Discrete Field Theory: Symmetries and Conservation Laws

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Abstract
We present a general algorithm constructing a discretization of a classical field theory from a Lagrangian. We prove a new discrete Noether theorem relating symmetries to conservation laws and an energy conservation theorem not based on any symmetry. This gives exact conservation laws for several theories, e.g., lattice electrodynamics and gauge theory. In particular, we construct a conserved discrete energy–momentum tensor, approximating the continuum one at least for free fields. The theory is stated in topological terms, such as coboundary and products of cochains.

Keywords Discrete field theory · Discrete differential geometry · Conservation law · Noether’s theorem

Mathematics Subject Classification 49M25 · 49S05 · 55N45 · 81T25

1 Introduction
This work is a try to build a general discrete field theory. This has the following motivation:

• getting effective numeric algorithms for field theory;
• putting field theory to a mathematically rigorous basis;
• creating an alternative candidate for a fundamental field theory.

Numerous discretizations of particular field theories are known [1, 10–12, 15–17, 20]. Our aim is not to invent new discretizations but to extract and study the best among the known ones. Discretizations exhibiting exact (not just approximate) conservation
laws have been proved to be most successful for computational purposes [15]. This leads us to the following principles of discretization:

- keep approximation of continuum theory;
- keep conservation laws exact;
- drop spatial symmetries easily.

These principles have a built-in difficulty: we have to drop most continuous symmetries, but usually, conservation laws are obtained just from such symmetries using the Noether theorem. We develop a new method to get discrete conservation laws. Compared to [13, 15, 18, 19, 24], it allows to write the conservation laws explicitly as one-line formulae (using standard topological notation) in numerous examples.

The following basic warm-up results of discrete field theory are obtained in the present paper:

- discretization of several field theories in a similar fashion keeping conservation laws exact (Sect. 2);
- a new discrete Noether theorem relating symmetries to conservation laws (Theorems 1.4.7 and 3.1.8);
- a new discrete energy conservation theorem not based on symmetry (Theorems 1.4.9 and 2.2.9).

1.1 Quick Start

We start with an elementary and informal description of one result (Theorem 2.2.9), in the simplest unknown particular case. It is an energy conservation theorem for lattice electrodynamics in 2 spatial and 1 time dimensions. For these small dimensions, we just draw everything. The more realistic case of 3 spatial and 1 time dimensions is analogous; see Sect. 2.2, where we state the result precisely.

Recall briefly the energy conservation theorem in continuum electrodynamics (the Poynting theorem). Let $x, y, t$ be the Cartesian coordinates in space; see Fig. 1. Electric and magnetic fields are arbitrary smooth vector-valued functions $\vec{E}(x, y, t)$ and $\vec{B}(x, y, t)$.
\vec{B}(x, y, t) such that \vec{E} \perp Ot and \vec{B} \parallel Ot. The \textit{energy density} and the \textit{energy flux} (the Poynting vector) are the functions \( \frac{1}{2} (\vec{E}^2 + \vec{B}^2) \) and \( \vec{E} \times \vec{B} \). The Poynting theorem asserts that under Maxwell’s equations (where \( \vec{E} =: (E_x, E_y, 0) \) and \( \vec{B} =: (0, 0, B_t) \))

\[
\begin{align*}
\frac{\partial B_t}{\partial t} + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= 0; \\
\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} &= 0; \\
\frac{\partial B_t}{\partial x} + \frac{\partial E_y}{\partial t} &= 0; \\
\frac{\partial B_t}{\partial y} - \frac{\partial E_x}{\partial t} &= 0;
\end{align*}
\]

the following identity holds for each cube with the edges parallel to the coordinate axes:

\[
\int \frac{\vec{E}^2 + \vec{B}^2}{2} \, dA - \int \frac{\vec{E}^2 + \vec{B}^2}{2} \, dA = \int \vec{E} \times \vec{B} \, d\vec{n}.
\]

Here the cube is shown by dotted lines, and the faces which a particular integral is taken over are in bold. The first two integrals mean the total energy contained in the same square in the \( Oxy \) plane at two different moments of time \( t \). The third integral means the total inward energy flux through the boundary between these two moments. Thus the equation means energy conservation.

Let us discretize. Dissect the unit cube into \( N \times N \times N \) equal cubes. Throughout this subsection by \textit{cubes} we mean the latter cubes, by \textit{faces} and \textit{edges}—their faces and edges. A discrete \textit{electromagnetic field} \( F \) is any real-valued function on the set of faces. Informally, its values \( F(\square), F(\square), F(\square) \) discretize \( -B_t, E_y, E_x \) respectively, depending on face direction (for exterior-calculus fans: \( F \) itself discretizes the \textit{electromagnetic field} \( F = -B_t \, dx \wedge dy + E_x \, dt \wedge dx + E_y \, dt \wedge dy) \). Hereafter a particular face at which the function is evaluated is in bold, and one of the adjacent cubes is shown by dotted lines to identify the face position. The well-known discrete Maxwell’s equations are

\[
\begin{align*}
F(\square) - F(\square) - F(\square) + F(\square) + F(\square) - F(\square) &= 0; \\
F(\square) - F(\square) - F(\square) + F(\square) - F(\square) &= 0; \\
F(\square) - F(\square) - F(\square) + F(\square) &= 0; \\
F(\square) - F(\square) + F(\square) - F(\square) &= 0.
\end{align*}
\]

Here we sum the values of \( F \) at the faces of a particular cube (in the first equation) or the faces containing a particular edge (in the other equations), with appropriate signs. We write one equation per cube and one per non-boundary edge and impose no boundary conditions.
It’s time for our new definition. Let $T$ be the function on the set of non-boundary faces given by

\[
T(\overline{\Box}) = \frac{1}{2} \left[ F(\overline{\Box}) \cdot F(\overline{\Box}) + F(\overline{\Box}) \cdot F(\overline{\Box}) + F(\overline{\Box}) \cdot F(\overline{\Box}) \right];
\]

\[
T(\overline{\Box}) = \frac{1}{2} \left[ F(\overline{\Box}) \cdot F(\overline{\Box}) + F(\overline{\Box}) \cdot F(\overline{\Box}) \right];
\]

\[
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\]

For instance, the latter equality expresses the value of $T$ at a vertical face parallel to the $x$-axis through the values of $F$ at the same face and the two horizontal faces right behind it. The value of $T$ at a horizontal (respectively, vertical) face discretizes energy density (respectively, flux). Proposition 2.2.13 below asserts that under a natural choice of $F$ we have uniform convergence as $N \to \infty$:

\[
T(\overline{\Box}) \Rightarrow N^2 \int \frac{1}{2}(\vec{E}^2 + \vec{B}^2) \, dA, \quad T(\overline{\Box}) \Rightarrow -N^2 \int \vec{E} \times \vec{B} \, d\vec{n}, \quad T(\overline{\Box}) \Rightarrow N^2 \int \vec{E} \times \vec{B} \, d\vec{n}.
\]

The desired discrete Poynting theorem (particular case of Theorem 2.2.9 below) asserts that assuming only Maxwell’s equations (1), we have the following identity for each non-boundary cube:

\[
T(\overline{\Box}) - T(\overline{\Box}) - T(\overline{\Box}) + T(\overline{\Box}) + T(\overline{\Box}) - T(\overline{\Box}) = 0.
\]

Properties (2)–(3) are exactly what one requests from a discretization of energy density and flux according to the above discretization principles; it is nontrivial to satisfy both properties simultaneously. A proof in pictures is in §4.2. And we proceed to a systematic discussion of discrete field theory.

### 1.2 Background

Discrete field theory is actually at least as old as the continuum one. In 1847, Kirchhoff stated the laws of an electrical network, which is the simplest model of the theory; see Sect. 2.1. In the continuum limit, the laws approximate the Laplace equation; thus the model perfectly serves for the numerical solution of the latter. Remarkable approximation theorems were proved by Lusternik [21], Courant et al. [10] in 1920s and later generalized, e.g., in [4, 6, 8, 30, 33]. Planar networks lead to the discretization

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of complex analysis having applications in statistical physics (e.g., obtained in the 2010s by Smirnov et al. [6]) and even computer graphics [16].

Discrete field theory was closely related to topology from the youth of both subjects. The Kirchhoff laws are naturally stated in terms of the boundary and the coboundary operators; see Sect. 2.1 for an elementary introduction. Such formulation is usually attributed to Weyl; see [17, Sect. 1F, p. 31] for an elaborate historical survey. In the 1930s, de Rham established a correspondence between these operators and the exterior derivative and its dual; see [1] for a survey and [29] for general philosophy. This lead to the above discrete Maxwell equations (1); see also Sect. 2.2 and [5, 17, 20, 28].

The next major step was done by Kolmogorov and Alexander in the 1930s, who invented a product discretizing the exterior product in a sense. Kolmogorov commented that such discretization was his original motivation. The construction was soon modified by Whitney [31] and others to give the famous cup product. The original product was anticommutative, whereas the cup product was associative. One cannot get both properties simultaneously (this fact is crucial for rational homotopy theory). This reflects a general phenomenon that not all properties survive under discretization. We choose the associative cup product as a discretization of the exterior product, in contrast to [3, 16, 33]. This requires vertices ordering in discrete field theory, a structure introduced for the first time.

Later there appeared discrete models for other classical fields: e.g., Feynman checkerboard from the 1940s and Regge calculus from the 1960s for the Dirac and the gravitational field respectively; see [26] for an elementary introduction and survey of the former model.

In the 1970s, Wegner and Wilson introduced lattice gauge theory as a computational tool for gauge theory describing all known interactions; see [22] or Sect. 2.3 for an elementary introduction and [11] for details. This culminated in determining the proton mass theoretically with an error < 2%.

In the 1980s, Connes developed a formalism, dealing (to some extent) uniformly with continuous and discrete geometries [9]. Using it, Dimakis et al. discretized the Yang–Mills equations [12, Eq. (4.15)]. Corollary 2.3.5 extends their result by adding sources and the crucial unitarity constraint. Compare this with the efforts put to achieve the gauge covariance in the remarkable survey [2, §9].

In the 1990s, Forman [14] introduced a different discretization of differential forms, using cochain maps rather than cochains. Wilson [33] and Berbatov et al. [3] equipped them with the cup product (again, non-associative) to discretize Riemann surfaces and the Riemannian metric tensor respectively. Although their setup is different, generalized Forman’s forms appear in our energy conservation law.

In the late 1990s, Marsden et al. discretized basic general theorems of field theory: the Euler–Lagrange equations and the Noether theorem on a 2-dimensional grid; see [24, Eqs. (5.2) and (5.7)], cf. [19, Eq. (60) and (72)], [18, Theorem 5.2.37], [23, Theorem 5.5], [13, Theorems 7.1 and 8.1 in Ch. III]. These results extended the ones obtained earlier for 1-dimensional difference equations; see [18] for references. The discrete Euler–Lagrange equations in Sect. 1.4 are straightforward generalizations of the known ones. But Discrete Noether Theorem 1.4.7 is different: we construct a conserved current through edges just as in the Kirchhoff law, whereas in previous works, the current was defined on vertices [13], pairs (triangle, its vertex) [24], and
pairs (square, its vertex) [19]. In [23], the conservation law was stated in a global form, bypassing a construction of a current. This all led to rather technical statements of the conservation laws, to our knowledge, never applied to a particular field theory; see [23] for a survey. For the first time we use vertices ordering and cap product, making the statement, proof, and applications of the discrete Noether theorem particularly simple.

In the 2010s, Kraus et al. have stepped beyond the Lagrangian formulation [19]. A discretization of hydrodynamics was introduced by Gawlik et al. [15, Sect. 4]. They derived general Euler–Poincare equations and Kelvin–Noether theorem [15, Sect. 3]. Their approach was based on the discretization of the diffeomorphism group, thus applied to a rather specific class of models.

There was a folklore belief that no conserved discrete energy–momentum tensor exists in this framework. e.g., in 2016 Chelkak, Glazman, and Smirnov introduced a “halfway” conserved tensor [7, Corollary 2.12(1)]; cf. [27]. Even the notion of a rank 2 symmetric tensor itself is hard to discretize [1, Sect. 7]. But in the 2000s, Dorodnitsyn discretized energy and momentum conservation in some particular cases [13, Example in Sect. 8 of Ch. III]. His construction, like the other known ones, was based on moving the points of a 2-dimensional lattice. We extend it using a new approach not relying on any continuous motion or symmetry. As a result, in Theorem 2.2.9 we construct an exactly conserved discrete energy–momentum tensor, approximating the continuum one at least for free fields.

The great success of discrete models forces us to search for a general discretization method and even to build the whole field theory starting from discrete rather than continuous space and time [2].

1.3 Main Idea

We propose the following discretization algorithm for field theories:

1. Take a continuum Lagrangian written in terms of exterior calculus operations from Table 1.
2. Replace the exterior calculus operations with cochain operations using Table 1 literally.
3. Get equations of motions/conservation laws from discrete Euler–Lagrange/Noether theorems.

This idea is well-known but the realization is new. In Tables 1 and 2, in contrast to the rest of the paper, we assume familiarity with the basics of exterior calculus and continuum field theory.

We stress that Part I of Table 1 gives an algorithm, not just an analogy (as Part II). The algorithm provides conservation laws only for symmetries that are preserved by the discretization. Thus we usually guarantee charge conservation (based on the automatically preserved gauge symmetry) and energy-momentum conservation (not based on any symmetry in our setup).

Results of applying the algorithm to basic field theories are summarized in Table 2 and discussed in Sect. 2. The output discrete theories are usually simpler than the input continuum ones; knowledge of the latter is not required for understanding the former.
| Continuum                                                                 | Discrete                                                                 | Definition |
|--------------------------------------------------------------------------|--------------------------------------------------------------------------|------------|
| Algorithmic part I. Replacement in Lagrangian and action                 |                                                                          |            |
| Differentiable manifold (spacetime)                                      | Simplicial or cubical complex M                                          | 1.4.1      |
| with a fixed vertices ordering                                           | M                                                                       |            |
| k-form, $\mathbb{R}$- or $\mathbb{C}^m \times n$-valued                | $k$-cochain, $\mathbb{R}$- or $\mathbb{C}^m \times n$-valued $\phi$     | 1.4.1      |
| Exterior derivative                                                      | Coboundary $\delta$                                                    | 1.4.2      |
| Exterior product                                                         | Cup product $\sim$                                                     | 3.1.1      |
| Interior product                                                         | Cap product $\sim$                                                     | 3.1.1      |
| Connection 1-form, Lie-algebra-valued                                    | Connection, not Lie-algebra-valued $A$                                  | 2.3.9      |
| Curvature 2-form, Lie-algebra-valued                                     | Curvature, not Lie-algebra-valued $F$                                   | 2.3.9      |
| Covariant exterior derivative                                            | Covariant coboundary $D_A$                                             | 3.1.1      |
| Raising all indices                                                      | Sharp-operator (new notion) # $\#$                                     | 2.2.1      |
| Function on $\mathbb{R}$ or $\mathbb{C}^m \times n$ (e.g., log or Tr) | The same function on $\mathbb{R}$ or $\mathbb{C}^m \times n$ $f$       |            |
| Spacetime integration of a 0-form                                        | Sum of the values of a 0-chain $\epsilon$                              | 2.1.1      |
| Informal part II. Correspondence in equations of motion and conservation laws |                                                                          |            |
| Codifferential, $\sharp$-conjugated                                      | Boundary $\delta$                                                     | 1.4.2      |
| Covariant codifferential, $\sharp$-conjugated                            | Covariant boundary $D_A^\ast$                                          | 3.1.1      |
| Interior product                                                         | Cop product (new notion) $\ast$                                        | 3.1.1      |
| Tensor product over $C^\infty(M)$                                        | Chain–cochain cross product $\times$                                   | 1.4.8      |
| Type $(1, 1)$ tensor                                                     | Type $(1, 1)$ tensor (new notion) $T$                                   | 1.4.8      |
| Integration of its $k$-th component                                      | Flux (new notion) $\langle T, h \rangle_k$                             | 1.5.1      |
| Integration of a $k$-form                                                | Pairing $\langle \phi, h \rangle$                                     | 4.3.1      |
| Field theory | 0-form | Discrete Lagrangian | Equation of motion | Conserved current | Energy-momentum tensor |
|--------------|--------|---------------------|--------------------|------------------|----------------------|
| Electric network | φ \( \mathbb{R} \)-valued | \( \frac{1}{2} \partial \phi \cdot \partial \phi - s \partial \phi \) | \( \delta \delta \phi = s \) | \( \delta \phi \) | \( \delta \phi \times \delta \phi \) |
| | 1-form A | \( -\frac{1}{2} \partial A \cdot \partial A - j \partial A \) | \( -\partial \# \delta A = j \) | \( j \) | \( -\# \delta A \times \delta A \) |
| Electrodynamics | φ \( \mathbb{R} \)-valued | \( -\frac{1}{2} \sharp \partial A \cdot \partial A - j \partial A \) | \( -\partial \# \delta A = j \) | \( j \) | \( -\# \delta A \times \delta A \) |
| | 1-form A | \( \Re \mathrm{Tr} \left[ \frac{1}{2} \sharp \partial A \cdot \partial A + j \partial A \right] \) | \( \Re \mathrm{Tr} \left[ \frac{1}{2} \# \partial A \times \partial A + j \right] \) | \( j \) | \( -\# \delta A \times \delta A \) |
| Gauge theory | Connection | \( -\Re \mathrm{Tr} \left[ \frac{1}{2} \sharp \partial A \cdot \partial A + j \partial A \right] \) | \( \Re \mathrm{Tr} \left[ \frac{1}{2} \# \partial A \times \partial A + j \right] \) | \( j \) | \( -\# \delta A \times \delta A \) |
| Klein–Gordon | φ \( \mathbb{C} \)-valued | \( \# \partial \phi \times \partial \phi - m^2 \partial \phi \times \partial \phi \) | \( \partial \# \delta \phi = m^2 \phi \) | \( -2 \Im \left[ \# \delta \phi \times \phi \right] \) | \( 2 \Re \left[ \# \delta \phi \times \phi \right] \) |
| Klein–Gordon | 0-form φ | \( \sharp \partial \phi \times \partial \phi - m^2 \partial \phi \times \partial \phi \) | \( \partial \# \delta \phi = m^2 \phi \) | \( -2 \Im \left[ \# \delta \phi \times \phi \right] \) | \( 2 \Re \left[ \# \delta \phi \times \phi \right] \) |
| In a gauge field | 0-form φ | \( m^2 \partial \phi \times \partial \phi \) | \( = m^2 \phi \) | \( = m^2 \phi \) | \( = m^2 \phi \) |
All the output theories of Sect. 2 are known, but some obtained conservation laws are new. As a tool, we use discrete covariant differentiation (see Sect. 3.1 and [12]) and build a new discretization of tensor calculus involving non-antisymmetric tensors (see Sect. 1.4). This is done in terms of cochain operations from Table 1, which appear naturally in examples. A reader looking for a zero-knowledge introduction can now proceed directly to Sect. 2.

**Remark 1.3.1** In Table 1 we intentionally include no discretization for the Hodge star or products other than exterior, interior, tensor products. In all the examples, we have succeeded to avoid them.

Continuum and discrete notations fit not that well. But both are commonly used in their contexts (except for a few new discrete objects, for which we keep the continuum notation in a different font).

Putting a continuum Lagrangian to the required input form is not always possible and can be ambiguous: For instance, in Table 2, electrodynamics can be also viewed as a gauge theory with a $u(1)$-valued connection. This leads to the same continuum Lagrangian but different discretizations.

### 1.4 Statements

Let us state the main new results precisely in their simplest form. This subsection is a technical summary. The introduced notions are all motivated in Sect. 2, where they appear little by little in examples. Further generalizations are postponed until Sect. 3.

**Definition 1.4.1** Dissect the hypercube $0 \leq x_0, x_1 \ldots, x_{d-1} \leq N$ in $\mathbb{R}^d$ into $N^d$ unit hypercubes; see Fig. 2. By $k$-dimensional faces we mean the $k$-dimensional faces of those unit hypercubes. The collection of all those faces is called the $d$-dimensional grid $I_N^d$. In what follows we denote $M = I_N^d$, unless the values of $d$ and $N$ need to be shown explicitly (this is convenient for generalizations).

A $k$-dimensional field or $k$-cochain or function on $k$-dimensional faces is a real-valued function defined on the set of $k$-dimensional faces of $M$. Denote by $C^k(M; \mathbb{R}) = C_k(M; \mathbb{R})$ the set of all $k$-dimensional fields; see Remark 3.2.3 for comparison with the other definitions in literature.

A Lagrangian is a function $\mathcal{L} : C^k(M; \mathbb{R}) \rightarrow C_0(M; \mathbb{R})$. The action $S : C^k(M; \mathbb{R}) \rightarrow \mathbb{R}$ is the sum of the values of the Lagrangian over all the vertices. A field $\phi \in C^k(M; \mathbb{R})$ is an extremal or a critical point or stationary for the action functional, if $\frac{\partial}{\partial t} S[\phi + t \Delta] |_{t=0} = 0$ for each $\Delta \in C^k(M; \mathbb{R})$.

Discrete field theory studies extremals of action functionals. Now we introduce additional structure: the dictionary order of vertices, the natural orientation of faces, the boundary and coboundary operators. The latter is an analog of the (exterior) derivative.

**Definition 1.4.2** Fix the dictionary order of the vertices of the grid $I_N^d$: set $(x_0, x_1, \ldots, x_{d-1}) < (y_0, y_1, \ldots, y_{d-1})$ if and only if $x_0 = y_0$, ..., $x_{k-1} = y_{k-1}$, and $x_k < y_k$ for some $0 \leq k \leq d - 1$. 

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Fig. 2 The $3 \times 3$ grid $I_3^2$ with the dictionary order of vertices; the vertices are enumerated in increasing order. Orientation of 1- and 2-dimensional faces. The 1-dimensional ($e_1$ and $e_2$) and 2-dimensional ($f_1$) faces with the maximal vertex $v$. The 1-dimensional face $e_1'$ of $f_1$ containing the vertex $u$ and not containing the vertex $v$

Fig. 3 Boundary and coboundary (see Definition 1.4.2). A non-boundary 3-dimensional face (to the left) is shown again by dashed lines (to the right). The face at which a particular function is evaluated is in bold. The signs in the expression for $\partial \phi$ are different from the ones in (1) because the latter depicts the different equation $\partial \# F = 0$ introduced in Sect. 2.2

Denote by max $f$ (min $f$) the maximal (minimal) vertex of a face $f$ of $M$. (On the grid, it is the vertex with the maximal (minimal) sum of the coordinates).

Fix the following orientation of $k$-dimensional faces of $M$. A positively oriented basis in a face is formed by the $k$ vectors starting at the minimal vertex of the face, going along the edges of the face, and listed in the order opposite to the order of the endpoints. E.g., a positively oriented basis in a $d$-dimensional face of $I_d^N$ is $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$, because $(1, 0, \ldots, 0) > (0, 1, \ldots, 0) > \cdots > (0, 0, \ldots, 1).$ A $(k+1)$-dimensional face $f$ and a $k$-dimensional face $e \subset f$ are cooriented (respectively, opposite oriented), if the ordered set consisting of the outer normal to $e$ in $f$ and a positive basis in $e$ is a positive (respectively, negative) basis in $f$.

The boundary $\partial \phi$ and the coboundary $\delta \phi$ of a function $\phi$ on $k$-dimensional faces $e$ are the functions on $(k-1)$- and $(k+1)$-dimensional faces $v$ and $f$ respectively given by (see Figs. 3 and 4)

$$[\partial \phi](v) = \sum_{e \supset v \text{ cooriented with } v} \phi(e) - \sum_{e \supset v \text{ oriented opposite to } v} \phi(e),$$

$$[\delta \phi](f) = \sum_{e \subset f \text{ cooriented with } f} \phi(e) - \sum_{e \subset f \text{ oriented opposite to } f} \phi(e).$$

Hereafter an empty sum is set to be 0, and $C^k(M; \mathbb{R}) := \{0\}$ for $k < 0$ or $k > \dim M$. 
**Remark 1.4.3** (For specialists) The values of a $k$-cochain at the $k$-dimensional faces with a common maximal vertex $v$ discretize the components of a $k$-form $\sum_{i_1 < \cdots < i_k} \phi_{i_1, \ldots, i_k} \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ at the point $v$. The special choice of orientations makes the coboundary consistent with the exterior derivative. The coboundary treats different spatial directions differently just like the exterior derivative.

Informally, a Lagrangian is **local** or **first-order**, if its value at a vertex depends only on the values of the field $\phi$ and the coboundary $\delta \phi$ at the faces for which the vertex is maximal. **Partial derivatives** with respect to $\phi$ and $\delta \phi$ are fields of dimension $k$ and $k+1$ respectively, obtained by differentiating the Lagrangian as if $\phi$ and $\delta \phi$ were independent variables. The precise definition is as follows.

**Definition 1.4.4** A Lagrangian $L : C^k(M; \mathbb{R}) \to C^0(M; \mathbb{R})$ is **local**, if for each vertex $v \in M$ there is a smooth function $L_v(\phi_1, \ldots, \phi_p, \phi'_1, \ldots, \phi'_q) \in C^1(\mathbb{R}^{p+q})$ such that for each $\phi \in C^k(M; \mathbb{R})$ we have

$$L(\phi)(v) = L_v(\phi(e_1), \ldots, \phi(e_p), [\delta \phi](f_1), \ldots, [\delta \phi](f_q)), \quad (4)$$

where $e_1, \ldots, e_p$ and $f_1, \ldots, f_q$ are all the faces of dimension $k$ and $k+1$ respectively with the maximal vertex $v$; see Fig. 2. Define

$$\frac{\partial L[\phi]}{\partial \phi} \in C_k(M; \mathbb{R}) \quad \text{and} \quad \frac{\partial L[\phi]}{\partial (\delta \phi)} \in C_{k+1}(M; \mathbb{R})$$

by the following formulae for each $l = 1, \ldots, p$ and $m = 1, \ldots, q$:

$$\frac{\partial L[\phi]}{\partial \phi}(e_l) := \frac{\partial L_v}{\partial \phi_l}(\phi(e_1), \ldots, \phi(e_p), [\delta \phi](f_1), \ldots, [\delta \phi](f_q)), \quad (5)$$

$$\frac{\partial L[\phi]}{\partial (\delta \phi)}(f_m) := \frac{\partial L_v}{\partial \phi'_m}(\phi(e_1), \ldots, \phi(e_p), [\delta \phi](f_1), \ldots, [\delta \phi](f_q)). \quad (6)$$

The following theorem is a straightforward generalization of known ones; cf. [24, Eq. (5.2)].

**Theorem 1.4.5** (The discrete Euler–Lagrange equations) Let $L : C^k(M; \mathbb{R}) \to C^0(M; \mathbb{R})$ be a local Lagrangian. Then a field $\phi \in C^k(M; \mathbb{R})$ is an extremal, if and only if the following equation holds:

$$\frac{\partial}{\partial (\delta \phi)} \frac{\partial L[\phi]}{\partial \phi} + \frac{\partial L[\phi]}{\partial (\delta \phi)} = 0. \quad (7)$$

(Here a plus sign stands because the boundary operator $\partial$ for $k = 0$ discretizes minus divergence.)

The Noether theorem gives a **conserved current** for each continuous symmetry of the Lagrangian. It is most nicely stated in terms of the **cap product** relying on the ordering of the vertices; see Fig. 4.
\[ \partial \psi(1) = \psi(1) - \psi(2); \quad [\psi \circ \phi](1) = \psi(1) \phi(0); \]
\[ [\partial \phi](1) = \phi(1) - \phi(0); \quad [\psi \circ \phi](1) = \psi(1) \phi(1). \]

**Fig. 4** Cochain operations on the 1-dimensional grid \( I_1^1 \). Here \( \phi \) and \( \psi \) are functions on vertices and edges respectively. Bold font is used for edge numbers. Cf. Definitions 1.4.2, 1.4.6, 3.1.1

**Definition 1.4.6** A *current* is an arbitrary function on edges. A current \( j \) is *conserved*, if \( \partial j = 0 \).

The (particular case of) *cap product* \( \psi \circ \phi \) of functions \( \psi \) and \( \phi \) on \((k + 1)\)- and \( k \)-dimensional faces respectively is the function on edges \( uv \) given by

\[
[\psi \circ \phi](uv) = (-1)^k \sum_{i=1}^{q} \lambda_i \psi(f_i) \phi(e_i'),
\]
where \( u < v \); \( f_1, \ldots, f_q \) are all the \((k + 1)\)-dimensional faces containing \( uv \) and having the maximal vertex \( v \); \( e_i' \) is the \( k \)-dimensional face of \( f_i \) containing \( u \) and not containing \( v \) (see Fig. 2);

\[
\lambda_i := \begin{cases} 
+1, & \text{if } f_i \text{ and } e_i' \text{ are cooriented;} \\
-1, & \text{if } f_i \text{ and } e_i' \text{ are opposite oriented.}
\end{cases}
\]

**Theorem 1.4.7** (Discrete Noether theorem) Let \( L: C^k(M; \mathbb{R}) \to C_0(M; \mathbb{R}) \) be a local Lagrangian and \( \phi \in C^k(M; \mathbb{R}) \) be an extremal. The Lagrangian is invariant under an infinitesimal transformation \( \Delta = C^k(M; \mathbb{R}) \), i.e.,

\[
\frac{\partial}{\partial t} L[\phi + t \Delta] \bigg|_{t=0} = 0,
\]
if and only if the following current is conserved:

\[
 j[\phi] = \frac{\partial L[\phi]}{\partial(\delta \phi)} \circ \Delta.
\]

This theorem is different from known discretizations of the Noether theorem in [13, 18, 19, 24].

Discrete spacetime has no continuous symmetries, but there is still a corresponding conserved tensor. *Conserved tensors* are functions on faces of the *Cartesian square* \( M \times M \) rather than of \( M \) itself; see Fig. 5. We shall see that such functions appear naturally in examples in Sect. 2.

**Definition 1.4.8** Let \( I_N^d \times I_N^d \) be the Cartesian square of the \( d \)-dimensional grid. It is a \( 2d \)-dimensional grid with the faces of the form \( e \times f \), where \( e \) and \( f \) are faces of \( I_N^d \) of arbitrary dimension.

A *tensor* of type \((q, 1)\), where \( q = 1 \) or \( 0 \), is a function on all faces \( e \times f \) such that \( \dim f - \dim e = 1 - q \). The *chain-cochain cross product* of fields \( \phi \) and \( \psi \) with
Fig. 5 The Cartesian square of the 1-dimensional grid $I_1^4$ and the equation of a conserved tensor; see Definition 1.4.8. This is a well-known discretization of the Cauchy–Riemann equations [4, Eq. (2.2)], up to orientation. Thus tensor conservation means one-half of the Cauchy–Riemann equations (for vertical edges only), like in [7, Corollary 2.12(1)], although our setup is very different from theirs.

$$\dim \phi - \dim \psi = 1 - q$$

is the tensor

$$[\psi \times \phi](e \times f) = \begin{cases} 
\psi(e)\phi(f), & \text{if } \dim e = \dim \psi \text{ and } \dim f = \dim \phi; \\
0, & \text{if } \dim e \neq \dim \psi \text{ or } \dim f \neq \dim \phi.
\end{cases}$$

The boundary operator $\partial$ is the unique linear map between the spaces of type $(1, 1)$ and $(0, 1)$ tensors such that for each fields $\phi, \psi$ with $\dim \phi = \dim \psi$ we have

$$\partial(\psi \times \phi) = \partial \psi \times \phi + \psi \times \delta \phi.$$ 

(Beware that this is not the boundary operator on $I_2^d$.) A type $(1, 1)$ tensor $T$ is conserved, if $\partial T = 0$.

**Theorem 1.4.9** (Energy–momentum conservation) For each local Lagrangian $\mathcal{L}: C^k(M; \mathbb{R}) \to C_0(M; \mathbb{R})$ and each extremal $\phi \in C^k(M; \mathbb{R})$ the following energy-momentum tensor is conserved:

$$T[\phi] = \frac{\partial \mathcal{L}[\phi]}{\partial (\delta \phi)} \times \delta \phi + \frac{\partial \mathcal{L}[\phi]}{\partial \phi} \times \phi. \quad (10)$$

The notion of discrete tensors extends [14]; see Remark 2.2.7. This theorem is completely new.

**1.5 Summit**

We conclude the summary of main results with the most technical one: an integral form of energy conservation, sketched already in Sect. 1.1. To tensor (10) defined on $M \times M$ we now assign a conserved quantity defined on the grid $M$ itself. This allows us to compare tensors with their continuum analogs.

**Definition 1.5.1** Let $e_k$, where $k = 0, \ldots, d - 1$, be the vector of length $\frac{1}{2}$ pointing in the direction of the $x_k$-axis. Each linear combination of such vectors with coefficients in the set $\{0, 1, \ldots, 2N\}$ is the center of a unique face of $I_N^d$. We use the same notation for a face $f$ and its center. In particular, $f + e_k$ denotes the face with the center at the point obtained from the center of $f$ by the translation by the vector $e_k$ (the dimensions of $f$ and $f + e_k$ always differ by 1). A hyperface is a $(d - 1)$-dimensional face.
A type \((1, 1)\) tensor \(T\) is partially symmetric, if \(T(e \times f) = T(f \times e)\) for each pair of parallel faces \(e \parallel f\) (hereafter any two vertices are viewed as parallel faces). Take a partially symmetric tensor \(T\), a non-boundary hyperface \(h\), a number \(k \in \{0, \ldots, d - 1\}\), and the unique \(l \in \{0, \ldots, d - 1\}\) such that \(e_l \perp h\). Then the \(k\)-th component of the flux of \(T\) across \(h\) in the positive normal direction is

\[
\langle T, h \rangle_k = \frac{1}{2} \sum_{f: f \subset h, f \supset \max h,f \| e_k \text{ for } h \| e_k} (-1)^{\text{dim } Pr(f,k,l)+l+1} \cdot \begin{cases} 
T((f + e_l - e_k) \times f) + T((f + e_l + e_k) \times f), & \text{if } h \parallel e_k; \\
T(f \times f) - T((f + e_k) \times (f - e_k)), & \text{if } h \perp e_k,
\end{cases}
\]

where the sum is over faces \(f\) of arbitrary dimension (we set \(f \| e_k\), if \(f\) is a vertex), and \(Pr(f,k,l)\) is the orthogonal projection of \(f\) to the linear span of all \(e_m\) with \(\min\{k,l\} \leq m \leq \max\{k,l\}\).

The \(k\)-th component of the flux of \(T\) across the boundary of a \(d\)-dimensional face \(g\) is

\[
\langle T, \partial g \rangle_k := \sum_{h \subset \partial g \text{ cooriented with } g} \langle T, h \rangle_k - \sum_{h \subset \partial g \text{ oriented opposite to } g} \langle T, h \rangle_k.
\]

**Theorem 1.5.2** (Integral energy–momentum conservation) If a partially symmetric type \((1, 1)\) tensor \(T\) is conserved, then for each \(d\)-dimensional face \(g\) disjoint with \(\partial I_d^N\) and each \(k\) we get \(\langle T, \partial g \rangle_k = 0\).

In particular, if tensor \((10)\) is partially symmetric, then its flux across any closed hypersurface composed of non-boundary hyperfaces vanishes exactly. In many examples, \((9)-(10)\) approximate their continuum analogues; see Theorem 2.1.17, Proposition 2.2.13, Remark 2.3.10, and [25, Propositions A.2.4 and A.3.6]. Thus we have established the discretization principles from Sect. 1.

### 1.6 Limitations

So far the proposed general discrete field theory has no applications (as a mathematical theory) and is not refutable (as a candidate for a fundamental physical theory).

Most of the technical issues concern the discretization of energy conservation and tensor calculus:

On the one hand, the new notion of energy–momentum tensor \((10)\) seems to be too abstract and too general. It discretizes not the continuum energy–momentum tensor precisely but a related object mapped to the latter; see Remark 2.2.8. Depending on a particular Lagrangian, \((10)\) approximates either the nonsymmetric canonical energy-momentum tensor, or the symmetric Belinfante–Rosenfeld one, or even a non-conserved tensor; see Remark 2.2.10.

On the other hand, discrete non-antisymmetric tensor calculus from Sects. 1.4–1.5 seems to be too restrictive: it includes only type \((1, 1)\) tensors and only the trivial
connection; the flux is defined only on a grid. The way of further generalization is unclear: e.g., for lattice gauge theory from Sect. 2.3, a naive way to define a real energy-momentum tensor leads to a nonconserved tensor; cf. Remark 2.3.10.

Concerning approximation of continuum theories by discrete ones, only the following warm-up results are proved: First, for electrical networks the known approximation result is recalled in Sect. 2.1. Second, for the completely new discrete energy–momentum tensor the continuum limit is found in Sect. 2.

Some other limitations are stated as open problems in Sect. 5.

1.7 Overview

In Sect. 2 we give basic examples of discrete field theories. It contains an exposition of known results and a few new ones for nonspecialists; Sect. 2 is independent from Sect. 1 (except that Definitions 1.4.1, 1.4.2, and 1.4.8 are cited and used in Sect. 2.2 after they become motivated). In Sect. 3 we state the main results in full generality. The only prerequisites for Sect. 3 are Definitions 1.4.1, 1.4.2, 2.3.1, 2.3.4, 2.3.9. In Sect. 4 we prove the results of Sects. 1–3. In Sect. 5 we state open problems. More examples are given in [25, Sect. A].

The paper is written in a mathematical level of rigor, i.e., all the definitions, conventions, and theorems (including corollaries, propositions, lemmas) should be understood literally. Theorems remain true, even if cut out from the text. The proofs of theorems use the statements but not the proofs of the other ones. Most statements are much less technical than the proofs and the order of statements is different from the logical order of proofs; thus the proofs are kept in a separate section. Remarks are informal and are not used elsewhere (hence skippable) unless the opposite is explicitly indicated.

2 Examples

2.1 Electrical Networks

Basic Model

We start with the simplest discrete field theory to illustrate and motivate the main concepts. Consider an $N \times N$ grid of unit resistors; see Fig. 6. A standard problem is to find currents in the grid, given the current sources at the boundary. It is solved using the following mathematical model.

**Definition 2.1.1** Each of the $N^2$ unit squares of the $N \times N$ grid is called a *face*. Orient the boundary $\partial f$ of each face $f$ counterclockwise. Assume that the coordinate axes are parallel to the edges, and orient edges in the directions of the axes. A *function on vertices/edges/faces* is a real-valued function defined on the set of vertices/edges/faces of the grid.

A *source* $s$ is a function on vertices vanishing at all the non-boundary vertices. The *current generated* by the source $s$, or the *stationary current*, is the function on edges satisfying two equations:
**the Kirchhoff current law or charge conservation law:** $\partial j = -s$;

**the Kirchhoff voltage law in the case of unit resistances:** $\delta j = 0$.

Here the boundary $\partial j$ and the coboundary $\delta j$ of a function $j$ on edges are the functions on vertices and faces respectively given by the following formulae (see Fig. 6 to the middle and the right):

$$[\partial j](v) = \sum_{e \text{ ending at } v} j(e) - \sum_{e \text{ starting at } v} j(e),$$

$$[\delta j](f) = \sum_{e \text{ oriented along } \partial f} j(e) - \sum_{e \text{ oriented opposite to } \partial f} j(e),$$

for each vertex $v$ and face $f$, where the sums are over edges $e$ containing $v$ and contained in $\partial f$ respectively. Denote by $\epsilon s := \sum_v s(v)$ the sum over all vertices $v$ (the operator $\epsilon$ is defined only for functions on vertices).

The following existence and uniqueness result is well-known.

**Proposition 2.1.2** A current generated by a source $s$ exists, if and only if $\epsilon s = 0$. If a current generated by the source $s$ exists, then it is unique.

**Remark 2.1.3** It could be more conceptual to write the Kirchhoff voltage law in the form $\delta R j = 0$, where $R$ is a map between 1-chains and 1-cochains depending on the resistances. In our setup, chains and cochains are identified and the resistances equal 1, hence $R$ is the identity map and is omitted.

**Electrical Potential**

Let us state a least-action principle for electrical networks. Throughout Sect. 2.1 $j$ is a stationary current.

**Definition 2.1.4** An electrical potential $\phi$ is a function on vertices satisfying

**the Ohm law in the case of unit resistances:** $j = -\delta \phi$.

Here the coboundary $\delta \phi$ is the function on edges given by the formula

$$[\delta \phi](uv) = \phi(v) - \phi(u),$$

where $uv$ denotes an oriented edge starting at $u$ and ending at $v$ hereafter.
The following well-known existence and uniqueness result is straightforward.

**Proposition 2.1.5** For each stationary current there is a unique up to additive constant electrical potential.

The following properties of an electrical potential $\phi$ may serve as equivalent definitions:

- *the Laplace equation with the Neumann boundary condition:* $\partial \delta \phi = s$;
- *the least action principle:* among all the functions on vertices, $\phi$ minimizes the functional
  \[
  S[\phi] = \frac{1}{2} \sum_{\text{edges } uv} (\phi(u) - \phi(v))^2 - \sum_{\text{vertices } v} s(v)\phi(v) = \epsilon L[\phi],
  \]
  where
  \[
  L[\phi] = \frac{1}{2} \delta \phi \wr \delta \phi - s \wr \phi.
  \]

Here the (particular case of) cap-product $\wr$ is defined as follows; see Fig. 6 to the middle.

**Definition 2.1.6** Denote by max $f$ the vertex of a face $f$ or an edge $f$ having the maximal sum of the coordinates. Set max $f := f$, if $f$ is a vertex. The cap-product $\phi \wr \psi$ of two functions $\phi$ and $\psi$ on faces (respectively, edges or vertices) is the function on vertices given by

\[
[\phi \wr \psi](v) = \sum_{f: \max f = v} \phi(f)\psi(f),
\]

where the sum is over faces (respectively, edges or vertices) $f$ such that max $f = v$.

**Remark 2.1.7** The Euler–Lagrange equation (given by Theorem 1.4.5) for the Lagrangian $L[\phi]$ is the Laplace equation. Apart the grid boundary, the Lagrangian is invariant under the transformation $\phi \mapsto \phi - t$, where $t \in \mathbb{R}$. The resulting conserved current (given by Theorem 1.4.7) is $j = -\delta \phi$.

### Magnetic Field

There is one more discrete field in an electrical network: the current $j$ generates a magnetic field.

**Definition 2.1.8** A magnetic field $F$ (or magnetic flux through faces in the $(0, 0, -1)$-direction) generated by a current $j$ is a function on faces satisfying the following equation apart the boundary:

- *the Ampère law in the case of unit-area faces:* $-\partial F = j$.

Here the boundary $\partial F$ is the function on edges given by the formula

\[
[\partial F](e) = F(f) - F(g).
\]
for each pair of adjacent faces \( f \) and \( g \) such that \( \partial f \) (respectively, \( \partial g \)) is oriented along (respectively, opposite to) the common edge \( e \); see Fig. 6 to the left. (The definition of \([\partial F](e)\) for boundary edges \( e \) is not required for this subsection.)

The following well-known existence and uniqueness result is straightforward.

**Proposition 2.1.9** For a stationary current there is a unique up to additive constant magnetic field.

Throughout Sect. 2.1 the functions \( \phi \) and \( F \) are an electrical potential and a magnetic field respectively.

**Remark 2.1.10** The pair \((\phi, F)\) and \(-j\) discretize an analytic function and its derivative [4, 6].

**Definition 2.1.11** A magnetic vector potential \( A \) of the field \( F \) is a function on edges such that \( \delta A = F \).

A magnetic vector potential \( A \) has the following properties (proved similarly to the ones from Sect. 2.2):

- **the source equation**: \(-\delta \delta A = j\) apart the grid boundary;
- **gauge invariance**: \( A + \delta g \) is a vector potential of the same field for any function \( g \) on vertices;
- **the least action principle**: among all functions on edges, \( A \) minimizes \( S[A] = \epsilon L[A] \), where
  \[
  L[A] = \frac{1}{2} \delta A \cdot \delta A + j \cdot A.
  \]

**Energy and Momentum**

Let us state energy and momentum conservation in an electrical network in a simple heuristic form. This is a visual motivation for more abstract Definition 1.4.8 (not used in this subsection).

For functions \( \phi \), \( \psi \) on faces (respectively, edges or vertices), denote by \( \langle \phi, \psi \rangle = \sum_f \phi(f) \psi(f) \) the sum over all faces \( f \) (respectively, edges or vertices). The obvious identity \( \langle \delta \phi, j \rangle = \langle \phi, \partial j \rangle \) implies

- **the Tellegen theorem or global energy conservation**: \( \langle \delta \phi, j \rangle + \langle \phi, s \rangle = 0 \).

Now we study local conservation and the flow of energy. Energy flows in the direction of the Poynting vector, hence transversely to (not along) the resistors. Thus we assign energy flow to bisectors of edges. The cross-product formula for the Poynting vector is then discretized directly.

**Definition 2.1.12** The doubling is the \( 2N \times 2N \) grid with the vertices at vertices, edge midpoints, and face centers of the initial \( N \times N \) grid. Orient all the edges still in the direction of the coordinate axes.
The heat power $W$ is the function on the vertices $v$ of the doubling given by the formula (Fig. 7)

$$W(v) = \begin{cases} 
-\delta\phi(e) j(e), & \text{if } v \text{ is the midpoint of an edge } e; \\
0, & \text{if } v \text{ is the center of a face or a vertex of the initial grid.} 
\end{cases}$$

The Poynting vector or energy flux $S$ is the function on edges $uv$ of the doubling, $\max uv = v$, given by

$$S(uv) = \begin{cases} 
[\delta\phi](e) F(f), & \text{if } u \text{ and } v \text{ are the centers of a vertical edge } e \text{ and a face } f \text{ or vice versa;} \\
-\delta\phi(e) F(f), & \text{if } u \text{ and } v \text{ are the centers of a horizontal edge } e \text{ and a face } f \text{ or vice versa;} \\
0, & \text{if } u \text{ or } v \text{ is a vertex of the initial grid.} 
\end{cases}$$

The Lorentz force $L$ is defined analogously to $S$, only $\delta\phi$ is replaced by $-j/2$ (thus $L = S/2$ in our basic model). The magnetic pressure $P$ (or momentum flux of the magnetic field towards the edges in the normal direction) is the function on non-boundary vertices $v$ of the doubling given by the formula

$$P(v) = \begin{cases} 
F(f) F(f)/2, & \text{if } v \text{ is the center of a face } f; \\
F(f) F(g)/2, & \text{if } v \text{ is the midpoint of the common edge of faces } f \text{ and } g; \\
0, & \text{if } v \text{ is a vertex of the initial grid.} 
\end{cases}$$

The straightforward consequences of these definitions and the Kirchhoff laws are:

- **Energy conservation**: $\partial S - W = 0$.
- **Momentum conservation for the magnetic field**: $\delta P + L = 0$ on those edges of the doubling that contain the face-centers of the initial grid.

In Sect. 2.2 we introduce a more conceptual form of the two laws, explaining the latter restriction.

Now we state a less visual momentum conservation law for the electric field. This is essentially [13, Example in Sect. 8 of Ch. III]. One expects the following properties of the momentum flux $\sigma(e)$ across edges $e$ of the initial grid (the latter property is required by the discretization principles from Sect. 1):

- $\sigma(e)$ equals the momentum flux of a continuum electric field across $e$, if the potential is linear;
• $\sigma(e)$ depends only on the values of $\delta \phi$ at the edges intersecting $e$ and is bilinear in these values; 
• $\delta \sigma = 0$ apart the grid boundary: the momentum flux across the boundary of each face vanishes.

The simplest function $\sigma$ satisfying these properties is defined as follows; cf. Fig. 8.

**Definition 2.1.13** The momentum flux of the electric field across edges in the negative normal direction, or the electric part of the Maxwell stress tensor, is the pair $\sigma = (\sigma_1, \sigma_2)$ of functions on edges disjoint with the grid boundary given by the following formula for each $k = 1, 2$:

$$
\sigma_k(uv) = \frac{(-1)^{k+1}}{2} \begin{cases} 
\delta \phi(uu_+)\delta \phi(uv) + \delta \phi(vv_+)\delta \phi(uv), & \text{if } uu \parallel Ox_k; \\
\delta \phi(uv)\delta \phi(uv) - \delta \phi(vv_+)\delta \phi(vv), & \text{if } uu \perp Ox_k,
\end{cases}
$$

where $uu_+, vv_+$ are the edges orthogonal to $uv$ with the maximal vertices $u_+, v, v_+$; see Fig. 8.

**Corollary 2.1.14** (Momentum conservation for the electric field) (Cf. [13, Example in Sect. 8 of Ch. III]) For each electric potential $\phi$ we have $\delta \sigma_1 = \delta \sigma_2 = 0$ on each face not intersecting the grid boundary.

This corollary and all the other corollaries in Sect. 2 are particular cases of the results of Sects. 1.4 and 3.1, and are directly deduced from those. We keep detailed proofs in Sect. 4.5 for the reader’s reference.

**Remark 2.1.15** The function $\sigma_k$ is the flux (given by Definition 1.5.1) of the energy-momentum tensor $T[\phi] = \delta \phi \times \delta \phi$ (given by Theorem 1.4.9 for the Lagrangian $L[\phi] = \frac{1}{2} \delta \phi \sim \delta \phi$).

**Approximation**

The basic network model indeed converges to a continuum one, as the grid becomes finer and finer.

The continuum model is a homogeneous conducting plate defined as follows. Let $I^2$ be the unit square, $\vec{n}$ be the unit inner normal vector field on $\partial I^2$ besides the corners, $\ast$ be the counterclockwise rotation through $\pi/2$ about the origin (the Hodge star),

$$
\delta_{kl} = \delta^l_k := \begin{cases} 
1, & \text{if } k = l; \\
0, & \text{if } k \neq l.
\end{cases}
$$

A source $s$ is a continuous function on $\partial I^2$. The fields $\vec{j}, \phi, F, W, \vec{S}, \vec{L}, P, L, \sigma$ generated by $s$ are continuous scalar/vector/matrix fields on $I^2$, being $C^1$ and satisfying the following conditions apart $\partial I^2$:

$$
-\nabla \phi = \vec{j}, \quad W = -\nabla \phi \cdot \vec{j}, \quad \vec{S} = -\ast \nabla \phi \cdot F, \quad L = \frac{1}{2}(\nabla \phi)^2 \quad (= \frac{1}{2} d\phi \sim d\phi), \\
\ast \nabla F = \vec{j}, \quad \vec{L} = \ast \vec{j} \cdot F, \quad P = \frac{1}{2} F \cdot F, \quad \sigma_{kl} = \frac{\partial \phi}{\partial x^l} \frac{\partial \phi}{\partial x^l} - \frac{1}{2} \delta_{kl}(\nabla \phi)^2.
$$
\[
\sigma_2(v) = -\frac{1}{2} \left[ \delta \phi(v) \cdot \delta \phi(v) + \delta \phi(v) \cdot \delta \phi(v) \right]
\]

\[
\sigma_2(\square) = -\frac{1}{2} \left[ \delta \phi(\square) \cdot \delta \phi(\square) + \delta \phi(\square) \cdot \delta \phi(\square) \right]
\]

Fig. 8  Notation in Definition 2.1.13 of discrete momentum flux. The square $uu_+v_+$ is shown by dotted lines to the right. The edge at which a particular function is evaluated is in bold. Cf. §1.1
and the following boundary condition on $\partial I^2$ besides the corners:

$$\vec{j} \cdot \vec{n} = s.$$ 

In other words, $\phi + i F$ is an analytic function such that $\frac{\partial}{\partial n} \phi = -s$; the other fields are expressions in it.

Let the unit square $I^2$ be dissected into $N^2$ equal squares. Given a source $s_N$, define the fields $j_N, \phi_N, F_N, W_N, S_N, L_N, P_N, L_N, \sigma_N$ on the resulting grid literally as above on the grid of size $N \times N$.

**Remark 2.1.16** It would be somewhat more conceptual to modify the above Ampère law for the resulting grid because the faces are not unit squares anymore. This leads just to the normalization of the fields by powers of $N$. We avoid such modification for simplicity.

It would be more conceptual to write the Lagrangian as $L = \frac{1}{2} d\phi \wedge d\phi - s \wedge \phi$ but the second term vanishes apart the boundary anyway.

The continuum model has more symmetries than the discrete one: e.g., $L$ is rotational-invariant whereas $L_N$ is not, at least in a naive sense; cf. [18, Definition 5.2.36].

Dissect each side of $\partial I^2$ into $N + 1$ (not $N$) equal segments called *auxiliary segments*. Write $a_N(x) \Rightarrow b_N(x)$ for functions $a_N, b_N$ on a set $M_N$, if $\max_{x \in M_N} |a_N(x) - b_N(x)| \rightarrow 0$ as $N \rightarrow \infty$.

**Theorem 2.1.17** (Approximation theorem) Let $s: \partial I^2 \rightarrow \mathbb{R}$ be a continuous source with $\int_{\partial I^2} s \, dl = 0$. Dissect $I^2$ into $N^2$ equal squares and define a discrete source $s_N$ on the resulting grid by the formula

$$s_N(v) := \int_{v_-v_+} s \, dl,$$

where $v_-v_+ \subset \partial I^2$ is the arc formed by 1 or 2 auxiliary segments containing a vertex $v \in \partial I^2$.

Take continuous fields $\vec{j}, \phi, F, W, S, L, P, \sigma$ and discrete ones $j_N, \phi_N, F_N, W_N, S_N, L_N, P_N, \sigma_N = (\sigma_{N,1}, \sigma_{N,2})$ generated by the sources. Assume that $\phi, F$ and $\phi_N, F_N$ vanish at the center of $I^2$ and at one of the vertices or faces closest to the center respectively. Take $r > 0$. Then on the set of all vertices $v$, edges $e$, faces $f$, edge-midpoints $e'$, and face-centers $f'$ at distance $\geq r$ from $\partial I^2$ we have:

$$\phi_N(v) \Rightarrow \phi(v), \quad N j_N(e) \Rightarrow N \int_e \vec{j} \cdot d\vec{l}, \quad N^2 W_N(e') \Rightarrow W(e'),$$

$$N S_N(e'f') \Rightarrow 2N \int_{e'f'} \vec{S} \cdot d\vec{l},$$

$$N^2 L_N(v) \Rightarrow L(v), \quad F_N(f) \Rightarrow N^2 \int_f F \, dA, \quad P_N(e') \Rightarrow P(e'),$$

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\[ NL_N(e^i f^i) \rightarrow N \int_{e^i f^i} \vec{L} \cdot d\vec{l}, \]
\[ N^2 \sigma_{N,k}(e) \rightarrow N \int_{e} \left( \sigma_{k2} dx^1 - \sigma_{k1} dx^2 \right) \text{ as } N \rightarrow \infty. \]

The theorem is essentially known; it is easily deduced from highly nontrivial known results in §4.

### 2.2 Lattice Electrodynamics

A standard problem in electrodynamics is to find forces between given charges and currents. This is done in two steps: first the field generated by the charges and currents is computed, then—the action of the field upon them. For a discretization, continuum spacetime is replaced by a 4-dimensional grid.

#### Generation of the Field by the Current

The discrete theory is nicely stated in terms of Definitions 1.4.1 and 1.4.2, which we use hereafter.

**Definition 2.2.1** The Minkowski sharp operator \( \# \) applied to a function \( F \) on \( k \)-dimensional faces \( f \) of the grid \( I_{dN} \), where \( k \neq 0 \), is

\[
[#F](f) := \begin{cases} 
(-1)^{k-1} F(f), & \text{if } f \parallel (1, 0, \ldots, 0), \\
(-1)^k F(f), & \text{if } f \perp (1, 0, \ldots, 0). 
\end{cases}
\]

An electromagnetic vector potential \( A \) generated by a current \( j \) is a function on edges satisfying

- **The source equation**: \(- \partial \# \delta A = j\).

**Remark 2.2.2** We do not discuss conditions under which the vector potential exists and is unique.

The operator \( \# \) is new. It is a discrete analog of raising all indices in the metric of signature \(+, -, \ldots, -\). We use it instead of a discrete Hodge star [28] to avoid working with the dual lattice, which would complicate the theory and its generalization to arbitrary complexes.

The following 3 properties of an electromagnetic vector potential \( A \) generated by a current \( j \) immediately follow from the well-known identities \( \delta \delta = 0 \) and \( \partial \partial = 0 \); cf. (1):

- **Maxwell’s equations**: \( \delta F = 0 \) and \(- \partial \# F = j\), where \( F := \delta A \) is the electromagnetic field;
- **Gauge invariance**: \( A + \delta g \) is generated by the same current \( j \) for any function \( g \) on vertices;
• Charge conservation: $\partial j = 0$, if there exists a vector potential generated by the current $j$.

**Corollary 2.2.3** An electromagnetic vector potential $A$ is generated by a current $j$, if and only if $A$ is an extremal of the functional $S[A] = \epsilon L[A]$, where

$$L[A] = -\frac{1}{2} \# A \cdot \delta A - j \cdot A.$$ 

**Remark 2.2.4** Electrodynamics in linear nondispersive media is discretized analogously, only the Minkowski sharp operator is replaced by a linear operator depending on the media.

To convince the reader that lattice electrodynamics is a realistic model, let us informally sketch a network model for it [20]. Set $d = 4$. For each edge of the grid $I_{d-1}^N$, take an oscillatory circuit consisting of one (nonconstant) current source, one unit capacitor, and as many unit-transformer coils as there are faces containing the edge; see Fig. 9 to the bottom-left. Join the obtained circuits in the shape of the grid, join the transformer cores in the shape of the 1-dimensional skeleton of the dual grid, and join the capacitor dielectric cores in the shape of the 2-dimensional skeleton of the dual grid. We get an electric, a magnetic, and a dielectric network coupled together; a part is shown in Fig. 9. We conjecture that the integrals of appropriate currents and voltages over time intervals $[n, n + 1]$, where $n \in \mathbb{Z}$, satisfy the discrete Maxwell equations above.

**Action of the Field on the Current**

The field acts on the current by the Lorenz force, which we are going to discretize now. The rest of Sect. 2.2 contains completely new notions and results; cf. [5] and [14].

We start with an informal motivation. The formula for the Lorenz force $L$ in Sect. 2.1 involves the product of the values of fields at edges and faces. Thus it is reasonable
to view it as a “projection” of a more fundamental quantity defined on the Cartesian square of the grid. More precisely, the set of faces of the Cartesian square is naturally mapped to the set of faces of the doubling: to a face $e \times f$ assign the face of the doubling with the center at the midpoint of the segment joining the centers of $e$ and $f$. Up to sign and factor $1/2$, the fields $W, S, L, P$ from Sect. 2.1 are “induced” by the latter map from the cross products $j \times \delta \phi, F \times \delta \phi, j \times F, F \times F$ respectively. This naturally leads to Definition 1.4.8, which we use hereafter. These heuristic fields $W, S, L, P$ are now replaced by tensors.

**Definition 2.2.5** Let $A$ be a vector potential generated by a current $j$, and $F = \delta A$. The Lorentz force is the type $(0, 1)$ tensor $L = j \times F$. It has support on faces $e \times f \subset I_N^d \times I_N^d$ such that $\dim e = 1$, $\dim f = 2$.

The energy-momentum tensor, or stress-energy tensor, of the electromagnetic field (respectively, of both the field and the current) is the type $(1, 1)$ tensor $T' = -\#F \times F$ (respectively, $T = -\#F \times F - j \times A$). The tensor $T'$ has support on 4-dimensional faces $e \times f \subset I_N^d \times I_N^d$ such that $\dim e = \dim f = 2$.

An immediate consequence of these definitions, Maxwell’s equations, and charge conservation is:

- **Energy and momentum conservation**: $\partial T' = L$ and $\partial T = 0$.

**Remark 2.2.6** The latter is a particular case of Theorem 1.4.9 for the Lagrangian from Corollary 2.2.3.

**Remark 2.2.7** Let us give general comments on the discretization of tensors in Definition 1.4.8.

In contrast to continuum theory, type $(0, 1)$ tensors are not 1-dimensional fields.

Our tensors can be alternatively viewed as functions on faces of $I_N^d$, with two arguments. We prefer working with functions on $I_N^d \times I_N^d$ to get elegant expressions with chain cross product such as (10).

Although $I_N^d \times I_N^d$ is naturally identified with $I_N^{2d}$, the boundary operator on tensors is not the boundary operator on $I_N^{2d}$. To avoid confusion, we distinguish between $I_N^d \times I_N^d$ and $I_N^{2d}$ below.

A type $(q, 1)$ tensor can be equivalently defined as an element of $C^{d+q-1}(I_N^d \times I_N^{d*}; \mathbb{R})$, where $I_N^{d*}$ is the dual grid. Then the boundary operator on tensors is exactly the boundary operator on $I_N^d \times I_N^{d*}$. We avoid working with dual grids for simplicity and easier generalization to arbitrary complexes.

A type $(q, 1)$ tensor can also be viewed a collection of linear maps $C^k(I_N^d; \mathbb{R}) \to C^{k+q-1}(I_N^d; \mathbb{R})$ for all $k = 0, \ldots, d$: the values of the tensor at faces $e \times f$ comprise the coefficients of those maps. This shows that our tensors generalize Forman’s forms [14], with the same role of doubling.

It would be more conceptual to restrict the domain of a tensor to a “neighborhood of the diagonal” in $I_N^d \times I_N^d$: the values at the other faces do not contribute to the tensor flux. Type $(0, 1)$ tensors can be restricted to the set of faces $e \times f$ such that $e \subset f$; this makes the definition completely equivalent to Forman’s discretization of 1-forms. Concerning type $(1, 1)$ tensors, their natural domain is faces $e \times f$ such
that either \( e = f \) or \( e \cap f \) is a codimension 1 face in both \( e \) and \( f \). This is not equivalent to Forman’s discretization (type (1, 1) tensors are not 0-forms, as expected), but is a natural generalization. We avoid such restriction for simplicity, especially in computations involving cross products.

**Remark 2.2.8** (For specialists.) Let us clarify the relation of our tensor calculus to continuum theory. The set of type \((q, 1)\) tensors is naturally isomorphic to \( \bigoplus_{p=0}^{d} \mathbb{C}_p(\mathcal{I}^d_N); \mathbb{R} \) \( \otimes \mathbb{R} \mathbb{C}^{p-q+1}(\mathcal{I}^d_K); \mathbb{R} \). Thus it discretizes the space \( \bigoplus_{p=0}^{d} \Omega^p(\mathcal{I}^d)^* \otimes \Omega^{p-q+1}(\mathcal{I}^d) \) rather than the space \( T_1^q(\mathcal{I}^d) \) of continuum type \((q, 1)\) tensors. (Here \( \Omega^p(\mathcal{I}^d) \) denotes the set of \( C^\infty \) \( p \)-forms on the unit hypercube \( \mathcal{I}^d \) and \( \otimes \) denotes the tensor product over \( \Omega^0(\mathcal{I}^d) \)). But the former space is mapped to the latter by the ‘contraction’ map

\[
T_{m_1 \ldots m_p}^{\ m_1 \ldots m_p \ldots m_{p-q+1}} \mapsto \begin{cases} T_{km_1 \ldots m_p}^{\ km_1 \ldots m_p}, & \text{if } q = 0; \\ T_{km_2 \ldots m_p}^{\ km_2 \ldots m_p} - \frac{1}{2p} \partial_k^l \ T_{m_1 \ldots m_p}, & \text{if } q = 1, p > 0. \end{cases}
\]

(Summation over repeating indices is understood.)

Since no discretization of the image is available (at least for \( q = 1 \)), the discretization of the domain is proclaimed to be space of type \((q, 1)\) tensors. Here the role of the \( \partial_k^l \)-term is the same as in the Einstein tensor: it makes the ‘contraction’ map commute with certain codifferentials when \( T \) has certain symmetry properties (namely, \( \varepsilon T \) is symmetric wrt interchanging \( m_j \) and \( n_j \) but antisymmetric wrt interchanging \( m_i \) and \( m_j \)):

\[
\frac{\partial \mathcal{L}}{\partial (d\phi)} \otimes d\phi + \frac{\partial \mathcal{L}}{\partial \phi} \otimes \phi \in \bigoplus_{p=0}^{d} \Omega^p(\mathcal{I}^d)^* \otimes \Omega^p(\mathcal{I}^d) \xrightarrow{\partial^* \otimes \text{id} + \text{id} \otimes \partial^*} \bigoplus_{p=0}^{d-1} \Omega^p(\mathcal{I}^d)^* \otimes \Omega^{p+1}(\mathcal{I}^d) \xrightarrow{\text{divergence}} \bigoplus_{p=0}^{d-1} \Omega^{p}(\mathcal{I}^d)^* \otimes \Omega^{p+1}(\mathcal{I}^d).
\]

Similarly, (10) discretizes \( \frac{\partial \mathcal{L}}{\partial (d\phi)} \otimes d\phi + \frac{\partial \mathcal{L}}{\partial \phi} \otimes \phi \) rather than the continuum energy-momentum tensor \( T_k^l \), but the former is usually taken to the latter by the ‘contraction’ map. Here \( \left( \frac{\partial \mathcal{L}}{\partial (d\phi)} \right)^{m_1 \ldots m_p}_{m_1 \ldots m_p} := \frac{\partial \mathcal{L}}{\partial (d\phi)_{m_1 \ldots m_p}} \). The former is conserved (i.e. taken to 0 by \( d^* \otimes \text{id} + \text{id} \otimes d \)) regardless of symmetries of \( \mathcal{L}[\phi] \).

In particular, \( L \) and \( T \) discretize the tensors \( j^l F_{kn} \) and \( -\mathcal{L}^{lm} F_{kn} \), but the latter two are taken to the continuum Lorenz force and energy-momentum tensor by the ‘contraction’ maps. In contrast to \( T \), the tensor \( T \) has no conserved continuum analogue.

The formula for the discrete energy-momentum tensor \( T' \) is even simpler than the continuum analogue. This is achieved at the cost of rather subtle Definition 1.5.1 of discrete tensor integration.

**Integral Conservation Laws**

To compare discrete tensors with their continuum analogues, we need their integration. This naturally leads to Definition 1.5.1, which we use in the rest of Sect. 2.2. Actually,
we have already applied it in the particular cases $d = 3, k = 0$, $T = -\# F \times F$ and $d = 2, k = 1, 2$, $T = \delta \phi \times \delta \phi$ in Sect. 1.1 and Definition 2.1.13 respectively. A more general setup when this construction is well-applicable to energy–momentum tensor (10) is a free field.

**Theorem 2.2.9** (Integral energy–momentum conservation for a free field) *If the Lagrangian is $\mathcal{L}[\phi] = -\frac{1}{2} \# \delta \phi \sim \delta \phi - \frac{1}{2} m^2 \phi \sim \phi$, where $m \geq 0$, and $\phi \in C^k(I_d^d; \mathbb{R})$ satisfies Euler–Lagrange equation (7) on all non-boundary faces, then for each $d$-dimensional face $g$ disjoint with $\partial I_d^d$ and each $0 \leq l < d$ the flux of energy–momentum tensor (10) across $\partial g$ vanishes, i.e., $\langle T[\phi], \partial g \rangle_l = 0$.

Here we have dropped the Euler–Lagrange equations on the boundary, otherwise the system degenerates; the boundary faces do not contribute to the tensor flux anyway.

Electrodynamics (without currents) is the particular case of a free field theory with $m = 0, k = 1$. Further specification to $d = 3$ gives (3), where the function $T(h)$ on faces $h$ is actually $\langle T', h \rangle_0$. The case $m = 0, k = 1, d = 2$ was established in [13, Example in Sect. 8 of Ch. III] by a different method.

**Remark 2.2.10** Let us give general comments on the discretization of tensor flux in Definition 1.5.1.

There are many other ways to define a tensor flux; we have chosen the simplest one.

Our definition has the following informal motivation. Values of a tensor are “sitting” on the faces of the doubling; see the paragraph before Definition 2.2.5. The flux across a hyperface is then the sum of these values over the faces adjacent to the hyperface from the appropriate “side”.

For nonconserved tensors, an analog of the Stokes formula holds; see Proposition 4.2.4.

Unlike continuum theory, the 0-th component of the flux of the energy–momentum tensor $T'$ (see Definition 2.2.5) across a hyperface $h \perp (1, 0, \ldots, 0)$ is not necessarily positive, thus cannot be interpreted as energy density. This is a higher-order effect with respect to the discretization step $1/N$.

We use the notation $\langle T, h \rangle_k$, with literally the same definition, even if $T$ is not partially symmetric. This makes no sense in a discrete setup but is useful for the continuum limit.

The energy–momentum tensor $T$ of both the field and the current (see Definition 2.2.5) is not partially symmetric. In a sense, it still approximates some continuum tensor, but the latter is not conserved. We know neither an integral conservation law nor a conserved continuum analog for $T$.

The energy–momentum tensor $T'$ is symmetric in a sense (after “raising an index”). In particular, we shall see that it approximates the symmetric Belinfante–Rosenfeld energy–momentum tensor rather than the nonsymmetric canonical energy–momentum tensor. In other field theories, e.g., for the Dirac field, the discrete energy–momentum tensor approximates the nonsymmetric canonical energy–momentum tensor rather than the Belinfante–Rosenfeld one; see [25, Proposition A.3.6].
Let us illustrate analogy between tensor (10) and the continuum canonical energy–momentum tensor

\[ T_{kl} = \frac{\partial L}{\partial (\partial \phi / \partial x_l)} \frac{\partial \phi}{\partial x_k} - \delta^{li}_{k} L. \]

**Proposition 2.2.11** Let a local Lagrangian \( L : C^0(I^d_N; \mathbb{R}) \to C^0(I^d_N; \mathbb{R}) \) be homogeneous quadratic in \( \phi \) and \( \delta \phi \). Let \( \phi \) be a 0-dimensional field (not necessarily an extremal) and \( T[\phi] \) be the tensor (not necessarily partially symmetric) given by (10). Then for each \( 0 \leq k, l < d \) and each hyperface \( h \perp e_l \) having maximal vertex \( v \) and disjoint with the grid boundary we have

\[
(-1)^l \langle T[\phi], h \rangle_k = \frac{1}{2} \left( \frac{\partial L[\phi]}{\partial (\delta \phi)} (v + e_l) + \frac{\partial L[\phi]}{\partial (\delta \phi)} (v + e_l - 2e_k) \right) \cdot \delta \phi(v - e_k) - \delta^i_{k} L[\phi](v).
\]

**Approximation**

The discrete energy–momentum tensor \( T' \) indeed approximates the continuum one, as we show now. In continuum theory, an electromagnetic field is a continuous antisymmetric matrix field \( F_{mn} \) on the unit hypercube \( I^d \). The (Belinfante–Rosenfeld) energy-momentum tensor of the field (for the metric of signature \((+,-,\ldots,-)\)) is the matrix field

\[
T'_k = -F^{lm}F_{km} + \frac{1}{4} \delta^i_{k} F^{mn} F_{mn},
\]

where summation over repeating indices is understood and

\[
F^{mn} := \begin{cases} 
- F_{mn}, & \text{if } m = 0 \text{ or } n = 0; \\
F_{mn}, & \text{if } m \neq 0 \text{ and } n \neq 0.
\end{cases}
\]

Let \( I^d \) be dissected into \( N^d \) equal hypercubes. Given an arbitrary discrete 2-dimensional field \( F \), define the energy–momentum tensor \( T' = -\#F \times F \) on the resulting grid literally as on the grid \( I^d_N \).

**Remark 2.2.12** It is somewhat more natural to modify the definition of the operator \( \# \) by the factor \( N^{2k-d} \) because the faces are not unit hypercubes anymore. This leads just to the normalization of the energy–momentum tensor \( T' \) by a power of \( N \). We avoid such modification for simplicity.

**Proposition 2.2.13** (Approximation property) Let \( F_{mn} \) be a continuous electromagnetic field on \( I^d \). Dissect \( I^d \) into \( N^d \) equal hypercubes and define a discrete 2-dimensional field \( F_N \) on faces \( f \) of the resulting grid by the formula

\[
F_N(f) := F_{mn} \left( \text{max } f \right),
\]
where the integers $m < n$ are determined by the conditions $e_m, e_n \parallel f$. Let $T_k^l$ and $T'_N = -\#F_N \times F_N$ be the continuous and discrete energy-momentum tensor respectively. Take $0 \leq k, l < d$. Then on the set of all hyperfaces $h \perp e_l$ not intersecting $\partial I^d$ we have (under the notation before Theorem 2.1.17)

$$(-1)^l \langle T_N', h \rangle_k \Rightarrow T_k^l (\max h) \quad \text{as } N \to \infty.$$

**Remark 2.2.14** Here $F_{mn}$ and $F_N$ do not necessarily satisfy Maxwell’s equations (and typically $F_N$ cannot, even if $F_{mn}$ does). The approximation of a smooth solution of Maxwell’s equations by discrete ones, a standard question of computational electrodynamics, is not discussed in the paper.

### 2.3 Lattice Gauge Theory

Classical gauge theory generalizes electrodynamics. It is a basis for quantum gauge theory describing all known interactions except gravity. The idea is simple, as shown by the following toy model; cf. [22].

#### Toy Model

Several cities are joined by roads in the shape of an $M \times N$ grid; see Fig. 10. Each city has its own type of goods in an unlimited quantity. E.g., city $a$ has apples and city $b$ has bananas. For two neighboring cities $a$ and $b$ an exchange rate $U(ab) > 0$ is fixed, e.g., 2 banana for an apple. The rate is symmetric, i.e., $U(ba) = U(ab)^{-1}$: one gets back an apple for 2 banana.

A cunning citizen can travel and exchange along a square $abcd$ to multiply his initial amount of goods by a factor of $U(ab)U(bc)U(cd)U(da)$. The total speculation profit
is measured by the quantity
\[ S[U] := \sum_{\text{all faces } abcd} \log^2(U(ab)U(bc)U(cd)U(da)). \]

Here \( \log^2(x) \) is chosen as a function vanishing at \( x = 1 \) and positive for \( x \neq 1 \).

The king can set exchange rates except those on the boundary of the grid. He sets them to minimize the quantity \( S[U] \). The resulting collection of rates is a stationary Abelian gauge group field.

A stationary gauge group field is far from being unique. For an interior city, one can change the units, e.g., exchange dozens of apples instead of single ones. Such gauge transformation multiplies the rates for all the roads starting from the city by the same value but preserves \( S[U] \).

A similar model on a \( d \)-dimensional grid (with an additional minus sign for each summand in \( S[U] \) such that \( abcd \) is parallel to \( (1, 0, \ldots, 0) \)) is equivalent to lattice electrodynamics discussed in Sect. 2.2. This follows from Corollary 2.2.3, if one sets \( A(ab) = \log U(ab) \) and \( j = 0 \); see also Remark 2.3.6.

**Currents**

Now modify the model by introducing production of goods. For each pair of neighboring cities \( a \) and \( b \) fix a production rate \( j(ab) \geq 0 \); e.g., if \( a \) has apples and \( b \) has jam, then one produces \( j(ab) \) units of jam from one apple. The rate is not at all symmetric: one cannot produce apples from jam. Assume that production always goes in the direction of the coordinate axes.

There is a new way to profit: producing jam and exchanging back to apples, one multiplies the initial amount of apples by \( j(ab)U(ba) \). The total profit is now measured by the quantity \( S[U, j] = S[U] + \sum_{ab} (j(ab)U(ba) - 1) \). A collection of rates \( U \) minimizing \( S[U, j] \) for fixed \( j \) is called generated by \( j \). These rates may not exist, and the total profit can be negative.

These rates satisfy the conservation law \(-j(1)U(1)^{-1} - j(2)U(2)^{-1} + U(3)^{-1} j(3) + U(4)^{-1} j(4) = 0\) for each interior city \( v \), where we use the notation from Fig. 6 to the middle (this law is a version of Corollary 2.3.5). This is a “gauge-invariant” equation, which coincides with the usual charge conservation \( \partial j = 0 \) in the case when \( U = 1 \) everywhere.

**Non-Abelian Gauge Theory**

In non-Abelian gauge theory, the goods become vectors and the rates become matrices. To catch the idea, one can start with the case when \( d = 2, n = 1, G = \{ g \in \mathbb{C} : |g| = 1 \} \), and drop all \#-operators.

**Definition 2.3.1** Denote by \( \mathbb{C}^{m \times n} \) the set of matrices with complex entries having \( m \) rows and \( n \) columns. For \( u \in \mathbb{C}^{m \times n} \) denote by \( u^* \in \mathbb{C}^{n \times m} \) the conjugate transpose matrix.
A gauge group $G$ is a Lie group represented by unitary transformations of $\mathbb{C}^n$. A gauge group field $U$ and a covariant current $j$ are functions on edges of $M$ assuming values in $G$ and $\mathbb{C}^{n \times n}$ respectively.

The operator of parallel transport along a simple oriented path $\pi$ going along the edges is

$$U(\pi) := \prod_{e} U(e)^{(e, \pi)},$$

where the product is over all the edges $e$ of the path $\pi$, and $(e, \pi) = +1$ if $e$ is cooriented with $\pi$, and $-1$ otherwise. In particular, the trace $\text{Tr} U(\partial f)$ is a well-defined complex-valued function on 2-dimensional faces $f$. A gauge group field $U$ generated by a covariant current $j$ is an extremal for the functional (for fixed $j$)

$$S[U] = \sum_{\text{faces } f} \#(\text{Re Tr } U(\partial f) - n) - \sum_{\text{edges } e} \text{Re Tr } [j^*(e)U(e)].$$  \hspace{1cm} (11)

Since $S[U]$ is a continuous function on a compact set, we get the following existence theorem.

**Proposition 2.3.2** For each covariant current there exists a gauge group field generated by it.

Now we state the Yang–Mills equation (necessary and sufficient for $U$ to be generated by $j$) and a conservation law. This is a new Corollary 2.3.5 extending [12, Eq. (4.15)]. It involves projection to certain tangent space of the Lie group $G$. In gauge theory the role of the (co)boundary is played by the covariant (co)boundary, which is a “gauge covariant” operator equal the (co)boundary for $U = 1$.

**Definition 2.3.3** Fix a gauge group field $U$. Let $j$ be a $\mathbb{C}^{n \times n}$-valued function on edges. Its covariant boundary $D^*_{A}j$ is a $\mathbb{C}^{n \times n}$-valued function on vertices $v$ given by

$$[D^*_{A}j](v) = \sum_{e \text{ ending at } v} \text{U}(e)^{-1}j(e) - \sum_{e \text{ starting at } v} j(e)\text{U}(e)^{-1}. \hspace{1cm} (12)$$

Denote by $D^*_{A}\#F$ the $\mathbb{C}^{n \times n}$-valued function on edges $e$ given by

$$[D^*_{A}\#F](e) = \sum_{2\text{-faces } f \supset e} \#(\text{U}(e) - \text{U}(\partial f - e)), \hspace{1cm} (13)$$

where $\partial f - e$ is the path starting at the vertex $\text{min} e$, consisting of the 3 edges of $\partial f - e$, and ending at $\text{max} e$. E.g., in Fig. 10 we have $[D^*_{A}\#F](dc) = \text{U}(dabc) + \text{U}(dfe c) - 2\text{U}(dc)$.

So far the notations $D^*_{A}j$ and $D^*_{A}\#F$ should be viewed as indivisible. Separate conceptual definitions of $A, F, D^*_{A}$ are given in Definitions 2.3.9 and 3.1.1, so that (12)–(13) become easy propositions.
Definition 2.3.4  The scalar product of $u, v \in \mathbb{C}^{n \times n}$ is $\langle u, v \rangle := \text{Re Tr}[u^*v]$. Let $T_u G \subset \mathbb{C}^{n \times n}$ be the linear subspace parallel to the tangent subspace to $G$ at a point $u \in G$. Let $\text{Pr}_{T_u G}: \mathbb{C}^{n \times n} \to T_u G$ be the orthogonal projection and $\text{Pr}_{T_u G} j$ be the function on edges $e$ given by $[\text{Pr}_{T_u G} j](e) = \text{Pr}_{T_u G(j)}(e)$. A covariant current $j$ is conserved, if $D^*_A \text{Pr}_{T_u G} j = 0$.

Corollary 2.3.5  A gauge field $U$ generated by a covariant current $j$ satisfies the following equations:

- the Yang–Mills equation: $-\text{Pr}_{T_u G} D^*_A F = \text{Pr}_{T_u G} j$;
- Charge conservation law: $D^*_A \text{Pr}_{T_u G} j = 0$.

Remark 2.3.6  The latter form of charge conservation, different from the usual $\partial j = 0$, reflects the fact that non-Abelian gauge fields are themselves charged. In contrast to continuum theory, this remains true even if $G$ is Abelian (the deep reason is that the cup product is non-Abelian). Also, $D^*_A j \neq 0$ in general: e.g., if $j$ vanishes on all edges except one, then $D^*_A j \neq 0$ whatever $U$ is.

But for the Abelian group $G = \{e^{i\phi}: \phi \in \mathbb{R}\}$ and $d = 2$, the action can be modified so that charge conservation returns to the form $\partial j = 0$ (here $j \in C^1(I^2_N; \mathbb{R})$ is not a covariant current anymore):

$$S^{\text{Ab}}[U] = -\frac{1}{2} \sum_{\text{2-faces } f} \arccos^2 \text{Re} U(f) + i \sum_{\text{edges } e} \text{arccos}^2 \text{Re} U(e).$$

The range of $U$ must be restricted to $\{e^{i\phi}: -\pi/4 < \phi < \pi/4\}$ to keep the action single-valued and differentiable. The resulting theory is equivalent to lattice electrodynamics of §2.2, also with restricted range, because $S^{\text{Ab}}[e^{i\phi}] = \epsilon \left[-\frac{1}{2} \# \delta \phi \sim \delta \phi - j \sim \phi\right]$ for $\phi \in C^1(I^2_N; \mathbb{R})$ with $|\phi| < \pi/4$.

Connection and Curvature

Definition 2.3.7  Let $g$ and $\phi$ be $G$- and $\mathbb{C}^{n \times n}$-valued functions on vertices and $k$-faces respectively. The gauge transformation of $\phi$ by $g$ is the function $g^* \sim \phi \sim g$ on $k$-faces $f$ given by (cf. Table 3)

$$[g^* \sim \phi \sim g](f) := g^*(\min f) \phi(f) g(\max f).$$

Corollary 2.3.8  (Gauge invariance) Each simultaneous gauge transformation of $U$ and $j$ by the same element $g$ preserves $S[U]$. If $U$ is generated by $j$, then $g^* \sim U \sim g$ is generated by $g^* \sim j \sim g$.

Definition 2.3.9  The unit gauge group field $1$ equals the unit $n \times n$ matrix at each edge. For a gauge group field $U$, the connection (or gauge potential) is the $\mathbb{C}^{n \times n}$-valued function $A = A[U] = U - 1$. The curvature (or field strength) is the $\mathbb{C}^{n \times n}$-valued function $F = F[U]$ on 2-dimensional faces given by

$$F[U](abcd) := U(ab)U(bc) - U(ad)U(dc)$$

(14)
Table 3  Products of 0- and 1-(co)chains ($ab$ denotes an edge with $a < b$; the sums are over edges)

|dim $\phi$ = 1, dim $\psi$ = 0| dim $\phi$ = 0, dim $\psi$ = 1| dim $\phi$ = dim $\psi$ = 1|
|---|---|---|
| $[\phi \sim \psi](ab) = \phi(ab) \psi(b)$ | $[\phi \sim \psi](ab) = \phi(a) \psi(ab)$ | $\phi \sim \psi$ is defined in Figure 6 |
| $\phi \sim \psi = 0$ | $\phi \sim \psi = 0$ | $\phi \sim \psi(b) = \sum_{bc:a < b} \phi(ab) \psi(bc)$ |
| $[\phi \sim \psi](ab) = \phi(ab) \psi(ab)$ | $[\phi \sim \psi](ab) = \phi(ab) \psi(ab)$ |

for each face $abcd$ with the vertices listed counterclockwise starting from the minimal one; see Fig. 6.

Remark 2.3.10  On a grid, a gauge group field $U$ is a gauge transformation of the unit gauge group field, if and only if the curvature $F[U]$ vanishes (this is proved by a standard “homological” argument.)

In contrast to continuum theory, the connection and curvature assume values not in the Lie algebra of the Lie group $G$ but in certain other subsets of $\mathbb{C}^{n \times n}$ approximating the Lie algebra in a sense. The fields $A$ and $F$ from Sects. 2.1–2.2 are neither connection nor curvature for no gauge group field.

Similarly to Proposition 2.2.13, the tensor $-\text{Re Tr} \left[ #F^* \times F \right]$ approximates the continuum Belinfante–Rosenfeld energy–momentum tensor. But the former is not conserved and not even gauge invariant.

Proposition 2.3.11  There is the following expression for action (11):

$$S[U] = \epsilon \text{Re Tr} \left[ -\frac{1}{2} #F^* \sim F - j^* \sim U \right].$$

The latter is the one given by the algorithm from Sect. 1.3 up to an additive constant; see Table 2.

3 Generalizations

In this section we state the main results in their full generality, i.e., for general connections and arbitrary simplicial and cubical complexes. The results of Sect. 1.4 are obtained in the particular case when the complex is a grid, the gauge group is trivial, i.e., $G = \{1\}$, and the fields are real-valued. Most of the results of Sect. 2 are obtained from these general results by substituting specific Lagrangians.

3.1 General Connections

Interaction with a gauge field is introduced by replacement of (co)boundary by covariant (co)boundary. The latter is defined in terms of cochain products as follows; see Table 3 and cf. [12, Sects. IV–V]. Let $U \in C^1(M; G)$, $A = U - 1$, $F$ be a gauge group field, the connection, and the curvature respectively.

Definition 3.1.1  Denote by $C^k(M; V)$ the set of functions defined on the set of $k$-dimensional faces and assuming values in a set $V$. Here $V$, and hence $C^k(M; V)$, is a set, not necessarily a group.
Denote by \( a \ldots b \) the face \( f \) such that \( \min f = a, \max f = b \) (if such face \( f \) exists, then it is unique). An ordered triple of faces \( a \ldots b, b \ldots c \subset a \ldots c \) of dimensions \( k, l, k + l \) respectively is cooriented (respectively, opposite oriented), if the ordered set consisting of a positive basis in \( a \ldots b \) and a positive basis in \( b \ldots c \) is a positive (respectively, negative) basis in \( a \ldots c \). Write

\[
\langle a, b, c \rangle = \begin{cases} +1, & \text{if } a \ldots b, b \ldots c, a \ldots c \text{ are cooriented}, \\ -1, & \text{if } a \ldots b, b \ldots c, a \ldots c \text{ are oppositely oriented}. \end{cases}
\]

The cup, cap, and cop product of functions \( \Phi \in C^k(M; \mathbb{C}^{p \times q}) \) and \( \Psi \in C^l(M; \mathbb{C}^{q \times r}) \) are the \( \mathbb{C}^{p \times r} \)-valued functions on \( (k + l)-\), \( (k - l)-\), and \( (l - k)\)-dimensional faces respectively given by

\[
[\Phi \prec \Psi](a \ldots c) = \sum_{b: \dim(a \ldots b) = k, \dim(b \ldots c) = l} \langle a, b, c \rangle \Phi(a \ldots b) \Psi(b \ldots c); \\
[\Phi \prec \Psi](b \ldots c) = \sum_{a: \dim(a \ldots c) = k, \dim(a \ldots b) = l} \langle a, b, c \rangle \Phi(a \ldots c) \Psi(a \ldots b); \\
[\Phi \prec^* \Psi](a \ldots b) = \sum_{c: \dim(b \ldots c) = k, \dim(a \ldots c) = l} \langle a, b, c \rangle \Phi(b \ldots c) \Psi(a \ldots c),
\]

where the sums are over all the vertices such that there exist 3 faces \( a \ldots b, b \ldots c \subset a \ldots c \) of the indicated dimensions.

For \( \Phi \in C^k(M; \mathbb{C}^n) \), the covariant coboundary and the covariant boundary are respectively

\[
DA \Phi := \delta \Phi + A \prec \Phi - (-1)^k \Phi \prec A; \\
D^*A \Phi := \partial \Phi + (\Phi \prec^* A)^* + (-1)^k (A \prec^* \Phi^*)^*.
\]

For \( \phi \in C^k(M; \mathbb{C}^1) \), the gauge transformation by \( g \in C^0(M; G) \) is the field \( \phi \prec g \), and the covariant coboundary and the covariant boundary are respectively

\[
DA \phi := \delta \phi - (-1)^k \phi \prec A; \\
D^*A \phi := \partial \phi + (\phi^* \prec A^*)^*.
\]

**Remark 3.1.2** Definitions of a gauge transformation and covariant (co)boundary crucially depend on the set of field values (more precisely, on the representation of \( G \)): compare (15)–(16) and (17)–(18). For \( n = 1 \) there is a minor conflict of notation between these pairs of equations, cleared up by context.

Informally, (17)–(18) mean the following. Think of the field value at a face \( e \) as sitting at the maximal vertex \( \text{max} e \). Then the covariant (co)boundary value at a face \( v \) is defined just as the ordinary (co)boundary, but all the involved field values are parallelly transported to the maximal vertex \( \text{max} v \).
The definition of the cup product is equivalent to [31, (22.3)] but not [32, Chapter IX, §14, Eq. (7)].

Up to sign and factors interchange, the cop product is the cap product in the same grid but with reversed vertices ordering. The cap and cop products vanish for \( k < l \) and \( k > l \) respectively, and do not coincide for \( k = l \neq 0 \). Usually both are denoted in the same way, which does not lead to a conflict until one identifies chains and cochains (hence the domains of the products). Since we have performed such identification, we need to introduce new notation \( \langle \rangle \) and new term “cop product”.

**Proposition 3.1.3** For each gauge group field \( U \) we have \( F = \delta A + A \sim A, \, D_A F = 0 \), and (12)–(13).

**Definition 3.1.4** (Cf. Definition 1.4.4) A map

\[
\mathcal{L} : C^k(M; \mathbb{C}^{1 \times n}) \times C^1(M; \mathbb{C}^{n \times n}) \rightarrow C_0(M; \mathbb{R})
\]

is a local Lagrangian, if for each vertex \( v \in M \) there is a smooth function \( L_v(\phi_1, \ldots, \phi_p, \phi'_1, \ldots, \phi'_q) \in C^1(\mathbb{C}^{1 \times n}) \) such that for each \( \phi \in C^k(M; \mathbb{C}^{1 \times n}) \) and \( U \in C^1(M; \mathbb{C}^{n \times n}) \) we have

\[
\mathcal{L}[\phi, U](v) = L_v(\phi(e_1), \ldots, \phi(e_p), [D_A[U] \phi](f_1), \ldots, [D_A[U] \phi](f_q)),
\]

where \( e_1, \ldots, e_p \) and \( f_1, \ldots, f_q \) are all the faces of dimension \( k \) and \( k+1 \) respectively with the maximal vertex \( v \); see Fig. 2. Define \( \frac{\partial L_v}{\partial \phi_i} : (\mathbb{C}^{1 \times n})^{p+q} \rightarrow \mathbb{C}^{n \times n} \) by \( \left( \frac{\partial L_v}{\partial \phi_i} \right)_m = \frac{\partial L_v}{\partial (\text{Re} \phi_i^m)} - i \frac{\partial L_v}{\partial (\text{Im} \phi_i^m)} \), where \( \phi_i = (\phi_i^1, \ldots, \phi_i^n) \in \mathbb{C}^{1 \times n} \). Define \( \frac{\partial L_v}{\partial \phi_m^i} \) analogously. Define

\[
\frac{\partial \mathcal{L}}{\partial \phi}(\phi, U) \in C_k(M; \mathbb{C}^{n \times 1}) \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial (D_A \phi)} \in C_{k+1}(M; \mathbb{C}^{n \times 1})
\]

by the following formulae for each \( l = 1, \ldots, p \) and \( m = 1, \ldots, q \):

\[
\frac{\partial \mathcal{L}}{\partial \phi}(\phi, U) (e_l) := \frac{\partial L_v}{\partial \phi_l}(\phi(e_1), \ldots, \phi(e_p), [D_A[U] \phi](f_1), \ldots, [D_A[U] \phi](f_q)),
\]

\[
\frac{\partial \mathcal{L}}{\partial (D_A \phi)} (f_m) := \frac{\partial L_v}{\partial \phi_m}(\phi(e_1), \ldots, \phi(e_p), [D_A[U] \phi](f_1), \ldots, [D_A[U] \phi](f_q)).
\]

A field \( \phi \in C^k(M; \mathbb{C}^{1 \times n}) \) is an extremal or stationary for the functional \( S[\phi, U] = \epsilon \mathcal{L}[\phi, U], \) if \( \frac{\partial}{\partial t} S[\phi + t \Delta, U] \big|_{t=0} = 0 \) for each \( \Delta \in C^k(M; \mathbb{C}^{1 \times n}) \) and given fixed \( U \in C^1(M; G) \).

**Proposition 3.1.5** For fixed current or covariant current \( j \), each of the Lagrangians in Table 4 to the left is local and the partial derivatives are given by the two columns to the right.
Table 4  Partial derivatives of basic Lagrangians

| Lagrangian $L[\phi]$ | $L_v(\phi_1, \ldots, \phi_p, \phi'_1, \ldots, \phi'_q)$ | $\frac{\partial L}{\partial \phi}$ | $\frac{\partial L}{\partial (D_A \phi)}$ |
|----------------------|---------------------------------|----------------|----------------|
| 1  $j \sim \phi$  
(dim $\phi = 1$)  
$\sum_{i=1}^j e_i$ | $j$ | 0 | |
| 2  $\# \delta \phi \sim \delta \phi$  
$\sum_{i=1}^q g(k, l) \phi_l' \phi_i'$ | 0 | 2$\# \delta \phi$ | |
| 3  $\phi \sim \phi^*$  
$\sum_{i=1}^p \phi_i \phi_i^*$ | $2\phi$ | 0 | |
| 4  $\# D_A \phi \sim (D_A \phi)^*$  
$\sum_{i=1}^q g(k, l) \phi_l'(\phi_i')^*$ | 0 | 2$\# D_A \phi$ | |
| 5  Re Tr $[j^* \sim U]$  
$\sum_{i=1}^p f^*(e_i) U_l$ | $j$ | 0 | |
| 6  Re Tr $[F^* \sim F]$  
$\sum_{i=1}^q g(1, l)(U_l')^* U_l'$ | 0 | 2$\# F$ | |

Theorem 3.1.6  (The Euler–Lagrange equation) Let $L : C^k(M; \mathbb{C}^{1 \times n}) \times C^1(M; \mathbb{C}^{n \times n}) \to C_0(M; \mathbb{R})$ be a local Lagrangian, $A \in C^1(M; \mathbb{C}^{n \times n})$ be a connection. Then $\phi \in C^k(M; \mathbb{C}^{1 \times n})$ is an extremal if and only if

$$D_A^* \left( \frac{\partial L[\phi, 1 + A]}{\partial (D_A \phi)} \right)^* + \left( \frac{\partial L[\phi, 1 + A]}{\partial \phi} \right)^* = 0. \tag{19}$$

A local Lagrangian $L : C^1(M; \mathbb{C}^{n \times n}) \to C_0(M; \mathbb{R})$ and the partial derivatives $\frac{\partial L}{\partial U} \in C_1(M; \mathbb{C}^{n \times n})$, $\frac{\partial L}{\partial (F[U])} \in C_2(M; \mathbb{C}^{n \times n})$ are defined analogously to Definition 3.1.4, only the fields $\phi$ and $D_A \phi$ are replaced by a gauge group field $U$ and the curvature $F[U]$ respectively (notice that $F[U] \neq D_A U$). A gauge group field $U$ is an extremal, if it is stationary for the functional $S[U] = \epsilon L[U]$ under the constraint $U \in C^1(M; G)$. For fixed $\phi \in C^k(M; \mathbb{C}^{1 \times n})$, a local Lagrangian $L[\phi, U]$ in the sense of Definition 3.1.4 is a local Lagrangian in the sense of this paragraph (by the second paragraph of Remark 3.1.2). The latter is the reason for using row-vectors $\phi$ rather than column-vectors.

Theorem 3.1.7  (The Euler–Lagrange equation) Let $L : C^1(M; \mathbb{C}^{n \times n}) \to C_0(M; \mathbb{R})$ be a local Lagrangian. Then a gauge group field $U \in C^1(M; G)$ is an extremal, if and only if

$$\Pr_{T,U,G} \left[ D_A^* \left( \frac{\partial L[U]}{\partial (F[U])} \right)^* + \left( \frac{\partial L[U]}{\partial U} \right)^* \right] = 0. \tag{20}$$

Theorem 3.1.8  (Noether’s theorem) If a local Lagrangian $L[\phi, U]$ satisfies (8) for some $\Delta \in C^k(M; \mathbb{C}^{1 \times n})$ and fixed $U \in C^1(M; G)$, then for each extremal $\phi$ the edgewise scalar product of the covariant current $j[\phi, U] = \left( \frac{\partial L[\phi, U]}{\partial (D_A \phi)} \right)^* \Delta$ with $U$ is conserved, i.e. $\partial \langle j[\phi, U], U \rangle = 0$.

A Lagrangian $L[\phi, U]$ is gauge invariant, if $L[\phi \sim g, g^* \sim U \sim g] = L[\phi, U]$ for each $\phi \in C^k(M; \mathbb{C}^{1 \times n})$, $U \in C^1(M; G)$, $g \in C^0(M; G)$. For gauge invariant Lagrangians the numerous Noether currents are combined together as follows.
Theorem 3.1.9 (Charge conservation) If a local Lagrangian $\mathcal{L}[\phi, U]$ is gauge invariant, then for each gauge group field $U$ and each extremal $\phi$ the following covariant current is conserved:

$$j[\phi, U] = \left( \frac{\partial \mathcal{L}[\phi, U]}{\partial (D_A \phi)} \right)^* \phi = \left( \frac{\partial \mathcal{L}[\phi, U]}{\partial U} \right)^* \rightarrow \Pr_T U G j[\phi, U] = 0,$$

i.e., $D^* A \Pr_T U G j[\phi, U] = 0$.

Theorem 3.1.10 (Charge conservation) Let $\mathcal{L}[U] = \mathcal{L}'[U] - \text{Re Tr}[j^* \rightarrow U]$ be a local Lagrangian, where $j \in C_1(M; \mathbb{C}^{n \times n})$ is fixed and $\mathcal{L}'[U]$ is gauge invariant and does not depend on $j$. Then for each extremal $U \in C^1(M; G)$ the covariant current $j$ is conserved, i.e., $D^* A \Pr_T U G j[\phi, U] = 0$.

The last three theorems are not completely obvious even for a $1 \times 1$ grid. The crucial gauge invariance is usually guaranteed by the following result.

Proposition 3.1.11 (Gauge covariance, see [12]) For each $U \in C^1(M; G)$, $\Phi \in C^k(M; \mathbb{C}^{n \times n})$, $\phi \in C^k(M; \mathbb{C}^1 \times n)$, $g \in C^0(M; G)$ we have:

- $A[g^* \rightarrow U \rightarrow g] = g^* \rightarrow A[U] \rightarrow g + g^* \rightarrow \delta g$
- $F[g^* \rightarrow U \rightarrow g] = \Phi \rightarrow F[U] \rightarrow g$;
- $D_A[g^* \rightarrow U \rightarrow g] = g^* \rightarrow (D_A[U] \Phi) \rightarrow g$;
- $D^*_A[g^* \rightarrow U \rightarrow g] = g^* \rightarrow (D^*_A[U] \Phi) \rightarrow g$.

The Lagrangians in the left column and rows 3, 4, 6 of Table 4 are gauge invariant.

3.2 Simplicial and Cubical Complexes

Definition 3.2.1 A finite simplicial (respectively, cubical) complex is a finite set of simplices (respectively, hypercubes) in a Euclidean space of some dimension satisfying the following properties:

- the intersection of any two simplices (respectively, hypercubes) from the set is either empty or their common face (a simplex/hypercube itself is also viewed as its own face);
- all the faces of a simplex (respectively, a hypercube) from the set belong to the set as well.

Spacetime $M$ is an arbitrary finite simplicial or cubical complex with a fixed vertices ordering. For a cubical complex, we require that the minimal and the maximal vertex of each 2-dimensional face are opposite (this is essential for the definition of products and curvature). The simplices/hypercubes of $M$ are called faces of $M$.

Remark 3.2.2 While vertices ordering is required, a particular choice is not that important. For an arbitrary ordering, the discretization algorithm from Sect. 1.3...
automatically produces a local Lagrangian for all field theories we considered (cf. Proposition 3.1.5). Changing the ordering is like changing the lattice: combinatorial relations are changed but the underlying physical theory remains the same.

Until this subsection, spacetime was a grid with the dictionary vertices ordering. Passing to general spacetime is like passing from a coordinate chart to a coordinate-free formulation. The paper is intentionally designed to make this almost automatic. For an arbitrary spacetime $M$, all notions in the middle column of Table 1 except $\#$ and $\langle T, h \rangle_k$ are defined literally as above (see the right column for definition numbers) up to the following modifications required for simplicial complexes $M$ only:

**Definition 1.4.8:** The Cartesian square $M \times M$ is now a cell complex (rather than simplicial or cubical complex) with faces of the form $e \times f$, where $e$ and $f$ are faces of $M$.

**Definition 2.3.9:** The curvature is no longer defined by (14) but now by the formula

$$F[U](abc) = U(ab)U(bc) - U(ac)$$

for each face $abc$ with the vertices listed in increasing order $a < b < c$.

**Definition 3.1.1:** A face is no longer determined by just the minimal and the maximal vertices. Thus we denote by $a_1a_2\ldots a_{s+1}$ the $s$-dimensional face with the vertices $a_1 < a_2 < \cdots < a_{s+1}$. Then $a\ldots b, b\ldots c, a\ldots c$ are replaced by $a_1\ldots a_s b, a_1\ldots a_s b c_1 \ldots c_l, a_1\ldots a_s b c_1 \ldots c_l$ respectively, summation over $b$ is omitted, and summation over $a$ and $c$ is replaced by summation over all collections $(a_1, \ldots, a_s)$ and $(c_1, \ldots, c_l)$ respectively.

With these modifications, all the theorems in Sects. 1.4 and 3.1, as well as their proofs, remain literally true for an arbitrary spacetime $M$. (Propositions 3.1.3 and 3.1.5 remain true, once one drops all $\#$-operators; see the proofs.) We do not use and do not define $\#$ and $\langle T, h \rangle_k$ for $M \neq I^d_N$.

**Remark 3.2.3** To make the definition of fields more accessible to nonspecialists, we took the liberty to use equivalent definitions of some commonly used notions and to identify spaces connected by the unique fixed isomorphism. Now we compare Definition 1.4.1 with the other ones in literature.

Often simplicial (or cubical) $k$-chains are defined in a more abstract way, as the elements of the linear space $C_k(M; \mathbb{R})$ generated by the $k$-dimensional faces of $M$ (with somehow fixed orientation); and $k$-cochains are defined as elements of the dual space $C^k(M; \mathbb{R})$. But space $C_k(M; \mathbb{R})$ comes with the obvious unique distinguished basis: the basis consists of all the $k$-dimensional faces; the orientation of the faces is determined by the order of their vertices in spacetime $M$ as specified in Definition 1.4.2; the faces are listed in the dictionary order with respect to the ordered lists of their vertices. The distinguished basis identifies both $C_k(M; \mathbb{R})$ and $C^k(M; \mathbb{R})$ with the set of real-valued functions defined on the set of $k$-dimensional faces, that is, $k$-cochains in the sense of Definition 1.4.1. Notice that this identification is not related to spacetime metric.
Thus we do not distinguish between chains and cochains. Inserting the obvious isomorphism between their spaces in our formulae would give no advantage but would only complicate notation. However, to make notation compatible with the commonly used one, we sometimes switch between different notation $C^k(M; \mathbb{R})$ and $C_k(M; \mathbb{R})$ for the same object (in our setup).

We do distinguish between row- and column-vectors. This makes clear, if the product of two vectors is a number or a matrix. Some of our results depend on the type of vectors used as field values.

We do not assume that $M$ is a manifold. In fact, faces of $M$ of dimension $> 2$ have never appeared at all in the examples from Sect. 2. The whole ambient spacetime is not that important: think of an electric network lying on a table; is spacetime of the model 1-, 2-, 3- or 4-dimensional? This is why we avoid dual grids and the Hodge star. However dimension-like properties of $M$ like the average vertex degree are of course important.

4 Proofs

4.1 Basic Results

First we prove the results of Sect. 1.4. The statements are recalled right before the proofs for the convenience. Throughout Sect. 4.1 $\mathcal{L}: C^k(M; \mathbb{R}) \to C_0(M; \mathbb{R})$ is a local Lagrangian and $\phi, \Delta \in C^k(M; \mathbb{R})$ (they are not necessarily extremals). Besides the notation from Sect. 1.4, we only use the following one:

- $\epsilon$ is the sum of the values of a 0-dimensional field over all the vertices;
- $\phi \sim \Delta$ is the 0-dimensional field given by

$$[\phi \sim \Delta](v) = \sum_{f: \text{dim } f = k, \text{max } f = v} \phi(f) \Delta(f),$$

where the sum is over all the $k$-dimensional faces $f$ with the maximal vertex $v$; cf. Definition 1.4.6.

Lemma 4.1.1 (Lagrangian functional derivative) For arbitrary fields $\phi, \Delta \in C^k(M; \mathbb{R})$ we have

$$\frac{\partial \mathcal{L}[\phi + t \Delta]}{\partial t} \bigg|_{t=0} = \left( \frac{\partial \mathcal{L}[\phi]}{\partial \phi} + \partial \frac{\partial \mathcal{L}[\phi]}{\partial (\delta \phi)} \right) \sim \Delta - (-1)^k \partial \left( \frac{\partial \mathcal{L}[\phi]}{\partial (\delta \phi)} \sim \Delta \right).$$

Proof Take a vertex $v \in M$. Starting with (4)–(6), then using the chain rule, and finally the well-known ‘integration by parts’ identity [31] (which holds for any $\psi \in C^{k+1}(M; \mathbb{R})$)

$$\partial(\psi \sim \phi) = (-1)^{\text{dim } \phi} (\partial \psi \sim \phi - \psi \sim \delta \phi) \quad (21)$$

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we get
\[
\frac{\partial L[\phi + t\Delta]}{\partial t}(v) \bigg|_{t=0} = \frac{\partial}{\partial t} L_v \left( [\phi + t\Delta](e_1), \ldots, [\phi + t\Delta](e_p), [\delta \phi + \delta t\Delta](f_1), \ldots, [\delta \phi + \delta t\Delta](f_q) \right) \bigg|_{t=0} \\
= \sum_{l=1}^{p} \frac{\partial L_v}{\partial \phi_l} (\phi(e_1), \ldots, \phi(e_p), [\delta \phi](f_1), \ldots, [\delta \phi](f_q)) \cdot \frac{\partial}{\partial t} [\phi + t\Delta](e_l) \bigg|_{t=0} \\
+ \sum_{m=1}^{q} \frac{\partial L_v}{\partial \phi_m} (\phi(e_1), \ldots, \phi(e_p), [\delta \phi](f_1), \ldots, [\delta \phi](f_q)) \cdot \frac{\partial}{\partial t} [\delta \phi + \delta t\Delta](f_m) \bigg|_{t=0} \\
= \sum_{l=1}^{p} \frac{\partial L_v}{\partial \phi} (e_l) \Delta(e_l) + \sum_{m=1}^{q} \frac{\partial L_v}{\partial (\delta \phi)} (f_m) [\delta \Delta](f_m) \\
= \left[ \frac{\partial L_v}{\partial \phi} \right] \Delta + \frac{\partial L_v}{\partial (\delta \phi)} \delta \Delta \bigg] (v) \\
= \left[ \left( \frac{\partial L_v}{\partial \phi} + \frac{\partial L_v}{\partial (\delta \phi)} \right) \Delta \right] (v) - (-1)^k \varepsilon \left( \frac{\partial L_v}{\partial (\delta \phi)} \Delta \right) (v).
\]

\[\square\]

Lemma 4.1.2 Fix \( \phi \in C_k(M; \mathbb{R}) \). If \( \varepsilon [\phi \sim \Delta] = 0 \) for each \( \Delta \in C^k(M; \mathbb{R}) \), then \( \phi = 0 \).

Proof Take \( \Delta = \phi \). Then \( 0 = \varepsilon [\phi \sim \phi] = \sum_{f: \dim f=k} \phi(f)^2 \). Thus \( \phi = 0 \). \qed

Theorem 4.1.3 (Restatement of Theorem 1.4.5) A field \( \phi \) is an extremal if and only if
\[
\frac{\partial}{\partial (\delta \phi)} \frac{\partial L[\phi]}{\partial \phi} + \frac{\partial L[\phi]}{\partial \phi} = 0.
\]

Proof of the Euler–Lagrange Theorem 4.1.3 A field \( \phi \) is an extremal if and only if for any \( \Delta \) we have
\[
0 = \varepsilon \frac{\partial L[\phi + t\Delta]}{\partial t} \bigg|_{t=0} = \varepsilon \left[ \left( \frac{\partial L_v}{\partial \phi} + \frac{\partial L_v}{\partial (\delta \phi)} \right) \Delta \right] - (-1)^k \varepsilon \left( \frac{\partial L_v}{\partial (\delta \phi)} \Delta \right).
\]

The latter two equalities follow from Lemma 4.1.1 and the obvious identity \( \varepsilon \partial = 0 \) respectively. Since \( \Delta \) is arbitrary, by Lemma 4.1.2 the resulting equation is equivalent to desired equation (7). \qed
Theorem 4.1.4 (Restatement of Theorem 1.4.7) An extremal \( \phi \) satisfies the condition \( \left. \frac{\partial}{\partial t} \mathcal{L} [\phi + t \Delta] \right|_{t=0} = 0 \) if and only if the current \( j[\phi] = \frac{\partial \mathcal{L}[\phi]}{\partial (\delta \phi)} \sim \Delta \) is conserved.

Proof of the Noether Theorem 4.1.4 By Lemma 4.1.1 and Theorem 4.1.3 for an extremal \( \phi \) we get

\[
\left. \frac{\partial \mathcal{L}[\phi + t \Delta]}{\partial t} \right|_{t=0} = \left( \frac{\partial \mathcal{L}[\phi]}{\partial \phi} + \frac{\partial \mathcal{L}[\phi]}{\partial (\delta \phi)} \right) \sim \Delta - (-1)^k \partial \left( \frac{\partial \mathcal{L}[\phi]}{\partial (\delta \phi)} \sim \Delta \right) = -(-1)^k \partial j[\phi].
\]

Thus \( j[\phi] \) is a conserved current, if and only if the left-hand side vanishes. \( \square \)

Remark 4.1.5 This is immediately generalized to symmetries of the action \( S \) rather than the Lagrangian \( \mathcal{L} \): if \( \left. \frac{\partial}{\partial t} S[\phi + t \Delta] \right|_{t=0} = 0 \) then \( \left. \frac{\partial}{\partial t} \mathcal{L} [\phi + t \Delta] \right|_{t=0} = \partial \Lambda[\phi, \Delta] \) for some \( \Lambda[\phi, \Delta] \in C_1(M; \mathbb{R}) \) because \( M \) is connected. Then \( j[\phi] + (-1)^k \Lambda[\phi, \Delta] \) is a conserved current.

Theorem 4.1.6 (Restatement of Theorem 1.4.9) For each extremal \( \phi \) the energy-momentum tensor \( T[\phi] = \frac{\partial \mathcal{L}[\phi]}{\partial (\delta \phi)} \times \delta \phi + \frac{\partial \mathcal{L}[\phi]}{\partial \phi} \times \phi \) is conserved.

Proof of Energy–momentum conservation Theorem 4.1.6 By Theorem 4.1.3, Definition 1.4.8, and the known identity \( \partial \partial = \delta \delta = 0 \), for each extremal \( \phi \) we have

\[
\partial T[\phi] = \partial \left( \frac{\partial \mathcal{L}[\phi]}{\partial (\delta \phi)} \times \delta \phi + \frac{\partial \mathcal{L}[\phi]}{\partial \phi} \times \phi \right) = \partial \left( \frac{\partial \mathcal{L}[\phi]}{\partial (\delta \phi)} \times \delta \phi - \partial \frac{\partial \mathcal{L}[\phi]}{\partial (\delta \phi)} \times \phi \right) = \frac{\partial \mathcal{L}[\phi]}{\partial (\delta \phi)} \times \delta \phi + \frac{\partial \mathcal{L}[\phi]}{\partial (\delta \phi)} \times \delta \phi - \partial \frac{\partial \mathcal{L}[\phi]}{\partial (\delta \phi)} \times \phi
\]

\( \square \)

4.2 Summit

Now we prove the result of Sect. 1.5. We start with a visual heuristic proof of a particular case from Sect. 1.1.

Proof of identity (3). By definition, twice the left-hand side of (3) equals

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & 8 & 9 & 10 & 11 & 12
\end{pmatrix}
\begin{pmatrix}
\begin{array}{c}
F(\phi) + F(\phi) \\
F(\phi) + F(\phi) \\
F(\phi) - F(\phi) \\
F(\phi) - F(\phi) \\
- F(\phi) + F(\phi) \\
+ F(\phi) + F(\phi) \\
- F(\phi) - F(\phi) \\
+ F(\phi) + F(\phi) \\
\end{array}
\end{pmatrix}
\begin{array}{c}
-3a+b \\
-3a+b \\
1-a \\
1-a \\
1-a+c \\
1-a+c \\
1-a \\
1-a \\
\end{array}
\begin{pmatrix}
\begin{array}{c}
F(\phi) \\
F(\phi) \\
F(\phi) \\
F(\phi) \\
F(\phi) \\
F(\phi) \\
F(\phi) \\
F(\phi)
\end{array}
\end{pmatrix}
\begin{array}{c}
a+b \\
a+b \\
c+d \\
c+d \\
c+d \\
c+d \\
c+d \\
c+d \\
\end{array}
\]

\( \square \) Springer
Here the terms labeled by letters cancel each other; the terms in square brackets vanish by (1).

To proceed, we are going to integrate tensors defined on $I^d_N \times I^d_N$ over the faces of the doubling.

**Definition 4.2.1** Dissect the hypercube $0 \leq x_0, x_1, \ldots, x_{d-1} \leq N$ in $\mathbb{R}^d$ into $(2N)^d$ equal hypercubes. The cubical complex consisting of all the faces of the resulting hypercubes is the *doubling* of $I^d_N$. For a vertex $f$ of the doubling, define $f_0, \ldots, f_{d-1} \in \mathbb{Z}$ by the formula $f = f_0e_0 + \cdots + f_{d-1}e_{d-1}$. The face of the initial grid $I^d_N$ with the center $f$ is denoted by $f$ as well.

Let $T$ be a partially symmetric type $(1, 1)$ tensor, $g$ be a non-boundary face of the doubling, $e_l \perp g$, $f = \max g$. The *k-th component of the flux of $T$ across $g$ in the positive normal direction* is

\[
\langle T, g \rangle_k = \frac{1}{2}(-1)^{\frac{1}{2}(l+1+\sum_{m=k}^{\max(k,l)} f_m)} \begin{cases} 
-T((f - e_k) \times (f + e_l)), & \text{if } l \neq k, 2 \nmid f_k, 2 \nmid f_l; \\
T((f + e_l - e_k) \times f), & \text{if } l \neq k, 2 \nmid f_k, 2 \mid f_l; \\
T(f \times (f + e_l - e_k)), & \text{if } l \neq k, 2 \mid f_k, 2 \nmid f_l; \\
-T((f + e_l) \times (f - e_k)), & \text{if } l \neq k, 2 \mid f_k, 2 \mid f_l; \\
T(f \times f) - T((f + e_l) \times (f - e_k)), & \text{if } l = k.
\end{cases}
\]

Let $L$ be a type $(0, 1)$ tensor, $g$ be a $d$-dimensional face of the doubling, $f = \max g$. Denote

\[
\langle L, g \rangle_k = \frac{1}{2}(-1)^{1+\sum_{m<k} f_m} \begin{cases} 
L(f \times (f - e_k)), & \text{if } 2 \mid f_k; \\
L((f - e_k) \times f), & \text{if } 2 \nmid f_k.
\end{cases}
\]

**Proposition 4.2.2** The flux of a partially symmetric type $(1, 1)$ tensor across a hyperface $h$ of the grid $I^d_N$ (see Definition 1.5.1) is the sum of fluxes across all the hyperfaces of the doubling contained in $h$.

**Proof** Compare the $k$-th components of the fluxes. Take $e_l \perp h$. Consider the 2 cases: $l = k$ and $l \neq k$.

For $l = k$, the map $g \mapsto \max g$ is a 1–1 map between the set of hyperfaces of the doubling contained in $h$ and the set of faces of the initial grid $I^d_N$ contained in $h$ and containing $\max h$. (Recall that the vertex $\max g$ is identified with the face $f$ of the
initial grid with the center at max \(g\). Since \(\dim \Pr(f, k, k) = 0 = f_k\) (mod 2), by Definitions 1.5.1 and 4.2.1 the case \(l = k\) follows.

For \(l \neq k\), the map \(g \mapsto \begin{cases} \max g, & \text{if } 2 \nmid (\max g), \\ \max g - e_k, & \text{if } 2 \mid (\max g); \end{cases}\) is a 2–1 map between the set of hyperfaces of the doubling in \(h\) and the set of faces \(f\) of the initial grid \(I^d_N\) such that \(f \subset h, f \ni \max h, f \parallel e_k\). The contribution of a pair of hyperfaces mapped to the same face \(f\) to the sum of fluxes is

\[
\frac{1}{2} (-1)^{l+1} \left( 1 + \sum_{0 \leq m \leq \max(k,l)} f_m \right) T((f + e_l - e_k) \times f)
+ \frac{1}{2} (-1)^{l+1} \left( 1 + \sum_{0 \leq m \leq \max(k,l)} (f_m + \delta m_k) \right) [-T((f + e_k + e_l) \times (f + e_l - e_k))]
= \frac{1}{2} (-1)^{\dim \Pr(f,k,l)+l+1} \left[ T((f + e_l - e_k) \times f) + T((f + e_l + e_k) \times f) \right]
\]

because \(2 \nmid f_k\) and \(2 \mid f_l\) by the assumptions \(f \subset h \perp e_l\) and \(f \parallel e_k\). Summation over all such pairs proves the case \(l \neq k\).

Now let us prove an analogue of the Stokes formula; cf. (3). For that we need a lemma.

**Lemma 4.2.3** For each \(k\)-dimensional face \(f\) of the \(d\)-dimensional grid \(I^d_N\) denote by \([f] \in C^k(I^d_N; \mathbb{R})\) the field, which equals 1 at \(f\), and equals 0 at all the other faces. Then

\[
\partial[f] = \sum_{1:2 \mid f_l} (-1)^{\sum_{0 \leq m \leq l} f_m} \cdot ([f - e_l] - [f + e_l]);
\]

\[
\delta[f] = \sum_{1:2 \mid f_l} (-1)^{\sum_{0 \leq m \leq l} f_m} \cdot ([f - e_l] - [f + e_l]).
\]

**Proof** This is a direct computation using Definition 1.4.2. It suffices to prove that \(f\) and \(f - e_l\) are cooriented, if and only if \(2 \mid \sum_{0 \leq m \leq l} f_m\). Assume that \(2 \mid f_l\); the opposite case is analogous. A positive basis in \(f\) is the sequence formed by all the vectors \(e_m\) such that \(2 \nmid f_m\) in a natural order. A positive basis in \(f - e_l\) is obtained by insertion of \(e_l\) into the sequence. Adding the outer normal to the former basis means adding \(e_l\) at the beginning of the sequence instead. Since moving \(e_l\) to the beginning of the sequence requires \(\sum_{0 \leq m < l} f_m\) (mod 2) transpositions, the lemma follows.

**Proposition 4.2.4** (The Stokes Formula) Let \(0 \leq k < d\). For each partially symmetric type \((1, 1)\) tensor \(T\) and each \(d\)-dimensional face \(g\) of the doubling of \(I^d_N\) we have \(\langle T, \partial g \rangle_k = \langle \partial T, g \rangle_k\).

**Proof** This is a direct computation; a technical difficulty is signs. Set \(f = \max g\). Assume that \(2 \mid f_k\); the opposite case is discussed at the end of the proof. For any fields \(\phi\) and \(\psi\) denote \(T(\psi \times \phi) = \sum_{e,f} T(e \times f) \psi(e)\phi(f)\). Then by Definition 1.4.8 we have \(\partial T(e \times f) = T([e] \times \partial[f]) + T(\delta[e] \times [f])\) and by Lemma 4.2.3 we have
\[2(-1)^{1+\sum_{m<k} f_m}(\partial T, g)_k = \partial T(f \times (f - e_k))\]

\[= T([f] \times \partial [f - e_k]) + T(\delta [f] \times (f - e_k))\]

\[= \sum_{l:2|f_l - \delta_{kl}} (-1)\sum_{m \leq l}(f_m - \delta_{mk})\]

\[\cdot \left[ T(f \times (f - e_k - e_l)) - T(f \times (f - e_k + e_l)) \right]\]

\[+ \sum_{l:2|f_l} (-1)\sum_{m \leq l} f_m\]

\[\cdot \left[ T((f - e_l) \times (f - e_k)) - T((f + e_l) \times (f - e_k)) \right].\]

It remains to show that here the \(l\)-th summand multiplied by \((-1)^{1+\sum_{m<k} f_m}\) equals twice the difference of the fluxes across the two opposite hyperfaces of \(g\) orthogonal to \(e_l\) multiplied by \((-1)^l\). (The latter sign factor is required to get the right contribution of the two faces into the whole flux across \(\partial g\) in the positive normal direction; see Lemma 4.2.3 for \(k = d\)). Denote \(f' = f - e_l\), \(k' = \min\{k, l\}\), \(l' = \max\{k, l\}\). Denote by \(g + e_l/2\) and \(g - e_l/2\) the hyperfaces of \(g\) orthogonal to \(e_l\) such that \(\max(g + e_l/2) = f\) and \(\max(g - e_l/2) = f'\) respectively.

Consider the following 3 cases: 1) \(l = k\); 2) \(l \neq k\) and \(2 \nmid f_l\); 3) \(l \neq k\) and \(2 \nmid f_l\).

For \(l = k\) (hence \(2 \nmid f_k = f_l\)) the \(l\)-th summands in the two sums multiplied by \((-1)^{1+\sum_{m<k} f_m}\) add up to

\[(-1)^{1+\sum_{m<k} f_m} (-1)\sum_{m \leq k}(f_m - \delta_{mk}) \cdot \left[ T(f \times (f - 2e_k)) - T(f \times f) \right]\]

\[+ (-1)^{1+\sum_{m<k} f_m} (-1)\sum_{m \leq k} f_m\]

\[\cdot \left[ T((f - e_k) \times (f - e_k)) - T((f + e_k) \times (f - e_k)) \right]\]

\[= (-1)^{f_k+1} \cdot \left[ T(f \times f) - T((f + e_k) \times (f - e_k)) \right]\]

\[\cdot \left[ T((f - e_k) \times (f - e_k)) - T((f - e_k + e_k) \times (f - 2e_k)) \right]\]

\[= (-1)^k \cdot (-1)^{k+1+f_k} \cdot \left[ T(f \times f) - T((f + e_k) \times (f - e_k)) \right]\]

\[\cdot \left[ T((f - e_k) \times (f - e_k)) - T((f - e_k + e_k) \times (f - 2e_k)) \right]\]

\[= (-1)^k 2(T, g + e_k/2)_k - (-1)^k 2(T, g - e_k/2)_k;\]

see Definition 4.2.1 applied for \(l = k\). We have found the contribution of the \(l\)-th summands for \(l = k\).

For \(l \neq k\) and \(2 \nmid f_l\) the \(l\)-th summand multiplied by \((-1)^{1+\sum_{m<k} f_m}\) is

\[(-1)^{1+\sum_{m<k} f_m} (-1)\sum_{m \leq l}(f_m - \delta_{mk})\]

\[\cdot \left[ T(f \times (f - e_k - e_l)) - T(f \times (f - e_k + e_l)) \right]\]

\[\overset{(*)}{=} (-1)^{1+\sum_{m' \leq l'} f_m}\]

\[\cdot \left[ T(f \times (f + e_l - e_k)) - T((f - e_l + e_l) \times (f - e_l - e_k)) \right].\]
\[
(-1)^l \cdot (-1)^{l+1+\sum_{k' \leq m \leq l'} f_m} \cdot T (f \times (f + e_l - e_k)) \\
- (-1)^l \cdot (-1)^{l+1+\sum_{k' \leq m \leq l'} f_m} \cdot [-T ((f' + e_l) \times (f' - e_k))] \\
= (-1)^l 2(T, g + e_l/2)_k - (-1)^l 2(T, g - e_l/2)_k;
\]

see Definition 4.2.1 applied for \( l \neq k, 2 \mid f_k, 2 \nmid f_l \) and \( l \neq k, 2 \mid f'_k, 2 \mid f'_l \). Here (*) follows from

\[
1 + \sum_{m < k} f_m + \sum_{m \leq l} (f_m - \delta_{mk}) \\
= \begin{cases} 
\sum_{k \leq m \leq l} f_m, & \text{if } k < l; \\
\sum_{l < m < k} f_m + 1, & \text{if } k > l;
\end{cases} = \sum_{k' \leq m \leq l'} f_m \pmod{2},
\]

where we used the conditions \( 2 \mid f_l \) and \( 2 \mid f_k \) to change the range of summation over \( m \).

For \( l \neq k \) and \( 2 \mid f_l \) the \( l \)-th summand multiplied by \((-1)^{l+\sum_{m < k} f_m} \) is

\[
(-1)^{l+\sum_{m < k} f_m} (-1) \sum_{m \leq l} f_m \cdot [T ((f - e_l) \times (f - e_k)) - T ((f + e_l) \times (f - e_k))] \\
= (-1)^l \sum_{k' \leq m \leq l'} f_m \cdot [T ((f + e_l) \times (f - e_k)) - T ((f - e_l) \times (f - e_l + e_l - e_k))] \\
= (-1)^l \cdot (-1)^{l+1+\sum_{k' \leq m \leq l'} f_m} \cdot [-T ((f + e_l) \times (f - e_k))] \\
- (-1)^l \cdot (-1)^{l+1+\sum_{k' \leq m \leq l'} f_m} \cdot T (f' \times (f' + e_l - e_k)) \\
= (-1)^l 2(T, g + e_l/2)_k - (-1)^l 2(T, g - e_l/2)_k.
\]

Summation of the expressions obtained in the three cases completes the proof in the case when \( 2 \mid f_k \).

For \( 2 \mid f_k \) the proof is analogous and starts from the evaluation of \( 2(-1)^{l+\sum_{m < k} f_m} \langle \partial T, g \rangle_k = \partial T ((f - e_k) \times f) \). For \( l = k \) one ends up with an expression involving \( T ((f - e_k) \times (f + e_k)) \) rather than \( T ((f + e_k) \times (f - e_k)) \). But the latter two values are equal because \( T \) is partially symmetric. \( \square \)

**Theorem 4.2.5 (Restatement of Theorem 1.5.2)** If a partially symmetric type \((1, 1)\) tensor \( T \) is conserved, then for each \( d \)-dimensional face \( g \) disjoint with \( \partial I^d_N \) and each \( k \) we get \( \langle T, \partial g \rangle_k = 0 \).

**Proof of Theorem 4.2.5** This follows directly from Propositions 4.2.2 and 4.2.4 because \( g \) can be filled by \( d \)-dimensional faces of the doubling. \( \square \)

**Remark 4.2.6** Here the assumption that \( T \) is conserved can be relaxed to \( [\partial T](e \times f) = 0 \) for all faces \( e, f \not\subset \partial I^d_N \), because the boundary faces do not contribute to the flux when \( g \cap \partial I^d_N = \emptyset \).
4.3 Identities

For the sequel we need several identities for cochain operations, most of which are well-known. Throughout this subsection $M$ is an arbitrary simplicial or cubical complex unless otherwise indicated.

**Definition 4.3.1** The *pairing* of fields $\phi, \psi \in C^k(M; \mathbb{C}^{m \times n})$, where $m=1$ or $m=n$, is defined by

$$\langle \phi, \psi \rangle = \text{Re} \text{Tr} \sum_{f} \phi(f)\psi^*(f) = \epsilon \text{Re} \text{Tr}[\phi \sim \psi^*] = \epsilon \text{Re} \text{Tr}[\phi^* \sim \psi].$$

Given $U \in C^1(M; G)$, denote by $C^1(M; T_U G)$ the set of all $\Delta \in C^1(M; \mathbb{C}^{n \times n})$ such that $\Delta(e)$ belongs to $T_{U(e)}G$ for each edge $e$. For $\phi \in C_k(M; \mathbb{C}^{n \times m})$, where $m=1$ or $m=n$, denote

$$\tilde{D}_A^* \phi = (D_A^* \phi^*)_* = \partial \phi + (-1)^k A \sim \phi + \delta_{mn} \cdot \phi \sim A.$$  \hspace{1cm} (22)

**Lemma 4.3.2** (Pairing nondegeneracy) Fix $\phi \in C_k(M; \mathbb{C}^{m \times n})$, $\psi \in C_0(M; \mathbb{C}^{n \times n})$, $\chi \in C_1(M; \mathbb{C}^{n \times n})$.

If $\langle \phi, \Delta \rangle = 0$ for each $\Delta \in C^k(M; \mathbb{C}^{m \times n})$, then $\phi = 0$.

If $\langle \psi, \Delta \rangle = 0$ for each $\Delta \in C^0(M; T_U G)$, then $\text{Pr}_{T_U G} \psi = 0$.

If $\langle \chi, \Delta \rangle = 0$ for each $\Delta \in C^1(M; T_U G)$, then $\text{Pr}_{T_U G} \chi = 0$.

**Proof** For the first assertion, take $\Delta = \phi$. Then $0 = \langle \phi, \phi \rangle = \sum_f \text{Re} \text{Tr}[\phi^*(f)\phi(f)] = \sum_f \sum_{i,j=1}^{m,n} |\phi_{ij}(f)|^2.$ Thus $\phi = 0$.

For the third assertion, take $\Delta = \text{Pr}_{T_U G} \chi$. Then $0 = \langle \chi, \text{Pr}_{T_U G} \chi \rangle = \sum_e \langle \chi(e), \text{Pr}_{T_U(e)} G \chi(e) \rangle = \sum_e \langle \text{Pr}_{T_{U(e)} G} \chi(e), \text{Pr}_{T_{U(e)} G} \chi(e) \rangle$, where the sums are over all edges $e$, because $\text{Pr}_{T_{U(e)} G}$ is an orthogonal projection. Since the pairing $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^{n \times n}$ is nondegenerate, it follows that $\text{Pr}_{T_U G} \chi = 0$.

The second assertion is proved analogously. \hfill \Box

**Lemma 4.3.3** In a cubical complex $M$, for each $U \in C^1(M; G)$ and $\Phi \in C^k(M; \mathbb{C}^{n \times n})$ we have

$$D_A \Phi = U \sim \Phi - (-1)^k \Phi \sim U; \hspace{1cm} F = U \sim U;$$

$$\tilde{D}_A^* \Phi = \Phi \sim U + (-1)^k U \sim \Phi.$$ The two identities in the 1st column hold for a simplicial complex $M$ for $k=0$ and $k=1$ respectively.

**Proof** By Definitions 1.4.2 and 3.1.1 it follows that

$$[\delta \Phi](a \ldots c) = \sum_{\substack{b: \dim(a \ldots b) = 1, \\ \dim(b \ldots c) = k}} \langle a, b, c \rangle \Phi(b \ldots c) - (-1)^k$$
For each $\phi \in C^k(M; C^{p \times q})$, $\psi \in C^l(M; C^{q \times r})$, $\chi \in C^m(M; C^{r \times s})$ we have

\begin{align*}
\delta \delta &= 0; \\
\delta(\phi \sim \psi) &= (\delta \phi) \sim \psi + (-1)^{\dim \phi} \phi \sim (\delta \psi); \quad (\phi \sim \psi) \sim \chi = \phi \sim (\psi \sim \chi); \\
\partial \partial &= 0; \\
\partial(\phi \sim \psi) &= (-1)^{\dim \psi} (\partial \phi \sim \psi - \phi \sim \delta \psi); \quad (\phi \sim \psi) \sim \chi = \phi \sim (\psi \sim \chi); \\
\epsilon \partial &= 0; \\
\partial(\phi^* \sim \psi) &= \phi^* \sim \partial \psi + (-1)^{\dim \psi - \dim \phi} \delta \phi \sim \psi + \phi^* (\psi \sim \chi) = (\phi \sim \psi)^* \sim \chi; \\
(\phi \sim \psi)^* &= \begin{cases} 
\psi^* \sim \phi^*, & \text{if } \dim \phi = 0; \\
\psi^* \sim \phi^*, & \text{if } \dim \psi = 0;
\end{cases} \quad (\phi^* \sim \psi) \sim \chi = \phi^* (\psi \sim \chi).
\end{align*}

For each $\phi \in C^k(M; C^{n \times m})$, $\psi \in C^l(M; C^{m \times n})$, $U \in C^1(M; G)$, where $m = 1$ or $m = n$, we have

\begin{align*}
D_A D_A \psi &= -\psi \sim F + \delta_{mn} \cdot F \sim \psi; \\
D_A (\phi \sim \psi) &= D_A \phi \sim \psi + (-1)^{\dim \phi} \phi \sim D_A \psi, \quad \text{if } m = n; \\
\check{D}_A \check{D}_A \phi &= -F \sim \phi + \delta_{mn} \cdot \phi \sim F; \\
\check{D}_A (\phi \sim \psi) &= (-1)^{\dim \psi} (\check{D}_A \phi \sim \psi - \phi \sim D_A \psi); \\
\Re \Tr \epsilon \check{D}_A \phi &= 0, \quad \text{if } m = n \text{ and } \dim \phi = 1; \\
\check{D}_A (\phi^* \sim \psi) &= \phi^* \sim \check{D}_A \psi + (-1)^{\dim \psi - \dim \phi} D_A \phi \sim \psi, \quad \text{if } m = n.
\end{align*}

For each $\phi \in C^k(M; C^{m \times n})$, $\psi \in C^l(M; C^{n \times m}$ or $C^{m \times n})$, $\chi \in C_{k+l}(M; C^{m \times n})$, $U \in C^1(M; G)$, where $m = 1$ or $m = n$ (and $l = 1$ for the 1st, 3rd, 4th, and 6th identities below), we have:

\[(\chi, \delta \phi) = \langle \partial \chi, \phi \rangle;\]
\begin{align*}
\langle \chi, \psi \sim \phi \rangle &= \langle (\chi^* \sim \psi)^*, \phi \rangle; \\
\text{Re Tr} \ D_A^* \psi &= \partial \text{Re Tr} [U^* \cdot \psi]; \\
\langle \chi, D_A \phi \rangle &= \langle D_A^* \chi, \phi \rangle; \\
\langle \chi, \phi \sim \psi \rangle &= \langle (\psi \sim \chi^*)^*, \phi \rangle; \\
\text{Pr}_{T_1G} D_A^* \psi &= D_A^* \text{Pr}_{T_1G} \psi.
\end{align*}

In the 3rd and 6th identities, \( m = n \) and \( \sim \) is the edgewise product: \( [U^* \cdot \psi](e) := U^*(e) \psi(e) \) for each edge \( e \).

**Proof** The identities involving neither cop product nor covariant (co)boundary are well-known in the case when the functions assume values in a commutative ring; cf. [12]. Without the commutativity the proof is literally the same. Let us prove the remaining identities.

For an ordered 4-ple of faces \( a \ldots b, b \ldots c, c \ldots d \subset a \ldots d \) write \( \langle a, b, c, d \rangle = +1 \), if the ordered set consisting of positive bases in \( a \ldots b, b \ldots c, c \ldots d \) is a positive basis in \( a \ldots d \). Otherwise write \( \langle a, b, c, d \rangle = -1 \). Clearly, \( \langle a, b, c, d \rangle = \langle a, b, c \rangle \langle a, c, d \rangle = \langle a, b, d \rangle \langle b, c, d \rangle \). Thus by Definition 3.1.1

\[
[\phi \sim^* (\psi \sim^* \chi)](a \ldots b) = \sum_{c : \dim(b \ldots c) = k, \dim(a \ldots c) = m - l} \langle a, b, c \rangle \phi(b \ldots c)[\psi \sim^* \chi](a \ldots c)
\]

\[
= \sum_{c,d : \dim(b \ldots c) = k, \dim(c \ldots d) = l, \dim(a \ldots d) = m} \langle a, b, c \rangle \langle a, c, d \rangle \phi(b \ldots c) \psi(c \ldots d) \chi(a \ldots d)
\]

\[
= \sum_{c,d : \dim(b \ldots c) = k, \dim(c \ldots d) = l, \dim(a \ldots d) = m} \langle a, b, d \rangle \langle b, c, d \rangle \phi(b \ldots c) \psi(c \ldots d) \chi(a \ldots d)
\]

\[
= [(\phi \sim \psi) \sim^* \chi](a \ldots b).
\]

Setting \( m = k + l \), changing the notation \( \chi \) to \( \chi^* \), and applying the operator \( \varepsilon \text{ Re Tr} \), we obtain \( \langle (\psi \sim^* \chi^*)^*, \phi \rangle = \langle \chi, \phi \sim \psi \rangle \). Taking \( \psi = A, \phi \in C^k(M; \mathbb{C}^{m \times n}) \), \( \chi \in C_{k+1}(M; \mathbb{C}^{m \times n}) \), multiplying by \( (-1)^\dim \phi = -(-1)^\dim \chi \), adding the known identity \( \langle \partial \chi, \phi \rangle = \langle \chi, \delta \phi \rangle \) (and for \( m = n \) also the known identity \( \langle (\chi \sim^* \psi)^*, \phi \rangle = \langle \chi, \phi \sim \psi \rangle \)), and using (15)–(18), we get \( \langle D_A^* \chi, \phi \rangle = \langle \chi, D_A \phi \rangle \).

The formula for \( (\phi \sim^* \psi) \sim \chi \) is proved analogously.

Next, the formula for \( \hat{D}_A^*(\phi \sim \psi) \) for a cubical complex and \( m = n \) follows from

\[
(-1)^l \hat{D}_A^*(\phi \sim \psi) = (-1)^l (-1)^{k-l} U \sim (\phi \sim \psi) + (-1)^l (\phi \sim \psi) \sim U
\]

\[
= (-1)^k (U \sim \phi) \sim \psi + (\phi \sim U) \sim \psi - \phi \sim (U \sim \psi)
\]

\[
+ (-1)^l \phi \sim (\psi \sim U)
\]

\[
= (\hat{D}_A^* \phi) \sim \psi - \phi \sim D_A \psi.
\]
where we used Lemma 4.3.3 and the identities not involving (covariant) (co)boundary. Alternatively, the formula for $\tilde{D}_A^*(\phi \rightsquigarrow \psi)$ can be deduced from the formula for $\delta(\phi \rightsquigarrow \psi)$ by pairing with an arbitrary field $\Delta$ and applying Lemma 4.3.2 and the identities from the paragraph before the previous one; this works for a simplicial complex and for $m = 1$ as well.

The formulae for $D_A(\phi \rightsquigarrow \psi)$, $\tilde{D}_A^*(\phi \rightsquigarrow \psi)$, $D_A D_A$, $\tilde{D}_A^* \tilde{D}_A^*$ are proved analogously.

Finally, for each vertex $v$ by Lemma 4.3.3 we have (where $\langle U, \psi \rangle$ is the edgewise scalar product)

$$[\operatorname{Re} \operatorname{Tr} D_A^* \psi](v) = \operatorname{Re} \operatorname{Tr}[\psi^* \rightsquigarrow U - U \rightsquigarrow \psi]^*(v)$$

$$= \sum_{e: \max e = v} \langle \psi(e), U(e) \rangle - \sum_{e: \min e = v} \langle \psi(e), U(e) \rangle = [\partial(\langle U, \psi \rangle)](v);$$

$$D_A^* \operatorname{Pr}_{T_G} G \psi = (\operatorname{Pr}_{T_G} G \psi)^* \rightsquigarrow U - U \rightsquigarrow (\operatorname{Pr}_{T_G} G \psi)^*$$

$$= \operatorname{Pr}_{T_G} G (\psi^* \rightsquigarrow U)^* - \operatorname{Pr}_{T_G} G (U \rightsquigarrow \psi^*)^* = \operatorname{Pr}_{T_G} G D_A^* \psi.$$

Applying the operator $\epsilon$ we get $\operatorname{Re} \operatorname{Tr} \epsilon \tilde{D}_A^* \psi = \epsilon \operatorname{Re} \operatorname{Tr} D_A^* \psi^* = \epsilon \operatorname{Re} \operatorname{Tr} [U \cdot \psi] = 0. \quad \Box$

### 4.4 Generalizations

Now we prove the results of Sect. 3. Starting from Lemma 4.4.1 below, the proof is parallel to that of Sect. 4.1.

**Proof of Proposition 3.1.3** By the formulae of Lemma 4.3.3 for $F$ and $D_A \Phi$ in the case when $(\Phi, A) = (A, 0)$, we get $F = (1 + A) \rightsquigarrow (1 + A) = 0 + D_0 A + A \rightsquigarrow A = \delta A + A \rightsquigarrow A$. By Lemma 4.3.3 and the associativity of the cup product, $D_A F = U \rightsquigarrow F - F \rightsquigarrow U = U \rightsquigarrow (U - U) - (U - U) = 0$. By Lemma 4.3.3 and the 3rd column of Table 3 we get (12).

Let us prove (13). For each face $f \supset e$ we have either $\min f = \min e$ or $\max f = \max e$ (cf. Definition 3.2.1). Consider a face $f = abcd$ containing $e = ab$ such that $\min f = a$. Then $U(e) - U(\partial f - e) = U(ab) - U(abcd) = (F(abcd)^* U(bc))^*$. Applying $\#$ and summing the obtained expression over all such faces $f$, we get $(\#F^* \rightleftharpoons U)^*$. Analogous sum over all the faces $f$ such that $\max f = b$ gives $(U \rightsquigarrow \#F^*)^*$. Then Lemma 4.3.3 implies (13).

For a cubical complex the proof is the same, only one drops all $\#$-operators. For a simplicial complex, in addition, references to Lemma 4.3.3 should be replaced by a direct checking. \( \Box \)

**Proof of Proposition 3.1.5** This is a straightforward computation using the explicit expression for the function $L_v$ given in the middle part of Table 4. There we use notation from Definition 3.1.4 and

$$g(k, l) = \begin{cases} (-1)^k, & \text{if } f_i \parallel (1, 0, \ldots, 0), \\ (-1)^{k-1}, & \text{if } f_i \perp (1, 0, \ldots, 0). \end{cases}$$
For a cubical/simplicial complex the proof is the same, only one drops all # and $g(k, l)$.

\textbf{Lemma 4.4.1} (Lagrangian functional derivative) \textit{For a local Lagrangian $L: C^k(M; \mathbb{C}^{1 \times n}) \times C^1(M; \mathbb{C}^{n \times n}) \to C_0(M; \mathbb{R})$ and arbitrary fields $\phi, \Delta \in C^k(M; \mathbb{C}^{1 \times n})$, $U \in C^1(M; G)$ we have}

\[
\left. \frac{\partial L[\phi + t\Delta, U]}{\partial t} \right|_{t=0} = \text{Re Tr} \left[ \left( \frac{\partial L[\phi]}{\partial \phi} + \tilde{D}_A^* \frac{\partial L[\phi]}{\partial (D_A\phi)} \right) \wedge \Delta \right. \left. -(-1)^k \tilde{D}_A^* \left( \frac{\partial L[\phi, U]}{\partial (D_A\phi)} \wedge \Delta \right) \right].
\]

\textbf{Proof} This is proved literally as Lemma 4.1.1 with $\delta$ and $\partial$ replaced by $D_A$ and $\tilde{D}_A^*$ respectively, and Re Tr applied to each summand. Instead of (21) use the formula for $\tilde{D}_A^* (\phi \sim \psi)$ from Lemma 4.3.4.

\textbf{Proof of Theorem 3.1.6} A field $\phi$ is an extremal, if and only if $\frac{\partial S[\phi + t\Delta, U]}{\partial t} \bigg|_{t=0} = 0$ for each $\Delta \in C^k(M; \mathbb{C}^{1 \times n})$. By Lemmas 4.4.1 and 4.3.2 this is equivalent to (19) because $\epsilon \text{Re Tr} \tilde{D}_A^* = 0$ by Lemma 4.3.4.

\textbf{Lemma 4.4.2} (Lagrangian functional derivative) \textit{For a local Lagrangian $L: C^1(M; \mathbb{C}^{n \times n}) \to C_0(M; \mathbb{R})$ and arbitrary fields $U \in C^1(M; G)$, $\Delta \in C^1(M; T_UG)$ we have}

\[
\left. \frac{\partial L[U + t\Delta]}{\partial t} \right|_{t=0} = \text{Re Tr} \left[ \left( \frac{\partial L[U]}{\partial U} + \tilde{D}_A^* \frac{\partial L[U]}{\partial (F[U])} \right) \wedge \Delta + \tilde{D}_A^* \left( \frac{\partial L[U]}{\partial (F[U])} \wedge \Delta \right) \right].
\]

\textbf{Proof} This is proved analogously to Lemma 4.1.1 with $\phi$ and $\delta\phi$ replaced by $U$ and $F = \delta A + A \sim A$ (see Proposition 3.1.3), using the formula for $\tilde{D}_A^* (\phi \sim \psi)$ from Lemma 4.3.4 instead of (21), and

\[
\left. \frac{\partial}{\partial t} F[U + t\Delta] \right|_{t=0} = \frac{\partial}{\partial t} [\delta(U + t\Delta - 1) + (U + t\Delta - 1) \wedge (U + t\Delta - 1)] \bigg|_{t=0} = \delta \Delta + (U - 1) \sim \Delta + \Delta \sim (U - 1) = D_A \Delta.
\]

\textbf{Proof of Theorem 3.1.7} A gauge group field $U$ is an extremal, if and only if $\frac{\partial S[U + t\Delta]}{\partial t} \bigg|_{t=0} = 0$ for each $\Delta \in C^1(M, T_UG)$. By Lemmas 4.4.2, 4.3.4, and 4.3.2 this is equivalent to (20).

\textbf{Proof of Theorem 3.1.8} This follows from $\partial \langle j[\phi, U] \rangle = \text{Re Tr} D_A^* j[\phi, U] = \left. \frac{\partial L[\phi + t\Delta, U]}{\partial t} \right|_{t=0} = 0$. Here the 1st equality is given by Lemma 4.3.4. The 2nd one is proved as in the proof of Theorem 4.1.4 with $\delta$, $\partial$ replaced by $D_A$, $\tilde{D}_A^*$, and Re Tr applied to each summand. The 3rd one is (8).
Remark 4.4.3 If (8) holds in a subset of $M$, then the current $\langle j[\phi, U], U \rangle$ is conserved on the subset.

Lemma 4.4.4 (Lagrangian functional derivative) For a local Lagrangian $L: C^k(M; \mathbb{C}^{1 \times n}) \times C^1(M; \mathbb{C}^{n \times n}) \to C_0(M; \mathbb{R})$ and arbitrary fields $\phi \in C^k(M; \mathbb{C}^{1 \times n})$ and $U, \Delta \in C^1(M; \mathbb{C}^{n \times n})$ we have

$$\frac{\partial L[\phi, U + t\Delta]}{\partial t} \bigg|_{t=0} = \text{Re} \text{ Tr} \left[ \left( \frac{\partial L[\phi, U]}{\partial (DA\phi)} \sim \phi \right) \sim \Delta \right] \quad \text{and}$$

$$\frac{\partial L[\phi, U]}{\partial U} = \frac{\partial L[\phi, U]}{\partial (DA\phi)} \sim \phi.$$ 

Proof Analogously to the proof of Lemma 4.1.1 using (17) and Lemma 4.3.4 we get

$$\frac{\partial L[\phi, U + t\Delta]}{\partial t} \bigg|_{t=0} = \text{Re} \text{ Tr} \left[ \left( \frac{\partial L[\phi, U]}{\partial \phi} \sim \frac{\partial \phi}{\partial t} + \frac{\partial L[\phi, U]}{\partial (DA\phi)} \sim \frac{\partial (DA[U+t\Delta]\phi)}{\partial t} \right) \bigg|_{t=0} \right]$$

$$= 0 + \text{Re} \text{ Tr} \left[ \frac{\partial L[\phi, U]}{\partial (DA\phi)} \sim \frac{\partial [\delta \phi + \phi \sim (U - 1 + t\Delta)]}{\partial t} \bigg|_{t=0} \right]$$

$$= \text{Re} \text{ Tr} \left[ \frac{\partial L[\phi, U]}{\partial (DA\phi)} \sim (\phi \sim \Delta) \right] = \text{Re} \text{ Tr} \left[ \left( \frac{\partial L[\phi, U]}{\partial (DA\phi)} \sim \phi \right) \sim \Delta \right].$$

A local Lagrangian $L[\phi, U]$ is also local with respect to $U$ and does not depend on $F[U]$. Since $\Delta \in C^1(M; \mathbb{C}^{n \times n})$ is arbitrary, by Lemmas 4.4.2 and 4.3.2 it follows that $\frac{\partial L[\phi, U]}{\partial U} = \frac{\partial L[\phi, U]}{\partial (DA\phi)} \sim \phi.$

Lemma 4.4.5 (Infinitesimal form of gauge invariance) For each gauge invariant differentiable function $L: C^k(M; \mathbb{C}^{1 \times n}) \times C^1(M; \mathbb{C}^{n \times n}) \to C_0(M; \mathbb{R})$ and each $\Delta \in C^0(M, T_1G)$ we have

$$\frac{\partial}{\partial t} L[\phi + t\phi \sim \Delta, U + tDA\Delta] \bigg|_{t=0} = 0.$$

Proof Since $L[\phi, U]$ is gauge invariant and differentiable, by Lemma 4.3.3 up to first order in $t$

$$L[\phi, U] = L[\phi \sim \exp(t\Delta), \exp(-t\Delta) \sim U \sim \exp(t\Delta)]$$

$$= L[\phi + t\phi \sim \Delta, U + t(U \sim \Delta \sim U) + o(t)]$$

$$= L[\phi + t\phi \sim \Delta, U + tDA\Delta] + o(t) \quad \text{as } t \to 0.$$ 

Subtracting $L[\phi, U]$ from both sides and dividing by $t$, we get the required result. □

Lemma 4.4.6 (Local covariant constants) For each $U \in C^1(M; G)$, $g_0 \in T_1G$, and each vertex $v$ there is $g \in C^0(M, T_1G)$ such that $g(v) = g_0$ and $[DAg](uv) = 0$ for each neighbor $u$ of $v$. ☵ Springer
Proof Set \( g(v) = g_0, g(u) = U(uv)g(v)U(vu) \) at each neighbor \( u \) of \( v \), and let \( g \) be arbitrary at the other vertices. By Lemma 4.3.3 we have \( [DA_g](uv) = U(uv)g(v) - U(uv)g(v)U(vu)U(uv) = 0 \).

Proof of Theorem 3.1.9 Take an arbitrary vertex \( v \) and \( g_0 \in T_1 G \). Let \( g \in C^0(M; T_1 G) \) be given by Lemma 4.4.6. Apply Lemma 4.4.5 for \( \Delta = g \). Since \( [DA_g](uv) = 0 \) for each neighbor \( u \) of \( v \), we obtain that equation (8) holds at the vertex \( v \) with \( \Delta = \phi \sim g \) (notice that the connection in (8) does not depend on \( t \)). By Theorem 3.1.8, Remark 4.4.3, and Lemma 4.3.4, we have

\[
0 = \partial \text{Re Tr} \left[ \left( \frac{\partial L[\phi, U]}{\partial (DA\phi)} \sim (\phi \sim g) \right) \cdot U \right](v)
\]

\[
= \text{Re Tr} \left[ \tilde{D}_A^* \left( \left( \frac{\partial L[\phi, U]}{\partial (DA\phi)} \sim \phi \right) \sim g \right) \right](v)
\]

\[
= \text{Re Tr} \left[ \tilde{D}_A^* j[\phi, U]^* \sim g - j[\phi, U]^* \sim DA_g \right](v)
\]

\[
= \text{Re Tr} \left[ D_A^* j[\phi, U](v) \cdot g_0^* \right].
\]

Here we used that \( [DA_g](uv) = 0 \) for each edge \( uv \) containing \( v \). Since the vertex \( v \) and \( g_0 \in T_1 G \) are arbitrary, by Lemma 4.3.2 it follows that \( \text{Pr}_{T_1 G} D_A^* j[\phi, U] = 0 \). Then by Lemma 4.3.4 we have \( D_A^* \text{Pr}_{T_1 G} j[\phi, U] = 0 \). Finally, by Lemma 4.4.4 we have \( \frac{\partial L[\phi, U]}{\partial (DA\phi)} \sim \phi = \frac{\partial L[\phi, U]}{\partial U} \).

Proof of Theorem 3.1.10 Denote \( S[U] = \epsilon L[U] \) and \( S'[U] = \epsilon L'[U] \). Take arbitrary \( \Delta \in C^0(M, T_1 G) \). By Lemmas 4.4.5 (with \( L[\phi, U] \) replaced by \( L'[U] \)) and 4.3.4 we get

\[
\frac{\partial}{\partial t} S[U + tDA\Delta] \bigg|_{t=0} = \frac{\partial}{\partial t} \left( S'[U + tDA\Delta] + (j, U + tDA\Delta) \right) \bigg|_{t=0}
\]

\[
= 0 + (j, DA\Delta) = (D_A^* j, \Delta).
\]

For a stationary gauge group field \( U \) the left side vanishes, because \( DA\Delta = U \sim \Delta - \Delta \sim U \in C^1(M, T_1 G) \) is a possible variation of \( U \). Thus \( D_A^* j, \Delta = 0 \) for arbitrary \( \Delta \in C^0(M, T_1 G) \). By Lemmas 4.3.2 and 4.3.4 we get \( 0 = \text{Pr}_{T_1 G} D_A^* j = D_A^* \text{Pr}_{T_1 G} j \), as required.

Proof of Proposition 3.1.11 Let us present the proof for a cubical complex. For a simplicial complex the argument is literally the same, only each instance of the fourth vertex “d” is just removed.

Since the group \( G \) consists of unitary matrices, for each edge \( uv \) and each face \( abcd \) we have

\[
A[g^* \sim U \sim g](uv) = g^*(u)U(uv)g(v) - 1 = g^*(u)(U(uv) - 1)g(v) + g^*(u)(g(v) - g(u))
\]
Now we apply the general results of Sects. 1.4, 1.5, and 3.1 to prove particular results.

### 4.5 Proofs of Examples

**Proof of Corollary 2.1.14**

By using (17)–(18) and Lemma 4.3.4 we get

\[
D_{A[g^* \sim U \sim g]}(\phi \sim g) = g^* (A)[U] \sim g + g^* \sim \delta g
\]

Now, using (17)–(18) and Lemma 4.3.4 we get

\[
D_{A[g^* \sim U \sim g]}(\phi \sim g) = (\delta \phi) \sim g + (-1)^k \phi \sim g = g^* (A)[U] \sim g + \delta g
\]

\[
= (D_{A[U]} \phi) \sim g;
\]

\[
(D_{A[g^* \sim U \sim g]}(\phi \sim g))^* = \partial (\phi \sim g) + (-1)^k [g^* \sim A[U] \sim g - \delta g^* \sim g]
\]

\[
= \partial (g^* \sim \phi^*) + (-1)^k [g^* \sim A[U] \sim g - \delta g^* \sim g]
\]

\[
= g^* \sim \partial \phi^* + (-1)^k \delta g^* \sim \phi^* + (-1)^k (g^* \sim A[U] \sim \delta g^*)
\]

\[
= g^* \sim (\partial \phi^* + (-1)^k A[U] \sim \phi^*)
\]

The formulae involving \( \Phi \in C^k(M; \mathbb{C}^n \times \mathbb{C}^n) \) are proved analogously. Gauge invariance of the Lagrangians in rows 3, 4, and 6 of Table 4 is a straightforward consequence. \( \square \)

### 4.5 Proofs of Examples

Now we apply the general results of Sects. 1.4, 1.5, and 3.1 to prove particular results of Sect. 2.

**Proof of Corollary 2.1.14** This follows from Theorem 1.5.2 and Remark 4.2.6 applied to \( T[\phi] = \delta \phi \times \delta \phi \), because \( \partial T[\phi] = \partial \delta \phi \times \delta \phi + \delta \phi \times \delta \phi = 0 \) apart \( \partial I^2_2 \), since \( \partial \delta \phi = s \) and \( \delta \delta = 0 \). \( \square \)

**Proof of Theorem 2.1.17** First let us prove the “convergence” of \( F_N \) to \( F \). It is convenient to modify the grid slightly. Consider the auxiliary grid \( M \) obtained by dissection of \( I^2 \) into \( (N + 1)^2 \) equal squares and its dual \( N \times N \) grid \( M' \) with the vertices at face-centers of \( M \). Consider all the discrete fields in question as defined on \( M' \) instead of the initial \( N \times N \) grid; this does not affect approximation.

Let \( F_N ' \) be the function on vertices of \( M \) such that \( \partial \delta F_N ' = 0 \) apart \( \partial I^2 \) and \( F_N ' = F \) on \( \partial I^2 \). The restriction of \( F_N ' \) to non-boundary vertices can be considered as a function on faces of \( M' \). Actually, it is a magnetic field on \( M' \) generated by the source \( s_N \) (in
particular, it exists by Proposition 2.1.9). Indeed, the condition $\partial \delta F_N' = 0$ implies that it is a magnetic field generated by some source. The source is exactly $s_N$ because for each boundary vertex $v$ of the initial $N \times N$ grid we have $F_N'(v) - F_N'(v_+ - ) = F(v_+) - F(v-) = \int_{v_-, v_+} s \, dl = s_N(v)$, where $v_-, v, v_+$ are in the counterclockwise order along $\partial I^2$. By Propositions 2.1.2 and 2.1.9 the function $F_N' - F_N$ on faces of $M'$ is a constant (depending on $N$).

By [6, Proposition 3.3] on the set of vertices $v$ of $M$ at distance $\geq r$ from $\partial I^2$, we have $F_N'(v) \equiv F(v)$ as $N \to \infty$. In particular, for one of the faces $F_N$ closest to $c := (\frac{1}{2}, \frac{1}{2})$ we have $F_N(f_N) \to F(c) = 0 = F_N(f_N)$ as $N \to \infty$. Since $F_N' - F_N$ is a constant function on faces $f$ of $M'$, it follows that

$$F_N(f) \Rightarrow F_N'(f) \Rightarrow F(\max f) \Rightarrow N^2 \int_f F \, dS.$$ 

The convergence of $j_N = -\partial F_N$ follows immediately from the second part of [6, Proposition 3.3].

To prove the convergence of $\phi_N$, join a vertex $v$ with the vertex $u$ closest to $c$ such that $\phi_N(u) = 0$ by a shortest grid path $uv$. By the convergence of $j_N$ we get $\phi_N(v) = \sum_{e \subseteq uv} (u, v) j_N(e) \Rightarrow \int_{\partial v} \vec{j} \cdot \vec{l} = \phi(v)$.

The convergence of the other fields is a straightforward consequence. For instance, let $e = uv$ be a horizontal edge with the midpoint $e'$ and $f \supset e$ be a face with the center $f'$. Then

\[
NL_N(e' f') = \frac{N}{2} j_N(e) F_N(f) \Rightarrow F_N(f) \frac{N}{2} \int_e \vec{j} \cdot \vec{l} \\
\Rightarrow F(e') \vec{j}(e') \cdot \frac{N}{2} \int_e \vec{l} \Rightarrow * \vec{j}(e') F(e') \cdot N \int_{e' f'} d\vec{l} \Rightarrow N \int_{e' f'} L \cdot d\vec{l},
\]

\[
N^2 \sigma_{N, 2}(uv) = -\frac{N^2}{2} \left[ \delta \phi(uv)^2 - \delta \phi(vv_+) \delta \phi(vv_-) \right] \\
\Rightarrow \frac{1}{2} \frac{\delta \phi}{\delta x_2}(v)^2 - \frac{1}{2} \frac{\delta \phi}{\delta x_1}(v)^2 = \sigma_{22}(v) \Rightarrow N \int_e \left( \sigma_{22} \, dx_1^2 - \sigma_{21} \, dx_2^2 \right),
\]

as required (in the latter formula the notations $v_+$ and $v_-$ from Definition 2.1.13 are used).

**Proof of Corollary 2.2.3** Use Theorem 1.4.5 and Proposition 3.1.5; see rows 1–2 of Table 4.

**Proof of Theorem 2.2.9** Tensor (10) is partially symmetric for this particular Lagrangian $L[\phi]$; see rows 2–3 of Table 4. Thus the result follows from Theorems 1.4.9, 1.5.2, and Remark 4.2.6.

**Proof of Proposition 2.2.11** Consider the cases when $l \neq k$ and $l = k$ separately.

For $l \neq k$ the only nonvanishing contribution to the flux of $T$ comes from the edge $f = v - e_l$ because $f \parallel e_k$ otherwise. We have $\dim \text{Pr}(f, k, l) = 1$. Thus by Definition 1.5.1 and (10) we get

\[
(-1)^l (T, h)_k = \frac{1}{2} (-1)^l (-1)^l \left[ T((v + e_l) \times (v - e_k)) \right]
\]
\[
+ T (v + e_l - 2e_k) \times (v - e_k)) \]
\[
= \frac{1}{2} \left[ \frac{\partial L}{\partial (\delta \phi)}(v + e_l) + \frac{\partial L}{\partial (\delta \phi)}(v + e_l - 2e_k) \right] \delta \phi(v - e_k).
\]

For \(l = k\) the contribution to the flux comes from \(f = v\) and \(f = v - e_m\) for each \(m \neq k\). Thus
\[
(-1)^k \langle T, h \rangle_k = \frac{1}{2} (-1)^k (-1)^{k+1} \left[ T(v \times v) - T((v + e_k) \times (v - e_k)) \right] \\
+ \sum_{m \neq k} T((v - e_m) \times (v - e_m)) \\
= -\frac{1}{2} \left[ \frac{\partial L}{\partial \phi}(v)(v) - \frac{\partial L}{\partial (\delta \phi)}(v + e_k)[\delta \phi](v - e_k) \right] \\
+ \sum_{m \neq k} \frac{\partial L}{\partial (\delta \phi)}(v - e_m)[\delta \phi](v - e_m) \\
= \frac{1}{2} \left[ \frac{\partial L}{\partial (\delta \phi)}(v + e_k) + \frac{\partial L}{\partial (\delta \phi)}(v - e_k) \right] \delta \phi(v - e_k) \\
- \frac{1}{2} \frac{\partial L}{\partial \phi}(v)(v) - \frac{1}{2} \sum_m \frac{\partial L}{\partial (\delta \phi)}(v - e_m) \delta \phi(v - e_m) \\
= \frac{1}{2} \left[ \frac{\partial L}{\partial (\delta \phi)}(v + e_k) + \frac{\partial L}{\partial (\delta \phi)}(v + e_k - 2e_k) \right] \delta \phi(v - e_k) - L[\phi](v).
\]

The latter equality is proved as follows. Since \(L\) is homogeneous quadratic, it follows that \(\frac{\partial L}{\partial \phi_1} \phi_1 + \frac{\partial L}{\partial \phi'_1} \phi'_1 + \cdots + \frac{\partial L}{\partial \phi'_d} \phi'_d = 2L(\phi_1, \phi'_1, \ldots, \phi'_d)\). Hence \(\frac{\partial L}{\partial \phi} \sim \phi + \frac{\partial L}{\partial (\delta \phi)} \sim \delta \phi = 2L[\phi]\), as required.

**Proof of Proposition 2.2.13** First note that \(F_N(f) = F_{mn}(\max f) = F_{mn}(\max h)\) on the set of all pairs \((f, h)\) having common vertices, because \(F_{mn}\) is continuous on \(I^d\), hence uniformly continuous.

Denote \(v = \max h\). Consider the cases when \(l = k\) and \(l \neq k\) separately.

Assume that \(l = k\). For a 1- or 2-dimensional face \(f \subset h \perp e_k\) we have \(\dim \Pr(f, k, k) = 0\). Thus
\[
(-1)^k \langle T'_N, h \rangle_k = -\frac{1}{2} \left[ \sum_{f : f \subset h, f \ni v, \dim f = 2} T'_N(f \times f) \right] \\
- \sum_{f : f \subset h, f \ni v, \dim f = 1} T'_N((f + e_k) \times (f - e_k)) \\
= \frac{1}{2} \left[ \sum_{f : f \subset h, f \ni v, \dim f = 2} #F_N(f)F_N(f) \right].
\]
\[-\sum_{f : f \subset h, f \ni v, \dim f = 1} \#F_N(f + e_k)F_N(f - e_k)\]

\[\Rightarrow \frac{1}{2} \left[ \sum_{m, n \neq k : m < n} F^{mn}F_{mn} - \sum_{m \neq k} F^{km}F_{km} \right] (v)\]

\[= \left[ \frac{1}{4} \sum_{m, n} F^{mn}F_{mn} - \sum_{m \neq k} F^{km}F_{km} \right] (v)\]

\[= T^k(v).\]

Assume that \(l \neq k\). For a 2-dimensional face \(f \parallel e_k, e_m\), where \(m \neq k, l\), we have \(\dim Pr(f, k, l) = 2\) or 1 depending on if \(m\) is between \(k\) and \(l\) or not. Thus

\[(-1)^l (T^k_N, h)_k = \frac{1}{2} \sum_{f : f \subset h, f \ni v, \dim f = 2, f \parallel e_k} (-1)^{\dim Pr(f, k, l)} \left[ \#F_N(f + e_l - e_k) + \#F_N(f + e_l + e_k) \right] F_N(f)\]

\[\Rightarrow - \sum_{m \neq k} \text{sgn}(m - k)\text{sgn}(m - l)F^{\min[l, m], \max[l, m]}(v)F^{\min[k, m], \max[k, m]}(v)\]

\[= - \sum_{m \neq k} F^{lm}(v)F_{km}(v)\]

\[= T^l(v).\]

\[\square\]

**Proof of Proposition 2.3.11** Let \(abcd\) be a face with the vertices listed in the order compatible with the positive orientation of its boundary (see Definition 1.4.2), starting from the minimal vertex. Then

\[\Re \text{Tr} \left[ \#F^*(abcd)F (abcd) \right]\]

\[= \#\Re \text{Tr} \left[ (U(abc) - U(adc))^*(U(abc) - U(adc)) \right]\]

\[= \#\Re \text{Tr} \left[ U(cbabc) - U(cdbbc) - U(cabc) + U(c dbc) \right]\]

\[= \#\Re \text{Tr} \left[ 1 - U(abcd) - U(abcd)^* + 1 \right]\]

\[= 2\#(n - \Re \text{Tr} U(abcd)).\]

Multiplying by \(-1/2\) and summing over all the faces \(abcd\), we get the required expression. \[\square\]

**Proof of Corollary 2.3.5** Use notation \(F, D_A^*, S[U]\) from Definitions 2.3.9, 3.1.1, and Proposition 2.3.11. By Propositions 3.1.3 and 2.3.11, this notation is compatible with (11)–(13). Then the Yang–Mills equation follows from Theorems 3.1.7 and 3.1.5; see rows 5–6 of Table 4. Proposition 3.1.11 and Theorem 3.1.10 imply charge conservation. \[\square\]

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**Proof of Corollary 2.3.8** This follows directly from Propositions 2.3.11 and 3.1.11 (see row 5 of Table 4) because Re Tr[$j^* \sim U$] is preserved under simultaneous gauge transformation of $U$ and $j$. □

5 Open Problems

- Expand the suggested discretization algorithm to:
  - quantum field theories via path integral formalism;
  - general relativity via discretizing the raising-index operator $\sharp$ for nonflat spacetimes;
  - hydrodynamics via discretizing the fluid energy-momentum tensor.

- Extend the suggested discretization algorithm to involve the following conservation laws:
  - energy conservation in nontrivial connection via making the cross product gauge invariant;
  - angular momentum conservation via discretizing the radius vector;
  - integral-form energy conservation in general complexes via discretizing tensor integration.

- Prove the conservation of the discrete covariant chiral current. Generally, is the covariant current from Theorem 3.1.8 times $i$ conserved for each gauge invariant Lagrangian satisfying (8)?

- Prove analogous conservation laws in statistical field theory. E.g., is the expectation of a covariant current conserved, if the gauge group field is random with the probability density proportional to the exponential of the action from Definition 2.3.9?

- Apply the discretization algorithm to characteristic classes to obtain invariants of piecewise-linear homeomorphisms or rational homotopy type.

- Construct a “second-generation” discretization algorithm for field theories, in which not only spacetime, but also the set of field values becomes discrete; e.g., as in the Feynman checkerboard.

- Prove that the discussed discrete field theories approximate continuum ones in a sense. Even no analogue of Theorem 2.1.17 for planar graphs with faces not being inscribed is known [6, 30].

- State and prove a “reciprocal Noether theorem” giving a symmetry of the continuum limit for each discrete conservation law.

- Find an experimentally measurable quantity in our discretization not converging to the continuum counterpart; this would make the discretization refutable against the continuum theory.

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