pp Waves of Conformal Gravity with Self-Interacting Source

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Abstract: Recently, Deser, Jackiw and Pi have shown that three-dimensional conformal gravity with a source given by a conformally coupled scalar field admits \textit{pp} wave solutions. In this letter, we consider this model with a self-interacting potential preserving the conformal structure. A \textit{pp} wave geometry is also supported by this system and, we show that this model is equivalent to topologically massive gravity with a cosmological constant whose value is given in terms of the potential strength.

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1. Introduction

In three dimensions, the matrix gauge connection \( (A_\mu)^\alpha_{\beta} \) from which non-Abelian Chern-Simons Lagrangian is constructed can be replaced by the Christoffel connection \( \Gamma^\alpha_{\mu\beta} \). The resulting Chern-Simons action reads

\[
S(\Gamma) = \frac{1}{2\kappa} \int d^3x \epsilon^{\alpha\beta\gamma} \left( \frac{1}{2} \Gamma^\rho_{\alpha\sigma} \partial_\beta \Gamma^\sigma_{\gamma\rho} + \frac{1}{3} \Gamma^\rho_{\alpha\sigma} \Gamma^\sigma_{\beta\tau} \Gamma^\tau_{\gamma\rho} \right),
\]

where \( \kappa \) is a dimensionless constant. The variation of this action with respect to the metric leads to

\[
\delta S(\Gamma) = -\frac{1}{2\kappa} \int d^3x \sqrt{-g} C^{\mu\nu} \delta g_{\mu\nu},
\]

where \( C^{\mu\nu} \) is the so-called Cotton tensor\(^1\) whose expression after using the Bianchi identity is given by

\[
C^{\mu\nu} = \frac{1}{\sqrt{-g}} \epsilon^{\mu\alpha\beta} D_\alpha \left( R^\nu_{\beta} - \frac{1}{4} \delta^\nu_{\beta} R \right).
\]

Note that this tensor is symmetric, traceless and identically conserved.

Recently, Deser, Jackiw and Pi\(^1\) have considered a three-dimensional model with a conformal scalar field for which gravity is governed by the Cotton tensor. They found that this system supports a pp wave ansatz geometry where the source also behaves like a wave. They also showed that in a special frame their system is equivalent to topologically massive gravity\(^2\). In this letter, we extend this work by including a self-interacting potential which does not spoil the conformal invariance. We show that the resulting theory still admits pp wave solutions for which the source does not behave like a wave, in accordance with its self-interacting nature. In addition, we prove the equivalence of our system with topologically massive gravity with a cosmological constant whose value is given in term of the potential strength.

The paper is organized as follows. We first present the action of three-dimensional conformal gravity with source given by a conformally coupled scalar field with a conformally invariant self-interacting potential. Then, we solve the field equations for a pp wave ansatz, and we find two different classes of solutions depending on the ratio between the coupling constants. We establish the correspondence between this model and topologically massive gravity with a cosmological constant. Finally, exploiting this analogy, we derive explicit solutions of the later theory.

2. Three-dimensional conformal gravity with a self-interacting potential

We consider the three-dimensional conformal gravity with source given by a conformally invariant scalar field with self-interacting potential whose action reads

\(^1\)Our conventions are the following: the signature is \((- + +)\), \( \epsilon_{012} = +1 \), the Riemann tensor is \( R^\sigma_{\mu\nu\rho} = +\partial_\tau \Gamma^\sigma_{\mu\rho} - \ldots \) and the Ricci tensor is \( R_{\mu\nu} = +\partial_\tau \Gamma^\sigma_{\mu\nu} - \ldots \).
\[ S(\psi, g) = S(\Gamma) + I(\psi, g) = S(\Gamma) - \frac{1}{2} \int d^3 x \sqrt{-g} \left( g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi + \frac{1}{8} R \psi^2 + \lambda \psi^6 \right). \] (2.1)

Here the potential strength \( \lambda \) is a dimensionless constant and \( R \) stands for the scalar curvature. The associated field equations are

\[ C_{\mu\nu} = \kappa T_{\mu\nu} = \kappa \left( \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} \left( \partial_\alpha \psi \partial^\alpha \psi + \lambda \psi^6 \right) + \frac{1}{8} (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu + G_{\mu\nu}) \psi^2 \right), \] (2.2)

\[ \Box \psi - \frac{1}{8} R \psi - 3 \lambda \psi^5 = 0, \] (2.3)

where \( \Box = \nabla_\mu \nabla^\mu \). For this model, the gravity is governed only by the Cotton tensor and the matter source, in accordance with its conformal invariance, has a traceless energy-momentum tensor. We now show that equations (2.2-2.3) admit \( pp \) wave gravitational fields.

### 2.1 The \( pp \) wave solutions

The plane-fronted gravitational waves with parallel rays (\( pp \) waves) are characterized by the existence of a covariantly constant null vector field \( k^\mu \) (see e.g. [3]). This allows to write the geometry in the three-dimensional case as

\[ ds^2 = -F(u, y) du^2 - 2 du dv + dy^2, \] (2.4)

where \( k^\mu \partial_\mu = \partial_\nu \). The covariantly constant field is as well a Killing field and hence it is appropriate to demand the same symmetry on the matter field, which in turn implies that \( \psi = \psi(u, y) \). In fact, the same conclusion is reached assuming a full dependence on the scalar field, modulo the gravitational equations, as was carefully shown in Ref. [4] in the free case. As was pointed in the same reference, the only nonvanishing component of the Cotton tensor (1.2) for the above metric is \( C_{uu} = \frac{1}{2} \partial^3_{yyy} F \), which implies that all the components of the energy-momentum tensor must vanish except \( T_{uu} \). In particular, the vanishing of the \( yy \)-component gives

\[ T_{yy} = \frac{1}{8\sigma^2} \left[ (\partial_y \sigma)^2 - 4\lambda \right] = 0, \] (2.5)

where we have introduced \( \sigma = \psi^{-2} \) for convenience. For a nontrivial self-interaction, this equation is solved only for a strictly positive coupling constant \( \lambda > 0 \)

\[ \sigma(u, y) = 2\sqrt{\lambda} y + f(u), \] (2.6)

where \( f \) is an undetermined function of the retarded time. No other restriction follows from the energy-momentum tensor in order to specify the scalar field. We would like to point out that equation (8) also solves the scalar field equation (5) and, hence \( \psi = \sigma^{-1/2} \) is the general solution for the scalar field.
Note that in the absence of self-interaction \((\lambda = 0)\), Eq. (2.3) would imply that the scalar field just depends arbitrarily on the retarded time \(u\), Ref. [1]. This argument is no longer valid in the presence of a self-interaction potential since this later causes the field to be inhomogeneous on the spacelike direction orthogonal to the wave. It remains only to solve the \(uu\)–component of Eqs. (2.2) that reads

\[
\frac{1}{2} \partial^3_{y y} F = \frac{\kappa}{16} \left[ \partial_y \left( \frac{\partial_y F}{2\sqrt{\lambda y} + f} \right) + \frac{d^2 f}{du^2} \left( \frac{2}{(2\sqrt{\lambda y} + f)^2} \right) \right].
\]  

(2.7)

The general solution of this equation for a coupling constant \(\lambda \neq (\kappa/16)^2\) is

\[
F(u, y) = F_1(u) \left( 2\sqrt{\lambda y} + f \right)^{1 + \kappa/(16\sqrt{\lambda})} + F_2(u) \left( 2\sqrt{\lambda y} + f \right)^2 + \frac{1}{2\lambda} \frac{d^2 f}{du^2} \left( 2\sqrt{\lambda y} + f \right) + F_3(u),
\]

(2.8)

where \(F_1\), \(F_2\), and \(F_3\) are undetermined functions of the retarded time. In the derivation of the above result one of the terms appearing after the second integration is a power in \(y\) depending on \(\lambda\). For the special coupling constant \(\lambda = (\kappa/16)^2\) this power is \(-1\) and the third integration of the related term gives a logarithmic contribution. Hence, for \(\lambda = (\kappa/16)^2\) the solution is

\[
F(u, y) = \left[ F_1(u) \ln \left( \frac{\kappa}{8} y + f \right) + F_2(u) \right] \left( \frac{\kappa}{8} y + f \right)^2 + \frac{128}{\kappa^2} \frac{d^2 f}{du^2} \left( \frac{\kappa}{8} y + f \right) + F_3(u),
\]

(2.9)

with \(F_1\), \(F_2\), and \(F_3\) being again integration functions.

Since the curvature is just characterized by the only nontrivial component of the Ricci tensor \(R_{uu} = \frac{1}{2} \partial^2_{yy} F\), any linear dependence on \(y\) does not appear in the curvature which suggests that such dependence is removable by coordinate transformations. This is a common feature of the \(pp\) wave gravitational fields [3]. In our case the appropriated coordinate change is

\[
(u, v, y) \mapsto \left( u, v + \frac{1}{2\sqrt{\lambda}} \frac{df}{du} y + \frac{f}{4\lambda} \frac{df}{du}, - \int \frac{1}{8\lambda} \left( \frac{df}{du} \right)^2 - \frac{F_3}{2} \right) \), \quad y + \frac{f}{2\sqrt{\lambda}},
\]

(2.10)

which is equivalent to put \(f = 0 = F_3\) in the expressions (2.8) and (2.9). Finally, the \(pp\) wave solutions of conformal gravity with a self-interacting source are

\[
ds^2 = - \left[ F_1(u) y^{\kappa/(16\sqrt{\lambda}) - 1} + F_2(u) \right] y^2 du^2 - 2 dudv + dy^2, \quad \psi = \frac{1}{\sqrt{2\sqrt{\lambda} y}},
\]

(2.11)

for self-interaction coupling constant \(\lambda \neq (\kappa/16)^2\), and

\[
ds^2 = - \left[ F_1(u) \ln y + F_2(u) \right] y^2 du^2 - 2 dudv + dy^2, \quad \psi = \frac{\sqrt{\frac{8}{\kappa y}}}{\sqrt{2\sqrt{\lambda} y}},
\]

(2.12)

for the value \(\lambda = (\kappa/16)^2\), where in both cases we have redefined appropriately the undetermined functions. It is interesting to note that, contrary to the free case [4], the scalar field
does not depend on retarded time. In other words, the self-interaction breaks the wave-like behavior of the source present in the free case.

Since the integration of Eq. (2.7) is different for \( \lambda = 0 \), a natural fair question is to ask whether there is a link between our self-interacting solution at the zero \( \lambda \) limit and the free one of Ref. [1]. At the first sight it seems that the self-interacting solution (2.8) is singular at this limit. However, after redefining the arbitrary functions of Eq. (2.8) as

\[
F_1 = \tilde{F} f^{-(1 + \kappa/16\sqrt{\lambda})}, \quad F_2 = \frac{\alpha}{4\sqrt{\lambda}} f f - \frac{1}{4\lambda} \frac{d^2 f}{du^2}, \quad F_3 = \beta - \frac{\alpha}{4\sqrt{\lambda}} f - \frac{1}{4\lambda} f \frac{d^2 f}{du^2},
\]

the free case \([1]\) is exactly recovered in the zero \( \lambda \) limit, where \( \tilde{F} \), \( \alpha \), and \( \beta \) are the new arbitrary functions of the retarded time.\(^2\) A different situation occurs when one tries to eliminate the linear dependence in \( y \) in the \( pp \) wave geometries, since the coordinate change in the self-interacting case (2.10) is singular as \( \lambda \) goes to zero.

A study of Killing equations allows to conclude that for generic wave profiles \( F_i(u) \), \( i = 1, 2 \), the above geometries have only the original Killing field \( \partial_v \). There exists another Killing field for the special election \( F_i(u) = A_i/u^2 \), \( i = 1, 2 \), given by

\[ u \partial_u - v \partial_v, \quad (2.13) \]

whose commutator with \( \partial_v \) closes on \( \partial_v \). The corresponding isometry is the scaling \( (u, v, y) \mapsto (\alpha u, \alpha^{-1} v, y) \).

3. Correspondence to topologically massive gravity with a cosmological constant

We now establish the connection between the model previously studied and the topologically massive gravity [2] with a cosmological constant.

It is interesting to note that, up to a boundary term, in the frame \( g'_{\mu\nu} = (\psi/\psi_0)^4 g_{\mu\nu} \) the scalar field action \( I(\psi, g) \) can be seen as the Einstein-Hilbert action with a cosmological constant, i.e.,

\[
I(\psi, g) = -\frac{\psi_0^2}{16} \int \left[ R(g') - 2(-4\lambda \psi_0^4) \right] \sqrt{-g'} d^3 x.
\]

Here, \( \psi_0^2 \) is a constant with dimension of the inverse of the length in order to have dimensionless actions in both sides of this relation. For convenience, we choose this constant to be \( \psi_0^2 = 1/(\pi G) \) where \( G \) is the Newton’s constant. Moreover, since the Chern-Simons action (1.1) is conformally invariant, the full action \( S \) defined in (2.1) becomes in the new frame

\[
S = S(\Gamma) - \frac{1}{16\pi G} \int (R - 2\Lambda) \sqrt{-g} d^3 x,
\]

\(^2\)We thank R. Jackiw for providing us these redefinitions.
where we have dropped the primes and $\Lambda$ is the cosmological constant given in terms of the potential strength by

$$\Lambda = -\frac{4\lambda}{\pi^2 G^2}.$$  \hfill (3.2)

The field equation associated to (3.1) reproduce those of topologically massive gravity with a cosmological constant, i.e.,

$$\frac{1}{\mu} C_{\mu\nu} + G_{\mu\nu} + \Lambda g_{\mu\nu} = 0,$$  \hfill (3.3)

where the topological mass is expressed as $\mu = -\frac{\kappa}{8\pi G}$. Note that, in order to make contact with topologically massive gravity the signs of the topological mass and the dimensionless parameter $\kappa$ must be different. This ambiguity is irrelevant when the constants of the problem are not tied, since we are just mapping a solution of Eqs. (2.2-2.3) to one of Eqs. (3.3). Indeed, using the correspondence together with the $pp$ wave solutions (2.11) found previously for $\lambda > 0$ and $\lambda \neq (\kappa/16)^2$, it is straightforward to see that the following spacetimes

$$ds^2 = l^2 \left\{ - \left[ F_1(u)e^{-(\mu+1)y} + F_2(u) \right] du^2 - 2e^{-2y}dudv + dy^2 \right\},$$  \hfill (3.4)

are solutions of topologically massive gravity with a negative cosmological constant $\Lambda = -1/l^2$. As anticipated, the sign of the topological mass is irrelevant in this case and can be chosen to be positive. This last remark is no longer valid for the other $pp$ wave solution (2.12) since the potential strength $\lambda$ and the dimensionless constant $\kappa$ are not independent constants. Indeed, the correspondence to topologically massive gravity with a negative cosmological constant exists provided the topological mass to be negative and given by $\mu = -1/l$. In this case, the spacetime geometry given by

$$ds^2 = l^2 \left\{ - \left[ F_1(u)y + F_2(u) \right] du^2 - 2e^{-2y}dudv + dy^2 \right\},$$  \hfill (3.5)

is solution of equations (3.3) with a negative cosmological constant $\Lambda = -1/l^2$ and a negative topological mass.

To end with this section, we would like to stress that the solutions of topologically massive gravity obtained through the correspondence are no longer $pp$ waves, since these spacetimes do not admit covariantly constant null fields.

4. Discussion

Here, we have shown that $(2 + 1)$–dimensional conformal gravity with source given by a conformally invariant self-interacting scalar field admits $pp$ wave solutions. The corresponding scalar sources do not behave like a wave, since they are independent of the retarded time and inhomogeneous in the spacelike direction orthogonal to the wave. Obviously, this is a manifestation of its self-interacting character. We found just one particular wave profile
supporting an additional isometry of the $pp$ waves, which corresponds to a special scaling of
the retarded time and the null coordinate. We have established a correspondence between
this model and topologically massive gravity with a cosmological constant. Finally, using this
analogy we have obtained explicit solutions of the later theory for a negative cosmological
constant. It would be interesting to explore whether the conformal gravity supports other
physical interesting configurations.

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