ON THE CLASSIFICATION OF 3-DIMENSIONAL
$SL_2(\mathbb{C})$-VARIETIES

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Abstract. In the present work we describe 3-dimensional complex $SL_2$-
varieties where the generic $SL_2$-orbit is a surface. We apply this result to
classify the minimal 3-dimensional projective varieties with Picard-number 1
where a semisimple group acts such that the generic orbits are 2-dimensional.
This is an ingredient of the classification [Keb98] of the 3-dimensional relat-
ively minimal quasihomogeneous varieties where the automorphism group is
not solvable.

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1. Introduction

In [Keb98] we give a classification of the 3-dimensional relatively minimal quasi-
homogeneous projective varieties where the automorphism group is linear algebraic
and not solvable. By “relatively minimal” we mean varieties having at most $\mathbb{Q}$-
factorial terminal singularities and allowing an extremal contraction of fiber type.
These varieties always occur at the end of the minimal model program if one starts
with a projective rational quasihomogeneous manifold whose automorphism group
is not solvable.

Certain aspects of this project utilize results on non-transitive $SL_2(\mathbb{C})$-actions
which in our opinion are of separate interest. We have chosen to present these here
as opposed to including them in the midst of the classification work, where the
methods are essentially different.

The aim of the first part of this paper is to describe 3-dimensional complex
$SL_2$-varieties where the generic $SL_2$-orbit is a surface. More precisely, we give
elementary criteria for the fibers of the categorical quotient to be irreducible or
normal and describe neighborhoods of reduced fibers (see proposition 3.1). We
reduce to this case by using concretely constructed equivariant GALOIS coverings
which are étale in codimension one. Under certain restrictions on the isotropy
group, a stronger classification is known —see [Arz98].
In the main part of the paper we apply these results to yield the following ingredient of the classification in [Keb98].

**Theorem 1.1.** Let $X$ be a $\mathbb{Q}$-factorial projective 3-dimensional variety with Picard-number $\rho(X) = 1$ having at most terminal singularities. Assume that a semisimple linear algebraic group $S$ acts algebraically on $X$ such that generic $S$-orbits are 2-dimensional. Then $X$ is isomorphic to the smooth 3-dimensional quadric or to one of the (weighted) projective spaces $\mathbb{P}_3$, $\mathbb{P}_{(1,1,1,2)}$ or $\mathbb{P}_{(1,1,2,3)}$.

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2. **On the Normality of Fibers of the Categorical Quotient**

Recall that for an affine variety the quotient is defined as the spectrum of the ring of invariant functions. The following are the results of this section:

**Proposition 2.1.** Let $X$ be an irreducible complex affine 3-dimensional normal $SL_2$-variety. Then all fibers of the categorical quotient map $q : X \to Y$ are irreducible. If $X$ is additionally Cohen-Macaulay, then a $q$-fiber is normal if it is reduced.

Under additional assumptions on the singularities, the claim is true for non-reduced fibers as well.

**Proposition 2.2.** In the setting of proposition 2.1 assume additionally that $X$ has at most canonical singularities. Then every fiber of the categorical quotient is normal with it’s reduced structure.

Before proceeding with the proofs we recall two elementary facts: First, the only normal affine complex $SL_2$-surfaces with non-trivial action are

- the smooth affine quadric $\mathbb{Q}^2_2$: this space is $SL_2$-homogeneous. The isotropy group of a point is a torus.
- $\mathbb{P}_2$ minus a quadric curve: this is a quotient of $\mathbb{Q}^2_2$ by $\mathbb{Z}_2$. We denote it by $\mathbb{Q}^2_2/\mathbb{Z}_2$. The isotropy group is the normalizer of a torus.
- the affine cone over a rational normal curve: we denote this by $\mathbb{F}^n$, where $n$ is the degree of the curve. The isotropy is generated by a unipotent part and a cyclic group, isomorphic to $\mathbb{Z}_n$. This space contains an open $SL_2$-orbit and an $SL_2$-fixed point.

See [Huc86] for a more detailed description.

Second, if $X$ is a 3-dimensional $SL_2$-variety with non-trivial action and $D_1 \subset X$ is a divisor, then $SL_2$ acts non-trivially on $D_1$. This follows directly from a linearization argument; see [HO80, I.1.5] for matters concerning linearization. In particular, if $X$ is affine and $D_2$ is another divisor, then $D_1 \cap D_2$ must be a single point.

**Proof of proposition 2.1.** Assume without loss of generality that $\dim Y = 1$, for the proposition is trivial otherwise. Since all $q$-fibers are connected, we must rule out the possibility that there is a point $y \in Y$ such that $q^{-1}(y)$ is connected and not irreducible. If this was the case, then the irreducible components of $q^{-1}(y)$ can only meet in the unique $SL_2$-fixed point in $q^{-1}(y)$, i.e. $q^{-1}(y)$ is not connected in codimension one. On the other hand, HARTSHORNE’s connectedness theorem states that $X$ is connected in dimension 2 (see [Eis95, Thm. 18.12 and the preceding discussion]). Now $Y$ is normal, hence smooth, so that $q^{-1}(y)$ is CARTIER. In this situation GROTHENDIECK’s connectedness theorem shows that $q^{-1}(y)$ must be connected in dimension 1 (see [Gro62, exp. XIII]), a contradiction.
If $X$ is Cohem-Macaulay, then every $q$-fiber automatically satisfies Serre’s condition $S_2$ (see [Rei87]). If it is reduced, it’s singular set is either the unique $SL_2$-fixed point or empty. The normality follows directly from Serre’s criterion.

Proof of proposition 2.2. Again it is sufficient to consider the case that $\dim Y = 1$. If $y \in Y$ is a point such that $q^{-1}(y)$ has multiplicity $m > 1$, let $\Delta$ be an analytic neighborhood of $y$, isomorphic to a disk and let $\tilde{q} : \tilde{X} \to \Delta$ be the $m$th root fibration, associated to the restriction of $q$ to $\Delta$. If $\tilde{y}$ denotes the (reduced) preimage if $y$ in $\Delta$, then $\tilde{q}^{-1}(\tilde{y})$ is reduced. The map $\tilde{X} \to X$ is an $SL_2$-equivariant cyclic cover, branched only over the unique $SL_2$-fixed point point in $q^{-1}(y)$, if at all. This has two consequences: first, [Rei80, prop. 1.7] applies, showing that $\tilde{X}$ has canonical singularities, so that $\tilde{X}$ is Cohem-Macaulay and $\tilde{q}^{-1}(\tilde{y})$ is normal. Secondly, because the induced map $\tilde{q}^{-1}(\tilde{y}) \to q^{-1}(y)$ is just the quotient by the action of the Galois group, $q^{-1}(y)_{\text{red}}$ must also be normal.

There exists a preprint of I. V. Arzhantsev where, using the techniques of [LV83], a proof of proposition 2.2 is indicated for arbitrary normal singularities.

3. Neighborhoods of fibers

Now we consider the neighborhood of reduced fibers.

**Proposition 3.1.** In the setting of proposition 2.1, if $Y$ is a curve and $y \in Y$ is a point such that $q^{-1}(y)$ is reduced, then there exists a Zariski-open neighborhood $\Delta$ of $y$ such that $q^{-1}(\Delta)$ is equivariantly isomorphic to one of the following:

- a product $\mathbb{F}_a^n \times \Delta$ where $SL_2$ acts on $\mathbb{F}_a^n$ only
- $\{(x,y,z), \delta \in \mathbb{C}^3 \times \Delta | 4xz - y^2 = P(\delta)\}$, where $P \in O(\Delta)$, having zeros only at $y$ and $SL_2$ acts on $\mathbb{C}^3$ via the 3-dimensional irreducible representation.
- a quotient of the latter by $\mathbb{Z}_2$, acting with weights $(1,1,1)$ on $\mathbb{C}^3$ and trivially on $\Delta$.

The proof follows from two technical considerations. Recall from [Kra85, II.2.4] that there is an equivariant embedding $i : X \to \oplus V_{k_i}$, where the $V_{k_i}$ are irreducible $SL_2$-representation spaces.

**Lemma 3.2.** There exists a $j \in \mathbb{N}$ such that the projection $\pi : \oplus V_{k_i} \to V_{k_j}$ is a closed embedding if restricted to $q^{-1}(y)$.

**Proof.** We consider the possibilities for the central fiber separately:

- if $q^{-1}(y) \cong \mathbb{Q}_2^n / \mathbb{Z}_2$: then every non-trivial equivariant map is a closed embedding because the isotropy of $\mathbb{Q}_2^n / \mathbb{Z}_2$ is maximal.
- if $q^{-1}(y) \cong \mathbb{Q}_2^n$: the only possible images of an $SL_2$-equivariant morphism which is not an embedding are $\mathbb{Q}_2^n / \mathbb{Z}_2$ and $\{0\}$. Both have normalizers of tori in their isotropy groups, but $\mathbb{Q}_2^n$ has not. Thus, there must be a projection with image $\mathbb{Q}_2^n$. This must be an embedding.
- if $q^{-1}(y) \cong \mathbb{F}_a^n$: one has to rule out that all projection map $q^{-1}(y)$ to $\{0\}$ or to $\mathbb{F}_a^n$, $k > 1$, this being the only possible images. Assume to the contrary and let $U < SL_2$ be a unipotent subgroup. It’s fixed point set is a line $C$, isomorphic to $\mathbb{C}$, and all projections map $C$ to $\{0\}$ or are branched covers, ramified at zero. Thus, the rank of the Jacobian of $i|_C$ drops at zero —a contradiction to $i$ being an embedding.

Having embedded the central fibers, we show that the restriction to a neighboring $q$-fiber is injective as well.
Lemma 3.3. There exists a Zariski-open neighborhood $U$ of $y$ such that for all $\eta \in U$ the restriction of $\pi$ a the $q$-fiber $X_\eta = q^{-1}(y)$ is a closed embedding.

Proof. Choose $U$ to be a maximal neighborhood of $y \in Y$ such that $\pi(X_\eta) \neq 0$ for all $\eta \in Y$ and such that all $q$-fibers over $U \setminus \{y\}$ are isomorphic. Use the classification of the 2-dimensional algebraic subgroups of $SL_2$ to see that this is always possible. Again we perform a case-by-case check:

$q^{-1}(y) \cong F^2_n$: in this case $V_{k_n}$ must be $\mathbb{C}^2$. As $\pi(X_\eta) \neq \{0\}$, we have $\pi(X_\eta) \cong F^2_1$ and $X_\eta$ must be isomorphic to $F^2_1$ itself, there being no $SL_2$-equivariant cover.

$q^{-1}(y) \cong F^3_n$: here $V_{k_n}$ is the irreducible 3-dimensional representation space. The only $SL_2$-invariant divisors in here are $F^3_2$ and smooth quadrics. Arguing as above, one must show that the generic $q$-fiber $X_\eta$ is not isomorphic to a cover of $F^3_2$ or $Q_2^3$, i.e. $X_\eta \not\cong F^3_1$. If this was the case, then linearize the center $Z$ of $SL_2$ at a smooth point of $q^{-1}(y)$. This gives an analytic curve germ $C \subset X$, invariant under $Z$ and intersecting $q^{-1}(y)$ transversally in a single point. As $Z$ is not contained in the isotropy group of any point in $X_\eta$ other than 0, $C$ must intersect the neighboring fiber twice. This is a contradiction to $q^{-1}(y)$ being reduced.

$q^{-1}(y) \cong F^a_n$ where $n = 3$ or $n > 4$: a similar linearization argument as above, using a $\mathbb{Z}_n$ from the isotropy group of a generic point in $q^{-1}(y)$, shows that the generic $X_\eta$ must contain a $\mathbb{Z}_n$-fixed curve. Classification yields that $X_\eta \cong F^a_{k_n}$ for one $k \in \mathbb{N}$. But $k$ must be 1: every $X_\eta$ contains a curve which is $\mathbb{Z}_{k_n}$-fixed and $q^{-1}(y)$ must, too.

$q^{-1}(y) \cong F^3_4$: here $V_{k_4}$ is the irreducible 5-dimensional representation space where the only $SL_2$-invariant surfaces are $F^a_4$ or are isomorphic to $Q_2^3/\mathbb{Z}_2$. The linearization argument used above rules out that $X_\eta \cong Q_2^3$ or a cover of $F^a_4$.

$q^{-1}(y) \cong Q_2^3$: here $V_{k_3}$ contains two types of 2-dimensional $SL_2$-orbits: $Q_2^3$ and $F^3_1$. We know that $\pi(X_\eta) \cong Q_2^3$, as otherwise $\pi(X_\eta)$ must contain a $U$-pointwise fixed curve and $\pi(q^{-1}(y))$ must, too. A contradiction. Again $\pi(X_\eta)$ must be injective as there is no $SL_2$-equivariant cover of $Q_2^3$.

$q^{-1}(y) \cong Q_2^3/\mathbb{Z}_2$: apply the linearization argument involving a generic isotropy group, i.e. the normalizer of a torus to see that the neighboring $q$-fibers cannot be isomorphic to $Q_2$. Now argue as in the last case.

With this information we start the

Proof of proposition 3.1. Choose $\Delta \subset Y$ as in lemma 3.3. Then the map $(\pi \circ \iota) \times q : X \to V_{k_2} \times Y$ is injective if restricted to $q^{-1}(\Delta)$.

Recall that every irreducible representation space of $SL_2$ contains a unique $SL_2$-orbit whose closure is isomorphic to $F^a_n$. Thus the claim of proposition 3.1 holds if all $q$-fibers are isomorphic to $F^a_n$.

If $q^{-1}(y) \cong F^3_2$, and the generic fiber is a smooth quadric, then $q^{-1}(\Delta)$ can be equivariantly embedded into $V_2 \times \Delta$. Equip $V_2$ with coordinates $(x, y, z)$, fix a torus $T < SL_2$ and note that it’s fixed point set $V_2^T$ is a 1-dimensional linear subspace. If $y$ is the linear coordinate on $V_2^T$, then the intersection $X^T := X \cap (V_2^T \times \Delta)$ is given by $\{ -y^2 = P(\delta) \}$ where $P \in O(\Delta)$. This is because $X^T$ is 2:1 over $\Delta$ and invariant under multiplication of $V_2$ with -1. By choice of $\Delta$, $P$ has no zero on $\Delta \setminus \{0\}$. Now $X$ being uniquely determined by $X^T$ as $X = SL_2 \cdot X^T$ shows that $X$ is given by $\{(x, y, z), \delta \in \mathbb{C}^3 \times \Delta \mid 4xz - y^2 = P(\delta) \}$, all $SL_2$-invariant surfaces in $V_2$ being given as $4xz - y^2 = \text{const}$ after proper choice of coordinates. Thus, the claim is shown as well.
If all fibers are isomorphic to \( \mathbb{Q}^n_2 \) and \( V_{k_j} \neq V_2 \), then argue similarly: \( V_{k_j}^T \) is 1-dimensional and \( X^T = X \cap (V_{k_j}^T \times \Delta) \) is given by \( \{ -y^2 = P(\delta) \} \) where \( P \in \mathcal{O}^*(\Delta) \).

We show that \( X \) is isomorphic to \( \{(x, y, z, \delta) \in \mathbb{C}^3 \times \Delta | 4xz - y^2 = P(\delta) \} \). \( X_2 \subset V_2 \cap \Delta \). A linear identification of \( V_2^T \) and \( V_{k_j}^T \) yields an isomorphism between \( X^T \) and \( X_{k_j}^T \). Let \( \Gamma^T \subset (V^T_2 \times \Delta) \times (V^T_{k_j} \times \Delta) \subset (V_2 \times \Delta) \times (V_{k_j} \times \Delta) \) be the graph and set \( \Gamma := SL_2 \Gamma \subset (V_2 \times \Delta) \times (V_{k_j} \times \Delta) \). Now \( X^T \) and \( X_{k_j}^T \) both having isotropy group \( T \) at any point implies that \( \Gamma \) is the graph of a bijective morphism, i.e. an isomorphism between the (normal) varieties \( X \) and \( X_2 \).

If \( q^{-1}(y) \equiv \mathbb{F}_3^a \) or all fibers are isomorphic to \( \mathbb{Q}^n_2/\mathbb{Z}_2 \) one uses the same line of argumentation with the only difference that the \( SL_2 \)-invariant surfaces in the 5-dimensional representation space are given by the ideal

\[
3d^2 - 8ce + 4\delta e, \quad cd - 6be + \delta d
\]
\[
3bd - 48ae + 26c + 2\delta^2, \quad c^2 - 36ae + 2\delta e + \delta^2
\]
\[
bc - 6ad + \delta b, \quad 3b^2 - 8ac + 4\delta a.
\]

This variety is a quotient of \( \{4xz - y^2 = \delta \} \) by \( \mathbb{Z}_2 \), where the \( SL_2 \)-equivariant quotient map is given by \((x, y, z) \mapsto (x^2, 2xy, 2xz + y^2, 2yz, z^2)\).

The next lemma covers a special case which we will need to consider later.

**Proposition 3.4.** In the setting of proposition 3.1, if \( Y \cong \mathbb{C} \) and all fibers over \( Y \setminus \{0\} \) are isomorphic, then \( X \) is equivariantly isomorphic to

- \( \mathbb{P}^n_n \times \mathbb{C} \) where \( SL_2 \) acts on \( \mathbb{P}^n_n \) only
- \( X_k := \{(x, y, z) \in \mathbb{C}^3 \times \Delta | 4xz - y^2 = \delta^k \} \) where \( k \in \mathbb{N} \) and \( SL_2 \) acts on \( \mathbb{C}^3 \) via the 3-dimensional irreducible representation.
- a quotient of the latter by \( \mathbb{Z}_2 \), acting with weights \((1,1,1)\) on \( \mathbb{C}^3 \) and trivially on the base.

**Proof.** If all \( q \)-fibers are isomorphic to \( \mathbb{F}_3^a \), then proposition 3.1 shows the local triviality. Note that the only automorphisms of \( \mathbb{F}_3^a \) commuting with the \( SL_2 \)-action are in \( \mathbb{C}^* \). But \( H^1(\mathbb{C}, \mathbb{C}^*) \) is trivial so that the local trivializations glue together to give a global one.

If the generic fiber is \( \mathbb{Q}^n_2 \), then employ the same methods as in the proof of proposition 3.1: embed \( X \rightarrow \oplus V_{k_j} \times \mathbb{C} \) and assume that \((\pi_0 \times Id) : \oplus V_{k_j} \times \mathbb{C} \rightarrow V_{k_0} \times \mathbb{C} \) is an embedding, if restricted to \( q^{-1} \) of a neighborhood of \( y \). Choose a torus \( T \) and let \( X^T \subset V^T_{k_0} \) be the \( T \)-fixed point set. Assuming without loss of generality that \( y = 0 \), \((\pi_0 \times Id)(X^T) \) is given as \( \{-y^2 = c \cdot \delta^k \cdot \prod (y_j - \delta)^{m_j} \} \) where \( y_j \neq 0 \), \( \delta \) is to coordinate on \( Y \cong \mathbb{C} \) and \( c \neq 0 \) is a constant. We know that \( X^T \) is locally (analytically) reducible over each of the \( y_j \). It’s \((\pi_0 \times Id)\)-image is, as well. Thus, the \( m_j \) are even.

Let \( U_0 \subset \mathbb{C} \) be the maximal set such that all fibers are isomorphic to \( \mathbb{Q}^n_2 \). To construct an isomorphism \( X_k \rightarrow X \) over \( U_0 \), it is necessary to find an isomorphism between \( X^T \cap q^{-1}(U_0) \) and \( X^T_{k_j} := \{-y^2 = \delta^k \mid \delta \in U_0 \} \) and then apply the construction from the proof of proposition 3.1, involving the graph \( \Gamma \). Note that \( X^T \cap q^{-1}(U_0) \) and \( X^T_{k_j} \) are both smooth and have a birational morphism onto \((\pi_0 \times Id)(X^T)\), the latter being given by

\[
X^T_k \rightarrow (\pi_0 \times Id)(X^T)
\]
\[
(y, \delta) \mapsto \left( y \prod (y_j - \delta)^{m_j}, \sqrt{\delta} \right).
\]

Thus, they must be isomorphic. Now the construction gives an isomorphism over \( U_0 \).
If $U_0 \subset \mathbb{C}$ is not the whole of $\mathbb{C}$, then set $U_1 := \mathbb{C} \setminus \{y_1, \ldots, y_k\}$. Recall that $V_{k_0}$ is necessary 3-dimensional, equip it with coordinates $x$, $y$ and $z$ and set

$$
\{4xz - y^2 = \delta^k\} \to \{4xz - y^2 = \delta^k \cdot \prod (y_j - \lambda)^{m_j}\}
$$

$$
((x, y, z), \delta) \mapsto (\prod (y_j - \delta)^{m_j} (x, y, z), \delta)
$$

where $((x, y, z), \delta)$ are coordinates on $\mathbb{C}^3 \times \mathbb{C}$.

We have to show that the two local isomorphisms over $U_0$ and $U_1$ agree. Note that the only automorphisms of $\mathbb{P}^2_2$ commuting with the $SL_2$-action are in $\mathbb{Z}_2$. Now $H^1(\mathbb{C}, \mathbb{Z}_2)$ being trivial shows that after multiplying one of the local isomorphisms with $(-1)$, if necessary, we can always glue.

Again the analogous construction works if the generic fiber is isomorphic to $\mathbb{P}^2_2/\mathbb{Z}_2$.

\[\square\]

**Remark 3.5.** Propositions 3.1 and 3.4 could also be proved using elementary deformation theory, see [Pin74].

Now we describe certain quasi-projective varieties which will play an important role in the next chapter. For this, a categorical quotient of a quasi-projective variety is an invariant affine surjective morphism $q : X \to Y$ which is a categorical quotient on an affine cover of the base.

**Proposition 3.6.** Let $X$ be a quasi-projective normal complex $SL_2$-variety with at most terminal singularities and categorical quotient $q : X \to \mathbb{P}_1$. If every $q$-fiber over $\mathbb{P}_1^* \subset \mathbb{P}_1$ is isomorphic to $\mathbb{C}^2$, then the singularities of $X$ are of type $\frac{1}{2} (1, 1, -1)$ and $\frac{1}{2} (1, 1, -1)$ (i.e. are locally isomorphic to $\mathbb{C}^2/\mathbb{Z}_n$, where $\mathbb{Z}_n$ acts with weights $(1, 1, -1)$) and $X$ is toric. Here $n$ and $m$ are the multiplicities of the exceptional $q$-fibers.

**Proof.** Set $X^* := q^{-1}(\mathbb{P}_1^*)$. By proposition 3.1, $X^*$ is a locally trivial $\mathbb{C}^2$-bundle. Since the transition functions must commute with $SL_2$, they are in $O^*$, and $X^*$ must be the sum of two line bundles. However, Pic$(\mathbb{P}_1^*) = 0$, so that $X^*$ is isomorphic to the trivial bundle. Choose two different unipotent subgroups $U_1$, $U_2 < SL_2$ and two sections $\sigma_1$ and $\sigma_2$ in $X^*$ over $\mathbb{P}_1^*$ which do not have zeros and such that $\sigma_i$ is pointwise $U_i$-fixed. The $\sigma_i$ yield an isomorphism $X^* \to \mathbb{C}^2 \times \mathbb{P}_1^*$. Let $(\mathbb{P}_1^*)^3$ act on $X^*$ in these coordinates by

$$(r, s, t)((x, y), \lambda) \mapsto ((rx, sy), t\lambda)$$

Set $X^0 := q^{-1}(\mathbb{C})$ and let $n$ be the multiplicity of $q^{-1}(0)$. Let $\gamma : \tilde{X}^0 \to X^0$ be the nth root fibration. By proposition 3.4, $X^0$ is the trivial $\mathbb{C}^2$-bundle over $\mathbb{C}$. Choosing sections $\tau_i$ as above yields coordinates $((\tilde{x}, \tilde{y}), \lambda)$ on $\tilde{X}^0$. Then the closures of the pull-back of the $\sigma_i$ are given by $\tilde{\gamma}^{-1}(\sigma_1) = \{x = 0, y = \lambda^{k_1}\}$ and $\tilde{\gamma}^{-1}(\sigma_2) = \{x = \lambda^{k_2}, y = 0\}$ and the $(\mathbb{P}_1^*)^3$-action on $X^*$ pulls back to $(r, s, t)((\tilde{x}, \tilde{y}), \lambda) \mapsto ((t^{k_2}rx, r^{k_2}sy, t\lambda)$. In particular, the $(\mathbb{P}_1^*)^3$-action can be extended to the whole of $\tilde{X}^0$.

By construction, $X^0$ is a cyclic quotient of $\tilde{X}^0$ by the group $\mathbb{Z}_n$. If $(f, \lambda) \in \mathbb{C}^2 \times \mathbb{C}$ and $\xi \in \mathbb{Z}_n$ is a primitive root, then we may write without loss of generality $\xi(f, \lambda) = (a(\xi, \lambda)f, \xi \cdot \lambda)$, where $a(\xi, \lambda) \in Aut(\mathbb{P}_n_0)$, commuting with $SL_2$, i.e. $a(\xi, \lambda) \in \mathbb{P}_1^*$. Since there is no non-constant algebraic morphism from $\mathbb{C}$ to $\mathbb{C}^*$, $a(\xi, \lambda)$ does not depend on $\lambda$. We may thus view $\mathbb{Z}_n$ as acting on $\mathbb{C}^2 \times \mathbb{C}$ with weights $(a, a, 1)$. Note that the $\mathbb{Z}_n$-quotient map commutes with the action of $(\mathbb{P}_1^*)^3$, i.e. $X^0$ is toric. A similar argument over $\mathbb{P}_1 \setminus \{0\}$ shows that $X$ is toric.

In order to show that $X$ has singularities of type $\frac{1}{n}(1, 1, -1) = \frac{1}{a}(-1, -1, 1)$, i.e. that $a = 1$, note that the $a$ and $n$ must be coprime (this is because $\gamma$ is etale in codimension one). The classification theory of terminal singularities asserts that
only $a = -1$ is possible if the quotient is supposed to be terminal, see [Rei87, sect. 5.1]. Again, the same argument applies to $q^{-1}(\mathbb{P}_1 \setminus \{0\})$.

4. An Application

Our main goal here is to prove theorem 1.1, but first we construct the example which is actually realized.

**Example 4.1.** Let $X$ be the weighted projective space $\mathbb{P}_{(1,1,2,3)}$. By [Ful93, p. 35], identify $X$ with the toric variety $X$ whose fan is constructed from the vectors $e_1$, $e_2$, $e_3$ and $v = (-1, -2, -3) \in \mathbb{Z}^3$, i.e. whose cones are generated by any three of these vectors. We claim that $X$ has $\mathbb{Q}$-factorial terminal singularities and Picard-number $\rho(X) = 1$. Furthermore, $SL_2$ acts on $X$ such that the generic orbit is 2-dimensional.

**X has $\mathbb{Q}$-factorial terminal singularities:** The cones generated by $(e_1, e_2, e_3)$ and $(e_2, e_3, v)$ describe smooth varieties. The cone generated by $(e_1, e_2, v)$ can be brought into the form $(e_1, e_2, (-1, -2, 3))$ using the matrix $\text{Diag}(1, 1, -1) \in GL(2, \mathbb{Z})$. The latter cone is known (see [Ful93, p. 35]) to describe the singularity $\mathbb{C}^3/\mathbb{Z}_3$, where $\mathbb{Z}_3$ acts with weights $(1, 1, -1)$. Analogously,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} (e_1, e_3, v) = (e_1, e_3, (-1, 2, -1)).$$

This describes the singularity $\mathbb{C}^3/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts with weights $(1, 1, 1)$. Both singularities are terminal —compare [Rei87, thm. on p. 379].

**X has Picard-number one:** By [Ful93, p. 64f], $\text{Pic}(X) = \mathbb{Z}$.

**SL$_2$-action:** If $[x : y : z : u]$ are weighted homogeneous coordinates associated to the weights $(1, 1, 2, 3)$, let $SL_2$ act on $x$ and $y$ via the 2-dimensional representation.

Since the proof of theorem 1.1 is rather long, we subdivide it into a number of lemmata. For the rest of this paper, we use the notation of theorem 1.1 without mentioning it further.

**Lemma 4.2.** In the situation of theorem 1.1, $S \cong SL_2$.

**Proof.** If it was not, then the generic $S$-orbit is a homogeneous divisor. Two of them don’t intersect, a contradiction to $\rho(X) = 1$.

**Lemma 4.3.** Every irreducible $S$-invariant divisor $D_\alpha \subset X$ is $S$-quasihomogeneous and it’s normalization is isomorphic to either

1. the projective cone over the rational normal curve of degree $n$, $\mathbb{P}_n$, where $S$ acts with a fixed point (this includes $\mathbb{P}_2$),
2. $\mathbb{P}_2$, where $S$ acts via the 3-dimensional irreducible $SL_2$-representation, or
3. the Hirzebruch surface $\Sigma_0$, where $S$ acts diagonally

There exists a curve $C \subset X$, $C \cong \mathbb{P}_1$ such that (set-theoretically) $C = D_\alpha \cap D_\beta$ for all $S$-invariant divisors $D_\alpha$ and $D_\beta$. Furthermore, every irreducible $S$-invariant divisor is locally (analytically) irreducible at any point of $C$.

**Proof.** We have remarked at the very beginning that $D_\alpha$ cannot be pointwise $S$-fixed. Suppose that the generic $S$-orbit in $D_\alpha$ was 1-dimensional. Then $D_\beta$ intersects $D_\alpha$ in finitely many orbits and therefore has empty intersection with a generic $S$-orbit in $D_\alpha$, a contradiction to $\rho(X) = 1$. Consequently, $D_\alpha$ is $S$-quasihomogeneous and if $\eta : D_\alpha \to D_\alpha$ is the normalization, then classification
(see [Huc86]) shows that $\tilde{D}_\alpha$ must be isomorphic to a variety in the list, or to $\Sigma_n$, $n > 0$ with $S$ stabilizing two sections.

Due to $\rho(X) = 1$ there is a number $k \in \mathbb{N}$ such that $kD_3$ is CARTIER and ample. In particular, $\eta^*(kD_3)$ is an effective ample divisor with $S$-invariant support. This is possible in all cases save $\Sigma_n$, $n > 0$. In the allowed cases, there is a unique irreducible $S$-invariant curve $C \subset \tilde{D}_\alpha$ which yields the assertion. \hfill $\square$

One of the key points in the proof of theorem 1.1 is the following local description of the $D_\alpha$ in the neighborhood of $C$.

**Lemma 4.4.** Let $x \in C$ and $T < S$ be a torus fixing $x$. Then there exists a 2-dimensional $T$-representation space $E$ with positive weights, a neighborhood $V$ of $0 \in E$, a neighborhood $\Delta$ of $0 \in \mathbb{C}$ and an immersion $\phi : V \times \Delta \to U \subset X$ with the following properties:

1. $\phi^{-1}(C) = \{0\} \times \Delta$
2. $D_\alpha \subset X$ is an $S$-invariant divisor if and only if there exists a $T$-invariant curve $N \subset E$ such that $\phi^{-1}(D_\alpha) = N \times \Delta$.

**Proof.** Let $U \subset T_xX$ be a sufficiently small neighborhood of $0$ and let $\lambda : U \to X$ be a linearization of the $T$-action. As $C$ is smooth, the tangent space $T_xC \subset T_xX$ coincides with one of the weight spaces. Take two weight vectors $v_1$ and $v_2 \in T_xX$ which, together with $T_xC$ span the whole space $T_xX$. Let $E$ be the space spanned by $v_1$ and $v_2$.

As a first step, we claim that after replacing $T$ by $T^{-1}$, if necessary, all weights of the $T$-action on $T_xX$ are positive. In order to show this it is sufficient to show that for generic $y \in U$ the $T$-orbit $Ty$ contains $x$ in the closure. This, however, is true because $\lambda(y)$ is contained in an $S$-invariant divisor and $x \in T_y$ by the classification of lemma 4.3.

In order to construct the map $\phi$, note that $\lambda|_V$ is immersive. The image of the tangential map $T(\lambda|_V)$ is transversal to $T_xC$. Let $H < S$ be a unipotent one-parameter group not fixing $x$. The associated vector field, evaluated at $x$ is contained in $T_xC$ so that the map $\phi : V \times H \to X \quad (v, h) \mapsto h \cdot \lambda(v)$ has maximal rank at $(0, 0)$. Thus, $\phi$ is invertible in a small neighborhood. Property (1) holds by construction.

In order to show property (2) it is sufficient to consider irreducible $D_\alpha$. Claim that $D_\alpha$ is $S$-invariant if and only if there exists a point $y \in D_\alpha \cap U$ such that $D_\alpha = \overline{H.T.y}$. Indeed, if $D_\alpha$ is $S$-invariant, then it contains $C$, and therefore also a point $y \in D_\alpha \cap \lambda(V) \setminus C$ and $\overline{T.y}$ is necessarily a curve containing $x$ in the closure. Therefore $\overline{T.y}$ is not $H$-invariant and $\overline{H.T.y}$ is an irreducible component of $D_\alpha$, hence equal to $D_\alpha$. This shows already that if we set $N := T.\phi^{-1}(y)$, then $\phi(N \times \Delta)$ is contained in $D_\alpha$. If $\phi^{-1}(D_\alpha) \neq N \times \Delta$, then it contains another irreducible component, a contradiction to the local irreducibility of $D_\alpha$. \hfill $\square$

We utilize the local description to draw conclusions concerning the global configuration of the divisors $D_\alpha$.

**Corollary 4.5.** There are at least two different $S$-invariant divisors $D_0$ and $D_\infty$ in $X$ which are smooth along $C$. Unless $X$ is isomorphic to the smooth 3-dimensional quadric $\mathbb{Q}_3$, to $\mathbb{P}_3$ or to $\mathbb{P}(1,1,1,2)$, the normalizations are isomorphic to $\tilde{D}_0 \cong \mathbb{F}_n$ and $\tilde{D}_\infty \cong \mathbb{F}_m$ with $n, m > 1$. If $\tilde{D}_\alpha$ is the normalization of a generic $S$-invariant divisor, then either

1. $m$ and $n$ are coprime and $\tilde{D}_\alpha \cong \mathbb{P}_2$ where $S$ acts with a fixed point, or
2. $m$ and $n$ are even, $\frac{m}{q}$, $\frac{n}{q}$ are coprime and $\tilde{D}_\alpha \cong \Sigma_0$, or
3. $m$ and $n$ are divisible by four, $\frac{m}{q}$, $\frac{n}{q}$ are coprime and $\tilde{D}_\alpha \cong \mathbb{P}_2$ where $S$ acts via the 3-dimensional irreducible representation.

**Proof.** Taking $N$ to be one of the weight spaces in $E$, lemma 4.4 immediately yields $D_0$ and $D_\infty$. Note that by lemma 4.3 two $S$-invariant divisors intersect in $C$ only so that the generic $S$-invariant divisor $D_\alpha$ does not meet the singular set of $X$. Use the standard argument linearizing the $S$-action at a fixed point to exclude the possibility that $D_\alpha \cong \mathbb{P}_n$ where $n > 1$. Thus, $D_\alpha$ is smooth away from $C$.

Secondly, remark that if $X$ is a cone then the classification from [Mor82, thm. 3.3 and cor. 3.4] yields that $X \cong \mathbb{P}_3$ or $\mathbb{P}_{(1,1,1,2)}$ if $X$ is assumed to have $\mathbb{Q}$-factorial and terminal singularities; note that a cone over $\Sigma_0$ is never $\mathbb{Q}$-factorial as there are Weyl-divisors intersecting in a single point.

Recall that $X$ is isomorphic to a cone or to $\mathbb{Q}_3$ if there is a CARTIER divisor in $X$ which is isomorphic to $\mathbb{P}_2$ or $\Sigma_0$; see [Bád82, thms. 1 and 5] for the cases that $X$ is smooth or that $D_\alpha \cong \mathbb{P}_2$ and [Bád84, thm. 3] for the remaining case.

Thus, excluding this case amounts to saying that the normalizations of $D_0$ and $D_\infty$ are isomorphic to $\mathbb{P}_*$, since otherwise $D_\alpha \setminus C$ would be homogeneous, would not intersect the (finite) singular set of $X$ and would thus be CARTIER. The indices $n$ and $m$ are exactly the weights of the $T$-action on $E$, as given by lemma 4.4. The possible weights of the $T$-action on the $SL_2$-quasihomogeneous surfaces $D_\alpha$ (see the classification of lemma 4.3) and the fact that $N$ must be singular give conditions (1)–(3).

Note that the set of semi-stable points with respect to the unique lifting of the $SL_2$-action to $\mathcal{O}(D_\alpha)$ is $X \setminus C$. Let $q : X \setminus C \to Y$ denote the resulting quotient in the sense of geometric invariant theory.

**Corollary 4.6.** We have $Y \cong \mathbb{P}_1$ and either $X \cong \mathbb{Q}_3$, $\mathbb{P}_3$ or $\mathbb{P}_{(1,1,1,2)}$ or and there are points $0, \infty \in \mathbb{P}_1$ such that $q^{-1}(0) = n'D_0$, $q^{-1}(\infty) = m'D_\infty$ and all other $q$-fibers are reduced. Here $n' = n$, $\frac{n}{2}$ or $\frac{n}{4}$, according to the cases of corollary 4.5; $m'$ similarly.

**Proof.** The description of lemma 4.4 guarantees that the quotient map extend to a rational map $X \dasharrow Y$ which becomes regular if we perform a weighted blow-up of $C$ with weights $n$ and $m$. Since the exceptional set of this blow-up is rational, $Y$ is, as well. Thus $Y \cong \mathbb{P}_1$. In particular, all $q$-fibers are linearly equivalent, and the $D_\alpha$ are linearly equivalent up to positive multiplicities.

In order to see that all the $D_\alpha$ have multiplicity 1 as $q$-fibers, it is sufficient to see that the divisors $D_\alpha$ are linearly equivalent. By lemma 4.4, $D_\alpha$ is locally given by a curve $N$ having the equation $x^{(n')} = y^{(m')}$, $D_0 = \{x = 0\}$ and $D_\infty = \{y = 0\}$. Thus,

$$D_0.D_\alpha = n'C, \quad D_1.D_\alpha = n'C, \quad D_\alpha.D_\alpha = n'm'C.$$  

Consequently

$$D_0 \sim \frac{1}{n'}D_\alpha \quad \text{and} \quad D_\infty \sim \frac{1}{m'}D_\alpha$$

as $\mathbb{Q}$-divisors. This finishes the proof.

**4.1. Proof the Theorem 1.1.** With these preparations we start the proof the main theorem 1.1. If $X \cong \mathbb{Q}_3$, $\mathbb{P}_3$ or $\mathbb{P}_{(1,1,1,2)}$, we can stop here. Otherwise, we are in one of the cases (1)–(3) or corollary 4.5. We treat these cases separately.

**Proof of 1.1 in case (1) of corollary 4.5.** By proposition 3.1, all $q$-fibers over $\mathbb{P}_1 \setminus \{0, \infty\}$ are isomorphic to $\mathbb{C}^2$. By proposition 3.6, $X$ has two singularities of type $\frac{1}{m}(1,1,-1)$ and $\frac{\infty}{m}(1,1,-1)$, and $X \setminus C$ is toric. Since $X$ is smooth along $C$, the associated vector fields extend to $X$, showing that $X$ is toric, too.
Consequence: $X$ can be given as a fan in $\mathbb{Z}^3$. Let $\sigma_1$ and $\sigma_2 \subset \mathbb{Z}^3$ be the cones describing the smooth $(\mathbb{C}^*)^3$-fixed points on $C$, $\sigma_3$ describe the point $\frac{1}{n}(1,1,-1)$ and $\sigma_4$ be associated to $\frac{1}{m}(1,1,1)$. There can be no further fixed points.

Choose coordinates such that $\sigma_1$ is spanned by the unit vectors $(e_1,e_2, e_3) \subset \mathbb{Z}^3$. Because every cone is spanned by four rays, there must be a vector $v = (a, b, c) \in \mathbb{Z}^3$ such that $\sigma_2, \sigma_3$ and $\sigma_4$ are spanned by $2$ unit vectors and $v$ each. After renaming the $e_i$, if necessary, assume that $\sigma_2 = (e_1, e_2, v), \sigma_3 = (e_1, e_3, v)$ and $\sigma_4 = (e_2, e_3, v)$.

We will find out the possibilities for $v$. First, note that two cones must not intersect in anything but a face. Thus, $a$, $b$ and $c$ must be negative. We use the local description of the singularities:

- $\sigma_2$ is smooth: consequently, $(e_1, e_2, v)$ must be a basis of $\mathbb{Z}^3$ and $c = -1$.
- $\sigma_3$ is $\frac{1}{n}(1,1,-1)$: It is known (see [Ful93, p. 35]) that the cone generated by $(e_1, e_3, -(n-1)e_1 + ne_2 - e_3)$ corresponds to a singularity of type $\frac{1}{n}(1,1,-1)$. Thus, there exists a $g \in GL(3, \mathbb{Z})$ such that $g(e_1, e_3, -(n-1)e_1 + ne_2 - e_3) = (e_1, e_3, (a, b, -1))$. Calculating the product

\[
\begin{pmatrix}
1 & \alpha & 0 \\
0 & \beta & 0 \\
0 & \gamma & 1 \\
\end{pmatrix}
\begin{pmatrix}
-(n-1) \\
1 \\
-1 \\
\end{pmatrix}
= \begin{pmatrix}
-n + 1 + \alpha n \\
\beta n \\
\gamma n - 1 \\
\end{pmatrix}
\]

yields $\gamma = 0$, and $a \in \mathbb{Z}n + 1$. Since $\text{det } g = \pm 1$, $\beta \in \pm 1$. The inequality $b < 0$ gives $b = -n$.

- $\sigma_4$ is $\frac{1}{m}(1,1,1)$: Similar to the above there is a $g \in GL(3, \mathbb{Z})$ such that $g(e_2, e_3, me_3 - (m-1)e_2 - e_3) = (e_1, e_3, (a, b, -1))$. The same calculation shows $a = -m$ and $b \in \mathbb{Z}m + 1$.

Summarizing the above, we need to find all $n$ and $m$ such that there are numbers $\mu, \nu \in \mathbb{Z}$ with

\[
\begin{align*}
\text{(1)} & \quad m = \mu n - 1 \\
\text{(2)} & \quad n = \nu m - 1 \\
\end{align*}
\]

By assumption, $n$ and $m$ are coprime so that we can always assume without loss of generality that $n > m > 1$. Then equation 1 holds iff $\mu = 1$ and $m = n - 1$. Inserting this into equation 2 gives $m(\nu - 1) = 2$ which in turn implies $m = 2$. Now compare $v$ to the description of example 4.1.

Proof of 1.1 in case (2) of corollary 4.5. Let us begin by giving a detailed description of this case over a trivialization. Set $X_0 := X \setminus D_\infty$. By corollary 4.6, $q^{-1}(0)$ has support on $D_0$ and multiplicity $n'$; this is the only $q$-fiber with non-trivial multiplicity over $\mathbb{C}$. Let $\tilde{q} : X_0 \to \mathbb{C}$ be the $n'$th root fibration associated to $q : X^0 \to \mathbb{C}$. Now $X_0$ is a quotient of $X_k$ by the cyclic group $\mathbb{Z}_{n'}$ acting freely in codimension 1. Choose an analytic disk $\Delta \subset \mathbb{C}$ around 0 which is $\mathbb{Z}_{n'}$-invariant. Then, after proper choice of coordinates, $\tilde{q}^{-1}(\Delta) \cong \{4x^2 - y^2 = \delta^k\}$ as ensured by proposition 3.1.

It is elementary to see that every automorphism of $\tilde{q}^{-1}(\Delta)$ over $\Delta$ commuting with $SL_2$ is given by $((x, y, z), \delta) \to (\pm t^\delta(x, y, z), t\delta)$ for some $t \in \mathbb{C}^*$. Thus, the action of $\mathbb{Z}_{n'}$ on $X_k$ extends to $\mathbb{C}^3 \times \Delta$ where $\tilde{q}^{-1}(\Delta)$ is $SL_2$-equivariantly embedded. Since the action of $\mathbb{Z}_{n'}$ must commute with $SL_2$, the weights of the $\mathbb{Z}_{n'}$-action on $\mathbb{C}^3 \times \{0\}$ must be equal.

An analogous construction can be given at $\infty$. We now show that $n'$ and $m'$ are not coprime. This contradicts the assumption.

Consider the following cases:

- $k = 0$: In this case there is no $SL_2$-fixed point in $\tilde{q}^{-1}(\Delta)$, and consequently none on $D_0$. Thus, $X$ must be a cone or $\mathbb{Q}_3$; see the proof of corollary 4.5 for this. A contradiction to the assumption.
\( k = 1 \): In this case \((x, y, z)\) are coordinates for \(\tilde{X}\). The quotient is terminal iff the weights are of the form \((a, -a, 1)\) (see [Rei87, sect. 5.1]). Thus, \(n' = 2\).

\( k > 1 \): Note that there is no \(\mathbb{Z}_{n'}\)-fixed subspace in \(\mathbb{C}^3 \times \Delta\). In this situation [Mor85, Thm. 12] shows that \(n'\) must be 4.

Now apply the same argumentation to \(D_\infty\) and realize that the coprimeness assertion of corollary 4.5 is necessarily violated. This yields the claim. \(\square\)

**Proof of 1.1 in case (3) of corollary 4.5.** As before, set \(X^0 := X \setminus D_\infty\) and consider the divisor \(L := K_{X^0} - D_0\). By adjunction formula, \(L|_{D_0} = K_{D_0}\) which has index \(\frac{q}{2}\). Thus, the index of \(L\) in \(X^0\) is in \(\frac{q}{2}\mathbb{N}\). Now perform the cyclic cover associated to \(L\) (see [Rei80, cor. 1.9] for details): \(\gamma : X^0 \to X^0\). Stein factorization gives a diagram

\[
\begin{array}{ccc}
\tilde{X}^0 & \xrightarrow{\gamma} & X^0 \\
\downarrow q & & \downarrow q \\
\tilde{Y} & \xrightarrow{\gamma} & Y
\end{array}
\]

where we can choose \(\tilde{Y}\) to be normal, hence smooth. We are interested in the preimage of \(D_0\). First, note that every vector field on \(X^0 \setminus \text{Sing}(X^0)\) can be lifted to \(\tilde{X} \setminus q^{-1}(\text{Sing}(X^0))\). Since \(\tilde{X}\) is normal, we obtain an action of the associated 1-parameter group on \(\tilde{X}\). In particular, since \(S\) is simply connected, \(S\) acts on \(\tilde{X}^0\) in a way that \(\gamma\) is equivariant.

As a next step we need to show that \(q^{-1}(0)\) is reduced. By corollary 4.6, \(q^{-1}(0)\) has multiplicity \(n' = \frac{q}{2}\). On the other hand, generic \((\gamma \circ q)\)-fibers have at least \(\frac{q}{2}\) components. This is due to the fact that \(D_\alpha \cong \mathbb{P}_2\), where \(SL_2\) acts without fixed point, admits only \(\Sigma_0\) as a connected \(S\)-equivariant cover. Consequence: \(q^{-1}(0) = \gamma^{-1}(D_0)\) is reduced and isomorphic to \(\mathbb{P}_2^1\). Now apply the argumentation from the proof in case (2). \(\square\)

**References**

[Arz98] I. V. Arzhantsev. On \(SL_2\)-actions of complexity one. Izvestiya RAN, 61(4):685–698, 1998.

[Bád82] L. Bădescu. On Ample Divisors. Nagoya Math. J., 86:155–171, 1982.

[Bád84] L. Bădescu. Hyperplane Sections and Deformations. In L. Bădescu and D. Popescu, editors, *Algebraic Geometry Bucharest 1982*. volume 1056 of Lecture Notes in Mathematics. Springer, 1984.

[Eis95] D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*, volume 150 of Graduate Texts in Mathematics. Springer, 1995.

[Ful93] W. Fulton. *Introduction to Toric Varieties*. Number 131 in Annals of Mathematics Studies. Princeton University Press, 1993.

[Ge82] A. Grothendieck. Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA2), volume 2 of Advanced studies in pure mathematics. North-Holland Publ. Co., Amsterdam, 1962.

[HO80] A. Huckleberry and E. Oeljeklaus. Classification Theorems for Almost Homogeneous Spaces. Institut Elie Cartan, 1980.

[Huc86] A. Huckleberry. The classification of homogeneous surfaces. Expo. Math., 4, 1986.

[Keb98] S. Kebekus. Relatively Minimal Quasihomogeneous Projective 3-Folds. to appear, 1998.

[Kra85] H. Kraft. Geometrische Methoden in der Invariantentheorie, volume D1 of Aspekte der Mathematik. Vieweg, 1985.

[LV83] D. Luna and T. Vust. Plongements d’espaces homogènes. Comment Math. Helv., 58:245–245, 1983.

[Mor82] S. Mori. Threefolds whose canonical bundles are not numerically effective. Ann. of Math., 116, 1982.

[Mor85] S. Mori. On 3-Dimensional Terminal Singularities. Nagoya Math. J., 98:43–66, 1985.

[Pin74] H. Pinkham. Deformations of Cone with Negative Grading. Journal of Algebra, 30, 1974.

[Rei80] M. Reid. Canonical 3-folds. In Arnaud Beauville, editor, *Algebraic Geometry Angers 1979*. Alphen aan den Rijn, 1980. Sijthoff & Noordhoff.

[Rei87] M. Reid. Young Person’s Guide to Canonical Singularities. *Proceedings of Symposia in Pure Mathematics*, 46, 1987.
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