Is the Second Independent Diagnostic Test in Medical Diagnosis Useful?

Kasumi Iwamoto*, Ryo Tajiri*, Kei Fujikawa* and Takashi Yanagawa*

*Department of Biostatistics, Graduate School of Medicine, Kurume University
**The Biostatistics Center, Kurume University
e-mail:k.iwamoto375@gmail.com

We consider usefulness of the second diagnostic test in medical diagnosis from the point of view of positive predictive value (PPV) and negative predictive value (NPV). We assume in this paper that medical diagnosis is given by the result of a single diagnostic test or group discussions of experts. We call the process the diagnostic test and consider the situation where an individual has chance to undergo two diagnostic tests. When the second diagnostic test is undertaken, two decision rules, Rule 1 and Rule 2, may be considered. Rule 1 is judge positive if the both tests are positive and negative otherwise. Rule 2 is judge negative if both tests are negative and positive otherwise. The test is called reasonable if, and only if it selects diseased person with higher probability than it does non-diseased persons. It is shown that when the first and second tests are reasonable, usefulness of the second test depends on one's priority on PPV or NPV and whether one takes Rule 1 or Rule 2.

Key words: medical diagnostic test; negative predictive value; positive predictive value; sensitivity; specificity.

1. Introduction

A medical method that aims at determining whether a patient is affected by a disease is called the diagnostic test. Accurate diagnosis of patients is crucial when planning the treatment of a disease. Diagnosis is given by the result of a single instrument such as biomarker, mammography, CT scan and others, or group discussion of experts regarding the results obtained by those single instruments and clinical symptoms. We call the process the diagnostic test in this paper.

Particularly, diagnostic tests that evaluate the strength of suspicion of certain diseases on binary (positive and negative) are called the binary diagnostic tests. We discuss in this paper a binary diagnostic test.

The characteristic of a binary diagnostic test is represented by sensitivity and specificity (Pepe 2003, Zhou et al., 2002). Sensitivity and specificity are defined by the following equation:
Sensitivity = \( P(T = 1 \mid D = 1) \), \[ \text{Specificity} = P(T = 0 \mid D = 0) \]

where \( T \) indicates the diagnostic results according to the binary diagnostic test, and \( D \) indicates the actual condition of the disease with \( D = 1 \) positivity, and \( D = 0 \) disease-free. Sensitivity is the conditional probability for patients who are actually disease to be diagnosed as positive, and specificity is the conditional probability for patients who are not actually disease to be diagnosed as negative. Often \textit{false positive rate} and \textit{false negative rate} are used instead of sensitivity and specificity to represent the characteristic of a diagnostic test, which are defined by

\[
\text{False positive rate: } \alpha = P(T = 1 \mid D = 0) = 1 - \text{Specificity},
\]
\[
\text{False negative rate: } \beta = P(T = 0 \mid D = 1) = 1 - \text{Sensitivity}.
\]

Those error rates are supposed to be small, but could be substantial, in particular, in disease at an early stage, depending on such factors as carcinoma, size of tumors, overestimation of cellular atypia and others. For examples, Wendie A. Berg \textit{et al.} (2004) assessed that the false positive and false negative rates of mammography test alone for detecting breast cancer were 32.2\% and 25\%, respectively; Kevin R. Loughlin (2009) reported that the false positive and false negative rates of PSA (Prostate Specific Antigen) test for prostate cancer that became popular since 1986 were 0.70 \sim 0.80 and 0.20.

Because of these errors, results of diagnostic test are not conclusive. ‘Is tested positive truly contracted with disease?’, or ‘Is tested negative truly disease-free?’ is thus the serious concern of the individual who undergoes medical diagnosis. The positive predictive value (PPV) is the probability that the tested positive is truly contracted with disease and the negative predictive value (NPV) is the probability that the tested negative is truly disease-free. The PPV and NPV are defined by the following equations.

\[
\text{PPV} = P(D = 1 \mid T = 1), \quad \text{NPV} = P(D = 0 \mid T = 0),
\]

In both cases, values closer to 1 mean that the diagnostic test is accurate.

Using the definition of the conditional probability it is straightforward to show the following equalities \cite{Pepe2003, Ioannidis2005}.

\[
\text{PPV} = \frac{R(1 - \beta)}{R(1 - \beta) + \alpha}, \quad (1)
\]
\[
\text{NPV} = \frac{1 - \alpha}{R\beta + (1 - \alpha)}, \quad (2)
\]

where \( R = P(D = 1) / P(D = 0) \).

Often, two or more single diagnostic tests are combined, or single diagnostic test is combined with clinical examinations by aiming to get accurate diagnosis. For example, combinatory use of ultrasound and mammography was studied by Ohuchi \textit{et al.} (2011). From the side of patients, an individual is often imposed a burden to determine whether to undertake the second diagnostic
test from an independent medical institution. For example, those individuals who live in local provinces and undertake a screening test, that is, a quick simple diagnostic test, often have to determine whether to visit hospitals in big cities by making expensive trips. Those hospitals estimate their diagnostic systems are advanced and conduct an independent diagnostic test again. The second diagnostic test could commit errors. If this is the case, the second test result works the other way.

The purpose of this paper is to explore the decision procedures that make the second independent diagnostic test useful.

In Section 2, we formulate the problem mathematically. When the second diagnostic test is obtained, two decision rules, Rule 1 and Rule 2, may be considered. Rule 1 is judge positive if both tests are positive and negative otherwise. Rule 2 is judge negative if both tests are negative and positive otherwise. In Section 3, we obtain PPV and NPV for two decision rules, and compare them to show the usefulness of the second diagnostic test. In Section 4, discussion is given to those results obtained in Section 3 from the viewpoint of practical aspects.

2. Mathematical Development

2.1 PPV and NPV when an individual undertakes the second test

Let $T_i = 1(0)$ ($i = 1, 2$) iff the $i$-th test positive (negative) and

$$\alpha_i = P(T_i = 1 \mid D = 0), \quad \beta_i = P(T_i = 0 \mid D = 1)$$

be the false negative and false positive rates of the $i$-th test. The positive predictive value (PPV), and negative predictive value (NPV) of the $i$-th diagnostic test are obtained as

$$PPV_i = \frac{R(1 - \beta_i)}{R(1 - \beta_i) + \alpha_i},$$

$$NPV_i = \frac{1 - \alpha_i}{R\beta_i + (1 - \alpha_i)}.$$  

(3)  

(4)

where $R = P(D = 1)/P(D = 0)$.

The following two decision rules may be considered when the second test is conducted.

(Rule 1) Judge positive if both tests are positive i.e. $T_1 = T_2 = 1$, and negative otherwise.

(Rule 2) Judge negative if both tests are negative i.e. $T_1 = T_2 = 0$, and positive otherwise.

We assume in this paper that the second test is carried out independently of the first test, namely, we assume the following equation.

$$P[(T_1 = 1) \cap (T_2 = 1) \mid D = 1] = P(T_1 = 1 \mid D = 1)P(T_2 = 1 \mid D = 1).$$

The PPV and NPV for Rule 1 are obtained under the assumption as

$$PPV^{(R1)} = \frac{R(1 - \beta_1)(1 - \beta_2)}{R(1 - \beta_1)(1 - \beta_2) + \alpha_1\alpha_2}.$$  

(5)
\[ NPV^{(R_1)} = \frac{1 - \alpha_1 \alpha_2}{R(\beta_1 + \beta_2 - \beta_1 \beta_2) + (1 - \alpha_1 \alpha_2)}. \]  

Similarly the PPV and NPV for Rule 2 are obtained as follows.

\[ PPV^{(R_2)} = \frac{R(1 - \beta_1 \beta_2)}{R(1 - \beta_1 \beta_2) + \alpha_1 + \alpha_2 - \alpha_1 \alpha_2}, \]
\[ NPV^{(R_2)} = \frac{(1 - \alpha_1)(1 - \alpha_2)}{R \beta_1 \beta_2 + (1 - \alpha_1)(1 - \alpha_2)}. \]

We may show the following theorems whose proofs are given in Appendix.

**Theorem 1** It follows the following relationships for the positive predictive value in Rule 1 and Rule 2.

(i) (Rule 1)

\[ 1 - \beta_2 < \alpha_2 \Leftrightarrow PPV^{(R_1)} < PPV_1, \]
\[ 1 - \beta_2 > \alpha_2 \Leftrightarrow PPV^{(R_1)} > PPV_1. \]

(ii) (Rule 2)

\[ \alpha_1 \beta_1 (1 - \alpha_2 - \beta_2) < \alpha_2 (1 - \alpha_1 - \beta_1) \Leftrightarrow PPV^{(R_2)} < PPV_1, \]
\[ \alpha_1 \beta_1 (1 - \alpha_2 - \beta_2) > \alpha_2 (1 - \alpha_1 - \beta_1) \Leftrightarrow PPV^{(R_2)} > PPV_1. \]

**Theorem 2** It follows the following relationships for the negative predictive value in Rule 1 and Rule 2.

(i) (Rule 1)

\[ \alpha_1 \beta_1 (1 - \alpha_2 - \beta_2) < \beta_2 (1 - \alpha_1 - \beta_1) \Leftrightarrow NPV^{(R_1)} < NPV_1, \]
\[ \alpha_1 \beta_1 (1 - \alpha_2 - \beta_2) > \beta_2 (1 - \alpha_1 - \beta_1) \Leftrightarrow NPV^{(R_1)} > NPV_1. \]

(ii) (Rule 2)

\[ 1 - \beta_2 < \alpha_2 \Leftrightarrow NPV^{(R_2)} < NPV_1, \]
\[ 1 - \beta_2 > \alpha_2 \Leftrightarrow NPV^{(R_2)} > NPV_1. \]

### 2.2 Implication of theorems

We call the diagnostic tests reasonable if, and only if it selects diseased persons with higher probability than it does non-diseased persons, that is the same as requiring \( 1 - \beta_i > \alpha_i \). This would be the minimum assumption if the test is to be of practical value (Yanagawa and Gladen (1984)). Based on above theorems, we obtain several corollaries.

**Corollary 1** When the first test is unreasonable and the second test is reasonable, we have the following relationships for Rule 1 and Rule 2.

(i) (Rule 1)

\[ PPV^{(R_1)} > PPV_1, \quad NPV^{(R_1)} > NPV_1. \]
(ii) (Rule 2)

\[ PPV^{(R2)} > PPV_1, \quad NPV^{(R2)} > NPV_1. \]

The corollary shows the usefulness of the second opinion for PPV and NPV in both rules when the first test is unreasonable, giving the result as anticipated.

**Corollary 2** When the first test is reasonable and the second test is unreasonable, we have

(i) (Rule 1)

\[ PPV^{(R1)} < PPV_1, \quad NPV^{(R1)} < NPV_1. \]

(ii) (Rule 2)

\[ PPV^{(R2)} < PPV_1, \quad NPV^{(R2)} < NPV_1. \]

The corollary shows that the second opinion is not useful for PPV and NPV in both rules when the second test is unreasonable. It is also the result as anticipated.

**Corollary 3** When the first and second tests are reasonable, we have

(i) (Rule 1)

(a) For PPV

\[ PPV^{(R1)} > PPV_1 \]

(b) For NPV

\[ \alpha_1 \beta_1 (1 - \alpha_2 - \beta_2) < \beta_2 (1 - \alpha_1 - \beta_1) \iff NPV^{(R1)} < NPV_1, \]
\[ \alpha_1 \beta_1 (1 - \alpha_2 - \beta_2) > \beta_2 (1 - \alpha_1 - \beta_1) \iff NPV^{(R1)} > NPV_1, \]

(ii) (Rule 2)

(a) For PPV

\[ \alpha_1 \beta_1 (1 - \alpha_2 - \beta_2) < \alpha_2 (1 - \alpha_1 - \beta_1) \iff PPV^{(R2)} < PPV_1, \]
\[ \alpha_1 \beta_1 (1 - \alpha_2 - \beta_2) > \alpha_2 (1 - \alpha_1 - \beta_1) \iff PPV^{(R2)} > PPV_1, \]

(b) For NPV

\[ NPV^{(R2)} > NPV_1. \]

We have the following corollary from Corollary 3.

**Corollary 4** When the first and second tests are reasonable and \( \alpha_1 = \alpha_2, \beta_1 = \beta_2 \), it follows that

\[ PPV^{(R1)} > PPV_1 \quad \text{and} \quad NPV^{(R1)} < NPV_1, \]
\[ PPV^{(R2)} < PPV_1 \quad \text{and} \quad NPV^{(R2)} > NPV_1. \]
We examined the inequality in Corollary 4 when $R=1$ and changing the values of $\alpha_1, \alpha_2 = \alpha_1 + f$, $\beta_1$ and $\beta_2 = \beta_1 + h$, for $0 < \alpha_1 \leq 0.2$, $0 < \beta_1 \leq 0.5$, $f = 0, \pm 0.05, \pm 0.1$, $h = \pm 0.1, \pm 0.3, \pm 0.4$. Figure 1 shows the profiles of $NPV_1$ and $NPV^{(R1)}$ and Figure 2 shows the profiles $PPV_1$ and $PPV^{(R2)}$ when $\alpha_1 = \alpha_2 = 0.05$. The profiles of $NPV_1$, $NPV^{(R1)}$, $PPV_1$ and $PPV^{(R2)}$ for other combinations of $\alpha_1, \beta_1$ and $f$ are almost unchanged.

Corollary 4 and these figures indicate that if the characteristics of the second test are similar to the first test, the second diagnostic test is useful for PPV, but not useful for NPV in Rule 1, and it is useful for NPV, but not useful for PPV in Rule 2.

3. Summary and Discussion

In this paper we compared PPV and NPV when an individual underwent the second diagnostic test with the corresponding PPV and NPV when it did not, and evaluated the usefulness
Is the Second Independent Diagnostic Test in Medical Diagnosis Useful? 59

of the second test. It is shown that if one of the diagnostic tests is unreasonable we get the result as anticipated, namely, if the first test is unreasonable, the second test is useful, but if the second test is unreasonable, the second opinion is useless. Note that no conventional diagnostic tests are unreasonable as pointed up by a reviewer of this paper. We have given those results from the view point of mathematical completeness.

It is also shown in this paper that when the first and second tests are reasonable the usefulness of the second test depends on one’s priority of PPV or NPV and whether one takes Rule 1 or Rule 2; namely, to make the second diagnostic test useful take decision Rule 1 if one’s priority is PPV and take decision Rule 2 if one’s priority is NPV.

Special interest is in undergoing the second test that is reasonable and has similar characteristics as the first test. Intuitively, it seems useless to undertake such test again, but the paper shows that it does useful when priority is put on PPV and Rule 1 is employed, and priority is on NPV and Rule 2 is employed. Often the benefit could be substantial. For example, when $R=0.1$, $\alpha_1 = \alpha_2 = 0.1$ and $\beta_1 = \beta_2 = 0.2$, the values of $PPV_1$ and $PPV^{(R1)}$ are 0.44 and 0.87, respectively.

The above findings do not depend on the value of $R$. However, the values of PPV themselves largely depend on $R$. Table 1 gives the values of $PPV_1$, $PPV^{(R1)}$, $PPV^{(R2)}$, $NPV_1$, $NPV^{(R1)}$ and $NPV^{(R2)}$ when $\alpha_1 = \alpha_2 = 0.1, 0.2$, $\beta_1 = \beta_2 = 0.1, 0.2$ and $R = 0.001, 0.01, 0.1, 0.4$. The table shows that $PPV_1 = 0.01$, $PPV^{(R1)} = 0.06$ and $PPV^{(R2)} = 0.01$ when $R = 0.001$ and $\alpha_1 = \alpha_2 = 0.1$, $\beta_1 = \beta_2 = 0.2$, but those values jump up to 0.76, 0.96 and 0.67, respectively, when $R = 0.4$.

For individuals who put priority on the PPV it would be useless to undertake diagnostic tests when values of PPV are very small. However, Table 1 indicates that such diagnostic tests are still useful for individuals who put priority on NPV. Actually, the table shows that those values of NPV’s are closer to 1 for any $R = 0.001 \sim 0.4$, and $\alpha, \beta = 0.1 \sim 0.2$, but if this is the case it would be not necessary to undergo the second diagnostic tests since values of $NPV_1$ are already close to 1.

In summary we suggest individuals to compute the value of PPV’s and NPV’s using the formula given in this paper and evaluate the value of undertake the second diagnostic test. The characteristics, $\alpha_i$ and $\beta_i$, and the values of $R$ could be obtained from the Internet these days.

Often individuals that undergo the second diagnostic test are those people whose results of the first diagnostic test are positive, but wishing to deny it. If this is the case it would be most natural to decide positive if the second test results positive and negative otherwise. This decision rule is equivalent to Rule 1 thus the positive and negative predictive values equal to $PPV^{(R1)}$ and $NPV^{(R1)}$, respectively. Thus it follows from the present study that to undergo the second diagnostic test that has similar characteristic as the first test is useless for the individual who want to deny the result of the first test. Of course, it is not the case if the second diagnostic test
Table 1. PPV and NPV for selected values of $\alpha_1, \alpha_2, \beta_1, \beta_2$ and R

| R  | $\alpha$ | $\beta$ | $PPV^{(R1)}$ | $PPV^{(R2)}$ | $NPV^{(R1)}$ | $NPV^{(R2)}$ |
|----|---------|---------|-------------|-------------|-------------|-------------|
| 0.001 | 0.1  | 0.2 | 0.008 | 0.060 | 0.005 | 1.000 |
|      | 0.2  | 0.1 | 0.004 | 0.020 | 0.003 | 1.000 |
| 0.01 | 0.1  | 0.2 | 0.074 | 0.390 | 0.048 | 0.998 |
|      | 0.2  | 0.1 | 0.043 | 0.168 | 0.027 | 0.999 |
| 0.1  | 0.1  | 0.2 | 0.444 | 0.865 | 0.336 | 0.978 |
|      | 0.2  | 0.1 | 0.310 | 0.669 | 0.216 | 0.988 |
| 0.4  | 0.1  | 0.2 | 0.762 | 0.962 | 0.669 | 0.918 |
|      | 0.2  | 0.1 | 0.643 | 0.890 | 0.524 | 0.952 |

is nearly perfect i.e. $\alpha_2 \approx 0, \beta_2 \approx 0$, such as biopsy.

Acknowledgements

We would like to thank the reviewers for their constructive comments that have strengthened this paper.

REFERENCES

Ioannidis, J.P.A. (2005). Why most published research findings are false. *PLOS Medicine*, 2(8), 696–701.

Loughlin, K.R. (2009). How reliable is the prostate-specific antigen (PSA) test when it comes to detecting prostate cancer? *Prostate Knowledge*, Harvard Medical School+Harvard Health Publications, http://www.harvardprostateknowledge.org/

Ohuchi, N., Ishida, T., Kawai, M., Narikawa, Y., Yamamoto, S. and Sobue, T. (2011). Randomized Controlled Trial on Effectiveness of Ultrasonography Screening for Breast Cancer in Women Aged 40–49 (J-START): Research Design. *Japanese Journal of Clinical Oncology*. 41, 275–277.

Pepe, M.S. (2003). The Statistical Evaluation of Medical Tests for Classification and Prediction. Oxford University Press, New York.

Yanagawa, T. and Gladen, B.C. (1984). Estimating disease rates from a diagnostic test. *American Journal of Epidemiology*, 119(6), 1015–1023.

Berg, W.A., Gutierrez, L., NessAiver, M.S., Carter, W.B., Bhargavan, M., Lewis, R.S. and Ioffe, O.B. (2004). Diagnostic accuracy of mammography, clinical examination, US, and MR imaging in preoperative assessment of breast cancer. *Radiology*, 233, 830–849.

Zhou, X.H., Obuchowski, N.A. and McClish, D.K. (2002). Statistical Methods in Diagnostic Medicine. Wiley, New York.
Appendix

(Proof of Theorem 1)

From (1) and (3), we have

\[
PPV^{(R1)} - PPV_1 = \frac{R(1 - \beta_1)(1 - \beta_2)}{R(1 - \beta_1)(1 - \beta_2) + \alpha_1 \alpha_2} - \frac{R(1 - \beta_1)}{R(1 - \beta_1) + \alpha_1}
\]

\[
= \frac{R(1 - \beta_1)(1 - \beta_2)\{R(1 - \beta_1) + \alpha_1\} - R(1 - \beta_1)\{R(1 - \beta_1)(1 - \beta_2) + \alpha_1 \alpha_2\}}{\{R(1 - \beta_1)(1 - \beta_2) + \alpha_1 \alpha_2\}\{R(1 - \beta_1) + \alpha_1\}}.
\]

Since the denominator is positive, we check the numerator. The numerator is represented as follows:

\[
\text{(numerator)} = R(1 - \beta_1)(1 - \beta_2)\{R(1 - \beta_1) + \alpha_1\} - R(1 - \beta_1)\{R(1 - \beta_1)(1 - \beta_2) + \alpha_1 \alpha_2\}
\]

\[
= R\alpha_1(1 - \beta_1)(1 - \beta_2) - R\alpha_1 \alpha_2(1 - \beta_1)
\]

\[
= R\alpha_1(1 - \beta_1)(1 - \alpha_2 - \beta_2).
\]

Since \(R\alpha_1\) and \(1 - \beta_1\) are always positive, we have

\[
1 - \beta_2 < \alpha_2 \iff PPV^{(R1)} < PPV_1,
\]

\[
1 - \beta_2 > \alpha_2 \iff PPV^{(R1)} > PPV_1.
\]

Similarly, from (1) and (5), we have

\[
PPV^{(R2)} - PPV_1 = \frac{R(1 - \beta_1 \beta_2)}{R(1 - \beta_1 \beta_2) + \alpha_1 + \alpha_2 - \alpha_1 \alpha_2} - \frac{R(1 - \beta_1)}{R(1 - \beta_1) + \alpha_1}
\]

\[
= \frac{R(1 - \beta_1 \beta_2)\{R(1 - \beta_1) + \alpha_1\} - R(1 - \beta_1)\{R(1 - \beta_1 \beta_2) + \alpha_1 + \alpha_2 - \alpha_1 \alpha_2\}}{\{R(1 - \beta_1 \beta_2) + \alpha_1 + \alpha_2 - \alpha_1 \alpha_2\}\{R(1 - \beta_1) + \alpha_1\}}.
\]

Since the denominator is positive, we check the numerator. The numerator is represented as follows:

\[
\text{(numerator)} = R(1 - \beta_1 \beta_2)\{R(1 - \beta_1) + \alpha_1\} - R(1 - \beta_1)\{R(1 - \beta_1 \beta_2) + \alpha_1 + \alpha_2 - \alpha_1 \alpha_2\}
\]

\[
= R\alpha_1(1 - \beta_1 \beta_2) - R(1 - \beta_1)(\alpha_1 + \alpha_2 - \alpha_1 \alpha_2)
\]

\[
= R\{-\alpha_1 \beta_1 \beta_2 - \alpha_2 + \alpha_1 \alpha_2 + \alpha_1 \beta_1 + \alpha_2 \beta_1 - \alpha_1 \alpha_2 \beta_1\}
\]

\[
= R\{\alpha_1(-\beta_1 \beta_2 + \beta_1 - \alpha_2 \beta_1) + \alpha_2(-1 + \alpha_1 + \beta_1)\}
\]

\[
= R\{(\alpha_1 \beta_1(1 - \alpha_2 - \beta_2) - \alpha_2(1 - \alpha_1 - \beta_1))\}.
\]

Since \(R\) is always positive, we have

\[
\alpha_1 \beta_1(1 - \alpha_2 - \beta_2) < \alpha_2(1 - \alpha_1 - \beta_1) \iff PPV^{(R2)} < PPV_1,
\]

\[
\alpha_1 \beta_1(1 - \alpha_2 - \beta_2) > \alpha_2(1 - \alpha_1 - \beta_1) \iff PPV^{(R2)} > PPV_1.
\]
(Proof of Theorem 2)

From (2) and (4), we have
\[
NPV^{(R1)} - NPV_1 = \frac{1 - \alpha_2 \alpha_2}{R(\beta_1 + \beta_2 - \beta_1 \beta_2) + (1 - \alpha_2 \alpha_2)} - \frac{1 - \alpha_1}{R \beta_1 + 1 - \alpha_1}
\]
\[
= \frac{(1 - \alpha_1 \alpha_2)(R \beta_1 + 1 - \alpha_1) - (1 - \alpha_1)(R(\beta_1 + \beta_2 - \beta_1 \beta_2) + (1 - \alpha_1 \alpha_2))}{(R(\beta_1 + \beta_2 - \beta_1 \beta_2) + (1 - \alpha_1 \alpha_2))}(R \beta_1 + 1 - \alpha_1).
\]

Since the denominator is positive, we check the numerator. The numerator is represented as follows:

\[
\text{(numerator)} = (1 - \alpha_1 \alpha_2)(R \beta_1 + 1 - \alpha_1) - (1 - \alpha_1)(R(\beta_1 + \beta_2 - \beta_1 \beta_2) + (1 - \alpha_1 \alpha_2))
\]
\[
= R \beta_1(1 - \alpha_1 \alpha_2) - (1 - \alpha_1)(R(\beta_1 + \beta_2 - \beta_1 \beta_2))
\]
\[
= R(\beta_1 - \alpha_1 \alpha_2 \beta_1 - \beta_1 - \beta_2 + \beta_1 \beta_2 + \alpha_1 \beta_1 + \alpha_1 \beta_2 - \alpha_1 \beta_1 \beta_2)
\]
\[
= R\{\beta_1(- \alpha_1 \alpha_2 + \alpha_1 \beta_2) + \beta_2(-1 + \alpha_1 + \beta_1)\}
\]
\[
= R\{\alpha_1 \beta_1(1 - \alpha_2 - \beta_2) - \beta_2(1 - \alpha_1 - \beta_1)\}.
\]

Since \( R \) is always positive, we have
\[
\alpha_1 \beta_1(1 - \alpha_2 - \beta_2) < \beta_2(1 - \alpha_1 - \beta_1) \Leftrightarrow NPV^{(R1)} < NPV_1,
\]
\[
\alpha_1 \beta_1(1 - \alpha_2 - \beta_2) > \beta_2(1 - \alpha_1 - \beta_1) \Leftrightarrow NPV^{(R1)} > NPV_1.
\]

Similarly, from (2) and (6), we have
\[
NPV^{(R2)} - NPV_1 = \frac{(1 - \alpha_1)(1 - \alpha_2) R \beta_1 + 1 - \alpha_1}{R \beta_1 + 1 - \alpha_1}(1 - \alpha_1 - \alpha_2) - \frac{1 - \alpha_1}{R \beta_1 + 1 - \alpha_1}
\]
\[
= \frac{R \beta_1(1 - \alpha_1)(1 - \alpha_2) - (1 - \alpha_1)(R \beta_1 \beta_2 + (1 - \alpha_1)(1 - \alpha_2))(1 - \alpha_1)}{R \beta_1 \beta_2 + (1 - \alpha_1)(1 - \alpha_2)}(R \beta_1 + 1 - \alpha_1).
\]

Since the denominator is positive, we check the numerator. The numerator is represented as follows:

\[
\text{(numerator)} = (1 - \alpha_1)(1 - \alpha_2)(R \beta_1 + 1 - \alpha_1) - (1 - \alpha_1)(R \beta_1 \beta_2 + (1 - \alpha_1)(1 - \alpha_2))
\]
\[
= R \beta_1(1 - \alpha_1)(1 - \alpha_2 - \beta_2).
\]

Since \( R \beta_1 \) and \( 1 - \alpha_1 \) are always positive, we have
\[
1 - \alpha_2 - \beta_2 < 0, \text{ namely, } 1 - \beta_2 < \alpha_2 \Leftrightarrow NPV^{(R2)} < NPV_1,
\]
\[
1 - \alpha_2 - \beta_2 > 0, \text{ namely, } 1 - \beta_2 > \alpha_2 \Leftrightarrow NPV^{(R2)} > NPV_1.
\]