Certain Metric Properties of Level Hypersurfaces

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Abstract. This note establishes several integral identities relating certain metric properties of level hypersurfaces of Morse functions.

1 Introduction

Let $f$ be a $C^2$ Morse function on an open connected subset $\Omega$ of $\mathbb{R}^{n+1}$ where $n \geq 2$. Suppose that $a$ and $b$ are values of $f$ such that $f^{-1}([a, b])$ is compact. For $t \in [a, b]$, let $\nu(t)$ be the ($n$-dimensional) volume of the level-$t$ set $f^{-1}(t)$. Note that, since $f$ is a Morse function, $\nu(t)$ is well-defined even if $t$ is a critical value and that $\nu : [a, b] \to \mathbb{R}$ is continuous. At each regular point (i.e., noncritical point) on $f^{-1}(t)$, let $N = -\nabla f / |\nabla f|$. This choice of unit normal induces a Gauss map $G$ on the set of regular points on $f^{-1}(t)$, with $G(p) = N(p) \in S^n$.

The mean curvature $H$ and the Gaussian curvature $K$ are defined on the set of regular points on $f^{-1}(t)$ by the standard definitions

$$H = \frac{1}{n} \text{Tr} dG \quad \text{and} \quad K = \det dG.$$

We henceforth view $H$ and $K$ as functions on the set of regular points of $f^{-1}([a, b])$; i.e., $H(p)$ and $K(p)$ are the mean curvature and Gaussian curvature of $f^{-1}(f(p))$ at $p$.

We now state our main results, in which $d\mu$ is the Lebesgue measure on $\mathbb{R}^{n+1}$ and $\partial_i$ denotes the $i$-th partial derivative.

**Theorem** Assume the preceding assumptions and notation.

(a) $\nu(b) - \nu(a) = n \int_{f^{-1}([a, b])} H \, d\mu$.

(b) $\int_a^b \nu(t) \, dt = \int_{f^{-1}([a, b])} |\nabla f| \, d\mu$.

(c) $\int_{f^{-1}([a, b])} K \partial_i f \, d\mu = 0$ for each $i \in \{1, \cdots, n+1\}$.

Implicit in these results is the assertion that the functions $H$ and $K \partial_i f$ are integrable on $f^{-1}([a, b])$. This is a consequence of $f$ being a Morse function, as we shall demonstrate.)

2 Two Preparatory Results

In many results, we assume the following hypothesis.

**Hypothesis** $\dagger$: $f$ is a $C^2$ Morse function on an open connected subset $\Omega$ of $\mathbb{R}^{n+1}$ where $n+1 \geq 3$; $a$ and $b$ are values of $f$ such that $f^{-1}([a, b])$ is compact.

**Lemma 1** Assume Hypothesis $\dagger$. Suppose that $g$ is a function that is continuous on the set of regular points in $f^{-1}([a, b])$ and integrable on $f^{-1}([a, b])$. Then

$$\int_{f^{-1}([a, b])} g \, d\mu = \int_a^b \left( \int_{f^{-1}(t)} \frac{g}{|\nabla f|} \, d\sigma \right) \, dt,$$
where $d\sigma$ is the (n-dimensional) volume form on $f^{-1}(t)$ and $\int_{f^{-1}(t)}(g/|\nabla f|)\,d\sigma$ is only defined for $t$ a regular value.\

**Proof.** There are two cases, according as whether $[a, b]$ contains a critical value.

Case 1: $[a, b]$ is free of critical values. For each $p \in f^{-1}(a)$, let $t \mapsto F(p, t)$ be the integral curve for the field $\nabla f / |\nabla f|^2$. The map $F : f^{-1}(a) \times [a, b] \to f^{-1}(a)$ is then a diffeomorphism, providing the transformation of variables that results in the claimed formula. (In detail, take a coordinate patch $U$ on $f^{-1}(a)$ and apply Fubini’s theorem to $U \times [a, b]$ $F|_{U \times [a, b]} : F(U \times [a, b])$.

Case 2: $[a, b]$ contains a critical value. Let $S$ be the (finite) set of critical values in $[a, b]$. Then, $(a, b) \setminus S$ is a disjoint union of finitely many intervals $I_j := (c_j, c_{j+1})$ of regular values. As $f^{-1}([a, b]) = \bigcup_j f^{-1}(I_j) \cup f^{-1}(S \cup \{a, b\})$ and $f^{-1}(S \cup \{a, b\})$ has Lebesgue measure zero (as a subset of $\mathbb{R}^{n+1}$),

$$\int_{f^{-1}([a, b])} g\,d\mu = \sum_j \int_{f^{-1}(I_j)} g\,d\mu.$$ 

Applying Case 1 to $f^{-1}([c_j + \epsilon, c_{j+1} - \delta])$ and letting $\epsilon, \delta \to 0^+$, we have

$$\int_{f^{-1}(I_j)} g\,d\mu = \lim_{\epsilon, \delta \to 0^+} \int_{f^{-1}([c_j + \epsilon, c_{j+1} - \delta])} g\,d\mu = \lim_{\epsilon, \delta \to 0^+} \int_{c_j + \epsilon}^{c_{j+1} - \delta} \left( \int_{f^{-1}(t)} \frac{g}{|\nabla f|}\,d\sigma \right) dt = \int_{c_j}^{c_{j+1}} \left( \int_{f^{-1}(t)} \frac{g}{|\nabla f|}\,d\sigma \right) dt.$$

Summing these integrals over $j$ proves the assertion. $lacksquare$

Recall from §1 the mean curvature $H$ and Gaussian curvature $K$, both regarded as functions on the set of regular points of $f$. Explicit formulæ are known for $H$ and $K$. To state them, let $Q$ be the Hessian quadratic form associated with $f$ and define the quadratic form $Q^*$ to be the one whose standard matrix is the adjugate (or “classical adjoint”) of the standard matrix for $Q$: we shall regard the two quadratic forms $Q$ and $Q^*$ as real-valued functions of one vector variable. Then,

$$H = \frac{|\nabla f|^2 \text{Tr} Q - Q(\nabla f)}{n |\nabla f|^3} \quad \text{and} \quad K = \frac{Q^*(\nabla f)}{|\nabla f|^{n+2}}.$$

These are implicit in [3, p. 204] and made explicit in [2]. (In both of these references, $f^{-1}(t)$ is oriented by $\nabla f / |\nabla f|$, the opposite of our choice of $\mathbf{N}$.)

**Lemma 2** For a $C^2$ Morse function $f$ on an open set $\Omega \subset \mathbb{R}^{n+1}$, the functions $H$, $K\partial_i f$, and $K |\nabla f|$ are all integrable on any compact subset of $\Omega$.

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1 The “outer” integral $\int_0^\infty \cdots dt$ on the right may first be interpreted as an improper Riemann integral. Once the formula is proven, applying it to $|g|$ shows that the one-variable function $\varphi(t) := \int_{f^{-1}(t)} (|g|/|\nabla f|)\,d\sigma$ is absolutely integrable over $[a, b]$, since $|\varphi(t)| \leq h(t) := \int_{f^{-1}(t)} (|g|/|\nabla f|)\,d\sigma$ and $\int_a^b h(t)\,dt = \int_{f^{-1}([a, b])} |g|\,d\mu$. Hence, $\int_0^\infty \varphi(t)dt$ may also be interpreted as a Lebesgue integral.
Proof. It suffices to show that they are integrable “near” each critical point \( p \), i.e., on a closed ball \( D \) centered at \( p \) in which \( p \) is the only critical point. Without loss of generality, assume that \( p \) is the origin \( 0 \in \mathbb{R}^{n+1} \). We notate a typical point in \( \mathbb{R}^{n+1} \) by writing its position vector \( r \) and we let \( r = \|r\| \). Then, for \( r \) near \( 0 \),

\[
f(r) = f(0) + P(r) + o(r^2)
\]

where \( P(r) \) is the quadratic polynomial \( \frac{1}{2}Q(r) \). For each \( i \in \{1, \cdots, n+1\} \),

\[
\partial_i f = \partial_i P + r \epsilon_i,
\]

where \( \epsilon_i \to 0 \) as \( r \to 0 \), and for \( r \in D' := D \setminus \{0\} \),

\[
\partial_i P(r) = r \alpha_i(r/r)
\]

where \( \alpha_i \) is a function on \( S^n \). Hence, on \( D' \),

\[
|\nabla f(r)|^2 = r^2 \sum_{i=1}^{n+1} (\alpha_i(r/r) + \epsilon_i)^2.
\]

As \( f \) is a Morse function, \( 0 \) is the only critical point of \( P \) and thus \( \sum_i \alpha_i(r)^2 > 0 \) for \( r \in S^n \). Letting

\[
m = \min_{r \in S^n} \sum_{i=1}^{n+1} \alpha_i(r)^2,
\]

we have, for sufficiently small \( r \), \( \frac{1}{2}mr^2 \leq |\nabla f(r)|^2 \leq 2mr^2 \). Hence, there are positive numbers \( C, M_1, M_2, \delta \) such that, whenever \( r \leq \delta \),

\[
|\nabla f(r)| \geq Cr
\]

as well as

\[
|\nabla f|^2 \text{Tr} Q - Q(\nabla f)(r) \leq M_1 r^2 \quad \text{and} \quad |Q^*(\nabla f)(r) \leq M_2 r^2.
\]

Therefore, for \( r \leq \delta \),

\[
|H(r)| = \frac{|\nabla f|^2 \text{Tr} Q - Q(\nabla f)(r)}{n|\nabla f(r)|^3} \leq M_1 \frac{1}{nC^3 r^3}
\]

and

\[
|K(r)\partial_i f(r)| \leq |K(r)\nabla f(r)| = \frac{|Q^*(\nabla f)(r)|}{|\nabla f|^{n+1}} \leq M_2 \frac{1}{Cn+1 r^{n-1}}.
\]

It is a standard fact that, for any \( c > 0 \), \( 1/r^{n+1-c} \) is integrable on any origin-centered ball in \( \mathbb{R}^{n+1} \). Hence, \( H, K\partial_i f, \) and \( K|\nabla f| \) are all integrable on \( D \). 

### 3 Main Results

We establish the main results of the article.

**Theorem 3** Under Hypothesis \( \dagger \), \( \nu(b) - \nu(a) = n \int_{f^{-1}(a,b)} H \, d\mu \).
Proof. First recall (from [1, p. 142]) that 

\[ H = -\frac{1}{n} \text{div} N. \]

With \( N := -\nabla f / |\nabla f| \), \( H = \frac{1}{n} \text{div} \frac{\nabla f}{|\nabla f|} \).

In the following, let \( R = f^{-1}([a, b]) \). There are two cases according as whether \([a, b]\) contains a critical value.

Case 1: \([a, b]\) is free of critical values. Then, \( R \) is an \((n + 1)\)-manifold with boundary \( f^{-1}(a) \cup f^{-1}(b) \). Let \( n \) denote the unit outward normal (relative to \( R \)) on \( \partial R \); then \( n = -\nabla f / |\nabla f| \) on \( f^{-1}(a) \) and \( n = \nabla f / |\nabla f| \) on \( f^{-1}(b) \). Now,

\[ \nu(b) - \nu(a) = \int_{\partial R} \left\langle \frac{\nabla f}{|\nabla f|}, n \right\rangle d\sigma = \int_{R} \text{div} \frac{\nabla f}{|\nabla f|} d\mu = \int_{R} nH d\mu. \]

Case 2: \([a, b]\) contains a critical value. Let \( S \) be the (finite) set of critical values in \([a, b]\). Then, \( (a, b) \setminus S \) is a disjoint union of finitely many intervals \( I_j = (c_j, c_j+1) \) of regular values. As \( R = \bigcup_j f^{-1}(I_j) \cup f^{-1}(S \cup \{a, b\}) \) and \( f^{-1}(S \cup \{a, b\}) \) has Lebesgue measure zero,

\[ \int_{R} H d\mu = \sum_j \int_{f^{-1}(I_j)} H d\mu. \]

It remains to note that, for each \( j \),

\[ \int_{f^{-1}(I_j)} H d\mu = \lim_{\epsilon \to 0^+} \int_{f^{-1}([c_j + \epsilon, c_{j+1} - \epsilon])} H d\mu \quad \text{(by integrability of } H) \]

\[ = \lim_{\epsilon \to 0^+} \frac{1}{n} \left( \nu(c_{j+1} - \epsilon) - \nu(c_j + \epsilon) \right) \quad \text{(by Case 1)} \]

\[ = \frac{1}{n} (\nu(c_{j+1}) - \nu(c_j)) \quad \text{(by continuity of } \nu). \]

With the aid of Lemma [1] Theorem [3] easily yields a formula for \( \nu' \), which would take considerable effort to obtain otherwise.

Corollary 4 Assume Hypothesis †. For any regular value \( t_0 \in [a, b] \),

\[ \nu'(t_0) = n \int_{f^{-1}(t_0)} \frac{H}{|\nabla f|} d\sigma. \]

Proof. For a regular value \( t_0 \in (a, b) \),

\[ \nu'(t_0) = \frac{d}{dt} \bigg|_{t_0} \int_{f^{-1}([a, t])} nH d\mu \quad \text{(by Theorem [3])} \]

\[ = \frac{d}{dt} \bigg|_{t_0} \int_{a}^{t} \left( \int_{f^{-1}([a, \tau])} nH d\sigma \right) d\tau \quad \text{(by Lemma [1])} \]

\[ = \int_{f^{-1}(t_0)} \frac{nH}{|\nabla f|} d\sigma \quad \text{(by fundamental theorem of calculus).} \]

We show more applications of Lemma [1] with a certain choice of \( g \).
Theorem 5 Under Hypothesis †, \( \int_{f^{-1}(a,b)} (h \circ f) \cdot |\nabla f| \, d\mu = \int_a^b h(t) \nu(t) \, dt \) for any integrable function \( h \) on \([a, b]\). In particular, for any \( t_0 \in [a, b]\),

\[
\int_a^{t_0} \nu(t) \, dt = \int_{f^{-1}(a,t_0)} |\nabla f| \, d\mu,
\]
or equivalently,

\[
\nu(t_0) = \frac{d}{dt} \bigg|_{t_0} \int_{f^{-1}(a,t)} |\nabla f| \, d\mu.
\]

Proof. The first assertion follows from Lemma 1 by letting \( g = (h \circ f) \cdot |\nabla f| \).

The second assertion results from letting \( h \) be the indicator function for \([a,t_0]\).

Continuity of \( \nu \) makes applicable the fundamental theorem of calculus, yielding the last assertion. ■

Proposition 6 Assume Hypothesis †.

(a) \( \int_{f^{-1}(a,b)} K \partial_i f \, d\mu = 0 \) for \( i \in \{1, \cdots, n+1\} \).

(b) If, in addition, \( n \) is even and \([a,b]\) is free of critical values, then

\[
\int_{f^{-1}(a,b)} K |\nabla f| \, d\mu = \frac{1}{2}(b-a)\chi(f^{-1}(a))\nu(S^n),
\]

where \( \nu(S^n) \) is the \((n\text{-dimensional}) \) volume of the unit sphere \( S^n \) and \( \chi(f^{-1}(a)) \) is the Euler characteristic of \( f^{-1}(a) \).

Proof. For Part (a), let \( g \) in Lemma 1 be the vector-valued function \( K\nabla f \).

Then,

\[
\int_{f^{-1}(a,b)} K \nabla f \, d\mu = \int_a^b \left( \int_{f^{-1}(t)} K \frac{\nabla f}{|\nabla f|} \, d\sigma \right) \, dt.
\]

Now, note that

\[
\int_{f^{-1}(t)} K \frac{\nabla f}{|\nabla f|} \, d\sigma = -\int_{f^{-1}(t)} K \nabla f \, d\sigma = 0.
\]

For detail of the last equality, let \( M \) denote \( f^{-1}(t) \) and define the vector-valued \( n \)-form \( \omega \) on \( S^n \) by letting \( \omega = \text{Id}_{S^n} \, d\sigma_{S^n} \), where \( d\sigma_{S^n} \) is the volume form on \( S^n \). Then, with \( G \) being the Gauss map \( p \mapsto N(p) \), \( G^* \omega = K \nabla f \, d\sigma \) as can be verified pointwise. Hence,

\[
\int_M K \nabla f \, d\sigma = \int_M G^* \omega = \deg G \cdot \int_{S^n} \omega.
\]

But

\[
\int_{S^n} \omega = \int_{S^n} \text{Id}_{S^n} \, d\sigma_{S^n} = 0
\]

due to cancellation of antipodal contributions.

Under the hypothesis of Part (b), \( f^{-1}(t) \) is diffeomorphic to \( f^{-1}(a) \) for \( t \in [a,b] \). By Gauss-Bonnet theorem,

\[
\int_{f^{-1}(t)} K \, d\sigma = \frac{1}{2} \chi(f^{-1}(t))\nu(S^n) = \frac{1}{2} \chi(f^{-1}(a))\nu(S^n)
\]

Letting \( g = K |\nabla f| \) in Lemma 1 then proves Part (b). ■
References

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