Enumeration of (0,1)-matrices avoiding some $2 \times 2$ matrices

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Abstract
We enumerate the number of (0,1)-matrices avoiding $2 \times 2$ submatrices satisfying certain conditions. We also provide corresponding exponential generating functions.

1 Introduction
Let $M(k,n)$ be the set of $k \times n$ matrices with entries 0 and 1. It is obvious that the number of elements in the set $M(k,n)$ is $2^{kn}$. It would be interesting to consider the number of elements in $M(k,n)$ with certain conditions. For example, how many matrices of $M(k,n)$ do not have $2 \times 2$ submatrices of the forms $(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix})$ and $(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix})$? In this article we will give answers to the previous question and other questions.

Consider $M(2,2)$, the set of all possible $2 \times 2$ submatrices. For two elements $P$ and $Q$ in $M(2,2)$, we denote $P \sim Q$ if $Q$ can be obtained from $P$ by row or column exchanges. It is obvious that $\sim$ is an equivalence relation on $M(2,2)$. With this equivalence relation, $M(2,2)$ is partitioned with seven equivalent classes having the following seven representatives.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$
$$T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$
Here \( C, T, \) and \( L \) mean “corner”, “top”, and “left”, respectively. Let \( S \) be the set of these representatives, i.e.,
\[
S := \{I, \Gamma, C, T, L, J, O\}.
\]

Given an element \( S \) in \( S \), a matrix \( A \) is an element of the set \( M(S) \) if and only if for \( \text{every} \) permutation \( \pi_1 \) of the rows and \( \pi_2 \) of the columns, the resulting matrix does not have the submatrix \( S \). Equivalently, \( A \in M(S) \) means that \( A \) has no submatrices in the equivalent class \([S]\). For a subset \( \alpha \) of \( S \), \( M(\alpha) \) is defined by the set \( \cap_{S \in \alpha} M(S) \). Note that the definition of \( M(S) \) (also \( M(\alpha) \)) is different from that in [13,17]. If \( A \) belongs to \( M(\alpha) \), then we say that \( A \) avoids \( \alpha \). We let \( \phi(k, n; \alpha) \) be the number of \( k \times n \) \((0,1)\)-matrices in \( M(\alpha) \).

Our goal is to express \( \phi(k, n; \alpha) \) in terms of \( k \) and \( n \) explicitly for each subset \( \alpha \) of the set \( S \). For \(|\alpha| = 1\), We can easily notice that \( \phi(k, n; \Gamma) = \phi(k, n; C) \) and \( \phi(k, n; J) = \phi(k, n; O) \) by swapping 0 and 1. We also notice that \( \phi(k, n; T) = \phi(n, k; L) \) by transposing the matrices. The number \( \phi(k, n; I) \) is well known (see [2,7,8]) and \((0,1)\)-matrices avoiding type \( I \) are called \((0,1)\)-lonesum matrices (we will define and discuss this in 2.2). In fact, lonesum matrices are the primary motivation of this article and its corresponding work. The study of \( M(J) \) (equivalently \( M(O) \)) appeared in [9,13,17], but finding a closed form of \( \phi(k, n; J) = \phi(k, n; O) \) is still open. The notion of “\( \Gamma \)-free matrix” was introduced by Spinrad [16]. He dealt with a totally balanced matrix which has a permutation of the rows and columns that are \( \Gamma \)-free. We remark that the set of totally balanced matrices is different from \( M(\Gamma) \).

In this paper we calculate \( \phi(k, n; \alpha) \), where \( \alpha \)'s are \( \{\Gamma\} \) (equivalently \( \{C\} \)) and \( \{T\} \) (equivalently \( \{L\} \)). We also enumerate \( M(\alpha) \) where \( \alpha \)'s are \( \{\Gamma, C\}, \{T, L\}, \) and \( \{J, O\} \). For the other subsets of \( S \), we discuss them briefly in the last section. Note that some of our result (subsection 3.5) is an independent derivation of some of the results in [11, section 3] by Kitaev et al.; for other relevant papers see [10,12].

# 2 Preliminaries

## 2.1 Definitions and Notations

A matrix \( P \) is called \((0,1)\)-matrix if all the entries of \( P \) are 0 or 1. From now on we will consider \((0,1)\)-matrices only, so we will omit \“(0,1)\” if it causes no confusion. Let \( M(k, n) \) be the set of \( k \times n \)-matrices. Clearly, if \( k, n \geq 1 \), \( M(k, n) \) has \( 2^{kn} \) elements. For convention we assume that \( M(0,0) = \{\emptyset\} \) and \( M(k,0) = M(0,n) = \emptyset \) for positive integers \( k \) and \( n \).

Given a matrix \( P \), a submatrix of \( P \) is formed by selecting certain rows and columns from \( P \). For example, if \( P = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{pmatrix} \), then \( P(2,3;2,4) = \begin{pmatrix} f & h \\ j & l \end{pmatrix} \).

Given two matrices \( P \) and \( Q \), we say \( P \) contains \( Q \), whenever \( Q \) is equal to a submatrix of \( P \). Otherwise say \( P \) avoids \( Q \). For example, \( \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 \end{pmatrix} \) contains \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) but avoids \( \begin{pmatrix} 1 & 0 \end{pmatrix} \).

For a matrix \( P \) and a set \( \alpha \) of matrices, we say that \( P \) avoids the type set \( \alpha \) if \( P \) avoids all the matrices in \( \alpha \). If it causes no confusion we will simply say that \( P \) avoids \( \alpha \).
Given a set $\alpha$ of matrices, let $\phi(k, n; \alpha)$ be the number of $k \times n$ matrices avoiding $\alpha$. From the definition of $M(k, n)$, for any set $\alpha$, we have $\phi(0, 0; \alpha) = 1$ and $\phi(k, 0; \alpha) = \phi(0, n; \alpha) = 0$ for positive integers $k$ and $n$. Let $\Phi(x, y; \alpha)$ be the bivariate exponential generating function for $\phi(k, n; \alpha)$, i.e.,

$$
\Phi(x, y; \alpha) := \sum_{n \geq 0} \sum_{k \geq 0} \phi(k, n; \alpha) \frac{x^k y^n}{k! n!} = 1 + \sum_{n \geq 1} \sum_{k \geq 1} \phi(k, n; \alpha) \frac{x^k y^n}{k! n!}.
$$

Let $\Phi(z; \alpha)$ be the exponential generating function for $\phi(n, n; \alpha)$, i.e.,

$$
\Phi(z; \alpha) := \sum_{n \geq 0} \phi(n, n; \alpha) \frac{z^n}{n!}.
$$

Given $f, g \in \mathbb{C}[[x, y]]$, we denote $f \equiv g$ if the coefficients of $x^k y^n$ in $f$ and $g$ are the same, for each $k, n \geq 2$.

### 2.2 $I$-avoiding matrices (Lonesum matrices)

This is related to the lonesum matrices. A lonesum matrix is a $(0, 1)$-matrix determined uniquely by its column-sum and row-sum vectors. For example, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} is a lonesum matrix since it is a unique matrix determined by the column-sum vector $(2, 0, 3)$ and the row-sum vector $(2, 1, 2)^t$. However \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} is not, since \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} has the same column-sum vector $(2, 0, 2)$ and row-sum vector $(2, 1, 1)^t$.

**Theorem 2.1** (Brewbaker [2]). A matrix is a lonesum matrix if and only if it avoids $I$.

Theorem 2.1 implies that $\phi(k, n; I)$ is equal to the number of $k \times n$ lonesum matrices.

**Definition 2.2.** Bernoulli number $B_n$ is defined as following:

$$
\sum_{n \geq 0} B_n \frac{x^n}{n!} = \frac{x e^x}{e^x - 1}.
$$

Note that $B_n$ can be written explicitly as

$$
B_n = \sum_{m=0}^{n} (-1)^{m+n} \frac{m! S(n, m)}{m + 1},
$$

where $S(n, m)$ is the Stirling number of the second kind. The poly-Bernoulli number, introduced first by Kaneko [7], is defined as

$$
\sum_{n \geq 0} B_n^{(k)} \frac{x^n}{n!} = \frac{\text{Li}_k(1 - e^{-x})}{1 - e^{-x}},
$$
where the polylogarithm $\text{Li}_k(x)$ is the function $\text{Li}_k(x) := \sum_{m \geq 1} \frac{x^m}{m^k}$. Bernoulli numbers are nothing but poly-Bernoulli numbers with $k = 1$. Sanchez-Peregrino [15] proved that $B_n^{(-k)}$ has the following simple expression:

$$B_n^{(-k)} = \sum_{m=0}^{\min(k,n)} (m!)^2 S(n+1, m+1) S(k+1, m+1).$$

Brewbaker [2] and Kim et al. [8] proved that the number of $k \times n$ lonesum matrices is the poly-Bernoulli number $B_n^{(-k)}$, which yields the following result.

**Proposition 2.3** (Brewbaker [2]; Kim, Krotov, Lee [8]). The number of $k \times n$ matrices avoiding $I$ is equal to $B_n^{(-k)}$, i.e.,

$$\phi(k, n; I) = \sum_{m=0}^{\min(k,n)} (m!)^2 S(n+1, m+1) S(k+1, m+1).$$

The generating function $\Phi(x, y; I)$, given by Kaneko [7], is

$$\Phi(x, y; I) = e^{x+y} \sum_{m \geq 0} \left[ (e^x - 1)(e^y - 1) \right]^m = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}.$$  \hspace{1cm} (2)

Also, $\Phi(z; I)$ can be easily obtained as follows:

$$\Phi(z; I) = \sum_{n \geq 0} \phi(n, n; I) \frac{z^n}{n!} = \sum_{n \geq 0} \sum_{m \geq 0} (-1)^{m+n} m! S(n, m) (m+1)^n \frac{z^n}{n!}$$

$$= \sum_{m \geq 0} (-1)^m m! \sum_{n \geq 0} S(n, m) \frac{(-m-1)z^n}{n!}$$

$$= \sum_{m \geq 0} (1 - e^{-(m+1)z})^m.$$  \hspace{1cm} (3)

3 Main Results

3.1 $\Gamma$-avoiding matrices (or $C$)

By row exchange and column exchange we can change the original matrix into a block matrix as in Figure 1. Here $[0]$ (resp. $[1]$) stands for a 0-block (resp.1-block) and $[0^*]$ stands for a 0-block or an empty block. Diagonal blocks are $[1]$’s except for the last block $[0^*]$, and the off-diagonal blocks are $[0]$’s.

**Theorem 3.1.** The number of $k \times n$ matrices avoiding $\Gamma$ is given by

$$\phi(k, n; \Gamma) = \sum_{m=0}^{\min(k,n)} m! S(n+1, m+1) S(k+1, m+1).$$  \hspace{1cm} (4)
**Figure 1:** A matrix avoiding $\Gamma$ can be changed into a block diagonal matrix.

**Proof.** Let $\mu = \{C_1, C_2, \ldots, C_{m+1}\}$ be a set partition of $[n+1]$ into $m+1$ blocks. Here the block $C_i$’s are ordered by the largest element of each block. Thus $n+1$ is contained in $C_{m+1}$. Likewise, let $\nu = \{D_1, D_2, \ldots, D_{m+1}\}$ be a set partition of $[k+1]$ into $m+1$ blocks. Choose $\sigma \in S_{m+1}$ with $\sigma(m+1) = m+1$, where $S_{m+1}$ is the set of all permutations of length $m+1$. Given $(\mu, \nu, \sigma)$ we define a $k \times n$ matrix $M = (a_{i,j})$ by

$$a_{i,j} := \begin{cases} 1, & (i, j) \in C_l \times D_{\sigma(l)} \text{ for some } l \in [m] \\ 0, & \text{otherwise} \end{cases}.$$ 

It is obvious that the matrix $M$ avoids the type $\Gamma$.

Conversely, let $M$ be a $k \times n$ matrix avoiding type $\Gamma$. Set $(k+1) \times (n+1)$ matrix $\widetilde{M}$ by augmenting zeros to the last row and column of $M$. By row exchange and column exchange we can change $\widetilde{M}$ into a block diagonal matrix $B$, where each diagonal is 1-block except for the last diagonal. By tracing the position of columns (resp. rows) in $\widetilde{M}$, $B$ gives a set partition of $[n+1]$ (resp. $[k+1]$). Let $\{C_1, C_2, \ldots, C_{m+1}\}$ (resp. $\{D_1, D_2, \ldots, D_{m+1}\}$) be the set partition of $[n+1]$ (resp. $[k+1]$). Note that the block $C_i$’s and $D_i$’s are ordered by the largest element of each block. Let $\sigma$ be a permutation on $[m]$ defined by $\sigma(i) = j$ if $C_i$ and $D_j$ form a 1-block in $B$.

The number of set partitions $\pi$ of $[n+1]$ is $S(n+1, m+1)$, and the number of set partitions $\pi'$ of $[k+1]$ is $S(n+1, k+1)$. The cardinality of the set of $\sigma$’s is the cardinality of $S_m$, i.e., $m!$. Since the number of blocks $m+1$ runs through 1 to min($k, n$) + 1, the sum of $S(k+1, m+1) S(n+1, m+1) m!$ gives the required formula.

**Example 1.** Let $\mu = 4/135/26/7$ be a set partition of $[7]$ and $\nu = 25/6/378/149$ of $[9]$ into 4 blocks. Let $\sigma = 3124$ be a permutation in $S_4$ such that $\sigma(4) = 4$. From $(\mu, \nu, \sigma)$ we can construct the $6 \times 8$ matrix $M$ which avoids type $\Gamma$ as in Figure 2.

To find the generating function for $\phi(k, n; \Gamma)$ the following formula (see [5]) is helpful.

$$\sum_{n \geq 0} S(n+1, m+1) \frac{x^n}{n!} = e^x \frac{(e^x - 1)^m}{m!}. \quad (5)$$
From Theorem 3.1 and (5), we can express $\Phi(x, y; \Gamma)$ as follows:

$$
\Phi(x, y; \Gamma) = \sum_{n,k \geq 0} \sum_{m \geq 0} \frac{m! S(n + 1, m + 1) S(k + 1, m + 1)}{k! n!} \frac{x^k y^n}{k! n!} \\
= \sum_{m \geq 0} \frac{1}{m!} \sum_{k \geq 0} m! S(k + 1, m + 1) \frac{x^k}{k!} \sum_{n \geq 0} m! S(n + 1, m + 1) \frac{y^n}{n!} \\
= \sum_{m \geq 0} \frac{1}{m!} e^x (e^x - 1)^m e^y (e^y - 1)^m \\
= \exp[(e^x - 1)(e^y - 1) + x + y].
$$

(6)

Remark 1. It seems to be difficult to find a simple expression of $\Phi(z; \Gamma)$. The sequence $\phi(n, n; \Gamma)$ is not listed in the OEIS [14]. The first few terms of $\phi(n, n; \Gamma)$ ($0 \leq n \leq 9$) are as follows:

1, 2, 12, 128, 2100, 48032, 1444212, 54763088, 2540607060, 140893490432, \ldots

3.2 $\{\Gamma, C\}$-avoiding matrices

Given the equivalence relation $\sim$ on $M(2, 2)$, which is defined in Section 1, if we define the new equivalent relation $P \sim Q$ by $P \sim Q$ or $P \sim (\frac{1}{2} \frac{1}{2}) - Q$, then $[\Gamma] \cup [C]$ becomes a single equivalent class. Clearly $\phi(k, n; \{\Gamma, C\})$ is the number of $k \times n$ $(0, 1)$-matrix which does not have a submatrix in $[\Gamma] \cup [C]$. From now on we simply write $\phi(k, n; \Gamma, C)$, instead of $\phi(k, n; \{\Gamma, C\})$. The reduced form of a matrix $M$ avoiding $\{\Gamma, C\}$ is very simple as in Figure 3. In this case if the first row and the first column of $M$ are determined then the rest of the entries of $M$ are determined uniquely. Hence the number $\phi(k, n; \Gamma, C)$ of such matrices is

$$
\phi(k, n; \Gamma, C) = 2^{k+n-1} \quad (k, n \geq 1),
$$

(7)

and its exponential generating function is

$$
\Phi(x, y; \Gamma, C) = 1 + \frac{1}{2} (e^{2x} - 1)(e^{2y} - 1).
$$

(8)
Clearly, $\phi(n, n; \Gamma, C) = 2^{2n-1}$ for $n \geq 1$. Thus its exponential generating function is

$$\Phi(z; \Gamma, C) = \frac{1}{2} (e^{4z} + 1).$$ (9)

### 3.3 $T$-avoiding matrices (or $L$)

Given a $(0, 1)$-matrix, 1-column (resp. 0-column) is a column in which all entries consist of 1’s (resp. 0’s). We denote a 1-column (resp. 0-column) by $\mathbf{1}$ (resp. $\mathbf{0}$). A mixed column is a column which is neither $\mathbf{0}$ nor $\mathbf{1}$. For $k = 0$, we have $\phi(0, n; T) = \delta_{0,n}$. In case $k \geq 1$, i.e., there being at least one row, we can enumerate as follows:

- **case 1**: there are no mixed columns. Then each column should be 0 or 1. The number of such $k \times n$ matrices is $2^n$.

- **case 2**: there is one mixed column. In this case each column should be 0 or 1 except for one mixed column. The number of $k \times n$ matrices of this case is $2^n - 1 \cdot (2^k - 2)$.

- **case 3**: there are two mixed columns. As in case 2, each column should be 0 or 1 except for two mixed columns, say, $v_1$ and $v_2$. The number of $k \times n$ matrices of this case is the sum of the following three subcases:
  - $v_1 + v_2 = \mathbf{1}$: $2^{n-2\binom{n}{2}} 2! S(k, 2)$
  - $v_1 + v_2$ has an entry 0: $2^{n-2\binom{n}{2}} 3! S(k, 3)$
  - $v_1 + v_2$ has an entry 2: $2^{n-2\binom{n}{2}} 3! S(k, 3)$

- **case 4**: there are $m$ ($m \geq 3$) mixed columns $v_1, \ldots, v_m$. The number of $k \times n$ matrices of this case is the sum of the following four subcases:
  - $v_1 + \cdots + v_m = \mathbf{1}$: $2^{n-m\binom{n}{m}} m! S(k, m)$
  - $v_1 + \cdots + v_m = (m-1)\mathbf{1}$: $2^{n-m\binom{n}{m}} m! S(k, m)$
  - $v_1 + \cdots + v_m$ has an entry 0: $2^{n-m\binom{n}{m}} (m+1)! S(k, m+1)$
  - $v_1 + \cdots + v_m$ has an entry $m$: $2^{n-m\binom{n}{m}} (m+1)! S(k, m+1)$

Adding up all numbers in the previous cases yields the following theorem.
Theorem 3.2. For $k, n \geq 1$ the number of $k \times n$ matrices avoiding $T$ is given by

$$\phi(k, n; T) = 2 \sum_{l \geq 1} \binom{n}{l-1} t^k + (n^2 - n - 4) 2^{n-2} - n(n+3) 2^{n+k-3}. \quad (10)$$

Proof.

$$\phi(k, n; T) = 2^n + 2^{n-1} \binom{n}{1} (2^k - 2) + 2^{n-2} \binom{n}{2} (2! S(k, 2) + 3! 2 S(k, 3))$$

$$+ \sum_{m=3}^n 2^{n-m+1} \binom{n}{m} (m! S(k, m) + (m+1)! S(k, m+1))$$

$$= 2 \sum_{m=0}^n 2^{n-m} \binom{n}{m} m! S(k+1, m+1) + (n^2 - n - 4) 2^{n-2} - n(n+3) 2^{n+k-3}$$

$$= 2 \sum_{l \geq 1} \binom{n}{l-1} t^k + (n^2 - n - 4) 2^{n-2} - n(n+3) 2^{n+k-3}.$$

Note that in the proof of Theorem 3.2 we use the identity

$$\sum_{m \geq 0} \binom{n}{m} m! S(k, m) 2^{n-m} = \sum_{l \geq 0} \binom{n}{l} t^k,$$

where both sides count the number of functions $f$ from $[k]$ to $[n]$ such that each element of $[n] \setminus f([k])$ has two colors.

The generating function $\Phi(x, y; T)$ is given by

$$\Phi(x, y; T) = 1 + \sum_{n \geq 1} \sum_{k \geq 1} 2 \sum_{l \geq 1} \binom{n}{l-1} t^k \frac{x^k y^n}{k! n!}$$

$$+ \sum_{n \geq 1} \sum_{k \geq 1} (n^2 - n - 4) 2^{n-2} \frac{x^k y^n}{k! n!} - \sum_{n \geq 1} \sum_{k \geq 1} n(n+3) 2^{n+k-3} \frac{x^k y^n}{k! n!}$$

$$= 1 + \left(2 e^x \left(e^{y(x+1)} - 1\right) - 2 e^{2y} + 2\right)$$

$$+ \left(e^x - 1\right) \left((y^2 - 1) e^{2y} + 1\right) - \frac{1}{2} y(y+2) e^{2y} (e^{2x} - 1)$$

$$= 2 e^{y(x+1)+x} - \frac{y^2 + 2y}{2} e^{2x+2y} + (y^2 - 1) e^{x+2y} - e^x - \frac{y^2 - 2y + 2}{2} e^{2y} + 2. \quad (11)$$

Note that if we use the symbol “$\equiv$” introduced in Section 2.1 then

$$\Phi(x, y; T) \equiv 2 e^{y(x+1)+x} - \frac{y^2 + 2y}{2} e^{2x+2y} + (y^2 - 1) e^{x+2y}.$$
For the $n \times n$ matrices we have
\[
\phi(n, n; T) = 2 \sum_{l \geq 1} \binom{n}{l-1} l^n + (n^2 - n - 4) 2^{n-2} - n(n+3) 2^{2n-3}.
\]
Thus the generating function $\Phi(n, n; T)$ is given by
\[
\sum_{n \geq 0} \phi(n, n; T) \frac{z^n}{n!} = 2 \sum_{n \geq 0} \sum_{l \geq 1} \binom{n}{l} \frac{z^n}{(n+1)!} + \sum_{n \geq 0} \frac{n^2 - n - 4}{4} \frac{(2z)^n}{n!} - \frac{n(n+3)}{8} \frac{(4z)^n}{n!}
\]
\[
= \frac{2}{z} \sum_{l \geq 1} \frac{(lz)^l}{l!} \sum_{n \geq l-1} \frac{(lz)^{n-l+1}}{(n-l+1)!} + (z^2 - 1)e^{2z} - 2z(z+1)e^{4z}
\]
\[
= \frac{2}{z} \sum_{l \geq 1} \frac{l!(z^2)^l}{l!} + (z^2 - 1)e^{2z} - 2z(z+1)e^{4z}
\]
\[
= \frac{2}{z} (zez W'(-zez)) + (z^2 - 1)e^{2z} - 2z(z+1)e^{4z}
\]
\[
= \frac{-2 W(-zez)}{zez} + (z^2 - 1)e^{2z} - 2z(z+1)e^{4z}, \quad (12)
\]
where
\[
W(x) := \sum_{n \geq 1} (-n)^{n-1} \frac{x^n}{n!}
\]
is the Lambert $W$-function which is the inverse function of $f(W) = We^W$. See [3] for extensive study about the Lambert $W$-function.

**Remark 2.** The sequence $\phi(n, n; T)$ is not listed in the OEIS [14]. The first few terms of $\phi(n, n; T)$ ($0 \leq n \leq 9$) are as follows:

\[
1, 2, 14, 200, 3536, 67472, 1423168, 34048352, 927156224, 28490354432, \ldots
\]

### 3.4 \{T, L\}-avoiding matrices

Given the equivalence relation $\sim$ on $M(2, 2)$, which is defined in Section 1, if we define the new equivalent relation $P \sim' Q$ by $P \sim Q$ or $P \sim Q'$, then $\{T\} \cup \{L\}$ becomes a single equivalent class. Clearly $\phi(k, n; \{T, L\})$ is the number of $k \times n$ $(0,1)$-matrix which does not have any submatrix in $\{T\} \cup \{L\}$. By the symmetry of $\{T, L\}$, we have
\[
\phi(k, n; T, L) = \phi(n, k; T, L).
\]
So it is enough to consider the case $k \geq n$. For $k \leq 2$ or $n \leq 1$, we have
\[
\phi(0, n; T, L) = \delta_{0,n}, \quad \phi(1, n; T, L) = 2^n, \quad \phi(k, 0; T, L) = \delta_{k,0}, \quad \phi(k, 1; T, L) = 2^k, \quad \phi(2, 2; T, L) = 12.
\]
Given a $(0,1)$-vector $v$ with a length of at least $3$, $v$ is called 1-dominant (resp. 0-dominant) if all entries of $v$ are 1’s (resp. 0’s) except one entry.

**Theorem 3.3.** For $k \geq 3$ and $n \geq 2$, the number of $k \times n$ matrices avoiding $\{T, L\}$ is equal to twice the number of rook positions in the $k \times n$ chessboard. In other words,

$$\phi(k,n;T,L) = 2 \sum_{m=0}^{\min(k,n)} \left( \binom{k}{m} \binom{n}{m} m! \right).$$

(13)

**Proof.** Suppose $M$ is a $k \times n$ matrix avoiding $\{T, L\}$. It is easy to show each of the following steps:

(i) If $M$ has a mixed column $v$, then $v$ should be either 0-dominant or 1-dominant.

(ii) Assume that $v$ is 0-dominant. This implies that other mixed columns (if any) in $M$ should be 0-dominant.

(iii) Any non-mixed column in $M$ should be a 0-column.

(iv) The location of 1’s in $M$ corresponds to a rook position in the $k \times n$ chessboard.

If we assume $v$ is 1-dominant in (ii) then the locations of 0’s again corresponds to a rook position. The summand of RHS in (13) is the number of rook positions in the $k \times n$ chessboard with $m$ rooks. This completes the proof. $\Box$

The generating function $\Phi(x, y; T, L)$ is given by

$$\Phi(x, y; T, L) = 2e^{xy} + x + y - \frac{(xy)^2}{2} - 2xy + 3 - 2e^x - 2e^y + x(e^y - 2y - 1)(e^y - 1) + y(e^x - 2x - 1)(e^x - 1).$$

(14)

Here the crucial part of the equation (14) can be obtained as follows:

$$\sum_{k,n \geq 0} \left( \sum_{m \geq 0} \binom{k}{m} \binom{n}{m} m! \right) \frac{x^k y^n}{k! n!} = \sum_{m \geq 0} m! \left( \sum_{k \geq 0} \binom{k}{m} x^k \right) \left( \sum_{n \geq 0} \binom{n}{m} y^n n! \right) = \sum_{m \geq 0} m! \left( \frac{x^m}{m!} e^x \right) \left( \frac{y^m}{m!} e^y \right) = \exp(xy + x + y).$$

Note that if we use the symbol $\frac{2}{n}$ introduced in Section 2.1, then

$$\Phi(x, y; T, L) \equiv 2 e^{xy} + x + y - \frac{(xy)^2}{2}.$$

For the $n \times n$ matrices we have

$$\phi(0,0;T,L) = 1, \quad \phi(1,1;T,L) = 2, \quad \phi(2,2;T,L) = 12, \quad \text{and}$$

$$\phi(n,n;T,L) = 2 \sum_{m=0}^{n} \binom{n}{m}^2 m!. \quad (n \geq 3)$$
Thus the generating function $\Phi(z; T, L)$ is given by

$$\Phi(z; T, L) = \frac{2e^{\frac{z}{1-z}}}{1-z} - 1 - 2z - z^2. \quad (15)$$

Note that we use the equation

$$\sum_{n \geq 0} \left( \sum_{m=0}^{n} \binom{n}{m}^2 \frac{m!}{n!} \right) \frac{z^n}{n!} = \frac{e^{\frac{z}{1-z}}}{1-z},$$

which appears in [1] pp. 597–598.

### 3.5 \{J, O\}-avoiding matrices

Recall the equivalence relation $\sim'$ defined in subsection 3.2. With this relation, $\{J, O\}$ becomes a single equivalent class. Due to the symmetry of $\{J, O\}$ it is obvious that

$$\phi(k, n; J, O) = \phi(n, k; J, O).$$

The $k$-color bipartite Ramsey number $br(G; k)$ of a bipartite graph $G$ is the minimum integer $n$ such that in any $k$-coloring of the edges of $K_{n,n}$ there is a monochromatic subgraph isomorphic to $G$. Beineke and Schwenk [1] had shown that $br(K_{2,2}; 2) = 5$. From this we can see that

$$\phi(k, n; J, O) = 0 \quad (k, n \geq 5).$$

For $k = 1$ and $2$, we have

$$\phi(1, n; J, O) = 2^n, \quad \phi(2, n; J, O) = (n^2 + 3n + 4)2^{n-2}.$$

Note that the sequence $(n^2 + 3n + 4)2^{n-2}$ appears in [14] A007466] and its exponential generating function is $(1 + x)^2e^{2x}$.

For $k \geq 3$, we have

$$\phi(3, n; J, O) = \phi(4, n; J, O) = 0 \quad \text{for } n \geq 7,$$

$$\phi(5, n; J, O) = \phi(6, n; J, O) = 0 \quad \text{for } n \geq 5,$$

$$\phi(k, n; J, O) = 0 \quad \text{for } k \geq 7 \text{ and } n \geq 3.$$

For exceptional cases, due to the symmetry of $\{J, O\}$, it is enough to consider the followings:

$$\phi(3, 3; J, O) = 156, \quad \phi(3, 4; J, O) = 408, \quad \phi(4, 4; J, O) = 840,$$

$$\phi(3, 5; J, O) = \phi(3, 6; J, O) = \phi(4, 5, J, O) = \phi(4, 6; J, O) = 720.$$

The sequence $\phi(k, n; J, O)$ is listed in Table 1. Note that Kitaev et al. have already calculated $\phi(k, n; J, O)$ in [11, Proposition 5], but the numbers of $\phi(3, 3; J, O)$ and $\phi(4, 4; J, O)$ are different with ours.
Table 1: The sequence $\phi(k, n; J, O)$

| $k \backslash n$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | \cdots |
|------------------|----|----|----|----|----|----|----|---------|
| 1                | 1  | 2  | 4  | 8  | 16 | 32 | 64 | 128     |
| 2                | 4  | 14 | 44 | 128| 352| 928| 2368| \cdots |
| 3                | 8  | 44 | 156| 408| 720| 720| 0   | 0       |
| 4                | 16 | 128| 408| 840| 720| 720| 0   | 0       |
| 5                | 32 | 352| 720| 720| 0   | 0   | 0   | 0       |
| 6                | 64 | 928| 720| 720| 0   | 0   | 0   | 0       |
| 7                | 128| 2368| 0  | 0  | 0   | 0   | 0   | 0       |

The generating function $\Phi(x, y; J, O)$ is given by

$$
\Phi(x, y; J, O) = 1 + xe^{2y} + ye^{2x} + x^2(1+y)^2e^{2y} + y^2(1+x)^2e^{2x} \\
- \left( x + y + \frac{x^2}{2!} + \frac{y^2}{2!} + 2xy + 2x^2y + 2xy^2 + 14\frac{x^2y^2}{2!2!} \right) + 156\frac{x^3y^3}{3!3!} + 840\frac{x^4y^4}{4!4!} \\
+ 720\left( \frac{x^3y^5}{3!5!} + \frac{x^5y^3}{5!3!} + \frac{x^3y^6}{3!6!} + \frac{x^5y^3}{6!3!} + \frac{x^4y^5}{4!5!} + \frac{x^5y^4}{5!4!} + \frac{x^4y^6}{4!6!} + \frac{x^6y^4}{6!4!} \right). \tag{16}
$$

In particular, the generating function $\Phi(z; J, O)$ is given by

$$
\Phi(z; J, O) = 1 + 2z + 7z^2 + 26z^3 + 35z^4. \tag{17}
$$

4 Concluding remarks

Table 2 summarizes our results (except the $I$-avoiding case). Due to the amount of difficulty, we are not able to enumerate the number $\phi(k, n; J)$, hence $\phi(k, n; O)$. Note that $\phi(k, n; J)$ is equal to the following:

(a) The number of labeled $(k, n)$-bipartite graphs with girth of at least 6, i.e., the number of $C_4$-free labeled $(k, n)$-bipartite graphs, where $C_4$ is a cycle of length 4.

(b) The cardinality of the set $\{(B_1, B_2, \ldots, B_k) : B_i \subseteq [n] \forall i, |B_i \cap B_j| \leq 1 \forall i \neq j\}$.

For the other subsets $\alpha$ of $S$ which is not listed in Table 2, we have calculated $\phi(k, n; \alpha)$ in [6]. Note that if the size of $\alpha$ increases then enumeration of $M(\alpha)$ becomes easier.

For further research, we suggest the following problems.

1. In addition to $(0, 1)$-matrices, one can consider $(0, 1, \ldots, r)$-matrices with $r \geq 2$.

2. Consideration of the results of adding the line sum condition to each individual case given in the first column of Table 2.
\[ \alpha \quad \phi(k, n; \alpha) \quad \Phi(x, y; \alpha) \quad \Phi(z; \alpha) \]

| \( I \)          | (1) | (2) | (3) |
|------------------|-----|-----|-----|
| \( \Gamma \) (or \( C \)) | (4) | (5) | complicated |
| \( \{ \Gamma, C \} \) | (7) | (8) | (9) |
| \( T \) (or \( L \)) | (10) | (11) | (12) |
| \( \{ T, L \} \) | (13) | (14) | (15) |
| \( J \) (or \( O \)) | unknown | unknown | unknown |
| \( \{ J, O \} \) | Table 1 | (16) | (17) |

Table 2: Formulas and generating functions avoiding \( \alpha \).

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