Diffusion in the special theory of relativity

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The Markovian diffusion theory is generalized within the framework of the special theory of relativity using a modification of the mathematical calculus of diffusion on Riemannian manifolds (with definite metric) to describe diffusion on Lorentzian manifolds with an indefinite metric. A generalized Langevin equation in the fiber space of position, velocity and orthonormal velocity frames is defined from which the generalized relativistic Kramers equation in the phase space in external force fields is derived. The obtained diffusion equation is invariant under Lorentz transformations and its stationary solution is given by the Jüttner distribution. Besides a non-stationary analytical solution is derived for the example of force-free relativistic diffusion.

PACS numbers: 05.10.Gg, 03.30.+p, 04.20.-q

1. INTRODUCTION

The formulation of a consistent theory of Markovian diffusion within the framework of the relativity theory is a long-standing problem in physics. Over the years umpteen studies have been devoted to this issue (see e.g. [1] - [11]) with different and apparently irreconcilable points of view. At least up to the knowledge of the author, a general accepted consistent solution of this problem is still outstanding. However, besides its fundamental theoretical interest such theory is of particular importance in several applications such as in high energy collision experiments (see e.g. [12]), astrophysics (see e.g. [13]) and others.

An alternative physical approach for the description of a relativistic gas in a heat bath is given by statistical thermodynamics and the Boltzmann equation. Jüttner derived the thermal equilibrium distribution for a relativistic gas already in 1911 [14]. After that many authors have given important contributions to the development of the relativistic kinetic theory (for an introduction see e.g. [13], [18]). Recently numerical microscopic 1-dimensional simulations [20] and a critical analysis of alternative findings [21] yield arguments in favor the Jüttner distribution. Relativistic diffusion processes has been comprehensively reviewed [11] which besides the issue of stochastic relativistic diffusion theory also includes relativistic equilibrium thermostatics and microscopic models for Langevin-typ equations and where a complete list of references can be found.

Diffusion theory in the Euclidian space $\mathbb{R}^d$ is a well developed topic (see e.g. [22], [23]). However, the description of diffusion on a non-Euclidian manifolds $M^d$ is a subject containing several pitfalls. There exists a well developed rigorous mathematical theory of stochastic differential equations and diffusion processes on Riemannian manifolds with a definite metric signature (see e.g. [24], [25]). The stochastic calculus on Riemannian manifolds found considerable interest in mathematics and has played a central role in recent years within the analysis in path and loop spaces in topology and other fields. However, this mathematical approach can not be applied to describe diffusion on Lorentzian or Pseudo-Riemannian manifolds with an indefinite metric. In the present paper we derive a physically motivated modification of this calculus to describe diffusion in the phase space of the position and velocity, in which the difficulties in the description of diffusion on manifolds with indefinite metric signature are bypassed. In this approach a generalized relativistic Langevin equation in the fiber bundle of position, velocity and orthonormal velocity frames is defined from which the generalized non-relativistic Kramers equation in external force fields is derived. As will be shown the derived relativistic diffusion equation satisfies the general principle of special relativity and is invariant under Lorentz transformations. The steady-state solution of this equation for a heat bath with constant friction coefficient yields the Jüttner distribution.

The paper is organized as follows. In Chapter 2 the concept and main formulas of the mathematical stochastic calculus on Riemannian manifolds is presented. Since this approach is little-known in physics in the appendix B some details of this calculus and its relation with the stochastic calculus on Euclidian manifolds (appendix A) are presented. In Chapter 3 the generalized relativistic Langevin equation in the fiber bundle of position, velocity and orthonormal velocity frames is defined and the relativistic diffusion equation for the probability density function or the transition probability is derived. In Chapter 4 the steady-state solution for a relativistic gas in a heat bath with constant friction coefficient and in Chapter 5 the non-steady solution for the force-free case are derived and in chapter 6 the conclusions are presented.
2. MATHEMATICAL STOCHASTIC CALCULUS ON RIEMANNIAN MANIFOLDS

Stochastic differential equations in diffusion theory in a d-dimensional Euclidian space $\mathbb{R}^d$ with continuous pathway are defined by the fundamental d-dimensional Wiener process $W^a(t)$. On a Riemannian manifold $M^d$ the fundamental Wiener process is difficult to handle. By using an inadequately posed formulation of a stochastic differential equation it is not assured that its solution remains on the manifold $M^d$. Here the Riemannian metric and $\eta$ Euclidian metric where $\delta$ is the flat Euclidian metric with continuous values $\langle W^a(t) \rangle = 0$ and $\langle W^a(\tau) W^b(\tau + \delta) \rangle = D \delta_{ab}$.

Associated to each diffusion process there is a second order differential operator denoted as the generator $A$ of the diffusion. This operator is associated with the Kolmogorov backward equation and defined in appendix A and B for diffusion processes on Euclidian and Riemannian manifolds, respectively. For the stochastic process lifted to the fiber bundle $O(M)$ as defined in Eq.(1) the diffusion generator in the Stratonovich integral interpretation is given by Eq.(B4) as

$$A_{O(M)} = \frac{D}{2} \sum_{a=1}^{d} H_a H_a + H_0$$

where the fundamental vector fields $H_a$ and $H_0$ are given by Eq.(B3):

$$H_a = e_0^a(\mathbf{x}) \partial_a - \Gamma_{ml}^i(x)e_0^l \partial_a$$

$$H_0 = A^i(\mathbf{x}) \partial_i - \Gamma_{ml}^i(x) A^m(\mathbf{x}) \partial_a$$

The diffusion generator $A_{O(M)}$ on $O(M)$ can be projected to $M^d$ with $f(\mathbf{r}) = f(x,0)$, $\mathbf{r} = (x^i, e_i^a)$ using the relation $A_{O(M)} f(\mathbf{r}) = A_M f(\mathbf{x})$ where

$$A_M = \frac{D}{2} \sum_{a=1}^{d} (e_i^a \partial_i e_0^a \partial_j) + A^i \partial_i = \left( \frac{D}{2} \Delta_M + A^i \partial_i \right)$$

and $\Delta_M = g^{ij} \partial_i \partial_j - g^{ij} \Gamma_k^i \partial_k$ is the Laplace-Beltrami operator on the manifold $M^d$. The generalized Fokker-Planck equation is obtained by the adjoint of the diffusion generator $A_M^*$ which includes the volume element $\sqrt{g}$. $\Phi = \Phi(x, \tau | y, 0)$ is the transition probability with the initial condition $\Phi(x, 0 | y, 0) = \delta(\mathbf{x} - \mathbf{y})$ and adequate boundary conditions at infinity. The probability density $\phi(x^i, \tau)$ is determined by the same equation with the initial condition $\phi(x, \tau = 0) = \phi^0(\mathbf{x})$.
A remarkable particularity of Markovian diffusion on a Riemannian manifold is the supposition that for the diffusion coefficients in Eq. (1) only the orthonormal frame coefficients \( e^i_a(x) \) are admissible which are directly related with the geometry of the Riemannian manifold. In contrast on an Euclidian manifold a much more general class of diffusion coefficients is permitted.

### 3. RELATIVISTIC DIFFUSION IN THE PHASE SPACE

A direct application of the mathematical calculus of diffusion processes on general Riemannian manifolds for relativistic physics is not possible due to the supposition restricting the stochastic formalism to the special case of a Riemannian manifold with a definite metric signature, but in the relativistic theory the Lorentzian or Pseudo-Riemannian manifolds exhibit an indefinite metric signature \((\pm,+,+,-,+)\). Moreover, it has been proven that Markovian diffusion processes in the base manifold (position space) on a Lorentzian or Pseudo-Riemannian manifold do not exist \([3],[4]\). However, considering more carefully the mathematical model described by Eq. (1) one can recognize a significant difference to the physical non-relativistic diffusion model described by the Langevin equation. In Eq. (1) the noise term directly acts on the position variable in the base space, while the noise term in the Langevin equation operates like a force on the change of the velocity in the tangent space. This crucial difference in the mathematical model to the physically motivated Langevin approach is the central point which enables a generalization of the Markovian diffusion theory within the framework of the special relativity theory performed in the phase space of coordinates \(x^i\) and normalized velocity variables \(u^i\) \((i = 1, 2, 3)\). The 4-velocity space for a gas with massive particles is a hyperboloid (or pseudo-sphere) described by the relation \((u^0)^2 - (u^i)^2 - (u^3)^2 = 1\). This means that the relativistic velocity space is a noncompact hyperbolic 3-dimensional Riemannian manifold (not Pseudo-Riemannian). Therefore with an appropriate modification we can apply for the velocity space the mathematical stochastic calculus for Riemannian manifolds as presented in the appendix B. Since the stochastic force acts directly only on the change of the velocity and not on the position coordinates we can define the relativistic generalization of the Langevin equation in a fiber bundle here denoted by \(F(M_L) : \{x^i, u^i, E^i_a\} = F(M_L)\) where \(x^i\) belong to the Lorentzian base manifold \(M_L\), the relativistic velocity \(u^i\) to the tangent space \(TM_L\) and \(E^i_a(u)\) are the moving orthonormal frames in the hyperbolic velocity space (this means they belong to the second order tangent space \(TTM_L\)). Locally, this fiber bundle is simply the product space of these three sub-spaces. With a corresponding modification of Eq. (B1) in appendix B the generalized relativistic Langevin equations can be defined in the fiber bundle space \(F(M_L)\) by

\[
\begin{align*}
\frac{dx^i(\tau)}{d\tau} &= u^i(\tau)dr, \\
\frac{du^i(\tau)}{d\tau} &= F^i_a(\tau)dW^a(\tau) + F^i(\tau)dr, \\
\frac{dE^i_a(\tau)}{d\tau} &= -\gamma^{j}_{ml}(u)E^j_b(\tau)dW^m(\tau) + \gamma^{j}_{ml}(u)F^j_b(\tau)dr
\end{align*}
\]

(6)

Here \(\tau\) is an evolution parameter along the world lines of the particles which can be chosen as the proper time. The laboratory time \(t = \tau u^0/c\) is a function of the proper time \(\tau\) and \(u^0\) which here and below is defined by \(u^0 = [1 + (u^3)^2 + (u^2)^2 + (u^1)^2]^{1/2}. \gamma^{j}_{ml}(u)\) are the Christoffel connection coefficients on the hyperboloid and \(F^i = K^i/m\), where \(K^i\) are the spatial components of the 4-force, \(m\) is the rest mass of the particles and the indices \(a, b\) denotes the spatial components in the hyperbolic space \((a, b = 1, 2, 3)\). Since the stochastic force \(dW^a(\tau)\) do not act directly on the position variable \(x^i(\tau)\) the indefinite signature of the Lorentzian manifold here do not create any difficulty, as it arise for a stochastic differential equation as Eq. (1) for a manifold with indefinite metric. Sufficient conditions for the existence and uniqueness of the stochastic differential Eq. (6) are that the drift and diffusion coefficients satisfy the uniform Lipschitz condition and the stochastic process \(X(\tau) = \{x(\tau), u(\tau)\}\) is adapted to the Wiener process \(W^a(\tau)\), that is, the output \(X(\tau_2)\) is a function of \(W^a(\tau_1)\) up to that time \((\tau_1 \leq \tau_2)\) \([23]\). The moving frames in the hyperbolic velocity space are defined by the relation

\[
\sum_{a=1}^{3} E^i_a E^j_b = \delta_{ij},
\]

(7)

or equivalently

\[
G_{ij}E^i_a E^j_b = \delta_{ab},
\]

(8)

where \(G_{ij}\) is the Riemannian metric of the hyperbolic velocity space, \(G^{ij}\) the inverse matrix of \(G_{ij}\) and the Christoffel connection coefficients \(\gamma^{j}_{ml}(u)\) on the hyperboloid are given by

\[
\gamma^{j}_{jk}(u) = \frac{1}{2}G^{jm} \left[ \partial G_{jm}/\partial u^k + \partial G_{mk}/\partial u^j - \partial G_{jk}/\partial u^m \right].
\]

(9)

Since the manifold on the hyperboloid is embedded into the Minkowski space, the metric \(G_{ij}(u)\) can be calculated from the infinitesimal arc length given by \(ds^2 = -(du^0)^2 + (du^i)^2 + (du^j)^2 + (du^k)^2\) with \(u^0 = [1 + (u^i)^2 + (u^j)^2 + (u^k)^2]^{1/2}\). In this way we obtain \(ds^2 = G_{ij}(u)du^i du^j\) with \(G_{ij}(u) = \delta_{ij} - (u_i u_j)/(u^0)^2\), \(G = \det G_{ij} = (u^0)^{-2}\) and \(\gamma^{j}_{jk}(u) = -u^{i} G_{ik}\). Corresponding the definition of fundamental vector fields on \(O(M)\) in the appendix B one can now introduce the fundamental horizontal vector field \(H_a\) and \(H_0\) on the fiber bundle \(F(M_L)\). With
corresponding modifications we find from Eq. (B3):
\[
H_a = E^i_a \frac{\partial}{\partial x^i} - \gamma^i_m(u) E^i_a E^m_b \frac{\partial}{\partial E^b},
\]
(10)
\[
H_0 = u^i \frac{\partial}{\partial x^i} + F^i \frac{\partial}{\partial u^i} - \gamma^i_m(u) E^i_a(\tau) F^m \frac{\partial}{\partial E^a},
\]
and the diffusion operator \( A_{F(M)} \) for the stochastic process is given as in Eq. (B4) by
\[
A_{F(M)} = \frac{D}{2} \sum H_a H_a + H_0.
\]
(11)
We project the stochastic curve from the fiber space \( F(M_L) \) with coordinates \( x^i, u^i \) \( \to \) \( x^i, u^i \) \( \to \) \( x^i, u^i \) to the phase space with coordinates \( x^i, u^i \) : \( A_{F(M)} f(x, u, 0) = A_{K}(x, u) \), where the diffusion generator in the phase space \( A_{K} \) is given by
\[
A_{K} = \frac{D}{2} \sum_{a=1}^{3} E^i_a \frac{\partial}{\partial u^i} E^j_a \frac{\partial}{\partial u^j} + u^i \partial \partial x^i + F^i \frac{\partial}{\partial u^i}.
\]
(12)
The special relativistic diffusion equation in the phase space is given by the adjoint of the operator \( A_{K} \) and similar to Eq. (B7) the generalized relativistic Kramers equation takes the form
\[
\frac{\partial \Phi}{\partial \tau} = -u^i \frac{\partial \Phi}{\partial x^i} - \text{div}_v (F \Phi) + \frac{D}{2} \Delta_u \Phi,
\]
(13)
where \( \Delta_u \) is the Laplace Beltrami Operator of the hyperbolic velocity space given by
\[
\Delta_u = G^{ij} \frac{\partial^2}{\partial u^i \partial u^j} - G^{ij} \gamma^k_{ij} \frac{\partial}{\partial u^k} = \frac{1}{\sqrt{-G}} \frac{\partial}{\partial u^i} (\sqrt{G} G^{ij} \frac{\partial}{\partial u^j}),
\]
(14)
and the corresponding divergence operator is given by
\[
\text{div}_v (F \Phi) = \frac{1}{\sqrt{-G}} \frac{\partial}{\partial u^i} (\sqrt{G} F^i \Phi)
\]
(15)
with \( G = \det(G^{ij}) \).

Eq. (13) represents the relativistic generalization of the Kramers equation for the probability density function \( \Phi = \phi(\tau, x, u) \) with the initial condition \( \phi(\tau = 0; x, u) = \phi_0(x, u) \). The transition probability is determined by the same equation, but is defined by the initial condition \( \phi(x, u, 0 | x_0, u_0, 0) = \frac{1}{\sqrt{2\pi}} \delta(u^i - u_0^i) \delta(\Sigma_0(a)) \delta(x^i - x_0^i) \). If the force \( F^i \) depend on the time \( t \) we have to substitute \( t \) by \( t = \tau[1 + (u^i)^2 + (u^j)^2 + (u^k)^2]^{1/2} / c \). For an external electromagnetic field \( F^{\mu\nu} \) the normalized force \( F^i \) is given by \( F^i = e F^{\mu\nu} u^\nu \). For small velocities \( |u^i|^2 \ll 1 \) Eq. (13) pass over to the nonrelativistic Kramers equation [20].

In the relativistic framework, the Lorentz-invariance of the physical laws is one of the most fundamental property. Eq. (13) refers to a special inertial rest frame \( \Sigma \) of an observer. Now consider a second observer at rest in another inertial frame \( \Sigma' \) that moves with constant velocity \( w \) relative to \( \Sigma \), as shown and discussed by many authors (see e.g. [11, 100]) the probability density function \( \phi(\tau; x, u) \) transforms as a Lorentz scalar; i.e. it fulfills the condition
\[
\phi'(\tau', x', u') = \phi(\tau, x, u)
\]
(16)
where the variables \( \tau', x', u' \)
\[
x^i = \Lambda^i_j x^j + \Lambda^i_0 x^0, \quad u^i = \Lambda^i_j u^j + \Lambda^i_0 u^0, \quad \tau' = \tau
\]
(17)
are related by the Lorentz transformation. This requires that Eq. (13) is invariant with respect to a Lorentz transformation. By using \( x^0 = \tau u^0 \), the chain rule \( \partial / \partial x^i = \Lambda^i_j \partial / \partial x^j \) and the inverse transformation \( u^i = \sum_j \gamma^i_j \Lambda^j_0 \) with \( \Lambda^i_j \Lambda^j_0 = \delta^i_0 \) we find
\[
\frac{\partial \phi}{\partial \tau'} = \frac{\partial \phi}{\partial \tau} + u^i \Lambda^i_0 \frac{\partial \phi}{\partial x^i}
\]
(18)
On a Riemannian manifold the divergence operator and the Laplace-Beltrami operator are intrinsically invariant with respect to general coordinate transformations. Therefore Lorentz transformation on the pseudosphere do not change the explicit form of these operators. For the divergence operator this can be simply proven by the transformation property of the covariant differentiation \( D_j F^{ai} = \partial F^{ai} / \partial u^j + \gamma^a_{ijk} F^k \) given by
\[
D' j F^{ai} = (\partial u^i / \partial u^j)(\partial u^j / \partial u^{ai}) D_j F^{ai}.
\]
For the divergence operator \( \text{div}_v (F \Phi) = D_j (F^a \Phi) \) this yields the relation \( D'_j F^{aj} = D_j F^{aj} \). The invariance of the Laplace-Beltrami operator \( \Delta_u \phi = D_j (G^{aj} \partial_j \phi) \) can be similarly proven. Since \( G^{ij} \partial_j \phi = A \) transforms like a vector and the divergence \( D_j A^j \) is as shown above an invariant operator the relation \( \Delta_u \phi = \Delta_u \phi \) follows. By using Eq. (18) we express the variables \( x^i, u^i \) by the new variables \( x'^i, u'^i \) in the inertial frame \( \Sigma' \) and account the invariance of the divergence and Laplace-Beltrami operator. Then Eq. (13) takes the form
\[
\frac{\partial \phi}{\partial \tau'} = -u^i \frac{\partial \phi}{\partial x^i} - \text{div}_v (F \phi) + \frac{D}{2} \Delta_u \phi,
\]
(19)
Thus, the derived relativistic diffusion equation satisfies the general principle of special relativity and is invariant under Lorentz transformations. Note that Eq. (13) differs from previously derived relativistic diffusion equations. Debbasch et al. [6] introduced a phenomenological
relativistic Langevin equation in the phase space and derived from this a generalized Kramers equation of the classical Ornstein-Uhlenbeck process in which the diffusion term is given by the 3-dimensional Euclidian Laplacian in the momentum space. A different approach has been presented by Dunkel and Hänggi [8] but the final generalized Kramers equation and its steady-state solution also differ from Eq. (13). Both the derived equation in [8] as well those in [30] are not invariant under Lorentz transformations (see [11] and [27]).

For the solution of the relativistic diffusion equation it is convenient to introduce the hyperbolic coordinate system for the 4-velocity, defined by \( u^1 = \alpha \sin \vartheta \sin \varphi \), \( u^2 = \alpha \sin \vartheta \cos \varphi \), \( u^3 = \alpha \cos \vartheta \) and \( u^0 = c \alpha \). We denote the velocities in the non-Cartesian coordinates by \( \alpha^1 = \alpha, \alpha^2 = \theta, \alpha^3 = \varphi \), \( a = 1, 2, 3 \). The metric in this coordinates are simply to calculate and are given by \( G_{11} = 1, G_{22} = \alpha \sin \vartheta, G_{33} = \alpha \cos \vartheta \) and \( G_{ij} = 0 \) for \( i \neq j \). With the given metric the Laplace Beltrami Operator \( \Delta_\alpha \) in the hyperbolic velocity space takes the form

\[
\Delta = \frac{\partial^2}{\partial \alpha^2} + 2 \coth \alpha \frac{\partial}{\partial \alpha} - \frac{1}{(\sinh \alpha)^2} \left( \frac{\partial^2}{\partial \vartheta^2} + \cot \theta \frac{\partial}{\partial \vartheta} + \frac{1}{(\sin \vartheta)^2} \frac{\partial^2}{\partial \varphi^2} \right) = \frac{\partial^2}{\partial \alpha^2} + 2 \coth \alpha \frac{\partial}{\partial \alpha} - \frac{1}{(\sinh \alpha)^2} \left( \frac{\partial^2}{\partial \vartheta^2} + \cot \theta \frac{\partial}{\partial \vartheta} + \frac{1}{(\sin \vartheta)^2} \frac{\partial^2}{\partial \varphi^2} \right) \tag{20}
\]

and

\[
\text{div}_\alpha(F\Phi) = (\sinh \alpha)^{-2} \frac{\partial}{\partial \alpha} \left( (\sinh \alpha)^2 F^\alpha \Phi \right) - \frac{1}{(\sinh \alpha)^2} \left( \sin \vartheta F^{\theta} \Phi \right) - \frac{1}{(\sinh \alpha)^2} \left( \sin \varphi F^{\phi} \Phi \right), \tag{21}
\]

is the divergence operator in the hyperbolic velocity space. Here the force in the hyperbolic coordinate system \( F^\alpha, F^\theta, F^\phi \) is related with \( F^i \) by \( F^\alpha = (\sinh \alpha)^{-1} \left[ \sin \vartheta \cos \varphi F^1 + \sin \varphi F^2 \right] + \cos \vartheta \left[ \sin \varphi F^3 \right] \). \( F^\theta = (\sinh \alpha)^{-1} \left[ \cos \vartheta \cos \varphi F^1 + \cos \varphi F^2 \right] - \sin \vartheta F^3 \) and \( F^\phi = (\sinh \alpha)^{-1} \left[ \sin \vartheta \sin \varphi F^1 + \cos \varphi F^2 \right] \). The transition probability is determined by the initial condition \( \Phi(x, \alpha, \vartheta, \varphi, \alpha(0) = (x_0, \alpha_0, \vartheta_0, \varphi_0, 0) = (\sinh \alpha)^{-2} \left( \sin \vartheta \cos \varphi F^1 + \sin \varphi F^2 \right) \).\] (\( x_1 = x_0 - c_t \), \( x_2 = x_0 - c_t \)).

The diffusion of massless particles such as e.g. the diffusion of photons in random media (see e.g. [30]) can be described in analogous way, but with the condition \( u^0 = 0 \). For the velocity coordinates we choose \( u^0 = \vartheta, u^1 = \varrho \), \( u^2 = \varphi, u^3 = \sin \vartheta \), \( u^4 = \cos \vartheta \). The Laplace-Beltrami operator then takes the form

\[
\Delta = \frac{\partial^2}{\partial \varrho^2} + \frac{\partial}{\partial \varrho} - \frac{1}{\varrho^2} \left( \frac{\partial^2}{\partial \vartheta^2} + \cot \vartheta \frac{\partial}{\partial \vartheta} + \frac{1}{(\sin \vartheta)^2} \frac{\partial^2}{\partial \varphi^2} \right) \tag{22}
\]

4. STEADY STATE SOLUTION OF PARTICLES IN A HEAT BATH: THE JÜTTNER DISTRIBUTION

First, we consider particles with a rest mass \( m \) in a gas in an isotropic homogenous heat bath. The interaction of particles with the bath is described by a random noise force and a friction force. In the nonrelativistic theory the friction force is given by \( F^i = -\nu v^i \) where \( \nu \) is the friction coefficient and \( v^i \) are the components of the nonrelativistic velocity. The relativistic generalization of the friction force requires the introduction of a friction tensor \( \nu^\alpha_\beta \) similar to the pressure tensor in special relativity theory [8, 9]. The friction force is expressed as \( F^\alpha = \nu^\alpha_\beta [u^\beta - U^\beta] \), where \( U^\alpha \) is the 4-velocity of the heat bath. For an isotropic homogeneous heat bath the friction tensor is given by

\[
\nu^\alpha_\beta = \nu (\eta^\beta_\alpha + u^i u_\alpha), \tag{23}
\]

with \( \nu \) denoting the scalar friction coefficient measured in the rest frame of the particles. In the laboratory frame the heat bath is at rest described by \( U^\alpha = (1, 0, 0, 0) \). Therefore the friction force is given by \( F^i = -\nu u^i \) or in hyperbolic coordinates \( F^\alpha = -\nu \alpha \cos \varphi F^1 + \nu \alpha \cos \varphi F^2 - \nu \alpha \sin \varphi F^3 \) and \( F^\theta = -\nu \alpha \sin \vartheta F^1 + \nu \alpha \sin \varphi F^2 + \nu \alpha \cos \varphi F^3 \). We consider the spatial homogenous and isotropic solution of equation Eq.(13) with Eq.(20) and (21) described by

\[
\frac{\partial \phi(a)}{\partial \tau} = D \left[ \frac{\partial^2}{\partial \alpha^2} + 2 \coth \alpha \frac{\partial}{\partial \alpha} \right] \phi(a) + \nu (\sinh \alpha)^{-2} \frac{\partial}{\partial \alpha} \left[ (\sinh \alpha)^3 \phi(a) \right]. \tag{24}
\]

The steady-state solution of this equation is given by \( \phi(a) = C \exp(-\chi \alpha) \) with \( \chi = \nu/D \) and \( C = 4 \pi K_2(\chi) / \chi \). \( K_2(\chi) \) denotes the modified Hankel function. This distribution is identical with the Jüttner equilibrium distribution if the Einstein relation \( kT = \nu m^2 / \nu \) and consequently \( \alpha = \nu / m^2 \) is used. The above derived relativistic diffusion equation Eq.(13) yields for the 3D case the correct thermodynamic relativistic equilibrium distribution for a constant friction coefficient. Note that in a previously derived relativistic diffusion equation [8] the Jüttner equilibrium distribution for a relativistic gas only arise as the steady-state solution for a specifically adapted energy-dependent friction constant \( \nu = \nu_0(u^0)^2 \). The generalized Kramers equation in [8] yields for constant friction coefficients and in the Stratonovich interpretation a modified Jüttner distribution, but using the non-standard pre-point discretization rule the Jüttner distribution was found. Recently fully relativistic one-dimensional molecular dynamics simulations favored the Jüttner distribution in the 1D case [11].
5. NONSTEADY SOLUTION FOR THE FORCE-FREE CASE

Now we consider the unsteady solution of Eq. (13) for a spatial homogenous gas with vanishing force $F$. We use the Laplace transformation $\Phi(\alpha, \vartheta, \varphi, \tau) = \int_0^\infty \hat{\Phi}(\alpha, \vartheta, \varphi, \lambda) \exp(-\lambda \tau) d\lambda$ and for the eigenvalue functions the ansatz $\hat{\Phi}^I_M(\alpha, \vartheta, \varphi, \lambda) = g_J^\alpha(\alpha) Y_{\lambda}^J(\vartheta, \varphi)$ where $Y_{\lambda}^J(\vartheta, \varphi) = P_J^M(\vartheta) e^{i M \varphi}$ are the spherical harmonics with the associated Legendre functions $P_J^M(\vartheta)$. For $g_J^\alpha(\alpha)$ the following eigenvalue equation is derived:

$$\left\{ D \left[ \frac{\partial^2}{\partial \alpha^2} + 2 \frac{\partial}{\partial \alpha} \cosh \alpha - J(J+1) \frac{1}{\sinh^2 \alpha} \right] - \lambda \right\} g_J^\alpha(\alpha) = 0. \quad (25)$$

Here the discrete index $J$ takes the values $J = 0, 1, 2, \ldots$ and $M = -J, -J+1, \ldots, 0, 1, \ldots J$. The eigenfunctions for the Laplace-Beltrami operator with the eigenvalues $\lambda = D(z^2 + 1)$ satisfying the boundary condition are given by

$$g_J^\alpha(z) = C_J^\alpha (z^2 - 1)^J \left( \frac{d}{dz} \right)^{J+1} \cos(\alpha \text{ arch } z), \quad (26)$$

with $z = \cosh \alpha$ and $C_J^\alpha = (-1)^{J+1} \sum_{k=0}^{\infty} \frac{1}{2k+1} (z^2 + k^2)^{-\frac{J}{2}}$. The eigenfunctions $\hat{\Phi}^I_M(\alpha, \vartheta, \varphi, \lambda)$ satisfy the relations of orthogonality and completeness. The transition probability is determined by the initial condition $\Phi(\alpha, \vartheta, \varphi, \tau = 0 | \alpha_0, \vartheta_0, \varphi_0, 0) = (\sinh \alpha)^{-2} (\sin \vartheta)^{-1} \delta(\alpha - \alpha_0) \delta(\vartheta - \vartheta_0) \delta(\varphi - \varphi_0)$. Using the orthogonality relation we can write

$$\Phi(\alpha, \vartheta, \varphi, \tau | \alpha_0, \vartheta_0, \varphi_0, 0) = \sum_{M, J} \int_0^\infty \hat{\Phi}^I_M(\alpha, \vartheta, \varphi, \lambda) \Phi_J^\alpha(\alpha_0, \vartheta_0, \varphi_0, \lambda) \exp[-D(\lambda^2 + 1)\tau] d\lambda. \quad (27)$$

Let us now consider the fundamental solution $J = 0$. Substituting $g_0^\alpha(\alpha) = \frac{1}{\sqrt{2\alpha}} (\sinh \alpha)^{-1} \sin \alpha \kappa$ into Eq.(27) gives the transition probability

$$\Phi(\alpha, \tau | \alpha_0, 0) = C(\tau D)^{-\frac{1}{2}} \exp\left\{ -\frac{\tau}{4D} \right\} \times \exp\left\{ -\frac{\alpha^2 + \alpha_0^2}{4D\tau} \right\}, \quad (28)$$

with $C = 2(4\pi)^{-\frac{1}{2}}$. For $\alpha_0 \rightarrow 0$ this solution was first found in [28]. For small velocities ($\alpha \ll 1$) the transition distribution shows a remarkable behavior. If we solve the corresponding non-relativistic Kramers equation [20] substituting in Eq.(25) $\sinh \alpha \rightarrow \alpha, \cosh \alpha \rightarrow 1$ we find for $J=0$ $g_0^\alpha(\alpha) = -\frac{1}{\sqrt{2\pi}} (\alpha^{-1} \sin \alpha \kappa)$, but the eigenvalue is given by $\lambda = D\alpha^2$ and the solution now is

$$\Phi(\alpha, \tau | \alpha_0, 0) = C(\tau D)^{-\frac{1}{2}} \alpha^{-1} \alpha_0^{-1} \exp\left\{ -\frac{(\alpha - \alpha_0)^2}{4D\tau} \right\} - \exp\left\{ -\frac{(\alpha + \alpha_0)^2}{4D\tau} \right\}. \quad (29)$$

For $\alpha_0 = 0$ Eq.(29) pass to the known Wiener distribution in the velocity space. In the limit $\alpha \ll 1$ the short time behavior of Eq.(29) is up to an exponential small factor in agreement with Eq.(28), however in the long time behavior both solutions differ by the exponential factor $\exp(-D\tau)$. In Fig.1 the relativistic distribution Eq.(28) is presented by the solid lines for $\alpha_0 = 0$ and the Wiener distribution Eq.(29) by the dotted lines. As can be seen both distributions differs by orders of magnitudes even in the non relativistic region $a \ll 1$ for long times $D\tau \gg 1$. This discrepancy can be explained by the topological properties of the hyperbolic space (included by the boundary conditions) which are different from that of the Euclidian space in the non relativistic theory. The deep connection between local and global properties of diffusion processes is a central topic in the mathematical field of heat kernels on Riemannian manifolds [29]. The appearance of the factor $\exp(-D\tau)$ can also be explained by physical arguments, it comes from that in the hyperbolic coordinates the Jacobian is proportional to $\sinh^2 \alpha$ which is exponentially large for $\tau \rightarrow \infty$. For large $\tau$ the entire velocity space is explored and the small factor $\exp(-D\tau)$ cancels the exponentially large Jacobian and guarantees the probability conservation.

Let us still compare the solution for a massive particle with that of a massless one. With the Laplacian Eq.(22) the eigenvalues takes the form $\lambda = D\alpha^2$ and the eigenvalue solutions are $\hat{\Phi}^I_M(\alpha, \vartheta, \varphi, \lambda) = g_J^\alpha(\alpha) Y_{\lambda}^J(\vartheta, \varphi)$.

$$g_J^\alpha(z) = \sqrt{\frac{\alpha}{2\pi \tau}} (\alpha^{-2} \sin \vartheta)^{-1} \delta(\alpha - \alpha_0) \delta(\vartheta - \vartheta_0) \delta(\varphi - \varphi_0). \quad (29)$$

The photon transition probability for $J = 0$ then is given by

$$\Phi(\alpha, \tau | \alpha_0, 0) = (2\pi r_0 \sqrt{\pi})^{-1} (\tau D)^{-\frac{1}{2}} \sin\left( \frac{r_0}{2D\tau} \right) \exp\left\{ -\frac{r_0^2 + \alpha_0^2}{4D\tau} \right\}, \quad (30)$$

In comparison with Eq.28 the factor $\exp(-D\tau)$ here is absent since the Jacobian do not increase exponentially for $\tau \rightarrow \infty$.

6. CONCLUSIONS

In conclusion, a theory of Markovian diffusion processes within the framework of the special theory of relativity is formulated. In the derivation of the basic relativistic diffusion equation the mathematical calculus of
stochastic differential equations on Riemannian manifolds is used, which here is modified for the description of diffusion in the phase space on Lorentzian manifolds with an indefinite metric. A generalized Langevin equation in the fiber space of position, velocity and orthonormal velocity frames is defined and the generalized relativistic stochastic differential equations on Riemannian manifolds is used, which here is modified for the description of diffusion in the phase space on Lorentzian manifolds with an indefinite metric. A generalized Langevin equation in the fiber space of position, velocity and orthonormal velocity frames is defined and the generalized relativistic Kramers equation is derived. This diffusion equation is invariant under Lorentz transformations. In the case of a relativistic gas in a heat bath its steady-state solution is identical with the Čröttner distribution for constant friction coefficients. An analytical nonsteady solution for the transition probability is given for the special case of vanishing external fields. This solution differs from the Wiener velocity distribution even for small velocities due to topological reasons.

The formalism presented in this paper can be extended to a theory within the framework of general relativity. This will be done in a forthcoming paper.

**APPENDIX A: DIFFUSION ON EUCLIDIAN MANIFOLDS**

Using appropriate modifications the stochastic calculus on Riemannian manifolds utilizes basic theorems and formulas from the calculus on Euclidian manifolds. Therefore in appendix A some basic notations, formulas and theorems for the Euclidian space are summarized including some elementary proofs of the basic theorems.

Diffusion processes in a d-dimensional Euclidian space are described by stochastic differential equations of the form [22]–[24]

\[ dX^i = \sigma^i_a(\tau, X)dW^a + b^i(\tau, X)d\tau. \]  \( A1 \)

\[ X(x_1, \ldots, x_d) \in \mathbb{R}^d \] is a stochastic process with \( X(0) = x \). The diffusion coefficients \( \sigma^i_a(\tau, X) \) are given matrices and the drift coefficients \( b^i(\tau, X) \) are coefficients of a smooth vector field. \( W^a \) are the components of the elementary Wiener process \( dW^a = W^a(t+\Delta t) - W^a(t) \) with the probability density \( P(W^a) = (2\pi D\Delta t)^{-\frac{1}{2}} \exp\left(-\frac{(W^a(t))^2}{2D\Delta t}\right) \) and the expectation values \( \langle W^a \rangle = 0 \). \( D \) is the diffusion constant.

The stochastic integral in the second term of Eq.(A2) is defined as the limit \( \lim_{\Delta \to 0} \int_0^\tau \sigma^i_a(s, X)dW^a_s = \sum_{n=1}^\infty \sigma^i_a(s^n, X)(W^a(s^n) - W^a(s^{n-1})) \) as \( n \to \infty \). This integral depends on the choice of the intermediate point \( s^n \). With the choice \( s^n = s_{n-1} \) the Stratonovich stochastic integral is defined. The Ito integral is a Markovian process and plays a fundamental role in the theory of diffusion processes and most of general mathematical treatments can only rigorously proven by using this calculus. Alternatively, choosing \( s^n = s_{n-1} \) the Stratonovich integral is defined. The Stratonovich integral has the advantage of leading to ordinary chain rule formulas under a transformation. This property makes the Stratonovich integral natural to use for stochastic differential equations on Riemannian manifolds. However, in general Stratonovich integrals are not Markovian processes which hinders rigorous mathematical treatments in most cases.

A differential equation Eq.(A1) is defined only with respect to one of the both stochastic integrals, changing the interpretation of the integral refers to a differential problem with different solutions. Therefore the chosen interpretation should be denoted in the differential equation. The symbol \( \sigma^i_a(\tau, X)dW^a \) denotes the Ito integral interpretation and \( \sigma^i_a(\tau, X)\circ dW^a \) the Stratonovich interpretation.

With the Ito interpretation the solution \( X^i_\tau \) of Eq.(A1) is denoted as an Ito process if the diffusion and drift coefficients satisfy the Lipshift condition and \( \sigma^i_a(\tau, X) \) is adapted to the fundamental Wiener process \( W^a_\tau \). An Ito process has the important property of being Markovian. Then \( Y_\tau = f(X_\tau) \) is also an Ito process, and its stochastic differential equation (Ito formula) is

\[ df = \frac{D}{2} \sum_{a=1}^d (\sigma^i_a(\tau, X)\sigma^j_a(\tau, X)\partial_i\partial_j f + \] \[ + b^i(\tau, X)\partial_i f) d\tau + D\sigma^i_a(\tau, X)\partial_i f dW^a. \]  \( A3 \)
Associated to an Ito process is the diffusion generator $A$ of $X_r$, which is defined to act on a suitable function $f$ by:

$$ Af = \lim_{t \to 0} \frac{E^x[f(X_r)] - f(x)}{t}, $$

(A4)

where $x = X_0$ is the initial point of $X_r$. By using Eq. (A3) one can show that $A$ is given by

$$ Af = \frac{D}{2} \sum_{a=1}^d \sigma^a_i(\tau, X)\sigma^a_j(\tau, X) \partial_i \partial_j f + b^i(\tau, X) \partial_i f. $$

(A5)

The generator $A$ can be used in the derivation of Kolmogorov’s backward equation. This equation describes how the expected value $E^x[f(X_r)]$ of any smooth function $f$ of $X$ evolve in time. If we define

$$ u(\tau, x) = E^x[f(X_r)], $$

(A6)

then $u$ satisfy the following equation:

$$ \frac{\partial}{\partial \tau} u(\tau, x) = E^x\left[\frac{d}{d\tau} f(X_r)\right] = \frac{d}{d\tau} E^x[f(X_r)] $$

(A7)

and with Eq. (A3)

$$ \frac{\partial}{\partial \tau} u(\tau, x) = Au(\tau, x) $$

(A8)

with $u(0, \Omega) = f(x)$. The Fokker-Plack equation (or forward Kolmogorov equation) describes how the probability density function $\phi(\tau, x)$ of $X_r$ evolve with time. This density function is defined as a function of the $d$ stochastic variables, such that for any domain $\Omega$ in the $d$-dimensional space of the variables $X_1, ..., X_d$, the probability that a realization of a set of variables falls inside the domain $\Omega$ is

$$ \Pr(X_1^* \in \Omega) = \int_{\Omega} \phi(\tau, x)dx_1...dx_d. $$

(A9)

The probability density function can be used to calculate the expected value $E^x[f(X_r)]$ by $E^x[f(X_r)] = \int f(x) \phi(\tau, x)dx_1...dx_d$. We now consider the time derivative of such expectation value and use again Ito’s formula Eq. (A3):

$$ E^x\left[\frac{d}{d\tau} f(X_r)\right] = \frac{d}{d\tau} E^x[f(X_r)] = E^x[Af(X_r)] = $$

$$ = \int_{\Omega} \phi(\tau, x)Af(x)dx_1...dx_d = $$

$$ = \int_{\Omega} \phi(\tau, x)A^*f(x)dx_1...dx_d $$

(A10)

We integrate by parts and discard surface terms to obtain

$$ \int_{\Omega} f(x)A^*\phi dx_1...dx_d = $$

$$ = \int_{\Omega} f(x)\frac{\partial}{\partial \tau} \phi dx_1...dx_d, $$

(A11)

where $A^*$ denotes the Hermitian adjoint of $A$. Since $f(x)$ is arbitrary we find for the Fokker-Plack equation within the Ito integral interpretation:

$$ \frac{\partial}{\partial \tau} \phi(\tau, x) = A^*\phi(\tau, x) $$

(A12)

with the adjoint operator

$$ A^* f = \frac{D}{2} \sum_{a=1}^d \partial_i \partial_j \sigma^a_i(\tau, X)\sigma^a_j(\tau, X)f - \partial_i b^i(\tau, X)f. $$

(A13)

Since the stochastic calculus on Riemannian manifolds is naturally formulated in the Stratonovich integral interpretation, we will consider the connection between both types of integrals. Let us formulate the stochastic differential equation Eq. (A1) with the Ito interpretation by an corresponding equation with the Stratonovich interpretation

$$ dX = \tilde{A}^i(\tau, X) \circ dW^i + \tilde{b}^i(\tau, X)d\tau. $$

(A14)

In the Stratonovich interpretation the differentiation of a function $f$ yields the ordinary chain rule, e.g. in the Ito formula Eq. (A3) the first term with second-order derivatives does not appear and the above given treatment can not be done in a consistent way. However, there exist a connection between the Ito and the Stratonovich integrals (22, 23) of functions $\varphi_a(X(\tau), \tau)$ in which $X(\tau)$ is the solution of the Ito differential equation Eq. (A1):

$$ \int_0^\tau \varphi_a(s, X) dW^a_s = \int_0^\tau \varphi_a(s, X) \circ dW^a_s - $$

$$ - \frac{D}{2} \int_0^\tau \sigma^a_i(\tau, X) \partial_i \varphi(s, X) \circ dW^a_s $$

(A15)

If we now make the choice
\[ \tilde{b}^i(\tau, X) = b^i(\tau, X) - \sum_{a=1}^{d} \frac{D}{2} \sigma^2_a(\tau, X) \partial_i \sigma^i_a(\tau, X), \]

\[ \sigma^i_a(\tau, X) = \tilde{\sigma}^i_a(\tau, X) \tag{A16} \]

and substitute \( \tilde{b}^i(\tau, X) \) into Eq.(A5), the diffusion operator \( \mathbf{A} \) in the Stratonovich interpretation is

\[ \mathbf{A} = \frac{D}{2} \sum_{a=1}^{d} \sigma^2_a(\tau, X) \partial_i \sigma^i_a(\tau, X) \left[ \sigma^i_a(\tau, X) \phi(\tau, x) \right] - \partial_i \tilde{b}^i(\tau, x) \phi(\tau, x). \tag{A18} \]

We introduce the fundamental vector fields

\[ L_a = \sigma^i_a(\tau, x) \partial_i, \quad L_0 = \tilde{b}^i(\tau, x) \partial_i \tag{A19} \]

the generator \( \mathbf{A} \) of the stochastic process can be expressed by the fundamental vector fields \( L_a \)

\[ \mathbf{A} = \frac{D}{2} \sum_{a=1}^{d} L_a L_a + L_0. \tag{A20} \]

**APPENDIX B: DIFFUSION ON RIEMANNIAN MANIFOLDS**

Stochastic differential equations are defined by the driving Wiener process \( W^a(t) \) (or more generally by a semi-martingale), but this process is difficult to handle on a Riemannian manifold. In differential geometry for a general d-dimensional Riemannian manifold \( M^d \) (with definite metric signature) equipped with a Christoffel connection \( \Gamma^i_{ab} \) it is possible to lift a smooth curve \( c^i(t) \) in \( M^d \) to a horizontal curve in the tangent bundle \( TM \) which is endowed with an Euclidian structure by using the bundles of orthonormal frames \( e^i_a = e^i_a(x) \partial_i \) (i,a=1-d). The orthonormal frame bundle \( O(M) \) is described by the local coordinates \( \{ r = (x^i, e^i_a) \} = O(M). \) The infinitesimal motion of a smooth curve \( x^i(t) \) in \( M^d \) is that of \( \gamma^i(t) \) in \( O(M) \) described by the ordinary differential equations for a parallel transport

\[ dx^i = e^i_a(x) d\gamma^a, \]

\[ de^i_a(x) = -\Gamma^i_{ab} e^b_a(x) dx^a. \tag{B1} \]

Here \( \eta^{ab} e^i_a(x)e^j_b(x) = g^{ij}, \partial_i e^a = -\Gamma^i_{ab} e^b_a, g^{ij} \) is the Riemannian metric and \( \eta^{ab} = \delta^{ab} \) the flat Euclidian metric where \( \delta^{ab} \) is the Kronecker symbol. \( r^i(t) \) is called the horizontal lift of the curve \( x^i(t) \) to the orthonormal frame bundle \( O(M) \) and it lies in the Euclidian space \( \mathbb{R}^{d^2} \). The horizontal curve \( \gamma^i(t) \) corresponds uniquely to a smooth curve in the tangent space (which can be identified with an Euclidian space \( \mathbb{R}^d \)).

Stochastic differential equations on a Riemannian manifold can be defined by using the above described horizontal lift to the orthonormal frame bundle \( O(M) \) endowed with an Euclidian structure \cite{24, 25}. By using this approach a stochastic process on the Riemannian manifold can be constructed by using the fundamental Wiener process each component of which is a process in the Euclidian space \( \mathbb{R}^d \) and interpreting the corresponding stochastic integral in the sense of Stratonovich. This means that the manifold is moved along a stochastic curve in the tangent space by a parallel translation with the help of the orthonormal frame bundles and the Christoffel connection coefficients \( \Gamma^i_{ab} \). Correspondingly a random curve can be defined in the same way as in Eq.(B1) by using the canonical realization of a d-dimensional Wiener process and substituting \( du^a \rightarrow dW^a(t) \). Therefore the stochastic differential equation describing diffusion on a Riemannian manifold is

\[ dx^i = e^i_a(\tau) \odot dW^a + A^i dx^a, \tag{B2} \]

where the components of an arbitrary tangential vector \( A^i \) are additionally introduced for a more general situation with account of an external force field. The components of the elementary Wiener process \( dW^a(t+\Delta t)-W^a(t) \) are defined in the Euclidian space with the probability density \( P(W^a) = (2D\Delta t)^{-\frac{1}{2}} \exp(-\frac{1}{2}W^a(t)^2) \) with the expectation values \( \langle W^a \rangle = 0, \quad \langle W^a(t)W^b(\tau+s) \rangle = Ds\delta_{ab} \).

The derivation of the Kolmogorov backward equation with the definition of the diffusion operator \( A_{O(M)} \) can be performed by the same rules as in Euclidian space in the Stratonovich calculus. Corresponding the definition of the fundamental vector fields \( L_a \) and \( L_0 \) in Eq.(A19) one can now introduce the fundamental horizontal vector fields \( H_a \) and \( H_0 \) on \( O(M) \) for the extended stochastic differential system Eq.(B2):

\[ H_a = e^i_a(x) \frac{\partial}{\partial x^i} - \Gamma^i_{ml}(x)e^m_a \frac{\partial}{\partial e^l_b}, \]

\[ H_0 = A^i(\tau, X) \partial_i - \Gamma^i_{ml}e^l_a(\tau)A^m \frac{\partial}{\partial e^a}, \tag{B3} \]
and the operator $A_{O(M)}$ for the stochastic process in the orthonormal frame bundle is given by

$$\mathbf{A}_{O(M)} = \frac{D}{2} \sum_{a=1}^{d} H_a H_a + H_0. \quad (B4)$$

$A_{O(M)}$ is the horizontal lift of the diffusion generator $A_M$ on the manifold to the orthonormal frame bundle. Obviously, the projection of a function in $O(M)$ to $M$ with $f(r) = f(x, 0), r = (x^i, e_j^i)$ satisfy the relation

$$\mathbf{A}_{O(M)} f(r) = A_M f(x) \quad (B5)$$

where $A_M = \frac{D}{2} \sum_{a=1}^{d} \left( \eta^a \partial_a \eta^b \partial_b \right) + A_i \partial_i = \left( \frac{D}{2} \Delta + A^i \partial_i \right)$ and $\Delta_M = g^{ij} \partial_i \partial_j + g^{ij} \Gamma^k_{ij} \partial_k$ is the Laplace-Beltrami operator. The generalized Kolmogorov backward equation on a Riemannian manifold is obtained by

$$\frac{\partial}{\partial \tau} u(\tau, x) = \left( \frac{D}{2} \Delta_M + A^i \partial_i \right) u(\tau, x) \quad (B6)$$

As shown in the appendix A the generalized Fokker-Planck equation is given by the adjoint of the diffusion generator $A^*$ (which includes the volume element $\sqrt{g}$, $g = \det\{g_{ij}\}$). Since the Laplace-Beltrami operator is self-adjoint $\Delta_M = \Delta_M^*$ the generalized Fokker-Plack equation on a Riemannian manifold takes the form:

$$\frac{\partial \Phi}{\partial \tau} = -\text{div}_x (A \Phi) + \frac{D}{2} \Delta_M \Phi, \quad (B7)$$

where $\text{div}_x (A \Phi) = g^{-\frac{1}{2}} \partial_i (g^{\frac{1}{2}} A^i \Phi)$ is the divergence operator in the Riemannian manifold, $\Phi = \Phi(x, \tau | \gamma, 0)$ is the transition probability with the initial condition $\Phi(x, 0 | y, 0) = \delta(x-y)$ and adequate boundary conditions at infinity. The probability density $\varphi(x, \tau)$ is determined by the same equation with the initial condition $\varphi(x, \tau = 0) = \varphi^0(x)$.

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