Phases of SU(3) Gauge Theories with Fundamental Quarks via Dirac Spectral Density

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Abstract

We suggest that gauge interactions of SU(3) gluons and fundamental quarks produce three distinct types of infrared behavior in Dirac spectral density \( \rho(\lambda, V \to \infty) \) (Fig.1), effectively labeling three types of dynamical phases occurring in these theories. The two monotonic (standard) cases entail confinement with chiral symmetry breaking and the lack of both, respectively. The bimodal (anomalous) option signifies deconfined phase with broken chiral symmetry. This generalization rests on the following. (α) We show, via numerical simulation, that previously observed bimodal behavior in \( N_f=0 \) theory past deconfinement temperature \( T_c \) is (separately) stable with respect to both infrared and ultraviolet cutoffs, concluding that this prototypical anomalous phase indeed exists. The width of the anomalous peak, while small (few MeV at \( T/T_c = 1.12 \)), is non-zero in the infinite-volume limit. (β) We show in detail that transition to bimodal \( \rho(\lambda) \) in \( N_f=0 \) coincides with \( Z_3 \) deconfinement transition: anomalous phase occurs for \( T_c < T < T_{ch} \), with \( T_{ch} \) the temperature of valence chiral restoration. (γ) We present evidence for thermal anomalous phase in \( N_f=2+1 \) QCD at physical point. Its onset coincides with conventional crossover (“\( T_c \)”), and we conclude that anomalous regime \( T_c < T < T_{ch} \) is very likely a reality of nature’s strong interactions. (δ) Our past studies in the context of zero-temperature \( N_f=12 \) theories revealed that bimodal behavior is also generated by the light-quark effects. Given (γ) and (δ), we expect common occurrence of anomalous phase along a generic path to chiral restoration. As a special case, we predict its existence for \( N_f^c < N_f < N_f^{ch} \) with massless quarks at \( T = 0 \). Here \( N_f^{ch} \) coincides with the onset of conformal window \( N_f^{cr} \), and \( N_f^c \) could be low.

1. Introduction. SU(3) gauge theories with quarks in fundamental representation are important in elementary particle physics. Indeed, the strong dynamics of “real-world” quarks and gluons is believed to be described within this context, and the near-conformal infrared behavior in theories with many massless flavors (just below conformal window) may be relevant for theories of technicolor type. In addition, the systems within conformal window are attractive test beds of conformal dynamics in four dimensional quantum field theory.

Since the quarks of nature are not massless, it is desirable to consider all theories of the above type, i.e. including any number of arbitrarily massive quarks. Denoting such theory space at zero temperature as \( \mathcal{T}_0 \) and at any temperature as \( \mathcal{T} \supset \mathcal{T}_0 \), this wide landscape
Figure 1: Confinement/vSChSB structure of $\mathcal{T}$ via infrared behavior of Dirac spectral density.

involves various kinds of dynamics, expected to be well distinguished by corresponding vacuum properties. In that regard, confinement and spontaneous chiral symmetry breaking (SChSB) dominate the thinking about strongly interacting dynamics. Nevertheless, classifying elements of $\mathcal{T}$ in these terms involves both conceptual and practical issues.

One problem is that SChSB is well-defined only when at least a pair of quarks is massless. However, the vacuum of any theory in $\mathcal{T}$ can be probed for its ability to support long-range order via valence Goldstone pions (see e.g. [1, 2]). Formally, a pair of valence quarks and a pair of action-compensating pseudofermions is added to the system [3]. In the massless valence limit, flavored chiral rotations of valence fields $\eta, \bar{\eta}$ are elevated to symmetries, which the vacuum either respects or not. The latter possibility entails valence spontaneous chiral symmetry breaking (vSChSB), indicated by non-zero value of $\langle \bar{\eta} \eta \rangle$. SChSB and vSChSB are identical notions whenever SChSB is meaningful, but vSChSB provides for a symmetry-based distinction among vacua over the whole set $\mathcal{T}$.

The situation is different in case of confinement, whose meaning is quite intuitive throughout $\mathcal{T}$, but a satisfactory consensus on its precise definition in such general context (even within $\mathcal{T}_0$) is lacking. Moreover, existing definitions are technically difficult to verify in a given theory. Convenient symmetry-based interpretation is only available when quarks do not affect vacuum correlations: in pure glue ($N_f=0$) theories at finite temperature [4, 5]. In this restricted case, the breakdown of $Z_3$ symmetry, signaled by non–zero Polyakov line $\langle L \rangle \neq 0$ or its magnitude $\langle |L| \rangle > 0$, distinguishes confined and deconfined dynamics.

Despite of issues with its formal definition over $\mathcal{T}$, confinement can have simple and unambiguous dynamical signatures. Indeed, since the phenomenon shapes dynamics in a crucial manner, it is reasonable to expect that even simple vacuum observables are qualitatively affected. While details of this influence depend on the unknown specifics of confinement mechanism, the logic here is that of reverse engineering: if simple signatures can be empirically developed, consistently with existing results of lattice simulations and with theoretically clean $N_f=0$ case, then a valuable insight into the mechanism is obtained. In addition, practical benefits for investigating the phase structure of $\mathcal{T}$ are likely to be acquired.

Following such rationale, we propose that vSChSB/confinement structure of $\mathcal{T}$ may be inferred from low-energy behavior of Dirac spectral density $\rho(\lambda)$ in these theories. Gauge invariant $\rho(\lambda)$ is thus viewed as a vacuum object already known to encode vSChSB via its “infinitely infrared” behavior ($\rho(\lambda \rightarrow 0)$ in infinite volume) and Banks-Casher equivalence [6]. The suggestion is that properties of $\rho(\lambda)$ in finite infrared regime reflect vSChSB/confinement
combination as shown in Fig.1: case (A) corresponds to vSChSB with confinement, (B) to vSChSB without confinement, and (C,C’) to no vSChSB and no confinement. Note that (A) represents generic behavior observed in low-temperature “real-world” QCD simulations, and (C,C’) the standard possibilities for chirally symmetric vacuum. We refer to the atypical case (B) as anomalous. Few comments are in order.

(i) Dependencies of Fig.1 refer to limits approached in asymptotically large volumes. Non-monotonic ρ(λ) arising e.g. in small volumes at fixed topological charge are of no interest here. Physically relevant aspect of anomalous behavior relates to infinite-volume bimodality and the depletion of modes at intermediate scales. (ii) The proposed association between vacuum properties and ρ(λ) is dynamical, i.e. enabled by strong forces at work rather than a priori by the setup. (iii) Confinement without vSChSB is not among the possibilities in T, which aligns with conclusions of Ref. [7] regarding SchSB. (iv) The nature of proposed distinction among phases is such that it doesn’t matter whether bare or renormalized [8] spectral density is used. (v) Behavior (B) was first observed on Nf=0 backgrounds [9] past the deconfinement temperature Tc. However, its reality has not been systematically studied amidst suspicions (see e.g. [10]) that it may be an artifact specific to overlap quarks [11]. Substantiating the existence of anomalous phase in T is central to this discussion. (vi) Important input for current generalization was provided by recent observation of anomalous ρ(λ) in zero-temperature quark-gluon dynamics with many light flavors [2, 12]. Thus, both thermal and light-quark effects can induce anomalous phase, and it is reasonable to expect its common occurrence along a path in T leading to restoration of chiral symmetry.

In what follows, we first discuss new results from lattice QCD needed for this proposal, i.e. points (α)-(γ) outlined in the Abstract. We then proceed to generalize (Fig. 6), taking into account (δ), and to suggest the existence of anomalous (deconfined, chirally broken) phase preceding the conformal window. The most significant aspects of the proposed scheme, of the associated consequences, and of the physical results obtained, are elaborated upon last.

2. Reality of the Anomalous Phase. Basic requirement for viability of the proposed general scenario is establishing the existence of deconfined chirally broken dynamics with anomalous spectral density anywhere in T. Thermal Nf=0 theory is ideal for this purpose since all three elements involved are well-defined and lattice-accessible. Working with Wilson’s lattice theory at T/Tc=1.12 (fixed by r0Tc [13]), where we observed bimodality in spectral density previously [1, 2], our aim is to perform crucial stability tests with respect to infrared and ultraviolet cutoffs. Our preliminary work in this direction was reported in Ref. [14]. The results below refer to Z3-broken vacuum with “real Polyakov line” [15], which gets selected when deforming the theory away from Nf=0.

Overlap Dirac operator (ρ = 26/19) constructed from its Wilson counterpart (r = 1) is used to define valence quark dynamics throughout this work, making vSChSB an exact notion at regularized level. Spectral density ρ(λ) is the right derivative of cumulative function σ(λ) ≡ (∑0<λi<λ 1)/V, where V is the 4-volume and λi (real numbers) have magnitudes of Dirac eigenvalues and signs of their imaginary parts. Exclusion of exact zeromodes from counting is harmless since their effect vanishes in the infinite-volume limit. Given the finite statistics of any simulation, coarse-graining of ρ(λ) is unavoidable. In the absence of suspicion

Case (C’) involving strict “gap” in density is very difficult to infer from any practical simulation and may or may not exist in T. However, the distinction between (C) and (C’) is immaterial for our purposes.
for singularity, we use the symmetric definition away from origin, namely

$$\rho(\lambda, \delta) \equiv \frac{\sigma(\lambda + \delta/2) - \sigma(\lambda - \delta/2)}{\delta} \quad \lambda \geq \delta/2$$ (1)

When estimating the infinite-volume value of $\rho(\lambda \to 0)$ from finite systems, it is convenient to work with “right derivative” form, i.e. $\rho(\lambda = 0, \Delta) \equiv \sigma(\Delta) / \Delta$. Valence chiral condensate with overlap is identical to $\pi \lim_{\lambda \to 0} \lim_{V \to \infty} \rho(\lambda, V)$, as is formally in the continuum.

Focusing first on the infrared, we simulated $N^3 \times 7$ systems at $N = 20, 24, 32, 48$. To set $T = 1.12T_c$ requires lattice spacing $a = 0.085$ fm ($\beta = 6.054$, $r_0 = 0.5$ fm), giving linear size $L = 4.1$ fm to the largest system. Low-lying overlap eigensystems were computed and bimodal $\rho(\lambda)$, such as in the top left plot of Fig. 2 was found at all volumes. The spectrum exhibits excellent volume scaling, exemplified in the bottom left plot except for the most infrared point, which follows a growing trend ensuring the anomalous shape (B) in the infinite-volume limit. The coarse-graining $\delta = 20$ MeV is sufficiently fine except for the most infrared bin (0, 20) MeV. Here $\rho(\lambda)$ changes rapidly, as the closeup with $\delta = 4$ MeV (top middle) shows. Further lowering of $\delta$ (top right) only affects the most infrared point, and we have thus arrived at the resolution revealing the shape of the anomalous peak. Importantly, this shape is also stable under the change of infrared cutoff: comparing $N = 32$ to $N = 48$ (lower middle vs top middle), only accumulation in the lowest bin changes, growing again. We conclude that, in the infinite volume, $\rho(\lambda)$ has a positive (possibly infinite) local maximum at $\lambda = 0$, following behavior (B), with anomalous peak of finite width (few MeV).

While anomalous behavior (B) alone implies it, we address vSchSB explicitly (lower right plot of Fig. 2) via $\Delta$-dependence of $\rho(\lambda = 0, \Delta)$. Note that, in any finite volume, $\lim_{\Delta \to 0} \rho(0, \Delta, V) = 0$, and the associated downturn, shown for $N = 32$, will occur at sufficiently small $\Delta$. However, as expected in broken theory, the break is pushed toward zero with increasing volume. Moreover, the estimates at any fixed $\Delta$ are growing, making vSchSB all

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3 The horizontal lines guiding the eye are fits to a constant from three largest systems.
but certain. At $T = 1.12T_c$ the system is deconfined, and thus all elements of the proposed connection are in place for this case.

Turning to ultraviolet cutoff, we fixed the physical volume of $N = 24$ theory ($L = 2.04$ fm), and drove the system toward continuum limit, while keeping $T/T_c = 1.12$. This resulted in sequence $N \times N_1 = 24 \times 7$, $28 \times 8$, $34 \times 10$, $42 \times 12$ at $a = 0.0850$, $0.0744$, $0.0595$, $0.0496$ fm respectively. The bimodal $\rho(\lambda)$ was found at all cutoffs, with global view for the system closest to the continuum displayed in Fig. 3 (top left). The associated closeup with proper resolution is also shown (top middle). Note that, contrary to changing the volume, varying ultraviolet cutoff is not expected to affect very infrared scales significantly, unless a qualitative change of dynamics (phase transition) occurs, separating lattice and continuum-like behaviors. It is decisively the former scenario that is observed, and illustrated by comparisons to the situation at smaller cutoff (lower left and middle plots). The scaling of first and second data point at $\delta = 20$ MeV (top right) indicates that the bimodal shape of $\rho(\lambda)$ will be preserved in the continuum limit at this resolution. Moreover, the accumulation of very near-zero-modes, i.e. $\rho(0, \Delta)$ with resolution well within the natural width of the anomalous peak ($\Delta = 4$ MeV), is stable with cutoff (lower right plot), offering no hint of a qualitative change.

Given the demonstrated dramatic volume effects leading to anomalous phase (B) at physically relevant cutoff, and the stability of bimodality at fixed volume under ultraviolet cutoff, we conclude that continuum $N_f = 0$ theory at $T/T_c = 1.12$ is in the anomalous phase.

3. Confinement and Anomalous Spectral Density. An important aspect of the proposed general scheme is that the transition from confined to deconfined theory in $\mathcal{T}$ coincides with the transition from regular type (A) of $\rho(\lambda)$ to anomalous type (B). To show this for $N_f = 0$ thermal transition, we selected a system just below $T_c$. Replacing $N_1 = 8$ of our volume study with $N_1 = 7$ of our volume study at $T/T_c = 0.98T_c$. This nominal assignment uses universal continuum value $r_0T_c = 0.7498(50)$ of Ref. [13], and $T/T_c$ at finite cutoff is expected to shift slightly upwards. We thus checked the behavior of Polyakov loop in large volumes up to $N = 48$, and confirmed that the system at hand is
indeed confined and at the very edge of the transition.

Fig. 4 (left) shows spectral density for theory so tuned, on $20^3 \times 8$ lattice. Type (A) behavior is found, as predicted, with flat dependence at low energies indicating a system on the verge of the transition. To compare this to the situation at temperature just above deconfinement point, we increased the coupling slightly ($\beta = 6.0783$, $a = 0.0817$ fm) to set $T = 1.02T_c$. Spectral density for such theory on $36^3 \times 8$ lattice (middle) shows a dramatic change to sharp bimodal behavior: there is little doubt that the $Z_3$ deconfinement transition coincides with the transition to anomalous phase. For comparison, we also show the result for the system at $T = 1.12T_c$ analyzed above ($32^3 \times 7$, $\beta = 6.054$), residing well inside the anomalous region (right).

4. Thermal Anomalous Phase of “Real World” QCD. The above evidence of anomalous phase in $N_f=0$ QCD, and its association with deconfinement, gives an essential impetus for its existence in other corners of $T$. It is of particular interest to clarify whether anomalous dynamics is part of physical reality for nature’s quarks and gluons. We address this via $N_f=2+1$ QCD at physical quark masses, i.e. light mass of $(m_u + m_d)/2$ and heavy of $m_s$, since it is well established that this theory provides for very precise representation of relevant strongly interacting physics. Among other things, calculations in this context led to the conclusion that thermal transition of strong interactions is a crossover [16].

To probe the crossover region, we utilize $N_t=8$ lattice ensembles of Wuppertal-Budapest group (see [17] and references therein), used in precise determination of transition temperatures. On a technical side, the simulated theory involves Symanzik improved (tree level) gauge action and stout improved staggered fermions. The physical line of constant physics was defined via fixing $f_K/m_\pi$, $f_K/m_K$ and both scaling violations and taste-splitting staggered fermion effects were shown to be small at gauge couplings involved. Due to its crossover nature, transition temperature is not a unique concept and depends both on the observable used and the defining condition chosen. For reference, the temperatures associated with inflection points of (light) scalar density (“condensate”) and Polyakov line were reported as $T_c \approx 155$ MeV and 170 MeV respectively. Given that, we selected ensembles at temperatures $T = 150$, 175 and 200 MeV, to be close to both ends of the crossover, and to examine the possible presence of anomalous phase past the transition region.

The resulting overlap spectra are shown in Fig. 5. On the lower edge of the crossover (left plot), spectral density is extremely flat in the infrared, but following the standard behavior (A). On the other hand, just past the Polyakov line crossover temperature (middle plot), the anomalous behavior clearly sets in. There is thus little doubt that the appearance
of anomalous phase closely follows the conventional measures for expected transition to deconfinement, i.e. conventional "$T_c$". Well above the established transition region at $T = 200$ MeV, the anomalous phase fully develops with growing trend in the strength of the anomalous peak and the degree of depletion.

Given the extensive checks performed on the ensembles used, both with respect to continuum limit and sufficiency of volumes\(^4\) (see [17] and references therein), we conclude that anomalous phase $T_c < T < T_{ch}$, i.e. deconfined phase with broken valence chiral symmetry, exists in high-temperature dynamics of strong interactions, consistently with the general scheme proposed here.

5. Generic Anomalous Phase, $N_{f}^c$ and $N_{f}^{ch}$.

At this point, we have two relevant ingredients in place: convincing evidence that anomalous phase with claimed properties exists in $\mathcal{F}$, and evidence that thermal effects lead to bimodal $\rho(\lambda)$ rather generically. Regarding the latter, apart from results presented here, observation of bimodal behavior was reported under varied but always thermal circumstances, e.g. in Refs. [9, 10, 1, 19, 20, 2]. However, there is also an important third ingredient provided by the recent finding that light quark effects, without any thermal agitation, lead to very similar anomalous phases at sufficiently large number of flavors [2, 12]. Thermal and light-quark effects thus appear to be analogous in this regard. The importance of this stems from the fact thermal effects (increasing $T$), light-quark effects (decreasing masses, increasing $N_{f}$), and their combinations, are the only available freedoms to transit from broken chiral symmetry to its restoration in $\mathcal{F}$. This leads us to propose that anomalous phases commonly occur on generic paths to chirally symmetric vacua. Moreover, our evidence on precise association of deconfinement and anomalous $\rho(\lambda)$ in known cases, completes the proposed picture.

Note that, in the context of $N_{f}=12$, the above scenario was seen [2, 12] to imply the existence of a mass $m_c$ below which the anomalous phase appears. The phase may either extend down to $m = 0$ or to non-zero $m_{ch} < m_c$, namely the possible point of valence chiral restoration. Here we wish to point out another special case of the above general argument, concerning the path in $\mathcal{F}$ parametrized by the number of massless quark species $N_{f}$ at $T = 0$. Indeed, it is widely believed that the $N_{f}=2$ case represents a confining, chirally broken theory with type (A) spectral density. At the same time, since the work of Banks and Zaks [18], it is expected that there is a critical number of flavors $N_{f}^{cr} < 16.5$, such that theories in the window $N_{f}^{cr} < N_{f} < 16.5$ are both asymptotically free, and controlled at low energy by a conformal

\(^4\)The smallest linear size involved here is $L=3.9$ fm at $T = 200$ MeV.
Theories in the conformal window cannot generate low-energy scales (at least not below the scale of conformality), and the standard scenario is that $N_f^{cr}$ marks the common transition to deconfined, chirally symmetric phase with $\rho(\lambda)$ of type (C).

However, based on the proposed picture, we predict that when crossing from $N_f=2$ to $N_f^{cr}$, quark-gluon dynamics will first lose confinement (at $N_f^c$), and only then chiral symmetry breaking (at $N_f^{ch}$), generating anomalous phase for

$$2 < N_f^c < N_f < N_f^{ch} \equiv N_f^{cr}$$  \hspace{1cm} (2)

Note that $N_f=12$ could be in the anomalous phase $^{[2, 12]}$, and fewer flavors are even more conducive of the possibility that anomalous vSChSB survives masslessness (SChSB): they simply generate less of a condensate-destructive light-quark effect.

The structure of $\mathcal{T}$ we are proposing is shown schematically in Fig. 6 (left). Here at $T=0$ (set $T_0$) there are theories with all three types of $\rho(\lambda)$ and associated vacuum properties. We refer to type (A) as QCD-like for this purpose. Under the type (C) are theories from conformal window and others, e.g. those without asymptotic freedom. Heating up a QCD-like system generically involves two transitions ($T_c$ and $T_{ch}$), while anomalous type (B) theory only undergoes chiral restoration, and type (C) no transition of such kind. Moving in theory space at fixed temperature involves analogous transitions, such as those discussed above. Note that other kinds of changes, unrelated to vSChSB and confinement, may occur within the three regions. Also, we do not imply that phases (A) and (C) are strictly disconnected, with anomalous phase occurring along any path between them. We rather intend to convey a typical situation.

5. Most Important Points. In this work, we have proposed a very specific general association between infrared behavior of Dirac spectral density, and confinement/chiral symmetry breaking structure of SU(3) theories with fundamental quarks, represented by Fig. 1. There are few associated points to discuss and highlight.

I. A significant conceptual step implied by our analysis, represented by Fig. 6 is that chiral symmetry breaking without confinement is a commonplace for SU(3) theories with fundamental quarks: the schematic view on the left with many additional transitions is a
good representation of the reality, while the standard view shown on the right is not. Indeed, we provided a detailed evidence that anomalous phase in $\mathcal{T}$ does exist.

II. The proposed association of Dirac spectral density with vacuum properties makes $\rho(\lambda)$ into a valuable tool for investigating phase structure of $\mathcal{T}$. Indeed, recall that the standard methods used e.g. in QCD thermal studies, are typically of “transitional” type. In other words, it is necessary to investigate the region of parameter space and look for change. On the other hand, our method is as convenient as having an order parameter. Indeed, it gives a definite answer for any standalone theory. In fact, based on the proposed association, it is straightforward to define “order parameters” (single-number indicators) of phases discussed.

III. The concluded existence of anomalous, i.e. deconfined but chirally broken phase in the physics of quarks and gluons is very important. Indeed, its existence is relevant for understanding strongly interacting dynamics associated with creation of the plasma-like state currently studied at RHIC and LHC. We will elaborate on the ramifications of this with relevant details elsewhere.

IV. The predicted anomalous phase as an intermediate regime before reaching the conformal window may, if true, complicate certain scenarios associated with walking technicolor. Indeed, the near–conformal theories are utilized in this case. Note, however, that deconfinement doesn’t necessarily imply the complete absence of bound states.

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