Hidden Order of Boolean Networks

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Abstract—It is a common belief that the order of a Boolean network is mainly determined by its attractors, including fixed points and cycles. Using the semi-tensor product (STP) of matrices and the algebraic state-space representation (ASSR) of the Boolean networks, this article reveals that in addition to this explicit order, there is a certain implicit or hidden order, which is determined by the fixed points and limit cycles of their dual networks. The structure and certain properties of dual networks are investigated. Instead of a trajectory, which describes the evolution of a state, the hidden order provides a global horizon to describe the evolution of the overall network. We conjecture that the order of networks is mainly determined by the dual attractors via their corresponding hidden orders. Then these results about the Boolean networks are further extended to the k-valued case.

Index Terms—Boolean networks, dual networks, hidden order, semi-tensor product (STP) of matrices.

I. INTRODUCTION

BOOLEAN networks (BNs) were first proposed by Kauffman to describe genetic regulatory networks [17]. It has then been proved efficient and attracted considerable attention from biologists, computer and system scientists, etc. A BN is composed of a set of Boolean variables, which interact with each other through Boolean operators. The evolution of each variable is computed from Boolean functions which are generated by these operators. From the perspective of network topology, Boolean functions determine the connectivity of Boolean variables and hence make them the nodes of a network. BN is the simplest and the most representative model of finite logical systems. It is the simplest because the domain is binary; it is the most representative because almost all the properties can be generalized to the k-valued and mix-valued logical networks. Furthermore, BNs are powerful models not only in investigating gene regulatory networks but also show their unique advantages in the game theory [8], [12], [20], [34], multiagent systems [25], [33], vehicle engines [27], [29], and other theoretical/practical subjects [1], [15], [16].

Kauffman’s eventual purpose in proposing BN is “to answer the question, what are the sources of the overwhelming and beautiful order which graces the living world?” [18]. Roughly speaking, the answer is: Nature selection, as proposed by Darwin, plus the “emergent order” or “order for free” from self-organization that arises naturally.

How does self-organization emerge in a network? Kauffman’s viewpoint is: “Tiny attractors” lead to “vast, vast order.” It was proposed in [19] that: “Under the right conditions, these attractors can be the source of order in large dynamical systems. Since the system follows trajectories that inevitably flow into attractors, tiny attractors will ‘trap’ the system into tiny subregions of its state space. Among the vast range of possible behaviors, the system settles into an orderly few. The attractors, if small, create order. Indeed, tiny attractors are a prerequisite for the order for free that we are seeking.”

Roughly speaking, the attractors, consisting of the fixed points and limit cycles, make up the fundamental topological structure of the network. Since a BN has only finite nodes and hence finite states, any trajectory will converge to an attractor. The topological structures determine the “order” of mostly large-scale networks, which are particularly from metabolic networks or genetic regularity networks, etc. Searching for the order in lives through its attractors such as Kauffman did seem reasonable [18].

In his book “Hidden Order” [13], Holland described how the hidden order emerges from complex systems via adaptation, and how the hidden order determines the behavior of a complex system, which was called “complexity made simple.” For example, how does genome evolution affect character phylogeny? How does individual action give rise to social behavior? By introducing the concept of hidden order to BNs, the pattern of the macro phenomena generated by the underlying logical networks can be revealed.

Recently, we found a kind of hidden order of BNs, which may more significantly characterize certain properties of BNs. In the generalized sense, the hidden order of BNs considers how the dual BNs are generated by the original BNs, while in the narrow sense, it studies the relationship between the structure matrices of two kinds of BNs. Hidden orders come from the fixed points and cycles of logical functions under the action of the structure matrix of BNs. It has been discovered that this hidden order is critical for revealing certain properties of a large-scale BN by constructing a much smaller realization, which involves a smaller set of related states [9]. Furthermore,
the arguments about the hidden order of BNs also hold for the $k$-valued networks.

The main mathematical tool used in this article to model and analyze BNs is the semi-tensor product (STP) of matrices, which is a generalization of the conventional matrix product and has shown its power in dealing with logical systems by manipulating them in an algebraic state-space representation (ASSR). Compared with other methods, the ASSR allows the acquisition of centralized information, which makes a breakthrough in the field [3], [5]. Problems concerning controllability and observability [4], [10], [21], [30], optimal control [8], [11], probabilistic BNs [28], [32], disturbance decoupling [23], etc. are successfully investigated since then.

The main contributions of this article are as follows.

1) The hidden order of BNs is first proposed and investigated by building dual BNs whose states are the Boolean functions on BNs. Using the ASSR method, it is proven that the structure matrix of a BN uniquely determines the structure matrix of the dual BN and vice versa.

2) For an $n$-node BN, the structure matrix of its dual BN is $2^{2n} \times 2^{2n}$-dimensional, which makes it hard to analyze the dual BN in detail. Using the canonical form of BNs, the structure vectors of logical functions are decomposed into several independent parts. This decomposition eases the difficulty of inspecting the topological structure of dual spaces.

3) In a BN, its main topological structure consists of its attractors, that is, the fixed points and limit cycles, and the basins of attractors. As mentioned by Kauffman in [19], this topological structure determines the overall properties of the network. Our new results show that the proposed hidden order, i.e., the hidden topology, determined by the fixed points and limit cycles of the dual system also determines the properties of the overall network.

The rest of this article is organized as follows. The state space and its dual space of a BN are clarified in Section II. Section III explores the hidden order of BNs through their dual networks. The attractors and dual attractors of BNs are discussed in Section IV, using the canonical form of BNs. Section V considers the hidden order determined by dual networks. The attractors and dual attractors of BNs are determined by the fixed points and limit cycles of the proposed hidden order, i.e., the hidden topology, in [19], this topological structure determines the overall properties of the network. As mentioned by Kauffman in [19], this topological structure determines the overall properties of the network. Our new results show that the proposed hidden order, i.e., the hidden topology, determined by the fixed points and limit cycles of the dual system also determines the properties of the overall network.

The notations used in the text are shown in Table I.

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II. DUAL SPACE OF BNs

The mathematical tool used in this article is the STP of matrices, defined below.

Definition 1 [6]: Let $A \in \mathcal{M}_{m \times n}$, $B \in \mathcal{M}_{p \times q}$, and the least common multiple of $n$ and $p$ be $t = \text{lcm}(n, p)$. The STP of $A$ and $B$, denoted by $A \ltimes B$, is defined as

$$
(A \otimes I_{t/n})(B \otimes I_{t/p}).
$$

Consider an $n$-node BN, whose logical evolutionary dynamics is

$$
\begin{align*}
X_1(t + 1) &= f_1(X_1(t), \ldots, X_n(t)) \\
X_2(t + 1) &= f_2(X_1(t), \ldots, X_n(t)) \\
&\vdots \\
X_n(t + 1) &= f_n(X_1(t), \ldots, X_n(t))
\end{align*}
$$

where $X_i(t) \in \mathcal{D}$, $i \in [1, n]$ are the nodes, and $f_i : \mathcal{D}^n \rightarrow \mathcal{D}$ are the logical functions.

Identifying $1 \sim \delta^1_1$ and $0 \sim \delta^2_1$, $X_i$ can be expressed into its vector form as

$$
\tilde{X}_i := x_i = \begin{bmatrix} X_i \\ 1 - X_i \end{bmatrix} \quad i \in [1, n].
$$

Denote by $X = (X_1, X_2, \ldots, X_n) \in \mathcal{D}^n$ the overall state variable. Its vector form expression is

$$
\tilde{X} := x = \kappa_{i=1}^{n} x_i.
$$
Let $M = [\delta_{m}^{1}, \delta_{m}^{2}, \ldots, \delta_{m}^{n}] \in \mathcal{L}_{m \times n}$. We write it in a condensed form as $M = [\delta_{0}[i_1, i_2, \ldots, i_n]]$. The following results are borrowed from [6] and [7].

**Proposition 2** [6], [7]:

1) Let $f : \mathbb{D}^{n} \to \mathbb{D}$, expressed by $Y = f(X_1, X_2, \ldots, X_n)$, be a Boolean function. Then there exists a unique logical matrix $M_f \in \mathbb{L}_{2^{2n} \times 2}$, called the structure matrix of $f$, such that in vector form ($y = \tilde{Y}$) the Boolean function can be expressed by

$$y = M_f x.$$  

2) Let $M_f$ be the structure matrix of the Boolean function $f_i$, $i = 1, 2, \ldots, n$. Then there exists a unique logical matrix $M \in \mathbb{L}_{2^n \times 2^n}$ such that in vector form, BN (2) can be expressed by

$$x(t + 1) = Mx(t)$$

where $M = M_1 \ast M_2 \ast \cdots \ast M_n$ is called the structure matrix of BN (2).

Equation (4) is called the ASSR of BN (2).

Recall (3), $f$ is uniquely determined by the first (equivalently, second) row of $M_f$, and hence, from now on we will use this row, denoted by $V_f$, to represent a Boolean function.

**Example 3:** Consider $f(x_1, x_2, x_3) = (x_1 \land x_3) \lor x_2$. It is easy to calculate its ASSR as

$$f(x) = M \lor M_x x_1 x_3 x_2$$

where $M = \delta_{0,1}[1, 1, 2]$, $M_x = \delta_{2}[1, 2, 2, 2]$, and $M_f = M \lor M_x (I_2 \otimes W[2, 1]) x_1 x_2 x_3$.

Hence,

$$V_f = [1, 1, 1, 0, 1, 1, 0, 0].$$

**Remark 4:**

1) For an $n$-node BN, there are $2^n$ states, and hence, the number of the Boolean functions is $2^2^n$, which is much larger than that of the states.

2) A Boolean function can be considered as the indicator of a subset of states. For instance, (5) can be considered as a set $S = \{\delta_{0,1}^{1}, \delta_{0,1}^{2}, \delta_{0,1}^{3}, \delta_{0,1}^{4}\}$, which consists of the states satisfying certain property characterized by $f$ (here, the property is “$f(x_1, x_2, x_3)$ is true,” i.e., $f(x_1, x_2, x_3) = \delta_{0,1}^{1}$).

3) Each Boolean vector $V \in \mathbb{B}^{2^n}$ represents a Boolean function $f(x_1, x_2, \ldots, x_n)$. This is a one-to-one correspondence.

**Definition 5:** The state space of BN (2), denoted by $\mathcal{X}$, is defined as

$$\mathcal{X} := \{X_1, X_2, \ldots, X_n \mid X_i \in \mathbb{D}, i = 1, 2, \ldots, n\}.$$  

Equivalently, (6) can be expressed in its vector form as

$$\tilde{\mathcal{X}} := \{x \mid x \in \Delta_{2^n}\}.$$  

**Remark 6:** To avoid abusing mathematical notations, $\mathcal{X}$ will be used for denoting both the state space and its vector form.

**Definition 7:** The dual space of BN (2)’s state space, denoted by $\mathcal{X}^*$, is the set of Boolean functions over $\mathcal{X}$. That is,

$$\mathcal{X}^* := \{Z \mid Z : \mathcal{X} \to \mathbb{D}\}.$$  

Equivalently, $Z$ can be expressed by its vector form $V_Z \in \mathbb{B}^{2^n}$, that is,

$$\mathcal{X}^* := \{V_Z \mid V_Z \in \mathbb{B}^{2^n}\}.$$  

**Definition 8:** Let $f(X) \in \mathcal{X}^*$ be a Boolean function. The support of $f$, denoted by $\text{supp}(f)$, is defined as

$$\text{supp}(f) = \{X \mid f(X) \neq 0\} = \{x \mid f(x) \neq \delta_{2}^{1}\}.$$  

**Remark 9:**

1) According to [5] and the literature that follows, all the logical functions of $\{x_1, x_2, \ldots, x_n\}$, denoted by $\mathcal{F}(x_1, x_2, \ldots, x_n)$, are called the state space of BN (2). This definition is, in certain sense, confusing. In this article, the state space of BN (2), denoted by $\mathcal{X}$, is defined as (6) (or equivalently, (7)), and $\mathcal{F}(x_1, x_2, \ldots, x_n) = \mathcal{X}^*$ is the dual (state) space. The new definitions distinguish the “state space” from its “dual space.”

2) Since all the “elements” in $\mathcal{X}^*$ are logical functions, meanwhile $x_i$, $i = 1, \ldots, n$, are also logical functions, one has $\mathcal{X} \subseteq \mathcal{X}^*$.

3) There is no one-to-one correspondence between $\mathcal{X}$ and $\mathcal{X}^*$. In fact, $|\mathcal{X}| = 2^n$, and $|\mathcal{X}^*| = 2^{2^n}$.

4) [5] A set of Boolean functions $Z = \{z_1, z_2, \ldots, z_s\} \subseteq \mathcal{X}^*$ can be expressed by

$$z_i = G_ix, \quad i = 1, 2, \ldots, s$$

where $G_i \in \mathbb{L}_{2 \times 2^n}$. Setting $z = \sum_{i=1}^{s}z_i$, we have

$$z = Gx$$

where $G = G_1 \ast G_2 \ast \cdots \ast G_s \in \mathbb{L}_{2^n \times 2^n}$, and $Z$ is called a “subspace” of the state space. But to be precise, it is a subset of the dual space $\mathcal{X}^*$.

5) Assume $s = n$ and $G$ is nonsingular, then $Z$ can be viewed as equivalent to $\mathcal{X}$, and (10) is called a coordinate change.

6) [5] Consider $\{z_1, z_2, \ldots, z_n\} \subseteq \mathcal{X}^*$. If there exists another set of logical functions $\{z_{x+1}, z_{x+2}, \ldots, z_n\} \subseteq \mathcal{X}^*$, such that $\{z_1, z_2, \ldots, z_n\}$ is a coordinate change, then $\{z_1, z_2, \ldots, z_n\}$ is called a regular subspace.

The following result can be used to verify whether a given subspace $Z_0 = \{z_1, z_2, \ldots, z_n\}$ is regular.

**Proposition 10** [5]: Assume $Z_0 = \{z_1, z_2, \ldots, z_n\}$ has a structure matrix $M_0$. That is, $z_0 = M_0x$, where $M_0 \in \mathbb{L}_{2^n \times 2^n}$. Then $Z_0$ is a regular subspace, if and only if $M_0$ has $2^{n-s}$ columns equal $\delta_{2}^{i}$, $i = 1, \ldots, 2^n$. That is,

$$\left|\left\{ j \mid \text{Col}_j(M_0) = \delta_{2}^{i}\right\}\right| = 2^{n-s}, \quad i = 1, 2^n.\tag{11}$$
III. EXPLORING HIDDEN STRUCTURES OF BNS

A. Dual Networks

Consider BN (2). Let \( X^* = \{z_1, z_2, \ldots, z_{2^n}\} \) be the dual space of state space \( X \). Since each \( z_i \in X^* \) is a Boolean function of \( X \), \( z_i \) can be expressed by

\[
 z_i = G_i x, \quad i \in [1, 2^2]
\]

(12)

where \( G_i \in \mathcal{L}_{2 \times 2^n} \) is the structure matrix of \( z_i \).

Definition 11: Let \( z \in X^* \) be a Boolean function with structure matrix \( G_z \). Assume the first row of \( G_z \) is

\[
 \text{Row}_1(G_z) = [a_1, a_2, \ldots, a_{2^n}] \in \mathbb{B}^{2^n}
\]

which is called the structure vector of \( z \).

Then the vector form of \( z \), denoted by \( \tilde{z} \), is

\[
 \tilde{z} = \delta^{2^n}_{\phi}\n\]

(13)

where

\[
 i = 2^{2^n-1}a_1 + 2^{2^n-2}a_2 + \cdots + 2a_{2^n-1} + a_{2^n} + 1.
\]

Consequently, for \( i \in [1, 2^2] \), one has

\[
 z_i(t + 1) = G_i x(t + 1) = G_i M x(t) = z_{\phi(i)}(t) \in X^*
\]

(14)

where the structure matrix of \( z_{\phi(i)} \) is \( G_i M \).

In vector form, BN (14) is expressed by

\[
 \tilde{z}(t + 1) = \delta^{2^n}_{\phi}[\phi(1), \phi(2), \ldots, \phi(2^n)]\tilde{z}(t)
\]

(15)

which is the dynamic equation of logical functions.

Definition 12: Consider BN (2) with ASSR (4). The dynamic equation (15) is called the dual BN of (2).

Here is an example to demonstrate this.

Example 13: Consider the following 2-node BN:

\[
 \begin{align*}
 x_1(t + 1) &= x_2(t) \\
 x_2(t + 1) &= \neg(x_1(t) \lor x_2(t)).
\end{align*}
\]

(16)

The ASSR is \( x(t + 1) = M x(t) \), where \( M = \delta_4[2, 3, 2, 4] \).

The state transition graph of BN (16) is shown in Fig. 1.

In this example, there are 2^2 = 16 logical functions in total for state space \( X = \Delta_2 \). Using their structure vectors, we can arrange the functions in alphabetic order as

\[
 V_1 := [0, 0, 0, 0], \quad \tilde{z}_1 := \delta^{16}_1 \\
 V_2 := [0, 0, 0, 1], \quad \tilde{z}_2 := \delta^{16}_1 \\
 \vdots \\
 V_{16} := [1, 1, 1, 1], \quad \tilde{z}_{16} := \delta^{16}_1
\]

From (15), one has

\[
 V_{z(t+1)} = V_{z(t)} M.
\]

A straightforward computation shows that

\[
 V = \begin{bmatrix}
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 1 \\
 0 & 1 & 0 & 1 \\
 0 & 1 & 1 & 0 \\
 0 & 1 & 1 & 1 \\
 1 & 0 & 0 & 0 \\
 1 & 0 & 0 & 1 \\
 1 & 0 & 1 & 0 \\
 1 & 0 & 1 & 1 \\
 1 & 1 & 0 & 0 \\
 1 & 1 & 0 & 1 \\
 1 & 1 & 1 & 0 \\
 1 & 1 & 1 & 1
\end{bmatrix}
\]

(17)

\[
 M = \begin{bmatrix}
 V_1 & V_2 & \cdots & V_{16}
\end{bmatrix}.
\]

(18)

Setting \( \tilde{z}_i := \delta^{16}_i, \quad i = 1, 2, \ldots, 16 \), one can obtain the ASSR of the dual BN as follows:

\[
 \tilde{z}(t + 1) = M^* \tilde{z}(t)
\]

(19)

where

\[
 M^* = \delta^{16}[1, 2, 5, 6, 11, 12, 15, 16, 1, 2, 5, 6, 11, 12, 15, 16].
\]

Fig. 2 shows the state transition graph of the dual BN (19). Inside each oval is the structure vector of the corresponding logical function.

From Example 13, one can obtain the following result.

Proposition 14: The structure matrix \( M \) of a BN uniquely determines the structure matrix \( M^* \) of its dual BN, and vice versa.

Proof: For a given \( n \)-node BN, the dual space can be written in an ordered form as \( V := [V_1^T, V_2^T, \ldots, V_{2^n}^T]^T \). Then as in (18), after a one-step evolution of the network, this matrix \( V \) becomes

\[
 V_M := V M = [V_{i_1}^T, V_{i_2}^T, \ldots, V_{i_{2^n}}^T]^T.
\]

Thus, \( M^* := \delta^{2^n}_{\phi}[i_1, i_2, \ldots, i_{2^n}] \) will be the structure matrix of the evolution on the dual space, which is uniquely determined by \( M \).

Suppose that there is an unknown BN whose structure matrix is \( M \). Given the structure matrix of the dual BN \( M^* \), one can easily obtain \( V M \) by Row(j) \( (M^*)^T \) \( M \), i.e., finding the rows in \( V \) that are equal to \( (\delta^{2^n}_j)^T \), \( j = 1, 2, \ldots, 2^n \); then the same rows in \( V M \) are Row(j) \( (M^*)^T \), \( j = 1, 2, \ldots, 2^n \). Since \( M \) exists, one does not have to verify the other rows in \( V \), which completes the proof.

Note that \( V M = V_M \) is an overdetermined equation (\( M \) is unknown) and \( V \) is of full column rank. Using Proposition 14, it is easy to obtain the following result.

Corollary 15: A matrix \( M^* := \delta^{2^n}_{\phi}[i_1, i_2, \ldots, i_{2^n}] \) is the structure matrix of a dual BN, if and only if

\[
 (V^TV)^{-1} V_M \in \mathcal{L}_{2^n \times 2^n}
\]
**B. Dual Attractors**

Definition 17: Consider BN (2) with ASSR (4). Let \( f(x) \in X^* \) be a Boolean function with structure matrix \( G_f \), that is, \( f(x) = G_f x \).

1) \( f(x) \) is called a dual fixed point, if

\[
G_f \text{M} = G_f.
\]

2) \( (f_1(x), f_2(x), \ldots, f_{\ell+1}(x) = f_1(x)) \) is called a dual cycle with length \( \ell \), if their structure matrices \( G_i = G_{f_i}, \ i = 1, 2, \ldots, \ell + 1 \) satisfy

\[
G_i \neq G_j, \quad i \neq j, \quad 1 \leq i, j \leq \ell
\]

\[
G_1 = G_{t+1}; \quad G_{j+1} = G_j \text{M}, \quad j = 1, \ldots, \ell. \quad (21)
\]

**Remark 18:**

1) The dual fixed points and dual cycles are together called the dual attractors, as they are attractors of the dual BN. A BN has finite (to be precise, \( 2^2 \)) logical functions, and hence, each logical function will converge to an attractor.

2) In (20) and (21), the structure matrix of \( f \) can be replaced by its first row, i.e., the structure vector \( V_f \) shown in (18).

3) Similar to the attractors in BNs, the basin of a set of dual attractors \( A^* \subset X^* \) is defined as

\[
B_{A^*} := \{ z \in X^* | \exists N \in \mathbb{N}, \text{ s.t. } zM^s \in A^* \ \forall s \geq N \}. \quad (22)
\]

**Example 19:** Recall Example 13. From Fig. 2, one sees easily that there are four dual fixed points

\[
V_{z_1} = [0, 0, 0, 0]; \quad V_{z_2} = [0, 0, 1, 1];
V_{z_3} = [1, 0, 0, 0]; \quad V_{z_4} = [1, 0, 1, 1]
\]

and two dual cycles

\[
V_{z_{11}} = [1, 0, 0, 0] \Rightarrow V_{z_5} = [0, 0, 0, 0]
V_{z_{12}} = [1, 0, 0, 1] \Rightarrow V_{z_6} = [0, 0, 1, 0].
\]

The basin of each attractor is also obvious in Fig. 2.

**C. Invariant Subspaces**

Consider BN (2) with ASSR (4). Starting from any \( z_0 \in X^* \), we can construct a subset of \( X^* \) as \( \{ z_0, z_1 := M^*z_0, \ldots, z_N := M^Nz_{N-1} \} \), where \( M^* \) is the structure matrix of the dual BN shown in (19).

According to [9], a subset \( Z_0 := \{ z_0, z_1, \ldots, z_N \} \in X^* \) is called M-invariant, if

\[
\{ M^*z_i, \ i = 0, 1, \ldots, N \} \subset Z_0. \quad (23)
\]

If \( N \) is the smallest natural number satisfying (23), then the corresponding \( Z_0 \) is the smallest M-invariant subspace containing \( z_0 \) (equivalently, \( z_0 \)).

Similarly, we can construct a (smallest) M-invariant subspace \( V \) containing a subset \( Z \subset X^* \). Furthermore, \( V \) can be considered as a subspace of \( X \), only when \( V \) is a regular subspace.

Considering each dual attractor set \( A^*_i \) of BN (2), the following result is obvious.

**Proposition 20:** Let \( A^*_i, \ i = 1, \ldots, s \) be the sets of dual attractors of network (2), and the basin of attraction of \( A^*_i \) is \( B^*_i, \ i = 1, \ldots, s \). Let \( C^*_i := A^*_i \cup B^*_i \), then \( C^*_i, \ i = 1, \ldots, s \) are the M-invariant subspaces. Moreover, they form a partition of the dual space \( X^* \). That is,

\[
X^* = \bigcup_{i=1}^s C^*_i
\]

where \( C^*_i \cap C^*_j = \emptyset \) for \( i \neq j \).

From Example 13, one can obtain the following result.

**Proposition 21:** Consider BN (2).
1) Assume \( z \in \mathcal{X}^\ast \) is a dual fixed point, then \( \neg z \) is also a dual fixed point.
2) Assume \( (z_0, z_1, \ldots, z_\ell = z_0) \subset \mathcal{X}^\ast \) is a dual cycle, then 
\( (\neg z_0, \neg z_1, \ldots, \neg z_\ell) \) is also a dual cycle.
3) If \( B_i^+ \) is the basin of attraction for attractor set \( A_i^+ \), then 
\( \neg B_i^+ \) is the basin of attraction for attractor set \( \neg A_i^+ \).

**Proof:** Assume \( z \) is a dual fixed point whose structure matrix is \( G_z \). Then it is straightforward that 
\[
G_{\neg z} = G \cdot G_z
\]
where \( G_z = \delta_2[1,2] \) is the structure matrix of negation. Since \( z \) is a dual fixed point, we have \( G_z M = G_z \), then 
\[
G_{\neg z} M = (G \cdot G_z) M = G_z \cdot M = G \cdot G_z = G_{\neg z}
\]
which means \( \neg z \) is a dual fixed point. The proofs of other results follow similarly. \( \square \)

**IV. ATTRACTORS VERSUS DUAL ATTRACTORS**

The canonical form of BNs has been discussed in [10] and [24]. In the following, we use the framework provided in [24].

**Definition 22:**
1) [24] A matrix is called cyclic if it can be expressed by
\[
A = \delta_2[2,3,\ldots,k,1].
\]
2) [14] A matrix \( B \) is called nilpotent if there is an \( s \in \mathbb{N} \) such that \( B^s = 0 \).

**Proposition 23** [24]: Consider BN (2) with its ASSR (4). There exists a coordinate change \( \tilde{x} = T x \), such that under \( \tilde{x} \), (4) becomes
\[
\tilde{x}(t + 1) = \tilde{M} \tilde{x}(t) \equiv \text{diag}(C_1, \ldots, C_s) \tilde{x}(t)
\]
where
\[
C_i = \begin{bmatrix} A_i & E_i \\ 0 & B_i \end{bmatrix}, \quad A_i
\]
is cyclic and \( B_i \) is nilpotent, \( i = 1, \ldots, s \).

\( C_i \) corresponds to cycle \( A_i \) with other states in the basin of attraction \( B_i \). To be precise, if the \( i \)th subnetwork of (24) corresponding to \( C_i \) is expressed into a blockwise form as
\[
\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} A_i & E_i \\ 0 & B_i \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
\]
then the set \( \{x_1 \} \) forms the \( i \)th cycle and the set \( \{x_2 \} \) forms its basin of attraction.

(24) is called the canonical form of a BN. Using it, we have the following result.

**Proposition 24:** The length of the largest dual cycle is
\[
\text{lcm}(\|A_1\|_{\ell_0}, \|A_2\|_{\ell_0}, \ldots, \|A_s\|_{\ell_0})
\]
where \( \| \|_{\ell_0} \) is the \( \ell_0 \)-norm, i.e., the length of the attractor.

**Proof:** Consider a BN in its canonical form, then \( \forall f \in \mathcal{X}^\ast \). We can aggregate \( V_f \) as \( V_f = [v_1, v_2, \ldots, v_t] \), where \( v_t \in B_{1 \times \|C \|_{\ell_0}}, i = 1, 2, \ldots, s \).

Now we have \( V_f \tilde{M} = [v_1 C_1, v_2 C_2, \ldots, v_t C_t] \). Consider each \( v_i C_i, i = 1, \ldots, s \), from the structure of \( C_i \), obviously

**Fig. 3.** Transition graph of BN (26).

the largest length of the attractors produced by \( v_i C_i \) is \( \|A_i\|_{\ell_0} \), in this case, unless \( A_i = 1 \)
\[
[v_1, v_2, \ldots, v_t \|A_i\|_{\ell_0}] \neq 1 \text{ or } 0.
\]

It is straightforward that for some appropriate \( V_f \) [i.e., each \( v_i \) satisfies condition (25)], the length of the largest cycle that is produced by \( V_f \tilde{M} \) is \( \text{lcm}(\|A_1\|_{\ell_0}, \ldots, \|A_s\|_{\ell_0}) \).

Assume \( C_i \in \mathcal{M}_{r_i \times r_i}, i = 1, \ldots, s \), then \( \sum_{i=1}^s r_i = 2^n \).

Since each BN has its canonical form, in the following we assume \( x \) itself is a “canonical” coordinate frame such that the BN under \( x \) is in the canonical form.

Recall canonical form (24). Assume the states are partitioned into
\[
\mathcal{X} = \bigcup_{i=1}^s \mathcal{X}_i
\]
where \( \mathcal{X}_i := \{ \delta_i \mid \tilde{M} \delta_{2i} \} \) includes a column of \( C_i \) corresponds to the states in the \( i \)th block in (24).

Then we have the following result.

**Proposition 25:** Let \( \mathcal{X}^\ast = \{ f \in \mathcal{X}^\ast \mid \text{supp}(f) \subset \mathcal{X}_i \} \), then \( \mathcal{X}_i^\ast \) are the \( \tilde{M} \)-invariant subspaces, \( i = 1, \ldots, s \).

**Proof:** Consider \( \forall f \in \mathcal{X}^\ast \), denote by \( V_f \) its structure vector. For \( \forall \delta_{2i} \in \Delta_2, i = 1, \ldots, s \), if \( V_f \tilde{M} \tilde{\delta}_{2i} = 0 \), then \( \tilde{M} \delta_{2i} \in \mathcal{X}_i \), which means that \( \delta_{2i} \in \mathcal{X}_i \), hence \( f \in \mathcal{X}_i \), where \( V_f = V_f \tilde{M} \). Therefore, \( \mathcal{X}_i^\ast \) is an \( \tilde{M} \)-invariant subspace.

**Example 26:** Consider BN
\[
\begin{align*}
Z_1(t+1) &= Z_1(t) \cup (\neg Z_1(t) \land \neg Z_2(t) \land Z_3(t)) \\
Z_2(t+1) &= (\neg Z_2(t) \land Z_3(t) \leftrightarrow Z_5(t)) \\
Z_3(t+1) &= \neg Z_3(t) \land Z_2(t)
\end{align*}
\]
\[
(26)
\]
\( Z_3(t+1) = (Z_1(t) \land Z_5(t) \land \neg Z_3(t)) \land Z_2(t) \).

Its ASSR is calculated as \( z(t+1) = M z(t) \), where \( z(t) = \prod_{i=1}^s z_i(t) \), \( M = \delta_3[3,2,1,2,6,8,3,5] \). The state transition graph is depicted in Fig. 3.

According to the state transition graph Fig. 3, a coordinate transformation can be obtained as in [24]
\[
x = T z
\]
where \( T = \delta_8[1,4,2,5,6,7,3,8] \).

Under coordinates \( x \), the system becomes
\[
x(t+1) = \tilde{M} x(t)
\]
where \( \tilde{M} = T \ast M \ast T^T = \delta_8[2,1,2,4,4,7,8,6] \).

Now BN (28) is in the canonical form, and the structure
matrix is
\[ \tilde{M} = \text{diag}(C_1, C_2, C_3) \]
where
\[ C_i = \begin{bmatrix} A_i & E_i \\ 0 & B_i \end{bmatrix}, \quad i = 1, 2, 3 \]
with
\[ A_1 = \delta_2[2, 1], \quad E_1 = \delta_2, \quad B_1 = 0 \]
\[ A_2 = 1, \quad E_2 = 1, \quad B_2 = 0 \]
\[ A_3 = \delta_2[2, 3, 1]. \]

Finally, we consider the topological structure of the dual space. Since the dual subspaces \( X_i^* = \{ f(x) \in X^* \mid \text{supp}(f) \subset X_i \}, \quad i = 1, 2, 3, \) are invariant, attractors in \( X_i^* \) with their basins form a partition of \( X_i^* \). We discuss the topological structures of \( X_i^* \) one by one as follows.

1) Consider \( X_1^* \): Let \( f \in X_1^* \). Then \( V_f = [c_1, c_2, c_3, 0, 0, 0, 0, 0, 0] \). Using \( (c_1, c_2, c_3) \) to represent \( f \), we have dual structure on \( X_1^* \) as given in Fig. 4.

2) Consider \( X_2^* \): Let \( f \in X_2^* \). Then \( V_f = [0, 0, 0, c_4, c_5, 0, 0, 0, 0] \). Using \( (c_4, c_5) \) to represent \( f \), we have dual structure on \( X_2^* \) as given in Fig. 5.

3) Finally, consider \( X_3^* \): Let \( f \in X_3^* \). Then \( V_f = [0, 0, 0, 0, 0, c_6, c_7, c_8] \). Using \( (c_6, c_7, c_8) \) to represent \( f \), we have the dual structure of \( X_3^* \) as given in Fig. 6.

Remark 27: In the above discussions, we take the convention from Definition 8 and assign the entries in the first row of \( V_f \), whose corresponding states do not belong to the support of \( f_i \), to be zero.

V. HIDDEN ORDER

The topological structure on \( X^* = \mathcal{F}_\ell \) is not manifest. From Proposition 14, one sees that the dual structure matrix \( M^* \) is determined by the original structure matrix \( M \) and the dual space \( V \). However, it is not straightforward to figure out the structure of the dual space since there is no general pattern of how the topological structure of an arbitrary \( M \) determines that of its dual \( M^* \). Particularly, under the original coordinate frame which is the natural one, it is even murky. Because from Propositions 24 and 25, one can find that the dual attractors are generated by the attractors on \( X \). To be more specific, an \( \tilde{M} \)-invariant subspace \( X_\ell^* \) is generated by \( X_\ell \) corresponding to a block in the canonical form. If we use a structure matrix \( M \) under canonical coordinates to generate the dual structure matrix \( M^* \), then the structure vectors of the dual states in the dual attractors will turn out to be evolving only on certain components. Without the canonical form, the relationship between attractors on \( X \) and \( X^* \) cannot be addressed in such ways.

We use the following example, in which 20 structure vectors are enough to represent the dual attractors among all the 256 dual states, to illustrate this.

Example 28: Recall Example 26. Consider the functions in \( X_\ell^* \). Back to the original coordinate frame, it is easy to calculate their structure vectors; see Table II, where the underlined entries are the only ones that evolve.

\[
\begin{array}{c|c|c}
\hline
(c_1, c_2, c_3) & V_f & V_* = V_f^T \\
\hline
(0,0,0) & [0,0,0,0,0,0,0,0,0] & \\
(0,0,1) & [0,1,0,0,0,0,0,0,0] & \\
(0,1,0) & [0,0,0,0,1,0,0,0,0] & \\
(0,1,1) & [0,1,0,0,0,0,0,0,0] & \\
(1,0,0) & [1,0,0,0,0,0,0,0,0] & \\
(1,1,0) & [1,1,0,0,0,0,0,0] & \\
(1,1,1) & [1,1,0,0,1,0,0] & \\
\hline
\end{array}
\]

Combining Example 26 with Example 28, one sees easily that finding the order determined by dual logical functions is not easy. Hence, the order suggested by the topological
TABLE IV

| Structure Vector of Functions in D3 | V2 = VfT |
|-------------------------------------|----------|
| (c0, c7, c9) of Vf             | [0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0] |
| (0,0.0)                           | [0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0] |
| (0,0.1)                           | [0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0] |
| (0,1.0)                           | [0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0] |
| (0,1.1)                           | [0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0] |
| (1,0.0)                           | [0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0] |
| (1,0.1)                           | [0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0] |
| (1,1.0)                           | [0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0] |
| (1,1.1)                           | [0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0] |

structure of a dual BN (i.e., dynamics of logical functions) is called the hidden order. From Examples 26 and 28, one sees that the hidden order determines the behaviors of the BN. The relationship between attractors and dual attractors is depicted by Fig. 7, where D1 is the set of dual attractors generated by attractors contained in \( A_i \).

Recall (18). Without the canonical form, one should use \( V \in \mathcal{B}_{2^n \times 2^n} \) and \( VM \) to obtain the dual structure matrix \( M^\ast \), which is a tedious job, especially when \( n \) is large. From the above, we know that a logical function can be decomposed into several independent parts according to the support sets. This allows one to analyze the dual attractors without computing the \( 2^n \times 2^n \)-dimensional dual structure matrix and then doing some power operation, which is a common way (see [31]).

The following result is understandable from the above analysis.

**Proposition 29:** For a given BN (28), there are \( \prod_{i=1}^{n} ||C_i|| \) types of dual attractors, where \( ||C_i|| \) denotes the number of attractors in the \( i \)th block.

**Proof:** Similarly as in the proof of Proposition 24, under the canonical form, any logical function can be denoted by \( V_f = [\nu_1, \nu_2, \ldots, \nu_s] \), where \( \nu_i \in \mathcal{B}_{1 \times |C_i|_{0}} \). Then \( V_fM = [\nu_1C_1, \nu_2C_2, \ldots, \nu_sC_s] \). Because of the property of \( C_i \), any \( v_i \) will enter an attractor. As \( \nu_i \), \( i = 1, 2, \ldots, s \) are updated independently, the number of the dual attractors is consequently the number of permutations of the attractors in \( A_i \). □

**VI. BOOLEAN ALGEBRA ON \( \mathcal{X}^* \)**

Recall network (26) in Example 26, which has nodes \( n = 3 \). Hence, \( |\mathcal{X}^*| = 2^2 = 256 \). But in Example 28, only \( \mathcal{X}^*_i \), \( i = 1, 2, 3 \), which involve only 20 elements of \( \mathcal{X}^* \), have been investigated. Are \( \mathcal{X}^*_i \), \( i = 1, 2, 3 \) enough to determine the topological structure of \( \mathcal{X}^* \)? The answer is “yes.” The reason is that the elements in \( \mathcal{X}^* \) are not independent of each other, that is to say, some basic functions in \( \mathcal{X}^* \) can generate the whole dual space. There is a Boolean algebra structure over \( \mathcal{X}^* \), to which this section is devoted.

**Definition 30 [26]:** A Boolean algebra is \( B_A = \{\wedge, \vee, \neg, 1, 0\} \), satisfying

1) Commutative law
\[ x \vee y = y \vee x, \quad x \wedge y = y \wedge x. \]

2) Associative Law
\[ (x \vee y) \vee z = x \vee (y \vee z), \quad (x \wedge y) \wedge z = x \wedge (y \wedge z). \]

3) Distributive Law
\[ x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z). \]

4) Identity Law
\[ x \vee 0 = x, \quad x \wedge 1 = x. \]

5) Complement Law
\[ x \vee \neg x = 1, \quad x \wedge \neg x = 0. \]

**Definition 31:** Let \( f, g \in \mathcal{X}^* \) with structure vectors \( V_f \) and \( V_g \), respectively. Then for \( i = 1, 2, \ldots, 2^n \), we have

1) \( V_{f \wedge g}(i) = V_f(i) \wedge V_g(i) \).
2) \( V_{f \vee g}(i) = V_f(i) \vee V_g(i) \).
3) \( V_{\neg f}(i) = \neg f(i) \).
4) \( V_1 = 1^T_2, \quad V_0 = 0^T_2 \).

The following result comes from Definition 30 immediately.

**Proposition 32:** Consider BN (2) with its ASSR (4).

1) \( \mathcal{X}^* \) with \( \wedge, \vee, \neg, 1, 0 \), defined in Definition 31, is a Boolean algebra.

2) Let \( M \in \mathcal{L}_{2^n \times 2^n} \). Then \( \psi_M : \mathcal{X}^* \to \mathcal{X}^* \), defined as,
\[
V_{\psi_M(f)} = V_fM
\] (29)

is a \( B_A \) homomorphism. That is,
\[
\psi_M(1) = 1, \quad \psi_M(0) = 0
\]
\[
\psi_M(x^* \wedge y^*) = \psi_M(x^*) \wedge \psi_M(y^*)
\]
\[
\psi_M(x^* \vee y^*) = \psi_M(x^*) \vee \psi_M(y^*)
\]
\[
\psi_M(\neg x^*) = \neg \psi_M(x^*). \] (30)

**Definition 33:** Let
\[
d_i^*(x) = \begin{cases} 1, & x = \delta_{i0} \\ 0, & \text{Otherwise} \end{cases} \quad i = 1, 2, \ldots, 2^n.
\]

Then \( \{d_i^* \mid i = 1, 2, \ldots, 2^n\} \) is called a set of generators of \( \mathcal{X}^* \).
**Proposition 34:** Let

\[ V_{d_j} = \bigvee_{j=1}^{s} V_{d_{j}^*} . \]

Then

\[ f(x) = \bigvee_{j=1}^{s} d_{j}^*(x). \]  (31)

Moreover, if \( f(t) = \bigvee_{j=1}^{s} d_{j}^*(t) \), then

\[ f(t + 1) = \bigvee_{j=1}^{s} V_{d_{j}^*} M. \]  (32)

Equation (32) can be used to construct the dynamics on \( X^* \).

**Example 35:** Recall Example 26. It is easy to calculate that

\[
\begin{align*}
V_{d_{1}} M &= [1, 0, 0, 0, 0, 0, 0, 0] M = [0, 1, 0, 0, 0, 0, 0, 0] \quad M, \\
V_{d_{2}} M &= [0, 1, 0, 0, 0, 0, 0, 0] M = [1, 0, 1, 0, 0, 0, 0, 0] \quad M, \\
V_{d_{3}} M &= [0, 0, 0, 0, 0, 0, 0, 0] M = [0, 0, 0, 0, 0, 0, 0, 0] \quad M, \\
V_{d_{4}} M &= [0, 0, 0, 0, 0, 0, 0, 0] M = [0, 0, 0, 0, 0, 0, 0, 0] \quad M, \\
V_{d_{5}} M &= [0, 0, 0, 0, 0, 0, 0, 0] M = [0, 0, 0, 0, 0, 0, 0, 0] \quad M, \\
V_{d_{6}} M &= [0, 0, 0, 0, 0, 0, 0, 0] M = [0, 0, 0, 0, 0, 0, 0, 0] \quad M, \\
V_{d_{7}} M &= [0, 0, 0, 0, 0, 0, 0, 0] M = [0, 0, 0, 0, 0, 0, 0, 0] \quad M.
\end{align*}
\]

Now assume

\[ V_{f(t)} = [1, 0, 0, 0, 0, 0, 0, 0, 0] = V_{d_{1}} \bigvee V_{d_{2}} \bigvee V_{d_{3}}. \]

Then

\[ V_{f(t+1)} = V_{d_{1}} M \bigvee V_{d_{2}} M \bigvee V_{d_{3}} M = [0, 1, 0, 0, 0, 0, 0, 1, 0]. \]

Using the vector form provided in (13), we have

\[
\begin{align*}
V_{f(t)} &= V_{x_{139}} \quad M, \\
V_{f(t+1)} &= V_{x_{138}} \quad M
\end{align*}
\]

hence

\[ x_{139}(t + 1) = x_{138}(t). \]

All the dynamic equations of \( X^* \) can be obtained similarly.

**Remark 36:** According to this Boolean algebra structure, among \( 2^{2^n} \) logical equations in \( X^* \), only \( 2^{2^n} \) of them are independent, which compose a coordinate transformation of \( X^* \).

**VII. REALIZATION OF BCNs**

Consider a BCN

\[
\begin{align*}
X_1(t + 1) &= f_1(X_1(t), \ldots, X_n(t), U_1(t), \ldots, U_m(t)), \\
& \vdots \\
X_n(t + 1) &= f_n(X_1(t), \ldots, X_n(t), U_1(t), \ldots, U_m(t)), \\
Y_j(t) &= g_j(X_1(t), \ldots, X_n(t)), \quad j \in [1, p]
\end{align*}
\]

(33)

where \( X_i(t), i \in \{1, n\} \) are the state variables, \( U_k(t), k \in \{1, m\} \) are the controls, and \( Y_j(t), j \in \{1, p\} \) are the outputs. Its ASSR is described as

\[ x(t + 1) = Lu(t)x(t) \]

where \( x(t) = \bigwedge_{i=1}^{s} X_i(t), u(t) = \bigwedge_{k=1}^{m} U_k(t), y(t) = \bigwedge_{j=1}^{p} Y_j(t), \)

\( L \in \mathcal{L}_{2^n \times 2^n} \), and \( E \in \mathcal{L}_{2^n \times 2^n} \).

**Definition 37:** [9]: Consider BCN (33) with its ASSR (34).

Let \( \Xi^* \subset X^* \). If \( \mathcal{V}^* \subset X^* \) satisfies

1) \( \Xi^* \subset \mathcal{V}^* \).
2) \( \mathcal{V}^* \) is \( M_i \)-invariant, where \( M_i = L \delta^*_M, \quad i \in [1, 2^m] \),

then \( \mathcal{V}^* \) is called a control-invariant subspace (CIS) containing \( \Xi^* \).

If \( \mathcal{V}^* \) is a CIS, and

1) \( \mathcal{V}^* \subset \mathcal{W}^* \) for any other CIS \( \mathcal{W}^* \),

then \( \mathcal{V}^* \) is called the smallest CIS containing \( \Xi^* \).

Now we provide an algorithm to calculate the smallest CIS containing a given \( \Xi^* \); see Algorithm 1.

| Algorithm 1: Smallest CIS Containing \( \Xi^* \) [9] |
|---|
| **Data:** \((\Xi^*, L, m, n)\) |
| a subset of the dual space \( X^* \), the structure matrix |
| of the BCN, the number of controls, the number of |
| state variables |
| **Result:** \( \mathcal{V}^* \) |
| the smallest CIS containing \( \Xi^* \) |
| \( j = 0, k = 0, \mathcal{V}_0^* := \Xi^*; \) |
| **while** \( k < 2^m \) **do** |
| \( \mathcal{V}_{k+1}^* := \mathcal{V}_k^* \bigcup \bigcup_{j=1}^{2^m} \mathcal{V}_k^* L \delta^*_M; \) |
| **if** \( \mathcal{V}_{k+1}^* = \mathcal{V}_k^* \) **then** |
| \( \mathcal{V}^* := \mathcal{V}_k^* \); |
| \( k := 2^m \); |
| **else** |
| \( k + + ; \) |
| **end** |
| **end** |

From Definition 37 and Algorithm 1, the following conclusion can be easily obtained.

**Proposition 38:** \( \mathcal{V}^* \) obtained by Algorithm 1 is the CIS containing \( \Xi^* \).

Assume \( \mathcal{V}^* = \{z_1, z_2, \ldots, z_s\} \) and

\[ z_i = G_i x, \quad i = 1, 2, \ldots, s. \]

Let \( z = \bigwedge_{i=1}^{s} z_i \). Then

\[ z = Gx \]

where \( G = G_1 \ast G_2 \ast \cdots \ast G_s \in \mathcal{L}_{2^n \times 2^n} \).

Next, consider

\[ z(t + 1) = Gx(t + 1) = GLu(t)x(t) = [GM_1, GM_2, \ldots, GM_{2^n}]u(t)x(t). \]

(35)

From [9] we know that \( \mathcal{V}^* \) is \( M_i \)-invariant, then there exists \( H_i \in \mathcal{L}_{2^n \times 2^n} \), such that

\[ GM_i = H_i G, \quad i = 1, 2, \ldots, 2^m. \]
Set $H = [H_1, H_2, \ldots, H_{2^n}]$. Then (35) becomes
\[ z(t+1) = Hu(t)z(t). \] (36)

Consider BCN (33) with its ASSR (34). Now we suppose that $\rho_i \in \mathcal{V}^*$, $i \in \{1, p\}$. Then the outputs can be expressed by
\[ y_i = \rho_i(x) = F_i z, \quad i = 1, 2, \ldots, p. \]

Hence
\[ y(t) = Fz(t) \] (37)

where $F = F_1 \ast F_2 \ast \cdots \ast F_p$.

Summarizing the above, we have the following concept as a matter of course.

**Definition 39** [9]: Let $\mathcal{V}^* \subset \mathcal{X}^*$ be the smallest CIS, which contains $y_j = \rho_j(x)$, $j = 1, 2, \ldots, p$ and is $M_i$-invariant, $i = 1, 2, \ldots, 2^m$. Then the corresponding system (36) and (37) is called the minimum realization of BCN (33).

**Remark 40**: 1) The minimum realization is on $\mathcal{X}^*$.

Unlike continuous-time (control) systems, since $|\mathcal{X}^*| >> |\mathcal{X}|$, the dimension of minimum realization may be larger than the dimension of the original BCN.

2) Using the Boolean algebraic structure, the dimension of the minimum realization can further be reduced, which is assumed to be less than or equal to the dimension of the original BCN.

**Example 41**: Consider the following BCN:
\[
\begin{align*}
  x(t+1) &= Lu(t)x(t) \\
  y(t) &= Ex(t)
\end{align*}
\] (38)

where $x(t) \in \Delta_8$, $u(t), y(t) \in \Delta_2$

$L = \delta_8[4, 2, 8, 8, 5, 6, 6, 3, 8, 7, 3, 1, 4, 6, 6, 4,$

$E = \delta_2[2, 2, 2, 1, 2, 2, 1].$

By Algorithm 1, the smallest CIS containing $y$ is
\[ \mathcal{V}^* = \{x_1^*, y, x_2^*, x_3^*\} \]

where
\[ x_1^* = \delta_2[2, 1, 2, 2, 2, 2, 2] \]
\[ x_2^* = \delta_2[2, 2, 2, 2, 2, 2, 2] \]

The minimum realization of (38) is
\[
\begin{align*}
  x_1^*(t+1) &= x_2^*(t) \\
  x_2^*(t+1) &= [x_1^*(t), x_2^*(t)]u(t) \\
  x_3^*(t+1) &= \delta_2^2.
\end{align*}
\] (39)

Consider a large-scale BN (refer to Fig. 8). A distributed realization is described as follows.

1) Inject some inputs $u^i$, $i = 1, \ldots, s$.
2) Observe some outputs $y^i$, $i = 1, \ldots, s$.
3) Consider the corresponding minimum realizations $\Sigma_i : u^i \Rightarrow y^i$, $i = 1, \ldots, s$.

We can investigate parts of the BN for some particular properties. This kind of distributed realization helps investigate large-scale BNs.

**VIII. $k$-Valued Logical Networks**

Denote
\[ D_k := \left\{ 0, \frac{1}{k-1}, \frac{2}{k-1}, \ldots, 1 \right\}, \quad k \geq 2. \]

When $k = 2$, $D_2 = D$. This subsection considers the case when $k \geq 3$.

Setting
\[ \frac{i}{k-1} \sim \delta_{k-1}^{k-1}, \quad i = 0, 1, \ldots, k-1 \]

we have the vector expression of $X \in D_k$ as $x = \tilde{X} \in \Delta_k$.

Equation (2) is a $k$-valued logical network, if $X_i \in D_k$ and $f_i : D_k^x \to D_k$, $i = 1, 2, \ldots, n$. Using vector form expression, we also obtain its ASSR (4), where $M \in L_{k^x \times k^x}$. 

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Fig. 8. Distributed realization.

Fig. 9. Chained gears’ structure of attractors.
The state space is

\[ \mathcal{X} = \{ (X_1, \ldots, X_n) : X_i \in \mathcal{D}_k, i = 1, 2, \ldots, n \} \sim \Delta_k. \]

Let \( f : \mathcal{D}_k^n \rightarrow \mathcal{D}_k \). Similar to the Boolean case, there exists a unique logical matrix \( M_f \in L_{k,k} \) such that

\[ f(x) = M_f x = \delta_k[i_1, i_2, \ldots, i_k] x. \]

The vector

\[ V_f := [i_1, i_2, \ldots, i_k] \]

is called the structure vector of \( f \).

Its dual space is

\[ \mathcal{X}^* = \{ f : f : \mathcal{D}_k^n \rightarrow \mathcal{D}_k \} \sim \{ [i_1, \ldots, i_k] : 1 \leq i_j \leq k, j \in [1, k^n] \}. \]

In an argument similar to the Boolean case, one sees that the aforementioned arguments about the Boolean (control) networks with certain obvious modifications remain true for the \( k \)-valued logical networks.

IX. CONCLUSION

By introducing dual spaces, dual BNs, and minimum realization of BCNs, the hidden order of a BN is first revealed and explored.

It was pointed out by Kauffman that the tiny attractors in a large-scale Boolean network determine the vast order [19]. In [2] and [3], the structure of chained gears was proposed to explain why tiny attractors determine the order of the overall BN. Fig. 9 depicts a set of chained gears, where each circle represents a cycle.

In this chained gear structure, the tiny ones can be considered the driving gears, and the large gears can be considered the following ones. Hence, the tiny gears determine the order of the overall system.

This structure appears only to dual space \( \mathcal{X}^* \). Consider Example 26 again. Assume \( z_i \in \mathcal{X}_i^*, i = 1, 2, 3 \) lie on (dual) cycles \( C^*_i \subset \mathcal{X}_i^* \), respectively, and \( |C^*_i| = \ell_i \). If \( z_{1+2} = z_1 \lor z_2 \), then it is obvious that \( z_{1+2} \) is on a cycle \( C^*_{1+2} \in \mathcal{X}_1^* \cup \mathcal{X}_2^* \), and \( |C^*_{1+2}| = \ell_1 + \ell_2 \). Furthermore, if \( z_{1+2+3} = z_1 \lor z_2 \lor z_3 \), then \( C^*_{1+2+3} \in \mathcal{X}_1^* \cup \mathcal{X}_2^* \cup \mathcal{X}_3^* \) can be obtained. In general, a large cycle in \( \mathcal{X}^* \) can be generated by tiny cycles within each invariant (dual) subspaces \( \mathcal{X}_i^* \). It leads to the conclusion that the tiny attractors determine the vast order. Moreover, this fact also reveals that the hidden order from dual BNs may play a more important role in determining the order of BNs when it is applied to modeling the live world.

It seems to us that the (explicit) order determined by the attractors of a BN is the inner order of a BN. The order observed by us may be one of the dual spaces, that is, logical functions on the BN. Furthermore, the results proposed in this article can be generalized to other logical systems such as the \( k \)-valued logical networks and mix-valued logical networks.

A DNA system with A, T, C, and G may be considered a four-valued network. We conjecture that the hidden order from its dual network might be the key to understanding it.

Several important control problems of BNs such as input–output decomposition, disturbance decoupling, and output tracking depend on related sets of logical functions. Since hidden order reveals the topological relation over all logical functions, it helps provide a guideline for designing required controls to solve the corresponding problems. As for minimum realization, it has been shown in this article that a minimum realization itself is a hidden order between the inputs and outputs.

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