Relative equilibria in curved restricted 4-and 5-body problems

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Abstract. We consider a 5-body problem on 2-dimensional surfaces of constant curvature $\kappa$, with four of the masses arranged at the vertices of a square and the fifth mass at the north pole of the sphere. The five-body set up is discussed for $\kappa > 0$ and for $\kappa < 0$. When the curvature is positive, it is shown that relative equilibria exists when the four masses at the vertices of the square are either equal or two of them are infinitesimal such that it doesn’t effect the motion of the remaining three masses. However with two pairs of masses at the vertices of the square, no relative equilibria exists. In the hyperbolic case, $\kappa < 0$, there exist two values for the angular velocity which produce negative elliptic relative equilibria when the masses at the vertices of the square are equal. We also show that the solutions with non-equal masses do not exist in $H^2$.

1. Introduction
The classical $N$-body problem has a long history. Isaac Newton first proposed it in 1687 in his first edition of \textit{Principia} in the context of the Moon’s motion. He assumed that universal gravitation acts between celestial bodies (reduced to point masses) in direct proportion with the product of the masses and in inverse proportion with the square of the distance. The study of the $N$-body problem was further advanced by the Bernoulis, Lagrange, Laplace, Euler, Cauchy, Jacobi, Dirichlet, Poincaré and many others. The idea of extending the gravitational force between point masses to spaces of constant curvature occurred soon after the discovery of hyperbolic geometry. A few years earlier, Gauss had interpreted Newton’s gravitational law as stating that the attracting force between bodies is inversely proportional with the area of the sphere of radius equal to the distance between the point masses (i.e. proportional to $1/r^2$, where $r$ is the distance). Using this idea, Bolyai and Lobachevsky suggested that, should space be hyperbolic, the attracting force between bodies must be inversely proportional to the hyperbolic area of the corresponding hyperbolic sphere (i.e. proportional to $1/\sinh^2(|\kappa|^{1/2}r)$, where $r$ is the distance and $\kappa < 0$ the curvature of the hyperbolic space). This
is equivalent to saying that, in hyperbolic space, the potential that describes the
gravitational force is proportional to $\coth(|\kappa|^{1/2}r)$.

The curved $N$-body problem became somewhat neglected after the birth of
general relativity, but was revived after the discretization of Einstein’s equation
showed that an $N$-body problem in spaces of variable curvature is too complicated
to be treated with analytical tools. In the 1990s, the Russian school of celestial
mechanics considered both the curved Kepler and the curved 2-body problem. More
recently, the work of Diacu, Santoprete, and Pérez-Chavela considered the curved
$N$-body problem for $N > 2$ in a new framework, leading to many interesting results,
[1–17]. In fact, solutions of the curved $N$-body problem in which the bodies move
without collisions help to understand the geometry of the universe and the motion of
the celestial bodies.

In this work we have considered the motion of five bodies on 2-dimensional surfaces
of constant curvature $\kappa$, namely spheres $S^2_\kappa$ for $\kappa > 0$ and the hyperbolic sphere $H^2_\kappa$
for $\kappa < 0$. In this paper, we discuss the existence of solutions for the 5-body problem
in which equilibrium solutions appear for particular geometrical configurations. We
also show that the relative equilibria of the 5-body problem with four equal masses
and an arbitrary mass for the spaces corresponding to $\kappa > 0$ and $\kappa < 0$ always
exist. Moreover, in section 3, we have proved that the square pyramidal solutions in $S^2$
for two pairs of equal masses do not exist. We have considered the planetary
problem when some masses are negligible. Then, we show that the square pyramidal
solutions in $S^2$ with two negligible masses do exist. However, in sections 4, we show
that the relative equilibria for two pairs of equal masses do not exist in $H^2$.

2. Equations of motion
Consider $N \geq 2$ point masses $m_1, \ldots, m_N > 0$ moving on the 2-sphere of constant
Gaussian curvature $\kappa > 0$

$$S^2_\kappa := \{(x, y, z) | x^2 + y^2 + z^2 = \kappa^{-1}\}.$$ or on the hyperbolic 2-sphere of curvature $\kappa < 0$,

$$H^2_\kappa := \{(x, y, z) | x^2 + y^2 - z^2 = \kappa^{-1}\},$$

embedded in the Minkowski space $\mathbb{R}^{2,1}$. The notation $\mathbb{R}^{2,1}$ expresses the signature
$(+,+,−)$ of the Lorentz inner product. Whereas $S^2_\kappa$ express the signature $(+,+,+)$
of the inner product. Let $\mathbf{q}_i = (x_i, y_i, z_i)$ be the coordinates of the point mass $m_i$, satisfying the constraint

$$x_i^2 + y_i^2 + \sigma z_i^2 = \kappa^{-1},$$

where $\sigma$ is the signum function

$$\sigma := \begin{cases} +1 & \text{for } \kappa > 0 \\ -1 & \text{for } \kappa < 0. \end{cases}$$

The inner product between the vectors $\mathbf{q}_i = (x_i, y_i, z_i)$ and $\mathbf{q}_j = (x_j, y_j, z_j)$ is given by

$$q^{ij} = \mathbf{q}_i \cdot \mathbf{q}_j := x_ix_j + y_iy_j + \sigma z_i z_j.$$
We define the distance between the bodies \( m_i \) and \( m_j \) in \( S^2_\kappa \) and \( \mathbb{H}^2_\kappa \) as
\[
q_{ij} := \begin{cases} 
   \left[ (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 \right]^{1/2} & \text{for } \kappa \geq 0 \\
   \left[ (x_i - x_j)^2 + (y_i - y_j)^2 - (z_i - z_j)^2 \right]^{1/2} & \text{for } \kappa < 0.
\end{cases}
\]

Consider five point masses, \( m_i > 0, \ i = 1, 2, 3, 4, 5 \), whose position vectors, velocities, and accelerations are given by
\[
\mathbf{q}_i = (x_i, y_i, z_i), \quad \dot{\mathbf{q}}_i = (\dot{x}_i, \dot{y}_i, \dot{z}_i), \quad \ddot{\mathbf{q}}_i = (\ddot{x}_i, \ddot{y}_i, \ddot{z}_i), \quad i = 1, 2, 3, 4, 5.
\]

Proof. When \( \kappa = 1 \), the equations of motion are given by
\[
\ddot{q}_i = \sum_{j=1, j \neq i}^N \frac{m_j \left[ q_j - \sigma(q_i \cdot q_j)q_i \right]}{[\sigma - \sigma(q_i \cdot q_j)]^{3/2}} - \sigma(q_i \cdot \dot{q}_i)q_i, \quad i = 1, 2, 3, 4, 5.
\] (1)

On components, the equations of motion and the constraints can be written as
\[
\begin{align*}
\ddot{x}_i &= \sum_{j=1, j \neq i}^N m_j \frac{x_j - \sigma(q_i \cdot q_j)x_j}{[\sigma - \sigma(q_i \cdot q_j)]^{3/2}} - \sigma(q_i \cdot \dot{q}_i)x_i, \\
\ddot{y}_i &= \sum_{j=1, j \neq i}^N m_j \frac{y_j - \sigma(q_i \cdot q_j)y_j}{[\sigma - \sigma(q_i \cdot q_j)]^{3/2}} - \sigma(q_i \cdot \dot{q}_i)y_i, \\
\ddot{z}_i &= \sum_{j=1, j \neq i}^N m_j \frac{z_j - \sigma(q_i \cdot q_j)z_j}{[\sigma - \sigma(q_i \cdot q_j)]^{3/2}} - \sigma(q_i \cdot \dot{q}_i)z_i,
\end{align*}
\] (2)

\[
x_i^2 + y_i^2 + \sigma z_i^2 = \sigma, \quad x_i \dot{x}_i + y_i \dot{y}_i + \sigma z_i \dot{z}_i = 0, \quad i = 1, \ldots, n.
\]

3. The positively curved 5-body problem with one mass at the north pole and four masses form a square in \( S^2_\kappa \)

In the positively curved 5-body problem, we assume that one body of mass \( m_5 \) is fixed at the north pole and the other four bodies of masses \( m_1, m_2, m_3, m_4 \) are located at the vertices of a rotating square which is orthogonal to \( z \)-axis and parallel with the equator \( z = 0 \). The bodies are moving on the unit sphere \( S^2 \), which has constant curvature 1.

**Lemma 1.** In the curved 5-body problem in \( S^2 \), if four bodies of masses \( m_1 = m_2 = m_3 = m_4 = m \) are located at the vertices of a rotating square and the fifth body of mass \( m_5 \) is fixed at the north pole \((0, 0, 1)\), then the angular velocity of the system depends on the masses of the five bodies and the distances between them.

**Proof.** When \( \kappa = 1 \), the equations of motion are given by
\[
\ddot{q}_i = \sum_{j=1, j \neq i}^N \frac{m_j \left[ q_j - (q_i \cdot q_j)q_i \right]}{[1 - (q_i \cdot q_j)]^{3/2}} - (q_i \cdot \dot{q}_i)q_i, \quad i = 1, 2, 3, 4, 5.
\] (3)

We are interested in solutions of the form
\[
\mathbf{q} = (q_1, q_2, q_3, q_4, q_5) \in S^2, \quad \mathbf{q}_i = (x_i, y_i, z_i), \quad i = 1, 2, 3, 4, 5.
\]
Figure 1. The case of 4 equal masses and one body at the north pole.

\[
\begin{align*}
&x_1 = r \cos \alpha t, \quad y_1 = r \sin \alpha t, \quad z_1 = \pm \sqrt{1 - r^2} \\
&x_2 = -r \cos \alpha t, \quad y_2 = -r \sin \alpha t, \quad z_2 = \pm \sqrt{1 - r^2} \\
&x_3 = -r \sin \alpha t, \quad y_3 = r \cos \alpha t, \quad z_3 = \pm \sqrt{1 - r^2} \\
&x_4 = r \sin \alpha t, \quad y_4 = -r \cos \alpha t, \quad z_4 = \pm \sqrt{1 - r^2} \\
&x_5 = 0, \quad y_5 = 0, \quad z_5 = 1,
\end{align*}
\]

where \( r \) denotes the radius of the circle in which the square configuration rotates, and \( \alpha \) denotes the angular velocity of the rotation.

**Definition 1.** Relative equilibria are solutions of the curved n-body problem in which the relative distances between particles remain constant during the motion. So the particle system rotates like a rigid body.

Due to the symmetry, without loss of generality, we will write the equations motion for \( x_i \) and ignore \( y_i \). Consider

\[
\begin{align*}
q_1 \cdot q_2 &= q_3 \cdot q_4 = 1 - 2r^2 = A, \\
q_1 \cdot q_3 &= q_1 \cdot q_4 = q_2 \cdot q_3 = q_2 \cdot q_4 = 1 - r^2 = B, \\
q_i \cdot q_5 &= \sqrt{1 - r^2} = C, \quad i = 1, 2, 3, 4.
\end{align*}
\]

Substituting these coordinates and expressions into the equations of motion (3) corresponding to the \( x_i \) coordinate, we get

\[
\ddot{x}_i = R x_i, \quad i = 1, 2, 3, 4, \tag{6}
\]

where

\[
R = -\frac{m}{4r^3\sqrt{B}} - \frac{2mB}{[r^2(2 - r^2)]^{3/2}} \pm \frac{m_5C}{r^3} - r^2 \alpha^2.
\]
We get the following condition for the existence of the square pyramid relative equilibria in $\mathbb{S}^2$ by comparing the above equations.

$$\alpha^2 = \frac{m^3}{4r^3(1-r^2)^{3/2}} + \frac{2m}{r^3(2-r^2)^{3/2}} \pm \frac{m_5}{r^3(1-r^2)^{1/2}}$$  \hspace{1cm} (7)

Equation (7) shows that the angular velocity of this system depends on the four equal masses $m$, the mass of the body at the north pole, and the distances between them. This completes the proof of lemma 1.

Theorem 2. Consider the positively curved square 5-body problem in $\mathbb{S}^2$ with four masses inscribed in a circle of radius $0 < r < 1$, parallel with the equator, while the fifth mass is fixed at the north pole.

a. When $m_i = m$, $i = 1, 2, 3, 4$ relative equilibria exists for all positive values of $m$ and $M$ when $z > 0$. However, when $z < 0$ relative equilibria exists only in

$$R_1(\Gamma, z) = \{ (\Gamma, z) | -1 < z < 0 \land \Gamma > 2\sqrt{\frac{z^2}{(z^2 + 1)^3} - \frac{1}{4z^2}} \}.$$

b. When $m_1 = m_2 = m$ and $m_3 = m_4 = M$, the square pyramid relative equilibria exists only if $m = M$.

c. In the restricted case when $m_1 = m_2 \approx 0$ then there is a circle of radius $r = \sqrt{2/3}$ parallel with the equator, such that a square configuration inscribed in this circle, with $m_1, m_2$ at the opposite ends of one diagonal and $m_3, m_4$ at the opposite ends of the other diagonal, forms a relative equilibrium and the mass at the north pole doesn’t influence the existence of relative equilibria.

3.1. Proof of theorem 2.(a)

Let $\Gamma = \frac{m_5}{m}$, then equation (7) can be written as:

$$\frac{\alpha^2}{m} = \pm \frac{1}{4z^3(1-z^2)^{3/2}} \pm \frac{2}{(1-z^4)^{3/2}} \pm \frac{\Gamma}{z(1-z^2)^{3/2}} = f^\pm(\Gamma, z)$$

where $z^3 = (1-r^2)^{3/2}$, $r^3 = \pm(1-z^2)^{3/2}$, $(2-r^2)^{3/2} = (z^2 + 1)^{3/2}$, and $(1+z^2)^{3/2}(1-z^2)^{3/2} = (1-z^4)^{3/2}$. To show the existence of Relative Equilibria we need to show that $f^\pm(\Gamma, z) \geq 0$. Let $z > 0$, then

$$f^+(\Gamma, z) = \frac{1}{4z^3(1-z^2)^{3/2}} + \frac{2}{(1-z^4)^{3/2}} + \frac{\Gamma}{z(1-z^2)^{3/2}}. \hspace{1cm} (8)$$

It is trivial to see that $f^+(\Gamma, z) > 0$ when $z \in (0, 1)$ which implies the existence of RE for all values of $z$ and $\Gamma$.

In the case when $z < 0$, we have

$$f^-(\Gamma, z) = -\frac{1}{4z^3(1-z^2)^{3/2}} - \frac{2}{(1-z^4)^{3/2}} - \frac{\Gamma}{z(1-z^2)^{3/2}}. \hspace{1cm} (9)$$
To find regions of RE when $z < 0$, we rewrite equation (9).

$$f^-(\Gamma, z) = \frac{\mu(\Gamma, z)}{\delta(z)},$$

where

$$\mu(\Gamma, z) = \sqrt{1-z^2} \left(8z^3 + (z^2 + 1)^{3/2} (4\Gamma z^2 + 1)\right),$$

$$\delta(z) = 4z^3 (1 - z^2) (z^2 - 1) (z^2 + 1)^{3/2}.$$

The function $\delta(z) > 0$ when $z \in (-1, 0)$ as $z^3$ and $(z^2 - 1)$ are negative and two of the remaining components are positive when $z \in (-1, 0)$. Using algebraic manipulation, it can be shown that $\mu(\Gamma, z) > 0$ in

$$R_1(\Gamma, z) = \{(\Gamma, z) | -1 < z < 0 \land \Gamma > 2 \sqrt{\frac{z^2}{(z^2 + 1)^3} - \frac{1}{4z^2}}\}.$$  \hspace{1cm} (10)

This shows the existence of RE in the region $R_1(\Gamma, z)$, shown in figure (2).

3.2. **Proof of theorem 2.(b)**

Let $m_1 = m_2 = m$ and $m_4 = m_3 = M$ are the two pairs of masses at vertices of a square, such that $m_1, m_2$ are at the opposite ends of one diagonal and $m_3, m_4$ are at the opposite ends of the other diagonal, where $m_5$ is at the north pole $(0, 0, 1)$. Using the configurations given by (4) and substituting these coordinates and expressions (5) into the equations of motion (3) corresponding to $x_1$ and $x_2$, we get

$$\alpha^2 = \frac{m}{4r^3(1-r^2)^{3/2}} + \frac{2M}{r^3(2-r^2)^{3/2}} \pm \frac{m_5}{r^3(1-r^2)^{1/2}}.$$

The equation corresponding to $x_3$ and $x_4$ yield

$$\alpha^2 = \frac{M}{4r^3(1-r^2)^{3/2}} + \frac{2m}{r^3(2-r^2)^{3/2}} \pm \frac{m_5}{r^3(1-r^2)^{1/2}}.$$
To have a relative equilibria, we must have

$$\frac{m}{4r^3(1-r^2)^{3/2}} + \frac{2M}{r^3(2-r^2)^{3/2}} = \frac{M}{4r^3(1-r^2)^{3/2}} + \frac{2m}{r^3(2-r^2)^{3/2}}.$$ 

This is equivalent to

$$\frac{m(\delta - 8\gamma)}{4\gamma \delta} = \frac{M(\delta - 8\gamma)}{4\gamma \delta},$$

where $\gamma = (1-r^2)^{3/2}$ and $\delta = (2-r^2)^{3/2}$. This equation can be satisfied only if $m = M$. Consequently, the solution for the square pyramidal problem in $S^2$ must have equal masses.

3.3. Proof of theorem 2.(c)

Suppose that the masses $m_1$ and $m_2$ are very small and do not influence the motion of other particles, but are affected by them. More precisely, we consider the configuration for the square, such that the negligible masses $m_1 = m_2 \approx 0$ are at the opposite ends of one diagonal and $m_3, m_4 = M$ are at the opposite ends of the other diagonal, where $m_5$ fixed at the north pole $(0, 0, 1)$. Substituting into the equations of motion (3) corresponding to $x$ coordinates we obtain

$$\ddot{x}_1 = -\frac{2MBr \cos \alpha t}{r^3(2-r^2)^{3/2}} \pm \frac{m_5Cr \cos \alpha t}{r^3} - r^3 \alpha^2 \cos \alpha t,$$

$$\ddot{x}_2 = -\frac{2MBr \cos \alpha t}{r^3(2-r^2)^{3/2}} \pm \frac{m_5Cr \cos \alpha t}{r^3} + r^3 \alpha^2 \cos \alpha t,$$

$$\ddot{x}_3 = \frac{MBr \sin \alpha t}{4r^3 B^{3/2}} \pm \frac{m_5Cr \sin \alpha t}{r^3} + r^3 \alpha^2 \sin \alpha t,$$
\[ \ddot{x}_4 = -\frac{MBr \sin \alpha t}{4r^3B^{3/2}} \pm \frac{m_5Cr \sin \alpha t}{r^3} - r^3 \alpha^2 \sin \alpha t, \]
\[ \ddot{x}_5 = 0. \]

From equations corresponding to \( m_1 \) and \( m_2 \), we obtain
\[ \alpha^2 = \frac{2M}{r^3(2 - r^2)^{3/2}} + \frac{m_5}{r^3(1 - r^2)^{1/2}} \]  \hspace{1cm} (11)

Similarly, equations of motion corresponding to \( m_3 \) and \( m_4 \), lead us to
\[ \alpha^2 = \frac{M}{4r^3(1 - r^2)^{3/2}} + \frac{m_5}{r^3(1 - r^2)^{1/2}} \] \hspace{1cm} (12)

For relative equilibria to exist the right hand side of equations (11) and (12) must be equal which implies that
\[ \frac{2M}{r^3(2 - r^2)^{3/2}} = \frac{M}{4r^3(1 - r^2)^{3/2}}. \]

Notice that we arrived at the condition for the existence of a relative equilibria for the restricted 4-body problem with 2 equal masses and 2 negligible masses. Thus the square pyramidal body problem of 2 equal masses and 2 negligible masses reduced to 4-body problem with two equal masses and 2 negligible masses. This is due to the force on the body at the north pole being cancel out. In fact, we treat the square pyramid of 2 negligible masses as the 4-body problem of 2 negligible masses. Hence, we recovered the same result studied in [5], in \( S^2 \), there is a circle of radius \( r = \sqrt{2/3}r^{-1/2} \), parallel with the equator, such that a square configuration inscribed in this circle, with opposite vertices having the same mass, forms a relative equilibrium. But here we restrict our study to the unit sphere of curvature 1.

**Remark 1.** Unlike the square pyramid problem with equal masses in \( S^2 \), the angular velocity \( \alpha \) depends only on \( M \) not \( m_5 \).

4. **The negatively curved 5-body problem with one mass at the vertex and four masses form a square in \( \mathbb{H}^2 \)**

In this section we study the dynamics of the 5-bodies that interact in \( \mathbb{H}^2 \) and will investigate the existence of elliptic relative equilibria of the square pyramidal configuration which has four masses at the vertices of a square and one mass at the vertex of \( \mathbb{H}^2 \).
Theorem 3. Consider the negatively curved square 5-body problem in $\mathbb{H}^2$ with four masses $m_i = m, i = 1, 2, 3, 4$ at the vertices of a square and the fifth mass fixed at the vertex of $\mathbb{H}^2$.

a. When $m_i = m, i = 1, 2, 3, 4$ relative equilibria exists for all positive values of $m$, $m_5$ and $r$ (or $z > 1$).

b. The negatively curved square pyramidal 5-body problem with two pairs of equal mass in $\mathbb{H}^2$ has no elliptic relative equilibria.

4.1. Proof of theorem 3.(a)

When $\kappa = -1$, the equations of motion are given by

$$\ddot{q}_i = \sum_{j=1, j\neq i}^N \frac{m_j [q_j + (q_i \cdot q_j)q_i]}{[1 + (q_i \cdot q_j)^2]^{3/2}} + (q_i \cdot \dot{q}_i)q_i, \quad i = 1, 2, 3, 4, 5. \tag{13}$$

Then, we search for solutions of the form

\[
\begin{align*}
x_1 &= r \cos \alpha t, \\
y_1 &= r \sin \alpha t, \\
z_1 &= \sqrt{1 + r^2} \\
x_2 &= -r \cos \alpha t, \\
y_2 &= -r \sin \alpha t, \\
z_2 &= \sqrt{1 + r^2} \\
x_3 &= -r \sin \alpha t, \\
y_3 &= r \cos \alpha t, \\
z_3 &= \sqrt{1 + r^2} \\
x_4 &= r \sin \alpha t, \\
y_4 &= -r \cos \alpha t, \\
z_4 &= \sqrt{1 + r^2} \\
x_5 &= 0, \\
y_5 &= 0, \\
z_5 &= 1
\end{align*}
\tag{14}
\]

Consider the Lorentz inner product between particles

\[
\begin{align*}
q_1 \Box q_2 &= q_3 \Box q_4 = -(1 + 2r^2), \\
q_i \Box q_5 &= -\sqrt{1 + r^2}, \quad i = 1, 2, 3, 4 \\
q_1 \Box q_3 &= q_1 \Box q_4 = q_2 \Box q_3 = q_2 \Box q_4 = -(1 + r^2)
\end{align*}
\tag{15}
\]

Substituting these coordinates and expressions into the equations of motion (13) corresponding to the $x_i$ coordinate, we get

$$\ddot{x}_i = R x_i, \quad \ddot{x}_5 = 0, \quad i = 1, 2, 3, 4 \tag{16}$$

where

$$R = -\frac{m}{4r^3 \sqrt{1 + r^2}} - \frac{2m(1 + r^2)}{r^3 (r^2 + 2)^{3/2}} - \frac{m_5 \sqrt{1 + r^2}}{r^3} + r^2 \alpha^2. \quad \text{Comparing the above equations to obtain conditions for the existence of relative equilibria, we obtain}
\]

$$\alpha^2 = \frac{m}{4r^3 (1 + r^2)^{3/2}} + \frac{2m}{r^3 (2 + r^2)^{3/2}} + \frac{m_5}{r^3 (1 + r^2)^{1/2}} =: f(r). \tag{17}$$

This shows the dependence of angular velocity of the system on the four equal masses, $m$, the mass of the body at $(0, 0, 1)$ and $r$. Thus for any $m$, $m_5$, and $r$ in $\mathbb{H}^2$, there are always two values of $\alpha$ satisfying the above equation, one corresponding to each direction of rotation.
Let \( \Gamma = \frac{m_5}{m} \), where \( r > 0 \) and \( z^2 = r^2 + 1 \) then equation (14) can be written as
\[
\frac{\alpha^2}{m} = \frac{1}{4z^3(z^2 - 1)^{3/2}} + \frac{2}{(z^2 - 1)^{3/2}} + \frac{\Gamma}{z(z^2 - 1)^{3/2}} =: f(z)
\]
The function \( f(z) \) is positive for \( z > 1 \). Thus, for \( m, \Gamma > 0 \) and \( z > 1 \), there are positive and negative relative equilibria \( \alpha \) that lead to an elliptic relative equilibrium. This completes the proof of Theorem 3(a).

4.2. Proof of theorem 3.(b)
Consider two pairs of masses \( m_1 = m_2 = m \) and \( m_3 = m_4 = M \), at opposite ends of the two diagonals of a square inscribed in a circle and the fifth mass \( m_5 \) at \((0, 0, 1)\).

We use the configurations given by (14) and substitute these coordinates into the equations of motion (2) with \( \kappa = -1 \). Then, for relative equilibria to exist, we must have
\[
\frac{m}{4r^3(1 + r^2)^{3/2}} + \frac{2M}{r^3(2 + r^2)^{3/2}} = \frac{M}{4r^3(1 + r^2)^{3/2}} + \frac{2m}{r^3(2 + r^2)^{3/2}}.
\]
This is equivalent to
\[
\frac{(m - M)}{4(1 + r^2)^{3/2}} = \frac{2(m - M)}{(2 + r^2)^{3/2}}
\]
This equation has no real solution for \( r \). Hence the negatively curved square pyramid 5-body problem with two pairs of masses has no relative equilibria.

5. Conclusions
We model a five-body problem which has four masses inscribed in a circle and the fifth mass \( (M) \) is at the north pole of the sphere. The radius of the circle is taken to be 1. The model is investigated for existence of relative equilibria in \( \mathbb{H}^2 \), and \( \mathbb{S}^2 \). We prove the existence for relative equilibria when the four masses at the vertices of the square are equal and \( z > 0 \) and derive a region of existence for relative equilibria in \( \mathbb{S}^2 \) when \( z < 0 \). It is shown that in the restricted case, the mass at the north pole has no effect on the existence of relative equilibria. It is also shown that the negatively curved square pyramidal 5-body problem with two pairs of equal masses in \( \mathbb{H}^2 \) has no elliptic relative equilibria.

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