Thermodynamics of Various Entropies in Specific Modified Gravity with Particle Creation

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Abstract

We consider the particle creation scenario in the dynamical Chern-Simons modified gravity in the presence of perfect fluid equation of state \( p = (\gamma - 1)\rho \). By assuming various modified entropies (Bekenstein, logarithmic, power law correction and Reiny), we investigate the first law of thermodynamics and generalized second law of thermodynamics on the apparent horizon. In the presence of particle creation rate, we discuss the generalized second law of thermodynamics and thermal equilibrium condition. It is found that thermodynamic laws and equilibrium condition remain valid under certain conditions of parameters.

1 Introduction

Recently, different observations such as cosmic microwave background radiations (CMBR) [1] and sloan digital sky survey (SDSS) [2] have confirmed the accelerated expansion of universe. It is predicted that the expansion of the universe is due the curious form of force, i.e, dark energy (DE). This discovery was unexpected, because before this invention cosmologists just
think that the expansion of the universe would be decelerating because of
the gravitational attraction of the matter in the universe. Scientists have
proposed various aspects of dynamical DE models such as quintessence [3],
K-essence [4], phantom [5], tachyon [7], holographic [8] and pil-
grim DE [9]-[13]. The simplest candidate of DE is cosmological constant but
its formation and mechanisms are unknown. In order to explain the cosmic
acceleration, various DE models and modified theories of gravity have been
predicted such as $f(R)$, $f(T)$ [14]-[16], $f(T, T_G)$ [17] [18], $f(T, T)$ [19] [20],
dynamical Chern-Simons modified gravity [21]-[23] etc.

In cosmology, attempts to disclose the connection between Einstein grav-
ity and thermodynamics were carried out in [25]-[35]. The basic concept of
thermodynamics comes from black hole physics. In general, there has been
some profound thought on the connection among gravity and thermodynam-
ics for a long time. The initial work was done by Jacobson who showed
that the gravitational Einstein equation can be derived from the relation
between the horizon area and entropy, together with the Clausius relation
$\delta Q = T \delta S$ (where $\delta Q$, $T$ and $\delta S$ represents the change in energy, temperature
and entropy change of the system respectively) [36]. Further, various gravity
theories has been investigated to study the deep connection between gravity
and thermodynamics [37]-[44]. Cosmological investigations of thermodynamics
in modified gravity theories have been executed in Refs. [45]-[51] (for a
recent review on thermodynamic properties of modified gravity theories, see,
e.g., [52]). Saha and Mondal [53] have studied thermodynamics on appar-
tent horizon with the help of gravitationally induced particle scenario with
constant specific entropy and particle creation rate $\Gamma$.

In the present work, we develop the particle creation scenario in the dy-
namical Chern-Simons modified gravity by assuming various modified en-
tropies (bekenstein entropy logarithmic entropy, power law correction and
Reyni entropy) on the apparent horizon. In the presence of particle creation
rate, we discuss the generalized second law of thermodynamics (GSLT). This
paper is organized as follows: In the next section, we will present the basic
equations of dynamical Chern-Simons modified theory and particle creation
rate. Also, we investigate the first law of thermodynamics. Section 3 contains
the illustration of GSLT. In section 4, we analyze the stability of thermody-
namical equilibrium for constant as well as variable $\Gamma$. Section 5 contains
the comparison of results with preceding works. In the last section, we sum-
marize our results.
2 Modified Entropies and Particle Creation Rate

It was shown that the differential form of the Friedmann equation in the FRW universe can be written in the form of the first law of thermodynamics on the apparent horizon. The profound connection provides a thermodynamical interpretation of gravity which makes it interesting to explore the cosmological properties through thermodynamics. It was proved that for any spherically symmetric spacetime, the field equations can be expressed as $TdS = dE + PdV$ for any horizon \[54\], where $E$, $P$ and $V$ represent the internal energy, pressure and volume of the spherical system respectively. The generalized second law of thermodynamics (GSLT) has been studied extensively in the behavior of expanding universe. According to GSLT, ” the sum of all entropies of the constituents (mainly DM and DE) and entropy of boundary (either it is apparent or event horizons) of the universe can never decrease.” \[55\]. Most of the researchers have discussed the validity of GSLT of different systems including interaction of two fluid components dark energy (DE) and dark matter (DM) \[56\] and interaction of three fluid components \[57\] in FRW universe.

To discuss the behavior of GSLT, scientists assumed the horizon entropy as $1/4$ of its area \[58\], power law correction \[59\], logarithmic entropy \[60\] and Reyni entropy. GSLT has been discussed on the basis of gravitationally induced particle scenario which was firstly introduced by Schrodinger \[61\] on microscopic level. Parker at al. \[62\]-\[66\] extend this mechanism towards quantum field theory in curved space. Prigogine at al. \[67\] introduced the macroscopic mechanism of gravitationally induced particle scenario. Afterward, covariant description and difference between particle creation and bulk viscosity of creation process was given \[68\]-\[70\]. The particle creation process can be predicted with the incorporation of backreaction term in the Einstein field equations whose negative weight may help in clarifying the cosmic acceleration. In such a way, most of the phenomenological models of particle creation have been granted \[71\]-\[76\]. In addition, it was proved that phenomenological particle creation help us to discuss the behavior of accelerating universe and paved the alternative way to the concordance ΛCDM model \[76\]-\[80\].

In the following discussion, we check the validity of first law of thermodynamics with Gibbs relation, GSLT and thermodynamical equilibrium by
assuming the following entropy corrections.

- **Bekenstein entropy:** The Bekenstein entropy and Hawking temperature of the apparent horizon are given by \((8\pi = G = 1)\)

\[
S_A = \frac{A}{4} = \frac{R_A^2}{8} \quad \text{and} \quad T_A = \frac{1}{2\pi R_A} = \frac{4}{R_A} \quad \text{where} \quad A = 4\pi R_A^2. \quad (1)
\]

- **Logarithmic corrected entropy:** To study the expansion of entropy of the universe, we discuss the addition of entropy related to the horizon. Quantum gravity allows the logarithmic corrections in the presence of thermal equilibrium fluctuations and quantum fluctuations [81]-[87]. The logarithmic entropy corrections can be defined as

\[
S_A = \frac{A}{4L_p^2} + \alpha \ln \frac{A}{4L_p^2} + \beta \frac{4L_p^2}{A}, \quad (2)
\]

where \(L_p\) is the Planck’s length and \(\alpha, \beta\) are constants whose values are still under consideration.

- **Power law corrected entropy:** Thermodynamics of apparent horizon in the standard FRW cosmology, the geometric entropy is assumed to be proportional to its horizon area \((S_A = \frac{A}{4})\). The quantum corrections provided to the entropy-area relationship lead to the curvature correction in the Einstein-Hilbert action and vice versa [88]. The power-law quantum correction to the horizon entropy motivated by the entanglement of quantum fields between inside and outside of the horizon is given by [89]

\[
S_A = \frac{A}{4L_p^2} \left(1 - K_\delta A^{1-\frac{\delta}{2}}\right), \quad (3)
\]

where \(K_\delta = \frac{\delta \left(\frac{4\pi}{4-\delta}\right)^{\frac{4}{4-\delta}}}{(4-\delta)r_c^{4-\delta}}\). Here \(\delta\) is a dimensionless constant and \(r_c\) is the crossover scale.

- **Renyi entropy:** A novel sort of Renyi entropy has been proposed and inspected [90]-[92]. In which not exclusively is the logarithmic corrected entropy of the original Renyi entropy utilized yet the Bekenstein
Hawking entropy $S_{BH}$ is thought to be a non-extensive Tsallis entropy $S_A$. One can obtain Renyi entropy $S_R$ as \[ S_R = \frac{\ln(1 + \eta S_A)}{\eta}. \] (4)

The action of Chern-Simons theory is given by [21]-[23] \[ S = \frac{1}{16\pi G} \int d^4x \left[ \sqrt{-g}R + \frac{\ell}{4} \theta^* R_{\rho\sigma\mu\nu} R_{\rho\sigma\mu\nu} - \frac{1}{2} g^{\mu\nu} \nabla_{\mu} \theta \nabla_{\nu} \theta + V(\theta) \right] + S_{\text{mat}}, \] (5)

where $R$, $R_{\rho\sigma\mu\nu} R_{\rho\sigma\mu\nu}$, $\ell$, $\theta$, $S_{\text{mat}}$, and $V(\theta)$ are Ricci scalar, a topological invariant called the Pontryagin term, the coupling constant, the dynamical variable, the action of matter and the potential, respectively. The Friedmann equation for flat universe turns out to be [24] \[ H^2 = \frac{1}{3\rho} + \frac{c^2}{6a^6}, \] (6)

here $c$ is constant, $H = \frac{\dot{a}(t)}{a(t)}$ is the Hubble parameter and $a(t)$ is the scale factor. The equation of continuity for this model can be described as \[ \dot{\rho} + \Theta (\rho + P + \Pi) = 0. \] (7)

The particle creation pressure ($\Pi$) representing the gravitationally induced process for particle creation and $\Theta = 3H$ is the fluid expansion. The total number of $n$—particles in an open thermodynamics are \[ \dot{n} + \Theta n = n\Gamma, \] (8)

where $\Gamma$ is the creation rate of number of particles in comoving volume i.e., $(N = na^3)$ having two phases negative and positive. The negative $\Gamma$ relates with the particle anhilation and the positive $\Gamma$ relates with production of particle. Equations (7) and (8) with Gibbs relations can be written as \[ Tds = d \left( \frac{\rho}{n} \right) + pd \left( \frac{1}{n} \right). \] (9)

The equation related to creation pressure $\Pi$ and $\Gamma$ can be determined as \[ \Pi = -\frac{\Gamma}{\Theta}(\rho + p). \] (10)
Under traditional assumption that the specific entropy of each particle is constant, i.e., the process is adiabatic or isentropic, which implies that dissipative fluid is similar to a perfect fluid with a non-conserved particle number. The respective EoS for this model represented by \( p = (\gamma - 1)\rho \). Differentiation of Eq.(6) gives
\[
\dot{H} = -\frac{1}{2} \left( (\rho + p + \Pi) - \frac{\Gamma}{a^6} \right).
\] (11)

Inserting Eqs.(10) and \( p = (\gamma - 1)\rho \) in above equation, we get
\[
\frac{\dot{H}}{H^2} = -\frac{1}{2H^2} \left( \gamma \left( 3H^2 - \frac{c^2}{2a^6} \right) \left( 1 - \frac{\Gamma}{3H} \right) + \frac{c^2}{a^6} \right).
\] (12)

For flat FRW universe, Hubble parameter relates with the apparent horizon as \( R_A = \frac{1}{H} \). Differentiating the apparent horizon with respect to time, we have
\[
\dot{R}_A = -\frac{\dot{H}}{H^2} = \frac{1}{2H^2} \left( \gamma \left( 3H^2 - \frac{c^2}{2a^6} \right) \left( 1 - \frac{\Gamma}{3H} \right) + \frac{c^2}{a^6} \right).
\] (13)

The deceleration parameter \( q \) is of the form
\[
q = -\frac{\dot{H}}{H^2} - 1 = \frac{1}{2H^2} \left( \gamma \left( 3H^2 - \frac{c^2}{2a^6} \right) \left( 1 - \frac{\Gamma}{3H} \right) + \frac{c^2}{a^6} \right) - 1.
\] (14)

### 2.1 First Law of Thermodynamics

Next, we investigate the first law of thermodynamics in the presence of modified entropies. The relation between thermodynamics and Einstein field equations was found by Jacobson with the help of clausius relation at apparent horizon described as
\[
-dE_A = T_A dS_A.
\] (15)

For the sake of convenance we consider \( X = T_A dS_A + dE_A \). The differential \( dE_A \) is the amount of energy crossing the apparent horizon can be evaluated as
\[
-dE_A = \frac{1}{2} R^3 (\rho + p) H dt = \frac{\gamma}{2H^2} \left( 3H^2 - \frac{c^2}{2a^6} \right).
\] (16)
Bekenstein entropy:

From Eq. (11), the differential of surface entropy at apparent horizon leads to

$$dS_A = \frac{1}{8H^3} \left( \gamma (3H^2 - \frac{c^2}{2a^6}) \left(1 - \frac{\Gamma}{3H}\right) + \frac{c^2}{a^6} \right). \quad (17)$$

The above equation with horizon temperature leads to

$$T_A dS_A = \frac{1}{2H^2} \left( \gamma (3H^2 - \frac{c^2}{2a^6}) \left(1 - \frac{\Gamma}{3H}\right) + \frac{c^2}{a^6} \right). \quad (18)$$

Hence $X$ becomes

$$X = -\frac{\gamma}{2H^2} (3H^2 - \frac{c^2}{2a^6}) + \frac{1}{2H^2} \left( \gamma (3H^2 - \frac{c^2}{2a^6}) \left(1 - \frac{\Gamma}{3H}\right) + \frac{c^2}{a^6} \right). \quad (19)$$

With the help of Eq. (19), we observe that the first law of thermodynamics holds (i.e., $X \to 0$) when

$$\Gamma = 3H \left( \frac{c^2}{2a^6} \left(3H^2 - \frac{c^2}{2a^6}\right) \right). \quad (20)$$

Logarithmic corrected entropy:

The differential form of Eq. (2) is given as

$$dS_A = \frac{\left( \gamma (3H^2 - \frac{c^2}{2a^6}) \left(1 - \frac{\Gamma}{3H}\right) + \frac{c^2}{a^6} \right) \left( \frac{1}{4H L_p^2} + 2\alpha H - 16\beta H^4 L_p^2 \right)}{2H^2} dt, \quad (21)$$

which leads to

$$T_A dS_A = \frac{1}{2H^2} \left( \gamma (3H^2 - \frac{c^2}{2a^6}) \left(1 - \frac{\Gamma}{3H}\right) + \frac{c^2}{a^6} \right) \left( \frac{1}{L_p^2} + 8H^2 \alpha - 64\beta H^4 L_p^2 \right). \quad (22)$$

Combining Eqs. (16) and (22), we get

$$X = \frac{1}{2H^2} \left( \gamma (3H^2 - \frac{c^2}{2a^6}) \left(1 - \frac{\Gamma}{3H}\right) + \frac{c^2}{a^6} \right) \left( \frac{1}{L_p^2} + 8H^2 \alpha - 64\beta H^4 L_p^2 \right)$$
\[ - \frac{\gamma}{2H^2} (3H^2 - \frac{c^2}{2a^6}). \] (23)

From above equation, we observe the validity of first law of thermodynamics if

\[ \Gamma = 3H \left( 1 - \frac{1}{\gamma} (3H^2 - \frac{c^2}{a^6})^{-1} \left( \gamma (3H^2 - \frac{c^2}{2a^6}) \left( \frac{1}{L_p^2} + 8H^2\alpha - 64\beta H^4 L_p^2 - \frac{c^2}{a^6} \right) \right) \right). \] (24)

**Power law corrected entropy:**

Differentiating Eq.(3), we get

\[ dS_A = \frac{1}{2H^2} \left( \gamma (3H^2 - \frac{c^2}{2a^6}) (1 - \frac{\Gamma}{3H}) + \frac{c^2}{a^6} \right) \left( \frac{1}{4HL_p^2} - \frac{K_\delta}{4L_p^2} (2 - \frac{\delta}{2}) \left( \frac{1}{H} \right)^{2-\delta} \right) dt. \] (25)

which yields

\[ T_A dS_A = \frac{1}{2H^2} \left( \gamma (3H^2 - \frac{c^2}{2a^6}) (1 - \frac{\Gamma}{3H}) + \frac{c^2}{a^6} \right) \left( \frac{1}{L_p^2} - (2 - \frac{\delta}{2}) \frac{K_\delta}{L_p^2} \left( \frac{1}{H} \right)^{2-\delta} \right). \] (26)

Hence, \( X \) takes the form

\[ X = \frac{1}{2H^2} \left( \gamma (3H^2 - \frac{c^2}{2a^6}) (1 - \frac{\Gamma}{3H}) + \frac{c^2}{a^6} \right) \left( \frac{1}{L_p^2} - (2 - \frac{\delta}{2}) \frac{K_\delta}{L_p^2} \left( \frac{1}{H} \right)^{2-\delta} \right) - \frac{\gamma}{2H^2} (3H^2 - \frac{c^2}{2a^6}). \] (27)

From Eq.(27), it can be analyzed that the first law of thermodynamics remains valid for the following particle creation rate

\[ \Gamma = 3H \left( 1 - \frac{1}{\gamma} (3H^2 - \frac{c^2}{2a^6})^{-1} \left( \gamma (3H^2 - \frac{c^2}{2a^6}) \left( \frac{1}{L_p^2} - (2 - \frac{\delta}{2}) \frac{K_\delta}{L_p^2} \left( \frac{1}{H} \right)^{2-\delta} \right)^{-1} \right) \right). \] (28)

**Renyi entropy:**

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Eq. (4) gives the differential of entropy as

$$dS_R = \frac{1}{H} \left( \gamma \left( 3H^2 - \frac{c^2}{2a^6} \right) \left( 1 - \frac{H}{3H} + \frac{c^2}{a^6} \right) \left( \frac{1}{\eta + 8H^2} \right) \right).$$

(29)

which gives rise to

$$T_A dS_R = \frac{4}{(\eta + 8H^2)} \left( \gamma \left( 3H^2 - \frac{c^2}{2a^6} \right) \left( 1 - \frac{H}{3H} + \frac{c^2}{a^6} \right) \right).$$

(30)

Using Eqs. (16) and (30) we get

$$X = \frac{4}{(\eta + 8H^2)} \left( \gamma \left( 3H^2 - \frac{c^2}{2a^6} \right) \left( 1 - \frac{H}{3H} + \frac{c^2}{a^6} \right) \right) - \frac{\gamma}{2H^2} \left( 3H^2 - \frac{c^2}{a^6} \right).$$

(31)

From Eq. (31), it can be seen that the first law of thermodynamics is showing the validity when

$$\Gamma = 3H \left( 1 - \frac{1}{\gamma} \left( 3H^2 - \frac{c^2}{2a^6} \right)^{-1} \left( \frac{\gamma(\eta + 8H^2)}{8H^2} (3H^2 - \frac{c^2}{2a^6}) - \frac{c^2}{a^6} \right) \right).$$

(32)

3 Generalized Second Law of Thermodynamics

We discuss the GSLT of an isolated macroscopic physical system where the total entropy $S_T$ must satisfies the following conditions $d(S_A + S_f) \geq 0$ i.e., entropy function cannot be decrease. In this relation, $S_A$ and $S_f$ appear as the entropy at apparent horizon and the entropy of cosmic fluid enclosed within the horizon, respectively. The Gibbs equation is of the form

$$T_f dS_f = dE_f + pdV,$$

(33)

where $T_f$ is the temperature of the cosmic fluid and $E_f$ is the energy of the fluid ($E_f = \rho V$). The evolution equation for fluid temperature having constant entropy can be described as

$$\frac{\dot{T}_f}{T_f} = (\Gamma - \Theta) \frac{\partial p}{\partial \rho}.$$  

(34)
Using Eq. (12), we get \((\Gamma - \Theta) = \frac{6H\dot{H} + \frac{c^2}{6a^6}}{\gamma (3H^2 - c^2 a^6)}\), hence, the above Eq. (34) leads to the integral

\[
\ln \left( \frac{T_f}{T_0} \right) = \frac{\gamma - 1}{\gamma} \int \frac{2\dot{H} + \frac{c^2}{6a^6}}{(H^2 - \frac{c^2}{6a^6})} dH.
\]  
(35)

Integration of above equation leads to

\[
T_f = T_0 \left( H^2 - \frac{c^2}{6a^6} \right)^{\frac{\gamma - 1}{\gamma}},
\]  
(36)

where \(T_0\) is the constant of integration. Equation (33) yields the differential of fluid entropy as

\[
dS_f = -\frac{T_0^{-1}}{H^2} \left( H^2 - \frac{c^2}{6a^6} \right)^{\frac{\gamma - 1}{\gamma}} \left( (2\dot{H} + \frac{c^2}{a^6}) - \frac{\gamma \dot{H}}{H^2} (3H^2 - \frac{c^2}{6a^6}) \right).
\]  
(37)

Next, we observe the validity GSLT by assuming Bekenstein entropy, logarithmic corrected entropy, power law correction and Renyi entropy.

**Bekenstein entropy:**

In present case, we get the differential of total entropy by using Eqs. (17) and (37) as

\[
\dot{S}_T = \frac{1}{8H^3} \left( \gamma (3H^2 - \frac{c^2}{2a^6}) (1 - \frac{\Gamma}{3H}) + \frac{c^2}{a^6} \right) + \frac{T_0^{-1}}{H^2} \left( H^2 - \frac{c^2}{6a^6} \right)^{\frac{\gamma - 1}{\gamma}} \times \left( (2\dot{H} + \frac{c^2}{a^6}) - \frac{\gamma \dot{H}}{H^2} (3H^2 - \frac{c^2}{2a^6}) \right),
\]  
(38)

where \(S_T = S_A + S_f\). The plot between \(\dot{S}_T\) versus \(\gamma\) is shown in Figure 1 for three values of \(T\). We observe the validity of GSLT at the present epoch by setting the constant values \(H = H_0 = 67, c = -1, a = a_0 = 1\) and \(q = -0.53\). It can be analyzed from figure that \(\dot{S}_T \geq 0\) for all values of \(T\) leads to the validity of GSLT.

**Logarithmic corrected entropy:**
Using Eqs. (21) and (37) the differential of total entropy is given by

\[
\dot{S}_T = \frac{1}{2H^2} \left( \gamma \left( 3H^2 - \frac{c^2}{2a^6} \right) \left( 1 - \frac{\Gamma}{3H} \right) + \frac{c^2}{a^6} \right) \left( \frac{1}{4HL_p^2} + 2\alpha H - 16\beta H^3L_p^2 \right) dt \\
+ \frac{T_0^{-1}}{H^2} \left( H^2 - \frac{c^2}{6a^6} \right)^{\frac{1-\gamma}{\gamma}} \left( 2\dot{H} + \frac{c^2}{a^6} - \frac{\gamma \dot{H}}{H^2} \left( 3H^2 - \frac{c^2}{2a^6} \right) \right) dt.
\]  

(39)

The plot of $\dot{S}_T$ versus $\gamma$ with respect to the three values of $T$ by fixing the constant values $\alpha = 1$, $\beta = -0.00001$ and $L_P = 1$ as shown in Figure 2. The trajectories in the plot remains which leads to the validity of GSLT in the presence of logarithmic entropy.

**Power law corrected entropy:**
From Eqs. (26) and (37), we get the differential of total entropy as
\[
\dot{S}_T = \frac{1}{2H^2} \left( \gamma (3H^2 - \frac{c^2}{2a^6}) (1 - \frac{\Gamma}{3H}) + \frac{c^2}{a^6} \right) \left( \frac{1}{4HL_p^2} - \frac{K_3}{4L_p^2} \left( 2 - \frac{\delta}{2} \right) \left( \frac{1}{H} \right)^{3-\delta} \right) dt \\
+ \frac{T_0^{-1}}{H^2} \left( H^2 - \frac{c^2}{6a^6} \right)^{\frac{1-\gamma}{\gamma}} \left( (2\dot{H} + \frac{c^2}{a^6}) - \frac{\gamma \dot{H}}{H^2} (3H^2 - \frac{c^2}{2a^6}) \right) dt. 
\]

The plot of $\dot{S}_T$ versus $\gamma$ for three values of $T$ by setting the constant values as $\delta = 1$, $r_c = \frac{1}{67}$, and $L_p = 1$ as shown in Figure 3. We can see $\dot{S}_T$ remains positive for all values of $T$ which leads to the validity of GSLT.

**Renyi entropy:**

We observe the validity of GSLT with $\dot{S}_T \geq 0$ for which Eqs. (29) and (37) takes the form
\[
\dot{S}_T = \frac{1}{H(\eta + 8H^2)} \left( \gamma (3H^2 - \frac{c^2}{2a^6}) (1 - \frac{\Gamma}{3H}) + \frac{c^2}{a^6} \right) + \frac{T_0^{-1}}{H^2} \left( H^2 - \frac{c^2}{6a^6} \right)^{\frac{1-\gamma}{\gamma}} \\
\times \left( (2\dot{H} + \frac{c^2}{a^6}) - \frac{\gamma \dot{H}}{H^2} (3H^2 - \frac{c^2}{2a^6}) \right). 
\]

The plot between $\dot{S}_T$ versus $\gamma$ for three values of $\eta$ is shown in Figure 4 for $\eta = 1$. It can be seen that GSLT is satisfying $\dot{S}_T \geq 0$ for all values of $T$ which leads to the validity of GSLT.
Figure 4: Plot of $\dot{S}_T$ versus $\gamma$ Renyi entropy.

## 4 Thermodynamical equilibrium

We discuss the two scenarios for thermodynamical equilibrium by taking particle creation rate $\Gamma=\text{constant}$ and $\Gamma = \Gamma(t)$ for which entropy function attains a maximum entropy state i.e., $d(S_A + S_I) < 0$.

### 4.1 $\Gamma = \text{constant}$

Firstly, we consider $\Gamma$ as constant and observe the stability of thermodynamical equilibrium.

**Bekenstein entropy:**

We get second order differential equation for constant $\Gamma$ by using Eq. (38)

$$
\dot{\dot{S}}_T = -\frac{3\dot{H}\left(\frac{c^2}{a^7} + \gamma(1 - \frac{\Gamma}{3H})\left(-\frac{c^2}{2a^6} + 3H^2\right)\right)}{8H^4} - 2\dot{H}\left(-\frac{c^2}{6a^6} + H^2\right)^{\frac{1-\gamma}{\gamma}}
\times \left(-\gamma(1 - \frac{\Gamma}{3H})\left(3H^2 - \frac{c^2}{2a^6}\right) + \gamma\left(\frac{3H^2 - \frac{c^2}{2a^6}}{2H^2} + \gamma\left(1 - \frac{\Gamma}{3H}\right)\left(3H^2 - \frac{c^2}{a^6}\right)\right)\right)
\times \left(\frac{H^3T_0}{\gamma H^2T_0} - \gamma(1 - \frac{\Gamma}{3H})\left(3H^2 - \frac{c^2}{2a^6}\right) + \gamma\left(\frac{3H^2 - \frac{c^2}{2a^6}}{2H^2} + \gamma\left(1 - \frac{\Gamma}{3H}\right)\left(3H^2 - \frac{c^2}{a^6}\right)\right)\right)
\times \left(\frac{c^2\dot{a}}{a^7} + 2H\dot{H}(1 - \gamma)(H^2 - \frac{c^2}{6a^6})^{-1 + \frac{1-\gamma}{\gamma}} + \gamma\Gamma\dot{H}\left(3H^2 - \frac{c^2}{2a^6}\right)\right)\frac{1}{24H^5}
$$
Figure 5: Plot of $\ddot{S}_T$ versus $\gamma$ for Bekenstein entropy when $\Gamma$ is constant.

$$\ddot{S}_T = -\frac{6c^2\dot{a}}{8H^3a^7} + \gamma\left(1 - \frac{\Gamma}{3H}\right)\left(\frac{3c^2\dot{a}}{a^7} + 6H\dot{H}\right) + \frac{\left(H^2 - \frac{c^2}{2\alpha}\right)\frac{1-\gamma}{\gamma}}{H^2T_0}\left(-\gamma\Gamma\dot{H}\right)$$

$$\times \left(3H^2 - \frac{c^2}{2a^6}\right) - \gamma\dot{H}\left(3H^2 - \frac{c^2}{2a^6}\right)\left(\frac{c^2}{2a^6} + \gamma\left(1 - \frac{\Gamma}{3H}\right)\left(3H^2 - \frac{c^2}{2a^6}\right)\right)$$

$$- \gamma\left(1 - \frac{\Gamma}{3H}\right)\left(\frac{3c^2\dot{a}}{a^7} + 6H\dot{H}\right) + \gamma\left(\frac{c^2}{2a^6} + \gamma\left(1 - \frac{\Gamma}{3H}\right)\left(3H^2 - \frac{c^2}{2a^6}\right)\right)$$

$$\times \left(\frac{3c^2\dot{a}}{a^7} + 6H\dot{H}\right) + \left(-\frac{6c^2\dot{a}}{a^7} + \frac{\gamma\dot{H}\left(3H^2 - \frac{c^2}{2a^6}\right)}{2H^2} + \frac{3c^2\dot{a}}{a^7} + 6H\dot{H}\right)$$

$$\times \gamma\left(1 - \frac{\Gamma}{3H}\right)\left(3H^2 - \frac{c^2}{2a^6}\right)\gamma. \quad (42)$$

The graphical behavior between $\ddot{S}_T$ versus $\gamma$ as shown in Figure 5 for different values of parameter $T$. We observe that $\ddot{S}_T < 0$ for all values of $T$ which leads to the validity of thermodynamical equilibrium.

**Logarithmic corrected entropy:**

In the presence of logarithmic corrected entropy second order differential equation is obtained from Eq.(39) as

$$\ddot{S}_T = -\frac{\dot{H}\left(\frac{1}{4H^3} + 2\alpha H - 16L^p\beta H^3\right)\left(\frac{c^2}{2a^6} + \gamma\left(1 - \frac{\Gamma}{3H}\right)\left(3H^2 - \frac{c^2}{2a^6}\right)\right)}{H^3}$$

$$- \left(-\gamma\left(1 - \frac{\Gamma}{3H}\right)\left(3H^2 - \frac{c^2}{2a^6}\right) + \frac{\gamma\left(3H^2 - \frac{c^2}{2a^6}\right)\left(\frac{c^2}{2a^6} + \gamma\left(1 - \frac{\Gamma}{3H}\right)\left(3H^2 - \frac{c^2}{2a^6}\right)\right)}{2H^2T_0}\right)$$
Figure 6: Plot of $\ddot{S}_T$ versus $\gamma$ for logarithmic corrected entropy when $\Gamma$ is a constant.

\begin{align*}
\times 2\dot{H}(H^2 - \frac{c^2}{6a^6})^{1-\gamma} + (1 - \gamma)\left(\frac{c^2\dot{a}}{a^7} + 2H\dot{H}\right)(H^2 - \frac{c^2}{6a^6})^{-1+\frac{1-\gamma}{2}} \\
\times \left(-\gamma(1 - \frac{\Gamma}{3H})(3H^2 - \frac{c^2}{2a^6}) + \frac{\gamma(3H^2 - \frac{c^2}{2a^6})}{2H^4}\gamma(1 - \frac{\Gamma}{3H})(3H^2 - \frac{c^2}{2a^6})\right) \\
\gamma H^2T_0 \\
+ \frac{(\frac{c^2}{a^6} + \gamma(1 - \frac{\Gamma}{3H})(3H^2 - \frac{c^2}{2a^6}))}{2H^2} \left(2\alpha \dot{H} - \frac{H}{4L_p^2 H^2} - 48L_p^2 \beta H^2 \dot{H}\right) \\
+ \frac{1}{4HL_p^2} + 2\alpha H - 16L_p^2 \beta H^3 \left(-\frac{6c^2\dot{a}}{a^7} + \gamma(1 - \frac{\Gamma}{3H})\left(\frac{3c^2\dot{a}}{a^7} + 6H\dot{H}\right) \right) \\
+ \frac{\gamma\Gamma \dot{H} (3H^2 - \frac{c^2}{2a^6})}{6H^4} + \frac{1}{H^2T_0} \left(H^2 - \frac{c^2}{6a^6}\right)^{1-\gamma} \left(-\gamma\Gamma \dot{H} (3H^2 - \frac{c^2}{2a^6}) \right) \\
- \frac{\gamma\dot{H} (3H^2 - \frac{c^2}{2a^6})}{H^3} \left(\frac{c^2}{a^6} + \gamma(1 - \frac{\Gamma}{3H})(3H^2 - \frac{c^2}{2a^6})\right) - \gamma(1 - \frac{\Gamma}{3H}) \\
\times \left(\frac{3c^2\dot{a}}{a^7} + 6H\dot{H}\right) + \frac{\gamma\left(\frac{3c^2\dot{a}}{a^7} + 6H\dot{H}\right)}{2H^2} \left(\frac{c^2}{a^6} + \gamma(1 - \frac{\Gamma}{3H})(3H^2 - \frac{c^2}{2a^6})\right) \\
+ \gamma(3H^2 - \frac{c^2}{2a^6}) \left(\frac{\gamma\Gamma \dot{H} (3H^2 - \frac{c^2}{2a^6})}{2H^2} + \gamma\left(1 - \frac{\Gamma}{3H}\right)\left(\frac{3c^2\dot{a}}{a^7} + 6H\dot{H}\right) \right) \\
- \frac{6c^2\dot{a}}{2H^2a^7}\right) \right). \\
\end{align*}

The graphical behavior between $\ddot{S}_T$ and $\gamma$ is shown in Figure 6 for three values of $T$. We observe that all the trajectories are showing the negative increasing behavior, which exhibit the validity of thermodynamical equilib-
thermodynamical equilibrium.

**Power law corrected entropy:**

To find the validity of thermodynamical equilibrium we take $\Gamma$ as constant for which second order differential equation can be expressed with the help of Eq. (40) as

\[
\tilde{S}_T = -\frac{\left(\frac{c^2}{a^\gamma} + \gamma \left(1 - \frac{\Gamma}{3H}\right) (3H^2 - \frac{c^2}{2a^\gamma})\right) \left(\frac{1}{4HL_p} - \left(\frac{2 - \frac{4}{3}}{1}\right) \frac{1}{4L_p} K_\delta\right) \dot{H}}{H^3}
- \left(\frac{\gamma \left(1 - \frac{\Gamma}{3H}\right) (3H^2 - \frac{c^2}{2a^\gamma}) + \gamma \left(\frac{2 - \frac{4}{3}}{1}\right) \frac{1}{3H^2} K_\delta\right) \frac{\dot{H} H}{H^3 T_0}
\times \left(\frac{\gamma \left(1 - \frac{\Gamma}{3H}\right) (3H^2 - \frac{c^2}{2a^\gamma}) + \gamma \left(\frac{2 - \frac{4}{3}}{1}\right) \frac{1}{3H^2} K_\delta\right) \frac{\dot{H} H}{H^3 T_0}
+ \left(\frac{1}{4HL_p} - \left(\frac{2 - \frac{4}{3}}{1}\right) \frac{1}{4L_p} K_\delta\right) \left(\frac{-\frac{6c^2\dot{a}}{a^\gamma}}{H^2} + \gamma \left(1 - \frac{\Gamma}{3H}\right) \frac{3c^2\dot{a}}{a^\gamma} + 6H \dot{H}\right)
+ \frac{\gamma \Gamma \dot{H} \left(3H^2 - \frac{c^2}{2a^\gamma}\right)}{6H^4} + \left(\frac{H^2 - \frac{c^2}{2a^\gamma}}{H^2 T_0}\right) \left(\frac{\gamma \Gamma \dot{H} \left(3H^2 - \frac{c^2}{2a^\gamma}\right)}{3H^2} - \gamma \frac{3c^2\dot{a}}{a^\gamma}\right)
+ 6H \dot{H} \left(1 - \frac{\Gamma}{3H}\right) - \gamma \dot{H} \left(3H^2 - \frac{c^2}{2a^\gamma}\right) + \gamma \left(1 - \frac{\Gamma}{3H}\right) \left(3H^2 - \frac{c^2}{2a^\gamma}\right)
+ \frac{\gamma \left(\frac{3c^2\dot{a}}{a^\gamma} + 6H \dot{H}\right)}{2H^2} \left(\frac{c^2}{a^\gamma} + \gamma \left(1 - \frac{\Gamma}{3H}\right) \left(3H^2 - \frac{c^2}{2a^\gamma}\right)\right) + \gamma \left(3H^2 - \frac{c^2}{2a^\gamma}\right)
\times \left(\frac{-\frac{6c^2\dot{a}}{a^\gamma} + \gamma \Gamma \dot{H} \left(3H^2 - \frac{c^2}{2a^\gamma}\right)}{2H^2} + \gamma \left(1 - \frac{\Gamma}{3H}\right) \left(\frac{3c^2\dot{a}}{a^\gamma} + 6H \dot{H}\right)\right). (44)
\]

The plot of $\tilde{S}_T$ versus $\gamma$ as shown in Figure 7 for three values of $T$. We analyze that thermodynamical equilibrium is satisfying the condition $\tilde{S}_T < 0$ for all values of $T$, which leads to validity of thermodynamical equilibrium.
Figure 7: Plot of $\ddot{S}_T$ versus $\gamma$ for power law corrected entropy when $\Gamma$ is constant.

**Renyi entropy:**

In present scenario, we observe the validity of thermodynamical equilibrium by keeping $\Gamma$ as constant for which second order differential equation is given by using Eq. (11)

$$\ddot{S}_T = -\frac{16\dot{H}}{(\eta + 8H^2)^2}\left(\frac{c^2}{a^6} + \gamma(3H^2 - \frac{c^2}{2a^6})(1 - \frac{\Gamma}{3H})\right) - \frac{\dot{H}}{H^2(\eta + 8H^2)} \times \left(\frac{c^2}{a^6} + \gamma(3H^2 - \frac{c^2}{2a^6})(1 - \frac{\Gamma}{3H})\right)$$

$$\times \left(1 - \frac{\Gamma}{3H}\right)(-\gamma) + \frac{\gamma(3H^2 - \frac{c^2}{2a^6})(\frac{c^2}{a^6} + \gamma(3H^2 - \frac{c^2}{2a^6})(1 - \frac{\Gamma}{3H}))}{2H^2} - \gamma(3H^2 - \frac{c^2}{2a^6})$$

$$\times \left(1 - \frac{\Gamma}{3H}\right)(1 - \gamma)(\frac{c^2}{a^6} + 2H\dot{H})(H^2 - \frac{c^2}{6a^6})^{-1 + \frac{1}{\gamma}} \gamma H^2 T_0 + \frac{1}{H(\eta + 8H^2)}$$

$$\times \left(-\frac{6c^2\dot{a}}{a^7} + \gamma\Gamma\dot{H}(\frac{3H^2 - \frac{c^2}{2a^6}}{3H^2}) + \gamma(1 - \frac{\Gamma}{3H})(\frac{3c^2\dot{a}}{a^7} + 6H\dot{H})\right) + \frac{1}{H^2 T_0}$$

$$\times \left(H^2 - \frac{c^2}{6a^6}\right)^{\frac{1}{1 - \gamma}} \left(-\gamma\dot{H}(\frac{3H^2 - \frac{c^2}{2a^6}}{H^3})(\frac{c^2}{a^6} + \gamma(3H^2 - \frac{c^2}{2a^6})(1 - \frac{\Gamma}{3H}))\right)$$
The graphical behavior of $\ddot{S}_T$ versus $\gamma$ for three values of $T$ as shown in Figure 8. It can be seen that $\ddot{S}_T$ is negative for all values of $T$ satisfying the condition $d^2S_T/dt^2 < 0$ which exhibits the thermodynamical equilibrium.

4.2 $\Gamma = \Gamma(t)$

Secondly, we consider particle creation rate $\Gamma$ as a variable, i.e., $\Gamma = \Gamma(t)$.

**Bekenstein entropy:**

The differentiation of Eq. (38) leads to

$$\ddot{S}_T = -\frac{3\dot{H}(\frac{c^2}{a^7} + \gamma(1 - \frac{\Gamma}{3H})(\frac{c^2}{2a^6} + 3H^2))}{8H^4} - \frac{2\dot{H}}{6a^6 + H^2} \frac{1}{1 - \gamma}$$

$$\times \left( -\gamma(H^2 - \frac{c^2}{2a^6})(1 - \frac{\Gamma}{3H}) + \frac{\gamma(3H^2 - \frac{c^2}{2a^6})(\frac{c^2}{2a^6} + \gamma(3H^2 - \frac{c^2}{2a^6})(1 - \frac{\Gamma}{3H}))}{2H^2} \right)$$

$$\times \frac{H^3T_0}{H^3T_0}$$

$$+ \left( -\gamma(H^2 - \frac{c^2}{2a^6})(1 - \frac{\Gamma}{3H}) + \frac{\gamma(3H^2 - \frac{c^2}{2a^6})(\frac{c^2}{2a^6} + \gamma(3H^2 - \frac{c^2}{2a^6})(1 - \frac{\Gamma}{3H}))}{2H^2} \right)$$

$$\times \frac{H^3T_0}{\gamma H^2T_0}$$
Figure 9: Plot of $\dot{S}_T$ versus $\gamma$ for Bekenstein entropy when $\Gamma = \Gamma(t)$.

\[
\begin{align*}
\dot{S}_T &= (1 - \gamma) \left( \frac{c^2 a}{\dot{a}} + 2H \dot{H} \right) \left( H^2 - \frac{c^2}{6a^6} \right)^{-1 + \frac{1 - \gamma}{H^2}} - \frac{6c^2 a}{8H^3 a^7} + \frac{(3c^2 a + 6H \dot{H})}{8H^3} \\
\times \gamma \left( 1 - \frac{\Gamma}{3H} \right) + \gamma \left( \frac{3H^2 - \frac{c^2}{2a^6} \left( \frac{\Gamma H}{3H^2} - \frac{\dot{H}}{3H} \right)}{8H^3} \right) + \frac{(H^2 - \frac{c^2}{2a^6})^{1 - \gamma}}{H^2 T_0} \\
\times \left( \frac{3c^2 a}{a^7} + 6H \dot{H} \right) + \gamma \left( \frac{c^2}{2a^6} + \frac{\gamma (3H^2 - \frac{c^2}{2a^6} \left( 1 - \frac{\Gamma}{3H} \right)) (3c^2 a + 6H \dot{H})}{2H^2} \right) - \frac{\gamma (3H^2 - \frac{c^2}{2a^6} \left( \frac{\Gamma H}{3H^2} - \frac{\dot{H}}{3H} \right))}{2H^2} \left( \frac{\dot{H}}{3H^2} - \frac{\dot{H}}{3H} \right) + \left( -\frac{6c^2 a}{a^7} + \gamma \left( 1 - \frac{\Gamma}{3H} \right) \frac{3c^2 a + 6H \dot{H}}{2H^2} \right) \\
+ \gamma \left( 3H^2 - \frac{c^2}{2a^6} \left( \frac{\Gamma H}{3H^2} - \frac{\dot{H}}{3H} \right) \right) \left( 3H^2 - \frac{c^2}{2a^6} \right) \gamma. \quad (46)
\end{align*}
\]

The plot between $\dot{S}_T$ versus $\gamma$ for three values of $T$ as shown in Figure 9 by keeping the constant values same as for constant $\Gamma$. It is observed that the thermodynamical is obeying the condition $\dot{S}_T < 0$ which leads to the validity of thermodynamical equilibrium.

**Logarithmic corrected entropy:**

We discuss the stability analysis of thermal equilibrium in the presence of logarithmic corrected entropy by taking $\Gamma$ as variable for which Eq. (39) reduces
to second order differential equation as

\[ \dot{S}_T = -\frac{\dot{H}(\frac{1}{4H^2} + 2\alpha H - 16L_p^2\beta H^3)\left(\frac{c^2}{a^6} + \gamma\left(1 - \frac{\Gamma}{3H}\right)(3H^2 - \frac{c^2}{2a^6})\right)}{H^3} \]

\[ - \left(-\gamma\left(1 - \frac{\Gamma}{3H}\right)(3H^2 - \frac{c^2}{2a^6}) + \frac{\gamma(3H^2 - \frac{c^2}{a^6})\left(\frac{c^2}{a^6} + \gamma\left(1 - \frac{\Gamma}{3H}\right)(3H^2 - \frac{c^2}{2a^6})\right)}{2H^2}\right) \]

\[ \times 2\dot{H}\left(H^2 - \frac{c^2}{6a^6}\right)^{\frac{1-\gamma}{2}} + (1 - \gamma)(\frac{c^2}{a^6} + 2H\dot{H})\left(H^2 - \frac{c^2}{6a^6}\right)^{-1+\frac{1-\gamma}{2}} \]

\[ \times \left(-\gamma\left(1 - \frac{\Gamma}{3H}\right)(3H^2 - \frac{c^2}{2a^6}) + \frac{\gamma(3H^2 - \frac{c^2}{a^6})\left(\frac{c^2}{a^6} + \gamma\left(1 - \frac{\Gamma}{3H}\right)(3H^2 - \frac{c^2}{2a^6})\right)}{2H^2}\right) \]

\[ \times \left(\frac{c^2}{a^6} + (1 - \frac{\Gamma}{3H})(3H^2 - \frac{c^2}{2a^6})\right)\left(2\alpha\dot{H} - \frac{H}{4L_p^2H^2} - 48L_p^2\beta H^2 \dot{H}\right) \]

\[ + \frac{2\dot{H}}{\gamma H^2 T_0} \]

\[ + \left(\frac{1}{4H^2} + 2\alpha H - 16L_p^2\beta H^3\right)\left(-\frac{6c^2}{a^6} + (1 - \frac{\Gamma}{3H})(\frac{3c^2}{a^6} + 6H\dot{H})\right) \]

\[ + \frac{\gamma(3H^2 - \frac{c^2}{2a^6})\left(\frac{\dot{H}}{H^2} - \frac{\dot{H}}{3H}\right)}{H^2 T_0} + \frac{1}{H^2 T_0}\left(H^2 - \frac{c^2}{6a^6}\right)^{\frac{1-\gamma}{2}} \left(-\left(3H^2 - \frac{c^2}{2a^6}\right)\right) \]

\[ \times \gamma\dot{H}\left(\frac{c^2}{a^6} + (1 - \frac{\Gamma}{3H})(3H^2 - \frac{c^2}{2a^6})\right) - \gamma\left(1 - \frac{\Gamma}{3H}\right)(\frac{3c^2}{a^6} + 6H\dot{H}) \]

\[ + \frac{\gamma\left(\frac{c^2}{a^6} + (1 - \frac{\Gamma}{3H})(3H^2 - \frac{c^2}{2a^6})\right)(\frac{3c^2}{a^6} + 6H\dot{H})}{2H^2} \]

\[ \times \left(\frac{\dot{H}}{3H^2} - \frac{\dot{H}}{3H}\right) + \frac{\gamma(3H^2 - \frac{c^2}{2a^6})\left(-\frac{6c^2}{a^6} + (1 - \frac{\Gamma}{3H})(\frac{3c^2}{a^6} + 6H\dot{H})\right)}{2H^2} \]

\[ + \frac{\gamma^2(3H^2 - \frac{c^2}{2a^6})^2}{2H^2} \left(\frac{\dot{H}}{3H^2} - \frac{\dot{H}}{3H}\right) \right). \]

(47)

The plot of $\dot{S}_T$ versus $\gamma$ as shown in Figure 10 for three values of $T$ by keeping the same values as in above case, we observe that thermal equilibrium condition $\dot{S}_T < 0$ fulfill which leads to the validity of thermodynamical equilibrium.

**Power law corrected entropy:**

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Figure 10: Plot of $\tilde{S}_T$ versus $\gamma$ when logarithmic corrected entropy $\Gamma = \Gamma(t)$.

For variable $\Gamma$, the differentiation of Eq. (40) turns out to be

$$
\tilde{S}_T = -\left(\frac{1}{4HL_p} - \frac{\left(\frac{2 - \delta}{4}\right)}{2}\frac{1}{2}\frac{3 - \delta}{K_3}\right)\left(\frac{2}{3}\pi + \gamma \left(1 - \frac{\Gamma}{3H}\right) \left(3H^2 - \frac{c^2}{2a^2}\right)\right) \dot{H}
$$

$$
- \left(-\gamma \left(1 - \frac{\Gamma}{3H}\right) \left(3H^2 - \frac{c^2}{2a^2}\right) + \frac{\gamma \left(3H^2 - \frac{c^2}{2a^2}\right)}{H^3} + \frac{\gamma \left(3H^2 - \frac{c^2}{2a^2}\right)}{2H^2} \right) \dot{H}
$$

$$
\times 2\dot{H} \left(H^2 - \frac{c^2}{2a^2}\right)^{\frac{1}{\gamma}} \left(1 - \frac{\Gamma}{3H}\right) + \frac{\gamma \left(3H^2 - \frac{c^2}{2a^2}\right)}{H^3T_0} \left(\frac{c^2}{a^2} + \frac{\gamma \left(3H^2 - \frac{c^2}{2a^2}\right)}{2H^2} \left(1 - \frac{\Gamma}{3H}\right)\right)
$$

$$
+ \frac{1}{4HL_p^2} - \frac{\left(\frac{2 - \delta}{4}\right)}{2}\frac{1}{2}\frac{3 - \delta}{K_3}\left(\frac{2}{3}\pi + \gamma \left(1 - \frac{\Gamma}{3H}\right) \left(3H^2 - \frac{c^2}{2a^2}\right)\right) \dot{H}
$$

$$
+ \frac{\gamma \left(3H^2 - \frac{c^2}{2a^2}\right)}{2H^2} \left(H^2 - \frac{c^2}{2a^2}\right)^{\frac{1}{\gamma}} \dot{H} \left(H^2 - \frac{c^2}{2a^2}\right)^{\frac{1}{\gamma}} + \frac{\gamma \left(3H^2 - \frac{c^2}{2a^2}\right)}{H^3} \left(\frac{c^2}{a^2} + \frac{\gamma \left(3H^2 - \frac{c^2}{2a^2}\right)}{2H^2} \left(1 - \frac{\Gamma}{3H}\right)\right)
$$

$$
\times \gamma \left(1 - \frac{\Gamma}{3H}\right) - \frac{\gamma \dot{H} \left(3H^2 - \frac{c^2}{2a^2}\right)}{H^3} \left(\frac{c^2}{a^2} + \frac{\gamma \left(3H^2 - \frac{c^2}{2a^2}\right)}{2H^2} \left(1 - \frac{\Gamma}{3H}\right)\right) - \gamma \left(3H^2 - \frac{c^2}{2a^2}\right)\right)\right)
$$
$S_T$ versus $\gamma$ for power law correction when $\Gamma = \Gamma(t)$

\[
\begin{align*}
\times \left( \frac{\dot{\Gamma}H}{3H^2} - \frac{\dot{\Gamma}}{3H} \right) + \frac{\gamma(3H^2 - \frac{c^2}{2a^6})}{2H^2} \left( + \gamma(1 - \frac{\Gamma}{3H}) \left( \frac{3c^2\dot{a}}{a^7} + 6H\dot{H} \right) \right. \\
+ \left. \gamma(3H^2 - \frac{c^2}{2a^6}) \left( \frac{\dot{\Gamma}H}{3H^2} - \frac{\dot{\Gamma}}{3H} \right) - \frac{6c^2\dot{a}}{a^7} \right). 
\end{align*}
\] (48)

The plot of $\ddot{S}_T$ versus $\gamma$ for three values of $T$ by keeping the same values as above mentioned (Figure 11), one can observe easily the validity of thermodynamical equilibrium for all values of $T$ with $\ddot{S}_T < 0$.

**Renyi entropy:**

For variable $\Gamma$, Eq. (41) gives

\[
\begin{align*}
\ddot{S}_T &= -\frac{16\dot{H}}{(\eta + 8H^2)^2} \left( \frac{c^2}{a^6} + \gamma(3H^2 - \frac{c^2}{2a^6})(1 - \frac{\Gamma}{3H}) \right) - \frac{\dot{H}}{H^2(\eta + 8H^2)} \\
&\times \left( \frac{\dot{\Gamma}H}{3H^2} - \frac{\dot{\Gamma}}{3H} \right) \left( -\gamma \right) + \frac{\gamma(3H^2 - \frac{c^2}{2a^6})(\frac{c^2}{a^6} + \gamma(3H^2 - \frac{c^2}{2a^6})(1 - \frac{\Gamma}{3H}))}{2H^2} \\
&\times \left( \frac{\dot{\Gamma}H}{3H^2} - \frac{\dot{\Gamma}}{3H} \right) \left( 1 - \frac{\Gamma}{3H} \right) \left( \frac{c^2}{a^6} + \gamma(3H^2 - \frac{c^2}{2a^6})(1 - \frac{\Gamma}{3H}) \right) \\
&\times \left( \frac{\dot{\Gamma}H}{3H^2} - \frac{\dot{\Gamma}}{3H} \right) \left( 1 - \gamma \right) \left( \frac{\dot{a}}{a^7} + 2H\dot{H} \right) \left( \frac{H^2 - \frac{c^2}{6a^6}}{\gamma H^2T_0} \right)^{1+\frac{1}{\gamma}} + \frac{1}{H(\eta + 8H^2)} \\
&+ \frac{1}{H(\eta + 8H^2)} \\
\end{align*}
\]
The graphical behavior of $\ddot{S}_T$ versus $\gamma$ for three values of $T$ as shown in Figure 12 by keeping the constant values same as in above. One can see that $\ddot{S}_T$ is negative for all values of $T$ satisfying the condition $d^2S_T < 0$ which leads to the validity of thermodynamical equilibrium.

5 Comparison

Here we provide some literature on underlying scenario for comparison and summary of present work.

Harko et al. [95] considered the possibility of a gravitationally induced particle production through the mechanism of a non-minimal curvature-matter coupling. Firstly, they have reformulated the model in terms of an equivalent scalartensor theory, with two arbitrary potentials. The particle number
creation rates, the creation pressure, and the entropy production rates have explicitly obtained as functions of the scalar field and its potentials as well as of the matter Lagrangian. The temperature evolution laws of the newly created particles have also obtained. The cosmological implications of the model have briefly investigated and it is shown that the late-time cosmic acceleration may be due to particle creation process. Furthermore, it has also shown that due to the curvature-matter coupling, during the cosmological evolution a large amount of comoving entropy is also produced.

Mitra et al. [96] have studied thermodynamics laws by assuming flat FRW universe enveloped by by apparent and event horizon in the framework of RSII brane model and DGP brane scenario. Assuming extended Hawking temperature on the horizon, the unified first law is examined for perfect fluid (with constant equation of state) and Modified Chaplygin Gas model. As a result there is a modification of Bekenstein entropy on the horizons. Further the validity of the GSLT and thermodynamical equilibrium have also been investigated. By assuming the gravitationally induced particle scenario with constant specific entropy and arbitrary particle creation rate ($\Gamma$), thermodynamics on the apparent horizon for FRW universe has been discussed [97]. They have investigated the first law, GSLT and thermodynamical equilibrium by assuming the EoS for perfect fluid and put forward various constraints on $\Gamma$ for which thermodynamical laws hold.

Recently, we have done the thermodynamical analysis for gravitationally induced particle creation scenario in the framework of DGP braneworld model [98] by assuming usual entropy as well as its entropy corrections (power law and logarithmic) in the flat FRW universe. We have extracted EoS parameter and obtained its various constraints with respect to quintessence, vacuum and phantom era of the universe. For variable as well as constant particle creation rate ($\Gamma$), the first law of thermodynamics, GSLT and thermal equilibrium condition is satisfied in all the cases of entropy forms within some specific ranges of $\gamma$. In the present work, we have extended this work in the dynamical Chern-Simons modified gravity taking equation of state for perfect fluid as $p = (\gamma - 1)\rho$. By assuming various modified entropies (Bekenstein, logarithmic, power law correction and Reyni), we investigate the first law of thermodynamics, equilibrium condition and generalized second law of thermodynamics on the apparent horizon in the presence of particle creation rate. It is concluded that the GSLT and thermodynamical equilibrium are satisfying the conditions $\frac{dS_T}{dt} \geq 0$ and $\frac{d^2S_T}{dt^2} < 0$ for all values of $T$ throughout the range $\frac{2}{3} \leq \gamma \leq 2$.  

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In section 4, we have discussed the thermal equilibrium phenomenon for $\Gamma$ as variable and constant. The cosmic history is well-established through different observational sources that the radiation phase was followed by a matter dominated era which eventually passed through to a second de Sitter phase. Accordingly, it can be expected that in the radiation dominated era the entropy increased but the thermodynamic equilibrium was not achieved [99]. If this were not true, the universe would have attained a state of maximum entropy and would have stayed in it forever unless acted upon by some external agent. However, it is a well-known fact [100] that the production of particles was suppressed during the radiation phase, so in this model there would be no external agent to remove the system from thermodynamic equilibrium. In the present work, it is very difficult to find the analytical constraints to meet the equilibrium condition as discussed in the [101] due to lengthy expressions of $s_T$. Therefore, we checked these conditions graphically taking specific values of model parameters.

Since the prefect fluid is the simplest model in the cosmological studies, we study the prefect fluid case. In fact, this model can provide suitable results in the Einstein theory as well as modified theories [102, 103]. Moreover, the motivation of the present work in the framework of the particle creation mechanism comes from some recent related works. It has been shown in [104, 105] that the entire cosmic evolution from inflationary stage can be described by particle creation mechanism with some specific choices of the particle creation rates. As these works show late-time acceleration without any concept of dark energy, so, it is very interesting to think of the particle creation mechanism as an alternative way of explaining the idea of dark energy.

6 Conclusion

In this work, we have investigated the validity of first law of thermodynamics, GSLT and thermodynamical equilibrium for particle creation scenario in the presence of perfect fluid EoS $p = (\gamma - 1)\rho$ by assuming the different entropy corrections such as Bakenstein entropy, logarithmic corrected entropy and power law corrected entropy and Renyi entropy in a newly proposed dynamical Chern-Simons modified gravity. We have summarized our results as follows:

- For Bekenstein entropy
We have analyzed that first law of thermodynamics is showing the validity for \( \Gamma = 3H \left( \frac{c^2}{\gamma a^6} \left( 3H^2 - \frac{c^2}{2a^6} \right) \right) \). However, GSLT remains valid for all values of \( T \) with \( \frac{2}{3} \leq \gamma \leq 2 \). Further, we have analyzed the validity of thermodynamical equilibrium for constant and variable \( \Gamma \). From Figures (5 and 9), we observe that thermodynamical equilibrium is satisfying the condition \( \frac{d^2 S_T}{dt^2} < 0 \) for all values of \( T \) with \( \frac{2}{3} \leq \gamma \leq 2 \).

- **For Logarithmic corrected Entropy**
  In the presence of logarithmic corrected entropy it can be seen that the first law of thermodynamics is valid on the apparent horizon when 
  \[ \Gamma = 3H \left( 1 - \frac{1}{\gamma} (3H^2 - \frac{c^2}{2a^6}) \right) \left( \frac{\gamma (3H^2 - \frac{c^2}{2a^6})}{(1 + \frac{8H^2 a - 64BH^4 L^2}{L^2})} - \frac{c^2}{a^6} \right) \]. We have also investigated the validity of GSLT on apparent horizon satisfying the condition \( \frac{dS_T}{dt} \geq 0 \) (Figure 2). The graphical behavior of \( \dot{S}_T \) versus \( \gamma \) as shown in Figures (6 and 10). We observe the validity of thermodynamical equilibrium for all values of \( T \) for all values of \( \gamma \) for constant as well as variable \( \Gamma \).

- **Power law corrected entropy**
  For power law corrected entropy we have investigated that first law of thermodynamics is hold at apparent horizon for 
  \[ \Gamma = 3H \left( 1 - \frac{1}{\gamma} (3H^2 - \frac{c^2}{2a^6}) \right) \left( \frac{\gamma (3H^2 - \frac{c^2}{2a^6})}{(1 + \frac{8H^2 a - 64BH^4 L^2}{L^2})} - \frac{c^2}{a^6} \right) \]. From Figure 3 we can analyzed that the GSLT is valid for all values of \( T \) with \( \frac{2}{3} \leq \gamma \leq 2 \). Further, we have investigated the validity of thermodynamical equilibrium obeying the condition \( \dot{S}_T < 0 \) as shown in Figures (7 and 11) for all values of \( T \) with \( \frac{2}{3} \leq \gamma \leq 2 \) for both constant and variable \( \Gamma \).

- **For Renyi Entropy**
  In this entropy we have observed that first law of thermodynamics is holds when 
  \[ \Gamma = 3H \left( 1 - \frac{1}{\gamma} (3H^2 - \frac{c^2}{2a^6}) \right) \left( \frac{\gamma (3H^2 - \frac{c^2}{2a^6})}{(1 + \frac{8H^2 a - 64BH^4 L^2}{L^2})} - \frac{c^2}{a^6} \right) \]. The Graphical behavior of Figure 12 shows that all trajectories remains positive for all values of \( T \) with \( \frac{2}{3} \leq \gamma \leq 2 \) which leads to the validity of GSLT. Moreover, thermodynamical equilibrium condition satisfied for all values of \( T \) with all values of \( \gamma \) for constant and variable \( \Gamma \).
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