Abstract. Sommese has conjectured a classification of smooth projective varieties $X$ containing, as an ample divisor, a $\mathbb{P}^d$-bundle $Y$ over a smooth variety $Z$. This conjecture is known if $d > 1$, if $\dim(X) \leq 4$, or if $Z$ admits a finite morphism to an Abelian variety. We confirm the conjecture if the Picard rank $\rho(Z) = 1$, or if $Z$ is not uniruled. In general we reduce the conjecture to a conjectural characterization of projective space: namely that if $W$ is a smooth projective variety, $E$ is an ample vector bundle on $W$, and $\text{Hom}(E, T_W) \neq 0$, then $W \simeq \mathbb{P}^n$.

1. Introduction

Beltrametti and Sommese give a conjectural classification of smooth projective varieties $X$ containing a $\mathbb{P}^d$-bundle as an ample divisor [4, Conjecture 5.5.1]. The main goal of this paper is to prove this conjecture in the case where $X$ has minimal Picard rank,

$$\rho(X) = 2.$$ 

Throughout the paper we work over $\mathbb{C}$; the phrase “$\mathbb{P}^d$-bundle,” will be taken to mean a $\mathbb{P}^d$-bundle locally trivial in the analytic topology.

The conjecture is:

Conjecture 1. Let $X$ be a smooth projective variety and $Y \subset X$ a smooth ample divisor. Suppose that $p : Y \to Z$ is a morphism exhibiting $Y$ as a $\mathbb{P}^d$-bundle over a $b$-dimensional manifold $Z$. Then one of the following holds:

1. $X \simeq \mathbb{P}^3$, $Y \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is a smooth quadric, and $p$ is one of the projections to $\mathbb{P}^1$.
2. $X \simeq Q^3 \subset \mathbb{P}^4$ is a smooth quadric threefold, $Y \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is a hyperplane section, and $p$ is a projection to one of the factors.
3. $Y \simeq \mathbb{P}^1 \times \mathbb{P}^b$, $Z \simeq \mathbb{P}^b$, $p : Y \to Z$ is the projection to the second factor, and $X$ is the projectivization of an ample vector bundle $E$ on $\mathbb{P}^1$.
4. $X \simeq \mathbb{P}(E)$ for an ample vector bundle $E$ on $Z$, and $\mathcal{O}_X(Y) \simeq \mathcal{O}_{\mathbb{P}(E)}(1)$ (i.e. $Y$ is a fiberwise hyperplane).
Sommese has proven Conjecture 1 in the case where $d \geq 2$ (see e.g. [4, Theorem 5.5.2]). The conjecture has also been proven under the assumption that $d = 1$ and $b := \dim(Z) \leq 2$, due to the work of several authors (see e.g. [3, Theorem 7.4] and the references therein). We prove the conjecture in the case where

$$\rho(Z) = 1,$$

(if $\dim(X) \geq 4$, this is equivalent to $\rho(X) = 2$) and in general reduce it to a plausible conjectural improvement of a result of Andreatta–Wisniewski [1], namely

**Conjecture 2.** Let $X$ be a smooth projective variety and $\mathcal{E}$ an ample vector bundle on $X$. If

$$\text{Hom}(\mathcal{E}, T_X) \neq 0,$$

then $X \cong \mathbb{P}^n$.

We also prove the Conjecture 1 in the case that $Z$ is not uniruled.

**Remark 3.** By [1], the existence of a map $\mathcal{E} \to T_X$ of constant rank implies $X \cong \mathbb{P}^n$; likewise, [2, Corollary 4.3] proves the conjecture if $\rho(X) = 1$. One may also use the methods of [2, Section 4] to prove the conjecture if there exists a map $\mathcal{E} \to T_X$ generically of maximal rank.

The idea of our argument is to show (via an analysis of the deformation theory of the map $p : Y \to Z$) that either $p$ extends to a map $\tilde{p} : X \to Z$ (using results of [7,8]), or there is an ample vector bundle $\mathcal{E}$ on $Z$ and a map $\mathcal{E} \to T_Z$. In the former case, we are done by work of Sommese; in the latter case, we may apply Conjecture 2 to proceed.

**2. The Proof**

We first show:

**Lemma 4.** As before, let $X$ be a smooth projective variety, $Y \subset X$ a smooth ample divisor, and $p : Y \to Z$ a $\mathbb{P}^1$-bundle. Let $\tilde{Y}$ be the formal scheme obtained by completing $X$ at $Y$. If $p : Y \to Z$ does not extend to a morphism $\tilde{p} : \tilde{Y} \to Z$, then there exists an ample vector bundle $\mathcal{E}$ on $Z$ and a non-zero morphism $\mathcal{E} \to T_Z$.

**Proof.** Let $\mathcal{I}_Y$ be the ideal sheaf of $Y$, and let $Y_n$ be the subscheme of $X$ cut out by $\mathcal{I}_Y^n$. Then the obstruction to extending a map

$$p_n : Y_n \to Z$$

(1)

to a map

$$Y_{n+1} \to Z$$

(2)

lies in

$$\text{Ext}^1(p^*\Omega^1_Z, \mathcal{I}_Y^n / \mathcal{I}_Y^{n+1}) = H^1(Y, p^*T_Z \otimes \mathcal{I}_Y^n / \mathcal{I}_Y^{n+1}).$$
(See e.g. [8, Theorem 4.3] for this deformation-theoretic computation.) This last is equal to

$$H^1(\mathcal{R}p_*(p^*T_Z \otimes J^n_Y / J^{n+1}_Y))$$

which is the same as

$$H^1(T_Z \otimes R^1p_*J^n_Y / J^{n+1}_Y)$$

by the projection formula. As $J^n_Y / J^{n+1}_Y$ is anti-ample, this last equals

$$H^0(T_Z \otimes R^1p_*J^n_Y / J^{n+1}_Y).$$

Applying Serre duality, we see that this is the same as

$$H^0(T_Z \otimes (p_*(\mathcal{O}(nY)|_Y \otimes \omega_{Y/Z}))^\vee) \cong \text{Hom}(p_*(\omega_{Y/Z} \otimes \mathcal{O}(nY)|_Y), T_Z).$$

But by [9, Theorem 1.2],

$$p_*(\omega_{Y/Z} \otimes \mathcal{O}(nY)|_Y)$$

is either zero or ample. Thus either the problem of extending $p$ to $\widehat{Y}$ is unobstructed, or the obstruction is a non-zero map from an ample vector bundle $\mathcal{E}$ on $Z$ to $T_Z$, as desired.

We will also require:

**Lemma 5.** Let $X$ be a smooth projective variety of dimension at least 3, and $Y \subset X$ an ample divisor. Let $Z$ be a smooth variety with $\dim(Z) < \dim(Y)$. Then the restriction map

$$\text{Hom}(X, Z) \to \text{Hom}(\widehat{Y}, Z)$$

is a bijection. Here $\widehat{Y}$ is, as before, the formal scheme obtained by completing $X$ at $Y$.

**Proof.** This is a combination of two results from [8]. First, by [8, Corollary 2.10] or [7, Corollary 2.2.3], applied to the projection $X \times Z \to X$, a map $p : \widehat{Y} \to Z$ extends uniquely to some Zariski-open neighborhood $U$ of $Y$. Second, by [8, Corollary 3.3] or [7, Corollary 3.1.3], this rational map to $Y$ is in fact regular.

**Corollary 6.** Let $X, Y, Z, p$ be as in Conjecture 1. Suppose that either

1. $Z$ is not uniruled, or
2. $\rho(Z) = 1$.

Then Conjecture 1 is true for $X, Y, Z, p$. 

Proof. Without loss of generality, $p$ has relative dimension 1 (i.e. it exhibits $Y$ as a $\mathbb{P}^1$-bundle over $Z$) as the case of relative dimension greater than 1 is already known [4, Theorem 5.5.2]. We may also assume $\dim(Z) > 2$, as again, if $\dim(Z) \leq 2$, the result is already known [3, Theorem 7.4].

By Lemma 4, either $p$ extends to a map $\hat{p} : \hat{Y} \to Z$ or $T_Z$ contains an ample subsheaf, namely the image of $\mathcal{O}$ from Lemma 4. In the former case, the map $p$ extends to a map $\tilde{p} : X \to Z$ by Lemma 5 and we are done by [3, Theorem 5.5(ii)] (in particular, we are in case (4) of the conjecture). In the latter case, we consider the situations

1. $Z$ not uniruled, or
2. $\rho(Z) = 1$

separately.

(1) Suppose $Z$ is not uniruled. Then $T_Z$ contains no ample subsheaves by a result of Miyaoka (see e.g. [6, IV.1.16]), so we have a contradiction.

(2) Alternately, suppose $\rho(Z) = 1$. Then as $T_Z$ contains an ample subsheaf, by [2, Corollary 4.3], $Z \cong \mathbb{P}^n$. By [5, Theorem 2.1] (using that $\dim(Z) > 2$) we conclude the result, namely that we are in case (3) of the conjecture. □

Corollary 7. Suppose that Conjecture 2 is true. Then Conjecture 1 holds as well.

Proof. This is the same argument as in the $\rho(Z) = 1$ case above, replacing the reference to [2] with Conjecture 2. □

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