Onium-Onium scattering at fixed impact parameter: exact equivalence between the color dipole model and the BFKL Pomeron.

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Abstract

We compute the onium-onium scattering amplitude at fixed impact parameter in the framework of the perturbative QCD dipole model. Relying on conformal properties of the dipole cascade and of the elementary dipole-dipole scattering amplitude, we obtain an exact result for this onium-onium scattering amplitude, which is proven to be identical to the BFKL result, and which exhibits the frame invariance of the calculation. The asymptotic expression for this amplitude and for the dipole distribution in an onium at fixed impact parameter agree with previous numerical simulations. We show how it is possible to describe onium-$e^\pm$ deep inelastic scattering in the dipole model, relying on $k_T$-factorization properties. The elementary scattering amplitudes involved in the various processes are computed using eikonal techniques.

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1 Introduction

The HERA experiments [1, 2] have focused attention on the Balitsky-Fadin- Kuraev-Lipatov (BFKL) Pomeron [3], which should be relevant for describing the small-$x_{bj}$ behaviour of the proton structure function. More generally, this perturbative QCD hard Pomeron describes the behaviour of hadronic scattering amplitudes at very high energy $s$ and fixed momentum transfer $t \sim -m^2$ ($m$ being an hadronic mass scale). In this Leading Logarithmic Approximation, one takes into account the exchange of a bound state of two reggeized gluons in $t$-channel. This resummation, based on perturbative Regge physics, predicts an increase of the amplitude

$$A_{BFKL} \propto s^{\alpha_P},$$

where

$$\alpha_P = 1 + \frac{\alpha_S N_c \pi}{4 \ln 2} > 1.$$\hspace{1cm}(1.1)\hspace{1cm}(1.2)

Using the optical theorem, it follows that this behaviour violates the Froissart bound at very high $s$

$$\sigma_{tot} \leq c \ln^2 s.$$\hspace{1cm}(1.3)

This violation is directly related to the unitarization problem of QCD, which is one of the main problems to be solved in the theory of strong interaction. Various approaches have been recently proposed in order to restore unitarity. In the multiregge approach, it has been shown that the Generalized Leading Logarithmic Approximation [4, 5, 6], where one takes into account the exchange of any fixed number of reggeized gluons in $t$-channel, is equivalent to the non-compact Heisenberg XXX spin chain [7, 8] in the multicolor limit of QCD. The solution of this integrable model is still an open problem [9, 10, 11, 12, 13]. In the model recently developed by Mueller et al [14, 15, 16, 17] and separately by Nikolaev et al [18, 19], in order to control the perturbative approach, one deals with onia, which are heavy quark-antiquark bound states, so that their transverse size naturally provides an infra-red cut-off. The relevant degrees of freedom at high energy are then made of color dipoles. These color dipoles produce a classical cascade in the multicolor limit, which reveals a Pomeron type dynamics. This approach, combined with $k_T$-factorization [20, 21, 22], has been successfully applied to deep inelastic $e^\pm - p$ scattering for describing HERA data for $F_2$ [23, 24, 25]. In the more general case of onium-onium scattering, the BFKL approximation corresponds to the exchange of one pair of gluons between two excited dipoles, each one being extracted from one of the two onia. The unitarization problem can also be studied in this dipole model. In real physics, unitarity and analyticity of the $S$ matrix and the finite range of strong interaction lead to the Froissart bound [23] for total cross-section. Unitarity implies in particular at fixed impact parameter that the probability of any event cannot exceed 1, that is

$$|S(b)| \leq 1.$$\hspace{1cm}(1.4)

One way of enforcing unitarity in the framework of perturbative QCD would then be to take into account two and more left moving dipoles scattering off an equal number of right moving dipoles. As long as the relative rapidity $Y$ of the two onia is not too high, it is possible to deal with unitarization effects without facing saturations effects, that is one can consider each dipole in the wave function of the onia as still dilute [16, 26]. However, because of non-trivial dynamics in transverse space, the usual Glauber multiple scattering series, which corresponds to summing up multiple Pomeron exchanges, diverges factorially. A possible way of escaping this problem is to sum over to the number of exchanged Pomerons, and only after this summation is performed,
averaging over configurations of dipoles in the onia wave functions. Such an approach, except for a toy model where the transverse dynamics is absent [16], has not been yet carried out analytically. Numerically, such an approach is possible, and Monte Carlo simulations indeed show that QCD unitarizes in the dipole framework [27, 28].

Our aim is to study how this can be carried out analytically. In this paper, we will focus on the onium-onium cross-section at fixed impact parameter. It is organized as follows. In section 2, we calculate the onium-onium scattering amplitude at fixed impact parameter. The method developed here is rather general, and is based on expansions over conformal three points correlation functions. Using this representation, we calculate the distribution of dipoles inside an onium, at fixed impact parameter, and obtain its precise dependence with respect to the transverse size of the excited dipole. Then, the cross-section at fixed impact parameter is obtained using the elementary dipole-dipole cross-section, which is calculated by eikonal methods and expanded on a conformal basis. Our result shows the exact equivalence between the dipole and the BFKL approaches for inclusive processes. In section 3, we study some physical applications of the dipole picture, namely $e^\pm - \text{Onium}$ deep inelastic scattering. We rely on $k_T$-factorization and apply eikonal techniques for computing the elementary dipole-gluon cross-section.

2 Onium-onium cross-section at fixed impact parameter

In this section, we calculate the onium-onium scattering amplitude in the BFKL approximation. This calculation has been first carried out in Ref. [13, 14]. It used an expansion on a conformal three points correlation function basis. Asymptotic expressions of this basis were used in this calculation. The calculation which is presently developed gives the correct dependence with respect to the transverse sizes of onia. This dependence is highly non trivial and could not be obtained without a careful treatment of the conformal basis.

We thus consider onium-onium scattering in the leading logarithmic approximation. The infinite-momentum wave function of an onium, in the large $N_c$ limit, was calculated perturbatively in Ref. [14]. This calculation is based on an approximation of eikonal type, due to the ordering of longitudinal momenta. This allows one to compute the probability of emitting a gluon from a quark, and then from a quark-antiquark pair, that is from a color dipole. In the multicolor limit, the cascade of emitted soft gluons decouples, leading to a semi-classical cascade of color dipoles, in term of probability, since interference terms cancels in this limit. The onium-onium elastic scattering amplitude at fixed impact parameter $A(Y, b)$ can then be expressed using a parton type formulation. It involves the number of dipoles in each onium and the elementary cross-section of two such dipoles. For a relative rapidity $Y$ and impact parameter $b$, $A(Y, b)$ can be expressed as

$$A(Y, b) = -i \int d^2 x_1 \int d^2 x_2 \int_0^1 dz_1 \int_0^1 dz_2 \Phi(x_1, z_1) \Phi(x_2, z_2) F(x_1, x_2, Y, b). \quad (2.5)$$

$\Phi(x_i, z_i)$ is the square of the heavy quark-antiquark part of the onium wavefunction, $x_i$ being the transverse size of the quark-antiquark pair and $z_i$ the longitudinal momentum fraction of the antiquark. The momentum $p_{1T}$ and $p_{2T}$ of the two onia are supposed to be large, with $p_{1T} = p_{2T} = 0$. $Y$ is related to $\tilde{Y}$ by $\tilde{Y} = Y + \ln z_1 z_2$, due to the fact that the perturbative dipole cascade originates from the quark-antiquark pair. The distribution $\Phi(x, z)$ of this pair cannot be computed perturbatively, and goes far beyond the purpose of the present approach. The
scattering amplitude $F$ is then evaluated in term of the perturbative dipole cascade. Following Ref. [14, 15], we define $n(x_0, x, b, \bar{Y})$ such that

$$N(x, b, \bar{Y}) = \int d^2x \int_0^1 dz_1 \Phi(x, z_1) n(x, x', \bar{Y}, b)$$

(2.6)

is the number density of dipoles of transverse size $x'$, at a transverse distance $b$ from the center of the quark-antiquark pair, where the momentum fraction of the softest of the two gluons (or quark or antiquark) which compose the dipole is larger or equal to $e^{-\bar{Y}}$. $\bar{Y}$ is the relative rapidity with respect to the heavy quark given by $\bar{Y} = Y + \ln z_1$. In the leading logarithm approximation (noted $F^{(1)}$) where the scattering is due to the exchange of a single pair of gluons between the two dipoles extracted from the left and right moving onia, $F^{(1)}$ reads

$$F^{(1)}(x_1, x_2, \bar{Y}, b) = -\frac{1}{2} \int \frac{d^2 x_1'}{2\pi x_1'^2} \frac{d^2 x_2'}{2\pi x_2'^2} d\rho_0 d\rho_2 d^2 b_1 d^2 b_2 d^2 (b'_1 - b_1) \delta^2(b_1 - b_2 - b'_1 + b'_2 - b')$$

$$\times n(x_1, x_1', \bar{Y}, b_1) n(x_2, x_2', \bar{Y}, b_2) \sigma_{DD}(x_1', x_2', b'_1 - b'_2).$$

(2.7)

The rapidities $\bar{Y}_1$ and $\bar{Y}_2$ are such that $\bar{Y} = \bar{Y}_1 + \bar{Y}_2$. Eq. (2.7) involves the elementary dipole-dipole cross-section at fixed impact parameter $\sigma_{DD}$, which has been evaluated in [15], and which is calculated in appendix A.2 using eikonal techniques. For two dipoles of transverse sizes $x_1'$ and $x_2'$, whose centers are located at $b_1'$ and $b_2'$, one obtains

$$\sigma_{DD}(x_1', x_2', b'_1 - b'_2) = \alpha_s \left\{ \ln \frac{|b'_1 - b'_2 + x_1' + x_2'| |b'_1 - b'_2 - x_1' + x_2'|}{|b'_1 - b'_2 + x_1' - x_2'| |b'_1 - b'_2 - x_1' - x_2'|} \right\}^2.$$  

(2.8)

The forward scattering amplitude can then be evaluated by integration over impact parameter, namely

$$A(Y) = \int d^2 b A(Y, b).$$  

(2.9)

$A(Y)$ is normalized so that the optical theorem reads

$$\sigma(Y) = 2 \text{Im} A.$$  

(2.10)

In order to get the expression for $n(x, x', \bar{Y}, b)$, one relies on the global conformal invariance of the dipole emission kernel, related to the absence of scale. We thus decompose this distribution on the basis of conformally invariant three points holomorphic and antiholomorphic correlation functions [29, 30]. Introducing complex coordinates in the two-dimensional transverse space

$$\rho = (\rho_x, \rho_y)$$  

(2.11)

$$\rho = \rho_x + i\rho_y$$ and $\rho^* = \rho_x - i\rho_y,$  

(2.12)

the complete set of eigenfunctions $E^{n, \nu}$ of the dipole emission kernel is

$$E^{n, \nu}(\rho_{10}, \rho_{20}) = (-1)^n \left( \frac{\rho_{12}}{\rho_{10}\rho_{20}} \right)^h \left( \frac{\rho_{12}^*}{\rho_{10}^*\rho_{20}^*} \right)^{\bar{h}},$$  

(2.13)

with

$$h = \frac{1 - n}{2} + i\nu$$  

$$\bar{h} = \frac{1 + n}{2} + i\nu$$  

(2.14)
being the corresponding conformal weights, with \( n \) integer and \( \nu \) real. This set constitutes a unitary irreducible representation of \( \text{SL}(2, \mathbb{C}) \) [31].

We get rid of the longitudinal degrees of freedom by using a Mellin transform with respect to \( \tilde{Y} \), namely

\[
n(x, x', \tilde{Y}, b) = \int \frac{d\omega}{2\pi i} e^{\omega \tilde{Y}} n_\omega(x, x', b).
\]

Expanding the dipole distribution on the conformal basis, one writes

\[
n_\omega(x, x', b) = \sum_{n=-\infty}^{n=+\infty} 8 \int \frac{d\nu}{(2\pi)^3} \frac{d^2w}{x'^2} \left( \nu^2 + \frac{n^2}{4} \right) n_{\{\nu, n\}_\omega} \times E^{n,\nu} \left( b + \frac{x'}{2} - w, b - \frac{x'}{2} - w \right) E^{n,\nu*} \left( \frac{x'}{2} - w, -\frac{x'}{2} - w \right).
\]

In this expression, the transverse integration is done with respect to the coordinate \( w \) of the center of mass of the quark-antiquark pair. The longitudinal dynamics give rise to the term \( n_{\{\nu, n\}_\omega} \), which was computed in Ref. [14], and has the following expression

\[
n_{\{\nu, n\}_\omega} = \frac{2}{\omega - \frac{2\alpha_s N_c}{\pi} \chi(n, \nu)},
\]

where

\[
\chi(n, \nu) = \psi(1) - \frac{1}{2} \psi \left( \frac{|n| + 1}{2} + i\nu \right) - \frac{1}{2} \psi \left( \frac{|n| + 1}{2} - i\nu \right) = \psi(1) - \text{Re} \psi \left( \frac{|n| + 1}{2} + i\nu \right).
\]

Going back to the longitudinal space, the distribution of dipoles takes the form

\[
n(x, x', \tilde{Y}, b) = \sum_{n=-\infty}^{+\infty} 16 \int \frac{d\nu}{(2\pi)^3} \frac{d^2w}{x'^2} \left( \nu^2 + \frac{n^2}{4} \right) \exp \left( \frac{2\alpha_s N_c}{\pi} \chi(n, \nu) \tilde{Y} \right) \times E^{n,\nu} \left( b + \frac{x'}{2} - w, b - \frac{x'}{2} - w \right) E^{n,\nu*} \left( \frac{x'}{2} - w, -\frac{x'}{2} - w \right).
\]

This distribution can be more easily computed by using a Fourier transform with respect to the impact parameter, that is by fixing the \( t \) channel exchanged momentum. We thus define

\[
n(x, x', \tilde{Y}, b) = \int \frac{d^2q}{(2\pi)^2} e^{-i\frac{q}{b} \cdot b} n(x, x', \tilde{Y}, q),
\]

and introduce, following Ref. [29], the corresponding mixed representation of \( E^{n,\nu} \), namely

\[
E^{n,\nu}_q(\rho) = \frac{2\pi^2}{b_{n,\nu}} \int \frac{d^2R}{|\rho|} e^{i\frac{q}{b} \cdot R} E^{n,\nu} \left( R + \frac{\rho}{2}, R - \frac{\rho}{2} \right).
\]

The normalisation term \( b_{n,\nu} \) is given by

\[
b_{n,\nu} = 4^3 \pi^3 \frac{1}{-i\nu + |n|/2} 2^{4i\nu} \Gamma(-i\nu + (1 + |n|)/2) \Gamma(i\nu + |n|/2) \Gamma(-i\nu + |n|/2) \Gamma(i\nu + (1 + |n|)/2).
\]

In this Fourier representation the dipole distribution reads

\[
n(x, x', \tilde{Y}, b) = \sum_{n=-\infty}^{+\infty} \int \frac{d\nu}{2\pi} \int \frac{d^2q}{(2\pi)^2} e^{-i\frac{q}{b} \cdot b} E_{n,\nu*}^{q}(x) E_{n,\nu}^{q}(x') \frac{x}{x'} \exp \left( \frac{2\alpha_s N_c}{\pi} \chi(n, \nu) \tilde{Y} \right).
\]
2.1 The large rapidity dipole density at fixed impact parameter

At very large $\tilde{Y}$, the term corresponding to $n = 0$ dominates the exponential term in the expansion (2.23) because of the expression of $\chi(n, \nu)$ (see Eq. (2.18)), and we will restrict ourselves to this case in this subsection.

We first give a useful representation for $E_q^{0, \nu}$ (a general representation for $E_q^{n, \nu}$ can also be obtained [32]). From Eqs. (2.13) and (2.21), we get

$$E_q^{0, \nu}(\rho) = \frac{2\pi^2}{b_{0, \nu}} \rho^{2i\nu} \int \frac{d^2 R e^{2i\nu \rho R}}{\left( |R + \rho/2| |R - \rho/2| \right)^{1+2i\nu}}$$

This finally yields

$$E_q^{0, \nu}(\rho) = \frac{2\pi^2}{b_{0, \nu}} \rho^{2i\nu} \int \frac{d^2 R e^{2i\nu \rho R}}{\left( |R + \rho/2| |R - \rho/2| \right)^{1+2i\nu}}. \quad (2.24)$$

Introducing the Feynman representation of the integrand

$$\frac{1}{\left( \frac{1}{2} \right)^{1+2i\nu}} = \Gamma(1+2i\nu) \int_0^1 \frac{d\alpha}{D_{\alpha}^{1+2i\nu}}, \quad (2.25)$$

where

$$D_{\alpha} = (1-\alpha)\rho^2 + \alpha(\rho + 1)^2 = \rho^2 + \alpha(1-\alpha), \quad (2.26)$$

and setting $R = \rho + \alpha$, Eq. (2.24) now reads

$$E_q^{0, \nu}(\rho) = \frac{2\pi^2}{b_{0, \nu}} \frac{\Gamma(1+2i\nu)}{\Gamma^2 \left( \frac{1}{2} i + \nu \right)} \int_0^1 d\alpha \left( 1 - \alpha \right)^{i\nu-1/2} e^{-\frac{1}{2}\nu^2 \rho^2 (\rho + 1)^2} \int \frac{d^2 R e^{2i\nu \rho R}}{\left( \rho^2 + \alpha(1-\alpha) \right)^{1+2i\nu}}. \quad (2.27)$$

The integration with respect to the angle $(q, R)$ leads to a Bessel function $J_0(qR)$. The integration with respect to $R = |R|$ can then be carried out, using the formula (6.565) of Ref. [33]

$$\int \frac{J_{\nu}(q x) x^{\nu+1}}{[x^2 + \alpha^2]^{\mu+1}} dx = \frac{\alpha^{\nu-\mu} q^\mu K_{\nu-\mu}(aq)}{2^\mu \Gamma(\mu + 1)}. \quad (2.28)$$

This finally yields

$$E_q^{0, \nu}(\rho) = \frac{4\pi^3}{b_{0, \nu}} e^{2i\nu \rho/2} \left( \frac{q}{2} \right)^{2i\nu} \frac{1}{\Gamma^2 \left( \frac{1}{2} + i\nu \right)} \int_0^1 d\alpha \left( 1 - \alpha \right)^{i\nu-1/2} e^{-i\nu \rho^2 K_{\nu}(\rho \sqrt{\alpha(1-\alpha)})}. \quad (2.29)$$

This last integration can be performed and yields (see appendix A.3)

$$E_q^{0, \nu}(\rho) = \left( \frac{q}{2} \right)^{2i\nu} 2^{-2i\nu} \Gamma(1 - i\nu) \left[ J_{\nu} \left( \frac{pq}{4} e^{i\Psi} \right) J_{\nu} \left( \frac{pq}{4} e^{-i\Psi} \right) - J_{-\nu} \left( \frac{pq}{4} e^{i\Psi} \right) J_{-\nu} \left( \frac{pq}{4} e^{-i\Psi} \right) \right], \quad (2.30)$$

where $\Psi$ is the angle $(q, \rho)$. We will not need this explicit form in the following calculation.

Let us now compute the contribution to the density of dipole (2.23) corresponding to $n = 0$. Since the longitudinal and the transverse degrees of freedom can be easily separated
(see formula (2.23)), we will forget the exponential term depending on the rapidity \( \tilde{Y} \), and will restore this dependence at the very end. We thus define \( n_{(n,\nu)}(x, x', b) \) such that

\[
n(x, x', \tilde{Y}, b) = \sum_{n=-\infty}^{+\infty} \int \frac{d\nu}{2\pi} n_{(n,\nu)}(x, x', b) \exp \left( \frac{2\alpha N_c}{\pi} \chi(n, \nu) \tilde{Y} \right). \tag{2.31}
\]

In the asymptotic case we are interested in,

\[
n(x, x', \tilde{Y}, b) \sim \int \frac{d\nu}{2\pi} n_{(0,\nu)}(x, x', b) \exp \left( \frac{2\alpha N_c}{\pi} \chi(0, \nu) \tilde{Y} \right). \tag{2.32}
\]

We can now express the distribution of dipole, taking into account Eq. (2.32). When \( \tilde{Y} \rightarrow \infty \), one can use a saddle approximation for the \( \nu \)-integration. The function \( \chi(0, \nu) \) is maximum at \( \nu = 0 \), which thus defines the saddle-point. We will thus expand \( n_{(0,\nu)} \) around \( \nu = 0 \). Using the expression (2.29), \( n_{(0,\nu)} \) then reads

\[
n_{(0,\nu)}(x, x', b) = \frac{16\pi^6}{|b_{0,\nu}|^2 x' x} \Gamma(\frac{1}{2} + i\nu) \Gamma(\frac{1}{2} - i\nu) \int \frac{d^2q}{(2\pi)^2} \int_0^1 d\alpha \int_0^1 d\beta [\alpha(1-\alpha)]^{-\frac{1}{2}} [\beta(1-\beta)]^{-\frac{1}{2}} \times e^{iqx} \frac{x}{2}(1-2\alpha) - iqx \frac{x'}{2}(1-2\beta) e^{-iqb} \frac{b}{2} K_{2\nu}(q x \sqrt{\alpha(1-\alpha)}) K_{-2\nu}(q x' \sqrt{\beta(1-\beta)}). \tag{2.33}
\]

Setting \( \nu = \frac{x}{2}(1-2\alpha) - \frac{x'}{2}(1-2\beta) + b \), the integration with respect to the angle \( (q, \nu) \) gives a Bessel function \( J_0(qv) \). The integration with respect to \( q \) can be carried out using the following formula \[34\]

\[
\int_0^\infty x^{\mu+1} K_\lambda(a_1 x) K_\lambda(a_2 x) J_\mu(a_3 x) dx = \frac{1}{2} \sqrt{\frac{\pi}{2}} \left( \frac{a_3}{a_1 a_2} \right)^{\mu+1} \mathcal{P}_{\lambda - \frac{1}{2}}^{-\mu - \frac{1}{2}}(z)(z^2 - 1)^{-\frac{\mu}{2} - \frac{1}{4}} \Gamma(\mu + \lambda + 1) \Gamma(\mu - \lambda + 1), \tag{2.34}
\]

where \( z = \frac{1}{2} a_1^2 + a_2^2 + a_3^2 \), and \( \mathcal{P} \) is a Legendre function. Here \( \mu = 0, \lambda = 2i\nu \) and the corresponding Legendre function is

\[
\mathcal{P}_{\gamma - \frac{1}{2}}^{\frac{1}{2}}(z) = (2\pi)^{-\frac{1}{4}} \left( \frac{1}{2} + \gamma \right)^{-\frac{1}{2}} (z^2 - 1)^{\frac{1}{4}} \left\{ [z + (z^2 - 1)^{\frac{1}{2}}]^{\frac{1}{2}} - [z - (z^2 - 1)^{\frac{1}{2}}]^{\frac{1}{2}} \right\}. \tag{2.35}
\]

In the case we are interested in, \( a_1 = x \sqrt{\alpha(1-\alpha)}, a_2 = x' \sqrt{\beta(1-\beta)} \) and \( a_3 = b \). In the domain \( b \gg x, x' \) (where the total cross-section gets its dominant contribution: see subsection 2.2),

\[
z \simeq \frac{b^2}{2ax'x} [\alpha(1-\alpha)\beta(1-\beta)]^{-\frac{1}{2}} \gg 1, \tag{2.36}
\]

and thus

\[
\int q K_{2i\nu} \left( q x \sqrt{\alpha(1-\alpha)} \right) K_{-2i\nu} \left( q x' \sqrt{\beta(1-\beta)} \right) J_0(qv) dq \simeq \frac{1}{2} \frac{1}{2i\nu} \Gamma(1 + 2i\nu) \Gamma(1 - 2i\nu) \times \frac{1}{b^2} \left\{ \left[ \frac{b^2}{xx'} \right]^{2i\nu} [\alpha(1-\alpha)\beta(1-\beta)]^{-i\nu} - \left[ \frac{b^2}{xx'} \right]^{-2i\nu} [\alpha(1-\alpha)\beta(1-\beta)]^{i\nu} \right\}. \tag{2.37}
\]
Using Ref. \[29\],
\[ a_{n,\nu} = \frac{\pi^4/2}{\nu^2 + n^2/4} = \frac{b_{n,\nu}^4}{2\pi^2}, \] (2.38)
which gives for \(n_{(0,\nu)}\)
\[ n_{(0,\nu)}(x, x', b) \simeq -\frac{2i\nu}{\pi} \frac{x}{x' b^2} \left\{ \frac{\Gamma(1 + 2i\nu) \Gamma(1 - 2i\nu)}{\Gamma(1 - 2i\nu) \Gamma(1/2 + i\nu)} \int_0^\infty d\alpha \int_0^\infty d\beta \right. \]
\[ \times \left\{ \left(\frac{b^2}{xx'} \right)^{2i\nu} \left[\alpha(1 - \alpha)\beta(1 - \beta)\right]^{-\frac{1}{2} - i\nu} - \left(\frac{b^2}{xx'} \right)^{-2i\nu} \left[\alpha(1 - \alpha)\beta(1 - \beta)\right]^\frac{1}{2} + i\nu \right\}. \] (2.39)
The integration with respect to \(\alpha\) and \(\beta\) can now be carried out. This yields
\[ n_{(0,\nu)}(x, x', b) \simeq -\frac{2i\nu}{\pi} \frac{x}{x' b^2} \left\{ \left(\frac{16b^2}{xx'} \right)^{2i\nu} - \left(\frac{16b^2}{xx'} \right)^{-2i\nu} + \mathcal{O}(\nu^3) \right\}. \] (2.40)
Using the doubling formula, one finally obtains
\[ n_{(0,\nu)}(x, x', b) \simeq -\frac{2i\nu}{\pi} \frac{x}{x' b^2} \left\{ \ln(16b^2/xx') - \ln(16b^2/xx') - \mathcal{O}(\nu^3) \right\}. \] (2.41)
This can now be inserted in Eq. (2.32) to calculate the large \(\tilde{Y}\) limit of the density, when \(b^2/xx' \ll 1\). We develop \(\chi(0, \nu)\) around \(\nu = 0\)
\[ \chi(0, \nu) \sim 2\ln 2 - 7\zeta(3)\nu^2 + \mathcal{O}(\nu^4), \] (2.42)
and take into account that \(\chi(0, -\nu) = \chi(0, \nu)\). One has then to compute
\[ n(x, x', \tilde{Y}, b) \simeq -\frac{x}{x' b^2} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} \frac{4i\nu}{2\pi} \exp \left\{ \frac{2\alpha_s N_c}{\pi} \frac{\ln(16b^2/xx')}{2\ln 2 \tilde{Y} - \frac{2\alpha_s N_c}{\pi} 7\zeta(3)\tilde{Y} \nu^2 + 2i\nu \ln(16b^2/xx')} \right\}. \] (2.43)
After making the replacement \(\nu' = \nu - i(\ln(16b^2/xx') / (2\alpha_s N_c/7\zeta(3)Y)\), the gaussian integration with respect to \(\nu'\) finally yields
\[ n(x, x', \tilde{Y}, b) \simeq \frac{x}{4b^2 x'} \ln(16b^2/xx') \left(\frac{7\alpha_s N_c \zeta(3)}{\pi} \right)^{3/2} \exp \left\{ \frac{4\alpha_s N_c}{\pi} \ln 2 \tilde{Y} \right\} \exp \left\{ -\ln^2(16b^2/xx') - \frac{\ln^2(16b^2/xx')}{14\alpha_s N_c \zeta(3)Y} \right\}. \] (2.44)
This result is valid in the domain
\[ \frac{2\alpha_s N_c}{\pi} 7\zeta(3)\tilde{Y} \ll \ln(16b^2/xx') \ll 1. \] (2.45)
This expression corrects the expression (8) of Ref. \[10\] when one considers the distribution in transverse space. Numerical simulations and approximate analytical calculations indeed confirm this factor 16 in the transverse distribution \[35\]. Note that it is only in this asymptotic regime that \(n(x, x', \tilde{Y}, b)\) has no angular dependence.
2.2 Exact result for the onium-onium cross-section at fixed impact parameter and equivalence with BFKL Pomeron

Let us now compute the onium-onium scattering amplitude at fixed impact parameter according to the process displayed in figure 1. We denote $x_{a1}$ ($x_{b1}$) the transverse coordinate of the heavy quark (antiquark) making up the right moving onium and $x_{a2}$ ($x_{b2}$) the coordinates of the corresponding quark (antiquark) making the left moving onium. These onia of transverse sizes $x_1 = x_{a1} - x_{b1}$ and $x_2 = x_{a2} - x_{b2}$ scatter through the exchange of a pair of gluons between two elementary dipoles, respectively of transverse size $x_1'$ and $x_2'$, located at $b_1$ and $b_2$ with respect to the reference point 0 (which is arbitrary due to translation invariance). These two elementary dipoles are produced by the two heavy quark-antiquark pairs at a distance $b_1$ and $b_2$ from their center of mass.

![Figure 1: Onium-Onium scattering at leading order.](image)

In order to compute $F(1)$ as given by Eq. (2.7), we now use the result (2.15, 2.16) for the dipole density and the following expansion of the dipole-dipole cross-section

$$
\sigma_{DD}(x_1', x_2', b_1 - b_2) = \frac{2\alpha_s^2}{(2\pi)^2} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dv \int d^2w \left( v^2 + \frac{n^2}{4} \right) \frac{1 + (-1)^n}{(v^2 + \frac{n^2}{4})^2} \left( \int d^2w_1 \int d^2w_2 \right) E_{n,v}^*(b_1 - b_2 - b_1')w_1 \cdot E_{n,v}(b_2 - b_2' - b_2')w_2',
$$

(2.46)

which is proved in appendix A.2. The full expression obtained is

$$
F^{(1)}(x_1, x_2, \bar{Y}, b) = -\frac{\alpha_s^2(16)}{(2\pi)^2} \sum_{n_1=-\infty}^{+\infty} \sum_{n_2=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dv_1}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dv_2}{(2\pi)^3} \int_{-\infty}^{+\infty} dv \int d^2w_1 \int d^2w_2 \left( v_1^2 + \frac{n_1^2}{4} \right) \left( v_2^2 + \frac{n_2^2}{4} \right) \exp \left( \frac{2\alpha_s N_c}{\pi} \left( \chi(n_1, v_1)\bar{Y}_1 + \chi(n_2, v_2)\bar{Y}_2 \right) \right)
$$

\times \left( v^2 + \frac{n^2}{4} \right) \left( \frac{1 + (-1)^n}{(v^2 + \frac{n^2}{4})^2} \right) \left( \frac{1 + (-1)^n}{(v^2 + \frac{n^2}{4})^2} \right) \exp \left( \frac{2\alpha_s N_c}{\pi} \left( \chi(n_1, v_1)\bar{Y}_1 + \chi(n_2, v_2)\bar{Y}_2 \right) \right)
$$

(2.46)

In order to compute $F^{(1)}$ as given by Eq. (2.7), we now use the result (2.15, 2.16) for the dipole density and the following expansion of the dipole-dipole cross-section $\sigma_{DD}$
\[ \times E^{n_{1}, \nu_{1}} \left( b_{1} - \frac{x'_{1}}{2} - w_{1}, b_{2} - \frac{x'_{2}}{2} - w_{2} \right) E^{n_{1}, \nu_{1}^{*}} \left( \frac{x_{1}}{2} - w_{1}, -\frac{x_{1}}{2} - w_{1} \right) \]

\[ \times E^{n_{2}, \nu_{2}^{*}} \left( b_{1} - \frac{x'_{1}}{2} - w_{1}, b_{2} - \frac{x'_{2}}{2} - w_{2} \right) E^{n_{2}, \nu_{2}} \left( \frac{x_{2}}{2} - w_{2}, -\frac{x_{2}}{2} - w_{2} \right) \]

\[ \times E^{n_{\nu}} \left( b_{1} + \frac{x'_{1}}{2} - w_{1}, b_{2} - \frac{x'_{2}}{2} - w_{2} \right) E^{n_{\nu}^{*}} \left( b_{\text{nt}} + b_{1} + \frac{x'_{1}}{2} - w_{1}, b_{\text{nt}} + b_{2} - \frac{x'_{2}}{2} - w_{2} \right) \]  

(2.47)

where we have used the fact that \( \sigma_{DD}(x', x'', b'_{1} - b'_{2}) \) is translationally invariant and only depends on \( b_{\text{nt}} = b_{0}' - b_{0} \). The quantum numbers \( n_{1}, \nu_{1} \) and \( n_{2}, \nu_{2} \) correspond respectively to the dipole distributions \( n(x_{1}, x'_{1}, Y_{1}, b_{1}) \) and \( n(x_{2}, x'_{2}, Y_{2}, b_{2}) \).

Note the existence of a term \( 1 + (-1)^{n} \) which arises from the \( s \leftrightarrow u \) symmetry of the elementary dipole-dipole cross-section since the gluon is a massless vector boson (see appendix A.2). As a consequence only the even \( n \) will contribute to the following scattering amplitude.

The integration with respect to \( b_{3} \) can be performed through the delta distribution. Then, the remaining expression can be strongly simplified when using the following orthonormalization condition for the functions \( E^{n, \nu} \) (see Eq. (A.16) of Ref. [9]):

\[
\int \frac{d^{2}\rho_{1} d^{2}\rho_{2}}{|\rho_{12}|^{4}} E^{n, \nu}(\rho_{10}, \rho_{20}) E^{m, \nu^{*}}(\rho_{10}', \rho_{20}') = a_{n, \nu} a_{m, \nu} \delta(\nu - \mu) \delta^{2}(\rho_{00}^{\nu} - \rho_{00}^{\nu'}) + (-1)^{n} b_{n, \nu} |\rho_{00}^{\nu} - \rho_{00}^{\nu'}|^{2} \delta(\mu - \mu') \delta_{n,-m} \delta(\nu + \mu) .
\]

(2.48)

Note that this equation corrects Eq. (A.16) of Ref. [9] since the factor \( (-1)^{n} \) in the second term of the right-hand side was missing. This term arises from the fact that

\[
\left( \frac{z^{2} - 1}{z^{2*} - 1} \right)^{\frac{n}{2}} = (-1)^{n} \left( \frac{1 - z^{2}}{1 - z^{2*}} \right)^{\frac{n}{2}},
\]

(2.49)

which has to be taken into account when performing the last transformation in Eq. (A.19) of Ref. [9]. Applying the relation (2.48) for \( \rho_{1} = b_{1} + \frac{x_{1}'}{2} \), \( \rho_{2} = b_{1} - \frac{x_{1}'}{2} \), \( \rho_{3} = w_{1} \) and \( \rho_{4} = w_{2} \) on one hand, for \( \rho'_{1} = b_{1} + b_{\text{nt}} + \frac{x_{2}'}{2} \), \( \rho'_{2} = b_{1} + b_{\text{nt}} - \frac{x_{2}'}{2} \), \( \rho'_{3} = w_{2} + b \) and \( \rho'_{4} = w \) on the other hand, and using the equality \( d^{2}b_{1} d^{2}b_{\text{nt}} d^{2}x_{1}' d^{2}x_{2}' = d^{2}b_{1} d^{2}b_{\text{nt}} d^{2}x_{1} d^{2}x_{2} \), the integration over these transverse variables gives

\[
F^{(1)}(x_{1}, x_{2}, Y, b) = -\frac{\alpha_{s}^{2}}{(2\pi)^{2}} \frac{16}{2(2\pi)^{2}} \sum_{n_{1}=-\infty}^{\infty} \sum_{n_{2}=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\nu_{1}}{(2\pi)^{3}} \int_{-\infty}^{\infty} \frac{d\nu_{2}}{(2\pi)^{3}} \int_{-\infty}^{\infty} \frac{d\nu}{(2\pi)^{3}} \int d^{2}w_{1} \int d^{2}w_{2} \times
\]

\[
\times \int d^{2}w \left( \frac{n_{1}^{2}}{4} \right) \left( \frac{n_{2}^{2}}{4} \right) \left( \nu^{2} + \frac{n^{2}}{4} \right) \frac{1 + (-1)^{n}}{\left( \nu^{2} + \frac{1}{4} \right)^{2} \left( \nu^{2} + \frac{n^{2}}{2} \right)^{2}} \times
\]

\[
\times \exp \left( \frac{2\alpha_{s} N_{c}}{\pi} \left( \lambda(n_{1}, \nu_{1}) \tilde{Y}_{1} + \lambda(n_{2}, \nu_{2}) \tilde{Y}_{2} \right) \right) \times
\]

\[
\left[ a_{n_{1}, \nu_{1}} \delta_{n_{1}, n} \delta(\nu_{1} - \nu) \delta^{2}(w - w_{1}) + b_{n_{1}, \nu_{1}} |w - w_{1}| \right]^{n_{1}} \left( \frac{w - w_{1}}{w^{*} - w_{1}^{*}} \right)^{n_{1}} \delta_{n_{1}, -n} \delta(\nu_{1} + \nu) \left( -1 \right)^{n_{1}}
\]

\[
\times \left[ a_{n_{2}, \nu_{2}} \delta_{n_{2}, n} \delta(\nu_{2} - \nu) \delta^{2}(w - w_{2}) + b_{n_{2}, \nu_{2}} |w - w_{2}| \right]^{n_{2}} \left( \frac{w - w_{2}}{w^{*} - w_{2}^{*}} \right)^{n_{2}} \delta_{n_{2}, -n_{2}}
\]

\[
\times \delta(\nu_{2} + \nu) \left( -1 \right)^{n} E^{n_{1}, \nu_{1}^{*}} \left( \frac{x_{1}}{2} - w_{1}, -\frac{x_{1}}{2} - w_{1} \right) E^{n_{2}, \nu_{2}^{*}} \left( \frac{x_{2}}{2} - w_{2}, -\frac{x_{2}}{2} - w_{2} \right). \quad (2.50)
\]
The two terms involving the normalisation factors $b_{n,\nu}$ and $b_{n,\nu}$ can be reexpressed using the fact that $E^{n,\nu}$ and $E^{n,\nu*}$ are related by the following relation, which corrects Eq. (A.12) of Ref. [29] (see appendix A.7)

$$E^{n,\nu}(\rho_{10}, \rho_{20}) = \frac{b_{n,\nu}}{a_{n,\nu}} \int d^2 \rho_0' E^{n,\nu}(\rho_{10}', \rho_{20}') |\rho_{00}'|^{-2+4i\nu} \left( \frac{\rho_{00}'}{\rho_{00}} \right)^n (-1)^n. \tag{2.51}$$

Note that this relation arises from the equivalence between the two corresponding representations of $SL(2, C)$ [31]. The integration with respect to $\underline{w}_1$ and $\underline{w}_2$ can then be performed. It gives

$$\int d^2 \underline{w}_1 E^{n,\nu} \left( \frac{x_1}{2} - \underline{w}_1, -\frac{x_1}{2} - \underline{w}_1 \right) |\underline{w}_1 - \underline{w}_1|^{-2+4i\nu} \left( \frac{\underline{w}_1 - \underline{w}_1}{\underline{w}_1^* - \underline{w}_1} \right)^n (-1)^n = \frac{a_{n,\nu}}{b_{n,\nu}} E^{n,\nu*} \left( \frac{x_1}{2} - \underline{w}, -\frac{x_1}{2} - \underline{w} \right). \tag{2.52}$$

and

$$\int d^2 \underline{w}_2 E^{-n,-\nu} \left( \frac{x_2}{2} - \underline{w}_2, -\frac{x_2}{2} - \underline{w}_2 \right) |\underline{w}_2 - \underline{w}_2 - b|^{-2-4i\nu} \left( \frac{\underline{w}_2 - \underline{w}_2 - b}{\underline{w}_2^* - \underline{w}_2 - b} \right)^n (-1)^n = \frac{a_{-n,-\nu}}{b_{-n,-\nu}} E^{-n,-\nu*} \left( \frac{x_2}{2} - \underline{w} + b, -\frac{x_2}{2} - \underline{w} + b \right), \tag{2.53}$$

where we have used the fact that $E^{-n,-\nu*}(\rho_{10}, \rho_{20}) = E^{n,\nu}(\rho_{10}, \rho_{20})$. Since $b_{-n,-\nu*} = b_{n,\nu}$ (see Eq. (2.22)) and $a_{-n,-\nu} = a_{n,\nu}$ (see Eq. (2.38)), the contributions of the four terms obtained after expanding the brackets in Eq. (2.50) are identical and one finally gets, using $\tilde{Y} = Y_1 + \tilde{Y}_2$,

$$F^{(1)}(x_1, x_2, \tilde{Y}, b) = -\frac{\alpha_s^2}{(2\pi)^2} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu \int d^2 \underline{w} \left( \nu^2 + \frac{n^2}{4} \right) \frac{1 + (-1)^n}{\nu^2 + \left( \frac{n+1}{2} \right)^2 \nu^2 + \left( \frac{n+1}{2} \right)^2} \times \exp \left( \frac{2\alpha_s N_c}{\pi} \chi(n, \nu) \tilde{Y} \right) E^{n,\nu*} \left( \frac{x_1}{2} - \underline{w}, -\frac{x_1}{2} - \underline{w} \right) E^{n,\nu} \left( \frac{x_2}{2} - \underline{w} + b, -\frac{x_2}{2} - \underline{w} + b \right). \tag{2.54}$$

Some comments are in order about this result, obtained without any approximation. First, it is clearly independent of the choice of the reference frame, since the result only depends on the total rapidity $\tilde{Y}$. In appendix A.4 we explicitly show that this formula describes in an equivalent way, in the laboratory frame of the left-moving onium (which is defined to be the frame where this onium is moving relativistically, but has not enough rapidity to reveal its soft gluon emissions), the scattering of this non evolved heavy quark-antiquark pair off one excited dipole at a distance $b$ from the center of mass of the fast right-moving onium.

Second, and more importantly, it explicitly proves the exact equivalence between the dipole and the BFKL approaches at leading order. Indeed, taking into account form factors when coupling the $t$-channel bound state of reggeized gluons to the external quark-antiquark pairs (see appendix A.2 for details) and the difference of definition of amplitudes ($A_{\text{dipole}} = \frac{1}{2\pi} A_{\text{BFKL}}$), we should have the following relation between the dipole and the BFKL result

$$F^{(1)}(x_1, x_2, \tilde{Y}, b) = \alpha_s^2 \int \frac{d\omega}{2\pi i} \exp(\omega \tilde{Y}) \left[ f_\omega(\underline{x}_{\nu 1}, \underline{x}_{\nu 1}, \underline{x}_{\nu 2}, \underline{x}_{\nu 2}) + f_\omega(\underline{x}_{\nu 1}, \underline{x}_{\nu 1}, \underline{x}_{\nu 2}, \underline{x}_{\nu 2}) \right], \tag{2.55}$$
where \( f_\omega(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_0) \) is defined by equation (26) of Ref. [25]

\[
\begin{align*}
  f_\omega(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_0, \mathbf{x}_2, \mathbf{x}_0) &= \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d^2 \mathbf{x}_0 \left( \nu^2 + \frac{n^2}{4} \right) \frac{1}{(\nu^2 + \frac{n-1}{2})^2 (\nu^2 + \frac{n+1}{2})^2} \\
  &\times \frac{1}{\omega - \frac{2\alpha_s N_c}{\pi} \chi(n, \nu)} E^{n,\nu} (\mathbf{x}_2 - \mathbf{x}_0, \mathbf{x}_0 - \mathbf{x}_0) E^{n,\nu} (\mathbf{x}_0 - \mathbf{x}_0, \mathbf{x}_0 - \mathbf{x}_0).
\end{align*}
\]  

(2.56)

In this formula the integrand gets a factor \((-1)^n\) when permuting \(x_{a1} \leftrightarrow x_{b1}\). Performing the changes of variable \(w = \mathbf{x}_0 - x_{a1} x_{b1}\) and \((n, \nu) \rightarrow (-n, -\nu)\), one then recovers exactly the expansion (2.54) with the factor \(1 + (-1)^n\), which proves the result. Note that the equivalence between the BFKL and dipole kernel can be also proven by comparing the real and virtual graphs in covariant and light-cone quantization. The result is that the sum of real and virtual contributions is identical in both case, although each of these terms differs. Thus, this result is true only for inclusive quantities [17].

Defining \(F_{(n,\nu)}^{(1)}\) as

\[
F_{(n,\nu)}^{(1)}(\mathbf{x}_1, \mathbf{x}_2, \bar{Y}, \bar{b}) = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} F_{(n,\nu)}^{(1)}(\mathbf{x}_1, \mathbf{x}_2, \bar{b}) \exp \left( \frac{2\alpha_s N_c}{\pi} \chi(n, \nu) \bar{Y} \right),
\]

(2.57)

and using Eqs. (2.19) and (2.31), equation (2.54) can be rewritten as

\[
F_{(n,\nu)}^{(1)}(\mathbf{x}_1, \mathbf{x}_2, \bar{b}) = \frac{\pi \alpha_s^2 x_2^2}{8} \frac{1 + (-1)^n}{(\nu^2 + \frac{n-1}{2})^2 (\nu^2 + \frac{n+1}{2})^2} n_{(n,\nu)}(\mathbf{x}_1, \mathbf{x}_2, \bar{b}).
\]

(2.58)

In the asymptotic regime where one can keep only the term corresponding to \(n = 0\), this relation simplifies to

\[
F_{(0,\nu)}^{(1)}(\mathbf{x}_1, \mathbf{x}_2, \bar{b}) = -\frac{\pi \alpha_s^2 x_2^2}{4} \frac{1}{(\nu^2 + \frac{1}{4})^2} n_{(0,\nu)}(\mathbf{x}_1, \mathbf{x}_2, \bar{b}).
\]

(2.59)

Computing \(F_{(1,\nu)}^{(1)}(\mathbf{x}_1, \mathbf{x}_2, \bar{Y}, \bar{b})\) by a saddle point method as we did for \(n(\mathbf{x}_1, \mathbf{x}_2, \bar{Y}, \bar{b})\) (see Eq. (2.43)), one has to expand the prefactor around \(\nu = 0\), which yields

\[
F_{(1,\nu)}^{(1)}(\mathbf{x}_1, \mathbf{x}_2, \bar{Y}, \bar{b}) \approx -4\pi \alpha_s^2 x_2^2 n(\mathbf{x}_1, \mathbf{x}_2, \bar{Y}, \bar{b})
\]

\[
\approx -\pi \alpha_s^2 x_2^2 \ln(16 b^2/x_1 x_2) \ln(\frac{4\alpha_s N_c}{\pi}) \exp \left\{ \ln^2(16 b^2/x_1 x_2) \right\} \exp \left\{ \frac{\ln^2(16 b^2/x_1 x_2)}{4\alpha_s N_c} \right\}
\]

(2.60)

in the domain

\[
\frac{2\alpha_s N_c}{\pi} \zeta(3) \bar{Y} \ll \ln \frac{16 b^2}{x_1 x_2} \ll 1.
\]

(2.61)

This result, which differs from Eq. (10) of Ref. [16] by a factor 16, is in agreement with numerical simulations [33].
2.3 Calculation of the onium-onium total cross-section

In this subsection we compute the onium-onium total cross-section. It is related to the onium-onium cross-section at fixed impact parameter by

\[ F_{\text{tot}}^{(1)}(\vec{x}_1, \vec{x}_2, \vec{Y}) = \int d^2 \vec{b} \ F^{(1)}(\vec{x}_1, \vec{x}_2, \vec{Y}, \vec{b}). \]  

(2.62)

We show that, provided the elementary dipole-dipole cross section is integrated over distances but not averaged over angles, one can get the onium-onium total cross-section by a simple direct calculation.

Combining formulae (2.7) and (2.62), one obtains

\[ F_{\text{tot}}^{(1)}(\vec{x}_1, \vec{x}_2, \vec{Y}) = \frac{1}{2} \int \frac{d^2 \vec{x}_1'}{2\pi x_1'^2} \frac{d^2 \vec{x}_2'}{2\pi x_2'^2} \frac{d^2 b_1}{2\pi} \frac{d^2 b_2}{2\pi} \frac{d^2 (b_2' - b_1')}{2\pi} \times n(x_1, x_1'; b_1, Y_1) n(x_2, x_2'; b_2, Y_2) \sigma_{DD}(x_1', x_2', b_1' - b_2') \]

\[ = \frac{1}{2} \int \frac{d^2 \vec{x}_1'}{2\pi x_1'^2} \frac{d^2 \vec{x}_2'}{2\pi x_2'^2} n(x_1, x_1', Y_1) n(x_2, x_2', Y_2) \int \frac{d^2 b_2}{2\pi} \frac{d^2 b_1}{2\pi} \sigma_{DD}(x_1', x_2', b_1' - b_2'), \]

(2.63)

where \( n(x_j, x_j', Y_i) \) is the integrated density of dipoles

\[ n(x, x', Y) = \int d^2 \vec{b} \ n(x, x', Y, \vec{b}). \]

(2.64)

Thus, since \( b \) and \( q \) are Fourier conjugated,

\[ n(x, x', Y) = \sum_{n=-\infty}^{+\infty} \int \frac{d\nu}{2\pi} \lim_{q \to 0} E_n^{\nu}(x) E_n^{\nu}(x') \frac{|x|}{|x'|} \exp \left( \frac{2\alpha N_c}{\pi} \chi(n, \nu) \tilde{Y} \right) \]

\[ = \sum_{n=-\infty}^{+\infty} \int \frac{d\nu}{2\pi} \left( \frac{x^* x'}{xx'^*} \right)^{n/2} \left( \frac{x'}{x} \right)^{-2i\nu} \exp \left( \frac{2\alpha N_c}{\pi} \chi(n, \nu) \tilde{Y} \right), \]

(2.65)

where we have used Eq. (2.23) and the expansion (A.32) of \( E_n^{\nu} \) for \( q \to 0 \). The integration of \( \sigma_{DD}(x_1', x_2', b_1' - b_2') \) with respect to the distance \( b_2' - b_1' \) is performed in appendix A.2 and is given by formula (A.33), which still depends on the orientation of the elementary dipoles. One now gets for \( F_{\text{tot}}^{(1)} \)

\[ F_{\text{tot}}^{(1)}(\vec{x}_1, \vec{x}_2, \vec{Y}) = -\frac{1}{2} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu_1 \sum_{n_1=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu_2 \sum_{n_2=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu_1 \sum_{n_2=-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu_2 \int \frac{d^2 \vec{x}_1'}{2\pi x_1'^2} \frac{d^2 \vec{x}_2'}{2\pi x_2'^2} \]

\[ \times x_1 \left( \frac{x^*_1 x_1'}{x_1 x_1'} \right)^{n_1/2} x_2 \left( \frac{x^*_2 x_2'}{x_2 x_2'} \right)^{n_2/2} \left( \frac{x_2'}{x_2} \right)^{-2i\nu_2} \exp \left( \frac{2\alpha N_c}{\pi} \chi(n_1, \nu_1) \tilde{Y}_1 + \chi(n_2, \nu_2) \tilde{Y}_2 \right) \]

\[ \times \alpha_s \left( \frac{x_1 x_2'}{4} \right) \frac{1 + (-1)^n}{\nu_2 + \left( \frac{n-1}{2} \right)^2} \left( \frac{x_1 x_2'}{x_1' x_2'} \right)^{-n/2} \left( \frac{x_1' x_2'}{x_1 x_2} \right)^{2i\nu_2} \].

(2.66)

Integrating with respect to \( x_1' \) and \( x_2' \) leads to a lot of \( \delta \) distributions since

\[ \frac{1}{2\pi} \int \frac{dx_1'}{2x_1' x_1} \frac{dx_2'}{2x_2' x_2} \frac{x_1 - x_1'}{-i(\nu_1 - \nu)} \frac{x_2 - x_2'}{-i(\nu_2 - \nu)} = \pi \delta_{n_1 n} \delta(\nu_1 - \nu) \]

(2.67)
and a similar integral over $x'_2$. Defining $F^{(1)}_{tot(n,\nu)}(x_1, x_2)$ and $n_{(n,\nu)}(x_1, x_2)$ via

$$F^{(1)}_{tot(n,\nu)}(x_1, x_2) = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} F^{(1)}_{tot(n,\nu)}(x_1, x_2) \exp\left(\frac{2\alpha_s N_c}{\pi} \chi(n, \nu)\tilde{Y}\right)$$

(2.68)

and

$$n_{(n,\nu)}(x_1, x_2) = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi} n_{(n,\nu)}(x_1, x_2) \exp\left(\frac{2\alpha_s N_c}{\pi} \chi(n, \nu)\tilde{Y}\right),$$

(2.69)

this finally yields

$$F^{(1)}_{tot(n,\nu)}(x_1, x_2) = -\frac{\pi \alpha_s^2}{4} \frac{1 + (-1)^n}{\left(p^2 + \left(\frac{n-1}{2}\right)^2\right)\left(p^2 + \left(\frac{n+1}{2}\right)^2\right)} \left(\frac{x_1^*x_2}{x_1x_2^*}\right)^{n/2} |x_1|^{1+2\nu} |x_2|^{1-2\nu}$$

(2.70)

$$= -\frac{\pi \alpha_s^2 x_2^2}{8} \frac{1 + (-1)^n}{\left(p^2 + \left(\frac{n-1}{2}\right)^2\right)\left(p^2 + \left(\frac{n+1}{2}\right)^2\right)} n_{(n,\nu)}(x_1, x_2)$$

Comparison with Eq. (2.58) provides a check of this calculation when integrating both sides with respect to $b$. In the asymptotic regime, corresponding to $n = 0$,

$$F^{(1)}_{0,\nu}(x_1, x_2) = -\frac{\pi \alpha_s^2 x_2^2}{4} \frac{1}{\left(p^2 + \frac{1}{4}\right)^2} n_{(0,\nu)}(x_1, x_2),$$

(2.71)

which could be obtained from Eq. (2.59). Using Eq. (2.65), the corresponding asymptotic integrated dipole distribution reads

$$n(x_1, x_2, \tilde{Y}) = \int \frac{d\nu}{2\pi} x_1 \left(\frac{x_1}{x_2}\right)^{-2\nu} \exp\left(\frac{2\alpha_s N_c}{\pi} \chi(n, \nu)\tilde{Y}\right),$$

(2.72)

which gives, in the saddle point approximation at large $\tilde{Y}$,

$$n(x_1, x_2, \tilde{Y}) = \frac{1}{2} x_1 x_2 \exp\left\{\frac{4\alpha_s N_c}{\pi} \ln 2 \tilde{Y}\right\} \exp\left\{-\frac{\ln^2(x_1/x_2)}{16 \alpha_s N_c \zeta(3) \tilde{Y}}\right\}.$$

(2.73)

Thus, one gets for the total cross-section

$$F^{(1)}(x_1, x_2, \tilde{Y}) \simeq -4\pi \alpha_s^2 x_2^2 n(x_1, x_2, \tilde{Y})$$

$$\simeq -2\pi \alpha_s^2 x_1 x_2 \exp\left\{\frac{4\alpha_s N_c}{\pi} \ln 2 \tilde{Y}\right\} \exp\left\{-\frac{\ln^2(x_1/x_2)}{16 \alpha_s N_c \zeta(3) \tilde{Y}}\right\},$$

(2.74)

in agreement with formula (26) of Ref. [13].

Let us integrate the scattering amplitude $F^{(1)}(b)$ with respect to the impact parameter $b$ in the domain (2.61) where formula (2.60) is valid (neglecting the fact that the upper bound is not infinite). The corresponding integration then gives

$$F^{(1)}(x_1, x_2, \tilde{Y}) \simeq -2\pi \alpha_s^2 x_1 x_2 \exp\left\{\frac{4\alpha_s N_c}{\pi} \ln 2 \tilde{Y}\right\} \exp\left\{-\frac{\ln^2(x_1/x_2)}{16 \alpha_s N_c \zeta(3) \tilde{Y}}\right\},$$

(2.75)
which is identical to Eq. (2.74). From the gaussian distribution obtained in Eq. (2.75), it is clear, comparing with Eq. (2.60), that the total cross section at BFKL order is dominated by impact parameter configuration much larger than the transverse sizes of the two scattering onia, corresponding to

\[ \ln \left( \frac{16b^2}{x_1x_2} \right) \sim \sqrt{\frac{14\alpha_s N_c}{\pi}} \zeta(3) \tilde{Y}. \]  

(2.76)

Note that this dominant contribution is inside the domain (2.61). These dominant configurations are much more central than what was claimed in Ref. [16]. It confirms previous numerical simulations [27]. Thus, the calculation, based on perturbative QCD, is expected to remain valid for high values of \( \tilde{Y} \).

3 Electron-Onium Deep Inelastic Scattering

In this section, we perform an analysis of \( e^\pm - onium \) deep inelastic scattering at low \( x_{bj} \), based on \( k_T \)-factorization and dipole color model, as illustrated in figure 2. Our aim is to compute various structure functions in the small \( x_{bj} \) regime, where the onium wave function is dominated by a color dipole cascade. In the Regge limit, one can apply the \( k_T \)-factorization tool \[20, 21, 22\] in order to extract a photon of virtuality \( Q^2 \) off an onium. It involves the elementary Born cross-section \( \hat{\sigma}_{\gamma g}/Q^2 \) of
the process $\gamma g(k) \rightarrow q \bar{q}$. Here the gluon is off-shell, quasi transverse, with a virtuality $k^2 \simeq k_0^2$. One also has to introduce the unintegrated gluon distribution density at a factorization scale $Q_0^2$, which is related to the usual gluon distribution by

$$G(x_{bj}, Q^2, Q_0^2) = \int_0^{Q^2} d^2 k \mathcal{F}(x_{bj}, k, Q_0^2).$$

We first deal with a dipole of transverse size $x_{01}$, which can be either part of a heavy onium (i.e. a heavy $q\bar{q}$ pair) or extracted from a proton as will be emphasized later. The $k_T$-factorization implies, for the total $\gamma^* - \text{dipole}$ cross-section $\sigma_{\gamma^*}^d$,

$$Q^2 \sigma_{\gamma^*}^d(x_{bj}, Q^2; x_{01}^2) = \int d^2 k \int_0^1 \frac{dz}{z} \tilde{\sigma}_{\gamma g}(x_{bj}/z, k^2/Q^2) \mathcal{F}(z, k; x_{01}^2),$$

$\mathcal{F}(z, k; x_{01}^2)$ being the Fourier transform of $\mathcal{F}(z, k; Q_0^2)$ in transverse space. We next evaluate $\mathcal{F}$ by coupling the gluon to the softest dipole which arises in the cascade. This is achieved by using a second $k_T$-factorization. It involves the elementary Born cross-section $\tilde{\sigma}_{gd}/k^2$ of the process $d(x) g(k) \rightarrow d(x)$ for a dipole of transverse size $x$ and a soft gluon of virtuality $k^2$. This $k_T$-factorization can be expressed by

$$k^2 \mathcal{F}(z, k; x_{01}^2) = \int \frac{d^2 x}{(2\pi)^2 x^2} \int z \frac{dz'}{z'} n(x_{01}, x, \ln \frac{z_1}{z'}) \tilde{\sigma}_{gd}(z/z', x^2 k^2) \delta(z/z' - 1),$$

where the distribution density $n(x_{01}, x, z)$ was defined in section 2. As previously $z_1 p_+ \ln \chi(k)$ is the light-cone momentum of the quark part of the dipole $(x_{01})$. $\tilde{\sigma}_{gd}$ is computed in appendix A.3 using eikonal techniques. Defining $Y = \ln z_1/z$, one gets for the dipole-photon cross-section

$$Q^2 \sigma_{\gamma^*}^d(x_{bj}, Q^2; x_{01}^2) = \int d^2 k \int_0^1 \frac{dz}{z} \tilde{\sigma}_{\gamma g}(x_{bj}/z, k^2/Q^2) \int \frac{d^2 x}{x^2} n(x_{01}, x, Y)$$

$$\times 4 \pi \alpha_s \frac{N_c}{(2\pi)^4} (2 - e^{i k_+ x} - e^{-i k_+ x}) \frac{1}{k^2}.$$  

Let us now compute the convolutions in longitudinal and transverse spaces. As in section 2, we introduce a double Mellin-transform in these both variables, namely

$$n(x_{01}, x, Y) = \int \frac{d\omega}{2i\pi} e^{\omega Y} n_\omega(x_{01}, x)$$

and

$$n_\omega(x_{01}, x) = \int \frac{d\gamma}{2i\pi} \left( \frac{x_{01}}{x} \right)^{2\gamma} n_\omega(\gamma).$$

Here the Mellin variable in transverse space is $\gamma = \frac{1}{2} + iv$, which is introduced here rather than $\nu$ since it plays the role of an anomalous dimension. We consider only the dominant Regge trajectory, that is $n = 0$. Formula (2.17) then reads

$$n_\omega(\gamma) = \frac{2}{\omega - 2\alpha_s N_c/\pi} \chi(\gamma)$$

where

$$\chi(\gamma) = \chi(0, \frac{1}{2} + iv) = \Psi(1) - \frac{1}{2} \Psi(\gamma) - \frac{1}{2} \Psi(1 - \gamma)$$

and

\begin{align*}
\chi = \chi(0, \frac{1}{2} + iv) = \Psi(1) - \frac{1}{2} \Psi(\gamma) - \frac{1}{2} \Psi(1 - \gamma) \quad (3.84)
\end{align*}

\text{15}
The quantity \( \hat{\sigma}_{\gamma_0} \) has been calculated for different polarizations of the incoming photon in [21]. We introduce the corresponding double Mellin transform of this cross-section, namely

\[
4\pi^2\alpha_{e.m} h_\omega(\gamma) = \gamma \int_0^\infty \frac{dk^2}{k^2} \left( \frac{k^2}{Q^2} \right)^\gamma \hat{\sigma}_\omega \left( \frac{k^2}{Q^2} \right)
\] (3.85)

or equivalently

\[
\hat{\sigma}_\omega \left( \frac{l^2}{Q^2} \right) = 4\pi^2\alpha_{e.m} \int \frac{d\gamma}{2i\pi} \left( \frac{l^2}{Q^2} \right)^{-\gamma} \frac{1}{\gamma} h_\omega(\gamma),
\] (3.86)

with

\[
\hat{\sigma}_\omega \left( \frac{k^2}{Q^2} \right) = \int_0^1 dz \, z^{\omega-1} \hat{\sigma} \left( z, \frac{k^2}{Q^2} \right).
\] (3.87)

The expression for \( \sigma^{d*}_{\gamma_0} \) now reads, after performing the convolution in longitudinal space,

\[
Q^2 \sigma^{d*}_{\gamma_0}(x_{bj}, Q^2; x^2_{01}) = 4\pi^2\alpha_{e.m} \frac{\alpha_s N_c}{4\pi^3} \int dk \int d\gamma' \int \frac{d\gamma}{2i\pi} \int \frac{d\omega}{2i\pi} \exp \left( \omega \ln \frac{z_1}{x_{bj}} \right) \left( \frac{x_{01}}{x} \right)^{2\gamma}
\]

\[
\times \frac{2}{\omega - \frac{2\alpha_s N_c}{\pi} \chi(\gamma)} \int \frac{d^2k}{k^2} \left( \frac{k^2}{Q^2} \right)^{-\gamma'} (2 - e^{i\mathbf{k} \cdot \mathbf{x}} - e^{-i\mathbf{k} \cdot \mathbf{x}}) \frac{1}{k^2}.
\] (3.88)

The integration with respect to the polar angle of \( x \) leads to a Bessel function. One has then to integrate over \( x \), namely

\[
\int \frac{dx}{x} 4\pi(1 - J_0(kx)) \left( \frac{x}{x_{01}} \right)^{-2\gamma} = 4\pi(kx_{01})^{2\gamma} \frac{2\gamma - 1 - 2\gamma}{\gamma} \Gamma(1 - \gamma) \Gamma(1 + \gamma) = 4\pi(kx_{01})^{2\gamma} v(\gamma).
\] (3.89)

The integration over \( k \) gives \( \gamma = \gamma' \). Since \( n_\omega(\gamma) \) (formula (3.83)) exhibits a pole at \( \omega_p = \frac{\alpha_s N_c}{\pi} \chi(\gamma) \), the \( \omega \) integral finally yields

\[
\frac{Q^2}{4\pi^2\alpha_{e.m}} \sigma^{d*}_{\gamma_0}(x_{bj}, Q^2; x^2_{01}) = \frac{2\alpha_s N_c}{\pi} \int \frac{d\gamma}{2i\pi} h_{\omega_p}(\gamma) \frac{v(\gamma)}{\gamma} (x^2_{01} Q^2)^\gamma \exp \left( \frac{2\alpha_s N_c}{\pi} \chi(\gamma) \ln \frac{z_1}{x_{bj}} \right).
\] (3.90)

In the regime we are interested in, \( \omega_p \ll \gamma \) and the dependence of \( h_{\omega_p}(\gamma) \) on \( \omega_p \) can be neglected, replacing \( \omega_p \) by 0 [20]. We then get for the total cross-section \( \gamma^* - dipole(x_{01}) \) and for the related structure function (we assume \( R = F_L/F_T \) to be small in the relation between the total cross-section and the structure function)

\[
\frac{Q^2}{4\pi^2\alpha_{e.m}} \sigma^{d*}_{\gamma_0}(x_{bj}, Q^2; x^2_{01}) = F^{d}(x_{bj}, Q^2; x^2_{01}) = \frac{2\alpha_s N_c}{\pi} \int \frac{d\gamma}{2i\pi} (Q^2 x_{01}^2)^\gamma h(\gamma) \frac{v(\gamma)}{\gamma} \exp \left( \frac{2\alpha_s N_c}{\pi} \chi(\gamma) \ln \frac{z_1}{x_{bj}} \right).
\] (3.91)

The dipole \( (x_{01}) \) being part of a bound state, for example extracted from an onium, one has now to average with respect to the corresponding wave function, \( \Phi^{(0)}(x_{01}, z_1) \). The initial dipole state is supposed to be well localized in transverse space. Namely, its transverse size, given by the scale \( M^2 \) which is defined via

\[
(M^2)^{-\gamma} = \int d^2x_{01} (x_{01}^2)^\gamma dz_1 \Phi^{(0)}(x_{01}, z_1),
\] (3.92)

is assumed to be perturbative. One then obtains for the onium structure function

\[
F^{\text{onium}}(x_{bj}, Q^2; M^2) = \frac{2\alpha_s N_c}{\pi} \int \frac{d\gamma}{2i\pi} h(\gamma) \frac{v(\gamma)}{\gamma} \left( \frac{Q^2}{M^2} \right)^\gamma \exp \left( \frac{2\alpha_s N_c}{\pi} \chi(\gamma) \ln \frac{1}{x_{bj}} \right).
\] (3.93)
It is then possible to apply formula (3.93) to peculiar structure functions, namely

\[
\begin{bmatrix}
F_T \\
F_L \\
F_G
\end{bmatrix} = \frac{2\alpha_s N_c}{\pi} \int \frac{d\gamma}{2i\pi} \left( \frac{Q^2}{M^2} \right)^\gamma \exp \left( \frac{2\alpha_s N_c}{\pi} x_{bj} \ln \frac{1}{x_{bj}} \right) \left( \begin{array}{c} h_T \\ h_L \\ 1 \end{array} \right) \frac{v(\gamma)}{\gamma} \tag{3.94}
\]

where \(F_{T(L)}\) is the structure function corresponding to transverse (longitudinal) photons and \(F_G\) the gluon structure function. The coefficient functions

\[
\begin{bmatrix}
h_T \\
h_L \\
1
\end{bmatrix} = \frac{\alpha_s}{3\pi \gamma} \frac{1}{\Gamma(2-2\gamma)\Gamma(2+2\gamma)} \left( \frac{1}{\gamma(1-\gamma)} \right)
\tag{3.95}
\]

were computed in ref [20]. When \(x_{bj}\) is small, the \(\gamma\)-integration can be performed by the steepest-descent method. The corresponding asymptotic saddle point is located at \(\gamma = \frac{1}{2}\), which defines the BFKL anomalous dimension. Expanding the \(\chi\) function around \(\frac{1}{2}\), we obtain a saddle point at

\[
\gamma_s = \frac{1}{2} \left( 1 - a \ln \frac{Q}{Q_0} \right), \quad \text{where} \quad a = \left( \frac{\bar{\alpha} N_c \zeta(3) \ln \frac{1}{x_{bj}}}{\pi} \right)^{-1}.
\tag{3.96}
\]

The approximation of expanding \(\chi(\gamma)\) around \(\frac{1}{2}\) is valid when

\[
a \ln \left( \frac{Q}{M} \right) \approx \ln \frac{Q}{M} / \ln \frac{1}{x_{bj}} \ll 1,
\tag{3.97}
\]

that is the small \(x_{bj}\), moderate \(Q/M\) kinematical domain. This yields

\[
F_2 \equiv F_T + F_L = Ca^{1/2} \frac{Q}{M} \exp \left( (\alpha_P - 1) \ln \frac{1}{x_{bj}} - \frac{a}{2} \ln^2 \frac{Q}{Q_0} \right),
\tag{3.98}
\]

where \(\alpha_P\) is defined by Eq. (1.2). Thus, \(F_2\) depends only on 3 parameters, \(C\), \(M\) and \(\alpha_P\). Suppose we can fit \(F_2\) with this form. Then, get a prediction for \(F_G^\prime\) and \(R = F_L/F_T\) without any free parameter. Namely,

\[
\frac{F_G}{F_2} \bigg|_{\gamma = \gamma_s} = \frac{1}{h_T + h_L} \frac{3\pi \gamma_s}{\alpha_s} \frac{1 - \frac{2}{3} \gamma_s}{1 + \frac{3}{2} \gamma_s - \frac{3}{2} \gamma_s^2} \frac{\Gamma(2-2\gamma_s)\Gamma(2+2\gamma_s)}{\Gamma(1-\gamma_s)\Gamma(1+\gamma_s)} \tag{3.99}
\]

and

\[
R = \frac{h_L}{h_T} = \frac{\gamma_s(1-\gamma_s)}{(1+\gamma_s)(1-\frac{\gamma_s}{2})},
\tag{3.100}
\]

where \(\gamma_s\) is given by the expression (3.96). Note that the overall non-perturbative normalization \(C\) does not enter \(R\) and \(F_G/F_2\). It is possible to apply this analysis to the proton [24]. This requires some additional assumptions when considering the coupling of the dipole cascade to the proton. It leads to a successful description of the HERA data [1]. It also provides a prediction for the gluon density based on the BFKL dynamics, and a prediction for the ratio \(R\), using formulas (3.99) and (3.100). All the previous analysis was done by performing an expansion around the small \(x_{bj}\) behaviour of the BFKL Pomeron, that is by considering \(\ln 1/x_{bj}\) as a big parameter, which leads to a saddle point close to \(\gamma = \frac{1}{2}\) (see Eq. (3.96)). A different analysis of the modification, due to the BFKL dynamics, of the anomalous dimension of the double
logarithmic approximation common to DGLAP and BFKL can be done by considering now \( \ln Q^2/M^2 \) as a big parameter. The corresponding saddle point of Eq. (3.94) is now around \( \gamma = 0 \). In Mellin space, the double logarithmic expression of the anomalous dimension is given by \( \gamma_\omega(\alpha_s) = 3\alpha_s/(\pi\omega) \), and expansion of the \( \chi \) function (3.84) around \( \gamma = 0 \) leads to corrections given by powers of \( 3\alpha_s/(\pi\omega) \). This method provides an extension of the domain of applicability (3.97) [36].

As it has been seen in this section, the dipole model can be safely applied when the two scales of the process are both perturbative, as it is the case for \( e^\pm - \text{onium} \) scattering. The application to \( e^\pm - p \) scattering requires some assumptions for the coupling to the proton. Because of the well-known diffusion in transverse momentum space, such an application of the dipole model, although successful [23, 24, 25], cannot be considered as a clean test of high-energy perturbative regge dynamics. A possible test of such dynamics could be based on single jets events in DIS [37] or double jets events in hadron-hadron collision [38]. Another interesting test of BFKL dynamics would be the \( \gamma^* - \gamma^* \) events in \( e^+ - e^- \) colliders at high energy in the center of mass of the virtual photon pair and with high (perturbative) photons virtualities. This has been already proposed in the framework of the original BFKL equation [39]. Such a process can equivalently be described in the dipole picture of BFKL dynamics. This will be developed elsewhere [40].

4 Conclusion

In this article we have shown the exact equivalence between BFKL and dipole approaches for the onium-onium cross-section at fixed impact parameter. This proof relies on conformal properties of the dipole cascade and of the elementary dipole-dipole cross section. We have also obtained asymptotic expressions for the dipole distribution inside an onium and for the onium-onium cross-section at fixed impact parameter. These results agrees with previous numerical simulations. We also apply the dipole model to onium-\( e^\pm \) deep inelastic scattering, using the \( k_T \)-factorization, and obtain predictions for various structure functions in the BFKL dynamics. The different elementary cross-sections used in this paper are computed using eikonal techniques.

Relying on the same conformal properties, it should be possible to get analytical expressions for the multipomeronic contributions to the onium-onium cross-section, which are expected to be important for large rapidities.

From a phenomenological point of view, the dipole framework could be applied to other inclusive processes. The application of this technique for exclusive quantities remains however an open question, due to the use of light-cone quantization, in which the intermediate states are unphysical.

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A Appendices

A.1 Calculation of $\sigma_{DD}$ using eikonal methods

In this appendix we compute the dipole-dipole cross-section using eikonal techniques. In QCD, the eikonal current due to a fast quark of momentum $p$, responsible for the emission of a soft gluon of momentum $k$ ($k \ll p$) and color $a$ reads

$$j^\nu(k) = -igT^a \frac{p^\nu}{p.k + i\epsilon}.$$  \hfill (A.1)

Let us consider the scattering of two dipoles. Let $x_0$ ($x_1$) be the transverse coordinate of the free quark (antiquark) making the right moving dipole and $x'_0$ ($x'_1$) the coordinates of the corresponding quark (antiquark) making the left moving dipole. In this mixed representation where the fast radiating particule is represented in impact parameter space, and the radiated gluon is represented in momentum space, the eikonal current corresponding to a quark of transverse coordinate $x$ then reads

$$j^a\nu(k) = -igT^a \tilde{\eta}^\nu \tilde{\eta}.k + i\epsilon e^{-ix_0.k} - e^{-ix_1.k},$$  \hfill (A.2)

where $\tilde{\eta} = (\hat{1}, \tilde{0}, \frac{1}{\perp})$. This current is responsible for a term $-ij^a\nu(k)\epsilon_\nu$ when computing the amplitude of emission of a gluon by the quark. This terms arises when evaluating the evolution operator $T \exp \{-i \int d^4x j^a\nu(x)A_{a\nu}(x)\}$ due to the usual hamiltonian $\int d^3x j^a\nu(x)A_{a\nu}(x)$. Note that this current is completely described by the transverse size of the dipole and does not require any additional information about its internal structure.

The eikonal current corresponding to the right moving dipole then reads

$$j^a\nu(k) = -igT^a \tilde{\eta}^\nu \tilde{\eta}.k + i\epsilon e^{-ix_0.k} - e^{-ix_1.k},$$  \hfill (A.3)

and a corresponding formula for the left moving dipole, replacing $\tilde{\eta}$ by $\eta = (\hat{1}, \tilde{0}, \frac{1}{\perp})$.

Let us now compute the graph $A_1$ represented in figure 3. It reads, in Feynman covariant gauge,

$$A_1 = \frac{1}{N_c^2 \sum_{ab} \frac{1}{2p^+ + 2p^-} \int \frac{d^4k}{(2\pi)^4} \left\{ \frac{2(-p.k)}{i} (-ij^{b+}(-k)) \right\} \left\{ \frac{2(p'.k)}{i} (-ij^{b-}(k)) \right\} \times (-ij^{b+}(k))(-ij^{b-}(-k)) \left( \frac{-i}{k^2} \right)^2.$$  \hfill (A.4)
The terms $\frac{1}{2\nu^2} \frac{1}{2p^2}$ are related to the normalisation of the initial dipole states and $\frac{2(-p,k)}{k}$ and $\frac{2(p',k)}{k'}$ are due to the fact that the considered amplitude is computed for amputated propagators. The color factor reads

$$\frac{1}{N_c^2} \sum_{ab} Tr T^a T^b Tr T^a T^b = \frac{N_c^2 - 1}{4N_c^2},$$

(A.5)

which equals $\frac{1}{4}$ in the large $N_c$ limit.

Using the expression (A.3), this yields

$$A_1 = g^4 \int \frac{d^4 l}{(2\pi)^4} \frac{1}{k^- + i\epsilon} \frac{1}{k^+ - i\epsilon} \times (e^{ix_0.k} - e^{ix_1.k}) (e^{-ix_0.k} - e^{-ix_1.k}) (e^{ix_0'.k} - e^{ix_1'.k}) (e^{-ix_0'.k} - e^{-ix_1'.k}).$$

(A.6)

One has also to consider the crossed diagram $A_2$. In order to integrate with respect to $k^-$ and $k^+$ we consider two equivalent representations of $A_1$ and $A_2$, obtained by changing the sign of $k$ (see figure 4). Forgetting the phase factor for a moment, the forward amplitude $A$ then reads

\begin{equation}
A = A_1 + A_2 = g^4 \int \frac{d^2 k}{(2\pi)^2} \frac{d k^-}{2\pi} \frac{d k^+}{2\pi} \frac{1}{(k^2)^2} \frac{1}{2} \left( \frac{1}{k^- + i\epsilon} + \frac{1}{-k^- + i\epsilon} \right) \left( \frac{1}{k^+ + i\epsilon} + \frac{1}{-k^+ + i\epsilon} \right).
\end{equation}

(A.7)

The two last terms reads $(-2\pi i\delta(k^-))(-2\pi i\delta(k^+))$, which finally yield, performing the integration with respect to $k^-$ and $k^+$

\begin{equation}
A = -\frac{\alpha_s^2}{2} \int \frac{d^2 k}{(k^2)^2} \left( 2 - e^{ik.x_01} - e^{-ik.x_01} \right) \left( 2 - e^{ik.x_01'} - e^{-ik.x_01'} \right),
\end{equation}

(A.8)

which was obtained in Ref. [15] by computing elementary Feynman diagrams. If we neglect the dependence of $\Phi(x_{01}, z)$ and $n(x_{01}, x, Y)$ with respect to the dipoles orientation (see appendix A.5), it is possible to average with respect to the angle of these dipoles. $A$ then reads

\begin{equation}
A = -4\pi \alpha_s^2 \int_0^\infty \frac{dk}{k^3} [1 - J_0(kx_{01})] [1 - J_0(kx_{01}')]\] (A.9)
Using the identity
\[ \int_0^\infty \frac{dk}{k^3} [1 - J_0(ke)] [1 - J_0(k\tilde{e})] = \frac{1}{4} x^2_0 [1 + \ln \frac{x_0}{x_0}] \]  
(A.10)
where \( x_0 = \text{Min}(x_0, x_0') \) and \( x_0 = \text{Max}(x_0, x_0') \), this finally yields for the corresponding cross-section
\[ \sigma_{DD}(x_0, x_0') = -2A = 2\pi\sigma_0^2 x_0^2 \left[ 1 + \ln \frac{x_0}{x_0} \right] . \]  
(A.11)
This average result is however sufficient when computing the total cross-section (see appendix A.5).

Another representation of this elementary cross-section is very useful. Consider
\[ I = \frac{\alpha^2_s}{2} \int_{-\infty}^{\infty} \frac{d\nu}{(\nu^2 + \frac{1}{4})^2} x_1^{1+2i\nu} x_2^{1-2i\nu} . \]  
(A.12)
When \( x_1 > x_2 \) \((x_1 < x_2)\) one can close the integration contour around \(+i\infty\) \((-i\infty)\), so that one pick up the (simple) pole at \( \nu = i/2 \) \((\nu = -i/2)\). Thus,
\[ I = \frac{\alpha^2_s}{2} 2i\pi \frac{d}{d\nu} \left. \frac{e^{2i\nu \ln \frac{x_0}{x_0}}}{(\nu \pm i/2)^2} \right|_{\nu = \pm \frac{i}{2}} x_0 x_0 , \]  
(A.13)
where \( x_0 = \text{Min}(x_1, x_2) \) et \( x_0 = \text{Max}(x_1, x_2) \). This finally yields the expected result:
\[ \sigma_{DD}(x_0', x_0) = \frac{\alpha^2_s}{2} \int_{-\infty}^{\infty} \frac{d\nu}{(\nu^2 + \frac{1}{4})^2} x_1^{1+2i\nu} x_2^{1-2i\nu} . \]  
(A.14)
Note that this can be written equivalently as (see the following section)
\[ \sigma_{DD}(x_1', x_2') = \frac{\alpha^2_s}{4} x_1' x_2' \int_{-\infty}^{\infty} \frac{d\nu}{(\nu^2 + \frac{1}{4})^2} \lim_{q\to 0} \left[ E_q^{0\nu}(x_1') E_q^{0\nu}(x_2') \right] . \]  
(A.15)

### A.2 Calculation of \( \sigma_{DD}(\vec{x}, \vec{x}', \vec{b} - \vec{b}') \)

In this appendix we compute the elementary dipole-dipole cross-section at fixed impact parameter. As in the appendix A.1, we consider two dipoles of transverse sizes \( \vec{x} = \vec{x}_0 - \vec{x}_1 \) and \( \vec{x}' = \vec{x}_0' - \vec{x}_1' \), whose centers are situated at \( \vec{b} = \frac{\vec{x}_0 + \vec{x}_1}{2} \) and \( \vec{b}' = \frac{\vec{x}_0' + \vec{x}_1'}{2} \).

Let us compute the non-forward scattering amplitude of these two dipoles. In the high-energy limit \(-t << s = 2p.p'\), the exchanged momentum is quasi-transverse
\[ q \simeq (0, 0, \bar{q}). \]  
(A.16)
The graphs to be computed are displayed in figure 3. These graphs are computed using the rules of appendix A.1. Since \( q^- = q^+ = 0 \), the expression for the corresponding amplitude is similar to the one obtained in equation (A.8), except for phase factors. It reads
\[ A = -\frac{\alpha^2_s}{2} \int \frac{d^2q}{(2\pi)^2} \int \frac{d^2k}{k^2(k + q)^2} \left( e^{i\vec{x}_0 \cdot \vec{k}} - e^{i\vec{x}_1 \cdot \vec{k}} \right) \left( e^{-i\vec{x}_0 \cdot (\vec{k} + q)} - e^{-i\vec{x}_1 \cdot (\vec{k} + q)} \right) \times \left( e^{-i\vec{x}_0' \cdot \vec{k}} - e^{-i\vec{x}_1' \cdot \vec{k}} \right) \left( e^{i\vec{x}_0' \cdot (\vec{k} + q)} - e^{i\vec{x}_1' \cdot (\vec{k} + q)} \right) . \]  
(A.17)
Figure 5: Contributions to the elementary dipole-dipole scattering with non zero exchanged momentum.

Note that this expression is overall translationally invariant. However, it will depend on the relative distance between the two dipoles and on their angles. Expanding the previous expression and integrating with respect to the angles of \( k \) and \( k + q \), one obtains several Bessel functions

\[
A = -\frac{\alpha_s^2}{2} \left\{ I(|x_0 - x_{0'}|) - I(|x_0 - x_{1'}|) - I(|x_1 - x_{0'}|) + I(|x_1 - x_{1'}|) \right\}^2, \tag{A.18}
\]

where

\[
I(x) = \int_{\rho}^{+\infty} \frac{dk}{k} J_0(kx) \tag{A.19}
\]

\( \rho \) is an infrared cut-off which regularize the divergency at \( k = 0 \). In the limit \( \rho \to 0 \), \( I \) can be computed (see Ref. [14])

\[
I = \lim_{\lambda \to 0} \left[ \int_{0}^{+\infty} \frac{dk}{k^{1-x}} J_0(kx) - \int_{0}^{\rho} \frac{dk}{k^{1-x}} J_0(kx) \right] = \psi(1) + \ln 2 - \ln x - \ln \rho. \tag{A.20}
\]

When evaluating \( A \), the constant \( \psi(1) + \ln 2 \) and the infrared divergent term \( \ln \rho \) cancels. \( A \) then reads

\[
A = -\frac{\alpha_s^2}{2} \left\{ \ln \frac{|x_0 - x_{0'}|}{|x_0 - x_{1'}|} \right\}^2 \tag{A.21}
\]

Defining

\[
b = \frac{x_0 + x_1}{2} \quad \text{and} \quad b' = \frac{x_{0'} + x_{1'}}{2}, \tag{A.22}
\]

the non-forward cross-section finally reads

\[
\sigma_{DD}(x, x', b - b') = -2A = \alpha_s^2 \left\{ \ln \frac{|b' - b + \frac{x_0 + x_{0'}}{2}|}{|b' - b + \frac{x_0 + x_{0'}}{2}|} \right\}^2, \tag{A.23}
\]

as quoted in Ref. [27]. Note that it only depends on the relative distance \( b - b' \) because of the overall translational invariance. This expression can also be obtained by evaluating
this scattering amplitude in the laboratory frame of one of the two onia. In this frame, this
amplitude is expressed as an eikonal phase (computed in terms of a Wilson-loop), due to the
change of the wave function of the slow moving onium in the color field of the fast moving onium
[26]. Let us now show that this cross-section can be expanded on the basis of the functions
$E^{n,\nu}$. In the standard Regge calculation, instead of considering the scattering of two dipoles,
one considers the scattering of two gluons of momenta $-k$ and $k + q$ (see figure [A.24]). In the

\[
\begin{align*}
\text{Figure 6: Gluon-Gluon diffusion in the Regge framework.}
\end{align*}
\]

lowest order approximation, the contribution of this graph is

\[
A_{gg} = \frac{\delta^2(k - k')}{k^2(k + q)^2},
\]

which gives, after Fourier transform in impact parameter space [41]

\[
a(x_0, x_1, x_0', x_1') = (2\pi)^2 \ln(|x_{00'}|\lambda) \ln(|x_{11'}|\lambda),
\]

where $\lambda$ is the mass of the gluon introduced in order to remove the infrared divergency. Since
one is interested in the coupling with color neutral states, this expression can equivalently be
replaced by

\[
a(x_0, x_1, x_0', x_1') = 2\pi^2 \ln \left| \frac{x_{00'} x_{11'}}{x_{01} x_{01'}} \right| \ln \left| \frac{x_{00'} x_{11'}}{x_{01} x_{01'}} \right|
\]

because of the conservation of color current [41]. In the case of the dipole model, instead of
labeling the gluons, one labels the dipoles. Thus, the contributions of the figure 5 should be
equal to the contribution of figure 6 plus permutations ($x_0 \leftrightarrow x_1$) and ($x_0' \leftrightarrow x_1'$) (except for
normalisation factors). Indeed, one can check that

\[
\sigma_{DD}(x, x', b - b') = \frac{\alpha_s}{(2\pi)^2} \left[ a(x_0, x_1, x_0', x_1') + (x_0 \leftrightarrow x_1) + (x_0' \leftrightarrow x_1') + (x_0 \leftrightarrow x_1, x_0' \leftrightarrow x_1') \right]
\]

\[
= 2\alpha_s [a(x_0, x_1, x_0', x_1') + (x_0 \leftrightarrow x_1)].
\]

(A.27)
Using the expression (26) of Ref. [29] in the Born approximation, that is making $g = 0$, one finally obtains
\[
\sigma_{DD}(x, x', b - b') = 2\alpha_s^2 \left(\frac{d^2 q}{2\pi^2}\right)^2 \sum_{n=\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu \int d^2 w \left(\nu^2 + \frac{n^2}{4}\right) \frac{1 + (-1)^n}{\left(\nu^2 + \left(\frac{n-1}{2}\right)^2\right) \left(\nu^2 + \left(\frac{n+1}{2}\right)^2\right)}
\times E^{n,\nu*}(b + \frac{x}{2} - w, b - \frac{x}{2} - w) \ E^{n,\nu}(b' + \frac{x'}{2} - w, b' - \frac{x'}{2} - w),
\]
where
\[
E^{n,\nu}(x_{b0}, x_{b0}) = (-1)^n E^{n,\nu}(x_{a0}, x_{a0}).
\]

One can check on this expression that when integrating over $b' - b$ and averaging over angles, one recovers the total cross-section (A.14). Indeed, using the mixed representation (2.21), one obtains
\[
\sigma_{DD}(x, x') = \int d^2(b' - b) \sigma_{DD}(x, x', b - b')
= 2\alpha_s^2 \frac{x x'}{16} \sum_{n=\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu \frac{1 + (-1)^n}{\left(\nu^2 + \left(\frac{n-1}{2}\right)^2\right) \left(\nu^2 + \left(\frac{n+1}{2}\right)^2\right)}
\times \lim_{q \to 0} \left[ E^{n,\nu*}(x') E^{n,\nu}(x) \right].
\]

Using the expansion (A.1) of Ref. [29], one obtains
\[
E^{n,\nu*}(x') E^{n,\nu}(x) \bigg|_{q \equiv 1/x, 1/x'}
= \frac{(x x')^{n/2}}{(x^* x')^{n/2}} \left[ \left| \frac{x}{x'} \right|^{-2i\nu} \left(1 + \frac{(q x^*)^n}{q x'} \right) \left| q x' \right|^{-4i\nu} e^{i\delta(n,\nu)} \right]
\times \left[ \left| \frac{x}{x'} \right|^{-2i\nu} \left(1 + \frac{(q^* x')^n}{q^* x} \right) \left| q x' \right|^{-4i\nu} e^{i\delta(n,\nu)} \right]
\]
\[
= \frac{(x x')^{n/2}}{(x^* x')^{n/2}} \left[ \left| \frac{x}{x'} \right|^{-2i\nu} \left(1 + \frac{(q x^*)^n}{q x'} \right) \left| q x' \right|^{-4i\nu} e^{i\delta(n,\nu)} + \frac{(q^* x')^n}{q^* x} \right]
\]
\[
\left| \left| \frac{x}{x'} \right|^{-2i\nu} \left(1 + \frac{(q x^*)^n}{q x'} \right) \left| q x' \right|^2 \right| \left(\frac{x x'}{x^* x'}\right)^{n/2} \left| \frac{x}{x'} \right|^{-2i\nu}.
\]

Averaging over the angles, the only remaining term is $n = 0$. Thus,
\[
\sigma_{DD}(x, x') = \frac{\alpha_s^2}{2} \int_{-\infty}^{+\infty} d\nu \left| x \right|^{1+2i\nu} \left| x' \right|^{-1-2i\nu} \frac{1}{\left(\nu^2 + \frac{1}{4}\right)^2},
\]
which is identical to Eq. (A.13).
A.3 Calculation of $\hat{\sigma}_{gd}$ using eikonal methods

In this section we compute the elementary Born cross-section $\hat{\sigma}_{gd}/k^2$ of the process

$$d(x) \ g(k) \rightarrow d(x)$$ \hspace{1cm} (A.35)

for a dipole of transverse size $x$ and a soft gluon of virtuality $k^2$, in light-cone gauge. This process is illustrated in figure 7. Consider a dipole of transverse size $x = x_a - x_b$. The corresponding eikonal current has the same expression than in covariant gauge (A.3). This would not be the case for a more complicate system for which the current would be rotated in color space (see for example Ref. [42]). In light-cone perturbation theory, $k^2 = 2k^+k^- - k_z^2 = 0$. The corresponding current then reads

$$j^{a+}(k) = -ig T^a \frac{2k^+}{k^2} \left( e^{-ik_a \cdot k} - e^{-ik_b \cdot k} \right).$$ \hspace{1cm} (A.36)

The Born cross-section then reads, summing over color and polarization of the emitted gluon and averaging over the color of the dipole,

$$\hat{\sigma}_{gd} = \frac{1}{2(2\pi)^2 N_c} Tr |j^+ e^{-\lambda}|^2$$

$$= \frac{1}{2(2\pi)^2 N_c} \sum_{\lambda = 1,2} \sum_a g^2 Tr (T^a T^a) \left( e^{-ik_a \cdot x_a} - e^{-ik_b \cdot x_b} \right) \left( e^{ik_a \cdot x_a} - e^{ik_b \cdot x_b} \right) 4 \left( \frac{k_i \epsilon^\lambda}{k^2} \right)^2.$$ \hspace{1cm} (A.37)

Here one takes into account the fact that the dominant polarization of the soft emitted gluon in light-cone gauge corresponds to

$$\epsilon^\mu = \left( 0, \frac{k^\lambda}{k_+}, 1^{\lambda} \right) \sim \left( 0, \frac{k^\lambda}{k_+}, 0 \right)$$ \hspace{1cm} (A.38)

in the soft limit. This finally yields

$$\hat{\sigma}_{gd} = \frac{g^2 C_F}{2(2\pi)^2} \left( 2 - e^{ik_a \cdot x} - e^{-ik_b \cdot x} \right) \frac{4}{k^2} = \frac{\alpha_s N_c}{\pi} \left( 2 - e^{ik_a \cdot x} - e^{-ik_b \cdot x} \right) \frac{1}{k^2}.$$ \hspace{1cm} (A.39)

Figure 7: Amplitudes contributing to the elementary dipole-gluon cross-section.
Let us make two comments. First, the dominant polarization of the exchanged gluon is longitudinal, due to the expression (A.38). However, the polarization tensor in light-cone gauge reads

$$d_{\mu\nu} = g_{\mu\nu} - \frac{\eta^\mu k^\nu + \eta^\nu k^\mu}{\eta \cdot k}. \quad (A.40)$$

In the Regge limit, since $j_{\nu}$ is proportional to $\tilde{\eta}$, the relevant component of this tensor is $d_{\mu-}$, and since $\epsilon_{+} = 0$ in light cone gauge, it turns out that $\epsilon_{\mu} d_{\mu-} = \epsilon d_{\nu}^{-}$, and thus the exchanged gluon has a physical polarization, as it is expected in high-energy factorization.

Secondly, when integrating this cross-section over $k$, one can recover the dipole emission kernel which was constructed in Ref. [14].

### A.4 Calculation of the onium-onium cross section in the laboratory frame of one onium

In this appendix we illustrate the frame invariance of the onium-onium cross section by an explicit calculation in the laboratory frame where the left-moving heavy quark-antiquark pair is close to be at rest. Let us first consider as in section 2.2 the scattering of two heavy quark-antiquark pairs of transverse sizes $x_1$ and $x_2$ which scatter through the exchange of a pair of gluons between two elementary excited dipoles of transverse sizes $x'_1$ and $x'_2$. The corresponding expression for the scattering amplitude $F^{(1)}$ is displayed in formula (2.47). Using the same trick which led us to formula (2.50), we integrate over $d^2b_{\mu n}d^2x'_1$ and $d^2w_2$, which yields

$$F^{(1)}(x_1, x_2, \tilde{Y}, \tilde{b}) = -\frac{\alpha_s^2 (16\pi)^2}{(2\pi)^6} \sum_{n_1=0}^{+\infty} \sum_{n=0}^{+\infty} \int_{-\infty}^{+\infty} \frac{d\nu_1}{(2\pi)^3} \int_{-\infty}^{+\infty} dv \int \frac{d^2w_1}{x'_1^2} \frac{d^2w_1'}{x'_2^2}$$

$$\times \int d^2w \int \frac{d^2x'_1}{2\pi x'_1^2} \int d^4b_1 \left( \nu^2 + \frac{n^2}{4} \right) \left( \frac{1}{\nu^2 + \frac{n}{2}} - \frac{1}{\nu^2 + \frac{n+1}{2}} \right) \exp \left( \frac{2\alpha_s N_c}{\pi} (\chi(n_1, \nu_1)\tilde{Y}_1 + \chi(n, \nu)\tilde{Y}_2) \right)$$

$$\times E^{n_1,\nu_1} \left( b_1 + \frac{x'_1}{2} - w_1, b_1 + \frac{x'_1}{2} - w_1 \right) E^{n_1,\nu_1*} \left( \frac{x'_1}{2} - w + \frac{x}{2} - w + b \right)$$

$$\times E^{n,\nu} \left( b_1 + \frac{x'}{2} - w_1, b_1 + \frac{x'}{2} - w_1 \right) E^{n,\nu*} \left( \frac{x}{2} - w + b, \frac{x}{2} - w + b \right) \quad (A.41)$$

Integrating over $d^2x'_1 d^2b_1$ and $d^2w$ would give the result (2.54). It would show in particular that in the previous formula $n_1$ can be replaced by $n$ and $\nu_1$ by $\nu$. Since $\tilde{Y} = \tilde{Y}_1 + \tilde{Y}_2$, the previous formula can be written as

$$F^{(1)}(x_1, x_2, \tilde{Y}, b) = -\frac{1}{2} \int d^2b_1 \int \frac{d^2x'_1}{2\pi x'_1^2} n(x_1, x'_1, \tilde{Y}, b_1) \sigma_{DD}(x'_1, x_2, b_1 - b), \quad (A.42)$$

which is exactly what one would obtain when computing this process in the boosted frame where the left-moving onium is close to be at rest. Indeed, let us consider the dipole distribution inside the left-moving onium in its laboratory frame, namely $n(x_2, x'_2, \tilde{Y} = 0, b_2)$. Using expression (2.19), it reads, using the completeness condition for the functions $E^{n,\nu}$ (see equation (25) of
Calculating the scattering amplitude $F$ was computed considering the interaction of two elementary dipoles of transverse size $x$. In this limit it is important to consider the exact expression (A.23) of the elementary dipole pairs of transverse size $x'$ and compare the result to the exact formula (2.59). Let us consider two heavy quark-antiquark only the contributions $n$ only contribute through the quantum number $n$, which is exact only when calculating the approximation of integrating the elementary dipole-dipole cross section over impact parameter. However, we will see below that the distribution of dipoles is automatically implied (see Eqs. (A.33-A.34)).

We thus calculate first the cross-section at fixed impact parameter in these approximations and compare the result to the exact formula (2.59). Let us consider two heavy quark-antiquark pairs of transverse size $x_1'$ and $x_2'$, which scatter through the exchange of a pair of gluons between two elementary dipoles, respectively of transverse size $x_1'$ and $x_2'$, and at a distance.
from the center of the quark-antiquark pair \( b_1 \) and \( b_2 \). In this approximation, the corresponding scattering amplitude \( F^{(1)} \) given in Eq. \( (2.7) \) simplifies in

\[
F^{(1)}(\vec{x}_1, \vec{x}_2, \vec{Y}, b) = -\frac{1}{2} \int_0^{\infty} \frac{dx'_1}{x'_1} \frac{dx'_2}{x'_2} \frac{d^2b}{x'_1 x'_2} n(\vec{x}_1, \vec{x}_1', \vec{x}_2, \vec{Y}_1) n(\vec{x}_2, \vec{x}_2', \vec{b} - \vec{b}_1, \vec{Y}_2) \sigma_{DD}(x'_1, x'_2). \tag{A.45}
\]

The rapidity \( \vec{Y}_1 \) and \( \vec{Y}_2 \) are such that \( \vec{Y} = \vec{Y}_1 + \vec{Y}_2 \), \( F^{(1)} \) involves the elementary dipole-dipole total cross-section, which has been evaluated in [13], and which is calculated in appendix A.1 using eikonal techniques. For two dipoles of transverse size \( x'_1 \) and \( x'_2 \), the elementary forward dipole-dipole cross-section reads (see Eq. \( (A.11) \))

\[
\sigma_{DD}(x'_1, x'_2) = 2\pi \alpha_s^2 x'_2 \left[ 1 + \ln \frac{x'_2}{x'_1} \right], \tag{A.46}
\]

where \( x_1 = \text{Min}(x'_1, x'_2) \) et \( x_2 = \text{Max}(x'_1, x'_2) \).

We write \( F^{(1)} \) given by Eq. \( (A.45) \) as

\[
F^{(1)}(x_1, x_2, b, \vec{Y}) = \int_{-\infty}^{\infty} \frac{dv}{2\pi} F^{(1)}_v(x_1, x_2, b). \tag{A.47}
\]

Using the expression \( (A.14) \) of the elementary dipole-dipole cross section \( \sigma_{DD} \), \( F^{(1)}_v \) then reads

\[
F^{(1)}_v = -\frac{\pi \alpha_s^2}{2} \frac{1}{(\nu^2 + \frac{1}{4})^2} \int_{-\infty}^{\infty} \frac{dv_1}{2\pi} \int_{-\infty}^{\infty} \frac{dv_2}{2\pi} \exp \left( \frac{2\alpha_s N_c}{\pi} (\chi(0, \nu_1) \vec{Y}_1 + \chi(0, \nu_2) \vec{Y}_2) \right) \times \int \frac{d^2x'_1}{2\pi x'_1} \int \frac{d^2x'_2}{2\pi x'_2} \int d^2b_1 \int d^2b_2 \delta^2(b - b_1 - b_2) n_{\nu_1}(x_1, x'_1, b_1) n_{\nu_2}(x_2, x'_2, b_2) (x'_1 + 2i\nu)(x'_2)^{-2i\nu}. \tag{A.48}
\]

Using the representation

\[
\delta^2(b - b_1 - b_2) = \int \frac{d^2q}{(2\pi)^2} \exp(-iq.(b - b_1 - b_2)), \tag{A.49}
\]

and the expression \( (2.23) \) of \( n(\vec{x}, x', \vec{Y}, b) \), one gets

\[
F^{(1)}_v = -\frac{\pi \alpha_s^2}{2} \frac{1}{(\nu^2 + \frac{1}{4})^2} \int \frac{d^2q}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{dv_1}{2\pi} \int_{-\infty}^{\infty} \frac{dv_2}{2\pi} \exp \left( \frac{2\alpha_s N_c}{\pi} (\chi(0, \nu_1) \vec{Y}_1 + \chi(0, \nu_2) \vec{Y}_2) \right) \times \int \frac{d^2x'_1}{2\pi x'_1} \int \frac{d^2x'_2}{2\pi x'_2} \int d^2b_1 \int d^2b_2 \delta^2(b - b_1 - b_2) n_{\nu_1}(x_1, x'_1, b_1) n_{\nu_2}(x_2, x'_2, b_2) (x'_1 + 2i\nu)(x'_2)^{-2i\nu}. \tag{A.50}
\]

Let us compute the integrations with respect to \( x'_i \). Defining

\[
f_{\nu}(\nu, q) = \int \frac{d^2x'_i}{2\pi} E_q^{\nu}(x'_i)^{-2+2i\nu}, \tag{A.51}
\]

where \( \epsilon_1 = 1 \) and \( \epsilon_2 = -1 \), this function reads, taking into account the representation \( (2.23) \) and introducing a dimensional regularization \( d = 2+2\epsilon \) because of the divergency when \( x'_i \to 0 \),

\[
f_{\nu}(\nu, q) = \frac{4\pi^3}{b_{0\nu}^{\nu}} \left( \frac{q}{2} \right)^{2i\nu} \frac{1}{\Gamma^2(\frac{1}{2} + i\nu)} \int_0^1 d\alpha_i (\alpha_i(1 - \alpha_i))^{-\frac{1}{2}} \times \frac{\pi^\epsilon}{\Gamma(1+\epsilon)} \int_0^{+\infty} dx'_i (x'_i)^{-1+2i\nu+2\epsilon} K_{-2i\nu}(q x'_i \sqrt{\alpha_i(1 - \alpha_i)})) J_0 \left( \frac{x'_i}{2}(1 - 2\alpha_i) \right), \tag{A.52}
\]

28
where we have performed the integration with respect to the angle \((q, x')\). One can now perform the integration with respect to \(x_i\), using the formula (6.576) of Ref. 33:

\[
\int_{0}^{\infty} x^{-\lambda} K_\rho(ax)J_\sigma(bx) \, dx = \frac{1}{4} \left( \frac{b}{a} \right)^\sigma \left( \frac{a}{2} \right)^{\lambda-1} \Gamma \left( \frac{\sigma-\lambda+1+\rho}{2} \right) \Gamma \left( \frac{\sigma-\lambda+1-\rho}{2} \right) \Gamma(\sigma+1) \times \frac{1}{2} F_1 \left( \frac{\sigma - \lambda + 1 + \rho}{2}, \frac{\sigma - \lambda + 1 - \rho}{2}; \sigma + 1; -\frac{b^2}{a^2} \right). \tag{A.53}
\]

Here \(\sigma = 0, \rho = -2i\nu, b = \frac{q}{2}(1 - 2\alpha_i), a = q\sqrt{\alpha_i(1 - \alpha_i)}\) and \(-\lambda = 2i\epsilon_i\nu - 1 + 2\epsilon\). Thus,

\[
f_{\nu_i}(\nu, q) = \frac{4\pi^3}{b_{0,\nu_i}} \frac{1}{\Gamma^2(\frac{1}{2} + i\nu_i)} \frac{\pi^\epsilon}{\Gamma(1 + \epsilon)} \frac{1}{4} \int_{0}^{1} d\alpha_i(\alpha_i(1 - \alpha_i)) \times \Gamma(i\epsilon_i\nu - i\nu_i + \epsilon, i\epsilon_i\nu + i\nu_i + \epsilon) 2F_1 \left( i\epsilon_i\nu - i\nu_i + \epsilon, i\epsilon_i\nu + i\nu_i + \epsilon; 1; -\frac{1}{4} \frac{(1 - 2\alpha_i)^2}{\alpha_i(1 - \alpha_i)} \right). \tag{A.54}
\]

One can now perform the integration with respect to \(\nu_i\). We thus define

\[
g^i(\nu, x, q) = \int_{-\infty}^{\infty} \frac{d\nu_i}{2\pi} E_q^{0,\nu_i*}(x_i) f_{\nu_i}(\nu, q) \exp \left( \frac{2\alpha_s N_c}{\pi} \chi(0, \nu_i) \tilde{Y}_i \right). \tag{A.55}
\]

The analytic structure of the integrand is very simple: it has poles at \(\nu_i = \pm(\epsilon_i\nu - i\epsilon)\). The integration contour can be closed either in the upper plane or in the lower plane, and give the same contribution. For \(g^1\) we close the contour in the lower plane so as to pick up the residue at \(\nu_1 = \nu\), and for \(g^2\) we close the contour in the lower plane so as to pick up the residue at \(\nu_2 = -\nu\).

Thus,

\[
2F_1 \left( 0, 2i\nu; 1; -\frac{1}{4} \frac{(1 - 2\alpha_1)^2}{\alpha_1(1 - \alpha_1)} \right) = 2F_1 \left( -2i\nu, 0; 1; -\frac{1}{4} \frac{(1 - 2\alpha_2)^2}{\alpha_2(1 - \alpha_2)} \right) = 1, \tag{A.56}
\]

one gets

\[
g^i(\nu, x, q) = \frac{\pi^3}{b_{0,\nu_i}} \frac{\Gamma(2i\epsilon_i\nu)}{\Gamma^2(\frac{1}{2} + i\epsilon_i\nu)} \int_{0}^{1} d\alpha_i(\alpha_i(1 - \alpha_i)) \times \frac{1}{2} \Gamma(-i\epsilon_i\nu + \frac{1}{2}) \frac{1}{\Gamma(-2i\epsilon_i\nu + 1)} E_q^{0,\nu_i*}(x_i) \exp \left( \frac{2\alpha_s N_c}{\pi} \chi(0, \nu) \tilde{Y}_i \right), \tag{A.57}
\]

since \(\chi(0, \nu)\) is an even function of \(\nu\). Performing the integration with respect to \(\alpha_i\)

\[
\int_{0}^{1} d\alpha_i(\alpha_i(1 - \alpha_i)) \times \frac{1}{2} \Gamma(-i\epsilon_i\nu + \frac{1}{2}) = \frac{\Gamma^2(-i\epsilon_i\nu + \frac{1}{2})}{\Gamma(-2i\epsilon_i\nu + 1)}, \tag{A.58}
\]

\(F^{(1)}_\nu\) now reads

\[
F^{(1)}_\nu = -\frac{\pi\alpha_s^2}{2} \frac{x_1 x_2}{(\nu^2 + \frac{1}{4})^2} \int \frac{d^2 q}{(2\pi)^2} e^{-i\epsilon q \cdot b} g^1(\nu, x_1, q) g^2(\nu, x_2, q)
\]

\[
= -\frac{\pi\alpha_s^2}{2} \frac{x_1 x_2}{(\nu^2 + \frac{1}{4})^2} \int \frac{d^2 q}{(2\pi)^2} e^{-i\epsilon q \cdot b} \frac{\pi^6}{b_{0,\nu} b_{0,-\nu}} \frac{\Gamma(2i\nu)}{\Gamma(1 + 2i\nu)} \frac{\Gamma(-2i\nu)}{\Gamma(1 - 2i\nu)}
\]

\[
\times E_q^{0,\nu*}(x_1) E_q^{0,\nu*}(x_2) \exp \left( \frac{2\alpha_s N_c}{\pi} \chi(0, \nu) \tilde{Y} \right), \tag{A.59}
\]

29
when taking into account that \( \tilde{Y} = \tilde{Y}_1 + \tilde{Y}_2 \). Note that formula (A.59) could be equivalently obtained from Eq. (A.54) by performing the change of variable \( \alpha_i \rightarrow z_i = (1 - 2\alpha_i)^2 \) and using the Watson theorem [43]. Taking the conjugate of formula (2.29) and then performing the change of variable \( \alpha = \alpha' - 1 \), one can verify that
\[
E_q^{0,\nu}(\rho) = E_{-q}^{0,\nu}(\rho) = E_q^{0,-\nu}(\rho). \tag{A.60}
\]
Using formula (2.38), one finally gets
\[
F_{q}^{(1)} = -\frac{\pi \alpha_s^2}{8} \frac{x_1 x_2}{(\nu^2 + \frac{1}{4})^2} \int \frac{d^2 q}{(2\pi)^2} e^{-i q \cdot b} E_q^{0,\nu}(x_1) E_q^{0,\nu}(x_2) \exp \left( \frac{2\alpha_s N_c}{\pi} \chi(0, \nu) \tilde{Y} \right), \tag{A.61}
\]
that is
\[
F_{q}^{(1)} = -\frac{\pi \alpha_s^2}{8(\nu^2 + \frac{1}{4})^2} x_2^2 n_\nu(x_1, x_2, b, \tilde{Y}) = -\frac{\pi \alpha_s^2}{8(\nu^2 + \frac{1}{4})^2} x_1^2 n_\nu(x_2, x_1, b, \tilde{Y}). \tag{A.62}
\]
which differs from the exact result (2.59) by a factor \( \frac{1}{2} \). Thus, the approximation of using the elementary dipole-dipole total cross-section only gives half the correct result.

One expects that the previous approximations should be correct when calculating the total cross-section. Indeed, integrating (A.43) with respect to \( b \), one gets
\[
F^{(1)}(x_1, x_2, \tilde{Y}) = -\frac{1}{2} \int_{0}^{\infty} \frac{dx_1'}{x_1'} \frac{dx_2'}{x_2'} n(x_1', x_2', \tilde{Y}_1) n(x_2', x_1', \tilde{Y}_2) \sigma_{DD}(x_1', x_2'). \tag{A.63}
\]
From formula (2.65) and (A.14), this yields
\[
F^{(1)}(x_1, x_2, \tilde{Y}) = -\frac{1}{2} \int_{-\infty}^{+\infty} \frac{dv_1}{2\pi} \sum_{n_1=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dv_2}{2\pi} \sum_{n_2=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dv_2}{2\pi} \int_{x_2}^{+\infty} \frac{d^2 q_1}{2 \pi x_1^2} \frac{d^2 q_2}{2 \pi x_2^2} x_1^2 x_2 \left( \frac{x_1 x_2}{x_1 x_2'} \right)^{n_1/2} \left( 1 \right) \left( \frac{x_2}{x_2'} \right)^{1+2iv_2} \left( \frac{x_1}{x_1'} \right)^{1-2iv_1}.
\]
Integrating with respect to \( x_1' \) and \( x_2' \) and using relation (2.67), the only remaining terms are \( n_1 = n_2 = 0 \), due to the conservation of conformal weight, and one gets
\[
F^{(1)}(x_1, x_2, \tilde{Y}) = \frac{\pi \alpha_s^2}{2} \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \frac{1}{(\nu^2 + \frac{1}{4})^2} x_1 \left| x_1 \right|^{1-2iv} \left| x_2 \right|^{1+2iv} \exp \left( \frac{2\alpha_s N_c}{\pi} (\chi(n_1, \nu_1) \tilde{Y}_1 + \chi(n_2, \nu_2) \tilde{Y}_2) \right), \tag{A.65}
\]
that is
\[
F^{(1)}_{0,\nu}(x_1, x_2) = -\frac{\pi \alpha_s^2 x_2}{4} \frac{1}{(\nu^2 + \frac{1}{4})^2} n_{0,\nu}(x_1, x_2), \tag{A.66}
\]
which is exactly the result (2.71).

### A.6 Expression of \( E_q^{0,\nu} \) in terms of Bessel functions

In this appendix we prove the expression (2.30) of \( E_q^{0,\nu} \). Starting from equation (2.29) and making the change of variable \( \alpha = \sin^2 \frac{t}{2} \), the mixed function \( E_q^{0,\nu} \) reads
\[
E_q^{0,\nu}(\rho) = \frac{4\pi^3}{b_{0,\nu}} \left( \frac{q_0}{2} \right)^{2i\nu} \frac{1}{\Gamma^2 \left( \frac{1}{2} + i\nu \right)} \int_{0}^{\pi} \frac{e^{i\rho \cos t}}{2} K_{-2i\nu} \left( \frac{q_0}{2} \sin t \right) dt. \tag{A.67}
\]
Let us calculate
\[ C = \int_0^{\frac{\pi}{2}} e^{\frac{i q \rho}{2} \cos t} K_{-2i\nu} \left( \frac{q \rho}{2} \sin t \right) dt + \int_{\frac{\pi}{2}}^\pi e^{\frac{i q \rho}{2} \cos t} K_{-2i\nu} \left( \frac{q \rho}{2} \sin t \right) dt. \]  
(A.68)

Making the change of variable \( t' = \pi - t \) in the second integral, one gets
\[ C = 2 \int_0^{\frac{\pi}{2}} \cos \left( \frac{q \rho}{2} \cos t \right) K_{-2i\nu} \left( \frac{q \rho}{2} \sin t \right) dt. \]  
(A.69)

Combining
\[ K_\mu(\lambda) = \frac{\pi}{2 \sin(\pi \mu)} [I_{-\mu}(\lambda) - I_\mu(\lambda)] \]  
(A.70)

and
\[ I_\mu(\lambda) = e^{-i \frac{\pi}{2} \mu} J_\mu(e^{i \frac{\pi}{2} \lambda}), \]  
(A.71)

the Bessel function \( K \) can be written as
\[ K_\mu(\lambda) = \frac{1}{2} \Gamma(\mu) \Gamma(1 - \mu) \left[ e^{i \frac{\pi}{2} \mu} J_{-\mu}(e^{i \frac{\pi}{2} \lambda}) - e^{-i \frac{\pi}{2} \mu} J_\mu(e^{i \frac{\pi}{2} \lambda}) \right]. \]  
(A.72)

\( C \) can then be computed using relation (6.688) of Ref. [33]
\[ \int_0^{\frac{\pi}{2}} J_\mu(z \sin t) \cos(x \cos t) dt = \frac{\pi}{2} J_\mu(y_+) J_\mu(y_-), \]  
(A.73)

where \( y_{\pm} = \frac{\sqrt{x^2 + z^2} \pm x}{2} \).

In our case \( \mu = -2i\nu, x = \frac{q \rho}{2} \) and \( z = e^{i \frac{\pi}{2} \frac{q \rho}{2}} \). Defining \( \Psi \) the angle \((q, \rho)\), and computing \( y_{\pm} \),
\[ y_{\pm} = \pm \frac{q \rho}{4} e^{\pm i \Psi}, \]  
(A.74)

one gets
\[ C = \frac{\pi}{2} \Gamma(-2i\nu) \Gamma(1 + 2i\nu) \left[ J_{i\nu} \left( \frac{q \rho}{4} e^{i \Psi} \right) J_{i\nu} \left( \frac{q \rho}{4} e^{-i \Psi} \right) - J_{-i\nu} \left( \frac{q \rho}{4} e^{i \Psi} \right) J_{-i\nu} \left( \frac{q \rho}{4} e^{-i \Psi} \right) \right], \]  
(A.75)

where we have used the relation
\[ J_\mu \left( e^{i \pi \frac{z}{2}} \right) = e^{i \pi \mu} J_\mu(z) \]  
(A.76)
in order to get rid of the minus sign arising from the \( y_- \) contribution. Combining expressions \((A.67)\) and \((A.73)\), and using the expression (2.22) for \( b_{n,\nu} \), one finally gets
\[ E_{q,\nu}^0(\rho) = \left( \frac{q}{2} \right)^{2i\nu} 2^{-2i\nu} \Gamma^2(1-i\nu) \left[ J_{i\nu} \left( \frac{q \rho}{4} e^{i \Psi} \right) J_{i\nu} \left( \frac{q \rho}{4} e^{-i \Psi} \right) - J_{-i\nu} \left( \frac{q \rho}{4} e^{i \Psi} \right) J_{-i\nu} \left( \frac{q \rho}{4} e^{-i \Psi} \right) \right], \]  
(A.77)
in agreement with the more general result (10) of Ref. [32].
A.7 Properties of the three-points correlation functions $E^{n,\nu}$ and $E_q^{n,\nu}$

In this appendix we derive various useful mathematical formulae for the functions $E^{n,\nu}$ and $E_q^{n,\nu}$. Let us first show that $E^{n,\nu}$ and $E^{-n,-\nu}$ are related by the following expression

$$E^{n,\nu}(\rho_{10}, \rho_{20}) = \frac{b_{n,\nu}^*}{a_{n,\nu}} \int d^2\rho_0' E^{n,\nu}(\rho_{10}', \rho_{20}') |\rho_{00}'|^{-2+4i\nu} \left(\frac{\rho_{00}^*}{\rho_{00}'}\right)^n (-1)^n,$$  \hspace{1cm} (A.78)

which corrects formula (A.12) of [29]. Consider

$$T = \int d^2\rho_0' E^{n,\nu}(\rho_{10}', \rho_{20}') |\rho_{00}'|^{-2+4i\nu} \left(\frac{\rho_{00}^*}{\rho_{00}'}\right)^n.$$  \hspace{1cm} (A.79)

Using conformal invariance, we can take $\rho_2 \to \infty$ in order to simplify the calculation and restore the $\rho_2$ dependence afterwards by requiring the correct conformal transformation property. $T$ then reads

$$T = \int d^2\rho_0' \left(\frac{1}{\rho_0^*}\right)^n |\rho_{10}'|^n |\rho_{00}'|^{-2+4i\nu} \left(\frac{\rho_{00}^*}{\rho_{00}'}\right)^n \left|\frac{1}{\rho_{10}^*}\right|^{1+2i\nu}$$

$$= \int d^2R' \left(\frac{R' + \frac{\rho}{2}}{R^* + \frac{\rho}{2}}\right)^n |R'| - R|^{-2+4i\nu} \left(\frac{R^* - R}{R' - R}\right)^n \left|\frac{1}{R' + \frac{\rho}{2}}\right|^{1+2i\nu},$$  \hspace{1cm} (A.80)

where $\rho_{10} = R + \frac{\rho}{2}$, $\rho_{20} = R - \frac{\rho}{2}$, $\rho_{10}' = R' + \frac{\rho}{2}$ and $\rho_{20}' = R' - \frac{\rho}{2}$. After performing the change of variable $R' = R - R$ and introducing $R' = (R + \frac{\rho}{2})z$, $T$ reads

$$T = \left(\frac{R^* + \frac{\rho}{2}}{R^* + \frac{\rho}{2}}\right)^n |R + \frac{\rho}{2}|^{-1+2i\nu} K,$$  \hspace{1cm} (A.81)

where

$$K = \int d^2z \left(\frac{z + 1}{z^* + 1}\right)^n |z|^{-2+4i\nu} \left(\frac{z^*}{z}\right)^n \left|\frac{1}{z + 1}\right|^{1+2i\nu}.$$  \hspace{1cm} (A.82)

Using the techniques developed in Ref. [44], this integral can be computed after performing a Wick rotation for the $y$ integration. The corresponding replacement $y \to iye^{-2i\nu}$ reads $z + z^* = \alpha + \beta$ and $z - z^* = (\alpha - \beta)e^{-2i\nu}$ with $d^2z = dx dy = \frac{1}{2}dz dz^* = \frac{1}{2}d\alpha d\beta$. Separating the integration in $\alpha$ in three domains, the non-zero contribution is obtained for $\alpha \in [-1, 0]$ due to the $i\epsilon$. Closing the integration contour for the $\beta$ integration around the singularity $\beta = 0$, this yields

$$K = \int d\alpha d\beta \frac{(\alpha + 1)^n}{[(\alpha + 1)(\beta + 1) + i\epsilon]^{n+1+2i\nu}} \beta^{2|n|}$$

$$= \int_0^1 d\alpha (1 - \alpha)^{-\frac{|n| - 1}{2} + i\nu} \alpha^{-|n| - 1 + 2i\nu} \sin(|n| + 1 - 2i\nu) \int_0^{+\infty} d\beta \frac{\beta^{-|n| - 1 + 2i\nu}}{(\beta + 1)^{\frac{n+2}{2} + i\nu}},$$  \hspace{1cm} (A.83)

where we have performed the change of variable $\alpha \to -\alpha$. These integrals lead to $\beta$ functions. After some straightforward calculations, one gets

$$K = \frac{\pi}{2} \frac{1}{-i\nu + \frac{|n|}{2}} 2^{4i\nu} \frac{\Gamma\left(-i\nu + \frac{|n|+1}{2}\right)}{\Gamma\left(i\nu + \frac{|n|+1}{2}\right)} \frac{\Gamma\left(i\nu + \frac{|n|}{2}\right)}{\Gamma\left(-i\nu + \frac{|n|}{2}\right)} (-1)^n = \frac{b_{n,\nu}}{a_{n,\nu}} (-1)^n = \frac{b_{n,\nu}^*}{a_{n,\nu}} (-1)^n.$$  \hspace{1cm} (A.84)
Combining formulae (A.78) and using

\[ E^{n,\nu}(\rho_{10}, \rho_{20}) = \left( \frac{R^* + \rho_{20}^*}{R + \rho_{20}} \right)^{\frac{3}{2}} \left[ R + \rho_{20} \right]^{-1+2i\nu}, \quad \text{(A.85)} \]

one finally gets formula (A.78) after restoring the correct dependence in \( \rho_2 \).

Note that from Eq. (A.78) one can obtain the corresponding relation between \( E_q^{n,\nu} \) and \( E_q^{n,\nu} \). Indeed, performing the Fourier transform of both sides, one gets

\[ E_q^{n,\nu}(\rho_1 - \rho_2) = \frac{2\pi^2}{b_{n,\nu}} \int \frac{d^2 \rho_{10} + \rho_{20}}{|\rho_1 - \rho_2|} e^{-i\frac{1}{2} (\rho_{10} + \rho_{20}) \cdot q} E_q^{n,\nu}(\rho_{10}, \rho_{20}) \]

\[ = \frac{b_{n,\nu}}{a_{n,\nu}} \frac{2\pi^2}{b_{n,\nu}} \int \frac{d^2 \rho_{10} + \rho_{20}}{|\rho_1 - \rho_2|} e^{-i\frac{1}{2} (\rho_{10} + \rho_{20}) \cdot q} E_q^{n,\nu}(\rho_{10}, \rho_{20}) \]

\[ \times \int d^2 \rho_{20} e^{i(|\rho_{10} - \rho_{20}|)^2 - 2 + 4i\nu} (\frac{\rho_{20}^*}{\rho_{20}}) \]

\[ = \frac{b_{n,\nu}}{a_{n,\nu}} E_q^{n,\nu}(\rho_1 - \rho_2) \]

\[ \frac{b_{n,\nu}}{2\pi^2} |q|^{-4i\nu} \left( \frac{q^*}{q} \right)^n e^{-i\delta(n,\nu)}, \quad \text{(A.86)} \]

where the last integral is computed performing the integration \( d^2 \rho_{20} = d^2 \rho_{20} \) with radial coordinates \( \rho_{20} = |\rho_{20}| \exp(i\phi) \) (see the result (A.11) of Ref. [29]). Using the definition (2.21),

\[ E_q^{n,\nu}(\rho) = \frac{2\pi^2}{b_{n,\nu}} \int \frac{d^2 R}{|R|} e^{-i\frac{q}{2} R} E_q^{n,\nu} \left( R + \frac{\rho}{2}, R - \frac{\rho}{2} \right) \]

\[ = \frac{2\pi^2}{b_{n,\nu}} \int \frac{d^2 R}{|R|} e^{i\frac{q}{2} R} E_q^{n,\nu} \left( R + \frac{\rho}{2}, R - \frac{\rho}{2} \right) = E_q^{n,\nu}(\rho), \quad \text{(A.87)} \]

where we have used the fact that

\[ E_q^{n,\nu}(\rho_1, \rho_2) = (-1)^n E_q^{n,\nu}(\rho_2, \rho_1) = E_q^{n,\nu}(-\rho_2, -\rho_1). \quad \text{(A.88)} \]

Thus, using formula (2.38), one finally gets from Eq. (A.86)

\[ E_q^{n,\nu}(\rho) = |q|^{-4i\nu} \left( \frac{q^*}{q} \right)^n e^{-i\delta(n,\nu)} E_q^{n,\nu}(\rho), \quad \text{(A.89)} \]

in agreement with formula (A.15) of Ref. [29].

Note that the function \( E_q^{n,\nu} \) also possesses the following property

\[ E_q^{n,\nu}(\rho) = \frac{2\pi^2}{b_{n,\nu}} \int \frac{d^2 R}{|R|} e^{-i\frac{q}{2} R} E_q^{n,\nu} \left( R + \frac{\rho}{2}, R - \frac{\rho}{2} \right) \]

\[ = \frac{2\pi^2}{b_{n,\nu}} \int \frac{d^2 R}{|R|} e^{i\frac{q}{2} R} E_q^{n,\nu} \left( -R + \frac{\rho}{2}, -R - \frac{\rho}{2} \right) = E_q^{n,-\nu}(\rho), \quad \text{(A.90)} \]

where we have used the relation

\[ E_q^{n,\nu}(\rho_1, \rho_2) = E_q^{n,-\nu}(-\rho_2, -\rho_1). \quad \text{(A.91)} \]
Let us now prove the following orthonormalisation property for the three-points correlation functions $E_q^{n,\nu}$

$$
\int \frac{d^2 x'_1}{2\pi x'_1^2} E_{q,n,\nu}(x'_1) E_{q,n,\nu^*}(x'_1) = \pi \delta_{n_1,n} \delta(\nu_1 - \nu) + \pi \delta_{n_1,-n} \delta(\nu_1 + \nu) \left( \frac{q}{q^*} \right)^{n_1} |q^2|^{2\nu_1} e^{i\delta(n_1,\nu_1)}.
$$

(A.92)

Using the definition (2.21), the left-handside reads

$$
\int \frac{d^2 x'_1}{2\pi x'_1^2} E_{q,n,\nu}(x'_1) E_{q,n,\nu^*}(x'_1) = \frac{2\pi^2}{2\pi x'_1^2 b_{n,\nu} b_{n,\nu^*}} \{(R + x'_1/2, R - x'_1/2) \} E_{q,n,\nu}(R + x'_1/2, R - x'_1/2). 
$$

(A.93)

Performing the changes of variables

$$
\rho_1 - \rho_0 = R + \frac{x'_1}{2}, \quad \rho_2 - \rho_0 = R - \frac{x'_1}{2}, \quad \rho_1 - \rho_0^* = R' + \frac{x'_1}{2}, \quad \rho_2 - \rho_0^* = R' - \frac{x'_1}{2},
$$

(A.94)

and using the equality

$$
d^2 R \, d^2 R' \, d^2 x'_1 = d^2 \rho_1 \, d^2 \rho_2 \, d^2 \rho_0^*,
$$

(A.95)

one can apply the orthonormalization condition (2.48), which yields

$$
\int \frac{d^2 x'_1}{2\pi x'_1^2} E_{q,n,\nu}(x'_1) E_{q,n,\nu^*}(x'_1) = \frac{2\pi^2}{2\pi x'_1^2} \frac{1}{b_{n,\nu} b_{n,\nu^*}} \left[ a_{n,\nu^*} \delta_{n,1,n} \delta(\nu_1 - \nu) \int d^2 \rho_0^* e^{i q \rho_0^*} \delta^2(\rho_0) \right.

+ b_{n,\nu} \delta_{n,-1,n} \delta(\nu + \nu_1)(-1)^n \int d^2 \rho_{0^*} \left| \rho_{0^*} \right|^{-2-4i\nu} \left( \frac{\rho_{0^*}}{\rho_{0^*}^*} \right)^{n_1} e^{i q \rho_{0^*}} \right].
$$

(A.96)

Using the result (A.11) of Ref. [29] for the last integral, one finally gets the expected result (A.92).

### A.8 Calculation of $F^{(1)}$ in the mixed representation

In this appendix we show how to get the onium-onium scattering amplitude at fixed impact parameter using the mixed representation for the dipole distribution (see Eq. (2.23)) and for the elementary dipole-dipole cross-section (see Eq. (A.30)). Using the Fourier representation of the $\delta$ distribution, formula (2.7) reads

$$
F^{(1)}(x_1, x_2, \bar{Y}, b) = -\frac{1}{2} \int \frac{d^2 x'_1}{2\pi x'_1^2} \frac{d^2 x'_2}{2\pi x'_2^2} e^{i q \cdot h} \int d^2 \rho_{1^*} d^2 \rho_2 d^2 \rho_{0^*} \int \frac{d^2 q}{(2\pi)^2} e^{i q \cdot (h - b - b'_1 + b'_2 - b)}

\times n(x_1, x'_1, b, \bar{Y}_1) n(x_2, x'_2, b_2, \bar{Y}_2) \sigma_{DD}(x'_1, x'_2, b'_1 - b'_2).
$$

(A.97)

Combining this expression with formulae (2.23) and (A.30), one gets

$$
F^{(1)}(x_1, x_2, \bar{Y}, b) = -\frac{1}{2} \int \frac{d^2 q}{(2\pi)^2} e^{i q \cdot h} \int \frac{d^2 x'_1}{2\pi x'_1^2} \frac{d^2 x'_2}{2\pi x'_2^2} \sum_{n_1 = -\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{d \nu_1}{2 \pi} x_1 \left( E_{q,n_1,\nu_1}(x_1) E_{q,n_1,\nu_1^*}(x_1) + \delta_{n_1,1} \right)

\times \sum_{n_2 = -\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{d \nu_2}{2 \pi} x_2 \left( E_{q,n_2,\nu_2}(x_2) E_{q,n_2,\nu_2^*}(x_2) + \delta_{n_2,1} \right) \left( \frac{1 + (-1)^n}{\nu^2 + (n-1/2)^2} \right) \left( \nu^2 + (n+1/2)^2 \right)

\times E_{q,n,\nu^*}(x'_1) E_{q,n,\nu}(x'_2) \exp \left( \frac{2\alpha_s^2 N_c}{\pi} (\chi(n_1, \nu_1) \bar{Y}_1 + \chi(n, \nu) \bar{Y}_2) \right)
$$

(A.98)
From the property (A.90), one has $E_{q}^{n_2, \nu_2} = E_{q}^{-n_2, -\nu_2*}$. We can now apply the orthonormalization condition (A.92), which yields

$$F^{(1)}(x_1, x_2, \tilde{Y}, b) = -\frac{\pi \alpha_s^2 x_1 x_2}{8} \int \frac{d^2 q}{(2\pi)^2} e^{-i q \cdot b} \sum_{n = -\infty}^{+\infty} \int_{-\infty}^{+\infty} d\nu E_{q}^{n_2, \nu_2}(x_1) E_{q}^{n, \nu}(x_2)$$

$$\times \frac{1 + (-1)^n}{(\nu^2 + \left(\frac{n-1}{2}\right)^2)(\nu^2 + \left(\frac{n+1}{2}\right)^2)} \exp\left(\frac{2\alpha_s N_c}{\pi} (\chi(n_1, \nu_1) \tilde{Y}_1 + \chi(n, \nu) \tilde{Y}_2)\right). \quad (A.99)$$

From the expression of the dipole density (2.23), this finally reads

$$F^{(1)}_{\{n, \nu\}}(\bar{x}_1, \bar{x}_2; \bar{b}) = -\frac{\pi \alpha_s^2 x_1^2}{8} \frac{1 + (-1)^n}{(\nu^2 + \left(\frac{n-1}{2}\right)^2)(\nu^2 + \left(\frac{n+1}{2}\right)^2)} n_{\{n, \nu\}}(\bar{x}_1, \bar{x}_2; \bar{b}). \quad (A.100)$$

which is identical to Eq. (2.58) as expected.
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