11-COLORED KNOT DIAGRAM WITH FIVE COLORS

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Abstract. We prove that any 11-colorable knot is presented by an 11-colored diagram where exactly five colors of eleven are assigned to the arcs. The number five is the minimum for all non-trivially 11-colored diagrams of the knot. We also prove a similar result for any 11-colorable ribbon 2-knot.

1. Introduction

The \( n \)-colorability introduced by Fox [3] is one of the elementary notion in knot theory, and its properties have been studied in many papers. In 1999, Harary and Kauffman [5] defined a kind of minimal invariant, \( C_n(K) \), of an \( n \)-colorable knot \( K \). It is essential to consider the case that \( n \) is an odd prime; in fact, for composite \( n \), it is reduced to the cases of odd prime factors of \( n \). In this case, we can define a modified version by restricting “effective” \( n \)-colorings (cf. [6, 12]).

Let \( p \) be an odd prime. A non-trivial \( p \)-coloring \( C \) of a knot diagram \( D \) is regarded as a non-constant map

\[ C : \{ \text{arcs of } D \} \rightarrow \mathbb{Z}/p\mathbb{Z} = \{0, 1, \ldots, p-1\} \]

with a certain condition. For a \( p \)-colorable knot \( K \), the number \( C_p(K) \) is defined to be the minimum number of \( \#\text{Im}(C) \) for all non-trivially \( p \)-colored diagrams \((D, C)\) of \( K \). This number has been studied in some papers [2, 4, 7, 8, 10, 11, 13, 15, 17]. In particular, it is shown in [11] that

\[ C_p(K) \geq \lfloor \log_2 p \rfloor + 2 \]

for any \( p \)-colorable knot \( K \), and the equality holds for \( p = 3, 5, 7 \) [13, 17].

For \( p = 11 \), we have \( C_{11}(K) \geq 5 \) by the above inequality or [10, Theorem 2.4]. On the other hand, it is proved in [2] that \( C_{11}(K) \leq 6 \). If an 11-colored diagram \((D, C)\) satisfies \( \#\text{Im}(C) = 5 \), then there are two possibilities

\[ \text{Im}(C) = \{1, 4, 6, 7, 8\}, \{0, 4, 6, 7, 8\} \]

up to isomorphisms induced by affine maps of \( \mathbb{Z}/11\mathbb{Z} \). This split phenomenon is quite different from the cases \( p = 3, 5, 7 \).

Theorem 1.1. Any 11-colorable knot \( K \) satisfies the following.

(i) There is an 11-colored diagram \((D_1, C_1)\) of \( K \) with \( \text{Im}(C_1) = \{1, 4, 6, 7, 8\} \).
(ii) There is an 11-colored diagram \((D_2, C_2)\) of \( K \) with \( \text{Im}(C_2) = \{0, 4, 6, 7, 8\} \).

We remark that these two sets are common 11-minimal sufficient sets of colors but not universal ones in the sense of [4]. By Theorem 1.1 we have the following immediately.

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Corollary 1.2. Any 11-colorable knot $K$ satisfies $C_{11}(K) = 5$. □

This paper is organized as follows. In Section 2, we review the palette graph associated with a subset of $\mathbb{Z}/p\mathbb{Z}$ and its fundamental properties. In Section 3, we prove Theorem 1.1(i). The starting point of the proof is a modified version of the theorem in [2]: For any 11-colorable knot $K$, there is an 11-colored diagram $(D, C)$ of $K$ with $\text{Im}(C) = \{0, 1, 4, 6, 7, 8\}$. By applying Reidemeister moves to $(D, C)$ suitably, we remove the color 0 from the diagram. Sections 4–6 are devoted to proving Theorem 1.1(ii). We first remove the color 1 from $(D, C)$ as above by allowing the birth of new colors 3 and 10 in Section 4, and then remove the colors 10 and 3 in Sections 5 and 6, respectively. In the last section, we prove a similar result for an 11-colorable ribbon 2-knot.

2. Preliminaries

Throughout this section, $p$ denotes an odd prime.

Definition 2.1. Let $S$ be a subset of $\mathbb{Z}/p\mathbb{Z}$. The palette graph $G(S)$ of $S$ is a simple graph such that

(i) the vertex set of $G(S)$ is $S$, and
(ii) two vertices $a$ and $b \in S$ are connected by an edge if and only if $\frac{a+b}{2} \in S$.

By assigning $\frac{a+b}{2}$ to every edge joining $a$ and $b$, we regard $G(S)$ as a labeled graph. Such an edge is denoted by $\{a | a+b/2 | b\}$.

Definition 2.2. For two subsets $S$ and $S' \subset \mathbb{Z}/p\mathbb{Z}$, the palette graphs $G(S)$ and $G(S')$ are said to be isomorphic if there is a bijection $f : S \to S'$ such that $\frac{a+b}{2} \in S$ if and only if $\frac{f(a)+f(b)}{2} \in S'$. We denote it by $G(S) \cong G(S')$.

Lemma 2.3. If $S \subset S' \subset \mathbb{Z}/p\mathbb{Z}$, then $G(S)$ is a subgraph of $G(S')$, which is obtained from $G(S')$ by deleting the vertices in $S' \setminus S$ and the edges whose labels belong to $S' \setminus S$.

Proof. This follows from definition immediately. □

Theorem 2.4 ([11]). If the palette graph $G(S)$ is connected with $\#S > 1$, then we have $\#S \geq \lceil \log_2 p \rceil + 2$. □
Lemma 2.5. Let $S$ be a subset of $\mathbb{Z}/p\mathbb{Z}$ such that $G(S)$ is connected with $\#S = [\log_2 p] + 2$. Put $U = \{S' \subset \mathbb{Z}/p\mathbb{Z}|G(S') \cong G(S)\}$. Then we have $\#U = p(p-1)$.

Proof. Let $T$ be a maximal tree of $G(S)$. Let $v_1, v_2, \ldots, v_k$ be the vertices of $T$, and $e_1, e_2, \ldots, e_{k-1}$ the edges of $T$, where $k = \#S = [\log_2 p] + 2$. Let $A = (a_{ij})$ be the $(k-1) \times k$ matrix with $\mathbb{Z}$-entries defined by

$$a_{ij} = \begin{cases} 1 & (e_i \text{ is incident to } v_j), \\ -2 & (\text{the label of } e_i \text{ is } v_j), \\ 0 & (\text{otherwise}). \end{cases}$$

Let $A'$ be the $(k-1) \times (k-1)$ matrix obtained from $A$ by deleting the $k$th column. It is known in [11] that

(i) $\det(A')$ is odd,

(ii) $|\det(A')| < 2^{k-1}$, and

(iii) $\det(A')$ is divisible by $p$.

Since $2^{k-2} < p \leq |\det(A')| < 2^{k-1}$, we have $|\det(A')| = p$. This implies that the corank of $A$ with $\mathbb{Z}/p\mathbb{Z}$-entries is exactly equal to 2.

Let $V = \{x|Ax = 0 \pmod{p}\}$ denote the solution space. By the above argument, we have

$$V = \{\lambda \cdot v_1 + \mu \cdot v_2 + \cdots + \nu \cdot v_k | \lambda, \mu, \nu \in \mathbb{Z}/p\mathbb{Z}\}.$$

Since the elements of $U$ are identified with the vectors of $V$ whose entries are all distinct. Such a vector is obtained by the condition $\lambda \neq 0 \pmod{p}$. Therefore, we have $\#U = p(p-1)$. \hfill $\Box$

Theorem 2.6. Let $S$ and $S'$ be subsets of $\mathbb{Z}/p\mathbb{Z}$. Suppose that $G(S)$ and $G(S')$ are connected with $\#S = \#S' = [\log_2 p] + 2$. Then the following are equivalent.

(i) The palette graphs $G(S)$ and $G(S')$ are isomorphic.

(ii) There exist $\alpha \neq 0 \pmod{p}$ and $\beta \in \mathbb{Z}/p\mathbb{Z}$ such that the affine map $f(x) = \alpha x + \beta$ satisfies $f(S) = S'$.

Proof. (ii)$\Rightarrow$(i). Since $\alpha \neq 0 \pmod{p}$, $f : S \rightarrow S'$ is a bijection. Furthermore, $\frac{\alpha x + \beta}{\alpha} \in S$ holds if and only if $f\left(\frac{\alpha x + \beta}{\alpha}\right) = \frac{f(\alpha x + \beta)}{\alpha} \in f(S) = S'$ holds.

(i)$\Rightarrow$(ii). By the above argument, we have

$$U \supset \{f(S)|f(x) = \alpha x + \beta, \alpha \neq 0, \beta \in \mathbb{Z}/p\mathbb{Z}\},$$

where $U$ is the set in Lemma 2.5. Since these two sets have the same number of elements by Lemma 2.5, they are the same set. \hfill $\Box$

Let $D$ be a diagram of a knot $K$. We regard $D$ as a disjoint union of arcs whose endpoints are under-crossings. Fox [3] introduced the notion of $p$-colorings: A map $C : \{\text{arcs of } D\} \rightarrow \mathbb{Z}/p\mathbb{Z}$ is a $p$-coloring if $a + b \equiv 2c \pmod{p}$ holds at every crossing, where $a$ and $b$ are the elements assigned to the under-arcs by $C$, and $c$ is the one to the over-arc. The triple $\{a|b|c\}$ is called the color of the crossing. The assigned element of an arc of $D$ is called the color of the arc. If the color of an arc is $a$, then the arc is called an $a$-arc.

In a $p$-colored diagram $(D, C)$, the crossing of color $\{a|a|a\}$ is called trivial, and otherwise non-trivial. If $C$ is a constant map, it is called a trivial $p$-coloring, and otherwise, non-trivial. In other words, a $p$-coloring $C$ is non-trivial if and only if $\#\text{Im}(C) > 1$. If a knot $K$ admits a non-trivially $p$-colored diagram $(D, C)$, $K$ is called $p$-colorable.
For a $p$-colorable knot $K$, we denote by $C_p(K)$ the minimum number of $\#\text{Im}(C)$ for all non-trivially $p$-colored diagram $(D, C)$ of $K$. For the study of this number, it is helpful to use the palette graph $G(\text{Im}(C))$ of the image $\text{Im}(C) \subset \mathbb{Z}/p\mathbb{Z}$ in the following sense.

**Lemma 2.7.** If $\{a|c|b\}$ is a non-trivial color of a crossing of a $p$-colored diagram $(D, C)$, then the palette graph $G(\text{Im}(C))$ has an edge $\{a|c|b\}$.

*Proof.* Since $a + b \equiv 2c \pmod{p}$ holds, the lemma follows by definition. □

**Lemma 2.8.** The palette graph $G(\text{Im}(C))$ of a $p$-colored diagram $(D, C)$ of a knot is connected.

*Proof.* Let $a$ and $b$ be vertices of $G(\text{Im}(C))$. By definition, we have an $a$-arc and a $b$-arc of $D$. Since $D$ is a diagram of a knot (not a link), we can walk along $D$ from the $a$-arc to the $b$-arc. Let $\{a_i|c_i|a_{i+1}\} (1 \leq i \leq k-1)$ be the colors of non-trivial under-crossings on the path such that $a_1 = a$ and $a_k = b$. Then the vertices $a$ and $b$ in the palette graph are connected by a sequence of edges $\{a_i|c_i|a_{i+1}\} (1 \leq i \leq k-1)$. □

**Theorem 2.9** ([11]). Any non-trivial $p$-colored diagram $(D, C)$ of a knot satisfies $\#\text{Im}(C) \geq \lfloor \log_2 p \rfloor + 2$. Therefore, we have $C_p(K) \geq \lfloor \log_2 p \rfloor + 2$ for any $p$-colorable knot $K$.

*Proof.* This follows from Theorem 2.4 and Lemma 2.8. □

**Lemma 2.10.** Let $(D, C)$ be a non-trivially $p$-colored diagram of a knot $K$, and $f : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ an affine map defined by $f(x) = \alpha x + \beta$ with $\alpha \neq 0$ and $\beta \in \mathbb{Z}/p\mathbb{Z}$. Then there is a non-trivially $p$-colored diagram $(D, C')$ of $K$ such that $\text{Im}(C') = f(\text{Im}(C))$.

*Proof.* It is easy to see that the composition $C' = f \circ C$ is also a non-trivial $p$-coloring of $D$. □

Now, we consider the case $p = 11$. By Theorem 2.4 if the palette graph $G(S)$ of a subset $S \subset \mathbb{Z}/11\mathbb{Z}$ is connected with $\#S > 1$, then $\#S \geq 5$.

**Theorem 2.11** ([11] Theorem 12). Let $S$ be a subset of $\mathbb{Z}/11\mathbb{Z}$. If the palette graph $G(S)$ is connected with $\#S = 5$, then $G(S)$ is isomorphic to $G(\{1, 4, 6, 7, 8\})$ or $G(\{0, 4, 6, 7, 8\})$ as shown in Figure 1. □

![Figure 1](image1.png)

By Theorem 2.9 or [11] Theorem 2.4, we have $C_{11}(K) \geq 5$. The following theorem implies that $C_{11}(K) = 5$ or 6.
Theorem 2.12 [2]. For any 11-colorable knot \( K \), there is a non-trivially 11-colored diagram \((D, C)\) of \( K \) with \( \text{Im}(C) \subset \{0, 1, 4, 6, 7, 8\} \).

Figure 2 shows the palette graph \( G(\{0, 1, 4, 6, 7, 8\}) \). By Lemma 2.3, the two graphs in Theorem 2.11 are obtained from this graph by deleting the vertex \( a \) and the edges labeled \( a \) for \( a = 0, 1, \) respectively.

Lemma 2.13. For any 11-colorable knot \( K \), there is an 11-colored diagram \((D, C)\) of \( K \) with \( \text{Im}(C) = \{0, 1, 4, 6, 7, 8\} \).

Proof. We may assume that \((D, C)\) satisfies Theorem 2.12, that is, it is a non-trivially 11-colored diagram with \( \text{Im}(C) \subset \{0, 1, 4, 6, 7, 8\} \). We remark that \( \#\text{Im}(C) \geq 5 \) by Theorem 2.9.

\( 4, 6, 7 \in \text{Im}(C) \). Assume that \( 4 \notin \text{Im}(C) \). It follows that \( \text{Im}(C) = \{0, 1, 6, 7, 8\} \). The palette graph \( G(\text{Im}(C)) \) is as shown in the left of Figure 3 by Lemma 2.3, which contradicts to the connectivity in Lemma 2.8. We can also prove \( 6, 7 \in \text{Im}(C) \) by a similar argument. See the center and right of the figure.

\( 0 \in \text{Im}(C) \). Assume that \( 0 \notin \text{Im}(C) \). It follows that \( \text{Im}(C) = \{1, 4, 6, 7, 8\} \) and its palette graph is as shown in the left of Figure 3. Then we see that \((D, C)\) has a crossing of color \( \{6|1\} \) or \( \{1|8\} \). In fact, if we delete the corresponding edges both, the resulting graph becomes disconnected. By deforming the diagram near these crossings as shown in Figure 4 we can produce a 0-arc. We replace the original diagram with the new one as \((D, C)\).

\( 1 \in \text{Im}(C) \). Assume that \( 1 \notin \text{Im}(C) \). Then we have \( \text{Im}(C) = \{0, 4, 6, 7, 8\} \) and its palette graph is as shown the right of Figure 3. Since \((D, C)\) must have a crossing.
of color \{0|4|8\} by a similar reason to the above case, we deform the diagram near the crossing to make a 1-arc. See Figure 5.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5}
\caption{Figure 5.}
\end{figure}

\begin{align*}
8 \in \text{Im}(C). & \quad \text{Assume that } 8 \not\in \text{Im}(C). \quad \text{Then we have } \text{Im}(C) = \{0, 1, 4, 6, 7\} \text{ and its palette graph is as shown in the left of Figure 6.} \quad \text{We remark that the map } f : \mathbb{Z}/11\mathbb{Z} \to \mathbb{Z}/11\mathbb{Z} \text{ defined by } f(x) = 7x + 6 \text{ induces the isomorphism between } G(\{0, 4, 6, 7, 8\}) \text{ and } G(\{0, 1, 4, 6, 7\}). \quad \text{The existence of such a map is guaranteed by Theorem 2.6.} \quad \text{Since } (D, C) \text{ has a crossing of color } \{4|0|7\}, \text{ we deform the diagram near the crossing as shown in the right of the figure so that we obtain an 8-arc.} \quad \Box
\end{align*}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6}
\caption{Figure 6.}
\end{figure}

3. Proof of Theorem 1.1

**Lemma 3.1.** For any 11-colorable knot $K$, there is an 11-colored diagram $(D, C)$ of $K$ such that

(i) $\text{Im}(C) = \{0, 1, 4, 6, 7, 8\}$, and

(ii) there is no crossing of color $\{\ast|0|\ast\}$.

**Proof.** We may assume that $(D, C)$ satisfies Lemma 2.13. There are two types of crossings of $(D, C)$ whose over-arc is a 0-arc; that is, $\{0|0|0\}$ and $\{4|0|7\}$. In fact, in the palette graph $G(\{0, 1, 4, 6, 7, 8\})$, the only edge labeled 0 connects 4 and 7.

First, we assume that $(D, C)$ has crossings of color $\{4|0|7\}$. By deforming the diagram near the crossings as shown in Figure 4, we can eliminate all the crossings.
of color \{4|0|7\}. We remark that the set of colors which are appeared in the diagram does not change.

Next, we assume that \((D,C)\) has a crossing of color \{0|0|0\}, say \(x\). Walking along the diagram from \(x\), let \(y\) be the non-trivial crossing which we meet first. If there are crossings of color \{0|0|0\} between \(x\) and \(y\), we replace the original \(x\) with the nearest one to \(y\). Therefore, we may assume that there is no crossing between \(x\) and \(y\).

There are two cases with respect to the color of \(y\). In fact, in the palette graph \(G(\{0,1,4,6,7,8\})\), there are two edges incident to the vertex 0, which implies that the color of \(y\) is \{0|6|1\} or \{0|4|8\}. In each case, we deform the diagram \((D,C)\) near \(x\) and \(y\) as shown in Figure 8, so that the number of crossings of \{0|0|0\} is decreased. By repeating this process, we obtain a diagram with no crossing of \{0|0|0\} finally.

\[\text{Figure 8.}\]

Proof of Theorem 1.1(i). We may assume that \((D,C)\) satisfies Lemma 3.1. If there is a 0-arc, it is not an over-arc of any crossing, and its endpoints are the under-crossings of color \{0|4|8\} or \{0|6|1\}. In fact, there are two edges incident to the vertex 0 in \(G(\{1,4,6,7,8\})\). We have three cases with respect to the colors of the crossings of the endpoints of a 0-arc:

(i) \{0|4|8\} and \{0|6|1\},
(ii) \{0|4|8\} both, and
(iii) \{0|6|1\} both.

For the case (i), we deform the 6-arc over the crossing of \{0|4|8\} to eliminate the 0-arc. See the top of Figure 9. For the case (ii), we deform one of the crossings of color \{0|4|8\} as shown in the figure so that we reduce this case to (i). Similarly, for the case (iii), we deform one of the crossings of color \{0|6|1\} as shown in the figure so that we reduce this case to (i). See the bottom of the figure.

Corollary 3.2. For any 11-colorable knot \(K\) and \(a \neq b \in \mathbb{Z}/11\mathbb{Z}\), there is an 11-colored diagram \((D,C)\) of \(K\) with
\[\text{Im}(C) = \{a, b, 3a + 9b, 6a + 6b, 10a + 2b\}.\]
Figure 9.

Proof. Let $f : \mathbb{Z}/11\mathbb{Z} \to \mathbb{Z}/11\mathbb{Z}$ be the affine map defined by $f(x) = 4(b - a)(x - 1) + a$. Since the map $f$ satisfies

$$f(1) = a, \ f(4) = b, \ f(6) = 3a + 9b, \ f(7) = 10a + 2b, \ f(8) = 6a + 6b,$$

we have the conclusion by Lemma 2.10 and Theorem 1.1(i).

4. Proof of Theorem 1.1(ii) – Part I

Lemma 4.1. For any 11-colorable knot $K$, there is an 11-colored diagram $(D, C)$ of $K$ such that

(i) $\text{Im}(C) = \{0, 1, 4, 6, 7, 8\}$, and

(ii) there is no crossing of color $\{6|6\}$.

Proof. We may assume that $(D, C)$ satisfies Lemma 2.13. Assume that $(D, C)$ has a crossing of color $\{6|6\}$, say $x$. Walking along the diagram from $x$, let $y$ be the first non-trivial under-crossing. If there are crossings of color $\{6|6\}$ between $x$ and $y$, then we replace the original $x$ with the nearest one to $y$. Then we have the following:

(i) There is no crossing of $\{6|6\}$ between $x$ and $y$ by assumption.

(ii) Every crossing between $x$ and $y$ is of color $\{0|6\}$ or $\{4|6\}$: for there are exactly two edges labeled 6 in the palette graph $G(\{0, 1, 4, 6, 7, 8\})$.

(iii) The color of $y$ is $\{6|7\}$ or $\{7|8\}$: for there are exactly two edges incident to the vertex 6 in the palette graph, which are labeled 1 and 7, respectively.

Assume that there are crossings between $x$ and $y$. Let $z$ be the nearest crossing to $x$ among them. We deform the diagram near $x$ and $z$ as shown in the upper row.
of Figure 10 so that the number of crossings between $x$ and $y$ is decreased. By repeating this process, we may assume that there is no crossing between $x$ and $y$. Then we deform the diagram near $x$ and $y$ as shown in the lower row of the figure to eliminate the color $\{6|6|6\}$. By repeating this process, we obtain a diagram with no crossing of $\{6|6|6\}$ finally.

**Lemma 4.2.** For any 11-colorable knot $K$, there is an 11-colored diagram $(D, C)$ of $K$ such that

(i) $\text{Im}(C) = \{0, 1, 4, 6, 7, 8\}$, and

(ii) there is no crossing of color $\{1|1|1\}$ or $\{6|6|6\}$.

**Proof.** We may assume that $(D, C)$ satisfies Lemma 4.1. Assume that $(D, C)$ has a crossing of color $\{1|1|1\}$, say $x$. Walking along the diagram from $x$, let $y$ be the first non-trivial crossing. If there are crossings of $\{1|1|1\}$ between $x$ and $y$, then we replace the original $x$ with the nearest one to $y$.

In the palette graph $G(\{0, 1, 4, 6, 7, 8\})$, there are exactly three edges incident to the vertex 1 whose labels are 4, 6, and 8, and there is only one edge whose label is 1. Therefore, the color of the crossing $y$ is $\{6|1|7\}$, $\{1|6|0\}$, $\{1|8|4\}$, or $\{6|1|7\}$.

We deform the diagram near $x$ and $y$ as shown in Figure 11 so that the number of crossings of color $\{1|1|1\}$ is decreased. By repeating this process, we obtain a diagram with no crossing of $\{1|1|1\}$.

**Lemma 4.3.** For any 11-colorable knot $K$, there is a non-trivially 11-colored diagram $(D, C)$ of $K$ such that

(i) $\text{Im}(C) = \{0, 1, 4, 6, 7, 8\}$, and

(ii) there is no crossing of color $\{*|1|*\}$ or $\{6|6|6\}$.

**Proof.** We may assume that $(D, C)$ satisfies Lemma 4.2. Assume that $(D, C)$ has a crossing of color $\{*|1|*\}$. Since there is only one edge labeled 1 in the palette graph $G(\{0, 1, 4, 6, 7, 8\})$, the color of the corresponding crossing is $\{6|1|7\}$.

There is a 4-arc in $(D, C)$. We will pull the 4-arc toward each crossing of $\{6|1|7\}$. In the process, we can assume that the 4-arc crosses over several arcs whose colors are 0, 1, 4, 7, 8 missing 6. In fact, since there is no crossing of $\{6|6|6\}$, the set of 6-arcs is a disjoint union of intervals in the plane, and the complement in the plane
is connected. When the 4-arc crosses over an $a$-arc for $a = 0, 1, 4, 7, 8$, we have a pair of new crossings of color
\[
\{a|4|8-a\} = \{0|4|8\}, \{1|4|7\}, \{4|4|4\}, \{7|4|1\}, \{8|4|0\},
\]
respectively. See the left of Figure 12. We remark that any vertex of the palette graph $G(\{0, 1, 4, 6, 7, 8\})$ other than 6 is 4 itself or incident to an edge labeled 4.

By deforming the diagram near every crossing of $\{6|1|7\}$ with a 4-arc as shown in the right of the figure, we obtain a diagram with no crossing of $\{6|1|7\}$. Then the arcs in the obtained diagram are colored by 0, 1, 4, 6, 7, 8 and there is no crossing of $\{\ast|\ast|\ast\}$ or $\{6|6|6\}$. \hfill $\square$

**Lemma 4.4.** For any 11-colorable knot $K$, there is an 11-colored diagram $(D, C)$ of $K$ such that
(i) $\text{Im}(C) = \{0, 3, 4, 6, 7, 8, 10\}$,
(ii) there is no crossing of color $\{6|6|6\}$, and
(iii) if $\{a|b|c\}$ is the color of a crossing and at least one of $a, b, c$ is 3 or 10, then it is one of
\[
\{0|3|6\}, \{0|7|3\}, \{3|0|8\}, \{4|7|10\}, \{7|3|10\}, \{3|3|3\}.
\]
Proof. We may assume that \((D, C)\) satisfies Lemma 4.3. Since there are three edges incident to the vertex 1 in the palette graph \(G(\{0, 1, 4, 6, 7, 8\})\), every crossing with a 1-arc is of color \(\{1|4|7\}\), \(\{1|6|0\}\), or \(\{1|8|4\}\). If there is a crossing of \(\{1|8|4\}\), we deform the 4-arc near the crossing as shown in Figure 13 to replace the crossing with the one of color \(\{1|4|7\}\). Therefore, we may assume that there is no crossing of \(\{1|8|4\}\).

\begin{figure}[h]
\centering
\includegraphics[scale=0.5]{figure13.png}
\caption{}
\end{figure}

There is a 0-arc in \((D, C)\). We will pull the 0-arc toward each crossing of \(\{1|4|7\}\). In the process, we can assume that the 0-arc crosses over several \(a\)-arcs for \(a \in \{0, 1, 4, 7, 8\}\) missing 6 by the same reason in the proof of Lemma 4.3; that is, there is no crossing of \(\{6|6|6\}\). When the 0-arc crosses over an \(a\)-arc, we have a pair of new crossings of color
\[\{a|0| - a\} = \{0|0|0\}, \{1|0|10\}, \{4|0|7\}, \{7|0|4\}, \{8|0|3\},\]
respectively. We remark that the new colors 3 and 10 appear at the crossings of \(\{1|0|10\}\) and \(\{3|0|8\}\). See Figure 14.

\begin{figure}[h]
\centering
\includegraphics[scale=0.5]{figure14.png}
\caption{}
\end{figure}

By deforming the diagram near every crossing of \(\{1|4|7\}\) with a 0-arc as shown in Figure 15, we remove all the crossings of \(\{1|4|7\}\) and produce the color 10 at the crossings of \(\{1|0|10\}\) and \(\{4|7|10\}\).

\begin{figure}[h]
\centering
\includegraphics[scale=0.5]{figure15.png}
\caption{}
\end{figure}

There is a 7-arc in \((D, C)\). We will pull the 7-arc toward each 0-arc. In the process, we can assume that the 7-arc crosses over several \(a\)-arcs for \(a \in \)
\{0, 3, 4, 6, 7, 8, 10\} missing 1; for there is no crossing of \{1|1|1\}. Then we have

\[ \{a|7|3 - a\} = \{0|7|3\}, \{3|7|0\}, \{4|7|10\}, \{6|7|8\}, \{7|7|7\}, \{8|7|6\}, \{10|7|4\}, \]

respectively. We remark that the colors 3 and 10 appear at the crossings of \{0|7|3\} and \{4|7|10\}.

Now, every crossing with a 1-arc is of color \{1|6|0\} or \{1|0|10\}. The endpoints of every 1-arc are under-crossings of color

(i) \{1|6|0\} both,
(ii) \{1|0|10\} both, or
(iii) \{1|0|10\} and \{1|6|0\}.

For every 1-arc of type (i), we deform the diagram near the 1-arc equipped with a 7-arc into type (ii) as shown in the left of Figure 16. Here, the colors 3 and 10 appear at the crossings of \{0|7|3\}, \{7|3|10\}, \{0|3|6\}, \{3|3|3\}, and \{3|0|8\}.

![Figure 16.](image-url)

For every 1-arc of type (ii) or (iii), we deform the diagram near the 1-arc with a 7-arc as shown in the center and right of the figure, so that we can remove all the 1-arcs from the diagram. We remark that the colors 3 and 10 appear at the crossings of \{4|7|10\} for (ii) and \{3|0|8\} and \{7|3|10\} for (iii).

Since the original diagram has a 1-arc, at least one of deformations (i), (ii), and (iii) must happen. Therefore, the obtained diagram has a 10-arc. If the diagram has no 3-arc, the case (ii) must happen. By deforming a neighborhood of a crossing of \{4|0|7\} similarly to Figures 4 and 5, we can make a pair of crossings of \{0|7|3\} so that we have \(\text{Im}(C) = \{0, 3, 4, 6, 7, 8, 10\}\).

We remark that the 11-colored diagram \((D, C)\) in Lemma 4.4 has no crossing of color \{3|10|6\}, \{6|8|10\}, or \{10|10|10\}. In particular, there is no crossing whose over-arc is colored 10.
5. Proof of Theorem 1.1—Part II

Let \( G_1 \) be the graph obtained from the palette graph \( G(\{0, 4, 6, 7, 8\}) \) by adding two vertices 3 and 10 and five edges
\[
\{0|3|6\}, \{0|7|3\}, \{3|0|8\}, \{4|7|10\}, \{7|3|10\}.
\]
See Figure 17. In other words, \( G_1 \) is obtained from \( G(\{0, 3, 4, 5, 6, 8, 10\}) \) by deleting the edges \{3|10|6\} and \{6|8|10\}.

![Figure 17](image)

Assume that \((D, C)\) satisfies Lemma 5.1. If \(\{a|c|b\}\) is the non-trivial color of a crossing of \( (D, C) \), then the palette graph \( G_1 \) has the corresponding edge \( \{a|c|b\} \).

**Lemma 5.1.** For any 11-colorable knot \( K \), there is an 11-colored diagram \((D, C)\) of \( K \) such that

1. \( \text{Im}(C) = \{0, 3, 4, 6, 7, 8\} \), and
2. there is no crossing of color \( \{6|6|6\} \).

**Proof.** We may assume that \((D, C)\) satisfies Lemma 4.4. Since the graph \( G_1 \) has no edge whose label is 10 and \((D, C)\) has no crossing of \( \{10|10|10\} \), we see that there is no crossing of color \( \{6|10|6\} \).

Since there are two edges incident to the vertex 10 in \( G_1 \), every crossing with a 10-arc is of color \( \{4|7|10\} \) or \( \{7|3|10\} \). If there is a crossing of \( \{7|3|10\} \), we deform the 7-arc near the crossing as shown in the left of Figure 18 to replace the crossing with one of \( \{4|7|10\} \). We remark that the crossings of \( \{0|7|3\} \) and \( \{4|0|7\} \) are also produced. Therefore, we may assume that there is no crossing of \( \{7|3|10\} \).

![Figure 18](image)

There is a 0-arc in \((D, C)\). We will pull the 0-arc toward each 10-arc. In the process, we can assume that the 0-arc crosses over several arcs whose colors are 0, 3, 4, 7, 8 missing 6 and 10. In fact, since there is no crossing of color
\[
\{3|10|6\}, \{2|6|10\}, \{6|8|10\}, \{6|6|6\}, \{10|10|10\},
\]
the set of 6- and 10-arcs is a disjoint union of intervals, and the complement in the plane is connected. When the 0-arc crosses over an a-arc for $a = 0, 3, 4, 7, 8$, we have a pair of new crossings of color $\{a|0|−a\} = \{0|0|0\}, \{3|0|8\}, \{4|0|7\}, \{7|0|4\}, \{8|0|3\}$, respectively. We remark that any vertex of $G_1$ other than 6 and 10 is 0 itself or incident to an edge labeled 0.

We deform the diagram near every 10-arc with a 0-arc as shown in the right of the figure, so that we remove all the 10-arcs from the diagram. We remark that the crossings of $\{0|7|3\}, \{4|0|7\}$, and $\{7|7|7\}$ are produced. □

6. Proof of Theorem 1.1 (ii)–Part III

Lemma 6.1. For any 11-colorable knot $K$, there is an 11-colored diagram $(D, C)$ of $K$ such that

(i) $\text{Im}(C) = \{0, 3, 4, 6, 7, 8\}$,
(ii) there is no crossing of color $\{3|3|3\}, \{4|4|4\}$, or $\{6|6|6\}$.

Proof. We may assume that $(D, C)$ satisfies Lemma 5.1 with $\text{Im}(C) = \{0, 3, 4, 6, 7, 8\}$. Figure 19 shows the palette graph $G(\{0, 3, 4, 6, 7, 8\})$, which is obtained from $G_1$ by deleting the vertex 10 and its incident edges $\{4|7|10\}$ and $\{7|3|10\}$.

There is a 0-arc in $(D, C)$. Similarly to the proof of Lemma 5.1 we can pull the 0-arc freely without producing new colors. We remark that any vertex of $G(\{0, 3, 4, 6, 7, 8\})$ other than 6 is 0 itself or incident to an edge labeled 0. Then we deform the diagram near every 3- or 4-arc with a 0-arc as shown in Figure 20 so that there is no crossing of $\{3|3|3\}$ or $\{4|4|4\}$. □

Lemma 6.2. For any 11-colorable knot $K$, there is an 11-colored diagram $(D, C)$ of $K$ such that

(i) $\text{Im}(C) = \{0, 3, 4, 6, 7, 8\}$,
(ii) there is no crossing of color \{\ast|\ast\}, \{4|4\}, or \{6|6\}.

Proof. We may assume that \((D, C)\) satisfies Lemma 6.1. In the palette graph \(G(\{0, 3, 4, 6, 7, 8\})\), there is only one edge whose label is 3. Therefore, every crossing whose over-arc is 3 has the color \{0|3\}.

There is a 7-arc in \((D, C)\). We will pull the 7-arc toward each crossing of \{0|3\}. Since there is no crossing of \{4|4\}, we can assume that the 7-arc crosses over several arcs whose colors are 0, 3, 6, 7, 8 missing 4. If the 7-arc crosses an \(a\)-arc for \(a \in \{0, 3, 6, 7, 8\}\), then we have a pair of new crossings of color \\
\{a|7|3 - a\} = \{0|7|3\}, \{3|7|0\}, \{6|7|8\}, \{7|7|7\}, \{8|7|6\},
\nrespectively. We remark that any vertex of \(G(\{0, 3, 4, 6, 7, 8\})\) other than 4 is 7 itself or incident to an edge labeled 7. We deform the diagram near every crossing of \{0|3\} equipped with a 7-arc as shown in Figure 21 to remove all the crossings of \{0|3\}.

\[ \begin{array}{c}
0 & 7 & 3 & 6 & 7 & 0 & 3 & 8 & 6 \\
\end{array} \]

\textbf{Figure 21.}

Proof of Theorem 1.1(ii). We may assume that \((D, C)\) satisfies Lemma 6.2. Since there are two edges incident to the vertex 3 in \(G(\{0, 3, 4, 6, 7, 8\})\), every crossing with a 3-arc is of color \{3|0|8\} or \{0|7|3\}. Therefore, the endpoints of every 3-arc are under-crossings of color \\
\{3|0|8\} and \{0|7|3\}, \\
(ii) \{3|0|8\} both, or \\
(iii) \{0|7|3\} both.

For every 3-arc of type (i), we deform the diagram near the crossing of \{0|7|3\}, which reduces a 3-arc of type (ii). See the left of Figure 22. Therefore, we may assume that there is no 1-arc of type (i).

To remove a 3-arc of type (ii), We will pull a 7-arc toward the 3-arc. Since there is no crossing of \{4|4\}, the 7-arc can cross over several arcs whose colors are 0, 3, 6, 7, 8 missing 4 similarly to the proof of Lemma 6.2. We remark that when the 7-arc crosses over an 0- or 3-arc, then we have a pair of new crossings of color \{0|7|3\}. We deform the diagram near every 3-arc of type (ii) with a 7-arc to remove all the 3-arcs of type (ii). See the center of the figure.

Now, since every crossing with a 3-arc is of color \{0|7|3\}, every 3-arc is of type (iii). We deform the diagram near every 3-arc of type (iii) with a 7-arc as shown in the right of the figure so that we obtain a diagram with no 3-arc.

\[ \begin{array}{c}
0 & 7 & 3 & 6 & 7 & 0 & 3 & 8 & 6 \\
\end{array} \]

\textbf{Figure 22.}

\textbf{Corollary 6.3.} For any 11-colorable knot \(K\) and \(a \neq b \in \mathbb{Z}/11\mathbb{Z}\), there is an 11-colored diagram \((D, C)\) of \(K\) with
\[ \text{Im}(C) = \{a, b, 5a + 7b, 2a + 10b, 10a + 2b\}. \]
Figure 22.

Proof. Let $f : \mathbb{Z}/11\mathbb{Z} \to \mathbb{Z}/11\mathbb{Z}$ be the affine map defined by $f(x) = 3(b - a)x + a$. Since the map $f$ satisfies

$$f(0) = a, \ f(4) = b, \ f(6) = 5a + 7b, \ f(7) = 2a + 10b, \ f(8) = 10a + 2b,$$

we have the conclusion by Lemma 2.10 and Theorem 1.1(ii).

7. 11-colorable ribbon 2-knot

A ribbon 2-knot [3] is a kind of knotted 2-sphere embedded in $\mathbb{R}^4$. Such a 2-knot is presented by a diagram in $\mathbb{R}^3$ with only double point circles [18], the $n$-colorability is defined similarly to the classical case by assigning an element of $\mathbb{Z}/n\mathbb{Z}$ to each sheet of the diagram. Refer to [1] for a diagram of a knotted surfaces.

Lemma 7.1. Let $K$ be an 11-colorable ribbon 2-knot. For each set $S = \{1, 4, 6, 7, 8\}$ or $\{0, 4, 6, 7, 8\}$, there is an 11-colored diagram $(\mathcal{D}, C)$ of $K$ which satisfies the following.

(i) Every double point circle has a neighborhood as shown in Figure 23, and all the sheets of the diagram other than the small shaded disks are colored by $S$.

(ii) While the color $2a - b$ of the shaded disk may not belong to $S$, the pair $(a, b)$ must satisfy $2b - a \in S$.

Proof. Let $A$ be a virtual arc which presents $K$ [16]. Since $K$ is 11-colorable, so is $A$. Then there is an 11-colored diagram $(\mathcal{D}, C)$ of $A$ with $\text{Im}(C) = S$ by a similar argument in the proof of Theorems 1.1. The diagram of $K$ associated to $(\mathcal{D}, C)$ is the desired one [13, 17].

Theorem 7.2. Any 11-colorable ribbon 2-knot satisfies the following.

(i) There is an 11-colored diagram $(\mathcal{D}_1, C_1)$ of $K$ with $\text{Im}(C_1) = \{1, 4, 6, 7, 8\}$.

(ii) There is an 11-colored diagram $(\mathcal{D}_2, C_2)$ of $K$ with $\text{Im}(C_2) = \{0, 4, 6, 7, 8\}$.

Proof. (i) We may assume that $(\mathcal{D}, C)$ satisfies Lemma 7.1 for $S = \{1, 4, 6, 7, 8\}$. In the left of Figure 23, the shaded disk is colored $2b - a$. The pair $(a, b)$ with $a, b, 2b - a \in S$ and $2a - b \notin S$ is one of the following:

$$(a, b) = (4, 1), (4, 7), (1, 7), (1, 6), (7, 6), (7, 8), (6, 8), (6, 4), (8, 4), (8, 1).$$
In fact, each edge \( \{x|y|z\} \) in the palette graph \( G(S) \) produces such two pairs \((y,x)\) and \((y,z)\).

First, we consider the case \((a,b) = (4,1)\), where the shaded sheet is colored \(9\). There is an \(8\)-sheet in \((D,C)\). We pull the \(8\)-sheet toward the \(9\)-sheet without introducing new double points and deform the diagram as shown in the left of Figure 24 to remove the \(9\)-sheet. We remark that the figure shows a cross-section of the neighborhood of the \(9\)-sheet. Next, we consider the case \((a,b) = (4,7)\), where the shaded sheet is colored \(10\). We deform the horizontal \(4\)-sheet by surrounding the \(10\)-sheet, that reduces the case \((a,b) = (4,1)\). See the right of the figure.

Let \( f : \mathbb{Z}/11\mathbb{Z} \to \mathbb{Z}/11\mathbb{Z} \) be the affine map defined by \( f(x) = 9x + 9 \). Since we have
\[
f(1) = 7, \quad f(4) = 1, \quad f(6) = 8, \quad f(7) = 6, \quad \text{and} \quad f(8) = 4,
\]
the cases \((a,b) = (1,7), (7,6), (6,8), \) and \((8,4)\) are obtained from \((a,b) = (4,1)\) by applying \( f \) repeatedly, and the cases \((a,b) = (1,6), (7,8), (6,4), \) and \((8,1)\) are obtained from \((a,b) = (4,7)\) similarly.

(ii) We may assume that \((D,C)\) satisfies Lemma 7.1 for \( S = \{0, 4, 6, 7, 8\} \). The pair \((a,b)\) with \(a, b, 2b − a \in S\) and \(2a − b \not\in S\) is one of the following:
\[
(a,b) = (4,8), \quad (7,6), \quad (7,8), \quad (6,8), \quad (6,4), \quad \text{and} \quad (0,7).
\]
In fact, each edge \( \{x|y|z\} \) in the palette graph \( G(S) \) produces such two pairs \((y, x)\) and \((y, z)\) other than \((4, 8)\) from \{0|4|8\} and \((0, 4)\) from \{4|0|7\}.

For the case \((a, b) = (4, 8)\), we deform the horizontal 4-sheet by surrounding the shaded 1-sheet as shown in the left of Figure 25 so that we can remove the 1-sheet. The case \((a, b) = (0, 7)\) can be similarly proved. See the right of the figure.

For the case \((a, b) = (7, 6)\), we pull a 0-sheet and deform the diagram as shown in the left of Figure 26. Then we can remove the 5-sheet without introducing new colors. For the case \((a, b) = (7, 8)\), we first deform the horizontal 7-sheet by surrounding the shaded 9-sheet, that reduces to the case \((a, b) = (7, 6)\).

For the case \((a, b) = (6, 8)\), we pull a 7-sheet and surround the shaded 10-sheet by the 7-sheet as shown in the left of Figure 27 so that the color 10 is removed.
For the case \((a, b) = (6, 4)\), we pull a 7-sheet toward the shaded 2-sheet and deform the horizontal 6-sheet to surround the 2-sheet. Then this case reduces to the case \((a, b) = (6, 8)\). □

![Figure 27.](image)

For a \(p\)-colorable 2-knot \(K\), we denote by \(C_p(K)\) the minimum number of \(#\text{Im}(C)\) for all non-trivially \(p\)-colored diagrams \((D, C)\) of \(K\) [17]. Then the following is an immediate consequence of Theorem 7.2.

**Corollary 7.3.** Any 11-colorable ribbon 2-knot \(K\) satisfies \(C_{11}(K) = 5\). □

The proof of the following is as same as that of Corollaries 3.2 and 6.3.

**Corollary 7.4.** For any 11-colorable ribbon 2-knot \(K\) and \(a \not\equiv b \in \mathbb{Z}/11\mathbb{Z}\), there are 11-colored diagrams \((D_1, C_1)\) and \((D_2, C_2)\) of \(K\) with

\[
\text{Im}(C_1) = \{a, b, 3a + 9b, 10a + 2b, 6a + 6b\}, \quad \text{and} \quad \text{Im}(C_2) = \{a, b, 5a + 7b, 2a + 10b, 10a + 2b\}.
\]

□

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