A two-dimensional limit theorem for Lerch zeta-functions. II

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Abstract. We prove a two-dimensional limit theorem for Lerch zeta-functions with transcendental and rational parameters.

Keywords: Lerch zeta-function, probability measure, weak convergence.

Let $s = \sigma + it$ be a complex variable, and $0 < \lambda < 1$ and $0 < \alpha \leq 1$ be fixed parameters. The Lerch-zeta function $L(\lambda, \alpha, s)$ is defined, for $\sigma > 1$, by the Dirichlet series $L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i m \lambda}}{(m+\alpha)^s}$, and, because of $0 < \lambda < 1$, is analytically continued to an entire function.

Probabilistic limit theorems for the function $L(\lambda, \alpha, s)$ with transcendental and rational parameter $\alpha$ were proved in [2] while the case of algebraic irrational parameter $\alpha$ was considered in [3, 4, 6, 7].

In [5], we proved a limit theorem on the complex plane for a pair $(L(\lambda_1, \alpha_1, s), L(\lambda_2, \alpha_2, s))$, when $\alpha_1$ and $\alpha_2$ are transcendental and algebraic irrational numbers, respectively. The aim of this note is to prove a limit theorem of such a kind when the number $\alpha_2$ is rational. To state the theorem we need some notation and definitions.

Denote by $B(S)$ the class of Borel sets of the space $S$, by $\text{meas}\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and let

$$\nu_T(\cdots) = \frac{1}{T} \text{meas}\{t \in [0, T]; \cdots\},$$

where in place of dots a condition satisfied by $t$ is to be written. Let $\gamma = \{s \in \mathbb{C}: |s| = 1\}$. Define $\Omega_1 = \prod_{m=0}^{\infty} \gamma_m$ and $\Omega_2 = \prod_{p} \gamma_p$, where $\gamma_m = \gamma$ and $\gamma_p = \gamma$ for all $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and primes $p$, respectively. Denote by $\omega_1(m)$ and $\omega_2(p)$ the projections of $\omega_1 \in \Omega_1$ to $\gamma_m$ and of $\omega_2 \in \Omega_2$ to $\gamma_p$, respectively. Moreover, we extend the function $\omega_2(p)$ to the set $\mathbb{N}$ by the formula $\omega_2(m) = \prod_{p\|m} \omega_2(p), m \in \mathbb{N}$, where $p\|m$ means that $p^j|m$ but $p^{j+1} \nmid m$.

Let $\Omega = \Omega_1 \times \Omega_2$. Then $\Omega$ is a compact topological Abelian group, therefore, on $(\Omega, B(\Omega))$ the probability Haar measure $m_H$ can be defined. This gives a probability space $(\Omega, B(\Omega), m_H)$. Suppose that $\alpha_2 = \frac{a}{q}$, $0 < a < q$, $(a, q) = 1$. Denote by $\omega = (\omega_1, \omega_2)$ the elements of $\Omega$, and put, for brevity, $\underline{\alpha} = (\alpha_1, \alpha_2)$, $\underline{\lambda} = (\lambda_1, \lambda_2)$, $\underline{\sigma} = (\sigma_1, \sigma_2)$. On the probability space $(\Omega, B(\Omega), m_H)$, define the $\mathbb{C}^2$-valued random element $L(\underline{\lambda}, \underline{\alpha}, \underline{\sigma}, \omega)$, for $\min(\sigma_1, \sigma_2) > \frac{1}{2}$, by

$$L(\lambda, \alpha, \sigma, \omega) = (L(\lambda_1, \alpha_1, \sigma_1, \omega_1), L(\lambda_2, \alpha_2, \sigma_2, \omega_2)), $$
where
\[ L(\lambda_1, \alpha_1, \sigma_1, \omega_1) = \sum_{m=0}^{\infty} e^{2\pi i \lambda_1 m} \omega_1(m) \frac{\sigma_1}{(m + \alpha_1)^{\sigma_1}} \]
and
\[ L(\lambda_2, \alpha_2, \sigma_2, \omega_2) = e^{\frac{-2\pi i m}{q}} q^s \omega_2(q) \sum_{m \equiv 1 \mod(q)} e^{\frac{2\pi im}{m^{\sigma_2}}}. \]

Let \( L(\lambda, \alpha, \sigma + it) = (L(\lambda_1, \alpha_1 + it), L(\lambda_2, \alpha_2, \sigma + it)) \).

**Theorem 1.** Suppose that the number \( \alpha_1 \) is transcendental, \( \alpha_2 = \frac{a}{q} \), \( 0 < a < q \), \( (a, q) = 1 \), and \( \min(\sigma_1, \sigma_2) > \frac{1}{2} \). Then the probability measure
\[ P_T(A) \equiv \nu_T(L(\lambda, \alpha, \sigma + it) \in A), \quad A \in \mathcal{B}(\mathbb{C}), \]
converges weakly to the distribution of the random element \( L(\lambda, \alpha, \sigma, \omega) \) as \( T \to \infty \).

Let \( \mathcal{P} \) denote the set of all prime numbers. Since \( \alpha_1 \) is transcendental, the set \( \{\log(m + \alpha_1) : m \in \mathbb{N}_0\} \) is linearly independent over the field of rational numbers \( \mathbb{Q} \). The set \( \{\log p : p \in \mathcal{P}\} \) is also linearly independent over \( \mathbb{Q} \). Therefore, it is not difficult to prove that the set
\[ L(\alpha_1) \equiv \{\log(m + \alpha_1) : m \in \mathbb{N}_0\} \cup \{\log p : p \in \mathcal{P}\} \]
is linearly independent as well. This leads to the following lemma.

**Lemma 1.** (See [8].) Suppose that the number \( \alpha_1 \) is transcendental. Then the probability measure
\[ Q_T(A) \equiv \nu_T(((m + \alpha_1)^{-it} : m \in \mathbb{N}_0), (p^{-it} : p \in \mathcal{P}) \in A), \quad A \in \mathcal{B}(\Omega), \]
converges weakly to the Haar measure \( m_H \) as \( T \to \infty \).

Let \( \sigma_1 > \frac{1}{2} \) be a fixed number, and \( v_n(m, \alpha_1) = \exp\{-\left(\frac{m + \alpha_1}{m + \alpha_1}^\sigma_1\right)^s\} \), \( v_n(m) = \exp\{-\left(\frac{m}{m}^\sigma_1\right)^s\} \). Define \( L_n(\lambda, \alpha, s) = (L_n(\lambda_1, \alpha_1, s), L_n(\lambda_2, \alpha_2, s)) \), where
\[ L_n(\lambda_1, \alpha_1, s) = \sum_{m=0}^{\infty} e^{2\pi i \lambda_1 m} v_n(m, \alpha_1) \frac{\sigma_1}{(m + \alpha_1)^{\sigma_1}} \]
and
\[ L_n(\lambda_2, \alpha_2, s) = \sum_{m \equiv 1 \mod(q)} e^{\frac{2\pi im}{m^{\sigma_2}}} e^{\frac{-2\pi im}{q}} q^s v_n(m). \]

By contour integration it is proved, see, for example, [2], that the series for \( L_n(\lambda_1, \alpha_1, s) \) and \( L_n(\lambda_2, \alpha_2, s) \) both converge absolutely for \( \sigma > \frac{1}{2} \).
A two-dimensional limit theorem for Lerch zeta-functions. II

Let, for \( \omega = (\omega_1, \omega_2) \in \Omega \),
\[
L_n((\omega_1, \omega_2), s) = (L_n(\lambda_1, \alpha_1, \omega_1, s), L_n(\lambda_2, \alpha_2, \omega_2, s)),
\]
where
\[
L_n(\lambda_1, \alpha_1, \omega_1, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_1 m \omega_1} e^{m \sigma_1}}{(m + \alpha_1)^s}
\]
and
\[
L_n(\lambda_2, \alpha_2, \omega_2, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_2 m \omega_2} e^{m \sigma_2}}{(m + \alpha_2)^s}.
\]
Since \(|\omega_1(m)| = |\omega_2(m)| = 1\), the later two series also converge absolutely for \( \sigma > \frac{1}{2} \).

On \((C^2, B(C^2))\), define the probability measures \( P_{T,n}(A) = \nu_T(L_n(\lambda, \alpha, \omega, \sigma + it) \in A) \) and \( \tilde{P}_{T,n}(A) = \nu_T(L_n(\lambda, \alpha, \omega, \sigma + it) \in A) \).

**Lemma 2.** Suppose that \( \min(\sigma_1, \sigma_2) > \frac{1}{2} \). Then on \((C^2, B(C^2))\), there exists a probability measure \( P_n \) such that both the measures \( P_{T,n} \) and \( \tilde{P}_{T,n} \) converge weakly to \( P_n \) as \( T \to \infty \).

**Proof.** Define the function \( h_n : \Omega \to C^2 \) by the formula \( h_n(\omega) = L(\lambda, \alpha, \omega, \sigma) \). Then the function is continuous, and
\[
h_n((m + \alpha_1)^{-it}; m \in \mathbb{N}_0, (p^{-it}; p \in P)) = L(\lambda, \alpha, \omega, \sigma + it).
\]
Therefore, \( P_{T,n} = Q_T h_n^{-1} \). This, the continuity of \( h_n \), Lemma 1, and Theorem 5.1 of [1] show that \( P_{T,n} \) converges weakly to \( m_H h_n^{-1} \) as \( T \to \infty \).

By the same arguments, using the invariance of the Haar measure \( m_H \), we obtain that the measure \( \tilde{P}_{T,n} \) also converges weakly to \( m_H h_n^{-1} \) as \( T \to \infty \). \( \square \)

For \( z_1 = (z_{11}, z_{12}) \), \( z_2 = (z_{21}, z_{22}) \in C^2 \), let \( \rho_2(z_1, z_2) = (\sum_{j=1}^{2} |z_{j1} - z_{j2}|^2)^{\frac{1}{2}} \).

**Lemma 3.** Suppose that \( \min(\sigma_1, \sigma_2) > \frac{1}{2} \). Then
\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho_2(L(\lambda, \alpha, \omega, \sigma + it), L_n(\lambda, \alpha, \omega, \sigma + it)) dt = 0,
\]
and, for almost all \( \omega \in \Omega \),
\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho_2(L(\lambda, \alpha, \omega, \sigma + it), L_n(\lambda, \alpha, \omega, \sigma + it)) dt = 0.
\]

**Proof.** The lemma follows from corresponding one-dimensional relations, see [2], and from the definition of the metric \( \rho_2 \). \( \square \)

Define one more probability measure
\[
\tilde{P}_T(A) = \nu_T(L(\lambda, \alpha, \omega, \sigma + it) \in A), \quad A \in B(C^2).
\]
Lemma 4. Suppose that \( \min(\sigma_1, \sigma_2) > \frac{1}{T} \). Then on \( (\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2)) \), there exists a probability measure \( P \) such that both the measures \( P_T \) and \( \hat{P}_T \) converge weakly to \( P \) as \( T \to \infty \).

Proof. We remind that \( P_n \) is the limit measure in Lemma 2. First we observe that the family of probability measures \( \{P_n; n \in \mathbb{N}_0\} \) is tight. This is obtained by using Lemma 2 and the fact that

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T |L_n(\lambda_1, \alpha_1, \sigma_1 + it)|^2 \, dt = \sum_{m=0}^{\infty} \frac{v_n^2(m\alpha_1)}{(m + \alpha_1)^2\sigma_1} \leq \sum_{m=0}^{\infty} \frac{1}{(m + \alpha_1)^2\sigma_1} < \infty
\]

and

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T |L_n(\lambda_2, \alpha_2, \sigma_2 + it)|^2 \, dt = \sum_{m=1}^{\infty} \frac{q^2\sigma^2_v(m\alpha)}{m^{2\sigma_2}} \leq \sum_{m=1}^{\infty} \frac{\sigma^2_v}{m^{2\sigma_2}} < \infty.
\]

By the Prokhorov theorem, the tightness implies a relative compactness. Therefore, there exists a subsequence \( \{P_{n_k}\} \subset \{P_n\} \) such that \( P_{n_k} \) converges weakly to a certain probability measure \( P \) on \( (\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2)) \) as \( k \to \infty \).

Let \( \theta \) be a random variable defined on a certain probability space \( (\Omega_0, \mathcal{B}(\Omega_0), \mathbb{P}) \) and uniformly distributed on \([0, 1]\). Define \( \hat{X}_{T,n}(\lambda, \alpha, \sigma) = L_n(\lambda, \alpha, \sigma + i\theta T) \), and denote by \( D \) the convergence in distribution. Then, by Lemma 2, we have that

\[
\hat{X}_{T,n}(\lambda, \alpha, \sigma) \xrightarrow{D} \hat{X}_n(\lambda, \alpha, \sigma),
\]

where \( \hat{X}_n(\lambda, \alpha, \sigma) \) is the \( \mathbb{C}^2 \)-valued random element with distribution \( P_n \). Moreover, by the above remark, we have that

\[
\hat{X}_n(\lambda, \alpha, \sigma) \xrightarrow{D} P.
\]

Let \( \hat{X}_T(\lambda, \alpha, \sigma) = L(\lambda, \alpha, \sigma + i\theta T) \). Then we deduce from Lemma 3 that, for every \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} \limsup_{T \to \infty} \mathbb{P}(\rho_2(\hat{X}_T(\lambda, \alpha, \sigma), \hat{X}_{T,n}(\lambda, \alpha, \sigma)) \geq \varepsilon)
\]

\[
\leq \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho_2(L(\lambda, \alpha, \sigma + it), L_n(\lambda, \alpha, \sigma + it)) \, dt = 0.
\]

This, (1), (2) and Theorem 4.2 of [1] show that \( \hat{X}_T(\lambda, \alpha, \sigma) \xrightarrow{D} P \), or, in other words, the measure \( P_T \) converges weakly to \( P \) as \( T \to \infty \). The latter relation also shows that the measure \( P \) is independent on the choice of the sequence \( \{P_{n_k}\} \). Thus, we have that

\[
\hat{X}_n(\lambda, \alpha, \sigma) \xrightarrow{D} P.
\]

Define \( \hat{X}_{T,n}(\lambda, \alpha, \omega, \sigma) = L_n(\lambda, \alpha, \omega, \sigma + i\theta T) \) and \( \hat{X}_T(\lambda, \alpha, \omega, \sigma) = L(\lambda, \alpha, \omega, \sigma + i\theta T) \). Then, repeating the above arguments for the random elements \( \hat{X}_{T,n}(\lambda, \alpha, \omega, \sigma) \),
A two-dimensional limit theorem for Lerch zeta-functions. II

ω, σ and \( \hat{X}(\lambda, \alpha, \omega, \sigma) \) with using of Lemmas 2 and 3, and the relation (3), we obtain that the measure \( \hat{P}_T \) also converges weakly to \( P \) as \( T \to \infty \).

Proof of Theorem 1. In view of Lemma 4, it remains to show that the measure \( P \) coincides with the distribution of the random element \( L \).

Let \( A \) be a continuity set of the measure \( P \). Then by Lemma 4 we have that
\[
\lim_{T \to \infty} \nu_T \left( L(\lambda, \alpha, \omega, \sigma + it) \in A \right) = P(A).
\]

On the probability space \((\Omega, \mathcal{B}(\Omega), m_H)\), define the random variable \( \xi \) by
\[
\xi(\omega) = \begin{cases} 
1 & \text{if } L(\lambda, \alpha, \omega, \sigma) \in A, \\
0 & \text{otherwise.}
\end{cases}
\]

Then, clearly, the expectation
\[
\mathbb{E}[\xi] = \int_{\Omega} \xi(\omega) \, dm_H = m_H(\omega \in \Omega: L(\lambda, \alpha, \omega, \sigma) \in A) = P_L(A),
\]
where \( P_L \) is the distribution of the random element \( L \).

Let, for \( t \in \mathbb{R} \), \( \alpha_t = ((m + \alpha_1)^{-it}; m \in \mathbb{N}_0), (p^{-it}; p \in \mathcal{P}) \), and \( \varphi_t(\omega) = \omega \alpha_t, \omega \in \Omega \). Then \( \{\varphi_t: t \in \mathbb{R}\} \) is a one-parameter group of measurable measure preserving transformations on \( \Omega \). Since the set \( L(\alpha_1) \) is linearly independent over \( \mathbb{Q} \), by a standard method can be proved that the group \( \{\varphi_t: t \in \mathbb{R}\} \) is ergodic. Hence, the random process \( \xi(\varphi_t(\omega)) \) is ergodic as well. Therefore, the Birkhoff–Khintchine theorem shows that
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \xi(\varphi_t(\omega)) \, dt = \mathbb{E}[\xi].
\]

On the other hand, by the definition of \( \xi \) and \( \varphi_t \) we find that
\[
\frac{1}{T} \int_0^T \xi(\varphi_t(\omega)) \, dt = \nu_T \left( L(\lambda, \alpha, \omega, \sigma + it) \in A \right).
\]

This, (5) and (6) yield
\[
\lim_{T \to \infty} \nu_T \left( L(\lambda, \alpha, \omega, \sigma + it) \in A \right) = P_L(A).
\]

Therefore, by (4), \( P(A) = P_L(A) \) for all continuity sets \( A \) of \( P \). Hence, \( P(A) = P_L(A) \) for all \( A \in \mathcal{B}(\mathbb{C}^2) \).

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**REZIUMĖ**

**Dvimatė ribinė teorema Lercho dzeta funkcijoms. II**

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Straipsnyje įrodoma dvimatė ribinė teorema Lercho dzeta funkcijoms su transcendentčia ir racional–irraţionalų parametrais.

*Raktiniai žodžiai*: Lercho dzeta funkcija, tikimybinis matas, silpnasis konvergavimas.