On the Diagonal Susceptibility of the 2D Ising Model

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Abstract
We consider the diagonal susceptibility of the isotropic 2D Ising model for temperatures below the critical temperature. For a parameter $k$ related to temperature and the interaction constant, we extend the diagonal susceptibility to complex $k$ inside the unit disc, and prove the conjecture that the unit circle is a natural boundary.

I. Introduction

For the 2D Ising model \cite{15,16,23}, after the zero-field free energy \cite{20} and the spontaneous magnetization \cite{21,28}, the most important zero-field thermodynamic quantity is the magnetic susceptibility $\chi$. Since the free energy is known only in zero magnetic field, the susceptibility is usually studied through its relation with the zero-field spin-spin correlation function:

$$\beta^{-1} \chi = \sum_{M,N \in \mathbb{Z}} \{ \langle \sigma_{0,0} \sigma_{M,N} \rangle - M^2 \}$$  \hspace{1cm} (1)

where $\beta = (k_B T)^{-1}$, $T$ is temperature, $k_B$ is Boltzmann’s constant and $\mathcal{M}$ is the spontaneous magnetization. If $T_c$ denotes the critical temperature, we recall that for the isotropic 2D Ising model, i.e. horizontal and vertical interaction constants have the same value $J$, the spontaneous magnetization is given for $T < T_c$ by

$$\mathcal{M} = (1 - k^2)^{1/8}$$  \hspace{1cm} (2)

where $k := (\sinh 2/\beta J)^{-1/2}$ and $\mathcal{M}$ is zero for $T > T_c$. (For $0 < T < T_c$ we have $0 < k < 1$.)
The analysis of $\chi = \chi(T)$ in the neighborhood of the critical temperature $T_c$ has a long history. We refer the reader to McCoy et al. [17] for a review of these developments. The analysis of $\chi$ for complex temperatures was initiated by Guttmann and Enting [13] and by Nickel [18, 19]. (For further developments see [10, 22].) Nickel’s analysis takes as its beginning the (commonly called) form-factor or particle expansion of the spin-spin correlation function [27]. For $T < T_c$ this expansion is an infinite sum whose $n$th summand is a $2n$-dimensional integral. From an asymptotic analysis of these integrals, Nickel was led to conjecture that $|k| = 1$ is a natural boundary for $\chi$. As Nickel himself noted, the analysis is nonrigorous since one must show that there are no cancellations of singularities in the sum. This has turned out to be a difficult problem to resolve rigorously.

In Boukraa et al. [7], these authors, building on results of [14], introduce a simplified model for $\chi$, called the diagonal susceptibility $\chi_d$, which is defined by having “a magnetic field which acts only on one diagonal of the lattice.” (See [2] for further developments.) Thus the analogue of (1) is

$$
\beta^{-1} \chi_d = \sum_{N \in \mathbb{Z}} \left\{ \langle \sigma_{0,0} \sigma_{N,N} \rangle - M^2 \right\}.
$$

In this paper we consider $\chi_d$ only for $T < T_c$, in which case $k < 1$. Then we extend $\chi_d$ to $k$ complex with $|k| < 1$. Using the Toeplitz determinant representation of the diagonal correlations [15, 24], we first derive the known representation of $\chi_d$ in terms of a sum of multiple integrals $S_n$. The derivation is different from those in [8, 9, 14] and [26]. As in [26] we use the identity of Geronimo-Case [12] and Borodin-Okounkov [5] relating a Toeplitz determinant to the Fredholm determinant of a product of Hankel operators. (For simplified proofs of the GCBO formula, see [3, 6].) But here we go from there to the multiple integral representation directly using a general identity for the integral of a product of determinants [11] (see eqn. (1.3) in [25]). For further background on the relationship between Toeplitz determinants and Ising correlations, see [4, 11].

In Section 4 we show that for each root of unity $\epsilon \neq \pm 1$ a certain derivative of a certain $S_n$ is unbounded as $k \to \epsilon$ radially, while the same derivative of the sum of the other $S_n$ remains bounded (Lemma 4). Thus, the unit circle $|k| = 1$ is a natural boundary for $\chi_d$. This proves the conjecture by Boukraa et al. [7]. We note that in [7] the authors present an argument that the singularity of $S_n$ at an $n$th root of unity $\epsilon$ is of the form $(k - \epsilon)^{2n^2 - 1} \log(k - \epsilon)$. Lemma 2 in Section IV formalizes this statement and fills in details of the proof.
II. Toeplitz determinant representation

It was shown in [15, 24] that for \( N > 1 \) the diagonal correlation has a representation as an \( N \times N \) Toeplitz determinant:

\[
\langle \sigma_{0,0} \sigma_{N,N} \rangle = \text{det} \left( \varphi_{m-n} \right)_{1 \leq m,n \leq N}.
\]

Here

\[
\varphi(\xi) = \left[ \frac{1 - k\xi^{-1}}{1 - k\xi} \right]^{1/2},
\]

and

\[
\varphi_m = \frac{1}{2\pi i} \int \varphi(\xi) \xi^{-m-1} d\xi,
\]

with integration over the unit circle. (We have \( \langle \sigma_{0,0}^2 \rangle = 1 \).)

As in [26] we invoke the formula of Geronimo-Case [12] and Borodin-Okounkov [5] to write the Toeplitz determinant in terms of the Fredholm determinant of a product of Hankel operators. We have \( \varphi(\xi) = \varphi_+(\xi) \varphi_-(\xi) \), where

\[
\varphi_+(\xi) = (1 - k\xi)^{-1/2} \quad \text{and} \quad \varphi_-(\xi) = (1 - k\xi^{-1})^{1/2}.
\]

Since \( |k| < 1 \) these extend analytically inside and outside the unit circle, respectively. The square roots are determined by \( \varphi_+(0) = \varphi_-(\infty) = 1 \).

The Hankel operator \( H(\psi) \) is the operator on \( \ell^2(\mathbb{Z}^+) \) with kernel \( (\psi_{i+j+1})_{i,j \geq 0} \), where \( \psi_m \) given in analogy with [14]. The operator \( H_N(\psi) \) has kernel \( (\psi_{N+i+j+1}) \).

Using \( \varphi_{\pm}(\xi) = 1/\varphi_{\mp}(\xi^{-1}) \), we find that the formula of GCBO gives

\[
\text{det} \left( \varphi_{m-n} \right)_{1 \leq m,n \leq N} = M^2 \text{ det} \left( I - H_N \left( \frac{\varphi_-}{\varphi_+} \right) H_N \left( \frac{\varphi_+}{\varphi_-} \right) \right).
\]

Thus, if we define

\[
\Lambda(\xi) = \frac{\varphi_-(\xi)}{\varphi_+(\xi)} = \sqrt{(1 - k\xi)(1 - k/\xi)}, \quad K_N = H_N(\Lambda) H_N(\Lambda^{-1}),
\]

then

\[
\beta^{-1} \chi_d = 1 - M^2 + 2M^2 \sum_{N=1}^{\infty} [\text{det}(I - K_N) - 1] = 1 + M^2 (2S - 1),
\]

where

\[
S = \sum_{N=1}^{\infty} [\text{det}(I - K_N) - 1].
\]

In what follows we extend \( \Lambda \) to be holomorphic in the complex plane cut along \([0,k] \cup [k^{-1}, \infty] \).
III. Formula for $S$

We use a slightly different notation for Hankel operators here.

**Proposition.** Let $H_N(du)$ and $H_N(dv)$ be two Hankel matrices acting on $\ell^2(\mathbb{Z}^+)$ with $i, j$ entries

\[
\int x^{N+i+j} du(x), \quad \int y^{N+i+j} dv(y),
\]

respectively, where $u$ and $v$ are measures supported inside the unit circle. Set $K_N = H_N(du) H_N(dv)$. Then

\[
\sum_{N=1}^{\infty} \left[ \det(I - K_N) - 1 \right] = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \int \cdots \frac{1}{1 - \prod_i x_i y_i} \left( \det \left( \frac{1}{1 - x_i y_j} \right) \right)^2 \prod_i du(x_i) dv(y_i),
\]

where indices in the integrand run from 1 to $n$.

**Proof.** The Fredholm expansion is

\[
\det(I - K_N) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \sum_{p_1, \ldots, p_n \geq 0} \det(K_N(p_i; p_j)).
\]

Therefore its suffices to show that

\[
\sum_{N=1}^{\infty} \sum_{p_1, \ldots, p_n \geq 0} \det(K_N(p_i; p_j))
\]

\[
= \frac{1}{n!} \int \cdots \int \frac{1}{1 - \prod_i x_i y_i} \left( \det \left( \frac{1}{1 - x_i y_j} \right) \right)^2 du(x_1) \cdots du(x_n) dv(y_1) \cdots dv(y_n).
\]

We have

\[
K_N(p_i, p_j) = \int \int \frac{x^{N+p_i} y^{N+p_j}}{1 - xy} \ du(x) \ dv(y).
\]

It follows by a general identity [[1]] (eqn. (1.3) in [25]) that

\[
\det(K_N(p_i, p_j)) = \frac{1}{n!} \int \cdots \int \det(x_i^{N+p_i}) \ det(y_i^{N+p_j}) \prod_i \frac{1}{1 - x_i y_i} \prod_i du(x_i) dv(y_i)
\]

\[
= \frac{1}{n!} \int \cdots \int \left( \prod_i x_i y_i \right)^N \ det(x_i^{p_j}) \ det(y_i^{p_j}) \prod_i \frac{1}{1 - x_i y_i} \prod_i du(x_i) dv(y_i).
\]

Summing over $N$ gives
\[
\sum_{N=1}^{\infty} \det(K_N(p_i, p_j)) =
\frac{1}{n!} \int \cdots \int \frac{\prod_i x_i y_i}{1 - \prod_i x_i y_i} \det(x_i^{p_j}) \det(y_i^{p_j}) \prod_i \frac{1}{1 - x_i y_i} \prod_i du(x_i) dv(y_i).
\]

(Interchanging the sum with the integral is justified since the supports of \(u\) and \(v\) are inside the unit circle.)

Now we sum over \(p_1, \ldots, p_n \geq 0\). Using the general identity again (but in the other direction) gives

\[
\sum_{p_1, \ldots, p_n \geq 0} \det(x_i^{p_j}) \det(y_i^{p_j}) = n! \det \left( \sum_{p \geq 0} x_i^p y_j^p \right) = n! \det \left( \frac{1}{1 - x_i y_j} \right).
\]

We almost obtained the desired result. It remain to show that

\[
\det \left( \frac{1}{1 - x_i y_j} \right) \prod_i \frac{1}{1 - x_i y_i},
\]

which we obtain in the integrand, may be replaced by

\[
\frac{1}{n!} \left( \det \left( \frac{1}{1 - x_i y_j} \right) \right)^2.
\]

This follows by symmetrization over the \(x_i\). (The rest of the integrand is symmetric.) For a permutation \(\pi\), replacing the \(x_i\) by \(x_{\pi(i)}\) multiplies the determinant in (8) by \(\text{sgn} \, \pi\), so to symmetrize we replace the other factor by

\[
\frac{1}{n!} \sum_{\pi} \text{sgn} \, \pi \frac{1}{1 - x_{\pi(i)} y_i} = \frac{1}{n!} \det \left( \frac{1}{1 - x_i y_j} \right).
\]

Thus, symmetrizing (8) gives (9). \(\square\)

We apply this to the operator \(K_N = H_N(\Lambda) \, H_N(\Lambda^{-1})\) given by (5). The matrix for \(H_N(\Lambda)\) has \(i, j\) entry

\[
\frac{1}{2\pi i} \int \Lambda(\xi) \, \xi^{-N-i-j-2} \, d\xi,
\]

where the integration may be taken over a circle with radius in \((1, |k|^{-1})\). Setting \(\xi = 1/x\) and using \(\Lambda(1/x) = \Lambda(x)\) we see that the entries of \(H_N(\Lambda)\) are given as in (7) with

\[
du(x) = \frac{1}{2\pi i} \Lambda(x) \, dx,
\]

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and integration is over a circle $C$ with radius in $(\lvert k \rvert, 1)$. Similarly $H_N(\Lambda^{-1}) = H_N(v)$ where in (7)

$$d\psi(y) = \frac{1}{2\pi i} \Lambda(y)^{-1} dy,$$

with integration over the same circle $C$.

Hence the Proposition gives

$$S = \sum_{n=1}^{\infty} S_n,$$

with all integrations over $C$.

We deform $C$ to the contour back and forth along the interval $[0, k]$, and then make the substitutions $x_i \to k x_i$, $y_i \to k y_i$. We obtain

$$S_n = \frac{(-1)^n}{(n!)^2} \frac{1}{(2\pi i)^{2n}} \int \cdots \int \prod x_i y_i \left( \det \left( \frac{1}{1 - x_i y_j} \right) \right)^2 \prod \frac{\Lambda(x_i)}{\Lambda(y_i)} \prod dx_i dy_i,$$

where we have set

$$\kappa = k^2, \quad \Lambda_1(x) = \sqrt{\frac{(1 - x)(1 - \kappa x)}{x}}.$$

Using the fact that the determinant in the integrand is a Cauchy determinant we obtain the alternative expression

$$S_n = \frac{1}{(n!)^2} \frac{\kappa^{2n}}{\pi^{2n}} \int \cdots \int \prod x_i y_i \left( \det \left( \frac{1}{1 - \kappa x_i y_j} \right) \right)^2 \prod \frac{\Lambda_1(x_i)}{\Lambda_1(y_i)} \prod dx_i dy_i,$$

where $\Delta(x)$ and $\Delta(y)$ are Vandermonde determinants. Clearly, $S_n$ is holomorphic in $\kappa$ for $\lvert \kappa \rvert < 1$. It is straightforward to prove that the sum (10) converges uniformly in $\kappa$ for $\lvert \kappa \rvert \leq r$ for all $0 < r < 1$; and hence, $S$ is holomorphic in the $\kappa$ unit disc.

IV. Natural boundary

**Theorem.** The unit circle $\lvert \kappa \rvert = 1$ is a natural boundary for $S$.

There will four lemmas. In these, $\epsilon \neq 1$ will be an $n$th root of unity and we consider the behavior of $S$ as $\kappa \to \epsilon$ radially.
For $\ell \geq 0$ we use the representation (12) and look at
\[
\int_0^1 \cdots \int_0^1 \frac{\prod_i x_i y_i}{(1 - \kappa^n \prod_i x_i y_i)^{\ell+1}} \frac{\Delta(x)^2 \Delta(y)^2}{\prod_{i,j} (1 - \kappa x_i y_j)^2} \prod_i \frac{\Lambda_1(x_i)}{\Lambda_1(y_i)} \prod_i dx_i dy_i,
\]
where all indices run from 1 to $n$. This will be the main contribution to $d^\ell S_n/d\kappa^\ell$.

**Lemma 1.** The integral (13) is bounded when $\ell < 2n^2 - 1$ and it is of the order $\log(1 - |\kappa|)^{-1}$ when $\ell = 2n^2 - 1$.

**Proof.** First we establish the first part of the statement. The numerator in the first factor is bounded and the denominator in the second factor is bounded away from zero as $\kappa \to \epsilon$ since $\epsilon \neq 1$.

If $\prod_i x_i y_i < 1 - \delta$ then the rest of the integrand is bounded except for the last quotient, and the integral of that is $O(1)$.

If $\prod_i x_i y_i > 1 - \delta$ then each $x_i, y_i > 1 - \delta$ and the integrand has absolute value at most a constant times
\[
\frac{\Delta(x)^2 \Delta(y)^2}{|\kappa^n - \prod_i x_i y_i|^\ell+1} \prod_i \sqrt{\frac{1 - x_i}{1 - y_i}}.
\]

We assumed that $\kappa \to \epsilon$ along a radius, so $\kappa^{-n} > 1$. Therefore we get an upper bound if we replace $\kappa^{-n}$ by 1. Then in the integral we make the substitutions $x_i = 1 - \xi_i, y_i = 1 - \eta_i$ (so $\xi_i, \eta_i < \delta$) and we obtain
\[
\frac{\Delta(\xi)^2 \Delta(\eta)^2}{(1 - \prod_i (1 - \xi_i)(1 - \eta_i))^\ell+1} \prod_i \sqrt{\frac{\xi_i}{\eta_i}}.
\]

Whenever $z_i \in [0, 1]$ ($i = 1, \ldots, m$) we have $z_1 \cdots z_m \leq z_i$ for each $i$, and so averaging gives
\[
z_1 \cdots z_m \leq \left( \sum_j z_j \right)/m,
\]
and therefore
\[
1 - z_1 \cdots z_m \geq \sum_j (1 - z_j)/m.
\]

It follows that
\[
1 - \prod_i (1 - \xi_i)(1 - \eta_i) \geq \sum_i (\xi_i + \eta_i)/2n. \quad (14)
\]

Therefore the integrand above is at most $(2n)^{\ell+1}$ times
\[
\frac{\Delta(\xi)^2 \Delta(\eta)^2}{(\sum_i (\xi_i + \eta_i))^\ell+1} \prod_i \sqrt{\frac{\xi_i}{\eta_i}}.
\]

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This is homogeneous of degree $2n(n-1) - \ell - 1$. We first integrate over the region $\sum_i (\xi_i + \eta_i) = r$ and then over $r$. The resulting integral is at most a constant times
\[
\int_0^{2n\delta} r^{2n^2 - \ell - 2} \, dr.
\]
This is finite when $\ell < 2n^2 - 1$, and so the first statement of the lemma is established. We note that the $(2n-1)$-dimensional volume of the region $\sum_i (\xi_i + \eta_i) = 1$ is $1/\Gamma(2n)$, another nice factor which we can use if needed. But it won’t be.

We now consider the integral when $\ell = 2n^2 - 1$. As before, the integral over the region $\prod_i x_i y_i < 1 - \delta$ is $O(1)$, so we assume $\prod_i x_i y_i > 1 - \delta$. In particular each $x_i, y_i > 1 - \delta$. The factors $1 - \kappa x_i y_j$ in the second denominator equal $1 - \kappa (1 + O(\delta)) = (1 - \kappa)(1 + O(\delta))$ since $\kappa$ is bounded away from 1. From this we see that if we factor out $\kappa^{2n^3}$ from the first denominator and $(1 - \kappa)^{n^2}$ from the second, the integrand becomes
\[
\Delta(x)^2 \Delta(y)^2 \prod_i \sqrt{\frac{1 - x_i}{1 - y_i}} (1 + O(\delta)).
\]

We again make the substitutions $x_i = 1 - \xi_i$, $y_i = 1 - \eta_i$ and set $r = \sum_i (\xi_i + \eta_i)$. Then since $\prod_i (1 - \xi_i)(1 - \eta_i) = 1 - r + O(r^2)$ the integrand becomes
\[
\Delta(\xi)^2 \Delta(\eta)^2 \prod_i \sqrt{\frac{\xi_i}{\eta_i}} (1 + O(\delta)).
\]
The integration domain $\prod_i x_i y_i > 1 - \delta$ becomes $r + O(r^2) < \delta$, which is contained in $r < 2\delta$ and contains $r < \delta/2$. The integral without the $O(\delta)$ term is at least a constant times
\[
\int_0^{\delta/2} \frac{r^{2n^2 - 1}}{(\kappa^{-n} - 1 + 2r)^{2n^2}} \, dr,
\]
which is asymptotically a constant independent of $\delta$ times $\log(\kappa^{-n} - 1)^{-1}$ as $\kappa \to \epsilon$. Similarly the integral of the $O(\delta)$ term is at most a constant independent of $\delta$ times $\delta \log(\kappa^{-n} - 1)^{-1}$. Since $\delta$ is arbitrarily small, this proves the lemma. \qed

**Lemma 2.** We have
\[
\left( \frac{d}{d\kappa} \right)^{2n^2-1} S_n \approx \log(1 - |\kappa|)^{-1}.
\]

**Proof.** To compute the derivative of the integral in (12) one integral we get is a constant depending on $n$ times (13) with $\ell = 2n^2 - 1$. The other integrals are similar but in each the power in the denominator is less than $2n^2 - 1$ while we get extra factors obtained by differentiating the rest of the integrand for $S_n$. These factors are of the form $(1 - \kappa x_i y_i)^{-1}, (1 - \kappa x_i)^{-1},$ or $(1 - \kappa y_i)^{-1}$. By an obvious modification
of the first statement of Lemma 1 we see that these other integrals are all bounded. The lemma follows.

Lemma 3. If \( \epsilon^m \neq 1 \) then

\[
\left( \frac{d}{d\kappa} \right)^{2n^2 - 1} S_m = O(1).
\]

Proof. If \( \epsilon^m \neq 1 \) all integrands obtained by differentiating the integral in (12) are bounded as \( \kappa \to \epsilon \).

Lemma 4. We have

\[
\sum_{m>n} \left( \frac{d}{d\kappa} \right)^{2n^2 - 1} S_m = O(1).
\]

Proof. We shall show that for \( \kappa \) sufficiently close to \( \epsilon \) all integrals we get by differentiating the integral for \( S_m \) are at most \( A^m \), where \( A \) is some constant. Note that the value of \( A \) will change with each of its appearances. It may depend on \( n \), but not on \( m \). Because of the \( 1/(m!)^2 \) appearing in front of the integrals this will show that the sum is bounded.

As before, we first use (12) (with \( n \) replaced by \( m \)) and consider the integral we get when the first factor in the integrand is differentiated \( 2n^2 - 1 \) times. All indices in the integrands now run from 1 to \( m \).

First,

\[
|1 - \kappa^m \prod x_i y_i| = |\kappa^m| |\kappa^{-m} - \prod x_i y_i| \geq |\kappa|^m (1 - \prod x_i y_i).
\]

Next, \( |1 - \kappa x_i| \leq 2 \). Since \( y_i \in [0, 1] \) and \( \kappa \in [0, \epsilon] \) we also have \( \kappa y_i \in [0, \epsilon] \). Therefore \( |1 - \kappa y_i| \geq a \), where \( a = \text{dist}(1, [0, \epsilon]) \). Hence the integrand in (12) after differentiating the first factor has absolute value at most \( A^m \) times

\[
\frac{1}{(1 - \prod x_i y_i)^{2n^2}} \prod_{i,j} |1 - \kappa x_i y_j|^2 \prod_i \sqrt{\frac{1 - x_i y_i}{1 - y_i x_i}} \tag{15}
\]

Since we also have

\[
|1 - \kappa x_i y_j| \geq a, \tag{16}
\]

(15) is at most

\[
a^{-m^2} \frac{\Delta(x)^2 \Delta(y)^2}{(1 - \prod x_i y_i)^{2n^2}} \prod_i \sqrt{\frac{1 - x_i y_i}{1 - y_i x_i}}.
\]
With \(x_i = 1 - \xi_i, y_i = 1 - \eta_i\) and \(r = \sum_i (\xi_i + \eta_i)\) again, we first integrate over \(r < \delta\), where the small \(\delta\) will be chosen below. Using (14) again, we see that the integrand is at most \(A_m^m\) times

\[
a^{-m^2} \frac{\Delta(x)^2 \Delta(y)^2}{(\sum_i (\xi_i + \eta_i))^{2m^2}} \prod_i \sqrt{\xi_i/\eta_i}.
\]

(The factor \(m^{2n^2}\) coming from using (14) and a bound for \(\prod \sqrt{y_i/x_i}\) appearing in (15) were absorbed into \(A_m^m\).) When \(\xi_i, \eta_i < 1\) we have \(\Delta(x)^2, \Delta(y)^2 < 1\), so integrating with respect to \(r\) over \(r < \delta\), using homogeneity, gives at most a constant times

\[
a^{-m^2} \int_0^\delta r^{2m(m-1)-2n^2+2m-1} dr = a^{-m^2} \int_0^\delta r^{2m^2-2n^2-1} dr.
\]

(The integral of the last factor over \(r = 1\) equals \((\pi/2)^m/\Gamma(2m)\).) The integral is \(O(\delta^{2m^2})\) since \(m > n\), and so the above is exponentially small in \(m\) if we choose \(\delta^2 < a\).

There remains the integral over the region \(r > \delta\), and for this we use the representation (11). We are led to (15) with the second factor replaced by the absolute value of

\[
\left( \text{det} \left( \frac{1}{1 - \kappa x_i y_j} \right) \right)^2.
\]

From (14) we see that in this region the first factor in (15) is at most \((2m/\delta)^{2n^2}\). By (16) and the Hadamard inequality the square of the determinant has absolute value at most \(a^{-2m} m^m\). Therefore the integral over this region has absolute value at most

\[
\left( \frac{2m}{\delta} \right)^{2n^2} a^{-2m} m^m \int_0^1 \cdots \int_0^1 \prod_i \sqrt{1 - x_i y_i \over 1 - y_i x_i} \prod_i dx_i dy_i.
\]

The integral here is \(A_m^m\), and so we have shown that the integral in the region \(r > \delta\) is at most \(A_m^m m^m\).

This is a bound for only one term we get when we differentiate \(2n^2 - 1\) times the integrand for \(S_m\). The number of factors in the integrand involving \(\kappa\) is \(O(m^2)\) so if we differentiate \(2n^2 - 1\) times we get a sum of \(O(m^{4n^2})\) terms. In each of the other terms the denominator in the first factor has a power no larger than \(2n^2\) and at most \(2n^2\) extra factors appear which are of the form \((1 - \kappa x_i y_i)^{-1}\), \((1 - \kappa x_i)^{-1}\), or \((1 - \kappa y_i)^{-1}\). Each has absolute value at most \(a^{-1}\), so their product is \(O(1)\). It follows that we have the bound \(A_m^m m^m\) for the sum of these integrals. Lemma 4 is established.

\(\square\)

**Proof of the Theorem.** Suppose \(\kappa \to \epsilon\) radially, where \(\epsilon \neq 1\) is a root of unity. It is a primitive \(n\)-th root of unity for some \(n\). Then \(\epsilon^m \neq 1\) when \(m < n\) so Lemma 3
applies for these \( m \). Combining this with Lemma 4 and Lemma 2 gives

\[
\left( \frac{d}{d\kappa} \right)^{2n^2-1} S \approx \log(1 - |\kappa|)^{-1}.
\]

Therefore \( S \) cannot be analytically continued beyond any such \( \epsilon \), and these are dense in the unit circle. \( \Box \)

**Remark.** From the proofs of Lemma 3 and 4, with \( 2n^2 - 1 \) replaced by \( \ell \) in both, one can see that \( S \) extends to a \( C^\ell \) function of \( \kappa \) up to the boundary except for the \( m \)th roots of unity with \( m \leq \sqrt{(\ell + 1)/2} \). In particular, \( S \) extends to a function of class \( C^6 \) up to the boundary except for \( \kappa = 1 \).

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