The Bohnenblust–Hille inequality combined with
an inequality of Helson

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Abstract

We give a variant of the Bohnenblust-Hille inequality which, for certain families of polynomials, leads to constants with polynomial growth in the degree.

1 Introduction

Hardy and Littlewood showed in [8] that there exists a constant $K > 0$ such that for every $f \in H^1$ we have

$$\left( \int_{\mathbb{D}} |f(z)|^2 \, dm(z) \right)^{1/2} \leq K \int_{\mathbb{T}} |f(w)| \, d\sigma(w),$$

where $dm$ and $d\sigma$ denote respectively the normalised Lebesgue measures on the complex unit disk $\mathbb{D}$ and the torus (or unit circle) $\mathbb{T}$. Equivalently, this means that the Hardy space $H_1(\mathbb{T})$ is contained in the Bergman space $B_2(\mathbb{D})$. Shapiro [13, p. 117-118] showed that the inequality holds with $K = \pi$ and Mateljević [11] (see also [12, 14]) showed that actually the constant could be taken $K = 1$. A simple reformulation of the Bergman norm then gives that if $\sum_{n=0}^{\infty} a_n z^n$ is the Fourier series expansion of $f \in H^1(\mathbb{D})$ we have

$$\left( \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} \right)^{1/2} \leq \int_{\mathbb{T}} |f(w)| \, d\sigma(w).$$

A few years later Helson in [10] generalised this inequality to functions in $N$ variables. For $n \in \mathbb{N}$ denote by $d(n)$ the number of divisors and by $p^a = p_1^{a_1} \cdots p_k^{a_k}$...
the prime decomposition of \( n \). Then we have that for every \( f \in H^1(\mathbb{T}^N) \) with Fourier series expansion \( \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha z^\alpha \)
\[
\left( \sum_{\alpha \in \mathbb{N}_0^N} \frac{|c_\alpha|^2}{d(p^\alpha)} \right)^{1/2} \leq \int_{\mathbb{T}^N} |f(w)| d\sigma(w).
\] (1)

Given a multiindex \( \alpha \), we write \( \alpha + 1 = (\alpha_1 + 1) \cdots (\alpha_k + 1) \). Note that, with this notation, we have \( d(p^\alpha) = \alpha + 1 \).

On the other hand, by the Bohnenblust-Hille inequality [4] as presented in [5] there is a constant \( C > 0 \) such that for every \( m \)-homogeneous polynomial in \( N \) variables \( P(z) = \sum_{|\alpha| = m} c_\alpha z^\alpha \) with \( z \in \mathbb{C}^N \) we have
\[
\left( \sum_{|\alpha| = m} |c_\alpha|^m \right)^{m+1} \leq C^m \sup_{z \in \mathbb{D}^N} |P(z)|.
\] (2)

The proof of this inequality given in [5] consists basically of two steps: first to decompose the sum in (2) as the product of certain mixed sums and second to bound each one of these sums by a term including \( \|P\| \), the supremum of \( |P| \) in \( \mathbb{D}^N \). For this second step usually the following result of Bayart [1] is used: for every \( m \)-homogeneous polynomial in \( N \) variables we have
\[
\left( \sum_{|\alpha| = m} |c_\alpha|^m \right)^{1/2} \leq 2^{m/2} \int_{\mathbb{T}^N} \left| \sum_{|\alpha| = m} c_\alpha w^\alpha \right| d\sigma(w).
\] (3)

Very recently, it was proved in [2, Corollary 5.3] that for every \( \epsilon > 0 \) there exists \( \kappa > 0 \) such that we can take \( \kappa(1 + \epsilon)^m \) as the constant in (2). Our aim in this note is get a variant of (2) by using (1) instead of (3). With this variant, we see that for polynomials \( P \) each of whose monomials involve a uniformly bounded number of variables, the obtained constants have polynomial growth in \( m \).

2 Main result and some remarks

The following is our main result.

**Theorem 2.1.** Let \( \Lambda \subseteq \{ \alpha \in \mathbb{N}_0^N : |\alpha| = m \} \) be an indexing set. Then for every family \( \{c_\alpha\}_{\alpha \in \Lambda} \) we have
\[
\left( \sum_{\alpha \in \Lambda} \frac{|c_\alpha|^m}{\sqrt{\alpha + 1}} \right)^{m+1} \leq m^{m-1} \left( 1 - \frac{1}{m-1} \right)^{m-1} \sup_{z \in \mathbb{D}^N} \left| \sum_{\alpha \in \Lambda} c_\alpha z^\alpha \right|.
\]

We give several remarks before we present the proof.

**Remark 2.2.**
1. It is easy to see that $\sqrt{\alpha + 1} \leq \sqrt{2m}$. Hence the preceding inequality includes the hypercontractive version of the Bohnenblust-Hille inequality from \([2]\) as a special case.

2. Thanks to the term $\sqrt{\alpha + 1}$, the constants in the previous inequality grow much more slowly than the constants in \([2]\). Actually, we have

$$m^{\frac{m-1}{m}} \left(1 - \frac{1}{m-1}\right)^{m-1} = \frac{\sqrt{m}}{e} \left(1 + o(m)\right).$$

3. Let $\text{vars}(\alpha)$ denote the numbers of different variables involved in the monomial $z^\alpha$. In other words, $\text{vars}(\alpha) = \text{card}\{j : \alpha_j \neq 0\}$. Given $M$ we consider the set

$$\Lambda_{N,M} = \{\alpha \in \mathbb{N}_0^N : |\alpha| = m \text{ and } \text{vars}(\alpha) \leq M\},$$

(note that if $M \geq N$, then $\Lambda_{N,M} = \Lambda_{N,N}$). An application of Lagrange multipliers gives that for any $\alpha \in \Lambda_{N,M}$ we have for every $N$ and $M$

$$\alpha + 1 = (\alpha_1 + 1) \cdots (\alpha_k + 1) \cdots \leq \left(\frac{m}{M} + 1\right)^M.$$

Combining this with Theorem 2.1 we obtain for every $m, N, M$

$$\left( \sum_{\alpha \in \Lambda_{N,M}} |c_\alpha|^2 \right)^{\frac{m+1}{2m}} \leq \left( \frac{m}{M} + 1\right)^{M/2} \left( \sum_{\alpha \in \Lambda_{N,M}} \left( \frac{|c_\alpha|}{\sqrt{\alpha + 1}} \right)^{\frac{2m}{m+1}} \right)^{\frac{m+1}{m}}$$

$$\leq \left( \frac{m}{M} + 1\right)^{M/2} m^{-\frac{m}{2m}} \left(1 - \frac{1}{m-1}\right)^{m-1} \sup_{z \in \mathbb{B}^N} \left| \sum_{\alpha \in \Lambda_{N,M}} c_\alpha z^\alpha \right|,$$

hence

$$\left( \sum_{\alpha \in \Lambda_{N,M}} |c_\alpha|^2 \right)^{\frac{m+1}{2m}} \leq 2^{\frac{M}{m}} M^{\frac{M+1}{m}} \sup_{z \in \mathbb{B}^N} \left| \sum_{\alpha \in \Lambda_{N,M}} c_\alpha z^\alpha \right|. \quad (4)$$

This means that for polynomials whose monomials have a uniformly bounded number $M$ of different variables, we get a Bohnenblust-Hille type inequality with a constant of polynomial growth in $m$. We remark that the dimension $N$ plays no role in this inequality, the only important point here is the number of different variables in each monomial. As a consequence, an analogue of \((4)\) holds for $m$-homogeneous polynomials on $c_0$:

Let $P : c_0 \rightarrow \mathbb{C}$ be an $m$-homogeneous polynomial and

$$\Lambda_M = \{\alpha \in \mathbb{N}_0^{|\mathbb{N}|} : |\alpha| = m \text{ and } \text{vars}(\alpha) \leq M\}.$$

Then for every $M$ and $m$

$$\left( \sum_{\alpha \in \Lambda_M} |c_\alpha(P)|^{2m} \right)^{\frac{m+1}{2m}} \leq 2^{\frac{M}{m}} m^{\frac{M+1}{m}} \|P\|,$$

where the $c_\alpha(P)$ are the coefficients of $P$ and $\|P\|$ is the supremum of $|P|$ on the unit ball of $c_0$. 

3
4. In [6] Theorem 5.3] a very general version of the Bohnenblust-Hille inequality is given, involving operators with values on a Banach lattice. A straightforward combination of the proof of Theorem 2.1 (see the final section) and the arguments presented in [6 Theorem 5.3] easily gives a version of Theorem 2.1 in that setting.

3 The proof

Let us fix some notation before we prove our main result. We are going to use the following indexing sets

\[ \mathcal{M}(m, N) = \{ i = (i_1, \ldots, i_m) : 1 \leq i_j \leq N, j = 1, \ldots, m \} \]

\[ \mathcal{J}(m, N) = \{ i \in \mathcal{M}(m, N) : 1 \leq i_1 \leq \cdots \leq i_m \leq N \}. \]

In \( \mathcal{M}(m, N) \) we define an equivalence relation by \( i \sim j \) if there is a permutation \( \sigma \) of \( \{1, \ldots, N\} \) such that \( j_k = i_{\sigma(k)} \) for every \( k \). With this, if \( \{a_{i_1, \ldots, i_m}\} \) are symmetric then we have

\[ \sum_{i \in \mathcal{M}(m, N)} a_i = \sum_{i \in \mathcal{J}(m, N)} \sum_{J \in \mathcal{J}(m, N)} \text{card}(J) a_i. \]

Also, given \( i \in \mathcal{M}(m-1, N) \) and \( j \in \{1, \ldots, N\} \), for \( 1 \leq k \leq m-1 \) we define \( (i_k, j) = (i_1, \ldots, i_{k-1}, j, i_k, \ldots, i_{m-1}) \in \mathcal{M}(m, N) \) (that is, we put \( j \) in the \( k \)-th position, shifting the rest to the right).

There is a one-to-one correspondance between \( \mathcal{J}(m, N) \) and \( \{ \alpha \in \mathbb{N}_0^N : |\alpha| = m \} \) defined as follows. For each \( i \) we define \( \alpha = (a_1, \ldots, a_N) \) by \( a_r = \text{card}(j : i_j = r) \) (i.e. \( a_r \) counts how many times \( r \) comes in \( i \)); on the other hand, given \( \alpha \) we define \( i = (1, a_1, 1, \ldots, N, a_N, N) \in \mathcal{J}(m, N) \).

Each \( m \)-homogeneous polynomial on \( N \) variables has a unique symmetric \( m \)-linear form \( L : \mathbb{C}^N \times \cdots \times \mathbb{C}^N \rightarrow \mathbb{C} \) such that \( P(z) = L(z, \ldots, z) \) for every \( z \). If \( (c_\alpha) \) are the coefficients of the polynomial and \( a_{i_1, \ldots, i_m} = L(e_{i_1}, \ldots, e_{i_m}) \) is the matrix of \( L \) we have \( c_\alpha = \text{card}(i_1) a_{i_1} \), where \( \alpha \) and \( i \) are related to each other.

Finally, if \( \alpha \) and \( i \) are related and \( p_1 < p_2 < \cdots \) denotes the sequence of prime numbers, we will write \( p_\alpha = p_1^{a_1} \cdots p_N^{a_N} = p_{i_1} \cdots p_{i_m} = p_i. \)

Proof of Theorem 2.1. We follow essentially the guidelines of the proof of the Bohnenblust-Hille inequality as presented in [5]. First of all let us assume that \( c_\alpha = 0 \) for every \( \alpha \notin \Lambda \); then we have

\[
\left( \sum_{\alpha \in \Lambda} \left( \frac{|c_\alpha|}{\sqrt{\alpha + 1}} \right)^{2m} \right)^{\frac{1}{2m}} = \left( \sum_{i \in \mathcal{J}(m, N)} \left| \text{card}(i) \frac{a_i}{\sqrt{d(p_i)}} \right|^{2m} \right)^{\frac{1}{2m}} = \left( \sum_{i \in \mathcal{J}(m, N)} \left| \text{card}(i) \frac{a_i}{\sqrt{d(p_i)}} \right|^{2m} \right)^{\frac{1}{2m}}.
\]

4
We now use an inequality due to Blei [3, Lemma 5.3] (see also [5, Lemma 1]): for any family of complex numbers \((b_i)_{i \in \mathcal{M}(m,N)}\) we have
\[
\sum_{i \in \mathcal{M}(m,N)} |b_i| \leq \prod_{k=1}^{m} \left( \sum_{j=1}^{N} \left( \sum_{i \in \mathcal{M}(m-1,N)} |b(i,k,j)|^2 \right)^{1/2} \right)^{1/m}.
\] (5)

Using this and the fact that \(\text{card}[(i,k,j)] \leq m \text{card}[i]\) we get
\[
\left( \sum_{\alpha \in \Lambda} \frac{|c_{\alpha}|}{\sqrt{\alpha + 1}} \right)^{2m+1} \leq \prod_{k=1}^{m} \left( \sum_{j=1}^{N} \left( \sum_{i \in \mathcal{M}(m-1,N)} |\text{card}[i,k,j]| \frac{|a(i,k,j)|}{\sqrt{d(p(i,k,j))}} \right)^{2} \right)^{1/2} \prod_{k=1}^{m} \left( \sum_{j=1}^{N} \left( \sum_{i \in \mathcal{M}(m-1,N)} |\text{card}[i]| \frac{|a(i,k,j)|}{\sqrt{d(p(i,k,j))}} \right)^{2} \right)^{1/2} \leq \prod_{k=1}^{m} \left( \sum_{j=1}^{N} \left( \sum_{i \in \mathcal{M}(m-1,N)} |\text{card}[i]| \frac{|a(i,k,j)|}{\sqrt{d(p(i,k,j))}} \right)^{2} \right)^{1/2} \]

We now bound each one of the sums in the product. We use the fact that the coefficients \(a_j\) are symmetric. Also, if \(q\) divides \(p_{i_1} \cdots p_{i_m} = p_i\), then it also divides \(p_{i_1} \cdots p_{i_m} p_j = p(i,k,j)\); hence \(d(p_i) \leq d(p(i,k,j))\) for every \(i\) and every \(j\). This altogether gives
\[
\sum_{j=1}^{N} \left( \sum_{i \in \mathcal{M}(m-1,N)} |\text{card}[i]| \frac{|a(i,k,j)|}{\sqrt{d(p(i,k,j))}} \right)^{2} \right)^{1/2} \leq \sum_{j=1}^{N} \left( \sum_{i \in \mathcal{M}(m-1,N)} \frac{|\text{card}[i]| a(i,k,j)^2}{d(p(i,k,j))} \right)^{1/2} \]

Let us note that what we have here are the coefficients of an \((m-1)\)-homogeneous
polynomial in $N$ variables, we use now (1) to conclude our argument

$$\sum_{j=1}^{N} \left( \sum_{t \in \mathcal{J}(m-1, N)} |\text{card}(i) a(i, j)|^2 \right)^{1/2}$$

$$\leq \sum_{j=1}^{N} \int_{\mathcal{T}_N} \left| \sum_{t \in \mathcal{J}(m-1, N)} a(i, j) w_i \cdots w_{i_{m-1}} \right| d\sigma(w)$$

$$\leq \int_{\mathcal{T}_N} \sum_{j=1}^{N} \left| \sum_{t \in \mathcal{J}(m-1, N)} a(i, j) w_i \cdots w_{i_{m-1}} \right| d\sigma(w)$$

$$\leq \sup_{z \in \mathbb{D}^N} \sum_{j=1}^{N} \left| \sum_{t \in \mathcal{J}(m-1, N)} a(i, j) z_i \cdots z_{i_{m-1}} \right|$$

$$= \sup_{z \in \mathbb{D}^N} \sup_{y \in \mathbb{D}^N} \left| \sum_{j=1}^{N} \left| \sum_{t \in \mathcal{J}(m-1, N)} a(i, j) z_i \cdots z_{i_{m-1}} y_j \right| \right|$$

$$\leq \left(1 - \frac{1}{m-1}\right)^{m-1} \sup_{z \in \mathbb{D}^N} \left| \sum_{a \in \Lambda} c_a z^a \right|,$$

where the last inequality follows from a result of Harris [9, Theorem 1] (see also [5, (13)]). This completes the proof.

As we have already mentioned, very recently [2, Corollary 5.3] has shown that for every $\epsilon > 0$ there exists $\kappa > 0$ such that (2) holds with $\kappa (1 + \epsilon)^m$. The main idea for the proof is to replace (5) by a similar inequality in which we have mixed sums with $k$ and $m - k$ indices (instead of 1 and $m - 1$, as we have here). This allows to use instead of (3) the following inequality:

$$\left( \sum_{|a| = m} |c_a|^2 \right)^{1/2} \leq c_p \left( \int_{\mathcal{T}_N} \left| \sum_{|a| = m} c_a w^a \right|^p d\sigma(w) \right)^{1/p}$$

for $1 \leq p \leq 2$.

A good control on the constants $c_p$ (that tend to 1 as $p$ goes to 2) gives the improvement on the constant in (2) presented in [2]. In our setting, by dividing by $a + 1$, we are using (1), which already has constant 1. Hence this new approach does not improve the constants in our setting.

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