Balanced pairs and recollements in extriangulated categories with negative first extensions

Jian He and Panyue Zhou

Abstract

A notion of balanced pairs in an extriangulated category with a negative first extension is defined in this article. We prove that there exists a bijective correspondence between balanced pairs and proper classes $\xi$ with enough $\xi$-projectives and enough $\xi$-injectives. It can be regarded as a simultaneous generalization of Fu-Hu-Zhang-Zhu and Wang-Li-Huang. Besides, we show that if $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a recollement of extriangulated categories, then balanced pairs in $\mathcal{B}$ can induce balanced pairs in $\mathcal{A}$ and $\mathcal{C}$ under natural assumptions. As a application, this result generalizes a result by Fu-Hu-Yao in abelian categories. Moreover, it highlights a new phenomena when it applied to triangulated categories.

Keywords: extriangulated categories; proper class; balanced pair; recollement.

2020 Mathematics Subject Classification: 18G80; 18E10; 18E40.

1 Introduction

The recollement of triangulated categories was introduced first by Beilinson, Bernstein, and Deligne, see [BBD]. A fundamental example of a recollement situation of abelian categories appeared in the construction of perverse sheaves by MacPherson and Vilonen [MV]. Recollements of abelian and triangulated categories play an important role in ring theory, representation theory and geometry of singular spaces.

Relative homological algebra has been formulated by Hochschild in categories of modules and later by Heller, Butler and Horrocks in more general categories with a relative abelian structure. Beligiannis [B] studied a homological algebra in a triangulated category which parallels the homological algebra in an exact category in the sense of Quillen, by specifying a class of triangles $\xi$ which is called a proper class of triangles.

The notion of extriangulated categories was introduced by Nakaoka and Palu in [NP] as a simultaneous generalization of exact categories and triangulated categories. Exact categories (abelian categories are also exact categories) and extension closed subcategories of an extriangulated category are extriangulated categories, while there are some other examples of extriangulated categories which are neither exact nor triangulated, see [NP, ZZ, HZZ, NP1]. Hence many results hold on exact categories and triangulated categories can be unified in the same framework. Recently, Adachi, Enomoto and Tsukamoto [AET] introduced the notion

Panyue Zhou was supported by the National Natural Science Foundation of China (Grant No. 11901190) and the Scientific Research Fund of Hunan Provincial Education Department (Grant No. 19B239).
of extriangulated categories with negative first extensions. They also showed that exact categories and triangulated categories naturally admit negative first extension structures. Based on Beligiannis’s idea, Hu, Zhang and Zhou [HZZ] defined a proper class of extriangulated categories, they proved that if an extriangulated category \((C, E, s)\) was equipped with a proper class of \(E\)-triangles \(\xi\), then \(C\) had a new extriangulated structure. This construction gives extriangulated categories which are neither exact nor triangulated. Wang, Wei, and Zhang [WWZ] introduced the recollement of extriangulated categories, which is a simultaneous generalization of recollements of abelian categories and triangulated categories.

Chen [C] introduced the notion of balanced pair of additive subcategories in an abelian category. He studied relative homology with respect to balanced pairs in abelian category. Let \((A, B, C)\) be a recollement of abelian categories. Fu, Hu and Yao [FHY] proved that balanced pairs in \(B\) can induce balanced pairs in \(A\) and \(C\). Wang, Li, and Huang [WLH] showed that there exists a one-to-one correspondence between balanced pairs and Quillen exact structures \(\xi\) with enough \(\xi\)-projectives and enough \(\xi\)-injectives for abelian categories. Recently, the notion of balanced pair in triangulated categories was introduced by Fu, Hu, Zhang and Zhu [FHZZ], and proved a similar result to Wang, Li, and Huang [WLH]. More precisely, they showed there exists a bijective correspondence between balanced pairs and proper classes \(\xi\) with enough \(\xi\)-projectives and enough \(\xi\)-injectives. Inspired by this, we have a natural question whether their results of [WLH] and [FHZZ] can be unified under the framework of extriangulated categories. In this article, we give an affirmative answer in extriangulated categories with negative first extensions.

Let \(C\) be an extriangulated category with a negative first extension. Our first main result constructs a bijective correspondence between balanced pairs in \(C\) and proper classes \(\xi\) with enough \(\xi\)-projectives and enough \(\xi\)-injectives, see Theorem 3.10. This unifies their results of Fu, Hu, Zhang and Zhu [FHZZ] and Wang, Li and Huang [WLH] in the framework of extriangulated categories with negative first extensions. Suppose that \(B\) admits a recollement relative to extriangulated categories \(A\) and \(C\). Our second main result show that balanced pairs in \(B\) can induce balanced pairs in \(A\) and \(C\) under natural assumptions, see Theorem 4.6. This generalizes a result of Fu, Hu and Yao [FHY].

This article is organized as follows. In Section 2, we give some terminologies and some preliminary results. In Section 3, we prove our first main results. In Section 4, we prove our second main result.

## 2 Preliminaries

We briefly recall some definitions and basic properties of extriangulated categories from [NP]. We omit some details here, but the reader can find them in [NP].

Let \(C\) be an additive category equipped with an additive bifunctor

\[
E : C^{\text{op}} \times C \to \text{Ab},
\]

where Ab is the category of abelian groups. For any objects \(A, C \in C\), an element \(\delta \in E(C, A)\)
is called an E-extension. Let \( s \) be a correspondence which associates an equivalence class
\[
s(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]
\]
to any E-extension \( \delta \in \mathcal{E}(C, A) \). This \( s \) is called a realization of \( \mathcal{E} \), if it makes the diagrams in \([\text{NP}, \text{Definition 2.9}]\) commutative. A triplet \((C, \mathcal{E}, s)\) is called an extriangulated category if it satisfies the following conditions.

- \( \mathcal{E}: \text{C}^{\text{op}} \times \text{C} \rightarrow \text{Ab} \) is an additive bifunctor.
- \( s \) is an additive realization of \( \mathcal{E} \).
- \( \mathcal{E} \) and \( s \) satisfy the compatibility conditions in \([\text{NP}, \text{Definition 2.12}]\).

We collect the following terminology from \([\text{NP}]\).

**Definition 2.1.** Let \((C, \mathcal{E}, s)\) be an extriangulated category.

1. A sequence \( A \xrightarrow{x} B \xrightarrow{y} C \) is called a conflation if it realizes some E-extension \( \delta \in \mathcal{E}(C, A) \). In this case, \( x \) is called an inflation and \( y \) is called a deflation.
2. If a conflation \( A \xrightarrow{x} B \xrightarrow{y} C \) realizes \( \delta \in \mathcal{E}(C, A) \), we call the pair \((A \xrightarrow{x} B \xrightarrow{y} C, \delta)\) an E-triangle, and write it in the following way.

\[
A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta}
\]

We usually do not write this “\( \delta \)” if it is not used in the argument.

3. Let \( A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \) and \( A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{\delta'} \) be any pair of E-triangles. If a triplet \((a, b, c)\) realizes \((a, c): \delta \rightarrow \delta'\), then we write it as

\[
\begin{array}{c}
A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \\
\downarrow a \quad \downarrow b \quad \downarrow c \\
A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{\delta'}
\end{array}
\]

and call \((a, b, c)\) a morphism of E-triangles.

**Proposition 2.2.** \([\text{NP}, \text{Proposition 3.3}]\) Let \( C \) be an extriangulated category. For any E-triangle \( A \rightarrow B \rightarrow C \xrightarrow{\delta}, \) we have the following exact sequences:

\[
\mathcal{C}(C, -) \rightarrow \mathcal{C}(B, -) \rightarrow \mathcal{C}(A, -) \rightarrow \mathcal{E}(C, -) \rightarrow \mathcal{E}(B, -); \\
\mathcal{C}(-, A) \rightarrow \mathcal{C}(-, B) \rightarrow \mathcal{C}(-, C) \rightarrow \mathcal{E}(-, A) \rightarrow \mathcal{E}(-, B).
\]

Let us recall the definition of a negative first extension structure on an extriangulated category form \([\text{AET}]\).

**Definition 2.3.** Let \( \mathcal{C} \) be an extriangulated category. A negative first extension structure on \( \mathcal{C} \) consists of the following data:
(NE1) $E^{-1}: C^\text{op} \times C \to \text{Ab}$ is an additive bifunctor.

(NE2) For each $\delta \in \mathbb{E}(C, A)$, there exist two natural transformations

$$
\delta_{-1}^{-1}: E^{-1}(\cdot, C) \to C(\cdot, A), \\
\delta_{1}^{-1}: E^{-1}(A, \cdot) \to C(C, \cdot)
$$

such that for each $\mathbb{E}$-triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\delta}$

$$
E^{-1}(C, \cdot) \xrightarrow{\delta_{-1}^{-1}} E^{-1}(B, \cdot) \xrightarrow{\delta_{1}^{-1}} E^{-1}(A, \cdot) \xrightarrow{\delta} C(C, \cdot) \xrightarrow{\delta_{-1}^{-1}} C(B, \cdot); \\
E^{-1}(\cdot, A) \xrightarrow{\delta_{-1}^{-1}} E^{-1}(\cdot, B) \xrightarrow{\delta_{1}^{-1}} E^{-1}(\cdot, C) \xrightarrow{\delta} C(\cdot, A) \xrightarrow{\delta_{-1}^{-1}} C(\cdot, B);
$$

are exact.

Then we call $C = (C, \mathbb{E}, \delta, E^{-1})$ an extriangulated category with a negative first extension.

**Remark 2.4.** Typical examples of extriangulated categories are triangulated categories and exact categories (see [NP, Example 2.13]). Adachi, Enomoto and Tsukamoto [AET] show that both categories naturally admit negative first extension structures (see [AET, Example 2.4]).

Let $(C, \mathbb{E}, \delta)$ be an extriangulated category. Since $\mathbb{E}$ is a bifunctor, for any $a \in C(A, A')$ and $c \in C(C', C)$, we have $\mathbb{E}$-extensions

$$
\mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A') \quad \text{and} \quad \mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A).
$$

We simply denote them by $a \cdot \delta$ and $c^* \delta$.

**Lemma 2.5.** [NP, Proposition 3.15] Let $(C, \mathbb{E}, \delta)$ be an extriangulated category,

$$
A_1 \xrightarrow{x_1} B_1 \xrightarrow{y_1} C \xrightarrow{\delta_1} \quad \text{and} \quad A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C \xrightarrow{\delta_2}
$$

be any pair of $\mathbb{E}$-triangles. Then there exists a commutative diagram in $C$

$$
\begin{array}{ccc}
A_2 & \xrightarrow{m_2} & A_2 \\
\downarrow{m_1} & & \downarrow{x_2} \\
A_1 & \xrightarrow{e_1} & B_2 \\
\downarrow{e_2} & & \downarrow{y_2} \\
A_1 & \xrightarrow{x_1} & B_1 \xrightarrow{y_1} C
\end{array}
$$

which satisfies $\mathbb{E}(y_2^* \delta_1) = [A_1 \xrightarrow{m_1} M \xrightarrow{e_1} B_2]$ and $\mathbb{E}(y_1^* \delta_2) = [A_2 \xrightarrow{m_2} M \xrightarrow{e_2} B_1]$.

We now review the notion of proper classes of $\mathbb{E}$-triangles and its related properties from [HZZ]. From now on, assume that $(C, \mathbb{E}, \delta)$ is an extriangulated category.

A class of $\mathbb{E}$-triangles $\xi$ is closed under base change if for any $\mathbb{E}$-triangle

$$
A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \in \xi
$$
and any morphism \( c: C' \to C \), then any \( \mathbb{E}\)-triangle \( A \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{c'} \delta' \) belongs to \( \xi \).

Dually, a class of \( \mathbb{E}\)-triangles \( \xi \) is closed under isomorphisms and direct summands.

Definition 2.6. \( \mathbb{HZZ} \), Definition 3.1] Let \( \xi \) be a class of \( \mathbb{E}\)-triangles which is closed under isomorphisms. \( \xi \) is called a proper class of \( \mathbb{E}\)-triangles if the following conditions hold:

1. \( \xi \) is closed under finite coproducts and \( \Delta_0 \subseteq \xi \).
2. \( \xi \) is closed under base change and cobase change.
3. \( \xi \) is saturated.

We fix a proper class \( \xi \) of \( \mathbb{E}\)-triangles in the \( \mathcal{C} \).

Definition 2.7. \( \mathbb{HZZ} \), Definition 4.1] An object \( P \in \mathcal{C} \) is called \( \xi\)-projective if for any \( \mathbb{E}\)-triangle

\[
A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \to \in \xi
\]

and any morphism \( a: A \to A' \), then any \( \mathbb{E}\)-triangle \( A' \xrightarrow{x'} B' \xrightarrow{y'} C' \xrightarrow{a'} \delta' \) belongs to \( \xi \).

A class of \( \mathbb{E}\)-triangles \( \xi \) is called saturated if in the situation of Lemma 2.5, whenever

\[
A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C \xrightarrow{\delta_2} \to \quad \text{and} \quad A_1 \xrightarrow{m_1} M \xrightarrow{e_1} B_2 \xrightarrow{y_2\delta_1} \to
\]

belong to \( \xi \), then the \( \mathbb{E}\)-triangle

\[
A_1 \xrightarrow{x_1} B_1 \xrightarrow{y_1} C \xrightarrow{\delta_1} \to
\]

belongs to \( \xi \).

An \( \mathbb{E}\)-triangle \( A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{\delta} \to \) is called split if \( \delta = 0 \). It is easy to see that it is split if and only if \( x \) is section or \( y \) is retraction. The full subcategory consisting of the split \( \mathbb{E}\)-triangles will be denoted by \( \Delta_0 \).

An extriangulated category \( (\mathcal{C}, \mathbb{E}, \mathfrak{s}) \) is said to have enough \( \xi\)-projectives (respectively enough \( \xi\)-injectives) provided that for each object \( A \) there exists an \( \mathbb{E}\)-triangle \( K \xrightarrow{\varphi} P \xrightarrow{\alpha} A \xrightarrow{\delta} \) (respectively \( A \xrightarrow{\varphi} I \xrightarrow{\alpha} K \xrightarrow{\delta} \) ) in \( \xi \) with \( P \in \mathcal{P}(\xi) \) (respectively \( I \in \mathcal{I}(\xi) \)).

Definition 2.8. \( \mathbb{HZZ} \), Definition 4.4] An unbounded complex \( X \) is called \( \xi\)-exact complex if \( X \) is a diagram

\[
\cdots \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0 \xrightarrow{d_0} X_{-1} \xrightarrow{\delta} \cdots
\]

in \( \mathcal{C} \) such that for each integer \( n \), there exists an \( \mathbb{E}\)-triangle \( K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \xrightarrow{\delta_n} \) in \( \xi \) and \( d_n = g_{n-1}f_n \).

Definition 2.9. \( \mathbb{HZZ} \), Definition 4.5] Let \( \mathcal{W} \) be a class of objects in \( \mathcal{C} \). An \( \mathbb{E}\)-triangle

\[
A \xrightarrow{\varphi} B \xrightarrow{\psi} C \xrightarrow{\alpha} \to
\]
in $\xi$ is called to be $C(\cdot, W)$-exact (respectively $C(W, \cdot)$-exact) if for any $W \in W$, the induced sequence of abelian group $0 \to C(C, W) \to C(B, W) \to C(A, W) \to 0$ (respectively $0 \to C(W, A) \to C(W, B) \to C(W, C) \to 0$) is exact in Ab.

3 On the relation between proper classes of $E$-triangles and balanced pairs

In the rest of this article, unless otherwise stated, we always regard $C$ as an extriangulated category with a negative first extension.

Definition 3.1. Let $C$ be an extriangulated category. A subcategory $T$ of $C$ is called strongly exact-contravariantly finite, if for any object $C \in C$, there exists an $E$-triangle $K \to T \to C \delta \to 0$, $T \in T$, such that the induced sequence of abelian group $0 \to C(T', K) \to C(T', T) \to C(T', C) \to 0$ is exact in Ab, where $T' \in T$. i.e. the above $E$-triangle is $C(T, -)$-exact.

Dually, a subcategory $T$ of $C$ is called strongly exact-covariantly finite, if for any object $C \in C$, there exists an $E$-triangle $C \to T \to L \delta' \to 0$, $T \in T$, such that the induced sequence of abelian group $0 \to C(L, T') \to C(T, T') \to C(C, T') \to 0$ is exact in Ab, where $T' \in T$. i.e. the above $E$-triangle is $C(-, T)$-exact.

Remark 3.2. Let $T \subseteq C$ be a $\Sigma$-stable (i.e. $\Sigma T = T$) subcategory of triangulated category $C$. $T$ is strongly exact-contravariantly finite in $C$ if and only if $T$ is contravariantly finite in $C$, $T$ is strongly exact-covariantly finite in $C$ if and only if $T$ is covariantly finite in $C$ by Lemma 2.6 in [HYF].

Assume that $X$ is a full additive subcategory of $C$. For any $E$-triangle in $C$

$$T: X \to Y \to Z \delta \to 0,$$

We write

$$\xi_X = \left\{ \begin{array}{c|c} \text{the classes of } & \text{the } E\text{-triangle } T \text{ is } \\ E\text{-triangle } T & C(X, -)\text{-exact} \end{array} \right\}$$

Proposition 3.3. Let $X$ be a full additive subcategory of $C$ which is closed under direct summands. Then $\xi_X$ is a proper class of $C$. Moreover, $X$ is strongly exact-contravariantly finite in $C$ if and only if $C$ has enough $\xi_X$-projectives and $X = P(\xi_X)$. 
**Proof.** It is easy to see that $\xi_X$ is closed under isomorphisms, finite coproducts and containing all split $\mathbb{E}$-triangles. Next we claim that $\xi_X$ is closed under base change and cobase change.

Consider the following commutative diagram of $\mathbb{E}$-triangles:

$$
\begin{array}{ccc}
A & \xrightarrow{x'} & B' \\
\downarrow b & & \downarrow c \\
A & \xrightarrow{x} & B \\
\end{array}
$$

then we have the following commutative by Proposition 2.2

$$
\begin{array}{ccc}
C(X, A) & \xrightarrow{C(X,x')} & C(X, B') \\
\downarrow & & \downarrow \\
C(X, b) & \xrightarrow{C(X,y')} & C(X, c) \\
\end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & \longrightarrow \\
\xrightarrow{E(X, A)} & \xrightarrow{E(X, B')} & \xrightarrow{E(X, b)} \\
\end{array}
\begin{array}{ccc}
E(X, A) & \xrightarrow{E(X, x')} & E(X, B') \\
\downarrow & & \downarrow \\
E(X, A) & \xrightarrow{E(X, x)} & E(X, B) \\
\end{array}
$$

for any $X \in \mathcal{X}$. If the $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{-\delta} \Delta$ belongs to $\xi_X$, then $C(X, b)C(X, x') = C(X, x)$ is monic, thus $C(X, x')$ is monic. Similarly, since $C(X, y)$ is epic, then $E(X, b)E(X, x') = E(X, x)$ is monic, thus $E(X, x')$ is monic, i.e. $C(X, y')$ is epic. Hence $\xi_X$ is closed under base change.

Consider the following commutative diagram of $\mathbb{E}$-triangles:

$$
\begin{array}{ccc}
A & \xrightarrow{x'} & B' \\
\downarrow a & & \downarrow b \\
A' & \xrightarrow{x'} & B' \\
\end{array}
$$

then we have the following commutative by (NE2)

$$
\begin{array}{ccc}
\mathbb{E}^{-1}(X, B) & \xrightarrow{\mathbb{E}^{-1}(X,x)} & \mathbb{E}^{-1}(X, C) \\
\downarrow \mathbb{E}^{-1}(X,b) & & \downarrow \\
\mathbb{E}^{-1}(X, A') & \xrightarrow{\mathbb{E}^{-1}(X,x')} & \mathbb{E}^{-1}(X, C) \\
\end{array}
\begin{array}{ccc}
C(X, A) & \xrightarrow{C(X,x)} & C(X, B) \\
\downarrow & & \downarrow \\
C(X, A') & \xrightarrow{C(X,x')} & C(X, B') \\
\end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & \longrightarrow \\
\xrightarrow{E(X, A)} & \xrightarrow{E(X, B')} & \xrightarrow{E(X, b)} \\
\end{array}
\begin{array}{ccc}
E(X, A) & \xrightarrow{E(X, x)} & E(X, B) \\
\downarrow & & \downarrow \\
E(X, A) & \xrightarrow{E(X, x)} & E(X, B) \\
\end{array}
$$

for any $X \in \mathcal{X}$. If the $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{-\delta} \Delta$ belongs to $\xi_X$, then $C(X, b)C(X, y') = C(X, x)$ is epic, thus $C(X, y')$ is epic. Similarly, since $C(X, x)$ is monic, then $\mathbb{E}^{-1}(X, x')\mathbb{E}^{-1}(X, b) = \mathbb{E}^{-1}(X, x)$ is epic, thus $\mathbb{E}^{-1}(X, x')$ is epic, i.e. $C(X, x')$ is monic. Hence $\xi_X$ is closed under cobase change.

Consider the following commutative diagram of $\mathbb{E}$-triangles of Lemma 2.5:

$$
\begin{array}{ccc}
A_2 & \xrightarrow{A_2} & A_2 \\
\downarrow m_2 & & \downarrow x_z \\
A_1 & \xrightarrow{m_1} & M \\
\downarrow e_1 & & \downarrow B_2 \\
A_1 & \xrightarrow{x_1} & B_1 \\
\downarrow e_2 & & \downarrow y_2 \\
A_1 & \xrightarrow{x_1} & B_1 \\
\end{array}
\begin{array}{ccc}
C & \xrightarrow{C} & C \\
\end{array}
$$

\[ \text{for any } X \in \mathcal{X}. \]
then we have the following commutative by (NE2)

\[
\begin{array}{cccccc}
E^{-1}(X,M) & \xrightarrow{E^{-1}(X,e_1)} & E^{-1}(X,B_2) & \xrightarrow{C(X,A_1)} & C(X,M) & \xrightarrow{C(X,e_1)} & C(X,B_2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E^{-1}(X,e_2) & \xrightarrow{E^{-1}(X,y_2)} & E^{-1}(X,C) & \xrightarrow{C(X,z_1)} & C(X,B_1) & \xrightarrow{C(X,y_2)} & C(X,C)
\end{array}
\]

for any \(X \in \mathcal{X}\). If the \(\mathbb{E}\)-triangle \(A_1 \xrightarrow{m_1} M \xrightarrow{e_1} B_2 \rightarrow \) and \(A_2 \xrightarrow{x_2} B_2 \xrightarrow{y_2} C \rightarrow \) belong to \(\xi_X\), then we have \(C(X,e_1), E^{-1}(X,e_1), C(X,y_2)\) and \(E^{-1}(X,y_2)\) are epic. It is easy to see \(E^{-1}(X,y_1)\) and \(C(X,y_1)\) are epic, which implies that \(\mathbb{E}\)-triangle \(A_1 \xrightarrow{x_1} B_1 \xrightarrow{y_1} C \rightarrow \) belongs to \(\xi_X\), Hence \(\xi_X\) is saturated.

Finally, if \(\mathcal{C}\) has enough \(\xi_X\)-projectives and \(\mathcal{X} = \mathcal{P}(\xi_X)\) then it is easy to check that \(\mathcal{X}\) is strongly exact-contravariantly finite in \(\mathcal{C}\). Conversely, assume that \(\mathcal{X}\) is strongly exact-contravariantly finite in \(\mathcal{C}\). It is clear that \(\mathcal{X} \subseteq \mathcal{P}(\xi_X)\). On the other hand, let \(P \in \mathcal{P}(\xi_X)\). Then there exists an \(\mathbb{E}\)-triangle \(K \xrightarrow{} X \xrightarrow{} P \rightarrow \) in \(\xi_X\) with \(X \in \mathcal{X}\). Thus this \(\mathbb{E}\)-triangle is split, and hence \(P \in \mathcal{X}\). This completes the proof. \(\square\)

Assume that \(\mathcal{Y} \subseteq \mathcal{C}\) is a full additive subcategory of \(\mathcal{C}\). For any \(\mathbb{E}\)-triangle in \(\mathcal{C}\)

\[
T: X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\delta} ,
\]

We write

\[
\xi^\mathcal{Y} = \left\{ \text{the classes of } \mathbb{E}\text{-triangle } T \bigg| \text{the } \mathbb{E}\text{-triangle } T \text{ is } C(-,\mathcal{Y})\text{-exact} \right\}
\]

**Proposition 3.4.** Let \(\mathcal{Y}\) be a full additive subcategory of \(\mathcal{C}\) which is closed under direct summands. Then \(\xi^\mathcal{Y}\) is a proper class of \(\mathcal{C}\). Moreover, \(\mathcal{Y}\) is strongly exact-covariantly finite in \(\mathcal{C}\) if and only if \(\mathcal{C}\) has enough \(\xi^\mathcal{Y}\)-injectives and \(\mathcal{Y} = \mathcal{T}(\xi^\mathcal{Y})\).

**Proof.** This follows from the dual of Proposition 3.3 and Lemma 4.10 (1) in [HZZ]. \(\square\)

**Definition 3.5.** Let \(C \in \mathcal{C}\), \(\mathcal{X}\) and \(\mathcal{Y}\) be full additive subcategories of \(\mathcal{C}\).

1. An \(\mathcal{X}\)-resolution of \(C\) is a diagram \(X^\bullet \rightarrow C\) such that \(X^\bullet := \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow 0\) is a complex with \(X_i \in \mathcal{X}\) for all \(i \geq 0\), and \(\cdots \rightarrow X_1 \rightarrow X_0 \rightarrow C\) is a \(\xi^\mathcal{X}\)-exact complex. Moreover, the \(\mathcal{X}\)-resolution \(X^\bullet \rightarrow C\) of \(C\) is called \(C(-,\mathcal{Y})\)-exact if its \(\xi^\mathcal{X}\)-exact complex is \(C(-,\mathcal{Y})\)-exact.

2. An \(\mathcal{Y}\)-coresolution of \(C\) is a diagram \(C \rightarrow Y^\bullet\) such that \(Y^\bullet := 0 \rightarrow Y_0 \rightarrow Y_{-1} \rightarrow \cdots\) is a complex with \(Y_i \in \mathcal{Y}\) for all \(i \leq 0\), and \(C \rightarrow Y_0 \rightarrow Y_1 \rightarrow \cdots\) is a \(\xi^\mathcal{Y}\)-exact complex. Moreover, the \(\mathcal{Y}\)-resolution \(C \rightarrow Y^\bullet\) of \(C\) is called \(C(\mathcal{X},-\))\)-exact if its \(\xi^\mathcal{Y}\)-exact complex is \(C(\mathcal{X},-\))\)-exact.

**Remark 3.6.** Using standard arguments from relative homological algebra, one can prove a version of the comparison theorem for \(\mathcal{X}\)-resolution (resp. \(\mathcal{Y}\)-coresolution). It follows that any two \(\mathcal{X}\)-resolutions (resp. \(\mathcal{Y}\)-coresolutions) of an object \(C\) are homotopy equivalences.

Proposition 3.7. Let $\mathcal{X}$ and $\mathcal{Y}$ be full additive subcategory of $\mathcal{C}$ which are closed under direct summands. Then the following statements are equivalent:

1. $\xi_{\mathcal{X}} = \xi_{\mathcal{Y}}$, $\mathcal{X} = \mathcal{P}(\xi_{\mathcal{X}})$, $\mathcal{Y} = \mathcal{I}(\xi_{\mathcal{X}})$, and every object in $\mathcal{C}$ has enough enough $\xi_{\mathcal{X}}$-projectives and enough $\xi_{\mathcal{Y}}$-injectives.

2. The pair $(\mathcal{X}, \mathcal{Y})$ satisfies

   (B1) $\mathcal{X}$ is strongly exact-contravariantly finite and $\mathcal{Y}$ is strongly exact-covariantly finite in $\mathcal{C}$.

   (B2) For any object $M \in \mathcal{C}$, there exists an $\mathcal{X}$-resolution $X^\bullet \to M$ such that it is $C(-, \mathcal{Y})$-exact.

   (B3) For any object $N \in \mathcal{C}$, there exists a $\mathcal{Y}$-coresolution $N \to Y^\bullet$ such that it is $C(\mathcal{X}, -)$-exact.

Proof. The proof is an adaption of [FHZZ, Proposition 3.7].

(1) $\Rightarrow$ (2) It follows from Proposition 3.3 and Proposition 3.4.

(2) $\Rightarrow$ (1) By Proposition 3.3 and Proposition 3.4, it suffices to show $\xi_{\mathcal{X}} = \xi_{\mathcal{Y}}$. Let $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an $E$-triangle in $\xi_{\mathcal{X}}$. By hypothesis, there is an $\mathcal{X}$-resolution $X^\bullet \to C$ of $C$, such that it is $C(-, \mathcal{Y})$-exact. Then there exists a $\xi_{\mathcal{X}}$-exact complex

$$\cdots \xrightarrow{d_1} X_1 \xrightarrow{d_0} X_0 \xrightarrow{d_0} C$$

in $\mathcal{C}$ which is $C(-, \mathcal{Y})$-exact. This gives us an $E$-triangle $K_1 \xrightarrow{g_0} X_0 \xrightarrow{f_0} C \to 0$ which is $C(-, \mathcal{Y})$-exact, and hence we have the following commutative diagram of $E$-triangles

$$\begin{array}{ccc}
K_1 & \xrightarrow{g_0} & X_0 \\
| & & | \\
\alpha & \downarrow & \beta \\
A & \xrightarrow{f} & B \\
\end{array} \xrightarrow{g} \begin{array}{ccc}
\cdots & \xrightarrow{d_1} & X_1 \\
& & \xrightarrow{d_0} X_0 \\
& & \xrightarrow{d_0} C \\
\end{array} \to 0.$$

Let $Y \in \mathcal{Y}$, applying $C(-, \mathcal{Y})$ to the commutative diagram above, we have the following commutative diagram by Proposition 2.2

$$\begin{array}{cccccccc}
C(C, Y) & \xrightarrow{C(g, Y)} & C(B, Y) & \xrightarrow{C(f, Y)} & C(A, Y) & \xrightarrow{E(C, Y)} & E(C, Y) & \xrightarrow{E(g, Y)} & E(B, Y) \\
\| & & \| & & \| & & \| & & \| \\
C(C, Y) & \xrightarrow{C(f_0, Y)} & C(X_0, Y) & \xrightarrow{C(g_0, Y)} & C(K_1, Y) & \xrightarrow{E(C, Y)} & E(C, Y) & \xrightarrow{E(f_0, Y)} & E(X_0, Y).
\end{array}$$

Note that $C(f_0, Y)$ and $E(f_0, Y)$ are monic, then we have that $C(g, Y)$ and $E(g, Y)$ are monic. Thus the $E$-triangle $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\xi_{\mathcal{X}}$ is $C(-, \mathcal{Y})$-exact and it belongs to $\xi_{\mathcal{Y}}$. This implies that $\xi_{\mathcal{X}} \subseteq \xi_{\mathcal{Y}}$. Dually, one can show that $\xi_{\mathcal{Y}} \subseteq \xi_{\mathcal{X}}$. 

Now we introduce the notion of balanced pairs.
**Definition 3.8.** Suppose that \( \mathcal{X} \) and \( \mathcal{Y} \) are full additive subcategories of \( \mathcal{C} \) which are closed under direct summands. The pair \( (\mathcal{X}, \mathcal{Y}) \) is called a balanced pair if it satisfies one of the equivalence conditions of of Proposition 3.7.

**Remark 3.9.** (1) By Remark 3.2, if the category \( \mathcal{C} \) are triangulated, \( \mathcal{X} \) and \( \mathcal{Y} \) are \( \Sigma \)-stable, then Definition 3.8 coincides with the definition of balanced pairs of triangulated categories (cf. [FHZZ]). If the category \( \mathcal{C} \) are abelian, then Definition 3.8 coincides with the definition of balanced pairs of abelian categories (cf. [C]).

(2) By Remark 3.6, the condition (B1) may be rephrased as: any \( \mathcal{X} \)-resolution of \( M \) is \( \mathcal{C}(\mathcal{X}, \mathcal{Y}) \)-exact. Similarly, the condition (B2) may be rephrased as: any \( \mathcal{Y} \)-coresolution of \( N \) is \( \mathcal{C}(\mathcal{X}, \mathcal{Y}) \)-exact.

Our first main result is the following.

**Theorem 3.10.** Let \( \mathcal{C} \) be an extriangulated category with a negative first extension. The assignments

\[
\Psi : (\mathcal{X}, \mathcal{Y}) \mapsto \xi_X = \xi^Y \quad \text{and} \quad \Phi : \xi \mapsto (P(\xi), I(\xi))
\]

give mutually inverse bijections between the following classes:

(1) Balanced pairs \( (\mathcal{X}, \mathcal{Y}) \) in \( \mathcal{C} \).

(2) Proper classes \( \xi \) in \( \mathcal{C} \) with enough \( \xi \)-projectives and enough \( \xi \)-injectives.

**Proof.** Let \( (\mathcal{X}, \mathcal{Y}) \) be a balanced pair in \( \mathcal{C} \). Then \( \xi_X = \xi^Y \) is the desired proper class such that \( \mathcal{X} = P(\xi_X) \) and \( \mathcal{Y} = I(\xi^Y) \) by Proposition 3.7. Conversely, if \( \xi \) is a proper class in \( \mathcal{C} \) with enough \( \xi \)-projectives and enough \( \xi \)-injectives. We put \( (\mathcal{X}, \mathcal{Y}) = (P(\xi), I(\xi)) \) is a balanced pair by Proposition 3.7.

For any balanced pair \( (\mathcal{X}, \mathcal{Y}) \), it is easy to check that

\[
\Phi \Psi (\mathcal{X}, \mathcal{Y}) = \Phi (\xi_X = \xi^Y) = (P(\xi_X), I(\xi^Y)) = (\mathcal{X}, \mathcal{Y}).
\]

On the other hand, if \( \xi \) is a proper class in \( \mathcal{C} \) with enough \( \xi \)-projectives and enough \( \xi \)-injectives, it is easy to see that

\[
\Psi \Phi (\xi) = \Psi (P(\xi), I(\xi)) = (P(\xi_X), I(\xi^Y)) = \xi.
\]

This completes the proof.

By applying Theorem 3.10 to triangulated categories, we get the following.

**Corollary 3.11.** [FHZZ, Theorem 1.1] Let \( \mathcal{C} \) as a triangulated category. The assignments \( \Psi : (\mathcal{X}, \mathcal{Y}) \mapsto \xi_X = \xi^Y \) and \( \Phi : \xi \mapsto (P(\xi), I(\xi)) \) give mutually inverse bijections between the following classes:

(1) Balanced pairs \( (\mathcal{X}, \mathcal{Y}) \) in \( \mathcal{C} \).

(2) Proper classes \( \xi \) in \( \mathcal{C} \) with enough \( \xi \)-projectives and enough \( \xi \)-injectives.
Proof. This follows from Theorem 3.10 and Remark 2.4 and Remark 3.9.

By applying Theorem 3.10 to abelian categories, we get the following.

**Corollary 3.12.** [WLH, Theorem 2.2] Let $\mathcal{C}$ be an abelian category, there exists a one-to-one correspondence between balanced pairs and Quillen exact structures $\xi$ with enough $\xi$-pojectives and enough $\xi$-injectives.

**Proof.** This follows from Theorem 3.10 and Remark 2.4 and Remark 3.9.

**Corollary 3.13.** Let $(\mathcal{X}, \mathcal{Y})$ be a balanced pair in an extriangulated category $\mathcal{C}$ with a negative first extension. Then $\xi := \xi_{\mathcal{X}} = \xi_{\mathcal{Y}}$ is a proper class in $\mathcal{C}$. With the notation above, we set $E_\xi := E|_{\xi}$, $E^{-1}_\xi := E^{-1}|_{\xi}$, that is,

$$E_\xi(C, A) = \{\delta \in E(C, A) \mid \delta \text{ is realized as an E-triangle } A \xrightarrow{x} B \xrightarrow{y} C \xrightarrow{-\delta} \text{ in } \xi\}$$

for any $A, C \in \mathcal{C}$, and $s_\xi := s|_{E_\xi}$. Hence $(\mathcal{C}, E_\xi, E^{-1}_\xi, s_\xi)$ is an extriangulated category with a negative first extension.

**Proof.** This follows from Theorem 3.10 and Theorem 3.2 in [HZZ].

### 4 Glued balanced pairs

We always assume that any extriangulated category satisfies the (WIC) condition, see [NP, Condition 5.8]. We briefly recall of the concepts and basic properties of recollements of extriangulated categories from [WWZ]. We omit some details here, but the reader can find them in [WWZ].

**Definition 4.1.** [WWZ, Definition 3.1] Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three extriangulated categories. A **recollement** of $\mathcal{B}$ relative to $\mathcal{A}$ and $\mathcal{C}$, denoted by $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, is a diagram

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{i_*} & \mathcal{B} \\
\downarrow{i^!} & & \downarrow{j^!} \\
\mathcal{C} & \xleftarrow{j_*} & \mathcal{B}
\end{array}
$$

(4.1)

given by two exact functors $i_*, j^*$, two right exact functors $i^*, j_!$ and two left exact functors $i^!, j_*$, which satisfies the following conditions:

(R1) $(i^*, i_*, i^!)$ and $(j_!, j^*, j_*)$ are adjoint triples.

(R2) $\text{Im } i_* = \text{Ker } j^*$.

(R3) $i_*$, $j_!$ and $j_*$ are fully faithful.

(R4) For each $X \in \mathcal{B}$, there exists a left exact $E_{\mathcal{B}}$-triangle sequence

$$i_* i^! X \xrightarrow{\theta_X} X \xrightarrow{\vartheta_X} j_* j^* X \xrightarrow{i_* A}$$

(4.2)

with $A \in \mathcal{A}$, where $\theta_X$ and $\vartheta_X$ are given by the adjunction morphisms.
(R5) For each $X \in \mathcal{B}$, there exists a right exact $\mathcal{E}_B$-triangle sequence

$$i_* A' \longrightarrow j_! j^* X \longrightarrow \nu_X \longrightarrow i_* i^* X$$

(4.3)

with $A' \in \mathcal{A}$, where $\nu_X$ and $\nu_X$ are given by the adjunction morphisms.

**Remark 4.2.** (1) If the categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are abelian, then Definition 4.1 coincides with the definition of recollement of abelian categories (cf. [MH, P, FP]).

(2) If the categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are triangulated, then Definition 4.1 coincides with the definition of recollement of triangulated categories (cf. [BBD]).

(3) There is an example of recollement of an extriangulated category which is neither abelian nor triangulated, for more details, see [WWZ] and [HHZ].

We collect some properties of a recollement of extriangulated categories, which will be used in the sequel.

**Lemma 4.3.** [WWZ, Proposition 3.3] Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a recollement of extriangulated categories as (4.1). Then

1. all the natural transformations

   $$i^* i_* \Rightarrow \text{Id}_\mathcal{A}, \text{Id}_\mathcal{A} \Rightarrow i^! i^*, \text{Id}_\mathcal{C} \Rightarrow j_* j^!, \ j^* j_* \Rightarrow \text{Id}_\mathcal{C}$$

   are natural isomorphisms.

   1. $i^* j_! = 0$ and $i^! j_* = 0$.

   3. if $i^*$ is exact, then $j_!$ is exact.

   3′. if $i^!$ is exact, then $j_*$ is exact.

**Lemma 4.4.** Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a recollement of extriangulated categories.

1. If $\mathcal{X}$ is a strongly exact-contravariantly finite subcategory of $\mathcal{B}$ and $j_! j^* \mathcal{X} \subseteq \mathcal{X}$, then $j^* \mathcal{X}$ is a strongly exact-contravariantly finite subcategory of $\mathcal{C}$.

2. If $\mathcal{Y}$ is a strongly exact-covariantly finite subcategory of $\mathcal{B}$ and $j_* j^* \mathcal{Y} \subseteq \mathcal{Y}$, then $j^* \mathcal{Y}$ is a strongly exact-covariantly finite subcategory of $\mathcal{C}$.

**Proof.** (1) For any $C$ in $\mathcal{C}$, $j_! C \in \mathcal{B}$. There exists an $\mathcal{E}_B$-triangle

$\begin{array}{ccc} K & \xrightarrow{f} & X_0 \\ & \xrightarrow{g} & j_* C \rightarrow \rightarrow \end{array}$

with $X_0 \in \mathcal{X}$, such that the sequence of abelian group

$$0 \rightarrow C(X, K) \rightarrow C(X, X_0) \rightarrow C(X, j_* C) \rightarrow 0$$

is exact in Ab, for any $X \in \mathcal{X}$, since $\mathcal{X}$ is a strongly exact-contravariantly finite subcategory of $\mathcal{B}$. Because $j^*$ is exact, note that $j^* j_* \Rightarrow \text{Id}_C$ is a natural isomorphism, applying $j^*$ to the above $\mathcal{E}_B$-triangle, we obtain an $\mathcal{E}_C$-triangle

$\begin{array}{ccc} j^* K & \xrightarrow{j^* f} & j^* X_0 \\ & \xrightarrow{j^* g} & C \rightarrow \rightarrow \end{array}$

with $j^* X_0 \in j^* \mathcal{X}$. We need to show that the sequence of abelian group

$$0 \rightarrow C(j^* X, j^* K) \rightarrow C(j^* X, j^* X_0) \rightarrow C(j^* X, C) \rightarrow 0$$
is exact in Ab, for any \( X \in \mathcal{X} \). Since \((j_1, j^*, j_\ast)\) is a adjoint triple, there exists a commutative diagram

\[
\begin{align*}
\mathcal{C}(j_1 j^* X, K) & \xrightarrow{\mathcal{C}(j_1 j^* f)} \mathcal{C}(j_1 j^* X, X_0) \\
\cong & \mathcal{C}(j^* X, j^* K) \xrightarrow{\mathcal{C}(j^* j^* f)} \mathcal{C}(j^* X, j^* X_0) \xrightarrow{\mathcal{C}(j^* j^* g)} \mathcal{C}(j^* X, C) \\
\cong & \mathcal{C}(X, j_\ast j^* X_0) \xrightarrow{\mathcal{C}(X, j_\ast j^* g)} \mathcal{C}(X, j_\ast C).
\end{align*}
\]

Therefore, it suffices to show that the first row is monic and the third row is epic by the above commutative diagram. Since the \( \mathcal{E}_B\)-triangle

\[
\begin{array}{c}
K \\
\xrightarrow{f}
\end{array}
\begin{array}{c}
X_0 \\
\xrightarrow{g}
\end{array}
\begin{array}{c}
j_\ast C
\end{array}
\]

is \( \mathcal{C}(\mathcal{X}, -)\)-exact, \( j_1 j^* \mathcal{X} \subseteq \mathcal{X} \), then \( \mathcal{C}(j_1 j^* X, f) \) is monic. On the other hand, we need to show that there exists a morphism \( h : X \rightarrow j_\ast j^* X_0 \), such that \( j_\ast j^* g \circ h = \gamma \) for any \( \gamma : X \rightarrow j_\ast C \).

Suppose that \( \nu_{X_0} \) is the unit of the adjoint pair \((j^*, j_\ast)\), we have the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{s} & X_0 \\
\downarrow{h} \quad \downarrow{g} & & \downarrow{\nu_{X_0}} \\
& j_\ast C & \\
\downarrow{\gamma} & & \downarrow{
\begin{array}{c}
\nu_{X_0} \\
\xrightarrow{j_1 j^*}
\end{array}
\}
\end{array}
\]

Since \( K \xrightarrow{f} X_0 \xrightarrow{g} j_\ast C \) is \( \mathcal{C}(\mathcal{X}, -)\)-exact, then there exists a morphism \( s : X \rightarrow X_0 \) such that \( gs = \gamma \). Therefore we just need to take \( h = \nu_{X_0} s \), that is, \( \mathcal{C}(X, j_\ast j^* g) \) is epic. This shows that \( j^* \mathcal{X} \) is a strongly exact-contravariantly finite subcategory of \( \mathcal{C} \).

(2) For any \( C \in \mathcal{C} \) and \( j_1 C \in \mathcal{B} \), there exists an \( \mathcal{E}_B\)-triangle

\[
\begin{array}{c}
j_1 C \\
\xrightarrow{f}
\end{array}
\begin{array}{c}
Y_0 \\
\xrightarrow{g}
\end{array}
\begin{array}{c}
L
\end{array}
\]

with \( Y_0 \in \mathcal{Y} \), such that the sequence of abelian group

\[
0 \rightarrow \mathcal{C}(L, Y) \rightarrow \mathcal{C}(Y_0, Y) \rightarrow \mathcal{C}(j_1 C, Y) \rightarrow 0
\]

is exact in Ab, for any \( Y \in \mathcal{Y} \), since \( \mathcal{Y} \) is a strongly exact-covariantly finite subcategory of \( \mathcal{B} \).

Because \( j^* \) is exact, note that \( \text{Id}_\mathcal{C} \Rightarrow j_1 j^* \) is a natural isomorphism, applying \( j^* \) to the above \( \mathcal{E}_B\)-triangle, we obtain an \( \mathcal{E}_C\)-triangle

\[
\begin{array}{c}
C \\
\xrightarrow{j^* f}
\end{array}
\begin{array}{c}
j^* Y_0 \\
\xrightarrow{j^* g}
\end{array}
\begin{array}{c}
j^* L
\end{array}
\]

with \( j^* Y_0 \in j^* \mathcal{Y} \). We need to show that the sequence of abelian group

\[
0 \rightarrow \mathcal{C}(j^* L, j^* Y) \rightarrow \mathcal{C}(j^* Y_0, j^* Y) \rightarrow \mathcal{C}(C, j^* Y) \rightarrow 0
\]
is exact in $\text{Ab}$, for any $Y \in \mathcal{Y}$. Since $(j_!, j^*, j_*)$ is an adjoint triple, there is a commutative diagram

$$
\begin{array}{ccc}
C(L, j_* j^* Y) & \xrightarrow{C(g,j^* j^* Y)} & C(Y_0, j_* j^* Y) \\
\cong & & \cong \\
C(j_* L, j^* Y) & \xrightarrow{C(j^* f, j^* Y)} & C(j_* Y_0, j^* Y) \\
\cong & & \cong \\
C(j_* j^* Y_0, Y) & \xrightarrow{C(j_* j^* j^*)} & C(j_* C, Y).
\end{array}
$$

Therefore, it suffices to show that the first row is monic and the third row is epic by the above commutative diagram. Since the $E_B$-triangle $j_* C \xrightarrow{f} Y_0 \xrightarrow{g} L \rightarrow -$ is $\mathcal{C}(−, \mathcal{Y})$-exact, $j_* j^* \mathcal{Y} \subseteq \mathcal{Y}$, then $\mathcal{C}(g, j_* j^* Y)$ is monic. On the other hand, we need to show that there exists a morphism $\beta : j_* j^* Y_0 \rightarrow Y$, such that $\beta \circ j_* j^* f = \delta$ for any $\delta : j_* C \rightarrow Y$. Suppose that $\mu_{Y_0}$ is the counit of the adjoint pair $(j_!, j^*)$, we have the following commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\delta} & j_* j^* Y_0 \\
\downarrow{\beta} & & \downarrow{\mu_{Y_0}} \\
j_* C & \xrightarrow{f, Y} & j_* C \\
\downarrow{g} & & \downarrow{j_* f} \\
Y_0.
\end{array}
$$

Since $j_* C \xrightarrow{f} Y_0 \xrightarrow{g} L \rightarrow -$ is $\mathcal{C}(−, \mathcal{Y})$-exact, then there exists a morphism $t : Y_0 \rightarrow Y$, such that $tf = \delta$. Therefore we just need to take $\beta = t \mu_{Y_0}$, that is, $\mathcal{C}(j_* j^* f, Y)$ is epic.

This shows that $j^* \mathcal{Y}$ is a strongly exact-covariantly finite subcategory of $\mathcal{C}$. \hfill \Box

**Lemma 4.5.** Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a recollement of extriangulated categories.

1. If $\mathcal{X}$ is a strongly exact-contravariantly finite subcategory of $\mathcal{B}$ and $E_{\mathcal{A}}^{-1}(i^* \mathcal{X}, \mathcal{A}) = 0$, $i^*$ is exact, then $i^* \mathcal{X}$ is a strongly exact-contravariantly finite subcategory of $\mathcal{A}$;

2. If $\mathcal{Y}$ is a strongly exact-covariantly finite subcategory of $\mathcal{B}$ and $E_{\mathcal{A}}^{-1}(\mathcal{A}, i^! \mathcal{Y}) = 0$, $i^!$ is exact, then $i^! \mathcal{Y}$ is a strongly exact-covariantly finite subcategory of $\mathcal{A}$.

**Proof.** It is similar to the proof of Lemma 4.4. \hfill \Box

Our second main result is the following.

**Theorem 4.6.** Let $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ be a recollement of extriangulated categories. Suppose that $(\mathcal{X}, \mathcal{Y})$ is a balanced pair in $\mathcal{B}$ with $j_* j^* \mathcal{X} \subseteq \mathcal{X}$, $j_* j^* \mathcal{Y} \subseteq \mathcal{Y}$, $i_* i^* \mathcal{X} \subseteq \mathcal{X}$, $i_* i^! \mathcal{Y} \subseteq \mathcal{Y}$.

1. If $E_{\mathcal{C}}^{-1}(\mathcal{C}, j^* \mathcal{Y}) = E_{\mathcal{C}}^{-1}(j^* \mathcal{X}, \mathcal{C}) = 0$, then $(j^* \mathcal{X}, j^* \mathcal{Y})$ is a balanced pair in $\mathcal{C}$;

2. If $i^*$ and $i^!$ are exact, $E_{\mathcal{A}}^{-1}(i^* \mathcal{X}, \mathcal{A}) = E_{\mathcal{A}}^{-1}(\mathcal{A}, i^! \mathcal{Y}) = 0$, then $(i^* \mathcal{X}, i^! \mathcal{Y})$ is a balanced pair in $\mathcal{A}$.

**Proof.** We check the conditions of balanced pair.

1. (B1) This follows form Lemma 4.4.
(B2) For any $C \in \mathcal{C}$, $j_* C \in \mathcal{B}$. Since $(\mathcal{X}, \mathcal{Y})$ is a balanced pair in $\mathcal{B}$, it follows that there exists a $\mathcal{X}$-resolution as follows

$$
\cdots \xrightarrow{} X_2 \xrightarrow{} X_1 \xrightarrow{} X_0 \xrightarrow{} j_* C \tag{4.4}
$$

where $K_0 \xrightarrow{} X_0 \xrightarrow{} j_* C \xrightarrow{} 0$ and $K_i \xrightarrow{} X_i \xrightarrow{} K_{i-1} \xrightarrow{} 0$ in $\xi_{\mathcal{X}}$, $i \geq 1$. Moreover, the $\mathcal{X}$-resolution is $C(-, \mathcal{Y})$-exact.

Since $j^*$ is exact, note that $j^* j_* \Rightarrow \text{Id}_C$ is a natural isomorphism. Applying $j^*$ to (4.4), we obtain a complex as follows

$$
\cdots \xrightarrow{} j^* X_2 \xrightarrow{} j^* X_1 \xrightarrow{} j^* X_0 \xrightarrow{} C \tag{4.5}
$$

We claim that (4.5) is $C(j^* \mathcal{X}, -)$-exact. In fact, for any $X \in \mathcal{X}$, we have the following commutative diagram

$$
C(j^* X, j^* K_i) \xrightarrow{} C(j^* X, j^* X_i) \xrightarrow{} C(j^* X, j^* K_{i-1}) \xrightarrow{} \cdots \xrightarrow{} 0 \xrightarrow{} C(j^* j^* X, K_i) \xrightarrow{} C(j^* j^* X, X_i) \xrightarrow{} C(j^* j^* X, K_{i-1}) \xrightarrow{} 0.
$$

Note that $j^* j^* \mathcal{X} \subseteq \mathcal{X}$, then the first row is exact since the second row is exact in the above diagram. Besides, the sequence $0 \to C(j^* X, j^* K_0) \to C(j^* X, j^* X_0) \to C(j^* X, C) \to 0$ is exact by Lemma 4.4 (1).

Next, we need to show (4.5) is $C(-, j^* \mathcal{Y})$-exact by Remark 3.9 (2). For any $Y \in \mathcal{Y}$, we have the following commutative diagram

$$
C(j^* K_{i-1}, j^* Y) \xrightarrow{} C(j^* X_i, j^* Y) \xrightarrow{} C(j^* K_i, j^* Y) \xrightarrow{} \cdots \xrightarrow{} 0 \xrightarrow{} C(K_{i-1}, j_\ast j^* Y) \xrightarrow{} C(X_i, j_* j^* Y) \xrightarrow{} C(K_i, j_* j^* Y) \xrightarrow{} 0.
$$

Note that $j_* j^* \mathcal{Y} \subseteq \mathcal{Y}$, then the first row is exact since the second row is exact in the above diagram. Moreover, consider the following commutative diagram

$$
E_C^{-1}(j^* K_0, j^* Y) \xrightarrow{} C(C, j^* Y) \xrightarrow{} C(j^* X_0, j^* Y) \xrightarrow{} C(j^* K_0, j^* Y) \xrightarrow{} \cdots \xrightarrow{} 0 \xrightarrow{} C(X_0, j_* j^* Y) \xrightarrow{} C(K_0, j_* j^* Y) \xrightarrow{} 0.
$$

Since the second row is exact in the above diagram, note that $E_C^{-1}(C, j^* \mathcal{Y}) = 0$, then the sequence $0 \to C(C, j^* Y) \to C(j^* X_0, j^* Y) \to C(j^* K_0, j^* Y) \to 0$ is exact.

(B3) It is a dual of the proof of (B2). This completes the proof of (1).
(2) (B1) This follows form Lemma 4.5.

(B2) Since \( i^*_\mathcal{X} \) is a strongly exact-contravariantly finite subcategory of \( \mathcal{A} \), there exists a \( i^*_\mathcal{X} \)-resolution as follows

\[
\cdots \rightarrow i^*X_2 \rightarrow i^*X_1 \rightarrow i^*X_0 \rightarrow A
\]  

(4.6)

where \( K_0 \rightarrow i^*X_0 \rightarrow A \rightarrow \rightarrow \) and \( K_i \rightarrow i^*X_i \rightarrow K_{i-1} \rightarrow \rightarrow \) in \( \xi_{i^*_\mathcal{X}}, \) \( i \geq 1, \) for any \( A \in \mathcal{A} \). It suffices to show that the resolution (4.6) is \( \mathcal{C}(\mathcal{X}, -)^{-}\)-exact. Since \( i_* \) is exact, applying \( i_* \) to (4.6), we obtain a complex as follows

\[
\cdots \rightarrow i_*i^*X_2 \rightarrow i_*i^*X_1 \rightarrow i_*i^*X_0 \rightarrow i_*A
\]  

(4.7)

Note that \( i_*i^*_\mathcal{X} \subseteq \mathcal{X} \), We claim that (4.7) is a \( \mathcal{X} \)-resolution of \( i_*A \). In fact, it is easy to see the resolution (4.7) is \( \mathcal{C}(\mathcal{X}, -)^{-}\)-exact since the resolution (4.6) is \( \mathcal{C}(i^*_\mathcal{X}, -)^{-}\)-exact and \( (i^*, i_*) \) is an adjoint pair. Since \( (\mathcal{X}, \mathcal{Y}) \) is a balanced pair in \( \mathcal{B} \), it follows that (4.7) is \( \mathcal{C}(-, \mathcal{Y})^{-}\)-exact by Remark 3.9 (2). Moreover, since \( (i_*, i^!) \) is an adjoint pair, there exits the following two commutative diagrams:

\[
\begin{array}{ccc}
\mathcal{C}(A, j^!\mathcal{Y}) & \rightarrow & \mathcal{C}(i^*X_0, i^!\mathcal{Y}) \\
\downarrow \cong & & \downarrow \cong \\
0 & \rightarrow & \mathcal{C}(i_*A, Y) \\
\end{array}
\]  

(4.8)

and

\[
\begin{array}{ccc}
\mathcal{C}(K_{i-1}, i^!\mathcal{Y}) & \rightarrow & \mathcal{C}(i^*X_i, i^!\mathcal{Y}) \\
\downarrow \cong & & \downarrow \cong \\
0 & \rightarrow & \mathcal{C}(i_*K_{i-1}, Y) \\
\end{array}
\]  

(4.9)

where the second rows are exact of (4.8) and (4.9). Thus, the resolution (4.6) is \( \mathcal{C}(-, i^!\mathcal{Y})^{-}\)-exact.

(B3) It is a dual of the proof of (B2). This completes the proof of (2).

By applying Theorem 4.6 to abelian categories, we have the following.

**Corollary 4.7.** [FHY, Proposition 2.3] Let \( (\mathcal{A}, \mathcal{B}, \mathcal{C}) \) be a recollement of abelian categories. If \( (\mathcal{X}, \mathcal{Y}) \) is a balanced pair in \( \mathcal{B} \) with \( j_!j^*_\mathcal{X} \subseteq \mathcal{X}, \ j_*j^*\mathcal{Y} \subseteq \mathcal{Y}, \ i_*i^*_\mathcal{X} \subseteq \mathcal{X}, \ i_*i^!\mathcal{Y} \subseteq \mathcal{Y}, \) then

1. \( (j^*_\mathcal{X}, j^*\mathcal{Y}) \) is a balanced pair in \( \mathcal{C}; \)
2. \( (i^*_\mathcal{X}, i^!\mathcal{Y}) \) is a balanced pair in \( \mathcal{A}. \)

**Proof.** Since \( \mathcal{C}(\mathcal{X}, -) \) and \( \mathcal{C}(-, \mathcal{Y}) \) are left exact, this assumptions of Theorem 4.6 are not necessary in abelian categories. This follows from Theorem 4.6 and Remark 3.9. \( \square \)
References

[AET] T. Adachi, H. Enomoto, M. Tsukamoto. Intervals of s-torsion pairs in extriangulated categories with negative first extensions, arXiv: 2013.09549v1, 2021.

[B] A. Beligiannis. Relative homological algebra and purity in triangulated categories. J. Algebra 227: 268–361, 2000.

[BBD] A. Beilinson, J. Bernstein, P. Deligne. Faisceaux pervers, in: Analysis and topology on singular spaces, I, Luminy, 1981, Astérisque, vol. 100, Soc. Math. France, Paris, 5–171, 1982.

[C] X. Chen. Homotopy equivalences induced by balanced pairs. J. Algebra 324: 2718–2731, 2010.

[FHY] X. Fu, Y. Hu, H. Yao. The resolution dimensions with respect to balanced pairs in the recollement of abelian categories. J. Korean Math. Soc. 56(4): 1031–1048, 2019.

[FHZZ] X. Fu, J. Hu, D. Zhang, H. Zhu. Balanced pairs on triangulated categories. arXiv: 2109.00932, 2021.

[FP] V. Franjou, T. Pirashvili. Comparison of abelian categories recollements. Doc. Math. 9: 41–56, 2004.

[HYF] Y. Hu, H. Yao, X. Fu. Tilting objects in triangulated categories. Comm. Algebra 48: 410–429, 2020.

[HZZ] J. Hu, D. Zhang, P. Zhou. Proper classes and Gorensteinness in extriangulated categories. J. Algebra 551: 23–60, 2020.

[HHZ] J. He, Y. Hu, P. Zhou. Torsion pairs and recollements of extriangulated categories. arXiv: 2014.04924v1, 2021.

[MH] X. Ma, Z. Huang. Torsion pairs in recollements of abelian categories, Front. Math. China, 13(4): 875–892, 2018.

[MV] R. MacPherson, K. Vilonen. Elementary construction of perverse sheaves. Invent. Math. 84(2): 403–435, 1986.

[NP] H. Nakaoka, Y. Palu. Extriangulated categories, Hovey twin cotorsion pairs and model structures. Cah. Topol. Géom. Différ. Catég. 60(2): 117–193, 2019.

[NP1] H. Nakaoka, Y. Palu. External triangulation of the homotopy category of exact quasi-category. arXiv: 2004.02479, 2020.

[P] C. Psaroudakis. Homological theory of recollements of abelian categories. J. Algebra, 398: 63–110, 2014.
[WLH] J. Wang, H. Li, Z. Huang, Applications of exact structures in abelian categories. Publ. Math. Debrecen, 88: 269-286, 2016.

[WWZ] L. Wang, J. Wei, H. Zhang, Recollements of extriangulated categories. arXiv: 2012.03258v1, 2020.

[ZZ] P. Zhou, B. Zhu. Triangulated quotient categories revisited. J. Algebra 502: 196–232, 2018.

Jian He
Department of Mathematics, Nanjing University, 210093 Nanjing, Jiangsu, P. R. China
E-mail: jianhe30@163.com

Panyue Zhou
College of Mathematics, Hunan Institute of Science and Technology, 414006 Yueyang, Hunan, P. R. China.
E-mail: panyuezhou@163.com