Integrable theories in any dimension 
and homogenous spaces

Luiz A. Ferreira and Erica E. Leite

Instituto de Física Teórica - IFT/UNESP
Rua Pamplona 145
01405-900 São Paulo-SP, BRAZIL

Abstract

We construct local zero curvature representations for non-linear sigma models on homogeneous spaces, defined on a space-time of any dimension, following a recently proposed approach to integrable theories in dimensions higher than two. We present some sufficient conditions for the existence of integrable submodels possessing an infinite number of local conservation laws. Examples involving symmetric spaces and group manifolds are given. The $CP^N$ models are discussed in detail.
1 Introduction

The development of techniques to study non-perturbative aspects of physical theories is of crucial importance in practically all areas of Physics. Many open problems in high energy physics can not be studied with conventional perturbative methods, and they are in fact related to the non-linear character of the Lorentz invariant field theories describing the fundamental interactions of Nature.

It is perhaps correct to say that many of the developments obtained so far in such area involve, in one way or the other, soliton solutions. The most recent and striking examples are the exact results obtained about the strong coupling regime of supersymmetric gauge theories [1]. They involve a new version of the electromagnetic duality [2] which interchanges the role played by the two types of fundamental particles of the theory, namely the excitations of the weakly coupled fields (gauge and matter particles) and the solitons (magnetic monopoles and dyons).

One of the main features of such duality is that the solitons involved saturate a lower bound for the mass, the so-called Bogomolny bound [3]. The classical solutions for these monopoles can be calculated exactly because they satisfy some self-duality first order differential equations known as the Bogomolny-Prasad-Sommerfield (BPS) equations [3]. They define a kind of integrable submodel of the full theory, which present very interesting properties. They are the counterpart in Minkowski space-time of the self-duality condition for the Euclidean Yang-Mills theory containing the instanton solutions.

In order to develop techniques to study those types of phenomena one needs a deep understanding of the structures and symmetries of the corresponding theories. However, it is well known that soliton solutions are associated to integrability properties of the model, like infinite number of conservation laws and exact integration of the equations of motion. In two dimensional space-time, such relationship is now quite well understood and several techniques have been developed, based specially on the zero curvature or Lax pair equation for the theory. Therefore, it is of great importance to attempt to understand the non-perturbative aspects of non-linear field theories relevant for high energy physics, like gauge theories, using their integrability properties.

Recently, it has been proposed a new approach to construct and study integrable theories on a space-time of any dimension [4]. The central point of that approach is to generalize the zero curvature condition in two dimensions guided by the fact that it embodies conservation laws. The extension of integrability concepts to higher dimensions is a long standing problem. The main difficulties are associated to non
locality issues that rise when dealing with higher rank connections. Those problems can be circumvented by the introduction of auxiliary connections that allow for parallel transport. Indeed, it has been shown in [4] how to obtain local zero curvature conditions in space-time of any dimension. The self-dual Yang-Mills theory and the BPS sector of spontaneously broken gauge theories, discussed above, have been shown to be examples of theories admitting such local zero curvature representations.

One of the interesting aspects of [4] is that many theories presenting the local zero curvature are not integrable in the sense of possessing an infinite number of conservation laws. However, some of those theories contain integrable submodels that do present an infinite number of conserved currents.

The aim of the present paper is to clarify some sufficient conditions for the appearance of such integrable submodels. For that, we study Lorentz invariant field theories in space-time of any dimension, defined on homogeneous spaces. Basically, we treat the non-linear sigma models on coset spaces $G/K$, and show how to construct the local zero curvature representation for them using the approach of [4]. We argue that the equations of motion are determined by the representation $R^S$ of the subgroup $K$ defined by the tangent space of $G/K$. The construction of integrable submodels is then shown to be related to the representations of $G$ which contain $R^S$ in their branching in terms of representations of $K$. The submodel is in fact determined by the constraints that the zero curvature condition, based on those representations, imposes on the original theory. The number of conservation laws of the submodel is in fact equal to the sum of the dimensions of the representations of $G$ containing $R^S$ and leading to the same set of constraints. In many cases, the number of conserved currents is infinite.

The paper is organized as follow. In section 2 we summarize the ideas involved in the approach of [4] to integrable theories in any dimension. In section 3 we construct the zero curvature representation for the models defined on coset spaces $G/K$. The conditions for the existence of integrable submodels are discussed in section 4. The singlets of the subgroup $K$ play an important role in the construction of such submodels and their conservation laws. That is discussed subsection 4.1. The coset spaces which are symmetric spaces are considered in section 5. The cases of the group manifold and non-compact symmetric spaces are studied in sections 6 and 7 respectively, with some explicit examples given. Finally, the $CP^N$ models are presented in great detail in section 8 with the construction of their submodels and corresponding conservation laws.

We point out that the criteria for the construction of integrable submodels discussed here does not exhaust all possibilities. However, we believe it points towards some
very relevant and interesting structures that are certainly important for the study of integrable theories in higher dimensions. In particular, the constraints leading to the submodels can perhaps have an interpretation as a self-duality condition for the full theory.

2 The approach to integrable theories in any dimension

The central point of the approach of [4] is to generalize the zero curvature condition in two dimensions guided by the fact that it embodies conservation laws. Indeed, consider a connection $A_{\mu}$ and a curve $\Gamma$ on a two dimensional space time, and define the quantity $W$ through the equation

$$\frac{dW}{d\sigma} + A_{\mu} \frac{dx^\mu}{d\sigma} W = 0$$

(2.1)

with $\sigma$ parametrizing $\Gamma$. Then the zero curvature condition

$$[\partial_0 + A_0, \partial_1 + A_1] = 0.$$ 

(2.2)

is the sufficient condition for the quantity $W$ to be path independent as long as its end points are kept fixed. Therefore, if suitable boundary conditions are imposed on the fields, like periodic ones where space-time can be taken as $\mathbb{R} \times S^1$ for instance, then any power $N$ of the path ordered exponential $\text{Tr} (P \exp(\int_{S^1} A_x(x,t)dx))^N$ is conserved in time.

The basic idea in [4] to bring such concepts to higher dimensions, is to introduce quantities integrated over hypersurfaces and to find the conditions for them to be independent of deformations of the hypersurfaces which keep their boundaries fixed. Such an approach will certainly lead to conservation laws in a manner very similar to the two dimensional case. However, the main problem of that it is how to introduce non-linear zero curvatures keeping things as local as possible. The way out is to introduce auxiliary connections to allow for parallel transport. The number of possibilities of implementing those ideas increase with the dimensionality of space-time. However, the simplest scenario is that where, in a space-time of dimension $d + 1$, one introduces a rank $d$ antisymmetric tensor $B_{\mu_1\mu_2...\mu_d}$ and a vector $A_{\mu}$. The idea can perhaps be best stated using a formulation in “loop space”. On a $d + 1$ dimensional space-time $M$ one considers the space $\Omega^{d-1}(M, x_0)$ of $d - 1$ dimensional closed hypersurfaces based at a fixed point $x_0 \in M$. One then introduces on such “higher loop space” a 1-form $\mathcal{A}$ which is basically the quantity $W^{-1}B_{\mu_1\mu_2...\mu_d} W$ integrated over the closed hypersurfaces (see
for details). The quantity $W$ is defined in terms of the vector $A_\mu$ through (2.1). However, for $W$ to be independent of the way one integrates it from $x_0$ to a given point on the hypersurface, one has to assume that $A_\mu$ is flat, i.e.

$$F_{\mu\nu} = [\partial_\mu + A_\mu, \partial_\nu + A_\nu] = 0 ; \quad \mu, \nu = 0, 1, 2 \ldots d$$  \hfill (2.3)

Roughly speaking a $d$ dimensional closed hypersurface in $M$, based at $x_0$, corresponds to a (one dimensional) loop in $\Omega^{d-1}(M, x_0)$. Therefore, the condition to have things independent of deformation of hypersurfaces translates in such “higher loop space” to the zero curvature condition for $A$, namely

$$\mathcal{F} = \delta A + A \wedge A = 0$$  \hfill (2.4)

The relation (2.4) (together with (2.3)) is the generalization of the zero curvature (2.2) to higher dimensions proposed in [4]. Although (2.4) is local in $\Omega^{d-1}(M, x_0)$, it is highly non-local in the space-time $M$. Again in [4] it is presented some basic manners of introducing local conditions which are sufficient for the vanishing of $\mathcal{F}$. The relevant local conditions for the applications in this paper are the following.

Let $\mathcal{G}$ be a Lie algebra and $R$ be a representation of it. We introduce the non-semisimple Lie algebra $\mathcal{G}_R$ as

$$[T_a, T_b] = f_{ab}^c T_c$$
$$[T_a, P_i] = P_j R_{ji} (T_a)$$
$$[P_i, P_j] = 0$$  \hfill (2.5)

where $T_a$ constitute a basis of $\mathcal{G}$ and $P_i$ a basis for the abelian ideal $P$ (representation space). The fact that $R$ is a matrix representation, i.e.

$$[R (T_a), R (T_b)] = R ([T_a, T_b])$$  \hfill (2.6)

follows from the Jacobi identities.

We take the connection $A_\mu$ to be in $\mathcal{G}$ and the rank $d$ antisymmetric tensor $B_{\mu_1 \mu_2 \ldots \mu_d}$ to be in $P$, i.e.

$$A_\mu = A_\mu^a T_a , \quad B_{\mu_1 \mu_2 \ldots \mu_d} = B_{\mu_1 \mu_2 \ldots \mu_d}^i P_i$$  \hfill (2.7)

Then a set of sufficient local conditions for the vanishing of the curvature $\mathcal{F}$ in (2.4) is given by

$$D_\mu \tilde{B}^\mu = 0 ; \quad F_{\mu\nu} = 0$$  \hfill (2.8)
where we have introduced the covariant derivative
\[ D_\mu \cdot \equiv \partial_\mu \cdot + [A_\mu, \cdot] \] (2.9)
and the dual of \( B_{\mu_1 \mu_2 \ldots \mu_d} \) as
\[ \tilde{B}_\mu \equiv \frac{1}{d!} \varepsilon^{\mu_1 \mu_2 \ldots \mu_d} B_{\mu_1 \mu_2 \ldots \mu_d} \] (2.10)

The relations (2.8) constitute the local generalization to higher dimensions of the zero curvature condition (2.2). They lead to local conservation laws. Indeed, since the connection \( A_\mu \) is flat it can be written as
\[ A_\mu = -\partial_\mu W W^{-1} \] (2.11)
and consequently (2.8) imply that the currents
\[ J_\mu \equiv W^{-1} \tilde{B}_\mu W \] (2.12)
are conserved
\[ \partial_\mu J_\mu = 0 \] (2.13)

The zero curvature conditions (2.8) are invariant under the gauge transformations
\[ A_\mu \rightarrow g A_\mu g^{-1} - \partial_\mu g g^{-1} \]
\[ \tilde{B}_\mu \rightarrow g \tilde{B}_\mu g^{-1} \] (2.14)
and
\[ A_\mu \rightarrow A_\mu \]
\[ \tilde{B}_\mu \rightarrow \tilde{B}_\mu + \varepsilon_{\mu_1 \ldots \mu_d} D^{\mu_1} \alpha^{\mu_2 \ldots \mu_d} \equiv \tilde{B}_\mu + D^\nu \tilde{\alpha}_{\nu \mu} \] (2.15)
where we have introduced the dual \( \tilde{\alpha}_{\mu \nu} \equiv \varepsilon_{\mu \nu \mu_2 \ldots \mu_d} \alpha^{\mu_2 \ldots \mu_d} \). In (2.14) \( g \) is an element of the group obtained by exponentiating the Lie algebra \( \mathcal{G} \). The transformations (2.15) are symmetries of (2.8) as a consequence of the fact that the connection \( A_\mu \) is flat, i.e. \([D_\mu, D_\nu] = 0\). In addition, the parameters \( \alpha^{\mu_1 \ldots \mu_{d-1}} \) take values in the abelian ideal \( P \).

The currents (2.12) are invariant under the transformations (2.14), but under (2.15) they transform as
\[ J_\mu \rightarrow J_\mu + \varepsilon_{\mu_1 \ldots \mu_d} \partial^{\mu_1} \left( W^{-1} \alpha^{\mu_2 \ldots \mu_d} W \right) = J_\mu + \partial_\nu \left( W^{-1} \tilde{\alpha}_{\mu \nu} W \right) \] (2.16)

The transformations (2.14) and (2.15) do not commute and their algebra is isomorphic to the non-semisimple algebra \( \mathcal{G}_R \) introduced in (2.5).
3 Integrable theories on coset spaces

Consider a Lie group $G$ with Lie algebra $\mathcal{G}$ and a subgroup $K$ with Lie algebra $\mathcal{K}$. Then we have the decomposition

$$\mathcal{G} = \mathcal{S} + \mathcal{K} \quad (3.1)$$

where we have denote by $\mathcal{S}$ the orthogonal complement of $\mathcal{K}$ in $\mathcal{G}$. We then have

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K} \quad [\mathcal{K}, \mathcal{S}] \subset \mathcal{S} \quad [\mathcal{S}, \mathcal{S}] \subset \mathcal{S} + \mathcal{K} \quad (3.2)$$

We shall denote by $\Pi$ and $(1 - \Pi)$ the orthogonal projections of $\mathcal{G}$ onto $\mathcal{S}$ and $\mathcal{K}$ respectively

$$\Pi : \mathcal{G} \rightarrow \mathcal{S} \quad (1 - \Pi) : \mathcal{G} \rightarrow \mathcal{K} \quad (3.3)$$

We are interested in defining models on the coset space $G/K$. The fields of such models will be taken to be a set of local coordinates $\zeta^i$ on $G/K$, $i = 1, 2, \ldots \dim G/K$. Locally one can think of $G$ as the direct product of $G/K$ and $K$ and therefore a set of local coordinates on $G$ can be taken as the coordinates $\zeta^i$ of $G/K$ and some set of local coordinates on $K$.

We shall consider theories on a $d + 1$ dimensional space-time $M$, with coordinates $x^\mu$, $\mu = 0, 1, \ldots d$, and therefore the fields $\zeta^i$ will be mappings from $M$ to $G/K$.

Following (2.5) let us introduce a non-semisimple Lie algebra constructed out of $\mathcal{G}$ and its adjoint representation

$$[T_a, T_b] = f_{ab}^c T_c$$

$$[T_a, P^\psi (T_b)] = f_{ab}^c P^\psi (T_c)$$

$$[P^\psi (T_a), P^\psi (T_b)] = 0 \quad (3.4)$$

with $T_a$ being a basis for $\mathcal{G}$ and $P^\psi$ denotes the vector space of the adjoint representation (where the highest weight is the highest root $\psi$ of $\mathcal{G}$, $R^{\psi}_{ch} (T_a) = f_{ab}^c$).

Let us denote by $S_i$ and $K_r$ the generators of the subspace $\mathcal{S}$ and subalgebra $\mathcal{K}$ respectively ($i = 1, 2, \ldots \dim G/K$, $r = 1, 2, \ldots \dim K$). We then introduce the potentials

$$A_\mu \equiv g^{-1} \partial_\mu g = g^{-1} \frac{\partial g}{\partial \zeta^i} \frac{\partial \zeta^i}{\partial x^\mu} \equiv A^\alpha_\mu T_a$$

$$\tilde{B}_\mu \equiv P^\psi \left( \Pi \left( g^{-1} \partial_\mu g \right) \right) = A^i_\mu P^\psi (S_i) \quad (3.5)$$

where $g$ is an element of $G$. 

Since the connection $A_\mu$ is “pure gauge”, the flatness condition $F_{\mu\nu} = 0$ in (2.8) is automatically satisfied. Therefore, in order to get the local zero curvature conditions, we have just to impose that the covariant divergence of $\tilde{B}_\mu$ vanishes. That will be taken as the equations of motion of our field theory on $G/K$. Indeed, the number of such equations of motion is equal to the number of fields $\zeta^i$, i.e. the dimension of $G/K$. So, one gets

$$D^\mu \tilde{B}_\mu = P^\psi \left( \Pi \left( \partial^\mu \left( g^{-1} \partial_\mu g \right) \right) \right) + \left[ (1 - \Pi) \left( g^{-1} \partial^\mu g \right), P^\psi \left( \Pi \left( g^{-1} \partial_\mu g \right) \right) \right] = 0$$

(3.6)

where, since we are working with the adjoint representation, we have used the fact that

$$\left[ \Pi \left( g^{-1} \partial^\mu g \right), P^\psi \left( \Pi \left( g^{-1} \partial_\mu g \right) \right) \right] = A_{\mu,i}^i A_{\mu,j}^j \left[ S^i, P^\psi \left( S^j \right) \right] = A_{\mu,i}^i A_{\mu,j}^j P^\psi \left[ [S^i, S^j] \right] = 0$$

(3.7)

The action corresponding to (3.6) is

$$S = \frac{1}{2} \int d^{d+1}x \ Tr \left( \Pi \left( g^{-1} \partial_\mu g \right) \right)^2 = \frac{1}{2} \int d^{d+1}x A_{\mu,i}^i A^{j,\mu} \ Tr (S_i S_j)$$

(3.8)

Eq. (3.6) can be written as

$$\left( \partial^\mu A^i_{\mu} + A^{\mu,r}_{\mu} A^j_{\mu} R^S_{ij} (K_r) \right) P^\psi (S_i) = 0$$

(3.9)

where $R^S_{ij} (K_r)$ are the matrices of the representation of the subalgebra $\mathcal{K}$ defined by the subspace $S$

$$\left[ K_r, P^\psi (S_j) \right] = P^\psi (S_i) R^S_{ij} (K_r)$$

(3.10)

In fact, the adjoint representation $R^\psi$ of $\mathcal{G}$ decomposes, in terms of representations of the subalgebra $\mathcal{K}$, as

$$R^\psi = R^S + R^K$$

(3.11)

where $R^S$ and $R^K$ are the representations of $\mathcal{K}$ defined by the subspaces $S$ and $\mathcal{K}$ respectively. In fact, $R^K$ is the adjoint of $\mathcal{K}$. Notice those are not necessarily irreducible.

According to (2.12), the conserved currents for such theory are given by (comparing (2.14) and (3.5) one sees that $W \equiv g^{-1}$)

$$J_\mu = A^i_{\mu} g P^\psi (S_i) g^{-1} = A^i_{\mu} R^\psi_{ai} (g) P^\psi (T_a) \equiv J^a_\mu P^\psi (T_a)$$

(3.12)
The construction of integrable submodels

Although the theory defined above possesses a representation in terms of the local zero curvature (2.8), it does not present an infinite number of conserved currents. In fact, as shown in (3.12) the number of currents is equal to the dimension of $G$. Notice however, that the equations of motion (3.9) are determined by the branching of the adjoint representation of $G$ into representations of the subgroup $K$. More precisely, as shown in (3.10), what counts is the representation of $K$ defined by the subspace $S$. Therefore, any representation of $G$ which contains, in its branching rule, that representation of $K$ given by $S$, can be used to write a zero curvature representation for the model. The way to implement that is the following.

Let $R^\lambda$ be a representation of $G$ that when decomposed into representations of the subgroup $K$ presents the representation $R^S$ of $K$ defined by the subspace $S$ at least once, i.e.

$$R^\lambda = R^S + \text{anything}$$

(4.1)

Introduce the non-semisimple Lie algebra

$$[T_a , T_b] = f_{ab}^c T_c$$

$$[T_a , P^\lambda_\alpha] = P^\beta_\alpha R^\lambda_{\beta\alpha} (T_a)$$

$$[P^\lambda_\alpha , P^\lambda_\beta] = 0$$

(4.2)

with $P^\lambda_\alpha$, $\alpha = 1, 2, \ldots \dim R^\lambda$, being a basis of the representation space of $R^\lambda$.

Following (3.5), define the potentials

$$A_\mu \equiv g^{-1} \partial_\mu g \equiv A^\mu_a T_a$$

$$\bar{B}^\lambda_\mu \equiv A^i_\mu P^\lambda_i$$

(4.3)

where $P^\lambda_i$ correspond to a basis of the subspace of $R^\lambda$ which carries the representation $R^S$ of $K$ defined by (3.10), and which transforms exactly as $P^\psi (S_i)$, i.e.

$$[K_r , P^\lambda_j] = P^\lambda_i R^S_{ij} (K_r)$$

(4.4)

Notice that if $R^S$ is reducible one can rescale the basis of each irreducible component independently without changing the relation between (3.10) and (4.4).

Therefore, one gets

$$D^\mu \bar{B}^\lambda_\mu = \left( \partial^\mu A^i_\mu P^\lambda_i + A^r_\mu A^i_\mu \left[ K_r , P^\lambda_i \right] \right) + A^i_\mu A^j_\mu \left[ S_i , P^\lambda_j \right]$$

(4.5)

\[\text{It does not have to be irreducible}\]
Notice that the first two terms on the r.h.s. of (4.5) are identical to (3.6) (or (3.9)) and therefore to the equations of motion of the theory on $G/K$ defined above. However, contrary to (3.7) which is an identity, the last term on the r.h.s. of (4.5) does not vanish in general.

Therefore, the submodel of (3.9) defined by the equations

$$\partial^{\mu} A_{\mu}^{i} + A^{\mu,r} A_{\mu}^{j} p_{ij}^{S}(K_{r}) = 0 \quad (4.6)$$

$$A_{\mu}^{i} A^{j,\mu} \left( [S_{i}, P_{j}^{\lambda}] + [S_{j}, P_{i}^{\lambda}] \right) = 0 ; \quad i, j = 1, 2, \ldots \dim G/K \quad (4.7)$$

admits a representation in terms of the zero curvature

$$D^{\mu} \tilde{B}_{\mu}^{\lambda} = 0 \quad F_{\mu\nu} = 0 \quad (4.8)$$

and therefore possesses the conserved currents

$$J_{\mu}^{\lambda} \equiv A_{\mu}^{i} g P_{i}^{\lambda} g^{-1} = P_{\alpha}^{\lambda} R_{\alpha i}^{\lambda}(g) \quad A_{\mu}^{i} \equiv J_{\mu}^{\lambda,\alpha} P_{\alpha}^{\lambda} \quad (4.9)$$

where $P_{\alpha}^{\lambda}, \alpha = 1, 2, \ldots \dim R^{\lambda}$.

Since $S_{i}$ and $P_{i}^{\lambda}$ transform under the same representation $R^{S}$ of $K$, it follows that

$$\left( [S_{i}, P_{j}^{\lambda}] + [S_{j}, P_{i}^{\lambda}] \right)$$

transforms under $\left( R^{S} \otimes R^{S} \right)_{s}$, where the subscript $s$ stands for the symmetric part of the tensor product. Consider now the branchings

$$\left( R^{S} \otimes R^{S} \right)_{s} = \sum_{\gamma} R^{\gamma}(K) \quad (4.10)$$

and

$$R^{\lambda} = R^{S}(K) + \sum_{\beta} R^{\beta}(K) \quad (4.11)$$

where $R^{\gamma}(K)$ and $R^{\beta}(K)$ are irreducible representations of $K$.

Since $\left( [S_{i}, P_{j}^{\lambda}] + [S_{j}, P_{i}^{\lambda}] \right)$ corresponds to a given state in $R^{\lambda}$, it follows that it will have to vanish whenever such state belongs to a representation $R^{\gamma}(K)$ in (4.10) which do not appear in (4.11). Consequently, the constraints (4.7) on the fields which are really effective are those corresponding to the representations $R^{\gamma}(K)$ in (4.10) which coincide with one of the $R^{\beta}(K)$’s in (4.11).

Consequently, if the group $G$ possesses a number of representations $R^{\lambda}$’s (which may be infinite) fulfilling the following two requirements

1. The branching of such representations of $G$ into representations of $K$ presents, at least once, the representation $R^{S}$ of $K$ defined by the subspace $S$ (see (3.10))
2. The relation (4.7), in any of such representations, implies the same constraints on the fields. In other words, the representations \( R^\gamma (K) \)'s in (4.10), appearing in the branching of \( R^\lambda \) in (4.11), are the same for all \( R^\lambda \)'s.

then the submodel defined in (4.6)-(4.7) possesses a number of local conserved currents, given by (4.9), equal to the sum of the dimensions of such representations \( R^\lambda \)'s.

4.1 The role of singlet states

We now discuss a very special case where one can easily construct integrable submodels with an infinite number of local conservation laws. Suppose that \( G \) possesses a representation \( R^\lambda \) which when decomposed into representations of \( K \) presents \( R_S \) like in (4.1), but it also presents a singlet state \( P^\lambda_\Lambda \) of the subalgebra \( K \), i.e.

\[
[K, P^\lambda_\Lambda] = 0 \quad (4.12)
\]

By considering representations which are tensor products of \( R^\lambda \) with itself one then obtains several representations of \( K \) equivalent to \( R_S \), which are given by the tensor product of \( R_S \) with copies of the singlet \( P^\lambda_\Lambda \). For instance, in the case of \( R^\lambda \otimes R^\lambda \) one has that \( R_S \otimes P^\lambda_\Lambda \) and \( P^\lambda_\Lambda \otimes R_S \) are equivalent to \( R_S \). Indeed from (4.4)

\[
\left[ 1 \otimes K_r + K_r \otimes 1, P^\lambda_\Lambda \otimes P^\lambda_j \right] = \left( P^\lambda_\Lambda \otimes P^\lambda_i \right) R^S_{ij} (K_r) \quad (4.13)
\]

For the case of \((\otimes R^\lambda)^n\) any representation of the form \((\otimes P^\lambda_\Lambda)^l \otimes R^S (\otimes P^\lambda_\Lambda)^{n-l-1}\) is equivalent to \( R_S \). Therefore, following (4.3) one introduces the potentials

\[
A^\mu_{\alpha}^{(n)} \equiv A^\mu_\alpha \sum_{l=0}^{n-1} (\otimes 1)^l \otimes T_a (\otimes 1)^{n-l-1}
\]

\[
\tilde{B}^\lambda_{\mu}^{(n)} \equiv A^\mu_i \sum_{l=0}^{n-1} c_{n,l} \left( \otimes P^\lambda_\Lambda \right)^l \otimes P^\lambda_i \left( \otimes P^\lambda_\Lambda \right)^{n-l-1} \quad (4.14)
\]

where \( c_{n,l} \) are constants. We introduce such constants because, as we have pointed out below (4.4), one can rescale the basis of each irreducible component of the representations of \( K \) independently, without affecting the equations of motion. Only the constraints, defining the submodel, are affected by the constants \( c_{n,l} \).

The corresponding zero curvature conditions (4.8) leads in this case, to the following equations of motion (see (4.6)-(4.7))

\[
\partial^\mu A^i_\mu + A^a_{\mu \nu} A^{\mu \nu} R^S_{ij} (K_r) = 0 \quad (4.15)
\]

\(^2\)Clearly, for the cases where \( K \) is abelian, \( R^\lambda \) decomposes into singlet states only. We then require \( P^\lambda_\Lambda \) to be a charge zero singlet.
and constraints
\[
A^i_\mu A^{i\mu} \left[ \left( \sum_{m=0}^{n-1} (\otimes 1)^m \otimes S_i (\otimes 1)^{n-m-1} \right) \otimes \left( \sum_{l=0}^{n-1} c_{n,l} (\otimes P^\lambda_i)^l \otimes P^\lambda_j (\otimes P^\lambda_i)^{n-l-1} \right) \right] = 0
\]
(4.16)
with \(i, j = 1, 2, \ldots \dim G/K\).

Therefore, since (4.15) are the same equations as (3.9) we have a submodel of the non-linear sigma model on \(G/K\). The subclass of solutions is determined by the constraints (4.16).

The conserved currents obtained from the zero curvature are (see (4.9))
\[
J^\lambda \mu \equiv A^i_\mu \left[ (\otimes g) \left( \sum_{l=0}^{n-1} c_{n,l} V_{\alpha_1} (g) \cdots V_{\alpha_l} (g) R^\Lambda_{\alpha_1} (g) V_{\alpha_{l+1}} (g) \cdots V_{\alpha_n} (g) \right) P^\lambda_{\alpha_1} \otimes \cdots \otimes P^\lambda_{\alpha_n} \right] = J^{\lambda, (\alpha_1 \ldots \alpha_n)} \otimes P^\lambda_{\alpha_1} \otimes \cdots \otimes P^\lambda_{\alpha_n} \quad (4.17)
\]
where
\[
g P^\lambda \alpha g^{-1} = P^\lambda_{\alpha} V_{\alpha} (g) \quad (4.18)
\]
Consequently, if one can choose the constants \(c_{n,l}\) in such a way that (4.16) imply for any \(n\), the same constraints on the model, one has an infinite number of local conserved currents for the submodel. Notice that in such case one has
\[
J^{\lambda, (\alpha_1 \ldots \alpha_n)} = \sum_{l=0}^{n-1} c_{n,l} V_{\alpha_1} (g) \cdots V_{\alpha_l} (g) J^{\lambda, \alpha_{l+1}} \otimes \cdots \otimes V_{\alpha_n} (g) \quad (4.19)
\]
where \(J^{\lambda, \alpha} = A^i_\mu R^\lambda_{\alpha i}\) are the conserved currents for the case \(n = 1\).

Clearly, if there exists additional singlet states satisfying (4.12), one can use them to construct new currents and submodels. In fact, the relevant algebraic concept here is that of the kernel of the adjoint action of the subalgebra \(K\) on the non-semisimple algebra (1.2), since \(P^\lambda_{\alpha} \in Ker (Ad_K)\). We will discuss examples of such construction on the following sections.
5 The case of symmetric spaces

We now consider the coset spaces $G/K$ which are symmetric spaces $[3]$. In such cases there exists an involutive authomorphism $\sigma$, $\sigma^2 = 1$, such that $K$ is the invariant subgroup. Then, one decomposes the algebra of $G$ as in (3.1) such that $S$ correspond to the odd subspace, i.e.

$$\sigma(S) = -S; \quad \sigma(K) = K$$  \hspace{1cm} (5.1)

Therefore, instead of (3.2) one has

$$[K, K] \subset K \quad [K, S] \subset S \quad [S, S] \subset K$$  \hspace{1cm} (5.2)

The projection $\Pi$, introduced in (3.3), can now be performed by the automorphism $\sigma$. Indeed, $(1 - \sigma)$ and $(1 + \sigma)$ map $G$ into $S$ and $K$ respectively.

For any element $g \in G$ we define the so called principal variable $[4, 4]$ as

$$y(g) \equiv g \sigma(g)^{-1}$$  \hspace{1cm} (5.3)

One observes that $y(gk) = y(g)$ for $k \in K$, and so $y(g)$ is defined on the cosets $G/K$. There exists in fact a one to one correspondence between the cosets and the variable $y$, and therefore $y$ can be used to parametrize $G/K$. Notice that, $\sigma(y) = y^{-1}$.

The non-linear sigma model on the symmetric space $G/K$, defined on a space-time $M$ of dimension $d + 1$, is given by the action

$$S \equiv \frac{1}{2} \int d^{d+1}x \ Tr \left(y^{-1} \partial_\mu y\right)^2$$  \hspace{1cm} (5.4)

which corresponds to the equations of motion

$$\partial^\mu \left(y^{-1} \partial_\mu y\right) = 0$$  \hspace{1cm} (5.5)

Such theories admit a quite simple representation in terms of the local zero curvature conditions (2.8). Consider the non-semisimple Lie algebra (3.4) and introduce

$$A_\mu \equiv y^{-1} \partial_\mu y$$
$$\tilde{B}_\mu \equiv P^\psi \left(y^{-1} \partial_\mu y\right)$$  \hspace{1cm} (5.6)

Clearly, $F_{\mu\nu} = 0$, and the condition $D^\mu \tilde{B}_\mu = 0$ is equivalent to (5.5).

Notice that

$$y^{-1} \partial_\mu y = \sigma(g) \left(g^{-1} \partial_\mu g - \sigma(g^{-1} \partial_\mu g)\right) \sigma(g)^{-1}$$  \hspace{1cm} (5.7)
Therefore, performing the gauge transformation (2.14) with $\sigma(g)^{-1}$ one obtains
\[
 A_\mu \rightarrow g^{-1} \partial_\mu g \\
 \tilde{B}_\mu \rightarrow P^\psi \left( (1 - \sigma) \left( g^{-1} \partial_\mu g \right) \right)
\]
(5.8)

Notice that these potentials are in the same gauge as those in (3.5). Therefore, the analysis presented in (3.6)-(3.12), as well as the discussion about integrable submodels in section 4, hold true in the present case.

6 The case of the group manifold

The non-linear sigma model on a group manifold $G$, defined on a space-time $M$ of dimension $d + 1$, is given by

\[
 S \equiv \frac{1}{2} \int dx^{d+1} \text{Tr} \left( g^{-1} \partial_\mu g \right)^2 ; \quad g \in G
\]
(6.1)

and the corresponding equations of motion are

\[
 \partial^\mu \left( g^{-1} \partial_\mu g \right) = 0 ; \quad \text{or} \quad \partial^\mu \left( \partial_\mu gg^{-1} \right) = 0
\]
(6.2)

These models have already been studied in [8] using the same zero curvature approach proposed in [4] and some interesting integrable submodels as well as the corresponding conserved currents were constructed. Any group $G$, however, is a symmetric space [7] and therefore the theory (6.1) can be studied using the techniques of sections 3 and 5. That may help making more systematic the construction of integrable submodels. The relevant symmetric space is $G \otimes G/G_D$, where the elements of the tensor group $G \otimes G$ are of the form $g_1 \otimes g_2$, with $g_1, g_2 \in G$, and $G_D$ is the diagonal subgroup with elements $g \otimes g$, with $g \in G$. The involutive automorphism is

\[
 \sigma(g_1 \otimes g_2) = g_2 \otimes g_1
\]
(6.3)

and indeed $G_D$ is the invariant subgroup under $\sigma$.

The group $G$ is diffeomorphic to $G \otimes G/G_D$, with the diffeomorphism $G \otimes G \rightarrow G$ being given by $g_1 \otimes g_2 \rightarrow g_1g_2^{-1}$. Obviously the kernel is $G_D$ itself.

The principal variable $y$ introduced in (5.3) is given by

\[
 y(g_1 \otimes g_2) = g_1 \otimes g_2 \sigma(g_1 \otimes g_2)^{-1} = g_1g_2^{-1} \otimes \left( g_1g_2^{-1} \right)^{-1}
\]
(6.4)

Notice that $y$ is always the tensor product of a given element with its inverse. Since $y$ parametrizes $G \otimes G/G_D$ and since that has the same dimension as $G$, one can always...
choose a gauge where \( y = g \otimes g^{-1} \), with \( g \in G \). Therefore, the equation of motion (5.3) becomes
\[
\partial^\mu \left( y^{-1} \partial_\mu y \right) = \partial^\mu \left( g^{-1} \partial_\mu g \right) \otimes 1 - 1 \otimes \partial^\mu \left( \partial_\mu gg^{-1} \right) = 0
\] (6.5)
Therefore the non-linear sigma models defined on \( G \otimes G/G_D \) and \( G \) are the same, since (5.2) and (5.3) are equivalent.

Following (5.6) we can then introduce the potentials \( A_\mu \) and \( \tilde{B}_\mu \), which in the gauge (5.8) are given by
\[
A_\mu = p^{-1} \partial_\mu p \otimes 1 - 1 \otimes \partial_\mu p p^{-1} = \left( p^{-1} \otimes p \right) \partial_\mu \left( p \otimes p^{-1} \right)
\]
\[
\tilde{B}_\mu = P^\psi \left( \left( 1 - \sigma \right) \left( p^{-1} \partial_\mu p \otimes 1 - 1 \otimes \partial_\mu p p^{-1} \right) \right)
\] (6.6)
where \( p \) is such that \( pp = g \), with \( g \) being the group element in the definition of \( y \) (indeed \( y \left( p \otimes p^{-1} \right) = g \otimes g^{-1} \), see (5.3) and (5.3)).

The local zero curvature conditions (2.8) then imply the equation of motion (6.5), because \( F_{\mu\nu} = 0 \) is trivially satisfied since \( A_\mu \) is of the pure gauge form, and
\[
D^\mu \tilde{B}_\mu = (1 - \sigma) \left( \left( \partial^\mu \left( p^{-1} \partial_\mu p \right) + \partial^\mu \left( \partial_\mu pp^{-1} \right) \right) \otimes 1 \right) = 0
\] (6.7)
is equivalent to (6.5).

The conserved currents (2.12) are given by
\[
J_\mu = \left( p \otimes p^{-1} \right) \tilde{B}_\mu \left( p^{-1} \otimes p \right) = \left( \partial_\mu gg^{-1} \right) \otimes 1 - 1 \otimes \left( g^{-1} \partial_\mu g \right)
\] (6.8)
which correspond to the Noether currents associted to the invariance of (6.1) under the global right and left translations by elements of \( G \).

Let us now consider the construction of integrable submodels of the theory (6.1), which possess a larger number of conserved currents, using the ideas of section 4. The algebra \( \mathcal{G} \oplus \mathcal{G} \) of \( G \otimes G \) decomposes under \( \sigma \) as (see (6.1))
\[
\mathcal{G} \oplus \mathcal{G} = S + \mathcal{K}
\] (6.9)
with
\[
S \equiv \{ T_a^S \equiv 1 \otimes T_a - T_a \otimes 1 \} ; \quad \mathcal{K} \equiv \{ T_a^K \equiv 1 \otimes T_a + T_a \otimes 1 \} \] (6.10)
where \( T_a, a = 1, 2, \ldots \dim G \), are the generators of the algebra \( \mathcal{G} \) of \( G \) \( [T_a, T_b] = f_{ab}^c T_c \). Therefore
\[
\left[ T_a^K, T_b^K \right] = f_{ab}^c T_c^K ; \quad \left[ T_a^K, T_b^S \right] = f_{ab}^c T_c^S ; \quad \left[ T_a^S, T_b^S \right] = f_{ab}^c T_c^K \] (6.11)
In fact, denoting
\[ p^{-1} \partial_\mu p \pm \partial_\mu p p^{-1} \equiv A^\pm_\mu \equiv A^{\pm,a}_\mu T_a \] (6.12)
one obtains from (6.6) that
\[ A_\mu = \frac{1}{2} A^{-,a}_\mu T^\kappa_a + \frac{1}{2} A^{+,a}_\mu T^S_a \]
\[ \tilde{B}_\mu = A^{+,a}_\mu P^{\psi} (T^S_a) \] (6.13)

The equations of motion are then written as
\[ \partial^\mu A^{+,a}_\mu + \frac{1}{2} f_{bc}^a A^{-,b}_\mu A^{+,c,\mu} = 0 \] (6.14)

As we have discussed in section 4, the part of \( A_\mu \) that really contribute to the
 equations of motion is that in \( \mathcal{K} \). In addition, those equations are determined by the
 representation of \( \mathcal{K} \) defined by the subspace \( \mathcal{S} \). But \( \mathcal{K} \), the algebra of \( G_D \), is isomorphic
to \( G \) and therefore that representation is the adjoint. Consequently, as pointed out in
4, any representation of \( G \otimes G \) that contains the adjoint of \( G \) in its branching rule
 can be used to write a zero curvature for submodels of the theory (6.1). Given two
 representations \( R^\lambda \) and \( R^{\lambda'} \) of \( G \) one can construct a representation of \( G \otimes G \) by taking
the tensor product of them. Therefore, one should look for representations \( R^\lambda \) and \( R^{\lambda'} \) such
\[ R^\lambda \otimes R^{\lambda'} = \text{adjoint of } G + \text{anything} \] (6.15)

The construction of the zero curvature for submodels of (6.1) (with infinite number of
 conserved currents) is done by following the ideas described in section 4.

### 6.1 The example of SU(2)

Let us illustrate those ideas with the example of SU(2) where the commutation relations
 are given by
\[ [T_i, T_j] = i \varepsilon_{ijk} T_k \quad ; \quad i, j, k = 1, 2, 3 \] (6.16)

The equations of motion are those of (6.14) with \( f_{ab}^c \) replaced by \( i \varepsilon_{ijk} \). We now use
the fact that the adjoint (triplet) of SU(2) can be obtained by the tensor product of
two doublets, i.e.
\[ 2 \otimes 2 = 3 + 1 \] (6.17)

Denoting the basis of the doublet by \( P^{(1/2)} \pm_{1/2} \) one has \( (T_\pm \equiv T_1 \pm i T_2) \)
\[ [T_3, P^{(1/2)} \pm_{1/2}] = \pm \frac{1}{2} P^{(1/2)} \pm_{1/2} \quad ; \quad [T_\pm, P^{(1/2)} \pm_{1/2}] = P^{(1/2)} \pm_{1/2} \] (6.18)
For the tensor product representation space we take the basis

\[ P_{i}^{(\frac{1}{2}, \frac{1}{2})} = i \left( P_{1/2}^{(1/2)} \otimes P_{1/2}^{(1/2)} - P_{-1/2}^{(1/2)} \otimes P_{-1/2}^{(1/2)} \right) \]

\[ P_{2}^{(\frac{1}{2}, \frac{1}{2})} = P_{1/2}^{(1/2)} \otimes P_{1/2}^{(1/2)} + P_{-1/2}^{(1/2)} \otimes P_{-1/2}^{(1/2)} \]

\[ P_{3}^{(\frac{1}{2}, \frac{1}{2})} = -i \left( P_{1/2}^{(1/2)} \otimes P_{-1/2}^{(1/2)} + P_{-1/2}^{(1/2)} \otimes P_{1/2}^{(1/2)} \right) \]

\[ P_{A}^{(\frac{1}{2}, \frac{1}{2})} = P_{-1/2}^{(1/2)} \otimes P_{1/2}^{(1/2)} - P_{1/2}^{(1/2)} \otimes P_{-1/2}^{(1/2)} \]  \hspace{1cm} (6.19)

One can check that they satisfy

\[ \left[ \mathcal{T}^{K}_{i}, P_{j}^{(\frac{1}{2}, \frac{1}{2})} \right] = i \varepsilon_{ijk} P_{k}^{(\frac{1}{2}, \frac{1}{2})} \]

\[ \left[ \mathcal{T}^{S}_{i}, P_{j}^{(\frac{1}{2}, \frac{1}{2})} \right] = i \delta_{ij} P_{A}^{(\frac{1}{2}, \frac{1}{2})} \]

\[ \left[ \mathcal{T}^{K}_{i}, P_{A}^{(\frac{1}{2}, \frac{1}{2})} \right] = 0 \]

\[ \left[ \mathcal{T}^{S}_{i}, P_{A}^{(\frac{1}{2}, \frac{1}{2})} \right] = -i P_{i}^{(\frac{1}{2}, \frac{1}{2})} \]  \hspace{1cm} (6.20)

We then introduce the potential

\[ \tilde{B}_{\mu}^{(\frac{1}{2}, \frac{1}{2})} \equiv A^{+;i}_{\mu} P_{i}^{(\frac{1}{2}, \frac{1}{2})} \]  \hspace{1cm} (6.21)

which, like (6.13), contains the states transforming under the adjoint (triplet).

One can easily verify that the equation

\[ D^{\mu} \tilde{B}_{\mu}^{(\frac{1}{2}, \frac{1}{2})} = 0 \]  \hspace{1cm} (6.22)

with the same potential \( A_{\mu} \) as in (6.13), gives the same equations of motion (6.14).

However, it has a component in the direction of the singlet state which imposes the following constraint on the model

\[ A^{+;i}_{\mu} A^{+;j;\mu} \left[ \mathcal{T}^{S}, P_{j}^{(\frac{1}{2}, \frac{1}{2})} \right] = i A^{+;i}_{\mu} A^{+;i;\mu} P_{A}^{(\frac{1}{2}, \frac{1}{2})} = 0 \Rightarrow A^{+;i}_{\mu} A^{+;i;\mu} = 0 \]  \hspace{1cm} (6.23)

Using (6.12), and the fact that \( g^{-1} \partial_{\mu} g = p^{-1} (p^{-1} \partial_{\mu} p + \partial_{\mu} p p^{-1}) p \), such constraint can be written as

\[ \text{Tr} \left( p^{-1} \partial_{\mu} p + \partial_{\mu} p p^{-1} \right)^{2} = \text{Tr} \left( g^{-1} \partial_{\mu} g \right)^{2} = 0 \]  \hspace{1cm} (6.24)

where we have used the fact that \( \text{Tr} (T_{i} T_{j}) \sim \delta_{ij} \). Therefore, such constraint implies that the action (6.1) vanishes when evaluated on the solutions of such submodel.

The corresponding four conserved currents are

\[ J_{\mu}^{(\frac{1}{2}, \frac{1}{2})} = \left( p \otimes p^{-1} \right) \tilde{B}_{\mu}^{(\frac{1}{2}, \frac{1}{2})} \left( p^{-1} \otimes p \right) = J_{\mu}^{(\frac{1}{2}, \frac{1}{2}), \alpha \beta} P_{\alpha}^{(1/2)} \otimes P_{\beta}^{(1/2)} \]  \hspace{1cm} (6.25)
However, for

dimensions: 612.0x792.0

[72x366]In such cases we have to impose

(6.29) imply that

(6.32) to construct submodels with larger conservation laws. We then introduce the potentials

\[ P^{(1/2)} \rho p^{-1} = P^{(1/2)}_\alpha D^\beta_\alpha(p) \]  

(6.27)

Since we have a \( \mathcal{K} \)-singlet in such representation we can use the ideas of section 4.1 to construct submodels with larger conservation laws. We then introduce the potentials

\[ A^{(n)}_\mu \equiv \frac{1}{2} \sum_{l=0}^{n-1} (\otimes 1)^l \otimes T^K_i (\otimes 1)^{n-l-1} + \frac{1}{2} \sum_{l=0}^{n-1} (\otimes 1)^l \otimes T^S_i (\otimes 1)^{n-l-1} \]

(6.28)

As we have argued in section 4.1 the zero curvature condition for these potentials give the same equations of motion as those of (6.21), i.e. (6.14). However, the constraints correspond to those given in (4.16). One can easily check that for the case \( n = 2 \) no constraints are imposed on the fields if we choose

\[ c_{2,0} + c_{2,1} = 0 \]

(6.29)

However, for \( n > 2 \) there are no choices for \( c_{n,l} \) which can make the constraints weaker. In such cases we have to impose

\[ A^{+,-i} A^{+,-j,\mu} = 0 \quad \text{for any } i, j \]

(6.30)

Denoting the parameters of the group by \( \zeta^i, \quad i = 1, 2, 3 \), one gets from (6.12) that \( A^{+,-i} = \tilde{\mathcal{M}}^{(+)}_j(p) \partial_p \zeta^j \), with \( \tilde{\mathcal{M}}^{(+)}_j(p) \) being an invertible matrix. Therefore, the constraints (6.30) imply that \( \partial_{\mu} \zeta^{i} \partial^{\mu} \zeta^{j} = 0 \). Now, one can also write \( A^{-,i} = \mathcal{M}^{-}\jmath(p) \partial_{\mu} \zeta^{\jmath} \), and consequently \( A^{-,i} A^{+,-j,\mu} = 0 \). In addition,

\[ \partial^{\mu} A^{+,-i} = \tilde{\mathcal{M}}^{(+)}_j(p) \partial^{2} \zeta^{j} + \partial_{k} \left( \mathcal{M}^{(+)}_j \right) \partial^{\mu} \zeta^{k} \partial_{\mu} \zeta^{j} \]

(6.31)

Therefore, from the equations of motion (6.14) and constraints (6.30), we get that the submodel is defined by

\[ \partial^{2} \zeta^{j} = 0 \quad \partial_{\mu} \zeta^{i} \partial^{\mu} \zeta^{j} = 0 \]

(6.32)

If one allows the fields to be complex (i.e. work with \( \text{SL}(2, \mathbb{C}) \)), then (6.32) is the same as that submodel of \( CP^3 \) we discuss in section 8.1.1. The conserved currents can be evaluated using (4.19). However, we do not discuss them in more detail here because we shall treat such type of submodel in section 8.1.1.
7 The example of non-compact symmetric spaces

We now consider the symmetric spaces \( G/K \) where \( G \) is a real non-compact simple Lie group furnished with a Cartan involution \( \sigma, \sigma^2 = 1 \), with \( K \), invariant under \( \sigma \), being the maximal compact subgroup of \( G \). The Cartan property of \( \sigma \) means that if \( \text{Tr} \) is a \( \sigma \)-invariant bilinear form for the algebra \( \mathcal{G} \) of \( G \) \( \text{Tr} (TT') = \text{Tr} (\sigma (T) \sigma (T')) \) then \( \text{Tr} (T \sigma (T')) \) is negative definite. That implies that \( \text{Tr} (TT') \) is: 

i) positive definite if \( T, T' \in S \);

ii) negative definite if \( T, T' \in K \);

iii) zero if \( T \in S \) and \( T' \in K \).

Such symmetric spaces have some very special properties due to the so-called Iwasawa decomposition of \( G \). Let \( A \) denote the maximal abelian subspace of \( S \). It then follows that the adjoint action of \( A \) in \( G \) can be simultaneously diagonalized. We denote \( G_\gamma \equiv \{ T \in G \mid [H, T] = \gamma (H) T, \; \text{for} \; H \in A \} \). We now define the nilpotent subalgebra \( \mathcal{N} \equiv \sum_{\gamma > 0} G_\gamma \). The Iwasawa decomposition corresponds to

\[
\mathcal{G} = \mathcal{N} + A + K; \quad g = n a k \equiv b k \quad (7.1)
\]

where \( k \in K \), and \( n \) and \( a \) are elements of the subgroups obtained by exponentiating \( \mathcal{N} \) and \( A \) respectively, and where we have introduced \( b \equiv na \).

It then follows that such symmetric spaces are endowed with a hidden group theoretic structure, since the elements of \( G/K \) can be put into a one to one correspondence with the elements \( b \) of the so-called Borel subgroup. Even though \( G/K \) is a not a group itself, one can parametrize it by the group elements \( b \).

Using the symmetry \((2.14)\) one can choose a gauge where the potentials \((5.8)\), in the case of such non-compact symmetric spaces, take the form

\[
A_\mu = b^{-1} \partial_\mu b = a^{-1} \partial_\mu a + a^{-1} \left( n^{-1} \partial_\mu n \right) a \quad (7.2)
\]

\[
\tilde{B}_\mu = P^\psi \left( (1 - \sigma) \left( b^{-1} \partial_\mu b \right) \right) = P^\psi \left( 2 a^{-1} \partial_\mu a + a^{-1} \left( n^{-1} \partial_\mu n \right) a - a \sigma \left( n^{-1} \partial_\mu n \right) a^{-1} \right)
\]

where we have used the fact that \( \sigma (a) = a^{-1} \), since \( A \in S \) and so \( \sigma (A) = -A \).

7.1 The case where \( G \) is the normal real form

Consider the case where the algebra \( \mathcal{G} \) of \( G \) is spanned by real linear combinations of the Chevalley basis \( H_a, a = 1, 2, \ldots \text{rank} \; \mathcal{G}, \; E_\alpha \) and \( E_{-\alpha} \), with \( \alpha \) being the positive roots of \( \mathcal{G} \). That is the maximally non-compact real form of the corresponding complex simple Lie group, and its called the normal form. The Cartan involution we consider is given by \((\sigma^2 = 1)\)

\[
\sigma (H_a) = -H_a; \quad \sigma (E_\alpha) = -E_{-\alpha} \quad (7.3)
\]
Therefore
\[ S = \{ H_a, a = 1, 2, \ldots \text{rank } G ; E_\alpha + E_{-\alpha}, \text{ for any positive root } \alpha \} \]
\[ K = \{ E_\alpha - E_{-\alpha}, \text{ for any positive root } \alpha \} \] (7.4)
and
\[ A = \{ H_a, a = 1, 2, \ldots \text{rank } G \} ; \]
\[ \mathcal{N} = \{ E_\alpha, \text{ for any positive root } \alpha \} \] (7.5)

Parametrizing the group elements as
\[ a = \exp \left( -\frac{1}{2} \sum_{a=1}^{\text{rank } G} \varphi^a H_a \right) ; \quad n = \exp \left( \sum_{\alpha > 0} \zeta^\alpha E_\alpha \right) \] (7.6)

one gets from (7.2)
\[ A_\mu = -\frac{1}{2} \sum_{a=1}^{\text{rank } G} \partial_\mu \varphi^a H_a + \sum_{\alpha, \beta > 0} \partial_\mu \zeta^\alpha \mathcal{V}_{\alpha \beta} (\zeta) e^{\frac{1}{2} K_{\beta a} \varphi^a} E_\beta \]
\[ \tilde{B}_\mu = P^\psi \left( -\sum_{a=1}^{\text{rank } G} \partial_\mu \varphi^a H_a + \sum_{\alpha, \beta > 0} \partial_\mu \zeta^\alpha \mathcal{V}_{\alpha \beta} (\zeta) e^{\frac{1}{2} K_{\beta a} \varphi^a} (E_\beta + E_{-\beta}) \right) \] (7.7)
where \( K_{\beta a} \equiv \frac{2 \beta \cdot \alpha_a}{\alpha_a} \), with \( \alpha_a \) being a simple root of \( G \), and
\[ n^{-1} \frac{\partial n}{\partial \zeta^\alpha} \equiv \sum_{\beta > 0} \mathcal{V}_{\alpha \beta} (\zeta) E_\beta \] (7.8)

The conserved currents (2.12) are given by \((W^{-1} \equiv na)\)
\[ J_\mu = na \tilde{B}_\mu a^{-1} n^{-1} \]
\[ = P^\psi \left( n \left( -\sum_{a=1}^{\text{rank } G} \partial_\mu \varphi^a H_a + \sum_{\alpha, \beta > 0} \partial_\mu \zeta^\alpha \mathcal{V}_{\alpha \beta} (\zeta) \left( E_\beta + e^{K_{\beta a} \varphi^a} E_{-\beta} \right) \right) n^{-1} \right) \] (7.9)

7.1.1 The example of \( sl(2) \)

In such case there is just one positive root, and therefore we denote \( a = e^{-\frac{1}{2} \varphi^H} \), \( n = e^{\zeta E^+} \). The commutation relations for \( sl(2) \) are
\[ [ H, E_\pm ] = \pm 2 E_\pm ; \quad [ E_+, E_- ] = H \] (7.10)

We have \( n^{-1} \frac{\partial n}{\partial \zeta} = E_+ \), and so \( \mathcal{V}_{\alpha \beta} \equiv 1 \). Then, from (7.4) one gets
\[ D^\mu \tilde{B}_\mu = P^\psi \left( -\partial^2 \varphi + e^{2 \varphi} (\partial_\mu \zeta)^2 \right) H + e^{\varphi} \left( \partial^2 \zeta + 2 \partial_\mu \varphi \partial^\mu \zeta \right) (E_+ + E_-) \] (7.11)
Consequently, the local zero curvature conditions (2.8) imply the equations of motion
\[ \partial^2 \varphi - e^{2\varphi} (\partial_{\mu} \zeta)^2 = 0 \] (7.12)
\[ \partial^2 \zeta + 2 \partial_{\mu} \varphi \partial^{\mu} \zeta = 0 \] (7.13)

The conserved currents (7.9) are given by
\[ J_{\mu} = J_{\mu}^+ P^\psi (E_+) + J_{\mu}^0 P^\psi (H) + J_{\mu}^- P^\psi (E_-) \] (7.14)
with
\[ J_{\mu}^+ = \left( 1 - \zeta^2 e^{2\varphi} \right) \partial_{\mu} \zeta + 2 \zeta \partial_{\mu} \varphi \]
\[ J_{\mu}^0 = -\partial_{\mu} \varphi + e^{2\varphi} \zeta \partial_{\mu} \zeta \]
\[ J_{\mu}^- = e^{2\varphi} \partial_{\mu} \zeta \] (7.15)

Following the discussion of section 4, we now construct a submodel of (7.13) that possesses an infinite number of local conserved currents. In the notation of that section, the subgroup \( K \) here is \( SO(2) \) (or \( U(1) \)) and it is generated by \( (E_+ - E_-) \). The subspace \( S \) is generated by \( H \) and \( (E_+ + E_-) \). Since those generators do not diagonalize the action of the \( SO(2) \) subgroup, we shall work with the basis\footnote{Notice that formally, the \( sl(2) \) generated by \( T_3 \) and \( T_\pm \) is not the same as that generated by \( H \) and \( E_\pm \), since they are related by complex linear combinations. They are in fact distinct real forms of the same complex \( sl(2, \mathbb{C}) \).}
\[ T_3 \equiv \frac{1}{2i} (E_+ - E_-) ; \quad T_\pm \equiv \frac{1}{2} (H \pm i (E_+ + E_-)) \] (7.16)
which satisfy
\[ [T_3, T_\pm] = \pm T_\pm ; \quad [T_+, T_-] = 2 T_3 \] (7.17)

Therefore, the potentials (7.1) become
\[ A_{\mu} = -\frac{1}{2} (\partial_{\mu} \varphi + ie^{\varphi} \partial_{\mu} \zeta) T_+ - \frac{1}{2} (\partial_{\mu} \varphi - ie^{\varphi} \partial_{\mu} \zeta) T_- + ie^{\varphi} \partial_{\mu} \zeta T_3 \equiv A_{\mu}^i T_i \]
\[ \tilde{B}_{\mu} = -P^\psi ((\partial_{\mu} \varphi + ie^{\varphi} \partial_{\mu} \zeta) T_+ + (\partial_{\mu} \varphi - ie^{\varphi} \partial_{\mu} \zeta) T_-) \] (7.18)

Obviously, the adjoint of \( SL(2) \) possesses a singlet state of the \( SO(2) \) subalgebra, namely \( P^\psi (T_3) \). Therefore, using the ideas of section 4.1, we can construct submodels with large number of conservation laws. So, following (4.14) we introduce
\[ A_{\mu}^{(n)} \equiv A_{\mu}^i \sum_{l=0}^{n-1} (\otimes 1)^l \otimes T_i (\otimes 1)^{n-l-1} \] (7.19)
\[ \tilde{B}_{\mu}^{\psi(n)} \equiv \sum_{l=0}^{n-1} \left( \otimes P^\psi (T_3) \right)^l \otimes \left( A_{\mu}^+ P^\psi (T_+) + A_{\mu}^- P^\psi (T_-) \right) \left( \otimes P^\psi (T_3) \right)^{n-l-1} \]
According to the comments below (4.4) and (4.14) we could have rescaled the basis of the irreducible representations of $SO(2)$, independently. However, such freedom would not produce more submodels with infinite number of conserved currents (see discussion below (8.33) for a similar situation).

As we have argued in section 4.1 the zero curvature for these potentials lead to the equations of motion (7.13) and the constraints corresponding to (4.16). One can verify that such constraints for any value of $n$ correspond to $(A_+^\mu)^2 = (A_-^\mu)^2 = 0$, which is equivalent to

$$\left(\partial_\mu \varphi + ie^\varphi \partial_\mu \zeta\right)^2 = 0 \quad (7.20)$$

Therefore the equations of motion and constraints of the submodel defined by (7.13) and (7.20) can be written as

$$\partial^2 \varphi - (\partial_\mu \varphi)^2 = 0; \quad \partial^2 \zeta = 0; \quad (\partial_\mu \varphi)^2 - e^{2\varphi} (\partial_\mu \zeta)^2 = 0; \quad \partial_\mu \zeta \partial^\mu \varphi = 0 \quad (7.21)$$

Introducing

$$\phi \equiv e^{-\varphi}; \quad \text{and so } \phi \geq 0 \quad (7.22)$$

it becomes

$$\partial^2 \phi = 0; \quad \partial^2 \zeta = 0; \quad \partial_\mu \zeta \partial^\mu \phi = 0; \quad (\partial_\mu \phi)^2 = (\partial_\mu \zeta)^2 \quad (7.23)$$

Now, using the results of section 4, we can find an infinite number of conserved currents for the submodel defined by equations (7.13). These currents will have the form of (4.19); since for this model

$$b = na = e^{\zeta E_+} e^{-\frac{i}{2} \varphi H} = \begin{pmatrix} e^{-\frac{1}{2} \varphi} & \zeta e^{\frac{1}{2} \varphi} \\ 0 & e^{\frac{1}{2} \varphi} \end{pmatrix}, \quad (7.24)$$

the $V_\alpha(b)$'s take the form

$$b P^\psi (T_3) b^{-1} = P^\psi \begin{pmatrix} V^0 & V^+ \\ V^- & -V^0 \end{pmatrix} \quad (7.25)$$

with

$$V^+ = \left(1 + \frac{e^{2\varphi}}{2}\right) e^{-\varphi} = \left(1 + \frac{\zeta^2}{\phi^2}\right) \phi$$

$$V^0 = -\zeta e^\varphi = -\frac{\zeta}{\phi}$$

$$V^- = -e^\varphi = -\frac{1}{\phi}$$

\footnote{There is an additional choice which would lead to the constraint $|\partial_\mu \varphi + ie^\varphi \partial_\mu \zeta|^2 = 0$, instead of (7.20). However, the method of section 4.1 would lead to conserved currents for the case $n = 2$ only.}
So, for the case \( n = 2 \) in (4.19), one gets
\[
J_{\mu}^{(+,+)} \equiv J_{\mu}^0 V^+ = e^{-\varphi} \left( 1 + \zeta^2 e^{2\varphi} \right) \left[ 2 \zeta \partial_\mu \varphi + \left( 1 - \zeta^2 e^{2\varphi} \right) \partial_\mu \zeta \right];
\]
\[
J_{\mu}^{(0,0)} \equiv J_{\mu}^0 V^0 = \zeta e^{\varphi} \partial_\mu \varphi - \zeta^2 e^{3\varphi} \partial_\mu \zeta ;
\]
\[
J_{\mu}^{(-,-)} \equiv J_{\mu}^0 V^- = -e^{\varphi} \partial_\mu \zeta ;
\]
\[
J_{\mu}^{(0,-)} \equiv J_{\mu}^0 V^- + J_{\mu}^0 V^0 = e^{\varphi} \partial_\mu \varphi - 2 \zeta e^{3\varphi} \partial_\mu \zeta ;
\]
\[
J_{\mu}^{(0,+) \equiv J_{\mu}^0 V^+ + J_{\mu}^0 V^0 = 2 \zeta^3 e^{3\varphi} \partial_\mu \zeta - \left( 1 + 3 \zeta^2 e^{2\varphi} \right) e^{-\varphi} \partial_\mu \phi ;
\]
\[
J_{\mu}^{(+-)} \equiv J_{\mu}^+ V^- + J_{\mu}^- V^+ = 2 \zeta^2 e^{3\varphi} \partial_\mu \zeta - 2 \zeta e^{\varphi} \partial_\mu \varphi ;
\]

The models discussed in section 7.1 and in particular the example of \( sl(2) \) given by (7.13), have been discussed in the literature \[9, 10\] in the context of dualities in supergravity theories. It would be interesting to investigate the role of such infinite set of conserved currents in those theories.

8 The \( CP^N \) models

The \( CP^N \) model contains \( N \) complex scalar fields \( u_i, i = 1, 2, \ldots N \), and on a space-time of \( d + 1 \) dimensions it is defined by the action
\[
S \equiv \int d^{d+1}x \frac{\left( 1 + u^\dagger \cdot u \right) \left( \partial_\mu u^\dagger \cdot \partial^\mu u \right) - \left( u^\dagger \cdot \partial_\mu u \right) \left( \partial^\mu u^\dagger \cdot u \right)}{\left( 1 + u^\dagger \cdot u \right)^2} \quad (8.1)
\]
where we have denote by \( u \) the \( N \)-dimensional column matrix with components \( u_i \), and by \( u^\dagger \) the complex conjugate of its transpose. The corresponding equations of motion are\[
\left( 1 + u^\dagger \cdot u \right) \partial^2 u_i = 2 \left( u^\dagger \cdot \partial_\mu u \right) \partial^\mu u_i \quad (8.2)
\]
and the corresponding complex conjugates.

The \( CP^N \) model corresponds in fact to the non-linear sigma model on the symmetric space \( SU(N + 1)/SU(N) \times U(1) \), defined in the manner discussed in section 3, and therefore possesses a local zero curvature representation as discussed there. See \[11, 12\] for alternative formulations. Let \( \alpha_i \) and \( \lambda_i, i = 1, 2, \ldots N \), denote the simple roots and

\[^5\]Actually the equation of motion following from (8.1) is \( (1 + u^\dagger \cdot u) \partial^2 u_i + 2 \frac{\left(u^\dagger \cdot \partial u \right)^2 u_i}{\left( 1 + u^\dagger \cdot u \right)^2} - 2 \left(u^\dagger \cdot \partial_\mu u \right) \partial^\mu u_i - \left(u^\dagger \cdot \partial^2 u \right) u_i = 0 \). However, such equation (as well as (8.2)) implies, by contraction with \( u_i^* \), that \( (u^\dagger \cdot \partial^2 u) = 2 \frac{\left(u^\dagger \cdot \partial u \right)^2}{\left( 1 + u^\dagger \cdot u \right)^2} \). Those two relations leads to (8.2).
fundamental weights respectively, of $SU(N+1)$. They satisfy $\frac{2\lambda_i \cdot \alpha_j}{\alpha_j^2} = \delta_{ij}$. The relevant involutive automorphism is given by

$$\sigma(T) \equiv \Omega T \Omega^{-1}; \quad \Omega \equiv e^{i\pi \Lambda}; \quad \Lambda \equiv \frac{2\lambda_N \cdot H}{\alpha_N^2} \quad (8.3)$$

with $T$ being an element of the algebra $su(N+1)$, and $H_i$ being the basis of its Cartan subalgebra. Therefore, the subalgebra of $su(N+1)$ invariant under $\sigma$ is generated by the Cartan subalgebra and the step operators $E_{\pm \alpha}$ corresponding to roots which are orthogonal to $\lambda_N$, or in other words, which do not contain $\alpha_N$ in its expansion in terms of simple roots. Therefore, it corresponds to the subalgebra $su(N) \oplus u(1)$, where the simple roots of such $su(N)$ are the first $N-1$ simple roots of $su(N+1)$, i.e. $\alpha_a, a = 1, 2, \ldots N-1$. The $u(1)$ factor is obviously generated by $\Lambda$ defined in (8.3). Following the notation of (5.1) one has

$$S \equiv \{ S_{\pm i} \equiv E_{\pm(\alpha_i+\alpha_{i+1}+\ldots \alpha_N)} \; ; \; i = 1, 2, \ldots N \}$$

$$K \equiv su(N) \oplus u(1) \quad (8.4)$$

The action and equations of motion of the $CP^N$ model can then be written in the form (5.4) and (5.5) respectively. The main problem is to find the correct parametrization in terms of the fields $u_i$ of the $SU(N+1)$ group element $g$, in (5.3), such that (5.5) reproduces (8.2). The answer to it is

$$g = e^{iS} e^{\varphi[S, S^\dagger]} e^{iS^\dagger}; \quad \varphi \equiv \log \frac{\sqrt{1 + u^\dagger \cdot u}}{u^\dagger \cdot u} \quad (8.5)$$

where we have defined

$$S \equiv u_i S_i \quad S^\dagger \equiv u_i^* S_{-i} \quad (8.6)$$

with $S_{\pm i}$ introduced in (8.4), and where we have used the fact that in any finite dimensional representation we can choose the basis such that $H_i^\dagger = H_i$ and $E_\alpha^\dagger = E_{-\alpha}$.

In the $(N+1)$-dimensional defining representation of $SU(N+1)$, $g$ is given by

$$g \equiv \frac{1}{\vartheta} \left( \begin{array}{c} \Delta \\ iu^\dagger \\ 1 \end{array} \right); \quad \vartheta \equiv \sqrt{1 + u^\dagger \cdot u} \quad (8.7)$$

where $\Delta$ is a $N \times N$ hermitian matrix given by

$$\Delta_{ij} \equiv \vartheta \delta_{ij} - \frac{u_i u_j^*}{1 + \vartheta}; \quad i, j = 1, 2, \ldots N \quad (8.8)$$

It then follows that $g$ is indeed an unitary matrix.
Notice that \( u \) is an eigenvector of \( \Delta \) with unit eigenvalue

\[
\Delta \cdot u = u \quad (8.9)
\]

In the defining representation, \( \Lambda \) and \( \Omega \) leading to the automorphism \( \sigma \) of \((8.3)\) are given by

\[
\Lambda = \frac{1}{N + 1} \begin{pmatrix} \mathbb{1}_{N \times N} & 0 \\ 0 & -N \end{pmatrix} \quad \Omega = e^{i\pi/(N+1)} \begin{pmatrix} \mathbb{1}_{N \times N} & 0 \\ 0 & -1 \end{pmatrix} \quad (8.10)
\]

and therefore\(^6\)

\[
\sigma (g) = \Omega g \Omega^{-1} = g^{-1} \quad \text{and} \quad y (g) = g^2 \quad (8.11)
\]

where \( y (g) \) is defined in \((5.3)\).

One can check that

\[
g^{-1} \partial_{\mu} g = \frac{1}{\vartheta^2} \begin{pmatrix} \kappa_{\mu} & i \Delta \cdot \partial_{\mu} u \\ i \left( \partial_{\mu} u^\dagger \right) \cdot \Delta & v_{\mu} \end{pmatrix} \quad (8.12)
\]

where

\[
\kappa_{ij} \equiv \frac{\vartheta}{1 + \vartheta} \left( u_i \partial^\mu u_j^* - (\partial^\mu u) u^*_j \right) + \frac{1}{2} \left( u^\dagger \cdot \partial^\mu u - (\partial^\mu u^\dagger) \cdot u \right) \frac{u_i u_j^*}{(1 + \vartheta)^2}
\]

\[
v_{\mu} \equiv \frac{1}{2} \left( u^\dagger \cdot \partial_{\mu} u - (\partial_{\mu} u^\dagger) \cdot u \right) \quad (8.13)
\]

One can write \((8.12)\) as a linear combination of a basis of \( su(N + 1) \) using the odd generators \( S_{\pm i} \) introduced in \((8.4)\). In order to simplify the notation we introduce the covariant derivative

\[
\nabla_{\mu} u_i \equiv \Delta_{ij} \partial_{\mu} u_j = \left( \vartheta \partial_{\mu} - \frac{u^\dagger \cdot \partial_{\mu} u}{1 + \vartheta} \right) u_i \quad (8.14)
\]

The potentials \((5.8)\) can then be written as

\[
A_{\mu} = \ g^{-1} \partial_{\mu} g
\]

\[
= \frac{1}{\vartheta^2} \left( \vartheta \nabla_{\mu} S + i \left( \nabla_{\mu} S \right)^\dagger + \left[ S, (\partial_{\mu} S)^\dagger \right] - \left[ \partial_{\mu} S, S^\dagger \right] \right) \frac{1}{1 + \vartheta} - v_{\mu} \left[ S, S^\dagger \right] \frac{1}{(1 + \vartheta)^2} \quad (8.15)
\]

\[
\tilde{B}_{\mu} = \ P^\psi (1 - \sigma) \left( g^{-1} \partial_{\mu} g \right)
\]

\[
= \frac{2i}{\vartheta^2} P^\psi \left( \nabla_{\mu} S + (\nabla_{\mu} S)^\dagger \right) = \frac{2i}{\vartheta^2} \left( \nabla_{\mu} u_i P^\psi (S_i) + (\nabla_{\mu} u_i)^\dagger P^\psi (S_{-i}) \right)
\]

\(^6\)Notice that, from \((8.3)\), \((8.5)\) and \((8.11)\), one has (in the defining representation at least) that \( \Omega g \Omega^{-1} = e^{-iS^\dagger} e^{\psi [S, S^\dagger]} e^{-iS} = g^{-1} \), and so one can also write \( g = e^{iS} e^{-\psi [S, S^\dagger]} e^{iS} \).
Notice that in the even part under $\sigma$ of $A_\mu$, i.e. the terms involving commutators of $S$'s, the ordinary derivative can be replaced by the covariant derivatives (8.14) due to the antisymmetry of the terms.

By imposing the local zero curvature condition (2.8) on these potentials one obtains the $CP^N$ equations of motion (8.2). Indeed, the flatness condition $F_{\mu\nu} = 0$ is trivially satisfied since $A_\mu$ is of the pure gauge form. The condition $D_\mu \tilde{B}_\mu = 0$ leads to $2N$ equations which are equivalent to (8.2).

According to (2.12) (or (3.12)) the conserved currents of the $CP^N$ model are given by

$$J_\mu = g \tilde{B}_\mu g^{-1} = 2 P^\psi \left( \begin{pmatrix} J^{ij}_\mu \quad iJ^i_\mu \end{pmatrix} \right) = P^\psi \left( J^{ij}_\mu [S_i, S_{-j}] + iJ^i_\mu S_i + iJ^i_{\mu} S_{-i} \right)$$

$$= \frac{1}{1 + u^\dagger \cdot u} P^\psi \left( \left[ \partial_\mu \left( S + S^\dagger \right), S + S^\dagger \right] + i\partial_\mu \left( S + S^\dagger \right) \right)$$

$$- \frac{u^\dagger \cdot \partial_\mu u - \partial_\mu u^\dagger \cdot u}{1 + u^\dagger \cdot u} \left( i \left( S - S^\dagger \right) + \left[ S, S^\dagger \right] \right)$$

(8.16)

with $i, j = 1, 2, \ldots N$, and

$$J^{ij}_\mu = \frac{(1 + u^\dagger \cdot u) \left( \partial_\mu u_i u_j^\dagger - u_i \partial_\mu u_j^\dagger \right) - u_i u_j^\dagger \left( u^\dagger \cdot \partial_\mu u - \partial_\mu u^\dagger \cdot u \right)}{(1 + u^\dagger \cdot u)^2}$$

$$J^i_\mu = \frac{(1 + u^\dagger \cdot u) \partial_\mu u_i - u_i \left( u^\dagger \cdot \partial_\mu u - \partial_\mu u^\dagger \cdot u \right)}{(1 + u^\dagger \cdot u)^2}$$

$$J^\mu = \frac{u^\dagger \cdot \partial_\mu u - \partial_\mu u^\dagger \cdot u}{(1 + u^\dagger \cdot u)^2}$$

(8.17)

In (8.16) $J^{ij}_\mu$, $J^i_\mu$, $J^i_{\mu}$ and $J_\mu$ stand for matrices $(N \times N)$, $(N \times 1)$, $(1 \times N)$ and $(1 \times 1)$ respectively. Notice that the number of conserved currents is indeed equal to the dimension of $SU(N + 1)$, i.e. $(N^2 + 2N)$, since $\sum_{i=1}^N J^{ii}_\mu = J_\mu$.

\footnote{Where we have used the fact that in the defining representation of $SU(N + 1)$, one has $(S_i)_{rs} = \delta_{sr} \delta_{s,N+1}$ and $(S_{-i})_{rs} = \delta_{r,N+1} \delta_{is}$, with $r, s = 1, 2, \ldots N + 1$, and $i = 1, 2, \ldots N$.}
8.1 Integrable submodels of \( CP^N \)

We now follow the strategy of section 4 to construct submodels of \( CP^N \) which presents an infinite number of conserved currents.

Since \( su(N + 1) \) has no roots containing twice \( \pm \alpha_N \) in their expansions in terms of simple roots, it follows that \( S \) defined in (8.4) splits into two abelian subspaces generated by \( S_i \) and \( S_{-i} \), i.e.

\[
S = S_+ + S_- ; \quad S_\pm \equiv \{ S_{\pm i} ; \ i = 1, 2, \ldots N \} \tag{8.18}
\]

and

\[
[S_i, S_j] = [S_{-i}, S_{-j}] = 0 ; \quad \text{any } i, j \tag{8.19}
\]

It follows that \( S_+ \) and \( S_- \) transform under the representations \( N(1) \) and \( \bar{N}(-1) \) respectively, of the subalgebra \( K = su(N) \oplus u(1) \), i.e.

\[
\left[ K, P^\psi (S_i) \right] = P^\psi (S_j) R^{N(1)}_{ji} (K) \\
\left[ K, P^\psi (S_{-i}) \right] = P^\psi (S_{-j}) R^{\bar{N}(-1)}_{ji} (K) \tag{8.20}
\]

Therefore, according to the discussion of section 4 we have to look for representations \( P^\lambda \) of \( su(N + 1) \) such that its branching in terms of \( su(N) \oplus u(1) \) possesses the representations \( N(1) \) and \( \bar{N}(-1) \) at least once, i.e.

\[
P^\lambda = N(1) + \bar{N}(-1) + \text{anything} \tag{8.21}
\]

If that happens let us denote by \( P^\lambda_i \) and \( P^\lambda_{-i} \), \( i = 1, 2, \ldots N \), the basis of the subspaces corresponding to \( N(1) \) and \( \bar{N}(-1) \) respectively, that transform exactly like \( P^\psi (S_i) \) and \( P^\psi (S_{-i}) \), i.e.

\[
\left[ K, P^\lambda_i \right] = P^\lambda_j R^{N(1)}_{ji} (K) \\
\left[ K, P^\lambda_{-i} \right] = P^\lambda_{-j} R^{\bar{N}(-1)}_{ji} (K) \tag{8.22}
\]

As we have commented below (4.4), we can rescale the basis \( P^\lambda_i \) and \( P^\lambda_{-i} \) of \( N(1) \) and \( \bar{N}(-1) \) respectively, independently without changing the relation between (8.21) and (8.22). Then following (8.15), we introduce the potential

\[
\tilde{B}^\lambda_\mu = \frac{2i}{\beta^2} \left( \nabla_\mu u_i P^\lambda_i + \beta (\nabla_\mu u_i) \right) P^\lambda_{-i} \tag{8.23}
\]

where \( \beta \) is the parameter accounting for the freedom of rescaling the basis of the irreducible components.
Again, according to the arguments of section 4 the zero curvature condition

\[ D^\mu \tilde{B}^\lambda_\mu = 0 \]  

(8.24)

where the covariant derivative is w.r.t. the same potential \( A_\mu \) as in (8.15), leads to the equations of motion of the \( CP^N \) model (8.2) plus the constraints (see (4.7))

\[ \partial_\mu u_i \partial^\mu u_j \left( \left[ S_i, P^\lambda_j \right] + \left[ S_j, P^\lambda_i \right] \right) = 0 \]  

(8.25)

\[ \partial_\mu u_i^* \partial^\mu u_j^* \left( \left[ S_i, P^\lambda_j \right] + \left[ S_j, P^\lambda_i \right] \right) = 0 \]  

(8.26)

\[ \partial_\mu u_i \partial^\mu u_j^* \left( \beta \left[ S_i, P^\lambda_j \right] + \left[ S_j, P^\lambda_i \right] \right) = 0 \]  

(8.27)

In the above calculation we have used that \( \nabla_\mu u_i = \Delta_{ij} \partial_\mu u_j \), together with the fact that \( \Delta_{ij} \) is invertible, i.e.

\[ \Delta^{-1}_{ij} \equiv \frac{1}{\vartheta} \left( \delta_{ij} + \frac{u_i u_j}{1 + \vartheta} \right) \]  

(8.28)

Therefore, any eq. of the type \( \nabla_\mu u_i \nabla^\mu u_j M_{ij} = 0 \) can be written as \( \partial_\mu u_i \partial^\mu u_j M_{ij} = 0 \), for a generic tensor \( M_{ij} \).

We have that the terms involving commutators in (8.25), (8.26), and (8.27) transform under \( K = \text{su}(N) \oplus u(1) \) as \( (N \times N)_s (2) = \frac{N(N+1)}{2}(2), \ (\bar{N} \times \bar{N})_s (-2) = \frac{N(N+1)}{2}(-2), \) and \( (N \times \bar{N}) (0) = (1 + \text{adjoint}) (0) \), respectively. Therefore, as we discussed in section 4 the constraints (8.25)-(8.27) will only be effective if such representation appear in the branching of \( P^\lambda \) in terms of representations of \( K = \text{su}(N) \oplus u(1) \).

In any case, the model defined by the equations (8.2) and constraints (8.25)-(8.27) possesses the conserved currents (see (2.12))

\[ J^\lambda_\mu \equiv g \tilde{B}^\lambda_\mu g^{-1} \]  

(8.29)

with \( g \) given by (8.5). If the number of representations \( P^\lambda \) satisfying the conditions discussed above is infinite, one obviously gets an infinite number of conserved currents.

### 8.1.1 The singlet states and infinite number of currents

The adjoint representation of \( SU(N+1) \) decomposes into representations of \( SU(N) \otimes U(1) \) as

\[ \text{Adj} (SU(N+1)) = N(1) + \bar{N}(-1) + \text{Adj} (SU(N)) (0) + 1(0) \]  

(8.30)

Therefore it possesses a singlet state satisfying (4.12). That singlet corresponds to the \( U(1) \) generator \( \Lambda \) defined in (8.3) and (8.1).
Consequently, we can apply the ideas of section 4.1 to construct an infinite number of conserved currents for submodels of $CP^N$. Denoting the generators of $K = su(N) \oplus u(1)$, by $K_r$, $r = 1, 2, \ldots N^2$, one can write the potential $A_\mu$ in (8.17) as

$$A_\mu = i \left( A_\mu^r K_r + b_\mu + \tilde{b}_\mu^r \right) ; \quad b_\mu \equiv \frac{\nabla_\mu S}{1 + \bar{u} \cdot u}$$

Then, following (4.14) we define

$$A_\mu^{(n)} \equiv i \sum_{l=0}^{n-1} (\otimes 1)^l \otimes \left( A_\mu^r K_r + b_\mu + \tilde{b}_\mu^r \right) (\otimes 1)^{n-l-1} \quad (8.32)$$

$$\tilde{B}_\mu^{(n)} \equiv 2i \sum_{l=0}^{n-1} (\otimes \psi^* (\Lambda))^l \otimes \psi (c_{n,l} b_\mu + \bar{c}_{n,l} \tilde{b}_\mu^l) (\otimes \psi^* (\Lambda))^{n-l-1}$$

Since the representation $R^S = N(1) + N(-1)$, is reducible we can rescale each irreducible component independently (see comments below (4.4) and (4.14)). The constants $c_{n,l}$ and $\bar{c}_{n,l}$ account for such freedom.

We now impose that these potentials should satisfy the zero curvature conditions (4.8). Obviously $A_\mu^{(n)}$ satisfy $F_{\mu\nu} = 0$. As we have argued in section 4.1, the components of the condition $D_\mu \tilde{B}_\mu^{(n)} = 0$, involving $\partial_\mu \tilde{B}_\mu^{(n)}$ and the commutator of $\tilde{B}_\mu^{(n)}$ with the $K$-part of $A_\mu^{(n)}$ lead to the equations of motion of the $CP^N$ model (8.2). The commutator of $\tilde{B}_\mu^{(n)}$ with the $S$-part of $A_\mu^{(n)}$ leads to the constraints defining the submodel. We analyze those constraints by collecting the linearly independent terms in the tensor product. The terms involving $b_\mu$’s in the $l$ and $m$ positions of the tensor product are ($l < m$)

$$\left( c_{n,l} + c_{n,m} \right) \left( \otimes \psi^* (\Lambda) \right)^{l-1} \otimes \psi (b_\mu) \left( \otimes \psi (\Lambda) \right)^{m-l} \psi (b_\mu) \left( \otimes \psi (\Lambda) \right)^{n-m}$$

$$- \left( \bar{c}_{n,l} + \bar{c}_{n,m} \right) \left( \otimes \psi^* (\Lambda) \right)^{l-1} \otimes \psi \left( \tilde{b}_\mu^l \right) \left( \otimes \psi (\Lambda) \right)^{m-l} \psi \left( \tilde{b}_\mu^l \right) \left( \otimes \psi (\Lambda) \right)^{n-m}$$

$$- \left( c_{n,l} - \bar{c}_{n,m} \right) \left( \otimes \psi^* (\Lambda) \right)^{l-1} \otimes \psi \left( \bar{b}_\mu^l \right) \left( \otimes \psi (\Lambda) \right)^{m-l} \psi \left( \bar{b}_\mu^l \right) \left( \otimes \psi (\Lambda) \right)^{n-m}$$

$$+ \left( \bar{c}_{n,l} - c_{n,m} \right) \left( \otimes \psi^* (\Lambda) \right)^{l-1} \otimes \psi \left( \bar{b}_\mu^l \right) \left( \otimes \psi (\Lambda) \right)^{m-l} \psi \left( \bar{b}_\mu^l \right) \left( \otimes \psi (\Lambda) \right)^{n-m} = 0$$

where we have used the fact that $[\Lambda, b_\mu + \tilde{b}_\mu^l] = b_\mu - \tilde{b}_\mu^l$.

The terms involving commutators of $b_\mu$’s are

$$\left( c_{n,l} - \bar{c}_{n,l} \right) \left( \otimes \psi^* (\Lambda) \right)^l \otimes \psi \left( \left[ b_\mu, \tilde{b}_\mu^l \right] \right) \left( \otimes \psi (\Lambda) \right)^{n-l-1} = 0 \quad (8.34)$$

Therefore, if we choose

$$c_{n,l} = \bar{c}_{n,l} = 1 ; \quad \text{for any } n \text{ and } l \quad (8.35)$$

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we get the constraints $\nabla_\mu u_i \nabla^\nu u_j = 0$ and $\left(\nabla_\mu u_i\right)^\dagger \left(\nabla^\nu u_j\right)^\dagger = 0$, for any $i$ and $j$. However, from (8.14) we have that $\nabla_\mu u_i = \Delta_{ij} \partial_\mu u_j$, and since $\Delta_{ij}$ is invertible (see (8.28)), it follows that the constraints become just $\partial_\mu u_i \partial^\mu u_j = 0$ and their complex conjugates. Using such constraints on the equation of motion of $CP^N$ (8.2) one gets that $\partial^2 u_i = 0$. Consequently, the submodel we obtain is defined by the equations

$$\partial^2 u_i = 0; \quad \partial_\mu u_i \partial^\mu u_j = 0$$

(8.36)

and the corresponding complex conjugate equations.

According to the discussions of section 4.1 such submodel possesses an infinite number of currents given by (4.19). The quantities $V_\alpha$, defined in (4.18), are given by

$$V_{ij} \equiv \frac{\delta_{ij}}{N+1} - \frac{u_i u_j^\dagger}{1 + u^\dagger \cdot u}$$

$$V^i \equiv \frac{u_i}{1 + u^\dagger \cdot u}$$

$$V \equiv - \frac{1}{N+1} + \frac{1}{1 + u^\dagger \cdot u}$$

(8.38)

In (8.37) we are using the same notation as in (8.16). The number of independent quantities $V$’s is the dimension of $SU(N+1)$, since $\sum_{i=1}^N V_{ii} = V$.

One can easily check that the currents (8.17) can be written in terms of (8.38) as

$$J^i_{\mu} = - \left( \frac{\delta V^i}{\delta u_m} \partial_\mu u_m - \frac{\delta V^i}{\delta u_m^\dagger} \partial_\mu u_m^\dagger \right)$$

$$J^i_{\mu} = \frac{\delta V^i}{\delta u_m} \partial_\mu u_m - \frac{\delta V^i}{\delta u_m^\dagger} \partial_\mu u_m^\dagger$$

$$J^i_{\mu} = - \left( \frac{\delta V^i}{\delta u_m^\dagger} \partial_\mu u_m - \frac{\delta V^i}{\delta u_m} \partial_\mu u_m^\dagger \right)$$

(8.39)

Since we have chosen all the $c_{n,l}$’s to be unity (see (8.35)), it follows that all the conserved currents (4.19) are of the form

$$J^0_{\mu_{\alpha_1 \alpha_2 \ldots \alpha_n}} \equiv \frac{\delta F_{\mu_{\alpha_1 \alpha_2 \ldots \alpha_n}}}{\delta u_m} \partial_\mu u_m - \frac{\delta F_{\mu_{\alpha_1 \alpha_2 \ldots \alpha_n}}}{\delta u_m^\dagger} \partial_\mu u_m^\dagger$$

(8.40)
where
\[ F^{\alpha_1 \alpha_2 \ldots \alpha_n} \equiv - \prod_{l=1}^{n} V^{\alpha_l} \]  
(8.41)

with \( V^{\alpha_l} \)'s being any of the quantities in (8.38), i.e \( V^{ij}, V^i \) and \( V^{it} \).

In fact, any quantity of the form
\[ J_\mu = M_i \partial_\mu u_i + N_i \partial_\mu u_i^\dagger \]  
(8.42)

with \( M_i \) and \( N_i \) being functionals of \( u_j \) and \( u_j^\dagger \), is a conserved current of the submodel (8.36) provided
\[ \frac{\delta M_i}{\delta u_j} + \frac{\delta N_j}{\delta u_i} = 0 \]  
(8.43)
The conserved quantities we have obtained above, using the ideas of section 4.1, are particular examples of the cases where \( M_i = \frac{\delta F}{\delta u_i} \) and \( N_i = -\frac{\delta F}{\delta u_i^\dagger} \).

It is worth mentioning that the amount of conservation laws the submodel (8.36) possesses is due (at least partially) to the huge symmetry group it presents. Indeed, the submodel (8.36) is invariant under the transformations
\[ u_i \rightarrow u_i' \equiv \omega_i^{(1)} (u) + \omega_i^{(2)} (u^*) \]
\[ u_i^* \rightarrow u_i^{*'} \equiv \omega_i^{(1)} (u) + \omega_i^{(2)} (u^*) \]  
(8.44)

provided
\[ \omega_i^{(1)} (u) \omega_j^{(2)} (u^*) + \omega_i^{(2)} (u^*) \omega_j^{(1)} (u) = h_{ij}^{(1)} (u) + h_{ij}^{(2)} (u^*) \]
\[ \omega_i^{(1)} (u) \omega_{-j}^{(2)} (u^*) + \omega_i^{(2)} (u^*) \omega_{-j}^{(1)} (u) = h_{-i,-j}^{(1)} (u) + h_{-i,-j}^{(2)} (u^*) \]  
(8.45)

for any \( i, j, k, l = 1, 2, \ldots N \), and where \( (\omega_{\pm i}^{(1)}, h_{\pm i, \pm j}^{(1)}) \) and \( (\omega_{\pm i}^{(2)}, h_{\pm i, \pm j}^{(2)}) \) are functions of \( u_i \)'s and \( u_i^* \)'s only, respectively. Particular solutions for (8.45) are obtained by taking \( \omega_i^{(1)} = \text{const.} \), and \( \omega_j^{(2)} \) arbitrary, or vice-versa. The same being true for the negative \( \omega \)'s. Therefore, starting with a suitable small set of currents, a large amount of other currents can be construct using the transformations (8.44).
In the case $n = 2$, the currents (8.40) are given by

\[
J^{ij}_{\mu} V^{kl} + V^{ij} J^{kl}_{\mu} = \frac{1}{(1 + u^\dagger \cdot u)^2} \left\{ \frac{2(u^\dagger \cdot \partial_\mu u - \partial_\mu u^\dagger \cdot u)}{1 + u^\dagger \cdot u} u_i u_j u_k u_l^\dagger \right. \\
+ \frac{(1 + u^\dagger \cdot u)}{(N + 1)} \left[ \delta_{kl}(\partial_\mu u_i u_j^\dagger - u_i \partial_\mu u_j^\dagger) + \delta_{ij}(\partial_\mu u_k u_l^\dagger - u_k \partial_\mu u_l^\dagger) \right] \\
- u_i u_j^\dagger (\partial_\mu u_k u_l^\dagger - u_k \partial_\mu u_l^\dagger) - u_k u_l^\dagger (\partial_\mu u_i u_j^\dagger - u_i \partial_\mu u_j^\dagger) \\
- \frac{(u^\dagger \cdot \partial_\mu u - \partial_\mu u^\dagger \cdot u)}{N + 1} \left( \delta_{kl} u_i u_j^\dagger + \delta_{ij} u_k u_l^\dagger \right) \right\}
\]

\[
J^{ij}_{\mu} V^k - V^{ij} J^k_{\mu} = \frac{1}{(1 + u^\dagger \cdot u)^2} \left[ u_k (\partial_\mu u_i u_j^\dagger - u_i \partial_\mu u_j^\dagger) + u_i u_j^\dagger \partial_\mu u_k \right] \\
- 2 \left( u^\dagger \cdot \partial_\mu u - \partial_\mu u^\dagger \cdot u \right) \frac{u_i u_j^\dagger u_k}{1 + u^\dagger \cdot u} \\
- \frac{1 + u^\dagger \cdot u}{N + 1} \delta_{ij} \partial_\mu u_k + \frac{(u^\dagger \cdot \partial_\mu u - \partial_\mu u^\dagger \cdot u)}{N + 1} u_k \delta_{ij}
\]

\[
J^{ij}_{\mu} V^k + V^{ij} J^k_{\mu} = \frac{1}{(1 + u^\dagger \cdot u)^2} \left[ u_k^\dagger (\partial_\mu u_i u_j^\dagger - u_i \partial_\mu u_j^\dagger) \right] \\
- 2 \left( u^\dagger \cdot \partial_\mu u - \partial_\mu u^\dagger \cdot u \right) \frac{u_i u_j^\dagger u_k^\dagger}{1 + u^\dagger \cdot u} - u_i u_j^\dagger \partial_\mu u_k^\dagger \\
+ \frac{1 + u^\dagger \cdot u}{N + 1} \delta_{ij} \partial_\mu u_k^\dagger + \frac{(u^\dagger \cdot \partial_\mu u - \partial_\mu u^\dagger \cdot u)}{N + 1} \delta_{ij} u_k^\dagger \right]
\]

\[
J^{ij}_{\mu} V^j - V^{ij} J^j_{\mu} = \frac{1}{(1 + u^\dagger \cdot u)^3} \left\{ \left( 1 + u^\dagger \cdot u \right) \left( \partial_\mu u_i u_i^\dagger - u_i \partial_\mu u_i^\dagger \right) \right. \\
+ \left. 2 u_i u_j \left( u^\dagger \cdot \partial_\mu u - \partial_\mu u^\dagger \cdot u \right) \right\} \\
\]

\[
J^{ij}_{\mu} V^j + V^{ij} J^j_{\mu} = \frac{\partial_\mu u_i u_j^\dagger + u_i \partial_\mu u_j^\dagger}{(1 + u^\dagger \cdot u)^2} - \frac{2 u_i u_j^\dagger}{(1 + u^\dagger \cdot u)^2} \left( \partial_\mu u_i u^\dagger - u^\dagger \cdot \partial_\mu u \right)
\]

Notice that with the choice (8.35), the operator $\bar{B}_\mu^{\psi(n)}$ given in (8.32), belongs to the symmetric part of the tensor product. Therefore, the above currents are associated to the irreducible representations of $SU(N + 1)$ in the symmetric part of the tensor product of the adjoint representation with itself. For instance, in the case of $N = 1$ one gets that the adjoint (triplet) of $SU(2)$ satisfies $3 \otimes 3 = 5_s + 3_a + 1_s$. Indeed, for $N = 1$ one can easily check that (8.46) gives 6 currents and that 5 of them coincide
with the spin $j = 2$ (5) currents calculated in [4] for the example of $CP^1$. The sixth one coincides with one of the spin $j = 1$ currents of [4].

### 8.1.2 Further submodels

When analyzing the constraints (8.33) and (8.34) we looked for a solution valid for any $n$ in order to have an infinite number of currents. However, for the case $n = 2$ there is an additional solution, besides (8.35), which corresponds to

$$c_{2,0} + c_{2,1} = 0 \quad \bar{c}_{2,0} + \bar{c}_{2,1} = 0$$

(8.47)

Such choice leads to the submodel of $CP^N$ defined by

$$\left(1 + u^\dagger \cdot u\right) \partial^2 u_i = 2 \left(u^\dagger \cdot \partial_\mu u\right) \partial^\mu u_i \quad \partial_\mu u_i \partial^\mu u_j^* = 0$$

(8.48)

with $i, j = 1, 2, \ldots N$. Therefore, using the same procedures of section 8.1.1 one obtains conserved currents of the type (8.46). In such case, the currents will depend upon one parameter which is the ratio $\bar{c}_{2,0}/c_{2,0}$.

Additional conserved currents for the submodel (8.48) can be constructed using the ideas of section 8.1. As an example consider the case of $CP^2$. The representations 10 and $\bar{10}$ of $SU(3)$ break into irreducibles of $SU(2) \otimes U(1)$ as

$$10 = 4(1) + 3(0) + 2(-1) + 1(-2)$$

$$\bar{10} = 4(-1) + 3(0) + 2(1) + 1(2)$$

(8.49)

Therefore, $10 + \bar{10}$ contains the representation $R^{S} = 2(1) + 2(-1)$ discussed in (8.21), and therefore one can defined an operator $\tilde{B}_\mu$ like in (8.23) using such representations. One can check that the constraints (8.25)-(8.27) can be solved by imposing $\partial_\mu u_i \partial^\mu u_j^* = 0$, $i, j = 1, 2$. Then through (8.29) one obtains 20 conserved currents for the corresponding submodel (8.48) of $CP^2$. A more careful analysis is necessary to work out all the conserved currents of such type of submodels.

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