A numerical study of the dispersion and dissipation properties of virtual element methods for the Helmholtz problem

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Abstract

We study numerically the dispersion and dissipation properties of the plane wave virtual element method [27] and the nonconforming Trefftz virtual element method [23,24] for the Helmholtz problem. Whereas the former method is based on a conforming virtual partition of unity approach in the sense that the local (implicitly defined) basis functions are given as modulations of lowest order harmonic virtual element functions with plane waves, the latter one represents a pure Trefftz method with local edge-related basis functions that are eventually glued together in a nonconforming fashion. We will see that the qualitative and quantitative behavior of dissipation and dispersion of the method hinges upon the level of conformity and the use of Trefftz basis functions. To this purpose, we also compare the results to those obtained in [15] for the plane wave discontinuous Galerkin method [11,19], and to those for the standard polynomial based finite element method.

AMS subject classification: 35J05, 65N30, 65N25

Keywords: Helmholtz equation, virtual element methods, conforming and nonconforming methods, Trefftz methods, plane waves, dispersion and dissipation

1 Introduction

The numerical approximation of time-harmonic wave propagation problems by standard Galerkin discretizations encounters intrinsic difficulties due to the oscillatory nature of the analytical solutions. In this paper, we focus on the 2D Helmholtz problem, a scalar-valued representative of this class of problems, which is given by

$$
\begin{aligned}
-\Delta u - k^2 u &= 0 \quad \text{in } \Omega \\
\text{boundary conditions} &= \text{on } \Gamma := \partial \Omega,
\end{aligned}
$$

where $k > 0$ is the wave number, and $\Omega \subset \mathbb{R}^2$ is a bounded domain with sufficiently smooth boundary. Since this problem can be seen as the scalar-valued analogous of the time-harmonic Maxwell problem, it has been attracting a vast amount of attention throughout the last years.

Among the above-mentioned difficulties, falls the so-called pollution effect [6], which describes the widening discrepancy between the best approximation error and the discretization error for large values of the wave number $k$.

This effect is directly linked to numerical dispersion, representing the failure of the numerical method to reproduce the correct oscillating behavior of the analytical solution. More precisely, for a given wave number $k$, a continuous problem with plane wave solution is considered. Its numerical approximation delivers an approximate solution, which can be interpreted as a wave with a deviated wave number $k_n$. This mismatch of the continuous and discrete wave numbers, $k$ and $k_n$, respectively, can be measured separately in terms of the real part and the imaginary part with the following interpretation. The term $|\text{Re} (k - k_n)|$ represents the deviation (shift) of the phase (dispersion), and the term $|\text{Im} (k - k_n)| = |\text{Im} (k_n)|$ refers to the damping of the amplitude (dissipation) of the computed discrete solution. Moreover, the difference $|k - k_n|$ measures the

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The general strategy for a dispersion analysis can be summarized in the following two steps:

1. Consider the discretization scheme of the numerical method applied to \(-\Delta u - k^2 u = 0\) in \(\mathbb{R}^2\) using infinite meshes which are invariant under a discrete group of translations. Due to translation invariance, it is then possible to reduce the infinite mesh to a finite one.

2. Given a plane wave with wave number \(k\) traveling in a fixed direction, seek a so-called discrete Bloch wave solution, which can be regarded as a generalization of the given continuous plane wave based on the underlying approximating spaces, and determine for which (discrete) wave number \(k_n\) this Bloch wave solution actually solves the discrete variational formulation. This procedure leads to small nonlinear eigenvalue problems, which need to be solved.

In the framework of standard conforming finite element methods (FEM) for the Helmholtz problem, a full dispersion analysis was done in [13] for dimensions one to three. Furthermore, in [6] it was shown that the pollution effect can be avoided in 1D, but not in higher dimensions, and a generalized pollution-free FEM in 1D was constructed. Moreover, we highlight the work in [21], where a link between the results of the dispersion analysis and the numerical analysis was established for FEM, and the work in [1], where quantitative, fully explicit estimates for the behavior and decay rates of the dispersion error were derived in dependence on the order of the method relative to the mesh size and the wave number. Also in the context of non-conforming methods, dispersion analyses have been performed for the discontinuous Petrov-Galerkin (DPG) method [17], and the plane wave discontinuous Galerkin method (PWDG) [15]. Recently, a dispersion analysis for hybridized DG (HDG)-methods has been carried out in [18], including an explicit derivation of the wave number error for lowest order single face HDG methods.

In order to reduce the computational cost in terms of the number of degrees of freedom, and to mitigate the strong pollution of standard FEM, a series of so-called Trefftz methods have been the object of intensive research throughout the last years. These methods are characterized by the use of approximating spaces having the property that the basis functions are elementwise solutions to the homogeneous Helmholtz equation. Among those are the ultra weak variational formulation [11], PWDG [16, 19], discontinuous methods based on Lagrange multipliers [14], wave based methods [12], the least square formulation [26], and the variational theory of complex rays [28]. For an overview, we refer to [20]. As already mentioned, a numerical study of the dispersion properties was carried out for PWDG in [15].

Despite the novelty of the virtual element methodology [7, 8], the construction, design and analysis of numerical methods for the Helmholtz problem have already been tackled within the virtual element method (VEM) framework, giving rise to two methods. The first one is the plane wave virtual element method (PWVEM) introduced in [27], and the second the nonconforming Trefftz virtual element method (ncTVEM), which was introduced in [23] and extended to the case of piecewise constant wave number in [25]. Whereas the former is characterized by the fact that the local basis functions are obtained by modulating lowest order VE functions with plane waves (in this sense, it is a virtual version of the classical partition of unity method [5]), the latter is a pure Trefftz method by construction; its counterpart for the Laplace problem was introduced in [22].

In this paper, for the 2D Helmholtz problem, we investigate numerically the dispersion and dissipation properties of PWVEM and ncTVEM, and compare the results to those obtained in [15] for PWDG, and to those for standard polynomial based FEM. We highlight that, in contrast to some polynomial based methods, but similarly to PWDG in [15], an explicit analysis in the sense of fully explicit dispersion relations is not possible for PWVEM and ncTVEM, due to the use of plane wave related basis functions.

The outline of this paper is as follows. In Section 2 the abstract setting for the dispersion analysis is described. Then, in Section 3 the set of basis functions and the sesquilinear forms defining the numerical discretization schemes are specified for PWVEM and ncTVEM, and are recalled for PWDG. Finally, in Section 4 dispersion and dissipation are studied numerically for the different methods and a comparison of the results is given.
Notation. Throughout this paper, we denote by $\mathbb{N}_0$ and $\mathbb{N}_{\geq r}$ the sets of all natural numbers including zero and of all natural numbers larger than or equal to $r$, respectively. Moreover, we will use the notation $H^s(\mathcal{D})$ for the Sobolev space of functions on $\mathcal{D} \subset \mathbb{R}^2$ with square integrable weak derivatives up to order $s \in \mathbb{N}_0$, and $\mathcal{P}_t(\mathcal{D})$ for the space of polynomials on $\mathcal{D}$ of degree at most $t \in \mathbb{N}_0$. Finally, $C^0(\mathcal{D})$ denotes the space of continuous functions on $\mathcal{D}$.

2 Abstract dispersion analysis

In this section, we fix the abstract setting for the dispersion analysis employing the notation of [15].

To this purpose, in order to remove possible dependencies of the dispersion on the boundary conditions of the problem, we consider problem (1) on the unbounded domain $\Omega = \mathbb{R}^2$. Let $\mathcal{T}_n := \{K\}$ be a translation-invariant partition of $\Omega$ into polygons with mesh size $h := \max_{K \in \mathcal{T}_n} h_K$, where $h_K := \text{diam}(K)$, i.e. there exists a set of elements $K_1, \ldots, K_r$, $r \in \mathbb{N}$, such that the whole infinite mesh can be covered in a non-overlapping way by shifts of the “reference” patch $\hat{K} := \bigcup_{j=1}^r \hat{K}_j$. In other words, this assumption implies the existence of translation vectors $\xi_1, \xi_2 \in \mathbb{R}^2$, such that every element $K \in \mathcal{T}_n$ can be written as a linear combination with coefficients in $\mathbb{N}_0$ of one of the reference polygons $\hat{K}_\ell$, $\ell = 1, \ldots, r$. Some examples for translation-invariant meshes are shown in Figure 1. Moreover, we denote by $\mathcal{E}^K$ the set of edges belonging to $K$.

Let now $u(x) = e^{ikd \cdot x}$, $d \in \mathbb{R}^2$ with $|d| = 1$, be a plane wave with wave number $k$ and traveling in direction $d$. We denote by $\mathcal{V}_n$ the global approximation space resulting from the discretization of (1) using a Galerkin based numerical method, and by $\hat{\mathcal{V}}_n \subset \mathcal{V}_n$ a minimal subspace generating $\mathcal{V}_n$ by translations with

$$\xi_n := n_1 \xi_1 + n_2 \xi_2, \quad n = (n_1, n_2) \in \mathbb{Z}^2.$$  \hspace{1cm} (2)

More precisely, depending on the structure of the method, $\hat{\mathcal{V}}_n$ is determined as follows.

1. **Vertex-related** basis functions: In this case, $\hat{\mathcal{V}}_n$ is the span of all basis functions related to a minimal set of vertices $\{v_i\}_{i=1}^{\lambda^{(0)}}, \lambda^{(0)} \in \mathbb{N}$, such that all the other mesh vertices are obtained by translations with $\xi_n$ of the form (2). Examples are FEM and PWVEM [27].

2. **Edge-related** basis functions: Similarly as above, the space $\hat{\mathcal{V}}_n$ is in this case the span of all basis functions related to a minimal set of edges $\{e_i\}_{i=1}^{\lambda^{(1)}}, \lambda^{(1)} \in \mathbb{N}$, such that all the other edges of the mesh are obtained by translations with $\xi_n$ of the form (2). This is, for instance, the case of ncTVEM [23,24].

3. **Element-related** basis functions: Here, the space $\hat{\mathcal{V}}_n$ is simply given as the span of all basis functions related to a minimal set of elements $\{\sigma_i\}_{i=1}^{\lambda^{(2)}}, \lambda^{(2)} \in \mathbb{N}$, such that all other elements of the mesh are obtained by a translation with a vector $\xi_n$ of the form (2). One representative of this category is PWDG [16,19].
In the following, we will refer to these minimal sets of vertices \( \nu_i \), edges \( \nu_i \), and elements \( \nu_i \) as fundamental sets of vertices, edges, and elements, respectively.

As a direct consequence, every \( v_n \in \mathcal{V}_n \) can be written as
\[
v_n(x) = \sum_{n \in \mathbb{Z}^2} \tilde{v}_n(x - \xi_n), \quad \tilde{v}_n \in \tilde{\mathcal{V}}_n.
\]

Next, we define the discrete Bloch wave with wave number \( k_n \) and traveling in direction \( d \) by
\[
u_n(x) = \sum_{n \in \mathbb{Z}^2} e^{ik_n \cdot d} \xi_n \hat{u}_n(x - \xi_n),
\]
where \( \hat{u}_n \in \tilde{\mathcal{V}}_n \) and \( k_n \in \mathbb{C} \) with Re\( (k_n) > 0 \). Note that, since \( \hat{u}_n \in \tilde{\mathcal{V}}_n \), the infinite sum in (3) is in fact finite. Furthermore, given \( d \in \mathbb{R}^2 \) with \( |d| = 1 \), the Bloch wave \( u_n \) in (3) satisfies
\[
u_n(x + \xi_\ell) = e^{ik_n \cdot d} \xi_\ell u_n(x),
\]
for all \( \ell \in \mathbb{Z}^2 \). This property follows directly by using the definition of the Bloch wave:
\[
u_n(x + \xi_\ell) = \sum_{n \in \mathbb{Z}^2} e^{ik_n \cdot d} \xi_n \hat{u}_n(x + \xi_\ell - \xi_n) = \sum_{n \in \mathbb{Z}^2} e^{ik_n \cdot d} \xi_n \hat{u}_n(x - \xi_n - \ell)
\]
\[
= e^{ik_n \cdot d} \xi_\ell \sum_{m \in \mathbb{Z}^2} e^{ik_m \cdot d} \xi_m \hat{u}_n(x - \xi_m) = e^{ik_n \cdot d} \xi_\ell u_n(x).
\]

Therefore, Bloch waves can be regarded as discrete counterparts, based on the approximation spaces, of continuous plane waves.

We introduce the global (continuous) sesquilinear form
\[
a(u, v) := \sum_{K \in \mathcal{T}_n} a^K(u, v) := \sum_{K \in \mathcal{T}_n} \left[ \int_K \nabla u \cdot \nabla v \, dx - k^2 \int_K u \bar{v} \, dx \right] \quad \forall u, v \in H^1(\mathbb{R}^2),
\]
and we denote by \( a_n(\cdot, \cdot) \) the global discrete sesquilinear form defining the numerical method under consideration. In Section 3 \( \tilde{\mathcal{V}}_n \) and \( a_n(\cdot, \cdot) \) will be specified for PWVEM, ncTVEM, and PWDG, respectively.

Next, we define the discrete wave number \( k_n \in \mathbb{C} \) as follows.

**Definition 2.1.** Given \( k > 0 \) and \( d \in \mathbb{R}^2 \) with \( |d| = 1 \), the discrete wave number \( k_n \in \mathbb{C} \) is the number with minimal \( |k - k_n| \), for which a discrete Bloch wave \( u_n \) of the form (3) is a solution to the discrete problem
\[
a_n(u_n, \tilde{v}_n) = 0 \quad \forall \tilde{v}_n \in \tilde{\mathcal{V}}_n.
\]

Due to the scaling invariance of the mesh, we can assume that \( h = 1 \). Notice that the wave number \( k \) on a mesh with \( h = 1 \) corresponds to the wave number \( k_0 = \frac{k}{h_0} \) on a mesh with mesh size \( h_0 \).

Having this, the general procedure in the dispersion analysis now consists in finding those discrete wave numbers \( k_n \in \mathbb{C} \) and coefficients \( \hat{u}_n \in \tilde{\mathcal{V}}_n \), for which a Bloch wave solution of the form (3) satisfies (5), and to measure the deviation of \( k_n \) from \( k \) afterwards. This strategy results in solving small nonlinear eigenvalue problems. In fact, by plugging the Bloch wave ansatz (3) into (5) and using the sesquilinearity of \( a_n(\cdot, \cdot) \), we obtain
\[
\sum_{n \in \mathbb{Z}^2} e^{ik_n \cdot d} \xi_n a_n(\hat{u}_n(\cdot - \xi_n), \tilde{v}_n) = 0 \quad \forall \tilde{v}_n \in \tilde{\mathcal{V}}_n.
\]

Let \( \{ \tilde{\chi}_s \}_{s=1}^\infty \subset \tilde{\mathcal{V}}_n \) be a set of basis functions for the space \( \tilde{\mathcal{V}}_n \) that are related to fundamental elements, vertices, or edges, depending on the method. Then, we can expand \( \hat{u}_n \) in terms of this basis as
\[
\hat{u}_n = \sum_{s=1}^\infty u_s \tilde{\chi}_s.
\]
Plugging this ansatz into (6), testing with $\hat{\chi}_s$, $s = 1, \ldots, \Xi$, and interchanging the sums (this can be done since the infinite sum over $n$ is in fact finite) yields

$$\sum_{t=1}^{\Xi} u_t \left( \sum_{n \in \mathbb{Z}^2} e^{ik_n \cdot \xi_n} a_n(\hat{\chi}_t(\cdot - \xi_n), \hat{\chi}_s) \right) = 0 \quad \forall s = 1, \ldots, \Xi,$$

which can be represented as

$$\sum_{t=1}^{\Xi} T_{s,t}(k_n) u_t = 0 \quad \forall s = 1, \ldots, \Xi,$$

with

$$T_{s,t}(k_n) := \sum_{n \in \mathbb{Z}^2} e^{ik_n \cdot \xi_n} a_n(\hat{\chi}_t(\cdot - \xi_n), \hat{\chi}_s).$$

The matrix problem corresponding to (8) has the form

$$T(k_n)u = 0,$$

where $T : \mathbb{C} \rightarrow \mathbb{C}^{\Xi \times \Xi}$ is defined via (9), and $u = (u_1, \ldots, u_{\Xi})^T \in \mathbb{C}^{\Xi}$. We highlight that $T$ is a holomorphic map, and that (10) is a small nonlinear eigenvalue problem, which can be solved using e.g. an iterative method as done in [15], or a direct method based on a rational interpolation procedure [29] or on a contour integral approach [4,10]. For the numerical experiments presented in Section 4, we will make use of the latter, which we will denote by contour integral method (CIM) in the sequel. We stress that, due to the use of plane wave related basis functions (see next section), deriving an exact analytical solution to (10) is not even be possible for the lowest order case.

3 Minimal generating subspaces and sesquilinear forms for PWVEM, ncTVEM, and PWDG

In this section, we specify the minimal generating subspaces $\hat{\mathcal{V}}_n$, the corresponding sets of basis functions $\{\hat{\chi}_s\}_{s=1}^{\Xi}$, and the sesquilinear forms $a_n(\cdot, \cdot)$ for PWVEM [27] and ncTVEM [23, 24], and we recall them from [15] for PWDG [16, 19]. The basis functions for these three methods are vertex-related, edge-related, and element-related, respectively. In Figures 2-4 in Section 3.4, the stencils related to the fundamental sets of vertices, edges, and elements are depicted for these three methods and the meshes in Figure 1.

Before doing that, we need to fix some additional notation. Let $\{d_j\}_{j=1}^{p}$, $p = 2q + 1$, $q \in \mathbb{N}$, be a set of equidistributed plane wave directions. We denote by

$$w_j(x) := e^{ikd_j \cdot x}, \quad j = 1, \ldots, p,$$

the plane wave with wave number $k$ and traveling along the direction $d_j$. Furthermore, for every $K \in \mathcal{T}_n$, we set $w^K_j := w_{j|K}$, and we introduce the bulk place waves space

$$\mathbb{P}^{\psi}_{p}(K) := \text{span}\{w^K_j, j = 1, \ldots, p\}.\quad (12)$$

3.1 Plane wave virtual element method (PWVEM)

We first recall the structure of PWVEM introduced in [27], using the notation employed there.

To this purpose, given $K \in \mathcal{T}_n$, the lowest order local VE space is defined as

$$\hat{\mathcal{V}}_n^{(0)}(K) := \{v \in H^1(K) : v|_{\partial K} \in C^0(\partial K), v|_e \in P_1(e) \forall e \in \mathcal{E}^K, \Delta v = 0 \text{ in } K\},$$

where we recall that $\mathcal{E}^K$ denotes the set of edges of $K$. We underline that $\hat{\mathcal{V}}_n^{(0)}(K)$ includes $P_1(K)$, the space of linear polynomials over $K$, as a subspace. Moreover, it contains functions which cannot be written down explicitly in closed form, giving rise to the term virtual in the name of the method.
The space \( \mathcal{V}_n^{(0)}(K) \) is endowed with the local set of degrees of freedom given by the point values at the vertices \( V_K \), \( s = 1, \ldots, n_K \), of \( K \), where \( n_K \) denotes their number. Due to the unsolvability of the degrees of freedom, a set of canonical basis functions \( \{ \phi^K_j \}_{j=1}^{n_K} \) can be defined by duality, i.e. \( \phi^K_j (V_K) = \delta_{s,j} \), \( s, j = 1, \ldots, n_K \), with \( \delta \) denoting the standard Kronecker delta. It can be easily shown that these basis functions actually form a partition of unity.

Using (11) and (12), the local PWVE space is given by the modulation of the local canonical basis functions with plane waves:

\[
\mathcal{V}_n^{(0)}(K) := \left\{ v \in H^1(K) : v = \sum_{r=1}^{n_K} \sum_{j=1}^{p} a^K_{rj} \phi^K_{(r,j)}, \ a^K_{rj} \in \mathbb{C} \right\} \cap \mathcal{PW}_p(K),
\]

where \( \phi^K_{(r,j)} := \phi^K_r w^K_j \). Note that the inclusion in (14) is a direct consequence of the properties of the canonical basis functions and is in fact essential for deriving best approximation estimates needed in the error analysis of the method.

The global plane wave VE space is defined in terms of the local ones:

\[
\mathcal{V}_n^{(0)} := \left\{ v_n \in C^0(\mathbb{R}^2) : v_n|_K \in \mathcal{V}_n^{(0)}(K) \quad \forall K \in T_n \right\}.
\]

In the spirit of the pioneering works on VEM in [7,8], the global sesquilinear form is given by

\[
a_n^{(0)}(u_n, v_n) := \sum_{K \in T_n} a^K_p (\Pi^K_p u_n, \Pi^K_p v_n) + S^K((I - \Pi^K_p)u_n, (I - \Pi^K_p)v_n) \quad \forall u_n, v_n \in \mathcal{V}_n^{(0)},
\]

where, for every \( K \in T_n \), the computable projector \( \Pi^K_p : \mathcal{V}_n^{(0)}(K) \to \mathcal{PW}_p(K) \), is defined by

\[
a^K_p (\Pi^K_p u_n, w^K) = a^K_p (u_n, w^K) \quad \forall u_n, w^K \in \mathcal{V}_n^{(0)},
\]

with \( a^K_p (\cdot, \cdot) \) as in (10), and \( S^K (\cdot, \cdot) \) is a suitable computable sesquilinear form, see Remark 1 below. It can be shown that the projector \( \Pi^K_p \) is well-defined and continuous, under certain assumptions on the wave number \( k \) and the mesh size \( h \), see [27] Propositions 2.1 and 2.3.

We mention here that, in the framework of VEM, a quantity is called computable if it can be computed exactly, without need of numerical integration, only in terms of its degrees of freedom.

Remark 1. Since \( u_n \) and \( v_n \) are virtual functions, whose explicit representations inside each element are not known in closed form, \( a^K_p (u_n, v_n) \) is not computable. By making use of the Pythagorean theorem, such term is split into two parts:

\[
a^K_p (u_n, v_n) = a^K_p (\Pi^K_p u_n, \Pi^K_p v_n) + a^K_p ((I - \Pi^K_p)u_n, (I - \Pi^K_p)v_n).
\]

The first term on the right-hand side is computable, but the second is not and is approximated by a computable sesquilinear form, referred to as stabilization in the sequel, leading to (15).

In order to guarantee well-posedness of the numerical method, conditions on the choice of the stabilization are needed, see [23,27]. Roughly speaking, it has to be guaranteed that \( S^K (\cdot, \cdot) \) scales like \( a^K (\cdot, \cdot) \), to ensure continuity and the validity of a Garding inequality. In Section 4 below, the dispersion and dissipation properties of the different choice of stabilizations that work fine in practice.

Given a fundamental set of vertices \( \{ \nu_i \}_{i=1}^{\lambda^{(0)}} \), the set of basis functions \( \{ \chi^{(0)}_{s,i} \}_{s=1}^{\Xi} \subset C^0(\mathbb{R}^2) \) is defined as follows. Let \( s \leftrightarrow (i,j) \), with \( i \in \{ 1, \ldots, \lambda^{(0)} \} \) and \( j \in \{ 1, \ldots, p \} \), i.e. we identify \( s \) with the vertex index \( i \) and the direction index \( j \). Then,

\[
\chi^{(0)}_{s,i} := \Phi_{\nu_i} w_j \in C^0(\mathbb{R}^2),
\]

where \( w_j \) is the plane wave given in (11) and \( \Phi_{\nu_i} \) is defined elementwise as follows. If \( K \in T_n \) is an element abutting the fundamental vertex \( \nu_i \), then \( \Phi_{\nu_i|_K} \) coincides with the local canonical basis
function in $K$ which is associated with the (global) vertex $\nu_i$; otherwise $\Phi_{\nu_i|_e}$ is set to zero. Taking into account the definitions of the degrees of freedom and of $\mathcal{V}^{(0)}_n(K)$ in (13), it can be easily seen that $\hat{\chi}^{(0)}_i$ is in fact globally continuous with compact support. Clearly, $\Xi = \lambda^{(0)} p$.

To conclude, for PWVEM, the minimal generating subspace $\hat{\mathcal{V}}^{(0)}_n$ of $\mathcal{V}^{(0)}_n$ is given as the span of the basis functions $\{\hat{\psi}_j^{(0)}\}$, and the employed sesquilinear form is $a_n^{(0)}(\cdot, \cdot)$ in (15).

### 3.2 Nonconforming Trefftz virtual element method (ncTVEM)

As above, we start by recalling the structure of ncTVEM introduced in [23, 24].

To this purpose, defining, on each edge $e$ of the mesh, the plane wave trace $w^\mu_j := w^\mu_{j,\nu}$, one first computes a set of $L^2(e)$ orthogonal functions $\{\hat{w}^\mu_j\}_{j=1}^p$, obtained from $\{w^\mu_j\}_{j=1}^p$ by applying the orthogonalization-and-filtering process described in [24, Algorithm 2], which is based on an eigendecomposition of the edge plane wave mass matrix, and the removal of eigenfunctions associated with eigenvalues smaller than a fixed threshold. Clearly, $\hat{p}_e \leq p$. This procedure allows for a reduction of the condition numbers of the edge plane wave mass matrices, which is crucial for the convergence of the method. We set $\mathcal{J}_e := \{1, \ldots, \hat{p}_e\}$.

Given an element $K \in \mathcal{T}_n$, on each edge $e \in \mathcal{E}_K$, we define the impedance trace of a function $\tilde{v} \in H^1(K)$ by

$$\gamma^K_{\tilde{v}}(\cdot)_{\nu} := (\nabla \tilde{v} \cdot \mathbf{n}_K + i k \tilde{v})_{\nu},$$

and the space of filtered $L^2(e)$ orthogonalized plane wave traces by

$$\mathbb{P} \mathbb{W}(e) := \text{span}\{\tilde{w}^\mu_j, \ell \in \mathcal{J}_e\}.$$ 

With these definitions, the local Trefftz-VE space related to an element $K \in \mathcal{T}_n$ is given by

$$\mathcal{V}^{(1)}_n(K) := \{v_n \in H^1(K) \mid \Delta v_n + k^2 v_n = 0 \text{ in } K, \quad \gamma^K_{\tilde{v}}(v_n)_{\nu} \in \mathbb{P} \mathbb{W}(e) \forall e \in \mathcal{E}_K\}. \quad (19)$$

Clearly, $\mathbb{P} \mathbb{W}_p(K) \subset \mathcal{V}^{(1)}_n(K)$ holds true. The space (19) is endowed with the set of local degrees of freedom given, for any $v_n \in \mathcal{V}^{(1)}_n(K)$, by

$$\text{dof}_{e,\ell}(v_n) = \frac{1}{\Delta e} \int_{\Delta e} v_n \tilde{w}^\mu_{\ell,\nu} \, ds \quad \forall e \in \mathcal{E}_K, \forall \ell \in \mathcal{J}_e.$$ 

It can be shown that these degrees of freedom are unisolvent for $\mathcal{V}^{(1)}_n(K)$, provided that $k^2$ is not a Dirichlet-Laplace eigenvalue on $K$, see [23, Lemma 3.1]. The set of canonical basis functions $\{\hat{\psi}^{K}_{(e,\ell)}\}_{e \in \mathcal{E}_K, \ell \in \mathcal{J}_e} \subset \mathcal{V}^{(1)}_n(K)$ is defined by duality:

$$\text{dof}_{e,\ell} \left(\hat{\psi}^{K}_{(e,\ell)}(\ell)\right) = \delta_{(e,\ell),(\ell,\ell)} \begin{cases} 1 & \text{if } (e, \ell) = (\ell, \ell) \\ 0 & \text{otherwise} \end{cases} \quad \forall e, \ell \in \mathcal{E}_K, \forall \ell \in \mathcal{J}_e.$$

The global nonconforming Trefftz-VE space $\mathcal{V}^{(1)}_n$ is obtained by coupling the local counterparts (19) in a nonconforming fashion:

$$\mathcal{V}^{(1)}_n := \{v_n \in H^1,nc(\mathcal{T}_n) : v_n|_K \in \mathcal{V}^{(1)}_n(K) \quad \forall K \in \mathcal{T}_n\},$$

where $H^1,nc(\mathcal{T}_n)$ is the global nonconforming Sobolev space

$$H^1,nc(\mathcal{T}_n) := \left\{v_n \in H^1(\mathcal{T}_n) : \int_{\mathcal{E}_n} \|v_n\|_N \cdot \mathbf{n}^+ \tilde{w}^\nu \, ds = 0 \quad \forall w^\nu \in \mathbb{P} \mathbb{W}(e), \forall e \in \mathcal{E}_n\right\},$$

with the jump $\|\cdot\|_N$ defined by

$$\|v_n\|_N := v_n|_{K^+} \mathbf{n}^+ + v_n|_{K^-} \mathbf{n}^-,$$ 

(20)

$n^e$ is a fixed unit normal vector to the edge $e$, $\mathcal{E}_n$ denotes the set of edges, and $H^1(\mathcal{T}_n)$ is the broken Sobolev space, i.e. $H^1(\mathcal{T}_n) := \prod_{K \in \mathcal{T}_n} H^1(K)$. 

7
Let now \( \{ \eta_i \}_{i=1}^{\lambda} \) be a fundamental set of edges. Then, the set of basis functions \( \{ \tilde{\chi}_s^{(1)} \}_{s=1}^{\Xi} \) spanning the minimal generating subspace \( \tilde{V}_n^{(1)} \) is given by the union of the canonical basis functions related to \( \{ \eta_i \}_{i=1}^{\lambda} \). More precisely, for \( s \leftrightarrow (i,j) \), \( i \in \{1, \ldots, \lambda \} \) and \( j \in \mathcal{T}_n \), i.e. we identify \( s \) with the edge index \( i \) and the index \( j \) associated with the \( j \)-th orthogonalized plane wave basis function on this edge as above,
\[
\tilde{\chi}_s^{(1)} = \tilde{\chi}_{i,j}^{(1)} := \Psi(\eta_i, j),
\]
where \( \Psi(\eta_i, j) \) is defined elementwise as follows. If \( K \in \mathcal{T}_n \) is an element abutting the edge \( \eta_i \), then \( \Psi(\eta_i, j) \) coincides with the local canonical basis function associated with the (global) edge \( \eta_i \) and the \( j \)-th orthogonalized edge plane wave basis function; otherwise \( \Psi(\eta_i, j) \) is zero. Clearly, \( \Xi = \sum_{i=1}^{\lambda} \hat{p}_i \).

Concerning the sesquilinear form \( a_n^{(1)}(\cdot, \cdot) \), it coincides with \( a_n^{(0)}(\cdot, \cdot) \) in (15), where the projector \( \Pi_p^K \) is defined analogously as in (16), with the only difference that, this time, it maps from \( V_n^{(1)}(K) \) in (19) (instead of from \( V_n^{(0)}(K) \) in (14)) to \( \mathbb{P}_p(K) \).

### 3.3 Plane wave discontinuous Galerkin method (PWDG)

For PWDG, we refer to [15], where a complete dispersion analysis was carried out. Nevertheless, for the sake of completeness, we shortly recall here the definitions of the minimal generating subspace and the sesquilinear form adapted to our setting.

The global approximation space \( V_n^{(2)} \) is given by
\[
V_n^{(2)} := \{ v_n \in L^2(\mathbb{R}^2) : v_n|_K \in \mathbb{P}_p(K) \quad \forall K \in \mathcal{T}_n \}.
\]

Moreover, the global sesquilinear form \( a_n^{(2)}(\cdot, \cdot) \) is defined by
\[
a_n^{(2)}(u_n, v_n) := \sum_{K \in \mathcal{T}_n} a^K(u_n, v_n) - \int_{\mathcal{F}_n} \langle \nabla_n u_n \rangle_N \cdot \langle \nabla_n v_n \rangle_N \, ds - \frac{\beta}{ik} \int_{\mathcal{F}_n} \langle \nabla_n u_n \rangle_N \cdot \langle \nabla_n v_n \rangle_N \, ds
\]
\[
\quad - \int_{\mathcal{F}_n} \langle \nabla_n u_n \rangle \cdot \langle \nabla_n v_n \rangle_N \, ds + \frac{i k a}{N} \int_{\mathcal{F}_n} \langle u_n \rangle_N \cdot \langle v_n \rangle_N \, ds, \quad \forall u_n, v_n \in V_n^{(2)},
\]
where \( a^K(\cdot, \cdot) \) is given in (14), \( \mathcal{F}_n \) is the mesh skeleton, \( \alpha, \beta > 0 \) are the flux parameters, \( \nabla_n \) is the broken gradient, \( \langle \cdot \rangle_N \) is the standard trace jump as defined in (20), and, for a given edge \( e \), denoting by \( K^- \) and \( K^+ \) its adjacent elements,
\[
\langle \nabla_n u_n \rangle := \frac{1}{2} \left( \nabla u_n|_{K^+} + \nabla u_n|_{K^-} \right)
\]
is the trace average.

Let now \( \{ \sigma_j \}_{j=1}^{\lambda(p)} \) be a fundamental set of elements. Then, the basis functions \( \{ \tilde{\chi}_s^{(2)} \}_{s=1}^{\Xi} \) are given by \( \{ u_j^{(p)} \}_{i=1}^{\lambda(\sigma_j)} \) where \( s \leftrightarrow (i,j) \), i.e. \( s \) is identified with the element index \( i \) and the plane wave direction index \( j \), and \( \Xi = \lambda^{(p)} \). As mentioned above, the minimal generating subspace \( V_n^{(2)} \subset V_n^{(2)} \) is simply the span of the basis functions \( \{ \tilde{\chi}_s^{(2)} \}_{s=1}^{\Xi} \), and the sesquilinear form \( a_n^{(2)}(\cdot, \cdot) \) is given in (21).

### 3.4 Overview of the stencils generating the minimal subspaces

In Figures (24) we illustrate the stencils of the basis functions for PWVEM, ncTVEM, and PWDG, as introduced in Sections 3.1, 3.3, employing the meshes made of squares, triangles, and hexagons, respectively, depicted in Figure (1). The fundamental sets of vertices, edges, and elements are displayed in dark-blue, and the translation vectors \( \xi_1 \) and \( \xi_2 \) in red. Furthermore, the supports of the basis functions spanning the minimal generating subspaces are colored in light-blue for PWVEM and ncTVEM. Due to the locality of the basis functions, only those associated with the vertices, edges, and elements displayed in dark-blue and dark-yellow contribute to the sum (7). Integration only has to be performed over the elements \( K^\xi \) and the adjacent edges.
4 Numerical results

In this section, after fixing some parameters for the different methods in Sections 3.1-3.3 and specifying the quantities to be compared, we present a series of numerical tests using the meshes portrayed in Figure 1. Firstly, in Section 4.1, we investigate the qualitative behavior of dispersion and dissipation depending on the Bloch wave angle $\theta$ in Definition 3. Then, in Section 4.2, we compare the dispersion and dissipation errors against the effective plane wave degree $q$ and against the dimensions of the minimal generating subspaces. Finally, in Section 4.3, the dependence of the errors on the wave number is studied.

Choice of the parameters in PWDG, and the stabilizations in PWVEM and ncTVEM.

We use the choice of the flux parameters of the ultra weak variational formulation (UWVF), i.e. $\alpha = \beta = 1/2$, in PWDG, and we employ the stabilization terms suggested in [27] and [23, 24], for PWVEM and ncTVEM, respectively.

More precisely, in the framework of the two virtual element methods, the mentioned standard stabilizations can be written (locally) in matrix form as follows. For every $K \in T_n$,

$$S^K = (I^K - \Pi^K)^T M^K (I^K - \Pi^K),$$

where $\Pi^K$ is the matrix representation of the composition of the embedding of $\mathbb{P}_n^2(K)$ into $V^0_n(K)$ (PWVEM) or $V^1_n(K)$ (ncTVEM), after the projection $\Pi^K_p$ in [16], see also [24, 27]. The matrices $M^{(0),K}$ and $M^{(1),K}$ are suitable approximations of the matrices with entries given by

$$[(r,j), (s,\ell)] \mapsto \int_K \left( \nabla \varphi^K_{(r,j)} \cdot \nabla \varphi^K_{(s,\ell)} - k^2 \varphi^K_{(r,j)} \varphi^K_{(s,\ell)} \right) dx$$

and

$$[(e,\ell), (\tilde{e},\tilde{\ell})] \mapsto \int_K \left( \nabla \psi^K_{(e,\ell)} \cdot \nabla \psi^K_{(\tilde{e},\tilde{\ell})} - k^2 \psi^K_{(e,\ell)} \psi^K_{(\tilde{e},\tilde{\ell})} \right) dx,$$
Figure 4: Stencils of the basis functions related to the fundamental sets of vertices (PWVEM), edges (ncTVEM), and elements (PWDG), respectively, from left to right, when employing the meshes made of hexagons in Figure 1.

for PWVEM and ncTVEM, respectively, see also Remark 1. In the case of PWVEM, using the notation of (14), we can compute

\[
\nabla \phi^K_{(r,j)} \cdot \nabla \phi^K_{(s,\ell)} - k^2 \phi^K_{(r,j)} \phi^K_{(s,\ell)} = (\nabla \phi^K_{(s,\ell)} \cdot \nabla \phi^K_{(r,j)}) w^K_j w^K_\ell + ik(d_j \cdot \nabla \phi^K_{(r,j)}) \phi^K_{(s,\ell)} w^K_j w^K_\ell

- ik(d_\ell \cdot \nabla \phi^K_{(r,j)}) \phi^K_{(s,\ell)} w^K_j w^K_\ell + k^2 (d_j \cdot d_\ell - 1) \phi^K_{(s,\ell)} \phi^K_{(r,j)} w^K_j w^K_\ell.
\]

Then, due to scaling considerations, the last three terms on the right-hand side are neglected, and the first one is simplified obtaining

\[
M^{(0),K}_{(s,\ell), (r,j)} = \delta_{r,s} h_K^2 \int_K w^K_j w^K_\ell \, dx,
\]

where \( \delta \) is the usual Kronecker delta.

For ncTVEM, it was proposed in [23,24] to take the so-called the modified D-recipe stabilization

\[
M^{(1),K}_{(\tilde{e},\tilde{\ell}), (e,\ell)} = a^K(\Pi^K_{p} \psi_{\tilde{e},\tilde{\ell}}, \Pi^K_{p} \psi_{e,\ell}) \delta_{e,\ell} \delta_{\tilde{e},\tilde{\ell}},
\]

where \( a^K(\cdot,\cdot) \) is given in (4). Note that this stabilization is in fact a modification of the original D-recipe stabilization introduced in [9].

We highlight that, by taking the analogue of (24) for PWVEM, one does not recover numerically the expected theoretical rate of convergence of the method. On the other hand, (23) cannot be used directly in ncTVEM due to the fact that plane wave directions are filtered out on each edge, but are not removed in the bulk, which would lead to dimensional inconsistencies when using (23).

Numerical quantities. Given a wave number \( k > 0 \) and letting \( k_n \) be the discrete wave number in Definition 2.1, we will study the following quantities:

- **dispersion error** \( |\text{Re}(k - k_n)| \), which describes the difference of the propagation velocities of the continuous and discrete plane wave solutions;
- **dissipation error** \( |\text{Im}(k_n)| = |\text{Im}(k - k_n)| \), which represents the difference of the amplitudes (damping) of the continuous and discrete plane wave solution;
- **total error** \( |k - k_n| \), which measures the total deviation of the continuous and discrete wave numbers.

4.1 Dependence of dispersion and dissipation on the Bloch wave angle \( \theta \)

In this section, we study dispersion and dissipation of the different methods in dependence on the angle \( \theta \) of the direction \( d \) in the definition of the Bloch wave in (3). Importantly, we are here interested in a qualitative comparison of the methods, rather than a quantitative one, which
should be performed in terms of the dimensions of the minimal generating subspaces instead of
the effective degrees, and which is discussed in Section 4.2.

To this purpose, in Figures 5-7, the numerical quantities \(|\text{Re}(k - k_n)|\) and \(|\text{Im}(k_n)|\) are plotted
against \(\theta\) for the meshes made of squares, triangles, and hexagons, respectively, shown in Figure 4.
We took \(k = 3\) and \(q = 7\) for all those types of meshes (Figures 5-7, left). Moreover, for \(k = 10\),
we chose \(q = 10\) for the squares (Figure 5, right) and the triangles (Figure 6, right), and \(q = 13\)
for the hexagons (Figure 7, right). We remark that the latter choice for \(q\) on the meshes made
of hexagons is purely for demonstration purposes, in order to obtain a reasonable range for the
errors, where one can see the behavior more clearly. Moreover, we recall that the wave number \(k\)
here (mesh size \(h = 1\)) corresponds to the wave number \(k_0 = \frac{k}{k_0}\) on a mesh with mesh size \(h_0\).

Figure 5: Dispersive and dissipative behavior of PWDG, PWVEM, and ncTVEM in dependence on the polar
angle \(\theta\) of the Bloch wave direction \(\mathbf{d}\) in (3) on the meshes made of squares in Figure 4, with \(k = 3\)
and \(q = 7\) (left), and \(k = 10\) and \(q = 10\) (right). The color legend is the same as in Figure 5.

Figure 6: Dispersive and dissipative behavior of PWDG, PWVEM, and ncTVEM in dependence on the polar
angle \(\theta\) of the Bloch wave direction \(\mathbf{d}\) in (3) on the meshes made of triangles in Figure 4, with \(k = 3\)
and \(q = 7\) (left), and \(k = 10\) and \(q = 1\) (right). We notice that dispersion and dissipation are zero, up to machine precision, for choices of
the Bloch wave direction \(\mathbf{d}\) in (3) coinciding with one of the plane wave directions \(\{\mathbf{d}_j\}_{j=1}^p\) (here we always took equidistributed directions \(\mathbf{d}_j\), where \(\mathbf{d}_1 = (1, 0)\)). This follows directly from
the fact that, in this case, the Bloch wave satisfying (5) coincides with the corresponding plane
wave traveling along the direction \(\mathbf{d}\). Moreover, we observe that, for PWVEM and ncTVEM,
Figure 7: Dispersive and dissipative behavior of PWDG, PWVEM, and ncTVEM in dependence on the polar angle $\theta$ of the Bloch wave direction $\mathbf{d}$ in (3) on the meshes made of hexagons in Figure 1, with $k = 3$ and $q = 7$ (left), and $k = 10$ and $q = 13$ (right). The color legend is the same as in Figure 5.

Remark 2. We highlight that, in the case of VEM, the dissipation and dispersion behavior also hinges upon the choice of stabilization. To this purpose, for PWVEM, we compare the results obtained when employing the standard stabilization in (23) with two alternative stabilizations that also lead to the correct convergence behavior for the discretization error in practice. More precisely, let $\Pi_{\nabla,K}^1 : \widetilde{\mathcal{V}}_0^n(K) \to \mathcal{P}_1(K) \subset \widetilde{\mathcal{V}}_0^n(K)$ be the projector onto polynomials of degree at most one, defined by

$$
\int_K \nabla(\Pi_{\nabla,K}^1 v_n) \cdot \nabla p_1 \, dx = \int_K \nabla v_n \cdot \nabla p_1 \, dx \quad \forall p_1 \in \mathcal{P}_1(K)
$$

for all $v_n \in \widetilde{\mathcal{V}}_0^n(K)$, where $V^K_i$, $i = 1, \ldots, n_K$, are the vertices of $K$; see [7,8]. We consider

- **standard**, which is the stabilization defined in (23);
- **stab 1**, which is the stabilization one gets by replacing $\phi^K_r$ and $\phi^K_s$ on the right-hand side of (22) with $\Pi_{\nabla,K}^1 \phi^K_r$ and $\Pi_{\nabla,K}^1 \phi^K_s$, respectively;
- **stab 2**, the resulting stabilization after substituting the right-hand side of (22) by

$$
\delta_{r,s} \left[ \nabla(\Pi_{\nabla,K}^1 \phi^K_r) \cdot \nabla(\Pi_{\nabla,K}^1 \phi^K_s) w^K_j w^K_l + k^2 \langle \mathbf{d}_j \cdot \mathbf{d}_k - 1 \rangle (\Pi_{\nabla,K}^1 \phi^K_r)(\Pi_{\nabla,K}^1 \phi^K_s) w^K_j w^K_l \right].
$$

In Figure 8, we plot the dispersion error $| \text{Re}(k - k_n) |$ for the three stabilizations, $k = 3$ and $q = 6$, on the meshes made of squares and triangles. The dissipation is zero, up to machine precision, in all cases and is thus not shown. One can observe a different behavior between stab 1 and the other stabilizations.

4.2 Exponential convergence of the dispersion error against the effective degree $q$

Here, we investigate the dependence of dispersion and dissipation on the effective plane wave degree $q$ (namely, $p = 2q + 1$ bulk plane waves). For fixed wave number $k$, we will observe exponential convergence of the total error for increasing $q$, as already seen in [15] for PWDG. This result is not unexpected since also the $p$-versions for the discretization errors have exponential convergence, provided that the exact analytical solution is smooth; see [19] for PWDG, and the
numerical experiments in [27] and in [24] for PWVEM and ncTVEM, respectively. Moreover, we will make a comparison of these methods in terms of the total error versus the dimensions of the minimal generating subspaces.

To this purpose, we consider the following range for the wave number: $k \in \{2, 3, 4, 5\}$. We recall again that $k$ here corresponds in fact to $k_0 = \frac{k}{k_0}$ on a mesh with mesh size $h_0$.

**Dispersion and dissipation vs. effective degree $q$.** In Figures 9-11, the relative dispersion error $|\text{Re}(k - k_n)|/k$ and the relative damping error $|\text{Im}(k_n)|/k$ are displayed against $q$, for the meshes made of squares, triangles, and hexagons, respectively. The maxima of the relative dispersion and the relative dissipation, respectively, are taken over a large set of Bloch wave directions $d$. One can observe, after some preasymptotic regime, exponential convergence of the dispersion error for all methods, and of the dissipation error for PWDG. Apart from some instabilities, the dissipation is close to machine precision for PWVEM. Furthermore, the dispersion error is consistently smaller for PWDG than for PWVEM and ncTVEM.
Dispersion and dissipation vs. dimensions of minimal generating subspaces. From a computational point of view, it is also important to consider a comparison of the dispersion errors in terms of the dimensions of the minimal generating subspaces (density of the degrees of freedom). We directly compare the relative total errors $|k_n - k|/k$, thus measuring the total deviation of the discrete wave number from the continuous one. As above, the maxima over a large set of Bloch wave directions are taken. In Figure 12 those errors are displayed for the meshes in Figure 1. For ncTVEM, we can recognize the cliff effect, meaning that, at some point, the dispersion error decreases without increase of the dimension of the minimal generating subspace. This effect has already been noticed in [24, 25] for the discretization error and is a peculiarity of the orthogonalization-and-filtering process mentioned in Section 3.2. Moreover, one can observe a direct correlation between the density of the degrees of freedom, which depends on the shape of the meshes, see Figures 2 and the error plots (larger cardinalities of the fundamental sets lead to larger errors; as mentioned above, for ncTVEM, the filtering process leads to dimensionality reductions).

Comparison with standard FEM. Here, we highlight the advantages of using full Trefftz methods (ncTVEM, PWDG) or methods that make use of Trefftz functions (PWVEM) in com-
Comparison to standard polynomial based methods, such as FEM, whose dispersion properties were studied in e.g. [1, 6, 13, 21]. We focus for simplicity on the meshes made of squares in Figure 1, since, in this case, the basis functions in FEM have a tensor product structure and an explicit dispersion relation can be derived [1, Theorem 3.1]:

$$\cos(k_n) = R_q(k),$$

where, denoting by $\lfloor \cdot \rfloor_z \cot z$ and $\lfloor \cdot \rfloor_z \tan z$ the Padé approximants to the functions $z \cot z$ and $z \tan z$, respectively,

$$R_q(2z) := \frac{2N_0/2N_0 - 2}{2N_e/2N_e - 2} z \cot z - \frac{[2N_0 + 2/2N_e] z \tan z}{[2N_0 + 2/2N_e] z \cot z + [2N_e + 2/2N_e] z \tan z},$$

with $N_0 := \lfloor (q + 1)/2 \rfloor$ and $N_e := \lfloor q/2 \rfloor$. From (25), one can see that only dispersion plays a role in FEM. In Figure 13, we display the relative total errors against the effective degree $q$ (left) and against the dimensions of the minimal generating subspaces (right) for fixed $k = 3$. Similar results are obtained for other values of $k$ and are not shown. One can clearly notice that the dispersion error for FEM is lower than for the other methods, when comparing it in terms of $q$, but higher, when comparing it in terms of the dimensions of the minimal generating subspaces.

4.3 Algebraic convergence of the dispersion error against the wave number $k$

We study the dispersion and dissipation properties of the three methods with respect to the wave number $k$. Due to the fact that $h = 1$, and $k$ is related to the wave number $k_0$ on a mesh with
Figure 13: Comparison of the relative total errors for PWVEM, ncTVEM, PWDG, and the standard polynomial based FEM on a mesh made of squares as in Figure 1 for fixed wave number $k = 3$, in dependence on the effective/polynomial degree $q$ (left) and the dimension of the minimal generating subspaces (right). The maxima over a large set of Bloch wave directions $d$ are taken.

For numerical experiments, we fix the effective degrees $q = 3, 5, 7$. We employ once again the meshes made of squares and triangles in Figure 1. Similar results have been obtained on the mesh made of hexagons. In Figure 14, the relative total errors $|k - k_n|/k$ determined over a large set of Bloch wave directions $d$ are depicted against $k$. Algebraic convergence can be observed. Furthermore, larger values of $q$ lead to smaller errors. The peaks occurring in the convergence regions of PWVEM and ncTVEM could be related to the presence of Neumann eigenvalues, and Dirichlet and Neumann eigenvalues, that have to be excluded in the construction of PWVEM and ncTVEM, respectively, in order to have a well-posed variational formulation, see Sections 3.1 and 3.2, or also [27] and [23, 24]. Moreover, the oscillations for larger and smaller values of $k$ are related to the pre-asymptotic regime and the instability regime, which are typical of wave based methods.

In Table 1, we list some relative total errors for different values of $k$. They indicate a convergence behavior of

$$\max |k - k_n|/k \approx O(k^\eta), \quad k \to 0,$$

where $\eta \in [2q - 1, 2q]$. This was already observed in [15] for PWDG.

| $q = 3$ | squares | | | triangles | | |
|---|---|---|---|---|---|---|
| **method** | $k$ | $|k - k_n|$ | $k$ | $|k - k_n|$ | rate | $k$ | $|k - k_n|$ | $k$ | $|k - k_n|$ | rate |
| PWVEM | 2 | 1.50e-03 | 0.3 | 4.50e-08 | 5.48 | 2 | 2.71e-04 | 0.3 | 3.42e-09 | 5.95 |
| ncTVEM | 2 | 9.04e-03 | 0.3 | 3.69e-07 | 5.33 | 2 | 1.07e-03 | 0.3 | 4.09e-08 | 5.36 |
| PWDG | 2 | 1.71e-03 | 0.3 | 1.04e-07 | 5.11 | 2 | 3.87e-04 | 0.3 | 3.04e-08 | 4.98 |

| $q = 5$ | squares | | | triangles | | |
|---|---|---|---|---|---|---|
| PWVEM | 2 | 3.68e-06 | 0.8 | 5.09e-10 | 9.70 | 3 | 2.17e-05 | 2 | 4.54e-07 | 9.53 |
| ncTVEM | 2 | 6.48e-06 | 0.8 | 1.21e-09 | 9.37 | 3 | 5.91e-06 | 2 | 1.47e-07 | 9.11 |
| PWDG | 2 | 4.56e-07 | 0.8 | 1.47e-10 | 8.77 | 3 | 7.75e-07 | 2 | 1.97e-08 | 9.06 |

| $q = 7$ | squares | | | triangles | | |
|---|---|---|---|---|---|---|
| PWVEM | 4 | 1.55e-05 | 2 | 2.23e-09 | 12.76 | 6 | 7.79e-05 | 4 | 5.57e-07 | 12.19 |
| ncTVEM | 4 | 5.93e-06 | 2 | 6.54e-10 | 13.15 | 6 | 6.01e-06 | 4 | 3.39e-08 | 12.77 |
| PWDG | 4 | 2.92e-07 | 2 | 2.33e-11 | 13.62 | 6 | 7.10e-07 | 4 | 2.76e-09 | 13.69 |
Figure 14: Relative total dispersion in dependence on the wave number $k$ for fixed effective degrees $q = 3, 5, 7$. The maxima over a large set of Bloch wave directions $d$ are taken. As meshes, those made of squares (left) and triangles (right) in Figure 1 are employed.

Remark 3. Clearly, similarly as above, dispersion and dissipation can be investigated again separately from each other. Here, we only show the results, depicted in Figure 15 for fixed $q = 5$ and varying $k$ on the meshes made of squares. As already observed, one can deduce that PWVEM and ncTVEM are dispersion dominated, whereas dissipation plays a major role for PWDG.

Figure 15: Relative dispersion (left) and relative dissipation (right) in dependence on the wave number $k$ for fixed $q = 5$ on the meshes made of squares in Figure 1. The maxima over a large set of Bloch wave directions $d$ are taken.

5 Conclusions

We investigated numerically the dispersion and dissipation properties of the (conforming) plane wave virtual element method of [27] and of the nonconforming Trefftz virtual element method in [23, 24]. Moreover, we compared the results to those obtained in [15] for the plane wave discontinuous Galerkin method. Similarly to what already noticed there for PWDG, dispersion and dissipation hinge upon the choice of the Bloch wave direction also in the cases of PWVEM and ncTVEM. Furthermore, we observed a link to the level of conformity. Whereas the dissipation error is zero (up to machine precision) in the convergence regime for conforming methods, such as PWVEM and FEM, it is much larger and even dominates the dispersion error for the fully discontinuous PWDG. For ncTVEM, dispersion dominates dissipation, and the dissipation error is in general not zero, but is in most cases lower than for PWDG. In the case of PWVEM, we have also seen
that the dispersion error depends on the choice of stabilization. Additionally, we noticed for all methods exponential convergence of the relative total error with respect to the effective plane wave degree \( q \), for \( q \to \infty \). The dispersion error is consistently smaller for PWDG than for PWVEM and ncTVEM, when measured in terms of \( q \), however, when compared to the dimensions of the minimal generating subspaces, the results depend on the element geometry, and thus on the density of the degrees of freedom. Concerning the comparison of the total error with respect to the wave number \( k \), as \( k \to 0 \), algebraic convergence was observed. There, larger values of \( q \) lead to smaller errors. Finally, the comparison with the standard polynomial based FEM highlighted the advantages of employing Trefftz based methods, such as ncTVEM and PWDG, or methods that make use of Trefftz functions, like PWVEM, over standard polynomial based methods.

Acknowledgements The authors have been funded by the Austrian Science Fund (FWF) through the project F 65 (I.P.) and the project P 29197-N32 (I.P. and A.P.), and by the Doctoral Program (DK) through the FWF Project W1245 (A.P.).

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