A CONCISE FORMULA FOR THE HESSIAN DETERMINANT
OF A FUNCTION PARAMETERISING A QUADRATIC
HYPER SURFACE

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Abstract. We will give a concise formula for the Hessian determinant of
a smooth function $y : \mathbb{R}^n \supseteq \Omega \to \mathbb{R}$ such that its graph is contained in
a quadratic hypersurface. The proof will make heavy use of matrix algebra.

The aim of this work is to prove the following theorem:

Theorem 1. Let $y : \mathbb{R}^n \supseteq \Omega \to \mathbb{R}$ be a function of class $C^2$ defined on an open
subset of $\mathbb{R}^n$ and satisfying a quadratic equation

$$v(x)^\top Q v(x) = 0,$$

where

$$v(x) := \begin{bmatrix} x \\ y(x) \\ 1 \end{bmatrix}, \quad x \in \Omega$$

is an augmented $(n+2)$-dimensional column vector and $Q$ is an arbitrary $(n+2) \times (n+2)$ matrix. Then the following formula holds:

$$| \pm H_y(x) | \cdot \Delta_y(x)^{n/2+1} = - | Q + Q^\top |,$$

where $H_y$ is the Hessian matrix of $y$, $\Delta_y$ is the discriminant of the left hand side
of (1) with respect to the variable $y$ and the sign $\pm$ depends on the selected branch
of the square root function.

Here by Hessian matrix of $y$ we mean a square $n \times n$ matrix defined as follows:

$$H_y := \begin{bmatrix}
\frac{\partial^2 y}{\partial x_1^2} & \frac{\partial^2 y}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 y}{\partial x_1 \partial x_n} \\
\frac{\partial^2 y}{\partial x_2 \partial x_1} & \frac{\partial^2 y}{\partial x_2^2} & \cdots & \frac{\partial^2 y}{\partial x_2 \partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 y}{\partial x_n \partial x_1} & \frac{\partial^2 y}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 y}{\partial x_n^2}
\end{bmatrix}.$$

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and by *discriminant* of the quadratic polynomial $ay^2 + by + c$ with $a \neq 0$ with respect to the variable $y$ we mean

$$\Delta_y := b^2 - 4ac.$$ 

In the course of the proof we will also use a vector differential operator, usually represented by the nabla symbol $\nabla$. It is defined in terms of partial derivative operators as

$$\nabla = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix}.$$ 

In a convenient mathematical notation, we can consider e.g. a formal product of nabla with a scalar and a formal tensor product of nabla with a vector field. Namely,

$$\nabla y := \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \vdots \\ \frac{\partial y}{\partial x_n} \end{bmatrix}, \quad \nabla \otimes \vec{v} := \begin{bmatrix} \frac{\partial \vec{v}}{\partial x_1} & \cdots & \frac{\partial \vec{v}}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \vec{v}}{\partial x_1} & \cdots & \frac{\partial \vec{v}}{\partial x_n} \end{bmatrix}.$$ 

In particular,

$$H_y = \nabla \otimes \nabla y.$$ 

**Proof.** To simplify the notation, we will henceforth skip the dependence on $x$ whenever it is clear from the context.

Since the formula (2) depends on $Q$ only through $Q + Q^T$, without loss of generality we may assume that $Q$ is symmetric by taking its symmetric part

$$\frac{1}{2} (Q + Q^T).$$ 

Thus we can write

$$Q := \begin{bmatrix} A = A^T & b & c \\ b^T & d & e \\ c^T & e & f \end{bmatrix}$$ 

as a symmetric block matrix, where $A$ is a symmetric $n \times n$ matrix, $b, c$ are $n$-dimensional column vectors and $d, e, f$ are scalars.

In terms of these new variables, equation (1) reads

$$dy^2 + 2(b^T x + e)y + (x^T A x + 2c^T x + f)$$ 

and hence its discriminant with respect to the variable $y$ is itself a quadratic polynomial in $x$ given by

$$\Delta_y = 4(b^T x + e)^2 - 4d(x^T A x + 2c^T x + f)$$

$$= 4x^T (bb^T - dA) x + 8(eb^T - dc^T) x + 4(e^2 - df).$$ 

Denote its coefficients by

$$\Lambda := dA - bb^T, \quad \mu := eb - dc, \quad \nu := df - e^2.$$
respectively, so that the equality

\[ \Delta_y = -4x^\top \Lambda x + 8\mu^\top x - 4\nu \]

holds.

Since the formula (2) is continuous in both \( x \) and \( Q \), it is enough to prove it for a dense subset of pairs \((x, Q)\). Hence without loss of generality we may assume that \( d \neq 0 \) and \( \Delta_y > 0 \). Solving (4) for \( y \) yields

\[ y = \frac{-2(b^\top x + e) \pm \sqrt{\Delta_y}}{2d}, \]

where the sign \( \pm \) depends on the selected branch of the square root function and is opposite to that in (2). Since the Hessian matrix does not depend on the linear part, we have

\[ \pm H_y = \frac{\nabla \otimes \nabla \sqrt{\Delta_y}}{2d}, \]

with

\[ |\pm H_y| = 2^{-n}d^{-n} |\nabla \otimes \nabla \sqrt{\Delta_y}|. \]

Now, for an arbitrary function \( \Delta_y \), the Hessian matrix of \( \sqrt{\Delta_y} \) is given by

\[ \nabla \otimes \nabla \sqrt{\Delta_y} = \nabla \otimes \frac{\nabla \Delta_y}{2\Delta_y^{1/2}} = \frac{2\Delta_y \nabla \otimes \nabla \Delta_y - \nabla \Delta_y \otimes \nabla \Delta_y}{4\Delta_y^{3/2}} \]

and thus is a linear combination of a square matrix \( \nabla \otimes \nabla \Delta_y \) and a rank one matrix \( \nabla \Delta_y \otimes \nabla \Delta_y \). Its determinant can be computed using the following simple fact from linear algebra:

**Lemma 2** ([1, Theorem 18.1.1]). Let \( R \) represent an \( n \times n \) matrix, \( S \) an \( n \times m \) matrix, \( T \) an \( m \times m \) matrix, and \( U \) an \( m \times n \) matrix. If \( R \) and \( T \) are non-singular, then

\[ |R + STU| = |R| |T| |T^{-1} + UR^{-1}S|. \]

Recall that \( \Delta_y \) is a quadratic polynomial (7), in which case

\[ \nabla \Delta_y = -8\Lambda x + 8\mu, \quad \nabla \otimes \nabla \Delta_y = -8\Lambda. \]

Again, without loss of generality we may assume that \( \Lambda \) is non-singular. Applying Lemma 2 to

\[ R := 2\Delta_y \nabla \otimes \nabla \Delta_y, \quad S := -(\nabla \Delta_y), \quad T := I_1, \quad U := (\nabla \Delta_y)^\top \]

yields

\[ \left| \nabla \otimes \nabla \sqrt{\Delta_y} \right| \]

\[ = 4^{-n} \Delta_y^{-3n/2} |R| \left( 1 - (\nabla \Delta_y)^\top R^{-1}(\nabla \Delta_y) \right) \]

\[ = 2^{-n-1} \Delta_y^{-n/2-1} |\nabla \otimes \nabla \Delta_y| \left( 2\Delta_y - (\nabla \Delta_y)^\top (\nabla \otimes \nabla \Delta_y)^{-1}(\nabla \Delta_y) \right). \]

Further, without loss of generality we may assume that \( \Lambda \) is non-singular. Applying Lemma 2 to

\[ R := d\Lambda, \quad S := -b, \quad T := I_1, \quad U := b^\top \]
yields
\[ |\nabla \otimes \nabla \Delta y| = (-8)^n |R| (1 - b^T R^{-1} b) \]
\[ = (-8)^n d^{n-1} |A| (d - b^T A^{-1} b) . \]

Moreover,
\[ (\nabla \Delta y)^T (\nabla \otimes \nabla \Delta y)^{-1} (\nabla \Delta y) = -8x^T \Lambda x + 16\mu^T x - 8\mu^T \Lambda^{-1} \mu \]
and consequently
\[ 2\Delta_y - (\nabla \Delta y)^T (\nabla \otimes \nabla \Delta y)^{-1} (\nabla \Delta y) = -8\nu + 8\mu^T \Lambda^{-1} \mu \]
is a constant independent of \( x \). Since \( \Lambda = dA - bb^T \) is a linear combination of a square matrix \( A \) and a rank one matrix \( bb^T \), its inverse can be computed using another simple fact from linear algebra:

**Lemma 3** (Corollary 18.2.10). Let \( R \) represent an \( n \times n \) non-singular matrix, and let \( s \) and \( u \) represent \( n \)-dimensional column vectors. Then \( R + su^T \) is non-singular if and only if \( u^T R^{-1} s \neq -1 \), in which case
\[ (R + su^T)^{-1} = R^{-1} - (1 + u^T R^{-1} s)^{-1} R^{-1} s u^T R^{-1} . \]

Applying Lemma 3 to
\[ R := dA, \quad s := -b, \quad u := b \]
yields
\[ \Lambda^{-1} = R^{-1} + (1 - b^T R^{-1} b)^{-1} R^{-1} bb^T R^{-1} \]
\[ = d^{-1} \left( A^{-1} + (d - b^T A^{-1} b)^{-1} A^{-1} bb^T A^{-1} \right) \]
\[ = d^{-1} \left( d - b^T A^{-1} b \right)^{-1} \left( (d - b^T A^{-1} b) A^{-1} + A^{-1} bb^T A^{-1} \right) . \]

By combining (10) and (11) we obtain
\[ |\nabla \otimes \nabla \Delta y| (2\Delta_y - (\nabla \Delta y)^T (\nabla \otimes \nabla \Delta y)^{-1} (\nabla \Delta y)) \]
\[ = (-8)^n d^{n-1} |A| (d - b^T A^{-1} b) (-8\nu + 8\mu^T \Lambda^{-1} \mu) \]
\[ = (-8)^{n+1} d^{n-2} |A| \cdot (d - b^T A^{-1} b) (\nu - \mu^T \Lambda^{-1} \mu) . \]

Further, using (12) gives us
\[ d \left( d - b^T A^{-1} b \right) (\nu - \mu^T \Lambda^{-1} \mu) \]
\[ = d \left( d - b^T A^{-1} b \right) \nu - \mu^T \left( d \left( d - b^T A^{-1} b \right) A^{-1} \mu \right) \]
\[ = d \left( d - b^T A^{-1} b \right) \nu - \mu^T \left( (d - b^T A^{-1} b) A^{-1} + A^{-1} bb^T A^{-1} \right) \mu , \]
which after applying the definitions \( (6) \) expands to
\[ d^3 f - d^2 c^2 - d^2 f (b^T A^{-1} b) + dc^2 (b^T A^{-1} b) - de^2 (b^T A^{-1} b) \]
\[ + c^2 (b^T A^{-1} b) (b^T A^{-1} b) - e^2 (b^T A^{-1} b) (b^T A^{-1} b) + d^2 e (c^T A^{-1} b) \]
\[ - de (b^T A^{-1} b) (c^T A^{-1} b) + de (c^T A^{-1} b) (b^T A^{-1} b) + d^2 e (b^T A^{-1} c) \]
\[ - de (b^T A^{-1} b) (b^T A^{-1} c) + de (b^T A^{-1} b) (b^T A^{-1} c) - d^3 (c^T A^{-1} c) \]
\[ + d^2 (b^T A^{-1} b) (c^T A^{-1} c) - d^2 (c^T A^{-1} b) (b^T A^{-1} c) \]
and then after cancellation can be put in the form \( d^2 \xi \), where

\[
\xi := \begin{vmatrix}
  d - b^\top A^{-1} b & e - b^\top A^{-1} c \\
  e - c^\top A^{-1} b & f - c^\top A^{-1} c
\end{vmatrix}.
\]

Thus we have eventually arrived at

\[
|\nabla \otimes \nabla \Delta_y| \left(2 \Delta_y - (\nabla \Delta_y)^\top (\nabla \otimes \nabla \Delta_y)^{-1} (\nabla \Delta_y)\right)
\]

\[
= (-8)^{n+1} d^{n-2} |A| \cdot d^2 \xi
\]

On the other hand, since we assumed \( A \) to be non-singular, by means of a finite sequence of elementary column operations we are able to eliminate \( b \) and \( c \) from the last two columns of \( Q \). Indeed, both

\[
\begin{bmatrix}
  A \\
  b^\top \\
  c^\top
\end{bmatrix} A^{-1} b = \begin{bmatrix}
  b \\
  b^\top A^{-1} b \\
  c^\top A^{-1} b
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
  A \\
  b^\top \\
  c^\top
\end{bmatrix} A^{-1} c = \begin{bmatrix}
  c \\
  b^\top A^{-1} c \\
  c^\top A^{-1} c
\end{bmatrix}
\]

are linear combinations of leading \( n \) columns of \( Q \) with vectors of coefficients \( A^{-1} b \) and \( A^{-1} c \), respectively. Now, since the determinant is invariant under column addition, it follows that

\[
|Q| = \begin{vmatrix}
  A & b & c \\
  b^\top & d & e \\
  c^\top & e & f
\end{vmatrix} = \begin{vmatrix}
  A & 0 & 0 \\
  b^\top & d - b^\top A^{-1} b & e - b^\top A^{-1} c \\
  c^\top & e - c^\top A^{-1} b & f - c^\top A^{-1} c
\end{vmatrix}.
\]

Observe that the latter matrix is block-lower-triangular, in which case its determinant is simply equal to \( |A| \xi \). Therefore \((13)\) reads

\[
|\nabla \otimes \nabla \Delta_y| \left(2 \Delta_y - (\nabla \Delta_y)^\top (\nabla \otimes \nabla \Delta_y)^{-1} (\nabla \Delta_y)\right) = (-8)^{n+1} d^n |Q|.
\]

Finally, combining \((8), (9)\) and \((14)\) yields

\[
|\pm H_y| = 2^{-n} d^{n} \cdot 2^{-n-1} \Delta_y^{-n/2-1} \cdot (8)^{n+1} d^n |Q|
\]

which for symmetric matrix \( Q \) is equivalent to \((2)\). This concludes the proof. \( \square \)

Remark. Observe that if \( |H_y| \) or \( \Delta_y \) takes a zero value anywhere on \( \Omega \), then the matrix \( Q \) is singular and thus \( |H_y| \) vanishes identically. Indeed, if \( \Delta_y \) vanishes identically, then \( Q \) is a rank-one matrix, in which case \( y \) is affine and thus its Hessian determinant is zero. Otherwise, as a non-zero quadratic polynomial, \( \Delta_y \) does not vanish on an open dense subset of \( \Omega \), which implies that \( |H_y| \) must be zero there. Hence, by continuity, it vanishes on the whole \( \Omega \).

Remark. The sign \( \mp \) which depends on the selected branch of the square root function has any significance only for odd \( n \).

Remark. Since the proof of Theorem 1 was purely algebraic, the same result holds if we consider e.g. the field \( \mathbb{C} \) of complex numbers instead of the field \( \mathbb{R} \) of real numbers. However, note that both \((1)\) and \((3)\) should then be interpreted in the holomorphic sense, i.e. without conjugation.
References

[1] D. A. Harville, *Matrix algebra from a statistician’s perspective*, Springer New York, 2011.