Abstract

We investigate the possible regular solutions of the boundary Yang-Baxter equation for the vertex models associated with the $C_n^{(1)}$, $D_n^{(1)}$ and $A_{2n-1}^{(2)}$ affine Lie algebras. We find three types of solutions with $n$, $n-1$ and 1 free parameters, respectively. Special cases and all diagonal solutions are presented separately.

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1 Introduction

The search for integrable models through solutions of the Yang-Baxter equation \[1, 2, 3\]

\[ R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v) \]  

(1.1)

has been performed by the quantum group approach in \[4\], where the problem is reduced to a linear one. Indeed, $R$-matrices corresponding to vector representations of all non-exceptional affine Lie algebras were determined in this way in \[5\].

A similar approach is desirable for finding solutions of the boundary Yang-Baxter equation \[6, 7\] where the boundary weights follow from

\[ R_{12}(u - v)K_1^{-}(u)R_{12}^{t_1 t_2}(u + v)K_2^{-}(v) = K_2^{-}(v)R_{12}(u + v)K_1^{-}(u)R_{12}^{t_1 t_2}(u - v), \]  

(1.2)

and the dual reflection equation

\[ R_{12}(-u + v)(K_1^{+})^{t_1}(u)M_1^{-1}R_{12}^{t_1 t_2}(-u - v - 2\rho)M_1(K_2^{+})^{t_2}(v) \]

\[ = (K_2^{+})^{t_2}(v)M_1R_{12}(-u - v - 2\rho)M_1^{-1}(K_1^{+})^{t_1}(u)R_{12}^{t_1 t_2}(-u + v). \]  

(1.3)

In this case the duality supplies a relation between $K^{-}$ and $K^{+}$ \[8\]:

\[ K^{-}(u) \rightarrow K^{+}(u) = K^{-}(-u - \rho)^{t}M, \quad M = U^{t}U = M^{t}. \]  

(1.4)

Here $t_i$ denotes transposition in the $i$-th space, $\rho$ is the crossing parameter and $U$ is the crossing matrix, both specific to each model \[9\].

With this goal in mind, the study of boundary quantum groups was initiated in \[10\]. These boundary quantum groups have been used to determine $A_{1}^{(1)}$ reflection matrices for arbitrary spin \[11\], and the $A_{1}^{(2)}$ and some $A_{n}^{(1)}$ reflection matrices were re-derived in \[12\]. More recently, reflection solutions from $R$-matrices corresponding to vector representations of Yangians and super-Yangians were presented in \[13\]. However, as observed by Nepomechie \[12\], an independent systematic method of constructing the boundary quantum group generators is not yet available. In contrast to the bulk case \[5\], one can not exploit boundary affine Toda field theory, since appropriate classical integrable boundary conditions are not yet known \[14\]. Therefore, it is still an open question whether it is possible to find all solutions of the reflection equations by using quantum group generators.

Independently, there has been an increasing amount of effort towards the understanding of two-dimensional integrable theories with boundaries via solutions of the reflection equation \[12\]. In field theory, attention is focused on the boundary $S$-matrix. In statistical mechanics, the emphasis has been laid on deriving all solutions of \[12\] because different $K$-matrices lead to different universality classes of surface critical behavior \[15\] and allow the calculation of various surface critical phenomena, both at and away from criticality \[10\].

Although being a hard problem, the direct computation has been used to derive the solutions of the boundary Yang-Baxter equation \[12\] for given $R$. For instance, we mention the solutions of the reflection equations of two-component system \[17, 18\], for 19-vertex models \[19, 20, 21, 22\], for $A_{n-1}^{(1)}$ models \[23, 24, 25\], for $D_{n+1}^{(2)}$ models \[26, 27\], for $A_{2n+1}^{(2)}$ and $B_{n}^{(1)}$ models \[28\]. Many diagonal solutions for face and vertex models associated with affine Lie algebras were presented in \[16\]. For A-D-E interaction-round face models, diagonal and some non-diagonal solutions were presented in \[29\]. Reflection matrices for Andrews-Baxter-Forrester models in the RSOS/SOS representation were presented in \[30\].

Here we will again touch this issue in order to include the last three non-exceptional Lie algebras, namely $C_{n}^{(1)}$, $D_{1}^{(1)}$, and $A_{2n-1}^{(2)}$, in our classification list \[25, 27, 28\].

We have organized this paper as follows. In Section 2 we choose the reflection equations and in Section 3 and 4 their solutions are derived. In Section 5 all diagonal solutions are presented and in Section 6 the special cases are discussed. The last Section is reserved for the conclusion.
2 Reflection Equations

Quantum $R$-matrices for the vertex models associated to the $C_n^{(1)}$, $D_n^{(1)}$ and $A_{2n-1}^{(2)}$ affine Lie algebras as presented by Jimbo in [5] have the form

$$R = a_1 \sum_{i \neq j, j'} E_{ii} \otimes E_{ij} + a_2 \sum_{i \neq j, j'} E_{ii} \otimes E_{jj} + a_3 \sum_{i,j, i \neq j} E_{ij} \otimes E_{ji}$$

$$+ a_4 \sum_{i,j, j'} E_{ij} \otimes E_{ji} + \sum_{i,j} a_{ij} E_{ij} \otimes E_{i'j'}$$  \hspace{1cm} (2.1)

where $E_{ij}$ denotes the elementary $2n \times 2n$ matrices $([E_{ij}]_{ab} = \delta_{ia} \delta_{jb})$ and the Boltzmann weights with functional dependence on the spectral parameter $u$ are given by

$$a_1(u) = (e^u - q^2)(e^u - \xi), \quad a_2(u) = q(e^u - 1)(e^{2u} - \xi),$$

$$a_3(u) = -(q^2 - 1)(e^u - \xi), \quad a_4(u) = e^u a_3(u)$$  \hspace{1cm} (2.2)

and

$$a_{ij}(u) = \begin{cases} (q^2 e^u - \xi)(e^u - 1) & (i = j) \\ (q^2 - 1) \xi i \xi j q^{-i-j} (e^u - 1) - \delta_{ij} (e^u - \xi) & (i < j) \\ (q^2 - 1) e^u \xi i \xi j q^{-i-j} (e^u - 1) - \delta_{ij} (e^u - \xi) & (i > j) \end{cases}$$  \hspace{1cm} (2.3)

where $q = e^{-2\eta}$ denotes an arbitrary parameter, $i = 2n + 1 - i$ and $\varepsilon_i = 1$ $(1 \leq i \leq n)$, $= -1$ $(n + 1 \leq i \leq 2n)$ for $C_n^{(1)}$ and $\varepsilon_i = 1$ for $A_{2n-1}^{(2)}$ and $D_n^{(1)}$.

Here $\xi = q^{2n+2}$ for $C_n^{(1)}$, $\xi = q^{2n-2}$ for $D_n^{(1)}$ and $\xi = -q^{2n}$ for $A_{2n-1}^{(2)}$ and

$$\tilde{i} = \begin{cases} i - 1/2 & (1 \leq i \leq n) \\ i + 1/2 & (n + 1 \leq i \leq 2n) \end{cases}$$  \hspace{1cm} (2.4)

for $C_n^{(1)}$, and

$$\tilde{i} = \begin{cases} i + 1/2 & (1 \leq i \leq n) \\ i - 1/2 & (n + 1 \leq i \leq 2n) \end{cases}$$  \hspace{1cm} (2.5)

for $A_{2n-1}^{(2)}$ and $D_n^{(1)}$.

These $R$-matrices are regular satisfying PT-symmetry and unitarity:

$$R(0) = a_1(0) P, \quad R_{21}(u) = P_{12} R_{12}(u) P_{12}, \quad R_{12}(u) R_{21}(-u) = a_1(u) a_1(-u) 1,$$  \hspace{1cm} (2.6)

where $P$ is the permutation matrix.

After normalizing the Boltzmann weights by a factor $\sqrt{e^u}$, the crossing-unitarity symmetry

$$R_{12}(u) = (U \otimes 1) R_{12}^t(-u - \rho)(U \otimes 1)^{-1},$$  \hspace{1cm} (2.7)

holds with the crossing matrices $U$ and crossing parameters $\rho$ given by

$$U_{ij} = \delta_{ij} q^{(\tilde{i} - \tilde{j})/2}, \quad \rho = -\ln \xi.$$  \hspace{1cm} (2.8)

Regular solutions of the reflection equation mean that the matrix $K^{-}(u)$ in the form

$$K^{-} = \sum_{i,j=1}^{2n} k_{i,j}(u) E_{ij}$$  \hspace{1cm} (2.9)
satisfies the condition
\[ k_{i,j}(0) = \delta_{i,j}, \quad i, j = 1, 2, \ldots, 2n. \] (2.10)

Substituting (2.8) and (2.10) into (1.2), we will get \((2n)^4\) functional equations for the \(k_{i,j}\) matrix elements, many of which are dependent. In order to solve them, we shall proceed in the following way. First we consider the \((i,j)\) component of the matrix equation (1.2). By differentiating it with respect to \(v\) and taking \(v = 0\), we get algebraic equations involving the single variable \(u\) and \(4n^2\) parameters
\[ \beta_{i,j} = \frac{dk_{i,j}(v)}{dv} \bigg|_{v=0} \quad i, j = 1, 2, \ldots, 2n. \] (2.11)

Next we denote these equations by \(E[i,j] = 0\) and collect them into blocks \(B[i,j], i = 1, \ldots, 2n^2\) and \(j = i, i + 1, \ldots, 2n^2 - i\), defined by
\[
B[i,j] = \begin{cases}
E[i,j] = 0, & i, j = 1, 2, \ldots, 2n^2 - i, \\
E[i,j] = 0, & i, j = 2n^2 - i + 1, \ldots, 2n^2 - 1,
\end{cases}
\] (2.12)

For a given block \(B[i,j]\), the equation \(E[4n^2 + 1 - i, 4n^2 + 1 - j] = 0\) can be obtained from the equation \(E[i,j]\) by interchanging
\[ k_{i,j} \leftrightarrow k_{i',j'}, \quad \beta_{i,j} \leftrightarrow \beta_{i',j'}, \quad a_3 \leftrightarrow a_4, \quad a_{ij} \leftrightarrow a_{i'j'}. \] (2.13)

and the equation \(E[j,i] = 0\) is obtained from the equation \(E[i,j] = 0\) by interchanging
\[ k_{i,j} \leftrightarrow k_{j,i}, \quad \beta_{i,j} \leftrightarrow \beta_{j,i}, \quad a_{ij} \leftrightarrow a_{j'i'}. \] (2.14)

Since each \(R\)-matrix (2.1) satisfies unitarity, P and T invariances and crossing symmetry, the correspondent \(K^+\) matrix is obtained from the \(K^-\) matrix using (1.4) with the following \(M\) matrix
\[ M_{ij} = \delta_{ij} q^{2n+1-2\gamma}, \quad i, j = 1, 2, \ldots, 2n. \] (2.15)

### 3 General Solutions

The challenge at this point is to find the full set of solutions of the reflection equations (1.2) and (1.3) for \(R\)-matrices associated to \(C_n^{(1)}\), \(D_n^{(1)}\) and \(A_{2n-1}\) affine Lie algebras. We will start by looking at \(K\)-matrix solutions with all entries different from zero which will be named general solutions.

#### 3.1 Non-diagonal matrix elements

Analyzing the reflection equation (1.2) with the \(R\)-matrix (2.1), we can see that the simplest functional equations are those involving only two matrix elements of the type \(k_{i,i'}\) (secondary diagonal). They belong to the blocks \(B[1,2n+3], B[1,4n+5], B[1,6n+7], \ldots\), and we choose to express their solutions in terms of the element \(k_{1,2n}\) with \(\beta_{1,2n} \neq 0:\)
\[ k_{i,i'} = \frac{\beta_{i,i'}}{\beta_{1,2n}} k_{1,2n}. \] (3.1)

Next, we look at the last blocks of the collection \(\{B[1,j]\}\). Here we can write the matrix elements of the first row \(k_{1,j}\) \((j \neq 1, 2n)\) in terms of the element \(k_{1,2n}\) and their transpose in terms of the element \(k_{2n,1}\). From the last blocks of the collection \(\{B[2n+3,j]\}\), the matrix elements of the second row \(k_{2,j}\) \((j \neq 2, 2n)\)
are expressed in terms of $k_{2n-1}$ and their transpose in terms of $k_{2n-2}$. Following this procedure with the collections \{B[4n+5, j]\}, \{B[6n+7, j]\}, ..., we will be able to write all non-diagonal matrix elements as:

\[
k_{i,j} = \left( \frac{a_1a_{11} - a_2^2}{a_3a_4a_{11}^2 - a_2^2a_1a_2} \right) \left( \beta_{i,j}a_3a_{11} - \beta_{j',i}'a_2a_{ij}' \right) \frac{k_{1,2n}}{\beta_{1,2n}} \quad (j < i')
\]

and

\[
k_{i,j} = \left( \frac{a_1a_{11} - a_2^2}{a_3a_4a_{11}^2 - a_2^2a_1a_2} \right) \left( \beta_{i,j}a_4a_{11} - \beta_{j',i}'a_2a_{ij}' \right) \frac{k_{1,2n}}{\beta_{1,2n}} \quad (j > i')
\]

where we have used (3.1) and the identities

\[
a_{ij} = a_{j',i}' \quad \text{and} \quad a_{1,1} = a_{12a_{21}} \quad (j \neq 1).
\]

Taking into account the Boltzmann weights of each model, we substitute these expressions in the remaining reflection equations and look at those without diagonal entries $k_{i,i}$, in order to fix some parameters $\beta_{i,j}$ ($i \neq j$). For instance, from the blocks $B[i, i]$ one can see that their equations are solved by the relations

\[
\beta_{i,j}k_{j,i} = \beta_{j,i}k_{i,j} \implies \beta_{i,j}\beta_{j',i}' = \beta_{j,i}\beta_{i',j'}.
\]

From the equations $E[2, j]$ and $E[j, 2]$ we can find two possibilities to express the parameters for the matrix elements below the secondary diagonal ($\beta_{i,j}$ with $j > i'$) in terms of those above of the secondary diagonal

\[
\beta_{i,j} = \begin{cases} 
\pm \theta_1 \frac{1}{\sqrt{\xi}} q^{(\beta_{j',i}') + j - n - 1} \beta_{j',i}' & \text{for} \quad j > n + 1 \\
\pm \theta_1 \frac{1}{\sqrt{\xi}} q^{-n} \beta_{j',i}' & \text{for} \quad j \leq n + 1
\end{cases}
\]

where $\theta_1 = q\epsilon_i$ for $C_n^{(1)}$, $\theta_1 = 1$ for $A_{2n-1}^{(2)}$ and $D_n^{(1)}$.

These relations simplify the expressions for the non-diagonal matrix elements (3.2) and (3.3):

\[
k_{i,j}(u) = \begin{cases} 
\beta_{i,j}G^{(\pm)}(u) & (j < i') \\
\beta_{i,j} weaken(\beta_{i,j}'(u)) & (j = i') \\
\beta_{i,j} weaken(G^{(\pm)}(u)) & (j > i')
\end{cases}
\]

where $G^{(\pm)}(u)$ are defined conveniently by a normalization of $k_{1,2n}(u)$:

\[
G^{(\pm)}(u) = \frac{1}{\beta_{1,2n}} \left( \frac{q \pm \sqrt{\xi}}{qe^u \pm \sqrt{\xi}} \right) k_{1,2n}(u).
\]

Now, we substitute these expressions in the remaining equations $E[2, j]$ and $E[j, 2]$, and look at the equations of the type

\[
F(u)G^{(\pm)}(u) = 0
\]

where $F(u) = \sum_k f_k(\{\beta_{i,j}\})e^{ku}$. The constraint equations $f_k(\{\beta_{i,j}\}) \equiv 0, \forall k$, can be solved in terms of the $2n$ parameters. Of course the expressions for $k_{ij}$ will depend on our choice of these parameters. The choice $\beta_{12}, \beta_{13}, ..., \beta_{1,2n}$ and $\beta_{21}$ is the most appropriate for our purpose.

Taking into account all fixed parameters in terms of these $2n$ parameters, we can rewrite the $k_{ij}$ ($i \neq j$) matrix elements of $K^-(u)$ for $n > 2$ in the following way:
The secondary diagonal has the following entries

\[ k_{i,i'}(u) = \varepsilon_{i} \beta_{1,2n}^{2} \frac{q^{i-i'} (q \pm \sqrt{\xi})}{\xi (q + 1)^{2}} \left( \frac{q u^{n} \pm \sqrt{\xi}}{q \pm \sqrt{\xi}} \right) G(\pm)(u) \quad (i \neq 1, 2n) \] (3.10)

and

\[ k_{2n,1}(u) = \varepsilon_{2n} \beta_{1,2n}^{2} \frac{q^{2n-1} (q \pm \sqrt{\xi})}{\xi (q + 1)^{2}} \left( \frac{q u^{n} \pm \sqrt{\xi}}{q \pm \sqrt{\xi}} \right) G(\pm)(u). \] (3.11)

The first row (column) and the last row (column) with \( i, j \neq 1, 2n \) are:

\[ k_{1,j}(u) = \beta_{1,j} G(\pm)(u), \quad k_{1,1}(u) = \varepsilon_{i} \beta_{21} \frac{\beta_{1,2n}^{2}}{\beta_{1,2n}^{2}} q^{i-2} G(\pm)(u), \] (3.12)

\[ k_{2i}(u) = \varepsilon_{i} \beta_{1,2n}^{2} \frac{q^{i-1}}{\pm \sqrt{\xi}} e^{u} G(\pm)(u), \quad k_{2i,j}(u) = \beta_{21} \frac{\beta_{1,j} q^{i-2} q^{2n-2}}{\beta_{1,2n} \pm \sqrt{\xi}} e^{u} G(\pm)(u), \] (3.13)

and the remaining non-diagonal matrix elements are given by

\[ k_{ij}(u) = \varepsilon_{i} \beta_{1,j} \frac{\beta_{1,i'} q^{i-i'} \pm \sqrt{\xi}}{\beta_{1,2n}^{2}} \left( \frac{q \pm \sqrt{\xi}}{q + 1} \right) G(\pm)(u), \quad (j < i') \] (3.14)

\[ k_{ij}(u) = \varepsilon_{i} \beta_{1,j} \frac{\beta_{1,i'} q^{i-i'} \pm \sqrt{\xi}}{\beta_{1,2n}^{2}} \left( \frac{q \pm \sqrt{\xi}}{q + 1} \right) e^{u} G(\pm)(u), \quad (j > i') \] (3.15)

These relations solve all reflection equations without diagonal entries \( k_{ii}(u), i = 1, 2, ..., 2n \).

### 3.2 Diagonal matrix elements

At this point the remaining reflection equations involve 2n diagonal matrix elements \( k_{ii}(u) \), the function \( G(\pm)(u) \) and 4n parameters. From the equations \( E[1, 2] \) and \( E[1, 2n + 1] \) we can find \( k_{11}(u) \) and \( k_{22}(u) \) and from the equations \( E[4n^{2}, 4n^{2} - 1] \) and \( E[4n^{2}, 4n^{2} + 2n] \) we can find \( k_{21,2n}(u) \) and \( k_{2n-1,2n-1}(u) \).

Next, from the equations \( E[2, j], j = 3, 4, ..., 2n - 2 \) we can find the \( k_{jj}(u) \) matrix elements.

The expressions obtained in this way for the diagonal elements are too large. However, after finding the following \( n - 1 \) non-diagonal parameters

\[ \beta_{21} = -\frac{q}{\xi} \left( \frac{q \pm \sqrt{\xi}}{q + 1} \right)^{2} \frac{\beta_{12} \beta_{2}^{2}}{\beta_{1,2n}^{2}}, \quad \beta_{1,j} = (-1)^{n+j} \frac{\beta_{1,n} \beta_{1,n+1}}{\beta_{1,2n+1-j}}, \quad j = 2, 3, ..., n - 1 \] (3.16)

they are related with \( k_{11}(u) \) in a very simple way:

\[ k_{i,i}(u) = k_{11}(u) + (\beta_{i,i} - \beta_{11}) G(\pm)(u) \quad (2 \leq i \leq n) \] (3.17)

\[ k_{n+1,n+1}(u) = k_{n,n}(u) + (\beta_{n+1,n+1} - \beta_{n,n}) e^{u} G(\pm)(u) + H(\pm)(u) \] (3.18)

\[ k_{n+1,n+1}(u) = k_{n+1,n+1}(u) + (\beta_{i,i} - \beta_{n+1,n+1}) e^{u} G(\pm)(u) \quad (n + 2 \leq i \leq 2n) \] (3.19)

where

\[ H(\pm)(u) = -\Delta_{n}(\pm)(-q)^{n-1} \frac{\beta_{n+1}^{2} - \varepsilon_{n+1}}{(q + 1)^{2}} (e^{u} - 1) G(\pm)(u) \] (3.20)
with

$$\Delta_n^{(\pm)} = (-1)^n \frac{\beta_{1,n} \beta_{1,n+1}}{\beta_{1,2n}} \left( \frac{q \pm \sqrt{\xi}}{q \pm \sqrt{\xi}} \right)$$  \hspace{1cm} (3.21)$$

Here we note that $\mathcal{H}^{(\pm)}(u) = 0$ for $A_{2n-1}^{(2)}$ and $D_n^{(1)}$ models.

An important simplification occurs when we consider the equation $E[2n+1,4n]$ separately. This equation gives an additional relation between $k_{2n,2n}(u)$ and $k_{11}(u)$:

$$k_{2n,2n}(u) = e^{2u} k_{11}(u) + (\beta_{2n,2n} - \beta_{11} - 2)e^u \left( \frac{q \epsilon u \pm \sqrt{\xi}}{q \pm \sqrt{\xi}} \right) g^{(\pm)}(u).$$  \hspace{1cm} (3.22)$$

Taking $i = 2n$ into (3.22) and comparing with (3.20) we can find the following expression for $k_{11}(u)$:

$$k_{11}(u) = \frac{\mathcal{H}^{(\pm)}(u)}{e^{2u} - 1} + (\beta_{n,n} - \beta_{11} + (\beta_{2n,2n} - \beta_{n,n})e^u
\right)$$

$$-(\beta_{2n,2n} - \beta_{11} - 2)e^u \left( \frac{q \epsilon u \pm \sqrt{\xi}}{q \pm \sqrt{\xi}} \right) \right) G^{(\pm)}(u) \right) (e^{2u} - 1)^{-1}$$  \hspace{1cm} (3.23)$$

Substituting these expressions into the reflection equations we will get constraint equations to fix some of the $3n - 1$ remaining parameters. In order to do this we recall the equations $E[2,2n+j]$ to find $\beta_{jj}$, $j = 3,4,...,2n-2$ in terms of $\beta_{22}$. Then the $\beta_{22}$ is given by the equations $E[2,2n+1]$. After performing this we can use the equation $E[2,2n-1]$ to find $\beta_{2n-2,2n-1}$ and $\beta_{2n,2n}$. These parameters can be written in terms of $\beta_{11}, \beta_{1,n}, \beta_{1,n+1}$ and $\beta_{1,2n}$ in the following way:

$$\beta_{i,i} = \beta_{11} + \Delta_n^{(\pm)} \sum_{j=0}^{i-2} (-q)^j  \hspace{1cm} (1 < i \leq n)$$  \hspace{1cm} (3.24)$$

$$\beta_{n+1,n+1} = \beta_{11} + \Delta_n^{(\pm)} \left[ \frac{1 - (-q)^{n-1}}{q + 1} + (-q)^{n-1} \left( \frac{\theta_{n+1}^2 - \varepsilon_{n+1}}{\pm \sqrt{\xi}(q + 1)^2} \left( \frac{q \pm \sqrt{\xi}}{q \pm \sqrt{\xi}} \right) \right) \right]$$  \hspace{1cm} (3.25)$$

$$\beta_{i,i} = \beta_{n+1,n+1} + \Delta_n^{(\pm)} \left[ \frac{\theta_{n+1}^2 \varepsilon_{n+1}}{\pm \sqrt{\xi}} \sum_{j=n-1}^{i-3} (-q)^j \right]  \hspace{1cm} (n+2 < i \leq 2n-1)$$  \hspace{1cm} (3.26)$$

$$\beta_{2n,2n} = \beta_{11} + 2 + \Delta_n^{(\pm)} \frac{\left( q \pm \sqrt{\xi} \right)}{\xi} \frac{(\xi - \varepsilon_{2n} q^{n-1})}{(q + 1)^2}$$  \hspace{1cm} (3.27)$$

Note that $\beta_{n+1,n+1} = \beta_{n,n}$ or that $k_{n+1,n+1}(u) = k_{n,n}(u)$ for $A_{2n-1}^{(2)}$ and $D_n^{(1)}$.

Finally, we can use, for instance, the equation $E[2,4n]$ to fix $\beta_{1,n}$:

$$\beta_{1,n} = (-1)^n \frac{2 \xi (q + 1)^2}{(1 \mp \sqrt{\xi}) \theta_{1} q^n \mp (q \mp \sqrt{\xi}) \theta_{n+1} q^n} \frac{\beta_{1,2n}}{\beta_{1,n+1}}$$  \hspace{1cm} (3.28)$$

At this point it was possible to treat these solutions simultaneously. But now we needed separate them in order to take into account the existence of the amplitude $k_{1,n}(u)$ for each model:

- For $A_{2n-1}^{(2)}$ models $\xi = q^{2n}$ and $\theta_k = \varepsilon_k = 1, \forall k$. For these models there are no restrictions in (3.28). It follows that the solution with $G^{(\pm)}(u)$ (up sign) is related to the solution with $G^{(\pm)}(u)$ (down sign) by complex conjugation.
For $D^{(1)}_n$ models we have $\xi = q^{2n-2}$, $\theta_k = \epsilon_k = 1, \forall k$. It means that the factors $[q^{n-1} \mp (-1)^n \sqrt{\xi}]$ are different from zero for the solution with $G^{(+)}(u)$ only if $n$ is odd and for the solution with $G^{(-)}(u)$ if $n$ is even.

For $C^{(1)}_n$ models $\xi = q^{2n+2}$ and $\theta_1 = -\theta_{n+1} = q$. In this case the factors $[-q^{n+1} \mp (-1)^n \sqrt{\xi}]$ are different from zero for the solution with $G^{(+)}(u)$ if $n$ is even and for the solution with $G^{(-)}(u)$ if $n$ is odd.

Therefore, we have find two general solution for the $A^{(2)}_{2n-1}$ models and only one general solution for $C^{(1)}_n$ and $D^{(1)}_n$ models.

Substituting all fixed parameters into (3.23) we will find the following expressions for the amplitude $k_{11}(u)$:

$$k_{11}(u) = \frac{2e^{u}G^{(\pm)}(u)}{e^{2u} - 1} \left\{ \frac{\xi \left( 1 + q - (-q)^{n-1} + (-q)^n \right) + qe^{u}(\xi - q^{2n-2})}{(1 + \sqrt{\xi}) \left[ q^{n-1} - (-1)^n \sqrt{\xi} \right]} \right\}$$

(3.29)

for $A^{(2)}_{2n-1}$ and $D^{(1)}_n$ models and

$$k_{11}(u) = \frac{2e^{u}G^{(\pm)}(u)}{e^{2u} - 1} \left\{ \frac{\xi \left( 1 + q + (-q)^n + (-q)^{n+1} \right) + qe^{u}(\xi + q^{2n})}{(1 + \sqrt{\xi}) \left[ q^{n+1} - (-1)^n \sqrt{\xi} \right]} \right\}$$

(3.30)

for $C^{(1)}_n$ models.

From (3.29) we can see that $k_{11}(u)$ are quite simple for the $D^{(1)}_n$ models

$$k_{11}(u) = \frac{G^{(+)}(u)}{e^{u} - 1} \quad (n - odd), \quad k_{11}(u) = \frac{G^{(-)}(u)}{e^{u} - 1} \quad (n - even).$$

(3.31)

Moreover, substituting (3.30) into (3.27) we will find a simple relation between the diagonal matrix elements for $C^{(1)}_n$ models:

$$k_{n+i,n+i}(u) = e^{u}k_{i,i}(u) \implies \beta_{n+i,n+i} = \beta_{i,i} + 1, \quad 1 \leq i \leq n$$

(3.32)

Now, let us summarize these results: First, we have from (3.14) to (3.17) all non-diagonal matrix elements after substituting the $n$ fixed non-diagonal parameters $\beta_{21}$ and $\beta_{1,j} \,(j = 2, \ldots, n)$ given by (3.16) and (3.28).

Second, the diagonal matrix elements are obtained by using (3.17), (3.18) and (3.19) with $k_{11}(u)$ given by (3.29) for $A^{(2)}_{2n-1}$ and $D^{(1)}_n$ models and by (3.30) for $C^{(1)}_n$ models and substituting the diagonal parameters given by (3.24), (3.25), (3.26) and (3.27). This sequence results in two solutions with $n + 1$ parameters $\beta_{1,n+1}, \beta_{1,n+2}, \ldots, \beta_{1,2n}$ and $\beta_{11}$ for $A^{(2)}_{2n-1}$ models and one solution for $C^{(1)}_n$ and $D^{(1)}_n$ models.

The number of free parameters in these general solutions is $n$ because we still have to use the regular condition (2.10), which will fix the parameter $\beta_{11}$.

Here we note that $n = 1$ and $n = 2$ are special cases and will be treated in the section 6.
4 Reduced Solutions

In the previous section we have considered reflection $K$-matrices with all entries different from zero. In particular, for $C_n^{(1)}$ and $D_n^{(1)}$ models these matrices depend on the parity of $n$: one general solution with $G^+(u)$ for $n$ odd and one general solution with $G^-(u)$ for $n$ even for $D_n^{(1)}$ models and the opposite for the $C_n^{(1)}$ models. However, we have found that there exist solutions with $G^+(u)$ for $n$ even and solutions with $G^-(u)$ for $n$ odd for the $D_n^{(1)}$ models and the opposite for the $C_n^{(1)}$ models provided that we allow some matrix elements to be equal to zero. In order to show this we recall (3.1–3.3) to see that the vanishing of the secondary diagonal element $k_{n+1,n}(u)$ implies that $k_{i,n}(u) = 0 (i \neq n)$ and $k_{n+1,j}(u) = 0 (j \neq n + 1)$. Therefore we can consider the case $k_{n+1,n}(u) = 0$ which implies that $k_{n,n+1}(u) = k_{i,n}(u) = k_{n,j}(u) = 0$ for these models. It means that we are working out with $K$-matrices that contain $2(4n - 3)$ null entries. In particular, with $k_{1,n}(u) = 0$.

The non-diagonal matrix elements for these solutions can be directly obtained from (4.10) (4.11) taking into account the limits $\beta_{1,n} \to 0$ and $\beta_{1,n+1} \to 0$.

The secondary diagonal is

$$
\begin{align*}
  k_{i',i}(u) &= \varepsilon_{i,j'} \beta_{1,n}^2 G^+(u), \\
  k_{n+1,n}(u) &= k_{n+1,n}(u) = 0, \\
  k_{2n,1}(u) &= \varepsilon_{2n,1} \beta_{1,n}^2 \beta_{1,2n-1} G^+(u),
\end{align*}
$$

The boundary rows and columns are

$$
\begin{align*}
  k_{1,1}(u) &= \beta_{1,1} G^+(u), \\
  k_{1,n}(u) &= k_{1,n+1}(u) = k_{n+1,n}(u) = 0, \\
  k_{i,2n}(u) &= \varepsilon_{i,2n} \beta_{1,2n} G^+(u), \\
  k_{n,2n}(u) &= k_{n+1,2n}(u) = k_{2n,n}(u) = k_{2n,n+1}(u) = 0,
\end{align*}
$$

and the remaining non-diagonal matrix elements are

$$
\begin{align*}
  k_{i,j}(u) &= \varepsilon_{i,j} \beta_{1,n}^2 G^+(u), \\
  k_{i,j}(u) &= \varepsilon_{i,j} \beta_{1,n}^2 \xi G^+(u), \\
  k_{i,n}(u) &= k_{i,n+1}(u) = k_{n,j}(u) = k_{n+1,j}(u) = 0.
\end{align*}
$$

In order to find the corresponding diagonal elements we follow the steps presented previously, but now using the equations $E[n,2n(n-1)+2]$ and $E[n+1,2n^2+2]$ to find $k_{n,n}(u)$ and $k_{n+1,n+1}(u)$, respectively. Next we find the following $n - 2$ non-diagonal parameters

$$
\beta_{21} = -\frac{q}{\xi} \left( \frac{q \pm \sqrt{\xi}}{q + 1} \right)^2, \\
\beta_{1,j} = (-1)^{n-1+j} \beta_{1,n-1} \beta_{1,n+2} / \beta_{1,2n+1-j}, \\
  j = 2, 3, \ldots, n - 2
$$
with $n > 3$, they are also related with $k_{11}(u)$ in a very simple way:

\[
k_{ii}(u) = k_{11}(u) + (\beta_{i,i} - \beta_{11})G^{(\pm)}(u), \quad 1 < i \leq n - 1
\]

\[
k_{n,n}(u) = k_{n+1,n+1}(u) = k_{n-1,n-1}(u) + (\beta_{n,n} - \beta_{n-1,n-1})e^uG^{(\pm)}(u) + \mathcal{F}^{(\pm)}_{n-1}(u)
\]

\[
k_{ii}(u) = k_{n,n}(u) + (\beta_{i,i} - \beta_{n,n})e^uG^{(\pm)}(u) - \varepsilon_q\theta_0^2q^2\mathcal{F}^{(\pm)}_{n-1}(u), \quad n + 2 \leq i \leq 2n
\]

where

\[
\mathcal{F}^{(\pm)}_{n-1}(u) = (-q)^{n-2}\Delta^{(\pm)}_{n-1}(e^u - 1)/(q + 1)^2G^{(\pm)}(u)
\]

with

\[
\Delta^{(\pm)}_{n-1} = (-1)^{n-1}\beta_{1,n-1}\beta_{1,n+2}/\beta_{1,2n} \left( q \pm \sqrt{\xi} \right).
\]

Here we note that $\Delta^{(\pm)}_{n-1}$ can be understood as a limit ($\beta_{1,n} \to -\beta_{1,n-1}; \beta_{1,n+1} \to \beta_{1,n+2}$) of $\Delta_n^{(\pm)}$ given by \[22\].

Again, the equation $E[2n + 1, 4n]$ gives us another relation between $k_{2n,2n}(u)$ and $k_{11}(u)$:

\[
k_{2n,2n}(u) = e^{2u}k_{11}(u) + (\beta_{2n,2n} - \beta_{11} - 2)e^u \left( \frac{ge^u \pm \sqrt{\xi}}{q \pm \sqrt{\xi}} \right)G^{(\pm)}(u),
\]

which allows write $k_{11}(u)$ as

\[
k_{11}(u) = \left(1 - \varepsilon_{2n}\theta_0^2q^2\right)\mathcal{F}^{(\pm)}_{n-1}(u) + \left\{ \beta_{n-1,n-1} - \beta_{11} + (\beta_{2n,2n} - \beta_{n-1,n-1})e^u \right. \\
- (\beta_{2n,2n} - \beta_{11} - 2)e^u \left( \frac{ge^u \pm \sqrt{\xi}}{q \pm \sqrt{\xi}} \right) \left\} G^{(\pm)}(u)/e^{2u} - 1
\]

Substituting these expressions into the reflection equations we will get constraint equations to fix some of the $3n - 1$ remaining parameters. In order to do this we recall the equations $E[2, 2n + j], (j \neq n, n + 1)$ to find $\beta_{jj}, j = 3, 4, \ldots, 2n - 2$ in terms of $\beta_{22}$. Then the $\beta_{22}$ is given by the equations $E[2, 2n]$ and $E[2, 2n + j], (j \neq n, n + 1)$ to find $\beta_{jj}, j = 3, 4, \ldots, 2n - 2$ in terms of $\beta_{22}$. The parameters $\beta_{n,n}$ and $\beta_{n+1,n+1}$ can be fixed in terms of $\beta_{22}$ using the equations $E[n, 2n^2 - n + 2]$ and $E[n + 1, 2n^2 + n + 2]$, respectively. After this we can use the equation $E[2, 2n - 1]$ to find $\beta_{2n-1,2n-1}$ and $\beta_{2n,2n}$. These parameters can be written in terms of $\beta_{11}, \beta_{1,n-1}, \beta_{1,n+1}$ and $\beta_{1,2n}$ in the following way:

\[
\beta_{i,i} = \beta_{11} + \Delta^{(\pm)}_{n-1} \sum_{j=0}^{i-2} (-q)^j, \quad 1 < i \leq n - 1
\]

\[
\beta_{n,n} = \beta_{n+1,n+1} = \beta_{11} + \Delta^{(\pm)}_{n-1} \left[ \frac{1 - (-q)^{n-2}}{q + 1} \right] - q^{n-2}\Sigma^{(\pm)}_{n-1}
\]

\[
\beta_{n+2,n+2} = \beta_{n+1,n+1} = \beta_{11} + \Delta^{(\pm)}_{n-1} \left[ \frac{1 - (-q)^{n-2}}{q + 1} \right] - (q^{n-2} - \varepsilon_{n+2}\theta_0^2q^n)\Sigma^{(\pm)}_{n-1}
\]

\[
\beta_{ii} = \beta_{n+2,n+2} + \Delta^{(\pm)}_{n-1} \left[ \frac{\varepsilon_{n+2}\theta_0^2q^2}{\pm\sqrt{\xi}} \sum_{j=n-1}^{i-3} (-q)^j \right], \quad n + 3 \leq i \leq 2n - 1
\]
and
\[ \beta_{2n,2n} = \beta_{11} + 2 + \Delta_{n-1}^{(\pm)} \left( \frac{q \pm \sqrt{\xi}}{\xi} \right) \left( \frac{\xi - \varepsilon_{2n} q^{2n-1}}{(q + 1)^2} \right) \]  \hspace{1cm} (4.23)

where
\[ \sum_{n-1}^{(\pm)} = \frac{\beta_{1,n-1} \beta_{1,n+2}}{\beta_{1,2n}} \left( \frac{q \pm \sqrt{\xi}}{\xi} \right)^2 \]  \hspace{1cm} (4.24)

Next, we can use, for instance, the equation \( E[2,4n] \) to fix \( \beta_{1,n-1} \):
\[ \beta_{1,n-1} = (-1)^{n-1} \frac{2 \xi (\pm \sqrt{\xi})(q + 1)^2}{(1 \mp \sqrt{\xi}) [q^{n-1} \pm (-1)^n \sqrt{\xi}] [\varepsilon_{n+2} \theta^2_{n+2} q^n \pm (\xi + 2q^n)] (q \pm \sqrt{\xi}) \beta_{1,2n}} \]  \hspace{1cm} (4.25)

Following the discussion made for the general solutions, we also need to separate these solutions in order to take into account the existence of \( k_{1,n-1}(u) \) for each model:

- For \( A_{2n-1}^{(2)} \) models, \( \xi = -q^{2n} \) and \( \theta_k = \varepsilon_k = 1, \forall k \). In this case there are no restrictions in \( E[2n] \). It follows that the solution with \( G^{(+)}(u) \) (up sign) is related to the solution with \( G^{(-)}(u) \) (down sign) by complex conjugation.

- For \( D_{n}^{(1)} \) models, we have \( \xi = q^{2n-2}, \varepsilon_k = 1, \forall k \). It means that the factors \( [q^{n-1} \pm (-1)^n \sqrt{\xi}] \) are different from zero for the solution with \( G^{(+)}(u) \) only if \( n \) is even and for the solution with \( G^{(-)}(u) \) if \( n \) is odd.

- For \( C_{n}^{(1)} \) models, \( \xi = q^{2n+2}, \varepsilon_{n+2} = -1 \) and \( \theta_{n+2}^2 = q^2 \). In this case there are also no restrictions because both factors \( [q^{n-1} \pm (-1)^n \sqrt{\xi}] \) and \( [-q^{n+2} \pm (-1)^n \sqrt{\xi}] \) are different from zero. It means that we have two independent solutions, one with \( G^{(+)}(u) \) and another with \( G^{(-)}(u) \), for all \( n > 3 \).

Comparing these results with those flowed in the previous section, one could conclude that we have simply made a reduction of the general solution through an appropriate choice of the free parameters. Nevertheless, new solutions are appearing for \( C_{n}^{(1)} \) and \( D_{n}^{(1)} \) models.

Substituting all fixed parameters into \( E[4n] \) we will find the following expressions for the amplitude \( k_{11}(u) \):
\[ k_{11}(u) = \frac{2e^n G^{(\pm)}(u)}{e^{2n} - 1} \left( \frac{2G^{(\pm)}(u)}{e^n + 1} \left\{ \frac{\xi}{(1 \mp \sqrt{\xi}) [q^{n-1} \pm (-1)^n \sqrt{\xi}] [q^n \pm (\xi - 2n-2) q^{2n-2] [q^{n+2} \mp (\xi + 2q^n]}} \right\} \right) \]  \hspace{1cm} (4.26)

for \( A_{2n-1}^{(2)} \) and \( D_{n}^{(1)} \) models and
\[ k_{11}(u) = \frac{2e^n G^{(\pm)}(u)}{e^{2n} - 1} + \frac{2G^{(\pm)}(u)}{e^n + 1} \left( \frac{\xi}{(1 \mp \sqrt{\xi}) [q^{n-1} \pm (-1)^n \sqrt{\xi}] [q^{n+2} \mp (\xi + 2q^n]}) \right) \]  \hspace{1cm} (4.27)

for \( C_{n}^{(1)} \) models.
For $D_n^{(1)}$ case, we still have the simplified expression for $k_{11}(u)$ but now interchanging the parity of the solutions with $G^{(\pm)}(u)$:

\[ k_{11}(u) = \begin{cases} \frac{G^{(+)}(u)}{e^u - 1} & (n - \text{even}), \\ \frac{G^{(-)}(u)}{e^u - 1} & (n - \text{odd}). \end{cases} \quad (4.28) \]

For $C_n^{(1)}$ models we lose the relations between the diagonal entries, but defines a new solution with $G^{(+)}(u)$ when $n$ is odd and another one with $G^{(-)}(u)$ when $n$ is even.

Now, let us summarize these results: First, we have from (4.1) to (4.10) all non-diagonal matrix elements after substituting the $n - 2$ fixed non-diagonal parameters $\beta_{21}$ and $\beta_{1,j}$ ($j = 2, ..., n - 2$) given by (4.11) and (4.25).

Second, the diagonal matrix elements are obtained by using (4.11), (4.12) and (4.14) with $k_{11}(u)$ given by (4.28) for $D_n^{(1)}$ models and given by (4.27) for $C_n^{(1)}$ models and by substituting the diagonal parameters given by (4.10) - (4.27). This sequence results in two solutions with $n$ parameters $\beta_{1,n+2}, \beta_{1,n+3}, ..., \beta_{1,2n}$ and $\beta_{11}$ for these models.

The number of free parameters in these new solutions is $n - 1$ because we still have to use the regular condition (2.10), which will fix the parameter $\beta_{11}$.

The general solutions which we have found in the previous section have $n$ free parameters. Therefore, solutions with $n - 1$ free parameters can be understood as reductions of them through a complicated limit procedure as was described in this section. This is true for $A_{2n-1}^{(2)}$ models. For $D_n^{(1)}$ models our general solutions with $G^{(+)}(u)$ have $n$ free parameters but only for $n = 3, 5, 7, ...$, and our general solutions with $G^{(-)}(u)$ are defined only for $n = 4, 6, 8, ...$. The limit procedure derived here gives us solutions with $G^{(+)}(u)$ and $n - 1$ free parameters provided that $n = 4, 6, 8, ...$ and solutions with $G^{(-)}(u)$ and $n - 1$ free parameters if $n = 5, 7, 9, ...$. Similar considerations hold for $C_n^{(1)}$ models interchanging the parity of $n$.

Here we note that $n = 3$ are solutions with 3 free parameters and will be treated in the section 6.

We should continue with our reduction procedure in order to verify if exists other solutions for those models. The next step is to consider $k_{1,n-1}(u) = 0$, in addition to $k_{1,n}(u) = 0$. In this case we will find solutions with $16n - 20$ null entries and with $n - 2$ free parameters for all $n > 4$. However, all these solutions are reductions of those solutions with $n$ and $n - 1$ free parameters. So, in this step we have not found any new solutions. Here the cases with $n = 4$ will have $n - 1$ free parameters.

The rational limit of some of these reductions was presented in (1.13).

After exhausting all the possible reductions we will arrive to the last reduction which is obtained after applying $n - 1$ reduction steps. The final reduction for $A_{2n-1}^{(2)}$ and $D_n^{(1)}$ is the non-diagonal $K$-matrix solution of the reflection equation with the following entries different from zero

\[
\begin{align*}
k_{11}(u) &= 1, \quad k_{2n,2n}(u) = e^{2u}, \\
k_{22}(u) &= k_{33}(u) = \cdots k_{2n-1,2n-1}(u) = \frac{q^{2n-2} - e^{2u}}{q^{2n-2} - 1}, \\
k_{1,2n}(u) &= \frac{1}{2} \beta_{1,2n} (e^{2u} - 1), \quad k_{2n,1}(u) = \frac{2}{\beta_{1,2n} (q^{2n-2} - 1)^2} (e^{2u} - 1),
\end{align*}
\]

while for $C_n^{(1)}$ the corresponding reduction is

\[
\begin{align*}
k_{11}(u) &= 1, \quad k_{2n,2n}(u) = e^{2u}, \\
k_{22}(u) &= k_{33}(u) = \cdots k_{2n-1,2n-1}(u) = \frac{q^{2n} + e^{2u}}{q^{2n} + 1}, \\
k_{1,2n}(u) &= \frac{1}{2} \beta_{1,2n} (e^{2u} - 1), \quad k_{2n,1}(u) = -\frac{2}{\beta_{1,2n} (q^{2n} + 1)^2} (e^{2u} - 1).
\end{align*}
\]
The main goal here is to present all diagonal solutions from the reflection equations (1.2) and (1.3) for 5 Diagonal K-Matrix Solutions and non-diagonal entries. Nevertheless, special attention is reserved for reductions which give us diagonal 1-parameter solution.

Solving the reflection equations we find one trivial solution which is proportional to the identity, two (n-1)-parameter solution with all entries different from zero, one (n-1)-parameter solution with 8n - 6 null entries both n-parity dependent, and one 1-parameter solution given by (1.2a).

In conclusion we have the following picture: for A

• For A

• For D

• For C

These results give us all independent solutions of the reflection equations (1.2) with at least two non-diagonal entries. Nevertheless, special attention is reserved for reductions which give us diagonal K-matrix solutions.

5 Diagonal K-Matrix Solutions

The main goal here is to present all diagonal solutions from the reflection equations (1.2) and (1.3) for \( R \)-matrices associated to \( C^{(1)}_n \), \( D^{(1)}_n \) and \( A_{2n-1} \) affine Lie algebras. Instead of using a reduction procedure, we choose to find the diagonal solutions by solving again the reflection equations. In this section we will consider these solutions for each model separately.

5.1 \( A^{(2)}_{2n-1} \) Diagonal K-matrices

Solving the reflection equations we find one trivial solution which is proportional to the identity, two 1-parameter solution \( K_\beta \) with the following normalized matrix elements:

\[
\begin{align*}
k_{11}(u) &= k_{22}(u) = \cdots = k_{n-1,n-1}(u) = 1, \\
k_{n,n}(u) &= \left( \frac{\beta (e^u - 1) - 2}{\beta (e^{-u} - 1) - 2} \right), \\
k_{n+1,n+1}(u) &= e^{2u} \left( \frac{\beta (e^{-u} + q^{2n-2}) + 2q^{2n-2}}{\beta (e^u + q^{2n-2}) + 2q^{2n-2}} \right), \\
k_{2n,2n}(u) &= k_{2n-1,2n-1}(u) = \cdots = k_{n+2,n+2}(u) = e^{2u},
\end{align*}
\] (5.1)

where \( \beta \) is the free parameter. The second solution is obtained from (5.1) using the symmetry \( k_{n,n}(u) \leftrightarrow k_{n+1,n+1}(u) \). Moreover, for \( n > 2 \) we find \( 2n - 4 \) solutions \( \mathbb{K}[p] \), \( p = 2, 3, \ldots, n - 1 \) without any free parameters and with the following entries:

\[
\begin{align*}
k_{11}(u) &= k_{22}(u) = \cdots = k_{p-1,p-1}(u) = 1, \\
k_{p,p}(u) &= k_{p+1,p+1}(u) = \cdots = k_{2n-p+1,2n-p+1}(u) = e^{2u} \frac{e^{-u} \pm iq^{2p-n-1}}{e^u \pm q^{2p-n-1}}, \\
k_{2n,2n}(u) &= k_{2n-1,2n-1}(u) = \cdots = k_{2n-p+2,2n-p+2}(u) = e^{2u}, \\
p &= 2, 3, \ldots, n - 1.
\end{align*}
\] (5.2)
Here we observe that: the cases \( p = n \) are not computed as solutions because the \( K^{[p-n]} \) solutions are obtained from \( K_\beta \) by the choice of \( \beta \) such that \( k_{n,n} = k_{n+1,n+1} \). Therefore, for the \( A_{2n-1}^{(2)} \) models we have found \( 2n - 1 \) regular diagonal solutions.

We mention that the case \( n = 2 \) has been first discussed by Martins in the context of coupled six vertex models [31].

### 5.2 \( D_3^{(1)} \) Diagonal K-matrices

For these models we also have the symmetry \( k_{n,n}(u) \leftrightarrow k_{n+1,n+1}(u) \) in the \( K \)-matrix solutions.

The \( D_3^{(1)} \) diagonal solutions are special. In addition to the identity we have found two solutions with two free parameters

\[
K_{\alpha\beta} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \frac{\alpha(e^u-1)}{(e^u-1)^2} & 0 & 0 & 0 \\
0 & 0 & \frac{\beta(e^u-1)}{(e^u-1)^2} & 0 & 0 \\
0 & 0 & 0 & \frac{\alpha(e^u-1)}{(e^u-1)^2} & \frac{\beta(e^u-1)}{(e^u-1)^2} \\
0 & 0 & 0 & 0 & \frac{\alpha(e^u-1)}{(e^u-1)^2}
\end{pmatrix}
\]

(5.3)

where \( \alpha \) and \( \beta \) are the free parameters. This \( so(4) \) solution, in the isotropic limit, was already presented in [13].

For \( n > 2 \) we have found the identity and seven 1-parameter solutions \( K^{[i]}_\beta \) \( (i = 1, 2, \ldots, 7) \):

- The \( K^{[1]}_\beta \) matrix has the following entries:

\[
\begin{align*}
k_{11}(u) &= 1, \\
k_{22}(u) &= k_{33}(u) = \cdots = k_{2n-1,2n-1}(u) = \frac{\beta(e^u-1) - 2}{\beta(e^u-1) - 2}, \\
k_{2n,2n}(u) &= \frac{\beta(e^u-1) - 2}{\beta(e^u-1) - 2} \frac{q^{2n} - 4}{q^{2n} - 4}.
\end{align*}
\]

(5.4)

- The \( K^{[2]}_\beta \) matrix has the following entries:

\[
\begin{align*}
k_{11}(u) &= k_{22}(u) = \cdots = k_{n,n}(u) = 1, \\
k_{n+1,n+1}(u) &= k_{n+2,n+2}(u) = \cdots = k_{2n,2n}(u) = \frac{\beta(e^u-1) - 2}{\beta(e^u-1) - 2}.
\end{align*}
\]

(5.5)

- The \( K^{[3]}_\beta \) matrix has the following entries:

\[
\begin{align*}
k_{11}(u) &= k_{22}(u) = \cdots = k_{n-1,n-1}(u) = k_{n+1,n+1}(u) = 1, \\
k_{n,n}(u) &= k_{n+2,n+2}(u) = \cdots = k_{2n,2n}(u) = \frac{\beta(e^u-1) - 2}{\beta(e^u-1) - 2}.
\end{align*}
\]

(5.6)

- The \( K^{[4]}_\beta \) matrix has the following entries:

\[
\begin{align*}
k_{11}(u) &= 1, \\
k_{22}(u) &= k_{33}(u) = \cdots = k_{n,n}(u) = \frac{\beta(e^u-1) - 2}{\beta(e^u-1) - 2}, \\
k_{n+1,n+1}(u) &= k_{n+2,n+2}(u) = \cdots = k_{2n-1,2n-1}(u) = e^{2u}, \\
k_{2n,2n}(u) &= e^{2u} \frac{\beta(e^u-1) - 2}{\beta(e^u-1) - 2}.
\end{align*}
\]

(5.7)
The $K_{\beta}^{[5]}$ matrix has the following entries:

\[
\begin{align*}
    k_{11}(u) &= 1, \\
    k_{22}(u) &= k_{33}(u) = \cdots = k_{n-1,n-1}(u) = k_{n+1,n+1}(u) = \frac{\beta (e^u - 1) - 2}{\beta (e^{-u} - 1) - 2}, \\
    k_{n,n}(u) &= k_{n+2,n+2}(u) = \cdots = k_{2n-1,2n-1}(u) = e^{2u}, \\
    k_{2n,2n}(u) &= e^{2u} \frac{\beta (e^u - 1) - 2}{\beta (e^{-u} - 1) - 2}.
\end{align*}
\]

(5.8)

The $K_{\beta}^{[6]}$ matrix has the following entries:

\[
\begin{align*}
    k_{11}(u) &= k_{22}(u) = \cdots = k_{n-1,n-1}(u) = 1, \\
    k_{n,n}(u) &= e^{2u} \frac{\beta (e^u - q^{2n-4}) - 2q^{2n-4}}{\beta (e^u - q^{2n-4}) - 2q^{2n-4}}, \\
    k_{n+1,n+1}(u) &= \frac{\beta (e^u - 1) - 2}{\beta (e^{-u} - 1) - 2}, \\
    k_{n+2,n+2}(u) &= k_{n+3,n+3}(u) = \cdots = k_{2n,2n}(u) = e^{2u}.
\end{align*}
\]

(5.9)

The $K_{\beta}^{[7]}$ matrix has the following entries:

\[
\begin{align*}
    k_{11}(u) &= k_{22}(u) = \cdots = k_{n-1,n-1}(u) = 1, \\
    k_{n,n}(u) &= \frac{\beta (e^u - 1) - 2}{\beta (e^{-u} - 1) - 2}, \\
    k_{n+1,n+1}(u) &= e^{2u} \frac{\beta (e^u - q^{2n-4}) - 2q^{2n-4}}{\beta (e^u - q^{2n-4}) - 2q^{2n-4}}, \\
    k_{n+2,n+2}(u) &= k_{n+3,n+3}(u) = \cdots = k_{2n,2n}(u) = e^{2u}.
\end{align*}
\]

(5.10)

Moreover, for $n > 3$ we have found $n - 3$ solutions without any free parameters $K^{[p]}$, $p = 3, 4, ..., n-1$, with the following matrix elements

\[
\begin{align*}
    k_{11}(u) &= k_{22}(u) = \cdots = k_{p-1,p-1}(u) = 1, \\
    k_{p,p}(u) &= k_{p+1,p+1}(u) = \cdots = k_{2n-p+1,2n-p+1}(u) = e^{2u} \frac{e^{-u} + \epsilon_p q^{2p-n-2}}{e^{u} + \epsilon_p q^{2p-n-2}}, \\
    k_{2n-p+2,2n-p+2}(u) &= k_{2n-p+3,2n-p+3}(u) = \cdots = k_{2n,2n}(u) = e^{2u},
\end{align*}
\]

(5.11)

where $\epsilon_p = \pm 1$ for $2p \neq n + 2$ and $\epsilon_p = 1$ for $2p = n + 2$. Therefore, for $n \geq 3$ the $D_n^{(1)}$ models have $2n + 1$ diagonal solutions if $n$ is odd and $2n$ diagonal solutions if $n$ is even.

Here we note that the cases $K^{[p=n]}$ are not computed here because they are reductions of the cases $K_{\beta}^{[6]}$ and $K_{\beta}^{[7]}$ by the choice of $\beta$ such that $k_{n,n}(u) = k_{n+1,n+1}(u)$. Moreover, the cases $K^{[p=2]}$ are reductions of $K_{\beta}^{[1]}$ with the choice of $\beta$ such that $k_{2n,2n}(u) = e^{2u}$.

### 5.3 $C_n^{(1)}$ Diagonal K-matrices

The $K$-matrix solutions for these models do not enjoy the symmetry $k_{n,n}(u) \leftrightarrow k_{n+1,n+1}(u)$. After solving the corresponding reflection equations we have found one solution with one free parameter $K_{\beta}$,
with the following entries:

\[
  k_{11}(u) = k_{22}(u) = \cdots = k_{n,n}(u) = 1,
  \\
  k_{n+1,n+1}(u) = k_{n+2,n+2}(u) = \cdots = k_{2n,2n}(u) = \frac{\beta (e^u - 1) - 2}{\beta (e^{-u} - 1) - 2},
\]

(5.12)

where \( \beta \) is the free parameter. For \( n > 2 \), in addition to the identity we also have found \( n - 1 \) solutions without any free parameters \( K^{[p]} \), \( p = 2, 3, \ldots, n \), with the following matrix elements:

\[
  k_{11}(u) = k_{22}(u) = \cdots = k_{p-1,p-1}(u) = 1,
  \\
  k_{p,p}(u) = k_{p+1,p+1}(u) = \cdots = k_{2n-p+1,2n-p+1}(u) = e^{2u} e^{-u} + \epsilon_p q^{2p-n-2},
  \\
  k_{2n-p+2,2n-p+2}(u) = k_{2n-p+3,2n-p+3}(u) = \cdots = k_{2n,2n}(u) = e^{2u},
\]

(5.13)

where \( \epsilon_p = \pm 1 \) for \( 2p \neq n + 2 \) and \( \epsilon_p = 1 \) for \( 2p = n + 2 \). Therefore, for the \( C_n^{(1)} \) models we have \( 2n \) diagonal solutions if \( n \) is odd and \( 2n - 1 \) diagonal solutions if \( n \) is even.

Here we would like to note that our \( C_n^{(1)} \) diagonal solutions are contained in the set of the \( D_n^{(1)} \) diagonal solutions.

A final note about these diagonal solutions is that almost all those with free parameters degenerate, after an appropriate choice of the free parameters, into the two type of solutions presented in [16]. Namely the identity \((s = 1)\) and the \( K \) matrix \((s = 0)\) with the following normalized entries:

\[
  k_{11}(u) = k_{22}(u) = \cdots = k_{n,n}(u) = 1
  \\
  k_{n+1,n+1}(u) = k_{n+2,n+2}(u) = \cdots = k_{2n,2n}(u) = e^{2u}
\]

(5.14)

6 Special Cases

In this section we will present solutions which are out of our classification scheme. They are the case \( n = 1 \), the case \( n = 2 \), the solution with \( G^{-}(u) \) for \( D_{3}^{(1)} \) and the solution with \( G^{+}(u) \) for \( C_{3}^{(1)} \):

6.1 \( C_{1}^{(1)}, D_{1}^{(1)} \) and \( A_{1}^{(2)} \) K-matrix solutions

These models have one common general solution with three free parameters \( \beta_{11}, \beta_{12} \) and \( \beta_{21} \)

\[
  K^{-} = \begin{pmatrix}
    1 + \beta_{11} (e^u - 1) & \frac{1}{2} \beta_{12} (e^{2u} - 1) \\
    \frac{1}{2} \beta_{21} (e^{2u} - 1) & e^{2u} - \beta_{11} e^u (e^u - 1)
  \end{pmatrix}
\]

and one common diagonal solution which can be obtained from (6.1) when \( \beta_{12} = \beta_{21} = 0 \) and the identity.

6.2 \( A_{3}^{(2)} \) K-matrix solutions

In this model we have found one general solution with four free parameters, \( \beta_{12}, \beta_{13}, \beta_{14} \) and \( \beta_{24} \). The \( K^{-} \) matrix has the form

\[
  K^{-} = \begin{pmatrix}
    k_{11} & k_{12} & k_{13} & k_{14} \\
    k_{21} & k_{22} & k_{23} & k_{24} \\
    k_{31} & k_{32} & k_{33} & k_{34} \\
    k_{41} & k_{42} & k_{43} & k_{44}
  \end{pmatrix}
\]

(6.2)
where the normalized diagonal entries are given by

\[
k_{22} = e^u + \frac{\beta_{12} e^u (e^u - 1)}{2q^2 \beta_{14} \beta_{24}} \left\{ \beta_{24} \left[ q^2 (\beta_{13} + \beta_{24}) - \beta_{13} \right] (e^u + q^2) + \beta_{13}^2 (q^2 e^u + 1) \right\}
\]

\[
k_{33} = e^u - \frac{\beta_{12} e^u (e^u - 1)}{2q^2 \beta_{14} \beta_{24}} \left\{ \beta_{13} \left[ (\beta_{13} + \beta_{24}) - q^2 \beta_{24} \right] (e^u + q^2) + \beta_{13}^2 q^2 (q^2 e^u + 1) \right\}
\]

\[
k_{44} = e^u + \frac{\beta_{12} e^u (e^u - 1)}{2q^2 \beta_{14} \beta_{24}} \left\{ \beta_{13} \beta_{24} \left[ (q^2 + e^u)^2 - q^2 (e^u - 1) \right] + (\beta_{13}^2 - q^2 \beta_{24}^2) e^u (q^2 + e^u) \right\}
\]

(6.3)

and the non-diagonal matrix elements are

\[
k_{12} = \frac{\beta_{12}}{2 \beta_{24}} f(u), \quad k_{13} = \frac{1}{2} g(u), \quad k_{14} = \frac{1}{2} \beta_{14} (e^{2u} - 1),
\]

\[
k_{21} = -\frac{1}{2} \Omega f(u), \quad k_{23} = \frac{\beta_{14}}{2 \beta_{12}} \Omega (e^{2u} - 1), \quad k_{24} = \frac{1}{2} e^u f(u),
\]

\[
k_{31} = \frac{\beta_{12}}{2q^2 \beta_{24}} \Omega g(u), \quad k_{32} = -\frac{\beta_{12} \beta_{14}}{2q^2 \beta_{24}} \Omega (e^{2u} - 1), \quad k_{34} = -\frac{\beta_{12}}{2q^2 \beta_{24}} e^u g(u),
\]

\[
k_{41} = -\frac{\beta_{14}}{2q^2} \Omega^2 (e^{2u} - 1), \quad k_{42} = \frac{\beta_{12}}{2q^2 \beta_{24}} \Omega e^u f(u), \quad k_{43} = \frac{1}{2q^2} \Omega e^u g(u).
\]

(6.4)

Here we have defined two scalar functions \(f(u)\) and \(g(u)\) different from our \(G^{(\pm)}(u)\) functions:

\[
f(u) = \left[ \beta_{13} (e^u - 1) + \beta_{24} (e^u + q^2) \right] \left( \frac{e^{2u} - 1}{e^{2u} + q^2} \right),
\]

\[
g(u) = \left[ \beta_{13} (e^u + q^2) - q^2 \beta_{24} (e^u - 1) \right] \left( \frac{e^{2u} - 1}{e^{2u} + q^2} \right),
\]

\[
\Omega = \frac{\beta_{12} \beta_{13} (q^2 + 1) - 2q^2 \beta_{14}}{\beta_{14} (q^2 + 1)}.
\]

(6.5)

This solution can be unfolded in the two parameter general solutions with \(G^{(\pm)}(u)\), by an appropriate choice of \(\beta_{12}\) and \(\beta_{24}\). It follows, by the reduction procedure, the solution (5.21) and the two non-trivial diagonal matrices given by (5.21).

### 6.3 \( C_2^{(1)} \) K-matrix Solutions

For this case we have one general solution with three free parameters. The \( K^- \) matrix has the form

\[
K^- = \begin{pmatrix}
k_{11} & k_{12} & k_{13} & k_{14} \\
k_{21} & k_{22} & k_{23} & e^u k_{24} \frac{e^{2u}}{q} k_{13} \\
k_{31} & k_{32} & e^u k_{33} & e^u k_{14} \\
k_{41} & e^u k_{42} & -e^u k_{31} & e^u k_{22}
\end{pmatrix},
\]

(6.6)
with the following remaining diagonal entries:

\[ k_{11} = 1 - \frac{\beta_{12} \beta_{13}}{q^2 \beta_{14}} f(u), \quad k_{22} = 1 + \frac{\beta_{12} \beta_{13}}{\beta_{14}} f(u). \]  

\[ k_{12} = \beta_{12} f(u), \quad k_{13} = \beta_{13} f(u), \quad k_{14} = \beta_{14} (e^u - 1), \]

\[ k_{21} = -\frac{\beta_{13}}{q^2 \beta_{14}} \Gamma f(u), \quad k_{23} = \frac{\beta_{13}}{q^2 \beta_{12}} \Gamma (e^u - 1), \]

\[ k_{31} = \frac{\beta_{12}}{\beta_{14}} \Gamma f(u), \quad k_{32} = -\frac{\beta_{12}}{\beta_{13}} \Gamma (e^u - 1), \quad k_{41} = -\frac{\Gamma^2}{q^2 \beta_{14}} (e^u - 1). \]  

(6.7)

Here we also have used a new scalar function \( f(u) \), a little bit different from \( G^+(u) \):

\[ f(u) = \left( \frac{q^2 + 1}{q^2 + e^u} \right) (e^u - 1) \quad \text{and} \quad \Gamma = \frac{\beta_{12} \beta_{13}}{\beta_{14}} - \frac{q^2}{q^2 + 1}. \]  

(6.8)

This solution can be identified with the two parameter general solution with \( G^+(u) \) by an appropriate choice of \( \beta_{12} \). It follows, by the reduction procedure, the solution (4.30) and the three diagonal matrices given by (5.12) and (5.13).

### 6.4 D\(^{(1)}\)_2 K-matrix solutions

This is a very special case because we do not have any solution with all entries different from zero. Here we have the identity and one 2-parameter diagonal solution which is given by (5.3). In addition to the reduced solution (4.29) we also have found the following \( K^- \) matrix

\[
K^- = \begin{pmatrix}
  e^{-u} & 0 & 0 & 0 & 0 \\
  0 & e^u & \frac{1}{2} \beta (e^{2u} - 1) & 0 & 0 \\
  0 & \frac{e^{2u} - e^{2u-1}}{\beta (q^2 - 1)} & e^u & 0 & 0 \\
  0 & 0 & 0 & e^{u \frac{q^2 - e^{2u}}{q^2 - 1}} & 0 \\
  0 & 0 & 0 & 0 & e^{u \frac{q^2 - e^{2u}}{q^2 - 1}}
\end{pmatrix} \quad (6.10)
\]

where \( \beta \) is a free parameter.

In this way we have listed all solutions of (1.2) for \( C^{(1)}_2, D^{(1)}_2 \) and \( A^{(2)}_3 \) models. The corresponding \( K^+ \) solutions are obtained using (1.4) and taking into account the \( M \) matrix (2.15) for each model.

### 6.5 The D\(^{(1)}\)_3 solution with \( G^-(u) \)

In section 4 we have found one 3-parameter general solution with \( G^+(u) \) for this model. The reduction procedure gives us another solution with \( G^-(u) \) which also has three free parameters. The corresponding \( K^- \) matrix has the form

\[
K^- = \begin{pmatrix}
k_{11} & k_{12} & 0 & 0 & k_{15} & k_{16} \\
k_{21} & k_{22} & 0 & 0 & k_{25} & k_{26} \\
0 & 0 & k_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & k_{44} & 0 & 0 \\
k_{51} & k_{52} & 0 & 0 & k_{55} & k_{56} \\
k_{61} & k_{62} & 0 & 0 & k_{65} & k_{66}
\end{pmatrix} \quad (6.11)
\]
with the non-normalized diagonal matrix elements given by

\[
k_{11} = \left( \frac{2(e^u - q)}{1 - q(e^u - 1)} \right) \frac{G(-)(u)}{e^u + 1},
\]

\[
k_{22} = \left( \frac{2(e^u - q)}{1 - q(e^u - 1)} \right) \frac{[e^u(q - 1) + q(1 + q)] \beta_{12} \beta_{15}}{q \beta_{16}} \frac{G(-)(u)}{e^u + 1},
\]

\[
k_{33} = k_{44} = \left( \frac{2(e^u - q)^2}{1 - q^2(e^u - 1)} \right) \frac{e^u + q G(-)(u)}{1 - q(e^u + 1)},
\]

\[
k_{55} = \left( \frac{2(e^u - q)e^u}{1 - q(e^u - 1)} \right) \frac{[e^u(q + 1) + q(1 - q)] \beta_{12} \beta_{15}}{q \beta_{16}} \frac{G(-)(u)}{e^u + 1},
\]

\[
k_{66} = e^{2u} k_{11}.
\]

(6.12)

and the non-diagonal entries are

\[
k_{12} = \beta_{12} G(-)(u), \quad k_{15} = \beta_{15} G(-)(u), \quad k_{16} = \beta_{16} \frac{e^u - q}{1 - q} G(-)(u),
\]

\[
k_{21} = \beta_{21} G(-)(u), \quad k_{25} = -\beta_{21} \frac{\beta_{16} e^u - q}{\beta_{12}} G(-)(u), \quad k_{26} = -\frac{\beta_{15}}{q} e^u G(-)(u),
\]

\[
k_{51} = \beta_{51} \frac{q^2 \beta_{12} \beta_{15}}{\beta_{15}} G(-)(u), \quad k_{52} = -\beta_{21} \frac{q^2 \beta_{12} \beta_{16} e^u - q}{\beta_{15}} G(-)(u), \quad k_{56} = -q \beta_{12} e^u G(-)(u),
\]

\[
k_{61} = \beta_{61} \frac{q^2 \beta_{16} e^u - q}{1 - q} G(-)(u), \quad k_{62} = -\beta_{21} \frac{q \beta_{12} \beta_{15}}{\beta_{15}} e^u G(-)(u), \quad k_{65} = -q \beta_{21} e^u G(-)(u).
\]

(6.13)

where

\[
G(-)(u) = \frac{1}{\beta_{16}} \left( \frac{1 - q}{e^u - q} \right) \frac{G(-)(u)}{e^u + 1}, \quad \beta_{21} = \frac{\beta_{12} \beta_{15}}{q \beta_{16}} \frac{2}{q^2 - 1} \beta_{16}
\]

(6.14)

Here we note that for \( n > 3 \), this type of solution has \( n - 1 \) free parameters and follows the classification scheme presented in section 4.

6.6 The \( C_3^{(1)} \) solution with \( G^{(+)}(u) \)

For this model the general solution is given in terms of \( G^{(-)}(u) \) and also has three free parameters. Here we have another 3-parameter solution in terms of \( G^{(+)}(u) \). It has the form (6.11) with the following
solutions with all entries different from zero we have found two conjugated solutions for the and one solution for the.

The non-normalized diagonal entries are given by

\[ k_{11} = \frac{2e^u G^{(+)}(u)}{e^{2u} - 1} - \left( \frac{\beta_{12} \beta_{15}}{\beta_{16}} \frac{1 + q^3}{q^3} - \frac{q \beta_{21} \beta_{16} [(1 + q^2)e^u - q^2(1 + q^4)]}{\beta_{15}(1 + q^3)} \right) \frac{G^{(+)}(u)}{e^u + 1}, \]

\[ k_{22} = \frac{2e^u G^{(+)}(u)}{e^{2u} - 1} + \left( \frac{\beta_{12} \beta_{15}}{\beta_{16}} \frac{1 + q^3}{q^3} + \frac{q \beta_{21} \beta_{16} [(1 + q^2)e^u - q^2(1 + q^4)]}{\beta_{15}(1 + q^3)} \right) \frac{G^{(+)}(u)}{e^u + 1}, \]

\[ k_{33} = k_{44} = \frac{2e^u G^{(+)}(u)}{e^{2u} - 1} + \left( \frac{\beta_{12} \beta_{15} (1 + q^3)e^u}{\beta_{16} q^3} + \frac{q \beta_{21} \beta_{16} [(1 + q^4)(e^u + q^3) + (1 + q)(1 + q^3)]}{\beta_{15}(1 + q^3)} \right) \frac{e^u G^{(+)}(u)}{e^u + 1}, \]

\[ k_{55} = \frac{2e^u G^{(+)}(u)}{e^{2u} - 1} - \left( \frac{\beta_{12} \beta_{15}}{\beta_{16}} \frac{1 + q^3}{q^3} + \frac{q \beta_{21} \beta_{16} [(1 + q^2)(e^u + q^3) + q^2(1 + q)(1 + q^3)e^u]}{\beta_{15}(1 + q^3)} \right) \frac{e^u G^{(+)}(u)}{e^u + 1}, \]

\[ k_{66} = \frac{2e^{3u} G^{(+)}(u)}{e^{2u} - 1} - \left( \frac{\beta_{12} \beta_{15}}{\beta_{16}} \frac{1 + q^3}{q^3} + \frac{q \beta_{21} \beta_{16} [(1 + q^2)(e^u + q^3) + q^2(1 + q)(1 + q^3)e^u]}{\beta_{15}(1 + q^3)} \right) \frac{e^u G^{(+)}(u)}{e^u + 1}, \]

(6.15)

and the non-diagonal terms are given by

\[ k_{12} = \beta_{12} G^{(+)}(u), \quad k_{15} = \beta_{15} G^{(+)}(u), \quad k_{16} = \beta_{16} \frac{e^u}{1 + q^3} G^{(+)}(u), \]

\[ k_{21} = \beta_{21} G^{(+)}(u), \quad k_{25} = -\beta_{21} \frac{\beta_{16}}{\beta_{12}} \frac{e^u + q^3}{1 + q^3} G^{(+)}(u), \quad k_{26} = \beta_{16} \frac{e^u}{1 + q^3} G^{(+)}(u), \]

\[ k_{51} = -\beta_{21} \frac{q^2 \beta_{12}}{\beta_{15}} G^{(+)}(u), \quad k_{52} = \beta_{21} \frac{q^2 \beta_{12} \beta_{16} e^u + q^3}{\beta_{15}} G^{(+)}(u), \quad k_{56} = -q \beta_{12} e^u G^{(+)}(u), \]

\[ k_{61} = -\beta_{21} \frac{q^2 \beta_{12}}{\beta_{15}} G^{(+)}(u), \quad k_{62} = -\beta_{21} \frac{q \beta_{12} \beta_{16} e^u}{\beta_{15}} G^{(+)}(u), \quad k_{65} = -q \beta_{21} e^u G^{(+)}(u), \]

(6.16)

where

\[ G^{(+)}(u) = \frac{1}{\beta_{16}} \left( \frac{1 + q^3}{e^u + q^3} \right) k_{16} \quad \text{and} \quad \beta_{21} = -\frac{\beta_{12} \beta_{15}}{q^3 \beta_{16}} + \frac{2}{(q + 1)(q^3 + 1)} \beta_{16}. \]

(6.17)

For \( n > 3 \) this type of solution follows the classification scheme presented in section 4.

7 Conclusion

In this work we have made a systematic study of the reflection equations for the vertex models associated with \( C_n^{(1)} (sp(2n)) \), \( D_n^{(1)} (o(2n)) \) and \( A_{2n-1}^{(1)} (sl(n)) \) affine Lie algebras. After looking for \( K \)-matrix solutions with all entries different from zero we have found two conjugated solutions for the \( sl(2n) \) models and one solution for the \( sp(2n) \) and also one solution for \( o(2n) \) models. In both cases these solutions have \( n \) free parameters.
From the reduction procedure developed in section 4 it has been possible to find a second solution for the $\text{sp}(2n)$ and $\text{o}(2n)$ models. These new solutions have the entries $k_{i,n} = k_{i,n+1} = k_{n,j} = k_{n,j+1} = 0$ with $i, j \neq n, n+1$, $k_{n,n} = k_{n+1,n+1} \neq 0$ and $n - 1$ free parameters. A third solution with one free parameter has also been obtained from this reduction procedure. Therefore, for $n > 2$ we have found three types of solutions for these models. For a given general solution there are $n - 1$ possible reductions of this type but only the first and the last gives us new solutions.

The special cases (with emphasis in the diagonal cases) were discussed separately in the other sections of this paper.

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