Exact Classical and Quantum Solutions for a Covariant Oscillator Near the Black Hole Horizon in Stueckelberg-Horwitz-Piron Theory

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We found exact solutions for canonical classical and quantum dynamics for general relativity in Horwitz general covariance theory. These solutions can be obtained by solving the generalized geodesic equation and Schrödinger-Stückelberg-Horwitz-Piron (SHP) wave equation for a simple harmonic oscillator in the background of a two dimensional dilaton black hole spacetime metric. We proved the existence of an orthonormal basis of eigenfunctions for generalized wave equation. This basis functions form an orthonormal and normalized (orthonormal) basis for an appropriate Hilbert space. The energy spectrum has a mixed spectrum with one conserved momentum \(P\) according to a quantum number \(n\). To find the ground state energy we used a variational method with appropriate boundary conditions. A set of mode decomposed wave functions and calculated for the Stueckelberg-Schrödinger equation on a general five dimensional blackhole spacetime in Hamilton gauge.

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I. INTRODUCTION

General relativity (GR) is considered so far as the best classical gauge theory for gravity. It has passed many observational tests locally and generally (for a complete description of gravity as gauge theory see [1]). A fundamental basis to build any modification to GR must preserve the equivalence principle where two frames can be related to each other just by infinitesimal transformations. Building a relativistic quantum mechanics in a canonical formalism has an old history, started by the works of Stueckelberg [2], who defined the covariant canonical classical and quantum theory for one body systems, with its covariant Stückelberg-Schrödinger wave equation. The theory was generalized to be applicable to many body systems by Horwitz and Piron [3] (which we will refer to as SHP theory) and developed more fully by Horwitz [3].

Field theories on curved spaces have been studied, for example, by Birrell and Davies [1]. A significant work in the field, because he only used local coordinate transformations as a general coordinates transformation between frames. In his derivation he followed the original procedure of Einstein to build his ground breaking GR theory. The point is that he built both classical and quantum version of dynamics of a test particle subjected to the potential. In Horwitz’s derivation of dynamical equations the potential appears as a spacetime mass distribution. He extended his ideas to electromagnetism using the same method as Yang’s pioneering development of electromagnetism as a \(U(1)\) fiber bundle in his theory [4]. In Horwitz’s canonical dynamics for electromagnetic fields, \(U(1)\) field emerged as a gauge field on the quantum mechanical Hilbert space. In this work I study exact solutions for classical and quantum dynamical equations in his theory for a simple harmonic oscillator as an external force field.

II. CLASSICAL SOLUTIONS

As pointed out in the SHP theory, the classical dynamics of a test particle in a potential field \(V(x)\) can be derived using a local coordinate transformanion from \(\xi^\mu \rightarrow x^\mu\) transformation:

\[
H = \frac{M}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + V(x).
\]

Here \(V(x)\) on a manifold can be the potential at point \(\xi\) as dual coordinate to \(x\), and \(g_{\mu\nu}\) is the spacetime metric. In SHP theory, two aspects of time are distinguished: the chronological or historical time \(\tau\), and the coordinate time \(t\). The latter is a dynamical variable in the manifold, while the former parameterizes motion of an
event in the manifold. We consider only $\tau$ independent
metrics of spacetime i.e. those static spacetime metrics
with $\frac{\partial g_{\mu \nu}}{\partial \tau} = 0$. The exterior region is supposed to be an
asymptotically flat Einsteins manifold, i.e., a solution
to the Einstein gravitational field equation can be studied.
Here our aim is to study exact solutions for dynamical
field equations derived from the Hamiltonian \(2.1\).
It is easy to derive a modified geodesics equation using
least action principle and it yields to the following set of
coupled nonlinear second order differential equations for
the coordinates $x^\mu$:

$$\ddot{x}^\sigma + \Gamma^\sigma_{\lambda \gamma} \dot{x}^\lambda \dot{x}^\gamma = - \frac{1}{M} g^{\sigma \lambda} \frac{\partial V}{\partial x^\lambda}. \quad (2.2)$$

Note that Eq. \(2.2\) gives orbits different from the clas-
sical geodesic equations. We consider a simple harmonic
oscillator near the horizon of a two dimensional black
hole, possessing an asymptotically flat metric defined by
the line element $ds^2 = g_{\mu \nu} dx^\mu dx^\nu$:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)}. \quad (2.3)$$

It is adequate to use null coordinates $(x_+, x_-)$ where the metric \(2.3\) converts to the following conformally flat metric:

$$ds^2 = f(x_+, x_-)dx_+dx_-.. \quad (2.4)$$

The reason to use this low dimensional gravity toy model
is first of all its simplicity, secondly this gravitational
model can be derived from dimensional reduction of the
Einstein-Hilbert action from higher dimensional space-
times as well as the near horizon geometry of four di-

mensional blackholes. In metric \(2.4\), the pair of null coordinates are defined as

$$dx_\pm = \frac{dr}{f(r)} \mp dt. \quad (2.5)$$

Note that $f(r) > 0$ for exterior region of the blackhole for
$r > r_h$, where $r_h$ is location of horizon of blackhole. We
are working on the exterior region of blackhole spacetime
metric. In our toy model \(2.4\), we consider $V(r) = V(x_+, x_-)$ as a simple harmonic oscillator as an external
force to modify geodesic equation in the curved background $g_{\mu \nu}$. By integrating Eq. \(2.4\) we find:

$$x_+ = \int \frac{dr}{f(r)} - t + c_+, \quad (2.6)$$

$$x_- = \int \frac{dr}{f(r)} + t + c_. \quad (2.7)$$

here $(c_+, c_-)$ are integration constants. If two dimen-
sional (2D) gravity posses asymptotic flat regime, i.e, $r \to \infty, f(r) \to 1$ then

$$x_+ \approx r - t + c_+, \quad (2.8)$$

$$x_- \approx r + t + c_. \quad (2.9)$$

In the asymptotic geometry of \(2.2\), we take the particle
mass squared to be the "on shell" value $M^2$. We mention
here that $c_\pm$ play the roles of a zero point energy(chemical
potential) for the system. The reason is that when the
system remains near asymptotic limit ,

$$V(r) \approx \frac{1}{2k}(\frac{x_+ + x_- - c_0}{2})^2 \quad (2.10)$$

where $k$ is the spring constant and $c_0$ is the location of the
equilibrium point, i.e., $x'^2_+ + x'^2_- = c_0$. This equilibrium
point $(x'^0_+, x'^0_-)$ is obtained for $\frac{\partial f}{\partial r}(x'^0_+, x'^0_-) = 0$.

Note that now the metric \(2.3\) looks like just a con-
formal flat metric:

$$ds^2 = f(x^A, x^B)\eta_{AB}dx^A dx^B. \quad (2.11)$$

here $\eta_{AB} = diag(-1, +1)$ and $x^A = (x_+, x_-)$. Solving
\(2.2\) for metric \(2.11\) needs all non zero Christoffel sym-
\(\Gamma_{BC}^A = \frac{1}{2}g^{AD}(g_{BD,C} + g_{CD,B} - g_{BC,D}). \quad (2.12)\)

here $g_{BD,C} \equiv \frac{\partial g_{BD}}{\partial x^C}$ and etc. We use a formula relating
two conformal metrics; if $\tilde{g} = e^{2\Omega}g$,

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + \left( \delta_i^k \frac{\partial \Omega}{\partial x^j} + \delta_j^k \frac{\partial \Omega}{\partial x^i} - g_{ij}g^{kl} \frac{\partial \Omega}{\partial x^l} \right). \quad (2.13)$$

where in our case $\Omega = \frac{1}{2} \ln f$ and $\tilde{\Gamma}^k_{ij} = 0$. The po-
tential function $V$ is given in Eq. \(2.10\) the gradient components are:

$$\frac{\partial V}{\partial x^A} = \sqrt{\frac{fV}{2}} (1, 1). \quad (2.14)$$

Finally we obtain geodesic equations \(2.2\) as a pair of
coupled nonlinear differential equations:

$$\ddot{x}_\pm + \frac{1}{2} \dot{x}_\pm \frac{d(ln f)}{dt} - \dot{x}_\mp \dot{x}_\pm \frac{\partial ln f}{\partial x_\pm} + \frac{1}{Mf} \sqrt{2kV} = 0.. \quad (2.15)$$
The asymptotic solution for Eq. (2.15) is given as follows:

\[
x_+ - x_- \approx v_0 \tau + w_0, \quad (2.16)
\]

\[
t(\tau) \approx \frac{\tau}{t_0} + t_0. \quad (2.17)
\]

The goal is to find exact solution for Eq. (2.15) with potential \( f(r) \). For 2D blackhole a possible metric function \( f(r_h) = 0 \) as follows:

\[
f(x_+, x_-) = \frac{2m}{\lambda} e^{-\gamma} - 1, \quad (2.18)
\]

\[
y \equiv 2r_h \lambda^2 \sum_{a=\pm} (x_a - c_0). \quad (2.19)
\]

It is remarkable that Eq. (2.19) looks like the Bose-Einstein distribution functions of density of states with energy \( E_i \). We represent the potential \( V \) as \( V = \frac{1}{2} k \rho_{h}^2 \coth^2(y/2) \). If we define a new dependent variable \( w \equiv \sum_{a=\pm} x_a \) and geodesic equation (2.15) reduces to :

\[
\ddot{w} + \frac{r_h \lambda^2}{1 - e^{2r_h \lambda^2 (w - c_0)}} \dot{w}^2 = 0. \quad (2.20)
\]

If we substitute \( p = \dot{w} \), we obtain (one is given implicitly) two types of solutions for \( w(\tau) \):

\[
w(\tau) : \begin{cases}
= \text{constant} & \text{if } \dot{w} = 0 \\
= e^{-\gamma} \sqrt{1 - e^{2\gamma}} + \sin^{-1}(e^{\gamma}) = c_2 - \left(\frac{\lambda^2 e^{\gamma} \rho_{h}^2}{c_1}\right) \tau & \text{if } \dot{w} \neq 0
\end{cases}
\]

(2.21)

Finally we find a classical geodesic solution for SHP classical force on a 2D blackhole background. There is a well formulated quantum mechanical wave equation for quantum dynamics on a curved general relativity background well formulated in [6].

\[
i\hbar \frac{\partial}{\partial \tau} \Psi_{\tau}(x^\mu) = \hat{H} \Psi_{\tau}(x^\mu) \quad (3.1)
\]

where \( \hat{H} \) is quantum mechanical operator form of Eq. (2.11). It is defined on a Hilbert space with scalar product : 

\[
(\psi, \chi) = \int d^4x \sqrt{-g} \Psi^\dagger_{\tau}(x^\mu) \chi_{\tau}(x^\mu) \quad (3.2)
\]

where the volume element is written as a normal scalar and \( \dagger \) denotes complex conjugate. For a quantum mechanical Hamiltonian \( \hat{H} \) we follow the convention of indexes given in ref. [6]. We define the Hamiltonian as:

\[
\hat{H} = \frac{1}{2M \sqrt{-g}} p_{\mu} g^{\mu\nu} p_{\nu} + V(x). \quad (3.3)
\]

The potential is given as a quantum harmonic oscillator i.e., Eq. (2.11). On the background given in the spacetime metric (2.11) we find \( \sqrt{-g} = f/2 \), using the momentum operator given by \( p_{\mu} = -i\hbar \partial_{\mu} \), \( x^\mu = (x_+, x_-) \); supposing that \( \Psi_{\tau} = \Psi_{\tau}(x_+, x_-) \), we reduce the wave equation Eq. (3.1) to the following time independent schrödinger-Stueckelberg -Horwitz-Piron wave equation in the null coordinates :

\[
\text{III. QUANTUM MECHANICS VIA SCHRÖDINGER-STUECKELBERG -HORWITZ-PIRON WAVE EQUATION ON 2D BLACKHOLE BACKGROUND}
\]

In the previous section we found the classical trajectory of a test particle subjected to a harmonic oscillator
By substituting the ansatz (3.10) in Eq. (3.8), it is assumes the form

\[ -\frac{\hbar^2}{2M f^2} \left( 2\partial_+ \partial_- \phi(x_+, x_-) - \left( \partial_+ (\ln f) \partial_- \phi(x_+, x_-) + \partial_- (\ln f) \partial_+ \phi(x_+, x_-) \right) \right) + V(x_+, x_-) \phi(x_+, x_-) = E\phi(x_+, x_-) \]  

(3.4)

where we use separation of the variables as \( \Psi_\tau(x_+, x_-) = e^{-\frac{i}{\hbar}\log_f \phi(x_+, x_-)} \). Note that \( \partial_\pm f(x_+, x_-) = \frac{2\pi\lambda^2}{1-e^y} \) where \( y \) is defined in Eq. (2.19) at temperature \( T \). We then obtain the following partial differential equation for the time independent wave function \( \phi \equiv \phi(x_+, x_-) \):

\[ -\frac{\hbar^2\lambda^2}{2mM} (e^{-y} - 1) \left[ \partial_+ \partial_- - \frac{\hbar^2\lambda^2}{1-e^y}(\partial_+ + \partial_-) \right] \phi + \left[ \frac{1}{2} kr^2 \coth^2 \left( \frac{y}{2} \right) \right] \phi = E\phi \]  

(3.5)

If we use new coordinates \( \tilde{\xi} \equiv x_+ + x_- \), \( \tilde{\eta} \equiv x_+ - x_- \) and using separation of variables; (note that \( \tilde{\eta} \) is a cyclic variable, the momentum \( p \) corresponding to \( \tilde{\xi} \) is a conserved quantity), \( \phi(x_+, x_-) = e^{\frac{\tilde{\xi}}{\hbar}} F(\tilde{\xi}) \) and we end up with the following second order differential equation:

\[ \frac{d^2 F(y)}{dy^2} - \frac{\tilde{r}^2}{1-e^y} \frac{dF(y)}{dy} + \left[ \tilde{k}^2_{p,n} - \tilde{m}^2 \frac{e^y \coth^2 y}{1-e^y} \right] F(y) = 0 \]  

(3.6)

After a linear transformation in the form \( F(y) = \chi(y) \exp \left( e^{-y} - 1 \right) \tilde{r}^2 \) the resulting differential equation for \( \chi(y) = \chi \) is

\[ \chi'' + \left( \tilde{k}^2_{p,n} - \frac{\tilde{r}^4}{4(1-e^y)^2} - \tilde{m}^2 \frac{e^y \coth^2 y}{1-e^y} \right) + \frac{\tilde{r}^2 e^y}{2(1-e^y)^2} \right] \chi = 0 \]  

(3.8)

The differential equation (3.8) has two asymptotic solutions given by

\[ \chi(y) = \begin{cases} Ay \log y & \text{if } y \to 0 \\ B \sin(k_{p,n} y + \theta_0) & \text{if } y \to -\infty \end{cases} \]  

(3.9)

respectively. Thus, the general solution of equation (3.8) assumes the form

\[ \chi(y) = y \log y \sin(k_{p,n} y + \theta_0) g(y) \]  

(3.10)

By substituting the ansatz (3.10) in Eq. (3.8), it is straightforward to show that the auxiliary function \( g(y) \) satisfies another second order differential equation which cannot be solved analytically.

In the absence of such analytic solutions, we investigate the differential equation Eq. (3.8) using Sturm-Liouville theory. The energy spectrum for Eq. (3.8) is a mixed spectrum given by

\[ E_{p,n} = \frac{r_h \hbar^2 \lambda^4}{mM} \left( k_{p,n}^2 - \frac{\tilde{r}^2}{2r_h \lambda^2 \hbar^2} \right) \]  

(3.11)

We can multiply (3.8) by a quantity that converts it into self-adjoint form. The Sturm-Liouville eigenvalue problem of Eq. (3.8) becomes:
It is then straightforward to show that the equation (3.12) satisfies the self-adjoint condition. The solutions satisfy

\[
\int_{-\infty}^{0} F_{p,n}(y)F_{p,m}(y)(e^{-y} - 1)^{-\hat{r}^2} dy = \begin{cases} 0 & \text{if } m \neq n \\ N_{p,n} & \text{if } m = n \end{cases} \tag{3.13}
\]

The wave function of a general quantum state is given by the following superposition expression:

\[
\Psi_{\tau}(x_+, x_-) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dp e^{i\frac{\hbar}{p}(E_{p,n\tau} + p(x_+ - x_-))} C_{p,n} F_{p,n}(y)(e^{-y} - 1)^{-\hat{r}^2} dy, \tag{3.14}
\]

here \( C_{p,n} \) is the Fourier coefficient for the series and can be obtained using initial conditions \( \Psi_{\tau}(x_+, x_-)|_{\tau=0} \). This quantum mechanical wave function deserves to be investigated in details. First of all, as we mentioned the form of the wave function can be fixed if we know enough initial condition about the shape of the wave function on an initial chronological time i.e., \( \Psi_{\tau=0}(x_+, x_-) \). This initial wave packet specifies the initial state of the quantum state on two dimensional background. The probability of finding a state with momentum \( p \) and on the level \( n \) is given by \( |C_{p,n}|^2 \). This probability amplitude can be obtained using the orthogonalization condition (3.13) as follows:

\[
C_{p,n} = \int_{-\infty}^{\infty} \Psi_{\tau=0}(y)e^{-\frac{i}{\hbar}p(x_+ - x_-)} F_{p,n}(y)(e^{-y} - 1)^{-\hat{r}^2} dy, \tag{3.15}
\]

Furthermore, we can use the above amplitude to compute different types of probabilities, for example if we are looking for the probability of finding the system in ground state \( n = 0 \) with large momenta \( p_{UV} \) is:

\[
C_{p_{UV}} = \int_{-\infty}^{\infty} \Psi_{\tau=0}(y)e^{-\frac{i}{\hbar}p_{UV}(x_+ - x_-)} F_{p_{UV},0}(y)(e^{-y} - 1)^{-\hat{r}^2} dy, \tag{3.16}
\]

The quantum mechanical amplitude on the above line can be used to find the most probable large momentum state which can be used to explain high energies excitations in the vicinity of the curved background. A direct application may be to use this amplitude to find the probability of the near horizon radiation effects. A simple formu-
luation of such radiation phenomena can be used to find
an appropriate quantum mechanical description of the
radiation near the black holes horizons.

A. Finding ground state energy via a variational

method

The wave vector $\Psi_\tau(x_+, x_-)$ in the previous line can
not be obtained in closed form. As an attempt to get
more physics of the quantum system, we will try find the
ground state energy $E_\text{g} \leq E_{p,n}$ via a variational method.
The method is based on finding the value of a variation
parameter $\alpha$ for a trial wave function $\Psi_\tau(x_+, x_- | \alpha)$ that
minimizes the expectation value of the Hamiltonian:

$$< \Psi_\tau(x_+, x_- | \alpha) | \hat{H} | \Psi_\tau(x_+, x_- | \alpha) > \quad (3.17)$$

In our case it reduces to minimize the following func-
tional:

$$\bar{k}_0^2 = \text{Minimize} \left[ \int_{-\infty}^{0} dy (e^{-y} - 1)^{r^2} (F'_\alpha(y))^2 + \bar{m}^2 (e^y \coth y)(e^{-y} - 1)^{r^2} F^2_\alpha(y) \right] \quad (3.18)$$

where the trial function $F_\alpha(y)$ should satisfy the essential
boundary condition presented in the equation 3.17 for
the general wave function $\Psi_\eta(y, \zeta)$:

$$F_\alpha(y) = \begin{cases} 0 & \text{if } y \to 0 \\ 0 & \text{if } y \to \infty \end{cases} \quad (3.19)$$

An unnormalized trial function on the interval $y \in (-\infty, 0)$ with an unweighted scalar product is $F_\alpha(y) = \alpha y e^y$ because it satisfies the boundary condition given in
equation 3.17, can be used to minimize the functional
Eq. 3.18. We evaluated the integral in Eq. 3.18 and
plugging the result in Eq. 3.11 we obtain for the ground
state energy:

$$E_{\text{g}, 0} \approx \left( \bar{h} \alpha \right)^2 \left( H_{r+1} + \gamma - 1 \right) \psi(0) \left( r^2 + 2 \right) + \psi(1) \left( r^2 + 2 \right) - \frac{\gamma}{2} (\gamma - 1) \gamma + \frac{3}{2} - \frac{\lambda^2 p_{UV}^2}{2mM} \quad (3.20)$$

here $p_{UV}$ is the ultraviolet cutoff and take it to be a
momentum, the computations performed up to $O(\bar{m}^2)$, $\gamma$
is the Euler-Mascheroni constant defined as

$$\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \ln n \right) \quad (3.21)$$

with numerical value $\gamma \approx 0.577216$. $\psi^{(n)}(x)$ denotes
the PolyGamma function and gives as the digamma function
given as:

$$\psi^{(n)}(x) = \frac{d}{dx} \ln \Gamma(x) \quad (3.22)$$

Define $H_n = \sum_{k=1}^{n} \frac{1}{k}$. The harmonic number gives the
$\gamma h$ harmonic number. We ignored the effect of mass $\bar{m}^2$,
and the result is represented as the first order approxima-
tion. Note that in the semi classical limit where $\hbar \to 0$,
the second term in the bracket becomes constant and the
first term gives the first order $O(\hbar^2)$ correction to $E_0$.

IV. MODE DECOMPOSITION FOR
STUECKELBERG-SCHRÖDINGER EQUATION
ON FIVE DIMENSIONAL BLACKHOLE
SPACETIME

Blackholes in higher dimensional spacetimes have been
investigated as direct examples for violation of uniqueness
of the blackholes theorem in four dimensions (see [8] for a
brief review). In five dimensional Riemanninan manifolds
it is possible to find a doubly spinning black hole with a
pair of angular momenta as a five dimensional analog
of the axially stationary exterior Kerr metric in four di-
mensions [9]. Not only blackholes but also black rings
and strings have been investigated with different topolo-
gies for the horizon in higher dimensional GR and many
interesting properties studied , for example by describ-
thermodynamics of such higher dimensional black
objects using differential geometry [10] (see [3] for a com-
prehensive review on higher dimensional black objects ).
If we focus on asymptotically anti-de Sitter black holes
the gauge invariance of the \( \tau \) metric in the presence of the scalar field of the potential model. \( V(r) \) is now replaced by the generally \( \tau \) dependent function \( V_5(\tau, x^\sigma) \). The metric is given as \( [13] \):

\[
\frac{ds^2}{d\Omega_3} = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega_3
\]

where \( d\Omega_3 \) is the line element on the 3-dimensional unit sphere metric \( [17] \):

\[
d\Omega_3 = d\theta^2 + \sin^2 \theta d\phi^2 + \sin^2 \theta d\phi^2.
\]

The starting point is to write a gauge transformed version of the eigenvalue-eigenfunction for Hamiltonian Eq. \( [4.3] \) using a relativistic Hamiltonian with gauge invariant form:

\[
\Psi_\tau(t, r, \Omega_3) = \sum_{n=0}^{\infty} \int d\Omega_3 \int_{-\infty}^{\infty} d\omega e^{-\frac{1}{M} (E_n + \hbar \omega t)} Y_n(\Omega_3) R_n(r, \omega)
\]

with the above wave function in Eq. \( [4.4] \), we can write

\[
R_n''(r, \omega) + \left( 2ihe + \frac{2}{r} - \frac{f'}{f} \right) R_n'(r, \omega) + \left[ (\omega f)^2 + \frac{n(n+2)f}{r^2} + ihe(A' - \frac{A(r)f'}{f}) + e^2A(r)^2 + eV(r)f - \frac{2MEf}{\hbar^2} \right] R_n(r, \omega) = 0.
\]

where we introduce the spherical harmonics \( Y_n(\Omega_3) \) as eigenfunctions of the Laplacian operator on \( \Omega_3 \) satisfying the following eigenvalue-eigenfunction linear equation \( [17] \):

\[
\Delta_3 Y_n(\Omega_3) = -n(n+2)Y_n(\Omega_3)
\]

The presence of the cosmological constant strongly changes the black hole properties, for example, making them unstable \( [12] \).

In comparison to the uncharged wave equation solved in the previous section for a 2D toy blackhole, we now study a possible mode decomposition for \( \Psi_\tau \) in a general five dimensional neutral static-spherically symmetric metric. It was demonstrated that there exists a conformal transformation of the Hamiltonian which maps the system at one classical level to another conformal metric structure.

Here the \( a_\mu \), may depend on the affine parameter \( \tau \) as well as coordinate \( x^\sigma \), \( e \) is charge and the \( V_5(\tau, x^\sigma) \) is the gauge invariance of the \( \tau \) derivative in the canonical quantum mechanical Stueckelberg-Schrödinger equation. It was demonstrated that there exists a conformal transformation of the Hamiltonian which maps the system at one classical level to another conformal metric structure.

where the new metric is cast into a Kaluza-Klein effective metric \( [14] \). Using this effective conformal dual metric it was demonstrated that the conformally modified metric emerges from the Hamilton equations \( [13] \). For the sake of simplicity and preserving spherical symmetry we assume that \( V_5(\tau, x^\sigma) = V(r) \) and \( a_\mu(\tau, x^\sigma) = A(r) \delta^\tau_\mu \) and use a seperation of variables as

\[
\Psi_\tau(t, r, \Omega_3) = \sum_{n=0}^{\infty} \int d\Omega_3 \int_{-\infty}^{\infty} d\omega e^{-\frac{1}{M} (E_n + \hbar \omega t)} Y_n(\Omega_3) R_n(r, \omega)
\]

we can find approximate solutions for the radial equation \( [4.6] \) near horizon \( f(r_h) = 0 \) and at infinity \( r \to \infty \) if we suppose that \( A \) near the horizon , remains finite; calling it \( A_h \), it vanishes at infinity as well as \( V(r) \).
\[ \{ A(r), V(r) \} = \begin{cases} \{ A_h, V_h \} & \text{if } r \to r_h \\ 0 & \text{if } r \to \infty \end{cases} \] (4.8)

The approximated solutions are represented as follows:

\[
R_n(r, \omega) = \begin{cases} \frac{A_h e^{i(r-r_h)}}{\sqrt{2}} & \text{if } r \to r_h \\ e^{-iehr} \left( c_n e^{-kr} + d_n e^{kr} \right) & \text{if } r \to \infty \end{cases}
\]

where \( k = \sqrt{\frac{2ME}{\hbar^2} - \omega^2 - (\hbar)^2 - iA'e \hbar} \) and \( I_n(x), K_n(x) \) give the modified Bessel function of the first and second kinds. Note that \( R_n(r_h, \omega) = \frac{A_h}{2\sqrt{2}} \) remains finite. The asymptotic radial eigenfunctions can be used to find asymptotic Green functions for Eq. (4.4). It will be interesting if we can use this radial solution to find the most probable distance of the test particle from the horizon \( r_h \).

V. SUMMARY

There are still several attempts to obtain a unified theory in which gravity as a classical gauge theory can be naturally quantized in the canonical way (as well as path integral method). It is clear for me that any such consistent quantum gravity should be initiated from same steps as in the mathematical formulation of the quantum theory, passing through classical Poisson brackets to Dirac’s commutators and then studying the time evolution equation both for the state vector (in Schrödinger picture ) and quantum operator (in Heisenberg approach ). A simple and physically understandable approach where gravity is quantized canonically was recently proposed by Horwitz. In our letter we present exact solutions for classical and quantum dynamics of GR in this canonical method. As a curved background I focused on a two dimensional toy model blackhole as an integrable system. A classical geodesic equation with corrections was explicitly derived and integrated for a test particle in a simple radial potential field. Using the classical solution obtained in my letter, we can support the idea of classical fields as effective fields. By investigating the linear perturbations of the metric , one is able to explain the early (or pre-early) inflation and thermal spectrum of the matter contents. Furthermore, this solution can be used to build the appropriate forms for the Green function. It will provide the chronological time propagators and using those propagators we can study the causal structures of the spacetime metric at early universe. In the quantum mechanical version of this canonical theory, we investigated exact solutions for the modified Schrödinger equation on the curved background. The complete set of the exact mode solutions were calculated for the modified wave equation as well as an estimated ground state energy level. The quantum mechanical amplitudes can be used to explain the high energy excitations in the vicinity of the curved background. A direct application will be to find the probability of near horizon radiation effects and to justify the origin of the radiation near the black holes horizons. As an attempt to extend the theory to electromagnetism we solved the wave equation for a test charge particle. Asymptotic solutions obtained can be used to construct Green functions and causal propagators. This work will be continued in a forthcoming paper on Green functions and relevant field theoretical aspects.

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