Chapter 1
Relaxation of Periodic and Nonstandard Growth Integrals by means of Two-scale convergence

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Abstract An integral representation result is obtained for the variational limit of the family functionals \( \int_{\Omega} f(\frac{x}{\varepsilon}, Du) dx, \varepsilon > 0 \), when the integrand \( f = f(x,v) \) is a Carathéodory function, periodic in \( x \), convex in \( v \) and with nonstandard growth.

1.1 Introduction

In [TN1], the authors extended the notion of two-scale convergence introduced by [N] (see also [A], [CDG], [FZ], [V] among a wider literature for extensions and related notions) to the Orlicz-Sobolev setting and obtained, under strict convexity assumption on \( f \) and suitable boundary conditions, the existence of a unique minimizer for a suitable limit functional as the limit of the minimizers of the original functionals \( \int_{\Omega} f(\frac{x}{\varepsilon}, Du) dx \) as \( \varepsilon \rightarrow 0 \).

In particular they proved (cf. [TN1, Corollary 5.2]) that for every sequence \( (u_\varepsilon)_\varepsilon \in W^{1,L^B}(\Omega; \mathbb{R}) \) such that \( (Du_\varepsilon)_\varepsilon \) weakly 2s-converges to \( Du_0 = Du + Du_1 \),

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where \( u_0 = (u, u_1) \in W^1_\text{Lip}(\Omega) \times L^1(\Omega; W^1_\text{Lip}(Y)) \). Then

\[
\iint_{\Omega \times Y} f(y, \nabla u_0) \, dx \, dy \leq \liminf_{\varepsilon \to 0} \int_{\Omega} f\left( \frac{x}{\varepsilon}, \nabla u_\varepsilon \right) \, dx,
\]

where \( Y := (0, 1)^d \) \((d \in \mathbb{N})\) and \( \nabla u_0 := Du + D_1u_1 \) (see Section 2 for the notations adopted in this introduction).

On the other hand, by the very nature of two-scale converge they obtain, under homogeneous boundary conditions on \( \partial \Omega \), (the same proof can be performed for any boundary conditions), for \( u \) and \( u_1 \) regular, the existence of a suitable sequence \((u_\varepsilon)_{\varepsilon} \subseteq W^1_\text{Lip}(\Omega)\) such that \( u_\varepsilon \rightharpoonup u \) weakly in \( W^1_\text{Lip}(\Omega) \) and the opposite inequality holds:

\[
\lim_{\varepsilon \to 0} \int_{\Omega} f\left( \frac{x}{\varepsilon}, \nabla u_\varepsilon \right) \, dx = \iint_{\Omega \times Y} f(y, \nabla u_0) \, dx \, dy.
\]

Here, by means of two scale convergence we aim to extend their result to any couple of functions \((u_0 \equiv (u, u_1)) \in W^1_\text{Lip}(\Omega) \times L^1(\Omega; W^1_\text{Lip}(Y))\), and also to obtain an integral representation result for

\[
\inf \left\{ \liminf_{\varepsilon \to 0} \int_{\Omega} f\left( \frac{x}{\varepsilon}, \nabla u_\varepsilon \right) \, dx : u_\varepsilon \rightharpoonup u \text{ weakly in } W^1_\text{Lip}(\Omega) \right\}.
\]

Indeed, after stating preliminary results in Section 1.2 on Orlicz-Sobolev spaces and homogenization theory, in Section 1.3 we will prove the following theorem:

**Theorem 1.** Let \( \Omega \) be a bounded open set with Lipschitz boundary and let \( f : \Omega \times \mathbb{R}^d \to \mathbb{R} \) be a Carathéodory function such that

\[
f(x, \cdot) \text{ is convex for a.e. } x \in \Omega,
\]

and there exist constants \( c, c' \) and \( C \in \mathbb{R}^+ \) such that for a.e. \( x \in \Omega \) and every \( \xi \in \mathbb{R}^d \),

\[
cB'(|\xi|) - c' \leq f(x, \xi) \leq C(1 + B(|\xi|))
\]

with \( B, B' \) equivalent N-functions which satisfy the \( \Delta_2 \) condition. Then, it results that for every \( u \in W^1_\text{Lip}(\Omega) \),

\[
\inf \left\{ \liminf_{\varepsilon \to 0} \int_{\Omega} f\left( \frac{x}{\varepsilon}, \nabla u_\varepsilon \right) \, dx : u_\varepsilon \rightharpoonup u \text{ weakly in } W^1_\text{Lip}(\Omega) \right\}
\]

\[
= \inf \left\{ \limsup_{\varepsilon \to 0} \int_{\Omega} f\left( \frac{x}{\varepsilon}, \nabla u_\varepsilon \right) \, dx : u_\varepsilon \rightharpoonup u \text{ weakly in } W^1_\text{Lip}(\Omega) \right\} = \int_{\Omega} f_{\text{hom}}(Du) \, dx,
\]

where \( f_{\text{hom}} : \mathbb{R}^d \to \mathbb{R} \) is the density defined by

\[
f_{\text{hom}}(\xi) := \inf \left\{ \int_Y f(y, \xi + Du) \, dy : u \in W^1_\text{Lip}(Y) \right\}.
\]
We underline that the analysis presented in this paper, holds also in the vectorial case, i.e. fields \( u \in W^{1,1}(\Omega; \mathbb{R}^m) \), with the exact same techniques, provided that \( f(x, \cdot) \) is convex.

Furthermore, in order to prove (1.3), we also obtain for every \( u_0 \in W^{1,1}(\Omega) \times L^1(\Omega; W^{1,1}_{per}(Y)) \), the following two-scale representation:

\[
\inf \left\{ \liminf_{\varepsilon \to 0} \int_{\Omega} f \left( \frac{x}{\varepsilon}, Du_{\varepsilon} \right) \, dx : u_{\varepsilon} \rightharpoonup u_0 \right\} = \inf \left\{ \limsup_{\varepsilon \to 0} \int_{\Omega} f \left( \frac{x}{\varepsilon}, Du_{\varepsilon} \right) \, dx : u_{\varepsilon} \rightharpoonup u_0 \right\} = \iint_{\Omega \times Y} f(y, D u_0) \, dx \, dy.
\]

1.2 Preliminaries

This section is devoted to fix notation adopted in the sequel and state preliminary results on Orlicz-Sobolev spaces and homogenization results that will be exploited in the next section. For more details concerning these latter results, for the sake of brevity, we will refer directly to [TN1].

\( \Omega \subset \mathbb{R}^d \) \((d \in \mathbb{N})\) denotes a bounded open set with Lipschitz boundary.

1.2.1 Orlicz-Sobolev spaces

Let \( B : [0, +\infty) \to [0, +\infty] \) be an \( N \)-function as in [Ad], i.e., \( B \) is continuous, convex, with \( B(t) > 0 \) for \( t > 0 \), \( \frac{B(t)}{t} \to 0 \) as \( t \to 0 \), and \( \frac{B(t)}{t} \to \infty \) as \( t \to \infty \).

Equivalently, \( B \) is of the form \( B(t) = \int_0^t b(\tau) \, d\tau \), where \( b : [0, +\infty] \to [0, +\infty] \) is non decreasing, right continuous, with \( b(0) = 0 \), \( b(t) > 0 \) if \( t > 0 \) and \( b(t) \to +\infty \) if \( t \to +\infty \). We denote by \( \bar{B} \), the Fenchel’s conjugate, also called the complementary \( N \)-function of \( B \) defined by

\[
\bar{B}(t) = \sup_{s \geq 0} \{ st - B(s) \}, \ t \geq 0.
\]

It can be proven that (see [TN1] Lemma 2.1) if \( B \) is an \( N \)-function and \( \bar{B} \) is its conjugate, then for all \( t > 0 \), it results

\[
\frac{tb(t)}{B(t)} \geq 1 (> 1 \text{ if } b \text{ is strictly increasing}), \quad \bar{B}(b(t)) \leq tb(t) \leq B(2t).
\]

An \( N \)-function \( B \) is of class \( \triangle_2 \) (denoted \( B \in \triangle_2 \)) if there are \( \alpha > 0 \) and \( t_0 \geq 0 \) such that \( B(2t) \leq \alpha B(t) \) for all \( t \geq t_0 \).
In all what follows $B$ and $\tilde{B}$ are conjugates $N-$functions satisfying the delta-2 ($\triangle_2$) condition and $c$ refers to a constant that may vary from line to line.

The Orlicz-space $L^B(\Omega) = \left\{u : \Omega \to \mathbb{C} \text{ measurable, } \lim_{\delta \to 0^+} \int_{\Omega} B(\delta |u(x)|) \, dx = 0 \right\}$ is a Banach space for the Luxemburg norm:

$$\|u\|_{B,\Omega} = \inf \left\{k > 0 : \int_{\Omega} B\left(\frac{|u(x)|}{k}\right) \, dx \leq 1 \right\} < +\infty.$$ 

It follows that: $C_c^\infty(\Omega)$ is dense in $L^B(\Omega), L^\infty(\Omega)$ is separable and reflexive, the dual of $L^B(\Omega)$ is identified with $L^B(\Omega)$, and the norm induced on $L^B(\Omega)$ as a dual space is equivalent to $\|\cdot\|_{\tilde{B},\Omega}$.

Analogously one can define the Orlicz-Sobolev functional space as follows:

$$W^{1,L^B}(\Omega) = \left\{u \in L^B(\Omega) : \frac{\partial u}{\partial x_i} \in L^B(\Omega), 1 \leq i \leq d \right\},$$

where derivatives are taken in the distributional sense on $\Omega$. Endowed with the norm $\|u\|_{W^{1,L^B}(\Omega)} = \|u\|_{B,\Omega} + \sum_{i=1}^d \left\|\frac{\partial u}{\partial x_i}\right\|_{B,\Omega}, u \in W^{1,L^B}(\Omega)$, $W^{1,L^B}(\Omega)$ is a reflexive Banach space. We denote by $W^{1}_0 L^B(\Omega)$, the closure of $C_c^\infty(\Omega)$ in $W^{1,L^B}(\Omega)$ and the semi-norm $u \to \|u\|_{W^{1,L^B}(\Omega)} = \|Du\|_{B,\Omega} = \sum_{i=1}^d \left\|\frac{\partial u}{\partial x_i}\right\|_{B,\Omega}$ is a norm on $W^{1}_0 L^B(\Omega)$ equivalent to $\|\cdot\|_{W^{1,L^B}(\Omega)}$.

### 1.2.2 Homogenization

In order to deal with periodic integrands we will adopt the following notation.

Let $Y := (0,1)^d$. The letter $\varepsilon$ throughout will denote a family of positive real numbers converging to 0. The set $\mathbb{R}^d_\gamma$ will denote $\mathbb{R}^d$, but the subscript $\gamma$ emphasizes the fact that this is the set where the space variable $y$ is. We also define

$$C_{\text{per}}(Y) = \{v \in C(\mathbb{R}^d_\gamma) : Y \text{ periodic}\},$$

and

$$L^B_{\text{per}}(Y) := \{v \in L^B_{\text{loc}}(\mathbb{R}^N_\gamma) : Y \text{ periodic}\}.$$ 

Moreover we observe that $L^B_{\text{per}}(Y)$ is a Banach space under the Luxemburg norm $\|\cdot\|_{B,Y}$ and $C_{\text{per}}(Y)$ is dense in $L^B_{\text{per}}(Y)$ (see [TN1] Lemma 2.1).

For $v \in L^B_{\text{per}}(Y)$ let

$$v^\varepsilon(x) = v\left(\frac{x}{\varepsilon}\right), x \in \mathbb{R}^d.$$ 

Given $v \in L^B_{\text{loc}}(\Omega \times \mathbb{R}^N_\gamma)$ and $\varepsilon > 0$, we put
v^f(x) = v\left(x, \frac{x}{\varepsilon}\right), x \in \mathbb{R}^d \text{ whenever it makes sense.}

We define the vector space

\[ L^B(\Omega \times Y_{\text{per}}) := \{ u \in L^B_{\text{loc}}(\Omega \times \mathbb{R}_Y^N) : \text{ for a.e. } x \in \Omega, u(x, \cdot) \text{ is } Y \text{ - periodic} \}, \]

and observe that the embedding \( L^B(\Omega, C_{\text{per}}(Y)) \to L^B(\Omega \times Y_{\text{per}}) \) is continuous.

Moreover we will make use of the space

\[ W^{1,B}_{\text{per}}(Y) := \left\{ u \in W^1L^B_{\text{loc}}(\mathbb{R}_Y^N) : u, \frac{\partial u}{\partial x_i}, i = 1, \ldots, N, \text{ Y - periodic} \right\} \]

where the derivative \( \frac{\partial u}{\partial x_i} \) is taken in the distributional sense on \( \mathbb{R}_Y^N \), and we endow it with the norm \( \| u \|_{W^{1,B}_{\text{per}}(Y)} = \| u \|_{B,Y} + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{B,Y} \), which makes it a Banach space.

We also consider the space

\[ W^{1,B}_{\text{per}}(Y) = \left\{ u \in W^{1,B}_{\text{per}}(Y) : \int_Y u(y)dy = 0 \right\}, \]

and we endow it with the gradient norm

\[ \| u \|_{W^{1,B}_{\text{per}}(Y)} = \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{B,Y}. \]

Denoting by \( C^\infty_{\text{per}}(Y) = C_{\text{per}}(Y) \cap C^\infty(\mathbb{R}^N) \), and recalling that the space \( C^\infty_{L,\text{per}}(Y; \mathbb{R}) = \{ u \in C^\infty_{\text{per}}(Y; \mathbb{R}) : \int_Y u(y)dy = 0 \} \) is dense in \( W^1_{L,\text{per}}(Y) \), one can deduce (cf. [TN1]) the density of the embedding

\[ C^\infty_c(\Omega; \mathbb{R}) \otimes C^\infty_{L,\text{per}}(Y; \mathbb{R}) \subseteq L^1(\Omega; W^1_{L,\text{per}}(Y)). \tag{1.5} \]

In [TN1] the notion of two-scale convergence introduced by [N] and developed by [A] (see also, among a wide literature, [CDG], [FZ], [NG], [Y] for further developments and related notions like periodic unfolding method), has been extended to the Orlicz-Sobolev setting.

**Definition 1.** A sequence of functions \((u^\varepsilon)\varepsilon \in L^B(\Omega)\) is said to be:

- weakly two-scale convergent in \( L^B(\Omega) \) to a function \( u_0 \in L^B(\Omega \times Y_{\text{per}}) \) if for every \( \varepsilon \to 0 \), we have

\[ \int_\Omega u^\varepsilon f^\varepsilon dx \to \int_\Omega u_0 f dx, \text{ for all } f \in L^B(\Omega; C_{\text{per}}(Y)) \tag{1.6} \]

- strongly two-scale convergent in \( L^B(\Omega) \) to \( u_0 \in L^B(\Omega \times Y_{\text{per}}) \) if for \( \eta > 0 \) and \( f \in L^1(\Omega; C_{\text{per}}(Y)) \) verifying \( \| u_0 - f \|_{L^B(\Omega \times Y)} \leq \frac{\eta}{2} \), there exist \( \rho > 0 \) such that \( \| u^\varepsilon - f^\varepsilon \|_{L^B(\Omega)} \leq \eta \) for all \( 0 < \varepsilon \leq \rho \).
When \( f \in L^\infty(\Omega; C_{\text{per}}(Y)) \) we denote it by "\( u_\varepsilon \rightharpoonup u_0 \) in \( L^B(\Omega) \) weakly" or simply "\( u_\varepsilon \rightharpoonup u_0 \) in \( L^B(\Omega) \) two-scale weakly" and we will say that \( u_0 \) is the weak two-scale limit in \( L^B(\Omega) \) of the sequence \( \{u_\varepsilon\}_\varepsilon \). In order to denote strong two scale convergence of \( u_\varepsilon \to u_0 \) we adopt the symbol \( \|u_\varepsilon - u_0\|_{2\varepsilon - L^B(\Omega \times Y)} \to 0 \).

The following result, whose proof can be found in [TN1], allows to extend the notion of weak two-scale convergence at Orlicz-Sobolev functions, guaranteeing, at the same time, a compactness result.

**Proposition 1.** Let \( \Omega \) be a bounded open set in \( \mathbb{R}^d \) and let \( \{u_\varepsilon\}_\varepsilon \) be bounded in \( W^1L^B(\Omega) \). There exist a subsequence, still denoted in the same way, and \( u \in W^1L^B(\Omega) \), \( u_1 \in L^1(\Omega; W^1_{\text{per}}L^B(Y)) \) such that:

(i) \( u_\varepsilon \rightharpoonup u \) in \( L^B(\Omega) \),

(ii) \( D_{i\varepsilon}u_\varepsilon \rightharpoonup D_iu + D_iu_1 \) in \( L^B(\Omega) \), \( 1 \leq i \leq d \).

In the sequel we denote by \( u_0(x,y) \) the function \( u(x) + u_1(x,y) \), and by \( \nabla u_0 \) the vector \( Du + D_iu_1 \).

For the sake of brevity, we cannot explicitly quote all the results used throughout the paper but we will refer to [TN1] for further necessary properties of Orlicz-Sobolev spaces, two-scale convergence and homogenization in the Orlicz setting.

### 1.3 Proof of Theorem [1]

This section is devoted to the proof of Theorem [1]. To this aim recall the definition of \( f_{\text{hom}} \) given by (1.4).

**Proof (of Theorem [1]).** We start observing that the coercivity assumptions on \( f \), the compactness result, given by Proposition 1, and (1.4) guarantee that for every \( u_\varepsilon \rightharpoonup u \in W^1L^B(\Omega) \)

\[
\liminf_{\varepsilon \to 0} \int_\Omega f(\frac{x}{\varepsilon}, Du_\varepsilon) \, dx \geq \int\int_{\Omega \times Y} f(y, Du_0) \, dx \, dy, \tag{1.7}
\]

where \( u_0(x,y) = u(x) + u_1(x,y) \) is the weak two scale limit of \( \{u_\varepsilon\}_\varepsilon \). Clearly passing to the infimum on both sides of the equation (1.7), and recalling that \( Du_0 = Du + D_iu_1 \), we obtain

\[
\inf \left\{ \liminf_{\varepsilon \to 0} \int_\Omega f(\frac{x}{\varepsilon}, Du_\varepsilon) \, dx : u_\varepsilon \rightharpoonup u \in W^1L^B(\Omega) \right\} \\
\geq \inf \left\{ \int\int_{\Omega \times Y} f(y, Du + D_iu_1) \, dx \, dy : u_1 \in L^1(\Omega; W^1_{\text{per}}L^B(Y)) \right\} = \int_\Omega f_{\text{hom}}(Du) \, dx,
\]

where one can replicate the same proof as [CDDA Lemma 2.2] replacing \( t \) by 1 and \( f_1 \) by \( f_{\text{hom}} \) in (1.4) therein and exploit the convexity of \( f \) to replace functions with null boundary datum on \( \partial Y \), with periodic ones (see also end of [Ne] Chapter 3).
The upper bound exploits an argument very similar to the one presented in [Ne], relying, in the present context, on the density result in [1.5]. Indeed we can first observe that, as in [TN1, Corollary 5.1] for any given 
relying, in the present context, on the density result in (1.5). Indeed we can first

\[ \lim_{\varepsilon \to 0} \int_{\Omega} f \left( \frac{x}{\varepsilon}, D\phi_{\varepsilon}(x) \right) dx = \int_{\Omega \times Y} f(y, Du + D_y \phi_1(x, y)) dxdy. \]  

(1.8)

On the other hand, given \( u \in W^1L^B(\Omega) \) and \( u_1 \in L^1(\Omega; W^1L^B_{per}(Y)) \), [1.5] guarantees that for each \( \delta > 0 \) we can find maps \( u_\delta \in C^\infty(\overline{\Omega}) \) and \( v_\delta \in C^\infty(\overline{\Omega}; C^0_{per}(Y)) \) (this latter with zero averageb) such that

\[ ||u - u_\delta||_{W^1L^B(\Omega)} + ||u_1 - v_\delta||_{L^1(\Omega; W^1L^B_{per}(Y))} \leq \delta \]  

(1.9)

Next defining, for every \( \delta \), and for every \( x \in \Omega \),

\[ u_{\delta, \varepsilon}(x) := u_\delta(x) + \varepsilon v_\delta(x), \]

one has

\[ Du_{\delta, \varepsilon}(x) = Du_\delta(x) + \varepsilon D_x v_\delta \left( x, \frac{x}{\varepsilon} \right) + D_y v_\delta \left( x, \frac{x}{\varepsilon} \right). \]

Clearly, as \( \varepsilon \to 0 \), it results

\[ u_{\delta, \varepsilon} \to u_\delta \text{ in } L^B(\Omega), \]

\[ Du_{\delta, \varepsilon}(x) \rightharpoonup Du_\delta(x) + D_y v_\delta(x, y) \text{ strongly in } L^B(\Omega \times Y_{per}). \]

Now we define

\[ c_{\delta, \varepsilon} := ||u_{\delta, \varepsilon} - u||_{W^1L^B(\Omega)} + \left| \left| \left| Du_{\delta, \varepsilon} \right| \right|_{L^B(\Omega)} - \left| \left| Du - D_y u_1 \right| \right|_{L^B(\Omega \times Y)} \right|, \]  

(1.10)

having the aim of constructing, via a diagonalizing argument, a sequence strongly
two scale convergent to \( u_0 = u + u_1 \).

Thus, it is easily seen that

\[ \lim_{\delta \to 0} \lim_{\varepsilon \to 0} c_{\delta, \varepsilon} = 0, \]

which allows us to apply H. Attouch Diagonalization Lemma, thus detecting a se-
sequence \( \delta(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), such that \( c_{\delta(\varepsilon), \varepsilon} \to 0 \) and \( u_{\delta(\varepsilon), \varepsilon} \to u \) in \( L^B(\Omega) \), with

\[ Du_{\delta(\varepsilon), \varepsilon}(x) \rightharpoonup Du(x) + D_y u_1(x, y) \text{ strongly in } L^B(\Omega \times Y_{per}). \]

This latter convergence, and [TN1 Remark 4.1] ensure that \( Du_{\delta(\varepsilon), \varepsilon} \rightharpoonup Du \) weakly in \( L^B(\Omega) \), thus, [1.3], the continuity of \( f \) in the second variable, [1.2], guarantee that for every \( u_1 \in L^1(\Omega; W^1L^B_{per}(Y)) \),
\[
\lim_{\varepsilon \to 0} \int_{\Omega} f \left( \frac{x}{\varepsilon}, Du(\varepsilon, x) \right) dx = \iint_{\Omega \times Y} f(y, Du + D_y u_1(x, y)) dxdy.
\]

as desired.

Thus we can conclude that

\[
\inf \left\{ \limsup_{\varepsilon \to 0} \int_{\Omega} f \left( \frac{x}{\varepsilon}, Du_\varepsilon \right) dx : u_\varepsilon \rightharpoonup u \text{ in } W^1L^B(\Omega) \right\} 
\leq \lim_{\varepsilon \to 0} \int_{\Omega} f \left( \frac{x}{\varepsilon}, Du_\varepsilon \right) dx \leq \iint_{\Omega \times Y} f(y, Du(x) + D_y u_1(x, y)) dxdy.
\]

Hence

\[
\inf \left\{ \limsup_{\varepsilon \to 0} \int_{\Omega} f \left( \frac{x}{\varepsilon}, Du_\varepsilon \right) dx : u_\varepsilon \rightharpoonup u \text{ in } W^1L^B(\Omega; \mathbb{R}) \right\} 
\leq \inf \left\{ \iint_{\Omega \times Y} f(y, Du + D_y u_1) dxdy : u_1 \in L^1(\Omega; W^1L^B_{per}(Y)) \right\}.
\]

This, together with the last equality in (1.7) concludes the proof.

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