Equilibrium Relativistic Mass Distribution

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Abstract

The relativistic Maxwell-Boltzmann distribution for the system of $N$ events with motion in space-time parametrized by an invariant “historical time” $\tau$ is considered without the simplifying approximation $m^2 \approx M^2$, where $M$ is a given intrinsic property of the events. The relativistic mass distribution is obtained and the average values of $m$ and $m^2$ are calculated. The average value of the energy in nonrelativistic limit gives a correction of the order of 10% to the Dulong-Petit law. Expressions for the pressure and the density of events are obtained and the ideal gas law is recovered.

Key words: special relativity, relativistic Maxwell-Boltzmann, mass distribution
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1 Introduction

This article is the natural continuation of the work of Horwitz, Shashoua and Schieve [1], which in the present text will be cited as $I$. In that work a derivation of a

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manifestly covariant relativistic Boltzmann equation for the system of \( N \) events was made in the framework of a manifestly covariant classical and quantum mechanics \[2\]. These events, considered as the fundamental dynamical objects of the theory, move in an \( 8N \)-dimensional phase space. Their motion is parametrized by a continuous Poincaré invariant parameter \( \tau \) called the “historical time”. The collection of events (called “concatenation” \[3\]) along each world line corresponds to a particle in the usual sense; e.g., the Maxwell conserved current is an integral over the history of the charged event \[4\]. Hence the evolution of the state of the \( N \)-event system describes the history in space and time of an \( N \)-particle system. The relativistic Boltzmann equation was used to prove the \( H \)-theorem for evolution in \( \tau \). In the equilibrium limit a covariant form of the Maxwell-Boltzmann distribution was obtained. Since this distribution is the distribution of the 4-momenta of the events, \( m^2 = -p^2 = -p^\mu p_\mu \) is a random variable in a relativistic ensemble. In order to obtain a simple analytic result the authors restricted themselves to a narrow mass shell \( p^2 = -m^2 \cong -M^2 \) where \( M \) is a given fixed parameter, with the dimension of mass (an intrinsic property of the events), assumed to be the same for all the events of the system. The results obtained in this approximation are in agreement with the well-known results of Synge \[5\] from an on-mass-shell relativistic kinetic theory.

As Hakim \[6\] has remarked, some problems of relativistic statistical theory require consideration of relativistic particles endowed with variable masses. Thus, in statistical cosmology the universe is constituted of a gas of particles with equal masses. The particles are, in turn, considered as being galaxies and possibly clusters of galaxies or stars. However, galaxies and clusters of galaxies \[1\] do not appear to have identical masses. Therefore, a consistent approach to statistical cosmology as well as the statistical treatment of clusters of galaxies themselves should involve a mass distribution.

Moreover, in the statistical bootstrap model of Hagedorn \[7\] and Frautschi \[8\] for multiple production of particles in high energy reactions a mass spectrum of the asymptotic form \( \rho(m) \sim c m^a \exp(bm) \) (where \( a, b, c \) are constants) lies in the foundations of their theory and gives a good qualitative agreement with relevant experiments in high-energy physics. Miller and Suhonen \[9\] have discussed a possible correlation of the grand canonical distribution function of ref.\[10\] characterized by the mass fluctuations, with the hadronic spectrum of the Hagedorn-Frautschi form.

In this paper, we study such a relativistic system without the simplifying approximation \( m^2 \cong M^2 \), i.e., we consider a relativistically covariant Maxwell-Boltzmann distribution with mass parameter \( m \) taking any value within the range \( 0 \leq m < \infty \). We obtain the relativistic mass distribution by integration of the Maxwell- Boltzmann distribution over angular and hyperbolic angular variables. Calculation of average value of the energy and study of its nonrelativistic approximation gives a

\[1\] A number of clusters, such as Coma or Virgo, contains large numbers of galaxies of unequal masses.
relativistic correction of the order of 10% to the Dulong-Petit law for the free-particle gas assuming that the mass distribution remains valid in this approximation of our theory. We remark that, as is well known, the structure of the Galilean group, the symmetry of non-relativistic systems, implies that the mass of a particle must be a constant intrinsic property. We recognize, however, that the applicability of the Galilean group is an idealization of a world which seems to be more correctly described by the Poincaré group. The result that we have found follows from equilibrium thermodynamics without imposing the geometrical restriction of the precise Galilean group to an infinitely sharp mass shell.

By examining the energy-momentum tensor we obtain expressions for the pressure and the density of events and recover the ideal gas law previously obtained in $I$.

Since there was no fundamental theory available to Hakim, his development of the theory followed the phenomenological methods initiated by Synge. In this way he obtained a relativistic statistical mechanics containing many useful results. In particular, he used the Jüttner-Synge form (coinciding with the sharp-mass or low-temperature limit of the theory which we discuss here) to obtain a mass distribution. As we shall see, this construction leads to a normalization of the distribution function, which differs from ours. The theoretical framework which we use, permitting a derivation of the relativistic Boltzmann equation from first principles, provides a condition on the equilibrium ensemble which leads to a well-defined mass distribution (independent of the Jüttner-Synge result, although consistent with it), and a normalization condition consistent with a quantum-mechanical positive definite density.

2 Relativistic mass distribution

We begin with the Maxwell-Boltzmann distribution function which we write down in the form (differing from one obtained in [1, (3.14)] by the signs in the exponent) we

\[ \ln f_0(q, p) = A(p + p_c)^2 + \ln C(q) \]

possesses the property [1, (3.10)]

\[ \ln f_0(q, p) = \chi_1(q, p) + \chi_2(q, p) + ..., \]

where the quantities

\[ \chi_1(q, p_1) + \chi_1(q, p_2) \]

are conserved in collisions. For the sharp mass approximation $m^2 \approx M^2$ the difference between the two expressions manifests itself only in the normalization factor.
also use the metric \( g^{\mu\nu} = (-, +, +, +) \) and \( q \equiv q^\mu, \ p \equiv p^\mu \)

\[ f_0(q, p) = C(q)e^{A(q)(p^+ p^-)^2}, \quad A(q) > 0. \quad (1) \]

The function (1) must be normalized, according to \([1, (2.14)]\), as

\[ n(q) = \int f_0(q, p)d^4p = C(q) \int d^4p \exp\{A(p + p_c)^2\}, \quad (2) \]

where \( n(q) \) is the total number of events per unit space-time volume in the system in the neighborhood of the point \( q \). By introducing hyperbolic variables \([1, (3.16), (3.17)]\)

\[ \Omega^4: m \geq 0, 0 \leq \theta \leq \pi, 0 \leq \varphi < 2\pi, -\infty < \beta < \infty, \]

we can rewrite (2) as follows:

\[ n(q) = C(q)e^{-A_m^2} \int_{\Omega^4} m^3 \sinh^2 \beta \sin \theta dm d\beta d\theta d\varphi e^{-A_m^2 - 2Amc m \cosh \beta}. \quad (3) \]

After some calculations \([14]\) we obtain the normalization relation

\[ n(q) = C(q)e^{-Am_c^2} \frac{\pi}{2A^2} \Psi(2, 2; Am_c^2). \quad (4) \]

Here \( \Psi(a, b; z) \) is the confluent hypergeometrical function of \( z \) (ref.[15], p.257, sec.6.6). After integration over angular and hyperbolic angular variables we obtain from (3) the expression

\[ n(q) = C(q) \frac{2\pi e^{-Am_c^2}}{Am_c} \int_0^\infty dm m^2 e^{-Am_c^2} K_1(2Am_cm), \quad (5) \]

from which we identify the mass distribution function

\[ f(m) = Dm^2 e^{-Am_c^2} K_1(2Am_c m), \quad (6) \]

where

\[ D^{-1} = \frac{m_c}{4A} \Psi(2, 2; Am_c^2), \quad (7) \]

so that \( f(m) \) is normalized according to

\[ \int_0^\infty f(m) dm = 1. \quad (8) \]

In (5) and (6) \( K_1 \) is the Bessel function of the third kind (imaginary argument), where, in general,

\[ K_\nu(z) = \frac{\pi i}{2} e^{-\pi i \nu/2} H^{(1)}_\nu(iz). \]
With the help of this distribution one can obtain the local average value of an arbitrary function of mass $\phi(m)$:

$$\langle \phi(m) \rangle_q = \int_0^\infty \phi(m)f(m)dm.$$  \hfill (9)

Let us obtain, for example, the local average values of mass and mass squared in relativistic gas which represent the first two moments of the distribution (6):

$$\langle m \rangle_q = \frac{3\pi}{8A^2} \frac{\Psi(\frac{5}{2}, 2; Am_c^2)}{\Psi(2, 2; Am_c^2)},$$  \hfill (10)

$$\langle m^2 \rangle_q = \frac{2}{A} \frac{\Psi(3, 2; Am_c^2)}{\Psi(2, 2; Am_c^2)}.$$  \hfill (11)

More generally,

$$\langle m^\ell \rangle_q = \Gamma(\frac{\ell}{2} + 1)\Gamma(\frac{\ell}{2} + 2)A^{-\frac{\ell}{2}} \frac{\Psi(\frac{\ell}{2} + 2, 2; Am_c^2)}{\Psi(2, 2; Am_c^2)}.$$  \hfill (12)

Now, as in I, we define absolute temperature through the relation [1,(3.26)]

$$2Am_c = \frac{1}{k_B T},$$  \hfill (13)

which implies that in thermal equilibrium $Am_c$ is independent of $q$. Hence

$$\langle m \rangle = \frac{3\pi}{8}\sqrt{2m_c k_B T} \frac{\Psi(\frac{5}{2}, 2; \frac{m_c}{2k_B T})}{\Psi(2, 2; \frac{m_c}{2k_B T})}.$$  \hfill (14)

In the limit $T \to 0$ it follows from the asymptotic formula for $z \to \infty$ [15]

$$\Psi(a, b; z) \sim z^{-a}[1 + \sum_{n=1}^\infty (-1)^n a(a+1) \cdots (a+n-1)(1+a-b)(2+a-b) \cdots (n+a-b)]$$

that

$$\langle m \rangle \approx \frac{3\pi}{4} k_B T.$$  \hfill (16)

One can also obtain in this limit

$$\langle m^2 \rangle \approx 8(k_B T)^2.$$  \hfill (17)

Let us now calculate the first two moments of the distribution (1):

$$\langle p^\mu \rangle_q = \frac{\int d^4pp^\mu e^{A(p+p_c)^2}}{\int d^4pe^{A(p+p_c)^2}}.$$  \hfill (18)
\[ \langle p^\mu p^\nu \rangle_q = \frac{\int d^4pp^\mu p^\nu e^{A(p+p_c)^2}}{\int d^4p e^{A(p+p_c)^2}}. \]  

(19)

If we introduce the “free energy” of the relativistic gas through the relation

\[ \int d^4p e^{A(p+p_c)^2} = e^{-AF}, \]  

(20)

we will obtain the following expressions:

\[ \langle p^\mu \rangle_q = p^\mu_c (F' - 1), \]  

(21)

\[ \langle p^\mu p^\nu \rangle_q = p^\mu_c p^\nu_c \left[ (F' - 1)^2 - \frac{F''}{A} \right] + g^{\mu\nu} \frac{F' - 1}{2A}, \]  

(22)

\[ F' \equiv \frac{\partial F}{\partial m_c^2}. \]  

(23)

The value of \( F \), obtained from (20) and (4), is

\[ F = -\frac{1}{A} \ln \left[ \frac{\pi}{2A^2} e^{-Am_c^2} \Psi(2, 2; Am_c^2) \right]. \]

A simple calculation with the help of the relation (ref.[15], p.257, sec.6.6)

\[ \frac{d}{dz} \Psi(a, b; z) = -a \Psi(a + 1, b + 1; z) \]

gives

\[ \langle p^\mu \rangle_q = 2p^\mu_c \frac{\Psi(3, 3; Am_c^2)}{\Psi(2, 2; Am_c^2)}, \]  

(24)

\[ \langle p^\mu p^\nu \rangle_q = 6 \frac{\Psi(4, 4; Am_c^2)}{\Psi(2, 2; Am_c^2)} p^\mu_c p^\nu_c + g^{\mu\nu} \frac{\Psi(3, 3; Am_c^2)}{A \Psi(2, 2; Am_c^2)}. \]  

(25)

As in I, we make a Lorentz transformation to the local average motion rest frame moving with the relative velocity

\[ \mathbf{u} = \frac{\mathbf{p}_c}{E_c}, \]

in order to obtain the local energy density.

The rest frame energy is

\[ \langle E' \rangle_q = \frac{\langle E \rangle_q - \mathbf{u} \cdot \mathbf{p}}{\sqrt{1 - \mathbf{u}^2}}; \]

so that

\[ \langle E' \rangle_q = 2m_c \frac{\Psi(3, 3; Am_c^2)}{\Psi(2, 2; Am_c^2)}. \]  

(26)
Using the asymptotic formula (15) and expression (16), we have for $T \to 0$,

$$\langle E' \rangle - \langle m \rangle \cong (4 - \frac{3\pi}{4})k_B T = \gamma \frac{3}{2}k_B T,$$

$$\gamma = \frac{16 - 3\pi}{6} \approx 1.1.$$  \hfill (27)

This result differs from the nonrelativistic result $\frac{3}{2}k_B T$. The coefficient $\gamma$ represents a relativistic correction, determined by the relativistic mass distribution, to the classical value $\frac{3}{2}$. The existence of such corrections was first shown by Horwitz, Schieve and Piron [10] in their work on study of relativistic Gibbs ensembles even in the case of small fluctuations of mass over its sharp value $M_0$.

In the limit $T \to \infty$ we use the asymptotic formulas for $z \to 0$ (ref.[15], p.262, sec.6.8)

$$\Psi(a, b; z) = \Gamma(b - 1) \Gamma(a) z^{1-b} + O(|z|^{|Re b - 2|}), \quad Re b \geq 2, \quad b \neq 2,$$  \hfill (29)

and obtain

$$\langle E' \rangle \cong 2k_B T,$$  \hfill (31)

the result obtained also in $I$.

In this limit one can find from (10),(11)

$$\langle m \rangle \cong \sqrt{\frac{\pi m_c k_B T}{2}},$$  \hfill (32)

$$\langle m^2 \rangle \cong 2m_c k_B T.$$  \hfill (33)

To obtain the pressure and density distributions in our ensemble, as in $I$, we study the particle energy-momentum tensor defined by the $R^4$ density

$$T^{\mu\nu}(q) = \sum_i \int d\tau \frac{p_i^\mu p_i^\nu}{M} \delta^4(q - q_i(\tau)).$$  \hfill (34)

Using the result of the previous work $[1,(3.37)]$

$$\langle T^{\mu\nu}(q) \rangle_q = T_{\Delta V} \int d^4p_0(q, p) \frac{p^\mu p^\nu}{M}$$  \hfill (35)

and the expression (25) for $\langle p^\mu p^\nu \rangle_q$, we obtain

$$\langle T^{\mu\nu}(q) \rangle_q = \frac{T_{\Delta V} n(q)}{M} \left[ g^{\mu\nu} \Psi(3, 3; Am_0^2) + 6p_c^\mu p_c^\nu \Psi(4, 4; Am_0^2) \right] \frac{\Psi(2, 2; Am_0^2)}{A}.$$  \hfill (36)
In this expression $T_{\Delta V}$ is the average passage interval in $\tau$ for the events which pass through the small four-volume $\Delta V$ over the point $q$ of $R^4$.

The formula for the stress-energy tensor of a perfect fluid has the form [1,(3.39)]

$$\langle T^{\mu\nu}(q) \rangle_q = pg^{\mu\nu} - (p + \rho) \frac{\langle p^\mu \rangle_q \langle p^\nu \rangle_q}{\langle p^\lambda \rangle_q \langle p_\lambda \rangle_q}, \quad (37)$$

where $p$ is the pressure and $\rho$ is the density of energy at $q$.

According to (24),

$$\frac{\langle p^\mu \rangle_q}{\sqrt{\langle p^\lambda \rangle_q \langle p_\lambda \rangle_q}} = \frac{p^\mu_c}{m_c}, \quad (38)$$

hence

$$p = \frac{T_{\Delta V} n(q) \Psi(3, 3; Am^2_c)}{AM \Psi(2, 2; Am^2_c)}, \quad (39)$$

and

$$p + \rho = \frac{6T_{\Delta V} n(q)m^2_c \Psi(4, 4; Am^2_c)}{M \Psi(2, 2; Am^2_c)}, \quad (40)$$

To interpret these results, as in $I$, we should calculate the average (conserved) particle four-current, which has the microscopic form

$$J^\mu(q) = \sum_i \int \frac{p^\mu_i}{M} \delta^4(q - q_i(\tau)) d\tau. \quad (41)$$

Using the result of $I$ [(3.45),(3.59)]

$$\langle J^\mu(q) \rangle_q = T_{\Delta V} \int d^4p \frac{p^\mu}{M} f_0(q, p) \quad (42)$$

and expression (24) for $\langle p^\mu \rangle_q$, we obtain

$$\langle J^\mu(q) \rangle_q = \frac{2T_{\Delta V} n(q)}{M} \frac{p^\mu_c}{M} \Psi(3, 3; Am^2_c). \quad (43)$$

In the local rest frame $p^\mu_c = (m_c, 0)$,

$$\langle J^0(q) \rangle_q = \frac{2T_{\Delta V} n(q)m_c}{M} \Psi(3, 3; Am^2_c). \quad (44)$$

Defining the density of particles per unit space volume [1,(3.48)] as

$$N_0(q) = \langle J^0(q) \rangle_q, \quad (45)$$

we obtain the ideal gas law [1,(3.49)]

$$p = \frac{N_0}{2Am_c} = N_0k_BT. \quad (46)$$
Now we shall show how the general expression for the distribution function (6) can be simplified in two limiting cases $T \to 0$ and $T \to \infty$.

1. $T \to 0$. In this case we can see from (17) and (27) that $\langle m^2 \rangle \sim (k_B T)^2 \ll \langle E' \rangle m_c \sim m_c (k_B T)$. We therefore neglect $m^2 = -p^2$ in comparison to $2A_{\text{app}} c$ in the exponent of the Maxwell-Boltzmann distribution (1); hence, we begin with

$$f_0^{(1)}(p, q) \equiv C(q) e^{-Am_c^2 e^{2A_{\text{app}}}}.$$  \hspace{1cm} (47)

and after integration and normalization obtain

$$f_0(m) = \frac{(2Am_c)^3}{2} m^2 K_1(2Am_m).$$  \hspace{1cm} (48)

It is easy to see that the distribution function (48) produces results for $\langle m \rangle$ and $\langle m^2 \rangle$ in this limit in agreement with (16) and (17):

$$\langle m \rangle^0 = \frac{3\pi}{4} k_B T, \quad \langle m^2 \rangle^0 = 8(k_B T)^2.$$  

More generally,

$$\langle m^\ell \rangle^0 = \Gamma(\ell + 1) \Gamma(\frac{\ell}{2} + 2) (Am_m)^{-\ell} = \Gamma(\frac{\ell}{2} + 1) \Gamma(\frac{\ell}{2} + 2) (2k_B T)^\ell,$$ \hspace{1cm} (49)

which coincides with the low-temperature limit of (12). Similarly, if we introduce the “free energy” for the distribution (47), we obtain

$$\langle p^\mu \rangle^0 = \frac{p^\mu}{m_c} 4k_B T,$$ \hspace{1cm} (50)

which coincides with the low-temperature limit of (24).

2. $T \to \infty$. It is seen from (31) and (33) that $\langle m^2 \rangle$ and $\langle E' \rangle m_c$ are of the same order in $T$. But the argument of the function $K_1$ in the expression of the mass distribution function (6) is of the lower order of $T$ in this limit [(32)], i.e., $2Am_m m \sim \frac{3}{2} (k_B T)^\frac{3}{2}$. Since we use the asymptotic formula [16]

$$K_\nu(z) \sim \frac{1}{2} \Gamma(\nu) \left( \frac{z}{2} \right)^{-\nu}, \quad z \to 0,$$

which gives $K_1(2Am_m m) \cong 1/2Am_m m$, and obtain the (normalized) distribution

$$f^\infty(m) = 2Am e^{-Am^2}.$$ \hspace{1cm} (51)

One can easy check that the distribution (51) gives results for $\langle m \rangle$ and $\langle m^2 \rangle$ in this limit in agreement with (32) and (33):

$$\langle m \rangle^\infty = \sqrt{\frac{\pi m_c k_B T}{2}}, \quad \langle m^2 \rangle^\infty = 2m_c k_B T.$$  

More generally,
\[
\langle m^\ell \rangle_\infty = \Gamma(\frac{\ell}{2} + 1) A^{-\frac{\ell}{2}} = \Gamma(\frac{\ell}{2} + 1) (2mck_BT)^{\frac{\ell}{2}},
\]
which coincides with the high-temperature limit of (12).

In this way one can also obtain
\[
\langle p^\mu \rangle_\infty = \frac{p^\mu}{m_c} 2k_BT,
\]
the high-temperature limit of (24).

Summarizing, we can write down the expression for the mass distribution function, taking into account the limiting cases considered:

\[
f(m) = \begin{cases} 
\{ (2Am_c)^3 / 2 \} m^2 K_1(2Am_cm), & T \to 0 \\
\{ 4A/m_c \Psi(2, 2; Am_c^2) \} e^{-Am^2} m^2 K_1(2Am_cm), & \text{intermediate case} \\
2Amce^{-Am^2}, & T \to \infty
\end{cases}
\]

Synge [5], following phenomenological methods in his study of an on-mass-shell relativistic ensemble, used the normalization relation (the only normalization relation available to him) for the equilibrium distribution function for such a system,

\[
f_0(q, p) = G(q)e^{2Apce},
\]
called the “Jüttner-Synge distribution function”,

\[
\int f_0(q, p)d^4p = G(q) \int d^4p e^{2Apce} = N_0(q),
\]
where \(N_0(q) = \langle J_0(q) \rangle_q\) is the density of particles per unit space-volume (45), the 0-component of the average (conserved) particle four-current; in contrast to the normalization relation (56), we have the condition (2),

\[
\int f_0(q, p)d^4p = n(q),
\]
where \(n(q)\) is the density of events per unit space-time volume, a quantum-mechanical positive-definite density, which follows from the theory discussed in I.

In the theory of Synge
\[
\langle J^\mu(q) \rangle_q = \int f_0(q, p) \frac{p^\mu}{m} d^4p = \frac{G(q)}{m} \frac{\partial \Phi}{\partial (2Ap^\mu_c)},
\]
\footnote{in his notation, \(2Ap_c^\mu = \xi^\mu\).}
where $m$ is a (constant) mass of the particles of the system and (ref. [5], p.35)

$$
\Phi = \int e^{2Ap_c}d^4p = \frac{4\pi mK_1(2Am_c m)}{2Am_c}.
$$

(57)

In this way he obtained the normalization relation (ref. [5], p.36)

$$
N_0 = \frac{4\pi Gm^2K_2(2Am_c m)}{2Am_c},
$$

(58)

differing from the one obtained in $I$ [1,(3.18)] in the mass-sharp-approximation $m^2 \cong M^2$ for the same distribution function (55) within the framework of the theory discussed in $I$:

$$
n_0(q) = \frac{4\pi GM^2K_1(2AM_c M)2\triangle m}{2AM_c},
$$

(59)

where $G = C(q) \exp\{A(M + M_c)^2\}$ and $\triangle m$ is the fluctuation of mass around its sharp value $M$.

Hakim introduced variation of mass into the Jüttner-Synge distribution and, using the normalization condition (58), found the mass distribution [6,(3.3)]

$$
f(m) = \frac{2(2Am_c)^3}{3\pi}m^2K_2(2Am_c m).
$$

(60)

This distribution differs from our low-temperature limit (48)

$$
f^0(m) = \frac{(2Am_c)^3}{2}m^2K_1(2Am_c m),
$$

because it is obtained by assuming a different normalization of the initial Jüttner-Synge distribution. This is why the results of Hakim differ quantitatively from the results obtained within the framework of the theory discussed in the present paper in the low-temperature limit.

3 Concluding remarks

We have considered an equilibrium relativistic ensemble, described by the equilibrium relativistic Maxwell-Boltzmann distribution, with variable mass. We have found that the equilibrium state of such a system is characterized by a well-defined mass distribution (consistent with the Jüttner-Synge one in the sharp-mass or low-temperature limit), following directly from the Maxwell-Boltzmann distribution, by integration over angular and hyperbolic angular variables and satisfying a normalization condition consistent with a quantum-mechanical positive definite density.

The results of Hakim, who introduced variation of mass into the equilibrium relativistic Jüttner-Synge distribution and found a relativistic mass distribution [6], differ
quantitatively from the results obtained within the framework of the theory discussed in the present paper in the low-temperature limit (e.g., he did not obtain a relativistic correction to the Dulong-Petit law) because the initial Jüttner-Synge distribution, from which the mass distribution of Hakim is obtained, has a normalization differing from ours, based on the counting of world lines (particles) rather than events.

The relativistic mass distribution obtained in this paper may find applications in relativistic cosmology and astrophysics (when dealing with clusters of galaxies or stars), in the statistical mechanics of hadrons, in physics of relativistic plasmas and relativistic liquids.

In the present paper as well as in the previous one (I) the simplest case of relativistic equilibrium is treated. The nonequilibrium case may be treated separately, e.g., by application of the Grad method of moments adopted for the relativistic theory.

The case of indistinguishable particles will be also treated separately [17], in order to obtain their equilibrium relativistic distribution function and the corresponding equilibrium relativistic mass distribution.
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