Two-photon form factors of the $\pi^0$, $\eta$ and $\eta'$ mesons in the chiral theory with resonances

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We have developed a phenomenological approach which describes very well the $\pi^0$, $\eta$ and $\eta'$ meson two-photon form factors. The simultaneous description of the $\pi^0$, $\eta$ and $\eta'$ meson two-photon form factors is consistent with data in the space-like region. The obtained form factors are implemented in the event generator EKHARA and the simulated cross sections are presented. Uncertainties in the measured form factors coming from the model dependence in Monte Carlo simulations are studied. The model predictions for the form factor slopes at the origin are given and the high-$Q^2$ limit is also discussed.

I. INTRODUCTION

The two-photon transition form factors of the pseudoscalar mesons $\pi^0$, $\eta$ and $\eta'$ have received a great attention lately – both from the experimental and the theoretical side. The recent BaBar experiments [1, 2] have provided us with important information in the high-$Q^2$ region of the photon virtuality and have triggered new insight into the structure of mesons [3–10]. Hopefully, results from Belle experiment will soon be available, and will provide a very important cross-check of the BaBar data and boost a progress in the form factor phenomenology. A new experiment KLOE-2 at Frascati [11, 12] will soon be able to provide us with data and boost a progress in the form factor phenomenology. The recent BaBar experiments [1, 2] have provided us with important information in the high-$Q^2$ region of the photon virtuality and have triggered new insight into the structure of mesons [3–10].

The Monte Carlo generators based on reliable models are needed for data analysis and feasibility studies. One of the tools in this field is the Monte Carlo generator EKHARA [14, 15], which is already in use by KLOE-2 Collaboration [12]. A realiable simulation has to account for both photon virtualities in the form factor even for a “single-tag” experiment. Therefore, the formulae for the form factors as functions of two photon virtualities are needed. This criterion considerably reduces the choice for the form factor, because the majority of the published formulae within different theoretical approaches hold only for the case with one photon being real and the other — space-like and virtual.

It is worthwhile to stress that the knowledge of the transition form factors is important in itself, but it is also required for the calculation of the hadronic light-by-light (hLbyL) scattering part of the anomalous magnetic moment of the muon ($\alpha^{\text{hLbyL}}_\mu$), see, e.g., [10, 11].

In order to take full advantage from the newly planned $g - 2$ experiments at Fermilab [20] and JPARC [21], it is mandatory to improve the accuracy of the hLbyL contribution. This subject has been recently discussed in detail during the dedicated workshop in Seattle (http://www.int.washington.edu/PROGRAMS/11-47w/).

Many important issues related to the $\gamma^*\gamma^*P$ interaction have recently been discussed in [22].

It has not been feasible so far to develop a rigorous QED/QCD based theoretical description of the two-photon interaction of mesons, which would be applicable at an arbitrary energy scale. Various methods have been used, depending on the aim of a research: the Brodsky-Lepage (BL) high-$Q^2$ limit and interpolation formula [23]; the Operator Product Expansion (OPE) approach to Vector-Vector-Pseudoscalar (VVP) and Vector-Vector-Axial (VVA) three-point functions of QCD [24]; the Vector Meson Dominance (VMD) models [25, 26]; the holographic QCD approaches [5, 27–29]; the QCD sum rules [7–9, 30, 31]; the modified perturbative approach [1]; the Regge models [32]; the Dyson-Schwinger equation [33]; the Nambu-Jona-Lasinio model [34, 35]; the constituent quark models [3, 36]; the Resonance Chiral Theory approach [37, 38]; and others. The research in this field is mainly dedicated to the high-$Q^2$ region of the form factor with one real and one virtual photon. When one needs to cover a wide range of the photon virtuality, both high-energy and low-energy methods have to be merged in some appropriate way.

The purpose of this paper is twofold: to develop a reliable model able to describe the two-photon form factors of $\pi^0$, $\eta$ and $\eta'$ mesons with a very small number of parameters and, then, to implement these form factors in the generator EKHARA.

Our approach is described in Section [11]. We start from the formalism of chiral effective theory with resonances [39–41] as a phenomenological model. The masses...
II. OUR APPROACH AND THE RESULTS

A. Formulae for \( F_{\gamma^*\gamma^*P} \)

The two-photon form factor \( F_{\gamma^*\gamma^*P}(t_1, t_2) \) for the meson of type \( P = \pi^0, \eta, \eta' \) encodes the dependence of the amplitude \( \mathcal{M}(\gamma^*\gamma^* \to P) \) on the virtuality of the photons \((q_1^2 = t_1, q_2^2 = t_2)\):

\[
\mathcal{M}[\gamma^*(q_1, \nu) \gamma^*(q_2, \beta) \to P] = e^2 \epsilon_{\mu\nu\sigma\beta} q_1^\mu q_2^\nu F_{\gamma^*\gamma^*P}(t_1, t_2),
\]

where \( \epsilon_{\mu\nu\sigma\beta} \) is the totally antisymmetric Levi-Civita tensor. Note that \( F_{\gamma^*\gamma^*P}(t_1, t_2) = F_{\gamma^*\gamma^*P}(t_2, t_1) \) due to Bose symmetry of the photons. We obtain the formulae for the form factors \( F_{\gamma^*\gamma^*P}(t_1, t_2) \) on the basis of the effective chiral Lagrangian \((\text{ECL})\) extended to multi-octet resonance contributions, with the \( \eta - \eta' \) mixing accounted for as in \([43, 44]\). A brief summary of the model is given in Appendix A. We would like to remark that a similar approach was applied in the context of other processes in \([45, 46]\).

For simplicity we neglect the mixing between the octets, which can be added if required by the data. The diagrams describing \( \gamma^*\gamma^*P \) transition are presented in Fig. 1. The form factors read

\[
F_{\gamma^*\gamma^*\pi^0}(t_1, t_2) = -\frac{N_c}{12\pi^2 f_\pi} + \sum_{i=1}^n \frac{4 \sqrt{\beta} f_{V_i}}{3 f_\pi} \left( D_{\rho_i}(t_1) + D_{\omega_i}(t_1) \right)
+ \sum_{i=1}^n \frac{4 \sqrt{\beta} f_{V_i}}{3 f_\pi} t_2 \left( D_{\rho_i}(t_2) + D_{\omega_i}(t_2) \right) - \sum_{i=1}^n \frac{4 \sqrt{\beta} f_{V_i}^2}{3 f_\pi} t_2 t_1 \left( D_{\rho_i}(t_2) D_{\omega_i}(t_1) + D_{\rho_i}(t_1) D_{\omega_i}(t_2) \right),
\]

\[
F_{\gamma^*\gamma^*\eta}(t_1, t_2) = -\frac{N_c}{12\pi^2 f_\pi} \left( \frac{5}{3} C_q - \frac{\sqrt{3}}{3} C_s \right)
+ \sum_{i=1}^n \frac{4 \sqrt{\beta} f_{V_i}}{3 f_\pi} t_1 \left( 3 C_q D_{\rho_i}(t_1) + \frac{1}{3} C_q D_{\omega_i}(t_1) - \frac{2 \sqrt{3}}{3} C_s D_{\phi_i}(t_1) \right)
+ \sum_{i=1}^n \frac{4 \sqrt{\beta} f_{V_i}}{3 f_\pi} t_2 \left( 3 C_q D_{\rho_i}(t_2) + \frac{1}{3} C_q D_{\omega_i}(t_2) - \frac{2 \sqrt{3}}{3} C_s D_{\phi_i}(t_2) \right)
- \sum_{i=1}^n \frac{8 \sigma_{V_i} f_{V_i}^2}{f_\pi} t_2 t_1 \left( \frac{1}{2} C_q D_{\rho_i}(t_2) D_{\rho_i}(t_1) + \frac{1}{18} C_q D_{\omega_i}(t_2) D_{\omega_i}(t_1) - \frac{\sqrt{3}}{9} C_s D_{\phi_i}(t_2) D_{\phi_i}(t_1) \right),
\]

\[
F_{\gamma^*\gamma^*\eta'}(t_1, t_2) = F_{\gamma^*\gamma^*\eta}(t_1, t_2) \bigg| \begin{array}{c} C_q \to C_q' \\ C_s \to -C_s' \end{array},
\]

where \( n \) is a number of the vector meson resonance octets. The definitions of all couplings can be found in the Ap-
The vector meson propagators $D_V$ are

$$D_V(Q^2) = [Q^2 - M_V^2 + i \sqrt{Q^2 \Gamma_{tot,V}(Q^2)}]^{-1}. \quad (5)$$

In this paper we consider only the data in the space-like region of photon virtuality, thus the modeling of the vector resonance energy dependent widths $\Gamma_{tot,V}(Q^2)$ is not relevant as the widths are equal to zero. We take the values of the masses of all particles according to PDG [42].

We require that the form factors $F_{\gamma\gamma\gamma\gamma}(t_1, t_2)$ given in [2], [3] and [4] vanish when the photon virtuality $t_1$ goes to infinity for any value of $t_2$:

$$\lim_{t_1 \to -\infty} F_{\gamma\gamma\gamma\gamma}(t_1, t_2) \big|_{t_2 = \text{const}} = 0. \quad (6)$$

Notice, that in this case the conditions

$$\lim_{t \to \infty} F_{\gamma\gamma\gamma\gamma}(t, t) = 0, \quad (7)$$

$$\lim_{t \to \infty} F_{\gamma\gamma\gamma\gamma}(t, 0) = 0 \quad (8)$$

are automatically satisfied, which is considered as a correct short-distance behavior of the form factors (see, for example, discussion in [24]). The constraint (6) leads to the following relations for the couplings:

$$\sqrt{2} h_{V_i} f_{V_i} - \sigma_{V_i} f_{V_i} = 0, \quad i = 1, \ldots, n, \quad (9)$$

$$- \frac{N_c}{4 \pi^2} + 8 \sqrt{2} \sum_{i=1}^{n} h_{V_i} f_{V_i} = 0. \quad (10)$$

Therefore, for an ansatz with $n$ vector resonance octets the two-photon form factors $F_{\gamma\gamma\gamma\gamma}(t_1, t_2)$ are determined by $2n$ parameters (i.e., the products of the couplings: $f_{V_i} h_{V_i}$ and $\sigma_{V_i} f_{V_i}^2$, $i = 1, \ldots, n$), from which $n - 1$ are to be determined by experiment and the rest $n + 1$ are fixed by (9) and (10). For the one octet ansatz there are no free parameters and in case of the two octets ansatz there is one free parameter.

One of the main objectives of this paper was to develop a reliable model for the $\gamma\gamma\gamma\gamma$ transition form factors in the space-like region reflecting the experimental data and theoretical constrains and in the same time being as simple as possible. Even if we know that the $SU(3)$ flavor symmetry is broken we start our investigations using an $SU(3)$-symmetric model (apart from the masses of the mesons, which are fixed at their PDG [12] values) and try to see how many resonance octets we have to include in order to describe the data well. The existing data for the transition form factors in space-like region [1, 2, 3, 40] come from single-tag experiments, where one of the invariants is very close to zero (the one associated with the “untagged” lepton), thus we have information only about $F_{\gamma\gamma\gamma\gamma}(t, 0)$. It is common to define the $\gamma\gamma\gamma\gamma$ form factor $F_{\gamma\gamma\gamma\gamma}(Q^2, 0) \equiv F_{\gamma\gamma\gamma\gamma}(t, 0)$ with $Q^2 \equiv -t$ (associated with the “tagged” lepton). From Eqs. (2), (3) and (4) we see that $F_{\gamma\gamma\gamma\gamma}(Q^2, 0)$ is driven by $n$ parameters (i.e., the products of the couplings: $f_{V_i} h_{V_i}$, $i = 1, \ldots, n$) and there is always only one constraint (10) for any $n$. Therefore, the number of parameters in $F_{\gamma\gamma\gamma\gamma}(Q^2, 0)$ to be determined by experiment (“free parameters”) equals to $n - 1$ (similarly to the case of the $F_{\gamma\gamma\gamma\gamma}(t_1, t_2)$). In case of the one octet ansatz there are no free parameters and in the two octets case there is one free parameter.

**B. The one octet ansatz for the form factors**

Let us consider first the one octet ansatz. In this case

$$f_{V_i} h_{V_i} = \frac{3}{32 \pi^2 \sqrt{2}}, \quad (11)$$

and the model gives a prediction for the form factors $F_{\gamma\gamma\gamma\gamma}(Q^2, 0)$ without any possibility for adjustment. The predictions of this model are compared with experimental data in Figs. 2-4 (dotted line). To quantify the quality of the agreement of the model predictions we have calculated the $\chi^2$ values for each data set. For the pion transition form factor the model agrees with CELLO [15] and CLEO [40] and disagrees with the BaBar data [1], as can be seen from Table II which shows the $\chi^2$ values per experiment. For the $\eta$ and $\eta'$ transition form factor the model is in a perfect agreement with CELLO, however for CLEO and BaBar the $\chi^2$ is not good. In total, for the one octet ansatz we obtain $\chi^2 \approx 358$ for 116 experimental points.

Even though the overall agreement of this simple model with the data is not bad, there is a way to improve it, as will be discussed below.
FIG. 3: Transition form factor $\gamma^*\gamma\eta$ compared to the data. The high-$Q^2$ limit is shown as a bold solid straight line at $2 \times f_\eta = 2 \times 0.0975$ GeV, according to [43] and [44] (BL). The limit according to the two-angle $\eta-\eta'$ mixing scheme [43, 144] (FK) is shown as a shaded box (green online) at 0.1705 . . . 0.1931 GeV, accounting for the parameter ambiguities [43]. The high-$Q^2$ limit in our 1 octet ansatz and 2 octets ansatz are marked as (1) and (2), correspondingly.

FIG. 4: Transition form factor $\gamma^*\gamma\eta'$ compared to the data. The high-$Q^2$ limit is shown as a bold solid straight line at $2 \times f_\eta' = 2 \times 0.0744$ GeV, according to [43] and [44] (BL). The limit according to the two-angle $\eta-\eta'$ mixing scheme [43, 44] (FK) is shown as a shaded box (green online) at 0.29 . . . 0.31 GeV, accounting for the parameter ambiguities [43]. The high-$Q^2$ limit in our 1 octet ansatz and 2 octets ansatz are marked as (1) and (2), correspondingly.

C. The two octet ansatz for the form factors

In order to make the model more flexible, we include the second vector meson multiplet contributions. We would like to note that there are many known cases when in order to improve the model predictions one needs to account for the excited vector resonances, the charged form factor of the pion is among the most famous examples.

In the two octet ansatz we chose $h_{V_i}$ as a free parameter and determine the value of $f_{V_i} = 0.20173(86)$ using the PDG [42] value for the width

$$\Gamma(\rho \to ee) = \frac{e^4 M_\rho f_{V_i}^2}{12\pi}.$$  (12)

The fit to the data gives $\chi^2 \approx 140$ for 116 experimental points. The obtained value of $h_{V_i}$ is

$$h_{V_i} = 0.03121(14),$$  (13)

where the error is the parabolic error given by MINOS package from MINUIT CERNLIB program. The remaining coupling is given by

$$h_{V_2} f_{V_2} = \frac{3}{32\pi^2 \sqrt{2}} - h_{V_1} f_{V_1} = 0.42(5) \times 10^{-3}.$$  (14)

The comparison of the two octet ansatz with the data is also shown in Figs. 2,4 (solid line) and the $\chi^2$ values per experiment are given in Table I. The only data sample which is not in consistency with the model is the CLEO 98 sample which is not in consistency with the model.

In principle the parameter $h_{V_1}$ can be estimated by experiment via the value of the width

$$\Gamma(\rho^0 \to \pi^0 \gamma) = \frac{4\alpha M_\rho^2 h_{V_1}^2}{27 f_\pi^2} \left(1 - \frac{m^2}{M_\rho^2}\right)^3.$$  (15)

Using the PDG [42] values for the width [45], one obtains

$$h_{V_1} = 0.041(3),$$  (16)

which is in tension with the value given by fit [13]. This might be a result of neglecting the higher octets or omission of the $SU(3)$ flavor breaking effects. In order to check this we have added the third octet to the model and fitted two free parameters: $h_{V_1}$ and $h_{V_2} f_{V_2}$. The fit to the experimental data gives $\chi^2 \approx 136$ for 116 experimental points, so there is no essential improvement in the description of data by the model. The fit gives $h_{V_1} = 0.03279(75)$ and $h_{V_2} f_{V_2} = -0.73(54) \times 10^{-3}$.

Finally, for the couplings of the third octet one gets

$$h_{V_3} f_{V_3} = \frac{3}{32\pi^2 \sqrt{2}} - h_{V_1} f_{V_1} - h_{V_2} f_{V_2} \approx 7.45 \times 10^{-3}.$$  (17)

In this fit we observe a very high correlation between the
TABLE I: The $\chi^2$ per experiment and the total $\chi^2$. Number of data points (n.d.p.) is also given for each experiment. In all given experiments the pseudoscalar meson is produced in a two-photon process $e^+e^- \rightarrow e^+e^- P$, but the decay channels for $P$ identification vary. The “2 octets” column is calculated with the parameter values given by the global fit.

| experiment | 1 octet $\chi^2$/n.d.p. | 2 octets $\chi^2$/n.d.p. |
|------------|--------------------------|--------------------------|
| CELLO ($\pi^0 \rightarrow \gamma\gamma$) | 0.29/5 | 0.47/5 |
| CLEO ($\pi^0 \rightarrow \gamma\gamma$) | 6.27/15 | 20.96/15 |
| BaBar ($\pi^0 \rightarrow \gamma\gamma$) | 124.83/17 | 55.85/17 |
| CLEO ($\eta \rightarrow \gamma\gamma$) | 0.24/4 | 0.13/4 |
| CLEO ($\eta \rightarrow \pi^+\pi^-\pi^0$) | 19.28/6 | 11.13/6 |
| CLEO ($\eta \rightarrow \gamma\gamma$) | 8.55/8 | 2.10/8 |
| CLEO ($\eta \rightarrow \pi^+\pi^-\pi^0$) | 10.91/5 | 5.63/5 |
| BaBar ($\eta \rightarrow \gamma\gamma$) | 89.02/11 | 9.34/11 |
| CELLO ($\eta' \rightarrow \gamma\gamma$) | 0.11/5 | 0.29/5 |
| CLEO ($\eta' \rightarrow \gamma\gamma\pi^+\pi^-$) | 19.90/6 | 7.48/6 |
| CLEO ($\eta' \rightarrow \gamma\gamma\pi^+\pi^-\pi^+$) | 2.61/5 | 1.44/5 |
| CLEO ($\eta' \rightarrow \gamma\pi^+\pi^-$) | 14.01/6 | 4.64/6 |
| CLEO ($\eta' \rightarrow 6\gamma$) | 21.54/5 | 12.62/5 |
| CLEO ($\eta' \rightarrow 10\gamma$) | 0.49/2 | 0.23/2 |
| CLEO ($\eta' \rightarrow \pi^+\pi^06\gamma$) | 5.93/5 | 4.80/5 |
| BaBar ($\eta' \rightarrow \gamma\gamma$) | 33.87/11 | 3.10/11 |
| total | 357.87/116 | 140.22/116 |

TABLE II: The $\rho \rightarrow \pi\gamma$ decay width uncertainty and the corresponding values of $h_{V_3}$. Parameters $h_{V_3}$ and $h_{V_3}f_{V_3}$ with the off-diagonal correlation coefficient equal to $-0.99$. Notice that $h_{V_3}$ is almost unchanged as compared to [13] and we conclude that in order to accommodate the value [16] in this model we would need to allow for couplings which break the SU(3) flavor symmetry. This is however beyond the scope of the present paper. We leave the possible refinements of the model for further investigations.

| decay          | width       | reference | $h_{V_3}$ |
|----------------|-------------|-----------|------------|
| $\rho^0 \rightarrow \pi^0\gamma$ | 89(12) keV | PDG [42] | 0.041(3)   |
| $\rho^0 \rightarrow \pi^+\gamma$ | 77(20) keV | SND [50] | 0.038(5)   |
| $\rho^+ \rightarrow \pi^+\gamma$ | 68(7) keV  | PDG [42] | 0.036(2)   |

D. The Monte Carlo simulation

We have implemented the transition form factors obtained within the two octet model described above into the Monte Carlo generator EKHARA [http://prac.us.edu.pl/~ekhara]. From the technical point of view of the event generation, it is a straightforward generalization because the mappings used in [14] for $\pi^0$ work similarly well also for $\eta$ and $\eta'$. We simulate the cross sections $d\sigma/dQ^2$ for the process $e^+e^- \rightarrow e^+e^- P$ and compare it with existing “single-tag” data from the CELLO [45], CLEO [46] and BaBar [12] experiments. In a single-tag experiment, the “tagged” lepton fixes the value of $Q^2 = -t_1$ and the 4-momentum squared of the “untagged” lepton $t_2 = -q_2^2$ is kinematically restricted nearby zero. For example, in the BaBar experiment, the actual thresholds for $q_2^2$ are 0.18 GeV$^2$ for pions [1] and 0.38 GeV$^2$ for $\eta$ and $\eta'$ due to the imposed event selection. The experimental $d\sigma/dQ^2$ is given within these cuts, and, therefore, the simulated $d\sigma/dQ^2$ is computed within the similar event selection. As expected, a good agreement between the generator predictions and the data is observed, see Figs. 4, 5.

An important note here is in order. The values of the $d\sigma/dQ^2$ are the primary results of the experiment. The form factor $F_{P}(Q^2, 0)$ is calculated then on the basis of the measured $d\sigma/dQ^2$ and the simulation. It is known that the model dependence in the simulation leads to the uncertainty in the form factor, which is “measured” in this way. In the BaBar analyses, the corresponding uncertainties in cross section are estimated to be at the level of 3.5 % for pions [1] and 4.6 % for $\eta$ and $\eta'$ [2] (based on the simulation with the $q_2^2$-dependent and $q_2^2$-independent form factors). As stressed in [1], this uncertainty is very sensitive to the actual $q_2^2$ cut.

Recently, the effects of the $q_2^2$ cut were emphasized on the level of the form factor considerations [26]. The Monte Carlo generator in hand allows us to perform a more conclusive study, namely to investigate the magnitude of the cross section uncertainties, discussed above. Similarly to the method used in the BaBar [12] analysis, we perform two simulations: the first ($d\sigma[full]/dQ^2$) with the exact form factor $F_{\gamma\gamma\gamma\gamma}(t_1, t_2)$ and the second ($d\sigma[approx]/dQ^2$) with the approximated form factor $F_{\gamma\gamma\gamma\gamma}(t_1, t_2) = F_{\gamma\gamma\gamma}\gamma(t_1, t_2)$, i.e., neglecting the momentum transfer to the untagged lepton in the form factor. The relative difference of the corresponding cross sections is then plotted in Fig. 4. Our estimations for the uncertainty are in a rough agreement with that of BaBar [1, 2]. However, in contrast to the estimate of BaBar, a dependence of this uncertainty on $Q^2$ is observed in our simulation. If this effect is not accounted for in the data it might result in inducing a fake $Q^2$ dependence of the form factor.

In order to investigate the impact of the event selection on the error estimate, we perform the simulation for $\eta$ and $\eta'$ mesons with the direct cut on the second (untagged) invariant ($q_2^2$) and separately with the cut on
the angle between the initial and final untagged lepton ($|\cos \theta^*_{lep}| > 0.99$), which effectively induces the cut on $q_T^2$. From Fig. 7 we see that the error estimate and its $Q^2$ dependence is very sensitive to the event selection.

III. THE LIMITS OF THE FORM FACTORS

A. The high-$Q^2$ limit of the form factors

The issue of the asymptotic behavior of the form factors usually deserves an attention. In our approach, the high-$Q^2$ limits ($t \rightarrow -\infty$) are following. In case of the one octet ansatz, we obtain from Eqs. (2), (3) and (4):
\[ F_{\gamma^*\pi^0}(t,0) = \frac{1}{8\pi^2 f_\pi} \frac{1}{t} (M_\rho^2 + M_\omega^2) + O\left(\frac{1}{t^2}\right), \]  
(17)

\[ F_{\gamma^*\gamma^*\pi^0}(t,t) = \frac{1}{4\pi^2 f_\pi} \frac{1}{t^2} M_\rho^2 M_\omega^2 + O\left(\frac{1}{t^3}\right), \]  
(18)

\[ F_{\gamma^*\gamma\eta}(t,0) = \frac{1}{8\pi^2 f_\pi} \frac{1}{t} \left(3C_q M_\rho^2 + \frac{1}{3} C_q M_\omega^2 - \frac{2\sqrt{2}}{3} C_s M_\rho^2\right) + O\left(\frac{1}{t^2}\right), \]  
(19)

\[ F_{\gamma^*\gamma^*\eta}(t,t) = \frac{1}{8\pi^2 f_\pi} \frac{1}{t^2} \left(3C_q M_\rho^4 + \frac{1}{3} C_q M_\omega^4 - \frac{2\sqrt{2}}{3} C_s M_\rho^4\right) + O\left(\frac{1}{t^3}\right). \]  
(20)

In case of the two octets ansatz, we obtain

\[ F_{\gamma^*\gamma^*\pi^0}(t,0) = \frac{4\sqrt{2}}{3f_\pi} \frac{1}{t} \left[h_{V_1} f_{V_1} (M_\rho^2 + M_\omega^2) + h_{V_2} f_{V_2} (M_\rho^2 + M_\omega^2)\right] + O\left(\frac{1}{t^2}\right), \]  
(21)

\[ F_{\gamma^*\gamma^*\pi^0}(t,t) = \frac{8\sqrt{2}}{3f_\pi} \frac{1}{t^2} \left[h_{V_1} f_{V_1} M_\rho^4 + h_{V_2} f_{V_2} M_\rho^4 M_\omega^2 + h_{V_2} f_{V_2} M_\rho^4 M_\omega^2\right] + O\left(\frac{1}{t^3}\right), \]  
(22)

\[ F_{\gamma^*\gamma\eta}(t,0) = \frac{4\sqrt{2}}{3f_\pi} \frac{1}{t} \left[h_{V_1} f_{V_1} (3C_q M_\rho^2 + \frac{1}{3} C_q M_\omega^2 - \frac{2\sqrt{2}}{3} C_s M_\rho^2) + h_{V_2} f_{V_2} (3C_q M_\rho^2 + \frac{1}{3} C_q M_\omega^2 - \frac{2\sqrt{2}}{3} C_s M_\rho^2)\right] + O\left(\frac{1}{t^2}\right), \]  
(23)

\[ F_{\gamma^*\gamma^*\eta}(t,t) = \frac{8\sqrt{2}}{3f_\pi} \frac{1}{t^2} \left[h_{V_1} f_{V_1} (3C_q M_\rho^4 + \frac{1}{3} C_q M_\omega^4 - \frac{2\sqrt{2}}{3} C_s M_\rho^4) + h_{V_2} f_{V_2} (3C_q M_\rho^4 + \frac{1}{3} C_q M_\omega^4 - \frac{2\sqrt{2}}{3} C_s M_\rho^4)\right] + O\left(\frac{1}{t^3}\right). \]  
(24)

The limits for the \( \eta' \) form factor can be obtained from the above formulae according to [1].

The expressions [21] and [23] guide the high-\( Q^2 \) behavior of the form factors measured in single-tag exper-
iments (shown in Figs. 2 [3 4]). We see that in our approach the asymptotic value of $Q^2 |F_P(Q^2, 0)|$ depends not only on the mixing parameters and decay constants but also on the masses of the vector resonances. Numerically, for $\pi^0$ transition form factor, the value of $Q^2 |F_{\pi^0}(Q^2, 0)|$ in our approach with two octets is very close to that of the Brodsky-Lepage [23] high-$Q^2$ limit $Q^2 |F_{\pi^0}(Q^2, 0)| \to 2f_{\pi^0}$ shown as a bold solid line (BL) in Fig. 3.

The perturbative QCD prediction for the asymptotic of the $\eta$ and $\eta'$ form factors is often given in a simple approach in terms of the parameters $f_0 = 0.9075$ GeV and $f_{\eta'} = 0.0744$ GeV [4 24 25]. These values are shown as bold solid line (BL) in Figs. 2 [4 4] and in this case one can notice no coincidence with the values given by [23]. Sometimes it is also called the Brodsky-Lepage limit, with a reference to [51]. However, we would like to remark that in [51] the SU(3) flavor breaking effects are not considered and the assumed $\eta - \eta'$ mixing may be not consistent with modern data. An attempt to interpret the results of [51] by means of $f_{\eta'} = 0.0744$ GeV is in tension with the data for $\eta'$ form factor, as one can see in Fig. 2 and as noticed in [51].

In principle, there are other ways to apply the formulae of [51] to the form factors of physical states $\eta$ and $\eta'$. For example, the limit for the $\eta$ and $\eta'$ transition form factors can be calculated according to the two-angle $\eta - \eta'$ mixing scheme [4 44]. The latter values are shown as a shaded box (FK) in Figs. 2 [4 4] (green online). Notice, that numerically this limit for $\eta$ meson is very close to that of BL approach and also to the value given by our model, however for $\eta'$ all the three values are different.

B. The slope of the form factor at the origin

Sometimes it is convenient to define the so-called slope of the transition form factor at the origin $a_P$ ("slope parameter"): \[ a_P = \frac{1}{N_C} \sum_{i=1}^{n} h_{V_i} f_{V_i} \left( \frac{1}{M_{\rho_i}^2} + \frac{1}{M_{\omega_i}^2} \right), \] (26) where $x \equiv t/m_P^2$. Notice that being defined this way, $a_P$ is a dimensionless quantity, which is related to the effective region of the $\gamma\gamma^*P$ interaction ($r_P^2 = 6a_P/m_P^2$). The average experimental value for $a_\pi$ listed in PDG [42] (linear coefficients of the $\pi^0$ electromagnetic form factor) is mainly driven not by a direct measurement, but by an extrapolation done in Ref. [45]. The direct measurements of $a_\pi$ are less precise [53 54]. The experimental knowledge of $a_\eta$ is much better and recently the new experiments contributed: MAMI-C [55] and NA60 [56 57]. For $a_{\eta'}$, we were not able to find a result of a direct measurement.

From Eqs. (4) [2 3] and (4) one obtains the following model prediction for the slope parameters: \[ a_\eta = \frac{16\sqrt{2}\pi^2 m_\eta^2}{N_C} \sum_{i=1}^{n} h_{V_i} f_{V_i} \left( \frac{1}{M_{\rho_i}^2} + \frac{1}{M_{\omega_i}^2} \right), \] (27) \[ a_{\eta'} = \frac{16\sqrt{2}\pi^2 m_{\eta'}^2}{N_C} \sum_{i=1}^{n} h_{V_i} f_{V_i} \left( \frac{1}{M_{\rho_i}^2} + \frac{1}{M_{\omega_i}^2} \right), \] (28)

The numerical values for $a_\pi$ are listed in Table III On its basis we conclude that there is a reasonable agreement between model predictions and experiments.

We would like to remark that in the limit of the equal masses for vector resonances within the octet, Eqs. (23) [27, 25] lead to the following relation between the slope parameters: $a_\pi/m_\pi^2 = a_\eta/m_\eta^2 = a_{\eta'}/m_{\eta'}^2$.

IV. SUMMARY

Using the scheme of the $\eta - \eta'$ mixing with two decay parameters ($f_0$ and $f_8$) and two mixing angles ($\theta_0$, $\theta_8$) [45 44] and following the approach of chiral effective theory with resonances, [23 41], we derive the expressions for the two-photon transition form factors of the $P = \pi^0, \eta, \eta'$ mesons. The tree-level contributions within this effective field theory approach are considered. For the case of the one octet ansatz there are no free parameters and we obtain the model prediction for the form factors. For the case of the two octet ansatz the model parameter is fitted to the data. We find that the two octet calculation is consistent with the bulk of available data. The high-$Q^2$ limits of the form factors in our approach are compared to those of the Brodsky and Lepage [3 23 46] and to those of Feldmann and Kroll [4 44]. The slope of the transition form factor at the origin, $a_P$, is calculated and compared to available data. A reasonable agreement between model predictions and experiments is found.

The obtained form factors are implemented in the EKHARA Monte Carlo generator. As a test of the generator, the cross-section $d\sigma/dQ^2$ is simulated for the process $e^+e^- \to e^+e^-P$ and compared to data using the event selections, which mimic the “single-tag” experimental conditions.

Using the Monte Carlo simulation we investigate the impact of neglecting the momentum transfer to the untagged lepton ($t_2$) on the cross section and form factor measurements. The uncertainty in the visible cross section due to the simplification of the form factor $F_{\gamma\gamma^*P}(t_1, t_2) \approx F_{\gamma\gamma^*P}(t_1, 0)$ is estimated for the phase space cuts similar to the experimental ones.

Due to very small number of free parameters and good agreement with data, the approach presented in this work
is a good starting point for further model adjustments, e.g., for including the SU(3) flavor symmetry breaking in the couplings. Using the developed generator one will be able to study, e.g., a possible manifestation of such effects in the cross section \( d\sigma/dQ^2 \) within a realistic phase space cuts.

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### Appendix A: Formalism

The lightest pseudoscalar mesons are supposed to play a role of the (pseudo-)Nambu-Goldstone boson fields of spontaneously \( G = SU(3)_L \times SU(3)_R \) to \( H = SU(3)_Y \) broken symmetry. To introduce the physical states \( \eta \) and \( \eta' \) we choose the scheme with two mixing angles \( (\theta_0, \theta_8) \), see [43, 44]. The nonet of the pseudoscalar mesons reads

\[
u = \left( \frac{i}{\sqrt{2} f_\pi} \left( \frac{\pi^0 + C_\eta \eta' + C'_\eta \eta'}{\sqrt{2}} \exp \left( \frac{f_\pi}{f_K} K^+ \right) \frac{\pi^+}{\sqrt{2} f_K K^0} C_\eta \eta + C'_\eta \eta' \right) \right)
\]

where \( f_\pi \) and \( f_K \) are the pion and kaon decay constants and the following notation is used

\[
C_q \equiv \frac{f_\pi}{\sqrt{3} \cos(\theta_8 - \theta_0)} \left( \frac{1}{f_8} \cos \theta_0 - \frac{1}{f_0} \sqrt{2} \sin \theta_0 \right),
\]

\[
C'_q \equiv \frac{f_\pi}{\sqrt{3} \cos(\theta_8 - \theta_0)} \left( \frac{1}{f_8} \sqrt{2} \cos \theta_0 + \frac{1}{f_0} \sin \theta_0 \right),
\]

\[
C_s \equiv \frac{f_\pi}{\sqrt{3} \cos(\theta_8 - \theta_0)} \left( \frac{1}{f_8} \sqrt{2} \cos \theta_0 - \frac{1}{f_0} \sin \theta_0 \right),
\]

\[
C'_s \equiv \frac{f_\pi}{\sqrt{3} \cos(\theta_8 - \theta_0)} \left( \frac{1}{f_8} \cos \theta_0 - \frac{1}{f_0} \sqrt{2} \sin \theta_0 \right).
\]

Fixing the angles \( \theta_0, \theta_8 \) and constants \( f_0, f_8 \) [43, 44]

\[
\theta_0 = -21.2^0 \pm 1.6^0, \quad \theta_8 = -9.2^0 \pm 1.7^0,
\]

\[
f_8 = (1.26 \pm 0.04) f_\pi, \quad f_0 = (1.17 \pm 0.03) f_\pi
\]

and taking \( f_\pi = 92.4 \) MeV, one obtains \( C_q \approx 0.720, C_s \approx 0.471, C'_q \approx 0.590 \) and \( C'_s \approx 0.576 \). Notice that accordingly to notation [A1] the couplings of the \( \eta' \) meson by means of substitution

\[
C_q \rightarrow C'_q,
\]

\[
C_s \rightarrow -C'_s
\]

Obviously, this pattern also holds in the expressions for the form factors in our approach.

At the lowest order the Wess-Zumino-Witten Lagrangian [59, 60], that describes the interaction of pseudoscalar mesons with two photons, can be written down in the terms of the physical fields as

\[
\mathcal{L}_{\gamma\gamma P} = \frac{\epsilon^2 N_c}{24 \pi^2 f_\pi} \epsilon^{\mu\nu\alpha\beta} \partial_\mu B_\nu \partial_\alpha B_\beta \times \left[ \pi^0 + \eta \left( \frac{5}{3} C_q - \frac{\sqrt{2}}{3} C_s \right) + \eta' \left( \frac{5}{3} C'_q + \frac{\sqrt{2}}{3} C'_s \right) \right]
\]
where $N_c = 3$ is the number of quark colors and the electromagnetic field is denoted by $B_\mu$.

Assuming the SU(3) symmetry for the coupling constants of the vector mesons, the $\gamma V$ interaction is written as

$$L_{\gamma V} = -e \sum_{i=1}^{n} f_V \partial_\mu B_\nu (\rho_i^\mu + \frac{1}{3} \omega_i^\mu - \frac{\sqrt{2}}{3} \phi_i^\mu) (A6)$$

where we have summed over octets of the vector mesons, $V_i^\mu \equiv \partial_\mu V_i - \partial_i V_\mu$, $f_V$ is the (dimensionless) coupling for the vector representation of the spin-1 fields for a fixed octet.

The Lagrangians that describes vector-photon-pseudoscalar and two vector mesons interactions with pseudoscalar [41] in the terms of the physical fields read

$$L_{V V P} = \sum_{i=1}^{n} \frac{4 \sqrt{2} h_i}{3 f_\pi} \epsilon_{\mu \nu \alpha \beta} \partial^\alpha B^\beta \left[ (\rho_i^\mu + 3 \omega_i^\mu) \partial^\nu \pi^0 \right.$$

$$\left. + [(3 \rho_i^\mu + \omega_i^\mu) C_q + 2 \phi_i^\mu C_s] \partial^\nu \eta \right]$$

$$+ [(3 \rho_i^\mu + \omega_i^\mu) C_q' - 2 \phi_i^\mu C_s'] \partial^\nu \eta^\prime \right], \quad (A7)$$

where $h_i$ and $\sigma_i$ are the corresponding (dimensionless) coupling constants for a given $i$-th octet. For simplicity we neglect any mixing between the octets.

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