Perfect NOT and conjugate transformations

Fengli Yan\textsuperscript{1,2} Ting Gao\textsuperscript{1} and Zhichao Yan\textsuperscript{3}

\textsuperscript{1} College of Physics Science and Information Engineering and Hebei Advanced Thin Films Laboratory, Hebei Normal University, Shijiazhuang 050024, China
\textsuperscript{2} College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang 050024, China
\textsuperscript{3} College of Software, Hebei Normal University, Shijiazhuang 050024, China

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This paper reports on a study of the perfect NOT, probabilistic perfect NOT and conjugate transformations. The perfect NOT transformation criteria, two necessary and sufficient conditions for realizing a perfect NOT transformation on a quantum state set \(S\) of a qubit, are obtained. Furthermore, this paper discusses a probabilistic perfect NOT transformation (gate) when there is no perfect NOT transformation on a state set \(S\) and the construction of a probabilistic perfect NOT machine (gate) by a general unitary-reduction operation is shown. With a postselection of measurement outcomes, the probabilistic NOT gate yields perfect orthogonal complements of the input states. We also generalize the perfect NOT transformation to the conjugate transformation in the multi-level quantum system and a lower bound of the best possible efficiencies attained by a probabilistic perfect conjugate transformation are obtained.

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1. INTRODUCTION

The basic building block of any classical information processor is the single bit, which is prepared in one of two possible states, denoted 0 or 1. However, quantum information consists of qubits, each of which has the luxury of being in a superposition of the 0 and 1 states. Since there are an infinite number of superposition states, quantum systems have a much richer and more interesting existence than their classical counterparts. The superposition of states also makes the properties of quantum information quite different from that of its classical counterpart. Whereas the copying of classical information presents no difficulties, owing to the linearity of quantum mechanics, there is a quantum no-cloning theorem\textsuperscript{1,2} which asserts that it is impossible to construct a device that will perfectly copy an arbitrary (unknown) state of a two-level particle. However, the quantum no-cloning theorem does not rule out the possibility of either imperfect cloning\textsuperscript{3,4} or probabilistic cloning\textsuperscript{5}. Some applications of cloning have been presented\textsuperscript{6,7}. With the progress of quantum information theory, quantum cloning has become a quite interesting field.

There is another difference between classical and quantum information systems. It is very easy to complement a classical bit, i.e., to change the value of a bit, a 0 to a 1 and vice versa. Usually this operation can be accomplished by a NOT transformation (gate). However, in quantum information systems, changing an unknown state \(|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle\) of a qubit to its orthogonal complement \(|\Psi^\perp\rangle = \alpha^*|1\rangle - \beta^*|0\rangle\) that is orthogonal to \(|\Psi\rangle\) (i.e. inverting the state of a two-level quantum system) is impossible\textsuperscript{4,11}. The result is that one can not design a device that will take an arbitrary qubit and transform it into its orthogonal qubit. This is because complex conjugation of the coefficients in the NOT transformation of a qubit must be accomplished by an antiunitary transformation and cannot be performed by a unitary one. In other words, it is impossible to achieve the perfect NOT gate in quantum information systems.

However, the NOT transformation can be achieved on some states while leaving other states unchanged. Alternatively, there can be a transformation operation that approximates, at best, the NOT gate on all states, called the universal NOT gate\textsuperscript{8,9}. In fact, the output of a quantum cloning machine, the ancilla, carries the optimal anticlone of the input state so the universal NOT gate can be accomplished as a by-product of cloning\textsuperscript{4}.

A combination of unitary evolution together with measurements is an important method in quantum information processing and often achieves very interesting results. It has been used in quantum programming\textsuperscript{11}, the purification of entanglement\textsuperscript{12}, quantum teleportation\textsuperscript{13} and the preparation of quantum states\textsuperscript{14}. Recently, by using this method, Duan and Guo designed a probabilistic quantum cloning machine\textsuperscript{5}. With a postselection of the measurement results, the machine outputs perfect copies of the input states.

In this paper, the perfect NOT, probabilistic perfect NOT and conjugate transformations are investigated. We present the criteria for a perfect NOT transformation on a quantum state set \(S\) of a qubit. Two necessary and sufficient conditions for realizing a perfect NOT transformation on \(S\) are derived and this paper discusses how to build a device to achieve probabilistic perfect NOT transformations when there is no perfect NOT transformation on the state set \(S\). With certain nonzero probabilities of success, this device transforms an arbitrary unknown input state into its orthogonal complement. We also generalize the probabilistic NOT transformation to
next, we discuss the case in which \( \langle \Psi_i | \Psi_j \rangle \neq 0 \) for arbitrary \( i, j = 1, 2, \ldots, n \). Let \( \langle \Psi_i | \Psi_j \rangle = t_{ij} e^{i\phi_{ij}} \), and \( \langle P(i) | P(j) \rangle = p_{ij} e^{i\varphi_{ij}} \). Here \( 0 < t_{ij}, p_{ij} \leq 1 \), and \( 0 \leq \theta_{ij}, \varphi_{ij} < 2\pi \). Eq. (4) is then equivalent to

\[
t_{ij} e^{i\theta_{ij}} = t_{ij} e^{-i\theta_{ij}} p_{ij} e^{i\varphi_{ij}}.
\] (6)

It implies that

\[
p_{ij} = 1, \quad 2\theta_{ij} = \varphi_{ij} + 2k_{ij}\pi,
\] (7)

where \( k_{ij} = 0 \) or 1. Because \( p_{ij} = 1 \), there must be

\[
|P(i)\rangle = e^{i\varphi_i}|P^{(1)}\rangle
\] (8)

for \( i = 2, 3, \ldots, n \), and \( \varphi_i = 0 \). It follows that

\[
\varphi_{ij} = \varphi_j - \varphi_i + 2m_{ij}\pi,
\] (9)

where \( m_{ij} = 0 \) or 1. By Eq. (7), we obtain

\[
\varphi_j - \varphi_i = 2\theta_{ij} - 2(k_{ij} + m_{ij})\pi.
\] (10)

Thus,

\[
\varphi_j = 2\theta_{ij} - 2(k_{ij} + m_{ij})\pi.
\] (11)

Therefore,

\[
\theta_{ij} = \theta_{ij} - \theta_{il} + (k_{ij} + m_{ij} - k_{ij} - m_{ij} + k_{il} + m_{il})\pi,
\] (12)

\[i, j, l = 1, 2, \ldots, n; \quad k_{ij}, m_{ij} = 0 \text{ or } 1.\]

(13)

Furthermore, starting from Eq. (13) we can reverse the process. This means that if the quantum state set \( S = \{|\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_n\rangle\} \) satisfies \( \langle \Psi_i | \Psi_j \rangle \neq 0 \) and Eq. (13), one can find a unitary transformation \( U \) such that Eq. (5) hold and a perfect NOT transformation can be realized.

Based on the above argument we obtain following conclusion:

**Theorem 2.** Suppose that the quantum state set \( S = \{|\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_n\rangle\} \) satisfies \( \langle \Psi_i | \Psi_j \rangle \neq 0 \). Then, a perfect NOT transformation (gate) on the state set \( S \) can be realized by a unitary transformation (i.e., there is a unitary transformation \( U \) such that \( U |\Psi_i\rangle = |\Psi_i^\perp\rangle \)) if and only if

\[
\langle \Psi_i | \Psi_j \rangle = |\Psi_i^\perp\rangle \langle \Psi_j^\perp\rangle, \quad i, j = 1, 2, \ldots, n.
\] (14)

It turns out that if \( S \) contains all points of a Bloch sphere \([3, 10]\) for a qubit, one can not realize the perfect NOT transformation on the set \( S \).

Obviously, in Theorem 1 we only consider the case without an ancilla (probe). Now, let us introduce a probe \( P \) with the initial state \( |P^{(0)}\rangle \). By Lemma, if

\[
\langle P^{(0)} | \langle \Psi_i | \Psi_j \rangle |P^{(0)}\rangle = \langle \Psi_i | \Psi_j \rangle
\]

\[
= \langle \Psi_i^\perp | \Psi_j^\perp \rangle \langle P^{(i)} | P^{(j)} \rangle
\]

\[
= \langle \Psi_i^\perp | \Psi_j^\perp \rangle \langle P^{(0)} | P^{(i)} \rangle \langle P^{(0)} | P^{(j)} \rangle
\] (4)

for arbitrary \( i, j = 1, 2, \ldots, n \), then there exists a unitary transformation \( U \), such that

\[
U(|\Psi_i\rangle |P^{(0)}\rangle) = |\Psi_i^\perp\rangle |P^{(i)}\rangle.
\] (5)

It means we can realize the perfect NOT transformation on the quantum state set \( S = \{|\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_n\rangle\} \) with the assistance of the ancilla (probe).
3. PROBABILISTIC PERFECT NOT TRANSFORMATION

The definition of a probabilistic perfect NOT gate is that for a quantum state set \( S = \{ |\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_n\rangle \} \), there is a unitary transformation together with a measurement, which when combined with a postselection of measurement results, makes an arbitrarily unknown input quantum state \( |\Psi_i\rangle \) transform into its orthogonal complement \( |\Psi_i^\perp\rangle \) with certain nonzero probability of success. That is, for a quantum state set \( S = \{ |\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_n\rangle \} \), if there exists a unitary operation \( U \) and a measurement \( M \), which together yield the following evolution:

\[
|\Psi_i\rangle U + M \rightarrow |\Psi_i^\perp\rangle,
\]

then a probabilistic NOT gate is said to have been built. The combination of a unitary evolution operation and a measurement is very general and can be used to describe any operation in quantum mechanics \( [16] \).

Obviously, we can not build a probabilistic NOT gate for any arbitrary quantum state set \( S = \{ |\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_n\rangle \} \), so it is very important to find the conditions that the quantum state set \( S \) should be satisfied in order to construct a probabilistic perfect NOT gate.

The unitary evolution of the qubit \( A \) and probe \( P \) can be described by the following equation

\[
U(|\Psi_i\rangle|P_0\rangle) = \sqrt{\gamma_i}|\Psi_i^\perp\rangle|P(i)\rangle + \sqrt{1-\gamma_i}|\Phi_{AP}^i\rangle, \quad (i = 1, 2, \ldots, n),
\]

where \( |P_0\rangle \) and \( |P(i)\rangle \) are normalized states of the probe \( P \) (not generally orthogonal) and \( |\Phi_{AP}^1\rangle, |\Phi_{AP}^2\rangle, \ldots, \) and \( |\Phi_{AP}^n\rangle \) are \( n \) normalized states of the composite system \( AP \) (not generally orthogonal). We assume that in Eq.15 the coefficients before the states \( |\Psi_i^\perp\rangle|P(i)\rangle \), and \( |\Phi_{AP}^i\rangle \) are positive real numbers. Let \( S_0 \) be the subspace spanned by the states \( |P(1)\rangle, |P(2)\rangle, \ldots, |P(n)\rangle \). In order to realize the probabilistic perfect NOT transformation, we must require that after the unitary evolution a measurement of the probe with a postselection of the measurement results should project its state into the subspace \( S_0 \). After this projection, the state of the system \( A \) should be \( |\Psi_i^\perp\rangle \). Therefore, all of the states \( |\Phi_{AP}^i\rangle \) lie in a space orthogonal to \( S_0 \) and can be represented by the following equation

\[
|P(i)\rangle\langle P(i)|\Phi_{AP}^j\rangle = 0, \quad (i, j = 1, 2, \ldots, n).
\]

With above restriction, inter-inner-products of Eq.15 yield the following matrix equation

\[
X^{(i)} = \sqrt{\Gamma}X^{(1)} + \sqrt{E_n - \Gamma Y}E_n - \Gamma^+, \quad (i = 1, 2, \ldots, n),
\]

where \( X^{(i)} = [(|\Psi_i\rangle|\Psi_j\rangle) \), \( Y = [(|\Phi_{AP}^i\rangle|\Phi_{AP}^j\rangle) \), \( X^{(1)} = [(|\Psi_i^\perp\rangle|\Psi_j^\perp\rangle) \), \( X^{(1)} = [(|\Phi_{AP}^i\rangle|\Phi_{AP}^j\rangle) \), and \( E_n \) is the \( n \times n \) identity matrix. The diagonal efficiency matrix \( \Gamma \) is defined by \( \Gamma = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n) \); therefore, \( \sqrt{\Gamma} = \sqrt{\Gamma Y}E_n - \Gamma^+ \) is also a positive-semidefinite matrix. Conversely, if \( X^{(1)} - \sqrt{\Gamma}X^{(1)} + \sqrt{E_n - \Gamma^+} \) is a positive-semidefinite matrix, one can choose \( |\Phi_{AP}^1\rangle \) such that Eq.14 holds. By Lemma the states \( |\Psi_1\rangle, |\Psi_2\rangle, \ldots, \) and \( |\Psi_n\rangle \) are able to be probabilistically transformed to their respective orthogonal complement states. Thus we have the following theorem:

**Theorem 3.** The states \( |\Psi_1\rangle, |\Psi_2\rangle, \ldots, \) and \( |\Psi_n\rangle \) can be probabilistically perfectly transformed to their respective orthogonal complement states if and only if there exist a diagonal positive-definite matrix \( \Gamma \) and \( |P(i)\rangle \) \( (i = 1, 2, \ldots, n) \) such that the matrix \( X^{(1)} - \sqrt{\Gamma}X^{(1)} + \sqrt{E_n - \Gamma^+} \) is positive-semidefinite. Here \( X^{(1)} = [(|\Psi_i\rangle|\Psi_j\rangle) \) and \( X^{(1)} = [(|\Psi_i^\perp\rangle|\Psi_j^\perp\rangle) \) are \( n \times n \) matrices, and \( |P(i)\rangle \) \( (i = 1, 2, \ldots, n) \) are quantum states of a probe.

**Theorem 3** is very general, and for the linearly independent quantum state set \( S \) we have the conclusion:

**Theorem 4.** The states secretly chosen from the set \( S = \{ |\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_n\rangle \} \) can be probabilistically transformed into their respective orthogonal complements by a general unitary-reduction operation, if \( |\Psi_1\rangle, |\Psi_2\rangle, \ldots, \) and \( |\Psi_n\rangle \) are linearly independent.

**Proof:** Suppose the Hilbert space of the probe \( P \) is an \( n_p \)-dimensional space, where \( n_p \geq n + 1 \). We use \( |P_0\rangle, |P_1\rangle, \ldots, \) and \( |P_n\rangle \) to denote \( n + 1 \) orthonormal states of a probe. If there exists a unitary operator \( U \) that satisfies

\[
U(|\Psi_i\rangle|P_0\rangle) = \sqrt{\gamma_i}|\Psi_i^\perp\rangle|P_0\rangle + \sum_{j=1}^{n} c_{ij} |\Phi_{P}^j\rangle|P_0\rangle, \quad (i = 1, 2, \ldots, n),
\]

where \( |\Phi_{P}^j\rangle \) \( (j = 1, 2, \ldots, n) \) stand for \( n \) normalized states of the system (not generally orthogonal) and \( \varphi_i \) are real numbers, then after the evolution a measurement of the probe \( P \) is followed. Eq.14 is a special case of Eq.15. The NOT transformation is successful, and the output state of the system is \( |\Psi_i^\perp\rangle \), if and only if the measurement outcome of the probe is \( |P_0\rangle \). Evidently, the probability of success \( (\text{obtaining } |P_0\rangle) \) is \( \gamma_i \). For any input state \( |\Psi_i\rangle \), the probabilistic NOT device should succeed with a nonzero probability. This, in turn, implies that all of the \( \gamma_i \) must be positive real numbers. Hence, the evolution \( \text{Eq.14} \) can be realized if Eq.15 holds with positive efficiencies \( \gamma_i \). The \( n \times n \) inter-inner-products of Eq.15 yield the equation

\[
X^{(1)} = \sqrt{\Gamma}X^{(1)} + \sqrt{E_n - \Gamma^+} + CC^+, \quad (19)
\]

where the \( n \times n \) matrices \( C = [c_{ij}] \), \( X^{(1)} = [(|\Psi_i\rangle|\Psi_j\rangle) \),

\[
X^{(1)} = \sqrt{\Gamma}X^{(1)} + \sqrt{E_n - \Gamma^+} + CC^+, \quad (19)
\]

where the \( n \times n \) matrices \( C = [c_{ij}] \), \( X^{(1)} = [(|\Psi_i\rangle|\Psi_j\rangle) \),

\[
X^{(1)} = \sqrt{\Gamma}X^{(1)} + \sqrt{E_n - \Gamma^+} + CC^+, \quad (19)
\]

where the \( n \times n \) matrices \( C = [c_{ij}] \), \( X^{(1)} = [(|\Psi_i\rangle|\Psi_j\rangle) \),
and $X^{(\perp)} = [\mathbf{e}^{i(\varphi - \varphi_i)} \langle \Psi_1^+ | \Psi_j^+ \rangle] = [\mathbf{e}^{i(\varphi - \varphi_i)} \langle \Psi_j | \Psi_j^\ast \rangle]$. By considering Lemma we know that if there exists a diagonal positive-definite matrix $\Gamma$ satisfied Eq. (19), then one can realize the unitary evolution (14).

Duan and Guo [3] have shown: If $n$ states $|\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_n\rangle$ are linearly independent, the matrix $X^{(\perp)} = [\langle \Psi_j | \Psi_j \rangle]$ is positive definite.

Suppose that the minimum eigenvalue of $X^{(1)}$ is $c$ and the maximum eigenvalue of $X^{(\perp)}$ is $d$. Then there must exist a positive number $\varepsilon$ such that

$$c - \varepsilon d > 0.$$  \hspace{1cm} (20)

Let $B = (b_1, b_2, \ldots, b_n)^T$ be an arbitrary nonzero $n$ dimensional vector. Then

$$B^+(c - \varepsilon d)B > 0.$$  \hspace{1cm} (21)

It also follows that

$$B^+(cE - \varepsilon dE)B > 0,$$  \hspace{1cm} (22)

where $E$ is the $n \times n$ identity matrix. Presume that $X^{(1)}$ and $X^{(\perp)}$ are diagonalized by the unitary matrices $U$ and $V$, respectively. Eq. (22) can then be rewritten as

$$B^+(U^c u - \varepsilon V^d v)B > 0.$$  \hspace{1cm} (23)

We use $c_1, c_2, \ldots, c_n$ and $d_1, d_2, \ldots, d_n$ to denote the eigenvalues of matrices $X^{(1)}$ and $X^{(\perp)}$, respectively. It is easy to deduce

$$B^+[U^c \text{diag}(c_1, \ldots, c_n)U - \varepsilon V^d \text{diag}(d_1, \ldots, d_n)V]B > 0.$$  \hspace{1cm} (24)

That is,

$$B^+[X^{(1)} - \varepsilon X^{(\perp)}]B > 0.$$  \hspace{1cm} (25)

Obviously, there must be a diagonal matrix $\Gamma = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$ with $\gamma_i > 0$ that satisfies

$$\varepsilon X^{(\perp)} = \sqrt{\Gamma} X^{(\perp)} \sqrt{\Gamma}.$$  \hspace{1cm} (26)

Therefore, there is a diagonal matrix $\sqrt{\Gamma}$ such that

$$X^{(1)} - \sqrt{\Gamma} X^{(\perp)} \sqrt{\Gamma}$$  \hspace{1cm} (27)

is positive definite.

Suppose that the unitary matrix $W$ diagonalizes the Hermitian matrix $X^{(1)} - \sqrt{\Gamma} X^{(\perp)} \sqrt{\Gamma}$, that is,

$$W (X^{(1)} - \sqrt{\Gamma} X^{(\perp)} \sqrt{\Gamma}) W^+ = \text{diag}(m_1, m_2, \ldots, m_n),$$  \hspace{1cm} (28)

where all of the eigenvalues $m_1, m_2, \ldots, m_n$ are positive real numbers. We can then choose the matrix $C$ in Eq. (19) to be

$$C = W^+ \text{diag}(\sqrt{m_1}, \sqrt{m_2}, \ldots, \sqrt{m_n})W.$$  \hspace{1cm} (29)

Thus, there exists a diagonal positive definite efficiency matrix $\Gamma$ such that Eq. (19) holds and the proof of Theorem 4 is complete.

Next we consider probabilistic perfect NOT transformation of the quantum state set $\{ |\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle \}$. Suppose that $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are linearly independent and that

$$|\Psi_3\rangle = \alpha |\Psi_1\rangle + \beta |\Psi_2\rangle.$$  \hspace{1cm} (30)

Here $\alpha$ and $\beta$ satisfy the normalizing condition

$$\alpha^* \beta + \alpha \beta^* \alpha |\Psi_1\rangle + \alpha \beta^* |\Psi_2\rangle + \alpha^* \beta |\Psi_2\rangle |\Psi_1\rangle = 1.$$  \hspace{1cm} (31)

From Theorem 4, there exists a unitary transformation $U$ such that

$$U(|\Psi_1\rangle |P_0\rangle) = \sqrt{\gamma_1} |\Psi_1^+\rangle |P_0\rangle + \sqrt{1 - \gamma_1} |\Phi^{(1)}_{AP}\rangle,$$

$$U(|\Psi_2\rangle |P_0\rangle) = \sqrt{\gamma_2} |\Psi_2^+\rangle e^{i\varphi} |P_0\rangle + \sqrt{1 - \gamma_2} |\Phi^{(2)}_{AP}\rangle.$$  \hspace{1cm} (32)

The linearity of $U$ implies that

$$U(|\Psi_3\rangle |P_0\rangle)$$

$$= (\alpha \sqrt{\gamma_1} |\Psi_1^+\rangle + \beta \sqrt{\gamma_2} |\Psi_2^+\rangle e^{i\varphi}) |P_0\rangle + \alpha \sqrt{1 - \gamma_1} |\Phi^{(1)}_{AP}\rangle + \beta \sqrt{1 - \gamma_2} |\Phi^{(2)}_{AP}\rangle.$$  \hspace{1cm} (33)

Hence, if

$$\alpha \sqrt{\gamma_1} |\Psi_1^+\rangle + \beta \sqrt{\gamma_2} |\Psi_2^+\rangle e^{i\varphi} = \sqrt{\gamma_3} |\Psi_3^+\rangle e^{i\chi},$$  \hspace{1cm} (34)

one obtains

$$U(|\Psi_3\rangle |P_0\rangle) = \sqrt{\gamma_3} |\Psi_3^+\rangle |P_0\rangle + \sqrt{1 - \gamma_3} |\Phi^{(3)}_{AP}\rangle.$$  \hspace{1cm} (35)

Here $\chi$ is a real number. Therefore, we can realize the probabilistic perfect NOT transformation on the set $\{ |\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle \}$ in case of Eq. (34) being satisfied.

Now, suppose that

$$|\Psi_1\rangle = \cos \frac{\theta_1}{2} |0\rangle + \sin \frac{\theta_1}{2} e^{i\phi_1} |1\rangle,$$

$$|\Psi_2\rangle = \cos \frac{\theta_2}{2} |0\rangle + \sin \frac{\theta_2}{2} e^{i\phi_2} |1\rangle.$$  \hspace{1cm} (36)

Eq. (21) then becomes

$$|\alpha|^2 + |\beta|^2 + \alpha^* \beta (\cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} + \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i(\phi_2 - \phi_1)})$$

$$+ \alpha \beta^* (\cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} + \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-i(\phi_2 - \phi_1)}) = 1,$$  \hspace{1cm} (37)
and Eq. (33) changes to

\[
\begin{align*}
\alpha \sqrt{\gamma} \cos \frac{\theta_1}{2} + \beta e^{i\varphi} \sqrt{\gamma} \sin \frac{\theta_2}{2} \\
= \sqrt{\gamma} \sqrt{3} e^{i\chi}(\alpha^* \cos \frac{\theta_1}{2} + \beta^* \cos \frac{\theta_2}{2}), \\
\alpha \sqrt{\gamma} \sin \frac{\theta_1}{2} e^{-i\varphi_1} + \beta e^{i\varphi} \sqrt{\gamma} \sin \frac{\theta_2}{2} e^{-i\varphi_2} \\
= \sqrt{\gamma} \sqrt{3} e^{i\chi}(\alpha^* \sin \frac{\theta_1}{2} e^{-i\varphi_1} + \beta^* \sin \frac{\theta_2}{2} e^{-i\varphi_2}).
\end{align*}
\]

(38)

Therefore, if Eqs. (37) and (38) are satisfied by \(|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle\), then probabilistic perfect NOT transformation on this quantum state set can be realized.

For example, we can realize a probabilistic perfect NOT transformation with probability \(\gamma\) on the quantum states

\[
|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),
|\Psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle),
|\Psi_3\rangle = \frac{1}{\sqrt{2}}[(q + re^{-i\frac{\pi}{2}})|0\rangle + (q + i re^{-i\frac{\pi}{2}})|1\rangle],
\]

(39)

where \(q, r, \gamma, \varphi\) are real and satisfy

\[
\begin{align*}
q^2 + r^2 + \sqrt{2}qr \cos(\frac{\varphi}{2} - \frac{\pi}{2}) &= 1, \\
\frac{1}{2} - 2\gamma + q^2 + \gamma \sin \varphi &\geq 0,
\end{align*}
\]

(40)

Obviously, \(\varphi = \frac{\pi}{2}\) corresponds to the perfect NOT transformation case.

### 4. CONJUGATE TRANSFORMATION OF A MULTI-LEVEL QUANTUM SYSTEM

In this section we discuss conjugate transformation of a multi-level quantum system (qudit). Suppose that the dimension of a Hilbert space for the quantum system is \(d\). An arbitrary quantum state of the system can be written as

\[
|\Psi\rangle = \sum_{i=0}^{d-1} \alpha_i |i\rangle,
\]

(41)

where \(\alpha_i\) are complex numbers and \(|\{i\}\rangle\) is an orthonormal basis. Let us define a conjugate transformation \(T\) as

\[
T|\Psi\rangle = T(\sum_{i=0}^{d-1} \alpha_i |i\rangle) = \sum_{i=0}^{d-1} \alpha^*_i |i\rangle \equiv |\Psi^T\rangle.
\]

(42)

Obviously, a perfect NOT transformation equals \(UT\) for a qubit, where \(U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) is a unitary transformation. We call \(|\Psi^T\rangle\) the conjugate state of quantum state \(|\Psi\rangle\). Evidently, one can not design a machine that will take an arbitrary quantum state \(|\Psi\rangle\) and transform it into its conjugate state \(|\Psi^T\rangle\) because of the need for complex conjugation of the coefficients in the transformation, which must be accomplished by an antunitary transformation and cannot be performed by a unitary one. By Lemma, we can also assert that this kind transformation is impossible on a general quantum state set \(S = \{|\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_n\rangle\}\) of a qudit, since \(\langle \Psi_i | \Psi_j \rangle \neq \langle \Psi_i | \Psi_j^T \rangle = \langle \Psi_i | \Psi_j^* \rangle\) for two arbitrary quantum states in the set \(S\).

However, by the argument similar to the qubit case, we do have the following conclusions:

**Theorem 1’**. A perfect conjugate transformation on the state set \(S = \{|\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_n\rangle\}\) of a qudit can be realized by a unitary transformation acting on the system and a probe if and only if all inner-products of the quantum states in the set \(S\) are real.

**Theorem 2’**. Suppose that the quantum state set \(S = \{|\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_n\rangle\}\) of a qudit satisfies \(\langle \Psi_i | \Psi_j \rangle \neq 0\). Then a conjugate transformation on the state set \(S\) can be realized by a unitary transformation acting on the system and a probe if and only if Eq. (13) hold.

**Theorem 3’**. The states \(|\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_n\rangle\) can be probabilistically perfectly transformed to their respective conjugate states if and only if there exist a diagonal positive-definite matrix \(\Gamma\) and \(|P(i)\rangle\) \((i = 1, 2, \ldots, n)\) such that the matrix \(X^{(1)} = \sqrt{\Gamma} X^{(T)} \sqrt{\Gamma}\) is positive-semidefinite. Here \(X^{(1)} = \langle [\Psi_i | \Psi_j]\rangle\) and \(X^{(T)} = \langle [\Psi_i^T | \Psi_j^T]\rangle\) are \(n \times n\) matrices, and \(|P(i)\rangle\) \((i = 1, 2, \ldots, n)\) are quantum states of a probe.

**Theorem 4’**. The states secretly chosen from the set \(S = \{|\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_n\rangle\}\) of a qudit can be probabilistically transformed into their respective conjugate states by a general unitary-reduction operation if \(|\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_n\rangle\) are linearly independent.

Next we investigate the best possible efficiencies \(\gamma_i\) attained by a probabilistic conjugate transformation.

For the sake of simplicity, we only discuss the special case \(S = \{|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle\}\), where \(|\Psi_1\rangle, |\Psi_2\rangle, |\Psi_3\rangle\) are linearly independent. In this case, \(X^{(1)} = \sqrt{\Gamma} X^{(T)} \sqrt{\Gamma}\) becomes
and semidefinite condition of Eq.(44) requires the maximum of $t$ becomes
\[
\langle \Psi_1 | \Psi_2 \rangle - \sqrt{\gamma_1 \gamma_2} \langle \Psi_1 | \Psi_2 \rangle^* \langle P^{(1)} | P^{(2)} \rangle - \sqrt{\gamma_2 \gamma_3} \langle \Psi_3 | \Psi_2 \rangle^* \langle P^{(3)} | P^{(2)} \rangle = \frac{1 - \gamma_2}{1 - \gamma_3} \langle \Psi_2 | \Psi_3 \rangle - \sqrt{\gamma_1 \gamma_3} \langle \Psi_1 | \Psi_3 \rangle^* \langle P^{(1)} | P^{(3)} \rangle .
\]

Let $\langle \Psi_1 | \Psi_2 \rangle = t_{12} e^{i \theta_{12}}$, $\langle \Psi_1 | \Psi_3 \rangle = t_{13} e^{i \theta_{13}}$, and $\langle \Psi_2 | \Psi_3 \rangle = t_{23} e^{i \theta_{23}}$. We choose $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$, so Eq. (44) becomes
\[
(1 - \gamma) \begin{pmatrix}
1 & \langle \Psi_1 | \Psi_2 \rangle & \langle \Psi_1 | \Psi_3 \rangle \\
\langle \Psi_2 | \Psi_1 \rangle & 1 & A \\
\langle \Psi_3 | \Psi_1 \rangle & A^* & 1
\end{pmatrix} \geq 0,
\]
where $A = (\langle \Psi_2 | \Psi_3 \rangle - \gamma_1 \langle \Psi_2 | \Psi_3 \rangle^* e^{i (\theta_{13} - \theta_{12})})$. The positive-semidefinite condition of Eq. (44) requires
\[
\det \begin{pmatrix}
1 & \langle \Psi_1 | \Psi_2 \rangle & \langle \Psi_1 | \Psi_3 \rangle \\
\langle \Psi_2 | \Psi_1 \rangle & 1 & A \\
\langle \Psi_3 | \Psi_1 \rangle & A^* & 1
\end{pmatrix} \geq 0,
\]
and
\[
\det \begin{pmatrix}
1 & A \\
A^* & 1
\end{pmatrix} \geq 0.
\]
Let $\delta = t_{12} - t_{13} + t_{23}$, $a = -\det X = -1 + t_{12}^2 + t_{13}^2 + t_{23}^2 - 2t_{12}t_{13}t_{23} \cos \delta$, $b = t_{23}^2 - 1$. Eq. (44) means
\[
ar^2 + 2\gamma(2t_{23}^2 \sin^2 \delta - a) + a \leq 0.
\]
So we have
\[
0 < \gamma \leq 1 + \frac{2 \sqrt{t_{12}^4 - a t_{23}^2 \sin^2 \delta - 2t_{23}^2 \sin^2 \delta}}{a}.
\]
By Eq. (46) we obtain
\[
b \gamma^2 + 2\gamma(2t_{23}^2 \sin^2 \delta - b) + b \leq 0.
\]
Therefore, $\gamma$ satisfies
\[
0 < \gamma \leq 1 + \frac{2 \sqrt{t_{12}^4 - b t_{23}^2 \sin^2 \delta - 2t_{23}^2 \sin^2 \delta}}{b}.
\]
Since $a > b$, and $1 + 2 \frac{\sqrt{t_{12}^4 - x}}{x}$ is a monotone function, the maximum of $\gamma$ in this special case is
\[
\gamma_{\text{max}} = 1 + \frac{2 \sqrt{t_{12}^4 - a t_{23}^2 \sin^2 \delta - 2t_{23}^2 \sin^2 \delta}}{a}.
\]

5. SUMMARY

In conclusion, we have investigated a perfect NOT transformation on a quantum state set $S$ of a qubit and derived two necessary and sufficient conditions for realizing a perfect NOT transformation on $S$. A probabilistic perfect NOT transformation (gate) was constructed by a general unitary-reduction operation. With a post-selection of the measurement outcomes, the probabilistic NOT gate was shown to yield perfect respective orthogonal complements of the input states. We also show that one can construct a probabilistic perfect NOT gate of the input states secretly chosen from a certain set $S = \{|\Psi_1\rangle, |\Psi_2\rangle, \ldots, |\Psi_n\rangle\}$ if $|\langle \Psi_1 | \Psi_2 \rangle|, \ldots, |\langle \Psi_1 | \Psi_n \rangle|$ are linearly independent. Furthermore, we generalize the probabilistic NOT transformation to the conjugate transformation in a multi-level quantum system. The lower bound of the best possible efficiencies attained by a probabilistic perfect conjugate transformation was obtained.

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