POLYNOMIAL ALGEBRAS AND THEIR APPLICATIONS

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ABSTRACT: A way to construct and classify the three dimensional polynomially deformed algebras is given and the irreducible representations is presented. for the quadratic algebras 4 different algebras are obtained and for cubic algebras 12 different classes are constructed. Applications to quantum mechanical systems including supersymmetric quantum mechanics are discussed
1. Introduction

Nonlinear algebras have been useful in describing the algebraic structure of many physical systems during the last decade. The q-deformation of the Lie algebras, the quantum groups and $W$ algebras have been studied extensively both in physics as well as in mathematics\cite{1,2,3,4,5}. They are infinite dimensional algebras belonging to the class of universal enveloping algebras of standard Lie algebras. Besides these, algebras which are finite polynomial deformations of the Lie algebras, known as nonlinear polynomial algebras have also been observed as the dynamical symmetry algebras and as spectrum generating algebras of many quantum mechanical systems. In a large class of these systems the spectra are not linear, but quadratic or of higher orders and if the Casimir operator is taken as a Hamiltonian (as in most cases), the advantages of using Lie algebras as a dynamical symmetries are lost. To use the dynamical symmetry properties effectively, it is necessary to extend the Lie algebra to polynomial algebras in which the Hamiltonian is one of the diagonal generators. In this way, one can relate the degree of degeneracy to the dimensions of the representation and find out how the additional symmetry generators transform one degenerate eigenstate to another. For this it is useful also that additional symmetry generators close on a finite algebra, which in most cases is a quadratic algebra.

Polynomial Algebras with three generators $N_i, N_j, N_k$ are defined by the relations

$$[N_i, N_j] = C_{ij} (N_k),$$  \hspace{1cm} (1.1)
where the functions $C_{ij}$ are polynomial functions of the generators $\{N_k\}$ constrained by the Jacobi identity
\[ [N_i, C_{jk}] + [N_j, C_{ki}] + [N_k, C_{ij}] = 0. \] (1.2)

In many cases these non-linear algebras admit a coset structure i.e. among the generators there is a linear subalgebra and the commutator of the remaining generators give a symmetric function of the generators of the linear algebra. $p$ of the $d$ generators $N_i$ satisfy
\[ [N_i, N_j] = f_{ij}^k N_k, \quad i, j = 1...p. \] (1.3)
The remaining $d - p$ generators satisfy,
\[ [N_i, N_\alpha] = f_{i\alpha}^\beta N_\beta, \]
\[ [N_\alpha, N_\beta] = f_{\alpha\beta}(N_\gamma), \] (1.4)

A simplest example of such algebra is the case with $d = 1$.
\[ [N_0, N_\pm] = \pm N_\pm \]
\[ [N_+, N_-] = f(N_0) \] (1.5)
This is also known as polynomial $SU(2)$ or $SU(1, 1)$ algebra, a three dimensional polynomial algebra. A Casimir operator for this algebra can be taken as
\[ C = N_- N_+ + g(N_0) = N_+ N_- + g(N_0 - 1) \] (1.6)
\[ f(N_0) = g(N_0) - g(N_0 - 1) \] (1.7)
where $g(N_0)$ is a polynomial in $N_0$ having one degree higher than $f(N_0)$.

Three dimensional quadratic algebras are defined by the relations
\[ [Q_0, Q_\pm] = \pm Q_\pm \]
\[ [Q_+, Q_-] = a Q_0^3 + b Q_0 + c, \]
The Casimir operator for the algebra is given by,
\[ C = Q_+ Q_- + g(Q_0 - 1) \]
\[ C = Q_+ Q_- + \frac{1}{3} a Q_0^3 - \frac{1}{2} (a - b) Q_0^2 + \frac{1}{6} (a - 3b + 6c) Q_0 - c \]
A general three dimensional cubic algebra with a coset structure is given by,
\[ [C_0, C_\pm] = \pm C_\pm \]
\[ [C_+, C_-] = a C_0^3 + b C_0^2 + c C_0 + d, \]
The structure constants $a, b, c$ and $d$ are constants. The Casimir operator for this algebra is given by
\[ C = C_- C_+ + \frac{a}{4} C_0^4 + (\frac{b}{3} - \frac{a}{2}) C_0^3 + (\frac{a}{4} + \frac{b}{2} + \frac{c}{2}) C_0^2 + (\frac{b}{6} - \frac{c}{2} + d) C_0 \] (1.8)
An \(n^{th}\) order three dimensional polynomial algebra, generated by \(P_n^+, P_n^-, P_0^{(n)}\) is defined as

\[
\begin{align*}
[ P_0^{(n)}, P_\pm^{(n)} ] &= P_\pm^{(n)} \\
[ P_+^{(n)}, P_-^{(n)} ] &= f_n(P_0^{(n)}) \tag{1.9}
\end{align*}
\]

Here \(f_n(P_0)\) is an \(n^{th}\) order polynomial in \(P_0^{(n)}\).

2. Jordan Schwinger Like Construction Of Polynomial Algebra

Let \((P_\pm^{(n)}, P_0^{(n)})\) and \((P_\pm^{(m)}, P_0^{(m)})\) be the two sets of two dimensional polynomial algebras of order \(n\) and \(m\) respectively. Then the following operators defined as

\[
\begin{align*}
\Pi_+ &= P_+^{(m)}P_+^{(n)}, \quad \Pi_- = P_-^{(m)}P_-^{(n)}, \\
\Pi_0 &= \frac{P_0^{(m)} + P_0^{(n)}}{2} \tag{2.1}
\end{align*}
\]

will always satisfy a polynomial algebra of order \(m + n + 1\) if \(\Pi = \frac{P_0^{(m)} - P_0^{(n)}}{2}\) is a constant of motion.

Similarly

\[
\begin{align*}
\Pi_+ &= P_+^{(m)}P_-^{(n)}, \quad \Pi_- = P_-^{(m)}P_+^{(n)}, \\
\Pi_0 &= \frac{P_0^{(m)} - P_0^{(n)}}{2}, \tag{2.2}
\end{align*}
\]

will also satisfy an polynomial algebra \(m + n + 1\) if \(\Pi = \frac{P_0^{(m)} + P_0^{(n)}}{2}\) is a constant of motion.

Proof:

\[
\begin{align*}
[\Pi_0, \Pi_\pm] &= \frac{1}{2} \left[ P_\pm^{(m)} + P_\pm^{(n)}, P_\pm^{(m)}P_\pm^{(n)} \right] \\
&= \pm P_\pm^{(m)}P_\pm^{(n)} \\
&= \pm \Pi_\pm \tag{2.3}
\end{align*}
\]

\[
\begin{align*}
[\Pi_+, \Pi_-] &= \left[ P_+^{(m)}P_+^{(n)}, P_-^{(m)}P_-^{(n)} \right] \tag{2.4} \\
&= P_+^{(m)}P_-^{(m)} f_n(P_0^{(n)}) + P_-^{(m)}P_+^{(m)} f_m(P_0^{(m)}) \\
&= \left( C^{(m)} - g_{m+1}(P_0^{(m)}) - 1 \right) f_n(P_0^{(n)}) \\
&\quad + \left( C^{(n)} - g_{n+1}(P_0^{(n)}) - 1 \right) f_m(P_0^{(m)}) \tag{2.5}
\end{align*}
\]
\(C^{(m)}\) and \(C^{(n)}\) are the Casimir operators. Writing \(P_0\) in terms of \(\Pi_0\) and \(\Pi\) we get

\[
RHS = C^{(m)} \sum_{l=0}^{n} C_l (\Pi_0 - \Pi)^l
- \sum_{l=0}^{m+1} \sum_{s=0}^{n} a_l c_s (\Pi_0 + \Pi)^l (\Pi_0 - \Pi)^s
+ C^{(k)} \sum_{l=0}^{m} d_l (\Pi_0 + \Pi)^l
- \sum_{l=0}^{n+1} \sum_{s=0}^{m} b_l d_s (\Pi_0 - \Pi)^l (\Pi_0 + \Pi)^s
= f_{m+n+1}(\Pi_0^{m+n+1})
\]

\(f_{m+n+1}(\Pi_0^{m+n+1})\) a polynomial in \(\Pi_0^{m+n+1}\) of order \(m + n + 1\). Using the above construction the tables 1 and 2 give the classification of Quadratic and cubic algebras.

**Classification of Quadratic Algebras**

| Case | J-S rep. | Algebra | Constants |
|------|----------|---------|-----------|
| \(Q^{-(2)}\) | \(Q_0 = \frac{1}{2}(J_0 - N)\); \(Q_+ = J_\pm a; Q_- = J_- a^\dagger\) | \([Q_0, Q_{\pm}] = \pm Q_{\pm}\), \([Q_+, Q_-] = -3Q_0^2 - (2L - 1)Q_0 + (J + L(L + 1))\) | \(L = \frac{1}{2}(J_0 + N)\) |
| \(Q^{+(2)}\) | \(Q_0 = \frac{1}{2}(J_0 + N)\); \(Q_+ = J_\pm a^\dagger; Q_- = J_- a\) | \([Q_0, Q_{\pm}] = \pm Q_{\pm}\), \([Q_+, Q_-] = -3Q_0^2 + (2L + 1)Q_0 - (J + L(L - 1))\) | \(L = \frac{1}{2}(J_0 - N)\) |
| \(Q^{-(1,1)}\) | \(Q_0 = \frac{1}{2}(K_0 - N)\); \(Q_+ = K_\pm a^\dagger; Q_- = K_- a^\dagger\) | \([Q_0, Q_{\pm}] = \pm Q_{\pm}\), \([Q_+, Q_-] = +3Q_0^2 - (2L - 1)Q_0 + (K + (L(L + 1)))\) | \(L = \frac{1}{2}(K_0 + N)\) |
| \(Q^{+(1,1)}\) | \(Q_0 = \frac{1}{2}(K_0 + N)\); \(Q_+ = K_\pm a^\dagger; Q_- = J_- a\) | \([Q_0, Q_{\pm}] = \pm Q_{\pm}\), \([Q_+, Q_-] = -3Q_0^2 - (2L - 1)Q_0 + (K + (L(L + 1)))\) | \(L = \frac{1}{2}(K_0 - N)\) |
Classification of Cubic Algebras

| case    | J-S rep. | Algebra                                | Constants   |
|---------|----------|----------------------------------------|-------------|
| $C^-(11,11)$ | $C_0 = \frac{1}{2}(L_0 - M_0)$; $C_+ = \mu L_+ M_+; C_- = L_- M_-$ | $|C_0, C_\pm\rangle = \pm C_\pm, \quad [C_+, C_-] = -4\mu^2 C_0^3 - C_0(K^2\mu^2 - \sigma) + \lambda K$ | $K = \frac{1}{2}(L_0 + M_0)$ |
| $C^+(11,11)$ | $C_0 = \frac{1}{2}(L_0 + M_0)$; $C_+ = \lambda L_+ M_+; C_- = L_- M_-$ | $|C_0, C_\pm\rangle = \pm C_\pm, \quad [C_+, C_-] = -4\mu^2 C_0^3 + C_0(K^2\mu^2 - \sigma) + \lambda K$ | $K = \frac{1}{2}(L_0 - M_0)$ |
| $C^-(2,2)$ | $C_0 = \frac{1}{2}(J_0 - P_0)$; $C_+ = J_+ P_+; C_- = J_- P_-$ | $|C_0, C_\pm\rangle = \pm C_\pm, \quad [C_+, C_-] = 4C_0^3 - (4K^2 + \sigma)C_0 + \lambda K$ | $K = \frac{1}{2}(J_0 + P_0)$ |
| $C^+(2,2)$ | $C_0 = \frac{1}{2}(J_0 + P_0)$; $C_+ = J_+ P_+; C_- = J_- P_-$ | $|C_0, C_\pm\rangle = \pm C_\pm, \quad [C_+, C_-] = 4C_0^3 - 4C_0K^2 + 2(J + C_1)$ | $K = \frac{1}{2}(J_0 - P_0)$ |
| $C^-(2,11)$ | $C_0 = \frac{1}{2}(Q_0 - N)$; $C_+ = Q_+a^1; C_- = Q_-a$ | $|C_0, C_\pm\rangle = \pm C_\pm, \quad [C_+, C_-] = 4C_0^3 - 4C_0K^2 + 2(J - C_1)$ | $K = \frac{1}{2}(Q_0 - L_0)$ |
| $C^+(2,11)$ | $C_0 = \frac{1}{2}(Q_0 + N)$; $C_+ = Q_+a^1; C_- = Q_-a$ | $|C_0, C_\pm\rangle = \pm C_\pm, \quad [C_+, C_-] = 4C_0^3 - 6K - L - 2C_0^2 + EC_0 + F$ | $K = \frac{1}{2}(Q_0 + N)$ |

3. Construction Of Representations

Since we have constructed a higher order algebra from a lower algebra the representation of the higher order algebra can also be constructed from the lower order algebra. For example, let $|m, \lambda^{(m)}\rangle$ and $|n, \lambda^{(n)}\rangle$ be the basis states for the irreducible unitary representations of the polynomial algebras $(\mathcal{P}_0^{(m)}, \mathcal{P}_\pm^{(m)})$ and $(\mathcal{P}_0^{(n)}, \mathcal{P}_\pm^{(n)})$ respectively, where $\lambda^{(m)}$ and $\lambda^{(n)} \in \mathbb{R}$, labeling the unitary irreducible representations,

$$\mathcal{P}_0^{(m)} |q, \lambda^{(m)}\rangle = q + \lambda^{(m)} |q, \lambda^{(m)}\rangle$$

$$\mathcal{P}_+^{(m)} |q, \lambda^{(m)}\rangle = i\sqrt{t_q} |q + 1, \lambda^{(m)}\rangle$$

$$\mathcal{P}_-^{(m)} |q, \lambda^{(m)}\rangle = i\sqrt{t_q} |q - 1, \lambda^{(m)}\rangle$$

(3.1)

$$\mathcal{P}_0^{(n)} |s, \lambda^{(n)}\rangle = s + \lambda^{(n)} |s, \lambda^{(n)}\rangle$$

$$\mathcal{P}_+^{(n)} |s, \lambda^{(n)}\rangle = i\sqrt{d_s} |s + 1, \lambda^{(n)}\rangle$$

$$\mathcal{P}_-^{(n)} |s, \lambda^{(n)}\rangle = i\sqrt{d_s} |s - 1, \lambda^{(n)}\rangle$$

(3.2)

where $q, s = 0, \pm 1, ...$

The fact that $\Pi$ is a constant over the product state $|q, \lambda^{(m)}\rangle \ast |s, \lambda^{(n)}\rangle$ gives the condition,

$$2\Pi = \lambda^{(m)} + \lambda^{(n)} + q + s$$

(3.3)
If we impose the above constraint in the product states the following representations are possible:

\[ \Pi_0^{(m)} |q, \lambda^{(m)}, \lambda^{(n)}, \Pi\rangle = q + \lambda^{(m)} - \Pi |q, \lambda^{(m)}, \lambda^{(n)}, \Pi\rangle \]
\[ \Pi_+^{(m)} |q, \lambda^{(m)}, \lambda^{(n)}, \Pi\rangle = \sqrt{t_0^{(m)} \lambda^{(n)} d^{(n)}_{2\Pi - \lambda^{(m)} - \lambda^{(n)} - q - 1}} |q + 1, \lambda^{(m)}, \lambda^{(n)}, \Pi\rangle \]
\[ \Pi_-^{(m)} |q, \lambda^{(m)}, \lambda^{(n)}, \Pi\rangle = \sqrt{t_0^{(m)} \lambda^{(n)} d^{(n)}_{2\Pi - \lambda^{(m)} - \lambda^{(n)} + q - 1}} |q - 1, \lambda^{(m)}, \lambda^{(n)}, \Pi\rangle \]  

(3.4)

The dimension of the representations are decided by the condition

\[ t_0^{(m)} d^{(n)}_{2\Pi - \lambda^{(m)} - \lambda^{(n)} - q} \geq 0 \]
\[ 2\Pi - \lambda^{(m)} - \lambda^{(n)} - q \geq 0 \]  

(3.5)

As an example of our method we construct the representation of \( Q^- (1, 1) \), given by

\[ Q_0 = -\frac{1}{2} \left( K_0 - a_3^\dagger a_3 \right), \]
\[ Q_+ = K_+ a_3, \quad Q_- = Q_+^\dagger = K_- a_3^\dagger, \]
\[ K = \frac{1}{4} \left[ 1 - \left( a_3^\dagger a_1 - a_2^\dagger a_2 \right)^2 \right] = K^2, \]
\[ L = \frac{1}{2} \left( K_0 + a_3^\dagger a_3 \right), \]

where \((K_0, K_+, K_-)\) generate su(1, 1) with \( K^2 \) as the Casimir operator. \((Q_0, Q_+, Q_-)\) generate a quadratic algebra:

\[ [K, L] = 0, \quad [K, Q_{0, \pm}] = 0, \quad [L, Q_{0, \pm}] = 0, \]
\[ [Q_0, Q_{\pm}] = \pm Q_{\pm}, \]
\[ [Q_+, Q_-] = 3Q_0^2 + (2L - 1)Q_0 \]

with \( K \) and \( L \) taking constant values in any irreducible representation. The Casimir operator of this algebra is given by

\[ C = Q_+Q_- + Q_0^3 + (L - 2)Q_0^2 + (K - L^2 - 2L + 1)Q_0, \]

(3.6)

The condition that \( K \) and \( L \) take constant values in an irreducible representation fixes the basis to be the set of three-mode Fock states

\[ |k, l, n\rangle = |n, n + 2k - 1, 2l - k - n\rangle, \quad n = 0, 1, 2, \ldots, (2l - k), \]

(3.7)

with

\[ 2l - k = 0, 1, 2, \ldots, \quad k = 1/2, 1, 3/2, \ldots, \]

(3.8)
and
\[ K |k, l, n\rangle = k(1-k) |k, l, n\rangle, \quad L |k, l, n\rangle = l |k, l, n\rangle. \quad (3.9) \]

The basis states carry the \((2l-k+1)\)-dimensional unitary irreducible representation of the quadratic algebra which can be labeled by the values of the pair \((k, l)\). Explicitly, the \((k, l)\)-th representation is:
\[
\begin{align*}
Q_0 |k, l, n\rangle &= (k - l + n) |k, l, n\rangle, \\
Q_+ |k, l, n\rangle &= \sqrt{(n+1)(n+2k)(2l-n-k)} |k, l, n+1\rangle, \\
Q_- |k, l, n\rangle &= \sqrt{n(n+2k-1)(2l-n-k+1)} |k, l, n-1\rangle, \\
K |k, l, n\rangle &= k(1-k) |k, l, n\rangle, \quad L |k, l, n\rangle = l |k, l, n\rangle, \\
n &= 0, 1, 2, \ldots, (2l-k). \quad (3.10)
\end{align*}
\]

The Casimir operator has the value \((l^3 + (l+1)[k(1-k) - 1] + 1)\) in this representation.

For example, for each value of \(k = 1/2, 1, 3/2, \ldots\), there is a 2-dimensional representation of the algebra given by
\[
\begin{align*}
Q_0 &= \frac{1}{2} \begin{pmatrix} k - 1 & 0 \\ 0 & k + 1 \end{pmatrix}, \\
Q_+ &= \begin{pmatrix} 0 & 0 \\ \sqrt{2k} & 0 \end{pmatrix}, \\
Q_- &= \begin{pmatrix} 0 & \sqrt{2k} \\ 0 & 0 \end{pmatrix}, \\
K &= k(1-k), \quad L = \frac{1}{2} (k + 1), \quad (3.11) \\
C &= \frac{1}{8} (-3k^3 - 5k^2 + 11k - 3).
\end{align*}
\]

If we make the association
\[ |k, l, n\rangle \rightarrow \frac{z_2^{2k-1} z_3^{2l-k} (z_1 z_2 / z_3)^n}{\sqrt{n!(n+2k-1)!(2l-k-n)!}}. \quad (3.12) \]

Since \(k\) and \(l\) are constants for a given representation we can take
\[ \phi_{k,l,n}(z) = \frac{z^n}{\sqrt{n!(n+2k-1)!(2l-k-n)!}}, \quad n = 0, 1, 2, \ldots, (2l-k) \quad (3.13) \]

With this we get the Fock-Bargmann representation of \(Q(1,1)\)
\[
\begin{align*}
Q_0 &= z \frac{d}{dz} + k - l, \quad Q_+ = -z \frac{d}{dz} + (2l-k)z, \quad Q_- = z \frac{d^2}{dz^2} + 2k \frac{d}{dz}. \quad (3.14)
\end{align*}
\]

Using this method one can get the Bargmann schwinger representations of all the 16 algebras given in tables 1 and 2. These are useful in construction of Barut girardello coherent states of polynomial algebras as discussed in ref \[12\].
4. Polynomial Algebras In Supersymmetric Quantum Mechanics

Recent work on supersymmetric methods for the construction of conditionally exactly solvable problems has shown that these systems have a non-linear algebraic structure, with the symmetry algebra being a polynomial algebra [?]. Symmetry algebra of SUSY partner of Linear Oscillator has a Quadratic algebra structure, while the SUSY partner of the radial oscillator has a Cubic algebra structure. These algebras belong to our classification. To illustrate this consider the standard construction of the Susy partner of the Hamiltonian. Given the Hamiltonian

$$H_0 = -\frac{1}{2}(\frac{d^2}{dx^2} + V_0(x))$$  \hspace{1cm} (4.1)$$

we construct its supersymmetric partner

$$H_- = -\frac{1}{2}(\frac{d^2}{dx^2} + V_1(x))$$  \hspace{1cm} (4.2)$$

by defining

$$A = \frac{1}{\sqrt{2}}(-\frac{d}{dx} + W(x))$$  \hspace{1cm} (4.3)$$

such that

$$H_1A^\dagger = A^\dagger H_0$$  \hspace{1cm} (4.4)$$

One makes the ansatz $W(x) = \phi(x) + f(x)$, where $f(x)$ satisfies Riccati eqn. and $\phi(x)$ is the shape invariant susy potential and

$$f^2(x) + 2\phi(x)f(x) + f'(x) = 2(\epsilon - 1)$$  \hspace{1cm} (4.5)$$

. For the case of the harmonic oscillator $\phi(x) = x$, Junker and Roy construct Ladder operators for $H_-$, $B = A^\dagger a A$ $B^\dagger = A^\dagger a^\dagger A$ which along with $H_-$ satisfy the algebra:

$$[H_-, B] = -B \quad [H_-, B^\dagger] = -B^\dagger \quad [B^\dagger, B] = -3H_-^2 + 4\epsilon H_- - \epsilon^2$$  \hspace{1cm} (4.6)$$

. This corresponds to algebra $Q^+(1,1)$ and the corresponding constituent Lie algebras are the Heisenberg algebra generated by $A, A^\dagger, A^\dagger A$ and the SU(1,1) algebra generated by $K_+ = A^\dagger a^\dagger, K_- = Aa, K_0 = \frac{1}{2}(a^\dagger a + A^\dagger A + 1)$.

Similarly for the radial oscillator one has $\phi(x) = x - \frac{2\epsilon + 1}{x}$. If $c, c^\dagger$ are the creation and annihilation operators for $H_+$ such that

$$[H_+, c] = -2c \quad [H_+, c^\dagger] = -2c^\dagger \quad [c^\dagger, c] = -4H_+ + \gamma + \epsilon - \frac{3}{2}$$  \hspace{1cm} (4.7)$$

then for $H_-$ the operators $D = A^\dagger c A$ $D^\dagger = A^\dagger c^\dagger A$ together with $H_-$ satisfy the algebra:

$$[H_-, D] = -D \quad [H_-, D^\dagger] = -D^\dagger \quad [D^\dagger, D] = -8H_-^3 - 12(\gamma + \epsilon + \frac{1}{2})H_-^2 + 4(\epsilon^2 + \epsilon + 1 + 2\epsilon\gamma)$$  \hspace{1cm} (4.8)$$
This corresponds to algebra $C^-(q,1,\hbar)$ of our classification.

This method has been generalized by Hussin, Fernandez and Nieto using the method of intertwining operators to construct higher order SUSY potentials all of which belong to our classification and therefore the coherent states and representations can be read off the list.

5. Further Applications

We now consider some interesting applications of polynomial algebras to systems which are linked to supersymmetric quantum mechanics.

5.1 Quadratic oscillator

An interesting possibility is suggested by the structure of the algebra $Q^-(1,1)$. Let us define

$$N = Q_0, \quad A = \frac{1}{\sqrt{L(L+1) - K}} Q_-, \quad A^\dagger = \frac{1}{\sqrt{L(L+1) - K}} Q_+.$$  \hspace{1cm} (5.1)

Then the algebra $Q^-(1,1)$ becomes

$$[N, A] = -A, \quad [N, A^\dagger] = A^\dagger,$$
$$[A, A^\dagger] = 1 - \frac{2L - 1}{L(L+1) - K} N - \frac{3}{L(L+1) - K} N^2.$$  \hspace{1cm} (5.2)

We may consider this as the defining algebra of a quadratic oscillator, corresponding to a special case of the general class of deformed oscillators:

$$[N, A] = -A, \quad [N, A^\dagger] = A^\dagger, \quad [A, A^\dagger] = F(N).$$  \hspace{1cm} (5.3)

The quadratic oscillator (5.2) belongs to the class of generalized deformed parafermions [?]. It should be interesting to study the physics of assemblies of quadratic oscillators. In fact, the canonical fermion, with

$$N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$  \hspace{1cm} (5.4)

is a quadratic oscillator! Observe that

$$[N, f] = -f, \quad [N, f^\dagger] = f^\dagger, \quad [f, f^\dagger] = 1 - \frac{1}{2} N - \frac{3}{2} N^2.$$  \hspace{1cm} (5.5)

5.2 Two Body Calogero-Sutherland Model

The Hamiltonian of the two body Calogero-Sutherland model is given by

$$H_c = \frac{1}{2} \sum_{i=1}^{2} \left( \frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right) + \sum_{j<i} \frac{\lambda(\lambda-1)}{(x_i - x_j)^2}.$$  \hspace{1cm} (5.6)
defining the creation and annihilation operators,

\[ a_i^+ = \frac{1}{\sqrt{2}} (-D_i + \omega x_i) \]
\[ a_i = \frac{1}{\sqrt{2}} (D_i + \omega x_i) \] (5.7)

Where \( D_i \) is the Dunkel derivative

\[ D_i = \frac{\partial}{\partial x} + \lambda \sum_{i \neq j} \frac{1}{(x_i - x_j)} (1 - \sigma_{ij}) \] (5.8)

In the centre of mass coordinates defined by

\[ y_i = M_{ij} x_j, \]
\[ M_{ij} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \] (5.9)

the corresponding creation and annihilation operators are given by

\[ \tilde{A}_i = M_{ij} a_j, \quad \tilde{A}_i^+ = M_{ij} a_j^+ \] (5.11)

Defining:

\[ A_1 = \frac{\tilde{A}_1}{\sqrt{1 + 2\lambda\sigma}} \]
\[ A_1^+ = \frac{\tilde{A}_1}{\sqrt{1 + 2\lambda\sigma}} \]
\[ A_2 = \tilde{A}_2, \quad A_2^+ = \tilde{A}_2^+ \] (5.12)

The operators

\[ C_0 = \frac{1}{2} (A_1^+ A_1 - A_2^+ A_2) C_+ = \frac{1}{2} (A_1^+ A_2)^2 \] (5.13)
\[ C_- = \frac{1}{2} (A_2^+ A_1)^2 J = \frac{1}{2} (A_1^+ A_1 + A_2^+ A_2) \] (5.14)

satisfy the cubic algebra

\[ [C_0, C_\pm] = C_\pm \] (5.15)
\[ [C_+, C_-] = -2 C_0^3 + (2 C(J) - 1) C_0 \] (5.16)

The Hamiltonian in terms of the operator can written as

\[ H = 2\lambda\sigma)C_0 + (1 + \lambda\sigma)(2J + 1) \] (5.17)

This is significantly different from the algebra generated by Vinet et. al. ??, as the Hamiltonian is one of the generators. Here \( C(J) = J(J+1) \) is the Casimir of the underlying \( SU(2) \) algebra. Note that the above algebra has a one mode \( SU(2) \) realization, given by \( C_0 = J_0, C_+ = \frac{1}{2} J_+^2, C_- = \frac{1}{2} J_-^2 \)
6. Conclusion

We have seen that polynomial algebras emerge as the dynamical symmetry, invariance and spectrum generating algebras of many interesting physical systems. Among these, a special position is occupied by three dimensional polynomial algebras with a coset structure. Thus, a systematic study and the proper classification of these algebras and their irreducible representations was warranted. Such a comprehensive study has been carried out in detail.

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