Quasitoric Manifolds with Invariant Almost Complex Structure

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Abstract

We prove that any quasitoric manifold $M$ admits a $T^n$-invariant almost complex structure if and only if $M$ admits a positive omniorientation. In particular, we show that all obstructions to existence of $T^n$-invariant almost complex structure on $M$ arise from cohomology of underlying polytope - and hence are trivial.

1 Introduction

Let $M^{2n}$ be an oriented closed compact manifold. We say that $M^{2n}$ is quasitoric over simple polytope $P$ ([8],[11]) if:

1. there is locally standard action of torus $T^n$ on $M^{2n}$ and

2. there exists a map $\pi : M^{2n} \rightarrow P^n$ which is a superposition of projection $M^{2n} \rightarrow M^{2n}/T^n$ and diffeomorphism of manifolds with corners $M^{2n}/T^n \rightarrow P$.

Any smooth projective toric variety gives the example of quasitoric manifold, but not vice versa. Any quasitoric manifold admits a canonical smooth structure ([11]).

Submanifolds of the form $\pi^{-1}(F_j)$, $j = 1 \ldots m$, where $F_j$ is a codimension one facet, are called characteristic submanifolds. An omniorientation of $M^{2n}$ is the orientation of $M^{2n}$ and all of its characteristic submanifolds. Every omniorientation determines a canonical $T^n$-invariant weakly complex structure on $M^{2n}$ ([11]), which may also be constructed in terms of characteristic map $\lambda : \mathbb{Z}^m \rightarrow \mathbb{Z}^n$.

Every smooth projective toric variety admits canonical $T^n$-invariant complex structure. One may ask the following question ([8], Prob. 7.6): find the
criterion for existence of $T^n$-invariant almost complex structure on quasitoric manifold $M^{2n}$ in terms of $\lambda$. In this paper we present the solution of this problem.

$T^n$-invariant weakly complex structure on manifold $M^{2n}$ with torus action $\alpha$ is an isomorphism $c_{\tau}$ of bundle $\tau(M^{2n}) \oplus \mathbb{C}^{m-n}$ with complex vector bundle $\xi$. The map

$$\xi \xrightarrow{c_{\tau}^{-1}} \tau(M^{2n}) \oplus \mathbb{C}^{m-n} \xrightarrow{d\alpha(t) \oplus I} \tau(M^{2n}) \oplus \mathbb{C}^{m-n} \xrightarrow{c_{\tau}} \xi$$

is complex-linear for every $t$.

If $v$ is a fixed point for $\alpha$, then there is a map $\tau_v(M^{2n}) \xrightarrow{i} \xi_v \xrightarrow{\pi'} \mathbb{C}^n$, where $i$ is an injection and $\pi'$ is a projection map along fixed subspace $\mathbb{C}^{m-n} \subset \xi_v$. The sign of $v$ is the sign of determinant of the map $\pi' \circ i$.

An equivalent definition of sign is as follows: we set $\text{sign}(v) = +1$, if the orientations of $\tau(M)_v$ determined by orientation of $M$ and orientations of characteristic submanifolds coincide, and $\text{sign}(v) = -1$, if not.

We call omniorientation positive, if signs of all fixed points are positive.

One can describe notion sign of a fixed point in terms of characteristic function $\lambda$. Let $e_j$ be a normal vector to facet $F_j$ directed to interior of $P$. Then $\text{sign}(v)$ is equal to $\det(e_j) \cdot \det \lambda_v$, where $\lambda_v$ is a matrix composed of columns of $\lambda$ that correspond to codimension one facets meeting in $v$.

**Theorem 1** Quasitoric manifold $M^{2n}$ admits a $T^n$-invariant almost complex structure if and only if it admits a positive omniorientation.

Moreover, the following holds.

**Theorem 2** Any $T^n$-invariant almost complex structure on $M$ is equivariantly equivalent to canonical equivariant weakly complex structure $[11]$.\[11\]

This paper only contains the proof of theorem 1; the proof of theorem 2 will be published later. (Up to date, the proof exists, but only in Russian).

### 2 Remarks and examples

As follows from [8], quasitoric manifold $M$ with positive omniorientation and corresponding weakly complex structure $J$ satisfies $c_n(J) = \chi(M)$. So by [2], the structure $J$ is equivalent to some almost complex structure on $M$, not necessarily $T^n$-invariant.
Every smooth projective toric variety $M$ admits canonical $T^n$-invariant complex structure and symplectic form. So by [4], there’s a moment map $\pi : M \to P$, where $P$ is a simple convex polytope. Since all characteristic submanifolds of $M$ are complex subvarieties, $M$ has positive omniorientation.

Note that in real dimension 6 there exist examples of non-projective smooth toric varieties (see [10]). Nevertheless, the quotient by the action of $T^3$ is still isomorphic to simple convex polytope. Finding an example of non-projective non-quasitoric smooth toric variety still remains an open problem.

Now we turn to examples of non-algebraic quasitoric manifolds. Denote by $\mathbb{C}P^2_k$ a connected sum of $k$ copies of $\mathbb{C}P^2$ with standard orientation and smooth structure. As follows from equivariant connected sum construction ([III]), all $\mathbb{C}P^2_k$’s are quasitoric. Also, if $k$ is odd, then $\mathbb{C}P^2_k$ admits a positive omniorientation. If $k$ is even, then $\mathbb{C}P^2_k$ doesn’t admit an almost complex structure.

**Theorem 2.1** ([III]) Let $M^4$ be an oriented smooth manifold with no boundary. Then $M$ admits an almost complex structure if and only if the number $td(M) = \frac{1}{4}(\chi(M) + \text{sign}(M))$ is integer. Here $\text{sign}(M) = (b_2^+(M) - b_2^-(M))$ is the signature of a cohomology intersection form on $M$.

We have $td(\mathbb{C}P^2_k) = \frac{k+1}{2}$ and this implies non-existence of almost complex structure on $\mathbb{C}P^2_k$ for even $k$.

From Theorem 1 we obtain that the manifold $\mathbb{C}P^2_k$ admits $T^2$-invariant almost complex structure, if $k$ is odd. But if $k \geq 3$, $\mathbb{C}P^2_k$ can’t be a toric variety since all toric varieties are rational and their Todd genus is equal to 1.

As follows from [9], the manifold $\mathbb{C}P^2_k$ doesn’t admit $T^2$-invariant symplectic structure, if $k \geq 2$.

**Theorem 2.2** ([9]) Let $M$ be a manifold equipped with a symplectic circle action with only isolated fixed points. Then $td(M) = 1$ if the action is Hamiltonian, and $td(M) = 0$ if not.

The proof of following result is based on Seiberg-Witten theory of four-dimensional manifolds.

**Theorem 2.3** ([7]) The manifold $\mathbb{C}P^2_k$ doesn’t admit a symplectic structure, if $k \geq 2$. 

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Hence, manifolds $\mathbb{C}P_k^2$ with $k \geq 3$, $k$ odd, are neither symplectic manifolds nor toric varieties – but they admit $T^n$-invariant almost complex structure by Theorem 1.

3 Proof of theorem 1

The “only if” part of Theorem 1 is obvious. If $J$ is an invariant almost complex structure on $M$, then all characteristic submanifolds of $M$ are $J$-invariant and all signs of fixed points are positive. Therefore, it remains to show that existence of positive omniorientation implies the existence of invariant almost complex structure.

3.1 Notations

Denote by $sk_i(P)$ the set of all $i$-dimensional facets of $P$. Let $\iota : P \to M$ be an embedding of $P$ to $M$ satisfying the following conditions:

1. $\pi \circ \iota = id$;

2. restriction of $\iota$ on $Int G$ is smooth for any facet $G \subset P$.

An example of such $\iota$ is given by composition $P \to P \times T^n \to M$, where the last map is factorization map (10).

We assume that $M$ admits $T^n$-invariant metric $g$ (see 3). Then our problem is equivalent to constructing an operator $J$ on $\tau(M)$ with following properties:

1. $J^2 = -1$;

2. $J$ commutes with $d\alpha(t)$ for every $t \in T^n$;

3. $J$ is orthogonal with respect to the metric $g$.

Recall that the space of orthogonal complex structures on oriented vector space $\mathbb{R}^{2i}$ is homeomorphic to $SO(2i)/U(i)$.

The words “structure on $X$” will mean “$T^n$-invariant complex structure on vector bundle $\tau(M)$ restricted on $X$”.

If $M_G = \pi^{-1}(G)$ is a $2i$-dimensional quasitoric submanifold in $M$, then we denote by $\xi_1 \ldots \xi_{n-i}$ two-dimensional vector bundles over $M$ corresponding to $(n-i)$ one-dimensional stationary toric subgroups $T_1 \ldots T_{n-i}$ of $M_G$. The fibers of $\xi_1 \ldots \xi_{n-i}$ over $M_G$ are $J$-invariant since $J$ commutes with $d\alpha$.

We’ll say that structure $J$ on $\pi^{-1}(Int G)$ respects omniorientation $o$ if it agrees with orientations of $\xi_1, \ldots, \xi_{n-i}$ determined by $o$. 
Let $V$ be a real oriented Euclidean vector space of even dimension. Denote by $\mathbb{J}(V)$ the space of all complex structures of $V$ respecting orientation and metric. The case of $V = \tau(M_G)|_x$, where $G \subset P$ is a face, $x \in \iota(G)$, will be the most important. Recall that choosing a basis in $V$ identifies $\mathbb{J}(V)$ with $SO(2i)/U(i)$, where $i = \dim V$. In particular, $\mathbb{J}(V)$ is simply connected.

We’ll denote by $\mathbb{J}_G$ the bundle with fiber $\mathbb{J}(\tau(M_G)|_x)$ over $\iota(G)$, which is associated with $\tau(M_G)|_{\iota(G)}$. Clearly, $\mathbb{J}_G$ is trivial. Consider the space $\text{Aut}(\mathbb{J}(\tau(M_G)|_x))$ of homeomorphisms $\mathbb{J}(\tau(M_G)|_x)$ onto itself induced by change of basis in the space $\tau(M_G)|_x$.

Let us fix an arbitrary trivialization of $\tau(M_G)$ over $\iota(G)$. Then it determines also a trivialization of associated bundle $\mathbb{J}_G$. Any other trivialization of $\mathbb{J}_G$ is then determined by an arbitrary continious map $G \to \text{Aut}(\mathbb{J}(\tau(M_G)|_x))$, where $x \in \iota(G)$ is some fixed point. Since $SO(2i)$ is connected, $\text{Aut}(\mathbb{J}(\tau(M_G)|_x))$ is also connected. We obtain that the space of trivializations of $\mathbb{J}_G$ determined by trivialization of $\tau(M_G)$ over $\iota(G)$ is also connected.

### 3.2 Positivity and one-dimensional facets

**Lemma 3.1** If $M_G \subset M$ is a quasitoric submanifold, then $\xi_j \perp M_G$ and $\xi_j \perp \xi_k$, if $j \neq k$.

□ Let $v \in \tau(M_G), v_j \in \xi_j$ and $v_k \in \xi_k$ be nonzero vectors in the point $x \in M_G$, and $t_\pi \in T_j$ be an element of toric subgroup $T_j$ corresponding to multiplication by $-1$. Since $T_j$ acts trivially on $\xi_k$ and $\tau(M_G)$, we have $g(v, v_j) = g(v, t_\pi v_j) = g(v, -v_j) = 0$ and $g(v_j, v_k) = g(t_\pi v_j, v_k) = g(-v_j, v_k) = 0$. □

Let $o$ be any omniorientation of $M$, not necessarily positive. Then $o$ determines structure $J$ on $\pi^{-1}(sk_0(P))$ in the following way: $J$ is rotation by angle $\pi/2$ in the direction specified by $o$ on every fiber $\xi_1|_x, \ldots, \xi_n|_x$, where $x \in \pi^{-1}(sk_0(P))$.

**Proposition 3.2** Let $J$ be a structure on $\pi^{-1}(sk_0(P))$ determined by $o$. Then $J$ may be extended over $\pi^{-1}(sk_1(P))$ if and only if $o$ is positive.

□ Let us prove the “if” part first. Consider $I \subset P$ – an edge connecting vertices $x_0 \in I$ and $x_1, T_1 \ldots T_{n-1}$ – toric subgroups corresponding to $I$. Then $\tau(M) \simeq \tau(\pi^{-1}(I)) \oplus \xi_1 \oplus \ldots \oplus \xi_{n-1}$ over $\pi^{-1}(I)$. The orientation of bundles $\xi_1 \ldots \xi_{n-1}$ over $\pi^{-1}(I)$ is determined by $o$.

Denote by $W_0$ and $W_1$ orthogonal complements to $\xi_1 \ldots \xi_{n-1}$ at the points $\pi^{-1}(x_0)$ and $\pi^{-1}(x_1)$ respectively. Since $J$ was constructed by $o$, $J$ is a
rotation by $\pi/2$ in spaces $W_0$ and $W_1$. Note that $W_0$ and $W_1$ are tangent spaces to $\pi^{-1}(I)$ at $\pi^{-1}(x_0)$ and $\pi^{-1}(x_1)$. The positivity of $o$ implies that orientations of $W_0$ and $W_1$ agree as orientations of $\tau(\pi^{-1}(I)) = \tau(S^2)$. So we can define $J$ on $\tau(\pi^{-1}(I))$ as rotation by $\pi/2$ in the direction specified by orientations of $W_0$ and $W_1$. On the bundles $\xi_1, \ldots, \xi_{n-1}$ structure $J$ is defined as rotation by $\pi/2$ in the direction specified by $o$.

The "only if" part of Prop. 3.2 is proved in similar fashion. If $J$ is defined on entire $\pi^{-1}(I)$, then the orientations of $M$ at $\pi^{-1}(x_0)$ and $\pi^{-1}(x_1)$ agree – and that holds for every pair of adjacent vertices $(x_0, x_1)$. □

### 3.3 Triviality of higher obstructions

Now we consider the case $i > 1$. Suppose that $J$ is defined on $\pi^{-1}(sk_{i-1}(P))$ and we’re trying to extend it over $\pi^{-1}(sk_i(P))$.

**Lemma 3.3** Let $G \subset P$ be an $i$-dimensional facet. The space of positive structures $J$ over $\pi^{-1}(\text{Int } G)$ is homeomorphic to space of continuous mappings $\text{Map}(\text{Int } G, \mathbb{J}(\tau(M_G)|_x))$, where $x \in \iota(\text{Int } G)$ is a fixed point.

□ The positivity of $J$ implies that $J$ is uniquely defined on bundle $\tau(\pi^{-1}(\text{Int } G))^\perp \simeq \xi_1 \oplus \ldots \oplus \xi_{n-i}$. So it suffices to define $J$ on the tangent vector bundle $\tau(\pi^{-1}(\text{Int } G)) \simeq \tau(M_G)|_{\text{Int } G}$.

Denote by $T^i$ any $i$-dimensional toric subgroup complementary to $T^{n-i}$ in $T^n$. Then $T^i$ acts freely on $\pi^{-1}(\text{Int } G)$ and the quotient by the action is homeomorphic to $\text{Int } G$ itself. Since $J$ is $T^n$-invariant structure, it suffices to define $J$ only on $\iota(\text{Int } G)$. The bundle $\tau(M_G)$ is trivial oriented 2$i$-dimensional real vector bundle over $\iota(\text{Int } G)$ – and so $J$ may be chosen in an arbitrary way over it (the only condition is that $J$ must be orthogonal with respect to the metric $g$). If we fix some trivialization of associated bundle $\mathbb{J}_G$ over $\iota(\text{Int } G)$, then $J$ is determined by an arbitrary continuous map $\text{Int } G \rightarrow \mathbb{J}(\tau(M_G)|_x)$, where $x \in \iota(G)$ is a fixed point. □

Note that the bundle $\tau(M_G)$ over $\iota(G)$ is also trivial. This implies existence of a canonical isomorphism of homotopy groups $\pi_*(\mathbb{J}(\tau(M_G)|_x))$ and $\pi_*(\mathbb{J}(\tau(M_G)|_y))$ for any points $x, y \in \iota(G)$. Recall that the homotopy groups $\pi_*(\mathbb{J}(\tau(M_G)|_x))$ are independent of choice of starting point if $\mathbb{J}(\tau(M_G)|_x)$, since $J(\tau(M_G)|_x)$ is simply connected.

Let us fix a point $x \in \iota(G)$ and trivialization of bundle $\tau(M_G)$ over $\iota(G)$. Then the bundle $\mathbb{J}_G$ is also trivialized. Since $J$ is already defined over $\tau(M)|_{\partial G}$ and $\tau(M_G) \subset \tau(M)$ is $J$-invariant subbundle, we obtain a continuous map $f : \partial G \rightarrow \mathbb{J}(\tau(M_G)|_x)$. Denote by $C_G$ the homotopy class of spheroid $f$ in $\pi_{i-1}(J(\tau(M_G)|_x))$. 

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Now we’ll show that the class $C_G$ is well-defined. Clearly, $C_G$ is independent of the choice of $x$. Suppose that we have changed trivialization of $\tau(M_G)$ over $\iota(G)$. Then new trivialization is given by a continuous map $\phi : G \to \text{Aut}(\mathbb{I}(\tau(M_G)|_x))$ (see subsection 3.1). The map $f$ is changed to $g : \partial G \to J(\tau(M_G)|_x)$, where $g(y) = \phi(y) \circ f(y)$.

Let $\phi_t : G \to \text{Aut}(\mathbb{I}(\tau(M_G)|_x))$ be a continuous family of maps satisfying $\phi_0 = id$, $\phi_1 = \phi$ (recall that $\text{Aut}(\mathbb{I}(\tau(M_G)|_x))$ is connected). Then family of maps $\phi_t(y) \circ f(y)$ provides a homotopy between $f$ and $g$. This completes the proof of that $C_G$ is well-defined.

Note that $C_G$ depends on the orientation of $G$.

We summarize all in the following statement.

**Lemma 3.4** Let $J$ be a structure on $\pi^{-1}(sk_{i-1}(P))$ respecting $o$. Then $J$ may be extended over $\pi^{-1}(sk_{i-1}(P)) \cup M_G$ if and only if $C_G = 0$ in the group $\pi_{i-1}(SO(2i)/U(i))$.

Every polytope $P$ has a canonical cellular decomposition, with facets playing the role of cells. We define cellular cochain $\sigma^i_j \in C^i(P, \pi_{i-1}(SO(2i)/U(i)))$ by the rule $\sigma^i_j(G) = C_G$. By definition, $\sigma^i_j$ is zero if and only if $J$ may be extended from $\pi^{-1}(sk_{i-1}(P))$ to $\pi^{-1}(sk_i(P))$.

Strictly speaking, to call $\sigma^i_j$ a cocycle we have to identify all homotopy groups $\pi_{i-1}(\mathbb{I}(\tau(M_G)|_x))$ for all $i$-dimensional facets $G \subset P$.

**Lemma 3.5** Let $\dim G = i$, $j \leq 2i - 2$, $x \in \iota(G)$, $y \in \iota(P)$. Then homotopy groups $\pi_j(\mathbb{I}(\tau(M_G)|_x))$ and $\pi_j(\mathbb{I}(\tau(M)|_y))$ are canonically isomorphic.

\[\square\]

Note that lemma 3.5 implies that $\sigma^i_j$ is a cochain, since $i - 1 \leq 2i - 2$. Since homotopy groups $\pi_j(\mathbb{I}(\tau(M)|_x))$ and $\pi_j(\mathbb{I}(\tau(M)|_y))$ are canonically isomorphic, one can assume that $x = y$.

Consider an arbitrary embedding of two facets $H \subset L$ such that $\dim H \geq i$, $\dim L = \dim H + 1$. Let $x \in \iota(H)$. Then there is an embedding of spaces of complex structures

$$C(H, L) : \mathbb{I}(\tau(M_H)|_x) \to \mathbb{I}(\tau(M_L)|_x),$$

defined by formula $J \to J \oplus t_{\pi/2}$. Here $t_{\pi/2}$ is a rotation by $\pi/2$ in two-dimensional orthogonal complement $\tau(M_H)^{\perp} \subset \tau(M_L)$. The direction of $t_{\pi/2}$ is specified by coorientation of $\tau(M_H)|_x$ in $\tau(M_L)|_x$.

If we fix the basis in $\tau(M_H)|_x$ and then add two vectors to obtain a basis in $\tau(M_L)|_x$, then $c(H, L)$ turns into a canonical embedding of homogeneous spaces $SO(2r)/U(r) \to SO(2r + 2)/U(r + 1)$, where $r = \dim H$.
Lemma 3.6 The map 
\[ c_* : \pi_j(SO(2r)/U(r)) \to \pi_j(SO(2r+2)/U(r+1)) \]
is an isomorphism for every \( j \leq 2r - 2 \).

The embedding \( c \) may be viewed as the composition 
\[ SO(2i)/U(i) \to SO(2i+2)/U(i) \to SO(2i+2)/U(i+1), \]
where first map is an embedding of fiber to bundle space over \( SO(2i+2)/SO(2i) \) and the second map is a projection map of bundle space with fiber \( S^{2i+1} \). An exact homotopy sequence now implies that \( c \) induces isomorphism of corresponding homotopy groups up to dimension \( 2i - 2 \). □

Now we consider an arbitrary chain of embeddings \( G = G_0 \subset \ldots \subset G_{n-i} = P \), where \( \dim G_{s+1} = \dim G_s + 1 \) for every \( s \). Define an isomorphism \( c_*(G, P) : \pi_j(\mathbb{J}(\tau(M_G)|_x)) \to \pi_j(\mathbb{J}(\tau(M)|_x)) \) by formula \( c_*(G, P) = c_*(G_{n-i-1}, G_{n-i}) \circ \ldots \circ c_*(G_0, G_1) \). Then \( c_*(G, P) \) is just what we need; it suffices to prove following statement.

Lemma 3.7 Isomorphism \( c_*(G, P) \) is independent on the chain \( G_0 \subset \ldots \subset G_{n-i} \).

Consider an arbitrary subchain of the form \( G_{s-1} \subset G_s \subset G_{s+1} \). Then there exists a unique facet \( Q \) such that \( G_{s-1} \subset Q \subset G_{s+1} \) and \( G_s \neq Q \). If \( x \in G_{s-1} \), then the following diagram of embeddings 
\[ \mathbb{J}(\tau(M_{G_{s-1}})|_x) \longrightarrow \mathbb{J}(\tau(M_G)|_x) \]
\[ \downarrow \]
\[ \mathbb{J}(\tau(M_Q)|_x) \longrightarrow \mathbb{J}(\tau(M_{G_{s+1}})|_x) \]
is commutative. So if we replace \( G_s \) with \( Q \), the resulting isomorphism \( c_*(G, P) \) won’t change.

We’ll call two chains connecting \( G \) and \( P \) equivalent, if one may be obtained from another by sequence of operations we’ve just described. Let us prove by induction on \( n - i \) that any two chains connecting \( G \) and \( P \) are equivalent. The base statement \( (n - i = 1) \) is obvious since the chain is unique. Now consider two different chains \( G = G_1^1 \subset \ldots \subset G_{n-i}^1 = P \) and \( G = G_0^2 \subset \ldots \subset G_{n-i}^2 = P \). Let \( Q \subset P \) be a facet satisfying \( \dim Q = i + 2 \), \( G_1^1 \subset Q \) and \( G_1^2 \subset Q \). Consider an arbitrary chain \( \zeta \) connecting \( Q \) and \( P \). By induction hypothesis, two chains \( G_1^1 \subset G_2^1 \subset \ldots \subset P \)
and \( G_1 \subset Q \subset \ldots \subset P \), where \( Q \) is connected with \( P \) by \( \zeta \), are equivalent. This means \( G \subset G_1 \subset G_2 \subset \ldots \subset P \) is equivalent to the chain \( G \subset G_1 \subset Q \subset \ldots \subset P \). Moreover, \( G \subset G_1 \subset Q \subset \ldots \subset P \) is equivalent to \( G \subset G_2 \subset Q \subset \ldots \subset P \), due to the choice of \( Q \). Finally, applying the induction hypothesis once more, we obtain that \( G \subset G_2 \subset Q \subset \ldots \subset P \) and \( G \subset G_2 \subset G_3 \subset \ldots \subset P \) are equivalent. \( \square \)

This lemma completes the proof of that \( \sigma^i_J \) is well-defined. We summarize this in the following statement.

**Lemma 3.8** Let \( J \) be a structure on \( \pi^{-1}(sk_{i-1}) \) respecting \( o \). Then one can define an obstruction cochain \( \sigma^i_J \in C^i(P, \pi_{i-1}(SO(2i)/U(i))) \) which is zero iff \( J \) may be extended to \( \pi^{-1}(sk_i(P)) \).

Our next aim is to prove that \( \sigma^i_J \) is a cocycle.

**Lemma 3.9.** Suppose that \( J \) is a structure that respects \( o \) and is defined on \( \pi^{-1}(sk_{i-1}(P)) \), \( Q \) is an \((i+1)\)-dimensional facet of \( P \). Then

\[
\sum_{G \subset \partial Q} \sigma^i_J(G) = 0.
\]

\( \square \) Lemma 3.5 guarantees that we can use a single notation \( c_\ast(G, Q) \) for all isomorphisms \( c_\ast(G, Q) : \pi_{i-1}(J(M_G)|_x) \to \pi_{i-1}(J(M_Q)|_y) \), \( x \in \iota(G) \), \( y \in \iota(Q) \).

The structure \( J \) on the bundle \( \tau(M_G)|_{\iota(G)} \) automatically defines structure \( c(J) \) on \( \tau(M_Q)|_{\iota(G)} \) by the formula \( J \to J \oplus t_{\pi/2} \). The bundle \( \tau(M_Q) \) may be trivialized over \( \iota(Q) \). Since \( \iota(G) \subset \iota(Q) \) for every \( G \), there is a well-defined continuous map \( f_J : Q \cap sk_{i-1}(P) \to \iota(Q) \), \( y \in \iota(Q) \) a fixed point. By its construction, homotopy class of \( f_J|_{\partial G} \) is equal to \( c_\ast \sigma^i_J \).

Let \( U_\varepsilon \) be closed \( i \)-dimensional tubular neighbourhood of \( Q \cap sk_{i-1}(P) \) in \( \partial Q \) (such that \( G \cap U_\varepsilon \neq \emptyset \) for every \( G \)). The set \( Q \cap sk_{i-1}(P) \) is a deformational retract of \( U_\varepsilon \), and the retraction determines a map \( f_J : U_\varepsilon \to SO(2i+2)/U(i+1) \). The space \( U_\varepsilon \) is homeomorphic to \( i \)-dimensional sphere with \( k \) holes; let \( S_1, \ldots, S_k \) be the boundary spheres of the these holes.
Then any homotopy class \( \tilde{f}_J|_{S_j} \) coincides with \( c_\ast \sigma_j^i(G) \), if \( S_j \subset G \). Since the sum of spheroids \( \tilde{f}_J|_{S_j} \) must be zero in \( \pi_{i-1}(SO(2i+2)/U(i+1)) \), this completes the proof. □

Lemma 3.10 Suppose that \( J \) is a structure on \( \pi^{-1}(sk_{i-1}(P)) \) that respects \( o \) and \( \sigma_j^i \) is a coboundary. Then one can change \( J \) on \( \pi^{-1}(sk_{i-1}(P)\setminus sk_{i-2}(P)) \) and obtain new structure \( J' \) on \( \pi^{-1}(sk_{i-1}(P)) \) such that \( \sigma_j^i = 0 \).

□ By Lemma 3.6 the map \( c : SO(2i-2)/U(i-1) \to SO(2i)/U(i) \) induces an isomorphism of homotopy groups up to dimension \( 2i-4 \). If \( i > 2 \), then \( i-1 \leq 2i-4 \), and if \( i = 2 \), then \( SO(4)/U(2) \) and \( SO(2)/U(1) \) are simply connected – so the map \( c_\ast : \pi_{i-1}(SO(2i-2)/U(i-1)) \to \pi_{i-1}(SO(2i)/U(i)) \) is an isomorphism for \( i > 1 \).

Let \( \sigma_j^i = \delta \beta \) and \( H \) be some \((i-1)\)-dimensional facet. The proof of lemma 3.5 implies that we can use a single notation \( c_\ast \) for every isomorphism of the form \( c_\ast(H,G_i) : \pi_{i-1}(\mathcal{J}(\tau(M_H)|_x)) \to \pi_{i-1}(\mathcal{J}(\tau(M_{G_i})|_y)) \), where \( x \in H, \ y \in G_i, \ \dim G_i = i \). Now apply lemma 3.3: the space of structures on \( \pi^{-1}(Int H) \) respecting \( o \) is homeomorphic to space of continuous maps \( Int H \to \mathcal{J}(\tau(M_H)|_x) \). Let \( f_H : Int H \to \mathcal{J}(\tau(M_H)|_x) \) be the map corresponding to \( J \).

Let us identify \( Int H \) with open \((i-1)\)-dimensional ball in \( \mathbb{R}^i \) of radius 1. Consider the map \( \tilde{f}_H : Int H \to \mathcal{J}(\tau(M_H)|_x) \) satisfying following conditions:

- \( \tilde{f}_H(x) = f_H((2|x| - 1) \cdot x), \) if \( 1/2 \leq |x| \leq 1 \);
- the homotopy class of spheroid \( \tilde{f}_H|_{\{|x| \leq 1/2\}} \) is \( (-c_\ast^{-1}\beta(H)) \):
If we replace \( f_H \) with \( \tilde{f}_H \), we’ll obtain new structure \( J' \) on \( \pi^{-1}(sk_i(P)) \). Then \( \sigma^i_J'(G) = \sigma^i_J(G) - \beta(H) \) for every \( i \)-dimensional facet \( G \) such that \( H \subset G \). So if we replace \( f_H \) with \( \tilde{f}_H \) for every \((i - 1)\)-dimensional facet \( H \subset P \), we’ll have \( \sigma^i_J' = 0 \), since \( \sigma^i_J = \delta \beta \). \( \square \)

3.4 Proof of the main theorem

Now we can prove theorem 1. Recall that we construct \( J \) by induction on \( i \), starting from \( \pi^{-1}(sk_0(P)) \). Proposition 3.2 and the positivity of \( o \) guarantee that \( J \) may be extended from \( \pi^{-1}(sk_0(P)) \) to \( \pi^{-1}(sk_1(P)) \). Lemmas 3.8 and 3.9 imply that for \( i > 1 \) obstruction to extending \( J \) from \( \pi^{-1}(sk_{i-1}(P)) \) to \( \pi^{-1}(sk_i(P)) \) is a cocycle \( \sigma^i_J \in C^i(P, \pi_{i-1}(SO(2i)/U(i))) \). Since \( P \) has trivial homology, \( \sigma^i_J \) is a coboundary, so by Lemma 3.10 there exists structure \( J' \) on \( \pi^{-1}(sk_{i-1}(P)) \) that may be extended to \( \pi^{-1}(sk_i(P)) \).

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References

[1] W.-T. Wu. Sur le classes caracteristique des structures fibrees spheriques // Actualites Sci. Industr. 1183 (1952).

[2] E. Thomas. Complex Structures on Real Vector Bundles // Amer. J. Math. 1967. V.89. P.887-908.

[3] Glen E. Bredon. Introduction to Compact Transformation Groups. Pure and Applied Math. 46, Academic Press, 1972.
[4] M. F. Atiyah. Convexity and Commuting Hamiltonians // Bull. London Math. Soc. 1982. V.14, N1. P. 1-15.

[5] V. Guilemin, S. Sternberg. Convexity properties of the moment mapping // Inv. Math. 67 (1982), N3, P. 491-513.

[6] M. Davis, T. Januskiewicz. Convex Polytopes, Coxeter Orbifolds and Torus Actions // Duke Math J. 1991. V.62 N2 P. 417-451.

[7] C.H.Taubes. The Seiberg-Witten Invariants and Symplectic Forms // Math. Res. Lett. 1994. V.1. P.809-822.

[8] T. E. Panov. Hirzebruch Genera of Manifolds with Torus Action // Izv. RAN. Ser. Mat., 65:3 (2001), 123138

[9] K.E.Feldman. Hirzebruch Genera of Manifolds Equipped with a Hamiltonian Circle Action. arXiv:math/0110028v2

[10] Victor M. Buchstaber, Taras E. Panov. Torus Actions and Their Applications in Topology and Combinatorics. AMS Bookstore, 2002.

[11] V. Buchstaber, T. Panov, N. Ray. Spaces of Polytopes and Cobordism of Quasitoric Manifolds // Moscow Math. J (2007) V.7 N2.