Modular Constraints on
Conformal Field Theories with Currents

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Geometry of String and Gauge Theories
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Two-dimensional CFTs with $c \geq 1$

- The two-dimensional CFTs with $c < 1$ are classified by minimal model. The CFT data (spectrum and OPE coefficients) for these models were analyzed.

- The situation of $c \geq 1$ CFTs different, because no degenerate state appears in the unitary irreducible representation.

- Examples of two-dimensional CFTs: Minimal models, Liouville theory, WZW models, etc.

- We consider two-dimensional CFTs on the torus, and investigate how the modular property constrains the structure of partition function.

**GOAL:**

- **Modular Differential Equation**
  - Modular Bootstrap

  $\oplus$

  - Spectrum analysis,
  - Find the modular invariant partition function
The Character Decomposition

- The Virasoro characters are defined by

\[ \chi_0(\tau) = \frac{1}{\eta(\tau)} q^{-\frac{c-1}{24}} (1 - q), \quad \chi_h(\tau) = \frac{1}{\eta(\tau)} q^{h - \frac{c-1}{24}} \]

For convenience, we mainly use the reduced character.

\[ \hat{\chi}_0(\tau) = \tau^{\frac{1}{4}} \eta(\tau) \chi_0(\tau), \quad \hat{\chi}_h(\tau) = \tau^{\frac{1}{4}} \eta(\tau) \chi_h(\tau) \]

The torus partition function of unitary CFT admit the character decomposition,

\[ Z(\tau, \bar{\tau}) = \chi_0(\tau) \bar{\chi}_0(\bar{\tau}) + \sum_{h, \bar{h}} d(h, \bar{h}) \chi_h(\tau) \bar{\chi}_{\bar{h}}(\bar{\tau}) + \sum_{j=1}^{\infty} \left[ d(j) \chi_j(\tau) \bar{\chi}_0(\bar{\tau}) + \bar{d}(j) \chi_0(\tau) \bar{\chi}_j(\bar{\tau}) \right]. \]

Constraint from the modular invariance

- S- transformation : \( Z(\tau, \bar{\tau}) = Z\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right) \)

\[ \hat{G}_0(\tau, \bar{\tau}) + \sum_{j=1}^{\infty} \left[ d(j) \hat{G}_j(\tau, \bar{\tau}) + \bar{d}(j) \hat{G}_{\bar{j}}(\tau, \bar{\tau}) \right] + \sum_{h, \bar{h}} d(h, \bar{h}) \hat{G}_{h, \bar{h}}(\tau, \bar{\tau}) = 0 \]

where the function \( \hat{G}_\lambda(\tau, \bar{\tau}) \) is defined as \( \hat{\chi}_\lambda(\tau) \hat{\chi}_\lambda(\bar{\tau}) = \hat{\chi}_\lambda\left(-\frac{1}{\tau}\right) \hat{\chi}_\lambda\left(-\frac{1}{\bar{\tau}}\right). \)
The Modular Bootstrap Equation (MDE)

- Idea: $n$ characters of rational conformal field theory (RCFT) are solutions of $n$-th order modular differential equation, [Mathur, Muhki, Sen 88]

$$D^n_\tau \chi(\tau) + \sum_{k=0}^{n-1} \phi_k(\tau) D^k_\tau \chi(\tau) = 0,$$

with $D_\tau \chi(\tau) \equiv \partial_\tau \chi(\tau) - \frac{\pi i r}{6} \chi(\tau)$.

- Second Order Modular Differential Equation
  - Solve the second order differential equation,

$$D^2_\tau \chi(\tau) + \hat{\mu} E_4(\tau) \chi(\tau) = 0,$$

with an ansatz $\chi_\lambda(q) = q^\alpha (a_0 + a_1 q + a_2 q^2 + a_3 q^3 + a_4 q^4 + \cdots)$.

- The coefficients are positive integer only for $c \in \left\{ \frac{2}{5}, 1, 2, \frac{14}{5}, 4, \frac{26}{5}, 6, 7, \frac{38}{5}, 8 \right\}$.

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[Modular Constraints on Conformal Field Theories with Currents] [Mathur, Muhki, Sen 88], [Tuite 08]
Third Order Modular Differential Equation

- Solve the third order differential equation

\[ D_\tau^3 \chi(\tau) + \mu_1 E_4(\tau) D_\tau \chi(\tau) + \mu_2 E_6(\tau) \chi(\tau) = 0, \]

with an ansatz \( \chi(q) = q^\alpha (a_0 + a_2 q^2 + a_3 q^3 + a_4 q^4 + \cdots) \).

- Each coefficients are positive integer only for [Mathur, Muhki, Sen 88], [Tuite 08]

\[ c \in \left\{ -\frac{44}{5}, 8, 16, \frac{47}{2}, 24, 32, \frac{164}{5}, \frac{236}{7}, 40 \right\}. \]

- The primary characters have the form of

\[ \chi_{h\pm}(\tau) = q^{h\pm - \frac{c}{24}} \left[ b_0 + b_1 q + b_2 q^2 + \cdots \right] \]

with \( h_{\pm}(c) = \frac{c+4}{16} \pm \sqrt{\frac{368+24c-c^2}{16\sqrt{31}}} \).

- The structure of primary characters is undetermined from the modular differential equation.
Modular Bootstrap - Basic Strategy

- Apply the linear functional $\alpha \left[ \hat{G}(z, \bar{z}) \right] \equiv \sum_{m,n}^{m+n=N} \alpha_{m,n} \partial_z^m \partial_{\bar{z}}^n \hat{G}(z, \bar{z})$ to the modular bootstrap equation. ($\tau \equiv ie^z$, the crossing point: $z = 0$)

$$\alpha \left[ \hat{G}_0(z, \bar{z}) \right] + \sum_{j=1}^{j_{\text{max}}} \left( d(j) \alpha \left[ \hat{G}^j(z, \bar{z}) \right] + \bar{d}(j) \alpha \left[ \hat{G}^{\bar{j}}(z, \bar{z}) \right] \right) + \sum_{h, \tilde{h} \in \mathcal{P}} d(h, \tilde{h}) \alpha \left[ \hat{G}^{h, \tilde{h}}(z, \bar{z}) \right] = 0.$$  

- Find $\alpha_{m,n}$ such that,

$$\alpha \left[ \hat{G}_0(z, \bar{z}) \right] > 0,$$

and $$\alpha \left[ \hat{G}^j(z, \bar{z}) \right] \geq 0, \quad \alpha \left[ \hat{G}^{\bar{j}}(z, \bar{z}) \right] \geq 0 \quad \text{for} \quad j \in \mathbb{Z},$$

and $$\alpha \left[ \hat{G}^{h, \tilde{h}}(z, \bar{z}) \right] \geq 0 \quad \text{for} \quad (h, \tilde{h}) \in \mathcal{P}$$

If we find such $\alpha_{m,n}$, then we conclude no modular invariant partition function exist.
• **Inputs** [Collier, Lin, Yin 16]

  • Scalar Gap Problem

  In this problem, we impose a gap $\Delta_s$ only to the scalar operator. Namely,

  $$\Delta \geq \Delta_s \text{ for } j = 0, \quad \Delta \geq j \text{ for } j \neq 0.$$

  • Maximal Gap Problem

  In this problem, we impose a gap $\Delta_m$.

  $$\Delta \geq \text{Max}(j, \Delta_m).$$

  • Twist Gap Problem

  In this problem, we impose a gap $\Delta_t$ to the twist, $t \equiv \Delta - j$.

  $$\Delta \geq j + \Delta_t.$$
The Numerical Result ($c \leq 8$)

\[
\begin{array}{c}
\Delta_{\text{gap}} \\
\text{Scalar Gap} \\
\text{Maximal Gap (Without Conserved Currents)} \\
\text{Maximal Gap (With Conserved Currents)} \\
\text{Twist Gap (Without Conserved Currents)} \\
\text{Twist Gap (With Conserved Currents)}
\end{array}
\]
The Numerical Result \((c \leq 8)\), Twist Gap
Expected CFTs on the bound (Twist Gap)

- For Wess-Zumino-Witten model,

\[ c = \frac{k \dim \hat{g}}{k + g}, \quad h_{\lambda} = \frac{(\lambda, \lambda + 2\rho)}{2(k + g)} \]

- The twist gap problem realize level-1 WZW models on the boundary!

| Central Charge | Lowest Primary | Expected CFT         |
|----------------|----------------|----------------------|
| \( c = 1 \)   | \( \Delta_t = 1/2 \) | \( SU(2)_1 \) WZW model |
| \( c = 2 \)   | \( \Delta_t = 2/3 \) | \( SU(3)_1 \) WZW model |
| \( c = 14/5 \)| \( \Delta_t = 4/5 \) | \( (G_2)_1 \) WZW model |
| \( c = 4 \)   | \( \Delta_t = 1 \)  | \( SO(8)_1 \) WZW model |
| \( c = 26/5 \)| \( \Delta_t = 6/5 \) | \( (F_4)_1 \) WZW model |
| \( c = 6 \)   | \( \Delta_t = 4/3 \) | \( (E_6)_1 \) WZW model |
| \( c = 7 \)   | \( \Delta_t = 3/2 \) | \( (E_7)_1 \) WZW model |
| \( c = 8 \)   | \( \Delta_t = 2 \)  | \( (E_8)_1 \) WZW model |

- They are two-channel RCFTs, solution of the second order MDE.
The Numerical Result (Twist Gap)

Conserved Currents with $j \geq 1$
- $E_8$
- $E_8 \times E_8$

Conserved Currents with $j \geq 2$

Conserved Currents with $j \geq 3$

$E_8$

$E_8 \times E_8$

$\Delta_{\text{gap}}$ vs $c$

$\Delta_{\text{gap}}$ vs $c$

$k=1$ ECFT

$k=2$ ECFT

$c=32$

$\text{Baby Monster}$

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Modular Constraints on Conformal Field Theories with Currents
The Numerical Result (Twist Gap)

- When the holomorphic currents are included from $j = 1$,

| Central Charge | Lowest Primary $\Delta_t$ | Expected CFT |
|----------------|---------------------------|--------------|
| $c = 16$       | $2$                       | $(E_8 \times E_8)_1$ WZW model |
| $c = 24$       | $4$                       | Monster CFT  |
| $c = 32$       | $4$                       | $k = 4/3$ ECFT |
| $c = 48$       | $6$                       | $k = 2$ ECFT |

- The unique modular invariant partition function at $c = 24$ is,

$$Z_{k=1}(q, \bar{q}) = (j(q) - 744)(\bar{J}(\bar{q}) - 744) = (1 + 196884q^2 + \cdots)(1 + 196884\bar{q}^2 + \cdots)$$

- When the holomorphic currents are included from $j = 2$,

| Central Charge | Lowest Primary $\Delta_{gap}$ | Automorphism       |
|----------------|-------------------------------|--------------------|
| $c = 8$        | $1$                           | $2 \cdot O^+(10, 2)$ |
| $c = 16$       | $2$                           | $2^{16} \cdot O^+(10, 2)$ |
| $c = 47/2$     | $3$                           | Baby Monster       |
Finding the degeneracy bound  

Rewrite the modular bootstrap equation as

\[ \alpha \left[ \hat{G}_0(z, \bar{z}) \right] + d(h^*, \bar{h}^*) \alpha \left[ \hat{G}^{h^*, \bar{h}^*}(z, \bar{z}) \right] + \alpha \left[ \hat{G}^{\text{rest}}(z, \bar{z}) \right] = 0, \]

\[ \alpha \left[ \hat{G}^{\text{rest}}(z, \bar{z}) \right] \equiv \sum_{j=1}^{j_{\text{max}}} \left( d(j) \alpha \left[ \hat{G}^j(z, \bar{z}) \right] + \bar{d}(j) \alpha \left[ \hat{G}^\dagger(z, \bar{z}) \right] \right) + \sum_{h, \bar{h} \in \mathcal{P}} d(h, \bar{h}) \alpha \left[ \hat{G}^{h, \bar{h}}(z, \bar{z}) \right], \]

and solve the following problem.

Maximize \[ \alpha \left[ \hat{G}_0(z, \bar{z}) \right] \], such that \[ \alpha \left[ \hat{G}^{h^*, \bar{h}^*}(z, \bar{z}) \right] = 1 \]
and \[ \alpha \left[ \hat{G}^j(z, \bar{z}) \right] \geq 0, \alpha \left[ \hat{G}^\dagger(z, \bar{z}) \right] \geq 0 \] for \( j \in \mathbb{Z} \),
and \[ \alpha \left[ \hat{G}^{h, \bar{h}}(z, \bar{z}) \right] \geq 0 \] for \((h, \bar{h}) \in \mathcal{P}\).

This gives the maximum bound of the degeneracy of state with \((h^*, \bar{h}^*)\).

\[ d(h^*, \bar{h}^*) \leq -\alpha \left[ \hat{G}_0(z, \bar{z}) \right] \]
Extremal Functional Method [Paulos, El-Showk 14]

- Suppose the degeneracy saturate the maximum bound. Then,

\[
\sum_{j=1}^{j_{\text{max}}} \left( d(j) \alpha^* \left[ \hat{G}^j(z, \bar{z}) \right] + \bar{d}(j) \alpha^* \left[ \hat{G}^j(z, \bar{z}) \right] \right) + \sum_{h, \bar{h} \in \mathcal{P}} d(h, \bar{h}) \alpha^* \left[ \hat{G}^{h, \bar{h}}(z, \bar{z}) \right] = 0
\]

- Therefore,

\[
d(h, \bar{h}) \alpha^* \left[ \hat{G}^{h, \bar{h}}(z, \bar{z}) \right] = 0, \quad \text{for } \forall (h, \bar{h}) \in \mathcal{P}.
\]

Idea: Find the states such that \( \alpha^* \left[ \hat{G}^{h, \bar{h}}(z, \bar{z}) \right] = 0! \)

Spectrum Analysis

1. Apply EFM and find the states such that makes \( \alpha^* \left[ \hat{G}^{h, \bar{h}}(z, \bar{z}) \right] = 0. \)
2. For those states, repeat the degeneracy analysis.
3. Find the consistent character decomposition.
- \( F_4 \) example
- The EFM analysis applied to \((F_4)_1\) WZW gives,

\[
\begin{align*}
\Delta_{j=0} &\in \left\{ \frac{6}{5} + 2n, 2 + 2n \middle| n \in \mathbb{Z}^+ \right\}, \\
\Delta_{j=1} &\in \left\{ \frac{11}{5} + 2n, 3 + 2n \middle| n \in \mathbb{Z}^+ \right\}.
\end{align*}
\]

- From the EFM analysis, the low-lying spectrum of spin-0 and spin-1 sector are,

\[
\begin{align*}
\chi_{[0]}^{F_4}(q) &\equiv \chi_{1;0,0,0,0}(q) = 1 + 52q + 377q^2 + 1976q^3 + 7852q^4 + \mathcal{O}(q^5), \\
\chi_{[4]}^{F_4}(q) &\equiv \chi_{0;0,0,0,1}(q) = q^{\frac{3}{5}} \left(26 + 299q + 1702q^2 + 7475q^3 + 27300q^4 + \mathcal{O}(q^5)\right).
\end{align*}
\]
$F_4$ example (continued)

- For the each low-lying spectrum, the maximum degeneracies are,

| $(h, \bar{h})$ | Max. Deg | $(h, \bar{h})$ | Max. Deg | $(h, \bar{h})$ | Max. Deg |
|----------------|----------|----------------|----------|----------------|----------|
| $(\frac{3}{5}, \frac{3}{5})$ | 676.0000 | $(1, 1)$ | 2704.0000 | $(1, 0)$ | 52.00028 |
| $(\frac{3}{5}, \frac{8}{5})$ | 7098.0001 | $(2, 1)$ | 16848.001 | $(2, 0)$ | 324.0007 |
| $(\frac{3}{5}, \frac{13}{5})$ | 35802.002 | $(3, 1)$ | 80444.061 | $(3, 0)$ | 1547.0091 |
| $(\frac{8}{5}, \frac{8}{5})$ | 74529.0001 | $(2, 2)$ | 104976.005 | $(4, 0)$ | 5499.0126 |

- The relation between partition function and reduced partition function is given by,

$$ \hat{Z}_{F_4}^{W_2} (q, \bar{q}) = |\tau|^{\frac{1}{2}} \eta(\tau)^2 \bar{\eta}(\bar{\tau})^2 Z_{F_4} (q, \bar{q}) - (1 - q)(1 - \bar{q}) \text{ Vacuum contribution} $$

- Our numerical result agrees to the following diagonal form partition function.

$$ Z_{F_4} (q, \bar{q}) = |\chi_{[0]}^{F_4} (q)|^2 + |\chi_{[4]}^{F_4} (q)|^2 $$
The Result Summary ($c \leq 8$)

- The numerical result confirms the structure of modular invariant partition function, in terms of the character. In case of $(G_2)_1, (F_4)_1$ and $(E_7)_1$, it have the diagonal form.

$$
Z_{G_2}(q, \bar{q}) = |\chi_{[0]}^{G_2}(q)|^2 + |\chi_{[1]}^{G_2}(q)|^2 \\
Z_{F_4}(q, \bar{q}) = |\chi_{[0]}^{F_4}(q)|^2 + |\chi_{[4]}^{F_4}(q)|^2 \\
Z_{E_7}(q, \bar{q}) = |\chi_{[0]}^{E_7}(q)|^2 + |\chi_{[6]}^{E_7}(q)|^2
$$

- In case of $E_6$,

$$
Z_{E_6}(q, \bar{q}) = \chi_{[0]}^{E_6}(q)\bar{\chi}_{[0]}^{E_6}(\bar{q}) + \chi_{[1]}^{E_6}(q)\bar{\chi}_{[5]}^{E_6}(\bar{q}) + \chi_{[5]}^{E_6}(q)\bar{\chi}_{[1]}^{E_6}(\bar{q})
$$

$\chi_{[1]}^{E_6}(q)$ and $\chi_{[5]}^{E_6}(q)$ are complex conjugate to each other, their characters are identical.

- The $(A_1)_1, (A_2)_1, (G_2)_1, (D_4)_1$ and $(E_8)_1$ WZW models are realized via the scalar gap problem. [Collier, Lin, Yin 16]

- The twist gap problem realize WZW models with Deligne’s exceptional series on the numerical boundary!
• \((E_{7,1/2})_1 WZW\) model?
  • \(E_{7,1/2}\) is non-simple Lie algebra, its subalgebra is \(E_7\). It splits into \(E_7 \oplus 56 \oplus \mathbb{R}\).
  • The degeneracy analysis at \(c = \frac{38}{5}\) gives,

| \((h, \bar{h})\)   | Max. Deg | \((h, \bar{h})\)   | Max. Deg | \((h, \bar{h})\)   | Max. Deg |
|-------------------|----------|-------------------|----------|-------------------|----------|
| \((\frac{4}{5}, \frac{4}{5})\) | 3249.0004 | \((1, 1)\) | 36100.0000 | \((1, 0)\) | 190.00412 |
| \((\frac{4}{5}, \frac{9}{5})\) | 59565.012 | \((2, 1)\) | 501600.0000 | \((2, 0)\) | 2640.0481 |
| \((\frac{9}{5}, \frac{9}{5})\) | 1092025.06 | \((2, 2)\) | 6969600.0100 | \((3, 0)\) | 19285.021 |

• From the second order MDE, structure of the vacuum character and primary character of \(h = \frac{4}{5}\) are given by,

\[
\chi_{0}^{E_{7,1/2}}(q) = 1 + 190q + 2831q^2 + 22306q^3 + 129276q^4 + 611724q^5 + \mathcal{O}(q^5),
\]
\[
\chi_{4/5}^{E_{7,1/2}}(q) = q^{4/5} \left(57 + 1102q + 9367q^2 + 57362q^3 + 280459q^4 + 1181838q^5 + \mathcal{O}(q^6)\right).
\]

• If there is \((E_{7,1/2})_1\) WZW model, the modular invariant partition function may have the following diagonal form.

\[
\mathcal{Z}_{E_7}(q, \bar{q}) = \chi_{0}^{E_{7,1/2}}(q)\chi_{0}^{E_{7,1/2}}(\bar{q}) + \chi_{4/5}^{E_{7,1/2}}(q)\chi_{4/5}^{E_{7,1/2}}(\bar{q})
\]
Cousins of Extremal Conformal Field Theories

- The extremal conformal field theory is originally suggested by Witten, as a candidate for dual CFT of pure gravity in $AdS_3$. It admits holomorphic factorization, and the central charge is quantized by $c = 24k$. [Witten 07]

- The building block of the partition function is Klein-$j$ function, defined by Eisenstein series of weight 4 and weight 6.

$$J(q) = j(q) - 744 = 1728 \frac{E_4^3}{E_4^3 - E_6^2} - 744$$

$$= q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

- For $c = 24(k = 1)$, the unique modular invariant partition function is given by $Z_{k=1}(q, \bar{q}) = J(q)\bar{J}(\bar{q})$. For $c = 48(k = 2)$, the modular invariant partition function is given by $Z_{k=2}(q, \bar{q}) = (J(q)^2 - 393767)(\bar{J}(\bar{q})^2 - 393767)$.

- Extended to $c = 8m$, the modular invariant partition functions are constructed by fractional power of $j$-function. It has the structure of $[Avramis, Kehagias, Mattheopoulou 07]$

$$Z_{8m}(\tau) = j^{m/3}(\tau) \sum_{r=0}^{[m/3]} a_r j^{-r}(\tau).$$
Examine ECFTs via the modular bootstrap

- The twist gap problem realizes the ECFTs with \( c = 24, 32, 48 \) on its boundary.
- The EFM analysis suggests that all of them have the states with integer \( \Delta \).

Reading the degeneracy, we confirm the following structure of the modular invariant partition function.

\[
\begin{align*}
c = 24 : \mathcal{Z}_{k=1}(q, \bar{q}) &= J(q)\bar{J}(\bar{q}) \\
c = 32 : \mathcal{Z}_{k=\frac{4}{3}}(q, \bar{q}) &= (J(q)^{\frac{4}{3}} - 992J(q)^{\frac{1}{3}})(\bar{J}(\bar{q})^{\frac{4}{3}} - 992\bar{J}(\bar{q})^{\frac{1}{3}}) \\
c = 48 : \mathcal{Z}_{k=2}(q, \bar{q}) &= (J(q)^2 - 393767)(\bar{J}(\bar{q}) - 393767)
\end{align*}
\]
Gapped CFT

- Recall the vacuum character from third order differential equation. with an ansatz

\[ \chi_\lambda(q) = q^\alpha (a_0 + a_2 q^2 + a_3 q^3 + a_4 q^4 + \cdots) \]. We refer those CFTs as gapped CFTs.

\[
\chi_0^{c=8} (q) = 1 + 156 q^2 + 1024 q^3 + 6790 q^4 + 32768 q^5 + O(q^6)
\]
\[
\chi_0^{c=16} (q) = 1 + 2296 q^2 + 65536 q^3 + 1085468 q^4 + 12320768 q^5 + O(q^6)
\]
\[
\chi_0^{c=47/2} (q) = 1 + 96256 q^2 + 9646891 q^3 + 366845011 q^4 + 8223700027 q^5 + O(q^6)
\]

- In the mathematics, the corresponding vertex operator algebra was constructed.

Exceptional Vertex Operator Algebras and the Virasoro Algebra

Michael P. Tuite

- \( C = 8, d_2 = 155 \): This can be realized as the fixed point free lattice VOA \( V_L^+ \) (fixed under the automorphism lifted from the reflection isometry of the lattice \( L \)) for the rank 8 even lattice \( L = \sqrt{2}E_8 \). The automorphism group is \( O_{10}^+(2).2 \) [G].
- \( C = 16, d_2 = 2295 \): The VOA \( V_L^+ \) for the rank 16 Barnes-Wall even lattice \( L = \Lambda_{16} \) whose automorphism group is \( 2^{16}.O_{10}^+(2) \) [S].
- \( C = 23\frac{1}{2}, d_2 = 96255 \): This can be realized as the integrally graded subVOA of Höhn’s Baby Monster Super VOA \( VB^2 \) whose automorphism group is the Baby Monster group \( \mathbb{B} \) [Ho2].
The partition function of $c = 8$ gapped CFT

- The degeneracy analysis with imposing conserved currents from $j = 2$ gives,

| $(h, \bar{h})$ | Max. Deg | $(h, \bar{h})$ | Max. Deg | $(h, \bar{h})$ | Max. Deg |
|----------------|----------|----------------|----------|----------------|----------|
| $(\frac{1}{2}, \frac{1}{2})$ | 496.00000000 | $(1, 1)$ | 33728.000000 | $(2, 0)$ | 155.0000000 |
| $(\frac{1}{2}, \frac{3}{2})$ | 17360.000000 | $(2, 1)$ | 505920.000000 | $(3, 0)$ | 868.0000000 |
| $(\frac{3}{2}, \frac{3}{2})$ | 607600.0009 | $(2, 2)$ | 7612825.000000 | $(4, 0)$ | 5610.000000 |

- The MDE determines primary character up to the overall constants $a_0$ and $a_1$.

$$\chi_{c=8}^1(\tau) = a_0 q^{1/2} \left( 1 + 36q + 394q^2 + 2776q^3 + 15155q^4 + 69508q^5 + O(q^6) \right),$$

$$\chi_{c=8}^2(\tau) = a_1 q^1 \left( 1 + 16q + 136q^2 + 832q^3 + 4132q^4 + 17696q^5 + O(q^6) \right)$$

- Finally, we suggest that modular invariant partition function reads,

$$Z_{c=8} = \chi_{c=8}^0(\tau)\bar{\chi}_{c=8}^0(\bar{\tau}) + 496\chi_{c=8}^1(\tau)\bar{\chi}_{c=8}^1(\bar{\tau})|_{a_0=1} + 33728\chi_{c=8}^2(\tau)\bar{\chi}_{c=8}^2(\bar{\tau})|_{a_1=1}.$$

$$= 1 + \underbrace{496}_{1 + 155 + 340} q^{\frac{1}{2}} \bar{q}^{\frac{1}{2}} + \underbrace{17856}_{2 \times 155 + 2 \times 868 + 15810} q^{\frac{3}{2}} \bar{q}^{\frac{1}{2}} + \underbrace{33728}_{2108 + 31620} q\bar{q} + \underbrace{539648}_{539648} q^2 \bar{q} + \cdots$$
The partition function of $c = 16$ gapped CFT

The degeneracy analysis with imposing conserved currents from $j = 2$ gives,

| $(h, \tilde{h})$ | Max. Deg | $(h, \tilde{h})$ | Max. Deg | $(h, \tilde{h})$ | Max. Deg |
|-----------------|----------|-----------------|----------|-----------------|----------|
| $\left(\frac{3}{2}, \frac{3}{2}\right)$ | 32505856.0032 | $\left(1, 1\right)$ | 134912.0000 | $\left(2, 0\right)$ | 2295.0000 |
| $\left(\frac{3}{2}, \frac{5}{2}\right)$ | 1657798656.0001 | $\left(2, 1\right)$ | 18213120.0000 | $\left(3, 0\right)$ | 63240.00000 |
| $\left(\frac{3}{2}, \frac{7}{2}\right)$ | 34228666368.005 | $\left(2, 2\right)$ | 2464038225.0003 | $\left(4, 0\right)$ | 1017636.0000 |

The MDE determines primary character up to the overall constants $b_0$ and $b_1$.

\[
\chi_1^{c=16} = b_0 q^1 \left(1 + 136q + 4132q^2 + 67712q^3 + 770442q^4 + 6834240q^5 + \mathcal{O}(q^6)\right), \\
\chi_2^{c=16} = b_1 q^{3/2} \left(1 + 52q + 1106q^2 + 14808q^3 + 147239q^4 + 1183780q^5 + \mathcal{O}(q^6)\right)
\]

Finally, we suggest that modular invariant partition function reads,

\[
\mathcal{Z}_{c=16} = \chi_0^{c=16}(\tau)\tilde{\chi}_0^{c=16}(\bar{\tau}) + 134912\chi_1^{c=16}(\tau)\tilde{\chi}_1^{c=16}(\bar{\tau})|_{b_0=1} + 32505856\chi_2^{c=16}(\tau)\tilde{\chi}_2^{c=16}(\bar{\tau})|_{b_1=1}
\]

\[
= 1 + 2296q^2 + 65536q^3 + 134912q\bar{q} + \cdots \\
= 1 + \underbrace{2296}_{2 \times 1 + 186 + 2108} q^2 + \underbrace{65536}_{2 \times 1 + 186 + 14756 + 50592} q^3 + \underbrace{134912}_{186 + 340 + 868 + 22858 + 110670} q\bar{q} + \cdots
\]
### Baby Monster CFT

[Höhn 07]

- The degeneracies with \( c = \frac{47}{2} \) gives,

| \((h, \tilde{h})\)       | Max. Deg \((h, \tilde{h})\) | \((h, \tilde{h})\)       | Max. Deg         |
|--------------------------|-----------------------------|--------------------------|------------------|
| \((\frac{3}{2}, \frac{3}{2})\) | 19105641.026984403127       | \((\frac{5}{2}, \frac{5}{2})\) | 1298173112605.3499336 |
| \((2, 2)\)               | 9265025041.322733803        | \((\frac{31}{16}, \frac{31}{16})\) | 9265217540.6086142750 |
| \((\frac{5}{2}, \frac{3}{2})\) | 4980203754.2560961756       | \((\frac{47}{16}, \frac{31}{16})\) | 1011288637613.8107313 |

- The character from 3rd MDE is given by,

\[
\begin{align*}
\chi_0^{c=47/2} &= q^{48/48} a_0 \left( (1 + 96256q^2 + 9646891q^3 + 366845011q^4 + O(q^5)) \right) \\
\chi_1^{c=47/2} &= q^{25/48} a_1 \left( 1 + \frac{785}{3} q + \frac{44393}{3} q^2 + 418441q^3 + \frac{23301881}{3} q^4 + O(q^5) \right) \\
\chi_2^{c=47/2} &= q^{23/24} a_2 \left( 1 + \frac{5177}{47} q + 4372q^2 + 100627q^3 + 1625207q^4 + O(q^5) \right)
\end{align*}
\]

- Corresponding modular invariant partition function reads,

\[
\mathcal{Z}^{c=47/2} = \chi_0^{c=47/2}(\tau) \bar{\chi}_0^{c=16}(\bar{\tau}) \big|_{a_0=1} + \chi_1^{c=47/2}(\tau) \bar{\chi}_1^{c=16}(\bar{\tau}) \big|_{a_1=4371} + \chi_2^{c=47/2}(\tau) \bar{\chi}_2^{c=16}(\bar{\tau}) \big|_{a_2=96256}
\]

\[
= 1 + \frac{96256}{1+96255} q^2 + \frac{9646891}{2 \times 1 - 4371 + 2 \times 96255 + 945875} q^3 + \frac{19105641}{1 + 96255 + 9458750 + 9550635} q^{3/2} \bar{q}^{3/2} + \cdots
\]
Bootstrapping with $\mathcal{W}$-algebra

- We consider the following reduced $\mathcal{W}$-algebra character.

$$\chi_{h,w}(q) = \text{Tr}_{h,w}(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}})$$

- In case of the $\mathcal{W}_{2,3}$-algebra,

$$\chi_0(\tau) = \frac{q^{-\frac{c-2}{24}}(1 - q)^3(1 + q)}{\eta(\tau)^2}, \quad \chi(\tau) = \frac{q^{h - \frac{c-2}{24}}}{\eta(\tau)^2}$$

because of

$$\langle 0|L_1L_{-1}|0 \rangle = 0, \quad \langle 0|W_1W_{-1}|0 \rangle = 0, \quad \langle 0|W_2W_{-2}|0 \rangle = 0.$$  

- For general rank-$k$ case, the character for general rank-$k$ $\mathcal{W}_{f_1,f_2,\ldots,f_k}$-algebra is,

$$\chi(\tau) = \frac{q^{h - \frac{c-N+1}{24}}}{\eta(\tau)^{N-1}}, \quad \chi_0(\tau) = \frac{q^{-\frac{c-N+1}{24}}}{\eta(\tau)^{N-1}} \prod_{j=1}^k \prod_{i=1}^{f_j-1} (1 - q^i).$$
**The Numerical Bounds (Twist Gap)**

- The behavior of $c \geq k$ is identical to the Virasoro results. On the other hand, the numerical bound at $c \leq k$ is collapsed.
**Numerical bound with Rank-2 \( \mathcal{W} \)-algebra**

\[
\begin{array}{c}
\Delta_{\text{gap}} \\
\end{array}
\]

\[
\begin{array}{c}
(A_2)_1 \text{ WZW} \\
\end{array}
\]

\[
\begin{array}{c}
\mathcal{W}_{2,3} \\
\mathcal{W}_{2,4} \\
\mathcal{W}_{2,6} \\
\end{array}
\]
Rank-2 $\mathcal{W}$-algebra and $(A_2)_1$ WZW model

- In case of $\mathcal{W}_{2,3}$ and $\mathcal{W}_{2,4}$ they realize the $(A_2)_1$ WZW model at the end of the unitary world.
- For both cases, the reduced partition functions are,

\begin{align*}
\hat{Z}_{c=2}^{\mathcal{W}_{2,3}} &= 8q + 4q^3 + 7q^4 + 12q^7 + \cdots \\
\hat{Z}_{c=2}^{\mathcal{W}_{2,4}} &= 8q + 5q^3 + 5q^4 + 2q^6 + 11q^7 + \cdots 
\end{align*}

Namely, the character decomposition with positive integer coefficients is available.
- On the other hand, $\mathcal{W}_{2,6}$ rule out the $(A_2)_1$ WZW model.
- In the eye of character decomposition,

\begin{align*}
\hat{Z}_{c=2}^{\mathcal{W}_{2,6}} &= 8q + 5q^3 + 6q^4 - q^5 + 12q^7 + 2q^8 + \cdots 
\end{align*}

The decomposition have negative integer coefficient at $q^5$ order. Hence, character decomposition with $\mathcal{W}_{2,6}$ algebra do not consistent with $(A_2)_1$ WZW model.
• Numerical bound with Rank-3 $\mathcal{W}$-algebra

- Among the level-1 WZW models with Delign’s exceptional series, no CFT with $c = 3$, while rank-3 $\mathcal{W}_{2,3,4}$ algebra make cliff around $c \sim 3$. Useless $\mathcal{W}_{2,3,4}$?
- The numerical boundary pass through $(c = 3, \Delta = \frac{3}{4})$, the data for $(A_3)_1$ WZW model. The character of $(A_3)_1$ is solution of third order modular differential equation.
- **CLAIM**: The $\mathcal{W}_{2,3,4}$ algebra EXCLUSIVELY realize $(A_3)_1$ WZW model!
• Analysis on \((A_3)_1\) WZW model
  
• The numerical analysis on the degeneracy of low-lying states gives

| \((h, \bar{h})\) | Max. Deg | \((h, \bar{h})\) | Max. Deg | \((h, \bar{h})\) | Max. Deg |
|-----------------|----------|-----------------|----------|-----------------|----------|
| \((\frac{3}{8}, \frac{3}{8})\) | 32.00000 | \((\frac{1}{2}, \frac{1}{2})\) | 36.00000 | \((1, 1)\) | 225.00714 |
| \((\frac{3}{8}, \frac{11}{8})\) | 96.00000 | \((\frac{1}{2}, \frac{3}{2})\) | 48.00000 | \((1, 2)\) | 75.00020 |
| \((\frac{11}{8}, \frac{11}{8})\) | 288.01585 | \((\frac{3}{2}, \frac{3}{2})\) | 64.11818 | \((2, 2)\) | 25.00500 |

• The characters of \((A_3)\) affine Lie algebra reads,

\[
\chi_{[0]}^{A_3}(q) = 1 + 15q + 51q^2 + 172q^3 + 453q^4 + 1128q^5 + \mathcal{O}(q^6),
\]

\[
\chi_{[1]}^{A_3}(q) = q^{\frac{3}{6}} \left( 4 + 24q + 84q^2 + 248q^3 + 648q^4 + 1536q^5 + \mathcal{O}(q^3) \right),
\]

\[
\chi_{[2]}^{A_3}(q) = q^{\frac{1}{2}} \left( 6 + 26q + 102q^2 + 276q^3 + 728q^4 + 1698q^5 + \mathcal{O}(q^3) \right),
\]

\[
\chi_{[3]}^{A_3}(q) = q^{\frac{3}{6}} \left( 4 + 24q + 84q^2 + 248q^3 + 648q^4 + 1536q^5 + \mathcal{O}(q^3) \right).
\]

• Again, the numerical analysis is consistent with the following diagonal structure.

\[
\mathcal{Z}_{A_3}(q, \bar{q}) = |\chi_{[0]}^{A_3}(q)|^2 + |\chi_{[1]}^{A_3}(q)|^2 + |\chi_{[2]}^{A_3}(q)|^2 + |\chi_{[3]}^{A_3}(q)|^2
\]
Conclusion and Outlook

- The two-channel RCFTs ($k = 1$ WZW models with Deligne’s exceptional series) and the three-channel RCFTs (cousins of extremal conformal field theories) are analyzed via modular bootstrap. It turns out that twist gap problem with holomorphic currents realize those theories on the numerical bound.

- The modular invariant partition function for special class of three-channel RCFTs ($c = 8$, $c = 16$, $c = \frac{47}{2}$) are suggested. The coefficients in partition function decomposed in terms of $O^+(10, 2)$ or baby monster group.

- Extension to the $\mathcal{W}$-algebra cases. We expect the refined unitary bound will be $c \geq k$.

- Application to the supersymmetric cases: Super WZW models, super extremal conformal field theory?