisfied at least eventually. Such a sequence \(r''_n\) exists because \(\lim_{\nu \to 0} R_0(\nu) = +\infty\) and (2.4) is assumed.

We have (eventually) \(r'_n, r''_n \in (0, \frac{\theta(\mu_n)}{4d})\) and \(r'_n, r''_n \in [C_3 \eta(\mu_n, \nu_n), \eta(\mu_n, \nu_n) R_0(\nu_n)]\), hence for both \(r_n := r'_n\) and \(r_n := r''_n\) we have
\[
C_1 r_n^\alpha \leq S_2^{\mu_n}(r_n) \leq C_2 r_n^\alpha, \quad \frac{1}{4d} \beta_n r_n^2 \leq S_2^{\mu_n}(r_n) \leq \beta_n r_n^2
\]
where we have set \(\beta_n = \mu_n \left[ \int_T ||Du(x)||^2 dx \right] \). The contradiction will come from the fact that, if it could happen that \(\beta_n\) adjusts the factor \(r_n^2\) to produce \(r_n^\alpha\), this cannot happen simultaneously for the two sequences \(r'_n = r'_n\) and \(r''_n = r''_n\). Indeed, from the previous inequalities we must have
\[
C_1 r_n^\alpha \leq \beta_n r_n^2, \quad \beta_n r_n^2 \leq 4d C_2 r_n^\alpha
\]
and hence
\[
\beta_n \geq C_1 r_n^{\alpha - 2}, \quad \beta_n \leq 4d C_2 r_n^{\alpha - 2}
\]
for both \(r_n = r'_n\) and \(r_n = r''_n\). But the inequalities
\[
\beta_n \geq C_1 (r'_n)^{\alpha - 2}, \quad \beta_n \leq 4d C_2 (r''_n)^{\alpha - 2}
\]
and the assumption \(\alpha < 2\) imply
\[
r'_n \geq C r''_n
\]
eventually, for a suitable constant \(C > 0\). This is impossible since \(\lim_{n \to \infty} a_n = +\infty\). The proof is complete. ■

Remark 2.3 The divergent factor \(R_0(\nu)\) in the definition (2.2) of a scaling law is essential to have a non trivial definition. If, on the contrary, we simply ask that the scaling law holds on a bounded interval \(r \in [C_3 \eta_\nu, C_4 \eta_\nu]\), we have a definition without real interest. Let us explain this fact with a (useless) definition and an
necessary conditions for K41

Example. Let us say that a family \( M \subset P \times \mathbb{R}_+ \) satisfies a local \( \alpha \) property, \( \alpha < 2 \), if there is a function \( \tilde{\eta}(\mu, \nu) \) and constants \( C_2 \geq C_1 > 0, C_4 \geq C_3 > 0, \nu_0 > 0 \), such that

\[
C_1 r^\alpha \leq S_2^{\mu, \nu}(r) \leq C_2 r^\alpha \quad \text{for } r \in [C_3 \tilde{\eta}(\mu, \nu), C_4 \tilde{\eta}(\mu, \nu)]
\]

(2.5)

for every \( \nu \in (0, \nu_0) \) and every \( \mu \in M_\nu \). As an example, consider a case with the mapping \( \nu \mapsto M_\nu \) which is single valued and injective and

\[
S_2^{\mu, \nu}(r) = \nu^{-1} r^2
\]

where \( M_\nu = \{ \mu^\nu \} \). This function \( S_2^{\mu, \nu}(r) \) certainly does not have any interesting scaling exponent (different from 2) but satisfies the previous local \( \alpha \) property simultaneously for a continuum of values of \( \alpha \). Indeed, given any \( \alpha \in (0, 2) \) take \( \tilde{\eta}(\mu^\nu, \nu) = \nu^{\frac{1}{2} - \alpha} \); then given a choice of \( C_4 \geq C_3 > 0 \), for every \( r \in [C_3 \tilde{\eta}(\mu^\nu, \nu), C_4 \tilde{\eta}(\mu^\nu, \nu)] \), namely for \( \nu^{-\frac{1}{2} - \alpha} r \in [C_3, C_4] \), we have

\[
S_2^{\mu, \nu}(r) = \left( \nu^{-\frac{1}{2} - \alpha} r \right)^{2-\alpha} r^\alpha \in [C_1, C_2] \cdot r^\alpha
\]

with \( C_1 = C_3^{2-\alpha}, C_2 = C_4^{2-\alpha} \). This example shows that the local \( \alpha \) property is not a distinguished scaling property. Moreover, it shows that (2.1) and (2.5) are compatible: this is why a proof of theorem 2.2 is necessary.

Example 2.4 Since we have just given a negative example (artificial, but close to what happens in 2D), let us also give an example of a function of \( (\nu, r) \) which satisfies the properties of definition 1.3 and also 2.1 (to see that they are compatible). It may look artificial, but it was devised on the basis of the vortex model of [7], described also below. The function is

\[
S_2^{\mu, \nu}(r) = \int_\eta^1 \left( \frac{l^{2/3}}{l} \right)^2 \frac{dl}{l}
\]

with \( \eta = \nu^{3/4} \). We have

\[
r \leq \eta \Rightarrow S_2^{\mu, \nu}(r) = \int_\eta^1 \left( \frac{l^{2/3}}{l} \right)^2 \frac{dl}{l} = \frac{3}{4} r^2 [\nu^{-1} - 1]
\]

which is essentially the behaviour [2.7]. On the other hand,

\[
r \in [\eta, 1] \Rightarrow S_2^{\mu, \nu}(r) = \int_\eta^r \left( \frac{l^{2/3}}{l} \right)^2 \frac{dl}{l} + \int_r^1 \left( \frac{l^{2/3}}{l} \right)^2 \frac{dl}{l}
\]

\[
= \frac{9}{4} r^{2/3} - \frac{3}{2} [l^{1/2} - \frac{3}{4} r^{2/3}]
\]

which is bounded above and below by the order \( r^{2/3} \) since \( r \in [\nu^{3/4}, 1] \) (\( \nu^{1/2} \leq r^{2/3} \)).
Let us finally state two general consequences of the previous theorem, that we shall apply to stochastic Navier-Stokes equations.

**Corollary 2.5** Given a family \( M \subset \mathcal{P} \times \mathbb{R}_+ \), if

\[
\inf_{(\mu, \nu) \in M} \theta(\mu) > 0
\]

then no scaling law in the sense of the previous theorem may hold true with a length scale \( \tilde{\eta}(\mu, \nu) \) such that

\[
\lim \inf_{\nu \to 0} \left( \inf_{\mu \in M_{\nu}} \tilde{\eta}(\mu, \nu) \right) = 0.
\]

We shall see that this simple corollary applies to the 2D stochastic Navier-Stokes equation and the Stokes problem, so K41 scaling law is ruled out for these systems.

Let us apply the theorem to the case of K41 scaling law. We take, in the previous theorem,

\[
\tilde{\eta}(\mu, \nu) = \eta(\mu, \nu) = \nu^{3/4} \epsilon(\mu, \nu)^{-1/4}
\]

as in the introduction. In the following result, \( \mu \left[ \int_T \| D^2 u(x) \|^2 \, dx \right] \) may be infinite.

**Corollary 2.6** Let \( M \subset \mathcal{P} \times \mathbb{R}_+ \) be a family with the K41 scaling law, in the sense of Definition 1.3. Then there exist \( \nu_0 > 0 \) and \( C > 0 \) such that

\[
\mu \left[ \int_T \| D^2 u(x) \|^2 \, dx \right] \geq C \epsilon^{3/2}(\mu, \nu) \cdot \nu^{-5/2}
\]

for every \( \nu \in (0, \nu_0) \) and every \( \mu \in M_{\nu} \).

**Proof.** From (2.3), the definition of \( \eta(\mu, \nu) \) and the definition of \( \theta^2(\mu) \) we have

\[
\lim \sup_{\nu \to 0} \left( \sup_{\mu \in M_{\nu}} \frac{\mu \left[ \int_T \| D^2 u(x) \|^2 \, dx \right]}{\nu^{3/2} \epsilon(\mu, \nu)^{-1/2} \left[ \int_T \| D^2 u(x) \|^2 \, dx \right]} \right) < \infty.
\]

Thus, from the definition of \( \epsilon(\mu, \nu) \),

\[
\lim \sup_{\nu \to 0} \left( \sup_{\mu \in M_{\nu}} \frac{\nu^{-5/2} \epsilon(\mu, \nu)^{3/2}}{\mu \left[ \int_T \| D^2 u(x) \|^2 \, dx \right]} \right) < \infty.
\]

This implies the claim of the Corollary. \( \blacksquare \)

**Remark 2.7** Dimensional analysis says that \( \nu \) has dimension \( [L]^2[T]^{-1} \), \( \epsilon \) has dimension \( [L]^2[T]^{-3} \), so \( \epsilon^{3/2}(\mu, \nu) \cdot \nu^{-5/2} \) has dimension \( [L]^{-2}[T]^{-2} \), the correct dimension of \( E^\mu \left[ \int_T \| D^2 u(x) \|^2 \, dx \right] \).
2.2 Application to stochastic Navier-Stokes equations

In this section we consider equation (1.1) in dimension 2 and 3 and also the corresponding linear equations (Stokes equations).

2.2.1 The noise

Since we are dealing with spaces of translation invariant measures, we wish to consider classes of noises that produce such measures. Every Gaussian translation invariant noise is ‘diagonal’ with respect to the Stokes operator $A$ in the sense that eigenmodes are all independent. In order to give a rigorous definition for our driving noise, we define

$$\Lambda^{(\infty)} := \left\{ k \in 2\pi \mathbb{Z}^d : |k| > 0 \right\}$$

and we assume that the noise of equation (1.1) has the form

$$\sum_{k \in \Lambda^{(\infty)}} \sigma_k \beta_k(t)e^{-ik \cdot x}$$

where $(\beta_k)_{k \in \Lambda^{(\infty)}}$ are independent $d$-dimensional Brownian motions and $(\sigma_k)_{k \in \Lambda^{(\infty)}}$ are $d \times d$ complex-valued matrices such that

$$k \cdot \sigma_k = 0$$

and

$$\sum_{k \in \Lambda^{(\infty)}} |\sigma_k|^2 < \infty.$$  

Moreover, in order to obtain real-valued noise, we assume that

$$\bar{\sigma}_k = \sigma_{-k}$$

for every $k \in \Lambda^{(\infty)}$. Additionally, the vector-valued random field

$$W(t, x) = \sum_{k \in \Lambda^{(\infty)}} \sigma_k \beta_k(t)e^{-ik \cdot x}$$

is, for every $t \geq 0$, partially isotropic if and only

$$|\sigma_k| = |\sigma_{Rk}|$$

for all $k \in \Lambda^{(\infty)}$ and for every coordinate rotation $R$.

Finally, in order to have measures with $\mu(D(A)) = 1$ we assume that

$$\sum_{k \in \Lambda^{(\infty)}} |k|^2 |\sigma_k|^2 < \infty,$$

since the values $|k|^2$ correspond to the eigenvalues of $A$. To summarise, we shall always assume that the noise (2.6) satisfies assumptions (2.7)-(2.11).
2.2.2 The two-dimensional case

The following result is well known.

**Lemma 2.8** Let $\mu$ be an invariant measure of (1.1) ($d = 2$) such that

$$\mu \int \| Du(x) \|^2 dx < \infty.$$  

Then $\mu \in P_0$ and

$$\nu \cdot \mu \int \| Du(x) \|^2 dx = \frac{1}{2} \sum_{k \in \Lambda^{(\infty)}} |\sigma_k|^2,$$

$$\nu \cdot \mu \int \| D \text{curl} u(x) \|^2 dx = \frac{1}{2} \sum_{k \in \Lambda^{(\infty)}} |k|^2 |\sigma_k|^2.$$

**Proof.**

Given $\mu$, consider the (product) filtered probability space $(\Omega, \mathcal{A}, (\mathcal{A}_t)_{t \geq 0}, P)$ supporting both a family of independent $d$-dimensional Brownian motions $\beta_k(t)$, $(k, \alpha) \in \Lambda^{(\infty)}$, and a non anticipating random variable $u_0 \in \mathcal{A}_0$ with law $\mu$. The corresponding strong solution $u(t, x)$ of (1.1) is a stationary process and satisfies, due to Itô formula, the balance relations

$$\frac{1}{2} E^P \int_T \| u(t, x) \|^2 dx + \nu E^P \int_0^t \int_T \| Du(s, x) \|^2 dx$$

$$= \frac{1}{2} E^P \int_T \| u_0(x) \|^2 dx + \frac{1}{2} \sum_{k \in \Lambda^{(\infty)}} |\sigma_k|^2 \cdot t$$

$$\frac{1}{2} E^P \int_T \| \text{curl} u(t, x) \|^2 dx + \nu E^P \int_0^t \int_T \| D \text{curl} u(s, x) \|^2 dx$$

$$= \frac{1}{2} E^P \int_T \| \text{curl} u_0(x) \|^2 dx + \frac{1}{2} \sum_{k \in \Lambda^{(\infty)}} |k|^2 |\sigma_k|^2 \cdot t.$$  

The result easily follows from stationarity. \[\blacksquare\]

**Corollary 2.9** There exists a positive constant $\theta_0$, independent of $\nu$, such that

$$\theta(\mu) \geq \theta_0$$

for every invariant measure $\mu \in P$ of (1.1).

**Proof.** The property $\theta(\mu) \geq \theta_0$ follows from the definition of $\theta(\mu)$ and the two identities of the previous lemma, since

$$\int_T \| D^2 u(x) \|^2 dx \leq C \int_T \| D \text{curl} u(x) \|^2 dx.$$
for a universal constant $C > 0$. 

In the next theorem, when we say that $\mathcal{M} \subset P \times \mathbb{R}_+$ is a family of invariant measures of (1.1), we clearly understand that each element $(\mu, \nu) \in \mathcal{M}$ has the property that $\mu$ is an invariant measure for the Markov semigroup associated to equation (1.1) with viscosity equal to $\nu$.

**Theorem 2.10** In dimension $d = 2$, a family of invariant measures $\mathcal{M} \subset P \times \mathbb{R}_+$ of (1.1) cannot have any scaling law (in the sense of (2.2)).

**Remark 2.11** Under our assumptions on the noise, invariant measures of (1.1) that belong to $P$ certainly exist. In principle there could exist invariant measures for (1.1) not belonging to $P$, but this has recently been excluded under very weak conditions on the driving noise (see [11] and the references therein).

**Remark 2.12** Consider equation (1.1) without the nonlinear term (called Stokes equations):

$$\frac{\partial u}{\partial t} + \nabla p = \nu \Delta u + \sum_{k \in \Lambda(\infty)} \sigma_k \dot{\beta}_k(t) e^{-ik \cdot x}$$

in dimension $d = 2, 3$. Let $\mathcal{M} \subset P \times \mathbb{R}_+$ be a family of invariant measures for it. Then the same results of the previous theorem hold true. The proof is the same. Alternatively, one may work componentwise in the Fourier modes and prove easily the claims.

### 2.2.3 The three-dimensional case

The lack of knowledge about the well posedness of the 3D stochastic Navier-Stokes equations has, among its consequences, the absence of the Markov property, and therefore of the usual notion of invariant measure. One may introduce several variants. Here we adopt the following concept.

Consider the usual Galerkin approximations, recalled in Appendix B. The equation with generic index $n$ in this scheme defines a Markov process, with the Feller property, and has invariant measures, by the classical Krylov-Bogoliubov method: if $X_n^x(t)$ is its solution starting from $x$ and $\nu_n^{n,x}$ is the law of $X_n^x(t)$ on $H$, by Itô formula it is easy to get a bound of the form (see for instance [5])

$$\sup_{T \geq 0} \frac{1}{T} \int_0^T E\left[\|X_n^x(t)\|^2\right] dt \leq C < \infty$$

which implies ([2] have been the first ones to use this elegant fast method) the necessary tightness in $T$ of the time averaged measures

$$\mu_T^{n,x} := \frac{1}{T} \int_0^T \nu_t^{n,x} dt.$$ 

If we choose the initial condition $x = 0$, then $\mu_T^{n,x} \in P$ (in particular it is space homogeneous and partially isotropic), so there exist invariant measures in $P$ for
the Galerkin equation. Denote by $S^n$ the set of all such invariant measures (thus $S^n \subset \mathcal{P}$).

The constant $C$ in the estimate above is also independent of $n$; it follows that the invariant measures of the class $S^n$ just constructed fulfill the bound

$$\mu^n \left[ \| \cdot \|_2^2 \right] \leq C.$$ 

In fact it is possible to show that every element of $S^n$ has this property, if we do not want to use this property, it is sufficient to restrict the definition of $S^n$ in the sequel). These facts imply that $\cup_n S^n$ is relatively compact in the weak topology of probability measures on $H$. We denote by $\mathcal{P}_{\text{NS}}^G(\nu)$ (the superscript $G$ will remind us that we use the particular procedure of Galerkin approximations) the set of limit points of $\cup_n S^n$, precisely defined as follows: a probability measure $\mu$ on $H$ belongs to $\mathcal{P}_{\text{NS}}^G(\nu)$ if there is a sequence $k_n \to \infty$ and elements $\mu_{k_n} \in S^{k_n}$ such that $\mu_{k_n}$ converges to $\mu$ in the weak topology of probability measures on $H$. The elements of the set $\mathcal{P}_{\text{NS}}^G(\nu)$ are space homogeneous and partially isotropic (these relations are stable under weak convergence). Furthermore, they have the other regularity properties required to belong to $\mathcal{P}$: finite second moment in $V$ comes from the previous estimates, $\mu(D(A)) = 1$ from a regularity result of $\mathcal{P}$, see also $\mathcal{P}$, summarized in the following lemma. Therefore $\mathcal{P}_{\text{NS}}^G(\nu) \subset \mathcal{P}$.

**Lemma 2.13** Given $\nu > 0$, there is a constant $C_\nu > 0$ (depending on $\nu$) such that

$$\mu^n \left( |A|^{2/3}_H \right) \leq C$$

for every $n$ and every invariant measure $\mu_n \in S^n$.

Given $u \in V$, let $S_u$ be the tensor with $L^2(T)$ components

$$S_u = \frac{1}{2} (Du + Du^T)$$

(called stress tensor). The scalar field

$$\langle S_u(x) \text{curl} u(x), \text{curl} u(x) \rangle$$

describes the stretching of the vorticity field. If we set $\xi = \text{curl} u$, then formally we have

$$\frac{\partial \xi}{\partial t} + (u \cdot \nabla) \xi = \nu \Delta \xi + S_u \xi + i \sum_{k \in \Lambda^{(\infty)}} k \times \sigma_k \beta_k e^{-ik \cdot x}.$$ 

A formal application of Itô formula yields the inequality

$$\nu \cdot \mu \int_T \| D\text{curl} u(x) \|_2^2 \, dx \leq \mu \int_T \langle S_u(x) \text{curl} u(x), \text{curl} u(x) \rangle \, dx$$

$$+ \frac{1}{2} \sum_{k \in \Lambda^{(\infty)}} |k|^2 |\sigma_k|^2.$$
for \( \mu \in \mathcal{P}_{NS}^G(\nu) \) (in fact formally the identity). Along with the general results of

the previous sections we would get

\[
\mu \left[ \int_T \langle S_u(x) \text{curl}(x), \text{curl}(x) \rangle dx \right] \geq C \varepsilon^{3/2}(\mu, \nu) \cdot \nu^{-3/2}. \quad (2.12)
\]

This would be the final result of this section, having an interesting physical interpretation. However we are not able to prove it in this form. We analyze the status of this inequality by presenting some related rigorous results. They are of two different natures: Corollary 2.15 reformulates it for the coarse graining scheme given by Galerkin approximations; Corollary 2.19 expresses the most natural statement directly for \( \mu \in \mathcal{P}_{NS}^G(\nu) \) but it requires an additional unproved regularity assumption.

**Lemma 2.14** Given \( \mu \in \mathcal{P}_{NS}^G(\nu) \), and \( \mu_{nk} \in S^{kn} \) such that \( \mu_{nk} \) converges to \( \mu \) in the weak topology of probability measures on \( H \), then

\[
\mu \left[ |A|^2_H \right] \leq \lim_{k \to \infty} \mu_{nk} \left[ |A|^2_H \right].
\]

The same is true for \( \mu \int_T \| D \text{curl}(x) \|^2 dx \) in place of \( \mu \left[ |A|^2_H \right] \).

**Proof.** Let \( \{ \varphi_m \}_{m \in \mathbb{N}} \in C_b(H) \) be a sequence that converges monotonically increasing to \( |A|^2_H \) for every \( x \in D(A) \), it is easy to construct it by cut-off and finite dimensional approximations). Since \( \mu(D(A)) = 1 \), by Beppo-Levi theorem \( \mu([\varphi_m]) \to \mu([A|^2_H]) \). Given \( \varepsilon > 0 \), let \( m_0 \) be such that \( \mu([\varphi_m]) \geq \mu([A|^2_H]) - \varepsilon \).

Since \( \mu_{nk}[\varphi_{m_0}] \to \mu[\varphi_{m_0}] \) as \( k \to \infty \), eventually in \( k \) we thus have \( \mu_{nk}[\varphi_{m_0}] \geq \mu([A|^2_H]) - 2\varepsilon \), and therefore also \( \mu_{nk}[|A|^2_H] \geq \mu([A|^2_H]) - 2\varepsilon \). This proves the first part of the lemma; the second one is similar. \( \blacksquare \)

**Corollary 2.15** Let \( \mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+ \), with \( \mathcal{M}_\nu \subset \mathcal{P}_{NS}^G(\nu) \), be a family with the K41 scaling law, in the sense of definition 7.3 Then there exist \( \nu_0 > 0 \) and \( C > 0 \) such that

\[
\liminf_{k \to \infty} \mu_{nk} \left[ \int_T \langle S_u(x) \text{curl}(x), \text{curl}(x) \rangle dx \right] \geq C \varepsilon^{3/2}(\mu, \nu) \cdot \nu^{-3/2}
\]

for every \( \nu \in (0, \nu_0) \), every \( \mu \in \mathcal{M}_\nu \) and every sequence \( \mu_{nk} \in S^{kn} \) such that \( \mu_{nk} \) converges to \( \mu \) in the weak topology of probability measures on \( H \).

**Proof.** From the previous section we know that

\[
\mu \int_T \| D^2 u(x) \|^2 dx \geq \varepsilon^{3/2}(\mu, \nu) \cdot \nu^{-5/2}.
\]

Since

\[
\langle Af, g \rangle_H = \langle \text{curl}f, \text{curl}g \rangle_H \quad (2.13)
\]
for every $f, g \in D(A)$, we have
\[
\mu \int_T \| D\text{curl}u(x) \|^2 dx \geq C \epsilon^{3/2}(\mu, \nu) \cdot \nu^{-5/2}
\]
for a suitable universal constant $C > 0$. From the previous lemma we have
\[
\liminf_{k \to \infty} \mu_{n_k} \int_T \| D\text{curl}u(x) \|^2 dx \geq C \epsilon^{3/2}(\mu, \nu) \cdot \nu^{-5/2}.
\]
Thus the claim of the corollary will follow from the inequality
\[
\nu \cdot \mu_{n_k} \int_T \| D\text{curl}u(x) \|^2 dx \leq \mu_{n_k} \int_T \langle S_u(x)\text{curl}u(x), \text{curl}u(x) \rangle dx
\]
\[
+ \frac{1}{2} \sum_{k \in \Lambda(n)} |k|^2 |\sigma_k|^2.
\]
Let us sketch the proof of this inequality (see [4] for more details). Consider the Galerkin approximations
\[
du^{(n)} + \left[ \nu A\left( u^{(n)} \right) + \pi^{(n)} B\left( u^{(n)}, u^{(n)} \right) \right] dt = \sum_{k \in \Lambda(n)} \sigma_k d\beta_k e^{-ikx}
\]
described in Appendix B. From Itô formula for $\langle Au^{(n)}(t), u^{(n)}(t) \rangle_H$ we get
\[
\langle Au^{(n)}(t), u^{(n)}(t) \rangle_H + \int_0^t 2 \langle A\left( u^{(n)} \right), \nu A\left( u^{(n)} \right) + \pi^{(n)} B\left( u^{(n)}, u^{(n)} \right) \rangle_H ds
\]
\[
= \left( \langle Au^{(n)}(0), u^{(n)}(0) \rangle_H + M^n_t \right) + \frac{1}{2} \sum_{k \in \Lambda(n)} |k|^2 |\sigma_k|^2
\]
where $M^n_t$ is a square integrable martingale. We have
\[
\langle Au^{(n)}, \pi^{(n)} B\left( u^{(n)}, u^{(n)} \right) \rangle_H = \langle Au^{(n)}, B\left( u^{(n)}, u^{(n)} \right) \rangle_H
\]
since $\pi^{(n)}$ is selfadjoint and commutes with $A$. Besides (2.13) we also have
\[
\langle Af, B(g, g) \rangle_H = \langle \text{curl} f, (g \cdot \nabla)\text{curl} g + S_g \text{curl} g \rangle_H
\]
hence
\[
\langle Af, B(f, f) \rangle_H = \langle \text{curl} f, S_f \text{curl} f \rangle_H
\]
for every $f, g \in D(A)$. Therefore we have
\[
\left| \text{curl}u^{(n)}(t) \right|_H^2 + \int_0^t \left( 2 \nu \| D\text{curl}u^{(n)} \|^2_H + \langle \text{curl}u^{(n)}, S_{u^{(n)}} \text{curl}u^{(n)} \rangle_H \right) ds
\]
\[
\leq \left| \text{curl}u^{(n)}(0) \right|_H^2 + M^n_t + \frac{1}{2} \sum_{k \in \Lambda(n)} |k|^2 |\sigma_k|^2.
\]
This implies (2.14) and the proof is complete. ■
Remark 2.16 We cannot conclude (2.12) from the previous corollary without further (unproved) assumptions on \( \mu \) or \( \{ \mu_{n_k} \} \). This could be just a technical point due to the present lack of better regularity estimates for the 3D Navier-Stokes equations, or it could be a facet of a deeper phenomenon. Let us explain it with a cartoon argument. First recall that it is easy to construct, say on the torus \( T \), a sequence \( \{ f_n \} \) of functions converging a.s. to zero, but with \( \int_T f_n \, dx = 1 \) (or even \( \int_T f_n \, dx \to \infty \)); just take the mollifiers of a Dirac delta distribution; if we like, the example can be modified so that \( f_n \) tend to develop singularities on a dense zero measure set in \( T \), but the a.s. limit is still zero. Thus we see that for the limit measure \( \mu \) we could have a small value of \( \mu \left[ \int_T (S u(x) \text{curl} u(x), \text{curl} u(x)) \, dx \right] \) even if some coarse graining procedure, here represented by the Galerkin approximations, could give us a large value of \( \mu_{n_k} \left[ \int_T (S u(x) \text{curl} u(x), \text{curl} u(x)) \, dx \right] \). Such arguments arise the question of the physical meaning of the true Navier-Stokes equations and possibly of its coarse graining approximations; this is not our aim, but we wanted to say that the previous corollary may be considered perhaps as a result of possible physical interest in itself, even if we cannot rewrite it in the form (2.12).

Lemma 2.17 Given \( \mu \in \mathcal{P}^G_{\text{NS}}(\nu) \), and every sequence \( \mu_{n_k} \in S^{k_n} \) such that \( \mu_{n_k} \) converges to \( \mu \) in the weak topology of probability measures on \( H \), we also have \( \mu_{n_k} \to \mu \) weakly on \( [W^{1,3}(T)]^3 \).

Proof. From the lemma above, \( \{ \mu_{n_k} \} \) is bounded in probability on \( D(A) \):

\[
\mu_{n_k}(|Ax|_H > R) = \mu_{n_k}\left(|Ax|^{2/3}_H > R^{2/3}\right) \leq R^{-2/3} \mu_{n_k}\left(|Ax|^{2/3}_H\right) \leq \frac{C}{R^{2/3}}.
\]

The embedding of \( D(A) \) into \( [W^{1,3}(T)]^3 \) is compact: recall that Sobolev embedding theorem gives us \( W^{2,2} \subset W^{\beta, \frac{6}{\beta-1}} \) for every \( \beta \in (1, 2) \), and the embedding of \( W^{\beta, \frac{6}{\beta-1}} \) in \( W^{1,3} \) is compact; choose then \( \beta = 3/2 \). Therefore \( \{ \mu_{n_k} \} \) is tight in \( [W^{1,3}(T)]^3 \). Easily we deduce that it converges weakly to \( \mu \) also in \( [W^{1,3}(T)]^3 \).

Corollary 2.18 If \( \mu \in \mathcal{P}^G_{\text{NS}}(\nu) \) is the weak limit (in \( H \) and thus in \( [W^{1,3}(T)]^3 \) of a sequence \( \mu_{n_k} \in S^{k_n} \) such that

\[
\mu_{n_k}\left[\|\cdot\|^{2+\varepsilon}_V\right] \leq C
\]

for some \( \varepsilon, C > 0 \), then

\[
\nu : \mu \int_T \|Du(x)\|^2 \, dx = \frac{1}{2} \sum_{k\in\Lambda(\infty)} |\sigma_k|^2.
\]
Necessary conditions for K41

If in addition

$$\mu_{n_k} \left[ \| \cdot \|^3 \right] \leq C$$

then

$$\nu \cdot \mu \int_T \| D \text{curl} u(x) \|^2 dx \leq \mu \int_T (S_u(x) \text{curl} u(x), \text{curl} u(x)) \ dx$$

$$+ \frac{1}{2} \sum_{k \in \Lambda^{(\infty)}} |k|^2 |\sigma_k|^2.$$  

**Proof.** It is sufficient to apply repeatedly the following fact: if $\mu_n \to \mu$ weakly in a Polish space $X$, $\varphi \in C(X)$ and $\mu_n \left[ | \varphi |^{1+\varepsilon} \right] \leq C$, then $\mu_n[\varphi] \to \mu[\varphi]$. This fact is well known but we provide the proof for completeness. Let $Y_n$ and $Y$ be r.v.’s with law $\mu_n$ and $\mu$ resp., with values in $X$, such that $Y_n \to Y$ a.s. in $X$. Then $\mu_n[\varphi] = E[\varphi(Y_n)]$, $\mu[\varphi] = E[\varphi(Y)]$, so by Vitali convergence theorem it is sufficient to prove that $\varphi(Y_n)$ is uniformly integrable. We have

$$E[\varphi(Y_n) 1_{\varphi(Y_n) \geq \lambda}] \leq (E[\varphi(Y_n)^p])^{1/p} P(\varphi(Y_n) \geq \lambda)^{1/q} \leq C \lambda^{-\delta}.$$  

Thus the uniform integrability is proved and the proof is complete.  

**Corollary 2.19** Let $\mathcal{M} \subset P \times \mathbb{R}_+$, with $\mathcal{M}_\nu \subset \mathcal{P}_{\text{NS}}(\nu)$, be a family with the K41 scaling law, in the sense of definition 1.3. Assume that every $\mu$ in $\mathcal{M}$ is the weak limit of a sequence $\mu_{n_k} \in \mathcal{S}^{k_n}$ such that

$$\mu_{n_k} \left[ \| \cdot \|^3 \right] \leq C$$

for some $\varepsilon, C > 0$. Then there exists $\nu_0 > 0$ and $C > 0$ such that (2.12) holds for every $\nu \in (0, \nu_0)$ and every $\mu \in \mathcal{M}_\nu$.

**Remark 2.20** If K41 scaling law holds then vortex stretching must be intense. Heuristically, no geometrical depletion of such stretching may occur (in contrast to the 2D case where the stretching term is zero because $\text{curl} u(x)$ is aligned with the eigenvector of eigenvalue zero of $S_u(x)$): indeed, if we extrapolate the behaviour $E \left[ |Du|^2 \right] \sim \frac{1}{\nu}$ as $Du \sim \frac{1}{\sqrt{\nu}}$, $\text{curl} u \sim \frac{1}{\sqrt{\nu}},$ then we get $E[S_u \text{curl} u \cdot \text{curl} u] \sim \frac{1}{\nu \sqrt{\nu}}$ if there is no help from the geometry. Another way to explain this idea is the following sort of generalised Hölder inequality.

**Corollary 2.21** Let $\mathcal{M} \subset P \times \mathbb{R}_+$, with $\mathcal{M}_\nu \subset \mathcal{P}_{\text{NS}}(\nu)$, be a family with the K41 scaling law, fulfilling the assumptions of corollary 2.19. Then there exists $\nu_0 > 0$ and $C > 0$ such that

$$\left( \mu \int_T \| Du \|^2 dx \right)^{1/2} \leq C \left( \mu \int_T \| S_u \text{curl} u \cdot \text{curl} u \|^2 dx \right)^{1/3}$$

for every $\nu \in (0, \nu_0)$ and every $\mu \in \mathcal{M}_\nu$.  

**Proof.** From the previous corollary and the definition of \( \epsilon(\mu, \nu) \) we have

\[
\left( \mu \left[ \int_{\mathcal{T}} \| \mathbf{S} u \cdot \text{curl} u \|^{2} dx \right] \right)^{1/3} \geq \left( C \epsilon^{3/2}(\mu, \nu) \cdot \nu^{-3/2} \right)^{1/3} \\
= C' \epsilon^{1/2}(\mu, \nu) \cdot \nu^{-1/2} \\
= C'' \left( \mu \int_{\mathcal{T}} \| D u \|^{2} dx \right)^{1/2}.
\]

The proof is complete. ■

3 Necessary and sufficient conditions for K41

We continue with the notations and concepts just introduced in the last section on the 3D case.

The result of this section can be formulated for definition 1.3, but the presence of the factor \( \epsilon(\mu, \nu)^{-1/4} \) in the definition of \( \eta(\mu, \nu) \) makes some statements much less direct. So, having in mind the exploratory character of these equivalent conditions, we prefer to adopt a simplified form of our definition of the K41 scaling law.

**Definition 3.1** We say that a scaling law of K41 type holds true for a set \( \mathcal{M} \subset \mathcal{P} \times \mathbb{R}^{+} \) if there exist \( \nu_{0} > 0, C > c > 0, C_{0} > 0, \) and a monotone function \( R_{0}: (0, \nu_{0}] \rightarrow \mathbb{R}^{+} \) with \( R_{0}(\nu) > C_{0} \) and \( \lim_{\nu \to 0} R_{0}(\nu) = +\infty \), such that the bound

\[
c \cdot r^{2/3} \leq S^{\mu}_{2}(r) \leq C \cdot r^{2/3}
\]

holds for every pair \( (\mu, \nu) \in \mathcal{M} \) and every \( r \) such that \( \nu \in (0, \nu_{0}] \) and

\[
C_{0} \nu^{3/4} < r < \nu^{3/4} R_{0}(\nu).
\]

Recalling that \( \eta(\mu, \nu) = \nu^{3/4} \epsilon(\mu, \nu)^{-1/4} \), we see that this definition is equivalent to if there exist \( \epsilon_{1} > \epsilon_{0} > 0 \) such that

\[
\epsilon_{0} \leq \epsilon(\mu, \nu) \leq \epsilon_{1}
\]

for all \( (\mu, \nu) \in \mathcal{M} \). Unfortunately, in 3D only the upper bound can be proven. However, this could be just a technical problem due to the fact that we can only use weak solutions (for slightly more regular solutions Corollary 2.18 implies that \( \epsilon(\mu, \nu) \) would be bounded from above and below).

Consider the auxiliary stochastic Navier-Stokes equations

\[
\frac{\partial \tilde{u}}{\partial t}(t, x) + (\tilde{u}(t, x) \cdot \nabla) \tilde{u}(t, x) + \nabla \bar{p}(t, x) = \tilde{v} \Delta \tilde{u}(t, x) + \sum_{k \in \Lambda_{L}^{(\infty)}} \sigma_{k} \hat{\beta}_{k}(t) e^{-ik \cdot x}
\]

(3.2)
on the torus $[0, L]^3$ with $\text{div} \, \tilde{u} = 0$ and periodic boundary conditions (the set $\Lambda^{(\infty)}$ is defined in (B.1)). As we shall see below (see the next section and lemma B.1), we obtain this equation when we perform the following scaling transformation on the solutions $u$ of the original equation (1.1):

$$\tilde{u}(t, x) = L^{1/3} u(L^{-2/3} t, L^{-1} x)$$

(and a suitably defined $\tilde{p}(t, x)$). The value of $\tilde{\nu}$ under this transformation is

$$\tilde{\nu} = \nu L^{4/3}.$$

This scaling transformation has been introduced in the mathematical-physics literature, see [1]. What makes it special is that no coefficient depending on the scale parameter appears in front of the noise, so the energy input per unit of time and space is the same for every $L$. Heuristically, if we believe in a cascade picture of the energy (without essential inverse cascade), this invariance of the energy input should imply that the small scale properties of (1.1) and (3.2) are the same, namely that they are invariant under this transformation; this should lead to the K41 scaling law.

Similarly to the case $L = 1$, we may introduce the (non empty) set $\mathcal{P}_{NS}^G(\tilde{\nu}, L)$ of limit points of the (homogeneous and isotropic) invariant measures of the corresponding Galerkin approximations.

Let us denote by $\mathcal{P}_{NS}^G$ the set of all pairs $(\mu, \tilde{\nu})$ such that $\mu \in \mathcal{P}_{NS}^G(\nu)$. Similarly, let us denote by $\tilde{\mathcal{P}}_{NS}^G$ the set of all triples $(\mu, \tilde{\nu}, L)$ such that $\mu \in \mathcal{P}_{NS}^G(\tilde{\nu}, L)$.

### 3.1 Basic equivalent condition

The following condition seems interesting since it looks rather qualitative, in contrast to Definition 3.1, and shows that the exponent $2/3$ arises from the scaling properties of the stochastic Navier-Stokes equations.

Let us introduce the notation $\mathcal{P}_L$ for the set of probability measures analogous to $\mathcal{P}$, but on the torus $[0, L]^3$. Denote by $\mathcal{P} \times \mathbb{R}_+^2$ the set of all triples $(\nu, \tilde{\nu}, L)$ such that $(\tilde{\nu}, L) \in \mathbb{R}_+^2$ and $\mu \in \mathcal{P}_L$. In the next definition and later on we use the notation $\mu \left[ \|u(e) - u(0)\|^2 \right]$ when $\mu \in \mathcal{P}_L$ (and other similar mean values): this means

$$\mu \left[ \|u(e) - u(0)\|^2 \right] = \int_{H_L} \|u(e) - u(0)\|^2 d\mu(u),$$

where $H_L$ has been introduced in section 1.1.

**Definition 3.2** We call admissible region a set $D \subset \mathbb{R}_+^2$ of the following form:

$$D = \{ (\tilde{\nu}, L) \in \mathbb{R}_+^2; \tilde{\nu} \in (0, \nu_0), L > \tilde{R}_0(\tilde{\nu}) \}$$

where $\tilde{\nu}_0 > 0$ and $\tilde{R}_0: (0, \tilde{\nu}_0) \to [1, \infty)$ is a strictly decreasing function with $\tilde{R}_0(\tilde{\nu}) \to \infty$ as $\tilde{\nu} \to 0$. 

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**NECESSARY AND SUFFICIENT CONDITIONS FOR K41**

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An admissible region is depicted in the left-hand side of Figure 1 below.

**Condition A** A subset \( \tilde{\mathcal{M}} \subset \mathcal{P} \times \mathbb{R}^2_+ \) is said to satisfy Condition A if there exist an admissible region \( D \subset \mathbb{R}^2_+ \) and two constants \( C > c > 0 \) such that

\[
c \leq \mu \left[ \| u(e) - u(0) \|_2 \right] \leq C \tag{3.3}
\]

for every \((\mu, \tilde{\nu}, L) \in \tilde{\mathcal{M}}\) with \((\tilde{\nu}, L) \in D\).

**Proposition 3.3** The set \( \tilde{\mathcal{P}}_{\text{NS}}^G \) satisfies Condition A if and only if the set \( \mathcal{P}_{\text{NS}}^G \) has a scaling law of K41 type, in the sense of Definition 3.1.

**Proof.** Given \( R > 0 \), consider the mapping \( S_R : \mathcal{H}_R \to \mathcal{H} \) defined by

\[
(S_R u)(x) = R^{1/3} u(Rx) . \tag{3.4}
\]

This mapping induces a mapping \( S \) from \( \mathcal{P} \times \mathbb{R}^2_+ \) to \( \mathcal{P} \times \mathbb{R}_+ \) by

\[
S(\mu, \tilde{\nu}, \tilde{r}) = (S^\ast \tilde{\nu} \mu, \tilde{\nu} \tilde{r}^{-4/3}) . \tag{3.5}
\]

It follows immediately from Theorem B.2 that one has

\[
\mathcal{P}_{\text{NS}}^G = S(\tilde{\mathcal{P}}_{\text{NS}}^G) \text{, and } \tilde{\mathcal{P}}_{\text{NS}}^G = S^{-1}(\mathcal{P}_{\text{NS}}^G) . \tag{3.6}
\]

Furthermore, it follows immediately from the above definitions that if \( (\mu, \nu) = S(\tilde{\mu}, \tilde{\nu}, \tilde{r}) \), then

\[
S_2^\nu(r) = r^{2/3} \int_{H_{\tilde{r}}} \| u(e) - u(0) \|^2 d\tilde{\mu}(u) . \tag{3.7}
\]

It therefore follows that, in order to prove the equivalence between Condition A and K41, it suffices to show that the domains of validity of eq. 3.3 and of eq. 3.1 are the same (with possibly different constants and functions \( R_0 \) and \( \tilde{R}_0 \)), provided that \((\nu, r) \) and \((\tilde{\nu}, \tilde{r}) \) are related by

\[
\tilde{\nu} = \nu r^{-4/3} , \quad \tilde{r} = r^{-1} . \tag{3.8}
\]

We denote by \( K : (\nu, r) \mapsto (\tilde{\nu}, \tilde{r}) \) the above map.

**Condition A implies K41.** The domain of validity of eq. 3.3 is given by

\[
\tilde{\nu} \leq \tilde{\nu}_0 , \quad \tilde{r} \geq \tilde{R}_0(\tilde{\nu}) . \tag{3.9}
\]

Under the map \( K^{-1} \), this becomes

\[
r \geq \left( \frac{\nu}{\nu_0} \right)^{3/4} \equiv C_0 \nu^{3/4} , \quad \frac{1}{r} \geq \tilde{R}_0(\nu r^{-4/3}) . \tag{3.10}
\]

Both domains are shown in Fig. 1.
Necessary and Sufficient Conditions for K41

\[ \nu - \frac{3}{4} \geq F(\nu r^{-4/3}) . \] (3.11)

This condition (as can be inferred from the Fig. 1), can only be satisfied simultaneously with the first condition in eq. 3.10 if \( \nu \leq \nu_0 \equiv F(\tilde{\nu}_0)^{-4/3} \). On \((0, \nu_0]\) this domain, eq. 3.11 is equivalent to

\[ r \leq \left( \frac{\nu}{F^{-1}(\nu^{-3/4})} \right)^{3/4} \equiv \nu^{3/4} R_0(\nu) , \] (3.12)

where \( R_0(x) = (F^{-1}(x^{-3/4}))^{-3/4} \). Additionally \( R_0 \) is well-defined on \((0, \nu_0]\) and that it is greater than \( C_0 \) on this domain. Furthermore, since \( F \) is decreasing, \( R_0 \) is strictly decreasing and it is easy to check that \( \lim_{x \to 0} R_0(x) = \infty \) because the same property holds for \( F \).

**K41 implies Condition A** The domain of validity of K41 is given by

\[ \nu \leq \nu_0 , \quad r \nu^{-3/4} \in [C_0, R_0(\nu)] . \] (3.13)

Under the map \( K \), this becomes

\[ \tilde{\nu} r^{-4/3} \leq \nu_0 , \quad \tilde{\nu}^{-3/4} \in [C_0, R_0(\tilde{\nu} r^{-4/3})] . \] (3.14)

The second condition can be rewritten as

\[ \tilde{\nu} \in [G(\tilde{\nu} r^{-4/3}), \tilde{\nu}_0] , \] (3.15)
Necessary and sufficient conditions for K41

Parameter domain for K41

Image of the previous domain

Figure 2: Effect of $K$ on a domain of the type (3.13)

where we defined $\tilde{\nu}_0 = C_0^{-4/3}$ and $G(x) = R_0(x)^{-4/3}$. Both of these domains are shown in Figure 2.

We can rewrite as above the condition $\tilde{\nu} \geq G(\tilde{\nu}\tilde{r}^{-4/3})$ as

$$\tilde{r} \geq \left( \frac{\tilde{\nu}}{G^{-1}(\tilde{\nu})} \right)^{3/4} \equiv \tilde{R}_0(\tilde{\nu}) . \quad (3.16)$$

Again, it is an easy exercise to show that $\tilde{R}_0$ as defined above is monotone and satisfies $\lim_{x \to 0} \tilde{R}_0(x) = \infty$. The only points that remain to be clarified are:

a. We haven’t taken the first equation in eq. 3.14 into account.

b. The domain of definition of $R_0$ may not extend to $\tilde{\nu}_0$.

Both problems can be solved at once by simply choosing a smaller value for $\tilde{\nu}_0$.

3.2 Necessary and sufficient conditions in terms of high and low modes

Although Condition A contains only bounds (at finite distance points) and not scaling exponents (with small distant points), and thus in principle it represents a progress in the direction of analysis of K41, it still looks difficult to verify or disprove it for Navier-Stokes equations, since it is rather unusual to work with the difference of a solution at two points. This is the main motivation for the following new necessary and sufficient conditions.

Looking at them on the other direction, as necessary conditions for K41, they declare that under K41 the energy of high modes is bounded and the enstrophy of low modes is bounded, an information with a certain physical content.
In this section, for notational simplicity, we drop the tildes in our notation. Recall that an admissible region is defined by

\[ D = \{ (\nu, L) \in \mathbb{R}^2_+; \nu \in (0, \nu_0), L > R_0(\nu) \} , \]

and that Condition \( A \) requires

\[ c \leq \mu \left[ \| u(e) - u(0) \|^2 \right] \leq C \]

for every \( (\mu, \nu, L) \) with \( (\nu, L) \in D \).

We start with a preparatory lemma which depends on the scaling properties of stochastic Navier-Stokes equation in an essential way. This is the only point in this section where specific informations about the measures are used.

**Lemma 3.4** If \( \hat{P}^G_{NS} \) satisfies Condition \( A \) then there exist constants \( C' > c' > 0 \) and an admissible region \( D' \) such that

\[ c' \leq \sum e \int \frac{3}{4} \mu \left[ \| u(e) - u(0) \|^2 \right] d\lambda \leq C' \]

for every \( (\mu, \nu, L) \in \hat{P}^G_{NS} \) with \( (\nu, L) \in D' \). The sum \( \sum_e \) is extended to all coordinate unitary vectors. We simply have \( C' = (1.5^{2/3} d) \cdot C, c' = (0.5^{2/3} d) \cdot c, D' \) defined by \( 0.5^{4/3} \cdot \nu_0 \) and \( 1.5 R_0(1.5^{-4/3} \nu) \), where \( \nu_0 \) and \( R_0(\nu) \) define \( D \).

**Proof.** Given \( \lambda \in \left[ \frac{1}{2}, \frac{3}{2} \right] \) and \( (\mu, \nu, L) \in \hat{P}^G_{NS} \), namely \( \mu \in P^G_{NS}(\nu, L) \), consider the measure \( \mu_\lambda \) that corresponds to \( \mu \) under the transformation \( u \mapsto \lambda^{-1/3} u(\lambda) \) used in the previous section, having the property

\[ \mu \left[ \| u(\lambda e) - u(0) \|^2 \right] = \lambda^{2/3} \mu_\lambda \left[ \| u(e) - u(0) \|^2 \right] . \]

By Theorem B.2 we know that \( \mu_\lambda \in P^G_{NS}(\nu \lambda^{-4/3}, L/\lambda) \), hence \( (\mu_\lambda, \nu \lambda^{-4/3}, L/\lambda) \in \hat{P}^G_{NS} \). Thus Condition \( A \) implies

\[ c \leq \mu_\lambda \left[ \| u(e) - u(0) \|^2 \right] \leq C \]

if \( \nu \lambda^{-4/3} < \nu_0 \) and \( L/\lambda > R_0(\nu \lambda^{-4/3}) \). The first condition is true if \( \nu < 0.5^{4/3} \nu_0 \). The second one if \( L > 1.5 R_0(1.5^{-4/3} \nu) \). The proof can now be easily completed.

Let us use some Fourier analysis on the torus \( T_L = [0, L]^d \) (see also Appendix B). Every \( u \in H_L \) is given by

\[ u(x) = \sum_{k \in \Lambda^{(\infty)} L} e^{-i k \cdot x} \hat{u}(k) \]
where
\[ \hat{u}(k) := L^{-3} \int_{T_L} e^{ik \cdot x} u(x) \, dx \]
and we have Parseval identity
\[ L^{-3} \int_{T_L} \| u(x) \|^2 \, dx = \sum_{k \in \Lambda_L^{(\infty)}} \| \hat{u}(k) \|^2. \]

We introduce another condition which requires the sum of the enstrophy of low modes and energy of high modes to be finite and bounded away from zero.

**Condition B** A subset \( \tilde{\mathcal{M}} \subset \mathcal{P} \times \mathbb{R}_+^2 \) is said to satisfy **Condition B** if there exist an admissible region \( D \subset \mathbb{R}_+^2 \) and two constants \( C > c > 0 \) such that
\[
c \leq \sum_{\substack{k \in \Lambda_L^{(\infty)} \\|k\| \leq 1}} \|k\|^2 \mu \left[ \|\hat{u}(k)\|^2 \right] + \sum_{\substack{k \in \Lambda_L^{(\infty)} \\|k\| > 1}} \mu \left[ \|\hat{u}(k)\|^2 \right] \leq C
\]
for every \( (\mu, \nu, L) \in \tilde{\mathcal{P}}_{\text{NS}}^G \) such that \( (\nu, L) \in D \).

With this definition, we may establish a first basic theorem as a corollary of the previous lemma.

**Theorem 3.5** **Condition A** implies **Condition B**

**Remark 3.6** We understand that constants and admissible regions involved in **Conditions A** and **B** are not necessarily the same.

**Proof.** For every \( u \in H_L \) we have
\[
\|u(\lambda e) - u(0)\|^2 = L^{-3} \int_{T_L} \|u(x + \lambda e) - u(x)\|^2 \, dx
\]
\[
= \sum_{k \in \Lambda_L^{(\infty)}} \left| e^{ik \cdot \lambda e} - 1 \right|^2 \|\hat{u}(k)\|^2
\]
and thus, for every \( \mu \in \mathcal{P}_{\text{NS}}^G(\nu, L) \) we have
\[
\sum_{e} \int_{\frac{1}{2}}^{\frac{3}{2}} \mu \left[ \|u(\lambda e) - u(0)\|^2 \right] d\lambda
\]
\[
= \sum_{k \in \Lambda_L^{(\infty)}} \left( \sum_{e} \int_{\frac{1}{2}}^{\frac{3}{2}} \left| e^{ik \cdot \lambda e} - 1 \right|^2 d\lambda \right) \mu \left[ \|\hat{u}(k)\|^2 \right].
\]
But there exist universal constants $C' > c' > 0$ such that
\[
c'(\|k\|^2 \wedge 1) \leq \sum_{e} \int_{1/2}^{3} |e^{ik \cdot \lambda e} - 1|^2 d\lambda \leq C'(\|k\|^2 \wedge 1).
\]
Therefore, the quantities
\[
\sum_{e} \int_{1/2}^{3} \mu \left( \|u(\lambda e) - u(0)\|^2 \right) d\lambda
\]
and
\[
\sum_{k \in \Lambda^L_{\infty}} \left( \|k\|^2 \wedge 1 \right) \mu \left[ \|\hat{u}(k)\|^2 \right]
\]
are “equivalent”, up to universal constants. This proves the claim. 

We have at least a partial converse of the previous result if we require that in the admissible region the enstrophy of high modes is by itself bounded away from zero. Then we introduce the following condition:

**Condition C** A subset $\tilde{\mathcal{M}} \subset \mathcal{P} \times \mathbb{R}^2_+$ is said to satisfy **Condition C** if there exist an admissible region $D \subset \mathbb{R}^2_+$ and two constants $C > c > 0$ such that
\[
c \leq \sum_{k \in \Lambda^L_{\infty}, \|k\| \leq 1/2} \|k\|^2 \mu \left[ \|\hat{u}(k)\|^2 \right] \leq \sum_{k \in \Lambda^L_{\infty}, \|k\| \leq 1} \|k\|^2 \mu \left[ \|\hat{u}(k)\|^2 \right] + \sum_{k \in \Lambda^L_{\infty}, \|k\| > 1} \mu \left[ \|\hat{u}(k)\|^2 \right] \leq C
\]
for every $(\mu, \nu, L) \in \tilde{\mathcal{P}}^G_{NS}$ such that $(\nu, L) \in D$.

Note that **Condition C** implies directly **Condition B**. What is more interesting is the following:

**Proposition 3.7** **Condition C** implies **Condition A**

**Proof.** We have
\[
\sum_{e} |e^{ik \cdot e} - 1|^2 \leq C \left( \|k\|^2 \wedge 1 \right).
\]
for every $k$. Moreover if $\|k\| \leq 1/2$ we have
\[
c \|k\|^2 \leq \sum_{e} |e^{ik \cdot e} - 1|^2
\]
for some constant $c > 0$. The claim then follows from the next lemma and the following inequality
\[
\sum_{k \in \Lambda^L_{\infty}} \left( \sum_{e} |e^{ik \cdot e} - 1|^2 \right) \mu \left[ \|\hat{u}(k)\|^2 \right] \geq \sum_{k \in \Lambda^L_{\infty}, \|k\| \leq 1/2} \left( \sum_{e} |e^{ik \cdot e} - 1|^2 \right) \mu \left[ \|\hat{u}(k)\|^2 \right] \geq c \sum_{k \in \Lambda^L_{\infty}, \|k\| \leq 1/2} \|k\|^2 \mu \left[ \|\hat{u}(k)\|^2 \right].
\]
Lemma 3.8 \( \tilde{P}^G_{NS} \) satisfies Condition A if and only if it satisfies the following Condition A': there exist \( C > c > 0 \), and an admissible region \( D \) such that

\[
c \leq \sum_{k \in \Lambda_L^{(\infty)}} \left( \sum_{e} \left| e^{i k \cdot e} - 1 \right|^2 \right) \mu \left[ \| \hat{\mu}(k) \|^2 \right] \leq C
\]

for every \((\mu, \nu, L) \in \tilde{P}^G_{NS}\) such that \((\nu, L) \in D\).

**Proof.** From previous computations, we know that for every \( \mu \in P^G_{NS} (\nu, L) \) we have

\[
\sum_{e} \mu \left[ \| u(e) - u(0) \|^2 \right] = \sum_{k \in \Lambda_L^{(\infty)}} \left( \sum_{e} \left| e^{i k \cdot e} - 1 \right|^2 \right) \mu \left[ \| \hat{\mu}(k) \|^2 \right].
\]

This proves the claim. ■

4 A random eddy model

We now exhibit a model having the property stated in the conjecture, and other heuristically meaningful properties for a turbulent velocity field. The model is mathematically rigorous but it is not derived from the Navier-Stokes equations, it is just a cartoon of what we believe to resemble the turbulent 3D field given by the Navier-Stokes equations. Therefore the only merit of the following result is to show that there exists a field with the property stated in the conjecture, and such a field is not just an artificial example but it is strongly inspired by numerical and physical observations of turbulent fluids.

For simplicity we work in the full three-dimensional space \( \mathbb{R}^3 \), instead of the torus \( T \).

The model should be thought of as a random collection of vortex filaments, i.e. concentrations of vorticity around one-dimensional continuous curves. The filaments will be of various kind, from very elongated ones, whose existence is well documented in numerical observations of fully developed turbulence, to other more “eddy-like” and symmetric.

The basic ingredient of the construction is a vortex filament of length \( T \), thickness \( \ell \) and core velocity \( U \), which is stochastically modelled around a “Brownian” core: consider a 3d-Brownian motion \( \{ X_t \}_{t \in [0,T]} \) starting from a point \( X_0 \). This is the backbone of the vortex filament whose vorticity field is given by

\[
\xi_{\text{single}}(x) = \frac{U}{\ell^2} \int_0^T \varrho e(x - X_t) \circ dX_t.
\]
where $\circ dX$ denote Stratonovich integration. The letter $t$, that sometimes we shall also call time, is not physical time but just the parameter of the curve. We assume that $\varrho_t(x) = \varrho(x/\ell)$ for a radially symmetric measurable bounded (smooth) function $\varrho$ with compact support in the ball $B(0,1)$ (the unit ball in 3d Euclidean space). Heuristically $\xi_{\text{single}}(x)$ is an average of the “directions” $dX_t$ for points $X_t$ in the ball $B(x,\ell)$. The various parameters $U,\ell,T$ have to be thought of as giving the “typical” magnitudes of the respective properties. It should be noted that $\xi$ is not a “real” vorticity field (since in this model its divergence is not zero) but should be understood as providing the contribution to the fluid vorticity coming from the eddies.

The velocity field $u$ is generated from $\xi$ according to the Biot-Savart relation

$$u_{\text{single}}(x) = \frac{U}{\ell^2} \int_0^T K_\ell(x - X_t) \circ dX_t$$

(4.2)

where the vector kernel $K_\ell(x)$ is defined as

$$K_\ell(x) = \frac{1}{4\pi} \int_{B(0,\ell)} \varrho(y) \frac{x - y}{|x - y|^3} dy.$$  

(4.3)

We want to describe a random superposition of infinitely many independent Brownian vortex filaments, uniformly distributed in space, each of which will be associated with intensity-thickness-length parameters $(U,\ell,T)$ “randomly drawn” according to a measure $\gamma$. The total vorticity of the fluid is the sum of the vorticities of the single filaments, so, by linearity of the relation vorticity-velocity, the total velocity field will be the sum of the velocity fields of the single filaments.

The correct mathematical implementation of this heuristic picture is given by the construction of a Poisson random measure on a suitable space. Let $\Xi$ be the metric space

$$\Xi = \{(U,\ell,T,X) \in \mathbb{R}^3_+ \times C([0,1];\mathbb{R}^3) : 0 < \ell \leq \sqrt{T} \leq 1\}$$

with its Borel $\sigma$-field $\mathcal{B}(\Xi)$. Let $(\Omega,\mathcal{A},P)$ be a probability space, with expectation denoted by $E$, and let $\mu_\omega, \omega \in \Omega$, be a Poisson random measure on $\mathcal{B}(\Xi)$, with intensity $\nu$ (a $\sigma$-finite measure on $\mathcal{B}(\Xi)$) given by

$$d\nu(U,\ell,T,X) = d\gamma(U,\ell,T)dW(X).$$

for $\gamma$ a $\sigma$-finite measure on the Borel sets of $\{(U,\ell,T) \in \mathbb{R}^3_+ : 0 < \ell \leq \sqrt{T} \leq 1\}$ and $dW(X)$ the $\sigma$-finite measure defined by

$$\int_{C([0,1];\mathbb{R}^3)} \psi(X)dW(X) = \int_{\mathbb{R}^3} \left[ \int_{C([0,1];\mathbb{R}^3)} \psi(X)dW_{x_0}(X) \right] dx_0$$

for any integrable test function $\psi : C([0,1];\mathbb{R}^3) \to \mathbb{R}$. Here $dW_{x_0}(X)$ is the Wiener measure on $C([0,1];\mathbb{R}^3)$ starting at $x_0$ and $dx_0$ is the Lebesgue measure on $\mathbb{R}^3$. Heuristically the measure $\mathcal{W}$ describes a Brownian path starting from an
uniformly distributed point in all space. The assumptions on \( \gamma \) will be specified at due time.

The random measure \( \mu_\omega \) is uniquely determined by its characteristic function

\[
\mathbb{E} \exp \left( \int_\Xi \varphi(\zeta) \mu_\omega(d\zeta) \right) = \exp \left( \int_\Xi (e^{\varphi(\zeta)} - 1) \nu(d\zeta) \right),
\]

for any bounded measurable function \( \varphi \) on \( \Xi \) with support in a set of finite \( \nu \)-measure. In particular, for example, the first two moments of \( \mu \) read

\[
\mathbb{E} \int_\Xi \varphi(\xi) \mu(d\xi) = \int_\Xi \varphi(\xi) \nu(d\xi)
\]

and

\[
\mathbb{E} \left[ \int_\Xi \varphi(\xi) \mu(d\xi) \right]^2 = \left[ \int_\Xi \varphi(\xi) \nu(d\xi) \right]^2 + \int_\Xi \varphi^2(\xi) \nu(d\xi).
\]

Given the Poisson random measure \( \mu \) we can introduce our random velocity field as

\[
u(x) = \int_\Xi u^\zeta_{\text{single}}(x) \mu(d\zeta) = \mu(u^\cdot_{\text{single}}(x)). \tag{4.4}
\]

for any \( x \in \mathbb{R}^3 \), where \( \zeta = (U, \ell, T, X) \) and

\[
\zeta \mapsto u^\zeta_{\text{single}}(x) := \frac{U}{\ell^2} \int_0^T K_\ell(x - X_t) \wedge dX_t
\]

can be shown to be a well defined \( \mu \)-measurable function.

In plain words, given \( \omega \in \Omega \), the point measure \( \mu_\omega \) specifies the parameters and locations of infinitely many filaments: formally

\[
\mu = \sum_{\alpha \in \mathbb{N}} \delta_{\zeta^\alpha} \tag{4.5}
\]

for a sequence of i.i.d. random points \( \{\zeta^\alpha\} \) distributed in \( \Xi \) according to \( \nu \) (this fact is not rigorous since \( \nu \) is only \( \sigma \)-finite, but can be justified by a localisation procedure). Since the total velocity at a given point \( x \in \mathbb{R}^3 \) should be the sum of the contributions from each single filament, i.e. in heuristic terms

\[
u(x) = \sum_{\alpha} u^\zeta_{\text{single}}^\alpha(x) \tag{4.6}
\]

this justifies, physically, the above formula.

To end the construction of the model it remains to choose a suitable measure \( \gamma \) for the distribution of the parameters. Lacking physically motivated choices of \( \gamma \) we resorted in \[17\] to show that it is possible to fix \( \gamma \) in such a way to recover statistics which corresponds to multifractal scaling of the velocity increments, for any possible choice of the multifractal spectrum. In this way we showed how to build a random field with prescribed multifractal spectrum which also possess
some geometric properties of real turbulent fields. In particular we can choose \( \gamma \) to recover K41 behaviour of the velocity increments. In the following we will fix this particular choice and show that, for our random field

\[
\mathbb{E}|Du(0)|^2 \sim \eta^{-4/3} \quad \mathbb{E}|D^2u(0)|^2 \sim \eta^{-10/3}
\] (4.7)

where \( \eta \), in the context of this section will be a UV cutoff scale for the vortex model, i.e. we will not allow vortices with thickness \( \ell \) smaller than \( \eta \) which physically models the “viscous” (or Kolmogorov) scale which determines the lower end of the inertial range. The scaling (4.7) implies that \( \theta \sim \eta \) for \( \eta \to 0 \).

So we stipulate that

\[
d\gamma(U, \ell, T) = \delta_{\ell^{1/3}}(U)\delta_{\ell^2}(T)\ell^{-4}1_{\ell(\eta, 1)}d\ell
\] (4.8)

where we ignore a possible constant prefactor which will not play any role in our discussion. This choice of \( \gamma \) corresponds to force the vortex filaments with thickness \( \ell \) to have length proportional to \( \ell^2 \) and to have typical velocity of the order of \( \ell^{1/3} \), the “density” \( \ell^{-4} \) is chosen to roughly have “space-filling” vortices at all scales. Moreover vortices can have thickness going from the small scale \( \eta \) to a large “integral” scale of order 1.

Let us state the result. For technical reasons we will assume that there exists positive constants \( c, C, \lambda \) and \( u \in B(0, 1) \) such that the following bounds on \( K \) holds

\[
c1_{x \in B(u, \lambda)} \leq |DK_1(x)| \leq C1_{x \in B(0, 1)}
\] (4.9)

and

\[
c1_{x \in B(u, \lambda)} \leq |D^2K_1(x)| \leq C1_{x \in B(0, 1)},
\] (4.10)

for any \( x \in B(0, 1) \). A sufficient condition for the upper-bounds is that \( \rho \) is bounded.

**Proposition 4.1** With the above definitions, we have

\[
\mathbb{E}|Du(0)|^2 \asymp \eta^{-4/3} \quad \mathbb{E}|D^2u(0)|^2 \asymp \eta^{-10/3}
\] (4.11)

as \( \eta \to 0 \).

**Proof.** By a small abuse of notation, we have

\[
\mathbb{E}|Du(0)|^2 = \nu \left[ |Du_{\text{single}}(0)|^2 \right] = \nu \left[ \frac{U^2}{\ell^4} \int_0^T |DK_1(0 - X_\ell)|^2 dt \right]
\]

since \( \nu(Du_{\text{single}}(0)) = 0 \) being \( u_{\text{single}} \) a \( \nu \)-Itô integral and where we used the energy-identity for the Itô integral. Note that \( |DK_1(x)| = |DK_1(x/\ell)| \) and that, when \( x \notin B(0, 1) \) we have

\[
|DK_1(x)| \leq C|x|^{-2}, \quad |D^2K_1(x)| \leq C|x|^{-3}
\] (4.12)
by direct estimation from the Biot-Savart formula.

Now we use the Lemma 4.2 below together with the bounds (4.9) and (4.12) to get
\[ \mathbb{E}|\xi(0)|^2 \propto \gamma \left[ U^2 \ell^{-1} T \right] = \gamma \left[ \ell^{5/3} \right] \]
where we used the fact that, under \( \gamma \), \( U = \ell^{1/3} \) and \( T = \ell^2 \). Then easily we conclude that
\[ \mathbb{E}|Du(0)|^2 \propto \eta^{-4/3}. \]

Analogously, from the bounds (4.10) and (4.12) we have that
\[ c \ell^{-1} 1_{x \in B(u, \lambda \ell)} \leq \left| D^2 K_{\ell}(x) \right| \leq C \ell^{-1} \left[ 1_{x \in B(0, \ell)} + 1_{x \notin B(0, \ell)} |x/\ell|^{-3} \right] \]
\[ \leq C' \ell^{-1} \left[ 1_{x \in B(0, \ell)} + 1_{x \notin B(0, \ell)} |x/\ell|^{-2} \right] \]
so using again Lemma 4.2 we can obtain that
\[ \mathbb{E}|D^2 u(0)|^2 \propto \gamma \left[ U^2 \ell^{-3} T \right] = \gamma \left[ \ell^{-1/3} \right] \propto \eta^{-10/3} \]
ending the proof. \( \blacksquare \)

**Lemma 4.2** We have the estimate
\[ \mathcal{W} \left[ \int_0^T 1_{X_t \in B(0, \ell)} dt \right] \propto \ell^3 T \] (4.13)
and, if
\[ |\varphi_\ell(x)| \leq C \left( 1_{x \in B(0, \ell)} + 1_{x \notin B(0, \ell)} |x/\ell|^{-2} \right) \] (4.14)
we have
\[ \mathcal{W} \left[ \int_0^T |\varphi_\ell(x - X_t)|^2 dt \right] \leq \ell^3 T \] (4.15)

**Proof.** These results are particular cases of more general bounds proved in [7]: the first is proved in Lemma 14 of the reference, while eq.(4.15) is proved in Lemma 3: the proof refers to the particular case in which \( \varphi_\ell = K_\ell \) but it is easy to see that a sufficient condition is given by eq.(4.14). \( \blacksquare \)

**Appendix A Mollification of measures**

Some computations of the paper with Taylor formula require more regularity than that of typical fields under \( \mu \in \mathcal{P} \). For this reason we introduce mollifications of measures \( \mu \in \mathcal{P} \). Let us remark that this technical effort is useless if the noise is more regular, since one can prove more regularity of the typical elements under \( \mu \in \mathcal{P} \).

Let \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) be a smooth function with compact support, symmetric, non negative, strictly positive at zero, with \( \int_{\mathbb{R}^d} \varphi(||x||) dx = 1 \). Set \( \varphi_\varepsilon(x) = \)
MOLLIFICATION OF MEASURES

\[ \varepsilon^{-d} \varphi(\|x/\varepsilon\|) \], so \( \int_{\mathbb{R}^d} \varphi_\varepsilon(x) dx = 1 \); \( \{ \varphi_\varepsilon \}_{\varepsilon > 0} \) is a family of usual smooth mollifiers. For every \( u \in H \) set

\[ u_\varepsilon(x) = \int_{\mathbb{R}^d} \varphi_\varepsilon(x - y) u(y) dy. \]

Given \( \mu \in \mathcal{P}_0 \), the mapping \( u \mapsto u_\varepsilon \) in \( H \) induces an image measure \( \mu_\varepsilon \in \mathcal{P}_0 \) which is in fact supported on smooth fields.

**Lemma A.1** If \( \mu \in \mathcal{P} \), then \( \mu_\varepsilon \in \mathcal{P} \).

**Proof.** We have

\[
u_\varepsilon(x - a) = \int_{\mathbb{R}^d} \varphi_\varepsilon(x - a - y) u(y) dy \quad y' = y + a \quad \int_{\mathbb{R}^d} \varphi_\varepsilon(x - y') u(y' - a) dy'  
\]

where the last equality is understood in law under \( \mu \), and it holds true as processes in \( x \). Hence \( u_\varepsilon(\cdot - a) \equiv u_\varepsilon(\cdot) \). This means

\[
\int_{H} f(u_\varepsilon(\cdot - a)) d\mu(u) = \int_{H} f(u_\varepsilon) d\mu(u)
\]

for bounded continuous \( f \)'s, and therefore

\[
\int_{H} f(u(\cdot - a)) d\mu_\varepsilon(u) = \int_{H} f(u) d\mu_\varepsilon(u)
\]

so the space homogeneity of \( \mu_\varepsilon \) is proved.

Similarly, we have

\[
u_\varepsilon(Rx) = \int_{\mathbb{R}^d} \varphi_\varepsilon(R(x - R^{-1} y)) u(y) dy = \int_{\mathbb{R}^d} \varphi_\varepsilon(x - R^{-1} y) u(y) dy
\]

form the symmetry of \( \varphi_\varepsilon \), hence

\[
u_\varepsilon(Rx) \quad y' = R^{-1} y \quad \int_{\mathbb{R}^d} \varphi_\varepsilon(x - y') u(R y') dy'  
\]

\[
\equiv \int_{\mathbb{R}^d} \varphi_\varepsilon(x - y') Ru(y') dy' 
\]

hence

\[
\int_{H} f(u_\varepsilon(R \cdot)) d\mu(u) = \int_{H} f(R u_\varepsilon(\cdot)) d\mu(u)
\]

and finally

\[
\int_{H} f(u(R \cdot)) d\mu_\varepsilon(u) = \int_{H} f(R u(\cdot)) d\mu_\varepsilon(u).
\]

The proof is complete. \( \blacksquare \)
Lemma A.2 For every $\mu \in \mathcal{P}$, if
\[
\int_H \int_T |Du(x)|^2 \, dx \, d\mu(u) < \infty, \quad \int_H \int_T |D^2 u(x)|^2 \, dx \, d\mu(u) < \infty,
\]
then
\[
\int_H \|u(re) - u(0)\|^2 \, d\mu(u) < \infty
\]
and
\[
\lim_{\varepsilon \to 0} \int_H |Du(0)|^2 d\mu_\varepsilon(u) = \int_H \int_T |Du(x)|^2 \, dx \, d\mu(u) \\
\lim_{\varepsilon \to 0} \int_H |D^2 u(0)|^2 d\mu_\varepsilon(u) = \int_H \int_T |D^2 u(x)|^2 \, dx \, d\mu(u) \\
\lim_{\varepsilon \to 0} \int_H \|u(re) - u(0)\|^2 d\mu_\varepsilon(u) = \int_H \|u(re) - u(0)\|^2 d\mu(u).
\]

Proof. There exists $C > 0$ such that
\[
\int_T |Du_\varepsilon(x)|^2 \, dx \leq C \int_T |Du(x)|^2 \, dx
\]
for every $u \in D(A)$; and $\int_T |Du_\varepsilon(x)|^2 \, dx \to \int_T |Du(x)|^2 \, dx$ as $\varepsilon \to 0$ for every $u \in D(A)$. Hence, by Lebesgue theorem,
\[
\lim_{\varepsilon \to 0} \int_H \left[ \int_T |Du(x)|^2 \, dx \right] d\mu_\varepsilon(u) = \int_H \int_T |Du(x)|^2 \, dx \, d\mu(u).
\]
But $\mu_\varepsilon$ is space homogeneous, hence
\[
\int_H \left[ \int_T |Du(x)|^2 \, dx \right] d\mu_\varepsilon(u) = \int_H |Du(0)|^2 d\mu(u).
\]
This proves the first claim. The proof of the second one is entirely similar. For the third one, we have
\[
\|u_\varepsilon(x + re) - u_\varepsilon(x)\|^2 = \left\| r \int_0^1 Du_\varepsilon(x + \sigma e) \, d\sigma \right\|^2 \\
\leq r^2 \int_0^1 \|Du_\varepsilon(x + \sigma e)\|^2 \, d\sigma
\]
for every $u \in D(A)$, hence
\[
\int_T \|u_\varepsilon(x + re) - u_\varepsilon(x)\|^2 \, dx \leq r^2 \int_0^1 \int_T \|Du_\varepsilon(x + \sigma e)\|^2 \, dx \, d\sigma \\
= r^2 \int_0^1 \int_T \|Du_\varepsilon(x)\|^2 \, dx \, d\sigma \\
\leq C r^2 \int_T |Du(x)|^2 \, dx.
\]
Therefore, again by Lebesgue theorem,
\[
\lim_{\varepsilon \to 0} \int_H \int_T \|u(x + re) - u(x)\|^2 dx \, d\mu_\varepsilon(u) = \int_H \int_T \|u(x + re) - u(x)\|^2 dx \, d\mu(u) .
\]
The third claim follows now from the space homogeneity of both $\mu_\varepsilon$ and $\mu$. ■

We are now in the position to prove a quantitative consequence of isotropy, that we shall use in the sequel. In the next statement we understand that both terms in the equality are either finite and equal, or both infinite.

**Lemma A.3** For every $\mu \in \mathcal{P}$ and every coordinate unitary vector $e$ we have
\[
\int_H \int_T \|Du(x)\|^2 dx \, d\mu(u) = d \int_H \int_T \|Du(x) \cdot e\|^2 dx \, d\mu(u).
\]
For $\mu_\varepsilon$, we have the same identity and also
\[
\int_H \|Du(0)\|^2 d\mu_\varepsilon(u) = d \int_H \|Du(0) \cdot e\|^2 d\mu_\varepsilon(u).
\]

**Proof.** **Step 1.** Denote by $e_1, \ldots, e_d$ the coordinate unitary vectors. For $u \in D(A)$ we have
\[
\|Du(x)\|^2 = \sum_{ij} \left| \frac{\partial u_i}{\partial x_j}(x) \right|^2, \quad \|Du(x) \cdot e_j\|^2 = \sum_i \left| \frac{\partial u_i}{\partial x_j}(x) \right|^2
\]
and thus
\[
\|Du(x)\|^2 = \sum_j \|Du(x) \cdot e_j\|^2.
\]
Therefore
\[
\int_H \|Du(0)\|^2 d\mu_\varepsilon(u) = \sum_j \int_H \|Du(0) \cdot e_j\|^2 d\mu_\varepsilon(u)
\]
and
\[
\int_H \int_T \|Du(x)\|^2 dx \, d\mu(u) = \sum_j \int_H \int_T \|Du(x) \cdot e_j\|^2 dx \, d\mu(u).
\]
It is then sufficient to prove that all terms of the sums on the right-hand-sides are equal, in order to prove the first and last claim of the lemma; we shall prove this below in steps 2 and 3. Finally, the first assertion for $\mu_\varepsilon$ is a particular case of the first claim of the lemma ($\mu_\varepsilon$ is an element of $\mathcal{P}$).
Step 2. Now, given \( j = 1, \ldots, d \), take a rotation \( R \) as in the definition of \( \mathcal{P} \) such that \( Re_1 = e_j \). Given \( N > 0 \),

\[
\int_H \left( \|Du(0) \cdot e_j\|^2 \wedge N \right) d\mu_\varepsilon(u) \\
= \lim_{r \to 0} \int_H \left( r^{-2}\|u(re_j) - u(0)\|^2 \wedge N \right) d\mu_\varepsilon(u) \\
= \lim_{r \to 0} \int_H \left( r^{-2}\|u(Rre_1) - u(R0)\|^2 \wedge N \right) d\mu_\varepsilon(u) \\
= \lim_{r \to 0} \int_H \left( r^{-2}\|u(re_1) - u(0)\|^2 \wedge N \right) d\mu_\varepsilon(u) \\
= \int_H \left( \|Du(0) \cdot e_1\|^2 \wedge N \right) d\mu_\varepsilon(u).
\]

By monotone convergence in \( N \), we get that \( \int_H \|Du(0) \cdot e_j\|^2 d\mu_\varepsilon(u) \) is independent of \( j \). This proves one of the claims.

Step 3. From the previous step and homogeneity we have that \( \int_H \int_T \|Du(x) \cdot e_j\|^2 dx \mu_\varepsilon(u) \) is also independent of \( j \). Arguing as in the proof of the previous lemma, this integral converges to \( \int_H \int_T \|Du(x) \cdot e_j\|^2 dx \mu(u) \), which is therefore also independent of \( j \). The proof is complete.

Appendix B  Scaling theorems

The torus, \( T_L = [0, L]^d \), the energy space \( H_L \) with norm \( \| \cdot \|_{H_L} \), the spaces \( V_L, D(A_L), D_L \) and the Stokes operator \( A_L \) on \( T_L \) have been already introduced in section 1.1. We define

\[
\Lambda_L^{(\infty)} = \left\{ k \in \frac{2\pi}{L}\mathbb{Z}^d : |k|^2 > 0 \right\},
\]

and, for the purpose of Galerkin approximations, we introduce also

\[
\Lambda_L^{(n)} = \left\{ k \in \frac{2\pi}{L}\mathbb{Z}^d : 0 < |k|^2 \leq \left( \frac{2\pi}{L}n \right)^2 \right\}
\]

so that \( \Lambda_L^{(\infty)} = \cup_n \Lambda_L^{(n)} \). In particular, \( \Lambda^{(\infty)} = \Lambda_1^{(\infty)} \).

B.1  Scaling theorem for Galerkin approximations

Let \( V'_L \) be the dual of \( V_L \); with proper identifications we have \( V_L \subset H_L \subset V'_L \) with continuous injections. Let \( B_L(.,.) : V_L \times V_L \to V'_L \) be the bilinear operator defined for all \( u, v, w \in D_L \) as

\[
\langle w, B_L(u, v) \rangle_\mathcal{H}_L = \sum_{i,j=1}^d \frac{1}{L^d} \int_{T_L} u_i \frac{\partial v_j}{\partial x_i} w_j dx = \sum_{h+l=k} (l \cdot \hat{u}(h)) \hat{v}(l) \cdot \hat{w}(k).
\]
Given $L > 0$, $\nu > 0$ and $\theta > 0$, consider (formally) the equation in $H_L$

$$du + [\nu A_L u + B_L(u, u)]dt = 0 \sum_{k \in \Lambda_L^{(\infty)}} \sigma_k \cdot d\beta_k e^{-ik \cdot x},$$

where $\beta_k^L = \beta_{LK}$ and $\sigma_k^L = \sigma_{LK}$, and $(\beta_k^L)_{k \in \Lambda^{(\infty)}}$ and $(\sigma_k^L)_{k \in \Lambda^{(\infty)}}$ have been introduced in Section 2.2.1 and are subject to the assumptions imposed therein, so that the random fields

$$W_L^{(n)}(t, x) = \sum_{k \in \Lambda_L^{(n)}} \sigma_k^L \beta_k^L(t) e^{-ik \cdot x}$$

and the field $W_L^{(\infty)}(t, x)$ similarly defined, are space-homogeneous and partially (in the sense of the rotations of the torus) isotropic.

Let $H_L^{(n)}$ be the subspace of $H_L$ corresponding to the modes with wavelengths in $\Lambda_L^{(n)}$ and consider the equation in $H_L^{(n)}$

$$du^{(n)} + [\nu A_L u^{(n)} + \pi_L(u^{(n)}, u^{(n)})]dt = 0 \sum_{k \in \Lambda_L^{(n)}} \sigma_k \cdot d\beta_k e^{-ik \cdot x} \quad (B.3)$$

where $\pi_L^{(n)}$ is the orthogonal projection of $H_L$ onto $H_L^{(n)}$.

**Lemma B.1** If $u^{(n)}$ is a solution in $H_L$ of (B.3), with initial condition $u^{(n)}(0)$ and parameters $(\nu, L, \theta)$, then

$$\overline{u}^{(n)}(t, x) := \lambda^\beta u^{(n)}(\lambda^{1+\beta} t, \lambda^{1+\beta} x)$$

is a solution in $H_{L/\lambda}$ of equation (B.3) with initial condition $\overline{u}^{(n)}(0)$ and parameters $\left(\nu \lambda^{\beta-1}, L/\lambda, \lambda^{\frac{1+\beta}{2}} \theta\right)$ (but with new Brownian motions).

**Proof.** This statement is not clear a priori, especially because of the scaling transformation of the nonlinear term, so we give all the details. The solution $u^{(n)}$, as a Fourier series, is given by

$$u^{(n)}(t, x) = \sum_{k \in \Lambda_L^{(n)}} \hat{u}^{(n)}(t, k) e^{-ik \cdot x},$$

and the solution $\overline{u}^{(n)}$, as a process in $H_{L/\lambda}^{(n)}$, is given by

$$\overline{u}^{(n)}(t, x) = \sum_{k \in \Lambda_L^{(n)/\lambda}} \hat{\overline{u}}^{(n)}(t, k) e^{-ik \cdot x},$$

The Fourier coefficients of $u^{(n)}$ and $\overline{u}^{(n)}$ are related by the scaling

$$\overline{\hat{u}}^{(n)}(t, k) = \frac{\lambda^d}{L^d} \int_{T_L/\lambda} \overline{u}^{(n)}(t, x) e^{ik \cdot x} dx = \frac{\lambda^{d+\beta}}{L^d} \int_{T_L/\lambda} u^{(n)}(\lambda^{1+\beta} t, \lambda^{1+\beta} x) e^{ik \cdot x} dx$$

$$= \lambda^\beta \lambda^{\beta/\lambda} \int_{T_L} u^{(n)}(\lambda^{1+\beta} t, x') e^{ik \cdot x'} dx' = \lambda^\beta \hat{u}(\lambda^{1+\beta} t, k'). \quad (B.4)$$
From the equation (B.3) in integral form,
\[ u^{(n)}(t) + \int_0^t \left[ \nu A_L u^{(n)} + \pi_L^{(n)} B_L \left( u^{(n)}, u^{(n)} \right) \right](s) \, ds = u^{(n)}(0) + \theta \sum_{k \in \Lambda_L^{(n)}} \sigma_k^{(L,\lambda)}(t) e^{-ik \cdot x}, \]
we have
\[
\lambda^\beta u^{(n)}(\lambda^{1+\beta} t, \lambda x) + \lambda^{1+2\beta} \int_0^t \left[ \nu A_L u^{(n)} + \pi_L^{(n)} B_L \left( u^{(n)}, u^{(n)} \right) \right](\lambda^{1+\beta} s, \lambda x) \, ds = \\
\lambda^\beta u^{(n)}(0, \lambda x) + \lambda^{1+2\beta} \theta \sum_{k \in \Lambda_L^{(n)}} \sigma_k^{L,\lambda} \bar{\beta}_k^{L,\lambda}(t) e^{-ik \cdot x}
\]
where \( \bar{\beta}_k^{L,\lambda}(t) := \lambda^{- \frac{1+\beta}{2} \frac{L}{\lambda}} \beta_k^{L,\lambda}(\lambda^{1+\beta} t) \) are new Brownian motions. The first term on the l.h.s. is \( \bar{u}^{(n)}(t, x) \), and the first term on the r.h.s. is \( \bar{u}^{(n)}(0, x) \). In addition, we have
\[ A_{L,\lambda} \bar{u}^{(n)}(t, x) = \lambda^{2+\beta} \left( A_L u^{(n)} \right)(\lambda^{1+\beta} t, \lambda x). \]
The proof of the claim will be complete if we show that
\[ \lambda^{1+2\beta} \left[ \pi_L^{(n)} B_L \left( u^{(n)}, u^{(n)} \right) \right](\lambda^{1+\beta} t, \lambda x) = \left[ \pi_{L,\lambda}^{(n)} B_{L,\lambda} \left( \bar{u}^{(n)}, \bar{u}^{(n)} \right) \right](t, x). \]
For every \( \varphi \in V_{L,\lambda} \), by using the Fourier expression (B.2) of the non-linear term and the scaling of Fourier coefficients (B.4),
\[ \langle \pi_{L,\lambda}^{(n)} B_{L,\lambda} \left( \bar{u}^{(n)}, \bar{u}^{(n)} \right)(t, \cdot), \varphi \rangle_{H_{L,\lambda}} = \langle B_{L,\lambda} \left( \bar{u}^{(n)}, \bar{u}^{(n)} \right)(t, \cdot), \pi_{L,\lambda}^{(n)} \varphi \rangle_{H_{L,\lambda}} = \sum_{h+l=k} \left( l \cdot \hat{\varphi}(k) \right) \hat{u}^{(n)}(t, l) \cdot \overline{\varphi(k)}. \]
\[ = \lambda^{1+2\beta} \sum_{h+l=k} \left( l \cdot \hat{\varphi}(k) \right) \hat{u}^{(n)}(\lambda^{1+\beta} t, \lambda^{-1} l) \cdot \overline{\varphi(k)} = \lambda^{1+2\beta} \langle B_L(u^{(n)}, u^{(n)})(\lambda^{1+\beta} t, \lambda \cdot h), \pi_{L,\lambda}^{(n)} \varphi \rangle_{H_{L,\lambda}} = \lambda^{1+2\beta} \langle \pi_L^{(n)} B_L(u^{(n)}, u^{(n)})(\lambda^{1+\beta} t, \lambda \cdot h), \varphi \rangle_{H_{L,\lambda}}, \]
where the sums above are extended to all wavelengths \( h, l \) and \( k \in \Lambda_L^{(n)} \) such that \( h + l = k \).■

**B.2 Scaling theorem for stationary measures**

Similarly to section 2.2.3 denote by \( \mathcal{P}_{\mathcal{NS}}^G(\nu, L, \theta) \) the set of probability measures that are limit of homogeneous isotropic invariant measures of equations (B.3).
Given $\lambda > 0$ and $\beta \in \mathbb{R}$ and $\mu \in \mathcal{P}_{NS}^G(\nu, L, \theta)$, let $u$ be a random field on $T_L$ with law $\mu$, define the random field $\tilde{u}$ on $T_{L/\lambda}$ as

$$\tilde{u}(x) = \lambda^\beta u(\lambda x)$$

and let $\tilde{\mu}$ be the law of $\tilde{u}$ on $H_{L/\lambda}$. More intrinsically, $\tilde{\mu}$ is defined by the relation

$$\int_{H_{L/\lambda}} f(u) d\tilde{\mu}(u) = \int_{H_L} f(\lambda^\beta u(\lambda \cdot)) d\mu(u)$$

for every bounded continuous $f$ on $H_{L/\lambda}$.

**Theorem B.2** If $\mu \in \mathcal{P}_{NS}^G(\nu, L, \theta)$ then $\tilde{\mu} \in \mathcal{P}_{NS}^G\left(\nu\lambda^{\beta-1}, L/\lambda, \lambda^{1+3\beta/2} \theta\right)$.

**Proof.** The measure $\mu$ of the theorem is the weak limit of a sequence $\{\mu_{n_k}\}$ of invariant measures on $H_{L^{(n_k)}}$ of the Galerkin problems with indexes $n_k$. For each $n_k$, let $u^{(n_k)}$ be a stationary solution (on some probability space) of (B.3), with parameters $(\nu, L, \theta)$ and marginal $\mu_{n_k}$. Let $\tilde{u}^{(n_k)}$ be the rescaled process as above, which is a solution of (B.3) with parameters $\left(\nu\lambda^{\beta-1}, L/\lambda, \lambda^{1+3\beta/2} \theta\right)$ (by the lemma above) and is a stationary process. Its marginal $\tilde{\mu}_{n_k}$ is the scaling of $\mu_{n_k}$, similarly to the relation defined above between $\mu$ and $\tilde{\mu}$. Moreover $\tilde{\mu}_{n_k}$ is an invariant measure for equation (B.3) with parameters $\left(\nu\lambda^{\beta-1}, L/\lambda, \lambda^{1+3\beta/2} \theta\right)$. From the weak convergence of $\mu_{n_k}$ to $\mu$ it is now easy to deduce the weak convergence of $\tilde{\mu}_{n_k}$ to $\tilde{\mu}$. Therefore $\tilde{\mu} \in \mathcal{P}_{NS}^G\left(\nu\lambda^{\beta-1}, L/\lambda, \lambda^{1+3\beta/2} \theta\right)$. The proof is complete. ■

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