ON WINNING SETS AND NON-NORMAL NUMBERS

BILL MANCE

ABSTRACT. In [9], W. Schmidt proved that the set of non-normal numbers in base $b$ is a winning set. We generalize this result by proving that many sets of non-normal numbers with respect to the Cantor series expansion are winning sets. As an immediate consequence, these sets will be shown to have full Hausdorff dimension.

1. Introduction

1.1. Winning sets. In [9], W. Schmidt proposed the following game between two players: Alice and Bob. Let $\alpha \in (0,1)$, $\beta \in (0,1)$, $S \subset \mathbb{R}^n$, and let $\rho(I)$ denote the radius of a set $I$. Bob first picks any closed interval $B_1 \subset \mathbb{R}^n$. Then Alice picks a closed interval $A_1 \subset B_1$ such that $\rho(A_1) = \alpha \rho(B_1)$. Bob then picks a closed interval $B_2 \subset W_1$ with $\rho(B_2) = \beta \rho(A_1)$. After this, Alice picks a closed interval $A_2 \subset B_2$ such that $\rho(A_2) = \alpha \rho(B_2)$, and so on. We say that the set $S$ is $(\alpha, \beta)$-winning if Alice can play so that

$$\bigcap_{n=1}^{\infty} B_n \subset S.$$ (1.1)

The set $S$ is $(\alpha, \beta)$-losing if it is not $(\alpha, \beta)$-winning and $\alpha$-winning if it is $(\alpha, \beta)$-winning for all $0 < \beta < 1$. Winning sets satisfy the following properties:

1. If $S \subset \mathbb{R}^n$ is an $\alpha$-winning set, then the Hausdorff dimension of $S$ is $n$.
2. The intersection of countably many $\alpha$-winning sets is $\alpha$-winning.
3. Bi-Lipschitz homeomorphisms of $\mathbb{R}^n$ preserve winning sets.

1.2. Normal numbers.

Definition 1.1. Let $b$ and $k$ be positive integers. A block of length $k$ in base $b$ is an ordered $k$-tuple of integers in $\{0, 1, \ldots, b-1\}$. A block of length $k$ is a block of length $k$ in some base $b$. A block is a block of length $k$ in base $b$ for some integers $k$ and $b$.

Definition 1.2. Given an integer $b \geq 2$, the $b$-ary expansion of a real $x$ in $[0, 1)$ is the (unique) expansion of the form

$$x = \sum_{n=1}^{\infty} \frac{E_n}{b^n} = 0.E_1E_2E_3\ldots$$

Date: October 12, 2010 and, in revised form, —–, —–.

2000 Mathematics Subject Classification. Primary 11K16, 11A63.

Key words and phrases. Cantor series, Normal numbers, Schmidt games, Winning sets.

I would like to thank Vitaly Bergelson, Dimitry Kleinbock, and Jim Tseng for helpful discussions.

1See [5] and [9].
such that $E_n$ is in \{0, 1, \ldots, b - 1\} for all $n$ with $E_n \neq b - 1$ infinitely often.

Denote by $N^b_n(B, x)$ the number of times a block $B$ occurs with its starting position no greater than $n$ in the $b$-ary expansion of $x$.

**Definition 1.3.** A real number $x$ in $[0, 1)$ is normal in base $b$ if for all $k$ and blocks $B$ in base $b$ of length $k$, one has

$$\lim_{n \to \infty} \frac{N^b_n(B, x)}{n} = b^{-k}.$$

Borel introduced normal numbers in 1909 and proved that almost all (in the sense of Lebesgue measure) real numbers in $[0, 1)$ are normal in all bases. The best known example of a number that is normal in base 10 is due to Champernowne [4].

The number $H_{10} = 0.123456789101112\ldots$, formed by concatenating the digits of every natural number written in increasing order in base 10, is normal in base 10. Any $H_b$, formed similarly to $H_{10}$ but in base $b$, is known to be normal in base $b$. Since then, many examples have been given of numbers that are normal in at least one base. One can find a more thorough literature review in [6] and [7].

Suppose that $X = \{x_n\}_{n=1}^\infty$ is a sequence of real numbers. For a positive integer $N$ and some $I \subset [0, 1)$, we define $A_N(I, X)$ to be the number of terms $x_n$ with $1 \leq n \leq N$, for which $x_n - \lfloor x_n \rfloor \in I$. Thus, we may write

$$A_N(I, X) = \# \{n \in [1, N] : x_n - \lfloor x_n \rfloor \in I \}.$$

**Definition 1.4.** The sequence $X = \{x_n\}_{n=1}^\infty$ is uniformly distributed mod 1 if for every pair $a, b$ of real numbers with $0 \leq a < b < 1$, we have

$$\lim_{N \to \infty} \frac{A_N([a, b], X)}{N} = \lambda([a, b]) = b - a.$$

**Theorem 1.5.** A real number $x \in [0, 1)$ is normal in base $b$ if and only if the sequence $\{b^n x\}_{n=0}^\infty$ is uniformly distributed mod 1.

**Theorem 1.6.** Let $\alpha' < \alpha < 1$. Then every $\alpha$-winning set is $\alpha'$-winning.

**Theorem 1.7.** The only $\alpha$-winning set $S \subset \mathbb{R}$ with $\alpha > 1/2$ is $S = \mathbb{R}$.

W. Schmidt proved the following in [9]:

**Theorem 1.8.** Let $0 < \alpha < 1$, $0 < \beta < 1$, $\gamma = 1 + \alpha \beta - 2\alpha > 0$. Let $b$ be an integer so large that $b > 4/(\alpha \beta \gamma)$ and let $d$ be an integer in $[0, b-1]$. Then the set of all real numbers whose $b$-ary expansion has finitely many occurrences of the digit $d$ is $(\alpha, \beta)$-winning.

**Corollary 1.9.** Let $b \geq 2$ be an integer and let $S$ be the set of numbers not normal in base $b$. Then $S$ is a $1/2$-winning set.

Based on Theorem 1.6 and Theorem 1.7, Corollary 1.9 is as strong as we could hope for.
1.3. Conventions and definitions. The following conventions and definitions will hold for the rest of this paper. Put

\[ S = \{(x, y) \in (0, 1)^2 : 1 + xy - 2x > 0\}. \]

Given any \((\alpha, \beta) \in S\), we set \(\gamma = \gamma(\alpha, \beta) = 1 + \alpha\beta - 2\alpha\). If \(I = (a, b)\) and \(J = (c, d)\) with \(c > b\), then we say that the distance between \(I\) and \(J\) is \(c - b\). We will use a similar definition for closed and half-open intervals. Additionally, we will say that \(I\) intersects \(J\) if \(I \cap J \neq \emptyset\).

Let \(\mathcal{C} = \{\mathcal{C}_k\}_{k=1}^{\infty}\) where \(\mathcal{C}_k = \{C_{k,n} \}_{n=-\infty}^{\infty}\) is a sequence of finite disjoint intervals all separated by some positive distance where \(C_{k,n-1}\) is positioned to the left of \(C_{k,n}\). Additionally, we require that every finite interval contained in \(\mathbb{R}\) intersects at most finitely many members of \(\mathcal{C}_k\) and that for every interval \(I\), there exists an integer \(K_I\) such that for all \(k > K_I\), \(I\) intersects at least two members of \(\mathcal{C}_k\). Let \(\mathcal{D} = \{\mathcal{D}_k\}_{k=1}^{\infty}\) where \(\mathcal{D}_k = \{D_{k,n} \}_{n=-\infty}^{\infty}\) is the sequence of intervals between each adjacent member of \(\mathcal{C}_k\) such that \(D_{k,n}\) is between \(C_{k,n-1}\) and \(C_{k,n}\). Set

\[ W_{k,n} = \{C_{k+1,m} : C_{k+1,m} \subset D_{k,n}\} \quad \text{and} \quad V_{k,n} = \{D_{k+1,m} : D_{k+1,m} \subset D_{k,n}\} \]

and assume that for all \(k\) and \(n\), \(C_{k+1,n}\) is contained in some member of \(\mathcal{C}_k\) or some member of \(\mathcal{D}_k\) and \(D_{k+1,n}\) is also contained in some member of \(\mathcal{C}_k\) or some member of \(\mathcal{D}_k\). Then we say that \(\mathcal{C}\) is \((\alpha, \beta)\)-friendly if each member of \(\mathcal{D}_k\) contains at least three members of \(\mathcal{C}_{k+1}\) and there exists an integer \(K_{\mathcal{C}}\) such that for all \(k > K_{\mathcal{C}}\), integers \(n\), and intervals \(D \in V_{k,n}\), we have

\[ \lambda(D_{k,n}) > \frac{1}{\alpha\beta\gamma} \cdot \max(\lambda(C_{k,n}), \lambda(C_{k,n+1})) \]

and

\[ \lambda(D) < \frac{(\alpha\beta)^2\gamma}{6} \min(\lambda(D_{k,n-1}), \lambda(D_{k,n}), \lambda(D_{k,n+1})). \]

Put

\[ X_{\mathcal{C}} = \left\{ x \in \mathbb{R} : \left\{ k \in \mathbb{N} : x \in \bigcup_{n=-\infty}^{\infty} C_{k,n} \right\} \text{is finite} \right\}. \]

**Theorem 1.10.** If \(\mathcal{C}\) is \((\alpha, \beta)\)-friendly, then \(X_{\mathcal{C}}\) is an \((\alpha, \beta)\)-winning set.

Although it isn’t a direct generalization of Theorem 1.8, Theorem 1.10 may be used to study a much wider variety of sets than Theorem 1.8. We use Theorem 1.10 to prove the following:

1. The set of numbers not normal in some base \(b\) is 1/2-winning. This result also follows directly from Theorem 1.8.
2. (Theorem 3.13 and Theorem 3.15) If \(Q\) is infinite in limit, then the set of real numbers that are not \(Q\)-ratio normal of order 2 and the set of numbers that are not \(Q\)-distribution normal are both 1/2-winning. If \(Q\) is 1-divergent, then the set of numbers that are not simply \(Q\)-normal is 1/2-winning. For every basic sequence \(Q\), the set of numbers that is not strongly \(Q\)-distribution normal is 1/2-winning.
2. Proof of Theorem [1,10]

We will need the following lemma from [9]:

**Lemma 2.1.** Suppose that \((\alpha, \beta) \in S\) and let the integer \(t\) satisfy \((\alpha \beta)^t < \gamma/2\). Assume that a ball \(B_k\) with radius \(r_k\) and center \(b_k\) occurs in some \((\alpha, \beta)\)-play. Then Alice needs to worry about avoiding at most one interval, \(C\), more than \(g\).

**Lemma 2.2.** If \((\alpha, \beta) \in S\), then \(\alpha \beta \gamma < 1/4\).

**Proof.** Let \(f(x, y) = xy(1 + xy - 2x) = xy + x^2y^2 - 2x^2y\) for \((x, y) \in [0, 1]^2\). By a routine computation, the maximum of \(f(x, y)\) is 1/4 and occurs only at \((x, y) = (1/2, 1)\). Since \(S \subset [0, 1]^2\) and \((1/2, 1) \notin S\), \(\alpha \beta \gamma < 1/4\) for \((\alpha, \beta) \in S\).

**Lemma 2.3.** Suppose that \(C\) is \((\alpha, \beta)\)-friendly and that Bob has chosen \(B_n\). If the positive integer \(k > K_c\) is such that \(B_m\) intersects at least two distinct members of \(\mathcal{C}_k\), then there exists an integer \(s > n\) so that Alice can play such that \(B_s\) intersects no members of \(\mathcal{C}_k\) and \(B_s\) intersects at least two members of \(\mathcal{C}_{k+1}\).

**Proof.** Alice and Bob can play however they want until step \(m > n\) where \(B_{m-1}\) has a non-empty intersection with adjacent intervals \(C_{k,e-1}\) and \(C_{k,e}\) in \(\mathcal{C}_k\), but \(B_m\) intersects at most one of \(C_{k,e-1}\) and \(C_{k,e}\) and no other members of \(\mathcal{C}_k\). Let \(g = \max(\lambda(C_{k,e-1}), \lambda(C_{k,e}))\). Thus, \(\lambda(B_m) \geq \lambda(D_{k,e})\) and

\[
\lambda(B_m) \geq \frac{1}{\alpha \beta \gamma} g \text{ and } \lambda(B_m) > \alpha \beta \left(\frac{1}{\alpha \beta \gamma} g\right) = \frac{1}{\gamma} g, \text{ so (2.1)}
\]

Thus, Alice needs to worry about avoiding at most one interval, \(C\), of length no more than \(g\). Without loss of generality, assume that the center of \(C\) is less than or equal to the center \(b_m\) of \(B_m\) and that \(C \subset C_{k,e-1}\). Then by (2.1),

\[
C \subset (-\infty, b_m + g/2] \subset (-\infty, b_m + \lambda(B_m) \gamma/2).
\]

Let \(t\) be the positive integer that satisfies \(\frac{\alpha \beta \gamma}{2} \leq (\alpha \beta)^t < \frac{\gamma}{2}\) and set \(s = m + t\). By Lemma 2.1 Alice can play in such a way that

\[
B_s \subset (b_m + \lambda(B_m) \gamma/2, \infty),
\]

so \(B_s\) does not intersect any member of \(\mathcal{C}_k\). We further note that

\[
\lambda(B_s) = (\alpha \beta)^t \lambda(B_m) \geq \frac{\alpha \beta \gamma}{2} \lambda(B_m) = \frac{(\alpha \beta)^2 \gamma}{2} \lambda(D_{k,e}).
\]

Thus, if \(D \in \mathcal{V}_{k,e}\), then by (1.4)

\[
\lambda(B_s) > \frac{(\alpha \beta)^2 \gamma}{2} \left(\frac{6}{(\alpha \beta)^2 \gamma} \lambda(D)\right) = 3\lambda(D).
\]

By (1.3) and Lemma 2.2 the length of each member of \(W_{k,e}\) is less than one fourth of the length of each adjacent member of \(\mathcal{V}_{k,e}\). Since the length of \(B_s\) is at least 3 times bigger than every gap between members of \(W_{k,e}\), \(B_s\) must intersect at least two members of \(W_{k,e}\). \(\square\)

We may now prove Theorem [1,10].

---

2This is safe to do as (1.3) will allow us control over the members of \(V_{k,e-1}, V_{k,e}, \text{ and } V_{k,e+1}\) and we will still be able to guarantee that \(B_s\) intersects two members of \(\mathcal{C}_{k+1}\).
Proof: First, Bob will pick a closed interval $B_1$. Since $\mathcal{C}$ is $(\alpha, \beta)$-friendly, there is a $k > K_\mathcal{C}$ such that $B_1$ will intersect at least two members of $\mathcal{C}_k$. By Lemma 2.3 there exists a positive integer $s$ such that Alice may force some $B_s \cap C_{k,t} = \emptyset$ for all $t$. But Lemma 2.3 guarantees that $B_s$ will satisfy the hypotheses of Lemma 2.3. So Alice may force $B_r \cap C_{r,t} = \emptyset$ for all $r \geq k$ and integers $t$. Thus, $\mathcal{X}_\mathcal{C}$ is $(\alpha, \beta)$-winning. \qed

3. Applications

We will need the following lemma:

Lemma 3.1. If $S$ is an $(\alpha, \beta)$-winning set for all $(\alpha, \beta) \in D$, then $S$ is 1/2-winning.

Proof. Suppose that $(1/2, \beta) \in (0, 1)^2$. Then $\gamma = \beta/2 > 0$, so $(1/2, \beta) \in D$. Thus, we may conclude that $S$ is a 1/2-winning set. \qed

3.1. The $b$-ary expansions. As a warmup, we prove that the set of non-normal numbers with respect to the $b$-ary expansion is 1/2-winning as a consequence of Theorem 1.10. This result was originally found in [9].

Theorem 3.2. Suppose that $b \geq 2$ is an integer. Then the set of numbers that are not normal in base $b$ is 1/2-winning.

Proof. Let $(\alpha, \beta) \in S$ and let $m$ be large enough so that $b^m > \frac{6}{(\alpha \beta)^2 \gamma^2}$. Put $\eta = b^m$. Define $T_\eta : \mathbb{R} \to [0, 1)$ by $T_\eta x = \eta x \pmod{1}$. Note that for $I = [c, d) \subset [0, 1)$,

$$T_\eta^{-k}(I) = \bigcup_{n = -\infty}^{\infty} \left[ \frac{n + c}{\eta^k}, \frac{n + d}{\eta^k} \right].$$

Set $I = [0, 1/\eta)$, so $\lambda(I) = 1/\eta < \frac{\alpha \beta^2}{6} < \frac{\alpha \beta^2}{(1 + \alpha \beta) \gamma^2}$. Let $\mathcal{C}_k$ consist of the intervals in (3.1) and put $\mathcal{C} = \{\mathcal{C}_k\}$. The intervals in (3.1) all have length of $\eta^{-k} \lambda(I)$ and are separated by a distance of $\eta^{-k}(1 - \frac{\lambda(I)}{1})$. Thus, for all $k$ and $n$, $\lambda(C_{k,n}) = \eta^{-k} \lambda(I)$ and $\lambda(D_{k,n}) = \eta^{-k}(1 - \lambda(I))$, so (1.3) and (1.4) both hold. So, $\mathcal{X}_\mathcal{C}$ is $(\alpha, \beta)$-winning. Since $(\alpha, \beta) \in S$ was arbitrary and $\mathcal{X}_\mathcal{C}$ is always contained in the set of numbers not normal in base $b$, this set is 1/2-winning by Lemma 3.1. \qed

3.2. The Cantor series expansion. The $Q$-Cantor series expansion, first studied by Georg Cantor in [3], is a natural generalization of the $b$-ary expansion.

Definition 3.3. $Q = \{q_n\}_{n=1}^\infty$ is a basic sequence if each $q_n$ is an integer greater than or equal to 2.

Definition 3.4. Given a basic sequence $Q$, the $Q$-Cantor series expansion of a real number $x$ is the (unique) expansion of the form

$$x = [x] + \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \cdots q_n}$$

such that $E_n$ is in $\{0, 1, \ldots, q_n - 1\}$ for all $n$ with $E_n \neq q_n - 1$ infinitely often.\footnote{The proof of this theorem is similar to the proof of Corollary 1.9 found in [9].} \footnote{Uniqueness can be proven in the same way as for the $b$-ary expansion.}
Clearly, the b-ary expansion is a special case of \([3,2]\) where \(q_n = b\) for all \(n\). If one thinks of a b-ary expansion as representing an outcome of repeatedly rolling a fair \(b\)-sided die, then a \(Q\)-Cantor series expansion may be thought of as representing an outcome of rolling a fair \(q_1\) sided die, followed by a fair \(q_2\) sided die and so on. For example, if \(q_n = n + 1\) for all \(n\), then the \(Q\)-Cantor series expansion of \(e - 2\) is

\[
e = 2 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots
\]

If \(q_n = 10\) for all \(n\), then the \(Q\)-Cantor series expansion for \(1/4\) is

\[
\frac{1}{4} = \frac{2}{10} + \frac{5}{10^2} + \frac{0}{10^3} + \frac{0}{10^4} + \cdots
\]

For a given basic sequence \(Q\), let \(N^Q_n(B, x)\) denote the number of times a block \(B\) occurs starting at a position no greater than \(n\) in the \(Q\)-Cantor series expansion of \(x\). Additionally, define

\[
Q^{(k)}_n = \sum_{j=1}^{n} \frac{1}{q_j q_{j+1} \cdots q_{j+k}}.
\]

**Definition 3.5.** A real number \(x\) is \(Q\)-normal of order \(k\) if for all blocks \(B\) of length \(k\),

\[
\lim_{n \to \infty} \frac{N^Q_n(B, x)}{Q^{(k)}_n} = 1.
\]

We say that \(x\) is \(Q\)-normal if it is \(Q\)-normal of order \(k\) for all \(k\). A real number \(x\) is \(Q\)-ratio normal of order \(k\) if for all blocks \(B\) and \(B'\) of length \(k\), we have

\[
\lim_{n \to \infty} \frac{N^Q_n(B, x)}{N^Q_n(B', x)} = 1.
\]

\(x\) is \(Q\)-ratio normal if it is \(Q\)-ratio normal of order \(k\) for all positive integers \(k\). \(x\) is simply \(Q\)-normal if it is \(Q\)-normal of order 1 and simply \(Q\)-ratio normal if it is \(Q\)-ratio normal of order 1.

Let \(x\) be a real number and let \(Q\) be a basic sequence. Define the 1-periodic function \(T_{Q, n} : \mathbb{R} \to [0, 1)\) by \(T_{Q, n}(x) = q_1 \cdots q_n x \pmod{1}\). Given a basic sequence \(Q\) and an interval \(I \subset [0, 1)\), we let \(E_{Q, I, k}\) consist of the intervals of \(T_{Q, k}^{-1}(I)\) and put \(E_{Q, I} = \{E_{Q, I, k}\}_{k=1}^{\infty}\).

**Definition 3.6.** A real number \(x\) is \(Q\)-distribution normal if the sequence \(\{T_{Q, n}(x)\}_{n=1}^{\infty}\) is uniformly distributed mod 1. \(x\) is strongly \(Q\)-distribution normal if for all positive integers \(k\) and \(p \in [1, k]\), the sequence \(\{T_{Q, kn+p}(x)\}_{n=1}^{\infty}\) is uniformly distributed mod 1.

**Definition 3.7.** A basic sequence \(Q\) is \(k\)-divergent if \(\lim_{n \to \infty} Q^{(k)}_n = \infty\). \(Q\) is fully divergent if \(Q\) is \(k\)-divergent for all \(k\). \(Q\) is infinite in limit if \(q_n \to \infty\).

\[\text{All strongly } Q\text{-distribution normal numbers are also } Q\text{-distribution normal. This follows as the superposition of a finite number of sequences that are uniformly distributed mod 1 is also uniformly distributed mod 1 (see [2]). For some basic sequences } Q, \text{ there exist numbers that are } Q\text{-distribution normal, but not strongly } Q\text{-distribution normal. For example, put } E = (0, 1, 0, 2, 1, 3, 0, 3, 1, 4, 2, 5, \ldots), \; Q = (2, 2, 4, 4, 4, 4, 6, 6, 6, 6, 6, \ldots), \text{ and } x = \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \cdots q_n}.
\]

Then \(x\) is \(Q\)-distribution normal, but not strongly \(Q\)-distribution normal as \(T_{Q, 2n}(x) > 1/2\) for all \(n\).
For $Q$ that are infinite in limit, it has been shown that the set of all real numbers $x$ that are $Q$-normal of order $k$ has full Lebesgue measure if and only if $Q$ is $k$-divergent \cite{8}. Therefore, if $Q$ is infinite in limit, then the set of all $x$ that are $Q$-normal has full Lebesgue measure if and only if $Q$ is fully divergent. We need the following from \cite{11}:

**Theorem 3.8.** Let $\{a_n\}_{n=1}^{\infty}$ be a given sequence of distinct integers. Then the sequence $\{a_n x\}_{n=1}^{\infty}$ is uniformly distributed mod 1 for almost all real numbers $x$.

The typicality of strongly $Q$-distribution normal numbers follows from Theorem 3.8. Clearly, all numbers that are $Q$-normal of order $k$ are also $Q$-ratio normal of order $k$. However, unlike the $b$-ary expansion, neither $Q$-normality or $Q$-distribution normality imply each other (see \cite{1} for explicit constructions).

**Lemma 3.9.** Suppose that $(\alpha, \beta) \in S$ and $Q$ is a basic sequence such that $q_k > \frac{6}{(\alpha \beta)^2 \gamma}$ for large enough $k$ and that $I = [a, b] \subset [0, 1)$ with $\lambda(I) < \frac{\alpha \beta \gamma}{1 + \alpha \beta \gamma}$. Then $C_{Q, I}$ is $(\alpha, \beta)$-friendly.

**Proof.** Note that

$$T_k^{-1}(I) = \bigcup_{n=-\infty}^{\infty} \left[ \frac{n + a}{q_1 q_2 \cdots q_k}, \frac{n + b}{q_1 q_2 \cdots q_k} \right]$$

is the union of intervals of length $\lambda(I) \cdot \frac{1}{q_1 q_2 \cdots q_k}$ separated by a distance of $(1 - \lambda(I)) \cdot \frac{1}{q_1 q_2 \cdots q_k}$. Let $K$ be large enough such that for $k \geq K$, we have $q_{k+1} > \frac{6}{(\alpha \beta)^2 \gamma}$. Then

$$\frac{1}{1 - \lambda(I)} = \frac{1}{q_1 q_2 \cdots q_k} \cdot \frac{1}{q_{k+1}} < \frac{(\alpha \beta)^2 \gamma}{6 \cdot q_1 q_2 \cdots q_k} \cdot \frac{1}{1 - \lambda(I)}$$

so (1.3) holds. Next, we see that

$$(1 - \lambda(I)) \cdot \frac{1}{q_1 q_2 \cdots q_k} > \left( 1 - \frac{\alpha \beta \gamma}{1 + \alpha \beta \gamma} \right) \frac{1}{q_1 q_2 \cdots q_k} = \frac{1}{q_1 q_2 \cdots q_k} \cdot \frac{1}{1 + \alpha \beta \gamma} \cdot \frac{1}{q_1 q_2 \cdots q_k} \cdot \lambda(I) \cdot \frac{1}{q_1 q_2 \cdots q_k}$$

so (1.3) holds and $C_{Q, I}$ is $(\alpha, \beta)$-friendly. \hfill \Box

**Lemma 3.10.** If $Q$ is infinite in limit, $x$ is $Q$-ratio normal of order 2, and $t$ is a non-negative integer, then

$$\lim_{n \to \infty} N^Q_n((t), x) = \infty.$$

**Proof.** Since $Q$ is infinite in limit and $x$ is $Q$-ratio normal of order 2, for all $i, j \geq 0$, we have

$$\lim_{n \to \infty} \frac{N^Q_n((t, i), x)}{N^Q_n((t, j), x)} = 1.$$

So, for all $j$ there is an $n$ such that $N^Q_n((t, j), x) \geq 1$. Since there are infinitely many choices for $j$, the lemma follows. \hfill \Box

Given $(\alpha, \beta) \in S$, put $I(\alpha, \beta) = \left[ 0, \frac{\alpha \beta \gamma}{1 + \alpha \beta \gamma} \right]$. Let $P_Q$ be the set of real numbers whose $Q$-Cantor series expansion contains finitely many copies of the digit 0.

**Lemma 3.11.** If $Q$ is infinite in limit, then $P_Q$ is 1/2-winning.
Theorem 3.13. Suppose that \( Q \) is infinite in limit. Then the set of numbers that are not \( Q \)-ratio normal of order 2 is a 1/2-winning set. The set of real numbers that are not \( Q \)-distribution normal is 1/2-winning. If \( Q \) is 1-divergent, then the set of numbers that are not simply \( Q \)-normal is 1/2-winning.

Proof. The first conclusion follows directly from Lemma 3.11 and Lemma 3.12. If \( Q \) is 1-divergent and \( x \) is simply \( Q \)-normal, then every digit occurs infinitely often in the \( Q \)-Cantor series expansion of \( x \). Thus, the set of numbers that is not \( Q \)-normal is 1/2-winning. Additionally, the set of real numbers that are not \( Q \)-distribution normal contains \( \mathcal{X}_{C_{Q,I,(\alpha,\beta)}} \) for all \( (\alpha,\beta) \in S \) and is 1/2-winning as well.

Corollary 3.14. If \( Q \) is infinite in limit, then the set of numbers that are not \( Q \)-normal of order 2 is a 1/2-winning set.

Theorem 3.15. If \( Q \) is any basic sequence, then the set of numbers that are not strongly \( Q \)-distribution normal is a 1/2-winning set.

Proof. Suppose that \( (\alpha,\beta) \in S \). Put \( t = 1 + \lceil \log_2 \frac{6}{(\alpha \beta)^{1/2}} \rceil \). Let the basic sequence \( \Psi_{Q,t} = \{ \psi_{t,j} \}_{j=1}^{\infty} \) be given by \( \psi_{t,j} = q_{(j-1)t+1} \cdot q_{(j-1)t+2} \cdot \cdots q_{jt} \). Thus, if \( q_n \geq 2 \) for all \( n \), then \( \psi_{t,j} \geq 2^t > 6/(\alpha \beta)^{1/2} \). Then \( \mathcal{X}_{\Psi_{Q,t,(\alpha,\beta)}} \) is \( (\alpha,\beta) \)-winning. But \( \mathcal{X}_{\Psi_{Q,t,(\alpha,\beta)}} \) is contained in the set of numbers that are not strongly \( Q \)-distribution normal, so the conclusion follows.

Corollary 3.16. The Hausdorff dimension of all of the sets considered in Theorem 3.13, Corollary 3.14, and Theorem 3.15 is 1.

Remark 3.17. The proof of Theorem 3.13 also shows that if \( Q \) is infinite in limit, then the set of real numbers \( x \) such that \( \{ q_1 q_2 \cdots q_n x \pmod{1} \}_{n=1}^{\infty} \) is not dense in \([0,1)\) is a 1/2-winning set. Similar results can be stated for the sets considered in Theorem 3.13 and Theorem 3.15.

References

1. C. Altomare, B. Mance, Cantor series constructions contrasting two notions of normality, Monatsch. Math. (to appear).
2. E. Borel, Les probabilités dénombrables et leurs applications arithmétiques, Rend. Circ. Mat. Palermo 27 (1909), pp. 247–271.
3. G. Cantor, Über die einfachen Zahlensysteme, Zeitschrift für Math. und Physik 14 (1869), pp. 121–128.
4. D. G. Champernowne, *The construction of decimals normal in the scale of ten*, Journal of the London Mathematical Society 8 (1933), pp. 254–260.

5. S. G. Dani. *On badly approximable numbers, Schmidt games and bounded orbits of flows*. In *Number theory and dynamical systems*, London Math. Soc. Lecture Note Ser. 134, pp. 69–86, York, 1987.

6. M. Drmota, R. F. Tichy, *Sequences, Discrepancies and Applications*, Springer-Verlag, Berlin Heidelberg (1997).

7. L. Kuipers, H. Niederreiter, *Uniform Distribution of Sequences*, Dover, Mineola, NY, 2006.

8. A. Rényi, *On the distribution of the digits in Cantor’s series*, Mat. Lapok 7 (1956), pp. 77–100.

9. W. M. Schmidt, *On badly approximable numbers and certain games*, Trans. A.M.S. 123 (1966), pp. 27–50.

Department of Mathematics, The Ohio State University, Columbus, Ohio 43210-1174

E-mail address: mance@math.ohio-state.edu