ABSTRACT

Next step is reported in the program of Racah matrices extraction from the differential expansion of HOMFLY polynomials for twist knots: from the double-column rectangular representations $R = [rr]$ to a triple-column and triple-hook $R = [333]$. The main new phenomenon is the deviation of the particular coefficient $f_{[333]}^{[21]}$ from the corresponding skew dimension, what opens a way to further generalizations.

1 Introduction

Calculation of Racah matrices is the long-standing, difficult and challenging problem in theoretical physics [1]. It is further obscured by the basis-dependence of the answer in the case of generic representations, but this "multiplicity problem" is absent in the case of rectangular representations. The modern way [2]–[7] to evaluate the most important "exclusive" matrices $\bar{S}^{R}_{\mu\nu}$:

$$(R \otimes \bar{R}) \otimes R \rightarrow R$$

is based on the combination of two very different expressions for $R$-colored HOMFLY polynomials [8] of the double-braid knots, coming one from the arborescent calculus of [9] and another from the differential expansion theory [10]–[12] in the case of rectangular $R = [rr]$ with $s$ columns of length $r$:

$$H^{(m,n)}_{R} = \sum_{\mu,\nu \subset R} \frac{\sqrt{D_{\mu}D_{\nu}}}{d_{R}} \bar{S}^{R}_{\mu\nu} \Lambda^{m}_{\mu} \Lambda^{n}_{\nu} = \sum_{\lambda \subset R} \chi^{(r)}_{\lambda}(r) \chi^{(s)}_{\lambda}(s) \cdot (q)^{2|\lambda|} \cdot \frac{\chi^{(N+r)}_{\lambda}(N+r)\chi^{(N-s)}_{\lambda}(N-s)}{\chi^{(N)}_{\lambda}^{2}} \cdot \sum_{\mu,\nu \in \lambda} f_{\lambda}^{\mu} f_{\lambda}^{\nu} \cdot \Lambda^{m}_{\mu} \Lambda^{n}_{\nu}.$$ (1)
Here the sums go over sub-diagrams of the Young diagram \( R \), and \( \chi^*(N) \) denote the corresponding dimensions for the algebra \( sl_N \), i.e. the values of Schur functions \( \chi \{ p_k \} \) at the topological locus \( p_k = p_k^N = \frac{(A^k)^{\lambda}}{(q^k)!} \) with \( \{ x \} = x - x^{-1} \) and \( A = q^N \). The other ingredients of the formula come from the evolution method \[11\] applied to the family of twist knots (double braids with \( n = 1 \)): as functions of the "eigenvalue" \( \Lambda_\mu \) of the \( R \)-matrix, a \( q \)-power of the Casimir or cut-and-join operator \[13\]. In arborescent calculus the weights are made from the elements of Racah matrix \( \bar{S} \) while in the theory of differential expansions they are composed into amusing generating functions \[11\] with \( f_\lambda^0 = 1 \). Each term in the sum has a non-trivial denominator, however the full sum is a Laurent polynomial in \( A \) and \( q \) for all \( m \). Moreover, it vanishes for \( m = 0 \) (unknot), equals one for \( m = -1 \) (figure eight knot \( 4_1 \)) and is a monomial at \( m = 1 \) (trefoil). According to \[3\] and \[5\] the \( F \)-functions are best described in a peculiar hook parametrization of Young diagrams:

In particular,

\[
\Lambda_\mu = \Lambda_{(i_1,j_1|i_2,j_2|...)} = \prod_{k=1}^{\infty} \left( A \cdot q^{i_k-j_k} \right)^{2(i_k+j_k+1)} \tag{3}
\]

the overall coefficients

\[
c_\lambda = c_{(a_1,b_1|a_2,b_2|...)} = \prod_{k=1}^{\infty} \left( A \cdot q^{a_k-b_k} \right)^{(a_k+b_k+1)} \tag{4}
\]

and

\[
F_\lambda^{(-1)} = 1, \quad F_\lambda^{(0)} = \delta_{\lambda,\emptyset}, \quad F_\lambda^{(1)} = (-)^{\sum (a_k+b_k+1)} c_\lambda^2 \tag{5}
\]

Clearly, \( c_\lambda \) drops away from the r.h.s. of \[1\].

The shape of the coefficients \( f_\lambda^k \) strongly depends on the number of hooks in \( \lambda \) and \( \mu \). Currently they are fully known for \( \lambda = (a_1,b_1|a_2,0) \) – what is enough to get the Racah matrices \( \bar{S} \) for the case \( R = [r,r] \) (actually, for this purpose \( b_1 = 0, 1 \) is sufficient).

- As already mentioned, for the empty diagram \( \mu \) always

\[
f_\lambda^0 = 1 \tag{6}
\]

- For the single-hook \( \lambda \) and thus single-hook \( \mu \subset \lambda \) expressions are still relatively simple and fully factorized:

\[
f_{(i,j)}^{(i,j)} = g_{(a,b)}^{(i,j)} \cdot K_{(a,b)}^{(i,j)} = (-)^{i+j+1} \cdot \frac{[a]!}{[a-i]! [i]!} \cdot \frac{[b]!}{[b-j]! [j]!} \cdot \frac{[a+b+1]}{[i+j+1]} \cdot \frac{D_0! D_i!}{D_{a+i+1}! D_{b+j+1}!} \cdot \frac{\tilde{D}_i! \tilde{D}_i!}{\tilde{D}_b! \tilde{D}_{b+j+1}! \tilde{D}_{b-j+1}!} \tag{7}
\]

with

\[
g_{(i,j)}^{(i,j)} = (-)^{i+j+1} \cdot \frac{D_{2i+1} \tilde{D}_{2j+1}}{D_0 D_{i-j}} \cdot \frac{(D_0!)^2}{D_{a+i+1}! D_{a-i-1}!} \cdot \frac{(\tilde{D}_b!)^2}{\tilde{D}_{b+j+1}! \tilde{D}_{b-j+1}!} \tag{8}
\]

and

\[
K_{(a,b)}^{(i,j)}(N) = \frac{\chi_{a/b}^* (N) \chi_{a}^*(N)}{\chi_{a}^*(N)} \tag{9}
\]
This $K$ involves skew characters, defined by

$$\sum_{\mu \subseteq \lambda} \chi_{\lambda/\mu}(p_k') \cdot \chi_{\mu^{tr}}(p_k'') = \chi_{\lambda}(p_k' + p_k'')$$

(10)

and satisfying the sum rule

$$\sum_{\mu \subseteq \lambda} (-)^{|\mu|} \cdot \chi_{\lambda/\mu} \cdot \chi_{\mu^{tr}} = \delta_{\lambda,\emptyset}$$

(11)

which follows from (10) and the transposition law $\chi_{\mu}(-p_k) = (-)^{|\mu|} \chi_{\mu^{tr}}(p_k)$, and can be considered as a prototype of $\chi^{q}$. The other notation in (7) and (8) is:

$$D_a = \{ Aq^a \} = \{ q \} \cdot [N + a], \quad \hat{D}_b = \{ A/q^b \} = \{ q \} \cdot [N - b]$$

(12)

and

$$D_a! = \prod_{k=0}^{a} D_k = (q)^{a+1} \cdot \frac{[N + a]!}{[N - 1]!}, \quad D_b! = \prod_{k=0}^{b} D_k = (q)^{b+1} \cdot \frac{[N]!}{[N - b - 1]!}$$

(13)

(note that these products start from $k = 0$ and include respectively $a + 1$ and $b + 1$ factors).

- For two-hook $\lambda = (a_1, b_1 | a_2, b_2)$ the formulas are far more involved, and they are different for different number of hooks in $\mu$:

$$f(i_1, j_1)_{(a_1, b_1 | a_2, b_2)} = f^{(a_1, b_1)}_{(i_1, j_1)} \cdot \xi^{(i_1, j_1)}_{(a_1, b_1 | a_2, b_2)} = g^{(i_1, j_1)}_{(a_1, b_1)} \cdot K^{(i_1, j_1)}_{(a_1, b_1)}(N) \cdot \xi^{(i_1, j_1)}_{(a_1, b_1 | a_2, b_2)}$$

(14)

$$f^{(i_1, j_1)}_{(a_1, b_1 | a_2, b_2)} = \frac{[N + i_1 + i_2 + 1][N - j_1 - j_2 - 1]}{[N + i_1 - j_2][N + i_2 - j_1]} \cdot g^{(i_1, j_1)}_{(a_1, b_1)} g^{(i_2, j_2)}_{(a_2, b_2)} \cdot K^{(i_1, j_1)}_{(a_1, b_1)}(N) K^{(i_2, j_2)}_{(a_2, b_2)}(N) \xi^{(i_1, j_1)}_{(a_1, b_1 | a_2, b_2)}$$

(15)

Non-trivial are the correction factors, true for $a_2 \cdot b_2 = 0$:

$$\xi^{(i_1, j_1)}_{(a_1, b_1 | a_2, b_2)} = \frac{[N + a_2 - j_1][N - b_2 + i_1]}{[N + a_2 + i_1 + 1][N - b_2 - j_1 - 1]} \cdot K^{(i_1, j_1)}_{(a_1, b_1 | a_2, b_2)}(N + i_1 - j_1) \cdot \delta_{i_1, j_1} +$$

$$+ K^{(i_1, j_1)}_{(a_1, b_1 | a_2, b_2)}(N + (i_1 + 1) \delta_{b_2} - (j_1 + 1) \delta_{a_2}) \cdot (1 - \delta_{i_1, j_1})$$

(16)

and

$$\xi^{(i_1, j_1)}_{(i_2, j_2)}_{(a_1, b_1 | a_2, b_2)} = K^{(i_1, j_1)}_{(a_1, b_1 | a_2, b_2)}(N + (i_1 + i_2 + 2) \cdot \delta_{b_2} - (j_1 + j_2 + 2) \cdot \delta_{a_2})$$

(17)

where $\delta_x = \{ 1 \text{ for } x = 0 \text{ and } 0 \text{ for } x \neq 1 \}$

and

$$K^{(i_1, j_1)}_{(a_1, b_1 | a_2, b_2)}(N) = \frac{K^{(i_1, j_1)}_{(a_1, b_1)}(N)}{K^{(i_1, j_1)}_{(a_1, b_1 | a_2, b_2)}}$$

(18)

$$K^{(i_1, j_1)}_{(a_1, b_1 | a_2, b_2)}(N) = \frac{K^{(i_1, j_1)}_{(a_1, b_1 | a_2, b_2)}(N)}{K^{(i_1, j_1)}_{(a_1, b_1)}(N) \cdot K^{(i_2, j_2)}_{(a_2, b_2)}(N)}$$

(19)

Thus corrections involve a natural modification of $K$-factors and somewhat strange shifts of the argument $N$, i.e. multiplicative shift of $A$ by powers of $q$. These formulas were found in [3][15] for the case when $a_2 \cdot b_2 = 0$ (i.e. when either $b_2 = 0$ or $a_2 = 0$). Sufficient for all the simplest non-symmetric rectangular representations $R = [r, r]$ and $R = [2']$ are respectively $b_2 = 0$ and $a_2 = 0$. Note that underlined expression are the arguments of $K$-functions – *not* additional algebraic factors. Boxes contain projectors on sectors with particular values of $i_1$ and $j_1$.

Our goal in this paper is to make the first step towards lifting the restriction $a_2 \cdot b_2 = 0$. Namely, we consider the case of the simplest 3-hook $R = [333]$, which has 20 Young sub-diagrams, of which there are two, $\lambda = [332] = (2, 2|1, 1)$ and $\lambda = [333] = (2, 2|1, 1|0, 0)$ with $a_2 \cdot b_2 \neq 0$. 
The new function \( F_{(22|11)}^{(m)} = F_{(332)}^{(m)} \)

The diagram \([332] = (22|11)\) is still two-hook, but both \(a_2 = b_2 = 1\) are non-vanishing. If we apply just the same formulas \([13], [19]\) in this case, the answer will be non-polynomial. However, one can introduce additional correction factors \(\eta^\mu\) for all the items in the sum over \(\mu\) and adjust them to cancel all the singularities. Of 19 factors non-trivial (different from unity) are just 8 (we omit the subscript \(\lambda = (22|11)\) to simplify the formulas):

\[
\eta^0 = \eta^{(00)} = \eta^{(01)} = \eta^{(10)} = \eta^{(02)} = \eta^{(20)} = \eta^{(22)} = 1
\]

\[
\eta^{(11)} = \frac{[4]^3 D_1 D_{-1}}{[5]^2 D_2 D_{-2}} \cdot \frac{K_{(11)}^{(11)}(N)}{K_{(22|11)}^{(11)}(N)}, \quad \eta^{(12)} = \frac{K_{(12)}^{(10)}(N+2)}{K_{(22|11)}^{(11)}(N)}, \quad \eta^{(21)} = \frac{K_{(21)}^{(21)}(N-2)}{K_{(22|11)}^{(11)}(N)}
\]

\[
\eta^{(11)(00)} = \frac{D_2 D_0 D_{-2}}{D_3 D_1 D_{-1} D_{-3}}, \quad \eta^{(12)(00)} = \frac{D_2 D_0 D_{-2}}{D_3 D_{-1}}, \quad \eta^{(21)(00)} = \frac{D_0 D_{-2}}{D_3 D_{-1}}
\]

\[
\eta^{(22)(00)} = \eta^{(22)(10)} = \eta^{(22)(11)} = 1 \quad (20)
\]

and the resulting expression is

\[
A^{-8} F_{(22|11)}^{(m)} = \frac{1}{D_2 D_0 D_0 D_1 D_{-2} D_{-3} D_{-2} D_{-3} D_{-3}} - \frac{[4][2]}{[5][3]} \cdot \Lambda_{m}^{(00)} + \frac{[4][2]}{[3]} \cdot \left( A (q^4 + 2q^2 + 2 + q^{-2} + q^{-4}) \right) \cdot \Lambda_{m}^{(01)} + \frac{[4][2]}{[3]} \cdot \left( A (q^4 + 2q^2 + 2 + q^{-2} + q^{-4}) \right) \cdot \Lambda_{m}^{(10)} - \frac{[5][4]}{[3]} \cdot \Lambda_{m}^{(02)} - \frac{[5][4]}{[3]} \cdot \Lambda_{m}^{(20)} - \frac{[4][2]}{[3]} \cdot \Lambda_{m}^{(11)} + \frac{[5]}{[3]} \cdot \left( A (q^4 + 2q^2 + 2 + q^{-2} + q^{-4}) \right) \cdot \Lambda_{m}^{(12)} + \frac{[5]}{[3]} \cdot \left( A (q^2 + 2 + q^{-2} + q^{-4}) \right) \cdot \Lambda_{m}^{(21)} - \frac{[4]^2}{[5][4]} \cdot \Lambda_{m}^{(22)} + \frac{[5][4]}{[2]} \cdot \Lambda_{m}^{(11)(00)} - \frac{[5][4]}{[2]} \cdot \Lambda_{m}^{(12)(00)} - \frac{[5][4]}{[2]} \cdot \Lambda_{m}^{(21)(00)} + \frac{[4][2]}{[3]} \cdot \Lambda_{m}^{(12)(10)} + \frac{[4][2]}{[3]} \cdot \Lambda_{m}^{(21)(10)} - \frac{[4][2]}{[3]} \cdot \Lambda_{m}^{(22)(00)} + \frac{[4][2]}{[3]} \cdot \Lambda_{m}^{(22)(10)} + \frac{[4][2]}{[3]} \cdot \Lambda_{m}^{(22)(11)}
\]

It nicely satisfies the sum rules \([3]\).

3 Extension to \(F_{(a_1 b_1|11)}\)

We can now develop the success with \(F_{(22|11)}\) and extend it to other 2-hook diagrams with \(a_2 \cdot b_2 \neq 0\). We actually restrict our attention to the case of \(a_2 \cdot b_2 = 1\), i.e. \(a_2 = b_2 = 1\).
In the next case of \( F_{(33|11)} \) the correction factors are (again we write just \( \eta^\mu \) instead of \( \eta^\mu_{(33|11)} \)):

\[
\begin{align*}
\eta^{0} &= \eta^{(00)} = \eta^{(01)} = \eta^{(10)} = \eta^{(02)} = \eta^{(20)} = \eta^{(30)} = \eta^{(22)} = \eta^{(03)} = \eta^{(32)} = \eta^{(23)} = \eta^{(33)} = 1 \\
\eta^{(11)} &= \frac{u_{(33)}}{K^{(11)}_{(33|11)}(N)} \\
\eta^{(12)} &= \frac{K^{(12)}_{(33|11)}(N+2)}{K^{(12)}_{(33|11)}(N)} \\
\eta^{(13)} &= \frac{K^{(13)}_{(33|11)}(N+2)}{K^{(13)}_{(33|11)}(N)} \\
\eta^{(21)} &= \frac{K^{(21)}_{(33|11)}(N-2)}{K^{(21)}_{(33|11)}(N)} \\
\eta^{(31)} &= \frac{K^{(31)}_{(33|11)}(N-2)}{K^{(31)}_{(33|11)}(N)} \\
\eta^{(11)(00)} &= \frac{D_2 D_0}{D_3 D_1} \\
\eta^{(12)(00)} &= \frac{D_2 D_0}{D_3 D_1} \\
\eta^{(13)(00)} &= \frac{D_2 D_0}{D_3 D_1} \\
\eta^{(21)(00)} &= \frac{D_2 D_0}{D_3 D_1} \\
\eta^{(21)(10)} &= \frac{D_2 D_0}{D_3 D_1} \\
\eta^{(21)(00)} &= \frac{D_2 D_0}{D_3 D_1} \\
\eta^{(22)(00)} &= \frac{D_2 D_0}{D_3 D_1} \\
\eta^{(22)(10)} &= \frac{D_2 D_0}{D_3 D_1} \\
\eta^{(23)(00)} &= \frac{D_2 D_0}{D_3 D_1} \\
\eta^{(23)(10)} &= \frac{D_2 D_0}{D_3 D_1} \\
\eta^{(32)(00)} &= \frac{D_2 D_0}{D_3 D_1} \\
\eta^{(32)(10)} &= \frac{D_2 D_0}{D_3 D_1} \\
\eta^{(33)(00)} &= \frac{D_2 D_0}{D_3 D_1} \\
\eta^{(33)(10)} &= \frac{D_2 D_0}{D_3 D_1} \\
\eta^{(33)(11)} &= \frac{D_2 D_0}{D_3 D_1}
\end{align*}
\]

This implies a simple extension of (16) and (17) to arbitrary diagrams \((a_1, b_1|1, 1)\), i.e. true for \( a_2 \cdot b_2 = 0, 1 \) are:

\[
\xi^{(i_1, j_1)}_{(a_1, b_1|a_2, b_2)} = \frac{[N + a_2 - j_1][N - b_2 + i_1]}{[N + a_2 + i_1 + 1][N - b_2 - j_1 - 1]} \cdot K^{(i_1, j_1)}_{(a_1, b_1|a_2, b_2)}(N + i_1 - j_1) \cdot \delta_{i_1, j_1} + \frac{K^{(i_1, j_1)}_{(a_1, b_1|a_2, b_2)}(N + i_1 - j_1)}{K^{(11)}_{(a_1, b_1|a_2, b_2)}}(1 - \delta_{a_2, b_2}) \cdot \delta_{i_1, j_1 - 1}
\]

with

\[
u_{(a_1, b_1|1, 1)} = \left( K^{(1, 1)}_{(a_1, b_1|a_2, b_2)} - \frac{[a_2 + 2][b_1 + 1]}{[a_1][b_1]} \cdot \frac{[3][2^2\{q^2\}^2]}{D_0} \right) \cdot \frac{D_1 D_2 D_3 D_1}{D_0 D_2 D_3}
\]

and

\[
\xi^{(i_1, j_1|a_2, b_2)}_{(a_1, b_1|a_2, b_2)} = K^{(i_1, j_1)}_{(a_1, b_1|a_2, b_2)}(N + (i_1 + i_2 + 2) \cdot \delta_{b_2} - (j_1 + j_2 + 2) \cdot \delta_{a_2}) \cdot \left( \frac{D_2 D_0}{D_3 D_1} \right)^{\delta_{i_1 - 1, 1} - \delta_{a_1, b_1}} \cdot \left( \frac{D_2 D_0}{D_3 D_1} \right)^{\delta_{i_1 - 1, -1} - \delta_{a_2, b_2}}
\]

Formula (23) means that the coefficient \( \xi^{(11)}_{(1)} \) is no longer proportional to the skew character \( \chi^{a|11}_{\lambda} \). Interpretation of this deviation remains to be found. Note that for \( a_2 \cdot b_2 = 0 \) we have just

\[
\xi^{(i_1, j_1|a_2, b_2)}_{(a_1, b_1|a_2, b_2)} = K^{(1, 1)}_{(a_1, b_1|a_2, b_2)} \quad \text{for} \quad a_2 \cdot b_2 = 0
\]

instead of (23) – as one more manifestation of discontinuity of the formulas, expressed in terms of hook variables.

4 The new function \( F_{(33|33)}^{(m)} = F_{(22|11|00)}^{(m)} \)

This \( F \)-factor is the first, associated with the triple-hook diagram \( \lambda \). To get an explicit formula we impose the polynomiality requirement on the correction factors \( \eta^\mu_{(22|11|00)} \) to the naive analogue of (14)–(19) for 3-hook diagrams:

\[
\begin{align*}
f^{(i_1, j_1)}_{(a_1, b_1|a_2, b_2|a_3, b_3)} &= f^{(a_1, b_1)}_{(i_1, j_1)} \cdot g^{(i_1, j_1)}_{(a_1, b_1|a_2, b_2)} = g^{(i_1, j_1)}_{(a_1, b_1)} \cdot K^{(i_1, j_1)}_{(a_1, b_1)}(N) \cdot \xi^{(i_1, j_1)}_{(a_1, b_1|a_2, b_2|a_3, b_3)} \\
f^{(i_1, j_1|a_2, b_2)}_{(a_1, b_1|a_2, b_2|a_3, b_3)} &= \frac{[N + i_1 + i_2 + 1][N - j_1 - j_2 - 1]}{[N + i_1 - j_2][N + i_2 - j_1]} \cdot g^{(i_1, j_1)}_{(a_1, b_1)} g^{(i_2, j_2)}_{(a_2, b_2)} \cdot K^{(i_1, j_1)}_{(a_1, b_1)}(N) K^{(i_2, j_2)}_{(a_2, b_2)}(N) \cdot \xi^{(i_1, j_1|a_2, b_2)}_{(a_1, b_1|a_2, b_2|a_3, b_3)}
\end{align*}
\]

5
\[
\begin{align*}
\tilde{f}(i_1,j_1|i_2,j_2|i_3,j_3) &= \frac{[N + i_1 + i_2 + 1][N - j_1 - j_2 - 1]}{[N + i_1 - j_2][N + i_2 - j_1]} \cdot \frac{[N + i_1 + i_3 + 1][N - j_1 - j_3 - 1]}{[N + i_1 - j_3][N + i_3 - j_1]} \cdot \frac{[N + i_2 + i_3 + 1][N - j_2 - j_3 - 1]}{[N + i_2 - j_3][N + i_3 - j_2]} \cdot g_{(i_1,j_1)}(D_{(i_2,j_2)} g_{(i_3,j_3)} - K_{(i_1,j_1)}(N) K_{(i_2,j_2)}(N) K_{(i_3,j_3)}(N)) \cdot \xi_{(i_1,j_1,i_2,j_2,i_3,j_3)}^{(i_1,j_1)} \cdot \xi_{(i_1,j_1,i_2,j_2,i_3,j_3)}^{(i_1,j_1)} \cdot \xi_{(i_1,j_1,i_2,j_2,i_3,j_3)}^{(i_1,j_1)},
\end{align*}
\]

(25)

In the first approximation the correction factors in the 3-hook case are (they are never literally true, before \(\eta\)-factors are introduced):

\[
\begin{align*}
\xi_{(i_1,j_1,i_2,j_2)}^{(i_1,j_1)} &= K_{(a_1,b_1|[a_2,b_2],[a_3,b_3])}^{(i_1,j_1)}(N) \cdot (1 - \delta_{i_1,j_1}) + \\
&+ \frac{[N + a_2 - j_1][N - b_2 + i_1]}{[N + a_2 + i_1 + 1][N - b_2 - j_1 - 1]} \cdot \frac{[N + a_3 - j_1][N - b_3 + i_1]}{[N + a_3 + i_1 + 1][N - b_3 - j_1 - 1]} \cdot K_{(a_1,b_1|[a_2,b_2],[a_3,b_3])}^{(i_1,j_1)}(N + i_1 - j_1) \cdot \delta_{i_1,j_1}.
\end{align*}
\]

(26)

(note that \(a_2 > 0\) and \(b_2 > 0\) for 3-hook diagrams thus the shifts like \(N \rightarrow N + (i_1 + 1)\delta_{b_2} - (j_1 + 1)\delta_{a_2}\) do not matter) and

\[
\xi_{(i_1,j_1,i_2,j_2,i_3,j_3)}^{(i_1,j_1)} = K_{(a_1,b_1|[a_2,b_2],[a_3,b_3])}^{(i_1,j_1,i_2,j_2,i_3,j_3)}(N + (i_1 + i_2 + i_3 + 3) \cdot \delta_{a_1} - (j_1 + j_2 + j_3 + 3) \cdot \delta_{a_2})
\]

with

\[
\begin{align*}
K_{(a_1,b_1|[a_2,b_2],[a_3,b_3])}^{(i_1,j_1)}(N) &= \frac{K_{(a_1,b_1|[a_2,b_2],[a_3,b_3])}^{(i_1,j_1)}(N)}{K_{(a_1,b_1)}^{(i_1,j_1)}(N)} \\
K_{(a_1,b_1|[a_2,b_2],[a_3,b_3])}^{(i_1,j_1|i_2,j_2)}(N) &= \frac{K_{(a_1,b_1|[a_2,b_2],[a_3,b_3])}^{(i_1,j_1|i_2,j_2)}(N)}{K_{(a_1,b_1)}^{(i_1,j_1)}(N) \cdot K_{(a_2,b_2)}^{(i_2,j_2)}(N)} \\
K_{(a_1,b_1|[a_2,b_2],[a_3,b_3])}^{(i_1,j_1|i_2,j_2,i_3,j_3)}(N) &= \frac{K_{(a_1,b_1|[a_2,b_2],[a_3,b_3])}^{(i_1,j_1|i_2,j_2,i_3,j_3)}(N)}{K_{(a_1,b_1)}^{(i_1,j_1)}(N) \cdot K_{(a_2,b_2)}^{(i_2,j_2)}(N) \cdot K_{(a_3,b_3)}^{(i_3,j_3)}(N)}
\end{align*}
\]

(27)

Correction factors \(\eta^\mu_{(22|11|00)}\) appear to be

\[
\eta^{(11)} = \frac{D_1 D_3 D_1 D_3}{D_2 D_2 D_2}, \quad \eta^{(12)} = \frac{D_3^2 D_1}{D_2 D_2} = \frac{K_{(22|11|00)}^{(12)}(N + 2)}{K_{(22|11|00)}^{(12)}(N)}, \quad \eta^{(21)} = \frac{D_2 D_1}{D_2 D_2} = \frac{K_{(22|11|00)}^{(21)}(N - 2)}{K_{(22|11|00)}^{(21)}(N)}
\]

(28)

\[
\begin{align*}
\eta^{(00)} &= \eta^{(01)} = \eta^{(10)} = \eta^{(20)} = \eta^{(02)} = \eta^{(22)} = 1 \\
\eta^{(11)} &= \frac{D_1^4 D_1 D_1 D_1}{D_3 D_3 D_3 D_3}, \\
\eta^{(12)} &= \frac{D_3 D_3 D_3 D_3}{D_2 D_1 D_1 D_1} = \frac{K_{(22|11|00)}^{(12)}(N + 1)}{K_{(22|11|00)}^{(12)}(N)} \cdot D_2^2 \frac{D_3 D_3}{D_3 D_3} \\
\eta^{(21)} &= \frac{D_3 D_3 D_3 D_3}{D_2 D_1 D_1 D_1} = \frac{K_{(22|11|00)}^{(21)}(N - 1)}{K_{(22|11|00)}^{(21)}(N)} \cdot D_2^2 \frac{D_3 D_3}{D_3 D_3} \\
\eta^{(12)} &= \frac{D_3 D_3 D_3 D_3}{D_2 D_1 D_1 D_1} = \frac{K_{(22|11|00)}^{(12)}(N + 1)}{K_{(22|11|00)}^{(12)}(N)} \cdot D_2^2 \frac{D_3 D_3}{D_3 D_3} \\
\eta^{(21)} &= \frac{D_3 D_3 D_3 D_3}{D_2 D_1 D_1 D_1} = \frac{K_{(22|11|00)}^{(21)}(N - 1)}{K_{(22|11|00)}^{(21)}(N)} \cdot D_2^2 \frac{D_3 D_3}{D_3 D_3} \\
\eta^{(22)} &= \eta^{(22)} = \eta^{(22)} = \frac{D_3 D_3}{D_2 D_2} = \frac{D_3 D_3}{D_3 D_3} \\
\eta^{(22)} &= \eta^{(22)} = \eta^{(22)} = \frac{D_3 D_3}{D_2 D_2} = \frac{D_3 D_3}{D_3 D_3}
\end{align*}
\]
and the answer for the $F$-function is

\[
A^{-9} \cdot F^{(m)}_{(22|11)(00)} =
\]

\[
\frac{1}{D_2 D_1^2 D_0^3 D_{-2}^7 D_{-3}^2} \left( 1 - A_{(22|11)(00)}^m \right) - \frac{[3]^2}{D_2 D_1 D_0^3 D_{-1} D_{-2} D_{-3}} \left( A_{(00)}^m - A_{(22|11)}^m \right) + \frac{[4]^2[3]^2}{D_2^4 D_1^2 D_0^2 D_{-1}^2 D_{-2} D_{-3}} \left( A_{(01)}^m - A_{(22|10)}^m \right) + \frac{[5][4]^2[3]^2}{D_2^5 D_1^3 D_0 D_{-1}^2 D_{-2}^2 D_{-3}} \left( A_{(10)}^m - A_{(22|01)}^m \right) - \frac{[6]^2[5]^2[4]^2}{D_2^6 D_1^4 D_0^2 D_{-1}^3 D_{-3} D_{-4}} \left( A_{(21)}^m - A_{(22|12)}^m \right) - \frac{[7][6]^2[5]^2[4]^2}{D_2^7 D_1^5 D_0^3 D_{-1}^4 D_{-3}^2 D_{-4}} \left( A_{(22)}^m - A_{(11)(00)}^m \right)
\]

This is actually a Laurent polynomial at all $m$, satisfying (5).

5 Extension to $F_{(a_1,b_1|11)(00)}$

Again, we can easily extend this result to arbitrary $a_1$ and $b_1$: the substitute of (23), true for $a_2 \cdot b_2 = 1$, $a_3 \cdot b_3 = 0$, is

\[
\xi_{(a_1,b_1|a_2,b_2|a_3,b_3)}^{(i_1,j_1)} = K_{(a_1,b_1|a_2,b_2|a_3,b_3)}^{(i_1,j_1)} \left( N + 2(\delta_{i_1,1} - \delta_{j_1,1}) \right) \cdot \frac{u}{K_{(a_1,b_1|a_2,b_2|a_3,b_3)}}^{\delta_{i_1,j_1}} \cdot \left( 1 - \delta_{i_1,j_1} \right) + \frac{[N + a_2 - j_1][N - b_2 + i_1]}{[N + a_2 + i_1][N - b_2 - j_1 - 1]} \cdot \frac{[N + a_3 - j_1][N - b_3 + i_1]}{[N + a_3 + i_1][N - b_3 - j_1 - 1]} \cdot K_{(a_1,b_1|a_2,b_2|a_3,b_3)}^{(i_1,j_1)} \left( N + i_1 - j_1 \right) \cdot \delta_{i_1,j_1}
\]

The shift $N \rightarrow N + (i_1 + i_2 + i_3 + 3) \cdot \delta_{i_3} - (j_1 + j_2 + j_3 + 3) \cdot \delta_{j_3}$ in the last line is not actually tested by these formulas, because the associated $K_{(a_1,b_1|11)(00)}^{(i_1,j_1)}$ do not depend on $A$.

The quantity $u_{(a_1,b_1|11)(00)}$ is given by a literal analogue of (23):

\[
u_{(a_1,b_1|11)(00)} = \left( K_{(a_1,b_1|1,1)(0,0)}^{(1,1)} - \frac{[a_1 + 2][b_1 + 2]}{[a_1][b_1]} \right) \cdot \frac{[3]^2[2]^2(q)^2}{D_0^2} \cdot \frac{D_2 D_0^3 D_{-1} D_{-2} D_{-3}}{D_3 D_2 D_{-2} D_{-3}}
\]

6 Racah matrix $\bar{S}$ for representation $R = [333]$

Coming back to the case of $R = [333]$ we can now use (11) to get the matrix elements $S_{\mu\nu}^{[333]}$. For this purpose it is technically convenient to substitute the expansion in $A^m_{\mu} A^n_{\nu}$ by that in $A_{\mu} A_{\nu}$ with independent $A$ and $\lambda$ instead of arbitrary $m$ and $n$. To get a $20 \times 20$ matrix we need to enumerate the subdiagrams of $R = [333]$, which are also in one-to-one correspondence with the 20 irreducible representations in $R \otimes \bar{R} = [333] \otimes [333]$. 

\[
\begin{array}{ccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\emptyset & [1] & [11] & [111] & [2] & [21] & [211] & [22] & [221] & [222] \\
\emptyset & (00) & (01) & (02) & (10) & (11) & (12) & (1100) & (1200) & (1201)
\end{array}
\]
Dimensions $D_\mu$ of these representations are obtained from the terms with $\nu = \emptyset$ in (11), because $S^{\mu\emptyset} = \sqrt{D_\mu/D_\nu}$.

$$D_\mu = d_R \cdot \text{coeff}(H_R, \Lambda_\mu \Lambda_{\emptyset})$$

(32)

After that

$$S^{\mu\nu} = \frac{d_R}{\sqrt{D_\mu D_\nu}} \cdot \text{coeff}(H_R, \Lambda_\mu \Lambda_{\nu \emptyset})$$

(33)

The simplest test of the result is that $S$ is orthogonal matrix,

$$\sum_{\nu=1}^{20} S^{\mu\nu} S^{\mu'\nu} = \delta_{\mu\mu'}$$

(34)

It is also symmetric.

The second exclusive matrix $S^{[333]}$ is then the diagonalizing matrix of $\bar{T}\bar{S}\bar{T}$ [9]:

$$\bar{T}\bar{S}\bar{T} = ST^{-1}S^t$$

(35)

with the known diagonal $T$ and $\bar{T}$, made from the $q$-powers of Casimir. This is actually a linear equation for $S$,

$$\left(\bar{T}\bar{S}\bar{T}\right) S = ST^{-1}$$

(36)

which is practically solvable, though explicit calculation is somewhat tedious. The resulting matrix $S^{[333]}$ should be orthogonal – what fixes the normalization of the solution to [36]. In variance with $\bar{S}$, this $S$ is not symmetric.

A typical example of the matrix element is

$$S^{[333]}_{15,15} = \frac{\sqrt{3}}{\sqrt{D_2 D_3}} \cdot \frac{\sqrt{2}}{\sqrt{D_4}} \cdot \sqrt{D_5 D_6}$$

$$\left( A^6 q^{-2} - A^4 (2q^8 + 3q^6 + 2q^4 + q^2) - 3 - 5q^{-2} - 2q^{-4} + 2q^{-6} + 2q^{-8} + 3q^{-10} + q^{-12} + A^2 (q^{18} + 3q^{16} + 4q^{14} + 4q^{12} - 6q^8 - 9q^6 - 5q^4 + 2q^2 + 12 + 10q^{-2} + 5q^{-4} - 9q^{-10} - 7q^{-11} - q^{-12} + 4q^{-14} + 4q^{-16} + 3q^{-18} + q^{-20}) - (q^8 - 2q^{10} + 6q^8 - 9q^6 - 5q^4 + 2q^2 + 12 + 13q^2 + 5q^4 - 9q^6 - 7q^{10} - q^{12} + 4q^{14} + 4q^{16} + 3q^{18} + q^{20}) - (3q^6 - 2q^4 + 2q^2 + 2q^4 + 2q^4 + 2q^6 + 2q^{10} + q^{12} + A^{-6} q^2) \right)$$

The polynomial in brackets reduces to $D(0)^6 = \{A\}^6$ at $q = 1$ and to $-([4][3][2])^3 \{q\}^6$ at $A = 1$. A better quantity for practical calculations is unnormalized $\bar{S}^{\mu\nu} = \bar{S}_{\mu\nu} \cdot \sqrt{D_\mu D_\nu}$, which does not contain square roots.

7 Conclusion

The main result of the present letter is explicit expression for the two previously unknown $F$-functions $F_{(22010)}^{(m)}$ and $F_{(221100)}^{(m)}$. Most important is the deviation from the coefficient $f_{(11)}^{(11)}$, from the skew dimension, even shifted – what is expressed by eq. [23], see also [31]. This new phenomenon explains the failure of previous naive attempts to write down an explicit general expression for $F$ in arbitrary representation: an adequate substitute of the skew characters and appropriate generalization of the corresponding conjecture in [5] is needed for this. The next step in this study should be further extension to $a_2 \cdot b_2 > 1$.

The two newly-found functions, if combined with the other 18, associated with 0,1,2-hook diagrams $\lambda$ with the property $a_2 \cdot b_2 = 0$, provide explicit expression for [333]-colored HOMFLY for all twist and double braid knots. Moreover, from [1] one can read all the elements of the Racah matrix $S^{[333]}$, while $S^{[333]}$ is then found from [36]. Thus this paper solves the long-standing problem to evaluate $S^{[333]}$ and $S^{[333]}$. Explicit expressions
for these Racah matrices as well as for the [333]-colored HOMFLY for the simplest twist and double-braid knots are available at [14].

It still remains to evaluate the twist-knot polynomials and Racah matrices for generic rectangular representations – the new step, made in the present paper, provides the essential new knowledge about this problem which can help to overcome the existing deadlock.

For additional peculiarities of non-rectangular case see [6]. The main point there is that representations in $R \otimes \bar{R}$ are no longer in one-to-one correspondence with the sub-diagrams of non-rectangular $R$. Still, factorization of the coefficients in the differential expansion for double braids persists, and thus the Racah matrices $\bar{S}$ can still be extracted from knot polynomials – though the procedure becomes more tedious [7].

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