ON THE EQUILIBRIA AND QUALITATIVE DYNAMICS OF A FORCED NONLINEAR OSCILLATOR WITH CONTACT AND FRICTION

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Abstract. After previous works related to the equilibrium states, this paper goes deeper into the study of the effect of coupling between smooth and non-smooth non-linearities on the qualitative behavior of low dimensional dynamical systems. The non-smooth non-linearity is due to non-regularized unilateral contact and Coulomb friction while the smooth one is due to large strains of a simple mass spring system, which lead to a nonlinear restoring force. The main qualitative differences with the case of a linear restoring force are due to the shape of the set of equilibrium states.

1. Introduction. This paper follows several studies concerning the equilibria, the stability and the qualitative dynamics of simple oscillators involving non regularized unilateral contact and Coulomb friction. Let us first say a few words about the works which constitute the motivation of this present paper. Several papers such as [1] or [6], dealt with the equilibria and the stability of a mass-spring system at small strains initially proposed in [7]. In particular, the main qualitative features of its behavior were investigated in [8]. At the same time several papers dealt with the dynamics of a similar oscillator, under large strains but with bilateral contact conditions without friction (see [2] or other papers of the same authors). Finally, maintaining the nonlinear restoring force due to large strains and introducing unilateral contact and Coulomb friction, the system involves a coupling between smooth and non-smooth non-linearities as pointed out in [9] where a full investigation of the sets of equilibria under constant loading was presented.

Analysing the sets of equilibria, we put emphasis on the difference with the case where the restoring force was linear due to small strains. We present here some aspects of the answer of the system under unilateral contact, Coulomb friction and large strains when submitted to a periodic excitation. As classically done in qualitative analysis of dynamical systems, this work aims at an investigation of the \{period,amplitude\}-plane of the excitation, which will generalize the study presented in [3] where a particular value of the load was chosen.

We now outline the content of the paper:

- Section 2 presents the mass-spring system together with the basic equations and inequalities of the dynamics. After that, we recall the sets of equilibria
under constant loadings and present two preliminary results which will be important tools to compute the trajectory.

- In section 3 we focus on a particular case where the set of equilibria is close to the one obtained with the linear restoring force. This section aims at showing that in some cases the previous analysis gives a sufficiently accurate approximation of the dynamics so that nothing new is necessary regardless of whether there exists an additional non-linearity or not. This could be very useful for quick applications. But we comment on the fact that there nevertheless remains some memory of this non-linearity for which the approximation may fail.

- The main part of this work is to be found in sections 4, 5 and 6. Section 4 discusses the evolution of the set of equilibria first under an increasing load and then under an oscillating load. In particular the existence of a value of the amplitude of the oscillating load beyond which there no longer exist equilibrium states is emphasized.

- The qualitative dynamics is explored in sections 5 and 6. We first study the case when the load is an oscillating tangential excitation of sufficiently small amplitude. In this case it is found that all the trajectories reach an equilibrium state. However, if the amplitude of the oscillating load is larger, then the dynamics involves periodic solutions which may coexist with a chaotic attractor, depending on the period of the load, the latter point being studied in section 6.

2. The model and the equations.

2.1. The model. The model we are studying is represented on Figure 1. It is a simple mass-spring system submitted to large strains in addition to some external force and to unilateral contact with a horizontal plane and Coulomb friction. The notations for the reference and the deformed configurations are to be found on the figure. The origin of the axis corresponds to a trivial equilibrium solution in absence of any external force, the point \((-X_0, 0)\) corresponds to the case when the spring is vertical and the point \((-2X_0, 0)\) corresponds to the other trivial equilibrium solution.

The tangential and normal components of the displacement are denoted by \(u_t\) and \(u_n\) respectively. \(R_t\) and \(R_n\) are unknowns and denote respectively the tangential and normal components of the reaction of the obstacle and \(\mu\) is the friction coefficient. The unilateral contact and friction conditions are classically expressed by:

\[
\begin{align*}
  u_n &\leq 0,  \\
  R_n &\leq 0,  \\
  u_n R_n & = 0,
\end{align*}
\]  

Figure 1. The simple mass-spring model
and
\[ \mu R_n \leq R_t \leq -\mu R_n, \]
\[ |R_t| < -\mu R_n \implies \dot{u}_t = 0, \]
\[ |R_t| = -\mu R_n \implies \exists \lambda \geq 0 \text{ such that } R_t = -\lambda \dot{u}_t. \]  

(2)

To be realistic from a physical point of view, the formulation of the dynamics must allow the possibility of impacts, so that the reaction is not a function but a measure with respect to time and the velocity is a function of bounded variation. The unilateral contact statement (1) can be understood in the sense of measures but the Coulomb friction law (2) must be replaced by a variational formulation.

Let \( F_t \) and \( F_n \) be the components of the external force, then the trajectory is the solution of the following system where the initial data is compatible with the obstacle:

\[ \begin{cases} 
\text{i) Equations of motion} : \\
m\ddot{u}_t = N_t(u_t, u_n) + F_t + R_t, \\
m\ddot{u}_n = N_n(u_t, u_n) + F_n + R_n, \quad t > 0 \\
\text{ii) Initial data} : \\
u_t(0) = u_0^0, \quad u_n(0) = u_n^0, \quad \dot{u}_t(0) = \dot{u}_n(0) = 0, \\
\text{iii) Unilateral contact} : \\
u_n \leq 0, \quad R_n \leq 0, \quad u_n R_n = 0, \\
\text{iv) Coulomb’s friction} : \\
\forall \varphi \in C^0([0, T] ; \mathbb{R}^{n-1}), \\
\int_{[0, T]} R_t(\varphi - \dot{u}_t^+) - \mu R_n(\varphi - |\dot{u}_n^+|) \geq 0, \\
\text{v) Impact} : \\
\text{Let } \tau \text{ such that } u_n(\tau) = 0, \text{ then } \dot{u}_n^+(\tau) = -e\dot{u}_n^-(\tau) \quad e \in [0, 1]. 
\end{cases} \]  

(3)

The impact law (3-v) is well defined since \( \dot{u} \) is a function of bounded variation (see [10]).

Through simple geometrical calculations according to Figure 1, taking \( k \) as the linear stiffness of the spring, the components \( N_t(u_t, u_n) \) and \( N_n(u_t, u_n) \) of the restoring force read

\[ \begin{cases} 
N_t(u_t, u_n) = -k(X_0 + u_t) \left[ 1 - \frac{\sqrt{X_0^2 + h^2}}{(X_0 + u_t)^2 + (h + u_n)^2} \right], \\
N_n(u_t, u_n) = -k(h + u_n) \left[ 1 - \frac{\sqrt{X_0^2 + h^2}}{(X_0 + u_t)^2 + (h + u_n)^2} \right]. 
\end{cases} \]  

(4)

At small strains, due to the fact that \( N_t(0, 0) = N_n(0, 0) = 0 \) and to straightforward calculations these nonlinear terms reduce to \(-k(\sin^2 \varphi \ u_t + \sin \varphi \cos \varphi \ u_n) \) and \(-k(\sin \varphi \cos \varphi \ u_t + \cos^2 \varphi \ u_n) \) where \( \varphi \) represents the angle between the spring and the normal to the obstacle (see [1]). So that we obtain the familiar equations

\[ \begin{cases} 
m\ddot{u}_t + K_t u_t + W u_n = F_t + R_t, \\
m\ddot{u}_n + W u_t + K_n u_n = F_n + R_n. 
\end{cases} \]  

(5)

2.2. The set of equilibria under constant loading. The set of equilibrium solutions of problem (3) has been completely investigated in [9]. It consists of states in contact and out of contact, which may either coexist or exist separately according to the values of the forces. Since equilibrium states out of contact mean that the model reduces to a classical nonlinear dynamical system, we restrict our attention...
to equilibrium states in contact, which obviously implies some bounds on the force \((F_t, F_n)\). More precisely, inserting \(u_n \equiv 0\) into equations (3-i,ii) and looking for equilibria, we obtain that \(u_t\) and \(R_n\) are related by the following expression:

\[ u_t(R_n) = -X_0 \pm h \sqrt{\frac{k^2 h^2 \beta^2}{(F_n + R_n - kh)^2} - 1}, \]

from which we obtain the following relation between the tangential and normal component of the reaction \(R_t\) and \(R_n\) of the equilibrium solution

\[ (R_t + F_t)^2 = (F_n + R_n)^2 \left( \frac{k^2 h^2 \beta^2}{(F_n + R_n - kh)^2} - 1 \right), \]

where \(\beta := \sqrt{1 + \left(\frac{X_0}{h}\right)^2}\) is a geometrical constant introduced for simplicity’s sake.

Finding the equilibrium solutions in contact then amounts to finding the intersection, which obviously depends on the choice of the parameters, of the graph of equation (7) with the Coulomb cone. The generic situation for the set of equilibria in contact is represented as follows on Figure 2. Several points have been plotted on this figure which shall be useful in the forthcoming discussion. Their coordinates are easily calculated:

\[
\begin{align*}
A & \quad (F_t, -F_t + \frac{F_n}{kh}) \quad B \quad (F_t, F_n), \\
C & \quad (F_t, F_n - \sqrt{\frac{\beta^2}{(\frac{F_n}{kh} - 1)^2}} - 1) \quad 0, \\
D & \quad 0, \quad E \quad \left( -kh(\frac{F_t}{kh}) + (\beta^{2/3} - 1)^{3/2} \right) \quad -kh\left(\frac{F_t}{kh} - 1 + \beta^{2/3}\right).
\end{align*}
\]

The thick curve is the graph given by equation (7), that is the equilibrium equation obtained from problem (3) in the case \(u_n \equiv 0\), and the thin lines delimit the friction cone given by (3-iv). From the analysis presented in [9], we recall that increasing \(F_t\)
amounts to translating the thick curve to the left, while increasing $F_n$ amounts to pushing the curve downwards on the $R_n$ axis. It will also be useful to recall that for any force $(F_t, F_n)$ equation (7) defines implicitly the graph represented in Figure 2 and that this graph can be divided into two parts having the same asymptote. Coming back to the mechanics of the model represented on Figure 1, these two parts of the graph correspond respectively to equilibria around the trivial positions $(0, 0)$ and $(-2X_0, 0)$.

2.3. Preliminary results. The following result has been proved in [4]:

**Lemma 2.1.** Under the hypothesis that the restoring force $N(u)$ is an analytic function of $u$ satisfying a global Lipschitz condition, and that the driving force $F(t)$ is a piecewise analytic function of $t$, problem (3) has a unique solution for any given initial condition that is compatible with the unilateral constraint.

We now give a technical result which shall be essential for the computation of a trajectory.

**Lemma 2.2.** Let us consider a solution of problem (3) under constant loading during a time interval in which contact with the obstacle is always achieved. When the tangential velocity becomes equal to zero after a sliding phase, either the velocity stays equal to zero for all time or the velocity changes sign (i.e. the sliding direction changes).

**Proof.** The first step consists in recalling the basic equations of sliding motions. Trajectories in contact have been widely studied in [8]. They involve equations $i)$ and $ii)$ of system (3) in the particular case $u_n \equiv 0$, together with the conditions $R_t = \mu R_n$ when the unknown sliding velocity is positive and $R_t = -\mu R_n$ when it is negative, or the condition $|R_t| < \mu |R_n|$ when the tangential velocity is zero (that is when the mass is clamped by friction). This easily leads to the following equations

\[
m\ddot{u} + k\left(\dot{X}_0 + \dot{u} - \mu h\right) \left[1 - \frac{h\beta}{\sqrt{(\dot{X}_0 + \dot{u})^2 + h^2}}\right] = F_t - \mu F_n,
\]

for positive sliding, and

\[
m\ddot{u} + k\left(\dot{X}_0 + \dot{u} + \mu h\right) \left[1 - \frac{h\beta}{\sqrt{(\dot{X}_0 + \dot{u})^2 + h^2}}\right] = F_t + \mu F_n,
\]

for negative sliding, keeping in mind that in both cases the normal component of the reaction is given by

\[
k h \left[1 - \frac{h\beta}{\sqrt{(\dot{X}_0 + \dot{u})^2 + h^2}}\right] - F_n = R_n.
\]

Assume the tangential velocity becomes zero at some time $t = t^*$, then the normal component of the reaction $R_n(t^*) = -F_n - N'_n(u_t(t^*), 0)$ either belongs to the set of $R_n$ which correspond to equilibrium solutions or not.

- If $R_n(t^*)$ belongs to the set of $R_n$ which correspond to equilibrium solutions then there is a jump of the tangential component of the reaction and the unique solution of problem (3) is given by the trajectory obtained up to the time $t^*$ when the tangential velocity reaches zero and the constant function $(u_t(t), u_n(t)) = (u_t(t^*), 0)$ for $t > t^*$. 
If on the other hand \( R_n(t^*) \) does not belong to the set of \( R_n \) which correspond to equilibrium solutions, then \((u_t(t^*),0)\) is not an equilibrium solution. We know thanks to [4] that there can be no jump in the tangential velocity, so that this tangential velocity is continuous at time \( t^* \). It cannot be equal to zero because \((u_t(t^*),0)\) is not an equilibrium solution, therefore the velocity for \( t > t^* \) is either strictly positive or strictly negative (indeed an accumulation of zeros to the right of \( t^* \) is excluded by the analyticity result [4]). Let us suppose that for an interval of time on the left of \( t^* \) the velocity was positive (we could of course assume that the velocity was negative and the argument would easily be transposed). If the velocity stays positive for an interval of time to the right of \( t^* \) then the sliding continues on the same side of the cone, therefore \( u_t \) is the solution of a classical ordinary differential equation, equation (9) in this case, and there can be no discontinuity of the second derivative. There is consequently a minimum of the function \( \dot{u}_t \) at \( t = t^* \) and the second derivative of \( u_t \) is therefore equal to zero at \( t = t^* \). But in that case we would have:

\[
\begin{align*}
m\ddot{u}_n(t^*) &= 0 = N_n(u_t(t^*),0) + F_t + R_n(t^*), \\
m\ddot{u}_n(t^*) &= 0 = N_n(u_t(t^*),0) + F_n + R_n(t^*),
\end{align*}
\]

which would imply that \((u_t(t^*),0)\) is an equilibrium solution, which contradicts the fact that we are in the case where \( R_n(t^*) \) does not belong to the set of \( R_n \) which correspond to equilibrium solutions.

We thus conclude that the velocity must be strictly negative for an interval of time to the right of \( t^* \), so that the velocity changes sign and the sliding continues on the other side of the cone. This implies that for \( t > t^* \) equation (9) no longer governs the movement but that equation (10) does.

\[\square\]

3. A particular case: \( F_n > 0 \) large enough and \( |F_t| \) large enough. The aim of this section is to show that although the set of equilibria is given by a complicated graph and involves a lot of different cases, there are some cases where a good approximation of the dynamics can be obtained by the same calculations as those of the linear case. In section 2.1 we have already checked that under the hypothesis of small displacements the equations of motion reduce to the classical equations (5). So that at small strains this linearization leads exactly to the cases that have been explored in [8]. However the shape of the graph of Figure 2 suggests that more interesting linearizations can be undertaken.

3.1. Linearization around an equilibrium \((u^0_t,0)\). We first consider the normal component of the external force \( F_n \) such that \( F_n \geq kh \) which ensures that the horizontal asymptote of the graph of the equilibrium solutions given by equation (7) is in the negative half plane. We then consider the point \( M := (R_t = 0, R_n = -\frac{F_n}{2}) \) in the \( (R_t, R_n) \)-plane which is in the Coulomb cone and compute the values \((F^0_t, F^0_n)\) to ensure that the point \( M \) is on the graph of the equilibrium solutions. We obtain \( F^0_n = kh \), \( F^0_t = \frac{kh}{2} \sqrt{4\beta^2 - 1} \) and let \((u^0_t, u^0_n = 0)\) be the associated equilibrium solution of problem (3) with a reaction \((R^0_t, R^0_n)\), i.e. \( u^0_t = -X_0 + h \sqrt{4\beta^2 - 1} \).

Let us now assume that a small change \( \tilde{F}_t \) of the tangential component of the external force modifies the tangential motion from \( u^0_t \) to \( u^0_t + \dot{u}_t \) and the reaction \( R^0_t, R^0_n \) to \( \tilde{R}_t, \tilde{R}_n \). Going through Taylor expansions, identifying the
terms in equilibrium with \((F_0^t, F_0^n)\), and removing higher order terms, we obtain that this linearization of equations \((3-i, ii)\) around point \(M\) reads:

\[
\begin{align*}
  i) & \quad m\ddot{u}_t + K_t\dot{u}_t = \dot{F}_t + \dot{R}_t, \\
  ii) & \quad W\dot{u}_t = F_n + \dot{R}_n,
\end{align*}
\]

(12)

where the coefficients \(K_t\) and \(W\) are given by the partial derivatives of \(N_t(u_t, u_n)\) and \(N_n(u_t, u_n)\) at \((u_t^0, 0)\) that is:

\[
K_t = k\left(-1 + \frac{1}{8\beta^2}\right), \quad W = -\frac{k}{8\beta^2}\sqrt{4\beta^2 - 1}.
\]

These coefficients are those of a stiffness matrix which is the linear approximation of the nonlinear restoring force \((N_t(u_t, u_n), N_n(u_t, u_n))\) in the neighborhood of point \((u_t^0, 0)\). Figure 3 represents the equilibrium solutions with a nonlinear restoring force and its linearization in the neighborhood of point \((u_t^0, 0)\).

**Remark 1.** It is important to keep in mind that linearizing problem (3) would be meaningless, since the right-hand sides of lines \(i)\) and \(ii)\) involve measures and lines \(iii)\) and \(iv)\) involve inequalities.

From now on, “linear” or “linearized” will be used without ambiguity for the problem where the equations of the dynamics have been linearized but where the non-linearities due to the contact and friction constraints remain. We shall refer to the initial problem as the “nonlinear” problem when both the equations of the dynamics and the contact and friction constraints are nonlinear. We show in the remaining part of this section that solving the linearized problem gives a very good approximation of the solution to the nonlinear one when the situation corresponds to the one represented in Figure 3. Unfortunately this is not always the case, so that we end the section by explaining that this linearization must be used with caution, emphasizing the situations where nonlinear and linearized solutions differ.

3.2. **Approximating the qualitative dynamics.** It was obtained in subsection 3.1 that in order to calculate the sliding trajectories around point \(M\), we can replace problem (3) by a linearized problem which involves equations (12). The dynamics of this linearized problem has already been completely investigated. We only give

![Figure 3. An example of a situation where the set of equilibria in contact is close to a straight line.](image-url)
two examples comparing the behavior of the solution of the initial nonlinear system with that of the solution of equations (12) with the same initial data and unilateral contact and friction conditions. In both cases we apply a periodic perturbation of amplitude $\varepsilon$ and period $2T$ to the tangential force. In the first case, represented on Figure 4, where the trajectory goes to equilibrium in finite time, we observe the proximity of the final times and of the positions at arrival. In the second case, represented on Figure 5, we compare periodic solutions where the perturbing force is of larger amplitude than that of Figure 4.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{nonlinear_problem}
\includegraphics[width=0.4\textwidth]{linearized_problem}
\caption{Comparison of the convergence to an equilibrium solution for the fully nonlinear problem and its linearization when $T = 4$ and $u_t(0) = 1.3$. The nonlinear equilibrium $u_t = 2.832528624$ is obtained at $t = 23.107$ while the trajectory reaches the equilibrium $u_t = 2.801836470$ at time $t = 23.110$ using the linearized equations.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{small_period}
\includegraphics[width=0.4\textwidth]{large_period}
\caption{Comparison of periodic trajectories: a - for a very small period, $T = 0.3$: For $\varepsilon = 1.5$, a periodic solution is found in the case of the nonlinear strain for an initial value $(u_0 = 3.4225, \dot{u}_0 = 0)$, while the linearized problem gives $(u_0 = 3.2079, \dot{u}_0 = 0)$; b - for a large period, $T = 10$: For $\varepsilon = 0.7$, a periodic solution is found in the case of the nonlinear strain for an initial value $(u_0 = 2.4767, \dot{u}_0 = 0)$, while the linearized problem gives $(u_0 = 2.4860, \dot{u}_0 = 0)$.}
\end{figure}

Remark 2. The algorithms used to compute the trajectories represented on Figures 4 and 5 were obtained in [8] for the linear case and shall be given in the next section for the nonlinear case.
Let us now add a few comments about this approximation.

- This analysis will hold for any normal component \( F_n \geq h k \); it can be transposed if \( F_t \) is changed into \(-F_t\) by symmetry with respect to the \( R_n \) axis in the \( \{R_t, R_n\}\)-plane. In particular, the symmetry means that the slope of the straight line which approximates the nonlinear graph changes sign which simply implies that the rotation around the equilibrium changes sign.
- As \(|F_t|\) increases, the quality of the approximation given by problem (12) increases because the part of the graph of equation (7) that intersects the Coulomb cone becomes close to a straight line.
- Through this approximation, we observe that the nonlinear system behaves in the same way as the linear one concerning the existence of a transition between a range where there exist infinitely many equilibrium states and no periodic solutions and another range where there exist periodic solutions but no equilibrium states. The study of this transition is one of the topics of the next sections.

3.3. Where the qualitative dynamics of the linearized problem differs from the nonlinear one. There are two main qualitative differences in the behavior of the nonlinear problem and its linearization. These differences appear even in a small neighborhood of the point around which we have linearized.

3.3.1. Loosing contact. The first important difference in the behavior of the solutions is that for the linear problem a sufficiently large perturbation provokes a loss of contact whereas for a whole range of nonlinear problems (in fact as soon as \( F_n \geq h k \)) no loss of contact appears however large the perturbation. Indeed for the linearized system the set of equilibria is represented by a straight line of nonzero slope that will obviously intersect the \( R_t \) axis. But for the nonlinear system the graph of the set of equilibria in the \( \{R_t, R_n\}\)-plane never intersects the \( R_t \) axis when the horizontal asymptote of the graph of equation (7) is in the negative half plane (as is the case when \( F_n \geq h k \)).

3.3.2. Period doubling solutions. We refer now to Figure 5-b. We know from [8] that in the range of the \( \{T, \varepsilon\}\)-plane where such a trajectory exists, the linearized system exhibits infinitely many periodic solutions around the one represented on Figure 5-b. Moreover, in the linear case, all the periodic trajectories in a neighborhood of the trajectory of Figure 5-b have a period twice that of the excitation. Numerical experiments in the fully nonlinear case have never exhibited such a set of periodic solutions. On the contrary non-periodic trajectories are obtained and they all converge to the periodic solution of Figure 5-b.

4. Qualitative study of the set of equilibria.

4.1. Changes of the set of equilibria under an increasing load. In order to understand how the set of equilibrium solutions evolves when the load increases we have plotted in Figure 6 the graph of the equilibrium solutions under three different loads in the \( \{R_t, R_n\}\)-plane and the corresponding equilibria. Figure 6-a shows the modification of the intersection of the graph of equation (7) with the friction cone when the graph is translated from the right to the left that is under an increasing load. The set of equilibria corresponding to the three different situations are represented in Figure 6-b where the tangential component \( u_t \) has been translated.
in order to have the three trivial equilibrium solutions either side of the $u_n$ axis. In all the remaining Figures we have chosen this representation.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{image}
\caption{Variation of the set of equilibrium states under increasing tangential load: - box: $F_t = 0$, - diamond: $F_t = F_t^* > 0$, - circle: $F_t = F_t^{**} > F_t^*$.
}
\end{figure}

4.2. **Some preliminary calculations.** We consider equilibria around $u_t(R_n) = 0$ (the observations and the analysis would be exactly the same around $u_t(R_n) = -2X_0$), so that, from equation (7), we have

\begin{equation}
R_t(R_n) = -F_t + (R_n + F_n)\sqrt{\frac{h^2k^2\beta^2}{R_n^2} - 1}.
\end{equation}

In order to characterize the different situations that shall occur the following computations are needed. We begin by determining in the case where $F_t = 0$ (that is when the graph of equilibria is centered on the $R_n$ axis) and $F_n = hk$ (that is when the horizontal asymptote of the graph of equilibria is the $R_t$ axis), the value of the friction coefficient that ensures that the graph is tangent to the left border of the Coulomb cone. Writing that $R_t(R_n)$ given by equation (13) is equal to $\mu R_n$, and that their first derivatives are equal we obtain:

\begin{align}
&i) \quad \mu R_n = (R_n + kh)\sqrt{\frac{h^2k^2\beta^2}{R_n^2} - 1}, \\
&ii) \quad \mu = \frac{-R_n^3 - h^3k^3\beta^2}{R_n^3\sqrt{\frac{h^2k^2\beta^2}{R_n^2} - 1}},
\end{align}

from which, going through straightforward calculations, we obtain that $R_n$ must be a solution to

\begin{equation}
R_n^2 - kh\beta^2 R_n - 2h^2k^2\beta^2 = 0,
\end{equation}
which in turn gives that the only negative root of this equation is:

$$\hat{R}_n = \frac{hk\beta^2}{2} \left( 1 - \sqrt{1 + \frac{8}{\beta^2}} \right),$$

and the corresponding $R_t$ is given by equation (13). Inserting this value into equation (14) we obtain that the graph is tangent to the Coulomb cone if the coefficient of friction is equal to

$$\mu_c = \left( 1 + \frac{kh}{R_n} \right) \sqrt{\frac{h^2k^2\beta^2}{R_n^2} - 1}.$$ 

We note that these values, although intricate, depend only on the given geometry and are easily computed.

From a more practical point of view, let us now assume that the geometry and the friction coefficient are given, and let us look for the amplitude of the tangential perturbation which leads to the tangency on the left border of the cone. Again we start from equation (13) now written as

$$\mu R_n + F_t = (R_n + kh) \sqrt{\frac{h^2k^2\beta^2}{R_n^2} - 1}$$

from which we obtain

$$F_t = \frac{kh(-R_n^2 + kh\beta^2 R_n + 2h^2k^2\beta^2)}{R_n^2 \sqrt{\frac{h^2k^2\beta^2}{R_n^2} - 1}}$$

(15)

where the value of $R_n$ at the tangency satisfies equation (14-ii) that is:

$$\mu R_n^3 \sqrt{\frac{h^2k^2\beta^2}{R_n^2} - 1} + R_n^3 + h^3k^3\beta^2 = 0.$$ 

From the coordinates of point $E$ given in formula (8) the value of $R_n$ at the tangency belongs to the interval $[-hk\beta^{2/3}, 0]$ and as the function (13) is monotone in this interval, this equation has a single root in the interval $[-hk\beta^{2/3}, 0]$, so that the value of $R_n$ is easily given by a symbolic calculation software. Inserting this value into formula (15) gives the amplitude of the tangential component of the force for which the set of equilibria is tangent to the left border of the cone. Let us denote by $F_{t1}$ the numerical value of this amplitude.

Let us now look at the case where the tangency holds on the right border of the cone. The border of the cone is now given by the equation $R_t(R_n) = -\mu R_n$ and the equilibrium point at tangency by:

$$-\mu R_n + F_t = (R_n + kh) \sqrt{\frac{h^2k^2\beta^2}{R_n^2} - 1}.$$ 

It is then trivial that we obtain the same equation as (15) but, as the point of tangency is this time situated between point $A$ and point $E$ due to formula (8) the value of $R_n$ at the tangency belongs to the interval $[-hk\beta, -hk\beta^{2/3}]$ and the function (13) being monotone in this interval, this equation has a single root in the interval $[-hk\beta, -hk\beta^{2/3}]$. This leads to a value of $F_t$ which we denote by $F_{t2}$. Through elementary geometrical properties, we have $|F_{t1}| < |F_{t2}|$. 


These results can be summarized as follows where we assume, without restriction, that \( \mu \) is large enough for the set of equilibria to be simply connected for \( F_t = 0 \). This gives a quantitative analysis of what was observed on Figure 6.

- \( F_t < F_{t2} < 0 \): The graph of equation (13) does not intersect the cone, so that there is no equilibrium point in contact associated with this branch.
- \( F_t = F_{t2} \): The graph of equation (13) is tangent to the right border of the cone, so that a single equilibrium point in contact appears which is in imminent sliding to the left.
- \( F_{t2} < F_t < F_{t1} \): As \( F_t \) increases, the set of equilibrium points in contact increases monotonically.
- \( F_t = F_{t1} \): This is where the set of equilibria is the largest while one equilibrium point appears in imminent sliding to the right.
- \( F_t > F_{t1} > 0 \): A hole appears in the set of equilibria from the point in imminent sliding to the right, and extends monotonically inside the set of equilibria. So that the set of equilibria is no longer connected, which is an important qualitative difference with the case of linear stiffness.

![Figure 7](image.png)

**Figure 7.** The three limit cases of equilibrium sets. On the left \( F_t = F_{t1} > 0 \), in the middle \( F_t = 0 \) and on the right \( F_t = F_{t2} < 0 \).

It is trivial that this analysis applies to the equilibria associated to the other branch

\[
R_t(R_n) = -F_t - (R_n + F_n) \sqrt{\frac{k^2 k^2 \beta^2}{R_n^2} - 1} \tag{16}
\]

of the set of equilibria in contact after a symmetry with respect to the \( R_n \) axis.

4.3. **Equilibria under tangential oscillations.** From now on, the normal component \( F_n(t) \) of the loading is kept constant, equal to or larger than \( hk \). This insures that there always exist equilibrium solutions in contact whatever the tangential component (see [9]). Up to the end of the paper we shall in fact choose \( F_n(t) = hk \). We shall consider that the tangential component \( F_t(t) \) is written in the following way:

\[
F_t(t) = F_t + P_t(t),
\]

and we shall restrict our attention, as was the case in [8], to rectangular wave form perturbations defined, for \( i = 0, 1, \ldots \) and \( \varepsilon > 0 \), by:

\[
P_t(t) = \varepsilon \text{ if } t \in [2iT, (2i + 1)T] \text{ and } P_t(t) = 0 \text{ if } t \in [(2i + 1)T, (2i + 2)T]. \tag{17}
\]
4.3.1. The set of equilibria. In the nonlinear case explicit closed-form calculations cannot be performed as far as in the linear case but the set of equilibrium solutions under such a perturbation is nevertheless very easy to compute. The sets of equilibrium solutions submitted to an oscillation are given by

$$\Sigma = \bigcap_{t>0} \Sigma(t),$$

where $\Sigma(t)$ consists of the set of tangential components of the equilibrium positions at time $t$, their normal components being equal to zero as these equilibria are in contact. Due to the fact that $F_n(t) = h\kappa$, the set $\Sigma(t)$ is never empty. In the case of a rectangular wave form perturbation given by (17), $\Sigma(t)$ is constant equal to $\Sigma_0$ for $t \in [(2i+1)T, (2i+2)T]$, and constant equal to $\Sigma_\epsilon$ for $t \in [2iT, (2i+1)T]$, so that $\Sigma$ reduces to

$$\Sigma = \Sigma_0 \cap \Sigma_\epsilon.$$ 

Therefore, $\Sigma$ may consist either of an interval, of several disjoint intervals, of isolated points, or of an empty set, depending on the value of $\epsilon$. For example, referring to Figure 6-b there are two disjoint sets of equilibria corresponding to the intersection of the “box” curve and the “diamond” curve, a single equilibrium point corresponding to the intersection of the “circle” curve and the “diamond” curve or an empty set corresponding to the intersection of the “box” curve and the “circle” curve.

4.3.2. Transition to nonexistence of equilibria. As long as the set of equilibria $\Sigma$ is non empty there exist equilibrium solutions for the oscillating load. But the measure of $\Sigma$, is maximum for $\epsilon = 0$ and decreases monotonically to zero as $\epsilon$ increases. We can therefore define $\bar{\epsilon}$ as the amplitude of the perturbing force for which $\Sigma$ reduces to isolated points. For any $\epsilon > \bar{\epsilon}$, we obtain $\Sigma = \emptyset$. In other words for $\epsilon > \bar{\epsilon}$ there no longer exist any equilibrium solutions. The value of $\bar{\epsilon}$ depends on all the parameters of the problem, its computation is rather intricate and shall be omitted.

5. Qualitative dynamics when $\epsilon$ is smaller than $\bar{\epsilon}$. All the Figures and numerical results presented in Sections 5 and 6 have been obtained with the following values of the parameters: $m = 1$, $h = 1$, $X_0 = 1$, $k = 1$ and $\mu = 0.5$.

5.1. Comments about the calculations. Numerical calculations are presented in the following sections, either in order to illustrate theoretical results, or simply as numerical experiments when theoretical results are not yet available. The manner in which these calculations were performed is described and justified below. Both the non smoothness due to frictional contact and the non-linearity of the dynamics must be handled with extreme care.

From a practical point of view, by introducing $y := \frac{u_t + X_0}{h}$, equations (9) and (10) become

$$\begin{cases}
(i) \quad \ddot{y} + \frac{k}{m}(y - \mu) \left[ 1 - \frac{\beta}{\sqrt{y^2 + 1}} \right] = \frac{F_t - \mu F_n}{mh}, \\
(ii) \quad \ddot{y} + \frac{k}{m}(y + \mu) \left[ 1 - \frac{\beta}{\sqrt{y^2 + 1}} \right] = \frac{F_t + \mu F_n}{mh},
\end{cases}$$

(18)
so that the classical phase plane \( \{ u_t, \dot{u}_t \} \) is changed into the \( \{ y, \dot{y} \} \)-plane (see from Figure 8 onwards). It is in fact easier to read the figures in the \( \{ y, \dot{y} \} \)-plane because \( y = 0 \) corresponds to the case where the spring is vertical.

The existence and uniqueness result given in Lemma 2.1 ensures that given initial data, a numerical scheme provides an approximation of the solution to these equations. A fourth order Runge-Kutta method was used to integrate (18) while symbolic calculations were used to check that a sliding phase holds to the left or to the right. In fact the basic structure of the algorithm used to solve this problem is the same as in the case of linear strains, the only difference is that in the latter case we were able to have a closed form expression for the solution of these sliding phases whereas here we must use a numerical scheme. Note that we are here in presence of very smooth nonlinear differential equations in both sliding phases so that the error estimates on the results we present are close to \( 10^{-12} \).

When the loading is constant Lemma 2.2 determines the trajectory after a time \( t^* \) at which the velocity is equal to zero: either the velocity stays equal to zero for all future time i.e. the mass has attained an equilibrium and is clamped by friction, or the velocity changes sign i.e. the mass starts sliding in the opposite direction. When the loading is oscillating due to a perturbation of the form (17), Lemma 2.2 also determines the trajectory after a time \( t^* \) at which the velocity is equal to zero, but only during a half period where the loading is constant. If the velocity has remained equal to zero for some time before the end of a half period, i.e. the mass has been at equilibrium for that value of the loading, how can the velocity be determined when the load changes at the beginning of the next half period. Does the mass stay motionless or does it start sliding to the right or to the left? The answer is given by computing the tangential and normal components of the new reaction corresponding to the position of the mass at time \( t^* \) with the new load.

We then obtain, as easy consequences of Coulomb’s friction law:
- if \( R_t(t^*) < \mu R_n(t^*) \) then the mass starts sliding to the right,
- if \( R_t(t^*) > -\mu R_n(t^*) \) then the mass starts sliding to the left,
- and of course if \( \mu R_n(t^*) \leq R_t(t^*) \leq -\mu R_n(t^*) \) the mass stays motionless which means that an equilibrium solution has been obtained for this oscillating load.

5.2. Preliminary: The case of constant loading. Let us start by showing a few trajectories. Figures 8, 9-a and 9-b represent sets of trajectories issued from initial data in the four quadrants of the phase plane and corresponding respectively to, \( (F_n = kh, \ F_t = 0) \), \( (F_n = kh, \ F_t = F^*_t) \) and \( (F_n = kh, \ F_t = F^{**}_t) \) as defined in Figure 6, where the sets of equilibria are composed of two disjoint sets in Figure 9-a, (the diamond case of Figure 6-b) or of a single isolated set in Figure 9-b (the circle case of Figure 6-b).

Remark 3. It is interesting to note that due to the fact that the set of equilibria is not connected in general, we observe an extreme sensitivity to initial data, as can be seen on Figure 10.

5.3. The qualitative dynamics. We can now study the dynamics when the tangential loading is oscillating according to formula (17). The assumption on the normal loading implies that there are always equilibrium solutions in contact if the tangential loading is constant. The first result given here is formally close to the one obtained for the oscillator with a linear stiffness presented in [8]. It can be stated as:
**Figure 8.** A set of trajectories in the phase space in the case $F_t = 0$

**Figure 9.** Phase plane under increasing sufficiently small tangential load

**Figure 10.** Sensitivity to initial data: trajectories starting from initial data in the same small neighborhood (the small circle in the down left quarter of the phase space) diverge and reach very different equilibrium states.

**Proposition 1.** There exists a value $\varepsilon$ of the amplitude of the oscillating load such that, for any $\varepsilon < \varepsilon$: 
i) infinitely many equilibrium states exist whatever the period of the excitation, 

ii) all trajectories lead to equilibrium.

Proof. Point i) is a simple consequence of section 4.3. Indeed Σ is defined as the smallest value of $\varepsilon$ such that the measure of $\Sigma = \Sigma_0 \cap \Sigma_{\varepsilon}$ is equal to zero. For any value of $\varepsilon$ strictly smaller than $\Sigma$, the measure of $\Sigma$ is strictly positive so that $\Sigma$ consists of one or several segments of equilibria.

The proof of point ii) is more intricate. For the paper to be self-contained, we first recall some basic tools.

Having a second order ordinary differential equation in $\mathbb{R}$ of the form

$$\frac{d^2u}{dt^2} + f(u) = G,$$

where $f(u)$ is some smooth nonlinear function having a primitive $F(u)$, and where $G$ depends neither on $t$ nor on $u$ (although the method could be adapted), a classical integration process, which we just recall briefly, reads as follows:

Let $\frac{d^2u}{dt^2} + f(u) = G$,

then $\frac{d^2u}{dt^2} \frac{du}{dt} + f(u) \frac{du}{dt} = G \frac{du}{dt},$

then $\frac{1}{2} \left( \frac{du}{dt} \right)^2 + \frac{d F(u)}{dt} = G \frac{du}{dt},$

so that $\frac{1}{2} \left( \frac{du}{dt} \right)^2 + F(u) = Gu + C,$

where $C$ is an integration constant which is given by the initial conditions. We then get:

$$\frac{du}{dt} = \pm \sqrt{C + 2(Gu - F(u))},$$

and $t = \pm \int_{u_0}^{u} \frac{ds}{\sqrt{C + 2(Gs - F(s))}}.$

In the present case the calculations are performed using the functions $f(u)$ given in equations (18) successively in the cases of positive sliding and negative sliding, the forcing varying as in formula (17). Recalling the change of variables introduced in equations (18) and assuming that sliding holds in the positive direction, we obtain explicitly

$$\begin{align*}
a) \quad \dot{y} &= \sqrt{C + \frac{2k}{m} \left[ \left( \mu + \frac{F_t - \mu F_n}{kh} \right) y - \frac{1}{2} y^2 - \mu \beta argsh(y) + \beta \sqrt{1 + y^2} \right]}, \\
and \quad t &= \int_{y_0}^{y} \frac{ds}{\sqrt{C + \frac{2k}{m} \left[ \left( \mu + \frac{F_t - \mu F_n}{kh} \right) y - \frac{1}{2} y^2 - \mu \beta argsh(y) + \beta \sqrt{1 + y^2} \right]}}. 
\end{align*}$$

When the external force is perturbed by a positive tangential component of amplitude $\varepsilon$, according to equation (17), only the term $F_t - \mu F_n$ is changed into
$F_t + \varepsilon - \mu F_n$. Let the pair $(y_0, v_0)$ be the initial data at $t = 0$, then equation (21-a) gives:

$$C = \frac{v_0^2}{2} - 2\left(\frac{\mu k}{m} + \frac{F_t - \mu F_n}{m h}\right)y_0 + \frac{k}{m} y_0^2 + 2\frac{\mu k \beta}{m} \tanh(y_0) - 2\frac{k \beta}{m} \sqrt{1 + y_0^2}. \quad (22)$$

Similar expressions are obtained for negative sliding when the tangential force is equal to $F_t + \varepsilon$ or to $F_t$.

Then point ii) will result from the two following lemmas:

**Lemma 5.1.** Let $y_1(t)$ and $y_2(t)$ be two trajectories starting from the same initial data $(y(0) = y_0, \dot{y}(0) = v_0)$, $y_1(t)$ being associated with a tangential component of the force $F_t + \varepsilon$ for $\varepsilon < \varepsilon$, and $y_2(t)$ being associated with a tangential component of the force $F_t$.

Then $y_1(t) > y_2(t)$ as long as $\dot{y}_1(t)$ and $\dot{y}_2(t)$ do not reach zero.

**Lemma 5.2.** Whatever the external force $F_t$, let $y_1(t)$ be a solution of equation (20) corresponding to positive sliding, who reaches the value $\bar{y}_1$ at zero velocity. Assume $\bar{y}_1$ is out of equilibrium and let $y_2(t)$ be solution of equation (20) corresponding to negative sliding starting from the initial data $(\bar{y}_1, 0)$.

Then $|\dot{y}_1(t)| > |\dot{y}_2(t)|$ as long as $\dot{y}_1(t)$ and $\dot{y}_2(t)$ do not change sign.

The trajectories $y_1$ and $y_2$ defined in these two lemmas are schematically represented in the phase space on Figure 11.

![Diagram](Figure 11. Qualitative illustration of Lemmas 5.1 and 5.2.)

**Proof of lemma 5.1.** Let us start with the case of positive sliding, that is $(y(0) = y_0, \dot{y}(0) = v_0 > 0)$. We have that $y_1$ is a solution of $(\dot{y}(t))^2 := g_1(y)$ and $y_2$ is a solution of $(\dot{y}(t))^2 := g_2(y)$ where $g_1$ and $g_2$ are given by equation (21-a) in which the tangential component of the force is $F_t + \varepsilon$ for $g_1$ and $F_t$ for $g_2$. We then obtain:

$$g_1(y) - g_2(y) = \frac{2}{m} \varepsilon (y - y_0),$$

which gives directly the result since the right hand side is positive. The initial data $(y_0, v_0)$ was assumed to be in the positive half-plane. In the case of negative sliding, that is when $(y(0) = y_0; \dot{y}(0) = v_0 < 0)$, we obtain:

$$g_1(y) - g_2(y) = \frac{2}{m} \varepsilon (y - y_0),$$

which is negative since negative sliding implies $y < y_0$.

**Proof of lemma 5.2.** The two mappings $y_1$ and $y_2$ are defined as being solution to the two differential equations $(\dot{y}(t))^2 := g_1(y)$ and $(\dot{y}(t))^2 := g_2(y)$ where $g_1$ and $g_2$...
are now defined by:

\[
\begin{align*}
g_1(y) &= \frac{2k}{m} \left[ \left( \mu + \frac{F_i - \mu F_n}{kh} \right) (y - y_1) - \frac{1}{2} (y^2 - y_1^2) 
- \mu \beta (\text{argsh}(y) - \text{argsh}(y_1)) + \beta (\sqrt{1+y^2} - \sqrt{1+y_1^2}) \right] \\
g_2(y) &= \frac{2k}{m} \left[ \left( - \mu + \frac{F_i + \mu F_n}{kh} \right) (y - y_1) - \frac{1}{2} (y^2 - y_1^2) 
+ \mu \beta (\text{argsh}(y) - \text{argsh}(y_1)) + \beta (\sqrt{1+y^2} - \sqrt{1+y_1^2}) \right].
\end{align*}
\] (23)

As long as the trajectories associated with equations (23) do not intersect the y axis we have:

\[ g_1(y) - g_2(y) = - \frac{4\mu F_n}{mh} (y - y_1) + \frac{4k \mu}{m} (y - y_1) - \frac{4\mu \beta}{m} (\text{argsh}(y) - \text{argsh}(y_1)). \]

The growth of the function \( \text{argsh}(\cdot) \) is slower than linear, and \( \beta \) is strictly larger than one, which makes it possible to have an estimate of this difference, but for the contact to hold strictly for any \( F_i \) we have chosen \( F_n = hk \), which gives:

\[ g_1(y) - g_2(y) = - \frac{4\mu \beta}{m} (\text{argsh}(y) - \text{argsh}(y_1)). \]

Therefore \( g_1(y) - g_2(y) \) is strictly positive since \( y \) is always strictly smaller than \( y_1 \) for both trajectories.

We observe that these qualitative results are obtained without any integration but only by comparing the right hand side of two differential equations. \( \square \)

**Lemma 5.3.** Assume that under a tangential force \( F_i = \varepsilon \) a first phase of positive sliding from an initial data \((y_0,0)\), denoted by \( y_1(t) \), holds up to a point out of equilibrium \((\bar{y}_1,0)\) at \( t = t_1 \) where the velocity passes through zero for the first time. The trajectory then continues by a second phase \( y_2(t) \) in negative sliding. This trajectory would reach a zero velocity again at some point \( \bar{y}_2 \) at \( t = t_2 \) if \( F_i \) remained unchanged. But at some time \( t_2 \) strictly between \( t_1 \) and \( \bar{t}_2 \) the external force \( F_i \) is set equal to zero. Let \( y_3(t) \) be this last phase, still in negative sliding and \( \bar{y}_3 \) its value when the velocity reaches zero again. Then there exists \( \varepsilon_0 \) such that:

\[
\forall t_2 \in ]t_1, \bar{t}_2[, \begin{cases} 
\bar{y}_3 > y_0 & \text{if } 0 < \varepsilon < \varepsilon_0, \\
\bar{y}_3 \leq y_0 & \text{if } \varepsilon \geq \varepsilon_0.
\end{cases}
\] (24)

The trajectories described in Lemma 5.3 are represented on Figure 12 in the case \( \varepsilon < \varepsilon_0 \). In fact Lemma 5.3 shows that the amplitude of the perturbing force \( \varepsilon \) controls the arrival point after the complete loop, in particular whether this point is smaller than the initial point \( y_0 \) or not. It is therefore the amplitude of the perturbing force \( \varepsilon \) which implies that either all the trajectories go to equilibrium or that the amplitude of the trajectories increases, which in turn allows the existence of periodic solutions.

Since we already know how to compare \( \dot{y}_1(t) \) to \( \dot{y}_2(t) \) and \( \dot{y}_2(t) \) to \( \dot{y}_3(t) \), the calculations are essentially the same as those of Lemmas 5.1 and 5.2 but they are rather more tedious so shall be omitted here.
These results can now be used to complete the proof of point \( ii \) of Proposition 1.

- Let us first assume that the period of the oscillating load is very large. Then everything happens as if the forcing was constant. During a whole half period, the calculation involves only Lemma 5.2 and the trajectory is represented by any trajectory of Figures 8 or 9.

- As the period decreases the calculation of the trajectory cannot avoid taking into account the change of the force during a phase of positive sliding or of negative sliding. Moreover, the smaller the period, the larger the number of such changes. What happens at these changes is qualitatively given either by Lemma 5.1 or by Lemma 5.3.

- Calculating a complete trajectory implies accumulating any number of successive phases of positive sliding with the perturbing force or without the perturbing force and of negative sliding with or without the force. This does not seem to lead to new theoretical difficulties, but of course leads to increasing complexity as the period decreases, so that the calculations require a symbolic computation software.

Numerical computations suggest that \( \varepsilon_0 = \tau \), which implies that the behavior of the system involves a single transition.

Trajectories associated with the perturbation of rectangular wave shape defined above are represented on Figure 13. By comparison with Figures 8 and 9 a-b, the non-regular points correspond to the discontinuities of the external perturbation.

6. **Periodic solutions when \( \varepsilon \) is greater than \( \tau \).** The amplitude of the tangential perturbation is chosen sufficiently large for the set of equilibria to be empty (see subsection 4.3).

The main result of this section can qualitatively be stated as follows: When submitted to a periodic forcing of sufficiently large amplitude, the behavior of the oscillator is such that there exist periodic solutions whatever the period of the excitation. Moreover:

- \( i \) for small periods, a single periodic solution exists, the period of which is that of the excitation and as the period increases the amplitude of the periodic solutions increases;
ii) for large periods, there also exists a single periodic solution, the period of which is that of the excitation but its amplitude does not depend on the period (in fact the amplitude is constant);

iii) between these two ranges there exists an interval on the period axis in which at least one periodic solution coexists with another solution which may exhibit chaos.

Some parts of the proof of this global result are essentially in progress and the justifications rely on numerical experiments, but some others parts are already based upon theoretical tools. We present now the state of these justifications.

6.1. For large periods. When the period of the forcing is large, we shall prove that the single periodic solution can be constructed from an initial data with a zero velocity. The orbit of this periodic orbit involves a cusp and is represented on Figure 14.

**Proposition 2.** Assume the amplitude of the forcing $\varepsilon$ is greater than $\varepsilon$, which means that the set of equilibria is empty. Then, when the period of the forcing is sufficiently large:

i) there exists a periodic solution having the period of the excitation,

ii) the orbit of the periodic solution does not depend on $T$.

**Proof.**

• First step: There exists a single periodic solution of period $2T$.

Numerical observations guided the direct construction which is used in the proof of this first step. It is formally very close to the proof of the corresponding result obtained in [8] in the case of linear stiffness, except for the important fact that because of the non-linearity, implicit equations are to be solved.

In the proof of Proposition 1, we recalled an integration procedure for the equation of the dynamics, either for positive or for negative sliding, with or without the perturbation of amplitude $\varepsilon$. We shall now use the same tools to build a periodic solution.

Starting in positive sliding from the initial condition $(y_0, 0)$, where $y_0$ belongs to the set $\Sigma_0$, and assuming that the value of the perturbing tangential force remains equal to $\varepsilon$ for a sufficiently long time, the velocity comes to zero.
again at time $t_1$ at a position $y_1$ which satisfies the equation:

$$(y_0^2 - y_1^2) - 2(\varepsilon/h)(y_0 - y_1) - 2\beta(\sqrt{1 + y_0^2} - \sqrt{1 + y_1^2}) + 2\beta\mu(\text{argsh}(y_0) - \text{argsh}(y_1)) = 0.$$  

Then starting from the initial condition $(y_1, 0)$, for a negative sliding with the same value $\varepsilon$ of the perturbation, the velocity comes to zero again at time $t_2$ and at position $y_2$. So that $y_2$ is the solution of the following equation:

$$(y_1^2 - y_2^2) - 2(\varepsilon/h)(y_1 - y_2) - 2\beta(\sqrt{1 + y_1^2} - \sqrt{1 + y_2^2}) - 2\beta\mu(\text{argsh}(y_1) - \text{argsh}(y_2)) = 0.$$  

Finally starting from the initial condition $(y_2, 0)$, still in negative sliding, but where the perturbation has changed to zero, the velocity comes to zero again at time $t_3$ and at position $y_3$ given by the equation:

$$(y_2^2 - y_3^2) - 2\beta(\sqrt{1 + y_2^2} - \sqrt{1 + y_3^2}) - 2\beta\mu(\text{argsh}(y_2) - \text{argsh}(y_3)) = 0.$$  

This construction is valid as long as $T \geq (t_1 + t_2)$ and $y_2$ belongs to the set $\Sigma_\varepsilon$, and as long as $y_3$ belongs to the set $\Sigma_0$. We thus define a mapping $\phi$ from $\Sigma_0$ into $\Sigma_\varepsilon$ such that $\phi(y_0) = y_3$. It is clear that a periodic solution exists if and only if there exists $\tilde{y}_0$ such that $\phi(\tilde{y}_0) = \tilde{y}_0$, i.e. $\tilde{y}_0$ is a fixed point of the mapping $\phi$. As the mapping $\phi$ is continuous from the compact set $\Sigma_0$ into itself, there exists a fixed point to $\phi$.

From a numerical point of view this fixed point is obtained by usual fixed point iterations. The mapping $\phi$ is strictly decreasing in $\Sigma_0$ so that this fixed point is unique. The numerical values chosen at the beginning of section 5 lead to $\tilde{y}_0 = 0.8736596$.

**Second step:** There exists a lower bound for the period of the forcing to lead to such a trajectory. As observed above this construction is possible only if $T \geq (t_1 + t_2)$. Let us denote this lower bound by $T^{**}$. The value of $T^{**}$ can
be explicitly calculated through equation (21):

\[ T^{**} = \int_{y_0}^{y_1} \frac{ds}{\sqrt{C_1 + \frac{2k}{m} \left( \left( \mu + \frac{F_t + \varepsilon - \mu F_n}{kh} \right)s - \frac{s^2}{2} - \mu \beta \text{argsh}(s) + \beta \sqrt{1 + s^2} \right)}} + \int_{y_1}^{y_2} \frac{-ds}{\sqrt{C_2 + \frac{2k}{m} \left( \left( \mu + \frac{F_t + \varepsilon + \mu F_n}{kh} \right)s - \frac{s^2}{2} + \mu \beta \text{argsh}(s) + \beta \sqrt{1 + s^2} \right)}} \]

(25)

The numerical values chosen at the beginning of section 5 give \( T^{**} = 6.874539 \). The time \( t_3 \) is of course smaller than \( t_1 + t_2 \).

• Third step: As soon as \( T \geq T^{**} \), the shape of this periodic orbit in the phase plane does not depend on \( T \).

Since \( y_2 \) and \( y_3 \) are respectively equilibrium states when the loading is perturbed by \( \varepsilon \) and when it is not, the mass remains at rest at these two points until the external force changes. The mass shall remain at rest at point \( y_2 \) until the external force changes that is during \( T - t_1 - t_2 \). And the mass shall stay at rest at point \( y_3 \) during \( T - t_3 \). So that the time intervals during which the mass stays at rest increase with \( T \). The trajectories of the solutions depend on the value of the half period \( T \), see for example Figure 15 where the trajectories for \( T = 8 \) and \( T = 16 \) are represented. However the orbits in the phase plane of these two solutions are identical and given by Figure 14.

\[ \begin{align*}
\text{Figure 15.} & \quad \text{The trajectory involves time intervals where the mass} \\
& \quad \text{remains at rest.}
\end{align*} \]

6.2. For small periods.

**Proposition 3.** Assume the forcing still has a sufficiently large amplitude for the set of equilibria to be empty, but now has a sufficiently high frequency. Then

i) there exists a single periodic solution which has the period of the excitation.

ii) the amplitude of this periodic solution increases with \( T \), as long as \( T \) does not overpass a critical value denoted by \( T^* \).

A few trajectories illustrating Proposition 3 are represented on Figure 16.
A few ideas towards the proof.

- Point i) states that there exists a single periodic solution of period $2T$. The idea is to use the same direct construction as for Proposition 2. The numerical experiments suggest that we are looking for a solution with a phase difference between the loading and the answer. Thus the method used in the previous section in the case of large periods is now applied with an initial data at $t = 0$ equal to $(y(0) = y_0, \dot{y}(0) = v_0)$ where $v_0$ is strictly positive. The procedure is nevertheless essentially the same:
  - let $y_1$ be the value of the displacement at time $t_1$ when the velocity of the positive sliding with perturbation $\varepsilon$ reaches zero;
  - let $t_1$ be smaller than $T$, then the motion continues in negative sliding with the perturbation $\varepsilon$ up $(y_2(t_2), \dot{y}_2(t_2))$, where $t_2 = T$;
  - let the perturbation $\varepsilon$ be removed at time $t_2$ and $y_3$ be the value of the displacement at time $t_3$ when the velocity reaches zero again after a phase of negative sliding;
  - finally the motion continues in positive sliding without perturbation on the loading up to time $2T$. Let $(y_4, \dot{y}_4)$ be the displacement and velocity at that time $2T$.

Therefore a periodic solution of period $2T$ exists if

$$y_4(y_0, v_0) = y_0, \quad \dot{y}_4(y_0, v_0) = v_0.$$

This procedure defines a mapping from $\mathbb{R}^2$ into $\mathbb{R}^2$, and a periodic solution exists when this mapping has a fixed point.

- The second point of Proposition 3, that is that the amplitude of the periodic solution increases from $T = 0$ monotonically with $T$, is observed from numerical computations together with the existence of an upper bound of the range of the periods where this behavior is observed. Let $T^*$ be this upper bound, which is observed to be strictly smaller than the lower bound $T^{**}$ obtained in the previous subsection. The behavior of solutions for periods larger than $T^*$ is described in the next subsection.

In the case of the numerical values adopted in this paper, the half period $T^*$ is found to be around $T^* = 3.2$. Indications for the reason of the very weak accuracy of this result will be given in the next subsection.
6.3. **In the interval** \([T^*, T^{**}]\). This last subsection is based only on numerical experiments, but it seems that things become really more intricate in the range of periods \([T^*, T^{**}]\). The calculations are performed very rigorously as described in subsection 5.1, but the question of whether we may have theoretical foundations for what is observed is open for the moment, so that this last short subsection could be considered as a first step towards future work. The main qualitative features that have been observed are the following.

1. There are apparently two simultaneous phenomena that appear at \(T = T^*\).
   The first one is that the periodic solution of period equal to the period of the excitation computed in Proposition 3 splits itself in two so that its period becomes twice that of the excitation. The second one is that a loss of uniqueness of the periodic solution is observed, characterized by the occurrence of a solution of very large amplitude, consequently involving very large velocities. The corresponding global behavior is represented on Figure 17. It is very difficult to compute precisely this value \(T^*\) because it is difficult to determine exactly when the period becomes double (in fact we can only give a lower bound for \(T^*\)) and because the computational time needed to compute solutions of large amplitude increases drastically as \(T\) approaches \(2T^*\) (so that here we can only give an upper bound for \(T^*\)).

2. As \(T\) increases, the answer of the system changes in the following way:
   - The amplitude of the very large amplitude solution decreases quickly, as represented on Figure 18, in particular Figure 18-b suggests that the amplitude tends to infinity as the period decreases to \(2T^*\).
   - On the other hand, the evolution of the double period solution seems to involve an accumulation of period doubling which, in the case of smooth dynamical systems, would announce a transition towards chaos. This is qualitatively represented on Figures 19 and 20. Moreover, the computations reveal that this qualitative behavior appears, disappears and appears again several times in small sub-intervals of \([T^*, T^{**}]\).

3. For larger periods, the very complicated attractor seems to disappear, while a cusp appears on the large amplitude solution which tends to the single solution given in Proposition 2 when \(T\) reaches \(T^{**}\).
Figure 18. Decreasing of the size of the orbit of the large amplitude periodic solutions

Figure 19. From the periodic solution to the strange attractor, where successively $T = 3.3$, $T = 3.5$, $T = 3.55$, $T = 3.59$.

7. Concluding remarks and open problems. A part of the new qualitative features that have been exhibited in this paper have been observed numerically. What occurs in the range $(T^*, T^{**})$ is far from having been fully investigated. This is the case for the second point of Proposition 3 together with the whole subsection which presents the transition to chaos and the shape of the attractors. Although the numerical investigation has been carried out with special care, it is not a proof, not much more than guides for future work.

An important result of this observation can be stated as follows:

In a previous work (see [8]), we stressed the fact that we never observed transitions to chaos as long as the oscillator was studied at small strains even if it involved the strong non-linearities of the non-regularized contact and friction laws.
Now we observe that the same oscillator submitted to the same non-smooth non-linearities seems to exhibit chaotic areas as soon as large strains are taken into account, which might have some relation with the coupling of smooth and non-smooth non-linearities as studied in the case of large strains and plasticity [5].

Several points have been stated only from numerical observation in the present work. Theoretical proofs of some of them are probably possible at short term. Others are long term works, the key of this being the need of a general theory of qualitative non-smooth dynamics. Some technical points remain necessary to complete the present analysis:

The oscillations of the external loading have been taken into account around $F_t = 0$. It might be interesting to study the effects of an oscillating perturbation around any value of $F_t$ for which there exist equilibrium states in contact.

Only oscillations of the tangential component of the load have been taken into account; it might also be interesting to study the effects of oscillations of the normal component of the load.

We have considered oscillations of rectangular wave shape in order, as it was stressed in the paper, to allow analytical calculations as far as possible. More general loadings including a sinusoidal oscillation might lead to new interesting features.

REFERENCES

[1] S. Basseville, A. Léger and E. Pratt, Investigation of the equilibrium states and their stability for a simple model with unilateral contact and Coulomb friction, Arch. Appl. Mech., 73 (2003), 409–420.
[2] Q. J. Cao, M. Wiercigroch, E. Pavlovskaja, C. Grebogi, J. Thompson, An archetypal oscillator for smooth and discontinuous dynamics, Phys. Review, 74 (2006), 046218, 5pp.
[3] Q. J. Cao, A. Léger and Z. X. Li, The equilibrium stability of a smooth to discontinuous oscillator with dry friction, J. of Computational and Nonlinear Dynamics, (2013).
[4] A. Charles and P. Ballard, Existence and uniqueness of solution to dynamical unilateral contact problems with Coulomb friction: the case of a collection of points, Mathematical Modelling and Numerical Analysis, 48 (2014), 1–25.
[5] A. Cimetière and A. Léger, Some problems about elastic-plastic post-buckling, Int. J. Solids Structures, 32 (1996), 1519–1533.
[6] M. Jean, The nonsmooth contact dynamics method, Computer Methods Appl. Mech. Engn, 177 (1999), 235–257.
[7] A. Klarbring, Examples of nonuniqueness and nonexistence of solutions to quasistatic contact problems with friction, Ing. Arch., 60 (1990), 529–541.

[8] A. Léger and E. Pratt, Qualitative analysis of a forced nonsmooth oscillator with contact and friction, Annals of Solid and Structural Mechanics, 2 (2011), 1–17.

[9] A. Léger, E. Pratt and Q. J. Cao, A fully nonlinear oscillator with contact and friction, Nonlinear Dynamics, 70 (2012), 511–522.

[10] J. J. Moreau, Unilateral contact and dry friction in finite freedom dynamics, in Nonsmooth Mechanics and Applications (eds. J. J. Moreau and P. D. Panagiotopoulos), CISM Courses and Lectures, 302, Springer-Verlag, Vienne-New York, 1988, 1–82.

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