RECOGNISING THE SMALL REE GROUPS IN THEIR NATURAL REPRESENTATIONS

HENRIK BÅARNHELM

Abstract. We present Las Vegas algorithms for constructive recognition and constructive membership testing of the Ree groups $2G_2(q) = \text{Ree}(q)$, where $q = 3^{2m+1}$ for some $m > 0$, in their natural representations of degree 7. The input is a generating set $X \subseteq \text{GL}(7,q)$.

The constructive recognition algorithm is polynomial time given a discrete logarithm oracle. The constructive membership testing consists of a preprocessing step, that only needs to be executed once for a given $X$, and a main step. The latter is polynomial time, and the former is polynomial time given a discrete logarithm oracle.

Implementations of the algorithms are available for the computer algebra system Magma.

1. Introduction

This paper will consider algorithmic problems for a class of finite simple groups, as matrix groups over finite fields, given by sets of generators. The most important problems under consideration are the following:

(1) The constructive membership problem. Given $G = \langle X \rangle \leq \text{GL}(d,q)$ and $g \in U \supseteq G$, decide whether or not $g \in G$, and if so express $g$ as a straight line program in $X$.

(2) The constructive recognition problem. Given $G = \langle X \rangle \leq \text{GL}(d,q)$, construct an effective isomorphism from $G$ to a standard copy $H$ of $G$, together with an effective inverse isomorphism. An isomorphism $\psi : G \to H$ is effective if $\psi(g)$ can be computed efficiently for every $g \in G$.

In [1] we considered these problems for the Suzuki groups. Here we consider the Ree groups $2G_2(q) = \text{Ree}(q)$, $q = 3^{2m+1}$ for any $m > 0$. We only consider the natural representations, which have dimension 7. Our standard copy is $\text{Ree}(q)$, defined in Section 3.

The primary motivation for considering these problems comes from the matrix group recognition project [3, 22, 28].

The ideas used here for the constructive recognition and membership testing of $\text{Ree}(q)$ are similar to those used in [1] and [11] for $\text{Sz}(q)$ and $\text{SL}(2,q)$, respectively. The results are also similar in the sense that we reduce these problems to the discrete logarithm problem.

In Section 4.2 we solve the constructive membership problem for $\text{Ree}(q)$. In Section 4.3 we consider conjugates of $\text{Ree}(q)$ and show how to construct effective isomorphisms to $\text{Ree}(q)$, hence solving constructive recognition in the natural representation.

The main objective of this paper is to prove the following:

Theorem 1.1. Let $\sigma_0(d)$ be the number of divisors of $d \in \mathbb{N}$. Assume an oracle for the discrete logarithm problem in $\mathbb{F}_q$, with time complexity $O(\chi D)$ field operations, and a random element oracle for subgroups of $\text{GL}(7,q)$, with time complexity $O(\xi)$ field operations.


1
• There exists a Las Vegas algorithm that for each \( \langle X \rangle \leq \text{GL}(7,q) \), with \( q = 3^{2m+1} \) for some \( m > 0 \), such that \( \langle X \rangle \cong \text{Ree}(q) \), constructs an effective isomorphism \( \Psi : \langle X \rangle \to \text{Ree}(q) \), such that \( \Psi^{-1} \) is also effective. The algorithm has expected time complexity \( O(\xi \log \log(q) + \log(q)(\sigma_0(\log(q)) + \log(q)) + \chi_D) \) field operations.

• There exists a Las Vegas algorithm that for each \( \langle X \rangle \leq \text{GL}(7,q) \), with \( q = 3^{2m+1} \) for some \( m > 0 \), such that \( \langle X \rangle \cong \text{Ree}(q) \), solves the constructive membership problem for \( \langle X \rangle \). The algorithm has expected time complexity \( O(\xi + \log(q)^3) \) field operations and also has a pre-processing step, which only needs to be executed once for a given \( X \), with expected time complexity \( O((\xi \log \log(q) + \log(q)^3 + \chi_D) \log \log(q)^2) \) field operations. The length of the returned SLP is \( O(\log(q) \log \log(q)^2) \).

Implementations of the algorithms have been done in MAGMA [6].

A version of the material in this paper appeared in [2], relying on a few conjectures. Advice by Bill Kantor and Gunter Malle has led to proofs of the conjectures, for which we are very grateful. In particular, the central idea behind the algorithm in Section 4.3 is due to Bill Kantor.

We thank John Bray, Peter Brooksbank, Alexander Hulpke, Charles Leedham-Green, Eamonn O’Brien, Maud de Visscher and Robert Wilson for their helpful comments.

2. Preliminaries

We will now briefly discuss some general concepts that are needed later.

2.1. Complexity. Time complexity is measured in field operations. Basic matrix arithmetic requires \( O(1) \) field operations. Raising a matrix to an \( O(q) \) power requires \( O(\log(q)) \) field operations, for example using [24, Lemma 10.1].

We never need to compute large precise orders of matrices. It is sufficient to compute pseudo-orders [5, Section 8]. This can be done using [10], in \( O(\log(q) \log \log(q)) \) field operations.

We shall assume an oracle for the discrete logarithm problem in \( \mathbb{F}_q \) [33, Chapter 3], requiring \( O(\chi_D) \) field operations.

2.2. Straight line programs. For constructive membership testing, we want to express an element of a group \( G = \langle X \rangle \) as a straight line program in \( X \), abbreviated to SLP. An SLP is a data structure for a word, which allows for efficient computations [32, Section 1.2.3].

2.3. Random group elements. Our algorithms need to construct (nearly) uniformly distributed random elements of a group \( G = \langle X \rangle \leq \text{GL}(d,q) \). The algorithm of [4] solves this task in polynomial time, but it is not commonly used in practice. The product replacement algorithm of [9] also solves this task. It is fast in practice and polynomial time [29].

We shall assume that we have a random element oracle, which produces a uniformly random element of \( \langle X \rangle \) using \( O(\xi) \) field operations, and returns it as an SLP in \( X \).

An important issue is the length of the SLPs that are computed. The length of the SLPs must be polynomial, otherwise evaluation would not be polynomial time. We assume that SLPs of random elements have length \( O(n) \) where \( n \) is the number of random elements that have been selected so far during the execution of the algorithm.

In [28], a variant of the product replacement algorithm is presented that constructs random elements of the normal closure of a subgroup. This will be used
here to construct random elements of the derived subgroup of a group $\langle X \rangle$, using the fact that this is precisely the normal closure of $\langle [x, y] : x, y \in X \rangle$.

2.4. **Probabilistic algorithms.** The algorithms we consider are probabilistic of the type known as Las Vegas algorithms. This type of algorithm is discussed in [13 Section 3.2.1]. We present Las Vegas algorithms in the same way as in [1].

2.5. **Recognition of $\text{PSL}(2, q)$.** In [11], an algorithm for constructive recognition and constructive membership testing of $\text{PSL}(2, q)$ is presented.

We will use [11] since $\text{PSL}(2, q)$ arise as a subgroup of $\text{Ree}(q)$. Because of this, we state the main result here.

**Theorem 2.1.** Assume an oracle for the discrete logarithm problem in $\mathbb{F}_q$. There exists a Las Vegas algorithm that, given $\langle X \rangle \leq \text{GL}(d, q)$, which acts absolutely irreducibly and cannot be written over a smaller field, with $\langle X \rangle \cong \text{PSL}(2, q)$ and $q = p^e$, constructs an effective isomorphism $\varphi : \langle X \rangle \to \text{PSL}(2, q)$ and performs pre-processing for constructive membership testing. The algorithm has expected time complexity

$$O((\xi + d^3 \log(q) \log(q^d)) \log \log(q) + d^3 \sigma_0(d) |X| + d\chi_D + \xi(d) d)$$

field operations.

The inverse of $\varphi$ is also effective. Each image of $\varphi$ can be computed using $O(d^3)$ field operations, and each pre-image using $O(d^3 \log(q) \log(q^d) + d^3)$ field operations. After the algorithm has executed, constructive membership testing of $g \in \text{GL}(d, q)$ requires $O(d^3 \log(q) \log(q^d) + d^3)$ field operations, and the resulting SLP has length $O(\log(q) \log(q^d))$.

2.6. **Notation.** Some notation will be fixed throughout the paper.

- For a group $G$ and a prime $p$, let $O_p(G)$ denote the largest normal $p$-subgroup of $G$.
- If $G$ acts on a set $O$ and $P \in O$, then $G_P$ denotes the stabiliser in $G$ of $P$.
- Let $q = 2^{2m+1}$, where $m > 0$, be the size of the finite field $\mathbb{F}_q$. Let $t = 3^m = \sqrt{3q}$ and let $\omega$ be a fixed primitive element of $\mathbb{F}_q$.
- Let

$$\text{antidiag}(x_1, \ldots, x_7) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & x_1 \\
0 & 0 & 0 & 0 & 0 & x_2 & 0 \\
0 & 0 & 0 & 0 & x_3 & 0 & 0 \\
0 & 0 & 0 & x_4 & 0 & 0 & 0 \\
0 & 0 & x_5 & 0 & 0 & 0 & 0 \\
x_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_7 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

- For a module $M$ or a matrix $g$, we denote the symmetric square of $M$ or $g$ by $S^2(M)$ and $S^2(g)$, respectively.
- The time complexity in field operators for an invocation of a random element oracle on a group $G$ will be denoted $\xi$.
- The time complexity in field operators for an invocation of a discrete logarithm oracle on $\mathbb{F}_q$ will be denoted $\chi_D$.
- We will use $\sigma_0$ as defined in Theorem [13]. Note that from [13 pp. 64, 359, 262], for every $\varepsilon > 0$, if $d$ is sufficiently large, then $\sigma_0(d) < 2^{(1+\varepsilon) \log_q(d) / \log \log_q(d)}$.
- For a vector space $V$, we denote the corresponding projective space by $\mathbb{P}(V)$.
- We denote the standard $n$-dimensional vector space over $\mathbb{F}_q$ by $\mathbb{F}_q^n$, and the corresponding projective space by $\mathbb{P}^n(\mathbb{F}_q)$. 

3. The small Ree groups

The small Ree groups were first described in [30, 31]. An elementary construction is given in [25, Chapter 4].

3.1. Definition and properties. We now define our standard copy of the Ree groups. The generators we use are those described in [20]. For \( x \in \mathbb{F}_q \) and \( \lambda \in \mathbb{F}_q^* \), define the matrices

\[
\alpha(x) = \begin{bmatrix}
1 & x & 0 & 0 & -x^{3t+1} & -x^{3t+2} & x^{4t+2} \\
0 & 1 & x^t & x^{t+1} & -x^{2t+1} & 0 & -x^{3t+2} \\
0 & 0 & 1 & x^t & -x^{2t} & 0 & x^{3t+1} \\
0 & 0 & 0 & 1 & x^t & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]  

(3.1)

\[
\beta(x) = \begin{bmatrix}
1 & 0 & -x^t & 0 & -x & 0 & -x^{t+1} \\
0 & 1 & 0 & x^t & 0 & -x^{2t} & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & x \\
0 & 0 & 0 & 1 & 0 & x^t & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & x^t \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]  

(3.2)

\[
\gamma(x) = \begin{bmatrix}
1 & 0 & 0 & -x^t & 0 & -x & -x^{2t} \\
0 & 1 & 0 & 0 & -x^t & 0 & x \\
0 & 0 & 1 & 0 & 0 & x^t & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & x^t \\
0 & 0 & 0 & 0 & 1 & 0 & x^t \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]  

(3.3)

\[
h(\lambda) = \text{diag}(\lambda^t, \lambda^{1-t}, \lambda^{2t-1}, 1, \lambda^{1-2t}, \lambda^{t-1}, \lambda^{-t})
\]

(3.4)

\[
\Upsilon = \text{antidiag}(-1, -1, -1, -1, -1, -1, -1)
\]

(3.5)

and define the Ree group as

\[
\text{Ree}(q) = \langle \alpha(x), \beta(x), \gamma(x), h(\lambda), \Upsilon \mid x \in \mathbb{F}_q, \lambda \in \mathbb{F}_q^* \rangle.
\]

(3.6)

Also, define the subgroups of upper triangular and diagonal matrices:

\[
U(q) = \langle \alpha(x), \beta(x), \gamma(x) \mid x \in \mathbb{F}_q \rangle
\]

(3.7)

\[
H(q) = \{h(\lambda) \mid \lambda \in \mathbb{F}_q^*\} \cong \mathbb{F}_q^x.
\]

(3.8)

From [25] we know that each element of \( U(q) \) can be expressed uniquely as

\[
S(a, b, c) = a(x)\beta(b)\gamma(c)
\]

(3.9)

so \( U(q) = \{S(a, b, c) \mid a, b, c \in \mathbb{F}_q\} \), and it follows that \( |U(q)| = q^3 \). Also, \( U(q) \) is a Sylow 3-subgroup of \( \text{Ree}(q) \), and direct calculations show that

\[
S(a_1, b_1, c_1)S(a_2, b_2, c_2) =
\]

\[
= S(a_1 + a_2, b_1 + b_2 - a_1a_2^{3t}, c_1 + c_2 - a_2b_1 + a_1a_2^{3t+1} - a_1^2a_2^3),
\]

(3.10)

\[
S(a, b, c)^{-1} = S(-a, -(b + a^{3t+1}), -(c + ab - a^{3t+2})),
\]

(3.11)

\[
S(a_1, b_1, c_1)S(a_2, b_2, c_2) =
\]

\[
= S(a_1, b_1 - a_1a_2^{3t} + a_2a_1^{3t}, c_1 + a_1b_2 - a_2b_1 + a_1a_2^{3t+1} - a_2a_1^{3t+1} - a_1^2a_2^3 + a_2^2a_1^3).
\]

(3.12)
and
\[ S(a, b, c)^{h(\lambda)} = S(\lambda^{3t-2}a, \lambda^{1-3t}b, \lambda^{-1}c). \] (3.13)

It follows that \( \text{ Ree}(q) = \langle S(1, 0, 0), h(\omega), \Upsilon \rangle \), and these are our standard generators. The group preserves a symmetric bilinear form on \( \mathbb{F}_q^6 \), represented by the matrix
\[ J = \text{antidiag}(1, 1, 1, -1, 1, 1, 1) \] (3.14)

From [34, Chapter 11] and [35 Chapter 4] we obtain the following.

**Proposition 3.1.** Let \( G = \text{ Ree}(q) \).

1. \(|G| = q^3(q^3 + 1)(q - 1)\) where \( \gcd(q^3 + 1, q - 1) = 2 \).
2. Conjugates of \( U(q) \) intersect trivially.
3. The centre \( Z(U(q)) = \{ S(0, 0, c) \mid c \in \mathbb{F}_q \} \).
4. The derived group \( U(q)' = \{ S(0, b, c) \mid b, c \in \mathbb{F}_q \} \), and its elements have order 3.
5. The elements in \( U(q) \setminus U(q)' = \{ S(a, b, c) \mid a \neq 0 \} \) have order 9 and their cubes form \( Z(U(q)) \setminus \{ 1 \} \).
6. \( N_G(U(q)) = U(q)H(q) \) and \( G \) acts doubly transitively on the right cosets of \( N_G(U(q)) \), i.e. on a set of size \( q^3 + 1 \).
7. \( U(q)H(q) \) is a Frobenius group with Frobenius kernel \( U(q) \).
8. The proportion of elements of order \( q - 1 \) in \( U(q)H(q) \) is \( \phi(q - 1)/(q - 1) \), where \( \phi \) is the Euler totient function.
9. \( N_G(H(q)) = \langle h(\omega), \Upsilon \rangle \cong D_{2(q - 1)} \).

For our purposes, we want another set to act (equivalently) upon.

**Proposition 3.2.** There exists \( \mathcal{O} \subseteq \mathbb{P}^6(\mathbb{F}_q) \) on which \( G = \text{ Ree}(q) \) acts faithfully and doubly transitively. Namely,
\[ \mathcal{O} = \{ (0 : 0 : 0 : 0 : 0 : 0 : 1) \} \cup \]
\[ \{ (1 : a^t : -b^t : (ab)^t : c^t : -b - a^{3t+1} - (ac)^t : c - (bc)^t : a^t b^{2t} : a^t c - b^{t+1} + a^{4t+2} - c^{2t} - a^{3t+1} b' - (abc)^t) \} \] (3.15)

Moreover, the stabiliser of \( P_\infty = (0 : 0 : 0 : 0 : 0 : 0 : 1) \) is \( U(q)H(q) \), the stabiliser of \( P_0 = (1 : 0 : 0 : 0 : 0 : 0 : 0) \) is \( (U(q)H(q))^\Upsilon \) and the stabiliser of \( (P_\infty, P_0) \) is \( H(q) \).

**Proof.** Notice that \( \mathcal{O} \setminus \{ P_\infty \} \) consists of the first rows of the elements of \( U(q)H(q) \).
From Proposition 3.1 it follows that \( G \) is the disjoint union of \( U(q)H(q) \) and \( U(q)H(q) \Upsilon U(q)H(q) \). Define a map between the \( G \)-sets as \( (U(q)H(q))g \mapsto P_\infty g \).

If \( g \in U(q)H(q) \) then \( P_\infty g = P_\infty \) and hence the stabiliser of \( P_\infty \) is \( U(q)H(q) \). If \( g \notin U(q)H(q) \) then \( g = xy \) where \( x, y \in U(q)H(q) \). Hence \( P_\infty g = P_0 y \in \mathcal{O} \) since \( P_0 y \) is the first row of \( y \). It follows that the map defines an equivalence between the \( G \)-sets.

**Proposition 3.3.** Let \( G = \text{ Ree}(q) \).

1. The stabiliser in \( G \) of any two distinct points of \( \mathcal{O} \) is conjugate to \( H(q) \), and the stabiliser of any triple of points has order 2.
2. The number of elements in \( G \) that fix exactly one point is \( q^6 - 1 \).
3. All involutions in \( G \) are conjugate in \( G \).
4. An involution fixes \( q + 1 \) points.

**Proof.** (1) Immediate from [34 Chapter 11, Theorem 13.2(d)].
(2) A stabiliser of a point is conjugate to \( U(q)H(q) \), and there are \(|\mathcal{O}|\) conjugates. The elements fixing exactly one point are the non-trivial elements of \( U(q) \). Therefore the number of such elements is \(|\mathcal{O}|(|U(q)| - 1) = (q^2 + 1)(q^3 - 1) = (q^6 - 1)\).

(3) Immediate from [18, Chapter 11, Theorem 13.2(e)].

(4) Each involution is conjugate to \( h(-1) = \text{diag}(-1, 1, -1, 1, -1, 1, -1) \).

Evidently, \( h(-1) \) fixes \( P_\infty \) since \( h(-1) \in H(q) \). If \( P = \langle p_1 : \cdots : p_7 \rangle \in \mathcal{O} \) with \( p_1 \neq 0 \), then \( P \) is fixed by \( h(-1) \) if and only if \( p_2 = p_4 = p_6 = 0 \). But then \( P \) is uniquely determined by \( p_1 \), so there are \( q \) possible choices for \( P \).

Thus the number of points fixed by \( h(-1) \) is \( q + 1 \).

\[ \square \]

We shall need the following general result, whose easy proof we omit.

**Lemma 3.4.** Let \( g \in G \leq \text{GL}(d, F) \), where \( d \) is odd, and \( F \) a finite field, and assume that \( G \) preserves a non-degenerate bilinear form and \( \det(g) = 1 \). Then \( g \) has 1 as an eigenvalue.

**Proposition 3.5.** All cyclic subgroups of \( G = \text{Ree}(q) \) of order \( q - 1 \) are conjugate to \( H(q) \) and hence each is a stabiliser of two points of \( \mathcal{O} \).

**Proof.** Let \( C = \langle g \rangle \leq G \) be cyclic of order \( q - 1 \) and let \( p \) be an odd prime such that \( p \mid q - 1 \). Then there exists \( k \in \mathbb{Z} \) such that \( |g^k| = p \). Since \( q^2 + 1 \equiv 2 \pmod{p} \) and \( |g^k| > 2 \), the cycle structure of \( g^k \) on \( \mathcal{O} \) must be a number of \( p \)-cycles and 2 fixed points \( P \) and \( Q \). Since \( G \) is doubly transitive there exists \( x \in G \) such that \( P^x = P_\infty \) and \( Q^x = Q_0 \).

Now either \( g \) fixes \( P \) and \( Q \) or interchanges them, so \( g^x \in N_G(H(q)) = \langle H(q), \Upsilon \rangle \cong \text{D}_2(q-1) \). Hence \( \langle g^x \rangle = H(q) \), the unique cyclic subgroup of order \( q - 1 \) in \( \langle H(q), \Upsilon \rangle \). \[ \square \]

**Proposition 3.6.** Let \( G = \text{Ree}(q) \) and let \( \phi \) be the Euler totient function.

(1) The centraliser of an involution \( j \in G \) is isomorphic to \( \langle j \rangle \times \text{PSL}(2, q) \) and hence has order \( q(q^2 - 1) \).

(2) The number of involutions in \( G \) is \( q^2(q^2 - q + 1) \).

(3) The number of elements in \( G \) of order \( q - 1 \) is \( \phi(q - 1)q^3(q^3 + 1)/2 \).

(4) The number of elements in \( G \) of even order is \( q^2(7q^5 - 23q^4 + 8q^3 + 23q^2 - 39q + 24)/24 \).

(5) The number of elements in \( G \) that fix at least one point is \( q^2(q^3 - q^4 + 3q^2 - 5q + 2)/2 \).

**Proof.**

(1) Immediate from [18, Chapter 11].

(2) All involutions are conjugate, and the index in \( G \) of the involution centraliser is \( \frac{q^2(q^2 - 1)}{q(q^2 - 1)} = q^2(q^2 - q + 1) \).

(3) By Proposition 3.5 each cyclic subgroup of order \( q - 1 \) is a stabiliser of two points and is uniquely determined by the pair of points that it fixes. Hence the number of cyclic subgroups of order \( q - 1 \) is \( \binom{2}{2} = \frac{q^2(1+1)}{2} \).

By Proposition 3.5 the intersection of two distinct subgroups has order 2, so the number of elements of order \( q - 1 \) is the number of generators of all these subgroups.

(4) By [25, Lemma 2], every element of even order lies in a cyclic subgroup of order \( q - 1 \) or \( (q + 1)/2 \). In each cyclic subgroup of order \( q - 1 \) there is a unique involution and hence \( (q - 3)/2 \) non-involutions of even order. Similarly, there are \( (q - 3)/4 \) non-involutions in a cyclic subgroup of order \( q - 1 \).
(q + 1)/2. By Proposition 3.3 the total number of elements of even order is therefore

\[(q - 3)(q^3 + 1)q^3/4 + (q - 3)(q - 1)(q^2 - q + 1)q^2/q^2 - q + 1)\]

\[= q^2(7q^5 - 23q^4 + 8q^3 + 23q^2 - 39q + 24)/24 \quad (3.16)\]

(5) The only non-trivial elements of G that fix more than 2 points are involutions. Hence in each cyclic subgroup of order q - 1 there are q - 3 elements that fix exactly 2 points, so by Proposition 3.3 the number of elements that fix at least one point is \(q^6 + \frac{(q - 3)(q^2 + 1)q^2}{2} + q^2(q^2 - q + 1) = \frac{q^2(q^6 - q^4 + 3q^2 - 5q + 1)}{2} \).

\[\square\]

**Proposition 3.7.** A maximal subgroup of \(G = \text{Ree}(q)\) is conjugate to one of the following subgroups:

- \(N_G(U(q)) = U(q)H(q)\), the point stabiliser,
- \(C_G(j) \cong (j) \times \text{PSL}(2, q)\), the centraliser of an involution \(j\),
- \(N_G(A_0) \cong (C_2 \times C_2 \times A_0):C_6\), where \(A_0 \leqslant \text{Ree}(q)\) is cyclic of order \((q + 1)/4\),
- \(N_G(A_1) \cong A_1C_6\), where \(A_1 \leqslant \text{Ree}(q)\) is cyclic of order \(q + 1 - 3t\),
- \(N_G(A_2) \cong A_2C_6\), where \(A_2 \leqslant \text{Ree}(q)\) is cyclic of order \(q + 1 + 3t\),
- Ree(s) where \(q\) is a proper power of \(s\).

Moreover, all maximal subgroups except the last are reducible.

**Proof.** The structure of the maximal subgroups follows from [21] and [25]. Hence it is sufficient to prove the final statement.

Clearly the point stabiliser is reducible. By Proposition 3.3 \(j\) is conjugate to \(h(-1) = \text{diag}(-1, 1, -1, 1, -1, 1, -1)\) so it has two eigenspaces \(S_j\) and \(T_j\) for 1 and -1 respectively. Clearly \(\dim S_j = 3\) and \(\dim T_j = 4\).

Let \(v \in S_j\) and \(g \in \text{PSL}(2, q)\). Then \((vg)j = (vj)g = vg\) since \(g\) centralises \(j\) and \(j\) fixes \(v\), which shows that \(vg \in S_j\), so this subspace is fixed by \(\text{PSL}(2, q)\). Similarly, \(T_j\) is also fixed. Hence \(S_j\) and \(T_j\) are submodules and the involution centraliser is reducible.

Let \(H\) be a normaliser of a cyclic subgroup and let \(x\) be a generator of the cyclic subgroup that is normalised. Since \(G \leqslant \text{SO}(7, q)\), by Lemma 3.4 \(x\) has an eigenspace \(E\) for the eigenvalue 1, where \(E\) is a proper non-trivial subspace of \(V\). If \(v \in E\) and \(h \in H\), then \((vh)x^h = vh\) so that \(vh\) is fixed by \(\langle x^h \rangle = \langle x \rangle\). This implies that \(vh \in E\) and thus \(E\) is a proper non-trivial \(H\)-invariant subspace, so \(H\) is reducible. \(\square\)

**Proposition 3.8.** Let \(G = \text{Ree}(q)\) with natural module \(V\), let \(j \in G\) be an involution and let \(C = C_G(j)\).

1. \(C' \cong \text{PSL}(2, q)\)
2. \(V|_{C'} \cong V_3 \oplus V_4\) where \(\dim V_i = i\).
3. \(V_3\) and \(V_4\) are absolutely irreducible. Moreover, \(V_4 \cong S^{\phi} \otimes S^{\phi^k}\) and \(V_3 \cong S^2(S)\) where \(S\) is the natural module of \(\text{PSL}(2, q)\), \(\phi\) is the Frobenius automorphism and \(1 \leqslant i < k \leqslant 2m + 1\).
4. When \(j = h(-1)\), the forms preserved on \(V_3\) and \(V_4\) are \(J_3 = \text{antidiag}(1, -1, 1, 1)\) and \(J_4 = \text{antidiag}(1, 1, 1, 1)\), up to scalar multiples.

**Proof.**

1. Immediate from Proposition 3.3
2. From the proof of Proposition 3.7 we see that \(V|_{C'}\) has submodules \(V_3\) and \(V_4\), so this is also true of \(V|_{C'}\), since \(C' \cong \text{PSL}(2, q)\).
Proposition 3.9. Let $C_3 \cong \text{PSL}(2,q)$ be the group acting on $V_3$. Since $\text{PSL}(2,q)$ has no irreducible module of dimension 2, if $V_3$ is reducible, it must have three 1-dimensional constituents. But $\text{PSL}(2,q)$ is simple, so $O_3(C_3) = 1$, hence $V_3 \cong 1 \oplus 1 \oplus 1$. By [14] Theorem 4.4, the 1-dimensional representations must be trivial, so $C_3 = 1$. But this is clearly false, since $[\mathcal{Y}, h(\omega)] \in C_G(h(-1))'$ and acts non-trivially on its corresponding 3-dimensional submodule.

Similarly, let $C_4$ be the group acting on $V_4$, and assume it is reducible. Then $V_4 \cong 1 \oplus V_3'$, where $V_3'$ is irreducible of dimension 3 and the 1-dimensional module is trivial. This implies that every $g \in C_4$ has 1 as an eigenvalue. Again, $[\mathcal{Y}, h(\omega)]$ provides a contradiction.

The result follows from the structure of irreducible modules of $\text{PSL}(2,q)$ [7] \S 30.

Proposition 3.10. Let $G = \text{Ree}(q)$ and let $C = C_G(j)$ for some involution $j \in G$. Let $N = N_{\Omega(3,q)}(C')$. Then $[N : C'] = 2$.

Proof. By Proposition 3.9 $C' \cong \text{PSL}(2,q)$, its module splits up as a 3-space and a 4-space, and $C'$ acts diagonally on these submodules. The normaliser must preserve this decomposition, and from Proposition 3.9 it is clear that the form preserved on the 4-space is of $\delta$-type, so $N$ embeds in $\Omega(3, q) \times \Omega(4, q)$. Let $C_3$ and $N_3$ be the images of $C'$ and $N$ on the 3-space. Define $C_4$ and $N_4$ analogously.

Now $[\Omega(3, q)] = [\text{PSL}(2, q)]$, so $[N_3 : C_3] = 1$. Since $\Omega(4, q) \cong \text{SL}(2, q) \circ \text{SL}(2, q)$ [3] Chapter 4, it acts as a tensor product on the 4-space. By Proposition 3.9 $C'$ acts diagonally as a tensor product on the 4-space. Clearly, the only element in $\Omega(4, q) \setminus C_4$ which can normalise $C_4$ is the central element $-1$, so $[N_4 : C_4] = 2$. This proves the result.

Lemma 3.11. If $g \in G = \text{Ree}(q)$ is uniformly random, then

\[
\begin{align*}
\Pr[|g| = q - 1] &= \frac{\phi(q - 1)}{2(q - 1)} > \frac{1}{12 \log \log(q)} \quad (3.17) \\
\Pr[|g| \text{ even}] &= \frac{7q^2 - 9q - 24}{24q(q + 1)} > 1/4 \quad (3.18) \\
\Pr[|g| \text{ fixes a point}] &= \frac{-2 + 3q + q^4}{2(q + q^3)} \geq 1/2 \quad (3.19)
\end{align*}
\]
Proof. In each case, the first equality follows from Proposition 3.6 and Proposition 3.1. In the first case, the inequality follows from [26, Section II.8], and in the other cases the inequalities are clear since $q \geq 27$.

Corollary 3.12. In $\text{Ree}(q)$, the expected number of random selections required to obtain an element of order $q - 1$ is $O(\log \log q)$. Similarly, the expected number of random selections required obtain an element that fixes a point, or an element of even order, is $O(1)$.

Proof. Clearly the number of selections is geometrically distributed, where the success probabilities for each selection are given by Lemma 3.11. Hence the expectations are as stated.

Proposition 3.13. Elements in $\text{Ree}(q)$ of order prime to 3, with the same trace, are conjugate.

Proof. From [34], the number of conjugacy classes of non-identity elements of order prime to 3 is $q - 1$. Observe that for $\lambda \in \mathbb{F}_q^\times$, $\text{Tr}(S(0,0,1)\Upsilon h(\lambda)) = \lambda^2 - 1$ and $|S(0,0,1)\Upsilon h(\lambda)|$ is prime to 3 if also $\lambda \neq -1$.

Moreover, $h(-1)$ has order 2 and trace $-1$ so there are $q - 1$ possible traces for non-identity elements of order prime to 3, and elements with different trace must be non-conjugate. Thus all conjugacy classes must have different traces.

We omit the proof of the following well-known result.

Proposition 3.14. Let $G = \text{PSL}(2,q)$. If $x, y \in G$ are uniformly random, then $\text{Pr}[\langle x, y \rangle = G] = 1 - O(\sigma_0(\log(q))/q)$ (3.20)

The following result is analogous to [1, Proposition 5.1], so the proof is omitted.

Proposition 3.15. If $g_1, g_2 \in U(q)H(q)$ are uniformly random and independent, then $\text{Pr}[|[g_1, g_2]| = 9] = 1 - \frac{1}{q - 1}$ (3.21)

3.2. Alternative definition. The definition of $\text{Ree}(q)$ that we have given is the one that best suits most of our purposes. However, to deal with some aspects of constructive recognition, we need the more common definition.

Following [35, Chapter 4], the exceptional group $G_2(q)$ is constructed by considering the Cayley algebra $\mathbb{O}$ (the octonion algebra), which has dimension 8, and defining $G_2(q)$ as the automorphism group of $\mathbb{O}$. Thus each element of $G_2(q)$ fixes the identity and preserves the algebra multiplication, and it follows that $G_2(q)$ is isomorphic to a subgroup of $\Omega(7,q)$.

Furthermore, when $q$ is an odd power of 3, $G_2(q)$ has a certain outer automorphism, sometimes called the exceptional outer automorphism, whose set of fixed points forms a group, and $\text{Ree}(q) = \Omega^2 G_2(q)$.

4. Algorithms

Here we present the algorithms mentioned in the introduction. We first give an algorithm to identify our standard copy, which is needed later.

Theorem 4.1. There exists a Las Vegas algorithm that, given $\langle X \rangle \leq \text{GL}(7,q)$, decides whether or not $\langle X \rangle = \text{Ree}(q)$. The algorithm has expected time complexity $O(\sigma_0(\log(q))(|X| + \log(q)))$ field operations.

Proof. Let $G = \text{Ree}(q)$. The algorithm proceeds as follows:

(1) Determine if $X \subseteq G$: all the following steps must succeed in order to conclude that a given $g \in X$ also lies in $G$. 

Proof.
(a) Determine if \( g \in \Omega(7, q) \), which is true if \( \det g = 1 \), if \( gJg^T = J \), where \( J \) is given by \( (3, 14) \), and if the spinor norm of \( g \) is 0. The spinor norm is calculated using \( (37) \) Theorem 2.10.

(b) Determine if \( g \in G_2(q) \), which from Section \( 3.2 \) is true if \( g \) preserves the octonion algebra multiplication \( \cdot \). Hence test if \( (e_i \cdot e_j)g = (e_i, g) \cdot (e_j, g) \) for each \( i, j = 1, \ldots, 7 \), where \( M = \langle e_1, \ldots, e_7 \rangle \) is the natural module of \( G_2(q) \). The multiplication table for \( \cdot \) can be pre-computed using \( (35) \) Chapter 4.

(c) Determine if \( g \) is a fixed point of the exceptional outer automorphism of \( G_2(q) \), mentioned in Section \( 3.2 \). Computing the automorphism amounts to extracting a submatrix of the exterior square of \( g \) and then replacing each matrix entry \( x \) by \( x^{3m} \).

(2) Determine if \( \langle X \rangle \) is a proper subgroup of \( G \), or equivalently if \( \langle X \rangle \) is contained in a maximal subgroup. By Proposition \( 3.7 \) it is sufficient to determine if \( \langle X \rangle \) can be written over a smaller field or if \( \langle X \rangle \) is reducible. This can be done using the algorithms described in \( [12] \) and the MeatAxe \( [17, 19] \).

The first step takes \( O(|X| \log(q)) \) field operations. The expected time of the algorithms in \( [12] \) and of the MeatAxe is \( O(\sigma_0(\log(q))(|X| + \log(q))) \) field operations. Hence our recognition algorithm has the stated expected time, and it is Las Vegas since the MeatAxe is Las Vegas.

\[ \square \]

4.1. Finding a \( \text{element of a stabiliser} \). Let \( G = \text{Ree}(q) = \langle X \rangle \). The algorithm for constructive membership testing needs to obtain independent random elements of \( G_P \), for a given point \( P \), as SLPs in \( X \). This is straightforward if, for any pair of points \( P, Q \in \mathcal{O} \), we can construct \( g \in G \) as an SLP in \( X \) such that \( Pg = Q \).

We first give an overview of the algorithm. The general idea is to obtain an involution \( j \in G \) by random search, and then compute \( C_\theta(j) \cong (j) \times \text{PSL}(2, q) \) using \( [8] \). The given module restricted to the centraliser splits up as in Proposition \( 3.8 \) and the points \( P, Q \in \mathcal{O} \) restrict to points \( P_3, Q_3 \) in the 3-dimensional submodule. If the restrictions satisfy certain conditions, then we can write down \( g \in C_\theta(j) \) that maps \( P_3 \) to \( Q_3 \), and obtain \( g \) as an SLP in the generators of \( C_\theta(j) \) using the maps from Theorem \( 2.1 \). With high probability, we can then multiply \( g \) by an element that fixes \( P_3 \) so that it also maps \( P \) to \( Q \). A discrete logarithm oracle is needed in that step. When using \( [8] \), we can easily keep track of SLPs of the centraliser generators, hence we obtain \( g \) as an SLP in \( X \).

By Corollary \( 3.13 \) it is easy to find elements of even order by random search, which we can power up to obtain involutions.

To use \( [8] \) we need an algorithm that determines if the whole centraliser has been generated. Since its derived group should be \( \text{PSL}(2, q) \), by Proposition \( 3.14 \) with high probability it is sufficient to compute two random elements of the derived group. Random elements of the derived group can be obtained as described in Section \( 2.6 \).

Let us now describe the algorithm in more detail. First we fix some notation.

- \( j \in G = \langle X \rangle = \text{Ree}(q) \) is an involution, and \( C = C_\theta(j) = \langle Y \rangle \),
- \( V \cong V_3 \oplus V_4 \) is the module of \( C' \),
- \( \varphi_\nu : V \rightarrow V_3 \) is the natural projection homomorphism,
- \( \varphi_\phi : P(V_3) \rightarrow P(V_3) \) is the induced projective map,
- \( \varphi_\theta : \text{GL}(7, q) \rightarrow \text{GL}(3, q) \) is the corresponding homomorphism of group elements and \( C_3 = \varphi_\theta(C') \),
- \( \pi_3 : \text{PSL}(2, q) \rightarrow \text{GL}(3, q) \) is the symmetric square map, so \( \pi_3 : g \mapsto S^2(g) \),
- \( \rho_\theta : C_3 \rightarrow \text{PSL}(2, q) \) is the map to the standard copy from Theorem \( 2.1 \).
\begin{itemize}
  \item $\pi_7 : C_3 \to C'$ is calculated by first using $\rho_O$ to map an element to the standard copy, then expressing at as an SLP, which is then evaluated on $Y$.
  \item $c_3$ is a change-of-basis from $V_3$ to $S^2(A)$, where $A$ is the natural module of $\text{PSL}(2,q)$. Hence $C_3^{O_3} = \text{Im} \pi_3$.
\end{itemize}

Clearly, an application of the MeatAxe on $V$ provides a change-of-basis which allows us to set up the maps $\varphi_V$, $\varphi_O$ and $\varphi_G$. An application of Theorem 4.2 on $C_3$ allows us to set up the maps $\pi_3$ and $\pi_7$, and to obtain $c_3$.

4.1.1. Constructing a mapping element. We now consider the algorithm that constructs elements that map one point of $O$ to another. If we identify $A$ as $\langle x \rangle \oplus \langle y \rangle$ for indeterminates $x, y$, then we can identify the module $U = \mathbb{P}(S^2(A))$ with the space of quadratic forms in $x$ and $y$ modulo scalars, so that $U = \langle x^2 \rangle \oplus \langle xy \rangle \oplus \langle y^2 \rangle$. Then $C_3^{O_3}$ acts projectively on $U$ and $|U| = |\mathbb{P}(V_3)| = |\mathbb{P}^2(\mathbb{F}_q)| = (q^2 - 1)/(q - 1) = q^2 + q + 1$.

**Proposition 4.2.** Under the action of $H = C_3^{O_3}$, the set $U$ splits into 3 orbits.

(1) The orbit containing $xy$, i.e. the non-degenerate quadratic forms that represent 0, which has size $q(q + 1)/2$.

(2) The orbit containing $x^2 + y^2$, i.e. the non-degenerate quadratic forms that do not represent 0, which has size $q(q - 1)/2$.

(3) The orbit containing $x^2$ (and $y^2$), i.e. the degenerate quadratic forms, which has size $q + 1$.

The pre-image in $\text{SL}(2,q)$ of $\rho_O((H_{xy})_{c_3}^{-1})$ is dihedral of order $2(q - 1)$, generated by the matrices

\[
\begin{bmatrix}
\omega & 0 \\
0 & \omega^{-1}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\]

(4.1)

**Proof.** This is elementary theory of quadratic forms, except that we work projectively.

\[]

**Proposition 4.3.** Use the notation above.

(1) The number of points of $O$ that are contained in $\text{Ker}(\varphi_V)$ is $q + 1$.

(2) Let $P \in O$ be uniformly random. The probability that $P \notin \text{Ker}(\varphi_V)$, and that $\varphi_O(P)c_3$ is both non-degenerate and represents 0 is at least $1/2 + O(1/q)$.

**Proof.**

(1) The map $\varphi_V$ projects onto $V_3$, so the kernel consists of those vectors that lie in $V_4$. From the proof of Proposition 3.7 with respect to a suitable basis, $V_4$ is the $-1$-eigenspace of $h(-1)$. Hence by an argument similar to the proof of Proposition 3.5 a point $P = (p_1, \ldots, p_r) \in V_4$ if $p_2 = p_4 = p_6 = 0$ and there are $q + 1$ such points in $O$.

(2) Since $P$ is uniformly random and $\varphi_O$ is chosen independently of $P$, it follows that $\varphi_O(P)$ is uniformly random from $\varphi_O(O)$. Without loss of generality we can take $c_3$ to be the identity. Using the notation above, $\varphi_O(P) = p_2x^2 + p_4xy + p_6y^2$, which is degenerate if $p_2 = 0$. This happens with probability $(q + 1)/(q^2 + 1)$. If $p_2 \neq 0$, then $P \notin \text{Ker}(\varphi_V)$, and we can then express the point as $\varphi_O(P) = x^2 + bxy + cy^2$ where $(1 : b : c)$ is uniformly distributed in $\mathbb{P}^2(\mathbb{F}_q)$. It is degenerate if $b^2 - c = 0$, which happens with probability $1/q$. If it is not degenerate, it represents 0 when $b^2 - c$ is a square in $\mathbb{F}_q$, which happens with probability $1/2$. The result follows.

\[]

The algorithm that maps one point to another is given as Algorithm 4.1.
Algorithm \[\text{FindMappingElement}(X, P, Q)\]

1. **Input:** Generating set \(X\) for \(G = \text{Ree}(q)\), \(H = C_3\).
   Points \(P \neq Q \in O\) such that \(P, Q \not\in \text{Ker}(\varphi_V)\), and \(\varphi_O(P)\) and \(\varphi_O(Q)\) are non-degenerate and represent 0.
2. **Output:** \(g_2 \in G\), written as an SLP in \(X\), such that \(Pg_2 = Q\).
   \(P_1 := \varphi_O(P)c_3; Q_3 := \varphi_O(Q)c_3\)
4. Construct upper triangular \(g \in \text{PSL}(2, q)\) such that \(P_3 \pi_3(g) = Q_3\)
5. \(R_3 := \varphi_O(P \pi_7(\pi_3(g)^{c_3^1}))c_3\)
6. \(/ / \) Now \(R_3 := Q_3\)
7. Construct \(c \in \text{GL}(3, q)\) such that \((xy)c = R_3\)
8. Let \(D\) be the image in \(\text{PSL}(2, q)\) of the diagonal matrix in \((4.1)\)
9. \(s := \pi_7(\pi_3(D)^{c_3^1})\)
10. \(/ / \) Now \(s \leq \varphi_G^{-1}(H_{R_3})\)
11. \(\delta, \lambda \in \text{Diagonalise}(s)\)
12. \(/ / \) Now \(\delta = s^2\)
13. \(\text{if } \exists \lambda \in \mathbb{F}_q^*\) such that \((P \pi_7(\pi_3(g)^{c_3^1}))\lambda h(\lambda) = Qz\)
    \(\text{then}\)
14. \(i := \text{DiscreteLog}(\delta, h(\lambda))\)
15. \(/ / \) Now \(\delta^i = h(\lambda)\)
16. \(\text{return } \pi_7(\pi_3(g)^{c_3^1})s^i\)
17. \(\text{return } \text{FAIL}\)

4.1.2. Constructing a stabilising element. Let \(G = \langle X \rangle\), \(P \in O\) be given. The complete algorithm that constructs a random element of \(GP\) proceeds as follows.

1. Find a random involution \(j \in G\).
2. Compute generators \(Y\) for \(C = C_G(j)\) using [8], and generators for \(C'\) as described in Section 2.3.
3. Use the MeatAxe to verify that the module for \(C'\) splits up only as in Proposition 3.8.
4. Return to the first step if \(P\) lies in the kernel of \(\varphi_V\), if \(\varphi_O(P)c_3\) is degenerate, or if it does not represent 0.
5. Use Theorem 2.1 to verify that we have the whole of \(C'\) and to set up maps listed at the start of of Section 4.1. Return to the second step if this fails.
6. Take random \(g_1 \in G\) and let \(Q = Pg_1\). Repeat until \(P \neq Q\, Q\) does not lie in the kernel of \(\varphi_V\) and \(\varphi_O(Q)c_3\) is not degenerate and represents 0.
7. Use Algorithm \[\text{FindMappingElement}\] to find \(g_2 \in C'\) such that \(Q = Pg_2\). Return to the previous step if it fails, otherwise return \(g_1g_2\).

4.1.3. Correctness and complexity.

**Lemma 4.4.** Let \(P_3, Q_3 \in \varphi_O(O)\) be non-degenerate and represent 0. There exists \(g \in \text{PSL}(2, q)\) such that the pre-image of \(g\) in \(\text{SL}(2, q)\) is upper triangular and \(P_3 \pi_3(g)^{c_3^1} = Q_3\).

**Proof.** Without loss of generality, we can take \(c_3 = 1\). Since \(P_3\) and \(Q_3\) are non-degenerate, \(P_3 = x^2 + axy + by^2\) and \(Q_3 = x^2 + lxy + ny^2\) where \((1 : a : b)\) and \((1 : l : n)\) are in \(\mathbb{P}^2(\mathbb{F}_q)\). Also,
\[
g = \begin{bmatrix} u & v \\ 0 & 1/u \end{bmatrix}
\]
where \(u, v \in \mathbb{F}_q\) and \(u \neq 0\).
We want to determine $u, v$ such that $P\pi_3(g) = Q$. Note that $g$ is the pre-image in $\text{SL}(2, q)$ of an element in $\text{PSL}(2, q)$ and therefore $\pm u$ determine the same element of $\text{PSL}(2, q)$. The map $\pi_3$ is the symmetric square map, so
\[
\pi_3(g) = \begin{bmatrix}
u^2 & -uv & v^2 \\
0 & 1 & v/u \\
0 & 0 & 1/u^2
\end{bmatrix}
\quad(4.3)
\]
This leads to the following equations:
\[
u^2 = C 
\quad(4.4)
-uv + a = C\ell 
\quad(4.5)
v^2 + avu^{-1} + bu^{-2} = Cn 
\quad(4.6)
\]
for some $C \in \mathbb{F}_q^\times$. We can solve for $u$ in (4.4) and for $v$ in (4.5), so that (4.6) becomes
\[
C^2(n - l^2) + a^2 - b = 0 
\quad(4.7)
\]
This quadratic equation has a solution if the discriminant $(l^2 - n)(a^2 - b) \in (\mathbb{F}_q^\times)^2$. But the latter is true since both $(a^2 - b)$ and $(l^2 - n)$ are non-zero squares. The result follows.

**Theorem 4.5.** If Algorithm 1 returns an element $g_2$, then $Pg_2 = Q$. If $P$ and $Q$ are chosen as in Section 4.1.2, then the probability that Algorithm 1 finds such an element is at least $1/2 + O(1/q)$.

**Proof.** By Proposition 4.2, the point $R_3$ is in the same orbit as $xy$, so the element $c$ at line 4 can easily be found by diagonalising the form corresponding to $R_3$.

Let $H = C_3$. Then $\pi_3(D)^{-1} \in H_{R_3}$ has order $(q - 1)/2$. Hence $s$ also has order $(q - 1)/2$, and $s \in \phi_G^{-1}(H_{R_3})$.

We choose $Q$ such that there exist $g_1 \in C'$ with $Pg_1 = Q$. If we let $g_3 = \pi_3(g)^{-1}$, with $g$ as in the algorithm, and $R = P\pi_7(g_3)$, then $R\pi_7(g_3)^{-1}g_1 = Q$ and $\phi_G(R) = R_3 = Q_3$. Hence $\phi_G(\pi_7(g_3)^{-1}g_1) \in H_{Q_3}$, and therefore $\pi_7(g_3)^{-1}g_1 \in \phi_G^{-1}(H_{R_3})$.

By Proposition 4.2, $\phi_G^{-1}(H_{R_3})$ is dihedral of order $q - 1$, and $s$ generates a subgroup of index 2. Therefore $Pr[\pi_7(g_3)^{-1}g_1 \in \langle s \rangle] = 1/2$, which is the success probability of line 13.

It is straightforward to determine if $\lambda$ exists, since $h(\lambda)$ is diagonal. Hence the success probability of the algorithm is as stated. □

**Theorem 4.6.** Assume an oracle for the discrete logarithm problem in $\mathbb{F}_q$. The time complexity of Algorithm 1 is $O(\log(q)^3 + \chi_D)$ field operations. The length of the returned SLP is $O(\log(q) \log \log(q))$.

**Proof.** By Lemma 4.3, line 4 involves solving a quadratic equation in $\mathbb{F}_q$, and hence uses $O(1)$ field operations. Evaluating the maps $\pi_3$ and $\pi_7$ uses $O(\log(q)^3)$ field operations, and it is clear that the rest of the algorithm can be done using $O(\chi_D)$ field operations.

By Theorem 2.1, the length of the SLP from the constructive membership testing in $\text{PSL}(2, q)$ is $O(\log(q) \log \log(q))$, which is therefore also the length of the returned SLP. □

**Corollary 4.7.** Assume an oracle for the discrete logarithm problem in $\mathbb{F}_q$. There exists a Las Vegas algorithm that, given $\langle X \rangle \leq \text{GL}(7, q)$ such that $G = \langle X \rangle = \text{Ree}(q)$ and $P \in \mathcal{O}$, constructs a random element of $GP$ as an SLP in $X$. The expected time complexity of the algorithm is $O(\xi \log \log(q) + \log(q)^3 + \chi_D)$ field operations. The length of the returned SLP is $O(\log(q) \log \log(q))$. 
Proof. The algorithm is given in Section 4.1.2.

An involution is found by obtaining a random element of even order and then raising it to an appropriate power. Hence by Corollary 4.1.2, the expected time to find an involution is $O(\xi + \log(q) \log \log(q))$ field operations.

By [14, Theorem 7], we can use [8] to obtain generators of the centraliser, using $O(1)$ field operations. As described in Section 2.3, we can obtain uniformly random elements of its derived group. By Proposition 3.14, two random elements will generate $\text{PSL}(2, q)$ with high probability. This implies that the expected time to obtain generators for $\text{PSL}(2, q)$ is $O(1)$ field operations.

By Proposition 4.2, $P$ is equal to $Q$ with probability $2/(q(q+1))$. By Proposition 4.3, $P$ has the required properties with probability $1/2$, and similarly for $Q$, so the expected time of the penultimate step is $O(1)$ field operations.

Since $P$ and $Q$ can be considered uniformly random and independent in Algorithm 11, the element returned by that algorithm is uniformly random. Hence the element returned by the algorithm in Section 4.1.2 is uniformly random.

The expected time complexity of the last step is given by Theorem 4.5 and 4.6. It follows from the above argument that the expected time complexity of the algorithm in Section 4.1.2 is as stated.

The algorithm is clearly Las Vegas, since it is straightforward to check that the element we compute fixes the point $P$. \hfill \square

4.2. Constructive membership testing. We now describe the constructive membership algorithm for our standard copy $\text{Ree}(q)$. Given a set of generators $X$, such that $G = \langle X \rangle = \text{Ree}(q)$, and given $g \in G$, we want to express $g$ as an SLP in $X$. To decide if $g \in G$, we use the first step of the algorithm in Theorem 4.1.

The general structure of the algorithm mirrors the corresponding algorithm for the Suzuki groups $H$. It consists of a pre-processing step and a main step.

4.2.1. Pre-processing. The pre-processing step constructs “standard generators” for $O_3(G_{P_\infty}) = U(q)$ and $O_3(G_{R_0})$. For $O_3(G_{P_\infty})$ the standard generators are matrices

\begin{equation}
\{S(a_i, x_i, y_i)\}_{i=1}^n \cup \{S(0, b_i, z_i)\}_{i=1}^n \cup \{S(0, 0, c_i)\}_{i=1}^n,
\end{equation}

for some unspecified $x_i, y_i, z_i \in \mathbb{F}_q$, such that $\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_n\}, \{c_1, \ldots, c_n\}$ form vector space bases of $\mathbb{F}_q$ over $\mathbb{F}_3$ (so $n = \log_3 q = 2m + 1$). The standard generators for $O_3(G_{R_0})$ are analogous.

We shall need the following elementary result, which we state without proof.

Lemma 4.8. There exist algorithms for the following row reductions.

1. Given $g = h(\lambda)S(a, b, c) \in G_{P_\infty}$, construct $x \in O_3(G_{P_\infty})$ expressed as an SLP in the standard generators, such that $gx = h(\lambda)$.

2. Given $g = S(a, b, c)h(\lambda) \in G_{P_\infty}$, construct $x \in O_3(G_{P_\infty})$ expressed as an SLP in the standard generators, such that $xg = h(\lambda)$.

3. Given $P_\infty \neq P \in \mathcal{O}$, construct $g \in O_3(G_{P_\infty})$ expressed as an SLP in the standard generators, such that $Pg = F_0$.

The SLPs of the constructed elements have length $O(\log(q))$. The algorithms have time complexity $O(\log(q)^2)$ field operations. Analogous algorithms exist for $G_{R_0}$.

Theorem 4.9. Given an oracle for the discrete logarithm problem in $\mathbb{F}_q$, the pre-processing step is a Las Vegas algorithm that constructs standard generators for $O_3(G_{P_\infty})$ and $O_3(G_{R_0})$ as SLPs in $X$ of length $O(\log(q) \log \log(q))^2$. It has expected time complexity $O((\xi \log \log(q) + \log(q)^3 + \chi_D) \log \log(q))$ field operations.

Proof. The pre-processing algorithm consists of the following steps:

1. Obtain random $a_1, a_2 \in G_{P_\infty}$ and $b_1, b_2 \in G_{R_0}$ using the algorithm from Corollary 4.7. Let $c_1 = [a_1, a_2], c_2 = [b_1, b_2]$. 

\end{document}
(2) Determine if there exists $d_1 \in \{a_1, a_2\}$ that can be diagonalised to $h(\lambda) \in G$, where $\lambda \in F_q^\times$ does not lie in a proper subfield of $F_q$. Similarly determine existence of a $d_2$ from $b_1$ and $b_2$. Determine if $|c_1| = |c_2| = 9$. Return to the first step if any of these tests fail.

(3) As standard generators for $O_3(G_{P_0})$ we take $U = U_1 \cup U_2$ where

\[
U_1 = \bigcup_{i=1}^{2m+1} \left\{ \alpha_i^{d_1}, (\alpha_i^2)^{d_1} \right\}
\]

\[
U_2 = \bigcup_{1 \leq i < j \leq 2m+1} \left\{ [\alpha_i^{d_1}, \alpha_i^{d_1}] \right\}
\]

From $c_2$ and $d_2$ we similarly obtain standard generators $L$ for $O_3(G_{P_0})$ 

It follows from (3.10) and (3.13) that $U$ is uniformly distributed independent random elements. The elements of order dividing $q - 1$ can be diagonalised as required. By Proposition 3.1 the proportion of elements of order $q - 1$ in $G_{P_0}$ and $G_{P_0}$ is $\phi(q - 1)/(q - 1)$.

It is straightforward to determine if $a_i$ or $b_i$ diagonalise to some $h(\lambda)$, since they are triangular. To determine if $\lambda$ lies in a proper subfield, it is sufficient to determine if $|\lambda| \mid 3^n - 1$, for some proper divisor $n$ of $2m + 1$.

Hence by Proposition 3.16 the expected time for the first two steps is

$O((\xi \log \log(q) + \log(q)^3 + \chi_D) \log \log(q))$

field operations.

By the remark preceding the theorem, $U$ determines three sets of field elements $\{a_1, \ldots, a_{2m+1}\}$, $\{b_1, \ldots, b_{2m+1}\}$ and $\{c_1, \ldots, c_{2m+1}\}$. By (3.13), in this case each $a_i = a\lambda_i$, $b_i = b\lambda_i^{(t+2)}$ and $c_i = c\lambda_i^{(t+3)}$, for some fixed $a, b, c \in F_q^\times$, where $\lambda$ is as in the algorithm. Since $\lambda$ does not lie in a proper subfield, these sets form vector space bases of $F_q$ over $F_3$. Hence $U$ and $L$ are standard generators and the algorithm is Las Vegas.

\[\square\]

4.2.2. Main algorithm. We now present the algorithm to express an arbitrary $g \in G$ as an SLP. It is given as Algorithm 2

4.2.3. Correctness and complexity.

**Theorem 4.10.** Algorithm 2 is correct, and is a Las Vegas algorithm.

**Proof.** First observe that since $r$ is randomly chosen, we obtain it as an SLP.

The elements $z_1$ and $z_2$ can be constructed using Lemma 4.8 so we can obtain them as SLPs.

The element $u$ constructed at line 13 clearly has trace $x$. Because $u$ can be computed using Lemma 4.8 we obtain it as an SLP. From Proposition 4.13 we know that $u$ is conjugate to $h(\lambda)^{\pm 1}$, for some $\lambda \in F_q^\times$, and therefore fixes two points of $O$. Hence the elements found at lines 16 and 17 can be computed using Lemma 4.8 so we obtain them as SLPs.

Finally, the elements that determine $w$ have been constructed as SLPs, and it is clear that if we evaluate $w$ we obtain $g$. Hence the algorithm is Las Vegas and the theorem follows. \[\square\]

**Theorem 4.11.** Algorithm 2 has expected time complexity $O(\xi + \log(q)^3)$ field operations and the length of the returned SLP is $O((\log(q) \log \log(q))^2)$. 
Algorithm 2. ElementToSLP($U, L, g$)

1. **Input:** Standard generators $U$ for $G_{P_{\infty}}$ and $L$ for $G_{P_0}$. Matrix $g \in \langle X \rangle = G$.
2. **Output:** SLP for $g$ in $X$
3. **repeat**
   4. **repeat**
   5. $r := \text{RANDOM}(G)$
   6. **until** $gr$ has an eigenspace $Q \in \mathcal{O}$ and $P \neq Q$
   7. Construct $z_1 \in G_{P_{\infty}}$ using $U$ such that $Qz_1 = P_0$.
   8. // Now $(gr)^{z_1} \in G_{P_0}$
   9. Construct $z_2 \in G_{P_0}$ using $L$ such that $(gr)^{z_2} = h(\lambda)$ for some $\lambda \in \mathbb{F}_q^\times$
10. $x := \text{Tr}(h(\lambda))$
11. **until** $x - 1$ is a square in $\mathbb{F}_q^\times$
12. // Express diagonal matrix as SLP
13. Construct $u = S(0, 0, \sqrt{(x - 1)\mathbb{F}_q})S(0, 1, 0)^T$ using $U \cup L$
14. // Now Tr$(u) = x$
15. Let $P_1, P_2 \in \mathcal{O}$ be the fixed points of $u$
16. Construct $a \in G_{P_{\infty}}$ using $U$ such that $P_1 a = P_0$
17. Construct $b \in G_{P_0}$ using $L$ such that $(P_2 a)b = P_{\infty}$
18. // Now $u^{ab} \in G_{P_{\infty}} \cap G_{P_0} = H(q)$, so $u^{ab} \in \{h(\lambda)\}^\pm$
19. **if** $u^{ab} = h(\lambda)$
   20. **then**
   21. Let $w$ be the SLP for $(u^{ab}z_2^{-1})^{z_1^{-1}}r^{-1}$
   22. **return** $w$
   23. **else**
   24. Let $w$ be the SLP for $((u^{ab})^{-1}z_2^{-1})^{z_1^{-1}}r^{-1}$
25. **return** $w$

**Proof.** It follows immediately from Lemma 4.8 that lines 11, 16, 19, 21 and 24 use $O(\log(q)^3)$ field operations.

From Corollary 4.12 the expected time to find $r$ is $O(\chi)$ field operations. Half of the elements of $\mathbb{F}_q^\times$ are squares, and $x$ is uniformly random, hence the expected time of the outer repeat statement is $O(\chi + \log(q)^3)$ field operations.

Obtaining the fixed points of $a$, and performing the check at line 19 only amounts to considering eigenvectors, hence uses $O(\log(q))$ field operations. Thus the expected time complexity of the algorithm is $O(\chi + \log(q)^3)$ field operations.

From Theorem 4.10 each standard generator SLP has length $O(\log(q)(\log \log(q))^2)$ and hence $w$ has length $O(\log(q)(\log \log(q))^2)$. $\square$

4.3. Conjugates of the standard copy. Assume that we are given a conjugate $G$ of $\text{Ree}(q)$. We consider the problem of constructing $g \in \text{GL}(7, q)$ such that $G^g = \text{Ree}(q)$, thus obtaining an algorithm that constructs effective isomorphisms from any conjugate of $\text{Ree}(q)$ to the standard copy.

**Theorem 4.12.** Assume an oracle for the discrete logarithm problem in $\mathbb{F}_q$. There exists a Las Vegas algorithm that, given a conjugate $G = \langle X \rangle$ of $\text{Ree}(q)$, constructs $g \in \text{GL}(7, q)$ such that $\langle X \rangle^g = \text{Ree}(q) = S$. The algorithm has expected time complexity $O(\chi \log \log(q) + \log(q)(\sigma_0(\log(q)) + \log(q)) + \chi_D)$ field operations.

**Proof.** We prove the result by exhibiting the algorithm. Let $V = \langle e_1, \ldots, e_7 \rangle$ be the given module.
(1) Find a random involution \( j_G \in G \). Let \( j_S = h(-1) \in S \). By Corollary 3.12 the expected time is \( O(\xi + \log(q) \log(\log(q))) \).

(2) Compute generators for \( C_G(j_G) \) using \[3\], and generators for \( C^G = C_G(j_G)' \) by taking commutators of the generators of \( C_G(j_G) \). Observe that \( C_S(j_S) = (T, h(\omega), S(0, 1, 0)) [30] \) and similarly compute generators for \( C^S = C_S(j_S)' \). Similarly as in the proof of Corollary 4.1 the expected time is \( O(1) \).

(3) Use the MeatAxe to decompose the module of \( C^G \) into its direct summands \( V_3^G \) and \( V_1^G \) of dimension 3 and 4. Decompose the module of \( C^S \) into \( V_3^S = \langle e_2, e_4, e_6 \rangle \) and \( V_1^S = \langle e_1, e_3, e_5, e_7 \rangle \). Hence obtain change-of-bases \( c_G \) and \( c_S \) which exhibit the direct sums, with the 3-dimensional submodules coming first. Let \( C_3^G \) and \( C_4^G \) be the projections of \( C^G \) acting on the 3-space and 4-space, respectively, and similarly define \( C_3^S \) and \( C_4^S \). Since we have \( O(1) \) generators for \( C^G \) and \( C^S \), the expected time is \( O(\log(q)) \). Note that we also obtain a bijection between the generators of \( C^G \) and \( C_3^G \) or \( C_4^G \), respectively, and similarly for \( C^S \).

(4) Use Theorem 4.1 to constructively recognise \( C_4^G \) and \( C_4^S \) and obtain standard generators \( Y_4^G \) and \( Y_4^S \) for these groups as SLPs in the input generators. Evaluate the SLPs on the generators of \( C^G \) and \( C^S \) and use \( c_G \) and \( c_S \) to project the resulting matrices to the 3-spaces. Hence also obtain standard generators \( Y_3^G \) and \( Y_3^S \) for \( C_3^G \) and \( C_3^S \). The expected time is \( O((\xi + \log(q) \log(\log(q))) \log(\log(q)) + \chi_D) \). Note that \( |Y_3^G| = |Y_3^G| = |Y_3^S| = |Y_3^S| \).

(5) Let \( A \) be the natural module of PSL(2, q). By Proposition 3.8 \( V_3^S \cong A^{\phi|s} \otimes A^{\phi|s} \), for some \( 1 \leq i < j \leq k \leq 2m + 1 \), where \( \phi \) is the Frobenius automorphism. Similarly, \( V_4^G \cong A^{\phi|G} \otimes A^{\phi|G} \). Use the MeatAxe together with \( Y_4^G \) and \( Y_4^S \) to obtain \( 1 \leq k \leq 2m + 1 \) such that \( V_4^G \cong (V_4^S)^{\phi|k} \). Hence obtain a change-of-basis \( c_4 \) between these. Then \( (C_4^G)^{c_4} = C_3^S \). The expected time is \( O(\log(q))^2 \).

(6) Similarly, use the MeatAxe together with \( Y_3^G \) and \( Y_3^S \) to construct a change-of-basis \( c_3 \) from \( V_3^G \) to \( (V_3^S)^{\phi|k} \). Then \( (C_3^G)^{c_3} = C_3^S \). The expected time is \( O(\log(q)) \).

(7) Let \( c_T = diag(c_3, c_4) \). Let \( c = c_T^{-1} \). Then \( (C^G)^{c} = C_S \).

(8) Now \( C_S \leq G^c \cap S \), so \( G^c \) must preserve a form which is preserved by \( C_S \). Use the MeatAxe to construct the form \( K \) preserved by \( G^c \). By Proposition 3.10 \( K = antidiag(1, a, 1, -a, 1, a, 1) \) for some \( a \in (F^*)^2 \), up to a scalar multiple. Let \( x = \sqrt{a} \) and \( c_J = diag(1, x, 1, x, 1, x, 1) \in C_S \). Then \( G^{c_J} \) preserves the form \( J \). The expected time is \( O(1) \).

(9) Now \( G^{c_J} \leq \Omega(7, q) \) and \( C_S \leq G^{c_J} \). By Proposition 3.11 \( C_S \) is contained in at most two \( \Omega(7, q) \)-conjugates of \( S \), so \( G^{c_J} = S \) with probability at least \( 1/2 \). Use Theorem 4.1 to test this. The expected time is \( O(\sigma_0(\log(q)) \log(q)) \).

If any of the tests or Las Vegas algorithms used fail, we start again from the beginning. In total, the expected time complexity is \( O(\xi \log(q) + \log(q)\sigma_0(\log(q)) + \log(q) + \chi_D) \) field operations. This proves the result. \( \square \)

4.4. Main theorem.

Proof of Theorem 4.1. The algorithm providing \( \Psi \) follows from Theorem 4.1.2 since \( \Psi \) and \( \Psi^{-1} \) are just conjugations, they can be computed using \( O(1) \) field operations, so they are effective.

Constructive membership testing in \( \text{Ree}(q) \) follows from Theorem 4.1.9 and Algorithm 2. For constructive membership testing in \( (X) \) we first map the element to \( \text{Ree}(q) \) using \( \Psi \), then express it as an SLP. \( \square \)
5. Implementation and performance

Implementations of the algorithms are available in MAGMA. The implementations use the existing MAGMA implementations of the algorithms described in [8], [9], [10], [11], [12] and [17, 19].

We have benchmarked the computation of generating sets for stabilisers, in other words most of the algorithm from Theorem 4.9. This is shown in Figure 5.1. For each field size $q = 3^{2m+1}$, generating sets for stabilisers of 100 random points were computed, and the average running time for each call is listed. The amount of this time that was spent in discrete logarithm computations outside [11], SLP evaluations and in [11] is also indicated. Note that the algorithm of [11] also uses a discrete logarithm oracle.

When $2m + 1$ has a “small” prime divisor, finite field arithmetic in $\mathbb{F}_q$ in MAGMA is particularly fast. This is because MAGMA uses Zech logarithms for finite fields up to a certain size, and for larger fields it tries to find a subfield smaller than this size. If this is possible the arithmetic in the larger field will be very fast. To avoid jumps in the figure, and to properly measure field operations, we have turned off this optimisation, and have in each case divided by the time required for $10^6$ multiplications of random pairs of field elements.

![Figure 5.1. Benchmark of stabiliser computation](image)

In the same fashion, we have benchmarked the conjugation algorithm from Theorem 4.12. This is shown in Figure 5.2.

All benchmarks were carried out using MAGMA V2.18-2, Intel64 flavour, on a PC with an Intel Core2 CPU running at 2 GHz, and with 2 GB of RAM. The largest
The value of $m$ in the tests was 20, since discrete logarithm computations became very slow in $\mathbb{F}_{3^{43}}$.

### References

1. Henrik Bäärnhielm, *Recognising the Suzuki groups in their natural representations*, J. Algebra 300 (2006), no. 1, 171–198. MR 2228642
2. , *Algorithmic problems in twisted groups of Lie type*, Ph.D. thesis, Queen Mary, University of London, 2007.
3. Henrik Bäärnhielm, Derek Holt, C.R. Leedham-Green, and E.A. O’Brien, *A new model for computation with matrix groups*, (2011), submitted.
4. László Babai, *Local expansion of vertex-transitive graphs and random generation in finite groups*, STOC ’91: Proceedings of the twenty-third annual ACM Symposium on Theory of Computing (New York, NY, USA), ACM Press, 1991, pp. 164–174.
5. László Babai and Robert Beals, *A polynomial-time theory of black box groups. I*, Groups St. Andrews 1997 in Bath, I, London Math. Soc. Lecture Note Ser., vol. 260, Cambridge Univ. Press, Cambridge, 1999, pp. 30–64. MR 1676609 (2000h:20089)
6. Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. 24 (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993). MR 1484478
7. R. Brauer and C. Nesbitt, *On the modular characters of groups*, Ann. of Math. (2) 42 (1941), 556–590. MR 0004042 (2.309c)
8. John N. Bray, *An improved method for generating the centralizer of an involution*, Arch. Math. (Basel) 74 (2000), no. 4, 241–245. MR 1742633 (2001c:20063)
9. Frank Celler, Charles R. Leedham-Green, Scott H. Murray, Alice C. Niemeyer, and E.A. O’Brien, *Generating random elements of a finite group*, Comm. Algebra 23 (1995), no. 13, 4931–4948. MR 1356111 (96k:20115)
10. Frank Celler and C.R. Leedham-Green, *Calculating the order of an invertible matrix*, Groups and computation, II (New Brunswick, NJ, 1995), DIMACS Ser. Discrete Math. Theoret.
35. Robert A. Wilson, The finite simple groups.
34. Harold N. Ward, On Ree’s series of simple groups.
33. Igor E. Shparlinski, Finite fields: theory and computation.
32. ´Akos Seress, Writing projective representations over subfields.
31. S.P. Glasby, C.R. Leedham-Green, and E.A. O’Brien, An introduction to the theory of numbers.
30. Rimhak Ree, The product replacement algorithm is polynomial.
29. Igor Pak, A new construction of the Ree groups of type $^2G_2$.
28. E.A. O’Brien, Permutation group algorithms.
27. Scott H. Murray and Colva M. Roney-Dougal, Treating the exceptional cases of the MeatAxe.
26. D. S. Mitrinovi´c, J. S´andor, and B. Crstici, Handbook of number theory.
25. V. M. Levchuk and Ya. N. Nuzhin, The structure of Ree groups.
24. C.R. Leedham-Green and E.A. O’Brien, Constructive recognition of classical groups in odd characteristic.
23. C.R. Leedham-Green and Scott H. Murray, Variants of product replacement.
22. Charles R. Leedham-Green, The computational matrix group project.
21. Peter B. Kleidman, The maximal subgroups of the Chevalley groups.
20. Gregor Kemper, Frank L¨ubeck, and Kay Magaard, Constructive recognition of $G_2(q)$.
19. G´abor Ivanyos and Klaus Lux, Treating the exceptional cases of the MeatAxe.
18. Bertram Huppert and Norman Blackburn, Finite groups. III.
17. Derek F. Holt and Sarah Rees, Handbooks of number theory.
16. Derek F. Holt, Bettina Eick, and Eamonn A. O’Brien, Handbook of computational group theory.
15. P.E. Holmes, S.A. Linton, E.A. O’Brien, A.J.E. Ryba, and R.A. Wilson, Algorithms for matrix groups.
14. Morton E. Harris and Christoph Hering, On the smallest degrees of projective representations.
13. G. H. Hardy and E. M. Wright, An introduction to the theory of numbers.
12. S.P. Glasby, C.R. Leedham-Green, and E.A. O’Brien, Writing projective representations over subfields.
11. M.D.E. Conder, C.R. Leedham-Green, and E.A. O’Brien, An introduction to the theory of numbers.
10. David P. Garman, Peter B. Kleidman, and Martin W. Liebeck, The maximal subgroups of the Chevalley groups.
9. Richard A. Wilson, The finite simple groups.
8. G. J. Janusz, Fields and Galois theory.
7. James E. Humphreys, Reflection groups and Coxeter groups.
6. Robert W. Gomberg, Relations between isomorphic subgroups.
5. Donald G. Higman, Automorphisms of regular graphs.
4. I. N. Herstein, Topics in algebra.
3. David S. Dummit and Richard M. Foote, Abstract algebra.
2. I.N. Herstein, Topics in algebra.
1. L. E. Dickson, History of the theory of numbers.
