On the stability of Bresse system with one discontinuous local internal Kelvin–Voigt damping on the axial force

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Abstract. In this paper, we investigate the stabilization of a linear Bresse system with one discontinuous local internal viscoelastic damping of Kelvin–Voigt type acting on the axial force, under fully Dirichlet boundary conditions. First, using a general criteria of Arendt–Batty, we prove the strong stability of our system. Finally, using a frequency domain approach combined with the multiplier method, we prove that the energy of our system decays polynomially with different rates.

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1. Introduction

In this paper, we investigate the stability of Bresse system with only one discontinuous local internal Kelvin–Voigt damping on the axial force. More precisely, we consider the following system:

\[
\begin{cases}
\rho_1 \varphi_{tt} - k_1 (\varphi_x + \psi + lw)_x - lk_3 (w_x - l \varphi) - ld(x)(w_{tx} - l \varphi_t) = 0, & (x, t) \in (0, L) \times (0, \infty), \\
\rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1 (\varphi_x + \psi + lw) = 0, & (x, t) \in (0, L) \times (0, \infty), \\
\rho_1 w_{tt} - [k_3 (w_x - l \varphi) + d(x)(w_{tx} - l \varphi_t)]_x + lk_1 (\varphi_x + \psi + lw) = 0, & (x, t) \in (0, L) \times (0, \infty),
\end{cases}
\]

(1.1)

with the following Dirichlet boundary conditions

\[
\varphi(0, t) = \varphi(L, t) = \psi(0, t) = \psi(L, t) = w(0, t) = w(L, t) = 0, \quad t > 0,
\]

(1.2)

and the following initial conditions

\[
\begin{cases}
\varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), \quad \psi(x, 0) = \psi_0(x), \quad x \in (0, L), \\
\psi_t(x, 0) = \psi_1(x), \quad w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), \quad x \in (0, L),
\end{cases}
\]

(1.3)
where \( \rho_1, \rho_2, k_1, k_2, k_3, l \) and \( L \) are positive real numbers. We suppose that there exists \( 0 < \alpha < \beta < L \) and a positive constant \( d_0 \) such that
\[
d(x) = \begin{cases} 
  d_0 & \text{if } x \in (\alpha, \beta), \\
  0 & \text{if } x \in (0, \alpha) \cup (\beta, L).
\end{cases} \tag{1.4}
\]

The Bresse system is a model for arched beams, see [29, Chap. 6]. It can be expressed by the equations of motion:
\[
\begin{align*}
\rho_1 \varphi_{tt} &= Q_x + lN, \\
\rho_2 \psi_{tt} &= M_x - Q, \\
\rho_1 w_{tt} &= N_x - lQ,
\end{align*}
\tag{1.5}
\]
where \( N = k_3(w_x - l\varphi) + d(x)(w_{tx} - l\varphi_t) \) is the axial force, \( Q = k_1(\varphi_x + \psi + lw) \) is the shear force, and \( M = k_2\psi_x \) is the bending moment. The functions \( \varphi, \psi, \) and \( w \) are, respectively, the vertical, shear angle, and longitudinal displacements. Here, \( \rho_1 = \rho A, \rho_2 = \rho I, k_1 = kGA, k_3 = EA, k_2 = EI \) and \( l = R^{-1}, \) in which \( \rho \) is the density of the material, \( E \) the modulus of the elasticity, \( G \) the shear modulus, \( k \) the shear factor, \( A \) the cross-sectional area, \( I \) the second moment of area of the cross section, \( R \) the radius of the curvature, and \( g \) the curvature.

There are several publications concerning the stabilization of Bresse system with different kinds of damping (see [1,2,5,13-17,22,23,32,35,37,43]). We note that by neglecting \( w \) \((l \to 0)\) in (1.5), the Bresse system reduces to the following conservative Timoshenko system:
\[
\begin{align*}
\rho_1 \varphi_{tt} - k_1(\varphi_x + \psi)x &= 0, \\
\rho_2 \psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi) &= 0.
\end{align*}
\]

There are also several publications concerning the stabilization of Timoshenko system with different kinds of damping (see [4,9,10,42]).

In the recent years, many researchers showed interest in problems involving Kelvin–Voigt damp ing where different types of stability, depending on the smoothness of the damping coefficients, have been showed (see [6,7,24,25,27,30,33,34,38,40]). Moreover, there is a number of new results concerning systems with local Kelvin–Voigt damping and non-smooth coefficients at the interface (see [3,19-21,26,36,41]).

Among this vast literature, let us recall some specific results on the Bresse systems.

In 2017, Guesmia in [22] studied the stability of Bresse system with one infinite memory in the longitudinal displacement (i.e., third equation) under Dirichlet–Neumann–Neumann boundary conditions, he established some stability results under a smallness condition on \( l \) and on \( \int_0^\infty g(s)ds, \) where \( l \) is the curvature and \( g \) is the memory kernel. In 2018, Afilal et al. in [2] studied the stability of Bresse system with global frictional damping in the longitudinal displacement, by considering the following system on \((0, 1) \times (0, \infty)\):
\[
\begin{align*}
\rho_1 \varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - l\varphi) &= 0, \\
\rho_2 \psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) &= 0, \\
\rho_1 w_{tt} - k_3(w_x - l\varphi) + lk_1(\varphi_x + \psi + lw) + \delta w_t &= 0,
\end{align*}
\tag{1.6}
\]
with the initial conditions (1.3) where \( L = 1 \) and under mixed boundary conditions of the form:
\[
\begin{align*}
\varphi(0, t) = \varphi_x(0, t) = w_x(0, t) &= 0, \quad \text{in} \quad (0, \infty), \\
\varphi_x(1, t) = \psi(1, t) = w(1, t) &= 0, \quad \text{in} \quad (0, \infty),
\end{align*}
\]
where \( \delta \) is a positive real number, they assumed that:
\[
l \neq \frac{\pi}{2} + m\pi, \quad \forall m \in \mathbb{N}. \tag{1.7}
\]
They proved under (1.7) the strong stability of system (1.6) provided that the curvature \( l \) satisfies:
\[
l^2 \neq \frac{\rho_2 k_3 + \rho_1 k_2}{\rho_2 k_3} \left( \frac{\pi}{2} + m\pi \right)^2 + \frac{\rho_1 k_1}{\rho_2 (k_1 + k_3)}, \quad \forall m \in \mathbb{Z}. \tag{1.8}
\]
Also, they established under (1.7) and (1.8) the exponential stability of system (1.6) if and only if \( \frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} = \frac{k_3}{\rho_1} \). Otherwise, they established polynomial energy decay rate of order \( t^{-\frac{1}{2}} \). In 2019, Fatori et al. in [15] proved under (1.9), (1.10) and (1.11) the exponential stability of Bresse system with indefinite memory in the longitudinal displacement under Dirichlet–Neumann–Neumann boundary conditions.

Moreover, they used the previous results (i.e., strong and exponential stability of (1.6) on \((0,L) \times (0,\infty)\)) to obtain under (1.9), (1.10) and (1.11) the exponential stability of Bresse system with indefinite memory in the longitudinal displacement under Dirichlet–Neumann–Neumann boundary conditions.

In 2019, El Arwadi and Youssef in [14] studied the stabilization of the Bresse beam with three global Kelvin–Voigt dampings under fully Dirichlet boundary conditions, and they established an exponential energy decay rate. In 2020, Gerbi et al. in [18] studied the stabilization of non-smooth transmission problem involving Bresse systems with fully Dirichlet or Dirichlet–Neumann–Neumann boundary conditions, by considering system (1.5) on \((0,L) \times (0,\infty)\) with

\[
N = k_3(w_x - l\varphi) + D_3(w_{xt} - l\varphi_t), \quad Q = k_1(\varphi_x + \psi + lw) + D_1(\varphi_{xt} + \psi_t + lw_t), \quad M = k_2\psi_x + D_2\psi_{xt},
\]

where \( D_1, D_2 \) and \( D_3 \) are bounded positive functions over \((0,L)\). They established:

- Analytic stability in the case of three global Kelvin–Voigt dampings (i.e., \( D_i \in L^{\infty}(0,L), D_i \geq d_0 > 0 \) in \((0,L), i = 1,2,3\)).
- Exponential stability in the case of three local Kelvin–Voigt dampings with smooth coefficients at the interface (i.e., \( D_i \in W^{1,\infty}(0,L), D_i \geq d_0 > 0 \) in \( \emptyset \neq \omega := (\alpha, \beta) \subset (0,L), i = 1,2,3 \)).
- Polynomial energy decay rate of order \( t^{-1} \) in the case of three local Kelvin–Voigt dampings with non-smooth coefficients at the interface (i.e., \( D_i \in L^{\infty}(0,L), D_i \geq d_0 > 0 \) in \((\alpha_i, \beta_i) \subset (0,L), i = 1,2,3, \) and \( \bigcap_{i=1}^3 (\alpha_i, \beta_i) = \omega \)).

- Polynomial stability energy decay rate of order \( t^{-\frac{1}{2}} \) in the case of one local Kelvin–Voigt damping on the bending moment with non-smooth coefficient at the interface (i.e., \( D_1 = D_3 = 0, D_2 \in L^{\infty}(0,L) \) and \( D_2 \geq d_0 > 0 \) in \( \omega \)).

But to the best of our knowledge, it seems that no result in the literature exists concerning the case of Bresse system with only one discontinuous local internal Kelvin–Voigt damping on the axial force, especially under fully Dirichlet boundary conditions and without any condition on the curvature \( l \). The goal of the present paper is to fill this gap by studying the stability of system (1.1)–(1.3).

This paper is organized as follows: In Sect. 2, we prove the well-posedness of our system by using semigroup approach. In Sect. 3, following a general criteria of Arendt Batty, we show the strong stability of our system in the absence of the compactness of the resolvent. Finally, in Sect. 4, by using the frequency
domain approach combining with a specific multiplier method, we prove that the energy of our system decays polynomially with the rates:

\[
\begin{align*}
& \frac{t^{-1}}{2} \quad \text{if} \quad \frac{k_1}{\rho_1} = \frac{k_2}{\rho_2}, \\
& \frac{t^{-1}}{2} \quad \text{if} \quad \frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2}.
\end{align*}
\]

2. Well-posedness of the system

In this section, we will establish the well-posedness of system (1.1)–(1.3) by using semigroup approach. The energy of system (1.1)–(1.3) is given by

\[
E(t) = \frac{1}{2} \int_0^L \left( \rho_1 |\varphi_t|^2 + \rho_2 |\psi_t|^2 + \rho_1 |w_t|^2 + k_1 |\varphi_x + \psi + lw|^2 + k_2 |\psi_x|^2 + k_3 |w_x - l\varphi|^2 \right) dx.
\]

Let \((\varphi, \varphi_t, \psi, \psi_t, w, w_t)\) be a regular solution of system (1.1)–(1.3). Multiplying the equations in (1.1) by \(\varphi_t, \psi_t, w_t\), respectively, Then, using the boundary conditions (2.14) and the definition of \(d(x)\) (see (1.4) and Fig. 1), we obtain

\[
E'(t) = -\int_0^L d(x)|w_{tx} - l\varphi_t|^2 dx = -d_0 \int_\alpha^\beta |w_{tx} - l\varphi_t|^2 dx \leq 0. \tag{2.1}
\]

From (2.1), system (1.1)–(1.3) is dissipative in the sense that its energy is non-increasing with respect to time. Now, we define the following Hilbert space \(\mathcal{H}\) by:

\[
\mathcal{H} := \left( H_0^1(0, L) \times L^2(0, L) \right)^3.
\]

The Hilbert space \(\mathcal{H}\) is equipped with the following inner product and norm

\[
(U, U^1)_{\mathcal{H}} = \int_0^L \left\{ k_1(v^1_x + v^3 + lv^5)(v^1_x + v^3 + lv^5) + \rho_1 v^2 v^2 + k_2 v^3 v^3 + \rho_2 v^4 v^4 \\
+ k_3(v^5_x - lv^1)(v^5_x - lv^1)dx + \rho_1 v^6 v^6 \right\} dx
\]

and

\[
\|U\|_{\mathcal{H}}^2 = \int_0^L \left( k_1|v^1_x + v^3 + lv^5|^2 + \rho_1|v^2|^2 + k_2|v^3|^2 + \rho_2|v^4|^2 + k_3|v^5_x - lv^1|^2 + \rho_1|v^6|^2 \right) dx. \tag{2.2}
\]
where $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in \mathcal{H}$ and $\bar{U} = (\bar{v}^1, \bar{v}^2, \bar{v}^3, \bar{v}^4, \bar{v}^5, \bar{v}^6)^\top \in \mathcal{H}$. Now, we define the linear unbounded operator $A : D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ by:

$$D(A) = \left\{ U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in \mathcal{H} \mid v^1, v^3 \in H^2(0, L) \cap H^3_0(0, L), \quad \left[ k_3 v^5_x + d(x)(v^6_x - lv^2) \right]_x \in L^2(0, L) \right\}$$  \hspace{1cm} (2.3)

and

$$A \begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \\ v^5 \\ v^6 \end{pmatrix} = \begin{pmatrix} k_1 (v^1_x + v^3 + lv^5)_x + \frac{lk_3}{\rho_1} (v^5_x - lv^1) + \frac{ld(x)}{\rho_1} (v^6_x - lv^2) \\ k_2 v^3_x - \frac{k_3}{\rho_2} (v^1_x + v^3 + lv^5) \\ 1 \frac{k_3 (v^5_x - lv^1) + d(x)(v^6_x - lv^2)}{\rho_1} \left( v^1_x + v^3 + lv^5 \right) \end{pmatrix},$$  \hspace{1cm} (2.4)

for all $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(A)$.

In this sequel, $\| \cdot \|$ will denote the usual norm of $L^2(0, L)$.

**Remark 2.1.** From Poincaré inequality, we deduce that there exists a positive constant $c_1$ such that

$$k_1 \| v^1_x + v^3 + lv^5 \|^2 + k_2 \| v^3 \|^2 + k_3 \| v^5_x - lv^1 \|^2 \leq c_1 \left( \| v^1_x \|^2 + \| v^3 \|^2 + \| v^5 \|^2 \right), \quad \forall (v^1, v^3, v^5) \in (H^3_0(0, L))^3.$$

Moreover, we can show by a contradiction argument that there exists a positive constant $c_2$ such that

$$c_2 \left( \| v^1_x \|^2 + \| v^3 \|^2 + \| v^5 \|^2 \right) \leq k_1 \| v^1_x + v^3 + lv^5 \|^2 + k_2 \| v^3 \|^2 + k_3 \| v^5_x - lv^1 \|^2, \quad \forall (v^1, v^3, v^5) \in (H^3_0(0, L))^3.$$

Therefore, the norm defined in (2.2) is equivalent to the usual norm of $\mathcal{H}$. \hfill \Box

Now, if $U = (\varphi, \varphi_t, \psi, \psi_t, w, w_t)^\top$, then system (1.1)–(1.3) can be written as the following first-order evolution equation

$$U_t = AU, \quad U(0) = U_0,$$  \hspace{1cm} (2.5)

where $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1)^\top \in \mathcal{H}$.

**Proposition 2.1.** The unbounded linear operator $A$ is $m$-dissipative in the Hilbert space $\mathcal{H}$.

**Proof.** For all $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(A)$, we have

$$\Re(AU, U)_{\mathcal{H}} = -\int_0^L d(x) \left| v^5_x - lv^2 \right|^2 \, dx = -d_0 \int_\alpha^\beta \left| v^6_x - lv^2 \right|^2 \, dx \leq 0.$$  \hspace{1cm} (2.6)

which implies that $A$ is dissipative. Let us prove that $A$ is maximal. For this aim, let $F = (f^1, f^2, f^3, f^4, f^5, f^6)^\top \in \mathcal{H}$, we look for $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(A)$ unique solution of

$$-AU = F.$$  \hspace{1cm} (2.7)

Detailing (2.7), we obtain

$$-v^2 = f^1,$$  \hspace{1cm} (2.8)

$$-k_1 (v^1_x + v^3 + lv^5)_x - lk_3 (v^5_x - lv^1) - ld(x)(v^6_x - lv^2) = \rho_1 f^2,$$  \hspace{1cm} (2.9)

$$-v^4 = f^3,$$  \hspace{1cm} (2.10)

$$-k_2 v^3_x + k_1 (v^1_x + v^3 + lv^5) = \rho_2 f^4,$$  \hspace{1cm} (2.11)

$$-v^6 = f^5,$$  \hspace{1cm} (2.12)

$$-k_3 (v^5_x - lv^1) + d(x)(v^6_x - lv^2)_x + lk_1 (v^1_x + v^3 + lv^5) = \rho_4 f^6.$$  \hspace{1cm} (2.13)
with the following boundary conditions
\[ v^1(0) = v^1(L) = v^3(0) = v^3(L) = v^5(0) = v^5(L) = 0. \] (2.14)

By inserting (2.8) and (2.12) in (2.9) and (2.13), system (2.8)–(2.13) implies:
\[ -k_1 \left( v_x^1 + v^3 + lv^5 \right)_x - l k_3 (v_x^5 - lv^1) = \rho_1 f^2 + l d(x)(-f^5_x + l f^1), \] (2.15)
\[ -k_2 v_x^3 + k_1 (v_x^1 + v^3 + lv^5) = \rho_2 f^4, \] (2.16)
\[ - \left[ k_3 (v_x^5 - lv^1) + d(x)(-f^5_x + l f^1) \right]_x + l k_1 (v_x^1 + v^3 + lv^5) = \rho_1 f^6. \] (2.17)

Let \((\phi^1, \phi^2, \phi^3) \in (H_0^1(0, L))^3\). Multiplying (2.15), (2.16) and (2.17) by \(\overline{\phi^1}, \overline{\phi^2}, \overline{\phi^3}\), respectively, integrating over \((0, L)\), then using formal integrations by parts, we obtain
\[ \mathcal{B}((v^1, v^3, v^5), (\phi^1, \phi^2, \phi^3)) = \mathcal{L}((\phi^1, \phi^2, \phi^3)), \quad \forall (\phi^1, \phi^2, \phi^3) \in (H_0^1(0, L))^3, \] (2.18)
where
\[
\mathcal{B}((v^1, v^3, v^5), (\phi^1, \phi^2, \phi^3)) = k_1 \int_0^L (v_x^1 + v^3 + lv^5) \overline{\phi_x^1} \, dx - l k_3 \int_0^L (v_x^5 - lv^1) \overline{\phi_x^1} \, dx + k_2 \int_0^L v_x^3 \overline{\phi_x^3} \, dx
\]
\[ + k_1 \int_0^L (v_x^1 + v^3 + lv^5) \overline{\phi_x^2} \, dx + k_3 \int_0^L (v_x^5 - lv^1) \overline{\phi_x^2} \, dx + l k_1 \int_0^L (v_x^1 + v^3 + lv^5) \overline{\phi_x^3} \, dx \]

and
\[
\mathcal{L}((\phi^1, \phi^2, \phi^3)) = \rho_1 \int_0^L f^2 \overline{\phi_x^1} \, dx + l \int_0^L d(x)(-f^5_x + l f^1) \overline{\phi_x^1} \, dx + \rho_2 \int_0^L f^4 \overline{\phi_x^2} \, dx
\]
\[ + \int_0^L d(x)(f^5_x - l f^1) \overline{\phi_x^3} \, dx + \rho_1 \int_0^L f^6 \overline{\phi_x^3} \, dx. \]

It is easy to see that \(\mathcal{B}\) is a sesquilinear and continuous form on \((H_0^1(0, L))^3 \times (H_0^1(0, L))^3\) and \(\mathcal{L}\) is a linear and continuous form on \((H_0^1(0, L))^3\). In fact, from Remark 2.1, we deduce that there exists a positive constant \(c\) such that
\[
\mathcal{B}((v^1, v^3, v^5), (v^1, v^3, v^5)) = k_1 \|v_x^1 + v^3 + lv^5\|^2 + k_2 \|v_x^3\|^2 + k_3 \|v_x^5 - lv^1\|^2
\]
\[ \geq c \left( \|v_x^1\|^2 + \|v_x^3\|^2 + \|v_x^5\|^2 \right) \]
\[ = c \|v^1, v^3, v^5\|_{(H_0^1(0, L))^3}^2. \] (2.19)

Thus, \(\mathcal{B}\) is a coercive form on \((H_0^1(0, L))^3 \times (H_0^1(0, L))^3\). Then, it follows by Lax–Milgram theorem that (2.18) admits a unique solution \((v^1, v^3, v^5) \in (H_0^1(0, L))^3\). By taking test-functions \((\phi^1, \phi^2, \phi^3) \in (D(0, L))^3\), we see that (2.15)–(2.17) hold in the distributional sense, from which we deduce that \((v^1, v^3) \in (H^2(0, L) \cap H_0^1(0, L))^2\), while \([k_3 v_x^5 + d(x)(v_x^5 - lv^1)]_x \in L^2(0, L)\). Consequently, \(U = (v^1, -f^1, v^3, -f^3, v^5, -f^5) \in D(A)\) is the unique solution of (2.7). Then, \(A\) is an isomorphism and since \(\rho (A)\) is open set of \(C\) [see Theorem 6.7 (Chapter III) in [28]], we easily get \(R(\lambda I - A) = \mathcal{H}\) for a sufficiently small \(\lambda > 0\). This, together with the dissipativeness of \(A\), implies that \(D(A)\) is dense in \(\mathcal{H}\) and that \(A\) is m-dissipative in \(\mathcal{H}\) (see Theorems 4.5, 4.6 in [39]). The proof is thus complete. \(\square\)
According to Lumer–Phillips theorem (see [39]), Proposition 2.1 implies that the operator $A$ generates a $C_0$-semigroup of contractions $e^{tA}$ in $\mathcal{H}$ which gives the well-posedness of (2.5). Then, we have the following result:

**Theorem 2.1.** For all $U_0 \in \mathcal{H}$, system (2.5) admits a unique weak solution

$$U(t) = e^{tA}U_0 \in C^0(\mathbb{R}^+, \mathcal{H}).$$

Moreover, if $U_0 \in D(A)$, then the system (2.5) admits a unique strong solution

$$U(t) = e^{tA}U_0 \in C^0(\mathbb{R}^+, D(A)) \cap C^1(\mathbb{R}^+, \mathcal{H}).$$

### 3. Strong stability

In this section, we will prove the strong stability of system (1.1)–(1.3). The main result of this section is the following theorem.

**Theorem 3.1.** The $C_0$-semigroup of contraction $(e^{tA})_{t \geq 0}$ is strongly stable in $\mathcal{H}$; i.e., for all $U_0 \in \mathcal{H}$, the solution of (2.5) satisfies

$$\lim_{t \to +\infty} \|e^{tA}U_0\|_\mathcal{H} = 0.$$

According to Theorem A.2, to prove Theorem 3.1, we need to prove that the operator $A$ has no pure imaginary eigenvalues and $\sigma(A) \cap i\mathbb{R}$ is countable. The proof of Theorem 3.1 has been divided into the following two Lemmas.

**Lemma 3.1.** For all $\lambda \in \mathbb{R}$, $i\lambda I - A$ is injective, i.e.,

$$\ker(i\lambda I - A) = \{0\}, \quad \forall \lambda \in \mathbb{R}.$$

**Proof.** From Proposition 2.1, we have $0 \in \rho(A)$. We still need to show the result for $\lambda \in \mathbb{R}^*$. For this aim, suppose that $\lambda \neq 0$ and let $U = (v^1, v^2, v^3, v^4, v^5, v^6)^T \in D(A)$ such that

$$AU = i\lambda U. \tag{3.1}$$

Equivalently, we have the following system

$$v^2 = i\lambda v^1, \tag{3.2}$$

$$k_1(v^1_x + v^3 + lv^5) + k_3(v^5_x - lv^1) + ld(x)(v^6_x - lv^2) = i\lambda \rho_1 v^2, \tag{3.3}$$

$$v^4 = i\lambda v^3, \tag{3.4}$$

$$k_2v^3_x - k_1(v^1_x + v^3 + lv^5) = i\lambda \rho_2 v^4, \tag{3.5}$$

$$v^6 = i\lambda v^5, \tag{3.6}$$

$$[k_3(v^5_x - lv^1) + d(x)(v^6_x - lv^2)] - l k_1(v^1_x + v^3 + lv^5) = i\lambda \rho_1 v^6. \tag{3.7}$$

From (2.6), (3.1) and the definition of $d(x)$, we obtain

$$0 = \Re(i\lambda U, U)_{\mathcal{H}} = \Re(AU, U)_{\mathcal{H}} = -\int_0^L d(x) |v^6_x - lv^2|^2 \, dx = -d_0 \int_{\alpha}^{\beta} |v^6_x - lv^2|^2 \, dx. \tag{3.8}$$

Thus, we have

$$v^6_x - lv^2 = 0 \quad \text{in} \quad (\alpha, \beta). \tag{3.9}$$

Inserting (3.2) and (3.6) in (3.9) and using the fact that $\lambda \neq 0$, we get

$$v^5_x - lv^1 = 0 \quad \text{in} \quad (\alpha, \beta). \tag{3.10}$$
Now, inserting (3.9) and (3.10) in (3.3) and (3.7), then inserting (3.2), (3.4) and (3.6) in (3.3), (3.5) and (3.7), respectively, we deduce that

\[
\begin{align*}
\rho_1 \lambda^2 v^1 + k_1 (v_x^1 + v^3 + lv^5) &= 0 \text{ in } (\alpha, \beta), \\
\rho_2 \lambda^2 v^3 + k_2 v_{xx}^3 - k_1 (v_x^1 + v^3 + lv^5) &= 0 \text{ in } (\alpha, \beta), \\
\rho_1 \lambda^2 v^5 - lk_1 (v_x^1 + v^3 + lv^5) &= 0 \text{ in } (\alpha, \beta).
\end{align*}
\]  

(3.11)

(3.12)

(3.13)

Deriving (3.13) with respect to \(x\), we get

\[
\rho_1 \lambda^2 v_x^5 - lk_1 (v_x^1 + v^3 + lv^5) = 0 \text{ in } (\alpha, \beta).
\]

Inserting (3.11) in the above equation, we get

\[
\rho_1 \lambda^2 (v_x^5 + lv^1) = 0 \text{ in } (\alpha, \beta) \text{ and consequently as } \lambda \neq 0, \text{ we get } v_x^5 + lv^1 = 0 \text{ in } (\alpha, \beta).
\]  

(3.14)

Now, adding (3.10) and (3.14), we obtain

\[
v_x^5 = 0 \text{ in } (\alpha, \beta) \text{ and consequently } v^1 = 0 \text{ in } (\alpha, \beta).
\]  

(3.15)

Inserting (3.15) in (3.11), we get

\[
v_x^3 = 0 \text{ in } (\alpha, \beta).
\]  

(3.16)

Now, system (3.2)–(3.7) can be written in \((0, \alpha) \cup (\beta, L)\) as the following:

\[
\begin{align*}
\rho_1 \lambda^2 v^1 + k_1 (v_x^1 + v^3 + lv^5) + lk_3 (v_x^5 - lv^1) &= 0 \text{ in } (0, \alpha) \cup (\beta, L), \\
\rho_2 \lambda^2 v^3 + k_2 v_{xx}^3 - k_1 (v_x^1 + v^3 + lv^5) &= 0 \text{ in } (0, \alpha) \cup (\beta, L), \\
\rho_1 \lambda^2 v^5 - k_3 (v_x^5 - lv^1) - lk_1 (v_x^1 + v^3 + lv^5) &= 0 \text{ in } (0, \alpha) \cup (\beta, L).
\end{align*}
\]  

(3.17)

(3.18)

(3.19)

Let \(V = (v_x^1, v_{xx}^1, v_x^3, v_{xx}^3, v_x^5, v_{xx}^5)^\top\). From (3.15), (3.16) and the regularity of \(v^i, i \in \{1, 3, 5\}\), we have \(V(\alpha) = 0\). Now, by deriving system (3.17)–(3.19) with respect to \(x\) in \((0, \alpha)\), we deduce that

\[
V_x = A_\lambda V \text{ in } (0, \alpha),
\]  

(3.20)

where

\[
A_\lambda = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
\frac{i^2 k_3 - \lambda^2 \rho_1}{k_1} & 0 & 0 & -1 & 0 & -l(1 + \frac{k_3}{k_1}) \\
0 & \frac{k_3}{k_1} & \frac{k_3 - \rho_2 \lambda^2}{k_1} & 0 & \frac{l k_1}{k_3} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & \frac{l (k_1 + 1)}{k_3} & \frac{l k_1}{k_3} & 0 & \frac{i^2 k_1 - \rho_1 \lambda^2}{k_3} & 0
\end{pmatrix}
\]  

(3.21)

The solution of the differential equation (3.20) is given by

\[
V(x) = e^{A_\lambda (x - \alpha)} V(\alpha).
\]  

(3.22)

Thus, from (3.22) and the fact that \(V(\alpha) = 0\), we get

\[
V = 0 \text{ in } (0, \alpha).
\]  

(3.23)

From (3.23) and the fact that \(v^1(0) = v^3(0) = v^5(0) = 0\), we get

\[
v^1 = 0 \text{ in } (0, \alpha), \quad v^3 = 0 \text{ in } (0, \alpha) \quad \text{and} \quad v^5 = 0 \text{ in } (0, \alpha).
\]  

(3.24)

From (3.24), (3.2), (3.4), (3.6) and the fact that \(\lambda \neq 0\), we obtain

\[
U = 0 \text{ in } (0, \alpha).
\]  

(3.25)

From (3.25) and the regularity of \(v^i, i \in \{3, 5\}\), we obtain

\[
v^3(\alpha) = 0 \quad \text{and} \quad v^5(\alpha) = 0,
\]
consequently, from (3.15) and (3.16), we get
\[ v^1 = 0 \text{ in } (\alpha, \beta), \quad v^3 = 0 \text{ in } (\alpha, \beta) \text{ and } v^5 = 0 \text{ in } (\alpha, \beta), \]
consequently, from (3.2), (3.4), (3.6) and the fact that \( \lambda \neq 0 \), we obtain
\[ U = 0 \text{ in } (\alpha, \beta). \tag{3.26} \]
Now, let \( W = (v^1, v^1_x, v^3, v^5, v^5_x) \). From (3.26) and the regularity of \( v^i, i \in \{1, 3, 5\} \), we have \( W(\beta) = 0 \) and system (3.17)–(3.19) in \( (\beta, L) \) implies:
\[ W_x = A_\lambda W \text{ in } (\beta, L), \]
where \( A_\lambda \) is defined before (see (3.21)). Thus, we have
\[ W(x) = e^{A_\lambda(x-\beta)}W(\beta) = 0, \]
consequently, from (3.2), (3.4) and (3.6), we deduce that
\[ U = 0 \text{ in } (\beta, L). \tag{3.27} \]
Finally, from (3.25), (3.26) and (3.27), we obtain
\[ U = 0 \text{ in } (0, L). \]
The proof is thus complete.

\[ \square \]

**Lemma 3.2.** For all \( \lambda \in \mathbb{R} \), we have
\[ R(i\lambda I - A) = \mathcal{H}. \]

**Proof.** From Proposition 2.1, we have \( 0 \in \rho(A) \). We still need to show the result for \( \lambda \in \mathbb{R}^* \). For this aim, let \( F = (f^1, f^2, f^3, f^4, f^5, f^6)^\top \in \mathcal{H} \), we want to find \( U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(A) \) solution of
\[ (i\lambda I - A)U = F. \tag{3.28} \]
Detailing (3.28), we obtain
\[ i\lambda v^1 - v^2 = f^1, \tag{3.29} \]
\[ i\lambda v^2 - \frac{k_1}{\rho_1} (v^1_x + v^3 + lv^5)_x - \frac{lk_3}{\rho_1} (v^5_x - lv^1) - \frac{ld(x)}{\rho_1} (v^6_x - lv^2) = f^2, \tag{3.30} \]
\[ i\lambda v^3 - v^4 = f^3, \tag{3.31} \]
\[ i\lambda v^4 - \frac{k_2}{\rho_2} v^3_{xx} + \frac{k_1}{\rho_2} (v^1_x + v^3 + lv^5) = f^4, \tag{3.32} \]
\[ i\lambda v^5 - v^6 = f^5, \tag{3.33} \]
\[ i\lambda v^6 - \frac{1}{\rho_1} \left[ k_3 (v^5_x - lv^1) + d(x)(v^6_x - lv^2) \right]_x + \frac{lk_1}{\rho_1} (v^1_x + v^3 + lv^5) = f^6, \tag{3.34} \]
with the following boundary conditions
\[ v^1(0) = v^1(L) = v^3(0) = v^3(L) = v^5(0) = v^5(L) = 0. \tag{3.35} \]
Inserting \( v^2 = i\lambda v^1 - f^1, v^4 = i\lambda v^3 - f^3 \) and \( v^6 = i\lambda v^5 - f^5 \) in (3.30), (3.32) and (3.34), respectively, we obtain
\[ -\lambda^2 v^1 - \frac{k_1}{\rho_1} (v^1_x + v^3 + lv^5)_x - \frac{lk_3}{\rho_1} (v^5_x - lv^1) - \frac{i\lambda d(x)}{\rho_1} (v^5_x - lv^1) = g^1, \tag{3.36} \]
\[ -\lambda^2 v^3 - \frac{k_2}{\rho_2} v^3_{xx} + \frac{k_1}{\rho_2} (v^1_x + v^3 + lv^5) = g^2, \tag{3.37} \]
\[ -\lambda^2 v^5 - \frac{1}{\rho_1} \left[ k_3 (v^5_x - lv^1) + i\lambda d(x)(v^5_x - lv^1) \right]_x + \frac{lk_1}{\rho_1} (v^1_x + v^3 + lv^5) = g^3, \tag{3.38} \]
\begin{equation}
\begin{aligned}
g^1 := i\lambda f^1 + f^2 + \frac{ld(x)}{\rho_1}(-f^5 + lf^1) \in H^{-1}(0, L), \\
g^2 := i\lambda f^3 + f^4 \in H^{-1}(0, L), \\
g^3 := i\lambda f^5 + f^6 + \frac{\rho_1}{\rho_1^*}[d(x)(-f^5 + lf^1)]_x \in H^{-1}(0, L).
\end{aligned}
\tag{3.39}
\end{equation}

For all \( U = (v^1, v^3, v^5)^\top \in \mathbb{H} := (H^1_0(0, L))^3 \), we define the linear operator \( L : \mathbb{H} \mapsto \mathbb{H}' := (H^{-1}(0, L))^3 \) by:

\[
L U = \begin{pmatrix}
-\frac{k_1}{\rho_1} (v^1_x + v^3 + lv^5)_x - \frac{\rho_1}{k_1} (v^5_x - lv^1) - \frac{1}{\rho_1} k_3 (v^5_x - lv^1) \\
\frac{2k_2}{\rho_2} v^3_x + \frac{k_1}{\rho_2} (v^1_x + v^3 + lv^5) \\
-\frac{1}{\rho_1} k_3 (v^5_x - lv^1) + i\lambda d(x)(v^5_x - lv^1)_x + \frac{1}{\rho_1} k_3 (v^1_x + v^3 + lv^5)
\end{pmatrix}.
\tag{3.40}
\]

Let us prove that the operator \( L \) is an isomorphism. For this aim, take the duality bracket \( \langle \cdot, \cdot \rangle_{\mathbb{H}', \mathbb{H}} \) of (3.40) with \( \Psi := (\rho_1 \psi^1, \rho_2 \psi^2, \rho_1 \psi^3)^\top \in \mathbb{H} \), we obtain

\[
\langle L U, \Psi \rangle_{\mathbb{H}', \mathbb{H}} = \langle -k_1 (v^1_x + v^3 + lv^5)_x - k_3 (v^5_x - lv^1) - i\lambda d(x)(v^5_x - lv^1), \psi^1 \rangle_{H^{-1}(0, L), H^1_0(0, L)} \\
+ \langle -k_2 v^3 + k_1 (v^1_x + v^3 + lv^5), \psi^2 \rangle_{H^{-1}(0, L), H^1_0(0, L)} \\
+ \langle - k_3 (v^5_x - lv^1) + i\lambda d(x)(v^5_x - lv^1)_x + k_1 (v^1_x + v^3 + lv^5), \psi^3 \rangle_{H^{-1}(0, L), H^1_0(0, L)}.
\]

Consequently, we obtain

\[
\langle L U, \Psi \rangle_{\mathbb{H}', \mathbb{H}} = k_1 \int_0^L (v^1_x + v^3 + lv^5) \psi^1 \, dx - k_3 \int_0^L (v^5_x - lv^1) \psi^1 \, dx - i\lambda \int_0^L d(x)(v^5_x - lv^1) \psi^1 \, dx \\
+ k_2 \int_0^L v^3 \psi^2 \, dx + k_1 \int_0^L (v^1_x + v^3 + lv^5) \psi^2 \, dx + k_3 \int_0^L (v^5_x - lv^1) \psi^2 \, dx \\
+ i\lambda \int_0^L d(x)(v^5_x - lv^1) \psi^2 \, dx + k_1 \int_0^L (v^1_x + v^3 + lv^5) \psi^3 \, dx
\]

defines a continuous sesquilinear form which is coercive on \( \mathbb{H} \). Indeed, from Remark 2.1, we deduce that there exists a positive constant \( c' \) such that

\[
\Re \langle L U, U \rangle_{\mathbb{H}', \mathbb{H}} = k_1 \|v^1_x + v^3 + lv^5\|^2 + k_2 \|v^3\|^2 + k_3 \|v^5_x - lv^1\|^2 \\
\geq c' \left( \|v^1_x\|^2 + \|v^3\|^2 + \|v^5_x\|^2 \right) \\
= c' \|v^1, v^3, v^5\|_{\mathbb{H}}^2 = c' \|U\|^2_{\mathbb{H}}.
\]

Therefore, by using Lax–Milgram theorem, we deduce that \( L \) is an isomorphism from \( \mathbb{H} \) onto \( \mathbb{H}' \).

Now, let \( U = (v^1, v^3, v^5)^\top \) and \( G = (g^1, g^2, g^3)^\top \), then system (3.36)–(3.38) can be transformed into the following form:

\[
(I - \lambda^2 L^{-1}) U = L^{-1} G.
\tag{3.41}
\]

Since \( I \) is compact operator from \( \mathbb{H} \) onto \( \mathbb{H}' \) and \( L^{-1} \) is an isomorphism from \( \mathbb{H}' \) onto \( \mathbb{H} \), the operator \( I - \lambda^2 L^{-1} \) is Fredholm of index zero. Then, by Fredholm’s alternative, (3.41) admits a unique solution
Thus, by using Lemma 3.1, we obtain
\[ V - \lambda^2 L^{-1} V = 0 \iff \lambda^2 V - LV = 0. \] (3.42)

Equivalently, we have
\[
-\lambda^2 v' - \frac{k_1}{\rho_1} \left( v_x + v^2 + lv^5 \right)_x - \frac{lk_3}{\rho_1} (v_x^5 - lv^4) - \frac{i\lambda d(x)}{\rho_1} (v^5_x - lv^4) = 0, \\
-\lambda^2 v^3 - \frac{k_3}{\rho_2} v^3_x + \frac{k_1}{\rho_2} (v^4_x + v^3 + lv^5) = 0, \\
-\lambda^2 v^5 - \frac{1}{\rho_2} \left[(k_3 + i\lambda d(x))v^5_x - l(k_3 + i\lambda )v^4 \right]_x + \frac{lk_1}{\rho_1} (v^3_x + v^3 + lv^5) = 0. 
\] (3.43) (3.44) (3.45)

It is easy to see that if \( V = (v^1, v^3, v^5)^T \) is a solution of (3.43)–(3.45), then the vector \( W \) defined by
\[
W = (v^1, i\lambda v^1, v^3, i\lambda v^3, v^5, i\lambda v^5)^T 
\]
belongs to \( D(A) \) and satisfies
\[
i\lambda W - AW = 0. 
\]

Thus, by using Lemma 3.1, we obtain \( W = 0 \) and consequently \( I - \lambda^2 L^{-1} \) is injective. Thanks to Fredholm’s alternative, (3.41) admits a unique solution \( U \in \mathbb{H} \) and
\[
v^1, v^3 \in H^2(0, L), \ [k_3 v^5_x + d(x)(i\lambda v^5 - f^5_x - l(i\lambda v^4 - f^4))]_x \in L^2(0, L). 
\]

Finally, by setting \( v^2 = i\lambda v^1 - f^1 \), \( v^4 = i\lambda v^3 - f^3 \) and \( v^6 = i\lambda v^5 - f^5 \), we deduce that \( U \in D(A) \) is a unique solution of (3.28). The proof is thus complete. \( \square \)

**Proof of Theorem 3.1.** From Lemma 3.1, we obtain that the operator \( A \) has no pure imaginary eigenvalues (i.e., \( \sigma_p(A) \cap i\mathbb{R} = \emptyset \)). Moreover, from Lemmas 3.1 and 3.2, \( i\lambda I - A \) is bijective for all \( \lambda \in \mathbb{R} \) and since \( A \) is closed, we conclude with the help of the closed graph theorem that \( i\lambda I - A \) is an isomorphism for all \( \lambda \in \mathbb{R} \), hence that \( \sigma(A) \cap i\mathbb{R} = \emptyset \). Therefore, according to Theorem A.2, we get that the \( C_0 \)-semigroup \( (e^{tA})_{t \geq 0} \) is strongly stable. The proof is thus complete. \( \square \)

### 4. Polynomial stability

In this section, we will prove the polynomial stability of system (1.1)–(1.3) with different rates. The main results of this section are the following theorems.

**Theorem 4.1.** If
\[ \frac{k_1}{\rho_1} = \frac{k_2}{\rho_2}, \]
then there exists \( C > 0 \) such that for every \( U_0 \in D(A) \), we have
\[
E(t) \leq \frac{C}{t} \| U_0 \|^2_{D(A)}, \quad t > 0. 
\]

**Theorem 4.2.** If
\[ \frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2}, \]
then there exists \( C > 0 \) such that for every \( U_0 \in D(A) \), we have
\[
E(t) \leq \frac{C}{\sqrt{t}} \| U_0 \|^2_{D(A)}, \quad t > 0. 
\]
Since $i\mathbb{R} \subset \rho(A)$ (see Sect. 3), according to Theorem A.3, to prove Theorems 4.1 and 4.2, we still need to prove the following condition

$$\sup_{A \in \mathbb{R}} \left\| (i\lambda I - A)^{-1} \right\|_{\mathcal{L}(\mathcal{H})} = O \left( |\lambda|^{\ell} \right), \quad \text{with} \quad \ell = 2 \text{ or } \ell = 4. \quad (H)$$

We will prove condition (H) by a contradiction argument. For this purpose, suppose that (H) is false, then there exists \( \{(\lambda^n, U^n) := (v^{1,n}, v^{2,n}, v^{3,n}, v^{4,n}, v^{5,n}, v^{6,n})^\top \}_{n \geq 1} \subset \mathbb{R}^* \times D(A) \) with

$$|\lambda^n| \to \infty \quad \text{and} \quad \|U^n\|_{\mathcal{H}} = \|(v^{1,n}, v^{2,n}, v^{3,n}, v^{4,n}, v^{5,n}, v^{6,n})^\top\|_{\mathcal{H}} = 1,$$  

such that

$$(\lambda^n)^\ell (i\lambda^n I - A)U^n = F^n := (f^{1,n}, f^{2,n}, f^{3,n}, f^{4,n}, f^{5,n}, f^{6,n})^\top \to 0 \quad \text{in} \quad \mathcal{H}. \quad (4.2)$$

For simplicity, we drop the index $n$. Equivalently, from (4.2), we have

$$i\lambda v^1 - v^2 = \lambda^{-\ell} f^1, \quad (4.3)

i\lambda \rho_1 v^2 - k_1(v^1_x + v^3 + lv^5) - lk_3(v^5_x - lv^1) = \rho_1 \lambda^{-\ell} f^2, \quad (4.4)

i\lambda v^3 - v^4 = \lambda^{-\ell} f^3, \quad (4.5)

i\lambda \rho_2 v^4 - k_2 v_{xx}^3 + k_1(v^1_x + v^3 + lv^5) = \rho_2 \lambda^{-\ell} f^4, \quad (4.6)

i\lambda v^5 - v^6 = \lambda^{-\ell} f^5, \quad (4.7)

By inserting (4.3) in (4.4), (4.5) in (4.6) and (4.7) in (4.8), we deduce that

$$\lambda^2 \rho_1 v^1 + k_1(v^1_x + v^3 + lv^5)_x + lk_3(v^3_x - lv^1) + ld(x)(v^5_x - lv^1) = -\rho_1 \lambda^{-\ell} f^2 - i\rho_1 \lambda^{-\ell+1} f^1, \quad (4.9)

\lambda^2 \rho_2 v^3 + k_2 v_{xx}^3 - k_1(v^1_x + v^3 + lv^5) = -\rho_2 \lambda^{-\ell} f^4 - i\rho_2 \lambda^{-\ell+1} f^3, \quad (4.10)

\lambda^2 \rho_1 v^5 + [k_3(v^3_x - lv^1) + d(x)(v^5_x - lv^2)]_x - lk_1(v^1_x + v^3 + lv^5) = -\rho_1 \lambda^{-\ell} f^6 - i\rho_1 \lambda^{-\ell+1} f^5. \quad (4.11)$$

Here, we will check the condition (H) by finding a contradiction with (4.1) by showing \( \|U\|_{\mathcal{H}} = o(1) \). For clarity, we divide the proof into several Lemmas. From the above system and the fact that \( \ell \in \{2, 4\} \), \( \|U\|_{\mathcal{H}} = 1 \) and \( \|F\|_{\mathcal{H}} = o(1) \), we remark that

$$\begin{align*}
\|v^1\| &= O \left( |\lambda|^{-1} \right), \\
\|v^3\| &= O \left( |\lambda|^{-1} \right), \\
\|v^5\| &= O \left( |\lambda|^{-1} \right), \\
\|v_{xx}^5\| &= O \left( |\lambda|^{-1} \right),
\end{align*}$$

(4.12)

Also, from Poincaré inequality and the fact that \( \|F\|_{\mathcal{H}} = o(1) \), we remark that

$$\|f^1\| \lesssim \|f_x^1\| = o(1), \quad \|f^3\| \lesssim \|f_x^3\| = o(1) \quad \text{and} \quad \|f^5\| \lesssim \|f_x^5\| = o(1). \quad (4.13)$$

**Lemma 4.1.** If \( \left( \frac{k_1}{\rho_1}, \frac{k_2}{\rho_2} \right) \neq \left( \frac{k_1}{\rho_1}, \frac{k_2}{\rho_2} \right) \) and \( \ell = 2 \) or \( \ell = 4 \). Then, the solution \( U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(A) \) of (4.3)–(4.8) satisfies the following estimations

$$\begin{align*}
\int_\alpha^\beta |v^6_x - lv^2|^2 \, dx &= \frac{O(1)}{\lambda^\ell}, \\
\int_\alpha^\beta |v^5_x - lv^1|^2 \, dx &= \frac{O(1)}{\lambda^{\ell+2}}, \\
\int_\alpha^\beta |v^6|^2 \, dx &= O(1) \quad \text{and} \quad \int_\alpha^\beta |v^5_x|^2 \, dx = \frac{O(1)}{\lambda^\ell},
\end{align*}$$

(4.14)

**Proof.** First, taking the inner product of (4.2) with \( U \) in \( \mathcal{H} \) and using (2.6), we get

$$\int_0^L d(x) |v^6_x - lv^2|^2 \, dx = d_0 \int_\alpha^\beta |v^6_x - lv^2|^2 \, dx = -\Re(AU, U)_{\mathcal{H}} = \lambda^{-\ell} \Re(F, U)_{\mathcal{H}} \leq \lambda^{-\ell} \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}.$$

(4.15)
Thus, from (4.15) and the fact that \(\|F\|_{\mathcal{H}} = o(1)\) and \(\|U\|_{\mathcal{H}} = 1\), we obtain the first estimation in (4.14). Deriving (4.7) with respect to \(x\) and multiply (4.3) by \(l\), then subtract the resulting equations, we deduce that

\[
i\lambda(v_x^5 - lv^1) - (v_x^6 - lv^2) = \lambda^{-\ell}(f_x^5 - lf^1).
\]

From the above equation, we obtain

\[
\int_{\alpha}^{\beta} |v_x^5 - lv^1|^2 \, dx \leq \frac{2}{\lambda^2} \int_{\alpha}^{\beta} |v_x^6 - lv^2|^2 \, dx + \frac{2}{\lambda^{2\ell+2}} \int_{\alpha}^{\beta} |f_x^5 - lf^1|^2 \, dx
\]

\[
\leq \frac{2}{\lambda^2} \int_{\alpha}^{\beta} |v_x^6 - lv^2|^2 \, dx + \frac{4}{\lambda^{2\ell+2}} \|f_x^5\|^2 + \frac{4l^2}{\lambda^{2\ell+2}} \|f^1\|^2.
\]

(4.16)

From (4.16), the first estimation in (4.14) and the fact that \(\ell \in \{2, 4\}\), \(\|f^1\| = o(1)\) (see (4.13)), \(\|f_x^5\| = o(1)\), we get the second estimation in (4.14). Now, it is easy to see that

\[
\int_{\alpha}^{\beta} |v_x^6|^2 \, dx = \int_{\alpha}^{\beta} |v_x^6 - lv^2 + lv^2|^2 \, dx \leq 2 \int_{\alpha}^{\beta} |v_x^6 - lv^2|^2 \, dx + 2l^2 \int_{\alpha}^{\beta} |v^2|^2 \, dx.
\]

From the above estimation, the first estimation in (4.14) and the fact that \(v^2\) is uniformly bounded in \(L^2(0, L)\), we get the third estimation in (4.14). From (4.7), we deduce that

\[
\int_{\alpha}^{\beta} |v_x^5|^2 \, dx \leq \frac{2}{\lambda^2} \int_{\alpha}^{\beta} |v_x^6|^2 \, dx + \frac{2}{\lambda^{2\ell+2}} \int_{\alpha}^{\beta} |f_x^5|^2 \, dx.
\]

Finally, from the above estimation, the third estimation in (4.14) and the fact that \(\|f_x^5\| = o(1)\), we obtain the fourth estimation in (4.14). The proof is thus complete. \(\Box\)

For all \(0 < \varepsilon < \frac{\beta - \alpha}{10}\), we fix the following cut-off functions

- \(p_j \in C^2([0, L]), j \in \{1, \ldots, 5\}\) such that \(0 \leq p_j(x) \leq 1\), for all \(x \in [0, L]\) and

\[
p_j(x) = \begin{cases} 1 & \text{if } x \in [\alpha + j\varepsilon, \beta - j\varepsilon], \\ 0 & \text{if } x \in [0, \alpha + (j - 1)\varepsilon] \cup [\beta + (1 - j)\varepsilon, L]. \end{cases}
\]

- \(q_1, q_2 \in C^1([0, L])\) such that \(0 \leq q_1(x) \leq 1, 0 \leq q_2(x) \leq 1\), for all \(x \in [0, L]\) and

\[
q_1(x) = \begin{cases} 1 & \text{if } x \in [0, \gamma_1], \\ 0 & \text{if } x \in [\gamma_2, L], \end{cases} \quad q_2(x) = \begin{cases} 0 & \text{if } x \in [0, \gamma_1], \\ 1 & \text{if } x \in [\gamma_2, L], \end{cases}
\]

with \(0 < \alpha < \gamma_1 < \gamma_2 < \beta < L\).

**Lemma 4.2.** If \(\left(\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2}, \ell = 2\right)\) or \(\left(\frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2}, \ell = 4\right)\). Then, the solution \(U = (v^1, v^2, v^3, v^4, v^5, v^6) \in D(\mathcal{A})\) of (4.3)–(4.8) satisfies the following estimations

\[
\int_{\alpha + \varepsilon}^{\beta - \varepsilon} |v^1|^2 \, dx = o(1) \quad \text{and} \quad \int_{\alpha + \varepsilon}^{\beta - \varepsilon} |\lambda v^5|^2 \, dx = o(1).
\]

(4.17)

**Proof.** First, multiplying (4.8) by \(-i\lambda^{-1}p_1 \overline{v^6}\) and integrating over \((\alpha, \beta)\), then using the fact that \(v^6\) is uniformly bounded in \(L^2(0, L)\) and \(\|f^6\| = o(1)\), we obtain

\[
\rho_1 \int_{\alpha}^{\beta} p_1 |v^6|^2 \, dx = -\frac{i}{\lambda} \int_{\alpha}^{\beta} p_1 \left[k_3(v_x^5 - lv^1) + d(x)(v_x^6 - lv^2)\right] \overline{v^6} \, dx
\]
\[
\frac{ilk_1}{\lambda} \int_{\alpha}^{\beta} p_1(v_1^1 + v^3 + lv^5)v^6 dx + \frac{o(1)}{|\lambda|^\ell+1},
\]

using the fact that \((v_1^1 + v^3 + lv^5), v^6\) are uniformly bounded in \(L^2(0, L)\), we get
\[
\frac{ilk_1}{\lambda} \int_{\alpha}^{\beta} p_1(v_1^1 + v^3 + lv^5)v^6 dx = o(1),
\]

consequently, as \(\ell \in \{2, 4\}\), we obtain
\[
\rho_1 \int_{\alpha}^{\beta} p_1|v^6|^2 dx = \frac{i}{\lambda} \int_{\alpha}^{\beta} -p_1 \left[k_3(v_1^5 - lv^1) + d(x)(v_1^6 - lv^2)\right] v^6 dx + o(1). \tag{4.18}
\]

Using integration by parts and the fact that \(p_1(\alpha) = p_1(\beta) = 0\), then using the definition of \(d(x)\), we get
\[
I_1 \equiv \frac{i}{\lambda} \int_{\alpha}^{\beta} p_1 \left[k_3(v_1^5 - lv^1) + d_0(v_1^6 - lv^2)\right] v^6 dx + \frac{i}{\lambda} \int_{\alpha}^{\beta} p_1' \left[k_3(v_1^5 - lv^1) + d_0(v_1^6 - lv^2)\right] v^6 dx,
\]

using Lemma 4.1 and the fact that \(v^6\) is uniformly bounded in \(L^2(0, L)\), \(\ell \in \{2, 4\}\), we get
\[
I_1 = \frac{o(1)}{|\lambda|^\ell+1}. \tag{4.19}
\]

Inserting (4.19) in (4.18) and using the fact that \(\ell \in \{2, 4\}\), we obtain
\[
\rho_1 \int_{\alpha}^{\beta} p_1|v^6|^2 dx = o(1).
\]

From the above estimation and the definition of \(p_1\), we obtain the first estimation in (4.17). Next, from (4.7), we deduce that
\[
\int_{\alpha+\varepsilon}^{\beta-\varepsilon} |\lambda v^6|^2 dx \leq 2 \int_{\alpha+\varepsilon}^{\beta-\varepsilon} |v^6|^2 dx + 2\lambda^{-2\ell} \int_{\alpha+\varepsilon}^{\beta-\varepsilon} |f^5|^2 dx.
\]

Finally, from the above inequality, the first estimation in (4.17) and the fact that \(\|f^5\| = o(1)\), \(\ell \in \{2, 4\}\), we obtain the second estimation in (4.17). The proof is thus complete. \(\square\)

**Lemma 4.3.** If \(\frac{k_1}{p_1} = \frac{k_2}{p_2}\) and \(\ell = 2\) or \(\frac{k_1}{p_1} \neq \frac{k_2}{p_2}\) and \(\ell = 4\). Then, the solution \(U = (v_1^1, v_2^1, v_3^1, v_4^1, v_5, v^6) \in D(A)\) of (4.3)–(4.8) satisfies the following estimations
\[
\int_{\alpha+\varepsilon}^{\beta-2\varepsilon} |v^1|^2 dx = o(1) \quad \text{and} \quad \int_{\alpha+\varepsilon}^{\beta-2\varepsilon} |\lambda v^1|^2 dx = o(1). \tag{4.20}
\]

**Proof.** First, multiplying (4.8) by \(p_2 v_2^1\), integrating over \((\alpha+\varepsilon, \beta-\varepsilon)\), using the fact that \(v_1^1\) is uniformly bounded in \(L^2(0, L)\) and \(\|f^6\| = o(1)\), we get
\[
\begin{align*}
i\lambda \rho_1 \int_{\alpha+\varepsilon}^{\beta-\varepsilon} p_2 v^6 v_2^1 dx + \int_{\alpha+\varepsilon}^{\beta-\varepsilon} -p_2 \left[k_3(v_1^5 - lv^1) + d(x)(v_1^6 - lv^2)\right] v_2^1 dx \\
+ l k_1 \int_{\alpha+\varepsilon}^{\beta-\varepsilon} p_2 |v_1^1|^2 dx + l k_1 \int_{\alpha+\varepsilon}^{\beta-\varepsilon} p_2 (v_3^1 + lv^5) v_2^1 dx = o(\lambda^{-\ell}),
\end{align*}
\]

\[
\begin{align*}
i\lambda \rho_1 \int_{\alpha+\varepsilon}^{\beta-\varepsilon} p_2 v^6 v_2^1 dx + \int_{\alpha+\varepsilon}^{\beta-\varepsilon} -p_2 \left[k_3(v_1^5 - lv^1) + d(x)(v_1^6 - lv^2)\right] v_2^1 dx \\
+ l k_1 \int_{\alpha+\varepsilon}^{\beta-\varepsilon} p_2 |v_1^1|^2 dx + l k_1 \int_{\alpha+\varepsilon}^{\beta-\varepsilon} p_2 (v_3^1 + lv^5) v_2^1 dx = o(\lambda^{-\ell}),
\end{align*}
\]
using the fact that \( \nu_2 \) is uniformly bounded in \( L^2(0, L) \), \( \|v^3\| = O(|\lambda|^{-1}) \), \( \|v^5\| = O(|\lambda|^{-1}) \) (see (4.12)),
we get
\[
lk_1 \int_{\alpha+\epsilon}^{\beta-\epsilon} p_2 (v^3 + lv^5) \nu_2^2 dx = o(1),
\]
consequently, as \( \ell \in \{2, 4\} \), we obtain
\[
lk_1 \int_{\alpha+\epsilon}^{\beta-\epsilon} p_2 \left| v_2^1 \right|^2 dx + I_2 + I_3 = o(1). \tag{4.21}
\]

Now, using integration by parts and the definition of \( p_2 \), then using Lemma 4.2 and the fact that \( \|v^1\| = O(|\lambda|^{-1}) \), we get
\[
I_2 = -i \lambda \rho_1 \int_{\alpha+\epsilon}^{\beta-\epsilon} p_2 v_2^6 \nu_2^1 dx - i \lambda \rho_1 \int_{\alpha+\epsilon}^{\beta-\epsilon} p_2 (v_2^6 - lv^2 + lv^2) \nu_2^1 dx
\]
\[
= -i \lambda \rho_1 \int_{\alpha+\epsilon}^{\beta-\epsilon} p_2 (v_2^6 - lv^2) \nu_2^1 dx - i \lambda \rho_1 \int_{\alpha+\epsilon}^{\beta-\epsilon} p_2 v_2^2 \nu_2^1 dx,
\]
using Lemma 4.1 and the fact that \( \|v^1\| = O(|\lambda|^{-1}) \), we get
\[
-I \lambda \rho_1 \int_{\alpha+\epsilon}^{\beta-\epsilon} p_2 v_2^6 \nu_2^1 dx = -i \lambda \rho_1 \int_{\alpha+\epsilon}^{\beta-\epsilon} p_2 v_2^2 \nu_2^1 dx + o(1) \tag{4.22}
\]

Now, it is easy to see that
\[
-i \lambda \rho_1 \int_{\alpha+\epsilon}^{\beta-\epsilon} p_2 v_2^6 \nu_2^1 dx = -i \lambda \rho_1 \int_{\alpha+\epsilon}^{\beta-\epsilon} p_2 (v_2^6 - lv^2 + lv^2) \nu_2^1 dx
\]
\[
= -i \lambda \rho_1 \int_{\alpha+\epsilon}^{\beta-\epsilon} p_2 (v_2^6 - lv^2) \nu_2^1 dx - i \lambda \rho_1 \int_{\alpha+\epsilon}^{\beta-\epsilon} p_2 v_2^2 \nu_2^1 dx,
\]
using Lemma 4.1 and the fact that \( \|v^1\| = O(|\lambda|^{-1}) \), we get
\[
-I \lambda \rho_1 \int_{\alpha+\epsilon}^{\beta-\epsilon} p_2 v_2^6 \nu_2^1 dx = -i \lambda \rho_1 \int_{\alpha+\epsilon}^{\beta-\epsilon} p_2 v_2^2 \nu_2^1 dx + o(1) \tag{4.23}
\]

Inserting (4.23) in (4.28) and using the fact that \( \ell \in \{2, 4\} \), we get
\[
I_2 = -i \lambda \rho_1 \int_{\alpha+\epsilon}^{\beta-\epsilon} p_2 |v_2^1|^2 dx + o(1) \tag{4.24}
\]

Next, using integration by parts and the definition of \( p_2 \), we get
\[
I_3 = \int_{\alpha+\epsilon}^{\beta-\epsilon} p_2 \left[ k_3 (v_2^5 - lv^1) + d_0 (v_2^6 - lv^2) \right] \nu_2^1 dx + \int_{\alpha+\epsilon}^{\beta-\epsilon} p_2 \left[ k_3 (v_2^5 - lv^1) + d_0 (v_2^6 - lv^2) \right] \nu_2^1 dx \tag{4.25}
\]
using Lemma 4.1 and the fact that \( v_2 \) is uniformly bounded in \( L^2(0, L) \), \( \|v_2^1\| = O(|\lambda|) \) (see (4.12)), \( \ell \in \{2, 4\} \), we get
\[
I_3 = o(|\lambda|^{-\frac{\ell}{2} + 1}) \tag{4.26}
\]
Inserting (4.24) and (4.26) in (4.21) and using the fact that \( \ell \in \{2, 4\} \), we get
\[
lk_1 \int_{\alpha+\epsilon}^{\beta-\epsilon} p_2 \left| v_2^1 \right|^2 dx + l \rho_1 \int_{\alpha+\epsilon}^{\beta-\epsilon} p_2 |v_2^1|^2 dx = o(1) \tag{4.27}
\]
Finally, from the above estimation and the definition of $p_2$, we obtain (4.20). The proof is thus complete. □

**Lemma 4.4.** If $\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2}$ and $\ell = 2$. Then, the solution $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(\mathcal{A})$ of (4.3)–(4.8) satisfies the following estimations

$$
\int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} |v_x^3|^2 \, dx = o(1) \quad \text{and} \quad \int_{\alpha+4\varepsilon}^{\beta-4\varepsilon} |\lambda v_3|^2 \, dx = o(1).
$$

(4.28)

**Proof.** First, take $\ell = 2$ in (4.9) and multiply it by $\frac{k_1}{\rho_1} p_3 v_x^3$, integrating over $(\alpha + 2\varepsilon, \beta - 2\varepsilon)$, using the definition of $d(x)$ and fact that $v_x^3$ is uniformly bounded in $L^2(0, L)$, $\|f^1\| = o(1)$, $\|f^2\| = o(1)$, we obtain

$$
\begin{align*}
\frac{k_1}{\rho_1} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 |v_x^3|^2 \, dx &= -\lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v_1^2 v_x^3 \, dx - \frac{k_1}{\rho_1} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v_{xx}^3 v_x^3 \, dx - \frac{k_1}{\rho_1} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v_2^5 v_x^3 \, dx \\
\quad - \frac{lk_3}{\rho_1} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 (v_x^3 - l v^1) v_x^3 \, dx - \frac{l d_0}{\rho_1} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 (v_x^3 - l v^2) v_x^3 \, dx + o(1).
\end{align*}
$$

(4.29)

Using Lemma 4.1 with $\ell = 2$, the definition of $p_3$ and the fact that $v_x^3$ is uniformly bounded in $L^2(0, L)$, we get

$$
\begin{align*}
-\frac{lk_3}{\rho_1} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 (v_x^3 - l v^1) v_x^3 \, dx &= o(1), \\
- \frac{l k_3}{\rho_1} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 (v_x^3 - l v^2) v_x^3 \, dx &= o(1),
\end{align*}
$$

(4.30)

Inserting (4.30) in (4.29), we get

$$
\frac{k_1}{\rho_1} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 |v_x^3|^2 \, dx = -\lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v_1^2 v_x^3 \, dx - \frac{k_1}{\rho_1} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v_{xx}^3 v_x^3 \, dx + o(1).
$$

(4.31)

Now, taking $\ell = 2$ in (4.10), we deduce that

$$
\lambda^2 \rho_2 v_x^3 + k_2 v_x^3 - k_1 (v_x^1 + v_x^3 + lv_3^5) = -\rho_2 \lambda^{-2} f_3 + i \rho_2 \lambda^{-1} f_3.
$$

(4.32)

Multiplying (4.32) by $\frac{k_1}{\rho_2} p_3 v_x^3$, integrating over $(\alpha + 2\varepsilon, \beta - 2\varepsilon)$, we obtain

$$
\lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v_x^3 v_x^3 \, dx + \frac{k_1}{\rho_2} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v_1^2 v_x^3 \, dx - \frac{k_1}{\rho_2} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v_{xx}^3 v_x^3 \, dx = o(1).
$$

(4.33)

Using integration by parts to the first two terms in the above equation, we get

$$
\begin{align*}
-\lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v_1^2 v_x^3 \, dx - \frac{k_1}{\rho_2} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v_{xx}^3 v_x^3 \, dx = \lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v_{xx}^3 v_x^3 \, dx + \frac{k_1}{\rho_2} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v_{xx}^3 v_x^3 \, dx \\
+ \frac{k_1}{\rho_2} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v_x^3 (v_x^1 + v_x^3 + lv_3^5) \, dx = o(1).
\end{align*}
$$

(4.34)

Using Lemma 4.3 and the fact that $v_x^3$, $(v_x^1 + v_x^3 + lv_3^5)$ are uniformly bounded in $L^2(0, L)$ and $\|v_x^3\| = O(\varepsilon)$, we get

$$
\lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v_1^2 v_x^3 \, dx = o(1), \quad \frac{k_2}{\rho_2} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v_2^5 v_x^3 \, dx = o(1), \quad \frac{k_1}{\rho_1} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v_x^3 (v_x^1 + v_x^3 + lv_3^5) \, dx = o(1).
$$

(4.35)

Inserting (4.35) in (4.34), then using the fact that $\frac{k_2}{\rho_2} = \frac{k_1}{\rho_1}$, we get

$$
-\lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v_1^2 v_x^3 \, dx - \frac{k_1}{\rho_1} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v_{xx}^3 v_x^3 \, dx = o(1).
$$
Inserting the above estimation in (4.31), then using the definition of \( p_3 \), we obtain the first estimation in (4.28). Next, multiplying (4.32) by \( p_4 v^3 \), integrating over \((\alpha + 3\varepsilon, \beta - 3\varepsilon)\), using integration by parts and the definition of \( p_4 \) and the fact that \( \|v^3\| = O(|\lambda|^{-1}) \), \( \|f^3\| = o(1) \) and \( \|f^3\| = o(1) \), we get

\[
\rho_2 \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} \rho_1 |\lambda v^3|^2 dx = k_2 \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} p_4 |v_x^2|^2 dx + k_2 \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} p_4 v^3 v_x^3 dx + k_1 \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} p_4 (v_x^4 + v_x^3 l v^5) v_x^3 dx + o(\lambda^{-2}).
\]

From the above estimation, the first estimation in (4.28) and the fact that \( (v^4_x + v^3_x + l v^5) \) is uniformly bounded in \( L^2(0, L) \) and \( \|v^3\| = O(|\lambda|^{-1}) \), we obtain

\[
\rho_2 \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} p_4 |\lambda v^3|^2 dx = o(1).
\]

Finally, from the above estimation and the definition of \( p_4 \), we obtain the second estimation desired. The proof is thus complete.

**Lemma 4.5.** If \( \frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2} \) and \( \ell = 4 \). Then, the solution \( U = (v^1, v^2, v^3, v^4, v^5, v^6)^T \in D(A) \) of (4.3)–(4.8) satisfies the following estimation

\[
\int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} |\lambda v^1|^2 dx = o(\lambda^{-2}).
\]  

**Proof.** For clarity, we divide the proof into five steps:

**Step 1:** In this step, we will prove that:

\[
l \rho_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 |\lambda v^1|^2 dx - l k_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 |v_x^1|^2 dx - \Re \left\{ l k_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v^3 v_x^3 dx \right\} \\
- \Re \left\{ l^2 k_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v^5 v_x^5 dx \right\} = o(\lambda^{-2}).
\]

For this aim, take \( \ell = 4 \) in (4.9) and multiply it by \( l p_3 \frac{v^1}{\lambda} \), integrating over \((\alpha + 2\varepsilon, \beta - 2\varepsilon)\), using the fact that \( \|v^1\| = O(|\lambda|^{-1}) \), \( \|f^1\| = o(1) \) and \( \|f^2\| = o(1) \), then taking the real part, we get

\[
l \rho_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 |\lambda v^1|^2 dx + \Re \left\{ l k_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 (v_x^1 + v^3 + l v^5) v_x^3 v_x^3 dx \right\} \\
+ \Re \left\{ l^2 k_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 (v_x^5 - l v^1) v_x^5 v_x^5 dx \right\} = o(\lambda^{-4}).
\]

Using integration by parts and the definition of \( p_3 \), we obtain

\[
I_4 = - \Re \left\{ l k_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 (v_x^1 + v^3 + l v^5) v_x^3 v_x^3 dx \right\} \\
= - l k_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 \left( |v_x^1| \right)^2 dx - \Re \left\{ l k_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v^3 v_x^3 v_x^3 dx \right\} \\
- l k_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 |v_x^3|^2 dx - \Re \left\{ l^2 k_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 v^5 v_x^5 v_x^5 dx \right\}.
\]
Using integration by parts and the fact that \( p_3'(\alpha + 2\varepsilon) = p_3'/(\beta - 2\varepsilon) = 0 \), then using Lemma 4.3, we obtain
\[
- \frac{lk_1}{2} \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3'(v^1)^2 \, dx = \frac{lk_1}{2} \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3'' |v^1|^2 \, dx = o(\lambda^{-2}). \tag{4.40}
\]

Using the definition of \( p_3 \), Lemmas 4.1, 4.3 with \( \ell = 4 \) and the fact that \( \|v^3\| = O(|\lambda|^{-1}), \|v^5\| = O(|\lambda|^{-1}) \), we obtain
\[
- \mathcal{R} \left\{ lk_1 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3^* v^3 v^4 \, dx \right\} = o(\lambda^{-2}), \quad - \mathcal{R} \left\{ l^2 k_1 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3^* v^5 v^7 \, dx \right\} = o(\lambda^{-2}). \tag{4.41}
\]

Inserting (4.40) and (4.41) in (3.39), we obtain
\[
\mathcal{I}_4 = -lk_1 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3 |v_x|^2 \, dx - \mathcal{R} \left\{ lk_1 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3 v^3 v^4 \, dx \right\} - \mathcal{R} \left\{ l^2 k_1 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3 v^5 v^7 \, dx \right\} + o(\lambda^{-2}). \tag{4.42}
\]

Moreover, from Lemmas 4.1, 4.3 and the fact that \( \ell = 4 \), we obtain
\[
\mathcal{R} \left\{ l^2 k_3 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3 (v^5 - lv^1)v^4 \, dx \right\} = o(\lambda^{-4}) \quad \text{and} \quad \mathcal{R} \left\{ l^2 d_0 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3 (v^6 - lv^2)v^7 \, dx \right\} = o(|\lambda|^{-3}). \tag{4.43}
\]

Inserting (4.42) and (4.43) in (3.38), we obtain (4.37).

**Step 2:** In this step, we will prove that:
\[
2l \rho_1 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3 |v^1|^2 \, dx = \mathcal{R} \left\{ i\lambda \rho_1 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3^* v^6 v^4 \, dx \right\} - \mathcal{R} \left\{ d_0 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3 (v^6 - lv^2)v^7 \, dx \right\} + o(\lambda^{-2}). \tag{4.44}
\]

For this aim, multiplying (4.8) by \( p_3 v^7_x \), integrating over \((\alpha + 2\varepsilon, \beta - 2\varepsilon)\), using the fact that \( v^4_x \) is uniformly bounded in \( L^2(0, L) \) and \( \|f^6\| = o(1) \), then taking the real part, we get
\[
\mathcal{R} \left\{ i\lambda \rho_1 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3^* v^6 v^4 \, dx \right\} + \mathcal{R} \left\{ - \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3 \left[ k_3 (v^5_x - lv^1) + d(x)(v^6_x - lv^2) \right] v^7_x \, dx \right\}
\]
\[
+ lk_1 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3 |v_x|^2 \, dx + \mathcal{R} \left\{ lk_1 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3^* v^3 v^7_x \, dx \right\} + \mathcal{R} \left\{ l^2 k_1 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3^* v^5 v^7_x \, dx \right\} = o(\lambda^{-4}). \tag{4.45}
\]

Adding (4.37) and (4.45), we obtain
\[
l \rho_1 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3 |v^1|^2 \, dx + \mathcal{I}_5 + \mathcal{I}_6 = o(\lambda^{-2}). \tag{4.46}
\]

Using integration by parts and the fact that \( p_3(\alpha + 2\varepsilon) = p_3(\beta - 2\varepsilon) = 0 \), we obtain
\[
\mathcal{I}_5 = -\mathcal{R} \left\{ i\lambda \rho_1 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3^* v^6 v^4 \, dx \right\} - \mathcal{R} \left\{ i\lambda \rho_1 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3^* v^6 v^7 \, dx \right\}. \tag{4.47}
\]
Now, it is easy to see that
\[
\Re \left\{ -i\lambda p_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \rho_3 v_x^6 v_x^1 dx \right\} = \Re \left\{ -i\lambda p_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \rho_3 (v_x^6 - l v^2 + l v^2) v_x^1 dx \right\} \\
= \Re \left\{ -i\lambda p_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \rho_3 (v_x^6 - l v^2) v_x^1 dx \right\} - \Re \left\{ i\lambda p_1 l \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \rho_3 v_x^2 v_x^1 dx \right\},
\]
using Lemma 4.1 and the fact that \( \|v^1\| = O(|\lambda|^{-1}) \), we get
\[
\Re \left\{ -i\lambda p_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \rho_3 v_x^6 v_x^1 dx \right\} = \Re \left\{ -i\lambda p_1 l \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \rho_3 v_x^2 v_x^1 dx \right\} + o(\lambda^{-2}).
\]
Inserting \( v^2 = i\lambda v^1 - \lambda^{-4} f^1 \) in the above estimation, then using the fact that \( \|v^1\| = O(|\lambda|^{-1}) \) and \( \|f^1\| = o(1) \), we get
\[
\Re \left\{ -i\lambda p_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \rho_3 v_x^6 v_x^1 dx \right\} = l p_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \rho_3 |\lambda v^1|^2 dx + o(\lambda^{-2}),
\]
Inserting the above estimation in (4.47), we obtain
\[
I_5 = l p_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \rho_3 |\lambda v^1|^2 dx - \Re \left\{ i\lambda p_1 l \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \rho_3 v_x^6 v_x^1 dx \right\} + o(\lambda^{-2}). \tag{4.48}
\]
Now, using integration by parts and the fact that \( \rho_3(\alpha + 2\varepsilon) = \rho_3(\beta - 2\varepsilon) = 0 \), then using the definition of \( d(x) \), we obtain
\[
I_6 = \Re \left\{ \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \rho_3 \left[ k_3 (v_x^5 - l v^1) + d(x) (v_x^6 - l v^2) \right] v_x^1 dx \right\} \\
+ \Re \left\{ \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \rho_3 \left[ k_3 (v_x^5 - l v^1) + d(x) (v_x^6 - l v^2) \right] v_x^1 dx \right\} \\
= \Re \left\{ k_3 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \rho_3 (v_x^5 - l v^1) v_x^1 dx \right\} - \Re \left\{ d_0 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \rho_3 (v_x^6 - l v^2) v_x^1 dx \right\} \\
+ \Re \left\{ k_3 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \rho_3 (v_x^5 - l v^1) v_x^1 dx \right\} + \Re \left\{ d_0 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \rho_3 (v_x^6 - l v^2) v_x^1 dx \right\},
\]
consequently, by using Lemma 4.1 with \( \ell = 4 \) and the fact that \( v_x^1 \) is uniformly bounded in \( L^2(0, L) \), \( \|v_x^1\| = O(|\lambda|) \), we get
\[
I_6 = \Re \left\{ d_0 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \rho_3 (v_x^6 - l v^2) v_x^1 dx \right\} + o(\lambda^{-2}). \tag{4.49}
\]
Thus, by inserting (4.48) and (4.49) in (4.46), we obtain (4.44).

**Step 3:** In this step, we will prove that:
\[
\Re \left\{ i\lambda p_1 l \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \rho_3 v_x^6 v_x^1 dx \right\} = o(\lambda^{-2}). \tag{4.50}
\]
For this aim, take \( \ell = 4 \) in (4.8) and multiply it by \( p_3' v^7 \), integrating over \((\alpha + 2\varepsilon, \beta - 2\varepsilon)\), using the fact that \( \|v^1\| = O(|\lambda|^{-1}) \), \( \|f^0\| = o(1) \), then taking the real part, we get
\[
\Re \left\{ i\lambda \rho_1 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3'(v^6 v^7)dx \right\} + \Re \left\{ - \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3' \left[ k_3(v^6_x - lv^1) + d(x)(v^6_x - lv^2) \right] v^7 dx \right\}
\]
\[
+ \frac{lk_1}{2} \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3'(|v^1|^2) dx + \Re \left\{ \frac{l k_1}{2} \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3'(v^3 + lv^5) v^7 dx \right\} = o(|\lambda|^{-5}),
\]
using (4.40), Lemma 4.3 and the fact that \( \|v^3\| = O(|\lambda|^{-1}) \), \( \|v^5\| = O(|\lambda|^{-1}) \), we obtain
\[
\frac{l k_1}{2} \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3'(|v^1|^2) dx = o(\lambda^{-2}) \quad \text{and} \quad \Re \left\{ \frac{l k_1}{2} \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3'(v^3 + lv^5) v^7 dx \right\} = o(\lambda^{-2}).
\]
Consequently, (4.51) implies
\[
\Re \left\{ i\lambda \rho_1 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3'(v^6 v^7) dx \right\} + \Im \gamma = o(\lambda^{-2}). \tag{4.52}
\]
Using integration by parts and the fact that \( p_3'(\alpha + 2\varepsilon) = p_3'(\beta - 2\varepsilon) = 0 \), then using Lemma 4.1 and the fact that \( v^1 \) is uniformly bounded in \( L^2(0, L) \), \( \|v^1\| = O(|\lambda|^{-1}) \), we obtain
\[
\Im \gamma = \Re \left\{ \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3' \left[ k_3(v^6_x - lv^1) + d(x)(v^6_x - lv^2) \right] v^7 dx \right\}
\]
\[
+ \Re \left\{ \frac{l k_1}{2} \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3'(v^3 + lv^5) v^7 dx \right\} = o(\lambda^{-2}).
\]
Therefore, from the above estimation and (4.52), we obtain (4.50).

**Step 4:** In this step, we will prove that:
\[
\Re \left\{ d_0 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3(v^6_x - lv^2) v^7 dx \right\} = -\Re \left\{ \frac{d_0 \rho_1}{k_1} \lambda^2 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3(v^6_x - lv^2) v^7 dx \right\} + o(\lambda^{-2}). \tag{4.53}
\]
For this aim, take \( \ell = 4 \) in (4.9) and multiply it by \( \frac{d_0}{k_1} p_3(v^6_x - lv^7) \), integrating over \((\alpha + 2\varepsilon, \beta - 2\varepsilon)\) and taking the real part, then using Lemmas 4.1 and the fact that \( \|f^1\| = o(1) \), \( \|f^2\| = o(1) \), we get
\[
\Re \left\{ \frac{d_0 \rho_1}{k_1} \lambda^2 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3 v^1(v^6_x - lv^2) dx \right\} + \Re \left\{ d_0 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3 v^1 v^6_x (v^6_x - lv^2) dx \right\}
\]
\[
+ \Re \left\{ \frac{d_0 l k_3}{k_1} \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3 v^5 (v^6_x - lv^2) dx \right\} + \Re \left\{ d_0 l \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3 v^5 x^6_x (v^6_x - lv^2) dx \right\}
\]
\[
= o(|\lambda|^{-5}),
\]
consequently, by using Lemma 4.1 and the fact that \( v^7_x \) is uniformly bounded in \( L^2(0, L) \), we get
\[
\Re \left\{ d_0 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3 v^1 x^6_x (v^6_x - lv^2) dx \right\} = -\Re \left\{ \frac{d_0 \rho_1}{k_1} \lambda^2 \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} p_3 v^1 (v^6_x - lv^2) dx \right\} + o(\lambda^{-2}).
\]
Thus, from the above estimation, we obtain (4.53).
Step 5: In this step, we conclude the proof of (4.36). For this aim, inserting (4.50) and (4.53) in (4.44), then using Young’s inequality, Lemma 4.1 and the fact that \( \ell = 4 \), we get
\[
2l\rho_1 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 |\lambda v^1|^2 \, dx = \Re \left\{ \frac{d_0\rho_1}{k_1} \lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3(v_x^6 - lv^2)v^1 \, dx \right\} + o(\lambda^{-2})
\]
\[
\leq \frac{d_0\rho_1}{k_1} \lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3|v_x^6 - lv^2||v^1| \, dx + o(\lambda^{-2})
\]
\[
= \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} \left( \frac{d_0\sqrt{\rho_1}}{k_1} \lambda \sqrt{p_3}\alpha |v_x^6 - lv^2| \right) \left( \sqrt{l\rho_1} \lambda \sqrt{p_3}|v^1| \right) \, dx + o(\lambda^{-2})
\]
\[
\leq \frac{\rho_1 d_0^2}{2k_1^2} \lambda^2 \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3|v_x^6 - lv^2|^2 \, dx + \frac{l\rho_1}{2} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 |\lambda v^1|^2 \, dx + o(\lambda^{-2}),
\]
consequently, we obtain
\[
\frac{3l\rho_1}{2} \int_{\alpha+2\varepsilon}^{\beta-2\varepsilon} p_3 |\lambda v^1|^2 \, dx = o(\lambda^{-2}).
\]
Finally, from the above estimation and the definition of \( p_3 \), we obtain (4.36). The proof is thus complete. \( \square \)

Lemma 4.6. If \( k_1 \rho_1 \neq k_2 \rho_2 \) and \( \ell = 4 \). Then, the solution \( U = (v^1, v^2, v^3, v^4, v^5, v^6)^T \in D(A) \) of system (4.3)-(4.8) satisfies the following estimations
\[
\int_{\alpha+4\varepsilon}^{\beta-4\varepsilon} |v^3|^2 \, dx = o(1) \quad \text{and} \quad \int_{\alpha+5\varepsilon}^{\beta-5\varepsilon} |\lambda v^3|^2 \, dx = o(1). \tag{4.54}
\]

Proof. First, take \( \ell = 4 \) in (4.9) and multiply it by \( k_1^{-1}p_4v_x^2 \), integrating over \((\alpha + 3\varepsilon, \beta - 3\varepsilon)\), using the definition of \( d(x) \) and the fact that \( v_x^3 \) is uniformly bounded in \( L^2(0, L) \), \( \|f^1\| = o(1) \), \( \|f^2\| = o(1) \), then taking the real part, we obtain
\[
\Re \left\{ \frac{\lambda^2 \rho_1}{k_1} \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} p_4v^1 v_x^2 \, dx + \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} p_4 v_x^1 v_x^2 \, dx + \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} p_4 v_x^3|^2 \, dx + l\int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} p_4v_x^5 v_x^2 \, dx \right\} = o(|\lambda|^{-3}),
\]
consequently, from Lemmas 4.1, 4.5 with \( \ell = 4 \) and the fact that \( v_x^3 \) is uniformly bounded in \( L^2(0, L) \), we obtain
\[
\Re \left\{ \int_{\alpha+\varepsilon}^{\beta-\varepsilon} p_4v_x^1 v_x^2 v_x^2 \, dx \right\} + \int_{\alpha+\varepsilon}^{\beta-\varepsilon} p_4 |v_x^3|^2 |v_x^2| \, dx = o(1). \tag{4.55}
\]
Now, take \( \ell = 4 \) in (4.10) and multiply it by \( k_2^{-1}p_4v_x^2 \), integrating over \((\alpha + 3\varepsilon, \beta - 3\varepsilon)\) and integrating by parts, using the fact that \( v_x^3 \) is uniformly bounded in \( L^2(0, L) \) and \( \|f^3\| = o(1) \), \( \|f^4\| = o(1) \), then taking the real part, we obtain
\[
\Re \left\{ -\frac{\lambda^2 \rho_2}{k_2} \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} p_4 v_x^3 v^1 \, dx - \frac{\lambda^2 \rho_2}{k_2} \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} p_4 v_x^3 v^1 \, dx - \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} p_4 v_x^3 v_x^2 \, dx - \int_{\alpha+3\varepsilon}^{\beta-3\varepsilon} p_4 v_x^3 v_x^2 \, dx \right\} = o(|\lambda|^{-3}),
\]
consequently, from Lemmas 4.3, 4.5 with \( \ell = 4 \) and the fact that \( v_3^2, (v_x^3 + v^3 + lv^5) \) are uniformly bounded in \( L^2(0, L) \) and \( \|v^3\| = O(|\lambda|^{-1}) \), we obtain

\[
\Re \left\{ - \int_{\alpha+3\epsilon}^{\beta-3\epsilon} p_4 v_3^2 \overline{v_x^2} \, dx \right\} = o(1). \tag{4.56}
\]

Adding (4.55) and (4.56), then using the definition of \( p_4 \), we obtain the first estimation in (4.54). Next, take \( \ell = 4 \) in (4.10) and multiply it by \( p_5 v^3 \), integrating over \((\alpha + 4\epsilon, \beta - 4\epsilon)\) and integrating by parts, then using the fact that \( \|v^3\| = O(|\lambda|^{-1}) \), \( \|f^3\| = o(1) \), \( \|f^4\| = o(1) \), we obtain

\[
\rho_2 \int_{\alpha+4\epsilon}^{\beta-4\epsilon} p_5 |v^3|^2 \, dx = k_2 \int_{\alpha+4\epsilon}^{\beta-4\epsilon} p_5 |v_x^3|^2 \, dx + k_2 \int_{\alpha+4\epsilon}^{\beta-4\epsilon} p_5 v_x^3 \overline{v^3} \, dx + k_1 \int_{\alpha+4\epsilon}^{\beta-4\epsilon} p_5 (v_x^1 + v^3 + lv^5) \overline{v^3} \, dx + o(\lambda^{-4}).
\]

From the above estimation, the first estimation in (4.54) and the fact that \( (v_x^1 + v^3 + lv^5) \) is uniformly bounded in \( L^2(0, L) \) and \( \|v^3\| = O(|\lambda|^{-1}) \), we obtain

\[
\rho_2 \int_{\alpha+4\epsilon}^{\beta-4\epsilon} p_5 |v^3|^2 \, dx = o(1).
\]

Finally, from the above estimation and the definition of \( p_5 \), we obtain the second estimation in (4.54). The proof is thus complete. \( \square \)

**Lemma 4.7.** Let \( h \in C^1([0, L]) \) such that \( h(0) = h(L) = 0 \). If \( \left( \frac{k_1}{p_1} = \frac{k_2}{p_2} \text{ and } \ell = 2 \right) \) or \( \left( \frac{k_1}{p_1} \neq \frac{k_2}{p_2} \text{ and } \ell = 4 \right) \), then the solution \( U = (v^1, v^2, v^3, v^4, v^5, v^6)^T \in D(A) \) of system (4.3)–(4.8) satisfies the following estimation

\[
\int_0^L h' \left( \rho_1 |\lambda v^1|^2 + k_1 |v_x^1|^2 + k_2 |v^3|^2 + k_2 |v_x^3|^2 + \rho_1 |\lambda v^5|^2 + k_3^{-1} |v_x^5 + d(x)(v_x^6 - lv^2)|^2 \right) \, dx = o(1).
\]

**Proof.** First, multiplying (4.9) by \( 2hv_x^3 \), integrating over \((0, L)\), taking the real part, then using Lemma 4.1, the fact that \( v_x^1 \) is uniformly bounded in \( L^2(0, L), \|v^1\| = O(|\lambda|^{-1}), \|f^1\| = o(1) \) and \( \|f^2\| = o(1) \), we obtain

\[
\int_0^L h \left( \rho_1 |\lambda v^1|^2 + k_1 |v_x^1|^2 \right) \, dx + \Re \left\{ 2k_1 \int_0^L h v_x^1 \overline{v_x^1} \, dx \right\} + \Re \left\{ 2l_k k_2 \int_0^L h v_x^1 \overline{v_x^1} \, dx \right\} = o(1) + |\lambda|^{-\ell+1}.
\]

Now, multiplying (4.10) by \( 2hv_x^3 \), integrating over \((0, L)\), taking the real part, then using the fact that \( v_x^3 \) is uniformly bounded in \( L^2(0, L), \|v^3\| = O(|\lambda|^{-1}), \|f^3\| = O(|\lambda|^{-1}), \|f^4\| = o(1) \) and \( \|f^5\| = o(1) \), we obtain

\[
\int_0^L h \left( |\rho_2 \lambda v^3|^2 + k_2 |v_x^3|^2 \right) \, dx - \Re \left\{ 2k_1 \int_0^L h v_x^3 \overline{v_x^3} \, dx \right\} - \Re \left\{ 2k_1 \int_0^L h (v^3 + lv^5) \overline{v_x^3} \, dx \right\} = o(1).
\]

Let \( S := k_3 v_x^5 + d(x)(v_x^6 - lv^2) \), from Lemma 4.1, the definition of \( d(x) \) and the fact that \( v_x^5 \) is uniformly bounded in \( L^2(0, L) \), we get \( S \) is uniformly bounded in \( L^2(0, L) \). Now, multiplying (4.11) by \( 2k_3^{-1} hS \),
integrating over $(0, L)$, taking the real part, then using the fact that $\|v^3\| = O(|\lambda|^{-1})$, $\|v^5\| = O(|\lambda|^{-1})$, $\|f^5\| = o(1)$ and $\|f^6\| = o(1)$, we obtain

$$\Re \left\{ \frac{2\lambda^2 \rho_1}{k_3} \int_0^L h v^5 \mathcal{S}_{dx} - k_3^{-1} \int_0^L h \left( |S|^2 \right)_x \ dx - \Re \left\{ \frac{2l(k_1 + k_3)}{k_3} \int_0^L h v_{x}^1 \mathcal{S}_{dx} \right\} \right\}_{= o(1)}$$

Moreover, from the definition of $S$ and $d(x)$, Lemma 4.1 and the fact that $v^1_x$ is uniformly bounded in $L^2(0, L)$, $\|v^5\| = O(|\lambda|^{-1})$, we obtain

$$\int_0^L h \left( \rho_1 |\lambda v^5|_x^2 + k_3^{-1} |S|_x^2 \right) \ dx - \Re \left\{ \frac{2\lambda^2 \rho_1 d_0}{k_3} \int_0^L h v^5 (v_{x}^2 - \overline{v^2}) \ dx \right\}_{= o(|\lambda|^{-\frac{\ell}{2}})}$$

Inserting the above estimations in (4.59) and using the fact that $\ell \in \{2, 4\}$, we obtain

$$\int_0^L h \left( \rho_1 |\lambda v^5|_x^2 + k_3^{-1} |S|_x^2 \right) \ dx - \Re \left\{ \frac{2l(k_1 + k_3)}{k_3} \int_0^L h v_{x}^1 v_{x}^2 \ dx \right\}_{= o(1)}.$$  

Adding (4.57), (4.58), (4.60) and using the fact that $\ell \in \{2, 4\}$, then using integration by parts, we obtain (4.7). The proof is thus complete.

**Lemma 4.8.** The solution $U = (v^1, v^2, v^3, v^4, v^5, v^6)^T \in D(A)$ of system (4.3)–(4.8) satisfies the following estimations

$$J(\alpha + 4\varepsilon, \beta - 4\varepsilon) = o(1) \quad \text{if} \quad \frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \quad \text{and} \quad \ell = 2, \quad (4.61)$$

$$J(\alpha + 5\varepsilon, \beta - 5\varepsilon) = o(1) \quad \text{if} \quad \frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2} \quad \text{and} \quad \ell = 4, \quad (4.62)$$

where

$$J(\gamma_1, \gamma_2) := \int_0^{\gamma_1} \left( \rho_1 |\lambda v^1|^2 + k_1 |v_{x}^1|^2 + \rho_2 |\lambda v^3|^2 + k_2 |v_{x}^3|^2 + \rho_1 |\lambda v^5|^2 \right) \ dx + k_3 \int_0^\alpha |v_{x}^5|^2 \ dx$$

$$+ \int_{\gamma_2}^L \left( \rho_1 |\lambda v^1|^2 + k_1 |v_{x}^1|^2 + \rho_2 |\lambda v^3|^2 + k_2 |v_{x}^3|^2 + \rho_1 |\lambda v^5|^2 \right) \ dx + k_3 \int_{\beta}^L |v_{x}^5|^2 \ dx,$$

for all $0 < \alpha < \gamma_1 < \gamma_2 < \beta < L$. 


Proof. First, take $h = xq_1 + (x - L)q_2$ in (4.7), then using the definition of $d(x)$ and the fact that $0 < \alpha < \gamma_1 < \gamma_2 < \beta < L$, we obtain

$$
\int_{\gamma_1}^{\gamma_2} \left( \rho_1 |\lambda v|^2 + k_1 |v_x|^2 + k_2 |\lambda v|^2 + \rho_1 |\lambda v|^2 \right) dx + k_3 \int_0^\alpha |v_x|^2 dx
$$
\[
+ \int_{\gamma_2}^L \left( \rho_1 |\lambda v|^2 + k_1 |v_x|^2 + k_2 |\lambda v|^2 + \rho_1 |\lambda v|^2 \right) dx + k_3 \int_\beta^L |v_x|^2 dx
\]
\[
= -\int_{\gamma_1}^{\gamma_2} (q_1 + xq_1) \left( \rho_1 |\lambda v|^2 + k_1 |v_x|^2 + k_2 |\lambda v|^2 + \rho_1 |\lambda v|^2 \right) dx
\]
\[
+ k_3^{-1} \int_{\gamma_1}^{\gamma_2} \left( k_3v_x^5 + d_0(v_x^6 - lv^3)^2 \right) dx
\]
\[
+ k_3^{-1} \int_{\gamma_1}^{\gamma_2} q_1|k_3v_x^5 + d_0(v_x^6 - lv^3)^2| dx + k_3^{-1} \int_{\gamma_1}^\alpha \left( k_3v_x^5 + d_0(v_x^6 - lv^3)^2 \right) dx.
\]

Now, take $\gamma_1 = \alpha + 4\varepsilon$ and $\gamma_2 = \beta - 4\varepsilon$ in the above equation, then using Lemmas 4.1-4.4 in case of $\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2}$ and $\ell = 2$, we obtain (4.61). Finally, take $\gamma_1 = \alpha + 5\varepsilon$ and $\gamma_2 = \beta - 5\varepsilon$ in the above equation, then using Lemmas 4.1-4.3, 4.6 in case of $\frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2}$ and $\ell = 4$, we obtain (4.62). The proof is thus complete. \qed

Proof of Theorem 4.1. First, from Lemmas 4.1-4.4 and the fact that $\ell = 2$, we obtain

$$
\begin{cases}
\int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} |\lambda v|^2 dx = o(1), & \int_{\alpha + \varepsilon}^{\beta - \varepsilon} |v|^6 dx = o(1), \quad \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} |v_x|^2 dx = o(1) \\
\int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} |\lambda v|^2 dx = o(1), & \int_{\alpha + 3\varepsilon}^{\beta - 3\varepsilon} |v_x|^2 dx = o(1) \quad \text{and} \quad \int_{\alpha + 4\varepsilon}^{\beta - 4\varepsilon} |\lambda v|^2 dx = o(1).
\end{cases}
$$

(4.63)

Now, from (4.61), (4.63) and the fact that $0 < \varepsilon < \frac{\beta - \alpha}{10}$, we deduce that $\|U\|_{H^1} = o(1)$ in $(0, L)$, which contradicts (H). This implies that

$$
\sup_{\lambda \in \mathbb{R}} \| (i\lambda I - A)^{-1} \|_{H} = O \left( \lambda^2 \right).
$$

Finally, according to Theorem A.3, we obtain the desired result. The proof is thus complete. \qed

Proof of Theorem 4.2. First, from Lemmas 4.1, 4.2, 4.3, 4.6 and the fact that $\ell = 4$, we obtain

$$
\begin{cases}
\int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} |\lambda v|^2 dx = O(\lambda^{-2}), & \int_{\alpha + \varepsilon}^{\beta - \varepsilon} |v|^6 dx = o(1), \quad \int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} |v_x|^2 dx = o(\lambda^{-2}) \\
\int_{\alpha + 2\varepsilon}^{\beta - 2\varepsilon} |\lambda v|^2 dx = o(1), & \int_{\alpha + 4\varepsilon}^{\beta - 4\varepsilon} |v_x|^2 dx = o(1) \quad \text{and} \quad \int_{\alpha + 5\varepsilon}^{\beta - 5\varepsilon} |\lambda v|^2 dx = o(1).
\end{cases}
$$

(4.64)

Now, from (4.62), (4.64) and the fact that $0 < \varepsilon < \frac{\beta - \alpha}{10}$, we deduce that $\|U\|_{H} = o(1)$ in $(0, L)$, which contradicts (H). This implies that

$$
\sup_{\lambda \in \mathbb{R}} \| (i\lambda I - A)^{-1} \|_{H} = O \left( \lambda^4 \right).
$$

Finally, according to Theorem A.3, we obtain the desired result. The proof is thus complete. \qed
5. Conclusion

We have studied the stabilization of a Bresse system with one discontinuous local internal viscoelastic damping of Kelvin–Voigt type acting on the axial force under fully Dirichlet boundary conditions. We proved the strong stability of the system by using Arendt–Batty criteria. We proved that the energy of our system decays polynomially with the rates:

\[
\begin{cases}
t^{-1} & \text{if } \frac{k_1}{\rho_1} = \frac{k_2}{\rho_2}, \\
t^{-\frac{1}{2}} & \text{if } \frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2}.
\end{cases}
\]

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Appendix A. Some notions and stability theorems

In order to make this paper more self-contained, we recall in this short appendix some notions and stability results used in this work.

Definition A.1. Assume that \( A \) is the generator of \( C_0 \)-semigroup of contractions \( (e^{tA})_{t \geq 0} \) on a Hilbert space \( H \). The \( C_0 \)-semigroup \( (e^{tA})_{t \geq 0} \) is said to be

1. Strongly stable if
   \[
   \lim_{t \to +\infty} \| e^{tA}x_0 \|_H = 0, \quad \forall x_0 \in H.
   \]
2. Exponentially (or uniformly) stable if there exists two positive constants \( M \) and \( \varepsilon \) such that
   \[
   \| e^{tA}x_0 \|_H \leq Me^{-\varepsilon t} \| x_0 \|_H, \quad \forall t > 0, \; \forall x_0 \in H.
   \]
3. Polynomially stable if there exists two positive constants \( C \) and \( \alpha \) such that
   \[
   \| e^{tA}x_0 \|_H \leq Ct^{-\alpha} \| Ax_0 \|_H, \quad \forall t > 0, \; \forall x_0 \in D(A).
   \]

To show the strong stability of the \( C_0 \)-semigroup \( (e^{tA})_{t \geq 0} \), we rely on the following result due to Arendt–Batty [8].

Theorem A.2. Assume that \( A \) is the generator of a \( C_0 \)-semigroup of contractions \( (e^{tA})_{t \geq 0} \) on a Hilbert space \( H \). If \( A \) has no pure imaginary eigenvalues and \( \sigma (A) \cap i\mathbb{R} \) is countable, where \( \sigma (A) \) denotes the spectrum of \( A \), then the \( C_0 \)-semigroup \( (e^{tA})_{t \geq 0} \) is strongly stable.

Concerning the characterization of polynomial stability stability of a \( C_0 \)-semigroup of contraction \( (e^{tA})_{t \geq 0} \), we rely on the following result due to Borichev and Tomilov [12] (see also [11,31])

Theorem A.3. Assume that \( A \) is the generator of a strongly continuous semigroup of contractions \( (e^{tA})_{t \geq 0} \) on \( \mathcal{H} \). If \( i\mathbb{R} \subset \rho(A) \), then for a fixed \( \ell > 0 \) the following conditions are equivalent

\[
\sup_{\lambda \in \mathbb{R}} \left\| (i\lambda I - A)^{-1} \right\|_{\mathcal{L}(\mathcal{H})} = O \left( |\lambda|^\ell \right), \quad (5.1)
\]

\[
\| e^{tA}U_0 \|^2_{\mathcal{H}} \leq \frac{C}{t^\ell} \| U_0 \|^2_{D(A)}, \; \forall t > 0, \; U_0 \in D(A), \; \text{for some } C > 0. \quad (5.2)
\]

\( \square \)
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