BULK UNIVERSALITY AND QUANTUM UNIQUE ERGODICITY FOR RANDOM BAND MATRICES IN HIGH DIMENSIONS

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ABSTRACT. We consider Hermitian random band matrices $H = (h_{xy})$ on the $d$-dimensional lattice $(\mathbb{Z}/L\mathbb{Z})^d$, where the entries $h_{xy} = \overline{h_{yx}}$ are independent centered complex Gaussian random variables with variances $s_{xy} = \mathbb{E}|h_{xy}|^2$. The variance matrix $S = (s_{xy})$ has a banded profile so that $s_{xy}$ is negligible if $|x-y|$ exceeds the band width $W$. For dimensions $d \geq 7$, we prove the bulk eigenvalue universality of $H$ under the condition $W \gg L^\varepsilon$ for a small constant $\varepsilon > 0$, we also prove the quantum unique ergodicity for the bulk eigenvectors of $H$ and a sharp local law for the Green’s function $G(z) = (H-z)^{-1}$ up to $\mathbb{I}m z \gg W^{-5}L^{5-d}$. The local law implies that the bulk eigenvector entries of $H$ are of order $O(W^{-5/2}L^{-d/2+5/2})$ with high probability.

1. INTRODUCTION

A $d$-dimensional random band matrix describes a Hamiltonian on a $d$-dimensional lattice with random hoppings in a band of width $W$. In this paper, we consider a large finite lattice $\mathbb{Z}^d_L := \{1, \ldots, L\}^d$ with $N := L^d$ many lattice sites and define a random band matrix ensemble $H = (h_{xy})$ on it. The entries of $H$ are independent (up to the Hermitian condition) centered random variables with a translation invariant variance profile $s_{xy} := \mathbb{E}|h_{xy}|^2 = f(|x-y|/W)$ for a rapidly decaying function $f$. In particular, the variance is negligible when $|x-y| \gg W$. Without loss of generality, we can normalize $f$ so that

$$\sum_x s_{xy} = \sum_y s_{xy} = 1. \quad (1.1)$$

It is well-known that under (1.1), the global eigenvalue distribution of $H$ converges weakly to the Wigner’s semicircle law supported in $[-2,2]$ [71]. The random band matrix ensemble naturally interpolates between the mean-field Wigner ensemble [71] and the famous Anderson model [5] as $W$ varies. In particular, a sharp Anderson metal-insulator transition is conjectured to occur when $W$ crosses a critical band width $W_c$: if $W \ll W_c$, $H$ has localized bulk eigenvectors and Poisson statistics for bulk eigenvalues; if $W \gg W_c$, $H$ has delocalized bulk eigenvectors and GOE/GUE statistics for bulk eigenvalues. Based on simulations and some non-rigorous supersymmetry arguments, the critical band width is conjectured to be $W_c = \sqrt{\log L}$ when $d = 2$, and $W_c \sim 1$ when $d \geq 3$ [16, 23, 24, 46, 47, 67, 68, 69, 72].

Despite the importance of the Anderson metal-insulator transition in physics, establishing its rigorous theory for a concrete model (including the Anderson model and random band matrices) remains one of the major open problems in mathematical physics. There have been many partial results concerning the localization or delocalization of one-dimensional (1D) random band matrices [6, 9, 18, 20, 22, 25, 29, 32, 33, 35, 36, 49, 57, 59, 60, 61, 62, 63, 64, 65, 66, 75] and random band matrices in dimensions $d \geq 2$ [30, 31, 32, 33, 35, 49, 73, 74, 75]. We refer the reader to [16, 22, 29, 73] for a brief review of some of these results. So far, the best delocalization result for high-dimensional band matrices was obtained in [73, 74], which proved that if $d \geq 8$ and $W \geq L^\varepsilon$ for an arbitrary small constant $\varepsilon > 0$, the bulk eigenvectors of random band matrices are (weakly) delocalized in the following senses: with high probability, most bulk eigenvectors have localization length of order $L$ and the $\ell^\infty$-norm of every bulk eigenvector of $H$ is at most $W/L$. On the other hand, the eigenvalue statistics of random band matrices are much harder to study. To

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the best of our knowledge, the GOE/GUE statistics of band matrices with general variance profiles have only been proved in 1D under \( W = \Omega(L) \) in [18] and under \( W \gg L^{3/4} \) in [22].

One main goal of this paper is to answer this important question for high-dimensional band matrix ensembles. We show that the bulk eigenvalue universality and quantum unique ergodicity (QUE) hold for random band matrices in dimensions \( d \geq 7 \) as long as the band width \( W \) is reasonably large. More precisely, we obtain the following results in this paper:

(i) If \( W \gg L^{5/(d+95)} \), the bulk eigenvalue statistics of \( H \) match those of GOE/GUE asymptotically.
(ii) If \( W \geq L^{\delta} \) for an arbitrary small constant \( \delta > 0 \), the QUE of bulk eigenvectors holds.
(iii) If \( W \geq L^{\delta} \) for an arbitrary small constant \( \delta > 0 \), an optimal local law holds for the Green’s function (or resolvent) of \( H \),

\[
G(z) = (H - z)^{-1}, \quad z \in \mathbb{C}_+ := \{ z \in \mathbb{C} : \text{Im} \, z > 0 \},
\]
down to the scale \( \text{Im} \, z \gg \eta_* := W^{-5}L^{5-d} \).

The bulk universality established here gives a rare example of random matrix statistics arising from a “local” model. Roughly speaking, the band width is viewed as an equivalent of the range of interaction, so for a band matrix ensemble to be considered as a local model, \( W \) needs to be finite. In very high dimensions \( d \), our assumption \( W \gg L^{5/(d+95)} \) is only slightly worse than the conjectured optimal assumption that \( W \) is large but finite. Notice that QUE is a stronger notion than delocalization by saying that every bulk eigenvector of \( H \) is asymptotically uniformly distributed on subsets of microscopic scales \( o(N) \).

Equivalently, it states that given an eigenvector \( u_\alpha \) and a real diagonal matrix \( \Pi \) with \( \text{tr} \, (\Pi) = 0 \), \( N|\langle u_\alpha, \Pi u_\alpha \rangle| \) is much smaller than its typical size \( \sum_x |\Pi_{xx}| \), where the inner product \( \langle \cdot, \cdot \rangle \) is defined as \( \langle u, v \rangle := u^*v \). The local law of \( G(z) \) says that the resolvent entries \( G_{xy} \) are well-approximated by \( m(z)\delta_{xy} \) for \( z = E + i\eta \) with \( E \) in the bulk and \( \eta \gg \eta_* \), where \( m(z) \) is the Stieltjes transform of Wigner’s semicircle law,

\[
m(z) := -z + \frac{\sqrt{z^2 - 4}}{2} = \frac{1}{2\pi} \int_{-2}^2 \frac{\sqrt{4 - \xi^2}}{\xi - z} \, d\xi, \quad (1.2)
\]

An immediate consequence of the local law is the following delocalization estimate: in dimensions \( d \geq 7 \), the \( \ell^\infty \)-norm of every bulk eigenvector of \( H \) is at most \( W^{-5/2}L^{-d/2+5/2} \) with high probability provided that \( W \geq L^{\delta} \). Compared with [73, 74], this result covers a new dimension \( d = 7 \) and improves the \( \ell^\infty \) bound \( W/L \) to \( W^{-5/2}L^{-d/2+5/2} \), which is already very close to the optimal bound \( L^{-d/2+7} \)—the extra factor is only \( (L/W)^{5/2} \).

One main strategy for the proof of the bulk universality of \( H \) roughly follows the three-step strategy initiated in [38, 41]. The reader can refer to [42] for an overview of this strategy. Our local law in Theorem 1.5 below completes the first step for \( \eta \gg \eta_* \). In the second step, we consider the matrix Brownian motion \( H_t := \sqrt{1-t}H + B_t \), where \( B_t \) is a Hermitian matrix whose entries are independent Brownian motions with variances \( t/N \) (i.e., for any fixed time \( t \), \( B_t \) has the law of \( \sqrt{t} \text{GOE} \) or \( \sqrt{t} \text{GUE} \)). With the local law as the input, it was proved in [52, 53] that the local bulk statistics of \( H_t \) converge to those of GOE/GUE at any time \( t \gg \eta_* \), which provides the optimal second step given the local law. In the third step, we need to show that the local bulk statistics of the original matrix \( H \) are well-approximated by those of \( H_t \), which is achieved by comparing the moments of \( \text{Im} \, g(z) \) with those of \( \text{Im} \, g_t(z) \), where \( g(z) := \frac{1}{N} \text{tr} \, (G(z)) \) and \( g_t(z) := \frac{1}{N} \text{tr} \, (G_t(z)) \) with \( G_t(z) := (H_t - z)^{-1} \). This presents a main obstacle to the three-step strategy, because previous Green’s function comparison arguments for this step require matching the variances of the entries of \( H \) with those of \( H_t \). However, the variances of the entries of a random band matrix (almost) vanish outside the band, so the variance matching will never hold.

To overcome the obstacle in the third step mentioned above, a mean-field reduction method was introduced in [18, 22]. Roughly speaking, it reduces the study of some spectral properties of \( H \) to that of a \( W \times W \) mean-field random matrix \( Q \) expressed as a rational function of certain blocks of \( H \). A key observation in [18, 22] is that the QUE, bulk universality and local law of \( Q \) imply the QUE and bulk universality of the original band matrix \( H \). However, the local law of \( Q \) was established via the generalized resolvent of \( H \) [20, 75]. Besides the fact that generalized resolvents seem to be extremely difficult to estimate in high dimensions, it looks like that the mean-field reduction approach is applicable only when the band width is large, while we aim to deal with narrow band matrices in this work.

In this paper, we prove the bulk universality of \( H \) using the three-step strategy but without using the mean-field reduction argument. Roughly speaking, a key observation is that in controlling \( \mathbb{E}(\text{Im} \, g_t(z))^k \) —
\[ \mathbb{E}(\text{Im} g(z))^k, \] the leading quantities governing its evolution with respect to \( t \) will contain factors of the form \( N^{1/2} (u_\alpha, \Pi u_\alpha)^2 \), where \( u_\alpha \) denotes a bulk eigenvector and \( \Pi \) is a real diagonal matrix with \( \text{tr}(\Pi) = 0 \) and \( \sum_\alpha |\Pi_{\alpha\alpha}| = 1 \). Applying the QUE for \( H \), these factors are much smaller than 1, which implies that, from \( t = 0 \) to \( t = L^c \eta_\alpha \), \( g_t(z) \) changes by a power of \( W^{-(d+9\varepsilon)} L^{95+C_\varepsilon} \) for some absolute constant \( C > 0 \). Hence, as long as \( W^{-(d+9\varepsilon)} L^{95+C_\varepsilon} \ll 1 \), the local spectral statistics essentially do not change along the DBM from \( t = 0 \) to \( t = L^c \eta_\alpha \). Together with the bulk universality for \( H_t \), it gives the bulk universality of the original band matrix \( H \).

Regarding the proof of QUE, previous arguments are either based on an eigenvector moment flow method \cite{21, 22} or some multi-resolvent local laws for \( G(z) \) \cite{26, 27, 28}. Our current proof of QUE is instead based on an extension of the ideas in \cite{73, 74}. More precisely, proving the QUE amounts to bounding high moments of \( P := \text{tr}(\Pi(\text{Im} G)\Pi(\text{Im} G)\Pi) \) for \( \Pi = \frac{1}{N} (G - G^\dagger) \). A key tool developed in \cite{73, 74} is a \( T \)-expansion of the \( T \)-variable, defined by \( T_{xy} := |m|^2 \sum_\alpha |s_{x\alpha}| |G_{y\alpha}|^2 \) \cite{35}. In this paper, we extend it to expansions of high moments of \( P \), that is, we can expand any \( \mathcal{P}^k \) into a sum of deterministic terms with arbitrary high order error. Then, we can explore some important cancellations for these deterministic terms, which lead to the proof of the QUE.

Some key ideas used in the proof will be briefly discussed in Section 1.2 below. We expect that the argument in this paper can be extended to random band matrices in lower dimensions and with smaller band width, and we will pursue this direction in future works.

Finally, we mention several advantages of the current proof of the bulk universality and QUE. (1) The fact that the QUE of \( H \) implies the bulk universality of \( H \) is very transparent in our current argument. In particular, we do not need to introduce an extra mean-field matrix \( Q \) to establish such a connection. As a consequence, the proofs of the QUE and bulk universality are simplified greatly. (2) Our current proof only uses the conventional resolvent \( G(z) \), whose local law is much easier to prove than the generalized resolvent in the mean-field reduction approach. (3) If we can prove the local law of \( G(z) \) up to the optimal scale \( \text{Im} z \gg \eta_\alpha = L^{-d} \) (instead of the current \( \eta_\alpha = W^{-5} L^{5-d} \)), then our current argument yields the bulk universality of random band matrices with band width \( W \geq L^5 \). (This is because the change of \( g_t(z) \) from \( t = 0 \) to \( t = L^c \eta_\alpha \) would become \( W^{-d} L^{C_\varepsilon} \) instead of \( W^{-(d+9\varepsilon)} L^{95+C_\varepsilon} \).) In addition, optimal QUE and complete delocalization of bulk eigenvectors would also follow. Hence, proving the local law of \( G(z) \) down to \( \text{Im} z \gg L^{-d} \) is now the key open problem in the study of band matrices.

### 1.1. Main results

In this subsection, we state the main results of this paper, including the bulk universality, QUE, local law, and delocalization of bulk eigenvectors. We consider \( d \)-dimensional random band matrices indexed by a cube of linear size \( L \) in \( \mathbb{Z}^d \), i.e.,

\[ \mathbb{Z}^d_L := (\mathbb{Z} \cap (-L/2, L/2]^d). \]  

We will view \( \mathbb{Z}^d_L \) as a torus and denote by \([x - y]_L \) the representative of \( x - y \) in \( \mathbb{Z}^d_L \), i.e.,

\[ [x - y]_L := [(x - y) + L\mathbb{Z}^d] \cap \mathbb{Z}^d_L. \]

Clearly, \( \|x - y\|_L := \|[x - y]_L\| \) is a periodic distance on \( \mathbb{Z}^d_L \) for any norm \( \| \cdot \| \) on \( \mathbb{Z}^d \). For definiteness, we use the \( \ell^\infty \)-norm in this paper, i.e. \( \|x - y\|_L := \|[x - y]_L\|_\infty \). In this paper, we consider the following class of \( d \)-dimensional random band matrices.

**Assumption 1.1** (Random band matrix). Fix any \( d \in \mathbb{N} \). For \( L \gg W \gg 1 \) and \( N := L^d \), we assume that \( H = H_{d,f,w,L} \) is an \( N \times N \) complex Hermitian random matrix whose entries (\( \text{Re} h_{xy}, \text{Im} h_{xy} : x, y \in \mathbb{Z}^d_L \)) are independent Gaussian random variables (up to symmetry \( H = H^\dagger \)) such that

\[ \mathbb{E}h_{xy} = 0, \quad \mathbb{E}(\text{Re} h_{xy})^2 = \mathbb{E}(\text{Im} h_{xy})^2 = s_{xy}/2, \quad x, y \in \mathbb{Z}^d_L, \]

where the variances \( s_{xy} \) satisfy that

\[ \text{Im} h_{xy} = f_{w,L}([x - y]_L) \]  

for a positive symmetric function \( f_{w,L} \) satisfying Assumption 1.2 below. We say that \( H \) is a \( d \)-dimensional random band matrix with linear size \( L \), band width \( W \) and variance profile \( f_{w,L} \). Denote the variance matrix by \( S := (s_{xy})_{x,y \in \mathbb{Z}^d_L} \), which is an \( N \times N \) doubly stochastic symmetric matrix.
Assumption 1.2 (Variance profile). We assume that \( f_{W,L} : \mathbb{Z}^d_L \to \mathbb{R}_+ \) is a positive symmetric function on \( \mathbb{Z}^d_L \) that can be expressed by the Fourier transform

\[
f_{W,L}(x) := \frac{1}{(2\pi)^d Z_{W,L}} \int \psi(Wp)e^{ip \cdot x} \, dp.
\] (1.7)

Here, \( Z_{W,L} \) is the normalization constant so that \( \sum_{x \in \mathbb{Z}^d_L} f_{W,L}(x) = 1 \), and \( \psi \in C^\infty(\mathbb{R}^d) \) is a symmetric smooth function independent of \( W \) and \( L \) and satisfies the following properties:

(i) \( \psi(0) = 1 \) and \( \|\psi\|_\infty \leq 1 \);

(ii) \( \psi(p) \leq \max\{1 - c_\psi|p|^2, 1 - c_\psi\} \) for a constant \( c_\psi > 0 \);

(iii) \( \psi \) is in the Schwartz space, i.e.,

\[
\lim_{|p| \to \infty} (1 + |p|)^k |\psi^{(k)}(p)| = 0, \quad \text{for any } k, l \in \mathbb{N}.
\] (1.8)

Clearly, \( f_{W,L} \) is of order \( O(W^{-d}) \) and decays faster than any polynomial, that is, for any fixed \( k \in \mathbb{N} \), there exists a constant \( C_k > 0 \) so that

\[
|f_{W,L}(x)| \leq C_k W^{-d} \|x\|_L/W^{-k}.
\] (1.9)

In other words, \( f_{W,L} \) is a Schwartz function of \( x/W \). Hence, the variance profile \( S \) defined in (1.6) has a banded structure, namely, for any constants \( \tau, D > 0 \),

\[
1_{|x-y| \geq W^{1+\tau} |s_{xy}|} \leq W^{-D}.
\] (1.10)

Combining (1.8) and (1.9) with the Poisson summation formula, we obtain that

\[
Z_{W,L} = \psi(0) + O(W^{-D}) = 1 + O(W^{-D}),
\] (1.11)

for any large constant \( D > 0 \) as long as \( L \geq W^{1+\varepsilon} \) for a constant \( \varepsilon > 0 \).

Denote the eigenvalues and normalized eigenvectors of \( H \) by \( \{\lambda_\alpha\} \) and \( \{u_\alpha\} \). Our first main result gives the bulk universality of random band matrices in dimensions \( d \geq 7 \). Define the \( k \)-point correlation function of \( H \) by

\[
\rho^{(k)}_H(\alpha_1, \alpha_2, \ldots, \alpha_k) := \int_{\mathbb{R}^{N-k}} \rho^{(N)}_H(\alpha_1, \alpha_2, \ldots, \alpha_N) \, d\alpha_{k+1} \cdots d\alpha_N,
\]

where \( \rho^{(N)}_H(\alpha_1, \alpha_2, \ldots, \alpha_N) \) is the joint density of all unordered eigenvalues of \( H \).

Theorem 1.3 (Bulk universality). Under Assumptions 1.1 and 1.2, fix any \( d \geq 7 \) and an arbitrary small constant \( c_0 > 0 \). If \( W^{d+95} \geq L^{95+c_0} \), then for any fixed \( k > 0 \), the \( k \)-point correlation function of \( H \) converges to that of GUE in the following sense. Fix a parameter \( b = N^{\varepsilon_1}/N \) for some small constant \( c_1 > 0 \). For any \( |E| \leq 2 - \kappa \) and \( C^1 \) test function \( \mathcal{O} \) with compact support, we have

\[
\lim_{N \to \infty} \frac{\int_{E-b}^{E+b} \, dE'}{2b} \int_{\mathbb{R}^k} \, d\alpha \, \mathcal{O}(\alpha) \left\{ \left( \rho^{(k)}_H - \rho^{(k)}_{\text{GUE}} \right)(E' + \alpha/N) \right\} = 0.
\]

As we will explain at the end of this subsection, this result can be readily extended to non-Gaussian random band matrices after some straightforward technical modifications. Hence, this theorem indeed proves the important bulk universality conjecture for random band matrices, i.e., the local eigenvalue statistics for large random matrices with independent entries are universal and do not depend on the particular distribution of matrix elements and the variance profile. The bulk universality was first proved for Wigner matrices [17, 38, 41, 43, 44, 45, 70] and later extended to 1D random band matrices [18, 22] and many other mean-field random matrix and random graph ensembles. We refer the reader to [42] for a more detailed review of the universality conjecture and related results in the literature.

As mentioned before, the bulk universality is a consequence of the QUE of bulk eigenvectors and the local law of the Green’s function. We state the QUE as our second main result. Roughly speaking, it shows that all bulk eigenvectors in dimensions \( d \geq 12 \) and most bulk eigenvectors in dimensions \( 7 \leq d \leq 11 \) are asymptotically uniformly distributed on microscopic scales. In particular, it implies that the bulk eigenvectors are not localized in any small subset of volume \( o(N) \), so their localization length must be \( L \).

Theorem 1.4 (Quantum unique ergodicity). Let \( \kappa, \delta > 0 \) be arbitrary small constants. Suppose \( W \geq L^\delta \) and Assumptions 1.1 and 1.2 hold.\]
(i) For $d \geq 12$ and any $I_N \subset \mathbb{Z}_L^d$ with $|I_N| \geq (L/W)^{10d/(d-2)}$, the following event occurs with probability tending to one:

$$\frac{1}{|I_N|} \sum_{x \in I_N} (N|u_\alpha(x)|^2 - 1) \to 0 \quad \text{uniformly for all } \alpha \text{ such that } |\lambda_\alpha| \leq 2 - \kappa. \quad (1.12)$$

(ii) For $d \geq 7$, the following event occurs with probability tending to one for any small constant $\epsilon > 0$ and $\ell \geq W^{-3/d}L^{2+\epsilon}$:

$$\frac{1}{N} \left\{ \alpha : |\lambda_\alpha| < 2 - \kappa, \left| \frac{1}{|I|} \sum_{x \in I} (N|u_\alpha(x)|^2 - 1) \right| \geq \epsilon \text{ for some } I \in \mathcal{I} \right\} \to 0, \quad (1.13)$$

where $\mathcal{I} := \{ I_{k,\ell} : k \in \mathbb{Z}_L^d, -L/\ell \leq k_i \leq L/\ell \}$ is a collection of boxes that covers $\mathbb{Z}_L^d$, with $I_{k,\ell} := \{ [y] : y_i \in [(k_i - 1)\ell/2, (k_i + 1)\ell/2) \cap \mathbb{Z}_L^d \}$. (recall the notation (1.4)).

Notice that the left-hand side of (1.12) is a special case of $|\langle u_\alpha, \Pi u_\alpha \rangle|^2$ with the diagonal matrix $\Pi$ defined by $\Pi_{xx} = N^{\frac{1}{2d}}1_{x \in I_N} - 1$. In part (ii), $\mathcal{I}$ can be any cover of $\mathbb{Z}_L^d$ consisting of subsets of size $\ell \delta$ and with cardinality $|\mathcal{I}| = O(N/\ell^d)$. The above probabilistic QUE was first proved for Wigner matrices [21] and later extended to 1D random band matrices [18, 22] and many other types of mean-field random matrices and random graphs [1, 7, 10, 11, 13, 15, 19, 56], to name a few. Recently, a stronger notion of QUE called the eigenstate thermalization hypothesis was also established for Wigner matrices [26, 27, 28].

Both the proofs of bulk universality and QUE are crucially based on the following (essentially sharp) local law up to the scale $\text{Im } \xi \gg W^{-5}L^{5-d}$.

**Theorem 1.5** (Local law). Let $\kappa, \delta, \delta_0 \in (0, 1)$ be arbitrary small constants. Under Assumptions 1.1 and 1.2, fix any $d \geq 7$ and suppose $W \geq L^3$. For any constants $\tau, D > 0$, we have the following estimate on $G(z)$ for $z = E + i\eta$ and all $x, y \in \mathbb{Z}_L^d$:

$$\mathbb{P}\left( \sup_{|E| \leq 2-\kappa} \sup_{\eta, \eta_2 \leq 1} |G_{xy}(z) - m(z)\delta_{xy}|^2 \leq W^\tau \left( B_{xy} + \frac{1}{N\eta} \right) \right) \geq 1 - L^{-D} \quad (1.14)$$

for large enough $L$, where $\eta_\star := W^{-5+\delta_0}L^{5-d}$ and we denote

$$B_{xy} := W^{-2}(\|x - y\|_L + W)^{-d+2}. \quad (1.15)$$

An immediate corollary of this local law is the following delocalization estimate on bulk eigenvectors.

**Corollary 1.6** (Delocalization). Under the assumptions of Theorem 1.5, for any constants $\tau, D > 0$, we have that

$$\mathbb{P}\left( \sup_{\alpha : |\lambda_\alpha| \leq 2-\kappa} \|u_\alpha\|_\infty^2 \leq W^{-5+\tau}L^{5-d} \right) \geq 1 - L^{-D} \quad (1.16)$$

for large enough $L$.

**Proof.** Taking $E = \lambda_\alpha$ and $\eta = W^{-5+\delta_0}L^{5-d}$ with $\delta_0 < \tau$ in the spectral decomposition

$$\text{Im } G_{xx}(E + i\eta) = \sum_\alpha \frac{\eta}{|\lambda_\alpha - E|^2 + \eta^2} |u_\alpha(x)|^2$$

gives that $|u_\alpha(x)|^2 \leq \text{Im } G_{xx}(E + i\eta)$. Then, using the local law (1.14) and $\text{Im } m(z) \leq 1$, we conclude the proof. \qed

Theorem 1.5 improves the local law, Theorem 1.4, in [73] from $d \geq 8$ to a lower dimension $d \geq 7$ and from $\eta \gg W^2/L^2$ to a smaller scale $\eta \gg W^{-5}L^{5-d}$ which has the correct leading dependence in $L^{-d}$. As a consequence, it yields a better delocalization estimate than [73], which proved that $\|u_\alpha\|_\infty^2 \leq W^{2+\epsilon}/L^2$ when $d \geq 8$. We believe that the local law (1.14) should hold for all $\eta \gg L^{-d}$, which will lead to the following complete delocalization of bulk eigenvectors:

$$\mathbb{P}\left( \sup_{\alpha : |\lambda_\alpha| \leq 2-\kappa} \|u_\alpha\|_\infty^2 \leq L^{-d+\tau} \right) \geq 1 - L^{-D}.$$
As has been explained in [73, 74], our proof can be readily adapted to non-Gaussian random band matrices after some technical modifications. More precisely, we have used Gaussian integration by parts in expanding resolvent entries (see Section A.2), and this can be replaced by certain cumulant expansion formulas (see e.g., [54, Proposition 3.1] and [51, Section II]) for general distributions. For simplicity of presentation, we choose to stick to the complex Gaussian cases to avoid technical complexities associated with non-Gaussian distributions. The proof for the real Gaussian case is also similar to the complex case except that the number of terms will double whenever applying the Gaussian integration by parts formula to resolvents (which is due to the fact that $\mathbb{E}h_{xy}^2 = 0$ in the complex case but not in the real case).

1.2. Some key ideas. As discussed before, one main challenge in proving the bulk universality for $H$ is how to compared $g(z)$ with $g_t(z)$ for some $t \gg W^{-5}L^{5-d}$. When $H$ is a mean-field Wigner matrix, we can find another Wigner matrix $H'$ so that the first four moments of $H$ and $H' + B_t$ are almost identical entrywise. Then, in the Green’s function comparison, the matching moments imply that if we replace the entries of $H$ by those of $H' + B_t$ one by one, the Green’s function will not change by much. The moment matching condition, however, is no longer true for the setting of band matrices. Our proof is instead based on the key observation that the stability of the Green’s function still holds, provided with QUE. As a simple example, we compare $\mathbb{E}\text{Im} g_t$ with $\mathbb{E}\text{Im} g$. With Ito’s formula, we find that $\partial_t \mathbb{E}(\text{Im} g_t - \text{Im} g)$ is dominated by terms containing QUE factors $|\langle u_\alpha, \Pi u_\alpha \rangle|^2$ for a real diagonal $\Pi$ with zero trace. For example, one of these terms is

$$-rac{1}{2N} \mathbb{E} \sum_{a,b} (G_{1})_{aa}(G_{1})_{bb} \left( s_{ab} - \frac{1}{N} \right).$$

(1.17)

On the right hand, each summand does not vanish unless $s_{ab} = N^{-1}$, but they are expected to be small after averaging due to QUE:

$$\sum_a (G_{1})_{aa} \left( s_{ab} - \frac{1}{N} \right) = \sum_a \frac{1}{\lambda_a(t) - z} \sum_a |u_\alpha(a)|^2 \left( s_{ab} - \frac{1}{N} \right),$$

(1.18)

where, fixing $b$, the diagonal matrix with $(a,a)$-entry given by $s_{ab} - N^{-1}$ has zero trace.

With the spectral decomposition of $\text{Im} G$, it is easy to see that proving the QUE amounts to bounding high moments of $P = \text{tr} ((\text{Im} G)\Pi(\text{Im} G)\Pi)$. A key ingredient for the proof of QUE is a complete expansion of the $k$-moment of $P$ for any fixed $k \in \mathbb{N}$. Roughly speaking, with a carefully designed expansion strategy, we can express $\mathbb{E}P^k$ as a sum of deterministic expressions, which satisfy some “nice graphical properties”. We can then control these deterministic expressions by exploring some cancellations in them.

Similar to [73, 74], our general complete expansions are based on a simpler and more basic $T$-expansion of the $T$-variable. Roughly speaking, we will show that

$$T_{xy} = |m|^2 \Theta_{xy}^m + |m|^2 \text{Im} m \frac{\text{Im}}{N_\eta} + (\text{fluctuations and higher order errors}),$$

(1.19)

where, roughly speaking, $\Theta_{xy}^m$ (cf. (2.4)) is the Green’s function of a random walk on $\mathbb{Z}_L^d$ with transition probability matrix $S$ and with zero mode removed, and the second term on the right-hand side gives the zero mode. The propagator $\Theta_{xy}$ describes the diffusive behavior of the random walk and is of order $W^{-2}(\|x - y\|_L + W)^{2-d}$ (cf. (2.11)). Hence, when $\text{Im} z \gg W^2/L^2$, the zero mode is negligible and can be absorbed into the diffusive term in [73, 74]. On the other hand, when $\text{Im} z$ gets below $W^2/L^2$, the zero mode will play a crucial role and make some expressions in [73, 74] diverge. An important step in the $T$-expansion of this paper is to separate the zero mode, so that all expressions are under control as long as $\eta \gg W^{-5}L^{5-d}$. This key idea will be explained in Section 2.1. If we pick out the zero mode $\frac{\text{Im} m}{N_\eta}$ from $T_{xy}$, then in the expansions of $\mathbb{E}P^k$, the “leading terms” will contain the vanishing factor $\sum_x \Pi_x \frac{\text{Im} m}{N_\eta} = 0$. This type of cancellation is crucial in the proof of QUE and is proved rigorously by a careful analysis of the deterministic expressions from complete expansions.

The rest of this paper is organized as follows. In Section 2, we introduce some graphical notations representing expressions of resolvent entries and the concept of self-energies. We then use them to define the $T$-expansion. In Section 3, we define the concept of complete $T$-expansions and complete expansions of general graphs. The main results, Theorems 1.3, 1.4 and 1.5, will be proved in Section 4 based on the tools developed in Sections 2 and 3. The proof of the local law requires a “sum zero property” for self-energies, whose proof is presented in Section 5. Two key lemmas in Section 3 regarding bounding
complete expansions of general graphs will be proved in Section 6. Finally, constructions of the $T$-expansion, the complete $T$-expansion, and complete expansions of general graphs are included into Appendix A–C.

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2. $T$-Expansion

Similar to [73, 74], our proofs in this paper are crucially based on an important tool—the $T$-expansion up to arbitrary high order. However, in order to decrease $\eta$ from $\eta \gg W^2/L^2$ in [73, 74] to $\eta \geq W^{-5 + \delta_0} L^{5-d}$, we need to derive a different $T$-expansion with zero mode removed. We now use the second order $T$-expansion to explain this key idea.

2.1. Second order $T$-expansion. Our $T$-expansion will be formulated in terms of the following $T$-variables with three subscripts:

$$T_{x,yy} := |m|^2 \sum_{a} s_{xa} G_{ax} G_{ay}' \quad \text{and} \quad T_{yy',x} := |m|^2 \sum_{a} G_{ya} G_{y'x} s_{ax}. \quad (2.1)$$

When $y = y'$, we have the conventional $T$-variable $T_{xy} \equiv T_{x,yy}$. In this subsection, we derive the second order $T$-expansion of $T_{xy}$ using the expansion

$$G = - \frac{1}{z + m} + \frac{1}{z + m} (H + m) G \quad \Rightarrow \quad G - m = -(H + m) G. \quad (2.2)$$

The formula (2.2) is derived easily from the definition of $G$ and the self-consistent equation $(z + m)m = -1$. Define $E_x$ as the partial expectation with respect to the $x$-th row and column of $H$, i.e., $E_x(\cdot) := E(\cdot | H(x))$, where $H(x)$ denotes the $(N - 1) \times (N - 1)$ minor of $H$ obtained by removing the $x$-th row and column. For simplicity, in this paper we will use the notations

$$P_x := E_x, \quad Q_x := 1 - E_x.$$

Taking the partial expectation of $HG$ and applying Gaussian integration by parts to the $H$ entries, we can obtain the following lemma. The corresponding $T$-expansion of $T_{xy}'$ can be obtained by considering the transposition of $T_{xy}'$.

Lemma 2.1 (Second order $T$-expansion). Under Assumption 1.1, we have

$$T_{a,b_1 b_2} = m \Theta_a^\circ b_1 G_{b_1 b_2} + \frac{|m|^2}{2m \eta} G_{b_2 b_1} - G_{b_1 b_2} + \sum_x \Theta_a^x \left[ A_{x,b_1 b_2}^{(2)} + Q_{x,b_1 b_2}^{(2)} \right], \quad (2.3)$$

for $a, b_1, b_2 \in \mathbb{Z}_L^d$, where

$$\Theta^\circ := \frac{|m|^2 S^\circ}{1 - |m|^2 S^\circ}, \quad \text{with} \quad S^\circ := P^\perp S P^\perp, \quad P^\perp := I - ee^\top, \quad e := N^{-1/2}(1, \ldots, 1)^\top, \quad (2.4)$$

and

$$A_{x,b_1 b_2}^{(2)} := m \sum_y s_{xy} (G_{yy} - m) G_{xb_1} G_{xb_2} - m \sum_y s_{xy} (G_{xx} - m) G_{yb_1} G_{yb_2}$$

$$Q_{x,b_1 b_2}^{(2)} := Q_x (G_{xb_1} G_{xb_2}) - m \delta_{xb_1} Q_{b_1} (G_{b_1 b_2}) - m \sum_y s_{xy} Q_x \left[ (G_{yy} - m) G_{xb_1} G_{xb_2} \right] - m \sum_y s_{xy} Q_x \left( G_{xx}, G_{yb_1}, G_{yb_2} \right).$$

Proof. Note that $e$ is the Perron-Frobenius eigenvector of $S$ with eigenvalue 1. Hence, we have $s_{yy}^\circ = s_{xy} - N^{-1}$ and $\Theta^\circ e = 0$. Furthermore, we recall the following classical Ward’s identity, which follows from a simple algebraic calculation:

$$\sum_x \mathcal{G}_{xy}'(z) G_{xy}(z) = \frac{G_{yy}'(z) - G_{yy}(z)}{2i\eta}, \quad \sum_x \mathcal{G}_{y'x}(z) G_{yx}(z) = \frac{G_{yy}(z) - G_{yy}'(z)}{2i\eta}. \quad (2.5)$$

As a special case, if $y = y'$, we have

$$\sum_x |G_{xy}(z)|^2 = \sum_x |G_{yx}(z)|^2 = \frac{\text{Im} G_{yy}(z)}{\eta}. \quad (2.6)$$
Now, using (2.5), we get that

\[ T_{a,b_1 b_2} = |m|^2 \sum_x s_{ax} G_{x b_1} G_{x b_2} = |m|^2 \sum_x s_{ax}^o G_{x b_1} G_{x b_2} + |m|^2 \frac{1}{N} \sum_x G_{x b_1} G_{x b_2} \]

\[ = |m|^2 \sum_x s_{ax}^o G_{x b_1} G_{x b_2} + \frac{|m|^2}{2iN\eta} (G_{b_2 b_1} - \overline{G}_{b_1 b_2}) = T_{a,b_1 b_2}^o + \frac{|m|^2}{2iN\eta} (G_{b_2 b_1} - \overline{G}_{b_1 b_2}), \quad (2.7) \]

where

\[ T_{a,b_1 b_2}^o := \sum_x P_{ax} T_{x,b_1 b_2} = |m|^2 \sum_x s_{ax}^o G_{x b_1} G_{x b_2}. \quad (2.8) \]

Using (2.4) and (2.2), we can write that

\[ T_{a,b_1 b_2}^o = \sum_x \Theta_{ax}^o \Theta_{b_1}^o G_{x b_1} G_{x b_2} \]

\[ = \sum_x \Theta_{ax}^o P_x [(m\delta_{xb_1} - m(HG)_{xb_1} - m^2 G_{xb_1})G_{xb_2}] + \sum_x \Theta_{ax}^o Q_x (G_{xb_1}) G_{xb_2} - |m|^2 \sum_x \Theta_{ax}^o (G_{xb_1}) G_{xb_2}. \]

Then, applying Gaussian integration by parts to \( P_x[(HG)_{xb_1} G_{xb_2}] \), we can get that

\[ \sum_x [\Theta_{ax}^o - |m|^2 (\Theta^o S)_{ax}] G_{xb_1} G_{xb_2} = m \Theta_{b_1}^o G_{xb_1} G_{xb_2} + A^{(2)}_{b_1 b_2} + Q^{(2)}_{b_1 b_2}. \]

Since the calculation is exactly the same as the one for Lemma 2.4 of [73], we omit the details. This completes the proof of Lemma 2.1.

Compared with the second order T-expansion in Lemma 2.5 of [73], there are two differences: the second term on the RHS of (2.3) is new and the other two terms use \( \Theta_{ax}^o \) instead of \( \Theta_{ax} \) as in [73], where \( \Theta := |m|^2 S/(1 - |m|^2 S) \). Note that \( \Theta^o \) differs from \( \Theta \) by a zero mode, i.e.,

\[ \Theta_{xy}^o = (P^\perp \Theta P^\perp)_{xy} = \Theta_{xy} - \frac{|m|^2}{N(1 - |m|^2)} = \Theta_{xy} - \frac{\text{Im} m}{N\eta}, \quad (2.9) \]

where we used the identity

\[ \frac{|m(z)|^2}{1 - |m(z)|^2} = \frac{\text{Im} m(z)}{\eta}, \quad (2.10) \]

which can be derived by taking the imaginary part of the equation \( z = -m(z) - m^{-1}(z) \). It is well-known that for \( z = E + i\eta \) with \( E \in (-2 + \kappa, 2 - \kappa) \) and \( \eta > 0 \),

\[ |\Theta_{xy}^o(z)| \leq \frac{W^\tau 1_{|x-y| \leq -1/2^{W+\tau} W^{1+\tau}}}{W^2 (x-y)^{d-2}} + \frac{1}{(x-y)^{d-2}} \leq W^\tau B_{xy}, \quad (2.11) \]

for any constants \( \tau, D > 0 \). For simplicity, here and throughout the rest of this paper, we abbreviate

\[ |x-y| \equiv \|x-y\|_L, \quad \langle x-y \rangle \equiv \|x-y\|_L + W. \quad (2.12) \]

The reader can refer to e.g., [35, 75] for a proof of (2.11). (For example, the proof in [75] was written for \( \Theta(z) \) with \( \eta \geq W^2/L^2 \), in which case we can use \( \|S\| \leq 1 \) and \( 1 - |m(z)|^2 \geq W^2/L^2 \). For \( \Theta^o(z) \) with smaller \( \eta \), the same proof still works except that we will use \( |m(z)| \leq 1 \) and the spectral gap of \( S \) at 1, i.e., \( 1 - \|S\| \gtrsim W^2/L^2 \).) With (2.11) and (2.9), we get that

\[ \Theta_{xy}(z) \leq W^\tau B_{xy} + \frac{\text{Im} m}{N\eta}. \quad (2.13) \]

To explain heuristically why the new second order T-expansion can be applied to smaller \( \eta \) and what trouble the T-expansion in [73] may have, we take \( b_1 = b_2 = a \) and look at the \( \sum_x \Theta_{ax}^o A^{(2)}_{x,aa} \) term in (2.3). Suppose we already know that the local law (1.14) holds. Then, using (2.11), we can bound that

\[ \sum_x \Theta_{ax}^o A^{(2)}_{x,aa} \lesssim \sum_x B_{xa} \left( B_{xa} + \frac{1}{N\eta} \right) \lesssim W^{-d} + \frac{1}{N\eta} \frac{L^2}{W} \ll 1, \]

where
as long as \( \eta \gg W^{-2}L^{2-d} \). Here, “\(<\)” denotes stochastic domination, which will be defined in Definition 2.2 below. On the other hand, for the \( T \)-expansion in [73], we need to control \( \sum_x \Theta_{xx}A^{x,y}_{x,y} \). Using (2.13), we can bound that
\[
\sum_x \Theta_{xx}A^{x,y}_{x,y} < \sum_x \left( B_{xx} + \frac{1}{N\eta} \right)^2 \lesssim W^{-d} + \frac{1}{N\eta^2} \frac{L}{W^2} + \frac{1}{N\eta^2},
\]
where the RHS diverges when \( \eta \gg L^{-d/2} \). Hence, the \( T \)-expansion in [73] does not work well for \( \eta \ll L^{-d/2} \).

Of course, in the proof, we do not have the local law (1.14) a priori, and we can only use a worse continuity estimate for \( G \) entries. Hence, in the current work, we cannot reach the level \( W^{-2}L^{2-d} \) yet.

In this paper, we use the following convenient notion of stochastic domination introduced in [34].

**Definition 2.2 (Stochastic domination and high probability event).** (i) Let \( \xi = \left( \xi^{W}(u) : W \in \mathbb{N}, u \in U^{W} \right), \quad \zeta = \left( \zeta^{W}(u) : W \in \mathbb{N}, u \in U^{W} \right) \), be two families of non-negative random variables, where \( U^{W} \) is a possibly \( W \)-dependent parameter set. We say \( \xi \) is stochastically dominated by \( \zeta \), uniformly in \( u \), if for any fixed (small) \( \tau > 0 \) and (large) \( D > 0 \),
\[
\mathbb{P} \left[ \bigcup_{u \in U^{W}} \{ \xi^{W}(u) > W^{\tau}\zeta^{W}(u) \} \right] \leq W^{-D}
\]
for large enough \( W \geq W_{0}(\tau, D) \), and we will use the notation \( \xi \prec \zeta \). If for some complex family \( \xi \) we have \( |\xi| \prec \zeta \), then we will also write \( \xi \prec \zeta \) or \( \xi = O_{\prec}(\zeta) \).

(ii) As a convention, for two deterministic non-negative quantities \( \xi \) and \( \zeta \), we will write \( \xi \prec \zeta \) if and only if \( \xi \leq W^\tau \zeta \) for any constant \( \tau > 0 \).

(iii) We say that an event \( \Xi \) holds with high probability (w.h.p.) if for any constant \( D > 0 \), \( \mathbb{P}(\Xi) \geq 1 - W^{-D} \) for large enough \( W \). More generally, we say that an event \( \Omega \) holds w.h.p. in \( \Xi \) if for any constant \( D > 0 \), \( \mathbb{P}(\Xi \setminus \Omega) \leq W^{-D} \) for large enough \( W \).

### 2.2. Graphs, scaling order and doubly connected property.

Similar to some previous works (e.g., [34, 73, 74, 75]) on random matrices, we will organize our proofs using graphs. In this subsection, we introduce the basic concepts of atomic graphs, molecular graphs, and doubly connected property. Many graphical notations used in this paper have been defined in [73, 74], but we repeat them for the convenience of the reader.

Our graphs will consist of matrix indices as vertices and matrix entries as edges. In particular, the entries of \( S, \Theta^{o} \) and \( G \) will be represented by different types of edges. In addition, we also have edges representing the following two deterministic matrices
\[
S^{+}(z) := \frac{m^{2}(z)S}{1 - m^{2}(z)S}, \quad S^{-}(z) := S^{+}(z),
\]
which satisfy the following estimate (2.15). It is a folklore result and we omit its proof (a formal proof for the \( d = 1 \) case is given in equation (4.21) of [20]).

**Lemma 2.3.** Suppose Assumptions 1.1 and 1.2 hold and \( z = E + i\eta \) with \( E \in (-2 + \kappa, 2 - \kappa) \) for a constant \( \kappa > 0 \). Then, for any constants \( \tau, D > 0 \), we have that
\[
|S^{+}_{xy}(z)| \lesssim W^{-d}1_{|x-y| \leq W^{1+\tau}} + (x - y)^{-D}.
\]

We first introduce the most basic atomic graphs.

**Definition 2.4 (Atomic graphs).** Given a standard oriented graph with vertices and edges, we assign the following structures and call the resulting graph an atomic graph.

- **Atoms:** We will call the vertices atoms. Every graph has some external atoms and internal atoms. The external atoms represent external indices whose values are fixed, while internal atoms represent summation indices that will be summed over.
- **Weights:** A regular weight on an atom \( x \) represents a \( G_{xx} \) factor, drawn as a blue solid \( \Delta \), or a \( \Theta_{xx} \) factor, drawn as a red solid \( \Delta \). A light weight on atom \( x \) represents a \( G_{xx} - m \) factor, drawn as a blue hollow \( \Delta \), or a \( \Theta_{xx} - m \) factor, drawn as a red hollow \( \Delta \).
- **Edges:** The edges are divided into the following types.

(1) **Solid edges:**
- a blue oriented edge from atom \( x \) to atom \( y \) represents a \( G_{xy} \) factor;
- a red oriented edge from atom \( x \) to atom \( y \) represents a \( G_{xy} \) factor.

(2) **Waved edges:**
- a neutral waved edge between atoms \( x \) and \( y \) represents an \( s_{xy} \) factor;
- a positive blue waved edge between atoms \( x \) and \( y \) represents an \( S^+_{xy} \) factor;
- a negative red waved edge between atoms \( x \) and \( y \) represents an \( S^-_{xy} \) factor.

(3) **Diffusive edges:** A diffusive edge between atoms \( x \) and \( y \) represents a \( \Theta_{xy} \) factor. We draw it as a black double-line edge between atoms \( x \) and \( y \).

(4) **Free edges:** A purple solid edge between atoms \( x \) and \( y \) represents a \((N \eta)^{-1}\) factor, and we call it a free edge.

(5) **Dotted edges:** A dotted edge connecting atoms \( \alpha \) and \( \beta \) represents a factor \( 1_{\alpha=\beta} = \delta_{\alpha\beta} \), and
- \( \times \)-dotted edge represents a factor \( 1_{\alpha \neq \beta} = 1 - \delta_{\alpha\beta} \). There is at most one dotted or \( \times \)-dotted edge between each pair of atoms.

The orientations of non-solid edges do not matter. Edges between internal atoms are called internal edges, while edges with at least one end at an external atom are called external edges.

- **\( P \) and \( Q \) labels:** A solid edges or a weight may have a label \( P_x \) or \( Q_x \) for some atom \( x \) in the graph. Moreover, every edge or weight can have at most one \( P \) or \( Q \) label.
- **Coefficients:** There is a coefficient associated with each graph. Unless otherwise specified, the coefficient is of order \( O(1) \).

**Definition 2.5 (Subgraphs).** A graph \( \mathcal{G}_1 \) is said to be a subgraph of \( \mathcal{G}_2 \), denoted by \( \mathcal{G}_1 \subset \mathcal{G}_2 \), if every graphical component (except the coefficient) of \( \mathcal{G}_1 \) is also in \( \mathcal{G}_2 \). Moreover, \( \mathcal{G}_1 \) is a proper subgraph of \( \mathcal{G}_2 \) if \( \mathcal{G}_1 \subset \mathcal{G}_2 \) and \( \mathcal{G}_1 \neq \mathcal{G}_2 \). Given a subset \( S \) of atoms in a graph \( \mathcal{G} \), the subgraph \( \mathcal{G}|_{S} \) induced on \( S \) refers to the subgraph of \( \mathcal{G} \) with atoms in \( S \) as vertices, the edges between these atoms, and the weights on these atoms. Given a subgraph \( \mathcal{G} \), its closure \( \overline{\mathcal{G}} \) refers to \( \mathcal{G} \) union its external edges.

Along the proof, we will introduce some other types of weights and edges. To each graph, we assign a value as follows.

**Definition 2.6 (Values of graphs).** Given an atomic graph \( \mathcal{G} \), we define its value as an expression obtained as follows. We first take the product of all the edges, all the weights and the coefficient of the graph \( \mathcal{G} \). Then, for the edges and weights with the same \( P_x \) or \( Q_x \) label, we apply \( P_x \) or \( Q_x \) to their product. Finally, we sum over all the internal indices represented by the internal atoms. The values of the external indices are fixed by their given values. The value of a linear combination of graphs \( \sum_i c_i \mathcal{G}_i \) is naturally defined as the linear combination of the graph values of \( \mathcal{G}_i \).

For simplicity, throughout this paper, we will always abuse the notation by identifying a graph (a geometric object) with its value (an analytic expression). In this sense, noticing that free edges represent \((N \eta)^{-1}\) factors without indices, two graphs are equivalent if they have the same number of free edges and all other graph components are the same. In other words, we can move a free edge freely to another place without changing the graph.

**Definition 2.7 (Normal graphs).** We say an atomic graph \( \mathcal{G} \) is normal if it satisfies the following properties:

(i) it contains at most \( O(1) \) many atoms and edges;
(ii) all internal atoms are connected together or to external atoms through paths of waved, diffusive or dotted edges;
(iii) there are no dotted edges between internal atoms;
(iv) every pair of atoms \( \alpha \) and \( \beta \) in the graph are connected by a \( \times \)-dotted edge if and only if they are connected by a \( G \) edge.

All graphs appearing in our proof are normal (after some simple graph operations related to dotted edges, see Definition A.8). By this definition, every \( G \) edge in a normal graph is off-diagonal, while all diagonal \( G \) factors will be represented by weights. Given a normal graph, we define its scaling order as follows.

**Definition 2.8 (Scaling order).** Given a normal graph \( \mathcal{G} \), we define its scaling order as

\[
\text{ord}(\mathcal{G}) := \#\{\text{off-diagonal } G \text{ edges}\} + \#\{\text{light weights}\} + 2\#\{\text{waved edges}\} + 2\#\{\text{free edges}\}
\]
Here, every dotted edge in a normal graph means that an internal atom is equal to an external atom, so we lose one free summation index. The concept of scaling order can also be defined for subgraphs.

In this paper, whenever we say the order of a graph, we are referring to its scaling order. We arrange the graphs in \( T \)-expansions according to their scaling orders, and an \( n \)-th order \( T \)-expansion indicates that the “error term” contains graphs of scaling order \( > n \). However, we emphasize that in general the scaling order does not imply the real size of the graph value directly. In order to establish such a connection for a graph, it needs to be doubly connected. To define the doubly connected property, we first define molecular graphs.

**Definition 2.9 (Molecules).** We partition the set of all atoms into a disjoint union of subsets \( \{ \text{all atoms} \} = \bigcup_j M_j \), where every \( M_j \) is called a molecule. Two atoms belong to the same molecule if and only if they are connected by a path of neutral/plus/minus waved edges and dotted edges. Every molecule containing at least one external atom is called an external molecule.

By (1.9) and (2.15), if two atoms \( x \) and \( y \) are in the same molecule, then we essentially have \( |x - y| \leq W^{1+\tau} \) up to a negligible error \( O(W^{-D}) \). Given an atomic graph, we will call the subgraph inside a molecule the local structure of this molecule. On the other hand, the global structure of an atomic graph refers to its molecular graph, which is the quotient graph with each molecule regarded as a vertex.

**Definition 2.10 (Molecular graphs).** Molecular graphs are graphs consisting of

- external and internal molecules;
- solid, diffusive and free edges between molecules.

Given an atomic graph \( \mathcal{G} \), we define its molecular graph, denoted by \( \mathcal{G}_M \), as follows:

- merge all atoms in the same molecule and represent them by a vertex;
- keep solid, diffusive and free edges between molecules;
- discard all the other components in \( \mathcal{G} \) (including weights, dotted or \( \times \)-dotted edges, and edges inside molecules).

In this paper, molecular graphs are used solely to help with the analysis of graph structures, while all graph expansions are applied to atomic graphs only. The following doubly connected property is a crucial property defined using molecular graphs. It will allow us to establish a direct connection between the scaling order of a graph and a bound on its value. All graphs in the \( T \)-expansion will satisfy this property.

**Definition 2.11 (Doubly connected property).** An atomic graph \( \mathcal{G} \) without external molecules is said to be doubly connected if its molecular graph \( \mathcal{G}_M \) satisfies the following property. There exists a collection, say \( \mathcal{B}_{\text{black}} \), of diffusive edges, and another collection, say \( \mathcal{B}_{\text{blue}} \), of blue solid, diffusive and free edges such that

- \( \mathcal{B}_{\text{black}} \cap \mathcal{B}_{\text{blue}} = \emptyset \),
- each of \( \mathcal{B}_{\text{black}} \) and \( \mathcal{B}_{\text{blue}} \) contains a spanning tree that connects all molecules.

(Note that red solid edges are not used in either collection.) For simplicity of notations, we will call the edges in \( \mathcal{B}_{\text{black}} \) as black edges and the edges in \( \mathcal{B}_{\text{blue}} \) as blue edges. Correspondingly, \( \mathcal{B}_{\text{black}} \) and \( \mathcal{B}_{\text{blue}} \) are referred to as black net and blue net, respectively.

An atomic graph \( \mathcal{G} \) with external molecules is called doubly connected if its subgraph with all external molecules removed is doubly connected.

**2.3. Self-energies and renormalized diffusive edges.** In higher order \( T \)-expansions, new types of diffusive edges will also appear. They have the same behavior as \( \Theta^\circ \), but “renormalized by self-energies” in a sense which we will explain below.

Following [73], a \( 2l \)-th order self-energy is a linear combination of deterministic graphs satisfying the properties in the following definition.

**Definition 2.12 (Self-energies).** Fix any \( l \in \mathbb{N} \). \( \mathcal{E}_{2l}(z) \) is a deterministic matrix satisfying the following properties.
(i) For any \( x, y \in \mathbb{Z}_L^d \), \((E_2)_{xy}\) is a sum of \( O(1) \) many deterministic graphs of scaling order \( 2l \), with external atoms \( x \) and \( y \). These graphs consist of waved edges, dotted edges (there may be a dotted edge between \( x \) and \( y \)), diffusive edges, and \( \Theta_k^{(n)} \) edges with \( 2 \leq k \leq n \leq 2l-1 \) (defined in (2.22)), but do not contain free edges or \( \Theta^{(n)} \) edges with \( n \geq 3 \) (defined in (2.21)).

(ii) \( E_{2l}(z) \) satisfies the following properties for \( z = E + i\eta \) with \( |E| \leq 2 - \kappa \) and \( \eta \in [W^{-5}L^{5-d}, 1] \):

\[
E_{2l}(x, x + a) = E_{2l}(0, a), \quad E_{2l}(0, a) = E_{2l}(0, -a), \quad \forall \ x, a \in \mathbb{Z}_L^d, \tag{2.17}
\]

\[
|\sum_{x \in \mathbb{Z}_L^d} (E_{2l})_{0x}(z)| < W^{-(l-2)d}B_{0x}^2, \quad \forall \ x \in \mathbb{Z}_L^d, \tag{2.18}
\]

\[
\sum_{x \in \mathbb{Z}_L^d} (E_{2l})_{0x}(z) < \left[ \left( \eta + \frac{W^{2d-6}}{L^{2d-6}} \right) W^{-d} + \frac{1}{N} \frac{L^2}{W^2} \right] W^{-(l-2)d}. \tag{2.19}
\]

By Definition 2.8, the scaling order of a deterministic graph can only be even. Moreover, every nontrivial self-energy \( E_i \) in this paper has scaling order \( \geq 4 \). Hence, as a convention, we set

\[
E_1 = E_2 = E_3 = 0, \quad \text{and} \quad E_{2l+1} := 0, \quad l \in \mathbb{N}. \tag{2.20}
\]

We now define the \( n \)-th order renormalized diffusive matrix:

\[
\Theta^{(n)} := \frac{1}{1 - \Theta^{(1)}} \Theta^{(1)}, \quad \text{where} \quad \Sigma^{(n)}(z) := \sum_{2l=4}^{n} E_{2l}(z). \tag{2.21}
\]

Moreover, we define its \( k \)-th order part as \( \Theta^{(n)}_k := \Theta^{(1)} \) and

\[
\Theta_k^{(n)} := \Theta^{(1)} E_k^{(k)} + \sum_{l=2}^{k} \sum_{\mathbf{k}=(k_1, \ldots, k_l) \in \Omega^{(l)}_{n,k}} \Theta^{(1)} E_{k_1}^{(k_1)} \Theta^{(1)} E_{k_2}^{(k_2)} \cdots \Theta^{(1)} E_{k_l}^{(k_l)}, \quad k \geq 4, \tag{2.22}
\]

where \( \Omega^{(l)}_{n,k} \subset \mathbb{N}^l \) is the subset of vectors \( \mathbf{k} \) satisfying that

\[
4 \leq k_i \leq n, \quad \text{and} \quad \sum_{i=1}^{l} k_i - 2(l-1) = k. \tag{2.23}
\]

In other words, \( \Theta^{(n)}_k \) contains all graphs in the Taylor expansion of \( \Theta^{(n)} \) that have scaling order \( k \), with the second condition in (2.23) meaning that the subgraph \( (\Theta^{(1)} E_{k_1}^{(k_1)} \Theta^{(1)} E_{k_2}^{(k_2)})_{xy} \) has scaling order \( k \). Notice that Definition 2.12 (i) is a recursive condition, that is, after defining lower order self-energies, we then use them to define \( \Theta_k^{(n)} \), \( 2 \leq k \leq n \leq 2l-1 \), which are further used to define \( E_{2l} \).

We will regard \( \Theta^{(n)} \) and \( \Theta_k^{(n)} \) as diffusive edges with labels.

**Definition 2.13 (Labelled diffusive edges).** We represent \( \Theta^{(n)}_{xy} \) and \( \Theta_k^{(n)}_{xy} \) by diffusive edges between atoms \( x \) and \( y \) with labels \( (n) \) and \( (n; k) \), respectively. In graphs, every labelled diffusive edge is drawn as a double-line edge with a label but without any internal structure. Moreover, when calculating scaling orders, a \( \Theta^{(n)} \) edge is counted as an edge of scaling order 2, and a \( \Theta_k^{(n)} \) edge is counted as an edge of scaling order \( k \).

Using the properties (2.17)–(2.19), we can show that the labelled diffusive edges defined above satisfies a similar bound as \( \Theta^{(1)} \) (recall (2.11)).

**Lemma 2.14.** Fix \( d \geq 6 \) and \( n \in \mathbb{N} \). Let \( E_{2l}, 4 \leq 2l \leq n \), be a sequence of self-energies satisfying Definition 2.12. We have that for any \( x, y \in \mathbb{Z}_L^d \) and fixed \( k \in \mathbb{N} \),

\[
|\Theta^{(n)}_{xy}(z)| < B_{xy}, \quad |(\Theta_k^{(n)})_{xy}(z)| < W^{-(k-2)d/2}B_{xy}, \tag{2.24}
\]

for all \( z = E + i\eta \) with \( |E| \leq 2 - \kappa \) and \( \eta \in [W^{-5}L^{5-d}, 1] \).

**Proof.** The estimate (2.24) was proved in Lemma 6.2 of [73] with \( \eta \geq W^2/L^2 \) and \( \Theta^{(1)} \) replaced by \( \Theta \). But, as we remarked below (2.12), the same proof also works for our setting by using \( |m(z)| \leq 1 \) and the spectral gap of \( S^{(1)} \) at 1. So we omit the details. \( \square \)

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2.4. Definition of the T-expansion. With the above preparations, in this subsection, we define the n-th order T-expansion for any fixed n.

**Definition 2.15** (n-th order T-expansion). Fix any \( n \in \mathbb{N} \) and a large constant \( D > n \). For \( a, b_1, b_2 \in \mathbb{Z}_L^d \), an n-th order T-expansion of \( T_{a, b_1; b_2} \) with D-th order error is an expression of the following form:

\[
T_{a, b_1; b_2} = m \Theta^{(n)}_{ab_1, b_2} + \frac{|m|}{2 \Delta N} (G_{b_2} b_1 - C_{b_1; b_2}) + \sum_{x} \Theta^{(n)}_{ax} \left[ R^{(n)}_{x; b_1, b_2} + A^{(>n)}_{x; b_1, b_2} + W^{(n)}_{x; b_1, b_2} + Q^{(n)}_{x; b_1, b_2} + (E_{nr, D})_{x; b_1, b_2} \right],
\]

where the right side is a sum of \( O(1) \) many graphs satisfying the following properties.

1. Every graph in \( R^{(n)}_{x; b_1, b_2} \), \( A^{(>n)}_{x; b_1, b_2} \), \( W^{(n)}_{x; b_1, b_2} \), \( Q^{(n)}_{x; b_1, b_2} \) and \((E_{nr, D})_{x; b_1, b_2}\) is a normal graph (recall Definition 2.7) with external atoms \( x, b_1, b_2 \). Furthermore, in every graph,
   - there is an edge, blue solid or diffusive or dotted, connected to \( b_1 \);
   - there is an edge, red solid or diffusive or dotted, connected to \( b_2 \).
2. The sequence of self-energies \( \mathcal{E}_k \), \( 4 \leq k \leq n \), satisfy Definition 2.12.
3. \( R^{(n)}_{x; b_1, b_2} \) is a sum of recollision graphs (cf. Definition 2.16) of scaling order \( \geq 3 \) and without any \( P/Q \) label or free edge.
4. \( A^{(>n)}_{x; b_1, b_2} \) is a sum of higher order graphs of scaling order \( > n \) and without any \( P/Q \) label or free edge.
5. \( W^{(n)}_{x; b_1, b_2} \) is a sum of graphs of scaling order \( \geq 3 \), without any \( P/Q \) label and with exactly one free edge.
6. \( Q^{(n)}_{x; b_1, b_2} \) is a sum of Q-graphs (cf. Definition 2.16) without any free edge.
7. \((E_{nr, D})_{x; b_1, b_2}\) is a sum of error graphs of scaling order \( > D \) (these graphs may contain \( P/Q \) labels and hence are not included into \( A^{(>n)}_{x; b_1, b_2} \)).
8. The graphs in \( R^{(n)}_{x; b_1, b_2} \), \( A^{(>n)}_{x; b_1, b_2} \), \( W^{(n)}_{x; b_1, b_2} \), \( Q^{(n)}_{x; b_1, b_2} \) and \((E_{nr, D})_{x; b_1, b_2}\) are doubly connected in the sense of Definition 2.11. Moreover, the free edge in every graph of \( W^{(n)}_{x; b_1, b_2} \) is redundant, that is, removing the free edge does not break the doubly connected property.

The graphs in \( R^{(n)} \), \( A^{(>n)} \), \( W^{(n)} \) and \( Q^{(n)} \) actually satisfy some additional graphical properties, which will be given later in Definition A.6.

The concepts of recollision graphs and Q-graphs are defined as follows.

**Definition 2.16** (Recollision graphs and Q-graphs). (i) We say a graph is a recollision graph, if there is at least one dotted edge connecting an internal atom to an external atom. By this definition, a recollision graph in \( R^{(n)}_{x; b_1, b_2} \) represents an expression where we set at least one summation index (which can be \( x \)) to be equal to \( b_1 \) or \( b_2 \).

(ii) We say a graph is a Q-graph if all G edges and weights in the graph have the same \( Q_x \) label for a specific atom \( x \). In other words, the value of a Q-graph can be written as \( Q_x (\Gamma) \) for an external atom \( x \) or \( \sum_x Q_x (\Gamma) \) for an internal atom \( x \), where \( \Gamma \) is a graph without \( P/Q \) labels.

To prove the local law, Theorem 1.5, we need to construct the T-expansion up to arbitrary high order.

**Theorem 2.17** (Construction of T-expansions). Given any \( M \in \mathbb{N} \), we can construct a sequence of n-th order T-expansions satisfying Definition 2.15 for all \( 2 \leq n \leq M \).

As a special case, when \( b_1 = b_2 = b \), (2.25) gives an expansion of the T-variable \( T_{ab} \). Later in Section 4.4, using the doubly connected property of the graphs in (2.25), we will show that, very roughly speaking, the second term on the right side of (2.25) provides a factor \((N \eta)^{-1}\) and the other terms can be bounded by \( B_{ab} \). This leads to the local law, Theorem 1.5, since we know that \( T_{ab} \) controls the size of \(|G_{ab}|^2\) (see Lemma 4.14 below).

3. Complete expansions

Letting \( b_1 = b_2 = b \), we notice that (2.25) is actually an incomplete expansion of \( b_1 = b_2 = b \) due to the random graphs in \( R^{(n)} \), \( A^{(>n)} \) and \( W^{(n)} \). In fact, since the diagonal G entries are close to m by
the local law (1.14), the first two terms on the right side of (2.25) give the two leading deterministic terms: $|m|^2 \Theta^{(n)} + |m|^2 \frac{\ln m}{N^\gamma}$. The $Q$-graphs in $Q^{(n)}$ are fluctuation terms with zero expectation, and the error graphs in $\mathcal{E}rr_{n,0}$ are always negligible as long as $D$ is taken sufficiently large. We need to further expand the graphs in $R^{(n)}$, $A^{(>n)}$, and $W^{(n)}$ into sums of deterministic graphs, fluctuation graphs, and error graphs. We call the expansion thus obtained a complete $T$-expansion. In other words, the complete $T$-expansion give the exact expectation of $T_{ab}$ in terms of a sum of deterministic graphs plus a negligible error.

3.1. Complete $T$-expansion. Before stating the complete $T$-expansion, we introduce another type of edge.

**Definition 3.1** (Ghost edges). We use a dashed edge between atoms $x$ and $y$ to represent a $W^2/L^2$ factor and call it a ghost edge. We do not count ghost edges when calculating the scaling order of a graph, i.e., the scaling order of a ghost edge is 0. Moreover, the doubly connected property in Definition 2.11 is extended to graphs with ghost edges by including these edges into the blue net. Finally, every ghost edge is associated with a $L^2/W^2$ factor in the coefficient of a graph.

Like free edges, ghost edges have no indices and can be moved freely to other places without changing the graph. Both ghost and free edges are introduced to maintain the doubly connected property and some other graphical properties (e.g., the SPD property in Definition A.4 and the generalized doubly connected property in Definition 6.1).

For any graph $G$, let $k_{gh}(G)$ denote the number of ghost edges in $G$. Define

$$\text{size}(G) := \left( \frac{L^2}{W^2} \right)^{k_{gh}(G)} W^{-\text{ord}(G)d_\eta},$$  

(3.1)

where

$$d_\eta := \begin{cases} \delta_0 / 2, & W^{-5+\delta_0} L^{5-d} \leq \eta < (W/L)^2, \\ d/2, & \eta \geq (W/L)^2. \end{cases}$$  

(3.2)

Roughly speaking, $d_\eta$ is chosen such that $B_{xy} + (N\eta)^{-1} = O(W^{-2d_\eta})$ for $\eta \geq W^{-5+\delta_0} L^{5-d}$. The following lemma shows that as long as there is a $T$-expansion of sufficiently high order, we can construct a complete $T$-expansion.

**Lemma 3.2** (Complete $T$-expansion). Under the assumptions of Theorem 1.5, suppose the local law

$$|G_{xy}(z) - m(z)\delta_{xy}|^2 < B_{xy} + \frac{1}{N\eta}$$  

(3.3)

holds for a fixed $z = E + i\eta$ with $|E| \leq 2 - \kappa$ and $\eta \in [W^{-5+\delta_0} L^{5-d}, 1]$. Fix any $n \in \mathbb{N}$ sufficiently large such that

$$L^2/W^2 \leq W^{(n-1)d_n - c_0}$$  

(3.4)

for some constant $c_0 > 0$. Suppose we have an $n$-th order $T$-expansion satisfying Definition 2.15. Then, for any large constant $D > 0$, $T_{a,b_1,b_2}$ can be expanded into a sum of $O(1)$ many normal graphs (which may contain ghost and free edges):

$$T_{a,b_1,b_2} = m\tilde{\Theta}_{ab_1}G_{b_1,b_2} + \frac{|m|^2}{2iN\eta}(G_{b_2b_1} - G_{b_1b_2}) + \mathcal{E}rr_{a,b_1,b_2} + \sum_{\mu} \sum_{x} \tilde{\Theta}_{ax}D_{x,b_1}^{(\mu)}G_{b_1b_2}f_{\mu}(G_{b_1b_2}) + \sum_{\nu} \sum_{x} \tilde{\Theta}_{ax}D_{x,b_2}^{(\nu)}G_{b_2b_1}f_{\nu}(G_{b_2b_1}) + \sum_{\gamma} \sum_{x} \tilde{\Theta}_{ax}Q_{x,b_1,b_2}^{(\gamma)}(G_{b_1b_2}, G_{b_2b_1}, G_{b_2b_2}, G_{b_1b_2}, G_{b_2b_1}) + \sum_{\omega} \sum_{x} \tilde{\Theta}_{ax}Q_{x,b_1,b_2}^{(\omega)}.$$

(3.5)

The graphs on the right-hand side satisfy the following properties.

1. $\mathcal{E}rr_{a,b_1,b_2}$ is an error term satisfying $\mathcal{E}rr_{a,b_1,b_2} < W^{-D}$.
2. $|\tilde{\Theta}_{xy}| < B_{xy}$ and $D_{x,b_1}^{(\mu)}$, $D_{x,b_2}^{(\nu)}$, $Q_{x,b_1,b_2}^{(\gamma)}$ are deterministic doubly connected graphs with size $\leq W^{-c_0}$.
3. $f_{\mu}(\cdot)$, $f_{\nu}(\cdot)$, and $g_{\gamma}(\cdot)$ are monomials.
4. $Q_{a,b_1,b_2}^{(\omega)}$ are doubly connected $Q$-graphs. (They actually satisfy some additional graphical properties, which will be given in Lemma B.1.)
Roughly speaking, taking \( b_1 = b_2 = b \), the above lemma shows that we can write \( T_{ab} \) as a sum of two leading terms (i.e., the first two terms on the right side of (3.5)), a fluctuation term of mean zero, a negligible error term, and some deterministic terms (except for some external weights \( G_{bb} \) and \( \overline{G}_{bb} \)).

### 3.2. Complete expansions of graphs with multiple external atoms

We can also extend the complete \( T \)-expansion to complete expansions of more general graphs with multiple external atoms. In other words, given a graph consisting of external atoms and edges between them, we want to expand it into a sum of deterministic graphs, \( Q \)-graphs and error graphs. Moreover, we will show that the deterministic graphs are properly bounded.

To state the main result, Lemma 3.4, of this subsection, we need to introduce a new concept of simple auxiliary graphs.

**Definition 3.3.** A graph is a simple auxiliary graph if it only contains external atoms and the following few types of edges:

- **pseudo-waved edges**: a waved edge between \( x \) and \( y \) represents a \( W^{-d}1_{|x-y|≤W^{1+τ}} \) factor for an arbitrary small constant \( τ > 0 \);
- **pseudo-diffusive edges**: a double-line edge between \( x \) and \( y \) represents a \( B_{xy} \) factor;
- **silent diffusive edges**: a green double-line edge between \( x \) and \( y \) represents a factor \( B_{xy} := W^{-4}(x−y)^{−(d−4)} \);
- **free edges**;
- **silent free edges**: a green solid edge between \( x \) and \( y \) represents a factor \( \frac{1}{N^4} \frac{L^2}{W^2} \).

All these edges are counted as edges of scaling order \( 2 \).

Pseudo-waved edges give bounds for waved edges by (1.10) and (2.15), and pseudo-diffusive edges give bounds for diffusive and labelled diffusive edges by (2.11) and (2.24). Silent pseudo-diffusive and free edges come from summations over internal atoms in the deterministic graphs obtained from complete expansions. Essentially, they come from the following estimates:

\[
\sum_w B_{xw}B_{wy} \lesssim \tilde{B}_{xy}, \quad \sum_w B_{xw} \frac{1}{N^4} \lesssim \frac{1}{N^4} \frac{L^2}{W^2}.
\]

**Lemma 3.4.** Under the setting of Lemma 3.2, suppose we have a complete \( T \)-expansion (3.5) for a fixed \( z = E + iη \) with \( |E| ≤ 2 - κ \) and \( η \in [W^{−5+δ_p}L^{−d}, 1] \). Let \( G_X(z) \) be a graph consisting of external atoms \( x = (x_1, \ldots, x_p) \), all taking different values, and (non-ghost and non-silent) edges between them. For any constant \( D > 0 \), we have that

\[
\mathbb{E}[G_X] = \sum_\mu G_X^\mu + O(W^{−D}), \quad (3.6)
\]

where the right-hand side is a sum of \( O(1) \) many deterministic normal graphs \( G_X^\mu \) with internal atoms and without silent edges such that if \( x_i \) and \( x_j \) are connected in \( G_X \), then they are also connected in \( G_X^\mu \) through non-ghost edges. Furthermore, every \( G_X^\mu \) is bounded by a sum of \( O(1) \) many simple auxiliary graphs:

\[
|G_X^\mu| < \sum_\gamma c_\gamma(W, L)G_X^{(\mu, \gamma)} + W^{−D}, \quad (3.7)
\]

where \( c_\gamma(W, L) \) are positive \( (W, L) \)-dependent coefficients. These simple auxiliary graphs satisfy the following properties.

(a) If \( x_i \) and \( x_j \) are connected in \( G_X^\mu \), then they are also connected in \( G_X^{(\mu, \gamma)} \). Furthermore, if \( x_i \) and \( x_j \) are connected in \( G_X^\mu \) without using free or ghost edges, then they are also connected in \( G_X^{(\mu, \gamma)} \) without using free or ghost edges.

(b) If \( G_X^{(\mu, \gamma)} \) contains \( k \geq 2 \) non-isolated atoms (i.e., atoms which have at least one neighbor), then there are at least \( \lceil k/2 \rceil \) special atoms such that each of them is connected with a different non-silent edge.

(c) Every \( G_X^{(\mu, \gamma)} \) satisfies \( c_\gamma(W, L)|\text{size}(G_X^{(\mu, \gamma)})| ≤ W^{−2d_4(\rho−t)} \) under the definition (3.1), where \( t \) denotes the number of connected components in \( G_X^{(\mu, \gamma)} \).
Remark 3.5. The bound $c_4(W, L)\text{size}(S^2_{\mathbf{G}(z, \gamma)}) \leq W^{-2d}(p-t)$ is actually a quite conservative estimate for general inputs. For example, if the initial graph $\mathbf{G}_x$ already contains a lot of edges in it, then the trivial bound $c_4(W, L)\text{size}(S^2_{\mathbf{G}(z, \gamma)}) \leq \text{size}(\mathbf{G}_x)$ may be better. However, the property (c) is sufficient for our purpose in this paper.

With Lemma 3.4, we can derive the following estimate. It will be used in proving the continuity estimate, Proposition 4.12, which is needed for the proof of the local law.

Lemma 3.6. Suppose the assumptions of Lemma 3.2 hold. Fix any $p \in \mathbb{N}$, consider a $p$-gon graph

$$\mathbf{G}_x(z) = \prod_{i=1}^{p} G_{x_i, x_{i+1}}(z),$$

where $x := (x_1, \ldots, x_p)$, $x_{p+1} \equiv x_1$, and $s_i \in \{\pm\}$ with the conventions $G^+_{xy} := G_{xy}$ and $G^-_{xy} := G_{-xy}$. Let $\mathcal{I} \subset \mathbb{Z}_L^d$ be a subset with $|\mathcal{I}| \geq W^d$ and denote $K := |\mathcal{I}|^{1/4}$. Then, we have that

$$\frac{1}{|\mathcal{I}|^p} \sum_{x_i \in \mathcal{I}, i \in [p]} \mathbf{G}_x(z) \ll g(K, W, \eta)^{p-1},$$

where

$$g(K, W, \eta) := \left[ \frac{1}{W2^{Kd-2}} + \frac{1}{N \eta} + \frac{1}{K^{d/2}} \sqrt{\frac{1}{N \eta} \frac{L^2}{W^2}} \left( \frac{1}{W^4 Kd-4} + \frac{1}{N \eta} \frac{L^2}{W^2} \right) \right]^{1/2}.$$  

4. PROOF OF THE MAIN RESULTS

4.1. Proof of the quantum unique ergodicity. To prove Theorem 1.4, we first observe that the quantity of interest

$$\frac{1}{|\mathcal{I}|^p} \sum_{x \in \mathcal{I}} (N|u_\alpha(x)|^2 - 1)$$

can be controlled by $\text{tr}(\mathcal{A} \mathcal{L}_{\Pi})$, where $\mathcal{A} := \text{Im} G = \frac{1}{2i}(G - G^*)$ and $\Pi$ is a diagonal matrix of zero trace. Then, some bounds on high moments of $\text{tr}(\mathcal{A} \mathcal{L}_{\Pi})$ obtained using Lemma 3.4 will conclude Theorem 1.4. The key ingredient for the proof is identifying a cancellation in $\text{tr}(\mathcal{A} \mathcal{L}_{\Pi})$ from the graphical properties of its complete expansions.

Lemma 4.1. Let $z = E + i \eta$ and $\Pi = \text{diag}(\Pi_x)_{x \in \mathbb{Z}_L^d}$ be a real diagonal matrix. Then, for any $l \geq \eta$,

$$\sum_{\alpha, \beta: |\lambda_\alpha - E| \leq l, |\lambda_\beta - E| \leq l} |\langle u_\alpha, \Pi u_\beta \rangle|^2 \leq \frac{4l^4}{\eta^2} \text{tr}(\mathcal{A}(z) \mathcal{L}(z) \Pi), \quad (4.1)$$

$$\sum_{\alpha, \beta: |\lambda_\alpha - E| \leq l, |\lambda_\beta - E| \leq l} |\langle u_\alpha, \Pi u_\beta \rangle|^2 \leq \frac{4l^4}{\eta^2} \text{tr}(\mathcal{A}(z) \mathcal{L}(z) \Pi \mathcal{L}(\Pi\Pi \alpha(z) \Pi)), \quad (4.2)$$

Proof. Using the spectral decomposition of $\mathcal{A}$,

$$\mathcal{A} = \frac{1}{2i}(G - G^*) = \sum_{\alpha} \frac{\eta}{|\lambda_\alpha - z|^2} u_\alpha u_\alpha^*,$$

giving that

$$\text{tr}(\mathcal{A} \mathcal{L}_{\Pi}) = \sum_{\alpha, \beta} \frac{\eta^2}{|\lambda_\alpha - z|^2 |\lambda_\beta - z|^2} |\langle u_\alpha, \Pi u_\beta \rangle|^2.$$  

This yields (4.1) immediately. The proof of (4.2) is similar. \[\square\]

With Lemma 4.1, the proof of Theorem 1.4 follows easily from the following proposition.

Proposition 4.2. Under the setting of Lemma 3.4, if $\Pi = \text{diag}(\Pi_x : x \in \mathbb{Z}_L^d)$ is a real diagonal matrix with $\text{tr}(\Pi) = 0$, then for any fixed $p \in \mathbb{N}$, we have

$$\mathbb{E} \left[ \left| \text{tr}(\mathcal{A}(z) \mathcal{L}(z) \Pi) \right|^{2p} \right] \lesssim \left( \sum_y |\Pi y| \right)^{2p} \left( \max_x \sum_y B_{xy} |\Pi y| \right)^{2p}. \quad (4.4)$$

The same bound holds for $\mathbb{E} \left[ \left| \text{tr}(\mathcal{A}(z) \mathcal{L}(z) \Pi) \right|^{2p} \right]$.  

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Proof of Theorem 1.4. Under the assumptions of Theorem 1.4, we know that the local law (3.3) holds by Theorem 1.5. Moreover, in the proof of Theorem 1.5, we have constructed a sequence of $T$-expansions up to arbitrary high order $n$ by Theorem 2.17. We can choose $n$ large enough depending on $\delta$, $\delta_0$ and $c_0$ so that (3.4) holds. Then, both Lemma 3.2 and Lemma 3.4 hold, so we can use Proposition 4.2 in the following proof.

Now, define $b_x := \frac{N}{|I_N|} 1_{x \in I_N}$ and $\Pi_x := b_x - 1$. Then, $\text{tr} (\Pi) = 0$ and

$$|\langle u_\alpha, \Pi u_\alpha \rangle|^2 = \left( \sum_x \Pi_x |u_\alpha(x)|^2 \right)^2 = \left( \sum_x b_x |u_\alpha(x)|^2 - 1 \right)^2 = \left( \frac{1}{|I_N|} \sum_{x \in I_N} (N|u_\alpha(x)|^2 - 1) \right)^2. \quad (4.5)$$

Next, taking $z = E + i \eta$ with $\eta = W^{-5+\delta_0} L^{5-d}$ and applying Markov’s inequality to (4.4), we obtain that

$$\text{tr} (\text{AILAIP}) \prec N \left( |I_N|^{2/d} \frac{N}{|I_N|} + \frac{L^2}{W^2} \right) \leq 2N^2 W^{-2} |N|^{2/d-1}.$$

Therefore, it follows from (4.1) that

$$\sup_{\alpha : |\lambda_\alpha - E| \leq \eta} |\langle u_\alpha, \Pi u_\alpha \rangle|^2 \prec \eta^2 \text{tr} (\text{AILAIP}) \prec (N\eta)^2 W^{-2} |N|^{2/d-1} = \frac{L^{10} W^{2\delta_0}}{W^{12} |N|^{1-2/d}}, \quad (4.6)$$

$$\frac{1}{N} \sum_{\alpha : |\lambda_\alpha - E| \leq \eta} |\langle u_\alpha, \Pi u_\alpha \rangle|^2 \prec \frac{\eta}{N} \text{tr} (\text{AILAIP}) \prec N\eta W^{-2} |N|^{2/d-1} = \frac{L^{5} W^{6\delta_0}}{W^7 |N|^{1-2/d}}. \quad (4.7)$$

Combining (4.6) and (4.5), we conclude (1.12) since $\delta_0 < 1$. From (4.7), we obtain that

$$\frac{1}{N} \sum_{\alpha : |\lambda_\alpha| \leq 2-\kappa} |\langle u_\alpha, \Pi u_\alpha \rangle|^2 \prec \frac{L^{5} W^{6\delta_0}}{W^7 |N|^{1-2/d}},$$

which, together with Markov’s inequality, implies that

$$\frac{1}{N} |\{ \alpha : |\lambda_\alpha| \leq 2-\kappa, |\langle u_\alpha, \Pi u_\alpha \rangle| \geq \epsilon \}| \prec \frac{e^{-2L^5 W^{6\delta_0}}}{W^7 |N|^{1-2/d}}. \quad (4.8)$$

Combining this bound with (4.5), with a union bound over $I_N \in \mathcal{I}$, we conclude (1.13). \hfill \Box

Finally, the proof of Proposition 4.2 follows from Lemma 3.4.

Proof of Proposition 4.2. First, note that $|\text{tr} (\text{AILAIP})|^{2p}$ is a sum of terms of the form

$$\sum_{\mathcal{G}} \sum_{x \in x} g_{x,y} \prod_i \Pi_{y_i} \Pi_{x_i},$$

where $g_{x,y}$ are graphs of the form

$$G_{x,y} = c(\{s_i\}) \prod_{i=1}^{2p} G^{(s_{2i-1})}_x G^{(s_{2i})}_{y_i},$$

with $s_i \in \{\pm\}$ and $c(\{s_i\})$ denoting a deterministic coefficient of order $O(1)$. By Lemma 3.4, we have that

$$\mathbb{E}[G_{x,y}] = \sum_{\mu} G^{(\mu)}_{x,y} + O(W^{-D}),$$

where $G^{(\mu)}_{x,y}$ are deterministic graphs as in (3.6). This implies that

$$\mathbb{E}[|\text{tr} (\text{AILAIP})|^{2p}] = \sum_{\mu} \sum_{x \in x} G^{(\mu)}_{x,y} \prod_i \Pi_{y_i} \Pi_{x_i} + O(W^{-D}). \quad (4.9)$$

We observe that it suffices to consider $G^{(\mu)}_{x,y}$ in which the $4p$ external atoms belong to at most $2p$ connected components when we do not include free edges into the edge set; otherwise the graph vanishes. To see this, suppose there are at least $2p+1$ such connected components. Then, there must be a connected component that contains only one external atom. Without loss of generality, suppose this atom is $x_1$. Since our graphs are translation invariant, we know that $G^{(\mu)}_{x,y}$ does not depend on $x_1$. Hence, $\sum_{x_1} G^{(\mu)}_{x,y} \Pi_{x_1} = G^{(\mu)}_{x,y} \sum_{x_1} \Pi_{x_1} = 0.$
By (3.7), every $G_{x,y}^{(\mu,\gamma)}$ is bounded by a sum of $O(1)$ many simple auxiliary graphs $\tilde{G}_{x,y}^{(\mu,\gamma)}$, with a coefficient
\[ c_{\mu,\gamma} \leq W^{2d_\gamma(k_{\mu,\gamma}+t_{\mu,\gamma}-4p)}, \]
(4.10)
where $k_{\mu,\gamma}$ and $t_{\mu,\gamma}$ are respectively the number of edges and the number of connected components (without removing free edges) in $G_{x,y}^{(\mu,\gamma)}$, and we have used property (c) of Lemma 3.4. Now, consider a graph $\tilde{G}_{x,y}^{(\mu,\gamma)}$ obtained from the following procedure.

- Remove all silent/non-silent free edges from $G_{x,y}^{(\mu,\gamma)}$.
- Remove a set of silent/non-silent pseudo-diffusive edges from the graph until each connected component becomes a rooted tree (the choice of the root is arbitrary). We always remove silent pseudo-diffusive edges first whenever possible.

Denote by $F_{\mu,\gamma}$ and $D_{\mu,\gamma}$ the number of (silent/non-silent) free edges removed and the number of (silent/non-silent) pseudo-diffusive edges removed, respectively. It follows that
\[ G_{x,y}^{(\mu,\gamma)} \leq W^{-d} \cdot D_{\mu,\gamma} \left( \frac{L^2}{W^2} \right)^{F_{\mu,\gamma}} \tilde{G}_{x,y}^{(\mu,\gamma)}. \]

To bound $\sum_{x,y} [\tilde{G}_{x,y}^{(\mu,\gamma)}] \prod_i |x_i||y_i|$, we sum over atoms from the leaves to the chosen root of each tree. If an atom connects to a pseudo-diffusive edge, then we get a factor $\max y \sum_y B_{xy}|y_i|$ from the summation over it. If an atom connects to a silent diffusive edge, then we get a factor $\max x \sum_x B_{xy}|y_i|$. If an atom is a root, then we get a factor $\sum_y |y_i|$. In sum, we obtain that
\[ \sum_{x,y} [\tilde{G}_{x,y}^{(\mu,\gamma)}] \prod_i |x_i||y_i| \leq W^{-d} \cdot D_{\mu,\gamma} \left( \frac{L^2}{W^2} \right)^{F_{\mu,\gamma}} \max x \sum_y B_{xy}|y_i|^{PD_{\mu,\gamma}} \max x \sum_y B_{xy}|y_i|^{SD_{\mu,\gamma}} \sum_y |y_i|^{CC_{\mu,\gamma}}, \]
(4.11)
where the exponents $PD_{\mu,\gamma}$, $SD_{\mu,\gamma}$, and $CC_{\mu,\gamma}$ denote respectively the numbers of pseudo-diffusive edges, silent pseudo-diffusive edges, and connected components in $\tilde{G}_{x,y}^{(\mu,\gamma)}$. Since $SD_{\mu,\gamma} = 4p - PD_{\mu,\gamma} - CC_{\mu,\gamma}$, abbreviating $A := \max x \sum_y B_{xy}|y_i|$, we get that the right-hand side of (4.11) is
\[ W^{-d} \cdot D_{\mu,\gamma} \left( \frac{L^2}{W^2} \right)^{F_{\mu,\gamma}} \max x \sum_y B_{xy}|y_i|^{PD_{\mu,\gamma}} \max x \sum_y B_{xy}|y_i|^{SD_{\mu,\gamma}} \sum_y |y_i|^{CC_{\mu,\gamma}} \]
\[ \leq W^{-d} \cdot D_{\mu,\gamma} \left( \frac{L^4}{W^4} \right)^{F_{\mu,\gamma}} \max x \sum_y B_{xy}|y_i|^{2p} \sum_y |y_i|^{2p} \]
\[ \leq W^{-d} \cdot D_{\mu,\gamma} \left( \frac{L^4}{W^4} \right)^{F_{\mu,\gamma}} \max x \sum_y B_{xy}|y_i|^{2p} \sum_y |y_i|^{2p}. \]
(4.12)
Here, in the first step, we used $CC_{\mu,\gamma} \leq 2p$, $F_{\mu,\gamma} + PD_{\mu,\gamma} \geq 2p$ by Lemma 3.4 (b), and
\[ \frac{W^2}{L^2} \cdot A \leq \max x \sum_y B_{xy}|y_i|, \quad A \leq W^{-d} \sum_y |y_i|, \]
and in the second step we used the trivial bounds
\[ W^{-d} + \frac{L^4}{W^4} \leq W^{-d} + 2p - CC_{\mu,\gamma} \geq k_{\mu,\gamma} + t_{\mu,\gamma} - 4p, \]
due to $k_{\mu,\gamma} = D_{\mu,\gamma} + F_{\mu,\gamma} + PD_{\mu,\gamma} + SD_{\mu,\gamma}$ and $t_{\mu,\gamma} = CC_{\mu,\gamma} \leq 2p$. Plugging (4.12) into (4.11) and using (4.10), we obtain that
\[ \sum_{x,y} [G_{x,y}^{(\mu,\gamma)}] \prod_i |x_i||y_i| \leq \sum_{\gamma} c_{\mu,\gamma} \sum_{x,y} [\tilde{G}_{x,y}^{(\mu,\gamma)}] \prod_i |x_i||y_i| \leq \left( \max x \sum_y B_{xy}|y_i| \right)^{2p} \left( \sum_y |y_i| \right)^{2p}. \]
This yields (4.4) together with (4.9).
4.2. Proof of the bulk universality. Let $\delta_0 \in (0, 1)$ be a sufficiently small constant and set $\eta_* := W^{-5+\delta_0}L^{-5-d}$, $\eta_0 := W^{-\delta_0}L^{-d}$. We define the matrix Ornstein–Uhlenbeck process $H_t$ as the solution to

$$
\frac{dH_t}{dt} = -\frac{1}{2}H_t dt + \frac{1}{\sqrt{N}} dB_t, \quad \text{with} \quad H_0 = H,
$$

(4.13)

where $B_t$ is an Hermitian matrix-valued Brownian motion. Define its Green’s function $G_t(z) = (H_t - z)^{-1}$ and the averaged trace $m_t(z) := N^{-1} \text{tr}(G_t(z))$.

**Proposition 4.3.** Under the setting of Theorem 1.3, fix $n \in \mathbb{N}$ and $|E| \leq 2 - \kappa$. Consider a sequence $z_i = E_i + i \eta_i$, $i = 1, \ldots, n$, with $|E_i - E| \leq \eta_*$ and $\eta_0 \leq \eta_i \leq W^{\delta_0}L^{-d}$. If $W^{95+d} \geq L^{95+c}$ for a constant $c > 0$, then

$$
\sup_{0 \leq t \leq \eta_*} \left| \mathbb{E} \prod_{i=1}^{n} \text{Im} m_t(z_i) - \mathbb{E} \prod_{i=1}^{n} \text{Im} m_{\eta_*}(z_i) \right| \leq L^{-c/(2n+5)\delta_0},
$$

(4.14)

for any constant $\delta_0 \in (0, c/(12n + 30))$.

**Proof of Theorem 1.3.** With Theorem 1.5 as the input, applying [53, Theorem 2.4] gives the universality of the correlation functions of $H_t$ at $t = \eta_*$. Then, Theorem 1.3 follows from a standard correlation function comparison theorem [42, Theorem 15.3] and Proposition 4.3, as long as we take a sufficiently small $\delta_0$. \qed

To conclude Proposition 4.3, we will bound products of $\text{Im} m_t(z_i)$ using the following lemma.

**Lemma 4.4.** For any fixed $n \in \mathbb{N}$, we have that

$$
\left| \frac{\partial}{\partial t} \prod_{i=1}^{n} \text{Im} m_t(z_i) \right| \leq C \mathbb{E} \left[ \sum_{u} L_{1,t}(z_u) \prod_{i \neq u} \text{Im} m_t(z_i) + \sum_{u \neq v} L_{2,t}(z_u, z_v) \prod_{i \neq u, v} \text{Im} m_t(z_i) \right],
$$

(4.15)

for an absolute constant $C > 0$, where we denote

$$
L_{1,t}(z) := \sum_{G_1, G_2 \in \{G_1, G_t\}} \left| \frac{1}{N} \sum_{a,b} (G_t^2(z))_{aa} s_{ab}^0 (G_2(z))_{bb} \right|,
$$

$$
L_{2,t}(z_1, z_2) := \sum_{G_1, G_2 \in \{G_1, G_t\}} \left| \frac{1}{N^2} \sum_{a,b} (G_t^2(z_1))_{ab} s_{ab}^0 (G_2(z_2))_{ba} \right|.
$$

**Proof.** Using the Ito’s formula and Gaussian integration by parts, we get that

$$
\frac{\partial}{\partial t} \prod_{i=1}^{n} \text{Im} m_t(z_i) = \frac{e^{-t}}{2} \sum_{a,b} \mathbb{E} \left[ \prod_{i=1}^{n} \text{Im} m_t(z_i) \right] \left( \frac{1}{N} - s_{ab} \right)
$$

$$
= -\frac{e^{-t}}{2} \sum_{a,b} \mathbb{E} \left[ \prod_{u \neq u} (\partial_{ab} \partial_{ba} \text{Im} m_t(z_u)) \prod_{i \neq u, v} \text{Im} m_t(z_i) + \sum_{u \neq v} (\partial_{ba} \text{Im} m_t(z_u))(\partial_{ab} \text{Im} m_t(z_v)) \prod_{i \neq u, v} \text{Im} m_t(z_i) \right],
$$

where $\partial_{ab}$ means the partial derivative $\partial/\partial (H_t)_{ab}$. One can check that the terms coming from $\partial_{ab} \partial_{ba} \text{Im} m_t(z_u)$ can be bounded by $L_{1,t}$, while the terms coming from $(\partial_{ba} \text{Im} m_t(z_u))(\partial_{ab} \text{Im} m_t(z_v))$ can be bounded by $L_{2,t}$. This concludes the proof. \qed

With Lemma 4.4, it suffices to prove the following estimates on $L_{1,t}$ and $L_{2,t}$.

**Lemma 4.5.** Under the setting of Proposition 4.9, for $u, v \in \{1, \ldots, n\}$, we have that

$$
\sup_{0 \leq t \leq \eta_*} L_{1,t}(z_u) \prec W^{-(d+65)/6+5\delta_0} L^{65/6} N, \quad \sup_{0 \leq t \leq \eta_*} L_{2,t}(z_u, z_v) \prec W^{-(d+60)/6+6\delta_0} L^{10} N.
$$

(4.16)

**Proof of Proposition 4.3.** Under the condition $W^{95+d} \geq L^{95+c}$, by Lemma 4.5, we have

$$
L_{1,t}(z_u) + L_{2,t}(z_u, z_v) \prec L^{-c/6+7\delta_0}/\eta_*,
$$

uniformly in $0 \leq t \leq \eta_*$. Plugging it into Lemma 4.4, we obtain that

$$
\left| \frac{\partial}{\partial t} \prod_{i=1}^{n} \text{Im} m_t(z_i) \right| \prec \eta_*^{-1} L^{-c/6+7\delta_0} \left( \max_u \mathbb{E} \prod_{i \neq u} \text{Im} m_t(z_i) + \max_u \mathbb{E} \sum_{u \neq v} \prod_{i \neq u, v} \text{Im} m_t(z_i) \right).
$$
Denoting $B_k(t) := \max_{S \subseteq \{1, \ldots, n\}, |S| = k} \mathbb{E} \prod_{i \in S} \text{Im} m_i(z_i)$, the above estimate implies that

$$\sup_{0 \leq t \leq \eta_*} \left| \frac{1}{N} \sum_{i=1}^{n} \text{Im} m_i(z_i) - \mathbb{E} \prod_{i=1}^{n} \text{Im} m_{\eta_*}(z_i) \right| \leq L^{-e/6+7\delta_0} \sup_{0 \leq t \leq \eta_*} \left| [B_{n-1}(t) + B_{n-2}(t)] \right|. \quad (4.17)$$

Recall that an local law for $m_{\eta_*}(z)$ has been established in [53, Theorem 3.3] down to the scale $\tilde{\eta} := W^{6d}L^{-d}$, which gives that $\text{Im} m_{\eta_*}(E_t + i\tilde{\eta}) = O(1)$ with high probability. Then, using the simple fact $\eta_* \text{Im} m(E_t + i\eta_* \tilde{\eta}) \leq \eta\text{Im} m(E_t + i\tilde{\eta})$, we obtain that $B_k(\eta_*) \lesssim W^{2k\delta_0}$ for all $1 \leq k \leq n$. With this initial bound, iterating (4.17) for $n$ many times concludes the proof. \hfill \Box

The rest of this subsection is devoted to the proof of Lemma 4.5. It suffices to prove (4.16) for every fixed $0 \leq t \leq \eta_*$. After that, we can take an $L^{-c}$-net of $t \in [0, \eta_*]$ for a large enough constant $C > 0$, and then use a simple union bound and perturbation argument to conclude Lemma 4.5. We slightly abuse the notations and still denote the eigenvalues and eigenvectors of $H_t$ by $\{\lambda_a\}$ and $\{u_a\}$.

**Lemma 4.6.** $L_{1,t}$ and $L_{2,t}$ satisfy that

$$L_{1,t}(z) \leq 4m_{\eta_*}(z) \leq 4 \sum_{\alpha,\beta} \left| \frac{p_\alpha(z)}{N} \right| M_{\alpha,\beta}, \quad L_{2,t}(z_1, z_2) \leq 4 \sum_{\alpha,\beta} \left| \frac{p_\alpha(z_1)}{N} \right| \left| \frac{p_\beta(z_2)}{N} \right| M_{\alpha,\beta},$$

where $p_\alpha(z) := (\lambda_a - z)^{-1}$, $m_t := N^{-1} \sum |p_\beta|$ and

$$M_{\alpha} := \sup_a |N (u_a, S^o_a u_a)|, \quad M_{\alpha,\beta} := \sup_{a} \sup_{\tilde{a} \in \{u_a, \pi_a\} : i \in \{\alpha, \beta\}} |N (\tilde{u}_a, S^o_a \tilde{u}_\beta)|,$

with $S^o_a := \text{diag}(s_{ab}^o : b \in \mathbb{Z}_L^d)$.\hfill \Box

**Proof.** By the eigendecomposition of $G_t$, we have that

$$\sum_{a,b} (G^2_t(z_1))_{ab} s^o_{ab} (G^2_t(z_2))_{ba} = \sum_{\alpha,\beta} p^2_\alpha(z_1)p^2_\beta(z_2) \sum_a u_a(a)\tilde{u}_\beta(a) \sum_b \tilde{u}_a(b) u_\beta(b) s^o_{ab}.$$

Hence, we get

$$\left| \sum_{a,b} (G^2_t(z_1))_{ab} s^o_{ab} (G^2_t(z_2))_{ba} \right| \leq \sum_{\alpha,\beta} p^2_\alpha(z_1)p^2_\beta(z_2) \left| \sum_a |u_a(a)||\tilde{u}_\beta(a)||\tilde{u}_a(b) u_\beta(b) s^o_{ab} \right| \leq \frac{1}{N} \sum_{\alpha,\beta} |p^2_\alpha(z_1)p^2_\beta(z_2)||M_{\alpha,\beta} \sum_a |u_a(a)||\tilde{u}_\beta(a)||\tilde{u}_a(b) u_\beta(b) s^o_{ab} | \leq \frac{1}{N} \sum_{\alpha,\beta} |p^2_\alpha(z_1)p^2_\beta(z_2)||M_{\alpha,\beta}. $$

The other cases are similar. \hfill \Box

Next, we bound the quantities appearing in the previous lemma using the local law and QUE. We have only proved the local law and QUE at $t = 0$, but they can be extended to any $0 \leq t \leq \eta_*$.\hfill \Box

**Lemma 4.7.** For any $0 \leq t \leq \eta_*$, Theorem 1.5, Corollary 1.6, and Proposition 4.2 hold for $H_t$.\hfill \Box

**Proof.** We copy the proof for the $t = 0$ case verbatim, where the only difference is that $H_t$ has a different variance matrix $S(t)$ with entries $e^{-t}s_{xy} + (1 - e^{-t})N^{-1}$. Correspondingly, the matrices $\Theta^o$ and $S^\pm$ are replaced by (recall (2.4) and (2.14))

$$\Theta^o(t) := \frac{e^{-t}|m|^2 S^o}{1 - e^{-t}|m|^2 S^o}, \quad S^+_t := \frac{m^2 S(t)}{1 - m^2 S(t)}, \quad S^-_t = S^+_t.$$

It is not hard to see that $\Theta^o(t)$ has the same behavior as $\Theta^o$, while $S^\pm_t$ has an extra zero mode compared to (2.15):

$$(S^+_t)_{xy} \asymp W^{-d}1_{|x-y| \leq W^{1+\tau}} + t/N + (x-y)^{-D}.$$

Hence, we only need to show that if we replace some $\pm$ waved edges in the graphs appearing in the proof of the $t = 0$ case by $t/N$ factors, the resulting new graphs will not cause any trouble. In fact, this kind of replacement only affect the definition of molecules (i.e, two atoms in the same molecule may not satisfy $|x-y| \leq W^{1+\tau}$ anymore), which in turn may affect the doubly connected property. However, notice that we can split a $t/N$ edge between $x$ and $y$ as $\frac{1}{N} \leq B_{xy} \leq W^2L^2$ with an additional small $t$ factor in the coefficient, where $B_{xy}$ is a pseudo-diffusive edge that can be used in the black net and $W^2/L^2$ is a ghost edge that can
be used in the blue net. Hence, all the new graphs are still doubly connected if we regard a \(1/N\) edge as a “long” edge between molecules, while the additional small \(t\) factor in their coefficients make them harmless in most places. The only place where these new graphs may have an effect is that the property (2.18) now becomes
\[
|\xi(x)| \lesssim W^{-(l-2)d}B_{0x}^2 + tW^{-(l-1)d}W^2 \lesssim W^{-(l-1)d}W^2 \langle x \rangle^{-d/2},
\] (4.18)
for \(x \in \mathbb{Z}^d_L\). The property (2.18) is only relevant in establishing Lemma 2.14, whose proof only uses the last weaker bound in (4.18) (see the proof of [73, Lemma 6.2]). Once the bounds on labelled diffusive edges are proved, the rest of the proof in the \(t = 0\) case can be copied verbatim to the \(t > 0\) case, and we will not repeat them here.

**Lemma 4.8.** Under the setting of Proposition 4.3, for any small constant \(\kappa > 0\), the following estimates hold for \(\ell \geq \eta_*\) and \(z = E + i\eta\) with \(|E| \leq 2 - \kappa\) and \(\eta_0 \leq \eta \leq \eta_*\):
\[
\sum_{\alpha,|\lambda - E| \geq \ell} |p_\alpha(z)|^2 \lesssim \frac{N \eta_0}{\ell \eta_0},
\] (4.19)
\[
\sup_{\alpha \in [\kappa, N(1 - \kappa)]N} M_\alpha + \sup_{\alpha, \beta \in [\kappa, N(1 - \kappa)]N} M_{\alpha, \beta} \lesssim N \eta_0,
\] (4.20)
\[
\sum_{\alpha,|\lambda - E| \leq \ell} |M_\alpha|^2 + \sum_{\alpha, \beta,|\lambda - E| \leq \ell,|\lambda - E| \leq \ell} |M_{\alpha, \beta}|^2 \lesssim \frac{N^{2\ell^4}}{\eta_*^2} W^{-d},
\] (4.22)
\[
\sum_{\alpha,|\lambda - E| \leq \ell} |M_\alpha|^2 \lesssim \frac{N \eta_0}{\eta_*},
\] (4.23)
\[
\sum_{\alpha,|\lambda - E| \leq \ell} |M_{\alpha, \beta}|^2 \lesssim \frac{N \eta_0}{\eta_*}.
\] (4.24)

**Proof.** Let \(z_* := E + i\eta_*\). By Theorem 1.5, for any \(|E| \leq 2 - \kappa\), we have
\[
\#\{\alpha : |\lambda_\alpha - E| \leq \eta_0\} \leq \sum_{\alpha} \frac{2\eta_0^2}{|\lambda_\alpha - (E + i\eta_0)|^2} = 2N\eta_0 \text{Im} m_t(z) \leq 2N \eta_* \text{Im} m_t(z_*) \lesssim N \eta_*.
\]
Since \(E\) and \(\kappa\) are arbitrary, it gives that for any \(k \in \mathbb{N}\) so that \(k\eta_0 \leq (\log L)^{-1}\),
\[
\#\{\alpha : |\lambda_\alpha - E| \in [k\eta_0, (k + 1)\eta_0]\} \lesssim N \eta_*.
\]
With this estimate, we obtain (4.19) as follows:
\[
\sum_{\alpha,|\lambda - E| \geq \ell} |p_\alpha(z)|^2 \leq \sum_{k \geq 0} \frac{4}{(\ell + k\eta_0)^2} \#\{\alpha : |\lambda_\alpha - E| \in [k\eta_0, (k + 1)\eta_0]\} \leq \frac{N \eta_*}{\ell \eta_0}.
\]
With a similar method, we get that
\[
\sum_{\alpha} |p_\alpha(z)| = \sum_{\alpha,|\lambda_\alpha - E| \leq \eta_0} |p_\alpha(z)| + \sum_{\alpha,|\lambda_\alpha - E| > \eta_0} |p_\alpha(z)| \lesssim \frac{N \eta_*}{\eta_0} + \sum_{k \eta_0 \lesssim \eta_*} \frac{N \eta_*}{k \eta_0} \lesssim N \eta_*,
\]
which yields (4.20). Next, (4.21) follows from Corollary 1.6, and (4.22) follows easily from the fact that
\[
||S^2_a|| \lesssim W^{-d}.
\]
Finally, by Lemma 4.1, we have that
\[
\sum_{\alpha,|\lambda - E| \leq \ell} |M_\alpha|^2 + \sum_{\alpha, \beta,|\lambda - E| \leq \ell,|\lambda - E| \leq \ell} |M_{\alpha, \beta}|^2 \lesssim \sup_{\alpha} \frac{N^{2\ell^4}}{\eta_*^2} \left[ \text{tr} (A(z_*)S^2_aA(z_*)S^2_a) + \text{tr} (A(z_*)S^2_aA(z_*)S^2_a) \right].
\]
Applying Proposition 4.2 with \(\Pi = S^2_a\), we get
\[
\text{tr} (A(z_*)S^2_aA(z_*)S^2_a) \sim W^{-d}, \quad \text{tr} (A(z_*)S^2_aA(z_*)S^2_a) \sim W^{-d}.
\]
The previous two equations together give (4.23). \(\Box\)
Proof of Lemma 4.5. We first use Lemma 4.6 and then plug in the estimates in Lemma 4.8 to get that
\[
L_{1,t}(z_n) \leq \frac{4m_0}{N} \left( \sum_{\alpha: |\lambda_\alpha| > \frac{2}{\kappa}} |p_\alpha(z_n)|^2 M_\alpha + \sum_{\alpha: |\lambda_\alpha - E| > t, |\lambda_\alpha| \leq \frac{2}{\kappa}} |p_\alpha(z_n)|^2 M_\alpha + \sum_{\alpha: |\lambda_\alpha - E| \leq t} |p_\alpha(z_n)|^2 M_\alpha \right)
\]
\[
< \frac{\eta_*}{N\eta_0} \left( N^2 \omega^d - \frac{\eta_*}{\ell \eta_0} \cdot N\eta_* + \sqrt{\sum_{\alpha: |\lambda_\alpha - E| \leq t} |p_\alpha(z_n)|^2} \right)^{1/2} \left( \sum_{\alpha: |\lambda_\alpha - E| \leq t} M_\alpha^{1/2} \right)
\]
Here, in the second step we used (4.19)–(4.22) and the Cauchy-Schwarz inequality, in the third step we used (4.23) and
\[
\sum_{\alpha: |\lambda_\alpha - E| \leq t} |p_\alpha(z_n)|^4 \leq \frac{1}{\eta_0^3} \sum_\alpha |p_\alpha(z_n)|^2 = \frac{N \text{Im} m_t(z_n)}{\eta_0^3} \leq \frac{N \text{Im} m_t(z_n)}{\eta_0^3}
\]
and in the fourth step we used \( \eta_* \text{Im} m_t(z_n) \leq \eta_* \text{Im} m_t(E_n + \eta_*) \) and the local law (1.14). Taking \( \ell = W^{d/6} N^{1/3} \eta_0^{5/6} \) gives the first bound in (4.16). Similarly, we can bound \( L_{2,t} \) as
\[
L_{2,t}(z_n, z_v) \leq 4 \sum_{\alpha: |\lambda_\alpha - E| > \ell} \sum_{\beta} \frac{1}{N^3} \sum_{\alpha: |\lambda_\alpha - E| \leq 2\ell, |\lambda_\beta - E_v| > \ell} |p_\alpha(z_n)|^2 |p_\beta(z_v)|^2 M_{\alpha, \beta} + 4 \sum_{\alpha: |\lambda_\alpha - E| \leq 2\ell} |p_\alpha(z_n)|^2 |p_\beta(z_v)|^2 M_{\alpha, \beta}
\]
\[
< \frac{1}{N^3} \left( N^2 \omega^d + \sum_{|u, v|} \frac{N \text{Im} m_t(z_n)}{\eta_0} \eta_* \cdot N\eta_* + \frac{N \sqrt{\text{Im} m_t(z_n)}}{\eta_0^2} \cdot N\eta_* \cdot W^{d/2} \right)
\]
\[
< W^{-d} + \left( \frac{\eta_*}{\eta_0} \right)^3 \frac{1}{\ell} + \frac{\ell^2}{N\eta_0^4 W^{d/2}} = \frac{2\eta_*^2}{N^1/3 W^{d/6} \eta_0^{10/3}},
\]
where we have chosen \( \ell = W^{d/6} N^{1/3} \eta_0^{5/6} \eta_* \geq \eta_* \geq \max_{|u, v|} |E_i - E| \) and used \( \{ \alpha: |\lambda_\alpha - E_v| \leq \ell \} \subset \{ \alpha: |\lambda_\alpha - E| \leq 2\ell \} \). This concludes the second bound in (4.16). \( \square \)

4.3. Proof of the local law. Theorem 1.5 follows immediately from Proposition 4.9 and Theorem 2.17.

Proposition 4.9. Under the setting of Theorem 1.5, suppose we have an \( n \)-th order \( T \)-expansion satisfying Definition 2.15.

(i) Suppose for some constant \( c_0 > 0 \),
\[
L^2/W^2 \leq W^{(n-1)d/2-c_0}.
\]
Then, the following local law holds uniformly in all \( z = E + i\eta \) with \( |E| \leq 2 - \kappa \) and \( \eta \in [W^2/L^2, 1] \):
\[
|G_{xy}(z) - m(z)\delta_{xy}|^2 < B_{xy} + \frac{1}{N\eta}, \quad \forall x, y \in \mathbb{Z}_L^d.
\]

(ii) Suppose for some constant \( c_0 > 0 \),
\[
L^2/W^2 \leq W^{(n-1)\delta_0/2-c_0}.
\]
Then, the local law (4.25) holds uniformly in all \( z = E + i\eta \) with \( |E| \leq 2 - \kappa \) and \( \eta \in [W^{-5+\delta_0} L^{5-d}, 1] \).

The two conditions (4.24) and (4.26) come from (3.4) due to the two values of \( d_0 \). Moreover, the two parts of Proposition 4.9 are used at different stages of the proof. Part (i) will be used in the proof of Theorem 2.17 (see the proof of Proposition 5.2 in Section 5). Once we have a sequence of \( T \)-expansions up to any order by Theorem 2.17, taking \( n \) sufficiently large so that (4.26) holds, we conclude Theorem 1.5 by part (ii).
Similar to many previous proofs of local laws in the literature, we prove Proposition 4.9 through a multiscale argument in $\eta$, that is, we gradually transfer the local law at a larger scale of $\eta$ to a multiplicative smaller scale of $\eta$. We first have an initial estimate at $\eta = 1$, which holds in all dimensions $d \geq 1$.

**Lemma 4.10** (Initial estimate, Lemma 7.2 of [74]). Under the assumptions of Theorem 1.5, for any $z = E + i \eta$ with $|E| \leq 2 - \kappa$ and $\eta = 1$, we have that

$$|G_{xy}(z) - m(z)\delta_{xy}|^2 < B_{xy}, \quad \forall \, x, y \in \mathbb{Z}_L^d.$$  

(4.27)

Next, starting with a large $\eta$, Proposition 4.12 gives a key continuity estimate, which says that if the local law holds at $\eta$, then a weaker local law will hold at a multiplicative smaller scale than $\eta$. To state it, we need the following definition.

**Definition 4.11.** An edge $A_{xy}$ is said to be $\lambda$-bounded if $\|A\|_w \prec W^\lambda$, where

$$\|A\|_w := W^{d_0} \max_{x, y \in \mathbb{Z}^d_L} |A_{xy}| + \sup_{k \in [\mathbb{Z}^d_L]} \max_{x, y \in \mathbb{Z}_L^d} \frac{1}{K^d \sqrt{W(K, W, \eta)}} \sum_{|y - x_0| \leq K} (|A_{xy}| + |A_{yx}|),$$

where $g$ is defined in (3.10). An edge $A_{xy}$ is said to be $(\Phi, \lambda)$-bounded if it is $\lambda$-bounded and $\|A\|_{s, \Phi} \prec 1$, where

$$\|A\|_{s, \Phi} := \max_{x, y \in \mathbb{Z}_L^d} |A_{xy}|/\left(W^{-1}(x - y)^{1-d/2} + \Phi\right).$$

Here, the second stronger norm corresponds to the local law when $\Phi = (N\eta)^{-1/2}$, while the first weaker norm is due to the following continuity estimate.

**Proposition 4.12** (Continuity estimate). Under the setting of Proposition 4.9, suppose

$$\|G_{xy}(\tilde{z}) - m(\tilde{z})\delta_{xy}\|_w \prec (N\eta)^{-1/2} \prec 1,$$  

(4.28)

with $\tilde{z} = E + i\tilde{\eta}$ for $|E| \leq 2 - \kappa$ and $\tilde{\eta} \in [W^{-5+\delta_0}L^{5-d}, 1]$. Then, we have that

$$\|G_{xy}(z) - m(z)\delta_{xy}\|_w \prec \tilde{\eta}/\eta,$$

uniformly in $z = E + i \eta$ with $\max(W^{-5+\delta_0}L^{5-d}, W^{-d_0/20}) \leq \eta \leq \tilde{\eta}$.

**Proof.** Let $I = \{y : |y - x_0| \leq K\}$. Using equation (5.25) of [73] and (4.28), we get that

$$\sum_{y \in I} (|G_{xy}(z)|^2 + |G_{yx}(z)|^2) \prec \frac{K^2}{W^2} + \frac{K^d}{N \eta} + \left(\frac{\tilde{\eta}}{\eta}\right)^2 \|A_{I}\|_{\|z\|_{2}, \|z\|_{2}},$$  

(4.29)

where $A_{I}$ is the submatrix of $\frac{1}{\delta^2}(G - G^*)$ with row and column indices in $I$. Using Lemma 3.6, we get that for any $p \in 2\mathbb{N},$

$$E\|A_{I}\|^p_{\|z\|_{2}, \|z\|_{2}} \leq E\text{Tr}(A_{I}^p) \prec (K^d)^p g(K, W, \eta)^{p-1}.$$  

Together with Markov’s inequality, it yields that $\|A_{I}\|_{\|z\|_{2}, \|z\|_{2}} \prec K^d g(K, W, \eta)$ since $p$ is arbitrary. Plugging this estimate into (4.29) gives that

$$\max_{x, x_0} \frac{1}{K^d} \sum_{y : |y - x_0| \leq K} (|G_{yx}(z)|^2 + |G_{xy}(z)|^2) \prec \left(\frac{\tilde{\eta}}{\eta}\right)^2 g(K, W, \eta).$$  

(4.30)

Using the Cauchy-Schwarz inequality, we get from the above bound that

$$\max_{x, x_0} \frac{1}{K^d} \sum_{y : |y - x_0| \leq K} (|G_{yx}(z)| + |G_{xy}(z)|) \prec \frac{\tilde{\eta}}{\eta} \sqrt{g(K, W, \eta)}.$$  

It remains to prove that

$$\|G(z) - m(z)\|_{\max} \prec \frac{\tilde{\eta}}{\eta} W^{-d_0}.$$  

(4.31)

This can be proved using a standard $\varepsilon$-net and perturbation argument, see e.g., the proof of equation (5.8) in [73]. We omit the details. □

Finally, Proposition 4.13 and Lemma 4.14 will improve the weak continuity estimate in Proposition 4.12 to the stronger local law.
Proposition 4.13 (Entrywise bounds on $T$-variables). Under the setting of Proposition 4.9, fix any $z = E + i \eta$ with $|E| \leq 2 - \kappa$ and $\eta \in [W^{-5+\delta_0}L^{5-d}, 1]$. Suppose
\[ \|G(z) - m(z)\|_w < W^\lambda \] (4.32)
for some constant $\lambda$ sufficiently small depending on $d$, $\delta_0$, $n$ and $c_0$ in (4.24) or (4.26). Then, we have that
\[ T_{xy}(z) < B_{xy} + \frac{1}{N\eta}, \quad \forall \ x, y \in \mathbb{Z}_d^d. \] (4.33)

We postpone the proof of Proposition 4.13 to Section 4.4. Combining this proposition with the following lemma, we can obtain the local law (4.25). The bound (4.35) was proved in equation (3.20) of [75], while (4.36) was proved in Lemma 5.3 of [35].

Lemma 4.14. Suppose for a constant $\varepsilon > 0$, a deterministic parameter $W^{-d/2} \leq \Phi \leq W^{-\varepsilon}$ and a subset $D \subset \mathbb{C}_+$, we have that
\[ \|G(z) - m(z)\|_{\max} < W^{-\varepsilon}, \quad \|T\|_{\max} < \Phi^2, \quad \text{uniformly in } z \in D. \] (4.34)
Then, the following estimates hold:
\[ 1_{x \neq y}|G_{xy}(z)|^2 < T_{xy}(z) \quad \text{uniformly in } x \neq y \in \mathbb{Z}_d^d \text{ and } z \in D, \] (4.35)
\[ |G_{xx}(z) - m(z)| < \Phi \quad \text{uniformly in } x \in \mathbb{Z}_d^d \text{ and } z \in D. \] (4.36)

Proof of Proposition 4.9. We define a sequence of $z_k = E + i \eta_k$ with decreasing imaginary parts $\eta_k := \max (W^{-k\lambda/3}, W^{-5+\delta_0}L^{5-d})$ for a sufficiently small constant $\lambda > 0$. First, Lemma 4.10 shows that the local law (4.25) holds for $z_0 = E + i$. Now, suppose (4.25) holds for $z_k = E + i \eta_k$, then Proposition 4.12 yields that $\|G_{xy}(z) - m(z)\|_w < W^\lambda$ uniformly in $z = E + i \eta$ with $\eta_{k+1} \leq \eta \leq \eta_k$. Therefore, the condition (4.32) holds and we get (4.33) by Proposition 4.13, which, together with Lemma 4.14, implies that the local law (4.25) holds at $z_{k+1}$. By induction in $k$, the above arguments show that the local law (4.25) holds for any fixed $z = E + i \eta$ with $\eta \in [W^{-5+\delta_0}L^{5-d}, 1]$ (or $\eta \in [W^2/L^{2}, 1]$ if we only have (4.24)). The uniformity in $z$ follows from a standard $\varepsilon$-net and perturbation argument, see e.g., the proof of Theorem 2.16 in [73]. We omit the details. \hfill \Box

4.4. Proof of Proposition 4.13. Proposition 4.13 is a simple consequence of the following lemma.

Lemma 4.15. Suppose the assumptions of Proposition 4.13 hold. Assume that
\[ T_{xy} < B_{xy} + \tilde{\Phi}^2, \quad \forall \ x, y \in \mathbb{Z}_d^d, \] (4.37)
for a deterministic parameter $\tilde{\Phi}$ satisfying $0 \leq \tilde{\Phi} \leq W^{-\varepsilon}$ for some constant $\varepsilon > 0$. Then, for any fixed $p \in \mathbb{N}$, we have that
\[ \mathbb{E}T_{xy}(z)^p < \left( B_{xy} + W^{-c}\tilde{\Phi}^2 + \frac{1}{N\eta} \right)^p \] (4.38)
for some constant $c > 0$ depending only on $d$, $\delta_0$, $n$ and $c_0$ in (4.24) or (4.26).

Proof of Proposition 4.13. Starting with $T_{xy} < B_{xy} + \tilde{\Phi}^2$, where $\tilde{\Phi}_0 := W^{-d_0+\lambda}$ due to (4.32), we combine (4.38) with Markov’s inequality to obtain that
\[ T_{xy}(z) < B_{xy} + W^{-c}\tilde{\Phi}_0^2 + \frac{1}{N\eta}. \]
Hence, (4.37) holds with a smaller parameter $\tilde{\Phi} = \tilde{\Phi}_1 := W^{-c/2}\tilde{\Phi}_0 + (N\eta)^{-1/2}$. Repeating this argument for $[D/c]$ many times, we obtain that
\[ T_{xy}(z) < B_{xy} + \frac{1}{N\eta} + W^{-D}. \]
This concludes (4.33) as long as $D$ is large enough. \hfill \Box

It remains to prove Lemma 4.15. We first simplify the atomic graphs using their molecular structures. We will see (in Lemma 4.17) that it suffices to consider the following class of graphs, called auxiliary graphs, which are actually obtained as quotient graphs with molecules reduced to vertices.

Definition 4.16. An auxiliary graph consists of vertices and double-line or solid edges between them, where
- every vertex represents a molecule,
• every double-line edge between vertices \( x \) and \( y \) represents a \( B_{xy} \) factor and has scaling order 2,
• every solid edge represents a non-negative \( \lambda \)-bounded edge and has scaling order 1.

**Lemma 4.17.** Let \( \mathcal{G} \) be a normal graph. There exists an auxiliary graph \( \tilde{\mathcal{G}} \), whose vertices are representative atoms of the molecules in \( \mathcal{G} \), such that if two molecules in \( \tilde{\mathcal{G}} \) are connected by a diffusive (resp. blue solid) edge, then their representative atoms are also connected by a double-line (resp. solid) edge in \( \tilde{\mathcal{G}} \). Furthermore,

\[
\mathcal{G}_{\text{abs}} \prec W^{-[\text{ord}(\tilde{\mathcal{G}})]} d_{\text{abs}} \tilde{\mathcal{G}},
\]

where \( \mathcal{G}_{\text{abs}} \) is obtained by replacing each component (including edges, weights and coefficients) in \( \mathcal{G} \) with its absolute value and ignoring all the \( P \) or \( Q \) labels (if any). Finally, if \( G - mI \) is \( \lambda \)-bounded (resp. \( (\Phi, \lambda) \)-bounded), then so are the solid edges in \( \mathcal{G} \).

**Proof.** We first choose a representative atom for each molecule in \( \mathcal{G} \). Then, the key to the proof is to show that all diffusive and \( \lambda \)-bounded (or \( (\Phi, \lambda) \)-bounded) edges between two molecules can be bounded by double-line and \( \lambda \)-bounded (or \( (\Phi, \lambda) \)-bounded) solid edges between their representative atoms. After that, the local structures inside molecules will provide some \( W \) factors that contribute to \( W^{-[\text{ord}(\tilde{\mathcal{G}})]} d_{\text{abs}} \tilde{\mathcal{G}} \) in (4.39). The detailed reduction procedure has been described in the proof of Lemma 6.10 of [73], so we omit the details. \( \square \)

A key ingredient for the proof of Lemma 4.15 is the following lemma.

**Lemma 4.18.** Suppose \( d \geq 7 \) and \( \eta \geq W^{-5+\delta_0} L^{5-d} \). Given any two matrices \( A^{(1)} \) and \( A^{(2)} \) with non-negative entries, we have that

\[
\sum_{x_i} A^{(1)}_{x_i,\alpha} \prod_{j=1}^{k} B_{x_j,y_j} \prec W^{-d_\sigma} \Gamma(y_1, \cdots, y_k) \|A^{(1)}\|_w, \tag{4.40}
\]

where \( \Gamma(y_1, \cdots, y_k) \) is a sum of \( k \) different products of \((k-1)\) double-line edges:

\[
\Gamma(y_1, \cdots, y_k) := \sum_{i=1}^{k} \prod_{j \neq i} B_{y_i,y_j}. \tag{4.41}
\]

In addition, if \( A^{(1)} \) and \( A^{(2)} \) are \( \lambda \)-bounded (or \( (\Phi, \lambda) \)-bounded), then

\[
A_{\alpha,\beta} := \frac{W^{d_\eta - \lambda}}{\Gamma(y_1, \cdots, y_k)} \sum_{x_i} A^{(1)}_{x_i,\alpha} A^{(2)}_{x_i,\beta} \prod_{j=1}^{k} B_{x_j,y_j} \text{ is also } \lambda \text{ (or } (\Phi, \lambda) \text{-bounded).} \tag{4.42}
\]

In a doubly connected auxiliary graph, we choose a blue spanning tree of the blue net and sum over all internal atoms from leaves to a chosen root (which is usually chosen as an external atom). Let \( x_i \) be a leaf of the blue tree, \( A^{(1)}_{x_i,\alpha} \) be the blue edge in the blue tree, and \( B_{x_i,y_j} \) be edges in the black net. Then, the estimate (4.40) shows that the summation over an internal atom \( x_i \) can be bounded by a sum of doubly connected graphs obtained by removing the atom \( x \) and the edges attached to it, and then adding edges between its neighboring atoms. Furthermore, if \( x_i \) connects to an external atom, say \( \beta \), then the estimate (4.42) shows that the summation over an internal atom \( x_i \) can be bounded by a sum of doubly connected graphs, where the external atom \( \beta \) still connects to an internal atom \( \alpha \). After summing over all internal atoms, we can bound an auxiliary graph by a sum of graphs consisting of external atoms only. Hence, together with Lemma 4.17, Lemma 4.18 enables us to bound the graphs in the \( T \)-expansion, which leads to the proof Lemma 4.15.

**Proof of Lemma 4.15.** This proof is similar to that for Lemma 8.1 in [74]. Using the \( n \)-th order \( T \)-expansion (2.25) with \( b_1 = b_2 = b \), we write that

\[
\mathbb{E} T^p_{ab} = \mathbb{E} T^{p-1}_{ab} \left\{ m \Theta_{ab}^{(n)} \mathcal{G}_{bb} + \frac{|m|^2}{N \eta} \text{Im} G_{bb} \right. \\
+ \left. \sum_x \Theta_{ax}^{(n)} \left[ p_{x,bb}^{(n)} + A_{x,bb}^{(n)} + \mathcal{W}_{x,bb}^{(n)} + (\mathcal{E}rr_{n,D})_{x,bb} \right] \right\}. \tag{4.43}
\]
With Lemma 4.17 and Lemma 4.18 as inputs, we can show that
\[
\sum_x \Theta^{(n)}_{ax} \mathcal{R}^{(n)}_{x,bb} \prec W^{-d_n/2 + \lambda} / \sqrt{N},
\sum_x \Theta^{(n)}_{ax} A^{(\geq n)}_{x,bb} \prec W^{(n-1)(-d_n + \lambda)} L^2 / W^2 (B_{ab} + \bar{\varphi}^2),
\sum_x \Theta^{(n)}_{ax} \mathcal{W}^{(n)}_{x,bb} \prec \frac{1}{N \bar{\varphi}^2} L^2 (B_{ab} + \bar{\varphi}^2),
\sum_x \Theta^{(n)}_{ax} \mathcal{E}^{(n)}_{x,bb} \prec W^{d_n} L^2 (B_{ab} + \bar{\varphi}^2),
\] 
More precisely, the estimates in Lemma 4.18 will replace the role of estimates (8.10)–(8.12) in [74]. With these estimates, using exactly the same arguments as in [74, Section 8.2], we can prove all the bounds in (4.44) except the third one involving \( \mathcal{W}^{(n)}_{x} \). For any graph in \( \mathcal{W}^{(n)} \), removing the unique free edge in it still gives a doubly connected graph (see property 4 of Definition A.6 below), so it can be written as \((N \eta)^{-1} / G_{x,b}\) for a doubly connected graph \( G \). We can bound \( G_{x,b} \) using the same argument as the one for \( A^{(\geq n)} \) in Claim 8.6 of [74] and get that
\[
\sum_x \Theta^{(n)}_{ax} G_{x,b} \prec \frac{L^2}{W^2} (B_{ab} + \bar{\varphi}^2).
\]
This yields the third bound in (4.44). Substituting (4.44) into (4.43), the desired result (4.38) with \( c = \min(c_0 - (n-1) \lambda, d_n/2 - \lambda/2) \) follows from an application of Hölder’s inequality and Young’s inequality. \( \square \)

Finally, the proof of Lemma 4.18 involves a basic calculation using Definition 4.11.

**Proof of Lemma 4.18.** Define the subsets
\[
\mathcal{I}_l = \left\{ x \in \mathbb{Z}^d_L : (x - y_l) \leq \min_{j \neq l} (x - y_j) \right\},
\] 
i.e., \( B_{x,y_l} = \min_{j \neq l} B_{x,y_j} \) for \( x \in \mathcal{I}_l \). Then, we have
\[
\sum_{x_i \in \mathcal{I}_l} A^{(1)}_{x_i} \prod_{j=1}^k B_{x_i,y_j} \leq \sum_{x_i \in \mathcal{I}_l} A^{(1)}_{x_i} B_{x_i,y_i} \prod_{j \neq l} B_{y_i,y_j} \leq \sum_{x_i \in \mathbb{Z}^d_L} A^{(1)}_{x_i} B_{x_i,y_i} \prod_{j \neq l} B_{y_i,y_j}.
\] 
Without loss of generality, assume that \( A^{(1)} \) is 0-bounded, i.e., \( \| A^{(1)} \|_w < 1 \). With \( K_n := 2^n W \), we have that
\[
\sum_{x_i \in \mathbb{Z}^d_L} |A^{(1)}_{x_i}| |B_{x_i,y_i}| \leq \sum_{1 \leq n \leq \log_2 K_n} \sum_{K_{n-1} \leq (x_i - y_i) \leq K_n} |A^{(1)}_{x_i}| |B_{x_i,y_i}|
\] 
\[
\leq \sum_{1 \leq n \leq \log_2 K_n} \max_{K_{n-1} \leq (x_i - y_i) \leq K_n} B_{x_i,y_i} \cdot \sum_{K_{n-1} \leq (x_i - y_i) \leq K_n} |A^{(1)}_{x_i}| 
\] 
\[
\leq \max_{1 \leq n \leq \log_2 K_n} \left[ \frac{1}{W^7 K_n^{d-\frac{d}{2}}} + \frac{K_n^{10-d} L^2}{W^{10} N \eta W^2} \right] \left( \frac{1}{W^7 K_n^{d-\frac{d}{2}}} + \frac{K_n^5}{W^3 N \eta W^2} \right)^{\frac{1}{2}},
\] 
which implies that
\[
\sum_{x_i \in \mathbb{Z}^d_L} |A^{(1)}_{x_i}| |B_{x_i,y_i}| \prec \begin{cases} W^{- \delta_0/2}, & \text{if } d \geq 7, \eta \geq W^{-5+\delta_0} L^{5-d} \\ W^{-d/2}, & \text{if } d \geq 7, \eta \geq W^{2} L^{2} \end{cases}.
\] 
Combined with (4.46), it yields (4.40) (and also explains why we choose the \( d_0 \) in (3.2)). The result (4.42) is a simple consequence of (4.40), and we omit the details of the proof. \( \square \)
4.5. Construction of the $T$-expansion. In this section, we construct a sequence of $T$-expansions satisfying Definition 2.15 order by order. First, it is not hard to see that the $n$-th order $T$-expansion can be obtained by solving the $n$-th order $T$-equation defined as follows.

**Definition 4.19** ($n$-th order $T$-equation). Fix any $n \in \mathbb{N}$ and a large constant $D > n$. For $a, b_1, b_2 \in \mathbb{Z}_L^d$, an $n$-th order $T$-equation of $T_{a,b_1,b_2}$ with $D$-th order error is an expression of the following form:

$$T_{a,b_1,b_2} = m \Theta^{\circ}_{ab_1} \overline{r}_{b_1,b_2} + \frac{|m|^2}{2N}\sum_{x} \left( \Theta_{x \alpha}^{\circ} \Sigma^{(n)}_{\alpha \beta} \right)_{x_1} \xi_{x_1,b_1,b_2} + \sum_{x} \Theta^{\circ}_{ax} \left[ R^{(n)}_{x,b_1,b_2} + A^{(\alpha > n)}_{x,b_1,b_2} + W^{(n)}_{x,b_1,b_2} + Q^{(n)}_{x,b_1,b_2} + \left( \mathcal{E}_{rr,n,D} \right)_{x,b_1,b_2} \right],$$

where $R^{(n)}, A^{(\alpha > n)}, W^{(n)}, Q^{(n)}, \mathcal{E}_{rr,n,D}$ are the graphs in Definition 2.15.

Second, given the $(n-1)$-th order $T$-expansion, following the expansion strategy described in [74], we can construct a $n$-th order $T$-equation.

**Proposition 4.20** (Construction of the $T$-equation). Given any $n \in \mathbb{N}$, suppose we have constructed an $(n-1)$-th order $T$-expansion satisfying Definition 2.15. Then, we can construct an $n$-th order $T$-equation satisfying Definition 4.19, where $\Sigma^{(n)}$ is a deterministic matrix such that $\Sigma^{(n)} = E_n + \Sigma^{(n-1)}$ with $E_n$ being a sum of doubly connected deterministic graphs satisfying Definition 2.12 (i).

Third, we can prove that the deterministic matrix $E_n$ constructed in Proposition 4.20 indeed is a self-energy, that is, it satisfies properties (2.17)–(2.19).

**Proposition 4.21.** The deterministic matrix $E_n$ constructed in the $n$-th order $T$-equation in Proposition 4.20 satisfies the properties (2.17)–(2.19) with $l = n/2$ (recall the convention (2.20)).

Combining the above results, we can prove Theorem 2.17 by induction.

**Proof of Theorem 2.17.** Suppose we have constructed the $n$-th order $T$-equation. Using property (2.17) for $\Sigma^{(n)} = \sum_{2l=4}^{n} E_{2l}$, we get that

$$\sum_{x} \left( I - \Theta^{\circ} \Sigma^{(n)} \right)_{x} = \sum_{x} \Theta^{\circ}_{x \alpha} \Sigma^{(n)}_{\alpha \beta} = \sum_{\alpha} \Theta^{\circ}_{x \alpha} \cdot \sum_{\beta} \Sigma^{(n)}_{\alpha \beta} = 0.$$  

Using (4.49) and (2.7), we can rewrite (4.48) as

$$\sum_{x} \left( I - \Theta^{\circ} \Sigma^{(n)} \right)_{x} T^{\circ}_{x,b_1,b_2} = m \Theta^{\circ}_{a b_1} \overline{r}_{b_1,b_2} + \sum_{x} \Theta^{\circ}_{ax} \left[ R^{(n)}_{x,b_1,b_2} + A^{(\alpha > n)}_{x,b_1,b_2} + W^{(n)}_{x,b_1,b_2} + Q^{(n)}_{x,b_1,b_2} + \left( \mathcal{E}_{rr,n,D} \right)_{x,b_1,b_2} \right].$$

Solving (4.50) and recalling the definition (2.21), we obtain that

$$T^{\circ}_{a,b_1,b_2} = m \Theta^{(n)}_{a b_1} \overline{r}_{b_1,b_2} + \sum_{x} \Theta^{(n)}_{ax} \left[ R^{(n)}_{x,b_1,b_2} + A^{(\alpha > n)}_{x,b_1,b_2} + W^{(n)}_{x,b_1,b_2} + Q^{(n)}_{x,b_1,b_2} + \left( \mathcal{E}_{rr,n,D} \right)_{x,b_1,b_2} \right].$$

Substituting it back to (2.7) gives the $n$-th order $T$-expansion (2.25). The above argument together with Propositions 4.20 and 4.21 shows that given the $(n-1)$-th order $T$-expansion, we can construct the $n$-th order $T$-expansion. By mathematical induction, we conclude Theorem 2.17.

The proof of Proposition 4.20 is similar to that for Theorem 3.7 in [74], and we will describe an outline of it in Section A without giving all the details. The proof of Proposition 4.21 will be presented in Section 5.

5. Proof of Proposition 4.21

The property (2.17) for $E_n$ is a simple consequence of the fact that $S(x - y) := s_{xy}$ is symmetric and translation invariant (see Lemma A.1 of [73]). The property (2.18) is a simple consequence of the following estimate on doubly connected deterministic graphs.
Lemma 5.1. Suppose \( d \geq 6 \). Let \( \mathcal{G} \) be a deterministic doubly connected normal graph without external atoms. Pick any two atoms of \( \mathcal{G} \) and fix their values as \( x, y \in \mathbb{Z}^d \). The resulting graph \( \mathcal{G}_{xy} \) satisfies that
\[
(\mathcal{G}_{ab})_{xy} \prec \left( \frac{L^2}{W^2} \right)^{k_{gh}} \left( \frac{1}{N\eta W^2} \right)^{k_{fr}} W^{-(n_{xy} - 4 - 2k_{tr})d/2} \left( B_{xy}^2 + 1_{k_{gh} + k_{fr} \geq 1} B_{xy}^{W^2 - d/28} \right),
\]
where \( n_{xy} := \text{ord}(\mathcal{G}_{xy}) \), \( k_{fr} \) is the number of free edges and \( k_{gh} \) is the number of ghost edges.

Proof. The \( k_{gh} = k_{fr} = 0 \) case of (5.1) was proved as Corollary 6.12 of [73], while the case with nonzero \( k_{gh} \) or \( k_{fr} \) follows from Lemma 9.5 of [74]. \( \square \)

Applying (5.1) with \( k_{gh} = k_{fr} = 0 \) gives (2.18), because \( \mathcal{E}_n \) constructed in Proposition 4.20 consists of doubly connected graphs without free or ghost edges. The estimate (5.1) with nonzero \( k_{gh} \) or \( k_{fr} \) will be used in the proof of Lemma 3.2 in Appendix B.

To conclude Proposition 4.21, we still need to prove the key “sum zero property” (2.19).

Proposition 5.2. Let \( \mathcal{E}_n \) be the deterministic matrix constructed in Proposition 4.20. Fix any \( d \geq 7 \),
\[
\left| \sum_x (\mathcal{E}_n)_{0x} \left( m, S, S^\pm, \Theta^{[n-1]} \right) \right| \prec \left( (\eta + (W/L)^{2d-6}) W^{-d} + W^{-2d - 2d^{-2d}} \right) W^{-(n-4)d/2},
\]
for \( z = E + \eta \) with \(|E| \leq 2 - \kappa \) and \( \eta \in [W^{-5 + \delta} L^{-d/2}, 1] \), where we abbreviate \( \Theta^{[n-1]} := (\Theta^{(1)}_{2 \leq r \leq n-1}). \)

To this end, we will compare \( \mathcal{E}_n \) with its infinite space limit defined below, which is known to satisfy the sum zero property (Proposition 5.4).

Definition 5.3 (Infinite space limits). We first define the infinite space limits of \( m, S, S^\pm, \) and \( \Theta^\circ \) by keeping \( W \) fixed and taking \( L \to \infty \) and \( \eta \to 0 \):
\[
m(E) := \lim_{\eta \to 0^+} m(E + \eta), \quad (S_\infty)_{00} := \lim_{L \to \infty} f_{W,L}(\alpha - \beta), \quad S_\infty^+ (E) := \frac{m^2(E) S_\infty}{1 - m^2(E) S_\infty}, \quad S_\infty^- (E) := S_\infty^+(E), \quad (\Theta_\infty)_{00} := \sum_{k \geq 1} S_\infty^k.
\]

Given a deterministic graph \( \mathcal{G} \equiv \mathcal{G} \left( (m(z), S(z), S^\pm(z), \Theta^{[n-1]}(z)) \right) \) with \( z = E + \eta \), we define
\[
\mathcal{G}^\infty \equiv \mathcal{G}^\infty \left( m(E), S_\infty, S_\infty^\pm(E), \Theta^{[n-1]}_\infty(E) \right),
\]
in the following way.

1. Replace the \( s_{00} \) edges in \( \mathcal{G} \) with \( (S_\infty)_{00} \).
2. Replace the \( S^\pm_{00} \) edges in \( \mathcal{G} \) with \( (S_\infty^\pm)_{00} \).
3. Replace the \( \Theta^\circ_{00} \) edges in \( \mathcal{G} \) with \( (\Theta_\infty)_{00} \). Replace the labeled diffusive edges \( (\Theta^{(1)}_{00})_{00} \) in \( \mathcal{G} \) with \( (\Theta^{(1)}_{\infty})_{00} \) (obtained by replacing the \( S, S^\pm \) and \( \Theta^\circ \) edges in \( \Theta^{(1)}_{00} \) by their infinite space limits).
4. For all \( m(z) \) in the coefficient (that is, \( m(z) \)'s that do not appear in \( S^\pm(z) \) and \( \Theta^\circ(z) \)), we replace them with \( m(E) \).
5. Finally, we let all internal atoms take values over \( \mathbb{Z}^d \) and denote the resulting graph by \( \mathcal{G}^\infty \).

Note that \( \mathcal{G}^\infty \) (if exists) depends only on \( E, W \) and \( \psi \) in Assumption 1.2, but does not depend on \( L \) and \( \eta \).

Proposition 5.4. For \( d \geq 8 \), we have
\[
\sum_x (\mathcal{E}_n^\infty)_{0x} \left( m(E), S_\infty(E), S_\infty^\pm(E), \Theta^{[n-1]}_\infty(E) \right) = O(W^{-D}),
\]
for any large constant \( D > 0 \). In addition, (5.5) also holds for \( d = 7 \) assuming that for some \( L \) satisfying
\[
W^{(n-3)d/2 + \varepsilon} \leq L^2 / W^2 \leq W^{(n-2)d/2 - \varepsilon},
\]
we have the \((n - 1)\)-th order \( T \)-expansion and the local law (4.25) holds when \( z = E + i W^{2 + \varepsilon} / L^2 \) for a sufficiently small constant \( \varepsilon > 0 \).

Proof. For \( d \geq 8 \), (5.5) was proved in Section 5.4 of [73]. The same proof works for \( d = 7 \) provided with the assumptions in the statement. \( \square \)

With Proposition 5.4, Proposition 5.2 is a simple consequence of the next two lemmas.
Lemma 5.5. For $d \geq 6$, let $G$ be a deterministic graph in $E_n$. For any $x \in \mathbb{Z}_L^d$ and $z = E + i \eta$, we have

$$
|G_{0x}(m(z), S, S^\pm(z), \Theta^{[n-1]}(z)) - G_{0x}(m(E), S_\infty, S_\infty^\pm(E), \Theta^{[n-1]}_\infty(E))| < (\eta W^{-d} + W^{-2L^2-d})B^{3}_{0x} W^{-\frac{n-6}{2}d}.
$$

Lemma 5.6. For $d \geq 6$, let $G$ be an arbitrary deterministic graph in $E_n$. We have

$$
|\sum_{x \in \mathbb{Z}^d} G_{0x}^\infty(m(E), S_\infty, S_\infty^\pm(E), \Theta^{[n-1]}_\infty(E)) - \sum_{x \in \mathbb{Z}_L^d} G_{0x}(m(E), S_\infty, S_\infty^\pm(E), \Theta^{[n-1]}_\infty(E))| < (W/L)^{2d-6} W^{-\frac{n-2}{2}d}.
$$

Proof of Proposition 5.2. Note that in the setting of Proposition 4.21, we have constructed an $(n-1)$-th order $T$-expansions, and part (i) of Proposition 4.9 then shows that the local law (4.25) holds for $z = E + i(W^{2+\varepsilon}/L^2)$ under the condition (5.6). Hence, (5.5) holds for all $d \geq 7$ by Proposition 5.4. In addition, from Lemma 5.5, we get that

$$
|\sum_{x \in \mathbb{Z}_L^d} G_{0x}(m, S, S^\pm, \Theta^{[n-1]}) - \sum_{x \in \mathbb{Z}_L^d} G_{0x}(m(E), S_\infty, S_\infty^\pm(E), \Theta^{[n-1]}_\infty(E))| < (\eta W^{-d} + W^{-2L^2-d})W^{-(n-4)d/2}.
$$

Together with Lemma 5.6, it implies that

$$
|\sum_{x \in \mathbb{Z}_L^d} (E_{n\infty}(x), m, S, S^\pm, \Theta^{[n-1]}) - \sum_{x \in \mathbb{Z}_L^d} (E_{n\infty}(x), m(E), S_\infty, S_\infty^\pm(E), \Theta^{[n-1]}_\infty(E))| < [\eta + (W/L)^{2d-6}] W^{-d} + W^{-2L^2-d}] W^{-(n-4)d/2}.
$$

Combining it with (5.5), we conclude the proof. \qed

The proof of Lemma 5.6 uses a stronger globally standard property of $G$ (see Definitions A.5 and A.6 below). By Lemma A.7 below, removing any diffusive edge in $G$ still gives a doubly connected graph.

Proof of Lemma 5.6. Note that the graphs $G^\infty$ and $G$ are different in the sense that the atoms in $G^\infty$ take values over $\mathbb{Z}^d$ while the atoms in $G$ take values over $\mathbb{Z}_L^d$.

Suppose $0$ connects to an atom $y$ through a diffusive edge $e = (0, y)$ in $G$. Then, removing this edge still gives a doubly connected graph by Lemma A.7. Now, let $G_{0x}^\infty$, be the graph obtained by fixing $y$ in $G_{0x}^\infty$ as an external atom. Note that $x \in G_{0x}^\infty$ is a graph obtained by changing $x$ in $G_{0x}^\infty$ to an internal atom. Then, applying Lemma 5.1 to the doubly connected graph with $e$ removed from $G_{0x}^\infty$ gives that

$$
\sum_{|y| \geq L/\log L} |G_{0y}^\infty| < W^{-(n-6)d/2} \sum_{|y| \geq L/\log L} B_{0y}^3 \times (W/L)^{2d-6} W^{-(n-2)d/2}.
$$

This means that if we do not sum over $|y| \geq L/\log L$ in $\sum_x G_{0x}^\infty$, it gives an error of order

$$
O_{\infty}\left((W/L)^{2d-6} W^{-(n-2)d/2}\right).
$$

If $0$ connects to an atom $y$ through a waved edge, then by (1.10) and (2.15), we get that summing over $|y| \geq L/\log L$ gives a negligible error $O(W^{-D})$.

The above argument can be extended to any pair of neighbors. For example, suppose $y$ connects to an atom $w$. Then, for $|y| \leq L/\log L$, if we do not sum over $|w| \geq 2L/\log L$ in $\sum_x G_{0x}^\infty$, it gives an error of order (5.8). Continuing this argument, since the graph $G$ is connected and there are $O(1)$ many atoms, we obtain the desired result. \qed

The rest of this section is devoted to the proof of Lemma 5.5.

5.1. Proof of Lemma 5.5. Note that when $\eta$ is of order 1, Lemma 5.5 follows directly from Lemma 5.1. Hence, without loss of generality, we assume $\eta \ll 1$ in the following proof.

Lemma 5.7. For $x \in \mathbb{Z}_L^d$ and any large constant $D > 0$, we have that

$$
|G_{0x}(m(E), S_\infty, S_\infty^\pm(E), \Theta^{[n-1]}_\infty(E)) - G_{0x}(m(z), S, S^\pm(z), \Theta^{[n-1]}_\infty(E))| < \eta B^{3}_{0x} W^{-(n-4)d/2} + L^{-D}.
$$
Proof. It is an immediate consequence of (1.10), (2.15), the equation (7.8) of [73] which shows that
\[ |(S_{∞}^L)_{0x}(E) - S_{0x}^+ (z)| \sim \eta W_1 \frac{W_{1+}}{\tau} + L^{-D}, \]
and the simple estimate \(|m(z) - m(E)| = O(\eta)|.

Lemma 5.8. Under the setting of Proposition 5.2, let \(G\) be an arbitrary deterministic graph in \(E_{l+1}\) for a fixed \(2 \leq l \leq n - 1\). If (5.7) holds for every graph in \(E_k\) with \(k \leq l\), then for every \(x \in \mathbb{Z}_d^l\),
\[ |G_{0x}(m(z), S, S^k(z), \Theta^0_{l}(E)) - G_{0x}(m(z), S, S^k(z), \Theta^0_{l}(z))| \sim (\eta W - d + W_2 L^{-d}) B^2_0 \eta W^{-(l+1)-6}d/2. \]

Proof of Lemma 5.5. First, we trivially have \(E = \mathbb{R}\) \(\subseteq\) \(\mathbb{Z}\). If \(\Phi\) \(\subseteq\) \(\Theta\) \(\subseteq\) \(\mathbb{R}\), under the setting of Proposition 5.2, let \(\Phi\) be a graph obtained by replacing the edge \(e\) in \(\Theta\) by \(\Theta\) \(\subseteq\) \(\mathbb{R}\). Then, we have that
\[ (G_{abs})_{0x} \sim (\eta W - d + W_2 L^{-d}) B^2_0 \eta W^{-(l+1)-6}d/2. \]

Lemma 5.9. If (5.7) holds for every graph in \(E_k\) with \(k \leq l\), then for any \(l' \leq l\),
\[ \left| (\Theta^{(f)}_{l'})_{xy} - (\Theta^{(f)}_{l', \infty})_{xy} \right| \sim W^{-(l'-2)d/2} \left( W^{-2} L^{-d} + \frac{\eta}{W^4 (x - y)^{d-4}} \right), \]
(5.9)

With this lemma, Lemma 5.8 is a direct consequence of the following result, which controls the error in replacing every (labeled) diffusive edge by its infinite space limit.

Lemma 5.10. Under the setting of Lemma 5.8, let \(e = (a, b)\) be an arbitrary diffusive \(\Theta_{k'}^{(k)}\) edge in \(G\) with \(k' \leq k \leq l\). Let \(G^c\) be a graph obtained by replacing the edge \(e\) in \(G\) by \((\Theta_{k'}^{(k)} - \Theta_{k'}^{(k)}_{ab})_{0x}\). Then, we have that
\[ (G_{abs})_{0x} \sim (\eta W - d + W_2 L^{-d}) B^2_0 \eta W^{-(l+1)-6}d/2. \]
(5.10)

Proof. Let \(\hat{G}\) be the subgraph of \(G\) with \(e\) removed and \(\hat{G}(a, b)\) be the graph obtained by fixing \(a, b\) in \(\hat{G}\) as external atoms. It follows from Lemma A.7 that \(\hat{G}\) is still doubly connected. Therefore,
\[ (G^c_{abs})_{0x} \leq \sum_{a, b} \left| (\Theta_{k'}^{(k)} - \Theta_{k'}^{(k)}_{ab})_{0x} \right| (G^c_{abs})_{0x} \sim (W - 2 L^{-d} + \eta W - d) W^{-(l'-2)d/2} (G_{abs})_{0x} \sim (W - 2 L^{-d} + \eta W - d) B^2_0 \eta W^{-(l+1)-6}d/2, \]
where we used Lemma 5.9 in the second step and Lemma 5.1 in the third step.

It remains to prove Lemma 5.9. Let \(\Theta(Z^d_L, S, |m|)\) be the Green’s function for the random walk on \(Z^d_L\) that has transition matrix \(S\) and is killed at each step with probability 1 \(-|m|\), i.e.,
\[ \Theta(Z^d_L, S, |m|)_{xy} := \sum_{k=1}^{\infty} |m|^{2k}(S^k)_{xy} = \left( \frac{|m|^2 S}{1 - |m|^2 S} \right)_{xy}, \quad x, y \in \mathbb{Z}^d. \]
Similarly, we can define \(\Theta(Z^d_L, S, |m|)\). Then, we will prove Lemma 5.9 using the following bound.

Lemma 5.11. For \(x, y \in \mathbb{Z}^d_L\) and any constants \(\tau, D > 0\), we have that
\[ \left| (\Theta(Z^d_L, S, |m|)_{xy} - \Theta(Z^d, S, |m|)_{xy}) \right| \sim \frac{(x - y)^2}{W^2 L^{-d}} \frac{1}{\eta W^2 L^{-d}} \eta W + L^{-D}. \]

Proof. Noticing that
\[ \Theta(Z^d_L, S, |m|)_{xy} = \sum_{k \in \mathbb{Z}^d} \Theta(Z^d, S, |m|)_{x+y+kL}, \]
we have
\[ \Theta(Z^d_L, S, |m|)_{xy} - \Theta(Z^d, S, |m|)_{xy} = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \Theta(Z^d, S, |m|)_{x+y+kL}. \]
(5.12)

Hence, it suffices to prove that for any \(x \in \mathbb{Z}^d_L\),
\[ \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \left| \Theta(Z^d, S, |m|)_{0, x+kL} + \Theta(Z^d, S, |m|)_{0, -x+kL} - 2 \Theta(Z^d, S, |m|)_{0, kL} \right| \sim \frac{(x)^2}{W^2 L^{-d}} \frac{1}{\eta W^2 L^{-d}} \eta W + L^{-D}. \]
This is a direct consequence of the transition probability estimate for the random walk on $\mathbb{Z}^d$ with transition matrix $S$. For example, it is proved on [73, page 76] that

$$\left| \Theta(\mathbb{Z}^d, S, |m|)_{0,x+kL} + \Theta(\mathbb{Z}^d, S, |m|)_{0,-x+kL} - 2\Theta(\mathbb{Z}^d, S, |m|)_{0,kL} \right|$$

$$< \left( W^{-2}(kL)^{-d} (x)^2 + L^{-D} \right) 1_{|kL| \leq \eta^{-1/2} W^2} + (|k|L)^{-D}.$$ 

Summing this over $k \in \mathbb{Z}^d \setminus \{0\}$ yields the desired result since $\tau$ is arbitrarily small. \hfill \Box

**Proof of Lemma 5.9.** We first prove (5.9) for $\Theta^{(l)}_2 = \Theta^\circ$. By definition, we have

$$|\Theta^\circ_{xy} - (\Theta^\circ, xy)| \leq \left| \Theta(\mathbb{Z}^d, S, |m|)_{xy} - \Theta(Z^d, S, |m|)_{xy} \right| - \left( \Theta(\mathbb{Z}^d_L, S, |m|)_{00} - \Theta(\mathbb{Z}^d, S, |m|)_{00} \right)$$

$$+ \left| \Theta(\mathbb{Z}^d_L, S, |m|)_{00} - \Theta(\mathbb{Z}^d, S, |m|)_{00} - \frac{1}{L^d} \sum_{x \in \mathbb{Z}^d_L} \Theta(\mathbb{Z}^d_L, S, |m|)_{0x} \right|$$

$$+ \left| \Theta(Z^d, S, |m|)_{xy} - \Theta(Z^d, S, \infty, |m|)_{xy} \right| + \left| \Theta(Z^d, S, |m|)_{xy} - \Theta(Z^d, S, \infty, 1)_{xy} \right|.$$ 

We bound the first term using Lemma 5.11. The second term is bounded by

$$\frac{1}{L^d} \sum_{x \in \mathbb{Z}^d_L} \left| \sum_{k \neq 0} \Theta(\mathbb{Z}^d, S, |m|)_{0,kL} - \sum_{k \neq 0} \Theta(\mathbb{Z}^d, S, |m|)_{0,x+kL} \right| + \frac{1}{L^d} \sum_{x \in \mathbb{Z}^d_L} \Theta(\mathbb{Z}^d, S, |m|)_{0x} < W^{-2}L^{2-d},$$

where we used the facts that the summands in the first term are given by

$$\left| \Theta(Z^d, S, |m|)_{0x} - \Theta(Z^d, S, |m|)_{0x} - \Theta(Z^d, S, |m|)_{00} \right| < W^{-2}L^{2-d}$$

due to (5.12) and Lemma 5.11, while the summands in the second term are bounded by $\Theta(Z^d, S, |m|)_{0x} < W^{-2}(x)^{2-d}$, a standard result for the random walk in $\mathbb{Z}^d$. Finally, we have

$$|\Theta(Z^d, S, |m|)_{xy} - \Theta(Z^d, S, \infty, |m|)_{xy}| \leq L^{-D},$$

$$|\Theta(Z^d, S, \infty, |m|)_{xy} - \Theta(Z^d, S, \infty, 1)_{xy}| < \frac{\eta}{W^d (x-y)^d} + (x-y)^{-D},$$

where the first bound is due to (1.10) and the second bound was proved as equation (7.9) in [73]. Combining the above estimates yields (5.9) for $l' = 2$.

For the general case, we note that by (2.22), $\Theta^{(l)}_l$ is a sum of graphs of the form $\Theta^\circ E_{k_1} \Theta^\circ E_{k_2} \Theta^\circ \cdots \Theta^\circ E_{k_l} \Theta^\circ$ with $k_i \leq l' \leq l$. Hence, using the bound (5.9) for $\Theta^\circ - \Theta^\circ_\infty$ and the fact that (5.7) holds for the graphs in $E_{k_i}$, $k_i \leq l$, we can replace $\Theta^\circ$ and $E_{k_i}$ by $\Theta^\circ_\infty$ and $E_{k_i}$ one by one with a good control of the resulting errors. The relevant proof is similar to that below equation (7.22) of [73], so we omit the details. \hfill \Box

6. **Proof of Lemma 3.4 and Lemma 3.6**

We define the **generalized doubly connected property** of graphs with external molecules, which generalizes the doubly connected property in Definition 2.11.

**Definition 6.1** (Generalized doubly connected property). A graph $G$ with external molecules is said to be generalized doubly connected if its molecular graph satisfies the following property. There exist a collection, say $B_{\text{black}}$, of diffusive (or pseudo-diffusive) edges, and another collection, say $B_{\text{blue}}$, of blue solid, diffusive (or pseudo-diffusive), free or ghost edges such that: (a) $B_{\text{black}} \cap B_{\text{blue}} = \emptyset$, (b) every internal molecule connects to external molecules through two disjoint paths: a path of edges in $B_{\text{black}}$ and a path of edges in $B_{\text{blue}}$. Simply speaking, a graph is generalized doubly connected if merging all its external molecules into a single internal molecule gives a doubly connected graph in the sense of Definition 2.11.

We will show that expanding a graph without internal molecules in a proper way always generates generalized doubly connected graphs. Moreover, in these graphs, at most half of the external molecules are used in the generalized doubly connected property in the sense of property (b) of the following lemma.

**Lemma 6.2.** Under the setting of Lemma 3.4, the deterministic graphs $G^{(x)}_x$ in (3.6) satisfy the following properties:

(a) $G^{(x)}_x$ are generalized doubly connected normal graphs without silent edges.
Consider all external molecules that are neighbors of internal molecules on the molecular graph, say \( M_1, \ldots, M_n \). We can find at least \( r \geq \lceil n/2 \rceil \) of them, denoted by \( \{ M_{k_i} \}_{i=1}^r \), that are simultaneously connected with redundant diffusive or free edges in the following sense: (i) every \( M_{k_i}, 1 \leq i \leq r \), connects to an internal molecule through a diffusive or free edge \( e_{k_i} \), and (ii) after removing all these \( r \) edges \( e_{k_i}, 1 \leq i \leq r \), the resulting graph is still generalized doubly connected. (We will call these molecules \( M_{k_i} \) as special external molecules and these redundant diffusive or free edges as special redundant edges.)

(c) If \( x_i \) and \( x_j \) are connected in \( G_x \), then they are also connected in \( G^{(\mu)}_x \) through non-ghost edges.

(d) Every \( G^{(\mu)}_x \) satisfies \( \text{size}(G^{(\mu)}_x) \leq W^{-2d_\eta(p-1)} \) under the definition (3.1), where \( t \) denotes the number of connected components in \( G^{(\mu)}_x \).

The proof of Lemma 6.2 will be given in Section C. To help the reader to understand the statement of Lemma 6.2, we provide the following example, which describes a typical class of deterministic graphs that will naturally appear in the expansion of \( G_x \). The reader can also use this example to help to understand the proofs of Lemma 3.4 and Lemma 3.6 below.

This figure shows a molecular graph, where vertices \( x_i \) denote external molecules and vertices \( a_i \) denote internal molecules. The black double-line edges represent diffusive edges in the black net, and the blue solid lines represent diffusive or free edges in the blue net. Notice that among the two blue edges connected with an internal molecule \( a_i, 1 \leq i \leq s-1 \), one of them is redundant for the generalized doubly connected property, and \( a_i \) is connected with two redundant blue edges. Hence, we can choose \( x_1, x_3, \ldots, x_{2s+1} \) as special external molecules among the \( 2s + 2 \) external molecules in the above graph.

**6.1. Proof of Lemma 3.4.** In this subsection, we complete the proof of Lemma 3.4 using Lemma 6.2. It remains to prove (3.7) and the properties (a)-(c). Given a graph \( G^{(\mu)}_x \) satisfying Lemma 6.2, similar to Lemma 4.17, we first remove all local structures inside its molecules and bound it by an auxiliary graph consisting of external atoms \( x_1, \ldots, x_p \), internal atoms that are representatives of internal molecules, and edges between them. More precisely, we reduce \( G^{(\mu)}_x \) to an auxiliary graph together with some \( W \) factors as follows: First, we pick a representative atom for each internal and external molecule (note that two external atoms can be in the same molecule if they are connected through a path of waved edges). Second, in light of (1.10), (2.11), (2.15) and (2.24), we bound every waved edge inside a molecule by a pseudo-waved edge (up to a negligible error \( W^{-D} \) connected with the representative atom of the molecule and bound every diffusive edge by a pseudo-diffusive edge (with some \( W \) factors if the diffusive edge is labelled and of scaling order \( > 2 \)). Third, we bound every pseudo-diffusive edge between two molecules with representatives, say \( x \) and \( y \), by \( W^{(d-2)r} B_{xy} \). Finally, we bound the subgraph inside a molecule as follows: we keep one pseudo-waved edge between two neighboring external atoms; we bound every other pseudo-waved or pseudo-diffusive edge inside the molecule by \( W^{-d} \); we bound every free edge inside the molecule by \( (N\eta)^{-1} \); we bound every summation over a non-representative internal atom by \( W^d \).

Denote the auxiliary graph thus obtained by \( \tilde{G}_x \). It also satisfies the properties (a)-(c) of Lemma 6.2 and that

\[
|G^{(\mu)}_x| \leq W^C \tau c(W, L) \cdot \tilde{G}_x + W^{-D},
\]

where \( C > 0 \) is a constant that does not depend on \( \tau \) and \( c(W, L) > 0 \) is a \( (W, L) \)-dependent coefficient so that

\[
c(W, L) \text{size} \left( \tilde{G}_x \right) \leq \text{size} \left( G^{(\mu)}_x \right).
\]

We still need to bound the summations over the internal atoms in \( \tilde{G}_x \). With the generalized doubly connected property, we can choose a collection of blue trees such that every internal atom connects to an external atom through a unique path on a blue tree. These blue trees are disjoint, and each of them contains an external
atom as the root. Then, we will sum over all internal atoms from leaves of the blue trees to the roots. In bounding every summation, we will use the following two estimates with $k \geq 1$ and $r \geq 0$:

$$
\sum_{x} \prod_{j=1}^{k} B_{xy_{j}} \cdot \prod_{s=1}^{r} B_{xz_{s}} \cdot B_{xa} \leq \sum_{t=1}^{k} \prod_{i,j \neq t} B_{y_{j}y_{i}} \cdot \left( \tilde{B}_{y_{a}a} \prod_{s=1}^{r} \tilde{B}_{z_{a}z_{s}} + \tilde{B}_{y_{i}a} \prod_{s=1}^{r} \tilde{B}_{z_{i}z_{s}} + \sum_{t=1}^{r} \tilde{B}_{y_{i}z_{t}} \tilde{B}_{z_{i}z_{t}} \prod_{s, s \neq t} \tilde{B}_{z_{s}z_{t}} \right),
$$

(6.3)

$$
\sum_{x} \prod_{j=1}^{k} B_{xy_{j}} \cdot \prod_{s=1}^{r} \tilde{B}_{xz_{s}} \cdot \frac{W^{2}}{L^{2}} \leq \sum_{t=1}^{k} \prod_{i,j \neq t} B_{y_{j}y_{i}} \cdot \left( \prod_{s=1}^{r} \tilde{B}_{z_{i}z_{s}} + \sum_{t=1}^{r} \tilde{B}_{y_{i}z_{t}} \prod_{s, s \neq t} \tilde{B}_{z_{s}z_{t}} \right),
$$

(6.4)

The left-hand side of (6.3) is a star graph consisting of $k$ pseudo-diffusive edges (where at least one of them is in the black tree), a pseudo-diffusive edge in the blue tree, and $r$ silent pseudo-diffusive edges connected with $x_i$, while every graph on the right-hand side is a connected graph consisting of $(k - 1)$ pseudo-diffusive edges and $(r + 1)$ silent pseudo-diffusive edges. The estimate (6.4) can be applied to the case where the blue edge is a ghost or free edge. Of course, there may be some free edges connected with $x$, in which case we can still use (6.3) and (6.4) by multiplying some $(N \eta)^{-1}$ factors with them.

Using (6.3) or (6.4), we can bound a summation over an internal atom by a sum of new graphs. Some edges in the new graphs may be self-loops on atoms, which are picked out as $W$-dependent factors. Moreover, when we lose a ghost edge in the summation, we pick out the corresponding $L^2/W^2$ factor in its coefficient. It is easy to see that every new graph still satisfies the properties (a) and (c) of Lemma 6.2, and its size when we lose a ghost edge in the summation, we pick out the corresponding $W^2/L^2$ factors with them.

Suppose we sum over an internal atom $x$ as follows:

![Diagram](https://via.placeholder.com/150)

Here, $(a_1, x), \ldots, (a_s, x)$ indicate pseudo-diffusive or free edges, with one of them being the leaf of a blue tree, $(y_1, x), \ldots, (y_k, x)$ are pseudo-diffusive edges in the black net, green edges $(z_1, x), \ldots, (z_r, x)$ indicate silent pseudo-diffusive or silent free edges, and edges $(x_1, x), \ldots, (x_j, a)$ indicate special redundant edges connected with external atoms. When $(x_i, x)$ is a redundant free edge, we can treat it as a free edge between $x_i$ and another external atom. Hence, without loss of generality, we assume that $(x_1, x), \ldots, (x_j, a)$ are all pseudo-diffusive edges. Then, we apply (6.3) or (6.4) to bound the summation over $x$. In every resulting graph on the right-hand side, one of the following cases happens for a special redundant edge $(x_i, x)$.

- If $(x_i, x)$ becomes an edge connecting $x_i$ to another internal atom, it is still special redundant.
- If $(x_i, x)$ becomes an edge connecting $x_i$ to another external atom, it gives a non-silent pseudo-diffusive edge between external atoms and associated with $x_i$.
- Suppose $(x_i, x)$ is lost after the summation and there is no black pseudo-diffusive edge between $x_i$ and $x$ before the summation. Then, we apply (6.3) or (6.4) with the edge in the blue tree playing the role of $B_{x_0}$ in (6.3) or the factor $W^2/L^2$ in (6.4). In the new graph where $(x_i, x)$ is lost, $y_1, \ldots, y_k$ are still connected to each other through pseudo-diffusive edges and they are connected to $x_i$ through a black pseudo-diffusive edge, which is redundant in the black net.
- Suppose $(x_i, x)$ is lost after the summation and there is at least one black pseudo-diffusive edge between $x_i$ and $x$ before the summation. Then, we apply (6.3) with an $(x_i, x)$ edge playing the role of $B_{x_0}$ in (6.3). If the edge, say $e$, in the blue tree before the summation is a pseudo-diffusive or free edge, then in the new graph where $(x_i, x)$ is lost, the edge $e$ becomes a redundant non-silent blue edge connected with $x_i$. If the edge $e$ is a ghost edge, then in the new graph where $(x_i, x)$ is lost, we gain an extra $W^2/L^2$ factor in the coefficient. In final simple auxiliary graphs, this $W^2/L^2$ factor will turn a silent edge connected with $x_i$ into a non-silent edge.
To sum up, after each summation over an internal atom, every special redundant edge connected with an external atom \( x_i \) either becomes a non-silent edge between \( x_i \) and another external atom or becomes another special redundant edge connected with \( x_i \). Hence, after summing over all internal atoms, the special redundant edges all lead to non-silent edges between external atoms. Furthermore, we observe the simple fact that the edges between external atoms are not affected throughout the summations, so every non-isolated external atom that is not a neighbor of internal atoms in \( \tilde{G}_x \) is connected with a non-silent edge. The above arguments show that after summing over all internal atoms, we can bound \( \tilde{G}_x \) by a sum of simple auxiliary graphs satisfying the properties (a) and (b) in Lemma 3.4 and of size \( \leq \text{size}(\tilde{G}_x) \). Combining these facts with (6.1) and (6.2) yields (3.7) and the property (c), since the constant \( \tau \) can be arbitrarily small. This concludes the proof.

6.2. Proof of Lemma 3.6. In this subsection, we complete the proof of Lemma 3.6 with Lemma 3.4. We first decompose the sum over \( x \in \mathcal{A} \) into different cases according to whether every pair of atoms \( x_i \) and \( x_j \) take the same value or not. We fix one case and identify atoms that take the same value. This gives a connected graph with \( 1 \leq q \leq p \) external atoms. Without loss of generality, we denote these external atoms by \( \bar{x} = \{x_i : 1 \leq i \leq q \} \) and the graph by \( \tilde{G}_x \). Applying Lemma 3.4, we obtain that

\[
E\tilde{G}_x \leq \sum_{\gamma} c_\gamma(W,L)\tilde{G}_x^{(\gamma)} + O(W^{-D}),
\]

(6.5)

where every \( \tilde{G}_x^{(\gamma)} \) is a connected simple auxiliary graph satisfying the properties (b) and (c) in Lemma 3.4 with \( t = 1 \). By the property (c), we have that

\[
c_\gamma(W,L) \leq W^{2d_n(k_\gamma - p + 1)}, \quad k_\gamma := \#\{\text{edges in } \tilde{G}_x^{(\gamma)}\}.
\]

(6.6)

It remains to bound

\[
\frac{1}{|\mathcal{I}|^q} \sum_{x_i \in \mathcal{I}, i \in [q]} \tilde{G}_x^{(\gamma)}, \quad [q] := \{1, \cdots, q\}.
\]

(6.7)

By property (b), there are \( n \geq \lceil q/2 \rceil \) special external atoms, each of which is associated with a unique special non-silent edge. Since every atom is connected with at least one silent/non-silent pseudo-diffusive or free edge, its average over \( \mathcal{I} \) provides at least a factor

\[
\frac{1}{W^{4K^{d-4}}} + \frac{1}{L^2 N\eta W^2}.
\]

(6.8)

In addition, we will show that each average over a special external atom connected with a special non-silent edge contributes a better factor

\[
\frac{1}{W^{2K^{d-2}}} + \frac{1}{N\eta L^2} + \frac{1}{K^{d/2}} + \frac{1}{\sqrt{N\eta W^2}}.
\]

(6.9)

In bounding (6.7), we first estimate averages over the special external atoms. We use the following two estimates to bound the average over such an external atom \( x \) connected with a special diffusive edge: for \( k \geq 1 \),

\[
\frac{1}{|\mathcal{I}|} \sum_{x \in \mathcal{I}} B_{xy}^{2} \prod_{j=2}^{k} B_{xj} \prod_{s=1}^{r} \tilde{B}_{xz}^{s} < \frac{1}{W^{dK}} \sum_{t=1}^{k} \sum_{j=1}^{r} \prod_{l \neq t} B_{yj} \cdot \tilde{B}_{yl} \cdot \prod_{s:s \neq l} B_{xz}, \quad (6.10)
\]

\[
\frac{1}{|\mathcal{I}|} \sum_{x \in \mathcal{I}} \prod_{j=1}^{k} B_{xy} \prod_{s=1}^{r} \tilde{B}_{xz}^{s} < \frac{1}{W^{2dK^{d-2}}} \sum_{t=1}^{k} \sum_{j=1}^{r} \prod_{l \neq t} B_{yj} \cdot \tilde{B}_{yl} \cdot \prod_{s:s \neq l} B_{xz}, \quad (6.11)
\]

Of course, if there are free or silent free edges connected with \( x \), we can still use (6.10) and (6.11) by multiplying them with some \((N\eta)^{-1}\) and \(L^2/W^2\) factors. We apply the above two estimates in two different cases. In the first case, suppose the special diffusive edge of \( x \) is paired with the special diffusive edge of \( y \). Then, we use (6.10) to bound the average over \( x \) by a factor \( W^{-dK^{d}} \) times a sum of new graphs, each of which is still connected and has two fewer special external atoms. If the first case does not happen, then we use (6.11) to bound the average over \( x \) by a factor \( W^{-2K^{d-2}} \) times a sum of new graphs, each of which is still connected and has one fewer special external atom.
Second, we sum over special external atoms connected with non-silent free edges. Given such an external atom \( x \), we use the trivial identity \( |I|^{-1} \sum_{x \in I} 1 = 1 \) if \( x \) is only connected with silent/non-silent free edges, the estimate (6.11) if \( x \) is connected with at least one pseudo-diffusive edge, or the following estimate if \( x \) is connected with silent pseudo-diffusive edges and silent/non-silent free edges:

\[
\frac{1}{|I|} \sum_{x \in I} \prod_{s=1}^{r} \tilde{B}_{xz_s} < \frac{1}{W^4 K^{d-4}} \sum_{t=1}^{r} \prod_{s:t \neq t} \tilde{B}_{z_s z_t}.
\]

(6.12)

In this way, we can bound the average over \( x \) by a factor \((N\eta)^{-1}\) times a sum of new graphs, each of which is still connected and has one fewer special external atom. Next, summing over a non-special external atom using \(|I|^{-1} \sum_{x \in I} 1 = 1 \) or (6.12) yields a factor \((6.8)\) times a sum of new connected graphs. Finally, the average over the last atom is equal to 1.

In sum, we can bound (6.7) as

\[
\frac{1}{|I|^q} \sum_{x \in I, \gamma \in \{\eta\}} G_x(\gamma) \leq W^{-2d_\eta(k_\gamma - q + 1)} \left( \frac{1}{K^d} \right)^n \left( \frac{1}{W^2 K^{d-2}} + \frac{1}{N\eta} \right)^{n-2n_1} \left( \frac{1}{W^4 K^{d-4}} + \frac{1}{N\eta W^2} \right)^{q-1-(n-n_1)}
\]

\[
\leq W^{-2d_\eta(k_\gamma - q + 1)} \left( \frac{1}{W^2 K^{d-2}} + \frac{1}{N\eta} + \frac{1}{K^{d/2}} \sqrt{\frac{1}{N\eta W^2}} \right)^n \left( \frac{1}{W^4 K^{d-4}} + \frac{1}{N\eta W^2} \right)^{q-1-n},
\]

where \( q/2 \leq n < q - 1 \) is the number of special external atoms and \( 0 \leq n_1 \leq n/2 \) is the number of times that (6.10) has been applied. Here, the factor \( W^{-2d_\eta(k_\gamma - q + 1)} \) comes from \( k_\gamma - q + 1 \) silent/non-silent pseudo-diffusive and free edges that become self-loops during the summations (for example, the \( W^{-d} \) factor in (6.10) comes from a pseudo-diffusive edge that becomes a self-loop in the summation), and we have applied trivial bounds \( W^{-d} \leq W^{-2d_\eta} \) and \( \frac{1}{N\eta W^2} \leq W^{-2d_\eta} \) to them. Now, combining (6.13), (6.6) and (6.5), we get that

\[
\frac{1}{|I|^p} \sum_{x \in I, \gamma \in \{\eta\}} G_x(z) \leq \sum_{q \in \{p\}} \sum_{q/2 \leq n < q-1} \frac{1}{|I|^{p-q}} \left( \frac{1}{W^2 K^{d-2}} + \frac{1}{N\eta} + \frac{1}{K^{d/2}} \sqrt{\frac{1}{N\eta W^2}} \right)^n \left( \frac{1}{W^4 K^{d-4}} + \frac{1}{N\eta W^2} \right)^{q-1-n}
\]

\[
\leq \left[ \left( \frac{1}{W^2 K^{d-2}} + \frac{1}{N\eta} + \frac{1}{K^{d/2}} \sqrt{\frac{1}{N\eta W^2}} \right) \left( \frac{1}{W^4 K^{d-4}} + \frac{1}{N\eta W^2} \right) \right]^{p-1}.
\]

This concludes the proof of Lemma 3.6.

**Appendix A. Construction of the T-equation**

In this section, we present the proof of Proposition 4.20. Since it is very similar to that for Theorem 3.7 in [74], we only describe an outline of the proof and point out the main difference from the argument in [74].

**A.1. More graphical properties.** The graphs in the \( T \)-expansion and \( T \)-equation satisfy some stronger structural properties than the doubly connected property, which we define now one by one. All these properties have been defined in [74], but some minor modifications are needed to incorporate a new type of edges, i.e., free edges, in our graphs.

**Definition A.1.** Let \( G \) be a doubly connected graph and \( G_M \) be its molecular graph with all red solid edges removed. A subset of internal molecules in \( G \), say \( M \), is called isolated if and only if \( M \) is connected to its complement \( M^\complement \) exactly by two edges in \( G_M \)—a diffusive edge in the black net and a blue solid, free or diffusive edge in the blue net. An isolated subgraph of \( G \) is a subgraph induced on an isolated subset of molecules.

An isolated subgraph of \( G \) is said to be proper if it is induced on a proper subset of internal molecules of \( G \). An isolated subgraph is said to be minimal if it has no proper isolated subgraph. As a convention, if
a graph $G$ does not contain any proper isolated subgraph, then the minimal isolated subgraph (MIS) refers to the subgraph induced on all internal molecules. On the other hand, given a doubly connected graph $G$, an isolated subgraph is said to be maximal if it is not a proper isolated subgraph of $G$.

**Definition A.2** (Redundant edges). In a doubly connected graph, an edge is said to be redundant if after removing it, the resulting graph is still doubly connected. Otherwise, the edge is said to be pivotal.

**Definition A.3** (Pre-deterministic property). A doubly connected graph $G$ is said to be pre-deterministic if there exists an order of all internal blue solid edges, say $b_1 \leq b_2 \leq \ldots \leq b_k$, such that

(i) $b_1$ is redundant;

(ii) for $1 \leq i \leq k-1$, if we replace each of $b_1, \ldots, b_i$ by a diffusive or free edge, then $b_{i+1}$ becomes redundant.

**Definition A.4** (Sequentially pre-deterministic property). A doubly connected graph $G$ is said to be sequentially pre-deterministic (SPD) if it satisfies the following properties.

(i) All isolated subgraphs of $G$ that have non-deterministic closure (recall Definition 2.5) forms a sequence $(I_j)_{j=0}^{k}$ such that

$$ I_0 \supset I_1 \supset \cdots \supset I_k, $$

(A.1)

where $I_0$ is the maximal proper isolated subgraph and $I_k$ is the MIS.

(ii) The MIS $I_k$ is pre-deterministic. Let $G_M$ be the molecular graph without red solid edges. For any $0 \leq j \leq k-1$, if we replace $I_{j+1}$ and its two external edges in $G_M$ by a single diffusive or free edge, then $I_j$ becomes pre-deterministic.

By definition, a subgraph has a non-deterministic closure if it contains $G$ edges and weights inside it, or it is connected with at least one external $G$ edge. Moreover, by the definition of isolated subgraphs, the two external edges in property (ii) are exactly the black and blue external edges in the black and blue nets.

**Definition A.5** (Globally standard graphs). A doubly connected graph $G$ is said to be globally standard if it is SPD and its proper isolated subgraphs are all weakly isolated. Here, an isolated subgraph is said to be weakly isolated if it has at least two external red solid edges; otherwise, it is said to be strongly isolated.

We refer the reader to Sections 5 and 6 of [74] for the motivations and intuitions behind these definitions and some detailed explanations about their meanings. We are now ready to complete the definitions of the $T$-expansion and $T$-equation.

**Definition A.6** (T-expansion and T-equation: additional properties). The graphs in Definitions 2.15 and 4.19 satisfy the following properties:

1. Graphs of scaling order $k$ in $R^{(n)}_{x,b_1b_2}$, $W^{(n)}_{x,b_1b_2}$ and $Q^{(n)}_{x,b_1b_2}$ do not depend on $n$.

2. $R^{(n)}_{x,b_1b_2}$ is a sum of globally standard recollision graphs without free edges.

3. $A^{(n)}_{x,b_1b_2}$ is a sum of SPD graphs without free edges.

4. $W^{(n)}_{x,b_1b_2}$ is a sum of SPD graphs, each of which has exactly one redundant free edge in its MIS.

5. $Q^{(n)}_{x,b_1b_2}$ is a sum of SPD $Q$-graphs without free edges. Moreover, the atom in the $Q$-label of a $Q$-graph belongs to the MIS, i.e., all solid edges and weights have the same $Q$-label $Q_x$ for an atom $x$ inside the MIS.

6. Every $E_i$ is a sum of globally standard deterministic graphs of scaling order $l$ and without free edges. Moreover, the labelled diffusive edges in $E_i$ are all $Q^{(i)}_k$ edges (recall (2.22)) with $2 \leq k \leq i \leq l-1$.

The following lemma has been used in the proof of Lemma 5.6.

**Lemma A.7.** Let $G$ be a globally standard deterministic graph without ghost or free edges. Then, any diffusive edge in $G$, say $e = (a,b)$, is redundant in $G$.

**Proof.** The statement is trivial if $e$ is inside a molecule. If $e$ is between different molecules, then the conclusion follows from the fact that $G$ has no isolated subgraphs. \qed
A.2. Local expansions. In this subsection, we define several basic graph operations that will be used in the construction of the $T$-equation. We first define two operations related to dotted edges.

**Definition A.8** (Dotted edge operations). (i) For any pair of atoms $\alpha$ and $\beta$ in a graph $\mathcal{G}$, if there is at least one $G$ edge but no $\times$-dotted edge between them, then we write $1 = 1_{\alpha=\beta} + 1_{\alpha\neq\beta}$; if there is a $\times$-dotted edge $1_{\alpha\neq\beta}$ but no $G$ edge between them, then we write $1_{\alpha\neq\beta} = 1 - 1_{\alpha=\beta}$. Expanding the product of all these sums, we can expand $\mathcal{G}$ as

$$\mathcal{G} := \sum \text{Dot} \cdot \mathcal{G},$$

where each Dot is a product of dotted and $\times$-dotted edges together with a $\pm$ sign. In Dot $\cdot \mathcal{G}$, every $\times$-dotted edge is associated with at least one off-diagonal $G$ edge, and all diagonal $G$ edges become weights on atoms.

(ii) We merge internal atoms that are connected by a path of dotted edges (but we sometimes do not merge an external atom with an internal atom due to their different roles in graphs).

The only reason for introducing these two almost trivial operations is to write a graph into a sum of normal graphs (recall Definition 2.7), whose scaling orders are well-defined.

Using (2.2) and Gaussian integration by parts, we obtain the following three types of graph expansions.

**Lemma A.9** (Weight expansions, Lemma 3.5 of [73]). Suppose $f$ is a differentiable function of $G$. Then

$$(G_{xx} - m)f(G) = m \sum_\alpha s_{x\alpha}(G_{xx} - m)(G_{\alpha\alpha} - m)f(G) + m \sum_\alpha \frac{s_{x\alpha}}{s_{\alpha\alpha}}(G_{\alpha\alpha} - m)(G_{\beta\beta} - m)f(G)$$

$$- m \sum_\alpha s_{x\alpha}G_{\alpha\alpha}\partial_{h_{\alpha\alpha}}f(G) - m \sum_\beta \frac{s_{x\alpha}}{s_{\alpha\alpha}}G_{\beta\beta}\partial_{h_{\beta\beta}}f(G) + Q_w,$$  

where $Q_w$ is a sum of $Q$-graphs,

$$Q_w := Q_x [(G_{xx} - m)f(G)] + \sum_\alpha Q_{\alpha} \left[ \sum_\beta \frac{s_{x\alpha}}{s_{\alpha\alpha}}(G_{\alpha\alpha} - m)f(G) \right]$$

$$- mQ_x \left[ \sum_\alpha s_{x\alpha}(G_{xx} - m)G_{\alpha\alpha}f(G) \right] - m \sum_\alpha Q_{\alpha} \left[ \sum_\beta \frac{s_{x\alpha}}{s_{\alpha\alpha}}G_{\beta\beta}(G_{\beta\beta} - m)f(G) \right]$$

$$+ mQ_x \left[ \sum_\alpha s_{x\alpha}G_{\alpha\alpha}\partial_{h_{\alpha\alpha}}f(G) \right] + m \sum_\alpha Q_{\alpha} \left[ \sum_\beta \frac{s_{x\alpha}}{s_{\alpha\alpha}}G_{\beta\beta}\partial_{h_{\beta\beta}}f(G) \right].$$

**Lemma A.10** (Edge expansions, Lemma 3.10 of [73]). Suppose $f$ is a differentiable function of $G$. Consider a graph

$$\mathcal{G} := \prod_{i=1}^{k_1} G_{y_iy_i} \cdot \prod_{i=1}^{k_2} G_{y_i' y_i'} \cdot \prod_{i=1}^{k_3} G_{w_i w_i} \cdot \prod_{i=1}^{k_4} G_{w_i' w_i'},$$

where the atoms $y_i, y_i', w_i, w_i'$ are all not equal to $x$. If $k_1 \geq 1$, then we have that

$$\mathcal{G} := \sum_{i=1}^{k_2} |m|^2 \left( \sum_\alpha s_{x\alpha}G_{\alpha y_i}G_{\alpha y_i'} \right) \frac{\mathcal{G}_{G_{y_i y_i}}}{G_{y_i y_i}} + \sum_{i=1}^{k_3} m^2 \left( \sum_\alpha s_{x\alpha}G_{\alpha y_i}G_{w_i y_i} \right) \frac{\mathcal{G}_{G_{y_i y_i}}}{G_{w_i y_i}}$$

$$+ \sum_{i=1}^{k_4} m(G_{xx} - m) \left( \sum_\alpha s_{x\alpha}G_{\alpha y_i}G_{w_i y_i} \right) \frac{\mathcal{G}_{G_{y_i y_i}}}{G_{y_i y_i}} + (k_1 - 1)m \sum_\alpha s_{x\alpha}G_{xx}G_{\alpha y_i} \frac{\mathcal{G}_{G_{y_i y_i}}}{G_{y_i y_i}} + Q_e,$$

where $Q_e$ is a sum of $Q$-graphs,

$$Q_e := Q_x (\mathcal{G}) - mQ_x \left[ \sum_\alpha s_{x\alpha}(G_{\alpha\alpha} - m)\mathcal{G}_{G_{y_i y_i}} \right] - \sum_{i=1}^{k_3} mQ_x \left[ \mathcal{G}_{G_{y_i y_i}} \left( \sum_\alpha s_{x\alpha}G_{\alpha y_i}G_{\alpha y_i'} \right) \frac{\mathcal{G}_{G_{y_i y_i}}}{G_{y_i y_i}} \right].$$
Lemma A.11 (GG expansion. Lemma 3.14 of [73]). Consider a graph \( G = G_{xy}G_{y'x}f(G) \) where \( f \) is a differentiable function of \( G \) and \( y, y' \neq x \). We have that
\[
G = mS_{xy}^+G_{y'y}f(G) + m\sum_{\alpha} s_{x\alpha}G_{\alpha\gamma}G_{\gamma'y}f(G) + m\sum_{\alpha,\beta} S_{x\alpha}^+s_{\alpha\beta}(G_{\beta\gamma} - m)G_{\gamma\alpha}G_{y'y}\alpha f(G)
\]
\[
+ (G_{xx} - m)\sum_{\alpha} s_{x\alpha}G_{\alpha\gamma}G_{y'y}\alpha f(G) + m\sum_{\alpha,\beta} S_{x\alpha}^+s_{\alpha\beta}(G_{\alpha\gamma} - m)G_{\gamma\beta}G_{y'y}\beta f(G)
\]
\[
- m\sum_{\alpha} s_{x\alpha}G_{\alpha\gamma}G_{y'y}\gamma\partial_{h_{\alpha\gamma}}f(G) - m\sum_{\alpha,\beta} S_{x\alpha}^+s_{\alpha\beta}(G_{\gamma\beta} - m)G_{\gamma\alpha}G_{y'y}\alpha f(G) + \mathcal{Q}_{GG},
\]
where \( \mathcal{Q}_{GG} \) is a sum of \( Q \)-graphs,
\[
\mathcal{Q}_{GG} := Q_x(G) + \sum_{\alpha} Q_{\alpha} \left[ S_{x\alpha}^+G_{\alpha\gamma}G_{y'y}\gamma f(G) - Q_{y} \left[ S_{x\gamma}^+G_{x'y'}f(G) - Q_{x} \left[ \sum_{\alpha} s_{x\alpha}(G_{\alpha\gamma} - m)G_{\gamma'y}f(G) \right] \right] \right]
\]
\[
- m\sum_{\alpha} Q_{\alpha} \left[ S_{x\alpha}^+s_{\alpha\beta}(G_{\beta\gamma} - m)G_{\gamma\alpha}G_{y'y}\gamma f(G) - Q_{x} \left[ G_{xx} \sum_{\alpha} s_{x\alpha}G_{\alpha\gamma}G_{y'y}\gamma f(G) \right] \right]
\]
\[
- m\sum_{\alpha} Q_{\alpha} \left[ \sum_{\beta} S_{x\alpha}^+s_{\alpha\beta}G_{\alpha\gamma}G_{\gamma\beta}G_{y'y}'f(G) + \sum_{\alpha} s_{x\alpha}G_{\alpha\gamma}G_{y'y}\gamma f(G) \right]
\]
\[
+ m\sum_{\alpha} Q_{\alpha} \left[ \sum_{\beta} S_{x\alpha}^+s_{\alpha\beta}G_{\alpha\gamma}G_{y'y}\gamma f(G) \right].
\]

Using the above three lemmas, we can define the graph operations that represent weight, edge and GG expansions. We refer the reader to Section 3 of [73] for their precise definitions. All these operations are called local expansions on the atom \( x \), in the sense that they do not create new molecules (since all new atoms created in these expansions connect to \( x \) through paths of waved edges) in contrast to the global expansion that will be defined in Section A.3 below.

The goal of local expansions is to expand a graph into a sum of locally standard graphs. To explain this concept, we first define standard neutral atoms.

Definition A.12 (Standard neutral atoms). An atom in a normal graph is said to be standard neutral if it satisfies the following two properties:

- it has neutral charge, where the charge of an atom is defined by counting its incoming and outgoing plus \( G \) (i.e., blue solid) edges and minus \( G \) (i.e., red solid) edges:
  \[
  \# \{\text{incoming } + \text{and outgoing } - \text{G edges}\} - \# \{\text{outgoing } + \text{and incoming } - \text{G edges}\};
  \]
- it is only connected with three edges besides the \( x \)-dotted edges: a \( G \) edge, a \( \overline{G} \) edge, and a neutral waved edge (i.e., an \( S \) edge).

By definition, the edges connected with a standard neutral atom, say \( \alpha \), take the form
\[
t_{x,y_1y_2} := \sum_{\alpha} s_{x\alpha}G_{\alpha y_1}\overline{G}_{y_2}1_{\alpha \neq y_1}1_{\alpha \neq y_2}, \quad \text{or} \quad t_{y_1y_2,x} := \sum_{\alpha} G_{y_1\alpha}\overline{G}_{y_2\alpha}s_{\alpha x}1_{\alpha \neq y_1}1_{\alpha \neq y_2},
\]
which are essentially the \( T \)-variables (except for the two \( x \)-dotted edges and the missing coefficient \( |m|^2 \)).

We define locally standard graphs as graphs that only contain standard neutral atoms or atoms that are not connected with any \( G \) edge.

Definition A.13 (Locally standard graphs). A graph is locally standard if

(i) it is a normal graph without \( P/Q \) labels;
(ii) it has no weights or light weights;
(iii) any internal atom is either standard neutral or connected with no solid edge.
As discussed in Section 3.4 of [73], applying local expansions repeatedly, we can expand any normal graph into a linear combination of locally standard, recollision, higher order or Q graphs.

**Lemma A.14.** Let \( \mathcal{G} \) be a normal graph. Then, for any fixed \( n \in \mathbb{N} \), we can expand \( \mathcal{G} \) into a sum of \( O(1) \) many graphs:

\[
\mathcal{G} = \mathcal{G}_{\text{local}} + \mathcal{A}^{(\geq n)} + \mathcal{Q}^{(n)},
\]

(A.8)

where \( \mathcal{G}_{\text{local}} \) is a sum of locally standard graphs, \( \mathcal{A}^{(\geq n)} \) is a sum of graphs of scaling order \( > n \), and \( \mathcal{Q}^{(n)} \) is a sum of Q-graphs. Some of the graphs on the right-hand side may be recollision graphs, i.e., there is at least one dotted edge between a pair of internal and external atoms (recall Definition 2.16). In addition, every molecular graph on the right-hand side can be obtained by a composition of the operations (L1)–(L3) in Definition A.15 below acting on the molecular graph of \( \mathcal{G} \). As a consequence, if \( \mathcal{G} \) is doubly connected, then all graphs on the right-hand side of (A.8) are also doubly connected.

**Proof.** This lemma is a generalization of Lemma 3.22 in [73], which only considered graphs with three external atoms \( a, b_1, b_2 \) (i.e., graphs coming from expansions of \( T_{a,b_1,b_2} \)), while the graph \( \mathcal{G} \) in our lemma is more general. But the proof for [73, Lemma 3.22] can be applied to our case almost verbatim, so we omit the details.

\[
\square
\]

**Definition A.15** (Local molecular operations). We define the following operations on molecular graphs related to local expansions on \( x \). Let \( M_x \) be the molecule containing \( x \).

1. **(L1)** Merge \( M_x \) with another molecule.
2. **(L2)** Add two new solid edges of the same color between \( M_x \) and another molecule.
3. **(L3)** For a pair of molecules, say \( M_1 \) and \( M_2 \), remove a solid edge between \( M_1 \) and \( M_2 \), and then add two solid edges of the same color: one between \( M_1 \) and \( M_x \) and the other between \( M_2 \) and \( M_x \).

For simplicity of presentation, we will call this operation as “\( M_x \) pulls a solid edge between \( M_1 \) and \( M_2 \)”.

### A.3. Global expansions.

In this subsection, we define global expansions, which may create new molecules in contrast to local expansions. Suppose we have the \((n-1)\)-th order T-expansion by induction.

Given a locally standard graph, say \( \mathcal{G} \), a global expansion involves replacing the \( T \)-variable containing a standard neutral atom by the \((n-1)\)-th order T-expansion. More precisely, picking a standard neutral atom \( \alpha \) in a locally standard graph, so that the edges connected to it take one of the forms in (A.7). Then, we apply the \((n-1)\)-th order T-expansion in (2.25) to these variables in the following way:

\[
t_{x,y_1,y_2} = |m|^{-2}T_{x,y_1,y_2} - \sum_{\alpha} s_{x\alpha}G_{\alpha y_1}G_{\alpha y_2}1_{\alpha \neq y_1,1_{\alpha=y_2} + 1_{\alpha=y_1,1_{\alpha \neq y_2} + 1_{\alpha=y_1,1_{\alpha=y_2}}}}.
\]

(A.9)

The last term on the right-hand side gives one (if \( y_1 = y_2 \)) or two (if \( y_1 \neq y_2 \)) recollision graphs, so we combine it with \( |m|^{-2} \sum_x \Theta^{(n-1)}_{x\alpha}R^{(n-1)}_{\alpha,y_1,y_2} \) and denote the resulting expression by \( R^{(n-1)}_{x,y_1,y_2} \). This gives that

\[
t_{x,y_1,y_2} = m^{-1}\Theta^{(n-1)}_{x,y_1,y_2}G_{y_1,y_2} + \frac{G_{y_1,y_2} - C_{y_1,y_2}}{2i|\eta|} + R^{(n-1)}_{x,y_1,y_2} + |m|^{-2} \sum_{\alpha} \Theta^{(n-1)}_{x\alpha} \left[ A^{(n-1)}_{\alpha,y_1,y_2} + W^{(n-1)}_{\alpha,y_1,y_2} + Q^{(n-1)}_{\alpha,y_1,y_2} + (E_{rr,n-1,D})_{\alpha,y_1,y_2} \right].
\]

(A.10)

The expansion of \( t_{y_1,y_2,x} \) can be obtained by exchanging the order of matrix indices in the above equation.

In a global expansion, if we replace \( t_{x,y_1,y_2} \) with a graph on the right-hand side of (A.10) that is not in \( |m|^{-2} \sum_x \Theta^{(n-1)}_{x\alpha}Q^{(n-1)}_{\alpha,y_1,y_2} \), then we need to perform dotted edge operations in Definition A.8 to write it into a sum of normal graphs. For each resulting graph, we either stop expanding it or continue performing local and global expansions. On the other hand, if we replace \( t_{x,y_1,y_2} \) with a graph in \( |m|^{-2} \sum_x \Theta^{(n-1)}_{x\alpha}Q^{(n-1)}_{\alpha,y_1,y_2} \), we will get a graph of the form

\[
\mathcal{G} = \sum_y \Gamma \mathcal{Q}_y(\mathcal{G}_1),
\]

(A.11)

where both \( \Gamma \) and \( \mathcal{G}_1 \) are graphs without \( P/Q \) labels. Then, we need to perform the so-called Q-expansion to expand (A.11) into a sum of Q-graphs and graphs without \( P/Q \) labels. The Q-expansion is defined in Section 4.4 of [74]. Instead of stating the full definition, we only describe its key feature here.
Recall that \( H^{(y)} \) is the \((N-1) \times (N-1)\) minor of \( H \) obtained by removing the \( y \)-th row and column. We define the resolvent minor \( G^{(y)}(z) := (H^{(y)} - z)^{-1} \). Using the Schur complement formula, we obtain that
\[
G_{x_1x_2} = G^{(y)}_{x_1y} + \frac{G_{x_1y}G_{yy}G_{yy}G_{y_2}}{G_{yy}}, \quad x_1, x_2 \in \mathbb{Z}^n.
\] (A.12)

Applying this identity to expand the resolvent entries in \( \Gamma \) one by one, we get that
\[
\Gamma = \Gamma^{(y)} + \sum \omega \Gamma_{\omega},
\] (A.13)
where \( \Gamma^{(y)} \) is a graph whose weights and solid edges are all \( G^{(y)} \) entries, so that it is independent of the \( y \)-th row and column of \( H \), and every \( \Gamma_{\omega} \) has a strictly higher scaling order than \( \Gamma \), at least two new solid edges connected with \( y \), and a factor of the form \((G_{yy})^{-k}(G_{yy})^{-l}\) for some \( k, l \in \mathbb{N} \). Using (A.13), we can expand (A.11) as
\[
\mathcal{G} = \sum \omega \sum y \Gamma_{\omega}Q_y(\Gamma_1) + \sum y Q_y(\Gamma\Gamma_1) - \sum \omega \sum y \Gamma_{\omega}Q_y(\Gamma_\omega\Gamma_1),
\] (A.14)
where the second and third terms are sums of \( Q \)-graphs, and the graphs in the first term has the following key features: (i) the scaling order of \( \sum y \Gamma_{\omega}Q_y(\Gamma_1) \) is strictly higher than \( \text{ord}(\mathcal{G}) \); (ii) at least one weight or solid edge in \( \Gamma \) is pulled to atom \( y \) (i.e., replaced by two solid edges connected with \( y \)) in \( \Gamma_{\omega} \).

Next, we apply (A.12) in the reverse way to remove all \( G^{(y)} \) entries and apply Taylor expansion to \((G_{yy})^{-1} = [m + (G_{yy} - m)]^{-1}\) to remove all \((G_{yy}^{-1} - 1)\) and \((G_{yy}^{-1})^{-1}\) entries. In this way, we can write the right-hand side of (A.14) into a sum of graphs containing only regular \( G \) edges and \( G \) weights. Finally, we still need to remove the \( Q_y \) label in \( \sum y \Gamma_{\omega}Q_y(\Gamma_1) \), which can be achieved by applying the local expansions and (A.12) repeatedly. Since the full definition of the \( Q \)-expansion is tedious, we will not repeat it here and refer the reader to Section 4.4 of [74] for more details. We only record the following key lemma regarding the \( Q \)-expansion.

**Lemma A.16** (Lemma 4.15 of [74]). Let \( \mathcal{G}_0 := \sum x \Gamma_0Q_x(\tilde{\Gamma}_0) \), where \( \mathcal{G}_0, \Gamma_0 \) and \( \tilde{\Gamma}_0 \) are all normal graphs without \( P/Q \) labels and \( x \) is an internal atom. Then, for any large constant \( D > 0 \), \( \mathcal{G}_0 \) can be expanded into a sum of \( O(1) \) many graphs:
\[
\mathcal{G}_0 = \sum \omega \mathcal{G}_\omega + \sum \zeta Q_x(\tilde{\mathcal{G}}_\zeta) + \mathcal{G}_{\text{err}},
\] (A.15)
where \( \mathcal{G}_\omega \) and \( \tilde{\mathcal{G}}_\zeta \) are normal graphs without \( P/Q \) labels, and \( \mathcal{G}_{\text{err}} \) is a sum of normal graphs of scaling order > \( D \). Moreover, the following properties hold.

(i) Every graph on the right-hand side of (A.15) has scaling order \( \geq \text{ord}(\mathcal{G}_0) \).

(ii) If there is a new atom in a graph on the right-hand side of (A.15), then it is connected to \( x \) through a path of waved edges (i.e., the expansion (A.15) is a local expansion on \( x \)).

(iii) Every molecular graph on the right-hand side can be obtained by a composition of the operations \( \text{L1} \)–\( \text{L3} \) acting on the molecular graph of \( \mathcal{G}_0 \). As a consequence, if \( \mathcal{G}_0 \) is doubly connected, then all graphs on the right-hand side of (A.15) are also doubly connected.

(iv) If \( \Gamma_0 \) does not contain any weight or solid edge attached to \( x \), then the scaling order of \( \mathcal{G}_\omega \) is strictly higher than \( \text{ord}(\mathcal{G}_0) \) for every \( \omega \). Furthermore, \( \mathcal{G}_\omega \) contains at least one atom that belongs to the original graph \( \Gamma_0 \) and is connected to \( x \) through a solid, waved or dotted edge (before it is merged with \( x \) by a dotted edge operation).

To summarize, global expansions correspond to the following operations.

**Definition A.17** (Global operations). Given \( t_{x,y_1y_2} \), we define the following operations.

(G1) Replace \( t_{x,y_1y_2} \) by a diffusive edge between \( x \) and \( y_1 \) and a solid edge between \( y_1 \) and \( y_2 \), and then apply dotted edge operations if necessary.

(G2) Replace \( t_{x,y_1y_2} \) by a free edge between \( x \) and \( y_1 \) and a solid edge between \( y_1 \) and \( y_2 \), and then apply dotted edge operations if necessary.

(G3) Replace \( t_{x,y_1y_2} \) by a graph in \( A_{x,y_1y_2}^{(n-1)} \), and then apply dotted edge operations if necessary.

(G4) Replace \( t_{x,y_1y_2} \) by a graph in \( \sum \alpha \Theta_\alpha^{(n-1)} \Theta_\alpha^{(n-1)} A_{x,y_1y_2}^{(n-1)} \), and then apply dotted edge operations if necessary.

(G5) Replace \( t_{x,y_1y_2} \) by a graph in \( \sum \alpha \Theta_\alpha^{(n-1)} \Theta_\alpha^{(n-1)} W_{\alpha,y_1y_2}^{(n-1)} \), and then apply dotted edge operations if necessary.
(G6) Replace $t_{x,y_1y_2}$ by a graph in $\sum_\alpha \Theta_{x\alpha}^{(n-1)} G_{\alpha,y_1y_2}^{(n-1)}$, and then apply the $Q$-expansion if necessary.

(G7) Replace $t_{x,y_1y_2}$ by a graph in $\sum_\alpha \Theta_{x\alpha}^{(n-1)} (\mathcal{E}rr_{n-1,D})_{\alpha,y_1y_2}^{(n-1)}$, and then apply dotted edge operations if necessary.

Using the doubly connected property of the graphs in the $(n-1)$-th order $T$-expansion, it is not hard to see that if we apply the operations (G1)–(G7) to a redundant blue solid edge (or more precisely, a $t$-variable containing this edge), the resulting graphs are still doubly connected. On the other hand, expanding a pivotal blue solid edge may break the doubly connected property. Hence, we need to make sure that whenever performing a global expansion, there is at least one redundant blue solid edge so that the expansion process can be continued without breaking the doubly connected property. This is guaranteed by the SPD property in Definition A.4. We now show that this property is preserved not only in local expansions, but also in global expansions as long as we expand the first blue solid edge in a pre-deterministic order of the MIS.

**Lemma A.18.** Let $G$ be a SPD graph.

(i) Applying operations (L1)–(L3) to the molecular graph of $G$ still gives a SPD molecular graph.

(ii) Applying operations (G1)–(G6) to a $t_{x,y_1y_2}$ variable containing the first blue solid edge in a pre-deterministic order of the non-deterministic MIS still gives a sum of SPD graphs.

**Proof.** The first statement was proved in Lemma 5.10 of [74], but under a weaker definition of the SPD property: the property (ii) in Definition A.3 was “if we replace each of $b_1, ..., b_i$ by a diffusive edge, then $b_{i+1}$ becomes redundant” and the property (ii) in Definition A.4 was “if we replace $I_j + 1$ and its two external edges in $G_M$ by a single diffusive edge, then $I_j$ becomes pre-deterministic”. (This is because the concept of free edges was not introduced in [74].) In checking the SPD property, we change the blue solid edges or closures of isolated subgraphs into diffusive or free edges one by one and check whether the next blue solid edge is redundant. Based on Lemma 5.10 of [74], we only need to consider cases where some blue solid edge or the closure of some isolated subgraph is replaced by a free edge at a certain step. Notice that the free edge obtained at this step is redundant in the graph, say $G'$, and hence we can move the free edge to other places so that every remaining blue solid edge becomes redundant. For example, for a blue solid edge $b$ between atoms $x$ and $y$, we can treat the free edge as a blue free edge between $x$ and $y$ so that the edge $b$ becomes redundant. Then, the SPD property of $G'$ is trivial to check.

The second statement for operations (G1) and (G2) follows from the definition of the SPD property of $G$, because they replace the first blue solid edge of the MIS by a diffusive edge and a free edge, respectively. For the operations (G3)–(G6), if we replaced $t_{x,y_1y_2}$ by a graph in

$$G_{x,y_1y_2}^{(n-1)} + \sum_\alpha \Theta_{x\alpha}^{(n-1)} \left[ A_{\alpha,y_1y_2}^{(n-1)} + (\mathcal{E}rr_{n-1,D})_{\alpha,y_1y_2}^{(n-1)} + Q_{\alpha,y_1y_2}^{(n-1)} \right],$$

the resulting graph, say $G_{\text{new}}$, is SPD due to the SPD property of the graphs in the $T$-expansion (recall Definition A.6) and the fact that $t_{x,y_1y_2}$ contains the first blue solid edge of the MIS. Then, applying dotted edge operations or the $Q$-expansion to $G_{\text{new}}$ gives a sum of SPD graphs by the first statement.

With Lemma A.18, we can easily show that the globally standard property (recall Definition A.5) is also preserved in local and global expansions.

**Lemma A.19.** Let $G$ be a globally standard graph without $P/Q$ labels and let $I_k$ be its MIS with non-deterministic closure. For local expansions, we have that:

(i) Applying operations (L1)–(L3) to the molecular graph of $G$ still gives a globally standard molecular graph. Furthermore, if we apply the local expansions in Lemma A.9–A.11 on an atom in $I_k$, then in every new $Q$-graph, the atom in the $Q$-label also belongs to the MIS with non-deterministic closure.

Let $t_{x,y_1y_2}$ be a $t$-variable that contains the first blue solid edge in a pre-deterministic order of $I_k$. Then, we have that:

(ii) If we apply the operation (G1), then every new graph has no $P/Q$ label or free edge, is globally standard, and has one fewer blue solid edge.

(iii) If we apply the operation (G2), then every new graphs has no $P/Q$ label, is globally standard, and has exactly one redundant free edge in its MIS.

(iv) If we apply the operation (G3), then every new graph has no $P/Q$ label or free edge, is globally standard, and has a scaling order $\geq \text{ord}(G) + 1$. 

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(v) If we apply the operation \((G_4)\), then every new graph has no \(P/Q\) label or free edge, is SPD, and has a scaling order \(\geq \text{ord}(G) + n - 2\).

(vi) If we apply the operation \((G_5)\), then every new graph has no \(P/Q\) label, is SPD, and has exactly one redundant free edge in its MIS.

(vii) If we apply the operation \((G_6)\), then we get a sum of \(O(1)\) many new graphs:

\[
\sum_{\omega} G_{\omega} + Q + \mathcal{G}_{\text{err}},
\]

where every \(G_{\omega}\) has no \(P/Q\) label or free edge, is globally standard and has a scaling order \(\geq \text{ord}(G) + 1\);

\(Q\) is a sum of \(Q\)-graphs, each of which has no free edge, is SPD, and has a MIS containing the atom in the \(Q\)-label;

\(\mathcal{G}_{\text{err}}\) is a sum of doubly connected graphs of scaling orders \(> D\).

(viii) If we apply the operation \((G_7)\), then every new graph is doubly connected and has a scaling order \(\geq \text{ord}(G) + D - 1\).

Proof. Based on Lemma A.18, the statement (i) was proved in [74, Lemma 6.4], and the statements (ii), (iv), (v), (vii) and (viii) were proved in [74, Lemma 6.5]. The statement (iii) can be proved in the same way as statement (ii), while the statement (vi) can be proved in the same way as statement (v). We omit the details. \(\square\)

A.4. Proof of Proposition 4.20. Based on Lemma A.19, we can define the global expansion strategy for the proof of Proposition 4.20. Starting with the second order \(T\)-expansion (2.3), we continue to expand the term \(\sum_{x} \Theta_{ab}^c A_{x,b_1b_2}^{(>2)}\) with local and global expansions. We will stop expanding a graph if it is a normal graph and satisfies at least one of the following properties:

(S1) it is a \(b_1/b_2\)-recollision graph;

(S2) its scaling order is at least \(n + 1\);

(S3) it contains a redundant free edge;

(S4) it is a \(Q\)-graph;

(S5) it is non-expandable, that is, its MIS with non-deterministic closure is locally standard and has no redundant blue solid edge.

Note that if a graph \(G\) satisfies the property that its subgraph induced on all internal atoms is deterministic, then \(G\) is non-expandable. On the other hand, in a non-expandable graph that does not satisfy (A.17), we cannot expand a plus \(G\) edge in the MIS without breaking the doubly connected property.

Strategy A.20 (Global expansion strategy). Fix any \(n \in \mathbb{N}\) and large constant \(D > n\). Given the above stopping rules (S1)–(S5), we expand \(T_{a,b_1b_2}\) according to the following strategy.

Step 0: We start with the second order \(T\)-expansion (2.3), where we only need to expand \(\sum_{x} \Theta_{ab}^c A_{x,b_1b_2}^{(>2)}\) since all other terms already satisfy the stopping rules. We apply local expansions to \(G\) and obtain a linear combination of new graphs, each of which either satisfies the stopping rules or is locally standard. At this step, there is only one internal molecule in every graph. Hence, the graphs are trivially globally standard.

Step 1: Given a globally standard input graph, we perform local expansions on atoms in the MIS with non-deterministic closure. We send the resulting graphs that already satisfy the stopping rules (S1)–(S5) to the outputs. Every remaining graph is globally standard by Lemma A.19, and its MIS is locally standard (i.e. the MIS contains no weight and every atom in it is either standard neutral or connected with no solid edge).

Step 2: Given a globally standard input graph \(G\) with a locally standard MIS, say \(I_k\), we find a \(t_{x,y_1y_2}\) or \(t_{y_1y_2,x}\) variable that contains the first blue solid edge in a pre-deterministic order of \(I_k\). If we cannot find such a \(t\)-variable, then we stop expanding \(G\).

Step 3: We apply the global expansions \((G1)\)–\((G7)\) to the \(t_{x,y_1y_2}\) or \(t_{y_1y_2,x}\) variable chosen in Step 2. We send the resulting graphs that already satisfy the stopping rules (S1)–(S5) to the outputs. The remaining graphs are all globally standard by Lemma A.19, and we sent them back to Step 1.

Applying the above expansion strategy, we can expand \(T_{a,b_1b_2}\) into

\[
m \Theta_{ab_1}^c G_{b_1b_2} + \frac{|m|^2}{2N}\left(G_{b_2b_1} - G_{b_1b_2}\right)
\]
plus a sum of \(O(1)\) many graphs satisfying the stopping rules (S1)–(S5). The graphs satisfying stopping rules (S1)–(S4) can be included into
\[
\sum_x \Theta_{ax}^o \left[ \mathcal{R}_{x,b_1 b_2}^{(n)} + \mathcal{A}_{x,b_1 b_2}^{(>n)} + \mathcal{W}_{x,b_1 b_2}^{(n)} + (\mathcal{E}_{n,D})_{x,b_1 b_2} \right].
\] (A.18)

Now, suppose a graph, say \(\mathcal{G}_{a,b_1 b_2}\), is an output graph of Strategy A.20 and does not satisfy (S1)–(S4). Then, \(\mathcal{G}_{a,b_1 b_2}\) either is non-expandable or does not contain a \(t\)-variable required by Step 2 of Strategy A.20. In either case, \(\mathcal{G}_{a,b_1 b_2}\) contains a locally standard MIS \(I_k\), which, by the pre-deterministic property of \(I_k\), does not contain any internal blue solid edge. Suppose \(I_k\) is a proper isolated subgraph. Due to the weakly isolated property, \(I_k\) has at least two external red solid edges, at most one external blue solid edge and no internal blue solid edge. Hence, \(I_k\) cannot be locally standard, which gives a contradiction. Hence, \(I_k\) is indeed the subgraph of \(\mathcal{G}_{a,b_1 b_2}\) induced on all internal atoms. Then, \(I_k\) is locally standard and does not contain any internal blue solid edges, so it must satisfy (A.17). Furthermore, by the locally standard property, \(I_k\) contains a standard neutral atom connected with \(b_1\) and \(b_2\). Thus, the output graphs of Strategy A.20 that do not satisfy (S1)–(S4) can be written as
\[
|m|^2 \sum_x (\Theta_{ax}^o \Sigma_{n})_{x} t_{x,b_1 b_2} = \sum_x (\Theta_{ax}^o \Sigma_{n})_{x} T_{x,b_1 b_2} + \sum_x (\Theta_{ax}^o \Sigma_{n})_{x} \left( |m|^2 t_{x,b_1 b_2} - T_{x,b_1 b_2} \right),
\]
where \(\Sigma_{n}\) is a sum of deterministic graphs and the second term can be included into \(\sum_x \Theta_{ax}^o \mathcal{R}_{x,b_1 b_2}^{(n)}\) (recall (A.9)). Finally, we expand all \(\Theta^{(n-1)}\) edges in \(\Sigma_{n}\) using (2.22), i.e.,
\[
\Theta^{(n-1)} = \sum_{k=2}^{n-1} \Theta_k^{(n-1)} + \Delta^{(n-1)},
\]
where \(\Delta^{(n-1)}\) is regarded as a labelled diffusive edge of scaling order \(> n - 1\) between \(x\) and \(y\). Then, we collect all graphs of scaling order \(\leq n\) into \(\Sigma_{n}\) and all remaining graphs into \(\sum_x \Theta_{ax}^o \mathcal{A}_{x,b_1 b_2}^{(>n)}\).

To conclude Proposition 4.20, we still need to show that the property (1) in Definition A.6 holds and that the sum of graphs of scaling order \(\leq n - 1\) in \(\Sigma_{n}\) is equal to \(\Sigma^{(n-1)}\). This follows from the argument in Section 6.4 of [74], and we omit the details.

**APPENDIX B. CONSTRUCTION OF THE COMPLETE T-EXPANSION**

The \(Q\)-graphs in (3.5) satisfy some additional properties given in the following lemma.

**Lemma B.1** (Complete T-expansion: additional properties). Under the assumptions of Lemma 3.2, for any large constant \(D > 0\), \(T_{a,b_1 b_2}\) can be expanded as (3.5), where each graph \(Q^{(w)}_{x,b_1 b_2}\) satisfies the following properties.

1. It is a SPD graph (with ghost edges included into the blue net).
2. The atom in the \(Q\)-label of \(Q^{(w)}_{x,b_1 b_2}\) belongs to the MIS with non-deterministic closure.
3. The size of \(Q^{(w)}_{x,b_1 b_2}\) satisfies that size\((Q^{(w)}_{x,b_1 b_2})\) \(\leq W^{-2da}\).
4. There is an edge, red solid or diffusive or dotted, connected to \(b_1\); there is an edge, red solid or diffusive or dotted, connected to \(b_2\).

**Proof of Lemmas 3.2 and B.1.** Starting with the \(n\)-th order T-expansion (2.25), to conclude Lemma 3.2, we need to further expand the graphs in
\[
\sum_x \Theta_{ax}^o \left( \mathcal{R}_{x,b_1 b_2}^{(n)} + \mathcal{A}_{x,b_1 b_2}^{(>n)} + \mathcal{W}_{x,b_1 b_2}^{(n)} \right),
\] (B.1)

To describe the expansion strategy, we first define the stopping rules. Given the large constant \(D\) in Lemma 3.2, we stop expanding a graph \(\mathcal{G}\) if it is normal and satisfies at least one of the following properties:

1. The subgraph of \(\mathcal{G}\) induced on internal atoms is deterministic and locally standard (i.e., if there is an internal atom connected with external edges, it must be standard neutral);
2. Size\((\mathcal{G})\) \(\leq W^{-D}\);
3. \(\mathcal{G}\) is a \(Q\)-graph.
The following lemma shows that a graph from the expansions can be bounded by its size. It will also be used in the proof of Lemma 6.2 in Section C.

**Lemma B.2.** Under the setting of Lemma 3.2, let \( \mathcal{G} \) be a generalized doubly connected graph (recall Definition 6.1). Then, we have

\[
\mathcal{G} < \text{size}(\mathcal{G}).
\]

**Proof.** With Lemma 4.17 and Lemma 4.18, its proof is exactly the same as that for Lemma 9.11 of [74]. So we omit the details. \( \square \)

The doubly connected property of the graphs from our expansions trivially imply that they are generalized doubly connected. Hence, those graphs satisfying the stopping role (T2) will be errors in \( \mathcal{E}_{rr_{a,b_1 b_2}} \).

Similar to Strategy A.20, the core of the expansion strategy for the proof of Lemma 3.2 is still to expand the plus \( G \) edges according to a pre-deterministic order in the MIS with non-deterministic closure. However, at a certain step, we have to expand a pivotal blue solid edge, say \( b \), connected with an isolated subgraph. To deal with this issue, we add a ghost edge between the ending atoms of the pivotal edge \( b \) and multiply the coefficient by \( L^2/W^2 \). Then, the blue solid edge \( b \) becomes redundant, so we can expand it as in Step 3 of Strategy A.20. However, we need to make sure that every graph from the expansion does not have a diverging size.

**Strategy B.3.** Given a large constant \( D > 0 \) and a SPD graph \( \mathcal{G} \) without \( P/Q \) labels, we perform one step of expansion as follows. Let \( \mathcal{I}_k \) be the MIS with non-deterministic closure.

**Case 1:** Suppose \( \mathcal{I}_k \) is not locally standard. We then perform local expansions on atoms in \( \mathcal{I}_k \) and send the resulting graphs that already satisfy the stopping rules (T1)–(T3) to the outputs. Every remaining graph has a locally standard MIS with non-deterministic closure and satisfies the SPD property by Lemma A.18.

**Case 2:** Suppose \( \mathcal{I}_k \) is locally standard. We find a \( t_{x,y_1 y_2} \) or \( t_{y_1 y_2,x} \) variable that contains the first blue solid edge in a pre-deterministic order of \( \mathcal{I}_k \), and then apply the global expansions (G1)–(G7) in Definition A.17 to it (where we need to apply (A.10) with \( n-1 \) replaced by \( n \)). We send the resulting graphs that already satisfy the stopping rules (T1)–(T3) to the outputs, while the remaining graphs are all SPD by Lemma A.18.

**Case 3:** Suppose \( \mathcal{I}_k \) is deterministic, strongly isolated in \( \mathcal{I}_{k-1} \), and locally standard. In other words, \( \mathcal{I}_k \) contains a standard neutral atom, say \( \alpha \), connected with an external blue solid edge and an external red solid edge. Suppose the edge \( G_{\alpha y_1} \) or \( G_{y_1 \alpha} \) in a \( t_{x,y_1 y_2} \) or \( t_{y_1 y_2,x} \) variable is the pivotal external blue solid edge of \( \mathcal{I}_k \). If there is a redundant free edge, then we move it to \( (\alpha, y_1) \); otherwise, we add a ghost edge between \( (\alpha, y_1) \) and multiply the graph by \( L^2/W^2 \). In the resulting graph, the edge \( G_{\alpha y_1} \) or \( G_{y_1 \alpha} \) becomes redundant if we include the free edge or the added ghost edge into the blue net. Then, apply the global expansions (G1)–(G7) in Definition A.17 by using (A.10) with \( n-1 \) replaced by \( n \). We send the resulting graphs that already satisfy the stopping rules (T1)–(T3) to the outputs, while the remaining graphs are all SPD by Lemma A.18.

Applying the Strategy B.3 repeatedly to expand the graphs in (B.1), we get \( O(1) \) many output graphs that satisfy the stopping rules (T1)–(T3). Combining them together, we get an expansion of \( T_{a,b_1 b_2}^\circ \) as

\[
T_{a,b_1 b_2}^\circ = m \Theta_{ab_1}^a \Theta_{b_2}^b + \sum_x \Theta_{ax}^a \Theta_{b_2}^b + \sum \int \Theta_{ax}^a \Theta_{b_2}^b \Theta_{xy}^\mu T_{y,b_1 b_2}^\mu + \sum \int \Theta_{ax}^a \Theta_{b_2}^b \Theta_{xy}^\mu T_{y,b_1 b_2}^\mu
\]

\[
+ \sum \int \Theta_{ax}^a \Theta_{b_2}^b \Theta_{xy}^\mu T_{y,b_1 b_2}^\mu + \sum \int \Theta_{ax}^a \Theta_{b_2}^b \Theta_{xy}^\mu T_{y,b_1 b_2}^\mu
\]

where the graphs satisfying (T2) are included into \( \sum_x \Theta_{ax}^a \Theta_{b_2}^b \Theta_{xy}^\mu \), the graphs satisfying (T3) are included into \( \sum \int \Theta_{ax}^a \Theta_{b_2}^b \Theta_{xy}^\mu \), and the graphs satisfying (T1) give the third to sixth terms on the right-hand side (where we have included the term \( \sum_x \Theta_{ax}^a \Theta_{b_2}^b \Theta_{xy}^\mu \) in (2.25) into the fourth term on the right-hand side). Following the proof of [74, Lemma 9.7], we can show that the graphs \( D_{xy}^\mu \) satisfy the properties in Lemmas 3.2 and B.1, and that \( D_{xy}^\mu \) and \( \Theta_{xy}^\mu \) are doubly connected graphs satisfying

\[
\text{size}(D_{xy}^\mu) \leq \frac{W^2}{L^2} W^{-2d_n-c_0},
\]

**Proof.** With Lemma 4.17 and Lemma 4.18, its proof is exactly the same as that for Lemma 9.11 of [74]. So we omit the details. \( \square \)
under (3.4). In fact, the doubly connected property of the graphs in (B.2) follows from the fact that the SPD property is maintained in Strategy B.3. The main technical part is to ensure that the sizes of these graphs will not diverge due to the $L^2/W^2$ factors introduced alongside the ghost edges. Since the relevant argument is very similar to that in the proof of [74, Lemma 9.7], we omit the details.

Now, solving the equation (B.2) as in (4.50), we can get an expansion of $T_{a,b_1,b_2}^\circ$, which, together with (2.7), gives the expansion (3.5) with

$$\tilde{Θ} := \left(1 - \sum_\mu \Theta^\circ \tilde{D}^\circ(\mu)\right)^{-1} \Theta^\circ.$$

Using Lemma 5.1 and recalling the definition (3.1), we get that

$$\sum_\alpha \Theta^\circ_{xa} \tilde{D}_{ay}(\mu) \sim \sum_\alpha B_{xa} \left(B_{ay} + B_{gy} \frac{W^2-d}{L^2}\right) W^{4d+\text{size}(\tilde{D}_{ay}(\mu))} \leq \frac{W^{-c_0}}{(x-y)^d},$$

where we used (B.3) and $W^2/L^2 \cdot B_{xy} \leq (x-y)^d$ in the second step. Using this estimate and the Taylor expansion of $\tilde{Θ}$, we can get that $|\tilde{Θ}_{xy}| \prec B_{xy}$. This concludes the proof. □

Appendix C. Proof of Lemma 6.2

Corresponding to Definition 6.1, we also define the generalized SPD and globally standard properties for graphs with external molecules (recall Definitions A.4 and A.5).

Definition C.1. (i) A graph $G$ is said to satisfy the generalized SPD property with external molecules if merging all external molecules of $G$ into one single internal molecule yields an SPD graph.

(ii) A graph $G$ is said to be generalized globally standard with external molecules if it is generalized SPD in the above sense and every proper isolated subgraph with non-deterministic closure is weakly isolated.

We will expand $G_x$ by applying Strategy A.20 with the following modifications:

- we will use the generalized globally standard property in Definition C.1;
- for a global expansion, we will use the complete $T$-expansion (3.5) instead of the $n$-th order $T$-expansion;
- we will stop expanding a graph if it is deterministic, its size is less than $W^{-D}$, or it is a $Q$-graph.

More precisely, we expand $G_x$ according to the following strategy.

Strategy C.2. Given a large constant $D>0$ and a generalized globally standard graph without $P/Q$ labels, we perform one step of expansion as follows.

Case 1: Suppose we have a graph where all solid edges are between external molecules. Corresponding to Step 0 of Strategy A.20, we perform local expansions to get a sum of locally standard graphs plus some graphs that already satisfy the stopping rules. Then, we apply (3.5) to an arbitrary $t$-variable and get a sum of generalized globally standard graphs plus some graphs that already satisfy the stopping rules.

Case 2: As in Step 1 of Strategy A.20, we perform local expansions on atoms in the MIS with non-deterministic closure and get a sum of generalized globally standard graphs with locally standard MIS plus some graphs that already satisfy the stopping rules.

Case 3: Given a generalized globally standard input graph with a locally standard MIS, we find a $t$-variable that contains the first blue solid edge in a pre-deterministic order of the MIS as in Step 2 of Strategy A.20, and then apply (3.5) to it as in Step 3 of Strategy A.20 to get a sum of generalized globally standard graphs plus some graphs that already satisfy the stopping rules.

In the proof of Lemma 9.14 in [74], it has been shown that applying the Strategy C.2 repeatedly to expand $G_x$ will finally give a sum of graphs satisfying the stopping rules, where the deterministic graphs are denoted as $G_x^{(\mu)}$, the $Q$-graphs have zero expectation, and the remaining graphs have small enough sizes and give an error $O(W^{-D})$ by Lemma B.2. Similar to the proof of [74, Lemma 9.14], we can show that the property (a) of Lemma 6.2 follows from the generalized globally standard property. The property (d) can be shown easily by keeping track of the sizes of the graphs from expansions using property (2) of Lemma 3.2 and property (3) of Lemma B.1. The main goal of the rest of the proof is to show that properties (b) and
(c) of Lemma 6.2 hold. For this purpose, we define the following property, which coincides with property (b) for deterministic graphs. We will show that this property holds throughout the expansion process.

**Definition C.3.** (b’) After replacing the maximal (weakly) isolated subgraph by a diffusive or free edge in the molecular graph without red solid edges, we can find at least \( r \geq \lceil n/2 \rceil \) external molecules that are simultaneously connected with special redundant (diffusive or free) edges in the following sense: after replacing every blue solid edge in the graph with a diffusive or free edge, the resulting graph satisfies property (b) of Lemma 6.2.

In the above definition, by “replacing every blue solid edge in the graph with a diffusive or free edge”, we mean all possible assignments of the types (diffusive or free) for the blue solid edge instead of only one particular assignment.

Given a generalized doubly connected graph, say \( G \), satisfying properties (b’) and (c), we will call the \( r \) molecules in property (b’) as “special redundant molecules”, the corresponding \( r \) redundant edges as “special redundant edges”, other external molecules that are neighbors of internal molecules as “special pivotal molecules”, and other edges between external and internal molecules as “special pivotal edges”. We need to show that all new graphs from an expansion of \( G \) also satisfy properties (b’) and (c). In the following proof, whenever we say the “original graph”, we actually mean the original graph together with a certain assignment of the types (diffusive or free) for all blue solid edges. It will be clear from the context which assignment we are referring to. In the proof, we will use the following trivial but very convenient fact.

**Claim C.4.** Given a graph \( G \), let \( S \) be a subset of molecules such that the subgraph induced on \( S \) is doubly connected. Then, \( G \) is doubly connected if and only if the following quotient graph \( G/S \) is doubly connected: \( G/S \) is obtained by treating the subset of molecules \( S \) as one single vertex, and the edges connected with \( S \) in \( G \) are now connected with this vertex \( S \) in \( G/S \).

We divide the proof into three cases corresponding to the three cases in Strategy C.2, which are respectively given in Sections C.1–C.3. These proofs show that the properties (b’) and (c) are preserved throughout the expansions, and hence complete the proof of Lemma 6.2.

**C.1. Global expansions of blue solid edges between external molecules.** It is trivial to see that the local expansions in Case 1 of Strategy C.2 do not break properties (b’) and (c) of the input graph \( G \) since no internal molecule is generated in this process. We only need to consider the global expansion of a blue solid edge between external molecules. First, replacing the relevant \( t \)-variable with \( E_{rr_{a,b_1,b_2}} \) in (3.5) gives an error term \( O(W^{-D}) \). Second, by using Claim C.4, it is easy to check that replacing the relevant \( t \)-variable with other non-\( Q \) graphs in (3.5) does not break properties (b’) and (c). Hence, we only need to consider the case where the blue solid edge is replaced by a \( Q \)-graph in (3.5). Suppose we have expanded a blue solid edge between molecules \( x_i \) and \( x_j \). After \( Q \)-expansions, we obtain the following three possible cases:

All these graphs are molecular graphs, \( \Gamma \) indicates the subgraph component induced on newly generated internal molecules, and we have only drawn the relevant molecules and edges without showing all other details. In graph (1), a molecule in \( \Gamma \) is merged with an external molecule \( x_k \) due to the dotted edge between them; in graph (2), a blue solid edge between external molecules \( x_k \) and \( x_l \) is pulled to a molecule in \( \Gamma \); in graph (3), a red solid edge between external molecules \( x_k \) and \( x_l \) is pulled to a molecule in \( \Gamma \) and the cases (1) and (2) do not happen. Note that in case (3), \( \Gamma \) is a weakly isolated subgraph. After replacing its closure by a diffusive or free edge, the resulting graph trivially satisfies properties (b’) and (c). It remains to show that the graphs in cases (1) and (2) satisfy property (b’) and (c).

In case (1), the internal molecules in \( \Gamma \) are connected with \( x_k \) through paths of blue edges and paths of black edges, so both \( x_i \) and \( x_j \) are special redundant in the new graph. Hence, if either \( x_i \) or \( x_j \) is not
special redundant in the original graph, then we have at least one more special redundant molecule and at most one more special pivotal molecule \( x_k \) in the new graph. On the other hand, suppose \( x_i \) or \( x_j \) are both special redundant molecules in the original graph. Then, they are also special redundant in the new graph, and the internal molecules in \( \Gamma \) are connected with \( x_j \) through paths of black edges and connected with \( x_i \) through paths of blue edges. Hence, \( x_k \) is a special redundant molecule in the new graph. Together with the simple fact that \( x_i \) still connects to \( x_j \), it shows that the graph in case (1) satisfies properties (b') and (c).

In case (2), by Claim C.4, it is equivalent to consider the following graph (2.1) with \( \Gamma \) reduced to a vertex \( a \):

\[
(2.1) \quad x_i \quad \quad x_j \quad \quad x_j \quad \quad x_j
\]

Note that two of \( x_i, x_k \) and \( x_l \) can be chosen as special redundant. Hence, if at least two of them are not special redundant in the original graph, then the new graph will have two more special redundant molecules and at most two more special pivotal molecules. If only one of them is not special redundant in the original graph, the resulting graph will have one more special redundant molecule and at most one more special pivotal molecule \( x_j \). If all of them are special redundant in the original graph, then we need to consider two different cases depending on how we assign the types for the blue solid edges. First, suppose we replace one external blue solid edge, say \((a, x_i)\), with a diffusive edge as in graph (2.2). Then, putting \((a, x_i)\) into the black net and \((x_j, a)\) gives a graph where \( x_j \) becomes special redundant. Second, suppose we replace all external blue solid edges with free edges, then we rearrange them as in graph (2.3), where the molecule \( x_j \) becomes special redundant. Moreover, notice that in graphs (2.2) and (2.3), \( x_i \) still connects to \( x_j \) and \( x_k \) still connects to \( x_l \). Hence, the new graph in case (2) satisfies properties (b') and (c).

C.2. Local expansions. In this subsection, we consider the local expansions in Case 2 of Strategy C.2. We divide the following proof into three different cases.

Case I: Suppose that there are no isolated subgraphs in \( G \). It is easy to check that special redundant molecules after a local expansion are still special redundant. We only need to consider molecules that are not connected with internal molecules directly in the original graph, but later become neighbors of some internal molecules after local expansions. This may happen if: (i) an external molecule is merged with an internal molecule due to a newly added waved or dotted edges; (ii) a blue solid edge between external molecules is pulled to an internal molecule. In both cases, the property (c) for new graphs is simple to check, and we only need to examine the property (b').

In case (i), suppose an internal molecule \( a \) is merged with an external molecule \( x_k \). We claim that this molecule is connected with at least one redundant blue edge and one redundant black edge. Take the black net as an example. In the original graph, we pick a disjoint union of black spanning trees that connect all internal molecules to special pivotal external molecules. Suppose \( a \) connects to a special pivotal molecule \( x_i \) on a black tree with \( x_i \) being its root. Then, after merging \( a \) and \( x_k \), the black edge between \( a \) (i.e., \( x_k \)) and its parent on the tree is special redundant, because all children of \( a \) now connect to \( x_k \) and all ancestors of \( a \) still connect to \( x_i \) in the new graph. In case (ii), suppose a blue solid edge between external molecules is pulled to an internal molecule \( a \). Then, both the edges \((x_i, a)\) and \((x_a, a)\) are special redundant and hence \( x_i \) and \( x_j \) are special redundant molecules.

Case II: In this case, suppose we are performing local expansions on atoms in a maximal weakly isolated subgraph, say \( \Gamma \), between internal molecules. Then, we show that the operations (L1)-(L3) in Definition A.15 do not break the properties (b') and (c). If these operations do not involve molecules outside \( \Gamma \), then there is nothing to prove. Also, notice that operation (L2) is a special case of (L3) by taking \( M_1 = M_2 \). Hence, we only need to show that after the following five kinds of operations, the new graphs still satisfy properties (b') and (c). All these graphs are molecular graphs without red solid edges, \( \Gamma \) indicates the maximal isolated subgraph in the original graph, and the lower circle indicates the subgraph induced on molecules that are not in \( \Gamma \). Again, we have only drawn the relevant molecules and edges without showing all other details.
In graphs (1) and (2), a molecule in $\Gamma$ is merged with an internal molecule and an external molecule, respectively, where we have used a dotted edge to indicate this operation. In graph (3), a blue solid edge between internal molecules is pulled to a molecule in $\Gamma$. In graph (4), a blue solid edge between an internal molecule and an external molecule $x_i$ is pulled to a molecule in $\Gamma$. In graph (5), a blue solid edge between external molecules $x_i$ and $x_j$ is pulled to a molecule in $\Gamma$. For simplicity of presentation, in the following proof, whenever we refer to the “original graph”, we actually mean the original graph with the closure of $\Gamma$ replaced by a diffusive or free edge $(a,b)$.

Using Claim C.4, it is trivial to check that the graph in case (1) satisfies properties (b') and (c). In case (2), it is equivalent to consider the graph with $\Gamma$ reduced to a vertex:

Here, $x_i$, $x_j$ and $x_k$ are external molecules and we assume that $(x_j, d)$ is a special redundant edge and $(x_k, c)$ is a special pivotal edge in the original graph. We claim that the special redundant edges such as $(x_j, d)$ in the original graph are still redundant in the new graph, and one of the edges $(a, x_i)$ and $(b, x_i)$ is special redundant. First, if $(b, x_i)$ is replaced by a free edge, then it can play the role of a free edge $(a,b)$ in the original graph, so that $(x_j, d)$ and $(x_i, a)$ are special redundant in the new graph. Second, suppose $(b, x_i)$ is replaced by a diffusive edge. Assume that $(a,b)$ is used as a blue diffusive edge in the original graph for the property (b') to hold. Then, there exists a blue path between $a$ (or $b$) and a special pivotal edge (this is because by the generalized SPD property of the original graph, removing the closure of $\Gamma$ still gives a doubly connected graph). WLOG, suppose this path is between $b$ and $c$, denoted by the blue dashed edge in the above graph. Then, the edge $(x_i, b)$ is special redundant in the new graph, because the molecules $a$ and $b$ are connected through the edge $(a, x_i)$ from $a$ to (the equivalence class of) external molecules, the edge $(x_k, c)$, and the blue dashed path, where the edge $(b, x_i)$ is not used. The case where $(a,b)$ is used as a black diffusive edge in the original graph can be proved in the same way by using a black-blue symmetry (i.e., switching the colors of the diffusive edges in the above argument leads to a proof). Together with the fact that $a$ and $b$ are still connected in the new graph, the above arguments show that the properties (b') and (c) hold for case (2).

In case (3), reducing $\Gamma$ into a single vertex $y$ gives the following graph:

If the edge $(b, y)$ is replaced by a free edge, then it can play the role of a free edge $(a,b)$ in the original graph. If both $(c, y)$ and $(d, y)$ are replaced by free edges, then they can play the role of free edges $(a,b)$ and $(c,d)$ in the original graph. If $(b, y)$ is replaced by a diffusive edge while $(c, y)$ and $(d, y)$ are replaced by a free edge and a diffusive edge, respectively, then the free edge can play the role of a free edge $(c,d)$ and the two diffusive edges $(a,y)$ and $(b,y)$ replace the role of a diffusive edge $(a,b)$ in the original graph. With all these assignments, we can easily check that the special redundant molecules in the original graph are still special.
redundant in the new graph in all the above cases. It remains to consider the hardest case where the edges 
\((b, y), (c, y)\) and \((d, y)\) are all replaced by diffusive edges.

First, we assume that in the original graph, both \((a, b)\) and \((c, d)\) are used as blue diffusive edges in order for the property \((b')\) to hold. Without loss of generality, we assume that the molecule \(b\) connects to a special pivotal molecule, say \(x_j\).

We first consider a case such that in the original graph, \(c\) or \(d\) connects to a special pivotal molecule, say \(x_i\), through a blue path \((d, f)\) which does not pass the edge \((a, b)\); see graph (3.1) above. Then, if we put \((b, y)\) into the black net and \((a, y), (c, y)\) and \((d, y)\) into the blue net, \(a\) still connects to \(b\) through a blue path consisting of edges \((a, y), (d, y), (x_i, f)\) and \((x_j, e)\), the blue path from \(d\) to \(f\), and the blue path from \(c\) to \(b\). Thus, the graph (3.1) also satisfies property \((b')\) as the original graph. On the other hand, suppose in the original graph, every blue path from \(c\) or \(d\) to a special pivotal molecule, say \(x_j\), has to pass the edge \((a, b)\); see the graph (3.2) above. Then, if we put \((a, y)\) into the black net and \((b, y), (c, y)\) and \((d, y)\) into the blue net, \(a\) still connects to \(b\) through a blue path consisting of edges \((b, y), (c, y)\) and the blue path from \(a\) to \(c\). Thus, the graph (3.2) also satisfies property \((b')\) as the original graph. Second, suppose that in the original graph, \((a, b)\) and \((c, d)\) are used as blue diffusive edge and black diffusive edge, respectively, in order for the property \((b')\) to hold. Then, putting \((a, y)\) and \((b, y)\) into the blue net and \((c, y)\) and \((d, y)\) into the black net, the resulting graph still satisfies property \((b')\). Finally, the case where the original graph uses \((a, b)\) as a black diffusive edge can be handled in the same way by using a black-blue symmetry. Together with the simple fact that \(a, b\) are connected and \(c, d\) are connected in the new graphs, the above arguments show that the properties \((b')\) and \((c)\) hold for case (3).

In case (4), without loss of generality, we assume that the pulled edge, say \((x_i, c)\), is used as a special redundant edge or a special pivotal blue edge in the original graph. The case where \((x_i, c)\) is used as a special pivotal black edge in the original graph can be handled in the same way by using a black-blue symmetry. Reducing \(\Gamma\) into a single molecule \(y\), we get the following graph (4.1):

Second, if \((c, y)\) and \((x_i, y)\) are replaced by free edges and \((b, y)\) is replaced by a diffusive edge, then one free edge plays the role of a free edge \((a, b)\) and the other one plays the role of a free edge \((x_i, c)\) in the new graph (4.3), and we see that the original special redundant molecules are still special redundant. Third, if \((c, y)\) is replaced by a free edge and \((b, y), (x_i, y)\) are replaced by diffusive edges, then the free edge can play the role of a free edge \((x_i, c)\). If \((a, b)\) is used as a blue diffusive edge in the original graph, then we put \((a, y)\) and \((b, y)\) into the blue net and \((x_i, y)\) into the black net as in graph (4.4); otherwise, we put \((a, y)\) and \((b, y)\) into the black net and \((x_i, y)\) into the blue net. In this way, the original special redundant molecules are still special redundant in the new graph. Fourth, if \((x_i, y)\) is replaced by a free edge and \((b, y), (c, y)\) are replaced by diffusive edges, then the free edge can play the role of a free edge \((x_i, c)\). If \((a, b)\) is used as a blue diffusive edge in the original graph, we put \((a, y), (b, y)\) into the blue net and \((c, y)\) into the black net as in graph (4.5); otherwise, we put \((a, y), (b, y)\) into the black net and \((c, y)\) into the blue net. In this way, the original special redundant molecules are still special redundant in the new graph. Fifth, suppose \((x_i, y), (b, y)\) and \((c, y)\) are all replaced by diffusive edges, and \((a, b)\) is used as a black diffusive edge in the
original graph for the property (b’) to hold. Then, we put \((a, y), (b, y)\) into the black net and \((c, y), (x_i, y)\) into the blue net as in graph (4.6). We see that the original special redundant molecules are still special redundant in the new graph, because \((a, y)\) and \((b, y)\) play the same role as \((a, b)\) in the original graph, while \((x_i, y)\) and \((c, y)\) play the same role as \((x_i, c)\) in the original graph.

![Graphs](image)

Sixth, suppose \((x_i, y), (b, y)\) and \((c, y)\) are all replaced by diffusive edges, \((a, b)\) is used as a blue diffusive edge, and \((x_i, c)\) is used as a special redundant edge in the original graph for the property (b’) to hold. Then, we put \((a, y), (b, y)\) into the blue net and \((c, y)\) into the black net, and we see that the original special redundant molecules are still special redundant in the new graph. Seventh, suppose \((x_i, y), (b, y)\) and \((c, y)\) are all replaced by diffusive edges, \((a, b)\) is used as a blue diffusive edge, and \((x_i, c)\) is used as a special pivotal blue diffusive edge in the original graph for the property (b’) to hold. Without loss of generality, suppose \(a\) connects to a special pivotal edge \((x_j, d)\) through a blue path as in graph (4.7). (The molecule \(d\) can be \(c\), in which case the following proof still works.) If we put \((a, y)\) into the black net and \((b, y)\), \((c, y)\) and \((x_i, y)\) into the blue net, then \(a\) still connects to \(b\) through a blue path consisting of edges \((b, y)\), \((x_i, y)\) and \((x_j, d)\), and the blue path from \(a\) to \(d\). Moreover, \((x_i, y)\) and \((c, y)\) can play the same role as \((x_i, c)\) in the original graph. Hence, the original special redundant molecules are still special redundant in the new graph (4.7).

Combining all the above arguments with the simple fact that \(a, b\) are connected and \(x_i, c\) are connected in the new graphs, we obtain that the properties (b’) and (c) hold for case (4).

Finally, in case (5), we consider the following graph (5.1) with \(\Gamma\) reduced into a single molecule \(y\).

![Graphs](image)

First, if \((b, y)\) is replaced by a free edge, then it can play the same role as the free edge \((a, b)\) in the original graph. In the new graph (5.2), either \(x_i\) or \(x_j\) can be chosen as special redundant, and hence the property (b’) holds. Second, suppose \((b, y)\) is replaced by a diffusive edge and \((a, b)\) is used as a black diffusive edge in the original graph for the property (b’) to hold. Then, putting \((a, y), (b, y)\) into the black net gives a new graph (5.3) in which either \(x_i\) or \(x_j\) is special redundant, and hence the property (b’) holds. Third, suppose \((b, y)\) is replaced by a diffusive edge and \((a, b)\) is used as a blue diffusive edge in the original graph for the property (b’) to hold. Without loss of generality, suppose \(a\) connects to a special pivotal edge \((x_k, c)\) through a blue path as in graph (5.4). Then, putting \((a, y)\) into the black net and \((b, y)\) into the blue net gives a graph in which either \(x_i\) or \(x_j\) is special redundant, because \(a\) still connects to \(b\) through a blue path consisting of \((b, y)\), \((x_k, c)\), \((x_i, y)\) or \((x_j, y)\), and the blue path from \(a\) to \(c\). Hence, the graph (5.4) satisfies the property (b’). Together with the simple fact that \(a, b\) are connected and \(x_i, x_j\) are connected in the new graph, the above arguments show that the properties (b’) and (c) hold for case (5).

**Case III:** In this case, suppose we are performing local expansions on atoms in a maximal weakly isolated subgraph, say \(\Gamma\), between a pair of internal and external molecules. Similar to Case II, we need to show that the operations (L1)–(L3) involving molecules outside \(\Gamma\) do not break the properties (b’) and (c). More precisely, we will show that after the following five kinds of operations, the new graphs still satisfy properties (b’) and (c):
In these graphs, $\Gamma$ indicates the maximal isolated subgraph, the lower circle indicates the subgraph induced on molecules that are not in $\Gamma$, and we have only drawn the relevant molecules and edges without showing all other details. In graphs (1) and (2), a molecule in $\Gamma$ is merged with an internal molecule and an external molecule, respectively. In graph (3), a blue solid edge between internal molecules is pulled to a molecule in $\Gamma$. In graph (4), a blue solid edge between an internal molecule and the external molecule $x_i$ is pulled to a molecule in $\Gamma$. In graph (5), a blue solid edge between the external molecules $x_i$ and $x_j$ is pulled to a molecule in $\Gamma$. Again, for simplicity of presentation, the “original graph” in the following proof refers to the original graph where the closure of $\Gamma$ is replaced by a diffusive or free edge $(x,a)$.

The cases (1) and (2) are almost trivial by using Claim C.4, and the case (3) can be handled in a similar way as the above Case II-(4). If the edge $(x,a)$ is special redundant in the original graph, then treating the external molecule $x$ as an internal molecule, the current Case III becomes a special cases of Case II above. Furthermore, if the blue solid edge between $x$ and $\Gamma$ is replaced by a free edge, then it can be used as a free edge $(x,a)$ in the original graph and the proof will be simple, so we omit the details. It remains to study cases (4) and (5) with $(x,a)$ being special pivotal and the blue solid edge between $x$ and $\Gamma$ replaced by a diffusive edge.

In case (4), reducing $\Gamma$ into a single molecule $y$ gives the following graph (4.1), where a blue solid edge $(x_i,b)$ is pulled to $y$:

First, if $(b,y)$ and $(x_i,y)$ are replaced by free edges, then they can play the same role as the free edges $(x,a)$ and $(x_i,b)$ in the original graph. Then, putting $(x,y)$ and $(a,y)$ into the blue and black nets, respectively, gives a new graph (4.2) that satisfies the property (b'). Second, if $(b,y)$ is replaced by a diffusive edge and $(x_i,y)$ is replaced by a free edge, then the free edge plays the same role as the free edge $(x_i,b)$ in the original graph. Moreover, if $(x,a)$ is used as a blue diffusive edge in the original graph for the property (b') to hold, then we put $(x,y)$ and $(a,y)$ into the blue net and $(x_i,y)$ into the black net as in graph (4.3); otherwise, we put $(x,y)$ and $(a,y)$ into the black net and $(b,y)$ into the blue net. In this way, $(x,y)$ and $(a,y)$ together will replace the role of $(x,a)$ in the original graph, and hence the new graph will satisfy the property (b').

Now, suppose $(b,y)$ and $(x_i,y)$ are both replaced by diffusive edges. Without loss of generality, we assume that $(x,a)$ is used as a blue diffusive edge in the original graph, while the case with $(x,a)$ used as a black diffusive edge in the original graph can be handled in the same way by using a black-blue symmetry. On the one hand, if $(x_i,b)$ is used as a black diffusive edge in the original graph for the property (b') to hold, then we put $(x,y)$, $(a,y)$ into the blue net and $(x_i,y)$, $(b,y)$ into the black net. The new graph (4.5) will satisfy the property (b').
On the other hand, suppose \((x_i, b)\) is used as a blue diffusive edge in the original graph for the property \((b')\) to hold. Without loss of generality, assume that \(b\) is connected with a blue special pivotal edge \((x_j, c)\) in the original graph, which is represented by the blue path in graph (4.6) (this pivotal edge can be \((x, a)\) in the original graph, in which case the following proof still works). Then, putting \((a, y)\), \((x, y)\) and \((x_i, y)\) into the blue net and \((b, y)\) into the black net as in graph (4.6), the resulting graph will satisfy the property \((b')\). This is because in the new graph, \(b\) still connects to \(x_i\) (which belongs to the same equivalence class of external molecules as \(x_j\)) through a blue path consisting of the blue path from \(b\) to \(c\) and the edge \((x_j, c)\).

Combining the above arguments with the simple fact that \(x, a\) are connected and \(x_i, b\) are connected in the new graphs, we see that the properties \((b')\) and \((c)\) hold for case (4).

Finally, consider case (5) with \((x, a)\) being special pivotal in the original graph and the blue solid edge between \(x\) and \(\Gamma\) replaced by a diffusive edge. Reducing \(\Gamma\) into a single molecule \(y\) gives the following graph (5.1), where a blue solid edge \((x_i, x_j)\) is pulled to \(y\):

If \((x, a)\) is used as a black diffusive edge in the original graph for the property \((b')\) to hold, then it is obvious that either \(x_i\) or \(x_j\) can be chosen as special redundant. We now consider the harder case where \((x, a)\) is used as a blue diffusive edge in the original graph for the property \((b')\) to hold. First, if \((x_i, y)\) and \((x_j, y)\) are both replaced by free edges, then we treat them as free edges \((x_i, a)\) and \((x_j, a)\) as in graph (5.2). Note that one of them can play the same role as the free edge \((x, a)\) in the original graph, while the other one is redundant if we put \((x, y)\) into the blue net and \((y, a)\) into the black net. This shows that at least one of \(x_i\) and \(x_j\) is free redundant, and hence the new graph (5.2) satisfies the property \((b')\). Second, suppose one of \((x_i, y)\) and \((x_j, y)\) is replaced by a free edge and the other one is replaced by a diffusive edge. Without loss of generality, let the free edge be \((x_j, y)\). If we treat it as a free edge \((x_i, a)\) and put \((x, y)\) and \((a, y)\) into the blue net and \((x_i, y)\) into the black net as in graph (5.3), then the edge \((x_j, a)\) is special redundant. On the other hand, if we put \((x, y)\) into the black net and \((y, a)\) into the blue net, then \((x_j, a)\) can replace the role of \((x, a)\) in the original graph and the edge \((x_i, y)\) becomes special redundant. In either case, at least one of \(x_i\) and \(x_j\) is special redundant, and hence the new graph (5.3) satisfies the property \((b')\). Finally, suppose \((x_i, y)\) and \((x_j, y)\) are both replaced by diffusive edges. Then, putting \((x, y)\), \((a, y)\) into the blue net and \((x_i, y), (x_j, y)\) into the black net as in graph (5.4), one of \(x_i\) and \(x_j\) becomes special redundant, and hence the property \((b')\) still holds. Together with the simple fact that \(x, a\) are connected and \(x_i, x_j\) are connected in the new graphs, we see that the properties \((b')\) and \((c)\) hold for case (5).

C.3. Global expansions of blue solid edges. In Case 3 of Strategy C.2, the blue solid edge we are expanding is either between internal molecules or between a pair of external and internal molecules. First, replacing the relevant \(t\)-variable with \(Er\alpha\beta\gamma\beta\) in (3.5) gives an error term \(O(W^{-D})\). Second, using Claim C.4, we see that replacing the relevant \(t\)-variable with other non-Q graphs in (3.5) does not break properties \((b')\) and \((c)\). Finally, suppose we replace the \(t\)-variable by a Q-graph in (3.5). Then, it is easy to see that the resulting graph before applying Q-expansions satisfies properties \((b')\) and \((c)\) by replacing the closure of the maximal isolated subgraph by a diffusive or free edge. Since Q-expansions are local expansions, all new graphs after them satisfy properties \((b')\) and \((c)\) by the arguments in Section C.2.
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