Gauge Theory of Oriented Media.

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The concept of wave field is introduced to represent oriented media. The wave field is a tensor field of second rank, and directors are its eigenvectors. This exhibition of directors defines a natural gauge group inherent in continua and allows one to derive from variational principle general relativistic and gauge invariant equations for the wave field in question. Thus, the gauge-theoretical approach to continuum with internal degrees of freedom gives unambiguous and minimally coupled theory.

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I. INTRODUCTION

In generalized continuum mechanics dealing with oriented media the fundamental structure is a set of vectors adjoined to earth point of medium. These vectors which can deform independently of the displacements of points of media are called directors. They define a microstructure of continuum. In the theory of spinning fluids, the name tetrads is also used. At present, this part of mechanical science is fairly well accepted and is widely applied for the description of media possessing not only energy - momentum but also internal degrees of freedom.

There exists an extended literature on continua with internal degrees of freedom, see, for example, Eugene Cosserat and Francois Cosserat [1], Weyssenhoff and Raabe [2], Maugin and Eringen [3] and references herein, Halbwachs [4], Minkevich and Karakura [5], Berman [6]. In the theory of elastic continua with defects the gauge approach was successfully used by Osipov [8].The theory of spinning fluid in generalized space-time manifolds was developed by Ray and Smally [7] as an extension of the theory of spinning fluids in special relativity. The Cosserat brothers were first who introduced the notion of a 3-tuple of unit rigid directors and laid down the mathematical foundations of the theory now a day known as the Cosserat continuum.

The goal of present paper is to formulate the most general gauge-theoretical approach to the theory of oriented media. To do this, the notion of a wave field is introduced to represent generalized continua. The wave field is characterized so as to define directors and other properties of the system in question, in particular, the form of interaction of this matter with the gravitational field. The wave field allows one to derive a minimally coupled and simple theory from the first principles which have fundamental meaning in contemporary physics.

II. FORMULATION OF THE PROBLEM

The key idea of the present consideration is that directors are defined as solutions of the eigenvalues problem of the form

$$\Psi^j_i h^j = \lambda h^i,$$  \hspace{1cm} (2.1)

where the matrix $\Psi^j_i$, is a tensor field of the type (1,1) and, respectively, the wave field of oriented media. The internal state of media is then characterized by the number of linear independent eigenvectors (directors) $h^i$ which define the type of microstructure of continua. Now, the problem is to derive a wave equation for the field $\Psi^i_j$ from the first principles.

We remark that equations (1) are invariant under the transformations

$$\bar{\Psi}^i_j = S^i_k \Psi^k_l T^l_j,$$  \hspace{1cm} (2.2)

$$\bar{h}^i = S^i_j h^j,$$  \hspace{1cm} (2.3)

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where \( S^i_j \) are components of the tensor field \( S \) of type (1,1) that satisfy the condition \( \text{det}(S^i_j) \neq 0 \). In this case, there exists an inverse transformation \( S^{-1} \) with components \( T^i_j \) such that \( S^i_k T^k_j = \delta^i_j \). It is evident that substitutions (2) form a local group of transformations, and equations for the field \( \Psi \) should be invariant with respect to this gauge symmetry group. As it is well-known, the matrix \( \Psi \) by the gauge transformation can be reduced to the canonical or Jordan form that is defined by the characteristic equation

\[
|\Psi^i_j - \lambda \delta^i_j| = 0.
\]

The Einstein General Covariance is the other deep guiding principle that we have at our disposal. In this framework, the wave equation for \( \Psi \) is defined in fact uniquely.

To derive general covariant Lagrangian of first order for \( \Psi \), it is natural to take \( D_i \Psi^i_k \) to be a tensor. Of course, the second gauge-covariant derivative should not be a tensor, but if we deal with the general covariant Lagrangian of first order, then by varying we will get a combination of gauge-covariant derivatives such that equation of second order will be tensor equation.

Let

\[
\tilde{x}^i = \tilde{x}^i(x^0, x^1, x^2, x^3), \quad x^i = x^i(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)
\]

coordinate transformation. Put

\[
X = \left( \frac{\partial \tilde{x}^i}{\partial x^j} \right), \quad X^{-1} = \left( \frac{\partial x^i}{\partial \tilde{x}^j} \right),
\]

then the gauge potential transforms as follows

\[
\tilde{G}_i = XG_kX^{-1} \frac{\partial x^k}{\partial \tilde{x}^i} + X \frac{\partial}{\partial x^i}X^{-1}.
\]

Since the tensor field \( \Psi \) transforms under gauge and coordinate transformations by the law \( \tilde{\Psi} = S\Psi S^{-1} \), \( \tilde{\Psi} = X\Psi X^{-1} \), then the traces \( \tilde{\Psi}^j_i = \text{Tr} \Psi \) and \( \tilde{\Psi}^j_i \tilde{\Psi}^k_l = \text{Tr}(\Psi \Psi) \) are evidently invariants of the gauge group and scalar with respect to the general coordinate transformations. It is known from the theory of linear operators that there also exist other invariants, but in what follows we will only use these simplest ones.

To get an expression for the tensor of strength of the gauge field, consider the commutator of covariant derivatives \([D_i, D_j]\). From (4) it follows that

\[
[D_i, D_j]\Psi = [H_{ij}, \Psi], \tag{2.7}
\]

where

\[
H_{ij} = \partial_i G_j - \partial_j G_i + [G_i, G_j] \tag{2.8}
\]

is the strength tensor of the gauge field with the following properties

\[
\tilde{H}_{ij} = SH_{ij}S^{-1}, \quad [D_i, D_j]\tilde{H}_{kl} = [H_{ij}, H_{kl}]. \tag{2.9}
\]

From (8) it follows that \( H_{ij} = (H^k_{ij}) \) is a tensor of the type (1,3) with respect to the general coordinate transformation.
III. GAUGE-INVARIENT EQUATIONS

Now we have all that is required to derive the simplest general covariant and gauge-invariant equations for fields $G^i$ and $\Psi$. In what follows $g_{ij}$ are the Einstein gravitational potentials, $g^{ij}$ are components of tensor inverse to $g_{ij}$, $\det g = \delta_i^j$. As it is known, the determinant $|g_{ij}| \neq 0$, which actually allows one to obtain, for the tensor field $g_{ij}$, the equations invariant under the general coordinate transformations. By analogy, let us consider the case when the determinant $|\Psi^{ij}| \neq 0$. Under this condition the field $\Psi$ has the inverse one and the nonlinear gauge-invariant equations can be suggested. This means that we consider the case of non-linear continuum mechanics with internal degrees of freedom. The linear case can be consider then as an approximation.

Thus, the gauge-invariant and general covariant Lagrangian of first order has the form

$$L = -\frac{a}{2}\text{Tr}(D_i \Psi D^i \Psi^{-1}) - \frac{b}{4}\text{Tr}(H_{ij} H^{ij}),$$

(3.1)

where $a$ and $b$ are constants,

$$D^i = g^{ij} D_j \text{ and } H^{ij} = g^{ik} g^{jl} H_{kl}.$$  

The gauge potential has dimension $cm^{-1}$, $\Psi$ is dimensionless. Taking into account that

$$\delta \Psi = -\Psi (\delta \Psi^{-1}) \Psi$$

and varying (10) with respect to $\Psi$ and $G^i$, we obtain the following equations of second order for basic fields

$$D_i (\sqrt{|g|} \Psi^{-1} D^i \Psi) = 0,$$

(3.2)

$$D_i (\sqrt{|g|} H^{ij}) = \sqrt{|g|} J^i,$$

(3.3)

where $|g|$ is the absolute value of the determinant of the matrix $(g_{ij})$ and

$$J^i = [\Psi^{-1}, D^i \Psi].$$

The tensor current $J$ has to satisfy the equation

$$D_i (\sqrt{|g|} J^i) = 0$$

as in accordance with (9),

$$D_i D_j (\sqrt{|g|} H^{ij}) = 0.$$  

Since this is really so, the system of equations (11) and (12) is compatible.

Varying the Lagrangian (10) with respect to $g_{ij}$, we obtain the gauge-invariant metric tensor of energy-momentum of oriented media

$$T_{ij} = a\text{Tr}(D_i \Psi D_j \Psi^{-1}) + b\text{Tr}(H_{ik} H^k_j) + g_{ij} L$$

which satisfies the equation

$$T^{ij} ;_i = 0.$$  

(3.4)

The semicolon denotes the covariant derivative with respect to the Levi-Civita connexion belonging to the field $g_{ij}$

$$D^i \Psi = \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}).$$

When deriving (13), besides equations of fields, one should use the standard relations [9] for the Christoffel symbols $\{^i_{jk}\}$ and the identity

$$D_i H_{jk} + D_j H_{ki} + D_k H_{ij} = 0.$$
which can easily be obtained with the help of relation (7). From (13) and the gauge invariance of the metric tensor of energy-momentum it follows that the Einstein equations

\[ R_{ij} - \frac{1}{2} g_{ij} G = \frac{G}{c^4} T_{ij}, \]

derived from the Lagrangian \( L_f = \mathcal{L} + \mathcal{L}_g \), where \( \mathcal{L}_g = \frac{c^4}{8} R \) is the Einstein-Hilbert Lagrangian, will be compatible. In the linear approximation \( \Psi^i_j = \delta^i_j + M^i_j \), we have from (11) the following equation for the matrix \( M = (M^i_j) \)

\[ D_i(\sqrt{|g|} D^i M) = 0. \tag{3.5} \]

We will say that continuum admits elastic deformations if the vector field \( V^i \) exists such that

\[ D_i V^j = 0. \tag{3.6} \]

If equation (15) has a nontrivial solution, then from (14) and (15) it follows that the vector field \( U^i = M^i_k V^k \) obeys the equation

\[ D_i(\sqrt{|g|} D^i U^j) = 0 \tag{3.7} \]

that can be considered as a general covariant and gauge-covariant analog of the known equation of elastic deformations of media.

Remark some more special cases of the general picture outlined. The consideration of rigid directors or generalized Cosserat continua means, in the framework of the gauge approach, the reduction of the gauge group defined by the condition

\[ S^i_k S^j_l g_{ij} = g_{kl}. \]

If we should like to set aside some director then we have the following constraint on the group of gauge transformations

\[ S^i_j h^j = h^i. \]

At last, in absence of gravitational field we put \( g_{ij} = \text{diag}(1, -1, -1, -1) \). We do not see other cases to be mentioned.

Some very interesting gauge-invariant quantities can be constructed from the fields \( \Psi \) and \( H_{ij} \). We dwell here on some of them. The invariants, in particular, are

\[ \varphi = \text{Tr} \Psi, \quad \Delta = |\Psi^i_j|, \quad F_{ij} = \text{Tr} H_{ij}. \]

If \( \Psi \) obeys equation (11), then taking the trace of both sides of this equation we obtain that the invariant \( \Delta \) satisfies the equation

\[ \partial_i(\sqrt{|g|} g^{ij} \partial_j \ln |\Delta|) = 0. \]

From (12) it follows that the bivector \( F_{ij} \) satisfies the free Maxwell equations

\[ \partial_i(\sqrt{|g|} F^{ij}) = 0. \]

Let \( Q_i = tr G_i = G^k_{i;k} \). According to (6) and the differentiation rule for determinants, the transformation law for \( Q_i \) under gauge transformations has the form

\[ \bar{Q}_i = Q_i - \partial_i \ln |D|, \]

where \( D = \det(S^i_j) \). From (8) it follows that \( F_{ij} = \partial_i Q_j - \partial_j Q_i \).

A state of media \( (\Psi, H_{ij}) \) is said to be singlet if it is invariant under all the symmetry transformations. In our case a singlet state is given by the equations

\[ \Psi = S \Psi S^{-1}, \quad H_{ij} = S H_{ij} S^{-1} \]

to be satisfied at any \( S \). The first equation has the solution

\[ \Psi = e^{\alpha} E, \quad \Psi^{-1} = e^{-\alpha} E, \]

where \( \alpha \) is a scalar field. In this case all directions are interchangeable. If the gauge field obeys the equation \( H_{ij} = S H_{ij} S^{-1} \), it also obeys the equation \( H_{ij} = 1/4 F_{ij} E \). Thus, a singlet state of media is represented by the scalar field and 1-form \( Q_i dx^i \).
IV. CONCLUSION

Outline the main principles of gauge approaches to the continua with internal degrees of freedom. In the proposed theory directors are not fundamental objects and are treated as solutions of the algebraic equation (1) at given matrix $\Psi$ which is a wave field that corresponds to the continuum in question and define all important quantities of oriented media. The field $\Psi$ is deduced from equations (11) and (12). The gravitational interactions of oriented media are described by the Einstein equations with the energy-momentum tensor of continua given by equation (13). For equations presented, the gauge invariance holds necessary valid in a sense that if $\Psi, G_i$ is a solution then

$$\bar{\Psi} = S\Psi S^{-1}, \quad \bar{G}_i = SG_iS^{-1} + S\partial_i S^{-1}$$

is a solution as well, where $S$ is any nondegenerate tensor field of the type (1,1). Such a general realization of gauge principle presupposes not only direct application of the equations in question but allows one to put forward the conjecture that oriented media presented above are a new fundamental type of matter.
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