GREEDY APPROACHES TO ONLINE STOCHASTIC MATCHING

ALLAN BORODIN, CALUM MACRURY, AND AKASH RAKHEJA

ABSTRACT. Within the context of stochastic probing with commitment, we consider the online stochastic matching problem; that is, the one-sided online bipartite matching problem where edges adjacent to an online node must be probed to determine if they exist based on edge probabilities that become known when an online vertex arrives. If a probed edge exists, it must be used in the matching (if possible). We consider the competitiveness of online algorithms in the random order input model (ROM), when the offline vertices are weighted. More specifically, we consider a bipartite stochastic graph $G = (U, V, E)$ where $U$ is the set of offline vertices, $V$ is the set of online vertices and $G$ has edge probabilities $(p_e)_{e \in E}$ and vertex weights $(w_u)_{u \in U}$. Additionally, $G$ has patience values $(\ell_v)_{v \in V}$, where $\ell_v$ indicates the maximum number of edges adjacent to an online vertex $v$ which can be probed. We assume that $U$ and $(w_u)_{u \in U}$ are known in advance, and that the patience, adjacent edges and edge probabilities for each online vertex are only revealed when the online vertex arrives. If any one of the following three conditions is satisfied, then there is a conceptually simple deterministic greedy algorithm whose competitive ratio is $1 - \frac{1}{e}$.

- When the offline vertices are unweighted.
- When the online vertex probabilities are “vertex uniform”; i.e., $p_{u,v} = p_v$ for all $(u, v) \in E$.
- When the patience constraint $\ell_v$ satisfies $\ell_v \in \{1, |U|\}$ for every online vertex; i.e., every online vertex either has unit or full patience.

As a consequence, by a result of Karande et al. [11], the same bounds are achieved when the online (stochastic) vertices arrive i.i.d. from an unknown distribution.

Setting the probability $p_e = 1$ for all $e \in E$, the stochastic problem becomes the classical online bipartite matching problem. Our competitive ratios thus generalize corresponding results for the classical ROM bipartite matching setting. Our result for stochastic matching with arbitrary patience is motivated by the primal-dual analysis of Devanur et al. [7] for the classical vertex weighted case. The competitive ratio is with respect to an LP relaxation of an ideal benchmark (the optimum offline probing algorithm).

We contrast the simplicity of our efficient deterministic greedy algorithm with the non-greedy randomized algorithms based on solving LPs when the stochastic graph is known in advance.
1. Introduction

Stochastic probing problems are part of the larger area of decision making under uncertainty and more specifically, stochastic optimization. Unlike more standard forms of stochastic optimization, it is not just that there is some stochastic uncertainty in the set of inputs, stochastic probing problems involve inputs that cannot be determined without probing (at some cost and/or within some constraint) so as to reveal the inputs. Applications of stochastic probing occur naturally in many settings, such as in matching problems where compatibility (for example, in online dating and kidney exchange applications) cannot be determined without some trial or investigation. Amongst other applications, the online bipartite stochastic matching problem notably models online advertising where the probability of an edge can correspond to the probability of a purchase in online stores or to pay per click revenue in online searching.

The stochastic matching problem was introduced by Chen et al. [5]. In this problem, we are given an adversarially generated stochastic graph $G = (V, E)$ with a probability $p_e$ associated with each edge $e$ and a patience (or time-out) parameter $\ell_v$ associated with each vertex $v$. An algorithm probes edges in $E$ within the constraint that at most $\ell_v$ edges are probed incident to any particular vertex $v \in V$. Also, when an edge $e$ is probed, it is guaranteed to exist with probability exactly $p_e$. If an edge $(u, v)$ is found to exist, it is added to the matching and then $u$ and $v$ are no longer available. The goal is to maximize the expected size of a matching constructed in this way. This problem can be generalized to vertices or edges having weights. We shall refer to this setting as the known stochastic graph setting.

In addition to improving upon the results of Chen et al., Bansal et al. [2] introduced an i.i.d. bipartite version of the problem where nodes on one side of the partition arrive online and edges adjacent to that node are then probed. In their model, the “type” of each online node (i.e., the adjacent edge probabilities, edge weights and patience value) is determined i.i.d. from a known distribution and each offline node has unlimited patience. We can refer to this setting of a known distribution as bipartite matching with a known stochastic type graph. As in other online bipartite matching problems, the match for an online node must be made before the next online arrival. As in Chen et al., the first edge that is successfully probed must be included in the matching.

Mehta and Panigrahi [16] adapted the stochastic matching model for online bipartite matching as originally studied in the classical (non-stochastic) adversarial order online model. That is, they consider the setting where the stochastic graph is unknown and online vertices are determined by an adversary. More specifically, they studied the problem in the case of an unweighted stochastic graph $G = (U, V, E)$ where $U$ is the set of offline vertices and the vertices in $V$ arrive online without knowledge of future online node arrivals. They considered the special case of uniform edge probabilities (i.e., $p_e = p$ for all $e \in E$) and unit patience values, that is $\ell_v = 1$ for all $v \in V$. Mehta et al. [17] considered this online stochastic bipartite setting with arbitrary edge probabilities, and very recently, Huang and Zhang [10] additionally handled the case of arbitrary offline vertex weights, although both works are also restricted to unit patience values, and require edge probabilities which are vanishingly small. Brubach et al. [4] recently considered the problem of generalizing to arbitrary patience values, and our work most closely follows their results.

This problem is sometimes referred to as the stochastic rewards problem. Amongst other applications, the stochastic rewards problem notably models online advertising where the probability of

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1. Unfortunately, the term “stochastic matching” is also used to refer to more standard optimization where the inputs (i.e., edges or vertices) are drawn from some known or unknown distributions but no probing is involved.
2. If the order of edge probes is non-adaptive (i.e., does not depend on previous matches), then the edge is included only if the offline node is available; that is, not already in the current matching.
3. Vanishingly small edge probabilities must satisfy $\max_{e \in E} p_e \to 0$, where the asymptotics are with respect to the size of $G$.
4. The stochastic rewards problem is sometimes meant to imply unit patience but we will be careful to state whether or not we are considering unit, full, or arbitrary patience values.
an edge can correspond to the probability of a purchase in online stores or to pay per click revenue in online searching. As in all these settings, we will assume commitment; that is, when an edge is probed and found to exist, it must be included in the matching (if possible without violating the matching constraint). The patience constraint can be viewed as a simple form of a budget constraint for the online vertices.

As stated, we will assume the random order model (ROM), where the order of online vertex arrivals is determined uniformly at random. For random order or adversarial order of online vertex arrivals, results for the online Mehta and Panigrahi model (even for unit patience) generalize the corresponding classical non-stochastic models where edges adjacent to an online node are known upon arrival and do not need to be probed. It follows that any in-approximations in the classical setting apply to the corresponding stochastic setting. Further generalizing the classical setting, when the stochastic graph is unknown, a competitive ratio for the random order model implies that the same ratio is obtained in the stochastic i.i.d. model (for an unknown distribution) as proven in the classical setting by Karande et al. [11].

In a related paper [3], we consider the setting when the stochastic graph is unknown, but there is a known stochastic type graph which arrivals are drawn i.i.d. from, as in the model introduced by Bansal et al. [2]. We also consider the bipartite stochastic matching problem when the stochastic graph is known and online vertices arrive in random order. In the latter setting, we can achieve a $1 - \frac{1}{e}$ competitive ratio for edge weighted graphs. However, the algorithm we use in that setting requires full knowledge of the stochastic graph $G$, and involves solving an appropriate exponentially sized LP and then randomly rounding. While this procedure is implemented in poly-time using the ellipsoid algorithm, it is substantially less efficient than our deterministic greedy algorithm for the offline vertex weighted case which we present in Section 2. It is, of course, well known that even in the classical non-stochastic setting, we cannot achieve a ratio better than $\frac{1}{e}$ in the random order model when edges are weighted, and $G$ is unknown. Following the deterministic matching algorithm of Kesselheim et al. [13] for the classical non-stochastic setting, in [3] we provide a $\frac{1}{e}$ competitive randomized algorithm in the random order model for edge weighted unknown stochastic graphs. That result also requires solving LPs and randomized rounding. In stating competitive results for stochastic probing problems, we need to define what are the benchmarks against which an online algorithm is being measured. We defer this discussion until Section 2.

Finally, we note that there is an extensive literature for stochastic matching problems. An extended overview of related work appears in [3]. Research most directly relating to this paper will appear as we proceed.

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5It is not clear that competitive ratios for stochastic matching with a known stochastic graph imply the same ratios for the i.i.d setting with a known type graph.
2. Preliminaries and Techniques

Let $G = (U, V, E)$ be a bipartite graph with offline vertex weights $(w_u)_{u \in U}$ and edge probabilities $(p_e)_{e \in E}$. We assume that the online nodes of $V$ are equipped with integer values, denoted $(\ell_v)_{v \in V}$, which we refer to as the patience values of the online nodes. If all of these parameters are associated with $G$ then we refer to it as a (bipartite) stochastic graph.

Given a stochastic graph $G$, we draw an independent Bernoulli random variable of parameter $p_e$ for each $e \in E$. We refer to this Bernoulli as the state of the edge $e$, and denote it by $st(e)$. If $st(e) = 1$, then we say that $e$ is active, and otherwise we say that $e$ is inactive. It is convenient to assume that $E = U \times V$. In this way, if we wish to exclude a pair $(u, v) \in U \times V$ from existing as an edge in $G$, then we may set $p_{u,v} = 0$, thus ensuring that $(u, v)$ is always inactive.

In the unknown stochastic graph model, the online nodes of the stochastic graph are not initially available to an online probing algorithm. Instead, only $U$ and the vertex weights $(w_u)_{u \in U}$ are initially known to the algorithm. We think of $V$, as well as the relevant edges probabilities, as being generated by an adversary. An ordering on $V$ is then generated either through an adversarial process or uniformly at random. We refer to the former case as the adversarial setting and the latter case as the ROM setting.

Based on whichever ordering is generated on $V$, the nodes are then presented to the online probing algorithm one by one. When an online node $v \in V$ arrives, the online probing algorithm sees all the adjacent edges and their associated probabilities, as well as $\ell_v$. However, the algorithm must perform a probing operation on an adjacent edge $e$ to reveal/expose its state, $st(e)$. The patience parameter $\ell_v$ of the online node then bounds the number of probing operations that can be made to edges adjacent to $v$.

As in the classical problem, an online probing algorithm must decide on a possible match for an online node $v$ before seeing the next online node. In fact, the online probing algorithm must respect commitment. That is, if an edge $e = (u, v)$ is probed and turns out to be active, then $e$ must be added to the current matching, provided $u$ and $v$ are both currently unmatched. Observe that the online stochastic matching problem generalizes the classical online problem, even when restricted to the case of unit patience (i.e., $\ell_v = 1$ for all $v \in V$). The goal of the online probing algorithm is to return a matching whose expected weight is as large as possible.

In the classical online competitive analysis setting, we compare the value of the online algorithm to that of an optimal matching of the graph. For stochastic probing problems, it is easy to see we cannot hope to obtain a reasonable competitive bound for this type of comparison; that is, if we are comparing the expected value of an online probing algorithm to the expected value of an optimum matching of the stochastic graph. For example, consider a single online vertex with patience 1, and $n$ offline (unweighted) vertices where each edge $e$ has probability $\frac{1}{n}$ of being present. The expectation of an online probing algorithm will be at most $\frac{1}{n}$ while the expected size of an optimal matching (over all instantiations of the edge probabilities) will be $1 - (1 - \frac{1}{n})^n \to 1 - \frac{1}{e}$. This example clearly shows that no constant ratio is possible if the patience is sublinear (in $n = |U|$).

A reasonable approach is to force the benchmark to adaptively probe edges subject to the patience and commitment constraints of an online probing algorithm. Specifically, knowing the stochastic graph $G$, and the patience requirements of the online nodes, the benchmark can probe edges in any adaptive order but must satisfy the patience requirements of the online vertices. By adaptive order, we mean that the next edge to be probed will depend on all the edges that have currently been revealed and the current matching. We emphasize that this benchmark is not restricted to any ordering of the online vertices. In particular, we note that after probing some edge $(w_1, v_1)$, the next probed edge can be $(w_2, v_2)$ where $w_2$ and $v_2$ each may be distinct from $w_1$ and $v_1$, respectively. The benchmark also respects commitment, in that if it probes an edge $e = (u, v)$ which turns out to be active, then it must match $u$ to $v$ (if possible). The goal of the benchmark is to build a
matching whose weight is as large as possible in expectation. We refer to this benchmark as the **committal benchmark**, and denote the expected value of its matching by $\text{OPT}(G)$.

Another option is to compare an online probing algorithm’s performance to the **non-committal benchmark**. This benchmark must still adaptively probe edges subject to patience constraints, but it need not respect commitment. Specifically, it may decide upon which subset of edges to match after all its probes have been made. Once again, the probes of the benchmark are also not restricted to any ordering of the online vertices, and its goal is to build a matching of maximum expected weight. We let $\text{OPT}_{\text{non}}(G)$ denote the expected value of the matching that the non-committal benchmark constructs. Observe that in the case of full patience (i.e., $\ell_v = |U|$ for all $v \in V$), the benchmark may probe all the edges of $G$, and thus corresponds to the expected weight of the optimum matching of the stochastic graph.

Clearly the non-committal benchmark is stronger than the committal benchmark; that is,

$$\text{OPT}_{\text{non}}(G) \geq \text{OPT}(G)$$

for any stochastic graph $G$. In general, the ratio between these values can be as small as 0.856, even for the case when $G$ has a single online node (see Borodin et. al [3] for an example). The standard in the literature is to prove competitive guarantees against the committal benchmark. As such, we discuss past results against this benchmark, but we will also indicate when we can provide guarantees against the non-committal benchmark.

Let us now suppose that $\mathcal{A}$ is an online probing algorithm which executes on the stochastic graph $G$. Let us assume the vertices arrive in the ROM setting, and denote $A(G)$ as the (random) matching constructed by $\mathcal{A}$ after the vertices of $V$ arrive. If we denote the value or weight of $\mathcal{A}(G)$ as $\text{val}(\mathcal{A}(G))$, then

$$\mathbb{E}[\text{val}(\mathcal{A}(G))]$$

is the expected weight of the matching $\mathcal{A}(G)$, where the randomness is with respect to edge states $(st(e))_{e \in E}$, the random order on $V$, and any randomized decisions made by $\mathcal{A}$.

Given a collection of stochastic graphs, say $\mathcal{C}$, we define the **competitive ratio** of $\mathcal{A}$ in the **ROM setting** (against the committal benchmark), as the value

$$\inf_{G \in \mathcal{C}} \frac{\mathbb{E}[\text{val}(\mathcal{A}(G))]}{\text{OPT}(G)}.$$  

We can of course extend this definition to the adversarial setting by allowing the infimum to also be taken over the vertex ordering on $V$. By replacing $\text{OPT}(G)$ by $\text{OPT}_{\text{non}}(G)$, we can also define competitive ratios against the non-committal benchmark. Clearly, any competitive ratio proven against the non-committal benchmark implies that the same competitive ratio holds against the committal benchmark.

While it is of course most desirable to prove competitive ratios when $\mathcal{C}$ corresponds to the collection of all stochastic graphs, we can only guarantee the simplicity and effectiveness of our probing algorithm when $\mathcal{C}$ is restricted to a number of natural settings.

### 2.1. An Overview of Results

Let $G = (U, V, E)$ be a stochastic graph, and fix $v \in V$ and $R \subseteq U$. We can consider the **induced stochastic subgraph**, denoted $G[\{v\} \cup R]$, which corresponds to restricting the stochastic graph $G$ to the vertices $\{v\} \cup R$, and retaining the relevant edge probabilities between $v$ and $R$, as well as the offline vertex weights of $R$ and the patience value $\ell_v$.

In order to define a greedy algorithm, we need to ensure that when $v$ arrives, its probes are made in a way that $v$ gains as much value as possible (in expectation), provided the currently unmatched nodes of $U$ are equal to $R$. As such, we must follow the probing strategy of the committal benchmark when restricted to $G[\{v\} \cup R]$. It will be convenient to denote the value of this benchmark on $G[\{v\} \cup R]$ by $\text{OPT}(v, R, \ell_v)$, which we shorten to $\text{OPT}(v, R)$ when clear. Similarly, we define $\text{OPT}_{\text{non}}(v, R, \ell_v) = \text{OPT}_{\text{non}}(v, R)$ to be value the non-committal benchmark attains on $G[\{v\} \cup R]$. 


It will sometimes be useful to abuse terminology slightly, and also refer to \( \text{OPT}(v, R, \ell_v) \) as the algorithm the committal benchmark follows to obtain this optimum value. We similarly abuse terminology in the context of \( \text{OPT}_{\text{non}}(v, R, \ell_v) \) and the non-committal benchmark.

Observe that if \( v \) has unit patience (i.e, \( \ell_v = 1 \)), \( \text{OPT}(v, R, \ell_v) \) reduces to probing the edge \((u,v) \in R \times \{v\}\) such that the value \( w_u \cdot p_{u,v} \) is maximized. Moreover, when \( \ell_v \geq |R| \), \( \text{OPT}(v, R, \ell_v) \) corresponds to probing the edges of \( R \cup \{v\} \) in non-increasing order of the associated vertex weights. In the case of arbitrary patience, Brubach et al. [4] show how to devise an efficient probing strategy for \( v \) whose expected value matches \( \text{OPT}(v, R, \ell_v) \). This probing strategy is based on solving a dynamic program. Crucially, Brubach et al. argue the committal benchmark on \( G[\{v\} \cup R] \) may be assumed to probe a subset of the edges adjacent to \( v \) in non-increasing order of the corresponding vertex weights. This observation leads to the following DP, which we restate for completeness:

Consider \( h = h(u, i) \), where for \( u \in R \) and \( 1 \leq i \leq \min\{|R|, \ell_v\} \).

\[
h(u, i) := \begin{cases} p_{u,v} \cdot w_u & \text{if } i = 1 \\ p_{u,v} \cdot w_u + (1 - p_{u,v}) \max_{u^* \in R: u^* \leq u} h(u^*, i - 1) & \text{if } i \geq 2 \end{cases} \tag{2.1}
\]

Brubach et al. argue that \( h(u, i) \) encodes the maximum expected value that a probing algorithm can attain, provided it probes the edge \((u, v)\) first and has patience \( i \). Clearly, (2.1) can be solved efficiently, thus yielding the value of \( h(u, i) \) for each \( u \in R \) and \( 1 \leq i \leq k \), where \( k := \min\{|R|, \ell_v\} \). Observe then that \( \max_{u \in R} h(u, \ell_v) \) corresponds to the value of \( \text{OPT}(v, R, \ell_v) \). Moreover, by solving (2.1), one can recover a tuple \( u = (u_1, \ldots, u_k) \) which indicates the order of probes one should make to attain the value \( \text{OPT}(v, R, \ell_v) \) in expectation. We refer to this procedure as STAR-DP.

Given \( R \subseteq U \), we denote the ordered probes specified by executing \( \text{STAR-DP} \) on \( G[\{v\} \cup R] \) by \( \text{STAR-DP}(v, R, \ell_v) \). That is, \( \text{STAR-DP}(v, R, \ell_v) \) is a tuple of length \( k = \min\{|R|, \ell_v\} \), whose entries are all distinct. Based on the \( \text{STAR-DP} \) algorithm, Brubach et al. consider the following deterministic online probing algorithm in the adversarial setting, which they show attains a competitive ratio of 1/2 against the committal benchmark.

\begin{algorithm}
\caption{Greedy-DP}
\begin{algorithmic}[1]
\STATE Input \( U \) with offline vertex weights \( (w_u)_{u \in U} \).
\STATE \( M \leftarrow \emptyset \).
\STATE \( R \leftarrow U \).
\FOR{\( t = 1, \ldots, n \)}
\STATE Let \( v_t \) be the current online arrival node, with patience \( \ell_{v_t} \).
\STATE Set \( k \leftarrow \min\{|R|, \ell_{v_t}\} \) and \( u = (u_1, \ldots, u_k) \leftarrow \text{STAR-DP}(v_t, R, \ell_{v_t}) \).
\FOR{\( i = 1, \ldots, k \)}
\STATE Probe \((u_i, v_t)\).
\IF{\( \text{st}(u_i, v_t) = 1 \)}
\STATE Set \( M(v_t) = u_i \) and update \( R \leftarrow R \setminus \{u_i\} \).
\ENDIF
\ENDFOR
\ENDFOR
\STATE Return \( M \).
\end{algorithmic}
\end{algorithm}

Since Algorithm 1 is deterministic, the 1/2 competitive ratio is best possible in the adversarial input setting. We instead consider the same algorithm in the ROM setting. Unfortunately, our primal-dual argument does not yield a competitive ratio which holds against all vertex weighted stochastic graphs. Instead, we focus on instances in which the executions of \( \text{STAR-DP} \) are more tractable.
Given a vertex \( v \in V \), we say that \( v \) is **rankable**, provided there exists a fixed ordering \( \lambda_v \) on \( U \) with the property that for each \( R \subseteq U \),

\[
\text{Star-DP}(v, R, \ell_v) = (r_1, \ldots, r_k)
\]

where \( r = (r_1, \ldots, r_k) \) corresponds to the top \( k = \min(\ell_v, |R|) \) ranked vertices of \( R \), based on \( \lambda_v \). Crucially, if \( v \) is rankable, then when vertex \( v \) arrives while executing Algorithm 1 one can compute the ranking \( \lambda_v \) on \( U \) and probe the unmatched vertices \( R \subseteq U \) based on this order. By following this probing strategy, (2.2) ensures that the expected value assigned to \( v \) will be \( \text{OPT}(v, R, \ell_v) \).

Our reasoning for working with rankable stochastic graphs is that for arbitrary patience, \( \text{STAR-DP}(v, R, \ell_v) \) can change very much, depending on how \( R \) changes. For instance, if \( R = U \), then \( \text{STAR-DP}(v, U, \ell_v) \) may specify \( u \in U \) as the first vertex in its tuple - thus giving it highest priority - whereas by removing \( u^* \in U \) (where \( u^* \neq u \)), \( \text{STAR-DP}(v, U \setminus \{u^*\}, \ell_v) \) may not return \( u \) in its tuple at all. In fact, this behaviour isn’t just an artefact of how \( \text{STAR-DP} \) is defined, but of the committal benchmark in general.

**Example 2.1.** Let \( G = (U, V, E) \) be a bipartite graph with \( U = \{u_1, u_2, u_3, u_4\} \), \( V = \{v\} \) and \( \ell_v = 2 \). Set \( p_{u_1,v} = 1/3 \), \( p_{u_2,v} = 1 \), \( p_{u_3,v} = 1/2 \), \( p_{u_4,v} = 2/3 \).

Fix \( \varepsilon > 0 \), and let the weights of offline vertices be \( w_{u_1} = 1 + \varepsilon \), \( w_{u_2} = 1 + \varepsilon/2 \), \( w_{u_3} = w_{u_4} = 1 \). We assume that \( \varepsilon \) is sufficiently small - concretely, \( \varepsilon \leq 1/12 \). If \( R_1 := U \), then \( \text{OPT}(v, R_1) \) probes \( (u_1, v) \) and then \( (u_2, v) \) in order. On the other hand, if \( R_2 = R_1 \setminus \{v_2\} \), then \( \text{OPT}(v, R_2) \) does not probe \( (u_1, v) \). Specifically, \( \text{OPT}(v, R_2) \) probes \( (u_3, v) \) and then \( (u_4, v) \).

Clearly, Example 2.1 shows that the vertices of \( G = (U, V, E) \) may not be rankable. That being said, there are a number of natural settings in which we can guarantee rankability.

**Example 2.2.** Let \( G = (U, V, E) \) be a stochastic graph, and suppose that \( v \in V \). If \( v \) satisfies either of the following conditions, then \( v \) is rankable:

1. **(C1)** Vertex \( v \) has unit patience or full patience; that is, \( \ell_v \in \{1, |U|\} \).
2. **(C2)** The edge probabilities \( (p_{u,v})_{u \in U} \) are non-decreasing with respect to the vertex weights \( (w_u)_{u \in U} \); that is, for each \( u_1, u_2 \in U \), if \( p_{u_1,v} \leq p_{u_2,v} \) then \( w_{u_1} \leq w_{u_2} \).

If \( \ell_v = 1 \), then \( \lambda_v \) corresponds to ranking the vertices of \( U \) based on the values \( (w_u \cdot p_{u,v})_{u \in U} \). On the other hand, if \( \ell_v = |U| \), or **(C2) is satisfied**, then \( \lambda_v \) corresponds to ranking the vertices of \( U \) based on the offline vertex weights.

We remark that **(C2)** encompasses the case when the vertices of \( U \) are unweighted, as well as when the edge probabilities of \( v \) are **online vertex uniform**. That is, when there exists a probability \( p_v \), such that \( p_{u,v} \in \{0, p_v\} \) for each \( u \in U \).

**Remark.** Conditions **(C1)** and **(C2)** do not exhaustively describe the ways a vertex \( v \) may be rankable. For instance, suppose \( U = \{u_1, u_2, u_3\} \), where \( w_{u_1} > w_{u_2} > w_{u_3} \) and \( p_{u_1,v} < p_{u_2,v} < p_{u_3,v} \). If we additionally assume that \( p_{u_1,v} \cdot w_{u_1} > p_{u_2,v} \cdot w_{u_2} > p_{u_3,v} \cdot w_{u_3} \) and \( \ell_v = 2 \) then the indices of \( U \) induce a ranking for \( v \), yet this example neither satisfies **(C1)** nor **(C2)**.

We refer to the stochastic graph \( G \) as **rankable**, provided all of its vertices are themselves rankable. We emphasize that distinct vertices of \( V \) may each use their own separate rankings of the offline vertices. In particular, the vertices of \( V \) may each fall into separate cases of Example 2.2. In Section 3 we prove the following theorem:

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6If \( \lambda_v \) exists, then we can always find it efficiently by defining \( r_1 := \text{STAR-DP}(v, U, \ell_v) \) as the first \( \ell_v \) ordered vertices of \( U \), which we then follow by \( r_2 := \text{STAR-DP}(v, U \setminus r_1, \ell_v) \), and so on, until all the vertices of \( U \) are ordered. That being said, we consider specific settings in Example 2.2 in which \( \lambda_v \) is more easily described.
Theorem 2.3. Suppose we are presented a stochastic graph $G$ with edge probabilities $(p_e)_{e \in E}$, offline vertex weights $(w_u)_{u \in U}$ and patience values $(\ell_v)_{v \in V}$. If Algorithm 1 returns the matching $M$, then

$$\mathbb{E}[\text{val}(M)] \geq \left(1 - \frac{1}{e}\right) \cdot \text{OPT}(G),$$

provided the online vertices arrive uniformly at random and $G$ is rankable. In other words, Algorithm 1 achieves a competitive ratio of $1 - 1/e$ against the committal benchmark in the ROM setting, provided we restrict to the class of all rankable stochastic graphs.

The competitive ratio of $1 - 1/e$ also holds against the non-committal benchmark in the explicit settings of Example 2.2.

Theorem 2.4. Suppose Algorithm 1 returns the matching $M$ when executing on the stochastic graph $G = (U, V, E)$. If each vertex of $V$ satisfies $[C_1]$ or $[C_2]$ of Example 2.2 then

$$\mathbb{E}[\text{val}(M)] \geq \left(1 - \frac{1}{e}\right) \cdot \text{OPT}_{\text{non}}(G).$$

We remark that even restricted to these specific settings, it need not be the case that $G = (U, V, E)$ satisfies $\text{OPT}_{\text{non}}(G) = \text{OPT}(G)$. For instance, Costello et al. [6] showed that if $G$ is unweighted, $|U| = |V| = 2$, $p_{u,v} = 0.64$ for all $(u, v) \in U \times V$, and $\ell_v = 2$ for $v \in V$, then $\text{OPT}(G)/\text{OPT}_{\text{non}}(G) = 0.898$. As such, we can’t claim Theorem 2.4 by simply relating $\text{OPT}(G)$ to $\text{OPT}_{\text{non}}(G)$ in these specific settings. Instead, we argue that the greedy decisions made by Algorithm 1 are no worse than those made by a modified version of Algorithm 1 which executes $\text{OPT}_{\text{non}}(v, R, \ell_v)$ when presented $v \in V$ and the remaining nodes $R \subseteq U$. We provide the details of this argument in Section 4.

When working with an arbitrary stochastic graph $G$, proving competitive ratios against the non-committal benchmark is more challenging. To see this, suppose that the vertices $R \subseteq U$ remain available when an online vertex $v$ arrives. At this point, the optimum probing strategy for $G'[\{v\} \cup R]$ used by Algorithm 1 may not match the performance of $\text{OPT}_{\text{non}}(v, R)$; that is, the ratio $\text{OPT}(v, R, \ell_v)/\text{OPT}_{\text{non}}(v, R, \ell_v)$ may be small. For instance, in [3] the authors provide an example that shows this ratio can be as small as $0.856$.

We can still prove that Algorithm 1 achieves a non-trivial competitive ratio by lower bounding this ratio by $1 - 1/e$ via an application of the results of Asadpour [1], but this is not without loss.

Theorem 2.5. Suppose Algorithm 1 returns the matching $M$ when executing on the stochastic graph $G = (U, V, E)$. In this case,

$$\mathbb{E}[\text{val}(M)] \geq \left(1 - \frac{1}{e}\right)^2 \cdot \text{OPT}_{\text{non}}(G),$$

provided $G$ is rankable.

In Section 4 we prove Theorem 2.5 and discuss directions for improving upon the ratio of $(1 - 1/e)^2$.

3. Proving Theorem 2.3

In order to review the primal-dual framework of Devanur et al. [7], we begin with the unit patience setting of Theorem 2.3. While the techniques used mainly follow the work of Devanur et al. [7] (more specifically, Niazadeh and Kleinberg [18]), the $1 - 1/e$ competitive ratio we obtain shows that online stochastic matching with ROM arrivals is provably easier than online stochastic matching with adversarial arrivals, as we elaborate on at the end of the section. Moreover, we are able to apply a standard primal-dual charging argument, in contrast to the difficulties inherent in such an approach in the adversarial arrival setting, as discussed by Huang and Zhang [10].
For each \( v \in V \), draw \( Y_v \in [0,1] \) independently and uniformly at random. We assume that the vertices of \( V \) are presented to the algorithm in a non-decreasing order, based on the values of \( (Y_v)_{v \in V} \). In this way, we say that vertex \( v \in V \) arrives at time \( Y_v \). Observe that the vertices of \( V \) are presented to the algorithm in a uniformly at random order, so this interpretation is equivalent to the ROM setting described previously.

When working with unit patience, Algorithm 1 reduces to a very simple greedy algorithm, which we restate for completeness:

**Algorithm 2 Greedy-DP-Unit**

Input \( U \) with offline vertex weights \( (w_u)_{u \in U} \).

1. \( M \leftarrow \emptyset \).
2. \( R \leftarrow U \).
3. for \( v \in V \) in increasing order of \( Y_v \) do
   4. Let \( u \in U \) be such that \( w_u \cdot p_{u,v} \) is maximum amongst all vertices of \( R \).
   5. Probe the edge \((u,v)\).
   6. if \((u,v)\) is active then
      7. Set \( M(v) = u \) and update \( R \leftarrow R \setminus \{u\} \).
   8. end if
4. end for
5. Return \( M \).

Instead of directly comparing the performance of Algorithm 2 to the committal benchmark, we first consider an LP originally introduced in [2, 16], specialized for the case in which the stochastic graph \( G \) has unit patience and offline vertex weights.

\[
\text{maximize} \quad \sum_{u \in U, v \in V} w_u \cdot p_{u,v} \cdot x_{u,v} \quad \text{(LP-std)}
\]

subject to
- \( \sum_{v \in V} p_{u,v} \cdot x_{u,v} \leq 1 \quad \forall u \in U \) \hspace{1cm} (3.1)
- \( \sum_{u \in U} x_{u,v} \leq 1 \quad \forall v \in V \) \hspace{1cm} (3.2)
- \( x_{u,v} \geq 0 \quad \forall u \in U, v \in V \) \hspace{1cm} (3.3)

If we denote \( \text{LPOPT}_{\text{std}}(G) \) as the value of an optimum solution to this LP, then it was shown in [2, 16] that \( \text{OPT}(G) \leq \text{LPOPT}_{\text{std}}(G) \). In light of this property, we hereby refer to \( \text{LP-std} \) as a relaxation of the committal benchmark.

Observe that we can also take the dual of \( \text{LP-std} \), yielding the following LP:

\[
\text{minimize} \quad \sum_{u \in U} \alpha_u + \sum_{v \in V} \beta_v \quad \text{(LP-dual-std)}
\]

subject to
- \( p_{u,v} \alpha_u + \beta_v \geq w_u p_{u,v} \quad \forall u \in U, v \in V \) \hspace{1cm} (3.4)
- \( \alpha_u \geq 0 \quad \forall u \in U \) \hspace{1cm} (3.5)
- \( \beta_v \geq 0 \quad \forall v \in V \) \hspace{1cm} (3.6)

**Theorem 3.1.** Algorithm 2 is a deterministic probing algorithm which achieves a competitive ratio of \( 1-1/e \) in the ROM setting of unit patience, arbitrary edge probabilities and offline vertex weights.

In order to prove this theorem, we consider a dual-fitting argument. Specifically, let \( F := 1 - 1/e \) and define \( g : [0,1] \rightarrow [0,1] \) where \( g(z) := \exp(z-1) \) for \( z \in [0,1] \). We construct a dual solution \(((\alpha_u)_{u \in U}, (\beta_v)_{v \in V})\) where all the variables are initially set equal to 0.
Suppose that we now consider an online node \( v \in V \) and an offline node \( u \in U \). If \( v \) is matched to \( u \) when it arrives at time \( Y_v \), then we set \( \alpha_u \) to \( w_u \cdot (1 - g(Y_v))/F \) and \( \beta_v \) to \( w_u \cdot g(Y_v)/F \).

Based on these assignments, observe that

\[
\text{val}(\mathcal{M}) = \sum_{u \in U, v \in V} w_{u} 1[\mathcal{M}(v) = u] \\
= F \cdot \left( \sum_{u \in U} \alpha_u + \sum_{v \in V} \beta_v \right),
\]

where \( \mathcal{M} \) is the matching returned by Algorithm 2 when executing on \( G \).

Thus,

\[
\mathbb{E}[\text{val}(\mathcal{M})] = F \left( \sum_{u \in U} \mathbb{E}[\alpha_u] + \sum_{v \in V} \mathbb{E}[\beta_v] \right),
\]

where the expectation is over the random variables \((Y_v)_{v \in V} \)
and the states of the edges of \( G \).

Let us now set \( \alpha_u^* := \mathbb{E}[\alpha_u] \) and \( \beta_v^* := \mathbb{E} [\beta_v] \) for \( u \in U \) and \( v \in V \). Observe the following claim regarding the dual variables \(((\alpha_u^*)_{u \in U}, (\beta_v^*)_{v \in V})\).

**Lemma 3.2.** The variables \(((\alpha_u^*)_{u \in U}, (\beta_v^*)_{v \in V})\) satisfy

\[
\mathbb{E}[\text{val}(\mathcal{M})] = \left( 1 - \frac{1}{e} \right) \left( \sum_{u \in U} \alpha_u^* + \sum_{v \in V} \beta_v^* \right).
\]

Moreover, the solution \(((\alpha_u^*)_{u \in U}, (\beta_v^*)_{v \in V})\) satisfies all the constraints of \( \text{LP-dual-std} \) and thus is feasible.

The techniques needed to prove Lemma 3.2 closely follow the work of [7, 18], as is apparent based on how the primal-dual variables are charged. Moreover, Theorem 3.1 is subsumed by Theorem 2.3, the latter of which has a complete proof in the next section. As such, we omit a self-contained proof of Lemma 3.2 and instead show how we can use this lemma to prove Theorem 3.1:

**Proof of Theorem 3.1.** Let us suppose that \( G = (U, V, E) \) is an arbitrary stochastic graph of unit patience, with offline vertex weights \((w_u)_{u \in U}\). We hereby denote \( A(G) \) as the matching returned by Algorithm 2 when executing on \( G \).

By applying Lemma 3.2, the existence of the feasible dual solution \(((\alpha_u^*)_{u \in U}, (\beta_v^*)_{v \in V})\) ensures that

\[
\mathbb{E}[\text{val}(A(G))] = \left( 1 - \frac{1}{e} \right) \left( \sum_{u \in U} \alpha_u^* + \sum_{v \in V} \beta_v^* \right) \geq \left( 1 - \frac{1}{e} \right) \text{LPOPT}_{\text{std}}(G) \geq \left( 1 - \frac{1}{e} \right) \text{OPT}(G),
\]

where the first inequality follows from the weak duality between \( \text{LP-std} \) and \( \text{LP-dual-std} \) and the second inequality holds since \( \text{LP-std} \) is a relaxation of the committal benchmark. As the graph \( G \) is arbitrary, the proof for unit patience is complete.

We emphasize that the randomized \( .621 < 1 - \frac{1}{e} \) in-approximation in Mehta and Panigrahpi [16] is with regard to the adversarial input model with unit patience and applies to performance guarantees made against \( \text{LP-std} \). Theorem 3.1 implies that this in-approximation cannot apply to the setting of ROM arrivals.

\footnote{It is unknown whether this in-approximation applies to \( \text{OPT}(G) \), in addition to \( \text{LPOPT}_{\text{std}}(G) \).}
We remark that the analysis of Theorem 3.1 is tight since an execution of Algorithm 2 corresponds to the seminal Karp et al. [12] Ranking algorithm for unweighted non-stochastic (i.e., \( p_{u,v} \in \{0,1\} \) for all \((u,v) \in U \times V\)) bipartite matching. More precisely, in the classical (unweighted, non-stochastic) online matching problem, an execution of the randomized Ranking algorithm in the adversarial setting can be coupled with an execution of the deterministic greedy algorithm in the ROM setting - the latter of which is a special case of Algorithm 2. The tightness of the ratio \( 1 - \frac{1}{e} \) therefore follows since this ratio is tight for the Ranking algorithm.

Finally, observe that the \( 1 - \frac{1}{e} \) competitive ratio in Theorem 3.1 combined with the Mehta and Panigrahi in-approximation imply that there is no generalization of this coupling to the stochastic matching setting (at least for unit patience).

3.1. Generalizing to Arbitrary Patience. We now consider the unknown stochastic matching problem for the case of arbitrary patience in the ROM setting. Specifically, we show that Algorithm 2 attains a competitive ratio of \( 1 - \frac{1}{e} \) in the setting of Theorem 2.3. We once again use a primal-dual analysis to derive these results, but we provably cannot work with the standard LP of Bansal et al. [2].

Let us once again denote \( \text{OPT}(G) \) as the value of the committal benchmark on \( G \). In order to prove results against the committal benchmark, we once again consider LP relaxations to upper bound the value of \( \text{OPT}(G) \). However, unlike the unit patience setting, it is not as clear as to what the right relaxation is.

The most prevalent LP used in the literature was introduced by Bansal et al. [2], which reduces to \([\text{LP-std}]\) in the case of unit patience:

\[
\begin{align*}
\text{maximize} & \quad \sum_{u \in U, v \in V} w_{u,v} \cdot p_{u,v} \cdot x_{u,v} \quad \text{(LP-std)} \\
\text{subject to} & \quad \sum_{v \in V} p_{u,v} \cdot x_{u,v} \leq 1 \quad \forall u \in U \quad (3.8) \\
& \quad \sum_{u \in U} p_{u,v} \cdot x_{u,v} \leq 1 \quad \forall v \in V \quad (3.9) \\
& \quad \sum_{u \in U} x_{u,v} \leq \ell_v \quad \forall v \in V \quad (3.10) \\
& \quad 0 \leq x_{u,v} \leq 1 \quad \forall u \in U, v \in V. \quad (3.11)
\end{align*}
\]

We hereby denote \( \text{LPOPT}_{\text{std}}(G) \) as the value of an optimal solution to \([\text{LP-std}]\). By interpreting \( x_{u,v} \) as the probability that the committal benchmark probes the edge \((u,v)\), it was observed by Bansal et al. in [2] that

\[ \text{OPT}(G) \leq \text{LPOPT}_{\text{std}}(G). \quad (3.12) \]

One of the challenges of working with \([\text{LP-std}]\) is that the ratio between \( \text{OPT}(G) \) and \( \text{LPOPT}_{\text{std}}(G) \) can become quite small, depending on the values of \((\ell_v)_{v \in V}\) and the instance \( G \). In [4], Brubach et al. define the stochasticity gap as the infimum of this ratio across all stochastic graphs, namely \( \inf_G \text{OPT}(G)/\text{LPOPT}_{\text{std}}(G) \). They consider the following example, thus upper bounding this quantity.

**Example 3.3** ([4]). Fix \( n \geq 1 \), and construct \( G_n = (U,V,E) \). Suppose that \(|U| = |V| = n\) and \( \ell_v = n \) for all \( v \in V \). Set \( E := U \times V \), and define \( p_{u,v} := 1/n \) for each \((u,v) \in E\). Observe that \( G_n \) corresponds to the Erdős-Rényi random graph \( G_{n,n,1/n} \). In this case,

\[ \mathbb{E}[\text{OPT}(G_n)] \leq 0.544 (1 + o(1)) \text{LPOPT}_{\text{std}}(G_n), \]

where the asymptotics are over \( n \to \infty \).

We state the Brubach et al. [4] impossibility result as follows:
Proposition 3.4 ([4]). Any probing algorithm which proves a guarantee against $\text{LPOPT}_{\text{std}}(G)$ has a competitive ratio of at most 0.544.

In particular, Proposition 3.4 implies that our primal-dual proof from the unit patience setting will not extend if we work with $\text{LP-std}$.

In [4], Brubach et al. suggest an LP which assumes a number of extra constraints in addition to those of $\text{LP-std}$. Recall that given $v \in V$ and $R \subseteq U$, we defined the induced stochastic subgraph $G[\{v\} \cup R]$ (which inherits the patience value $\ell_v$), and $\text{OPT}(v, R, \ell_v)$ as the value of the committal benchmark when executing on $G[\{v\} \cup R]$.

For each $u \in U$ and $v \in V$, we again interpret $x_{u,v}$ as the probability that the committal benchmark probes the edge $(u, v)$. Brubach et al. modified $\text{LP-std}$ by adding the following constraint for each $R \subseteq U$ and $v \in V$:

$$\sum_{u \in R} p_{u,v} \cdot x_{u,v} \leq \text{OPT}(v, R).$$  \hspace{1cm} (3.13)

This added constraint can be viewed as ensuring that the expected stochastic reward of $v$, suggested by the solution $(x_{u,v})_{u \in U, v \in V}$, is actually attainable by the committal benchmark.

Brubach et al. showed that this modified LP is a relaxation of the committal benchmark. For the purposes of proving Theorem 2.3, we shall make use of constraint (3.13), though there are a number of constraints of $\text{LP-std}$ which aren’t needed in our analysis. As such, we remove these non-essential constraints from $\text{LP-std}$ thus leaving us with the following LP:

$$\text{maximize} \quad \sum_{u \in U} \sum_{v \in V} w_u \cdot p_{u,v} \cdot x_{u,v}$$  \hspace{1cm} (LP-DP)

subject to

$$\sum_{v \in V} p_{u,v} \cdot x_{u,v} \leq 1 \quad \forall u \in U$$  \hspace{1cm} (3.14)

$$\sum_{u \in R} w_u \cdot p_{u,v} \cdot x_{u,v} \leq \text{OPT}(v, R, \ell_v) \quad \forall v \in V, R \subseteq U$$  \hspace{1cm} (3.15)

$$x_{u,v} \geq 0 \quad \forall u \in U, v \in V$$  \hspace{1cm} (3.16)

We hereby denote $\text{LPOPT}_{\text{DP}}(G)$ as the optimum value of this LP. Observe that by definition, the LP introduced by Brubach et al. has more constraints than $\text{LP-DP}$ and so has an optimum value which is no greater than $\text{LPOPT}_{\text{DP}}(G)$. Thus, since the Brubach et al. LP is a relaxation of the committal benchmark, so is $\text{LP-DP}$ that is, $\text{OPT}(G) \leq \text{LPOPT}_{\text{DP}}(G)$.

3.1.1. Defining the Primal-Dual Charging Scheme. In order to prove Theorem 2.3 we once again employ a primal-dual charging argument, however we make use of the dual of $\text{LP-DP}$.

For each $u \in U$, define the variable $\alpha_u$ as in $\text{LP-std}$. Moreover, for each $v \in V$ and $R \subseteq U$, define the variable $\phi_{v,R}$ (these latter variables correspond to constraint (3.13)).

$$\text{minimize} \quad \sum_{u \in U} \alpha_u + \sum_{v \in V} \sum_{R \subseteq U} \text{OPT}(v, R, \ell_v) \cdot \phi_{v,R}$$  \hspace{1cm} (LP-dual-DP)

subject to

$$p_{u,v} \cdot \alpha_u + \sum_{R \subseteq U: u \in R} w_u \cdot p_{u,v} \cdot \phi_{v,R} \geq w_u \cdot p_{u,v} \quad \forall u \in U, v \in V$$  \hspace{1cm} (3.17)

$$\alpha_u \geq 0 \quad \forall u \in U$$  \hspace{1cm} (3.18)

$$\phi_{v,R} \geq 0 \quad \forall v \in V, R \subseteq U$$  \hspace{1cm} (3.19)

The dual-fitting argument used to prove Theorem 2.3 has an initial set-up which proceeds in the same way as in the unit patience setting. Specifically, let $F := 1 - 1/e$ and define $g : [0, 1] \to [0, 1]$ where $g(z) := \exp(z-1)$ for $z \in [0, 1]$. For each $v \in V$, draw $Y_v \in [0, 1]$ independently and uniformly
at random. We assume that the vertices of $V$ are presented to Algorithm 1 in a non-decreasing order, based on the values of $(Y_v)_{v \in V}$.

We now describe how the charging assignments are made while Algorithm 2 executes on $G$. Firstly, we initialize a dual solution $((\alpha_u)_{u \in U}, (\phi_{v, R})_{v \in V, R \subseteq U})$ where all the variables are initially set equal to 0.

Let us now take $v \in V, u \in U$, and $R \subseteq U$, where $u \in R$. If $R$ consists of the unmatched vertices of $v$ when it arrives at time $Y_v$, then suppose that Algorithm 2 matches $v$ to $u$ while making its probes to a subset of the edges of $R \times \{v\}$. In this case, we charge $w_u \cdot (1 - g(Y_v))/F$ to $\alpha_u$ and $w_u \cdot g(Y_v)/(F \cdot \text{OPT}(v, R, \ell_v))$ to $\phi_{v, R}$. Observe that each subset $R \subseteq U$ is charged at most once, as is each $u \in U$. Moreover, assigning $w_u \cdot g(Y_v)/(F \cdot \text{OPT}(v, R, \ell_v))$ to $\phi_{v, R}$ benefits all the dual constraints associated with the edges $R \times \{v\}$, opposed to just the constraint associated with $(u, v)$ as in the unit patience case.

Observe now that by definition,

$$\text{val}(M) = \sum_{u \in U, v \in V} w_u \cdot 1_{[M(v) = u]}$$

$$= \left(1 - \frac{1}{e}\right) \cdot \left(\sum_{u \in U} \alpha_u + \sum_{v \in V} \sum_{R \subseteq U} \text{OPT}(v, R, \ell_v) \cdot \phi_{v, R}\right).$$

As such,

$$\mathbb{E}[\text{val}(M)] = \left(1 - \frac{1}{e}\right) \cdot \left(\sum_{u \in U} \mathbb{E}[\alpha_u] + \sum_{v \in V} \sum_{R \subseteq U} \text{OPT}(v, R, \ell_v) \cdot \mathbb{E}[\phi_{v, R}]\right),$$

(3.20)

where the expectation is over the random variables $(Y_v)_{v \in V}$, and the states of the edges of $G$.

If we now set $\alpha_u^* := \mathbb{E}[\alpha_u]$ and $\phi_{v, R}^* := \mathbb{E}[\phi_{v, R}]$ for $u \in U$, $v \in V$ and $R \subseteq U$, then (3.20) implies the following lemma:

**Lemma 3.5.** Suppose $G = (U, V, E)$ is a stochastic graph for which Algorithm 1 returns the matching $M$. In this case, if the variables $((\alpha_u^*)_{u \in U}, (\phi_{v, R}^*)_{v \in V, R \subseteq U})$ are defined through the above charging scheme, then

$$\mathbb{E}[\text{val}(M)] = \left(1 - \frac{1}{e}\right) \cdot \left(\sum_{u \in U} \alpha_u^* + \sum_{v \in V} \sum_{R \subseteq U} \text{OPT}(v, R, \ell_v) \cdot \phi_{v, R}^*\right).$$

We also make the following claim regarding the feasibility of the variables $((\alpha_u^*)_{u \in U}, (\phi_{v, R}^*)_{v \in V, R \subseteq U})$:

**Lemma 3.6.** If $G = (U, V, E)$ is a rankable stochastic graph, then the solution $((\alpha_u^*)_{u \in U}, (\phi_{v, R}^*)_{v \in V, R \subseteq U})$ satisfies all the constraints of LP-dual-DP, and thus is feasible.

We defer the proof of Lemma 3.6 to the next section and instead show how it allows us to complete the proof of Theorem 2.3.

**Proof of Theorem 2.3.** Suppose that $G = (U, V, E)$ is a rankable stochastic graph, and $M$ is the matching returned by Algorithm 1 when executing on $G$. In this case, Lemma 3.5 implies that

$$\mathbb{E}[\text{val}(M)] = \left(1 - \frac{1}{e}\right) \cdot \left(\sum_{u \in U} \alpha_u^* + \sum_{v \in V} \sum_{R \subseteq U} \text{OPT}(v, R, \ell_v) \cdot \phi_{v, R}^*\right).$$

On the other hand, since $G$ is rankable, we may apply Lemma 3.6 to ensure that

$$\mathbb{E}[\text{val}(M)] = \left(1 - \frac{1}{e}\right) \cdot \left(\sum_{u \in U} \alpha_u^* + \sum_{v \in V} \sum_{R \subseteq U} \text{OPT}(v, R, \ell_v) \cdot \phi_{v, R}^*\right)$$

(3.21)
Proposition 3.8. For any stochastic graph $G$

\[
\geq \left(1 - \frac{1}{e}\right) \cdot \text{LPOPT}_{DP}(G)
\]

\[
\geq \left(1 - \frac{1}{e}\right) \cdot \text{OPT}(G),
\]

where the first inequality follows from the weak duality of \text{LP-DP} and \text{LP-dual-DP}, and the second follows since \text{LP-DP} is a relaxation of the commital benchmark. As $G$ was an arbitrary rankable stochastic graph, this completes the proof.

\[\Box\]

3.1.2. Proving Dual Feasibility. Let us suppose that the variables $((\alpha_u)_{u \in U}, (\phi_{v,R})_{v \in V, R \subseteq U})$ are defined as in the charging scheme of Section 3.1.1

In order to prove the dual feasibility of the solution proposed in Lemma 3.6, we must show that for each fixed $u_0 \in U$ and $v_0 \in V$, we have that

\[
\mathbb{E}[p_{u_0,v_0} \cdot \alpha_{u_0} + w_{u_0} \cdot p_{u_0,v_0} \sum_{R \subseteq U : u_0 \in R} \phi_{v,R}] \geq w_{u_0} \cdot p_{u_0,v_0}.
\]

Our strategy for proving (3.24) involves the same trick as used in Devanur et al. \[7\], though adapted to the stochastic matching setting. Specifically, we consider the stochastic graph $\tilde{G} := (U, \tilde{V}, \tilde{E})$, where $\tilde{V} := V \setminus \{v_0\}$ and $\tilde{E}$ is constructed by removing all edges of $E$ which are incident to $v_0$. We assume that $\tilde{G}$ has the same offline vertex weights as $G$, and that the edges probabilities of $\tilde{E}$ remain unchanged from those of $E$.

We wish to compare the execution of the algorithm on the instance $\tilde{G}$ to its execution on the instance $G$. It will be convenient to couple the randomness between these two executions by making the following assumptions:

1. For each $e \in \tilde{E}$, $e$ is active in $\tilde{G}$ if and only if it is active in $G$.
2. The same random variables, $(Y_v)_{v \in \tilde{V}}$, are used in both executions.

If we now focus on the execution of $\tilde{G}$, then define the random variable $\tilde{Y}_c$ where $\tilde{Y}_c := Y_{\tilde{v}}$ if $u_0$ is matched to some $v_c \in \tilde{V}$, and $\tilde{Y}_c := 1$ if $u_0$ remains unmatched after the execution on $G$. We refer to the random variable $\tilde{Y}_c$ as the critical time of vertex $u_0$ with respect to $v_0$.

We claim the following lower bound on $\alpha_{u_0}$ in terms of the critical time $\tilde{Y}_c$:

**Proposition 3.7.** If $G$ is rankable, then

\[
\alpha_{u_0} \geq \frac{w_{u_0}}{F} (1 - g(\tilde{Y}_c)).
\]

Moreover, by taking the appropriate conditional expectation, we can also lower bound the random variables $((\phi_{v_0,R})_{R \subseteq U : u_0 \in R})_{v \in V}$.

**Proposition 3.8.** For any stochastic graph $G$,

\[
\sum_{R \subseteq U : u_0 \in R} \mathbb{E}[\phi_{v_0,R} | (Y_v)_{v \in V, (st(e))_{e \in E}}] \geq \frac{1}{F} \int_0^{\tilde{Y}_c} g(z) dz.
\]

We first focus on proving Proposition 3.7. We emphasize that this is the only part of the argument which requires the rankability of $G$.

**Proof of Proposition 3.7.** For each $v \in V$, denote $R_v^a(G)$ as the unmatched (remaining) vertices of $U$ right after $v$ is processed (attempts its probes) in the execution on $G$. We emphasize that if a probe of $v$ yields an active edge, thus matching $v$, then this match is excluded from $R_v^a(G)$. Similarly, define $R_v^a(\tilde{G})$ in the same way for the execution on $\tilde{G}$ (where $v$ is now restricted to $\tilde{V}$).
Observe then that because of the coupling between the two executions, together with the rank-ability of $G$, we know that

$$R_v^\alpha(G) \subseteq R_v^\alpha(\tilde{G}),$$  \hspace{1cm} (3.25)

for each $v \in \tilde{V}$.

Now, since $g(1) = 1$ (by assumption), there is nothing to prove if $\tilde{Y}_c = 1$. Thus, we may assume that $\tilde{Y}_c < 1$, and as a consequence, that there exists some vertex $v_0 \in V$ which matches to $u_0$ at time $\tilde{Y}_c$ in the execution on $G$.

On the other hand, by assumption we know that $u_0 \notin R_v^\alpha(\tilde{G})$ and thus by (3.25), that $u_0 \notin R_v^\alpha(G)$. As such, there exists some $v' \in V$ which probes $(u_0, v')$ and succeeds in matching to $u_0$ at time $\tilde{Y}_c' \leq \tilde{Y}_c$. Thus, since $g$ is monotone,

$$\alpha_{u_0} \geq \frac{w_{u_0}}{F} (1 - g(\tilde{Y}_c')) \frac{1}_{[\tilde{Y}_c < 1]} \geq \frac{w_{u_0}}{F} (1 - g(\tilde{Y}_c)),$$

and so the claim holds.

We now prove Proposition 3.8.

Proof of Proposition 3.8 We first define $R_{v_0}$ as the unmatched vertices of $U$ when $v_0$ arrives (this is a random subset of $U$). We also once again use $\mathcal{M}$ to denote the matching returned by Algorithm 1 when executing on $G$.

If we now take a fixed subset $R \subseteq U$, then the charging assignment to $\phi_{v_0,R}$ ensures that

$$\phi_{v_0,R} = \text{val}(\mathcal{M}(v_0)) \cdot \frac{g(Y_v)}{F \cdot \text{OPT}(v_0, R, \ell_{v_0})} \cdot 1_{[R_{v_0} = R]},$$

where $\text{val}(\mathcal{M}(v_0))$ corresponds to the weight of the vertex matched to $v_0$ (which is zero if $v_0$ remains unmatched after the execution on $G$).

In order to make use of this relation, let us first condition on the values of $(Y_v)_{v \in V}$, as well as the states of the edges of $\tilde{E}$; that is, $(\text{st}(e))_{e \in \tilde{E}}$. Observe that once we condition on this information, we can determine $g(Y_v)$, as well as $R_{v_0}$. As such,

$$\mathbb{E}[\phi_{v_0,R} | (Y_v)_{v \in V}, (\text{st}(e))_{e \in \tilde{E}}] = \frac{g(Y_v)}{F \cdot \text{OPT}(v_0, R, \ell_{v_0})} \mathbb{E}[\text{val}(\mathcal{M}(v_0)) | (Y_v)_{v \in V}, (\text{st}(e))_{e \in \tilde{E}}] \cdot 1_{[R_{v_0} = R]}.$$

On the other hand, the only randomness which remains in the conditional expectation involving $\text{val}(\mathcal{M}(v_0))$ is over the states of the edges adjacent to $v_0$. Observe now that since Algorithm 1 behaves optimally on $G[\{v_0 \cup R_{v_0}\}]$, we get that

$$\mathbb{E}[\text{val}(\mathcal{M}(v_0)) | (Y_v)_{v \in V}, (\text{st}(e))_{e \in \tilde{E}}] = \text{OPT}(v_0, R_{v_0}, \ell_{v_0}),$$  \hspace{1cm} (3.26)

and so for the fixed subset $R \subseteq U$,

$$\mathbb{E}[\phi_{v_0,R} | (Y_v)_{v \in V}, (\text{st}(e))_{e \in \tilde{E}}] = \frac{g(Y_v)}{F} \cdot 1_{[R_{v_0} = R]}.$$

After multiplying each side of (3.26) by the indicator random variable $1_{[R_{v_0} = R]}$, we therefore get that

$$\mathbb{E}[\phi_{v_0,R} | (Y_v)_{v \in V}, (\text{st}(e))_{e \in \tilde{E}}] = \frac{g(Y_v)}{F} \cdot 1_{[R_{v_0} = R]}.$$
Let us now focus on the case when \( v_0 \) arrives before the critical time; that is, \( 0 \leq Y_{v_0} < \tilde{Y}_c \). Up until the arrival of \( v_0 \), the executions of the algorithm on \( G \) and \( G \) proceed identically, thanks to the coupling between the executions. As such, \( u_0 \) must be available when \( v_0 \) arrives.

We interpret this observation in the above notation as saying the following:

\[
1_{[Y_{v_0} < \tilde{Y}_c]} \leq \sum_{R \subseteq U: u_0 \in R} 1_{[R_{v_0} = R]}.
\]

As a result,

\[
\sum_{R \subseteq U: u_0 \in R} \mathbb{E}[\phi_{v_0, R} | (Y_v)_{v \in \tilde{V}}, (st(e))_{e \in \hat{E}}] \geq \frac{g(Y_{v_0})}{F} 1_{[Y_{v_0} < \tilde{Y}_c]}.
\]

Now, if we take expectation over \( Y_{v_0} \), while still conditioning on the random variables \( (Y_v)_{v \in \tilde{V}} \), then we get that

\[
\mathbb{E}[g(Y_{v_0}) \cdot 1_{[Y_{v_0} < \tilde{Y}_c]} | (Y_v)_{v \in \tilde{V}}, (st(e))_{e \in \hat{E}}] = \int_0^{\tilde{Y}_c} g(z) \, dz,
\]

as \( Y_{v_0} \) is drawn uniformly from \([0, 1]\), independently from \((Y_v)_{v \in \tilde{V}}\) and \((st(e))_{e \in \hat{E}}\). Thus, after applying the law of iterated expectations,

\[
\sum_{R \subseteq U: u_0 \in R} \mathbb{E}[\phi_{v_0, R} | (Y_v)_{v \in \tilde{V}}, (st(e))_{e \in \hat{E}}] \geq \frac{1}{F} \int_0^{\tilde{Y}_c} g(z) \, dz,
\]

and so the claim holds.

\[\square\]

With Propositions \([3.7]\) and \([3.8]\) the proof of Lemma \([3.6]\) follows easily.

**Proof of Lemma \([3.6]\).** We first observe that by taking the appropriate conditional expectation, Proposition \([3.7]\) ensures that

\[
\mathbb{E}[\alpha_{u_0} | (Y_v)_{v \in \tilde{V}}, (st(e))_{e \in \hat{E}}] \geq \frac{w_{u_0}}{F} \cdot (1 - g(\tilde{Y}_c)),
\]

where the right-hand side follows since \( \tilde{Y}_c \) is entirely determined from \((Y_v)_{v \in \tilde{V}}\) and \((st(e))_{e \in \hat{E}}\). Thus, combined with Proposition \([3.8]\),

\[
\mathbb{E}[p_{u_0, v_0} \cdot \alpha_{u_0} + w_{u_0} \cdot p_{u_0, v_0} \cdot \sum_{R \subseteq U: u_0 \in R} \phi_{v, R} | (Y_v)_{v \in \tilde{V}}, (st(e))_{e \in \hat{E}}],
\]

is lower bounded by

\[
\frac{w_{u_0} \cdot p_{u_0, v_0}}{F} \cdot (1 - g(\tilde{Y}_c)) + \frac{w_{u_0} \cdot p_{u_0, v_0}}{F} \int_0^{\tilde{Y}_c} g(z) \, dz.
\]

However, \( g(z) := \exp(z - 1) \) for \( z \in [0, 1] \) by assumption, so

\[
(1 - g(\tilde{Y}_c)) + \int_0^{\tilde{Y}_c} g(z) \, dz = \left(1 - \frac{1}{e}\right),
\]

no matter the value of the critical time \( \tilde{Y}_c \). As such, since \( F := 1 - 1/e \), we may apply the law of iterated expectations and conclude that

\[
\mathbb{E}[p_{u_0, v_0} \cdot \alpha_{u_0} + w_{u_0} \cdot p_{u_0, v_0} \cdot \sum_{R \subseteq U: u_0 \in R} \phi_{v, R}] \geq w_{u_0} \cdot p_{u_0, v_0}.
\]
As the vertices \( u_0 \in U \) and \( v_0 \in V \) were chosen arbitrarily, the proposed dual solution of Lemma 3.6 is feasible, and so the proof is complete.

\[ \square \]

4. Proving Theorems 2.4 and 2.5

In this section, we prove Theorems 2.4 and 2.5 and discuss their implications. Recall that for \( v \in V \) and \( R \subseteq U \), we defined \( \text{OPT}_{\text{non}}(v, R, \ell_v) = \text{OPT}_{\text{non}}(v, R) \) as the expected value of the non-committal benchmark when executing on the induced stochastic graph \( G\{v\} \cup R \). As we wish to prove competitive ratios against the non-committal benchmark, we first adjust constraint (3.15) of LP-DP to incorporate \( \text{OPT}_{\text{non}}(v, R, \ell_v) \), thereby allowing us to model the more powerful non-committal benchmark.

\[
\begin{align*}
\text{maximize} & \quad \sum_{u \in U} \sum_{v \in V} w_u \cdot p_{u,v} \cdot x_{u,v} \quad \text{(LP-DP-non)} \\
\text{subject to} & \quad \sum_{v \in V} p_{u,v} \cdot x_{u,v} \leq 1 \quad \forall u \in U \quad (4.1) \\
& \quad \sum_{u \in R} w_u \cdot p_{u,v} \cdot x_{u,v} \leq \text{OPT}_{\text{non}}(v, R, \ell_v) \quad \forall v \in V, R \subseteq U \quad (4.2) \\
& \quad x_{u,v} \geq 0 \quad \forall u \in U, v \in V \quad (4.3)
\end{align*}
\]

Let us denote \( LPOPT_{\text{DP}-\text{non}}(G) \) as the value of an optimum solution to LP-DP-non. We claim that LP-DP-non is a relaxation of the non-committal benchmark.

**Theorem 4.1.** For any stochastic graph \( G = (U, V, E) \),

\[ \text{OPT}_{\text{non}}(G) \leq LPOPT_{\text{DP}-\text{non}}(G). \]

The proof of Theorem 4.1 employs the same techniques the authors used in [3] for showing that LP-std relaxes the non-committal benchmark. As such, we defer the proof to Appendix A and instead discuss the implications of this theorem.

Given a stochastic graph \( G = (U, V, E) \) and \( v \in V \) with patience \( \ell_v \), if \( \delta > 0 \), then we say that \( G\{v\} \cup U \) is \( \delta \)-vertex approximable, provided for each \( R \subseteq U \),

\[ \text{OPT}(v, R, \ell_v) \geq \delta \cdot \text{OPT}_{\text{non}}(v, R, \ell_v). \quad (4.4) \]

We say that \( G \) itself is \( \delta \)-vertex approximable, provided \( G\{v\} \cup U \) is \( \delta \)-vertex approximable for each \( v \in V \); that is, (4.4) holds for all the vertices of \( V \).

Observe that if \( G \) is \( \delta \)-vertex approximable, then we can relate the value of \( LPOPT_{\text{DP}}(G) \) to \( LPOPT_{\text{DP}-\text{non}}(G) \):

**Proposition 4.2.** If \( G \) is \( \delta \)-vertex approximable for some \( \delta > 0 \), then

\[ LPOPT_{\text{DP}}(G) \geq \delta \cdot LPOPT_{\text{DP}-\text{non}}(G). \]

By combining Theorem 4.1 and Proposition 4.2 we get the following result:

**Lemma 4.3.** Suppose we are presented a stochastic graph \( G \) with edge probabilities \((p_e)_{e \in E}\), offline vertex weights \((w_u)_{u \in U}\) and patience values \((\ell_v)_{v \in V}\). If Algorithm \( I \) returns the matching \( M \), then

\[ \mathbb{E}[\text{val}(M)] \geq \left(1 - \frac{1}{e}\right) \cdot \delta \cdot \text{OPT}_{\text{non}}(G), \]

provided \( G \) is rankable and \( \delta \)-vertex approximable for \( \delta > 0 \).
Theorem 2.4 ensure that one of the conditions of Example 2.2 holds for each \( v \in \mathcal{OPT} \) each \( v \in V \).

Proof of Theorem 2.4. Given the stochastic graph \( G \), to that \( v \) by induction on the patience value \( \ell \). R the same value by probing the edges of \( G \).

Proof. In Section 3 we proved Theorem 2.3 by observing that for any rankable graph \( G = (U, V, E) \), it holds that
\[
\mathbb{E}[\text{val}(M)] \geq \left(1 - \frac{1}{e}\right) \cdot \text{LPOPT}_{DP}(G),
\]
where \( M \) is the matching returned by Algorithm 1 when executing on \( G \).

On the other hand, by applying Proposition 4.2 and Theorem 4.1, we know that
\[
\text{LPOPT}_{DP}(G) \geq \delta \cdot \text{LPOPT}_{DP-non}(G) \geq \delta \cdot \text{OPT}_{non}(G).
\]

By combining these inequalities, the result holds.

Using Lemma 4.3, the proof of Theorem 2.4 now follows:

Proof of Theorem 2.4. Given the stochastic graph \( G = (U, V, E) \), recall that the assumptions of Theorem 2.4 ensure that one of the conditions of Example 2.2 holds for each \( v \in V \). That is, each \( v \in V \) satisfies one of the following:

1. Vertex \( v \) has unit patience or full patience; i.e., \( \ell_v \in \{1, |U|\} \).
2. The edge probabilities \((p_{u,v})_{u \in U}\) are non-decreasing with respect to the vertex weights \((w_u)_{u \in U}\); that is, for each \( u_1, u_2 \in U \), if \( p_{u_1,v} \leq p_{u_2,v} \) then \( w_{u_1} \leq w_{u_2} \).

In light of Lemma 4.3, it suffices to show that \( G \) is 1-vertex approximable. In other words, for each \( v \in V \) and \( R \subseteq U \),
\[
\text{OPT}(v, R, \ell_v) = \text{OPT}_{non}(v, R, \ell_v).
\]

Suppose first that \( v \in V \) satisfies [C1]. If \( \ell_v = 1 \) and \( R \subseteq U \), then \( \text{OPT}(v, R, 1) \) and \( \text{OPT}_{non}(v, R, 1) \) each correspond to probing \( u \in R \) such that \( w_u \cdot p_{u,v} \) is maximized. Thus,
\[
\text{OPT}(v, R, 1) = \text{OPT}_{non}(v, R, 1).
\]

If \( \ell_v = |U| \), then \( \text{OPT}_{non}(v, R, |U|) \) corresponds to probing all the edges of \( R \times \{v\} \) and matching \( v \) to that \( u \in U \) for which \( w_u \) is maximized and satisfies \( st(u, v) = 1 \). Clearly, \( \text{OPT}(v, R) \) can attain the same value by probing the edges of \( R \times \{v\} \) in non-increasing order of the corresponding vertex weights - i.e., \((w_u)_{u \in R}\). Thus, in this scenario as well,
\[
\text{OPT}(v, R, |U|) = \text{OPT}_{non}(v, R, |U|).
\]

Now, consider the setting in which \( v \) satisfies [C2]. In order to prove this equality, we proceed by induction on the patience value \( \ell_v \) of \( v \). Clearly, if \( \ell_v = 1 \), then the desired inequalities hold, as we have just proven.

Let us now take \( 2 \leq \ell_v \leq |U| \), and assume that for each \( S \subseteq U \),
\[
\text{OPT}(v, S, \ell_v - 1) = \text{OPT}_{non}(v, S, \ell_v - 1). \quad (4.5)
\]

If \( R \subseteq U \), then observe that we may assume that the first probe made by \( \text{OPT}_{non}(v, R, \ell_v) \) is to the edge \((u_1, v) \in R \times \{v\}\) such that \( w_{u_1} \) is maximized. Moreover, if \( st(u_1, v) = 1 \), then we may assume that \( \text{OPT}_{non}(v, R, \ell_v) \) will return the edge \((u_1, v)\), as the other vertices of \( R \) are of weight no greater than \( w_{u_1} \). Thus,
\[
\text{OPT}_{non}(v, R, \ell_v) = w_{u_1} \cdot p_{u_1,v} + (1 - p_{u_1,v}) \cdot \text{OPT}_{non}(v, R \setminus \{u_1\}, \ell_v - 1).
\]

Similarly, the committal benchmark may also be assumed to probe the edge \((u_1, v)\) first, so
\[
\text{OPT}(v, R, \ell_v) = w_{u_1} \cdot p_{u_1,v} + (1 - p_{u_1,v}) \cdot \text{OPT}(v, R \setminus \{u_1\}, \ell_v - 1).
\]

The result now follows by applying the induction hypothesis (4.5).
We now consider the framework of Theorem 2.5 in which $G = (U, V, E)$ is rankable, yet may not satisfy the conditions of Theorem 2.4. It is necessary to first describe a stochastic probing problem known as ProblemMax, as studied in [1], [8], [19].

Suppose that $N \geq 1$ independent random variables are defined, say $X = (X_1, \ldots, X_N)$, each of which has a known distribution, yet whose instantiated values are hidden. Moreover, fix a parameter $1 \leq k \leq N$. A probing strategy for ProblemMax probes $k$ of the random variables $X_1, \ldots, X_N$ in some adaptive order, thus revealing their instantiated values. If $P \subseteq [N]$ corresponds to the indices of the $k$ random variables probed, then the goal of the strategy is to maximize

$$\mathbb{E}[\max_{i \in [N]} X_i \cdot 1_{[i \in P]}].$$

We emphasize that a strategy can be adaptive, i.e., the $i^{th}$ random variable probed may depend on the values of the previously probed random variables. We define $\text{OPT}_{\text{adapt}}(X, k)$ as the largest expected value attainable by an adaptive probing strategy on the instance specified by $X_1, \ldots, X_N$ and $1 \leq k \leq N$.

We can also consider non-adaptive probing strategies. These are strategies which decide upon which random variables to probe, prior to revealing any of the values of $X_1, \ldots, X_N$. Observe that we can encode such a strategy as a subset $S \subseteq [N]$, and then the value of the non-adaptive strategy is precisely $\mathbb{E}[\max_{i \in S} X_i]$.

Let us denote $\text{OPT}_{\text{non-adapt}}(X, k)$ as the largest expected value of a non-adaptive probing strategy on the specified instance. Observe then that

$$\text{OPT}_{\text{non-adapt}}(X, k) = \sup_{S \subseteq [N]: |S| = k} \mathbb{E}[\max_{i \in S} X_i].$$

In [1], Asadpour et al. proved that

$$\frac{\text{OPT}_{\text{non-adapt}}(X, k)}{\text{OPT}_{\text{adapt}}(X, k)} \geq 1 - \frac{1}{e},$$

no matter the instance $(X, k)$ of ProblemMax.

Returning to the stochastic matching problem on the graph $G = (U, V, E)$, we can view the stochastic matching problem on the subgraph $G([v] \cup U)$ for $v \in V$ as a special case of ProblemMax, thus allowing us to make use of (4.6) to complete the proof of Theorem 2.5.

**Proof of Theorem 2.5.** Suppose that $G = (U, V, E)$ is a rankable stochastic graph. In light of Lemma 4.3, it suffices to show that $G$ is $(1 - 1/e)$-vertex approximable. That is, for each $v \in V$ and $R \subseteq U$, we must show that

$$\text{OPT}(v, R, \ell_v) \geq \left(1 - \frac{1}{e}\right) \cdot \text{OPT}_{\text{non}}(v, R, \ell_v).$$

Since the patience value $\ell_v$ is arbitrary, it suffices to show that this inequality holds for $R = U$, and so we restrict our attention to this case for the remainder of the proof.

Observe that if we define the random variable $X_{u,v} := w_u \cdot \text{st}(u, v)$ for each $u \in U$, then the random variables $(X_{u,v})_{u \in U}$ together with $\ell_v$ correspond to an instance of ProblemMax. In particular,

$$\text{OPT}_{\text{adapt}}((X_{u,v})_{u \in U}, \ell_v) = \text{OPT}_{\text{non}}(v, U, \ell_v)$$

On the other hand, we also claim that

$$\text{OPT}_{\text{non-adapt}}((X_{u,v})_{u \in U}, \ell_v) = \text{OPT}(v, U, \ell_v)$$

In order to see this, suppose that $S \subseteq U$ corresponds to an non-adaptive probing strategy of the random variables $(X_{u,v})_{u \in U}$, where $|S| = \ell_v$. This yields a committal probing algorithm for the instance $G([v] \cup S)$, whose expected value is equal to $\text{OPT}_{\text{non-adapt}}((X_{u,v})_{u \in U}, \ell_v)$. Specifically,
probe the edges of $S \times \{v\}$ in non-increasing order of the associated vertex weights $(w_u)_{u \in S}$. As a result,

$$\text{OPT}(v, U, \ell_v) \geq \text{OPT}_{\text{non-adapt}}((X_{u,v})_{u \in U}, \ell_v).$$

On the other hand, the Star-DP algorithm of Brubach et al. [4] operates by first specifying a fixed tuple $(u_1, \ldots, u_{\ell_v})$ of distinct vertices of $U$ such that $w_{u_1} \geq \ldots \geq w_{u_{\ell_v}}$. It then probes the edges $(u_i, v)_{i=1}^{\ell_v}$ in this order, and matches $v$ to the first $u_j$ such that $st(u_j, v) = 1$ (see Section 2 for details). Clearly, this corresponds to a non-adaptive probing strategy for the ProblemMax instance: simply probe the random variables $(X_{u_i,v})_{i \in [\ell_v]}$ in any order. Moreover, the expected value attained by this probing strategy is equal to the expected value attained by Star-DP. As such,

$$\text{OPT}_{\text{non-adapt}}((X_{u,v})_{u \in U}, \ell_v) \geq \text{OPT}(v, U, \ell_v),$$

and so $\text{OPT}(v, U, \ell_v) = \text{OPT}_{\text{non-adapt}}((X_{u,v})_{u \in U}, \ell_v)$.

By combining this equation with (4.7) and (4.6), it holds that

$$\text{OPT}(v, U, \ell_v) \geq \left(1 - \frac{1}{e}\right) \cdot \text{OPT}_{\text{non}}(v, U, \ell_v),$$

and so the proof is complete.

We conclude the section by mentioning that it is not known whether $1 - 1/e$ is a tight lower bound on the ratio

$$\frac{\text{OPT}(v, U, \ell_v)}{\text{OPT}_{\text{non}}(v, U, \ell_v)},$$

for $G = (U, V, E), v \in V$ and patience $\ell_v$. In fact, this is not even known in the more general ProblemMax setting. Clearly, if we could improve the lower bound on this ratio, then this would allow us to strengthen the competitive ratio of Theorem 2.5.

We do however know that this ratio can be as large as 0.856, as demonstrated by an example in [3]. Thus, $0.856 \times (1 - 1/e) \approx 0.541$ is the best possible competitive ratio against the non-committal benchmark we could hope to prove via these techniques.

5. Conclusion and Open Problems

We considered the online stochastic bipartite matching (with arbitrary patience values and offline vertex weights) problem in the random order input model (ROM). We noted how the stochastic setting generalizes the classical (i.e., non-stochastic) setting. We introduced the non-committal benchmark, which arguably is the strongest meaningful benchmark. Our main result shows that for a broad class of stochastic graphs, we obtain a simple deterministic greedy algorithm with competitive ratio $1 - \frac{1}{e} \approx 0.632$. Currently, even for the classical (i.e., non-stochastic) setting, $1 - \frac{1}{e}$ is the best ratio known for deterministic algorithms with random order input arrivals. The strongest ROM in-approximation of 0.823 (due to Manshadi et al. [15]) comes from the classical i.i.d. setting for a known distribution.

Our work leaves open many questions:

- Is there a provable difference between what an optimal online algorithm can obtain against the committal benchmark versus the non-committal benchmark? Specifically, does Algorithm [1] achieve a competitive ratio of $1 - 1/e$ against the non-committal benchmark which holds for all rankable stochastic graphs or even for all stochastic graphs?
- What is the best ratio that a deterministic online algorithm can obtain for all stochastic graphs in the ROM setting? Brubach et al. achieve the optimal competitive ratio 1/2 for adversarial input arrivals and hence the same bound holds for ROM arrivals. Can we at least make an improvement on 1/2 which holds for all stochastic graphs?
• Is the optimal competitive ratio for the stochastic case (for either the committal or non-committal benchmark) worse than the optimal ratio for the classical setting? Note that in the classical setting the benchmark is the weight of an offline optimal matching.

• Can our $1 - \frac{1}{e}$ competitive ratio be improved by a randomized algorithm? Here we note that in the classical ROM setting, the Ranking algorithm achieves a 0.696 ratio for unweighted graphs (due to Mahdian and Yan [14]) and a 0.6534 ratio (due to Huang et al. [9]) for vertex weighted graphs. Thus, randomization does help in the classical ROM setting.

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Proof of Theorem 4.1. It will be convenient to instead work with a modified version of LP-DP-non, where each edge $e \in E$ is instead associated with two variables, namely $x_{e}$ and $z_{e}$. We interpret the former variable as the probability that the non-committal benchmark probes the edge $e$, whereas the latter variable corresponds to the probability that $e$ is included in the matching constructed by the non-committal benchmark.

$$\max \sum_{u \in U, v \in V} w_{u} \cdot z_{u,v} \text{ (LP-eq-DP-non)}$$

subject to

$$\sum_{v \in V} z_{u,v} \leq 1 \quad \forall u \in U \quad (A.1)$$

$$\sum_{u \in R} w_{u} \cdot z_{u,v} \leq \OPT_{\text{non}}(v, R, \ell_{v}) \quad \forall v \in V, R \subseteq U \quad (A.2)$$

$$z_{u,v} \leq p_{u,v} \cdot x_{u,v} \quad \forall u \in U, v \in V \quad (A.3)$$

$$x_{u,v}, z_{u,v} \geq 0 \quad \forall u \in U, v \in V \quad (A.4)$$

We denote $\LPOPT_{\text{eq}}^{\text{DP-non}}(G)$ as the value of an optimum solution to LP-eq-DP-non. It turns out that LP-eq-DP-non and LP-DP-non take the same optimum value, no matter the stochastic graph $G$. That is,

$$\LPOPT_{\text{eq}}^{\text{DP-non}}(G) = \LPOPT_{\text{DP-non}}(G). \quad (A.5)$$

To see this, first suppose we are presented a solution $(x_{u,v})_{u \in U, v \in V}$ to LP-DP. In this case, if $z_{u,v} := p_{u,v} \cdot x_{u,v}$ for $u \in U$ and $v \in V$, then $(x_{u,v}, z_{u,v})_{u \in U, v \in V}$ is clearly a feasible solution to LP-eq-DP-non. As such,

$$\LPOPT_{\text{DP-non}}(G) \leq \LPOPT_{\text{eq}}^{\text{DP-non}}(G).$$

On the other hand, suppose that $(x_{u,v}, z_{u,v})_{u \in U, v \in V}$ is now an arbitrary solution to LP-eq-DP-non. In this case, define $\bar{x}_{u,v} := z_{u,v}/p_{u,v}$ for each $u \in U, v \in V$. We claim that $(\bar{x}_{u,v})_{u \in U, v \in V}$ is a feasible solution to LP-DP-non.

First observe that for each $u \in U$,

$$\sum_{v \in V} p_{u,v} \cdot \bar{x}_{u,v} = \sum_{v \in V} z_{u,v} \leq 1,$$

and for each $R \subseteq U$ and $v \in V$,

$$\sum_{u \in R} w_{u} \cdot \bar{x}_{u,v} = \sum_{u \in R} p_{u,v} \cdot x_{u,v} \leq \OPT_{\text{non}}(v, R, \ell_{v}).$$

Thus, $(\bar{x}_{u,v})_{u \in U, v \in V}$ is a feasible solution to LP-DP-non.

Finally, observe that

$$\sum_{u \in U, v \in V} w_{u} \cdot z_{u,v} = \sum_{u \in U, v \in V} p_{u,v} \cdot w_{u} \cdot \bar{x}_{u,v},$$

so $\LPOPT_{\text{DP-non}}^{\text{eq}}(G) \leq \LPOPT_{\text{DP-non}}(G)$, and thus (A.5) holds.

In order to complete the proof, it suffices to show that

$$\OPT_{\text{non}}(G) \leq \LPOPT_{\text{eq}}^{\text{DP-non}}(G),$$

as we now know that $\LPOPT_{\text{eq}}^{\text{DP-non}}(G) = \LPOPT_{\text{DP-non}}(G)$.
Let us suppose that $\mathcal{M}$ is the matching returned by the non-committal benchmark when it executes on $G = (U, V, E)$. If we fix $u \in U$ and $v \in V$, then define $x_{u,v}$ as the probability the non-committal benchmark probes the edge $(u, v)$, and $z_{u,v}$ as the probability that it includes $e$ in $\mathcal{M}$. Observe then that

$$\text{OPT}_{\text{non}}(G) = \mathbb{E}[\text{val}(\mathcal{M})] = \sum_{u \in U, v \in V} w_u \cdot z_{u,v}.$$ 

Thus, we need only show that $(x_{u,v}, z_{u,v})_{u \in U, v \in V}$ is a feasible solution to LP-eq-DP-non as this will ensure that

$$\text{OPT}_{\text{non}}(G) = \sum_{u \in U, v \in V} w_u \cdot z_{u,v} \leq \text{LPOPT}_{\text{eq-DP-non}}(G).$$

If we first fix $u \in U$ and $v \in V$, then observe that in order for $(u, v)$ to be included in $\mathcal{M}$, $(u, v)$ must be probed and $(u, v)$ must be active. On the other hand, these two events occur independently of each other. As such,

$$z_{u,v} \leq p_{u,v} \cdot x_{u,v}.$$ 

Now, each $u \in U$ is matched at most once by the non-committal benchmark, thus

$$\sum_{v \in V} z_{u,v} \leq 1.$$ 

Finally, fix $v \in V$, and denote $\mathcal{M}(v)$ as the edge matched to $v$ (which is $\emptyset$ by convention if $v$ remains unmatched), and denote $\text{val}(\mathcal{M}(v))$ as the weight of the vertex $v$ is matched to (which is 0 provided $v$ remains unmatched). Observe first that

$$\sum_{u \in U} w_u \cdot z_{u,v} = \mathbb{E}[\text{val}(\mathcal{M}(v))].$$

Moreover, executing the non-committal benchmark on $G$ induces a probing strategy on $G[\{v\} \cup U]$, which we denote by $B_v$. However, observe that since the non-committal benchmark decides upon which edges to match after all its probes are made, so does $B_v$. Specifically, the match it makes to $v$ is determined once all its probes to $U \times \{v\}$ are made. Clearly, the expected value of this match is equal to $\mathbb{E}[\text{val}(\mathcal{M}(v))]$ and can be no larger than $\text{OPT}_{\text{non}}(v, U, \ell_v)$. Thus,

$$\sum_{u \in U} w_u \cdot z_{u,v} = \mathbb{E}[\text{val}(\mathcal{M}(v))] \leq \text{OPT}_{\text{non}}(v, U, \ell_v).$$

More generally, if we fix $R \subseteq U$, then

$$\sum_{u \in U} w_u \cdot z_{u,v} = \mathbb{E}[\text{val}(\mathcal{M}(v)) \cdot 1_{[\mathcal{M}(v) \in R \times \{v\}]\}} \leq \text{OPT}_{\text{non}}(v, R, \ell_v).$$

To see this, consider a modification of $B_v$, say $B_v(R)$, which matches $v$ to $u \in U$ if and only if $B_v$ matches $v$ to $u$ and $(u, v) \in R \times \{v\}$.

This shows that all the constraints of LP-eq-DP-non hold for $(x_{u,v}, z_{u,v})_{u \in U, v \in V}$, and so the proof is complete. \hfill \Box

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8 The strategy $B_v$ can be defined formally by first drawing (simulated) independent copies of the edge states which are not adjacent to $v$, say $\text{st}(e)_{e \in E \setminus v \notin e}$. By executing the non-committal benchmark on $G$ with $\text{st}(e)_{e \in E \setminus v \notin e}$ and $\text{st}(e)_{e \in U \times \{v\}}$, we get the desired strategy on $G[\{v\} \cup U]$. 