1. PCF theory and singular cardinals

The abbreviation “PCF” stands for “Possible Cofinalities”. PCF theory was invented by Saharon Shelah to prove upper bounds on exponents of singular cardinals.

The starting point of PCF theory is in the realization that the usual exponent function is too coarse for measuring the power set of singular cardinals.

Consider the cardinal $\aleph_\omega$, which is the smallest singular cardinal and has countable cofinality. The usual exponent $\left(\aleph_\omega\right)^{\aleph_0}$ measures the total number of countable subsets of $\aleph_\omega$ and is larger than $\aleph_\omega$ itself by straightforward diagonalization. The exponent $\left(\aleph_\omega\right)^{\aleph_0}$ clearly satisfies $\left(\aleph_\omega\right)^{\aleph_0} \geq 2^{\aleph_0}$ and therefore has no upper bound in the list $\{\aleph_\alpha : \alpha \in \text{On}\}$ of cardinal numbers (because $2^{\aleph_0}$ itself has none).

The main conceptual change that PCF theory has generated lies in the fact that the number of countable subsets of $\aleph_\omega$ which is required to cover all countable subsets of $\aleph_\omega$ is always bounded by $\aleph_\omega^4$.

**Definition 1.** Cov($\aleph_\omega$, $\omega$) is the least number of countable subsets of $\aleph_\omega$ required to cover all countable subsets of $\aleph_\omega$. Equivalently, Cov($\aleph_\omega$, $\omega$) is the cofinality of the partially ordered set $([\aleph_\omega]^{\aleph_0}, \subseteq)$.

Fixing a covering collection $\mathcal{F} \subseteq [\aleph_\omega]^{\aleph_0}$ of cardinality Cov($\aleph_\omega$, $\omega$) and replacing each $A \in \mathcal{F}$ by $[A]^{\aleph_0}$ one gets the following equality:

$$\left(\aleph_\omega\right)^{\aleph_0} = \text{Cov}(\aleph_\omega, \omega) \times 2^{\aleph_0}.$$

**Theorem 2** (Shelah). Cov($\aleph_\omega$, $\omega$) $< \aleph_{\omega_4}$.

The factor $2^{\aleph_0}$ in $\left(\aleph_\omega\right)^{\aleph_0}$ is not bounded; the factor Cov($\aleph_\omega$, $\omega$) is. The number Cov($\aleph_\omega$, $\omega$) cannot be changed by ccc forcing (since every new countable set will be contained in an old countable set) and, furthermore, to make Cov($\aleph_\omega$, $\omega$) greater than $\aleph_{\omega+1}$, its minimal possible value, requires the consistency of large cardinals.

From this point of view, the power $2^{\aleph_0}$ of the regular cardinal $\aleph_0$ generates some “noise” in the collection $[\aleph_\omega]^{\aleph_0}$, whose important properties are captured by the robust factor Cov($\aleph_\omega$, $\omega$).
2. Reduced products and scales

The way PCF theory computes covering numbers of singular cardinals is by reducing them to the algebra of reduced products (modulo ideals) of sets of regular cardinals $A \subseteq \text{Reg}$ with the condition $\min A > |A|$. For such a set $A$, the set of possible cofinalities is defined by:

$$\text{Pcf} A = \{ \text{cf} \prod A/U : U \subseteq \mathcal{P}(A) \text{ an ultrafilter} \}.$$ 

Reduced products of the form $\prod A/I$ where $A \subseteq \text{Reg}$ with $\min A > A$ modulo an ideal $I$ over $A$ behave very differently from products of $\lambda$ copies of a regular cardinal $\lambda$ modulo some ideal (e.g. $(\omega^\omega, <^*)$). The PCF theorem is the fundamental structure theorem for products $\prod A/I$ for such $A \subseteq \text{Reg}$. It is of course worth pointing out the fact that there exists a structure theorem at all in this case!

The PCF theorem states the existence of PCF generators. It shows how each of the possible cofinalities in $\text{Pcf} A$ can be represented as a scale:

**Theorem 3** (Shelah). For every $A \subseteq \text{Reg}$ with $\min A > A$ and every $\lambda \in \text{Pcf} A$ there exists a set $B_\lambda \subseteq A$ called the generator of $\lambda$ so that: the product $\prod B_\lambda$ modulo the ideal $\langle B_\theta : \theta \in \text{Pcf} A \land \theta < \lambda \rangle$ has an increasing and cofinal sequence of length $\lambda$.

In a less technical formulation: there is a sequence of ideals over $A$, each generated over the union of previous ones by a single set $B_\lambda$, $\lambda \in \text{Pcf} A$, so that the cardinal $\lambda \in \text{Pcf} A$ is represented as the true cofinality of the product of $B_\lambda$ over the ideal $J_{<\lambda}$ generated by all smaller generators. A product has true cofinality if it contains an increasing and cofinal sequence (which is, e.g. the situation in $(\omega^\omega, <^*)$ when $b = \emptyset$).

The witness to the true cofinality of $\prod B_\lambda/J_{<\lambda}$ is a sequence $f_\lambda = \langle f_\alpha : \alpha < \lambda \rangle$ which is $<_{J_{<\lambda}}$-increasing and cofinal in $\prod B_\lambda/J_{<\lambda}$. Such a sequence is called a $\lambda$-scale. Among the first important discoveries of PCF theory was the fact that there is always an $\aleph_{\omega+1}$-scale.

**Theorem 4** (Shelah). There exists an infinite set $B \subseteq \omega$ and a sequence $f = \langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle \subseteq \prod_{n \in B} \omega_n$ so that:

1. $\alpha < \beta < \aleph_{\omega+1} \Rightarrow f_\alpha <^* f_\beta$;
2. for all $f \in \prod_{n \in B} \omega_n$ there exists $\alpha < \aleph_{\omega+1}$ for which $f <^* f_\alpha$.

The relation $f <^* g$ means: $f(n)$ is strictly smaller than $g(n)$ in all but finitely many $n$s.

**Theorem 5** (Shelah). $\text{Pcf}\{\aleph_n : n\}$ is an interval of regular cardinals with a last element $\max \text{Pcf}\{\aleph_n : n < \omega\}$ and $\max \text{Pcf}\{\aleph_n : n < \omega\}$ is equal to $\text{Cov}(\aleph_\omega, \omega)$.

3. PCF theory and topology

As every other important development in set theory, also PCF theory has consequences in topology. The theory is useful in constructing new
spaces and understanding old space whose properties are related to singular cardinals. More importantly, the new way of seeing things which is suggested by PCF theory is helpful in phrasing new theorems.

A good example to an application of PCF theory in topology is related to M. E. Rudin’s Dowker space $X^R$, whose cardinality is $(\aleph_\omega)^{\aleph_0}$. Recall the definition of $X^R$:

$$X^R = \{ f \in \prod_{n>1} \omega_n + 1 : (\exists m)(\forall n > 1)(\aleph_0 < \text{cf} f(n) < \aleph_m) \}.$$  

The topology on $X^R$ is the box topology.

It turns out that neither of the two properties of $X^R$—collectionwise normality and the absence of countable paracompactness—has anything to do with the factor $2^{\aleph_0}$ in $(\aleph_\omega)^{\aleph_0}$. The fact that $X^R$ is Dowker is related only to the structure of PCF scales in $\prod_{n>1} \omega_n$.

Fix a continuous $\aleph_{\omega+1}$-scale $\mathbf{f} = \langle f_\alpha : \alpha < \aleph_{\omega+1} \rangle$ by Theorem 4. “Continuous” means that whenever $(f_\alpha : \alpha < \delta)$ for $\delta < \aleph_{\omega+1}$ with $\text{cf} \delta > \aleph_0$ has a least upper bound, then $f_\delta$ itself is such a least upper bound. A continuous scale is gotten easily from any scale.

Define the following subspace of $X^R$:

$$X = \{ f \in X^R : (\exists \alpha < \aleph_{\omega+1})(f =^* f_\alpha) \}.$$  

The space $X$ has cardinality $\aleph_{\omega+1}$, local character $\aleph_{\omega}$, and weight $\aleph_{\omega+1}$ even when the respective characteristics of $X^R$ are much larger than $\aleph_{\omega+1}$. But just like $X^R$, $X$ is collectionwise normal and not countably paracompact. Thus:

**Theorem 6** (Kojman, Shelah [7]). There exists a Dowker space of cardinality and weight $\aleph_{\omega+1}$.

Another example of an application of PCF techniques to topology is the computation of the Baire number of the space of all uniform ultrafilters over a singular cardinal of countable cofinality.

**Theorem 7** (Kojman, Shelah [8]). If $\mu$ is the a singular cardinal of countable cofinality, then exactly $\aleph_2$ nowhere-dense sets are required to cover the space of all uniform ultrafilters over $\mu$. An ultrafilter over $\mu$ is uniform if it does not contain a set of cardinality $< \mu$.

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