D-branes in a plane-wave background

M.R. Gaberdiel\textsuperscript{1}, M.B. Green\textsuperscript{2}

\textsuperscript{1} Department of Mathematics
King’s College, London
Strand, London, WC2R 2LS, UK

\textsuperscript{2} Department of Applied Mathematics and Theoretical Physics
Cambridge University
Wilberforce Road, Cambridge, CB3 0WA, UK

Abstract: The D-branes of the maximally supersymmetric plane-wave background are described.

1 Introduction

The study of type IIB superstring theory in plane wave backgrounds is a fertile area for investigating the properties of string theory with nontrivial Ramond–Ramond (RR) condensates. One of the best studied examples is the Penrose limit of \( AdS_5 \times S^5 \) \cite{1}, for which the dual CFT has a corresponding limit \cite{2}. In the original \( AdS_5 \times S^5 \) background this correspondence is difficult to check quantitatively since string theory in the \( AdS_5 \times S^5 \) background has not, so far, proved tractable. On the other hand, the plane-wave theory can be formulated as a free two-dimensional field theory, at least in the light-cone gauge, and is therefore exactly solvable \cite{3, 4}.

In this talk we shall describe the construction of supersymmetric D-branes for this string theory background. Since the world-sheet theory in light cone gauge is not conformally invariant, a number of modifications arise relative to the usual conformal field theory discussion. In particular, the functions that enter in the calculation of the cylinder amplitudes are not standard \( \theta \) functions, and the open-closed consistency condition follows from rather non-trivial modular transformation properties of deformed \( \theta \) functions.

This talk is based on \cite{5} and \cite{6}.

2 Notation and review

The \( pp \)-wave background is a ten-dimensional space-time with metric,

\[ ds^2 = 2dx^+dx^- - \mu^2 x^I x^I dx^+ dx^+ + dx^I dx^I, \]

where \( x^\pm = (x^0 \pm x^9)/\sqrt{2} \) and \( I = 1, \ldots, 8 \). The five-form \( RR \) field strength is given by

\[ F_{+1234} = F_{+5678} = 2\mu, \]
where $\mu$ is a constant. In light-cone gauge where $x^+ = 2\pi p^+ \tau$, the Lagrangian describes eight massive free scalar and eight massive free fermion fields

$$\mathcal{L} = \frac{1}{4\pi} \left( \partial_+ x^I \partial_- x^I - 2m^2 (x^I)^2 \right) + \frac{i}{2\pi} \left( \bar{S}^a \partial_+ S^a + \bar{\tilde{S}}^a \partial_- \tilde{S}^a - 2m S^a \Pi_{ab} \tilde{S}^b \right),$$

where $S^a$ and $\tilde{S}^a$ are $SO(8)$ spinors of the same chirality and $\Pi = \gamma^1 \gamma^2 \gamma^3 \gamma^4$. The mass parameter $m$ is defined by $m = 2\pi p^+ \mu$. The $8 \times 8$ matrices, $\gamma^I$, and $\gamma^{\dot{I}}$, are the off-diagonal blocks of the $16 \times 16$ $SO(8)$ $\gamma$-matrices and couple $SO(8)$ spinors of opposite chirality. The presence of $\Pi$ in the fermionic sector of the lagrangian breaks the symmetry from $SO(8)$ to $SO(4) \times SO(4)$.

The modes of the transverse bosonic coordinates, $x^I$, of a string in type IIB pp-wave light-cone gauge string theory $[3]$ are $\alpha^I_k$ and $\tilde{\alpha}^I_k$, where $I$ is a vector index of $SO(8)$, and $k \in \mathbb{Z}$ with $k \neq 0$. These modes satisfy the commutation relations

$$[\alpha^I_k, \alpha^J_l] = \omega_k \delta^{IJ} \delta_{k,-l}, \quad [\alpha^I_k, \tilde{\alpha}^J_l] = 0, \quad [\tilde{\alpha}^I_k, \alpha^J_l] = \omega_k \delta^{IJ} \delta_{k,-l},$$

where

$$\omega_k = \text{sign}(k) \sqrt{k^2 + m^2} \quad |k| > 0.$$  

In addition there are the bosonic zero modes that describe the centre of mass position $x^I_0$ and some generalised momentum $p^I_0$ with $[p^I_0, x^I_0] = -i \delta^{IJ}$. It is convenient to introduce the creation and annihilation operators

$$a^I_0 = \frac{1}{\sqrt{2m}} (p^I_0 + ix^I_0), \quad \tilde{a}^I_0 = \frac{1}{\sqrt{2m}} (p^I_0 - ix^I_0),$$

in terms of which the commutation relation is then $[\tilde{a}^I_0, a^J_0] = \delta^{IJ}$. The modes of the fermionic fields $S^a_k$ and $\tilde{S}^a_k$, where $a$ is a spinor index of $SO(8)$ and $k \in \mathbb{Z}$, satisfy the anti-commutation relations

$$\{ S^a_k, S^b_l \} = \delta^{ab} \delta_{k,-l}, \quad \{ S^a_k, \tilde{S}^b_l \} = 0, \quad \{ \tilde{S}^a_k, S^b_l \} = \delta^{ab} \delta_{k,-l}.$$  

It is convenient to introduce the zero-mode combinations $\theta^a_0 = \frac{1}{\sqrt{2}} (S^a_0 + i \tilde{S}^a_0)$ as well as $\bar{\theta}^a_0 = \frac{1}{\sqrt{2}} (S^a_0 - i \tilde{S}^a_0)$, and further

$$\theta_R = \frac{1}{2} (1 + \Pi) \theta_0, \quad \bar{\theta}_R = \frac{1}{2} (1 + \Pi) \bar{\theta}_0, \quad \theta_L = \frac{1}{2} (1 - \Pi) \theta_0, \quad \bar{\theta}_L = \frac{1}{2} (1 - \Pi) \bar{\theta}_0.$$  

In light-cone gauge the thirty-two components of the supersymmetries decompose into ‘dynamical’ and ‘kinematical’ components. The ‘dynamical’ supercharges, $Q_a$ and $\bar{Q}_a$ ($\dot{a} = 1, \ldots, 8$), anticommute to give the light-cone Hamiltonian

$$2p^+ H = m \left( a^{\dot{a}}_0 \bar{a}^a_0 + i S^a_0 \Pi_{ab} \bar{S}^b_0 + 4 \right) + \sum_{k=1}^{\infty} \left[ \alpha^I_k \alpha^I_{-k} + \tilde{\alpha}^I_k \tilde{\alpha}^I_{-k} + \omega_k \left( S^a_{-k} S^a_k + \tilde{S}^a_{-k} \tilde{S}^a_k \right) \right]$$

$$= m \left( a^{\dot{a}}_0 \bar{a}^a_0 + \theta_R^a \bar{\theta}_R^a + \tilde{\theta}_L^a \bar{\theta}_L^a \right) + \sum_{k=1}^{\infty} \left[ \alpha^I_k \alpha^I_{-k} + \tilde{\alpha}^I_k \tilde{\alpha}^I_{-k} + \omega_k \left( S^a_{-k} S^a_k + \tilde{S}^a_{-k} \tilde{S}^a_k \right) \right].$$

In addition the theory has sixteen ‘kinematical’ supercharges $Q_a \equiv S^a_0$ and $\bar{Q}_a \equiv \tilde{S}^a_0$, where $a = 1, \ldots, 8$. The dynamical supercharges commute with the light-cone Hamiltonian, but the kinematical supercharges do not.
3 The construction of the boundary states

D-branes in string theory can always be described by boundary states. This description makes the coupling of the D-brane to the closed string states of the theory manifest. In the following we shall only be discussing instantonic D-branes, which are defined by the embeddings of euclidean \((p + 1)\)-dimensional world-volumes. These are the cases in which the light-cone directions \(x^\pm\) are orthogonal to the brane. We will adopt a notation, following \([7]\), in which these instantonic branes are denoted \((r, s)\)-branes \((r + s = p + 1)\), where \(r\) and \(s\) are the numbers of Neumann directions associated with the two \(SO(4)\) factors in the transverse space.

The boundary states are (up to important normalisations) uniquely determined by the gluing conditions that describe how left- and right-moving fields are related at the boundary. Here we shall only consider D-branes that preserve half of the dynamical supersymmetries, i.e.

\[
\left( Q_a + i \eta M_{ab} \bar{Q}_b \right) \|(r, s), \eta \rangle = 0 ,
\]

where the value of \(\eta = \pm 1\) distinguishes a brane from an anti-brane. The matrix \(M\) is given by

\[
M_{ab} = \left( \prod_{I \in \mathcal{N}} \gamma^I \right)_{\bar{a} \bar{b}} ,
\]

where the product extends over the Neumann directions \(\mathcal{N}\). In order for the boundary state to define a standard D-brane one requires in addition that the bosonic degrees of freedom satisfy

\[
\left( \alpha^I_k - \bar{\alpha}^J_{-k} \right) \|(r, s), y_t \rangle = 0 , \quad k \in \mathbb{Z} ,
\]

\[
\left( \bar{a}^I_0 - a^I_0 + i \sqrt{2m y^I_t} \right) \|(r, s), y_t \rangle \equiv - i \sqrt{2m} (x^I_0 - \bar{y}^I_t) \|(r, s), y_t \rangle = 0 ,
\]

for each Dirichlet direction \(I\), as well as

\[
\left( \alpha^I_k + \bar{\alpha}^I_{-k} \right) \|(r, s), y_t \rangle = 0 , \quad k \in \mathbb{Z} ,
\]

\[
\left( \bar{a}^I_0 + a^I_0 \right) \|(r, s), y_t \rangle = 0
\]

for each Neumann direction \(J\). (For simplicity we are considering here the case without Wilson lines.) It is easy to see that \((10)\), together with \((12)\) and \((13)\) implies that

\[
\left( S^a_0 + i \eta M_{ab} \bar{S}^b_0 \right) \|(r, s), y_t \rangle = 0 ,
\]

where \(M_{ab}\) is the same product of \(\gamma\) matrices as in \((11)\). This last equation implies that a complex combination of the kinematical supersymmetries is preserved by the boundary state. The structure of the boundary states depends now crucially on the choice of \(M\).

Class I: The first class is the one that was studied in \([8, 9, 5]\) and arises when the matrix \(M_{ab}\) satisfies

\[
\Pi \Pi M = -1 ,
\]

where \(\Pi = \gamma^1 \gamma^2 \gamma^3 \gamma^4\). From a supergravity point of view, these branes were analysed in \([7]\). The condition \((15)\) is equivalent to \(|r - s| = 2\). The branes of this kind satisfy the standard fermionic gluing condition

\[
\left( S^a_n + i \eta M_{ab} \bar{S}^b_n \right) \|(r, s), y_t \rangle = 0 .
\]

\(^1\)As is explained in \([3, 4]\), the following discussion can be easily generalised for time-like branes.
They satisfy (10) if $f^I = 0$ for the Dirichlet directions, and the corresponding open string preserves eight components (i.e. half) of both the dynamical and kinematical supersymmetries. A characteristic feature of this class is that the kinematical conditions (14) are not preserved as a function of $x^+$ since the commutator with the light-cone hamiltonian has the form
\[
[H, Q_a + i\eta M_{ab} \tilde{Q}_b] = \frac{m \eta}{2p^+} (\Pi M')_{ab} \left( Q_b - i\eta M_{bc} \tilde{Q}_c \right).
\]
(17)

In this case the open-string theory has a mass term in its hamiltonian of the form $S_0 M \Pi S_0$, and the ground state is an unmatched boson. In particular, the open string one-loop amplitude (and the corresponding closed string cylinder diagram) therefore does not vanish.

**Class II:** The second class arises when the matrix $M_{ab}$ satisfies
\[
\Pi M \Pi M = 1,
\]
(18)
a possibility that was not considered in [8, 9, 5] but arose in the supergravity analyses of [7, 10] and was later analysed in detail in [6] (the open string description was independently worked out in [11]). In this case $|r - s| = 0, 4$. The only branes of this type that satisfy (10) are the $(0, 0)$ brane at an arbitrary position, for which the fermionic gluing conditions are
\[
\left( S^a_n + i\eta R^{ab}_n \tilde{S}_b^n \right) \langle (0, 0), y \rangle = 0,
\]
(19)
and $R_n$ is the matrix
\[
R_n = \frac{1}{n} \left( \omega_n \mathbb{1} - \eta m \Pi \right).
\]
(20)

In addition, the $(4, 0)$ and $(0, 4)$ branes satisfy (10) provided that the Neumann boundary conditions (13) are suitably modified (see [6]). The corresponding open strings preserve half of the dynamical supersymmetries, but none of the kinematical supersymmetries. However, in contradistinction to class I,
\[
[H, Q_a + i\eta M_{ab} \tilde{Q}_b] = -\frac{m \eta}{2p^+} (\Pi M')_{ab} \left( Q_b + i\eta M_{bc} \tilde{Q}_c \right),
\]
(21)
and thus the kinematical conditions (14) are preserved as a function of $x^+$. In this case $S_0 M \Pi S_0 = 0$ and the open-string mass term vanishes, thus giving true fermionic zero modes (see also [12]). The ground states then form a degenerate supermultiplet, and the one-loop open string amplitude (as well as the corresponding closed string cylinder diagram) vanishes.

## 4 Open-closed duality

One of the most important consistency conditions for D-branes is the so-called open-closed duality relation. It requires that the cylinder diagram, that describes the interaction energy between two D-branes, can be evaluated in two different ways. From the closed string point of view, the cylinder diagram is the tree-level diagram describing the exchange of closed string states between two external boundary states. On the other hand, the diagram can also be interpreted as a one-loop open string diagram. These two points of view are related by exchanging the roles of the world-sheet parameters $\tau$ and $\sigma$. This

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2 The same also holds for the $(4, 4)$ brane, but the corresponding boundary state does not have a sensible flat space limit and is therefore probably inconsistent.
transformation replaces the modular parameter $t$ of the cylinder by $\tilde{t} = 1/t$; it also replaces the mass-parameter $m$ by $\tilde{m} = mt$. (See [3] for more details.)

In the usual flat space background, the open-closed duality condition is satisfied since the cylinder amplitude can be written in terms of standard $\theta$ functions that have well understood transformation properties under the modular transformation $t \to \tilde{t}$. For the case of the plane-wave background, on the other hand, the cylinder amplitudes involve non-trivial deformations of $\theta$ functions. In the case of the class I branes, the relevant functions are

$$f_1^{(m)}(q) = q^{-\Delta_m} (1 - q^m)^{1/2} \prod_{n=1}^{\infty} \left( 1 - q^{m^2+n^2} \right),$$

$$f_2^{(m)}(q) = q^{-\Delta_m} (1 + q^m)^{1/2} \prod_{n=1}^{\infty} \left( 1 + q^{m^2+n^2} \right),$$

$$f_3^{(m)}(q) = q^{-\Delta'_m} \prod_{n=1}^{\infty} \left( 1 + q^{m^2+(n-1)^2} \right),$$

$$f_4^{(m)}(q) = q^{-\Delta'_m} \prod_{n=1}^{\infty} \left( 1 - q^{m^2+(n-1)^2} \right),$$

where $q = e^{-2\pi t}$, and $\Delta_m$ and $\Delta'_m$ are given as

$$\Delta_m = -\frac{1}{(2\pi)^2} \sum_{p=1}^{\infty} \int_0^{\infty} ds e^{-p^2 s} e^{-\pi^2 m^2/s},$$

$$\Delta'_m = -\frac{1}{(2\pi)^2} \sum_{p=1}^{\infty} (-1)^p \int_0^{\infty} ds e^{-p^2 s} e^{-\pi^2 m^2/s}.$$  

The quantities $\Delta_m$ and $\Delta'_m$ are the Casimir energies of a single (two-dimensional) boson of mass $m$ on a cylindrical world-sheet with periodic and anti-periodic boundary conditions, respectively. For $m = 0$, $\Delta_m$ and $\Delta'_m$ simplify to the usual flat-space values,

$$\Delta_0 = -\frac{1}{(2\pi)^2} \sum_{p=1}^{\infty} \frac{1}{p^2} = -\frac{1}{24},$$

$$\Delta'_0 = -\frac{1}{(2\pi)^2} \sum_{p=1}^{\infty} \frac{(-1)^p}{p^2} = \frac{1}{48}.$$  

Thus $f_2^{(m)}(q)$, $f_3^{(m)}(q)$ and $f_4^{(m)}(q)$ simply reduce to the standard $f_2(q)$, $f_3(q)$ and $f_4(q)$ functions [13] as $m \to 0$.

For the case of the class I branes the open-closed consistency condition is then a consequence of the remarkable transformation properties of these functions [3]

$$f_1^{(m)}(q) = f_1^{(\tilde{m})}(\tilde{q}), \quad f_2^{(m)}(q) = f_4^{(\tilde{m})}(\tilde{q}), \quad f_3^{(m)}(q) = f_3^{(\tilde{m})}(\tilde{q}),$$

where $\tilde{q} = e^{-2\pi t} = e^{-2\pi / t}$ and $\tilde{m} = mt$. In the limit $m \to 0$ the second and third equations in (28) reproduce standard $\theta$-function identities. The identity for $f_1$ (or $\eta$) can also be derived from the first equation of (23). In fact, both sides of the first equation tend to zero as $m \to 0$ since $(1 - q^m)^{1/2} = \sqrt{2\pi tm + O(m)}$ and $(1 - \tilde{q}^{\tilde{m}})^{1/2} = \sqrt{2\pi \tilde{m} + O(m)}$. Thus, after dividing the first equation by $\sqrt{m}$, the limit $m \to 0$ becomes

$$f_1(q) = t^{-3/2} f_1(\tilde{q}),$$

The functions that arise for the class II branes are yet more complicated; they are described in detail in [4], where their modular properties are also discussed.
thus reproducing the standard modular transformation property of the $f_1$ (or $\eta$) function.

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