HECKE OPERATORS ON WEIGHTED DEDEKIND SYMBOLS

SHINJI FUKUHARA

Abstract. Dedekind symbols generalize the classical Dedekind sums (symbols). The symbols are determined uniquely by their reciprocity laws up to an additive constant. There is a natural isomorphism between the space of Dedekind symbols with polynomial (Laurent polynomial) reciprocity laws and the space of cusp (modular) forms. In this article we introduce Hecke operators on the space of weighted Dedekind symbols. We prove that these newly introduced operators are compatible with Hecke operators on the space of modular forms. As an application, we present formulae to give Fourier coefficients of Hecke eigenforms. In particular we give explicit formulae for generalized Ramanujan’s tau functions.

1. Introduction and statement of results

This article is a continuation of our study ([5, 6, 7]) on Dedekind symbols and modular forms. Here we introduce and investigate Hecke operators on Dedekind symbols, and investigate their properties.

First let us recall a few definitions in [5] which are necessary for our subsequent discussions. A Dedekind symbol is a generalization of the classical Dedekind sums ([14]), and is defined as a complex valued function $D$ on $V := \{(p, q) \in \mathbb{Z}^+ \times \mathbb{Z} | \gcd(p, q) = 1\}$ satisfying

$$D(p, q) = D(p, q + p).$$

The symbol $D$ is determined uniquely by its reciprocity law:

$$D(p, q) - D(q, -p) = R(p, q)$$

up to an additive constant. The function $R$ is defined on $U := \{(p, q) \in \mathbb{Z}^+ \times \mathbb{Z}^+ | \gcd(p, q) = 1\}$, and is called a reciprocity function associated with the Dedekind symbol $D$. The function $R$ necessarily satisfies the equation:

$$R(p + q, q) + R(p, p + q) = R(p, q).$$

When the reciprocity function $R$ is a (Laurent) polynomial in $p$ and $q$, the symbol $D$ is called a Dedekind symbol with (Laurent) polynomial reciprocity law. Those symbols are particularly important because they naturally correspond to modular forms (explicit forms of such Dedekind symbols were given in [6, 7]).

The aim of this article is to define Hecke operators on Dedekind symbols which are compatible with Hecke operators on modular forms. We then apply those operators
to express Fourier coefficients of Hecke eigenforms. For this purpose it is necessary to extend the domain \( V = \{ (p, q) \in \mathbb{Z}^+ \times \mathbb{Z} \mid \gcd(p, q) = 1 \} \) for Dedekind symbols to \( \mathbb{Z}^+ \times \mathbb{Z} \). That is, we need to define Dedekind symbol \( D(p, q) \) when \( \gcd(p, q) > 1 \).

Thus we reach the following definition of weighted Dedekind symbols (hereafter we always assume an integer \( w \) to be even and positive).

**Definition 1.1.** A complex valued function \( E \) on \( \mathbb{Z}^+ \times \mathbb{Z} \) is called a weighted Dedekind symbol of weight \( w \) if it satisfies the following two conditions:

\[
E(h, k) = E(h, k + h) \tag{1.4}
\]

for any \( (h, k) \in \mathbb{Z}^+ \times \mathbb{Z} \);

\[
E(ch, ck) = c^w E(h, k) \tag{1.5}
\]

for any \( (h, k) \in \mathbb{Z}^+ \times \mathbb{Z} \) and \( c \in \mathbb{Z}^+ \).

Moreover, a weighted Dedekind symbol \( E \) is said to be even (resp. odd) if \( E \) satisfies

\[
E(h, -k) = E(h, k) \quad (\text{resp. } E(h, -k) = -E(h, k)) \tag{1.6}
\]

for any \( (h, k) \in \mathbb{Z}^+ \times \mathbb{Z} \).

Roughly speaking, a symbol \( E \) is determined by its reciprocity law

\[
E(h, k) - E(k, -h) = S(h, k)
\]

up to addition of scalar multiples of the “trivial” weighted Dedekind symbol. Here \( S \) is a complex valued function defined on \( \mathbb{Z}^+ \times \mathbb{Z}^+ \).

More precisely, let \( E \) and \( E' \) be Dedekind symbols of weight \( w \) which have the identical reciprocity function, namely

\[
E(h, k) - E(k, -h) = E'(h, k) - E'(k, -h)
\]

for any \( (h, k) \in \mathbb{Z}^+ \times \mathbb{Z} \); then it holds that

\[
E - E' = c G_w
\]

where \( c \) is a constant and \( G_w \) is the “trivial” Dedekind symbol (of weight \( w \)) defined by

\[
G_w(h, k) := \{ \gcd(h, k) \}^w \tag{1.7}
\]

for any \( (h, k) \in \mathbb{Z}^+ \times \mathbb{Z} \).

Next we would like to demonstrate the relationship between modular forms and weighted Dedekind symbols in order to define compatible Hecke operators (refer to [5] for the relationship between modular forms and non-weighted Dedekind symbols). However, the case of non-cusp forms seems to be too involved to treat here. Hence we consider only the case of cusp forms in this section, and leave the general case to the later sections. The statement of our results requires the following notation:

\[
\Gamma := SL_2(\mathbb{Z}) \text{ (the full modular group)},
\]

\[
S_{w+2} := \text{the space of cusp forms on } \Gamma \text{ with weight } w + 2,
\]

\[
\mathcal{W}_w := \{ W \mid W \text{ is a Dedekind symbol of weight } w \},
\]

\[
\mathcal{W}_w^- := \{ W \in \mathcal{W}_w \mid W \text{ is odd } \},
\]

\[
\mathcal{W}_w^+ := \{ W \in \mathcal{W}_w \mid W \text{ is even } \},
\]

\[
\mathcal{E}_w := \{ E \mid E \text{ is a Dedekind symbol of weight } w \text{ such that } E(h, k) - E(k, -h) \text{ is}
\]

...
a homogeneous polynomial in \( h \) and \( k \) of degree \( w \})

(an element of \( \mathcal{E}_w \) is essentially a period polynomial modulo \( h^w - k^w \) \( \mathbb{F}_k \)),

\[
\mathcal{E}^-_w := \{ E \in \mathcal{E}_w \mid E \text{ is odd} \},
\mathcal{E}^+_w := \{ E \in \mathcal{E}_w \mid E \text{ is even} \},
\mathcal{U}_w^- := \{ g \mid g \text{ is a homogeneous polynomial in } h \text{ and } k \text{ of degree } w 
\text{satisfying } g(h + k, k) + g(h, h + k) = g(h, k) \text{ and } g(1, 1) = 0 \},
\mathcal{U}^-_w := \{ g \in \mathcal{U}_w \mid g \text{ is an odd polynomial, i.e., } g(h, -k) = -g(h, k) \},
\mathcal{U}_w^+ := \{ g \in \mathcal{U}_w \mid g \text{ is an even polynomial, i.e., } g(h, -k) = g(h, k) \}.
\]

It is obvious that \( \mathcal{W}_w^+ \oplus \mathcal{W}_w^- = \mathcal{W}_w \), \( \mathcal{E}_w^+ \oplus \mathcal{E}_w^- = \mathcal{E}_w \), \( \mathcal{U}_w^+ \oplus \mathcal{U}_w^- = \mathcal{U}_w \) and \( \mathcal{E}_w^\pm \subset \mathcal{W}_w^\pm \).

For a cusp form \( f \in S_{w+2} \) and \((h, k) \in \mathbb{Z}^+ \times \mathbb{Z} \), we define \( E_f \) by

\[
E_f(h, k) = \int_{k/h}^{i\infty} f(z)(hz - k)^w dz.
\]

Furthermore we define \( E^-_f \) and \( E^+_f \), respectively, by

\[
E^-_f(h, k) = \frac{1}{2} \{ E_f(h, k) - E_f(h, -k) \}
\]
and

\[
E^+_f(h, k) = \frac{1}{2} \{ E_f(h, k) + E_f(h, -k) \}
\]

for any \((h, k) \in \mathbb{Z}^+ \times \mathbb{Z} \). Then it is shown that \( E_f \) is a Dedekind symbols of weight \( w \), and we can define maps

\[
\alpha_{w+2} : S_{w+2} \rightarrow \mathcal{W}_w, \quad \alpha^\pm_{w+2} : S_{w+2} \rightarrow \mathcal{W}^\pm_w
\]
by

\[
\alpha_{w+2}(f) = E_f, \quad \alpha^\pm_{w+2}(f) = E^\pm_f.
\]

Furthermore, we know that \( E_f \) and \( E^\pm_f \) have polynomial reciprocity laws, that is, \( E_f \in \mathcal{E}_w \) and \( E^\pm_f \in \mathcal{E}^\pm_w \). Hence we have the restricted maps

\[
\alpha^\pm_{w+2} : S_{w+2} \rightarrow \mathcal{E}^\pm_w
\]
(we use the same notation \( \alpha^\pm_{w+2} \) for the restricted maps). Using the trivial element \( F_w \in \mathcal{E}^+_w \) defined (for any \((h, k) \in \mathbb{Z}^+ \times \mathbb{Z} \)) by

\[
F_w(h, k) = h^w,
\]
we obtain the following:

**Theorem 1.1.** The map

\[
\alpha^-_{w+2} : S_{w+2} \rightarrow \mathcal{E}^-_w
\]
is an isomorphism (between vector spaces) and the map

\[
\alpha^+_w : S_{w+2} \rightarrow \mathcal{E}^+_w
\]
is a monomorphism such that the image \( \alpha^+_w(S_{w+2}) \) is a subspace of \( \mathcal{E}^+_w \) of codimension two, and that \( \alpha^+_w(S_{w+2}), F_w \) and \( G_w \) span \( \mathcal{E}^+_w \).
Next we will see how weighted Dedekind symbols are linked to reciprocity functions. For a weighted Dedekind symbol \( E \), let \( \beta_w(E) \) be defined by

\[
\beta_w(E)(h, k) = E(h, k) - E(k, -h)
\]

for any \((h, k) \in \mathbb{Z}^+ \times \mathbb{Z}\). In other words, \( \beta_w(E) \) is the reciprocity function of \( E \).

In the case of Dedekind symbol \( E_f \) associated with a cusp form \( f \), this has the following expression:

\[
\beta_w(E_f)(h, k) = \int_0^{i \infty} f(z)(hz - k)^w dz.
\]

Hence obviously \( \beta_w(E_f)(h, k) \) is a homogeneous polynomial in \( h \) and \( k \). Furthermore we know \( \beta_w(E_f) \in U_w \).

Thus we have a homomorphism \( \beta_w : E_w \rightarrow U_w \).

Our second result is:

**Theorem 1.2.** The homomorphism \( \beta_w : E_w \rightarrow U_w \) is an epimorphism such that \( \beta_w(E_w^\pm) = U_w^\pm \) and \( \ker \beta_w \) is one dimensional subspace of \( E_w \) spanned by \( G_w \).

In particular, the restricted map \( \beta_w^- : E_w^- \rightarrow U_w^- \) is an isomorphism, and \( \beta_w^+ : E_w^+ \rightarrow U_w^+ \) is an epimorphism such that \( \ker \beta_w^+ \) is one dimensional subspace of \( E_w^+ \) spanned by \( G_w \).

Here we examine the composed maps

\[
\beta_w^+ \alpha_w^+: S_{w+2} \rightarrow E_w^+ \rightarrow U_w^+.
\]

Note that \( \beta_w^+(E_f^\pm)(h, k) = \beta_w^+ \alpha_w^+(f)(h, k) \) is a homogenized form of the period polynomial (Kohnen-Zagier \[9, pp. 199–200\]) for \( f \). This means the composed maps

\[
\beta_w^- \alpha_w^- : S_{w+2} \rightarrow E_w^- \rightarrow U_w^- \]

and

\[
\beta_w^+ \alpha_w^+: S_{w+2} \rightarrow E_w^+ \rightarrow U_w^+
\]

can be identified with the Eichler-Shimura isomorphisms (refer to \[9, p. 200\], \[5, Theorem 7.3\]). In fact, \( \beta_w^- \alpha_w^- \) is an isomorphism, and \( \beta_w^+ \alpha_w^+ \) is an monomorphism such that the image \( \beta_w^+ \alpha_w^+(S_{w+2}) \) and \( h^w - k^w \) span \( U_w^+ \).

These facts may be summarized in the following commutative diagram:

\[
\begin{array}{ccc}
\alpha_{w+2}^\pm & \cong & \beta_w^\pm \\
\text{the space of odd (resp. even) Dedekind symbols of weight } w \text{ with polynomial reciprocity laws (mod } F_w \text{ and } G_w \text{ if even)} & \cong & \text{the space of odd (resp. even) period polynomials of degree } w \text{ (mod } h^w - k^w \text{ if even)}.
\end{array}
\]

Diagram 1: The case of cusp forms.
In the above diagram, we have Hecke operators for modular forms and period polynomials. Indeed, Manin [11] and Zagier [18] proved that there are well defined Hecke operators on period polynomials which are compatible with the Eichler-Shimura isomorphism. However, no such operators are yet known for Dedekind symbols. Under these circumstances, we introduce the following operators:

**Definition 1.2.** For any positive integer $n$, we define the operator $T_n$ on $W_w$ by

\[(T_nE)(h, k) := \sum_{\substack{ad=p \quad b \pmod{d} \quad \text{for some } d \leq n}} E(dh, ak + bh).\]

We may call the operator $T_n$ as the Hecke operator.

We will show that $T_n$ maps any weighted Dedekind symbol $E$ in $W_w$ onto another weighted Dedekind symbol in $W_w$. Furthermore we will show that $T_n$ preserve $W_w^\pm$, namely $T_n$ induces operators on $W_w^\pm$:

\[T_n : W_w^\pm \rightarrow W_w^\pm.\]

Then we have the following result which asserts that Hecke operators on Dedekind symbols are compatible with well known Hecke operators on cusp forms:

**Theorem 1.3.** The following diagram commutes:

\[\begin{array}{ccc}
S_{w+2} & \xrightarrow{\alpha_{w+2}^+} & W_w^+ \\
| & T_n & | \\
S_{w+2} & \xrightarrow{\alpha_{w+2}^-} & W_w^-.
\end{array}\]

To ease the notation, We will use the same notation $T_n$ for the Hecke operators on $S_{w+2}$ and also on $W_w$.

Finally, as an application of Hecke operators on Dedekind symbols, we present formulae giving Fourier coefficients of cusp forms which are Hecke eigenforms in terms of Dedekind symbols:

**Theorem 1.4.** Let

\[f(z) = \sum_{n=1}^{\infty} a_f(n)e^{2\pi inz} \in S_{w+2}\]

be a normalized Hecke eigenform having $a_f(n)$ as its $n$th Fourier coefficient. Then there exists $(h, k) \in \mathbb{Z}^+ \times \mathbb{Z}$ such that $E_f(h, k) \neq 0$, and for such $(h, k)$, it holds that

\[a_f(n) = \frac{T_nE_f(h, k)}{E_f(h, k)}.\]

When we have an explicit description for $E_f$, the expression (1.12) is very useful to calculate the Fourier coefficient $a_f(n)$ for any $n \geq 1$.

For example, for $f$ in $S_{\ell+2}$ ($\ell = 10, 14, 16, 18, 20, 24$), we can calculate $a_f(n)$ rather efficiently. Here we illustrate Theorem 1.4 in the special case, e.g. $\ell = 10$ and $f = \Delta$, where $\Delta$ is the well known Hecke eigenform of weight 12 for $\Gamma$ defined by

\[\Delta(z) := e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{-24}.\]
Let \( \tau(n) \) be the \( n \)th Fourier coefficient of \( \Delta \), namely
\[
\Delta(z) = \sum_{n=1}^{\infty} \tau(n)e^{2\pi inz}.
\]
Customarily \( \tau(n) \) is called Ramanujan’s tau function.

Using Theorem 1.4 together with explicit formula for \( E_f \) (Theorem 5.2, Lemma 6.1), we obtain a surprisingly elementary formula for Ramanujan’s tau function:

**Theorem 1.5.** Let \( n \) be a positive prime integer. Then the Ramanujan’s tau function \( \tau(n) \) is expressed as
\[
\tau(n) = 1 + n^{11} + \frac{691}{756}(n^5 - n^{11}) - \frac{691}{6}\sum_{i=0}^{n-1} \sum_{\begin{subarray}{c}
(a \ b \\
c \ d)
\end{subarray} \in \Gamma/\pm 1 \atop {\begin{subarray}{c}
ac \neq 0 \\
|a|, |b|, |c|, |d| \leq 2n \\
(i/n+b/a)(i/n+d/c) < 0
\end{subarray}}} \operatorname{sgn} \left( \frac{i}{n} + \frac{b}{a} \right) (ai + bn)^5(ci + dn)^5
\]
where \( \operatorname{sgn}(x) \) denotes \(+1\) or \(-1\) according to whether \( x \) is positive or negative.

Throughout the article, we use the following notation and conventions. We assume that \( w \) is an even integer with \( w \geq 2 \). For \( 0 \leq n \leq w \), the number \( \tilde{n} := w - n \) stands for \( n \). We write \( \sigma_k(n) \) for the sum of the \( k \)th powers of the positive divisors of \( n \). We denote by \( [x] \) the greatest integer not exceeding \( x \in \mathbb{R} \). We also use the notation \( \operatorname{sgn}(x) \) for the sign of \( x \), that is, \(+1\), \(-1\) or 0 depending on \( x \) is positive, negative or zero, respectively. We denote by \( B_m(x) \) (resp. \( \tilde{B}_m(x) \)) the \( m \)th Bernoulli polynomial (resp. number) and by \( \bar{B}_m(x) \) the \( m \)th Bernoulli function:
\[
\bar{B}_m(x) := -m! \sum_{k=-\infty}^{+\infty} \frac{e^{2\pi ikx}}{(2\pi ik)^m}.
\]
It is well-known that for \( 0 \leq x < 1 \), \( \bar{B}_m(x) \) reduces to \( B_m(x) \).

**Remark 1.1.** The Hecke operators for the classical Dedekind sums \([14]\) and the generalized Dedekind sums \([1]\) were already introduced by Knopp \([8]\) and Parson-Rosen \([13]\), respectively. The Hecke operators on \( W_w \) (Definition 1.2) can be regarded as natural generalizations of those operators.

2. **Weighted Dedekind Symbols associated with modular forms**

In the previous section, for the sake of simplicity, we restricted our discussions to cusp forms. However, in the subsequent sections, we would like to deal with non-cusp modular forms as well. In this section we investigate the relationship among modular forms, *weighted Dedekind symbols* and period polynomials.

Let \( M_{w+2} \) denote the space of modular forms, that is,
\[
M_{w+2} := \text{the space of modular forms on }\Gamma \text{ with weight } w+2.
\]
We already defined the spaces \( U_w, U_w^\pm, E_w, E_w^\pm \) associated with \( S_{w+2} \). Here we will define corresponding spaces \( \hat{U}_w, \hat{U}_w^\pm, \hat{E}_w, \hat{E}_w^\pm \) associated with \( M_{w+2} \) (they all are naturally regarded as vector spaces over \( \mathbb{C} \)).
Now, for a polynomial \( g \) in \( h \) and \( k \), we define “Laurent polynomial” \( \hat{g} \) by
\[
\hat{g}(h, k) := \frac{1}{hk} \left[ g(h, k) - g(1, 1) \{ \gcd(h, k) \}^{w+2} \right].
\]

Using this notation, we introduce the following spaces:
\[
\hat{U}_w := \{ \hat{g} \mid g \text{ is a homogeneous polynomial in } h \text{ and } k \text{ of degree } w + 2 \text{ satisfying } \gcd(h + k, k) + k \gcd(h, h + k) = (h + k)g(h, k) \},
\]
\[
\hat{U}_w^- := \{ \hat{g} \in \hat{U}_w \mid \hat{g} \text{ is odd, i.e., } \hat{g}(h, -k) = -\hat{g}(h, k) \},
\]
\[
\hat{U}_w^+ := \{ \hat{g} \in \hat{U}_w \mid \hat{g} \text{ is even, i.e., } \hat{g}(h, -k) = \hat{g}(h, k) \},
\]
\[
\hat{E}_w := \{ E \mid E \text{ is a Dedekind symbol such that } E(h, k) - E(k, -h) \in \hat{U}_w \},
\]
\[
\hat{E}_w^- := \{ E \in \hat{E}_w \mid E \text{ is odd } \},
\]
\[
\hat{E}_w^+ := \{ E \in \hat{E}_w \mid E \text{ is even } \}.
\]

It is obvious that \( \hat{E}_w \subset W_w \), \( \hat{E}_w^\pm \subset W_w^\pm \), \( \hat{E}_w^+ \oplus \hat{E}_w^- = \hat{E}_w \) and \( \hat{U}_w^+ \oplus \hat{U}_w^- = \hat{U}_w \).

One can easily check that \( \hat{g} \in \hat{U}_w \) satisfies
\[
\hat{g}(h + k, k) + \hat{g}(h, h + k) = \hat{g}(h, k) \quad \text{and} \quad \hat{g}(1, 1) = 0.
\]

One can also show that \( \hat{U}_w^+ = \hat{U}_w^- \) and that \( \hat{U}_w^- \) is a codimension one subspace of \( \hat{U}_w^- \) as consequences of \cite{19} Theorem 4.1 (5) and eqn. (7.8).

In this setting we reformulate the relationship between modular forms and weighted Dedekind symbols. Throughout the article, the \( n \)th Fourier coefficients of \( f \in M_{w+2} \) are expressed by \( a_f(n) \). Namely
\[
(2.1) \quad f(z) = \sum_{n=0}^{\infty} a_f(n) e^{2\pi inz}.
\]

Then, for a modular form \( f \), \( E_f \) is defined as follows:

**Definition 2.1.** Let \( f \in M_{w+2} \). Firstly, for \((p, q) \in V\), we define \( E_f(p, q) \) by
\[
E_f(p, q) := \int_{z_0}^{\infty} \left\{ f(z) - a_f(0) \right\} (pz - q)^w dz
\]
\[
+ \int_{q/p}^{z_0} \left\{ f(z) - \frac{a_f(0)}{(pz - q)^{w+2}} \right\} (pz - q)^w dz
\]
\[
- a_f(0) \left\{ \frac{1}{(w + 1)p} (pz_0 - q)^{w+1} + \frac{1}{p(pz_0 - q)} \right\}
\]
\[
(z_0 \in H \text{ arbitrary}).
\]

The right-hand side of (2.2) is independent of the choice of \( z_0 \) (refer to \cite{19}).

Secondly, for any \((h, k) \in \mathbb{Z}^* \times \mathbb{Z}\), we define \( E_f(h, k) \) by
\[
(2.3) \quad E_f(h, k) := \{ \gcd(h, k) \}^w E_f\left( \frac{h}{\gcd(h, k)}, \frac{k}{\gcd(h, k)} \right).
\]
More precisely,

\[ E_f(h, k) = \int_{z_0}^{z_\infty} \left\{ f(z) - a_f(0) \right\} (hz - k)^w dz \]

\[ + \int_{k/h}^{z_0} \left\{ f(z) - \frac{\gcd(h, k)^{w+2}a_f(0)}{(hz - k)^{w+2}} \right\} (hz - k)^w dz \]

\[ - a_f(0) \left\{ \frac{1}{(w+1)h} (hz_0 - k)^{w+1} + \frac{\gcd(h, k)^{w+2}}{h(hz_0 - k)} \right\} \]

(2.4)

(z_0 \in H \; \text{arbitrary}).

Note that, when \( f \) is a cusp form, the formula (2.4) reduces to the formula (1.8) in the previous section.

Here we will give an alternative expression for \( E_f \) which is necessary for our proof of Theorem 3.3. For \((h, k) \in \mathbb{Z}^+ \times \mathbb{Z}\), we introduce the function \( B_f \):

\[ B_f(s; h, k) := e^{\pi i s/2} h^{s-1} \int_0^\infty \left\{ f(it + \frac{k}{h}) - a_f(0) \right\} t^{s-1} dt. \]

(2.5)

It is easy to see that \( B_f(s; h, k) \) is well defined for \( \Re(s) > 0 \) and has a meromorphic continuation, say \( B_f^*(s; h, k) \), to the entire complex numbers. More explicitly, we have

\[ B_f^*(s; h, k) = e^{\pi i s/2} h^{s-1} \int_{t_0}^{\infty} \left\{ f(it + \frac{k}{h}) - a_f(0) \right\} t^{s-1} dt \]

\[ + e^{\pi i s/2} h^{s-1} \int_{t_0}^{0} \left\{ f(it + \frac{k}{h}) - \frac{\gcd(h, k)^{w+2}a_f(0)}{(hit)^{w+2}} \right\} t^{s-1} dt \]

\[ - e^{\pi i s/2} h^{s-1} a_f(0) \left\{ \frac{t_0^s}{s} + \frac{\gcd(h, k)^{w+2}t_0^{s-w-2}}{(h)^{w+2}(s-w-2)} \right\} \]

(2.6)

\( (t_0 > 0 \; \text{arbitrary}). \)

The right hand side of (2.6) is independent of the choice of \( t_0 \). It is plain that, using (2.5), \( E_f(h, k) \) can be rewritten as:

\[ E_f(h, k) = B_f^*(w + 1; h, k). \]

(2.7)

Our first task is to show the following lemma:

**Lemma 2.1.** \( E_f \) is a Dedekind symbol of weight \( w \), that is, \( E_f \in W_w \).

**Proof.** The equation \( E_f(h, k) = E_f(h, k + h) \) comes from modularity of \( f \), in particular from the formula \( f(z + 1) = f(z) \).

The second equation \( E_f(ch, ck) = c^w E_f(h, k) \) follows from (2.5). \( \Box \)

Our next task is to obtain reciprocity law for \( E_f \). We introduce a “Laurent polynomial” \( S_f \) which turn out to be a reciprocity function for \( E_f \).

**Definition 2.2.** Let \( f \in M_{w+2} \). Firstly, for \((p, q) \in V\), we define \( S_f(p, q) \) by

\[ S_f(p, q) := \int_{z_0}^{z_\infty} \left\{ f(z) - a_f(0) \right\} (pz - q)^w dz + \int_{0}^{z_0} \left\{ f(z) - \frac{a_f(0)}{z^{w+2}} \right\} (pz - q)^w dz \]

\[ - a_f(0) \left\{ \frac{1}{(w+1)p(z_0 - q)^{w+1}} - \frac{1}{(w+1)q(p - \frac{q}{z_0})^{w+1}} \right\} + a_f(0) \frac{1}{pq} \]

(2.8)

\( (z_0 \in H \; \text{arbitrary}). \)
Again the right-hand side of (2.8) is independent of the choice of $z_0$.

Secondly, for any $(h, k) \in \mathbb{Z}^+ \times \mathbb{Z}$, we define $S_f(h, k)$ by

$$S_f(h, k) := \{\gcd(h, k)\}^w S_f\left(\frac{h}{\gcd(h, k)}, \frac{k}{\gcd(h, k)}\right).$$

In other words,

$$S_f(h, k) = \int_{z_0}^{\infty} \{f(z) - a_f(0)\} (h z - k)^w dz + \int_{0}^{z_0} \left\{f(z) - \frac{a_f(0)}{z^{w+2}}\right\} (h z - k)^w dz - a_f(0) \left\{\frac{1}{(w + 1)h} (h z_0 - k)^{w+1} - \frac{1}{(w + 1)k} (h - \frac{k}{z_0})^{w+1}\right\} + a_f(0) \frac{\{\gcd(h, k)\}^{w+2}}{hk} (z_0 \in H \text{ arbitrary}).$$

Again note that, when $f$ is a cusp form, the formula (2.10) reduces to the formula (2.10).

Now we obtain the following reciprocity law for $E_f$:

**Proposition 2.2.** Let $f$ be a modular form of weight $w + 2$, that is, $f \in M_{w+2}$. Then

1. it holds that

$$(2.11) \quad E_f(h, k) - E_f(k, -h) = S_f(h, k);$$

2. $E_f$ belongs to $\mathcal{E}_w$.

**Proof.** First we express $E_f(k, -h)$ as follows:

$$E_f(k, -h) = \int_{z_0}^{\infty} \{f(z) - a_f(0)\} (k z + h)^w dz + \int_{-h/k}^{1/z_0} \left\{f\left(\frac{z}{h}\right) - \frac{\{\gcd(h, k)\}^{w+2}a_f(0)}{(k z + h)^{w+2}}\right\} \left(\frac{z}{h}\right)^w dz - a_f(0) \left\{\frac{1}{(w + 1)k} \left(\frac{-k}{z_0} + h\right)^{w+1} + \frac{\{\gcd(h, k)\}^{w+2}}{k\left(\frac{-k}{z_0} + h\right)}\right\}$$

$$= \int_{z_0}^{1/z_0} \left\{f\left(\frac{z}{z}\right) - a_f(0)\right\} \left(\frac{-k}{z} + h\right)^w d\frac{dz}{z^2} + \int_{k/h}^{1/z_0} \left\{f\left(\frac{z}{h}\right) - \frac{\{\gcd(h, k)\}^{w+2}a_f(0)}{(k z + h)^{w+2}}\right\} \left(\frac{k}{z} + h\right)^w d\frac{dz}{z^2} - a_f(0) \left\{\frac{1}{(w + 1)k} \left(\frac{-k}{z_0} + h\right)^{w+1} + \frac{\{\gcd(h, k)\}^{w+2}}{k\left(\frac{-k}{z_0} + h\right)}\right\}$$

(substituting $-1/z$ for $z$)

$$= \int_{z_0}^{1/z_0} \left\{f(z) - \frac{a_f(0)}{z^{w+2}}\right\} (h z - k)^w dz + \int_{k/h}^{1/z_0} \left\{f(z) - \frac{\{\gcd(h, k)\}^{w+2}a_f(0)}{(h z - k)^{w+2}}\right\} (h z - k)^w dz.$$
From this, we have

\[
E_f(h, k) - E_f(k, -h) = \int_{z_0}^{\infty} \{f(z) - a_f(0)\} (hz - k)^w dz \\
+ \int_{z_0}^{0} \left\{f(z) - a_f(0) \left(\frac{\gcd(h, k)}{z_0 + h}\right)^w\right\} (hz - k)^w dz \\
- a_f(0) \left\{\frac{1}{(w+1)h} (hz_0 - k)^w + \frac{\gcd(h, k)}{h(z_0 + h)}\right\} \\
- \int_{z_0}^{0} \left\{f(z) - a_f(0) \left(\frac{\gcd(h, k)}{z_0 + h}\right)^w\right\} (hz - k)^w dz \\
+ a_f(0) \left\{\frac{1}{(w+1)k} \left(-\frac{k}{z_0 + h}\right)^w + \frac{\gcd(h, k)}{k(z_0 + h)}\right\} \\
= \int_{z_0}^{\infty} \{f(z) - a_f(0)\} (hz - k)^w dz \\
+ \int_{0}^{z_0} \left\{f(v) - a_f(0) \left(\frac{\gcd(h, k)}{v_0 + h}\right)^w\right\} (hv - k)^w dv \\
- a_f(0) \left\{\frac{1}{(w+1)h} (hz_0 - k)^w - \frac{1}{(w+1)k} \left(-\frac{k}{z_0 + h}\right)^w\right\} \\
+ a_f(0) \frac{\gcd(h, k)}{hk} \}
\]

This proves the assertion (1).

Furthermore an easy exercise yields \(S_f \in \hat{U}_w\), and this implies \(E_f \in \hat{E}_w\). This proves the assertion (2).

For \(E_f, E_f^-\) and \(E_f^+\) are defined as before, namely

\[
E_f^\pm(h, k) = \frac{1}{2} \{E_f(h, k) \pm E_f(h, -k)\}.
\]

Similarly, for \(S_f, S_f^-\) and \(S_f^+\) are defined by

\[
S_f^\pm(h, k) = \frac{1}{2} \{S_f(h, k) \pm S_f(h, -k)\},
\]

and they satisfy, by virtue of Proposition 2.2 the following identities

\[
E_f^\pm(h, k) - E_f^\pm(k, -h) = S_f^\pm(h, k).
\]

Then it follows that \(E_f^\pm \in \hat{E}_w\) for \(f \in M_{w+2}\), and we arrive at the following definition:
Definition 2.3. (1) The map \( \hat{\alpha}_{w+2} : M_{w+2} \to W_w \) is defined by
\[
\hat{\alpha}_{w+2}(f) = E_f;
\]

(2) The maps \( \hat{\alpha}_{w+2}^\pm : M_{w+2} \to W_w^\pm \) are defined by
\[
\hat{\alpha}_{w+2}^\pm(f) = E^\pm_f.
\]

Since \( \hat{\alpha}_{w+2}^\pm(f) = E^\pm_f \in \hat{E}_w^\pm \), we have the restricted maps of \( \hat{\alpha}_{w+2}^\pm \) from \( M_{w+2} \) to \( \hat{E}_w^\pm \). These maps will also be denoted by \( \hat{\alpha}_{w+2}^\pm \).

The following theorem asserts that the maps \( \hat{\alpha}_{w+2}^\pm \) are almost bijective.

Theorem 2.3. The map
\[
\hat{\alpha}_{w+2}^- : M_{w+2} \to \hat{E}_w^-
\]
is an isomorphism (between vector spaces), and the map
\[
\hat{\alpha}_{w+2}^+: M_{w+2} \to \hat{E}_w^+
\]
is a monomorphism such that the image \( \hat{\alpha}_{w+2}^+(M_{w+2}) \) is a codimension two subspace of \( \hat{E}_w^+ \), and that \( \hat{\alpha}_{w+2}^+(M_{w+2}) \) and \( G_w \) span \( \hat{E}_w^+ \).

The proof of this theorem will be given after the proof of Theorem 2.4.

Next we define maps from \( \hat{E}_w^\pm \) to \( \hat{U}_w^\pm \). For this, we will make the following definition:

Definition 2.4. For Dedekind symbol \( E \in \hat{E}_w^\pm \), the maps \( \hat{\beta}_w^\pm \) are defined by
\[
\hat{\beta}_w^\pm(E)(h, k) := E(h, k) - E(k, -h).
\]
Then obviously we have \( \hat{\beta}_w^\pm(E) \in \hat{U}_w^\pm \), and this gives the homomorphisms
\[
\hat{\beta}_w^\pm : \hat{E}_w^\pm \to \hat{U}_w^\pm.
\]

The following theorem asserts that the maps \( \hat{\beta}_w^\pm \) are bijective modulo \( G_w \):

Theorem 2.4. The map
\[
\hat{\beta}_w^- : \hat{E}_w^- \to \hat{U}_w^-
\]
is an isomorphism (between vector spaces) and the map
\[
\hat{\beta}_w^+ : \hat{E}_w^+ \to \hat{U}_w^+
\]
is an epimorphism such that \( \ker(\hat{\beta}_w^+) \) is one-dimensional subspace of \( \hat{E}_w^+ \) spanned by \( G_w \).

To establish this theorem, we first prove the following lemma:

Lemma 2.5. Let \( E \) and \( E' \) be Dedekind symbols of weight \( w \) satisfying
\[
E(h, k) - E(k, -h) = E'(h, k) - E'(k, -h)
\]
for any \( (h, k) \in \mathbb{Z}^+ \times \mathbb{Z} \). Then it holds that
\[
E - E' = c G_w
\]
for a constant \( c \in \mathbb{C} \).
Proof. Let \( D : V \to \mathbb{C} \) and \( D' : V \to \mathbb{C} \) be the restricted maps of \( E : \mathbb{Z}^+ \times \mathbb{Z} \to \mathbb{C} \) and \( E' : \mathbb{Z}^+ \times \mathbb{Z} \to \mathbb{C} \), respectively. Then \( D \) and \( D' \) are (non-weighted) Dedekind symbols which have an identical reciprocity function. By [5, Theorem 5.1], we know

\[ D - D' = c \]

for a constant \( c \in \mathbb{C} \) (the appearance of a constant \( c \) stems from the fact that we do not assume \( D(1,0) = 0 \) nor \( D'(1,0) = 0 \)). Now we have

\[
E(h,k) - E'(h,k) = \{\gcd(h,k)\}^w D\left(\frac{h}{\gcd(h,k)}, \frac{k}{\gcd(h,k)}\right)
- \{\gcd(h,k)\}^w D'\left(\frac{h}{\gcd(h,k)}, \frac{k}{\gcd(h,k)}\right)
= \{\gcd(h,k)\}^w c
= c G_w(h,k).
\]

This completes the proof. \( \square \)

Now we are ready to give a proof of Theorem 2.4.

Proof of Theorem 2.4. First we show that \( \hat{\beta}^\pm \) are epimorphisms. Let \( S \in \hat{U}^\pm_w \). Then, by definition, \( S(h,k) \) is expressed as

\[
S(h,k) = \frac{1}{hk} \left[ g(h,k) - g(1,1) \{\gcd(h,k)\}^{w+2} \right]
\]

with a homogeneous polynomial \( g \) of degree \( w + 2 \) satisfying

\[ hg(h + k, k) + kg(h, h + k) = (h + k)g(h,k). \]

Then it holds that

\[ S(h + k, k) + S(h, h + k) = S(h,k). \]

By Theorem 5.1 in [5], there is a (non-weighted) Dedekind symbol \( D \) which satisfies \( D(p,q) - D(q,-p) = S(p,q) \) for any \( (p,q) \in V \). Then we define \( E \) by

\[
E(h,k) = \{\gcd(h,k)\}^w D\left(\frac{h}{\gcd(h,k)}, \frac{k}{\gcd(h,k)}\right)
\]

for any \( (h,k) \in \mathbb{Z}^+ \times \mathbb{Z} \). One can easily check that \( E \in \hat{E}^\pm_w \) and \( \hat{\beta}^\pm_w (E) = S \). This implies \( \hat{\beta}^\pm_w \) are epimorphisms.

Lemma 2.5 shows that \( \hat{\beta}^w \) is a monomorphism and that \( \ker(\hat{\beta}^w \pm 2) \) is one-dimensional subspace of \( \hat{E}^w_\pm \) spanned by \( G_w \). This completes the proof. \( \square \)

Next we give a proof of Theorem 2.3.

Proof of Theorem 2.3. We see that the composed maps

\[
\hat{\beta}^-_w \hat{\alpha}^-_{w+2} : M_{w+2} \to \hat{E}^-_w \to \hat{U}^-_w
\]

and

\[
\hat{\beta}^+_w \hat{\alpha}^+_w : M_{w+2} \to \hat{E}^+_w \to \hat{U}^+_w
\]

are nothing but the Eichler-Shimura isomorphisms ([9, p. 200], [5, Theorem 7.3]). Hence \( \hat{\beta}^+_w \hat{\alpha}^+_w \) are isomorphisms. This implies that \( \hat{\alpha}^-_{w+2} \) is a monomorphism. Furthermore, by Theorem 2.4 we know that that \( \hat{\alpha}^+_w(M_{w+2}) \) and \( G_w \) span \( \hat{E}^+_w \).

Finally, since \( \hat{\beta}^-_w \) is an isomorphism by Theorem 2.4, we know that \( \hat{\alpha}^-_{w+2} \) is also an isomorphism. This completes the proof. \( \square \)
Finally we summarize these facts in the following commutative diagram (compare with Diagram 1):

Diagram 2: The case of modular forms.

3. HECKE OPERATORS ON WEIGHTED DEDEKIND SYMBOLS

One of the most important features of modular forms is that they have Hecke operators. The Hecke operators $T_n$ on modular forms are defined as follows (see for example [2, 16]):

**Definition 3.1.** For any $n = 1, 2, \ldots$, the Hecke operator $T_n$ is defined on $M_{w+2}$ by the equation

$$ (T_n f)(z) := n^{w+1} \sum_{d|n} d^{-w-2} \sum_{b=0}^{d-1} f \left( \frac{nz + bd}{d^2} \right). $$

This can be rewritten as:

$$ (T_n f)(z) = \frac{1}{n} \sum_{ad=n} \sum_{\substack{b \mod d \atop 0 < d}} a^{w+2} f \left( \frac{az + b}{d} \right). $$

The operator $T_n$ maps the vector space $M_{w+2}$ onto itself. In view of Diagram 2, we would like to define Hecke operators on the space $W_w$ of Dedekind symbols with weight $w$, compatible with Hecke operators on modular forms.

**Definition 3.2.** For a positive integer $n$, the operator $T_n$ on $W_w$ is defined by the equation

$$ (T_n E)(h, k) := \sum_{ad=n} \sum_{\substack{b \mod d \atop 0 < d}} E(dh, ak + bh) $$

for any $(h, k) \in \mathbb{Z}^+ \times \mathbb{Z}$. We will call $T_n$ the Hecke the Hecke operator on weighted Dedekind symbols.

The fact that $T_n E$ is again a Dedekind symbol of weight $w$ will be shown by the following Lemma 3.1.

**Lemma 3.1.** Let $E$ be Dedekind symbol of weight $w$.

1. Let $a$ and $d$ be fixed positive integers. We define $\tilde{E}_{a,d}$ by

$$ \tilde{E}_{a,d}(h, k) = \sum_{\substack{b \mod d \atop b(\mod d)}} E(dh, ak + bh) $$

for any $(h, k) \in \mathbb{Z}^+ \times \mathbb{Z}$. Then $\tilde{E}_{a,d}$ is also a Dedekind symbol of weight $w$. 
Let \( n \) be a positive integer. Then \( T_nE \) is also a Dedekind symbol of weight \( w \).

Proof. For any \((h, k) \in \mathbb{Z}_+ \times \mathbb{Z}\), we see

\[
\tilde{E}_{a, d}(h, k + h) = \sum_{b \pmod{d}} E(dh, a(k + h) + bh) = \sum_{b \pmod{d}} E(dh, ak + (a + b)h)
\]

\[
= \sum_{b \pmod{d}} E(dh, ak + bh) = \tilde{E}_{a, d}(h, k).
\]

Furthermore, we have

\[
\tilde{E}_{a, d}(ch, ck) = \sum_{b \pmod{d}} E(dch, ack + bch)
\]

\[
= c^w \sum_{b \pmod{d}} E(dh, ak + bh) = c^w \tilde{E}_{a, d}(h, k).
\]

These yield the assertion (1).

The assertion (2) follows directly from (1) and the identity

\[
T_nE = \sum_{ad=n \atop 0<d} \tilde{E}_{a, d}.
\]

\[\square\]

By Lemma 3.1, we have just proved that \( T_n \) is a well defined operator on \( \mathcal{W}_w \):

\[T_n : \mathcal{W}_w \rightarrow \mathcal{W}_w.\]

Here we show that \( T_n \) preserves \( \mathcal{W}_w^\pm \):

**Lemma 3.2.** The Hecke operator \( T_n \) preserves \( \mathcal{W}_w^\pm \), namely it holds that

\[T_n(\mathcal{W}_w^\pm) \subset \mathcal{W}_w^\pm.\]

**Proof.** Let \( E^\pm \in \mathcal{W}_w^\pm \). Then we have

\[(T_nE^\pm)(h, -k) = \sum_{ad=n \atop 0<d} \sum_{b \pmod{d}} E^\pm(dh, -ak + bh)
\]

\[
= \sum_{ad=n \atop 0<d} \sum_{b \pmod{d}} E^\pm(dh, -ak - bh)
\]

\[
= \pm \sum_{ad=n \atop 0<d} \sum_{b \pmod{d}} E^\pm(dh, ak + bh)
\]

\[
= \pm (T_nE^\pm)(h, k).
\]

This implies \( T_nE^\pm \in \mathcal{W}_w^\pm \) completing the proof. \[\square\]

Now we formulate our theorem which asserts that Hecke operators on Dedekind symbols are compatible with Hecke operators on modular forms.

**Theorem 3.3.** Let \( f \) be a modular form of weight \( w + 2 \), that is, \( f \in M_{w+2} \). Then
(1) it holds that
\[ \hat{\alpha}_{w+2}(T_n f) = T_n \alpha_{w+2}(f). \]

In other words, the following diagram commutes
\[
\begin{array}{ccc}
M_{w+2} & \xrightarrow{\hat{\alpha}_{w+2}} & W_w \\
\downarrow{\vphantom{\alpha}}_{T_n} & \quad & \downarrow{\vphantom{\alpha}}_{T_n} \\
M_{w+2} & \xrightarrow{\hat{\alpha}_{w+2}} & W_w \\
\end{array}
\]

(2) the Hecke operator \( T_n \) preserves \( W_{w}^{\pm} \), and the following diagram commutes
\[
\begin{array}{ccc}
M_{w+2} & \xrightarrow{\hat{\alpha}_{w+2}^{\pm}} & W_{w}^{\pm} \\
\downarrow{\vphantom{\alpha}}_{T_n} & \quad & \downarrow{\vphantom{\alpha}}_{T_n} \\
M_{w+2} & \xrightarrow{\hat{\alpha}_{w+2}^{\pm}} & W_{w}^{\pm} \\
\end{array}
\]

Proof. Let \( f \in M_{w+2} \). We recall the definition of the Hecke operator \( T_n \) on \( f \):\[
(T_n f)(z) = \frac{1}{n} \sum_{\substack{a \in \mathbb{Z}^n \\ 0 < d}} \sum_{b \pmod{d}} f\left(\frac{az + b}{d}\right),
\]
and the alternative expression (2.7) for \( E_{T_n f} \):
\[
E_{T_n f}(h, k) = B_{T_n f}^{\ast}(w + 1; h, k).
\]

Now we have
\[
B_{T_n f}(s; h, k) = e^{\pi i s/2} h^{s-1} \int_{0}^{\infty} \left\{ T_n f(it + \frac{k}{h}) - a_{T_n f}(0) \right\} t^{s-1} dt
\]
\[
= e^{\pi i s/2} h^{s-1} \int_{0}^{\infty} \sum_{\substack{a \in \mathbb{Z} \\ 0 < d}} \sum_{b \pmod{d}} a^{w+2} n \left\{ f\left(\frac{a}{d} it + \frac{ak + bh}{dh}\right) - a_f(0) \right\} t^{s-1} dt
\]
\[
= \sum_{\substack{a \in \mathbb{Z} \\ 0 < d}} \sum_{b \pmod{d}} a^{w+1-s} e^{\pi i s/2} (dh)^{s-1} \int_{0}^{\infty} \left\{ f(\frac{a}{d} x + \frac{ak + bh}{dh}) - a_f(0) \right\} x^{s-1} dx
\]
(substituting \( at/d \) for \( x \))
\[
= \sum_{\substack{a \in \mathbb{Z} \\ 0 < d}} \sum_{b \pmod{d}} a^{w+1-s} B_f(s; dh, ak + bh).
\]

Next considering the meromorphic continuations \( B_{T_n f}^{\ast}(s; h, k) \) and \( B_f^{\ast}(s; dh, ak + bh) \) of \( B_{T_n f}(s; h, k) \) and \( B_f(s; dh, ak + bh) \) respectively to the entire complex numbers, and then taking limits as \( s \to w + 1 \), we have
\[
E_{T_n f}(h, k) = B_{T_n f}^{\ast}(w + 1; h, k)
\]
\[
= \sum_{\substack{a \in \mathbb{Z} \\ 0 < d}} \sum_{b \pmod{d}} B_f^{\ast}(w + 1; dh, ak + bh)
\]
\[
= \sum_{\substack{a \in \mathbb{Z} \\ 0 < d}} \sum_{b \pmod{d}} E_f(dh, ak + bh)
\]
\[
= T_n E_f(h, k).
\]
This implies that $\hat{\alpha}_{w+2}(T_n f) = T_n \hat{\alpha}_{w+2}(f)$ which proves the assertion (1). The assertion (2) follows directly from Lemma 3.2 and (1).

4. AN APPLICATION OF HECKE OPERATORS ON DEDEKIND SYMBOLS

As an application of Hecke operators on Dedekind symbols, we present formula which gives Fourier coefficients of Hecke eigenforms in terms of Dedekind symbols.

Theorem 4.1. Let

$$f(z) = \sum_{n=0}^{\infty} a_f(n) e^{2\pi inz} \in M_{w+2}$$

be a normalized Hecke eigenform. Then

(1) it holds that

$$T_n E_{f}^{\pm}(h, k) = a_f(n) E_{f}^{\pm}(h, k),$$

in other words,

$$\sum_{ad=n \pmod{d}} \sum_{0 < c < d} E_{f}^{\pm}(dh, ak + bh) = a_f(n) E_{f}^{\pm}(h, k)$$

for any $(h, k) \in \mathbb{Z}^+ \times \mathbb{Z}$;

(2) there exists $(h, k) \in \mathbb{Z}^+ \times \mathbb{Z}$ such that $E_{f}^{\pm}(h, k) \neq 0$. For such $(h, k)$, it holds that

$$a_f(n) = \frac{T_n E_{f}^{\pm}(h, k)}{E_{f}^{\pm}(h, k)}.$$

Proof. Since $f$ is a normalized Hecke eigenform, it follows directly from the definition of eigenform that

$$T_n f = a_f(n)f.$$

Thus we have

$$E_{T_n f}^{\pm} = a_f(n)E_{f}^{\pm}.$$

By Theorem 3.3 we know that

$$E_{T_n f}^{\pm} = \hat{\alpha}_{w+2}(T_n f) = T_n \hat{\alpha}_{w+2}(f) = T_n E_{f}^{\pm}.$$

From (4.2) and (4.3), we have

$$T_n E_{f}^{\pm}(h, k) = a_f(n)E_{f}^{\pm}(h, k)$$

for any $(h, k) \in \mathbb{Z}^+ \times \mathbb{Z}$. This proves the assertion (1).

Next recall that $\hat{\alpha}_{w+2}$ are defined by

$$\hat{\alpha}_{w+2}(f) = E_{f}^{\pm}$$

for $f \in M_{w+2}$, and that

$$\hat{\alpha}_{w+2} : M_{w+2} \rightarrow \hat{E}_{w}^{\pm}$$

are monomorphisms. If $f$ is an eigenform, then, in particular, $f$ is non-trivial. Hence $E_{f}^{\pm}$ are also non-trivial since $\hat{\alpha}_{w+2}$ are monomorphisms. This implies $E_{f}^{\pm}(h, k) \neq 0$ for some $(h, k) \in \mathbb{Z}^+ \times \mathbb{Z}$. 

\[\square\]
When \(E_f^\pm(h,k) \neq 0\), from (4.1), we have
\[
a_f(n) = \frac{T_n E_f^\pm(h,k)}{E_f^\pm(h,k)}.
\]
This proves the assertion (2) completing the proof. \(\square\)

5. Weighted Dedekind symbols with polynomial reciprocity laws

In this section we give explicit description for weighted Dedekind symbols with polynomial reciprocity laws. Most of the arguments here are parallel to that of the non-weighted case so that the reader should refer to [7, 9] for more details.

**Definition 5.1.** Let \(n\) be an integer such that \(0 < n < w\). We define a sum \(I_{w,n} : \mathbb{Z}^+ \times \mathbb{Z} \to \mathbb{C}\) by
\[
I_{w,n}(h,k) := \sum_{\substack{(a,b,c,d) \in \Gamma/\pm1 \\ ac \neq 0 \\ (k/h + b/a)(k/h + d/c) < 0}} \text{sgn} \left( \frac{k}{h} + \frac{b}{a} \right) (ak + bh)^\hat{n}(ck + dh)^n.
\]
This sum reduces to the following finite sum (7)
\[
I_{w,n}(h,k) = \sum_{\substack{(a,b,c,d) \in \Gamma/\pm1 \\ ac \neq 0 \\ |b + [k/h + 1/2]| \leq |a| \leq h \\ |d + [k/h + 1/2]| \leq |c| \leq h \\ (k/h + b/a)(k/h + d/c) < 0}} \text{sgn} \left( \frac{k}{h} + \frac{b}{a} \right) (ak + bh)^\hat{n}(ck + dh)^n.
\]
In fact, in the sum \(I_{w,n}(h,k)\), each term
\[
\text{sgn} \left( \frac{k}{h} + \frac{b}{a} \right) (ak + bh)^\hat{n}(ck + dh)^n
\]
is equal to zero unless \(\left( \frac{a}{b} \frac{c}{d} \right) \in \Gamma\) satisfies
\[
|b + [k/h + 1/2]| \leq |a| \leq h, \quad |d + [k/h + 1/2]| \leq |c| \leq h.
\]
Furthermore we define a function \(E_{w,n} : \mathbb{Z}^+ \times \mathbb{Z} \to \mathbb{C}\) as follows.

(1) for \(n\) odd, \(E_{w,n}\) is defined by
\[
E_{w,n}(h,k) := I_{w,n}(h,k) - \frac{\tilde{B}_{n+1}(\frac{k}{h})}{n+1} h^w - \frac{\tilde{B}_{\tilde{n}+1}(\frac{k}{h})}{\tilde{n}+1} h^w + \frac{w + 2 B_{n+1} B_{\tilde{n}+1}}{B_{w+2}} \frac{h^w}{n+1 \tilde{n}+1}.
\]

(2) for \(n\) even, \(E_{w,n}\) is defined by
\[
E_{w,n}(h,k) := I_{w,n}(h,k) + \frac{\tilde{B}_{n+1}(\frac{k}{h})}{n+1} h^w - \frac{\tilde{B}_{\tilde{n}+1}(\frac{k}{h})}{\tilde{n}+1} h^w.
\]
Since \(I_{w,n}(1,0) = 0\), we obtain the following directly:

**Lemma 5.1.** If \(n\) is odd, we have
\[
E_{w,n}(1,0) = -\frac{B_{n+1}}{n+1} - \frac{B_{\tilde{n}+1}}{\tilde{n}+1} + \frac{w + 2 B_{n+1} B_{\tilde{n}+1}}{B_{w+2}} \frac{1}{n+1 \tilde{n}+1}.
\]

Now we also define a function \(S_{w,n}(h,k)\) in \(h\) and \(k\), which plays a role of reciprocity function for \(E_{w,n}\).
Definition 5.2. Let \( n \) be an integer such that \( 0 < n < w \). We define a polynomial \( S_{w,n} \) in \( h \) and \( k \) as follows.

(1) for \( n \) odd, \( S_{w,n} \) is defined by

\[
S_{w,n}(h,k) := -\frac{B_{n+1}(\frac{k}{n})}{n+1} h^w + \frac{B_{n+1}(\frac{h}{n})}{n+1} k^w - \frac{B_{n+1}(\frac{k}{n})}{n+1} h^w + \frac{B_{n+1}(\frac{h}{n})}{n+1} k^w \\
+ w + 2 \frac{B_{n+1} B_{n+1}(h^w - k^w)}{B_{w+2} n + 1 n + 1}.
\]

(2) for \( n \) even, \( E_{w,n} \) is defined by

\[
S_{w,n}(h,k) := +\frac{B_{n+1}(\frac{k}{n})}{n+1} h^w + \frac{B_{n+1}(\frac{h}{n})}{n+1} k^w - \frac{B_{n+1}(\frac{k}{n})}{n+1} h^w + \frac{B_{n+1}(\frac{h}{n})}{n+1} k^w.
\]

Note that the right hand sides of above equations are homogeneous polynomials in \( h \) and \( k \) of degree \( w \). Moreover, we know that the polynomial \( S_{w,n}(h,k) \) is even or odd depending on \( n \) is odd or even. Here are a couple of examples of \( S_{w,n} \):

\[
S_{10,4}(h,k) = \frac{2h^3k}{35} + \frac{5h^7k^3}{14} - \frac{3h^5k^5}{5} + \frac{5h^3k^7}{14} - \frac{2hk^9}{35}.
\]

and

\[
S_{10,5}(h,k) = -\frac{6h^{10}}{691} + \frac{h^8k^2}{6} - h^6k^4 + \frac{4h^6k^6}{2} - \frac{h^2k^8}{6} + \frac{6k^{10}}{691}.
\]

The following is a “weighted version” of [7, Theorems 1.1, 1.2]. The proof is similar to that of the “non-weighted version”, and we omit it.

Theorem 5.2. Let \( n \) be an integer such that \( 0 < n < w \). Then the following assertions hold:

1. \( E_{w,n} \) is an odd (resp. even) Dedekind symbol of weight \( w \) for \( n \) even (resp. odd).
2. \( E_{w,n} \) has the following reciprocity law:

\[
E_{w,n}(h,k) - E_{w,n}(k,-h) = S_{w,n}(h,k).
\]
3. Let \( E^- \) be an odd Dedekind symbol of weight \( w \) whose reciprocity function is a polynomial. Then \( E^- \) is a linear combination of \( E_{w,n} \) (\( 0 < n < w; n \) even).
4. Let \( E^+ \) be an even Dedekind symbol of weight \( w \) whose reciprocity function is a polynomial. Then \( E^+ \) is a linear combination of \( E_{w,n} \) (\( 0 < n < w; n \) odd), \( F_w \) and \( G_w \).

6. Dedekind symbols associated with cusp forms of \( w \leq 24 \)

In this section we investigate Dedekind symbols associated with cusp forms \( f \in S_{w+2} \) with \( w \leq 24 \). In this case, the dimension of \( S_{w+2} \) is at most one. Then, using Theorem 5.2 we can give explicit formula for \( E_f \).

Let \( \ell \) be one of the integer in \( \{10, 14, 16, 18, 20, 24\} \), then \( S_{w+2} \) is one-dimensional for each \( \ell \) in this set. We define \( f_{\ell+2} \) to be a unique normalized eigenform in \( S_{\ell+2} \) \((a_{f_1}(0) = 0, a_{f_2}(1) = 1) \). Then it is well known that \( f_{\ell+2} \) are expressed by discriminant \( \Delta \) and Eisenstein series \( Q, R \). Here

\[
\Delta(z) = e^{2\pi i z} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})^{24} = \sum_{n=1}^{\infty} \tau(n)e^{2\pi i n z},
\]
Q(z) = E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)e^{2\pi inz}

and

R(z) = E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)e^{2\pi inz}.

Indeed for \( \ell = 10, 14, 16, 18, 20 \) or 24, \( f_{\ell+2} \) are given, respectively, by \( \Delta, Q\Delta, R\Delta, Q^2\Delta, QR\Delta, Q^2R\Delta \) (refer to [17]). We use the following notation for the Fourier coefficient of \( f_{\ell+2} \):

\[ f_{\ell+2}(z) = \sum_{n=1}^{\infty} \tau_{\ell+2}(n)e^{2\pi inz}. \]

Note that \( \tau_{12} \) is nothing but Ramanujan’s tau function, namely \( \tau_{12}(n) = \tau(n) \).

We express even Dedekind symbol associated with \( f_{\ell+2} \) in terms of \( E_{\ell,n} \) which was explicitly given in Definition 5.1.

**Lemma 6.1.** Let \( \ell = 10, 14, 16, 18, 20 \) or 24. Then

\[ E_{f_{\ell+2}}^+ = cE_{\ell,2[\ell+2]/4} - 1 \]

where \( c \) is a constant.

**Proof.** Let \( n_0 = 2[(\ell+2)/4] - 1 \), and let \( \tilde{n}_0 = \ell - n_0 \). We know that \( 0 < n_0 < \ell \) and \( n_0 \) is odd. Then there is \( f \in S_{\ell+2} \) such that \( S_f^+ (h,k) = S_{\ell,n_0}(h,k) \) ([9 Theorem 1]). By Theorem 5.2 (2), the reciprocity polynomial of \( E_{\ell,n_0} \) is \( S_{\ell,n_0} \) while the reciprocity polynomial of \( E_{f_{\ell+2}}^+ \) is \( S_f^+ \) by (2.12). Then \( S_f^+ (h,k) = S_{\ell,n_0}(h,k) \) implies that

\[ E_{f_{\ell+2}}^+ = E_{\ell,n_0} + cG_{\ell} \]

for some constant \( c \) by Lemma 5.1.

Now we will show \( c = 0 \). We have

\[ E_f^+(1,0) = \int_0^{\infty} f(z)z^w dz = S_f^+(1,0) = S_{\ell,n_0}(1,0) \]
\[ = \frac{B_{n_0+1}}{n_0+1} - \frac{B_{\tilde{n}_0+1}}{\tilde{n}_0+1} + \frac{\ell + 2 B_{n_0+1}}{B_{\ell+2}} \frac{B_{\tilde{n}_0+1}}{n_0+1} \frac{B_{\tilde{n}_0+1}}{\tilde{n}_0+1} \]

by Definition 6.1 (1).

On the other hand, by Lemma 5.1 we have

\[ E_{\ell,n_0}(1,0) = - \frac{B_{n_0+1}}{n_0+1} - \frac{B_{\tilde{n}_0+1}}{\tilde{n}_0+1} + \frac{\ell + 2 B_{n_0+1}}{B_{\ell+2}} \frac{B_{\tilde{n}_0+1}}{n_0+1} \frac{B_{\tilde{n}_0+1}}{\tilde{n}_0+1} \]

These imply \( E_f^+(1,0) = E_{\ell,n_0}(1,0) \). Hence we know that \( c = 0 \) in the equation (6.1), and then

\[ E_f^+ = E_{\ell,n_0}. \]

Finally, since \( \dim S_{\ell+2} = 1 \), there is a constant \( c' \) such that \( f_{\ell+2} = c' f \). This implies

\[ E_{f_{\ell+2}}^+ = c'E_f^+ = c'E_{\ell,n_0}. \]

This completes the proof. □
7. Formulae for generalized Ramanujan’s tau functions

In this section we calculate $T_mE_{\ell+2}(h, k)$ and obtain explicit formula for $\tau_{\ell+2}(m)$. We start with the following lemma.

Lemma 7.1. Let $m$ be a positive prime integer, and let $n$ be odd. Then $T_mE_{w,n}(h, k)$ is calculated as follows:

$$T_mE_{w,n}(h, k) = I_{w,n}(h, mk) + \sum_{b=0}^{m-1} I_{w,n}(mh, k + bh)$$

$$-\frac{1}{n+1} \left\{ \bar{B}_{n+1}(\frac{mk}{h}) + m\bar{n}\bar{B}_{n+1}(\frac{k}{h}) \right\} h^w$$

$$-\frac{1}{\bar{n}+1} \left\{ \bar{B}_{\bar{n}+1}(\frac{mk}{h}) + m\bar{n}\bar{B}_{\bar{n}+1}(\frac{k}{h}) \right\} h^w$$

$$+ \frac{(1 + m^{w+1})}{B_{w+2} n + 1} n + 1 h^w.$$ 

In particular we have

$$T_mE_{w,n}(1, 0) = \sum_{b=0}^{m-1} I_{w,n}(m, b) - \frac{B_{n+1}}{n+1}(1 + m\bar{n}) - \frac{B_{\bar{n}+1}}{\bar{n}+1}(1 + m\bar{n})$$

$$+ \frac{(1 + m^{w+1})}{B_{w+2} n + 1} n + 1 h^w.$$ 

Proof. Applying the formula for Bernoulli function

$$\sum_{c=0}^{c-1} \bar{B}_{n+1}(x + \frac{b}{c}) = e^{-n} \bar{B}_{n+1}(cx) \quad (c \in \mathbb{Z}^+),$$

we have

$$T_mE_{w,n}(h, k)$$

$$= E_{w,n}(h, mk) + \sum_{b=0}^{m-1} E_{w,n}(mh, k + bh)$$

$$= I_{w,n}(h, mk) - \frac{B_{n+1}(\frac{mk}{h})}{n+1} h^w - \frac{B_{\bar{n}+1}(\frac{mk}{h})}{\bar{n}+1} h^w + \frac{w + 2 B_{n+1} B_{\bar{n}+1}}{B_{w+2} n + 1} h^w + \sum_{b=0}^{m-1} \left\{ I_{w,n}(mh, k + bh) - \frac{B_{n+1}(\frac{mk+bh}{mh})}{n+1} (mh)^w - \frac{B_{\bar{n}+1}(\frac{mk+bh}{mh})}{\bar{n}+1} (mh)^w \right\}$$

$$+ \frac{w + 2 B_{n+1} B_{\bar{n}+1}}{B_{w+2} n + 1} (mh)^w$$

$$= I_{w,n}(h, mk) + \sum_{b=0}^{m-1} I_{w,n}(mh, k + bh)$$

$$- \frac{1}{n+1} \left\{ \bar{B}_{n+1}(\frac{mk}{h}) + m\bar{n}\bar{B}_{n+1}(\frac{k}{h}) \right\} h^w$$

$$- \frac{1}{\bar{n}+1} \left\{ \bar{B}_{\bar{n}+1}(\frac{mk}{h}) + m\bar{n}\bar{B}_{\bar{n}+1}(\frac{k}{h}) \right\} h^w$$

$$+ \frac{(1 + m^{w+1})}{B_{w+2} n + 1} n + 1 h^w.$$
+ (1 + m_{w+1})^w \left/ \frac{B_{n+1} B_{n+1}}{B_{w+2} n + 1 \hat{n} + 1} \right. w.

This completes the proof. □

Next we calculate $T_m E_{w,n}(1,0)/E_{w,n}(1,0)$.

**Lemma 7.2.** Let $m$ be a positive prime integer, and let $n$ be odd. Suppose that $E_{w,n}(1,0) \neq 0$. Then we have

$$T_m E_{w,n}(1,0)/E_{w,n}(1,0) = 1 + m_{w+1} + \frac{1}{E_{w,n}(1,0)} \left( \sum_{b=0}^{m-1} I_{w,n}(m, b) - \frac{B_{n+1}}{n+1} (1 + m_{w+1}) - \frac{B_{n+1}}{n+1} (1 + m_{w+1}) \right) + \frac{1}{E_{w,n}(1,0)} \left( \sum_{b=0}^{m-1} I_{w,n}(m, b) - \frac{B_{n+1}}{n+1} (1 + m_{w+1}) + \frac{B_{n+1}}{n+1} (1 + m_{w+1}) \right).$$

**Proof.** Applying Lemma 7.1 we have

$$T_m E_{w,n}(1,0)/E_{w,n}(1,0) = \frac{1}{E_{w,n}(1,0)} \left( \sum_{b=0}^{m-1} I_{w,n}(m, b) - \frac{B_{n+1}}{n+1} (1 + m_{w+1}) - \frac{B_{n+1}}{n+1} (1 + m_{w+1}) \right) + \frac{1}{E_{w,n}(1,0)} \left( \sum_{b=0}^{m-1} I_{w,n}(m, b) - \frac{B_{n+1}}{n+1} (1 + m_{w+1}) + \frac{B_{n+1}}{n+1} (1 + m_{w+1}) \right).$$

This completes the proof. □

Finally we arrive at explicit formulae for generalized Ramanujan’s tau functions:

$T_m E_{w,n}(1,0)/E_{w,n}(1,0) = 1 + m_{w+1} + \frac{1}{E_{w,n}(1,0)} \left( \sum_{b=0}^{m-1} I_{w,n}(m, b) - \frac{B_{n+1}}{n+1} (1 + m_{w+1}) - \frac{B_{n+1}}{n+1} (1 + m_{w+1}) \right) + \frac{1}{E_{w,n}(1,0)} \left( \sum_{b=0}^{m-1} I_{w,n}(m, b) - \frac{B_{n+1}}{n+1} (1 + m_{w+1}) + \frac{B_{n+1}}{n+1} (1 + m_{w+1}) \right).$
**Theorem 7.3.** Let $m$ be a positive prime integer. For $\ell = 10, 14, 16, 18, 20$ and $24$, $\tau_{\ell+2}(m)$ are expressed as

\[
\tau_{12}(m) = 1 + m^{11} - \frac{691}{6} \left\{ -\frac{1}{126} (m^5 - m^{11}) + \sum_{b=0}^{m-1} I_{10, 5}(m, b) \right\},
\]

\[
\tau_{16}(m) = 1 + m^{15} + \frac{3617}{30} \left\{ -\frac{1}{120} (m^7 - m^{15}) + \sum_{b=0}^{m-1} I_{14, 7}(m, b) \right\},
\]

\[
\tau_{18}(m) = 1 + m^{17} - \frac{43867}{150} \left\{ -\frac{1}{132} (m^7 - m^{17}) + \frac{1}{240} (m^9 - m^{17}) \right. \\
\left. + \sum_{b=0}^{m-1} I_{16, 7}(m, b) \right\},
\]

\[
\tau_{20}(m) = 1 + m^{19} - \frac{174611}{2646} \left\{ -\frac{1}{66} (m^9 - m^{19}) + \sum_{b=0}^{m-1} I_{18, 9}(m, b) \right\},
\]

\[
\tau_{22}(m) = 1 + m^{21} + \frac{77683}{1050} \left\{ \frac{691}{32760} (m^9 - m^{21}) - \frac{1}{132} (m^{11} - m^{21}) \right. \\
\left. + \sum_{b=0}^{m-1} I_{20, 9}(m, b) \right\},
\]

\[
\tau_{26}(m) = 1 + m^{25} - \frac{657931}{40950} \left\{ -\frac{1}{12} (m^{11} - m^{25}) + \frac{691}{32760} (m^{13} - m^{25}) \right. \\
\left. + \sum_{b=0}^{m-1} I_{24, 11}(m, b) \right\}.
\]

**Proof.** By Theorem 4.1, we know

\[
\tau_{\ell+2}(m) = \frac{T_m E_{\ell+2}^+ (h, k)}{E_{\ell+2}^+ (h, k)}.
\]

By Lemma 6.1 we know

\[
E_{\ell+2}^+ = c E_{\ell, 2(\ell+2)/4} (h, k),
\]

and thus we have

\[
\tau_{\ell+2}(m) = \frac{T_m E_{\ell, 2(\ell+2)/4} - 1 (h, k)}{E_{\ell, 2(\ell+2)/4} - 1 (h, k)}.
\]

for $(h, k)$ with $E_{\ell, 2(\ell+2)/4} - 1 (h, k) \neq 0$.

Direct calculations using Lemma 5.1 yield

\[
E_{\ell, 2(\ell+2)/4} - 1 (1, 0) = \frac{6}{691} \cdot \frac{30}{3617} - \frac{150}{43867} \cdot \frac{2646}{174611} - \frac{1050}{77683} - \frac{40950}{657931}
\]

for $\ell = 10, 14, 16, 18, 20, 24$, respectively. In particular we know $E_{\ell, 2(\ell+2)/4} - 1 (1, 0) \neq 0$.

Hence we have

\[
\tau_{\ell+2}(m) = \frac{T_m E_{\ell, 2(\ell+2)/4} - 1 (1, 0)}{E_{\ell, 2(\ell+2)/4} - 1 (1, 0)}.
\]
Then we use Lemmas 7.2 putting $w = \ell$ and $n = 2[(\ell + 2)/4] - 1$, to obtain the formulae in Theorem 7.3.

\[ \square \]

Remark 7.1. There are other formulae for $\tau_{\ell+2}$ different from ours, by MacDonald [10] (see also Dyson [4]) and by Manin [11].

From Theorem 7.3 and the fact that $I_{w,n}(h, k)$ is an integer, we rediscover the following congruences which are obtained by Ramanujan [15], Swinnerton-Dyer [17] and Manin [11].

\[
\begin{align*}
\tau_{12}(m) &\equiv \sigma_{11}(m) \pmod{691}, \\
\tau_{16}(m) &\equiv \sigma_{15}(m) \pmod{3617}, \\
\tau_{18}(m) &\equiv \sigma_{17}(m) \pmod{43867}, \\
\tau_{20}(m) &\equiv \sigma_{19}(m) \pmod{283 \cdot 617}, \\
\tau_{22}(m) &\equiv \sigma_{21}(m) \pmod{131 \cdot 593}, \\
\tau_{26}(m) &\equiv \sigma_{25}(m) \pmod{657931}.
\end{align*}
\]

(These congruences can be shown first for $m$ prime, and then for any positive integer due to multiplicative properties of $\tau_{\ell+2}$ and $\sigma_{\ell+1}$.)

8. Knopp-Parson-Rosen identities

In this final section we show how naturally Knopp-Parson-Rosen identities are derived from Hecke operators on Dedekind symbols.

Let $f \in M_{w+2}$ be a normalized Hecke eigenform, and let $E_f$ be the odd Dedekind symbol associated with $f$. Then, in Theorem 4.1 we showed the following identity:

\[
(8.1) \sum_{ad=n, b \text{(mod $d$)}} E_f^- (dh, ak + bh) = a_f(n)E_f^- (h, k)
\]

for any $(h, k) \in \mathbb{Z}^+ \times \mathbb{Z}$.

Taking $f$ to be Eisenstein series $G_{w+2}$ of weight $w + 2$, we can see the identity (8.1) is nothing but Knopp-Parson-Rosen identity [8, 12, 13].

Now let us recall generalized Dedekind sums $s_{w+1}(k, h)$ introduced by Apostol.

Definition 8.1 (Apostol [1, 3]).

\[
s_{w+1}(k, h) = \sum_{\mu=0}^{h-1} \frac{\mu}{h} B_{w+1}(\frac{\mu k}{h}).
\]

In [3 Lemma 6.4]), for Eisenstein series $G_{w+2}$ defined by

\[
G_{w+2}(z) = -\frac{B_{w+2}}{2(w + 2)} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} n^{w+1} e^{2\pi imnz},
\]

it was shown that

\[
(8.2) E^-_{G_{w+2}}(h, k) = -\frac{h^w}{2(w + 1)} s_{w+1}(k, h).
\]

(Though, in [3, 8.2] was proved for $(p, q) \in V$, the proof is also valid for any $(h, k) \in \mathbb{Z}^+ \times \mathbb{Z}$.)

Then we have
Theorem 8.1.\(\text{[12],[13].}\)
\[
\sum_{\substack{ad=n \ b(\text{mod} d) \ 0 < d}} d^w s_{w+1}(ak + bh, dh) = \sigma_{w+1}(n)s_{w+1}(k, h).
\]

Proof. Since \(G_{w+2}\) is a normalized eigenform for \(T_n\) with the eigenvalue \(\sigma_{w+1}(n)\), we have
\[
\sum_{\substack{ad=n \ b(\text{mod} d) \ 0 < d}} E_{G_{w+2}}(dh, ak + bh) = \sigma_{w+1}(n)E_{\tilde{G}_{w+2}}(h, k).
\]
From (8.4) and (8.2), we obtain (8.3).

\[\Box\]

Remark 8.1. The reader can easily see that Theorems 1.1, 1.2, 1.3, 1.4 and 1.5 are special cases of the corresponding Theorems 2.3, 2.4, 3.3, 4.1 and 7.3 for modular forms. If we restrict these latter theorems to cusp forms, we rediscover Theorems 1A to 1D. Hence we omit proofs of those theorems.

References

[1] Apostol, T. M.: Generalized Dedekind sums and transformation formulae of certain Lambert series. Duke Math. J. 17, 147–157 (1950).
[2] Apostol, T. M.: Modular functions and Dirichlet series in number theory (Grad. Texts in Math. No. 41). New-York: Springer 1990.
[3] Carlitz, L.: Some theorems on generalized Dedekind sums. Pacific J. Math. 3, 513–522 (1953).
[4] Dyson, F. J.: Missed opportunities. Bull. Amer. Math. Soc. 78, 635–652 (1972).
[5] Fukuhara, S.: Modular forms, generalized Dedekind symbols and period polynomials. Math. Ann. 310, 83–101 (1998).
[6] Fukuhara, S.: Generalized Dedekind symbols associated with the Eisenstein series. Proc. Amer. Math. Soc. 127, 2561–2568 (1999).
[7] Fukuhara, S.: Dedekind symbols with polynomial reciprocity laws. Math. Ann. 329, 315–334 (2004).
[8] Knopp, M. I.: Hecke operators and an identity for the Dedekind sums. J. Number Theory 12, 2–9 (1980).
[9] Kohnen, W., Zagier, D.: Modular forms with rational periods. In: Rankin, R. A. (ed.): Modular Forms, pp. 197–249, Chichester: Ellis Horwood 1984.
[10] MacDonald, I. G.: Affine root systems and Dedekind’s \(\eta\)-function. Invent. Math. 15, 91–143 (1972).
[11] Manin, Y.: Periods of parabolic forms and \(p\)-adic Hecke series. Math. Sbornik, AMS Translation 21, 371–393 (1973).
[12] Parson, L. A.: Dedekind sums and Hecke operators. Math. Proc. Camb. Phil. Soc. 88, 11–14 (1980).
[13] Parson, L. A., Rosen, K. H.: Hecke operators and Lambert series. Math. Scand. 49, 5–14 (1981).
[14] Rademacher, H., Grosswald, E.: Dedekind sums (Carus Math. Mono. No. 16). Math. Assoc. Amer. 1972.
[15] Ramanujan, Y.: On certain arithmetical functions. Trans. Camb. Phil. Soc. 22, 159–184 (1916).
[16] Serre, J.-P.: A course in arithmetic (Grad. Texts in Math. No. 7). New-York: Springer 1973.
[17] Swinnerton-Dyer, H. P. F.: On \(\ell\)-adic representations and congruences for coefficients of modular forms. In: Summer School on Modular Functions (Antwerp, 1972) (Lecture Notes in Math. No. 763), pp. 1–55, Springer 1973.
[18] Zagier, D.: Hecke operators and periods of modular forms. In: Israel Math. Conf. Proc. 3, Bar-Ilan Univ., Ramat Aviv, pp. 321–336, Weizmann, Jerusalem 1990.
[19] Zagier, D.: Periods of modular forms and Jacobi theta functions. Invent. Math. 104, 449–465 (1991).