Research Article
Viscosity Approximation Methods with Errors and Strong Convergence Theorems for a Common Point of Pseudocontractive and Monotone Mappings: Solutions of Variational Inequality Problems

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1. Introduction

In many problems, for example, convex optimization, linear programming, monotone inclusions, elliptic differential equations, and variational inequalities, it is quite often to seek a proximal point of a given nonlinear problem. The proximal point algorithm is recognized as a powerful and successful algorithm in finding a common point of the fixed points of pseudocontractive mappings and the solutions of monotone mappings. Let $C$ be a closed convex subset of a real Hilbert space $H$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We recall that a mapping $A : C \to H$ is called monotone if and only if

$$
\langle x - y, Ax - Ay \rangle \geq 0, \quad \forall x, y \in C.
$$

A mapping $A : C \to H$ is called $\alpha$-inverse strongly monotone if there exists a positive real number $\alpha > 0$ such that

$$
\langle x - y, Ax - Ay \rangle \geq \alpha \| Ax - Ay \|^2, \quad \forall x, y \in C.
$$

Obviously, the class of monotone mappings includes the class of the $\alpha$-inverse strongly monotone mappings. The class of monotone mappings is one of the most important classes of mappings among nonlinear mappings. The classical variational inequality problem is formulated as finding a point $u \in C$ such that $\langle v - u, Au \rangle \geq 0$, for all $v \in C$. The set of solutions of variational inequality problems is denoted by $\text{VI}(C, A)$.

A mapping $T : C \to H$ is called pseudocontractive if, for all $x, y \in C$, we have

$$
\langle Tx - Ty, x - y \rangle \leq \| x - y \|^2.
$$

A mapping $T : C \to H$ is called $\kappa$-strict pseudocontractive if there exists a constant $0 \leq \kappa \leq 1$ such that

$$
\langle x - y, Tx - Ty \rangle \leq \| x - y \|^2 - \kappa \| (I - T)x - (I - T)y \|^2,
$$

$$
\forall x, y \in C.
$$

A mapping $T : C \to C$ is called nonexpansive if

$$
\| Tx - Ty \| \leq \| x - y \|, \quad \forall x, y \in C.
$$

Clearly, the class of pseudocontractive mappings includes the class of strict pseudocontractive mappings and the class of nonexpansive mappings. We denote by $F(T)$ the set of fixed points of $T$; that is, $F(T) = \{ x \in C : Tx = x \}$. 

A mapping $f : C \to C$ is called contractive with a contraction coefficient if there exists a constant $\rho \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \rho \|x - y\|, \quad \forall x, y \in C. \quad (6)$$

Recently viscosity approximation methods for finding fixed points of pseudocontractive mappings have received vast investigations because of their extensive applications in a variety of applied areas of partial differential equations, image recovery, and signal processing. In Hilbert spaces, many authors have studied the fixed-point problems of the nonexpansive mappings and monotone mappings by the viscosity approximation methods and obtained a series of good results; see [1–18] and the reference therein.

For finding an element of the set of fixed points of the nonexpansive mappings, Halpern [1] was the first to study the convergence of the scheme in 1967:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T(x_n). \quad (7)$$

In 2000, Moudafi [2] introduced the viscosity approximation methods and proved the strong convergence of the following iterative algorithm under some suitable conditions:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T(x_n). \quad (8)$$

Takahashi et al. [19, 20] introduced the following scheme and studied the weak and strong convergence theorems of the elements of $F(T) \cap VI(C, A)$, respectively, under different conditions:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) TP_C(x_n - \lambda_n Ax_n), \quad (9)$$

where $T$ is a nonexpansive mapping and $A$ is an $\alpha$-inverse strong monotone operator. Recently, Zegeye and Shahzad [21] introduced the mappings as follows:

$$T_r (x) = \left\{ z \in C : \langle y - z, Tz \rangle + \frac{1}{r} \langle y - z, (1 + r) z - x \rangle \leq 0, \forall y \in C \right\}. \quad (10)$$

$$F_r (x) = \left\{ z \in C : \langle y - z, Az \rangle + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}. \quad (11)$$

Very recently, Tang [22] introduced the following sequence and obtained the strong convergence theorems:

$$y_n = \lambda_n x_n + (1 - \lambda_n) \sum_{i=1}^{m} \mu_i T_{r_\i} x_n, \quad (12)$$

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{m} \alpha_i T_{r_\i} y_n. \quad (13)$$

For other related results, see [11–13, 23–25]. On the other side, there are perturbations always occurring in the iterative processes because the manipulations are inaccurate. There is no doubt that researching the convergent problems of iterative methods with perturbation members is a significant job. Starting from any initial guess, $z_0 \in H$, the proximal point algorithm generates a sequence $\{z_k\}$ according to the inclusion:

$$z_k \in z_{k+1} + \epsilon_k Az_{k+1}, \quad (14)$$

where $A$ is a maximal monotone operator and $\epsilon_k > 0$ is a parameter. For solving the original problem of finding a solution to the inclusion $0 \in Az$, Rockafellar [23] introduced the following algorithm:

$$z_k + \epsilon_k \in z_{k+1} + \epsilon_k Az_{k+1}, \quad (15)$$

where $\{\epsilon_k\}$ is a sequence of errors. Rockafellar [23] obtained the weak convergence of the algorithm. Very recently Yao and Shahzad [24] proved that sequences generated from the method of resolvent are given by

$$x_m = P_C(\alpha_m u_m + (1 - \alpha_m) T x_m), \quad m \geq 0, \quad (16)$$

where $\{\alpha_m\}$ is a sequence in $[0, 1]$, the sequence $\{u_m\} \subset H$ is a small perturbation, and $T$ is a nonexpansive mapping.

The following is our concern now: Is it possible to construct a new sequence with general errors that converges strongly to a common element of fixed points of pseudocontractive mappings and the solution set of monotone mappings and converges strongly to the unique solution of certain variational inequality?

In this paper, motivated and inspired by the above results, we introduce two iterations with perturbations which converge strongly to a common element of the set of fixed points of a finite family of pseudocontractive mappings more general than nonexpansive mappings and the solution set of a finite family of monotone mappings more general than $\alpha$-inverse strongly monotone mappings or maximal monotone mappings. Our theorems presented in this paper improve and extend the corresponding results of Yao and Shahzad [24], Zegeye and Shahzad [21], and Tang [22] and some other results in this direction.

2. Preliminaries

In the sequel, we will use the following lemmas.

**Lemma 1** (see [6]). Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$\alpha_{n+1} \leq (1 - \theta_n) \alpha_n + \sigma_n, \quad n \geq 0, \quad (17)$$

where $\{\theta_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence such that

(i) $\sum_{n=0}^{\infty} \theta_n = \infty$;

(ii) $\limsup_{n \to \infty} (\sigma_n/\theta_n) \leq 0$ or $\sum_{n=0}^{\infty} \sigma_n < \infty$.

Then $\lim_{n \to \infty} \alpha_n = 0$.

Let $C$ be a nonempty closed and convex subset of a real Hilbert space $H$; a mapping $P_C : H \to C$ is called the metric
projection if, for all \( x \in H \), there exists a unique point in \( C \), denoted by \( P_Cx \) such that
\[
\|x - P_Cx\| \leq \|x - y\|, \quad \forall y \in C.
\] (16)
It is well known that \( P_C \) is a nonexpansive mapping.

**Lemma 2** (see [25]). Let \( C \) be a nonempty closed and convex subset of a real smooth Hilbert space \( H \). Let \( x \in H \); then \( P_Cx \) have the property as follows:
\[
\langle x - P_Cx, y - P_Cx \rangle \leq 0, \quad \forall x \in H, \quad y \in C,
\]
\[
\|x - y\|^2 \geq \|x - P_Cx\|^2 + \|y - P_Cx\|^2, \quad \forall x \in H, \quad y \in C.
\] (17)

**Lemma 3** (see [21]). Let \( C \) be closed convex subset of Hilbert space \( H \). Let \( A : C \rightarrow H \) be a continuous monotone mapping, and let \( T : C \rightarrow C \) be a continuous pseudocontractive mapping, and define the mappings \( T_r \) and \( F_r \) as follows: for \( x \in H, \ r \in (0, \infty) \),
\[
T_r(x) := \left\{ z \in C : \langle y - z, Tz \rangle - \frac{1}{r} \langle y - z, (1 + r)z - x \rangle \leq 0, \forall y \in C \right\},
\] (18)
\[
F_r(x) := \left\{ z \in C : \langle y - z, Az \rangle + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}.
\]

Then the following hold:
(i) \( T_r \) and \( F_r \) are single valued;
(ii) \( T_r \) and \( F_r \) are firmly nonexpansive mappings; that is, \( \|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle, \|F_r x - F_r y\|^2 \leq \langle F_r x - F_r y, x - y \rangle \);
(iii) \( F(T_r) = F(T), F(F_r) = VI(C, A) \);
(iv) \( F(T) \) and \( VI(C, A) \) are closed convex.

**Lemma 4** (see [11]). Let \( \{x_n\} \) and \( \{z_n\} \) be bounded sequences in a Banach space and let \( \{\beta_n\} \) be a sequence in \( [0, 1] \) which satisfies the following condition:
\[
0 < \liminf_{n \to \infty} \beta_n < \limsup_{n \to \infty} \beta_n < 1.
\] (19)
Suppose
\[
x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n, \quad n \geq 0,
\]
\[
\lim_{n \to \infty} \left( \|z_n - x_n\| - \|x_{n+1} - x_n\| \right) \leq 0.
\] (20)
Then \( \lim_{n \to \infty} \|z_n - x_n\| = 0 \).

**Lemma 5** (see [22]). Let \( C \) be a nonempty closed and convex bounded subset of a Hilbert space \( H \), and let \( \{\Gamma_i : C \rightarrow C, i = 1, 2, \ldots, m\} \) be a finite family of nonexpansive mappings such that \( \Gamma \circ F(\Gamma) \neq 0 \). Suppose that \( \alpha = \inf \{\alpha_i\} > 0 \) and \( \sum_{i=1}^{m} \alpha_i = 1 \). Then there exists nonexpansive mapping \( \Gamma : C \rightarrow C \) such that \( F(\Gamma) = \cap_{i=1}^{m} F(\Gamma) \).

3. Main Results
Let \( C \) be closed convex subset of Hilbert space \( H \). Let \( \{A_i : C \rightarrow H, i = 1, 2, \ldots, N\} \) be a finite family of continuous monotone mappings, and let \( \{T_i : C \rightarrow C, i = 1, 2, \ldots, N\} \) be a finite family of continuous pseudocontractive mappings. For the rest of this paper, \( T_{ir} : E \rightarrow C \) and \( F_{ir} : E \rightarrow C \) are mappings defined as follows: for \( x \in E, r_n \in (0, \infty) \),
\[
T_{ir}(x) := \left\{ z \in C : \langle y - z, T_{ir}z \rangle - \frac{1}{r_n} \langle y - z, (1 + r_n)z - x \rangle \leq 0, \forall y \in C \right\},
\] (21)
\[
F_{ir}(x) := \left\{ z \in C : \langle y - z, A_{ir}z \rangle + \frac{1}{r_n} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}.
\] (22)

By using Lemmas 2.3–2.6 in Zegeye and Shahzad [21], we have that the mappings \( T_{ir} \) and \( F_{ir} \) are well defined and they are nonexpansive and \( F(T_{ir}) = F(T_i) \cap F(F_{ir}) = VI(C, A_i) \) are closed convex. Denote \( F_1 = \cap_{i=1}^{N} F(T_{ir}), F_2 = \cap_{i=1}^{N} F(F_{ir}) \).

**Theorem 6.** Let \( C \) be a nonempty closed convex subset of uniformly smooth strictly convex real Hilbert space \( H \). Let \( \{T_i : C \rightarrow C, i = 1, 2, \ldots, N\} \) be a finite family of continuous pseudocontractive mappings, and \( \{F_i : C \rightarrow C \) be a contraction with a contraction coefficient \( \rho \in (0, 1) \). \( T_{ir} \) and \( F_{ir} \) are defined as (21) and (22), respectively. Let \( \{x_n\} \) be a sequence generated by \( x_0 \in C \):
\[
y_n = P_C \left( \epsilon_n u_n + (1 - \epsilon_n) \sum_{i=1}^{N} \mu_i F_{ir} x_n \right),
\]
\[
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{N} \sigma_i F_{ir} y_n,
\] (23)
where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\epsilon_n\} \) are sequences of nonnegative real numbers in \( [0, 1] \) and \( \mu_i \geq 0, \sigma_i \geq 0, i = 1, 2, \ldots, N, \) and the sequence \( \{u_n\} \subset H \) is a small perturbation such that
(i) \( \alpha_n + \beta_n + \gamma_n = 1, n \geq 0, \sum_{i=1}^{N} \mu_i = 1, \) and \( \sum_{i=1}^{N} \sigma_i = 1 \);
(ii) \( \lim_{n \to \infty} \alpha_n = 0, \sum_{i=1}^{N} \alpha_n = \infty, \) and \( \lim_{n \to \infty} \epsilon_n = 0 \);
(iii) \( 0 < \liminf_{n \to \infty} r_n < \limsup_{n \to \infty} r_n < 1 \);
(iv) \( \lim \sup_{n \to \infty} r_n < 0, \sum_{i=1}^{N} |r_{n+1} - r_n| < \infty, \)
\[
\lim_{n \to \infty} \|u_n\| = 0, \quad \text{and} \quad \sum_{i=1}^{N} \|u_n\| < \infty.
\]
Then the sequence \( \{x_n\} \) converges strongly to an element \( w = \Pi_{F_i} f(w) \) and also \( w \) is the unique solution of the variational inequality
\[
\langle (f-I)w, y - w \rangle \leq 0, \quad \forall y \in F.
\] (24)
Proof. By using Lemmas 3 and 5, the mappings $\sum_{i=1}^{N} \mu_i F_{i_{rs}}$ and $\sum_{i=1}^{N} \sigma_i T_{i_{rs}}$ are well defined. First we prove that $\{x_n\}$ is bounded. Take $p \in F$, because $F_{i_{rs}}, P_C$ are nonexpansive; then we have that

$$\|y_n - p\| = \|P_C \left( (1 - \epsilon_n) \sum_{i=1}^{N} \mu_i F_{i_{rs}} x_n - p \right)\|
\leq \epsilon_n \left( \|x_n\| + \|p\| \right) + (1 - \epsilon_n) \sum_{i=1}^{N} \|F_{i_{rs}} x_n - F_{i_{rs}} p\|
\leq \epsilon_n \left( \|x_n\| + \|p\| \right) + (1 - \epsilon_n) \|x_n - p\|.$$  

(25)

For $n \geq 0$, because $T_{i_{rs}}$ and $F_{i_{rs}}$ are nonexpansive and $f$ is contractive, we have from (25) that

$$\|x_{n+1} - p\| = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{N} \sigma_i T_{i_{rs}} y_n - p
\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\|
+ \beta_n \|x_n - p\| + \gamma_n \|y_n - p\|
\leq (\rho \alpha_n + \beta_n) \|x_n - p\| + \alpha_n \|f(p) - p\|
+ \gamma_n \left( \|u_n\| + \|p\| \right) + (1 - \epsilon_n) \|x_n - p\|
\leq (1 - \rho \alpha_n) \|x_n - p\| + \epsilon_n \|x_n - p\|
+ (1 - \rho \alpha_n) \left( \frac{1}{1 - \rho} \|f(p) - p\| \right) + \epsilon_n \|u_n\|
\leq \max \left\{ \|x_n - p\|, \|p\|, \frac{1}{1 - \rho} \|f(p) - p\| \right\}
+ \epsilon_n \|u_n\|.  

(26)

This implies that

$$\|x_n - p\| \leq \max \left\{ \|x_0 - p\|, \|p\|, \frac{1}{1 - \rho} \|f(p) - p\| \right\}
+ \sum_{i=1}^{N-1} \epsilon_i \|u_n\|.  

(27)

Notice condition (iv); therefore, $\{x_n\}$ is bounded. Consequently, we get that $\{F_{i_{rs}} x_n\}, \{T_{i_{rs}} y_n\}$ and $\{y_n\}, \{f(x_n)\}$ are bounded.

Next, we show that $\|x_{n+1} - x_n\| \to 0$. We have from (23) that

$$\|y_{n+1} - y_n\| \leq \epsilon_n \|u_{n+1} - u_n\|
+ (1 - \epsilon_n) \Sigma_{i=1}^{N-1} \|F_{i_{rs}} x_{n+1} - F_{i_{rs}} x_n\|
+ \|e_{n+1} - e_n\| \|u_n - \Sigma_{i=1}^{N-1} \mu_i F_{i_{rs}} x_n\|.$$  

(28)

Let $v_{i_{rs}} = F_{i_{rs}} x_n, v_{i_{rs+1}} = F_{i_{rs+1}} x_{i_{rs+1}}$; by the definition of mapping $F_{i_{rs}}$, we have that

$$\langle y - v_{i_{rs}}, A_{i_{rs}} v_{i_{rs}} \rangle + \frac{1}{r_n} \langle y - v_{i_{rs}}, v_{i_{rs}} - x_n \rangle \geq 0, \quad \forall y \in C.$$  

(29)

$$\langle y - v_{i_{rs+1}}, A_{i_{rs+1}} v_{i_{rs+1}} \rangle + \frac{1}{r_{n+1}} \langle y - v_{i_{rs+1}}, v_{i_{rs+1}} - x_{n+1} \rangle \geq 0, \quad \forall y \in C.$$  

(30)

Putting $y := v_{i_{rs+1}}$ in (29) and letting $y := v_{i_{rs}}$ in (30), we have that

$$\langle v_{i_{rs+1}} - v_{i_{rs}}, v_{i_{rs+1}} - v_{i_{rs}} \rangle + \frac{1}{r_n} \langle v_{i_{rs+1}} - v_{i_{rs+1}}, v_{i_{rs+1}} - x_{i_{rs}} \rangle \geq 0,$$  

(31)

$$\langle v_{i_{rs+1}} - v_{i_{rs+1}}, v_{i_{rs+1}} - v_{i_{rs+1}} - x_{i_{rs+1}} \rangle \geq 0.$$  

(32)

Adding (31) and (32), we have that

$$\langle v_{i_{rs+1}} - v_{i_{rs+1}}, v_{i_{rs+1}} - v_{i_{rs+1}} - x_{i_{rs+1}} \rangle \geq 0.$$  

(33)

Since $\{A_i, i = 1, 2, \ldots, N\}$ are monotone mappings, which implies that

$$\langle v_{i_{rs+1}} - v_{i_{rs}}, v_{i_{rs+1}} - v_{i_{rs+1}} - x_{i_{rs+1}} \rangle \geq 0,$$  

(34)

we have that

$$\langle v_{i_{rs+1}} - v_{i_{rs}} - v_{i_{rs+1}} - v_{i_{rs+1}} - x_{i_{rs+1}} \rangle \geq 0;$$  

(35)

that is,

$$\|v_{i_{rs+1}} - v_{i_{rs+1}}\|^2
\leq \langle v_{i_{rs+1}} - v_{i_{rs+1}} - v_{i_{rs+1}} - v_{i_{rs+1}} - x_{i_{rs+1}} \rangle,$$  

(36)

Without loss of generality, let $b$ be a real number such that $r_n > b > 0$, for all $n \in N$; then we have that

$$\|v_{i_{rs+1}} - v_{i_{rs+1}}\| \leq \|x_{i_{rs+1}} - x_{i_{rs+1}}\| + \left| 1 - \frac{r_n}{r_{n+1}} \right| \|v_{i_{rs+1}} - x_{i_{rs+1}}\| \leq \|x_{i_{rs+1}} - x_{i_{rs+1}}\| + \frac{1}{b} \left| r_{n+1} - r_n \right| K,$$  

(37)
where \( K = \sup \| w_{i,n+1} - x_{n+1} \|, i = 1, 2, \ldots, N \). Then we have from (37) and (28) that
\[
\| y_{n+1} - y_n \| \leq (1 - \varepsilon_{n+1}) \| x_{n+1} - x_n \| + \varepsilon_{n+1} \| u_{n+1} - u_n \| + \frac{(1 - \varepsilon_{n+1}) r_{n+1} - r_n}{b} K + \| \varepsilon_{n+1} - \varepsilon_n \| \| u_n - \sum_{i=1}^N \mu F_{i,n} x_n \| .
\] (38)

On the other hand, let \( w_{i,n} = T_{i,n} y_n, w_{i,n+1} = T_{i,n} y_{n+1} \); we have that
\[
\langle y - w_{i,n}, T_i w_{i,n} \rangle - \frac{1}{r_n} \langle y - w_{i,n}, (1 + r_n) w_{i,n} - y_n \rangle \leq 0, \\
\forall y \in C.
\] (39)

Let \( y := w_{i,n} \) in (39) and let \( y := w_{i,n} \) in (40); we have that
\[
\langle w_{n+1} - w_{i,n}, T_i w_{i,n} \rangle - \frac{1}{r_n} \langle w_{n+1} - w_{i,n}, (1 + r_n) w_{i,n} - y_n \rangle \leq 0,
\] (41)
\[
\langle w_{i,n} - w_{i,n+1}, T_i w_{i,n+1} \rangle - \frac{1}{r_{n+1}} \langle w_{i,n} - w_{i,n+1}, (1 + r_{n+1}) w_{i,n+1} - y_{n+1} \rangle \leq 0.
\] (42)

Adding (41) and (42) and because \( \{ T_i, i = 1, 2, \ldots, N \} \) are pseudocontractive mappings, we have that
\[
\left\langle w_{i,n+1} - w_{i,n}, \frac{w_{i,n} - y_n}{r_n} - \frac{w_{i,n+1} - y_{n+1}}{r_{n+1}} \right\rangle \geq 0.
\] (43)

Therefore we have
\[
\left\langle w_{i,n+1} - w_{i,n}, w_{i,n} - y_n - \frac{r_n w_{i,n+1} - y_{n+1}}{r_{n+1}} \right\rangle + w_{i,n+1} - w_{i,n+1} \geq 0.
\] (44)

Hence we have that
\[
\| w_{i,n+1} - w_{i,n} \| \leq \| y_{n+1} - y_n \| + \frac{1}{b} | r_{n+1} - r_n | M,
\] (45)

where \( M = \sup \| w_{i,n} - y_n \|, i = 1, 2, \ldots, N \).

Let \( x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n \). Hence we have that
\[
z_{n+1} - z_n = \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (f(x_{n+1}) - f(x_n)) + \left( \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right) f(x_n) + \frac{y_{n+1}}{1 - \beta_{n+1}} \sum_{i=1}^N \sigma_i (w_{i,n+1} - w_{i,n}) + \frac{y_n}{1 - \beta_n} \sum_{i=1}^N \sigma_i w_{i,n}.
\] (46)

Then we have from (46), (45), and (38) that
\[
\| z_{n+1} - z_n \| - \| x_{n+1} - x_n \|
\leq \frac{\rho}{1 - \beta_{n+1}} \| x_{n+1} - x_n \| + \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| \| f(x_n) \| + \| \sum_{i=1}^N \sigma_i w_{i,n} \|
+ \frac{y_{n+1}}{1 - \beta_{n+1}} | r_{n+1} - r_n | (1 - \varepsilon_{n+1}) K + M
+ \frac{y_n}{1 - \beta_n} \| u_n - \sum_{i=1}^N \mu F_{i,n} x_n \|
+ \frac{y_n}{1 - \beta_n} \| u_n - u_{n+1} \|.
\] (47)

Notice conditions (ii), (iii), and (iv); we have that
\[
\limsup_{n \to \infty} \| z_{n+1} - z_n \| - \| x_{n+1} - x_n \| = 0.
\] (48)

Hence we have from Lemma 4 that
\[
\limsup_{n \to \infty} \| z_n - x_n \| = 0.
\] (49)

Therefore we have that
\[
\| x_{n+1} - x_n \| = | 1 - \beta_n | \| z_n - x_n \| \to 0.
\] (50)

Hence we have from (37), (38), and (45) that
\[
\| y_{n+1} - y_n \| \to 0, \quad \| w_{i,n+1} - w_{i,n} \| \to 0,
\] (51)
\[
\| v_{i,n+1} - v_{i,n} \| \to 0.
\]
the monotonicity of $A_i$, the nonexpansivity of $T_{ir_i}$, and the convexity of $\| \cdot \|$ that

\[
\| x_{n+1} - p \| ^2 \\
= \| \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{N} \sigma_i w_{i,n} - p \|^2 \\
\leq \| \alpha_n (f(x_n) - p) + \beta_n (x_n - p) \|^2 + \gamma_n \sum_{i=1}^{N} \sigma_i w_{i,n} - p \|^2 \\
\leq \alpha_n \| f(x_n) - p \|^2 + \beta_n \| x_n - p \|^2 + \gamma_n \| y_n - p \|^2 \\
\leq \alpha_n \| f(x_n) - p \|^2 + \beta_n \| x_n - p \|^2 + \gamma_n \| x_n - y_n \|^2 \\
+ (1 - \epsilon_n) \gamma_n \sum_{i=1}^{N} \mu_i \| x_{n+1} - p \|^2 \\
\leq \alpha_n \| f(x_n) - p \|^2 + \| x_n - p \|^2 \\
- \| x_{n+1} - p \|^2 + \gamma_n \| x_n - y_n \|^2 \\
\leq \alpha_n \| f(x_n) - p \|^2 + \| x_n - x_{n+1} \|^2 \\
\times (\| x_n - p \| + \| x_{n+1} - p \|) \\
+ \gamma_n \| x_n - y_n \|^2. \\
\]  
(52)

So we have that

\[
(1 - \epsilon_n) \gamma_n \sum_{i=1}^{N} \mu_i \| x_n - v_{i,n} \|^2 \leq \alpha_n \| f(x_n) - p \|^2 + \| x_n - p \|^2 \\
- \| x_{n+1} - p \|^2 + \gamma_n \| x_n - y_n \|^2. \\
\]  
(53)

Since $\alpha_n \to 0$, $\epsilon_n \to 0$, we have from (50) that

\[
\| x_n - v_{i,n} \| \to 0. \\
(54)
\]

In a similar way, we have that

\[
\| x_n - w_{i,n} \| \to 0. \\
(55)
\]

Consequently, we have that

\[
\| y_n - x_n \| \leq \| (1 - \epsilon_n) \gamma_n \sum_{i=1}^{N} \mu_i \| x_n - v_{i,n} \| \to 0, \\
\| y_n - w_{i,n} \| \leq \| y_n - x_n + \| x_n - w_{i,n} \| \to 0. \\
(56)
\]

Since the sequence $\{ x_n \}$ is bounded, there exists a subsequence $\{ x_{n_k} \}$ of $\{ x_n \}$ and $w \in C$ such that $x_{n_k} \to w$ weakly. And because $x_n \to v_{i,n}$, $w_{i,n} \to w$ weakly. Next we show that $w \in F$.

Because $v_{i,n} = F_{ir_i} x_n$, by the definition of mapping $F_{ir_i}$, we have that

\[
\langle y - v_{i,n}, A_i v_{i,n} \rangle + \frac{1}{r_n} \langle y - v_{i,n}, v_{i,n} - x_n \rangle \geq 0, \quad \forall y \in C, \\
\]  
(57)

\[
\langle y - v_{i,n}, A_i v_{i,n} \rangle + \left( y - v_{i,n}, \frac{v_{i,n} - x_{i,n}}{r_n} \right) \geq 0, \quad \forall y \in C. \\
\]

Let $v_t = tv + (1 - t)w$, $t \in [0, 1]$, for all $v \in C$; we have that

\[
\langle v_t - v_{i,n}, A_i v_t \rangle \geq \langle v_t - v_{i,n}, A_i v_{i,n} \rangle - \langle v_t - v_{i,n}, \frac{v_{i,n} - x_{i,n}}{r_n} \rangle \\
= \langle v_t - v_{i,n}, \frac{v_{i,n} - x_{i,n}}{r_n} \rangle. \\
\]  
(58)

Because $\{ A_i, i = 1, 2, \ldots, N \}$ are monotone and $x_{i,n} \to 0$, we have that

\[
0 \leq \lim_{k \to \infty} \langle v_t - v_{i,n}, A_i v_t \rangle = \langle v_t - w, A_i v_t \rangle. \\
(59)
\]

Consequently we have that

\[
\langle v - w, A_i v_t \rangle \geq 0. \\
(60)
\]

If $t \to 0$, by the continuity of $A_i$, we have that $\langle v - w, A_i w \rangle \geq 0$; that is, $w \in VI(C, A_i)$ and then $w \in F$.

Similarly, because $w_{i,n} = T_{ir_i} y_n$, by the definition of mapping $T_{ir_i}$, we have that

\[
\langle y - w_{i,n}, T_{i} w_{i,n} \rangle - \frac{1}{r_n} \langle y - w_{i,n}, (1 + r_n) w_{i,n} - y_n \rangle \leq 0, \\
\forall y \in C. \\
\]  
(61)

Let $v_t = tv + (1 - t)w$, $t \in [0, 1]$, for all $v \in C$.

Because $\{ T_i, i = 1, 2, \ldots, N \}$ are pseudocontractive mappings, we have that

\[
\langle w_{i,n} - v_{i,n}, T_{i} v_t \rangle \\
\geq \langle w_{i,n} - v_{i,n}, T_{i} v_t \rangle + \langle v_t - w_{i,n}, T_{i} w_{i,n} \rangle \\
- \frac{1}{r_n} \langle v_t - w_{i,n}, (1 + r_n) w_{i,n} - y_{i,n} \rangle \\
= \langle v_t - w_{i,n}, T_{i} w_{i,n} - T_{i} v_t \rangle \\
- \langle v_t - w_{i,n}, \frac{1 + r_n}{r_n} w_{i,n} - \frac{1}{r_n} y_{i,n} \rangle \\
\geq -\| v_t - w_{i,n} \|^2 - \frac{1}{r_n} \langle v_t - w_{i,n}, w_{i,n} - y_{i,n} \rangle \\
- \langle v_t - w_{i,n}, w_{i,n} \rangle. \\
(62)
\]
Because $y_{nk} - w_{nk} \to 0$, so we have that
\[
\lim_{k \to \infty} \langle w_{nk} - v_i, T_i y_i \rangle \geq \lim_{k \to \infty} \langle w_{nk} - v_i, v_i \rangle .
\]
Consequently we have that
\[
\langle w - v_i, T_i y_i \rangle \geq \langle w - v_i, v_i \rangle ;
\]
that is,
\[
\langle v - w, T_i y_i \rangle \leq \langle v - w, v_i \rangle .
\]

If $f \to 0$, by the continuity of $T_i$, we have that $\langle v - w, T_i w - w \rangle \leq 0$, for all $v \in C$; we conclude that $w = T_i w$; that is, $w \in F(T_i)$ and then $w \in F_i$. Consequently $w \in F = F_1 \cap F_2$. Denote $x^* = \Pi_F f(w)$; then $x^* \in F$ is the unique element that satisfies $\inf_{x \in F} \|x - f(w)\| = \|x^* - f(w)\|$. From Lemma 1, we have that $\langle f(w) - x^*, y - x^* \rangle \leq 0$, for all $y \in C$. If we take $y = f(w)$, then $\langle f(w) - x^*, f(w) - x^* \rangle \leq 0$; consequently we have that $f(w) = x^*$.

By using the weakly lower semicontinuity of the norm on $H$, we get that
\[
\|x^* - f(w)\| \leq \|w - f(w)\| \leq \lim_{k \to \infty} \inf \|x_{nk} - f(w)\| \\
\leq \lim_{k \to \infty} \sup \|x_{nk} - f(w)\| \\
\leq \inf_{x \in F} \|x - f(w)\| = \|x^* - f(w)\| ,
\]
which implies that
\[
\lim_{n \to \infty} \|x_n - f(w)\| = \|x^* - f(w)\| = \|w - f(w)\| \\
= \inf_{x \in F} \|x - f(w)\| .
\]

Thus, from Lemma 1, we have that
\[
\langle z - x^*, x^* - f(w) \rangle \geq 0, \quad \forall z \in C ,
\]
\[
\langle z - w, w - f(w) \rangle \geq 0, \quad \forall z \in C .
\]

Putting $z := w$ in (68) and $z := x^*$ in (69), we get that
\[
\langle w - x^*, x^* - f(w) \rangle \geq 0, \quad \forall z \in C ,
\]
\[
\langle x^* - w, w - f(w) \rangle \geq 0, \quad \forall z \in C .
\]

Adding (70) and (71) we get that $(x^* - w, x^* - w) \leq 0$; that is, $\|x^* - w\|^2 \leq 0$; thus $w = x^*$. Furthermore, from (67), we get that the sequence $x_n \to w = P_F f(w)$ strongly and $w$ is the solution of the following variational inequality:
\[
\langle z - w, (f - I)w \rangle \leq 0, \quad \forall z \in C .
\]

Now we show that $w$ is the unique solution of the variational inequality $(z - w, (f - I)w) \leq 0$, for all $z \in C$. Suppose that $\bar{w} \in F$ is another solution of the variational inequality; that is,
\[
\langle z - \bar{w}, (f - I)\bar{w} \rangle \leq 0, \quad \forall y \in C .
\]

Let $z := \bar{w}$ in (72) and let $z := w$ in (73); we have that
\[
\langle \bar{w} - w, f(w) - w \rangle \leq 0 ,
\]
\[
\langle w - \bar{w}, f(w) - \bar{w} \rangle \leq 0 .
\]

Adding (74) and (75), we have that
\[
\langle \bar{w} - w - (f(w) - f(w)), \bar{w} - w \rangle \leq 0 .
\]

Hence
\[
(1 - \rho) \|\bar{w} - w\|^2 \leq 0 .
\]
Because $\rho \in (0, 1)$, we conclude that $\bar{w} = w$; the uniqueness of the solution is obtained. The proof is complete. 

Theorem 7. Let $C$ be a nonempty closed convex subset of a uniformly smooth strictly convex real Hilbert space $H$. Let $\{T_i : C \to C, i = 1, 2, \ldots, N\}$ be a finite family of continuous pseudomonotone mappings, let $\{A_i : C \to H, i = 1, 2, \ldots, N\}$ be a finite family of continuous monotone mappings such that $F = F_1 \cap F_2 \neq \emptyset$, and let $f : C \to C$ be a contraction with a contraction coefficient $\rho \in (0, 1)$. $T_{ir_n}$ and $F_{ir_n}$ are defined as (21) and (22), respectively. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$, $\begin{align*}
y_n = (1 - \lambda_n) x_n + \lambda_n P_C \left( \varepsilon_n u_n + (1 - \varepsilon_n) \sum_{i=1}^{N} \mu_i F_{ir_n} x_n \right), \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{N} \sigma_i T_{ir_n} y_n,
\end{align*}
\]
where $\{\alpha_n\}, \{\lambda_n\}, \{\beta_n\}, \{\gamma_n\}, \{\varepsilon_n\}$ are sequences of nonnegative real numbers in $[0, 1]$ and $\mu_i \geq 0, \sigma_i \geq 0, i = 1, 2, \ldots, N$, and the sequence $\{u_n\} \subset H$ is a small perturbation such that
\[
\begin{align*}
(i) \quad & \alpha_n + \beta_n + \gamma_n = 1, n \geq 0, \sum_{n=1}^{\infty} \mu_i = 1, \text{ and } \sum_{n=1}^{\infty} \sigma_i = 1; \\
(ii) \quad & \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \text{ and } \lim_{n \to \infty} \varepsilon_n = 0; \\
(iii) \quad & 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1; \\
(iv) \quad & \limsup_{n \to \infty} \gamma_n > 0, \sum_{n=1}^{\infty} \gamma_n \leq \infty, \quad \lim_{n \to \infty} \sigma_i T_{ir_n} y_n = 0, \text{ and } \sum_{n=1}^{\infty} \gamma_n \leq \infty .
\end{align*}
\]

Then the sequence $\{x_n\}$ converges strongly to an element $w = \Pi_F f(w)$ and also $\bar{x}$ is the unique solution of the variational inequality
\[
\langle (f - I)(w), y - w \rangle \leq 0, \quad \forall y \in F .
\]

Proof. Take $p \in F$; because $F_{ir_n}, P_C$ are nonexpansive, then we have that
\[
\begin{align*}
\|y_n - p\| \\
= \left\| (1 - \lambda_n) x_n + \lambda_n P_C \left( \varepsilon_n u_n + (1 - \varepsilon_n) \sum_{i=1}^{N} \mu_i F_{ir_n} x_n - p \right) \right\| \\
\leq \varepsilon_n \left\| u_n + \lambda_n \|p\| \right\| + (1 - \lambda_n \varepsilon_n) \|x_n - p\|. 
\end{align*}
\]
For $n \geq 0$, because $T_{i_{n}}$ and $F_{i_{n}}$ are nonexpansive and $f$ is contractive, we have from (28) that
\[
\|x_{n+1} - p\|
= \|\alpha_{n}f(x_{n}) + \beta_{n}x_{n} + \gamma_{n}\sum_{i=1}^{N}T_{i_{n}}y_{n} - p\|
\leq \alpha_{n}\|f(x_{n}) - f(p)\| + \alpha_{n}\|f(p) - p\|
+ \beta_{n}\|x_{n} - p\| + \gamma_{n}\|y_{n} - p\|
\leq (\rho\alpha_{n} + \beta_{n})\|x_{n} - p\| + \alpha_{n}\|f(p) - p\|
+ \gamma_{n}\|f(p) - p\| + \gamma_{n}\|y_{n} - p\|
\leq (1 - \alpha_{n} + \rho\alpha_{n})\|x_{n} - p\| + \alpha_{n}\|f(p) - p\|
+ \gamma_{n}\|f(p) - p\| + \gamma_{n}\|y_{n} - p\|
\leq (1 - \alpha_{n} + \rho\alpha_{n})\|x_{n} - p\| + \alpha_{n}\|f(p) - p\|
+ \gamma_{n}\|f(p) - p\| + \gamma_{n}\|y_{n} - p\|
\leq \max\left\{\|x_{n} - p\|, \|f(p) - p\|\right\} + \epsilon_{n}\|u_{n}\|.
\] (81)

This implies that
\[
\|x_{n} - p\| \leq \max\left\{\|x_{0} - p\|, \|f(p) - p\|, \frac{1}{1 - \rho}\right\} + \epsilon_{n}\|u_{n}\|.
\] (82)

Notice condition (iv); therefore, $\{x_{n}\}$ is bounded. Consequently, we get that $\{F_{i_{n}}x_{n}\}$, $\{T_{i_{n}}y_{n}\}$ and $\{y_{n}\}$, $\{f(x_{n})\}$, and $P_{\mathcal{C}}(\epsilon_{n},u_{n} + (1 - \epsilon_{n})\sum_{i=1}^{N} \mu_{i}F_{i_{n}}x_{n})$ are bounded.

Next, we show that $\|x_{n+1} - x_{n}\| \to 0$. Denote $\tau_{n} = \epsilon_{n}u_{n} + (1 - \epsilon_{n})\sum_{i=1}^{N} \mu_{i}F_{i_{n}}x_{n}$, then we get that
\[
\|y_{n+1} - y_{n}\| \leq (1 - \lambda_{n+1})\|x_{n+1} - x_{n}\| + \lambda_{n+1}\|\tau_{n+1} - \tau_{n}\|
+ \left|\lambda_{n+1} - \lambda_{n}\right|\|x_{n} - P_{\mathcal{C}}\tau_{n}\|,
\]
\[
\|\tau_{n+1} - \tau_{n}\| \leq \epsilon_{n+1}\left\|u_{n+1}\right\| + \epsilon_{n}\|u_{n}\|
+ (1 - \epsilon_{n+1})\sum_{i=1}^{N} \mu_{i}\left\|F_{i_{n+1}}x_{n+1} - F_{i_{n}}x_{n}\right\|
+ |\epsilon_{n+1} - \epsilon_{n}|\left\|u_{n} - \sum_{i=1}^{N} \mu_{i}F_{i_{n}}x_{n}\right\|.
\] (83)

Repeating equations from (29) to (38), we have that
\[
\left\|x_{n+1} - x_{n}\right\| \leq \epsilon_{n+1}\left\|u_{n+1}\right\| + \epsilon_{n}\|u_{n}\|
+ (1 - \epsilon_{n+1})\left\|x_{n+1} - x_{n}\right\|
+ \left|\epsilon_{n+1} - \epsilon_{n}\right|\left\|u_{n} - \sum_{i=1}^{N} \mu_{i}F_{i_{n}}x_{n}\right\|.
\] (84)

Therefore,
\[
\left\|y_{n+1} - y_{n}\right\| \leq (1 - \lambda_{n+1})\left\|x_{n+1} - x_{n}\right\|
+ \lambda_{n+1}\left\|\epsilon_{n+1}\left\|u_{n+1}\right\| + \epsilon_{n}\|u_{n}\| + (1 - \epsilon_{n+1})\left\|x_{n+1} - x_{n}\right\|
+ \left|\epsilon_{n+1} - \epsilon_{n}\right|\left\|u_{n} - \sum_{i=1}^{N} \mu_{i}F_{i_{n}}x_{n}\right\|.
\]

Similarly to the rest of the proof of Theorem 6, we obtain the result. $\Box$

If, in Theorems 6 and 7, we let $f \equiv u \in \mathcal{C}$ be a constant mapping, we have the following corollaries.

**Corollary 8.** Let $\mathcal{C}$ be a nonempty closed convex subset of a uniformly smooth strictly convex real Hilbert space $H$. Let $\{T_{i} : \mathcal{C} \to \mathcal{C}, i = 1, 2, \ldots, N\}$ be a finite family of continuous pseudocontractive mappings, let $\{A_{i} : \mathcal{H} \to \mathcal{H}, i = 1, 2, \ldots, N\}$ be a finite family of continuous monotone mappings such that $F = F_{1} \cap F_{2} \neq \emptyset$, and let $u \in \mathcal{C}$ be a constant.
The sequence \( \{x_n\} \) converges strongly to an element \( w = \Pi_F u \) and also \( \overline{x} \) is the unique solution of the variational inequality
\[ \langle u - w, y - w \rangle \leq 0, \quad \forall y \in F. \]  

**Remark 10.** If \( \{u_n\} \subset C \), then sequence (23) reduces to
\[ y_n = \varepsilon_n u_n + (1 - \varepsilon_n) \sum_{i=1}^{N} \mu_i F_{i_r} x_n, \]
\[ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n \sum_{i=1}^{N} \sigma_i T_{i_r} y_n, \]
and sequence (78) reduces to
\[ y_n = (1 - \lambda_n) x_n + \lambda_n \left( \varepsilon_n u_n + (1 - \varepsilon_n) \sum_{i=1}^{N} \mu_i F_{i_r} x_n \right), \]
\[ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{N} \sigma_i T_{i_r} y_n. \]

The conclusions of Theorems 6 and 7 are true under the same conditions.

**Remark 11.** Our theorems extend and unify some of the results that have been proved for these important classes of nonlinear operators. In particular, Theorem 6 extends Theorem 6 of Yao and Shahzad [24] in the sense that our convergence is for the more general class of continuous pseudocontractive and continuous monotone mappings. Theorem 6 also extends Theorem 3.2 of Tang [22] in the sense that our convergence is for the more general algorithm with perturbations.

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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