A Primal-Dual Algorithm for Link Dependent Origin Destination Matrix Estimation

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Abstract—Origin-Destination Matrix (ODM) estimation is a classical problem in transport engineering aiming to recover flows from every Origin to every Destination from measured traffic counts and a priori model information. Taking advantage of probe trajectories, whose capture is made possible by new measurement technologies, the present contribution extends the concept of ODM to that of Link dependent ODM (LODM). LODM also contains the flow distribution on links making specification of assignment models, e.g., by means of routing matrices, unnecessary. An original formulation of LODM estimation, from traffic counts and probe trajectories is presented as an optimisation problem, where the functional to be minimised consists of five convex functions, each modelling a constraint or property of the transport problem: consistency with traffic counts, consistency with sampled probe trajectories, consistency with traffic conservation (Kirchhoff’s law), similarity of flows having similar origins and destinations, and positivity of traffic flows. A proximal primal-dual algorithm is devised to minimise the designed functional, as the corresponding objective functions are not necessarily differentiable. A case study, on a simulated network and traffic, validates the feasibility of the procedure and details its benefits for the estimation of an LODM matching real-network constraints and observations.

Index Terms—convex optimisation, proximal primal-dual algorithm, traffic flows, origin destination matrices

I. INTRODUCTION

The estimation of traffic flows is a keystone for understanding network usage and behaviour in specific situations, e.g., network has a limited capacity or traffic may significantly vary with time or with particular events. Estimating traffic flows is thus needed for the network efficiency analysis, for traffic prediction, and traffic optimisation. Origin-Destination matrices (ODM) estimation is one of the classical problem in transport engineering [3] but also in the study of Internet traffic [4], [5], [6]. ODM are double entry tables indexed by network zones or major nodes, whose elements contain the demand of traffic from origins indexed by rows, to destinations, indexed by columns. ODM can be recovered from traveller interviews directly. This is however a long, difficult and costly process. Thus, since the 70’s and as a consequence of the generalisation, in occidental cities, of the access to link counts (e.g., by magnetic/inductive loops), many researches have sought to estimate the ODM with traffic counts as their primary source of data.

Estimating ODM from link counts. Formally, a road network is represented by a graph $G = (V, L)$ where the set of vertices $V$ consists of the road intersections (possible origin or destination) and the set of edges $L$ is the set of direct itineraries between intersections in $V$. The edges of this graph are directed. The corresponding ODM is $T$ of size $|V| \times |V|$. Magnetic loops, on links $l \in L$, produce $|L|$ measures represented by vector $q$. Thus, ODM estimation problem amounts to solving the following inverse problem:

$$q = F(T) + \epsilon$$

where the assignment function $F$ relates OD flows to network link, for comparisons against traffic counts $q$, and where $\epsilon$ models the measurement error. The two main difficulties in solving Problem (1) stem, first, from its being ill-conditioned: the size of the quantity to be estimated $T$ is larger than that of the available measures $q$ and second, from $F$ being unknown and thus often modelled.

To solve Problem (1), a common approach is to rely on the so-called four-step model [7], which consists of Trip Generation, Trip Distribution, Modal Split and Trip Assignment. This model has two major stages: first, Trip Generation and Distribution permit to design $T$; second, the Assignment amounts to specifying $F$, e.g., by means of a routing matrix. The present article aims for the direct estimation of the LODM, and does not need such distinction. It is therefore beyond the scope of the present contribution to review the many variations on the design of $T$ and of its assignment. Interested readers are referred to e.g., [3], [8], [9], [10], [11], [12], [13], [14] and references therein.

Goals, contributions and outline. Despite the fact that $T$ is of size $|V| \times |V|$, solving (1) is in fact an inverse problem of size $|V| \times |V| \times |L|$, because of the required assignment step that actually routes each OD on links of the network. The goal of the present contribution is to directly account for the real dimensionality of the problem by proposing a new and original description tool for traffic that directly includes assignment information: the Link dependent Origin Destination Matrix (LODM). LODM represents the OD flows already assigned to each link of the network, thus incorporating the assignment, or equivalently making

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1The Modal Split’s interest lies when one consider several modes of transport. Here however, the focus is on car trips only and this step is ignored.
its independent specification unnecessary. We also propose to estimate LODM as an inverse problem of dimension $|V| \times |V| \times |L|$. We rely on traffic counts $q$ and, in addition, on a new set of data: a set of trajectories, that is, for a fraction of the users, the set of roads used to travel from origin to destination. Trajectories are interpreted as a sample of the LODM. Trajectory collection is now made possible by new technologies such as GPS [15], Bluetooth [16], [17], [18]. Floating car data [19]. Section II formalises the transport problem, from its engineering perspective. Section III details five significant properties imposed either by the network or for consistency with the observed data; it turns them into five components of an objective function that formalises LODM estimation from traffic counts and sampled trajectories. Because these five functions are convex but not necessarily differentiable, a proximal primal-dual algorithm is devised to minimise the objective function. To finish with, the feasibility of the proposed approach and the assessment of its estimation performance are investigated in Section IV, on a case study consisting of network and traffic simulations, designed to match various parameters of road network and traffic in large western metropolitan cities.

**Notations.** The following notations are used throughout this article: $X$, $X$ and $X$ refer to vectors, matrices and tensors, respectively. The Hadamard product (element-wise product) of $Y$ and $X$ is denoted $Y \circ X$. Subscript indices are used for dimensions over the nodes of the graph and the index $i$ is used to label origins, $j$ to label destinations, $k$, $m$, $n$ and $p$ to label nodes in general. Superscript indices are used for dimensions over the links and the indexes $l$ and $e$ are favoured.

The symbol $\bullet$ is used to denote the dimension that does not contribute to a sum: e.g., the sum over first and third dimensions, indexed respectively with $i$ and $l$, is written $\sum_{i,l} X_{i,l}$.

We denote by $\| \cdot \|$ the element-wise norm for matrices: e.g., $\|X\|_1 = \sum_{i,j} |X_{i,j}|$ and $\|X\|_2 = \left( \sum_{i,j} X_{i,j}^2 \right)^{\frac{1}{2}}$.

### II. Road Network and Link-Dependent ODM

#### A. The problem

The network is described as a graph $G = (V, L)$. The number of road users is denoted $N$. A schematic (small) graph is illustrated in Fig. 1(a).

On such a graph, LODM consists of a tensor of size $|V| \times |V| \times |L|$, labelled $Q = (Q_{lj})_{(i,j) \in E \times l \in L}$. As illustrated in Fig. 1, each trajectory adds a count of 1 in $Q_{lj}$ if the link $l$ is on the origin-destination path $(i, j)$. Therefore, $Q$ consists, for each link $l \in L$, in an OD matrix of size $|V| \times |V|$.

To perform the estimation of $Q$, we use information stemming from probe trajectories as well as traffic count on each link. The set of trajectories can be measured from various sources (GPS, Bluetooth, ...) and the actual technology matters little in the procedure. We propose here, though, to refer to the Bluetooth technology, which is of great interest as it currently provides trajectory datasets with the highest penetration rates, compared to other technologies; the penetration rate is the fraction of vehicles equipped with the chosen technology and from which information needed to reconstruct trajectory can be collected [15]. Trajectory information is stored into a tensor labelled $B$, of size $|V| \times |V| \times |L|$. $B$ can be read as a LDM consisting in a sampled version of $Q$, for a fraction of the total traffic. Traffic counts, labelled $q$, consists of the total volume of traffic on each link $l \in L$. $q$ is of size $|L|$. Traffic counts can be, for instance, measured by magnetic loops.

A variational approach will now be devised to estimate $Q^*$, the real LDM, by means of non-smooth convex optimisation. $B$ and $q$. The involved criterion represents on the one hand the relationships between the tensor $Q$ and the measures $(B, q)$ and, on the other hand, properties of the road network and traffic constraints (e.g., car conservation at intersections).

#### B. Structure of the graph and of the traffic

The structure of the graph is given by the incidence and excidence matrices denoted respectively $I$ and $E$ of size $|V| \times |L|$. These matrices describe the relations between the nodes and the edges, such that, for every $(k, l) \in V \times L$,

$$I_{k,l} = \begin{cases} 1 & \text{if the link } l \text{ is arriving to the node } k, \\ 0 & \text{otherwise}, \end{cases}$$

$$E_{k,l} = \begin{cases} 1 & \text{if the link } l \text{ is starting from the node } k, \\ 0 & \text{otherwise}. \end{cases}$$

Note that in graph theory, it is customary to name the difference $(I - E)$ as Incidence Matrix; however we need both matrices separately in this work.
Let us also define the tensors $I_1$ and $I_2$ (resp. $E_1$ and $E_2$) corresponding to the replication of $I$ (resp. $E$) such that,

$$\forall m \in V \quad (I_1)_{km}^l = \begin{cases} 1 & \text{if link } l \text{ is arriving to node } k, \\
0 & \text{otherwise}, \end{cases}$$
$$\forall k \in V \quad (I_2)_{km}^l = \begin{cases} 1 & \text{if link } l \text{ is arriving to node } m, \\
0 & \text{otherwise}. \end{cases}$$

(3)

Using these notations, we relate the LODM $Q$ to the classical OD matrix $T$ of size $|V| \times |V|$ where each element $T_{ij}$ contains the traffic flow originating from the node $i$ and having $j$ for destination as follows:

$$T = \sum_{\star l} E_1 \circ Q = \sum_{\star l} I_2 \circ Q. \quad (4)$$

C. Model, Measures and Estimates

In realistic networks, roads are ranked by transport engineers depending on several parameters (e.g., speed limit, capacity and priority at intersections). Usually, when road monitoring is planned, roads of highest ranks only are equipped with monitoring devices. Low rank roads are excluded from traffic studies for their traffic is low and not crucial to urban mobility. Thus, we consider an urban road network limited to highest rank roads only.

The set of users with their trajectories on those roads, are represented through the tensor $Q$, as described above and that we wish to estimate.

First, we assume that every road is equipped with magnetic loops, counting the number of cars using it. It implies therefore that every element in $q$ is known. This assumption is realistic in our case considering highest rank roads only. The magnetic loops are usually subject to counting errors and it is modelled here by a noise $\varepsilon$. Hence the measured quantity $q$ reads:

$$q = q^* + \varepsilon \quad (5)$$

where $q^*$ is the true traffic volumes.

Second, we also assume that Bluetooth devices are not turned on and off while users are travelling. If this assumption holds, the penetration rate can be defined per OD as the number of Bluetooth equipped vehicles divided by the total traffic for this particular OD and is denoted $\eta$ of size $|V| \times |V|$. $B$ appears as a noisy version of $Q^*$ for which the noise level depends on the penetration rate. The relation between the tensors $B$ and $Q^*$ can thus be modelled by a Poisson law, typically involved in counting processes. This leads to a model

$$\forall i,j \in V \times V \times L \quad B_{ij} = \mathcal{P}((\eta)i,jQ^*_{ij}). \quad (6)$$

where $\mathcal{P}$ is a Poisson law of parameter $(\eta)i,jQ^*_{ij}$.

III. VARIATIONAL APPROACH

Instead of using the traditional four-step model resolution, iterating over a process involving a priori information, modelling of the traffic, estimating the variables of interest, comparing to the observed measures and tuning the models, we propose here the use of a variational approach. Both our knowledge of the network and of the traffic states are included within an objective function that combines together five terms to be jointly minimised.

A. Objective Function

The terms of the objective function can be classified in three types: The first type, composed of functions [III-A1] [III-A2] and [III-A3] is aiming for consistency between measures and estimates. The second type, with function [III-A4] stems from the network topology. The third and last type, with function [III-A5] comes from an additional assumption based on our knowledge of transport networks.

1) Traffic Count Data Fidelity $f_{TC}$: Ensuring the consistency with traffic counts would require that Eq. (5) is satisfied. Moreover:

$$q^* = \sum_{ij \star} Q^* \quad (7)$$

Therefore, Eq. (5) becomes

$$q = \sum_{ij \star} Q^* + \varepsilon \quad (8)$$

The noise $\varepsilon$ and the true traffic LODM $(Q^*)$ are unknown. $Q^*$ is the quantity that is to be estimated and to do so, assuming that $\varepsilon$ is a random unbiased Gaussian noise, we look for the variable $Q$ minimising the negative log-likelihood derived from Eq. (8). Thus the first term of the global objective function to be minimised is the function $f_{TC}$ defined as:

$$f_{TC}(Q) = \|q - \sum_{ij \star} Q^*\|^2. \quad \text{(9)}$$

2) Poisson Bluetooth Sampling Data Fidelity $f_P$: Second, the consistency with Bluetooth measures, as modelled in Equation (6) requires the knowledge of the OD-dependent penetration rate $\eta_{ij}$. This information, of size $|V| \times |V|$, is not directly available from $q$ and $B$, therefore we introduce an approximation of this penetration rate of size $|L|$, noted $\eta$ and calculated as:

$$\eta = \sum_{ij \star} \frac{B_{ij}}{B}. \quad (10)$$

The resulting data fidelity term, denoted $f_P$, models the negative log-likelihood associated with the Poisson model [20]:

$$f_P(Q) = \sum_{ijl} \psi(B_{ij}^l, \eta_{ij}^lQ_{ij}^l) \quad (11)$$

where,

$$\psi(B_{ij}^l, \eta_{ij}^lQ_{ij}^l) = \begin{cases} B_{ij}^l \log \eta_{ij}^lQ_{ij}^l + \eta_{ij}^lQ_{ij}^l & \text{if } \eta_{ij}^lQ_{ij}^l > 0 \text{ and } B_{ij}^l > 0, \\
\eta_{ij}^lQ_{ij}^l & \text{if } \eta_{ij}^lQ_{ij}^l \geq 0 \text{ and } B_{ij}^l = 0, \\
+\infty & \text{if otherwise.} \end{cases} \quad (12)$$
3) Definition Domain Constraint $f_C$: Third, another term ensuring data consistency models that the total flow should be greater than the flow of Bluetooth enabled vehicles. It consists thus in imposing that $Q$ belongs to the following convex set $C$:

$$C = \{ Q = (Q_{ij})_{(i,j) \in \mathcal{V} \times V \times L} \in \mathbb{R}^{\mathcal{V} \times \mathcal{V} \times \mathcal{L}} \mid Q_{ij} \geq 0 \}.$$  

(13)

The corresponding convex function is the indicator function $\iota_C$:

$$f_C(Q) = \iota_C(Q) = \begin{cases} 0 & \text{if } Q \in C, \\ +\infty & \text{otherwise}. \end{cases}$$  

(14)

4) Kirchhoff’s Law $f_K$: This property is the classical law for flows on network, describing the conservation of cars at intersections. For each OD pair and at every node of the network, the number of cars is conserved when properly accounting for origins and destinations. For every origin $i \in \mathcal{V}$, destination $j \in \mathcal{V}$ and node $k \in \mathcal{V}$ of the network, this yields to,

$$\sum_l E^l_k Q^l_{ij} - \delta_{ij} T^l_{ij} = \sum_l I^l_k Q^l_{ij} - \delta_{ij} T^l_{ij}. \quad (15)$$

where $\delta_{ij}$ is the Kronecker delta, that is:

$$\forall (i,j) \in \mathcal{V} \times \mathcal{V} \quad \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases} \quad (16)$$

This constraint can then be summarized as

$$\forall (i,j,k) \in \mathcal{V}^3 \quad \sum_l A^l_{ijk} Q^l_{ij} = 0 \quad (17)$$

where the $|\mathcal{V}| \times |\mathcal{V}| \times |\mathcal{V}| \times |\mathcal{L}|$ tensor $A$ is defined as

$$A^l_{ijk} = (E^l_k - I^l_k) - (\delta_{ik} - \delta_{jk}) E^l_i. \quad (18)$$

It results in a convex function to be minimised:

$$f_K(Q) = \sum_{ijk} \left( \sum_l A^l_{ijk} Q^l_{ij} \right)^2. \quad (19)$$

Compared to our previous works [1], [2], here the Kirchhoff’s law is applied per OD pair, and not simply at a global scale. Indeed, the Kirchhoff’s law needs also to be satisfied at each node, independently of the origin and destination of the cars. Satisfying Equation (17) automatically implies that the Kirchhoff’s law used in [1], [2] is satisfied.

5) Total Variation $f_{TV}$: Finally, combining a transport and graph perspective, we assume that trips leaving from adjacent origins to reach same destination, or leaving same origin to reach adjacent destinations, are likely to use similar roads. This leads us to formulate the Total Variation (TV) as

$$f_{TV}(Q) = \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}} \omega_{ij'} |Q^l_{ij} - Q^l_{ij'}| + \sum_{j \in \mathcal{N}_i} \sum_{k \in \mathcal{L}} \omega_{jj'} |Q^l_{ij} - Q^l_{ij'}|.$$  

(20)

where $\mathcal{N}_i$ models the neighbourhood of $i'$ and where $\omega_{ij'}$ are positive weights on edges detailed in Equation (22). The use of the $\ell_1$-norm is justified for its edge preservation properties. Indeed, it has been shown in [21], [22] that the $\ell_1$-norm is adapted for cases where one seeks for spatial correlations while allowing some irregularities, e.g., edges, in image analysis. In this case, we want to allow some irregularities, e.g., for nodes in between two corridors where both could be a possible choice. Those nodes can be interpreted as edges in image analysis.

Equation (20) can further be simplified using a weighted effective incidence matrix, denoted $J$ defined as

$$\forall (k,l) \in \mathcal{V} \times \mathcal{L} \quad J^l_k = W^l (I^l_k - E^l_k) \quad (21)$$

and thus having a size $|\mathcal{V}| \times |\mathcal{L}|$, where each element $W^l$ denotes the weight for the link $l$. If $l$ is a link between $k$ and $m$ then:

$$W^l = \omega_{km} e^{-\frac{d_{lm}}{\alpha}}, \quad (22)$$

$d_{l}$ models the length of the link $l$ and $d_{0}$ is the average distance of the nodes. For the simulated network:

$$d_{0} = \sqrt{\frac{\text{GridWidth} \cdot \text{GridHeight}}{|\mathcal{V}|}}. \quad (23)$$

Eq. (20) can then be rewritten as

$$f_{TV}(Q) = \sum_l \| J^T Q^l \|^1_1 + \sum_l \| J^T (Q^l)^T \|^1_1. \quad (24)$$

where $Q^l$ models the $l$-th extracted matrix from $Q$. Its dimension is thus $|\mathcal{V}| \times |\mathcal{V}|$.

B. Algorithm

To sum up, the objective is to find an estimate of $Q^*$ satisfying

$$\hat{Q} \in \text{Argmin} \{ \gamma_{TC} f_{TC} + \gamma_{FP} f_{FP} + \gamma_C f_C + \gamma_K f_K + \gamma_{TV} f_{TV} \} \quad (25)$$

where $\gamma_\cdot$ are positive weights applied to the objectives and model their relative importance within the global objective.

All the five functions involved in Eq. (25) follow the usual assumptions required when dealing with convex optimisation tools: they are convex, lower-semicontinuous (l.s.c.) and proper. Moreover, both the functions $f_{TC}$ and $f_K$ are differentiable and their gradients are given below:

$$\nabla f_{TC}(Q) = \left( -2 \left( Q^l_{km} \right) \right)_{(i,j) \in \mathcal{V} \times \mathcal{V} \times \mathcal{L}} \quad (26)$$

and

$$\nabla f_K(Q) = \left( 2 A^l_{ijk} - \sum_l W^l A^e_{ijk} Q^e_{ij} \right)_{(i,j) \in \mathcal{V} \times \mathcal{V} \times \mathcal{L}}. \quad (27)$$

Their Lipschitz constants are denoted $\beta_{TC}$ and $\beta_K$ respectively [23]. The other three functions however are not differentiable and $f_{TV}$ involves a linear transformation $H$ such as:

$$f_{TV}(Q) = \| H(Q) \|_1 \quad (28)$$
where $H$ satisfies:

$$H : \mathbb{R}^{|V| \times |V| \times |L|} \to \mathbb{R}^{|L| \times |V| \times |L|} \times \mathbb{R}^{|L| \times |V| \times |L|}$$

and whose adjoint is

$$H^* : (R, S) \mapsto (\frac{1}{|L|} R^T)_{i \in L} + (\frac{1}{|L|} S^T)^T_{i \in L}.$$  \hfill (29)

In the following, we denote $\chi$ the norm of this operator. For further details about the way to compute this norm, the reader can refer to [25].

This optimisation problem is solved by means of a proximal primal-dual algorithm, as in [24], [25], [26], [27], which is particularly suited when the objective combines differentiable and non-differentiable functions along with linear operators. In such an iterative scheme, the non-differentiable functions are involved through their proximity operator [28] defined as:

$$(\forall u \in \mathcal{H}) \quad \text{prox}_f(u) = \arg \min_{x \in \mathcal{H}} f(x) + \frac{1}{2} \|u - x\|^2$$ \hfill (31)

where $\mathcal{H}$ denotes a real Hilbert space and $f$ a convex, l.s.c., proper function from $\mathcal{H}$ to $]-\infty, +\infty]$. For further details on proximal algorithms, the reader could refer to [29], [30], [31].

The proximity operator of the indicator of the convex set $H$ has a closed form expression as a projection [32]:

$$\text{prox}_{\gamma \chi f_C} (Q) = \begin{cases} P_C(Q) = \max(Q, B) & \text{if } \gamma_C > 0 \\ \mathbb{I} = Q & \text{if } \gamma_C = 0. \end{cases}$$ \hfill (32)

The proximity operator of the function, $f_p$, also has a closed form expression [20]:

$$\text{prox}_{\gamma_p f_p} (Q) = \left( \text{prox}_{\gamma_p \psi} \left( B_{ij}^t, \eta_j^t \right) \right)_{(ijl) \in V \times V \times L} = \left( \frac{Q_{ij} - \gamma_p \eta_j^t + \sqrt{Q_{ij} - \gamma_p \eta_j^t}^2 + 4 \gamma_p B_{ij}^t}{2} \right)_{(ijl) \in V \times V \times L}$$ \hfill (33)

The proximal operator of the sum of these two functions satisfies the following property [33]:

$$\text{prox}_{\gamma_C f_C + \gamma_p f_p} (Q) = P_C(\text{prox}_{\gamma_p f_p} (Q))$$ \hfill (34)

Last, the $\ell_1$-norm, applied to $H$, as in Eq. (28), also has a closed form expression for its proximity operator [34], [35], [36], [37]:

$$\text{prox}_{\gamma \ell_1} (R, S) = \begin{cases} \text{sign}(R) \max\{|R| - \gamma \ell_1, 0\}, & \text{if } R \geq 0 \\ \text{sign}(S) \max\{|S| - \gamma \ell_1, 0\}, & \text{if } S < 0 \end{cases}$$ \hfill (35)

The primal-dual proximal iterations designed for minimizing Eq. (25) are described in Algorithm 1. Under some technical assumptions regarding the domain of definition and the following condition [26] theorem (3.11):

$$\frac{1}{\tau} - \sigma \chi_H \geq \frac{\beta}{2}.$$ \hfill (36)

where the $\beta = \gamma_{TC} \beta_{TC} + \gamma_K \beta_K$ denotes the Lipschitz constant of $\gamma_{TC} f_{TC} + \gamma_K f_K$ and $\sigma > 0$, the sequence $(Q^{k+1})_{k \in \mathbb{N}}$ converges to a minimizer of Eq. (25). Algorithm 1 has one stopping criterion based on the convergence of the estimates. Yet, to limit computation time, we added a limit at $10^5$ iterations. This limit is seldom reached and results for which it has been reached are considered as non-feasible solutions.

IV. SIMULATED CASE STUDY

A. Experimental setup

1) Simulation context: To test and validate the proposed method, a simplified road network model has been created. This has been preferred to a real case study for three reasons: tractability, the possibility to access the ground truth and the opportunity to explore the behaviour of the method for varied conditions. However, the connectivity, the number of users and their OD patterns have been chosen to be consistent with those of real networks.

The number of nodes of the simulated network is $|V| = 50$ nodes. This number is kept relatively low to allow for a thorough exploration of the possible weights $\gamma$ of problem (25). For comparison, the Brisbane Bluetooth scanner network has around 900 intersections equipped with vehicle identification devices. Other works on ODM estimation consider often few tens of nodes ($\approx 100$ OD flows) [38], while very recent works considered up to 300 nodes [29].

For the simulation, nodes are first located randomly on a grid and then links are created while aiming for an average connectivity of $6$, a value shared by most of real road networks [40]. This is done first, by means of a minimum spanning tree (computed by the Kruskal’s algorithm [41]), then, by adding links randomly to the nodes with lower degree (sum of in and out edges) provided that the added links do not cross or repeat an existing one.

The number of users is fixed to $N = 10^5$. This leads to a mean flow per link of 3000 users. In big cities, it would correspond to around one hour of traffic during peak hours.

An origin (resp. destination) is randomly associated to each node, according to probabilities $p_O$ (resp. $p_D$), where $p_O(i)$ (resp. $p_D(i)$) is the probability of node $i$ to be an origin (resp. destination). We simulate a preferred direction of travel, to mimic a commuting pattern. To this end, $p_O$ is decreasing linearly with the X-axis of the grid while $p_D$ is increasing linearly. The shortest path from origin to destination is then attributed to each user.

For each OD pair, a Bluetooth penetration rate is drawn from a Gaussian distribution of mean 30% and standard deviation of 10% (and truncated to be between 0 and 1). This choice accounts for the variability of the ownership distribution of Bluetooth devices (which is not known) from one node to another, depending, as an example, on the wealth of the neighbourhoods of the node. The average is consistent with global penetration rates observed in Brisbane [42]. Each user has a probability equal to the Bluetooth penetration rate drawn for its OD of being equipped with a Bluetooth device. This gives us $P$ while the full set of trajectories gives $Q^*$ for ground
Algorithm 1 Primal Dual algorithm

Choose: $\gamma_T \geq 0$, $\gamma_K \geq 0$, $\gamma_T \gamma_V \geq 0$, $\gamma_P \in [0, 1]$, $\gamma_D \in [0, 1]$

Compute: $\chi_H$, $\beta = \gamma_T C + \gamma_K B_K$, if $\gamma_T \gamma_V = 0$, $\tau = \frac{0.9 \beta}{2 + \sigma}$, else choose $(\tau, \sigma)$ such as $\tau = \frac{0.9}{2 + \sigma \chi_H} \in [\frac{2}{3}, \frac{3}{2}]$

Set : $Q^0 = 0$, $(R^0, S^0) = (0, 0)$

For $k = 0, \ldots$:
1. $Q^{k+1} = Q^k - \tau (\gamma_T \nabla f_T(Q^k) + \gamma_K \nabla f_K(Q^k)) - \tau H^*(R^k, S^k)$
2. $Q^{k+1} = \text{prox}_{\gamma_T f_T}(Q^k)$
3. $(\tilde{R}^{k+1}, \tilde{S}^{k+1}) = \gamma H(2Q^k - \tilde{Q}^k) + (R^k, S^k)$
4. $(\tilde{R}^{k+1}, \tilde{S}^{k+1}) = (\tilde{R}^{k+1}, \tilde{S}^{k+1}) - \sigma \cdot \text{prox}_{\gamma_T / \sigma, \ell_i}(\frac{1}{\sigma} R^{k+1}, \frac{1}{\sigma} S^{k+1})$

Stop if: $\frac{\|Q^{k+1} - Q^k\|_2}{\|Q^{k+1}\|} < 10^{-6}$ OR $k > 10^5$

The measured traffic flow per link $q$ is obtained from $Q^*$, assuming the addition a noise $\xi$, for which each independent component is drawn from a Gaussian distribution $\mathcal{N}(q^*, r \cdot q^*)$. For consistency with the noise usually measured on magnetic loops [43], we take $r = 5\%$. Simulations have been carried out with Matlab 2014a on a Intel i7 at 3.00GHz, 16GB of RAM and Figure 2 illustrates the simulated case study with traffic counts on links ($q$) and the realisation of $p_O$ and $p_D$ for the $10^5$ users on the nodes.

2) Algorithmic parameters setup: The objective function (25) depends on five parameters $\gamma \geq 0$ and it can be shown that exploring $\gamma_C \in \{0; 1\}$, $\gamma_P \in \{0; 1\}$ and $\gamma \geq 0$ for the three remaining terms is enough to explore all the possible setups.

3) Performance evaluation: The efficiency of the estimation algorithm is assessed by comparing its results to the ground truth and the relevance of using such algorithm is established by comparing its best results with two naive LODM, estimated directly from the data at hand.

To compare estimates to the ground truth $Q^*$, we propose two indicators: First, the RMSE:

$$\text{RMSE}(\tilde{Q}) = \frac{\|\tilde{Q} - Q^*\|}{\|Q^*\|}.$$  \hfill (37)

The RMSE measures the standard deviation between the estimates and the ground truth. Second, to go beyond the simple RMSE, we use the Earth Movers’ Distance (EMD) to compare the distribution of the traffic flows between the estimated LODM and the ground truth. It is a metric often used for image comparison and its definition can be found in [44].

The naive estimates that can be directly computed from the observed data are denoted $\hat{Q}^0$ and $\hat{Q}^1$, computed as the Bluetooth LODM multiplied by the mean Bluetooth penetration rate over the whole network ($\bar{n}$), or over each link ($\bar{n}_i$) respectively, for every $(i, j, l) \in V \times V \times L$,

$$\hat{(Q^0)}_{ij} = \hat{n} B_{ij}^l \quad \text{where} \quad \hat{n} = \sum_{i,j,l} q / \sum_{i,j,l} B_{ij}$$

$$\hat{(Q^1)}_{ij} = \eta B_{ij}^l \quad \text{where} \quad \eta = q / \sum_{i,j,l} B_{ij}$$ \hfill (38)

Note that $\hat{Q}^0$ is related to the solution usually proposed in the literature about ODM estimation from Bluetooth data [45].

B. Results

1) Finding the best estimates: Solutions to problem (25) have been explored through a systematic exploration of the $\gamma$ values within the positive real numbers. This exploration aims to find a set of parameters $\gamma$ for which the estimate has minimal criteria and we denote by $Q_{RMSE}$ the estimate $\hat{Q}$

Figure 2. Simulated road network with the projection of $q$ on the links: the width is proportional to the flows, also correlated with the color. For nodes, the color distinguishes between origin (blue) and destination flows (red) and the diameter of the nodes is proportional to their value in $p_O$ and $p_D$. 

\begin{itemize}
    \item \text{Origin}
    \item \text{Destination}
\end{itemize}
minimising the RMSE and by $\hat{Q}^{\text{EMD}}$ the one minimising the EMD. That is:

$$\hat{Q}^{\text{RMSE}} \in \text{Argmin } \text{RMSE}(\hat{Q}),$$

(39)

$$\hat{Q}^{\text{EMD}} \in \text{Argmin } \text{EMD}(\hat{Q}).$$

(40)

As an example, Figure 3 illustrates a one dimensional cut of the evolution of the criterion RMSE, that is, as a function of $\gamma_{TC}$, the others being fixed. Similarly, Figure 4 illustrates the evolution of the EMD when varying $\gamma_{TV}$ only.

For the four estimates $\hat{Q}^0$, $\hat{Q}^1$, $\hat{Q}$ and $\hat{Q}$, Table I presents the values of the RMSE, and EMD indicators, along with the values of $f_{TC}$ (consistency with observed counts) and $f_K$ (conformity with Kirchhoff’s law). These two functions are chosen because they are the most important from a transport perspective. When applicable, the corresponding values for the $\gamma$ are indicated.

Table II demonstrates first, that results achieved with the algorithm outperform the naive solutions for the two indicators RMSE and EMD. This demonstrates that the idea of involving additional information to the observed data helps indeed to reach better estimates and justifies that the proposed algorithm actually makes sense. Yet, a remaining question indeed to reach better estimates and justifies that the proposed algorithm outperform the naive solutions.

The question of which of both solutions $\hat{Q}^{\text{RMSE}}$ and $\hat{Q}^{\text{EMD}}$ is the best remains open and amounts to choosing the best $\gamma$, a question left for future work.

![Table I

| $\gamma_{TC}$ | $\gamma_K$ | $\gamma_{TV}$ | RMSE | EMD | $f_K$ | $f_{TC}$ |
|---------------|------------|---------------|------|-----|------|--------|
| $\hat{Q}^0$   | 0.320      | 0.086         | 0    | 55  |      |        |
| $\hat{Q}^1$   | 0.307      | 0.069         | 1142 | 84  |      |        |
| $\hat{Q}^{\text{RMSE}}$ | 31.6 | 0.008 | 0.015 | 0.239 | 0.047 | 289 | 1 |
| $\hat{Q}^{\text{EMD}}$ | 1     | 0.025 | 0.027 | 0.244 | 0.045 | 133 | 28 |

2) Impact of each objective: Now that we have justified why the solutions stemming from Algorithm II are best, the question of the importance of each function can be raised. To answer such a question, Tables II(a) (resp. (b)) summarises the best RMSE values (resp. EMD) when only the objectives indexed by the rows and column are not set to zeros. Thus diagonal elements correspond to a single term in the objective function while the four others are set to zero and non-diagonal elements involve at most the two terms indexed by the row and column. For example, element (1,2) of Table II(a) corresponds to the best value of RMSE achieved for $\gamma_{TV} = \gamma_K = \gamma_{TC} = 0$ and values of $\gamma_K$ and $\gamma_{TC}$ evaluated on a grid. For those tables, light-grey cells correspond to estimates that could not outperform the naive estimates, and darker grey elements, cases for which Algorithm II reached the $10^5$ steps limit without convergence.

The observation of both tables (for RMSE and EMD) leads to the conclusion that neither term gives satisfactory results by itself. None of the diagonal elements outperform the naive solutions.

![Figure 3. RMSE as a function of $\gamma_{TC}$ for $\gamma_K = \gamma_{TV} = 0$ and $\gamma_{TC} = \gamma_P = 1$. Note that the first point on the left is for $\gamma_{TC} = 0$, to be distinguished from the logarithmic scale from the second point and after.

![Figure 4. EMD as a function of $\gamma_{TV}$ for $\gamma_K = 0$ and $\gamma_{TC} = \gamma_C = \gamma_P = 1$. Again, note that the first point on the left is for $\gamma_{TV} = 0$ whereas the rest is drawn on a logarithmic scale.](image-url)
estimates. To improve on those values, one must involve the Poisson assumption and either the traffic counts function or the Kirchhoff’s law. This means that one cannot obtain a good estimate of the traffic flows while there are not at least one term ensuring data fidelity along with a regularization term. The fact that the Poisson function seems to be the most important can be expected as $B$ brings the most information and this confirms the importance of probe trajectories to solve such traffic problem.

Thus in a second step, the Poisson function has been imposed ($\gamma_P = 1$) with either one or two extra functions and similar results are gathered in Tables III (a) and (b). In those tables, the indicator function corresponding to the projection on the convex set $f_C$ has also been imposed ($\gamma_C = 1$) for three reasons: First its additional computational cost is negligible compared to the other functions, second, it accelerates the convergence speed of the algorithm by reducing the number of steps required while having little impact on the values of the criteria at convergence. Last, it corresponds to the weakest assumption of our model: that the total flow is greater than the measured probe trajectories.

In this second scenario (with $\gamma_P = \gamma_C = 1$ and one or two additional functions), any estimate performs better than the naive one but for the case where only the total variation (TV) is added. Depending on whether a minimum is sought for RMSE or for the EMD, it is either the pair Kirchhoff’s law and TV, or the pair Traffic Counts and TV, that are the best suited to complement the Poisson assumption. In any case, best results, as summarised in Table I, are achieved when all functions are involved. As a last element of discussion, implementing the TV brings a huge additional computation cost (convergence reached in $\sim 4$ hours instead of $\sim 1$ hours), due to the computation of the linear operator $H$ and its adjoint. Yet it does improve the results (RMSE from 0.262 without TV to 0.239 and EMD from 0.067 to 0.045) in particular as the TV favours solutions with traffic on links without Bluetooth samples.

C. Results with fewer users on the networks

An important question for transport engineers is the one of dynamic estimation of the traffic. The approach presented in this article is static, yet the number of users involved in the process is highly dependent, in real network, on the studied time interval. As a first step toward pseudo-static estimation of the LODM, we reduced the number of users to $N = 10^4$. This corresponds to an average flow of 300 vehicles per link, that is, in a big city, 5 to 10 minutes of traffic during peak hours. It also corresponds to $\sim 4$ users per OD or, in average, 1.3 probe trajectories per path. We expect several paths not being sampled by the Bluetooth and therefore a decreased impact of the Poisson model. Table IV summarises the results and very similar conclusions can be drawn to those of Table I. These results are very encouraging as even in the limit cases, the estimates achieved with the algorithm are still an improvement.
with respect to the naive estimates.

| Q" | Q | RMSE | EMD | Jk | Jfc |
|----|----|------|-----|----|-----|
| 1.78 | 0.25 | 0.027 | 0.341 | 0.013 | 10.7 | 9.8 |
| 1 | 0.45 | 0.026 | 0.342 | **0.012** | 5.8 | 13.7 |

V. CONCLUSION

We have shown that the Link dependent Origin-Destination matrix is an interesting tool for traffic representation. Moreover we have evidenced that its estimation can be performed with a proximal primal dual algorithm and that the objective function to be minimised can be partially derived from natural properties of the problem (the consistency between measured and estimated traffic counts, the domain of definition and the Kirchhoff’s law). Then by adding a few sensible relationships, as for example, the Poisson sampling assumption and the similarities between nearby flows computed as the total variation, one can obtain from such method, estimates that outperform the naive solutions and that are relevant from a transport perspective. Improvements can still be sought, especially in the design of the objective function. Future works will investigate new implementations of the TV (e.g., with different weights or an anisotropic TV) and demonstrate that, if available, traffic turn fractions at intersections can be involved as an additional term to achieve even better results. Another trail for developing this problem is to look for an online algorithm as the one presented in [46] Section 5.2. Last, one could think about implementing time dependencies. This could be done by adding new relationships that would link successive estimations of the LODM or, alternatively, by using other methods: for example, a Kalman filter similarly to what has been done on traffic counts based ODM estimation [47], or also with supplementary data (e.g., Bluetooth [48] or other sensors [49]).

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