POLITE ACTIONS OF NON-COMPACT LIE GROUPS

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ABSTRACT. Based mainly on examples of interest in mechanics, we define the notion of a polite group action. One may view this as not only trying to give a more general notion than properness of a group action, but also to more fully understand the role of invariant functions in describing just about everything of interest in reduction.

We show that a polite action of a symmetry group of a dynamical system admits reduction and reconstruction.

Dirac’s seminal 1950 paper (6) showed how to construct a reduced bracket on a Hamiltonian system with constraints, but did not focus on constraints generated by the action of a symmetry group. The first significant theory of reduction of a Hamiltonian system with symmetry was given by Meyer in 1973 (12), and this was followed by work of Marsden and Weinstein a year later (13). Since then there has been a veritable flood of papers endeavouring to understand reduction and various forms of singular behaviour. For example, many of these works have studied what happens when the action of the symmetry group is not free and quotient spaces are not manifolds. It is probably fair to say that a reasonably complete reduction theory now exists in the case that the group action is proper (see, for example, 2, 5, 11, 12, 13.) Here we make the case that since there are interesting, important examples in mechanics where the symmetry group does not act properly, a less restrictive notion of group action warrants consideration.

This paper defines the notion of a polite action, and gives some examples. In addition, it proves that a polite action of the symmetry group of a dynamical system admits reduction and reconstruction. This means that the dynamical vector field projects to a vector field on a reduced space, and that the original dynamics can be recovered from the dynamics on the reduced space. This is all done in the context of vector fields and differential equations on manifolds.

Since the possibility exists that our notion of a polite action is not the last word on group actions in mechanics, we hope that, in the spirit of this commemorative volume, others will provide even better solutions to the problem of ‘what’s next’.

1. MOTIVATING EXAMPLES

The following examples motivate why one needs to deal with problems where the group action is not proper, so that strictly speaking the usual reduction theories do not apply.

1
(1) The one-dimensional harmonic oscillator. Here the Hamiltonian is \( h(p,q) = \frac{1}{2}p^2 + \frac{1}{2}q^2 \) on the phase space \( P = T^*\mathbb{R} \). All solutions of Hamilton’s equations are periodic with period \( 2\pi \). The Hamiltonian flow \( \phi_t \) is

\[
\phi_t \left( \begin{array}{c} q_0 \\ p_0 \end{array} \right) = \left( \begin{array}{cc} \cos t & \sin t \\ -\sin t & \cos t \end{array} \right) \left( \begin{array}{c} q_0 \\ p_0 \end{array} \right) .
\]

The action of \( \mathbb{R} \) on \( P \) is not proper but is indistinguishable from the free proper action of the compact group \( \mathbb{R}/2\pi\mathbb{Z} \).

(2) The stiff spring. The Hamiltonian is, for \( \epsilon > 0 \),

\[
h(q,p) = \frac{1}{2}p^2 + \frac{1}{2}q^2 + \frac{\epsilon}{4}q^4 .
\]

Hamilton’s equations yield Duffing’s equation \( q'' + q + \epsilon q^3 = 0 \). This implies that the solution may be written in terms of the Jacobi elliptic function \( cn \) as

\[
q(t) = cn \left( \sqrt{1+\epsilon}t; \sqrt{\frac{\epsilon}{2(1+\epsilon)}} \right) .
\]

Here the parameters are chosen so that \( q(t) \) solves the initial value problem

\[
q'' + q + \epsilon q^3 = 0, \quad q(0) = 1, \quad q'(0) = 0,
\]

for \( \epsilon > 0 \). It follows that the period \( \tau \) is

\[
\tau = \frac{4}{\sqrt{1+\epsilon}} K \left( \sqrt{\frac{\epsilon}{2(1+\epsilon)}} \right) ,
\]

\[
= 2\pi \left( 1 - \frac{3}{8} \epsilon + \frac{57}{256} \epsilon^2 + \cdots \right) .
\]

where \( K(k) \) is the complete elliptic integral of the first kind. It is now easy to solve for other initial conditions to find the period as a function of the energy \( h \) and \( \epsilon \). The Hamiltonian flow \( \phi_t \), which is an action of \( \mathbb{R} \) on \( P = T^*\mathbb{R} \) is still periodic, but not proper. In this case there is no fixed subgroup \( G \) of \( \mathbb{R} \) with the flow \( \phi_t \) being a proper \( \mathbb{R}/G \) action (although we can do this individually for each orbit.) However, it is common practice in mechanics to rescale the Hamiltonian vector field \( X_h \) by the period \( \tau \) to produce a new vector field \( Y = \tau X_h \), all of whose integral curves are periodic of period 1. It is a theorem that the resulting vector field \( Y \) is still a Hamiltonian vector field, and we produce a new variable called the action (see, for example [4].) In this way a free proper action \( \psi_t \) of the compact group \( SO(2) \) is associated to the original nonproper action \( \phi_t \) by setting \( \psi_t := \phi_{\tau t} \).

(3) The champagne bottle. The Hamiltonian in this case is

\[
h = \frac{1}{2}(p_1^2 + p_2^2) + (q_1^2 + q_2^2)^2 - (q_1^2 + q_2^2)
\]

on the phase space \( P = T^*\mathbb{R}^2 \). This is a completely integrable system because of the rotational invariance. The Hamiltonian \( h \), together with the angular momentum \( j \) gives the construction of action variables \( (I_1, I_2) \).
that generate a torus action whose orbits contain the original quasiperiodic
trajectories of the Hamiltonian. In this way, a proper group action is asso-
ciated to the non-proper Hamiltonian action of $\mathbb{R}^2$ associated to the flow
of the commuting Hamiltonian vector fields of the energy and the angu-
lar momentum (see [3] for more details.) However, what is interesting in
this case is that the construction of the actions is only local because of the
presence of an obstruction called monodromy preventing the torus group
action being globally well-defined (see [1].)

(4) A nonabelian example. We construct an oriented $S^3$ bundle over $\mathbb{R}^3 \setminus \{0\}$. The fiber $S^3$ is diffeomorphic to the group $\text{Spin}(3)$, but the bundle is not a principal bundle. In a sense, we may view this example as a simply-connected version of the previous example.

To start, consider the two copies of the trivial bundle $D^2 \times S^3$, which we think of as local trivializations of our bundle over the upper and lower hemispheres of the sphere $S^2$. Viewing $S^3$ as the unit sphere in $\mathbb{R}^4$, we consider the gluing map from one hemisphere to another as a map from the equator into $\text{Diff}^+(S^3)$. By a theorem of Hatcher [7], this diffeomorphism group retracts onto the orthogonal group $\text{SO}(4)$. The orthogonal group is diffeomorphic to the product $\text{SO}(3) \times \text{Spin}(3)$, and has fundamental group $\mathbb{Z}_2$. The transition map from one hemisphere to the other is given by the map

$$S^1 \rightarrow \text{SO}(4) : \phi \mapsto \begin{pmatrix} \cos \phi & -\sin \phi & 0 & 0 \\ \sin \phi & \cos \phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  

This map is a generator of the fundamental group of $\text{SO}(4)$ because the matrix represented by the upper left $2 \times 2$ block is a generator of the fundamental group of $\text{SO}(2)$, and we have the natural inclusions

$$\text{SO}(2) \hookrightarrow \text{SO}(3) \hookrightarrow \text{SO}(4)$$

and thus a surjection in homotopy $\pi_1(\text{SO}(2)) \rightarrow \pi_1(\text{SO}(4))$. This im-
plies that the bundle is not a trivial bundle.

Observe that the south pole $(0, 0, 0, 1)$ on the sphere $S^3$ is fixed by the transition map, and this implies that the map $S^2 \rightarrow \text{‘south pole’}$ is a global section of the bundle. This fact, together with the nontriviality of the bundle implies that the bundle is not a principal $\text{Spin}(3)$ bundle, as any principal bundle with a global section must be globally trivial.

Reviewing this example from the point of view of classifying spaces suggests that many more such examples may be constructed by considering $\text{Spin}(3)$ bundles over the four-sphere $S^4$.

The bundle constructed here may be given a symplectic structure by embedding the sphere $S^2$ into $\mathbb{R}^3 \setminus 0$ in the usual way. In more detail, let
$S^2$ be $x_1^2 + x_2^2 + x_3^2 = 1$, and $\psi_1, \psi_2, \psi_3$ be the usual left-invariant one-forms on $\text{Spin}(3)$. Then the form

$$\omega = \psi_1 \wedge \psi_2 + d(z(x_3)\psi_3) + dx_1 \wedge dx_2$$

is a symplectic form on our bundle where $z(x_3)$ is a function that satisfies

1) $z'(x_3) > 0$ for all $x_3$, and
2) $|z(x_3)| < 1$ for all $x_3$. For example, we may take $z(x) = x/\sqrt{1 + x^2}$.

(5) Consider the Hamiltonian system given by the motion of the free particle in space (you can take any dimension $n \geq 2$ for space.) The Euclidean group $SE(n)$ acts in a Hamiltonian way on the phase space $T^*\mathbb{R}^n$ and preserves the level set $h^{-1}(1/2)$, which are the straight lines parametrized by arclength. We are of course taking the Hamiltonian to be $h = |p|^2/2$. The components of the momentum map for the Euclidean group are the linear and angular momentum, and as they commute with the Hamiltonian, they pass to an action on the quotient space $\bar{P} := h^{-1}(1/2)/\sim$, where the $\sim$ represents the quotient by the Hamiltonian flow $\phi_t(q,p) = (q + tp, p)$. The quotient manifold $\bar{P}$, which is the space of oriented lines in $\mathbb{R}^n$, is naturally endowed with a symplectic structure, as follows from the reduction theorem. Furthermore, the action of the Euclidean group on the quotient $\bar{P}$ is Hamiltonian. This action is not fixed point free and is not proper, as the stability subgroup of a point in the quotient contains the subgroup which corresponds to the translations along the line that it represents. More precisely, the space of lines is the homogeneous space $SE(n)/(SO(n) \times \mathbb{R}) \sim T^*S^n$. This construction is used when studying the Radon transform, as it involves integration along lines.

\section{Polite actions}

Consider a Hamiltonian system $(P, \omega, h)$ invariant under the action $\phi$ of a connected Lie group $G$. Given a closed subgroup $H$ of $G$, define

$$P_H := \{ p \in P \mid G_p = H \}.$$

Denote by $N^H$ the normalizer of $H$ in $G$; that is

$$N^H = \{ n \in G \mid n^{-1}hn \in H \text{ for all } h \in H \}.$$ 

The normalizer is a closed subgroup of $G$.

\textbf{Lemma 2.1.} The action of $N^H$ on $P$ preserves $P_H$.

\textbf{Proof.} For $p \in P_H$, and $n \in N^H$, the isotropy group $G_{np}$ of $np$ is given by

$$G_{np} = \begin{cases} \{ g \in G \mid gnp = np \} \\ \{ g \in G \mid n^{-1}gpn = p \} \\ \{ g \in G \mid n^{-1}gn \in H \}. \end{cases}$$

In other words, $g \in G_{np}$ if and only if $h = n^{-1}gn \in H$. Therefore, $g = nhn^{-1} \in H$, and $G_{np} = H$. Hence, $np \in P_H$. Thus, the action of $N^H$ on $P$ preserves $P_H$. \hfill q.e.d.
Since the action of $N^H$ on $P$ preserves $P_H$, it induces an action of $N^H$ on $P_H$. Let $G_H = N^H/H$. Since $H$ is closed in $G$, it is closed in $N^H$, and $G_H$ is a Lie group. Moreover, there is an action

$$G_H \times P_H \to P_H : ([n], p) = np,$$

where $[n]$ is the equivalence class of $n$ in $G_H = N^H/H$.

Warning: The group $G_H$ is not a subgroup of the group $G$.

**Proposition 2.2.** The action of $G_H$ on $P_H$ is free.

*Proof.* For $g \in N^H$, suppose $[g] \in G_H$ preserves a point $p \in P_H$; that is $gp = p$. This means that $g \in G_p = H$. Therefore, $[g]$ is the identity in $G_H$. q.e.d.

**Definition 2.3.** The action of $G$ on $P$ is polite if for each closed subgroup $H$ of $G$, the set $P_H$ is a manifold and the action of $G_H$ on $P_H$ is proper.

### 3. Examples of polite actions

It is straightforward to check that the group action in all of the following examples is polite.

**Example 3.1.** The actions in the motivating examples 1,2,3,5 are all polite.

**Example 3.2.** Every action of a compact group is polite because it is proper.

**Example 3.3.** The $\mathbb{R}$ action generated by the flow of the vector field $X = \sin x \partial_x + \cos x \partial_y$ on the plane is free but not polite.

**Example 3.4.** The coadjoint action of a compact connected Lie group is polite because this is just the action of the (finite) Weyl group $N^H/H$ on a coadjoint orbit.

**Example 3.5.** The co-adjoint action of $\text{SL}(2, \mathbb{R})$.

There are three co-adjoint orbits of interest, which we can label as parabolic, hyperbolic and elliptic since the group is semi-simple. The elliptic and hyperbolic ones correspond to Cartan subalgebras, so the corresponding stability groups are self-normalizing. The only nontrivial case is the parabolic one, and here the normalizer is the Borel subgroup which we may take as the upper triangular matrices. The quotient $N^H/H$ in this case acts by dilations on the cone (translations along the ruling), and the action is again seen to be proper.

**Example 3.6.** A special class of solvable groups of type $\mathcal{S}$.

Following Nomizu [10], we say that a group $G$ belongs to the class $\mathcal{S}$ if the Lie algebra $\mathfrak{g}$ of the Lie group $G$ contains a codimension one commutative ideal $\mathfrak{a}$ and an element $Y$ with the property that $[Y, X] = X$ for all $X$ in the ideal $\mathfrak{a}$. Let $X_1, \ldots, X_n$ be a basis for $\mathfrak{a}$, and let $X_1^*, \ldots, X_n^*, Y^*$ be the corresponding dual basis in the dual space $\mathfrak{g}^*$. Let $(a, b)$ be an element in the half-space $\mathbb{R}^+ \times \mathbb{R}$,
and consider the point \( \mu = aX_1^* + bY^* \in g^* \). Then the non-zero infinitesimal generators of the co-adjoint action are generated by

\[
\text{ad}^*_{X_1^*} |_{\mu} = -a \frac{\partial}{\partial Y^*}, \quad \text{ad}^*_Y |_{\mu} = -a \frac{\partial}{\partial X_1^*}.
\]

This implies that the co-adjoint orbit through the point \( \mu \) is the two-dimensional open half plane spanned by \( Y^* \) and \( \mu \). It follows that the Lie algebra \( h = \text{span} \{ X_2, \ldots, X_n \} \), and hence that the isotropy group \( H \sim \mathbb{R}^{n-1} \). Thus the normalizer \( N^H = G \) and

\[
N^H / H \sim \text{Aff}^+(1, \mathbb{R})
\]

acts freely, transitively and properly on the co-adjoint orbit through \( \mu \). Note that the action is just the usual action of the affine group on the half-plane.

In light of the previous two examples we make the following

**Conjecture 3.7.** The coadjoint action of a Lie group on the dual of its Lie algebra is polite for any group in which the coadjoint orbits are locally closed.

### 4. Reduction and Reconstruction of Polite Symmetries

We consider a dynamical system given by a smooth vector field \( X \) on a manifold \( P \), called the phase space of the system. Evolutions of our dynamical system are integral curves \( \gamma : I \to P \) of \( X \), where \( I \) is an interval in \( \mathbb{R} \).

Let \( \Phi : G \times P \to P \) be an action of a Lie group \( G \) on \( P \). We say that \( G \) is a symmetry group of our dynamical system if the action \( \Phi \) preserves the vector field \( X \). The reduced phase space is the space \( \bar{P} = P / G \) of \( G \)-orbits in \( P \) endowed with a differential structure

\[
C^\infty(\bar{P}) = \{ f : \bar{P} \to \mathbb{R} | \rho^* f \in C^\infty(P)^G \},
\]

where \( \rho : P \to \bar{P} \) is the orbit map and \( C^\infty(P)^G \) is the ring of \( G \)-invariant smooth functions on \( P \). It should be noted that the orbit space \( \bar{P} \) has two topologies: the quotient space topology and the differential space topology. Here, we take the differential space topology. The reduced dynamical system is the derivation \( \rho_*X \) of \( C^\infty(\bar{P}) \) defined by

\[
(1) \quad \rho^*((\rho_*X)(f)) = X(\rho^* f)
\]

for every \( f \in C^\infty(\bar{P}) \).

**Proposition 4.1.** For every integral curve \( \gamma : I \to P \) of \( X \), the curve \( \rho \circ \gamma : I \to \bar{P} : t \mapsto \rho(\gamma(t)) \) satisfies the equation

\[
(2) \quad \frac{d}{dt} f(\rho(\gamma(t))) = ((\rho_*X)(f))(\rho(\gamma(t)))
\]

for each \( f \in C^\infty(\bar{P}) \) and \( t \in I \).

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1For applications of the theory of differential spaces to reduction of symmetries see [13].
**Proof.** It follows from equation (1) that

\[
\frac{d}{dt} f(\rho(\gamma(t))) = \frac{d}{dt} (\rho^* f)(\gamma(t)) = (X(\rho^* f))(\gamma(t)) = \rho^*((\rho_* X)(f))(\gamma(t)) = ((\rho_* X)(f))(\rho(\gamma(t))).
\]

q.e.d.

Equation (2) is called the *reduced equation*. A curve \( \rho \circ \gamma : I \to \bar{P} \) satisfying the reduced equation gives a reduced evolution of the system. Given a reduced evolution \( \bar{\gamma} : I \to \bar{P} \) of the system, the process of finding integral curves \( \gamma \) of \( X \) such that \( \rho \circ \gamma = \bar{\gamma} \) is called *reconstruction*. If the action \( \Phi \) of \( G \) on \( P \) is free and proper, the reduced equation as well as equations involved in reconstruction are ordinary differential equations on manifolds.

**Definition 4.2.** An action \( \Phi : G \times P \to P \) that preserves a vector field \( X \) on \( P \) admits reduction and reconstruction if the reduced equation and equations involved in reconstruction can be presented as differential equations on manifolds.

**Theorem 4.3.** A polite action \( \Phi : G \times P \to P \) that preserves a vector field \( X \) on \( P \) admits reduction and reconstruction.

We shall prove this theorem by a sequence of propositions.

**Proposition 4.4.** Let \( X \) be a vector field on \( P \) that is invariant under a polite action of a Lie group \( G \) on \( P \). For each closed subgroup \( H \) of \( G \), the flow of \( X \) preserves \( P_H \).

**Proof.** Let \( \exp tX \) be the local one-parameter group of local diffeomorphisms of \( P \) generated by \( X \), and \( H \) be a closed subgroup of \( G \). For each \( g \in H \) we have

\[
g \exp tX g^{-1} = \exp tX,
\]

because \( X \) is \( G \)-invariant and \( H \subseteq G \). Hence, for each \( p \in P_H \) and \( g \in H \),

\[
g \exp tX(p) = \exp tXgp = \exp tX(p),
\]

which implies that \( \exp tX(p) \in P_H \). q.e.d.

Let \( \bar{P}_H = \rho(P_H) \) and \( \rho_H : P_H \to \bar{P}_H \) be the restriction of \( \rho \) to \( P_H \). The following diagram

\[
\begin{array}{ccc}
P_H & \xrightarrow{i_H} & P \\
\rho_H \downarrow & & \downarrow \rho \\
\bar{P}_H & \xleftarrow{\iota_H} & \bar{P}
\end{array}
\]

where the horizontal arrows are the inclusion maps, commutes.

The space \( \bar{P}_H \) has the differential structure

\[
C^\infty_1(\bar{P}_H) = \{ h : \bar{P}_H \to \mathbb{R} \mid \rho^*_H h \in C^\infty(P_H) \}
\]

and a differential structure \( C^\infty_2(\bar{P}_H) \) generated by the restrictions to \( \bar{P}_H \) of smooth functions on \( \bar{P} \).
Proposition 4.5. The differential structures $C_2^\infty(\tilde{P}_H)$ and $C_1^\infty(\tilde{P}_H)$ are related by the inclusion

$$C_2^\infty(\tilde{P}_H) \subseteq C_1^\infty(\tilde{P}_H).$$

If the action of $G$ on $P$ is improper, $C_2^\infty(\tilde{P}_H)$ may be a proper subset of $C_1^\infty(\tilde{P}_H)$.

Proof. If $f \in C^\infty(\tilde{P})$, then $\epsilon_H^* f = f|_{\tilde{P}_H} \in C_2(\tilde{P}_H)$. On the other hand, $\rho^* f \in C^\infty(P)$ and the restriction of $\rho^* f$ to $P_H$ is an $N_H$-invariant smooth function $(\rho^* f)|_{P_H} = \iota_H^* \rho^* f$ on $P_H$. Moreover, $\rho \circ \iota_H = \epsilon_H \circ \rho_H$ implies that $\iota_H^* \rho^* f = \rho_H^* \epsilon_H^* f$. Therefore, $\epsilon_H^* f \in C_1^\infty(\tilde{P}_H)$.

Suppose now that $h : \tilde{P}_H \to \mathbb{R}$ is such that, for every $r \in \tilde{P}_H$, there exists a neighbourhood $U_r$ of $r$ in $\tilde{P}_H$ and a function $f_r \in C^\infty(P)$ such that $\epsilon_H^* f_r|_{U_r} = h|_{U_r}$. By definition of the differential structure generated by a family of functions, $f_r \in C_2^\infty(\tilde{P}_H)$. We have shown above that $\epsilon_H^* f_r \in C_1^\infty(\tilde{P}_H)$. Hence, $f_r|_{P_H \cap U_r} = f_r|_{U_r}$, which implies

$$C_2^\infty(\tilde{P}_H) \subseteq C_1^\infty(\tilde{P}_H).$$

On the other hand, suppose that $h \in C_1^\infty(\tilde{P}_H)$, which means that $\rho_H^* h \in C^\infty(P_H)^H$. The set

$$P_{(H)} = \{ gp \in P \mid g \in G, \ p \in P_H \}$$

is the union of the orbits of $G$ through points in $P_H$. We can extend the $H$-invariant function $\rho_H^* h$ on $P_H$ to a $G$-invariant function $k$ on $P_{(H)}$. If the action of $G$ on $P$ is not proper, we have no guarantee that a $G$-invariant function $k$ on $P_{(H)}$ extends to a $G$-invariant function on $P$, as may be seen in the following example. Let $X$ be the planar vector field

$$X = \sin x \partial_x + \cos x \partial_y.$$
By Proposition 4.4, for each closed subgroup $H$ of $G$, the flow \( \exp tX \) of the invariant vector field $X$ preserves $P_H$. The politeness of the action of $G$ on $P$ ensures that $P_H$ is a manifold and that the action of $G_H$ on $P_H$ is proper. Proposition 2.2 ensures that the action of $G_H$ on $P_H$ is free. Hence, $P_H/G_H$ is a quotient manifold of $P_H$, and $P_H$ has the structure of a left principal $G_H$-bundle over $P_H/G_H$. This implies that both the reduction and the reconstruction of the restriction of $X$ to $P_H$ is the same as in the case of a free and proper action. This completes the proof of Theorem 4.3.

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