DIOPHANTINE INHERITANCE AND DICHOTOMY FOR $p$-ADIC MEASURES

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Abstract. In this paper we prove complete $p$-adic analogues of Kleinbock’s theorems [19, 20] on inheritance of Diophantine exponents for affine subspaces as well as $0–1$ Diophantine dichotomies [21]. In particular, we answer in the affirmative (and in a stronger form), a conjecture of Kleinbock and Tomanov [24], as well as a question of Kleinbock [20]. Our main innovation is the introduction of a new $p$-adic Diophantine exponent which is better suited to homogeneous dynamics, and which we show to be closely related to the exponent considered by Kleinbock and Tomanov.

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1. Introduction

This paper is concerned with the study of $p$-adic Diophantine approximation on manifolds. We briefly recall the setting and basic results.

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from the paper [24] of Kleinbock and Tomanov.

For \( \mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{Z}^n \) and \( q_0 \in \mathbb{Z} \), set \( \tilde{\mathbf{q}} := (q_0, q_1, \ldots, q_n) \). Define the Diophantine exponent \( w(y) \) of \( y \in \mathbb{Q}_p^n \) to be the supremum of \( v > 0 \) such that there are infinitely many \( \tilde{\mathbf{q}} \in \mathbb{Z}^n \) satisfying

\[
|q_0 + \mathbf{q} \cdot y| \leq \|\tilde{\mathbf{q}}\|^{-v}.
\]

In view of Dirichlet’s theorem ([24] §11.2), \( w(y) \geq n + 1 \) for every \( y \in \mathbb{Q}_p^n \), with equality for Haar almost every \( y \) by the Borel-Cantelli lemma.

A vector \( y \in \mathbb{Q}_p^n \) is called very well approximable (VWA) if \( w(y) > n+1 \). The set of very well approximable vectors has zero Haar measure. Diophantine approximation on manifolds or “Diophantine approximation with dependent quantities” is concerned with the question of inheritance of Diophantine properties which are generic with respect to Lebesgue measure in \( \mathbb{R}^n \) or \( \mathbb{Q}_p^n \) with respect to Lebesgue measure, by appropriate proper subsets. The theory began with a conjecture of Mahler, that almost every point on the Veronese curve

\[
(x, x^2, \ldots, x^n) \subset \mathbb{R}^n
\]

is not very well approximable. Mahler made his conjecture in the context of a classification of numbers in terms of their approximation properties. We refer the reader to Bugeaud’s book [5] for details about developments in this area. Mahler’s conjecture was settled by Sprindzhuk, in the real, \( p \)-adic and positive characteristic (i.e. function fields in one variable over a finite field) contexts. It was conjectured by Sprindžuk in 1980 [31, 32] and proved by Kleinbock and Margulis [23] that real analytic manifolds not contained in any proper affine subspace of \( \mathbb{R}^n \) are extremal. They introduced methods from the ergodic theory of group actions into the subject and [23] has become a cornerstone of the subject. A further breakthrough was achieved by Kleinbock, Lindenstrauss and Weiss in [22], where a wide class of measures, including fractal measures as well as pushforwards of Lebesgue measure by nondegenerate maps which were previously introduced by Kleinbock and Margulis (the definition follows in the next paragraph) were studied in a unified manner. Subsequently, Kleinbock and Tomanov [24] proved an \( S \)-adic version of Sprindžuk’s conjectures and in [12], the second named author proved the positive characteristic version of Sprindžuk’s conjecture. In two important papers, [19, 20], D. Kleinbock systematically explored Diophantine approximation on affine subspaces and their nondegenerate submanifolds and the second named
The subject of $p$-adic Diophantine approximation started with the work of E. Lutz [25] and has seen numerous advances in different contexts over the years including early work of Mahler [26, 27]. We refer the reader to [5] for a comprehensive reference, to [24] and the references therein for work relating to $p$-adic metric Diophantine approximation on manifolds, and to [4, 1, 6, 8] for recent results.

In this paper, we provide a complete $p$-adic analogue of the results of D. Kleinbock [20] on Diophantine exponents of affine subspaces in both standard and multiplicative forms. Further, we also address the $p$-adic analogues of Kleinbock’s results on dichotomies in inheritance of Diophantine properties. This answers, in a stronger form, Conjecture IS of Kleinbock and Tomanov [24] as well as a question of D. Kleinbock [20].

While the methods of the present paper are heavily influenced by the work of D. Kleinbock, providing a complete $p$-adic analogue of his results poses substantial new difficulties. To circumvent these difficulties, one of the key innovations of the present paper is a new $p$-adic Diophantine exponent which measures approximation with respect to $\mathbb{Z}[1/p]$-points.

We now introduce some notation and state the main results in the present paper. For a Borel measure $\mu$ on $\mathbb{Q}_p^n$, we follow [20] in defining the Diophantine exponent $\omega(\mu)$ of $\mu$ to be

$$w(\mu) = \sup\{v : \mu(\{y \mid w(y) > v\}) > 0\}. \quad (1.2)$$

The exponent only depends on the measure class of $\mu$. Let $\lambda$ denote Haar measure on $\mathbb{Q}_p^n$ normalized so that $\mathbb{Z}_p^n$ has volume 1, the dimension being clear from the context. If $M \subset \mathbb{Q}_p^n$ is a $d$ dimensional analytic manifold, and $\mu$ is the pushforward of Haar measure on $\mathbb{Q}_p^d$ by a map parametrising $M$, then $\omega(M)$ is defined to be $\omega(\mu)$. Following Kleinbock [20], we say that a differentiable map $f : U \to \mathbb{Q}_p^n$, where $U$ is an open subset of $\mathbb{Q}_p^d$, is nondegenerate in an affine subspace $L$ of $\mathbb{Q}_p^n$ at $x \in U$...
if $f(U) \subset L$ and the span of all the partial derivatives of $f$ at $x$ up to some order coincides with the linear part of $L$. If $M$ is a $d$-dimensional submanifold of $L$, we will say that $M$ is nondegenerate in $L$ at $y \in M$ if there exists a diffeomorphism $f$ between an open subset $U$ of $Q^n_p$ and a neighbourhood of $y$ in $M$ is nondegenerate in $L$ at $f^{-1}(y)$. Finally, we will say that $f : U \to L$ (resp., $M \subset L$) is nondegenerate in $L$ if it is nondegenerate in $L$ at $\lambda$-a.e. point of $U$ (resp., of $M$, in the sense of the smooth measure class on $M$). Here is a special case of one of our main results.

**Theorem 1.1.** Let $L$ be an affine subspace of $Q^n_p$, and let $M$ be a submanifold of $L$ which is nondegenerate in $L$. Then

$$w(M) = w(L) = \inf \{ w(y) \mid y \in L \} = \inf \{ w(y) \mid y \in M \}. \quad (1.3)$$

In particular, this implies that if $L$ is an extremal subspace of $Q^n_p$, and $M$ is nondegenerate in $L$, then $M$ is extremal. As noted by Kleinbock in the context of real Diophantine approximation, the middle equality is non-obvious and non-trivial. The difficulty of proving it persists in the $p$-adic context and is in fact compounded by the difficulty, in the $p$-adic setting, of translating the Diophantine problem to dynamics and back. To overcome this challenge, we introduce the new idea of $\mathbb{Z}[1/p]$-exponents. We define these exponents $w_p(A)$ in the next section; the point is that these exponents seem to be better suited to techniques from homogeneous dynamics and in particular allow us to prove an “if and only if” Dani-type correspondence in the $p$-adic setting, which does not seem to be readily achieved whilst dealing with approximation of $p$-adic vectors by rational numbers. We suspect these exponents will find further use. We then explain how the $\mathbb{Z}[1/p]$-exponent is related to the usual exponent.

We will also provide a condition for $p$-adic extremality in terms of Diophantine properties of the parametrising matrix of an affine subspace.

**Theorem 1.2.** Let $L$ be an affine subspace, parametrized by a matrix $R_A$ as in (7.1). If all the rows are rational multiples of one row then one has

$$w_p(L) = \max(n, w_p(A)) \text{ and } w(L) = \max(n + 1, w(A)).$$

In [21], D. Kleinbock established a remarkable dichotomy with regards to certain Diophantine properties. In particular, he proved that for connected, analytic manifolds, having one not very well approximable point implies that almost every point is not very well approximable. We also address this in the $p$-adic context.
Theorem 1.3. Suppose $M$ is a connected analytic manifold of $\mathbb{Q}_p^n$. Let $v \geq n + 1$ and suppose $w(y) \leq v$ for some $y \in M$ then for almost every $y \in M$, $w(y) \leq v$.

Further, we address the topic of multiplicative Diophantine approximation in §9, proving, in particular, the multiplicative versions of Theorems 1.3 and 1.1 for the critical exponent. A special case of our results in the multiplicative setting is as follows.

Theorem 1.4. Let $\mathcal{L}$ be an affine subspace of $\mathbb{Q}_p^n$ and let $f: \mathbb{Q}_p^d \to \mathcal{L}$ be a $C^k$ map which is nondegenerate in $\mathcal{L}$ at $\lambda$ a.e. point. Suppose that the volume measure $\lambda$ on $\mathcal{L}$ is strongly extremal, then so is $f_\ast \lambda$.

We note that the multiplicative case also presents several additional difficulties in the $p$-adic setting.

Several interesting questions arise naturally including multiplicative inheritance theorems for arbitrary exponents, inhomogeneous theorems and Khintchine type theorems. We hope to return to some of these themes in subsequent work.

Diophantine approximation on affine subspaces plays in role in KAM theory, see [29, 33] and we hope that the present paper might also find applications. The setup of Diophantine approximation by $\mathbb{Z}[1/p]$-points has already been considered in considerable detail in the context of intrinsic Diophantine approximation on varieties by Ghosh, Gorodnik and Nevo [17, 18]. However, as we have mentioned, as far as we are aware the present paper is the first one to consider it in the context of (extrinsic) Diophantine approximation on manifolds, a problem with a distinct and different flavour to intrinsic approximation.

Structure of the paper. The main results of the paper are proved in the more general setting of Federer measures and nonplanar maps. In the next section we present these definitions and some more, leading up to the statement of Theorem 2.2 from which Theorem 1.1 follows. Section 3 introduces a $p$-adic exponent $w_p$ and discusses the relationship between $w_p$ and $w$. In §4 we prove an important dictionary in $p$-adic dynamics: namely the explicit connection between $p$-adic exponents and homogeneous dynamics. In §5 we extend the quantitative nondivergence theorem proved by Kleinbock to the $p$-adic setting. The subsequent section 6 deals with applications on nondivergence to $p$-adic exponents, in particular proving Theorem 2.2. The subsequent three
sections §’s 7, 8, 9 deal with higher Diophantine exponents, an almost all vs. no dichotomy and multiplicative Diophantine approximation.

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2. Preliminaries

**Measures and spaces.** A metric space $X$ is called Besicovitch [24] if there exists a constant $N_X$ such that the following holds: for any bounded subset $A$ of $X$ and for any family $B$ of nonempty open balls in $X$ such that

$$\forall x \in A \text{ is a center of some ball of } B,$$

there is a finite or countable subfamily $\{B_i\}$ of $B$ with

$$1_A \leq \sum_i 1_{B_i} \leq N_X.$$

As remarked in [24], any separable ultrametric space $X$ is Besicovitch with $N_X = 1$. We now define $D$-Federer measures following [22]. Let $\mu$ be a Radon measure on $X$, and $U$ an open subset of $X$ with $\mu(U) > 0$. We say that $\mu$ is $D$-Federer on $U$ if

$$\sup_{x \in \text{supp } \mu, r > 0} \frac{\mu(B(x, 3r))}{\mu(x, r)} < D.$$

Finally, we say that $\mu$ as above is Federer if for $\mu$-a.e. $x \in X$ there exists a neighbourhood $U$ of $x$ and $D > 0$ such that $\mu$ is $D$-Federer on $U$. We refer the reader to [22, 24] for examples of Federer measures.

Following, [20], for a subset $M$ of $\mathbb{Q}_p^n$, define its affine span $\langle M \rangle_a$ to be the intersection of all affine subspaces of $\mathbb{Q}_p^n$ containing $M$. Let $X$ be a metric space, $\mu$ a measure on $X$, $L$ an affine subspace of $\mathbb{Q}_p^n$ and $f$ a map from $X$ into $L$. Say that $(f, \mu)$ is nonplanar in $L$ if

$$L = \langle f(B \cap \text{supp } \mu) \rangle_a \forall \text{ nonempty open } B \text{ with } \mu(B) > 0.$$
\((C, \alpha)\)-good functions. In this section, we recall the notion of \((C, \alpha)\)-good functions on ultrametric spaces. We follow the treatment of Kleinbock and Tomanov [24]. Let \(X\) be a metric space, \(\mu\) a Borel measure on \(X\) and let \((F, \cdot|\cdot)\) be a valued field. For a subset \(U\) of \(X\) and \(C, \alpha > 0\), say that a Borel measurable function \(f : U \to F\) is \((C, \alpha)\)-good on \(U\) with respect to \(\mu\) if for any open ball \(B \subset U\) centered in \(\text{supp} \mu\) and \(\varepsilon > 0\) one has

\[
\mu \left( \{ x \in B \mid |f(x)| < \varepsilon \} \right) \leq C \left( \frac{\varepsilon}{\sup_{x \in B} |f(x)|} \right)^\alpha |B|.
\]

(2.1)

Where \(\|f\|_{\mu,B} = \sup \{ c : \mu(\{ x \in B : |f(x)| > c \}) > 0 \}\).

The following elementary properties of \((C, \alpha)\)-good functions will be used.

(G1) If \(f\) is \((C, \alpha)\)-good on an open set \(V\), so is \(\lambda f \forall \lambda \in \mathbb{R}\);

(G2) If \(f_i, i \in I\) are \((C, \alpha)\)-good on \(V\), so is \(\sup_{i \in I} |f_i|\);

(G3) If \(f\) is \((C, \alpha)\)-good on \(V\) and for some \(c_1, c_2 > 0\), \(c_1 \leq |f(x)| \leq c_2\) for all \(x \in V\), then \(g\) is \((C(c_2/c_1)^\alpha, \alpha)\)-good on \(V\).

(G4) If \(f\) is \((C, \alpha)\)-good on \(V\), it is \((C', \alpha')\)-good on \(V'\) for every \(C' \geq \max\{C, 1\}, \alpha' \leq \alpha\) and \(V' \subset V\).

One can note that from (G2), it follows that the supremum norm of a vector valued function \(f\) is \((C, \alpha)\)-good whenever each of its components is \((C, \alpha)\)-good. Furthermore, in view of (G3), we can replace the norm by an equivalent one, only affecting \(C\) but not \(\alpha\).

Polynomials in \(d\) variables of degree at most \(k\) defined on local fields can be seen to be \((C, 1/dk)\)-good, with \(C\) depending only on \(d\) and \(k\) using Lagrange interpolation. More generally, the following theorem shows that nondegenerate maps are good.

**Theorem 2.1.** [Theorem 4.3 [24]] Let \(F\) be either \(\mathbb{R}\) or an ultrametric valued field, and let \(f\) be a \(C^d\) map from an open subset \(U\) of \(F^d\) to \(F^n\). Then \(f\) is nonplanar and good at every point of \(U\) where it is nondegenerate.

As a corollary, we have

**Corollary 2.1.** Let \(\mathcal{L}\) be an affine subspace of \(\mathbb{Q}_p^n\) and let \(f = (f_1, \ldots, f_n)\) be a smooth map from an open subset \(U\) of \(\mathbb{Q}_p^d\) to \(\mathcal{L}\) which is nondegenerate in \(\mathcal{L}\) at \(x_0 \in U\). Then \(f\) is good at \(x_0\).

**Proof.** This follows from Theorem 2.1, see Corollary 3.2 in [19]. □
Main Theorem. Our main result is a complete $p$-adic analogue of Theorem 0.3 of [20].

Theorem 2.2. Let $\mu$ be a Federer measure on a Besicovitch metric space $X, \mathcal{L}$ an affine subspace of $\mathbb{Q}_p^n$, and let $f : X \to \mathcal{L}$ be a continuous map such that $(f, \mu)$ is good and nonplanar in $\mathcal{L}$. Then

$$w(f_*\mu) = w(\mathcal{L}) = \inf\{w(y) \mid y \in \mathcal{L}\} = \inf\{w(f(x)) \mid x \in \text{supp}\,\mu\}. \quad (2.2)$$

3. $p$-adic Diophantine exponents

We begin with some motivation for $p$-adic Diophantine approximation and the definition of $v$-approximable numbers in $\mathbb{Q}_p^n$, both using $\mathbb{Z}$ and $\mathbb{Z}[1/p]$ approximations. A natural starting point should be an analogue of Dirichlet’s theorem in this set up. In [24] a $p$-adic Dirichlet’s theorem using $\mathbb{Z}$ approximations has been discussed in detail. Here we observe that Dirichlet’s theorem using $\mathbb{Z}$ approximations does indeed give a Dirichlet theorem in case of $\mathbb{Z}[1/p]$ approximations.

We first discuss another way to conclude Dirichlet Theorem in $\mathbb{Q}_p^n$ when approximating by integers, namely using Minkowski’s theorem in the geometry of numbers. Take $y \in \mathbb{Q}_p^n$ such that $\|y\|_p \leq 1$ and consider the lattice

$$\Gamma_j = \left\{ (q_0, q) \in \mathbb{Z}^{n+1} \mid \|q_0 + q \cdot y\|_p \leq \frac{1}{p^j} \right\}. \quad (3.1)$$

In [8], we showed that the covolume of $\Gamma_j$ is $p^j$. Now consider the convex centrally symmetric box $[-p^{\frac{1}{n+1}}, p^{\frac{1}{n+1}}]$, then by Minkowski’s convex body Theorem we have a non-trivial lattice point inside this convex body. So we get that for every $j > 0$, the system of inequalities

$$\|q_0 + q \cdot y\|_p \leq \frac{1}{p^j}, \quad (3.1)$$

$$\|\tilde{q}\|_{\infty}^{n+1} \leq p^j \quad (3.2)$$

always has an integer solution. Here and below, we adopt the notation $\tilde{q} := (q_0, q)$ for $q_0, q \in \mathbb{Z}$ as well as $\mathbb{Z}[1/p]$. This also gives that for every $Q > 0$ there exists an integer solution for the system

$$\|q_0 + q \cdot y\|_p \leq \frac{p}{Q}, \quad (3.3)$$

$$\|\tilde{q}\|_{\infty}^{n+1} \leq Q. \quad (3.4)$$
Now observe that this implies that for every $Q > 0$ there exists an integer solution to the system

$$\|q\|\cdot q_0 + q \cdot y_p \leq \frac{p}{Q^n},$$

$$\|\tilde{q}\| \leq Q. \quad \text{(3.5)}$$

Note that $\|q\| \leq 1$ for integer vectors, so we have that for every $Q > 0$ there exists a $\mathbb{Z}[1/p]^{n+1}$ solution to the system

$$\|\tilde{q}\|\cdot q_0 + q \cdot y_p \leq \frac{p}{Q^n},$$

$$\|q\|\|\tilde{q}\| \leq Q. \quad \text{(3.6)}$$

While considering $p$-adic Diophantine approximation, a constant depending on $\|y\|_p$ occurs in Dirichlet’s theorem. So in a nutshell, for any $y \in \mathbb{Q}_p^n$ we have for every $Q > 0$ there exists a $\mathbb{Z}[1/p]^{n+1}$ solution to the system

$$\|\tilde{q}\|\cdot q_0 + q \cdot y_p \leq \frac{\text{const}(y)}{Q^n},$$

$$\|q\|\|\tilde{q}\| \leq Q. \quad \text{(3.7)}$$

**Definition 3.1. $v$-$\mathbb{Z}[1/p]$ approximable vectors:** $y \in \mathbb{Q}_p^n$ is $v$-$\mathbb{Z}[1/p]$-approximable if there exist $\tilde{q}$ with unbounded $\|q\|\|\tilde{q}\|$ such that

$$|q \cdot y + q_0|_p < \frac{1}{(\|q\|\|\tilde{q}\|)^v \|\tilde{q}\|\|\tilde{q}\|_\infty},$$

where we recall that $\tilde{q} = (q_0, q) \in \mathbb{Z}[1/p]^{n+1}$.

We will denote $v$-$\mathbb{Z}[1/p]$-approximable points by $W^v_p$ and also define

$$w_p(y) := \sup\{v \text{ appearing in (3.11)}\}. \quad \text{(3.9)}$$

Similarly, we will denote $v$-approximable points by $W_v$ and recall that we have defined

$$w(y) := \sup\{v \text{ appearing in (1.1)}\}. \quad \text{(3.10)}$$

We therefore have two Diophantine exponents:

1. $\omega$, defined in (1.1) which measures Diophantine approximation of $p$-adic vectors by rationals, and

2. $\omega_p$, defined in (3.12) which measures Diophantine approximation of $p$-adic vectors by $\mathbb{Z}[1/p]$ elements.
Later on, we will need higher Diophantine exponents of both kinds. Although the two types of approximations are a priori different, we will shortly prove that the Diophantine exponents are very closely related. The following lemma will come handy to compare these Diophantine exponents and further to relate Diophantine approximation to dynamics.

**Lemma 3.1.** Consider the set

$$E = \{ |q_0 + q \cdot y|_p \parallel \tilde{q} \parallel_\infty, \parallel \tilde{q} \parallel_p \parallel \tilde{q} \parallel_\infty \mid \tilde{q} = (q_0, q) \in \mathbb{Z}[1/p]^{n+1} \}. \quad (3.13)$$

If \((x_k, z_k) \in E\) such that \(z_k\) is bounded and \(x_k \to 0\) then \(x_k = 0\) for all but finitely many \(k\).

**Proof.** Suppose \(\|q_k\|_p \parallel \tilde{q}_k \parallel_\infty \leq M\) for some \(M > 0\) and

$$|q_{0k} + q_k \cdot y|_p \parallel \tilde{q}_k \parallel_\infty \to 0, \text{ as } k \to \infty,$$

where \(\tilde{q}_k = (q_{0k}, q_k) = (q_{ik})_{i=0}^n \in \mathbb{Z}[1/p]^{n+1}.\) Since

$$|q_{0k}| \parallel \tilde{q}_k \parallel_\infty \leq |q_{0k} + q_k \cdot y|_p \parallel \tilde{q}_k \parallel_\infty + \|q_k\|_p \parallel \tilde{q}_k \parallel_\infty \parallel y\|_p,$$

we can choose \(M\) such that \(|q_{0k}| \parallel \tilde{q}_k \parallel_\infty \leq M\) and \(\|q_k\|_p \parallel \tilde{q}_k \parallel_\infty \leq M\) and therefore \(\| \tilde{q}_k \|_p \parallel \tilde{q}_k \parallel_\infty \leq M\).

Note that there are only finitely many \(p\)-free integers in \(q_{ik}\), i.e. in \(\tilde{q}_k = (q_{ik})_{i=0}^n = (p^{m_{ik}} z_{ik})_{i=0}^n\) where \(p \nmid z_{ik}, \parallel z_{ik} \parallel_\infty\) is bounded. This follows from the fact that

$$|q_{ik}| \parallel \tilde{q}_k \parallel_\infty = p^{-m_{ik}} \cdot p^{m_{ik}} \parallel z_{ik} \parallel_\infty \leq \| \tilde{q}_k \|_p \parallel \tilde{q}_k \parallel_\infty \leq M.$$

So there are finitely many \(z_{ik}\).

We denote \(\| \tilde{q}_k \parallel_\infty = p^{m_k} \parallel z_k \parallel \) where \(p \nmid z_k, m_k \in \mathbb{Z}.\) If \(z_k = 0\) then \(\tilde{q}_k = 0.\)

Otherwise

$$p^{-m_{ik}} \cdot p^{m_k} \parallel z_k \parallel \leq \| \tilde{q}_k \|_p \parallel \tilde{q}_k \parallel_\infty \leq M,$$

and similarly,

$$p^{-m_k} \cdot p^{m_k} \parallel z_{ik} \parallel \leq \| \tilde{q}_k \|_p \parallel \tilde{q}_k \parallel_\infty \leq M.$$

So for nonzero elements \(q_{ik} = p^{m_{ik}} z_{ik}\) we have \(|m_k - m_{ik}|\) bounded since we have already noted that there are finitely many \(z_{ik}\). Therefore \(|q_{0k} + q_k \cdot y|_p \parallel \tilde{q}_k \parallel_\infty\) has only finitely many options. So it can only go to 0 if the terms are identically 0 for all but finitely many possibilities. \(\square\)

Now we can conclude the following relations between exponents.

**Proposition 3.1.** For any \(y \in \mathbb{Q}_p^n\) we have

$$w_p(y) = w(y) + 1.$$
Proof. Suppose \( y \in W_v \), then there exists infinitely many integer vectors \( \tilde{q} = (q_0, q) \in \mathbb{Z}^{n+1} \) such that

\[
|q \cdot y + q_0|_p < \frac{1}{(\|q\|_\infty)^v}. \tag{3.14}
\]

Since \( \|q\|_p \leq 1 \) for \( q \in \mathbb{Z}^{n+1} \), the above inequality is the same as

\[
|q \cdot y + q_0|_p < \frac{1}{(\|q\|_p \|\hat{q}\|_\infty)^v}. \tag{3.14}
\]

Lemma 3.1 assures us that the \( \|q\|_p \|\hat{q}\|_\infty \) appearing here are unbounded when \( v > 1 \). Hence, we have that \( W_v \subset W_{v-1}^w \) when \( v > 1 \). On the other hand, if \( y \in W_v^p \) then there are unbounded many \( \|q\|_p \|\hat{q}\|_\infty \) such that

\[
|q \cdot y + q_0|_p < \frac{1}{(\|q\|_p \|\hat{q}\|_\infty)^v}. \tag{3.14}
\]

where \( \tilde{q} = (q_0, q) \in \mathbb{Z}[1/p]^{n+1} \). This inequality can be rewritten as

\[
\|q\|_p(q \cdot y + q_0)|_p < \frac{1}{(\|q\|_p \|\hat{q}\|_\infty)^{v+1}}
\]

for unbounded many \( \|q\|_p \|\hat{q}\|_\infty \) where \( \tilde{q} = (q_0, q) \in \mathbb{Z}[1/p]^{n+1} \). But note that \( \|q\|_p \|\hat{q}\|_\infty \in \mathbb{Z}^{n+1} \). Hence the above criterion is the same as

\[
|(q \cdot y + q_0)|_p < \frac{1}{(\|\hat{q}\|_\infty)^{v+1}}
\]

for unbounded many \( \|\hat{q}\|_\infty \) where \( \tilde{q} = (q_0, q) \in \mathbb{Z}[1/p] \times \mathbb{Z}^n \). Now note that \( |q_0|_p \leq \max(\|y\|_p, 1) \). So when \( q_0 \notin \mathbb{Z} \), we may multiply by \( |q_0|_p > 1 \) and since we have an upper bound for \( |q_0|_p \) we get a \( \mathbb{Z} \) approximation but for slightly smaller \( v \). So we have for any \( \varepsilon > 0 \)

\[
|(q \cdot y + q_0)|_p < \frac{1}{(\|\hat{q}\|_\infty)^{v-\varepsilon+1}}
\]

for unbounded many \( \|\hat{q}\|_\infty \), and hence, for infinitely many \( \tilde{q} \in \mathbb{Z}^{n+1} \). Therefore \( W_v \subset W_{v+1-\varepsilon} \). Hence the conclusion follows. \( \square \)

A quick observation which follows from Dirichlet’s theorem and Proposition 3.1 is that \( w_p(y) \geq n \) for all \( y \in \mathbb{Q}^n_p \). We can define the Diophantine \( \mathbb{Z}[1/p] \) exponent more generally for a matrix \( A \) of order \( m, n \) and we will need these notions later in the paper. Define

\[
w_p(A) := \sup \left\{ v \mid \begin{array}{l}
\text{there exists unbounded many } \|q\|_p \|\hat{q}\|_\infty \\
s. t. \quad \|A \cdot q + q_0\|_p \leq \frac{1}{(\|q\|_p \|\hat{q}\|_\infty)^v \|\hat{q}\|_\infty} \\
\text{for some } \tilde{q} = (q_0, q) \in \mathbb{Z}[1/p]^m \times \mathbb{Z}[1/p]^n
\end{array} \right\}, \tag{3.15}
\]
and similarly the Diophantine $\mathbb{Z}$ exponent as

$$w(A) := \sup \left\{ v \mid \text{s.t. } \| Aq + q_0 \|_p \leq \frac{1}{(\|q\|_\infty)^v} \right\}.$$  

(3.16)

A similar reasoning as before shows that $w_p(A) = w(A) + 1.$

4. Connecting Diophantine approximation and homogeneous dynamics

We will weaken the hypothesis of Lemma 2.1 of [20] with the same conclusion.

**Lemma 4.1.** Suppose we are given a set $E \subset \mathbb{R}^2$ such that

- If the second coordinate of $E$ is bounded then the first coordinate cannot converge to 0 unless it is 0 ultimately, i.e. $(x_n, z_n) \in E$ such that $|z_n|$ is bounded and $x_n \to 0$ implies that $x_n = 0$ for all but finitely many $n$.
- $(0, z) \in E \implies (0, kz) \in E$ for infinitely many $k \in \mathbb{N}$.

Take $a, b > 0$ and $v > \frac{a}{b}$ and define

$$c := \frac{bv - a}{v + 1} \Leftrightarrow v = \frac{a + c}{b - c}.$$  

As before, $p$ is a prime. Then the following are equivalent:

1. There exists $(x, z) \in E$ with arbitrarily large $|z|$ such that $|x| \leq |z|^{-v}$.

2. There exists arbitrarily large $t > 0$ such that for some $(x, z) \in E \setminus \{0\}$ one has

$$\max(p^{at}|x|, p^{-bt}|z|) \leq p^{-ct}.$$  

**Proof.** We first show that (1) implies (2). There exists $(x, z) \in E$ with arbitrary large $|z|$ such that $|x| \leq |z|^{-v}$. Define $t > 0$ such that $p^{-bt}|z| = p^{-ct}$; this is possible since $b - c > 0$. Then

$$p^{at}|x| \leq p^{at}|z|^{-v} = p^{at}p^{(b-c)t(-v)} = p^{at}p^{-(a+c)t} = p^{-ct}$$

We now show that (2) implies (1). Accordingly, we assume that there exists a sequence of positives $\{t_n\} \to \infty$ such that

$$p^{at_n}|x_n| \leq p^{-ct_n} \text{ and } p^{-bt_n}|z_n| \leq p^{-ct_n}.$$  

Therefore

$$|x_n| \leq p^{-(a+c)t_n} = p^{-(b-c)vt_n} \leq |z_n|^{-v}.$$
If \( \{z_n\} \) is unbounded then (1) is proved. Suppose then, that \( \{z_n\} \) is bounded. Since \( \{x_n\} \to 0 \), by the hypothesis \( x_n = 0 \) for all but finitely many \( n \). Therefore we have that \((0, z_m) \in E \setminus \{0\} \). By the hypothesis, \((0, k z_m) \in E \setminus \{0\} \) for infinitely many \( k \in \mathbb{N} \), which will satisfy (1). \( \square \)

**Proposition 4.1.** For \( y \in \mathbb{Q}_p^n \), the following are equivalent

1. \( y \in W^p_v \), where \( v > n \),
2. there exists arbitrarily large \( t > 0 \) such that
   \[
   \max \{ p^{\frac{n+1}{n}} |q_0 + q . y|_p \|\bar{q}\|_\infty, p^{-\frac{t}{n+1}} \|q\|_p \|\bar{q}\|_\infty \} \leq p^{-ct}
   \]
   where \( a = \frac{n}{n+1}, b = \frac{1}{n+1}, c = \frac{n-n}{(n+1)(v+1)} \Leftrightarrow v = \frac{n(1+c)+c}{1-(n+1)c} \) and \( \bar{q} = (q_0, q) \in \mathbb{Z}[1/p]^{n+1} \).

**Proof.** We will apply Lemma 4.1 to the set

\[
E = \{ |q_0 + q . y|_p \|\bar{q}\|_\infty, \|q\|_p \|\bar{q}\|_\infty | \quad \bar{q} = (q_0, q) \in \mathbb{Z}[1/p]^{n+1} \}.
\]

By Lemma 3.1, this set satisfies the first hypothesis of Lemma 4.1. Suppose

\[
(0, \|q\|_p \|\bar{q}\|_\infty) \in E,
\]

then \( |q_0 + q . y|_p \|\bar{q}\|_\infty = 0 \). For any \( u \in \mathbb{N} \) such that \( p \nmid u \),

\[
|u . q_0 + u q . y|_p \|u \bar{q}\|_\infty = 0
\]

and

\[
\|u \bar{q}\|_p \|u \bar{q}\|_\infty = u \|q\|_p \|\bar{q}\|_\infty
\]

implies that \((0, u \|q\|_p \|\bar{q}\|_\infty) \in E \), giving the second hypothesis of Lemma 4.1. Now this Proposition follows directly from Lemma 4.1. \( \square \)

5. **Quantitative Nondivergence for flows on homogeneous spaces**

We begin this section by stating Theorem 2.1 of [20]. This theorem is an improvement of an original theorem of Kleinbock and Margulis ([23]). This improvement was the main tool in D. Kleinbock’s approach to studying Diophantine exponents of subspaces and their nondegenerate submanifolds. We will use a \( p \)-adic version of the Theorem, which in turn constitutes an improvement of the nondivergence theorem in [24]. Nondivergence estimates for flows on homogeneous spaces have a rich history, we refer the reader to [20] and the references therein.

**Theorem 5.1.** Let \( k, N \in \mathbb{Z}_+ \) and \( C, \alpha, D > 0 \). Suppose that we are given an \( N \)-Besicovitch metric space \( X \), a weighted poset \((B, \eta)\), a ball \( B = B(x, r) \) in \( X \), a measure \( \mu \) which is \( D \)-Federer on \( \tilde{B} = B(x, 3^m r) \), and a mapping \( \psi : B \to C(\tilde{B}), s \mapsto \psi_s \), such that the following holds:
$\ell(B) \leq k$;
(2) $\forall s \in B$, $\psi_s$ is $(C, \alpha)$ on $\tilde{B}$ with respect to $\mu$;
(3) $\forall s \in B$, $\|\psi_s\|_{\mu, B} \geq \eta(s)$;
(4) $\forall y \in \tilde{B} \cap \text{supp} \mu$, $\# \{s \in B \mid |\psi_s(y)| < \eta(s)\} < \infty$.

Then $\forall \varepsilon > 0$ one has

$$
\mu(B \setminus \Phi(\varepsilon, B)) \leq kC(ND^2)^k \varepsilon^\alpha \mu(B).
$$

In the following discussion we assume,

- $D$ is an integral domain, that is, a commutative ring with 1 and without zero divisors;
- $K$ is the quotient field of $D$;
- $R$ is a commutative ring containing $K$ as a subring.

The following Theorem is an improvement of Theorem 6.3 of [24] using the improved quantitative nondivergence i.e. Theorem 5.1 of D. Kleinbock. We refer the reader to loc.cit. for the definition of norm-like functions.

**Theorem 5.2.** Let $X$ be a metric space, $\mu$ a uniformly Federer measure on $X$, and let $D \subset K \subset R$ be as above, $R$ being a topological ring. For $m \in \mathbb{N}$, let a ball $B = B(x_0, r_0) \subset X$ and a continuous map $h : \tilde{B} \to \text{GL}(m, R)$ be given, where $\tilde{B}$ stands for $B(x_0, 3^m r_0)$. Also let $\nu$ be a norm-like function on $M(R, D, m)$. For any $\Delta \in \mathcal{P}(D, m)$ denote by $\psi_\Delta$ the function $x \mapsto \nu(h(x)\Delta)$ on $\tilde{B}$. Now suppose for some $C, \alpha > 0$ one has

(i) for every $\Delta \in \mathcal{P}(D, m)$, the function $\psi_\Delta$ is $(C, \alpha)$ on $\tilde{B}$ with respect to $\mu$,

(ii) for every $\Delta \in \mathcal{P}(D, m)$, $\|\psi_\Delta\|_{\mu, B} \geq \rho^{\text{rank} \Delta}$,

(iii) $\forall x \in \tilde{B} \cap \text{supp} \mu$, $\# \{\Delta \in \mathcal{P}(D, m) \mid \psi_\Delta(x) < \rho\} < \infty$.

Then for any positive $\varepsilon \leq \rho$ one has

$$
\mu\left(\left\{x \in B \mid \nu(h(x)\gamma) < \frac{\varepsilon}{C_\nu} \text{ for some } \gamma \in D^m \setminus \{0\}\right\}\right) \leq mC(NXD^2)^m \left(\frac{\varepsilon}{\rho}\right)^\alpha \mu(B).
$$

**Proof.** Take $\eta(\Delta) = \rho^{\text{rank}(\Delta)}$. We want to show that

$$
\Phi\left(\frac{\varepsilon}{\rho}, B\right) \subset \left\{x \in B \mid \nu(h(x)\gamma) \geq \frac{\varepsilon}{C_\nu}, \forall \gamma \in D^m \setminus \{0\}\right\}.
$$

Take $x \in \Phi(\frac{\varepsilon}{\rho}, B) \cap B$, so there exists a flag $\mathcal{F}_x$. Let

$\{0\} = \Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_l = D^m$

be all the elements of $\mathcal{F}_x \cup \{\{0\}, D^m\}$ such that the following conditions are satisfied:
(1) \( \varepsilon \eta(\Delta) \leq |\psi_\Delta(x)| \leq \eta(\Delta) \) \( \forall \delta \in \mathcal{F}_x \).

(2) \( |\psi_\Delta(x)| \geq \eta(\Delta) \) \( \forall \Delta \in \mathcal{B}(\mathcal{F}_x) \).

Pick any \( \gamma \in \mathcal{D}^m \setminus \{0\} \). Then there exists \( i, 1 \leq i \leq l \), such that \( \gamma \in \Delta_i \setminus \Delta_{i-1} \). Now \( \gamma \notin \Delta_{i-1} = \mathcal{R}\Delta_{i-1} \cap \mathcal{D}^m \) implies that \( \gamma \notin \mathcal{R}\Delta_{i-1} \) since \( \Delta_{i-1} \) is primitive, hence \( g\gamma \notin g\mathcal{R}\Delta_{i-1} = \mathcal{R}g\Delta_{i-1} \) for any \( g \in \text{GL}(m, \mathbb{R}) \). Therefore, if one defines \( \Delta' := \mathcal{D}\Delta_{i-1} + \mathcal{D}\gamma \), in view of (N2) one has

\[
\nu(h(x)\Delta') \leq C_\nu \nu(h(x)\Delta_{i-1}) \nu(h(x)\gamma).
\]

Further, let \( \Delta := K\Delta' \cap \mathcal{D}^m \). It is a primitive submodule containing \( \Delta' \) and of rank equal to \( \text{rank}(\Delta') \), so, by (N1),

\[
\nu(h(x)\Delta) \leq \nu(h(x)\Delta').
\]

Moreover, it is also contained in \( \Delta_i \) and contains \( \Delta_{i-1} \), since \( \Delta_{i-1} = K\Delta_{i-1} \cap \mathcal{D}^m \subset \Delta = K\Delta' \cap \mathcal{D}^m = K\Delta \cap \mathcal{D}^m \subset K\Delta_i \cap \mathcal{D}^m \). Therefore it is comparable to any element of \( \mathcal{F}_x \), i.e. belongs to \( \mathcal{F}_x \cup \mathcal{P}(\mathcal{F}_x) \). Then one can use properties (1) and (2) above to deduce that

\[
\nu(h(x)\gamma) \geq \nu(h(x)\Delta')/C_\nu \nu(h(x)\Delta_{i-1}) \geq \nu(h(x)\Delta)/C_\nu \nu(h(x)\Delta_{i-1}) \geq \varepsilon/\rho^{\text{rank}(\Delta_{i-1})} \geq \varepsilon/C_\nu.
\]

We recall Lemma 8.1 from [24]. It is proved for an arbitrary, finite set of places \( S \) of \( \mathbb{Q} \). We only need it for the case \( S = \{\infty, p\} \) for a prime \( p \).

**Lemma 5.1.** The function \( \nu : \mathcal{M}(\mathbb{Q}_S, \mathbb{Z}_S, m) \to \mathbb{R}_+ \) given by \( \nu(\Delta) = \text{cov}(\Delta) \), with \( \text{cov}() \) as defined earlier is norm-like, with \( C_\nu = 1 \).

As a consequence of Theorem 5.2 we get the following refined version of Theorem 8.3 of [24]

**Theorem 5.3.** Let \( X \) be a Besicovitch metric space, \( \mu \) a uniformly Federer measure on \( X \), and let \( S \) be as above. For \( m \in \mathbb{N} \), let a ball \( B = B(x_0, r_0) \subset X \) and a continuous map \( h : \tilde{B} \to \text{GL}(m, \mathbb{Q}_S) \) be
given, where $\tilde{B}$ stands for $B(x_0, 3^m r_0)$. Now suppose that for some $C, \alpha > 0$ and $0 < \rho < 1$ one has

(i) for every $\Delta \in \mathcal{P}(\mathbb{Z}_S, m)$, the function $\text{cov}(h(\cdot)\Delta)$ is $(C, \alpha)$ good on $\tilde{B}$ with respect to $\mu$;

(ii) for every $\Delta \in \mathcal{P}(\mathbb{Z}_S, m)$, $\sup_{x \in B \cap \text{supp} \mu} \text{cov}(h(x)\Delta) \geq \rho^{rk(\Delta)}$.

Then for any positive $\varepsilon \leq \rho$ one has

$$\mu\left(\left\{x \in B \mid \delta(h(x)\mathbb{Z}_S^m) < \varepsilon\right\}\right) \leq mC\left(N \chi D_\mu^2\right)^m \left(\frac{\varepsilon}{\rho}\right)^\alpha \mu(B).$$

**Proof.** The proof goes line by line as Theorem (8.3)\cite{24} using Theorem 5.2 in place of theorem (6.3) of [24]. \hfill \Box

### 6. Applying Nondivergence Estimates

For any $y \in \mathbb{Q}_p^{n+1}$ we associate a lattice $u_y \mathbb{D}^{n+1}$ in $(\mathbb{Q}_p \times \mathbb{R})^{n+1}$, where $u_y$ is defined as

$$u_y^p = \begin{bmatrix} 1 & y_1 & \cdots & y_n \\ & I_n \end{bmatrix}$$

and $u_y^\infty = I_{n+1}$. For $t \in \mathbb{N}$ define

$$g_t^p = \begin{bmatrix} p^{-t} & 0 \\ 0 & I_n \end{bmatrix} \quad \text{and} \quad g_t^\infty = \text{diag}(p^{-\frac{t}{n+1}}, \ldots, p^{-\frac{t}{n+1}}).$$

So $g_t u_y \in \text{GL}_{n+1}(\mathbb{Q}_p \times \mathbb{R})$.

We now proceed to connect dynamics with Diophantine properties. For any $v > n$ and $\frac{1}{n+1} > d > c = \frac{v-n}{(v+1)(v+1)}$, define $v_d = \frac{n(1+c)+d}{1-(n+1)c}$. Hence $\frac{n(1+c)+c}{1-(n+1)c} = v < v_d$. By Proposition 4.1,

$$\mathcal{W}_{v_d} = \left\{ y \in \mathbb{Q}_p^n \left| \max\{p^{-\frac{t}{n+1}}|q_0 + q_y|_p\|\tilde{q}\|_{\infty}, p^{-\frac{t}{n+1}}\|q\|_p\|\tilde{q}\|_{\infty}\} \leq p^{-dt} \right. \right\}.$$

Now define the set

$$\tilde{\mathcal{W}}_{v_d} := \left\{ y \in \mathbb{Q}_p^n \left| \exists \text{ arbitrarily large } t \in \mathbb{N} \right. \right\}.$$

Then we have

**Lemma 6.1.** Let $\nu$ be a measure on $\mathbb{Q}_p^n$. Then the following statements are equivalent:

(1) $\nu(\mathcal{W}_{v_d}) = 0 \forall d > c$. 
(2) $\nu(\widetilde{W_{vd}}) = 0 \forall d > c$. 

Proof. We first prove that (1) implies (2). Let $y \in \widetilde{W_{vd}}$, then there exists arbitrarily large $t \in \mathbb{N}$ such that

$$\delta(g_t u y \mathcal{D}^{n+1}) \leq p^{-dt}. \quad \text{(6.1)}$$

Observe that

$$(g_t u y \mathcal{D}^{n+1})_\infty = \{(p^{-\frac{t}{n+1}} q_0, \ldots, p^{-\frac{t}{n+1}} q_n) | q_i \in \mathcal{D}\},$$

$$(g_t u y \mathcal{D}^{n+1})_p = \{(p^{-t}(q_0 + q y), q_1, \ldots, q_n) | q_i \in \mathcal{D}\}$$

and

$$\delta(g_t u f(x) \mathcal{D}^{n+1}) = \min_{q \in \mathcal{D}^{n+1} \setminus \{0\}} c(g_t u f(x) \bar{q}).$$

So from (6.1) there exist infinitely many $t \in \mathbb{N}$ such that

$$\max\{p^i q y + q_0|p, \|q\|_p\} p^{-\frac{t}{n+1}} \|\bar{q}\|_\infty \leq p^{-dt}$$

which implies that

$$\max\{p^i q y + q_0|p, \|q\|_p\} p^{-\frac{t}{n+1}} \|\bar{q}\|_\infty, p^{-\frac{i}{n+1}} \|q\|_p, \|\bar{q}\|_\infty \leq p^{-dt}$$

for some $\bar{q} \in \mathcal{D}^{n+1}$. Hence $\widetilde{W_{vd}} \subset W_{vd} \forall d > c$.

We now prove (2) implies (1). We claim that $W_{vd} \subset \widetilde{W_{vd}}$ for some $d > d' > c$. Note that

$$\|q\|_p \|\bar{q}\|_\infty \leq p^{-(d - \frac{1}{n+1}) t}$$

$$\leq p^{-(d - \frac{1}{n+1}) p^{-(d - \frac{1}{n+1}) t]}$$

$$\leq \frac{p^{-(d - \frac{1}{n+1})}}{p^{(d - d') t]} p^{-(d' - \frac{1}{n+1}) t]} \text{ for some } d > d' > c$$

$$\leq p^{-(d' - \frac{1}{n+1}) t]} \text{ for large enough } t > 0.$$ 

Finally,

$$p^i q y + q_0|p, \|\bar{q}\|_\infty \leq p^i q y + q_0|p, \|\bar{q}\|_\infty \leq p^{-dt} \leq p^{-d'[t]} \leq p^{-d'[t]}$$

for some $\bar{q} \in \mathcal{D}^{n+1}$. Therefore we have that $W_{vd} \subset \widetilde{W_{vd}}$ and the conclusion follows. \hfill \Box

**Proposition 6.1.** Take $\mathcal{R} = \mathbb{Q}_p \times \mathbb{R}$ and $\mathcal{D} = \mathbb{Z}[1/p]$ and $v > n, c := \frac{v-n}{(n+1)(v+1)}$ as defined in Proposition 4.1. Let $X$ be a Besicovitch metric space and $\mu$ be a uniformly Federer measure on $X$. Denote $\tilde{B} := B(x, 3^{n+1} r)$. Suppose we are given a continuous function $f : X \mapsto \mathbb{Q}_p^n$ and $C, \alpha > 0$ with the following properties

(i) $x \mapsto \text{cov}(g_t u f(x) \Delta) \text{ is } (C, \alpha) \text{ good with respect to } \mu \text{ in } \tilde{B} \forall \Delta \in \mathcal{P}(\mathcal{D}, n+1)$,
(ii) for any $d > c$ there exists $T = T(d) > 0$ such that for any integer $t \geq T$ and any $\Delta \in \mathcal{P}(D, n + 1)$ one has
\[
\sup_{x \in B \cap \text{supp } \mu} \text{cov}(g_t u_{\mathbf{f}(x)} \Delta) \geq p^{-(\text{rank } \Delta)dt}.
\]
(6.2)

Then $w(f_* \mu|_B) \leq v$.

**Proof.** We will check that the map $h = g_t u_{\mathbf{f}}$ satisfies the assumptions of Theorem 5.3 with respect to the measure $\nu = f_* \mu|_B$ where (6.1) is the same as (i) of Theorem 5.3 with $m = n + 1$. To check the second condition take $\rho = p^{-\frac{c + dt}{2}}$ for $d > c$. Then (6.1) gives (ii) of Theorem 5.3 for all integer $t > T\left(\frac{c + dt}{2}\right)$. Therefore by Theorem 5.3
\[
\mu\left(\left\{ x \in B \left| \delta(g_t u_{\mathbf{f}(x)} D^{n+1}) < p^{-dt}\right.\right.\right) \leq (n + 1)C(N_X D^2 p^{n+1})\alpha \mu(B) \leq \text{const.} p^{-\alpha \frac{d-c}{2}t}
\]
for all but finitely many $t \in \mathbb{N}$.

Using the Borel-Cantelli lemma and (6.3) we have $\nu(W_v) = \nu(W_{v_d}) = 0 \forall d > c$, which again gives $\nu(W_v) = 0 \forall d > c$ by Lemma 6.1. Since $\lim_{k \to \infty} d_k = c$, we have that $\lim_{k \to \infty} v_{d_k} = v$ and now using the formula for $v_d$, we have that for any $u > v$ there exists $\frac{1}{n+1} > d > c$ such that $u > v_d$. So $\nu(W_u) = 0$ because $W_u \subset W_{d_k}$. Hence $w(\nu) = w(f_* \mu|_B) \leq v$. \qed

Let us now show that condition (2) in Proposition 6.1 is in fact necessary to conclude Proposition 6.1. This a $p$-adic version of Lemma (4.1) of [20].

**Lemma 6.2.** Let $\mu$ be a measure on $B$ and take $c, v > 0$ as before. Let $\mathbf{f} : B \mapsto \mathbb{Q}_p^n$ be such that condition (ii) in Proposition 6.1 does not hold. Then
\[
\mathbf{f}(B \cap \text{supp } \mu) \subset W_u^p \text{ for some } u > v.
\]
(6.6)

**Proof.** There exists $d > c$ such that condition (2) in Proposition 6.1 does not hold. So there exists a $1 \leq j \leq (n + 1)$ such that there exists a sequence of natural numbers $t_i \to \infty$ and corresponding $\Delta_i$ of rank $j$ such that
\[
\sup_{x \in B \cap \text{supp } \mu} \text{cov}(g_t u_{\mathbf{f}(x)} \Delta_i) \leq p^{-\text{rank } \Delta_i dt_i} = p^{-jdt_i}.
\]
Now consider the ball
\[ D = D_\infty \times D_p = \{ x^\infty \in \mathbb{R}^{n+1} \mid ||x^\infty||_\infty \leq p^{-dt_1} \} \times \left\{ x^{(p)} \in \mathbb{Q}_p^{n+1} \mid |x_1^{(p)}|_p \leq 1, |x_2^{(p)}|_p \leq 1, \ldots, |x_{n+1}^{(p)}|_p \leq 1 \right\}. \]

Denote \( \Delta = g_i u_{t(x)} \Delta_i \) and \( D \cap \mathbb{Q}_S \Delta = D_1 \), a ball in \( \mathbb{Q}_S \Delta \). Then \( \lambda_S(D_1) = \mu_S(\pi^{-1}(D_1)) \) where \( \mu_S \) is the Haar measure on \( \mathbb{Q}_S^j \) and
\[ \pi : \mathbb{Q}_S^j \to \mathbb{Q}_S \Delta = \mathbb{Q}_S v_1 + \cdots + \mathbb{Q}_S v_j \]
where the \( v_1, v_2, \ldots, v_j \) are taken such that \( v_1^\infty, \ldots, v_j^\infty \) form an orthonormal basis of \( (\mathbb{Q}_S \Delta)_\infty \) and
\[ (\mathbb{Q}_S \Delta)_p \cap \mathbb{Z}_p^{n+1} = \mathbb{Z}_p v_1^{(p)} + \cdots + \mathbb{Z}_p v_j^{(p)}. \]

Note that \( \lambda_S \) is the normalized Haar measure on \( \mathbb{Q}_S \Delta \). Now consider
\[ \pi^{-1} D_1 = \left\{ x \in \mathbb{Q}_S^j \mid ||x_1^\infty v_1^\infty + \cdots + x_j^\infty v_j^\infty||_\infty \leq p^{-dt_1} \right\} = \tilde{D}_1^\infty \times \tilde{D}_1^{(p)} \]
where,
\[ \tilde{D}_1^\infty = \{ x^\infty \in \mathbb{R}^j \mid ||x_1^\infty v_1^\infty + \cdots + x_j^\infty v_j^\infty||_\infty \leq p^{-dt_1} \}, \]
\[ \tilde{D}_1^{(p)} = \left\{ x^{(p)} \in \mathbb{Q}_p^j \mid |(x_1^{(p)} v_1^{(p)} + \cdots + x_j^{(p)} v_j^{(p)})_k|_p \leq 1 \ \forall k = 1, \ldots, n + 1 \right\}. \]

Since \( v_1^\infty, \ldots, v_j^\infty \) are orthonormal, we have that \( \mu_\infty(\tilde{D}_1^\infty) = 2^j p^{-jd t_1} \).

**What is A?** We can complete \( v_1^\infty, \ldots, v_j^\infty \) to an orthonormal basis, so
\[ ||(x_1^\infty, \ldots, x_j^\infty, 0, \ldots, 0)|| \leq ||A^{-1}|| ||A(x_1^\infty, \ldots, x_j^\infty, 0, \ldots, 0)|| \leq 1 p^{-dt_1} \]
and vice versa. On the other hand \( \mu_p(\tilde{D}_1^{(p)}) = \mu(E^{-1} \tilde{D}_1^{(p)}) \ \forall \ E \in \text{SL}_n(\mathbb{Q}_p) \). We can make the matrix by column operation in this form
\[ A = \begin{pmatrix} * & 0 & \cdots & 0 \\ * & * & \cdots & * \\ \vdots \\ * & * & \cdots & * \end{pmatrix} \]
where \( * \in \mathbb{Z}_p \). So this shows that
\[ 1 = \mu_p(B(0, 1) \times B(0, 1) \times \cdots B(0, 1)) \leq \mu_p(\tilde{D}_1^{(p)}). \]
So
\[ \mu(\pi^{-1}D_1) = \mu_\infty(\tilde{D}_1^\infty) \times \mu_p(\tilde{D}_1^{(p)}) \geq 2^j p^{-jd_t}, \]
and this gives
\[ \lambda_S(D_1) \geq 2^j p^{-jd_t} \geq 2^j \text{ cov}(g_t u_{f(x)} \Delta_i) = 2^j \text{ cov}(\Delta) \quad \forall x \in B \cap \text{ supp } \mu. \]
Then by the $S$-arithmetic version of Minkowski’s theorem ([24], Lemma 7.7) there is a nonzero vector $v \in g_t u_{f(x)} \Delta_i \cap D_1$ i.e. there exists $\tilde{q} = (q_0, q_1, \cdots, q_n) \in (\mathbb{Z}[1/p])^{n+1}$ such that
\[ p^{-\frac{j}{p^{t+1}}} \|\tilde{q}\|_\infty \leq p^{-dt}, \] (6.7)
and
\[ p^t|q_0 + q_1 f_1(x) + \cdots + f_n(x)|_p \leq 1 \]
\[ |q_1|_p \leq 1 \]
\[ \vdots \]
\[ |q_n|_p \leq 1. \] (6.8)
Therefore
\[ p^{-\frac{j}{p^{t+1}}} \|\tilde{q}\|_\infty \|q\|_p \leq p^{-\frac{j}{p^{t+1}}} \|\tilde{q}\|_\infty \leq p^{-dt} \text{ and} \]
\[ p^{\frac{j}{p^{t+1}}} |q_0 + q.f(x)|_p \|\tilde{q}\|_\infty \leq p^{-dt} \]
holds for infinitely many $t_i \in \mathbb{N}$. Hence by equivalence (4.1)
\[ \forall x \in \text{ supp } \mu \cap B \text{ such that } f(x) \in \mathcal{W}_{v_d}^p \text{ for some } d > c. \]
Thus there exists $v_d > v$ such that
\[ f(B \cap \text{ supp } \mu) \subset \mathcal{W}_{v_d}^p. \]

Throughout the rest of the paper we are going to denote $\mathcal{R} = \mathbb{Q}_p \times \mathbb{R}$ and $\mathcal{D} = \mathbb{Z}[1/p]$. One can associate to any nonzero submodule $\Delta \subset \mathcal{D}^{n+1}$ of rank $j$, an element $w$ of $\wedge^j(\mathcal{D}^{n+1})$ such that $\text{ cov}(\Delta) = c(w)$ and $\text{ cov}(g_t u_y \Delta) = c(g_t u_y w)$. We take $e_0, e_1, \cdots, e_n \in \mathbb{R}^{n+1}$ as the standard basis where $e_i = (e_i^p, e_i^\infty)$ and $\{e_i^p\}$, $\{e_i^\infty\}$ are the standard basis of $\mathbb{Q}_p^n$, $\mathbb{R}^n$ respectively. We will use the standard basis $\{e_I = e_{i_1} \wedge \cdots \wedge e_{i_j} | I \subset \{0, \cdots, n\} \text{ and } i_1 < i_2 < \cdots < i_j \}$ of $\wedge^j \mathbb{R}^{n+1}$. Thus we can write any $w \in \mathcal{D}^{n+1}$ as $w = \sum w_I e_I$, where $w_I \in \mathcal{D}$. Let us note the action of unipotent flows on the coordinates of $w$.

1. $u_y^p$ leaves $e_0^p$ invariant and sends $e_i^p$ to $y_i e_0^p + e_i^p$ for $i \geq 1$.

2. $u_y^\infty = \mathbb{I}_{n+1}$ leaves everything invariant.
Therefore

\[ u^p_y(e^p_I) = \begin{cases} 
  e^p_I & \text{if } 0 \in I \\
  e^p_I + \sum_{i \in I} \pm y_ie_{I \setminus \{i\} \cap \{0\}} & \text{if } 0 \notin I.
\end{cases} \]

Observe that under the action of \( g^p_t \), \( e^p_i \)'s are invariant for \( i \geq 1 \) and \( e^p_0 \) is an eigenvector with eigenvalue \( p^{-t} \). Therefore

\[ g^p_t u^p_y(e^p_I) = \begin{cases} 
  p^{-t} e^p_I & \text{if } 0 \in I \\
  e^p_I + p^{-t} \sum_{i \notin I} \pm y_ie_{I \setminus \{i\} \cap \{0\}} & \text{if } 0 \notin I.
\end{cases} \]

On the other hand \( u^\infty_y = \text{Id} \) and each \( e^\infty_i \) is an eigenvector with eigenvalue \( p^{-\frac{t}{n+1}} \). Thus

\[ g^\infty_t u^\infty_y e^\infty_I = p^{-\frac{t}{n+1}} e^\infty_I. \]

Therefore for \( w \in \bigwedge^j (\mathbb{D}^{n+1}) \), \( w = \sum w_I e_i \) where \( w_I \in \mathbb{D} \) we get

\[ (g_t u_y w)^p = \sum_{0 \notin I} w_I e^p_I + p^{-t} \sum_{0 \notin I} \left( w_I + \left( \sum_{i \notin I} \pm w_{I \setminus \{0\} \cup \{i\}} y_i \right) e_I \right), \tag{6.9} \]

and

\[ (g_t u_y w)^\infty = p^{-\frac{t}{n+1}} \sum w_I e^\infty_I. \tag{6.10} \]

It will be convenient for us to use the following notations,

\[ c(w) = \begin{pmatrix} 
  c(w)_0 \\
  c(w)_1 \\
  \vdots \\
  c(w)_n
\end{pmatrix}, \]

where \( c(w)_i = \sum_{\#J = j-1 \atop J \subset \{1, \ldots, n\}} w_{J \cup \{i\}} e^p_J \in \bigwedge^{j-1}(V_0) \) and \( V_0 \) is the subspace of \( \mathbb{Q}_p^{n+1} \) generated by \( e^p_1, \ldots, e^p_n \). We may therefore write

\[ (g_t u_y w)^p = \sum_{0 \notin I} w_I e^p_I + p^{-t} \left( e^p_0 \wedge \sum_{i=0}^n y_i c(w)_i \right) = \pi(w) + p^{-t} e^p_0 \wedge \tilde{y} c(w), \tag{6.11} \]

where \( y_0 = 1 \) and \( \pi \) is the orthogonal projection from \( \bigwedge^j(\mathbb{Q}_p^{n+1}) \mapsto \bigwedge^j(V_0) \). If \( w \) corresponds to \( \Delta \subset \mathbb{D}^{n+1} \), a submodule of rank \( j \) then
we have that
\[
\text{cov}(g_t u y \Delta) = c(g_t u y w) = \max \left( p^{-t} \| \sum_{i=0}^{n} y_i c(w)_i \|_p, \| \pi(w) \|_p \right) p^{-\frac{tj}{n+1}} \| w \|_\infty
\]
\[
= \max \left( p^{-t} \sum_{i=0}^{n} y_i c(w)_i \|_p \| w \|_\infty, p^{-\frac{tj}{n+1}} \| \pi(w) \|_p \| w \|_\infty \right).
\]

Thus,
\[
\sup_{x \in B \cap \text{supp } \mu} \text{cov}(g_t u f(x) \Delta)
= \max \left( p^{-t} \sup_{x \in B \cap \text{supp } \mu} \| \tilde{f}(x) c(w) \|_p \| w \|_\infty, p^{-\frac{tj}{n+1}} \| \pi(w) \|_p \| w \|_\infty \right),
\]
where \( \tilde{f} = (1, f_1, \cdots, f_n) \). Note that condition (6.2) can be written as
\[
\forall d > c \exists T > 0 \text{ such that } \forall t \geq T(d), \forall j = 1, \cdots, n \text{ and } \forall w. \quad (6.13)
\]

Now suppose that the \( \mathbb{Q}_p \)-span of the restrictions of \( 1, f_1, \cdots, f_n \) to \( B \cap \text{supp } \mu \) has dimension \( s + 1 \) and choose \( g_1, \cdots, g_s : B \cap \text{supp } \mu \mapsto \mathbb{Q}_p \) such that \( 1, g_1, \cdots, g_s \) form a basis of the space. Therefore there exists a matrix \( R = (r_{i,j})_{(s+1) \times (n+1)} \) such that \( \tilde{f}(x) = \tilde{g}(x) R \forall x \in B \cap \text{supp } \mu \) where \( \tilde{g} = (1, g_1, \cdots, g_s) \). We can rewrite
\[
\sup_{x \in B \cap \text{supp } \mu} \| \tilde{f}(x) c(w) \|_p = \sup_{x \in B \cap \text{supp } \mu} \| \tilde{g}(x) Rc(w) \|_p.
\]

Thus we have that (6.2) is equivalent to
\[
\forall d > c \exists T = T(d) > 0 \text{ such that for any integer } t \geq T
\]
\[
\forall j = 1, \cdots, n \text{ and } \forall w \in \bigwedge^j \mathcal{D}^{n+1}, \text{ one has}
\]
\[
\max \left( p^{-t} \| Rc(w) \|_p \| w \|_\infty, p^{-\frac{tj}{n+1}} \| \pi(w) \|_p \| w \|_\infty \right) \geq p^{-jd t},
\]
by equivalence of norms since \( 1, g_1, \cdots, g_s \) are linearly independent. In order to get rid of the auxiliary variable \( t \) we want to apply lemma 4.1.
Consider the components of $Rc(w)$:

$$
(Rc(w))_i = \sum_{k=0}^{n} r_{ik}c(w)_i
$$

$$
= \sum_{k=0}^{n} r_{ik} \sum_{\substack{J \subset \{1, \cdots, n\} \#J = j-1 \\ J \subseteq \{1, \cdots, n\}}} \sum_{k=0}^{n} w_{J \cup \{k\}} e_{J}^p
$$

$$
= \sum_{\substack{J \subset \{1, \cdots, n\} \#J = j-1}} \left( \sum_{k=0}^{n} r_{ik} w_{J \cup \{k\}} \right) e_{J}^p.
$$

So we have that

$$
\|Rc(w)\|_p = \max_{i=0, \cdots, s} \max_{\substack{J \subset \{1, \cdots, n\} \#J = j-1 \\ J \subseteq \{1, \cdots, n\}}} \left( \sum_{k=0}^{n} r_{ik} w_{J \cup \{k\}} \right) |p| w_m \|_{\infty}.
$$

Now consider the set

$$
E := \left\{ (\|Rc(w)\|_p w_m \|_{\infty}, \|\pi(w)\|_p w_m \|_{\infty}) \mid w_m \in S_{n+1,j} \right\}
$$

$$
= \left\{ \left( \max_{i=0, \cdots, s} \max_{\substack{J \subset \{1, \cdots, n\} \#J = j-1 \\ J \subseteq \{1, \cdots, n\}}} \left( \sum_{k=0}^{n} r_{ik} w_{J \cup \{k\}} \right) |p| w_m \|_{\infty}, \|\pi(w)\|_p w_m \|_{\infty} \right) \mid w_m \in S_{n+1,j} \right\}.
$$

**Lemma 6.3.** The set $E$ as above satisfies the hypotheses of Lemma 4.1.

**Proof.** Let $w_m \in S_{n+1,j}$ be a sequence such that

$$
\|\pi(w_m)\|_p w_m \|_{\infty} \leq M \forall m
$$

for some $M > 0$ and

$$
\max_{i=0, \cdots, s} \max_{\substack{J \subset \{1, \cdots, n\} \#J = j-1 \\ J \subseteq \{1, \cdots, n\}}} \left( \sum_{k=0}^{n} r_{ik} w_{J \cup \{k\}} \right) |p| w_m \|_{\infty} \to 0
$$

as $m \to \infty$ where $w_{m} = \sum w_{i}^{(m)} e_{J}$. For $J \subset \{1, \cdots, n\}$, such that $\#J = j-1$, denote $\tilde{w}_{J}^{(m)} := (w_{J \cup \{0\}}^{(m)}, \cdots, w_{J \cup \{n\}}^{(m)})$. We have that for every $m, J$ and $k = 1, \cdots, n$,

$$
|w_{J \cup \{k\}}^{(m)}| \|w_{J}^{(m)}\|_{\infty} \leq |w_{J \cup \{k\}}^{(m)}| \|w_{m}\|_{\infty} \leq M.
$$
Moreover for \( \varepsilon > 0 \) \( \exists N_\varepsilon \in \mathbb{N} \) such that \( \forall m \geq N_\varepsilon \),

\[
|r_{00}w_{J,\{0\}}^m|_p \|w_m\|_\infty \\
\leq \max(\varepsilon, \sum_{k=1}^{n} r_{im}w_{J,\{k\}}^m|_p \|w_m\|_\infty) \\
\leq \max(\varepsilon, \|R\|_p \|\pi(w_m)\|_p \|w_m\|_\infty) \\
\leq \max(\varepsilon, \|R\|_p M).
\]

Since \( \tilde{f}(x) = \tilde{g}(x)R \forall x \in B \cap \text{supp} \mu \) we have \( r_{00} = 1 \) and \( r_{i0} = 0 \) otherwise. This implies that

\[
|r_{00}w_{J,\{0\}}|_p \|\tilde{w}_{J,\{0\}}\|_\infty \leq |w_{J,\{0\}}^m|_p \|w_m\|_\infty \leq M_1
\]
for some \( M_1 > 0 \) and for every \( J \). Therefore, for \( \tilde{w}_{J,\{m\}} \),

\[
\|\tilde{w}_{J,\{m\}}\|_p \|\tilde{w}_{J,\{m\}}\|_\infty \leq M_1,
\]
for some \( M_1 > 0 \) and

\[
|\sum_{k=0}^{n} r_{ik}w_{J,\{k\}}^m|_p \|\tilde{w}_{J,\{m\}}\|_\infty \to 0.
\]

Now applying Lemma 3.1 we can conclude that first hypothesis is satisfied. The second hypothesis is satisfied since \((0, \|Rc(w)\|_p \|w\|_\infty) \in \mathcal{E}\) implies that for \( u \in \mathbb{N} \) with \( p \nmid u \), \((0, u \|Rc(w)\|_p \|w\|_\infty) \in \mathcal{E}\). \( \square \)

Here, in terms of Lemma 4.1, \( a = \frac{n+1-j}{n+1}, b = \frac{j}{n+1} \). Since the set \( \mathcal{E} \) satisfies the hypotheses of Lemma 4.1, we conclude that the condition (6.14) is equivalent to

\[
\forall j = 1, \cdots , n \text{ and } \forall w \in S_{n+1,j}, \forall d > c = \frac{v-n}{(n+1)(v+1)} \text{ where } v \geq n
\]

\[
\exists u_d = \frac{n+1-j}{n+1} + \frac{d j}{n+1} \text{ such that for arbitrarily large }
\]

\[
\|\pi(w)\|_p \|w\|_\infty, \text{ we have } \|Rc(w)\|_p \|w\|_\infty > (\|\pi(w)\|_p \|w\|_\infty)^{-u_d}.
\]

Moreover \( \lim_{k \to \infty} d_k = c \) implies that \( \lim_{k \to \infty} u_{d_k} = \frac{v-j+1}{j} \). Therefore condition (6.15) is equivalent to

\[
\forall j = 1, \cdots , n, \forall u > \frac{v-j+1}{j} \text{ and } \forall w \in S_{n+1,j}, \text{ for all arbitrary }
\]

\[
\text{large } \|\pi(w)\|_p \|w\|_\infty \text{ we have } \|Rc(w)\|_p \|w\|_\infty > (\|\pi(w)\|_p \|w\|_\infty)^{-u}.
\]
The proof of the Theorem 6.1 now goes as in [20], we repeat it for the sake of completeness. Note that we have that \( R \) is a matrix depending on the ball \( B \), the measure \( \mu \) and the map \( f \) such that

\[
(6.16) \text{ holds } \iff \text{ so does } (6.14) \iff \text{ so does } (6.2).
\]

Suppose
\[
\mathcal{L} = \langle f(B \cap \text{supp } \mu) \rangle_a.
\]

Let \( \text{dim } \mathcal{L} = s \) and let
\[
h : \mathbb{Q}_p^s \to \mathcal{L} \text{ be an affine isomorphism, and } \tilde{h}(x) = \tilde{x}R, x \in \mathbb{Q}_p^s,
\]

where as usual we have that \( \tilde{h} := (1, h_1, \ldots, h_n) \) and \( \tilde{x} := (1, x_1, \ldots, x_s) \). Then \( g = h^{-1} \circ f \) generates the space of \( \mathbb{Q}_p \)-span of the restrictions of \( 1, f_1, \ldots, f_n \) to \( B \cap \text{supp } \mu \) and satisfies \( \tilde{f}(x) = \tilde{g}(x)R \forall x \in B \cap \text{supp } \mu \).

Therefore condition \((6.16) \iff (6.14) \iff \) condition \((6.2)\) becomes a property of the subspace and in particular \( R \) can be chosen uniformly for all measures \( \mu \), ball \( B \) and measure \( \mu \) and \( f \) the function as long as \((6.17)\) holds.

**Theorem 6.1.** Let \( \mu \) be a Federer measure on a Besicovitch metric space \( X, \mathcal{L} \) an affine subspace of \( \mathbb{Q}_p^n \), and let \( f : X \to \mathcal{L} \) be a continuous map such that \( (f, \mu) \) is good and nonplanar i.e \((6.17)\) holds for all open balls \( B \) with \( \mu(B) > 0 \). Then the following are equivalent for \( v \geq n \) and \( c_v = \frac{v^n - n}{(n+1)(v+1)} \):

(i) \[
\{ x \in \text{supp } \mu | f(x) \notin W_p^u \} \text{ is nonempty for any } u > v. \quad (6.19)
\]

(ii) \[
w_p(f_* \mu) \leq v. \quad (6.20)
\]

(iii) \[
(6.16) \text{ holds for some ( } \implies \text{ for any ) } R \text{ satisfying (6.18).} \quad (6.21)
\]

(iv) \[
(6.14) \text{ holds for some ( } \implies \text{ for any ) } R \text{ satisfying (6.18).} \quad (6.22)
\]

**Proof.** We have already observed that \((6.22)\) holds if and only if so does \((6.21)\), and that \((6.20)\) implies \((6.19)\) by definition. It remains to show that \((6.21) \implies (6.20) \) and that \((6.19) \implies (6.22)\). Assume \((6.21)\) and since \( \mu \) is Federer and \( (f, \mu) \) is good, we have for \( \mu\text{-a.e } x \in X \) has a neighbourhood \( V \) such that \( (f, \mu) \) good and \( \mu \) is \( D\)-federer on \( V \). Choose a ball \( B = B(x, r) \) of positive measure such that the dilated ball \( \tilde{B} = B(x, 3^{n+1}r) \) is contained in \( V \). Since we have already noticed in \((6.14)\) that \( \text{cov}(g_{u_f(x)} \Delta) \) is max of norm of linear combinations of
1, $f_1, \ldots, f_n$ condition (i) of (6.1) is satisfied. And (6.21) is equivalent to second hypothesis (6.2) in (6.1). Therefore we can conclude (6.20). Suppose (6.22) does not hold then equivalently condition (6.2) does not hold and by Lemma 6.2 it follows that

$$f(B \cap \text{supp } \mu) \subset W_u$$

for some $u > v$. □

The upshot of the last theorem is that (6.21) and (6.22) do not involve any of $f, \mu, X$. So if conditions (6.19), (6.20) hold for some $f, \mu, X$ satisfying the hypotheses of the Theorem then they hold any other $f', \mu', X'$ satisfying the hypotheses of this theorem. So for any two $f, \mu, X$ and $f', \mu', X'$ satisfying the hypotheses of this theorem, we have that $w(f_*,\mu) = w(f_*'\mu')$.

**Theorem 6.2.** Let $\mu$ be a Federer measure on a Besicovitch metric space $X, \mathcal{L}$ an affine subspace of $\mathbb{Q}_p^n$, and let $f : X \to \mathcal{L}$ be continuous map such that $(f, \mu)$ is good and nonplanar in $\mathcal{L}$. Then

$$w_p(f_*\mu) = w_{Z_i}(\mathcal{L}) = \inf\{w_p(y) \mid y \in \mathcal{L}\} = \inf\{w_p(f(x)) \mid x \in \text{supp } \mu\}. \quad (6.23)$$

**Proof.** Let us take $\nu = \lambda$ (Haar measure) on $\mathbb{Q}_p^s$ and $h$ as in (6.18). Then by definition, $w(\mathcal{L}) = w_p(h_*\nu)$. Since $h, \nu$ and $\mathbb{Q}_p^s$ satisfy the hypotheses of Theorem 6.1, we have $w_p(f_*\mu) = w_p(\mathcal{L})$ by the previous discussion. And we already have from definitions that

$$\inf\{w_p(y) \mid y \in \mathcal{L}\} \leq \inf\{w_p(f(x)) \mid x \in \text{supp } \mu\} \leq w_p(f_*\mu).$$

Therefore to conclude the theorem it is enough to show that

$$w(\mathcal{L}) \leq \inf\{w(y) \mid y \in \mathcal{L}\}.$$

But this follows by taking $v = \inf\{w(y) \mid y \in \mathcal{L}\}$ and $h$ in Theorem 6.1; condition (6.19) automatically holds. □

**Corollary 6.1.** Let $\mathcal{L}$ be an $s$-dimensional affine subspace of $\mathbb{Q}_p^n$. Then

$$w_p(\mathcal{L}) = \max(n, \inf\{v \text{ for which (6.16) holds for } R \text{ as in (6.18)}\})$$

In view of Proposition 3.1, we have

**Corollary 6.2.** Let $\mu$ be a Federer measure on a Besicovitch metric space $X, \mathcal{L}$ an affine subspace of $\mathbb{Q}_p^n$, and let $f : X \to \mathcal{L}$ be continuous map such that $(f, \mu)$ is good and nonplanar in $\mathcal{L}$. Then

$$w(f_*\mu) = w(\mathcal{L}) = \inf\{w(y) \mid y \in \mathcal{L}\} = \inf\{w(f(x)) \mid x \in \text{supp } \mu\}. \quad (6.24)$$
7. Computation of Higher Diophantine Exponents

In this section we want to relate condition (6.16) in terms of Diophantine conditions of a parametrizing matrix of the $s$ dimensional subspace $\mathcal{L}$. Set

$$R = R_A := (I_{s+1} A) \in M_{s+1,n+1}$$

(7.1)

where $A = (a_{ij})_{i=0,\ldots,s}^{j=s+1,\ldots,n} \in M_{s+1,n-s}$. Then we can write

$$\|R_A c(w)\|_p = \max_{i=0,\ldots,s} \max_{J \subset \{1,\ldots,n\} \atop \#J = j-1} |\langle e_i^p + a_i, e_j^p, w \rangle|_p.$$ 

Let $\pi_\bullet$ be the projection from $\bigwedge^j \mathbb{Q}_p^{n+1} \bigoplus \bigwedge^j \mathbb{R}^{n+1}$ to

$$\bigwedge^j \mathbb{Q}_p \langle e_{s+1}^p, \ldots, e_n^p \rangle \bigoplus \bigwedge^j \mathbb{R} \langle e_{s+1}^\infty, \ldots, e_n^\infty \rangle$$

where $F \langle e_{s+1}, \ldots, e_n \rangle$ denotes the $F$ span of $e_{s+1}, \ldots, e_n$, the last $n-s$ vectors of standard basis of $F^{n+1}$ for a field $F$.

**Lemma 7.1.** Suppose that $\|R_A c(w)\|_p\|w\|_\infty \leq 1$ for some $w \in \mathcal{S}_{n+1,j}$. Then $\|w\|_p\|w\|_\infty \ll 1 + \|\pi_\bullet(w)\|_p\|w\|_\infty$.

**Proof.** Suppose the smallest index of $I \subset \{1, \ldots, n\}$ ($\#I = j$) is $m$. If $m > s$ then we have that

$$|\langle e_I, w \rangle|_p \leq \|\pi_\bullet(w)\|_p.$$ 

On the other hand, when $m \leq s$ we have

$$|\langle e_I^p, w \rangle|_p \|w\|_\infty$$

$$< |\langle e_m^p + a_m, e_I^\{\{m\}\}, w \rangle|_p \|w\|_\infty + |\langle a_m \wedge e_I^\{\{m\}\}, w \rangle|_p \|w\|_\infty$$

$$< \|R_A c(w)\|_p \|w\|_\infty + \max_{i=0,\ldots,s} \|a_i\|_p \max_{\#J = j} \max_{i=0,\ldots,s} J \subset \{m+1,\ldots,n\} |\langle e_J^p, w \rangle|_p \|w\|_\infty$$

$$< 1 + \max_{i=0,\ldots,s} \|a_i\|_p \max_{\#J = j} \max_{i=0,\ldots,s} J \subset \{m+1,\ldots,n\} |\langle e_J^p, w \rangle|_p \|w\|_\infty.$$ 

Now the same argument can be applied to the each of the components $|\langle e_I^p, w \rangle|_p \|w\|_\infty$ where the smallest index is at least $m+1$. After at most $s+1$ steps the process terminates and this concludes the proof. 

Note that for $w \in \mathcal{S}_{n+1,j}$ with $j > n-s$ we have that $\pi_\bullet(w) = 0$. Since

$$\|\pi(w)\|_p \|w\|_\infty \geq 1 \forall w \in \mathcal{S}_{n+1,j},$$
we have that
\[ \| R_A c(w) \|_p \| w \|_\infty \geq 1 \geq (\| \pi(w) \|_p \| w \|_\infty)^{-u} \]
for \( u > 0 \) and for all arbitrarily large \( \| \pi(w) \|_p \| w \|_\infty \). Thus condition (6.16) holds for \( j > n - s \). Therefore while computing the exponent \( w_{Z_s} (L) \), according to Corollary 6.1, the subgroups of rank greater than \( n - s \) are irrelevant. We now define higher Diophantine exponents.

**Definition 7.1.** For each \( j = 1, \ldots, n - s \), let
\[ w^p_j(A) := \sup \left\{ v \mid \exists w \in S_{n+1,j} \text{ with arbitrary large } \| \pi_*(w) \|_p \| w \|_\infty \text{ such that } \| R_A c(w) \|_p \| w \|_\infty < (\| \pi_*(w) \|_p \| w \|_\infty)^{-v + 1 - j} \right\}. \]

We now observe

**Corollary 7.1.** For an \( s \)-dimensional subspace \( L \) parametrized by \( A \) as in (7.1)
\[ w_p(L) = \max(n, w^p_j(A))_{j=1,\ldots,n-s}. \]

**Proof.** By Lemma 7.1 we can conclude that for some \( v \) condition (6.16) holds if and only if \( \max_{j=0,\ldots,n-s} w^p_j(A) \leq v \). And thus by (6.1), we can conclude this corollary. \( \square \)

**Lemma 7.2.** \( w^p_1(A) = w_p(A) \).

**Proof.** Take \( w = \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_s \\ q_{s+1} \\ \vdots \\ q_n \end{bmatrix} = \tilde{q} \in \mathbb{D}^{n+1} \setminus \{0\} = S_{n+1,1} \) and denote
\[ q_0 = \begin{bmatrix} q_0 \\ \vdots \\ q_s \end{bmatrix} \text{ and } q = \begin{bmatrix} q_{s+1} \\ \vdots \\ q_n \end{bmatrix}. \]
\[ R_A(c(w)) = q_0 + Aq, c(w) = w \text{ and } \pi_*(w) = q. \]
Hence the definition of both exponents coincide. \( \square \)

**Theorem 7.1.** Suppose \( L \) is an \( s = n - 1 \) dimensional subspace parametrized by \( R_A \) as in (7.1). Then
\[ w_p(L) = \max(n, w_p(A)). \]

**Proof.** The Theorem follows directly from Lemma 7.2 and Corollary 7.1. \( \square \)
Lemma 7.3. Let $\mathcal{L}$ be parametrized by $R_A$ as in (7.1). Then for any $v < w_{Z_s}(A)$ there exists arbitrary large $\|q\|_p \|\tilde{q}\|_\infty$ such that

$$|q \cdot y + q_0|_p \|\tilde{q}\|_\infty \leq (\|q\|_p \|\tilde{q}\|_\infty)^{-v}$$

for all $y \in \mathcal{L}$ and $\tilde{q} = (q_0, q) \in \mathcal{D}^{n+1}$.

(7.2)

Proof. Note that having arbitrarily large $\|q\|_p \|\tilde{q}\|_\infty$ is not the same as infinitely many $\tilde{q} \in \mathcal{D}^{n+1}$. Since $v < w_p(A)$ there exists $\varepsilon > 0$ such that

$$v < v + 2\varepsilon < w_p(A).$$

Hence there exists arbitrary large $\|q\|_p \|\tilde{q}\|_\infty$ such that

$$\|q_0 + Aq\|_p \|\tilde{q}\|_\infty \leq (\|q\|_p \|\tilde{q}\|_\infty)^{-v - 2\varepsilon}$$

and $\tilde{q} = (q_0, q) = \left(\begin{array}{c} q_0 \\ \vdots \\ q_n \end{array} \right)$ $\in \mathcal{D}^{s+1} \times \mathcal{D}^{n-s}$.

For any $y = (x, \tilde{x}A) \in \mathcal{L}$ where $\tilde{x} = (1, x) \in \mathcal{Q}_p^{s+1}$ we can write

$$\begin{vmatrix} q_0 + y \cdot \left[\begin{array}{c} q_1 \\ \vdots \\ q_n \end{array} \right] \\ q_0 + (x, \tilde{x}A) \cdot \left[\begin{array}{c} q_1 \\ \vdots \\ q_n \end{array} \right] \end{vmatrix}_p \|\tilde{q}\|_\infty = \left| q_0 + x \cdot \left[\begin{array}{c} q_1 \\ \vdots \\ q_n \end{array} \right] + \tilde{x}A \cdot \left[\begin{array}{c} q_1 \\ \vdots \\ q_n \end{array} \right] \right|_p \|\tilde{q}\|_\infty$$

$$= |\tilde{x}(Aq + q_0)|_p \|\tilde{q}\|_\infty$$

$$\leq \|x\|_p \|Aq + q_0\|_p \|\tilde{q}\|_\infty$$

$$\leq \|\tilde{x}\|_p (\|q\|_p \|\tilde{q}\|_\infty)^{-v - 2\varepsilon}$$

for arbitrary large $\|q\|_p \|\tilde{q}\|_\infty$. This implies that we also have

$$|q_0 + y \cdot q_1|_p \|\tilde{q}\|_\infty \leq (\|q\|_p \|\tilde{q}\|_\infty)^{-v - \varepsilon}.$$
Then we have that
\[ |q_0 + y \cdot q_1|_p \cdot \|q\|_\infty \]
\[ \leq \frac{1}{(\|q_1\|_p \cdot \|q\|_\infty)^{v+\varepsilon}} \cdot (\|q\|_p \cdot \|q\|_\infty)^{v+\varepsilon} \]
\[ \leq \frac{1}{(\|q_1\|_p \cdot \|q\|_\infty)^v} \]
for arbitrary large \( \|q\|_p \cdot \|q\|_\infty \), hence for arbitrary large \( \|q_1\|_p \cdot \|q\|_\infty \), thereby concluding the proof. \( \square \)

An immediate consequence of above lemma is the following corollary.

**Corollary 7.2.** Let \( \mathcal{L} \) is parametrized by \( R_A \) as in (7.1). Then
\[ w_p(A) \leq w_p(\mathcal{L}). \]

Another observation is when in the parametrizing matrix \( A \) with more than one column and all columns are rational multiple of one column then \( w_p(A) = \infty \) and so is \( w_p(\mathcal{L}) \). Combining this observation with Theorem 7.1 we can conclude the following:

**Theorem 7.2.** Let \( \mathcal{L} \) be an affine subspace parametrized by \( R_A \) as in (7.1). If all the columns are rational multiples of one column then
\[ w_p(\mathcal{L}) = \max(n, w_p(A)) \text{ and } w(\mathcal{L}) = \max(n + 1, w(A)). \]

The calculation of symmetries of higher order exponents is exactly the same as in case of \( \mathbb{R} \) as discussed in [20]. So we will just state them in the \( p \)-adic setup.

**Lemma 7.4.** For any \( A \in M_{s+1,n-s} \) and all \( w \in S_{n+1,j}, 2 \leq j \leq n-s \), one has
\[ \max_{i=0,\ldots,s} \max_{J \subseteq \{0,\ldots,n\}} \left| \langle (e_i^p + a_i) \wedge e_j^p, w \rangle \right|_p \ll \|R_A c(w)\|_p \]
\[ \|R_A c(w)\|_p \ll \max_{i=0,\ldots,s} \max_{J \subseteq \{i+1,\ldots,n\}} \left| \langle (e_i^p + a_i) \wedge e_j^p, w \rangle \right|_p. \]

The Lemma enables us to conclude that \( w_j^p(A) \) is symmetric under any row operation. The next lemma allows to consider other row operations namely the multiplication by nonzero rationals and adding one row to another and transposition of rows.

**Lemma 7.5.** Let \( A' = BA \) for some \( B \in GL_{s+1}(\mathbb{Q}) \); in other words, \( A' \) can be obtained from \( A \) by a sequence of elementary row operations with rational coefficients. Then \( w_j^p(A') = w_j^p(A) \) for all \( j \).
Now we will wrap up this section by stating the last lemma and a theorem similar to Theorem 7.2 for rational multiples of rows.

**Lemma 7.6.** Suppose that $A$ has more than one row, and let $A'$ be the matrix obtained from $A$ by removing one of its rows. Then $w^p_j(A') \geq w^p_j(A)$ for all $j$. If in addition the removed row is a rational linear combination of the remaining rows, then $w^p_j(A') = w^p_j(A)$ for all $j$.

As a consequence we have the following theorem.

**Theorem 7.3.** Let $\mathcal{L}$ be an affine subspace parametrized by $R_A$ as in (7.1). If all the rows are rational multiples of one row then one has

$$w_p(\mathcal{L}) = \max(n, w_p(A)) \quad \text{and} \quad w(\mathcal{L}) = \max(n+1, w(A)).$$

The lemmata and Theorem stated above are proved by a verbatim repetition of the arguments of Kleinbock [20], and so we omit the proofs.

8. Almost all vs. no

In this section, we address $p$-adic versions of D. Kleinbock’s paper [21] where he proved that analytic manifolds possess a remarkable dichotomy with regard to certain Diophantine properties, see also [7] and [28]. Set $S = \{\infty, p\}$. We begin with

**Lemma 8.1.** For any $\Delta \in \mathcal{P}(\mathcal{D}, n+1)$ and $g \in \text{GL}_{n+1}(\mathbb{Q}_p \times \mathbb{R})$ we have

$$\delta(g \mathbb{D}^{n+1}) \leq \text{cov}(g \Delta)^{\frac{1}{\pi(S)}}.$$

**Proof.** Note that $g \Delta$ is a lattice in $\mathbb{Q}_S g \Delta$. Set $j = \text{rank}(\Delta)$. Now consider the ball

$$D = D_\infty \times D_p$$

$$= \left\{ x^\infty \in \mathbb{R}^{n+1} \left| ||x^\infty||_\infty \leq (\text{cov}(g \Delta))^\frac{1}{j} \right. \right\} \times \left\{ x^{(p)} \in \mathbb{Q}_p^{n+1} \left| \begin{array}{c}
|x_1^{(p)}|_p \leq 1 \\
|x_2^{(p)}|_p \leq 1 \\
\vdots \\
|x_{n+1}^{(p)}|_p \leq 1
\end{array} \right. \right\}.$$

Denote $D \cap \mathbb{Q}_S g \Delta = D_1$, a ball in $\mathbb{Q}_S g \Delta$. The normalized Haar measure on $\mathbb{Q}_S \Delta_1$ where $\Delta_1 = g \Delta$ is defined as $\lambda_S(D_1) = \mu_S(\pi^{-1}(D_1))$ where $\mu_S$ is the Haar measure on $\mathbb{Q}_S$ and $\pi : \mathbb{Q}_S \mapsto \mathbb{Q}_S \Delta_1 = \mathbb{Q}_S v_1 + \cdots + \mathbb{Q}_S v_j$ and $v_1, v_2, \cdots, v_j$ are taken such that $v_1^\infty, \cdots, v_j^\infty$ form an orthonormal basis of $(\mathbb{Q}_S \Delta_1)_\infty$ and $(\mathbb{Q}_S \Delta_1)_p \cap \mathbb{Z}_p^{n+1} = \mathbb{Z}_p v_1^{(p)} + \cdots + \mathbb{Z}_p v_j^{(p)}$. So

$$\lambda_S(D_1) \geq 2^j \text{cov}(g \Delta).$$
Hence by Minkowski’s theorem there exists $g\gamma \in g\Delta$ i.e. $\gamma \in \mathbb{D}^{n+1}$ such that
\[
\|(g\gamma)^{\infty}\|_{\infty} \leq \text{cov}(g\Delta)^{\frac{1}{2}}
\]
\[
\|(g\gamma)(p)\|_{p} \leq 1.
\]
Therefore $c(g\gamma) \leq \text{cov}(g\Delta)^{\frac{1}{2}} + \text{rk}(\Delta) = \delta(g\mathbb{D}^{n+1}) \leq \text{cov}(g\Delta)^{\frac{1}{2}} \cdot \delta(g\mathbb{D}^{n+1})$.

**Proposition 8.1.** Let $U$ be a connected open subset of $\mathbb{Q}_{p}^{d}$ and let $\mathcal{F}$ be a finite-dimensional space of analytic $\mathbb{Q}_{p}$ valued functions on $U$. Then for any $x \in U$ there exists $C, \alpha > 0$ and a neighbourhood $W$ of $x$ such that every element of $\mathcal{F}$ is $(C, \alpha)$-good on $W$.

**Proof.** Without loss of generality we may assume that $\mathcal{F}$ contains constant functions. Let $1, f_1, \cdots, f_n$ be a basis of $\mathcal{F}$. Then $\mathbf{f} = (f_1, f_2, \cdots, f_n)$ is nonplanar. For analytic functions nondegeneracy is equivalent to nonplanarity. Therefore, the conclusion follows from Proposition 4.2 of [24].

The Corollary below now follows from the expression of $\text{cov}(g_{t}u_{f}(x)\Delta)$ in (6.12) as a maximum of norms of linear combinations of $f_i$’s.

**Corollary 8.1.** Let $U$ be a connected open subset of $\mathbb{Q}_{p}^{d}$, and let $\mathbf{f} : U \mapsto \mathbb{Q}_{p}^{n}$ be an analytic map. Then for any $x_0 \in U$ there exists $C, \alpha > 0$ and a neighbourhood $W$ of $x_0$ contained in $U$ such that for any $\Delta \in \mathcal{P}(\mathbb{D}, n+1)$ and $t \in \mathbb{N}$ the functions $x \mapsto \text{cov}(g_{t}u_{f(x)}\Delta)$ are $(C, \alpha)$-good on $W$.

Define $\gamma(y) := \sup\{c \geq 0 \mid \delta(g_{t}u_{y}\mathbb{D}^{n+1}) \leq p^{-ct} \text{ for infinitely many } t \in \mathbb{N}\}$ for $y \in \mathbb{Q}_{p}^{n}$. From Proposition 4.1 and Lemma 6.1 we can conclude that
\[
w_{p}(y) = \frac{n(1 + \gamma(y)) + \gamma(y)}{1 - (n+1)\gamma(y)}.
\]

We are now ready for

**Theorem 8.1.** Suppose $\mathbf{f} : U \mapsto \mathbb{Q}_{p}^{n}$ is an analytic map and $U$ is a connected open subset of $\mathbb{Q}_{p}^{d}$. Denote by $\lambda$ the Haar measure on $\mathbb{Q}_{p}^{d}$. Let $\gamma \geq 0$ and $x_0 \in U$ be such that $\gamma(f(x_0)) \leq \gamma$ then $\lambda$-almost every $x \in U$ we have $\gamma(f(x)) \leq \gamma$.

**Proof.** Let $\gamma \geq 0$ and $x_0 \in U$ be such that $\gamma(f(x_0)) \leq \gamma$. Consider the set $U_1 := \{x \in U \mid \gamma(f(x)) \leq \gamma\}$ which is nonempty since $x_0 \in U_1$ and define
\[
U_2 := \{x \in U \mid \mu(B \setminus U_1) = 0 \text{ for some neighbourhood } B \text{ of } x\}.
\]

We claim that $U_2 = U_1 \cap U$. Since $U_2$ is open and $U$ is connected so we have that $U_2 = U$. So for every point $x \in U$, there exists a
neighbourhood such that almost every point in that neighbourhood is inside $U_1$.

Now take a point $x_1 \in U_1 \cap U$ and a ball $B$ of $x_1$ such that $\tilde{B} := 3^{d-1}B$ is inside a neighbourhood appearing in Corollary 8.1. We want to apply Theorem 5.3 to the function $x \mapsto g_t u_f(x)$. The first condition (i) of Theorem 5.3 is satisfied. Since $x \in U_1$, there exists $x' \in B \cap U_1$ and this implies that $\gamma(f(x')) \leq \gamma$, which in turn implies that $\forall \gamma' > \gamma$, $\delta(g_t u_f(x') \mathbb{D}^{n+1}) \geq p^{-\gamma't}$ for all but finitely many $t \in \mathbb{N}$.

Now applying Lemma 8.1 we have that $\text{cov}(g_t u_f(x') \Delta)^{1/|\Delta|} \geq p^{-\gamma't}$ for all $\Delta \in \mathcal{P}(\mathbb{D}, n+1)$ and for all but finitely many $t \in \mathbb{N}$. Hence condition (ii) of Theorem 5.3 is satisfied. Taking $\varepsilon = p^{-\gamma''t}$ where $\gamma'' > \gamma'$ and applying Theorem 5.3 we have $\lambda \left\{ x \in B \mid \delta(g_t u_f(x') \mathbb{D}^{n+1}) < p^{-\gamma''t} \right\} \leq C' \left( \frac{p^{-\gamma''t}}{p^{-\gamma't}} \right)^{\alpha} \mu(B)$.

By the Borel-Cantelli lemma we immediately have that for $\mu$-a.e $x \in B$ we have that $\delta(g_t u_f(x') \mathbb{D}^{n+1}) \leq p^{-\gamma''t}$ for infinitely many $t \in \mathbb{N}$. Hence by definition for $\lambda$-a.e $x \in B$, $\gamma(f(x)) \leq \gamma$ as $\gamma'', \gamma'$ were taken arbitrary close to $\gamma$. This implies $x_1 \in U_2$ giving $U_1 \cap U \subset U_2$ and clearly $U_2 \subset \bar{U}_1$. $\square$

Now from formula 8.1 we can conclude the following:

**Corollary 8.2.** Suppose $f : U \mapsto \mathbb{Q}_p^n$ is an analytic map and $U$ is a connected open subset of $\mathbb{Q}_p^n$. Let $v \geq n$ and $x_0 \in U$ be such that $w_p(f(x_0)) \leq v$ then for almost every $x \in U$, we have $w_p(f(x)) \leq v$.

Since one can parametrize connected analytic manifolds by images of connected open neighbourhoods under analytic functions, we have the following Theorems.

**Theorem 8.2.** Suppose $M$ is a connected analytic manifold of $\mathbb{Q}_p^n$. Let $v \geq n$ and suppose $w_p(y) \leq v$ for some $y \in M$ then for almost every $y \in M$, $w_p(y) \leq v$.

Therefore we also have,

**Theorem 8.3.** Suppose $M$ is a connected analytic manifold of $\mathbb{Q}_p^n$. Let $v \geq n+1$ and suppose $w(y) \leq v$ for some $y \in M$ then for almost every $y \in M$, $\omega(y) \leq v$. 
So if one point in a connected analytic $p$-adic manifold is not very well approximable, then almost every point in the manifold is not very well approximable. Note that this phenomenon was already clear from Theorem 6.2 for the manifolds which were nondegenerate inside some affine subspace. The theorems above constitute $p$-adic analogues of Theorem 1.4 (a) of [21]. We have not pursued part (b) of the Theorem in loc. cit. which has to do with singular vectors.

9. Multiplicative Diophantine approximation

We define 

$$|q|_+ = \begin{cases} |q|_{\infty} & \text{if } q \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

for $q \in \mathcal{D}$. For $q \in \mathbb{Z}$ this definition matches with classical one. Further define 

$$\Pi_+(\tilde{q}) = \prod_{i=0}^{n} |q_i|_+.$$ 

Say that $y \in \mathbb{Q}_p^n$ is very well multiplicatively approximable (VWMA) if, for some $\varepsilon > 0$ there are infinitely many solutions $\tilde{q} = (q_0, q_1, \ldots, q_n) \in \mathbb{Z}^{n+1}$ to 

$$|q_0 + q \cdot y| \leq \Pi_+(\tilde{q})^{-(1+\varepsilon)}.$$ (9.1)

One can similarly define multiplicative Diophantine exponents and we will denote the corresponding sets by $\text{WM}_v$. The analogous multiplicative analogues of Sprindžuk’s conjectures were formulated by Baker and settled by Kleinbock and Margulis in [23]. In [19], D. Kleinbock proves his results for affine subspaces and their nondegenerate manifolds also in the multiplicative context. The setup is slightly more subtle but the dynamical approach is powerful enough to deal with this, one replaces the one-parameter diagonal action with a multiparameter action. In [14], the second named author proved a multiplicative version of a Khintchine type theorems for hyperplanes. Further in [24], the authors established the $S$-adic Baker-Sprindžuk conjectures, namely they also considered the multiplicative case. In §6.3 of [20], D. Kleinbock refers to the possibility of proving his improved exponent results also for subspaces, some of this was accomplished in [34].

9.1. A Dynamical correspondence for $p$-adic VWMA vectors.

In this section we define $g_t \in \text{GL}_{n+1}(\mathbb{Q}_p \times \mathbb{R})$ such that 

$$g_t^p := \text{diag}(p^{-t}, 1, \cdots, 1) \text{ and } g_t^\infty := \text{diag}(p^{-t_0}, p^{-t_1}, \cdots, p^{-t_n})$$

where $t = (t_0, \cdots, t_n) \in \mathbb{Z}_{+}^{n+1}$ and $t := \sum_{i=0}^{n} t_i$. 


Lemma 9.1. A vector \( \mathbf{y} \in \mathbb{Q}_p^n \) is very well multiplicatively approximable if and only if there exist unbounded \( t > 0 \) with \( t \in \mathbb{Z}_{n+1}^+ \) such that
\[
\delta(g_k \mathbf{y}, D^{n+1}) \leq p^{-\gamma t}
\]
for some \( 0 < \gamma < \frac{1}{n+1} \).

Proof. Suppose \( \delta(g_k \mathbf{y}, D^{n+1}) \leq p^{-\gamma t} \), this implies that for unbounded \( t > 0 \) and \( \mathbf{q} = (q_0, \mathbf{q}) \in D^{n+1} \), we have that
\[
\max(p^i |q_0 + q \cdot \mathbf{y}|_p, \|\mathbf{q}\|_p) \max_{i=0, \ldots, n} (p^{-t_i} |q_i|_\infty) \leq p^{-\gamma t}, \quad (9.2)
\]
which implies that
\[
\max(p^i |q_0 + \mathbf{q} \cdot \mathbf{y}|_p, \|\mathbf{q}\|_p) \max_{i=0, \ldots, n} (p^{-t_i} |q_i|_\infty) \leq p^{-\gamma t}. \quad (9.3)
\]
This implies that
\[
\max(p^i |q_0 + \mathbf{q} \cdot \mathbf{y}|_p, \|\mathbf{q}\|_p) \max_{i=0, \ldots, n} (p^{-t_i} |q_i|_\infty) \leq p^{-\gamma t} \quad (9.4)
\]
where \( \mathbf{q} = (q_0, \mathbf{q}) \). Since \( \gamma > 0 \) we have \( \mathbf{q} \neq 0 \). Hence for \( i = 1, \ldots, n \) we have \( |q_i|_p = \frac{|q_i|}{\|\mathbf{q}\|_p} \leq 1 \) giving \( q_i \in \mathbb{Z} \) for such \( i \). There exists \( 1 \leq k \leq n + 1 \) such that \( \# \{ i \mid t_i \geq \gamma t \} = k \), possibly considering a subsequence of \( t \). From \( \max_{i=0, \ldots, n} (p^{-t_i} |q_i|_\infty) \leq p^{-\gamma t} \) we have that
\[
|q_i|_\infty \leq p^{-\gamma t + t_i} \text{ and } 1 \leq p^{-\gamma t + t_i}
\]
for \( k \) no of \( i \). Otherwise we have \( |q_i|_\infty \leq p^{-\gamma t + t_i} < 1 \). Therefore
\[
\Pi_+(\mathbf{q}) \leq \prod_{i \text{ s.t } t_i \leq \gamma t} |q_i|_+
\]
\[
\leq p^{-k\gamma t + \sum_{i \text{ s.t } t_i \geq \gamma t} t_i}
\]
\[
\leq p^{(1-k)\gamma t}. \quad (9.5)
\]
Again possibly taking a subsequence, there exists \( 1 \leq m \leq n + 1 \) such that \( \# \{ i \mid q_i' \neq 0 \} = m \). By the definition \( q_i' \neq 0 \) implies \( |q_i|_+ = |q_i|_\infty \).

Now by (9.4) we have that
\[
|q_0 + \mathbf{q} \cdot \mathbf{y}|_p \leq p^{-\gamma t}
\]
Multiplying over all those \( i \) such that \( q_i' \neq 0 \),
\[
|q_0 + \mathbf{q} \cdot \mathbf{y}|_p^m \cdot \Pi_+(\mathbf{q}) \leq p^{-m\gamma t}
\]
\[
\leq p^{-m\gamma t}
\]
\[
\leq p^{-(m\gamma + m - 1)t}.
\]
Now from (9.5) it follows that
\[
|q'_0 + q' \cdot y|^m_p \leq \Pi_+(\bar{q}')^{-\left(\frac{2m+m-1}{1-k}\right)+1}
\leq \Pi_+(\bar{q}')^{-\frac{2m+m-k\gamma}{1-k\gamma}}.
\]

Hence we have that
\[
|q'_0 + q' \cdot y|_p \leq \Pi_+(\bar{q}')^{-\frac{m\gamma + m - k\gamma}{m(1-k\gamma)}}.
\]

Since \( \gamma < \frac{1}{n+1} < \frac{1}{k} \) we have
\[
\frac{(m\gamma + m - k\gamma)}{m(1-k\gamma)} - 1 = \frac{\gamma(m - k + mk)}{m(1-k\gamma)} = \varepsilon > 0.
\]

We now claim that the \( \Pi_+(\bar{q}') \) appearing here are unbounded. Note that
\[
|q_0 + q \cdot y|^m_p \Pi_+(\bar{q}) \to 0 \text{ due to the existence of unbounded many } t > 0.
\]
The same reason also gives
\[
1 < |q_0|_p|q_0|_\infty \leq \max(p^{-\varepsilon}, \|q\|_p|q_0|_\infty\|y\|_p)
\]
which implies that
\[
1 \leq \|q\|_p|q_0|_\infty\|y\|_p
\]
if \( q_0 \) is nonzero, which says that \( |q'_0|_+ \geq c \) where \( c > 0 \) depends only on \( y \). We denote \( q_i = p^{l_i}z_i \in \mathcal{D} \) where \( z_i \) are integers without any \( p \) factor. Then
\[
|q'_i|_+ = \begin{cases} |q_i|_\infty\|q\|_p & \text{if } q_i \neq 0 \\ 1 & \text{otherwise.} \end{cases}
\]

Suppose \( \Pi_+(\bar{q}') \leq M \) for some \( M > 0 \), i.e. \( \prod_{i=0}^n(|q'_i|_+) \leq M \) which in turn implies that
\[
|q_0|_+ \prod_{\{i>1 \mid q_i \neq 0\}} |z_i|_\infty \leq M.
\]

Here we are using that if \( \|q\|_p = p^l \) then \( -l \leq l \) for \( i > 1 \). Since \( |q'_0|_+ \) is bounded below by positive, there are finitely many options for the integers \( z_i \) occurring in \( q_i \) for \( i > 1 \). Hence we have that
\[
|q_0|_+ \prod_{\{i>1 \mid q_i \neq 0\}} (p^{l_i}\|q\|_p) \leq M'
\]
for some \( M' > 0 \). The above inequality gives
\[
|q_0|_+ \prod_{\{i>1 \mid q_i \neq 0\}} p^{l_i} \leq M'. \tag{9.6}
\]

Hence we have \( |q'_0|_+ \leq M' \) and arguing as in (9.4) we may conclude that
\[
|z_0|_\infty = |q'_0|_\infty|q'_0|_p \leq \max(1, M'||y||_p),
\]
and $\delta$ has only finitely many options. Now from (9.6)
\[ c \leq \prod_{\{i \mid q_i \neq 0\}} p^{l_i} \leq M'. \]

Thus we now have $-\alpha \leq \sum_{\{i \mid q_i \neq 0\}} l_i + ml \leq \alpha$ for some $\alpha > 0$. Since $c \leq |q_0|_+ \leq M'$ which gives $-\beta \leq (l_0 + l) \leq \beta$ for some $\beta > 0$. Hence for $i > 1$ and $q_i \neq 0$ we have $0 \leq l_i + l \leq \alpha'$ for some $\alpha' > 0$. But then only way $|q_0 + q \cdot y|_p \cdot \Pi_+ (\tilde{q})$ can go to 0 is if there exists some $\tilde{q}(\neq 0) \in \mathbb{Z}^{n+1}$ such that $q_0 + q \cdot y = 0$. In that case $y$ is very well multiplicatively approximable. So if $y$ is not such then $\Pi_+ (\tilde{q}')$ has to be unbounded and satisfies
\[ |q_0' + q' \cdot y|_p \leq \Pi_+ (\tilde{q}')^{-(1+\varepsilon)} \]

where $\tilde{q}' = (q_0', q') \in \mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}^n$. Another crucial observation now is that $|q_0'|_p$ is bounded above by a constant depending on $y$. So in case $q_0' \notin \mathbb{Z}$ we can write
\[ \left| |q_0|_p \cdot (q_0' + q' \cdot y) \right|_p |q_0'|_p \leq \Pi_+ (\tilde{q}')^{-(1+\varepsilon)}. \]

Now taking $q'' = |q_0|_p \tilde{q}'$ and using the upper bound on $|q_0|_p$ enables us to conclude
\[ |q'' + q'' \cdot y|_p \leq \Pi_+ (\tilde{q}'')^{-(1+\varepsilon')} \]

for infinitely many $\tilde{q}'' \in \mathbb{Z}^{n+1}$ with $\varepsilon' < \varepsilon$. Therefore $y$ is very well multiplicatively approximable.

We now prove the converse. Suppose we have that
\[ |q_0 + q \cdot y|_p \leq \Pi_+ (\tilde{q})^{-(1+\varepsilon)} \tag{9.7} \]

for infinitely many $\tilde{q} \in \mathbb{Z}^{n+1}$. Then choose $t_i > 0$ such that $|q_i|_+ = \Pi_+ (\tilde{q})^{-\frac{\varepsilon}{n+1}} p^{t_i}$. Multiplying these we get $p^t = \Pi_+ (\tilde{q})^{(1+\varepsilon)}$ which guarantees $t$ to be unbounded. The choice of $t_i$ gives the following condition
\[ p^{-t_i} |q_i|_\infty \leq p^{-t_i} |q_i|_+ \leq \Pi_+ (\tilde{q})^{-\frac{\varepsilon}{n+1}} = p^{-\gamma t} \forall i = 0, \ldots, n \]

where $\gamma = \frac{\varepsilon}{(n+1)(1+\varepsilon)}$. Hence $\|q'_t u y \tilde{q}\|_\infty \leq p^{-\gamma t}$. On the other hand,
\[ |p^{-t}(q_0 + q \cdot y)|_p = p^t |q_0 + q \cdot y|_p \leq p^t \cdot 1 = 1, \]

due to (9.7). Therefore we have $c(g_t u y \tilde{q}) \leq p^{-\gamma t}$. Now taking $\lfloor t \rfloor$ consisting of integer factors and observing that the ratio of $\delta(g_{\lfloor t \rfloor} u y \mathbb{D}^{n+1})$ and $\delta(g_{\lfloor t \rfloor} u y \mathbb{D}^{n+1})$ is bounded by uniform factor. Hence reducing $\gamma$ a bit we can conclude the lemma.

\[ \Box \]
Note one direction (namely, from VWMA vectors to dynamics) of the last lemma was already observed by Kleinbock and Tomanov in [24] with a slight variation. The main content of the previous lemma is that we can come back from dynamics to number theory using the reverse direction, which was earlier not known to our best knowledge.

**Theorem 9.1.** For any connected analytic manifold of \( \mathbb{Q}_p^n \) if one point is not very well multiplicatively approximable then almost every point on this manifold is not very well multiplicatively approximable.

**Proof.** Following the proof of theorem 8.1 and considering

\[
U_1 := \left\{ x \in U \left| \begin{array}{l}
\text{for any } 0 < \gamma < \frac{1}{n+1}, \quad \delta(g_t u_g D^{n+1}) \leq p^{-\gamma t} \\
\text{for all but finitely many } t \in \mathbb{Z}_+^{n+1}
\end{array} \right. \right\}
\]

one can show that \( U_1 \) has full measure if nonempty. Then the using the above Lemma 9.1 one can conclude that if one point is not very well multiplicatively approximable then \( U_1 \neq \emptyset \), hence of full measure. And then again using above Lemma 9.1 almost every point is not very well multiplicatively approximable.

Note that with same repetition of argument as Proposition 6.1 it can be proved that:

**Proposition 9.1.** Take \( \mathcal{R} = \mathbb{Q}_p \times \mathbb{R} \). Let \( X \) be a Besicovitch metric space and \( \mu \) be a uniformly Federer measure on \( X \). Denote \( \tilde{B} := B(x, 3^{n+1}r) \). Suppose we are given a continuous function \( f : X \mapsto \mathbb{Q}_p^n \) and \( C, \alpha > 0 \) with the following properties

(i) \( x \mapsto \text{cov}(g_t u_{f(x)} \Delta) \) is \((C, \alpha)\) good with respect to \( \mu \) in \( \tilde{B} \forall \Delta \in \mathcal{P}(\mathcal{D}, n+1) \),

(ii) for any \( d > 0 \) there exists \( T = T(d) > 0 \) such that for any \( t \in \mathbb{Z}_+^{n+1} \) with \( t \geq T \) and any \( \Delta \in \mathcal{P}(\mathcal{D}, n+1) \) one has

\[
\sup_{x \in B \cap \text{supp} \mu} \text{cov}(g_t u_{f(x)} \Delta) \geq p^{-(\text{rank} \Delta) dt} . \tag{9.8}
\]

Then \( \mu \) a.e every \( x \), \( f(x) \) is not VWMA.

Note that condition (ii) in the above Lemma is actually necessary. If (ii) does not hold then there exists unbounded \( t \) with \( t \in \mathbb{Z}_+^{n+1} \) such that

\[
\delta(g_t u_{f(x)} D^{n+1}) \leq p^{-dt}
\]

for some \( 0 \leq d < \frac{1}{n+1} \) for all \( x \in B \cap \text{supp} \mu \). Now from the above lemma we have that \( f(x) \in \text{WM} \frac{md + m - kd}{m(1-kd)} \) for some \( 1 \leq m, k \leq n+1 \). Since \( \frac{md + m - kd}{m(1-kd)} > \frac{1}{1-d} > 1 \) we have that \( f(B \cap \text{supp} \mu) \subset \text{WM} \frac{1}{1-a} \), i.e.
\( f(x) \) is VWMA for all \( x \in B \cap \text{supp } \mu \). Also condition (ii) in the above proposition is same as

for any \( d > 0 \) \( \exists T = T(d) > 0 \) such that for any \( t \in \mathbb{Z}_+^{n+1} \) with \( t \geq T \)

\[ \forall j = 1, \ldots, n \text{ and } j \in \bigwedge \mathcal{D}^{n+1} \text{, one has} \]

\[ \max \left( p^t \| Rc(w) \|_p, \| \pi(w) \|_p \right) \max_I p^{-t_I} \| w_I \|_\infty \geq p^{-jd_I}, \]

(9.9)

which is independent of \( f \) but depends on the affine subspace in which manifold is nondegenerate.

Combining all these previous observations and repeating the same arguments as in section \( \S 7 \) we have the following.

**Theorem 9.2.** Let \( \mu \) be a Federer measure on a Besicovitch metric space \( X, \mathcal{L} \) an affine subspace of \( \mathbb{Q}^n_p \), and let \( f : X \to \mathcal{L} \) be a continuous map such that \( (f, \mu) \) is good and nonplanar i.e. (6.17) holds for all open balls \( B \) with \( \mu(B) > 0 \). Then the following are equivalent:

(i) \( \{ x \in \text{supp } \mu \mid f(x) \text{ is not VWMA} \} \) is nonempty. \hspace{1cm} (9.10)

(ii) \( f(x) \) is not VWMA for \( \mu \text{ a.e } x. \) \hspace{1cm} (9.11)

(iii) Condition (9.9) holds . \hspace{1cm} (9.12)

Theorem 1.4 now follows as a corollary.

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