Abstract—This paper proposes a new analysis of graph using the concept of electric potential, and also proposes a graph simplification method based on this analysis. Suppose that each node in the weighted-graph has its respective potential value. Furthermore, suppose that the start and terminal nodes in graphs have maximum and zero potentials, respectively. When we let the level of each node be defined as the minimum number of edges/hops from the start node to the node, the proper potential of each level can be estimated based on geometric proportionality relationship. Based on the estimated potential for each level, we can re-design the graph for path-finding problems to be the electrical circuits, thus Kirchhoff’s Circuit Law can be directed applicable for simplifying the graph for path-finding problems.

I. INTRODUCTION

The graph as a research topic is still held important positions in various fields, such as mathematics, theoretical computer science [1]–[6], networks, machine learning, and quantum computing [7]–[9]. In particular, many researchers in various fields are working for graph simplification because human beings are faced with complex graphs problems on many issues. Recently, various research topics related to graph simplification have been proposed in many areas. The summary of related research areas about graph simplification is as follows, i.e., data mining [10], decentralized composite optimization [11], network model simplification [12]–[16], biological network analysis [17], scheduling [18], and clustering coefficient [19]. As such, interest in graph simplification dramatically increases. Therefore, this paper proposes a novel graph simplification algorithm.

The graph simplification algorithm, which is introduced in this paper is based on graph analysis using electric potential. Our graph analysis method uses the concept of node level, this means the minimum number of edges from the start node to the node. By using the concept of levels and potentials, we create various transformed graphs and present a methodology for analyzing them with a geometric approach. The ultimate purpose of the methodology that we introduce is to estimate the entire voltage of the graph. By estimating the entire voltage of the graph and mapping the edge cost of the graph to the resistance, we can estimate the values of each current flowing through each edge using Kirchhoff’s Circuit Law. Then, we can remove the edges, which have no current flow. This is an overview of the graph simplification algorithm.

II. BACKGROUND

In this section, we briefly describe two background concepts, i.e., electric potential and Kirchhoff’s Circuit Law.

A. Electric Potential

The electric field \( \mathbf{E} \) is a special vector function whose curl is always zero [20], [21] where the \( \mathbf{E} \) can represent as the vector sum of each electric field produced by each charge as:

\[
\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \cdots + \mathbf{E}_N,
\]

(1)

According to Stokes’ theorem [22],

\[
\oint \mathbf{E} \cdot d\mathbf{l} = 0
\]

(3)
is satisfied by (2). By (3), following scalar function is defined:

\[
V(r) = -\int_o^r \mathbf{E} \cdot d\mathbf{l},
\]

(4)

where \( o \) is a standard reference point, and \( r \) is a target point. The scalar function defined in (4) is called electric potential, and the differential form of (4) is as follows:

\[
\mathbf{E} = -\nabla V.
\]

(5)

In particular, in circuit theory, the difference of electric potential with these characteristics is called voltage.

B. Kirchhoff’s Circuit Law

Kirchhoff’s Circuit Law consists of Kirchhoff’s Current Law (KCL) and Kirchhoff’s Voltage Law (KVL) [23].

- Kirchhoff’s Current Law: KCL, also called Kirchhoff’s first law, states that the sum of incoming currents at a junction is equal to the sum of outgoing currents at the junction. If we define the sign of incoming currents at the junction as positive and the sign of outgoing currents
at the junction as negative, this law can be represented
that the sum of the currents at each junction is zero as:

$$\sum_{n=1}^{x} I_n = 0,$$

where $I_n$ is each current $n$ and $x$ is the number of
incoming or outgoing currents. In other words, KCL is
the same as the law of conservation of charge.

- Kirchhoff’s Voltage Law: KVL, also called Kirchhoff’s
second law, states that the sum of voltages of the closed-
circuit loop is zero. If we define the sign of voltage
similar to KCL’s currents case, this law can be as:

$$\sum_{n=1}^{y} V_n = 0,$$

where $V_n$ is each voltage $n$ and $y$ is the number of
voltages measured in the closed-circuit loop. In other
words, KVL is the same as the law of energy conservation
in a complete closed-circuit.

III. GRAPH ANALYSIS USING ELECTRIC POTENTIAL

This section describes the graph analysis method, which
combines the concept of each electric potential to each node,
before describing the graph simplification algorithm.

A. Motivation

Suppose that there is a graph $G(v, e)$ in two-dimensional
space where $v$ and $e$ represent nodes and edges, respectively.
We have to search for a logically dependable path from
the start to the terminal on $G$. We already know a variety of simple
methods for searching paths, e.g., greedy algorithms [24], [25].
However, if the original graph $G$ is too complex to find the
logically dependable path in a simple way, it is essential to
simplify the graph before searching for a logically dependable
path, e.g., the shortest path, the minimum cost path, and so
forth. Obviously, prior to simplifying the graph, we need to
analyze the graph with various perspectives. As one of the
new perspectives, we introduce the method of graph analysis
using electric potential in the next section.

B. Graph Analysis using Electric Potential

We can interpret the graph as an electric circuit by regarding
the electric resistances, junctions, and the voltage from the
power supply as edges, nodes, and electric potential difference
between the start and the terminal. If there is a voltage $\varepsilon$
applied to the graph, then the start has an electric potential $\varepsilon$
and the terminal has an electric potential 0. Then, eventually,
the rests of the nodes (i.e., non-start and non-terminal nodes)
have electric potentials between $\varepsilon$ and 0.

Each junction in the electric circuit has its own electric
potential, and no current flows through the resistance between
the junctions with the same electric potential. In other words,
if there are edges directly connected between nodes with the
same electric potential in the graph, no current will flow
through the edges. Thus, we may remove those edges in the
graph. By removing the edges between equipotential nodes,
columns in a straight line, has several analytical advantages. In Fig. 4 we can connect the highest point in the start node column where the edges first extend, the lowest point in the terminal node column where the edges finally gather, and the floor point of the start node column. These three points make a right triangle. Based on this right triangle, we can add some definitions to estimate the length of each side of the right triangle. And this allows us to determine the voltage of the entire graph.

**Voltage Estimation.** The entire voltage of the N-level graph can be estimated as a positive value less than the sum of the minimum costs between the level k − 1 and level k, where N and k are positive integers, 1 ≤ k ≤ N.

Before the description, suppose that there is an N-level graph, where N is a positive integer. Transform this graph in the form shown in Fig. 4 and then connect the highest point in the start node column, the lowest point in the terminal node column, and the floor point of the start node column to make a right triangle. The right triangle can be created as shown in Fig. 5. The entire voltage of the graph is equal to the electric potential of the level 0 node, which is equal to the length of \( \triangle SOT \). The straight line of each level represents each plane perpendicular to the plane where \( \triangle SOT \) is located. \( \triangle SOT \) represents the shortest straight distance passing through the ideal nodes, that maybe existed or not. \( \triangle SOT \) is shorter than any path, including the orange, red and green lines shown in

\[ \overline{ST} = \min C_{(0), (1)} + \min C_{(1), (2)} + \cdots + \min C_{(N-1), (N)} = \sum_{k=1}^{N} \min C_{(k-1), (k)}, \]  

where \( \min \) means the minimum value. Since \( \overline{ST} \) is the length of the longest side of \( \triangle SOT \), the entire voltage corresponding to the length of \( \overline{SO} \) should be smaller than \( \overline{ST} \). Thus, we can estimate the entire voltage of the graph as a positive value less than the sum of the minimum costs between adjacent levels.

**Definition 1.** Let \( \overline{ST} \) be the sum of the minimum costs between adjacent levels.

If we expressed the the edge costs between level \( k-1 \) and level \( k \) as \( C_{(k-1), (k)} \), then the length of \( \overline{ST} \) is as follows:

\[ \overline{ST} = \min C_{(0), (1)} + \min C_{(1), (2)} + \cdots + \min C_{(N-1), (N)} = \sum_{k=1}^{N} \min C_{(k-1), (k)}, \]  

where \( \min \) means the minimum value. Since \( \overline{ST} \) is the length of the longest side of \( \triangle SOT \), the entire voltage corresponding to the length of \( \overline{SO} \) should be smaller than \( \overline{ST} \). Thus, we can estimate the entire voltage of the graph as a positive value less than the sum of the minimum costs between adjacent levels.

**Definition 2.** Let the ratio of the lengths between adjacent levels which divide \( \overline{OT} \) be equal to the ratio of average costs between adjacent levels.

The \( \overline{OT} \) is the sum of the lengths between adjacent levels:

\[ \overline{OT} = \sum_{k=1}^{N} L_{(k-1), (k)}, \]  

where \( L_{(k-1), (k)} \) is the lengths between level \( k-1 \) and level \( k \). We do not estimate the length of \( \overline{OT} \) directly as a specific value, but we estimate the ratio of the lengths between adjacent levels that divide \( \overline{OT} \). The ratio of the lengths between adjacent levels is as follows:

\[ L_{(0), (1)} : L_{(1), (2)} : \cdots : L_{(N-1), (N)} = \text{avg}C_{(0), (1)} : \text{avg}C_{(1), (2)} : \cdots : \text{avg}C_{(N-1), (N)}, \]  

where \( \text{avg} \) means the average value. According to the similarity in geometry, this ratio is equal to the ratio of the lengths between adjacent ideal points which divide \( \overline{ST} \). Since we estimated the length of \( \overline{ST} \) from (8), each appropriate length between adjacent ideal points also can be estimated with this ratio. This can be interpreted as an appropriate small edge costs between adjacent levels.

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Fig. 3. 3-D potential graph. The original graph in Fig. 1 can be transformed into this graph form. The height of each node column corresponds to each node’s electric potential. Each level has a circular orbit, and each node column can be located only at the corresponding level orbit.

Fig. 4. Straight-line alignment by the position moving of each node column at each level orbit in Fig. 3. The three level 1 node columns are overlapped and the two level 2 node columns are overlapped.

Fig. 5. Geometric analysis of the N-level graph from Fig. 4. The light blue points S, T, and O represent the highest point in the start node column, the lowest point in the terminal node column, and the floor point of the start node column, respectively. The purple points represent the highest points of each node columns, and the gray points represent the ideal points at each level. The orange, red, and green lines represent examples of different paths, and the black area represents black-box sections between level 2 and level N − 1.
Algorithm 1 Graph Simplification

Volatage Estimation
Input: \( G(v, e) \)
Output: \( V_{\text{max}} \)
1: Determine the level of each node in the given graph \( G \).
2: Transform \( G \) to level unit graph \( G'(v', e') \).
3: Transform \( G' \) to 3-D potential graph and align in a straight line.
4: Construct a right triangle with three points: \( O, S, T \).
5: Place ideal nodes of each level on \( ST \).
6: Estimate \( ST = \sum_{k=1}^{N} \min C_{k-1}(k) \).
7: Choose one real number \( x \in \Pi \), where \( \Pi = \{x | 0 < x < ST\} \).
8: return \( x \).

Edge Removal via Sub-current Estimation
Input: \( G(v, e), V_{\text{max}} = x \)
Output: \( G'_{\text{removal}}(v, e') \)
9: Map \( G \) to an electric circuit as \( C(e_j) \rightarrow \text{Resistance}_j \).
10: Set the sub-currents of \( \text{Resistance}_j \) as \( I_j \).
11: Create the system of linear equations using KCL and KVL.
12: Solve the system of linear equations and obtain \( I_1 \cdots I_j \).
13: for int \( i = 1; i \leq j; i++ \) do
14: if \( I_i = 0 \) then
15: Remove \( e_i \) on \( G \).
16: \( G = G_{\text{removal}}, e_i \).
17: end if
18: end for
19: return \( G \).

At the end of this section, we’ve shown the various transformations and interpretations of graphs through the concept of hypothetical electric potentials and also described hypothetical voltage estimation and appropriate edge cost estimation using two definitions. The related example is in Sec. IV-B.

IV. GRAPH SIMPLIFICATION USING KIRCHHOFF’S CIRCUIT LAW

This section describes the graph simplification algorithm based on the graph analysis and the voltage estimation of the entire graph, that are described in the previous section.

A. Proposed Algorithm

The algorithm consists of two processes, voltage estimation and edge removal via sub-current estimation.

A summary of the first process, voltage estimation, which described in the previous section. At first, determine the level of each node in the given graph \( G(v, e) \), with \( |v| = i \) nodes and \( |e| = j \) edges, where \( i \) and \( j \) are positive integers. Construct a right triangle with three points: origin \( O = (0, 0, 0) \), start node \( S = (0, V_{\text{max}}, 0) \), and terminal node \( T = (L, 0, 0) \), where \( V_{\text{max}} \) and \( L \) are positive real numbers, and \( L < ST \). Place ideal nodes of each level within \( ST \) in level order. In other words, as shown in Fig. [5] ideal nodes should be placed sequentially on the diagonal of the right triangle. Calculate the estimated length of \( ST \) as shown in (8). Choose one of any positive real number smaller than the length of \( ST \), and determined as \( V_{\text{max}} \). Normally, the value of \( V_{\text{max}} \) is chosen from positive integer values.

The second process, edge removal via sub-current estimation, is performed as follows. Consider each edge costs (or weights) as each resistance, and consider \( V_{\text{max}} \), which is determined in the first process, as an entire voltage of graph. Set the sub-currents flowing at each edge of the graph as \( I_1 \cdots I_j \), and create the system of linear equations using KCL and KVL. Solve the system of linear equations and find the values of the sub-currents \( I_1 \cdots I_j \). To find the values of the sub-currents, the number of required linear equations is maximum \( j \). Remove the edges with zero sub-currents.

This algorithm is effective when there are many equipotential nodes with no current flowing between them. In a complex graph, there is also a high probability of many equipotential nodes, so utilization is expected.

B. Case Study

In this section, we explain the graph simplification algorithm with a simple example.

Suppose that, there is a weighted-graph, refer to Fig. [6]. We need to find the shortest path in a given graph \( G(5, 9) \), and we want to simplify this graph first. The levels of nodes \( S, a, b, c, \) and \( T \) in this graph correspond to 0, 1, 1, 1, and 2, respectively. Thus, we can convert graph \( G \) to level unit graph \( G'(3, 9) \). Through the process 3 to 6 of the algorithm a right triangle with points \( S = (0, V_{\text{max}}, 0), O = (0, 0, 0), \) and \( T = (L, 0, 0) \) can be formed, and the length of \( ST \) can be estimated as:

\[
ST = \min C_{(0),(1)} + \min C_{(1),(2)} = \min(2, 3, 1) + \min(4, 6, 3) = 1 + 3 = 4.
\]
Through the process 7 to 8 of Algorithm [1], \( V_{\text{max}} \) can be a real number where \( 0 < V_{\text{max}} < 4 \), i.e., we set \( V_{\text{max}} = 3 \).

We consider each edge cost as each resistance and find the sub-current flowing through each edge. Fig. 7 shows the sub-currents flowing through each edge by applying KCL. Since there are 6 sub-currents in this graph, we may create the system of linear equations with 6 closed-circuit loops according to KVL are as follows:

\[
\begin{align*}
3 &= 2I_2 + 4I_4, \\
3 &= I_1 + 6I_6, \\
3 &= I_1 + 3(I_1 - I_5 - I_6), \\
3 &= 3I_3 + 6(I_2 + I_3 - I_4 + I_5), \\
0 &= 3I_3 - 4I_5 - I_1, \\
0 &= 2I_2 + (I_2 - I_4) - 3I_3.
\end{align*}
\]  
(12)

The values of sub-currents \( I_1, I_2, I_3, I_4, I_5, \) and \( I_6 \) are obtained from (12) as \( 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \text{ and } \frac{1}{3} \), respectively. We confirm that there is no sub-current flowing through the edges \( ab \) and \( cb \) (i.e., \( I_{ab} = I_2 - I_4 = 0 \) and \( I_{cb} = I_5 = 0 \)), so they will be removed. Fig. 8 shows a simplified graph \( G^*(5, 7) \) with two edges removal. In \( G^* \), there are only four paths from \( S \) to \( T \), i.e., we can easily find the shortest path.

In addition, due to Definition 2, the ratio of the lengths between adjacent level which divides \( \overline{ST} \) is obtained as:

\[
L_{(0),(1)} : L_{(1),(2)} = \frac{\text{avg}C_{(0),(1)} : \text{avg}C_{(1),(2)}}{	ext{avg}(2, 3, 1) : \text{avg}(4, 6, 6, 3)} = \frac{19}{4} = 8 : 19.
\]  
(13)

This ratio is equal to the ratio of the lengths between adjacent ideal points which divide \( \overline{ST} \), so \( \overline{SD} \) and \( \overline{DT} \) are as:

\[
\begin{align*}
\overline{SD} &= \left( \frac{8}{8 + 19} \right) \overline{ST} \approx 1.185, \\
\overline{DT} &= \left( \frac{19}{8 + 19} \right) \overline{ST} \approx 2.815,
\end{align*}
\]  
(14)

where \( D \) is an ideal point of level 1 and \( \overline{TD} = 4 \) is an obtained value from (11). \( \overline{SD} \) and \( \overline{DT} \) can be considered as relatively small edge costs between level 0 to 1 and level 1 to 2, respectively. In Fig. 6, the closest to \( \overline{SD} = 1.185 \) among the edge costs between level 0 to 1 is 1 and the closest to \( \overline{DT} = 2.815 \) among the edge costs between levels 1 to 2 is 3. These two edges correspond to the actual shortest path, represented by the orange path in Fig. 8. This is because, in this example, all the edges between the same level nodes are removed by the graph simplification algorithm.

V. CONCLUSIONS AND FUTURE WORK

In this paper, we have proposed graph analysis methods using electric potentials, such as level unit transformed graph analysis, straight-line alignment 3-D potential graph analysis, and geometric analysis. Based on these analysis methods, a graph simplification algorithm using Kirchhoff’s Circuit Law has been proposed. The graph simplification algorithm consists of two processes, i.e., voltage estimation and edge removal via sub-current estimation. We have applied this algorithm to the example graph, and have demonstrated its usefulness as a preprocessing algorithm for path-finding problems. Thus, we also have confirmed the validity of the definitions underlying the proposed graph simplification algorithm.

As a future work direction, we will figure out which kinds of applications can be useful; and then conduct data-intensive performance evaluations.
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