PROPERTIES OF BOUNDED REPRESENTATIONS FOR G-FRAMES

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Abstract. Due to the importance of frame representation by a bounded operator in dynamical sampling, researchers studied the frames of the form \( \{ T^i f \}_{i \in \mathbb{N}} \), which \( f \) belongs to separable Hilbert space \( \mathcal{H} \) and \( T \in B(\mathcal{H}) \), and investigated the properties of \( T \). Given that \( g \)-frames include the wide range of frames such as fusion frames, the main purpose of this paper is to study the characteristics of the operator \( T \) for \( g \)-frames of the form \( \{ \Lambda T^i f \}_{i \in \mathbb{N}} \).

1. Introduction

Duffin and Schaeffer introduced an extension of orthonormal bases for separable Hilbert space \( \mathcal{H} \) named frames [13], which in spite of producing \( \mathcal{H} \), is not necessarily linearly independent. Frames are important tools in the signal/image processing [4, 5, 14], data compression [12, 23], dynamical sampling [1, 2] and etc.

Definition 1.1. A sequence \( F = \{ f_i \}_{i \in \mathbb{N}} \) in \( \mathcal{H} \) is called a frame for \( \mathcal{H} \), if there exist two constants \( A_F, B_F > 0 \) such that

\[
A_F \| f \|^2 \leq \sum_{i \in \mathbb{N}} |\langle f, f_i \rangle|^2 \leq B_F \| f \|^2, \quad f \in \mathcal{H}.
\]

For more on frames we refer to [9, 16].

Aldroubi et al. introduced the concept of dynamical sampling to examine sequences of the form \( \{ T^i f \}_{i \in \mathbb{N}} \subset \mathcal{H} \), that spans \( \mathcal{H} \) for \( T \in B(\mathcal{H}) \). As frames span the space, researchers have studied the frames \( F = \{ f_i \}_{i \in \mathbb{N}} \) for infinite dimensional Hilbert space \( \mathcal{H} \) that can be represented by \( T \), i.e. \( F = \{ T^i f_1 \}_{i \in \mathbb{N}} \) [12]. Christensen et al. have shown that the only frames with bounded representations are those which are linearly independent and the kernel of their synthesis operators is invariant under right-shift operator \( \mathcal{T} : \ell^2(\mathcal{H}, \mathbb{N}) \to \ell^2(\mathcal{H}, \mathbb{N}) \).

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defined by
\[ T(\{c_i\}_{i \in \mathbb{N}}) = (0, c_1, c_2, \ldots), \]
where \( \ell^2(\mathcal{H}, \mathbb{N}) = \{\{g_i\}_{i \in \mathbb{N}} : g_i \in \mathcal{H}, \sum_{i \in \mathbb{N}} \|g_i\|^2 < \infty\} \), such as orthonormal bases and Riesz bases [10]. They have also explored the relationship between frame representation and its duals. For the applications of frames, they established that frame representations were preserved under some perturbations. Results [2, Theorem 7] and [11, Proposition 3.5] are shown that the sequence \( \{T^{-1}f_i\}_{i \in \mathbb{N}} \) is not a frame, whenever \( T \) is unitary or compact. Also, Lemma 2.1 and Proposition 2.3 of [22] indicate \( \text{ran} \, T \) is close and give some equivalent conditions for \( T \) to be surjective.

In 2006, Sun introduced a generalization of frames, named \( g \)-frames [24] which are including some extensions and types of frames such as frames of subspaces [8], fusion frames [6, 7], oblique frames [3], a class of time-frequency localization operators and generalized translation invariant (GTI) [17]. Therefore, some concepts presented in frames such as duality, stability and Riesz-basis were also studied in \( g \)-frames [25].

Throughout this paper, \( J \) is countable set, \( \mathbb{N} \) is natural numbers and \( \mathbb{C} \) is complex numbers, \( \mathcal{H} \) and \( \mathcal{K} \) are separable Hilbert spaces, \( \text{Id}_H \) denotes the identity operator on \( \mathcal{H} \), \( B(\mathcal{H}) \) and \( GL(\mathcal{H}) \) denote the set of bounded linear operators and invertible bounded linear operators on \( \mathcal{H} \), respectively. Also, we will apply \( B(\mathcal{H}, \mathcal{K}) \) for the set of bounded linear operators from \( \mathcal{H} \) to \( \mathcal{K} \). We use \( \ker T \) and \( \text{ran} \, T \) for the null space and range \( T \in B(\mathcal{H}) \), respectively. Now, we summarize some facts about \( g \)-frames from [20, 24]. For more on related subjects to \( g \)-frames, we refer to [15, 19, 21].

**Definition 1.2.** We say that \( \Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i) : i \in \mathbb{N}\} \) is a generalized frame for \( \mathcal{H} \) with respect to \( \{\mathcal{K}_i : i \in \mathbb{N}\} \), or simply \( g \)-frame, if there are two constants \( 0 < A_{\Lambda} \leq B_{\Lambda} < \infty \) such that
\[
A_{\Lambda} \|f\|^2 \leq \sum_{i \in \mathbb{N}} \|\Lambda_i f\|^2 \leq B_{\Lambda} \|f\|^2, \quad f \in \mathcal{H}.
\]
(1.1)

We call \( A_{\Lambda}, B_{\Lambda} \) the lower and upper \( g \)-frame bounds, respectively. \( \Lambda \) is called a tight \( g \)-frame if \( A_{\Lambda} = B_{\Lambda} \), and a Parseval \( g \)-frame if \( A_{\Lambda} = B_{\Lambda} = 1 \). If for each \( i \in \mathbb{N}, \mathcal{K}_i = \mathcal{K} \), then, \( \Lambda \) is called a \( g \)-frame for \( \mathcal{H} \) with respect to \( \mathcal{K} \). Note that for a family \( \{\mathcal{K}_i\}_{i \in \mathbb{N}} \) of Hilbert spaces, there exists a Hilbert space \( \mathcal{K} = \bigoplus_{i \in \mathbb{N}} \mathcal{K}_i \) such that for all \( i \in \mathbb{N}, \mathcal{K}_i \subseteq \mathcal{K} \), where \( \bigoplus_{i \in \mathbb{N}} \mathcal{K}_i \) is the direct sum of \( \{\mathcal{K}_i\}_{i \in \mathbb{N}} \). A family \( \Lambda \) is called \( g \)-Bessel if the right hand inequality in (1.1) holds for all \( f \in \mathcal{H} \), in this case, \( B_{\Lambda} \) is called the \( g \)-Bessel bound.
Example 1.3. [24] Let \( \{f_i\}_{i \in \mathbb{N}} \) be a frame for \( \mathcal{H} \). Suppose that \( \Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathbb{C}) : i \in \mathbb{N}\} \), where

\[ \Lambda_i f = \langle f, f_i \rangle, \quad f \in \mathcal{H}. \]

It is easy to see that \( \Lambda \) is a \( g \)-frame.

For a \( g \)-frame \( \Lambda \), there exists a unique positive and invertible operator \( S_\Lambda : \mathcal{H} \to \mathcal{H} \) such that

\[ S_\Lambda f = \sum_{i \in \mathbb{N}} \Lambda_i^* \Lambda_i f, \quad f \in \mathcal{H}, \]

and \( A_\Lambda Id_\mathcal{H} \leq S_\Lambda \leq B_\Lambda Id_\mathcal{H} \). Consider the space

\[ \left( \sum_{i \in \mathbb{N}} \oplus \mathcal{K}_i \right)_{\ell^2} = \left\{ \{g_i\}_{i \in \mathbb{N}} : g_i \in \mathcal{K}_i, \ i \in \mathbb{N} \text{ and } \sum_{i \in \mathbb{N}} \|g_i\|^2 < \infty \right\}. \]

It is clear that, \( \left( \sum_{i \in \mathbb{N}} \oplus \mathcal{K}_i \right)_{\ell^2} \) is a Hilbert space with pointwise operations and with the inner product given by

\[ \langle \{f_i\}_{i \in \mathbb{N}}, \{g_i\}_{i \in \mathbb{N}} \rangle = \sum_{i \in \mathbb{N}} \langle f_i, g_i \rangle. \]

For a \( g \)-Bessel \( \Lambda \), the synthesis operator \( T_\Lambda : \left( \sum_{i \in \mathbb{N}} \oplus \mathcal{K}_i \right)_{\ell^2} \to \mathcal{H} \) is defined by

\[ T_\Lambda \left( \{g_i\}_{i \in \mathbb{N}} \right) = \sum_{i \in \mathbb{N}} \Lambda_i^* g_i. \]

The adjoint of \( T_\Lambda \), \( T_\Lambda^* : \mathcal{H} \to \left( \sum_{i \in \mathbb{N}} \oplus \mathcal{K}_i \right)_{\ell^2} \) is called the analysis operator of \( \Lambda \) and is as follow

\[ T_\Lambda^* f = \{\Lambda_i f\}_{i \in \mathbb{N}}, \quad f \in \mathcal{H}. \]

It is obvious that \( S_\Lambda = T_\Lambda T_\Lambda^* \). For a \( g \)-frame \( \Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i) : i \in \mathbb{N}\} \), the sequence \( \widetilde{\Lambda} = \{\widetilde{\Lambda} := \Lambda_i S_\Lambda^{-1} \in B(\mathcal{H}, \mathcal{K}_i) : i \in \mathbb{N}\} \) is a \( g \)-frame with lower and upper \( g \)-frame bounds \( \frac{1}{B_\Lambda} \) and \( \frac{1}{A_\Lambda} \), respectively, which is called canonical dual of \( \Lambda \). For \( g \)-Bessel sequences \( \Lambda \) and \( \Theta \), we consider \( S_{\Lambda \Theta} := T_\Lambda T_\Theta^* \).

**Definition 1.4.** Consider a sequence \( \Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i) : i \in \mathbb{N}\} \).

(i) We say that \( \Lambda \) is \( g \)-complete if \( \{f : \Lambda_i f = 0, i \in \mathbb{N}\} = \{0\} \).

(ii) We say that \( \Lambda \) is a \( g \)-Riesz sequence if there are two constants \( 0 < A_\Lambda \leq B_\Lambda < \infty \) such that for any finite set \( \{g_i\}_{i \in I_n} \),

\[ A_\Lambda \sum_{i \in I_n} \|g_i\|^2 \leq \| \sum_{i \in I_n} \Lambda_i^* g_i \|^2 \leq B_\Lambda \sum_{i \in I_n} \|g_i\|^2, \quad g_i \in \mathcal{K}_i. \]
(iii) We say that $\Lambda$ is a $g$-Riesz basis if $\Lambda$ is $g$-complete and $g$-Riesz sequence.

(iv) We say that $\Lambda$ is a $g$-orthonormal basis if it satisfies the following:

$$\langle \Lambda_i^* g_i, \Lambda_j^* g_j \rangle = \delta_{i,j} \langle g_i, g_j \rangle, \quad i, j \in \mathbb{N}, g_i \in K_i, g_j \in K_j,$$

$$\sum_{i \in \mathbb{N}} \|\Lambda_i f\|^2 = \|f\|^2, \quad f \in \mathcal{H}.$$ 

A $g$-Riesz basis $\Lambda = \{\Lambda_i \in B(\mathcal{H}, K_i) : i \in \mathbb{N}\}$ is $g$-biorthonormal with respect to its canonical dual $\tilde{\Lambda} = \{\tilde{\Lambda} := \Lambda_i S_{\Lambda}^{-1} \in B(\mathcal{H}, K_i) : i \in \mathbb{N}\}$ in the following sense

$$\langle \Lambda_i^* g_i, \tilde{\Lambda}_j^* g_j \rangle = \delta_{i,j} \langle g_i, g_j \rangle, \quad i, j \in \mathbb{N}, g_i \in K_i, g_j \in K_j.$$ 

**Theorem 1.5.** [21] Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, K_i) : i \in \mathbb{N}\}$ be a $g$-frame and $\Theta = \{\Theta_i \in B(\mathcal{H}, K_i) : i \in \mathbb{N}\}$ be a $g$-orthonormal basis. Then there is a bounded operator $V : \mathcal{H} \rightarrow \mathcal{H}$ such that $\Lambda_i = \Theta_i V^*$, for all $i \in \mathbb{N}$. If $\Lambda$ is a $g$-Riesz basis, then $V$ is invertible. If $\Lambda$ is a $g$-orthonormal bases, then $V$ is unitary.

**Theorem 1.6.** [24] Let for $i \in \mathbb{N}$, $\{e_{i,j}\}_{j \in J_i}$ be an orthonormal basis for $K_i$. Sequence $\Lambda = \{\Lambda_i \in B(\mathcal{H}, K_i) : i \in \mathbb{N}\}$ is a $g$-frame (respectively, $g$-Bessel family, $g$-Riesz basis, $g$-orthonormal basis) if and only if $\{\Lambda_i^* e_{i,j}\}_{i \in \mathbb{N}, j \in J_i}$ is a frame (respectively, Bessel sequence, Riesz basis, orthonormal basis).

Now we summarize some results of article [18] in which we generalize the results of articles [10, 11] to introduce the representation of $g$-frames with bounded operators.

**Remark 1.7.** Consider a frame $F = \{f_i\}_{i \in \mathbb{N}} = \{T^{i-1} f_1\}_{i \in \mathbb{N}}$ for $\mathcal{H}$ with $T \in B(\mathcal{H})$. For the $g$-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathbb{C}) : i \in \mathbb{N}\}$ where

$$\Lambda_i f = \langle f, f_i \rangle, \quad f \in \mathcal{H},$$

we have

$$\Lambda_{i+1} f = \langle f, f_{i+1} \rangle = \langle f, T f_i \rangle = \langle T^* f, f_i \rangle = \Lambda_i T^* f, \quad f \in \mathcal{H}.$$ 

Therefore, $\Lambda_i = \Lambda_1 (T^*)^{i-1}, i \in \mathbb{N}$. Conversely, if we consider a $g$-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathbb{C}) : i \in \mathbb{N}\} = \{\Lambda_1 T_i^{-1} : i \in \mathbb{N}\}$ for $T \in B(\mathcal{H})$, then by the Riesz representation theorem, $\Lambda_i f = \langle f, f_i \rangle, i \in \mathbb{N}$ and $f, f_i \in \mathcal{H}$, where $F = \{f_i\}_{i \in \mathbb{N}}$ is a frame that $f_i = (T^*)^{i-1} f_1, i \in \mathbb{N}$.

Now, we have been motivated to study $g$-frames $\Lambda = \{\Lambda_i \in B(\mathcal{H}, K) : i \in \mathbb{N}\}$, where $\Lambda_i = \Lambda_1 T_i^{-1}$ with $T \in B(\mathcal{H})$. 
Definition 1.8. We say that a g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ has a representation if there is a $T \in B(\mathcal{H})$ such that $\Lambda_i = \Lambda_1 T_i^{-1}, i \in \mathbb{N}$. In the affirmative case, we say that $\Lambda$ is represented by $T$.

The following theorem shows that for g-frames $\Lambda = \{\Lambda_i T_i^{-1} : i \in \mathbb{N}\}$, the boundedness of $T$ is equivalent to the invariance of $\ker T_\Lambda$ under the right-shift operator.

Theorem 1.9. Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ be a g-frame such that for every finite set $\{g_i\}_{i \in I_n} \subset \mathcal{K}$, $\sum_{i \in I_n} \Lambda_i^* g_i = 0$ for every $i \in I_n$. Suppose that $\ker T_\Lambda$ is invariant under the right-shift operator. Then, $\Lambda$ is represented by $T \in B(\mathcal{H})$, where $\|T\| \leq \sqrt{B_\Lambda A_\Lambda^{-1}}$.

Corollary 1.10. Every g-orthonormal and g-Riesz bases has a representation.

Remark 1.11. Consider a g-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ which is represented by $T$. For $S \in GL(\mathcal{H})$, the family $\Lambda S = \{\Lambda_i S \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ is a g-frame [21, Corollary 2.26], which is represented by $S^{-1} T S$.

In this paper, we generalize some recent results of [11, 22] to investigate properties of representations for g-frames with bounded operators.

2. G-Frame Representation Properties

In this section, we examine some properties of operator representations of g-frames, including being closed range, injective, unitary and compact.

In the following results, we first specify the range of adjoint of g-frame operator representations to indicates that the range of operator representations is closed. Then, we get necessary and sufficient conditions for g-frames to have injective operator representations.

Theorem 2.1. Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ be a g-frame that is represented by $T$. Then $\text{ran} T^* = \text{span} \{T^* \Lambda_i^* e_j\}_{i \in \mathbb{N}, j \in J}$, where $\{e_j\}_{j \in J}$ is an orthonormal basis for $\mathcal{K}$, and $\text{ran} T$ is close.

Proof. By Theorem 1.6, $\{\Lambda_i^* e_j\}_{i \in \mathbb{N}, j \in J}$ is a frame for $\mathcal{H}$, and so for every $f \in \mathcal{H}$, we have

$$T^* f = T^* \left( \sum_{i \in \mathbb{N}, j \in J} c_{ij} \Lambda_i^* e_j \right) = \sum_{i \in \mathbb{N}, j \in J} c_{ij} T^* \Lambda_i^* e_j.$$ 

Thus, $\text{ran} T^* \subseteq \text{span} \{T^* \Lambda_i^* e_j\}_{i \in \mathbb{N}, j \in J} := \mathcal{H}_0$. On the other hand, since $\{T^* \Lambda_i^* e_j\}_{i \in \mathbb{N}, j \in J}$, is a frame for $\mathcal{H}_0$, we have

$$g = \sum_{i \in \mathbb{N}, j \in J} d_{ij} T^* \Lambda_i^* e_j = T^* \left( \sum_{i \in \mathbb{N}, j \in J} c_{ij} \Lambda_i^* e_j \right), \quad g \in \mathcal{H}_0.$$
Then \( \text{ran}\ T^* = \mathcal{H}_0 \) is close and so \( \text{ran}\ T \) is close. \( \square \)

**Proposition 2.2.** Let \( \Lambda = \{ \Lambda_i T^{i-1} \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N} \} \) be a g-frame such that \( \| \Lambda_1 \| < \sqrt{\Lambda} \). Then \( T \) is injective.

**Proof.** For every \( f \in \mathcal{H} \),

\[
A_\Lambda \| f \|^2 \leq \sum_{i \in \mathbb{N}} \| \Lambda_i T^{i-1} f \|^2 \leq \| \Lambda_1 \|^2 (\| f \|^2 + \sum_{i \in \mathbb{N}} \| T^i f \|^2),
\]

thus \( \sum_{i \in \mathbb{N}} \| T^i f \|^2 \geq \left( \frac{A_\Lambda}{\| \Lambda_1 \|^2} - 1 \right) \| f \|^2 \) and since \( \frac{A_\Lambda}{\| \Lambda_1 \|^2} - 1 > 0 \), \( T \) is injective. \( \square \)

**Theorem 2.3.** Let \( \Lambda = \{ \Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N} \} \) be a g-frame that is represented by \( T \). Then the following are equivalent.

(i) \( T \) is injective.

(ii) \( \text{ran}(S^{-1}_\Lambda \Lambda_1^*) \cap \ker T = \{0\} \).

(iii) \( \text{ran} \Lambda_1^* \subseteq \text{ran} T^* \).

**Proof.** (i) \( \Rightarrow \) (ii) and (i) \( \Rightarrow \) (iii) are clear.

(ii) \( \Rightarrow \) (i) Suppose that \( T \) is not injective. Then there exists \( 0 \neq f \in \ker T \). We get

\[
f = \sum_{i \in \mathbb{N}} S^{-1}_\Lambda \Lambda_i^* \Lambda_i f = S^{-1}_\Lambda \Lambda_1^* \Lambda_1 f + \sum_{i \in \mathbb{N}} S^{-1}_\Lambda \Lambda_i^* \Lambda_i T f = S^{-1}_\Lambda \Lambda_1^* \Lambda_1 f.
\]

So \( f \in \text{ran}(S^{-1}_\Lambda \Lambda_1^*) \), which is a contradiction.

(iii) \( \Rightarrow \) (i) For any \( f \in \mathcal{H} \), we have

\[
f = \sum_{i \in \mathbb{N}} \Lambda_i^* \Lambda_i S^{-1}_\Lambda f = \Lambda_1^* \Lambda_1 S^{-1}_\Lambda f + \sum_{i \in \mathbb{N}} T^* \Lambda_i^* \Lambda_i S^{-1}_\Lambda f
\]

\[
= \Lambda_1^* \Lambda_1 S^{-1}_\Lambda f + T^* \left( \sum_{i \in \mathbb{N}} \Lambda_i^* \Lambda_i S^{-1}_\Lambda f \right).
\]

Since \( \text{ran} \Lambda_1^* \subseteq \text{ran} T^* \), \( f \in \text{ran} T^* \). Therefore \( T^* \) is surjective, and so \( T \) is injective. \( \square \)

The main purpose of the reminder of the paper is to show that the operator representation of g-frames can not be unitary and compact.

**Theorem 2.4.** Let \( \Lambda = \{ \Lambda_i T^{i-1} \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N} \} \) be a g-frame. Then for every \( f \in \mathcal{H} \), \( T^n f \to 0 \) as \( n \to \infty \).

**Proof.** For every \( n \in \mathbb{N} \) and \( f \in \mathcal{H} \), we have

\[
(2.1) \quad A_\Lambda \| T^n f \|^2 \leq \sum_{i \in \mathbb{N}} \| \Lambda_i T^{i-1+n} f \|^2 = \sum_{i=n}^{\infty} \| \Lambda_i T^i f \|^2.
\]
On the other hand, by $\sum_{i \in \mathbb{N}} \|\Lambda_i T_i^{-1} f\|^2 \leq B_\Lambda \|f\|^2$, we get $\sum_{i=n}^\infty \|\Lambda_i T_i f\|^2 \to 0$ as $n \to \infty$. Therefore, by the inequality (2.1), we conclude that $T^n f \to 0$ as $n \to \infty$. □

**Corollary 2.5.** For every unitary operator $T$ and every $\Lambda_1 \in B(\mathcal{H}, \mathcal{K})$, the sequence $\Lambda = \{\Lambda_1 T_i^{-1} \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ cannot be a $g$-frame.

**Proof.** For every $f \in \mathcal{H}$,

$$f \in B(\mathcal{H}, \mathcal{K}),$$

(2.2) \[ \|f\| = \|(T^*)^n T^nf\| \leq \|T^*\|^n \|T^n f\| = \|T^n f\|. \]

If $\Lambda$ is a $g$-frame, then by Theorem 2.4 $T^n f \to 0$ as $n \to \infty$, and so by the inequality (2.2), $\|f\| \to 0$, that is a contradiction. □

**Corollary 2.6.** Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ and $\Theta = \{\Theta_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ be two $g$-orthonormal bases. Then for every $\Gamma_i \in B(\mathcal{H}, \mathcal{K})$, the sequence $\Gamma = \{\Gamma_i S_{\Lambda_\Theta}^{-1} \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ is not a $g$-frame.

**Proof.** By Theorem 2.5 there exists a unitary operator $U \in B(\mathcal{H})$ such that $\Theta_i = \Lambda_i U$. We have

$$S_{\Lambda_\Theta} S_{\Lambda_\Theta}^* = T_\Lambda^* T_\Theta^* T_\Lambda = T_\Lambda^* U U^* T_\Lambda^* = S_{\Lambda_\Theta} \text{Id}_H S_{\Lambda_\Theta} = \text{Id}_H,$$

and similarly $S_{\Lambda_\Theta}^* S_{\Lambda_\Theta} = \text{Id}_H$. So $S_{\Lambda_\Theta}$ is a unitary operator on $\mathcal{H}$ and by Corollary 2.5 $\Gamma$ is not a $g$-frame for every $\Gamma_1 \in B(\mathcal{H}, \mathcal{K})$. □

**Proposition 2.7.** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. Assume that $T \in B(\mathcal{H}_1)$, $S \in B(\mathcal{H}_2)$ and $\Lambda \in B(\mathcal{H}_1, \mathcal{K})$, $\Theta \in B(\mathcal{H}_2, \mathcal{K})$ such that $T = V^{-1} S V$ and $\Theta V = \Lambda$ for some $V \in GL(\mathcal{H}_1, \mathcal{H}_2)$. Then $\{\Lambda T_i^{-1} \in B(\mathcal{H}_1, \mathcal{K}) : i \in \mathbb{N}\}$ is a $g$-frame, if and only if $\{\Theta S_i^{-1} \in B(\mathcal{H}_2, \mathcal{K}) : i \in \mathbb{N}\}$ is a $g$-frame. In the affirmative case $V$ is unique.

**Proof.** For every $f \in \mathcal{H}_1$, we have

$$\sum_{i \in \mathbb{N}} \|\Lambda T_i^{-1} f\|^2 = \sum_{i \in \mathbb{N}} \|\Theta V (V^{-1} S V) T_i^{-1} f\|^2 = \sum_{i \in \mathbb{N}} \|\Theta V V^{-1} S_i^{-1} V f\|^2 = \sum_{i \in \mathbb{N}} \|\Theta S_i^{-1} V f\|^2.$$

Since $V \in GL(\mathcal{H}_1, \mathcal{H}_2)$, the sequence $\{\Lambda T_i^{-1} \in B(\mathcal{H}_1, \mathcal{K}) : i \in \mathbb{N}\}$ is a $g$-frame, if and only if $\Lambda = \{\Theta S_i^{-1} \in B(\mathcal{H}_2, \mathcal{K}) : i \in \mathbb{N}\}$ is a $g$-frame.
Also, by Theorem 1.6 there exists \( \{c_{ij}\}_{i \in N, j \in J} \in \ell^2(\mathbb{C}, \mathbb{N}) \) such that

\[
(V^*)^{-1}f = (V^*)^{-1}\left( \sum_{i \in N, j \in J} c_{ij} (T^{i-1})^* \Lambda^* e_j \right)
\]

\[
= (V^*)^{-1}\left( \sum_{i \in N, j \in J} c_{ij} V^* (S^{i-1})^* (V^{-1})^* \Theta^* e_j \right)
\]

\[
= \sum_{i \in N, j \in J} c_{ij} (S^{i-1})^* \Theta^* e_j,
\]

which \( \{e_j\}_{j \in J} \) is an orthonormal basis for \( \mathcal{K} \).

\[ \square \]

**Proposition 2.8.** Let \( \Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\} \) be a g-frame. If for \( \Theta \in B(\mathcal{H}, \mathcal{K}) \) the sequence \( \{\Theta S^{-1}_\Lambda \} \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N} \) is a g-frame, then \( A_\Lambda < 1 \).

**Proof.** The proof is the same as the proof of the [22 Proposition 2.7].

\[ \square \]

In [22 Corollary 2.4], it has been shown that for Riesz basis \( \{T^{i-1} f_i\}_{i \in \mathbb{N}} \) the operator \( T \) cannot be surjective. While the following examples show that for the g-Riesz basis \( \{\Lambda_1 T^{i-1} \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\} \), the operator \( T \) can be injective.

**Example 2.9.**
(i) For \( \Lambda_1 \in GL(\mathcal{H}) \), the set \( \{\Lambda_1\} \) is a g-Riesz basis which is represented by \( Id_\mathcal{H} \).

(ii) By [22 Corollary 2.4], for a Riesz basis \( F = \{T^{i-1} f_i\} \), \( T \) is not surjective. Consider \( \Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathbb{C}) : i \in \mathbb{N}\} \), where \( \Lambda_i f = \langle f, f_i \rangle \). By Remark 1.7 \( \Lambda \) is represented by \( T^* \) which is not injective. On the other hand, for any \( i \in \mathbb{N}, \Lambda_i^*(1) = f_i \), and therefore by Theorem 1.6 \( \Lambda \) is a g-Riesz basis.

Theorem [11 Proposition 3.5] and [22 Proposition 2.2] show that for frame \( \{T^{i-1} f_i\}_{i \in \mathbb{N}} \), the operator \( T \) can not be compact. In the following, we show that in the finite space \( \mathcal{K} \) for g-frame \( \{\Lambda_1 T^{i-1} \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\} \) the operator \( T \) is not compact as well. By giving example, we show that this is not generally true.

**Proposition 2.10.** Let \( \Lambda = \{\Lambda_1 T^{i-1} \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\} \) be a g-frame, where \( \mathcal{K} \) is a finite-dimensional Hilbert space. Then \( T \) is not compact.

**Proof.** Let \( \{e_j\}_{j \in J} \) be an orthonormal basis for \( \mathcal{K} \) and \( T \) be compact. By Theorem 2.1 \( \text{ran} \ T^* = \text{Span} \{\Lambda_1^* e_j\}_{i \in N, j \in J} \), and therefore by [9 Lemma 2.5.1], there exists \( T^\dagger \in B(\mathcal{H}) \) such that \( T^* T^\dagger = Id_{\text{ran} \ T^*} \). Since \( T \) is compact, \( T^* \) is compact and so \( \text{ran} \ T^* \) is finite-dimensional. Consequently, \( \text{Span} \{\Lambda_1^* e_j\}_{i \in N, j \in J} \) is finite-dimensional and so by Theorem 1.6 \( \mathcal{H} \) is finite-dimensional, that is a contradiction. \[ \square \]
Example 2.11. Consider \( \Lambda_1 = 1d_{\ell^2(\mathcal{H}, \mathbb{N})} \) and \( T : \ell^2(\mathcal{H}, \mathbb{N}) \to \ell^2(\mathcal{H}, \mathbb{N}) \), that is defined by \( T\{a_j\}_{j \in J} = (\alpha a_1, 0, 0, ...) \) for a scalar \( \alpha \) with \( |\alpha| < 1 \). It is clear that \( T \) is compact and \( \Lambda = \{ \Lambda_1 T_i^{-1} \in B(\ell^2(\mathcal{H}, \mathbb{N})) : i \in \mathbb{N} \} \) is a \( g \)-frame. In fact, for every \( \{a_j\}_{j \in J} \in \ell^2(\mathcal{H}, \mathbb{N}) \), we have

\[
\|\{a_j\}_{j \in J}\|_2^2 \leq \sum_{i \in \mathbb{N}} \|\Lambda_1 T_i^{-1}\{a_j\}_{j \in J}\|_2^2 = \sum_{i \in \mathbb{N}} \|T_i^{-1}\{a_j\}_{j \in J}\|_2^2
= \|\{a_j\}_{j \in J}\|_2^2 + \sum_{i \in \mathbb{N}} \|(\alpha^i a_1, 0, 0, ...\|_2^2
\leq \frac{1}{1 - \alpha^2}\|\{a_j\}_{j \in J}\|_2^2.
\]

Theorem 2.12. Let \( \Lambda = \{ \Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N} \} \) be a \( g \)-Riesz sequence and \( \Theta = \{ \Theta_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N} \} \) be a sequence of operators, where \( \alpha := \sum_{i \in \mathbb{N}} \|\Lambda_i - \Theta_i\| \|\Lambda_i S_{\Lambda}^{-1}\| < 1 \) and \( \beta := \sum_{i \in \mathbb{N}} \|\Lambda_i - \Theta_i\|^2 < \infty \). Then \( \Theta \) is a \( g \)-Riesz sequence.

Proof. For every \( \{g_i\}_{i \in \mathbb{N}} \in \ell^2(\mathcal{K}, \mathbb{N}) \), we have

\[
\|\sum_{i \in \mathbb{N}} \Theta_i^* g_i\| = \|\sum_{i \in \mathbb{N}} (\Theta_i^* - \Lambda_i^*) g_i + \sum_{i \in \mathbb{N}} \Lambda_i^* g_i\|
\leq \sum_{i \in \mathbb{N}} \|\Theta_i^* - \Lambda_i^*\| \|g_i\| + \|\sum_{i \in \mathbb{N}} \Lambda_i^* g_i\|
\leq \left( \sum_{i \in \mathbb{N}} \|\Theta_i - \Lambda_i\|^2 \right)^{\frac{1}{2}} \|\{g_i\}_{i \in \mathbb{N}}\|_{\ell^2(\mathcal{K}, \mathbb{N})} + \sqrt{B_\Lambda} \|\{g_i\}_{i \in \mathbb{N}}\|_{\ell^2(\mathcal{K}, \mathbb{N})}
\leq (\sqrt{\beta} + \sqrt{B_\Lambda}) \|\{g_i\}_{i \in \mathbb{N}}\|_{\ell^2(\mathcal{K}, \mathbb{N})}.
\]

So for well-defined operator \( U : \mathcal{H} \to \mathcal{H} \), defined by

\[
U f = \sum_{i \in \mathbb{N}} \Theta_i^* \left( \sum_{j \in J} \langle f, S_{\Lambda}^{-1} \Lambda_i e_j e_j \rangle \right).
\]
we have
\[
\|Uf\| = \left\| \sum_{i \in \mathbb{N}} \Theta_i^* \left( \sum_{j \in J} \langle f, S_{\Lambda_i}^{-1} \Lambda_i^* e_j \rangle e_j \right) \right\|
\leq (\sqrt{\beta} + \sqrt{B_{\Lambda}}) \left\| \left\{ \sum_{j \in J} \langle f, S_{\Lambda_i}^{-1} \Lambda_i^* e_j \rangle e_j \right\}_{i \in \mathbb{N}} \right\|_{\ell^2(\mathbb{K}, \mathbb{N})}
\leq (\sqrt{\beta} + \sqrt{B_{\Lambda}}) \left\| \left\{ \sum_{j \in J} \langle f, P_M S_{\Lambda_i}^{-1} \Lambda_i^* e_j \rangle e_j \right\}_{i \in \mathbb{N}} \right\|_{\ell^2(\mathbb{K}, \mathbb{N})}
\leq \frac{\sqrt{\beta} + \sqrt{B_{\Lambda}}}{\sqrt{A_\Lambda}} \|P_M f\| \leq \frac{\sqrt{\beta} + \sqrt{B_{\Lambda}}}{\sqrt{A_\Lambda}} \|f\|,
\]
where \( M = \text{span}\{\Lambda_i^* e_j\}_{i \in \mathbb{N}, j \in J} \). Note that the operator \( U \) on \( M \) is equal to \( S_{\Theta_i} S_{\Lambda_i}^{-1} \). On the other hand, for every \( k \in \mathbb{N} \), we have
\[
\langle U \Lambda_k^* g, f \rangle = \sum_{i \in \mathbb{N}} \langle \Theta_i^* \Lambda_i S_{\Lambda_i}^{-1} \Lambda_i^* g, f \rangle = \sum_{i \in \mathbb{N}} \langle \Lambda_k^* g, S_{\Lambda_i}^{-1} \Lambda_i^* \Theta_i f \rangle
\leq \langle g, \Theta_k f \rangle = \langle \Theta_k^* g, f \rangle, \quad f \in \mathcal{H}, g \in \mathcal{K},
\]
which implies \( U \Lambda_k^* = \Theta_k^* f. \) Also
\[
\|f - Uf\| = \left\| \sum_{i \in \mathbb{N}} \Lambda_i^* \Lambda_i S_{\Lambda_i}^{-1} f - \sum_{i \in \mathbb{N}} \Theta_i^* \Lambda_i S_{\Lambda_i}^{-1} f \right\|
\leq \sum_{i \in \mathbb{N}} \|\Lambda_i - \Theta_i\| \|\Lambda_i S_{\Lambda_i}^{-1} f\|
\leq \sum_{i \in \mathbb{N}} \|\Lambda_i - \Theta_i\| \|\Lambda_i S_{\Lambda_i}^{-1}\| \|f\| = \alpha \|f\|, \quad f \in M,
\]
and so we get \( \|Uf\| \geq (1 - \alpha) \|f\| \). Consequently, for any finite sequence \( \{g_i\} \subseteq \mathcal{K} \)
\[
\left\| \sum_{i \in \mathbb{N}} \Theta_i^* g_i \right\| = \left\| \sum_{i \in \mathbb{N}} U \Lambda_i^* g_i \right\| = \|U \sum_{i \in \mathbb{N}} \Lambda_i^* g_i\|
\geq (1 - \alpha) \| \sum_{i \in \mathbb{N}} \Lambda_i^* g_i \| \geq (1 - \alpha) \sqrt{A_\Lambda} \left( \sum_{i \in \mathbb{N}} \|g_i\|^2 \right)^{1/2}.
\]
\[\square\]

**Theorem 2.13.** Let \( \Lambda = \{\Lambda_i T^{i-1} \in B(\mathcal{H}, \mathcal{K}); i \in \mathbb{N} \} \) be a g-Riesz sequence and for \( \Theta_1 \in B(\mathcal{H}, \mathcal{K}) \) there exists \( \mu \in [0, 1) \) such that \( \|\Theta_1 T^*\| \leq \mu \|\Theta_1\| \) and \( \|\Theta_1\| < (1 - \mu) \sqrt{A_\Lambda} \). Then \( \{\Lambda_1 + \Theta_1 T^{i-1} \in B(\mathcal{H}, \mathcal{K}); i \in \mathbb{N} \} \) is a g-Riesz sequence.
Proof. It is sufficient to examine the conditions of Theorem 2.12 for the sequence \( \{(\Lambda_1 + \Theta_i)T^{i-1} \in B(\mathcal{H}, \mathcal{K}); i \in \mathbb{N}\} \).

\[
\sum_{i \in \mathbb{N}} \| (\Lambda_1 + \Theta_i)T^{i-1} - \Lambda_1T^{i-1} \|^2 = \sum_{i \in \mathbb{N}} \| \Theta_i T^{i-1} \|^2 \\
\leq \sum_{i \in \mathbb{N}} \mu^{2i-2} \| \Theta_1 \|^2 = \frac{\| \Theta_1 \|^2}{1 - \mu^2}.
\]

Also, by Remark ??

\[
\sum_{i \in \mathbb{N}} \| \Theta_i T^{i-1} \| \| \Lambda_1 S_A^{-1} \| \leq \frac{\| \Theta_1 \|}{(1 - \mu) \sqrt{A}} < 1.
\]

\[\square\]

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