Mirror Manifolds: A Brief Review and Progress Report

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We first give a complete, albeit brief, review of the discovery of mirror symmetry in $N = 2$ string/conformal field theory. In particular, we describe the naturality arguments which led to the initial mirror symmetry conjectures and the subsequent work which established the existence of mirror symmetry through direct construction. We then review a number of striking consequences of mirror symmetry – both conceptual and calculational. Finally, we describe recent work which introduces a variant on our original proof of the existence of mirror symmetry. This work affirms classical–quantum symmetry duality as well as extends the domain of our initial mirror symmetry construction.

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1. Introduction

Classical solutions for string theory are described by conformal field theories. The most interesting solutions, for fundamental as well as “phenomenological” reasons, are those which yield spacetime supersymmetric theories: $N = 2$ superconformal models. Superconformal field theories which arise from sigma models on appropriately chosen target spaces comprise an especially interesting category of vacuum solutions. These solutions which admit a geometrical interpretation are extremely useful probes of the unusual and sometimes remarkable properties of string theory. The reason for this is clear: with a geometrical interpretation in hand, we can perform direct comparisons between the string description and more conventional analyses based on general relativity and/or quantum mechanics.

A striking feature of string theory which emerges from this analysis is the existence of duality transformations. These transformations are symmetries of string physics which, when interpreted as geometrical operations, relate distinct geometrical configurations. The simplest example of this notion is provided by $c = 1$ string theory. With the geometrical interpretation of this solution as string propagation on a circle, one finds the surprising conclusion that physics is invariant under $R \to 1/2R$ where $R$ is the radius of the circle [1]. This phenomenon has no analog in conventional point particle quantum mechanics nor in general relativity in such a context. The physical interpretation of this symmetry is related to the existence of “winding modes” in which the string wraps around the spacetime; this feature of string physics may be generalized to toroidal spacetimes and is an important signature of truly stringy physics.

Toroidal spacetimes are in fact a small subset of a class of classical string vacua for which a geometrical interpretation is known – the class of Calabi-Yau manifolds. Mirror symmetry provides an extremely robust generalization of such ideas into this far larger class of compactifications. It identifies topologically distinct string backgrounds which give rise to identical physics. By providing a link between a priori unrelated manifolds, mirror symmetry establishes a powerful conceptual and calculational tool from the points of view of both physics and mathematics. Although remarkable from the mathematical vantage point (and quite powerful, as we shall discuss) the existence of mirror manifolds is rather natural from the point of view of physics. In fact, several groups, based on ‘naturality’ arguments, had earlier speculated on the possibility of mirror symmetry. Let us now turn to the motivation for such speculation.
There are two types of moduli on a Calabi-Yau manifold: those associated with the complex structure and those associated with the Kähler structure. The former are described by the cohomology group $H^{2,1}$ while the latter by $H^{1,2}$. From the mathematical point of view these are vastly different objects. Intuitively, the complex structure moduli parametrize deformations of the shape of the Calabi-Yau manifold while the Kähler moduli determine its size. These deformations of the manifold are described by deformations of the associated conformal field theory by marginal operators. Like the deformations of the related geometrical structure, these come in two varieties: those with $(\Delta = \overline{\Delta} = 1/2; \quad Q = \overline{Q} = 1)$ and those with $(\Delta = \overline{\Delta} = 1/2; \quad Q = -\overline{Q} = 1)$, where $\Delta$ denotes the conformal weight and $Q$ denotes the charge under the $U(1)$ subgroup of the $N = 2$ superconformal algebra. Unlike their geometrical counterparts, the difference between these two kinds of operators has no intrinsic significance. The sign of the left moving $U(1)$ charge is essentially a matter of convention. When we identify this conformal field theory with string propagation on a Calabi-Yau manifold, we must identify one of these two sets of fields with elements in $H^{1,1}$ and the other set of fields with elements in $H^{2,1}$. It seems somewhat strange that geometry so greatly exaggerates the difference between these two kinds of conformal fields.

The above led Dixon to speculate that a more natural state of affairs would ensue if there were two geometrical interpretations of the conformal field theory which differ by interchanging the identifications of conformal fields with differential forms. In this way, both a priori possible identifications of conformal fields with differential forms would be realized. The naturalness of this speculation was elaborated upon by Lerche, Vafa and Warner (see also [4]) who found that the marginal operators discussed above (or rather, their lower components) are elements in two distinct rings which are present in any $N = 2$ superconformal theory. What these authors found is that in models with a geometrical interpretation, one of these two rings is isomorphic to a deformation of the cohomology.

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1 We emphasize Calabi-Yau threefolds for most of our discussion as this is the most natural dimension from the string theoretic point of view. Essentially all of our results, however, immediately extend to higher dimension.

2 We recall that when describing string propagation we are naturally led to a complexification of the Kähler cone when we augment the metric with the background antisymmetric tensor field $B$.

3 The quantum numbers given here refer to the lower component of the supermultiplet to which the given operator belongs.
ring of the underlying manifold. The other ring has no obvious geometric interpretation. As pointed out in [3], it seems strange that only one conformal field theory ring has a geometrical interpretation. A more natural situation would again follow if there were two Calabi-Yau manifold interpretations of such a conformal field theory in which the identification of fields with forms would be interchanged. Thus, each manifold would provide the geometrical interpretation for one of the two conformal field theory rings.

These speculations based on naturality from the point of view of physics should be juxtaposed with similar reasoning from the viewpoint of mathematics. The hypothesized pairing of manifolds described above implies the existence of Calabi-Yau manifolds whose (2, 1) and (1, 1) Hodge numbers are interchanged. For a variety of technical reasons (including the supposed unlikely existence of Calabi-Yau manifolds with $h^{1,1}$ as large as some of the known values of $h^{2,1}$) it was felt by a number of mathematicians [20] that it would be rather unnatural for such pairs to exist.

These conflicting intuitions were partially resolved by solid arguments in [5] in which the existence of conformal field theories with two geometrical interpretations, in the sense described above, was proven. For a particular class of conformal field theories (those based on products of minimal models and deformations thereof), we explicitly found the dual geometrical interpretations in terms of manifolds differing by the interchange of $H^{1,1}$ and $H^{2,1}$. Because this interchange corresponds to a reflection of the manifold’s Hodge diamond (which is equivalent to a ninety-degree rotation of the diamond), we gave the name mirror manifolds [5] to such a pair of Calabi-Yau spaces. We will review this work in the next section.

The arguments of [5] are valid for a rather restricted class of Calabi-Yau manifolds, but there is strong evidence that the existence of mirror manifolds is not. Simultaneous with the above work, Candelas, Lynker and Schimmrigk [6] completed a thorough computer search of Calabi-Yau manifolds in weighted projective four space initiated in [7]. They found that the set of all such constructions is almost invariant under the interchange of $h^{1,1}$ and $h^{2,1}$. While finding two manifolds differing by the interchange of these Hodge numbers

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4 These two rings are shown in [3] to be connected via spectral flow. Thus, another way of phrasing the speculation in [5] is that if spectral flow could be given a geometrical interpretation, then it would produce the desired pair of Calabi-Yau spaces. As yet, no such geometrical interpretation of spectral flow has been found.

5 Of course, giving rise to the same conformal field theory (the same physics) implies more than this, as we will discuss.
numbers by no means assures that the manifolds are a mirror pair (to be a mirror pair they must correspond to the same conformal field theory) this data indicates that mirror duality probably extends beyond the particular class of models treated in [5]. We have subsequently shown [21] that at least some of the candidate mirror pairs found in [3] do in fact correspond to isomorphic conformal field theories, but there are unexpected subtleties in the argument, which will be presented in section four.

The existence of the mirror manifold \( M' \) implies some rather strong and sometimes startling conclusions regarding the physics and mathematics of \( M \). Section three is devoted to a discussion of the most striking of these, and to a number of recent papers [8-10] in which these results have been verified and elaborated upon by various groups. Section five offers some brief conclusions and mentions some open issues in mirror symmetry.

2. Constructing Mirror Manifolds

The basic strategy utilized in [3] for the construction of mirror manifolds is straightforward to describe. We begin with a class of conformal field theories whose geometrical interpretation in terms of string propagation on Calabi-Yau spaces has been known for some time [11,12]: tensor products of \( N = 2 \) minimal model conformal field theories (with an appropriate projection onto states with integral \( U(1) \) charges [11]). These were shown in [12] to correspond to Calabi-Yau hypersurfaces in weighted projective space.

Our strategy for building mirror manifolds is to seek a nontrivial automorphism of the conformal field theory which is not an automorphism of the geometrical description. Rather, we hope that this sought for mapping will yield an isomorphic conformal theory while producing a new corresponding Calabi-Yau manifold, topologically distinct from the initial one.

How can we find such a mapping? Recall from our earlier discussion that it is only the relative sign of the left moving \( U(1) \) charge which distinguishes amongst the two kinds of conformal field theory moduli. Imagine the consequences of an operation \( O \) on the conformal field theory \( K \) which generates an isomorphic theory with the explicit isomorphism requiring flipping the sign of all left moving \( U(1) \) charges. To do so, let us further imagine that this operation descends to the geometrical description and hence has a legitimate interpretation as an action on the initial Calabi-Yau space, \( M \). By legitimate here we mean, of course, that the range of our operation is contained within the space of Calabi-Yau manifolds. Now we can extract the consequences: Both \( M \) and \( O(M) \) correspond to the same
conformal field theory since $K$ and $\mathcal{O}(K)$ are isomorphic. Furthermore, since the explicit isomorphism between $K$ and $\mathcal{O}(K)$ is the reversal of the sign of the left moving $U(1)$ charges, the identification of conformal fields with differential forms is reversed on $\mathcal{O}(M)$ relative to $M$. Thus $M$ and $\mathcal{O}(M)$ would constitute a mirror pair, if such an operation $\mathcal{O}$ could be found. In particular, we would have

$$h^{2,1}_M = h^{1,1}_{\mathcal{O}(M)} \quad (2.1a)$$
$$h^{1,1}_M = h^{2,1}_{\mathcal{O}(M)} \quad (2.1b)$$

In [5] such an operation $\mathcal{O}$ was constructed. Its existence relies on the notion of ‘orbifolding’. If a manifold $M$ is invariant under a group of symmetries $G$ (in our case these will generally be holomorphic automorphisms) we can consider the quotient space $M' = M/G$. We restrict attention to groups $G$ which ensure that $M'$ is also Calabi-Yau. Such $G$ are called ‘allowable’ [6]. Similarly, if a conformal theory $K$ respects a symmetry group $G$, we can consider the quotient theory $K/G$. The crucial property of these operations for our argument, is that the quotient conformal theory describes propagation on the quotient of the underlying Calabi-Yau manifold by the same allowable group actions $G^7$. The correspondence between group actions in the two pictures is furnished by the identification of homogeneous coordinates on the manifold with primary fields in the conformal field theory given by the arguments of [12].

We can now describe the sought for operation: $\mathcal{O}$ is taken to be orbifolding by the maximal allowable $G$ which is a subgroup of the group of holomorphic scaling symmetries. We will not reproduce the proof of this statement here, referring the interested reader to [5] for details, but will attempt to sketch the salient points. To begin, we leave geometry and focus on a single $N = 2$ minimal model at level $P$. Such a theory respects a $\mathbb{Z}_{P+2}$ scaling symmetry. Furthermore, the effect of orbifolding by this $\mathbb{Z}_{P+2}$ is to yield an isomorphic theory with the explicit isomorphism consisting of the reversal of all left moving $U(1)$ charges [22]. We can now apply this fact to $(U(1)$ projected) tensor products of minimal

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6 We note that $G$ is allowable if it preserves the holomorphic $n$–form present on the initial Calabi-Yau $n$–fold.

7 To be more precise, the quotient will have singularities at the fixed points of the $G$ action. The orbifold conformal field theory corresponds to a singular limit of an appropriate desingularization of the quotient. The condition on $G$ described above assures us of the existence of a desingularization which is itself a Calabi-Yau manifold.
models at levels, say, $P_1, \ldots, P_5$. (For more general cases see [5].) We denote the $U(1)$ projected theory by $(P_1, \ldots, P_5)$ and we recall [12] the conformal theory – Calabi-Yau identification

\[(P_1, P_2, P_3, P_4, P_5) \rightarrow \mathbb{Z}_{P_1+2} + \mathbb{Z}_{P_2+2} + \mathbb{Z}_{P_3+2} + \mathbb{Z}_{P_4+2} + \mathbb{Z}_{P_5+2} = 0 \quad (2.2)\]

where the right-hand side is a Calabi-Yau hypersurface in weighted projective four space $\mathbb{C}^4 \mathbb{P}^4_{d_1+2, d_2+2, d_3+2, d_4+2, d_5+2}$, where $d$ is the degree of homogeneity of the defining polynomial. Both sides respect a group of scaling symmetries $S = \mathbb{Z}_{d_1+2} \times \cdots \mathbb{Z}_{d_5+2}$. Let $G \subset S$ be the maximally allowable subgroup of $S$. We claim that $K/G$ is isomorphic to $K$ under the isomorphic mapping which flips the sign of all left-moving $U(1)$ charges. Furthermore, following our discussion above, $M' = M/G$ is a new topologically distinct Calabi-Yau manifold which corresponds to the same conformal theory; in particular $h_{M'}^{1,1} = h_M^{1,1}$ and $h_{M'}^{1,1} = h_M^{2,1}$. Our remarks in the previous paragraph regarding a single minimal model hint at the underlying justification of our assertion. However, passing from $S$ to $G$ and implementing the $U(1)$ projection are additional complications over the single model case which must be dealt with. In fact, each complication turns out to be crucial in resolving the other [3].

To be concrete, let us recall the simplest example of this mirror manifold construction. Consider the theory $(3, 3, 3, 3)$ which corresponds to the Fermat quintic hypersurface in $\mathbb{C}^4 \mathbb{P}^4$.

\[(3, 3, 3, 3) \rightarrow \mathbb{Z}_5^5 + \mathbb{Z}_2^5 + \mathbb{Z}_3^5 + \mathbb{Z}_4^5 + \mathbb{Z}_5^5 = 0 \quad (2.3)\]

It is immediate to check that $S = (\mathbb{Z}_5^5)$ while $G = (\mathbb{Z}_5)^3$. Thus, the results of [5] show that

\[\mathbb{Z}_5^5 + \mathbb{Z}_2^5 + \mathbb{Z}_3^5 + \mathbb{Z}_4^5 + \mathbb{Z}_5^5 = 0 \quad (2.4a)\]

and

\[\frac{\mathbb{Z}_5^5 + \mathbb{Z}_2^5 + \mathbb{Z}_3^5 + \mathbb{Z}_4^5 + \mathbb{Z}_5^5}{(\mathbb{Z}_5)^3} = 0 \quad (2.4b)\]

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8 This is in fact a bit naive. A $\mathbb{Z}_d$ subgroup of this group acts trivially on both sides; on the conformal field theory side this is because we have already projected to states invariant under this action to obtain a string vacuum, while on the geometry side this is result of the fact that the $Z_i$ are homogeneous coordinates on a projective space.
are mirror manifolds. Both of these Calabi-Yau spaces correspond to the conformal theory \((3,3,3,3,3)\) and as we see from table 2, their Hodge numbers are appropriately interchanged. In table 1 we list some other examples.

Before discussing the implications of mirror symmetry, we pause here to emphasize three important points. First, our construction works equally well on any orbifold of the theories under discussion. That is, the space of all orbifolds of a given theory can be partitioned into mirror pairs. We illustrate this with the quintic hypersurface in table 2. We note that the mirror of a theory \(M/H\) with \(H \subset G\) is given by \(M/H^*\) where \(H^*\) is the complement of \(H\) in \(G\) (that is, the smallest group containing \(H\) and \(H^*\) is \(G\)). Second, our arguments are not specific to complex dimension three and immediately generalize to other dimensions. Third, our discussion to this point has been tied to very special points in the respective Calabi-Yau moduli spaces. Namely, we have focused on Fermat points as these correspond to the well understood minimal model conformal field theories. By deformation arguments we can immediately extend our results to more general points in moduli space. For example, let \(M\) (a Fermat hypersurface) and \(M'\) be mirror manifolds, each of which corresponds to the conformal field theory \(K\). Let us deform, say, the complex structure of \(M\). This corresponds to deforming \(K\) by a particular and identifiable marginal operator. On \(M'\) this conformal field theory operator is interpretable as a Kähler modulus, due to the flipped sign of the \(U(1)\) charge, as discussed. Hence, the mirror of the complex structure deformed \(M\) is this Kähler deformation of \(M'\). Quite generally, then, we can follow this procedure to determine the mirror of any deformation of \(M\). Hence, modulo the caveat in the footnote, we have the main result of [5]:

For any Calabi-Yau hypersurface \(M\) which belongs to a moduli space that admits a Fermat point, there exists a mirror manifold (in the full sense of conformal field theory) \(M'\). At the Fermat point, \(M\) and \(M'\) are related by \(M' = M/G\) where \(G\) is the maximal allowable subgroup of the group of scaling symmetries.

We should note that even this statement is not the most general we could make, since for example, as shown in [5] and illustrated by the third line of table 1, the hypersurface constraint can be relaxed. As mentioned in the introduction, we have reason to believe

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9 The \((\mathbb{Z}_5)^3\) action is given in table 2.
10 We should acknowledge at this point the possibility of subtleties in performing this identification globally in the respective moduli spaces. Mathematically, then, it would be more precise to limit our analysis to local deformations only although we believe our statements to be true globally as well.
that even then the class of models for which the above argument works is not the largest for which mirror manifolds exist. Having outlined the arguments by which mirror symmetry was shown to exist, let us briefly turn to some consequences. In \[5\] we described in some detail a number of important implications of mirror symmetry. In the next section we review the arguments that lead to these claims, as well as some subsequent work by other groups which lead to a rather spectacular verification of their veracity as well as to some new applications.

3. Applications of Mirror Symmetry

Having reviewed the initial speculations and subsequent work which established the existence of mirror symmetry, we would now like to turn to a discussion of the implications of this phenomenon, as well as some recent work applying mirror symmetry to interesting and explicit examples.

Let \( M \) and \( M' \) be mirror Calabi-Yau manifolds each corresponding to the conformal field theory \( K \). Consider a (non–vanishing) three point function of conformal field theory operators corresponding to \((2,1)\) forms on \( M \). Mathematically, this correlation function is given by the simple integral on \( M \) \[14\]
\[
\int_M \omega^{abc} \tilde{b}_a^{(i)} \wedge \tilde{b}_b^{(j)} \wedge \tilde{b}_c^{(k)} \wedge \omega
\] (3.1)
where the \( \tilde{b}_a^{(i)} \) are \((2,1)\) forms (expressed as elements of \( H^1(M', T) \) with their subscripts being tangent space indices) and \( \omega \) is the holomorphic three form. Due to the nonrenormalization theorem proved in \[13\], we know that this expression (3.1) is the exact conformal field theory result. By mirror symmetry, these same conformal field theory operators correspond to particular and identifiable \((1,1)\) forms on the mirror \( M' \), which we can label \( b^{(i)} \). Mathematically, (due to the absence of a nonrenormalization theorem) the expression for such a coupling \[15\],\[16\] in terms of geometric quantities on \( M' \) is comparatively complicated:
\[
\int_{M'} b^{(i)} \wedge b^{(j)} \wedge b^{(k)} + \sum_n e^{-nR} \left( \sum_{I_n} \left( \int_{I_n} \left( \int_{I_n} X^*(b^{(i)}) \left( \int_{I_n} X^*(b^{(j)}) \left( \int_{I_n} X^*(b^{(k)}) \right) \right) \right) \right)
\] (3.2)
where \( I_n \) is a holomorphic instanton of charge \( n \), \( X \) is the map of the (worldsheet) instanton into \( M' \), and \( R \) is the radius of \( M' \). The first term in (3.2) is the topological triple intersection form on \( M' \).
Now, since both (3.1) and (3.2) correspond to the same conformal field theory correlation function, they must be equal; hence we have

$$\int_M \omega^{abc} \tilde{b}_a^{(i)} \wedge \tilde{b}_b^{(j)} \wedge \tilde{b}_c^{(k)} \wedge \omega = \int_{M'} b^{(i)} \wedge b^{(j)} \wedge b^{(k)} + \sum_n e^{-nR} \sum_{I_n} (\int_{I_n} X^*(b^{(i)}))(\int_{I_n} X^*(b^{(j)}))(\int_{I_n} X^*(b^{(k)})).$$

(3.3)

Notice the crucial role played by the underlying conformal field theory in deriving this equation. If we simply had two manifolds whose Hodge numbers were interchanged we could not, of course, make any such statement. This result is rather surprising and clearly very powerful. We have related expressions on a priori unrelated manifolds which probe rather intimately the structure of each. Furthermore, the left hand side of (3.3) is directly calculable while the right hand side requires, among other things, knowledge of the rational curves of every degree on the space.

Since written, (3.3) has been verified in several illuminating examples. We briefly discuss the results of these papers and some interesting implications. The authors of sought to verify (3.3) in a particular example through the direct computation of both sides. For ease of computation the authors chose to work with the mirror pair having $\chi = \pm 8$ in table 2. They also chose to work at large radius on the mirror manifold (the $\chi = +8$ member of the pair) so as to suppress the instanton contributions in (3.3) and leave only the topological intersection number on the right hand side. By direct computation, the authors of verified (3.3) in this limit. This work also uncovered an interesting and as yet incompletely understood subtlety: If a manifold has $h^{1,1} > 1$, one must be careful regarding precisely how the ‘large radius’ limit in this multidimensional moduli space is defined. It is possible that different limits will pick out different theories, possibly corresponding to the inequivalent ways in which orbifold singularities can be repaired. These ideas and further elaborations on them are discussed in, to which the reader is referred to for details.

The other work we would like to describe is [8]. In this paper, Candelas, de La Ossa, Green and Parkes analyze (3.3) for the case of the quintic–mirror-quintic pair. In particular, they consider a one parameter family of mirror manifolds given by deforming along the single Kähler modulus (complex structure modulus) on the quintic (mirror quintic), as we discussed in the general context in section 2. Through a careful analysis of the map between Kähler and complex structure deformations, the authors are able to

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11 For a recent mathematical presentation of this see [17].
extract information about the right hand side of (3.3) from direct calculation of the left hand side. In particular, they find the number of rational curves (holomorphic instantons) of any desired degree on the quintic hypersurface. This is to be compared with the fact that mathematicians have to this point only succeeded in computing these numbers up to degree three, and then only with great effort. Thus, a previously unsolved problem in mathematics (which is quite relevant, for example, to the manifold classification problem) is solved by mirror symmetry.

4. Recent Work on Extending Mirror Symmetry

The final topic we would like to mention is some recent work of ours [21] which offers the possibility of extending the mirror manifold construction reviewed in section 2 to a more general class of theories. At the same time the work also affirms another aspect of mirror symmetry which to this point we have not stressed. Namely, geometrical symmetries of a given Calabi-Yau manifold manifest themselves as quantum symmetries of the mirror manifold. Recall that a quantum symmetry [18] of a theory is one whose presence cannot be ascertained simply by studying ordinary classical geometrical actions. Rather, one must directly study symmetries of the full quantum Hilbert space. For example, the $R \to 1/2R$ symmetry of string propagation on a circle is a quantum symmetry: it certainly cannot be ascertained from the geometrical invariances of a circle, but nonetheless is a symmetry of the full quantum theory.

There are a number of reasons why we expect classical and quantum symmetries to be interchanged by mirror symmetry. One concrete reason is as follows. A conformal field theory is pieced together out of holomorphic and antiholomorphic fields. In left-right symmetric theories, then, if the holomorphic part of theory respects a symmetry group $G$ so will the antiholomorphic part. Thus, the full symmetry group is typically $G \times G$. If this conformal theory admits a geometrical interpretation, the classical geometrical symmetries are generally the diagonal or antidiagonal subgroup of $G \times G$. This is quite familiar, for example, from many studies of minimal model string vacua [23], [5]. Now, in the case of scaling symmetries, flipping the sign of the left moving $U(1)$ charge interchanges the diagonal and antidiagonal subgroups of $G \times G$. Hence, if a conformal theory admits a geometrical interpretation in terms of mirror manifolds, classical symmetries of one manifold will be quantum symmetries of the other and vice versa. We will see this explicitly in a moment.
The main point of [21] is that certain apparently ill defined nonlinear field transformations are not quite as bad as they initially seem. In fact, they can be quite useful. The basic idea is to introduce nonlinear field transformations which are only one–to–one after making certain global identifications. We aim to choose these transformations so that the latter global identifications are in fact identical to the orbifolding groups introduced earlier in our construction of mirror manifolds. This idea has also been suggested by [8] and [19]. However, these field transformations do not simply provide a new representation for the mirror construction of [5], as these authors had supposed – rather, we argue that one does not generate the mirror of the original theory but instead the mirror of a theory at a different point in the same moduli space as the original. One part of our argument relies heavily on the notion of quantum symmetries [18] and by its success gives us confidence that quantum symmetries of a manifold are an accurate guide to geometrical symmetries of its mirror (as we expect from our discussion above). This allows us to carry through our strategy for extending the class of mirror manifolds (as we shall see) to cover all Landau-Ginzburg theories for which there exists a (nonlinear) field redefinition taking the theory (up to global identifications) to Fermat form.

To illustrate these ideas, let’s return to the example of the quintic hypersurface and its mirror (2.4). Recall that the mirror manifold \( M' \) is given by a \((\mathbb{Z}_5)^3\) orbifold of the Fermat quintic \( M \). This quotient may be induced by the change of variables

\[
(X_1, X_2, X_3, X_4, X_5) = (Y_1 Y_3^{1/5}, Y_2 Y_5^{1/5}, Y_3 Y_4^{1/5} Y_2^{1/5}, Y_4 Y_4^{1/5} Y_2^{1/5}, Y_5^{1/5}).
\]

(4.1)

This is because the latter change of variables is only one–to–one if one performs a \((\mathbb{Z}_5)^3\) orbifolding on the \( X \) coordinates. Notice that this transformation is singular at the origin. In terms of these variables, the superpotential becomes

\[
M'' = Y_3 Y_1^5 + Y_5 Y_2^4 + Y_4 Y_3^4 + Y_2 Y_4^4 + Y_5^4.
\]

(4.2)

Furthermore, it is crucial to emphasize that making (4.1) one–to–one requires that we make global identifications on the \( Y \) variables as well as the \( X \) variables. It is straightforward to see that this identification on \( Y \)’s is a \( \mathbb{Z}_{256} \) action

\[
(Y_1, Y_2, Y_3, Y_4, Y_5) \sim (\rho Y_1, \rho^{176} Y_2, \rho^{251} Y_3, \rho^{20} Y_4, \rho^{64} Y_5)
\]

(4.3)

The reader will no doubt recall that this is essentially the program followed in [12] in which nonlinear field transformations were used to pass from Landau-Ginzburg theories to Landau-Ginzburg orbifolds – the latter of which can be identified with Calabi-Yau manifolds.
where $\rho$ is a nontrivial $256^{th}$ root of unity. These identifications are precisely implemented by embedding our equation (1.2) as a hypersurface $M''$ in $\mathbb{WP}^4_{41,48,51,52,64}$ -- the weighted projective space identifications induce precisely (4.3)\textsuperscript{14}. We see, therefore, that $M''$ is cut out by a polynomial of homogeneity degree 256 in this weighted projective space. One can explicitly verify that $M'$ and $M''$ each has a one dimensional moduli space of complex structure deformations, and 101 harmonic $(1,1)$ forms labeling Kähler deformations. The naive reasoning described above might lead one to conclude the isomorphism $M' \sim M''$.

In fact, $M''$ is not the mirror of the Fermat hypersurface. By general arguments [18] the conformal field theory based on $M''$ respects a quantum $\mathbb{Z}_{256}$ symmetry. The conformal field theory based on $M$, however, does not respect a $\mathbb{Z}_{256}$ -- classical or quantum. Thus, $M''$ and $M$ are not a mirror pair. As discussed in [21] a natural conclusion to draw is that $M''$ is the mirror of a quintic hypersurface with a complex structure differing from the Fermat $M$. If this is true we expect there to be a complex structure on the quintic which gives rise to a classical $\mathbb{Z}_{256}$ symmetry (which has the correct action on the cohomology as dictated by the conformal field theory [21]). Such a complex structure can be found: $M''$ is the mirror of

$$
\tilde{M} = X_1^5 + X_1X_2^4 + X_2X_3^4 + X_3X_4^4 + X_4X_5^4,
$$

where the symmetry acts on the fields as

$$(X_1, X_2, X_3, X_4, X_5) \rightarrow (X_1, \beta^{64}X_2, \beta^{176}X_3, \beta^{20}X_4, \beta^{251}X_5) \quad (4.5)$$

where $\beta$ is a primitive $256^{th}$ root of unity.

To understand what is going on here let us focus on the Landau-Ginzburg theory associated with the superpotential (1.2). From [12] we know that this theory (after the $U(1)$ projection) describes the Calabi-Yau sigma model at a particular Kähler structure, determined by the renormalization group flows.\textsuperscript{15} The corresponding mirror theory in

\textsuperscript{13} The subscripts here denote the degree of homogeneity of each of the homogeneous weighted projective space coordinates.

\textsuperscript{14} This is most easily seen by using the generator $\lambda = \rho^{25}$ in terms of which (4.3) are manifestly the weighted space identifications.

\textsuperscript{15} As emphasized in [18], this particular structure is in fact determined by the quantum symmetry.
quintic moduli space thus differs from the model with which we began by an adjustment of its complex structure (since the mirror of a Kähler field is a complex structure field). In this way, we wind up at (4.4). As we mentioned, the field transformation (4.1) is singular. Hence, to some extent it is surprising that this approach yields sensible results. Undoubtedly, from the physics viewpoint, this is due to the exceptionally well behaved operator products which arise in \( N = 2 \) conformal theories [3]; it is of interest to understand the mathematical justification of such manipulations.

The upshot of this discussion is that we have constructed an explicit mirror pair \( \tilde{M} \) and \( M'' \) away from any Fermat points. By our general deformation argument reviewed in section 2 we knew such a mirror pair existed; now we have explicitly constructed it. For the benefit of the skeptical reader, we hasten to point out that in [21] we verify these assertions by deriving the differential equations satisfied by the periods, as was done for the Fermat mirror pair in [8]. We find precisely the hypergeometric differential equation predicted by the above reasoning. Note also that \( M'' \), being a hypersurface in weighted four space, is in the list generated by [3]. We now learn that this space is in fact mirror to the quintic in \( \mathbb{CP}^4 \), although at an unexpected point in the moduli space of the latter. Following the same basic idea we find a number of surprising explicit mirror pairs [21]. For example, preliminary work indicates that we can build mirrors to a variety of quintic hypersurfaces by orbifolding by a host of unexpected group actions. We list some explicit examples in table 3.

More generally, if we are given a Calabi-Yau theory \( Q_1 \) which admits a change of variables to Fermat form (up to global identifications) then we can construct the mirror of \( Q_1 \). More precisely, if we begin with the model \( Q_1 \), then this change of variables yields the result

\[
Q_1/G_1 \sim Q_2/G_2,
\]

where the quotient groups represent the induced identifications on both sets of coordinates and \( Q_2 \) is of Fermat type, i.e. has a point in its moduli space corresponding to a minimal model construction. The \( \sim \) in (4.6) implies equivalence up to deformations by twisted fields, but for our purpose of establishing the existence of the mirror manifold, the precise point in its moduli space is inessential. The arguments of [3] allow us to find the mirror manifold of the orbifold appearing on the right hand side of (4.6), expressed as a quotient of \( Q_2 \) by the appropriate ‘dual’ subgroup to \( G_2 \). However, we are interested in constructing the mirror of \( Q_1 \), not of its quotient. It is here that the work of [21] as briefly described
above gives the solution. The left hand side of (4.6) respects a quantum $G_1$ symmetry \[18\]; its quotient by this is $Q_1$. As we observed, though, on the mirror manifold of (4.6), this will be realized as a geometrical symmetry at some point in the moduli space. At this point, the quotient by this symmetry will mirror the quotient by the quantum $G_1$, yielding the mirror $Q'_1$.

This clearly takes us a significant step beyond \[5\]; an interesting open question is to characterize precisely the set of theories for which this procedure will generate the mirror.

5. Conclusions

In this talk we have reviewed the initial speculations \[4,3\] and subsequent explicit demonstration \[5\] of the existence of mirror symmetry – a symmetry whereby two Calabi-Yau manifolds whose Hodge diamonds differ by a ninety-degree rotation are shown to correspond to the same conformal field theory. We have stressed that this symmetry interchanges the roles of complex structure and Kähler moduli on the two manifolds and, correspondingly, interchanges classical and quantum symmetries. We have also emphasized one especially important implication of mirror symmetry initially found in \[5\]– namely eqn. (3.3). This is a rather remarkable equality in that an infinite instanton sum is reexpressed as a simple geometrical integral. Although (3.3) follows directly from the general arguments of \[5\], it is most pleasing that it has been directly verified in a special case \[5\]. Furthermore, the power of (3.3), as briefly reviewed, has been recently exploited by \[8\] to solve a previously unresolved mathematical issue: the determination of the number of rational curves of arbitrary degree on the quintic threefold.

There are a number of compelling unresolved issues surrounding mirror symmetry. The two most prominent are: 1) Is mirror symmetry fully general in the sense that any Calabi-Yau $n$–fold has a mirror manifold whose Hodge diamond is rotated by ninety-degrees and which corresponds to the same conformal field theory? The speculative discussions based on naturality certainly seem to support this statement. At this time, as discussed, mirror symmetry has only been established for a particular subclass of Calabi-Yau manifolds. 2) What is the underlying mathematical reason for the existence of mirror manifolds? Associated with this is the question of the role played by our orbifolding prescription – is it fundamental to the existence of mirror manifolds or simply a useful tool which is central to the construction of all presently known examples? Such questions are under active study; with some luck insight into their answers will not be long in coming.

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The equations listed define $M$ as a hypersurface in an appropriate weighted projective 4-space (such that the equation is homogeneous). The last column gives the mirror manifold $M'$ as a quotient of $M$. The third entry defines a submanifold of $\mathbb{CP}^3 \times \mathbb{CP}^2$, demonstrating that the restriction to minimal models sometimes extends beyond hypersurfaces.

| $M$ | $h^{2,1}$ | $h^{1,1}$ | $S$ | $M'$ |
|-----|----------|----------|-----|-----|
| $z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0$ | 101 | 1 | $\mathbb{Z}_5^5$ | $M/(\mathbb{Z}_3^3)$ |
| $z_1^5 + z_2^{10} + z_3^{10} + z_4^{10} + z_5^2 = 0$ | 145 | 1 | $\mathbb{Z}_5 \times \mathbb{Z}_5^{10} \times \mathbb{Z}_2$ | $M/(\mathbb{Z}_2^{10})$ |
| $z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0$ | 35 | 8 | $\mathbb{Z}_3 \times \mathbb{Z}_9^3$ | $M/(\mathbb{Z}_3 \times \mathbb{Z}_9)$ |
| $z_1^3 + z_2^4 + z_3^5 + z_4^6 + z_5^{20} = 0$ | 47 | 23 | $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_{20}$ | $M/(\mathbb{Z}_2)$ |
| $z_1^3 + z_2^4 + z_3^4 + z_4^{12} + z_5^{12} = 0$ | 89 | 5 | $\mathbb{Z}_3 \times \mathbb{Z}_4^2 \times \mathbb{Z}_4^2$ | $M/(\mathbb{Z}_3 \times \mathbb{Z}_9^2)$ |
| $z_1^4 + z_2^6 + z_3^{21} + z_4^{28} + z_5^2 = 0$ | 52 | 28 | $\mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_{21} \times \mathbb{Z}_{28}$ | $M/(\mathbb{Z}_2)$ |
| $z_1^5 + z_2^6 + z_3^{10} + z_4^{30} + z_5^2 = 0$ | 91 | 7 | $\mathbb{Z}_5 \times \mathbb{Z}_6 \times \mathbb{Z}_{10} \times \mathbb{Z}_{30}$ | $M/(\mathbb{Z}_2 \times \mathbb{Z}_{10})$ |

Table 1

Mirrors $M'$ of Some Minimal Model Compactifications $M$
The generators of each symmetry group are represented by the powers of a fundamental fifth root of unity by which they multiply the homogeneous coordinates of \( \mathbb{CP}^4 \). Thus \([0, 0, 0, 1, 4]\), for example, represents
\[
(z_1, z_2, z_3, z_4, z_5) \rightarrow (z_1, z_2, z_3, \alpha z_4, \alpha^4 z_5)
\]
with \(\alpha^5 = 1\).
\[ \begin{array}{ccc}
M & M' & G \text{ Generators} \\
\begin{array}{c}
z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0 \\
z_1^4 z_2 + z_2^4 z_3 + z_3^4 z_4 + z_4^4 z_5 + z_5^4 z_1 = 0 \\
z_1^4 z_2 + z_2^4 z_3 + z_3^4 z_1 + z_4^5 + z_5^5 = 0 \\
z_1^4 z_2 + z_2^5 z_1 + z_3^5 + z_4^5 + z_5^5 = 0 \\
z_1^4 z_2 + z_2^4 z_1 + z_3^4 z_4 + z_4^4 z_5 + z_5^4 z_3 = 0
\end{array} & M/(\mathbb{Z}_5^3) & [0,0,1,4] \\
 & & [0,0,0,1,4] \\
 & M/(\mathbb{Z}_{41}) & [1,37,16,18,10] \\
 & M/(\mathbb{Z}_{51}) & [1,37,16,38,0] \\
 & M/(\mathbb{Z}_5 \times \mathbb{Z}_{13}) & [0,0,0,1,4] \\
 & & [1,9,3,0,0] \\
 & M/(\mathbb{Z}_5^2 \times \mathbb{Z}_3) & [0,0,1,0,4] \\
 & & [0,0,0,1,4] \\
 & M/(\mathbb{Z}_3 \times \mathbb{Z}_{13}) & [1,2,0,0,0] \\
 & & [0,0,1,9,3]
\end{array} \]

Table 3
Quintic M Hypersurfaces and Their Mirrors M′

The indicated quotient of the manifold M is its mirror M′. The group actions are denoted by the powers of an appropriate primitive root of unity by which they multiply the coordinates on \( \mathbb{C}P^4 \), as in table 2.
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