Duality of Width and Depth of Neural Networks

Feng-Lei Fan, Ge Wang

Contact: fanf2@rpi.edu, wangg6@rpi.edu

AI-based X-ray Imaging System (AXIS) Lab, Biomedical Imaging Center, Rensselaer Polytechnic Institute, Troy 12180, NY, USA

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Abstract

Here, we report that the depth and the width of a neural network are dual from two perspectives. First, we employ the partially separable representation to determine the width and depth. Second, we use the De Morgan law to guide the conversion between a deep network and a wide network. Furthermore, we suggest the generalized De Morgan law to promote duality to network equivalency.

1 Introduction

Recently, deep learning \cite{14,26} has become the mainstream of machine learning and achieved the state-of-the-art performance in many important applications \cite{7,10,25,42}. One of the key reasons that accounts for the successes of deep learning is the increased depth, which allows a hierarchical representation of features. There are a number of papers dedicated to explaining why deep networks are better than shallow ones. Encouraging progresses have been made along this direction. The idea to show the superiority of deep networks is basically to find a special family of functions that are very hard to be approximated by a shallow network but easy to be approximated by a deep network, or that a deep network can express more complicated functions than a wide network using some complexity measure \cite{4,9,12,28,30,31,34,38}. For example, in \cite{12} a special class of radial functions was so constructed that a one-hidden-layer network needs to use an exponential number of neurons to do a good approximation, but a two-hidden-layer network just needs a polynomial number of neurons. With the number of linear regions as the complexity measure, Montufar et al. \cite{31} showed that the number of linear regions grows exponentially with the depth of a network but only polynomially with the width of a network. In \cite{4}, a topological measure was utilized to characterize the complexity of functions. It was shown that deep networks can represent functions of much higher complexity than what the shallow counterparts can. Although both theoretical insights and real-world applications suggest that deep networks are better than shallow ones, the width effects of networks are also increasingly recognized.

In the last eighties, a one-hidden-layer network with sufficiently many neurons was shown to have a universal approximation ability \cite{17,20}. Clearly, either width and depth can enable a sufficient representation ability. In recent years, the term ‘wide/broad learning’ was coined to complement deep learning. In Cheng et al. \cite{8}, a wide network and a deep network were conjugated to realize the memorization (wide network) and generalization (deep network) in a recommender. Challenged by the long training time and large numbers of parameters of deep networks, Chen et al. \cite{6} proposed to use a broad random vector functional-link neural network for broad learning. Zagoruyko and Komodakis \cite{44} designed a novel architecture in which the depth of a residual network is decreased while the width of a residual network is increased, producing a far better performance than that of commonly used thin and deep networks. Intuitively speaking, a wide network and a deep network should be complementary. In this study, we demonstrate via analysis and simulation that width and depth are actually dual from two perspectives: functional representation and Boolean algebra.
In the first aspect, we correlate the width and depth of a network to the structure of a function to be approximated. The class of partially separable multivariate functions [29] allows that every continuous \(n\)-variable function \(f\) on \([0, 1]^n\) can be represented in the \(L_1\) sense:

\[
\int_{(x_1, \ldots, x_n) \in [0, 1]^n} |f(x_1, \ldots, x_n) - \sum_{i=1}^L \prod_{l=1}^n \phi_{i_l}(x_i)| < \sigma.
\]  

(1)

where \(\sigma\) is an arbitrarily small positive number, \(\phi_{i_l}\) is a continuous function, and \(L\) is the number of products.

Previously, our group designed the quadratic neuron that replaces the inner product in a conventional neuron with a quadratic function. The resultant quadratic neuron is more expressive than the conventional neuron. In a nutshell, each continuous function \(\phi_{i_l}\) can be approximated by a polynomial of some degree. Based on the Algebraic Fundamental Theorem [37], each polynomial can be factorized as the product of quadratic terms, which can be appropriately represented by quadratic neurons. As a consequence, in such a quadratic representation scheme, the width and depth of a network structure must reflect the complexity of \(\sum_{i=1}^L \prod_{l=1}^n \phi_{i_l}(x_i)\).

In the second perspective, we study the conversion between the width and depth of a neural network from the duality offered by the De Morgan law:

\[
A_1 \cup A_2 \cdots \cup A_k \cdots \cup A_n = \overline{A_1} \cap \overline{A_2} \cdots \cap \overline{A_k} \cdots \cap \overline{A_n},
\]  

(2)

where \(A_i\) is a propositional rule. It is a common sense that a neural network can be linked to a rule-based system such as a collection of propositional rules: IF (\(input \in [a_i, b_i]^m\)), THEN class is \(C_j\). We can illustrate that a deep network is to realize a union of propositional rules (left side) and such a network is equivalent to a wide network that realizes the complement of the intersection of those rules after complement (right side). Furthermore, each rule is a local function over a decision interval. Suppose that the decision intervals for any two rules do not overlap, that is, \(A_i \cup A_j = A_i + A_j, i \neq j\), then we can generalize the De Morgan law to

\[
\begin{align*}
A_1 \cup A_2 \cdots \cup A_k \cdots \cup A_n &= A_1 \cap A_2 \cdots \cap A_k \cdots \cap A_n, \\
= \sum_{i=1}^M A_i \cup \cdots \cup A_{i_q} + \sum_{j=1}^N A_j \cup \cdots \cup A_{j_q} \\
= \sum_{i=1}^M \overline{A_i} \cap \cdots \cap \overline{A_{i_q}} + \sum_{j=1}^N \overline{A_j} \cap \cdots \cap \overline{A_{j_q}} \\
= A_1 \cap A_2 \cdots \cap A_k \cdots \cap A_n
\end{align*}
\]  

(3)

To represent \(\sum_{i=1}^M \overline{A_i} \cap \cdots \cap \overline{A_{i_q}} + \sum_{j=1}^N A_j \cup \cdots \cup A_{j_q}\), the \(M\) wide sub-networks and the \(N\) deep sub-networks should be combined together, which demonstrates that a wide network and a deep network can be converted step by step, hence are actually equivalent to one another in the view of Boolean algebra. Therefore, with the generalized De Morgan law, we extend the network duality to the network equivalency.

In summary, our main contribution is the establishment of the width-depth duality of neural networks from two complementary angles. To put our contributions in perspective, we would like to mention relevant studies. Kawaguchi et al. [22] analyzed the effect of width and depth on the quality of local minima. They showed that the quality of local minima improves toward the global minima as depth and width becomes larger. Daniely et al. [11] shed light on the duality between neural networks and compositional kernel Hilbert spaces using an acyclic graph that can succinctly delineate neural networks and compositional kernels in a unified framework. From the physical point of view, Georgiev [18] discussed the duality of observables (for example, image pixels) and observations in machine learning. To our best knowledge, this work is the first that reveals the width-depth duality of neural networks.

## 2 Perspective I: Functional Approximation

Let us first introduce necessary preliminaries.
**Quadratic Neuron**\(^\text{[13-14,16]}\): The operation integrating \(n\) input variables to a quadratic/second-order neuron before being nonlinearily processed is expressed as

\[
 h(x) = \left( \sum_{i=1}^{n} w_{ir} x_i + b_r \right) \left( \sum_{i=1}^{n} w_{is} x_i + b_s \right) + \sum_{i=1}^{n} w_{ib} x_i^2 + c \tag{4}
\]

where \(x\) denotes the input vector, \(w_r, w_s, w_b\) are vectors of the same dimensionality as that of \(x\), and \(b_r, b_s, c\) are biases. Our quadratic function definition only utilizes \(3n\) parameters, which is sparser than the general second-order representation which requires \(\frac{n(n+1)}{2}\) parameters. While our quadratic neuron design is unique, other designs of quadratic neurons were also reported in the later literature; for example, Tsapanos et al.\(^\text{[40]}\) proposed a type of neurons with paraboloid decision boundaries, which is a special case of our design.

**Algebraic Structure:** Any univariate polynomial of degree \(N\) can be perfectly computed by a quadratic Rectified Linear Unit (ReLU) network with the depth of \(O(\log_2(N))\) and width of no more than \(N\).

Proof: According to the Algebraic Fundamental Theorem\(^\text{[37]}\), a univariate polynomial of degree \(N\) can be expressed as \(P_N(x) = C \prod_{i=1}^{N} (x - x_i) \prod_{j=1}^{l_1} (x^2 + p_j x + q_j)\), where \(l_1 + 2l_2 = N\). Because two ReLU neurons of opposite phases form the identical mapping, we use every pair of neurons of opposite phases in the first layer to compute \((x - x_i), (x - x_i)\); \((x - x_i)(x - x_{i+1})\); or \(x^2 + p_i x + q_i\) and the second layer has a half the number of neurons of the first layer to combine the outputs of the first layer. By repeating such a process, with the depth of \(O(\log_2(N))\) and the width of \(O(N)\) a quadratic network can exactly represent \(P_N(x)\).

**Weierstrass Approximation Theorem**\(^\text{[5]}\): Suppose that \(f\) is a continuous real-valued function supported on \([a, b]\), then for any given \(\sigma > 0\), there exists a polynomial \(P(x)\) satisfying that

\[
 \sup_{x \in [a, b]} |f(x) - P(x)| < \sigma \tag{5}
\]

**Extreme Value Theorem**\(^\text{[5]}\): For a continuous map \(f : [0, 1] \rightarrow R\), there exists a constant \(M > 0\) such that \(|f(x)| \leq M\) for all \(x \in [0, 1]\).

**Partially Separable Representation:** Approximating a multivariate function \(f(x_1, x_2, ..., x_n)\) by a set of functions of fewer variables is a basic problem in approximation theory\(^\text{[29]}\). Despite that some function \(f\) is directly separable, such as

\[
 f(x_1, x_2, ..., x_n) = \phi_1(x_1) \phi_2(x_2) \cdots \phi_n(x_n), \tag{6}
\]

a more general formulation to express a multivariate \(f\) is

\[
 f(x_1, x_2, ..., x_n) = \sum_{i=1}^{L} \prod_{k=1}^{n} \phi_{ik}(x_k). \tag{7}
\]

If \(L\) is permitted to be very large, then such a model is rather universal. For bivariate functions, an inspiring theorem has been proved\(^\text{[29]}\): Let \(\{u_n\}_{n \in N}\) and \(\{v_n\}_{n \in N}\) be orthonormal bases of \(L^2(X)\) and \(L^2(Y)\) respectively, then \(\{u_m v_n\}_{(m,n) \in \mathbb{N}^2}\) is an orthonormal basis of \(L^2(X \times Y)\).

For a general multivariate function, we relax the restrictive equality to the approximation in the \(L_1\) sense and assume that, for every continuous \(n\)-variable function \(f(x_1, ..., x_n)\) on \([0, 1]^n\), given any positive number \(\sigma\), there exists a group of \(\phi_{ik}\), satisfying:

\[
 \int_{[x_1, ..., x_n] \in [0,1]^n} |f(x_1, ..., x_n) - \sum_{i=1}^{L} \prod_{k=1}^{n} \phi_{ik}(x_k)| < \sigma. \tag{8}
\]

**Analysis:** Because \(f(x_1, ..., x_n)\) is continuous in a closed space, \(f(x_1, ..., x_n)\) is bounded, and \(f(x_1, ..., x_n)\) is Lebesgue-integrable. Non-trivially, we can construct a generic separable representation as \(\sum_{i=1}^{L} f(a_{i1}, a_{i2}, ..., a_{in}) h(x_1 - a_{i1})h(x_2 - a_{i2})...h(x_n - a_{in})\), where \(h(x - a_{il})\) can be adjusted by \(\delta\), and expressed as

\[
 \begin{align*}
 1 - \frac{a_{il} - a_{il} - x_i}{\delta} &\quad x_i \in [a_{il} - 2\delta, a_{il} - \delta] \\
 1 &\quad x_i \in [a_{il} - \delta, a_{il} + \delta] \\
 1 - \frac{x_i - a_{il} - \delta}{\delta} &\quad x_i \in [a_{il} + \delta, a_{il} + 2\delta]
\end{align*} \tag{9}
\]
\[ \prod_{i=1}^{n} h(x_i - a_i) \text{ forms a } n \text{-dimension trapezoidal nearly unit cube, and } L \text{ depends on the sampling rate. By increasing sampling more } a_{il} \text{ parameters with respect to } x_i \text{ and reducing } \delta, \sum_{i=1}^{L} f(a_{1i}, a_{2i}, \ldots, a_{ni})h(x_1 - a_{1l})h(x_2 - a_{2l})\ldots h(x_n - a_{nl}) \text{ can approach } f(x_1, \ldots, x_n) \text{ well in the } L_1 \text{ sense.} \]

Since a quadratic network can represent a univariate polynomial and a multiplication operation, the quadratic networks with the structure shown in Figure 1 should allow universal approximation of any continuous multivariate function. The logic goes as follows. First, we know that \( \sum_{i=1}^{L} L_{il} \phi_l(x_i) \) can represent \( f(x_1, \ldots, x_n) \), and every \( \phi_l \) can be approximated by a polynomial, which can be represented by a quadratic network through factorization. Therefore, these quadratic networks can represent any continuous multivariate function. Immediately, we have the following conjecture:

**Conjecture 1:** A quadratic network is a universal approximator.

**Analysis:** Within a given arbitrarily small margin \( \sigma_{il} \), every function \( \phi_{il}(x_i) \) corresponds to a polynomial \( P_{il}(x_i) \) expressed by a quadratic network according to Weierstrass Approximation Theorem:

\[
\sup_{x_i \in [0, 1]} |\phi_{il}(x_i) - P_{il}(x_i)| < \sigma_{il}, \quad (10)
\]

Since a quadratic neuron comprises multiplication operation, we can use quadratic networks to represent the product of \( \phi_{il}, i = 1, \ldots, n \). Suppose \( \phi_{il}(x_i) = P_{il}(x_i) + \alpha_{il} \), where \( |\alpha_{il}| < \sigma_{il} \) is an auxiliary constant, then it holds that, for \( (x_1, \ldots, x_n) \in [0, 1]^n \),

\[
\sup_{(x_1, \ldots, x_n) \in [0, 1]^n} \left| \prod_{i=1}^{n} \phi_{il}(x_i) - \prod_{i=1}^{n} P_{il}(x_i) \right| \leq \sup_{(x_1, \ldots, x_n) \in [0, 1]^n} \left| \prod_{i=1}^{n} (P_{il}(x_i) + \alpha_{il}) - \prod_{i=1}^{n} P_{il}(x_i) \right| < \sup_{(x_1, \ldots, x_n) \in [0, 1]^n} \left| \sum_{k \neq i} \prod_{k \neq i} P_{ik}(x_k) \alpha_{il} + \Delta \right| < \sup_{(x_1, \ldots, x_n) \in [0, 1]^n} \left| \sum_{k \neq i} \prod_{k \neq i} P_{ik}(x_k) \alpha_{il} \right| + |\Delta| \quad (11)
\]

where \( \Delta \) represents the sum of higher order terms of \( \alpha_{il} \). In reference to the Extreme Value Theorem, it is known that each \( P_{lk} \) is bounded. Thus, making \( \sigma_{il} \) sufficiently small will compress \( \sum_{i=1}^{n} |\prod_{k \neq i} P_{ik}(x_k) \alpha_{il}| \) and \( |\Delta| \), which brings us to:

\[
\sup_{(x_1, \ldots, x_n) \in [0, 1]^n} \left| \prod_{j=1}^{n} \phi_{lj}(x_j) - \prod_{j=1}^{n} P_{lj}(x_j) \right| < \sigma/L. \quad (12)
\]
Next, we integrate $L$ parts as shown in Figure 1 and obtain that

\[
\sup_{(x_1, \ldots, x_n) \in [0,1]^n} \left| \sum_{l=1}^{L} \prod_{i=1}^{n} \phi_l(x_i) - \sum_{l=1}^{L} \prod_{i=1}^{n} P_l(x_i) \right| < \sigma \tag{13}
\]

Finally, because each $P_l$ can be exactly represented by a quadratic neural network through factorization, and the partially separable representation can approximate any multivariate continuous function in the $L_1$ sense. It is concluded that a quadratic network is a universal approximator with a separable structure.

**Complexity:** Now, let us analyze the complexity of the aforementioned representation scheme. Figure 2 shows that such a universal approximator can be either wide or deep. Suppose that the polynomial $P_l$ is of degree $N_{li}$, then the representation of each $P_l$ can be done with a network with the width of $\sum_{l=1}^{L} \sum_{i=1}^{n} N_{li}$ and a depth of $\max_{l,i} \{ \log_2(N_{li}) \}$. Next, the multiplication demands an additional network with the width of $L n$ and a depth of $\log_2(n)$. Therefore, the overall quadratic network architecture will be of width $\max \left\{ \sum_{l=1}^{L} \sum_{i=1}^{n} N_{li}, L n \right\}$ and depth $\max_{l,i} \{ \log_2(N_{li}) \} + \log_2(n)$. Because the depth scales with a log function, which changes slowly when input variable is large. For simplicity, we take an approximation for depth $\max_{l,i} \{ \log_2(N_{li}) \} + \log_2(n) = \log_2(\max_{l,i} \{ N_{li} \}) + \log_2(n) \approx \log_2 \left( \sum_{l=1}^{L} \sum_{i=1}^{n} N_{li} \right) + \log_2(n)$. Let $\sum_{l=1}^{L} \sum_{i=1}^{n} N_{li} = N^\Sigma$, then the formulas to compute the width and depth are simplified as follows

\[
\begin{align*}
\text{Width} & = \max \{ N^\Sigma, L n \} \\
\text{Depth} & = \log_2(N^\Sigma) + \log_2(n) \tag{14}
\end{align*}
\]

One interesting point from Eq. [14] is that the lower bounds for depth and width to realize a partially separable representation are also suggested. As shown in Figure 3, we plot the dynamics of width and depth as $N^\Sigma$ changes. There are two highlights in Figure 3. The first is that the width is generally larger than the depth, which is different from the superficial impression on deep learning. The second is that, as the $N^\Sigma$ goes up, the width/depth ratio is accordingly increased.

**Remark 1:** Through the universal approximation and complexity analysis, we realize that the width and depth of a network depend on the structure/complexity of the function to be approximated. In other words, they are...
Table 1: Descriptions of different building blocks.

| Modules | Degree | Operation | Width | Depth |
|---------|--------|-----------|-------|-------|
| I       | $N_{li}$ | Express $\phi_{li}$ | $N_{li}$ | $\log_2(N_{li})$ |
| II      | $n$    | Express $\Phi_i$ | $n$  | $\log_2(n)$ |

Figure 3: The relation of width and depth as $N^\Sigma$ changes ($L = 4$, $n = 5$).

controlled by the nature of a specific task. As the task becomes complicated, the width and depth must increase accordingly. It is worthwhile mentioning that the partially separable representation is not the only vehicle for functional approximation. For example, we have the Kolmogorov-Arnold representation theorem [24]: Every continuous $n$-variable function on $[0, 1]^n$ can be represented in the form:

$$f(x_1, ..., x_n) = \sum_{i=1}^{2^n+1} \Phi_i \left( \sum_{j=1}^n \phi_{ij}(x_j) \right), \tag{15}$$

where $\phi_{ij}$ and $\Phi_i$ are continuous univariate functions.

**Functional fitting experiment:** Let us use a numerical example to investigate the effect of the ratio of width and depth on functional fitting. The 10,000 instances were sampled from the function $f(x, y) = \phi_1(x) + \phi_2(y)$ on $[0, 1] \times [0, 1]$, where $\phi_1, \phi_2$ are two polynomials of degree=8, and the coefficients of $\phi_1, \phi_2$ were randomly initialized by a Gaussian distribution (mean=0, variance=1). Because $f(x, y)$ is separated into two parts, we first used two concurrent quadratic sub-networks that respectively process two inputs and then aggregated the outcomes of two networks in the final layer. The utilized structures of each sub-network are 2-1-1, 4-2-1, 6-3-1, 8-4-1 and 10-5-1, which represent different width/depth ratios respectively. In the experiment, to examine the representation power of the networks thoroughly, we randomly initialized $\phi_1, \phi_2$ for 10 times. Figure 4 shows the mean square errors (MSEs) of different architectures by averaging results from multiple random initializations. It can be seen that the network 2-1-1, 4-2-1 and 6-3-1 yielded large MSE values. However, much improvements were made when the number of neurons in the first layer reaches 8, which is the degree of $\phi_1, \phi_2$. The MSE values of the networks 8-4-1 and 10-5-1 are 0.0542 and 0.0759 respectively. More experimental details are deferred to Supplementary Information. Such improvements show that width of the networks should match the complexity of the function to be approximated, which suggests that the width/depth ratio of a network should reflect the nature of a task for a good fitting.

**Perspective II: Boolean Algebra**

As an important direction for interpretable/explainable neural networks [1][15], rules can be extracted from an artificial neural network [35][36][39], i.e., using decompositional [39] and pedagogical methods [39]. Pedagogical methods decode a set of rules that imitate the input-output relationship of a network. These rules do not necessarily
correspond to the parameters of the network. One common type of rules are propositional in the IF-THEN format, where the preconditions are provided as a set of hypercubes with respect to some input:

\[
\text{IF} \ (\text{input} \in [a_i, b_i])^m, \ \text{THEN} \ \text{class is } C_j.
\]

The rule-based system reveals the linkage between the rule-based inference and the neural inference. Such a link goes even further for some special kind of networks. It is known that a radial-basis-function network is equivalent to a fuzzy Takagi-Sugeno rule system under mild modifications \[21\]. Therefore, it is rationale to consider a neural network in terms of propositional rules. Furthermore, we know that the De Morgan law holds true for propositional rules. Mathematically, the De Morgan law is formulated as:

\[
A_1 \cup A_2 \cup \cdots \cup A_k \cup \cdots \cup A_n = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k} \cap \cdots \cap \overline{A_n},
\]

where \(A_i\) is a propositional rule, the bar represents the negation operation. The De Morgan law gives a duality in the sense of binary logic that the operation \(\cap\) and \(\cup\) are dual, which means that for any propositional rule system described by \(A_1 \cup A_2 \cup \cdots \cup A_k \cup \cdots \cup A_n\), there exists an equivalent dual propositional rule system \(\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k} \cap \cdots \cap \overline{A_n}\). Since a neural network is considered as a propositional rule based system, the duality in propositional rule systems indicates that the duality should exists in neural networks. We further point out that the duality of the depth and width of a network is reminiscent of the duality in Eq. \[16\]. Specifically, we can always construct a deep network that implements the union of propositional rules and an equivalent wide network that implements the complement of the intersection of propositional rules after complement.

For simplicity, let us consider a single-variable binary classification problem, although our construction can be readily scaled to cases of multiple variables. Suppose that \(A_i\) is a rule defined as:

\[
\text{IF} \ (x \in [a_i, b_i]), \ \text{THEN} \ \text{class is } 1.
\]

Without loss of generality, we assume that any two intervals do not overlap with each other. As shown on the left side of Figure 5, the \(i^{th}\) network block (green rectangle) with three layers can represent the following function:

\[
1 - \frac{a_i - x}{\delta}, \quad x \in [a_i - \delta, a_i],
\]

\[
1, \quad x \in [a_i, b_i],
\]

\[
1 - \frac{x - b_i}{\delta}, \quad x \in [b_i, b_i + \delta],
\]

which gives a good approximation to \(A_i\) since \(\delta\) can be made arbitrarily small. The output of the \(i^{th}\) block is \(A_1 \cup A_2 \cup \cdots \cup A_i\). As the network goes deeper, more blocks are stacked together to include more propositional rules that lie in different intervals. In this process, these rules are integrated for the purpose that \(A_1 \cup A_2 \cup \cdots \cup A_k \cup \cdots \cup A_n\) is represented.

Now, we show how to use a wide network to construct \(\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k} \cap \cdots \cap \overline{A_n}\). As shown on the right
Figure 5: The deep network to implement $A_1 \cup A_2 \cup \cdots \cup A_n$ and the wide network to implement $\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$.

Side of Figure 5, the $i^{th}$ block (blue rectangle) that represents $\overline{A_i}$ can be defined as:

$$
\begin{cases}
1 & x \in (-\infty, a_i - \delta] \\
1 - \frac{x-a_i}{\delta} & x \in [a_i - \delta, a_i] \\
0 & x \in [a_i, b_i] \\
1 - \frac{b_i-x}{\delta} & x \in [b_i, b_i + \delta] \\
1 & x \in [b_i + \delta, \infty]
\end{cases}
$$

Concurrently, the $n$ blocks can represent $\overline{A_1}, \cdots, \overline{A_n}$ respectively. After all $\overline{A_i}$ rules are prepared, with the output neuron we perform the intersection and complement. To this end, a summation is conducted across all the inputs before the thresholding operation. After thresholding, a complement is conducted. The threshold is set as $n - 1$ for $n$ propositional rules. Specifically, we have

$$
\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k} \cap \cdots \cap \overline{A_n} = 1 - (\overline{A_1} + \overline{A_2} + \cdots + \overline{A_n} - (n - 1)).
$$

In summary, a deep network is dual to a wide network by our construction. As far as multi-variable propositional rules, IF ($input \in [a_i, b_i]^{m}$) THEN class is 1, are concerned, each rule can be decomposed as a set of single variable propositional rules and implemented by concatenating more network blocks.

**Remark 2:** In the light of the De Morgan law, the binary judgement serves as the language for the conversion of the width and depth. Suppose the input domain is bounded, it is easy to show that given sufficiently many rules, the union of these rules can tackle with any binary classification problem, which means that the duality presented here is quite general for pattern recognition tasks.

**Generalized De Morgan law:** Let us further examine $A_1 \cup A_2 \cdots \cup A_k \cdots \cup A_n$. Each rule $A_i$ is functionally equivalent to a local function over a decision interval, which we denote it as $A_i$. Due to the exclusiveness of these intervals, we have that $A_i \cup A_j = A_i + A_j$, $i \neq j$, indicating the union of two rules equals to the addition of two
Figure 6: The network architectures comprising of a deep network, a wide network and others between them were used in breast cancer example.

local functions. This fact immediately leads us to the generalized De Morgan law

\[
A_1 \cup A_2 \cdots \cup A_k \cdots \cup A_n \\
= \sum_{i=1}^{M} A_{i_1} \cup \cdots \cup A_{i_q} + \sum_{j=1}^{N} A_{j_1} \cup \cdots \cup A_{j_q} \\
= \sum_{i=1}^{M} \overline{A_{i_1} \cap \cdots \cap A_{i_q}} + \sum_{j=1}^{N} \overline{A_{j_1} \cup \cdots \cup A_{j_q}} \\
= \overline{A_1 \cap A_2 \cdots \cap A_k \cdots \cap A_n}
\]

(20)

For example,

\[
A_1 \cup A_2 \cdots \cup A_k \cdots \cup A_n \\
= A_1 \cup A_2 + A_3 \cup \cdots A_k \cdots \cup A_n \\
= \overline{A_1 \cap A_2 + A_3 \cup \cdots A_k \cdots \cup A_n}.
\]

(21)

Eq. [21] is interesting because it suggests the step-wise continuum of mutually equivalent networks, which comprises not only a deep network and a wide network shown in Figure 5 but also the transitional networks between them. In order to express \( \sum_{i=1}^{M} \overline{A_{i_1} \cap \cdots \cap A_{i_q}} + \sum_{j=1}^{N} \overline{A_{j_1} \cup \cdots \cup A_{j_q}} \), a generic network is constructed with \( n \) blocks, each block representing either \( \overline{A_{i_1}} \) or \( A_{j_q} \) to form the width and depth according to the generalized De Morgan law.

**Experiment:** The eight network architectures were constructed, including a deep network, a wide network and others between them. In reference to Figure 5, each green box contains four layers with three neurons in each layer while each blue box contains three layers with two neurons per layer. The breast cancer dataset covers 30 attributes characterizing two pathological features categories: "Benign" and "Malignant". With breast cancer
Wisconsin dataset as an example, we only used five attributes (radius, texture, perimeter, area, smoothness) to show the equivalency of networks. We randomly initialized all the parameters of each network configuration 20 times. The mean classification accuracy and the corresponding standard deviation of each network were tabulated in Table 2. The takeaway from Table 2 is that the performance difference between any two networks is no more than 0.0102, which means that eight networks performed rather comparably. We did the independent samples t-test for any pairs of experimental results of the networks, and the lowest p-value 0.0517 is from the results of networks II and V, which is not sufficient to reject that the networks II and V differently. More detailed discussions are in the Supplementary Information. Such similarity suggests the existence of equivalent networks. In this regard, duality between a wide network and a deep network becomes a special case of a family of equivalent networks in the light of Boolean algebra.

### Discussions

#### VC Dimension:

In addition to the above two perspectives, the duality of depth and width of a network can be appreciated in reference to VC dimension [41]. VC dimension is a measure on the capacity of a hypothesis space that can be learned by a neural network and other statistical learning models. The definition of VC dimension is based on the cardinality of the largest set of points that can be "shattered" by a model. Since VC dimension is originally geared towards binary classification, let us give the definition of VC dimension in the context of binary classification.

Given the hypothesis space $\mathcal{H}$ and the data collection $D = \{ x_1, x_2, \cdots, x_m \}$, every $h \in \mathcal{H}$ labels the instances in $D$, we denote the resultant labeling as

$$h|_D = \{ (h(x_1), h(x_2), \cdots, h(x_m)) \},$$

(22)

It is seen that, for $m$ instances, the maximum number of possible labeling schemes is $2^m$. If the hypothesis space $\mathcal{H}$ can realize all the possible labeling schemes on $D$, we say that $D$ can be shattered by $\mathcal{H}$. VC dimension of $\mathcal{H}$ is the cardinality of the dataset comprising the maximum number of instances that can be shattered.

The bound of VC dimension of a neural network has been extensively studied in the past decades. For example, let $W$ be the number of weights and $L$ be the number of layers, Goldberg and Jerrum [19] established the upper bound of VC dimension $O(W^2)$ for a network with piecewise polynomial activation. Furthermore, Bartlett et al. [3] found that an upper VC dimension bound is $O(WL^2 + W L \log(WL))$ and a lower bound is $O(WL)$. Recently, Bartlett et al. [2] derived a nearly tight VC dimension bound for ReLU networks:

$$c \cdot W L \log(W/L) \leq VCdim(W,L) \leq C \cdot W L \log(W),$$

(23)

where $c$ and $C$ are constants.

Eq. [23] implies that the above upper and lower bounds for VC dimension of a neural network are determined by the number of parameters and the number of layers. As a semi-quantitative estimate, let all the layers have the same number of neurons, then we have

$$VCdim(\text{Width}, \text{Depth}) \geq c \cdot \text{Width} \cdot \text{Depth} \cdot \log(\text{Width})$$

$$VCdim(\text{Width}, \text{Depth}) \leq C \cdot \text{Width} \cdot \text{Depth} \cdot \log(\text{Width})$$

(24)
Therefore, increasing either width or depth can boost the hypothesis space of a neural network. In other words, when it comes to promoting the expressive power of a network, increasing the width is essentially equivalent to increasing the depth in the sense of VC dimension, which also implies the duality of the width and depth of neural networks.

**Continued Fraction:** Mathematically, a continued fraction representation of a general polynomial can be expressed as:

\[ \sum_{i=0}^{n} a_i x^i = \frac{b_0}{1 + \frac{b_1 x}{1 + \frac{b_2 x}{1 + \cdots}}} \]

(25)

where \(a_i, b_i\) are related by \(b_0 = a_0, b_1 = a_1/a_{i-1}, i > 1\). By performing the polynomial factorization of the left side of Eq. [25], we have

\[ C \prod_{i} (x - x_i) \prod_{j} (x^2 + p_j x + q_j) = \frac{b_0}{1 + \frac{b_1 x}{1 + \frac{b_2 x}{1 + \cdots}}} \]

(26)

where \(x_i\) is the root of the polynomial, \(p_j, q_j\) are constants, and \(l_1 + 2l_2 = n\). Note that \(x_i, p_j, q_j\) and \(b_i\) are also related since \(x_i, p_i, q_i\) are related with \(a_i\). Eq. [26] is intriguing, as we can use a quadratic network to represent the left side of Eq. [26] based on the polynomial factorization. As far as the right side of Eq. [26] is concerned, the composition nature of a deep network represents the recursive structure of the continued fraction. With a wide network for the left side of Eq. [26], the basic computational block is \(Q_1(x) = x - x_i\) or \(Q_2(x) = x^2 + p_j x + q_j\).

With a deep network on the right side of Eq. [26], the basic computational block is \(f_{i+1}(x) = P_{i}(x) = \frac{b_{x} x^2}{1 + b_{2} x - f_{i}(x)}\) or \(f_1 = \frac{b_{x}}{1 - b_{0}}\). Furthermore, in terms of notations \(Q_i\) and \(P_i\), we condense Eq. [26] into a novel duality, referred to as the multiplication-composition duality:

\[ Q_1 \times Q_2 \times \cdots Q_l \times \cdots = P_1 \circ P_2 \circ \cdots P_l \circ \cdots \]

(27)

which can be viewed as a variant of the De Morgan law. In this sense, multiplication and composition operations correspond to \(\cup\) and \(\cap\) operations respectively, and the objects of those operations become global functions instead of local rules, as we have modeled above.

**Equivalent Networks:** In a broader sense, our duality studies demonstrate that there are mutually equivalent networks. The equivalency between two networks implies that given any input, two networks produce the same output. The network equivalency is useful in many ways, for example, in network optimization. Although deep networks manifest superb power, their applications can be constrained due to the computational complexity; i.e., when the memory capacity is low or when the application is time-critical. The goal of network optimization is to derive a best network that maintain as high accuracy as delivered by a complicated network, through quantization [43], pruning [27], distillation [32], binarification [33], low-rank approximation [45], etc. We envision that the duality of a deep network and a wide network suggests a new direction of network optimization in a task-specific fashion. Ideally, a wide network is able to replace the well-trained deep network without compromising the performance. Since the parameters of the wide network are paralleled, the wide network can be trained on a computing cluster with many machines, which facilitates the fast training and inference. Also, with a wide network of few layers, the number of feed forward operations becomes less for fast computation.

**Conclusion**

In conclusion, we have presented the duality between the depth and width of a neural network from two perspectives: functional approximation and Boolean algebra, and further we have extended duality to network equivalency.
Due to the great potential for this duality/equivalency theory to be scaled into real-world applications, more efforts will be put into revealing more important equivalent networks in the future.

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