Stability of generalized mixed type additive-quadratic-cubic functional equation in non-Archimedean spaces

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Abstract. In this paper, we prove generalized Hyres–Ulam–Rassias stability of the mixed type additive, quadratic and cubic functional equation

$$f(x + ky) + f(x - ky) = k^2 f(x + y) + k^2 f(x - y) + 2(1 - k^2)f(x)$$

for fixed integers $k$ with $k \neq 0, \pm 1$ in non-Archimedean spaces.

1. Introduction

We say that a functional equation ($\xi$) is stable if any function $g$ satisfying the equation ($\xi$) approximately is near to true solution of ($\xi$). We say that a functional equation ($\xi$) is superstable if every approximately solution is an exact solution of the equation ($\xi$) \cite{18,23,22}. The stability problem of functional equations originated from a question of Ulam \cite{27} in 1940, concerning the stability of group homomorphisms. Let $(G_1,.)$ be a group and let $(G_2,\ast)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(x),h(y)) < \delta$ for all $x,y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x),H(x)) < \epsilon$ for all $x \in G_1$? In the other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, D. H. Hyers \cite{13} gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \longrightarrow E'$ be a mapping between Banach spaces such that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta$$

for all $x,y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

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for all \( x \in E \). Moreover if \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in E \), then \( T \) is linear.

In 1978, Th. M. Rassias [24] provided a generalization of Hyers’ Theorem which allows the Cauchy difference to be unbounded.

The functional equation

\[
 f(x + y) + f(x - y) = 2f(x) + 2f(y),
\]

is related to symmetric bi-additive function [1], [2], [16] and [19]. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function \( f \) between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function \( B \) such that \( f(x) = B(x, x) \) for all \( x \) (see [1], [19]). The bi-additive function \( B \) is given by

\[
 B(x, y) = \frac{1}{4}(f(x + y) - f(x - y)).
\]

A Hyers–Ulam–Rassias stability problem for the quadratic functional equation (1.1) was proved by Skof for functions \( f : A \to B \), where \( A \) is normed space and \( B \) Banach space (see [25]). Cholewa [3] noticed that the Theorem of Skof is still true if relevant domain is replaced an abelian group (see also [4] and [12]).

Jun and Kim [17] introduced the following cubic functional equation

\[
 f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x),
\]

and they established the general solution and the generalized Hyers–Ulam–Rassias stability for the functional equation (1.2). The \( f(x) = x^3 \) satisfies the functional equation (1.2), which is called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. Jun and Kim proved that a function \( f \) between real vector spaces \( X \) and \( Y \) is a solution of (1.2) if and only if there exits a unique symmetric bi-additive function \( C : X \times X \times X \to Y \) such that \( f(x) = C(x, x, x) \) for all \( x \in X \), and \( C \) is symmetric for each fixed one variable and is additive for fixed two variables.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [8], [9], [12], [14], [15], [21], [22], [23] and [26]).

By a non-Archiimedean field we mean a field \( K \) equipped with a function (valuation) \( |\cdot| \) from \( K \) into \([0, \infty)\) such that \(|r| = 0 \) if and only if \( r = 0 \), \(|rs| = |r||s|\), and \(|r+s| \leq \max\{|r|, |s|\}\) for all \( r, s \in K \). Clearly \(|1| = |−1| = 1 \) and \(|n| \leq 1\) for all \( n \in N \).

**Definition 1.1.** Let \( X \) be a vector space over a scalar field \( K \) with a non-Archimedean non-trivial valuation \(|\cdot|\). A function \( \|\cdot\| : X \to \mathbb{R} \) is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) \( \|x\| = 0 \) if and only if \( x = 0 \);
(ii) \( \|rx\| = |r|\|x\| \) for all \( r \in K, x \in X \);
(iii) the strong triangle inequality (ultrametric); namely,

\[
 \|x + y\| \leq \max\{\|x\|, \|y\|\} \quad (x, y \in X)
\]

Then \( (X, \|\cdot\|) \) is called a non-Archimedean space.

Due to the fact that

\[
 \|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n - 1\} \quad (n > m)
\]

a sequence \( \{x_n\} \) is Cauchy if and only if \( \{x_{n+1} - x_n\} \) converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence...
M. S. Moslehian and Th. M. Rassias [20] proved the generalized Hyers–Ulam stability of the Cauchy functional equation and the quadratic functional equation in non-Archimedean spaces. M. Eshaghi Gordji and M. Bavand Savadkouhi [5], have obtained the generalized Hyers–Ulam–Rassias stability for cubic and quartic functional equation in non-Archimedean spaces.

Recently, M. Eshaghi Gordji and H. Khodaei [6], investigated the solution and stability of the generalized mixed type cubic, quadratic and additive functional equation

\[ f(x + ky) + f(x - ky) = k^2 f(x + y) + k^2 f(x - y) + 2(1 - k^2)f(x) \]  

for fixed integers \( k \neq 0, \pm 1 \) in quasi–Banach spaces. We only mention here the papers [7, 10] and [11] concerning the stability of the mixed type functional equations. In this paper, we prove the stability of functional equation (1.3) in non-Archimedean space.

2. Stability

Throughout this section, we assume that \( G \) is an additive group and \( X \) is a complete non-Archimedean space. Given \( f : G \to X \), we define the difference operator

\[ Df(x, y) = f(x + ky) + f(x - ky) - k^2 f(x + y) - k^2 f(x - y) - 2(1 - k^2)f(x) \]

for fixed integers \( k \neq 0, \pm 1 \) and for all \( x, y \in G \). We consider the following function inequality:

\[ \|Df(x, y)\| \leq \phi(x, y) \]

for an upper bound: \( \phi : G \times G \to [0, \infty) \).

**Theorem 2.1.** Let \( \phi : G \times G \to [0, \infty) \) be a function such that

\[ \lim_{n \to \infty} \frac{\phi(k^n x, k^n y)}{|k|^{2n}} = 0 \]  

(2.1)

\[ \lim_{n \to \infty} \frac{1}{|2, k^{2n}|} \phi(0, k^{n-1} x) = 0 \]  

(2.2)

for all \( x, y \in G \) and let for each \( x \in G \) the limit

\[ \lim_{n \to \infty} \max \left\{ \frac{1}{|k^j|} \phi(0, k^j x) : 0 \leq j < n \right\}, \]  

(2.3)

denoted by \( \tilde{\phi}_Q(x) \), exist. Suppose that \( f : G \to X \) is an even function satisfying

\[ \|Df(x, y)\| \leq \phi(x, y) \]  

(2.4)

for all \( x, y \in G \). Then there exist a quadratic function \( Q : G \to X \) such that

\[ \|Q(x) - f(x)\| \leq \frac{1}{|2, k^2|} \tilde{\phi}_Q(x) \]  

(2.5)

for all \( x \in G \). Moreover, if

\[ \lim_{i \to \infty} \lim_{n \to \infty} \max \left\{ \frac{1}{|k^j|} \phi(0, k^j x) : i \leq j < n + i \right\} = 0, \]

then \( Q \) is the unique quadratic function satisfying (2.5).
Proof. By putting $x = 0$ in (2.4), we get
\[
\|2f(ky) - 2k^2f(y)\| \leq \varphi(0, y)
\]  
for all $y \in G$. If we replace $y$ in (2.6) by $x$, and divide both sides of (2.6) by $2k^2$, we get
\[
\left\| \frac{f(kx)}{k^2} - f(x) \right\| \leq \frac{1}{|2k^2|} \varphi(0, x)
\]  
for all $x \in G$. Replacing $x$ by $k^{n-1}x$ in (2.7), we get
\[
\left\| \frac{f(k^n x)}{k^{2n}} - \frac{f(k^{n-1}x)}{k^{2(n-1)}} \right\| \leq \frac{1}{|2, k^{2n}|} \varphi(0, k^{n-1}x)
\]  
for all $x \in G$. It follows from (2.2) and (2.8) that the sequence $\left\{ \frac{f(k^n x)}{k^{2n}} \right\}$ is Cauchy. Since $X$ is complete, we conclude that $\left\{ \frac{f(k^n x)}{k^{2n}} \right\}$ is convergent. Set
\[
Q(x) := \lim_{n \to \infty} \frac{f(k^n x)}{k^{2n}}.
\]  
Using induction one can show that
\[
\left\| \frac{f(k^n x)}{k^{2n}} - f(x) \right\| \leq \frac{1}{|2, k^{2n}|} \max \left\{ \frac{1}{|k^{2i}|} \varphi(0, k^i x) : 0 \leq i < n \right\}
\]  
for all $n \in \mathbb{N}$ and all $x \in G$. By taking $n$ to approach infinity in (2.9) and using (2.3) one obtains (2.5). By (2.1) and (2.4), we get
\[
\|DQ(x, y)\| = \lim_{n \to \infty} \frac{1}{|k^{2n}|} \|f(k^n x, k^n y)\| \leq \lim_{n \to \infty} \frac{\varphi(k^n x, k^n y)}{|k^{2n}|} = 0
\]  
for all $x, y \in G$. Therefore the function $Q : G \to X$ satisfies (1.3). If $Q'$ is another quadratic function satisfying (2.5), then
\[
\|Q(x) - Q'(x)\| = \lim_{i \to \infty} |k^{-2i}| \|Q(k^i x) - Q'(k^i x)\|
\leq \lim_{i \to \infty} |k^{-2i}| \max \left\{ \|Q(k^i x) - f(k^i x)\|, \|f(k^i x) - Q'(k^i x)\| \right\}
\leq \frac{1}{|2, k^{2i}|} \lim_{i \to \infty} \lim_{n \to \infty} \max \left\{ \frac{1}{|k^i|} \varphi(0, k^j x) : i \leq j < n + i \right\}
= 0.
\]  
for all $x \in G$. Therefore $Q = Q'$. This completes the proof of the uniqueness of $Q$. \hfill \square

Theorem 2.2. Let $\varphi : G \times G \to [0, \infty)$ be a function such that
\[
\lim_{n \to \infty} \frac{1}{|2^n|} \max \left\{ \varphi(2^{n+1} x, 2^{n+1} y), 8|\varphi(2^n x, 2^n y)\right\} = 0
\]  
\[
\lim_{n \to \infty} \frac{1}{|2^n k^{2(k^2 - 1)}|} \max \left\{ \max \left\{ \max \left\{ [2(k^2 - 1)] \varphi(2^{n+1} x, 2^{n+1} x), k^2 \varphi(2^n x, 2^{n-1} x) \right\}, \max \left\{ \varphi(2^{n-1} x, 2^n x), \max \left\{ \varphi(2^{n-1} (k + 1) x, 2^{n-1} x), \varphi(2^{n-1} (k - 1) x, 2^{n-1} x) \right\} \right\}, \max \left\{ \varphi(2^{n-1} x, 2^{n-1} x), \varphi(2^{n-1} x, 2^n x) \right\} \right\}, \max \left\{ \max \left\{ \max \left\{ [2(k^2 - 1)] \varphi(2^{n-1} x, 2^{n-1} x), 3 \varphi(2^{n-1} x, 3^{n-1} x) \right\}, \max \left\{ 3 \varphi(2^{n-1} x, 3^{n-1} x), \varphi(2^{n-1} x, 2^{n-1} x) \right\} \right\}, \max \left\{ \varphi(2^{n-1} x, 2^{n-1} x), \varphi(2^{n-1} (2k - 1) x, 2^{n-1} x) \right\} \right\} = 0
\]
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for all $x, y \in G$ and let for each $x \in G$ the limit

$$\max \{ \frac{1}{|2|^i} \max \max \{ \max \{2(k^2 - 1)|\varphi(2^{i-1}x, 2^{j-1}x)|, |k^2|\varphi(2^i x, 2^{j-1}x) \} \}
, \max \{\varphi(2^{j-1} x, 2^j x), \max \{\varphi(2^{j-1} k + 1 x, 2^{j-1} x), \varphi(2^{j-1} (k - 1) x, 2^{j-1} x) \} \} \}
, \max \{ \max \{2(k^2 - 1)|\varphi(2^{j-1} x, 2^j x), \varphi(2^{j-1} (k - 1) x, 2^{j-1} x) \} \}
, \max \{\varphi(2^{j-1} (2k + 1) x, 2^{j-1} x), \varphi(2^{j-1} (2k - 1) x, 2^{j-1} x) \} \} \} : 0 \leq i < n \},$$

denoted by $\tilde{\varphi}_x(x)$. Suppose that $f : G \to X$ is an odd function satisfying

$$\|Df(x, y)\| \leq \varphi(x, y)$$

for all $x, y \in G$. Then there exist an additive function $A : G \to X$ such that

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{1}{|2|^{k^2 - 1}} \tilde{\varphi}_x(x)$$

for all $x \in G$. Moreover, if

$$\lim_{i \to \infty} \lim_{n \to \infty} \max \frac{1}{|2|^i} \max \max \{ \max \{2(k^2 - 1)|\varphi(2^{i-1} x, 2^{j-1} x)|, |k^2|\varphi(2^i x, 2^{j-1}x) \} \}
, \max \{\varphi(2^{j-1} x, 2^j x), \max \{\varphi(2^{j-1} k + 1 x, 2^{j-1} x), \varphi(2^{j-1} (k - 1) x, 2^{j-1} x) \} \} \}
, \max \{ \max \{2(k^2 - 1)|\varphi(2^{j-1} x, 2^j x), \varphi(2^{j-1} (k - 1) x, 2^{j-1} x) \} \}
, \max \{\varphi(2^{j-1} (2k + 1) x, 2^{j-1} x), \varphi(2^{j-1} (2k - 1) x, 2^{j-1} x) \} \} \} : i \leq j < n + i \}
= 0,$$

then $A$ is the unique additive function satisfying (2.14).

Proof. It follows from (2.13) and using oddness of $f$ that

$$\|f(2y + x) - f(2y - x) - k^2 f(x + y) - k^2 f(x - y) + 2(k^2 - 1)f(x)\| \leq \varphi(x, y)$$

(2.15)

for all $x, y \in G$. Putting $y = x$ in (2.15), we have

$$\|f((k + 1)x) - f((k - 1)x) - k^2 f(2x) + 2(k^2 - 1)f(x)\| \leq \varphi(x, x)$$

(2.16)

for all $x \in G$. It follows from (2.16) that

$$\|f((2k + 1)x) - f(2(k - 1)x) - k^2 f(4x) + 2(k^2 - 1)f(2x)\| \leq \varphi(2x, 2x)$$

(2.17)

for all $x \in G$. Replacing $x$ and $y$ by $2x$ and $x$ in (2.15), respectively, we get

$$\|f((k + 2)x) - f((k - 2)x) - k^2 f(3x) - k^2 f(x) + 2(k^2 - 1)f(2x)\| \leq \varphi(2x, x)$$

(2.18)

for all $x \in G$. Setting $y = 2x$ in (2.15), gives

$$\|f((2k + 1)x) - f((2k - 1)x) - k^2 f(3x) - k^2 f(-x) + 2(k^2 - 1)f(x)\| \leq \varphi(x, 2x)$$

(2.19)

for all $x \in G$. Putting $y = 3x$ in (2.15), we obtain

$$\|f((3k + 1)x) - f((3k - 1)x) - k^2 f(4x) - k^2 f(-2x) + 2(k^2 - 1)f(3x)\| \leq \varphi(x, 3x)$$

(2.20)

for all $x \in G$. Replacing $x$ and $y$ by $(k + 1)x$ and $x$ in (2.15), respectively, we get

$$\|f((2k + 1)x) - f(-x) - k^2 f((k + 2)x) - k^2 f(kx) + 2(k^2 - 1)f((k + 1)x)\| \leq \varphi((k + 1)x, x)$$

(2.21)
for all \( x \in G \). Replacing \( x \) and \( y \) by \((k - 1)x\) and \( x \) in (2.15), respectively, one gets

\[
\|f((2k - 1)x) - f(x) - k^2 f((k - 2)x) - k^2 f(kx) + 2(k^2 - 1)f((k - 1)x)\| \\
\leq \varphi((k - 1)x, x)
\]  

(2.22)

for all \( x \in G \). Replacing \( x \) and \( y \) by \((2k + 1)x\) and \( x \) in (2.15), respectively, we obtain

\[
\|f((3k + 1)x) - f(-(k + 1)x) - k^2 f(2(k + 1)x) - k^2 f(2kx) + 2(k^2 - 1)f((2k + 1)x)\| \\
\leq \varphi((2k + 1)x, x)
\]  

(2.23)

for all \( x \in G \). Replacing \( x \) and \( y \) by \((k - 1)x\) and \( x \) in (2.15), respectively, we have

\[
\|f((3k - 1)x) - f(-(k - 1)x) - k^2 f(2(k - 1)x) - k^2 f(2kx) + 2(k^2 - 1)f((2k - 1)x)\| \\
\leq \varphi((2k - 1)x, x)
\]  

(2.24)

for all \( x \in G \). It follows from (2.16), (2.18), (2.19), (2.21) and (2.22) that

\[
\|f(3x) - 4f(2x) + 5f(x)\| \leq \frac{1}{k^2(k^2 - 1)} \max\{ \max\{2(k^2 - 1)\varphi(x, x), k^2|\varphi(2x, x)|\} \\
, \max\{\varphi(x, 2x), \max\{\varphi((k + 1)x, x), \varphi((k - 1)x, x)\}\} \}
\]  

(2.25)

for all \( x \in G \). And, from (2.16), (2.17), (2.19), (2.20), (2.23) and (2.24), we conclude that

\[
\|f(4x) - 2f(3x) - 2f(2x) + 6f(x)\| \leq \frac{1}{k^2(k^2 - 1)} \max\{ \max\{2(k^2 - 1)|\varphi(2x, 2x)|, k^2|\varphi(2x, 3x)|\} \\
, \max\{\varphi(x, 2x), \max\{\varphi((k + 1)x, x), \varphi((k - 1)x, x)\}\} \}
\]  

(2.26)

for all \( x \in G \). Finally, by using (2.25) and (2.26), we obtain that

\[
\|f(4x) - 10f(2x) + 16f(x)\| \leq \frac{1}{k^2(k^2 - 1)} \max\{ \max\{2(k^2 - 1)|\varphi(2x, 2x)|, k^2|\varphi(2x, 3x)|\} \\
, \max\{\varphi(x, 2x), \max\{\varphi((k + 1)x, x), \varphi((k - 1)x, x)\}\} \}
\]  

(2.27)

for all \( x \in G \). Let \( g : G \rightarrow X \) be a function defined by \( g(x) : = f(2x) - 8f(x) \) for all \( x \in G \). From (2.27), we conclude that

\[
\left\| \frac{g(2x)}{2} - g(x) \right\| \leq \frac{1}{2k^2(k^2 - 1)} \max\{ \max\{2(k^2 - 1)|\varphi(2x, 2x)|, k^2|\varphi(2x, 3x)|\} \\
, \max\{\varphi(x, 2x), \max\{\varphi((k + 1)x, x), \varphi((k - 1)x, x)\}\} \}
\]  

(2.28)
for all $x \in G$. Replacing $x$ by $2^{n-1}x$ in (2.28), we get

\[ \frac{|g(2^n x) - g(2^{n-1} x)|}{2^n} \leq \frac{1}{2^n k^2 (k^2 - 1)} \]

\[ \max \left\{ \max \{ |2(k-1)| \varphi(2^{n-1} x, 2^{n-1} x), |k|^2 \varphi(2^n x, 2^{n-1} x) \} \right\} \]
\[ \max \left\{ \varphi(2^{n-1} x, 2^n x), \varphi(2^{n-1} (k+1) x, 2^{n-1} x), \varphi(2^{n-1} (k-1) x, 2^{n-1} x) \} \right\} \]
\[ \max \left\{ \varphi(2^{n-1} x, 2^n x), \varphi(2^{n-1} x, 2^{n-1} x) \right\} \]
\[ \max \left\{ \varphi(2^{n-1} (2k+1) x, 2^{n-1} x), \varphi(2^{n-1} (2k-1) x, 2^{n-1} x) \} \right\} \]

(2.29)

for all $x \in G$. It follows from (2.11) and (2.29) that the sequence $\{ \frac{g(2^n x)}{2^n} \}$ is Cauchy. Since $X$ is complete, we conclude that $\{ \frac{g(2^n x)}{2^n} \}$ is convergent. Set $A(x) := \lim_{n \to \infty} \frac{g(2^n x)}{2^n}$.

Using induction one can show that

\[ \frac{|g(2^n x) - g(x)|}{2^n} \leq \frac{1}{2^n k^2 (k^2 - 1)} \max \frac{1}{2^n} \]

\[ \max \left\{ \max \{ |2(k-1)| \varphi(2^{i-1} x, 2^{i-1} x), |k|^2 \varphi(2^i x, 2^{i-1} x) \} \right\} \]
\[ \max \left\{ \varphi(2^{i-1} x, 2^i x), \max \{ \varphi(2^{i-1} (k+1) x, 2^{i-1} x), \varphi(2^{i-1} (k-1) x, 2^{i-1} x) \} \right\} \]
\[ \max \left\{ \varphi(2^{i-1} x, 2^i x), |k|^2 \varphi(2^i x, 2^i x) \right\} \]
\[ \max \left\{ \max \{ |2(k-1)| \varphi(2^{i-1} x, 2^i x), \varphi(2^{i-1} (2k+1) x, 2^{i-1} x) \} \right\} \]
\[ \max \left\{ \varphi(2^{i-1} (2k+1) x, 2^{i-1} x), \varphi(2^{i-1} (2k-1) x, 2^{i-1} x) \} \right\} \]
\[ : 0 \leq i < n \]

(2.30)

for all $n \in \mathbb{N}$ and all $x \in G$. By taking $n$ to approach infinity in (2.30) and using (2.12) one obtains (2.14). By (2.10) and (2.13), we get

\[ \|D A(x, y)\| = \lim_{n \to \infty} \frac{1}{2^n} \|g(2^n x, 2^n y)\| = \lim_{n \to \infty} \frac{1}{2^n} \|D(f(2^{n+1} x, 2^{n+1} y) - 8f(2^n x, 2^n y))\| \]
\[ \leq \lim_{n \to \infty} \frac{1}{2^n} \max \{\|D(f(2^{n+1} x, 2^{n+1} y))\|, \|8f(2^n x, 2^n y))\|\} \]
\[ \leq \lim_{n \to \infty} \frac{1}{2^n} \max \{\varphi(2^{n+1} x, 2^{n+1} y), 8|\varphi(2^n x, 2^n y)\| \}
\[ = 0 \]
for all \( x, y \in G \). Therefore the function \( A : G \to X \) satisfies (1.3). If \( A' \) is another additive function satisfying (2.14), then

\[
\|A(x) - A'(x)\| = \lim_{i \to \infty} |2^{-i}||A(2^i x) - A'(2^i x)|
\]

\[
\leq \lim_{i \to \infty} |2^{-i}| \max\{ \|A(2^i x) - g(2^i x)\|, \|g(2^i x) - A'(2^i x)\| \}
\]

\[
\leq \frac{1}{2k^2(k^2 - 1)} \lim_{i \to \infty} \lim_{n \to \infty} \max\{ \frac{1}{|2^i|} \}
\]

\[
\max\{ \max\{ \max\{ |2(k^2 - 1)\varphi(2^{i-1} x, 2^i x^2), |k^2|\varphi(2^i x, 2^{i-1} x) \} \}
\]

\[
, \max\{ \max\{ |\varphi(2^{i-1} x, 2^i x), |k^2|\varphi(2^i x, 2^{i-1} x) \} \}
\]

\[
, \max\{ \max\{ |\varphi(2^{i-1} x, 2^i x), |k^2|\varphi(2^i x, 2^{i-1} x) \} \}
\]

\[
, \max\{ \max\{ |\varphi(2^{i-1} x, 2^i x), |k^2|\varphi(2^i x, 2^{i-1} x) \} \}
\] : \( i \leq j < n + i \}

\[
= 0
\]

for all \( x \in G \). Therefore \( A = A' \). This completes the proof of the uniqueness of \( A \). \( \square \)

**Theorem 2.3.** Let \( \varphi : G \times G \to [0, \infty) \) be a function such that

\[
\lim_{n \to \infty} \frac{1}{|8^n|} \max\{ \varphi(2^{n+1} x, 2^{n+1} y), |2|\varphi(2^n x, 2^n y) \} = 0 \quad (2.31)
\]

\[
\lim_{n \to \infty} \frac{1}{|8^n k^2(k^2 - 1)|} \max\{ \max\{ \max\{ |2(k^2 - 1)|\varphi(2^{i-1} x, 2^i x), |k^2|\varphi(2^i x, 2^{i-1} x) \} \}
\]

\[
, \max\{ \max\{ |\varphi(2^{i-1} x, 2^i x), |k^2|\varphi(2^i x, 2^{i-1} x) \} \}
\]

\[
, \max\{ \max\{ |\varphi(2^{i-1} x, 2^i x), |k^2|\varphi(2^i x, 2^{i-1} x) \} \}
\]

\[
, \max\{ \max\{ |\varphi(2^{i-1} x, 2^i x), |k^2|\varphi(2^i x, 2^{i-1} x) \} \}
\] : \( 0 \leq i < n \}, \quad (2.32)

for all \( x, y \in G \) and let for each \( x \in G \) the limit

\[
\max\{ \frac{1}{|8^n|} \max\{ \max\{ \max\{ |2(k^2 - 1)|\varphi(2^{i-1} x, 2^i x), |k^2|\varphi(2^i x, 2^{i-1} x) \} \}
\]

\[
, \max\{ \max\{ |\varphi(2^{i-1} x, 2^i x), |k^2|\varphi(2^i x, 2^{i-1} x) \} \}
\]

\[
, \max\{ \max\{ |\varphi(2^{i-1} x, 2^i x), |k^2|\varphi(2^i x, 2^{i-1} x) \} \}
\]

\[
, \max\{ \max\{ |\varphi(2^{i-1} x, 2^i x), |k^2|\varphi(2^i x, 2^{i-1} x) \} \}
\] : \( 0 \leq i < n \}, \quad (2.33)

denoted by \( \tilde{\varphi}_C(x) \), exist. Suppose that \( f : G \to X \) is an odd function satisfying

\[
\|Df(x, y)\| \leq \varphi(x, y) \quad (2.34)
\]

for all \( x, y \in G \). Then there exist a cubic function \( C : G \to X \) such that

\[
\|f(2x) - 2f(x) - C(x)\| \leq \frac{1}{|8^n k^2(k^2 - 1)|} \tilde{\varphi}_C(x) \quad (2.35)
\]
for all \( x \in G \). Moreover, if

\[
\lim_{i \to \infty} \lim_{n \to \infty} \frac{1}{|S_i|} \max \{ \max \{ \max \{ |2(k-1)|\varphi(2^{n-1}x, 2^{n-1}x), |k|^2\varphi(2^{n}x, 2^{n-1}x) \} , \max \{ \varphi(2^{n}x, 2^jx), \max \{ \varphi(2^{n-1}(k+1)x, 2^{n-1}x), \varphi(2^{n-1}(k-1)x, 2^{n-1}x) \} \} \} , \max \{ \varphi(2^{n-1}x, 2^jx), \max \{ \varphi(2^{n-1}(k+1)x, 2^{n-1}x), \varphi(2^{n-1}(k-1)x, 2^{n-1}x) \} \} \} : i \leq j < n + i \\
= 0,
\]

then \( C \) is the unique cubic function satisfying (2.35).

**Proof.** Similar to the proof of Theorem 2.2, we have

\[
||f(4x) - 10f(2x) + 16f(x)|| \leq \frac{1}{|k|^2(k^2 - 1)} \max \{ \max \{ |2(k-1)|\varphi(x, x), |k|^2\varphi(2x, x) \} , \max \{ \varphi(x, 2x), \max \{ \varphi((k+1)x, x), \varphi((k-1)x, x) \} \} , \max \{ \varphi(x, x), |k|^2\varphi(2x, 2x) \} , \max \{ |2(k-1)|\varphi(x, 2x), \varphi(x, 3x) \} , \max \{ |2x(2k+1)x, x), \varphi((2k-1)x, x) \} \} \} \}
\]

for all \( x \in G \). Let \( h : G \to X \) be a function defined by \( h(x) := f(2x) - 2f(x) \) for all \( x \in G \) then we have

\[
\frac{h(2x)}{8} - h(x) \leq \frac{1}{|k|^2(k^2 - 1)} \max \{ \max \{ |2(k-1)|\varphi(x, x), |k|^2\varphi(2x, x) \} , \max \{ \varphi(x, 2x), \max \{ \varphi((k+1)x, x), \varphi((k-1)x, x) \} \} , \max \{ \varphi(x, x), |k|^2\varphi(2x, 2x) \} , \max \{ |2(k-1)|\varphi(x, 2x), \varphi(x, 3x) \} , \max \{ |2x(2k+1)x, x), \varphi((2k-1)x, x) \} \} \}
\]

(2.36)

for all \( x \in G \). Replacing \( x \) by \( 2^{n-1}x \) in (2.36), we get

\[
\frac{h(2^n)x}{8^n} - h(2^{n-1}x) \leq \frac{1}{|k|^2(k^2 - 1)} \max \{ \max \{ |2(k-1)|\varphi(2^{n-1}x, 2^{n-1}x), |k|^2\varphi(2^n, 2^{n-1}x) \} , \max \{ \varphi(2^{n-1}x, 2^nx), \max \{ \varphi(2^{n-1}(k+1)x, 2^{n-1}x), \varphi(2^{n-1}(k-1)x, 2^{n-1}x) \} \} , \max \{ \varphi(2^{n-1}x, 2^{n-1}x), |k|^2\varphi(2^n, 2^n) \} , \max \{ |2(k-1)|\varphi(2^{n-1}x, 2^nx), \varphi(2^{n-1}x, 3.2^{n-1}x) \} , \max \{ \varphi(2^{n-1}(2k+1)x, 2^{n-1}x), \varphi(2^{n-1}(2k-1)x, 2^{n-1}x) \} \} \}
\]

(2.37)

for all \( x \in G \). It follows from (2.32) and (2.37) that the sequence \( \{ \frac{h(2^n)x}{8^n} \} \) is Cauchy. Since \( X \) is complete, we conclude that \( \{ \frac{h(2^n)x}{8^n} \} \) is convergent. Set \( C(x) := \lim_{n \to \infty} \frac{h(2^n)x}{8^n} \).
Using induction one can show that

\[
\| \frac{h(2^n x)}{8^n} - h(x) \| \leq \frac{1}{|8, k^2(k^2 - 1)|} \max \{ \frac{1}{|8|} \max \{ \max \{ |\varphi(2^{i-1} x, 2^{i-1} x)|, |k^2| |\varphi(2^i x, 2^i x)| \} \\
\max \{ |\varphi(2i-1 x, 2^i x)|, |k^2| |\varphi(2^{i-1} x, 2^i x)| \} \\
\max \{ |\varphi(2^{i-1} x, 2^i x)|, |k^2| |\varphi(2^{i-1} x, 2^i x)| \} \} \}
\]

for all \( n \in \mathbb{N} \) and all \( x \in G \). By taking \( n \) to approach infinity in (2.38) and using (2.33) one obtains (2.35). By (2.31) and (2.34), we get

\[
\| DC(x, y) \| = \lim_{n \to \infty} \frac{1}{|8^n|} \| h(2^n x, 2^n y) \| = \lim_{n \to \infty} \frac{1}{|8^n|} \| D(f(2^{n+1} x, 2^{n+1} y) - 2f(2^n x, 2^n y)) \|
\]

\[
\leq \lim_{n \to \infty} \frac{1}{|8^n|} \max \{ \| D(f(2^{n+1} x, 2^{n+1} y)) \|, \| 2f(2^n x, 2^n y) \| \}
\]

\[
\leq \lim_{n \to \infty} \frac{1}{|8^n|} \max \{ \varphi(2^{n+1} x, 2^{n+1} y), 2|\varphi(2^n x, 2^n y) \|
\]

\[
= 0
\]

for all \( x, y \in G \). Therefore the function \( C : G \to X \) satisfies (1.3). If \( C' \) is another cubic function satisfying (2.35), then

\[
\| C(x) - C'(x) \| = \lim_{i \to \infty} |8|^{-i} \| C(2^i x) - C'(2^i x) \|
\]

\[
\leq \lim_{i \to \infty} |8|^{-i} \max \{ |C(2^i x) - h(2^i x)|, |h(2^i x) - C'(2^i x)| \}
\]

\[
\leq \frac{1}{|8, k^2(k^2 - 1)|} \lim_{i \to \infty} \lim_{n \to \infty} \max \{ \frac{1}{|8|} \max \{ |\varphi(2^{i-1} x, 2^{i-1} x)|, |k^2| |\varphi(2^i x, 2^i x)| \} \\
\max \{ |\varphi(2^{i-1} x, 2^i x)|, |k^2| |\varphi(2^{i-1} x, 2^i x)| \} \} \}
\]

\[
\leq \frac{1}{|8, k^2(k^2 - 1)|} \lim_{i \to \infty} \lim_{n \to \infty} \max \{ \frac{1}{|8|} \max \{ |\varphi(2^{i-1} x, 2^{i-1} x)|, |k^2| |\varphi(2^i x, 2^i x)| \} \\
\max \{ |\varphi(2^{i-1} x, 2^i x)|, |k^2| |\varphi(2^{i-1} x, 2^i x)| \} \}
\]

\[
\leq \frac{1}{|8, k^2(k^2 - 1)|} \lim_{i \to \infty} \lim_{n \to \infty} \max \{ \frac{1}{|8|} \max \{ |\varphi(2^{i-1} x, 2^{i-1} x)|, |k^2| |\varphi(2^i x, 2^i x)| \} \}
\]

\[
= 0
\]

for all \( x \in G \). Therefore \( C = C' \). This completes the proof of the uniqueness of \( C \). \( \square \)

**Theorem 2.4.** Let \( \varphi : G \times G \to [0, \infty) \) be a function such that

\[
\lim_{n \to \infty} \frac{1}{|2^n|} \max \{ \varphi(2^{n+1} x, 2^{n+1} y), |8| \varphi(2^n x, 2^n y) \}
\]

\[
= \lim_{n \to \infty} \frac{1}{|8^n|} \max \{ \varphi(2^{n+1} x, 2^{n+1} y), |2| \varphi(2^n x, 2^n y) \} = 0
\] (2.39)
for all \(x, y \in G\) and let for each \(x \in G\) the limit
\[
\max\left\{\frac{1}{|\delta i|} \max\left\{\max\{\max\{2(k^2 - 1)|\varphi(2^{i-1}x,2^{j-1}x),|k|^2|\varphi(2^jx,2^{j-1}x)\}\},\max\{\varphi(2^{i-1}x,2^jx),\max\{\varphi(2^{i-1}(k+1)x,2^{j-1}x),\varphi(2^{i-1}(k-1)x,2^{j-1}x)\}\}\right\}\right\}
\]
then \(A\) denoted by 
\[
\lim_{n \to \infty} \frac{1}{|\delta i|} \max\left\{\max\{\max\{2(k^2 - 1)|\varphi(2^{i-1}x,2^{j-1}x),|k|^2|\varphi(2^jx,2^{j-1}x)\}\},\max\{\varphi(2^{i-1}x,2^jx),\max\{\varphi(2^{i-1}(k+1)x,2^{j-1}x),\varphi(2^{i-1}(k-1)x,2^{j-1}x)\}\}\right\}\right\}
\]
\[
\max\{\varphi(2^{i-1}(2k+1)x,2^{j-1}x),\varphi(2^{i-1}(2k-1)x,2^{j-1}x)\}\}\}} : 0 \leq i < n\}
\]
denoted by \(\bar{\varphi}_A(x)\), and
\[
\max\left\{\frac{1}{|\delta i|} \max\left\{\max\{\max\{2(k^2 - 1)|\varphi(2^{i-1}x,2^{j-1}x),|k|^2|\varphi(2^jx,2^{j-1}x)\}\},\max\{\varphi(2^{i-1}x,2^jx),\max\{\varphi(2^{i-1}(k+1)x,2^{j-1}x),\varphi(2^{i-1}(k-1)x,2^{j-1}x)\}\}\right\}\right\}\right\}
\]
\[
\max\{\varphi(2^{i-1}(2k+1)x,2^{j-1}x),\varphi(2^{i-1}(2k-1)x,2^{j-1}x)\}\}\}} : 0 \leq i < n\}
\]
denoted by \(\bar{\varphi}_C(x)\), exists. Suppose that \(f: G \to X\) is an odd function satisfying
\[
\|Df(x,y)\| \leq \varphi(x,y)
\]
for all \(x, y \in G\). Then there exists an additive function \(A: G \to X\) and a cubic function \(C: G \to X\) such that
\[
\|f(x) - A(x) - C(x)\| \leq \frac{1}{|k^2(k^2 - 1)|} \max\left\{\frac{1}{|\delta i|} \bar{\varphi}_A(x),\max\{\bar{\varphi}_C(x)\}\right\}
\]
for all \(x \in G\). Moreover, if
\[
\lim_{i \to \infty} \lim_{n \to \infty} \frac{1}{|\delta i|} \max\left\{\max\{\max\{2(k^2 - 1)|\varphi(2^{i-1}x,2^{j-1}x),|k|^2|\varphi(2^jx,2^{j-1}x)\}\},\max\{\varphi(2^{i-1}x,2^jx),\max\{\varphi(2^{i-1}(k+1)x,2^{j-1}x),\varphi(2^{i-1}(k-1)x,2^{j-1}x)\}\}\right\}\right\}\right\}
\]
\[
\max\{\varphi(2^{i-1}(2k+1)x,2^{j-1}x),\varphi(2^{i-1}(2k-1)x,2^{j-1}x)\}\}\}} : i \leq j < n + i\}
\]
= 0,
\[
\lim_{i \to \infty} \lim_{n \to \infty} \frac{1}{|\delta i|} \max\left\{\max\{\max\{2(k^2 - 1)|\varphi(2^{i-1}x,2^{j-1}x),|k|^2|\varphi(2^jx,2^{j-1}x)\}\},\max\{\varphi(2^{i-1}x,2^jx),\max\{\varphi(2^{i-1}(k+1)x,2^{j-1}x),\varphi(2^{i-1}(k-1)x,2^{j-1}x)\}\}\right\}\right\}\right\}
\]
\[
\max\{\varphi(2^{i-1}(2k+1)x,2^{j-1}x),\varphi(2^{i-1}(2k-1)x,2^{j-1}x)\}\}\}} : i \leq j < n + i\}
\]
= 0,
then \(A\) is the unique additive function and \(C\) is the unique cubic function satisfying (2.41).
Proof. By Theorems 2.2 and 2.3, there exists an additive function $A_1 : G \to X$ and a cubic function $C_1 : G \to X$ such that

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{1}{2k^2(k^2 - 1)} \tilde{\varphi}_A(x)$$

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{1}{8k^2(k^2 - 1)} \tilde{\varphi}_C(x)$$

for all $x \in G$. So we obtain (2.41) by letting $A(x) = \frac{1}{6}A_1(x)$ and $C(x) = \frac{1}{6}C_1(x)$ for all $x \in G$. Now it is obvious that (2.41) holds true for all $x \in G$, and the proof of theorem is complete. \qed

**Theorem 2.5.** Let $\varphi : G \times G \to [0, \infty)$ be a function such that

$$\lim_{n \to \infty} \frac{1}{2^n} \max \{\varphi(2^{n+1}x,2^{n+1}y), |8\varphi(2^n x, 2^n y)|\}$$

$$= \lim_{n \to \infty} \frac{1}{8^n} \max \{\varphi(2^{n+1}x,2^{n+1}y), |2\varphi(2^n x, 2^n y)|\}$$

$$= \lim_{n \to \infty} \frac{\varphi(2^n x, 2^n y)}{2^n} = 0$$ (2.42)

for all $x, y \in G$ and let for each $x \in G$ the limit

$$\max \{\frac{1}{2^n} \max \{\max \{|2(k^2 - 1)|\varphi(2^{i-1}x, 2^{i-1}x), |k^2|\varphi(2^i x, 2^i x)|\}$$

$$, \max \{|2(k^2 - 1)|\varphi(2^{i-1}(k + 1)x, 2^{i-1}x), \varphi(2^{i-1}(k - 1)x, 2^{i-1}x)|\} \} \}$$

$$, \max \{\max \{|2(k^2 - 1)|\varphi(2^{i-1}x, 2^i x), |k^2|\varphi(2^{i-1}x, 2^i x)|\}$$

$$, \max \{|2(k^2 - 1)|\varphi(2^{i-1}x, 2^{i-1}x), |k^2|\varphi(2^i x, 2^i x)|\} \} : 0 \leq i < n\},$$

denoted by $\tilde{\varphi}_A(x)$, and

$$\lim_{n \to \infty} \max \{\frac{1}{2^n} \varphi(0, k^j x) : 0 \leq j < n\},$$

denoted by $\tilde{\varphi}_Q(x)$, and

$$\max \{\frac{1}{8^n} \max \{\max \{|2(k^2 - 1)|\varphi(2^{i-1}x, 2^{i-1}x), |k^2|\varphi(2^i x, 2^i x)|\}$$

$$, \max \{|2(k^2 - 1)|\varphi(2^{i-1}(k + 1)x, 2^{i-1}x), \varphi(2^{i-1}(k - 1)x, 2^{i-1}x)|\} \} \}$$

$$, \max \{\max \{|2(k^2 - 1)|\varphi(2^{i-1}x, 2^i x), |k^2|\varphi(2^{i-1}x, 2^i x)|\}$$

$$, \max \{|2(k^2 - 1)|\varphi(2^{i-1}x, 2^{i-1}x), |k^2|\varphi(2^i x, 2^i x)|\} \} : 0 \leq i < n\}$$

denoted by $\tilde{\varphi}_C(x)$, exists. Suppose that $f : G \to X$ is a function satisfying

$$\|Df(x, y)\| \leq \varphi(x, y)$$ (2.43)
for all \( x, y \in G \). Then there exists an additive function \( A : G \to X \) and a quadratic function \( Q : G \to X \) and a cubic function \( C : G \to X \) such that

\[
\|f(x) - A(x) - Q(x) - C(x)\| \leq \max\left\{ \frac{1}{|2^k(k^2 - 1)|}, \frac{1}{|2^k|}, \frac{1}{|8|} \right\} \max\left\{ \max\{\hat{\varphi}_A(x), \frac{1}{|8|} \hat{\varphi}_C(x)\}, \max\{\hat{\varphi}_A(-x), \frac{1}{|8|} \hat{\varphi}_C(-x)\}\right\} \max\left\{ \frac{1}{|4, k^2|}, \frac{1}{|8|} \max\{\hat{\varphi}_Q(x), \hat{\varphi}_Q(-x)\}\right\}
\]

(2.44)

for all \( x \in G \). Moreover, if

\[
\lim_{i \to \infty} \lim_{n \to \infty} \max\left\{ \frac{1}{|2^k|}, \max\{\max\{\max\{\max\{\max\{\hat{\varphi}(0, 0)\}, 2(k^2 - 1)|\hat{\varphi}(2^{j-1}x, 2^{j-1}x)|, k^2|\hat{\varphi}(2^j x, 2^j x)|\}, \max\{\hat{\varphi}(2^{j-1}x, 2^j x), \varphi(2^{j-1}(k + 1) x, 2^{j-1}x)\}, \varphi(2^{j-1}(k - 1) x, 2^{j-1}x)\}\right\}\}
\]

\[
\lim_{n \to \infty} \max\left\{ \frac{1}{|2^k|}, \max\{\max\{\max\{\max\{\max\{\hat{\varphi}(0, 0)\}, 2(k^2 - 1)|\hat{\varphi}(2^{j-1}x, 2^{j-1}x)|, k^2|\hat{\varphi}(2^j x, 2^j x)|\}, \max\{\hat{\varphi}(2^{j-1}x, 2^j x), \varphi(2^{j-1}(k + 1) x, 2^{j-1}x)\}, \varphi(2^{j-1}(k - 1) x, 2^{j-1}x)\}\right\}\}
\]

(2.45)

then \( A \) is the unique additive function and \( Q \) is the unique quadratic function and \( C \) is the unique cubic function.

Proof. Let \( f_0(x) = \frac{1}{2}[f(x) - f(-x)] \) for all \( x \in G \). Then \( f_0(0) = 0, f_0(-x) = -f_0(x) \), and

\[
\|Df_0(x, y)\| \leq \frac{1}{|2|} \max\{\varphi(x, y), \varphi(-x, -y)\}
\]

for all \( x, y \in G \). From Theorem 2.3, it follows that there exists a unique additive function \( A : G \to X \) and a unique cubic function \( C : G \to X \) satisfying

\[
\|f_0(x) - A(x) - C(x)\| \leq \max\left\{ \frac{1}{|2^k(k^2 - 1)|}, \frac{1}{|2^k|}, \frac{1}{|8|} \right\} \max\left\{ \max\{\hat{\varphi}_A(x), \frac{1}{|8|} \hat{\varphi}_C(x)\}, \max\{\hat{\varphi}_A(-x), \frac{1}{|8|} \hat{\varphi}_C(-x)\}\right\}
\]

(2.45)

for all \( x \in G \).

Let \( f_0(x) = \frac{1}{2}[f(x) + f(-x)] \) for all \( x \in G \). Then \( f_0(0) = 0, f_0(-x) = f_0(x) \), and

\[
\|Df_0(x, y)\| \leq \frac{1}{|2|} \max\{\varphi(x, y), \varphi(-x, -y)\}
\]
for all \(x, y \in G\). From Theorem 2.1, it follows that there exist a unique quadratic function \(Q : G \to X\) satisfying
\[
\|f_e(x) - Q(x)\| \leq \frac{1}{|4k^2|} \max\{\tilde{\varphi}_Q(x), \tilde{\varphi}_Q(x)\}
\] (2.46)
for all \(x \in G\). Hence (2.44) follows from (2.45) and (2.46).

\[\square\]

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