Relationship between Conformal Geometrodynamics and Dirac Equations

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Abstract

The paper describes a unique phenomenon – the possibility of establishing, in certain space regions, the one-to-one correspondence between equations related to absolutely different physical phenomena: (1) phenomena associated with the Weyl degrees of freedom in plane space; (2) phenomena, which can be described in terms of half-integer spin particles and observed quantities corresponding to a full set of bispinors.

The phenomenon established opens wide prospects for resolving in future the “old” disputable issue concerning the physical meaning of the Weyl vector. The paper discusses, in particular, the possibility of identifying the Weyl vector with the current density vector of bispinors constituting a bispinor matrix included in the Dirac equation. Some other issues are also discussed.

1 Introduction

To date, there are two approaches that provided results, which are of interest as applied to the development of a unified theory of space and three fundamental interactions – strong, electromagnetic, and weak. Conformal geometrodynamics (CGD)\footnote{For explanations concerning CGD equations (including their history) see [1], [2] and the papers they refer to.} is one these approaches. This term means the theory based on the conformally invariant generalization of the general relativity (GR) equations. Another approach is studying the algorithm of constructing a bispinor system representing the given system of tensor fields. The algorithm has been developed and described in [3] - [5].

In [6] the authors made an attempt to use the algorithm above for correlating the new degrees of freedom in CGD equations and the bispinors following
the Dirac equation. The given paper proposes, in essence, an improved way (in comparison with [6]) of setting up the correlation and interpreting the physical meaning of the Weyl vector appeared in CGD equations.

The improvement consists, first of all, in ascertaining, what particular CGD equations we are speaking about. We operate equations following from equation \( T_{\varepsilon,\mu} = 0 \), where \( T_{\alpha\beta} \) is the energy-momentum tensor constructed from the Weyl degrees of freedom and included in the right-hand member of the conformally generalized GR equations. Space is considered to be a plane, the inverse impact of the Weyl vector field on the space curvature is ignored. Though we deal with plane space, the equations of interest describe the physics of the Weyl degrees of freedom of a general kind; the Weyl vector is not reduced to a scalar function gradient in our approach.

The second improvement is that incompleteness (in [6]) of the obtained set of equations for the observed tensors, which follow from the Dirac equation, has been avoided. It is assumed that the Dirac equation is related to a set of four bispinors reproducing the given tensor values, rather than the one bispinor. Such an approach follows the algorithm of mapping tensors onto bispinors published earlier in [3] - [5]. A new result of this study includes expressions for the matrix connectivity correlated with bispinor matrix equations.

These improvements allow us to assert that in certain spacetime regions, where the polarization matrix is positively defined, solutions of CGD equations can be interpreted in terms of solutions of the Dirac equation. This interpretation automatically leads to equations defining the matrix connectivity (in other terms, gauge field) and symmetry groups. Proving these facts constitutes the major result of this study. The result, which confirms the interpretation of the Weyl vector in [6] as a current density vector for the whole set of bispinors making up the bispinor matrix in the Dirac equation.

The results obtained are summarized at the end of the paper. We also discuss prospects of using the results of this study as a basis for a new approach to setting up the unified theory of gravitational and Weyl degrees of freedom and internal degrees of freedom, which are required for describing the whole diversity of half-integer spin particles.
2 Dirac Equation and Its Corollaries

2.1 Dirac Matrices. Dirac Equation

The Dirac matrices (DMs) $\tilde{\gamma}_\alpha$ are defined as

$$\tilde{\gamma}_\alpha \tilde{\gamma}_\beta + \tilde{\gamma}_\beta \tilde{\gamma}_\alpha = 2\eta_{\alpha\beta}. \quad (1)$$

They are constant all over the space. If an explicit form of DMs is required, we use the Majoran set of matrices

$$\begin{align*}
\tilde{\gamma}_0 &= -i\rho_2\sigma_1, \\
\tilde{\gamma}_1 &= \rho_1, \\
\tilde{\gamma}_2 &= \rho_2\sigma_2, \\
\tilde{\gamma}_3 &= \rho_3,
\end{align*} \quad (2)$$

the elements of which are integer real numbers$^2$.

Suppose that the field operator $Z$ in the general case is a $4 \times 4$ matrix satisfying the Dirac equation

$$\tilde{\gamma}^\nu (\nabla_\nu Z) = m \cdot Z. \quad (3)$$

The matrix operator $Z$ will be called a bispinor matrix. Bispinor states are separated from $Z$ by its multiplication on the right by projection operators. Together with Eq. (3), the following equation is satisfied:

$$\left( \nabla_\nu Z^+ \right) \tilde{\gamma}^\nu \tilde{D} = -m \cdot Z^+ \tilde{D}. \quad (4)$$

The matrix $\tilde{D}$ found in (4) is defined as

$$\tilde{D} \tilde{\gamma}_\mu \tilde{D}^{-1} = -\tilde{\gamma}^+_\mu. \quad (5)$$

The covariant derivatives of the bispinor matrix in (3), (4) are written as

$$\begin{align*}
\nabla_\alpha Z &= Z;_{;\alpha} - Z\Gamma_\alpha \\
\nabla_\alpha Z^+ &= Z^+;_{;\alpha}^+ + \Gamma_\alpha Z^+. \quad (6)
\end{align*}$$

The quantity $\Gamma_\alpha$ in (6) will denote matrix connectivity. It is a set of real skew-Hermitian matrices:

$$\Gamma_\alpha^* = \Gamma_\alpha, \quad \Gamma_\alpha^+ = -\Gamma_\alpha. \quad (7)$$

$^2$ Solutions of relationships (1) always include a real set of DMs, if the signature $(- + + +)$ is used.
The matrix skew $P_{\alpha\beta}$ is defined through the matrix connectivity $\Gamma_\alpha$ in much the same way as the Riemann tensor is defined through the Christoffel symbols.

$$ P_{\alpha\beta} = \Gamma_{\beta,\alpha} - \Gamma_{\alpha,\beta} + \Gamma_\alpha \Gamma_\beta - \Gamma_\beta \Gamma_\alpha. \quad (8) $$

The following equalities hold:

$$ \begin{align*}
(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) Z &= -Z P_{\alpha\beta} \\
(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) Z^+ &= P_{\alpha\beta} Z^+. \quad (9)
\end{align*} $$

Let us give some relationships that follow immediately from the Dirac equation. Eq. (3) on the left is multiplied by $\tilde{\gamma}^\alpha$.

$$ \tilde{\gamma}^\alpha \tilde{\gamma}^\nu (\nabla_\nu Z) = m \cdot \tilde{\gamma}^\alpha Z. \quad (10) $$

The product $\tilde{\gamma}^\alpha \tilde{\gamma}^\nu$ is written as

$$ \tilde{\gamma}^\alpha \tilde{\gamma}^\nu = \eta^{\alpha\nu} + \tilde{S}^{\alpha\nu} \quad (11) $$

(here, $\tilde{S}^{\mu\nu} = 1/2 (\tilde{\gamma}^\mu \tilde{\gamma}^\nu - \tilde{\gamma}^\nu \tilde{\gamma}^\mu)$) and (11) is substituted into (10).

$$ (\nabla_\alpha Z) = -\tilde{S}^\nu_\alpha (\nabla_\nu Z) + m \cdot \tilde{\gamma}^\alpha Z. \quad (12) $$

After Hermitian conjugation of Eq. (12) and multiplication by $D$, we obtain:

$$ \left( \nabla_\alpha Z^+ \right) \tilde{D} = \left( \nabla_\nu Z^+ \right) \tilde{D} \tilde{S}^\nu_\alpha - m \cdot Z^+ \tilde{D} \tilde{\gamma}^\alpha. \quad (13) $$

Then, Eq. (3) on the left is multiplied by $\tilde{\gamma}^\mu \nabla_\mu$.

$$ \tilde{\gamma}^\mu \nabla_\mu \tilde{\gamma}^\nu (\nabla_\nu Z) = m \cdot \tilde{\gamma}^\mu \nabla_\mu Z. \quad (14) $$

Note that within the formalism at issue

$$ \nabla_\mu \tilde{\gamma}^\alpha = 0. \quad (15) $$

As a result, we have:

$$ \left( \eta^{\mu\nu} + \tilde{S}^{\mu\nu} \right) (\nabla_\mu \nabla_\nu Z) = m^2 \cdot Z. \quad (16) $$

It follows from Eq. (16) that
\begin{equation}
\left\{ \begin{array}{c}
(\nabla^\nu \nabla_\nu Z) = \frac{1}{2} \tilde{\Gamma}^{\mu\nu} Z P_{\mu\nu} + m^2 \cdot Z \\
(\nabla^\nu \nabla_\nu Z^+) \tilde{D} = \frac{1}{2} Z P_{\mu\nu} Z^+ \tilde{D} \tilde{S}^{\mu\nu} + m^2 \cdot Z^+ \tilde{D}
\end{array} \right. \}
\end{equation}  \tag{17}

The second relationship in Eq. (17) is derived from the first one by the Hermitian conjugation.

2.2 Algorithm for Tensor Mapping onto Bispinor Matrix

We define the vector \( j_\alpha \) and the anti-symmetric tensor \( h_{\alpha\beta} \) as follows:
\begin{equation}
j_\alpha \equiv \frac{1}{4} \text{Sp} \left\{ Z^+ \tilde{D} \tilde{\gamma}_\alpha Z \right\}, \tag{18}\end{equation}
\begin{equation}h_{\alpha\beta} \equiv -\frac{1}{8} \text{Sp} \left\{ Z^+ \tilde{D} \tilde{S}_{\alpha\beta} Z \right\}. \tag{19}\end{equation}

Formulas (18), (19) make it possible to uniquely define the vector \( j_\alpha \) and the anti-symmetric tensor \( h_{\alpha\beta} \) based on the real bispinor matrix \( Z \).

As noted above, in Refs. [3] - [5], an inverse task was accomplished, i.e. the task of finding the bispinor matrix \( Z \) based on the given vector \( j_\alpha \) and antisymmetric tensor \( h_{\alpha\beta} \). Refs. [3]-[5] addressed a complex case, in which the sought bispinor matrix \( Z \) reproduced the whole given set of tensor quantities (scalar, pseudoscalar, vector, pseudovector, antisymmetric tensor). This paper deals with a particular case of DM and matrix \( Z \) implementations over the field of real numbers. In this case, only the vector \( j_\alpha \) and the antisymmetric tensor \( h_{\alpha\beta} \) should be defined. In order to find the matrix \( Z \) in this particular case, one should set up a Hermitian matrix \( M \) written as
\begin{equation}M = j^\alpha \cdot (\tilde{\gamma}_\alpha \tilde{D}^{-1}) + h^{\mu\nu} \cdot (\tilde{S}_{\mu\nu} \tilde{D}^{-1}) \tag{20}\end{equation}
and extract a square root from it, i.e. find the matrix \( Z \) satisfying the relationship
\begin{equation}M = ZZ^+. \tag{21}\end{equation}

There are cases, when it is not difficult to extract a square root from \( M \). Such cases include the situations, in which the matrix \( M \) is positively defined, i.e. when all the eigenvalues \( \{ \mu_1, \mu_2, \mu_3, \mu_4 \} \) are positive. Let us illustrate this. Let \( \psi_A \) be an eigenvector of the matrix \( M \) corresponding to
the eigenvalue \( \mu_A \) \((A = 1, 2, 3, 4)\). By definition, the vectors \( \psi_A \) satisfy the relationships

\[
M \psi_A = \psi_A \mu_A.
\]  

Each of the vectors \( \psi_A \) has four components,

\[
\psi_A = \begin{pmatrix} \psi_{1A} \\ \psi_{2A} \\ \psi_{3A} \\ \psi_{4A} \end{pmatrix}.
\]  

(23)

Since relationship (22) defines the vectors \( \psi_A \) to within the multiplication by a numerical factor, the vectors \( \psi_A \) can be normalized by the condition

\[
|\psi_A|^2 = |\psi_{1A}|^2 + |\psi_{2A}|^2 + |\psi_{3A}|^2 + |\psi_{4A}|^2 = 1.
\]  

(24)

Let us introduce a matrix \( U \), the columns in which consist of components of the normalized vectors \( \psi_A \), i.e. the matrix

\[
U = \begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} & \psi_{14} \\ \psi_{21} & \psi_{22} & \psi_{23} & \psi_{24} \\ \psi_{31} & \psi_{32} & \psi_{33} & \psi_{34} \\ \psi_{41} & \psi_{42} & \psi_{43} & \psi_{44} \end{pmatrix}.
\]  

(25)

The matrix \( U \) is unitary, i.e. \( U^+ = U^{-1} \). We also introduce four projection operators \( P_A \):

\[
P_1 = \frac{1}{2} (E + \rho_3) (E + \sigma_3), \quad P_2 = \frac{1}{2} (E + \rho_3) (E - \sigma_3),
\]

\[
P_3 = \frac{1}{2} (E - \rho_3) (E + \sigma_3), \quad P_4 = \frac{1}{2} (E - \rho_3) (E - \sigma_3).
\]  

(26)

The set of projection operators (26) is a full system in the sense that \( \sum_{A=1,...,4} P_A = E \). Therefore, using the projection operators, matrix (25) can be written as

\[
U = \sum_{A=1,...,4} (UP_A).
\]  

(27)

If normalized matrix bispinors \( u_A \) are introduced by means of the relationship

\[
uA \equiv UP_A,
\]  

(28)
Eq. (22) will transform into

\[ Mu_A = u_A \cdot \mu_A. \]  

(29)

Here, \( \mu_A \) is a matrix of the form

\[
\mu_A = \begin{pmatrix}
\mu_1 \\
\mu_2 \\
\mu_3 \\
\mu_4
\end{pmatrix}
\]  

(30)

Summing of (29) over \( A \) gives:

\[ MU = \sum_{A=1}^{4} (u_A \cdot \mu_A) = U \sum_{A=1}^{4} (\mu_A \cdot P_A). \]  

(31)

Multiplication of (31) in the right-hand member by \( U^{-1} \) gives:

\[ M = U \cdot \sum_{A=1}^{4} (\mu_A P_A) \cdot U^+ = 
\]

\[ = \left\{ U \cdot \sum_{A=1}^{4} (\sqrt{\mu_A} P_A) \right\} \cdot \left\{ U \cdot \sum_{A=1}^{4} (\sqrt{\mu_A} P_A) \right\}^+ = ZZ^+. \]  

(32)

As a result, we obtain relationship (21), which needed to be proved. \( Z \) is used here to denote the matrix

\[ Z = U \cdot \left( \sum_{A=1}^{4} (\sqrt{\mu_A} P_A) \right). \]  

(33)

The explicit matrix form of the matrix \( Z \) is

\[ Z = U \cdot \begin{pmatrix}
\sqrt{\mu_1} \\
\sqrt{\mu_2} \\
\sqrt{\mu_3} \\
\sqrt{\mu_4}
\end{pmatrix}. \]  

(34)

Note that eigenvalues of \( M \) may also contain negative eigenvalues. In these cases, factorization of the matrix \( M \) in the form of (21) is inapplicable. Such cases are beyond the scope of this paper.

3 Empty squares in the matrix contain zeros.
The matrix $Z$ in expansions of the type (21) is defined ambiguously. If some matrix $Z$ satisfies relationship (21), the matrix $Z' = Z \cdot V$, where $V$ is an arbitrary unitary matrix, will satisfy the same relationship. Using this freedom, one can get the matrix $Z$ in (21) to be Hermitian. It is easy to see that for this purpose it would suffice to assume that $V$ is equal to the matrix $U^{-1}$.

2.3 Theorem I

Theorem I will be used to denote the relationship

$$j_{\alpha}^{\beta} = 0, \quad (35)$$

which can be easily checked by means of Eqs.(3), (4).

2.4 Theorem II

Theorem II states that if a bispinor matrix is governed by the Dirac equation (3) and the quantities $j_{\alpha}, h_{\alpha\beta}$ are defined as (18), (19), the relationship

$$(j_{\beta, \alpha} - j_{\alpha, \beta}) = 4m \cdot h_{\alpha\beta} + E_{\alpha\beta\mu} \frac{1}{4} \text{Sp} \left\{ (\nabla_{\nu} Z^+) \tilde{D}_{\gamma_5} \tilde{\gamma}^\mu Z - Z^+ \tilde{D}_{\gamma_5} \tilde{\gamma}^\mu (\nabla_{\nu} Z) \right\}$$

holds. The theorem will be proven in several steps. First, let us try to calculate the quantity $(j_{\beta, \alpha} - j_{\alpha, \beta})$ based on relationships (12), (13). It follows from (18) that:

$$(j_{\beta, \alpha} - j_{\alpha, \beta}) = \frac{1}{4} \text{Sp} \left\{ (\nabla_{\nu} Z^+) \tilde{D}_{\gamma_{\alpha}} Z + Z^+ \tilde{D}_{\gamma_{\alpha}} (\nabla_{\nu} Z) \right\} - \frac{1}{4} \text{Sp} \left\{ (\nabla_{\beta} Z^+) \tilde{D}_{\gamma_{\alpha}} Z + Z^+ \tilde{D}_{\gamma_{\alpha}} (\nabla_{\beta} Z) \right\}. \quad (37)$$

We use relationships (12), (13).

$$(j_{\beta, \alpha} - j_{\alpha, \beta}) =$$

$$= \frac{1}{4} \text{Sp} \left\{ (\nabla_{\nu} Z^+) \tilde{D}_{\gamma_{\alpha}} Z - m \cdot Z^+ \tilde{D}_{\gamma_{\alpha}} \bar{\gamma}_{\beta} Z - Z^+ \tilde{D}_{\gamma_{\beta}} \tilde{\gamma}_{\alpha} (\nabla_{\nu} Z) +$$

$$+ m \cdot Z^+ \tilde{D}_{\gamma_{\beta}} \bar{\gamma}_{\alpha} Z \right\} - \frac{1}{4} \text{Sp} \left\{ (\nabla_{\nu} Z^+) \tilde{D}_{\gamma_{\beta}} \bar{\gamma}_{\alpha} Z -$$

$$- m \cdot Z^+ \tilde{D}_{\gamma_{\beta}} \bar{\gamma}_{\alpha} Z - Z^+ \tilde{D}_{\gamma_{\alpha}} \tilde{\gamma}_{\beta} (\nabla_{\nu} Z + m \cdot Z^+ \tilde{D}_{\gamma_{\alpha}} \bar{\gamma}_{\beta} Z \right\}. \quad (38)$$
Terms without derivatives and with derivatives are grouped separately in the right-hand member of (38).

\[
(j_{\beta,\alpha} - j_{\alpha,\beta}) = \frac{1}{2} m \cdot Sp \left\{ -Z^+ \tilde{D} \tilde{\gamma}_{\alpha} \tilde{\gamma}_{\beta} Z + Z^+ \tilde{D} \tilde{\gamma}_{\beta} \tilde{\gamma}_{\alpha} Z \right\} \\
+ \frac{1}{4} Sp \left\{ (\nabla_{\nu} Z^+) \tilde{D} \tilde{S}_{\alpha}^\nu \tilde{\gamma}_{\beta} Z - Z^+ \tilde{D} \tilde{\gamma}_{\beta} \tilde{S}_{\alpha}^\nu (\nabla_{\nu} Z) - (\nabla_{\nu} Z^+) \tilde{D} \tilde{S}_{\beta}^\nu \tilde{\gamma}_{\alpha} Z \right\} \\
+ Z^+ \tilde{D} \tilde{\gamma}_{\alpha} \tilde{S}_{\beta}^\nu (\nabla_{\nu} Z) \right\}.
\]

DM products, such as \( \tilde{S}_{\alpha}^\nu \tilde{\gamma}_{\beta} \), \( \tilde{\gamma}_{\beta} \tilde{S}_{\nu}^\alpha \), are replaced in (39) with the following relationships:

\[
\tilde{S}_{\alpha}^\nu \tilde{\gamma}_{\beta} = -\eta_{\alpha\beta} \tilde{\gamma}^\nu + \delta_{\gamma}^\nu \tilde{\gamma}_{\alpha} + E_{\nu,\beta,\gamma} \tilde{\gamma}^\mu \\
\tilde{\gamma}_{\beta} \tilde{S}_{\alpha}^\nu = \eta_{\alpha\beta} \tilde{\gamma}^\nu - \delta_{\gamma}^\nu \tilde{\gamma}_{\alpha} + E_{\alpha,\beta,\gamma} \tilde{\gamma}^\mu
\]

We have:

\[
(j_{\beta,\alpha} - j_{\alpha,\beta}) = -m \cdot Sp \left\{ Z^+ \tilde{D} \tilde{S}_{\alpha\beta} Z \right\} \\
+ \frac{1}{4} Sp \left\{ (\nabla_{\nu} Z^+) \tilde{D} \left( -\eta_{\alpha\beta} \tilde{\gamma}^\nu + \delta_{\gamma}^\nu \tilde{\gamma}_{\alpha} + E_{\nu,\beta,\gamma} \tilde{\gamma}^\mu \right) Z \right\} \\
+ \frac{1}{4} Sp \left\{ -Z^+ \tilde{D} \left( \eta_{\alpha\beta} \tilde{\gamma}^\nu - \delta_{\gamma}^\nu \tilde{\gamma}_{\alpha} + E_{\nu,\beta,\gamma} \tilde{\gamma}^\mu \right) (\nabla_{\nu} Z) \right\} \\
+ \frac{1}{4} Sp \left\{ - (\nabla_{\nu} Z^+) \tilde{D} \left( -\eta_{\alpha\beta} \tilde{\gamma}^\nu + \delta_{\gamma}^\nu \tilde{\gamma}_{\alpha} - E_{\nu,\beta,\gamma} \tilde{\gamma}^\mu \right) Z \right\} \\
+ \frac{1}{4} Sp \left\{ Z^+ \tilde{D} \left( \eta_{\alpha\beta} \tilde{\gamma}^\nu - \delta_{\gamma}^\nu \tilde{\gamma}_{\alpha} - E_{\nu,\beta,\gamma} \tilde{\gamma}^\mu \right) (\nabla_{\nu} Z) \right\}.
\]

The terms with the metric tensor in (40) are canceled. The rest give:

\[
(j_{\beta,\alpha} - j_{\alpha,\beta}) = +8m \cdot h_{\alpha\beta} \\
+ \frac{1}{4} Sp \left\{ (\nabla_{\nu} Z^+) \tilde{D} \tilde{\gamma}_{\alpha} Z \right\} + E_{\nu,\alpha,\beta,\mu} \frac{1}{4} Sp \left\{ (\nabla_{\nu} Z^+) \tilde{D} \tilde{\gamma}^\mu Z \right\} \\
+ \frac{1}{4} Sp \left\{ Z^+ \tilde{D} \tilde{\gamma}_{\alpha} (\nabla_{\nu} Z) \right\} - E_{\nu,\alpha,\beta,\mu} \frac{1}{4} Sp \left\{ Z^+ \tilde{D} \tilde{\gamma}^\mu (\nabla_{\nu} Z) \right\} \\
+ \frac{1}{4} Sp \left\{ - (\nabla_{\nu} Z^+) \tilde{D} \tilde{\gamma}_{\beta} Z \right\} + E_{\nu,\alpha,\beta,\mu} \frac{1}{4} Sp \left\{ (\nabla_{\nu} Z^+) \tilde{D} \tilde{\gamma}^\mu Z \right\} \\
+ \frac{1}{4} Sp \left\{ -Z^+ \tilde{D} \tilde{\gamma}_{\beta} (\nabla_{\nu} Z) \right\} - E_{\nu,\alpha,\beta,\mu} \frac{1}{4} Sp \left\{ +Z^+ \tilde{D} \tilde{\gamma}^\mu (\nabla_{\nu} Z) \right\}.
\]

The terms with \( \tilde{D} \tilde{\gamma}_{\alpha} \), \( \tilde{D} \tilde{\gamma}_{\alpha} \) are reduced to \(-(j_{\beta,\alpha} - j_{\alpha,\beta})\). The rest are combined into the expression \( E_{\alpha,\beta,\mu} \frac{1}{2} Sp \left\{ (\nabla_{\nu} Z^+) \tilde{D} \tilde{\gamma}^\mu (\nabla_{\nu} Z) \right\} \). As a result, we have

\[
(j_{\beta,\alpha} - j_{\alpha,\beta}) = 8m \cdot h_{\alpha\beta} - (j_{\beta,\alpha} - j_{\alpha,\beta}) \\
+ E_{\nu,\alpha,\beta,\mu} \frac{1}{2} Sp \left\{ (\nabla_{\nu} Z^+) \tilde{D} \tilde{\gamma}^\mu Z - Z^+ \tilde{D} \tilde{\gamma}^\mu (\nabla_{\nu} Z) \right\}.
\]

Identical transformations of relationship (41) give the same expression as (36). Thus, Theorem II is proven.
2.5 Theorem III

If the bispinor matrix $Z$ satisfies the Dirac equation (3), the quantity $h_{\alpha\beta}$ defined by Eq. (19) will satisfy the relationship

$$E^\lambda_{\alpha\beta\varepsilon}h_{\alpha\beta;\varepsilon} = \frac{1}{4}\cdot\text{Sp}\left\{\left(\nabla^\lambda Z^+\right)\bar{D}\gamma_5 Z - Z^+\bar{D}\gamma_5 \left(\nabla^\lambda Z\right)\right\}.$$  \hspace{1cm} (42)

Equality (42) will be below referred to as Theorem III.

To prove it, let us multiply (36) by $E^\lambda_{\alpha\beta\varepsilon}$ and differentiate it in the index $\varepsilon$.

$$E^\lambda_{\alpha\beta\varepsilon}(j_\beta;\alpha - j_\alpha;\beta)_\varepsilon = 4m\cdot E^\lambda_{\alpha\beta\varepsilon}h_{\alpha\beta;\varepsilon} + E^\lambda_{\alpha\beta\varepsilon}E^\nu_{\alpha\beta\mu}\frac{1}{4}\text{Sp}\left\{\left(\nabla_\nu Z^+\right)\bar{D}\gamma_5 \gamma^\mu Z - Z^+\bar{D}\gamma_5 \gamma^\mu \left(\nabla_\nu Z\right)\right\}_\varepsilon.$$  \hspace{1cm} (43)

The term in the left-hand member of (43) identically vanishes. In order to transform the right-hand member, we use the formula

$$-4m\cdot E^\lambda_{\alpha\beta\varepsilon}h_{\alpha\beta;\varepsilon} = \frac{1}{2}\text{Sp}\left\{\left(\nabla^\lambda Z^+\right)\bar{D}\gamma_5 \gamma^\lambda Z - Z^+\bar{D}\gamma_5 \gamma^\lambda \left(\nabla^\lambda Z\right)\right\}.$$  \hspace{1cm} (44)

As a result:

$$E^\lambda_{\alpha\beta\varepsilon}E^\nu_{\alpha\beta\mu} = -2\delta^\lambda_\mu\eta^\nu_\varepsilon + 2\delta^\varepsilon_\mu\eta^\lambda_\nu.$$  \hspace{1cm} (45)

Derivatives in (44) are developed.

$$-4m\cdot E^\lambda_{\alpha\beta\varepsilon}h_{\alpha\beta;\varepsilon} = \frac{1}{2}\text{Sp}\left\{\left(\nabla^\nu \nabla_\nu Z^+\right)\bar{D}\gamma_5 \gamma^\lambda Z - Z^+\bar{D}\gamma_5 \gamma^\lambda \left(\nabla^\nu \nabla_\nu Z\right)\right\} + \frac{1}{2}\text{Sp}\left\{\left(\nabla^\mu \nabla^\nu Z^+\right)\bar{D}\gamma_5 \gamma^\nu Z - Z^+\bar{D}\gamma_5 \gamma^\nu \left(\nabla^\mu \nabla^\nu Z\right)\right\} + \frac{1}{2}\text{Sp}\left\{\left(\nabla^\nu Z^+\right)\bar{D}\gamma_5 \gamma^\nu Z - Z^+\bar{D}\gamma_5 \gamma^\nu \left(\nabla^\nu Z\right)\right\}.$$  \hspace{1cm} (45)

Relationship (45) can be simplified if we use relationships (9) and (17).
4m \cdot E^{\alpha\beta\epsilon} h_{\alpha\beta\epsilon} = -\frac{1}{2} \text{Sp} \left\{ \frac{1}{2} P_{\mu\nu} Z^+ \tilde{D}_{\gamma_5} \tilde{S}^{\mu\nu} \gamma^\lambda Z - \frac{1}{2} Z^+ \tilde{D}_{\gamma_5} \tilde{g}^\lambda \tilde{S}^{\mu\nu} Z P_{\mu\nu} \right\} \\
+ \frac{1}{2} \text{Sp} \left\{ - \left( \nabla^\lambda \nabla_\mu Z^+ \right) \tilde{D}_{\gamma_5} \tilde{g}^\mu Z - Z^+ \tilde{D}_{\gamma_5} \tilde{g}^\mu \left( \nabla^\lambda \nabla_\mu Z \right) - 2 P^\lambda \cdot \left( Z^+ \tilde{D}_{\gamma_5} \tilde{g}^\mu Z \right) \right\} \\
+ \frac{1}{2} m \cdot \text{Sp} \left\{ \left( \nabla^\lambda Z^+ \right) \tilde{D}_{\gamma_5} Z - Z^+ \tilde{D}_{\gamma_5} \left( \nabla^\lambda Z \right) \right\}. 

\text{(46)}

The terms with second derivatives in (46) are transformed.

\[ E^{\alpha\beta\epsilon} h_{\alpha\beta\epsilon} = \frac{1}{4} \cdot \text{Sp} \left\{ \left( \nabla^\lambda Z^+ \right) \tilde{D}_{\gamma_5} Z - Z^+ \tilde{D}_{\gamma_5} \left( \nabla^\lambda Z \right) \right\}. \tag{47} \]

Thus, we obtain an equality, which coincides with the statement of Theorem III, i.e. with equality (42).

### 2.6 Theorem IV

Theorem IV proves that the relationship

\[ h_{\alpha\nu} = -\frac{1}{8} \text{Sp} \left\{ \left( \nabla_\alpha Z^+ \right) \tilde{D} Z - Z^+ \tilde{D} \left( \nabla_\alpha Z \right) \right\} - m \cdot j_\alpha \tag{48} \]

holds.

The proof will consist in the direct validation of (48).

\[ h_{\alpha\nu} = -\frac{1}{8} \text{Sp} \left\{ Z^+ \tilde{D} \tilde{S}_\alpha Z \right\} = \\
= -\frac{1}{8} \text{Sp} \left\{ \left( \nabla_\alpha Z^+ \right) \tilde{D} \tilde{S}_\alpha Z \right\} - \frac{1}{8} \text{Sp} \left\{ Z^+ \tilde{D} \tilde{S}_\alpha \left( \nabla_\alpha Z \right) \right\}. \tag{49} \]

The matrix \( \tilde{S}_\alpha \) in the first case is replaced with

\[ \tilde{S}_\alpha = \delta_\alpha^\nu - \tilde{g}_\nu \tilde{\gamma}_\alpha, \tag{50} \]

and in the second case, with

\[ \tilde{S}_\alpha = -\delta_\alpha^\nu + \tilde{\gamma}_\alpha \tilde{\gamma}_\nu. \tag{51} \]

Hence:
Using Eqs. (3), (4), we obtain:

\[ h_{\alpha \nu} = -\frac{1}{8} \text{Sp} \left\{ (\nabla_\nu Z^+) \tilde{D} (\delta^\nu - \tilde{\gamma}^\nu \tilde{\gamma}_\alpha) Z \right\} \]

\[ -\frac{1}{8} \text{Sp} \left\{ Z^+ \tilde{D} (-\delta^\nu + \tilde{\gamma}_\alpha \tilde{\gamma}^\nu) (\nabla_\nu Z) \right\} = \]

\[ = -\frac{1}{8} \text{Sp} \left\{ (\nabla_\nu Z^+) \tilde{D} (\delta^\alpha) Z \right\} - \frac{1}{8} \text{Sp} \left\{ (\nabla_\nu Z^+) \tilde{D} (-\tilde{\gamma}^\nu \tilde{\gamma}_\alpha) Z \right\} \]

\[ = -\frac{1}{8} \text{Sp} \left\{ Z^+ \tilde{D} (-\delta^\nu) (\nabla_\nu Z) \right\} - \frac{1}{8} \text{Sp} \left\{ Z^+ \tilde{D} (\tilde{\gamma}^\nu \tilde{\gamma}_\alpha) (\nabla_\nu Z) \right\} \]

\[ + \frac{1}{8} \text{Sp} \left\{ (\nabla_\nu Z^+) \tilde{D} \tilde{\gamma}^\nu \tilde{\gamma}_\alpha Z \right\} - \frac{1}{8} \text{Sp} \left\{ Z^+ \tilde{D} \tilde{\gamma}_\alpha \tilde{\gamma}^\nu (\nabla_\nu Z) \right\} \]  

(52)

Thus, it is proven that relationship (48) indeed follows from the Dirac equation, i.e. Theorem IV is proven.

### 3 CGD Equations in Plane Space

Conformal geometrodynamics equations in the general case are equations

\[ R_{\alpha \beta} - \frac{1}{2} g_{\alpha \beta} R = -2 A_{\alpha} A_{\beta} - g_{\alpha \beta} A^2 + 2 g_{\alpha \beta} A^\nu_{\mu} + A_{\alpha \beta} + A_{\beta \alpha} + \lambda g_{\alpha \beta}. \]  

(54)

The right-hand member in Eqs. (54), i.e. the tensor

\[ T_{\alpha \beta} = -2 A_{\alpha} A_{\beta} - g_{\alpha \beta} A^2 - 2 g_{\alpha \beta} A^\nu_{\mu} + A_{\alpha \beta} + A_{\beta \alpha} + \lambda g_{\alpha \beta}, \]  

(55)

is considered to be the energy-momentum tensor of geometrodynamic continuum. A corollary of Eq. (54) and Bianchi identities is that

\[ T_{\alpha ; \beta} = 0. \]  

(56)

It follows from Eqs. (56) that the tensor \( F_{\alpha \beta} \) defined as

\[ F_{\alpha \beta} = A_{\beta , \alpha} - A_{\alpha , \beta}, \]  

(57)

should satisfy the equation
\[ F_{\alpha\beta} = \lambda_\alpha - 2\lambda A_\alpha. \] (58)

Let us express all these relationships for plane space in Cartesian coordinates chosen as world coordinates for the case, when the metric tensor is given by

\[ g_{\alpha\beta} \equiv \eta_{\alpha\beta} = \text{diag} [-1, 1, 1, 1], \] (59)

We write the full set of dynamic equations and the gauge condition, which will be solved below for the flat Riemann space:

\[
\begin{align*}
A_{\beta,\alpha} - A_{\alpha,\beta} &= F_{\alpha\beta} \\
F_{\alpha\beta} &= \lambda_\alpha - 2\lambda A_\alpha \\
\lambda &= \text{Const}, \quad A_{\nu} = 0 \\
g_{\alpha\beta} &= \eta_{\alpha\beta}
\end{align*}
\] (60)

The gauge condition here is

\[ \lambda = \text{Const.} \] (61)

A special feature of condition (61) is that when it is fulfilled, the relationship

\[ A_{\nu} = 0, \] (62)

which is also part of (60), is fulfilled as well.

In principle, one can exclude \( F_{\alpha\beta} \)'s from (60) and reduce all the equations to the second-order equation for \( A_\alpha \) supplemented with the gauge condition. We will use a different approach, in which all the eleven functions of \( A_\alpha, F_{\alpha\beta}, \lambda \) versus four variables are treated as independent, and equations (60) are treated as a set of eleven first-order partial differential equations for these functions. Such an approach is similar to the representation of the Klein-Gordon equations as a set of first-order equations, see Section 4.4 in [16].

A direct check can prove that for each solution of the CGD equations one can introduce a dimensionless vector

\[ J^\alpha \equiv \frac{k}{\lambda^{3/2}} (\nabla^\alpha \lambda) = \frac{k}{\lambda^{3/2}} g^{\alpha\beta} (\lambda_\beta - 2\lambda A_\beta) \] (63)

with the Weyl weight of -1 and a dimensionless anti-symmetric tensor
\[ H^{\alpha \beta} \equiv \frac{\theta}{\lambda} F^{\alpha \beta} = \frac{\theta}{\lambda} g^{\alpha \mu} g^{\beta \nu} (A_{\nu, \mu} - A_{\mu, \nu}) \]  \hspace{1cm} (64)

with the Weyl weight of -2. The quantities \( \kappa \) in (63) and \( \theta \) in (64) are some constant coefficients.

As applied to the vector \( J_\alpha \), gauge condition (60) is written as

\[ J^\alpha_{;\alpha} = 0. \]  \hspace{1cm} (65)

Since \( \lambda = \text{Const} \) in such gauging, the vector \( J_\alpha \) and the tensor \( H_{\alpha \beta} \) are related to the initial quantities \( A_\alpha, F_{\alpha \beta}, \lambda \) as

\[ A_\alpha = -\frac{\lambda^{1/2}}{2\kappa} J_\alpha; \quad F_{\alpha \beta} = \frac{\lambda}{\theta} H_{\alpha \beta}, \]  \hspace{1cm} (66)

and to each other, as

\[ (J_{\beta;\alpha} - J_{\alpha;\beta}) = -\frac{2\kappa}{\theta} \lambda^{1/2} H_{\alpha \beta}, \]  \hspace{1cm} (67)

\[ H_{\alpha;\beta} = \frac{\theta}{\kappa} \lambda^{1/2} \cdot J_\alpha. \]  \hspace{1cm} (68)

It follows from equality (67) that the tensor \( H_{\alpha \beta} \) satisfies four identities

\[ E^{\alpha \beta \mu \nu} H_{\beta \mu, \nu} \equiv 0. \]  \hspace{1cm} (69)

Consider the Cauchy problem for the vector \( J_\alpha \) and tensor \( H_{\alpha \beta} \). For this purpose, relationships (65)-(69) are written in Cartesian coordinates with differentiation of space and time indices.

Dynamic equations:

\[ [J_{\alpha,0} - J_{0,\alpha}] = -\frac{2\kappa}{\theta} \lambda^{1/2} H_{0k}, \]  \hspace{1cm} (70)

\[ H_{0k,0} + H_{kp,p} = \frac{\theta}{\kappa} \lambda^{1/2} \cdot J_k, \]  \hspace{1cm} (71)

\[ J_{0,0} = J_{p,p}, \]  \hspace{1cm} (72)

\[ H_{pq,0} = -H_{0p,q} + H_{0q,p}. \]  \hspace{1cm} (73)

Coupling equations:
\[ [J_{n,m} - J_{m,n}] = -\frac{2\kappa}{\theta} \lambda^{1/2} H_{mn}, \quad (74) \]

\[ H_{0p,p} = \frac{\theta}{\kappa} \lambda^{1/2} \cdot J_0, \quad (75) \]

\[ H_{pq,k} = -H_{kp,q} + H_{kq,p}. \quad (76) \]

Due to their structure, relationships (70)-(73) can be considered a set of first-order differential equations with respect to the desired functions \( J_0, J_k, H_{0k}, H_{mn} \). These functions are basically Cauchy data. Indeed, time derivatives of these functions are defined by dynamic equations (70), (71), (72), (73). As for relationships (74), (75), (76), these are connections to the Cauchy data. Relationship (76), which is fulfilled automatically if relationship (74) is fulfilled, can be excluded from the list of connections.

Such a statement of the Cauchy problem would be correct if connections (74), (75) were "drawn" into volume by virtue of dynamic equations (70), (71), (72), (73). It is easy to show (we will not do this here) that this situation applies.

4 Conditions of Coincidence of CGD Equations and Dirac Equation Corollaries

4.1 Comparison of Equations

1 shows the CGD equations and respective equations, which are direct corollaries of the Dirac equation.

Let us question ourselves, what conditions should be satisfied for the corresponding (i.e. given in the same lines of 1) equations to coincide. It is clear that the following relationships should be fulfilled in the first place:

\[ H_{\alpha\beta} = h_{\alpha\beta}, \quad J_\alpha = j_\alpha. \quad (77) \]

and in the second place:

\[ m = -(\kappa/2\theta) \lambda^{1/2}, \quad m = -(\theta/\kappa) \lambda^{1/2}. \quad (78) \]
**Table 1:** – Comparison of CGD equations with Dirac equation corollaries

| Dirac equation corollaries                                                                 | CGD equations                                                                 |
|------------------------------------------------------------------------------------------|-------------------------------------------------------------------------------|
| $J_{\alpha} = 0$                                                                          | $J_{\alpha} = 0$                                                              |
| Relationship (35)                                                                         | Relationship (63)                                                             |
| $(j_{\beta,\alpha} - j_{\alpha,\beta}) = 4m \cdot h_{\alpha\beta} + E_{\alpha\beta\mu} \frac{1}{4} \text{Sp}\left\{ (\nabla_\nu Z^+) \tilde{D} \tilde{\gamma}_5 \tilde{\gamma}^\mu Z - Z^+ \tilde{D} \tilde{\gamma}_5 \tilde{\gamma}^\mu (\nabla_\nu Z) \right\}$ | $(J_{\beta\alpha} - J_{\alpha\beta}) = -\frac{2\kappa}{\theta} \lambda^{1/2} H_{\alpha\beta}$ |
| Relationships (56)                                                                        | Relationships (67)                                                            |
| $h_{\alpha,\nu} = -m \cdot j_{\alpha}$                                                   | $H_{\alpha,\beta} = \frac{\theta}{\kappa} \lambda^{1/2} \cdot J_{\alpha}$    |
| $-\frac{1}{8} \text{Sp}\left( (\nabla_\alpha Z^+) \tilde{D} Z - Z^+ \tilde{D} (\nabla_\alpha Z) \right)$ | Equations (48)                                                               |
| Equations (48)                                                                            | $E_{\alpha\beta\mu} H_{\beta\mu,\nu} \equiv 0$                             |
| $E_{\alpha\beta\mu} h_{\alpha\beta\nu} = \frac{1}{4} \text{Sp}\left\{ (\nabla^\lambda Z^+) \tilde{D} \tilde{\gamma}_5 Z - Z^+ \tilde{D} \tilde{\gamma}_5 (\nabla^\lambda Z) \right\}$ | Equations (69)                                                               |

Conditions (78) are satisfied if the following relationship between the coefficients $\kappa$ and $\theta$ is fulfilled:

$$\kappa = -\theta \sqrt{2}. \quad (79)$$

The mass and the quantity $\lambda$ should be related as

$$m = \sqrt{\lambda/2}. \quad (80)$$

The third condition is that the following three quantities should vanish:

$$E_{\alpha\beta\mu} \text{Sp}\left\{ (\nabla_\nu Z^+) \tilde{D} \tilde{\gamma}_5 \tilde{\gamma}^\mu Z - Z^+ \tilde{D} \tilde{\gamma}_5 \tilde{\gamma}^\mu (\nabla_\nu Z) \right\} = 0, \quad (81)$$

$$\text{Sp}\left( (\nabla_\alpha Z^+) \tilde{D} Z - Z^+ \tilde{D} (\nabla_\alpha Z) \right) = 0, \quad (82)$$

$$\text{Sp}\left\{ (\nabla^\lambda Z^+) \tilde{D} \tilde{\gamma}_5 Z - Z^+ \tilde{D} \tilde{\gamma}_5 (\nabla^\lambda Z) \right\} = 0. \quad (83)$$

In order to ascertain the meaning of relationships (81) - (83), we introduce a matrix

$$n_\alpha \equiv Z^{-1} Z_{,\alpha}. \quad (84)$$

These relationships will change to:
\[ \text{Sp} \{ \Gamma_\alpha D \gamma_5 \gamma^\mu \} = \text{Sp} \left\{ \frac{1}{2} \left( n_\alpha - n_\alpha^+ \right) D \gamma_5 \gamma^\mu \right\}, \quad (85) \]

\[ \text{Sp} \{ \Gamma_\alpha D \} = \text{Sp} \left\{ \frac{1}{2} \left( n_\alpha - n_\alpha^+ \right) D \right\}, \quad (86) \]

\[ \text{Sp} \{ \Gamma_\alpha D \gamma_5 \} = \text{Sp} \left\{ \frac{1}{2} \left( n_\alpha - n_\alpha^+ \right) D \gamma_5 \right\}. \quad (87) \]

The quantities \( D, \gamma_5, \gamma^\mu \) in relationships (85)-(87) mean \( D = Z + \tilde{D} Z, \gamma_5 = Z^{-1} \tilde{\gamma}_5 Z, \gamma^\mu = Z^{-1} \tilde{\gamma}^\mu Z \).

In the general case, the matrices \( \Gamma_\alpha \) can be expanded over the whole set of antisymmetric real matrices related with the DM \( \gamma^\mu = Z^{-1} \tilde{\gamma}^\mu Z \).

\[ \Gamma_\alpha = U_\alpha(x) \cdot D^{-1} + V_\alpha(x) \cdot \gamma_5 D^{-1} + W_\alpha^\beta(x) \cdot \gamma_5 \gamma_\beta D^{-1}. \quad (88) \]

The expansion coefficients \( U_\alpha(x), V_\alpha(x), W_\alpha^\beta(x) \) in (88) are expressed as

\[
\begin{align*}
U_\alpha(x) &= \frac{1}{8} \text{Sp} \{ \Gamma_\alpha D \} \\
V_\alpha(x) &= \frac{1}{8} \text{Sp} \{ \Gamma_\alpha D \gamma_5 \} \\
W_\alpha^\beta(x) &= \frac{1}{8} \text{Sp} \{ \Gamma_\alpha D \gamma_\beta \gamma_\gamma \}.
\end{align*}
\]

Comparing relationships (89) with (85)-(87) shows that relationships (85)-(87) define all the expansion coefficients of matrix connectivity, i.e. coefficients \( U_\alpha(x), V_\alpha(x), W_\alpha^\beta(x) \).

\[
\begin{align*}
U_\alpha(x) &= \frac{1}{8} \text{Sp} \left\{ \left( n_\alpha - n_\alpha^+ \right) D \right\} \\
V_\alpha(x) &= \frac{1}{8} \text{Sp} \left\{ \left( n_\alpha - n_\alpha^+ \right) D \gamma_5 \right\} \\
W_\alpha^\beta(x) &= \frac{1}{8} \text{Sp} \left\{ \left( n_\alpha - n_\alpha^+ \right) D \gamma_\beta \gamma_\gamma \right\}.
\end{align*}
\]

Thus, if we know \( Z \) and matrix (84) at some time, we can find the field of matrix connectivity \( \Gamma_\alpha \) using formulas (90).

Thus, we have proven that the quantities \( j_\alpha, h_{\alpha \beta} \) constructed based on formulas (18), (19) from the bispinor matrix \( Z \) governed by Dirac equation (3) can be identified with the quantities \( J_\alpha, H_{\alpha \beta} \), which define the solution of the CGD equations provided that the following conditions are fulfilled:

The dimensionless constants \( \kappa \) and \( \theta \) are related as (79); The mass \( m \) is related to the constant \( \lambda \) as (80); The matrix connectivity \( \Gamma_\alpha \) is defined by (88), (90).
In conclusion of this section, for convenience of reference, let us list the expressions, which relate the vector $J_\alpha$ and the tensor $H_{\alpha\beta}$ to the initial quantities $A_\alpha, F_{\alpha\beta}, \lambda$ and to each other, and which result from taking account of (79), (80), (88), (90).

$$J_\alpha = \frac{2\theta}{m} A_\alpha; \quad H_{\alpha\beta} = \frac{\theta}{2m^2} F_{\alpha\beta},$$

$$m = \text{Const}; \quad J_\alpha^\alpha = 0,$$

$$J_{\beta;\alpha} - J_{\alpha;\beta} = 4m \cdot H_{\alpha\beta},$$

$$H_{\alpha;\beta} = -m \cdot J_\alpha,$$

$$E^{\alpha\beta\mu\nu} H_{\beta\mu,\nu} \equiv 0.$$

4.2 Cauchy Problem

If conditions i-4.2 are fulfilled, the statement of the Cauchy problem for the whole set of fields differs from the problem statement used in the standard version of the quantum field theory (QFT). The major difference lies in the way of finding $\Gamma_\alpha$. In QFT, this quantity is found from the Yang-Mills equations, in which the sources are current vectors. In our case, the matrix connectivity $\Gamma_\alpha$ is derived from relationships (90). This is related to the following:

If conditions i-4.2 are fulfilled, equations for the quantities $j_\alpha, h_{\alpha\beta}$ at each time have an autonomous form. Moreover, these equations allow for the correct statement of the Cauchy problem for the quantities $j_\alpha, h_{\alpha\beta}$.

Using the quantities $j_\alpha, h_{\alpha\beta}$ obtained, the Hermitian matrix $M$ (see (20)) is set up, following which the matrix $Z$ is found as a solution of algebraic equation (21).

Using the explicit form of $Z$, we can calculate its derivatives and substitute these into relationship (90). These particular relationships yield expansion coefficients of the matrix connectivity $\Gamma_\alpha$ for the full set of antisymmetric matrices (88).
The matrix connectivity $\Gamma_\alpha$ calculated using formulas (88) possesses all
the attributes of a gauge field corresponding to the group $SO(4)$. Indeed, it
is easy to demonstrate that gauge transformations

$$Z \rightarrow Z' = ZU^{-1}, \quad Z^+ \rightarrow Z'^+ = UZ^+$$

(96)

($U$ are real unitary matrices) transform the matrix connectivity $\Gamma_\alpha$ in
accordance with the law

$$\Gamma_\alpha \rightarrow \Gamma'_\alpha = U\Gamma_\alpha U^{-1} + UU_{-\alpha}^{-1},$$

which is standard for gauge fields.

5 Discussion of Results

This paper proposes a solution to one of the matters long discussed by physi-
crists and mathematicians – the matter of physical interpretation of the Weyl
degrees of freedom of space. Different attempts have been made in this area,
but no significant progress has been achieved so far. One of the approaches
provided for relating the Weyl degrees of freedom either to the parameters of
dark matter and energy in the Universe, or to the cosmological red shift, or
with scaling of spacetime slice measurements ([7] - [13]). Another approach
treated new degrees of freedom as an attribute of the integrated Weyl space
(i.e. space, in which the Weyl vector is a scalar function gradient) leading to
the appearance of the Shroedinger equation ([14], [15]).

The interpretation of the Weyl degrees of freedom proposed in this paper
is fundamentally new. We propose interpreting these degrees of freedom as
those, which reproduce the polarization density matrix $M$ of the whole set
of half-integer spin fields obeying the Dirac equation. The coincidence of
the CGD equations with the Dirac equation corollaries, which is discussed
in this paper, in our opinion, is a strong proof of viability of the proposed
interpretation.

Dynamics of the Weyl degrees of freedom is described by equations (70)-
(76) for any structure of the polarization matrix $M$. However, the quantum-
field interpretation of solutions of these equations has limited applicability;
it can be used only in the cases when all the eigenvalues of the matrix $M$ are
positive. This condition corresponds to usual requirements applied to polar-
ization density matrices in quantum mechanics and quantum field theory.
The coincidence of the CGD equations with the Dirac equation corollaries allows us to draw a conclusion that bispinor states in the quantum field theory are just a different language to describe dynamics of the Weyl degrees of freedom in some regions of space, where all the eigenvalues of the matrix $M$ are positive. When we solve the CGD equations in these regions, we in fact solve a quantum field problem, in which half-integer spin particles reproduce two quantities of the CGD equations: the vector $J_\alpha$ and the anti-symmetric tensor $H_{\alpha\beta}$.

In conclusion, let us express our vision of where our results are among different attempts to unify physical interactions. In our opinion, the outcome of this study can be used in theoretical validation of the Standard Model (SM) of elementary particles and, in particular, the confinement model. Indeed, production and annihilation operators for particles with spin $\frac{1}{2}$ and different sets of other quantum numbers are obtained by multiplying the bispinor matrix on the right by projection operators. The set of projection operators includes $P_{\pm} = \frac{1}{2}[E \pm I]$, where $I$ is the reflection matrix in the group of invariant transformations $O(4)$ - see [17]. The projection operators $P_{\pm}$ have ranks 1 and 3, and allow splitting four bispinors $ZP_{\eta,\lambda}$ ($\eta, \lambda = \pm$) into two groups. One group comprises three states that differ in two quantum numbers. The second group includes one state with zero quantum numbers (the "sterile" state similar to the right neutrino). Such splitting of states coincides with splitting in each generation of leptons and quarks within the SM. In the context of understanding the confinement phenomenon, it is important that bispinor and matrix connectivities exist only in the space regions, where the polarization density matrix is positively defined. Apparently, different states in such space-localized fields of the bispinor matrix and matrix connectivity can be correlated with different elementary particles.

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