Bounded critical Fatou components are Jordan domains, for polynomials.

P. Roesch, Y. Yin *
Institut de Mathématiques de Toulouse, France; Fudan University, China

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Abstract

We prove that the boundary of any bounded Fatou component for a polynomial is a Jordan curve, except maybe for Siegel disks.

1 Introduction

The Riemann sphere, viewed as a dynamical space on which act the rational maps, is divided into two sets: the Fatou set $F(f)$ and the Julia set $J(f)$. The dynamics on the Fatou set is well understood but is chaotic on the Julia set. In this article we will restrict ourself to the action of polynomials. The connected component of the Fatou set containing $\infty$, usually called $B(\infty)$, is distinguished: its boundary is exactly $J(f)$. It can have a very complicated topology (for instance it can contain continua of the form of a "hedgehog" discovered by Perez-Marco). However, the bounded Fatou components have very simple boundary:

Theorem 1. If $f$ is a polynomial, any bounded Fatou component, which is not a Siegel disk, is a Jordan domain (i.e. a disk with Jordan curve boundary).

Concerning Siegel disks, i.e. Fatou components on which the map is conjugate to irrational rotations, the question is still open. One conjectures:

Conjecture. For a polynomial, every bounded Fatou component is a Jordan domain.

This conjecture is supported by the work of [PeZ] and Shishikura announced it for high type Siegel disks in degree 2. Note that Chéritat recently constructed a holomorphic map (that is not a polynomial) defined in an open disk of $C$ containing the closure of a fixed Siegel disk, whose boundary is a pseudo circle, in particular it is not locally connected (see [Ch]).

As a consequence of the proof of Theorem 1 we obtain a “description from inside” of the dynamics on the component containing $U$ of the filled Julia set, $K(f) = C \setminus B(\infty)$:

Theorem 2. Let $f$ be a polynomial and $U$ a periodic bounded Fatou component which is not a Siegel disk. Every point of $\partial U$ is the landing point of at least one external ray. Moreover if $K_U$
denotes the connected component containing $U$ of the filled Julia set $K(f)$, then $K_U = \overline{U} \cup \bigcup_{t \in S^1} L_t$ where the sets $L_t$ are “limbs” sprouting out of $\overline{U}$ with the following properties:

1. $L_t$ is connected;
2. $L_t$ intersects $\overline{U}$ at exactly one point called $\gamma_U(t)$;
3. $L_t \neq \gamma_U(t)$ if and only if $L_t$ either contains a critical point or is eventually mapped to a limb $L_t'$ containing a critical point.

**Corollary 1.** If $J(f)$ is connected, the only point of $\partial U$, where the local connectivity of $J(f)$ can fail are the eventually periodic ones, where a comb might be attached by one point to the boundary.

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### 1.1 Overview

Theorem 2 follows from the proof of Theorem 1 which is a consequence of the following.

**Theorem 3.** Let $f$ be a polynomial and $U$ a bounded Fatou component. Assume that $J(f)$ is connected and that $f$ fixes $U$ with exactly one critical point in $U$, then $\partial U$ is locally connected.

**Lemma 1.1.** Theorem 3 implies Theorem 1.

**Proof.** Let $f$ be a polynomial and $U$ a bounded Fatou component which is not a Siegel disk.

Denote by $K_U$ the connected component of $K(f)$ containing $U$. There exist neighborhoods $X, X'$ of $K_U$, an integer $r$ and a polynomial $g$ with connected Julia set, such that the iterate $f^r$ of $f$ maps $X$ to $X'$ and $f^r : X \to X'$ is conjugated by a quasi-conformal homeomorphism to $g$ on a disk $D(0, R)$ with large $R$. This follows from the classical theory of polynomial-like mappings applied to some disk of low potential (for the Green function) containing $K_U$. Let $V$ be the image of $U$ by the conjugacy; it is equivalent to prove that $\partial U$ is a Jordan curve.

By Sullivan’s non-wandering Theorem, $V$ is mapped by $g$ to some Fatou component $W$ which is fixed by some iterate $h = g^k$. Moreover, this Fatou component $W$ contains a critical point so cannot be a Siegel disk: a disk where the dynamics is conjugated to an irrational rotation. Hence by the classification result of the Fatou components, there is a fixed point $p = h(p)$ in $W$ which is either attracting when $p \in W$, or parabolic when $p \in \partial W$. By a surgery procedure $h$ is quasi-conformally conjugated on a neighborhood of the Julia set to a polynomial having only one critical point in the Fatou component $Y$ corresponding to $V$. This surgery is done for instance in Theorem 5.1 of [C-G] for the attracting case and, in the parabolic case in Proposition 6.8 of [McM] (where general case is in fact done).

Thus, since the Julia set of an iterate (here $g^k$) is equal to the original Julia set (here of $g$), we have just proved that there is an homeomorphism between neighborhoods of the boundary of the corresponding Fatou components, $U$ and $Y$. Therefore, applying Theorem 3 to the polynomial $h$ and the Fatou component $Y$, implies the local connectivity of $\partial U$.

Then it is classical to prove that a it is a Jordan domain using the maximum principal. The Riemann map $\Phi$, from $D$ to $U$ (which is a disk), extends continuously to $\overline{D}$ since $\partial U$ is locally connected (by Caratheodory’s Theorem). Therefore, $\partial U$ is the curve $\Phi(S^1)$. If it is not simple, there exists $t, t'$ such that $\Phi(\epsilon^{2\pi t}) = \Phi(\epsilon^{2\pi t'})$ so that the curve $C = \Phi([0, 1]e^{2\pi t}) \cup \Phi([0, 1]e^{2\pi t'})$ is in $\overline{U}$ and bounds points that are attracted by $\infty$, which contradicts the maximum principle. \(\square\)
1.2 About the Proof of Theorem 3

We find connected neighborhoods for the points \( x \in \partial U \) as pieces of the complement of some backward iterated graph (classically called puzzle pieces). These graphs are the union of internal and external rays as well as equipotentials (in the parabolic case we use "parabolic rays"). To prove that the diameter of the pieces tends to zero we use different techniques depending on the point \( x \) of \( \partial U \):

- If \( x \) is eventually periodic (see section 3) it might be the attach-point of some other part of the Julia set. We consider the dynamics on the intersection of the closure of the puzzle pieces containing this periodic point (following the ideas of Kiwi in [K1]) and prove that two external rays, landing at \( x \), separate this continuum from the closure of \( U \). This is the only case where the intersection of the puzzle pieces might not be reduced to one point and then we might have a renormalizable map.

- If \( x \) combinatorially accumulates an eventually periodic point \( y \), meaning for the topology generated by the set of the puzzle pieces, we consider the first entrance time in the periodic nest: the first time the orbit enters in each puzzle piece of the nest around \( y \) (see section 4). If \( y \) is repelling, it is easy to find a non degenerate annulus between consecutive puzzle pieces and to pull it back with bounded degree around \( x \). If \( y \) is parabolic, we use some distortion properties on the enlarged puzzle pieces.

- The last case to consider is when \( x \) does not combinatorially accumulate eventually periodic points. Here, the recurrence of the critical points plays a fundamental role. We consider the combinatorial accumulation of the different critical points on themselves:
  
  - The “non-recurrent case” (see section 4) corresponds to the situation where there is at least a critical point (in the combinatorial accumulation of \( x \)) whose combinatorial accumulation does not contain critical points. This case works as before, excepted that for finding a non degenerate annulus, where we need to look at deeper puzzle pieces in the nest.
  
  - In the “recurrent case” we have to distinguish between two strength of recurrences: the “reluctantly recurrent” case is dealt in section 4 using long iterates of bounded degree, and the “persistently recurrent” case (see section 6) where we have to introduce the enhance nest introduced by Koslovski, Shen and van Strien in [K-S-S]. It is a double sub-nest \( (K_n, K'_n) \) of the nest of a critical point and that has the property that 
    \( K'_n \setminus K_n \) avoids the orbit of the critical points, that the time to go from \( K_n \) to \( K_{n-1} \) is more than half of the time to go from \( K_n \) back to \( K_0 \), whereas the degree of the map from \( K_n \) to \( K_{n-1} \) is uniformly bounded. To prove that the diameter of the puzzle of this nest tends to zero, we use the Covering Lemma of Kahn and Lyubich (see [K-L]). The arguments are similar to those provided in the proof of the Branner-Hubbard conjecture (see [QY] and also [KS]), excepted that we do not have non degenerate annuli here.

    To have a different point of view on the enhanced nest we recommend highly to read [TY] (in the case of a unique critical point) and [P-Q-R-T-Y] (in the case of several critical points).

Excepted for section 3 the rest of the article is devoted to the proof of Theorem 3.
2 Puzzles and parabolic techniques

2.1 Notations

We fix a polynomial $f$ of degree $D$ and assume that it has a connected filled-in Julia set $K(f)$. We consider a bounded Fatou component $U$, that is fixed by $f$ and which contains exactly one critical point.

Since $K(f)$ is connected, its complement $B(\infty) \cup \{\infty\}$ is simply connected. Denote by $\Phi_{\infty} : \overline{\mathbb{C}} \setminus \overline{D} \to B(\infty) \cup \{\infty\}$ the Riemann map such that $\Phi_{\infty}(\infty) = \infty$ and which is tangent to the identity near $\infty$. It conjugates $f$ to $z \mapsto z^D$. The bounded Fatou component $U$ is a topological disk, let $c_0$ be the unique critical point of $f$ in $U$ and denote by $\Phi_U : \mathbb{D} \to U$ a Riemann map such that $\Phi_U(0) = c_0$.

In the attracting case, $c_0$ is fixed (it is the super-attracting fixed point of $U$). We choose the map $\Phi_U$ such that $\Phi_U'(0) = 1$. Therefore, $\Phi_U$ conjugates $f$ to $z \mapsto z^d$ where $d$ is the degree of the critical point $c_0$.

In the parabolic case, there is a fixed point $p$ on $\partial U$ and every point of $U$ tends to $p$ under the iteration of $f$. We choose $\Phi_U$ such that $\Phi_U(v_d) = f(c_0)$ where $v_d = \frac{d-1}{d}$ Therefore, $\Phi_U$ conjugates $f$ to the Blaschke model $B(z) = \frac{z^{d} + v_d}{1 + v_d z^d}$ on $\mathbb{D}$ and maps $p$ to 1.

2.2 Rays and equipotentials

In the attracting case:

Definition 2.1. The external, resp. attracting internal, ray of angle $\theta$ is the set

$$R_{\infty}(\theta) = \Phi_{\infty}(\{e^{2i\pi \theta + t} \mid t > 0\}), \text{ resp. } R_U(\theta) = \Phi_U(\{e^{2i\pi \theta + t} \mid t < 0\}).$$

The external, resp. internal, equipotential (in the attracting case), of potential $v > 0$ is the set

$$E_{\infty}(v) = \Phi_{\infty}(\{e^{2i\pi \theta + v} \mid \theta \in \mathbb{R}\}), \text{ resp. } E_U(v) = \Phi_U(\{e^{2i\pi \theta - v} \mid \theta \in \mathbb{R}\}).$$

Proposition 1 (Douady, Hubbard, Sullivan, Yoccoz). If $t$ is rational, the internal ray $R_U(t)$ lands at a (eventually) periodic point which is either repelling or parabolic. Moreover, there is at least one external ray landing at this point. All these rays (internal and external) have the same rotation number.

We say that a $q$ cycle for $f^k$ of rays $R_0, \ldots R_{q-1}$ landing on a common $k$ periodic point $z$ and numbered in the counter clockwise order around $z$ defines combinatorial rotation number $p/q$, $(p, q) = 1$ iff $f^k(R_j) = R_{(j+p) \mod q}$.

In the parabolic case:

We recall a definition of parabolic rays given in [PeR]. We first construct parabolic rays in the disk for the model map $B$ and then we pull them back by the conjugacy.

Let $z_0 = 0$ and $T_0 := B^{-1}([0, v]) = \bigcup_{j=0}^{d-1} [0, z_j]$, where $z_j = v_d^{-1/d} e^{i\pi/d} \omega^j$ and $\omega = e^{2i\pi/d}$. Moreover let $T_j$ denote the connected component of $B^{-1}(T_0)$ containing $z_j$. Define recursively on $n \in \mathbb{N}^*$ and for each $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \{0, 1, \ldots, (d-1)\}$ the point $z_{\epsilon_1, \epsilon_2, \ldots, \epsilon_n}$ as the unique point of the preimage $B^{-1}(z_{\epsilon_2, \ldots, \epsilon_n})$ belonging to $T_{\epsilon_1, \epsilon_2, \ldots, \epsilon_n-1}$. And define $T_{\epsilon_1, \epsilon_2, \ldots, \epsilon_n}$ to be the connected component of the preimage $B^{-1}(T_{\epsilon_2, \ldots, \epsilon_n})$ containing $z_{\epsilon_1, \epsilon_2, \ldots, \epsilon_n}$. 


Define for each $n$ a $(d$-adic) portion of tree $\mathcal{T}_n := \bigcup_{k=0}^n B^{-k}(T_0)$, so that

$$\mathcal{T}_n = \mathcal{T}_{n-1} \cup \bigcup_{(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \{0,1,\ldots,(d-1)\}^n} T_{\epsilon_1, \epsilon_2, \ldots, \epsilon_n}.$$ 

Moreover define an infinite $d$-adic tree $\mathcal{T} := \cup_{k=0}^{\infty} B^{-k}(T_0)$ with boundary (in) $S^1$.

**Definition 2.2** (Parabolic rays for $B$). For $\xi \in \Sigma_d$, where $\Sigma_d := \{0,1,\ldots,d-1\}^\infty$, define the parabolic ray $R_{\xi}$ as the minimal connected subset of $\mathcal{T}$ containing the sequence of points $(z_{\epsilon_1, \epsilon_2, \ldots, \epsilon_n})_{n \in \mathbb{N}}$ (interpreting $n = 0$ as $z_0$).

For each $0 \leq j < d$, let $S_j$ be the open sector spanned by the interval $I_j = [\omega^j, \omega^{j+1}] \subset S^1$, i.e. the interior of the convex hull of the union of $I_j$ and 0. The map $B$ is univalent from $S_j$ onto $D \setminus [v, 1] \supset T_0$. Its boundary arcs $[0, \omega_j]$ and $[0, \omega_{j+1}]$ are each mapped (homeomorphically) onto the forward invariant arc $[v, 1]$. As $z_j \in S_j$ for each $j$ it easily follows by induction, that $\mathcal{T}_n \cap S_j$ is connected and for any $n$ and any $(\epsilon_2, \ldots, \epsilon_n)$ contains both the point $z_j, \epsilon_2, \ldots, \epsilon_n$ and the set $T_{\epsilon_1, \epsilon_2, \ldots, \epsilon_n}$. Consequently $S_j \cup \{0\}$ contains also any of the rays $R_{\xi}$ with $\epsilon_1 = j$.

Consider the attracting Fatou coordinate $\Phi_+: D \to C$ for $B$ on $D$, normalized by $\Phi_+(0) = 0$. For any $\xi \in \Sigma_d$, $\Phi_+$ maps $R_{\xi}$ homeomorphically onto $\mathbb{R} = [-\infty, 0]$, and has a degree $d$ critical point at the preimage $z_{\epsilon_1, \epsilon_2, \ldots, \epsilon_n}$ of $-n$ for each $n \geq 0$. The extended ray $\tilde{R}_{\xi} := R_{\xi} \cup [0,1]$ is mapped homeomorphically to $\mathbb{R}$ by $\Phi_+$. Denote by $\tilde{R}_{\xi}(t)$ the point $\Phi_+^{-1}(t) \cap \tilde{R}_{\xi}$ and define similarly $R_{\xi}(t)$ for $t \leq 0$. By construction,

$$\forall \xi \in \Sigma_d : B(\tilde{R}_{\xi}(t)) = \tilde{R}_{\sigma(\xi)}(t+1), \text{ where } \sigma(\epsilon_1, \epsilon_2, \ldots) = (\epsilon_2, \epsilon_3, \ldots) \text{ is the shift map on } \Sigma_d.$$ 

Note that every ray $R_{\xi}(t)$ lands at some point $z_{\xi} \in S^1$, as $t \to -\infty$.

**Definition 2.3** (Parabolic rays in $U$). For $\xi \in \Sigma_d$, define the parabolic ray of argument $\xi$ in $U$ as $R_{U}[\xi] := \Phi_+^{-1}(R_{\xi})$ and the extended ray as $\tilde{R}_{U}[\xi] := \Phi_+^{-1}(\tilde{R}_{\xi})$.

By construction, $f(\tilde{R}_{U}[\xi](t)) = \tilde{R}_{U}[\sigma(\xi)](t+1), \forall \xi \in \Sigma_d$.

The correspondence given by the formula $\theta = \sum_{n=1}^{\infty} \frac{\theta_n}{d^n}$, between angles and itineraries ($\xi$), is a bijection for non $d$-adic angles. Therefore, we will denote by $R_{U}(\theta)$ the ray of angle $\theta$, and by $\tilde{R}_{U}(\theta)$ the extended ray, as soon as $\theta$ is not $d$-adic. We say that the parabolic ray $\tilde{R}_{U}[\xi]$ converges if $\tilde{R}_{U}[\xi](t)$ admits a limit when $t$ tends to $-\infty$.

**Theorem 4.** For any $(\text{pre})$-periodic argument $\xi \in \Sigma_d$, i.e. $\sigma^k(\sigma^l(\xi)) = \sigma^l(\xi)$, the parabolic ray $R = R_{U}[\xi]$ converges to an $f$ (pre-)periodic point $z \in \partial U$ with $f^k(f^l(z)) = f^l(z)$. Furthermore, if $\xi$ is periodic (i.e. $l = 0$), let $k'$ denote the exact period of $z$ and let $q = k/k'$. Then the ray $R$ defines a combinatorial rotation number $p/q$, $(p, q) = 1$ for $z$. The periodic point $z$ is repelling or parabolic with multiplier $e^{2\pi p/q}$. Moreover any other ray in $U$ landing at $z$ is also $k$ periodic and defines the same rotation number.

This is a standard result which in its initial form is due to Sullivan, Douady and Hubbard, for external rays of polynomials, see also [Pe] Th. A and Prop. 2.1.

**Definition 2.4.** Consider for $\nu > 0$, the closure of the union of the curves $\Phi_+(\Phi_+^{-1}(\{z \mid \Re(z) = \ln \nu/\ln d\}))$. Denote by $E_\nu(v)$ the component containing $p$. For $0 < \nu \leq 1$, there is only one component in this set, however for $\nu > 1$, this set contains several components.
For \( v \in \mathbb{R}^+ \setminus \{1/d^k \mid k \geq 0\} \), \( E_U(v) \) is a simple closed curve. Moreover, it satisfies the same relation as in the attracting case, namely that \( f(E_U(v)) = E_U(dv) \) for \( v > 0 \).

**Lemma 2.5.** If \( \theta \neq \theta' \) the rays \( R_U(\theta) \) and \( R_U(\theta') \) do not land at the same point (if they land).

**Proof.** Assume that the two rays land at the same point, then the union of their closure defines a closed curve \( \gamma \). There are points of the Julia set in the bounded connected component of \( \mathbb{C} \setminus \gamma \); take for instance the landing point of dyadic angles that are between \( \theta \) and \( \theta' \). This contradicts the maximum principle. \( \square \)

### 2.3 Puzzles

Let us define the puzzle associated to a graph defined by a periodic angle \( \theta \). Then taking the appropriated preimage will lead to the definition for a pre-periodic angle. If \( \theta \) is a \( k \)-periodic angle by multiplication by \( d \), but not fixed, the internal ray \( R_U(\theta) \) is well defined (in the case of parabolic ray) and lands at a periodic point \( z(\theta) \) of \( \partial U \). Denote by \( R_\infty(\xi) \) an external ray landing at \( z(\theta) \) (Proposition [1]). It as the same period and same rotation number as \( R_U(\theta) \).

**Definition 2.6.** Let \( \Gamma(\theta) \) be the graph

\[
\Gamma(\theta) = \left( \bigcup_{j \geq 0} R_U(d^j \theta) \cup z(d^j \theta) \cup R_\infty(D^j \xi) \right) \cup E_\infty(1) \cup E_U \left( \frac{1}{d^{k-1}-1} \right).
\]

We have the following stability property: \( f(\Gamma(\theta)) \cap \Omega \subset \Gamma(\theta) \), where \( \Omega \) is the connected component of \( \mathbb{C} \setminus E_\infty(1) \cup E_U \left( \frac{1}{d^{k-1}-1} \right) \) containing \( \partial U \) (or a portion of it).

The definition of \( \Gamma(\theta) \) (namely the value of the equipotential in \( U \)) is chosen so that two rays of \( \Gamma(\theta) \) in \( U \) do not touch in \( \Omega \).

**Definition 2.7.** Let \( \Gamma_n(\theta) \) be the graph \( f^{-n}(\Gamma(\theta)) \). The connected components of \( \mathbb{C} \setminus \Gamma_n(\theta) \) intersecting \( \partial U \) are called puzzle pieces of depth \( n \). For \( y \in \partial U \), let us denote by \( P_n(y) \) the puzzle piece containing \( y \), when it exists. We extend this notation to the parabolic point \( p \), denoting by \( P_n(p) \) the puzzle piece containing \( p \) in its closure.

The next properties follow from the definition of the puzzle pieces. The proof is classical (see [R1] for instance).

**Lemma 2.8.**

- For every point \( y \in \partial U \), which is not eventually mapped to \( p \), there exists a strictly periodic angle \( \theta \) such that the orbit of \( y \) does not intersect \( \Gamma(\theta) \). Hence \( P_n(y) \) is well defined for every \( n \geq 0 \). Moreover, we can chose \( \theta \) such that the orbit of the critical points also never cross the graph \( \Gamma(\theta) \) except maybe at the pre-images of \( p \);

- The puzzles pieces are either nested or disjoint;

- The image \( f(P_n(y)) \) is the puzzle piece \( P_{n-1}(f(y)) \);

- If \( y \in \partial U \), the boundary of \( P_n(y) \) intersects \( U \) along exactly two portions of internal rays and a portion of equipotential that might be touching \( \partial U \) at a pre-image of \( p \);

- The boundary of two nested distinct puzzle pieces can only touch at a point of \( \partial U \) which is fixed or eventually periodic.
Notation 2.9. Let \( \text{End}(y) \) denote the set \( \bigcap_{n \in \mathbb{N}} \mathcal{P}_n(y) \). Let \( \text{Crit} \) denotes the union of the critical points of \( f \) outside of \( \bigcup_{n \in \mathbb{N}} f^{-n}(U) \) and call critical ends the ends containing at least a critical point, \( i.e. \) the ends \( \text{End}(c) \), for \( c \in \text{Crit} \).

Assumption 1. Let \( Y \) be a finite set of points. We can always assume that there is no critical point in \( P_0(y) \setminus \text{End}(y) \) for every point \( y \in Y \).

Indeed, one can replace \( \theta \) by one of its iterated pre-image for instance. We will use this mainly with \( Y = \text{Crit} \cup \{x\} \) or some iterate of it where \( x \) is the point we focus on.

Remark 2.10. Under this assumption we can define the degree on a critical end \( \text{End}(c) \) as the degree on \( P_0(c) \). It does not depend on the choice of the critical point in the end. For any \( y \), we denote by \( \delta(y) \) the degree of \( f \) on \( P_0(y) \).

Lemma 2.11. Let \( V \) be the closure of a puzzle piece. Then \( V \cap \partial U \) is connected.

Proof. One easily sees by induction that the boundary of \( V \) intersects \( U \) along the closure of two rays \( R_U(t_1), R_U(t_2) \). Therefore, \( V \cap \partial U \) is the intersection of the sets \( S_n = \Phi_U(\Delta_n) \) where \( \Delta_n = \{ z \in \mathbb{C} \mid |z| \in [e^{1/n}, 1], \arg(z) \in [2\pi t_1, 2\pi t_2] \} \). Hence \( V \cap \partial U \) is a connected set as the decreasing intersection of the compact connected sets \( S_n \).

Corollary 2.12. Let \( I_n(x) = \overline{P_n(x)} \cap \partial U \). It suffices to prove that \( \bigcap_{n \in \mathbb{N}} I_n(x) = \{x\} \) in order to show that \( \partial U \) is locally connected at the point \( x \).

Definition 2.13. If \( x \) lies on some graph \( \Gamma_n \), one can define \( I_n(x) \) by \( Q_n(x) \cap \partial U \) where \( Q_n(x) \) is the closure of the union of the pieces containing \( x \) in their closure.

Lemma 2.14. Let \( x \in \partial U \), for any graph \( \Gamma(\theta) \) of the Definition 2.6 the set \( I_n(x) \) is well defined.

Moreover, the property \( \bigcap_{n \in \mathbb{N}} I_n(x) = \{x\} \) does not depend on the graph.

Proof. Consider two graphs \( \Gamma, \Gamma' \); denote \( P_n, P'_n \) the pieces of the puzzles constructed from these graphs. For any \( n \geq 0 \), there exists \( k(n) \) such that \( P'_{k(n)}(x) \) (resp. \( Q_{k(n)}(x) \)) is included in \( P_n(x) \) or in \( Q_n(x) \). Indeed, by construction the intersection of \( P'_k(x) \) with \( U \) is the sector delimited by two rays \( R_U(t_k), R_U(s_k) \); the difference \( s_k - t_k \) tending to 0. Therefore \( P'_k(x) \) will be included in \( P_n(x) \) for large values of \( k \) (or in \( Q_n(x) \) if \( x \) lies on \( \Gamma' \) or on some inverse image of \( \Gamma' \)).

3 The “periodic” case

The Julia set \( J(f) \) might contain Cremer points, \( i.e. \) periodic points with multiplier \( e^{2\pi it} \) for some irrational \( t \). Recall that a Cremer point can not be on \( \partial U \) by a result of Goldberg and Milnor (see [C-M]). But there is some continuum, containing the Cremer point, that possibly touches \( \partial U \) and is then the cause of non local-connectivity at that point.

Since our graphs \( \Gamma_n(\theta) \) will not cut such a continuum (it is indecomposable), the corresponding sequence of puzzle pieces does not shrink to a point, \( i.e. \) \( \text{End}(x) \neq \{x\} \). However we prove that \( \text{End}(x) \) is attached by (at most) one point to \( \partial U \), which implies that \( \bigcap_n I_n(x) = \{x\} \):

Proposition 3.1. Let \( x \in \partial U \) be a fixed point of \( f \) and let \( \Gamma(\theta) \) be a graph (as in Definition 2.6) with \( \theta \) any periodic angle that is not fixed by multiplication by \( d_\ast \). We can define \( P_n(x) \) as the unique puzzle piece containing \( x \) in its closure and \( \text{End}(x) := \bigcap P_n(x) \). Then

- either \( \text{End}(x) = \{x\} \);
• or there exist two rays \( R_\infty(\zeta), R_\infty(\zeta') \) which land at \( x \) and separate \( \overline{U} \) from \( \text{End}(x) \setminus \{ x \} \).

**Corollary 3.2.** For any eventually periodic point \( x \in \partial U, \cap I_n(x) = \{ x \} \). In particular, \( \partial U \) is locally connected at \( x \).

**Proof.** Let \( x \in \partial U \) be a fixed point of \( f \), take some graph \( \Gamma \) as defined in the Proposition. Suppose that \( \text{End}(x) \neq \{ x \} \), consider the curve \( \gamma = R_\infty(\zeta) \cup \overline{R_\infty(\zeta')} \). Let \( V_n \) be the closure of the connected component of \( P_n(x) \setminus \gamma \) intersecting with \( U \). Note that \( \cap_n(V_n \cap \partial U) = I(x) \). Since \( \gamma \) separates \( \overline{U} \setminus \{ x \} \) from \( \text{End}(x) = \cap \overline{P}_n(x) \), it follows that \( \cap V_n = \{ x \} \). So that \( \partial U \) is locally connected at \( x \) since \( (V_n \cap \partial U) \) is a decreasing sequence of connected neighborhoods of \( x \) in \( \partial U \).

If \( x \) is a periodic point on \( \partial U \), we consider an iterate \( g \) of \( f \) such that \( x \) is a fixed by \( g \). Then applying previous result to \( g \), we obtain a sequence of connected neighborhoods of \( x \) in \( \partial U \) (the Julia set and the basin are the same for \( f \) and the iterate \( g \) of \( f \)).

If \( x \) is an eventually periodic point on \( \partial U \), we pullback by some iterate of \( f \) the previous sequence of connected neighborhoods of the periodic point in the orbit of \( x \). \qed

**Proof of the Proposition 3.1:** Let \( \theta \) be any periodic angle that is not fixed by multiplication by \( d \). Consider the sequence of puzzle pieces \( P_n(x) \) defined by the graph \( \Gamma(\theta) \) and containing \( x \) in their closure. As noted in Lemma 2.11, the boundary of \( P_n(x) \) intersects \( U \) along two rays \( R_U(t_n), R_U(t'_n) \) where \( t_n, t'_n \) converge to the same value \( t \) (since \( t_n - t'_n \to 0 \)). Denote by \( R_\infty(\zeta_n), R_\infty(\zeta'_n) \) the external rays lying in \( \Gamma_n(\theta) \) in “front of” \( R_U(t_n), R_U(t'_n) \), respectively, i.e. landing at the same point of \( \partial U \). The sequences \( (\zeta_n), (\zeta'_n) \) are monotone and bounded because the puzzle pieces are nested, thus they converge to some limit \( \zeta, \zeta' \) respectively. The rays \( R_U(t), R_\infty(\zeta) \) and \( R_\infty(\zeta') \) are fixed by \( f \) (so they land). Indeed, \( f(P_n(x)) = P_{n-1}(x) \) implies that \( f(R_U(t_n)) = R_U(t_{n-1}) \) and \( f(R_U(t'_n)) = R_U(t'_{n-1}) \) so that \( f(R_\infty(\zeta_n)) = R_\infty(\zeta_{n-1}) \) and \( f(R_\infty(\zeta'_n)) = R_\infty(\zeta'_{n-1}) \). Therefore the limit angles \( t \) and \( \zeta, \zeta' \) are fixed by multiplication by \( d \) an \( D \) respectively. Let \( y, z \) and \( z' \) be the landing point of the rays \( R_U(t), R_\infty(\zeta) \) and \( R_\infty(\zeta') \) respectively. They lie in \( \text{End}(x) \), so if \( \text{End}(x) = \{ x \} \) they coincide.

Assume that \( \text{End}(x) \neq \{ x \} \). In order to prove that \( x = y = z = z' \), we look at the ”external class” of \( f \) on \( \text{End}(x) \) following an argument of J. Kiwi (see [Ki]).

* External class \( \overline{\gamma} \) of \( f \) on \( \text{End}(x) \): The set \( \text{End}(x) \) is a non trivial compact full connected set (as the intersection of a decreasing sequence of such sets), so we can consider a Riemann map \( \phi : \overline{C} \setminus \text{End}(x) \to \overline{C} \setminus D \). First of all, if \( \text{End}(x) \cap \partial P_0(x) \neq \emptyset \) (this happens only in the parabolic case), we consider a small enlargement \( U_0 \) of \( P_0(x) \) at those points such that \( U_0 \setminus \text{End}(x) \) does not contain critical values of \( f \). Let \( V_0 = \phi(U_0 \setminus \text{End}(x)), U_1 = f^{-1}(U_0 \setminus \text{End}(x)) \) and \( V_1 = \phi(U_1) \). Then, the map \( g = \phi \circ f \circ \phi^{-1} \) is well defined from \( V_1 \) to \( V_0 \), since there is no pre-image of \( \text{End}(x) \) in \( P_0(x) \) other than \( \text{End}(x) \). Applying Schwarz reflection principle on \( V_1 \) and \( V_0 \), we get neighborhoods \( \tilde{V}_1 \) and \( \tilde{V}_0 \) of \( S^1 \) and a map \( \tilde{g} : \tilde{V}_1 \to \tilde{V}_0 \) such that \( \tilde{g}_{|\tilde{V}_1} = g \). Since \( \tilde{g} \) is a holomorphic covering that preserves \( S^1 \) and each side, it has no critical point on \( S^1 \). Therefore, the map \( \overline{\gamma} = \tilde{g}_{|S^1} \) is a covering of \( S^1 \), it is called the external class of \( f \) on \( \text{End}(x) \).

* Fixed points of \( \overline{\gamma} \):

**Lemma 3.3.** The fixed points of \( \overline{\gamma} \) are weakly repelling, i.e. if \( p \) is a fixed point of \( \overline{\gamma} \) on \( S^1 \) then for any \( z \neq p \) which is close enough to \( p \), \( |\overline{\gamma}(z) - p|_{S^1} > |z - p|_{S^1} \).
Conjugate to a Blaschke product which is fixed by $g$. Therefore there exists an arc $\alpha \subset V_1$ from $q \neq r$ two points of $S^1$ (one is equal to $p$ in the parabolic case) which bounds an open set $\Omega_1$ in $C \setminus D$ such that $\tilde{g}(\Omega_1) \subset \Omega_1$ (it is in the attracting domain of $p$). Let $\Omega = \phi^{-1}(\Omega_1)$. Therefore, $f(\Omega) \subset \Omega$, so that the family $(f^n)$ is normal on $\Omega$. This gives a "half-neighbourhood" of points of $J$ on which $f$ is normal. We prove now that this is impossible for polynomials. Let $\tilde{\Omega}$ be the Fatou component containing $\Omega$; it is bounded and fixed by $f$, moreover, by the Denjoy-Wolf Theorem, $\tilde{\Omega}$ contains a fixed point which attracts every point of $\Omega$. Moreover $\partial \Omega \cap \text{End}(x) \subset \partial \tilde{\Omega}$ contains more than one point, so there is a cross cut $c$ of $\text{End}(x)$ in $\tilde{\Omega}$. Indeed, the map $\phi^{-1}$ admits limit points at almost every $\theta \in (p, q)$ by Fatou’s Theorem and these limit points are not all equal (see for instance Koebe Lemma page 31 of [Go]). So we can use two such part of rays to construct an arc in $\tilde{\Omega}$ whose end points are on $\text{End}(x)$. This cross cut $c$ bounds a domain $W$ such that every point of $\partial W \cap \partial \tilde{\Omega}$ is in $\text{End}(x)$. Note that $W$ is the trace in $\tilde{\Omega}$ of some open set $W'$ intersecting $J$ (i.e. $W = W' \cap \tilde{\Omega}$). There is some $i > 0$ such that $f^i(W') \supset J$ and therefore $f^i(\partial W \cap \partial \tilde{\Omega}) = \partial \tilde{\Omega}$. Hence, since $\partial W \cap \partial \tilde{\Omega} \subset \text{End}(x)$ and $f^i(\text{End}(x)) \subset \text{End}(x)$ it follows that $\partial \tilde{\Omega} \subset \text{End}(x)$. This is not possible since $\text{End}(x)$ is a full compact connected set and $\tilde{\Omega}$ is a topological disk. One can also notice that the map $f$ on $\tilde{\Omega}$ is conjugate to a Blaschke product $b : D \to D$ with Julia set $\partial D$ (so that at most one point of $\tilde{\Omega}$ is not in $\cup_i f^i(W)$).

**Claim 1.** Let $u$ be a fixed point of $f$ in $\text{End}(x)$ which is accessible from $\overline{C \setminus \text{End}(x)}$ by an access $\delta$ which is fixed by $f$. Then $\phi(\delta)$ is fixed by $g$ and lands at a point $v := \phi((u, \delta))$ of $S^1$ which is also fixed by $g$.

**Proof.** This follows from the classical theory of Riemann maps (see [C-G, Po]).

**Some fixed points of $\overline{\mathcal{F}}$:**

The points $y, z, z'$ are fixed points of $f$ and have respective accesses $R_U(t), R_\infty(\zeta), R_\infty(\zeta')$ which are fixed by $f$. We can choose $\phi$ up to composing with some rotation such that the landing point $\phi((y, R_U(t))) = 1$. Denote by $u, u'$ the images of the landing point of $\phi((z, R_\infty(\zeta))), \phi((z', R_\infty(\zeta')))$. Note that the point $y$ is also accessible by an external fixed ray, say $R_\infty(\eta)$ (since it is repelling or parabolic), and the point $\phi((y, R_\infty(\eta))) = 1$.

**Localization of the fixed points of $\overline{\mathcal{F}}$:**

**Claim 2.** Between two fixed points of $\overline{\mathcal{F}}$ there is a strict preimage of $1$ by $\overline{\mathcal{F}}$.

**Proof.** Consider a lift $\tilde{g}$ of $\overline{\mathcal{F}}$ from $R$ to $R$ such that $\tilde{g}(0) = 0$. It is a strictly monotone map that sends $[0, 1]$ to $[0, r]$ for some $r$. Since the fixed points of $\tilde{g}$ are weakly repelling, the graph of $\tilde{g}$ crosses the line $D_k$ of equation $y = x + k$ for $k \in \{0, \cdots, r - 1\}$ from below to above. Hence, between the crossing with $D_k$ and $D_{k+1}$, the graph of $\tilde{g}$ crosses the line of equation $y = k + 1$. The result follows.

**Corollary 3.4.** The rays $R_\infty(\eta), R_\infty(\zeta), R_\infty(\zeta')$ land at the same point.
Proof. Assume in order to get a contradiction that \( u \neq 1 \) (recall that \( u := \phi((z, R_\infty(\zeta))) \)) and that \( 1 := \phi((y, R_U(t))) \). Since the curve \( \phi(R_\infty(\eta)) \) is an access to 1 from \( C \setminus \overline{D} \), there is a preimage by \( \overline{g} \) of \( \phi(R_\infty(\eta)) \) in each connected component of \( C \setminus (\overline{D} \cup \phi(R_\infty(\eta)) \cup \phi(R_\infty(\zeta))) \), by the claim. Therefore, for the open set \( U_1 \setminus \text{End}(x) \) the following holds: there are preimages \( R_\infty(\eta') \), \( R_\infty(\eta'') \) of \( R_\infty(\eta) \) by \( f \) inside each connected component of \( C \setminus (\text{End}(x) \cup R_\infty(\eta) \cup R_\infty(\zeta)) \). Assume that \( \zeta < \zeta' \) so that \( \zeta_n < \zeta < \zeta' < \zeta_n' \) because the puzzle pieces are nested (the other case is identical). Since these rays enter every puzzle piece \( P_n(x) \), at least one of the angle \( \eta', \eta'' \) belongs to the intervals \( (\zeta_n, \zeta) \) for every \( n \geq 0 \). Therefore, it is equal to \( \zeta \) but this is impossible since it is strictly pre-fixed. Thus \( u = 1 \). By the same reason, \( u' = 1 \), so that \( R_\infty(\eta), R_\infty(\zeta) \) and \( R_\infty(\zeta') \) land at the same point \( z \).

* The curve \( R_\infty(\zeta) \cup R_\infty(\zeta') \cup \{z\} \) separates \( U \) from \( \text{End}(x) \setminus \{x\} \):

Let \( W_0 \) (resp. \( W_1 \)) be the union of the connected components of \( P_0(x) \setminus (R_\infty(\zeta) \cup R_\infty(\zeta') \cup \{z\}) \) (resp. of \( P_1(x) \)) intersecting \( U \). Assume to get a contradiction that \( I := \text{End}(x) \cap W_1 \) is not empty. Either the map \( f : W_1 \to W_0 \) is a homeomorphism or \( I \) contains a critical point (there is no critical point in \( P_0(x) \setminus \text{End}(x) \)). In the second case, there is a preimage \( R_\infty(\eta'') \) of \( R_\infty(\eta) \) landing at a preimage of \( z \) in \( I \) (since \( f(I) = I \)) and the angle \( \eta'' \) belongs to one of the intervals \( (\zeta_n, \zeta) \cup (\zeta', \zeta_n') \) whose diameters tend to 0, thus \( \zeta = \eta'' \) or \( \zeta' = \eta'' \), which gives the contradiction. In the first case, \( f_0 = f^{-1} : W_0 \to W_1 \) is a conformal map and since \( W_1 \subset W_0 \), by Denjoy-Wolff’s Theorem (see [S]), there exists a fixed point \( x \) in \( \overline{W}_0 \) to which \( f_0^n \) converges uniformly on every compact set of \( W_0 \). If \( x \) is a parabolic point, let \( P \) be an open repelling petal near \( x \). By definition for any some small \( \epsilon > 0 \), there exists some \( N > 0 \), such that \( f_0^n(P) \) is in the \( \epsilon \) neighbourhood of \( x \) for any \( n > N \). Now, since \( f^n_0 \) converges uniformly to \( x \) on the compact set \( I \setminus P \), there exists some \( M > 0 \) so that for \( n > M \), \( f^M_0(I \setminus P) \) is in the \( \epsilon \) neighbourhood of \( x \). This contradicts the fact that \( f_0(I) = I \) since \( f^n_0(I) = f^n_0(P) \cup f^n_0(I \setminus P) \) is in the \( \epsilon \) neighbourhood of \( x \). (If \( x \) is repelling the argument is even easier.) Hence the curve \( \overline{R_\infty(\zeta)} \cup \overline{R_\infty(\zeta')} \) separates \( U \) from \( \text{End}(x) \setminus \{x\} \).

This achieves the proof of Proposition 3.1. \( \square \)

Corollary 3.5. Let \( y \in \partial U \) be a point on a preimage \( \Gamma_N \) of the graph \( \Gamma \) defining the puzzle. Let \( (P_n) \) be a nest of puzzle pieces with \( y \) as a common boundary point. Then \( \cap \overline{P_n} \cap \partial U = \{y\} \).

Proof. For every \( n \geq 0 \), we have \( \overline{P_n} \cap \partial U \subset I_n(y) \), where \( I_n(y) \) is given by Definition 2.13 for the graph \( \Gamma \). Now, since \( y \) is on \( \Gamma \cap \partial U \), it is an (eventually) periodic point and Corollary 3.2 implies that \( \cap I_n(y) = \{y\} \) (and this holds for any graph by Lemma 2.14). Therefore \( \cap \overline{P_n} \cap \partial U = \{y\} \). \( \square \)

4 The classical techniques

Now in the rest of the paper we consider points \( x \in \partial U \) that are not eventually periodic. We will consider graphs \( \Gamma \) such that \( x \) is not on a preimage of \( \Gamma \), so that the puzzle pieces \( P_n(x) \) are well defined. We will prove that \( \bigcap_{n \in \mathbb{N}} \overline{P_n}(x) \) reduces to \( \{x\} \).
4.1 Modulus techniques

One classical way to obtain that \( \text{End}(x) = \{x\} \) is to consider the modulus of the annuli \( A_{nk} := P_{nk} \setminus \overline{P_{nk+1}} \) where \( P_{nk} \) is some subsequence of the puzzle pieces containing \( x \) (not necessarily with consecutive puzzle pieces) satisfying \( \overline{P_{nk+1}} \subset P_{nk} \).

**Definition 4.1.** Any open annulus \( A \) is conformally equivalent to \( D(0, R) \setminus \overline{D(0,1)} \) for some \( R > 1 \). Then its modulus is \( \text{mod}(A) = \frac{\log R}{2\pi} \).

The following property is classical (see [M2] or Lemma 1.17 of [R1])

**Lemma 4.2.** Let \( \Gamma \) be a graph defining a puzzle and let \( x \) be any point which is not on the iterated pre-images of \( \Gamma \); for some \( n > k \) integers, denote by \( D \) the degree of \( f^{n-k} : P_n(x) \to P_k(y) \), where \( y = f^{n-k}(x) \). If there exists some \( r > 0 \) such that \( \overline{P_{k+r}(y)} \subset P_k(y) \) then \( \overline{P_{n+r}(x)} \subset P_n(x) \) and \( \text{mod}(P_n(x) \setminus \overline{P_{n+r}(x)}) \geq 1/D \text{mod}(P_k(y) \setminus \overline{P_{k+r}(y)}). \) The equality holds if there is no critical points of \( f^{n-k} \) in \( P_n(x) \setminus \overline{P_{n+r}(x)}. \)

The following Lemma will be used several times in the text.

**Lemma 4.3.** If \( \lim \inf \text{mod}(A_{nk}) > 0 \) then \( \text{End}(x) = \{x\} \).

**Proof.** The annuli \( A_{nk} \) are disjoint and essential in \( P_0(x) \setminus \text{End}(x) \). Up to extracting a subsequence, there exist \( \epsilon > 0 \) such that for any \( k \geq 0 \) \( \text{mod}(A_{nk}) > \epsilon \). Therefore, Grötzsch inequality implies that \( \text{mod}(P_0(x) \setminus \text{End}(x)) \geq \sum_{k \in \mathbb{N}} \text{mod}(A_{nk}) = \infty \). Since \( P_0(x) \) is bounded in \( C \), it follows that \( \text{End}(x) = \{x\} \). See [Ah].

In order to apply this Lemma, we need first to construct annuli between puzzles pieces, i.e. there is no intersection between their boundaries. Then to control their moduli we will use the dynamics and look after long iterates of \( f \) with controlled degree.

4.2 Finding non degenerate annuli

**Lemma 4.4.** Let \( y \in \partial U \) be a fixed point. Any graph \( \Gamma \), that does not contain fixed rays, defines a puzzle such that:

- \( y \) is not on the graph and \( \overline{P_1(y)} \subset P_0(y) \) or;
- \( y \) is parabolic and is on the graph \( \Gamma \); denote by \( P_1(y), P_0(y) \) the puzzle pieces containing \( y \) in their closure of depth 1 and 0 respectively, then their closure meets only at \( y \).

**Proof.** If \( y \) is a repelling fixed point, it is the landing point of a fixed ray of \( U \). We can suppose (up to changing the coordinate in \( U \)) that it is the ray of angle 0. Then by Lemma 2.5 the point \( y \) is not on the graph \( \Gamma \). Now order the cycle of internal rays in \( U \) defining the graph by \( \theta_1 < \cdots < \theta_r \), assuming that \( 0 < \theta_1 < \theta_r < 1 \). Then it is immediate that the puzzle piece \( P_1(y) \) (resp. \( P_0(y) \)) is bounded in \( U \) by the rays of angle \( \theta_1/2 \) (resp. \( \theta_1 \)) and \( \theta_r/2 + 1/2 \) (resp. \( \theta_r \)). Since \( \theta_1/2 < \theta_1 \) and \( \theta_r/2 + 1/2 > \theta_r \), the boundaries of \( P_0(y) \) and \( P_1(y) \) do not touch in \( \Gamma \). Note that the boundary of \( P_0(y) \) consists only in these two internal rays together with two external rays landing at the corresponding points of \( \partial U \) and internal/external equipotentials. Then it is clear that \( \overline{P_1(y)} \subset P_0(y) \).

If \( y \) is a parabolic fixed point but its immediate basin is not \( U \), then \( y \) is also the landing point of a fixed ray in \( U \) and the proof goes exactly as before.
If $y$ is a parabolic fixed point and $U$ is its immediate basin. Then by definition of the graphs, $y$ belongs to any graph $\Gamma$. Since there is no external ray landing at $y$ (it would be fixed since $U$ is fixed), there is only one puzzle piece containing $y$ in its closure. And, as before the puzzle piece $P_0(y)$ is bounded by a parabolic equipotential, two parabolic rays in $U$ and two external rays with an external equipotential. Therefore, the closure of $P_1(y)$ and of $P_0(y)$ can meet only at $y$.

Lemma 4.5. Let $x \in \partial U$ and let $P$ be any puzzle piece containing infinitely many points in the orbit of $x$ that we denote by $\mathcal{O}^+_\langle k \rangle (x) = \{ f^{kn}(x) \mid n \geq 0 \}$. If there is no periodic point in the accumulation set of $\mathcal{O}^+_\langle k \rangle (x)$ then, there exists a puzzle piece $Q$ satisfying $\overline{Q} \subset P$ which contains infinitely many points of $\mathcal{O}^+_\langle k \rangle (x)$.

Proof. We assume, in order to get a contradiction, that for any $r > r_0$ and for any $z$ such that $P_r(z) \subset P$ and $\mathcal{O}^+_\langle k \rangle (x) \cap P_r(z)$ is infinite, the boundaries of the pieces intersect: $\partial P \cap \partial P_r(z) \neq \emptyset$.

Since the graph is of the form of $\Gamma(\theta)$ (definition 2.6), the puzzle pieces can only intersect either at the parabolic point or at the landing point of finitely many rays on $\partial P$, we deduce that for infinitely many values of $r$ they will intersect at the same point $v$. This point has to be an eventually periodic point of $\partial U$. Thus we get a nested sequence $(P_n)$ of puzzle pieces containing a common point $v$ in their boundary and by Corollary 3.5 we know that $\cap (\mathcal{T}_n \cap \partial U) = \{v\}$. Since, for any $r \geq 0$, $\mathcal{T}_r(z) \cap \partial U$ contains infinitely many points of $\mathcal{O}^+_\langle k \rangle (x)$ and $\cap_{r \geq r_0} \mathcal{T}_r(z) \cap \partial U = \{v\}$, $v$ is in the accumulation set of $\mathcal{O}^+_\langle k \rangle (x)$. This contradiction with the assumption implies the Lemma.

4.3 Distortion property

The property for an annulus to be non degenerate is difficult to obtain in the parabolic case where puzzle pieces touch at the pre-images of the parabolic point. Therefore, we have to enlarge the pieces to obtain a non degenerate annulus. Doing this we might lose the property that the annuli are disjoint. So we use the following distortion Lemma (see [Y, YZ]). Note that we still need to have some non degenerate annulus and also bounded degrees.

Lemma 4.6. Let $U, V, U_n, V_n$ be topological disks with $\overline{V} \subset U$ and such that for some $k_n$, $f^{kn}(U_n) = U$ and $f^{kn}(V_n) = V$. Moreover, if $V_n$ contains some point $z$ of $J(f)$ and if the degree of the maps $f^{kn}: U_n \to U$ is bounded independently of $n$, then the diameter of $V_n$ tends to 0.

Proof. Assume by contradiction that the diameter of $V_n$ does not tend to 0. Up to extracting a subsequence, there exists $L > 0$ such that $diam(V_n) \geq L$. Denote by $Shape(U, z)$ the ratio $\frac{\max_{t \in \partial U} d(t, z)}{\min_{t \in \partial U} d(t, z)} = \frac{\max_{t \in \partial U} d(t, z)}{d(z, \partial U)}$. We will prove below that the shape $Shape(U_n, z)$ is bounded above by some constant $S > 0$ independent of $n$. In particular, the internal diameter is bounded below by a positive constant independent of $n$: Since the maximum $\max_{t \in \partial U_n} d(z, t)$ is reached by some $t$ in $\partial U_n$, one has that $diam(U_n) \leq 2d(z, \partial U_n)$. Therefore, $\min_{t \in \partial U_n} d(z, t) = \max_{t \in \partial U_n} d(t, z)/Shape(U_n, z) \geq L/(2S)$. Hence there is a small disk $D$ around $z$ contained in every $U_n$. This contradicts with $J(f) \subset f^k(D)$ for large values of $k$ since $z$ is in $J(f)$.

To prove that the shape is bounded above, we use the following property on the shape (see Lemma 2 of [YZ]).
**Lemma:** Let $\Delta$ be the unit disk, $U, \tilde{U}, V$ and $\tilde{V}$ be topological disks and $g : (\Delta, U, \tilde{U}) \to (\Delta, V, \tilde{V})$ be a holomorphic proper map of degree $d$ with $0 \in \tilde{U} \subset U \subset \Delta$, $0 \in \tilde{V} \subset V \subset \Delta$. Suppose that $\text{deg}(g|_{\tilde{U}}) = \text{deg}(g|_V) = \text{deg}(g|_\Delta) = d \geq 2$. Then there exists a constant $K = K(m, d) > 0$, where $\text{mod}(V \setminus \tilde{V}) \geq m > 0$, such that $\text{Shape}(U, 0) \leq K \text{Shape}(V, 0)^{\frac{1}{d}}$.

The proof of this Lemma uses the classical Koebe distortion Theorem for univalent function and the Grötzsch Theorem.

### 4.4 Bounded degree, property $(\star)$ and the successors.

To get control on the moduli or to apply the distortion Lemma, we need to have maps of bounded degree.

**Definition 4.7.** Let $(\star)$ be the following property of $x$ relative to a graph $\Gamma$:

$(\star)$: There is a sequence $(k_n)_{n \geq 1}$ tending to $\infty$, a point $z$ and a level $k_0$ such that the degree of the maps $f^{k_n} : P_{k_n+k_0}(x) \to P_{k_0}(z)$ is bounded independently of $n$.

One simple way to get a bounded degree is to consider the first entrance or the first return to a puzzle piece:

**Definition 4.8.** The first entrance time (resp. the first return time) of a point $z$ in puzzle piece $P$, is the minimal $r \geq 0$ (resp. $r \geq 1$) such that $f^r(z) \in P$. We call the first entrance in $P$ (resp. the first return in $P$) the point $f^r(z)$.

**Notation 4.9.** Let $b$ denote the number of distinct critical ends, i.e. ends containing at least one critical point (see Notation 2.9). Let $\delta$ be the maximum of the degree of $f$ over all the critical ends (see Remark 2.10).

**Lemma 4.10.** Let $r \geq 0$ be the first entrance time (resp. the first return time) in some puzzle piece $P_n$ (of depth $n$) of a point $z$, then the degree of $f^r : P_{n+r}(z) \to P_n$ is bounded by $\delta^b$ (resp. $\delta^{b+1}$).

**Proof.** The Lemma follows from the fact that the sequence of puzzle pieces $\{f^i(P_{n+r}(z)) \mid 0 \leq i \leq r\}$ meets each of the $b$ critical ends at most once. Indeed, assume in order to get a contradiction, that a critical point $c$ belongs to $f^i(P_{n+r}(z)) = P_{n+r-i}(f^i(z))$ and to $f^j(P_{n+r}(z)) = P_{n+r-j}(f^j(z))$ for $0 \leq i < j \leq r$. Then the point $f^i(z)$ which is in $P_{n+r-i}(z) \subset P_{n+r-j}(z)$ is mapped by $f^{r-j}$ in the piece $P_n = f^{r-j}(P_{n+r-j}(c))$. This contradicts the fact that $r$ is the first entrance time of $z$ in $P_n$ since $0 \leq r - j + i < r$. The Lemma follows in this case. If $r$ is the first return time we get a contradiction if $r - j + i \geq 1$ which happen exactly when $j = r$ and $i = 0$. In this case, only one critical end twice so the degree is bounded by $\delta^{b+1}$.

One particular case where the bounded degree property appears is when we look at the map from a successors of a piece $P$ to the piece $P$. They are, in the case of several critical points, the generalisation of the notion of children introduced by Branner and Hubbard. Recall that:

- a piece $P_{n+k}(c)$ is called a child of $P_n(c)$ if the map $f^{k-1} : P_{n+k-1}(f(c)) \to P_n(c)$ is a homeomorphism where $c$ is a critical point;

- in the case of several critical points, it can be generalized by:

  a piece $P_{n+k}(c')$ is called a child of $P_n(c)$ if $f^{k-1} : P_{n+k-1}(f(c')) \to P_n(c)$ is a homeomorphism where $c, c'$ are critical points.
To define a nest around a given critical point, we need to come back to the same critical point.

**Definition 4.11.** Let $c$ be a critical point. The piece $P_{n+k}(c)$ is said to be a successor of $P_n(c)$ if $f^k(P_{n+k}(c)) = P_n(c)$ and each critical point meets at most twice the set of pieces \{ $f^i(P_{n+k}(c))$, $0 \leq i \leq k$ \}.

Note that $P_{n+k}(c)$ and $P_n(c)$ contain $c$, therefore the set of pieces \{ $f^i(P_{n+k}(c))$, $0 \leq i \leq k$ \} contains $c$ at least two times. For this reason, we cannot impose less than twice in the definition.

**Remark 4.12.** If $P_{n+k}(c)$ is a successor of $P_n(c)$, then the map $f^k : P_{n+k}(c) \to P_n(c)$ has a degree less than $\delta^{2b}$.

**Corollary 4.13.** Let $c$ be a critical point. If there exists $n_0 \geq 0$ such that the piece $P_{n_0}(c)$ has infinitely many successors, then $c$ has property $(*)$.

**Proof.** If $P_{n_0+k_n}(c)$ is an infinite sequence of successors, it follows from the last remark.

## 5 Combinatorial accumulations.

To find long iterates starting from any puzzle piece $P_n(x)$ with bounded degree, we need to consider the critical points which appear in the orbit of $P_n(x)$. More generally we will have to consider how the critical points accumulate themselves.

**Definition 5.1.** For a given graph $\Gamma$, defining a puzzle, we say that $z'$ is in the combinatorial accumulation of $z$ and we note $z' \in \omega_{\text{comb}}(z)$ if:

$$\forall n \geq 0 \ \exists k > 0 \ \text{such that} \ f^k(z) \in P_n(z').$$

**Remark 5.2.** The relation is clearly transitive. Moreover:

- The notion of combinatorial accumulation depends on the graph $\Gamma$, at least by the fact that a point $z' \in \Gamma$ cannot be in $\omega_{\text{comb}}(z)$;
- Any iterate $f^k(z)$ of $z$ (with $k > 0$) is in $\omega_{\text{comb}}(z)$;
- This notion coincides with the standard notion of $\omega$-limit set $\omega(z)$ but for the topology generated by the set of all the puzzle pieces, except as we just noticed, that it contains the orbit of the point $z$ and that it does not contains the points of $\omega(z)$ which are on the iterated preimages of $\Gamma$;
- In particular, $z' \in \omega(z)$ and $f^n(z') \notin \Gamma$ for all $n \geq 0$ implies that $z' \in \omega_{\text{comb}}(z)$;
- For the converse, if $z' \in \omega_{\text{comb}}(z)$ and if $\text{End}(z') = \{z'\}$ (resp. $\text{End}(z') \cap \partial U = \{z'\}$), then either $z'$ is in the orbit of $z$ or $z' \in \omega(z)$.

**Remark 5.3.** Let $x \in \partial U$, the combinatorial accumulation does not depend on the graph (see Lemma 2.11) in the following sense:

$$y \in \omega_{\text{comb}}(x) \iff \forall n \geq 0 \ \exists k \geq 0 \ \text{such that} \ f^k(x) \in I_n(y).$$

Moreover, if $y$ is eventually periodic by Corollary 4.2 implies that

$$y \in \omega_{\text{comb}}(x) \iff y \in \omega(x) \text{ or } y = f^k(x) \text{ for some } k \geq 0.$$
From the definition (Definition 5.1) it follows that one can always make the following assumption, up to replacing the graph $\Gamma$ defining the puzzle by some of its iterated pre-image $\Gamma_N$ (note that the combinatorial accumulation set is the same for $\Gamma$ and $\Gamma_N$):

**Assumption 2.** Let $Y$ be a finite set of points containing the critical points. We can assume that for any $y, y' \in Y$, if $y \notin \omega_{\text{comb}}(y')$ then $y \notin P_0(f^k(y'))$ for all $k \geq 0$ (equivalently $y \in \omega_{\text{comb}}(y')$ or $f^k(y') \notin P_0(y)$ for all $k \geq 0$).

**Proof.** By definition, for any two points $y, y'$, either $y \in \omega_{\text{comb}}(y')$ or there exists $N \geq 0$ such that $y \notin P_n(f^k(y'))$ for any $n \geq N$ and any $k \geq 0$. If $y$ is on some iterated preimage of $\Gamma$ we fall in the second case. Since the set $Y$ is finite, taking the largest $N$ when $y, y'$ belong to $Y$ and taking the graph defined by the $N$-th preimage of $\Gamma$, the puzzle pieces avoided become of level 0.

### 5.1 Periodic point in the accumulation

In this subsection we take a point $x \in \partial U$ which is not eventually periodic and which contains a periodic point $y$ in its $\omega$-limit set. We can assume that $y$ is fixed up to taking an iterate of $f$, since the Julia set (and also the boundary of the basin $U$) will be identical for $f$ and for the iterate.

**Proposition 5.4.** Let $x \in \partial U$. Assume that $x$ is neither periodic nor eventually periodic and that it contains a fixed point $y$ in its $\omega$-limit set. Then, there exists a graph $\Gamma$ defining a puzzle such that $\text{End}(x) \cap \partial U = \{x\}$. In particular, $\partial U$ is locally connected at $x$.

**Corollary 5.5.** Let $x \in \partial U$. Assume that $x$ is neither periodic nor eventually periodic and that it contains a periodic point $y$ in its $\omega$-limit set. Then $\partial U$ is locally connected at $x$.

**Proof.** The corollary follows directly from the proposition by taking some iterate $g$ of $f$ and for $y$ the adapted iterate forward image of $y$ by $f$.

**Proof of Proposition 5.4.** Since $y \in \partial U$, for any graph $\Gamma$ that do not contain fixed rays, we can define the sequence of puzzle pieces $(P_n(y))$ (that contains $y$ in its closure). Moreover, for this graph $P_1(y) \subset P_0(y)$ if $y$ is repelling and $P_1(y) \cap P_0(y) = \{y\}$ if $y$ is parabolic on $\Gamma$ (by Lemma 4.4). Note that if we have to take an iterated preimage of $\Gamma$ in order to satisfy Assumption 1 or 2, we will keep this property: the preimage (by $f^r$) containing $y$ of the puzzle pieces $P_1(y)$ and $P_0(y)$ are puzzle pieces $P_{r+1}(y)$ and $P_r(y)$ that satisfies $P_{r+1}(y) \subset P_r(y)$ in the first case and $P_{r+1}(y) \cap P_r(y)$ reduces to some preimages of $y$ by $f^r$ in the second case.

For any $n \geq 0$, let $r_n$ be the first entrance time of $x$ into $P_n(y)$. The sequence $r_n$ is not eventually constant. Otherwise, we would have $f^r(x) \in \text{End}(y)$ for $r = r_n$ with $n \geq N_0$, but $f^r(x) \in \partial U$ and $\text{End}(y) \cap \partial U = \{y\}$ (Corollary 4.2) would imply that $f^r(x) = y$ (contradiction).

Then, no iterate of $x$ is in $\text{End}(y)$, so for $n \geq 0$, there exists $s > n$ such that $f^s(x) \notin P_s(y)$. Choose $s_n$ the minimal $s > n$ with this property. Now, for any $i \geq 0$, $P_{s_i}(y)$ is the only preimage of $P_i(y)$ in $P_0(y)$ since there is no critical point in $P_0(y) \setminus \text{End}(y)$ (by Assumption 1). Therefore, $f^j(P_{s_n}(f^{s_n}(x))) \subset P_{s_{n-j-1}}(y) \setminus P_{s_{n-j}}(y)$ for $j \leq s_n - 1$ and $f^{s_n} : P_{s_n}(f^{s_n}(x)) \rightarrow P_0(f^{s_n+1}(x))$ is a homeomorphism; but $f^{s_n+1}(x) \notin P_0(y)$ (by the same reason as before).

We should return to $P_1(y)$ but with bounded degree, and also bounded degree on the puzzle that is mapped to $P_0(y)$. 

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1) We consider first the case where $y$ is a parabolic point. Taking the first return time of $f^{r_n+s_n}(x)$ to $P_0(y)$ give now a sequence $m_n \geq n$ such that the map $f^{m_n}: P_{m_n}(x) \to P_0(y)$ has a degree bounded by $\delta^{2n}$. Moreover, $y$ is not in the closure of $f^{m_n-1}(P_{m_n}(x))$ otherwise $f^{m_n-1}(P_{m_n}(x))$ would already be in $P_0(y)$. Therefore, $f^{m_n-1}(P_{m_n}(x))$ contains a preimage $y'$ of $y$ which lies in a preimage $U'$ of $U$. We enlarge this puzzle piece $P_1(y')$ to get $\tilde{P}_1(y')$ (an open set) which is the union of $P_1(y')$ with a small neighborhood of $y$ in $U'$ that avoids the orbits of the critical points. This is possible since $U'$ is a pre-image of the basin $U$, so the orbits of the critical points intersect under a finite set $U'$ where we enlarge the puzzle piece. Let $\tilde{P}_m(x)$ denote the iterated preimage of $\tilde{P}_1(y')$ by $f^{m_n}$ (it is an enlargement of $P_{m_n}(x)$). The degree of $f^{m_n-1}: \tilde{P}_m(x) \to \tilde{P}_1(y')$ is exactly the same as the degree of $f^{m_n-1}: P_{m_n}(x) \to P_1(y')$. Lemma 1.6 apply then, since we have a non degenerate annulus between $\tilde{P}_1(y')$ and $P_2(y')$. The local connectivity follows then from Lemma 2.11.

2) We consider now the case of a repelling fixed point $y$. Here we can not avoid the post-critical set that maybe accumulates on $y$. So we look at the critical points that appear in the orbit of the puzzle pieces which are mapped to $P_0(y)$ (by some iterate of $f$). More precisely, consider $A := \{c \in \omega_{comb}(x) \cap \text{Crit} \mid y \in \omega_{comb}(c)\}$. Let $c$ be a critical point such that some iterate $f^k(x) \in P_0(c)$ for some $n \geq 0$, then if for some $0 \leq i \leq n$ a puzzle piece $P_i(f^i(c))$ contains $y$ (for some $j \geq 0$) then $c \in A$ (by Assumption 1 and 2).

Assume first that $A = \emptyset$. Let $t_n$ is the first entrance time of $f^{r_n+s_n}(x)$ in $P_1(y)$. There is no critical point in $f^i(P_n(f^{r_n+s_n}(x)))$ for $0 \leq i \leq t_n$, otherwise this critical point would belong to $A$. Then the map $f^{r_n}: P_n(f^{r_n+s_n}(x)) \to P_0(y)$ is a homeomorphism.

We consider now the case of $A \neq \emptyset$. For any $c \in A$, we denote by $e_c$ the first entrance time of $c \in A$ in $P_1(y)$ (it exists since $y \in \omega_{comb}(c)$), and consider the maximum $N$ of $e_c$ for $c \in A$. For each $c \in A$, we want to consider the first entrance time of $f^{r_n+s_n}(x)$ in $P_0(c)$. Since $c \in \omega_{comb}(x)$, it is easy to define, if no image of $x$ is in $\text{End}(c)$. Indeed, for any $k \geq 0$ there exists $i_k$ such that $f^{i_k}(x) \in P_k(c)$, and if the sequence $i_k$ does not tend to infinity, one can extract a sub-sequence that is constant so that some iterate $f^i(x) \in \text{End}(c)$. In this case, if $f^i(x) \in \text{End}(c)$, we start again with $z = f^i(x)$, if the sequence $j_k > 0$, such that $f^{j_k}(x) \in P_k(c)$ for $k \geq 0$, does not tend to infinite, then it implies that $\text{End}(c)$ is periodic (since $\text{End}(c) = \text{End}(z)$). But since $y \in \omega_{comb}(c)$ we obtain that for some $j$, $f^j(\text{End}(c)) = \text{End}(y)$ and therefore $f^{i+j}(x) \in \text{End}(y) \cap \partial U$ so that $f^{i+j}(x) = y$ since $\text{End}(y) \cap \partial U = \{y\}$ by Corollary 3.2.

Now let $t_c$ be the first entrance time of $f^{r_n+s_n}(x)$ in $P_N(c)$ and denote by $t_n$ the minimal one for $c \in A$. Let $c_0$ be the critical point such that $f^{r_n+s_n}(x)$ meets the first in the level $N$ puzzles pieces, note $e_n = e_{c_0}$ to simplify. Then the puzzle piece $P_{t_n+e_n}(f^{r_n+s_n}(x))$ is mapped by $f^{i_n+e_{n}}$ to $P_0(y)$ since it is mapped by $f^{i_n}$ to $P_{e_{n}}(c_0)$, which is mapped by $f^{e_{n}}$ to $P_0(y)$. If $t_n + e_n \leq N$ the degree of the map $f^{i_n+e_{n}}$ is clearly bounded by $\delta^N$. If $t_n + e_n > N$, the map $f^{i_n+e_{n}}$ can decompose into $f^{N+t}$ with $0 < t \leq t_N$ (since $n_0 \leq N$). There is no critical point in $P_{t_n+e_{n}-1}(f^{r_n+s_n+i}(x))$ for $0 \leq i < t_n$, since any such critical point should be in $A$ and $t_n$ is the first time that $f^{r_n+s_n}(x)$ meets a critical point in level $N$ puzzle pieces. Therefore the map $f^i : P_{t_n+e_{n}}(f^{r_n+s_n}(x)) \to P_N(f^{r_n+s_n+i}(x))$ is a homeomorphism and then the degree of $f^{N+t} : P_{t_n+e_{n}}(f^{r_n+s_n}(x)) \to P_0(y)$ is bounded by $\delta^N$.

Finally, we get a sequence $u_n = r_n + s_n + t_n + e_n$ tending to infinity such that $f^{u_n}(x) \in P_1(y)$ and the degree of $f^{u_n} : P_{u_n}(x) \to P_0(y)$ is bounded by $D = \delta^{b+N}$. Then Lemma 1.3 applies since \[\text{mod}(P_{u_n}(x) \setminus P_{u_n}(x)) \geq \text{mod}(P_0(y) \setminus P_1(y))/D.\]
5.2 How to get property (⋆) and its consequence

Lemma 5.6. If $y$ has property (⋆) and $y \in \omega_{\text{comb}}(x)$, then $x$ also has property (⋆).

Proof. Suppose $y$ has property (⋆). Then there exist $z$ and $(k_n)_{n \in \mathbb{N}^*}$ such that the degree of the maps $f^{k_n} : P_{k_n+k_0}(y) \to P_{k_0}(z)$ is bounded. We look at the first entrance of $x$ to the nest of $y$. For $n > 0$, let $r_n$ be the first entrance time of $x$ in $P_{k_n+k_0}(y)$. The degree of the maps $f^{r_n} : P_{r_n+k_n+k_0}(x) \to P_{k_0}(z)$ is bounded by Lemma 4.10. Therefore, the degree of the maps $f^{r_n+k_n} : P_{r_n+k_n+k_0}(x) \to P_{k_0}(z)$ is bounded and property (⋆) follows for $x$ with the same puzzle piece $P_{k_0}(z)$ and the sequence $k_n + r_n$. □

Definition 5.7. Let $\omega\text{Crit}(z)$ denote the set of critical points which are in $\omega_{\text{comb}}(z)$: $\omega\text{Crit}(z) = \text{Crit} \cap \omega_{\text{comb}}(z)$.

Lemma 5.8. Suppose that $x \in \partial U$ has either one of the following properties:

1. $\omega\text{Crit}(x) = \emptyset$,
2. $\omega\text{Crit}(x) \neq \emptyset$ and there exists $c \in \omega\text{Crit}(x)$ such that $\omega\text{Crit}(c) = \emptyset$;
3. $\omega\text{Crit}(x) \neq \emptyset$ and there exist $c, c' \in \omega\text{Crit}(x)$ such that $c \notin \omega\text{Crit}(c')$ (c and c' can be the same).

Then $x$ satisfies property (⋆).

Proof. If $\omega\text{Crit}(x) = \emptyset$, the map $f^{k-1} : P_{k-1}(f(x)) \to P_0(f^k(x))$ is a homeomorphism since no critical point belongs to $P_{k-1}(f^{k+i}(x))$ (by Assumption 2). Therefore the degree of the map $f^k : P_k(x) \to P_0(f^k(x))$ is bounded by $\delta$ (as can be a critical point). Since there is a finite number of puzzle pieces of level 0, the property (⋆) follows.

For Point 2, since $\omega\text{Crit}(x) \neq \emptyset$, there is a point $c \in \omega\text{Crit}(x)$. If we assume that $\omega\text{Crit}(c) = \emptyset$, then $c$ has property (⋆) (by point 1.) and the statement is a consequence of Lemma 5.6. 

For Point 3, we suppose that Points 1. and 2. are not satisfied: $\omega\text{Crit}(x) \neq \emptyset$ and for every $c \in \omega\text{Crit}(x)$, $\omega\text{Crit}(c) \neq \emptyset$. Note that if there exists $c \in \omega\text{Crit}(x)$ such that $c \notin \omega\text{Crit}(c)$, then for any $c' \in \omega\text{Crit}(c)$, we also have $c \notin \omega\text{Crit}(c')$. Therefore in Point 3, we can always take $c \neq c'$. Moreover, for any $r \geq 0$, $f^r(x) \notin \text{End}(c')$, otherwise since $c \in \omega\text{Crit}(x)$ it would imply that $c \in \omega\text{Crit}(c')$. Hence for any $k \geq 0$, if we consider the first entrance time $r_k$ of $x$ in $P_n(c')$, there exists a minimal $n_k$ such that $c' \in P_{n_k}(f^{r_k}(x)) \setminus P_{n_k+1}(f^{r_k}(x))$. Now, by Assumption 2 for any $r_k \leq l \leq r_k + n_k$, $c \notin P_0(f^l(x))$ (since $c \notin \omega\text{Crit}(c')$). Therefore, the map $f^{l_k} : P_{l_k}(x) \to P_0(f^l(x))$ has its degree bounded independently of $k$. It follows, by Lemma 5.6, that $x$ has property (⋆). □

If $x$ does not satisfy neither 1), 2) nor 3) then the set $\omega\text{Crit}(x)$ is non empty and for any critical points $c, c' \in \omega\text{Crit}(x), c' \in \omega\text{Crit}(c)$ and $c \in \omega\text{Crit}(c')$ (c and c' can be the same point).

Definition 5.9. A critical point $c \in \text{Crit}$ is said critically self-recurrent if $\omega\text{Crit}(c) \neq \emptyset$ and every $c' \in \omega\text{Crit}(c)$ satisfies $c \in \omega\text{Crit}(c')$.

Remark 5.10. • Observe that if $c \in \text{Crit}$ is critically self recurrent, then $c \in \omega\text{Crit}(c)$.
If there is a point $c \in \omega \text{Crit}(x)$ that is not self-recurrent, then $x$ has property $(\ast)$.

Proof. The first point follows from the transitivity property. For the second point, let $c \in \omega \text{Crit}(x)$ be a critical point that is not critically self-recurrent. Then there exists $c' \in \omega \text{Crit}(c)$ such that $c \in \omega \text{Crit}(c')$. By the transitivity property, $c' \in \omega \text{Crit}(x)$ and Point 3. of Lemma 5.8 apply.

Corollary 5.11. Let $x \in \partial U$ be a non (eventually) periodic that does not accumulate a periodic point. If $x$ satisfies property $(\ast)$ then $\text{End}(x) = \{x\}$. Therefore $\partial U$ is locally connected at $x$.

Proof. Property $(\ast)$ gives a sequence $(k_n)$ tending to $\infty$ and a puzzle piece $P_{k_0}(z)$ such that the degree of the maps $f^{k_n} : P_{k_n+k_0}(x) \to P_{k_0}(z)$ is bounded independently of $n > 0$. Assume first that the accumulation set of $O^+_{(k)}(x)$ does not contain any periodic point. From Lemma 4.5 for some piece $Q$ with $\overline{Q} \subset P_{k_0}(z)$, we know that $Q$ contains infinitely many points of $\{f^{k_n}(x) \mid n > 0\}$. Looking at the component of $Q := (f^{k_n})^{-1}(Q)$ we deduce that $\text{End}(x) = \{x\}$. Indeed, we have coverings $f^{k_n}$ of degree bounded by some constant $D$ that sends $Q_n$ to $Q$, $P_n := P_{k_n+k_0}(x)$ to $P := P_{k_0}(z)$ with $\partial U \subset P_n$, $\overline{Q} \subset P$, so that $\mod(P_n \setminus Q_n) \geq \frac{1}{D} \mod(P \setminus Q)$.

6 The persistently recurrent case

6.1 Reduction of the set of points

After the work done in previous sections, we can restrict the set of points we are working with. More precisely, we consider points $x \in \partial U$ that are not eventually periodic (see section 3) and that do not accumulates on periodic points (see Corollary 5.5). For such a point $x$, if $\omega \text{Crit}(x) = \emptyset$ or if there exist $c, c' \in \omega \text{Crit}(x)$ such that $c \notin \omega \text{Crit}(c')$ (not self-recurrent) or if there exists $c \notin \omega \text{Crit}(x)$ that has infinitely many successors, then $x$ satisfies property $(\ast)$ and we get that $\text{End}(x) = \{x\}$ by Lemma 5.8 Corollary 3.2 4.13 5.5 and 5.11.

Therefore we consider only points $x \in \partial U$ such that any critical point $c$ of $\omega \text{Crit}(x)$ is critically self-recurrent.

Definition 6.1. A point $c$ is persistently recurrent if it is critically self-recurrent and for any point $c_1 \in \omega \text{Crit}(c)$, any $n_0 \geq 0$ the puzzle piece $P_{n_0}(c_1)$ has only finitely many successors.

Denote by $\text{perCrit}(x)$ the set of points $c$ of $\omega \text{Crit}(x)$ which are persistently recurrent.

As noticed in Remark 5.10 if $\text{perCrit}(x) \neq \omega \text{Crit}(x)$ then $x$ satisfies property $(\ast)$. For this reason, we consider for the rest of the paper only points $x$ satisfying:

1. $x \in \partial U$ is not eventually periodic;
2. $x$ does not accumulate on periodic points;
3. and $\text{perCrit}(x) = \omega \text{Crit}(x) \neq \emptyset$.

Remark 6.2. The critical points that belong to a same end $\text{End}(y)$ cannot be separated by puzzle pieces of any depth. Everything goes as if there were only one (at most) critical point in $\text{End}(y)$ and this point would have multiplicity equal to $\delta(y) - 1$ (see Remark 2.10). Our arguments in the proofs will be as if it were the case. We could have consider the relation on the set of critical points that identifies two critical points if they belong to the same end and then argue on the equivalences classes but we prefer not to have more notations.
Thus we can make the following assumption without lost of generality.

**Assumption 3.** There is at most one critical point in each puzzle piece (in particular at level 0).

### 6.2 Property of the successors

In the persistently recurrent case, there are only finitely many successors, but a very fundamental fact is that there are always at least two successors. It is the content of the following Lemma whose proof is postponed to section 8:

**Lemma 6.3.** Let $c$ be any critical point of $f$.

1. If $\text{End}(c)$ is not periodic, then each puzzle piece $P_n(c)$ has at least two successors.

2. If $c \in \omega_{\text{Crit}}(x)$ has an end $\text{End}(c)$ which is periodic and $x$ is a point of $\partial U$, then $\omega(x)$ contains a periodic point.

Note that we do not use in the proof of this Lemma that we have a point $c$ of $\text{perCrit}(x)$. Nevertheless, this Lemma found is utility only in this case (since we do not know if we have enough successors).

**Definition 6.4.** Let $A$ be a puzzle piece, denote by $r(A)$ the smallest return time of a point of $A$ in $A$:

$$r(A) = \inf_{z \in A} \{ k(z) > 0 \mid f^k(z) \in A \}.$$ 

**Remark 6.5.**

1. If a point of $A$ returns to $A$, a sub-piece returns by the same iterate to $A$;

2. If $A' \subset A$ are two puzzle pieces, then $r(A') \geq r(A)$;

3. Let $y$ be a point in a sub-piece $A'$ of $A$; assume that for some $k > 0$, $f^k(A') = A$ and that $y \notin f^i(A')$ for $0 < i < k$, then $r(A') \geq k$.

**Proof.** If a point comes back to $A'$ it comes back also to $A$ so $r(A') \geq r(A)$. For the second statement, let $z$ be any point of $A'$. Note that since $A'$ is a puzzle piece, either $f^i(A') \cap A' = \emptyset$ or $f^i(A') \supset A'$ (for level reason). Since $y \notin f^i(A')$ for $0 < i < k$, $f^i(A') \cap A' = \emptyset$, so the points $f^i(z)$ and $y$ do not belong to the same piece. Hence the first return time of $z$ to $A'$ is necessarily more than $k$.

**Notation 6.6.** Denote by $D(P)$ the last successor of the puzzle piece $P$ and by $\sigma(P)$ the integer such that $f^{\sigma(P)}(D(P)) = P$.

**Corollary 6.7.** Let $A$ be a puzzle piece, then $r(D(A)) \geq \sigma(A) \geq 2r(A)$.

**Proof.** The inequality $r(D(A)) \geq \sigma(A)$ follows from previous remark.

Let $A'$ be the first successor of $A$ and let $k$ be the integer such that $f^k(A') = A$. Then, by definition $k \geq r(A)$. Therefore, the second successor $A''$ of $A$ corresponds to the first return of $f^k(c)$ to $A$, where $c$ is the critical point in $A$. Thus, there exists $k'$ such that $f^{k'}(f^k(c)) \in A$ and $f^{k+k'}(A'') = A$ (since $A''$ is successor of $A$, he cannot meet the critical point $c$ more than twice). Thus $k' \geq r(f^k(A'')) \geq r(A)$. By definition of $\sigma(A)$, we conclude that $\sigma(A) \geq k + k'$ and the result follows.
6.3 Properties of the enhanced nest

A tool particularly well adapted for studying the persistently recurrent critical points is the enhanced nest introduced by Koslovski-Shen-vanStrien (see [K-S-S, QY, TY]). In our situation we will use a slightly simplified version of the enhanced nest (see [P-Q-R-T-Y]). If the critical point $c$ is persistently recurrent, the enhanced nest consists in two sub-nests $(\mathcal{K}_n, \mathcal{K}'_n)$ of the critical nest $(P_j(c))$. The precise construction of these nests will be explained in section 7. For the moment, we give some of their properties.

Remark 6.8. The construction in [K-S-S, QY] or [P-Q-R-T-Y] has the feature that the annuli between two consecutive pieces of the subnests are non-degenerate. However this property is really not necessary for the construction itself. It is needed only afterwards for the control of the moduli of these annuli. Here we do the construction without knowing about the annuli. Then we prove that deep enough in the nest the annuli are non degenerate.

The two sub-nests are constructed using three operators called $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$. As it is for $\mathcal{D}$, $\mathcal{A}$ and $\mathcal{B}$ are some pull-back by a conveniently chosen iterate of $f$. The operators act on the set of all critical puzzle pieces and "have bounded degree". The operators $\mathcal{A}$ and $\mathcal{B}$ are closely related and used for producing an annulus that avoids the closure of the orbit of the critical points in $\perCrit(x)$. On the other hand, $\mathcal{D}$ which is just the last successor map (Notation 6.6) is used for giving long iterates of bounded degree.

The subnests $\mathcal{K}_n$, $\mathcal{K}'_n$ are defined inductively by $\mathcal{K}_n = \mathcal{A}\mathcal{D}^\tau(\mathcal{K}_{n-1})$ and $\mathcal{K}'_n = \mathcal{B}\mathcal{D}^\tau(\mathcal{K}_{n-1})$, starting from some $\mathcal{K}_0$ containing the critical point of $\perCrit(x)$ we focus on. Here $\tau$ is an arbitrary integer.

In [P-Q-R-T-Y] we explain that taking $\tau = b + 1$ is enough and that for $b \geq 2$, $\tau = b$ works also.

We give here some properties of $\mathcal{A}$ and $\mathcal{B}$ that will be proved in section 7 (see also [K-S-S] or [P-Q-R-T-Y]). Recall that $b$ is the number of critical ends and that $\delta$ is the maximal degree of $f$ over the critical ends.

Definition 6.9. Denote by $\mathcal{P}_{\perCrit(c)} = \bigcup_{c \in \perCrit(c)} \bigcup_{n \in \mathbb{N}} f^n(c)$ generated by the critical points that are accumulated by $c \in \perCrit(x)$. Note that this set does not depend on the choice of the critical point in $\perCrit(x)$.

Notice that in the post-critical orbit only the critical point is to avoid. However, if a critical point belongs to $c'$ the image of a puzzle piece $P_n(c)$, necessarily $c' \in \omegaCrit(c)$. Hence, it is enough to control the presence of points of $\mathcal{P}_{\omegaCrit(c)}$ in the puzzle pieces. Also, any point of $\mathcal{P}_{\omegaCrit(c)}$ in a puzzle piece implies that an iterate of a critical point is inside.

Proposition 6.10. Let $I$ be a puzzle piece containing a critical point $c$. Then $\mathcal{A}(I)$ and $\mathcal{B}(I)$ are puzzle pieces with the following properties:

1. $c \in \mathcal{A}(I) \subset \mathcal{B}(I) \subset I$ and $\mathcal{B}(I) \setminus \mathcal{A}(I)$ is disjoint from the set $\mathcal{P}_{\omegaCrit(c)}$;
2. there exist integers $b(I)$, $a(I)$ such that $f^{b(I)}(\mathcal{B}(I)) = I$ and $f^{a(I)}(\mathcal{A}(I)) = I$;
3. $\# \{0 \leq j < b(I) \mid c \in f^j(\mathcal{B}(I))\} \leq b$, and the degree of $f^{b(I)} : \mathcal{B}(I) \rightarrow I$ is less than $\delta^b$;
4. $\# \{0 \leq j < a(I) \mid c \in f^j(\mathcal{A}(I))\} \leq b + 1$, and the degree of $f^{a(I)} : \mathcal{A}(I) \rightarrow I$ is less than $\delta^{b+1}$.  

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This Proposition has several consequences on the two sequences \((K_n), (K'_n)\). Let \(h_n\) (resp. \(h'_n\)) denote the height of \(K_n\) (resp. \(K'_n\)) in the nest: \(K_n = P_{h_n}(c)\) and \(K'_n = P_{h'_n}(c)\)

**Corollary 6.11.** The enhanced nests \((K_n), (K'_n)\) have the following properties:

1. \(K_n \subset K'_n \subset K_{n-1}\);
2. \(K_n, K'_n\) are both mapped to \(K_{n-1}\) by some iterate of \(f\): there exist integers \(p_n, p'_n\) such that: \(f^{p_n}(K_n) = K_{n-1}\), \(f^{p'_n}(K'_n) = K_{n-1}\);
3. \(\deg(f^{p_n}: K_n \to K_{n-1}) \leq C\) and \(\deg(f^{p'_n}: K'_n \to K_{n-1}) \leq C\) where \(C\) depends only on \(b\) and \(\delta\);
4. \(K'_n \setminus K_n\) is an annulus (possibly degenerate) that is disjoint from the set \(\mathcal{P}_{\omega\text{Crit}(c)}\);
5. \(h(K'_n) - h(K_n) \geq r(K_{n-1})\) the return time in \(K_{n-1}\). Moreover, \(r(K_{n+1}) \geq 2^r r(K_n)\). So, \(h(K'_n) - h(K_n)\) tend to infinity.

**Proof.** Point 1) follows directly from Proposition 6.10 1 and the fact that for any piece \(J, \mathcal{D}(J) \subset J\).

Point 2) follows directly from Proposition 6.10 2. Then Point 3) is exactly the second part of Proposition 6.10 1. The point 4) follows from points 3. and 4. of Proposition 6.10 and Remark 4.12. To prove 5), note that \(f^{p_n}(c)\) and \(f^{p'_n}(c)\) are both in \(K_{n-1}\), thus \(p_n-p'_n \geq r(K_{n-1})\).

The result follows from the fact that \(p_n = h(K_{n-1}) - h(K_n)\) and \(p'_n = h_n - h'_n\). The rest of the statement follows from Corollary 6.7 since \(K_{n+1} \subset \mathcal{D}^r(K_n)\) so that \(r(K_{n+1}) \geq r(\mathcal{D}^r(K_n))\), that is bigger than \(2^r r(K_n)\).

In section 8 we give the proof of the following Lemma that is a consequence of Lemma 6.3:

**Lemma 6.12.** \(p_n \geq 2p_{n-1}\).

**Remark 6.13.** Lemma 6.12 is fundamental to understand the power of the construction. We get this way long iterates of bounded degree: let \(t_n\) be the ”time” necessary to reach \(K_0\), i.e. such that \(f^{t_n}(K_n) = K_0\). Then, \(t_n = p_n + \ldots + p_1 \leq p_n + p_n/2 + \ldots + p_n/2^{n-1} < 2p_n\), so that the last step \((f^{p_n}: K_n \to K_{n-1})\) takes more than half of the global time and has a degree bounded by a constant \(C\) independent on \(n\) (see point 4.). Point 5 implies that the annulus is large in terms of the height.

### 6.4 Estimates on the moduli and bounds on the degrees

Our next goal is to control the modulus \(\mu_n\) of the annulus \(K'_n \setminus K_n\).

**Lemma 6.14.** There exists two puzzle pieces \(\tilde{P}, \tilde{Q}\) containing \(c\) such that \(\tilde{Q} \subset \tilde{P}\).

**Proof.** Recall that \(c \in \omega\text{Crit}(x)\) and that \(\omega(x)\) contains no periodic point. Assume to get a contradiction that \(P_0(c) \setminus P_n(c)\) is degenerate for every \(n \geq 1\). Then \(\text{End}(c) = \cap \mathcal{P}_n(c)\) contains an eventually periodic point \(y \in \partial U\). By extension of notations, \(\text{End}(c) = \text{End}(y)\). Corollary 3.2 implies that \(\text{End}(y) \cap \partial U = \{y\}\), so \(y \in \omega(c)\). This contradicts the fact that \(\omega(x)\) contains no periodic point.

**Definition 6.15.** We define the “simplified enhanced nests” starting from \(K_0 = \tilde{Q}\).
Lemma 4.10. A degree bounded by $f$ and non degenerate. 

**Proof.** Since $f^{t_n}(K_n) = K_0$, pulling back by the good inverse branch of $f^{t_n}$, we get a non degenerate annulus between a puzzle piece containing $c$ called $R_n$ and $K_n$. The difference of height is constant: $h(K_n) - h(R_n) = h(Q) - h(P)$. Thus, since $h(K_n) - h(K'_n)$ tends to infinite, $h(K'_n) < h(R_n)$ for large $n$ so that $K'_n \supset R_n$. It follows that $K'_n \setminus K_n \supset R_n \setminus K_n$, so that it is non degenerate. 

We can find small pieces in $K_n$ on which long iterates have bounded degree. Moreover, they are pull-back of $K_n$.

Lemma 6.17. Let $z = f^{\xi}(c) \in K_n$ for some $\xi$. Denote by $r_n$ the first entrance time of $f^{t_n}(z)$ in $K_n$. Let $A$ be the pull back of $K_n$ around $z$ by $f^{t_n+r_n}$. Then, the degree of $f^{t_n+r_n}: A \to K_n$ is bounded by $C_1 := 2(C + \delta^b)$, where $C$ is the constant of Corollary 6.14.

**Proof.** Since $z \in K_n$, the puzzle piece $A := P_{t_n+r_n}(z)$ is included in $K_n$. Therefore, the degree of $f^{t_n}$ on $A$ is bounded by $C$ since the degree $f^{t_n} : K_n \to K_{n-1}$ is bounded by $C$. Now, take $s_n$ the first entry of $f^{s_n}(z)$ in $K_n$. By definition $f^{s_n}(z)$ enters in $K_n$ before $f^{t_n}(z)$ does, so $p_n + s_n \leq t_n + r_n$. Lemma 4.10 implies that $f^{s_n} : f^{t_n}(A) \to f^{t_n+r_n}(A)$ has its degree bounded by $\delta^b$. Now since $f^{t_n+r_n}(z) \in K_n$ and since the height of $f^{s_n+1}(A) = P_{t_n}(f^{s_n+1}(z))$ is $t_n = h_n + t_n - p_n - s_n + r_n \geq h_n$, the puzzle piece $f^{s_n+1}(A) \subset K_n$; therefore $f^{s_n}$ has degree bounded by $C$ on it. Since $2p_n \geq t_n$ (Remark 6.13) it follows that the degree of $f^{t_n}$ on $A$ is bounded by $2C + \delta^b$. Finally, the degree of $f^{t_n} : f^{t_n}(A) \to K_n$ is bounded by $\delta^b$ by Lemma 4.10.

Corollary 6.18. Let $A'$ be the pull back of $K'_n$ around $z$ by $f^{t_n+r_n}$ Then mod($A' \setminus A$) $\geq$ mod($K'_n \setminus K_n$)/$C_1$ for $C_1 = 2(C + \delta^b)$.

**Proof.** Since $K'_n \setminus K_n$ does not intersect the set $P_\omega(Crit(c))$, the degree $f^{t_n+r_n}$ is the same on $A$ and on $A'$. The result then follows from the Lemma 6.17.

We explain now how we can compare the moduli between $K'_m \setminus K_m$ and another $K'_n \setminus K_n$. It follows indirectly from the fact that the height $h(K'_m) - h(K_m)$ between $K'_m$ and $K_m$ is going to infinite, so $K'_m \setminus K_m$ can contain sub-annuli which are related to $K'_n \setminus K_n$.

Lemma 6.19. Let $\xi_n$ be the time such that $f^{\xi_n}(K_{n+2}) = K_n$. Then $f^{\xi_n}(K_{n+2}) \subset A$ where $A$ is the pull back defined in Lemma 6.17 with $z = f^{\xi_n}(c)$.

**Proof.** Since $f^{\xi_n}(K_{n+2})$ and $A$ are both puzzle pieces containing $z = f^{\xi_n}(c)$, it suffices to compare $\alpha = \#\{0 < j < t_n + r_n \mid c \in f^j(A)\}$ with $\beta = \#\{0 < j < p_{n+1} + p_{n+2} - \xi_n \mid c \in f^j(f^{\xi_n}(K_{n+2}))\}$.

The iterates $f^i(K_n)$ for $0 \leq i \leq p_n$ meets at most $b + 1 + \tau$ times the point $c$ (by definition of $K_n = AD^T(K_{n-1})$ and using Proposition 6.10). We can apply Lemma 6.17 (and its proof), since $f^{p_n}(z)$ returns in $K_n$ because $c \in \omega Crit(c)$. Let $s_n$ be the return time of $f^{p_n}(z)$ in $K_n$, then the images of $P_{t_n}(f^{p_n}(z))$ by $f^i$ for $0 \leq i < s_n$ never contain $c$. Therefore using that $p_n + s_n \leq t_n + r_n$ we obtain $\alpha \leq 2(b + 1 + \tau)$.

On the other side, the number of iterates $p_{n+1} + p_{n+2} - \xi$ to bring $f^{\xi}(K_{n+2})$ to $K_n$ is less than $\sigma(K_n)(\beta - 1)$ since $\sigma(K_n)$ (the number of iterates for the last successor) is the largest time a point in $K_n$ take to come back to $K_n$. It is also exactly the difference of height between $K_n$ and $f^{\xi}(K_{n+2})$ which is equal to the difference of height between $K'_{n+2}$ and $K_{n+2}$. Recall that
The iterate $K_{n+2} = B(D^r(K_{n+1}))$ and $K_n = A(D^r(K_{n+1}))$, so that this difference of height is bigger than $r(D^r(K_{n+1}))$, the return time in $D^r(K_{n+1})$. Moreover, $r(D^r(K_{n+1})) \geq 2^r r(K_{n+1}) \geq 2^{2r} r(K_n)$ by Corollary 6.11. Now, since $K_{n+1} \subset D^r(K_n)$, we obtain that $r(K_{n+1}) \geq r(D^r(K_n)) \geq 2^{r+1} r(D(K_n))$. Then it follows from Remark 6.15 that $r(D(K_n)) \geq \sigma(K_n)$. Hence $\beta \geq 2^{2r-1}$. We can summarise this discussion in: $\sigma(K_n)(\beta - 1) \geq p_{n+1} + p_{n+2} - \xi \geq h(K_{n+2}) - h(K_{n+2}) \geq r(D^r(K_{n+1})) \geq 2^{2r} \sigma(K_n)$.

It is easy to see that for $\tau \geq b + 1$ we obtain that $\beta \geq \alpha$, and the result follows.

**Corollary 6.20.** Then $\text{mod}(K'_{n+2} \setminus \bar{K}_{n+2}) \geq \text{mod}(K_n \setminus A)/C^2 \geq \text{mod}(K'_{n} \setminus K_n)/C'$ where $C$ and $C'$ are independant on $n$ and $\tau$.

**Proof.** The map $f^x : K'_{n+2} \rightarrow K_n$ has degree bounded by $C^2$ and the annulus $A' \setminus A$ is included in $K_n \setminus f^x(K_{n+2})$, so that $\text{mod}(K'_{n+2} \setminus K_{n+2}) \geq \text{mod}(K_n \setminus f^x(K_{n+2}))/C^2 \geq \text{mod}(K_n \setminus A)/C^2 \geq \text{mod}(A' \setminus A)/C^2$. Finally, by Corollary 6.18 we get $C' = 2(C + \delta^2)C^2$.

### 6.5 The Kahn-Lyubich Lemma

The Covering Lemma due to Kahn and Lyubich (see [K-L]) is a very powerful tool: it gives an “estimate” of the modulus of the pre-image of an annulus under a ramified covering $g$ when one has some control on the degree over some sub-annulus. We state the Theorem now and refer the reader to [K-L] for the proof.

**Theorem 6.21** (The Kahn-Lyubich Covering Lemma). Let $g : U \rightarrow V$ be a degree $D$ holomorphic ramified covering. For any $\eta > 0$ and $A, A', B, B'$ satisfying:

- $A \subset A' \subset U$ and $B \subset B' \subset V$ are all disks;
- $g$ is a proper map from $A$ to $B$, and from $A'$ to $B'$ with degree $d$;
- $\text{mod}(B' \setminus B) \geq \eta \text{mod}(U \setminus A)$.

There exists $\epsilon = \epsilon(\eta, D) > 0$ such that

$$\text{mod}(U \setminus A) \geq \epsilon \quad \text{or} \quad \text{mod}(U \setminus A) \geq \frac{\eta}{2d^2} \text{mod}(V \setminus B).$$

### 6.6 Proof of Theorem

Using the results obtained in previous sub-sections on the bound of the degree of iterates of $f$ and the comparison on the moduli we get, we now apply Kahn-Lyubich Covering Lemma to prove.

**Lemma 6.22.** The modulus $\mu_n$ of $K'_n \setminus K_n$ satisfies: $\liminf \mu_n > 0$.

**Proof.** Recall that there exists some $N \geq 0$ such that $K'_n \setminus K_n$ is non degenerate for $n \geq N$ (Corollary 6.13). To simplify the notations let us assume that $N = 0$. Then the proof goes by contradiction. We assume that $\liminf \mu_n = 0$. Therefore, there exists a sequence $k_j \rightarrow \infty$ such that $\mu_i \geq \mu_{k_j}$ for every $i \leq k_j$ and $\mu_{k_j} \rightarrow 0$. Since we already have indexes, we will fix some $n := k_j - 2$, so that $\mu_{n+2} \leq \mu_i$ for $i \leq n + 2$. We would like to apply the Kahn-Lyubich Covering Lemma with $U = K_n$ and $V = K_0$, but then the constant $\epsilon(\eta, D)$ would depend on the degree $D$ of the iterate $f^{t_n}$ from $K_n$ to $K_0$ and this degree goes to infinity (recall that $f^{t_n}(K_n) = K_0$).
Therefore, we apply the Covering Lemma with \( U = K_n \) and \( V = K_{n-Z} \) for some integer \( Z \). Let \( \eta = 1/(C^2C_1) \). We show that for \( Z > \frac{2C^2C_1^3}{\eta} \) the second case of the conclusion of the Covering Lemma cannot be satisfied. The set \( A_n \), resp. \( A_n' \), defined in Lemma 6.17 resp. in Corollary 6.18 is the pullback of \( K_n \), resp. \( K_n' \) around \( z \) by \( f^{n+r_n} \). Recall that \( \xi \) is the time such that \( f^{\xi}(K_{n+2}) = K_n \), and that \( r_n \) is the first entrance time of \( f^n(z) \) in \( K_n \). Define \( B := f^{\xi_n}(A) \) and \( B' := f^{\xi_n}(A') \). The puzzle pieces \( B \), resp. \( B' \) is the pullback of \( K_n \), resp. \( K_n' \) around \( f^{\xi_n}(z) \) by \( f^{b_n} \) where \( b_n := t_n - z_n + r_n \).

We verify now the hypothesis of the Covering Lemma. The map \( g := f^{\xi_n}: U \rightarrow V \) is a covering of degree bounded by \( ZC \) (by Property 4. of Corollary 6.11) and the puzzle pieces, \( A \subset A' \subset U \) and \( B \subset B' \subset V \) are all disks. Then \( g \) is a proper map from \( A \) to \( B \) and from \( A' \) to \( B' \); its degree is bounded by \( C_1 \) since the degree of the maps \( f^{t_n+r_n}: A \rightarrow K_n \) and \( f^{t_n}: A' \rightarrow K_n' \) is bounded by \( C_1 = 2(C + \delta^h) \) independent of \( n \) (Lemma 6.17). Therefore, the maps \( f^{b_n}: B \rightarrow K_n \) and \( f^{b_n}: B' \rightarrow K_n' \) have degree also bounded by \( C_1 \) and \( \text{mod}(B' \setminus B) \geq \text{mod}(K_n' \setminus K_n)/C_1 \). Since \( \mu_n \geq \mu_{n+2} \) by assumption, we see that the last hypothesis is satisfied (using Corollary 6.20):

\[
\text{mod}(B' \setminus B) \geq \frac{\mu_{n+2}}{C_1} \geq \frac{\text{mod}(U \setminus A)}{C^2C_1} = \eta \text{mod}(U \setminus A) \quad \text{for } \eta := \frac{1}{C'}, \text{ with } C' = C^2C_1.
\]

Thus the Kahn-Lyubich Lemma implies that either \( \text{mod}(U \setminus A) > \frac{n}{2C^2} \text{mod}(V \setminus B) \) or there exists \( \epsilon = \epsilon(\eta, D) \) independent on \( n \) such that \( \text{mod}(U \setminus A) > \epsilon \).

We will prove that the first inequality cannot arise for \( Z > \frac{2C^2C_1^3}{\eta} \). The reason is that the annulus \( V \setminus B \) contains the pull back of \( B \) around \( f^{\xi_n}(z) \) of the annuli \( K_n' \setminus K_n \) for \( 0 \leq i \leq Z \) by appropriate iterates of \( f \). Take the first time \( f^{t_n}(z) \) enters in \( K_n' \), let \( B_i \) be the corresponding pull back around \( f^{t_n}(z) \), and \( B_i' \) the one of \( K_n' \). Applying Lemma 6.17 in this case (for the index \( n - i \)), we get that \( \text{mod}(B_i' \setminus B_i) \geq \frac{\mu_{n-i}}{C_1} \). Hence, using that \( \mu_{n+2} \leq \mu_j \) for \( j \leq n + 2 \) and that \( d \leq C_1 \), we obtain

\[
\text{mod}(V \setminus B) \geq \sum_{i=0}^{Z} \text{mod}(B_i' \setminus B_i) \geq \frac{Z\mu_{n+2}}{C_1}.
\]

Using the Kahn-Lyubich Lemma, it follows that

\[
\text{mod}(U \setminus A) > \frac{Z\eta}{2C^2} \mu_{n+2}.
\]

Combining this inequality with the one given in Corollary 6.20: \( \mu_{n+2} \geq \frac{\text{mod}(U \setminus A)}{C^2} \), leads to \( \mu_{n+2} > \frac{Z\eta}{2C^2C_1^2} \mu_{n+2} \), and by the choice of \( Z \) one gets the contradiction that \( \mu_{n+2} > \mu_{n+2} \).

Finally, there exists \( \epsilon = \epsilon(\eta, D) > 0 \) independent of \( n \) such that \( \text{mod}(U \setminus A) > \epsilon \). Hence,

\[
\mu_{n+2} \geq \text{mod}(U \setminus A)/C^2 > \epsilon/C^2.
\]

Since this quantity is independent on \( n \) we obtain that \( \liminf \mu_i > 0 \) and get the contradiction.

\[
\square
\]

**Corollary 6.23.** The diameter of \( K_n \) tends to 0.
Proof. The annuli $A_n = K'_n \setminus K_n$ are disjoint and essential in the annulus $A := K_0 \setminus (\cap K_n)$. By Lemma 6.22 there exists some $\epsilon' > 0$ such that for large $N$, every $\mu_n \geq \epsilon'/2$ for $n \geq N$. Therefore by Grötzsch inequality (see [Ah]), $\sum \mod(A) \leq \mod(A)$ and in particular $\mu(A) = \infty$. To conclude we use the following classical characterization (see [Ah]): A continuum $K$ contained in a disk $D$ is reduced to a single point if and only if $\mod(D \setminus K) = \infty$. \hfill $\Box$

Corollary 6.24. Let $x$ be a point of $\partial U$ satisfying $\omega_{\text{Crit}}(x) = \per_{\text{Crit}}(x)$. Then $\text{End}(x) = \{x\}$.

Proof. Let $c_0$ be a critical point in $\omega_{\text{Crit}}(x)$. By Lemma 6.14 there exist two puzzles pieces $\tilde{Q}$ and $\tilde{P}$ such that $c_0 \in \tilde{Q}$ and $\tilde{Q} \subset \tilde{P}$. We construct the simplified enhanced nest around $c_0$ starting with $K_0 = \tilde{Q}$. Then Lemma 6.22 implies that $\liminf \mu_n > 0$ so $\text{End}(c_0) = \cap K_n$ reduces to $c_0$.

For each $n \geq 0$, we consider the first entrance time of $x$ to $K_n$: there exists $k_n$ such that $f^{k_n}(x) \in K_n$. Then the puzzle piece $P_{\delta^{k_n} + k_n}(x)$ (resp. $P_{\delta^{k_n}}(x)$) is mapped to $K'_n$ (resp. $K_n$) by $f^{k_n}$. These two maps $f^{k_n}|_{P_{\delta^{k_n} + k_n}(x)}$ and $f^{k_n}|_{P_{\delta^{k_n}}(x)}$ have degree bounded by $\delta^k$. Therefore the annulus $A_n(x) := P_{\delta^{k_n} + k_n}(x) \setminus P_{\delta^{k_n}}(x)$ has modulus $\mu_n(x) \geq \frac{1}{\delta^k} \mu_n$. Moreover, the annuli $A_n(x)$ are disjoint and nested around $x$ since $K_{n+1}' \subset K_n$. Therefore we obtain that $\mod(P_0(x) \setminus \text{End}(x)) \geq \sum \mu_n(x) \geq \frac{1}{\delta^k} \sum \mu_n = \infty$, as in previous Corollary. The result follows. \hfill $\Box$

7 The operators $\mathcal{A}$ and $\mathcal{B}$

This section is devoted to the definition of the operators $\mathcal{A}$ and $\mathcal{B}$.

The definition and the proofs are exactly the same as in [K-S-S], [P-Q-R-T-Y], [QY]. However, this construction is always presented with sequence of puzzle pieces that do not touch at the boundary. Since in our situation the puzzle pieces may touch, we will give here the details of the construction.

Important: Through all this section we will call “annulus” the difference $U \setminus U'$ between two open disks (or between one open disk and the closure of a smaller one) whose boundaries possibly touch at finitely many points. Here these disks will be always puzzle pieces. (Notice that with this definition, an annulus is not always connected).

7.1 Understanding the pullback to avoid the part of the post-critical set

We fix some point $c_0 \in \per_{\text{Crit}}(x)$ where $x$ is a given point of $\partial U$. We look after annuli around $c_0$ that avoid the post-critical set $\mathcal{P}_c$. By assumption 2 if $y \in \mathcal{P}_c \cap P_0(c_0)$ then $y = f^i(c)$ for some $c \in \omega_{\text{Crit}}(x)$ then $c \in \omega_{\text{comb}}(c_0)$. Therefore we focus on the critical points of $\omega_{\text{Crit}}(c_0)$.

Before entering into the details of their definition, we sketch briefly the construction of the operators $\mathcal{A}$ and $\mathcal{B}$. These operators act on the set of puzzles pieces containing a given critical point, here we call it $c_0$. Let $I$ be such a puzzle piece, consider the pullback of $I$ by the iterates $f^k$ and $f^{k'}$ ($k, k' > 0$) corresponding to the first and to the second entrance of the orbit of a critical point $c \in \omega_{\text{Crit}}(c_0)$. Doing this for each critical point $c \in \omega_{\text{Crit}}(c_0)$ by induction with a union of puzzle pieces instead of $I$, we obtain a finite set of pieces denoted by $P_c$. Then, we take another pullback inside $P_c$ called $P'_c$. Now, consider among all the successors of $I$, those mapped to $P_c$. Then, $\mathcal{B}(I)$ is defined as one of these successors satisfying some maximality property. The
piece $A(I)$ is defined as the pullback by $f^k(I) : B(I) \to I$, of $W$, the first pullback of $I$ around $f^k(I)(c_0)$ ($W = L_{f^k(I)(c_0)}(I)$).

When the map $f$ has only one critical point $c_0$, $B(I)$ is simply the last successor $D(I)$ and $A(I)$ the pullback in $B(I)$ (by $f^a(I)$) of $W$ the first pullback of $I$ around $f^a(I)(c_0)$.

We enter now into some important remarks and properties about the pullback by iterate corresponding to the first entrance in a puzzle piece.

**Definition 7.1.**
- If $P$ is a puzzle piece and $z \in C$, denote by $L_z(P)$ the puzzle piece containing $z$ that is mapped by $f^k$ to $P$, where $k > 0$ is the first entrance time of $z$ in $P$, if it exists.
- If $H$ is a finite union of puzzle pieces, let $k > 0$ be the first entrance time of $z$ in $H$ and let $P$ the component of $H$ which contains $f^k(z)$. Then define $L_z(H) := L_z(P)$.
- We call $L_z(P)$, resp. $L_z(H)$, the first pullback of $P$ (resp. of $H$) around $z$.
- Let $f^k(z)$ the second entrance of $z$ in $H$. We note $L''_z(H)$ the pullback by $f^k$ around $z$ of the puzzle piece $P \subset H$ containing $f^k(z)$. We call it the second pullback of $H$ around $z$.

**Remark 7.2.** Note that $L''_z(H)$ is also the pullback by $f^l$ around $z$ of $L_{f^l(z)}(H)$, where $0 < l < k$ label the successive first and second entrance of $z$ in $H$. Thus $L''_z(H) = L_z(L_{f^l(z)}(H)).$

**Lemma 7.3.** Let $P$ be a puzzle piece. If $z' \notin L_z(P)$, then $L_z(P) \cap L_{z'}(P) = \emptyset$.

**Proof.** We argue by contradiction. Let $k, k'$ be the first entrance time of $z, z'$ in $P$ so that $f^k(L_z(P)) = P$ and $f^{k'}(L_{z'}(P)) = P$. By assumption, $L_z(P) \cap L_{z'}(P) \neq \emptyset$, so $L_z(P) \subset L_{z'}(P)$ since they are puzzle pieces and since $z' \notin L_z(P)$. It implies that the levels of the puzzle pieces satisfy $h(L_z(P)) > h(L_{z'}(P))$. Therefore, $k > k'$ since $k = h(L_z(P)) - h(P)$ and $k' = h(L_{z'}(P)) - h(P)$. The contradiction comes from the fact that $k \leq k'$ since it is the first entrance time in $P$ and $f^{k'}(z) \in P$. \qed

The property of Lemma 7.3 extends to finite unions of puzzle pieces $H$ which satisfy an extension property. In the litterature there exists already similar properties of $H$ called nice or strictly nice: recall that $H$ is nice, resp. strictly nice, if for every $n \geq 1$ and every $z \in \partial H$, the iterate $f^n(z) \notin H$, resp. $f^n(z) \notin \overline{H}$. The property we will use is of being decent.

**Definition 7.4.** Let $H$ be a union of finitely many open Jordan disks. One says that $H$ is decent if:

1. $H$ is nice;
2. no connected component of $H$ is mapped exactly to a component of $H$.

One can visualize this as “the iterates of any component of $H$ is strictly bigger (not equal) than a component of $H$ or doesn’t intersect $H$”. Note also that we do not require to have an non degenerate annulus between an image of a component of $H$ and the component of $H$ contained in this image.

We are going to prove that a finite union of disks which are decent satisfy the conclusion of Lemma 7.3. We begin with next Remark which follows from the definition of decent.

**Remark 7.5.** If $H$ is a finite union of puzzle pieces that is decent and if $z \in H$, then $L_z(H) \subset H$ for $z \in H$. 

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Proof. Suppose, in order to get a contradiction, that $L_z(H) \not\subset H$. Let $P$ be the connected component of $H$ which contains $f^k(z)$, the first return of $z$ in $H$, and let $R$ be the component of $H$ containing $z$. Since puzzle pieces are either disjoint or nested, $R$ is contained in $L_z(H)$. Therefore, $f^k(R) \subset f^k(L_z(H)) = P$. Since $H$ is nice, $f^k(R) = P$ and this contradicts the fact that $H$ is decent. \qed

Lemma 7.6. If $H$ is a finite union of puzzle pieces that is decent, then for any two points $z, z' \in C$, the pieces $L_z(H)$ and $L_{z'}(H)$ either are disjoint or equal.

Proof. Suppose that $L_z(H)$ and $L_{z'}(H)$ are not disjoint. Assume, in order to get a contradiction, that $L_z(H) \cap L_{z'}(H)$ since both are puzzle pieces. Let $k, k'$ be the respective entrance time of $z, z'$ in $H$. Then $f^k(z') \in f^k(L_z(H)) \subset H$, so $k' \leq k$. Now $f^k(L_{z'}(H))$ is equal to a connected component $P_i$ of $H$ so that $f^{k-k'}(P_i) \subset f^k(L_z(H))$ which is a connected component $P_j$ of $H$. Since $z \notin L_{z'}(H)$ then $k - k' \neq 0$. As noticed during the proof of the Remark 7.5, it is not possible that $f^{k-k'}(P_i) \subset P_j$, since $H$ is decent. From this contradiction we get that either $L_z(H) = L_{z'}(H)$ or $L_z(H)$ and $L_{z'}(H)$ are disjoint. \qed

7.2 Construction of preferred puzzle pieces ($P_c$) around the critical points

Let $x \in C$ be such that $\perCrit(x) \neq \emptyset$. Recall that $b$ is the number of distinct critical ends and $\delta$ is the maximum of the degree of $f$ over the critical ends (see Notation 4.19).

Proposition 7.7. Let $I$ be a puzzle piece around $c_0 \in \perCrit(x)$. There exists two sets of puzzle pieces:

$$\Lambda := \{P_c \mid c \in \omega\Crit(c_0)\} \text{ and } \Lambda' := \{P'_c \mid c \in \omega\Crit(c_0)\}$$

with the following properties:

1. $\forall c \in \omega\Crit(c_0)$, $c \in P'_c$ and $P'_c \subset P_c$;

2. if $b = 1$ then $P_c = I$ and $P'_c = \mathcal{L}_{c_0}(I)$;

3. $P_c$ is a pullback of $I$ by some iterate $f^p$ and $\#\{0 \leq i < p \mid c_0 \in f^i(P_c)\} \leq b - 1$.

4. The piece $P'_c$ is also a pullback of $I$. Moreover, if there is a point $z \in \mathcal{P}_{\omega\Crit(c_0)} \cap (P_c \setminus \overline{P'_c})$, then there exists a puzzle piece $V$ that is included in $P_c \setminus \overline{P'_c}$ and $f^k: V \to P_c$ is an homeomorphism.

Recall that $\mathcal{P}_{\omega\Crit(c_0)}$ is the closure of the forward orbit of the points in $\omega\Crit(c_0)$. Remark that if we note $H := \bigcup_{c \in \omega\Crit(c_0)} P_c$, the piece $V$ given by point 4) is mainly $L_z(H)$ or some pullback of such a piece and the map $f^k: V \to P_c$ is the first return map in $H$.

Proof. We define the set by induction on the number of critical points. Let $H_0 := I$ and $J_0 := \mathcal{L}_{c_0}(I)$.

1. Assume that every point of $\omega\Crit(c_0) \setminus \{c_0\}$ enters in $J_0$ when it enters in $H_0$ for the first time:

Then we take $P_{c_0} := H_0$ and $P'_{c_0} := J_0$, for $c \in \omega\Crit(c_0)$ we take $P_c := \mathcal{L}_c(H_0)$ and $P'_c := \mathcal{L}_c(J_0)$. The property on the degree is clear for $P_c$ and for $P'_c$ (see Lemma 4.10).
Assume that \( b = 1 \), i.e. there is only \( c_0 \) in \( \text{Crit} \). If \( P_{\omega \text{Crit}(c_0)} \cap (H_0 \setminus J_0) \neq \emptyset \) then it should be some iterate \( f^r(c_0) \). Let \( V := L_{f^r(c_0)}(H) \) where \( H := \bigcup_{c \in \omega \text{Crit}(c_0)} P_c \), then \( V \cap J_0 = \emptyset \) (by Lemma \[7.3\]). Moreover, \( f^k : V \rightarrow H \), the first return map in \( H \), is an homeomorphism because \( f^k \) has no critical points in \( V \). This step proves the point 2) of the Proposition.

2. In the case \( b > 1 \), we suppose that there exists \( r \geq 1 \) and two finite unions of puzzle pieces \( J_r, H_r \) such that:

1. \( \{c_0, \cdots, c_r\} \subset J_r \subset H_r \);
2. \( H_r \) is decent, the components of \( J_r \) are of the form \( L_c(H_r) \);
3. every \( c \in \mathcal{C} := \omega \text{Crit}(c_0) \setminus \{c_0, \cdots, c_r\} \) enters in \( J_r \) when it enters in \( H_r \) for the first time.

Then we define \( P_c \) and \( P'_c \) as follows: for \( 0 \leq i \leq r \), \( P_{c_i} := H^i_1 \), resp. \( P'_{c_i} := J^i_1 \), is the connected component of \( H_r \), resp. of \( J_r \), containing \( c_i \) and for \( r < i \) \( P_{c_i} := L_{c_i}(H_r) \), \( P'_{c_i} := L_{c_i}(J_r) \).

The proof of point 4) of the Proposition goes as follows. Let \( z \in \mathcal{P}_{\omega \text{Crit}(c_0)} \cap (H_r \setminus J_r) \) and let \( V = L_z(H) \). If \( V = L_z(H^i_1) \) for some \( i \leq r \), it is clear by Lemma \[7.6\] that \( V \cap J_r = \emptyset \) since the components of \( J_r \) are of the form \( L_c(J_r) \) and \( z \notin J_r \). Else, \( V = L_z(P_c) \) for some \( i > r \). Suppose to get a contradiction that \( V \cap J_r = \emptyset \). Then \( V \supset L_{c_i}(H_r) \) (but they are not equal) for some \( 0 \leq i \leq r \). Let \( l, k \) be the respective entrance time of \( z \) to \( H \): \( f^l(V) = P_c \) and \( f^k(P_c) = H^i_1 \) for some \( j \leq r \). Then \( f^{k+l}(L_{c_i}(H_r)) \subset H^j_1 \) but they are not equal. This contradicts the fact that \( L_{c_i}(H_r) \) is the pullback of the puzzle piece of the first entrance to \( H_r \) (so cannot be included in).

Note that since \( V \cap J_r = \emptyset \) there is no critical points in \( V \). Then, the first iterate \( m \) such that \( f^m(V) \) contains a critical point \( c_i \) (by the recurrence assumption it is necessarily in \( \omega \text{Crit}(c_0) \)), is contained in the corresponding connected component of \( H \) (it cannot be bigger by Lemma \[7.6\] since it is eventually mapped to such a component). Therefore, \( f^m(V) \) is exactly the component \( P_{c_i} \) of \( H \), so that \( f^m : V \rightarrow P_{c_i} \) is an homeomorphism.

Now we consider a point \( z \in \mathcal{P}_{\omega \text{Crit}(c_0)} \cap (P_c \setminus P'_c) \) for some \( c \in \mathcal{C} \). Let \( y \) be the first entrance of \( z \) in \( H_r \) and \( V \) be the pullback of \( L_y(H) \) around \( z \). Note that \( y \notin J_r \). Indeed, the map from \( P_c \setminus P'_c \) to \( H^i_1 \setminus J^i_1 \) is a non ramified covering since all the critical points are mapped into \( J_r \). Hence, by the previous discussion, \( L_y(H) \) is disjoint form \( J_r \). And pulling back by the covering, we get that \( V \cap P'_c = \emptyset \). Moreover, the pullback by a non ramified covering of a disk is an homeomorphism, so \( j^k : V \rightarrow P_c \) is an homeomorphism (using previous study).

This achieves the proof of point 4). We will see later that Point 3) is satisfied also. This will follow from the construction of \( H_r \) and \( J_r \).

3. The construction of \( H_r \) and \( J_r \):

The union of puzzle pieces \( H_r, J_r \) with the properties stated in 2) are constructed by induction. Assume that we have constructed a set \( H_m \) union of puzzle pieces \( H^i_m \) around \( c_i \in \omega \text{Crit}(c_0) \) for \( 0 \leq i \leq m \) such that \( H_m \) is decent. Let \( H_m = \bigcup_{i=1}^{m} L_{c_i}(H_m) \subset H_m \).

If every critical point of \( \omega \text{Crit}(c_0) \setminus \{c_0, \cdots, c_m\} \) enters in \( H_m \) the first time it enters in \( H_m \), then \( H_m \) and \( J_m \) have the properties required in 2) and we are done.

If not, there is a point \( c_{m+1} \in \omega \text{Crit}(c_0) \setminus \{c_0, \cdots, c_m\} \) that enters in \( H_m \) the first time it enters \( H_m \), we set \( H^{m+1}_m := L''_{c_{m+1}}(H_m) \) and \( H_{m+1} := H_m \cup H^{m+1}_m \). Define \( J_{m+1} := \bigcup_{i=0}^{m+1} L_{c_i}(H_{m+1}) \).

If we prove that \( H_{m+1} \) is decent, it will follow by Remark \[7,5\] that \( J_{m+1} \subset H_{m+1} \). After finitely many steps we will achieve the induction, the sets \( H_m \) and \( J_m \) will have the properties required in 2).
By definition, it is clear that $J_m$ is also decent. Hence, we only need to verify that no puzzle piece of $J_m$ can be mapped into $H^m_{m+1}$. Let $k$ be the iterate such that $f^k(H^m_{m+1})$ is a $H_m$ puzzle piece. If $f^i(J_m) \subset H^m_{m+1}$ but does not coincide, then $f^{k+i}(J_m)$ is included in a $H_m$ puzzle piece but does not coincide with this piece. This is in contradiction with the definition of $J_m$ that is the pullback of the $H_m$ puzzle piece of the first entrance.

Now consider the puzzle piece $H^m_{m+1}$. Indeed, none of its iterates can be contained in $H^m_{m+1}$ for level of piece reason. Assume now, to get a contradiction, that for some $J_f$ and $J_m$ is included in a $J_f$ component of $H_m$. Recall also that $H^m_{m+1} = \mathcal{L}_{f^i}H_m$, so that it is disjoint from $J_m$ by Lemma \[7.6\] Moreover, when it meets $H_m$ for the second time, it is exactly along a connected component of $H_m$, so it contains a component of $J_m$. This contradicts the fact that some iterate is included in a $J_m$ component.

4. Degree properties : Let us prove now the second point of the Proposition. We stop the induction at some step $r$. For all $m \leq r$, any puzzle piece $H^i_m$ with $i \neq m$ is mapped back to a $H^i_{m-1}$ piece, and meets at most one time the critical points of $\omega Crit(c_0) \setminus \{c_0, \ldots, c_{m-1}\}$ and only one in the set $\{c_0, \ldots, c_{m-1}\}$ so the degree is less than or equal to $\delta^{b-m-1}$. The puzzle piece $H^m_m$ does twice the turn, therefore the degree is $\leq \delta^{2(b-m)}$. Therefore the degree from $P_c$ to $I$ is less than the product of the $\delta^{b-m}$, for $0 \leq m \leq r$, which is $\leq \delta^{2b-b}$ since $r \leq b-1$.

Now since at each step one meets $c_0$ at most once, the iterates from $P_c$ to $I$ meet $c_0$ at most $r \leq b-1$ times. This finishes the proof of the Proposition \[7.7\]

### 7.3 Construction of $\mathcal{A}$ and $\mathcal{B}$

In this section we give the precise definition of the operators $\mathcal{A}$ and $\mathcal{B}$, using the previous construction of the puzzle pieces $P_c, P'_c$. Let $I$ be a puzzle piece around $c_0$. To get an annuli avoiding $P_{\omega Crit(c_0)}$ we take pullbacks and ask to go through a puzzle piece of the set $\{P_c \mid c \in \omega Crit(c_0)\}$. So we want to consider the successors of $I$ (since we need annuli around $c_0$) that goes through such puzzle pieces. But this set might be empty, as we have seen that we might meet several times the same critical point from $P_c$ to $I$. We want to look at puzzle pieces $S$ containing $c_0$, that are mapped to $P_c$ and which meet the critical points at most twice before $P_c$. This corresponds to the notion of successor of $P_c$ generalised because we go back to $c_0$ and not to $c$. We can also use the notion of child as follows. Consider the following collection of pieces (see Definition \[7.1\])

$\tilde{\Lambda}_{c_0} := \{\mathcal{L}_{c_0}(Q) \mid Q$ is a child of $P_c$ and $c \in \omega Crit(c_0)\}$.

We detail now the maximality property that characterizes $\mathcal{B}(I)$ in $\tilde{\Lambda}_{c_0}$. Denote by $\tau_c(Q)$ the iterate such that $f^{\tau_c(Q)}(Q) = P_c$ when $Q$ is a child of $P_c$.

**Remark 7.8.** The set $\mathcal{N} := \{\tau_c(Q) \mid Q$ is a child of $P_c, c \in \omega Crit(c_0)\}$ is finite.

**Proof.** It follows from the fact that $c_0 \in \text{per Crit}(x)$. Indeed, this implies that any piece $P_c$ with $c \in \omega Crit(c_0)$ has finitely many successors, and therefore finitely many children. \[\square\]

**Definition 7.9.** Let $\tau = \max \mathcal{N}$ and $c_{\mathcal{N}}$ a point of $\omega Crit(c_0)$ where the maximum is reached and $Q_{\mathcal{N}}$ the corresponding child of $P_{c_{\mathcal{N}}} : f^{\tau}(Q_{\mathcal{N}}) = P_{c_{\mathcal{N}}}$. Denote by $\tilde{c}$ the critical point of $Q_{\mathcal{N}}$.

**Lemma 7.10.** By the definition of $\Lambda$ and $\mathcal{N}$, we get the property that $f^r(\tilde{c}) \in P'_{c_{\mathcal{N}}}$. Moreover, if $Q'_{c_{\mathcal{N}}}$ is the pullback containing $\tilde{c}$ of $P'_{c_{\mathcal{N}}}$ by $f^r : Q_{c_{\mathcal{N}}} \to P_{c_{\mathcal{N}}}$, then $(Q_{c_{\mathcal{N}}} \setminus Q'_{c_{\mathcal{N}}}) \cap P_{\omega Crit(c_0)} = \emptyset$. 

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Proof. Assume in order to get a contradiction that \( f^n(c) \notin P_{cr}' \). Then by the third point of Proposition 7.11, there exists a puzzle piece \( V \) containing \( f^n(c) \) and an iterate \( f \) that is a homeomorphism from \( V \) to some piece of \( \Lambda \), say \( P_\tau \). Now by \( f^n \) in \( Q_{cr} \), \( c \) is a child of \( \tau \). Hence the pullback \( Q_0 \) of \( V \) by \( f^n \) in \( Q_{cr} \) is a child of \( \tau \). This contradicts the fact that \( \tau \) is at most one critical point in each piece (here it is in \( P_{cr} \)).

Hence, since \( f^n(c) \in P_{cr}' \), we can define \( Q_{cr}' \) the pullback of \( P_{cr}' \) containing \( c \). The map \( f^n \) is a non ramified covering from \( Q_{cr} \setminus Q_{cr}' \) to \( P_{cr} \setminus P_{cr}' \), since \( Q_{cr} \) is a child of \( P_{cr} \) and since there is at most one critical point in each piece (here it is in \( Q_{cr}' \)).

Suppose, in order to get a contradiction, that \( P_{\omega Crit(c_0)} \cap (Q_{cr} \setminus Q_{cr}') \neq \emptyset \). Let \( z \) then be a point in \( P_{\omega Crit(c_0)} \cap (Q_{cr} \setminus Q_{cr}') \), image of a critical point \( c: z = f^k(c) \). Since \( f^n(z) \in P_{\omega Crit(c_0)} \cap (P_{cr} \setminus P_{cr}') \), there exists a puzzle piece \( V \) containing \( z \) and a point \( \tau \in \omega Crit(c_0) \) such that the iterate from \( V \) to \( P_{\tau} \) is a homeomorphism. As before, the pullback of \( V \) by \( f^n \) is a disk \( V' \) in \( Q_{cr} \setminus Q_{cr}' \) on which \( f^n \) is an homeomorphism. Then, taking the pullbacks along the orbit of \( c \), the last iterate \( f^i(L(V')) \) is a critical point which contains \( z \) which gives a child of \( P_{\tau} \). The contradiction comes again from the fact that \( \tau \) is strictly greater than \( \tau \).

Definition 7.11. Let \( B(I) := L_{c_0}(Q_{cr}) \). Let \( A(I) \) be the pullback by \( f^{b(I)} : B(I) \to I \) of \( W = L_{f^{b(I)}}(c_0) (I) \).

Lemma 7.12. By construction, \( (B(I) \setminus A(I)) \cap P_{\omega Crit(c_0)} = \emptyset \).

Proof. There is an integer \( n \) such that \( f^n(B(I)) = Q_{cr} \). We will prove that \( f^n(A(I)) \supset Q_{cr}' \); it will follow that any point of \( (B(I) \setminus A(I)) \cap P_{\omega Crit(c_0)} \) will have its image under \( f^n \) which is contained in \( P_{\omega Crit(c_0)} \cap (Q_{cr} \setminus Q_{cr}') \). But this set is empty by the previous Lemma. In order to prove that \( f^n(A(I)) \supset Q_{cr}' \), notice that since \( P_{\omega Crit(c_0)} \cap (Q_{cr} \setminus Q_{cr}') = \emptyset \), the image \( f^n(c_0) \) cannot be in \( Q_{cr} \setminus Q_{cr}' \). So since this point belongs to both \( f^n(A(I)) \) and \( Q_{cr}' \), these two pieces are nested. Assume that \( f^n(A(I)) \subset Q_{cr}' \). Then, \( W \supset f^{-n}(Q_{cr}') \subset I \). By construction, \( Q_{cr}' \) is a pullback of \( I \). We get a contradiction, from the fact that \( f^{-n}(Q_{cr}') \) will be mapped to \( I \) before \( W = L_{f^{b(I)}}(c_0) (I) \). The Lemma the follows.

7.4 Proof of Proposition 6.10

Recall the Statement of the Proposition:

**Proposition 6.10.** Let \( I \) be a puzzle piece containing a critical point \( c \). Then the following holds:

1. \( c \in A(I) \subset B(I) \subset I \) and \( B(I) \setminus A(I) \) avoids the postcritical set \( P_{\omega Crit(c_0)} \);
2. There exists \( b(I) \), \( a(I) \) such that \( f^{b(I)}(B(I)) = I \) and \( f^{a(I)}(A(I)) = I \);
3. \( \# \{0 \leq j < b(I) \mid c \in f^j(B(I)) \} \leq b \), and \( \deg(f^{b(I)} : B(I) \to I) \leq \delta^{b^2} \);
4. \( \# \{0 \leq j < a(I) \mid c \in f^j(A(I)) \} \leq b + 1 \), and \( \deg(f^{a(I)} : A(I) \to I) \leq \delta^{b^2+b} \).

Proof. Point 1) of the Proposition is just Lemma 7.12. Then point 2) follows just from the definition. To prove the degree properties of point 3) and point 4), recall that the iterate of \( f \) that maps \( B(I) \) to \( P_{cr} \) has degree bounded by \( \delta^b \) (Lemma 4.10). Then, point 3) follows from
the fact that the map from $P_{cr}$ to $I$ has degree bounded by $\delta^{2-b}$. By Lemma 4.10 the degree from $W$ to $I$ is bounded by $\delta$.

Finally, if $l$ denotes the time such that $f^l(B(I)) = P_{cr}$, the pieces $f^i(B(I))$ for $0 \leq i \leq l$ meets $c_0$ only once since $B(I) = \mathcal{L}_{c_0}(Q_{cr})$ is the first pull back around $c_0$ of $Q_{cr}$ which is a child of $P_{cr}$ (Definition 7.1), and it happens only in $B(I)$. Therefore from the construction of $P_c$ we get easily Point 3) and Point 4).

8 The properties of the enhanced nest

This section is devoted to the proof of Lemmas 6.3 and 6.12 which were crucial in the proof of Lemma 6.22 via the Lemmas 6.17, 6.19 and its power is unlighted in Remark 6.13.

8.1 Proof of Lemma 6.3 (see also [P-Q-R-T-Y])

This result is similar to the one for cubic and quadratic maps obtained by Branner-Hubbard and Yoccoz (see [BII], [II] and [M2]). It is a crucial step for the estimates in Lemma 6.12 for instance.

Lemma 8.1 (Previously called Lemma 6.3). Let $c$ be any critical point of $f$.

1. If End($c$) is not periodic, then each puzzle piece $P_n(c)$ has at least two successors.

2. If $c \in \omega$Crit($x$) has an end End($c$) which is periodic and $x$ is a point of $\partial U$, then $\omega(x)$ contains a periodic point.

Proof. Let $P$ be a puzzle piece containing $c$. Denote by $n_0$ its height: $P = P_{n_0}(c)$. Let $k_0 = 0 < k_1 < \cdots < k_n < \cdots$ denote the successive entrance time of (the orbit of) $c$ in $P$. Note that this sequence is infinite since $c \in \omega$Crit($c$). The first entrance time of $c$ to $P$ is $k_1$, denote by $Q$ the pull back of $P$ by $f^{k_1}$, it is the first successor of $P$. Note that $Q = P_{n_0+k_1}(c)$.

Claim 1. Either $P$ has a second successor or $k_i = ik_1$ for all $i \geq 0$ and $f^{ik_1}(c) \in Q$.

Proof. Assume first that there exists $k_i$ such that $f^{k_i}(c) \notin Q$. It means that $c$ which is in $P_{n_0}(f^{k_k}(c))$ is not in $P_{n_0+k_1}(f^{k_k}(c))$. This implies that there is no critical points in $P_{n_0+k_1}(f^{k_k}(c))$ (else there would be two critical points in $P_{n_0}(f^{k_k}(c))$ which contradicts assumption 3). Then consider $Q'$ the pull back of $P$ containing $f^{k_k}(c)$ by $f^{k_k+1-k_1}: P \to f^{k_k+1-k_1}(P)$. If $k_i+1 - k_i \geq k_1$, we just show that $Q'$ contains no critical points. If $k_i+1 - k_i < k_1$, if $Q'$ contains a critical point it should be $c$ (since $Q' \subset P$) but then $Q'$ is the first pull back of $P$ in $P$ which is impossible. Therefore we can continue to pull back $Q'$ by the iterates of $f$ along the orbit $\{c, f(c), \ldots, f^{k_k}(c)\}$ until we reach a critical puzzle piece $R$ of depth greater than that of $Q'$ that is greater than $n_0 + k_1$. If $c \in R$ then $R$ is a second successor of $P$. Else, $\mathcal{L}_c(R)$ will be a second successor of $P$.

Now assume that for every $i \geq 1$, $f^{ik_1}(c) \in Q$. In particular, $f^{ik_1}(c) \in Q$ and therefore $f^{2k_1}(c) \in f^{k_1}(Q) = P$. It follows that $k_2 \leq 2k_1$. On the other side, since $f^{k_1}(c) \in Q$, it follows that $f^{k_2-k_1}(Q) = P$. In particular, $f^{k_2-k_1}(c) \in P$, so $k_1 \leq k_2 - k_1$. Hence, $k_2 = 2k_1$ and by induction $k_i = ik_1$ for all $i \geq 0$.

Claim 2. Assume that $P$ has only one successor, that $k_i = ik_1$ for all $i \geq 0$ and that $f^{ik_1}(c) \in Q$. Denote by $R$ the puzzle piece $P_{n_0+2k_1}(c)$, it is the pull back by $f^{k_1}$ of $Q$ around $c$. Then, for all $i \geq 0$, $f^{ik_1}(c) \in R$.  

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Claim 3. Either \( P \) of \( W \)

Proof. We have proved in Claim 1 that either \( \zeta \) obtain that the critical point we denoted by \( c \) with \( 0 < l \leq r \), denote by \( S_n := f^{j_n}(R) \).

First we shall prove by contradiction that there is no critical point in the puzzle pieces \( f^j(Q) \) for \( j_r < j < j_{r+1} \). Assume that a critical point \( \tilde{c} \) belongs to \( f^j(Q) \) with \( j_r < j < j_{r+1} \). Let \( k = j_{r+1} - j \). The point \( f^k(\tilde{c}) \) is included in \( P \setminus Q \) and \( \mathcal{L}_{f^k(\tilde{c})}(P) \) as well. Since the map \( f^k : \mathcal{L}_{\tilde{c}} \to P \) has only a critical point at \( \tilde{c} \), if \( \tilde{S} \) is the connected component of \( f^{-k}(\mathcal{L}_{f^k(\tilde{c})}(P)) \), the piece \( \mathcal{L}_{\tilde{c}}(\tilde{S}) \) is a second successor of \( P \). Indeed, the each map of the form \( f^i : \mathcal{L}_x(S) \to S \) meets each critical point at most once.

The puzzle piece \( S_r \) contains \( c_r \) and is of height \( n_r := n_0 + 2k_1 - j_r \). The piece \( S_r \) is the only pre-image of \( Q \) in \( f^j(Q) \) by \( f^{j_r} : f^j(Q) \to P \). Since \( f^{k_1+1-(k_1+j_r)} \) maps \( f^j(Q) \) to \( P \) and \( f^k(c) \in Q \) so that \( f^{k_i+j_r}(c) \in f^j(Q) \), we deduce that \( f^{k_i+j_r}(c) \in S_r \) for all \( i \geq 0 \) since \( f^{k_1+1}(c) \in Q \). With the same argument, we prove successively that \( f^{k_i+j_i}(c) \in S_i \), for \( l \) from \( r \) to 0, which corresponds to \( f^{k_1}(c) \in R \) for \( i \geq 0 \) (for \( l = 0 \)).

Claim 3. Either \( P \) has a second successor or \( \text{End}(c) \) is periodic.

Proof. We have proved in Claim 1 that either \( P \) has a second successor or the assumption of Claim 2 holds. Then after iterating the same argument on \( P_{n_0+jk_1}(c) \) for \( j \geq 2 \) as in Claim 2 we obtain that the critical point \( c \) belongs to every piece \( P_n(f^{ik_1}(c)) \). Hence, the entire nest \( P_n(c) \) with \( n \geq n_0 + k_1 \) is mapped by \( f^{k_1} \) to the same nest \( P_n(c) \) with \( n \geq n_0 \).

Claim 4. Assume \( \text{End}(c) \) is periodic for \( c \in \omega_{\text{comb}}(x) \) with \( x \in \partial U \) then \( x \) combinatorially accumulates a periodic point.

Proof. Since \( c \in \omega_{\text{Crit}}(x) \) with \( x \in \partial U \), every puzzle piece of the nest \( (P_n(c)) \) intersects \( U \). Consider the internal rays in \( U \) of \( \partial P_{n_k}(c) \): they are of the form \( R_U(\zeta_n), R_U(\zeta_n') \), where \( \zeta_n \) and \( \zeta_n' \) are two adjacent sequences converging to some angle \( \zeta \). Since the nest is “fixed” by \( f^{k_1} \), it follows that \( R_U(\zeta) \) is also fixed by \( f^{k_1} \); so it converges to a point \( y \) in \( \text{End}(c) \) that is fixed by \( f^{k_1} \). Hence the end \( \text{End}(c) = \cap_n P_n(c) \) contains a \( k_1 \)-periodic point. Therefore \( \text{End}(y) = \text{End}(c) \) and since \( c \in \omega_{\text{Crit}}(x) \), \( x \) accumulates combinatorially the point \( y \). Therefore \( y \in \partial U \) (because \( x \in \partial U \)) and since \( \text{End}(y) \cap \partial U = \{ y \} \) it follows that \( x \) accumulates \( y \) (i.e. \( y \in \omega(x) \)).

8.2 Proof of Lemma 6.12

Set \( I_n = D^\tau(K_{n-1}) \), where \( D(J) \) denote the last successor of a puzzle piece \( J \), and \( \tau \) is some number that can be for instance \( b + 1 \). Recall that \( K_n = A(I_n) \) and that \( p_n, a(I_n), \sigma(\tau)(K_n) \) are the times to go respectively from \( K_n \) to \( K_{n-1} \), from \( A(I_n) \) to \( I_n \) and from \( I_{n+1} \) to \( K_n \), i.e. they satisfy \( f^{p_n}(K_n) = K_{n-1}, f^{a(I_n)}(A(I_n)) = I_n \), and \( f^{\sigma(\tau)(K_n)}(I_{n+1}) = K_n \). Recall the statement of Lemma 6.12.2

Lemma 6.12.2 \( p_n \geq 2p_{n-1} \).

Proof. Notice that \( p_n = a(I_n) + \sigma(\tau)(K_{n-1}) \) and that \( p_{n+1} = a(I_{n+1}) + \sigma(K_n) \).

Claim 1. The following inequality holds: \( 2r(I_n) \leq a(I_n) \leq (b + 1)r(K_n) \).

Proof. The left inequality comes from the definition of \( A(I_n) \). Recall that \( A(I_n) \) is the pullback of \( W \) by \( f^{b(I_n)} : B(I_n) \to I_n \), where \( W \) is \( \mathcal{L}_{f^{b(I_n)}(c)}(I_n) \). Since \( B(I_n) \subset I_n \), \( b(I_n) \geq r(I_n) \). Let \( k \)
be the time such that $f^k(W) = I_n$, then $k \geq r(I_n)$ since $W \subset I_n$. Therefore $a(I_n) = k + b(I_n) \geq 2r(I_n)$.

The right inequality is a corollary of Remark 6.5 and Proposition 6.10. Indeed, the point $c_0$ is contained in at most $b + 1$ iterates of $A(I_n)$ in $\{ f^i(A(I_n)) | 0 \leq i \leq a(I_n) \}$. Denote these iterates $f^{k_i}(A(I_n))$ with $k_0 = 0 < k_1 < \cdots < k_j = a(I_n)$. By Remark 6.5.2), $k_{i+1} - k_i \leq r(f^{k_i}(K_n))$, and since $f^{k_i}(K_n) \supset K_n$, $r(f^{k_i}(K_n)) \leq r(K_n)$, so that $a(I_n) = \sum (k_{i+1} - k_i) \leq (b + 1)r(K_n)$.

**Claim 2.** The following inequality holds: $(2^{\tau+1} - 2)r(K_n) \leq \sigma(\tau(K_n)) \leq 2r(I_{n+1}).$

**Proof.** We prove first the right inequality. Let us apply Remark 6.5.2) at two successors $A = D^{j+1}(K_n)$ and $A' = D^{j}(K_n)$: we get $\sigma(D^{j}(K_n)) \leq r(D^{j+1}(K_n))$. Now successive applications of Corollary 6.7 gives $r(D^{j+1}(K_n)) \leq \frac{1}{2^{\tau-j-1}}r(D^{\tau}(K_n))$. Therefore, $\sigma(D^{j}(K_n)) \leq \frac{1}{2^{\tau-j-1}}r(I_{n+1})$.

Hence $\sigma(\tau(K_n)) = \sum_{j=0}^{\tau-1} \sigma(D^{j}(K_n)) \leq \sum_{j=0}^{\tau-1} \frac{1}{2^{\tau-j-1}} r(I_{n+1})$. Thus, $\sigma(\tau(K_n)) \leq 2r(I_{n+1})$.

For the left inequality, note first that $\sigma(\tau) \geq 2r(\tau)$ by Corollary 6.7. Applying this inequality successively, we get that $\sigma(\tau(K_n)) = \sum_{j=0}^{\tau-1} \sigma(D^{j}(K_n)) \geq \sum_{j=0}^{\tau-1} 2r(D^{j}(K_n))$. Then Corollary 6.7 implies that for all $j \geq 0$, $r(D^{j}(K_n)) \geq 2^j r(K_n)$. Therefore, $\sigma(\tau(K_n)) \geq \sum_{j=0}^{\tau-1} 2^j r(K_n) = (2^{\tau+1} - 2)r(K_n)$.

**Proof of Lemma 6.13.** By claim 1 and 2 above, $p_n = a(I_n) + \sigma(\tau(K_n-1)) \leq (b+1)r(K_n) + 2r(I_n)$. Thus $p_n \leq (b+3)r(K_n)$ since $K_n \subset I_n$. The minorations coming from these two Lemmas imply that $p_{n+1} = a(I_{n+1}) + \sigma(K_n) \geq 2r(I_{n+1}) + (2^{\tau+1} - 2)r(K_n) \geq (2^{\tau+2} - 2)r(K_n)$ since $I_{n+1} \subset K_n$.

Hence, the Lemma follows from the fact that for $\tau \geq b + 1$ we have $(2^{\tau+2} - 2) \geq 2(b+3)$.

**9 Proof of Theorem 2**

Up to replacing $f$ by some iterate, we can assume that the bounded component $U$ is fixed by $f$. We can also assume that $U$ contains only one critical point in it (up to doing some surgery).

We will also assume that $K_U = K$: to achieve this we use the fact that the restriction of $f$ to the domain bounded by an equipotential around $K_U$ is a polynomial-like mapping. The Douady-Hubbard theorem provides us with a polynomial whose dynamics on the filled Julia set is conjugated to that of $f$ on $K_U$. We will still denote $f$ the new polynomial to which these three changes have possibly been performed.

By Theorem 1 the boundary of $\partial U$ is a Jordan curve. Hence, the Riemann map $\Phi_U : D \to U$ extends continuously to the boundary as an homeomorphism $\gamma_U : S^1 \to \partial U$ that conjugates the dynamics (of the model map on $S^1$ to $f$ on $\partial U$). We consider one of the graphs used in the article for constructing the puzzle.

**Lemma 9.1.** Recall the notation $End(z) := \bigcap_{n \in \mathbb{N}} P_n(z)$.

1. If $z \in \partial U$ is evently periodic then either $End(z) = \{ z \}$ or there exist two external rays $R_{\infty}(\zeta), R_{\infty}(\zeta')$ landing at $z$ and separating $End(z) \setminus \{ z \}$ from $\overline{U}$;
2. if $z \in \partial U$ is not eventually periodic then $End(z) = \{ z \}$;
3. in both cases when \( \text{End}(z) = \{z\} \), there exists at least one external ray converging to \( z \).

**Proof.** When \( z \) is periodic, this is Proposition 5.1. The case when \( z \) is eventually periodic follows by taking pre-images. This proves 1). When \( z \in \partial U \) is not eventually periodic and satisfies \((\ast)\); this is Corollary 5.11. Lemma 5.8 together with Corollary 4.13 and Corollary 5.11 imply that the only points \( z \in \partial U \) where the local connectivity could fail are those such that \( \text{perCrit}(x) \) is not empty. Those points were considered in details in section 6, where Corollary 6.24 showed \( \text{End}(x) = \{x\} \). This proves 2).

For 3), let \( R_{\infty}(\zeta_n), R_{\infty}(\zeta'_n) \) be the two external rays of \( \partial P_n(z) \) landing on \( \partial U \) and let \( \zeta \) be the limit of the sequence \( (\zeta_n) \). Then the end of \( R_{\infty}(\zeta) \) enters every puzzle \( P_n(z) \), so it converges to \( z = \cap_n P_n(z) \). \( \square \)

**Definition 9.2.** For \( t \in S^1 \), let \( \mathcal{R} \) denote the union of all the external rays landing at \( z = \gamma_U(t) \). Let \( \tilde{U} \) denote the component of \( C \setminus (\mathcal{R} \cup \{z\}) \) containing \( U \). Then take \( L_t := K_U \cap (C \setminus \tilde{U}) \), it is called a limb.

**Properties.** The limbs have the following properties:

- The set \( L_t \) is connected: it follows from the fact that \( \gamma_U(t) \) belongs to the closure of every component of \( C \setminus \mathcal{R} \);
- \( L_t \cap \overline{U} = \{\gamma_U(t)\} \), this follows from Lemma 9.1;
- \( K_U = \bigcup_{t \in S^1} L_t \cup U \) by definition;
- \( L_t \) reduces to one point if and only if there is exactly one ray converging to \( z \).

**Lemma 9.3.** Let \( z \in \partial \text{partiu}U \) be a point such that two rays at least converge to \( z = \gamma_U(t) \). Then either \( L_t \) contains a critical point or \( L_t \) is mapped to some \( L_t' \) which contains a critical point.

**Proof.** By definition, there exist two external rays \( R_{\infty}(\zeta) \) and \( R_{\infty}(\zeta') \) which land at \( z \) and which separates \( L_t \setminus \{z\} \) from \( \overline{U} \). Assume that \( L_t \) does not contain a critical point. The sector \( C \setminus \tilde{U} \) is a disk between these two external rays, it contains no critical point (since the critical points are all in \( K_U \)), the boundary is mapped homeomorphically to \( R_{\infty}(D\zeta) \cup R_{\infty}(D\zeta') \cup \{f(z)\} \) (if the polynomial is of degree \( D \)). Thus it is mapped homeomorphically to the sector, which does not contain \( U \), and is bounded by \( R_{\infty}(D\zeta) \) and \( R_{\infty}(D\zeta') \). Therefore \( f(L_t) = L_{dt} \) if the degree in \( U \) is \( d \). But the multiplication by \( D \) will eventually cover \( S^1 \). So some Limb of an iterated image of \( z \) has to contain a critical point. \( \square \)

**Lemma 9.4.** If \( L_t \) contains a critical point then at least two external rays converge to \( \gamma_U(t) \).

**Proof.** If the critical point is \( z = \gamma_U(t) \), then the pull back of any external ray landing at \( f(z) \) consists in two external rays landing at \( z \). If the critical point is in \( L_t \setminus \{z\} \), then in particular \( L_t \) is not reduced to a point. The result follows then from the definition. \( \square \)

Note that Corollary 1 follows from Lemma 9.4 since at an eventually periodic point \( x \in \partial U \), \( \text{End}(x) = \{x\} \). So by construction \( (P_n(x) \cap J) \) form a sequence of connected neighborhoods of \( x \) in \( J \) whose diameter tends to 0.

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