Quasinormal modes of Schwarzschild black holes in four and higher dimensions

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We make a thorough investigation of the asymptotic quasinormal modes of the four and five-dimensional Schwarzschild black hole for scalar, electromagnetic and gravitational perturbations. Our numerical results give full support to all the analytical predictions by Motl and Neitzke, for the leading term. We also compute the first order corrections analytically, by extending to higher dimensions, previous work of Musiri and Siopsis, and find excellent agreement with the numerical results. For generic spacetime dimension number D the first-order corrections go as $rac{1}{(D-3)(D-2)}$. This means that there is a more rapid convergence to the asymptotic value for the five dimensional case than for the four dimensional case, as we also show numerically.

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I. INTRODUCTION

The study of quasinormal modes of black holes began more than thirty years ago, when Vishveshwara noticed that the signal from a perturbed black hole is, for most of the time, an exponentially decaying ringing signal. It turns out that the ringing frequency and damping timescale are characteristic of the black hole, depending only on its parameters (like the mass and angular momentum). We call these characteristic oscillations the quasinormal modes (QNMs) and the associated frequencies are termed quasinormal frequencies (QN frequencies), because they are really not stationary perturbations. Not surprisingly, QNMs play an important role in the dynamics of black holes, and consequently in gravitational wave physics. In fact, it is possible to extract the parameters of the black hole simply by observing these QN frequencies, using for example gravitational wave detectors. The discovery that QNMs dominate the answer of a black hole to almost any exterior perturbation was followed by a great effort to find, numerically and analytically, the QN frequencies. For excellent reviews on the status of QNMs, prior to 2000, we refer the reader to Kokkotas and Schmidt and Nollert. It is important to note that on the astrophysical aspect, the most important QN frequencies are the lowest ones, i.e., frequencies with smaller imaginary part, and the most important spacetimes are the asymptotically flat and perhaps now the asymptotically de Sitter. However, three years ago Horowitz and Hubeny pointed out that QNMs of black holes in anti-de Sitter space have a different importance. According to the AdS/CFT correspondence, a black hole in anti-de Sitter space may be viewed as a thermal state in the dual theory. Perturbing this black hole corresponds to perturbing the thermal state, and therefore the typical timescale of approach to thermal equilibrium (which is hard to compute directly in the dual theory) should as well be governed by the lowest QN frequencies. That this is indeed the case was proved by Birmigham, Sachs and Solodukhin for the BTZ black hole, taking advantage that this is one of the few spacetimes where one can compute exactly its QN frequencies, as showed by Cardoso and Lemos. A similar study was made by Kurita and Sakagami for the D-3 brane. This interpretation for the imaginary part of the QN frequencies in terms of timescales of approach to thermal equilibrium in the dual conformal field theory has motivated a generalized search for the quasinormal modes of different black holes in anti-de Sitter spacetime, over the last three years.

Recently, the motivation for studying QN modes of black holes has grown enormously with the conjectures relating the highly damped QNMs (i.e., QN frequencies with large imaginary parts) to black hole area quantization and to the Barbero Immirzi parameter appearing in Loop Quantum Gravity. The seeds of that idea were planted some time ago by Bekenstein. A semi-classical reasoning of the conjecture that the black hole area spectrum is quantized leads to

$$A_n = \gamma l_P^2 n, \quad n = 1, 2, \ldots$$

Here $l_P$ is the Planck length and $\gamma$ is an undetermined constant. However, statistical physics arguments impose a constraint on $\gamma$:

$$\gamma = 4 \log k,$$

where $k$ is an integer. The integer $k$ was left undetermined (although there were some suggestions for it), until Hod supported by some of Bekenstein’s ideas, put forward the proposal to determine $k$ via a version of Bohr’s correspondence principle, in which one admits...
that the real part of QN frequencies with a large imaginary part plays a fundamental role. It was seen numerically by Nollert [5, 21] that QN frequencies with a large imaginary part behave, in the Schwarzschild geometry as

$$\omega M = 0.0437123 + \frac{i}{8}(2n + 1), \quad (3)$$

where $M$ is the black hole mass, and $n$ the mode number. Hod first realized that $0.0437123 \sim \frac{\ln 3}{2\pi}$, and went on to say that, if one supposes that the emission of a quantum with frequency $\frac{\ln 3}{2\pi}$ corresponds to the least possible energy a black hole can emit, then the change in surface area will be (using $A = 16\pi M^2$)

$$\Delta A = 32\pi MdM = 32\pi M\omega = 4\hbar \ln 3. \quad (4)$$

Comparing with (1) we then get $k = 3$ and therefore the area spectrum is fixed to

$$A_n = 4\ln 3 l^2 pn; \quad n = 1, 2, ... \quad (5)$$

It was certainly a daring proposal to map $0.0437123$ to $\ln 3$, and even more to use the QN frequencies to quantize the black hole area, by appealing to “Bohr’s correspondence principle”. The risk paid off: recently, Dreyer [13] put forward the hypothesis that if one uses such a correspondence it is possible to fix a formerly free parameter, the Barbero-Immirzi parameter, appearing in Loop Quantum Gravity. Dreyer’s work implied that the entropy has contributions from $J = 1$ edges mainly. One interpretation (proposed by Dreyer), is that one changes the gauge group from $SU(2)$, which is the most natural one, to $SO(3)$. There are other explanations, too. For example, Corichi [15] presents an argument based on simple conservation principles to suggest that one can keep $SU(2)$ and still have consistency with QNM, while Ling and Zhang [16] propose to consider the supersymmetric extension of the theory.

All of these proposals and conjectures could be useless if one could not prove that the real part of the QN frequencies do approach $\frac{\ln 3}{2\pi}$ i.e., that the number $0.0437123$ in Nollert’s paper was exactly $\frac{\ln 3}{2\pi}$. This was accomplished by Motl [26] some months ago, using an ingenious technique, working with the continued fraction representation of the wave equation. Subsequently, Motl and Neitzke [26] have used it to compute the asymptotic values in the $D$-dimensional Reissner-Nordström geometry. Moreover, the first order corrections are also easily obtained, as has been showed by Musiri and Siopsis [27] for the four dimensional Schwarzschild geometry. Here we shall also generalize these first order corrections so that they can encompass a general $D$-dimensional Schwarzschild black hole. A body of work has been growing on the subject of computing highly damped QNMs, both in asymptotically flat and in de Sitter or anti-de Sitter spacetimes.

As an aside, we note that one of the most intriguing features of Motl and Neitzke’s technique is that the region beyond the event horizon, which never enters in the definition of quasinormal modes, plays an extremely important role. In particular, the singularity at $r = 0$ is decisive for the computation. At the singularity the effective potential for wave propagation blows up. Somehow, the equation, or the singularity, knows what we are seeking! It is also worth of note the following: for high frequencies, or at least frequencies with a large imaginary part, the important region is therefore $r = 0$ where the potential blows up, whereas for low frequencies, of interest for late-time tails [32], it is the other limit, $r \to \infty$ which is important.

The purpose of this paper is two-fold: first we want to enlarge and settle the results for the four dimensional Schwarzschild black hole. We shall first numerically confirm Nollert’s results for the gravitational case, and Berti and Kokkotas’ results for the scalar case, and see that the match between these and the analytical prediction is quite good. Then we shall fill-in a gap left behind in these two studies: the electromagnetic asymptotic QN frequencies, for which there are no numerical results, but a lot of analytical predictions. This will put the monodromy method on a firmer ground. One might say that in the
present context (of black hole quantization and relation with Loop Quantum Gravity) electromagnetic perturbations are not relevant, only the gravitational ones are. This is not true though, because when considering the Reissner-Nordström case gravitational and electromagnetic perturbations are coupled and it is therefore of great theoretical interest to study each one separately, in the case they do decouple, i.e., the Schwarzschild geometry. We find that all the analytical predictions are correct: scalar and gravitational QN frequencies asymptote to $\omega M = \ln(3 + i(2n + 1)\pi) with $T_H = \frac{1}{\sqrt{\pi M}}$ is the Hawking temperature. The corrections are well described by the existing analytical formulas. On the other hand, electromagnetic QN frequencies asymptote to $\omega M = 0 + 2n\pi$ with no $1/\sqrt{n}$ corrections, as predicted by Musiri and Siopsis. Indeed we find that the corrections seem to appear only at the $1/\sqrt{n}$ level.

The second aim of this paper is to establish numerically the validity of the monodromy method by extending the numerical calculation to higher dimensional Schwarzschild black holes, in particular to the five dimensional one. We obtain numerically in the five dimensional case the asymptotic value of the QN frequencies, which turns out to be $\ln(3 + i(2n + 1)\pi$ (in units of the black hole temperature), for scalar, gravitational tensor and gravitational vector perturbations. Equally important, is that the first order corrections appear as $1/n^{3/2}$ for these cases. We shall generalize Musiri and Siopsis’ method to higher dimensions and find that (i) the agreement with the numerical data is very good (ii) In generic $D$ dimensions the corrections appear at the $1/n^{D/2}$ level. Electromagnetic perturbations in five dimensions seem to be special: according to the analytical prediction by Motl and Neitzke, they should not have any asymptotic QN frequencies. Our numerics seem not to go higher enough in mode number to prove or disprove this. They indicate that the real part approaches zero, as the mode number increases, but we have not been able to follow the QN frequencies for very high overtone number. We also give complete tables for the lowest lying QN frequencies of the five dimensional black holes, and confirm previous results for the fundamental QN frequencies obtained using WKB-like methods. We find no purely imaginary QN frequencies for the gravitational QNMs, which may be related to the fact that the potentials are no longer superpartners.

II. QN FREQUENCIES OF THE FOUR-DIMENSIONAL SCHWARZSCHILD BLACK HOLE

The first computation of QN frequencies of four-dimensional Schwarzschild black holes was carried out by Chandrasekhar and Detweiler. They were only able to compute the lowest lying modes, as it was and still is quite hard to find numerically QN frequencies with a very large imaginary part. After this pioneer work, a number of analytical, semi-analytical and numerical techniques were devised with the purpose of computing higher order QNMs and to establish the low-lying ones using different methods. The most successful numerical method to study QNMs of black holes was proposed by Leaver. In a few words his method amounts to reducing the problem to a continued fraction, rather easy to implement numerically. Using this method Leaver was able to determine the first 60 modes of the Schwarzschild black hole, which at that time was about an order of magnitude better than anyone had ever gone before. The values of the QN frequencies he determined, and we note he only computed gravitational ones, suggested that there is an infinite number of QN modes, that the real part of the QN frequencies approach a finite value, while the imaginary part grows without bound. That the real part of the QN frequencies approach a finite value was however opened to debate: it seemed one still had to go higher in mode number in order to ascertain the true asymptotic behaviour. Nollert showed a way out of these problems: he was able to improve Leaver’s method in a very simple fashion so as to go very much higher in mode number: he was able to compute more than 2000 modes of the gravitational QNMs. His results were clear: the real part of the gravitational QN frequencies approaches a constant value $\omega M = 0.0437123 + \frac{i}{8}(2n + 1) + \frac{a}{\sqrt{n}}$, where the correction term $a$ depends on $l$. About ten years later, Motl proved analytically that this is indeed the correct asymptotic behaviour. To be precise he proved that

$$\frac{\omega}{T_H} = \ln(3 + i(2n + 1)\pi) with corrections. \quad (7)$$

Since the Hawking temperature $T_H$ is $T_H = \frac{1}{8\pi M}$, Nollert’s result follows. He also proved that this same result also holds for scalar QN frequencies, whereas electromagnetic QN frequencies asymptote to a zero real part. Motl and Neitzke, using a completely different approach, have rederived this result whereas Van den Brink was able to find the leading asymptotic correction term $a$ in for the gravitational case. Very recently, Musiri and Siopsis, leaning on Motl and Neitzke’s technique, have found the leading correction term for any field, scalar, electromagnetic and gravitational. This correction term is presented in section where we also generalize to higher dimensional black holes. In particular, for electromagnetic perturbations they find that the first order corrections are zero.
Here we use Nollert’s method to compute scalar, electromagnetic and gravitational QN frequencies of the four-dimensional Schwarzschild black hole. The details on the implementation of this method in four dimensions are well known and we shall not dwell on it here any more. We refer the reader to the original references [21, 29].

Our results are summarized in Figures 1-2 and Tables I-II, where we also show the analytical prediction. These values were obtained using the first 5000 modes.

| s = 0          | 1 | 2 | 3 | 4 |
|----------------|---|---|---|---|
| Corr$^A_0$    | 0 | 1.20-1.20i | 1.2190-1.2190i | 8.52-8.52i | 8.5332-8.5332i |
| Corr$^A_1$    | 1 | 23.15-23.15i | 23.1614-23.1614i | 8.5332-8.5332i |
|                | 2 | 44.99-44.99i | 45.1039-45.1039i | 23.1614-23.1614i |
|                | 3 | 74.34-74.34i | 74.3604-74.3605i | 45.1039-45.1039i |

We have also computed the correction terms for the scalar case, and we have obtained results in agreement with Berti and Kokkotas’ ones [25]. Our results are summarized in Tables I-II, where we also show the analytical prediction [23] (see section III B). To fix the conventions adopted throughout the rest of the paper, we write

\[
\frac{\omega}{T_H} = \ln 3 + i(2n + 1)\pi + \frac{\text{Corr}}{\sqrt{n}}. \tag{9}
\]

Thus, the correction term $a$ in (6) is related to Corr by $a = \text{Corr} \times T_H$.

As for the electromagnetic correction terms, we found it was not very easy to determine them. In fact, it is hard, even using Nollert’s method, to go very high in mode number for these perturbations (we have made it to $n = 5000$). The reason is tied to the fact that these frequencies asymptote to zero, and it is hard to determine QN frequencies with a vanishingly small real part. Nevertheless, there are some features one can be sure of. We have fitted the electromagnetic data to the following form

\[
\frac{\omega}{T_H} = i(2n + 1)\pi + \frac{\text{Corr}}{\sqrt{n}},
\]

and found that this gave very poor results. Thus one saw numerically that the first order correction term is absent, as predicted in [23] (see also section III B where we redefine these corrections, generalizing them to arbitrary dimension). Our data seems to indicate that the leading correction term is of the form

\[
\frac{k}{n^{3/2}}.
\]

This is more clearly seen in Fig. 2 where we plot the first electromagnetic QN frequencies on a ln plot. For frequencies with a large imaginary part, the slope is

\[
\frac{\omega}{T_H} = \ln 3 + i(2n + 1)\pi + \frac{\text{Corr}}{\sqrt{n}}. \tag{9}
\]
TABLE II: The correction coefficients for the four- dimensional Schwarzschild black hole, both numerical, here labeled as “Corr$^4_l$” and analytical, labeled as “Corr$^4_{A_l}$”. These results refer to gravitational perturbations. The analytical results are extracted from [29] (see also formula (29) below). Notice the very good agreement between the numerically extracted results and the analytical prediction. These values were obtained using the first 5000 modes.

| $l$ | Corr$^4_l$ | Corr$^4_{A_l}$ |
|-----|------------|---------------|
| 1   |            |               |
| 2   | 6.08-6.08i | 6.0951-6.0951i|
| 3   | 13.39-13.39i | 13.4092-13.4092i |
| 4   | 23.14-23.14i | 23.1614-23.1614i |
| 5   | 35.33-35.33i | 35.3517-35.3517i |
| 6   | 48.90-48.90i | 49.9799-49.9799i |

Our numerics indicate that $b$ has an $l$-dependence going like

$$b \sim c_1 l(l+1) + c_2 (l(l+1))^2 + c_3 [l(l+1)]^3,$$

where $c_1$, $c_2$, and $c_3$ are constants. This is also the expected behaviour, since a $1/l^{D-3}$ dependence means we have to go to third order perturbation theory, where we get corrections of the form $l^{D-3}$ as one can easily convince oneself.

### III. QN FREQUENCIES OF $D$-DIMENSIONAL SCHWARZSCHILD BLACK HOLES

#### A. Equations and conventions

The metric of the $D$-dimensional Schwarzschild black hole in $(t, r, \theta_1, \theta_2, \ldots, \theta_{D-2})$ coordinates is

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2_{D-2},$$

with

$$f = 1 - \frac{m}{r^{D-3}}.$$ 

The mass of the black hole is given by $M = \frac{\Omega_{D-2} m}{16\pi G}$, where $\Omega_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma[(D-1)/2]}$ is the area of a unit $(D-2)$ sphere, $d\Omega^2_{D-2}$ is the line element on the unit sphere $S^{D-2}$, and $G$ is Newton’s constant in $D$-dimensions. In the following it will prove more useful to rescale variables so that the form of the metric is (12) but with $f = 1 - \frac{1}{r^{D-3}}$, i.e., we shall choose $m = 1$, following the current fashion. We will only consider the linearized approximation, which means that we are considering wave fields outside this geometry that are so weak they do not alter this background. The evolution equation for a massless scalar field follows directly from the Klein-Gordon equation (see [43] for details). The gravitational evolution equations have recently been derived by Kodama and Ishibashi [44]. There are three kinds of gravitational perturbations, according to Kodama and Ishibashi’s terminology: the scalar gravitational, the vector gravitational and the tensor gravitational perturbations. The first two already have their counterparts in $D = 4$, which were first derived by Regge and Wheeler [45] and by Zerilli [46]. The tensor type is a new kind appearing in higher dimensions. However, it obeys exactly the same equation as massless scalar fields, as can easily be seen. Due to the complex form of the gravitational scalar potential, we shall not deal with it. Instead, we shall only consider the tensor and vector type of gravitational perturbations. In any case, if the analytic results are correct, then the gravitational scalar QN frequencies should have the same asymptotic form as the gravitational vector and tensor QN frequencies. The evolution equation for the electromagnetic field in the higher dimensional Schwarzschild geometry was arrived at for the first time by Crispino, Higuchi and Matsas [47]. It has recently been rederived by Kodama and Ishibashi, in a wider context of charged black hole perturbations. We shall follow Kodama and Ishibashi’s terminology. According to them, there are two kinds of electromagnetic perturbations: the vector and scalar type. If one makes the charge of the black hole $Q = 0$ in Kodama and Ishibashi’s equations one recovers the equations by Crispino, Higuchi and Matsas, although this seems to have been overlooked in the literature. The evolution equation for all kinds of fields (scalar, gravitational and electromagnetic) can be reduced to the second
order differential equation
\[
\frac{d^2 \Psi}{dr_+^2} + (\omega^2 - V) \Psi = 0, \tag{14}
\]
where \( r \) is a function of the tortoise coordinate \( r_+ \), defined through \( \frac{dr_+}{dr} = f(r) \), and the potential can be written in compact form as
\[
V = f(r) \left[ \frac{(l + D - 3)}{r^2} + \frac{(D - 2)(D - 4)}{4r^2} + \frac{(1 - j^2)(D - 2)^2}{4r^{D-1}} \right]. \tag{15}
\]
The constant \( j \) depends on what kind of field one is studying:
\[
j = \begin{cases} 
0, & \text{scalar and gravitational tensor perturbations.} \\
1, & \text{gravitational vector perturbations.} \\
\frac{2}{D-2}, & \text{electromagnetic vector perturbations.} \\
2 - \frac{2}{D-2}, & \text{electromagnetic scalar perturbations.}
\end{cases} \tag{16}
\]
Notice that in four dimensions, \( j \) reduces to the usual values given by Motl and Neitzke \[23\]. According to our conventions, the Hawking temperature of a \( D \)-dimensional Schwarzschild black hole is
\[
T_H = \frac{D - 3}{4\pi} \tag{17}
\]
Our purpose here is to investigate the QN frequencies of higher dimensional black holes. Of course one cannot study every \( D \). We shall therefore focus on one particular dimension, five, and make a complete analysis of its QN frequencies. The results should be representative. In particular, one feature that distinguishes an arbitrary \( D \) from the four dimensional case is that the correction terms come with a different power, i.e., whereas in four dimensions the correction is of the form \( \frac{1}{(D-3)/(D-2)} \), we shall find that in generic \( D \) it is of the form \( \frac{1}{\sqrt{\frac{(D-3)}{(D-2)}}} \). Thus, if one verifies this for \( D = 5 \) for example, one can ascertain it will hold for arbitrary \( D \). We shall now briefly sketch our numerical procedure for finding QN frequencies of a five dimensional black hole. This is just Leaver’s and Nollert’s technique with minor modifications.

The QNMs of the higher-dimensional Schwarzschild black hole are characterized by the boundary conditions of incoming waves at the black hole horizon and outgoing waves at spatial infinity, written as
\[
\Psi(r) \to \begin{cases} 
e^{-i\omega r_*} & \text{as } r_* \to \infty \\
e^{i\omega r_*} & \text{as } r_* \to -\infty
\end{cases}, \tag{18}
\]
where the time dependence of perturbations has been assumed to be \( e^{i\omega t} \).

B. Perturbative calculation of QNMs of \( D \)-dimensional Schwarzschild black holes

In this section we shall outline the procedure for computing the first order corrections to the asymptotic value of the QN frequencies of a \( D \)-dimensional Schwarzschild black hole. This will be a generalization of Musiri and Siopsis’ method \[22\], so we adopt all of their notation, and we refer the reader to their paper for further details. The computations are however rather tedious and the final expressions are too cumbersome, so we shall refrain from giving explicit expressions for the final result.

We start with the expansion of the potential \( V \) near the singularity \( r = 0 \). One can easily show that near this point the potential \( \text{eq} \) may be approximated as
\[
V \sim \frac{\omega^2}{4z^2} \left[ 1 - j^2 + \frac{A}{(z\omega)^{(D-3)/(D-2)}} \right], \tag{19}
\]
where we adopted the conventions in \[24\] and therefore
\[
z = \omega r_+. \tag{20}
\]
In \( D = 4 \) this reduces to the usual expression \[22\] \[23\] for the potential near \( r = 0 \). Expression \[19\] is a formal expansion in \( \frac{1}{\omega(D-3)/(D-2)} \), so we may anticipate that indeed the first order corrections will appear in the form \( \frac{1}{\omega(D-3)/(D-2)} \). So now we may proceed in a direct manner: we expand the wavefunction to first order in \( \frac{1}{\omega(D-3)/(D-2)} \) as
\[
\Psi = \Psi^{(0)} + \frac{1}{\omega(D-3)/(D-2)} \Psi^{(1)}, \tag{21}
\]
and find that the first-order correction obeys the equation
\[
\frac{d^2 \Psi^{(1)}}{dz^2} + (\frac{1 - j^2}{4z^2} + 1) \Psi^{(1)} = \omega(D-3)/(D-2) \delta V \Psi^{(0)}, \tag{22}
\]
with
\[
\delta V = -\frac{A}{\omega(D-3)/(D-2)(z\omega)^{(D-3)/(D-2)}}. \tag{23}
\]
All of Musiri and Siopsis’ expressions follow directly to the \( D \)-dimensional case, if one makes the replacement \( \sqrt{-\omega_0} \to \omega^{(D-3)/(D-2)} \) in all their expressions. For example, the general solution of \[22\] is
\[
\Psi^{(1)}_\pm = C \Psi^{(0)} \int_0^z \Psi^{(0)}_\pm \delta V \Psi^{(0)}_\pm - C \Psi^{(0)} \int_0^z \Psi^{(0)}_\pm \delta V \Psi^{(0)}_\pm, \tag{24}
\]
where
\[
C = \frac{\omega^{(D-3)/(D-2)}}{\sin j\pi/2}. \tag{25}
\]
and the wavefunctions $\Psi^{(0)}_\pm$ are

$$\Psi^{(0)}_\pm = \sqrt{\frac{\pi z}{2}} J_{\pm j/2}(z).$$  (26)

The only minor modification is their formula (30). In $D$-dimensions, it follows that

$$\Psi^{(1)} \propto z^{1+j/2+k} G_\pm(z),$$

where $k = \frac{D-4}{2(D-2)}$, and $G_\pm$ are even analytic functions of $z$. So we have all the ingredients to construct the first order corrections in the $D$-dimensional Schwarzschild geometry. Unfortunately the final expressions turn out to be quite cumbersome, and we have not managed to simplify them. We have worked with the symbolic manipulator Mathematica. It is possible to obtain simple expression for any particular $D$, but apparently not for a generic $D$. For generic $D$ we write

$$\frac{\omega}{T_H} = \ln(1 + 2 \cos \pi j) + i(2n+1)\pi + \frac{\text{Corr}_D}{\omega(D-3)/(D-2)}$$

(28)

So the leading term is $\frac{\omega}{T_H} = \ln 3 + i(2n+1)\pi$, for scalar, gravitational tensor and gravitational tensor perturbations (and the same holds for gravitational scalar). However, for electromagnetic perturbations (vector or scalar) in $D = 5$, $1 + 2 \cos \pi j = 1 + 2 \cos \pi = 0$. So there seem to be no QN frequencies as argued by Motl and Neitzke 23. We obtained for $D = 4$ the following result for the correction coefficient in (28),

$$\text{Corr}_4 = -\frac{(1-i)\pi^{3/2}(-1 + j^2 - 3(l + 1))\cos \pi}{3(1 + 2 \cos \pi \pi)\Gamma[3/4] \Gamma[3/4 - j/2] \Gamma[3/4 + j/2]}$$

(29)

which reduces to Musiri and Siopsis’ expression. For $D = 5$ it is possible also to find a simple expression for the corrections:

$$\text{Corr}_5 = \frac{i(9j^2 - (24 + 20l(l + 2)))\pi^{3/2}}{120 \times 3^{1/3} \Gamma[2/3] \Gamma[2/3 - j/2] \Gamma[2/3 + j/2]}$$

$$\times \Gamma[1/6] \frac{\sin((1/3+j/2)\pi)}{\sin((1/3+j/2)\pi)} - \frac{1}{1 + \epsilon^{i\pi}}.$$  (30)

The limits $j \to 0, 2$ are well defined and yield

$$\text{Corr}_{5, j=0} = \frac{(1-i\sqrt{3})(6 + 10l + 5l^2)\pi^{3/2}}{45 \times 3^{1/3} \Gamma[2/3]^3 \Gamma[1/6]},$$

(31)

$$\text{Corr}_{5, j=2} = \frac{(-1+i\sqrt{3})(-3 + 10l + 5l^2)\pi^{3/2}}{45 \times 3^{1/3} \Gamma[-1/3] \Gamma[2/3] \Gamma[5/3] \Gamma[1/6]},$$

(32)

We list in Tables VII-VIII some values of this five dimensional correction for some values of the parameter $j$ and $l$ and compare them with the results extracted numerically. The agreement is quite good. The code for extracting the generic $D$-dimensional corrections is available from the authors upon request. It should not come as a surprise that for $j = 2/3$ the correction term blows up: indeed already the zeroth order term is not well defined.

C. Numerical procedure and results

1. Numerical procedure

In order to numerically obtain the QN frequencies, in the present investigation, we make use of Nollert’s method 45, since asymptotic behaviors of QNMs in the limit of large imaginary frequencies are prime concern in the present study.

In our numerical study, we only consider the QNMs of the Schwarzschild black hole in the five dimensional spacetime, namely the $D = 5$ case. As we have remarked, this should be representative. The tortoise coordinate is then reduced to

$$r_* = x^{-1} + \frac{1}{2x_1} \ln(x - x_1) - \frac{1}{2x_1} \ln(x + x_1),$$  (33)

where $x = r^{-1}$ and $x_1 = r_h^{-1}$. Here, $r_h$ stands for the horizon radius of the black hole. The perturbation function $\Psi$ may be expanded around the horizon as

$$\Psi = e^{-i\omega r_*} (x - x_1)^\rho (x + x_1)^\eta \sum_{k=0}^{\infty} a_k \left( \frac{x - x_1}{x + x_1} \right)^k,$$

(34)

where $\rho = i\omega/2x_1$ and $a_0$ is taken to be $a_0 = 1$. The expansion coefficients $a_k$ in equation (34) are determined via the four-term recurrence relation (it’s just a matter of substituting expression (34) in the wave equation (14)), given by

$$a_0 a_1 + \beta_0 a_0 = 0,$$

$$a_1 a_2 + \beta_1 a_1 + \gamma_1 a_0 = 0,$$

$$\alpha_k a_{k+1} + \beta_k a_k + \gamma_k a_{k-1} + \delta_k a_{k-2} = 0, \quad k = 2, 3, \ldots,$$

where

$$\alpha_k = 2(2\rho + k + 1)(k + 1),$$

$$\beta_k = -5(2\rho + k)(2\rho + k + 1) - l(l + 2) - \frac{3}{4},$$

$$\gamma_k = 4(2\rho + k - 1)(2\rho + k + 1) - \frac{9}{4} (1 - j^2),$$

$$\delta_k = -(2\rho + k - 2)(2\rho + k + 1) - \frac{9}{4} (1 - j^2).$$

It is seen that since the asymptotic form of the perturbations as $r_* \to \infty$ is written in terms of the variable $x$ as

$$e^{-i\omega r_*} = e^{-i\omega x^{-1}} (x - x_1)^\rho (x + x_1)^\rho,$$

(36)

the expanded perturbation function $\Psi$ defined by equation (34) automatically satisfy the QNM boundary conditions if the power series converges for $0 \leq x \leq x_1$. Making use of a Gaussian elimination 45, we can reduce...
the four-term recurrence relation to the three-term one, given by
\begin{align*}
\alpha'_k a_1 + \beta'_0 a_0 &= 0, \\
\alpha'_k a_{k+1} + \beta'_k a_k + \gamma'_k a_{k-1} &= 0, \quad k = 1, 2, \ldots, (37)
\end{align*}
where \( \alpha'_k, \beta'_k, \) and \( \gamma'_k \) are given in terms of \( \alpha_k, \beta_k, \gamma_k \) and \( \delta_k \) by
\begin{align*}
\alpha'_k &= \alpha_k, \\
\beta'_k &= \beta_k - \alpha'_k \delta_k / \gamma'_k - 1, \\
\gamma'_k &= \gamma_k - \beta'_k \delta_k / \gamma'_k - 1, \quad \text{for } k \geq 2.
\end{align*}
(39)

Now that we have the three-term recurrence relation for determining the expansion coefficients \( a_k \), the convergence condition for the expansion [44], namely the QNM conditions, can be written in terms of the continued fraction as [33, 54, 55, 56]
\begin{equation}
\beta'_0 - \frac{\alpha'_0 \gamma'_1 \alpha'_2 \alpha'_3 \ldots}{\beta'_1 - \beta'_2 - \beta'_3 - \ldots} = 0, (40)
\end{equation}
where the first equality is a notational definition commonly used in the literature for infinite continued fractions. Here we shall adopt such a convention. In order to use Nollert’s method, with which relatively high-order QNMs with large imaginary frequencies can be obtained, we have to know the asymptotic behaviors of \( a_{k+1}/a_k \) in the limit of \( k \to \infty \). According to a similar consideration as that by Leaver [49], it is found that
\begin{equation}
\frac{a_{k+1}}{a_k} = 1 \pm 2 \sqrt{\rho} k^{-1/2} + \left(2 \rho - \frac{3}{4}\right) k^{-1} + \cdots, \quad (41)
\end{equation}
where the sign for the second term in the right-hand side is chosen so as to be
\begin{equation}
\text{Re}(\pm 2 \sqrt{\rho}) < 0. \quad (42)
\end{equation}

In actual numerical computations, it is convenient to solve the \( k \)-th inversion of the continue fraction equation [40], given by
\begin{align*}
\beta'_k &= \frac{\alpha'_k \gamma'_k a_{k+1} \gamma'_k+2}{\beta'_k+1 - \beta'_k+2 - \ldots}, \\
&= \frac{\alpha'_k a_{k+1} \gamma'_k+2}{\beta'_k+1 - \beta'_k+2 - \ldots}, \quad (43)
\end{align*}
The asymptotic form [44] plays an important role in Nollert’s method when the infinite continued fraction in the right-hand side of equation [43] is evaluated [5].

Table III: The first lowest QN frequencies for scalar and gravitational tensor \((j = 0)\) perturbations of the five dimensional Schwarzschild black hole. The frequencies are normalized in units of black hole temperature, so the Table really shows \( \frac{\omega}{T_H} \), where \( T_H \) is the Hawking temperature of the black hole.

| \( n \) | \( l = 0 \) | \( l = 1 \) | \( l = 2 \) |
|-------|---------|-------|-------|
| 0     | 3.3539 + 2.4089i | 6.3837 + 2.2764i | 9.4914 + 2.2462i |
| 1     | 2.3367 + 8.3101i | 5.3809 + 7.2734i | 8.7506 + 6.9404i |
| 2     | 1.8868 + 14.786i | 4.1683 + 13.252i | 7.5009 + 12.225i |
| 3     | 1.6927 + 21.219i | 3.4011 + 19.708i | 6.2479 + 18.214i |
| 4     | 1.5839 + 27.607i | 2.9544 + 26.215i | 5.3149 + 24.597i |

2. Numerical results

Using the numerical technique described in section [31, 52], we have extracted the 5000 lowest lying QN frequencies for the five dimensional Schwarzschild black hole, in the case of scalar, gravitational tensor and gravitational vector perturbations. For electromagnetic vector perturbations the situation is different: the real part rapidly approaches zero, and we have not been able to compute more than the first 30 modes.

(i) Low-lying modes

Low-lying modes are important since they govern the intermediate-time evolution of any black hole perturbation. As such they may play a role in TeV-scale gravity scenarios [51], and higher dimensional black hole formation [34, 52]. For example, it is known [34, 53, 54] that if one forms black holes through the high energy collision of particles, then the fundamental quasinormal frequencies serve effectively as a cutoff in the energy spectra of the gravitational energy radiated away. In Tables [III, IV] we list the five lowest lying QN frequencies for some values of the multipole index \( l \). The fundamental modes are in excellent agreement with the ones presented by Konoplya [33, 34] using a high-order WKB approach. Notice he uses a different convention so one has to be careful when comparing the results. For example, in our units Konoplya obtains for scalar and tensorial perturbations \((j = 0)\) with \( l = 2 \) a fundamental QN frequency \( \omega_0/T_H = 9.49089 + 2.24721i \), whereas we get, from Table [III] the number \( 9.4914 + 2.2462i \) so the WKB approach does in fact yield good results, at least for low-lying modes, since it is known it fails for high-order ones.

In the four dimensional case, and for gravitational vector perturbations, there is for is each \( l \), a purely imaginary QN frequency, which had been coined an “algebraically special frequency” by Chandrasekhar [33]. For further properties of this special frequencies we refer the reader to [55]. The existence of these purely imaginary frequencies translates, in the four dimensional case,
TABLE IV: The first lowest QN frequencies for gravitational vector \((j = 2)\) perturbations of the five dimensional Schwarzschild black hole. The frequencies are normalized in units of black hole temperature, so the Table really shows \(\omega / T_H\), where \(T_H\) is the Hawking temperature of the black hole.

| \(n\) | \(l = 2\) | \(l = 3\) | \(l = 4\) |
|---|---|---|---|
| 0 | 7.1251 + 2.0579i | 10.8408 + 2.0976i | 14.3287 + 2.1364i |
| 1 | 5.9528 + 6.4217i | 8.8819 + 1.0929i | 12.8506 + 1.0574i |
| 2 | 3.4113 + 12.9581i | 5.9528 + 6.4217i | 10.7694 + 11.6626i |
| 3 | 2.5106 + 26.2625i | 3.9426 + 28.6569i | 8.5638 + 27.4339i |
| 4 | 2.8362 + 19.3422i | 5.9190 + 16.3979i | 9.4316 + 16.0274i |

TABLE V: The first lowest QN frequencies for electromagnetic vector \((j = 2/3)\) perturbations of the five dimensional Schwarzschild black hole. The frequencies are normalized in units of black hole temperature, so the Table really shows \(\omega / T_H\), where \(T_H\) is the Hawking temperature of the black hole.

| \(n\) | \(l = 1\) | \(l = 2\) | \(l = 3\) |
|---|---|---|---|
| 0 | 5.9862 + 2.2038i | 9.2271 + 2.2144i | 12.4184 + 2.2177i |
| 1 | 4.9300 + 7.0676i | 8.4728 + 6.8596i | 11.8412 + 6.7660i |
| 2 | 3.6588 + 12.9581i | 7.1960 + 12.0804i | 10.7694 + 11.6626i |
| 3 | 2.8362 + 19.3422i | 5.9190 + 16.3979i | 9.4316 + 16.0274i |

a relation between the two gravitational wavefunctions (i.e., between the Regge-Wheeler and the Zerilli wavefunction, or between the gravitational vector and gravitational scalar wavefunctions, respectively). It is possible to show for example that the associated potentials are related through supersymmetry. Among other consequences, this relation allows one to prove that the QN frequencies of both potentials are exactly the same [33].

We have not spotted any purely imaginary QN frequency for this five-dimensional black hole. In fact the QN frequency with the lowest real part is a gravitational vector QN frequency with \(l = 4\) and overtone number \(n = 13\), \(\omega = 8.7560 \times 10^{-2} + 6.9934i\). This may indicate that in five dimensions, there is no relation between the wavefunctions, or put another way, that the potentials are no longer superpartner potentials. This was in fact already observed by Kodama and Ishibashi [44] for any dimension greater than four. It translates also in Tables IIIIV which yield different values for the QN frequencies. Although we have not worked out the gravitational scalar QN frequencies, a WKB approach can be used for the low-lying QN frequencies, and also yields different values [33, 32].

(ii) Highly damped modes

Our numerical results for the highly damped modes, i.e., QN frequencies with a very large imaginary part, are summarized in Figures 3 and in Tables VI-VII.

In Fig. 3 we show our results for scalar, gravitational tensor \((j = 0)\), gravitational vector \((j = 2)\) and electromagnetic vector \((j = 2/3)\) QN frequencies. For \(j = 0\), the real part approaches \(\ln 3\) in units of the black hole temperature, a result consistent with the analytical result in [22] (see also section III B). Electromagnetic vector \((j = 2/3)\) QN frequencies behave differently: the real part rapidly approaches zero, and this makes it very difficult to compute the higher modes. In fact, the same kind of problem appears in the Kerr geometry [23], and this is a major obstacle to a definitive numerical characterization of the highly damped QNMs in this geometry. We have only been able to compute the first 30 modes with accuracy. The prediction of [22] for this case (see section III B) is that there are no asymptotic QN frequencies. It is left as an open question whether this is true or not. Our results indicate that the real part rapidly approaches zero, but we cannot say whether the modes die there or not, or even if they perform some kind of oscillation. The imaginary part behaves in the usual manner, growing linearly with mode number.

Let us now take a more detailed look at our numerical data for scalar and gravitational QN frequencies. In Fig. 4 we show in a \(\ln\) plot our results for scalar and gravitational tensor \((j = 0)\) QN frequencies.

Our numerical results are very clear: asymptotically the QN frequencies for scalar and gravitational tensor perturbations \((j = 0)\) behave as

\[
\frac{\omega}{T_H} = \ln 3 + i (2n + 1)\pi + \frac{\text{Corr}_n}{n^{2/3}},
\]

\[(44)\]
FIG. 4: The QN frequencies of scalar and gravitational tensor perturbations \((j = 0)\) of a five-dimensional Schwarzschild black hole. Notice that for frequencies with a large imaginary part the slope is \(l\)-independent and equals \(3/2\).

TABLE VI: The correction coefficients for the five-dimensional Schwarzschild black hole, both numerical, here labeled as \(\text{Corr}^N_5\) and analytical, labeled as \(\text{Corr}^A_5\). These results refer to scalar or gravitational tensor perturbations \((j = 0)\). The analytical results are extracted from the analytical formula (31). Notice the very good agreement, to within 0.5% or less, between the numerically extracted results and the analytical prediction.

| \(l\) | \(\text{Corr}^N_5\) | \(\text{Corr}^A_5\) |
|------|----------------|-----------------|
| 0    | 1.155 - 1.993i | 1.15404-1.99886i |
| 1    | 4.058 - 6.956i | 4.03915-6.99601i |
| 2    | 8.894 - 15.26i | 8.84765-15.3246i |
| 3    | 15.64 - 26.79i | 15.5796-26.9847i |
| 4    | 24.33 - 41.67i | 24.2349-41.9761i |

So the leading terms \(\ln 3 + (2n + 1)i\) are indeed the ones predicted in [23]. Interestingly, the first corrections do not appear as \(1/\sqrt{n}\), but as \(1/n^{2/3}\). This was shown to be the expected analytical result in section III B where we generalized Musiri and Siopsis’ [29] results to higher dimensions. In Table VI we show the coefficient \(\text{Corr}^N_5\) extracted numerically along with the predicted coefficient (see expression (31)). The table is very clear: the numerical values match the analytical ones. Another confirmation that the corrections appear as \(1/n^{2/3}\) is provided by Fig. 4.

In this figure the QN frequencies are plotted in a ln plot for an easier interpretation. One sees that for large imaginary parts of the QN frequencies, the slope of the plot is approximately \(-3/2\), as it should be if the corrections are of the order \(1/n^{2/3}\).

In Fig. 5 we show in a ln plot our results for gravitational vector \((j = 2)\) QN frequencies. Again, vector QN frequencies have the asymptotic behaviour given by expression (32), with a different correction term \(\text{Corr}^N_5\). Again, the corrections show themselves in the ln plot of Fig. 5: for very large imaginary parts, the slope is \(-3/2\), as it should. The numerically extracted coefficient \(\text{Corr}^N_5\) for \(j = 2\) perturbations is listed in Table VII along with the analytically predicted value (see section III B). The agreement is very good.

IV. DISCUSSION OF RESULTS AND FUTURE DIRECTIONS

We have made an extensive survey of the QNMs of the four and five dimensional Schwarzschild black hole. The investigation presented here makes a more complete characterization of the highly damped QNMs in the Schwarzschild geometry. In the four-dimensional case, we confirmed previous numerical results regarding the scalar and gravitational asymptotic QN frequencies.
We found that both the leading behaviour and the first order corrections for the scalar and gravitational perturbations agree extremely well with existing analytical formulas. We have presented new numerical results concerning electromagnetic QN frequencies of the four-dimensional Schwarzschild geometry. In particular, this is the first work dealing with highly damped electromagnetic QNMs. Again, we find that the leading behaviour and the first order corrections agree with the analytical calculations. The first order corrections appear at the $\frac{1}{T^2}$ level. In the five-dimensional case this represents the first study on highly damped QNMs. We have seen numerically that the asymptotic behaviour is very well described by $\frac{1}{T^2} = \ln 3 + \pi i (2n + 1)$ for scalar and gravitational perturbations, and agrees with the predicted formula. Moreover the first order corrections appear at the level $\frac{1}{T^3}$, which is also very well described by the analytical calculations, providing one more consistency check on the theoretical framework. In generic $D$ dimensions the corrections appear at the $\frac{1}{T^{D-(3)/2}}$ level. We have been able to prove or disprove the analytic result for electromagnetic QN frequencies in five dimensions ($j = 2/3$). Other conclusions that can be taken from our work are: the monodromy method by Motl and Neitzke is correct. It is important to test it numerically, as we did, since there are some ambiguous assumptions in that method. Moreover, their method is highly flexible, since it allows an easy computation of the correction terms, as shown by Musiri and Siopsis, and generalized here for the higher dimensional Schwarzschild black hole. We have basically showed numerically that the monodromy method by Motl and Neitzke is correct and its extension by Musiri and Siopsis, to include for correction terms in overton number, are excellent techniques to investigate the highly damped QNMs of black holes.

Highly damped QNMs of black holes are the bedrocks that the recent conjectures [12, 13, 10] are built on, relating these to black hole area quantization. However, to put on solid ground the conjecture that QNMs can actually be of any use to black hole area quantization, one has to do much better then one has been able to do up to now. In particular, it is crucial to have a deeper understanding of the Kerr and Reissner-Nordström QNMs. These are, next to the Schwarzschild, the simpler asymptotically flat geometries. If these conjectures hold, they must to do also for these spacetimes. Despite the fact that there are convincing numerical results for the Kerr geometry [30], there seems to be for the moment no serious analytical investigation of the highly damped QNMs for this spacetime. Moreover, even though we are in possession of an analytical formula for the highly damped QNMs of the Reissner-Nordström geometry [24], and this has been numerically tested already [25], we have no idea what it means! In particular, can one use it to quantize the black hole area? In this case the electromagnetic perturbations are coupled to the gravitational ones. How do we proceed with the analysis, that was so simple in the Schwarzschild geometry? Is the assumption of equally spaced eigenvalues a correct one, in this case? Can these ideas on black hole area quantization be translated to non-asymptotically flat spacetimes, as the de Sitter or anti-de Sitter spacetime? These are the fundamental issues that remain to be solved in this field. Should any of them be solved satisfactorily, then these conjectures would gain a whole new meaning. As they stand, it may just be a numerical coincidence that the real part of the QN frequency goes to $\ln 3$.

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