A DUAL LEAF APPROACH TO FATNESS: CLASSIFYING RESULTS

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Abstract. This work arose from the author’s urge to revive the subject of fat bundles. Recently, new works have appeared dealing with this concept: [DVGAM22, CGS22b, OS22a, OS22b, BSTW16], while the best reference on this subject is W. Ziller’s unpublished notes on fat bundles ([Zil00]). In this way, it is natural to return to the topic. Here we take advantage of recent developments in the field of positive curvatures to obtain, among other things, a simple and geometric classification of fat homogeneous Riemannian foliations. Although a similar classification was established long ago by Bérard-Bergery in [BB75], we believe that the novelty of this paper lies in the combination of dual foliations and the Ambrose-Singer’s Theorem with metric deformations. For example, we combine the regularization properties of Cheeger deformations with the non-metric dependence of fatness to obstruct the description of the vertical bundle. In particular, we prove that the leaves on a fat Riemannian submersion are a symmetric space when the total space has non-negative sectional curvature, among other classifying results.

1. Introduction

Fat bundles were first introduced in [Wei80] to understand conditions for the existence of metrics with positive sectional curvature and totally geodesic fibers on the total space of fiber bundles. Let \( \pi : F \hookrightarrow (M, g) \rightarrow (B, h) \) be a Riemannian submersion with fiber \( F \) and base \( B \). If one decomposes \( TM = V \oplus \mathcal{H} \), where \( V \) stands for the bundle whose fibers are tangent to \( F \), called the vertical bundle of \( \pi \), its \( g \)-orthogonal complementary bundle, called the horizontal bundle, is denoted by \( \mathcal{H} \) and it holds that \( (\mathcal{H}, g|_\mathcal{H}) \) is isometric to \( (TB, h) \).

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The submersion $\pi$ is called fat provided if for every $x \in M$ and every $X \in \mathcal{H}_x \setminus \{0\}$ holds
\[(1) \quad [\tilde{X}, \mathcal{H}_x]^v = \mathcal{V}_x\]
for a horizontal local extension $\tilde{X}$ of $X$. In equation (1) the superscript $v$ stands for the vertical component of the taken bracket.

Note that the fatness condition is independent of the metric since it is strictly related to submersion, but the relationship between this condition and positive curvature on fiber bundles becomes clear when the bundle is considered with a Riemannian submersion metric with totally geodesic fibers. Following Gray [Gra67] or O'Neil [O'N66], it holds that for any nontrivial plane $X \wedge V$ for $X \in \mathcal{H}, V \in \mathcal{V}$, in the presence of totally geodesic fibers, the unreduced sectional curvature of $g$ computed at $X \wedge V$ is given by
\[(2) \quad K_g(X, V) = |A^*_X V|^2_g,\]
where $A^*_X$ is the $g$-dual of $A_X$. Thus, the fatness condition is necessary for the existence of a metric with positive sectional curvature and totally geodesic fibers on $M$. A plane $\sigma$ of the form $\sigma = X \wedge V$ is called a vertizontal plane.

The fat condition severely restricts the possible dimensions of $\mathcal{V}$ when compared to the dimensions of $\mathcal{H}$. Namely, for a submersion to be fat it must hold for any $x \in M$ that $\dim \mathcal{V}_x \leq \dim \mathcal{H}_x - 1$. For example, if $\dim \mathcal{V}_x = \dim \mathcal{H}_x - 1$ it follows that $\dim \mathcal{H}_x = 2, 4, 8$, among other obstructions, see [Zil00] Proposition 2.5, p. 8.

As pointed out in [Zil00] Proposition 2.5, p. 8, for principal bundles with fiber $F = S^3$ or $SO(3)$ the base manifold $B$ must have a dimension multiple of 4. Moreover, the only known examples of principal fat bundles with these fibers occur for 3-Sasakian manifolds. Recall, for example, that a manifold $M$ is called 3-Sasakian if it admits an isometric almost free action of $S^3$ or $SO(3)$ whose orbits are totally geodesic with curvature 1 and so, that $R_g(U^*, X)Y = g(U^*, Y)X - g(X, Y)U^*$ for all $U \in g$, the Lie algebra of $G$, where the superscript $U^*$ denotes the action vector field obtained from $U$. For these metrics, all vertizontal curvatures are equal to 1.

For 3-Sasakian manifolds, the quotient $M/G$ has the structure of an orbifold and the bundle $G \to M \to M/G$ is by definition an fat $G$-principal orbifold. Then the interdisciplinary nature of fat bundles is shown: if $(M, g) \to (B, h)$ is a fat $G$-principal orbifold, then $h$ is an orbifold quaternionic Kähler metric with positive scalar curvature. That is, there is a 3-dimensional subbundle $E$ of $O(TB)$ consisting of isometric linear maps of $TB$ which are invariant under parallel translations and such that for any $b \in B$ the fiber $E_b$ is spanned by orthogonal almost complex structures $J_i$ satisfying $J_1 \circ J_2 = -J_2 \circ J_1 = J_3$. It is worth noting that the converse holds, see Propositions 2.16 and 2.18 in [Zil00].

Problem 1 in W. Ziller’s notes on fat bundles reads as follows: If there exists a fat principal $G$-bundle with $\dim G = 3$ is it possible to change the metric of $B$ so that all vertical curvatures equal 1? To answer this, we try to understand if not only any two vertizontal curvatures are pointwise constant, but two-plane independent. In general this is false, Theorem 2.7 in [GZV] constructs a counterexample to this. However, we prove that this holds on locally symmetric spaces of compact type $(M, g)$ with Riemannian foliations with bounded holonomy, also assuming that for any $x \in M$, the isotropy representation $\rho_x$ in a neighborhood of $x$ acts transitively on the unit sphere in $T_xM$. Such a general result also answers a question put to
the first named author by W. Ziller: Is a $SO(3), S^3$-fat locally symmetric bundle the round sphere up to universal covering? The answer is provided by Theorem A next.

**Theorem A** (Fat Riemannian foliations on Lie groups with bi-invariant metrics). If $(G, Q)$ is a compact connected Lie group $G$ with bi-invariant metric $Q$, then any fat Riemannian foliation $F$ on $G$ is given by the coset foliation induced by a Lie subgroup $H$ of $G$. Moreover, the leaf space $G/H$ is an even dimensional symmetric space and $H$ is either a single point, $S^1$, $SO(3)$ or $S^3$.

In [BB75], Bérard-Bergery classified fat bundles of the form $K/H \hookrightarrow G/H \rightarrow G/K$ and found many interesting examples, which include, in particular, most homogeneous metrics with positive sectional curvature. Since then, however, no new examples of fat bundles have been discovered (except in the special cases where the fiber is one-dimensional). Theorem A is thus not new for bundles in the sense that a classification has already been obtained, although our approach is essentially geometric and involves several steps of self-interest.

We have succeeded in recovering the full description of homogeneous fat bundles from Bérard-Bergery. More importantly, we take a step forward in understanding the algebraic structure of fat submersion via curvature properties and conclude that:

**Theorem B.** The fiber $F$ of a fat Riemannian submersion with totally geodesic leaves $\pi : F \hookrightarrow (M, g) \rightarrow B$ is a symmetric space if $\sec(g) \geq 0$. Moreover, if $\dim F > 1$ then $F$ is a symmetric space of rank 1 and the following are the only possibilities to the holonomy group $H$ of $\pi$, with possible fibers $H/K$:

| $H$  | $K$       | $\dim H/K$ |
|-----|-----------|------------|
| $SO(3)$ | $\{e\}$ | 3          |
| $SO(3)$ | $SO(2)$  | 2          |
| $SO(2n+1)$ | $SO(2n)$ | $2n$       |
| $SO(n)$ | $U(n)$    | $n(n-1)$ with $n = 2, 3$ |
| $Spin(n)$ | $\{e\}$ | $n = 3$    |
| $SU(2)$ | $\mathbb{Z}_2$ | 3         |
| $SU(2)$ | $\{e\}$  | 3          |
| $SU(2)$ | $Sp(1)$   | 5          |
| $SU(2)$ | $SO(2)$   | 2          |
| $SU(n+1)$ | $S(U(n) \times U(1))$ | $2n$|
| $Sp(n+1)$ | $Sp(n) \times Sp(1)$ | $4n$|
| $Sp(1)$ | $U(1)$    | 2          |
| $Sp(1)$ | $\{e\}$  | 3          |
| $SO(2n+4)$ | $SO(2n+1) \times SO(3)$ | $3(2n+1)$ |
| $F_4$ | $Spin(9)$ | 16         |

We recall that a Riemannian foliation $\mathcal{F}$ consists of a foliation on $M$ in which the leaves are locally equidistant. The term bounded holonomy refers to a global bound on each holonomy field with respect to its fixed initial data. This condition

\footnote{so we assume that the leaves are connected}
always holds, for example, when \( \mathcal{F} \) is induced by the connected components of the fibers of a fiber bundle with compact structure group, see Proposition 3.4 in [Spe18].

On principal bundles, a holonomy field \( \xi(t) \) consists in the restriction of an action vector field along a horizontal geodesic \( t \mapsto c(t) \). In the general case, \( \xi \) is given by the equation

\[
\nabla_c \xi = -A_c^* \xi - S_c \xi,
\]

where \( S_c \xi \) denotes the second fundamental form of the leaf passing through \( c(t) \).

In contrast to holonomy fields, in [Spe18] dual-holonomy fields are introduced. Specifying an arbitrary horizontal geodesic \( c(t) \) these are characterized by

\[
\nabla_c \nu = -A_c^* \nu + S_c \nu.
\]

In particular, \( \langle \xi(t), \nu(t) \rangle \) is always constant for any holonomy field \( \xi \) and any dual holonomy field \( \nu \), see Proposition 4.1 in [Spe18]. It is also worth noting that dual holonomy fields are always “self-dual”, as are holonomy fields in the presence of totally geodesic fibers, in the sense that they have a constant norm along \( c \). The concepts described earlier are crucial to the results in this paper, before commenting on them, let us finish describing our results.

Theorem B shows that the true reason that the fiber of a fat Riemannian submersion with totally geodesic fibers is a symmetric space is the presence of a metric with nonnegative sectional curvature on the total space of the submersion. We note that this was already known when \( M \) was a homogeneous space, as Bérard-Bergery writes in [BB75]. Moreover, such a result leads naturally to the following pattern built from a rigid striped fabric:

**Theorem C** (Bérard-Bergery). Let \( G \) be a compact simple and connected Lie group and consider the following Lie subgroups \( K < H < G \). Consider the following fat Riemannian submersion with totally geodesic leaves

\[
H/K \hookrightarrow (G/K, b) \to (G/H, \bar{b}),
\]

where \( G/K \cong P \times_H (H/K) \), where \( H \hookrightarrow P \to G/H \) is the reduced holonomy principal bundle associated with the homogeneous Riemannian foliation \( \mathcal{F} = \{H/K\} \). Suppose that \( b \) has nonnegative sectional curvature and that \( \dim H/K > 1 \). Then the triple \( (K, H, G) \) can be one of the following:

| \( G \)     | \( H \)       | \( K \)            |
|------------|---------------|--------------------|
| SU(3)      | SO(3)         | \{e\}, SO(2)       |
| SU(3)      | SU(2)         | \{e\}, SO(2), Z_2, Sp(1) |
| SU(2n + 1) | SO(2n + 1)    | SO(2n)             |
| SU(n + 2)  | SU(n + 1)     | S(U(n) x U(1))     |
| SU(2n + 1) | Sp(n + 1)     | Sp(n) x Sp(1)      |
| SU(2)      | Sp(1)         | \{e\}, U(1)        |
| SU(2n + 4) | SO(2n + 4)    | SO(2n + 1) x SO(3) |
| G_2        | SO(4)         | U(4), SO(3)        |
| E_6        | F_4           | Spin(9)            |
We remark that a very refined invariant to symmetric spaces consists in their rank, which coincides with the maximal dimension of a subspace of the tangent space (to any point) on which the curvature is identically zero. The rank is always at least one, and it is the same whether the sectional curvature is positive or negative. If the curvature is positive, the space is of compact type, and if it is negative, it is of non-compact type. In this paper we focus only on the first class. For completeness, we recall that symmetric spaces of Euclidean type have the same rank as their dimension and are isometric to a Euclidean space of that dimension.

A classification of Riemannian symmetric spaces $G/H$ is reduced to the classification of irreducible simply connected Riemannian symmetric spaces of compact and non-compact type. In both cases there are two classes:

(A) $G$ is a real simple Lie group;

(B) $G$ is the product of a compact simple Lie group with itself (compact type), or a complexification of such a Lie group (non-compact type).

The examples of class [A] are completely described by the classification of non-compact simply connected real simple Lie groups, while the examples of class [B] are completely described by the classification of simple Lie groups. For the compact type, the symmetric space $M$ is a compact simply connected simple Lie group, $G = M \times M$ and $H = \Delta M$, i.e., the diagonal subgroup. The compact simply connected Lie groups are the universal coverings of the classical Lie groups $\text{SO}(n), \text{SO}(n), \text{SU}(n), \text{Sp}(n)$, and the five exceptional Lie groups $E_6, E_7, E_8, F_4, G_2$. The tables included in the results above result from the classification described.

**The dual leaf approach to fatness.** Since fatness is usually related to totally geodesic leaves, we constantly assume this, although Theorem 2.5 implies that we can drop such a hypothesis for any result in this paper. This is because a compact holonomy group always implies the existence of compatible foliated metrics preserving vertizontal curvature (assuming fatness). Moreover, such a metric always preserves positive sectional curvature in the leaf space if it is smooth.

A crucial new result presented here is Theorem B, which ensures that every leaf is a symmetric space for the foliations treated here. The strategy of our proof relies on linking local holonomy diffeomorphisms to curvature properties, as introduced by Sperança both in [Spe17b, Spe17a, Spe18]. Such control also plays a crucial role in the recent work [CGS22b].

We will very often consider dual holonomy fields defined by the equation (4). Similarly, we need to deal with basic horizontal fields along a vertical curve $\gamma$, characterized as horizontal solutions of

$$\nabla_\gamma X = -A_\gamma X - S_X(\dot{\gamma}).$$

Under totally geodesic fibers the former equation is the definition of basic horizontal fields, which characterizes Jacobi fields induced by holonomy transport.

The dual leaf ingredient in our proofs is the following: Let $(M, g)$ be a compact Riemannian manifold with a Riemannian foliation $\mathcal{F}$ with totally geodesic leaves and bounded holonomy. Then $A_\gamma$ is basic along the geodesic $\text{exp}(sV)$, where $V \in \mathcal{V}$ and $X \in \mathcal{H}$ is basic. Moreover, if $g$ has non-negative sectional curvature.
then for each leaf \( L_p \in H \),

\[
L_p^# \cap V_p = \text{span}_R \{ A_XY(p) : X, Y \in H_p \},
\]

where \( L_p^# \) is a dual leaf through \( p \):

\[
L_p^# := \{ q \in M : \text{there exists a piecewise smooth horizontal curve from } p \text{ to } q \},
\]

see [Wil07] for more details.

Sperança proved in [Spe18] that if the foliation \( F \) is fat, then \( L_p^# = M \), so we can completely describe the vertical space \( V_p \) for any \( p \in M \) in terms of the O’Neill’s integrability tensor. Such an explicit description helps us construct fiberwise reversing geodesics isometries, ensuring that each fiber is not only a homogeneous space, but also a symmetric space.

Finally, we remark that although the fact that the leaves of \( F \) were already known to be symmetric spaces whenever \( M \) was a homogeneous space, the generalization here shows that the real geometric aspect needed to provide the symmetric realization of the fibers is the fact that \( g \) has non-negative sectional curvature.

**This paper is organized as follows.** In Section 2.1 we first present some known results and characterizations of fat bundles. Using Proposition 2.1 combined with the to be proved Corollary 3.3 we conclude the first part of Theorem B, see Theorem 2.1.

Section 2.2 is devoted to a quick recall of the concept of Cheeger deformations on fiber bundles with compact structure groups, mainly following [CGS22a]. The curvature formulae presented there (equation (9)) makes it easier to relate the vertical curvature of an associated bundle to a principal bundle. Proposition 2.1 concludes the proof of Theorem B up to the table description, postponed to Section 4.

Theorem A is proved in Section 3, see Theorem 3.8. In Section 3.1 we mainly discuss the relation between fatness with obstruction to dual foliations, proving Corollary 3.3.

Finally, in Section 4 we discuss the remaining proofs, also providing the complete descriptions of holonomy groups and fibers associated with fat bundles. See for instance Theorem 4.1.

### 2. Fat Submersions and Cheeger Deformations

In this section we deal with some preliminary aspects of both fat bundles and Cheeger deformations. The main result in this section is Theorem 2.2 below, Theorem B in the Introduction. Some Propositions here stated are well known, at least to experts, such as Proposition 2.1. However, we also state some needed results due to L. Sperança, presented in [Spe17b]. For the sake of completeness, we sketch some of the proofs to these results in forthcoming sections.

#### 2.1. Fat Submersions

The main references on Fat Submersions are [Zil00] and [GW09]. One of the main results in the subject is:

**Proposition 2.1.** The fiber \( F \) of a fat submersion \( \pi : F \hookrightarrow M \to B \) is a homogeneous manifold: the holonomy group \( H \) acts transitively on \( F \), i.e., \( F = H/K \). Consequently, the total space \( M \) is diffeomorphic to \( P \times_H (H/K) \cong P/K \), where \( P \) is the \( \pi \)-associated holonomy principal bundle.
To the proof of Proposition 2.1, see [Z1001, Proposition 2.6, p.9].

We go further the former Proposition providing a refinement to it, namely, we prove that the fiber of a fat submersion is a symmetric space indeed whenever the ambient sectional curvature is non-negative. To this aim, we make use of the fact that: fat foliations with totally geodesic leaves combined to non-negative curvature restricts the vertical space pointwise: it is fully generated by the O'Neill integrability tensor, see Corollary 3.3. The proof is finished using such a fact to construct explicit local isometries reversing geodesics on each fiber, following a classical construction due to Cartan, see [Car29] or [Gar21] Theorem 1.3.3, p. 9] and [IC92] Chapter 8.2.

**Theorem 2.2.** The fiber $F$ of a fat Riemannian submersion with totally geodesic leaves $\pi : F \hookrightarrow (M, g) \to B$ is a symmetric space whenever $\sec (g) \geq 0$. In addition, if $\dim F > 1$ it follows that $F$ has positive sectional curvature at the normal homogeneous space metric.

**Proof.** Since the fibers are totally geodesic and $g$ has non-negative sectional curvature, it must holds that for every $x \in M$ (Corollary 3.3):

$$\mathcal{V}_x \cong T_x (K/H) \cong \text{span}_\mathbb{R}\{A_X Y (x) : X, Y \in \mathcal{H}_x\}.$$  

Now, given two sufficiently close points $x, h \cdot x \in F_x \cong H/K$, with $h \in H$, let $\gamma : (-\epsilon, \epsilon) \to F_x$ be the unit speed vertical geodesic joining $x$ to $h \cdot x$.

On the other hand, we know that $\mathcal{V}_{h \cdot x} \cong \text{span}_\mathbb{R}\{A_X Y (h \cdot x) : X, Y \in \mathcal{H}_{h \cdot x}\}$. On the other hand, since $F$ is a homogeneous space obtained out the holonomy group of the foliation, it must holds that (see Section 2.2 for further details on the construction explained next): $g$ is obtained out of a product metric $g_P + g_F$ on $P \times F$ such that $\pi : (P \times F, g_P + g_F) \to (M, g)$ is a Riemannian submersion. Moreover, any $X \in T_{(p, f)} (P \times F)$ can be written as $X = (X + V^v, X_F + W^*)$, where $X$ is orthogonal to the $H$-orbit on $P$, $X_F$ is orthogonal to the $H$-orbit on $F$ and, for $V, W \in h$, the vectors $V^v$ and $W^*$ are the action vectors relative to the $H$-actions on $P$ and $F$, respectively.

Let $L_\pi : T_{(p, f)} M \to T_{(p, f)} (P \times F)$ be the horizontal lift associated with $\pi$, with $\bar{\pi}(p, f) = x$. Then,

$$L_\pi (X + X_F + U^*) = (X - (O^{-1} \tilde{U})^v, X_F + (O_F^{-1} \tilde{U})^*)$$

where $O, O_F$ are the orbit tensors associated with $g_P$ and $g_F$, respectively. Moreover, $\tilde{O} := O_P (1 + O^{-1} O_F)^{-1} = (O_F^{-1} + O^{-1})^{-1}$.

Being $p_1 : TP \times TF \to TP$ the first factor projection one gets $p_1 \circ L_\pi (X + X_F) = X$. Therefore, $p_1 \circ L_\pi$ maps the horizontal $\mathcal{H}_x$ onto the horizontal space of $P$ at $p$ with respect to $g_P$. Note, however, that since $F$ is a homogeneous space, it holds that $X_F \equiv 0$ and hence, $p_1 \circ L_\pi$ defines an isomorphism, denoted by $I$, between the horizontal space $\mathcal{H}_x$ and the $g_F$-orthogonal complement to the $H$-orbit through $x$, that we denote by $\mathcal{H}_x^P$.

Now, since $(\rho_h)_* (\mathcal{H}_x^P) \cong \mathcal{H}_{h \cdot x}^P$, we can identify $X \in \mathcal{H}_x$ with $I (X) \in \mathcal{H}_x^P$ to conclude that $\mathcal{H}_{h \cdot x} = \{I^{-1} (\rho_h)_* IX : X \in \mathcal{H}_x\}$. This manner,

$$\mathcal{V}_{h \cdot x} \cong \text{span}_\mathbb{R}\{A_{\gamma}^{-1} (\rho_h)_* I X (h \cdot x) : X, Y \in \mathcal{H}_x\} \cong \text{span}_\mathbb{R}\{(\rho_h)_* (A_X Y (x)) : X, Y \in \mathcal{H}_x\}$$

Let $\phi : \mathcal{V}_x \to \mathcal{V}_{h \cdot x}$ the linear isometry obtained by extending linearly the map

$$\phi (A_X Y (x)) := - (\rho_h)_* (A_X Y (x)), X, Y \in \mathcal{H}_x.$$

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To see that this is an isometry, observe that
\[ |\phi(A_X Y(x))|_g = |(\rho_h)^*(A_X Y(x))|_g = |A_X Y(x)|_g, \]
since the $H$-action on the fibers is isometric.

Now let $\sigma(h \cdot x) := \exp_x^F \circ \varphi \circ (\exp_x^F)^{-1}(h \cdot x)$ where $\exp_x^F$ denotes the exponential map of the isometrically induced metric on $F$. We choose the domains on the composition in a way that $\exp_x^F$ is a diffeomorphism.

Now, if $\gamma : (-\epsilon, \epsilon) \to F$ is such that $\gamma$ is well defined in the domain of $\sigma$, we let $P_\gamma$ be the parallel transport along it and define $\tilde{\gamma}(s)$ as the unique geodesic joining $\sigma(x)$ to $\sigma(h \cdot x)$. If $P_\tilde{\gamma}$ denotes the corresponding parallel transport along $\tilde{\gamma}$, we shall verify that
\[ I_{\gamma, \tilde{\gamma}} := P_\gamma \circ \phi \circ P_\gamma \]
commutes with the Riemann tensor of $g$. This implies that $\sigma$ is not only a local isometry but that $d\sigma(h \cdot x) = I_{\gamma, \tilde{\gamma}}$. Since by definition $\sigma$ reverses geodesics, the claim shall follow.

Once $A$ is a tensor, let us extend $X, Y \in \mathcal{H}_x$ basically along $\gamma$, also requiring that if $\nabla_X Y$ are such extensions, then $\nabla_{\hat{\mathcal{H}}(\sigma)} Y(s) \equiv 0$. Now, since both $Y(s), A_{\nabla_X Y}(s)$ are basic along $\gamma$, we thus conclude that the inner product
\[ g(A_{\nabla_X Y}(s), \dot{\gamma}(s)) = g(Y(s), A_{\nabla_X Y}(s)) \]
is constant along $\gamma$ and hence, $\sum_s A_{\nabla_X Y}(s)$ is always orthogonal to $\dot{\gamma}(s)$. Also observe that since $A_{\nabla_X Y}(s)$ is a Jacobi field along $\gamma$ and $H$ has non-negative sectional curvature (it is a symmetric space of compact type), then Wilking’s Theorem [GW09, Theorem 1.7.1] (see also [Wil07]) ensures that either $\nabla_{\nabla_X Y}(s)$ vanishes at some instant $s$, in which case it vanishes for every $s$, or $A_{\nabla_X Y}(s)$ is parallel along $\gamma$. Assuming $A_X Y(x)$ is non-zero one thus concludes that $A_{\nabla_X Y}(s)$ is the parallel transport of $A_X Y$ along $\gamma$. This implies that $A_{\nabla_X Y}(s)$ is a Jacobi field with constant coefficients, chosen any parallel orthonormal frame along $\gamma$. The same argument can be applied to $P_\gamma$. Hence, $I_{\gamma, \tilde{\gamma}}$ is a linear isometry. We remark that the commutativity of $I_{\gamma, \tilde{\gamma}}$ also becomes apparent via the relation
\[ 0 = \frac{d^2}{ds^2} A_{\nabla_X Y}(s) \dot{\gamma}(s) = -R_g(A_{\nabla_X Y}(s), \dot{\gamma}(s))(\dot{\gamma}(s)), \]
what is the main ingredient generically to show that a map defined as $\sigma$ is a local isometry.

The last part of the result, concerning positive sectional curvature on the fibers, is the content of Proposition 2.3.

2.2. Cheeger deformations and their regularization properties. Here we recall some very basic facts on Cheeger deformations, especially on associated bundles, such as described in [CGS22a]. In particular, we finish the proof of Theorem 2.2 that is

**Proposition 2.3.** Let $(M,g)$ be a Riemannian manifold with a fat Riemannian foliation $\mathcal{F}$ of totally geodesic leaves obtained out of a Riemannian submersion. If the holonomy group associated with $\mathcal{F}$ is compact and $g$ has non-negative sectional curvature then $F$ has positive sectional curvature at the normal homogeneous metric provided if $\dim F > 1$. 


We first recall the procedure known as Cheeger deformations. The main formulae come from classical references, such as [Zil] and [MS7]. Consider the product manifold $M \times G$ with the product metric $g + t^{-1}Q$, where $G$ acts on $M$ via isometries and $Q$ is a bi-invariant metric on $G$. We see $M \times G$ with the following $\ast$-action, that is nothing but the associated bundle action on $M \times G$:

\begin{equation}
 r \ast (m, g) := (rm, rg).
\end{equation}

The quotient projection $\pi'(m, g) := g^{-1}m$ defines principal bundle, and the metric $g + t^{-1}Q$ induces via $\pi'$ the metric $g_t$, known as a $t$-Cheeger deformation of $g$. In this section we denoted by $\mathfrak{m}_p$ the $G$-orthogonal complement of $\mathfrak{g}_p$, the Lie algebra of $G_p$. We recall that $\mathfrak{m}_p$ is isomorphic to the tangent space to the orbit $Gp$ via action fields: for any $U \in \mathfrak{g}$ the corresponding action field out of $U$ is defined by the rule

$$
U^*_p = \frac{d}{dt} \bigg|_{t=0} e^{tU} p.
$$

It is straightforward to check that the map $U \mapsto U^*_p$ is a linear morphism whose kernel is $\mathfrak{g}_p$. This manner, any vector tangent to $T_pGp$ is said to be vertical, hence, such a space is named as the vertical space at $p$, being denoted by $\mathcal{V}_p$. For each $p \in M$ its orthogonal complement, denoted by $\mathcal{H}_p$, is named horizontal space. A tangent vector $X \in T_pM$ can be uniquely decomposed as $X = X^* + U^*_p$, where $X$ is horizontal and $U \in \mathfrak{m}_p$.

It can be checked that the following useful tensors associated with Cheeger deformations, see [Zil] for further clarifications, are related as

1. The orbit tensor at $p$ is the linear map $O : \mathfrak{m}_p \rightarrow \mathfrak{m}_p$ defined by $g(U^*, V^*) = Q(OU, V)$, $\forall U^*, V^* \in \mathcal{V}_p$
2. For each $t > 0$ we define $O_t : \mathfrak{m}_p \rightarrow \mathfrak{m}_p$ as $g_t(U^*, V^*) = Q(O_tU, V)$, $\forall U^*, V^* \in \mathcal{V}_p$
3. The metric tensor of $g_t$, $C_t : T_pM \rightarrow T_pM$ is defined as $g_t(X, Y) = g(C_tX, Y)$, $\forall X, Y \in T_pM$

All the three tensors above are symmetric and positive definite. The next proposition shows how they are related to each other and to the original metric quantities.

**Proposition 2.4** (Proposition 1.1 in [Zil]). The tensors above satisfy:

1. $O_t = (O^{-1} + t1)^{-1} = O(1 + tO)^{-1}$,
2. If $X = X + U^*$ then $C_t(X) = X + ((1 + tO)^{-1}U)^*$.

Now we consider the more general concept of Cheeger deformations, presented in [CGS22a]. Let $F \rightarrow M \xrightarrow{\pi} B$ be a fiber bundle from a manifold $M$, with fiber $F$ and compact structure group $G$ and base $B$. For each pair $g$ and $g_F$ of $G$-invariant metrics on $P$ and $F$, respectively, there exists a metric $h$ on $M$ induced by $\pi$.

Let $\mathcal{M}$ denote the set of all metrics obtained in this way. For instance, whenever the $G$-action on $F$ is transitive, every metric on $M$ such that the holonomy acts by isometries on each fiber belongs to $\mathcal{M}$. Observe that this is precisely the case when the submersion is fat.

**Definition 1** (The deformation). Given $h \in \mathcal{M}$, consider $g + g_F$ a product metric on $P \times F$ such that $\pi : (P \times F, g + g_F) \rightarrow (M, h)$ is a Riemannian submersion.
We define $h_t$ as the submersion metric induced by $g_t + g_F$, where $g_t$ is the time $t$ Cheeger deformation associated with $g$.

For the sake of completeness, we paraphrase [CGS22a]: Fix $(p, f) \in P \times F$. Any $X \in T(p, f)(P \times F)$ can be written as $X = (X + V', X_F + W^*)$, where $X$ is orthogonal to the $G$-orbit on $P$, $X_F$ is orthogonal to the $G$-orbit on $F$ and, for $V, W \in g, g'$ and $W^*$ are the action vectors relative to the $G$-actions on $P$ and $F$ respectively. Let $O, O_F$ and $O_t$ be the orbit tensors associated with $g, g_F$ and $g_t$, respectively.

Noting that the point $(p, f)$ is fixed, we abuse notation and denote

$$d\tilde{\pi}(p, f)(X, X_F + U^*) := X + X_F + U^*.$$ 

Defining the tensors $\tilde{O}_f, \tilde{C}_f : m_f \to m_f$,

$$\tilde{O}_t := O_F(1 + O_t^{-1}O_F)^{-1} = (O_F^{-1} + O_t^{-1})^{-1},$$

$$\tilde{C}_t := -C_tO_t^{-1}\tilde{O}_t = -O^{-1}\tilde{O}_t,$$

it can be checked that

**Claim 1.** Let $\mathcal{L}_\pi : T\pi(p, f)M \to T(p, f)(P \times F)$ be the horizontal lift associated with $\pi$. Then,

$$\mathcal{L}_\pi(X + X_F + U^*) = (X - (O_t^{-1}\tilde{O}_tU)^\vee, X_F + (O_F^{-1}\tilde{O}_tU)^*) .$$

As a last thing we need to recall, it is the expression for the sectional curvature of $h$, obtained in Theorem 3.1 in [CGS22a]:

$$K_h(\tilde{X}, \tilde{Y}) = \tilde{\kappa}_0(\tilde{X}, \tilde{Y})$$

$$= \kappa_0(X + U^\vee, Y + V^\vee) + K_{g_F}(X_F - (O_F^{-1}OU)^*, Y_F - (O_F^{-1}OV)^*) + \tilde{z}_0(\tilde{X}, \tilde{Y})$$

$$= K_{g}(X + U^\vee, Y + V^\vee) + K_{g_F}(X_F - (O_F^{-1}OU)^*, Y_F - (O_F^{-1}OV)^*) + \tilde{z}_0(\tilde{X}, \tilde{Y}) .$$

where $\tilde{z}_0$ is a non-negative term.

**Proof of Proposition 2.3.** On the other hand, the condition of fatness implies that $F$ is a homogeneous space of the form $H/K$ for $H$ being the Holonomy group of the foliation. On the other hand, Proposition 2.1 also implies that $M \cong P \times_H (H/K)$, where $H \hookrightarrow P$ is the reduced holonomy principal bundle.

Moreover, since the fibers of $P$ are diffeomorphic to $H$, meanwhile the fibers $F$ are to $H/K$, we can consider the Riemannian principal submersion for any bi-invariant metric $Q_H$ on $H$

$$K \hookrightarrow (H, Q_H) \to (H/K, \tilde{Q})$$

to observe that if $Q_H$ has non-negative sectional curvature, then so does $\tilde{Q}$. Moreover, according to Theorem 3.1 in [CGS22a], or equation (9) above, the unReduced sectional curvature of a two plane $\{X \land Y\} = \{X + X_F + U^\vee \land Y + Y_F + V^\vee\}$ with respect to $g$ is bounded from below by

$$K_g(\tilde{X}, \tilde{Y}) \geq K_{g_F}(X + U^\vee, Y + V^\vee) + K_{g_F}(X_F - (O_F^{-1}OU)^*, Y_F - (O_F^{-1}OV)^*).$$

Since $F$ is a homogeneous spaces it holds that $X_F = 0 = Y_F$. Hence,

$$K_g(X + U^*, Y + V^*) \geq K_{g_F}(X + U^\vee, Y + V^\vee) + K_{g_F}((O_F^{-1}OU)^*, (O_F^{-1}OV)^*).$$
A non degenerated vertizontal plane tangent to $M$ has the form $X \wedge V^*$ for $V \in \mathfrak{h} \oplus \mathfrak{t} := \mathfrak{h} \cap (\mathfrak{t})_{\perp g_H}$, moreover,

$$K_{\mathfrak{g}}(X, V^*) = K_{\mathfrak{g}_P}(X, V^*) = \langle (A^P)^*_X V^* \rangle_{g_P}^2,$$

where $TP = \mathcal{H}^P \oplus \mathcal{V}^P$ and $(A^P)^*_X$ is the $g_P$-dual to $\frac{1}{2}[X, \cdot]^{\mathcal{V}^P}$.

Therefore, it is necessary and sufficient to the foliation $\mathcal{F}^M := \{H/K\}$ to be fat in $\pi: M \cong P \times_H (H/K) \to P/H$ that the subfoliation $\mathcal{F}^P_K := \{H/K\}$ be fat in $\pi: P \to P/H$, that is, that the connection on $\pi: P \to P/H$ is $K$-fat. In other words, that the following two-form is non-degenerated for every non zero $U \in \mathfrak{h} \oplus \mathfrak{t}$

$$(X, Y) \mapsto Q_H(\Omega(X, Y), U),$$

where $\Omega$ is the curvature connection of $\pi: P \to P/H$, that is, $(\Omega(X, Y))^* = -[X, Y]^\mathcal{V}^P$, compare with Definition 2.8.2 in [GW09, p. 106].

Since $\dim(F=H/K) > 1$, let $U, V \in \mathfrak{h} \oplus \mathfrak{t}$ satisfying $|U|_{Q_H} = |V|_{Q_H} = 1$, $Q_H(U, V) = 0$. Now, once the first part of the thesis in Theorem 2.2 implies that $H/K$ is a symmetric space, then $[U, V]^{\mathfrak{h} \oplus \mathfrak{t}} = 0$. So O’Neill’s submersion formula implies that

$$\sec_Q(U, V) = \frac{1}{4}|[U, V]|_{Q_H}^2 + \frac{3}{4}|[U, V]|_{Q_H}^2 = |[U, V]|_{Q_H}^2.$$

We claim that $[U, V]^\mathfrak{t} \neq 0$. Indeed, if it was not the case it would follow that $U, V$ lie in the Cartan subalgebra of $\mathfrak{h}$ and that $H/K$ is a symmetric space of rank greater than 1. Since the fatness constraint on the dimensions imposes

$$2 \leq \text{rank} H/K \leq \dim(H/K) \leq \dim(P/H) - 1$$

then $\dim(P/H) \geq 3$ so $\dim(P/H)$ is a multiple of 4, see Proposition 2.5 in [Zil00].

If $\dim(P/H) = 4$ then $\dim(H/K) = 2, 3$. If $\text{rank} H/K = \dim H/K = 2$ then $H/K$ is diffeomorphic to an Euclidean space, contradicting compactness. Therefore, we must have $\dim H/K = 3$. This however imposes another obstruction: $H$ should be the Heisenberg group and $K = \{e\}$, see the table on page 307 in [Mil76], thus $H$ is non-compact, again a contradiction.

To the general case, take $\pi_{4(k+1)}: H \hookrightarrow P \to P/H$ a fat bundle such that $\dim(P/H) = 4(k + 1)$ and assume by contradiction that there exists two non-zero and $Q_H$-orthonormal vectors $U, V \in \mathfrak{h} \oplus \mathfrak{t}$ such that $[U, V] = 0$. Then, there is a totally geodesic flat torus $T^2$ with $\text{Lie}(T^2) = \text{span}_\mathbb{R}\{U, V\}$ inside $P$. Since the $H$ action on $P$ is free so as the $T^2$ action on $H$ we can take the quotient:

$$\pi_{T^2} : H/T^2 \twoheadrightarrow P/T^2 \to P/T^2 \times H,$$

which is also a $K/T^2$-fat bundle since $T^2$ is contained in every maximal torus in $H$.

Then, once $\pi_{T^2}$ is a fat bundle with basis of dimension $4k + 2$, which is not a multiple of 4, then $\dim H/K \times T^2 = 1$, that is, $H/K \times T^2 \cong S^1$. Hence, $\dim H/K = 3$ and $\text{rank} H = 2$, which is a case already obstructed in the previous part of the argument.

Theorem below shows that we never need to assume that the fibers of a fat Riemannian foliation is totally geodesic in order to obtain the results in this paper, reinforcing the non-metricity nature of fatness, but also preserving curvature properties. This is what we refer to a regularization property of Cheeger deformations, see [SSW15].
**Theorem 2.5.** Let $\pi : F \hookrightarrow M \rightarrow B$ be a fat Riemannian submersion with horizontal distribution $\mathcal{H}$, also assume that $F$ is connected. If the holonomy group $H$ associated with it is compact, then there is a Riemannian metric $\tilde{g}$ on $M$, compatible with $\mathcal{H}$ and with totally geodesic fibers. In particular, $\tilde{g}$ has positive vertizontal curvature. Moreover, if the starting metric on $B$ has positive sectional curvature, the same holds for the induced Riemannian submersion metric by $\tilde{g}$.

**Proof.** Once the Riemannian foliation $F$ given by $\{F_x\}_{x \in M}$ is fat it must holds both that, according to Proposition 2.1, $F$ is an homogeneous space, but more importantly, $F = H/K$ where $H$ is the holonomy group of $\pi$. Moreover, Proposition 2.1 also implies that $M \cong P \times_H (H/K)$ where $P$ is the $\pi$-associated holonomy bundle.

Denote by $\bar{\pi} : P \times (H/K) \rightarrow M$ the quotient projection. Since $F$ is a homogeneous manifold for which $H$ acts isometrically, it holds that there exists $g_P$ and $g_F$ such that $g$ decouples as a product by the $\bar{\pi}$-pullback:

$$\bar{\pi}^* (g) = g_P + g_F,$$

where $g_P$ is a $H$-invariant metric on $P$ and $g_F$ is an $\text{Ad}(K)$-invariant metric on $F$. Theorem 1.4 in [CGS22a] ensures the result. $\square$

3. **RIEMANNIAN FOLIATIONS ON LIE GROUPS WITH BI-INARIANT METRICS**

The main purpose of this section relies on classifying fat Riemannian foliations on Lie groups with bi-invariant metrics. Though this was already achieved by Bérard-Bergery [Ber76] for Riemannian submersions, we produce a very simplified proof to it, relying only on curvature properties. Namely, we prove Theorem A.

3.1. **The fatness condition and vertical obstructions.** The fatness condition, although metric independent, recovers a very geometric flavor under the presence of totally geodesic fibers: the non-degenerate vertizontal planes are always positively curved. Moreover, since it was firstly idealized to the setting of principal bundles, where the classical Ambrose–Singer’s Theorem provides a geometric control description of the tangent space of the reduced holonomy bundle. The homogeneous space nature of the fiber of any fiber bundle associated with a fat principal bundle naturally leads us to ask if their vertical spaces could be isomorphic, as vector spaces, to subspaces of the Lie algebra of holonomy group. Observe that this is a main ingredient in the proof of Theorem B.

Since Wilking’s introduction of the today called Dual Foliation in [Wil07], the idea of exploiting weaker curvature conditions leading to obstructions on such a foliation became to be explored. Of particular interest in this work is the following result of Speranča, see [Spe18, Theorem 6.2]:

**Theorem 3.1** (Speranča). Let $(M, g)$ be a compact connected Riemannian manifold with a Riemannian foliation $\mathcal{F}$ of bounded holonomy and positive vertizontal curvature. Then, the dual foliation $\mathcal{F}^\#$ has only one leaf.

On the other hand, in [Spe17b] it is proved that

**Theorem 3.2** (Speranča). Let $(M, g)$ be a compact Riemannian manifold with non-negative sectional curvature with a Riemannian foliation $\mathcal{F}$ of totally geodesic leaves. Then, for each $x \in M$, if $L_x^\#$ denotes the dual leaf at $x$, it holds that

$$T_x L_x^\# \cap \mathcal{V}_x = \text{span} \{ A_X Y(x) : X, Y \in \mathcal{H}_x \}.$$
Therefore, the combination of the two former results recovers a geometrical connection between the Ambrose–Singer’s Theorem and fatness:

**Corollary 3.3.** Let \((M,g)\) be a compact connected Riemannian manifold with a Riemannian foliation \(\mathcal{F}\) of bounded holonomy and positive vertizontal curvature. Then for each \(x \in M\) it holds that \(\mathcal{V}_x = \text{span}\{A_XY(x) : X,Y \in \mathcal{H}_x\}\).

Only for the sake of completeness we both prove Theorem 3.2 plus add some comments on Riemannian foliations on non-negatively curved manifolds. We do not claim originality for the remaining of this section, the content described are original ideas of Speranča.

**Lemma 1.** Let \(\mathcal{F}\) be a totally geodesic Riemannian foliation with compact holonomy group on a compact non-negatively curved Riemannian manifold \((M,g)\). Then, for every \(x \in M\), there is a neighborhood of \(x\) and a \(\tau > 0\) such that

\[
\tau |X||Z||A^*_x \xi| \geq |(\nabla_X A^*_x \xi)X, Z| \tag{12}
\]

for all horizontal vectors \(X, Z\) and vertical vector \(\xi\).

**Proof.** Given \(X, Z \in \mathcal{H}\) and \(\xi \in \mathcal{V}\), O’Neill’s equations ([GW09, page 44]) ensure that the unreduced sectional curvature \(K(X, \xi + tZ) = R(X, \xi + tZ, \xi + tZ, X)\) is given by

\[
K(X, \xi + tZ) = t^2 K(X, Z) + 2t \langle (\nabla_X A)X Z, \xi \rangle + |A^*_X \xi|^2. \tag{13}
\]

Once \(g\) has non-negative sectional curvature it follows that \(K(X, \xi + tZ) \geq 0\). Therefore, the discriminant of \((13)\) (seen as a polynomial on \(t\)) must be non-negative. That is

\[
0 \leq K(X, Z)|A^*_X \xi|^2 - \langle (\nabla_X A)X Z, \xi \rangle^2.
\]

On small neighborhoods, continuity of \(K\) guarantees some \(\tau > 0\) such that \(K(X, Z) \leq \tau |X|^2|Z|^2\). Extending \(\xi\) to a holonomy field to the computations one concludes that

\[
\langle (\nabla_X A)Z Y, \xi \rangle = -\langle (\nabla_X A^*_X \xi)Z, Y \rangle
\]

for all horizontal \(X, Y, Z\). \(\Box\)

**Proposition 3.4.** Let \(\mathcal{F}\) be a totally geodesic Riemannian foliation with compact holonomy group on a compact non-negatively curved Riemannian manifold \((M,g)\). Let \(X_0 \in \mathcal{H}_p\) and \(\xi_0 \in \mathcal{V}_p\) be such that \(A^*_X \xi_0 = 0\). Then, the holonomy field along \(\xi(t)\) defined along \(c(t) = \exp(tX_0)\) is such that \(A^*_c(t)\) \(\xi(t) = 0\) for all \(t\).

**Proof.** Taking \(|X_0| = 1\) and \(Z = A^*_X \xi\) in \((12)\), we get:

\[
\tau |A^*_X \xi|^2 \geq \langle \nabla_c A^*_X \xi, A^*_X \xi \rangle = \frac{1}{2} \frac{d}{dt} |A^*_X \xi|^2. \tag{14}
\]

The Gronwall’s inequality for \(u(t) = |A^*_X \xi|^2\) and implies that

\[
|A^*_X \xi|^2 \leq |A^*_X(0)\xi(0)|^2 e^{2\tau t}
\]

for all \(t > 0\). In particular, if \(A^*_X(0)\xi(0) = 0\), then \(A^*_c(t)\xi(t) = 0\) for all \(t > 0\). The same argument works for \(t < 0\), by replacing \(X_0\) by \(-X_0\). \(\Box\)

Fixed a holonomy field \(\xi(t)\), the proof of Theorem 3.2 is finished with understanding of the distribution \(\mathcal{D}(t) = \ker(A^*_X : \mathcal{H}_{c(t)} \rightarrow \mathcal{H}_{c(t)})\). With the help of the previous results it is possible to prove that
Proposition 3.5 (Speranča). Let $X_0, \xi_0$ satisfy $A^*_X X_0 \xi_0 = 0$. If $\lambda^2$ is a continuous eigenvalue of $D$ along $c(t) = \exp(tX_0)$, then either $\lambda$ vanishes identically, or $\lambda$ never vanishes.

Theorem 3.2 then follows from the classical Ambrose–Singer’s Theorem combined with Proposition 3.5:

Sketch of the proof of Theorem 3.2. Combining the dual leaf terminology with the results in [Spe18] it is not hard to see that the classical Ambrose–Singer’s Theorem can be stated as

Theorem 3.6 (Ambrose–Singer). Let $F$ be a Riemannian foliation on a path connected $M$. Denote $a_q = \text{span}_R \{ A_X Y \mid X,Y \in H_q \}$. Then, for every $p \in M$,

$$TL^#_p = H_p \oplus \text{span}_R \{ \hat{c}(1)^{-1}(a_{c(1)}) \mid c : [0,1] \to M \text{ horizontal} \},$$

where $\hat{c}(1)^{-1}$ is the holonomy transport along $c$ from the time $t = 1$ to the time $t = 0$.

Let $p \in M$. Observe that

$$a^\perp_p = \{ \xi \in V_p \mid A^*_X \xi = 0 \ \forall X \in H_p \}.$$

Claim 2. If $c$ is horizontal curve, then $\hat{c}(1)(a^\perp_p) = a^\perp_{c(1)}$.

Proof. It is sufficient to prove the claim for horizontal geodesics, since $c$ can be smoothly approximated by piecewise horizontal geodesics. Let $c$ be a horizontal geodesic, $c(0) = p$, and $\xi(t)$ be a holonomy field with $\xi(0) \in a^\perp_p$. Then $\ker A^* \xi(0) = H_p$ and $\dim \ker A^* \xi(t)$ is constant with respect to $t$ (Proposition 3.5). Thus $A^* \xi(t) = H_{c(t)}$ for all $t$. $\square$

The proof if finished using that $\hat{c}(1)$ is an isometry since Claim 2 implies $\hat{c}(1)(a_p) = a_{c(1)}$. $\square$

3.2. Fat Riemannian Foliations on Compact Lie groups. On the one hand, in 1986 Ranjan asked whether a Riemannian submersion $\pi : (G, Q) \to (B, h)$ from a compact simple Lie group with a bi-invariant metric $Q$ is a coset foliation provided the foliation induced by the submersion is totally geodesic. On the other hand, since Riemannian submersions are main examples of Riemannian foliations, the classification of Riemannian submersions from compact Lie groups with bi-invariant metrics was asked by Grove (Problem 5.1 in [Gro]).

Paraphrasing [Spe17b], his question can be motivated as: most examples of manifolds with positive sectional curvature are related to Riemannian submersions from Lie groups. On the other hand, given a compact Lie group $G$ with bi-invariant metric, it is known that homogeneous foliations are Riemannian and have totally geodesic leaves. Therefore, it is natural to ask whether every Riemannian foliation with totally geodesic leaves is homogeneous or not.

In [Spe17b] Speranča answered Ranjan’s question in a more general setting: when such a submersion is defined only in an open subset of $G$ (with further compactness hypotheses). To our interests, we need:

Theorem 3.7 (Speranča–Grove–Ranjan). Any totally geodesic Riemannian foliation on a compact connected Lie group $G$ with a bi-invariant metric $Q$ is isometric to the coset foliation induced by a subgroup $H$ of $G$. 
As the main result in this section we prove a sort of better classification to Riemannian foliations on Lie groups with bi-invariant metrics, with the additional assumption of fatness. Note that we do not ask for totally geodesic fibers though:

**Theorem 3.8.** If \((G, Q)\) is a compact connected Lie group \(G\) with a bi-invariant metric \(Q\), any fat Riemannian foliation \(\mathcal{F}\) on \(G\) is given by the co-set foliation induced by some Lie subgroup \(H\) of \(G\). Moreover, the leaf space \(G/H\) is an even dimensional symmetric space and \(H\) is either a single point, \(S^1\), \(SO(3)\) or \(S^3\).

**Proof.** Let \(G\) be a compact connected Lie group with a fixed bi-invariant metric \(Q\). Assume that there is a Riemannian foliation \(\mathcal{F}\) on \((G, Q)\). If such a foliation is fat, which is a metric independent condition, for the metric \(\tilde{Q}\) obtained accordingly to Theorem 2.5 we can suppose without loss of generality that \(\mathcal{F}\) has totally geodesic leaves, furthermore, that the vertizontal curvature computed with respect to \(\tilde{Q}\) is positive. Since such a metric is still a bi-invariant metric, Theorem 3.7 implies that there exists a subgroup \(H\) of \(G\) for which \(\mathcal{F}\) is recovered as the coset foliation induced by \(H\). Moreover, at the identity of \(G\) one writes the following Ad\((H)\)-invariant decomposition

\[ g = \mathfrak{h} \oplus \mathfrak{m}, \]

where \(\mathfrak{m} \perp \mathfrak{h}\). Observe that it also imposes the stronger restriction that the leaf space \(G/\mathcal{F} \cong G/H\) and then, if \(o = eH\) it holds that \(\mathfrak{m} \cong T_o(G/H)\) and \([\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}\). Now, since \(H \hookrightarrow G \rightarrow G/H\) is a Riemannian submersion with \(G/H\) a homogeneous space, to obtain that \(G/H\) is a symmetric space, we only need to check that \([\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}\), though it follows via directly applying Theorem 2.2.

Finally, since \(A^*\) is an anti-symmetric tensor, whenever the foliation has odd co-dimension in \(M\) it should follows that for each point \(x \in M\) and each \(X \in \mathcal{H}_x\) it does exist \(V \in \mathcal{V}_x\) for which \(A^*_x V = 0\), contradicting fatness. Hence, the dimension of \(G/H\) is even. The possible reductions to \(H\) is due to the last part of the thesis in Theorem B: if \(\dim H > 1\) it must be a positively curved Lie group, which results in \(H\) to be either \(S^3\) or \(SO(3)\).

\[\square\]

4. **RIEMANNIAN HOMOGENEOUS BUNDLES: A CLASSIFICATION UNDER THE FATNESS CONDITION**

In this section we achieve the final goal of this paper: recovering the Bérard-Bergery classification obtained in [BB75], plus adding some better understanding to the general theory of fat bundles which turns to be possible due to the new Theorem B.

So let us consider the following homogeneous Riemannian fat bundle with \(G\) compact and connected and \(K < H < G\):

\[ H/K \hookrightarrow (G/K, b) \rightarrow (G/H, \bar{b}) \]

where \(G/K \cong P \times_H (H/K)\) and \(H \hookrightarrow P \rightarrow G/H\) is the principal reduced holonomy bundle associated with the homogeneous Riemannian foliation \(\mathcal{F} = \{H/K\}\). We assume that \(b\) has non-negative sectional curvature.

Theorem B implies that if \(\dim H/K > 1\) then not only \(H/K\) is a symmetric space of compact type but it has positive sectional curvature, thus, its rank is 1. On the other hand, we do know that the dimension of \(G/H\) is even. More can be said indeed, if \(2 \leq \dim H/K \leq 3\) then \(\dim G/H\) is a multiple of 4, meanwhile \(\dim H/K \geq 4\) implies that \(\dim G/H\) is a multiple of 8, see Proposition 2.5 in [Zil00].
We shall prove, just as Béard-Bergery, that \( G/H \) is also a symmetric space of compact type. Moreover, the rank of \( H \) is the same as the rank of \( G \). These info combined then impose the following very strong restriction on the triples \( (K,H,G) \):

- \( H/K \) is a rank one symmetric space of positive sectional curvature, thus, following either [WZ15] or [Zil14], \( H/K \cong S^n, CP^n, HP^n, CaP^2 \) we get the following table where \( H \) is a simple Lie group, see [Gor21] for the needed tables classifying symmetric spaces of class \([A]\):

| \( H \)       | \( K \)       | \( \dim H/K \) |
|--------------|--------------|--------------|
| SO(3)        | \{e\}        | 3            |
| SO(3)        | SO(2)        | 2            |
| SO(2n + 1)   | SO(2n)       | 2n           |
| SO(n)        | U(n)         | \( n(n-1) \) with \( n = 2,3 \) |
| Spin(n)      | \{e\}        | \( n = 3 \)  |
| SU(2)        | \( \mathbb{Z}_2 \) | 3           |
| SU(2)        | \{e\}        | 3            |
| SU(2)        | Sp(1)        | 5            |
| SU(2)        | SO(2)        | 2            |
| SU(n + 1)    | S(U(n) \times U(1)) | 2n         |
| Sp(n + 1)    | Sp(n) \times Sp(1) | 4n        |
| Sp(1)        | U(1)         | 2            |
| Sp(1)        | \{e\}        | 3            |
| SO(2n + 4)   | SO(2n + 1) \times SO(3) | 3(2n + 1)  |
| \( F_4 \)    | Spin(9)      | 16           |

Table A: Positively curved rank 1 symmetric spaces.

- once \( G/H \) is a symmetric space of compact type with \( H \) as in Table A, one can infer the possible candidates to \( G \) out of Table B below: observe that the only possibilities for \( G \) are those which \( H \) appears in Table A. See for instance [Hel62, Table V, p.518] to the classification of classical symmetric spaces of compact type, or [Gor21] for a complete classification of symmetric spaces.

| \( G \)       | \( H \)       |
|--------------|--------------|
| SU(3)        | SO(3)        |
| SU(2n + 1)   | SO(2n + 1)   |
| SU(n + 2)    | SU(n + 1)    |
| SU(2(n + 1)) | Sp(n + 1)    |
| SU(2)        | Sp(1)        |
| SU(2n + 4)   | SO(2n + 4)   |
| \( G_2 \)    | SO(4)        |
| \( F_6 \)    | \( F_4 \)    |

Table B: Restrictions to \( G \).

Summing up the collected info, we have Theorem [C] which we choose to restate below:
Theorem 4.1. Let $G$ be a compact simple and connected Lie group and consider the following Lie subgroups $K < H < G$. Consider the following fat Riemannian submersion with totally geodesic leaves

$$H/K \hookrightarrow (G/K, b) \rightarrow (G/H, \bar{b}),$$

where $G/K \cong P \times_H (H/K)$ with $H \rightarrow P \rightarrow G/H$ being the principal reduced holonomy bundle associated with the homogeneous Riemannian foliation $\mathcal{F} = \{H/K\}$. Assume that $b$ has non-negative sectional curvature and that $\dim H/K > 1$. Then the triple $(K,H,G)$ can be one of the following:

| $G$       | $H$       | $K$            |
|-----------|-----------|----------------|
| SU(3)     | SO(3)     | $\{e\}, SO(2)$|
| SU(2n+1)  | SO(2n+1)  | SO(2n)         |
| SU(n+2)   | SU(n+1)   | $S(U(n) \times U(1))$ |
| SU(2(n+1))| Sp(n+1)   | $Sp(n) \times Sp(1)$ |
| SU(2n+4)  | SO(2n+4)  | $SO(2n+1) \times SO(3)$ |
| $G_2$     | SO(4)     | $U(4), SO(3)$  |
| $E_6$     | $F_4$     | Spin(9)        |

Proof. Assuming that $G/H$ is a symmetric space of positive sectional curvature, since the dimension of $G/H$ is even, Lemma 1.2 in [WZ15] implies that $\text{rank} G = \text{rank} H$. Let $e$ be the unit element in $G$. To see that $G/H$ is a symmetric space one observes that: $\pi : H \rightarrow G \rightarrow G/H$ principal bundle is $K$-fat. Indeed, observe that $H/K \hookrightarrow H \rightarrow G \rightarrow G/H$ is the associated bundle to $\pi$ with fiber $H/K$, just as in the proof of Proposition 2.3.

Moreover, considering an $\text{Ad}(H)$ invariant splitting $g = \mathfrak{h} \oplus \mathfrak{m}$ we can consider a principal connection $\omega$ on $\pi$ with $G$-invariant horizontal space given by $H \cong \mathfrak{m}$, and an induced horizontal homogeneous distribution $\mathcal{H}$ for the fibration $G/H \rightarrow G/K$, with $\mathcal{H}_e K \cong \mathfrak{m}$.

On the other hand, an $\text{Ad}(K)$-invariant splitting $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$ for a choice of an $\text{Ad}(K)$ invariant metric on $\mathfrak{p}$ and an $\text{Ad}(H)$-invariant metric on $\mathfrak{m}$ induces homogeneous metrics on $H/K$ and $G/H$ via the identifications $T_e K H/K \cong \mathfrak{p}$ and $T_e H G/H \cong \mathfrak{m}$. Therefore, it is possible to define a metric on $G/K$ such that $T_e K G/K \cong \mathfrak{p} \oplus \mathfrak{m}$.

That all gathered we see that for any non-degenerated plane $X \wedge U$ with $X \in \mathfrak{m}$ and $U \in \mathfrak{p}$ it holds that $0 \neq [X, U] \in \mathfrak{m}$. Therefore, denoting by $\langle \cdot, \cdot \rangle$ the fixed bi-invariant metric on $G$ from which $b$ descends from, we have

$$0 \neq \langle [X, U], \mathfrak{m} \rangle = \langle [X, \mathfrak{m}], U \rangle.$$

Thus, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{p} \subset \mathfrak{h}$, finishing the proof.

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