Multi-Task Determinantal Point Processes for Recommendation

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Abstract
Determinantal point processes (DPPs) have received significant attention in the recent years as an elegant model for a variety of machine learning tasks, due to their ability to elegantly model set diversity and item quality or popularity. Recent work has shown that DPPs can be effective models for product recommendation and basket completion tasks. We present an enhanced DPP model that is specialized for the task of basket completion, the multi-task DPP. We view the basket completion problem as a multi-class classification problem, and leverage ideas from tensor factorization and multi-class classification to design the multi-task DPP model. We evaluate our model on several real-world datasets, and find that the multi-task DPP provides significantly better predictive quality than a number of state-of-the-art models.

Introduction
Increasing the number of items in the average shopping basket is a major concern for online retailers. While there are a wide range of possibilities strategies, this work focuses on the algorithm responsible for proposing a set of items that is best suited to completing the current shopping basket of the user.

Basket analysis and completion is a very old task for machine learning. For many years association rule mining (Agrawal, Imieliński, and Swami 1993) has been the state-of-the-art. Even though there are different variants of this algorithm, the main principle involves computing the conditional probability of buying an additional product by counting co-occurrences in past observations. Due to computational cost and robustness, modern approaches favor item-to-item collaborative filtering (Linden, Smith, and York 2003), or using logistic regression to predict if a user will purchase an item based on binary purchase scores obtained from shopping baskets (Lee et al. 2005).

As reported in the Related Work section, standard collaborative filtering approaches need to be extended to correctly capture diversity among products. Practitioners often mitigate this problem by adding constraints to the recommended set of items. As an example, when using categorical information, it is possible to force the recommendation of a pair of matching shoes when trousers are added to the basket, even if natural co-sale patterns would lead to the recommendation of other trousers. In this situation the presence of diversity in the recommendations is not directly driven by the learning algorithm, but by side information and expert knowledge. Ref. (Teo et al. 2016) proposes an effective Bayesian method for learning the weights of the categories in the case of visual search when categories are known.

Sometimes we need to learn diversity without relying on extra information. Naive learning of diversity directly from the data without using side information comes at a high computational cost, because the number of possible sets (baskets) grows exponentially with the number of items. The issue is not trivial, even when we want to be able to add only one item to an existing set, and becomes even harder when we want to add more than one item with the idea of maximizing the diversity of the final recommended set.

Refs. (Gartrell, Paquet, and Koenigstein 2017; Gartrell, Paquet, and Koenigstein 2016) address this combinatorial problem using a model based on Determinantal Point Processes (DPPs) for basket completion. DPPs are elegant probabilistic models of repulsion from quantum physics, which are used for a variety of tasks in machine learning (Kulesza and Taskar 2012). They allow sampling a diverse set of points, with similarity and popularity encoded using a positive semi-definite matrix called the kernel. Efficient algorithms for marginalization and conditioning DPPs are available. From a practical perspective, learning the DPP kernel is a challenge because the associated likelihood is non-convex, and learning it from observed sets of items is conjectured to be NP-hard (Kulesza and Taskar 2012).

For basket completion it is natural to consider that sets are the baskets which converted to sales. In this setting, the DPP is parameterized by a kernel matrix of size $p \times p$, where $p$ is the size of the catalog. Thus the number of parameters to fit grows quadratically with $p$, and the computational complexity for learning, prediction, and sampling grows cubicly with $p$. As learning a full-rank DPP is hard, (Gartrell, Paquet, and Koenigstein 2017) proposes regularizing the DPP by constraining the kernel to be low rank. This regularization also improves generalization and offers more diversity in recommendations, without hurting predictive performance. In fact in many settings the predictive quality is also improved, making the DPP a very desirable tool for modeling baskets. Moreover, the low-rank assumption also enables substantially better runtime performance compared to a full-rank DPP.
Nevertheless, because of the definition of the DPP, as described in the Model section, this low-rank assumption for the kernel means that any possible baskets with more items than the chosen rank will receive a probability estimate of 0. This approach is thus impossible to use for large baskets, and some other regularizations of the DPP kernel may be more appropriate. The contributions of this paper are fourfold:

- We modify the constraints over the kernel to support large baskets.
- We model the probability over all baskets by adding a logistic function on the determinant computed from the DPP kernel. We adapt the training procedure to handle this nonlinearity, and evaluate our model on several real-world basket datasets.
- By leveraging tensor factorization, we propose a new way to regularize the kernel among a set of tasks. This approach also leads to enhanced predictive quality.
- We show that this new model, which we call the multi-task DPP, allows us to capture directed basket completion. That is, we can leverage the information regarding the order in which items are added to a cart to improve predictive quality.

Furthermore, we show that these ideas can be combined for further improvements to predictive quality, allowing our multi-task DPP model to outperform state-of-the-art models by a large margin.

We begin by introducing our proposed algorithm, and then proceed to evaluate its effectiveness on several real-world datasets. We then discuss related work before concluding and introducing possible future work.

Model

Background

Determinantal Point Processes (DPPs) were originally used to model a distribution over particles that exhibit a repulsive effect (Vershik and Yakubovich 2001). Recently, interest in leveraging this repulsive behavior has led to DPPs receiving increasing attention within the machine learning community. Mathematically, discrete DPPs are distributions over discrete sets of points, or in our case items, where the model assigns a probability to observing a given set of items. Let \( I \) denote a set of items, and \( L \) the kernel matrix associated with the DPP whose entries encode item popularity and the similarity between items. The probability of observing the set \( I \) is proportional to the determinant of the principal submatrix of \( L \) indexed by the items in \( I \): \( P(I) \propto \det L_I \).

Thus, if \( p \) denotes the number of items in the item catalog, the DPP is a probability measure on \( 2^p \) (the power set, or set of all subsets of \( p \)). The kernel \( L \) encodes item popularities and the similarities between items, where the diagonal entry \( L_{ii} \) represents the popularity of item \( i \), and the off-diagonal entry \( L_{ij} = L_{ji} \) represents the similarity between items \( i \) and \( j \). A determinant can be seen as a volume from a geometric viewpoint, and therefore more diverse sets will tend to have larger determinants. For example, the probability of selecting two items \( i \) and \( j \) together can be computed as

\[
P(i, j) \propto \frac{L_{ii} L_{jj} - L_{ij}^2}{L_{ii} L_{jj}}
\]

In equation [1] we can see that the more similar \( i \) and \( j \) are, the less likely they are to be sampled together. The definition of the entries \( L_{ij} \) will therefore determine the repulsive behavior of the kernel for the task. For instance, if similarity is defined using image descriptors, then images of differing appearance will be selected by a DPP. On the other hand, if the entries \( L_{ij} \) are learned using previously observed sets, such as e-commerce baskets (Gartrell, Paquet, and Koenigstein 2017), then co-purchased items \( i \) and \( j \) are likely to be sampled by the DPP, and thus the “similarity” \( L_{ij} \) will be low. In an application such as a search engine or in document summarization, the kernel may be defined using feature descriptors \( \psi_i \in \mathbb{R}^D \) (i.e tf-idf of the text), and a relevance score \( q_i \in \mathbb{R}^+ \) of each item \( i \) such that \( L_{ij} = q_i \psi_i^T \psi_j q_j \), which favors relevant items (large \( q_i \)) and discourages lists composed of similar items.

Logistic DPP

Our objective is to find a set of items that are most likely to be purchased together. We formulate this as a classification problem, where the goal is to predict if a specific set of items will generate a conversion from the user, which we denote as \( Y \in \{0, 1\} \). We model the class label \( Y \) as a Bernoulli random variable with parameter \( \phi(I) \), where \( I \) is the set of items and \( \phi \) is a function that we will define below:

\[
p(y|I) = \phi(I)^y (1 - \phi(I))^{1-y}
\]

We model the function \( \phi \) using a DPP.

We assume that there exists a latent space such that diverse items in this space are likely to be purchased together. Similarly to (Gartrell, Paquet, and Koenigstein 2017), we introduce a low-rank factorization of the kernel matrix \( L \in \mathbb{R}^{p \times p} \):

\[
L = VV^T + D^2
\]

where \( V \in \mathbb{R}^{p \times r} \) is a latent matrix where each row vector \( v \) encodes the \( r \) latent factors of item \( i \). \( D \) is a diagonal matrix that, and together with \( |V_i| \), represents the intrinsic quality or popularity of each item. The squared exponent on \( D \) ensures that we always have a valid positive semi-definite kernel. We then define \( \phi(I) \propto \det(V_I, V_I^T + D^2) \geq 0 \). Note that without the diagonal term, the choice of \( r \) would restrict the cardinality of the observable set, because \( |I| > r \) would imply \( \phi(I) = 0 \) when \( D \equiv 0 \). Using this term will ensure that the success probability of any set will be positive, but the cross-effects will be lower for sets of cardinality higher than \( r \). We also see that items with similar latent vectors are less likely to be sampled than items with different latent vectors, since similar vectors will produce a parallelopotope with a smaller volume. To normalize the probability and encourage separation between vectors we use a logistic function on \( \phi \) such that:

\[
\phi(I) = \frac{1}{1 + \exp(-w \det L_I)} 
\]

\[
\sigma(w \det L_I)
\]
Usually the logistic function is of the form \(1/(1 + \exp(-w \cdot \text{det} L_x))\). However, in our case the determinant is always positive, since \(L\) is positive semi-definite, which would result in \(\mathbb{P}(y = 1|z)\) always greater than 0.5 with such a function. By construction, our formulation allows us to obtain a probability between 0 and 1. Finally, \(w \in \mathbb{R}\) is a scaling parameter, to be chosen by cross-validation, that insures that the exponential does not explode, since the diagonal parameter will be approximately 1.

**Learning.** In order to learn the matrix \(V\) we assume the existence of historical data \(\{I_m, y_m\}_{1 \leq m \leq M}\), where \(I_m\) is a set of items, and \(y_m\) is a label set to 1 if the set has been purchased, or 0 otherwise. This training data allows us to learn the matrices \(V\) and \(D\) by maximizing the log-likelihood of the data. To do so, we first write the click probability for all \(y\) as

\[
\mathbb{P}(y|z) = \sigma(w \cdot \text{det} L_x)^y (1 - \sigma(w \cdot \text{det} L_x))^{1-y}
\] (6)

The log-likelihood \(f(V, D)\) can then be written as

\[
f(V, D) = \log \sum_{m=1}^{M} \mathbb{P}(y_m|I_m) - \frac{\alpha}{2} \sum_{i=1}^{p} \alpha_i (||V_i||^2 + ||D_i||^2)
\]

Following (Gartrell, Paquet, and Koenigstein 2017), \(\alpha_i\) is an item regularization weight that is inversely proportional to item popularity. The matrices \(V\) and \(D\) are learned by maximizing the log-likelihood using stochastic gradient ascent. Details on the optimization algorithms and the gradient equations are available in the supplementary material.

**Multi-task DPP**

We now propose a modification to the previously introduced model that is better suited for the basket completion task. To do so we enhance the logistic DPP for the basket completion scenario, where we model the probability that the user will purchase a specified additional item based on the items already present in the user’s shopping basket. We formulate this as a multi-task classification problem, where the goal is to predict whether the user will purchase a given target item based on the user’s basket. In this setting there are as many tasks as there are items in the catalog, \(p\) (minus the items already in the basket). Learning one kernel per task would be impossible in practice and suffer from sparsity issues. Indeed, with one kernel per task, each target item would be present in only a fraction of the baskets, and thus dramatically reduce the size of the training set per kernel. To solve this issue we utilize a low-rank tensor. We use a cubic tensor \(K \in \mathbb{R}^{p \times p \times p}\), where each slice \(\tau\) (noted \(K_\tau\)) of \(K\) is the task (low-rank) kernel. By assuming that the tensor \(K\) is low-rank, we are able to implement sharing of learned parameters between tasks, as shown in the following equation:

\[
K_\tau = V R_\tau^2 V^T + D^2
\] (7)

where \(V \in \mathbb{R}^{p \times r}\) are the item latent factors that are common to all tasks, and \(R_\tau \in \mathbb{R}^{r \times r}\) is a task specific matrix that models the interactions between the latent components of each task. In order to balance the degrees of freedom between tasks and items, we further assume that \(R_\tau\) is a diagonal matrix. Therefore, the diagonal vector of \(R_\tau\) models the latent factors of each task, and the latent factors of the item can be seen as the relevance of the product for each latent factor. As is the case for the matrix \(D\), the squared exponent on \(R_\tau\) ensures that we always have a valid kernel. This decomposition is similar to the RESCAL decomposition (Nickel, Tresp, and Kriegel 2011), without the additional bias term and a diagonal constraint on the slice specific matrix. We also use a different learning procedure due to the use of the logistic function. The probability that a set of items \(I\) will be successful for task \(\tau\) is

\[
\mathbb{P}(y_\tau = 1|z) = \sigma(w \cdot \text{det} K_\tau, I) = 1 - \exp(-w \cdot \text{det} K_\tau, I)
\] (8)

Therefore, the log-likelihood \(g(V, D, R) = g\) is

\[
g = \sum_{m=1}^{M} \log \mathbb{P}(y_m|I_m) - \frac{\alpha}{2} \sum_{i=1}^{p} \alpha_i (||V_i||^2 + ||D_i||^2 + ||R_i||^2)
\]

where each observation \(m\) is associated with a task, and \(I_m\) is the set of items associated with an observation. As previously described, matrices \(V\), \(D\), and \((R_\tau)_{\tau \in \{1, \ldots, p\}}\) are learned by maximizing the log-likelihood using stochastic gradient ascent. Details on the optimization algorithms and the gradient equations are available in the supplementary material.

**Experiments**

We evaluate the performance of our model on the basket completion problem on several real-world datasets, and compare to several state-of-the-art baselines.

**MODELS**

- **Our models.** To understand the impact of the different components of our model compared to the low-rank DPP model, we evaluated the following versions of our model:
  - **LOGISTIC DPP:** This version of our model is similar to the low-rank DPP model, with the addition of the logistic function. To determine what item to recommend we use a greedy approach, where we select the next item such that the probability of the basket completed with this item is the largest. We used \(w = 0.01\).
  - **MULTI-TASK LOG-DPP WITHOUT BIAS:** In this version of the model we set \(D = 0\), which allows us to measure the impact of capturing the item bias in a separate matrix. The matrix \(V\) encodes the latent factors of items present in the basket, while each matrix \(R_\tau\) encodes the latent factors of each target item \(\tau\) that can be added to a basket. We used \(w = 0.01\).
  - **MULTI-TASK LOG-DPP:** This is the full version of our model, with bias enabled. We used \(w = 0.01\).

Our datasets do not provide explicit negative information. To generate negative feedback for our models we created negatives targets from observed baskets by sampling a random item among those items not in the basket. This approach could be improved through better negative sampling strategies, but since this is not part of our primary contributions we leave this investigation for future work.
• **Baselines.** The primary goal of our work is to improve state-of-the-art results provided by DPPs and introduce new modeling enhancements to DPPs. However, for the sake of completeness we also compare with other strong baseline models provided by state-of-the-art collaborative filtering approaches.

  – **Poisson Factorization (PF)** (Gopalan, Hofman, and Blei 2013) is a probabilistic matrix factorization model generally used for recommendation applications with implicit feedback. Since our datasets contain no user id information, we consider each basket to be a different user, and thus there are as many users as baskets in the training set. In practice this can cause issues with high memory consumption, since the number of baskets can be very large.

  – **Factorization Machines (FMs)** (Rendle 2010) is a general approach that models $d$-th-order interactions using low-rank assumptions. FMs are usually used with $d = 2$, since this corresponds to classic matrix factorization and because complexity increases linearly with $d$. Additionally, there is no open-source FM implementation that supports $d > 2$. For these reasons, we use $d = 2$ in our experiments. As with PF, to learn the FM model we consider each basket as a unique user. For fairness in comparison with our models, we also tried FM with negative sampling based on item popularity. However, we did not see any substantial improvement in model performance when using this negative sampling approach.

  – **Low-Rank DPP** (Gartrell, Paquet, and Koenigstein 2017) is a low-rank DPP model, suitable for basket completion, where the determinant of the submatrix of the kernel corresponds to the probability that all the items will be bought together in a basket.

  – **Bayesian Low-Rank DPP** (Gartrell, Paquet, and Koenigstein 2016) is the Bayesian version of the low-rank DPP model.

  – **Associative Classifier (AC)** is an algorithm that computes the support of a purchased set of items in order to obtain completion rules. As in (Gartrell, Paquet, and Koenigstein 2017), we used the Classification Based on Associations (CBA) algorithm (Liu, Hsu, and Ma 1998), available at (Coenen 2005), with minimum support of 1.0% and maximum confidence thresholds of 20.0%. Unlike other models, AC does not provide estimates for all possible sets. Therefore, we cannot compute results for some metrics used in our evaluation, such as MPR (described below).

  – **Recurrent Neural Network** This RNN model (Hidasi et al. 2015) is adapted for session-based recommender systems. The RNN requires ordered sequences, and thus we only evaluate this model on the Instacart dataset (described below), where the order in which items were added to each basket is available. We use the implementation of this model available from (Songweiping 2017).

For all models we tried different hyperparameter settings, such as the number of latents factors and regularization strength, and report the best results here. In the interest of reproducibility, all code used for our experiments is available at ANONYMOUS_REF.

**DATASETS.** For our basket completion experiments we use the following four datasets. The first three ones contain undirected baskets, that is there is no notion of the order in which the items have been added to the basket, whereas the last contains directed basket, that is ordered sets:

  – **Amazon Baby Registries** is a public dataset consisting of 110,006 registries and 15 disjoint registry categories. For the purposes of comparison with (Gartrell, Paquet, and Koenigstein 2016), we perform two experiments. The first experiment is conducted using the diaper category, which contains 100 products and approximately 10,000 baskets, composed of 2.4 items per basket on average. The second experiment is performed on the concatenation of the diaper, apparel, and feeding categories (sometimes noted here as DAF for Diaper+Apparel+Feedings), which contains 300 products and approximately 17,000 baskets, composed of 2.6 items per basket on average. The item categories are disjoint; for example, no basket containing diaper products will contain apparel products. This concatenation of disjoint categories can present difficulties for classic matrix factorization models (Gartrell, Paquet, and Koenigstein 2016), which may prevent these models from learning a good embedding of items.

  – **Belgian Retail Supermarket** is a public dataset (Brijs et al. 1999) that contains 88,163 sets of items that have been purchased together, with a catalog of 16,470 unique items. Each basket contains 9.6 items on average. AC cannot be trained on this dataset because this approach does not scale to large item catalogs.

  – **UK retail dataset** is a public dataset (Chen, Sain, and Guo 2012) that contains 22,034 sets of items that have been purchased together, among a catalog of 4,070 unique items. This dataset contains transactions from a non-store online retail company that primarily sells unique all-occasion gifts, and many customers are wholesalers. Each basket contains 18.5 items on average, with a number of very large baskets. Modeling these large baskets requires using a very large number of latent factors for the low-rank DPP, leading to somewhat poor results for this model. This is not an issue for our model, due to the item bias that is captured in a separate matrix. However, for purposes of comparison, we removed all baskets containing more than 100 items from this dataset; note that the low-rank DPP still requires 100 latent factors to model these baskets. AC could not be trained on this dataset because it does not scale to large item catalogs.

  – **Instacart** is, to the best of our knowledge, the only public dataset that contains the order in which products were added to baskets. It is composed of three datasets containing online grocery shopping behavior for more than 200,000 Instacart users: a “train” dataset, a “test” dataset, and a “prior” dataset. We use only the “train” dataset in our experiments, and remove items that appear less than 15 times and baskets of size lower than 3. This results in a dataset containing 700,052 sets of items and 10,531 unique items.

  2https://www.instacart.com/datasets/grocery-shopping-2017
METRICS. To evaluate the performance of each model we compute the Mean Percentile Rank and precision@K for K = 5, 10, and 20:

- **Mean Percentile Rank (MPR)**: Given a basket \( B \), we compute the percentile rank \( \text{PR}_{iB} \) of the held-out item, \( i_B \). Let \( p_i = P(Y = 1 | B) \). Then

\[
\text{PR}_{iB} = \frac{\sum_{p=1}^{K} \mathbb{1}(p_{iB} \geq p_i)}{p} \times 100\% \tag{9}
\]

The MPR is the average PR over all baskets in the test set:

\[
\text{MPR} = \frac{\sum_{B \in \mathcal{T}} \text{PR}_{iB}}{|\mathcal{T}|} \tag{10}
\]

where \( \mathcal{T} \) is the set of all baskets in the test set. A MPR of 100% means that the held-out item always receives the highest predictive score, while a MPR of 50% corresponds to a random sorting. Higher MPR scores are better.

- **Precision@K**: the fraction of test baskets where the held-out item is in the top K ranked items.

\[
\text{precision@K} = \frac{\sum_{B \in \mathcal{T}} \mathbb{1}(\text{rank}_{iB} \leq K)}{|\mathcal{T}|} \tag{11}
\]

Higher precision@K scores are better.

We evaluated the predictive quality of our models for both undirected and directed basket completion. Recall that for undirected baskets, there is no information regarding the order in which items are added to baskets, while directed baskets do contain such ordering structure. We use the Amazon, Belgian retail, and UK retail datasets for our undirected basket experiments, while the Instacart dataset is used for our directed basket experiments. For all experiments we use a random split of 70% of the data for training, and 30% for testing.

**Results for Undirected Baskets**

For our undirected basket experiments, we remove one item at random from each basket in the test set. We then evaluate the model prediction according to the predicted score of this removed item using the metrics described below.

Looking at Table I we see that classic collaborative filtering models sometimes have difficulty giving good recommendations in the basket-completion setting. Perhaps more surprising, but already described in (Gartrell, Paquet, and Koenigstein 2017), is that for the Amazon datasets, PF provides MPR performance that is approximately equivalent to a random model. For the Amazon diaper dataset this poor performance may be a result of the small size of each basket (around 2.4 items per basket on average), thus each “user” is in a cold start situation, and it is therefore difficult to provide good predictions. Poor performance on the diaper+apparel+feedings Amazon’s dataset may result from the fact that, apart from the small basket size of 2.61 items on average, this dataset is composed of three disjoint categories. These disjoint categories can break the low-rank assumption for matrix factorization-based models, as discussed in (Gartrell, Paquet, and Koenigstein 2016). This issue is somewhat mitigated in FM, due to the integration of an item bias into the model. This item bias allows the model to capture item popularity and thus provide acceptable performance in some cases.

Finally, the DPP-based models generally outperform the FM model. This is likely due to the fact that DPP models are able to capture higher-order interactions within baskets, while FM is only able to capture second-order interactions, since \( d = 2 \) for this model.

**Low Rank DPP vs Multi-task DPP.** We now turn to a performance comparison between our primary baseline, the low-rank DPP model, and our multi-task DPP model. From Table[2] we see that our approaches provide a substantial increase in performance for both Amazon datasets, with relative improvements between 10% and 70%. One factor that accounts for this performance improvement is that unlike the low-rank DPP, which models the probability that a set of items will be bought together, our approach directly models the basket completion task. In our multi-task DPP model, the extra dimensions allow the model to capture the correlation between each item in the basket and the target item, as well as the global coherence of the set.

Regarding the three-category Amazon dataset, a good model should not be impacted by the fact the all of the three categories are disjoint. Therefore, the precision@K scores should be approximately the same for both the single-category and three-category datasets, since we observe similar performance for each category independently. Since the MPR is 78% for one category, the MPR on the three-category dataset should be approximately 93%, since on average for each category the target item is in the 22nd position over the 100 items in the single-category catalog. Therefore, the target item should be in the 22nd position over the 300 items in the three-category catalog, resulting in a MPR of \( 1 - 22/300 = 93\% \). Our models come close to these numbers, but still exhibit some small degradation for the three-category dataset. Finally, we note that for this dataset, a model that samples an item at random from the right category would have a precision@20 of 20%, since there are 100 items per category. The low-rank DPP model provides close to this level of performance. Taken together, these observations indicate our model is robust to the disjoint category problem, and explains the 70% relative improvement we see for our model on the precision@20 metric. On the UK Retail dataset the improvements of our algorithm are still substantial for precision@K, with a relative improvement of between 20% and 30% (MPR is down by 5%). We also observe the same decrease in MPR for our logistic DPP model, but precision@K is similar to the low-rank DPP. On the Belgian Retail dataset we see that all models provide similar performance. For this dataset, baskets come from an offline supermarket, where it is possible that customers commonly purchased similar products at specific frequencies. Consequently it may be easy to capture frequent associations between purchased items, but very difficult to discover more unusual associations, which may explain why all models provide approximately the same performance.

**Logistic DPP vs Multi-task DPP.** To better understand the incremental performance of our model, we focus on the results of the logistic DPP and the multi-task log-DPP models. We see that the logistic (single-task) model does not im-
prove over the low-rank DPP on average, indicating that the logistic component of the model does not contribute to improved performance. However, we argue that this formulation may still be valuable in other classification applications, such as those with explicit negative feedback. For the multi-task log-DPP model, we see that the version of this model without bias is responsible for almost all of the performance improvement. Some additional lift is obtained when capturing the item popularity bias in a separate matrix. Since most of the gain comes from the multi-task kernel, one may ask if we could use the multi-task kernel without the logistic function and obtain similar results. We believe that this is not the case for two reasons. First, since we are clearly in a classification setting, it is more appropriate to use a logistic model that is directly tailored for such applications. Second, without the logistic function, each slice of the tensor should define a probability distribution, meaning that the probability of purchasing an additional product should sum to one over all possible baskets. However, we could add an arbitrarily bad product that would never be purchased, resulting in a probability of zero for buying that item in any basket, which would break the distributional assumption.

**Results for Directed Baskets**

Recall that directed baskets contain ordering information which may provide substantial information for basket completion. In order to evaluate the ability of our model to capture directed basket completions, we performed three experimental protocols on the Instacart dataset, which contains ordered sequences of items added to baskets. Each protocol varies in the way that we remove the item to predict from the basket:

1. As with previous experiments, we remove one item at random from each basket. For the low-rank DPP and FM models this item removal is performed only for baskets in the test set. We do not remove items from baskets in the training set, since these baskets are used to learn inter-item correlation patterns that are applied to new baskets. For the multi-task DPP, we perform item removal for both the training and test sets. Item removal in the training set is appropriate for the multi-task DPP since the removed item corresponds to the target item for this model.

2. We remove the last item added to each basket. For the low-rank DPP and FM models, this is done only for the test set, and for the multi-task DPP this done for both the training and test sets. Since we consider ordered sequences, we also evaluate the RNN model using this protocol, where item removal is done only for the test set.

3. We remove one item at random from each basket in the training set, and the last item added to each basket in the test set. Here we evaluate only the performance of the multi-task DPP model, since it is the only model that involves item removal in the training set.

Looking at Table 2 and comparing the multi-task DPP results for protocols (2) and (3), we see that our model performs much better when predicting the last item added to a basket when training is also done by removing the last item added (protocol (3)). This allows us to conclude that the order in which items are added to the basket is important, otherwise both protocols would give similar results. Next, when comparing the results of protocols (1) and (2), we see that multi-task DPP performance is lower when the model is trained to predict a randomly removed item than when trained to predict the last item added, while we see that this pattern is reversed for the low-rank DPP. This indicates that the low-rank DPP, although well suited to modeling item co-occurrence probabilities, is unable to capture directed basket completion. Finally, we see, surprisingly, that the RNN model does not provide good performance for this task. This relatively poor performance may come from the fact that the basket lengths are too small in this dataset for the RNN to learn the item sequences within baskets correctly.

**Related Work**

The topics of DPPs, basket completion, and diversity have significant attention in recent years.

In addition to the previously discussed work, DPPs have been used for natural language processing in order to discover diverse threads of documents (Gillenwater, Kulesza, and Taskar 2012), and to enhance diversity in recommender systems (Chen, Zhang, and Zhou 2017). Unlike in our application where we learn the kernels, in these applications the kernel is constructed using previously obtained latent factors, for instance using tf-idf (Gillenwater, Kulesza, and Taskar 2012). These latent factors are scaled by a relevance score learned in a more conventional fashion. For example, these relevance scores may represent the predicted rating of a particular user, or the similarity between the text in a document and the user query. Ultimately, these applications involve sampling from the DPP specified by this kernel, where the kernel parameters trade off between relevance and diversity. However, sampling from such a DPP efficiently is difficult, and this has lead to work on different sampling techniques. Ref. (Gillenwater et al. 2014) relies on MCMC sampling, while (Chen, Zhang, and Zhou 2017) proposes a greedy solution based on Cholesky decomposition.

Several algorithms have been proposed for learning the DPP kernel matrix. Ref. (Gillenwater et al. 2014) uses an expectation-maximization (EM) algorithm to learn a non-parametric form of the DPP kernel matrix. Ref. (Mariet and Sra 2015) proposes a fixed-point algorithm called Picard iteration, which is much faster than EM, but still slower than (Gartrell, Paquet, and Koenigstein 2017). Bayesian learning methods have also been proposed to learn the DPP kernel (Gartrell, Paquet, and Koenigstein 2016; Affandi et al. 2014). Improving diversity in recommender systems has also been studied without the use of DPPs, including, among other work, (Christoffel et al. 2013; Puthiya Parambath, Usunier, and Grandvalet 2016; Vargas and Castells 2014). For instance, (Christoffel et al. 2013) relies on random walk techniques to enhance diversity. In (Puthiya Parambath, Usunier, and Grandvalet 2016), the authors propose trading off between the relevance of the recommendation and diversity by introducing a coverage function to force the algorithm to produce recommendations that cover different centers of the interests of each user. Finally, the au-
In this paper we have proposed an extension of the DPP model that leverages ideas from multi-class classification and tensor factorization. While our model can be applied to a number of machine learning problems, we focus on the problem of basket completion. We have shown through experiments on several datasets that our model provides significant improvements in predictive quality compared to a number of competing state-of-the-art approaches and can appropriately capture directed basket completion. In future work we plan to investigate other applications of our model, such as user conversion prediction, attribution, and adversarial settings in games. We also plan to investigate better negative sampling methods for positive-only and unlabelled data. Finally, we also plan to investigate other types of loss functions, such as hinge loss, and other types of link functions for DPPs, such as the Poisson function, to tailor DPPs for regression problems. We believe that this work will allow us to customize DPPs so that they are suitable for many additional applications.

**Conclusion and Future Work**

In this paper we have proposed an extension of the DPP model that leverages ideas from multi-class classification and tensor factorization. While our model can be applied to a number of machine learning problems, we focus on the problem of basket completion. We have shown through experiments on several datasets that our model provides significant improvements in predictive quality compared to a number of competing state-of-the-art approaches and can appropriately capture directed basket completion. In future work we plan to investigate other applications of our model, such as user conversion prediction, attribution, and adversarial settings in games. We also plan to investigate better negative sampling methods for positive-only and unlabelled data. Finally, we also plan to investigate other types of loss functions, such as hinge loss, and other types of link functions for DPPs, such as the Poisson function, to tailor DPPs for regression problems. We believe that this work will allow us to customize DPPs so that they are suitable for many additional applications.

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**Table 1**: Result of all models on all datasets. \( r \) denotes the number of latent factors. Best results within each dataset are in bold.

| model                           | dataset                      | \( r \) | MPR  | Prec. @5 | Prec. @10 | Prec. @20 |
|---------------------------------|------------------------------|--------|------|---------|-----------|-----------|
| ASSOCIATIVE CLASSIFIER          | Amazon (diaper)              | -      | -    | 16.66   | 16.66     | 16.66     |
| POISSON FACTORIZATION           | Amazon (diaper)              | 40     | 50.30| 4.78    | 10.03     | 19.90     |
| Factorization Machines          | Amazon (diaper)              | 5      | 67.92| 24.01   | 32.62     | 46.25     |
| LOW RANK DPP                    | Amazon (diaper)              | 30     | 71.65| 25.48   | 35.80     | 49.98     |
| BAYESIAN LOW RANK DPP           | Amazon (diaper)              | 30     | 72.38| 26.31   | 36.21     | 51.51     |
| LOGISTIC DPP                   | Amazon (diaper)              | 50     | 71.08| 23.7    | 34.01     | 48.44     |
| MULTI-TASK LOGDPP NO BIAS      | Amazon (diaper)              | 50     | 78.41| 34.73   | 47.42     | 62.58     |
| MULTI-TASK LOGDPP              | Amazon (diaper)              | 40     | 87.02| 21.46   | 23.06     | 23.90     |
| ASSOCIATIVE CLASSIFIER          | Belgian Retail Supermarket   | 10     | 65.08| 20.85   | 21.10     | 21.37     |
| Factorization Machines          | Belgian Retail Supermarket   | 76     | 88.52| 21.48   | 23.29     | 25.19     |
| LOW RANK DPP                   | Belgian Retail Supermarket   | 76     | 89.08| 21.43   | 23.10     | 25.12     |
| BAYESIAN LOW RANK DPP           | Belgian Retail Supermarket   | 75     | 87.35| 21.17   | 23.11     | 25.77     |
| LOGISTIC DPP                   | Belgian Retail Supermarket   | 75     | 87.42| 21.02   | 23.35     | 25.13     |
| MULTI-TASK LOGDPP NO BIAS      | Belgian Retail Supermarket   | 80     | 89.80| 20.53   | 30.86     | 45.79     |
| MULTI-TASK LOGDPP              | Belgian Retail Supermarket   | 100    | 73.12| 1.77    | 2.31      | 3.01      |
| Factorization Machines          | UK Retail                    | 5      | 56.91| 0.47    | 0.83      | 1.50      |
| LOW RANK DPP                   | UK Retail                    | 100    | 82.74| 3.07    | 4.75      | 7.60      |
| BAYESIAN LOW RANK DPP           | UK Retail                    | 100    | 61.31| 1.07    | 1.91      | 3.25      |
| LOGISTIC DPP                   | UK Retail                    | 100    | 75.23| 3.18    | 4.99      | 7.83      |
| MULTI-TASK LOGDPP NO BIAS      | UK Retail                    | 100    | 77.67| 3.82    | 5.98      | 9.11      |
| MULTI-TASK LOGDPP              | UK Retail                    | 100    | 78.25| 4.00    | 6.20      | 9.40      |

**Table 2**: Performance of the models on Instacart dataset for the three protocols \( \rho \). All models used 80 latent factors except FM that used 5 latent factors.

| model                           | \( \rho \) | MPR  | Prec. @5 | Prec. @10 | Prec. @20 |
|---------------------------------|------------|------|----------|-----------|-----------|
| FM                             | (1)        | 61.10| 4.55     | 6.3       | 16.61     |
| LOW RANK DPP                   | (1)        | 76.46| 7.37     | 8.07      | 9.23      |
| MULTI-TASK DPP                 | (1)        | 80.46| 4.62     | 7.23      | 10.51     |
| FM                             | (2)        | 62.47| 9.35     | 10.66     | 11.92     |
| LOW RANK DPP                   | (2)        | 61.16| 7.49     | 8.05      | 8.8       |
| RNN                            | (2)        | 73.31| 1.08     | 1.99      | 3.2       |
| MULTI-TASK DPP                 | (2)        | 90.07| 9.91     | 13.67     | 19.97     |
| MULTI-TASK DPP                 | (3)        | 80.65| 5.23     | 6.05      | 9.72      |

**Gartrell, Paquet, and Koenigstein (2016)** may now be considered among the class of models belonging to the new state-of-the-art for basket completion, in light of their effectiveness both in terms of accuracy and training speed. Finally, classic collaborative filtering models tailored for positive and unlabelled data (Hu, Koren, and Volinsky 2008) and (Gopalan, Hofman, and Blei 2013) may be effectively used for basket completion.

**Vargas and Castells (2014)** propose transforming the problem of recommending items to users into recommending users to items. They introduce a modification of nearest-neighbor methods, and a probabilistic model that allows isolation of the popularity bias and favors less popular items.

Regarding basket completion, associative classifiers have long been the state-of-the-art (Agrawal, Imielinski, and Swami 1993), despite requiring very heaving computational load for training, and manual tuning for key parameter choices such as lift and confidence thresholds. Later work focuses on the task of purchase prediction by adapting collaborative filtering methods. Ref. (Mild and Reutterer 2003) proposes a solution based on nearest-neighbor models, while (Lee et al. 2005) relies on binary logistic regression to predict if a user will purchase a given item. More recently, DPPs (Gartrell, Paquet, and Koenigstein 2017)
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Appendix

Logistic DPP
Recall that the logistic DPP log-likelihood is:
\[
f(V, D) = \log \prod_{m=1}^{M} \mathcal{P}(y_m | Z_m) - \frac{\alpha_0}{2} \sum_{i=1}^{p} \alpha_i (||V||^2 + ||D_i||^2)
\]
= \sum_{m=1}^{M} \log \mathcal{P}(y_m | Z_m) - \frac{\alpha_0}{2} \sum_{i=1}^{p} \alpha_i (||V||^2 + ||D_i||^2)

Optimization. We maximize the log-likelihood using stochastic gradient ascent with Nesterov’s Accelerated Gradient, which is a form of momentum. To simplify notation, we define \([m] = I_m\) and \(\sigma_m = \sigma(w \det L_m)\). Let \(i \in \{1, \ldots, p\}, k \in \{1, \ldots, r\}.

lemma When \(D\) is fixed, the gradient of (12) with respect to \(V_{ik}\) is
\[
\frac{\partial f}{\partial V_{ik}} = 2w \sum_{m,i \in [m]} ([L_{ik}^{-1}]_{s,i} V_{ik}) \frac{y_m - \sigma_m}{\sigma_m} \det L_m
- \alpha_0 \alpha_i \delta_{ik}, V_{ik}
\]

Proof Without the regularization term we have
\[
\frac{\partial f}{\partial V_{ik}} = \sum_{m,i \in [m]} \frac{y_m - \sigma_m}{\sigma_m} \frac{\partial \sigma_m}{\partial V_{ik}} + 1 - y_m \frac{\partial \sigma_m}{\partial V_{ik}} \frac{1}{\sigma_m}
\]
= \[
= w \sum_{m,i \in [m]} \frac{y_m - \sigma_m}{\sigma_m} \tr\left(L_{ik}^{-1} \frac{\partial L_{ik}}{\partial V_{ik}}\right) \det L_m
\]
= \[
= 2w \sum_{m,i \in [m]} ([L_{ik}^{-1}]_{s,i} V_{ik}) \frac{y_m - \sigma_m}{\sigma_m} \det L_m
\]
where (16) follows from
\[
\frac{\partial L_{ik}}{\partial V_{ik}} \in \mathbb{R}^{2 \times 2}
\]
Therefore,
\[
\tr\left(L_{ik}^{-1} \frac{\partial L_{ik}}{\partial V_{ik}}\right) = \sum_{s,t} \left(V_{ik} \delta_{s,t} + V_{ik} \delta_{t,s}\right) [L_{ik}^{-1}]_{s,t}
\]
= \[
= \sum_s [L_{ik}^{-1}]_{s,i} V_{sk} + \sum_t [L_{ik}^{-1}]_{i,t} V_{ik}
\]
= \[
= 2 \sum_s [L_{ik}^{-1}]_{s,i} V_{sk}
\]
adding the derivative of the regularization term concludes the proof. □

lemma When \(V\) is fixed, the gradient of (12) with respect to \(D_i\) is
\[
\frac{\partial f}{\partial D_i} = 2w \sum_{m,i \in [m]} ([L_{ik}^{-1}]_{i,i} D_{i,i}) \frac{y_m - \sigma_m}{\sigma_m} \det L_m
- \alpha_0 \alpha_i \delta_{i,i}, D_{i,i}
\]

Proof As shown previously, and without the regularization term, we have
\[
\frac{\partial f}{\partial V_{ik}} = w \sum_{m,i \in [m]} \frac{y_m - \sigma_m}{\sigma_m} \tr\left(L_{ik}^{-1} \frac{\partial L_{ik}}{\partial D_{i,i}}\right) \det L_m
\]
Since,
\[
\frac{\partial L_{ik}}{\partial D_{i,i}} = 2D_{i,i} \delta_{i,i}\delta_{i,i}
\]
\[
\tr\left(L_{ik}^{-1} \frac{\partial L_{ik}}{\partial D_{i,i}}\right) = 2[L_{ik}^{-1}]_{i,i} D_{i,i}
\]
adding the derivative of the regularization term concludes the proof. □

Multi-Task DPP
Recall that the multi-task DPP log-likelihood is:
\[
g = \sum_{m} \log \mathcal{P}(y_r | [m]) - \frac{\alpha_0}{2} \sum_{i=1}^{p} \alpha_i (||V||^2 + ||D_i||^2 + ||R||^2)
\]

Optimization. Since each observation \(m\) is attached to a task, we denote \(\tau_m\) as the task that corresponds to observation \(m\). Thus we have \(\sigma_m = \sigma(w \det K_{\tau_m,[m]})\). When there is no ambiguity, we also denote \(K_{[m]} = K_{[m],[m]}\). Let \(i \in \{1, \ldots, p\}, k \in \{1, \ldots, r\}.

lemma When \(D\) and \(R\) are fixed, the gradient of (22) with respect to \(V_{ik}\) is
\[
\frac{\partial g}{\partial V_{ik}} = 2w \sum_{m,i \in [m]} \frac{y_m - \sigma_m}{\sigma_m} R_{ik} \frac{K_{\tau,m}^{-1} \frac{\partial K_{[m]}}{\partial V_{ik}}}{\det K_{[m]}} - \alpha_0 \alpha_i \delta_{ik}, V_{ik}
\]

Proof Without the regularization term we have
\[
\frac{\partial g}{\partial V_{ik}} = \sum_{m,i \in [m]} \frac{y_m - \sigma_m}{\sigma_m} \frac{\partial \sigma_m}{\partial V_{ik}} + 1 - y_m \frac{\partial \sigma_m}{\partial V_{ik}} \frac{1}{\sigma_m}
\]
= \[
= w \sum_{m,i \in [m]} \frac{y_m - \sigma_m}{\sigma_m} \tr\left(K_{[m]}^{-1} \frac{\partial K_{[m]}}{\partial V_{ik}}\right) \det K_{[m]}
\]
= \[
= w \sum_{m,i \in [m]} \frac{y_m - \sigma_m}{\sigma_m} \tr\left(K_{[m]}^{-1} \frac{\partial K_{[m]}}{\partial V_{ik}}\right) \det K_{[m]}
\]
Since,
\[
[K_{[m]}^{-1}]_{s,i} D_{s,t}^2 = \sum_{j=1}^{r} \sum_{j=1}^{r} V_{s,j} R_{r,j}^2 \delta_{s,t} \delta_{r,j}
\]
Thus,
\[
\tr\left(K_{[m]}^{-1} \frac{\partial K_{[m]}}{\partial V_{ik}}\right) = 2R_{r,k}^2 \delta_{i,k} \delta_{s,k} + V_{s,k} \delta_{i,k}
\]
adding the regularization term concludes the proof. □

lemma When \(V\) and \(R\) are fixed, the gradient of (22) with respect to \(D_{i,i}\) is
\[
\frac{\partial g}{\partial D_{i,i}} = 2w \sum_{m,i \in [m]} \frac{y_m - \sigma_m}{\sigma_m} \tr\left(K_{[m]}^{-1} \frac{\partial K_{[m]}}{\partial D_{i,i}}\right) \det K_{[m]}
\]

Proof Similarly, without the regularization term, we have
\[
\frac{\partial g}{\partial D_{i,i}} = w \sum_{m,i \in [m]} \frac{y_m - \sigma_m}{\sigma_m} \tr\left(K_{[m]}^{-1} \frac{\partial K_{[m]}}{\partial D_{i,i}}\right) \det K_{[m]}
\]
Algorithm 1 Optimization algorithm.

**Input:** $\alpha_0 \in \mathbb{R}, \beta \in \mathbb{R}$ the momentum coefficient, $m \in \mathbb{N}$ the minibatch size, $\epsilon \in \mathbb{R}$ the gradient step, $t=0$ the iteration counter, $T=0$, past data $D=\{I_m, y_m\}_{1 \leq m \leq M}$.

**Initialization:** Compute item popularity and output regularization weights $\alpha_i$.

Set $D_0 \sim \mathcal{N}(1, 0.01)$ on the diagonal and $D_0 \equiv 0$ the gradient accumulation on $D$. 
Set $V_0 \sim \mathcal{N}(0, 0.01)$ everywhere and $\tilde{V}_0 \equiv 0$ the gradient accumulation on $V$.

**if** multitask **then**

Set $R_{r,0} \sim \mathcal{N}(1, 0.01)$ on the diagonal for each task and $\tilde{R}_{r,0} \equiv 0$ the gradient accumulation on $R_r$.

**end if**

**while** not converged **do**

**if** $m(t+1) > M(T+1)$ **then**

Shuffle $D$ and set $T = T + 1$

**end if**

**if** multitask **then**

Update $(\tilde{V}_{t+1}, \tilde{D}_{t+1}, (\tilde{R}_{r,t+1})_\tau) = \beta \left( \tilde{V}_t, \tilde{D}_t, (\tilde{R}_{r,t})_\tau \right) + (1 - \beta) \epsilon \nabla g(V_t + \beta \tilde{V}_t, D_t + \beta \tilde{D}_t, (R_{r,t} + \beta \tilde{R}_{r,t})_\tau)$ according to formulas (22), (27) and (31)

**else**

Update $(\tilde{V}_{t+1}, \tilde{D}_{t+1}) = \beta(\tilde{V}_t, \tilde{D}_t) + (1 - \beta) \epsilon \nabla f(V_t + \beta \tilde{V}_t, D_t + \beta \tilde{D}_t)$ according to formulas (12) and (20)

**end if**

Update $V_{t+1} = V_t + \tilde{V}_{t+1}$

Update $D_{t+1} = D_t + \tilde{D}_{t+1}$

**if** multitask **then**

Update $R_{r,t+1} = R_{r,t} + \tilde{R}_{r,t+1}$ for all $\tau$

**end if**

**end while**

Using (27)

\[
\frac{\partial K_{[m]}}{\partial D_{s,i}}_{s,t} = 2D_{s,i} \delta_{s,i} \delta_{t,i}
\]

thus

\[
\text{tr} \left( K_{[m]}^{-1} \frac{\partial K_{[m]}}{\partial D_{s,i}} \right) = 2K_{[m]}^{-1} D_{s,i} \delta_{s,i}
\]

adding the regularization term concludes the proof. □

**Lemma** When $V$ and $D$ are fixed, the gradient of (22) with respect to $R_{k,k}$ is

\[
\frac{\partial g}{\partial R_{k,k}} = 2w \sum_{m,r \in [m]} \frac{y_r - \sigma_m}{\sigma_m} R_{r,k,k} K_{[m]}^{-1} V_{s,k} - V_{s,k}^T \cdot \text{det} K_{[m]} - \alpha_0 \sigma_r R_{k,k}
\]

**Proof** Similarly, without the regularization term, we have

\[
\frac{\partial g}{\partial R_{k,k}} = w \sum_{m,i \in [m]} \frac{y_r - \sigma_m}{\sigma_m} \text{tr} \left( K_{[m]}^{-1} \frac{\partial K_{[m]}}{\partial R_{r,k,k}} \right) \cdot \text{det} K_{[m]} \quad (32)
\]

Using (27)

\[
\frac{\partial K_{[m]}}{\partial R_{k,k}} |_{s,t} = 2R_{r,k,k} V_{s,k} V_{t,k} 
\]