Asymptotic Results for Heavy-tailed Lévy Processes and their Exponential Functionals

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December 25, 2019

Abstract

In this paper we first provide several conditional limit theorems for Lévy processes with negative drift and regularly varying tail. Then we apply them to study the asymptotic behavior of expectations of some exponential functionals of heavy-tailed Lévy processes. As the key point, we observe that the asymptotics mainly depends on the sample paths with early arrival large jump. Both the polynomial decay rate and the exact expression of the limit coefficients are given. As an application, we give an exact description for the extinction speed of continuous-state branching processes in heavy-tailed Lévy random environment with stable branching mechanism.

AMS Subject Classification (2010): Primary 60G51, 60J55; Secondary 60K37, 60J80

Keywords: Lévy processes; Regular variation; Conditional limit theorem; Exponential functional; Branching processes; Random environment; Survival probability

1 Introduction

The long-run behavior of Lévy processes and their functionals has been explored in a significant literature in the past decades. This is mainly justified by the wide and important applications in various fields such as risk theory, mathematical finance, physics and population evolution. In this work, we mainly explore the asymptotic behavior of Lévy processes and their exponential functionals in the presence of regularly varying tails.

Functional limit theorems for random walk \( \{ Z_n : n = 0, 1, \cdots \} \) conditioned to stay positive have been studied by many authors. Let \( \tau_0^d \) denote its first passage time in \((-\infty, 0]\). For the oscillating case, under the Spitzer’s condition \([1, 17, 19]\) showed that for some regularly varying function \( b_n \) with index \( \alpha \in (0, 2] \) the rescaled process \( \{ b_n^{-1} Z_{n[t]} : t \in [0, 1] | \tau_0^d > n \} \) converges weakly to the meander of a strictly \( \alpha \)-stable process. Vatutin and Wachtel \([37]\) studied the asymptotic behavior of local probabilities \( \mathbb{P}\{ \tau_0^d = n \} \) and \( \mathbb{P}\{ Z_n \in [x, x + \delta] | \tau_0^d = n \} \) for large \( n \) and \( x \). For random walk with negative drift and regularly varying tail, Durrett \([20]\) proved, in the special case that \( Z_1 \) has finite variance, that \( \{ Z_{n[t]} / n, t \in [0, 1] | Z_0 > 0 \} \) and \( \{ Z_{n[t]} / n, t \in [0, 1] | \tau_0^d > 0 \} \) converge weakly to a non-degenerate limit with sample paths having a single jump at time 0 and decreasing linearly, which also have been partially extended by Doney and Jones \([18]\) to Lévy processes with infinite variance. In the first part of this work, we study the asymptotic behavior of heavy-tailed Lévy process \( \{ \xi_t : t \geq 0 \} \) with negative drift conditioned to stay positive, which will play an important role to study their exponential functionals. Denote by \( \tau_0 \) the first passage time of \( \xi \) in \((-\infty, 0]\). We first show that the \( \tau_0 > t \) for large \( t \) if and only if the process \( \xi \) can stay positive before the arrival of large jumps. Then we show that spatial-scaled process \( \{ \xi_s / t, s \geq 0 | \tau_0 > t \} \) converges weakly to a non-degenerate limit. This is analogous but different to that in Durrett \([20]\). Indeed, the limit process here has sample paths which are step functions with a single jump. The single jump occurs at random time which has a size-biased distribution. Moreover, we also provide exact expressions for the decay rate of the Laplace transforms \( \mathbb{E}[e^{-\lambda \xi_t}, \xi_t > 0] \) and \( \mathbb{E}[e^{-\lambda \xi_t}, \tau_0 > t] \).

As the second main contribution, we apply the conditional limit theorems of Lévy processes to analyze the long-term behavior of their exponential functionals

\[
A_t(\xi) = \int_0^t e^{-\xi s} ds, \quad 0 \leq t \leq \infty, \tag{1.1}
\]

\(^*\)Financial support from the Alexander-von-Humboldt-Foundation is gratefully acknowledged.

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Here $A_\infty(\xi) = \infty$ is allowed. The study of exponential functionals has drawn the attention of many researchers because of the considerable role it plays in mathematical finance [24], statistics physics [13] and population evolution in random environment [4, 11]. A necessary and sufficient condition for $A_\infty(\xi) < \infty$ a.s. was given in Bertoin and Yor [8], i.e., $A_\infty(\xi) < \infty$ if and only if $\xi$, drifts to infinity. In this case, the characterizations and Wiener-Hopf type factorization of the law of $A_\infty(\xi)$ can be founded in [11, 34, 35, 40]. For more interesting results and properties of $A_\infty(\xi)$, reader may refer to the recent works [6, 36, 39] and references therein. In the case of $A_\infty(\xi) = \infty$, we are usually interested in the asymptotic behavior of $F(A_t(\xi))$ for some positive, decreasing function $F$ on $(0, \infty)$ that vanishes at $\infty$. Specially, many attention has been drawn to the decay rate of the following expectation as $t \to \infty$:

$$E[F(A_t(\xi))] = E\left[F\left(\int_0^t e^{-\xi_s} ds\right)\right].$$  \hfill (1.2)

which is closely related to the long-term properties of random processes in random environment; see [11, 27]. To the best of my knowledge, almost all previous works, except of [36], usually considered the expectation (1.2) with $\xi$ satisfying some exponential moment condition, i.e., the Laplace exponent $\phi(\lambda) := \log E[\exp\{\lambda \xi_1\}] < \infty$ for some $\lambda > 0$. Here we give a brief summary for them. Readers may refer to [30, 33] and references therein for details. For Lévy processes with bounded variations, Carmona et al. [11] provided a precise asymptotic behavior for $E[(A_t(\xi))^-p]$ for some constant $p > 0$ satisfying $\phi'(p) > 0$. Applying the discretization technique and the asymptotic results proved in [25], Bansaye et al. [4] provided four regimes for the decay rate of the expectation (1.2) with $\xi$ is a compound Poisson processes and $F(z) \sim Cz^{-\lambda}$. Their approach was extended by Palau et al. [33]. By observing that the asymptotics only depends on the sample paths with slowly decreasing local inifimum, Li and Xu [30] provided not only four different regimes for the convergence rate of the expectation (1.2) but also the exact expressions for the limiting coefficients. Their proofs heavily rely on the fluctuation theory and limit theorems of Lévy processes conditioned to stay positive.

However, the exponential moment condition cannot be satisfied by Lévy processes with regularly varying tails, e.g. stable processes. To my knowledge, Patie and Savov [36] is the only one that provided an exact description for the asymptotic behavior of the expectation (1.2) with $\xi$ is oscillating and satisfies the Spitzer’s condition. In the second part of this work, we study the asymptotic behavior of the expectation (1.2) in the case where $\xi$ has negative drift and regularly varying tail. Like the observation in Li and Xu [30], the expectation (1.2) with large $t$ is mainly contributed by the sample paths with slowly decreasing local inifimum. Inspired by the analysis in Vatutin and Zheng [38] for the discrete time setting and the conditional limit theorems proved in the first part, we notice that these sample paths can be identified according to the arrival time of their first large jumps. This not only help us to establish the polynomial decay for the expectation (1.2), but also allows us to determine the limit coefficients by the conditional limit theorems of the functional of Lévy process starting from a large jump.

To illustrate the strength of our asymptotic results for the expectation (1.2), in the third part of this work we study the long-run behaviors of continuous-state branching processes in heavy-tailed random environment. Let $\{Z^\gamma_t : t \geq 0\}$ be a spectrally positive $(\gamma + 1)$-stable process with $0 < \gamma \in (0, 1]$. Here we consider $Z^\gamma$ as a Brownian motion when $\gamma = 1$. For any $x \geq 0$, we consider the unique strong solution to the following stochastic integral equation:

$$X_t(x) = x + \int_0^t \gamma + 1) cX_u(x) dZ^\gamma_u + \int_0^t X_u(x) dZ^\gamma_u. \hfill (1.3)$$

where $c \geq 0$ and $\{Z^\gamma_t : t \geq 0\}$ a Lévy process with no jump less than $-1$. The solution is called a continuous-state branching process in random environment (CBRE-process) with stable branching mechanism and Lévy random environment $Z^\gamma$; see [26, 32] for more details. The construction of CBREs as the scaling limit of rescaled Galton-Watson processes in random environment (GWRE-processes) can be found in [2, 28]. From Section 4 in [26], there is another Lévy process $\{\xi_t : t \geq 0\}$ determined by the environment so that the survival probability of the CBRE-process at time $t \geq 0$ is given by

$$P\{X_t(x) > 0\} = E\left[1 - \exp\{-x(c\gamma A_t(\xi))^\gamma\}\right], \hfill (1.4)$$

which clearly is a special case of (1.2). Compared to the abundant literature about survival probabilities of GWRE-processes; e.g. [1, 5, 23, 25, 38] and references therein, there are only several works about survival probabilities of CBRE-processes can be found. With the expression (1.4), the asymptotics of survival probability were studied in [4, 10, 30, 31, 33] with $\xi$ satisfying some exponential moment condition. Recently, Bansaye et al. [3] studied the extinction speed of CBRE-processes with general branching mechanism and oscillating environment $\xi$. They
provided the exact expression for the extinction rate under a necessary condition that is $E[e^{\theta^+ \xi_t}] < \infty$ for some $\theta^+ > 1$. This excludes the CBRE-processes with the right-tail of random environment is regularly varying. Here we provide an exact expression for asymptotic behavior of survival probabilities of CBRE-processes with stable branching mechanism and heavy-tailed environment. In details, the survival probabilities decrease to 0 at a polynomial rate. This is in sharp contrast with the exponential decay of CB-processes and CB-process in light-tailed Lévy random environment; see [30].

The remainder of this paper is organized as follows. In Section 2, we recall definitions and some basic elements of fluctuation theory for Lévy processes, which are necessary for the proofs. In Section 3, we give several conditional limit theorems for Lévy processes with negative drift and regularly varying tail. The exact expression of asymptotic behavior of the expectation (1.2) is given in Section 4. In Section 5, we study the decay rate of survival probabilities of CBRE-processes.

**Notation.** For any $x \in \mathbb{R}$, let $[x]$ be its integer part and $x^+ := x \vee 0$. We make the convention that for any $t_1 \leq t_2 \in \mathbb{R}$,

$$\int_{t_1}^{t_2} = \int_{(t_1, t_2]} \quad \text{and} \quad \int_{t_1}^{\infty} = \int_{(t_1, \infty]}.$$

**2 Preliminaries**

In this section, we recall some basic notation and elements of fluctuation theory for Lévy processes. The reader may refer to [7, 29] for details. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual hypotheses and $D([0, \infty), \mathbb{R})$ the space of càdlàg paths endowed with Skorokhod topology. Denote by $\xi := \{\xi_t : t \geq 0\}$ a one-dimensional Lévy process with characteristic exponent $\Phi(\lambda) := -\log E[\exp(\imath \lambda \xi_1)]$ has the following formula:

$$\Phi(\lambda) = \imath a \lambda + \frac{1}{2} \sigma^2 \lambda^2 + \int_{\mathbb{R}} \left(1 - e^{\imath \lambda u} + \imath \lambda u\right) \nu(du), \quad (2.1)$$

where $\nu(du)$ is called Lévy measure with $\int_{\mathbb{R}} (u^2 + \nu(du)) < \infty$. For any probability measure $\mu$ on $\mathbb{R}$, we denote by $P_\mu$ and $E_\mu$ the law and expectation of the Lévy process $\xi$ started from $\mu$, respectively. When $\mu = \delta_x$ is a Dirac measure at point $x$, we write $P_x = P_{\delta_x}$ and $E_x = E_{\delta_x}$. For simplicity, we also write $P = P_0$ and $E = E_0$.

We write $S = \{S_t : t \geq 0\}$ and $I = \{I_t : t \geq 0\}$ for the supremum and infimum processes,

$$S_t := \sup\{0 \vee \xi_s : 0 \leq s \leq t\} \quad \text{and} \quad I_t := \inf\{0 \wedge \xi_s : 0 \leq s \leq t\}.$$

Let $S - \xi := \{S_t - \xi_t : t \geq 0\}$ and $\xi - I := \{\xi_t - I_t : t \geq 0\}$ be the reflected processes at supremum and infimum respectively, which are Markov processes with Feller transition semigroups; see Proposition 1 in [7, p.156]. For any $t > 0$, the last passage times by $\xi$ at its supremum and at its infimum before $t$ are defined by

$$G_t := \sup\{s \leq t : \xi_s = S_t \text{ or } \xi_{s-} = S_t\} \quad \text{and} \quad g_t := \sup\{s \leq t : \xi_s = I_t \text{ or } \xi_{s-} = I_t\}.$$

For any $x \leq 0$, we also define the first passage time of $\xi$ in $(-\infty, x]$ by

$$\tau_x := \inf\{t > 0 : \xi_t \leq x\}.$$

Let $L = \{L_t : t \geq 0\}$ be the local time of $S - \xi$ at zero in the sense of [7, p.109]. Its inverse local time process $L^{-1} := \{L_t^{-1} : t \geq 0\}$ is defined by

$$L_t^{-1} = \begin{cases} \inf\{s > 0 : L_s > t\}, & t < L_\infty; \\ \infty, & \text{otherwise}. \end{cases}$$

The ladder height process $H = \{H_t : t \geq 0\}$ of $\xi$ is defined by

$$H_t = \begin{cases} \xi_{L_t^{-1}}, & t < L_\infty; \\ \infty, & \text{otherwise}. \end{cases}$$

By Lemma 2 in [7, p.157], the two-dimensional process $(L^{-1}, H)$ is a Lévy process (possibly killed at an exponential rate) and is well known as the ladder process of $\xi$. It is usually characterized as follows: for any $\lambda, u \geq 0$,

$$E[\exp(-\lambda L_t^{-1} - uH_t)] = \exp(-\kappa(\lambda, u)), \quad (2.2)$$


where the bivariate exponent $\kappa(\lambda, u)$ is given by

$$\kappa(\lambda, u) = k \exp \left\{ \int_0^\infty dt \int_{(0,\infty)} \left( e^{-t} - e^{-\lambda t - u x} \right) P\{ \xi_t \in dx \} \right\},$$

(2.3)

see Corollary 10 in [7, p.165-166]. Here the constant $k > 0$ is determined by the normalization of the local time. The renewal function $V$ associated to the ladder height process $H$ is defined by

$$V(x) = \int_0^\infty P\{ H_t \leq x \} dt = E \left[ \int_0^\infty 1_{\{S_t \leq x\}} dL_t \right], \quad x \geq 0;$$

(2.4)

see, [7, p.171]. For the reflected process $\xi - I$, we can define the local time at 0, the inverse local time process, the ladder height process and the renewal function in the same way as for $S - \xi$. They are denoted by $\hat{L}$, $\hat{L}^{-1}$, $\hat{H}$ and $\hat{V}$ respectively. Denote by $\hat{\kappa}(\lambda, u)$ the characteristic exponent of the ladder process $(\hat{L}^{-1}, \hat{H})$ with constant $k > 0$. In this work, we choose some suitable normalization of the local times such that $k = \hat{k} = 1$.

From (2.1), the Lévy process $\xi$ starting from $\xi_0$ admits the following Lévy-Itô’s decomposition:

$$\xi_t = \xi_0 - at + \sigma B_t + \int_0^t \int_{\mathbb{R}} u \tilde{N}(ds, du),$$

(2.5)

where $B$ is a standard Brownian motion, $N(ds, du)$ is a Poisson random measure with intensity $ds\nu(du)$ and $\tilde{N}(ds, du) := N(ds, du) - ds\nu(du)$. For any $x > 0$, we define $\xi^x := \{\xi_t^x : t \geq 0\}$ by removing from $\xi$ all jumps larger than $x$, i.e.

$$\xi_t^x := \xi_0 - \left( a + \int_x^\infty u \nu(du) \right) t + \sigma B_t + \int_0^t \int_{-\infty}^x u \tilde{N}(ds, du),$$

(2.6)

which again is a Lévy process with characteristic exponent $\Phi^x(\lambda) := -\log E[\exp\{i \lambda \xi^x_1\}]$ has the following formula:

$$\Phi^x(\lambda) = i \left( a + \int_x^\infty u \nu(du) \right) \lambda + \frac{1}{2} \sigma^2 \lambda^2 + \int_{-\infty}^x (1 - e^{i \lambda u} + \chi \Lambda u) \nu(du),$$

(2.7)

### 3 Asymptotic results for Lévy processes with negative drift

In this section, we provide several conditional limit theorems for Lévy process $\xi$ with negative drift ($a > 0$) under the following condition.

**Condition 3.1** There exit a constant $\alpha > 1$ and a slowly varying function $\ell(x)$ at $\infty$ such that as $x \to \infty$,

$$P\{ \xi_1 > x \} \sim x^{-\alpha} \cdot \ell(x).$$

(3.1)

Actually, from (2.5)-(2.6), we see that the right tail of $\xi_1$ is light and hence $\xi_1 \geq x$ for large $x$ if and only if there are large jumps happen in $[0, 1]$, i.e.,

$$P\{ \xi_1 > x \} = P\{ \xi_1 + \int_0^1 \int_{x}^\infty u N(ds, du) > x \} \sim P\{ \int_0^1 \int_x^\infty u N(ds, du) > x \}. $$

From this and [21, Proposition 4.1], we have $P\{ \xi_1 > x \} \sim 1 - \exp\{-\nu(x, \infty)\}$ and hence Condition 3.1 holds if and only if as $x \to \infty$,

$$\bar{\nu}(x) := \nu(x, \infty) \sim x^{-\alpha} \cdot \ell(x).$$

(3.2)

#### 3.1 Limit theorems conditioned on $\xi_t \geq 0$

In this section, we study the number and location of large jumps conditioned on $\xi_t > 0$ for large $t$. For any $x, t > 0$, let $N^x_t$ be the number of jumps of $\xi$ larger than $x$ up to time $t$, i.e.,

$$N^x_t := \# \{ s \leq t : \Delta \xi_s := \xi_s - \xi_{s-} > x \} = \int_0^t \int_x^\infty N(ds, du).$$

(3.3)
Denote by $J^x$ the arrival time of the first jump in $(x, \infty)$, i.e.,

$$J^x := \inf\{s \geq 0 : \Delta \xi_s > x\}.$$ 

From properties of Poisson processes, we see that for any $k \in \mathbb{Z}_+$ and $t > 0$,

$$\mathbb{P}\{N^x_t > 0\} = 1 - \exp\{-\bar{v}(x) \cdot t\} \quad \text{and} \quad \mathbb{P}\{N^x_t = k\} = \frac{|\bar{v}(x) \cdot t|^k}{k!} \cdot \exp\{-\bar{v}(x) \cdot t\}. \quad (3.4)$$

From these and Condition 3.1, we have as $t \to \infty$,

$$\mathbb{P}\{N^{\alpha \bar{t}}_t > 0\} \sim \mathbb{P}\{N^{\alpha \bar{t}}_t = 1\} \sim \bar{v}(at) \cdot t. \quad (3.5)$$

Moreover, [18, Theorem 4] provided the following asymptotic equivalences: as $t \to \infty$,

$$\mathbb{P}\{\xi_t > 0\} \sim t \cdot \mathbb{P}\{\xi_1 > at\} \sim \bar{v}(at) \cdot t. \quad (3.6)$$

The following proposition proves that $\xi_t > 0$ for large $t$ is and only if there has been one jump larger than $at$ occurred before time $t$.

**Proposition 3.2** The two events $N^{\alpha \bar{t}}_t = 1$ and $\xi_t > 0$ are asymptotically equivalent as $t \to \infty$, i.e.,

$$\mathbb{P}\{N^{\alpha \bar{t}}_t = 1 | \xi_t > 0\} \sim \mathbb{P}\{\xi_t > 0 | N^{\alpha \bar{t}}_t = 1\} \to 1. \quad (3.7)$$

**Proof.** From Condition 3.1, for any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that $\int_{t_\epsilon}^{\infty} u \nu(du) < \epsilon$. From (2.5) and (2.6), for any $b \in (a + 2\epsilon, a + 3\epsilon)$ and $t > t_\epsilon$, conditioned on the event $N^{\bar{t}}_t = 1$ we have

$$\xi_t \geq \xi^{bt}_t + \Delta \xi_{\bar{t}} \geq \xi^{bt}_t + bt.$$ 

Since $\xi^{bt}_t$ is independent of $N^{\bar{t}}_t$ for any $t > t_\epsilon$, we have

$$\mathbb{P}\{\xi_t > 0, N^{\alpha \bar{t}}_t = 1\} \geq \mathbb{P}\{\xi^{bt}_t + bt > 0, N^{\alpha \bar{t}}_t = 1\} = \mathbb{P}\{\xi^{bt}_t + bt > 0\} \mathbb{P}\{N^{\alpha \bar{t}}_t = 1\}.$$ 

Notice that $E[\xi^{bt}_t + b] \geq \epsilon$ and hence $\mathbb{P}\{\xi^{bt}_t + bt > 0\} \to 1$ as $t \to \infty$. From (3.5),

$$\lim_{t \to \infty} \inf \mathbb{P}\{\xi_t > 0 | N^{\alpha \bar{t}}_t = 1\} \geq \lim_{b \to a+} \lim_{t \to \infty} \frac{\mathbb{P}\{\xi_t > 0, N^{\alpha \bar{t}}_t = 1\} \mathbb{P}\{N^{\alpha \bar{t}}_t = 1\}}{\mathbb{P}\{N^{\alpha \bar{t}}_t = 1\}} \mathbb{P}\{\xi_t > 0\} = 1. \quad (3.8)$$

Here we have gotten that $\mathbb{P}\{\xi_t > 0 | N^{\alpha \bar{t}}_t = 1\} \to 1$ as $t \to \infty$. On the other hand, by (3.5), (3.6) and (3.8),

$$\lim_{t \to \infty} \inf \mathbb{P}\{N^{\alpha \bar{t}}_t = 1 | \xi_t > 0\} \geq \lim_{b \to a+} \lim_{t \to \infty} \frac{\mathbb{P}\{\xi_t > 0, N^{\alpha \bar{t}}_t = 1\} \mathbb{P}\{N^{\alpha \bar{t}}_t = 1\}}{\mathbb{P}\{N^{\alpha \bar{t}}_t = 1\}} \mathbb{P}\{\xi_t > 0\} = 1.$$ 

Here we have finished the proof. \qed

The following theorem shows that conditioned on $\xi_t > 0$ for large $t$, the only jump larger than $at$ occurs at a uniformly distributed time in $[0, t]$. This also can be gotten from [18, Theorem 6].

**Theorem 3.3** There exists a uniformly distributed random variable $U$ on $[0, 1]$ such that $\{J^t / |\xi_t > 0\} \to U$ in distribution as $t \to \infty$.

**Proof.** For any $U > 0$, it suffices to prove that $\mathbb{P}\{J^t / |\xi_t > 0\} \to (1 - U)^+$. From Proposition 3.2, we have for $t > 0$ large enough,

$$\mathbb{P}\{J^t / |\xi_t > 0\} \sim \mathbb{P}\{J^t / |\xi_t > 0, N^{\alpha \bar{t}}_t = 1\} \sim \mathbb{P}\{N^{\alpha \bar{t}}_t = 0 | N^{\alpha \bar{t}}_t = 1\},$$

which equals 0 if $U > 1$. When $U \in [0, 1]$, from the independent increments of $\xi$ and (3.4),

$$\mathbb{P}\{N^{\alpha \bar{t}}_t = 0 | N^{\alpha \bar{t}}_t = 1\} = \frac{\mathbb{P}\{N^{\alpha \bar{t}}_t = 0\} \mathbb{P}\{N^{\alpha \bar{t}}_t = 1\}}{\mathbb{P}\{N^{\alpha \bar{t}}_t = 1\}} = \frac{e^{-U t \cdot \bar{v}(at)} \cdot \bar{v}(at) (1 - U) \cdot e^{-\bar{v}(at)(1 - U)t}}{\bar{v}(at) e^{-\bar{v}(at)(1 - U)t}} = 1 - U.$$ 

\qed
3.2 Limit theorems conditioned to stay positive

In this section, we provide several conditional limit theorems for the Lévy process \( \xi \) conditioned to stay positive. For any \( x > 0 \), from Proposition 17 in [7, p.172] we have \( E_x[\tau_0] = C \cdot \bar{V}(x) < \infty \) for some constant \( C > 0 \). Denisov and Slafer [16, Theorem 3.2] proved that as \( t \to \infty \),

\[
P_x\{\tau_0 > t\} \sim E_x[\tau_0] \frac{P(\xi_t > 0)}{t} \sim E_x[\tau_0] P\{\xi_1 > at\} \sim E_x[\tau_0] \cdot \bar{\nu}(at).
\]

Comparing this to Proposition 3.2, we see that it is not enough to keep \( \xi \) staying positive by only knowing that a large jump has occurred. The following theorem shows that conditioned to stay positive the arrival time of first large jump is distributed as a size-biased distribution, which induces that the early arrival of the first large jump is necessary.

**Theorem 3.4** For any \( x > 0, T \geq 0 \) and \( b \geq a \), we have as \( t \to \infty \),

\[
P_x\{\Delta \xi_{jt} \geq bt, J^at \leq T, \tau_0 > t\} \to (b/a)^{-\alpha} \cdot \frac{E_x[\tau_0; \tau_0 > T]}{E_x[\tau_0]}.
\]

**Proof.** We first prove this result with \( b > a \). Notice that \( J^at \) is a stopping time for any \( t > 0 \). Let \( \hat{\xi} \) be an independent copy of \( \xi \). From the strong Markov property of \( \xi \),

\[
P_x\{\Delta \xi_{jt} > bt, J^at \leq T, \tau_0 > t\} = P_x\{\Delta \xi_{jt} > bt, J^at \leq T, \tau_0 > J^at, P_{\hat{\xi},t} \{\inf_{r \in [0,t-J^at]} \hat{\xi}_r > 0\}\}.
\]

Conditioned on \( J^at \leq T, \tau_0 > \hat{J}^at \) and \( \Delta \xi_{jt} > bt \), for large \( t \) we have \( \xi_{jt} > bt \) and

\[
P_{\hat{\xi},t} \{\inf_{r \in [0,t-J^at]} \hat{\xi}_r > 0\} \geq P \{\inf_{r \in [0,t]} \hat{\xi}_r > -\xi_{jt}\} \geq P\{I_t > -bt\},
\]

which goes to 1 as \( t \to \infty \) because of \( E[\xi_1] = a < b \) and hence

\[
P_x\{\Delta \xi_{jt} > bt, J^at \leq T, \tau_0 > t\} \geq P\{I_t > -bt\} \cdot P_x\{\Delta \xi_{jt} > bt, J^at \leq T, \tau_0 > J^at\}
\]

\[
\sim P_x\{\Delta \xi_{jt} > bt, J^at \leq T, \tau_0 > J^at\}.
\]

Together with the fact that \( P_x\{\Delta \xi_{jt} > bt, J^at \leq T, \tau_0 > t\} \leq P_x\{\Delta \xi_{jt} > bt, J^at \leq T, \tau_0 > J^at\} \) for any \( t > T \), we have as \( t \to \infty \),

\[
P_x\{\Delta \xi_{jt} > bt, J^at \leq T, \tau_0 > t\} \sim P_x\{\Delta \xi_{jt} > bt, J^at \leq T, \tau_0 > J^at\}.
\]

To get the desired result, it suffices to prove that

\[
\lim_{t \to \infty} \frac{P_x\{\Delta \xi_{jt} > bt, J^at \leq T, \tau_0 > J^at\}}{P_x\{\tau_0 > t\}} = (b/a)^{-\alpha} \cdot \frac{E_x[\tau_0; \tau_0 > T]}{E_x[\tau_0]}.
\]

Obviously, we have \( \{\mathcal{N}^b_T = 1\} \subset \{\mathcal{N}^b_{J^at} = 1, J^at \leq T\} = \{\Delta \xi_{jt} > bt, J^at \leq T\} \subset \{\mathcal{N}^b_T = 1\} \cup \{\mathcal{N}^b_{\bar{J}^at} > 1\} \). From (3.4) and (3.9), we have as \( t \to \infty \),

\[
P_x\{\mathcal{N}^b_T > 1\} = 1 - e^{-\bar{\nu}(bt)T} - \bar{\nu}(bt)Te^{-\bar{\nu}(bt)T} \sim \frac{|\bar{\nu}(bt) \cdot T|^2}{2} = o(P_x\{\tau_0 > t\})
\]

and

\[
\frac{P_x\{\Delta \xi_{jt} > bt, J^at \leq T, \tau_0 > J^at\}}{P_x\{\tau_0 > t\}} \sim \frac{P_x\{\mathcal{N}^b_T \geq 1, \tau_0 > J^bt\}}{P_x\{\tau_0 > t\}}.
\]

Let \( I^b = \{I^b_s : s \geq 0\} \) be the infimum process of \( \xi^b \). We see that \( \xi^b \) is independent of \( J^b \) and \( \xi^b_s = \xi_s \) for any \( s \leq \bar{J}^b \). These induce that \( \tau_0 > J^b \) if and only if \( I_s = I^b_s > 0 \) for any \( s \leq \bar{J}^b \). Thus

\[
P_x\{\mathcal{N}^b_T \geq 1, \tau_0 > J^b\} = P_x\left\{ \int_0^T \int_{bt} \mathbf{1}_{\{I_s^b \geq 0\}} N(dr, dx) \geq 1 \right\}
\]
= 1 - E_x \left[ \exp \left\{ -\bar{\nu}(bt) \int_0^T 1_{I^*_{bt} > 0} dr \right\} \right] \sim \bar{\nu}(bt) \int_0^T P_x \{ I^*_{bt} > 0 \} dr. \quad (3.12)

Since \( \sup_{s \in [0,T]} |I^*_{bt} - I_s| \to 0 \) a.s. as \( t \to \infty \), we have

\[
P_x\{ \bigwedge_{T^* \geq 1}, \tau_0 > J^{bt} \} \sim \bar{\nu}(bt) \int_0^T P_x \{ I_r > 0 \} dr = \bar{\nu}(bt) \int_0^T P_x \{ \tau_0 > r \} dr.
\] \quad (3.13)

Taking this back into (3.11), from (3.2) we have as \( t \to \infty \),

\[
\frac{P_x\{ \Delta \mathcal{X}_{J^{at}} > bt, J^{at} \leq t, \tau_0 > J^{at} \}}{P_x\{ \tau_0 > t \}} \sim \frac{\bar{\nu}(bt) \int_0^T P_x \{ \tau_0 > r \} dr}{\bar{\nu}(at)} \frac{E_x[\tau_0; \tau_0 \leq T]}{E_x[\tau_0]}. \quad (3.14)
\]

Now we consider the case with \( b = a \). From the previous result, we have

\[
\liminf_{t \to \infty} P_x\{ J^{at} \leq t | \tau_0 > t \} \geq \lim_{t \to a+} \lim_{t \to \infty} P_x\{ \Delta \mathcal{X}_{J^{at}} > ct, J^{at} \leq t | \tau_0 > t \} = \frac{E_x[\tau_0; \tau_0 \leq T]}{E_x[\tau_0]}. \]

Moreover, like the deduction in (3.12)-(3.14) we also have as \( t \to \infty \)

\[
\limsup_{t \to \infty} P_x\{ J^{at} \leq t | \tau_0 > t \} \leq \lim_{t \to a} \sup_{t \to \infty} P_x\{ J^{ct} \leq t | \tau_0 > t \}
\leq \lim_{t \to a} \sup_{t \to \infty} \frac{P_x\{ \bigwedge_{T^* \geq 1}, \tau_0 > J^{ct} \}}{P_x\{ \tau_0 > t \}}
\leq \lim_{t \to a} \frac{E_x[\tau_0; \tau_0 \leq T]}{E_x[\tau_0]} = \frac{E_x[\tau_0; \tau_0 \leq T]}{E_x[\tau_0]}.
\]

The desired result for \( b = a \) follows directly from these two results. \( \square \)

From (3.9), we see that \( P_x\{ \tau_0 > t \} \sim E_x[\tau_0] \cdot \bar{\nu}(at) \sim 1 \) for \( x > 0 \) large enough, which means that \( \xi \) will stay positive for a long time after jumping into a set far away from the origin. This recommends us that \( \xi \) would stay positive for a long time if and only if it can stay positive until the arrival of the first large jump; see the following lemma.

**Lemma 3.5** For any \( x > 0 \), we have as \( t \to \infty \),

\[
P_x\{ \tau_0 > J^{at} \Delta \{ \tau_0 > t \} \} = o\left( P_x\{ \tau_0 > t \} \right). \quad (3.15)
\]

**Proof.** From the fact that \( E_x[\int_0^\infty 1_{I_s > 0} ds] = E_x[\tau_0] < \infty \), as proved in (3.12)-(3.13) we also have as \( t \to \infty \),

\[
P_x\{ \tau_0 > J^{at}, J^{at} \leq t \} = P_x\{ \tau_0 > J^{at}, \bigwedge_{T^* \geq 1} \} \sim \bar{\nu}(at) \int_0^\infty P_x\{ \tau_0 > s \} ds \sim E_x[\tau_0] \cdot \bar{\nu}(at)
\]

and hence from this and (3.9),

\[
\lim_{t \to \infty} \frac{P_x\{ \tau_0 > J^{at}, J^{at} \leq t \}}{P_x\{ \tau_0 > t \}} = 1.
\]

Applying Theorem 3.4 with \( b = a \), we also have

\[
\liminf_{t \to \infty} \frac{P_x\{ J^{at} \leq t, \tau_0 > t \}}{P_x\{ \tau_0 > t \}} \geq \lim_{t \to \infty} \lim_{t \to \infty} P_x\{ J^{at} \leq t | \tau_0 > t \} = \lim_{t \to \infty} \frac{E_x[\tau_0; \tau_0 \leq T]}{E_x[\tau_0]} = 1, \quad (3.16)
\]

which immediately induces that

\[
\lim_{t \to \infty} \frac{P_x\{ \tau_0 > J^{at}, \tau_0 > t \}}{P_x\{ \tau_0 > t \}} = \lim_{t \to \infty} \frac{P_x\{ \tau_0 > J^{at}, J^{at} \leq t \}}{P_x\{ \tau_0 > t \}} = 1 \quad (3.17)
\]

and

\[
\lim_{t \to \infty} \frac{P_x\{ J^{at} \leq t \}}{P_x\{ \tau_0 > t \}} = \frac{P_x\{ \tau_0 > J^{at} \}}{P_x\{ \tau_0 > t \}} - \lim_{t \to \infty} \frac{P_x\{ J^{at} \leq t | \tau_0 > t \}}{P_x\{ \tau_0 > t \}} = 0. \quad (3.18)
\]
Combining these two results, we would get the desired result, i.e. as $t \to \infty$,
\[
\frac{\Pr_x \{\tau_0 > J^at\} \Delta \{\tau_0 > t\}}{\Pr_x \{\tau_0 > t\}} = \frac{\Pr_x \{J^at < \tau_0 \leq t\}}{\Pr_x \{\tau_0 > t\}} + 1 - \frac{\Pr_x \{\tau_0 > J^at, \tau_0 > t\}}{\Pr_x \{\tau_0 > t\}} \to 0.
\]

From the previous results, we see that if $\xi_s > 0$ for all $s \in [0, t]$ and large $t$, there must has been a jump larger than at occurred very early. Moreover, after the large jumps the sample paths will stay in the high position for a long time. To describe this phenomena, the following theorem provides a limit theorem for the spatial-scaled process conditioned to stay positive. It shows that compared to the large jump, the effect of downward drift on the process in the future almost can be ignored. The similar discrete version of this theorem for random walks with finite variance was established by Durrett [20, Theorem 3.1].

**Theorem 3.6** Fix $x > 0$, let $T_x$ and $P_x$ be two independent positive random variables with $P\{T_x \geq t\} = E_x[\tau_0; \tau_0 \geq t]/E_x[\tau_0]$ for any $t \geq 0$ and $P\{P_{a,\alpha} \geq z\} = (z/\alpha)^{-\alpha}$ for any $z \geq \alpha$. Then $\{t^{-1}\xi_s, s \geq 0 | \tau_0 > t, \xi_0 = x\}$ converges to $\{P_{a,\alpha}1_{T_x < s}, s \geq 0\}$ weakly in $D([0, \infty), \mathbb{R})$ as $t \to \infty$.

**Proof.** It is easy to see that the desired result follows directly from the following two statements:

(i) For any fixed $T > 0$ and $\varepsilon > 0$, we have as $t \to \infty$,
\[
\lim_{t \to \infty} \Pr_x \left\{\sup_{s \in [0, T]} \left| t^{-1} \xi(s) - t^{-1} \Delta \xi_{J^at} \cdot 1_{[J^at, \infty)}(s) \right| \geq \varepsilon \ \bigg| \tau_0 > t \right\} = 0;
\]

(ii) As $t \to \infty$ we have $\{t^{-1} \Delta \xi_{J^at} \cdot 1_{[J^at, \infty)}(s) : s \geq 0 | \tau_0 > t, \xi_0 = x\}$ converges to $\{P_{a,\alpha}1_{T_x}(s) : s \geq 0\}$ weakly in $D([0, \infty), \mathbb{R})$.

For (i), we first decompose $\xi$ at the stopping time $J^at$ as follows: for any $s \geq 0$,
\[
\xi_s = \xi_{s \wedge J^at -} + \Delta \xi_{J^at}1_{[J^at, \infty)}(s) + \xi_{s \vee J^at} - \xi_{J^at}.
\]

From this and Lemma 3.5, we have for $t > 0$ large enough,
\[
\Pr_x \left\{\sup_{s \in [0, T]} \left| \xi(s) - \Delta \xi_{J^at}1_{[J^at, \infty)}(s) \right| \geq \varepsilon t \ | \tau_0 > t \right\} \sim \Pr_x \left\{\sup_{s \in [0, T]} \left| \xi_{s \wedge J^at -} + \xi_{s \vee J^at} - \xi_{J^at} \right| \geq \varepsilon t \ | \tau_0 > J^at \right\}
\]
\[
\leq \Pr_x \left\{\sup_{s \in [0, T]} \left| \xi_{s \vee J^at} - \xi_{J^at} \right| \geq \varepsilon t/2 \ | \tau_0 > J^at \right\}
\]
\[
+ \Pr_x \left\{\sup_{s \in [0, T]} \left| \xi_{s \wedge J^at -} \right| \geq \varepsilon t/2 \ | \tau_0 > J^at \right\}. \quad (3.19)
\]

By the strong Markov property and the independent increments of $\xi$,
\[
\Pr_x \left\{\sup_{s \in [0, T]} \left| \xi_{s \vee J^at} - \xi_{J^at} \right| \geq \varepsilon t/2 \ | \tau_0 > J^at \right\} = \Pr_x \left\\{\sup_{s \in [0, T]} \left| \xi_{s \wedge J^at -} - \xi_{J^at} \right| \geq \varepsilon t/2 \right\} \leq \Pr_x \left\{\sup_{s \in [0, T]} |\xi_s| > \varepsilon t/2 \right\},
\]
which vanishes as $t \to \infty$. For the last term in (3.19), from the fact that $\xi_s = \xi_{\alpha s}$ for any $s < J^at$ and the independence between $\xi_{\alpha s}$ and $J^at$, we have for any $M > 0$,
\[
\Pr_x \left\{\sup_{s \in [0, T]} \left| \xi_{s \wedge J^at -} \right| \geq \varepsilon t/2 \ | \tau_0 > J^at \right\} \leq \Pr_x \left\{\sup_{s \in [0, T]} |\xi_{s \wedge J^at -}| \geq \varepsilon t/2, J^at < M \right\} + \Pr_x \{J^at > M, \tau_0 > J^at\}
\]
\[
\leq \Pr_x \left\{\sup_{s \in [0, T]} \left| \xi_{\alpha s} \right| \geq \varepsilon t/2 \right\} \cdot \Pr_x \{J^at < M\} + \Pr_x \{J^at > M, \tau_0 > J^at\}.
\]

From (3.4) and Lemma 3.5, as $t \to \infty$ we have $\Pr\{J^at < M\} = \Pr(N^at_M \geq 1) \sim M\tilde{\nu}(at) \sim M\Pr\{\tau_0 > J^at\}$ and hence
\[
\Pr_x \left\{\sup_{s \in [0, T]} \left| \xi_{\alpha s} \right| \geq \varepsilon t/2 \right\} \cdot \Pr\{J^at < M\} = o(\Pr\{\tau_0 > J^at\}).
\]
From Theorem 3.4 and Lemma 3.5, we also have
\[
\lim_{M \to \infty} \lim_{t \to \infty} \frac{P_x \{ J^a_t > M, \tau_0 > J^a_t \}}{P_x \{ \tau_0 > J^a_t \}} = \lim_{M \to \infty} \lim_{t \to \infty} E_x[ J^a_t > M | \tau_0 > t ] = 0.
\]
Putting these estimates together, we have
\[
\lim_{t \to \infty} P_x \left\{ \sup_{s \in [0, T]} | \xi_{s, J^a_s -} | \geq \varepsilon t / 2 \right\} = 0
\]
and (i) follows. Now we start to prove (ii). From Lemma 3.5, it suffices to prove that \( \{ Y^t_s := t^{-1} \Delta \xi_{J^a_s}, 1_{(J^a_s, \infty)}(s) : s \in [0, 1] \} \) converges to \( \{ P_{a, 1}(x, \infty), s \in [0, 1] \} \) weakly as \( t \to \infty \) in \( D([0, \infty), \mathbb{R}) \). The convergence in the sense of finite-dimensional distributions follows directly from Theorem 3.4. Here we just need to prove the tightness. For any \( 0 \leq r_1 \leq r_2 \leq r_3 \leq 1 \), we see that for \( k = 1, 2 \),
\[
Y^t_{r_{k+1}} - Y^t_{r_k} = \begin{cases} t^{-1} \Delta \xi_{J^a_s}, & J^a_t \in (r_k, r_{k+1}); \\ 0, & \text{otherwise.} \end{cases}
\]
Hence \( |Y^t_{r_2} - Y^t_{r_1}| \cdot |Y^t_{r_3} - Y^t_{r_2}| = 0 \) a.s. and \( P \{ |Y^t_{r_2} - Y^t_{r_1}| \cdot |Y^t_{r_3} - Y^t_{r_2}| \geq \lambda | \tau_0 > J^a_t, \xi_0 = x \} = 0 \) for any \( \lambda, t > 0 \).

From Theorem 3.5 in [9, p. 124], the sequence \( \{ Y^t | \tau_0 > J^a_t, \xi_0 = x \} \) is tight in \( D([0, \infty), \mathbb{R}) \).

**Remark 3.7** For the random walk \( \{ \xi_k : k = 0, 1, \cdots \} \). For any \( x \geq 0 \), let \( \tau^d_x := \inf \{ k > 0 : \xi_k \leq -x \} \) and \( J^d_\delta := \inf \{ k \geq 1 : \xi_k - \xi_{k-1} > \delta \} \). For any \( x \geq 0 \), from [16, Theorem 3.2] we have as \( n \to \infty \),
\[
P_x \{ \tau^d_0 > n \} \sim E_x[ \tau^d_0 ] \cdot P \{ \xi_1 \geq an \}.
\]
Following the previous argument, we also can establish the discrete versions of Theorem 3.4 and 3.6 for \( \{ \xi_k : k = 0, 1, \cdots \} \) under Condition 3.1. Here we show the results without detailed proofs.

1. For any \( x \geq 0, b \geq a \) and \( N > 0 \),
\[
P_x \{ J^a_{bn} - J^a_{bn-1} > bn, J^a_n \leq \tau^d_n \} \to (b/a)^{-a} \cdot E_x[ \tau^d_0 : \tau^d_0 \leq N].
\]

2. \( \{ n^{-1} \xi_s, s \geq 0 : \tau^d_0 > n, \xi_0 = x \} \) converges to \( \{ P_{a, \lambda}(x, \infty), s \geq 0 \} \) weakly in \( D([0, \infty), \mathbb{R}) \) as \( t \to \infty \). Here \( P \{ T_x^d \geq N \} = E_x[ \tau^d_0 : \tau^d_0 \geq N ] / E_x[ \tau^d_0 ] \) for any \( N \geq 0 \).

### 3.3 Asymptotic results for conditional Laplace transforms

In this section, we provide limit theorems for the reflected processes together with several asymptotic results for the Laplace transforms of \( \xi_t \) conditioned to \( \xi_t > 0 \) for large \( t \) or stay positive. In the sequel of this section, we always assume both Condition 3.1 and the following condition hold.

**Condition 3.8** For any \( \delta > 0 \), we have \( P \{ \xi_1 \in (x, x + \delta) \} = \alpha x^{-1} P \{ \xi_1 > x \} \cdot \delta + o(\delta) \) as \( x \to \infty \).

As a continuous analogue of Corollary 2.1 in [14], the following lemma shows the asymptotic behavior of local probabilities for \( \xi_t \) and its proof will be given in Appendix.

**Lemma 3.9** For any \( \epsilon, \delta > 0 \), we have as \( t \to \infty \),
\[
\sup_{x \geq (\epsilon - a)t} \left| \frac{P_x \{ \xi_t \in [x, x + \delta] \}}{t \cdot P \{ \xi_1 \in [at + x, at + x + \delta] \} - 1 \right| \to 0.
\]

**Proposition 3.10** For any \( \lambda > 0 \) we have as \( t \to \infty \),
\[
E[e^{-\lambda \xi_t}; \xi_t \geq 0] \sim \frac{\alpha}{a \lambda} P \{ \xi_1 > at \} \quad \text{and} \quad E[e^{\lambda \xi_t}; \xi_t \leq 0] \sim \frac{\alpha}{a \lambda} \cdot P \{ \xi_1 > at \}.
\]
Proof. Here we just prove the first result and the second one can be proved similarly. For large $t$ we have

$$\mathbf{E}[e^{-\lambda \xi_t}; \xi_t \geq 0] = \mathbf{E}[e^{-\lambda \xi_t}; \xi_t \in [0, \sqrt{t}]) + o(e^{-\lambda \sqrt{t}}).$$

From Lemma 3.9 and Condition 3.8, for large $n$ fixed we have as $t \to \infty$,

$$\mathbf{E}[e^{-\lambda \xi_t}; \xi_t \in [0, \sqrt{t}]) \sim \sum_{k=0}^{[n\sqrt{t}]} \int_k^{(k+1)/n} e^{-\lambda x} \mathbf{P}\{\xi_t \in dx\} \sim \sum_{k=0}^{[n\sqrt{t}]} \int_k^{(k+1)/n} e^{-\lambda x} \mathbf{P}\{\xi_t \in at + dx\} \sim at \int_0^{\sqrt{t}} e^{-\lambda x} \mathbf{P}\{\xi_t > at\} dx \sim \frac{\alpha}{a\lambda} \mathbf{P}\{\xi_t > at\}.$$

\[\Box\]

As a preparation to study the asymptotic behavior of reflected processes, we provide the following useful proposition, which follows directly from Theorem 4(iii) in [12].

**Proposition 3.11** Assume that $f(t) > 0$ is regularly varying at $\infty$. For any two integrable functions $f_1, f_2$ satisfying that $f_1(t) \sim c_1 f(t)$ and $f_2(t) \sim c_2 f(t)$ as $t \to \infty$ with $c_1, c_2 > 0$, we have as $t \to \infty$,

$$\int_0^t f_1(t-s)f_2(s)ds \sim \left( c_1 \int_0^\infty f_2(s)ds + c_2 \int_0^\infty f_1(s)ds \right) \cdot f(t). \quad (3.24)$$

For any $z > 0$ and $u, v > 0$, from Theorem 45.2 and 45.7 in Sato (1999, Chapter 9) we have

$$\int_0^\infty z e^{-zt} \mathbf{E}[e^{-uS_t-v(S_t-\xi_t)}]dt = \exp \left\{ \int_0^\infty \frac{e^{-zt}}{t} (\mathbf{E}[e^{-u\xi_t}; \xi_t \geq 0] + \mathbf{E}[e^{v\xi_t}; \xi_t < 0]) - 1 \right\}. \quad (3.25)$$

From the representations of $\kappa(z, u)$ and $\hat{\kappa}(z, u)$, we also have

$$\int_0^\infty z e^{-zt} \mathbf{E}[e^{-uS_t-v(S_t-\xi_t)}]dt = \frac{\kappa(z, 0) \hat{\kappa}(z, 0)}{\kappa(z, u) \hat{\kappa}(z, v)}. \quad (3.26)$$

Moreover, from Frullani’s identity, we also have $\kappa(z, 0)\hat{\kappa}(z, 0) \sim C_0 z$ as $z \to 0+$ with

$$C_0 := \exp \left\{ - \int_0^\infty (1 - e^{-t}) \mathbf{P}\{\xi_t = 0\} \frac{dt}{t} \right\}. \quad (3.27)$$

From the dominated convergence theorem,

$$\int_0^\infty \mathbf{E}[e^{-uS_t-v(S_t-\xi_t)}]dt = \lim_{z \to 0+} \frac{1}{z} \frac{\kappa(z, 0) \hat{\kappa}(z, 0)}{\kappa(z, u) \hat{\kappa}(z, v)} = \frac{C_0}{\kappa(0, u) \hat{\kappa}(0, v)}. \quad (3.28)$$

Applying Proposition 17 in [7, p.172] and then integration by parts, we have

$$\int_0^\infty \mathbf{E}[e^{-uS_t-v(S_t-\xi_t)}]dt = C_0 \int_0^\infty e^{-ux} V(dx) \cdot \int_0^\infty e^{-vy} \hat{V}(dy). \quad (3.29)$$

We also can prove the following result for $(I, I - \xi)$ in the same way,

$$\int_0^\infty \mathbf{E}[e^{uI_s+v(I_s-\xi_s)}]ds = C_0 \int_0^\infty e^{-ux} \hat{V}(dx) \cdot \int_0^\infty e^{-vy} V(dy). \quad (3.30)$$

Moreover, since Laplace transform is one-to-one, we have for any $x, y \geq 0$,

$$\int_0^\infty \mathbf{P}\{S_s \leq x, S_s - \xi_s \leq y\}ds = C_0 V(x) \hat{V}(y) \quad \text{and} \quad \int_0^\infty \mathbf{P}\{-I_s \leq x, \xi_s - I_s \leq y\}ds = C_0 \hat{V}(x) V(y). \quad (3.31)$$

In the following two lemmas, we provide asymptotic results for the joint laws of the supremum/infimum processes and their reflected processes.
Lemma 3.12 For any \( u, v > 0 \), we have as \( t \to \infty \),
\[
\frac{at}{\mathbb{P}\{\xi_1 > at\}} \mathbb{E}[e^{-uS_t-v(S_t-\xi_t)}] \to \alpha \left(\frac{1}{u} + \frac{1}{v}\right) \int_0^\infty \mathbb{E}[e^{-uS_s-v(S_s-\xi_s)}]ds, \tag{3.32}
\]
\[
\frac{at}{\mathbb{P}\{\xi_1 > at\}} \mathbb{E}[e^{uI_t+v(I_t-\xi_t)}] \to \alpha \left(\frac{1}{u} + \frac{1}{v}\right) \int_0^\infty \mathbb{E}[e^{uI_s+v(I_s-\xi_s)}]ds. \tag{3.33}
\]

Proof. Here we just prove the first result and the second one can be proved similarly. Differentiating both sides of (3.25) with respect to \( z \), we have
\[
\int_0^\infty (1-zt)e^{-zt} \mathbb{E}[e^{-uS_t-v(S_t-\xi_t)}]dt = -\int_0^\infty ze^{-zt} \mathbb{E}[e^{-uS_t-v(S_t-\xi_t)}]dt \int_0^\infty e^{-zs} \left( \mathbb{E}[e^{-u\xi_s}; \xi_s \geq 0] + \mathbb{E}[e^{v\xi_s}; \xi_s < 0] - 1 \right)ds
\]
and
\[
\int_0^\infty e^{-zt} \cdot t \cdot \mathbb{E}[e^{-uS_t-v(S_t-\xi_t)}]dt = \int_0^\infty e^{-zt} \mathbb{E}[e^{-uS_t-v(S_t-\xi_t)}]dt \int_0^\infty e^{-zs} \left( \mathbb{E}[e^{-u\xi_s}; \xi_s \geq 0] + \mathbb{E}[e^{v\xi_s}; \xi_s < 0] \right)ds
\]
\[
= \int_0^\infty e^{-zt} dt \int_0^t \mathbb{E}[e^{-uS_t-v(S_t-\xi_t)}] \left( \mathbb{E}[e^{-u\xi_s}; \xi_t-s \geq 0] + \mathbb{E}[e^{v\xi_s}; \xi_t-s < 0] \right)ds.
\]

Since Laplace transform is one-to-one, we have
\[
t \cdot \mathbb{E}[e^{-uS_t-v(S_t-\xi_t)}] = \int_0^t \mathbb{E}[e^{-uS_t-v(S_t-\xi_t)}] \left( \mathbb{E}[e^{-u\xi_s}; \xi_t-s \geq 0] + \mathbb{E}[e^{v\xi_s}; \xi_t-s < 0] \right)ds. \tag{3.34}
\]
Moreover, we also have
\[
\mathbb{E}[e^{-uS_t-v(S_t-\xi_t)}] \leq \mathbb{E}[e^{-uS_t}; \xi_t \geq 0] + \mathbb{E}[e^{-(u+v)S_t+v\xi_t}; \xi_t < 0] \leq \mathbb{E}[e^{-u\xi_t}; \xi_t \geq 0] + \mathbb{E}[e^{v\xi_t}; \xi_t < 0].
\]
From Proposition 3.10 and 3.11, there exists a constant \( C > 0 \) such that for large \( t \)
\[
\mathbb{E}[e^{-uS_t-v(S_t-\xi_t)}] \leq \frac{1}{t} \int_0^t \mathbb{E}[e^{-u\xi_s}; \xi_t \geq 0] \left( \mathbb{E}[e^{-u\xi_t-s}; \xi_t-s \geq 0] + \mathbb{E}[e^{v\xi_t-s}; \xi_t-s < 0] \right)ds + \frac{1}{t} \int_0^t \mathbb{E}[e^{v\xi_s}; \xi_t-s < 0] \left( \mathbb{E}[e^{-u\xi_t-s}; \xi_t-s \geq 0] + \mathbb{E}[e^{v\xi_t-s}; \xi_t-s < 0] \right)ds
\]
\[
\sim Ct^{-1} \mathbb{P}\{\xi_1 > at\}.
\]
Applying Proposition 3.11 again to (3.34), we have
\[
\mathbb{E}[e^{-uS_t-v(S_t-\xi_t)}] \sim \alpha \left(\frac{1}{u} + \frac{1}{v}\right) \int_0^\infty \mathbb{E}[e^{-uS_s-v(S_s-\xi_s)}]ds \cdot \frac{\mathbb{P}\{\xi_1 > at\}}{at}.
\]
Here we have gotten the desired result. \( \square \)

Lemma 3.13 For any \( x, y \geq 0 \), we have as \( t \to \infty \),
\[
\frac{at}{\mathbb{P}\{\xi_1 > at\}} \mathbb{P}\{S_t \leq x, S_t - \xi_t \leq y\} \to C_0 \alpha \left[ \tilde{V}(y) \int_0^x V(z)dz + V(x) \int_y^\infty \tilde{V}(z)dz \right], \tag{3.35}
\]
\[
\frac{at}{\mathbb{P}\{\xi_1 > at\}} \mathbb{P}\{-I_t \leq x, \xi_t - I_t \leq y\} \to C_0 \alpha \left[ V(y) \int_0^x \tilde{V}(z)dz + \tilde{V}(x) \int_y^\infty V(z)dz \right]. \tag{3.36}
\]

Proof. From (3.34) and the one-to-one property of Laplace transform,
\[
t \cdot \mathbb{P}\{S_t \leq x, S_t - \xi_t \leq y\} = \int_0^t \int_0^x \mathbb{P}\{S_s \leq x-z, S_s - \xi_s \leq y\}d\mathbb{P}\{0 \leq \xi_t-s \leq z\}ds
\]
Theorem 3.14

From Lemma 3.9 and Condition 3.8, we have

$$\mathbb{P}\{-y \leq \xi_t \leq x\} \sim \frac{\alpha(x+y)}{a}\mathbb{P}\{\xi_1 > at\}. \quad (3.38)$$

By (3.9) and Proposition 3.11, there exists a constant $C > 0$ such that for large $t$

$$\mathbb{P}\{S_t \leq x, S_t - \xi_t \leq y\} \leq \frac{1}{t} \int_0^t \mathbb{P}\{S_s \leq x, S_s - \xi_s \leq y\}\mathbb{P}\{-y \leq \xi_{t-s} \leq x\}ds$$

$$\leq \frac{1}{t} \int_0^t \mathbb{P}\{S_s \leq x\}\mathbb{P}\{-y \leq \xi_{t-s} \leq x\}ds \sim C(x+y)\frac{\mathbb{P}\{\xi_1 > at\}}{at}. \quad (3.39)$$

For any $\epsilon \in (0, 1)$, from (3.38) we have

$$\frac{1}{\mathbb{P}\{\xi_1 > at\}} \int_0^t \int_0^x \mathbb{P}\{S_s \leq x - z_1, S_s - \xi_s \leq y\}d\mathbb{P}\{0 \leq \xi_{t-s} \leq z_1\}ds$$

$$\leq \frac{1}{\mathbb{P}\{\xi_1 > at\}} \int_0^t \int_0^x \mathbb{P}\{S_s \leq x, S_s - \xi_s \leq y\}\mathbb{P}\{0 \leq \xi_{t-s} \leq x\}ds$$

$$\leq \frac{1}{\mathbb{P}\{\xi_1 > at\}} \int_0^t C(x+y)\frac{\mathbb{P}\{\xi_1 > at\}}{as}\mathbb{P}\{0 \leq \xi_{t-s} \leq x\}ds$$

$$\leq C(x+y)\frac{\mathbb{P}\{\xi_1 > at\}}{c_{\alpha t}} \int_0^{(1-\epsilon)t} \mathbb{P}\{0 \leq \xi_t \leq x\}ds \leq \frac{C}{x}. \quad (3.39)$$

Moreover, there exists a constant $C > 0$ such that for any $s \in [(1-\epsilon)t, t]$,

$$\frac{a}{\mathbb{P}\{\xi_1 > at\}} \mathbb{P}\{-y \leq \xi_s \leq x\} \leq C.$$

By (3.38) and the dominated convergence theorem, we have

$$\lim_{t \to \infty} \frac{a}{\mathbb{P}\{\xi_1 > at\}} \int_0^t \int_0^x \mathbb{P}\{S_s \leq x - z_1, S_s - \xi_s \leq y\}d\mathbb{P}\{0 \leq \xi_{t-s} \leq z_1\}ds$$

$$= \alpha \int_0^\infty \int_0^x \mathbb{P}\{S_t \leq x - z_1, S_t - \xi_t \leq y\}dz_1ds = C_0\alpha \hat{V}(y) \int_0^x V(z)dz.$$ 

From this and (3.39), we have

$$\frac{a}{\mathbb{P}\{\xi_1 > at\}} \int_0^t \int_0^x \mathbb{P}\{S_s \leq x - z, S_s - \xi_s \leq y\}d\mathbb{P}\{0 \leq \xi_{t-s} \leq z\}ds \to C_0\alpha \hat{V}(y) \int_0^x V(z)dz.$$

Similarly, we also have

$$\frac{a}{\mathbb{P}\{\xi_1 > at\}} \int_0^t \int_0^y \mathbb{P}\{S_s \leq x, S_s - \xi_s \leq y - z\}d\mathbb{P}\{0 < -\xi_{t-s} \leq z\}ds \to C_0\alpha V(x) \int_0^y \hat{V}(z)dz.$$ 

Taking these results back into (3.37), we can get the first desired result. The second one can be proved similarly. □

Theorem 3.14

Let $\tau_0^+ := \inf\{t > 0 : \xi_t \geq 0\}$. For any $x, \lambda > 0$, we have as $t \to \infty$,

$$\frac{at}{\mathbb{P}\{\xi_1 > at\}} \mathbb{E}_x[e^{-\lambda \xi_t}; \tau_0 > t] \to \mathbb{C}_0 \alpha \cdot \hat{V}(x) \int_0^\infty e^{-\lambda y}V(y)dy, \quad (3.40)$$

$$\frac{at}{\mathbb{P}\{\xi_1 > at\}} \mathbb{E}_x[e^{\lambda \xi_t}; \tau_0^+ > t] \to \mathbb{C}_0 \alpha \cdot V(x) \int_0^\infty e^{-\lambda y}\hat{V}(y)dy. \quad (3.41)$$

Proof. Here we still just prove the first statement and the second one can be proved similarly. Applying duality of $\xi$ to the second equality below, we have

$$\mathbb{E}_x[e^{-\lambda \xi_t}; \tau_0 > t] = \mathbb{E}[e^{-\lambda \xi_t}; I_t \geq -x] = \mathbb{E}\left[e^{-\lambda (\xi_t - \inf_{s \leq t} \xi_s)}; (\xi_t - \inf_{s \leq t} \xi_s) \leq x\right]$$

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Remark 3.15 Following the argument above, we also can prove the analogue of Theorem 3.14 for the random walk \( \{ \xi_n : n = 0, 1, \cdots \} \) under Condition 3.1 and 3.8, i.e. for any \( x \geq 0 \) and \( \lambda > 0 \), there exist a constant \( C > 0 \) such that as \( n \to \infty \),

\[
E_x[e^{-\lambda \xi_n ; I_t \geq -x}] \sim C \frac{P[\xi_1 > an]}{an}.
\]  

(3.42)

4 Asymptotic results for exponential functionals

In this section, we study the asymptotic behavior of exponential functionals (1.1) of heavy-tailed Lévy processes with the help of conditional limit results introduced in the last section. Bertoin and Yor [8, Theorem 1] showed that \( P\{A_\infty(\xi) < \infty \} = 1 \) if and only if \( P\{A_\infty(\xi) < \infty \} > 0 \); equivalently, if and only if \( a < 0 \). For any \( x > 0 \) and \( t \geq 0 \), we have

\[
(1 - e^{-t})P\{A_t(\xi) \leq x \} \leq \int_0^\infty e^{-s}P\{A_s(\xi) \leq x \} ds = P\{A_\infty(\xi) \leq x \},
\]  

(4.1)

where \( e \) is an exponentially distributed random variable independent of \( \xi \) with parameter 1. Patie and Savov [36, Theorem 2.19] proved that \( P\{A_\infty(\xi) \leq x \} \sim Cx \) as \( x \to 0+ \) and hence there exists a constant \( C_\tau > 0 \) such that for any \( x \geq 0 \),

\[
P\{A_t(\xi) \leq x \} \leq \frac{e^{t}}{e^{t} - 1}P\{A_\infty(\xi) \leq x \} \leq C_\tau x.
\]  

(4.2)

Moreover, they also proved that \( E[|A_t(\xi)|^{-\kappa}] < \infty \) for any \( \kappa \in (0, 1) \) and \( t > 0 \); see [36, Theorem 2.18].

We now start to study the asymptotic behavior of the expectation \( E[F(A_\tau(\xi))] \) as \( t \to \infty \) for some function \( F \) defined on \( (0, \infty) \). To simplify the presentation of the results, we also assume that \( F \) satisfies the following two conditions.
Condition 4.1 \( F \) is bounded, positive, non-increasing and \( \sup_{x > 0} x^\beta F(x) < \infty \) for some \( \beta \in (0, 1) \).

Condition 4.2 For any \( \delta > 0 \), there exists a constant \( K_\delta > 0 \) such that \( |F(x) - F(y)| \leq K_\delta |x - y| \) for any \( x, y \geq \delta \).

If \( a < 0 \), we have \( \P\{A_\infty(\xi) < \infty\} = 1 \) and hence \( \E[F(A_\infty(\xi))] \to \E[F(A_\infty(\xi))] < \infty \) as \( t \to \infty \). If \( a = 0 \) and \( \rho_0 := \lim_{t \to \infty} \P\{\xi_t < 0\} \in [0, 1] \), Patie and Savov [36, Theorem 2.18(2)] proved that \( \E[F(A_t(\xi))] \sim t^{-\rho_0} \rho_0(t) \) as \( t \to \infty \), where \( \rho_0(t) \) is a slowly varying function at \( \infty \). In this section, we consider the asymptotic behavior of \( \E[F(A_t(\xi))] \) with \( a > 0 \) and \( \xi \) satisfying Condition 3.1 and 3.8. Let \( \xi \) be an independent copy of \( \xi \). Let \( \mathbf{J} \) be an \( \mathbb{R}_+ \)-valued random variable independent of \( \xi \) and \( \xi \) with \( \P\{\mathbf{J} > x\} = \nu(x)^1 \) for large \( x \).

The main theorem of this section is the following:

**Theorem 4.3** Assume that \( a > 0 \) and Condition 3.1 and 3.8 hold, we have the finite and nonzero limit
\[
\lim_{t \to \infty} \frac{\E[F(A_t(\xi))]}{\P(\xi_t > at)} = \int_0^\infty \E[C_F(s)] ds < \infty, \tag{4.3}
\]
where \( C_F(s) := \lim_{t \to \infty} \E_{\theta^s_{\xi_t}} \left[ F(A_t(\xi) + e^{\xi_t} J_{t-s}(\hat{\xi})) \right] \) and \( \theta^s_{\xi_t} := \sigma(\xi_r : r \leq s) \).

Before showing the proof for this theorem, in the following lemma we study the effect of the initial state \( \xi_0 = -x \) on the expectation \( \E[F(A_t(\xi))] = \E[F(e^{xA_t(\xi)})] \). This offers us a criticality to identify the sample paths that make main contribution to the expectation (1.2).

**Lemma 4.4** For any \( t \geq 0 \) and \( q \geq 1 \), there exist two constants \( C, \lambda_0 > 0 \) such that for any \( x \in \mathbb{R} \),
\[
\E[F(e^{xA_t(\xi)})]^q] + \E[F(e^{xA_t(\xi)} \cdot \xi_t^x)] \leq Ce^{-\lambda_0 x}. \tag{4.4}
\]

**Proof.** Here we just prove this result with \( t = 1 \) and \( x \geq 0 \). Other cases can be proved in the same way. Firstly,
\[
\E[F(e^{xA_t(\xi)})^q] = \int_{-\infty}^{\infty} |F(e^{x-y})|^q d\P\{-\log A_t(\xi) \leq y\} = \int_{x/2}^{\infty} |F(e^{x-y})|^q d\P\{A_t(\xi) \geq e^{-y}\} + \int_{x/2}^{\infty} |F(e^{x-y})|^q d\P\{A_t(\xi) \geq e^{-y}\}.
\]
Since \( F \) is non-increasing, we have
\[
\int_{x/2}^{\infty} |F(e^{x-y})|^q d\P\{A_t(\xi) \geq e^{-y}\} \leq |F(e^{x/2})|^q \leq Ce^{-q^\beta x/2}.
\]
From the boundedness of \( F \) and (4.2), we also have
\[
\int_{x/2}^{\infty} |F(e^{x-y})|^q d\P\{A_t(\xi) \geq e^{-y}\} \leq CP\{A_t(\xi) \leq e^{-x/2}\} < Ce^{-x/2}.
\]
Putting all estimates above together, we have \( \E[F(e^{xA_t(\xi)})^q] \leq Ce^{-(q^\beta \wedge 1)x/2} \). By Hölder’s inequality, for any \( p, q \geq 1 \) satisfying that \( p < \alpha \) and \( \frac{1}{p} + \frac{1}{q} = 1 \) we have
\[
\E[F(e^{xA_t(\xi)} \cdot \xi_t^x)] \leq \E[F(e^{xA_t(\xi)})^q]^{1/q} \cdot \E\left[ \left| \xi_t^x \right|^p \right]^{1/p} \leq Ce^{-\lambda_0 x}
\]
with \( \lambda_0 := (q^\beta \wedge 1)/(2q) \).

This proposition shows that the asymptotics of the expectation (1.2) may mainly depend on the sample paths with slowly decreasing local infimum. From Lemma 3.5 and Theorem 3.6, we see that the sample paths decrease slowly if and only if there is a large jump occurs early. To show clearly the main ideas of the proof for Theorem 4.3, we write it into several steps with a series of lemmas. In Section 4.1, we prove that the contribution of sample paths with late arrival large jump to the expectation (1.2) is insignificant, i.e., for \( t, N > 0 \) large enough,
\[
\E[F(A_t(\xi)); \mathcal{J}^{at} > N] = o(\P\{\xi_t > at\}). \tag{4.5}
\]
In Section 4.2, we analyze the exact contribution of sample paths with early arrival large jump to the expectation. From Theorem 3.6, we observe that \( A_t(\xi) \) increases very slowly after the early arrival large jump, which results that \( F(A_t(\xi)) \) decreases so slowly that it can be well approximated by \( F(A_T(\xi)) \) with \( T = o(t) \), i.e., for large \( N \),
\[
\E[F(A_t(\xi)); \mathcal{J}^{at} < N] \sim \E[F(A_T(\xi)); \mathcal{J}^{at} < N] \sim C(T)\P\{\xi_t > at\}. \tag{4.6}
\]
Based on these estimates, in Section 4.3 we give the proof for Theorem 4.3.

\(^1\)Actually, all the following results hold for any \( \mathbf{J} \) satisfying that \( \P\{\mathbf{J} > x\} \sim \nu(x) \) as \( x \to \infty \).
4.1 Contribution of sample paths with late arrival large jump

In this section, we prove that the contribution of sample paths with late arrival large jump to the expectation (1.2) can be ignored. Recall $J_d$ defined in Remark 3.7, which represents the first movement of random walk $\{\xi_k : k = 0, 1, \cdots\}$ that larger than $x$. From the fact that $P\{N_1^{ax} > 0\} \sim P\{\xi_1 \geq ax\}$ as $x \to \infty$; see (3.1) and (3.5), for large $x$ the following proposition shows that $J_d > N$ if and only if $J_d > N$.

**Proposition 4.5** For any $N \geq 1$, we have as $x \to \infty$,

$$
\frac{P\{\{N_1^{ax} > 0\}\Delta\{\xi_1 > ax\}\}}{P\{\xi_1 > ax\}} \to 0 \text{ and } \frac{P\{\{J_d^{ax} \geq N\}\Delta\{J_d^{ax} \geq N\}\}}{P\{\xi_1 > ax\}} \to 0. \tag{4.7}
$$

**Proof.** From (3.9) we have as $x \to \infty$,

$$
P\{\{N_1^{ax} > 0\}\Delta\{\xi_1 > ax\}\} \sim \frac{2P\{\xi_1 \leq ax, N_1^{ax} > 0\}}{P\{\xi_1 > ax\}}.
$$

For any $b > a$,

$$
P\{\xi_1 \leq ax, N_1^{ax} > 0\} \leq P\{\xi_1 \leq ax, N_1^{ax} > 0\} + P\{N_1^{ax} - N_1^{bx} > 0\}.
$$

From (3.5) we have

$$
P\{N_1^{ax} - N_1^{bx} > 0\} \sim \bar{v}(ax) - \bar{v}(bx) \text{ and } \frac{P\{N_1^{ax} - N_1^{bx} > 0\}}{P\{\xi_1 > ax\}} \to 1 - (a/b)^\alpha.
$$

Moreover, since $\xi_1$ is independent of $N_1^{ax}$ for $bx > 1$, then

$$
P\{\xi_1 \leq ax, N_1^{ax} > 0\} \leq P\{\xi_1 \leq ax, N_1^{bx} > 0\} = P\{\xi_1 \leq (a - b)x\} \cdot P\{N_1^{bx} > 0\} = o(P\{N_1^{ax} > 0\}).
$$

Putting all results above together, we have

$$
\limsup_{x \to \infty} \frac{P\{\{N_1^{ax} > 0\}\Delta\{\xi_1 > ax\}\}}{P\{\xi_1 > ax\}} \leq 1 - (a/b)^\alpha,
$$

which vanishes as $b \to a^+$. We start to prove the second result. By the independent increments of $\xi$, we have for any $k < N$,

$$
P\{\{J_d^{ax} \geq N, J_d^{ax} = k\} = \prod_{i=0}^{k-1} P\left\{\int_{i}^{i+1} N(ds, du) = 0, \xi_{i+1} - \xi_i \leq ax\right\}
$$

$$
\times P\left\{\int_{k}^{k+1} N(ds, du) = 0, \xi_{k+1} - \xi_k > ax\right\} \times P\left\{\int_{k+1}^{N} N(ds, du) = 0\right\}
$$

$$
= |P\{N_1^{ax} = 0, \xi_1 \leq ax\}|^k \cdot P\{N_1^{ax} = 0, \xi_1 > ax\} \cdot P\{N_1^{ax} = 0\}^{k-1}.
$$

Both the first and the last probability on the right side of the last equality go to 1 as $x \to \infty$, which immediately induces that

$$
P\{\{J_d^{ax} \geq N, J_d^{ax} < N\} \sim N \cdot P\{N_1^{ax} = 0, \xi_1 > ax\}.
$$

Similarly, we also have as $x \to \infty$,

$$
P\{\{J_d^{ax} < N, J_d^{ax} \geq N\} \sim N \cdot P\{N_1^{ax} > 0, \xi_1 \leq ax\}.
$$

Putting these two estimates together, we have as $x \to \infty$,

$$
P\{\{J_d^{ax} \geq N\}\Delta\{J_d^{ax} \geq N\}\} \sim N \cdot P\{\{N_1^{ax} > 0\}\Delta\{\xi_1 > ax\}\}
$$

and the second result follows from this and the first result in (4.7). \hfill \Box

From the last proposition, it suffices to prove that $F(A_\xi(\xi)); J_d^{at} > N] = o(P\{\xi_1 > at\})$ for large $N$. For any $n \geq 1$, let $I_d := \inf\{\xi_k : k = 0, 1, \cdots n\}$ and $g_d := \inf\{0 \leq k \leq n : \xi_k = I_d\}$. According to the distance between the large jump and the local infimum, we split the sample paths with late arrival large jump into the following two classes: for $K > 0$,

$$
E[F(A_\xi(\xi)); J_d^{at} > N] = E[F(A_\xi(\xi)); J_d^{at} > N, g_d \leq K] + E[F(A_\xi(\xi)); J_d^{at} > N, g_d > K]. \tag{4.8}
$$

The following lemma shows that the local infimum mostly is close to the first large jump.
Lemma 4.6 For any fixed $K > 0$, we have
\[
\lim_{N \to \infty} \limsup_{n \to \infty} \frac{P\{J^a_n > N, g^d_n < K\}}{P\{\xi_1 > an\}} = 0.
\] (4.9)

Proof. From Proposition 4.5, it suffices to prove the following result with $K \in \mathbb{N}$:
\[
\lim_{N \to \infty} \limsup_{n \to \infty} \frac{P\{J^a_d > N, g^d_n < K\}}{P\{\xi_1 > an\}} = 0,
\]
By the duality of $\xi$, for $n > N > K$ we have
\[
P\{J^a_d > N, g^d_n < K\} = \sum_{k=0}^{K-1} P\{J^a_d > N, g^d_n = k\} \leq \sum_{k=0}^{K-1} P\{J^a_d > N - k, \tau^d_{\theta_n} \geq n - k\} P\{g^d_n = k\} \leq P\{\tau^d_n \geq n - K\} \cdot K \cdot P\{J^a_d > N - K, \tau^d_{\theta_n} \geq n - K\}.
\]
From (3.20) and (3.21), we have
\[
\lim_{n \to \infty} \frac{P\{J^a_d > N, g^d_n < K\}}{P\{\xi_1 > an\}} \leq C \cdot K \cdot \frac{E[\tau^d_{\theta_n} \geq n - K]}{E[\tau^d_{\theta_n}]} ,
\]
which vanishes as $N \to \infty$. □

Proposition 4.7 For any $\epsilon > 0$ satisfying that $\epsilon \cdot E[\tau^d_{\theta_n}] < 1$, there exists a constant $C_\epsilon > 0$ such that for any $n \geq 1$ and $x \in [0, en]$,
\[
P_x\{\tau^d_{\theta_n} \geq n\} \leq C_\epsilon (1 + x) \cdot P\{\xi_1 > an\}.
\]

Proof. Let $\{\tau^d_{\theta_n}\}_{i=1}^\infty$ be a sequence of i.i.d copies of $\tau^d_{\theta_n}$. From the strong Markov property of $\xi$, for any $\theta > 1$ with $\theta \cdot E[\tau^d_{\theta_n}] < 1$ we have
\[
P_x\{\tau^d_{\theta_n} \geq n\} \leq \sum_{i=1}^{[x]+1} P\{\tau^d_{\theta_n} \geq [x], \tau^d_{\theta_n} \geq n\} \leq \sum_{i=1}^{[x]+1} \left(\tau^d_{\theta_n} \geq n(1 - \theta \cdot E[\tau^d_{\theta_n}] \cdot (1 + [x])/n)\right)
\]
\[
\leq \sum_{i=1}^{[x]+1} \left(\tau^d_{\theta_n} \geq n(1 - \theta \cdot E[\tau^d_{\theta_n}] \cdot [en]/n)\right)
\]
\[
\sim \sum_{i=1}^{[x]+1} \left(\tau^d_{\theta_n} \geq n(1 - \epsilon \cdot E[\tau^d_{\theta_n}])\right).
\]
From the first result in [15, Theorem 2] and (3.20), there exists a constant $C > 0$ such that for any $x \geq 0$,
\[
P\left\{\sum_{i=1}^{[x]+1} \left(\tau^d_{\theta_n} \geq n(1 - \epsilon \cdot E[\tau^d_{\theta_n}])\right) \leq C(x + 1)P\{\tau^d_{\theta_n} \geq n(1 - \epsilon \cdot E[\tau^d_{\theta_n}])\}
\]
\[
\sim C(x + 1)P\{\xi_1 \geq an(1 - \epsilon \cdot E[\tau^d_{\theta_n}])\}
\]
\[
\sim C(1 - \epsilon \cdot E[\tau^d_{\theta_n}])^{-\alpha} \cdot (x + 1)P\{\xi_1 > an\}.
\]
Here we have gotten the desired result. □

The following lemma considers the second term on the right side of (4.8) with the observation that sample paths with late arrival local infimum usually result in the fast increasing of $A_t(\xi)$ and hence their contribution to the expectation (1.2) decreases fast.

Lemma 4.8 There exists a constant $C > 0$ such that for any $t \geq 0$,
\[
E[F(A_t(\xi))] \leq CP\{\xi_1 > at\}.
\] (4.10)

Moreover, we also have
\[
\lim_{K \to \infty} \limsup_{n \to \infty} \frac{E[F(A_{n+1}(\xi)), g^d_n \geq K]}{P\{\xi_1 > an\}} = 0.
\] (4.11)
Proof. It is easy to see that the first result follows directly from the second one with \( K = 0 \). We now start to prove the second result. By the monotonicity of \( F \), we have

\[
E[F(A_{n+1}(\xi)), g_n^d \geq K] \leq \sum_{k=K}^{n} E \left[ F \left( e^{-\xi_k} \int_0^1 e^{-(\xi_{k+s} - \xi_s)} \, ds \right), g_n^d = k \right]
\]

where \( \hat{\xi} \) is an independent copy of \( \xi \). For any \( \epsilon > 0 \) satisfying that \( \epsilon \cdot E[\tau_d^d] < 1 \), we have

\[
E_{\xi_k} \left[ F \left( e^{-\xi_k} A_1(\xi) \right), \sup_{k=1, \ldots, n-k} \hat{\xi}_i \geq 0 \right] = E_{\xi_k} \left[ F \left( e^{-\xi_k} A_1(\xi) \right), 0 \leq \hat{\xi}_i \leq \epsilon(n-k) \right] \sup_{k=2, \ldots, n-k} \hat{\xi}_i \geq 0
\]

\[
+ E_{\xi_k} \left[ F \left( e^{-\xi_k} A_1(\xi) \right), \hat{\xi}_i > \epsilon(n-k) \right] \sup_{k=2, \ldots, n-k} \hat{\xi}_i \geq 0 \right].
\]

(4.13)

From Condition 4.1, we see that the second term on the right side of above equation can be bounded by

\[
Ce^{\beta \xi_k} E \left[ (A_1(\xi))^\beta, \hat{\xi}_i > \epsilon(n-k) \right] = Ce^{\beta \xi_k} E \left[ (A_1(\xi))^\beta, \hat{\xi}_i > \epsilon(n-k) \right].
\]

From (3.5) and the first result in (4.7), we have \( \xi_1 > \epsilon(n-k) \) for large \( n \) if and only if \( N_1^{(n-k)} = 1 \). This induces that for large \( n \),

\[
E \left[ (A_1(\xi))^\beta, \xi_1 > \epsilon(n-k) \right] \sim E \left[ (A_1(\xi))^\beta, N_1^{(n-k)} = 1 \right] \leq E \left[ (A_{\xi(\epsilon(n-k))})^\beta, N_1^{(n-k)} = 1 \right] \cdot P \{ N_1^{(n-k)} = 1 \}.
\]

From the independence between \( \xi^{(n-k)} \) and \( \xi^{(n-k)} \), there exists a uniformly distributed random variable \( U \) on \([0, 1]\) independent of \( \xi \) such that

\[
E \left[ \left( A_{\xi^{(n-k)}}(\xi^{(n-k)}) \right)^\beta, N_1^{(n-k)} = 1 \right] = E \left[ (A_U(\xi^{(n-k)}))^\beta \right] \leq \int_0^1 E \left[ (A_U(\xi))^\beta \right] ds,
\]

which is finite because of \( E \left[ (A_U(\xi))^\beta \right] \sim s^{-\beta} \) as \( s \to 0 \); see Theorem 2.18(1) in [36]. Putting all result above together, we have

\[
E_{\xi_k} \left[ F \left( e^{-\xi_k} A_1(\xi) \right), \hat{\xi}_i > \epsilon(n-k) \right] \leq Ce^{\beta \xi_k} \cdot P \{ N_1^{(n-k)} = 1 \} \leq C(a/e)^\alpha \cdot e^{\beta \xi_k} \cdot P \{ \xi_1 > a(n-k) \}. \tag{4.14}
\]

From Markov property of \( \xi \), Proposition 4.7 and Lemma 4.4, there exists a constant \( \lambda_0 > 0 \) such that

\[
E_{\xi_k} \left[ F \left( e^{-\xi_k} A_1(\xi) \right), 0 \leq \hat{\xi}_i \leq \epsilon(n-k), \sup_{k=2, \ldots, n-k} \hat{\xi}_i \geq 0 \right]
\]

\[
= E_{\xi_k} \left[ F \left( e^{-\xi_k} A_1(\xi) \right), 0 \leq \hat{\xi}_i \leq \epsilon(n-k), P \left\{ \sup_{k=2, \ldots, n-k} \hat{\xi}_i \geq 0 \right\} \right]
\]

\[
\leq CE_{\xi_k} \left[ F \left( e^{-\xi_k} A_1(\xi) \right) \cdot (\hat{\xi}_i + 1) \right] \cdot P \left\{ \xi_1 > a(n-k - 1) \right\} \leq Ce^{\lambda_0 \xi_k} \cdot P \{ \xi_1 > a(n-k - 1) \}.
\]

Taking this and (4.14) back into (4.13), by the duality of \( \xi \) we have

\[
E \left[ E_{\xi_k} \left[ F \left( e^{-\xi_k} A_1(\xi) \right), \inf_{i=1, \ldots, n-k} \hat{\xi}_i \geq 0 \right], g_k^d = k \right] \leq CE \left[ e^{(\lambda_0 \wedge \beta) \xi_k}; g_k^d = k \right] \cdot P \{ \xi_1 > a(n-k - 1) \}
\]

\[
= CE \left[ e^{-(\lambda_0 \wedge \beta) \xi_k}; \tau_0^d > k \right] \cdot P \{ \xi_1 > a(n-k - 1) \}.
\]

By (3.42) and Proposition 3.11, we have for large \( n \),

\[
\sum_{k=K}^{n} E \left[ E_{\xi_k} \left[ F \left( e^{-\xi_k} A_1(\xi) \right), \inf_{i=1, \ldots, n-k} \hat{\xi}_i \geq 0 \right], g_k^d = k \right] \leq \frac{C}{K} \sum_{k=K}^{n} P \{ \xi_1 \geq ak \} \cdot P \{ \xi_1 > a(n-k - 1) \}
\]

\[
\sim \frac{C}{K} \sum_{k=0}^{\infty} P \{ \xi_1 \geq ak \} \cdot P \{ \xi_1 > an \}.
\]

Taking this back into (4.12), we have

\[
\limsup_{n \to \infty} \frac{E[F(A_{n+1}(\xi)); g_n^d \geq K]}{P \{ \xi_1 > an \}} \leq \frac{C}{K},
\]

which vanishes as \( K \to \infty \). \qed
4.2 Contribution of sample paths with early arrival large jump

We now start to analyze the contribution of sample paths with early arrival large jump to the expectation (1.2). In this case, we observe that the effect of their partial paths before the large jump on $A_t(\xi)$ is slight and hence it is the key step to clarify the increasing rate of $A_t(\xi)$ after the large jump. As we have showed in Theorem 3.6, the sample paths stay in the set far away from origin for a log time after the large jump. This suggests us that $A_t(\xi)$ can be well approximated by $A_T(\xi)$ with $T$ larger than the arrival time of the first large jump; see the following lemma.

Lemma 4.9 If $F$ is globally Lipschitz continuous on $(0, \infty)$, we have

$$
\lim_{T \to \infty} \limsup_{t \to \infty} E\left[|F(e^{-J}A_T(\xi)) - F(e^{-J}A_t(\xi))| \mid J > at\right] = 0. \tag{4.15}
$$

Proof. Since $F$ is bounded and globally Lipschitz continuous, there exist two constants $K_1, K_2 > 0$ such that for any $b > a$

$$
E\left[|F(e^{-J}A_T(\xi)) - F(e^{-J}A_t(\xi))| \mid J > at\right] \leq E\left[(K_1 e^{-\xi} \int_T^t e^{-\xi_s} ds \wedge K_2 \mid J > at\right] \leq \varepsilon_1(b,t) + \varepsilon_2(b,T,t),
$$

where

$$
\varepsilon_1(b,t) := \frac{K_2 P\{J \in (at, bt]\}}{P\{J > at\}} \quad \text{and} \quad \varepsilon_2(b,T,t) := \frac{E[(K_1 e^{-\xi} \int_T^t e^{-\xi_s} ds \wedge K_2, J > bt]}{P\{J > at\}}.
$$

From the definition of $J$, we have $\varepsilon_1(b,t) \to K_2[1 - (a/b)^\alpha]$ as $t \to \infty$. Moreover,

$$
\varepsilon_2(b,T,t) \leq E\left((K_1 \int_T^t e^{-(\xi_s + bs)} ds \wedge K_2 \mid P\{J > bt\} \leq E\left((K_1 \int_T^\infty e^{-(\xi_s + bs)} ds \wedge K_2\right],
$$

which vanishes as $T \to \infty$ because of $E[\xi_1] + b > 0$ and $\int_0^\infty e^{-(\xi_s + bs)} ds < \infty$ a.s. Putting these estimates together, we see that (4.15) follows as $b \to a^+$.

Lemma 4.10 For any $T > 0$, there exists a constant $C_{F,T} > 0$ such that

$$
\lim_{t \to \infty} E\left[F(e^{-J}A_T(\xi)) \mid J > at\right] = C_{F,T}. \tag{4.16}
$$

Moreover, $C_{F,T}$ decreases to $C_F > 0$ as $T \to \infty$ and

$$
\lim_{t \to \infty} E\left[F(e^{-J}A_t(\xi)) \mid J > at\right] = C_F. \tag{4.17}
$$

Proof. Since $F$ is bounded and non-increasing, we have $E[F(e^{-J}A_T(\xi)) \mid J \geq at]$ is non-decreasing in $t$ with upper bound and hence the limit (4.16) holds. Moreover, notice that $C_{F,T}$ is non-increasing in $T$ and hence $C_{F,T} \to C_F \in [0, \infty)$ as $T \to \infty$. We now show that $C_F > 0$. For any $b > a$, let $\tilde{\xi} = \xi + bt$, which drifts to $\infty$. We have

$$
E\left[F(e^{-J}A_t(\xi)) \mid J > at\right] \geq \frac{E[F(e^{-J}A_t(\xi), J > bt)]}{P\{J > at\}} \geq \frac{E[F(e^{-bt}A_t(\xi))] P\{J > bt\}}{P\{J > at\}} \geq C_F \frac{P\{J > bt\}}{P\{J > at\}}.
$$

Since $E[F(\xi)] \in (0, \infty)$, there exists $C_0 > 0$ such that for any $T > 0$,

$$
\liminf_{t \to \infty} E\left[F(e^{-J}A_T(\xi)) \mid J > at\right] \geq C_0,
$$

which immediately induces that $C_F > 0$. 

4.3 Proof for Theorem 4.3

We first consider the special case with \( F(x) \) is globally Lipschitz continuous. For any \( N > 0 \), we have

\[
\lim_{t \to \infty} \frac{\mathbb{E}[F(A_t(\xi))] \mathbb{P}\{\xi > at\}}{\mathbb{P}\{\xi > at\}} = \lim_{N \to \infty} \lim_{t \to \infty} \frac{\mathbb{E}[F(A_t(\xi)); J^at \leq N]}{\mathbb{P}\{\xi > at\}} + \lim_{N \to \infty} \limsup_{t \to \infty} \frac{\mathbb{E}[F(A_t(\xi)); J^at > N]}{\mathbb{P}\{\xi > at\}}.
\]

From Lemma 4.6 and 4.8, the second limit on the right side of above equation equals to 0. For large \( t > N \),

\[
\mathbb{E}[F(A_t(\xi)); J^at \leq N] = \mathbb{E} \left[ F \left( \int_{(0,J^at)} e^{-\xi^at} dr + e^{-\xi^at}_{J^at} \int_s^t e^{-(\xi - \xi^at)} dr; J^at \leq N \right) \right].
\]

From the independence between \( J^at \) and \( \Delta \xi^at \), and (3.4), we have

\[
\mathbb{P}\{J^at > ds, \Delta \xi^at > dy\} = 1_{\{y \geq at\}} e^{-\nu^a(t)s} ds \nu(dy).
\]

From the independent increments of \( \xi \), we have

\[
\mathbb{E} \left[ F(A_t(\xi)); J^at \leq N \right] = \int_0^N \mathbb{E} \left[ F \left( \int_{(0,s]} e^{-\xi^as} dr + e^{-\xi^as}_{s} \int_0^{t-s} e^{-\xi \hat{\nu}(s)} dr \right) \right] \mathbb{P}\{J^at \in ds, \Delta \xi^as \in dy\}
\]

\[
= \int_0^N \nu(at) e^{-\nu^a(at)s} ds \int_{\mathbb{R}_+} \mathbb{E} \left[ F \left( A_s - (\xi^at) + e^{-\xi^as}_{s} J^at \hat{\xi} \right) \right] \frac{\nu(dy)}{\nu(at)}
\]

\[
= \nu(at) \int_0^N e^{-\nu^a(at)s} ds \mathbb{E} \left[ \mathbb{P}_{\hat{\xi}} \left[ F \left( A_s - (\xi^at) + e^{-\xi^as}_{s} J^at \hat{\xi} \right) \right| J \geq at \right] ds.
\]

By the dominated convergence theorem and (3.1)-(3.2), we have

\[
\lim_{t \to \infty} \frac{\mathbb{E}[F(A_t(\xi)); J^at \leq N]}{\mathbb{P}\{\xi > at\}} = \int_0^N \mathbb{E} \left[ \lim_{t \to \infty} \mathbb{E}_{\hat{\xi}} \left[ F \left( A_s - (\xi^at) + e^{-\xi^as}_{s} J^at \hat{\xi} \right) \right| J \geq at \right] ds.
\]

From Lemma 4.10 and the fact that \( \sup_{t \in [0,P]} |\xi^at - \xi_t| \to 0 \) a.s., the limit above exists and hence by the stochastical continuity of \( A(\xi) \) and \( \xi \),

\[
\lim_{t \to \infty} \frac{\mathbb{E}[F(A_t(\xi))]}{\mathbb{P}\{\xi > at\}} = \int_0^\infty \mathbb{E}[C_F(s-)] ds = \int_0^\infty \mathbb{E}[C_F(s)] ds,
\]

which is finite; see (4.10). We now prove this theorem for general \( F \). For \( n \geq 1 \), let \( F_n(y) = F(1/n)1_{\{y \leq 1/n\}} + F(y)1_{\{y > 1/n\}} \), which is globally Lipschitz and \( F_n(y) \to F(y) \) increasingly. From the result above, we have

\[
\lim_{t \to \infty} \frac{\mathbb{E}[F_n(A_t(\xi))]}{\mathbb{P}\{\xi > at\}} = \int_0^\infty \mathbb{E}[C_{F_n}(s)] ds
\]

with \( \mathbb{E}[C_{F_n}(\cdot)] \) is non-decreasing in \( n \). From the monotone convergence theorem and (4.10), we have as \( n \to \infty \),

\[
\int_0^\infty \mathbb{E}[C_{F_n}(s)] ds \to \int_0^\infty \mathbb{E}[C_F(s)] ds < \infty.
\]

Let \( G_n(y) = F(y) - F_n(y) \). From Condition 4.1 and 4.2, it is easy to see for any \( n \geq 1 \) and \( x \geq 0 \) have

\[
G_n(x)/G_n(0) \leq 1 \wedge x^{-\beta}.
\]

From Lemma 4.8, we can prove that there exists a constant \( C > 0 \) such that for any \( n \geq 1 \) and \( t \geq 0 \),

\[
\mathbb{E}[G_n(A_t(\xi))/G_n(0)] \leq C \mathbb{P}\{\xi > at\} \quad \text{and} \quad \limsup_{t \to \infty} \frac{\mathbb{E}[G_n(A_t(\xi))]}{\mathbb{P}\{\xi > at\}} \leq CG_n(0),
\]

which goes to 0 as \( n \to \infty \). Putting all results together, we have

\[
\lim_{t \to \infty} \frac{\mathbb{E}[F(A_t(\xi))]}{\mathbb{P}\{\xi > at\}} = \lim_{n \to \infty} \lim_{t \to \infty} \frac{\mathbb{E}[G_n(A_t(\xi))]}{\mathbb{P}\{\xi > at\}} + \lim_{n \to \infty} \int_0^\infty \mathbb{E}[C_{F_n}(s)] ds = \int_0^\infty \mathbb{E}[C_F(s)] ds.
\]

Here we have finished the proof.
5 Asymptotic results for CB-processes in Lévy random environment

In this section, we apply the results in the last section to study the asymptotic behavior of survival probabilities of continuous-state branching processes in Lévy random environment. Let \( \{Z_t^\gamma : t \geq 0\} \) be a spectrally positive \((\gamma + 1)\)-stable process with \( 0 < \gamma \leq 1 \) and \( \{Z_t^e : t \geq 0\} \) be a Lévy process with no jump less than \(-1\). When \( \gamma = 1 \), we think of \( \{Z^\gamma(t) : t \geq 0\} \) as a Brownian motion. When \( 0 < \gamma < 1 \), we assume \( \{Z^\gamma(t) : t \geq 0\} \) has Lévy measure:

\[
m(dz) = \frac{\gamma 1_{\{z > 0\}} dz}{1(1 - \gamma) z^{2+\gamma}}.
\]

Associated to the Lévy processes \( \xi \) defined by (2.5), we may assume the Lévy process \( Z^e \) admits the following Lévy-Itô decomposition:

\[
Z^e_t = a_0 t + \sigma B_t + \int_0^t \int_{[-1,1]} (e^w - 1) \tilde{N}(ds, du) + \int_0^t \int_{[-1,1]^c} (e^w - 1) N(ds, du),
\]

(5.1)

where \([-1,1]^c = \mathbb{R} \setminus [-1,1] \), \( \tilde{N}(ds, du) = N(ds, du) - d\nu(du) \) and

\[
a_0 = a + \frac{\sigma^2}{2} + \int_{[-1,1]} (e^z - 1 - z) \nu(dz) - \int_{[-1,1]^c} z \nu(dz).
\]

Then \( Z^e \) has no jump smaller than \(-1\). Clearly, the two processes \( Z^e \) and \( \xi \) generate the same filtration. Let \( c \geq 0 \) be another constant. Given the initial value \( x \geq 0 \), by [22, Theorem 6.2], there exists a unique positive strong solution \( \{X_t(x) : t \geq 0\} \) to (1.3). The solution is called a continuous-state branching process in random environment (CBRE-process) with stable branching mechanism. Here the random environment is modeled by the Lévy process \( Z^e \). The reader may refer to [26, 32] for discussions of more general CBRE-processes.

Let \( \mathbf{P}^\xi \) denote the conditional law given \( Z^e \) or \( \xi \). Let \( Z_t(x) = X_t(x) \exp\{-\xi_t\} \) for any \( t \geq 0 \). For any \( \lambda \geq 0 \) and \( t \geq r \geq 0 \), by [4, Theorem 1] or [26, Theorem 3.4] we have

\[
\mathbb{E}^\xi[e^{-\lambda Z_t(x)} | \mathcal{F}_r] = \exp\{-Z_r(x) u^\xi_{r,t}(\lambda)\},
\]

(5.2)

where \( r \mapsto u^\xi_{r,t}(\lambda) \) is the solution to

\[
\frac{d}{dr} u^\xi_{r,t}(\lambda) = ce^{-\gamma \xi(r)} u^\xi_{r,t}(\lambda)^{\gamma + 1}, \quad u^\xi_{0,t}(\lambda) = \lambda.
\]

By solving the above equation, we get

\[
u^\gamma_{r,t}(\lambda) = \left( c \gamma \int_r^t e^{-\gamma \xi(s)} ds + \lambda^{-\gamma} \right)^{-1/\gamma};
\]

(5.3)

see the proof of [4, Proposition 4]. From (5.2) and (5.3) we see that the survival probability of the CBRE-process up to time \( t \geq 0 \) is given by

\[
\mathbf{P}(X_t(x) > 0) = \mathbf{P}(Z_t(x) > 0) = \lim_{\lambda \to \infty} \mathbb{E}\left[1 - e^{-\lambda Z_t(x)}\right]
\]

\[
= \lim_{\lambda \to \infty} \mathbb{E}\left[1 - \exp\{-x u^\xi_{0,t}(\lambda)\}\right] = \mathbb{E}\left[F_x\left(\int_0^t e^{-\gamma \xi(s)} ds\right)\right],
\]

(5.4)

where \( F_x(z) := 1 - \exp\{-x(c\gamma z)^{-1/\gamma}\} \) satisfies Condition 4.1 and 4.2. The following theorem is an immediate consequence of Theorem 4.3. Using the notation introduced there, it gives characterizations of the three regimes of the survival probability of the CB-process in heavy-tailed Lévy random environment:

**Theorem 5.1** We have the following three regimes of the survival probability:

(1) *(Supercritical)* If \( a > 0 \), then the following nonzero, finite limit exists:

\[
\lim_{t \to \infty} \mathbf{P}\{X_t(x) > 0\} = \mathbb{E}[F_x(A_\infty(\gamma \xi))];
\]

(2) *(Critical)* If \( a = 0 \), then the following limit exists:

\[
\lim_{t \to \infty} \mathbf{P}\{X_t(x) > 0\} = \mathbb{E}[F_x(A_\infty(\gamma \xi))];
\]

(3) *(Subcritical)* If \( a < 0 \), then the following limit exists:

\[
\lim_{t \to \infty} \mathbf{P}\{X_t(x) > 0\} = \mathbb{E}[F_x(A_\infty(\gamma \xi))].
\]
(2) (Critical) If \( a = 0 \) and \( \rho_0 := \lim_{t \to \infty} P\{\xi_t < 0\} \in [0, 1] \). There exists a constant \( C(x) > 0 \) and a slowly varying function \( \ell_0(t) \) at \( \infty \) such that
\[
\lim_{t \to \infty} t^n \ell_0(t) P\{X_t(x) > 0\} = C(x);
\]

(3) (Subcritical) If \( a < 0 \) and Condition 3.1 and 3.8 hold. Then
\[
\lim_{t \to \infty} \frac{(at)^n}{\ell(at)} P\{X_t(x) > 0\} = \int_0^\infty E[C_{F_x}(s)] ds < \infty,
\]
where \( C_{F_x}(s) := \lim_{t \to \infty} E_{\theta_x}[F_x(A_s(\gamma_x) + e^{-\gamma(\xi_s + s)}A_{t-s}(\hat{\gamma}_x))] \mid J > at] \).

6 Appendix

Proof for Lemma 3.9. Let \( \xi \) be an independent copy of \( \xi \). For any \( c < \epsilon/2 \), from the independence of increments we have \( P\{\xi_t \in [x, x + \delta]\} = P\{\xi_t + \xi_{-t} \in [x, x + \delta]\} \) for any \( t \geq 0 \) and
\[
\sup_{x \geq (\epsilon - a)t} \left| \frac{P\{\xi_t \in [x, x + \delta]\}}{t \cdot P\{\xi_1 \in [at + x, at + x + \delta]\}} - 1 \right| \leq \sup_{x \geq (\epsilon - a)t} \int_{-\epsilon}^{\epsilon} \left| \frac{P\{\xi_t + y \in [x, x + \delta]\}}{t \cdot P\{\xi_1 \in [at + x, at + x + \delta]\}} - 1 \right| dy
\]
\[
+ \sup_{x \geq (\epsilon - a)t} \int_{-\epsilon}^{\epsilon} \left| \frac{P\{\xi_t + y \in [x, x + \delta]\}}{t \cdot P\{\xi_1 \in [at + x, at + x + \delta]\}} - 1 \right| dy
\]
\[
+ \sup_{x \geq (\epsilon - a)t} \int_{-\epsilon}^{\epsilon} \left| \frac{P\{\xi_t + y \in [x, x + \delta]\}}{t \cdot P\{\xi_t \in [at + x, at + x + \delta]\}} - 1 \right| dy.
\]
Here we denote the three terms on the right side of the above inequality as \( I_1(c, t), I_2(c, t) \) and \( I_3(c, t) \) respectively. We first have
\[
I_1(c, t) \leq \sup_{x \geq (\epsilon - a)t} \sup_{y \leq \epsilon} \left| \frac{P\{\xi_1 \in [at + x - y, at + x + y]\}}{P\{\xi_1 \in [at + x, at + x + \delta]\}} - 1 \right| \leq \frac{c}{\epsilon},
\]
which goes to 0 as \( c \to 0 \). Moreover, from Corollary 2.1 in [14] we have
\[
\sup_{x \geq (\epsilon - a)t} \sup_{y \leq \epsilon} \left| \frac{P\{\xi_t \in [x, x + \delta]\}}{P\{\xi_1 \in [at + x, at + x + \delta]\}} - 1 \right| \leq \sup_{x \geq (\epsilon/2-a)t} \left| \frac{P\{\xi_t \in [x, x + \delta]\}}{P\{\xi_1 \in [at + x, at + x + \delta]\}} - 1 \right| \to 0.
\]
For \( I_2(c, t) \), from Corollary 2.1 in [14] we also have
\[
I_2(c, t) \leq \sup_{x \geq (c + c-a)t} \left| \frac{P\{\xi_t \in [x, x + \delta]\}}{t \cdot P\{\xi_1 \in [at + x, at + x + \delta]\}} - 1 \right|,
\]
\[\text{2 Recently, Bansaye et al.} \ [3] \text{also considered this case for the general branching mechanism with an additional exponential moment condition: } E[e^{\theta^+ \xi_1}] < \infty \text{ for some } \theta^+ > 1, \text{ which can not be satisfied by the random environment with regularly varying right tail.} \]
which vanishes as $t \to \infty$. For $I_3(c, t)$, it is easy to see that

$$I_3(c, t) \leq \frac{\mathbb{P}\{\hat{\xi}_{t-[t]} \geq ct\}}{t \cdot \mathbb{P}\{\xi_t \in [at + x, at + x + \delta]\}} + \mathbb{P}\{S_1 \geq ct\} \leq \frac{\mathbb{P}\{S_1 \geq ct\}}{t \cdot \mathbb{P}\{\xi_t \in [at + x, at + x + \delta]\}} + \mathbb{P}\{S_1 \geq ct\}.$$  

Here the second term on the right side of the equality above goes to 0 as $t \to \infty$. For the first term, from (2.5), there exists a constant $C > 0$ such that

$$S_1 \leq C + \sup_{s \in [0, 1]} |\sigma B_s + \int_0^s \int_{|x| \leq 1} x \tilde{N}(ds, dx)| + \int_1^\infty \int_1^\infty x N(ds, dx).$$

From [21, Proposition 4.1], we have as $t \to \infty$,

$$\mathbb{P}\{S_1 \geq ct\} \sim \mathbb{P}\left\{\int_1^\infty \int_1^\infty x N(ds, dx) \geq ct/2\right\} \sim \tilde{v}(ct/2)$$

and hence $I_3(c, t) \to 0$ as $t \to \infty$. □

Acknowledgements. The author would like to thank Professor Mladen Savov for enlightening comments. He also recommended the author his recent works about exponential functionals of Lévy processes, which helped a lot to simplify the proofs in Section 4.

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