Jordan Triple Higher \((\sigma, \tau)\)-Homomorphisms on Prime Rings

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Abstract

In this paper, the concept of Jordan triple higher \((\sigma, \tau)\)-homomorphisms on prime rings is introduced. A result of Herstein is extended on this concept from the ring \(R\) into the prime ring \(R'\). We prove that every Jordan triple higher \((\sigma, \tau)\)-homomorphism of ring \(R\) into prime ring \(R'\) is either triple higher \((\sigma, \tau)\)-homomorphism or triple higher \((\sigma, \tau)\)-anti-homomorphism of \(R\) into \(R'\).

Keywords: Jordan homomorphisms, triple homomorphism, Jordan triple higher \((\sigma, \tau)\)-homomorphism.

Introduction

The idea of Jordan homomorphism of rings initially appeared in Ancochea’s \([1]\) study of semi-automorphisms, the later investigated by Kaplansky, Jacobson and Rickart \([2, 3]\). Herstein \([4]\) studied Jordan homomorphisms in prime rings. He proved that a Jordan homomorphism onto prime ring of characteristic different from 2 and 3 is either a homomorphism or an anti-homomorphism. Bresar \([5]\) generalized Herstein’s work on semiprime rings.

Throughout this paper, \(R\) is a ring with the center \(Z(R)\) prime if \(aRb = (0)\) implies \(a = 0\) or \(b = 0\) with \(a, b \in R\), and is semiprime if \(aRa = (0)\) implies \(a = 0\). \(R\) is \(n\)-torsion free if \(na = 0; a \in R\), then \(a = 0\).

In this paper, we extend the result of Herstein to triple higher \((\sigma, \tau)\)-homomorphism and Jordan triple higher \((\sigma, \tau)\)-homomorphism. We show that every Jordan triple higher \((\sigma, \tau)\)-homomorphism, from prime ring \(R\) into prime ring \(R'\), is triple higher \((\sigma, \tau)\)-homomorphism or triple higher \((\sigma, \tau)\)-anti-homomorphism.

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2. Preliminaries

We begin by the following definition.

Definition 2.1. [3, 4, 5]
An additive mapping $\theta$ of a ring $R$ into a ring $R'$ is called,
(a) a homomorphism if $\theta(ab) = \theta(a)\theta(b)$ for all $a, b \in R$,
(b) anti-homomorphism if $\theta(ab) = \theta(b)\theta(a)$ for all $a, b \in R$,
(c) a Jordan homomorphism if $\theta(ab + ba) = \theta(a) \theta(b) + \theta(b)\theta(a)$ for all $a, b \in R$ and
(d) a Jordan triple homomorphism if $\theta(aba) = \theta(a)\theta(b)\theta(a)$ for all $a, b \in R$.

Obviously, every homomorphism or anti-homomorphism is a Jordan homomorphism and every Jordan homomorphism is Jordan triple homomorphism but the converse needs not to be true in general.

Definition 2.2. [6]
Let $\mathbb{N}$ be the set of natural numbers. A family of additive mappings $\theta = (\phi_i)_{i \in \mathbb{N}}$ of $R$ into $R'$ is called
(a) a higher homomorphism if for all $n \in \mathbb{N}, a, b \in R$,
$$\phi_n(ab) = \sum_{i=1}^{n} \phi_i(a) \phi_i(b),$$
(b) a higher anti-homomorphism if for all $n \in \mathbb{N}, a, b \in R$,
$$\phi_n(ab) = \sum_{i=1}^{n} \phi_i(b) \phi_i(a),$$
(c) a Jordan higher homomorphism if for all $n \in \mathbb{N}, a, b \in R$,
$$\phi_n(ab + ba) = \sum_{i=1}^{n} \phi_i(a) \phi_i(b) + \phi_i(b)\phi_i(a)$$
(d) a triple higher homomorphism if for all $n \in \mathbb{N}, a, b \in R$,
$$\phi_n(abc) = \sum_{i=1}^{n} \phi_i(a) \phi_i(b)\phi_i(c),$$
(e) a Jordan triple higher homomorphism if for all $n \in \mathbb{N}, a, b \in R$,
$$\phi_n(aba) = \sum_{i=1}^{n} \phi_i(a) \phi_i(b)\phi_i(a)\phi_i(a).$$

Definition 2.3. [7]
Let $\mathbb{N}$ be the set of natural numbers. A family of additive mappings $\theta = (\phi_i)_{i \in \mathbb{N}}$ of $R$ into $R'$ and $\sigma, \tau$ as two homomorphisms of $R$ is said to be
(a) a $(\sigma, \tau)$—higher homomorphism if for each $n \in \mathbb{N}$ and for all $a, b \in R$,
$$\phi_n(ab) = \sum_{i=1}^{n} \phi_i(\sigma^i(a)\phi_i(\tau^i(b))$$
(b) a $(\sigma, \tau)$—higher anti-homomorphism if for each $n \in \mathbb{N}$ and for all $a, b \in R$,
$$\phi_n(ab) = \sum_{i=1}^{n} \phi_i(\sigma^i(b)\phi_i(\tau^i(a))$$
(c) a Jordan $(\sigma, \tau)$—higher homomorphism if for each $n \in \mathbb{N}$ and for all $a, b \in R$,
$$\phi_n(ab + ba) = \sum_{i=1}^{n} \phi_i(\sigma^i(a)\phi_i(\tau^i(b)) + \phi_i(\sigma^i(b)\phi_i(\tau^i(a))$$
(d) a Jordan triple $(\sigma, \tau)$—higher homomorphism if for all $n \in \mathbb{N}, a, b \in R$,
$$\phi_n(aba) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a))$$
Definition 2.4.

Let \( \mathbb{N} \) be the set of natural numbers. A family of additive mappings \( \theta = (\phi_i)_{i \in \mathbb{N}} \) of \( R \) into \( R' \) and \( \sigma, \tau \) as two homomorphisms of \( R \) is said to be

(a) a triple \( (\sigma, \tau) \)-higher homomorphism if for all \( n \in \mathbb{N}, a, b \in R \),

\[
\phi_n(abc) = \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \phi_i(\sigma^i \tau^{n-i}(b)) \phi_i(\tau^i(c))
\]

(b) a triple \( (\sigma, \tau) \)-higher anti-homomorphism if for all \( n \in \mathbb{N}, a, b \in R \),

\[
\phi_n(abc) = \sum_{i=1}^{n} \phi_i(\sigma^i(c)) \phi_i(\sigma^i \tau^{n-i}(b)) \phi_i(\tau^i(a)).
\]

Now, we give an example of triple \( (\sigma, \tau) \)-higher homomorphism and Jordan triple \( (\sigma, \tau) \)-higher homomorphism.

Example 2.5:

Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a triple \( (\sigma, \tau) \)-higher homomorphism from \( R \) into \( R' \). Then for each \( n \in \mathbb{N} \) and for all \( a, b, c \in R \), we have:

\[
\phi_n(abc) = \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \phi_i(\sigma^i \tau^{n-i}(b)) \phi_i(\tau^i(c))
\]

Let \( T = R \times R \times R \) and \( T' = R' \times R' \times R' \). Then \( T \) and \( T' \) are rings. We define \( \theta' = (\phi'_i)_{i \in \mathbb{N}} \) to be a family of mappings from \( T \) to \( T' \) by:

\[
\phi'_n((a, b, c)) = (\phi_n(a), \phi_n(b), \phi_n(c))
\]

for all \( (a, b, c) \in T \).

Then \( \phi \) is a triple \( (\sigma, \tau) \)-higher homomorphism.

Let \( S \) be the subset \( \{ (a, a, a): a \in R \} \) of \( T \) and \( S' \) be the subset \( \{ (b, b, b): b \in R' \} \) of \( T' \). Then \( S \) and \( S' \) are rings and the family of mappings \( \theta' = (\phi'_i)_{i \in \mathbb{N}} \) from \( S \) to \( S' \) is defined in terms of the Jordan \( (\sigma, \tau) \)-higher homomorphism by

\[
\phi'_n((a, a, a)) = (\phi_n(a), \phi_n(a), \phi_n(a))
\]

for all \( (a, a, a) \in S \).

Then \( \phi \) is a Jordan triple \( (\sigma, \tau) \)-higher homomorphism from \( S \) to \( S' \).

Obviously, every triple \( (\sigma, \tau) \)-higher homomorphism or triple \( (\sigma, \tau) \)-higher anti-homomorphism is a Jordan triple \( (\sigma, \tau) \)-higher homomorphism but the converse needs not to be true in general.

In an earlier work[6], the author provided an example of Jordan higher homomorphism but not higher homomorphism on a ring. We extend it to triple \( (\sigma, \tau) \)-higher homomorphism on ring as follows.

Example 2.6.

Suppose that \( S \) is a ring with non-trivial involution \( * \), \( R = S \oplus S \oplus S \), \( a \in S \) such that \( a \in Z(S) \) and \( s_1 a s_2 = 0 \), for all \( s_1, s_2 \in R \). Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a family of mappings of \( R \) into itself defined, for each \( n \in \mathbb{N} \) and \( (s, t, s) \in R \), by:

\[
\phi_n(s, t, s) = \begin{cases} 
(2 - n)a \sigma^i(s), (n - 1)\sigma^i \tau^{n-i}(t'), (2 - n)a \sigma^i(s), & n = 1, 2 \\
0 & n \geq 3
\end{cases}
\]

Therefore, it is clear that \( \phi \) is a Jordan triple \( (\sigma, \tau) \)-higher homomorphism but not a triple \( (\sigma, \tau) \)-higher homomorphism.

Now, we will give the following lemmas which are used in the proofs of the main results.

Lemma 2.7: [5]

Let \( R \) be a 2-torsion free semiprime ring. If \( x, y \in R \) such that \( xry + yrx = 0 \), for all \( r \in R \), then \( xry = yrz = 0 \).

Lemma 2.8:

Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a Jordan triple \( (\sigma, \tau) \)-higher homomorphism of \( R \) into \( R' \). Then for each \( n \in \mathbb{N} \) and for all \( a, b, c \in R \),

\[
\phi_n(abc + cba) = \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \phi_i(\sigma^i \tau^{n-i}(b)) \phi_i(\tau^i(c)) + \phi_i(\sigma^i(c)) \phi_i(\sigma^i \tau^{n-i}(b)) \phi_i(\tau^i(a))
\]

Proof: Since \( \phi \) is a Jordan triple \( (\sigma, \tau) \)-higher homomorphism, hence
By linearizing \( a \), we get

\[
\phi_n((a + c)b(a + c)) = \sum_{i=1}^{n} \phi_i\left(\sigma^i(a)\right) \phi_i\left(\sigma^i\tau^{n-i}(b)\right) \phi_i\left(\tau^i(a)\right) + \phi_i\left(\sigma^i(c)\right) \phi_i\left(\sigma^i\tau^{n-i}(b)\right) \phi_i\left(\tau^i(c)\right)
\]

On the other hand:

\[
\phi_n((a + c)b(a + c)) = \phi_n(aba + abc + cba + cbc) = \phi_n(aba) + \phi_n(abc + cba) + \phi_n(cbc)
\]

By comparing (1) and (2), we achieve the result.

**Remark 2.9:**

Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a Jordan triple \((\sigma, \tau)-\)higher homomorphism from \( R \) into \( R' \). Then for each \( n \in \mathbb{N} \) and for all \( a, b \in R \), we will write

\[
A_n(a, b, c) = \phi_n(abc) - \sum_{i=1}^{n} \phi_i\left(\sigma^i(a)\right) \phi_i\left(\sigma^i\tau^{n-i}(b)\right) \phi_i\left(\tau^i(c)\right)
\]

\[
B_n(a, b, c) = \phi_n(abc) - \sum_{i=1}^{n} \phi_i\left(\sigma^i(c)\right) \phi_i\left(\sigma^i\tau^{n-i}(b)\right) \phi_i\left(\tau^i(a)\right)
\]

Note that \( A_n(a, b, c) = 0 \), if and only if \( \phi \) is a triple \((\sigma, \tau)-\)higher homomorphism, and \( B_n(a, b, c) = 0 \), if and only if \( \phi \) is a triple \((\sigma, \tau)-\)higher anti-homomorphism.

For the purpose of this paper, we can list the following elementary properties about the above:

1. \( A_n(a, b, c) + A_n(c, b, a) = 0 \),
2. \( B_n(a, b, c) + B_n(c, b, a) = 0 \),

**Lemma 2.10:**

If \( \theta = (\phi_i)_{i \in \mathbb{N}} \) is a Jordan triple \((\sigma, \tau)-\)higher homomorphism from a ring \( R \) into a ring \( R' \), then for all \( a, b \in R \) and \( n \in \mathbb{N} \),

\[
i) \quad A_n(a + b, c, d) = A_n(a, c, d) + A_n(b, c, d)
\]

\[
ii) \quad A_n(a, b + c, d) = A_n(a, b, d) + A_n(a, c, d)
\]

\[
iii) \quad A_n(a, b, c + d) = A_n(a, b, c) + A_n(a, b, d)
\]

**Proof:**

\[
i) \quad A_n(a + b, c, d) = \phi_n((a + b)cd) - \sum_{i=1}^{n} \phi_i\left(\sigma^i(a + b)\right) \phi_i\left(\sigma^i\tau^{n-i}(c)\right) \phi_i\left(\tau^i(d)\right)
\]

\[
= \phi_n(acd + bcd) - \sum_{i=1}^{n} \phi_i\left(\sigma^i(a)\right) \phi_i\left(\sigma^i\tau^{n-i}(c)\right) \phi_i\left(\tau^i(d)\right) - \sum_{i=1}^{n} \phi_i\left(\sigma^i(b)\right) \phi_i\left(\sigma^i\tau^{n-i}(c)\right) \phi_i\left(\tau^i(d)\right)
\]

Since \( \phi_n \) is an additive mapping for each \( n \), then

\[
= \phi_n(acd) - \sum_{i=1}^{n} \phi_i\left(\sigma^i(a)\right) \phi_i\left(\sigma^i\tau^{n-i}(c)\right) \phi_i\left(\tau^i(d)\right) + \phi_n(bcd) - \sum_{i=1}^{n} \phi_i\left(\sigma^i(b)\right) \phi_i\left(\sigma^i\tau^{n-i}(c)\right) \phi_i\left(\tau^i(d)\right) = A_n(a, c, d) + A_n(b, c, d)
\]

In a similarly way, we can prove (ii) and (iii).
3. **Main Results**

**Lemma 3.1:**

If \( \theta = (\phi_i)_{i \in \mathbb{N}} \) is a Jordan triple higher \((\sigma, \tau)\)-homomorphism of \( R \) into \( R' \), then for each \( n \in \mathbb{N} \) and for all \( a, b, c, r \in R \),

\[
A_n(\sigma^n(a, b, c))\phi_n(\sigma^n(r))B_n(\tau^n(a, b, c)) + B_n(\sigma^n(abc))\phi_n(\sigma^n(r))A_n(\tau^n(a, b, c)) = 0.
\]

**Proof:**

We proceed by the induction on \( n \in \mathbb{N} \). Assume that \( \theta \) is a Jordan triple higher \((\sigma, \tau)\)-homomorphism and take \( a, b, c, r \in R \).

If \( n = 1 \): Define \( w = abcrba + cbabarbc \), then we get the required result.

We can assume that the following equation is true for all \( a, b, c, r \in R, n \in \mathbb{N} \) and \( m < n \):

\[
A_m(\sigma^m(a, b, c))\phi_m(\sigma^m(r))B_m(\tau^m(a, b, c)) + B_m(\sigma^m(abc))\phi_m(\sigma^m(r))A_m(\tau^m(a, b, c)) = 0
\]

Now, we have

\[
\phi_n(w) = \phi_n(a(bcr\pi c)a + c(barab)c)
\]

\[
= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i(r)\pi(bcr\pi c))\phi_i(\tau^i(a)) + \phi_i(\sigma^i(c))\phi_i(\sigma^i(r)\pi(barab)c)\phi_i(\tau^i(c))
\]

\[
= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i(b)) \sum_{j=1}^{i} \phi_j(\sigma^j(r)\pi(crc)c)\phi_j(\tau^j(b))\phi_j(\tau^i(a))
\]

\[
+ \phi_i(\sigma^i(c))\phi_i(\sigma^i(b)) \sum_{j=1}^{i} \phi_j(\sigma^j(r)\pi(ar\pi a)c)\phi_j(\tau^j(b))\phi_j(\tau^i(c))
\]

\[
= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i(b)) \sum_{j=1}^{i} \phi_j(\sigma^j(r)\pi(crc)c)\phi_j(\tau^j(b))\phi_j(\tau^i(a))
\]

\[
+ \sum_{i=1}^{n} \phi_i(\sigma^i(c))\phi_i(\sigma^i(b)) \sum_{j=1}^{i} \phi_j(\sigma^j(r)\pi(ar\pi a)c)\phi_j(\tau^j(b))\phi_j(\tau^i(c))
\]

\[
= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i(b)) \sum_{j=1}^{i} \phi_j(\sigma^j(r)\pi(crc)c)\phi_j(\tau^j(b))\phi_j(\tau^i(a))
\]

\[
+ \sum_{i=1}^{n} \phi_i(\sigma^i(c))\phi_i(\sigma^i(b)) \sum_{j=1}^{i} \phi_j(\sigma^j(r)\pi(ar\pi a)c)\phi_j(\tau^j(b))\phi_j(\tau^i(c))
\]

\[
= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i(b)) \phi_i(\sigma^i(r)\pi(crc)c)\phi_i(\tau^i(b))\phi_i(\tau^i(a))
\]

\[
+ \sum_{i=1}^{n} \phi_i(\sigma^i(c))\phi_i(\sigma^i(b)) \phi_i(\sigma^i(r)\pi(ar\pi a)c)\phi_i(\tau^i(b))\phi_i(\tau^i(c))
\]

\[
= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i(b)) \phi_i(\sigma^i(r)\pi(crc)c)\phi_i(\tau^i(b))\phi_i(\tau^i(a))
\]

\[
+ \sum_{i=1}^{n} \phi_i(\sigma^i(c))\phi_i(\sigma^i(b)) \phi_i(\sigma^i(r)\pi(ar\pi a)c)\phi_i(\tau^i(b))\phi_i(\tau^i(c))
\]

\[
= \phi_n(w) = \phi_n(\sigma^n(abc)r(cb\pi a) + cb\pi ar\pi a(c))
\]

On the other hand

\[
\phi_n(w) = \phi_n((abc)r(cb\pi a) + cb\pi ar\pi a(c))
\]
\[= \sum_{i=1}^{n} \phi_i \left( \sigma^i(abc) \right) \phi_i \left( \sigma^i \tau^{n-i}(r) \right) \phi_i \left( \tau^i(cb) \right) + f_i \left( \sigma^i(cb) \right) \phi_i \left( \sigma^i \tau^{n-i}(r) \right) \phi_i \left( \tau^i(abc) \right)\]

Since \( \theta \) is a Jordan triple higher \((\sigma, \tau)\)-homomorphism, then

\[= \sum_{i=1}^{n} \phi_i \left( \sigma^i(abc) \right) \phi_i \left( \sigma^i \tau^{n-i}(r) \right) \left( \sum_{j=1}^{i} \phi_j \left( \tau^j(c) \right) \phi_j \left( \tau^j(b) \right) \phi_j \left( \tau^j(a) \right) \right)\]

\[+ \phi_j \left( \tau^j(a) \right) \phi_j \left( \tau^j(b) \right) \phi_j \left( \tau^j(c) \right) - \phi_j(\tau^j(abc))\]

\[+ \sum_{i=1}^{n} \sum_{j=1}^{i} \phi_j \left( \sigma^j(c) \right) \phi_j \left( \sigma^j \tau^{j-i}(b) \right) \phi_j \left( \tau^j(c) \right) - \phi_i(\sigma^i(abc)) \phi_i \left( \sigma^i \tau^{n-i}(r) \right) \phi_i \left( \tau^i(abc) \right)\]

\[= \sum_{i=1}^{n} \phi_i \left( \sigma^i(abc) \right) \phi_i \left( \sigma^i \tau^{n-i}(r) \right) \sum_{j=1}^{i} \phi_j \left( \tau^j(c) \right) \phi_j \left( \tau^j(b) \right) \phi_j \left( \tau^j(a) \right)\]

\[+ \sum_{i=1}^{n} \phi_i \left( \sigma^i(abc) \right) \phi_i \left( \sigma^i \tau^{n-i}(r) \right) \sum_{j=1}^{i} \phi_j \left( \tau^j(a) \right) \phi_j \left( \tau^j(b) \right) \phi_j \left( \tau^j(c) \right)\]

\[- \sum_{i=1}^{n} \phi_i \left( \sigma^i(abc) \right) \phi_i \left( \sigma^i \tau^{n-i}(r) \right) \phi_i \left( \tau^i(abc) \right)\]

\[+ \sum_{i=1}^{n} \sum_{j=1}^{i} \phi_j \left( \sigma^j(c) \right) \phi_j \left( \sigma^j \tau^{j-i}(b) \right) \phi_j \left( \tau^j(a) \right) \phi_i \left( \sigma^i \tau^{n-i}(r) \right) \phi_i \left( \tau^i(abc) \right)\]

\[+ \sum_{i=1}^{n} \sum_{j=1}^{i} \phi_j \left( \sigma^j(a) \right) \phi_j \left( \sigma^j \tau^{j-i}(b) \right) \phi_j \left( \tau^j(c) \right) \phi_i \left( \sigma^i \tau^{n-i}(r) \right) \phi_i \left( \tau^i(abc) \right)\]

\[- \sum_{i=1}^{n} \sum_{j=1}^{i} \phi_i \left( \sigma^i(abc) \right) \phi_i \left( \sigma^i \tau^{n-i}(r) \right) \phi_i \left( \tau^i(abc) \right)\]

\[= - \sum_{i=1}^{n} \phi_i \left( \sigma^i(abc) \right) \phi_i \left( \sigma^i \tau^{n-i}(r) \right) \left( \sum_{j=1}^{i} \phi_j(\tau^j(abc)) \phi_j(\tau^j(b)) \phi_j(\tau^j(a)) \right)\]

\[- \sum_{i=1}^{n} \phi_i \left( \sigma^i(abc) \right) \phi_i \left( \sigma^i \tau^{n-i}(r) \right) \left( \sum_{j=1}^{i} \phi_j(\tau^j(abc)) \phi_j(\tau^j(b)) \phi_j(\tau^j(c)) \right)\]

\[+ \sum_{i=1}^{n} \phi_i \left( \sigma^i \sigma^i(c) \right) \phi_i \left( \sigma^i \tau^{n-i} \tau^i \right) \phi_i \left( \tau^i(abc) \right)\]

\[+ \sum_{i=1}^{n} \phi_i \left( \sigma^i \sigma^i(a) \right) \phi_i \left( \sigma^i \tau^{n-i} \tau^i \right) \phi_i \left( \tau^i(abc) \right)\]
From equation (3) and (4), we get

\[ 0 = -\phi_n(\sigma^n(abc))\phi_n(\sigma^n(r))B_n(\tau^n(a, b, c)) - \phi_n(\sigma^n(abc))\phi_n(\sigma^n(r))A_n(\tau^n(a, b, c)) \]

\[ + \sum_{i=1}^{n-1} \phi_i(\sigma^i\sigma^i(c))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a))\phi_i(\sigma^i\tau^{n-i}(r))\phi_i(\tau^i(abc)) \]

\[ - \phi_i(\tau^i(abc)) \phi_i(\tau^i(c)) \]

\[ + \phi_n(\sigma^n(c))\phi_n(\sigma^n(b))\phi_n(\sigma^n(a))\phi_n(\sigma^n(r))\phi_n(\tau^n(abc)) \]

\[ + \sum_{i=1}^{n-1} \phi_i(\sigma^i\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c))\phi_i(\sigma^i\tau^{n-i}(r))\phi_i(\tau^i(abc)) \]

\[ + \phi_i(\tau^i(abc)) \phi_i(\tau^i(b)) \phi_i(\tau^i(a)) \]

\[ + \sum_{i=1}^{n-1} \phi_i(\sigma^i\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c))\phi_i(\sigma^i\tau^{n-i}(r))\phi_i(\tau^i(abc)) \]

\[ + \phi_i(\tau^i(abc)) \phi_i(\tau^i(b)) \phi_i(\tau^i(a)) \]
\[- \sum_{i=1}^{n-1} \phi_i (\sigma^i (abc)) \phi_i (\sigma^i \tau^{n-i} (r)) A_i (\tau^i (a, b, c)) \]
\[- \sum_{i=1}^{n-1} \phi_i (\sigma^i (abc)) \phi_i (\sigma^i \tau^{n-i} (r)) B_i (\tau^i (a, b, c)) \]

\[= -\phi_n (\sigma^n (abc)) \phi_n (\sigma^n (r)) B_n (\tau^n (a, b, c)) - \phi_n (\sigma^n (abc)) \phi_n (\sigma^n (r)) A_n (\tau^n (a, b, c)) \]
\[+ \phi_n (\sigma^n (c)) \phi_n (\sigma^n (b)) \phi_n (\sigma^n (a)) \phi_n (\sigma^n (r)) A_n (\tau^n (a, b, c)) + \phi_n (\sigma^n (a)) \phi_n (\sigma^n (b)) \phi_n (\sigma^n (c)) \phi_n (\sigma^n (r)) B_n (\tau^n (a, b, c)) \]

\[+ \sum_{i=1}^{n-1} \phi_i (\sigma^i \sigma^i (c)) \phi_i (\sigma^i \tau^{n-i} \sigma^i (b)) \phi_i (\tau^i \sigma^i (a)) \phi_i (\sigma^i \tau^{n-i} (r)) A_n (\tau^n (a, b, c)) \]
\[+ \sum_{i=1}^{n-1} \phi_i (\sigma^i \sigma^i (a)) \phi_i (\sigma^i \tau^{n-i} \sigma^i (b)) \phi_i (\tau^i \sigma^i (c)) \phi_i (\sigma^i \tau^{n-i} (r)) B_n (\tau^n (a, b, c)) \]

\[= -\phi_n (\sigma^n (abc) - \phi_n (\sigma^n (a)) \phi_n (\sigma^n (b)) \phi_n (\sigma^n (c)) \phi_n (\sigma^n (r)) B_n (\tau^n (a, b, c)) \]
\[- \phi_n (\sigma^n (abc) - \phi_n (\sigma^n (a)) \phi_n (\sigma^n (b)) \phi_n (\sigma^n (c)) \phi_n (\sigma^n (r)) A_n (\tau^n (a, b, c)) \]
\[- \sum_{i=1}^{n-1} \phi_i (\sigma^i (abc) - \phi_i (\sigma^i (a)) \phi_i (\sigma^i \tau^{n-i} \sigma^i (b)) \phi_i (\tau^i \sigma^i (a)) \phi_i (\sigma^i \tau^{n-i} (r)) A_n (\tau^n (a, b, c)) \]
\[- \sum_{i=1}^{n-1} \phi_i (\sigma^i (abc) - \phi_i (\sigma^i (a)) \phi_i (\sigma^i \tau^{n-i} \sigma^i (b)) \phi_i (\tau^i \sigma^i (c)) \phi_i (\sigma^i \tau^{n-i} (r)) B_n (\tau^n (a, b, c)) \]

\[= -A_n (\sigma^n (a, b, c) \phi_n (\sigma^n (r)) B_n (\tau^n (a, b, c)) - B_n (\sigma^n (a, b, c) \phi_n (\sigma^n (r)) A_n (\tau^n (a, b, c)) \]
\[- \sum_{i=1}^{n-1} B_n (\sigma^i (a, b, c)) \phi_i (\sigma^i \tau^{n-i} (r)) A_n (\tau^n (a, b, c)) \]
\[- \sum_{i=1}^{n-1} A_n (\sigma^i (a, b, c)) \phi_i (\sigma^i \tau^{n-i} (r)) B_n (\tau^n (a, b, c)) \]

Hence, we have
\[A_n (\sigma^n (a, b, c) \phi_n (\sigma^n (r)) B_n (\tau^n (a, b, c) + B_n (\sigma^n (a, b, c) \phi_n (\sigma^n (r)) A_n (\tau^n (a, b, c) = 0. \]

**Lemma 3.2:**

Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a Jordan triple higher \((\sigma, \tau)\)-homomorphism of \( R \) into \( R' \), then for each \( n \in \mathbb{N} \) and for all \( a, b, c, \tau \in R \),
\[A_n (\sigma^n (a, b, c) \phi_n (\sigma^n (r)) B_n (\tau^n (a, b, c) = B_n (\sigma^n (a, b, c) \phi_n (\sigma^n (r)) A_n (\tau^n (a, b, c) = 0. \]

**Proof.**

By Lemma 3.1 and Lemma 2.7, we achieve the result.

**Theorem 3.3:**

Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a Jordan triple higher \((\sigma, \tau)\)-homomorphism of ring \( R \) into prime ring \( R' \). Then for each \( n \in \mathbb{N} \) and for all \( a, b, c, r, x, y, z \in R \),
\[A_n (\sigma^n (a, b, c) \phi_n (\sigma^n (r)) B_n (\tau^n (x, y, z) = 0. \]

**Proof.**

By replacing \( a + x \) by \( a \) in Lemma 3.2, we get
\[ A_n \left( \sigma^3(a + x, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a + x, b, c) \right) = 0 \]

Hence
\[ A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) + A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) + A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) = 0 \]

By Lemma 3.2, we obtain
\[ A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) + A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) = 0 \]

Therefore, we get
\[ 0 = A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) \]
\[ = -A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) A_n \left( \tau^3(a, b, c) \right) \]

Since \( R' \) is prime, we obtain
\[ A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) = 0. \]

By replacing \( b + y \) for \( b \) in equation (5), we get
\[ A_n \left( \sigma^3(a, b + y, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b + y, c) \right) = 0 \]

Hence
\[ A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) + A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) + A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) = 0 \]

We can use equation (5), then we get
\[ A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) + A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) = 0 \]

Therefore, we get
\[ 0 = A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) \]
\[ = -A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) A_n \left( \tau^3(a, b, c) \right) \]

Since \( R' \) is prime, we obtain
\[ A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) = 0. \]

By replacing \( c + z \) for \( c \) in equation (6), we get
\[ A_n \left( \sigma^3(a, b, c + z) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c + z) \right) = 0 \]

Hence
\[ A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) + A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) + A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) = 0 \]

We can use equation (5), then we get
\[ A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) + A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) = 0 \]

Therefore, we get
\[ 0 = A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) \]
\[ = -A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) A_n \left( \tau^3(a, b, c) \right) \]

Since \( R' \) is prime, we obtain
\[ A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) = 0. \]

In the following theorem we give the conditions which make the Jordan triple higher \((\sigma, \tau)\)-homomorphism is either triple higher \((\sigma, \tau)\)-homomorphism or triple higher \((\sigma, \tau)\)-anti-homomorphism.

**Theorem 3.4:**

Every Jordan triple higher \((\sigma, \tau)\)-homomorphism of ring \( R \) into prime ring \( R' \) is either triple higher \((\sigma, \tau)\)-homomorphism or triple higher \((\sigma, \tau)\)-anti-homomorphism.

**Proof.**

Let \( \theta \) be a Jordan triple higher \((\sigma, \tau)\)-homomorphism. Then by Theorem 3.3, we have
\[ A_n \left( \sigma^3(a, b, c) \right) \phi_n \left( \sigma^3(r) \right) B_n \left( \tau^3(a, b, c) \right) = 0 \]

Since \( R' \) is prime, therefore either \( A_n \left( \sigma^3(a, b, c) \right) = 0 \) or \( B_n \left( \tau^3(a, b, c) \right) = 0 \), for each \( n \in \mathbb{N} \) and for all \( a, b, c, x, y, z \in R \).

If \( B_n \left( \tau^3(a, b, c) \right) = 0 \), then by Remark 2.9, we obtain \( \theta \) is triple higher \((\sigma, \tau)\)-anti-homomorphism.

But if \( A_n \left( \sigma^3(a, b, c) \right) = 0 \), then by Remark 2.9, we obtain \( \theta \) is triple higher \((\sigma, \tau)\)-homomorphism.
Proposition 3.5:
Let $\theta = (\phi_i)_{i \in \mathbb{N}}$ be a Jordan triple higher $(\sigma, \tau)$-homomorphism from prime ring $R$ into prime ring $R'$, then $\theta$ is higher $(\sigma, \tau)$-homomorphism.

Proof:
Since $\theta$ is a Jordan triple higher $(\sigma, \tau)$-homomorphism, then for all $a, r \in R$ and $n \in \mathbb{N}$, we have
\[
\phi_n(ar) = \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \phi_i(\sigma^i\tau^{n-i}(r)) \phi_i(\tau^i(a))
\]

By replacing $a$ by $ab$, we get
\[
\phi_n((ab)r(ab)) = \sum_{i=1}^{n} \phi_i(\sigma^i(ab)) \phi_i(\sigma^i\tau^{n-i}(r)) \phi_i(\tau^i(ab))
\]
\[= \phi_n(\sigma^n(ab))rab + ab \sum_{i=1}^{n} \phi_i(\sigma^i\tau^{n-i}(r)) \phi_i(\tau^i(ab)) \tag{7}
\]

On the other hand, we get
\[
\phi_n((ab)r(ab)) = \sum_{i=1}^{n} \phi_i(\sigma^i(ab)) \phi_i(\sigma^i\tau^{n-i}(r)) \phi_i(\tau^i(ab))
\]
\[= \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \phi_i(\sigma^i(b)) rab + ab \sum_{i=1}^{n} \phi_i(\sigma^i\tau^{n-i}(r)) \phi_i(\tau^i(ab)) \tag{8}
\]

By comparing (7) and (8), we get
\[
\left( \phi_n(\sigma^n(ab)) - \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \phi_i(\sigma^i(b)) \right) rab = 0.
\]

Since $R$ is prime and $ab \neq 0$, we get
\[
\phi_n(ab) = \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \phi_i(\tau^i(b))
\]

Hence $\theta$ is a higher $(\sigma, \tau)$-homomorphism.

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