Note on Sunflowers

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Abstract

A sunflower with $p$ petals consists of $p$ sets whose pairwise intersections are identical. Building upon a breakthrough of Alweiss, Lovett, Wu, and Zhang from 2019, Rao proved that any family of $(Cp\log(pk))^k$ distinct $k$-element sets contains a sunflower with $p$ petals, where $C > 0$ is a constant; this bound was reproved by Tao. In this note we record that, by a minor variant of their probabilistic arguments, any family of $(Cp\log k)^k$ distinct $k$-element sets contains a sunflower with $p$ petals, where $C > 0$ is a constant.

1 Introduction

A sunflower with $p$ petals is a family of $p$ sets whose pairwise intersections are identical (the intersections may be empty). Let $\text{Sun}(p,k)$ denote the smallest natural number $s$ with the property that any family of at least $s$ distinct $k$-element sets contains a sunflower with $p$ petals. In 1960, Erdős and Rado [2] proved that $(p−1)^k < \text{Sun}(p,k) \leq (p−1)^k! + 1 = O((pk)^k)$, and conjectured that for any $p \geq 2$ there is a constant $C_p > 0$ such that $\text{Sun}(p,k) \leq C_p^k$ for all $k \geq 2$. This well-known conjecture remains open, but there was a breakthrough in 2019: using iterative encoding arguments, Alweiss, Lovett, Wu, and Zhang [1] proved that $\text{Sun}(p,k) \leq (Cp^3 \log k \log \log k)^k$ for some constant $C > 0$. Using Shannon’s noiseless coding theorem, Rao [3] simplified the proof and obtained a slightly better bound. Frankston, Kahn, Narayanan, and Park [4] refined some key arguments from [1], and their ideas were utilized by Rao [5] to prove that $\text{Sun}(p,k) \leq (Cp\log(pk))^k$ for some constant $C > 0$, which in turn was reproved by Tao [7] using Shannon entropy arguments.

The aim of this note is to record, for the convenience of other researchers, that a minor variant of (the probabilistic part of) the arguments from [6,7] gives $\text{Sun}(p,k) \leq (Cp\log k)^k$ for some constant $C > 0$.

Theorem 1. There is a constant $C \geq 4$ such that $\text{Sun}(p,k) \leq (Cp\log k)^k$ for all integers $p,k \geq 2$.

Setting $r(p,k) = Cp\log k + \mathbb{1}_{\{k = 1\}}p$, we shall in fact prove $\text{Sun}(p,k) \leq r(p,k)^k$ for all integers $p \geq 2$ and $k \geq 1$. Similar to [1,6,7], this upper bound follows easily by induction on $k \geq 1$ from Lemma 2 below, where a family $\mathcal{S}$ of $k$-element sets is called $r$-spread if there are at most $r^{k−|\mathcal{T}|}$ sets of $\mathcal{S}$ that contain any non-empty set $\mathcal{T}$. (Indeed, the base case $k = 1$ is trivial due to $r(p,1) = p$, and the induction step $k \geq 2$ uses a simple case distinction: if $S$ is $r(p,k)$-spread, then Lemma 2 guarantees a sunflower with $p$ petals; otherwise there is a non-empty set $\mathcal{T}$ such that more than $r(p,k)^{k−|\mathcal{T}|} \geq r(p,k−|\mathcal{T}|)^{k−|\mathcal{T}|}$ sets of $\mathcal{S}$ contain $\mathcal{T}$, and among this family of sets we easily find a sunflower with $p$ petals using induction.)

Lemma 2. There is a constant $C \geq 4$ such that, setting $r(p,k) = Cp\log k$, the following holds for all integers $p,k \geq 2$. If a family $\mathcal{S}$ with $|\mathcal{S}| \geq r(p,k)^k$ sets of size $k$ is $r(p,k)$-spread, then $\mathcal{S}$ contains $p$ disjoint sets.

Inspired by [1, in [6,7]], probabilistic arguments are used to deduce Lemma 2 with $r(p,k) = \Theta(p\log(pk))$ from Theorem 3 below (based on the union bound or linearity of expectation, respectively). Here $X_\delta$ denotes the random subset of $X$ where each element is included independently with probability $\delta$.

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Lemma 4. Our proof of Lemma 2 only invokes Theorem 3 with $|S| \geq r^k$, then $\mathbb{P}(\exists S \in S : S \subseteq X_3) > 1 - \epsilon$.

The core idea of [1] [6] [7] is to randomly partition the set $X$ into $V_1 \cup \cdots \cup V_p$, by independently placing each element $x \in X$ into a randomly chosen $V_i$. Note that the marginal distribution of each $V_i$ equals the distribution of $X_3$ with $\delta = 1/p$. Invoking Theorem 3 with $\varepsilon = 1/p$ and $r = B\delta^{-1} \log(k/\varepsilon)$, a standard union bound argument implies that, with non-zero probability, all of the random partition-classes $V_i$ contain a set from $S$. Hence $p$ disjoint sets $S_1, \ldots, S_p \in S$ must exist, which proves Lemma 2 with $r(p, k) = Bp \log(pk)$.

We prove Lemma 2 with $r(p, k) = \Theta(p \log k)$ using a minor twist: by randomly partitioning the vertex-set into more than $p$ classes $V_i$, and then using linearity of expectation (instead of a union bound).

Proof of Lemma 4. Set $C = 4B$. We randomly partition the set $X$ into $V_1 \cup \cdots \cup V_{2p}$, by independently placing each element $x \in X$ into a randomly chosen $V_i$. Let $I_i$ be the indicator random variable for the event that $V_i$ contains a set from $S$. Since $V_i$ has the same distribution as $X_3$ with $\delta = 1/(2p)$, by invoking Theorem 3 with $\varepsilon = 1/2$ and $r = r(p, k) = 2Bp \log(k^2) \geq B\delta^{-1} \log(k/\varepsilon)$ we obtain $\mathbb{E} I_i > 1/2$. Using linearity of expectation, the expected number of partition-classes $V_i$ with $I_i = 1$ is thus at least $p$. Hence there must be a partition where at least $p$ of the $V_i$ contain a set from $S$, which gives the desired $p$ disjoint sets $S_1, \ldots, S_p \in S$.

Generalizing this idea, Theorem 3 gives $p > |1/\delta|(1 - \epsilon)$ disjoint sets $S_1, \ldots, S_p \in S$, which in the special case $|1/\delta| \varepsilon \leq 1$ (used in [1] [6] [7]) with $\delta = \epsilon = 1/p$ simplifies to $p \geq |1/\delta|$. 

2 Remarks

Our proof of Lemma 2 only invokes Theorem 3 with $\varepsilon = 1/2$, i.e., does not exploit the fact that Theorem 3 has an essentially optimal dependence on $\epsilon$ (see Lemma 4 below). In particular, this implies that we could alternatively also prove Lemma 2 and thus the Sun($p, k) \leq (Cp \log k)^k$ bound of Theorem 1 using the combinatorial arguments of Frankston, Kahn, Narayanan, and Park [3] (we have verified that the proof of Theorem 1.7 can be extended to yield Theorem 3 under the stronger assumption $r \geq B\delta^{-1} \max\{\log k, \log^2(1/\epsilon)\}$, say).

We close by recording that Theorem 3 is essentially best possible with respect to the $r$-spread assumption, which follows from the construction in [1] Section 4).

Lemma 4. For any reals $0 < \delta, \varepsilon \leq 1/2$ and any integers $k \geq 1$, $1 \leq r \leq 0.25\delta^{-1} \log(k/\varepsilon)$, there exists a $r$-spread family $S$ of $k$-element subsets of $X = \{1, \ldots, rk\}$ with $|S| = r^k$ and $\mathbb{P}(\exists S \in S : S \subseteq X_3) < 1 - \epsilon$.

Proof. Fix a partition $V_1 \cup \cdots \cup V_k$ of $X$ into sets of equal size $|V_i| = r$. Let $S$ be the family of all $k$-element sets containing exactly one element from each $V_i$. It is easy to check that $S$ is $r$-spread, with $|S| = r^k$. Focusing on the necessary event that $X_3$ contains at least one element from each $V_i$, we obtain

$$\mathbb{P}(\exists S \in S : S \subseteq X_3) \leq (1 - (1 - \delta)^r)^k \leq e^{-(1-\delta)^r} < e^{-2r}\delta^r k \leq e^{-\sqrt{rk}} \leq 1 - \epsilon$$

by elementary considerations (since $e^{-\sqrt{k}} \leq 1 - \epsilon$ due to $0 < \epsilon \leq 1/2$).

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References

[1] R. Alweiss, S. Lovett, K. Wu, and J. Zhang. Improved bounds for the Sunflower lemma. Preprint (2019). arXiv:1908.08483v1
[2] P. Erdős and R. Rado. Intersection Theorems for Systems of Sets. J. London Math. Soc. 35 (1960).
[3] K. Frankston, J. Kahn, B. Narayanan, and J. Park. Thresholds versus Fractional Expectation-thresholds. Preprint (2019). arXiv:1910.13433v2
[4] S. Janson, T. Luczak, and A. Ruciński. Random Graphs. Wiley-Interscience (2000).
[5] A. Rao. Coding for Sunflowers. Preprint (2019). arXiv:1908.04774v1
[6] A. Rao. Coding for Sunflowers. Discrete Analysis 2 (2020), 8 pp. arXiv:1909.04774v2
[7] T. Tao. The sunflower lemma via Shannon entropy. Blogpost (2020). https://terrytao.wordpress.com/2020/07/20/the-sunflower-lemma-via-shannon-entropy/
Appendix: Theorem 3

Theorem 3 follows from Tao’s proof of Proposition 5 in [7] (noting that any $r$-spread family $S$ with $|S| \geq r^k$ sets of size $k$ is also $r$-spread in the sense of [7]). We now record that Theorem 3 also follows from Rao’s proof of Lemma 4 in [6] (where the random subset of $X$ is formally chosen in a slightly different way).

Proof of Theorem 3 based on [6]. Set $\gamma = \delta/2$ and $m = \lceil \gamma |X| \rceil$. Let $X_i$ denote a set chosen uniformly at random from all $i$-element subsets of $X$. Since $X_\delta$ conditioned on containing exactly $i$ elements has the same distribution as $X_i$, by the law of total probability and monotonicity it routinely follows that $P(\exists S \in S : S \subseteq X_\delta)$ is at least $P(\exists S \in S : S \subseteq X_m) \cdot P(|X_\delta| \geq m)$. The proof of Lemma 4 in [6] shows that $P(\exists S \in S : S \subseteq X_m) > 1 - \epsilon^2$ whenever $r \geq \alpha \gamma^{-1} \log(k/\epsilon)$, where $\alpha > 0$ is a sufficiently large constant. Noting $|S| \leq |X|^k$ we see that $|S| \geq r^k$ enforces $|X| \geq r$, so standard Chernoff bounds (such as [4, Theorem 2.1]) imply that $P(|X_\delta| < m) \leq P(|X_\delta| \leq |X|\delta/2)$ is at most $e^{-|X|\delta/8} \leq e^{-r\delta/8} \leq \epsilon^2$ whenever $r \geq 16\delta^{-1} \log(1/\epsilon)$. This completes the proof with $B = \max\{2\alpha, 16\}$, say (since $(1 - \epsilon^2)^2 \geq 1 - \epsilon$ due to $0 < \epsilon \leq 1/2$). \qed