Non-Abelian Strings and the Lüscher Term

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Abstract

We calculate the Lüscher term for recently suggested non-Abelian flux tubes (strings). The main feature of the non-Abelian strings is the presence of orientational zero modes associated with rotation of their color flux inside a non-Abelian subgroup. The Lüscher term is determined by the number of light degrees of freedom on the string wordsheet. Unlike the standard $\pi/12$ we get $N\pi/12$ for non-Abelian strings in the $U(N)$ gauge theories. Thus, the Lüscher coefficient acquires a dependence on the rank of the gauge group. In the models with non-Abelian strings discussed in the literature there are two distinct scales: the string tension $\xi$ (the string thickness $\sim \xi^{-1/2}$) and the dynamical scale of strong interactions $\Lambda$. At weak coupling $\xi \gg \Lambda^2$. The Lüscher term for non-Abelian strings experiences a jump: at $\xi^{-1/2} \ll L \ll \Lambda^{-1}$ it is $N\pi/12$ while at $L \gg \Lambda^{-1}$ the orientational moduli are frozen out and the Lüscher coefficient approaches its “Lüscher” value $\pi/12$. We raise the question of possible extra (i.e. non-translational) light moduli on the worldsheets of QCD strings at large $N$. 
1 Introduction

The energy of a long string (flux tube) in confining gauge theories behaves as

\[ E(L) = TL + C - \frac{\gamma}{L} + ... \]  

(1)

where \( L \) is the string length, \( C \) is a constant of dimension of mass while \( \gamma \) is a dimensionless constant. The \( O(1/L) \) term is referred to as the Lüscher term \([1]\). Its value was calculated by Lüscher,

\[ \gamma = \frac{\pi}{12}, \]  

(2)

and is believed to be universal. In fact, the Lüscher coefficient measures the number of light (massless) degrees of freedom on the string world sheet. Equation (2) assumes that the only massless excitations of the string are due to two translational zero modes.

Recently discovered non-Abelian strings \([2]\) do not satisfy this assumption. What is the difference between Abelian \([3]\) and non-Abelian strings? In the former case the gauge group acting in the infrared and responsible for the flux tube formation is Abelian (i.e. \( U(1) \times U(1) \,... \)). In the latter case we deal with a non-Abelian group in the infrared. In addition to the position of the string center in the perpendicular plane, non-Abelian strings are characterized by internal moduli. The best-known example of the first type is the Seiberg–Witten string found in \([4]\) in a slightly deformed \( \mathcal{N} = 2 \) super-Yang–Mills theory. If the deformation parameter \( \mu \) is small,

\[ \mu \ll \Lambda, \]

the \( SU(N) \) gauge group is spontaneously broken, the group acting in the low-energy description is \( U(1)^{N-1} \), and the string obtained is a generalization of the good old Abrikosov flux tube \([3]\). It is Abelian.

In the opposite limit

\[ \mu \gg \Lambda, \]

the breaking of \( SU(N) \) down to \( U(1)^{N-1} \) does not occur. The infrared dynamics is determined by \( SU(N) \); the corresponding flux tube should be non-Abelian. Presumably, there is no phase transition in \( \mu \), and the Abelian and non-Abelian flux tubes are smoothly connected. A similar phenomenon takes place \([5]\) in QCD-like theories on \( R_3 \times S_1 \). The radius \( r \) of the compactified dimension plays the same role as \( \mu \). Unfortunately, the limit \( \mu \gg \Lambda \)
or $r \gg \Lambda^{-1}$ are not under theoretical control. The first non-supersymmetric example of a controllable situation, in which a non-Abelian string emerges at weak coupling, was discussed in [6]. In this model there are two distinct scales: the string tension $\xi$ (the string thickness $\sim \xi^{-1/2}$) and the dynamical scale of strong interactions $\Lambda$. At weak coupling $\xi \gg \Lambda^2$. The Lüscher coefficient measures the number of light degrees of freedom on the string worldsheet. The main feature of the non-Abelian strings is the presence of orientational zero modes associated with rotation of their color flux inside a non-Abelian subgroup. Unlike the standard $\pi/12$ we get $N \pi/12$ for the non-Abelian strings in the U($N$) gauge theories. Thus, the Lüscher coefficient acquires a dependence on the rank of the gauge group. In fact, the Lüscher term for non-Abelian strings experiences a jump: at $\xi^{-1/2} \ll L \ll \Lambda^{-1}$ it is $N \pi/12$ while at $L \gg \Lambda^{-1}$ the orientational moduli are frozen out and the Lüscher coefficient approaches its “Lüscher” value $\pi/12$.

At the end of this paper we discuss the question of possible extra (i.e. non-translational) light moduli on the worldsheet of QCD strings at large $N$.

### 2 A model supporting non-Abelian strings

The model discussed in [6] is a U($N$) gauge theory with $N$ flavors of complex scalars (squarks) $\varphi^A$ ($A = 1, \ldots, N$) in the fundamental representation of the gauge group. The action of this model is

$$S = \int d^4x \left\{ \frac{1}{4g_2^2} (F_{\mu\nu}^a)^2 + \frac{1}{4g_1^2} (F_{\mu\nu})^2 + |\nabla^\mu \varphi^A|^2 + \frac{g_2^2}{2} (\bar{\varphi}_A T^a \varphi^A)^2 + \frac{g_1^2}{8} (|\varphi^A|^2 - N\xi)^2 \right\}, \quad (3)$$

where

$$\nabla^\mu = \partial^\mu - i \frac{1}{2} A^\mu + i T^a A^a_\mu,$$

$A^\mu$ and $A_\mu^a$ are the U(1) and SU($N$) gauge fields, respectively; $T^a$ are the SU($N$) generators, while $g_1^2$ and $g_2^2$ are the U(1) and SU($N$) gauge couplings.

Squark fields develop vacuum expectation values (VEV’s) of the color-flavor locked form

$$\langle q^{kA} \rangle = \sqrt{\xi} \begin{pmatrix} 1 & \ldots & 0 \\ \ldots & \ldots & \ldots \\ 0 & \ldots & 1 \end{pmatrix},$$
where the squark fields are written as an $N \times N$ matrix in the color-flavor indices. The VEV’s \textit{(4)} spontaneously break both the gauge and flavor SU($N$)'s. A diagonal global SU($N$) survives, however,

\[ U(N)_{\text{gauge}} \times SU(N)_{\text{flavor}} \to SU(N)_{C+F}. \]

Thus, a color-flavor locking takes place in the vacuum. A version of this pattern of the symmetry breaking was suggested long ago \textit{[7]}. Since the gauge symmetry is broken, both the U(1) and SU($N$) gauge bosons acquire masses

\[ m_{U(1)} = g_1 \sqrt{\xi}, \quad m_{SU(N)} = g_2 \sqrt{\xi}, \]

respectively.

This model supports string solutions \textit{[6]}, which break global SU($N$)$_{C+F}$ symmetry present in the vacuum down to SU($N-1$) × U(1). This ensures appearance of orientational zero modes of the string making it non-Abelian. The phenomenon is quite similar to supersymmetric non-Abelian strings found in $\mathcal{N} = 2$ supersymmetric gauge theories \textit{[2]}. The tension of the elementary string is determined by the squark VEV’s \textit{[6]},

\[ T_{c_1} = 2\pi \xi, \]

where the subscript cl marks the classical approximation. The orientational moduli belong to the quotient

\[ CP(N-1) = \frac{SU(N)_{C+F}}{SU(N-1) \times U(1)}. \]

The low-energy effective theory on the string worldsheet is given by the $CP(N-1)$ model with the action

\[ S^{(1+1)} = \int d^2x \left\{ \frac{T_{c_1}}{2} (\partial_k z^i)^2 + 2\beta \left| \nabla_k n^l \right|^2 \right\}, \]

(see \textit{[6]} or the review paper \textit{[8]} for derivation). Here $k = 1, 2$ labels the worldsheet coordinates, $z^i$, $i = 1, 2$ are two real translational moduli (the string position in the plane orthogonal to the string) and $N$ complex fields $n^l$ ($l = 1, ..., N$) subject to the constraint $|n^l|^2 = 1$ denote orientational moduli. The axillary U(1) gauge field gauging the common U(1) phase of $n^l$ enters
without kinetic energy, and $\nabla_k = \partial_k - i A_k$. The two-dimensional coupling $\beta$ is related to the bulk coupling, $\beta = 2\pi/g_2^2$ at the scale $\sqrt{\xi}$. Overall we have $2N - 1 - 1 = 2(N - 1)$ real orientational moduli — we subtracted one constraint and one “eaten” phase. This is the number of degrees of freedom in the $CP(N - 1)$ model.

This example taught us that, besides two translational gapless excitations, other light modes associated with internal degrees of freedom, can exist on the string worldsheet. The $CP(N - 1)$ model part of the worldsheet theory gives rise to the scale parameter $\Lambda$ which determines the mass gap for orientational moduli. The tension of the string $\xi$ is much larger than $\Lambda^2$. This implies that there exists a window of distances,

$$\xi^{-1/2} \ll L \ll \Lambda^{-1}$$

in which all $2N$ moduli characterizing the non-Abelian string — two translational and $2N - 2$ orientational — can be considered as massless. Correspondingly, in this window the Lüscher coefficient will take the value

$$\gamma = \frac{N\pi}{12}.$$  

In the transitional domain $L \sim \Lambda^{-1}$ it must smoothly decrease eventually tending to $\langle 2 \rangle$ at $L \gg \Lambda^{-1}$.

### 3 The Lüscher coefficient

To derive (11) we calculate the quantum energy of the string of the length $L$ with nailed ends. We have

$$E_{\text{qu}} = \sum_k N \frac{\pi k}{L},$$

where the subscript qu indicates the inclusion of quantum corrections (we are interested only in the infrared corrections). The above expression is divergent. To regularize it we introduce a “lattice” spacing $\epsilon$ ($1/\epsilon$ is the ultraviolet (UV) cutoff). Then we have

$$E_{\text{qu}} = \sum_k N \frac{\pi k}{L} \exp \left( - \frac{\pi k}{L} \frac{1}{\epsilon} \right)$$

$$= \frac{N}{\pi \epsilon^2} L - \frac{N\pi}{12} \frac{1}{L} + O(\epsilon).$$


What is the meaning of the first divergent term here? It is easy to see that it presents renormalization of the classical string tension (7). To phrase it differently, the $O(1/\epsilon^2)$ term is the one-loop contribution in the vacuum energy in the worldsheet theory (9). The $CP(N-1)$ model (9) is an effective theory which describes dynamics of the string light modes at energies below the inverse thickness of the string given by the bulk theory masses (6). Thus the UV cutoff $1/\epsilon$ in this theory must be of the order of $g\sqrt{\xi}$, where we assume that both coupling constants in (3) are of the same order, $g_1^2 \sim g_2^2 \sim g^2$. Thus, we conclude that the total string energy is

$$E = \left(2\pi \xi + C g^2 \xi\right) L - \frac{N\pi}{12} \frac{1}{L} + O(1/L^2).$$

We can use this result at $L$ much larger than the string thickness, $L \gg 1/(g\sqrt{\xi})$. It has a different $L$ dependence as compared to the leading linear in $L$ term. Therefore, it makes sense to trace this term. At $L \gg \Lambda^{-1}$ the orientational degrees of freedom freeze out and no longer contribute to the string energy.

In fact, the Lüscher coefficient in non-Abelian strings can follow even a richer pattern of behavior. Indeed, the models [2] admit another dimensional parameter in the bulk $\Delta m$. This parameter manifests itself on the world sheet as a twisted mass. Instead of $CP(N-1)$ model for orientational moduli, we get $CP(N-1)$ with the twisted mass. If

$$\xi \gg |\Delta m| \gg \Lambda$$

the orientational modes acquire the mass gap equal to $|\Delta m|$ (see [2, 6]). This implies, in turn, that the window (10) is divided into two sub-windows, $[\xi^{-1/2}, |\Delta m|^{-1}]$ and $[|\Delta m|^{-1}, \Lambda^{-1}]$. In the first sub-window the Lüscher coefficient is given by (11) while in the second by (2). This parameter can be adjusted at will. As $|\Delta m| \to 0$ the second sub-window shrinks to nothing.

4 QCD strings

The most intriguing question is whether or not the QCD string and strings in other QCD-like theories has only two translational massless moduli on its worldsheet. Of course, at $N = 3$ the emergence of other light modes (with the mass mass gap $\ll \Lambda$) does not seem likely since the only dimensional parameter in this case is $\Lambda$. Moreover, lattice numerical data (see e.g. Greensite’s
review quoted in \[1\]) support the Lüscher value \(2\). However, at \(N \gg 1\) the answer does not seem so obvious. A priori it is not ruled out that in the multicolor limit the QCD string acquires an analog of the orientational moduli of Refs. \[2, 6\] with the mass gap suppressed by powers of \(1/N\). A heuristic motivation is provided by a consideration on \(R_3 \times S_1\) similar to \[5\].

If the value of the \(S_1\) radius \(r\) is small compared to \(\Lambda^{-1}\), we deal with \(N - 1\) distinct Abelian strings which must fuse into a single non-Abelian string at \(r > \Lambda^{-1}\). Additional (quasi) moduli might occur in the process of fusion.

Light modes are definitely present on the worldsheet of \(k\)-strings \[9\]. The issue of existence/nonexistence of “light” modes of the fundamental QCD strings, unrelated to the excitations of the internal flux-tube structure, is hard to investigate in a model-independent way since the underlying dynamics is that of strong coupling. The answer could be provided by precision lattice measurements in the multicolor Yang–Mills. The question of feasibility of such measurements remains open.

## 5 Discussion

A standard presumption in string theory is that the theory on the string worldsheet must be conformal. This assumption is also applied to long-distance dynamics of string solitons in four-dimensional field theories, see e.g. \[10\]. Non-Abelian strings discussed in this paper present a clear-cut counterexample. The low-energy dynamics on the string worldsheet is non-conformal because so is the \(CP(N - 1)\) model. The total number of the moduli fields is \(2N\) rather than two translational moduli. This implies that the standard Lüscher coefficient \(\pi/12\) changes to \(N\pi/12\) in the window \(\text{[10]}\).

An open and intriguing question is whether QCD strings at large \(N\) acquire light (quasi)moduli additional with regards to two translational moduli.

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References

[1] M. Lüscher, Nucl. Phys. B 180, 317 (1981); see also O. Alvarez, Phys. Rev. D 24, 440 (1981); for a review see J. Greensite, Prog. Part. Nucl. Phys. 51, 1 (2003) [arXiv:hep-lat/0301023].

[2] A. Hanany and D. Tong, JHEP 0307, 037 (2003) [hep-th/0306150]; R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, Nucl. Phys. B 673, 187 (2003) [hep-th/0307287]; M. Shifman and A. Yung, Phys. Rev. D 70, 045004 (2004) [hep-th/0403149]; A. Hanany and D. Tong, JHEP 0404, 066 (2004) [hep-th/0403158].

[3] A. Abrikosov, Sov. Phys. JETP 32 1442 (1957) [Reprinted in Solitons and Particles, Eds. C. Rebbi and G. Soliani (World Scientific, Singapore, 1984), p. 356]; H. Nielsen and P. Olesen, Nucl. Phys. B61 45 (1973) [Reprinted in Solitons and Particles, Eds. C. Rebbi and G. Soliani (World Scientific, Singapore, 1984), p. 365].

[4] N. Seiberg and E. Witten, Nucl. Phys. B 426, 19 (1994) [Erratum-ibid. B 430, 485 (1994)] [arXiv:hep-th/9407087].

[5] A. Armoni, M. Shifman and M. Ünsal, Planar Limit of Orientifold Field Theories and Emergent Center Symmetry, [arXiv:0712.0672 [hep-th]].

[6] A. Gorsky, M. Shifman and A. Yung, Phys. Rev. D 71, 045010 (2005) [arXiv:hep-th/0412082].

[7] K. Bardakci and M. B. Halpern, Phys. Rev. D 6, 696 (1972).

[8] M. Shifman and A. Yung, Supersymmetric Solitons, Rev. Mod. Phys. 79 1139 (2007) [arXiv:hep-th/0703267].

[9] A. Armoni and M. Shifman, Nucl. Phys. B 671, 67 (2003) [arXiv:hep-th/0307020].

[10] J. Polchinski and A. Strominger, Phys. Rev. Lett. 67, 1681 (1991).