Electroweak baryon and lepton number violating processes at high energies by the valley instanton

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Abstract

We calculate the cross sections of the baryon and lepton number violating processes. The proper valley method enables us to calculate the multi-boson processes, which have the possibility to observation. The mass corrections of the gauge boson, Higgs particle and top quark are evaluated and the group integration and the phase space integration are performed analytically and numerically.

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1 Introduction

It is well known that the baryon and lepton numbers are not exactly conserved in the standard model, due to the chiral anomaly. The topologically non-trivial configuration, especially, instanton induces the baryon and lepton number violating process as the following;

\[ q + q \rightarrow (3n_f - 2)\bar{q} + n_f\bar{l} + n_w W + n_h H, \]

where W represents gauge boson, W$^\pm$, Z and H represents the Higgs particle. We denote the number of generations by $n_f$ and the numbers of the bosons by $n_w, n_h$. As ’t Hooft first indicated[1], this process is highly suppressed by the factor \( \exp(-16\pi^2/g^2) \) at low energies. This is because the process is accompanied by quantum tunneling through the potential barrier that separates the topologically distinct vacua. The top of the potential, which is a saddle point in the functional space, corresponds to the static unstable solution, so-called sphaleron[2], and the height of the potential barrier is given by the energy of the sphaleron: \( E_{\text{sph}} \sim m_w/\alpha \). In naive picture, the tunneling rate grows up as the energy of the initial state becomes higher, and it enhances when the initial state has high energies comparable to \( E_{\text{sph}} \). Since \( E_{\text{sph}} \) is \( O(10) \) TeV, the suppression is expected to decrease or vanish and the baryon and lepton number violating processes be observable in TeV region.

In this viewpoint, the high energy behavior of the process (1.1) has been studied in recent years[3]. By the sphaleron-based calculation[4], it was suggested that the scattering processes involved with many gauge bosons and Higgs particles are not exponentially suppressed. Ringwald[5] and Espinosa[6] calculated the baryon and lepton number violating scattering amplitudes semi-quantitatively by the constrained instanton method[7], and they also suggested that the amplitudes of the multi-boson process increase with the energy and reach \( 1\text{pb} \) for a center of mass energy of \( O(10) \) TeV. Unfortunately, the high-energy behavior of the process with many bosons, which is most interesting, is out of the validity of their calculations. One of the problem is that they used the constrained instanton. The constrained instanton method is not available for the large-radius instanton, which plays an important role in multi-boson process. It is pointed out in Ref.[8] that the approximation breaks for \( n_w + n_h \sim 4 \) in the constrained instanton method.

Another problem of Ref.[5] is that the orientation effect of the instanton is not taken into account. The integration of the orientation in the Lorentz SU(2)$_R$ rotations is not trivial but very important especially for processes with many gauge bosons. Moreover, in considering the multi-boson processes at high energies, masses of the gauge boson, Higgs particle and top quark cannot be neglected. All the effect remains unevaluated.
In this paper, we overcome these problems and evaluate the cross section. Instead of the constrained instanton method, we use the proper valley method, which enables us to calculate the multi-boson process for $n_w + n_h \sim 40$. The integrations of the orientation and phase space integration are performed analytically in the extremely relativistic limit and non-relativistic limit, and performed numerically in the general cases. The mass corrections of the gauge boson, Higgs particle and top quark are taken into account.

In Section 2, we consider the valley instanton, which is constructed by the proper valley method. This configuration gives an important contribution to the path integral in topologically non-trivial sector of the functional space. In Section 3, we introduce the collective coordinates for the valley instanton. In Section 4, we calculate the multi-point Green function by the proper valley method. In performing the group integration and phase space integration, we obtain the cross sections of the baryon and lepton number violating processes. In Section 5, we give conclusion and discussion.

2 Valley instanton in the electroweak theory

We consider the standard model with $n_f$ generations of quarks and leptons. For simplicity, we set $SU(3)_c$ and $U(1)_Y$ coupling constants to zero, and set the Kobayashi-Maskawa matrix to the identity. Thus we treat the simplified standard model that reduces to an $SU(2)$ gauge-Higgs system with fermions. As for quarks, we have the left-handed doublets $q^{aL}_{i}$ and the right-handed singlets $q^{aR}_{i}$, where $a = 1, 2, 3$ is the color index and $i = 1, \cdots, n_f$ is the index of the generations. Similarly, we have the left-handed doublets $l^{iL}_{i}$ and the right-handed singlets $l^{iR}_{i}$ for leptons. The bosonic part of the action is given by

\begin{align}
S_g &= \frac{1}{2g^2} \int d^4x \, \text{tr} F_{\mu \nu} F_{\mu \nu}, \\
S_h &= \frac{1}{\lambda} \int d^4x \left\{ (D_\mu H)^\dagger (D_\mu H) + \frac{1}{8} (H^\dagger H - v^2)^2 \right\},
\end{align}

where $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu]$ and $D_\mu = \partial_\mu - i A_\mu$. The masses of the gauge boson and the Higgs boson are

\begin{align}
m_w = \sqrt{\frac{g^2}{2\lambda}} v, \quad m_h = \frac{1}{\sqrt{2}} v.
\end{align}

The fermionic part of the action is

\begin{align}
S_q &= \int d^4x (iq^{iL}_{L} \sigma_\mu D_\mu q_{L} - iu^{iR}_{R} \bar{\sigma}_\mu D_\mu u_{R} - id^{iR}_{R} \bar{\sigma}_\mu D_\mu d_{R})
\end{align}
\[ S_l = \int d^4 x (i l_L^\dagger \sigma_\mu D_\mu l_L - ie^\dagger_{\mu} \sigma_\mu e e_R - y l_L^\dagger H e_R + h.c.), \]

(2.5)

where \( \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} = i\sigma^2 \), \( \sigma_\mu = (\sigma, i) \) and \( \overline{\sigma}_\mu = (\sigma, -i) \). The masses of fermions are given by \( m_{u,d,l} = y_{u,d,l} v \). The total action is given by

\[ S = S_g + S_h + \sum_{i=1}^{n_f} \left( S_l + \sum_{a=1}^{3} S_q \right). \]

(2.6)

In order to calculate the Green function, \( \langle \psi^A^a L A^a_{\nu}^\mu H^{nk} \rangle \), we must construct a series of the configurations, which give a dominant contribution to the path integral in the topologically non-trivial functional subspace. They are not solutions of the equation of motion. The vacuum expectation value of the Higgs field breaks the classical scale invariance and therefore, except the zero-radius instanton, there exists no solution of the equation of motion in this model.

The dominating configurations make a valley of the action \([10,11]\). Therefore, the proper valley method is suitable to construct them \([9]\). The construction of this series of configurations is carried out in Ref.\([8]\). We dub these configurations valley instanton. The valley instanton is the solution of a generalized equation of motion, the proper valley equation which is called in Ref.\([8]\) by the name of the new valley equation. The proper valley equation for this system is given by,

\[ \begin{align*}
\frac{\delta^2 S}{\delta A_\mu \delta A_\nu} F^A_{\nu} + \frac{\delta^2 S}{\delta A_\mu \delta H} F^H + \frac{\delta^2 S}{\delta A_\mu \delta H} F^{H\dagger} &= \lambda_e F^A_{\mu}, \\
\frac{\delta^2 S}{\delta H^{\dagger} \delta A_\mu} F^A_{\mu} + \frac{\delta^2 S}{\delta H^{\dagger} \delta H} F^{H\dagger} + \frac{\delta^2 S}{\delta H^{\dagger} \delta H} F^H &= \lambda_e F^{H\dagger},
\end{align*} \]

(2.7)

where the integration over the space-time is implicit. The valley is parameterized by the eigenvalue \( \lambda_e \) that is identified with the zero mode corresponding to the scale invariance at \( v \to 0 \).

To simplify the equation, we adopt the following ansatz;

\[ A_\mu(x) = \frac{x_{\nu} \bar{\sigma}_{\mu\nu}}{x^2} \cdot 2a(r), \quad H(x) = v (1 - h(r)) \eta, \]

(2.8)

where \( \eta \) is a constant isospinor, and \( a \) and \( h \) are real dimensionless functions of dimensionless variable \( r \), which is defined by \( r = \sqrt{x^2/\rho} \). The matrix \( \bar{\sigma}_{\mu\nu} \) is defined, according to the
conventions of Ref. [6], as $\overline{\sigma}_{\mu
u} = \overline{\eta}_{\mu
u}\sigma^\alpha/2$. We have introduced the scaling parameter $\rho$. The tensor structure in (2.8) is the same as that of the instanton in the singular gauge [1].

Inserting this ansatz to (2.7), the structure of $F^A_\mu$ and $F^{H\dagger}$ is determined as the following:

$$
F^A_\mu(x) = \frac{x_\nu\overline{\sigma}_{\mu\nu}}{x^2} \cdot \frac{2v^2}{\lambda} f^a(r), \quad F^{H\dagger}(x) = -\frac{v^3}{\lambda} f^h(r) \eta.
$$

(2.9)

At $\rho v \to 0$, (2.7) reduces to the equation of motion and the equation for the zero-mode fluctuation around the instanton solution. The solution of this set of equations is the following:

$$
a_0 = \frac{1}{1 + r^2}, \quad h_0 = 1 - \left(\frac{r^2}{1 + r^2}\right)^{1/2},
$$

$$
f^a_0 = \frac{2Cr^2}{(1 + r^2)^2}, \quad f^h_0 = \frac{Cr}{(1 + r^2)^{3/2}},
$$

(2.10)

where $C$ is an arbitrary function of $\rho v$. Note that $a_0$ is the instanton solution in the singular gauge and $h_0$ is the Higgs configuration in the instanton background [1]. We have adjusted the scaling parameter $\rho$ so that the radius of the instanton solution is unity. The mode solutions $f^a_0$ and $f^h_0$ are obtained from $\partial a_0/\partial \rho$ and $\partial h_0/\partial \rho$, respectively.

We construct the valley instanton analytically for $\rho v \lesssim 1$. When $\rho v = 0$, it is given by the ordinary instanton configuration $a_0, h_0, f^a_0$ and $f^h_0$. When $\rho v$ is small but not zero, it is expected that small $\rho v$ corrections appear in the solution. On the other hand, at large distance from the core of the valley instanton, this solution is expected to decay exponentially, because gauge boson and Higgs boson are massive. Therefore, the solution is similar to the instanton near the origin and decays exponentially in the asymptotic region. We solve the valley equation in both regions and analyze the connection in the intermediate region. In this manner we find the solution.

In the asymptotic region, $a, h, f^a$ and $f^h$ become small and the valley equation can be linearized. The solution of equations is

$$
a(r) = C_1 r \frac{d}{dr} G_{\rho m_w}(r) + \frac{1}{\nu} f^a(r),
$$

$$
h(r) = C_2 G_{\rho m_h}(r) + \frac{1}{\nu} f^h(r),
$$

$$
f^a(r) = C_3 r \frac{d}{dr} G_{\rho m_w}(r),
$$

$$
f^h(r) = C_4 G_{\rho m_h}(r),
$$

(2.11)
where \( C_i \) are arbitrary functions of \( \rho v \), \( \nu \) is defined as \( \lambda = v^2 \nu / \lambda \) and \( \mu_{w,h} \) are defined as \( \mu_{w,h} = m_{w,h} \sqrt{1 - 2 \nu} \). The function \( G_m(r) \) is

\[
G_m(r) = \frac{m K_1(mr)}{(2\pi)^2 r},
\]

where \( K_1 \) is a modified Bessel function. As was expected above, these solutions decay exponentially at infinity. By matching two solutions (2.10) and (2.11) in the intermediate region, we obtain

\[
C_1 + \frac{C_3}{\nu} = -2 \pi^2, \quad C_2 + \frac{C_4}{\nu} = 2 \pi^2, \quad C_3 = -4 \pi^2 C, \quad C_4 = 4 \pi^2 C.
\]

By more detailed analysis, we find \( C, \nu \sim 1/4 \) at \( \rho v \to 0 \) and we observe \( C, \nu \sim 1/4 \) for \( \rho v < 1 \) numerically [8].

Inserting the valley instanton to the action, we obtain,

\[
S = \frac{8 \pi^2}{g^2} + \frac{2 \pi^2}{\lambda} (\rho v)^2 + \cdots.
\]

The fermionic zero mode around the valley instanton is given by,

\[
q_L^\alpha(x) = -\frac{1}{\pi \rho^3} x^\alpha \hat{\sigma}_\nu \left( -\epsilon_{rs} \eta^r \eta_{\beta} u_L(r) + \eta_r \epsilon_{\beta \gamma} \eta^r \eta_{\gamma} d_L(r) \right),
\]

\[
u_R(x) = \frac{i}{2\pi} \frac{m_u}{\rho} u_R(r) \eta_\alpha,
\]

\[
d_{R\alpha}(x) = \frac{i}{2\pi} \frac{m_d}{\rho} d_R(r) \epsilon_{\alpha \gamma} \eta^\gamma,
\]

where the Greek and the Roman letters denote indices of spinor and isospinor, respectively. For \( \rho v \lesssim 1 \), the solution is given approximately, by the following:

\[
u_L(r) = \begin{cases} 
\frac{1}{r(r^2 + 1)^{3/2}}, & \text{if } r \ll (\rho m_u)^{-1/2} ; \\
-2\pi^2 \frac{d}{dr} G_{\rho m_u}(r), & \text{if } r \gg (\rho m_u)^{-1/2} ;
\end{cases}
\]

\[
u_R(r) = \begin{cases} 
\frac{1}{r^2 + 1}, & \text{if } r \ll (\rho m_u)^{-1/2} ; \\
2\pi^2 G_{\rho m_u}(r), & \text{if } r \gg (\rho m_u)^{-1/2},
\end{cases}
\]

and \( d_L \) and \( d_R \) are obtained by replacing \( m_u \) with \( m_d \). The above approximative behavior is correct even for \( \rho v \sim 1 \), because the valley instanton keeps its shape unchanged for \( \rho v \sim 1 \). We obtain the zero mode for leptonic sector in the similar manner.

### 3 Collective coordinates of the valley instanton

In this section, we give the collective coordinate method for the valley instanton. Because the valley instanton breaks the invariance under the translations and the Lorentz \( SU(2)_R \) rotations,
we must introduce the collective coordinates to restore it\footnote{In Higgs phase, the collective coordinates for the global $SU(2)$ gauge rotations are not needed as they are not symmetries.}. However, as the valley instanton is not a classical solution except the zero radius one, the zero modes for the translations and the Lorentz $SU(2)_R$ rotations do not exist in general. Therefore the conventional collective coordinate method can not be used. The basic strategy of our collective coordinate method is to treat the translations and the Lorentz $SU(2)_R$ as a kind of “gauge symmetry” as well as the original gauge symmetry \[13\]. This guarantees automatically the manifest invariance under the translations and the Lorentz rotations through the following deformation and do not require the zero modes.

In addition to these collective coordinates, the valley direction, namely the $\rho$ integral, must be brought into. This is because along the valley direction the action of the system varies gently. The corresponding integration can not be approximated by the Gaussian integration and must be performed more exactly. The total number of the collective coordinates is eight (four collective coordinates from the translations, three from the Lorentz $SU(2)_R$ rotations and one from the valley direction) and the same as that of the BPST instanton.

First we rewrite the gauge field and the Higgs fields as

\begin{align}
A_\mu(x) &= A_\mu(x;\rho) + Q_\mu(x;\rho), \quad (3.1) \\
H(x) &= H(x;\rho) + Q_H(x;\rho). \quad (3.2)
\end{align}

Here $A_\mu(x;\rho)$ and $H(x;\rho)$ are the valley instanton and $Q_\mu(x;\rho)$ and $Q_H(x;\rho)$ are quantum fluctuations around them. As was shown in \[2\] 8, the tensor structures of the valley instanton are

\begin{align}
A_\mu(x;\rho) &= \frac{x_\nu \tilde{\sigma}_{\mu\nu}}{x^2} \cdot 2a(r), \quad (3.3) \\
H(x;\rho) &= v(1 - h(r))\eta, \quad (3.4)
\end{align}

where $r$ is $r = \sqrt{x^2}/\rho$. For the fermions, we consider them as the quantum fluctuations.

Substituting \[3.1\] and \[3.2\] for the original Lagrangian, we obtain a Lagrangian for the quantum fluctuations. Due to the invariance of the original Lagrangian, it has the invariance under the gauge rotations, the translations and the Lorentz $SU(2)_R$ rotations. The transformation law of the quantum fluctuations are as follows.
1. Gauge rotations: $g(x)$

\[
\begin{align*}
&u_R^g(x) = u_R(x), \quad u_R^\dagger(x) = u_R^\dagger(x), \\
d_R^g(x) = d_R(x), \quad d_R^\dagger(x) = d_R^\dagger(x), \\
&q_L^g(x) = g(x)q_L(x), \quad q_L^\dagger(x) = q_L^\dagger(x)g^\dagger(x), \\
Q_{\mu}^g(x;\rho) = -i\partial_\mu g(x)g^\dagger(x) + g(x)(A_\mu(x;\rho) + Q_\mu(x;\rho))g^\dagger(x) - A_\mu(x;\rho), \\
Q_{\mu}^H(x;\rho) = g(x) (H(x;\rho) + Q_H(x;\rho)) - H(x;\rho). 
\end{align*}
\] (3.5)

2. Translations: $x_0$

\[
\begin{align*}
&u_R^{x_0}(x) = u_R(x + x_0), \quad u_R^{\dagger x_0}(x) = u_R^\dagger(x + x_0), \\
d_R^{x_0}(x) = d_R(x + x_0), \quad d_R^{\dagger x_0}(x) = d_R^\dagger(x + x_0), \\
&q_L^{x_0}(x) = q_L(x + x_0), \quad q_L^{\dagger x_0}(x) = q_L^\dagger(x + x_0), \\
Q_{\mu}^{x_0}(x;\rho) = Q_{\mu}(x + x_0;\rho) + A_\mu(x + x_0;\rho) - A_\mu(x;\rho), \\
Q_{\mu}^{x_0}H(x;\rho) = Q_H(x + x_0;\rho) + H(x + x_0;\rho) - H(x;\rho). 
\end{align*}
\] (3.6)

3. Lorentz $SU(2)_R$ rotations: $U$

\[
\begin{align*}
&U_R^\alpha(x) = U_R^\alpha(x), \quad U_R^{\dagger \alpha}(x) = U_R^{\dagger \alpha}(x), \\
&d_R^\alpha(x) = d_R(x), \quad d_R^{\dagger \alpha}(x) = d_R^{\dagger \alpha}(x), \\
&q_L^\alpha(x) = q_L^\alpha(R^{-1}x), \quad q_L^{\dagger \alpha}(x) = q_L^{\dagger \alpha}(R^{-1}x)U_{\beta}^{\dagger \alpha}, \\
Q_{\mu}^{U}(x;\rho) = \frac{x_\mu}{x^2}U\bar{\sigma}_\mu U^\dagger \cdot 2a(r) + R_{\mu\nu}Q_{\nu}(R^{-1}x;\rho) - A_\mu(x;\rho), \\
Q_{\mu}^{U}H(x;\rho) = Q_H(R^{-1}x;\rho),
\end{align*}
\] (3.7)

where $R^{-1}x = R_{\mu\nu}^{-1}x_\nu$ and the $SO(4)$ matrix $R$ is defined as follows;

\[
R_{\mu\nu}\bar{\sigma}_\nu = \bar{\sigma}_\mu U, \tag{3.8}
\]

and we have used the following equations.

\[
R_{\mu\nu}A_\nu(R^{-1}x;\rho) = \frac{x_\nu}{x^2}U^\dagger \bar{\sigma}_\nu U \cdot 2a(r), \quad H(R^{-1}x;\rho) = H(x;\rho). \tag{3.9}
\]

The transformation law of $l_L$ and $e_R$ is the same as that of $q_L$ and $d_R$ respectively.

We will treat these transformations as gauge symmetries. To extract the “gauge orbits” from the path integral, we use the Faddeev-Popov technique. For simplicity, we abbreviate the
whole original fields by the single filed $\Phi$, the valley instanton by $\Phi_0$ and the whole quantum fluctuations by $Q$. We define the Faddeev-Popov determinant as

$$\Delta_{FP}(Q)^{-1} \equiv \int Dg dU dx_0 d\rho \delta \left( F(Q^{g_{x_0}U}(x; \rho)) \right) \delta \left( \int d^4x \frac{1}{N(\rho)} \frac{\delta S}{\delta \Phi_0(x; \rho)} Q^{g_{x_0}U}(x; \rho) \right), \quad (3.10)$$

where $Q^{g_{x_0}U}(x; \rho)$ is the quantum fluctuations transformed successively under the gauge rotations $g(x)$, the translations $x_0$ and the Lorentz $SU(2)_R$ rotations $U$, and

$$\delta \left( F(Q^{g_{x_0}U}(x; \rho)) \right) = \prod_x \delta \left( F_{\text{gauge}}(Q^{g_{x_0}U}(x; \rho)) \right) \cdot \prod_{\mu=1}^4 \delta \left( F_{\text{trans}}^\mu(Q^{g_{x_0}U}(\rho)) \right),$$

$$N(\rho) = \int d^4x \left( \frac{\delta S}{\delta \Phi_0(x; \rho)} \right)^2 = \int d^4x \left[ \frac{1}{2} \left( \frac{\delta S}{\delta A_\mu(x; \rho)} \right)^2 + \left( \frac{\delta S}{\delta H^1(x; \rho)} \right) \left( \frac{\delta S}{\delta H(x; \rho)} \right) \right], \quad (3.11)$$

$$\frac{\delta S}{\delta \Phi_0(x; \rho)} Q^{g_{x_0}U}(x; \rho) = \frac{\delta S}{\delta A_\mu(x; \rho)} Q^{g_{x_0}U}_\mu(x; \rho) + \frac{\delta S}{\delta H(x; \rho)} Q^{g_{x_0}U}_H(x; \rho) + \frac{\delta S}{\delta H^1(x; \rho)} Q^{g_{x_0}U}_{H^1}(x; \rho).$$

Here $F_{\text{gauge}}$, $F_{\text{Lorentz}}$ and $F_{\text{trans}}$ are gauge fixing functions, for example, which are given by

$$F_{\text{gauge}}(Q(x; \rho)) = (D_\mu Q_\mu)^a(x; \rho),$$

$$F_{\text{trans}}(Q(\rho)) = \int d^4x \text{tr} \left\{ f_\mu^\rho(x; \rho) Q_\mu(x; \rho) \right\}, \quad (3.12)$$

$$F_{\text{Lorentz}}^a(\rho) = \int d^4x \text{tr} \left\{ f_\mu^a(x; \rho) Q_\mu(x; \rho) \right\},$$

where $f_\mu^\rho(x; \rho)$ and $f_\mu^a(x; \rho)$ are appropriate functions for the gauge-fixing. The first delta-function in the right-hand side of (3.11) gives the gauge fixing and the second one restricts the quantum fluctuation to be orthogonal to the valley direction.

Now we consider the following Green function;

$$\langle \psi(x_1) \cdots \psi(x_{4n_f}) A_{\mu_1}(y_1) \cdots A_{\mu_{n_\omega}}(y_{n_\omega}) \phi(z_1) \cdots \phi(z_{n_\omega}) \rangle = \mathcal{N} \int D\Phi \psi(x_1) \cdots \psi(x_{4n_f}) A_{\mu_1}(y_1) \cdots A_{\mu_{n_\omega}}(y_{n_\omega}) \phi(z_1) \cdots \phi(z_{n_\omega}) e^{-S(\Phi)}, \quad (3.13)$$

where $\mathcal{N}$ is an appropriate normalization factor and $\psi(x)$ is the Dirac fermion corresponding
to quark or lepton, namely,

\[
\psi(x) = \begin{cases} 
(u_R(x), q_L(x)) & \text{for quark in the up sector} \\
(d_R(x), q_L(x)) & \text{for quark in the down sector} \\
(0, l_L(x)) & \text{for lepton in the up sector} \\
(e_R(x), l_L(x)) & \text{for lepton in the down sector}
\end{cases}
\]

(3.14)

and \( \phi(x) \) is the shifted Higgs field, \( \phi(x) = H(x) - \nu \eta \). Inserting the following identity

\[
1 = \int \mathcal{D}g dU d_0 d\rho \Delta_{FP}(Q) \delta \left( F(Q^{\rho \eta U}(x; \rho)) \right) \delta \left( \int d^4 x \frac{1}{N(\rho)} \frac{\delta S}{\delta \Phi_0(x; \rho)} Q^{\rho \eta U}(x; \rho) \right), \quad (3.15)
\]

we obtain

\[
\langle \psi(x_1) \cdots \psi(x_{4n_f}) A_{\mu_1}(y_1) \cdots A_{\mu_{nw}}(y_{nw}) \phi(z_1) \cdots \phi(z_{nh}) \rangle
\]

\[
= \mathcal{N} \int \mathcal{D}g \mathcal{D}U \mathcal{D}_0 \mathcal{D}\rho \int \mathcal{D}\Phi \Delta_{FP}(Q) \delta \left( F(Q^{\rho \eta U}(x; \rho)) \right) \delta \left( \int d^4 x \frac{1}{N(\rho)} \frac{\delta S}{\delta \Phi_0(x; \rho)} Q^{\rho \eta U}(x; \rho) \right)
\]

\[
\times \psi(x_1) \cdots \psi(x_{4n_f}) A_{\mu_1}(y_1) \cdots A_{\mu_{nw}}(y_{nw}) \phi(z_1) \cdots \phi(z_{nh}) e^{-S(\Phi)}. \quad (3.16)
\]

Substituting the inserted fields in the integral for

\[
u_{\alpha}(x_i) = U_{\beta}^0 u_{R\beta}^{\alpha}(R(x_i - x_0)), \quad u_{\alpha}^0(x_i) = u_{R\beta}^{\alpha}(R(x_i - x_0)),
\]

\[
d_{\alpha}(x_i) = U_{\beta}^0 d_{R\beta}^{\alpha}(R(x_i - x_0)), \quad d_{\alpha}^0(x_i) = d_{R\beta}^{\alpha}(R(x_i - x_0)),
\]

\[
q_L^\alpha(x_i) = g^\dagger(x_i) q_{L\alpha}^{\rho \eta U \alpha}(R(x_i - x_0)), \quad q_{L\alpha}^{\beta}(x_i) = q_{L\alpha}^{\rho \eta U \beta}(R(x_i - x_0))U_{\beta}^\alpha g(x_i),
\]

\[
e_{R\beta}(x_i) = U_{\beta}^0 e_{R\beta}^{\rho \eta U}(R(x_i - x_0)), \quad e_{R\beta}^\alpha(x_i) = e_{R\beta}^{\rho \eta U \beta}(R(x_i - x_0))U_{\beta}^\alpha g(x_i),
\]

\[
l_L^\alpha(x_i) = g^\dagger(x_i) l_{L\alpha}^{\rho \eta U \alpha}(R(x_i - x_0)), \quad l_{L\alpha}^{\beta}(x_i) = l_{L\alpha}^{\rho \eta U \beta}(R(x_i - x_0))U_{\beta}^\alpha g(x_i),
\]

\[
A_{\mu}(y_i) = \left( \frac{y_i - x_0}{(y_i - x_0)^2} \right) 2g^\dagger(y_i) U_{\mu \nu} U^\dagger g(y_i) \cdot 2a(y_i x_0) + ig^\dagger(y_i) \partial_\mu g(y_i)
\]

\[
+ g^\dagger(y_i) R_{\mu \nu}^{\rho \eta U}(R(y_i - x_0); \rho) g(y_i),
\]

\[
\phi(z_i) = g^\dagger(z_i) v(1 - h(r_{zi} x_0)) \eta - \nu \eta + g^\dagger(z_i) Q_H^{\rho \eta U}(R(z_i - x_0); \rho),
\]

where \( r_{yi} x_0 = \sqrt{(y_i - x_0)^2/\rho} \), and using the relations

\[
S(\Phi) = S(\Phi_0 + Q^{\rho \eta U}),
\]

\[
\Delta_{FP}(Q) = \Delta_{FP}(Q^{\rho \eta U}),
\]

\[
\mathcal{D} \Phi = \mathcal{D} Q = \mathcal{D} Q^{\rho \eta U}, \quad (3.18)
\]

all the integration variables which appear in the path integral become \( Q^{\rho \eta U} \). Then we can rename it back to \( Q \). Furthermore, since the physical observables are invariant under the \( SU(2) \)
gauge rotations, we can simply set \( g(x) = 1 \) in the path-integral instead of the integration of \( g(x) \). This may breaks the translation and Lorentz invariance of the Green functions which are not invariant under the \( SU(2) \) gauge rotations, however, the lack of the explicit symmetries in such quantities is a gauge artifact. For physical observables, the translation and Lorentz invariance are not broken. Finally, we obtain

\[
\langle \psi(x_1) \cdots \psi(x_{4n_f}) A_{\mu_1}(y_1) \cdots A_{\mu_{nw}}(y_{nw}) \phi(z_1) \cdots \phi(z_{nw}) \rangle
= \mathcal{N} \int dU dx_0 d\rho \int \mathcal{D}Q \Delta_{FP}(Q) \delta(F(Q(x;\rho))) \delta \left( \int d^4x \frac{1}{N(\rho)} \frac{\delta S}{\delta \Phi_0(x;\rho)}Q(x;\rho) \right)
\times \psi(x_1) \cdots \psi(x_{4n_f}) A_{\mu_1}(y_1) \cdots A_{\mu_{nw}}(y_{nw}) \phi(z_1) \cdots \phi(z_{nw}) e^{-S(\Phi_0+Q)},
\]

(3.19)

where the inserted fields in the integral are given by

\[
\psi(x_i) = \begin{cases} 
U_\alpha^\beta u_{R\beta}(R(x_i - x_0)) \\
q_{L1}^\alpha(R(x_i - x_0)) \\
U_\alpha^\beta d_{R\beta}(R(x_i - x_0)) \\
q_{L2}^\alpha(R(x_i - x_0)) \\
0 \\
l_{L1}^\alpha(R(x_i - x_0)) \\
U_\alpha^\beta e_{R\beta}(R(x_i - x_0)) \\
l_{L2}^\alpha(R(x_i - x_0)) 
\end{cases}, \quad \psi^\dagger(x_i) = \begin{cases} 
U_{R\alpha}^\dagger(R(x_i - x_0)) \\
q_{L1}^\dagger(R(x_i - x_0))U^\alpha \\
d_{R\alpha}^\dagger(R(x_i - x_0)) \\
l_{L2}^\dagger(R(x_i - x_0))U^\alpha \\
0 \\
l_{L1}^\dagger(R(x_i - x_0))U^\alpha \\
d_{R\alpha}^\dagger(R(x_i - x_0)) \\
l_{L2}^\dagger(R(x_i - x_0))U^\alpha 
\end{cases},
\]

\[
A_{\mu}(y_i) = \frac{(y_i - x_0)^\mu}{(y_i - x_0)^2} U_\mu^\nu U^\dagger \cdot 2a(r_{yi}, x_0) + R_{\mu\nu}^{-1} Q_\nu(R(y_i - x_0); \rho),
\]

\[
\phi(z_i) = -v \hbar (r_{zi} x_0) \eta + Q_H(R(z_i - x_0); \rho).
\]

### 4 Calculation of the cross section

In this section, we calculate the cross section for the process of the type \( \bar{q} + \bar{q} \rightarrow (3n_f - 2)q + n_l l + n_w W + n_h H \), using the valley instanton. From Section 3, the Fourier transform of the Green function is given by

\[
\tilde{G}(p, q, k) = e^{-8\pi^2/\rho^2} g^{-8-n_w} \lambda^{-n_h/2} \int dU \int d\rho e^{-2\pi^2(\rho v)^2/\lambda} \rho^{2n_f-5} F(\rho, \rho) 
\times \tilde{\psi}(p_1) \cdots \tilde{A}_\mu(q_1) \cdots \tilde{\phi}(k_1) \cdots,
\]

(4.1)

where \( \tilde{G}(p, q, k) \) is defined by

\[
(2\pi)^4 \delta^4(p_1 + \cdots + q_1 + \cdots + k_1 + \cdots) \tilde{G}(p, q, k) = \int d^4x_1 \cdots d^4y_1 \cdots d^4z_1 \cdots e^{ip_1 x_1 + \cdots + i q_1 y_1 + \cdots + ik_1 z_1 + \cdots} 
\times \langle \psi(x_1) \cdots A(y_1) \cdots \phi(z_1) \cdots \rangle.
\]

(4.2)
The dimensionless function $F(\rho v, \rho \mu)$ includes the contributions from the fermionic determinant, bosonic determinant and Jacobian factor:

$$g^{-8} e^{-8\pi^2/g^2} e^{-2\pi^2(\rho v)^2/\lambda} p^{2n_f-5} F(\rho v, \rho \mu) = \frac{1}{Z_0} \int DQ \Delta_{FP}(Q) \delta \left( F(Q(x; \rho)) \right) \delta \left( \int d^4x \frac{1}{N(\rho)} \frac{\delta S}{\delta \Phi_0(x; \rho)} Q(x; \rho) \right) e^{-S(\Phi_0+Q)}. \quad (4.3)$$

We introduce the renormalization point $\mu$, and $g$ and $\lambda$ in (4.1) are renormalized coupling constants. In the leading order, $\tilde{A}_\mu$ and $\tilde{\phi}$ in [4.1] are given by the valley instanton and $\tilde{\psi}$ is given by the zero mode around the valley instanton. The Fourier transforms of the fermionic zero mode, $\tilde{\psi}(p)$ is given by,

$$\tilde{\psi} = \left( \tilde{\psi}_R^\alpha \tilde{\psi}_L^\beta \right) = 2\pi i \rho \left( \frac{-m_f U^\dagger \chi}{\tilde{p} U^\dagger \chi} \right) \frac{1}{p^2 + m_f^2} + \cdots \quad (4.4)$$

where $\cdots$ includes the regular term in the limit: $p^2 + m_f^2 \to 0$, and $\chi_\alpha$ is the constant spinor, $(0, -1)$ for $\psi = u, \nu$ and $(1, 0)$ for $\psi = d, e$. We have used the notation: $\tilde{p} = p_{\mu} \sigma_\mu$. The Fourier transforms of the valley instanton, $\tilde{A}_\mu$ is given by

$$\tilde{A}_\mu = \frac{1}{\nu} \pi^2 \rho^2 \frac{q_\nu \sigma_{\mu \nu} U^\dagger}{q^2 + m_w^2} + \cdots \quad (4.5)$$

The Fourier transform of the shifted Higgs field $\tilde{\phi}$ is given by,

$$\tilde{\phi} = \frac{1}{2\nu} \pi^2 \rho^2 \frac{\nu}{k^2 + m_h^2} \eta + \cdots \quad (4.6)$$

By (4.1)-(4.6), we obtain

$$\tilde{G}(p, q, k) = (-i)^{n_w} 2^{4n_f-n_h} \pi^{4n_f+2n_w+2n_h} g^{-8-n_w} e^{-8\pi^2/g^2} \nu^{-6n_f-2n_w-n_h+4} \times D\left(\frac{\mu}{\nu}; \lambda\right) \int dUL(U; p, q, k) \quad (4.7)$$

where $D\left(\frac{\mu}{\nu}; \lambda\right)$ and $L(U; p, q, k)$ are defined by,

$$D\left(\frac{\mu}{\nu}; \lambda\right) = \lambda^{-n_h/2} \int d(\rho v) e^{-2\pi^2(\rho v)^2/\lambda} \rho^6 n_f + 2n_w + 2n_h \nu^{-6n_f-2n_w-n_h} F(\rho v, \rho \mu), \quad (4.8)$$

$$L(U; p, q, k) = \prod_{i=1}^{4n_f} \left( \frac{-m_f U^\dagger \chi}{\tilde{p}_i U^\dagger \chi} \right) \frac{1}{p_i^2 + m_f^2} \prod_{j=1}^{n_w} q_{ij} \sigma_{\mu \nu} U^\dagger \chi \prod_{k=1}^{n_h} \eta \frac{k^2 + m_h^2}{q_j^2 + m_w^2}. \quad (4.9)$$

We ignore the momentum dependence that goes to zero on mass-shell.

Performing the LSZ procedure, we obtain the invariant amplitude $T_{n_f, n_w, n_h}$. Summing up the polarization and charge state of the gauge field, isospinor and spinor of the fermion field,
we obtain the amplitude;
\[
\sum |T_{n_f,n_w,n_h}|^2 = 2^{8n_f+n_w-2n_h} \pi^{8n_f+4n_w+4n_h} (g^2)^{n_w-8} e^{-16\pi^2/g^2} v^2 - 2(6n_f+2n_w+n_h-4) 
\times D^2(\mu/v; \lambda) \int dU \prod_{i=1}^{4n_f} \text{tr}(\vec{p}_i U) \prod_{j=1}^{n_w} (-) g^{\mu\rho} q^\rho_j q_j^\sigma \text{tr}(\sigma_{\mu\nu} U_s \sigma_{\nu\rho} U^\dagger).
\]

(4.10)
The cross section is given by the following;
\[
\sigma = \frac{1}{n_w! n_h! 2^4} \int \prod_{i=3}^{4n_f} \frac{d^3 p_i}{(2\pi)^3 2E_{p_i}} \prod_{j=1}^{n_w} \frac{d^3 q_j}{(2\pi)^3 2E_{q_j}} \prod_{l=1}^{n_h} \frac{d^3 k_l}{(2\pi)^3 2E_{k_l}} (2\pi)^4 \delta(4)(p_{in} - p_{out}) 
\times \sum |T_{n_f,n_w,n_h}|^2, \quad \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} 
\times \frac{1}{2^{4n_w-6n_h-5} (2\pi)^{4n_f+n_w+n_h+10} (g^2)^{n_w-8} e^{-16\pi^2/g^2} v^2 - 2(6n_f+2n_w+n_h-4) 
\times D^2(\mu/v; \lambda) I_{2n_f,n_w,n_h}(s) \cdot s^{-1},
\]

(4.11)
where \( s = p_{in}^2 = (p_1 + p_2)^2 \) and the masses of fermions of the initial state are ignored. We denote the sum of the momentums of the final state by \( p_{out} \). All the information about phase space and group integral is encoded in the function \( I_{2n_f,n_w,n_h}(s) \). We define \( I_{l,m,n}(s) \) by,
\[
I_{l,m,n}(s) = \int \prod_{i=3}^{2l} \frac{d^3 p_i}{2E_{p_i}} \prod_{j=1}^{m} \frac{d^3 q_j}{2E_{q_j}} \prod_{l=1}^{n} \frac{d^3 k_l}{2E_{k_l}} \delta(4)(p_{in} - p_{out}) 
\times 2^m \int dU \prod_{i=1}^{2l} \text{tr}(\vec{p}_i U) \prod_{j=1}^{m} (-) g^{\mu\rho} q^\rho_j q_j^\sigma \text{tr}(\sigma_{\mu\nu} U_s \sigma_{\nu\rho} U^\dagger)
\]
\[
= \int \prod_{i=3}^{2l} \frac{d^3 p_i}{2E_{p_i}} \prod_{j=1}^{m} \frac{d^3 q_j}{2E_{q_j}} \prod_{l=1}^{n} \frac{d^3 k_l}{2E_{k_l}} \delta(4)(p_{in} - p_{out}) 
\times \int dU \prod_{i=1}^{2l} \text{tr}(\vec{p}_i U) \prod_{j=1}^{m} \left( \{\text{tr}(\vec{q}_j U)\}^2 - m_w^2 \right).
\]

(4.12)
To calculate the cross section, we must evaluate two functions, \( D(\mu/v; \lambda) \) and \( I_{l,m,n}(s) \), which are given by (4.8) and (4.12), respectively. The former includes \( \rho \) integral and the latter includes the phase space and group integral.

### 4.1 Phase space and group integral

We consider the Laplace transformation of \( I_{l,m,n}(s) \). At first, we assume that \( p_{in}, p_1 \) and \( p_2 \) are independent variables, and at last we input \( p_{in} = p_1 + p_2 \). The Laplace transform
\( \Phi_{l,m,n}(\alpha; p_1, p_2) \) is given by,

\[
\Phi_{l,m,n}(\alpha; p_1, p_2) = \int d^{4}p_m \, e^{-\alpha p_m} I_{l,m,n}(p_m; p_1, p_2)
\]

\[
= \int dU \text{tr}(\bar{p}_1 U) \text{tr}(\bar{p}_2 U) \phi_f(\alpha, U)^{2l-2} \phi_g(\alpha, U)^m \phi_h(\alpha)^n, \tag{4.13}
\]

where

\[
\phi_f(\alpha, U) = \int \frac{d^3p}{2E_p} \text{tr}(\bar{p} U) e^{-\alpha p} = 2\pi \frac{m^2}{\alpha^2} K_2(\alpha m_f) \text{tr}(\bar{\alpha} U), \tag{4.14}
\]

\[
\phi_g(\alpha, U) = \int \frac{d^3q}{2E_q} \left[ \left\{ \text{tr}(\bar{q} U) \right\}^2 - m_w^2 \right] e^{-\alpha q}
= \frac{2\pi m_w^3}{\alpha^3} K_3(\alpha m_w) \left\{ \text{tr}(\bar{\alpha} U) \right\}^2 - \alpha^2, \tag{4.15}
\]

\[
\phi_h(\alpha) = \int \frac{d^3k}{2E_k} e^{-\alpha k} = \frac{2\pi m_h}{\alpha} K_1(\alpha m_h). \tag{4.16}
\]

In the above, \( K_\nu \) is the modified Bessel function and \( \alpha = \sqrt{\alpha^2} \). Then we can perform the group integration easily,

\[
\int dU \text{tr}(\bar{p}_1 U) \text{tr}(\bar{p}_2 U) \left\{ \text{tr}(\bar{\alpha} U) \right\}^{2l-2} \left[ \left\{ \text{tr}(\bar{\alpha} U) \right\}^2 - \alpha^2 \right]^m
= \sum_{i=0}^{m} C_i \int dU \text{tr}(\bar{p}_1 U) \text{tr}(\bar{p}_2 U) \left\{ \text{tr}(\bar{\alpha} U) \right\}^{2l-2+2i} (-\alpha^2)^{n-i}
= \left\{ C_0 (p_1 \cdot \alpha)(p_2 \cdot \alpha) + C_1 (p_1 \cdot p_2) \alpha^2 \right\} \alpha^{2(l+m-2)}, \tag{4.17}
\]

where

\[
C_0 = 4 \sum_{i=0}^{m} (-)^{m-i} C_i \frac{(2(l + i - 1))!}{(l + i + 1)!(l + i - 2)!}, \tag{4.18}
\]

\[
C_1 = 2 \sum_{i=0}^{m} (-)^{m-i} C_i \frac{(2(l + i - 1))!}{(l + i + 1)!(l + i - 1)!}. \tag{4.19}
\]

Finally, using the inverse Laplace transformation, we obtain the function \( I_{l,m,n} \). We can evaluate \( I_{l,m,n} \) analytically in the case of the extremely relativistic limit and the non-relativistic limit. In the extremely relativistic limit, \( I_{l,m,n} \) is given by,

\[
I_{l,m,n} = C_{l,m,n}^{ER} s^{3l+2m+n-4}, \tag{4.20}
\]

where

\[
C_{l,m,n}^{ER} = 2^m \left( \frac{\pi}{2} \right)^{2l+m+n-3} \left\{ (3l + 2m + n - 3)!(3l + 2m + n - 5)! \right\}^{-1}
\times \sum_{i=0}^{m} (-)^{m-i} C_i \frac{(2(l + i - 1))!}{(l + i + 1)!(l + i - 1)!} \left( 1 + (l + i)(3l + 2m + n - 4) \right). \tag{4.21}
\]
In the non-relativistic limit, we obtain
\[ I_{l,m,n} = C_{l,m,n}^{NR} m_f^{3/2} m_w^{5/2} m_h^{n/2} (\sqrt{s} - M)^{6l+3(m+n)/2-10} \cdot s^{1/4}, \] (4.22)
where
\[ C_{l,m,n}^{NR} = \left\{ \frac{(6l + \frac{3}{2}m + \frac{3}{2}n - 10)!}{2^{4l+(m+n)/2-7} \pi^{2l+3(m+n)/2-3}} \right\}^{-1} \cdot \sum_{i=0}^{m} (-)^{m-i} \frac{2(l+i-1)}{(l+i+1)!(l+i-1)!}. \] (4.23)

We assume the fermions are massless except for top quark and define \( M = m_f + n_w m_w + n_h m_h \).

In the general cases, we can perform the inverse Laplace transformation numerically, using the steepest descent approximation.

### 4.2 \( \rho \) integral

We evaluate the function \( D(\mu/v; \lambda) \) defined by (4.8). In (4.8), \( F \) and \( \nu \) depends on \( \rho v \), non-trivially. However, according to Ref.[8], the valley instanton is nearly equal to BPST instanton in the region \( \rho v \lesssim 1 \) and therefore we approximate the forms of \( F \) and \( \nu \) by that of BPST instanton, if \( \rho \) integral is dominated by the region \( \rho v \lesssim 1 \).

For \( \rho v \lesssim 1 \),
\[ \nu \sim \frac{1}{4}, \quad F(\rho v, \rho \mu) \sim c(\rho \mu)^{(43-8n_f)/6}, \] (4.24)
where \( c \) is a numerical constant [1]
\[ c \sim 861.94 e^{-0.997n_f} 2^{10+6n_f} \pi^{6+4n_f}. \] (4.25)

We can calculate \( \rho \) integral by (4.24) and obtain
\[ D\left(\frac{\mu}{v}; \lambda\right) = 2^{2n_f+2n_h-1} c \lambda^{-n_h/2} \left( \frac{\lambda}{2\pi^2} \right)^{t} \left( \frac{\mu}{v} \right)^{(43-8n_f)/6} \Gamma(t), \] (4.26)
where \( t = 7n_f/3 + n_w + n_h + 19/12 \). The integral is dominated by the contribution around the saddle point;
\[ \rho v_s = \left\{ \frac{\lambda}{2\pi^2} \left( t - \frac{1}{2} \right) \right\}^{1/2}. \] (4.27)

Therefore, we can check the consistency of the approximation by this value of the saddle point. The saddle point \( \rho v_s \) depends on the number of bosons monotonously. The valley instanton
Figure 1: The energy dependence of the cross section for $n_w = 75$ and $n_h = 0$. The solid line denotes the result by the saddle point approximation. The dashed line and the dotted line denote the approximate result in the non-relativistic limit and extremely relativistic limit, respectively. We set $n_f = 3$, $\mu = m_w = 80 \text{GeV}$, $g^2 = 0.42$, $m_f = 180 \text{GeV}$ and $m_h = 200 \text{GeV}$ and neglect the other fermion masses.

is quite similar to the original instanton even at $\rho v \approx 1$, and therefore the approximation is valid for $n_w + n_h \approx 40$, where we assume $m_h = m_w$ and use $g^2 \approx 0.42$. On the other hand, the constrained instanton is deviated from the original instanton at $\rho v = 0.5$ and so the approximation breaks for $n_w + n_h \approx 4$. Therefore we can calculate the amplitude of the multi-boson process, using the valley instanton. Finally, we obtain

$$
\sigma = \frac{c^2}{n_w!n_h!} 2^{-n_h - 7}(2\pi)^{-40n_f/3 - 3n_w - 3n_h + 11/3}\Gamma(t)^2 \\
\times \left( g^2 \right)^{14n_f/3 + n_w + n_h - 29/6} e^{-16\pi^2/g^2} m_w^{-2(6n_f + 2n_w + n_h - 4)} \left( \frac{\mu}{m_w} \right)^{(43 - 8n_f)/3}.
$$

(4.28)
Figure 2: The cross section for the various $n_w$ and $n_h$ at $s^{1/2} = 15.55$ TeV.
Figure 3: The energy dependence of the cross section for \( n_w = 0, 20, 40, 75 \) and 100, and \( n_h = 0 \). A dashed line denotes the unitarity bound.
4.3 Results

We show the result in Fig.1 for the case where \( n_w = 75 \) and \( n_h = 0 \). In Fig.1, the solid line denotes the numerical result by the steepest descent method. The dashed line and the dotted line denote the approximate result in the non-relativistic limit and extremely relativistic limit, respectively. We show cross sections for the various values of \( n_w \) and \( n_h \) in Fig.2 at \( \sqrt{s} = 15.55 \) TeV. From this, we understand that the gauge boson plays an important role rather than the Higgs particle. In Fig.3, we show the energy dependence of the cross section for \( n_w = 0, 20, 40, 75 \) and \( 100 \), and \( n_h = 0 \). We show the unitarity bound as a dashed line in Fig.3. The approximation breaks in the region of the energy where the cross section overcomes the unitarity bound. In the region, an interaction between the gauge bosons in the final state \( 3 \) and interaction between multi-instantons \( 13 \) play important roles, which are not evaluated in our analysis.

5 Conclusion

In this paper, we have applied the valley instanton to the baryon and lepton number violating processes. The constrained instanton method was used in the past works, but the processes with many bosons, which are expected to be observed, are out of the validity of the method. By using the proper valley method, we have given a reliable calculation of the process with many bosons. First, by treating the Lorentz rotations and the translations as the gauge symmetries, we have introduced the collective coordinates for the valley instanton which is not a classical solution. Then we have calculated the cross section of the baryon and lepton number violating processes. Furthermore, we have taken into account the instanton orientation effect, boson and top quark mass effect, which are not considered yet. Then we have obtained more accurate result than earlier works. The final result matches the qualitative estimate \( 3 \). The cross section grows up, as total energy becomes higher, and the processes with many gauge bosons and no Higgs particle grow up more rapidly.

For high-energy processes associated with many bosons, we see the unitarity is broken, which means that our analysis is not appropriate in such cases. It is because we don’t take into account the multi-instanton effects \( 15 \) or the interaction among the gauge bosons. By the proper valley method, new phenomena might be revealed in the subjects. One of the reason is that there appear new terms in the interaction between the valley instanton and the anti valley instanton. Our present study is the first step toward these subjects.
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Appendix

A Faddeev-Popov determinant

If we choose the gauge conditions as (3.12) and ignore higher loop corrections, the Faddeev-Popov determinant becomes

\[
\Delta_{FP}(\rho) = 8\pi^2 \cdot \det \begin{vmatrix}
A^{ab}(x, y; \rho) & B^{\mu\nu}(x; \rho) & C^{ab}(x; \rho) & 0 \\
D^{\mu b}(y; \rho) & E^{\mu\nu}(\rho) & F^{\mu b}(\rho) & X^\mu(\rho) \\
G^{ab}(y; \rho) & H^{\mu\nu}(\rho) & I^{ab}(\rho) & Y^a(\rho) \\
0 & 0 & 0 & W(\rho)
\end{vmatrix}
= 8\pi^2 \cdot \det \begin{vmatrix}
A^{ab}(x, y; \rho) & B^{\mu\nu}(x; \rho) & C^{ab}(x; \rho) \\
D^{\mu b}(y; \rho) & E^{\mu\nu}(\rho) & F^{\mu b}(\rho) \\
G^{ab}(y; \rho) & H^{\mu\nu}(\rho) & I^{ab}(\rho)
\end{vmatrix} \times |W(\rho)|, \quad (A.1)
\]

where

\[
A^{ab}(x, y; \rho) = (D^0_\lambda D^0_\lambda)_{ab} \delta(x - y) \\
= \left\{ \partial^2 \delta_{ab} + 2\epsilon_{acb}A^c_\lambda(x; \rho)\partial_\lambda + \epsilon_{acd}\epsilon_{deb}A^{c}_\lambda(x; \rho)A^e_\lambda(x; \rho) \right\} \delta(x - y),
\]

\[
B^{\mu\nu}(x; \rho) = -2\text{tr} \left\{ \left[ i\frac{\sigma^a}{2}, A_\lambda(x; \rho) \right] \partial_\nu A_\lambda(x; \rho) \right\},
\]

\[
C^{ab}(x; \rho) = -2\text{tr} \left\{ \left[ i\frac{\sigma^a}{2}, A_\lambda(x; \rho) \right] \left[ i\frac{\sigma^b}{2}, A_\lambda(x; \rho) \right] \right\},
\]

\[
D^{\mu b}(y; \rho) = -\text{tr} \left\{ D^0_\lambda f^{\mu}(y; \rho) \frac{\sigma^b}{2} \right\},
\]

\[
E^{\mu\nu}(\rho) = \int d^4x \text{tr} \left\{ f^{\mu}(x; \rho) \partial_\nu A_\lambda(x; \rho) \right\},
\]

\[
F^{\mu b}(\rho) = \int d^4x \text{tr} \left\{ f^{\mu}(x; \rho) \left[ i\frac{\sigma^b}{2}, A_\lambda(x; \rho) \right] \right\},
\]

\[
G^{ab}(y; \rho) = -\text{tr} \left\{ D^0_\lambda f^{\mu}(y; \rho) \frac{\sigma^b}{2} \right\}, \quad (A.2)
\]
\[ H^{\alpha\nu}(\rho) = \int d^4x \text{tr} \left\{ f^a_{\lambda}(x;\rho) \partial_{\nu} A_{\lambda}(x;\rho) \right\}, \]
\[ I^{ab}(\rho) = \int d^4x \text{tr} \left\{ f^a_{\lambda}(x;\rho) \left[ i \frac{\sigma^b}{2}, A_{\lambda}(x;\rho) \right] \right\}, \]
\[ X^\mu(\rho) = -\int d^4x \text{tr} \left\{ f^\mu_{\lambda}(x;\rho) \frac{\partial A_{\lambda}(x;\rho)}{\partial \rho} \right\}, \]
\[ Y^a(\rho) = -\int d^4x \text{tr} \left\{ f^a_{\lambda}(x;\rho) \frac{\partial A_{\lambda}(x;\rho)}{\partial \rho} \right\}, \]
\[ W(\rho) = -\frac{1}{N(\rho)} \int d^4x \left[ \frac{\delta S}{\delta A^a_{\lambda}(x;\rho)} \frac{\partial A^a_{\lambda}(x;\rho)}{\partial \rho} + \frac{\delta S}{\delta H(x;\rho)} \frac{\partial H(x;\rho)}{\partial \rho} + \frac{\delta H^\dagger(x;\rho)}{\partial \rho} \frac{\delta S}{\delta H^\dagger(x;\rho)} \right]. \]

When \( \rho v \to 0 \), the instanton measure given by our collective coordinate method is the same as one given by 't Hooft\[1\]. To see this, it is convenient to take \( f^\nu_{\mu}(x;\rho) \) and \( f^a_{\mu}(x;\rho) \) as
\[ f^\nu_{\mu} = \partial_{\nu} A_{\mu}(x;\rho) - D^0_{\mu} A_{\nu}(x;\rho), \quad (A.3) \]
\[ f^a_{\mu}(x;\rho) = \left[ i \frac{\sigma^a}{2}, A_{\mu}(x;\rho) \right] - D^0_{\mu} \left[ \frac{\sigma^a}{2} \frac{\rho^2}{x^2 + \rho^2} \right], \quad (A.4) \]
where \( D^0_{\mu} \) is covariant derivative under the background of the valley instanton; \( D^0_{\mu} = \partial_{\mu} - i[A_{\mu}(x;\rho), \cdot] \). Then it holds
\[ E^{\mu\nu}(\rho) = \int d^4x \text{tr} \left\{ f^\mu_{\lambda}(x;\rho) f^\nu_{\kappa}(x;\rho) \right\} + \int d^4x \text{tr} \left\{ f^\mu_{\lambda}(x;\rho) D^0_{\nu} A_{\kappa}(x;\rho) \right\}, \quad (A.5) \]
\[ F^{\mu b}(\rho) = \int d^4x \text{tr} \left\{ f^b_{\lambda}(x;\rho) f^\mu_{\kappa}(x;\rho) \right\} + \int d^4x \text{tr} \left\{ f^b_{\lambda}(x;\rho) D^0_{\nu} \left[ \frac{\sigma^b}{2} \frac{\rho^2}{x^2 + \rho^2} \right] \right\}, \quad (A.6) \]
\[ H^{\alpha\nu}(\rho) = \int d^4x \text{tr} \left\{ f^a_{\lambda}(x;\rho) f^\nu_{\kappa}(x;\rho) \right\} + \int d^4x \text{tr} \left\{ f^a_{\lambda}(x;\rho) D^0_{\nu} A_{\kappa}(x;\rho) \right\}; \quad (A.7) \]
\[ I^{ab}(\rho) = \int d^4x \text{tr} \left\{ f^a_{\lambda}(x;\rho) f^b_{\kappa}(x;\rho) \right\} + \int d^4x \text{tr} \left\{ f^a_{\lambda}(x;\rho) D^0_{\nu} \left[ \frac{\sigma^b}{2} \frac{\rho^2}{x^2 + \rho^2} \right] \right\}. \quad (A.8) \]

When \( \rho v \to 0 \), the valley instanton becomes the BPST instanton and \( f^\nu_{\mu} \) and \( f^a_{\mu} \) become the zero modes corresponding to the global gauge rotations and the translations, which satisfy
\[ D^0_{\mu} f^\nu_{\mu} = 0, \quad (A.9) \]
\[ D^0_{\mu} f^a_{\mu} = 0. \]

Therefore, as \( \rho v \to 0 \), we obtain
\[ D^{ab}(y;\rho) \to 0, \quad G^{ab}(y;\rho) \to 0, \]
\[ E^{\mu\nu}(\rho) \to \int d^4x \text{tr} \left\{ f^\mu_{\lambda}(x;\rho) f^\nu_{\kappa}(x;\rho) \right\} = \delta_{\mu\nu} \| f^\mu_{\lambda} \|^2 \cdot \frac{1}{2}. \]
\[ F^{\mu a}(\rho) \rightarrow \int d^4x \text{tr} \left\{ f^a_{\lambda}(x; \rho) f^b_{\lambda}(x; \rho) \right\} = 0, \]  
(A.10)

\[ H^{\alpha \nu}(\rho) \rightarrow \int d^4x \text{tr} \left\{ f^a_{\alpha}(x; \rho) f^b_{\nu}(x; \rho) \right\} = 0, \]

\[ I^{ab}(\rho) \rightarrow \int d^4x \text{tr} \left\{ f^a_{\lambda}(x; \rho) f^b_{\lambda}(x; \rho) \right\} = \delta_{ab} \| f^a_{\lambda} \|^2 \cdot \frac{1}{2}. \]

From (2.9) and (2.10), it holds

\[ \frac{\delta S}{\delta A_\mu(x; \rho)} = F^A_\mu \rightarrow \frac{\rho v^2}{4\lambda} \frac{\partial A_\mu(x; \rho)}{\partial \rho}, \quad \frac{\delta S}{\delta H(x; \rho)} = F^H \rightarrow \frac{\rho v^2}{4\lambda} \frac{\partial H(x; \rho)}{\partial \rho}, \]  
(A.11)

as \( \rho v \rightarrow 0 \) and

\[ \int d^4x \left| \frac{\partial H(x; \rho)}{\partial \rho} \right|^2 = O(\rho v) \int d^4x \left( \frac{\partial A_\mu(x; \rho)}{\partial \rho} \right)^2. \]  
(A.12)

Then

\[ |W(\rho)| \rightarrow \sqrt{\int d^4x \left( \frac{\partial A_\mu^a(x; \rho)}{\partial \rho} \right)^2} = \left\| \frac{\partial A_\mu}{\partial \rho} \right\|. \]  
(A.13)

Therefore, we obtain

\[ \Delta_{FP}(\rho) \rightarrow 8\pi^2 \cdot 2^{-7} \cdot \left| \text{Det}A^{ab}(x, y; \rho) \right| \cdot \left\| \frac{\partial A_\mu}{\partial \rho} \right\| \cdot \| f^\nu \|^8 \cdot \| f^a_\mu \|^6, \]  
(A.14)

as \( \rho v \rightarrow 0 \). When we integrate the quantum fluctuations, the following factor appears:

\[ \left( \frac{1}{\sqrt{2\pi}} \right)^4 \cdot \left( \frac{2}{\| f^\nu \|} \right)^4 \cdot \left( \frac{2}{\| f^a_\mu \|} \right)^3. \]  
(A.15)

Then the Jacobian becomes

\[ 8\pi^2 \cdot \left( \frac{1}{\sqrt{2\pi}} \right)^4 \cdot \left\| \frac{\partial A_\mu}{\partial \rho} \right\| \cdot \| f^\nu \|^4 \cdot \| f^a_\mu \|^3. \]  
(A.16)

This coincide with one given by ’t Hooft, since

\[ \left\| \frac{\partial A_\mu}{\partial \rho} \right\| \rightarrow 4\pi, \quad \| f^\nu \| \rightarrow 2\sqrt{2\pi}, \quad \| f^a_\mu \| \rightarrow 2\pi \rho, \]  
(A.17)

when \( \rho v \rightarrow 0 \). Because the determinant parts coming from the path integral agree with that of BPST instanton when \( \rho v = 0 \), the instanton measure of the valley instanton coincides that of BPST instanton in this limit.
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