ORLICZ-LORENTZ SEQUENCE SPACES EQUIPPED WITH THE ORLICZ NORM*

In memory of Henryk Hudzik, a great man, our teacher and colleague

Yunan CUI (崔云安)
Department of Mathematics, Harbin University of Science and Technology, Harbin 150080, China
E-mail: cuiya@hrbust.edu.cn

Paweł FORALEWSKI† Joanna KOŃCZAK
Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Poznań,
Uniwersytetu Poznańskiego 4, 61-614 Poznań, Poland
E-mail: katon@amu.edu.pl; jkonczak@amu.edu.pl

Abstract In this article, we consider Orlicz-Lorentz sequence spaces equipped with the Orlicz norm $(\lambda_{\varphi,\omega}, \| \cdot \|_{\varphi,\omega}^O)$ generated by any Orlicz function and any non-increasing weight sequence. As far as we know, research on such a general case is conducted for the first time. After showing that the Orlicz norm is equal to the Amemiya norm in general and giving some important properties of this norm, we study the problem of existence of order isomorphically isometric copies of $l^\infty$ in the space $(\lambda_{\varphi,\omega}, \| \cdot \|_{\varphi,\omega}^O)$ and we find criteria for order continuity and monotonicity properties of this space. We also find criteria for monotonicity properties of $n$-dimensional subspaces $\lambda_{\varphi,\omega}^n$ $(n \geq 2)$ and the subspace $(\lambda_{\varphi,\omega})_a$ of order continuous elements of $\lambda_{\varphi,\omega}$. Finally, as an immediate consequence of the criteria considered in this article, the properties of Orlicz sequence spaces equipped with the Orlicz norm are deduced.

Key words Orlicz-Lorentz spaces; Banach symmetric spaces; Orlicz norm; order continuity and copies of $l^\infty$; monotonicity properties

2010 MR Subject Classification 46B20; 46B45; 46A45; 46A80; 46B42

1 Preliminaries

Let $l^0$ be the space of all sequences $x : \mathbb{N} \to (-\infty, \infty)$. For any $x, y \in l^0$, we write $x \leq y$ if $x(i) \leq y(i)$ for any $i \in \mathbb{N}$. Given any $x \in l^0$, we define its distribution function $\mu_x : [0, \infty) \to \{0, \infty\} \cup \mathbb{N}$ by

$$
\mu_x(\lambda) = m\{i \in \mathbb{N} : |x(i)| > \lambda\},
$$

where $m$ is the counting measure on $2^\mathbb{N}$ (see [2, 24, 26]), and its non-increasing rearrangement $x^* = (x^*(i))_{i=1}^\infty$ as

$$
x^*(i) = \inf\{\lambda \geq 0 : \mu_x(\lambda) < i\}
$$

Received September 18, 2020; revised May 31, 2021. The first author gratefully acknowledges the support of NSF of China (11871181).

†Corresponding author
for any \( \phi \) right derivatives of \( \pi \) also Remark 1.1). In fact, for the sequence of Orlicz functions (\( \phi \)) of \( N \), let us define a subdifferential \( \partial \phi \) by the formula

\[
\partial \phi = \{ v \geq 0 : \phi(u) + \psi(v) = uv \}.
\]

By \( l \) and \( p \) we denote the left and right derivative of \( \phi \) respectively. Then we have that

(i) \ if \( u \in [0,b_\phi] \); then \( \partial \phi(u) = [l(u), p(u)] \),

(ii) \ if \( u = b_\phi \) and \( l(b_\phi) < \infty \), then \( \partial \phi(b_\phi) = [l(b_\phi), \infty) \);

(iii) \ if either \( u = b_\phi \) and \( l(u) = \infty \) or \( u > b_\phi \), then \( \partial \phi(u) = \emptyset \).

Let \( \omega : N \to \mathbb{R} \) be a nonnegative, nontrivial and non-increasing sequence, called a weight sequence.

For any Orlicz function \( \phi \) and any weight sequence \( \omega \), the convex semimodular \( I_{\phi,\omega} : l^0 \to \mathbb{R}_+ = [0, \infty] \) is defined as follows:

\[
I_{\phi,\omega}(x) := \sum_{i=1}^{\infty} \phi(x^*(i)) \omega(i) = \sup_{\pi} \sum_{i=1}^{\infty} \phi(x(\pi(i))) \omega(i),
\]

where \( \pi \) denotes a permutation of the set \( N \) and the supremum is extended over all permutation of \( N \). Note that the second equality in the above formula follows from [11, Theorem 2.2] (see also Remark 1.1). In fact, for the sequence of Orlicz functions \( (\phi_i)_{i=1}^{\infty} \), where \( \phi_i(u) = \phi(u)\omega(i) \) for any \( i \in N \), we have \( p_i(u) = p(u)\omega(i) \) for the same \( i \), where \( p_i \) \( (i \in N) \) and \( p \) denote the right derivatives of \( \phi_i \) \((i \in N)\) and \( \phi \), respectively. In consequence, the sequence of Orlicz
functions \((\phi_i)_{i=1}^{\infty}\) satisfies condition (L1) from [11] if and only if \(a_\varphi = b_\varphi\) or the sequence \(\omega\) is non-increasing.

Given any Orlicz function \(\varphi\) and any weight sequence \(\omega\), we define the modular space
\[
\lambda_{\varphi, \omega} = \{ x \in l^0 : I_{\varphi, \omega}(\lambda x) < \infty \text{ for some } \lambda > 0 \},
\]
which is called the Orlicz-Lorentz sequence space. Recall that the subspace \(\lambda_{\varphi, \omega}^{\theta}\) of the space \(\lambda_{\varphi, \omega}\) is defined by the formula
\[
\lambda_{\varphi, \omega}^{\theta} = \{ x \in l^0 : \forall \lambda > 0 \exists k \in \mathbb{N} \sum_{i=k}^{\infty} \varphi(\lambda x^*(i)) \omega(i) < \infty \}.
\]
(1.4)

It is easy to show that if \(\sum_{i=1}^{\infty} \omega(i) < \infty\) or \(a_\varphi > 0\), then \(\lambda_{\varphi, \omega} = l^\infty\). In this case, by Theorem 4.2 in [11], we have additionally \(\lambda_{\varphi, \omega}^{\theta} = l^\infty\) if \(\sum_{i=1}^{\infty} \omega(i) < \infty\) and \(b_\varphi = \infty\) and \(\lambda_{\varphi, \omega}^{\theta} = c_0\) whenever \((\sum_{i=1}^{\infty} \omega(i) < \infty \text{ and } b_\varphi < \infty)\) or \((\sum_{i=1}^{\infty} \omega(i) = \infty \text{ and } a_\varphi > 0)\). On the other hand, if \(\sum_{i=1}^{\infty} \omega(i) = \infty\) and \(a_\varphi = 0\), we get \(\lambda_{\varphi, \omega} \subset c_0\). Moreover, in this case, \(\lambda_{\varphi, \omega} = \lambda_{\varphi, \omega}^{\theta}\) if and only if \(\varphi \in \Delta_2(0)\) (see [11, Theorem 3.1] and [4, Theorem 8]).

It is well known that the space \(\lambda_{\varphi, \omega}\) equipped with the Luxemburg norm \(\| \cdot \|_{\varphi, \omega}\), defined by
\[
\| x \|_{\varphi, \omega} = \inf \{ \lambda > 0 : I_{\varphi, \omega}(x/\lambda) \leq 1 \},
\]
(1.5)
is a Banach symmetric space (for basic properties of symmetric spaces, we refer to [2, 24, 26]).

By analogy to Orlicz spaces, we can define two additional norms (see Remark 1.1), namely the Orlicz norm
\[
\| x \|_{\varphi, \omega}^O = \sup \left\{ \sum_{i=1}^{\infty} x^*(i)y^*(i)\omega(i) : I_{\varphi, \omega}(y) \leq 1 \right\}
\]
(1.6)
and the Amemiya norm
\[
\| x \|_{\varphi, \omega}^A = \inf_{k > 0} \left( \frac{1}{k} (1 + I_{\varphi, \omega}(kx)) \right).
\]
(1.7)

In the past research mainly Orlicz-Lorentz spaces equipped with the Luxemburg norm were studied (see [1, 4, 9–14, 20, 21, 25, 30, 36–38]). Recently, Orlicz-Lorentz spaces equipped with the Orlicz norm have been considered in some papers (see [15, 34] and also [6]). However, in those papers only some special cases of Orlicz functions and weight sequences have been investigated. In our paper we admit any Orlicz function and any non-increasing weight sequence.

**Remark 1.1** Let \(\varphi\) be any Orlicz function and \(\omega\) any nonnegative and nontrivial weight sequence. If \(a_\varphi = b_\varphi\), then \(\lambda_{\varphi, \omega} = l^\infty\) and \(\| x \|_{\varphi, \omega} = \| x \|_{\varphi, \omega}^A = \| x \|_{l^\infty} / b_\varphi\) for any \(x \in \lambda_{\varphi, \omega}\). Let now \(a_\varphi < b_\varphi\). Then the following conditions are equivalent: the sequence \(\omega\) is non-increasing, the second equality in formula (1.3) holds and the semimodular \(I_{\varphi, \omega}\) is convex (see [11, Theorem 2.2]). The convexity of the semimodular \(I_{\varphi, \omega}\) is in turn essential to prove that the functionals defined by (1.5) and (1.7) are norms. Also, the functional defined by (1.6) is a norm if the sequence \(\omega\) is non-increasing.
2 Some Basic Results

We start with the following

Lemma 2.1 For any \( x \in \lambda_{\varphi, \omega} \), we have

\[
\| x \|_{\varphi, \omega} = \inf_{k > 0} \left\{ \max \left( \frac{1}{k}, \frac{1}{k} I_{\varphi, \omega}(kx) \right) \right\}.
\] (2.1)

Proof We recall the proof of this lemma for completeness. It is obvious that the above formula holds for \( x = 0 \). Assume now that \( x \neq 0 \) and take an arbitrary \( k > 0 \). If \( I_{\varphi, \omega}(kx) \leq 1 \), then \( \| x \|_{\varphi, \omega} \leq \frac{1}{k} \). On the other hand, if \( I_{\varphi, \omega}(kx) > 1 \), then

\[
I_{\varphi, \omega}\left(\frac{kx}{I_{\varphi, \omega}(kx)}\right) \leq \frac{1}{I_{\varphi, \omega}(kx)} \cdot I_{\varphi, \omega}(kx) = 1,
\]

so we have \( \| x \|_{\varphi, \omega} \leq \frac{1}{k} I_{\varphi, \omega}(kx) \). Hence, we obtain

\[
\| x \|_{\varphi, \omega} \leq \max \left( \frac{1}{k}, \frac{1}{k} I_{\varphi, \omega}(kx) \right).
\]

By the arbitrariness of \( k > 0 \), we get

\[
\| x \|_{\varphi, \omega} \leq \inf_{k > 0} \left\{ \max \left( \frac{1}{k}, \frac{1}{k} I_{\varphi, \omega}(kx) \right) \right\}.
\]

It only remains to prove the reversed inequality. If \( 0 < \| x \|_{\varphi, \omega} < \frac{1}{k_0} \), then \( I_{\varphi, \omega}(k_0 x) \leq 1 \), which means that

\[
\inf_{k > 0} \left\{ \max \left( \frac{1}{k}, \frac{1}{k} I_{\varphi, \omega}(kx) \right) \right\} \leq \max \left( \frac{1}{k_0}, \frac{1}{k_0} I_{\varphi, \omega}(k_0 x) \right) = \frac{1}{k_0}.
\]

Since \( \frac{1}{k_0} \) may be arbitrarily close to \( \| x \|_{\varphi, \omega} \), we conclude that

\[
\inf_{k > 0} \left\{ \max \left( \frac{1}{k}, \frac{1}{k} I_{\varphi, \omega}(kx) \right) \right\} \leq \| x \|_{\varphi, \omega}.
\]

\[\square\]

For any \( x \in \lambda_{\varphi, \omega} \), we define

\[
f(k) = f_x(k) = \frac{1}{k} (1 + I_{\varphi, \omega}(kx))
\] (2.2)

and

\[
\lambda_\infty = \lambda_\infty(x) = \sup \{ \lambda > 0 : I_{\varphi, \omega}(\lambda x) < \infty \}.
\] (2.3)

The function \( f \) is continuous on the interval \((0, \lambda_\infty)\) and left continuous at \( \lambda_\infty \) if \( \lambda_\infty < \infty \).

Let \( E \subset l^0 \) be a normed Riesz space, that is, a partially ordered normed vector space over the real numbers, such that

(i) \( x \leq y \) implies \( x + z \leq y + z \) for any \( x, y, z \in E \);

(ii) \( \alpha x \geq 0 \) for any \( x \geq 0 \) in \( E \) and any non-negative real \( \alpha \);

(iii) for all \( x, y \in E \), there exist a least upper bound \( x \lor y \) and a greatest lower bound \( x \land y \);

(iv) for any \( x, y \in E \), we have \( \| x \| \leq \| y \| \) whenever \( |x| \leq |y| \)

(see [35]). We say that \( E \) has the Fatou property if for any \( x \in l^0 \) and \( (x_n)_{n=1}^\infty \) in \( E_+ \) (the positive cone of \( E \)) such that \( x_n \not\nearrow |x| \) coordinatewise and \( \sup_n \| x_n \| < \infty \), we have \( x \in E \) and \( \| x \| = \lim_{n \to \infty} \| x_n \| \) (see [26]).

It is easy to show that both \((\lambda_{\varphi, \omega}, \| \cdot \|_O_{\varphi, \omega})\) and \((\lambda_{\varphi, \omega}, \| \cdot \|_A_{\varphi, \omega})\) are normed Riesz spaces.

We get also the following
Lemma 2.2 The space $\langle \lambda_{\varphi,\omega}, \| \cdot \|_{\varphi,\omega}^A \rangle$ has the Fatou property.

Proof Suppose that $x \in l_0$, $x_n \in \lambda_{\varphi,\omega}$ for any $n \in \mathbb{N}$, $0 \leq x_n \neq |x|$ coordinatewise, and $\sup_n \|x_n\|_{\varphi,\omega}^A = \alpha < \infty$. By Lemma 2.1, we have $\|x_n\|_{\varphi,\omega} \leq \|x_n\|_{\varphi,\omega}^A$ for all $n \in \mathbb{N}$, whence $\sup_n \|x_n\|_{\varphi,\omega} \leq \alpha$. By the definition of the Luxemburg norm, we have $I_{\varphi,\omega} \left( \frac{x_n}{\alpha+1} \right) \leq 1$ for any $n \in \mathbb{N}$. Since $\varphi \circ \frac{x_n}{\alpha+1} / \varphi \circ \frac{x}{\alpha+1}$ coordinatewise, by Beppo Levi’s theorem, we get

$$I_{\varphi,\omega} \left( \frac{x}{\alpha+1} \right) = \sum_{i=1}^{\infty} \varphi \left( \frac{x^*(i)}{\alpha+1} \right) \omega(i) \leq 1$$

and we conclude that $x \in \lambda_{\varphi,\omega}$. Since for any $n \in \mathbb{N}$

$$\|x_n\|_{\varphi,\omega}^A = \inf_{k>\alpha} \left( 1 + I_{\varphi,\omega}(kx_n) \right) \leq \inf_{k>\alpha} \left( 1 + I_{\varphi,\omega}(kx) \right) = \|x\|_{\varphi,\omega}^A,$$

the inequality $\|x\|_{\varphi,\omega}^A \geq \alpha$ holds. We will show that $\|x\|_{\varphi,\omega}^A = \alpha$. Assume to the contrary that $\|x\|_{\varphi,\omega}^A > \beta > \alpha$; that is, for any $k > 0$ we have

$$\frac{1}{k} \left( 1 + I_{\varphi,\omega}(kx) \right) > \beta. \quad (2.4)$$

Moreover, we get $\frac{1}{k} I_{\varphi,\omega}(kx) \geq \frac{\beta - \alpha}{2} I_{\varphi,\omega} \left( \frac{2}{\beta - \alpha} \cdot x \right) > \frac{\beta + \alpha}{2}$ for any $k \geq \frac{2}{\beta - \alpha}$. By Beppo Levi’s theorem, there exists $n_0$ such that

$$\frac{1}{k} I_{\varphi,\omega}(kx_n) \geq \frac{\beta - \alpha}{2} I_{\varphi,\omega} \left( \frac{2}{\beta - \alpha} \cdot x_n \right) \geq \frac{\beta + \alpha}{2} \quad (2.5)$$

for any $n \geq n_0$ and $k \geq \frac{2}{\beta - \alpha}$. Simultaneously, for $k \leq \frac{2}{\beta + \alpha}$ we get

$$\frac{1}{k} \left( 1 + I_{\varphi,\omega}(kx_n) \right) \geq \frac{1}{k} \geq \frac{\beta + \alpha}{2} \quad (2.6)$$

for any $n \in \mathbb{N}$. Hence, by (2.5) and (2.6) and the definition of the Amemiya norm, we conclude that for any $n \geq n_0$, there exists $k_n \in \left[ \frac{2}{\beta + \alpha}, \frac{2}{\beta - \alpha} \right]$ such that

$$\|x_n\|_{\varphi,\omega}^A = \frac{1}{k_n} \left( 1 + I_{\varphi,\omega}(k_nx_n) \right) \leq \alpha. \quad (2.7)$$

Without loss of generality, passing to a subsequence if necessary, we can assume that $\lim_{n \to \infty} k_n = k_0$. Using Beppo Levi’s theorem again, in virtue of inequality (2.4), we get that there exists $n_1 \geq n_0$ such that

$$\frac{1}{k_0} \left( 1 + I_{\varphi,\omega}(k_0x_n) \right) \geq \frac{\beta + \alpha}{2} \quad (2.8)$$

for any $n \geq n_1$. Since the function $f$ defined for $x_n$ (see formula (2.2)) is continuous on the interval $(0, \lambda_\infty(x_n))$ (see formula (2.3)) and (left continuous at $\lambda_\infty(x_n)$) and equal to infinity for $k > \lambda_\infty(x_n)$ whenever $\lambda_\infty(x_n) < \infty$, there exists $n_2 \geq n_1$ such that

$$\frac{1}{k_n} \left( 1 + I_{\varphi,\omega}(k_nx_n) \right) \geq \frac{\beta + \alpha}{2} \quad (2.9)$$

for any $n \geq n_2$. Hence, by

$$\frac{1}{k_n} \left( 1 + I_{\varphi,\omega}(k_nx_n) \right) \geq \frac{1}{k_n} \left( 1 + I_{\varphi,\omega}(k_nx_n) \right)$$

for any $n \geq n_1$, we get a contradiction with (2.7). \hfill \Box
Theorem 2.3  The Orlicz and Amemiya norms are equal; that is, for any \( x \in \lambda_{\psi, \omega} \), we have
\[
\| x \|_{\psi, \omega}^O = \| x \|_{\psi, \omega}^A. \tag{2.8}
\]

Proof  Let us take any \( x \in \lambda_{\psi, \omega} \). In virtue of the Young inequality (1.1), we get, for any \( k > 0 \) and any \( y \in \lambda_{\psi, \omega} \) with \( I_{\psi, \omega}(y) \leq 1 \), that
\[
\sum_{i=1}^{\infty} x^*(i)y^*(i)\omega(i) = \frac{1}{k} \sum_{i=1}^{\infty} k \cdot x^*(i) y^*(i) \omega(i) \\
\leq \frac{1}{k} \sum_{i=1}^{\infty} (\varphi(kx^*(i)) + \psi(y^*(i))) \omega(i) \\
\leq \frac{1}{k} (1 + I_{\psi, \omega}(kx)),
\]
whence
\[
\| x \|_{\psi, \omega}^O = \sup \left\{ \sum_{i=1}^{\infty} x^*(i)y^*(i)\omega(i) : I_{\psi, \omega}(y) \leq 1 \right\} \leq \frac{1}{k} (1 + I_{\psi, \omega}(kx)).
\]

By the arbitrariness of \( k > 0 \), we obtain \( \| x \|_{\psi, \omega}^O \leq \| x \|_{\psi, \omega}^A \).

Now, we will prove the opposite inequality. First, we will prove it for any nonnegative \( x \in \lambda_{\psi, \omega} \) such that \( m(\text{supp} x) = n < \infty \). We will consider three cases.

Case 1  We start with the assumption that \( b_\varphi = \infty \) and \( \lim_{u \to \infty} \frac{\varphi(u)}{u} = \lim_{u \to \infty} p(u) = \infty. \)
Then, there exist a constant \( k_1 > 0 \) and a non-increasing sequence \( s_1 = (s_1(i))_{i=1}^{\infty} \) with \( s_1(i) \in [l(k_1x^*(i)), p(k_1x^*)] \) for \( i = 1, \ldots, n \) satisfying \( I_{\psi, \omega}(s_1) = 1 \). Hence,
\[
\| x \|_{\psi, \omega}^O \geq \sum_{i=1}^{\infty} x^*(i)s_1(i)\omega(i) = \frac{1}{k_1} \sum_{i=1}^{\infty} k_1 \cdot x^*(i)s_1(i)\omega(i) \\
= \frac{1}{k_1} \sum_{i=1}^{\infty} (\varphi(k_1x^*(i)) + \psi(s_1(i))) \omega(i) \\
= \frac{1}{k_1} (1 + I_{\psi, \omega}(k_1x)) \geq \| x \|_{\psi, \omega}^A. \tag{2.9}
\]

Case 2  Assume now that \( \lim_{u \to \infty} \frac{\varphi(u)}{u} = \lim_{u \to \infty} p(u) = B < \infty \). Notice that this assumption implies that \( b_\varphi = \infty \). Moreover, in this case we have \( B = b_\psi \).

Case 2.1  Suppose that \( \psi(B) \cdot \sum_{i=1}^{n} \omega(i) > 1 \) (it is obvious that this inequality holds if \( \psi(B) = \infty \); in this case we have \( p(u) < \lim_{u \to \infty} p(u) = B \) for all \( u > 0 \) and in consequence \( \psi(p(u)) < \infty \) for the same \( u \)). Then, we have \( a_\psi = \infty \). Moreover, in this case we have \( B = b_\psi \).

Moreover, we have \( I_{\psi, \omega}(p(mx)) = \sum_{i=1}^{n} \psi(p(mx^*(i)))\omega(i) \geq \psi(B - \delta) \sum_{i=1}^{n} \omega(i) > 1. \)

\[\square\] Springer
Simultaneously, defining $z := (B, \ldots, B, 0, 0 \ldots)$, we get

$$I_{\psi, \omega}(z) = \sum_{i=1}^{n} \psi(B) \omega(i) = \psi(B) \sum_{i=1}^{n} \omega(i) \leq 1$$

and

$$\sum_{i=1}^{\infty} x^*(i) z^*(i) \omega(i) = B \sum_{i=1}^{n} x^*(i) \omega(i),$$

whence we obtain $\|x\|_{\psi, \omega}^{O} = B \sum_{i=1}^{n} x^*(i) \omega(i)$. On the other hand,

$$\|x\|_{\psi, \omega}^{A} = \inf_{k>0} \frac{1}{k} \left(1 + I_{\psi, \omega}(kx)\right) \leq \lim_{k \to \infty} \frac{1}{k} \left(1 + I_{\psi, \omega}(kx)\right)$$

$$= \lim_{k \to \infty} \frac{1}{k} I_{\psi, \omega}(kx) = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{n} \varphi(kx^*(i)) \omega(i)$$

$$= \sum_{i=1}^{n} x^*(i) \lim_{k \to \infty} \frac{\varphi(kx^*(i))}{k} \omega(i)$$

$$= B \sum_{i=1}^{n} x^*(i) \omega(i) = \|x\|_{\psi, \omega}^{O}.$$  

**Case 3** Finally, suppose that $b_\psi < \infty$. We will consider two subcases.

Case 3.1. First, we will assume that the equality $l(b_\psi) = \infty$ holds. The function $\varphi$ is finite-valued on $[0, b_\psi)$ and $p(u) \not\to \infty$ as $u \not\to b_\psi$. Therefore, there exists $0 < k < \frac{b_\psi}{x(1)}$ such that $\psi(p(kx^*(1))) \omega(1) > 1$. Moreover, $I_{\psi, \omega}(p(kx)) \leq n \cdot \psi(p(kx^*(1))) \omega(1) < \infty$ and thus we conclude that there exist $k_3 > 0$ and a non-increasing sequence $s_3 = (s_3(i))_{i=1}^{n}$ with $s_3(i) \in [l(k_3x^*(i)), p(k_3x^*(i))]$ for $i = 1, \ldots, n$ satisfying $I_{\psi, \omega}(s_3) = 1$. We may repeat the proceeding from Case 1.

Case 3.2. In the last subcase, we assume that $l(b_\psi) < \infty$. Let $k = \frac{b_\psi}{x(1)}$. If $I_{\psi, \omega}(l(kx)) \geq 1$, then we proceed analogously as in Case 3.1.

Assume now that $I_{\psi, \omega}(l(kx)) < 1$. Since $\psi$ is finite-valued, we can find a number $\beta > l(b_\psi)$ such that

$$\psi(\beta) \omega(1) + \sum_{i=2}^{n} \psi(l(kx^*(1))) \omega(i) = 1.$$  

Defining

$$y = (\beta, l(kx^*(2)), l(kx^*(3)), \ldots, l(kx^*(n)), 0, 0 \ldots),$$

we obtain $y = y^*$, $I_{\psi, \omega}(y) = 1$ and

$$kx^*(i) y^*(i) = \varphi(kx^*(i)) + \psi(y^*(i)) \text{ for any } i = 1, 2, \ldots, n.$$
This yields
\[
\|x\|_{\varphi,\omega} \geq \sum_{i=1}^{n} x^*(i) y^*(i) \omega(i) = \frac{1}{k} \sum_{i=1}^{n} kx^*(i) y^*(i) \omega(i)
\]
\[
= \frac{1}{k} \sum_{i=1}^{n} (\varphi(kx^*(i)) + \psi(y^*(i))) \omega(i) = \frac{1}{k} (1 + I_{\varphi,\omega}(kx)) \geq \|x\|_{A,\varphi,\omega}.
\]

To end this part of the proof, let us take any \(x \in \lambda_{\varphi,\omega}\) and let \(x_n = (|x(1)|, |x(2)|, \ldots, |x(n)|, 0, \ldots)\). Notice that by the definition of the Orlicz norm and Beppo Levi’s theorem, it follows that \(\|x\|_{\varphi,\omega} = \lim_{n \to \infty} \|x_n\|_{\varphi,\omega}\). Simultaneously, by Lemma 2.2, we have \(\|x\|_{A,\varphi,\omega} = \lim_{n \to \infty} \|x_n\|_{A,\varphi,\omega}\). Therefore, we eventually obtain
\[
\|x\|_{\varphi,\omega} = \lim_{n \to \infty} \|x_n\|_{\varphi,\omega} \geq \lim_{n \to \infty} \|x_n\|_{A,\varphi,\omega} = \|x\|_{A,\varphi,\omega}
\]
and the proof is finished. \(\square\)

Remark 2.4 (i) Let us recall that in paper [19], Hudzik and Maligranda proved the equality of norms \(\| \cdot \|_{\varphi,\omega}\) and \(\| \cdot \|_{A,\varphi,\omega}\) in the Orlicz spaces generated by any Orlicz function.

(ii) From now on, while considering the Orlicz norm, we will use norms (1.6) and (1.7) interchangeably. Since the Fatou property implies completeness (see [28, Theorem 1]), the space \((\lambda_{\varphi,\omega}, \| \cdot \|_{\varphi,\omega})\) is a Banach lattice (see [26, 35]). Moreover, it is easy to show that \((\lambda_{\varphi,\omega}, \| \cdot \|_{\varphi,\omega})\) is a Banach symmetric space.

Lemma 2.5 The following assertions are true:

(i) For any \(x \in \lambda_{\varphi,\omega}\), we have \(\|x\|_{\varphi,\omega} \leq \|x\|_{\varphi,\omega} \leq 2\|x\|_{\varphi,\omega}\);

(ii) If \(\|x\|_{\varphi,\omega} \leq 1\), then \(I_{\varphi,\omega}(x) \leq \|x\|_{\varphi,\omega}\);

(iii) If \(k < \lambda_{\omega} = \lambda_{\infty}(x)\), then \(I_{\varphi,\omega}(p(kx^*)) < \infty\);

(iv) Let \(x^*(1) \neq a_\varphi\) whenever \(a_\varphi > 0\) and \(p(a_\varphi) > 0\). If \(\|x\|_{\varphi,\omega} \leq 1\), then \(I_{\varphi,\omega}(p(x^*)) \leq 1\).

Proof (i) By Lemma 2.1 we get
\[
\|x\|_{\varphi,\omega} = \inf_{k > 0} \left\{ \max \left( \frac{1}{k} I_{\varphi,\omega}(kx) \right) \right\} \leq \inf_{k > 0} \left( \frac{1}{k} + I_{\varphi,\omega}(kx) \right) \]
\[
= \|x\|_{\varphi,\omega} \leq \inf_{k > 0} \left\{ 2 \max \left( \frac{1}{k} I_{\varphi,\omega}(kx) \right) \right\} = 2\|x\|_{\varphi,\omega}.
\]

(ii) In virtue of (i) we have \(\|x\|_{\varphi,\omega} \leq \|x\|_{\varphi,\omega} \leq 1\). By the properties of the Luxemburg norm, we get \(I_{\varphi,\omega}(x) \leq \|x\|_{\varphi,\omega}\); whence \(I_{\varphi,\omega}(x) \leq \|x\|_{\varphi,\omega}\).

(iii) Assume to the contrary that \(I_{\varphi,\omega}(p(kx^*)) = \infty\). Since \(\psi(p(kx^*(1))) < \infty\) (by the assumption \(k < \lambda_{\omega}\) we have \(kx^*(1) < b_\varphi\) whenever \(b_\varphi < \infty\), we get \(\sum_{i=1}^{\infty} \omega(i) = \infty\) and \(\lim_{i \to \infty} kx^*(i) = a_\varphi\). Hence, by the assumption \(k < \lambda_{\omega}\), we obtain \(\lim_{i \to \infty} kx^*(i) = a_\varphi = 0\). Let us define
\[
x_n = (x(1), x(2), \ldots, x(n), 0, 0, \ldots)
\]
for any \(n \in \mathbb{N}\). Since \(|x_n| / |x|\), we obtain \(x_n^* \neq x^*\). In consequence, \(\|x_n\|_{\varphi,\omega} / \|x\|_{\varphi,\omega}\) and \(I_{\varphi,\omega}(p(kx_n^*)) / I_{\varphi,\omega}(p(kx^*))\). So, there exists \(n_0 \in \mathbb{N}\) such that \(1 < I_{\varphi,\omega}(p(kx_n^*)) < \infty\) for all \(n \geq n_0\). For any fixed \(h \in (k, \lambda_{\omega})\) and any \(n \geq n_0\), we have that
\[
I_{\varphi,\omega}(hx_n) \geq \sum_{i=1}^{\infty} kx_n^*(i) p(kx_n^*(i)) \omega(i) - I_{\varphi,\omega}(p(kx_n^*))\]
\(\square\) Springer
\[ I_{\varphi, \omega} (kx_n) = \sum_{i=1}^{\infty} kx_n^*(i) p(kx_n^*(i)) \omega(i) - I_{\psi, \omega} (p(kx_n^*)) . \]

Therefore, for any \( n \geq n_0 \) we obtain
\[
\frac{1}{h} (1 + I_{\varphi, \omega} (hx_n)) - \frac{1}{k} (1 + I_{\varphi, \omega} (kx_n)) \\
= \frac{h - k}{h + k} \left( -1 + \frac{k}{h - k} (I_{\varphi, \omega} (hx_n) - I_{\varphi, \omega} (kx_n)) - I_{\varphi, \omega} (kx_n) \right) \\
\geq \frac{h - k}{h + k} \left( -1 + \frac{k}{h - k} \sum_{i=1}^{\infty} (h - k) x_n^*(i) p(kx_n^*(i)) \omega(i) - I_{\varphi, \omega} (kx_n) \right) \\
= \frac{h - k}{h + k} (I_{\psi, \omega} (p(kx_n^*)) - 1) > 0.
\]
(2.12)

Whence for the same \( n \), we get
\[ I_{\psi, \omega} (p(kx_n^*)) \leq \frac{k}{h - k} (1 + I_{\varphi, \omega} (hx)) + 1 < \infty, \]
which gives a contradiction with the fact that \( I_{\psi, \omega} (p(kx_n^*)) \not\rightarrow I_{\psi, \omega} (p(kx^*)) = \infty. \)

(iv) For \( x = 0 \), by \( \psi(p(0)) = \psi(a_{\varphi}) = 0 \), we have \( I_{\psi, \omega} (p(x^*)) = 0 \). Let now \( x \neq 0 \), that is \( x^*(1) > 0 \).

If \( 0 < x^*(1) \leq a_{\varphi} \), then \( f(1) = 1 \) (see formula (2.2)), so \( \|x\|_{\psi, \omega}^Q \leq 1 \). Moreover, by condition \( x^*(1) \neq a_{\varphi} \) whenever \( a_{\varphi} > 0 \) and \( p(a_{\varphi}) > 0 \), we get \( p(x^*(i)) = 0 \) for any \( i \in \mathbb{N} \), whence \( I_{\psi, \omega} (p(x^*)) = 0 \).

Now, assume that \( x^*(1) > a_{\varphi} \geq 0 \). Since \( f(k) > 1 \) for any \( k \in (0, 1] \), by \( \|x\|_{\psi, \omega}^Q \leq 1 \), we have \( 1 < \lambda_{\infty} \). Hence, by (iii), we get \( I_{\psi, \omega} (p(x^*)) \leq 1 \). If \( 1 < I_{\psi, \omega} (p(x^*)) \), then, using the convexity of the modular \( I_{\psi, \omega} \), we obtain
\[
I_{\psi, \omega} \left( \frac{p(x^*)}{I_{\psi, \omega} (p(x^*))} \right) \leq \frac{1}{I_{\psi, \omega} (p(x^*))} 
I_{\psi, \omega} (p(x^*)) = 1.
\]
Hence,
\[
1 \geq \|x\|_{\psi, \omega}^Q \geq \sum_{i=1}^{\infty} x^*(i) \frac{p(x^*(i))}{I_{\psi, \omega} (p(x^*))} \omega(i) \\
= \frac{1}{I_{\psi, \omega} (p(x^*))} \sum_{i=1}^{\infty} x^*(i) p(x^*(i)) \omega(i) \\
= \frac{1}{I_{\psi, \omega} (p(x^*))} (I_{\psi, \omega} (x) + I_{\psi, \omega} (p(x^*)) > 1, \]
a contradiction. \( \square \)

**Remark 2.6** Notice that if \( x^*(1) = a_{\varphi} \) whenever \( a_{\varphi} > 0 \) and \( p(a_{\varphi}) > 0 \), then the implication \( \|x\|_{\psi, \omega}^Q \leq 1 \Rightarrow I_{\psi, \omega} (p(x^*)) \leq 1 \) is not always true. This can be illustrated by the following example:

**Example 2.7** Define
\[
\varphi(u) = 0 \text{ for } u \in [0, 1) \text{ and } \varphi(u) = u^2 - 1 \text{ for } u \geq 1,
\]
\[
\omega_1(1) = 1 \text{ and } \omega_2(1) = \frac{1}{2}.
\] We have \( a_{\varphi} = 1 \) and
\[
p(u) = 0 \text{ for } u \in [0, 1) \text{ and } p(u) = 2u \text{ for } u \geq 1.
\]
For $x = (1, 0, 0, \ldots)$, we get $\|x\|^{(\varphi, \omega)} = 1$, $I_{\varphi, \omega_1}(p(x^*)) = \psi(p(1)) \cdot \omega_1(1) = \psi(2) \cdot \omega_1(1) = 2$ and $\|x\|^{(\varphi, \omega_2)} = 1$, $I_{\varphi, \omega_2}(p(x^*)) = \psi(p(1)) \cdot \omega_2(1) = \psi(2) \cdot \omega_2(1) = 1$.

**Lemma 2.8** Let $(x_n)$ be a sequence in $\lambda_{\varphi, \omega}$. The following assertions are true:

(i) If $\lim_{n \to \infty} \|x_n\|^{(\varphi, \omega)} = 0$, then $\lim_{n \to \infty} I_{\varphi, \omega}(x_n) = 0$.

(ii) Let $\sum_{i=1}^{\infty} \omega(i) = \infty$. Then the implication: if $\lim_{n \to \infty} I_{\varphi, \omega}(x_n) = 0$, then $\lim_{n \to \infty} \|x_n\|^{(\varphi, \omega)} = 0$ holds true if and only if $\varphi$ satisfies the condition $\Delta_2(0)$.

(iii) Let now $\sum_{i=1}^{\infty} \omega(i) < \infty$. Then the implication: if $\lim_{n \to \infty} I_{\varphi, \omega}(x_n) = 0$ then $\lim_{n \to \infty} \|x_n\|^{(\varphi, \omega)} = 0$ holds true if and only if $a_\varphi = 0$.

**Proof** We will give a short proof of the above lemma for the sake of completeness.

(i) If $\lim_{n \to \infty} \|x_n\|^{(\varphi, \omega)} = 0$, then we have $\|x_n\|^{(\varphi, \omega)} \leq 1$ for $n$ large enough. Whence by Lemma 2.5 (ii), we get $I_{\varphi, \omega}(x_n) \leq \|x_n\|^{(\varphi, \omega)}$ for the same $n$. This gives $\lim_{n \to \infty} I_{\varphi, \omega}(x_n) = 0$.

(ii) Assume that $\sum_{i=1}^{\infty} \omega(i) = \infty$ and the function $\varphi$ satisfies the condition $\Delta_2(0)$. By [31, Theorem 1.6] and Lemma 2.5(i), we only need to show that $\lim_{n \to \infty} I_{\varphi, \omega}(2x_n) = 0$ whenever $\lim_{n \to \infty} I_{\varphi, \omega}(x_n) = 0$. Take any fixed $\varepsilon > 0$. Without loss of generality, we may assume that $\varphi(u_0) \omega(1) \geq \varepsilon$, where $u_0$ is the constant from the condition $\Delta_2(0)$. Since $\lim_{n \to \infty} I_{\varphi, \omega}(x_n) = 0$, there exists $k \in \mathbb{N}$ such that $I_{\varphi, \omega}(x_n) < \frac{\varepsilon}{K}$ for any $n \geq k$, where $K$ is the constant from the condition $\Delta_2(0)$. Since $x_n(1) \leq u_0$ for any $n \geq k$, we obtain

$$I_{\varphi, \omega}(2x_n) = \sum_{i=1}^{\infty} \varphi(2x_n(i)) \omega(i) \leq K \sum_{i=1}^{\infty} \varphi(x_n(i)) \omega(i) = K I_{\varphi, \omega}(x_n) < \varepsilon$$

for the same $n$. By the arbitrariness of $\varepsilon > 0$, we conclude that $\lim_{n \to \infty} I_{\varphi, \omega}(2x_n) = 0$. Now we shall show the necessity of the condition $\Delta_2(0)$. Suppose first that $a_\varphi > 0$ and define $x_n = (a_\varphi, 0, 0, \ldots)$ for any $n \in \mathbb{N}$. We have that $I_{\varphi, \omega}(x_n) = 0$ and $\|x_n\|^{(\varphi, \omega)} := a > 0$ for all $n \in \mathbb{N}$. Assume now that $a_\varphi = 0$ and $\varphi$ does not satisfy the condition $\Delta_2(0)$. Then, we find a sequence $(u_n)_{n=1}^{\infty}$ decreasing to zero and such that

$$\varphi(2u_n) \geq 2^{n+1} \varphi(u_n), \quad \varphi(u_n) \omega(1) \leq \frac{1}{2^{n+1}}.$$

Let $j_n, n \in \mathbb{N}$ be such that $\frac{1}{2^{n+1}} < \varphi(u_n) \sum_{i=1}^{j_n} \omega(i) \leq \frac{1}{2^n}$. Define $x_n = \sum_{i=1}^{j_n} u_n e_i$ for any $n \in \mathbb{N}$. We have $I_{\varphi, \omega}(x_n) \leq \frac{\varepsilon}{K}$ for all $n \in \mathbb{N}$. Simultaneously $I_{\varphi, \omega}(2x_n) > 1$, whence $\|x_n\|^{(\varphi, \omega)} \geq \|x_n\|^{(\varphi, \omega)} \geq \frac{1}{2^n}$ for any $n \in \mathbb{N}$.

(iii) Let $\sum_{i=1}^{\infty} \omega(i) < \infty$. We may show the necessity of the condition $a_\varphi = 0$ analogously as in (ii). Now we will prove the sufficiency of this condition. In a fashion similar to (ii), we only need to show that $\lim_{n \to \infty} I_{\varphi, \omega}(2x_n) = 0$ whenever $\lim_{n \to \infty} I_{\varphi, \omega}(x_n) = 0$. Take any fixed $\varepsilon > 0$. Since $\lim_{n \to \infty} x_n(1) = 0$, we will find $k \in \mathbb{N}$ such that

$$\varphi(2x_n(1)) < \frac{\varepsilon}{\sum_{i=1}^{\infty} \omega(i)}$$
for any \( n \geq k \). Therefore, for the same \( n \) we obtain

\[
I_{\varphi,\omega} (2x_n) = \sum_{i=1}^{\infty} \varphi (2x^*_n (i)) \omega (i) \leq \varphi (2x^*_n (1)) \sum_{i=1}^{\infty} \omega (i) < \varepsilon.
\]

By the arbitrariness of \( \varepsilon > 0 \), it follows that \( \lim_{n \to \infty} I_{\varphi,\omega} (2x_n) = 0 \). \( \square \)

For any \( x \in \lambda_{\varphi,\omega} \setminus \{0\} \), we ask the question if the infimum in formula (1.7) is attained, that means if there exists \( k > 0 \) such that

\[
\|x\|_{\varphi,\omega}^O = \|x\|_{\varphi,\omega}^A = \frac{1}{k} (1 + I_{\varphi,\omega} (kx)) = \frac{1}{k} \left( 1 + \sum_{i=1}^{\infty} a^* (i) p (kx^* (i)) \omega (i) \right).
\]

In order to answer this question, we define constants as following:

\[
k^* = k^* (x) = \inf \{ k > 0 : I_{\varphi,\omega} (p (kx^*)) \geq 1 \},
\]

\[
k^{**} = k^{**} (x) = \sup \{ k > 0 : I_{\varphi,\omega} (p (kx^*)) \leq 1 \}.
\]

First, we shall prove two lemmas.

**Lemma 2.9** For any \( x \in \lambda_{\varphi,\omega} \setminus \{0\} \), we have

\[
0 < k^* \leq k^{**} \leq \lambda_\infty \leq \infty,
\]

where \( \lambda_\infty \) is defined by (2.3).

**Proof** The second and fourth inequalities are obvious. Now, we will prove that \( k^* > 0 \). If \( a_\varphi > 0 \), then for any \( k < \frac{a_\varphi}{x^*_n (1)} \), we have \( kx^* (i) < a_\varphi \) for all \( i \in \mathbb{N} \). Whence \( p (kx^* (i)) = 0 \) for the same \( i \). Therefore \( I_{\varphi,\omega} (p (kx^*)) = 0 \) for any \( k < \frac{a_\varphi}{x^*_n (1)} \) and consequently \( k^* \geq \frac{a_\varphi}{x^*_n (1)} \).

Suppose now that \( a_\varphi = 0 \). For \( k_1 = \min \left( 1, \frac{1}{\|x\|_{\varphi,\omega}^O} \right) \), we have \( \|k_1 x\|_{\varphi,\omega}^O \leq 1 \). By Lemma 2.5(iv), we obtain \( I_{\varphi,\omega} (p (k_1 x^*)) \leq 1 \). If \( I_{\varphi,\omega} (p (k_1 x^*)) < 1 \), then \( k^* \geq k_1 \). Assume now that \( I_{\varphi,\omega} (p (k_1 x^*)) = 1 \). Then \( \psi (p (k_1 x^* (1))) > \psi (p (0)) = \psi (a_\varphi) = 0 \) and in consequence \( p (k_2 x^* (1)) > p (0) = a_\varphi \). Since \( p \) is right-continuous, there exists \( k_2 \in (0, k_1) \) such that \( p (k_2 x^* (1)) \leq \frac{1}{2} (p (0) + p (k_1 x^* (1))) \). Thus \( \psi (p (k_2 x^* (1))) < \psi (p (k_1 x^* (1))) \), and this gives \( I_{\varphi,\omega} (p (k_2 x^*)) < 1 \). Therefore, we have \( k^* \geq k_2 \).

Finally, we will show that \( k^{**} \leq \lambda_\infty \). For any \( k \in (0, k^{**}) \), we have \( I_{\varphi,\omega} (p (kx^*)) \leq 1 \). This yields (see (2.2))

\[
f (k) \leq \frac{1}{k} \left( 1 + I_{\varphi,\omega} (kx) + I_{\varphi,\omega} (p (kx^*)) \right) = \frac{1}{k} \left( 1 + \sum_{i=1}^{\infty} kx^* (i) p (kx^* (i)) \omega (i) \right)
\]

\[
= \frac{1}{k} + \sum_{i=1}^{\infty} x^* (i) p (kx^* (i)) \omega (i) \leq \frac{1}{k} + \|x\|_{\varphi,\omega}^O < \infty
\]

for the same \( k \). As a result, we obtain \( k^{**} \leq \lambda_\infty \). Moreover, if \( k^{**} < \infty \), by the left continuity of the modular, we get \( f (k^{**}) \leq \frac{1}{k^{**}} + \|x\|_{\varphi,\omega}^O \). \( \square \)

As previously noted, the function \( f (k) \) is continuous on the interval \( (0, \lambda_\infty) \) and left-continuous at \( \lambda_\infty \) whenever \( \lambda_\infty < \infty \). We will also apply the following

**Lemma 2.10** For any \( x \in \lambda_{\varphi,\omega} \setminus \{0\} \), the function \( f (k) \) is strictly decreasing on the interval \( (0, k^*) \) and strictly increasing on the interval \( (k^{**}, \lambda_\infty) \) whenever \( k^{**} < \lambda_\infty \). Moreover, if \( k^* < k^{**} \), then \( f (k) \) is constant on the interval \( [k^*, k^{**}] \) and \( f (k) = \|x\|_{\varphi,\omega}^O \) for any \( k \in [k^*, k^{**}] \) (if \( k^{**} = \infty \), then we write \( [k^*, \lambda_\infty] \) in place of \( [k^*, k^{**}] \) respectively).
This gives
\[
\frac{1}{k_1} (1 + I_{\varphi,\omega}(k_1 x)) - \frac{1}{k_2} (1 + I_{\varphi,\omega}(k_2 x))
\]
\[
= \frac{k_2 - k_1}{k_1 \cdot k_2} \left( 1 + \frac{k_2}{k_2 - k_1} (I_{\varphi,\omega}(k_1 x) - I_{\varphi,\omega}(k_2 x)) + I_{\varphi,\omega}(k_2 x) \right)
\]
\[
\geq \frac{k_2 - k_1}{k_1 \cdot k_2} \left( 1 + \frac{k_2}{k_2 - k_1} \sum_{i=1}^{\infty} (k_1 - k_2) x^*(i) p(k_2 x^*(i)) \omega(i) + I_{\varphi,\omega}(k_2 x) \right)
\]
\[
= \frac{k_2 - k_1}{k_1 \cdot k_2} (1 - I_{\varphi,\omega}(p(k_2 x^*)))
\]
Hence, by \(k_2 < k^*\), we get
\[
f(k_1) - f(k_2) \geq \frac{k_2 - k_1}{k_1 \cdot k_2} (1 - I_{\varphi,\omega}(p(k_2 x^*))) > 0,
\]
whence, in virtue of the arbitrariness of \(k_1, k_2\), we conclude that the function \(f\) is strictly decreasing on the interval \((0, k^*)\).

Suppose now that \(k^* < \lambda_\infty\) and \(k^* < k_2 < k_1 < \lambda_\infty\). We have that \(I_{\varphi,\omega}(k_2 x) < \infty\) and \(I_{\varphi,\omega}(k_1 x) < \infty\). By Lemma 2.5(iii) and the definition of \(k_2\), we get \(1 < I_{\varphi,\omega}(p(k_2 x^*)) < \infty\). Proceeding analogously as in the previous case (see also (2.12)), we get
\[
f(k_1) - f(k_2) \geq \frac{k_1 - k_2}{k_1 \cdot k_2} (I_{\varphi,\omega}(p(k_2 x^*)) - 1) > 0,
\]
whence it follows that the function \(f\) is strictly increasing on the interval \((k^*, \lambda_\infty)\).

In the final part of the proof, we will assume that \(k^* < k^{**}\). By the definition of the constants \(k^*\) and \(k^{**}\), we have \(I_{\varphi,\omega}(p(k x^*)) = 1\) for any \(k \in (k^*, k^{**})\), and thus we find that
\[
\|x\|_{\varphi,\omega}^O \leq \frac{1}{k} (1 + I_{\varphi,\omega}(k x)) = f(k) = \frac{1}{k} (I_{\varphi,\omega}(p(k x^*)) + I_{\varphi,\omega}(k x))
\]
\[
= \sum_{i=1}^{\infty} x^*(i) p(k x^*(i)) \omega(i) \leq \|x\|_{\varphi,\omega}^O
\]
for the same \(k\). By the continuity (left continuity at \(k^{**}\) if \(k^{**} = \lambda_\infty < \infty\)) of the function \(f\), we get \(f(k^*) = \|x\|_{\varphi,\omega}^O\) and \(f(k^{**}) = \|x\|_{\varphi,\omega}^O\) whenever \(k^{**} < \infty\). \(\square\)

For any \(x \in \lambda_{\varphi,\omega} \setminus \{0\}\), we define \(K(x)\) as follows: If \(k^{**} < \infty\), then \(K(x) = [k^*, k^{**}]\); if \(k^* < k^{**} = \infty\), then \(K(x) = [k^*, \infty)\); and if \(k^* = \infty\), then \(K(x) = \emptyset\).

**Theorem 2.11** Let \(x \in \lambda_{\varphi,\omega} \setminus \{0\}\). If \(k^* < \infty\), then
\[
\|x\|_{\varphi,\omega}^O = \frac{1}{k} (1 + I_{\varphi,\omega}(k x))
\]
f for any \(k \in K(x)\). If \(k^* = \infty\), then we have
\[
\|x\|_{\varphi,\omega}^O = \lim_{k \to \infty} \frac{1}{k} (1 + I_{\varphi,\omega}(k x)). \tag{2.13}
\]
**Proof** By Theorem 2.3 we have
\[ \|x\|_{\varphi,\omega}^2 = \|x\|_{\varphi,\omega}^4 = \inf_{k > 0} \frac{1}{k} (1 + I_{\varphi,\omega}(kx)) = \inf_{k > 0} f(k). \]
If \( k^* < \infty \), then by Lemma 2.10 the function \( f \) attains its infimum for any \( k \in K(x) \). In the second case, if \( k^* = \infty \), the function \( f \) is decreasing on the interval \((0, \infty)\), whence we get (2.13).

**Remark 2.12** Let us notice that if \( b_\varphi = \infty \) and \( \lim_{u \to \infty} \frac{\varphi(u)}{u} = \lim_{u \to \infty} p(u) = \infty \), then for any \( x \in \lambda_{\varphi,\omega} \setminus \{0\} \) we have \( k^{**} < \infty \). Indeed, if \( \lambda_\infty = \infty \), then there exists \( k \) such that \( \psi(p(kx^*(1))) \omega(1) > 1 \) and \( k^{**} \leq k \). In the case when \( b_\varphi < \infty \), we have \( k^{**} \leq \lambda_\infty \leq \frac{b_\varphi}{\varphi(1)} \) for any \( x \in \lambda_{\varphi,\omega} \setminus \{0\} \).

Suppose now that \( \lim_{u \to \infty} \frac{\varphi(u)}{u} = \lim_{u \to \infty} p(u) = B < \infty \) and \( x \in \lambda_{\varphi,\omega} \setminus \{0\} \). First, we will show that if \( \psi(B) \sum_{i=1}^{m(\supp x)} \omega(i) > 1 \), then \( k^{**} < \infty \). For this end, let \( n_x \in \mathbb{N} \) be such that
\[ \psi(B) \sum_{i=1}^{n_x} \omega(i) > 1 \tag{2.14} \]
(in the case when \( m(\supp x) < \infty \), we take \( n_x = m(\supp x) \) and in the case when \( m(\supp x) = \infty \), we can find \( n_x \in \mathbb{N} \), for which inequality (2.14) holds). Since \( \lim_{u \to \infty} p(u) = B \), there exists \( u_x > 0 \) for which \( \psi(p(u_x)) \sum_{i=1}^{n_x} \omega(i) > 1 \). Defining \( k_x = \sup \frac{\varphi_x}{\varphi(i_x)}, \) we obtain that \( I_{\psi,\omega}(p(kx^*)) > 1 \), so \( k^{**} \leq k_x \).

In the case of \( \psi(B) \sum_{i=1}^{m(\supp x)} \omega(i) < 1 \) or \( \psi(B) \sum_{i=1}^{m(\supp x)} \omega(i) = 1 \) and \( p(u) < B \) for any \( u > 0 \), we get \( k^* = \infty \), which means \( K(x) = \emptyset \).

Finally, assume that \( \psi(B) \frac{n_x}{\sum_{i=1}^{m(\supp x)} \omega(i)} = 1 \) and \( p(u) = B \) starting from some \( u_0 > 0 \). If \( m(\supp x) < \infty \), then \( k^* \leq \frac{\varphi_x}{x'(n_x)} < k^{**} = \infty \), where \( n_x = m(\supp x) \) as above. Let now \( m(\supp x) = \infty \). If \( \psi(B) \frac{n_x}{\sum_{i=1}^{\infty} \omega(i)} < k^{**} = \infty \), then \( x \in \lambda_{\varphi,\omega} \setminus \{0\} \) such that \( \psi(1) = x \in \Phi(1), \) e.g. \( x = (x(1), 0, 0, \ldots), x(1) \in \mathbb{N} \). The case \( \psi(B) \omega(1) = 1 \) is more complicated, as the following example shows:

**Example 2.13** (i) Let \( \varphi(u) = 0 \) for \( u \in [0, 1] \) and \( \varphi(u) = u - 1 \) for \( u > 1 \). Then \( p(u) = 0 \) for \( u \in [0, 1] \) and \( p(u) = 1 \) for \( u > 1 \). We have \( B = 1 \) and \( \psi(B) = 1 \). Assuming that \( \omega(1) = 1 \), we obviously get \( \psi(B) \omega(1) = 1 \). Let us take an arbitrary \( x \in \lambda_{\varphi,\omega} \setminus \{0\} \). For \( k_x = \frac{1}{x'(1)} \), we get
\[ I_{\psi,\omega}(p(kx^*)) \geq \psi(p(kx \cdot x^*(1))) \omega(1) = 1, \]
whence \( k^* \leq k_x \). In particular, if \( m(\supp x) = 1 \), we get \( k^{**} = \infty \).

(ii) Define now \( p(u) = 1 - \frac{1}{u^2} \) for \( u \in [n - 1, n] \), \( n \in \mathbb{N} \), \( \varphi(u) = \int_0^u p(t) dt \) and \( \omega(1) = 1 \). We have that \( B = 1 \), \( \psi(B) = 1 \), and \( \psi(B) \omega(1) = 1 \). Since \( p(t) < B \) for any \( t \geq 0 \), we get \( \psi(p(t)) < \psi(B) \) for the same t. So, if \( m(\supp x) = 1 \), then \( I_{\psi,\omega}(p(kx^*)) = 1 \). Springer
\( \psi(p(kx^*(1)))\omega(1) < 1 \) for any \( k > 0 \), whence \( K(x) = 0 \). On the other hand, if \( \omega(2) > 0 \), we obtain \( k^{**} < \infty \) whenever \( m(\sup \chi) \geq 2 \).

(iii) Finally, assume that \( \varphi(u) = u \) for \( u \geq 0 \). We have that \( p(u) = 1 \) for \( u \geq 0 \), \( B = 1 \) and \( \psi(B) = 0 \). Hence, for any \( x \in \lambda_{\varphi,\omega} \), we get \( I_{\psi,\omega}(p(kx^*)) = 0 \) for every \( k > 0 \) and, consequently, \( K(x) = \emptyset \).

Lemma 2.14 Let \( A \subset N \) be such that \( \sum_{i=1}^{m(A)} \omega(i) < \infty \). If \( \lim_{u \to \infty} \frac{\varphi(u)}{u} = B \) and \( \psi(B) \sum_{i=1}^{m(A)} \omega(i) \leq 1 \), then

\[
\|\chi_A\|_{\psi,\omega}^O = B \sum_{i=1}^{m(A)} \omega(i). \tag{2.15}
\]

In any other case,

\[
\|\chi_A\|_{\psi,\omega}^O = \psi^{-1} \left( \frac{1}{\sum_{i=1}^{m(A)} \omega(i)} \right) \sum_{i=1}^{m(A)} \omega(i), \tag{2.16}
\]

where \( \psi^{-1} \) is the generalized inverse of the function \( \psi \) defined by \( \psi^{-1}(v) = \inf \{u \geq 0 : \psi(u) > v\} \) for \( v \in [0, \infty) \) and \( \psi^{-1}(\infty) = \lim_{v \to \infty} \psi^{-1}(v) \) (see [32] and [22, Lemma 3.1]).

Proof Let \( A \subset N \) be such that \( \sum_{i=1}^{m(A)} \omega(i) < \infty \) (if \( \sum_{i=1}^{\infty} \omega(i) = \infty \), then \( m(A) < \infty \)). In the next part of the proof, the symbols \( k^* \) and \( k^{**} \) denote \( k^*(\chi_A) \) and \( k^{**}(\chi_A) \), respectively. We will consider three cases.

Case 1 First let us assume that \( b_\varphi = \infty \) and \( \lim_{k \to \infty} \frac{\varphi(k)}{k} = \infty \). Then \( k^{**} < \infty \) (see Remark 2.12). If \( k^* < k^{**} \), then for any arbitrary fixed \( k_0 \in (k^*, k^{**}) \), we have

\[
I_{\psi,\omega}(p(k_0(\chi_A)^*)) = \psi(p(k_0)) \sum_{i=1}^{m(A)} \omega(i) = 1 \tag{2.17}
\]

and this gives

\[
p(k_0) = \psi^{-1} \left( \frac{1}{\sum_{i=1}^{m(A)} \omega(i)} \right). \tag{2.18}
\]

By the equality in the Young’s inequality (see (1.2)) and equalities (2.17) and (2.18), we get

\[
\|\chi_A\|_{\psi,\omega}^O = \frac{1}{k_0} \left\{ 1 + I_{\varphi,\omega}(k_0(\chi_A)) \right\} = \frac{1}{k_0} \left\{ 1 + \varphi(k_0) \sum_{i=1}^{m(A)} \omega(i) \right\}
\]

\[
= \frac{1}{k_0} \left\{ 1 + (k_0 \cdot p(k_0) - \psi(p(k_0))) \sum_{i=1}^{m(A)} \omega(i) \right\}
\]

\[
= \frac{1}{k_0} \left\{ 1 + k_0 \cdot p(k_0) \sum_{i=1}^{m(A)} \omega(i) - 1 \right\}
\]
As a consequence, we get (2.16) again.

such that replacing \( k_0 \) by \( k^{**} \). In the case when \( I_{\psi,\omega} (p(k^{**}(\chi_A)^*)) > 1 \), we get \( I_{\psi,\omega} (l(k^{**}(\chi_A)^*)) \leq 1 \). Hence, there exists \( v \in \partial \varphi (k^{**}) = [l(k^{**}), p(k^{**})] \) (see (1.2)) such that

\[
\psi (v) \sum_{i=1}^{m(A)} \omega (i) = 1.
\]

Proceeding analogously as above, we obtain again (2.16).

**Case 2** Suppose now that \( b_\varphi < \infty \). Then \( k^* \leq k^{**} \leq b_\varphi \) (see Remark 2.12). If \( k^* < b_\varphi \) we may again repeat the proof of Case 1 to get (2.16). Assume now that \( k^* = k^{**} = b_\varphi \). But then \( I_{\psi,\omega} (l(k^{**}(\chi_A)^*)) \leq 1 \) and we will find \( w \in \partial \varphi (k^{**}) = \partial \varphi (b_\varphi) = [l(b_\varphi), \infty) \) (see (1.2)) such that

\[
\psi (w) \sum_{i=1}^{m(A)} \omega (i) = 1.
\]

As a consequence, we get (2.16) again.

**Case 3** Finally, suppose that \( \lim_{u \to -\infty} \varphi(u) = B < \infty \). If \( \psi (B) \sum_{i=1}^{m(A)} \omega (i) > 1 \), then \( k^{**} < \infty \) and repeating the argumentation from Case 1, we get (2.16) (see Remark 2.12). In the case when \( \psi (B) \sum_{i=1}^{m(A)} \omega (i) \leq 1 \), by [7, Lemma 1], we get (2.15).

\[
\text{Remark 2.15} \quad \text{It is worth mentioning that if } \psi (B) \sum_{i=1}^{m(A)} \omega (i) = 1, \text{ then we have that}
\]

\[
\| \chi_A \|_{\psi,\omega}^{O} = B \sum_{i=1}^{m(A)} \omega (i) = \psi^{-1} \left( \frac{1}{\sum_{i=1}^{m(A)} \omega (i)} \right) \sum_{i=1}^{m(A)} \omega (i).
\]

On the other hand, if \( \psi (B) \sum_{i=1}^{m(A)} \omega (i) < 1 \), by the equality \( B = b_\varphi \) we have

\[
\psi (b_\varphi) = \psi (B) < \frac{1}{\sum_{i=1}^{m(A)} \omega (i)}
\]

and \( \psi (u) = \infty \) for \( u > b_\varphi \) and the function \( \psi \) does not assume the value \( 1/ \sum_{i=1}^{m(A)} \omega (i) \).

Let \( E \subset l^0 \) be a H Köthe sequence space. An element \( x \in E \) is said to be order continuous if for any sequence \( (x_n) \) in \( E_+ \) (the positive cone of \( E \)) with \( x_n \leq |x| \) and \( x_n \to 0 \) coordinatewise, there holds \( \| x_n \|_E \to 0 \). The subspace \( E_a \) of all order continuous elements in \( E \) is an order ideal in \( E \). The space \( E \) is called order continuous if \( E_a = E \) (see [26]).

Since both the Luxemburg and the Orlicz norms are equivalent (see Lemma 2.5(i)), the subspace of order continuous elements for both of these norms is the same (as a subset of the
elements of the space \( \lambda_{\varphi, \omega} \). We will denote it as \((\lambda_{\varphi, \omega})_a\). By Theorem 4.2 in [11], we get \((\lambda_{\varphi, \omega})_a = c_0\) whenever \(\sum_{i=1}^{\infty} \omega(i) < \infty\) or \(a_\varphi > 0\). On the other hand, if \(\sum_{i=1}^{\infty} \omega(i) = \infty\) and \(a_\varphi = 0\), then \((\lambda_{\varphi, \omega})_a = \lambda_{\varphi, \omega}^0\) (see formula (1.4)). We get also the following

**Theorem 2.16** The space \((\lambda_{\varphi, \omega}, \|\cdot\|_{\varphi, \omega}^O)\) is order continuous if and only if \(\sum_{i=1}^{\infty} \omega(i) = \infty\) and \(\varphi \in \Delta_2(0)\).

This theorem is a consequence of the fact that the space \((\lambda_{\varphi, \omega}, \|\cdot\|_{\varphi, \omega})\) (with the Luxemburg norm) is order continuous if and only if \(\sum_{i=1}^{\infty} \omega(i) = \infty\) and \(\varphi \in \Delta_2(0)\) (see [11, Theorems 2.4 and 4.3] and [4, Theorem 8], cf. also [20, Theorem 2.4]) as well as the equivalence of the Luxemburg and the Orlicz norms (see Lemma 2.5(i)).

### 3 Monotonicity Properties and Copies of \(l^\infty\)

In this section we study the monotonicity properties of the Orlicz-Lorentz spaces equipped with the Orlicz norm.

This problem was first studied by Gong and Zhang in [15]. However, in that paper, it has been assumed that the Orlicz function \(\varphi\) is an N-function (that means that \(a_\varphi = 0\), \(\lim_{n \to 0} \varphi(n) = 0\), \(b_\varphi = \infty\) and \(\lim_{n \to \infty} \varphi(n) = \infty\)) and the weight sequence \(\omega\) satisfies the condition \(\sum_{i=1}^{\infty} \omega(i) = \infty\). In this paper, we will present respective criteria for Orlicz-Lorentz spaces generated by any Orlicz function and any non-increasing weight sequence. Moreover, we will also study some subspaces of \(\lambda_{\varphi, \omega}\), namely \(n\)-dimensional Orlicz-Lorentz spaces \(\lambda_{\varphi, \omega}^n\) for \(n \geq 2\) and the subspace \((\lambda_{\varphi, \omega})_a\) of order continuous elements in \(\lambda_{\varphi, \omega}\). The proof methods, which we apply, are different from most of the methods used in [15]. We use, among other things, some relationships between monotonicity properties and other geometric and topological properties.

A Banach lattice \(E\) is said to be strictly monotone if for any \(x, y \in E_+\) (the positive cone of \(E\), \(y \leq x\) and \(y \neq x\) imply that \(\|y\| < \|x\|\) (see [3]). \(E\) is said to be lower (upper) locally uniformly monotone, whenever for any \(x \in E_+\) with \(\|x\| = 1\) and any \(\varepsilon \in (0, 1)\) (resp. \(\varepsilon > 0\)) there is \(\delta = \delta(x, \varepsilon) \in (0,1)\) (resp. \(\delta = \delta(x, \varepsilon) > 0\)) such that the conditions \(y \in E\), \(0 \leq y \leq x\) (resp. \(y \geq 0\)) and \(\|y\| \geq \varepsilon\) imply \(\|x - y\| \leq 1 - \delta\) (resp. \(\|x + y\| \geq 1 + \delta\)) (see [18]). We say that \(E\) is uniformly monotone if for any \(\varepsilon \in (0, 1)\), there is \(\delta(\varepsilon) \in (0, 1)\) such that \(\|x - y\| \leq 1 - \delta(\varepsilon)\) whenever \(x, y \in E_+,\ y \leq x,\ \|x\| = 1\) and \(\|y\| \geq \varepsilon\) (see [3]).

**Theorem 3.1** The space \((\lambda_{\varphi, \omega}, \|\cdot\|_{\varphi, \omega}^O)\) is strictly monotone if and only if \(\sum_{i=1}^{\infty} \omega(i) = \infty\) and \(a_\varphi = 0\).

**Proof** Sufficiency. Let \(x, y \in \lambda_{\varphi, \omega}\) be such that \(x \geq 0\), \(y \geq 0\) and \(y \neq 0\). Then \(x \leq x + y\) and \(x \neq x + y\). We will show that \(\|x\|_{\varphi, \omega}^O < \|x + y\|_{\varphi, \omega}^O\).

By \(x \leq x + y\), it follows that \(x^* \leq (x + y)^*\). Simultaneously, by the fact that \(x \neq x + y\) and Lemma 3.2 in [20], there exists \(j \in \mathbb{N}\) such that \(x^*(j) < (x + y)^*(j)\). Consequently, we get \(I_{\varphi, \omega}(kx) < I_{\varphi, \omega}(k(x + y))\) for any \(k \in (0, \lambda_\infty(x + y))\) (this inequality holds also for \(k = \lambda_\infty(x + y)\)) whenever \(I_{\varphi, \omega}(\lambda_\infty(x + y) \cdot x) < \infty\). Thus, if there exists \(h > 0\) such that...
\[ h \in K(x + y), \text{ then} \]
\[
\|x + y\|_{\varphi, \omega}^O = \frac{1}{h} \left( 1 + I_{\varphi, \omega}(h(x + y)) \right) \geq \frac{1}{h} \{1 + I_{\varphi, \omega}(hx)\} \\
\geq \inf_{k > 0} \frac{1}{k} \{1 + I_{\varphi, \omega}(kx)\} = \|x\|_{\varphi, \omega}^O. \tag{3.1}
\]

Assume now that \( K(x + y) = \emptyset \). By Remark 2.12, we have \( \lim_{u \to \infty} \frac{\omega(u)}{u} = B < \infty \) and
\[
(\psi(B) \sum_{i=1}^{m(\supp(x+y))} \omega(i) < 1 \text{ or } \psi(B) \sum_{i=1}^{m(\supp(x+y))} \omega(i) = 1 \text{ and } p(u) < B \text{ for any } u > 0).
\]
Therefore, \( K(x) = \emptyset, \lambda_{\infty}(x) = \lambda_{\infty}(x + y) = \infty \) and for any \( k \geq 1 \), we obtain
\[
\frac{1}{k} \left( 1 + I_{\varphi, \omega}(k(x + y)) \right) - \frac{1}{k} \{ 1 + I_{\varphi, \omega}(kx)\} \\
\geq \frac{1}{k} \left( \varphi(k(x + y)^*)(j) - \varphi(kx^*(j)) \right) \omega(j) \\
\geq \frac{1}{k} \int_{\mathbb{R}} \frac{1}{x^*(j) + (x+y)^*(j)} p(t) dt \cdot \omega(j) \\
\geq \frac{1}{k} \cdot k \left( (x + y)^*(j) - \frac{x^*(j) + (x+y)^*(j)}{2} \right) p \left( \frac{x^*(j) + (x+y)^*(j)}{2} \right) \omega(j) > 0. \tag{3.2}
\]

This gives
\[
\|x + y\|_{\varphi, \omega}^O = \lim_{h \to \infty} \frac{1}{h} \left( 1 + I_{\varphi, \omega}(h(x + y)) \right) \geq \lim_{h \to \infty} \frac{1}{h} \{1 + I_{\varphi, \omega}(hx)\} = \|x\|_{\varphi, \omega}^O.
\]

Necessity. Firstly, suppose that \( \sum_{i=1}^{\infty} \omega(i) < \infty \). Define
\[
x = (0, 1, 1, 1, \ldots) \quad \text{and} \quad y = (1, 0, 0, 0, \ldots).
\]

It is obvious that \( x, y \in \lambda_{\varphi, \omega}, x \geq 0, y \geq 0 \) and \( y \neq 0 \). Since
\[
x^* = (x + y)^* = (1, 1, 1, 1, \ldots),
\]
we get \( \|x\|_{\varphi, \omega}^O = \|y\|_{\varphi, \omega}^O \).

Finally, let us assume that \( \sum_{i=1}^{\infty} \omega(i) = \infty \) and \( a_{\varphi} > 0 \). For
\[
x = (0, a_{\varphi}, a_{\varphi}, a_{\varphi}, \ldots) \quad \text{and} \quad y = (a_{\varphi}, 0, 0, 0, \ldots),
\]
we obtain again that \( x, y \in \lambda_{\varphi, \omega}, x \geq 0, y \geq 0 \) and \( y \neq 0 \). Simultaneously,
\[
x^* = (x + y)^* = (a_{\varphi}, a_{\varphi}, a_{\varphi}, a_{\varphi}, \ldots),
\]
whence \( \|x\|_{\varphi, \omega}^O = \|x + y\|_{\varphi, \omega}^O. \qed \)

**Theorem 3.2** The space \( (\lambda_{\varphi, \omega}, \|\cdot\|_{\varphi, \omega}^O) \) contains an order linearly isometric copy of \( l^\infty \) if and only if \( \sum_{i=1}^{\infty} \omega(i) < \infty \) or \( a_{\varphi} > 0 \).

**Proof** Sufficiency. First assume that \( \sum_{i=1}^{\infty} \omega(i) < \infty \) and define \( x = (1, 1, 1, \ldots) \). It is obvious that \( x \in \lambda_{\varphi, \omega} \) and thus
\[
y = \frac{x}{\|x\|_{\varphi, \omega}^O} = \left( \frac{1}{\|x\|_{\varphi, \omega}^O}, \frac{1}{\|x\|_{\varphi, \omega}^O}, \frac{1}{\|x\|_{\varphi, \omega}^O}, \ldots \right).
\]
belongs to the unit sphere of \((\lambda_{\varphi, \omega}, \| \cdot \|_{\varphi, \omega})\). Let \((N_j)_{j=1}^{\infty}\) be a sequence of pairwise disjoint subsets of the set \(\mathbb{N}\) such that \(m(N_j) = \infty\) for any \(j \in \mathbb{N}\) and \(\mathbb{N} = \bigcup_{j=1}^{\infty} N_j\). Let us define
\[
y_j = y \chi_{N_j} \quad \text{for any} \quad j \in \mathbb{N}.
\]
Since \(y_j^* = y^*\) for any \(j \in \mathbb{N}\), we get \(\|y_j\|_{\varphi, \omega} = 1\) for the same \(j\). Hence, we conclude that the operator, defined by
\[
P(z) = \sum_{j=1}^{\infty} z_j y_j \quad \text{for any} \quad z = (z_j)_{j=1}^{\infty} \in l^\infty,
\]
which is linear and positive, is an order isometry of \(l^\infty\) onto the closed subspace \(P(l^\infty)\) of \((\lambda_{\varphi, \omega}, \| \cdot \|_{\varphi, \omega})\).

Suppose now that \(\sum_{i=1}^{\infty} \omega(i) = \infty\) and \(a_\varphi > 0\). Define \(y = (a_\varphi, a_\varphi, a_\varphi, \ldots)\). We have that \(1 + I_{\varphi, \omega}(y) = 1 + \frac{1}{\varphi}(1 + I_{\varphi, \omega}(ky)) = \frac{1}{\varphi} > 1\) for \(k \in (0, 1)\) and \(\frac{1}{\varphi}(1 + I_{\varphi, \omega}(ky)) = \infty\) for \(k > 1\), whence \(\|y\|_{\varphi, \omega} = 1\). Defining the sequence \((y_j)_{j=1}^{\infty}\) and the operator \(P\) in the same way as above, we get again that the space \((\lambda_{\varphi, \omega}, \| \cdot \|_{\varphi, \omega})\) contains an order linearly isometric copy of \(l^\infty\).

Necessity. By Theorem 3.1 the space \((\lambda_{\varphi, \omega}, \| \cdot \|_{\varphi, \omega})\) is strictly monotone and thus it does not contain an order linearly isometric copy of \(l^\infty\) whenever \(\sum_{i=1}^{\infty} \omega(i) = \infty\) and \(a_\varphi = 0\). \(\Box\)

By Theorems 3.1 and 3.2, we obtain the following

**Corollary 3.3** The following conditions are equivalent:

(i) \(\sum_{i=1}^{\infty} \omega(i) = \infty\) and \(a_\varphi = 0\);

(ii) the space \((\lambda_{\varphi, \omega}, \| \cdot \|_{\varphi, \omega})\) is strictly monotone;

(iii) the space \((\lambda_{\varphi, \omega}, \| \cdot \|_{\varphi, \omega})\) does not contain an order linearly isometric copy of \(l^\infty\).

**Remark 3.4** Let us notice that if \(\sum_{i=1}^{\infty} \omega(i) = \infty\) and \(a_\varphi = 0\) but \(\varphi \notin \Delta_2(0)\), then \((\lambda_{\varphi, \omega}, \| \cdot \|_{\varphi, \omega})\) it not order continuous but it does not contain an order \(H\) linearly isometric copy of \(l^\infty\) (see Theorems 2.16 and 3.2). However, as it was shown by Lozanovskii in [27], \((\lambda_{\varphi, \omega}, \| \cdot \|_{\varphi, \omega})\) contains an order isomorphic copy of \(l^\infty\). Moreover, it contains subspaces arbitrarily nearly isometric to \(l^\infty\) (see [33]).

**Theorem 3.5** The following conditions are equivalent:

(i) the function \(\varphi\) satisfies the condition \(\Delta_2(0)\) and \(\sum_{i=1}^{\infty} \omega(i) = \infty\);

(ii) the space \((\lambda_{\varphi, \omega}, \| \cdot \|_{\varphi, \omega})\) is lower locally uniformly monotone;

(iii) the space \((\lambda_{\varphi, \omega}, \| \cdot \|_{\varphi, \omega})\) is upper locally uniformly monotone.

**Proof** (i) \(\Rightarrow\) (ii). By Theorems 2.16 and 3.1, we conclude that the space \((\lambda_{\varphi, \omega}, \| \cdot \|_{\varphi, \omega})\) is order continuous and strictly monotone, whence in virtue of [12, Theorem 2.7] it is lower locally uniformly monotone. The implication (ii) \(\Rightarrow\) (i) follows from [8, Proposition 2.1] and Theorem 2.16.

(i) \(\Rightarrow\) (iii). By Theorems 3.1 and [7, Theorem 2], the space \((\lambda_{\varphi, \omega}, \| \cdot \|_{\varphi, \omega})\) is strictly monotone and it has the Kadec-Klee property with respect to the coordinatewise convergence. Hence, by Theorem 4.1 in [9], the space \((\lambda_{\varphi, \omega}, \| \cdot \|_{\varphi, \omega})\) is upper locally uniformly monotone.

\(\Box\) Springer
(iii) \(\Rightarrow\) (i). Since the upper local uniform monotonicity implies strict monotonicity, we obtain
\[
\sum_{i=1}^{\infty} \omega(i) = \infty \text{ and } a_\varphi = 0 \text{ by Theorem 3.1.}
\]
Now, we will show the necessity of the condition \(\Delta_2(0)\). Assume that \(\varphi \notin \Delta_2(0)\). We can find \(i_0 \in \mathbb{N}\) and \(u_0, k_0 > 0\) such that
\[
\|x\|_{\varphi, \omega}^O = \frac{1}{k_0}(1 + I_{\varphi, \omega}(k_0x)) = 1,
\]
where \(x := u_0x_{(1,2,\ldots,i_0)}\). Indeed, if \(b_\varphi < \infty\) or \((b_\varphi = \infty\) and \(\lim_{u \to \infty} \frac{\varphi(u)}{u} = \infty\)), then we can assume \(i_0 = 1\) (see Remark 2.12). On the other hand, in the case when \(B = \lim_{u \to \infty} \frac{\varphi(u)}{u} < \infty\), we get \(\psi(B) > 0\) (note here that the case when \(\psi(B) = 0\) is equivalent to the equality \(\varphi(u) = Bu\) for any \(u \geq 0\) and this function satisfies the condition \(\Delta_2(0)\)). Since \(\sum_{i=1}^{\infty} \omega(i) = \infty\), there exists \(i_0 \in \mathbb{N}\) such that \(\psi(B) \sum_{i=1}^{i_0} \omega(i) > 1\) and, in consequence, \(K(x) \neq \emptyset\).

Since \(\varphi\) does not satisfy the condition \(\Delta_2(0)\), we can find a sequence \((u_n)_{n=1}^{\infty}\) decreasing to 0 such that \(u_1 < \min(u_0, b_\varphi/2)\),
\[
\varphi(2u_n) \geq 2^{n+1}\varphi(u_n) \quad \text{and} \quad \varphi(u_n) \omega(1) \leq \frac{1}{2^{n+1}}.
\]
Let \(i_n\) be such that
\[
\frac{1}{2^{n+1}} < \varphi(u_n) \sum_{i=1}^{i_n} \omega(i) \leq \frac{1}{2^n}.
\]
Defining \(y_n = u_nx_{(i_n+i_0+1,\ldots,i_0+i_0+i_0)}\), \(n \in \mathbb{N}\), we get \(I_{\varphi, \omega}(y_n) \leq \frac{1}{2^n}\) and \(I_{\varphi, \omega}(2y_n) > 1\) for any \(n \in \mathbb{N}\), whence \(\|y_n\|_{\varphi, \omega}^O \geq \|y_n\|_{\varphi, \omega} \geq \frac{1}{2^n}\) for the same \(n\). Now let us define \(z_n = \frac{1}{k_0}y_n\). We get \(I_{\varphi, \omega}(k_0z_n) \leq \frac{1}{2^n}\) and \(\|z_n\|_{\varphi, \omega}^O \geq \frac{1}{2^{n+1}}\) for any \(n \in \mathbb{N}\). Simultaneously, by the orthogonal subadditivity of the modular, for any natural \(n\) we have
\[
\|x + z_n\|_{\varphi, \omega}^O \leq \frac{1}{k_0}(1 + I_{\varphi, \omega}(k_0(x + z_n))) \leq \frac{1}{k_0}(1 + I_{\varphi, \omega}(k_0x)) + \frac{1}{k_0}I_{\varphi, \omega}(k_0z_n) \leq 1 + \frac{1}{2^n k_0},
\]
which is a contradiction with the fact that \((\lambda_{\varphi, \omega}, \cdot\|_{\varphi, \omega}^O)\) is upper locally uniformly monotone.

\[\square\]

**Theorem 3.6** Let \(\lambda_{\varphi, \omega}^n\) be an \(n\)-dimensional subspace of \(\lambda_{\varphi, \omega}\) \((n \geq 2)\). The space \((\lambda_{\varphi, \omega}^n, \cdot\|_{\varphi, \omega}^O)\) is strictly monotone (equivalently uniformly monotone) if and only if \(\omega(i) > 0\) for \(i = 1, 2, \ldots, n\) and \((a_\varphi = 0\) whenever \(K(x_{n-1}) \neq \emptyset\), where \(x_{n-1} = (1, \ldots, 1, 0)\)).

**Proof** Sufficiency. If \(K(x_{n-1}) \neq \emptyset\), we can proceed analogously to the proof of sufficiency in Theorem 3.1. Suppose now that \(K(x_{n-1}) = \emptyset\). Such a situation takes place when \(\lim_{u \to \infty} \frac{\varphi(u)}{u} = B < \infty\) and \((\psi(B) \sum_{i=1}^{n-1} \omega(i) < 1\) or \((\psi(B) \sum_{i=1}^{n-1} \omega(i) = 1\) and \(p(u) < B\) for any \(u > 0\)); see also Remark 2.12 and Example 2.13. Take any \(x, y \in \lambda_{\varphi, \omega}^n\) such that \(x \geq 0, y \geq 0, y \neq 0\). First suppose that there exists \(h > 0\) such that \(h \in K(x + y)\). If there exists \(j\) \((1 \leq j \leq n)\) such that \(\varphi(hx^\ast(j)) < \varphi(h(x + y)^\ast(j))\), then proceeding analogously as in (3.1), we obtain
\[
\|x + y\|_{\varphi, \omega}^O > \|x\|_{\varphi, \omega}^O. \quad \text{Assume now that } \varphi(hx^\ast(i)) = \varphi(h(x + y)^\ast(i)) \text{ for any } i = 1, 2, \ldots, n.
\]
Since \(x^\ast \neq (x + y)^\ast\), there exists \(j\) \((1 \leq j \leq n)\), such that \(0 \leq hx^\ast(j) < h(x + y)^\ast(j) \leq \varnothing \text{ Springer}
Since $h \in K(x + y)$, then by the assumption $K(x_{n-1}) = \emptyset$ we get $h(x + y)^*(n) \geq a_\varphi$.
Conclusively, we have $0 \leq hx^*(n) < h(x + y)^*(n) = a_\varphi$. Let $g > h$ be such that $gx^*(n) < a_\varphi$.
We have $I_{\varphi,\omega}(p(h \cdot x)) \leq I_{\psi,\omega}(p(g \cdot x)) < 1$, hence, by Lemma 2.10, we get
\[
\|x + y\|^O_{\varphi,\omega} = \frac{1}{h} (1 + I_{\varphi,\omega}(h(x + y))) = \frac{1}{h} (1 + I_{\varphi,\omega}(hx)) > \frac{1}{g} (1 + I_{\varphi,\omega}(gx)) \geq \|x\|^O_{\varphi,\omega}.
\]
Finally let us assume that $K(x + y) = \emptyset$ and let $j$ be such that $x^*(j) < (x + y)^*(j)$. There exists $k_0$ such that
\[
k_0 \frac{x^*(j) + (x + y)^*(j)}{2} > a_\varphi.
\]
Repeating the argumentation from the proof of sufficiency of Theorem 3.1 (see (3.2)), for any $k \geq k_0$ we get
\[
\frac{1}{k} (1 + I_{\varphi,\omega}(k(x + y))) - \frac{1}{k} (1 + I_{\varphi,\omega}(kx)) \geq \frac{1}{2} \left( (x + y)^*(j) - x^*(j) \right) p \left( k_0 \left( \frac{x^*(j) + (x + y)^*(j)}{2} \right) \right) \omega(j)
\]
and in consequence $\|x + y\|^O_{\varphi,\omega} > \|x\|^O_{\varphi,\omega}$.
\[
Necessity. \text{ Assume that } \omega(n) = 0 \text{ and define } x = (1, \ldots, 1, 0) \text{ and } y = (0, \ldots, 0, 1). \text{ We have that } x + y = (1, \ldots, 1) \text{ and } I_{\varphi,\omega}(kx) = I_{\varphi,\omega}(k(x + y)) \text{ for any } k > 0, \text{ whence } \|x + y\|^O_{\varphi,\omega} = \|x\|^O_{\varphi,\omega}.
\]
Suppose now that $\omega(n) > 0$, $h \in K(x_{n-1})$ and $a_\varphi > 0$. Define $y = (0, \ldots, 0, \min(n, 1))$. Then $x_{n-1} + y = (1, \ldots, 1, \min(n, 1))$ and
\[
\|x_{n-1}\|^O_{\varphi,\omega} = \inf_{k > 0} \frac{1}{k} (1 + I_{\varphi,\omega}(kx_{n-1})) \leq \inf_{k > 0} \frac{1}{k} (1 + I_{\varphi,\omega}(k(x_{n-1} + y))) = \|x_{n-1} + y\|^O_{\varphi,\omega}.
\]
On the other hand,
\[
\|x_{n-1}\|^O_{\varphi,\omega} = \frac{1}{h} (1 + I_{\varphi,\omega}(hx_{n-1})) = \frac{1}{h} (1 + I_{\varphi,\omega}(h(x_{n-1} + y))) \geq \|x_{n-1} + y\|^O_{\varphi,\omega},
\]
whence $\|x_{n-1}\|^O_{\varphi,\omega} = \|x_{n-1} + y\|^O_{\varphi,\omega}$.

**Theorem 3.7** The following conditions are equivalent:

(i) $\omega(i) > 0$ for any $i \in \mathbb{N}$ and $(a_\varphi = 0$ whenever there exists $n \in \mathbb{N}$ such that $K(x_n) \neq \emptyset$, where $x_n = (1, \ldots, 1, 0, 0, \ldots)$);

(ii) the space $(\lambda_{\varphi,\omega}, \| \cdot \|^O_{\varphi,\omega})$ is strictly monotone;

(iii) the space $(\lambda_{\varphi,\omega}, \| \cdot \|^O_{\varphi,\omega})$ is lower locally uniformly monotone.

**Remark 3.8** Let $\omega(i) > 0$ for any $i \in \mathbb{N}$. Then the condition $K(x_n) = \emptyset$ for any $n \in \mathbb{N}$ is equivalent to the fact that $\lim_{u \to \infty} \frac{\omega(u)}{u} = B < \infty$ and $\psi(B) \sum_{i=1}^{\infty} \omega(i) \leq 1$; see also Remark 2.12 and Example 2.13.

In particular, if $\sum_{i=1}^{\infty} \omega(i) = \infty$, then the condition $K(x_n) = \emptyset$ for any $n \in \mathbb{N}$ is equivalent to the equality $\lim_{u \to \infty} \frac{\omega(u)}{u} = B$ and $\psi(B) = 0$; however, we have in that case $\varphi(u) = Bu$ for $u \geq 0$, and thereby $a_\varphi = 0$. Hence, if $\sum_{i=1}^{\infty} \omega(i) = \infty$, the space $(\lambda_{\varphi,\omega}, \| \cdot \|^O_{\varphi,\omega})$ is strictly monotone (equivalently lower locally uniformly monotone) if and only if $a_\varphi = 0$. }

\[
\square \text{ Springer}
\]
Proof (i)⇒(ii). If \( a_\varphi = 0 \), then the proof of this implication is analogous to the proof of
sufficiency in Theorem 3.1. Let us now assume that \( a_\varphi > 0 \). Then for any \( n \in \mathbb{N} \) we
have \( K(x_n) = \emptyset \). Take any \( x, y \in (\lambda_{\varphi, \omega})_a \), \( x \geq 0 \), \( y \geq 0 \), \( y \neq 0 \). First we shall show that
\( K(x + y) = \emptyset \). Let us take any \( k > 0 \). Since \((\lambda_{\varphi, \omega})_a = c_0\), there exists \( n(k) \in \mathbb{N} \) such that
\( k(x + y)^*(n) < a_\varphi \) for any \( n > n(k) \), whence
\[
I_{\psi, \omega}(p(k(x + y)^*)) = \sum_{i=1}^{n(k)} \psi(p(k(x + y)^*(i))) \omega(i) < 1.
\]
By the arbitrariness of \( k > 0 \), we get \( K(x + y) = \emptyset \). Using again the fact that \((\lambda_{\varphi, \omega})_a = c_0\), we
conclude that there exists \( j \in \mathbb{N} \) such that \( x^*(j) < (x + y)^*(j) \). We will also find \( k_0 > 0 \) such that
\[
k_0 \frac{x^*(j) + (x + y)^*(j)}{2} > a_\varphi.
\]
Repeating the reasoning from Theorem 3.6, we get again \( \|x + y\|_{\varphi, \omega} > \|x\|_{\varphi, \omega} \).

The proof of implication (ii)⇒(i) is analogous to the proof of necessity of Theorem 3.6.
The implication (ii)⇒(iii) follows from [12]*Theorem 2.7, whereas the implication (iii)⇒(ii) is
obvious. \( \square \)

**Theorem 3.9**

(i) If \( \sum \omega(i) = \infty \), then the space \((\lambda_{\varphi, \omega})_a, \| \cdot \|_{\varphi, \omega}\) is upper locally
uniformly monotone if and only if the function \( \varphi \) satisfies the condition \( \Delta_2(0) \).

(ii) If \( \sum \omega(i) < \infty \), then the space \((\lambda_{\varphi, \omega})_a, \| \cdot \|_{\varphi, \omega}\) is upper locally uniformly monotone
if and only if \( \omega(i) > 0 \) for any \( i \in \mathbb{N} \) and \( a_\varphi = 0 \) if there exists \( n \in \mathbb{N} \) such that \( K(x_n) \neq \emptyset \),
where \( x_n = (1, 1, \ldots, 1, 0, 0, \ldots) \).

Proof (i) If \( \sum \omega(i) = \infty \) and \( \varphi \in \Delta_2(0) \), then, using Theorem 2.16, we get \((\lambda_{\varphi, \omega})_a = \lambda_{\varphi, \omega}\). Whence by Theorem 3.5 we conclude that the space \((\lambda_{\varphi, \omega})_a, \| \cdot \|_{\varphi, \omega}\) is upper locally
uniformly monotone.

Let us now assume that \( \sum \omega(i) = \infty \) and \((\lambda_{\varphi, \omega})_a, \| \cdot \|_{\varphi, \omega}\) is upper locally uniformly
monotone. Hence, it is strictly monotone as well, and we obtain \( a_\varphi = 0 \) (see Remark 3.8).

The necessity of the condition \( \Delta_2(0) \) can be shown in the same way as in the proof of the
implication (iii)⇒(i) in Theorem 3.5. Notice that the elements \( x, y_n \) and \( z_n \), \( n \in \mathbb{N} \) defined in
that proof belong to \((\lambda_{\varphi, \omega})_a\).

(ii) By Theorem 3.7 and [7, Theorem 3], we get that the space \((\lambda_{\varphi, \omega})_a, \| \cdot \|_{\varphi, \omega}\) is strictly
monotone and it has the Kadec-Klee property with respect to the coordinatewise convergence,
whence by [9, Theorem 4.1] we obtain that it is upper locally uniformly monotone.

The necessity follows from Theorem 3.7. \( \square \)

**Example 3.10** Let \( \varphi(u) = 0 \) for \( u \in [0, 1] \) and \( \varphi(u) = u - 1 \) for \( u > 1 \). We have that
\( a_\varphi = 1 \), \( \lim_{u \to \infty} \frac{\varphi(u)}{u} = 1 \), and \( \psi(1) = 1 \).

First assume that \( \omega_1(i) = \frac{1}{i^2} \). Then \( \psi(1) \sum \omega_1(i) = 1 \). By Theorem 3.6, the space
\((\lambda_{\varphi, \omega_1}, \| \cdot \|_{\varphi, \omega_1}\) is uniformly monotone for any \( n \geq 2 \). Moreover, by Theorems 3.7 and 3.9, the
space \((\lambda_{\varphi, \omega_1}, \| \cdot \|_{\varphi, \omega_1}\) is lower and upper locally uniformly monotone.

\( \square \)
Let now ω₂ be such that ω₂(1) ≥ 1. Then ψ(1)ω₂(1) ≥ 1 and in consequence K(x₁) ≠ ∅, where x₁ = (1, 0, 0, ...), whence by Theorem 3.6 the space (λ₂,ω₂, ||·||₀) is not strictly monotone.

**Remark 3.11**  (i) Let us note that if ∑ᵢ₌₁ⁿ ω(i) = ∞, aₙ = 0 and φ ∉ ∆₂(0), then the space (λₙ,ωₙ, ||·||₀) is lower locally uniformly monotone but it is not upper locally uniformly monotone (see Theorems 3.7 and 3.9 and Remark 3.8). A similar problem was considered by Hudzik and Kurc in [18] for the Musielak-Orlicz space (and thereby also for the Orlicz space). It is still unknown if upper local uniform monotonicity implies lower local uniform monotonicity.

(ii) Let now ∑ᵢ₌₁ⁿ⁺¹ ω(i) < ∞. Then, as we stated before, (λₙ,ωₙ, ||·||₀) and ||·||₀ are equivalent (see [11, Theorem 4.2]). If additionally ω(i) > 0 for any i ∈ N and (aₙ = 0 if only there exists n ∈ N such that K(xₙ) ≠ ∅, where xₙ = (1, 1, ..., 1, 0, 0, ...)), then the space c₀ equipped with the norm ||·||₀ is not only strictly monotone but also lower and upper locally uniformly monotone. However, it is not uniformly monotone as the next Theorem shows.

The sequence ω is called regular if there exists η ∈ (0, 1] such that for any n ∈ N, the equality

$$\sum_{i=₁}^{₂n} ω(i) ≥ (1 + η) \sum_{i=₁}^{ₙ} ω(i)$$

holds. It is easy to show that if ω is regular, then ∑ᵢ₌₁ⁿ⁺¹ ω(i) = ∞. The converse implication is not true. We can namely take a sequence ω, where ω(i) = 1 for i ∈ N, which is not regular although ∑ᵢ₌₁ⁿ⁺¹ ω(i) = ∞.

**Theorem 3.12**  The following conditions are equivalent:

(i) the Orlicz function φ satisfies the condition ∆₂(0) and the weight sequence is regular;

(ii) the space (λₙ,ωₙ, ||·||₀) is uniformly monotone;

(iii) the space (λₙ,ωₙ, ||·||₀) is uniformly monotone.

**Proof**  (i)⇒(ii). In this part of the proof, we use the idea presented by Hudzik and Kamińska in [16] which was later modified in papers [9] and [15]. We will show the whole proof for clarity and completeness.

By Theorem 6 in [17], it is sufficient to show that for any ε > 0, there exists δ(ε) ∈ (0, 1) such that for any x ≥ 0, xₙ,0 = 1 and any set A ⊆ N such that ||xₙ,A||₀ ≥ ε, we have ||xₙ,N \ A||₀ ≤ 1 - δ(ε). The proof consists of three parts.

1. First assume that x is non-negative, non-increasing and it has a finite support, which means that there exists m ∈ N such that x(m) > x(m + 1) = 0, and ||xₙ||₀ ≤ 1. We will show that for any ε ∈ (0, 1), there exists β(ε) ∈ (0, 1) such that for any set A ⊆ {1, 2, ..., m}, we get ||xₙ,A \ {1,2,...,m}\A||₀ ≤ 1 - β(ε) whenever ||xₙ,A||₀ ≥ ε. As ||xₙ,A||₀ ≥ ε, by Lemma 2.8(ii) we have

$$Iₙ,A(xₙ) ≥ ν(ε)$$

where the constant 0 < ν(ε) ≤ ε depends only on ε. Let p ∈ N be the smallest natural number
such that the inequality

\[ \frac{1}{(1+\eta)^{p}} \leq \frac{\nu(\varepsilon)}{4} \leq \frac{\varepsilon}{4} \]

holds, where \( \eta \) is the constant from the definition of the regularity of the weight function.

Let \( a_1 := \sup \{ i = 1, 2, \ldots, m : x(i) = x(1) \} \). Then \( 1 \leq a_1 \leq m \) and \( x(a_1) > x(a_1 + 1) \). If \( a_1 < m \), that means \( x(a_1 + 1) > 0 \), analogously as above we define \( a_2, \ldots, a_l = m \) in such a way that

\[ x(a_{j-1}) > x(a_{j-1} + 1) = \ldots = x(a_{j}) > x(a_{j} + 1), \]

for \( j = 2, \ldots, l \). Let \( u_j = x(a_j) \) for \( j = 1, 2, \ldots, l \) and \( u_{j+1} = 0 \). Defining

\[ b_j = m(\{1, \ldots, a_j\} \cap A) \quad \text{and} \quad c_j = m(\{1, \ldots, a_j\} \backslash A) \]

for \( j = 1, \ldots, l \), we have \( b_j + c_j = a_j \) for \( j = 1, \ldots, l \). We define four subsets of \( \mathbb{N} \) by the following formulas:

\[
\begin{align*}
N_1 &= \{ j \in \{1, \ldots, l\} : c_j < b_j/2^p \}, \\
N_2 &= \{ j \in \{1, \ldots, l\} : b_j/2^p \leq c_j \leq b_j \}, \\
N_3 &= \{ j \in \{1, \ldots, l\} : b_j < c_j \leq 2^pb_j \}, \\
N_4 &= \{ j \in \{1, \ldots, l\} : 2^pb_j < c_j \}.
\end{align*}
\]

Obviously, the sets \( N_i (i = 1, \ldots, 4) \) are pairwise disjoint and

\[ \bigcup_{i=1}^{4} N_i = \{1, \ldots, l\}. \]

Suppose first that \( K(x) \neq \emptyset \), which means that there exists \( k_0 > 1 \) such that

\[ 1 \geq \|x\|_{\varphi,\omega}^0 = \frac{1}{k_0} \left\{ 1 + I_{\varphi,\omega}(k_0x) \right\} = \frac{1}{k_0} \left\{ 1 + \sum_{i=1}^{m} \varphi(k_0x^*(i))\omega(i) \right\} \]

\[ = \frac{1}{k_0} \left\{ 1 + \sum_{j=1}^{l} (\varphi(k_0u_j) - \varphi(k_0u_{j+1})) \sum_{i=1}^{a_j} \omega(i) \right\}. \quad (3.4) \]

By inequality (3.3), using the convexity of the modular, we obtain

\[ I_{\varphi,\omega}(k_0x^A) = \sum_{j=1}^{l} (\varphi(k_0u_j) - \varphi(k_0u_{j+1})) \sum_{i=1}^{b_j} \omega(i) \geq k_0\nu(\varepsilon). \quad (3.5) \]

Since \( b_j \leq a_j \), using the regularity of the weight \( H \) sequence, we get

\[
\begin{align*}
1 &\geq \frac{1}{k_0} \left\{ 1 + \sum_{j \in N_1} (\varphi(k_0u_j) - \varphi(k_0u_{j+1})) \sum_{i=1}^{a_j} \omega(i) \right\} \\
&\geq \frac{1}{k_0} \left\{ 1 + \sum_{j \in N_1} (\varphi(k_0u_j) - \varphi(k_0u_{j+1})) \sum_{i=1}^{b_j} \omega(i) \right\} \\
&\geq \frac{1}{k_0} \left\{ 1 + \sum_{j \in N_1} (\varphi(k_0u_j) - \varphi(k_0u_{j+1})) \sum_{i=1}^{2^pc_j} \omega(i) \right\} \\
&\geq \frac{1}{k_0} \left\{ 1 + (1+\eta)^p \sum_{j \in N_1} (\varphi(k_0u_j) - \varphi(k_0u_{j+1})) \sum_{i=1}^{c_j} \omega(i) \right\} \\
&\geq \frac{1}{k_0} \left\{ 1 + (1+\eta)^p \sum_{j \in N_1} (\varphi(k_0u_j) - \varphi(k_0u_{j+1})) \sum_{i=1}^{c_j} \omega(i) \right\}
\end{align*}
\]
and this yields
\[ \sum_{j \in N_1} (\varphi(k_0 u_j) - \varphi(k_0 u_{j+1})) \sum_{i=1}^{c_j} \omega(i) \leq \frac{k_0 - 1}{(1 + \eta)^p} \leq \frac{k_0 \nu(\varepsilon)}{4}. \] (3.6)

We have \( c_j \leq b_j \) for \( j \in N_2 \). Hence, by the equality \( b_j + c_j = a_j \), as well as the monotonicity and regularity of the weight \( H \) sequence, we obtain
\[ \sum_{j \in N_2} (\varphi(k_0 u_j) - \varphi(k_0 u_{j+1})) \sum_{i=1}^{a_j} \omega(i) - \sum_{j \in N_2} (\varphi(k_0 u_j) - \varphi(k_0 u_{j+1})) \sum_{i=1}^{c_j} \omega(i) \]
\[ \geq \sum_{j \in N_2} (\varphi(k_0 u_j) - \varphi(k_0 u_{j+1})) \sum_{i=1}^{2b_j} \omega(i) - \sum_{j \in N_2} (\varphi(k_0 u_j) - \varphi(k_0 u_{j+1})) \sum_{i=1}^{b_j} \omega(i) \]
\[ \geq \eta \sum_{j \in N_2} (\varphi(k_0 u_j) - \varphi(k_0 u_{j+1})) \sum_{i=1}^{b_j} \omega(i). \] (3.7)

Since \( b_j < c_j \leq 2^p b_j \) for \( j \in N_3 \), applying again the monotonicity and regularity of the weight \( H \) sequence, we have
\[ \sum_{j \in N_3} (\varphi(k_0 u_j) - \varphi(k_0 u_{j+1})) \sum_{i=1}^{a_j} \omega(i) - \sum_{j \in N_3} (\varphi(k_0 u_j) - \varphi(k_0 u_{j+1})) \sum_{i=1}^{c_j} \omega(i) \]
\[ \geq \sum_{j \in N_3} (\varphi(k_0 u_j) - \varphi(k_0 u_{j+1})) \sum_{i=1}^{a_j + b_j} \omega(i) - \sum_{j \in N_3} (\varphi(k_0 u_j) - \varphi(k_0 u_{j+1})) \sum_{i=1}^{a_j} \omega(i) \]
\[ = \sum_{j \in N_3} (\varphi(k_0 u_j) - \varphi(k_0 u_{j+1})) \sum_{i=a_j + 1}^{a_j + b_j} \omega(i) \]
\[ \geq \frac{1}{2^{p+1}} \sum_{j \in N_3} (\varphi(k_0 u_j) - \varphi(k_0 u_{j+1})) \sum_{i=a_j + 1}^{a_j + 2^{p+1} b_j} \omega(i) \]
\[ \geq \frac{1}{2^{p+1}} \sum_{j \in N_3} (\varphi(k_0 u_j) - \varphi(k_0 u_{j+1})) \sum_{i=a_j + 1}^{2^{a_j}} \omega(i) \]
\[ \geq \frac{\eta}{2^{p+1}} \sum_{j \in N_3} (\varphi(k_0 u_j) - \varphi(k_0 u_{j+1})) \sum_{i=1}^{a_j} \omega(i) \]
\[ \geq \frac{\eta}{2^{p+1}} \sum_{j \in N_3} (\varphi(k_0 u_j) - \varphi(k_0 u_{j+1})) \sum_{i=1}^{b_j} \omega(i). \] (3.8)

Proceeding analogously for \( N_4 \) as in case of \( N_1 \), we get
\[ 1 \geq \frac{1}{k_0} \left\{ 1 + \sum_{j \in N_4} (\varphi(k_0 u_j) - \varphi(k_0 u_{j+1})) \sum_{i=1}^{a_j} \omega(i) \right\} \]
\[ \geq \frac{1}{k_0} \left\{ 1 + \sum_{j \in N_4} (\varphi(k_0 u_j) - \varphi(k_0 u_{j+1})) \sum_{i=1}^{c_j} \omega(i) \right\} \]
\[ \geq \frac{1}{k_0} \left\{ 1 + \sum_{j \in N_4} (\varphi(k_0 u_j) - \varphi(k_0 u_{j+1})) \sum_{i=1}^{2^p b_j} \omega(i) \right\} \]
\[
\geq \frac{1}{k_0} \left\{ 1 + (1 + \eta)^p \sum_{j \in N_4} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{b_j} \omega(i) \right\},
\]
whence
\[
\sum_{j \in N_4} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{b_j} \omega(i) \leq \frac{k_0 - 1}{(1 + \eta)^p} \leq \frac{k_0 \nu(\varepsilon)}{4}. \tag{3.9}
\]

If \( \sum_{j \in N_1} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{b_j} \omega(i) \geq \frac{k_0 \nu(\varepsilon)}{2} \), then by inequality (3.6), we get
\[
\sum_{j=1}^{l} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{a_j} \omega(i) - \sum_{j=1}^{l} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{c_j} \omega(i)
\]
\[
\geq \sum_{j \in N_1} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{a_j} \omega(i) - \sum_{j \in N_1} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{c_j} \omega(i)
\]
\[
\geq \sum_{j \in N_1} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{b_j} \omega(i) - \sum_{j \in N_1} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{c_j} \omega(i)
\]
\[
\geq \frac{k_0 \nu(\varepsilon)}{2} - \frac{k_0 \nu(\varepsilon)}{4} = \frac{k_0 \nu(\varepsilon)}{4}. \tag{3.10}
\]

Let now \( \sum_{j \in N_1} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{b_j} \omega(i) < \frac{k_0 \nu(\varepsilon)}{2} \). Then, using inequality (3.5), we have
\[
\sum_{j \in N_2 \cup N_3} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{b_j} \omega(i) \geq \frac{k_0 \nu(\varepsilon)}{2},
\]
whence by inequality (3.9), we get \( \sum_{j \in N_2 \cup N_3} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{b_j} \omega(i) \geq \frac{k_0 \nu(\varepsilon)}{4} \). Thus, by (3.7) and (3.8), we obtain
\[
\sum_{j=1}^{l} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{a_j} \omega(i) - \sum_{j=1}^{l} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{c_j} \omega(i)
\]
\[
\geq \sum_{j \in N_2 \cup N_3} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{a_j} \omega(i) - \sum_{j \in N_2 \cup N_3} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{c_j} \omega(i)
\]
\[
\geq \frac{\eta}{2^{p+1}} \sum_{j \in N_2 \cup N_3} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{b_j} \omega(i) \geq \frac{\eta}{2^{p+1}} \cdot \frac{k_0 \nu(\varepsilon)}{4} = \frac{k_0 \eta \cdot \nu(\varepsilon)}{2^{p+3}}. \tag{3.11}
\]

Finally, by (3.4), (3.10) and (3.11), we have
\[
I_{\varepsilon, \omega}(k_0 x \chi_{1,2,...,m} \setminus A) = \sum_{j=1}^{l} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{c_j} \omega(i)
\]
\[
= \sum_{j=1}^{l} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{a_j} \omega(i)
\]
\[
- \left( \sum_{j=1}^{l} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{b_j} \omega(i) - \sum_{j=1}^{l} \left( \varphi(k_0 u_j) - \varphi(k_0 u_{j+1}) \right) \sum_{i=1}^{c_j} \omega(i) \right).
\]

\[\copyright\ Springer\]
\[ \leq k_0 - 1 - k_0 \frac{\eta \cdot \nu(\varepsilon)}{2p+3}, \]

whence
\[
\|x\chi_{(1,2,\ldots,m)\setminus A}\|_{\mathcal{O},\omega}^Q \leq \frac{1}{k_0} \left\{ 1 + I_{\phi,\omega}\left( k_0 x\chi_{(1,2,\ldots,m)\setminus A}\right) \right\} \leq 1 - \frac{\eta \cdot \nu(\varepsilon)}{2p+3}. \tag{3.12}
\]

Let us now assume that \( K(x) = \emptyset \). Then \( \lim_{u \to \infty} \frac{\varepsilon(u)}{u} = \lim_{u \to \infty} p(u) = B \) (and \( \psi(B) \sum_{i=1}^{m} \omega(i) < 1 \) or \( \psi(B) \sum_{i=1}^{m} \omega(i) = 1 \) and \( p(u) < B \) for any \( u > 0 \)); see Remark 2.12). By Lemma 1, from [7], we have
\[
1 \geq \|x\|_{\mathcal{O},\omega}^Q = B \sum_{i=1}^{m} x^*(i) \omega(i) = B \sum_{j=1}^{l} (u_j - u_{j+1}) \sum_{i=1}^{a_j} \omega(i) \tag{3.13}
\]
and
\[
\|x\chi_{A}\|_{\mathcal{O},\omega}^Q = B \sum_{j=1}^{l} (u_j - u_{j+1}) \sum_{i=1}^{b_j} \omega(i) \geq \varepsilon. \tag{3.14}
\]

Using the fact that \( b_j \leq a_j \) and regularity of the weight \( H \) sequence, we obtain
\[
1 \geq B \sum_{j \in N_1} (u_j - u_{j+1}) \sum_{i=1}^{a_j} \omega(i) \geq B \sum_{j \in N_1} (u_j - u_{j+1}) \sum_{i=1}^{b_j} \omega(i) \geq B \sum_{j \in N_1} (u_j - u_{j+1}) \sum_{i=1}^{c_j} \omega(i) \tag{3.15}
\]
and consequently,
\[
B \sum_{j \in N_1} (u_j - u_{j+1}) \sum_{i=1}^{c_j} \omega(i) \leq \frac{1}{(1 + \eta)^p} \leq \frac{\varepsilon}{4}. \tag{3.16}
\]

We proceed analogously as in (3.7) and (3.8) and we get
\[
B \sum_{j \in N_2} (u_j - u_{j+1}) \sum_{i=1}^{a_j} \omega(i) - B \sum_{j \in N_2} (u_j - u_{j+1}) \sum_{i=1}^{c_j} \omega(i) \geq \eta B \sum_{j \in N_2} (u_j - u_{j+1}) \sum_{i=1}^{b_j} \omega(i) \tag{3.17}
\]
and
\[
B \sum_{j \in N_3} (u_j - u_{j+1}) \sum_{i=1}^{a_j} \omega(i) - B \sum_{j \in N_3} (u_j - u_{j+1}) \sum_{i=1}^{c_j} \omega(i) \geq \frac{\eta B}{2^{p+1}} \sum_{j \in N_3} (u_j - u_{j+1}) \sum_{i=1}^{b_j} \omega(i). \tag{3.18}
\]

Proceeding in turn analogously as for \( N_1 \) (see (3.15) and (3.16)), we have
\[
B \sum_{j \in N_4} (u_j - u_{j+1}) \sum_{i=1}^{b_j} \omega(i) \leq \frac{1}{(1 + \eta)^p} \leq \frac{\varepsilon}{4}. \tag{3.19}
\]

If \( B \sum_{j \in N_1} (u_j - u_{j+1}) \sum_{i=1}^{b_j} \omega(i) \geq \frac{\varepsilon}{2} \), then by inequality (3.16) (proceeding similarly to (3.10)), we get
\[
B \sum_{j=1}^{l} (u_j - u_{j+1}) \sum_{i=1}^{a_j} \omega(i) - B \sum_{j=1}^{l} (u_j - u_{j+1}) \sum_{i=1}^{c_j} \omega(i) \geq \frac{\varepsilon}{4}. \tag{3.20}
\]
In the opposite case, that is, if \( B \sum_{j \in N_1} (u_j - u_{j+1}) \sum_{i=1}^{b_j} < \frac{\varepsilon}{2} \), by (3.14) and (3.19), we have \( B \sum_{j \in N_2 \cup N_3} (u_j - u_{j+1}) \sum_{i=1}^{b_j} \geq \frac{\varepsilon}{4} \). Therefore, by analogy with (3.11), by (3.17) and (3.18) we have

\[
B \sum_{j=1}^{l} (u_j - u_{j+1}) \sum_{i=1}^{a_j} \omega(i) - B \sum_{j=1}^{l} (u_j - u_{j+1}) \sum_{i=1}^{c_j} \omega(i) \geq \frac{\eta \cdot \varepsilon}{2p+3}, \tag{3.21}
\]

In virtue of (3.20) and (3.21), we obtain

\[
\|x \chi_{\{1,2,\ldots,m\} \setminus A}\|_{\mathcal{O}_{\phi,\omega}} = B \sum_{j=1}^{l} (u_j - u_{j+1}) \sum_{i=1}^{c_j} \omega(i) = B \sum_{j=1}^{l} (u_j - u_{j+1}) \sum_{i=1}^{a_j} \omega(i)
- \left( B \sum_{j=1}^{l} (u_j - u_{j+1}) \sum_{i=1}^{a_j} \omega(i) - B \sum_{j=1}^{l} (u_j - u_{j+1}) \sum_{i=1}^{c_j} \omega(i) \right)
\leq 1 - \frac{\eta \cdot \varepsilon}{2p+3},
\]

whence, and by (3.12), assuming \( \beta(\varepsilon) = \frac{\nu(\varepsilon)}{2p+3} (\nu(\varepsilon) \leq \varepsilon) \) we obtain

\[
\|x \chi_{\{1,2,\ldots,m\} \setminus A}\|_{\mathcal{O}_{\phi,\omega}} \leq 1 - \beta(\varepsilon), \tag{3.22}
\]

which ends the first part of the proof of the implication (i)\Rightarrow(ii).

2. Supposing now that \( x \) is non-negative, it has a finite support and \( \|x\|_{\mathcal{O}_{\phi,\omega}} \leq 1 \). Then there exists a permutation \( \sigma \) of the set \( N \) such that \( x^* (i) = x (\sigma(i)) \) for any \( i \in N \). Take any fixed \( \varepsilon \in (0,1) \) and let \( A \subset N \) be such that \( \|x_{\chi_A}\|_{\mathcal{O}_{\phi,\omega}} \geq \varepsilon \). Defining \( C = \sigma^{-1} A \), we have \( \|x_{\chi_C}\|_{\mathcal{O}_{\phi,\omega}} = \|x_{\chi_A}\|_{\mathcal{O}_{\phi,\omega}} \geq \varepsilon \), whence, by the first part of the proof, we get \( \|x_{\chi_{N \setminus A}}\|_{\mathcal{O}_{\phi,\omega}} = \|x^* \chi_{\{1,2,\ldots,m(\text{supp} x)\} \setminus C}\|_{\mathcal{O}_{\phi,\omega}} \leq 1 - \beta(\varepsilon) \).

3. Finally, let \( x \in (\Lambda_{\phi,\omega})_+ \) (the positive cone of \( \Lambda_{\phi,\omega} \)), \( \|x\|_{\mathcal{O}_{\phi,\omega}} = 1 \) and let \( A \subset N \) be such that \( \|x_{\chi_A}\|_{\mathcal{O}_{\phi,\omega}} \geq \varepsilon \). Let us define \( x_n = x \chi_{\{1,2,\ldots,n\}} \). By the Fatou property, there exists \( n_0 \in N \) such that \( \|x_{\chi_{\Lambda\setminus A}}\|_{\mathcal{O}_{\phi,\omega}} \geq \frac{\varepsilon}{2} \) for \( n \geq n_0 \). Therefore, by the second part of the proof, there exists \( \beta \left( \frac{\varepsilon}{2} \right) > 0 \) such that \( \|x_{\chi_{N \setminus \Lambda}}\|_{\mathcal{O}_{\phi,\omega}} \leq 1 - \beta \left( \frac{\varepsilon}{2} \right) \) for \( n \geq n_0 \). Using the Fatou property again, we have \( \|x_{\chi_{N \setminus A}}\|_{\mathcal{O}_{\phi,\omega}} \leq \frac{1}{2} - (1 - \delta(\varepsilon)) \), where \( \delta(\varepsilon) := \beta \left( \frac{\varepsilon}{2} \right) \).

The implication (ii)\Rightarrow(iii) is obvious. Now, we will prove the implication (iii)\Rightarrow(i). Since uniform monotonicity implies strict monotonicity, by Theorem 3.7 we have \( \omega(i) > 0 \) for any \( i \in N \). Suppose that a weight sequence \( \omega \) is not regular. Then, there exists an increasing sequence \( (i_n) \) such that \( \lim_{n \to \infty} i_n = \infty \) and

\[
\sum_{i=1}^{2i_n} \omega(i) \leq \left( 1 + \frac{1}{n} \right) \sum_{i=1}^{i_n} \omega(i).
\]

for any \( n \in N \). If \( K(\chi_{\{1,2,\ldots,i_n\}}) = \emptyset \) for any \( n \in N \), then \( \lim_{u \to \infty} \frac{\omega(u)}{u} = B \) and \( \psi(B) \sum_{i=1}^{\infty} \omega(i) \leq 1 \) (see Remark 2.12). By Lemma 2.14 we get

\[
\|x_{\chi_{\{1,2,\ldots,2i_n\}}}\|_{\mathcal{O}_{\phi,\omega}} = B \sum_{i=1}^{2i_n} \omega(i)
\]
for any $n \in \mathbb{N}$. Therefore, putting $x_n := u_n \chi_{\{1,2,\ldots,2i_n\}}$, where
\[
u_n = \frac{1}{B \sum_{i=1}^{2i_n} \omega(i)} ,
\]
we obtain $\|x_n\|_{\psi,\omega}^O = 1$ for any $n \in \mathbb{N}$. Defining $y_n := u_n \chi_{\{1,2,\ldots,i_n\}}$ for all $n \in \mathbb{N}$, we have that $y_n = y_n^* = (x_n - y_n)^*$ and
\[
\|x_n - y_n\|_{\psi,\omega}^O = \|y_n\|_{\psi,\omega}^O = B u_n \sum_{i=1}^{i_n} \omega(i) \geq \frac{n}{n+1} B u_n \sum_{i=1}^{2i_n} \omega(i) = \frac{n}{n+1}
\]
for the same $n$, and we conclude that $((\lambda_{\psi,\omega})_n, \| \cdot \|_{\psi,\omega}^O)$ is not uniformly monotone, a contradiction.

Now, suppose that $K(\chi_{\{1,2,\ldots,i_n\}}) \neq \emptyset$ starting from some $n$. Without loss of generality we may assume that $n = 1$. Hence, applying again Lemma 2.14, we obtain
\[
\|\chi_{\{1,2,\ldots,2i_1\}}\|_{\psi,\omega}^O = \psi^{-1} \left( \frac{1}{2i_1} \sum_{i=1}^{2i_1} \omega(i) \right) \sum_{i=1}^{2i_1} \omega(i)
\]
for all $n \in \mathbb{N}$. Therefore, defining for any $n \in \mathbb{N}$ elements $x_n = v_n \chi_{\{1,2,\ldots,2i_n\}}$ and $y_n = v_n \chi_{\{1,2,\ldots,i_n\}}$, where
\[
u_n = \frac{1}{\|\chi_{\{1,2,\ldots,2i_1\}}\|_{\psi,\omega}^O},
\]
we get that $\|x_n\|_{\psi,\omega}^O = 1$, $y_n = y_n^* = (x_n - y_n)^*$ and
\[
\|x_n - y_n\|_{\psi,\omega}^O = \|y_n\|_{\psi,\omega}^O = v_n \psi^{-1} \left( \frac{1}{\sum_{i=1}^{i_n} \omega(i)} \sum_{i=1}^{i_n} \omega(i) \right) \sum_{i=1}^{i_n} \omega(i) \geq v_n \psi^{-1} \left( \frac{1}{\sum_{i=1}^{2i_n} \omega(i)} \sum_{i=1}^{2i_n} \omega(i) \right) \sum_{i=1}^{2i_n} \omega(i) = \frac{n}{n+1},
\]
which again gives a contradiction. Thereby, we showed that the weight sequence is regular, whence we get $\sum_{i=1}^{\infty} \omega(i) = \infty$ in particular. Since uniform monotonicity implies upper local uniform monotonicity, by Theorem 3.9(i), we obtain $\varphi \in \Delta_2(0). \qed$

**Remark 3.13** It is worth noticing that the necessary and sufficient conditions for respective monotonicity properties of the Orlicz-Lorentz space (or its subspaces) equipped with the Orlicz norm are weaker than the analogous conditions for the Orlicz-Lorentz space (or its subspaces) equipped with the Luxemburg norm (see [4, Lemma 1], [9, Theorems 4.2–4.4] and [12, Corollary 4.5]). In the case of the Luxemburg norm, we always need to assume that $\varphi(b_{\psi,\omega}) \omega(1) \geq 1$. Moreover, the necessary condition for strict monotonicity is the condition $\Delta_2(0)$ for the space $((\lambda_{\psi,\omega})_n, \| \cdot \|_{\psi,\omega}^O)$ and the condition $a_{\varphi} = 0$ for the space $((\lambda_{\psi,\omega})_n, \| \cdot \|_{\psi,\omega}^O)$ (see Example 3.10).
4 Applications to Orlicz Sequence Spaces

Let us notice that in the case when \( \omega(i) = 1 \) for any \( i \in \mathbb{N} \), Orlicz-Lorentz sequence spaces become the well-known Orlicz sequence spaces.

On the basis of the results obtained in the previous part of this paper, we easily get respective criteria for Orlicz sequence spaces \( l_\varphi \) as well as for their infinitely dimensional subspaces \( h_\varphi \) (that is, subspaces of order continuous elements in \( l_\varphi \)) and their \( n \)-dimensional subspaces \( l_\varphi^n \) \( (n \geq 2) \).

**Corollary 4.1** (See Theorem 3.6) Let \( l_\varphi^n \) be an \( n \)-dimensional subspace of \( l_\varphi \). The space \( (l_\varphi^n, \| \cdot \|_{O_\varphi}) \) is strictly monotone (equivalently uniformly monotone) if and only if \( a_\varphi = 0 \) whenever \( K(x_{n-1}) \neq \emptyset \), where \( x_{n-1} = (1, \ldots, 1, 0) \).

**Remark 4.2** Note that in an Orlicz space, the condition \( K(x_{n-1}) = \emptyset \) is equivalent to the fact that \( \lim_{u \to \infty} \frac{\varphi(u)}{u} = B \) and \( (\psi(B) < \frac{1}{n-1} \text{ or } (\psi(B) = \frac{1}{n-1} \text{ and } p(u) < B \text{ for any } u > 0)) \).

**Corollary 4.3** (See Theorems 3.1, 3.2 and 3.7) The following conditions are equivalent:

(i) the Orlicz function \( \varphi \) vanishes only at zero \( (a_\varphi = 0) \);

(ii) the sequence Orlicz space \( (l_\varphi, \| \cdot \|_{O_\varphi}) \) does not contain an order linearly isometric copy of \( l^\infty \);

(iii) the sequence Orlicz space \( (l_\varphi, \| \cdot \|_{O_\varphi}) \) is strictly monotone;

(iv) the space \( (h_\varphi, \| \cdot \|_{O_\varphi}) \) is strictly monotone;

(v) the space \( (h_\varphi, \| \cdot \|_{O_\varphi}) \) is locally uniformly monotone.

**Corollary 4.4** (See Theorems 2.16, 3.5, 3.9(i) and 3.12) The following conditions are equivalent:

(i) the Orlicz function \( \varphi \) satisfies the condition \( \Delta_2(0) \);

(ii) the sequence Orlicz space \( (l_\varphi, \| \cdot \|_{O_\varphi}) \) is order continuous;

(iii) the sequence Orlicz space \( (l_\varphi, \| \cdot \|_{O_\varphi}) \) is lower locally uniformly monotone;

(iv) the sequence Orlicz space \( (l_\varphi, \| \cdot \|_{O_\varphi}) \) is upper locally uniformly monotone;

(v) the sequence Orlicz space \( (l_\varphi, \| \cdot \|_{O_\varphi}) \) is uniformly monotone;

(vi) the space \( (h_\varphi, \| \cdot \|_{O_\varphi}) \) is upper locally uniformly monotone;

(vii) the space \( (h_\varphi, \| \cdot \|_{O_\varphi}) \) is uniformly monotone.

**References**

[1] Astashkin S V, Sukochev F A, Wong C P. Distributionally concave symmetric spaces and uniqueness of symmetric structure. Adv Math, 2013, 232: 399–431

[2] Bennett C, Sharpley R. Interpolation of operators. Pure and Applied Mathematics. Vol 129. Boston, MA: Academic Press, Inc, 1988

[3] Birkhoff G. Lattice theory. Third edition. American Mathematical Society Colloquium Publications. Vol XXV. Providence, RI: American Mathematical Society, 1967

[4] Cerdá J, Hudzik H, Kamińska A, Mastyło M. Geometric properties of symmetric spaces with applications to Orlicz-Lorentz spaces. Positivity, 1998, 2: 311–337

[5] Chen S T. Geometry of Orlicz spaces. Dissertationes Math (Rozprawy Mat), 1996, 356: 204

[6] Cui Y, Foralewski P, Hudzik H. M-constants in Orlicz-Lorentz sequence spaces with applications to fixed point theory. Fixed Point Theory, 2018, 19: 141–166

[7] Cui Y, Foralewski P, Hudzik H, Kaczmarek K. Kadee-Klee properties of Orlicz-Lorentz sequence spaces equipped with the Orlicz norm. Positivity, 2021, 25: 1273–1294
[8] Dominguez T, Hudzik H, López G, Mastyło M, Sims B. Complete characterizations of Kadec-Klee properties in Orlicz spaces. Houston J Math, 2003, 29: 1027–1044

[9] Foralewski P. On some geometric properties of generalized Orlicz-Lorentz sequence spaces. Indag Math (NS), 2013, 24: 346–372

[10] Foralewski P, Hudzik H, Kołwicz P. Non-squareness properties of Orlicz-Lorentz sequence spaces. J Funct Anal, 2013, 264: 605–629

[11] Foralewski P, Hudzik H, Szymbaszkiewicz L. On some geometric and topological properties of generalized Orlicz-Lorentz sequence spaces. Math Nachr, 2008, 281: 181–198

[12] Foralewski P, Kołwicz P. Local uniform rotundity in Calderón-Lozanovskiǐ spaces. J Convex Anal, 2007, 14: 395–412

[13] Foralewski P, Kończak J. Local uniform non-squareness of Orlicz-Lorentz function spaces. Rev R Acad Cienc Exactas Fís Nat Ser A Mat RACSAM, 2019, 113: 3425–3443

[14] Gong W Z, Shi Z R. Points of monotonicity in Orlicz-Lorentz function spaces. Nonlinear Anal, 2010, 73: 1300–1317

[15] Gong W Z, Zhang D X. Monotonicity in Orlicz-Lorentz sequence spaces equipped with the Orlicz norm. Acta Mathematica Scientia, 2016, 36B: 1577–1589

[16] Hudzik H, Kamińska A. Monotonicity properties of Lorentz spaces. Proc Amer Math Soc, 1995, 123: 2715–2721

[17] Hudzik H, Kamińska A, Mastyło M. Monotonicity and rotundity properties in Banach lattices. Rocky Mountain J Math, 2000, 30: 933–950

[18] Hudzik H, Kurc W. Monotonicity properties of Musielak-Orlicz spaces and dominated best approximation in Banach lattices. J Approx Theory, 1998, 95: 353–368

[19] Hudzik H, Maligranda L. Amemiya norm equals Orlicz norm in general. Indag Math (NS), 2000, 11: 573–585

[20] Kamińska A. Some remarks on Orlicz-Lorentz spaces. Math Nachr, 1990, 147: 29–38

[21] Kamińska A, Leśnik K, Raynaud Y. Dual spaces to Orlicz-Lorentz spaces. Studia Math, 2014, 222: 229–261

[22] Kolwicz P, Leśnik K, Maligranda L. Pointwise multipliers of Calderón-Lozanovskiǐ spaces. Math Nachr, 2013, 286: 876–907

[23] Krasnosel’kii M A, Rutickiǐ Ja B. Convex functions and Orlicz spaces. Translated from the first Russian edition by Leo F Boron P. Groningen: Noordhoff Ltd, 1961

[24] Kreǐn S G, Petun̆in YŭI, Semenov E M. Interpolation of linear operators. Translations of Mathematical Monographs. Vol 54. Providence RI: American Mathematical Society, 1982

[25] Levi F E, Cuenya H H. Gateaux differentiability in Orlicz-Lorentz spaces and applications. Math Nachr, 2007, 280: 1282–1296

[26] Lindenstrauss J, Tzafriri L. Classical Banach spaces. II. Function spaces. Ergebnisse der Mathematik und ihrer Grenzgebiete. Vol 97. Berlin-New York: Springer-Verlag, 1979

[27] Lozanovskiǐ G Ja. Isomorphous Banach structures. Siberian Math J, 1969, 10: 64–68

[28] Luxemburg W A J. Banach function spaces. Thesis. Technische Hogeschool te Delft, 1955

[29] Maligranda L. Orlicz spaces and interpolation. Seminários de Matemática [Seminars in Mathematics]. Vol 5. Campinas: Universidade Estadual de Campinas, Departamento de Matemática, 1989

[30] Montgomery-Smith S. Comparison of Orlicz-Lorentz spaces. Studia Math, 1992, 103: 161–189

[31] Musielak J. Orlicz spaces and modular spaces//Lecture Notes in Mathematics. Vol 1034. Berlin: Springer-Verlag, 1983

[32] O’Neil R. Fractional integration in Orlicz spaces. I. Trans Amer Math Soc, 1965, 115: 300–328

[33] Partington J R. Subspaces of certain Banach sequence spaces. Bull London Math Soc, 1981, 13: 162–166

[34] Wang J, Ning Z. Rotundity and uniform rotundity of Orlicz-Lorentz sequence spaces equipped with the Orlicz norm. Math Nachr, 2011, 284: 2297–2311

[35] Zaanen A C. Introduction to operator theory in Riesz spaces. Berlin: Springer-Verlag, 1997

[36] Zlatanov B. Upper and lower estimates in weighted Orlicz sequence spaces and Lorentz-Orlicz sequence spaces. Plodoviv Univ Paisiǐ Khilendarski Nauchn Trud Mat, 2007, 35: 181–204

[37] Zlatanov B. Upper estimates for Gâteaux differentiability of bump functions in Orlicz-Lorentz spaces. JIPAM J Inequal Pure Appl Math, 2007, 8: Article 113, 8pp

[38] Zlatanov B. Kottman’s constant, packing constant and Riesz angle in some classes of Köthe sequence spaces. Carpathian J Math, 2019, 35: 103–124