GLOBAL SOLVABILITY OF 3D INHOMOGENEOUS NAVIER-STOKES EQUATIONS WITH DENSITY-DEPENDENT VISCOSITY

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Abstract. In this paper, we consider the three-dimensional inhomogeneous Navier-Stokes equations with density-dependent viscosity in presence of vacuum over bounded domains. Global-in-time unique strong solution is proved to exist when $\|\nabla u_0\|_{L^2}$ is suitably small with arbitrary large initial density. This generalizes all the previous results even for the constant viscosity.

Keywords: density-dependent viscosity, inhomogeneous Navier-Stokes equations, strong solution, vacuum.

AMS: 35Q35, 35B65, 76N10

1. Introduction

The Navier-Stokes equations are usually used to describe the motion of fluids. In particular, for the study of multiphase fluids without surface tension, the following density-dependent Navier-Stokes equations acts as a model on some bounded domain $\Omega \subset R^N (N = 2, 3)$,

\[
\begin{aligned}
\rho_t + \text{div} (\rho u) &= 0, \quad \text{in } \Omega \times (0, T], \\
(\rho u)_t + \text{div} (\rho u \otimes u) - \text{div} (2\mu(\rho)d) + \nabla P &= 0, \quad \text{in } \Omega \times (0, T], \\
\text{div} u &= 0, \quad \text{in } \Omega \times [0, T], \\
\rho|_{t=0} &= \rho_0, \quad u|_{t=0} = u_0, \quad \text{in } \Omega.
\end{aligned}
\]

Here $\rho, u, P$ denote the density, velocity and pressure of the fluid, respectively.

\[d = \frac{1}{2} [\nabla u + (\nabla u)^T]\]

is the deformation tensor.

$\mu = \mu(\rho)$ states the viscosity and is a function of $\rho$, which is assumed to satisfy

\[\mu \in C^1[0, \infty), \quad \mu \geq \underline{\mu} > 0 \quad \text{on } [0, \infty) \quad \text{for some positive constant } \underline{\mu}.\]

In this paper, we study the initial boundary value problem to the system (1.1)-(1.2).

The mathematical study for nonhomogeneous incompressible flow was initiated by the Russian school. They studied the case that $\mu(\rho)$ is a constant and the initial density $\rho_0$ is bounded away from $0$. In the absence of vacuum, global existence of weak solutions as well as local strong solution was established by Kazhikov [4, 21]. The uniqueness of local strong solutions was first established by Ladyzhenskaya-Solonnikov [22] for the initial boundary-value problem, see also [25]. Furthermore, unique local strong solution is proved to be global in
2D \cite{26}. In recent years, Danchin initiated the studies for solutions in critical spaces. He \cite{9,10} derived the global well-posedness for small initial velocity in critical spaces, where density is close to a constant. For some subsequent works, refer to \cite{1,24} and references therein. We remark that in the very interesting papers \cite{11,12}, Danchin-Mucha studied the case for which density is piecewise constant, see also some generalizations in 2D \cite{18}.

When initial vacuum is taken into account and $\mu(\rho)$ is still a constant, Simon \cite{26} proved the global existence of weak solutions. Later, Choe-Kim \cite{7} proposed a compatibility condition as (1.4) below to establish local existence of strong solution. Global strong solution allowing vacuum in 2D was recently derived by the authors \cite{20}. Meanwhile, some global solutions in 3D with small critical norms have been constructed, refer to the results in \cite{2,8} and references therein.

Finally, we come to the most general case: viscosity $\mu(\rho)$ depends on density $\rho$. Most results were concentrated on 2D case. Global weak solutions were derived by the revolutionary work \cite{14,23} of DiPerna and Lions. Later, Desjardins \cite{13} proved the global weak solution with more regularity for the two-dimensional case provided that the viscosity function $\mu(\rho)$ is a small pertubation of a positive constant in $L^\infty$- norm. Very recently, Abidi-Zhang \cite{3} generalized this 2D result to strong solutions. Regarding the 3D case, Cho-Kim \cite{6} constructed a unique local strong solution by imposing some initial compatibility condition. Their result is stated as follows:

**Theorem 1.1.** Assume that the initial data $(\rho_0, u_0)$ satisfies the regularity condition

\begin{equation}
0 \leq \rho_0 \in W^{1,q}, \quad 3 < q < \infty, \quad u_0 \in H^1_{0,\sigma} \cap H^2,
\end{equation}

and the compatibility condition

\begin{equation}
- \text{div} (\mu(\rho_0) \left[ \nabla u_0 + (\nabla u_0)^T \right]) + \nabla P_0 = \rho_0^{\frac{4}{3}} g,
\end{equation}

for some $(P_0, g) \in H^1 \times L^2$. Then there exists a small time $T$ and a unique strong solution $(\rho, u, P)$ to the initial boundary value problem (1.1) such that

\begin{align*}
\rho &\in C([0, T]; W^{1,q}), \quad \nabla u, P \in C([0, T]; H^1) \cap L^2(0, T; W^{1,r}), \\
\rho_t &\in C([0, T]; L^r), \quad \sqrt{\rho} u_t \in L^\infty(0, T; L^2), \quad u_t \in L^2(0, T; H^1_{0}),
\end{align*}

for any $r$ with $1 \leq r < q$. Furthermore, if $T^*$ is the maximal existence time of the local strong solution $(\rho, u)$, then either $T^* = \infty$ or

\begin{equation}
\sup_{0 \leq t < T^*} (\|\nabla \rho(t)\|_{L^3} + \|\nabla u(t)\|_{L^2}) = \infty.
\end{equation}

Motivated by the global existence result \cite{8} for the special case that $\mu$ is a constant, we aim to establish global well-posedness result for variable coefficient case. However, due to the strong coupling between viscosity coefficient and density, it’s more complicated and involved with variable coefficient $\mu(\rho)$ and requires more delicate analysis.

Our main result proves the existence of global strong solution, provided $\|\nabla u_0\|_{L^2}$ is suitably small. To conclude, we arrive at
Theorem 1.2. Assume that the initial data \((\rho_0, u_0)\) satisfies (1.3)-(1.4), and \(0 \leq \rho_0 \leq \overline{\rho}\). Then there exists some small positive constant \(\varepsilon_0\), depending on \(\Omega, q, \overline{\rho}, \underline{\rho} = \sup_{[0, \overline{\rho}]} \mu(\rho), \mu, \|\nabla \mu(\rho_0)\|_{L^q}(= M)\), such that if
\[
\|\nabla u_0\|_{L^2} \leq \varepsilon_0,
\]
then the initial boundary value problem (1.1) admits a unique global strong solution \((\rho, u)\), with
\[
\rho \in C([0, \infty); W^{1,q}), \quad \nabla u, P \in C([0, \infty); H^1) \cap L^2_{loc}(0, \infty; W^{1,r}),
\]
\[
\rho_t \in C([0, \infty); L^q), \quad \sqrt{\rho} u_t \in L^\infty_{loc}(0, \infty; L^2), \quad u_t \in L^2_{loc}(0, \infty; H^1_0),
\]
for any \(r, 1 \leq r < q\).

The main idea is to combine techniques developed by the authors in [8,19] and time weighted energy estimates successfully applied to compressible Navier-Stokes equation by Hoff [17].

Let’s briefly sketch the proof. First we assume that \(\|\nabla \mu(\rho)\|_{L^q}\) is less than \(4M\) and \(\|\nabla u\|_{L^2}^2\) is less than \(4\|\nabla u_0\|_{L^2}^2\) on \([0, T]\), then we prove that in fact \(\|\nabla \mu(\rho)\|_{L^q}\) is less than \(2M\) and \(\|\nabla u\|_{L^2}^2\) is less than \(2\|\nabla u_0\|_{L^2}^2\) on \([0, T]\), under the assumption \(\|\nabla u_0\|_{L^2} \leq \varepsilon_0 \leq \frac{1}{2}\). On the other hand, the control of \(\|\nabla \mu(\rho)\|_{L^q}\) and \(\|\nabla u\|_{L^2}\) lead to uniform estimates for other quantities, which guarantee the extension of local strong solutions. All the above procedures depends on a time independent bound of \(\|\nabla u\|_{L^1 L^\infty}\). Thanks to bounded domain, \(u\) indeed has exponentially decay rather than insufficient polynomial decay for the whole space, that’s the main reason why we can only treat system (1.1) in bounded domain.

Remark 1.1. Theorem 1.2 also hold for 2D case. Since the proof is quite similar, we omit it for simplicity.

The rest of the paper is organized as follows: Section 2 consists of some notations, definitions, and basic lemmas. Section 3 is devoted to the proof of Theorem 1.2.

2. Preliminaries

\(\Omega\) is a bounded smooth domain in \(\mathbb{R}^3\). Denote
\[
\int f \, dx = \int_\Omega f \, dx.
\]
For \(1 \leq r \leq \infty\) and \(k \in \mathbb{N}\), the Sobolev spaces are defined in a standard way,
\[
L^r = L^r(\Omega), \quad W^{k,r} = \left\{ f \in L^r : \nabla^k f \in L^r \right\},
\]
\[
H^k = W^{k,2}, \quad C^\infty_{0,\sigma} = \left\{ f \in C^\infty_0 : \text{div } f = 0 \right\}.
\]
\[
H^1 = \overline{C^\infty_0}, \quad H^1_{0,\sigma} = \overline{C^\infty_{0,\sigma}}, \text{ closure in the norm of } H^1.
\]

High-order a priori estimates rely on the following regularity results for density-dependent Stokes equations.
Lemma 2.1. Assume that $\rho \in W^{1,q}$, $3 < q < \infty$, and $0 \leq \rho \leq \bar{\rho}$. Let $(u, P) \in H^1_0 \times L^2$ be the unique weak solution to the boundary value problem

\begin{equation}
- \text{div} \ (2\mu(\rho)d) + \nabla P = F, \quad \text{div} \ u = 0 \quad \text{in} \ \Omega, \quad \text{and} \quad \int \frac{P}{\mu(\rho)} \ dx = 0,
\end{equation}

where $d = \frac{1}{2} [\nabla u + (\nabla u)^T]$ and

$$\mu \in C^1[0, \infty), \quad \underline{\mu} \leq \mu(\rho) \leq \bar{\mu} \quad \text{on} \ [0, \bar{\rho}].$$

Then we have the following regularity results:

1. If $F \in L^2$, then $(u, P) \in H^2 \times H^1$ and

\begin{equation}
\|u\|_{H^2} + \|P/\mu(\rho)\|_{H^1} \leq C \left( \frac{1}{\underline{\mu}} + \frac{\bar{\mu}}{\mu^{\frac{1}{\theta_2}}} \right) \|\nabla \mu\|_{L^q} \|F\|_{L^2},
\end{equation}

where $\theta_2$ satisfies

$$\frac{1}{2} - \frac{1}{q} = \frac{\theta_2}{3} + \frac{1}{6}, \quad \text{i. e.,} \quad \theta_2 = \frac{q}{q - 3}.$$

2. If $F \in L^r$ for some $r \in (2, q)$, then $(u, P) \in W^{2,r} \times W^{1,r}$ and

\begin{equation}
\|u\|_{W^{2,r}} + \|P/\mu(\rho)\|_{W^{1,r}} \leq C \left( \frac{1}{\underline{\mu}} + \frac{\bar{\mu}}{\mu^{\frac{1}{\theta_r}}} \right) \|\nabla \mu\|_{L^q} \|F\|_{L^r},
\end{equation}

where

$$\frac{1}{\theta_r} = \frac{\frac{5}{6} - \frac{1}{r}}{\frac{1}{3} - \frac{1}{q}}.$$

Here the constant $C$ in (2.9) and (2.10) depends on $\Omega$, $q$, $r$.

The proof of Lemma 2.1 has been given in [6], although the lemma is slightly different from the version in [6]. We sketch it here for completeness.

Proof. For the existence and uniqueness of the solution, please refer to Giaquinta-Modica [16]. We give the a priori estimates here. Assume that $F \in L^2$. Multiply the first equation of (2.8) by $u$ and integrate over $\Omega$, then by Poincaré’s inequality,

$$\int 2\mu(\rho)|d|^2 \ dx = \int F \cdot u \ dx \leq \|F\|_{L^2} \cdot \|u\|_{L^2} \leq C \|F\|_{L^2} \cdot \|\nabla u\|_{L^2}.$$

Note that

$$2 \int |d|^2 \ dx = \int |\nabla u|^2 \ dx,$$

hence

\begin{equation}
\|\nabla u\|_{L^2} \leq C \mu^{-1} \|F\|_{L^2}.
\end{equation}

Since $\int \frac{P}{\mu(\rho)} \ dx = 0$, according to Bovosgii’s theory, there exists a function $v \in H^1_0$, such that

$$\text{div} \ v = \frac{P}{\mu(\rho)},$$
and
\[ \| \nabla v \|_{L^2} \leq C \left\| \frac{P}{\mu(\rho)} \right\|_{L^2}. \]

Multiply the first equation of (2.8) by \(-v\), and integrate over \(\Omega\), then
\[
\int \frac{P^2}{\mu(\rho)} \, dx = - \int F \cdot v \, dx + 2 \int \mu(\rho) d : \nabla v \, dx \\
\leq \| F \|_{L^2} \cdot \| v \|_{L^2} + C \mu \cdot \| \nabla u \|_{L^2} \cdot \| \nabla v \|_{L^2} \\
\leq C \| F \|_{L^2} \cdot \| \nabla v \|_{L^2} + C \frac{\mu}{\mu} \cdot \| F \|_{L^2} \cdot \| \nabla v \|_{L^2} \\
\leq C \frac{\mu}{\mu} \cdot \| F \|_{L^2} \cdot \left\| \frac{P}{\mu(\rho)} \right\|_{L^2}.
\]

On the other hand side,
\[
\int \frac{P^2}{\mu(\rho)} \, dx \geq \mu \int \frac{P^2}{\mu(\rho)^2} \, dx.
\]
Hence,
\[ (2.12) \quad \left\| \frac{P}{\mu(\rho)} \right\|_{L^2} \leq C \frac{\mu}{\mu} \cdot \| F \|_{L^2}. \]

The first equation of (2.8) can be re-written as
\[
- \Delta u + \nabla \left( \frac{P}{\mu(\rho)} \right) = \frac{F}{\mu(\rho)} + \frac{2d \cdot \nabla \mu(\rho)}{\mu(\rho)} - \frac{P \nabla \mu(\rho)}{\mu(\rho)^2}.
\]
By virtue of the classical theory for Stokes equations and Gagliardo-Nirenberg inequality, we have
\[
\| u \|_{H^2} + \left\| \nabla \left( \frac{P}{\mu(\rho)} \right) \right\|_{L^2} \\
\leq C \left( \left\| \frac{F}{\mu(\rho)} \right\|_{L^2} + \left\| \frac{d \cdot \nabla \mu(\rho)}{\mu(\rho)} \right\|_{L^2} + \left\| \frac{P \nabla \mu(\rho)}{\mu(\rho)^2} \right\|_{L^2} \right) \\
\leq C \left( \mu^{-1} \| F \|_{L^2} + \mu^{-1} \| \nabla \mu \|_{L^\infty} \cdot \| \nabla u \|_{L^\frac{2q}{q-2}} \right) + \mu^{-1} \| \nabla \mu \|_{L^q} \cdot \left\| \frac{P}{\mu} \right\|_{L^\frac{2q}{q-2}} \\
\leq C \left( \mu^{-1} \| F \|_{L^2} + \mu^{-1} \| \nabla \mu \|_{L^\infty} \cdot \| \nabla u \|_{L^\frac{2q}{q-2}} \right) + \mu^{-1} \| \nabla \mu \|_{L^q} \cdot \left\| \frac{P}{\mu} \right\|_{L^2} \cdot \| \nabla u \|_{H^1}^{1-\theta_2} \\
+ \mu^{-1} \| \nabla \mu \|_{L^q} \cdot \left\| \frac{P}{\mu} \right\|_{L^2} \cdot \left\| \nabla \left( \frac{P}{\mu} \right) \right\|_{L^2}^{1-\theta_2}.\]

By Young’s inequality,

\begin{equation}
\|u\|_{H^2} + \left\| \nabla \left( \frac{P}{\mu(\rho)} \right) \right\|_{L^2} \\
\leq C \mu^{-1} \|F\|_{L^2} + C \mu^{-\frac{1}{q_2}} \|\nabla \mu\|_{L^q}^{\frac{1}{2}} \cdot \left( \|\nabla u\|_{L^2} + \frac{\|P\|_{L^2}}{\mu} \right) \\
\leq C \mu^{-1} \|F\|_{L^2} + C \mu^{-\frac{1}{q_2}} \cdot \left( 1 + \frac{\mu}{P} \right) \cdot \|\nabla \mu\|_{L^q}^{\frac{1}{2}} \cdot \|F\|_{L^2} \\
\leq C \left( \frac{1}{\mu} \right. + \frac{\mu}{\mu_{\frac{1}{q_2}}+2} \|\nabla \mu\|_{L^q}^{\frac{1}{2}} \left. \right) \|F\|_{L^2},
\end{equation}

where \(\theta_2\) satisfies

\[
\frac{1}{2} - \frac{1}{q} = \frac{\theta_2}{3} + \frac{1}{6}, \quad \text{or} \quad \theta_2 = \frac{q}{q - 3}.
\]

Similarly,

\begin{equation}
\|u\|_{W^{2,r}} + \|\nabla \left( \frac{P/\mu(\rho)}{\rho} \right)\|_{L^r} \leq C \left( \frac{1}{\mu} \right. + \frac{\mu}{\mu_{\frac{1}{q}}+2} \|\nabla \mu\|_{L^q}^{\frac{1}{2}} \left. \right) \|F\|_{L^r},
\end{equation}

where

\[
\frac{1}{\theta_r} = \frac{\frac{5}{6} - \frac{1}{q}}{\frac{1}{3} - \frac{1}{q}}.
\]

3. Proof of Theorem 1.2

The proof of Theorem 1.2 is composed of two parts. The first part contains a priori time-weighted estimates of different levels. Upon these estimates, the second part uses a contradiction induction process to extend the local strong solution. The two parts are presented in Subsections 3.1 and 3.2, respectively.

3.1. A Priori Estimates. In this subsection, we establish some a priori time-weighted estimates. The initial velocity belongs to \(H^1\), but some uniform estimates of higher order and independent of time are required. To achieve that, we take some power of time as a weight. The idea is based on the parabolic property of the system. In this subsection, the constant \(C\) will denote some positive constant which maybe dependent on \(\Omega, q\), but is independent of \(\rho_0\) or \(u_0\).

First, as the density satisfies the transport equation \((1.1)_{1}\) and making use of \((1.1)_{3}\), one has the following lemma.

**Lemma 3.1.** Suppose \((\rho, u, P)\) is the unique local strong solution to \((1.1)\) on \([0, T]\), with the initial data \((\rho_0, u_0)\), it holds that

\[
0 \leq \rho(x, t) \leq \bar{\rho}, \quad \text{for every} \quad (x, t) \in \Omega \times [0, T].
\]

Next, the basic energy inequality of the system \((1.1)\) reads
**Theorem 3.2.** Suppose \((\rho, u, P)\) is the unique local strong solution to (1.1) on \([0, T]\), with the initial data \((\rho_0, u_0)\), it holds that

\[
\int_0^t \rho |u(t)|^2 \, dx + \int_0^t \mu(|\rho|) |d|^2 \, dx \leq C \cdot \bar{\rho} \cdot \|u_0\|_{L^2}^2, \quad \text{for every } t \in [0, T],
\]
or in other words,

\[
\int_0^t \rho |u(t)|^2 \, dx + \mu \int_0^t |\nabla u|^2 \, dx \leq C \cdot \bar{\rho} \cdot \|u_0\|_{L^2}^2, \quad \text{for every } t \in [0, T].
\]

**Proof.** The proof is standard. Multiplying the momentum equation by \(u\) and integrating over \(\Omega\) yield that

\[
\frac{1}{2} \frac{d}{dt} \int \rho |u|^2 \, dx + 2 \int \mu(|\rho|) |d|^2 \, dx = 0.
\]

Then (3.16) is true owing to the fact \(2 \int |d|^2 \, dx = \int |\nabla u|^2 \, dx\) and \(\mu(|\rho|) \geq \underline{\mu}\). \(\square\)

Denote

\[
M = \|\nabla \mu(\rho_0)\|_{L^q},
\]

\[
M_2 = \frac{1}{\underline{\mu}} + \frac{\overline{\mu}}{\underline{\mu}} \cdot \frac{1}{\mu^{1/\theta_2 + 1} \cdot (4M)^{\frac{1}{2}}},
\]

and

\[
M_r = \frac{1}{\underline{\mu}} + \frac{\overline{\mu}}{\underline{\mu}} \cdot \frac{1}{\mu^{1/\theta_r + 1} \cdot (4M)^{\frac{1}{r}}}.
\]

**Theorem 3.3.** Suppose \((\rho, u, P)\) is the unique local strong solution to (1.1) on \([0, T]\), with the initial data \((\rho_0, u_0)\), and satisfies

\[
\sup_{t \in [0, T]} \|\nabla \mu(\rho(t))\|_{L^q} \leq 4M,
\]

and

\[
\sup_{t \in [0, T]} \|\nabla u(t)\|_{L^2} \leq 4\|\nabla u_0\|_{L^2} \leq 1.
\]

There exists a positive number \(C_1\), depending on \(\Omega, q\) such that if

\[
C_1 \underline{\mu}^{-2} (M_2^2 + M^4 M_2^6) \bar{\rho}^4 \cdot \|\nabla u_0\|_{L^2}^2 \leq \ln 2,
\]

then

\[
\frac{1}{\underline{\mu}} \int_0^T \|\sqrt{\rho} u_t\|_{L^2}^2 \, dt + \sup_{t \in [0, T]} \|\nabla u(t)\|_{L^2}^2 \leq 2\|\nabla u_0\|_{L^2}^2.
\]

Before the proof of Theorem 3.3, let us introduce an auxiliary lemma, which is a result of the \(W^{2,2}\)-estimates in Lemma 2.1.

**Lemma 3.4.** Suppose \((\rho, u, P)\) is the unique local strong solution to (1.1) on \([0, T]\) and satisfies

\[
\sup_{t \in [0, T]} \|\nabla \mu(\rho(t))\|_{L^q} \leq 4M.
\]

Then it holds that

\[
\|\nabla u\|_{H^1} \leq C M_2 \|\rho u_t\|_{L^2} + C M_2^2 \cdot \bar{\rho}^2 \cdot \|\nabla u\|_{L^2}^2.
\]
Proof. The momentum equation can be rewritten as follows,

\begin{equation}
-2\text{div} \left( \mu(\rho) d \right) + \nabla P = -\rho u_t - (\rho u \cdot \nabla) u.
\end{equation}

It follows from Lemma 2.1 and Gagliardo-Nirenberg inequality that

\[ \|\nabla u\|_{H^1} \leq CM_2 \left( \|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2} \right) \]
\[ \leq CM_2 \|\rho u_t\|_{L^2} + CM_2 \cdot \overline{\rho} \cdot \|u\|_{L^6} \cdot \|\nabla u\|_{L^3} \]
\[ \leq CM_2 \|\rho u_t\|_{L^2} + CM_2 \cdot \overline{\rho} \cdot \|\nabla u\|_{L^2}^{\frac{1}{2}} \cdot \|\nabla u\|_{H^1}^{\frac{1}{2}}. \]

By Young’s inequality,

\[ \|\nabla u\|_{H^1} \leq CM_2 \|\rho u_t\|_{L^2} + CM_2 \cdot \overline{\rho} \cdot \|\nabla u\|_{L^2} \]

\[ \Box \]

Proof of Theorem 3.3. Multiply the momentum equation by \( u_t \) and integrate over \( \Omega \), then

\[ \int \rho |u_t|^2 \, dx + \frac{d}{dt} \int \mu(\rho) |d|^2 \, dx \]
\[ \leq \left| \int \rho u \cdot \nabla u \cdot u_t \, dx \right| + C \int |\nabla \mu(\rho)| \cdot |u| \cdot |\nabla u|^2 \, dx. \]

Here we used the renormalized mass equation for \( \mu(\rho) \),

\[ \partial_t [\mu(\rho)] + u \cdot \nabla \mu(\rho) = 0, \]

which is derived due to the fact \( \text{div} \ u = 0 \).

Applying Gagliardo-Nirenberg inequality and Lemma 3.4

\[ \left| \int \rho u \cdot \nabla u \cdot u_t \, dx \right| \]
\[ \leq \frac{1}{8} \|\sqrt{\rho} u_t\|^2_{L^2} + C \|\sqrt{\rho} u_t\|^2_{L^6} \cdot \|\nabla u\|^2_{L^3} \]
\[ \leq \frac{1}{8} \|\sqrt{\rho} u_t\|^2_{L^2} + C \overline{\rho} \cdot \|\nabla u\|^3_{L^2} \cdot \|\nabla u\|_{H^1} \]
\[ \leq \frac{1}{8} \|\sqrt{\rho} u_t\|^2_{L^2} + C \overline{\rho} \cdot \|\nabla u\|^3_{L^2} \cdot \left[ CM_2 \|\rho u_t\|_{L^2} + CM_2^2 \overline{\rho}^2 \|\nabla u\|_{L^2} \right], \]

and similarly,

\[ C \int |\nabla \mu(\rho)| \cdot |u| \cdot |\nabla u|^2 \, dx \]
\[ \leq C \|\nabla \mu(\rho)\|_{L^3} \cdot \|u\|_{L^6} \cdot \|\nabla u\|^2_{L^4} \]
\[ \leq C \|\nabla \mu(\rho)\|_{L^3} \cdot \|\nabla u\|^2_{L^2} \cdot \|\nabla u\|_{H^1}^{\frac{1}{2}} \]
\[ \leq CM \|\nabla u\|^3_{L^2} \cdot \left[ CM_2 \|\rho u_t\|_{L^2} + CM_2^2 \overline{\rho}^2 \|\nabla u\|_{L^2} \right]. \]
Hence, by Young’s inequality,
\[
\int \rho |u_t|^2 \, dx + \frac{d}{dt} \int \mu(\rho)|d|^2 \, dx 
\leq \frac{1}{8} \|\sqrt{\rho} u_t\|^2_{L^2} + \frac{1}{8} \|\sqrt{\rho} u_t\|^2_{L^2} + CM_2^2 \bar{\mu}^3 \|\nabla u\|^6_{L^2} + \frac{1}{8} \|\sqrt{\rho} u_t\|^2_{L^2} 
+ C \left( M_2^2 \cdot \bar{\mu}^3 \|\nabla u\|^2_{L^2} \right)^4 + CM_2^3 \cdot \|\nabla u\|^6_{L^2} 
\leq \frac{3}{8} \|\sqrt{\rho} u_t\|^2_{L^2} + C \left( M_2^2 + M_4 M_2^6 \right) \cdot \bar{\mu}^3 \cdot \|\nabla u\|^6_{L^2}.
\]
So we have
\[
(3.22) \quad \int \rho |u_t|^2 \, dx + \frac{d}{dt} \int \mu(\rho)|d|^2 \, dx \leq C \left( M_2^2 + M_4 M_2^6 \right) \cdot \bar{\mu}^3 \cdot \|\nabla u\|^6_{L^2}.
\]
Integrate with respect to time on \([0, t] \times \Omega\),
\[
\frac{1}{\mu} \int_0^t \int \rho |u_t|^2 \, dx \, ds + \sup_{s \in [0, t]} \int |\nabla u|^2 \, dx \leq C \mu^{-1} \left( M_2^2 + M_4 M_2^6 \right) \cdot \bar{\mu}^3 \cdot \int_0^t \|\nabla u\|^6_{L^2} \, ds.
\]
Applying Gronwall’s inequality,
\[
\frac{1}{\mu} \int_0^T \int \rho |u_t|^2 \, dx \, dt + \sup_{t \in [0, T]} \int |\nabla u|^2 \, dx 
\leq \|\nabla u_0\|^2_{L^2} \cdot \exp \left\{ C \mu^{-1} (M_2^2 + M_4 M_2^6) \cdot \bar{\mu}^3 \cdot \int_0^T \|\nabla u\|^4_{L^2} \, dt \right\}.
\]
According to Theorem 3.2 and the assumption (3.18),
\[
(3.23) \quad \int_0^T \|\nabla u\|^4_{L^2} \, dt \leq \sup_{t \in [0, T]} \|\nabla u\|^2_{L^2} \cdot \int_0^T \|\nabla u\|^2_{L^2} \, ds 
\leq C \mu^{-1} \cdot \bar{\mu} \cdot \|u_0\|^2_{L^2} 
\leq C \mu^{-1} \cdot \bar{\mu} \cdot \|\nabla u_0\|^2_{L^2}.
\]
Hence, we arrive at
\[
(3.24) \quad \frac{1}{\mu} \int_0^T \int \rho |u_t|^2 \, dx \, dt + \sup_{t \in [0, T]} \int |\nabla u|^2 \, dx 
\leq \|\nabla u_0\|^2_{L^2} \exp \{ C \mu^{-2} (M_2^2 + M_4 M_2^6) \cdot \bar{\mu}^4 \cdot \|u_0\|^6_{L^2}. \}
\]
Now it is clear that (3.20) holds, provided (3.19) holds.

\[\Box\]

As a byproduct of the estimates in the proof, we have the following result.

**Theorem 3.5.** Suppose \((\rho, u, P)\) is the unique local strong solution to (1.1) on \([0, T]\), with the initial data \((\rho_0, u_0)\), and satisfies the assumptions (3.17) - (3.19) as in Theorem 3.3. Then
\[
(3.25) \quad \frac{1}{\mu} \int_0^T t \|\sqrt{\rho} u_t\|^2_{L^2} \, dt + \sup_{t \in [0, T]} t \|\nabla u\|^2_{L^2} \leq \frac{C \cdot \bar{\mu}}{\mu} \|\nabla u_0\|^2_{L^2}.
\]
Proof. Multiplying (3.22) by \( t \), as shown in the last proof, one has

\[
\frac{1}{\mu} \int_0^T t \|\nabla u_t\|_{L^2}^2 \, dt + \sup_{t \in [0,T]} t \|\nabla u\|_{L^2}^2 \leq \frac{1}{\mu} \int_0^T \mu(\rho)|d|^2 \, dx \cdot \exp\{C_1 \mu^{-2}(M_2^2 + M^4 M_2^6) \cdot \rho^4 \cdot \|\nabla u_0\|_{L^2}^2\}.
\]

According to Theorem 3.2,

\[
\int_0^T \int \mu(\rho)|d|^2 \, dx \, dt \leq C \cdot \rho \cdot \|\nabla u_0\|_{L^2}^2 \leq C \cdot \rho \cdot \|\nabla u_0\|_{L^2}^2.
\]

Hence,

\[
\frac{1}{\mu} \int_0^T t \|\sqrt{\rho u_t}\|_{L^2}^2 \, dt + \sup_{t \in [0,T]} t \|\nabla u\|_{L^2}^2 \leq \frac{1}{\mu} \int_0^T \int \mu(\rho)|d|^2 \, dx \cdot \exp\{C_1 \mu^{-2}(M_2^2 + M^4 M_2^6) \cdot \rho^4 \cdot \|\nabla u_0\|_{L^2}^2\}
\]

\[
\leq \frac{C}{\mu} \|\nabla u_0\|_{L^2}^2 \cdot \exp\{C_1 \mu^{-2}(M_2^2 + M^4 M_2^6) \cdot \rho^4 \cdot \|\nabla u_0\|_{L^2}^2\}
\]

\[
\leq \frac{C}{\mu} \|\nabla u_0\|_{L^2}^2.
\]

\(\square\)

**Theorem 3.6.** Suppose \((\rho, u, P)\) is the unique local strong solution to (1.1) on \([0, T]\), with the initial data \((\rho_0, u_0)\), and satisfies the assumptions \((3.17)-(3.19)\). Then

\[
\sup_{t \in [0,T]} t \int \rho |u_t|^2 \, dx + \mu \int_0^T t \|\nabla u_t\|_{L^2}^2 \leq C \|\nabla u_0\|_{L^2}^2 \Theta_1 \cdot \exp\{C \Theta_2\}
\]

and

\[
\sup_{t \in [0,T]} t^2 \int \rho |u_t|^2 \, dx + \mu \int_0^T t^2 \|\nabla u_t\|_{L^2}^2 \leq C \frac{\rho}{\mu} \|\nabla u_0\|_{L^2}^2 \Theta_1 \cdot \exp\{C \Theta_2\}.
\]

where

\[
\Theta_1 = \frac{M_2^4 \rho^8}{\mu^3} + \frac{M^2 M_2^8 \rho^{10}}{\mu^3} + \mu, \quad \Theta_2 = \frac{\rho^4}{\mu^4} + \frac{M_2^4 \rho^4}{\mu^2} + M^2 M_2^4 \rho^2.
\]

**Proof.** Take t-derivative of the momentum equation,

\[
\rho u_t + (\rho u) \cdot \nabla u - \div (2\mu(\rho)d_t) + \nabla P = -\rho_t u_t - (\rho u) \cdot \nabla u + \div (2\mu(\rho)d_t).
\]

Multiplying (3.31) by \(tu_t\) and integrating over \(\Omega\), we get after integration by parts that

\[
\frac{t}{2} \frac{d}{dt} \int \rho |u_t|^2 \, dx + 2t \int \mu(\rho)|d_t|^2 \, dx
\]

\[
= -t \int \rho_t |u_t|^2 \, dx - t \int (\rho u)_t \cdot \nabla u \cdot u_t \, dx - t \int 2\mu(\rho)d_t \cdot \nabla u_t \, dx.
\]
Let us estimate the terms on the righthand of (3.32). First, utilizing the mass equation and Poincaré’s inequality, one has

\[ -t \int \rho_t |u_t|^2 \, dx \]

\[ = -2t \int \rho u_t \cdot \nabla u_t \cdot u_t \, dx \]

\[ \leq C p^{\frac{1}{2}} \cdot t \cdot \| \sqrt{\rho} u_t \|_{L^3} \cdot \| \nabla u_t \|_{L^2} \cdot \| u \|_{L^6} \]

\[ \leq C p^{\frac{1}{2}} \cdot t \cdot \| \sqrt{\rho} u_t \|_{L^2}^{\frac{3}{2}} \cdot \| \sqrt{\rho} u_t \|_{L^6}^{\frac{1}{2}} \cdot \| \nabla u_t \|_{L^2} \cdot \| u \|_{L^2} \]

\[ \leq C p^{\frac{1}{2}} \cdot t \cdot \| \sqrt{\rho} u_t \|_{L^2}^{\frac{3}{2}} \cdot \| \nabla u_t \|_{L^2}^{\frac{3}{2}} \cdot \| \nabla u \|_{L^2} \]

\[ \leq \frac{1}{8} \mu \cdot t \| \nabla u_t \|_{L^2}^2 + C \mu^{-3} \cdot \| \nabla u_t \|_{L^2}^4 \cdot \| u \|_{L^2}^4. \]

Second, utilizing the renormalized mass equation for \( \mu(\rho) \),

\[ -2t \int \mu(\rho)_t \cdot d \cdot \nabla u_t \, dx \]

\[ \leq Ct \int |u| \cdot |\nabla \mu(\rho)| \cdot |d| \cdot |\nabla u_t| \, dx \]

\[ \leq Ct \cdot \| \nabla \mu(\rho) \|_{L^3} \cdot \| \nabla u_t \|_{L^2} \cdot \| d \|_{L^6} \cdot \| u \|_{L^6} \]

\[ \leq CMt \cdot \| \nabla u_t \|_{L^2} \cdot \| \nabla u \|_{H^1}. \]

It follows from Lemma 3.4 that

\[ -2t \int \mu(\rho)_t \cdot d \cdot \nabla u_t \, dx \]

\[ \leq \frac{1}{8} \mu \cdot t \| \nabla u_t \|_{L^2}^2 + \frac{CM^2}{\mu} \cdot t \left( M_2^4 \rho^2 \| \sqrt{\rho} u_t \|_{L^2}^4 + M_2^8 \rho^8 \| \nabla u \|_{L^2}^{12} \right) \]

\[ \leq \frac{1}{8} \mu \cdot t \| \nabla u_t \|_{L^2}^2 + \frac{CM^2 M_2^4 \rho^2}{\mu} \cdot t \| \sqrt{\rho} u_t \|_{L^2}^4 + \frac{CM^2 M_2^8 \rho^8}{\mu} \cdot t \| \nabla u \|_{L^2}^{12}. \]

Finally, taking into account the mass equation again, we arrive at

\[ -t \int (\rho u)_t \cdot \nabla u \cdot u_t \, dx \]

\[ = -t \int \rho u_t \cdot \nabla (u \cdot \nabla u_t) \, dx - t \int \rho u_t \cdot \nabla u \cdot u_t \, dx \]

\[ \leq t \int \rho |u| \cdot |\nabla u|^2 \cdot |u_t| \, dx + Ct \int \rho |u|^2 \cdot |\nabla^2 u| \cdot |u_t| \, dx \]

\[ + t \int \rho |u|^2 \cdot |\nabla u| \cdot |\nabla u_t| \, dx + t \int \rho |u_t|^2 \cdot |\nabla u| \, dx \]

\[ = \sum_{i=1}^{4} J_i. \]
Hence, it follows from Sobolev embedding inequality, Gagliardo-Nirenberg inequality, and Lemma 3.4 that

\begin{equation}
J_1 \leq C\overline{\rho} \cdot t \|u_t\|_{L^6} \cdot \|u\|_{L^6} \cdot \|\nabla u\|_{L^3}^2
\leq C\overline{\rho} \cdot t \|\nabla u_t\|_{L^2} \cdot \|\nabla u\|_{L^2} \cdot \|\nabla u\|_{H^1}
\leq C\overline{\rho} \cdot t \|\nabla u_t\|_{L^2} \cdot \|\nabla u\|_{L^2} \cdot \left(M_2\overline{\rho}^2 \|\sqrt{\rho} u_t\|_{L^2} + M_2^2\overline{\rho}^2 \|\nabla u\|_{L^2}^2\right)
\leq \frac{1}{8} \mu \cdot t \|\nabla u_t\|_{L^2}^2 + \frac{C M_2^2\overline{\rho}^3}{\mu} \cdot t \|\sqrt{\rho} u_t\|_{L^2}^2 \cdot \|\nabla u\|_{L^2}^4 + \frac{C M_2^2\overline{\rho}^3}{\mu} \cdot t \|\nabla u\|_{L^2}^{10}.
\end{equation}

Similarly, it holds that

\begin{equation}
J_2 \leq C\overline{\rho} \cdot t \|u_t\|_{L^6} \cdot \|\nabla^2 u\|_{L^2} \cdot \|u\|_{L^6}^2
\leq C\overline{\rho} \cdot t \|\nabla u_t\|_{L^2} \cdot \|\nabla u\|_{H^1} \cdot \|\nabla u\|_{L^2}^2
\leq \frac{1}{8} \mu \cdot t \|\nabla u_t\|_{L^2}^2 + \frac{C M_2^2\overline{\rho}^3}{\mu} \cdot t \|\sqrt{\rho} u_t\|_{L^2}^2 \cdot \|\nabla u\|_{L^2}^4 + \frac{C M_2^2\overline{\rho}^3}{\mu} \cdot t \|\nabla u\|_{L^2}^{10},
\end{equation}

and

\begin{equation}
J_3 \leq C\overline{\rho} \cdot t \|\nabla u_t\|_{L^2} \cdot \|\nabla u\|_{L^6} \cdot \|u\|_{L^6}^2
\leq \frac{1}{8} \mu \cdot t \|\nabla u_t\|_{L^2}^2 + \frac{C M_2^2\overline{\rho}^3}{\mu} \cdot t \|\sqrt{\rho} u_t\|_{L^2}^2 \cdot \|\nabla u\|_{L^2}^4 + \frac{C M_2^2\overline{\rho}^3}{\mu} \cdot t \|\nabla u\|_{L^2}^{10}.
\end{equation}

Owing to Lemma 3.4 and Sobolev embedding inequality,

\begin{equation}
J_4 \leq C t \|\sqrt{\rho} u_t\|_{L^4}^2 \cdot \|\nabla u\|_{L^2}
\leq C t \|\sqrt{\rho} u_t\|_{L^2}^2 \cdot \|\nabla u_t\|_{L^2}^2 \cdot \|\nabla u\|_{L^2}
\leq \frac{1}{8} \mu \cdot t \|\nabla u_t\|_{L^2}^2 + \frac{C\overline{\rho}^3}{\mu} \cdot t \|\sqrt{\rho} u_t\|_{L^2}^2 \cdot \|\nabla u\|_{L^2}^2.
\end{equation}

Combine all the above estimates (3.33)-(3.40),

\begin{equation}
\frac{d}{dt} \int \rho |u_t|^2 \, dx + \frac{1}{\mu} \cdot t \|\nabla u_t\|_{L^2}^2
\leq \frac{C\overline{\rho}^3}{\mu^3} \cdot t \|\sqrt{\rho} u_t\|_{L^2}^2 \cdot \|\nabla u_t\|_{L^2}^2 \cdot \|\nabla u\|_{L^2}^4 + \frac{C M_2^2\overline{\rho}^3}{\mu} \cdot t \|\sqrt{\rho} u_t\|_{L^2}^2 \cdot \|\nabla u\|_{L^2}^4
+ \frac{C M_2^2\overline{\rho}^3}{\mu} \cdot t \|\nabla u\|_{L^2}^{10} + \frac{C M_2^2\overline{\rho}^3}{\mu} \cdot t \|\nabla u\|_{L^2}^{12} + \int \rho |u_t|^2 \, dx.
\end{equation}
Applying Gronwall’s inequality,

\[
\sup_{t \in [0,T]} t \int \rho |u_t|^2 dx + \mu \int_0^T t \| \nabla u_t \|^2_{L^2} dt \\
\leq \left[ \int_0^T \left( \frac{CM_2^4 \rho^6}{\mu} \cdot t \| \nabla u \|^6_{L^2} + \frac{CM_2^2 \rho^8}{\mu} \cdot t \| \nabla u \|^8_{L^2} + \| \sqrt{\rho} u_t \|^2_{L^2} \right) dt \right] \\
\cdot \exp \left\{ \int_0^T \left( \left( \frac{C_1^3}{\mu^3} + \frac{CM_2^2 \rho^4}{\mu} \right) \| \nabla u \|^4_{L^2} + \frac{CM_2^2 \rho^8}{\mu} \| \sqrt{\rho} u_t \|^2_{L^2} \right) dt \right\}.
\]

Taking (3.20) and (3.23) into account,

\[
(3.42) \quad \sup_{t \in [0,T]} t \int \rho |u_t|^2 dx + \mu \int_0^T t \| \nabla u_t \|^2_{L^2} dt \\
\leq \left[ \int_0^T \left( \frac{CM_2^4 \rho^6}{\mu} \cdot t \| \nabla u \|^6_{L^2} + \frac{CM_2^2 \rho^8}{\mu} \cdot t \| \nabla u \|^8_{L^2} + \| \sqrt{\rho} u_t \|^2_{L^2} \right) dt \right] \\
\cdot \exp \left\{ C \left( \left( \frac{\rho^4}{\mu^4} + \frac{M_2^2 \rho^4}{\mu^2} + M^2 M_2^4 \rho^2 \right) \right) \right\}.
\]

According to Theorems 3.2, 3.5 and the assumption (3.18),

\[
\int_0^T t \| \nabla u \|^6_{L^2} dt \leq \sup_{t \in [0,T]} t \| \nabla u \|^6_{L^2} \cdot \sup_{t \in [0,T]} \| \nabla u \|^6_{L^2} \cdot \int_0^T \| \nabla u \|^2_{L^2} dt \\
\leq \frac{C \cdot \rho^2}{\mu^2} \| u_0 \|^2_{L^2} \leq \frac{C \cdot \rho^2}{\mu^2} \| \nabla u_0 \|^2_{L^2}.
\]

Similarly,

\[
\int_0^T t \| \nabla u \|^8_{L^2} dt \leq \frac{C \cdot \rho^2}{\mu^2} \| \nabla u_0 \|^2_{L^2}.
\]

And by virtue of Theorem 3.3

\[
\int_0^T \| \sqrt{\rho} u_t \|^2_{L^2} dt \leq \mu \cdot \| \nabla u_0 \|^2_{L^2}.
\]

Hence,

\[
(3.43) \quad \sup_{t \in [0,T]} t \int \rho |u_t|^2 dx + \mu \int_0^T t \| \nabla u_t \|^2_{L^2} dt \\
\leq C \| \nabla u_0 \|^2_{L^2} \left( \frac{M_2^2 \rho^8}{\mu^4} + \frac{M^2 M_2^8 \rho^{10}}{\mu^8} + \mu \right) \cdot \exp \left\{ \left( \frac{\rho^4}{\mu^4} + \frac{M_2^2 \rho^4}{\mu^2} + M^2 M_2^4 \rho^2 \right) \right\}.
\]
On the other hand, multiplying (3.41) by \( t \), one has

\[
\frac{d}{dt} \left( \frac{t^2}{2} \int \rho |u_t|^2 \, dx \right) + \mu t^2 \| \nabla u_t \|_{L^2}^2 \leq \frac{C\rho^3}{\mu^3} \cdot t^2 \| \sqrt{\rho} u_t \|_{L^2}^2 \cdot \| \nabla u \|_{L^2}^4 + \frac{CM^2\rho^3}{\mu} \cdot t^2 \| \sqrt{\rho} u_t \|_{L^2}^2 \cdot \| \nabla u \|_{L^2}^4
\]

(3.44)

\[
+ \frac{CM^2\rho^3}{\mu} \cdot t^2 \| \nabla u \|_{L^2}^4 + \frac{CM^2\rho^3}{\mu} \cdot t^2 \| \sqrt{\rho} u_t \|_{L^2}^4
\]

\[
+ CM^2 M^2 \rho^3 \cdot t^2 \| \nabla u \|_{L^2}^2 + t \int \rho |u_t|^2 \, dx.
\]

Applying Gronwall’s inequality,

\[
\sup_{t \in [0,T]} t^2 \int \rho |u_t|^2 \, dx + \mu \int_0^T t^2 \| \nabla u_t \|_{L^2}^2 \, dt \leq \left[ \int_0^T \left( \frac{CM^2\rho^3}{\mu} \cdot t^2 \| \nabla u \|_{L^2}^4 + \frac{CM^2\rho^3}{\mu} \cdot t^2 \| \sqrt{\rho} u_t \|_{L^2}^2 + \mu t \| \nabla u \|_{L^2}^4 \right) \, dt \right] \cdot \exp \left\{ C \left( \frac{\rho^4}{\mu^4} + \frac{M^2\rho^4}{\mu^2} + M^2 M^2 \rho^2 \right) \right\}.
\]

According to Theorems 3.2 and 3.5,

\[
\int_0^T t^2 \| \nabla u \|_{L^2}^4 \, dt \leq \sup_{t \in [0,T]} t^2 \| \nabla u \|_{L^2}^4 \cdot \sup_{t \in [0,T]} \| \nabla u \|_{L^2}^4 \cdot \int_0^T \| \nabla u \|_{L^2}^2 \, dt \leq \frac{C\rho^3}{\mu^3} \| \nabla u_0 \|_{L^2}^2.
\]

Similarly,

\[
\int_0^T t^2 \| \nabla u \|_{L^2}^4 \, dt \leq \frac{C\rho^3}{\mu^3} \| \nabla u_0 \|_{L^2}^2.
\]

Hence,

(3.46)

\[
\sup_{t \in [0,T]} t^2 \int \rho |u_t|^2 \, dx + \mu \int_0^T t^2 \| \nabla u_t \|_{L^2}^2 \, dt \leq C \| \nabla u_0 \|_{L^2}^2 \left( \frac{M^2\rho^4}{\mu^4} + \frac{M^2\rho^4}{\mu^2} + \rho \right) \cdot \exp \left\{ C \left( \frac{\rho^4}{\mu^4} + \frac{M^2\rho^4}{\mu^2} + M^2 M^2 \rho^2 \right) \right\}.
\]

Lemma 3.7. Suppose \((\rho, u, P)\) is the unique local strong solution to (1.1) on \([0,T]\), with the initial data \((\rho_0, u_0)\), and satisfies the assumptions (3.17) - (3.19).
Then for any \( r \in (3, \min\{q, 6\}) \)

\[
(3.47) 
\int_0^T \|\nabla u\|_{L^\infty} \, dt 
\leq C \|\nabla u_0\|_{L^2} \left[ M_r \rho^{5r-6} \mu^{\frac{3(r-2)}{4r}} \left( 1 + \frac{\overline{h}}{\mu} \right)^\frac{1}{2} \Theta_1^\frac{1}{2} \exp\{C\Theta_2\} + M_r^{5r-5} \rho^{\frac{6(r-1)}{r}} \right].
\]

**Proof.** By virtue of Lemma 2.1, one has for \( r \in (3, \min\{q, 6\}) \)

\[
\|\nabla u\|_{W^{1,r}} \leq CM_r (\|\rho u_t\|_{L^r} + \|\rho u \cdot \nabla u\|_{L^r}) 
(3.48) 
\leq CM_r \left( \|\rho u_t\|_{L^\infty}^{\frac{6-r}{2}} \cdot \|\rho u_t\|_{L^{6r/(6-r)}}^{\frac{3(r-2)}{2r}} + \overline{h}\|\nabla u\|_{L^{6r/(6-r)}} \right) 
\leq CM_r \left( \|\rho u_t\|_{L^\infty}^{\frac{6-r}{2}} \cdot \|\rho u_t\|_{L^{6r/(6-r)}}^{\frac{3(r-2)}{2r}} + \overline{h}\|\nabla u\|_{L^{6r/(6-r)}} \right). 

Applying Young's inequality and Sobolev embedding inequality,

\[
(3.49) 
\|\nabla u\|_{W^{1,r}} \leq CM_r \rho^{\frac{5r-6}{4r}} \cdot \|\sqrt{\rho} u_t\|_{L^2} \cdot \|\nabla u_t\|_{L^{2r}}^{\frac{3(r-2)}{2}} + CM_r^{\frac{5r-6}{r}} \rho^{\frac{5r-6}{r}} \cdot \|\nabla u\|_{L^2}^{\frac{6(r-1)}{r}}. 
\]

Hence,

\[
(3.50) 
\int_0^T \|\nabla u\|_{L^\infty} \, dt 
\leq C \int_0^T \|\nabla u\|_{W^{1,r}} \, dt 
\leq C \int_0^T \left( M_r \rho^{\frac{5r-6}{4r}} \|\sqrt{\rho} u_t\|_{L^2} \cdot \|\nabla u_t\|_{L^{2r}}^{\frac{3(r-2)}{2}} + M_r^{\frac{5r-6}{r}} \rho^{5r-6} \|\nabla u\|_{L^2}^{\frac{6(r-1)}{r}} \right) \, dt.
\]

If \( T \leq 1 \), according to Theorem 3.6

\[
\int_0^T \|\sqrt{\rho} u_t\|_{L^2}^{\frac{6-r}{2}} \cdot \|\nabla u_t\|_{L^{2r}}^{\frac{3(r-2)}{2}} \, dt 
\leq \int_0^T \left( t^\frac{1}{2} \|\sqrt{\rho} u_t\|_{L^2} \right)^{\frac{6-r}{2}} \cdot \left( t^\frac{1}{2} \|\nabla u_t\|_{L^2} \right)^{\frac{3(r-2)}{2}} \cdot t^{-\frac{1}{2}} \, dt 
\leq C \left( \sup_{t \in [0,T]} t^\frac{1}{2} \|\sqrt{\rho} u_t\|_{L^2} \right)^{\frac{6-r}{2}} \cdot \left( \int_0^T t \|\nabla u_t\|_{L^2}^2 \, dt \right)^{\frac{3(r-2)}{4r}} \cdot \left( \int_0^T t^{-\frac{2r-6}{4r}} \, dt \right)^{\frac{r+6}{4r}} 
\leq C \mu^{-\frac{3(r-2)}{4r}} \cdot \|\nabla u_0\|_{L^2} \Theta_1^\frac{1}{2} \exp\{C\Theta_2\}.
\]
If $T > 1$, applying Theorem 3.6 again,
\[
\int_0^T \left( t^\alpha \| \sqrt{\rho} u_t \|_{L^2} \cdot \| \nabla u_t \|_{L^2}^{\frac{3(r-2)}{2r}} \right) dt \\
\leq \int_0^T \left( t^\alpha \| \sqrt{\rho} u_t \|_{L^2} \right)^\frac{6-r}{2r} \cdot \left( t^\alpha \| \nabla u_t \|_{L^2} \right)^\frac{3(r-2)}{2r} \cdot t^{-\frac{r}{4}} dt \\
+ \int_1^T \left( t \| \sqrt{\rho} u_t \|_{L^2} \right)^\frac{6-r}{2r} \cdot \left( t \| \nabla u_t \|_{L^2} \right)^\frac{3(r-2)}{2r} \cdot t^{-1} dt \\
\leq C \left( \sup_{t \in [0,T]} t^\alpha \| \sqrt{\rho} u_t \|_{L^2} \right)^\frac{6-r}{2r} \cdot \left( \int_0^T t \| \nabla u_t \|_{L^2}^2 dt \right)^\frac{3(r-2)}{4r} \cdot \left( \int_0^T t^{-\frac{r}{4}} dt \right) \\
+ C \left( \sup_{t \in [0,T]} t \| \sqrt{\rho} u_t \|_{L^2} \right)^\frac{6-r}{4r} \cdot \left( \int_0^T t^2 \| \nabla u_t \|_{L^2}^2 dt \right)^\frac{3(r-2)}{4r} \cdot \left( \int_1^T t^{-\frac{r}{4}} dt \right)^\frac{6-r}{4r} \\
\leq C \mu^\frac{3(r-2)}{4r} \| \nabla u_0 \|_{L^2} \Theta_1^\frac{1}{2} \exp\{C \Theta_2\} + C \mu^\frac{3(r-2)}{4r} \left( \frac{\overline{\rho}}{\mu} \right)^\frac{1}{2} \| \nabla u_0 \|_{L^2} \Theta_1^\frac{1}{2} \exp\{C \Theta_2\} \\
\leq C \mu^\frac{3(r-2)}{4r} \left( 1 + \frac{\overline{\rho}}{\mu} \right)^\frac{1}{2} \| \nabla u_0 \|_{L^2} \Theta_1^\frac{1}{2} \exp\{C \Theta_2\}.
\]

On the other hand,
\[
(3.51) \quad \int_0^T \| \nabla u \|_{L^2}^{\frac{6(r-1)}{2r}} dt \leq \sup_{t \in [0,T]} \| \nabla u \|_{L^2} \cdot \int_0^T \| \nabla u \|_{L^2}^2 dt \leq \frac{C \cdot \overline{\rho}}{\mu} \| \nabla u_0 \|_{L^2}.
\]

Therefore,
\[
(3.52) \quad \int_0^T \| \nabla u \|_{L^\infty} dt \\
\leq C \| \nabla u_0 \|_{L^2} \left[ M_{\overline{\rho}, \overline{\mu}} \mu^{-\frac{3(r-2)}{2r}} \left( 1 + \frac{\overline{\rho}}{\mu} \right)^\frac{1}{2} \Theta_1^\frac{1}{2} \exp\{C \Theta_2\} + M_{\overline{\rho}, \overline{\mu}} \frac{5r-6}{r} \frac{\overline{\rho}^{6(r-1)} - r}{\mu} \right] \\
\triangleq C_2(M, \overline{\rho}, \overline{\mu}, \mu) \| \nabla u_0 \|_{L^2}.
\]

\[\Box\]

**Theorem 3.8.** Suppose $(\rho, u, P)$ is the unique local strong solution to (1.1) on $[0,T]$ and the assumptions (3.17)-(3.19). There exists a positive number $\epsilon_0$, depending on $\Omega$, $q$, $M$, $\overline{\rho}$, $\overline{\mu}$, $\mu$ such that if
\[
\| \nabla u_0 \|_{L^2} \leq \epsilon_0 ,
\]
then
\[
(3.53) \quad \sup_{t \in [0,T]} \| \nabla \mu(\rho) \|_{L^1} \leq 2M ,
\]
and
\[
\sup_{t \in [0,T]} \| \nabla \rho \|_{L^1} \leq 2 \| \nabla \rho_0 \|_{L^1} .
\]
Proof. Consider the $x_i$-derivative of the renormalized mass equation for $\mu(\rho)$,
\[
(\partial_t \mu(\rho))_i + (\partial_i u \cdot \nabla)\mu(\rho) + u \cdot \nabla \partial_i \mu(\rho) = 0. 
\]
It implies that for every $t \in [0, T]$, in view of Lemma 3.7 one has
\[
\|\nabla \mu(\rho)(t)\|_{L^q} \leq \|\nabla \mu(\rho_0)\|_{L^q} \cdot \exp \left\{ \int_0^t \|\nabla u\|_{L^\infty} \, ds \right\}
\leq \|\nabla \mu(\rho_0)\|_{L^q} \cdot \exp \left\{ C_2(M, \overline{\rho}, \overline{\mu}, \overline{\mu}) \cdot \|\nabla u_0\|_{L^2} \right\}.
\]
Choose some small positive constant $\epsilon_0$, which satisfies
\[
\epsilon_0 \leq \frac{1}{2}, \quad \epsilon_0^2 \cdot C_1 M^{-2} (M^2 + M^4 M_2^6) \leq \ln 2,
\]
and
\[
\epsilon_0 \cdot C_2(M, \overline{\rho}, \overline{\mu}, \overline{\mu}) \leq \ln 2.
\]
If $\|\nabla u_0\|_{L^2} \leq \epsilon_0$, (3.53) holds.

Similarly,
\[
\|\nabla \rho(t)\|_{L^q} \leq \|\nabla \rho_0\|_{L^q} \cdot \exp \left\{ \int_0^t \|\nabla u\|_{L^\infty} \, ds \right\}
\leq 2\|\nabla \rho_0\|_{L^q},
\]
Therefore, Theorem 3.8 is proved. \qed

3.2. Proof of Theorem 1.2. With the a priori estimates in Subsection 3.1 in hand, we are prepared for the proof of Theorem 1.2.

Proof. According to Theorem 1.1, there exists a $T_*>0$ such that the density-dependent Navier-Stokes system (1.1) has a unique local strong solution $(\rho, u, P)$ on $[0, T_*].$ We plan to extend the local solution to a global one.

Since $\|\nabla \mu(\rho_0)\|_{L^q} = M < 4M_i$, and due to the continuity of $\nabla \mu(\rho)$ in $L^q$ and $\nabla u_0$ in $L^2$, there exists a $T_1 \in (0, T_*]$ such that $\sup_{0 \leq t \leq T_1} \|\nabla \mu(\rho(t))\|_{L^q} \leq 4M_i$, and at the same time $\sup_{0 \leq t \leq T_1} \|\nabla u(t)\|_{L^2} \leq 2\|\nabla u_0\|_{L^2}$. 

Set
\[
T^* = \sup \left\{ T \mid (\rho, u, P) \text{ is a strong solution to } (1.1) \text{ on } [0, T] \right\}.
\]

\[
T^*_1 = \sup \left\{ T \mid \sup_{0 \leq t \leq T} \|\nabla \mu(\rho(t))\|_{L^q} \leq 4M_i, \text{ and } \sup_{0 \leq t \leq T} \|\nabla u(t)\|_{L^2} \leq 2\|\nabla u_0\|_{L^2} \right\}.
\]

Then $T^*_1 \geq T_1 > 0.$ Recalling Theorems 3.3 and 3.8 it’s easy to verify
\[
T^* = T^*_1,
\]
provided that $\|\nabla u_0\|_{L^2} \leq \epsilon_0$ as assumed.

We claim that $T^* = \infty$. Otherwise, assuming that $T^* < \infty$. By virtue of Theorems 3.3 and 3.8 for every $t \in [0, T^*)$, it holds that
\[
\|\nabla \rho(t)\|_{L^q} \leq 2\|\nabla \rho_0\|_{L^q}, \quad \text{and} \quad \|\nabla u(t)\|_{L^2} \leq \sqrt{2}\|\nabla u_0\|_{L^2},
\]
which contradicts to the blowup criterion (1.5). Hence we complete the proof for Theorem 1.2. \qed
Acknowledgement: The research of Xiangdi Huang is supported in part by NSFC Grant No. 11101392. The research of Yun Wang is supported in part by NSFC Grant No. 11301365.

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