NONSINGULAR RICCI FLOW ON A NONCOMPACT MANIFOLD IN DIMENSION THREE

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Abstract. We consider the Ricci flow $\frac{\partial}{\partial t} g = -2Ric$ on the 3-dimensional complete noncompact manifold $(M, g(0))$ with non-negative curvature operator, i.e., $Rm \geq 0$, $|Rm(p)| \to 0$, as $d(o, p) \to 0$. We prove that the Ricci flow on such a manifold is nonsingular in any finite time.

1. Introduction

The aim of this paper is to get a global existence of Ricci flow with bounded non-negative curvature operator in three dimensions. This kind of question was asked by Hamilton [9]. We remark that the local existence of the flow was obtained by Shi [12]. So we only need to show that the curvature is bounded in finite time. Our research is based on previous important results obtained by Hamilton and Perelman ([9], [3], and [4]), which will be recalled in next section.

The Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$$

was first introduced by Richard Hamilton [10]. Using it, Hamilton has obtained some interesting theorem, such as [10]. Hamilton’s program is to prove Poincaré conjecture and Thurston’s geometrization conjecture by Ricci flow. In three remarkable papers [6], [7], [8], Perelman significantly advanced the theory of the Ricci flow. Perelman introduced canonical neighborhood, and analyze the high curvature region. Perelman also analyzed one of the special solution to the Ricci flow, $\kappa$ solution, which is usually the limit solution of the blow up sequence. Before Perelman, Hamilton [9] had defined asymptotic volume and obtained that the asymptotic volume is constant under Ricci flow. By a induction argument, Perelman obtained that the asymptotic volume is zero when the solution is $\kappa$ solution. In order to analyze the high curvature region, Hamilton obtained some compactness of Ricci flow [11]. But in order to apply this compactness, one has to check the non-collapse and curvature bound assumption. We can use these ideas to prove the existence of a non-singular Ricci flow on a 3-manifold.

Our main result is the following

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Theorem 1.1. Assume that \((M, g(t)), t \in [0, T]\) is a Ricci flow on 3-dimensional complete noncompact Riemannian manifold. Suppose \(Rm(0) \geq 0, |Rm(p, 0)| \to 0, d(o, p) \to \infty\). Then \(T = \infty\), i.e Ricci flow is nonsingular in finite time on such a manifold.

We now give an interesting example.

Example 1.2. Consider the paraboloid of revolution \(x_4 = x_1^2 + x_2^2 + x_3^2, (x_1, \cdots, x_4) \in \mathbb{R}^4\). We know the curvature \(Rm(x) \to 0, x \to \infty\) and \(Rm > 0\). So Ricci flow on it will not blow up in finite time. We also know that the asymptotic volume is \(0\), so the paraboloid can't converge uniformly to flat \(\mathbb{R}^3\). We also note that when the infinite of the manifold is asymptotical to a cone, the Ricci flow doesn't blow up in finite time. But this time the asymptotic volume is between \(0\) and \(\omega\), where \(\omega\) is the volume of standard ball \(B(0, 1) \subset \mathbb{R}^3\).

We remark that in the radial symmetrical case, a similar result was obtained in [5], where another assumption such as the asymptotic flatness was used. One may also see our previous work [2] for more results in this direction.

2. Some famous results

In this section, we recall the following results of Hamilton and Perelman, which will be used in our proof of Theorem 1.1.

By monotonicity of \(W\) functional and reduced volume, Perelman proved twice the non-collapse of Ricci flow. Furthermore Perelman obtained the following convergence theorems about Ricci flow, which we will use many times (see [6], [3], and [4]).

Theorem 2.1. Fix canonical neighborhood constance \((C, \epsilon)\), and a non-collapsing constance \(r > 0, \kappa > 0\). Let \((\mathcal{M}_n, G_n, x_n)\) be a sequence of based generalized 3-dimensional Ricci flows. We set \(t_n = t(x_n)\) and \(Q_n = R(x_n)\). We denote by \(M_n\) the time \(t_n\) time slice of \(\mathcal{M}_n\). We suppose that:

1. Each \((\mathcal{M}_n, G_n)\) has time interval of definition contained in \([0, \infty)\) and has curvature pinched toward positive.
2. Every point \(y_n \in (\mathcal{M}_n, G_n)\) with \(t(y_n) \leq t_n\) and \(R(y_n) \geq 4R(x_n)\) has a strong \((C, \epsilon)\) canonical neighborhood.
3. \(\lim_{n \to \infty} Q_n = \infty\).
4. For each \(A < \infty\) the following holds for all \(n\) sufficiently large. The ball \(B(x_n, t_n, AQ_n^{-\frac{1}{2}})\) has compact closure in \(M_n\) and the flow is \(\kappa\) non-collapsed on scales \(\leq r\) at each point of \(B(x_n, t_n, AQ_n^{-\frac{1}{2}})\).
5. There is \(\mu > 0\) such that for every \(A < \infty\) the following holds for all \(n\) sufficiently large. For \(y_n \in B(x_n, t_n, AQ_n^{-\frac{1}{2}})\) the maximum flow line through \(y_n\) extends backwards for a time at least \(\mu(\max(Q_n, R(y_n)))^{-1}\).
Then after passing to a subsequence and shifting the times of each the Ricci flow so that \( t_n = 0 \) for every \( n \), there is a geometry limit \((M_\infty, g_\infty, x_\infty)\) of the sequence of based Riemannian manifolds \((M_n, Q_n G_n(0), x_n)\). The limit is a complete 3-dimensional Riemannian manifold of bounded, non-negative curvature. Furthermore, for some \( t_0 > 0 \) which depends on the curvature bound for \((M_\infty, g_\infty)\), there is a geometric limiting Ricci flow defined on \((M_\infty, g_\infty(t)), -t_0 \leq t \leq 0\), with \( g_\infty(0) = g_\infty \).

**Theorem 2.2.** Suppose that \( \{\mathcal{M}_n, Q_n, x_n\}_{n=1}^\infty \) is a sequence of 3-dimensional Ricci flow satisfying all the hypothesis of above theorem. Suppose in addition that there is \( T_0 \) with \( 0 < T_0 \leq \infty \) such that the following holds. For any \( T < T_0 \), for each \( A < \infty \), and all \( n \) sufficiently large, there is an embedding \( B(x_n, t_n, AQ_n^{-\frac{1}{2}}) \times (t_n - TQ_n^{-1}, t_n) \) into \( \mathcal{M}_n \) compatible with time and with the vector field and at every point of the image the flow is \( \kappa \) non-collapsed on scales \( \leq r \). Then, after shifting the times of the generalized flows so that \( t_n = 0 \) for all \( n \) and passing to a subsequence there is a geometric limiting Ricci flow \((M_\infty, g_\infty(t), x_\infty)\), \(-T_0 < t \leq 0\), for the rescaled flows \( (\mathcal{M}_n, Q_n G_n, x_n)\). This limiting flow is complete and of nonnegative curvature. Furthermore, the curvature is locally bounded on time. If in addition \( T_0 = \infty \), then it is a \( \kappa \) solution.

In this paper, we also need the following result of Hamilton on Ricci flow. By using barrier functions, Hamilton has proved the asymptotically property is preserved under Ricci flow (see [9]).

**Theorem 2.3.** Assume that we have a solution to the Ricci flow on a complete noncompact manifold with bounded curvature. If \( |Rm(p, 0)| \to 0, d(g_0)(o, p) \to \infty \), this remains true for \( t \geq 0 \).

**Remark 2.4.** One may see [2] for an improvement of this result.

### 3. Proof of Theorem 1.1

In the following, we will consider the 3-dimensional Ricci flow on \((M, g(0))\), with \( Rm \geq 0 \), \( |Rm(p)| \to 0 \), as \( d(o, p) \to \infty \), where \( o \) is a fixed point. In order to applied the above convergence theorem, we first show two lemmas. The method is from Perelman’s famous papers (see [6], [7]). But the condition is different from us here.

**Lemma 3.1.** For sufficiently small \( r > 0 \), there is \( \kappa > 0 \) such that the Ricci flow on this noncompact manifold is \( \kappa \) non-collapse on scales \( \leq r \).

**Proof.** Fix \((x, t_0) \in M \times [0, T)\). Since \( Rm \geq 0 \), Shi has proved that it’s preserved by Ricci flow [12]. By using a well known result of Gromoll and Meyer [1], we have an injectivity radius estimate

\[
inj(M^n, g(t)) \geq \frac{\pi}{\sqrt{R_{\text{max}}(t)}}.
\]
Since 
\[ R(x, t) < C_1, \quad (x, t) \in M \times [0, \frac{1}{2}t_0], \]
by the above injective estimate, we have
\[ \text{Vol}_B(x, t, r) \geq V' r^3, \quad (x, t) \in M \times [0, \frac{1}{2}t_0]. \]
where \( V' \) is a constant. By the inequality of reduced length,
\[ \frac{\partial l_x(q, \tau)}{\partial \tau} + \Delta l_x(q, \tau) \leq \frac{(\frac{n}{2}) - l_x(q, \tau)}{\tau}, \]
we know there is a point \((\tilde{q}, \tilde{\tau})\), \(\tilde{\tau} = \frac{1}{4}t_0\), such that \( l_x(\tilde{q}, \tilde{\tau}) \leq \frac{3}{2} \), where \( \tilde{\tau} = t_0 - \tilde{\tau} \). By the inequality (see [6] and [3])
\[ |\nabla l_x(q, \tau)|^2 \leq |\nabla l_x(q, \tau)|^2 + R(q, \tau) \leq \frac{(1 + 2n)l_x(q, \tau)}{\tau}, \]
we have that
\[ l_x(q, \tilde{\tau}) \leq \left( \sqrt{\frac{2n + 1}{2}d_{g(t_0 - \tilde{\tau})}(q, \tilde{q})} + \sqrt{\frac{n}{2}} \right)^2, \]
So, for any \( A < \infty \), we have \( l_x(q, \tilde{\tau}) < C(A) \), when \((q, t_0 - \tilde{\tau}) \in B(q, t_0 - \tilde{\tau}, A)\), where \( C(A) \) is a constant depend on \( r \). By Perelman’s non-collapse theorem, we know that if
\[ |Rm(p, t)| < r^{-2}, \quad (p, t) \in B(x, t_0, r) \times [t_0 - r^2, t_0] \]
then
\[ \text{Vol}_B(x, t_0, r) \geq kr^n. \]

\[
\text{Lemma 3.2.} \quad \text{Fix } 0 < \epsilon < 1. \quad \text{Then there is } r > 0 \text{ such that for any point } (x_0, t_0) \text{ in the flow with } R(x_0, t_0) \geq r^{-2} \text{ the following hold. } (x_0, t_0) \text{ has a strong canonical } (C(\epsilon), \epsilon) \text{ neighborhood.}
\]

\[
\text{Proof.} \quad \text{By Hamilton’s theorem 2.3, we know that for any } t'_n < T, \ t'_n \rightarrow T, \text{ the curvature is bounded}
\]
\[ |Rm(x, t)| < C(t'_n), \quad (x, t) \in M \times [0, t'_n]. \]

Set
\[ A_n = \{(x, t) \in M \times [0, t'_n] | (x, t) \text{ doesn’t have canonical neighborhood}\}. \]
Then there is also a up bound to curvature of point in \( A_n \),
\[ R(x, t) < \tilde{C}(t'_n), \quad (x, t) \in A_n. \]
We can pick point \((x_n, t_n) \in A_n\), such that
\[ R(x_n, t_n) > \frac{1}{2} \tilde{C}(t'_n), \quad t_n \leq t'_n \]
We need to prove that
\[ \lim_{n \to \infty} R(x_n, t_n) < \infty. \]
Suppose not. Then we may assume that \( Q_n = R(x_n, t_n) \to \infty \). From the construction, we know that \( \forall (x, t) \in M \times [0, t_n] \), if \( R(x, t) > 4R(x_n, t_n) \), \((x, t)\) have a canonical neighborhood. Since it’s a 3-dimensional Ricci flow, the curvature is pinching toward positive. By Lemma 3.1, we have the non-collapse assumption. Since \( Q_n \to \infty \), \( t_n \to T \), for any fix \( T > 0 \), we have \((t_\Omega - TQ_n^{-1}, t_n) \subset [0, t_n]\) for sufficiently small \( n \). That is the addition assumption of Theorem 2.2 is satisfied.

By Theorem 2.2, we have that \((M, Q_n g(t_n + \frac{1}{Q_n}), (x_n, t_n))\) converges to a limit flow \((M_\infty, g_\infty(t), (x_\infty, 0))\), which is a \( \kappa \) solution. So for sufficiently large \( n \), \((x_n, t_n)\) has a canonical neighborhood. This contradicts our assumption that none of the point \((x_n, t_n)\) has a canonical neighborhood. \( \square \)

**Proof of Theorem 1.1** Suppose that the Ricci flow blows up at time \( T \). By Theorem 2.3, we know there is a limit metric \( g(T) \) at infinity such that \( g(t)|_{M - K} \to g(T)|_{M - K} \), where \( K \) is a suitable compact set of \( M \). Since the Ricci flow blows up at the time \( t = T \), we have \( \sup_{x \in M} Rm(x, t) \to \infty \), \( t \to T \). Otherwise, we can extend the flow keeping the curvature bounded (That contradicts the maximum of existence time). So we have that there is a point \( p \in K \), such that the scalar curvature blows up at \( T \), that is \( R(p, t) \to \infty \), \( t \to T \). We pick a sequence \( t_n \to T \), such that \( Q_n = R(p, t_n) \to \infty \).

By Lemma 3.2, we have the assumptions of Theorem 2.2 is satisfied. So \((M, Q_n g(t_n + \frac{1}{Q_n}), (p, t_n))\) converges to a \( \kappa \) solution \((M_\infty, g_\infty(t), (x_\infty, 0))\), we know that the asymptotic volume of \( \kappa \) solution is zero, that is,

\[
\lim_{r \to \infty} \frac{Vol_B(x_\infty, r)}{r^3} = 0.
\]

Fixing any \( \epsilon > 0 \), there is a sufficient large \( r \) such that

\[
\frac{Vol_B(x_\infty, r)}{r^3} \leq \frac{\epsilon}{2}.
\]

Since \((M, Q_n g(t_n + \frac{1}{Q_n}), (p, t_n))\) converges to \((M_\infty, g_\infty(t), (x_\infty, 0))\), for large \( n \), we have

\[
\frac{Vol_{B_{Q_n g(t_n)}}((p, t_n), r)}{r^3} \leq \epsilon.
\]

That is

\[
\frac{Vol_{B_{g(t_n)}}((p, t_n), \frac{r}{Q_n})}{(\frac{r}{Q_n})^3} \leq \epsilon.
\]

On the other hand, there is a compact region \( \Omega \subset M - K \), such that \( g(t)|_{\Omega} \to g(T)|_{\Omega} \). Since \( R > 0 \), \( \frac{d}{dt} \int_\Omega d\mu = -\int_\Omega R d\mu \leq 0 \), we have

\[Vol_{g(t)} \Omega \geq \delta, \quad t \in [0, T].\]

Since \( \Omega \) is compact, we can find \( \tilde{r} > 0 \), such that

\[\Omega \subset B_{g(0)}(p, \tilde{r})\].
Since $\text{Ric} \geq 0$, the distance is decreasing. We have $\Omega \subset B_{g(t)}(p, \tilde{r})$, $t \in [0, T)$. So
$$\text{Vol} B_{g(t_n)}(p, \tilde{r}) \geq \delta.$$ 
Since $Q_n \to \infty$, for sufficiently large $n$, $\tilde{r} > \frac{r}{Q_n}$. We now choose $\epsilon < \frac{\delta}{\tilde{r}^3}$. Using $\text{Ric} \geq 0$, we have by the volume comparison theorem that
$$\frac{\delta}{\tilde{r}^3} > \epsilon > \frac{\text{Vol} B_{g(t_n)}((x_n, t_n), \frac{r}{Q_n})}{\left(\frac{r}{Q_n}\right)^3} > \frac{\text{Vol} B_{g(t_n)}(x_n, \tilde{r})}{\tilde{r}^3} \geq \frac{\delta}{\tilde{r}^3},$$
which is a contradiction. So the Ricci flow on $(M, g(0))$ is nonsingular at finite time. That is, $T = \infty$.

Remark 3.3. X. Dai and L. Ma proved that the Ricci flow on the asymptotically flat manifold cannot converge uniformly to flat manifold by ADM mass (see [2]).

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