LIFESPAN ESTIMATES FOR 2-DIMENSIONAL SEMILINEAR WAVE EQUATIONS IN ASYMPTOTICALLY EUCLIDEAN EXTERIOR DOMAINS

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Abstract. In this paper we study the initial boundary value problem for two-dimensional semilinear wave equations with small data, in asymptotically Euclidean exterior domains. We prove that if $1 < p \leq p_c(2)$, the problem admits almost the same upper bound of the lifespan as that of the corresponding Cauchy problem, only with a small loss for $1 < p \leq 2$. It is interesting to see that the logarithmic increase of the harmonic function in 2-D has no influence to the estimate of the upper bound of the lifespan for $2 < p \leq p_c(2)$. One of the novelties is that we can deal with the problem with flat metric and general obstacles (bounded and simple connected), and it will be reduced to the corresponding problem with compact perturbation of the flat metric outside a ball.

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1. Introduction

In this paper, we are interested in the investigation of the blow-up part of the analogs of the Strauss conjecture in two dimensional asymptotically Euclidean exterior domain $(\Omega, g)$. We assume $\partial \Omega$ is a smooth Jordan curve. By asymptotically Euclidean exterior domain, we mean that it is a submanifold of the asymptotically Euclidean space $(\mathbb{R}^2, g)$. For simplicity, we assume $\Omega = \mathbb{R}^2 \backslash \overline{B}_R$ for some $R > 0$, see however, Lemma 1.5.

Recall that, for $(\mathbb{R}^2, g)$, the metric $g$ is assumed to be of the form

\begin{equation}
\tag{1.1} g = g_1 + g_2,
\end{equation}

where $g_1$ is a spherically symmetric, long range perturbation of the flat metric $g_0$, and $g_2$ is a short range perturbation. With possibly changing the choice of $R$, we could write $g_1$, in terms of the polar coordinates $x = r(\cos \theta, \sin \theta) \in \Omega$, as follows

\begin{equation}
\tag{1.2} g_1 = K^2(r)dr^2 + r^2d\theta^2,
\end{equation}

where $d\theta^2$ is the standard metric on the unit circle $S^1$, and

\begin{equation}
\tag{1.3} |\partial_r^n(K - 1)| \lesssim (r)^{-m-\rho_1}, m = 0, 1, 2,
\end{equation}

for some given constant $\rho_1 \in (0, 1)$. Here and in what follows, $(x) = \sqrt{1 + |x|^2}$, and we use $A \lesssim B$ ($A \gtrsim B$) to stand for $A \leq CB$ ($A \geq CB$) where the constant $C$ may change from line to line. Concerning $g_2$, we have

\begin{equation}
\tag{1.4} g = g_{jk}(x)dx^jdx^k \equiv \sum_{j,k=1}^{2} g_{jk}(x)dx^jdx^k, \quad g_2 = g_{2,jk}(x)dx^jdx^k,
\end{equation}

where we have used the convention that Latin indices $j, k$ range from 1 to 2 and the Einstein summation convention for repeated upper and lower indices. Furthermore, we assume $g_2$ satisfies

\begin{equation}
\tag{1.4} \nabla^\beta g_{2,jk} = O((r)^{-\rho_2 - |\beta|}), |\beta| \leq 2,
\end{equation}

for some $\rho_2 > 1$. By these assumptions, it is clear that there exists a constant $\delta_0 \in (0, 1)$ such that

\begin{equation}
\tag{1.5} \delta_0 |\xi|^2 \leq g^{jk}(x)\xi_j\xi_k \leq \delta_0^{-1} |\xi|^2, \forall x, \xi \in \mathbb{R}^2, K(r) \in (\delta_0, 1/\delta_0),
\end{equation}

where $(g^{jk}(x))$ denotes the inverse of $(g_{jk}(x))$.

With these preparations in hand, we may write out our problem explicitly, that is, initial boundary value problem of semilinear wave equations with small initial data posed on asymptotically Euclidean manifolds $(\Omega, g)$ with $g$ satisfies (1.1)-(1.4)

\begin{equation}
\tag{1.6} \begin{cases}
    u_{tt} - \Delta_g u = |u|^p, & t > 0, \ x \in \Omega,
    u(0, x) = \varepsilon u_0(x), \ u_t(0, x) = \varepsilon u_1(x), & x \in \Omega,
    u(t, x) = 0, & t > 0, \ x \in \partial \Omega,
\end{cases}
\end{equation}

where, $\Delta_g = \nabla^j \partial_j$ is the standard Laplace-Beltrami operator, $\varepsilon > 0$ is a small parameter. Concerning the initial data, we assume

\begin{equation}
\tag{1.7} (u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega), \ \text{supp}(u_0, u_1) \subset B_{R_0}, \ u_0, u_1 \geq 0, u_0 \neq 0, u_1 \neq 0,
\end{equation}

for some $R_0 > R$. 


Such kind of problem is the generalization of the Strauss conjecture (see [33]): the following Cauchy problem of semilinear wave equation with small initial data with sufficient regularity and sufficient decay at infinity

\[
\begin{align*}
&u_{tt} - \Delta u = |u|^p, \quad (t, x) \in (0, T) \times \mathbb{R}^n, \\
u(0, x) = \varepsilon u_0(x), \quad u_t(0, x) = \varepsilon u_1(x), \quad x \in \mathbb{R}^n,
\end{align*}
\]

admits a critical exponent \( p_c(n) (n \geq 2) \), which means that if \( 1 < p \leq p_c(n) \) then problem (1.8) has no global solution in general, whereas the solution exists globally in time if \( p > p_c(n) \) and \( 0 < \varepsilon \ll 1 \). Here \( p_c(n) \) is the positive root of the quadratic equation

\[
(n - 1)p^2 - (n + 1)p - 2 = 0.
\]

This conjecture has been essentially verified, we list all of the corresponding results in the following table (one can also find it in [34]):

| \( n \) | \( 1 < p < p_c(n) \) | \( p = p_c(n) \) | \( p_c(n) < p \leq 1 + 4/(n - 1) \) |
|-------|-----------------|-----------------|-----------------|
| 2     | Glassey [7]     | Schaeffer [27]  | Glassey [8]     |
| 3     | John [11]       | Schaeffer [27]  | John [11]       |
| \( \geq 4 \) | Sideris [28] | Yordanov-Zhang [41], Zhou [44], indep. | Georgiev-Lindblad-Sogge [5] |

If there is no global solution, then it is interesting to estimate the time when the solution blows up, i.e., the lifespan. The results have been established for two cases: (I) subcritical power \( (1 < p < p_c(n)) \); (II) critical power \( (p = p_c(n)) \). For the former case, we now know that there exist two positive constants \( c \) and \( C \) such that the lifespan satisfies for \( n \geq 2 \) and \( \max(1, 2/(n - 1)) < p < p_c(n) \)

\[
c\varepsilon^{-\frac{2(p-1)}{3(n-1)}} \leq T(\varepsilon) \leq C\varepsilon^{-\frac{2(p-1)}{3(n-1)}},
\]

where \( \gamma(n, p) = 2 + (n + 1)p - (n - 1)p^2 > 0 \). We may read the facts from the following table:

| \( n \) | Lower bound | Upper bound |
|-------|-------------|-------------|
| 2     | Zhou [45]   | Zhou [45]   |
| 3     | Lindblad [20] | Lindblad [20] |
| \( \geq 4 \) | Lai-Zhou [13] | rescaling argument from Sideris [28] |

For \( 1 < p \leq 2 \) and \( n = 2 \), the lifespan estimate can be improved to ([20], [34])

\[
T(\varepsilon) \leq \begin{cases} C\varepsilon^{-\frac{2}{3}} & , 1 < p < 2, \\ C\varepsilon^{-1}(\log(1/\varepsilon))^{1/2} & , p = 2. \end{cases}
\]

For the critical case \( (p = p_c(n)) \), the lifespan is much longer and has the form:

\[
\exp(c\varepsilon^{-p(p-1)}) \leq T(\varepsilon) \leq \exp(C\varepsilon^{-p(p-1)}).
\]

We have the following table:

| \( n \) | Lower bound | Upper bound |
|-------|-------------|-------------|
| 2     | Zhou [45]   | Zhou [45]   |
| 3     | Zhou [46]   | Zhou [46]   |
| \( \geq 4 \) | Lindblad-Sogge [22] for \( n \leq 8 \) or radial solution | Takamura-Wakasa [35] |
As waves propagate to infinity of the space, hence besides the Cauchy problem in the whole space, it is also interesting to consider the corresponding obstacle problem, i.e., the initial boundary value problem in exterior domain. Due to the difficulty caused by the boundary, such kind of problem has not been well understood, particularly for the global existence in high dimensions \( n \geq 5 \). Anyway, we have the following results of global existence vs blow-up:

| \( n \) | \( 1 < p < p_c(n) \) | \( p = p_c(n) \) | \( p > p_c(n) \) |
|---|---|---|---|
| 2 | Li-Wang [18] | Lai-Zhou [16] | Smith-Sogge-Wang [30] |
| 3 | Zhou-Han [47] | Lai-Zhou [14] | Hidano et al [9] |
| 4 | Zhou-Han [47] | Sobajima-Wakasa [31] | Du et al [3], reproved by Hidano et al [9] |
| \( \geq 5 \) | Zhou-Han [47] | Lai-Zhou [15], reproved by Sobajima-Wakasa [31] | \( p = 2, \) Metcalfe-Sogge [25], reproved by Wang [40] |

Just like the Cauchy problem, it is meaningful to study the lifespan for the blow-up exponent. We expect the same estimate as that of the Cauchy problem, regardless of the boundary obstacle, at least when the obstacle is nontrapping. Denoting the expected sharp lower bound and upper bound by “L” and “U” respectively, we have the following known results:

| \( n \) | \( 1 < p < p_c(n) \) | \( p = p_c(n) \) |
|---|---|---|
| 2 | L : ? | U : This work |
| 3 | L : Du-Zhou [4](\( p = 2 \)), Yu [42] (\( 2 < p < 1 + \sqrt{2} \)) | L : ? |
| 4 | L : ? | U : Zhou-Han [47] |
| \( \geq 5 \) | L : ? | U : Zhou-Han [47] |

We also have to mention the generalization of problem (1.8) from Euclidean space to other manifold, such as asymptotically Euclidean manifolds (see [26], [32], [38] and references therein), and black hole spacetime (see [1], [19], [21], [24] and references therein). One can find a detailed description of such kind of generalization in a recent survey paper [39]. Another direction is to consider the initial boundary value problem in exterior domain with big initial data (see [17], [29] and references therein).

Before stating our results, we shall make a hypothesis.

**Hypothesis:** There exists \( \lambda_0 \in (0, 1/(2R)) \), such that for any \( \lambda \in (0, \lambda_0) \), we could find a solution \( \phi_\lambda \) solving

\[
\Delta_g \phi_\lambda = \lambda^2 \phi_\lambda, \ x \in \Omega, \ \phi_\lambda|_{\partial\Omega} = 0,
\]
which enjoys the following uniform estimates (independent of $\lambda$)

$$(H) \quad \phi_\lambda \sim \begin{cases} 
\ln r/R & r \leq \lambda^{-1}, \\
\langle r\lambda \rangle^{-1/2} e^{\lambda \int_0^r K(\tau) d\tau} & r \geq \lambda^{-1},
\end{cases}$$

for any $\lambda \in (0, \lambda_0)$.

**Theorem 1.1.** Let $p \in (1, p_c(2))$ and assume (H). Consider the problem (1.6) with initial data (1.7), posed on asymptotically Euclidean exterior domain $(\Omega, g)$ satisfying (1.1)-(1.4). Then we have the following

1. There exists a unique weak solution $u(t, x) \in C_t H^1_0 \cap C^1_t L^2$ to the initial boundary value problem (1.6) on $[0, T(\varepsilon)] \times \Omega$, where $T(\varepsilon)$ denotes the lifespan, i.e., the maximal time of existence.
2. There exist constants $C, \varepsilon_0 > 0$ independent of $\varepsilon$ such that for any $\varepsilon \in (0, \varepsilon_0)$, we have

$$T(\varepsilon) \leq \begin{cases} 
\exp \left( C\varepsilon^{-p(p-1)} \right), & p = p_c(2), \\
C\varepsilon^{-2(p-1)}, & 2 \leq p < p_c(2), \\
C(\varepsilon^{-1} \ln(\varepsilon^{-1}))(p-1)/(3-p), & 1 < p < 2.
\end{cases}$$

The local well-posed result follows from a standard energy argument. For the upper bound of lifespan estimates, it relies on the so-called test function method. For the subcritical case, we have to show the proper asymptotic behaviors of two test functions, that is, $(1.14)$ for $\phi_0$ and (H) for $\phi_\lambda$, with some fixed $\lambda_1 \in (0, \lambda_0)$. However, for the critical case, we need to use a family of test function $\phi_\lambda$ with $\lambda$ varying in $(0, \lambda_0)$, to construct another test function $b_\eta$ with more subtle asymptotic behavior as stated in (4.8) and (4.9) below. Concerning $\phi_0$, we have

**Lemma 1.2.** Let $(\Omega, g)$ be an asymptotically Euclidean exterior domain satisfying (1.1)-(1.4). Then there exists a solution $\phi_0$ to

$$\Delta g \phi_0 = 0 \ , \ x \in \Omega \ , \ \phi_0 |_{\partial \Omega} = 0$$

satisfying

$$\phi_0(x) \sim \ln r/R .$$

**Remark 1.3.** It will be clear from the proof that the first two upper bounds in (1.14) hold for non-negative data with either $u_0$ or $u_1$ nontrivial. While, for the last upper bound, we need only to assume non-negative data with $u_1 \not= 0$.

Notice that Theorem 1.1 is a conditional result. Let us review some cases where the assumption (H) is valid. It is well known for the Euclidean space $(\mathbb{R}^2, g_0)$, where $\phi_\lambda$ could be given by the spherical average of $e^{\lambda x \cdot \omega}$,

$$\phi_\lambda(x) = \int_{S^1} e^{\lambda x \cdot \omega} d\omega \sim \langle r\lambda \rangle^{-1/2} e^{\lambda r} ,$$

see Yordanov-Zhang [41]. When $g_3$ is an exponential perturbation, that is, there exists $\alpha > 0$ so that

$$|\nabla g_{3,k}| + |g_{3,k}| \leq \alpha e^{-\alpha \int_0^r K(\tau) d\tau} ,$$

the corresponding estimate for $(\mathbb{R}^2, g_1 + g_3)$ is recently obtained for $g = g_1 + g_3$ by Liu-Wang [23], while the case $g = g_0 + g_3$ was obtained by Wakasa-Yordanov in [37]. Based on these results, we could verify the hypothesis (H), in the case of $g = g_1 + g_3$, and thus obtain the following
Theorem 1.4. Let \( g = g_1 + g_3 \). Then the hypothesis (H) holds and we have the same results as that in Theorem 1.1.

In the above, we have considered exclusively on the asymptotically Euclidean exterior domains to disk. It is natural to ask what happens for the general asymptotically Euclidean exterior domains \((D, g)\). It turns out that, when \( \partial D \) is a smooth Jordan curve, the general problem could be reduced to the case of disk exterior. Actually, with the help of the Riemann mapping theorem, we could prove the following

Lemma 1.5. Let \( D \subset \mathbb{R}^2 \) be exterior domain to a smooth Jordan curve \( \partial D \). Then there exists a diffeomorphism mapping preserving the boundary

\[
A : D \rightarrow \mathbb{R}^2 \setminus \overline{B_R},
\]

for some \( R > 0 \), which is an identity map for \( r \gg 1 \).

With the help of this lemma, we could translate the problem (1.6) in the domain \((D, g)\), into the corresponding problem for \((\Omega, \tilde{g})\), by change of variables with the new unknown function \( w(x) = u(A^{-1}(x)) : \Omega \rightarrow \mathbb{R} \). Thus, whenever we have (H) for \((\Omega, \tilde{g})\), the lifespan estimates (1.14) apply for \((D, g)\).

In particular, the result applies for any exterior domains to smooth Jordan curves. More precisely, let \( \mathcal{K} \subset \mathbb{R}^2 \) be a domain interior to a smooth Jordan curve, (that is, \( \mathcal{K} \) is a nonempty, open, bounded, smooth, simple connected domain), we consider the following initial boundary value problem

\[
\begin{aligned}
&u_{tt} - \Delta u = |u|^p, \quad t > 0, \quad x \in D = \mathbb{R}^2 \setminus \overline{\mathcal{K}}, \\
&u(0, x) = \varepsilon u_0(x), \quad u_t(0, x) = \varepsilon u_1(x), \quad x \in D, \\
&u|_{\partial \mathcal{K}} = 0.
\end{aligned}
\]

Theorem 1.6. Let \( \mathcal{K} \subset \mathbb{R}^2 \) be nonempty bounded smooth simple connected domain. Consider (1.19) with initial data (1.7). Then we have the same lifespan estimates (1.14) for energy solutions.

Remark 1.7. When the spatial dimension is not greater than 4, all of the previous blow-up results and lifespan estimates for exterior problem with critical power heavily rely on the assumption that the obstacle is a ball (see [14], [16], [31]), under which they can construct some special test functions explicitly. In contrast, our results hold for very general obstacle in 2-D.

Outline. Our paper is organized as follows. In the next section, we sketch the proof of local well-posedness for the energy solutions, by a standard energy argument. In particular, it shows the finite speed of propagation (2.5), for the solution. In Section 3, we prove the existence of test function \( \phi_0 \), Lemma 1.2, for the Dirichlet problem of the Laplace equation on \((\Omega, g)\). With the help of \( \phi_0 \), as well as the hypothesis (H), in Section 4, we present the proof of Theorem 1.1. In Section 5, we show that the hypothesis (H) holds, at least when the metric \( g \) is exponential perturbation of a spherically symmetric, long range asymptotically Euclidean metric \( g = g_1 + g_3 \). At last, in Section 6, with the help of the Riemann mapping theorem, we prove Lemma 1.5, which enables us to reduce the problems with general obstacles to the problem exterior to a disk (keeping the metric near the spatial infinity).
2. Local well-posedness

Before going to the proof of blow-up results, we first show the local well-posedness for problem (1.6), based on energy estimates. By multiplying the equation in (1.6) with \( \partial_t u \), we get the energy estimate

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \left( u_t^2 + g^{ij}(x) u_{x_i} u_{x_j} \right) dV_g = \int_\Omega |u|^p u_t dV_g,
\]

where \( dV_g = \sqrt{\det(g_{jk}(x))} dx \) is the volume form with respect to the metric. Due to the assumption of \( g^{ij}(x) \) in (1.5), this implies

\[
\|\partial u\|_{L^\infty L^2([0,T] \times \Omega)} \lesssim \varepsilon + T\||u|^p\|_{L^\infty L^2([0,T] \times \Omega)}.
\]

On the other hand, it is easy to get

\[
\|u\|_{L^\infty L^2([0,T] \times \Omega)} \lesssim \varepsilon (1 + T) + T^2\||u|^p\|_{L^\infty L^2([0,T] \times \Omega)}.
\]

It then follows by combining (2.2) and (2.3) that

\[
\|u\|_{L^\infty H^1([0,T] \times \Omega)} \lesssim \varepsilon (1 + T) + (T + T^2)\||u|^p\|_{L^\infty L^2([0,T] \times \Omega)}.
\]

Recalling that

\[
H^1(\Omega) \hookrightarrow L^p(\Omega), \ 1 < p < \infty,
\]

with the help of all these estimates, a standard contraction mapping argument yields the desired local well-posed result.

In particular, as the solution is obtained through iteration, we see that the solution enjoys the finite speed of propagation:

\[
\text{supp } u(t) \subset \left\{ x \in \Omega : \int_R^t K(\tau) d\tau \leq t + R_1 \right\} := D_1(t) \subset \left\{ x : r \leq \frac{t + R_2}{\delta_0} \right\},
\]

for any data satisfying (1.7), where \( R_2 > R_1 = \int_R^{R_2} K(\tau) d\tau > 0 \).

3. Test function for the Dirichlet problem

In this section, we consider the Laplace equation posed on the asymptotically Euclidean exterior domain \((\Omega, g)\), that is, Lemma 1.2.

We observe that, for the purpose of the lemma, we need only to prove the following weaker estimate

\[
\phi_0(x) \simeq \ln r, \ r \gg 1.
\]

Actually, by the strong maximum principle, we know that \( \phi_0(x) \) is positive everywhere in \( \Omega \). Moreover, by Hopf’s lemma, we have

\[
\partial_r \phi_0(r, \theta)|_{r=R} > 0,
\]

which gives us that

\[
\phi_0(r, \theta) \sim r - R \sim \ln r/R,
\]

for any \( R \leq r \lesssim 1 \).
3.1. Kelvin transformation. At first, inspired by the classical Kelvin transform, we use the spatial inversion to introduce the conformal compactization. More specifically, we introduce new coordinate in the \( \bar{B}_1/R\{0\} \) for \( \Omega \).

\[
\Phi(x) = \phi_0(|x|^2) = \phi_0(x^*), \quad x \in \bar{B}_1/R\{0\}.
\]

Let \( ds^2 = g_{jk}(x^*)dx^jdx^{*k} = \tilde{h}_{jk}(x)dx^jdx^{k} \), where \( x = r\omega \cdot g_{jk}(x^*) = \delta_{jk} + f(x^*)\omega_j\omega_k + g_{2,jk}(x^*) \) and \( f(x^*) = K^2(x^*) - 1 \). By (1.3)-(1.4), we have

\[
f(x^*) = O(|x|^\rho_1), \quad g_{2,jk}(x^*) = O(|x|^\rho_2), \quad \rho_1 \in (0,1], \rho_2 > 1,
\]
as \( x \to 0 \). As

\[
\frac{\partial x^*j}{\partial x^k} = \frac{\delta_{jk} - 2\omega^j\omega^k}{r^2},
\]
we get

\[
\tilde{h}_{jk}(x) = \frac{1}{r^4}(\delta_{ij} - 2\omega^i\omega^j)g_{il}(x^*)(\delta_{lk} - 2\omega^l\omega^k) = \frac{h_{jk}}{r^4},
\]
where

\[
h_{jk}(x) = (\delta_{ij} - 2\omega^i\omega^j)g_{il}(x^*)(\delta_{lk} - 2\omega^l\omega^k) = \delta_{jk} + f(x^*)\omega_j\omega_k + g_{2,jl}(x^*)(\delta_{ji} - 2\omega^i\omega^j)(\delta_{lk} - 2\omega^l\omega^k).
\]

Then we obtain

\[
\Delta_g\phi_0(x^*) = \Delta_{h_0}\Phi(x) = r^4\Delta_k\Phi(x).
\]

In particular, if we have \( g = g_1 \), there is \( h_0 \) such that

\[
\Delta_{g_1}\phi(x^*) = r^4\Delta_{h_0}\Phi_0(x), \quad h_{0,jk}(x) = \delta_{jk} + f(x^*)\omega_j\omega_k.
\]

3.2. Test function for \( g_1 \). Considering

\[
\Delta_{g_1}\phi = 0, \quad x \in \Omega; \quad \phi = 0, \quad x \in \partial\Omega,
\]
it is easy to find a spherically symmetric solution

\[
\phi(r) = \int_R^r \frac{K(s)}{s}ds \simeq \ln \frac{r}{R},
\]
as

\[
\Delta_{g_1} = K^{-1}r^{-1}\partial_r K^{-1}r\partial_r + r^{-2}\partial_\theta^2.
\]

By (3.5), we have

\[
\Phi_0(x) = \phi\left(\frac{x}{|x|^2}\right) = \phi\left(\frac{1}{r}\right) = \int_R^r \frac{K(s)}{s}ds,
\]
satisfying

\[
\left\{
\begin{array}{l}
\Delta_{h_0}\Phi_0 = 0, \quad x \in B_1/R\{0\}, \\
\Phi_0 = 0, \quad x \in \partial B_1/R.
\end{array}
\right.
\]
3.3. **Proof of Lemma 1.2.** By (3.1)-(3.4), we are reduced to finding a solution in the region $B_{1/R} \setminus \{0\}$

\[
\begin{cases}
\Delta_h \Phi = 0, & x \in B_{1/R} \setminus \{0\}, \\
\Phi = 0, & x \in \partial B_{1/R}
\end{cases}
\]

(3.10)
satisfying $\Phi(x) \sim \ln \frac{1}{r}$ near $r = 0$. Let $u(x) = \Phi(x) - \Phi_0(x)$, it remains to construct a solution $u \in L^\infty(B_{1/R} \setminus \{0\})$, due to the fact that $\Phi_0 \sim \ln(1/r)$ near $r = 0$. Concerning $u$, it satisfies

\[
\begin{cases}
\Delta_h u = (\Delta_{h_0} - \Delta_h) \Phi_0, & x \in B_{1/R} \setminus \{0\}, \\
u = 0, & x \in \partial B_{1/R},
\end{cases}
\]

and we would like to view it as the Dirichlet problem in the ball $B_{1/R}$.

For that purpose, we set $h_{jk}(0) = \delta_{jk}$ so that it is continuous in $B_{1/R}$, in view of (3.2)-(3.3). Moreover, we claim that

\[
(\Delta_h - \Delta_{h_0}) \Phi_0 = O(r^{p_2 - 2}),
\]

(3.11) which could be written in the form $\partial_i F$ for some $F = O(r^{p_2 - 1}) \in L^\infty(B_{1/R})$. With the help of the claim, by standard elliptic existence theorems, there is a unique solution $u \in H^1_0(B_{1/R})$. In addition, as the equation is of divergence form,

\[
\partial_j (|h|^{1/2} h^{jk} \partial_k u) = |h|^{1/2} (\Delta_h - \Delta_{h_0}) \Phi_0 = \partial_1 F,
\]

with $F \in L^\infty(B_{1/R})$, an application of Meyer’s theorem (see, e.g., Taylor [36, Chapter 14, Proposition 12.2]) gives us $u \in W^{1,\beta}(B_{1/R})$ for some $q > 2$, which in turn gives us the desired result $u \in L^\infty(B_{1/R})$.

We are left to give the proof of the claim (3.11). By calculation, we have

\[
(\Delta_h - \Delta_{h_0}) \Phi_0 = (h^{jk} - h_0^{jk}) \partial_j \partial_k \Phi_0 + |h|^{-1/2} \partial_j (|h|^{1/2} h^{jk}) - |h_0|^{-1/2} \partial_j (|h_0|^{1/2} h_0^{jk}) \partial_k \Phi_0.
\]

By (3.8) and (1.3), we get

\[
\partial_j \Phi_0 = O(r^{-1}), \quad \partial_j \partial_k \Phi_0 = O(r^{-2}).
\]

Similarly, by (1.3)-(1.4), (3.3) and (3.5), we have

\[
|\nabla^\beta (h_{jk} - h_{0,jk})| + |\nabla^\beta (h^{jk} - h_0^{jk})| + |\nabla^\beta (|h| - |h_0|)| = O(r^{p_2 - 1/2}), |\beta| = 0, 1.
\]

In summary, it is easy to see that

\[
| (\Delta_h - \Delta_{h_0}) \Phi_0 | \lesssim \sum_{j,k} \sum_{1 \leq |\beta| \leq 2, |\alpha| + |\beta| = 2} | \nabla^\alpha g_2^{jk}(x^*) | | \nabla^\beta \Phi_0 | = O(r^{p_2 - 2}),
\]

which completes the proof.

3.4. **An integral estimate.** With $\phi_0$ and its asymptotic behavior, together with (H) for $\phi_\lambda$, we will need the following

**Lemma 3.1.** Let $\lambda_1 \in (0, \lambda_0)$, then we have

\[
\int_{\Omega \cap \{\int_{K(s)} ds \leq t + R_1\}} \phi_0^{\frac{p-1}{p'}} e^{-\lambda_1 t} \phi_\lambda^{p'} dV_0 \lesssim (\ln(t + 1))^{\frac{1}{p-1}} (t + 1)^{1 - \frac{p'}{p}},
\]

(3.12) where $p' = \frac{p}{p-1}$ and the implicit constant may depend on $\lambda_1$. 

Proof. Let us begin with the region with \( r \leq \lambda_1^{-1} \), in that case, by (1.16) and the first estimate in (H), we have \( \phi_0 \xrightarrow{r \to 0} \phi_\lambda \preceq \phi_0(r)(\phi_0(\lambda_1^{-1}))^{-\frac{1}{\lambda' t}} \preceq \ln(r/R) \) and the integral is controlled by

\[
\int_{R \leq r \leq \lambda_1^{-1}} \phi_0(r)e^{-\lambda' t r}dr \preceq e^{-\lambda' t}.
\]

For the remained case \( r \geq \lambda_1^{-1} \), we could use (1.16) and the second estimate in (H). Thus we have

\[
\int_{\{r \geq \lambda_1^{-1}\} \cap \{f_k K(s)ds \leq \frac{\lambda R}{2}\}} \phi_0 \xrightarrow{r \to 0} e^{-\lambda' t} \phi_\lambda dr \lesssim \int_{r \leq t + R_2} e^{-\frac{\lambda' t}{1-\frac{\lambda' t}{2}}} dr \lesssim e^{-\frac{\lambda' t}{1-\frac{\lambda' t}{2}}}(t+1)^2,
\]

and

\[
\int_{\frac{\lambda R}{2} \leq f_k K(s)ds \leq t + R_1} \phi_0 \xrightarrow{r \to 0} e^{-\lambda' t} \phi_\lambda dr \lesssim (\ln(1+t))^{-\frac{1}{\lambda' t}} \int_{\frac{\lambda R}{2} \leq f_k K(s)ds \leq t + R_1} e^{-\lambda' t + \lambda' t f_k K(s)ds} \to 1 - \frac{\lambda' t}{2} dr.
\]

In summary, we obtain (3.12).

4. PROOF OF THEOREM 1.1

In this section, we are devoted to the proof for Theorem 1.1, under the assumption (H).

4.1. Subcritical case. Let \( \eta(t) \in C^\infty([0, \infty)) \) be a decreasing function satisfying

\[
\eta(t) = \begin{cases} 1 & t \leq \frac{1}{2}, \\ 0 & t \geq 1, \end{cases} \quad \eta^*(t) = \eta(t)\chi_{[1/2, 1]}(t)
\]

and

\[
\eta_T(t) = \eta(t/T), \quad \eta_{TT}(t) = \eta^*(t/T), \quad T \in (1, T(\varepsilon)).
\]

As \( u \in C^1_H \cap C^1_t L^2 \) is the energy solution to (1.6), we have finite speed of propagation (2.5) and \( u \in C^2_t H^{-1} \) based on the equation. Using \( \eta_{TT}(t)\phi_0(x) \) as a test function, where \( \phi_0(x) \) is the test function in Lemma 1.2, we obtain

\[
\langle |u|^p \eta_{TT}^{2p'}(t)\phi_0(x) \rangle_{\mathcal{G}} = \langle u_{tt} - \Delta_g u, \eta_{TT}^{2p'}(t)\phi_0(x) \rangle_{\mathcal{G}}
\]

\[
= \frac{d}{dt} \langle \eta_{TT}^{2p'}(t)\phi_0(x) \rangle_{\mathcal{G}} - \langle u, \partial_t \eta_{TT}^{2p'}(t)\phi_0(x) \rangle_{\mathcal{G}} + \langle u, \square_g \eta_{TT}^{2p'}(t)\phi_0(x) \rangle_{\mathcal{G}}
\]

\[
= \frac{d}{dt} \langle \eta_{TT}^{2p'}(t)\phi_0(x) \rangle_{\mathcal{G}} - \langle u, \partial_t \eta_{TT}^{2p'}(t)\phi_0(x) \rangle_{\mathcal{G}} + \langle u, \partial_t^2 \eta_{TT}^{2p'}(t)\phi_0(x) \rangle_{\mathcal{G}}
\]

where \( \square_g = \partial_t^2 - \Delta_g \) denotes the dual relation between distribution \( u \) and test function \( v \), which agrees with \( \int_{\Omega} u(x)v(x)\sqrt{\det(g_{ij}(x))}dx = \int_{\Gamma} u(x)v(x)d\mathcal{G} \) for usual functions \( u, v \). Integrating over \([0, T]\), and observing that \( \eta_T(T) = \partial_t \eta_T(0) = \partial_t \eta_T(T) = 0 \), we get

\[
\int_0^T \int_{\Omega} |u|^p \eta_{TT}^{2p'}(t)\phi_0 d\mathcal{G} dt + \varepsilon \int_{\Omega} u_1 \phi_0 d\mathcal{G} = \int_0^T \int_{\Omega} u_2 \eta_{TT}^{2p'}(t)\phi_0 d\mathcal{G} dt.
\]
Noting that
\[ \frac{\partial_t^2 \eta_T^{2p'}}{\partial_t} = \frac{2p'(2p' - 1)}{T^2} \eta_T^{2p' - 2}|(\eta'|_T|^2 + \frac{2p'}{T^2} \eta_T^{2p' - 1}(\eta'|_T) = \mathcal{O}(T^{-2}, \eta_T^{2p' - 2}), \]
we obtain
\[ \int_0^T \int_\Omega u \partial_t \eta_T^{2p'} \phi_0 dV_0 dt \]
\[ \leq C \left( \int_0^T \int_\Omega |u|^p \eta_T^{2p'} \phi_0 dV_0 dt \right)^{\frac{1}{p}} \left( \int_0^T \int_{D_1(t)} T^{-2p'} \phi_0 dV_0 dt \right)^{\frac{1}{p'}} \]
\[ \leq CT^{3 - 2p'} \ln T + \frac{1}{3} \int_0^T \int_\Omega |u|^p \eta_T^{2p'} \phi_0 dV_0 dt, \]
where we have used Hölder’s inequality, (1.16) and Young’s inequality. Plugging it to (4.1), we know that
\[ (2.2) \quad \varepsilon \int_\Omega u_1(x) \phi_0 dV_0 + \int_0^T \int_\Omega |u|^p \eta_T^{2p'} \phi_0 dV_0 dt \leq CT^{3 - 2p'} \ln T. \]
This inequality implies the lifespan estimate for \(1 < p < 2\) in (1.14). Actually, from the last inequality we conclude that \(T^{3 - 2p'} \ln T \gtrsim \varepsilon\) for any \(T \in (1, T(\varepsilon))\), which gives us, for \(1 < p < 3\),
\[ (4.3) \quad T(\varepsilon) \lesssim (\varepsilon^{-1} \ln(\varepsilon^{-1}))^{(p - 1)/(3 - p)} . \]
But, comparing to the one
\[ (4.4) \quad T(\varepsilon) \lesssim \varepsilon^{-\frac{2p(p - 1)}{3 - 3p - p^2}}, \]
which we will get for \(1 < p < p_c(2)\), the lifespan estimate (4.3) is better than (4.4) only for \(1 < p < 2\).
To proceed, we introduce another test function
\[ \eta_T^{2p'} \psi(t, x) = \eta_T^{2p'} e^{-\lambda t} \phi_\lambda(x) \]
to get
\[ \int_0^T \int_\Omega |u|^p \eta_T^{2p'} \psi(t, x) dV_0 dt \]
\[ = \int_0^T \langle u_{tt} - D_3 u, \eta_T^{2p'} \psi(t, x) \rangle_0 dt, \]
\[ = \left. \langle u_t, \eta_T^{2p'} \psi(t, x) \rangle_0 - \langle u, \partial_t (\eta_T^{2p'} \psi(t, x)) \rangle_0 \right|_{t=0} + \int_0^T \langle u, \partial_t (\eta_T^{2p'} \psi(t, x)) \rangle_0 dt \]
\[ = -\varepsilon \int_\Omega (\lambda_1 u_0(x) + u_1(x)) \phi_\lambda(x) dV_0 \]
\[ + \int_0^T \int_\Omega u \partial_t^2 \eta_T^{2p'} dV_0 dt + 2 \int_0^T \int_\Omega u (\partial_t \eta_T^{2p'})(\partial_t \psi) dV_0 dt, \]
which yields

\begin{equation}
\varepsilon \int_\Omega (\lambda_1 u_0(x) + u_1(x)) \phi_{\lambda_1}(x) dV + \int_0^T \int_\Omega |u|^p \eta_t^{2p'} e^{-\lambda t} \phi_{\lambda_1}(x) dV dt \\
= \int_0^T \int_\Omega u e^{-\lambda t} \phi_{\lambda_1}(x) (\partial_t^2 \eta_t^{2p'} - 2\lambda_1 \partial_t \eta_t^{2p'}) dV dt.
\end{equation}

Noticing that

\[ \partial_t^2 \eta_t^{2p'} - 2\lambda_1 \partial_t \eta_t^{2p'} = O(T^{-1}(\eta_t^*)^{2p'-2}), \]

As above, by combining Hölder’s inequality and (3.12), the right hand side of (4.5) is controlled by

\begin{equation}
CT^{-1} \int_0^T \int_\Omega |u| (\eta_t^*)^{2p'-2} \phi_0^{1/p} \phi_0^{-1/p} e^{-\lambda t} \phi_{\lambda_1}(x) dV dt \\
\leq CT^{-1} \|u(\eta_t^*)^{2p'-2}\|_{L^p([0,T] \times \Omega)} \|\phi_0^{-1/p} e^{-\lambda t} \phi_{\lambda_1}\|_{L^p(\cup_{t \in [T/2,T]} D_t(1))} \\
\leq CT^{-1} + (2-\frac{2}{p'}) \frac{1}{p'} (\ln T)^{-\frac{p}{2}} \left( \int_0^T \int_\Omega |u|^p (\eta_t^*)^{2p'} \phi_0 dV dt \right)^{\frac{1}{p}}.
\end{equation}

We then conclude from (4.5), (4.6) and (4.2) that

\begin{equation}
\varepsilon^p T^{2-\frac{2}{p'}} \ln T \lesssim \int_0^T \int_\Omega |u|^p (\eta_t^*)^{2p'} \phi_0 dV dt \lesssim T^{3-2p'} \ln T,
\end{equation}

which gives us the desired lifespan estimate (4.4) for \( 1 < p < p_c(2) \).

4.2. Critical case. For the lifespan estimate of the critical power, we need one more lemma, which is similar to Lemma 4.2 in [12]. For completeness, we give a proof.

**Lemma 4.1.** Let \( q > 0, \lambda_1 \in (0, \lambda_0) \) and

\[ b_q(t, x) = \int_0^{\lambda_1} e^{-\lambda t} \phi_{\lambda}(x) \lambda^{q-1} d\lambda, \]

then we have

1. \( b_q(t, x) \) satisfies following identities
   \[ \partial_t b_q(t, x) = -b_{q+1}(t, x), \quad \partial_t^2 b_q(t, x) = b_{q+2}(t, x) = \Delta_b b_q(t, x). \]

2. For \( x \in D_1(t) \), we have

\begin{equation}
b_q(t, x) \lesssim \begin{cases} (t + R_1)^{-q} & \text{if } 0 < q < \frac{1}{2}, \\ (t + R_1)^{-\frac{1}{2}} & \text{if } q > \frac{1}{2}, \end{cases}
\end{equation}

and

\begin{equation}
b_q(t, x) \gtrsim \frac{\phi_0}{\ln(t + R_1)} (t + R_1)^{-q}, \quad \forall q > 0.
\end{equation}

**Proof.** The first part is trivial. Concerning (4.9), let \( \delta_1 < \min(\delta_0 R_1 / R_2, \lambda_1 R_1) \), such that \( \frac{\delta_1}{t + R_1} \leq \frac{\delta_1}{R_1} \leq \lambda_1 \), then for any \( \lambda \leq \frac{\delta_1}{R_1 + R_1} \), we have \( r \leq (t + R_2) / \delta_0 <
For the case \( q > 0 \), we have \( r \sim t + R_1 \) and \( \lambda \sim t + R_1 \). If \( q \leq 1/2 \), the estimate \((4.10)\) becomes
\[
 b_q(t,x) \lesssim \int_0^{\lambda_1} e^{-\lambda(t+R_1)} e^{\lambda \int_0^t K(s) ds} \lambda^{-q/2} d\lambda \lesssim (t + R_1)^{-q} \int_0^\infty e^{-\lambda^{q-1}} d\lambda \lesssim (t + R_1)^{-q}.
\]
For the case \( q < 1/2 \), we have \( r \sim t + R_1 \) and \( \lambda \sim t + R_1 \). If \( q \leq 1/2 \), the estimate \((4.10)\) becomes
\[
 b_q(t,x) \lesssim \int_0^{\lambda_1} (\lambda(t+R_1))^{-q/2} \lambda^{-q-1} d\lambda \lesssim (t + R_1)^{-q} \int_0^\infty (s)^{-q/2} ds \lesssim (t + R_1)^{-q}.
\]
It remains to consider the case \( q > 1/2 \) and \( r \sim t + R_1 \), for which we have
\[
 b_q(t,x) \lesssim (t + R_1)^{-q/2} \int_0^{\lambda_1} e^{-\lambda(t+R_1+1-\int_r^R K(s) ds)} \lambda^{-q-3/2} d\lambda
\]
\[
 \lesssim (t + R_1)^{-q/2} \left( t + R_1 + 1 - \int_r^R K(s) ds \right) \lambda^{-q} \int_0^\infty e^{-\lambda^{q-2}} d\lambda
\]
\[
 \lesssim (t + R_1)^{-q/2} \left( t + R_1 + 1 - \int_r^R K(s) ds \right) \lambda^{-q-3/2} d\lambda \]
This completes the proof of \((4.8)\). \( \square \)

For \( M \in [2,T] \subset [2,T(\varepsilon)] \), we set
\[
 Y[u]|b_q](M) = \int_1^M \left( \int_0^T \int_\Omega |u|^p(t,x)b_q(t,x)(\eta^*_M(t))^{2p'} dV_q dt \right) \sigma^{-1} d\sigma,
\]
with \( q = \frac{1}{2} - \frac{1}{p'} \). Then from \((4.9)\) and \((4.7)\), we know that
\[
 M \frac{dY}{dM} = \int_0^T \int_\Omega |u|^p \frac{\phi_0}{\ln M} M^{-q}(\eta^*_M)^{2p'} dV_q dt
\]
\[
 \geq \int_{M/2}^M \int_\Omega |u|^p \frac{\phi_0}{\ln M} M^{-q}(\eta^*_M)^{2p'} dV_q dt
\]
\[
 \gtrsim \varepsilon^p M^{2-\frac{3}{p}} \ln M \frac{M^{\frac{1}{p} - \frac{1}{p'}}}{\ln M} = \varepsilon^p M^{\frac{3}{2} - \frac{1}{p' + \frac{1}{p}}} = \varepsilon^p,
\]
where we used the fact that $p = p_c(2)$. On the other hand, as in (4.10) in [12], we have

$$
Y(M) = \int_1^M \left( \int_0^T \int_\Omega b_\theta |u|^p (\eta^p_\theta(t))^{2p'} dV_\theta dt \right) \sigma^{-1} d\sigma \\
= \int_0^T \int_\Omega b_\theta |u|^p \left( \int_1^t (\eta^p_\theta(s))^{2p'} \sigma^{-1} d\sigma dV_\theta dt \right) \\
= \int_0^T \int_\Omega b_\theta |u|^p \int_t^{\max(\frac{s}{2} \frac{M}{\sigma}, 1)} (\eta^p_\theta(s))^{2p'} s^{-1} ds dV_\theta dt \\
\leq \int_0^T \int_\Omega b_\theta |u|^p (\eta(t/M))^{2p'} \left( \int_t^{1} s^{-1} ds dV_\theta dt \right) \leq 2 \int_0^T \int_\Omega b_M^{2p'} |u|^p dV_\theta dt .
$$

As $(\partial_t^2 - \Delta) b_\theta = 0$, we get

$$
Y(M) \lesssim \int_0^T \langle |u|^p, \eta^{2p'}_M b_\theta \rangle_{\theta} dt = \int_0^T \langle \partial_t^2 u - \Delta u, b_\theta \eta^{2p'}_M \rangle_{\theta} dt \\
= -\left( \varepsilon \int_\Omega u_1(x) b_\theta(0, x) dV_\theta + \varepsilon \int_\Omega u_0(x) b_\theta(0, x) dV_\theta \right) \\
+ \int_0^T \int_\Omega u(\partial_t^2 - \Delta b_\theta) (b_\theta \eta^{2p'}_M) dV_\theta dt \\
\leq \int_0^T \int_\Omega u \left( 2\partial_t b_\theta \partial_t \eta^{2p'}_M + b_\theta \partial_t^2 \eta^{2p'}_M \right) dV_\theta dt := I_1 + I_2 .
$$

By (4.8) and (4.9), we know that

$$
\int_0^M \int_{D_1(t)} b_\theta^{-\frac{n}{n+1}} b_\theta^{p-1} r dr dt \\
\lesssim \int_0^M \int_{D_1(t)} \frac{(\ln(t + R_1))^{\frac{p}{n+1}}}{\ln r/R} \frac{(t + R_1)^{\frac{p}{n+1}}}{t + R_1 + 1 - \int_{R}^{t} K(s) ds} dr dt \\
\lesssim \int_0^M \left( \ln(t + R_1) \right)^{\frac{p}{n+1}} (t + R_1)^{\frac{p}{n+1}} \int_{\ln R}^{t + R_1} r^{-1} (\ln(r/R))^{-\frac{1}{n+1}} dr dt \\
+ \int_0^M (t + R_1)^{\frac{1}{n+1}} \int_{t + R_1}^{t + R_1} (t + R_1 + 1 - s)^{-1} ds dt \\
\lesssim M^{\frac{2(n+1)-1}{2(n+1)} - \frac{1}{n+1}} \ln M = M^{\frac{n+1}{n+1}} \ln M ,
$$
then $I_1$, $I_2$ can be estimated as

$$I_1 \lesssim M^{-1} \left( \int_0^T \int_\Omega |u|^p b_q (\eta^*_M)^{2p'} dV_\theta dt \right)^{\frac{1}{p}} \left( \int_{\frac{M}{R}}^{M} \int_{D_1(t)} b_q \frac{r^{-1}}{b_q r+1} dV_\theta dt \right)^{\frac{p-1}{p}}$$

$$\lesssim (\ln M)^{\frac{p-1}{p}} \left( \int_0^T \int_\Omega |u|^p b_q (\eta^*_M)^{2p'} dV_\theta dt \right)^{\frac{1}{p}}$$

and

$$I_2 \lesssim M^{-2} \left( \int_0^T \int_\Omega |u|^p b_q \eta^*_M^{2p'} dV_\theta dt \right)^{\frac{1}{p}} \left( \int_{\frac{M}{R}}^{M} \int_{D_1(t)} b_q dV_\theta dt \right)^{\frac{p-1}{p}}$$

$$\lesssim M^{-2} \left( \int_0^T \int_\Omega |u|^p b_q \eta^*_M^{2p'} dV_\theta dt \right)^{\frac{1}{p}} \left( \int_{\frac{M}{R}}^{M} \int_{K(s) ds \leq t + R_1} (t + R_1)^{-q} r dr dt \right)^{\frac{p-1}{p}}$$

$$\lesssim \left( \int_0^T \int_\Omega |u|^p b_q \eta^*_M^{2p'} dV_\theta dt \right)^{\frac{1}{p}} .$$

In summary, recalling (4.12), we obtain

$$MY'(M) = \int_0^T \int_\Omega |u|^p b_q \eta^*_M^{2p'} dV_\theta dt \gtrless (\log M)^{1-p} Y^p(M), MY'(M) \gtrless \varepsilon^p ,$$

for any $M \in [2, T] \subset [2, T(\varepsilon))$. Let $M = T$, we see that

$$Y^p(T) \leq CT (\log T)^{p-1} Y'(T), \varepsilon^p \lesssim TY'(T) ,$$

for any $T \in [2, T(\varepsilon))$. With the help of (4.14), the desired lifespan estimate

$$T(\varepsilon) \leq \exp(c \varepsilon^{-p(1-p)})$$

for $p = p_c(2)$ follows directly from the following lemma with $p_1 = p_2 = p_c(2)$ and $\delta = \varepsilon^p$.

**Lemma 4.2. (Lemma 3.10 in [10])** Let $2 < t_0 < T$, $0 \leq \phi \in C^1([t_0, T))$. Assume that

$$\begin{cases} 
\delta \leq K_1 t \phi'(t), & t \in (t_0, T), \\
\phi(t)^{p_1} \leq K_2 t (\log t)^{p_2} \phi'(t), & t \in (t_0, T)
\end{cases}$$

with $\delta, K_1, K_2 > 0$ and $p_1, p_2 > 1$. If $p_2 < p_1 + 1$, then there exist positive constants $\delta_0$ and $K_3$ (independent of $\delta$) such that

$$T \leq \exp \left( K_3 \delta^{-\frac{p_1-1}{p_1+p_2+\tau}} \right)$$

when $0 < \delta < \delta_0$.

5. **Test function $\Delta_{g_1 + \omega_1} \Phi_\lambda = \lambda^2 \Phi_\lambda$**

In this section, we consider the generalized eigenvalue problem of the Laplacian operator on $(\Omega, g)$, Theorem 1.4, when the metric $g$ is an exponential perturbation of the spherically symmetric, long range asymptotically Euclidean metric $g_1$. 


5.1. Spherically symmetric case. Let us begin with the spherically symmetric case, considering (1.13) with \( \Omega = B_R^c \) and \( g = g_1 \):

\[
\Delta_{B_R^c} \phi_\lambda = \lambda^2 \phi_\lambda, x \in \Omega; \quad \phi_\lambda = 0, x \in \partial \Omega.
\]

If we look at the corresponding problem on \( \mathbb{R}^2 \)

\[
\Delta_{B_R^c} h_\lambda = \lambda^2 h_\lambda, x \in \mathbb{R}^2; \quad h_\lambda = 1, r = \lambda^{-1},
\]

it is known from Liu-Wang [23] that there exists a small positive constant \( \lambda_2 \) such that

\[
h_\lambda(r) \simeq r|\lambda|^{-1/2} e^{\lambda \int_0^r K(s)ds}, \quad \forall \lambda \in (0, \lambda_2).
\]

By comparing the desired solution with \( h_\lambda \), we shall prove that there exists \( \phi_\lambda \) such that

\[
\phi_\lambda \sim \begin{cases} 
\phi_0 \leq h_\lambda, \phi_\lambda \simeq h_\lambda - h_\lambda(R), & \text{if } 0 < \lambda \leq \lambda_1, \\
\phi_0 \simeq \lambda^{-1/2} e^{\lambda \int_0^r K(s)ds}, & \text{if } \lambda \geq \lambda_1,
\end{cases}
\]

for \( \lambda \in (0, \min(1/(2R), \lambda_2)) \).

**Lemma 5.1.** Let \( n = 2 \). Then for any \( \lambda \in (0, \lambda_2) \), the following boundary value problem

\[
\begin{align*}
\Delta_{B_R^c} \phi_\lambda(x) &= \lambda^2 \phi_\lambda(x), \quad x \in \overline{B_R^c}, \\
\phi_\lambda(x) &= 0, \quad x \in \partial B_R, \\
\phi_\lambda(x) - h_\lambda(x) &\rightarrow 0, \quad |x| \rightarrow \infty
\end{align*}
\]

admits a spherically symmetric solution satisfying

\[
0 \leq h_\lambda(r) - h_\lambda(R) \leq \phi_\lambda(x) \leq h_\lambda(r), \quad x \in B_R^c.
\]

**Proof.** Firstly we consider the following boundary value problem

\[
\begin{align*}
\Delta_{B_R^c} \tilde{\phi}_\lambda(x) &= \lambda^2 \tilde{\phi}_\lambda(x), \quad x \in \overline{B_R^c}, \\
\tilde{\phi}_\lambda(x) &= -h_\lambda(x), \quad x \in \partial B_R, \\
\tilde{\phi}_\lambda(x) &\rightarrow 0, \quad |x| \rightarrow \infty.
\end{align*}
\]

By standard variational argument with functional \( I[u] = \int_{B_R^c} (\lambda^2 |u|^2 + g_1^j k \nabla_j u \nabla_k u) dV_{B_R^c} \), we see that there admits a unique solution \( \tilde{\phi}_\lambda(x) \in H^1(B_R^c) \cap C^\infty(B_R^c) \). Let

\[
\tilde{\phi}_\lambda(x) = \tilde{\phi}_\lambda(x) + h_\lambda(x),
\]

then \( \phi_\lambda(x) \) satisfies the boundary value problem (5.3).

It remains to prove

\[
\tilde{\phi}_\lambda(x) \in [-h_\lambda(R), 0], \quad x \in \overline{B_R^c}.
\]

For any \( C_0 > 0 \), by (5.5), we know that

\[
\begin{cases}
- \Delta_{B_R^c} \left( \tilde{\phi}_\lambda(x) - C_0 \right) + \lambda^2 \left( \tilde{\phi}_\lambda(x) - C_0 \right) = -\lambda^2 C_0 < 0, \quad x \in \overline{B_R^c}, \\
\left( \tilde{\phi}_\lambda(x) - C_0 \right) \bigg|_{\partial B_R} = (-h_\lambda(x) - C_0) \bigg|_{\partial B_R} < 0, \\
\tilde{\phi}_\lambda(x) - C_0 \rightarrow -C_0 < 0, \quad |x| \rightarrow \infty.
\end{cases}
\]
By maximum principle, we have
\[ \tilde{\phi}_\lambda(x) - C_0 \leq 0, \ x \in \overline{B_R}, \]
for any \( C_0 > 0 \), which means
\[ \tilde{\phi}_\lambda(x) \leq 0, \ x \in \overline{B_R}. \]

Similarly, it is clear that
\[
\begin{align*}
- \Delta_\partial \left( \tilde{\phi}_\lambda(x) + h_\lambda(R) \right) + \lambda^2 \left( \tilde{\phi}_\lambda(x) + h_\lambda(R) \right) &= \lambda^2 h_\lambda(R) > 0, \ x \in \overline{B_R}, \\
\left( \tilde{\phi}_\lambda(x) + h_\lambda(R) \right) &\bigg|_{\partial B_R} = (-h_\lambda(x) + h_\lambda(R)) \bigg|_{\partial B_R} = 0, \\
\tilde{\phi}_\lambda(x) + h_\lambda(R) &\to h_\lambda(R) > 0, \ |x| \to \infty.
\end{align*}
\]

By maximum principle, we have
\[ \tilde{\phi}_\lambda(x) + h_\lambda(R) \geq 0, \ x \in \overline{B_R}, \]
and this completes the proof of Lemma 5.1. \( \square \)

With the help of Lemma 5.1, we get
\[ (5.6) \quad \phi_\lambda(x) \simeq h_\lambda(r), \]
for any \( r \geq \lambda^{-1} \) with \( \lambda < \min(\lambda_2, (2R)^{-1}) \).

**Lemma 5.2.** Let \( \phi_\lambda(x) \) be the function in Lemma 5.1 and \( 0 < \lambda < \min(\lambda_2, (2R)^{-1}) \).
Then we have
\[ (5.7) \quad \phi_\lambda(x) \simeq \frac{\ln r/R}{\ln 1/(\lambda R)}, \forall R \leq \lambda^{-1}. \]

**Proof.** Let \( \delta > 0 \) to be determined and
\[ F_\lambda(x) = \delta \frac{\phi_0(x)}{\phi_0(\lambda^{-1})} h_\lambda(x) - \phi_\lambda(x), \]
then, by (3.7),
\[
\begin{align*}
- \Delta_\partial F_\lambda + \lambda^2 F_\lambda(x) &= -\frac{\delta}{K^2(r) \ln(\lambda^{-1})} \partial_\nu h_\lambda(x) \partial_\nu \phi_0 < 0, \ x \in \overline{B_R}, \\
F_\lambda(x) &= 0, \ |x| = R, \\
F_\lambda(x) &= \delta - \phi_\lambda(x) \leq 0, \ |x| = \lambda^{-1},
\end{align*}
\]
when \( \delta > 0 \) is sufficiently small such that \( \phi_\lambda(x) \geq \delta = \delta h_\lambda(r) \) for \( r = \lambda^{-1} \), in view of (5.6). By the maximum principle, one obtain
\[ F_\lambda(x) \leq 0, \phi_\lambda(x) \geq \delta \frac{\phi_0(x)}{\phi_0(\lambda^{-1})} h_\lambda(x) \geq \delta h_\lambda(0) \frac{\phi_0(x)}{\phi_0(\lambda^{-1})} \geq \delta' \frac{\ln r/R}{\ln(\lambda R)}, \forall R \leq \lambda^{-1}, \]
for some \( \delta' > 0 \), which yields the desired lower bound.

On the other hand, let
\[ G_\lambda(x) = \phi_\lambda(x) - \frac{\phi_0}{\phi_0(\lambda^{-1})}, \]

then
\[
\begin{align*}
-\Delta_{B_1} G_{\lambda} + \lambda^2 G_{\lambda}(x) &= -\lambda^2 \frac{\phi_0}{\phi_0(\lambda^{-1})} < 0, \quad x \in \overline{B_R}, \\
G_{\lambda}(x) &= 0, \quad |x| = R, \\
G_{\lambda}(x) &= \phi_{\lambda}(x) - 1 \leq 0, \quad |x| = \lambda^{-1},
\end{align*}
\]
which gives us \( G_{\lambda}(x) \leq 0 \). \hfill \Box

5.2. Derivative estimates. For future reference, we need to obtain more information concerning the behavior of \( \phi_{\lambda} \). At first, we claim that we have the following derivative estimates
\[
\partial_r \phi_{\lambda}(r) \leq D_0 \lambda^2(r - R) \phi_{\lambda}(r), \quad |\partial_r^2 \phi_{\lambda}(r)| \leq D_0 \lambda^2 \phi_{\lambda}(r), \quad \forall r \geq R
\]
for some constant \( D_0 \) independent of \( \lambda \in (0, 1/(2R)] \).

As \( \phi_{\lambda} \) is radial, by (1.2) and (3.7), we have
\[
\Delta_{B_1} \phi_{\lambda} = K^{-1} r^{-1} \partial_r (K^{-1} r \partial_r \phi_{\lambda}),
\]
and so
\[
\partial_r (K^{-1} r \partial_r \phi_{\lambda}) = \lambda^2 K r \phi_{\lambda}.
\]
As \( \phi_{\lambda} \) is increasing, we get
\[
K^{-1} r \partial_r \phi_{\lambda} \leq \int_{R}^{r} \lambda^2 \tau K \phi_{\lambda}(\tau) d\tau \leq \frac{1}{2} \lambda^2 (r^2 - R^2) \|K\|_{L^{\infty}} \phi_{\lambda}(r),
\]
that is, \( \partial_r \phi_{\lambda} \leq \|K\|_{L^{\infty}} \lambda^2 (r - R) \phi_{\lambda}(r) \).

For the second order derivative of \( \phi_{\lambda} \), by (5.9), we have
\[
|\partial_r^2 \phi_{\lambda}| = |\lambda^2 K^2 \phi_{\lambda} - \frac{1}{r} - \frac{K'}{K} | \partial_r \phi_{\lambda} | \leq \lambda^2 K^2 \phi_{\lambda} + C \frac{r}{r} \partial_r \phi_{\lambda} \lesssim \lambda^2 \phi_{\lambda}
\]
for some \( C > 0 \) due to (1.3).

5.3. Test function \( \Delta_{B_1 + B_3} \Phi_{\lambda} = \lambda^2 \Phi_{\lambda} \). Turning to the exponential perturbation of the spherically symmetric asymptotically Euclidean metric, we consider the following problem
\[
\Delta_{B_1 + B_3} \Phi_{\lambda} = \lambda^2 \Phi_{\lambda}, \quad x \in \Omega; \quad \Phi_{\lambda} = 0, \quad x \in \partial \Omega.
\]
We would like to obtain similar bounds as that for \( g_3 \).

Lemma 5.3. There exists \( \lambda_3 > 0 \) such that for any \( \lambda \in (0, \lambda_3) \), we can construct a solution \( \Phi_{\lambda} \) such that
\[
\Phi_{\lambda} \sim \phi_{\lambda} \sim \begin{cases} \frac{\phi_{\lambda}(r)}{h_{\lambda}} \geq \langle r \lambda \rangle^{-1/2} \mathcal{E} \int_{r}^{\infty} k(s) ds & 0 \leq \lambda^{-1}, \\ h_{\lambda} \geq \langle r \lambda \rangle^{-1/2} \mathcal{E} \int_{r}^{\infty} k(s) ds & \lambda \geq \lambda^{-1}. \end{cases}
\]

5.4. Proof of Lemma 5.3. As in [23, Lemma 3.1], see also Wakasa-Yordanov [37, Lemma 2.2], we introduce \( \psi = \phi_{\lambda} - \Phi_{\lambda} \) where \( \Delta_{B_1} \phi_{\lambda} = \lambda^2 \phi_{\lambda} \). Then we are reduced to prove existence of \( \psi \) and show smallness of \( \|\psi\|_{L^{\infty}} \).

In fact, \( \psi \) satisfies
\[
\Delta_\Theta \psi = \Delta_\Theta (\phi_{\lambda} - \Phi_{\lambda}) = \lambda^2 \phi_{\lambda} - \lambda^2 \Phi_{\lambda} = (\Delta_{\Theta} - \Delta_\Theta) \phi_{\lambda} = \lambda^2 \psi - W \phi_{\lambda},
\]
and \( \psi |_{\partial B_R} = 0 \). We claim that, for any \( \lambda \in (0, \min(\alpha/2, 1/(2R), \lambda_2)) \),
\[
\|W \phi_{\lambda}\|_{L^q} \lesssim \lambda^2, \quad \forall q \in [1, \infty).
\]
Actually, we have
\[ W\phi_\lambda = (g_1^{jk} - g^{jk}) \partial_j \partial_k \phi_\lambda + [g_1^{-1/2} \partial_j (g_1^{1/2} g_1^{jk}) - g^{-1/2} \partial_j (g^{1/2} g^{jk})] \partial_k \phi_\lambda . \]
By (5.8) and (1.17) with \( \lambda < \min(\alpha/2, 1/(2R), \lambda_2) \), it is easy to see that
\[ |W\phi_\lambda| \leq \sum_{j,k} \sum_{1 \leq |\beta| \leq 2} |\nabla^{\leq 1} g_1^{jk} \nabla^\beta \phi_\lambda| \leq \lambda^2 \|\nabla \phi_\lambda\|_{L^2} e^{(\lambda - \alpha) \int_0^R K(r) dr} \leq \lambda^2 \|\phi_\lambda\|_{L^2} e^{-\frac{\delta}{2} R} , \]
which gives us (5.12).
Standard elliptic theory ensures that there exists a unique weak \( H^1_0 \) solution \( \psi \) to (5.11), which satisfies \( \psi \in H^3 \cap C^\infty \). To show the smallness of \( \|\psi\|_{L^\infty} \), we first take the (natural) inner product of (5.11) with \( \psi \) to get
\[ -\langle \Delta g \psi, \psi \rangle_\partial + \lambda^2 \langle \psi, \psi \rangle_\partial = -(W \phi_\lambda, \psi)_\partial . \]
Thus by the Cauchy-Schwarz inequality and uniform elliptic condition (1.5) we have
\[ \delta_0 \|\nabla \psi\|_{L^2}^2 + \lambda^2 \|\psi\|_{L^2}^2 \lesssim \|\psi\|_{L^2}^2 , \]
which yields
\[ \|\psi\|_{L^2} \lesssim 1, \|\nabla \psi\|_{L^2} \lesssim \lambda . \]
In view of the equation, we obtain
\[ \|\Delta g \psi\|_{L^2} \lesssim \lambda^2 . \]
As \( \psi \in H^2 \cap H^1_0 \), we get from [6, Theorem 9.11-13] that
\[ \|\nabla^2 \psi\|_{L^2} \lesssim \|\psi\|_{H^1} + \|\Delta g \psi\|_{L^2} \lesssim 1 . \]

Applying the Gagliardo-Nirenberg inequality (for the exterior domain, see, e.g., [2]), we get
\[ \|\psi\|_{L^2} \lesssim \|\psi\|_{L^2}^{1/2} \lesssim \lambda^{1/2} , \|\nabla \psi\|_{L^2} \lesssim \|\nabla^2 \psi\|_{L^2} \lesssim \lambda^{1/2} . \]
By Sobolev embedding, we obtain
\[ \|\psi\|_{L^2} \lesssim \|\psi\|_{L^2} + \|\nabla \psi\|_{L^2} \lesssim \lambda^{1/2} . \]
Notice that \( \phi_\lambda \) tends to zero as \( r \to R \), the estimate (5.14) is not good enough to be controlled by \( \phi_\lambda \) near the boundary. For that purpose, we observe that, in view of the equations, (5.12) and (5.13), we have
\[ \|\Delta g \psi\|_{L^2} \lesssim \lambda^2 . \]
As \( \psi \in W^{2,4} \cap W^{1,4}_0 \), we get from [6, Theorem 9.11-13] that
\[ \|\nabla^2 \psi\|_{L^2} \lesssim \|\psi\|_{W^{1,4}} + \|\Delta g \psi\|_{L^2} \lesssim \lambda^{1/2} . \]
Another application of the Sobolev embedding gives us
\[ \|\nabla \psi\|_{L^2} \lesssim \|\nabla \psi\|_{L^2} + \|\nabla^2 \psi\|_{L^2} \lesssim \lambda^{1/2} , \]
which, together with the boundary condition \( \psi = 0 \) for \( r = R \), yields
\[ \|\psi(r, \theta)\|_{L^2} \lesssim \lambda^{1/2} (r - R) . \]

In summary, by (5.14), (5.16) and (5.2) we have obtained
\[ \|\psi(r, \theta)\|_{L^2} \lesssim \lambda^{1/2} \min\{|x| - R, R\} \lesssim \lambda^{1/2} \phi_0(r) \ll \phi_\lambda , \]
which gives us the desired estimate (5.10) for \( r \geq R \) and \( \lambda \ll \lambda_3 \ll \min(1, \lambda_2, \alpha/2, 1/(2R)) \).
6. General obstacles

In this section, we present the proof of Lemma 1.5, which enables us to reduce the problems with general obstacles to the problem exterior to a disk.

6.1. Star-shaped obstacles. It turns out that we could construct an explicit diffeomorphism, when the obstacle $K$ (the interior of the Jordan curve $\partial D$) is star-shaped. As $K$ is star-shaped, there exists a smooth $R : S^1 \to \mathbb{R}_+$:

$$K = \{(r \cos \theta, r \sin \theta), r < R(\theta) \in (0, \infty)\}.$$  

At first, we set $\delta_2 > 0$ such that

$$R(\theta) \in [2\delta_2, \delta_2^{-1}/2],$$

$R_3 = \delta_2$ and $R_4 = \delta_2^{-1}$. Let $\mu$ be a decreasing cut-off function such that $\mu = 0$ for $r > 2$ and $\mu = 1$ for $r < 1$. We set

$$(6.1) \quad f(r, \theta) = \mu \left(\frac{r}{R_4}\right) R_3 - \mu \left(\frac{r}{R_4}\right) R_4 R(\theta) + (1 - \mu \left(\frac{r}{R_4}\right)) r$$

and introduce

$$(6.2) \quad A : (r, \theta) \to (f(r, \theta), \theta).$$

Notice that $f(R(\theta), \theta) = R_3$, $f(r, \theta) = r$ for any $r > 2R_4$, and

$$\partial_r f(r, \theta) = \mu \left(\frac{r}{R_4}\right) R_3 - \mu \left(\frac{r}{R_4}\right) R_4 R(\theta) - R_3 \frac{R_4}{R(\theta)} R_4 \frac{R_4}{R(\theta)} \in \left[\frac{R_4}{R(\theta)} 1 + \frac{2R_4}{R(\theta)} C\right]$$

and so

$$(6.3) \quad \partial_r f(r, \theta) \in [2\delta_2^2, 1 + 2C], \forall r, \theta,$$

where we have denoted $C = \|\mu'\|_{L^\infty}$ and used the fact that $\mu' = 0$ unless $r \in [R_4, 2R_4]$.

It follows then that the map

$$A : D \to \Omega := \mathbb{R}^2 \setminus \overline{B_{R_3}}$$

is a smooth diffeomorphism mapping preserving the boundary, which is an identity map for $r > 2R_4$.

6.2. General obstacles. Let $K \subset \mathbb{R}^2$ be the interior of the Jordan curve $\partial D$, i.e., nonempty bounded smooth simple connected domain. Without loss of generality, we assume $0 \in K$. Let

$$\hat{D} = \{z; z = x + iy, (x, y) \in D\} \subset C, \quad \hat{D} = \{z; \frac{1}{z} \in \hat{D}\}.$$  

Then by the Riemann mapping theorem (see, e.g., [36, Chapter 5, Theorem 4.1]), there exist $R > 0$ and (holomorphic) diffeomorphism $B$ preserving the boundary such that

$$B : \overline{D \cup \{0\}} \to \overline{B_1/R}, \quad B(0) = 0, \quad B'(0) = 1,$$

which gives us a diffeomorphism mapping

$$B : \hat{D} \to B_1/R \setminus \{0\}.$$  

Let $\delta \in (0, 1)$ be a small parameter and $\mu(z)$ be a smooth cut-off function such that $\mu = 0$ when $|z| \leq 2$ and $\mu = 1$ when $|z| \geq 3$. Then we define the map

$$(6.4) \quad \mathcal{H}(z) = \mu_\delta(z) B(z) + (1 - \mu_\delta(z)) z, \quad \mu_\delta(z) = \mu\left(\frac{z}{\delta}\right).$$
We claim that there exists a small $\delta_1 > 0$ such that
\begin{equation}
\mathcal{H}(z) : \overline{D} \to B_{1/R}\setminus\{0\}
\end{equation}
is a diffeomorphism mapping when $\delta < \delta_1$. Thus it is easy to see that
\begin{equation}
\mathcal{A}(x, y) = \frac{1}{\mathcal{H}(\frac{1}{x+iy})} : D \to \Omega
\end{equation}
is a desired diffeomorphism mapping. Hence we are reduced to showing the claim (6.5).

By definition, we have $\mathcal{H} = \mathcal{B}$ when $|z| \geq 3\delta$ and $\mathcal{H}(z) = z$ when $|z| < 2\delta$. So we need only to consider the case $2\delta < |z| < 3\delta$. Let $\mathcal{B} : (x, y) \to (F, \tilde{G})$, then we have
\begin{equation}
\mathcal{J}(F, \tilde{G})(0) = \left( \begin{array}{c} \tilde{F}_x(0) \\ \tilde{F}_y(0) \\ \tilde{G}_x(0) \\ \tilde{G}_y(0) \end{array} \right) = \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right).
\end{equation}
Let $\mathcal{H} : (x, y) \to (F, G)$, by definition (6.4) we have
\[
F = x + \mu_\delta(\tilde{F} - x), \quad G = y + \mu_\delta(\tilde{G} - y)
\]
and by (6.6), for $r = \sqrt{x^2 + y^2} < 4\delta$ small enough, we get that
\[
F_x = 1 + \mathcal{O}(\delta), \quad F_y = \mathcal{O}(\delta), \quad G_x = \mathcal{O}(\delta), \quad G_y = 1 + \mathcal{O}(\delta).
\]
Then there exists $\delta_3 > 0$, such that for any $\delta \leq \delta_3$, we have $\mathcal{H}(B_{3\delta}) \subset B_{1/R}$ and
\[
\mathcal{J}(F, G)(x, y) \in \left( \frac{1}{2}, \frac{3}{2} \right), \quad \forall r < 4\delta_3.
\]
By inverse function theorem and compactness of $B_{3\delta}$, there exists a uniform $\varepsilon_1 > 0$ such that $\mathcal{H}$ is a local diffeomorphism on $B_{\varepsilon_1}(r_0, \theta_0)$ for any $r_0 \in [0, 3\delta_3]$ and $\theta_0 \in [0, 2\pi]$. Thus we are reduced to proving bijection of $\mathcal{H}$.

We firstly prove the injection. For that purpose, we replace $(x, y)$ by polar coordinates $(r, \theta)$ and $(F, G)$ by $(R, \varphi)$. Recall that $x\partial_y - y\partial_x = \partial_\theta$ and $x\partial_x + y\partial_y = r\partial_r$, we get
\begin{equation}
\varphi_\theta = 1 + \mathcal{O}(\delta), \quad R_\theta = \mathcal{O}(\delta^2), \quad \varphi_r = \mathcal{O}(1), \quad R_r = 1 + \mathcal{O}(\delta), \quad r < 4\delta.
\end{equation}
Let $\Gamma_r = \mathcal{H}(\partial B_r) \quad (r > 0)$ be closed curves. With possibly shrinking $\delta_3 > 0$, we could assume, for some $C > 0$,
\begin{equation}
|\mathcal{O}(t)| \leq Ct, \quad C\delta \leq \frac{1}{4}.
\end{equation}
Then by (6.7) we have $\varphi_\theta \in [3/4, 5/4]$ and thus the winding number of $\Gamma_r$ about the origin 0 must be 1, which shows that $\Gamma_r$ is diffeomorphic to the unit circle $S^1$.

To complete the proof of injection, we need only to show $\{\Gamma_r\}_{0 < r \leq 3\delta}$ are disjoint, for which it suffices to prove
\[
\Gamma_{r_1} \cap \Gamma_{r_2} = \emptyset, \quad 0 < |r_1 - r_2| < \frac{\delta_3}{2} < \varepsilon.
\]
In fact, if $\Gamma_{r_1} \cap \Gamma_{r_2} \neq \emptyset$ for some $r_1 \neq r_2$ then $\varphi(r_1, \theta_1) = \varphi(r_2, \theta_2)$ for some $\theta_1, \theta_2$. Without loss of generality, we assume $\theta_2 > \theta_1$. With $\theta_t = (1 - t)\theta_0 + t\theta_1, \quad r_t = (1 - t)r_0 + tr_1$, by (6.7) and (6.8), we have
\[
0 = \frac{d}{dt} \int_1^2 \varphi(r_t, \theta_t) dt = \int_1^2 \varphi_r \frac{d}{dt} r_t + \varphi_\theta \frac{d}{dt} \theta_t dt \geq (1 - C\delta)(\theta_2 - \theta_1) - C(r_2 - r_1),
\]
which yields \( \theta_2 - \theta_1 \leq \frac{C_1}{2} < \frac{\epsilon}{2} \), that is

\[
(r_2, \theta_2) \in B_\epsilon(r_1, \theta_1).
\]

This is a contradiction to the diffeomorphism of \( \mathcal{H} \) on \( B_\epsilon(r_1, \theta_1) \).

Let \( \Gamma^r \) denote the inner points of \( \Gamma_r \), as \( \Gamma^0 = \{0\} \) and \( \Gamma^\delta = B(\partial B_3^\delta) \), we see that \( \Gamma_r \) is increasing about \( r \) (\( \Gamma^r_1 \subset \Gamma^r_2 \), if \( r_1 < r_2 \)). As \( \Gamma_r \) depends continuously on \( r \), then it is clear that we have \( \mathcal{H}(B_3^\delta) = \Gamma^\delta = B(B_3^\delta) \). This completes the proof of (6.5).

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