R-MATRICES FOR QUANTUM AFFINE ALGEBRAS AND KHOVANOV-LAUDA-ROUQUIER ALGEBRAS, I

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Abstract. Let \( I \) be a finite set of pairs consisting of good \( U'_q(\mathfrak{g}) \)-modules and invertible elements in the base field \( \mathbb{C}(q) \). The distribution of poles of normalized \( R \)-matrices yields Khovanov-Lauda-Rouquier algebras \( R^I(n) \) for \( n \geq 0 \). We define a functor \( \mathcal{F} \) from the category of finite-dimensional \( R^I(n) \)-modules to the category of finite-dimensional \( U'_q(\mathfrak{g}) \)-modules. We show that the functor \( \mathcal{F} \) sends convolution products to tensor products and is exact if \( R^I(n) \) is of type \( A,D,E \).

Introduction

Let \( U_q(\mathfrak{g}) = U_q(\widehat{\mathfrak{sl}_N}) \) be the quantum affine algebra of type \( A^{(1)}_{N-1} \) and let \( V_{\text{aff}} \) be the affinization of \( V \), the vector representation of \( U'_q(\mathfrak{g}) \). We denote by \( H_{\text{aff}}^n \) the affine Hecke algebra. In [4, 5, 8], a functor \( \mathcal{F}_n : H_{\text{aff}}^n \text{-mod} \rightarrow U'_q(\mathfrak{g})\text{-mod} \) is introduced, which is given by

\[
M \mapsto (V_{\text{aff}})^{\otimes n} \otimes_{H_{\text{aff}}^n} M.
\]

It was shown that the functor \( \mathcal{F}_n \) is exact and that \( \mathcal{F}_n \) is fully faithful when \( N > n \).

In [18], this idea was generalized to the case when \( U'_q(\mathfrak{g}) \) is an arbitrary quantum affine algebra and \( V \) is a good module. That is, the poles of the \( R \)-matrix on \( (V_{\text{aff}})^{\otimes n} \) define a quiver, which in turn defines a Khovanov-Lauda-Rouquier algebra \( R(n) \). Then one can construct a functor

\[
\mathcal{F}_n : R(n)\text{-mod} \rightarrow U'_q(\mathfrak{g})\text{-mod}
\]

given by

\[
M \mapsto \widehat{V}^{\otimes n} \otimes_{R(n)} M,
\]

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where \( \widehat{V} \otimes^n \) is a certain completion of \( (V_{aff})^\otimes^n \). Moreover, it was shown that \( \mathcal{F}_n \) sends convolution products to tensor products. When \( U'_q(\mathfrak{g}) = U'_q(\mathfrak{sl}_N) \) and \( V \) is the vector representation, it coincides with the one given in [4].

In this article, we extend this construction to the case when we have a family of good modules. More precisely, let \( \{ V_s \}_{s \in S} \) be a family of good \( U'_q(\mathfrak{g}) \)-modules and let \( I \) be a finite subset of \( S \times \mathbb{C}(q)^\times \). An element \( i \in I \) is denoted by \( i = (S(i), X(i)) \).

Then we can define a quiver \( \Gamma_I \) with \( I \) as a set of vertices as follows. For \( i, j \in I \), let \( \mathcal{R}^\text{norm}_{V_{S(i)}(V_{S(j)})_u \otimes (V_{S(j)})_v} \rightarrow (V_{S(j)})_v \otimes (V_{S(i)})_u \) be the normalized \( R \)-matrix. We join \( i \) and \( j \) by edges if \( \mathcal{R}^\text{norm}_{V_{S(i)}(V_{S(j)})_u \otimes (V_{S(j)})_v} \) has a pole at \( u/v = X(i)/X(j) \). Then the quiver \( \Gamma_I \) defines a Khovanov-Lauda-Rouquier algebra \( R_I(n) \). For each sequence \( \nu \in I^n \), set \( V_\nu = (V_{S(\nu_1)})_{aff} \otimes \cdots \otimes (V_{S(\nu_n)})_{aff} \), and let \( \widehat{V} \otimes^n \) be a certain completion of \( \bigoplus_{\nu \in I^n} V_\nu \). Then \( \widehat{V} \otimes^n \) has a structure of a \( (U'_q(\mathfrak{g}), R_I(n)) \)-bimodule, and we can define the functor

\[
\mathcal{F}_n: R_I(n)\text{-mod} \rightarrow U'_q(\mathfrak{g})\text{-mod}
\]

given by

\[
M \mapsto \widehat{V} \otimes^n \otimes_{R_I(n)} M.
\]

In this paper, we first prove that \( \mathcal{F}_n \) sends convolution products to tensor products (Theorem 2.6). Moreover, when the Cartan datum associated with \( I \) is of type \( A, D, E \), we show that \( \mathcal{F}_n \) is an exact functor (Theorem 2.9).

In a forthcoming paper, we will study several applications of the above functor \( \mathcal{F}_n \). In particular, we will give a construction of categories whose Grothendieck groups have quantum cluster structures. We will also provide an interpretation of the isomorphism between \( \tau \)-deformed Grothendieck rings and negative parts of the quantum groups, which was established recently in [10].

1. Quantum groups and Khovanov-Lauda-Rouquier algebras

1.1. Quantum groups. In this section, we recall the definitions of the quantum groups. Let \( I \) be a finite index set. A Cartan datum is a quintuple \( (A, P, \Pi, P^\vee, \Pi^\vee) \) consists of

(a) an integer-valued matrix \( A = (a_{ij})_{i,j \in I} \), called the symmetrizable generalized Cartan matrix, which satisfies

(i) \( a_{ii} = 2 (i \in I) \),

(ii) \( a_{ij} \leq 0 (i \neq j) \),

(iii) \( a_{ij} = 0 \) if \( a_{ji} = 0 \) (\( i, j \in I \)),

(iv) there exists a diagonal matrix \( D = \text{diag}(s_i | i \in I) \) such that \( DA \) is symmetric, and \( s_i \) are positive integers.

(b) a free abelian group \( P \) of finite rank, called the weight lattice,
Definition 1.1. The quantum group $U_q(g)$ associated with a Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$ is the associative algebra over $\mathbb{Q}(q)$ with 1 generated by $e_i, f_i \ (i \in I)$ and $q^h \ (h \in P^\vee)$ satisfying following relations:

$$
\begin{align*}
q^0 &= 1, \quad q^h q^{h'} = q^{h+h'} \quad \text{for } h, h' \in P^\vee, \\
q^h e_i q^{-h} &= q^{(h,\alpha_i)} e_i, \quad q^h f_i q^{-h} = q^{-(h,\alpha_i)} f_i \quad \text{for } h \in P^\vee, i \in I, \\
e_i f_j - f_j e_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad \text{where } K_i = q^{s_i h_i}, \\
\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \begin{array}{c} 1 - a_{ij} \\ r \end{array} \right] \ e_{i}^{1-a_{ij} - r} e_j e_i^r &= 0 \quad \text{if } i \neq j, \\
\sum_{r=0}^{1-a_{ij}} (-1)^r \left[ \begin{array}{c} 1 - a_{ij} \\ r \end{array} \right] \ f_{i}^{1-a_{ij} - r} f_j f_i^r &= 0 \quad \text{if } i \neq j.
\end{align*}
$$

Here, we set $[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i$ and $[m]_i = \frac{[m]_i!}{[m-n]_i [n]_i!}$ for each $m, n \in \mathbb{Z}_{\geq 0}, \ i \in I.$

We denote by $\overline{q} = q^{-1}, \quad \overline{q^h} = q^{-h} \ (h \in P^\vee), \quad \overline{e_i} = e_i \ (i \in I), \quad \overline{f_i} = f_i \ (i \in I).$
We denote by $\varphi$ and $*$ the $\mathbb{Q}(q)$-algebra anti-automorphisms of $U_q(\mathfrak{g})$ given by
\[
(q^h)^* = q^{-h} \quad (h \in P^\vee), \quad e_i^* = e_i \quad (i \in I), \quad f_i^* = f_i \quad (i \in I),
\]
\[
\varphi(q^h) = q^h \quad (h \in P^\vee), \quad \varphi(e_i) = f_i \quad (i \in I), \quad \varphi(f_i) = e_i \quad (i \in I).
\]
We have two comultiplications $\Delta_\pm : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ given by
\[
\Delta_+(q^h) = q^h \otimes q^h, \quad \Delta_+(e_i) = e_i \otimes 1 + K_i \otimes e_i, \quad \Delta_+(f_i) = f_i \otimes K_i^{-1} + 1 \otimes f_i,
\]
\[
\Delta_-(q^h) = q^h \otimes q^h, \quad \Delta_-(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \quad \Delta_-(f_i) = f_i \otimes 1 + K_i \otimes f_i.
\]
Let $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) be the subalgebra of $U_q(\mathfrak{g})$ generated by $e_i$’s (resp. $f_i$’s), and let $U_q^0(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $q^h \ (h \in P^\vee)$. Then we have the triangular decomposition
\[
U_q(\mathfrak{g}) \simeq U_q^-(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}),
\]
and the weight space decomposition
\[
U_q(\mathfrak{g}) = \bigoplus_{\beta \in \mathbb{Q}} U_q(\mathfrak{g})_\beta,
\]
where $U_q(\mathfrak{g})_\beta := \{ x \in U_q(\mathfrak{g}) \ ; \ q^h x q^{-h} = q^{(h,\beta)} x \ \text{for any} \ h \in P^\vee \}$.

Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ and set
\[
e_i^{(n)} = e_i^n/[n]_i!, \quad f_i^{(n)} = f_i^n/[n]_i! \quad (n \in \mathbb{Z}_{\geq 0}).
\]
We define the $\mathcal{A}$-form $U_A(\mathfrak{g})$ to be the $\mathcal{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}, f_i^{(n)}$ ($i \in I, n \in \mathbb{Z}_{\geq 0}$), $q^h \ (h \in P^\vee)$. Let $U_A^+(\mathfrak{g})$ (resp. $U_A^-(\mathfrak{g})$) be the $\mathcal{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}$ (resp. $f_i^{(n)}$) for $i \in I, n \in \mathbb{Z}_{\geq 0}$.

1.2. Crystal bases and global bases.
Let $e'_i$ and $e''_i$ be the operator on $U_q^-(\mathfrak{g})$ defined by
\[
[e_i, x] = \frac{e''_i(x)K_i - K_i^{-1}e'_i(x)}{q_i - q_i^{-1}} \quad (x \in U_q^-(\mathfrak{g})).
\]

The operator $e'_i$ satisfies
\[
e'_i(f_j) = \delta_{ij} \quad (i, j \in I) \quad \text{and} \quad e'_i(xy) = e'_i(x)y + q_i^{(h_i,wt(x))}xe'_i(y) \quad (x, y \in U_q^-(\mathfrak{g})).
\]

Hence we have
\[
e'_if_j = q_i^{-(h_i,\alpha_j)}f_j e'_i + \delta_{ij},
\]
where $f_j \in \text{End}_{\mathbb{Q}(q)}(U_q^-(\mathfrak{g}))$ denotes the left multiplication by $f_j \in U_q^-(\mathfrak{g})$. There exists a unique non-degenerate symmetric bilinear form $(\cdot, \cdot)_-$ on $U_q^-(\mathfrak{g})$ satisfying
\[
(1,1)_- = 1 \quad \text{and} \quad (e'_i x, y)_- = (x, f_i y)_- \quad \text{for any} \ x, y \in U_q^-(\mathfrak{g}).
Any element \( x \in U_q^-(g) \) can be uniquely written as
\[
x = \sum_{n \geq 0} f_i^{(n)} x_n \quad \text{for some } x_n \in \text{Ker}(\epsilon'_i).
\]

We define the Kashiwara operators \( \tilde{e}_i \) and \( \tilde{f}_i \) on \( U_q^-(g) \) by
\[
\tilde{e}_i x = \sum_{n \geq 1} f_i^{(n-1)} x_n, \quad \tilde{f}_i x = \sum_{n \geq 0} f_i^{(n+1)} x_n.
\]

**Proposition 1.2** ([12]). Let \( A_0 = \{ f \in \mathbb{Q}(q) \mid f \text{ is regular at } q = 0 \} \). Define
\[
L(\infty) = \sum_{\ell \geq 0, i_1, \ldots, i_\ell \in I} A_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot 1 \subset U_q^-(g),
\]
\[
B(\infty) = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot 1 \mod qL(\infty) \mid \ell \geq 0, i_1, \ldots, i_\ell \in I \} \subset L(\infty)/qL(\infty).
\]

Then we have
(a) \( \tilde{e}_i L(\infty) \subset L(\infty) \) and \( \tilde{f}_i L(\infty) \subset L(\infty) \),
(b) \( B(\infty) \) is a basis of \( L(\infty)/qL(\infty) \),
(c) \( \tilde{f}_i B(\infty) \subset B(\infty) \) and \( \tilde{e}_i B(\infty) \subset B(\infty) \cup \{ 0 \} \).

We call the pair \( (L(\infty), B(\infty)) \) the crystal basis of \( U_q^-(g) \).

Set \( L(\infty)^- = \{ \bar{x} \mid x \in L(\infty) \} \). Then the triple \( (L(\infty), L(\infty)^-, U_{A_0}^-(g)) \) is balanced; i.e.,
\[
L(\infty) \cap L(\infty)^- \cap U_{A_0}^-(g) \to L(\infty)/qL(\infty)
\]
is a \( \mathbb{Q} \)-vector space isomorphism.

Let \( G^{\text{low}} \) be the inverse of the above isomorphism. Then
\[
B^{\text{low}} := \{ G^{\text{low}}(b) \mid b \in B(\infty) \}
\]
forms a basis of \( U_q^-(g) \). We call it the lower global basis. Let
\[
B^{\text{up}} := \{ G^{\text{up}}(b) \mid b \in B(\infty) \}
\]
be the dual basis of \( B^{\text{low}} \) with respect to the bilinear form \( (\cdot, \cdot)_- \). It is called the upper global basis of \( U_q^-(g) \).

### 1.3. Khovanov-Lauda-Rouquier algebras.

Now we recall the definition of the Khovanov-Lauda-Rouquier algebras associated with a given Cartan datum \( (A, P, P^\vee, P^{\vee}) \).

Let \( k \) be a base field. For \( i, j \in I \) such that \( i \neq j \), set
\[
S_{i,j} = \{ (p, q) \in \mathbb{Z}^2_{\geq 0} \mid (\alpha_i | \alpha_i)p + (\alpha_j | \alpha_j)q = -2(\alpha_i | \alpha_j) \}.
\]
Let us define the polynomials \((Q_{ij})_{i,j \in I}\) in \(k[u,v]\) by

\[
Q_{ij}(u,v) = \begin{cases} 
0 & \text{if } i = j, \\
\sum_{(p,q) \in S_{i,j}} t_{i,j;p,q} u^p v^q & \text{if } i \neq j.
\end{cases}
\]

They satisfy \(t_{i,j;p,q} = t_{j,i;q,p}\) (equivalently, \(Q_{i,j}(u,v) = Q_{j,i}(v,u)\)) and \(t_{i,j;\alpha_{ij},0} \in k^x\).

We denote by \(S_n = (s_1, \ldots, s_{n-1})\) the symmetric group on \(n\) letters, where \(s_i := (i,i+1)\) is the transposition of \(i\) and \(i+1\). Then \(S_n\) acts on \(I^n\) by place permutations.

**Definition 1.3.** The Khovanov-Lauda-Rouquier algebras \(R(n)\) of degree \(n\) associated with the Cartan datum \((\Lambda, P, \Pi, P^\vee, \Pi^\vee)\) and the matrix \((Q_{ij})_{i,j \in I}\) is the associative algebra over \(k\) generated by the elements \(\{e(\nu)\}_{\nu \in I^n}, \{x_k\}_{1 \leq k \leq n}, \{\tau_m\}_{1 \leq m \leq n-1}\) satisfying the following defining relations:

\[
e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad \sum_{\nu \in I^n} e(\nu) = 1,
\]

\[
x_k x_m = x_m x_k, \quad x_k e(\nu) = e(\nu) x_k,
\]

\[
\tau_m e(\nu) = e(s_m(\nu))\tau_m, \quad \tau_k \tau_m = \tau_m \tau_k \text{ if } |k-m| > 1,
\]

\[
\tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}) e(\nu),
\]

\[
(\tau_k x_m - x_{s_k(m)} \tau_k) e(\nu) = \begin{cases} 
-e(\nu) & \text{if } m = k, \nu_k = \nu_{k+1}, \\
e(\nu) & \text{if } m = k + 1, \nu_k = \nu_{k+1}, \\
0 & \text{otherwise},
\end{cases}
\]

\[
(\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) e(\nu) = \begin{cases} 
Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}) - Q_{\nu_k,\nu_{k+1}}(x_{k+2}, x_{k+1}) & \text{if } \nu_k = \nu_{k+2}, \\
0 & \text{otherwise}.
\end{cases}
\]

The above relations are homogeneous provided with

\[
\deg e(\nu) = 0, \quad \deg x_k e(\nu) = (\alpha_{\nu_k} | \alpha_{\nu_k}), \quad \deg \tau_k e(\nu) = -(\alpha_{\nu_k} | \alpha_{\nu_{k+1}}),
\]

and hence \(R(n)\) is a \((\mathbb{Z}_-)\)-graded algebra. For a graded \(R(\beta)\)-module \(M = \bigoplus_{k \in \mathbb{Z}} M_k\), we define \(qM = \bigoplus_{k \in \mathbb{Z}} (qM)_k\), where

\[
(qM)_k = M_{k-1} \ (k \in \mathbb{Z}).
\]

We call \(q\) the grade-shift functor on the category of graded \(R(n)\)-modules.

For \(n \in \mathbb{Z}_{\geq 0}\) and \(\beta \in \Pi_+\) such that \(|\beta| = n\), we set

\[
I^\beta = \{ \nu = (\nu_1, \ldots, \nu_n) \in I^n; \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta \},
\]

and

\[
R(\beta) = \bigoplus_{\nu \in I^\beta} R(n)e(\nu).
\]
Note that for each $\beta \in \mathbb{Q}^+$, the element $e(\beta) := \sum_{\nu \in I^\beta} e(\nu)$ is a central idempotent of $R(n)$. The algebra $R(\beta)$ is called the Khovanov-Lauda-Rouquier algebra at $\beta$.

For a graded $R(m)$-module $M$ and a graded $R(n)$-module $N$, we define the convolution product $M \circ N$ by

$$M \circ N = R(m + n) \otimes_{R(m) \otimes R(n)} (M \otimes N).$$

Let us denote by $R(\beta)$-proj (respectively, $R(\beta)$-gmod) the category of finitely generated graded projective (respectively, finite-dimensional over $k$ graded) $R(\beta)$-modules. When $K(R(\beta)$-proj) and $K(R(\beta)$-gmod) denote the corresponding Grothendieck groups, the spaces

$$\bigoplus_{\beta \in \mathbb{Q}^+} K(R(\beta)$-proj), \quad \bigoplus_{\beta \in \mathbb{Q}^+} K(R(\beta)$-gmod)

are $A$-algebras with multiplications given by convolution products and $A$-actions given by the grade-shift functor $q$.

A Khovanov-Lauda-Rouquier algebra categorifies the negative half of the corresponding quantum group. More precisely, we have the following theorem.

Theorem 1.4 ([16,19]). Let $U_q(\mathfrak{g})$ be the quantum group associated with a given Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$ and $R = \bigoplus_{n \geq 0} R(n)$ be the Khovanov-Lauda-Rouquier algebra associated with the same Cartan datum and a matrix $(Q_{ij})_{i,j \in I}$ given in (1.1). Then there exists an $A$-algebra isomorphism

$$U^-_A(\mathfrak{g}) \simeq \bigoplus_{\beta \in \mathbb{Q}^+} K(R(\beta)$-proj).

By duality, we have

$$U^-_A(\mathfrak{g})^{\vee} \simeq \bigoplus_{\beta \in \mathbb{Q}^+} K(R(\beta)$-gmod),

where $U^-_A(\mathfrak{g})^{\vee} := \{ x \in U_q(\mathfrak{g}) ; (x, U^-_A(\mathfrak{g})) \subset A \}$.

The Khovanov-Lauda-Rouquier algebras also categorify the global bases in the following sense:

Theorem 1.5 ([21,20]). Assume that $A$ is symmetric. Then under the isomorphism in Theorem 1.4, the lower global basis (respectively, upper global basis) corresponds to the set of isomorphism classes of indecomposable projective modules (respectively, the set of isomorphism classes of simple modules).

2. Quantum affine algebras and their representations

2.1. Quantum affine algebras.
In this section, we briefly review the representation theory of quantum affine algebras following [1, 14] and introduce a functor between the category of Khovanov-Lauda-Rouquier algebra modules and the category of quantum affine algebra modules. Hereafter, we take $\mathbb{C}(q)$ as the base field $k$.

Let $I = \{0, 1, \ldots, n\}$ be an index set and $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix of affine type; i.e., $A$ is positive semidefinite of corank 1. Here 0 is chosen as the leftmost vertices in the tables in [11, pages 48, 49]. We take a Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$ as follows.

The coweight lattice $P^\vee$ is given by

$$P^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d.$$

The element $d$ is called the scaling element. We define the simple roots $\alpha_i$’s ($i \in I$) and the fundamental weights $\Lambda_i$’s ($i \in I$) in the weight lattice $P := \text{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{Z})$ as follows:

$$\alpha_i(h_j) = a_{ji}, \quad \alpha_i(d) = \delta_{0,i}, \quad \text{and} \quad \Lambda_i(h_j) = \delta_{i,j}, \quad \Lambda_i(d) = 0.$$

We denote by $\Pi = \{\alpha_i ; i \in I\}$ and $\Pi^\vee = \{h_i ; i \in I\}$ the set of simple roots and the set of simple coroots, respectively.

Let us denote by $\mathfrak{g}$ the affine Kac-Moody algebra corresponding to the Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$. Consider the positive integers $c_i$’s and $d_i$’s determined by the conditions

$$\sum_{i=0}^{n} c_i a_{ij} = \sum_{j=0}^{n} a_{ji} d_i = 0 \quad \text{for all} \quad j \in I,$$

and $\{c_0, c_1, \ldots, c_n\}, \{d_0, d_1, \ldots, d_n\}$ are relatively prime positive integers (see [11, Chapter 4]). Then the center of $\mathfrak{g}$ is 1-dimensional and is generated by the canonical central element

$$c = c_0 h_0 + c_1 h_1 + \cdots + c_n h_n$$

([11, Proposition 1.6]). Also it is known that the imaginary roots of $\mathfrak{g}$ are nonzero integral multiples of the null root

$$\delta = d_0 \alpha_0 + d_1 \alpha_1 + \cdots + d_n \alpha_n$$

([11, Theorem 5.6]). Note that $d_0 = 1$ if $\mathfrak{g} \neq A_{2n}^{(2)}$ and $d_0 = 2$ if $\mathfrak{g} = A_{2n}^{(2)}$. Note also that $c_0 = 1$ in all cases.

Now the weight lattice can be written as

$$P = \mathbb{Z} \Lambda_0 \oplus \mathbb{Z} \Lambda_1 \oplus \cdots \oplus \mathbb{Z} \Lambda_n \oplus \mathbb{Z} d_0^{-1} \delta$$

(see [11, Chapter 4]). We have

$$\alpha_0 = \sum_{i=0}^{n} a_{ij} \Lambda_i + d_0^{-1} \delta, \quad \alpha_j = \sum_{i=0}^{n} a_{ij} \Lambda_i \quad \text{for} \quad j = 1, \ldots, n.$$
Let us denote by $U_q(\mathfrak{g})$ the quantum group associated with the affine Cartan datum $(A, P, \Pi, P, \Pi)$. We denote by $U'_q(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by $e_i, f_i, K_i^{\pm 1} (i = 0, 1, \ldots, n)$ and it is called the quantum affine algebra. Hereafter we extend the base field $\mathbb{Q}(q)$ of $U'_q(\mathfrak{g})$ to $k := \mathbb{C}(q)$ for convenience.

Set

$$P' = Zh_0 \oplus \cdot \cdot \cdot \oplus Zh_n \subset P'',$$

and $h_{cl} = \mathbb{Q} \otimes \mathbb{Z} P' \subset h$. Let $cl : h^* \rightarrow (h_{cl})^*$ be the projection thus obtained. Then $cl^{-1}(0) = \mathbb{Q} \delta$. We denote the classical weight lattice $cl(P)$ by $P_{cl}$. Set $\Pi_{cl} = cl(\Pi)$, and set $\Pi'_{cl} = \{h_0, \ldots, h_n\}$. Then $U'_q(\mathfrak{g})$ can be regarded as the quantum group associated with the quintuple $(A, P_{cl}, \Pi_{cl}, P'_{cl}, \Pi'_{cl})$.

Set $h^{(0)} = \{\lambda \in h^* \mid \lambda(c) = 0\}$, $h_{cl}^{(0)} = cl(h^{(0)})$ and $P_{cl}^{0} = cl(P) \cap cl(h^{(0)})$. We call the elements of $P_{cl}^{0}$ by the classical integral weight of level 0. Let $W$ be the Weyl group of $\mathfrak{g}$. It is the subgroup of Aut$(h^*)$ generated by the simple reflections $\sigma_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$ for $i = 0, 1, \ldots, n$. Since $\delta(h_i) = \alpha_i(c) = 0$ for $i = 0, 1, \ldots, n$, there exists a group homomorphism $W \rightarrow Aut(h_{cl}^{(0)})$. We denote the image by $W_{cl}$. Then $W_{cl}$ is a finite group and it is isomorphic to the subgroup of $W$ generated by $\sigma_1, \ldots, \sigma_n$.

A $U'_q(\mathfrak{g})$-module $M$ is called an integrable module if $M$ has a weight space decomposition

$$M = \bigoplus_{\lambda \in P_{cl}} M_{\lambda},$$

where $M_{\lambda} = \{u \in M ; q^h u = q^{(h, \lambda)} u \text{ for all } h \in P'_{cl}\}$, and if the actions of $e_i$ and $f_i$ on $M$ are locally nilpotent for any $i \in I$. In this paper, we mainly consider the category of finite-dimensional integrable $U'_q(\mathfrak{g})$-modules. Let us denote this category by $\mathcal{C}$. The objects in this category are called of type 1 (for example, see [3]).

**Definition 2.1.** Let $u$ be a weight vector of weight $\lambda \in P_{cl}$ of an integrable $U'_q(\mathfrak{g})$-module $M$. We call $u$ extremal, if we can find vectors $\{u_w\}_{w \in W}$ satisfying the following properties:

$$u_w = u \text{ for } w = e,$$

$$\text{if } \langle h_i, w\lambda \rangle \geq 0 \text{, then } e_i u_w = 0 \text{ and } f_i^{(h_i, w\lambda)} u_w = u_{s_i w},$$

$$\text{if } \langle h_i, w\lambda \rangle \leq 0 \text{, then } f_i u_w = 0 \text{ and } e_i^{(h_i, w\lambda)} = u_{s_i w}.$$
where \( d_i \in \mathbb{Z}_{>0} \) denotes the generator of the free abelian group \( \{ m \in \mathbb{Z} : \omega_i + m\delta \in W\omega_i \} \).

We define the \( U'_q(\mathfrak{g}) \)-module \( V(\omega_i) \) by
\[
V(\omega_i) = W(\omega_i)/(z_i - 1)W(\omega_i).
\]

It can be characterized as follows (\cite[Section 1.3]{[14]}):

1. The weights of \( V(\omega_i) \) are contained in the convex hull of \( W_{d\text{-cl}}(\omega_i) \).
2. \( \dim V(\omega_i)_{d\text{-cl}(\omega_i)} = 1 \).
3. For any \( \mu \in W_{d\text{-cl}}(\omega_i) \subset P_{d\text{-cl}}^0 \), we can associate a nonzero vector \( u_\mu \) of weight \( \mu \) such that
\[
 u_{s_i \mu} = \begin{cases} 
 f_i^{(h_i, \mu)} u_\mu & \text{if } \langle h_i, \mu \rangle \geq 0, \\
 e_i^{(-\langle h_i, \mu \rangle)} u_\mu & \text{if } \langle h_i, \mu \rangle \leq 0.
\end{cases}
\]

We call \( V(\omega_i) \) the fundamental representation of \( U'_q(\mathfrak{g}) \) of weight \( \omega_i \).

Let \( \bar{\Phi} \) be an involution of a \( U'_q(\mathfrak{g}) \)-module \( M \) satisfying \( \bar{a}a = \bar{a} \bar{u} \) for any \( a \in U'_q(\mathfrak{g}) \) and \( u \in M \). We call such an involution a bar involution.

We say that a finite crystal \( B \) with weight in \( P_{d\text{-cl}}^0 \) is a simple crystal if there exists \( \lambda \in P_{d\text{-cl}}^0 \) such that \( \#(B_\lambda) = 1 \) and the weight of any extremal vector of \( B \) is contained in \( W_{d\text{-cl}} \lambda \).

If a \( U'_q(\mathfrak{g}) \)-module \( M \) has a bar involution, a crystal base with simple crystal graph, and a global base, then we say that \( M \) is a good module (\cite[Section 8]{[14]}). For example, the fundamental representation \( V(\omega_i) \) is a good \( U'_q(\mathfrak{g}) \)-module. Any good module is an irreducible \( U'_q(\mathfrak{g}) \)-module.

Let \( A \) be a commutative \( k \)-algebra and let \( x \) be an invertible element of \( A \). For an \( A \otimes_k U'_q(\mathfrak{g}) \)-module \( M \), let us denote by \( \Phi_x(M) \) the \( A \otimes_k U'_q(\mathfrak{g}) \)-module constructed in the following: there exists an \( A \)-linear bijection \( \Phi_x : M \rightarrow \Phi_x(M) \) which satisfy
\[
 q^h \Phi_x(u) = \Phi_x(q^h u) \quad (h \in P_{d\text{-cl}}^0), \quad e_i \Phi_x(u) = x^{\delta_{i,0}} \Phi_x(e_i u), \quad f_i \Phi_x(u) = x^{-\delta_{i,0}} \Phi_x(f_i u).
\]

For invertible elements in \( x, y \) of \( A \) and \( A \otimes_k U'_q(\mathfrak{g}) \)-modules \( M, N \), we have
\[
\Phi_x \Phi_y(M) \simeq \Phi_{xy}(M)
\]
by \( \Phi_x(\Phi_y(u)) \leftrightarrow \Phi_{xy}(u) \), and
\[
\Phi_x(M \otimes_k N) \simeq \Phi_x(M) \otimes_k \Phi_x(N)
\]
by \( \Phi_x(u \otimes v) \leftrightarrow \Phi_x(u) \otimes \Phi_x(v) \).

For an integrable \( U'_q(\mathfrak{g}) \)-module \( M \), the affinization of \( M \) is given by
\[
M_{\text{aff}} := \Phi_z(k[z, z^{-1}] \otimes_k M).
\]

Note that by defining \( q^d \Phi_z(z^n \otimes u) = q^{d, n(\delta)} \Phi_z(z^n \otimes u) \) for \( u \in M \), \( M_{\text{aff}} \) becomes a \( U_q(\mathfrak{g}) \)-module. (We need a slight modification for \( \mathfrak{g} = A^{(2)}_{2n} \).) For example, we have
\[ V(\varpi_i)_{\text{aff}} \simeq k[z_i^{1/d_i}] \otimes_{k[z_i]} W(\varpi_i), \text{ and hence if } d_i = 1, \text{ then } W(\varpi_i) \simeq V(\varpi_i)_{\text{aff}} \] (Theorem 5.15).

For \( a \in k^\times \), we define \( U'_q(\mathfrak{g}) \)-module \( M_a \) by
\[
M_a := M_{\text{aff}}/(z-a)M_{\text{aff}} \simeq \Phi_a(M).
\]

It is called the evaluation module of \( M \) at \( a \).

2.2. \( R \)-matrices.

We recall the notion of the \( R \)-matrices of good modules following [14, Section 8].

Let \( M_1 \) and \( M_2 \) be good \( U'_q(\mathfrak{g}) \)-modules. Set \( (M_1)_{\text{aff}} = \Phi_{z_1}(k[z_1^{\pm 1}] \otimes M_1) \), \( (M_2)_{\text{aff}} = \Phi_{z_2}(k[z_2^{\pm 1}] \otimes M_2) \), and let \( u_1 \) and \( u_2 \) be the dominant extremal weight vectors in \( M_1 \) and \( M_2 \), respectively.

Then there exists a \( U'_q(\mathfrak{g}) \)-module homomorphism
\[
R_{M_1,M_2}^{\text{norm}} : (M_1)_{\text{aff}} \otimes (M_2)_{\text{aff}} \to k(z_1, z_2) \otimes_{k[z_1^{\pm 1}, z_2^{\pm 1}]} ((M_2)_{\text{aff}} \otimes (M_1)_{\text{aff}}),
\]
satisfying
\[
R_{M_1,M_2}^{\text{norm}} \circ z_i = z_i \circ R_{M_1,M_2}^{\text{norm}} \text{ for } i = 1, 2
\]
and
\[
R_{M_1,M_2}^{\text{norm}}(u_1 \otimes u_2) = u_2 \otimes u_1
\]
([14, Section 8]).

Let \( d_{M_1,M_2}(u) \in k[u] \) be a monic polynomial with the smallest degree such that the image of \( d_{M_1,M_2}(z_1/z_2)R_{M_1,M_2}^{\text{norm}} \) is contained in \( (M_2)_{\text{aff}} \otimes (M_1)_{\text{aff}} \). We call \( R_{M_1,M_2}^{\text{norm}} \) the normalized \( R \)-matrix and \( d_{M_1,M_2} \) the denominator of \( R_{M_1,M_2}^{\text{norm}} \). Since \( (M_1)_x \otimes (M_2)_y \) is irreducible for generic \( x, y \in k^\times \), we have
\[
R_{M_1,M_2}^{\text{norm}}(1) \circ R_{M_1,M_2}^{\text{norm}} = 1_{(M_1)_{\text{aff}} \otimes (M_2)_{\text{aff}}}.
\]

It also satisfies the Yang-Baxter equation
\[
(R_{M_1,M_2}^{\text{norm}} \otimes 1) \circ (1 \otimes R_{M_2,M_3}^{\text{norm}}) \circ (R_{M_2,M_3}^{\text{norm}} \otimes 1) = (1 \otimes R_{M_2,M_3}^{\text{norm}}) \circ (R_{M_1,M_2}^{\text{norm}} \otimes 1) \circ (1 \otimes R_{M_1,M_2}^{\text{norm}}).
\]

The following fact is proved in [14, Proposition 9.3].

Lemma 2.2. The zeroes of \( d_{M_1,M_2}(z) \) belong to \( \mathbb{C}[[q^{1/m}]]q^{1/m} \) for some \( m \in \mathbb{Z}_{>0} \).

Example 2.3. When \( \mathfrak{g} = \mathfrak{sl}_N \), the normalized \( R \)-matrices for the fundamental representations are given as follows (see, for example, [6]):
\[
R_{V(\varpi_k),V(\varpi_\ell)}^{\text{norm}}(\varpi) = \sum_{0\leq i \leq \min(k,\ell)} \prod_{s=1}^i \frac{1 - (-q)^{|k-\ell|+2s}z}{z - (-q)^{|k-\ell|+2s}} P_{\varpi_{\max(k,\ell)+s+\varpi_{\max(k,\ell)}-i}},
\]
where \( z = z_1/z_2 \) and \( P_\lambda \) denotes the projection from \( V(\varpi_k) \otimes V(\varpi_\ell) \) to the direct summand \( V(\lambda) \) as a \( U'_q(\mathfrak{sl}_N) \)-module.
Note that $P_{V(\varpi_k),V(\varpi_\ell)}^{\text{norm}}$ has simple poles at $z = (-q)^{|k-\ell|+2s}$ for $1 \leq s \leq \min\{k, \ell\}$.

2.3. The action of $R^l(n)$ on $\hat{V}^{\otimes n}$. Let $\{V_s\}_{s \in \mathcal{S}}$ be a family of good $U'_q(g)$-modules and let $\lambda_s$ be a dominant extremal weight of $V_s$ and $v_s$ a dominant extremal weight vector in $V_s$ of weight $\lambda_s$.

Let $\mathbb{T} = k^\times$ and let $I$ be a finite subset of $\mathcal{S} \times \mathbb{T}$. For each $i \in I$, let $X : I \rightarrow \mathbb{T}$ and $S : I \rightarrow \mathcal{S}$ be the maps defined by $i = (S(i), X(i))$.

For each $i, j \in I$, set

$$P_{ij}(u, v) = (v - u)^{d_{ij}},$$

where $d_{ij}$ denotes the order of the zero of $dV_{S(i)}(u)\nu_{S(j)}(z_1/z_2)$ at $z_1/z_2 = X(i)/X(j)$.

Let $R^l$ be the Khovanov-Lauda-Rouquier algebra associated with $Q_{ij}(u, v) = P_{ij}(u, v)P_{ji}(v, u)$ for $i, j \in I$.

**Remark 2.4.** Consider the quiver $\Gamma_I := (I, \Omega)$ with the set of vertices $I$ and the set of oriented edges $\Omega$ such that

$$\# \{h \in \Omega : s(h) = i, t(h) = j\} = d_{ij},$$

where $s(h)$ and $t(h)$ denote the source and the target of an oriented edge $h \in \Omega$.

Lemma 2.2 implies that $X(i)/X(j) \in \mathbb{C}[[q]]q$ if $d_{ij} > 0$. Hence we obtain

$$\text{if } d_{ij} > 0, \text{ then } d_{ji} = 0.$$ 

Thus the quiver $\Gamma_I$ has neither loops nor 2-cycles. The underlying unoriented graph of $\Gamma_I$ gives a symmetric Cartan datum and the polynomials in (2.4) coincide with the ones used in [21] associated with the symmetric Cartan datum of $\Gamma_I$.

Set

$$\mathbb{P}_n := \bigoplus_{\nu \in I^n} k[x_1, \ldots, x_n]e(\nu),$$

$$\hat{\mathbb{P}}_n := \bigoplus_{\nu \in I^n} \hat{\mathcal{O}}_{\mathbb{T}^n, X(\nu)} e(\nu),$$

$$\hat{\mathbb{K}}_n := \bigoplus_{\nu \in I^n} \hat{\mathbb{K}}_{\nu} e(\nu),$$

where

$$\hat{\mathcal{O}}_{\mathbb{T}^n, X(\nu)} = k[[X_1 - X(\nu_1), \ldots, X_n - X(\nu_n)]]$$

is the completion of the local ring of $\mathbb{T}^n$ at $X(\nu) := (X(\nu_1), \ldots, X(\nu_n))$ and $\hat{\mathbb{K}}_{\nu}$ is the field of quotients of $\hat{\mathcal{O}}_{\mathbb{T}^n, X(\nu)}$.

Then we have

$$\mathbb{P}_n \hookrightarrow \hat{\mathbb{P}}_n \hookrightarrow \hat{\mathbb{K}}_n.$$
as \( k \)-algebras, where the first arrow is given by
\[
x_k e(\nu) \mapsto (X(\nu_k)^{-1} X_k - 1)e(\nu).
\]
Note that
\[
k[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \subset \hat{O}_{T^n, X(\nu)} \text{ for all } \nu \in I^n.
\]
Let
\[
k[S_n] := \bigoplus_{w \in S_n} kr_w
\]
be the group algebra of \( S_n \); i.e., the \( k \)-algebra with the defining relations
\[
\begin{align*}
\tau_a^2 &= 1 & a &= 1, \ldots, n - 1 \\
\tau_a \tau_{a'} &= \tau_{a'}, \\
\tau_a \tau_{a+1} \tau_a &= \tau_{a+1} \tau_a \tau_{a+1} & a &= 1, \ldots, n - 2
\end{align*}
\]
where \( r_a = r_{s_a} \) (1 \( \leq \) a < n).

The symmetric group \( S_n \) acts on \( P_n, \hat{P}_n, \hat{K}_n \) from the left and we have
\[
P_n \otimes k[S_n] \hookrightarrow \hat{P}_n \otimes k[S_n] \hookrightarrow \hat{K}_n \otimes k[S_n]
\]
as algebras. Here the algebra structure on \( \hat{K}_n \otimes k[S_n] \) is given by
\[
r_w f = w(f)r_w \text{ for } f \in \hat{K}_n, \ w \in S_n.
\]
Then \( \hat{K}_n \) may be regarded as a right \( \hat{K}_n \otimes k[S_n] \)-module by \( a(f \otimes r_w) = w^{-1}(af) \) (\( a, f \in \hat{K}_n \) and \( w \in S_n \)).

Then \( \hat{K}_n \) may be regarded as a right \( \hat{K}_n \otimes k[S_n] \)-module by \( a(f \otimes r_w) = w^{-1}(af) \) (\( a, f \in \hat{K}_n \) and \( w \in S_n \)).

Set
\[
e(\nu) \tau_a = \begin{cases} e(\nu)r_a P_{\nu_a, \nu_{a+1}}(x_{a+1}, x_a) & \text{if } \nu_a \neq \nu_{a+1} \\
e(\nu)(r_a - 1)(x_a - x_{a+1})^{-1} & \text{if } \nu_a = \nu_{a+1}.
\end{cases}
\]
Then the subalgebra of \( \hat{K}_n \otimes k[S_n] \) generated by
\[
e(\nu) \ (\nu \in J^n), \quad x_a \ (1 \leq a \leq n), \quad e(\nu) \tau_a \ (1 \leq a \leq n - 1)
\]
is isomorphic to the Khovanov-Lauda-Rouquier algebra \( R^l(n) \) of degree \( n \) associated with \( Q_{ij}(u, v) = P_{ij}(u, v)P_{ji}(v, u) \) [19, Proposition 3.12], [16, Theorem 2.5].

For each \( \nu = (\nu_1, \ldots, \nu_n) \in I^n \), we set
\[
V_\nu = \Phi_{X_1}(k[X_1^{\pm 1}] \otimes V_{S(\nu_1)}) \otimes \cdots \otimes \Phi_{X_n}(k[X_n^{\pm 1}] \otimes V_{S(\nu_n)})
\]
which is a \( k[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \otimes U'_q(g) \)-module. Then we define
\[
\begin{align*}
\hat{V} \otimes n & := \bigoplus_{\nu \in J^n} \hat{O}_{T^n, X(\nu)} \otimes_{k[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]} V_\nu e(\nu), \\
\hat{V}^n_K & := \hat{K}_n \otimes_{F_n} \hat{V} \otimes n.
\end{align*}
\]
For each \( \nu \in I^n \) and \( a = 1, \ldots, n-1 \), there exists a \( U'_q(\mathfrak{g}) \)-module homomorphism

\[
R^\nu_{a,a+1} : V_\nu \to k(X_1, \ldots, X_n) \otimes_{k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]} V_{s_a(\nu)}
\]

which is given by

\[
v_1 \otimes \cdots \otimes v_a \otimes v_{a+1} \otimes \cdots \otimes v_n \mapsto v_1 \otimes \cdots \otimes R^\text{norm}_{V_{S(\nu a)}, V_{S(\nu a+1)}}(v_a \otimes v_{a+1}) \otimes \cdots \otimes v_n
\]

for \( v_k \in \Phi_{X_k}(V_{S(\nu k)}) \) (\( 1 \leq k \leq n \)).

It follows that

\[
R^\nu_{a,a+1} \circ X_k = X_{s_a(k)} \circ R^\nu_{a,a+1}
\]

from \((2.1)\),

\[
R^\nu_{a,a+1} = 1_{V_\nu}
\]

from \((2.2)\),

\[
R^\nu_{a,a+1} \circ R^\nu_{a+1,a+2} \circ R^\nu_{a,a+1} = R^\nu_{a+1,a+2} \circ R^\nu_{a,a+1} \circ R^\nu_{a+1,a+2}
\]

from \((2.3)\).

Set \( d_{\nu_a, \nu_{a+1}}(u) = d_{V_{S(\nu a)}, V_{S(\nu_{a+1})}}(u) \). Then,

\[
d_{\nu_a, \nu_{a+1}}(X_{a+1}/X_a) R^\nu_{a,a+1} : V_\nu \to V_{s_a(\nu)}.
\]

The algebra \( \hat{k}_n \otimes k[S_n] \) acts on \( \hat{V}_K^n \) from the right, where

\[
e(\nu)e_a : \hat{k}_\nu \otimes_{k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]} V_\nu
\]

\[
\mapsto \hat{k}_{s_a(\nu)} \otimes k(x_1, \ldots, x_n) \left( k(X_1, \ldots, X_n) \otimes_{k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]} V_{s_a(\nu)} \right)
\]

is given by

\[
(f \otimes v)e(\nu)e_a = s_a(f)e(s_a(\nu)) \otimes R^\nu_{a,a+1}(v)
\]

for \( f \in \hat{k}_\nu, v \in k(X_1, \ldots, X_n) \otimes_{k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]} V_\nu \). The subalgebra \( \hat{k}_n \) acts by the multiplication. The relations \((2.5)\) and \((2.6)\) follow from the properties of normalized \( R \)-matrices and hence we have a well-defined action of the algebra \( \hat{k}_n \otimes k[S_n] \) on \( \hat{V}_K^n \). Since the normalized \( R \)-matrices are \( U'_q(\mathfrak{g}) \)-module homomorphisms, the right action of \( \hat{k}_n \otimes k[S_n] \) commutes with the left action of \( U'_q(\mathfrak{g}) \) on \( \hat{V}_K^n \).

**Theorem 2.5.** The subspace \( \hat{V}_K^n \) of \( \hat{V}_K^n \) is stable under the action of the subalgebra \( R^I(n) \) of \( \hat{k}_n \otimes k[S_n] \). In particular, \( \hat{V}_K^n \) has a structure of \( (U'_q(\mathfrak{g}), R^I(n)) \)-bimodule.

**Proof.** It is obvious that \( \hat{V}_K^n \) is stable by the actions of \( e(\nu) \) (\( \nu \in I^n \)) and \( x_a \) (\( 1 \leq a \leq n \)). Thus it is enough to show that \( \hat{V}_K^n \) is stable under \( e(\nu)x_a \) (\( \nu \in I^n, 1 \leq a < n \)).
Assume \( \nu_a \neq \nu_{a+1} \). Then we have
\[
eq e(\nu) r_a d_{\nu_a, \nu_{a+1}}(X_{a+1}/X_a) (c(s_a(\nu)) \frac{P_{\nu_a, \nu_{a+1}}(x_{a+1}/x_a)}{d_{\nu_a, \nu_{a+1}}(X_{a+1}/X_a)})
\]
Since \( d_{\nu_a, \nu_{a+1}} \) is the multiplicity of the zero of the polynomial \( d_{\nu_a, \nu_{a+1}}(X_{a+1}/X_a) \) at \( X_{a+1}/X_a = X(\nu_a)/X(\nu_{a+1}) \), we have
\[
\frac{(X(\nu_{a+1})^{-1}X_a - X(\nu_a)^{-1}X_{a+1})d_{\nu_a, \nu_{a+1}}}{d_{\nu_a, \nu_{a+1}}(X_{a+1}/X_a)} \in \hat{\mathcal{O}}^n_{\text{ad}(\nu_a(\nu))}.
\]
It follows that
\[
(\hat{\mathcal{O}}^n_{\text{ad}} \otimes \mathbf{k}[X_1^{\pm 1}, \ldots X_n^{\pm 1}]) V_\nu \xrightarrow{e(\nu) \tau_a}
\]
\[
= (\hat{\mathcal{O}}^n_{\text{ad}} \otimes \mathbf{k}[X_1^{\pm 1}, \ldots X_n^{\pm 1}]) V_\nu \xrightarrow{e(\nu) r_a} P_{\nu_a, \nu_{a+1}}(x_{a+1}, x_a)
\]
\[
\subset (\hat{\mathcal{O}}^n_{\text{ad}} \otimes \mathbf{k}[X_1^{\pm 1}, \ldots X_n^{\pm 1}]) V_{sa(\nu)} \xrightarrow{(X(\nu_{a+1})^{-1}X_a - X(\nu_a)^{-1}X_{a+1})d_{\nu_a, \nu_{a+1}}}{d_{\nu_a, \nu_{a+1}}(X_{a+1}/X_a)} \]
\[
\subset \hat{\mathcal{O}}^n_{\text{ad}} \otimes \mathbf{k}[X_1^{\pm 1}, \ldots X_n^{\pm 1}] V_{sa(\nu)},
\]
as desired.

Assume \( \nu_a = \nu_{a+1} \). Then \( R_{V_{\nu_a(x)}}^{\text{norm}}(V_{\nu_a(x)}) \) does not have a pole at \( X_a = X_{a+1} \) by Lemma 2.2. Since \( \Phi_x(V_{\nu_a(x)}) \otimes \Phi_x(V_{\nu_a(x)}) \) is irreducible for any \( x \in \mathbf{k}^\times \), we obtain \( R_{V_{\nu_a(x)}}^{\text{norm}}(V_{\nu_a(x)})|_{X_a = X_{a+1}} = \text{id} \). Therefore, we have
\[
(\hat{\mathcal{O}}^n_{\text{ad}} \otimes \mathbf{k}[X_1^{\pm 1}, \ldots X_n^{\pm 1}]) V_\nu \xrightarrow{e(\nu) \tau_a}
\]
\[
= (\hat{\mathcal{O}}^n_{\text{ad}} \otimes \mathbf{k}[X_1^{\pm 1}, \ldots X_n^{\pm 1}]) V_\nu \xrightarrow{e(\nu)(r_a - 1)(x_a - x_{a+1})^{-1}}
\]
\[
= (\hat{\mathcal{O}}^n_{\text{ad}} \otimes \mathbf{k}[X_1^{\pm 1}, \ldots X_n^{\pm 1}]) V_\nu \xrightarrow{e(\nu)X(\nu_a)(r_a - 1)(X_a - X_{a+1})^{-1}}
\]
\[
\subset \hat{\mathcal{O}}^n_{\text{ad}} \otimes \mathbf{k}[X_1^{\pm 1}, \ldots X_n^{\pm 1}] V_\nu,
\]
as desired. \( \square \)

Since \( \hat{V}^n \) is a \((U'_q(g), R^l(n))\)-bimodule, we can construct the following functor:
\[
\mathcal{F}_n : R^l(n)-\text{gmod} \rightarrow U'_q(g)-\text{mod}
\]
\[
M \mapsto \mathcal{F}_n(M) := \hat{V}^n \otimes_{R^l(n)} M,
\]
where \( U'_q(g)-\text{mod} \) denotes the category of finite-dimensional \( U'_q(g) \)-modules.
Theorem 2.6. Let $M_1 \in R^I(n_1)$-gmod and $M_2 \in R^I(n_2)$-gmod. Then there exists a canonical isomorphism of $U_q(g)$-modules

$$\mathcal{F}_n(M_1 \circ M_2) \simeq \mathcal{F}_{n_1}(M_1) \otimes \mathcal{F}_{n_2}(M_2),$$

where $n = n_1 + n_2$.

Proof. For each $\nu = (\nu_1, \ldots, \nu_n) \in I^n$, set $\nu' = (\nu_1, \ldots, \nu_{n_1})$ and $\nu'' = (\nu_{n_1+1}, \ldots, \nu_n)$. Then we have an algebra homomorphism $\hat{\mathcal{O}}_{T_{n_1},X(\nu')} \otimes \hat{\mathcal{O}}_{T_{n_2},X(\nu'')} \to \hat{\mathcal{O}}_{T_n,X(\nu)}$. Moreover, for any finite-dimensional $\hat{\mathcal{O}}_{T_{n_1},X(\nu')}$-module $L_1$ and any finite-dimensional $\hat{\mathcal{O}}_{T_{n_2},X(\nu'')}$-module $L_2$, the induced morphism

$$L_1 \otimes L_2 \to \hat{\mathcal{O}}_{T_n,X(\nu')} \otimes \hat{\mathcal{O}}_{T_n,X(\nu'')} \otimes (L_1 \otimes L_2)$$

is an isomorphism. Hence for any finite-dimensional $\mathbb{P}_{n_1}$-module $L_1$ and any finite-dimensional $\mathbb{P}_{n_2}$-module $L_2$, the induced morphism

$$(\hat{V}^{n_1} \otimes \hat{V}^{n_2}) \otimes (L_1 \otimes L_2) \to \hat{V}^n \otimes (L_1 \otimes L_2)$$

is an isomorphism.

The module $\hat{V}^n \otimes_{R^I(n)} (M_1 \circ M_2) \simeq \hat{V}^n \otimes_{R^I(n_1) \otimes R^I(n_2)} (M_1 \otimes M_2)$ is the quotient of $\hat{V}^n \otimes_{\mathbb{P}_{n_1} \otimes \mathbb{P}_{n_2}} (M_1 \otimes M_2)$ by the submodule generated by $va \otimes u - v \otimes au$ where $a \in R^I(n_1) \otimes R^I(n_2)$, $v \in \hat{V}^n$, $u \in M_1 \otimes M_2$. A similar result holds also for $(\hat{V}^{n_1} \otimes \hat{V}^{n_2}) \otimes_{R^I(n_1) \otimes R^I(n_2)} (M_1 \otimes M_2)$. Thus we obtain the desired result

$$(\hat{V}^{n_1} \otimes \hat{V}^{n_2}) \otimes_{R^I(n_1) \otimes R^I(n_2)} (M_1 \otimes M_2) \simeq \hat{V}^n \otimes_{R^I(n_1) \otimes R^I(n_2)} (M_1 \otimes M_2).$$

The following propositions are key ingredients for proving our main theorem.

Proposition 2.7 ([13 Corollary 2.9], [2] Theorem 4.6). If the quiver associated with $R^I(n)$ is of type $A, D, E$, then $R^I(n)$ has finite global dimension.

Proposition 2.8. Let $A \to B$ be a homomorphism of algebras. We assume the following conditions:

(a) $B$ is a finitely generated projective $A$-module,
(b) $\text{Hom}_A(B, A)$ is a projective $B$-module,
(c) the global dimension of $B$ is finite.

Then we have:

(i) any $B$-module projective over $A$ is projective over $B$,
(ii) any $B$-module flat over $A$ is flat over $B$. 

\qed
Proof. Since the proof is similar, we give only the proof of (ii).

Let us denote by flat. dim$_A M$ the flat dimension of an $A$-module $M$. By (a) we have

$$\text{flat. dim}_A(M) \leq \text{flat. dim}_B(M)$$

for any $B$-module $M$.

By (b), Hom$_A(B, A) \otimes_A L$ is a flat $B$-module if $L$ is a flat $A$-module. Indeed, the functor $X \otimes_B \text{Hom}_A(B, A)$ is exact in $X \in \text{Mod}(A^{\text{opp}})$ and hence $X \otimes_B \text{Hom}_A(B, A) \otimes_A L$ is also exact in $X$.

On the other hand, for any $A$-module $L$, the canonical $B$-module homomorphism

$$\text{Hom}_A(B, A) \otimes L \to \text{Hom}_A(B, L), \quad f \otimes s \mapsto (B \ni b \mapsto f(b)s)$$

is an isomorphism by (a). Hence we conclude that Hom$_A(B, L)$ is a flat $B$-module for any flat $A$-module $L$. It immediately implies that

$$\text{flat. dim}_B(\text{Hom}_A(B, L)) \leq \text{flat. dim}_A(L) \quad \text{for any } A\text{-module } L.$$

Now, let $M$ be a $B$-module. Then there exists a canonical $B$-module homomorphism

$$\varphi_M : M \to \text{Hom}_A(B, M)$$

given by $\varphi_M(x)(b) = bx$. It is evidently injective.

In order to prove the proposition, it is enough to show the following statement for any $d \geq 0$:

$$\text{for any } B\text{-module } M, \text{flat. dim}_A(M) \leq d \implies \text{flat. dim}_B(M) \leq d.$$

We shall show it by the descending induction on $d$. If $d \gg 0$, it is a consequence of (c). Let $M$ be a $B$-module with flat. dim$_A(M) \leq d$. We have an exact sequence

$$0 \to M \xrightarrow{\varphi_M} \text{Hom}_A(B, M) \to N \to 0.$$

Then flat. dim$_A(\text{Hom}_A(B, M)) \leq \text{flat. dim}_B(\text{Hom}_A(B, M)) \leq \text{flat. dim}_A(M) \leq d$. Hence we have flat. dim$_A N \leq d + 1$, which implies that flat. dim$_B(N) \leq d + 1$ by the induction hypothesis. Finally we conclude that flat. dim$_B(M) \leq d$. Thus the induction proceeds. \hfill \Box

Theorem 2.9. If the quiver associated with $R^I(n)$ is of type $A, D, E$, then the functor $\mathcal{F}_n$ is exact.

Proof. Let us apply Proposition 2.8 with $A = \mathbb{P}_n$ and $B = R^I(n)$. The conditions (a) and (b) are well-known, and (c) is nothing but Proposition 2.7. Therefore, since $\hat{V}^n$ is a flat $\mathbb{P}_n$-module, it is a flat $R^I(n)$-module. \hfill \Box
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