T-COERCIVITY FOR SOLVING STOKES PROBLEM WITH NONCONFORMING FINITE ELEMENTS

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Abstract. We propose to analyse the discretization of the Stokes problem with nonconforming finite elements in light of the T-coercivity (cf. [1] for Helmholtz-like problems, see [2], [3] and [4] for the neutron diffusion equation). We propose explicit expressions of the stability constants. Finally, we give numerical results illustrating the importance of using divergence-free velocity reconstruction.

Keywords. Stokes problem, T-coercitivity, nonconforming finite elements, Fortin operator

2020 Mathematics Subject Classification. 65N30, 35J57, 76D07

Funding. CEA SIMU/SITHY project

1. Introduction

The Stokes problem describes the steady state of incompressible Newtonian flows. They are derived from the Navier–Stokes equations [5]. With regard to numerical analysis, the study of Stokes problem helps to build an appropriate approximation of the Navier–Stokes equations. We consider here a discretization with nonconforming finite elements [6, 7]. We propose to state the discrete inf-sup condition in light of the T-coercivity (cf. [1] for Helmholtz-like problems, see [2], [3] and [4] for the neutron diffusion equation), which allows to estimate the discrete error constant. In Section 2 we recall the T-coercivity theory as written in [1]. In Section 3 we apply it to the continuous Stokes Problem. We give details on the triangulation in Section 4 and we apply the T-coercivity to the discretization of Stokes problem with nonconforming mixed finite elements in Section 5. In Section 6 (resp. 7), we precise the proof of the well-posedness in the case of order 1 (resp. order 2) nonconforming mixed finite elements. Finally, we give numerical results illustrating the importance of using divergence-free velocity reconstruction.

2. T-coercivity

We recall here the T-coercivity theory as written in [1]. Consider first the variational problem, where $V$ and $W$ are two Hilbert spaces and $f \in V'$. Find $u \in V$ such that $\forall v \in W$, $a(u, v) = \langle f, v \rangle_V$.

(2.1) $a(u, v) = \langle f, v \rangle_V.$
Classically, we know that Problem (2.1) is well-posed if \( a(\cdot, \cdot) \) satisfies the stability and the solvability conditions of the so-called Banach–Nečas–Babuška (BNB) Theorem (see a.e. [8, Thm. 25.9]). For some models, one can also prove the well-posedness using the T-coercivity theory (cf. [1] for Helmholtz-like problems, see [2], [3] and [4] for the neutron diffusion equation).

**Definition 1.** Let \( V \) and \( W \) be two Hilbert spaces and \( a(\cdot, \cdot) \) be a continuous and bilinear form over \( V \times W \). It is T-coercive if
\[
\exists T \in \mathcal{L}(V, W), \text{ bijective, } \exists \alpha > 0, \forall v \in V, |a(v, Tv)| \geq \alpha \|v\|_V^2.
\]
If in addition \( a(\cdot, \cdot) \) is symmetric, it is T-coercive if
\[
\exists T \in \mathcal{L}(V, V), \exists \alpha > 0, \forall v \in V, |a(v, Tv)| \geq \alpha \|v\|_V^2.
\]
When the bilinear form \( a(\cdot, \cdot) \) is symmetric, the requirement that the operator \( T \) is bijective can be dropped. It is proved in [1] that the T-coercivity condition is equivalent to the stability and solvability conditions of the BNB Theorem. Whereas the BNB theorem relies on an abstract inf–sup condition, T-coercivity uses explicit inf–sup operators, both at the continuous and discrete levels.

**Theorem 1.** (well-posedness) Let \( a(\cdot, \cdot) \) be a continuous and bilinear form. Suppose that the form \( a(\cdot, \cdot) \) is T-coercive. Then Problem (2.1) is well-posed.

3. Stokes problem

Let \( \Omega \) be a connected bounded domain of \( \mathbb{R}^d, d = 2, 3 \), with a polygonal \((d = 2)\) or Lipschitz polyhedral \((d = 3)\) boundary \( \partial \Omega \). We consider Stokes problem:
\[
\begin{cases}
-\nu \Delta \mathbf{u} + \mathbf{grad} p &= \mathbf{f}, \\
\mathbf{div} \mathbf{u} &= 0.
\end{cases}
\]

with Dirichlet boundary conditions for the velocity \( \mathbf{u} \) and a normalization condition for the pressure \( p \):
\[
\mathbf{u} = 0 \text{ on } \partial \Omega, \quad \int_{\Omega} p = 0.
\]

The vector field \( \mathbf{u} \) represents the velocity of the fluid and the scalar field \( p \) represents its pressure divided by the fluid density which is supposed to be constant. Thus, the SI unit of the components of \( \mathbf{u} \) is \( \text{m} \cdot \text{s}^{-1} \) and the SI unit of \( p \) is \( \text{m}^2 \cdot \text{s}^{-2} \). The first equation of (3.1) corresponds to the momentum balance equation and the second one corresponds to the conservation of the mass. The constant parameter \( \nu > 0 \) is the kinematic viscosity of the fluid, its SI unit is \( \text{m}^2 \cdot \text{s}^{-1} \). The vector field \( \mathbf{f} \in \mathbf{H}^{-1}(\Omega) \) represents a body forces divided by the fluid density, its SI unit is \( \text{m} \cdot \text{s}^{-2} \).

Before stating the variational formulation of Problem (3.1), we provide some definition and reminders. Let us set \( \mathbf{L}^2(\Omega) = (L^2(\Omega))^d, \mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^d, \mathbf{H}^{-1}(\Omega) = (H^{-1}(\Omega))^d \) its dual space and \( L^2_{\text{div}}(\Omega) = \{ q \in L^2(\Omega) \mid \int_{\Omega} q = 0 \} \). We recall that \( \mathbf{H}(\text{div}; \Omega) = \{ \mathbf{v} \in \mathbf{L}^2(\Omega) \mid \text{div} \mathbf{v} \in L^2(\Omega) \} \). Let us first recall Poincaré–Steklov inequality:
\[
\exists C_{PS} > 0 \mid \forall \mathbf{v} \in H_0^1(\Omega), \quad \|\mathbf{v}\|_{L^2(\Omega)} \leq C_{PS} \|\mathbf{grad} \mathbf{v}\|_{L^2(\Omega)}.
\]
The SI unit of \( C_{PS} \) is \( m \).

Thanks to this result, in \( H_0^1(\Omega) \), the semi-norm is equivalent to the natural norm, so that the scalar product reads \( (\mathbf{v}, w)_{H_0^1(\Omega)} = (\mathbf{grad} \mathbf{v}, \mathbf{grad} w)_{L^2(\Omega)} \) and the norm
is \( \|v\|_{H^1_0(\Omega)} = \|\text{grad } v\|_{L^2(\Omega)} \). Let \( v, w \in H^1_0(\Omega) \). We denote by \((v_i)_{i=1}^d \) (resp. \((w_i)_{i=1}^d \)) the components of \( v \) (resp. \( w \)), and we set \( \text{Grad } v = (\partial_j v_i)_{i,j=1}^d \in L^2(\Omega) \), where \( L^2(\Omega) = [L^2(\Omega)]^{d \times d} \). We have:

\[
(\text{Grad } v, \text{Grad } w)_{L^2(\Omega)} = (v, w)_{H^1_0(\Omega)} = \sum_{i=1}^d (v_i, w_i)_{H^1_0(\Omega)}
\]

and:

\[
\|v\|_{H^1_0(\Omega)} = \left( \sum_{j=1}^d \|v_j\|^2_{H^1_0(\Omega)} \right)^{1/2} = \|\text{Grad } v\|_{L^2(\Omega)}.
\]

Let us set \( V = \{ v \in H^1_0(\Omega) \mid \text{div } v = 0 \} \). The space \( V \) is a closed subset of \( H^1_0(\Omega) \). We denote by \( V^\perp \) the orthogonal of \( V \) in \( H^1_0(\Omega) \). Let \( \nu_p > 0 \) be a kinematic viscosity. We recall that \([5, \text{cor. I.2.4}]:\)

**Proposition 1.** The operator \( \text{div } : H^1_0(\Omega) \to L^2(\Omega) \) is an isomorphism of \( V^\perp \) onto \( L^2_{\text{div}}(\Omega) \). Let \( \nu_p > 0 \) be a constant kinematic viscosity. We call \( C_{\text{div}} \) the constant such that:

\[
\forall p \in L^2_{\text{div}}(\Omega), \exists ! v_p \in V^\perp | \text{div } v_p = \frac{1}{\nu_p} p \text{ and } \|v_p\|_{H^1_0(\Omega)} \leq \frac{C_{\text{div}}}{\nu_p} \|p\|_{L^2(\Omega)}.
\]

Here, the constant \( C_{\text{div}} \) has no unit. It depends only on the domain \( \Omega \). Notice that we have: \( C_{\text{div}} = 1/\beta(\Omega) \) where \( \beta(\Omega) \) is the inf-sup condition (or Ladyzhenskaya–Babuška–Brezzi condition):

\[
\beta(\Omega) = \inf_{q \in L^2_{\text{div}}(\Omega) \setminus \{0\}} \sup_{v \in H^1_0(\Omega) \setminus \{0\}} \frac{(q, \text{div } v)_{L^2(\Omega)}}{\|q\|_{L^2(\Omega)} \|v\|_{H^1_0(\Omega)}}.
\]

Generally, the value of \( \beta(\Omega) \) is not known explicitly. In \([9]\), Bernardi et al established results on the discrete approximation of \( \beta(\Omega) \) using conforming finite elements. Recently, Gallistl proposed in \([10]\) a numerical scheme with adaptive meshes for computing approximations to \( \beta(\Omega) \). In the case of \( d = 2 \), Costabel and Dauge \([11]\) established the following bound:

**Theorem 2.** Let \( \Omega \subset \mathbb{R}^2 \) be a domain contained in a ball of radius \( R \), star-shaped with respect to a concentric ball of radius \( \rho \). Then

\[
\beta(\Omega) \geq \frac{\rho}{\sqrt{2} R} \left( 1 + \sqrt{1 - \frac{\rho^2}{R^2}} \right)^{-1/2} \geq \frac{\rho}{2 R}.
\]

Let us detail the bound for some remarkable domains. If \( \Omega \) is a ball, \( \beta(\Omega) \geq \frac{1}{2} \) and if \( \Omega \) is a square, \( \beta(\Omega) \geq \frac{1}{2 \sqrt{2}} \). Suppose now that \( \Omega \) is stretched in some direction by a factor \( k \), then \( \beta(\Omega) \geq \frac{1}{k \sqrt{k}} \). Finally, if \( \Omega \) is L-shaped (resp. cross-shaped) such that \( L = k l \), where \( L \) is the largest length and \( l \) is the smallest length of an edge, then \( \beta(\Omega) \geq \frac{1}{2 \sqrt{2} k} \) (resp. \( \beta(\Omega) \geq \frac{1}{3 l} \)).

The variational formulation of Problem (3.1) reads: Find \((u, p) \in H^1_0(\Omega) \times L^2_{\text{div}}(\Omega)\) such that

\[
\begin{cases}
p(u, v)_{H^1_0(\Omega)} - (p, \text{div } v)_{L^2(\Omega)} = (f, v)_{H^1_0(\Omega)} \\
(q, \text{div } u)_{L^2(\Omega)} = 0
\end{cases} \quad \forall v \in H^1_0(\Omega) ; \quad \forall q \in L^2_{\text{div}}(\Omega).
\]
Classically, one proves that Problem (3.6) is well-posed using Poincaré-Steklov inequality \([3.2]\) and Prop. 1. Check for instance the proof of [3, Thm. I.5.1].

Let us set \(\mathcal{X} = H^1_0(\Omega) \times L^2_{mez}(\Omega)\) which is a Hilbert space which we endow with the following norm:

\[
\|(v, q)\|_{\mathcal{X}} = \|v\|_{H^1_0(\Omega)} + \nu^{-1} \|q\|_{L^2(\Omega)}.
\]

We consider now the following bilinear symmetric and continuous form:

\[
a_S : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}
\]

\[
\langle (u', p'), (v, q) \rangle = \nu(u', v)_{H^1_0(\Omega)} - (p', \nabla v)_{L^2(\Omega)} - (q, \nabla u')_{L^2(\Omega)}.
\]

We can write Problem (3.1) in an equivalent way as follows:

\[
\text{(3.9) Find } (u, p) \in \mathcal{X} \text{ such that } a_S ((u, p), (v, q)) = (f, v)_{H^1_0(\Omega)} \quad \forall (v, q) \in \mathcal{X}.
\]

Let us prove that Problem (3.9) is well-posed using the T-coercivity theory.

**Theorem 3.** Problem (3.9) is well-posed. It admits one and only one solution such that:

\[
\forall f \in H^{-1}(\Omega), \quad \left\{ \begin{array}{l}
\|u\|_{H^1_0(\Omega)} \leq \nu^{-1} \|f\|_{H^{-1}(\Omega)}; \\
\|p\|_{L^2(\Omega)} \leq C_{\text{div}} \|f\|_{H^{-1}(\Omega)}.
\end{array} \right.
\]

**Proof.** We follow here the proof given in [12, 13]. Let us consider \((u', p') \in \mathcal{X}\) and let us build \((v^*, q^*) = T(u', p') \in \mathcal{X}\) satisfying (2.3) (with \(V = \mathcal{X}\)). We need three main steps.

1. According to Prop. 1 there exists \(v_{p'} \in H^1_0(\Omega)\) such that: \(\nabla v_{p'} = \nu^{-1} p'\) in \(\Omega\) and

\[
\|v_{p'}\|_{H^1_0(\Omega)}^2 \leq \left( \frac{C_{\text{div}}}{\nu} \right)^2 \|p'\|_{L^2(\Omega)}^2.
\]

Let us set \((v^*, q^*) := (\gamma u' - v_{p'}, -\gamma p')\), with \(\gamma > 0\). We obtain:

\[
a_S ((u', p'), (v^*, q^*)) = \nu \gamma \|u'\|_{H^1_0(\Omega)}^2 + \nu^{-1} \|p'\|_{L^2(\Omega)}^2 - \nu (u', v_{p'})_{H^1_0(\Omega)}.
\]

2. In order to bound the last term of (3.12), we use Young inequality and then inequality (3.11):

\[
(u', v_{p'})_{H^1_0(\Omega)} \leq \frac{\eta}{2} \|u'\|_{H^1_0(\Omega)}^2 + \frac{\eta^{-1}}{2} \left( \frac{C_{\text{div}}}{\nu} \right)^2 \|p'\|_{L^2(\Omega)}^2.
\]

3. Using the bound (3.13) in (3.12) and choosing \(\eta = \gamma\), we get:

\[
a_S ((u', p'), (v^*, q^*)) \geq \gamma \nu \|u'\|_{H^1_0(\Omega)}^2 + \nu^{-1} \left( 1 + \frac{\gamma^{-1}}{2} \left( \frac{C_{\text{div}}}{\nu} \right)^2 \right) \|p'\|_{L^2(\Omega)}^2.
\]

Consider now \(\gamma = (C_{\text{div}})^2\).

Noticing that \(\nu \|u\|_{H^1_0(\Omega)}^2 + \nu^{-1} \|p'\|_{L^2(\Omega)}^2 \geq \frac{\nu}{2} \|u', p'\|_{\mathcal{X}}^2\), we obtain:

\[
a_S ((u', p'), (v^*, q^*)) \geq \frac{\nu}{2} C_{\text{min}} \| (u', p') \|_{\mathcal{X}}^2 \text{ where } C_{\text{min}} = \frac{1}{2} \min( (C_{\text{div}})^2, 1).
\]
The operator $T$ such that $T(u', p') = (v^*, q^*)$ is linear and continuous:
\[
\|T(u', p')\|_X := \|v^*\|_{H^1_0(\Omega)} + \nu^{-1}\|q^*\|_{L^2(\Omega)} \\
\leq \gamma \|u'\|_{H^1_0(\Omega)} + \|v'\|_{H^1_0(\Omega)} + \gamma \nu^{-1}\|p'\|_{L^2(\Omega)}, \\
\leq \gamma \|u'^2\|_{H^1_0(\Omega)} + (C_{\text{div}} + \gamma) \nu^{-1}\|p'\|_{L^2(\Omega)}, \\
\leq C_{\text{max}} \|(u', p')\|_X,
\]
where $C_{\text{max}} = C_{\text{div}} (1 + C_{\text{div}})$.

The symmetric and continuous bilinear form $a(\cdot, \cdot)$ is then $T$-coercive and according to Theorem 1 Problem (3.9) is well-posed. Let us prove (3.10). Consider $(u, p)$ the unique solution of Problem (3.9). Choosing $v = 0$, we obtain that $\forall q \in L^2_{\text{div}}(\Omega)$, $(q, \text{div} u)_{L^2(\Omega)} = 0$, so that $u \in V$. Now, choosing $v = u$ and using Cauchy-Schwarz inequality, we have:
\[
\nu \|u\|_{H^1_0(\Omega)}^2 = (f, u)_{H^1_0(\Omega)} \leq \|f\|_{H^{-1}(\Omega)} \|u\|_{H^1_0(\Omega)},
\]
so that:
\[
\|u\|_{H^1_0(\Omega)} \leq \nu^{-1}\|f\|_{H^{-1}(\Omega)}.
\]
Next, we choose in (3.9) $v = v_p \in V^\perp$, where $\text{div} v_p = -\nu^{-1} p$ (see Prop. 1). Noticing that $u \in V$ and $v_p \in V^\perp$, it holds:
\[
-(p, \text{div} v_p)_{L^2(\Omega)} = \nu^{-1}\|p\|_{L^2(\Omega)}^2 = (f, v_p)_{H^1_0(\Omega)}, \\
\leq \|f\|_{H^{-1}(\Omega)} \|v_p\|_{H^1_0(\Omega)} \leq C_{\text{div}} \nu^{-1}\|f\|_{H^{-1}(\Omega)} \|p\|_{L^2(\Omega)},
\]
so that:
\[
\|p\|_{L^2(\Omega)} \leq C_{\text{div}}\|f\|_{H^{-1}(\Omega)}.
\]

Remark 1. We recover the first Banach–Nečas–Babuška condition \[\text{Thm. 25.9, (BNB1)}\]:
\[
a_S((u', p'), (v^*, q^*)) \geq \frac{\nu}{2} C_{\text{min}} (C_{\text{max}})^{-1} \|(u', p')\|_X \|(v^*, q^*)\|_X.
\]

Let us call $C_{\text{stab}} = \frac{\nu}{2} C_{\text{min}} (C_{\text{max}})^{-1}$ the stability constant. With the choice of our parameters, $C_{\text{stab}}$ is such that:
\[
C_{\text{stab}} = \begin{cases} 
\nu \frac{C_{\text{div}}}{4} & \text{if } 0 < C_{\text{div}} \leq 1, \\
\nu \frac{(C_{\text{div}})^{-1}}{4} & \text{if } 1 \leq C_{\text{div}}.
\end{cases}
\]

Thus, the T-coercivity approach allows to give an estimate of the stability constant. In our computations, it depends on the choice of the parameters $\eta$ and $\gamma$, so that it could be optimized.

If we were using a conforming discretization to solve Problem (3.9) (a.e. Taylor-Hood finite elements \[\text{[2]}\]), we would use the bilinear form $a_S(\cdot, \cdot)$ to state the discrete variational formulation. Let us call the discrete spaces $X_{c,h} \subset H^1_0(\Omega)$ and $Q_{c,h} \subset L^2_{\text{div}}(\Omega)$. Then to prove the discrete T-coercivity, we would need to state

\[\text{Remark that } (v^*, q^*) = (0, 0) \iff (u', p') = (0, 0): \text{ the operator } T \in \mathcal{L}(X, X) \text{ is bijective.}\]

\[\text{Remark that } \forall q \in L^2_{\text{div}}(\Omega), (q, \text{div} u)_{L^2(\Omega)} = 0.\]
the discrete counterpart to Proposition 1. To do so, we can build a linear operator \( \Pi_c : X \to X_h \), known as Fortin operator, such that (see a.e. [15 §8.4.1]):

\[
\exists C_c \forall v \in H^1(\Omega) \quad \| \text{Grad} \Pi_c v \|_{L^2(\Omega)} \leq C_c \| \text{Grad} v \|_{L^2(\Omega)},
\]

\[
\forall v \in H^1(\Omega) \quad (\text{div} \Pi_c v, q_h)_{L^2(\Omega)} = (\text{div} v, q_h)_{L^2(\Omega)}, \quad \forall q_h \in Q_{c,h}.
\]

Using a nonconforming discretization, we will not use the bilinear form \( a_S(\cdot, \cdot) \) to exhibit the discrete variational formulation, but we will need a similar operator to (3.14)-(3.15) to prove the discrete T-coercivity, which is stated in Theorem 4.

4. DISCRETIZATION

We call \((O, (x^d)_{d=1}^d)\) the Cartesian coordinates system, of orthonormal basis \((e^d)_{d=1}^d\). Consider \( (T_h)_{h} \) a simplicial triangulation sequence of \( \Omega \). For a triangulation \( T_h \), we use the following index sets:

- \( I_K \) denotes the index set of the elements, such that \( T_h := \bigcup_{\ell \in I_K} K_\ell \) is the set of elements.
- \( I_F \) denotes the index set of the facets\(^3\) such that \( F_h := \bigcup_{f \in I_F} F_f \) is the set of facets.

Let \( I_F = I_F^b \cup I_F^b \), where \( \forall f \in I_F^b, F_f \in \Omega \) and \( \forall f \in I_F^b, F_f \in \partial \Omega \).

- \( I_S \) denotes the index set of the vertices, such that \( \{S_j\}_{j \in I_S} \) is the set of vertices.

Let \( I_S = I_S^b \cup I_S^b \), where \( \forall j \in I_S^b, S_j \in \Omega \) and \( \forall j \in I_S^b, S_j \in \partial \Omega \).

We also define the following index subsets:

- \( \forall \ell \in I_K, I_{F,\ell} = \{ f \in I_F | F_f \subset K_\ell \}, \quad I_{S,\ell} = \{ j \in I_S | S_j \subset K_\ell \} \).
- \( \forall j \in I_S, I_{K,j} = \{ \ell \in I_K | S_j \subset K_\ell \}, \quad N_j := \text{card}(I_{K,j}) \).

For all \( \ell \in I_K \), we call \( h_\ell \) and \( \rho_\ell \) the diameters of \( K_\ell \) and its inscribed sphere respectively, and we let: \( \sigma_\ell = \frac{h_\ell}{\rho_\ell} \). When the \( (T_h)_{h} \) is a shape-regular triangulation sequence (see a.e. [10 def. 11.2]), there exists a constant \( \sigma > 1 \) such that for all \( h \), for all \( \ell \in I_K, \sigma_\ell \leq \sigma \).

For all \( f \in I_F, M_f \) denotes the barycentre of \( F_f \), and by \( n_f \) its unit normal (outward oriented if \( F_f \in \partial \Omega \)). For all \( j \in I_S, \) for all \( \ell \in I_{K,j}, \) \( \lambda_{j,\ell} \) denotes the barycentric coordinate of \( S_j \) in \( K_\ell \); \( F_{j,\ell} \) denotes the face opposite to vertex \( S_j \) in element \( K_\ell \), and \( x_{j,\ell} \) denotes its barycentre. We call \( S_{j,\ell} \) the outward normal vector of \( F_{j,\ell} \) and of norm \( |S_{j,\ell}| = |F_{j,\ell}| \). We remind the expression of \( \lambda_{j,\ell} \) and the integration formula (25.14) p. 187 of [17]:

\[
\forall x \in K_\ell, \lambda_{j,\ell}(x) = (d |K_\ell|)^{-1} (x_{j,\ell} - x) \cdot S_{j,\ell} ;
\]

\[
\int_{K_\ell} \prod_{i \in I_{S,\ell}} \lambda_{i,\ell}^{\alpha_{i,\ell}} = (d |K_\ell|)^{-1} \frac{d! \prod_{i \in I_{S,\ell}} \alpha_{i,\ell}!}{(d + \sum_{i \in I_{S,\ell}} \alpha_{i,\ell})!}.
\]

Let introduce spaces of piecewise regular elements:

We set \( P_hH^1 = \{ v \in L^2(\Omega) ; \forall \ell \in I_K, v_{|K_\ell} \in H^1(K_\ell) \} \), endowed with the scalar

\footnote{The term facet stands for face (resp. edge) when \( d = 3 \) (resp. \( d = 2 \)).}
We recall classical finite elements estimates \cite{16}. Let $\hat{K}$ be the reference simplex and $\hat{F}$ be the reference facet. For $\ell \in \mathcal{I}_K$ (resp. $f \in \mathcal{I}_F$), we denote by $T_\ell : K \to K_\ell$ (resp. $T_f : F \to F_\ell$) the geometric mapping such that $\forall \mathbf{x} \in \hat{K}$, $\mathbf{x}_K = T_\ell(\mathbf{x}) = \mathbf{B}_\ell \mathbf{x} + \mathbf{b}_\ell$ (resp. $\mathbf{x}_F = T_f(\mathbf{x}) = \mathbf{B}_f \mathbf{x} + \mathbf{b}_f$), and we set $J_\ell = \det(\mathbf{B}_\ell)$ (resp. $J_f = \det(\mathbf{B}_f)$). There holds:

$$|J_\ell| = d! |K_\ell|, \quad \|\mathbf{B}_\ell\| = \frac{h_\ell}{\rho_\ell}, \quad \|\mathbf{B}_\ell^{-1}\| = \frac{h_\ell}{\rho_\ell}, \quad |J_f| = (d-1)! |F_f|.$$ 

For $v \in L^2(K_\ell)$, we set $\hat{v}_\ell = v \circ T_\ell$. For $v \in v^2(F_\ell)$, we set: $\hat{v}_f = v \circ T_f$. Changing the variable, we get:

$$\|v\|_{L^2(K_\ell)}^2 = |J_\ell| \|\hat{v}_\ell\|_{L^2(\hat{K})}^2, \quad \text{and} \quad \|v\|_{L^2(F_\ell)}^2 = |J_f| \|\hat{v}_f\|_{L^2(\hat{F})}^2.$$ 

Let $v \in \mathcal{P}_h H^1$. By changing the variable, $\text{grad} v|_{K_\ell} = (\mathbf{B}_\ell^{-1})^T \text{grad}_{\hat{K}} \hat{v}_\ell$, and it holds:

$$\|\text{grad}_{\hat{K}} \hat{v}_\ell\|_{L^2(\hat{K})}^2 \leq \|\mathbf{B}_\ell\|^2 |J_\ell|^{-1} \|\text{grad} v\|_{L^2(K_\ell)}^2 \lesssim \sigma^2 (\rho_\ell)^{-(d-2)} \|	ext{grad} v\|_{L^2(K_\ell)}^2.$$ 

Let us recall some useful inequalities that we will need:

- The Poincaré-Steklov inequality in cells \cite[Lemma 12.11]{16}:
  for all $\ell \in \mathcal{I}_K$ ($K_\ell$ is a convex set), $\forall v \in H^1(K_\ell)$:

$$\|v\|_{L^2(K_\ell)} \leq \pi^{-1} h_\ell \|\text{grad} v\|_{L^2(K_\ell)}, \quad \text{where} \quad v_\ell = v|_{K_\ell} - \frac{\int_{K_\ell} v}{|K_\ell|}.$$ 

- The multiplicative trace inequality as written in the proof of \cite[Lemma 12.15]{16} for $p = 2$: for all $\ell \in \mathcal{I}_K$, for all $f \in \mathcal{I}_{F,\ell}$, $\forall v \in H^1(K_\ell)$:

$$\|v\|_{L^2(F_\ell)}^2 \leq \frac{|F_\ell|}{|K_\ell|} \|v\|_{L^2(K_\ell)} \left(\|v\|_{L^2(K_\ell)} + \frac{1}{d |(\ell,f)|} \|\text{grad} v\|_{L^2(K_\ell)}\right),$$

where $l_{(\ell,f)}$ is the largest length of an edge in $K_\ell$ and not belonging to $F_f$.

- Combining (4.4) and (4.5), we get that $\forall v \in H^1(K_\ell)$:

$$\|v\|_{L^2(F_\ell)}^2 \leq \frac{|F_\ell|}{|K_\ell|} \pi^{-1} h_\ell \left(\pi^{-1} h_\ell + \frac{2 l_{(\ell,f)}}{d |(\ell,f)|}\right) \|\text{grad} v\|_{L^2(K_\ell)}^2.$$
Notice that in the reference element, inequality (4.6) reads:
\[
\forall \ell \in \mathcal{I}_K, \forall f \in \mathcal{I}_F, \ell, \quad \|\nabla f\|_{L^2(F)}^2 \lesssim \|\text{grad}_h v\ell\|_{L^2(K)}^2.
\]
For all \(D \subset \mathbb{R}^d\), we call \(P_k(D)\) the set of order \(k\) polynomials on \(D\), \(P^k(D) = (P_k(D))^d\), and we consider the broken polynomial space:
\[
P_{\text{disc}}(T_h) = \{q \in L^2(\Omega); \quad \forall \ell \in \mathcal{I}_K, \ q_{|K_\ell} \in P^k(K_\ell)\}.
\]

5. The nonconforming mixed finite element method for Stokes

The nonconforming finite element method was introduced by Crouzeix and Raviart in [5] to solve Stokes Problem (3.1). We approximate the vector space (5.1) of order \(k\) component by component by piecewise polynomials of order \(k\):

Proposition 2. The broken norm \(v_h \rightarrow \|v_h\|_h\) is a norm over \(X_{0,h}\).

Proof. Let \(v_h \in X_{0,h}\) such that \(\|v_h\|_h = 0\). Then for all \(\ell \in \mathcal{I}_K\), \(v_h|_{K_\ell}\) is a constant. For all \(f \in \mathcal{I}_F\), the jump \([v_h]_{F_j}\) vanishes, so that \(v_h\) is a constant over \(\Omega\). We deduce from the discrete boundary condition that \(v_h = 0\). □

The space of nonconforming approximation of \(H^1(\Omega)\) (resp. \(H_0^1(\Omega)\)) of order \(k\) is \(X_h = (X_h)^d\) (resp. \(X_{0,h} = (X_{0,h})^d\)). We set \(X_h := X_{0,h} \times Q_h\) where \(Q_h = P_{\text{disc}}^k(T_h) \cap L^2_{\text{disc}}(\Omega)\). We deduce from Proposition 2 the

Proposition 3. The broken norm defined below is a norm on \(X_h\):
\[
\|\langle \cdot, \cdot \rangle\|_{X_h} : \begin{cases} 
X_h &\rightarrow \mathbb{R} \\
(v_h, q_h) &\rightarrow \|v_h\|_h + \nu^{-1}\|q_h\|_{L^2(\Omega)}.
\end{cases}
\]

Thus, the product space \(X_h\) endowed with the broken norm \(\|\cdot\|_{X_h}\) is a Hilbert space.

Proposition 4. The following discrete Poincaré–Steklov inequality holds: there exists a constant \(C_{PS}^{nc}\) independent of \(T_h\) such that
\[
\forall v_h \in X_{0,h}, \quad \|v_h\|_{L^2(\Omega)} \leq C_{PS}^{nc}\|v_h\|_h,
\]
where \(C_{PS}^{nc}\) is independent of \(T_h\) and is proportional to the diameter of \(\Omega\).

Proof. Inequality (5.3) is stated in [8] Lemma 36.6] for \(k = 1\), but one can check that the proof holds true for higher orders, thanks to the patch-test condition. An alternative proof is given in [18] Theorem C.1. □
We consider the discrete continuous bilinear form $a_{S,h}(\cdot, \cdot)$ such that:

\[
\begin{cases}
  a_{S,h} : X_h \times X_h &\to \mathbb{R} \\
  (u_h', p_h') \times (v_h, q_h) &\to \nu (u_h', v_h)_h - (p_h', \text{div} v_h) - (q_h, \text{div} u_h')
\end{cases}
\]

Let $\ell_f \in \mathcal{L}(X_h, \mathbb{R})$ be such that:

\[
\forall (v_h, q_h) \in X_h, \quad \ell_f ((v_h, q_h)) = \begin{cases}
  \langle f, v_h \rangle_{L^2(\Omega)} &\text{if } f \in L^2(\Omega) \\
  \langle f, \mathcal{I}_h(v_h) \rangle_{H^{1/2}(\Omega)} &\text{if } f \notin L^2(\Omega)
\end{cases}
\]

where $\mathcal{I}_h : X_{0,h} \to Y_{0,h}$, with $Y_{0,h} = \{ v_h \in H^1_0(\Omega) : \forall \ell \in \mathcal{I}_K, v_{h|K} \in P^k(K) \}$, is the averaging operator described in [10, Section 22.4.1]. There exists a constant $C_{T,h}^{nc} > 0$ independent of $\mathcal{T}_h$ such that:

\[
\| \mathcal{I}_h v_h \|_{H^1_0(\Omega)} \leq C_{T,h}^{nc} \| v_h \|_h, \quad \forall v_h \in X_{0,h}.
\]

The nonconforming discretization of Problem (3.9) reads:

Find $(u_h, p_h) \in X_h$ such that:

\[
a_{S,h} ((u_h, p_h), (v_h, q_h)) = \ell_f ((v_h, q_h)) \quad \forall (v_h, q_h) \in X_h.
\]

Let us prove that Problem (5.5) is well-posed using the T-coercivity theory.

**Theorem 4.** Suppose that there exists a Fortin operator $\Pi_{nc} : H^1(\Omega) \to X_h$ such that:

\[
\begin{align*}
  &\exists C_{nc} \forall v \in H^1(\Omega) \quad \| \Pi_{nc} v \|_h \leq C_{nc} \| \nabla v \|_{L^2(\Omega)}, \\
  &\forall v \in H^1(\Omega) \quad (\text{div} v, \Pi_{nc} v, q_h) = (\text{div} v, q_h)_{L^2(\Omega)}, \quad \forall q \in Q_h,
\end{align*}
\]

where the constant $C_{nc}$ does not depend on $h$. Then Problem (5.5) is well-posed. Moreover, it admits one and only one solution $(u_h, p_h)$ such that:

\[
\begin{cases}
  \| u_h \|_h \leq C_{PS}^{nc} \nu^{-1} \| f \|_{L^2(\Omega)} &\text{if } f \in L^2(\Omega) \\
  \| p_h \|_{L^2(\Omega)} \leq 2 C_{PS}^{nc} C_{\text{div}}^{nc} \| f \|_{L^2(\Omega)} &\text{if } f \notin L^2(\Omega)
\end{cases}
\]

\[
\begin{cases}
  \| u_h \|_h \leq C_{T,h}^{nc} \nu^{-1} \| f \|_{H^{-1}(\Omega)} &\text{if } f \notin L^2(\Omega) \\
  \| p_h \|_{L^2(\Omega)} \leq 2 C_{T,h}^{nc} C_{\text{div}}^{nc} \| f \|_{H^{-1}(\Omega)} &\text{if } f \notin L^2(\Omega)
\end{cases}
\]

where $C_{\text{div}}^{nc} = C_{\text{div}}^{nc} C_{nc}$. Additionally, we can compute classical a priori error estimates (see [8, Theorems 3, 4 and 6]). Suppose that $(u, p) \in H^{1+k}(\Omega) \times H^k(\Omega)$, we then have the estimate:

\[
\| u - u_h \|_{L^2(\Omega)} \leq C_{\sigma}^{\ell} h^{k+1} \left( |u|_{H^{k+1}(\Omega)} + \nu^{-1} |p|_{H^k(\Omega)} \right),
\]

where the constant $C > 0$ is independent of $h$, $\sigma$ is the shape regularity constant and the exponent $\ell \in \mathbb{N}^*$ depends on $k$.

**Proof.** Let us consider $(u_h', p_h') \in X_h$ and let us build $(v_h', q_h') \in X_h$ satisfying (2.3) (with $V = X_h$). We follow the three main steps of the proof of Theorem 1:

1. According to Proposition 1 there exists $v_{p_h'} \in V^\perp$ such that $\text{div} v_{p_h'} = \nu^{-1} p_h'$ in $\Omega$ and:

\[
\| v_{p_h'} \|^2_{H^1(\Omega)} \leq \left( \frac{C_{\text{div}}}{\nu} \right)^2 \| p_h' \|^2_{L^2(\Omega)}.
\]
Consider \( \mathbf{v}_{h,p_h'} = \Pi_{nc} \mathbf{v}_{p_h'} \), for all \( q_h \in Q_h \), we have: \( \langle \text{div}_h \mathbf{v}_{h,p_h'}, q_h \rangle = \nu^{-1} \langle p_h', q_h \rangle_{L^2(\Omega)} \) and

\[
(5.10) \quad \| \mathbf{v}_{h,p_h'} \|^2 \leq \left( \frac{C_{nc}^{\text{div}}}{\nu} \right)^2 \| p_h' \|_{L^2(\Omega)}^2 \text{ where } C_{nc}^{\text{div}} = C_{nc} C_{\text{div}}.
\]

Let us set \((\mathbf{v}_h^*, q_h^*) := (\gamma_{nc} \mathbf{u}_h^* - \mathbf{v}_{h,p_h}, -\gamma_{nc} p_h')\), with \( \gamma_{nc} > 0 \). We obtain:

\[
(5.11) \quad a_{S,h} ((\mathbf{u}_h', p_h'), (\mathbf{v}_h^*, q_h^*)) = \nu \gamma_{nc} \| \mathbf{u}_h^* \|^2 + \nu^{-1} \| p_h' \|_{L^2(\Omega)}^2 - \nu \langle \mathbf{u}_h', \mathbf{v}_{h,p_h'} \rangle.
\]

2. In order to bound the last term of (5.11), we use Young inequality and then inequality (5.10):

\[
(5.12) \quad \langle \mathbf{u}_h', \mathbf{v}_{h,p_h'} \rangle \leq \frac{\eta_{nc}}{2} \| \mathbf{u}_h^* \|^2 + \frac{\eta_{nc}^{-1}}{2} \left( \frac{C_{nc}^{\text{div}}}{\nu} \right)^2 \| p_h' \|_{L^2(\Omega)}^2.
\]

3. Using the bound (5.12) in (5.11) and choosing \( \eta_{nc} = \gamma_{nc} \), we get:

\[
(5.13) \quad \left\{ \begin{array}{l}
\| \mathbf{u}_h \|_h \leq \nu^{-1} C_{nc}^{PS} \| f \|_{L^2(\Omega)} \quad \text{if } f \in L^2(\Omega), \text{ using } (5.3); \\
\| \mathbf{u}_h \|_h \leq \nu^{-1} C_{nc}^{PS} \| f \|_{H^{-1}(\Omega)} \quad \text{if } f \not\in L^2(\Omega), \text{ using } (5.4).
\end{array} \right.
\]

Consider \((\mathbf{v}_h, q_h) = (\mathbf{v}_{h,p_h}, 0)\) in (5.5), where \( \mathbf{v}_{h,p_h} = \Pi_{nc} \mathbf{v}_{p_h} \) is built as \( \mathbf{v}_{h,p_h} \) in point 1, setting \( p_h' = p_h \). Suppose that \( f \in L^2(\Omega) \). Using the triangular inequality, Cauchy-Schwarz inequality, Poincaré-Steklov inequality (5.3), Theorem 1, and estimate (5.13), we have:

\[
\| p_h \|^2_{L^2(\Omega)} = \nu \langle \mathbf{u}_h, \mathbf{v}_{h,p_h} \rangle_h - \langle f, \mathbf{v}_{h,p_h} \rangle_{L^2(\Omega)},
\]

\[
\leq \nu \| \mathbf{u}_h \|_h \| \mathbf{v}_{h,p_h} \|_{L^2(\Omega)} + \| f \|_{L^2(\Omega)} \| \mathbf{v}_{h,p_h} \|_{L^2(\Omega)}
\]

\[
\leq 2 C_{nc}^{PS} \| f \|_{L^2(\Omega)} \| \mathbf{v}_{h,p_h} \|_{L^2(\Omega)} \leq 2 C_{nc}^{PS} C_{nc} \| f \|_{L^2(\Omega)} \| \text{Grad} \mathbf{v}_{p_h} \|_{L^2(\Omega)},
\]

\[
\leq 2 C_{nc}^{PS} C_{nc}^{\text{div}} \| f \|_{L^2(\Omega)} \| p_h \|_{L^2(\Omega)}.
\]

\footnote{Note that \((\mathbf{v}_h^*, q_h^*) = (0,0) \Leftrightarrow (\mathbf{u}_h^*, p_h^*) = (0,0)\), so that the operator \( T_h \in \mathcal{L}(X_h, X_h) \) is bijective.}
Applying the same reasoning when \( f \in H^{-1}(\Omega) \), we get that:

\[ \left\{ \begin{array}{ll}
\| p_h \|_{L^2(\Omega)} & \leq 2 C_{nC_{\text{div}}} \| f \|_{L^2(\Omega)} \quad \text{if } f \in L^2(\Omega), \text{ using } (5.3) ; \\
\| p_h \|_{L^2(\Omega)} & \leq 2 C_{nC_{\text{div}}} \| f \|_{H^{-1}(\Omega)} \quad \text{if } f \notin L^2(\Omega), \text{ using } (5.4) .
\end{array} \right. \]

The a priori error estimate corresponds to [6, Theorem 4]. □

Remark 2. Again, we recover the first Banach–Nečas–Babuška condition [8, Thm. 25.9, (BNB1)]:

\[ a_{S,h} \left( (u'_h, p'_h), (v'_h, q'_h) \right) \geq \frac{\nu}{2} (C_{\text{max}}^{-1} \| (u'_h, p'_h) \|_{X_h} \| (v'_h, q'_h) \|_{X_h} , \right. \]

Let us call \( C_{\text{stab}} = \frac{\nu}{2} C_{\text{min}} (C_{\text{max}})^{-1} \) the stability constant. With the choice of our parameters, \( C_{\text{stab}} \) is such that:

\[ C_{\text{stab}} = \begin{cases}
\frac{\nu}{4} \frac{C_{\text{div}}}{1 + C_{\text{div}}} & \text{if } 0 < C_{\text{div}} \leq 1, \\
\frac{\nu (C_{\text{div}})^{-1}}{4} \frac{C_{\text{div}}}{1 + C_{\text{div}}} & \text{if } 1 \leq C_{\text{div}} .
\end{cases} \]

The main issue with nonconforming mixed finite elements is the construction the basis functions. In a recent paper, Sauter explains such a construction in two dimensions [18, Corollary 2.4], and gives a bound to the discrete counterpart \( \beta_T(\Omega) \) defined in (3.4):

\[ (5.15) \quad \beta_T(\Omega) = \inf_{q_h \in Q_{h,\setminus \{0\}}} \sup_{v_h \in X_{0,h,\setminus \{0\}}} \frac{\| \text{div} v_h \|}{\| q_h \|_{L^2(\Omega)} \| v_h \|} \geq c_T k^{-\alpha} . \]

This bound is in \( c_T k^{-\alpha} \), where the parameter \( \alpha \) is explicit and depends on \( k \) and on the mesh topology; and the constant \( c_T \) depends only on the shape-regularity of the mesh.

6. Nonconforming Crouzeix-Raviart mixed finite elements

We study the lowest order nonconforming Crouzeix-Raviart mixed finite elements [6]. Let us consider \( X_{CR} \) (resp. \( X_{0,CR} \)), the space of nonconforming approximation of \( H^1(\Omega) \) (resp. \( H_0^1(\Omega) \)) of order 1:

\[ X_{CR} = \left\{ v_h \in P^1_{\text{disc}}(T_h) ; \quad \forall f \in I_F, \int_{I_F} [v_h] = 0 \right\} ; \]

\[ X_{0,CR} = \left\{ v_h \in X_{CR} ; \quad \forall f \in I_F^b, \int_{I_F} v_h = 0 \right\} . \]

The space of nonconforming approximation of of \( H^1(\Omega) \) (resp. \( H_0^1(\Omega) \)) of order 1 is \( X_{CR} = (X_{CR})^d \) (resp. \( X_{0,CR} = (X_{0,CR})^d \)). We set \( X_{CR} := X_{0,CR} \times Q_{CR} \) where \( Q_{CR} = P^0_{\text{disc}}(T_h) \cap L^2_{\text{sm}}(\Omega) \). We can endow \( X_{CR} \) with the basis \( (\psi_f)_{f \in I_F} \) such that: \( \forall \ell \in I_K, \psi_f|_{I_{F,\ell}} = \begin{cases} 1 - d \lambda_{i,\ell} & \text{if } f \in I_{F,\ell}, \\
0 & \text{otherwise}, \end{cases} \)
where \( S_i \) is the vertex opposite to \( F_i \) in \( K_i \). We then have \( \psi_{f|F_i} = 1 \), so that
\[
[\psi_f|F_i] = 0 \quad \text{if} \quad f \in \mathcal{I}_F (\text{i.e.} \ F_i \in \Omega), \quad \forall f' \neq f, \int_{F_i} \psi_f = 0.
\]
We have: \( X_{CR} = \text{vect} ((\psi_f)_{f \in \mathcal{I}_F}) \) and \( X_{0,CR} = \text{vect} ((\psi_f)_{f \in \mathcal{I}_F}) \).
The Crouzeix-Raviart interpolation operator \( \pi_{CR} \) for scalar functions is defined by:
\[
\pi_{CR} : \left\{ \begin{array}{l} H^1(\Omega) \\ \forall v \end{array} \right. \to X_{CR} \quad \text{such that:} \quad \pi_{CR}v = \frac{1}{\|F_f\|} \int_{F_f} v.
\]
\[
\text{Notice that \( \pi_{CR} \) preserves the constants, so that} \quad \pi_{CR}(\psi_f) = \psi_f.
\]
We recall the following result \([19, \text{Lemma 2}]):
\[
\text{Lemma 1. The Crouzeix-Raviart interpolation operator} \pi_{CR} \text{ is such that:}
\]
\[
(6.2) \quad \forall v \in H^1(\Omega), \quad \| \pi_{CR}v \|_h \leq \| \text{grad} \ v \|_{L^2(\Omega)}.
\]
\[
\text{Proof. We have, integrating by parts twice and using Cauchy-Schwarz inequality:}
\]
\[
\text{grad} \pi_{CR}v|_{K_i} = |K_i|^{-1} \int_{K_i} \text{grad} \pi_{CR}v = |K_i|^{-1} \sum_{f \in \mathcal{I}_F, f \neq f_i} \int_{F_f} \pi_{CR}v \ n_f,
\]
\[
= |K_i|^{-1} \sum_{f \in \mathcal{I}_F, f \neq f_i} \int_{F_f} v \ n_f = |K_i|^{-1} \int_{K_i} \text{grad} v,
\]
\[
\Rightarrow \| \text{grad} \pi_{CR}v|_{K_i} \|_{L^2(K_i)} \leq |K_i|^{-1/2} \| \text{grad} v \|_{L^2(K_i)}.
\]
\[
\text{Summing these local estimates over} \ \ell \in \mathcal{I}_K, \ \text{we obtain} \ (6.2). \quad \square
\]

For a vector \( v \in H^1(\Omega) \) of components \( (v_d)_{d=1}^d \), the Crouzeix-Raviart interpolation operator is such that: \( \Pi_{CR}v = (\pi_{CR}v_d)_{d=1}^d \). Let us set \( \Pi_f v = (\pi_f v_d)_{d=1}^d \).

\[
\text{Lemma 2. The Crouzeix-Raviart interpolation operator} \ Pi_{CR} \text{ can play the role of the Fortin operator:}
\]
\[
(6.3) \quad \forall v \in H^1(\Omega) \quad \| \Pi_{CR}v \|_h \leq \| \text{Grad} v \|_{L^2(\Omega)},
\]
\[
(6.4) \quad \forall v \in H^1(\Omega) \quad (\text{div}_h \Pi_{CR}v, q_h) = (\text{div} v, q_h)_{L^2(\Omega)}, \quad \forall q \in Q_h.
\]

\[
\text{Moreover, for all} \ \ell \in \mathcal{I}^1(\Omega), \ \Pi_{CR}v = v.
\]
\[
\text{Proof. We obtain} \ (6.3) \ \text{applying Lemma 1 component by component. By integrating by parts, we have} \ \forall v \in H^1(\Omega), \ \forall \ell \in \mathcal{I}_K:
\]
\[
\int_{K_i} \text{div} \Pi_{CR}v = \sum_{f \in \mathcal{I}_F, f \neq f_i} \int_{F_f} \Pi_{CR}v \cdot n_f = \sum_{f \in \mathcal{I}_F, f \neq f_i} \int_{F_f} \Pi_f v \cdot n_f,
\]
\[
= \sum_{f \in \mathcal{I}_F, f \neq f_i} \int_{F_f} v \cdot n_f = \int_{K_i} \text{div} v,
\]
so that \( (6.4) \) is satisfied. \quad \square
\]

We can apply the T-coercivity theory to show the next following result:

\[
\text{Theorem 5. Let} \ \chi_h = X_{CR}. \ \text{Then the continuous bilinear form} \ a_{S,h}(\cdot, \cdot) \text{ is} \ T_h-\text{coercive and Problem} \ (5.5) \text{ is well-posed}.
\]
\[
\text{Proof. Using estimates} \ (6.3) \ \text{and} \ (5.3), \ \text{we apply the proof of Theorem 4}. \quad \square
\]
Since the constant of the interpolation operator $\Pi_{CR}$ is equal to 1, we have $C_{min}^{CR} = C_{min}$ and $C_{max}^{CR} = C_{max}$: the stability constant of the nonconforming Crouzeix-Raviart mixed finite elements is independent of the mesh. This is not the case for higher order (see [20, Theorem 2.2]).

7. Fortin-Soulie mixed finite elements

We consider here the case $d = 2$ and we study Fortin-Soulie mixed finite elements [7]. We consider a shape-regular triangulation sequence $(\mathcal{T}_h)_h$.

Let us consider $X_{FS}$ (resp. $X_{0,FS}$), the space of nonconforming approximation of $H^1(\Omega)$ (resp. $H^1_0(\Omega)$) of order 2:

$$X_{FS} = \left\{ v_h \in P^2_{disc}(\mathcal{T}_h) : \forall f \in \mathcal{T}_f, \forall q_h \in P^1(F_f), \int_{F_f} [v_h] q_h = 0 \right\};$$

(7.1)

$$X_{0,FS} = \left\{ v_h \in X_{FS} : \forall f \in \mathcal{T}_f, \forall q_h \in P^1(F_f), \int_{F_f} v_h q_h = 0 \right\}.$$

The space of nonconforming approximation of $H^1(\Omega)$ (resp. $H^1_0(\Omega)$) of order 2 is $X_{FS} = (X_{FS})^2$ (resp. $X_{0,FS} = (X_{0,FS})^2$). We set $X_{FS} = X_{0,FS} \times Q_{FS}$ where $Q_{FS} := P^1_{disc}(\mathcal{T}_h) \cap L^2_{zmv}(\Omega)$.

The building of a basis for $X_{0,FS}$ is more involved than for $X_{0,CR}$ since we cannot use two points per facet as degrees of freedom. Indeed, for all $\ell \in K_\ell$, there exists a polynomial of order 2 vanishing on the Gauss-Legendre points of the facets of the boundary $\partial K_\ell$. Let $f \in \mathcal{T}_f$. The barycentric coordinates of the two Gauss-Legendre points $\left( p_{+f}, p_{-f} \right)$ on $F_f$ are such that:

$$p_{+f} = (c_+, c_-), \quad p_{-f} = (c_-, c_+), \quad \text{where} \quad c_{\pm} = (1 \pm 1/\sqrt{3})/2.$$

These points can be used to integrate exactly order three polynomials:

$$\forall g \in P^3(F_f), \quad \int_{F_f} g = \frac{|F_f|}{2} \left( g(p_{+f}) + g(p_{-f}) \right).$$

For all $\ell \in I_K$, we define the quadratic function $\phi_{K_\ell}$ that vanishes on the six Gauss-Legendre points of the facets of $K_\ell$ (see Fig. 1):

$$\phi_{K_\ell} := 2 - 3 \sum_{i \in I_3, \ell} \lambda_{i,\ell}^2 \text{ such that } \forall f \in \mathcal{T}_{F_\ell}, \forall q \in P^1(F), \quad \int_{F_f} \phi_{K_\ell} q = 0.$$

\[
\begin{align*}
\text{Figure 1. The six Gauss-Legendre points of an element } K_\ell \text{ and the elliptic function } \phi_{K_\ell}.\
\end{align*}
\]
We also define the spaces of $P^2$-Lagrange functions:

\[
X_{\text{LG}} := \left\{ v_h \in H^1(\Omega) \; \middle| \; \forall \ell \in \mathcal{I}_K, \quad v_h|_{K_\ell} \in P^2(K_\ell) \right\},
\]
\[
X_{0,\text{LG}} := \left\{ v_h \in X_{\text{LG}} \; \middle| \; v_h|_{\partial\Omega} = 0 \right\}.
\]

The Proposition below proved in [7, Prop. 1] allows to build a basis for $X_{0,FS}$:

**Proposition 5.** We have the following decomposition: $X_{FS} = X_{LG} + \Phi_h$ with $\dim(X_{LG} \cap \Phi_h) = 1$. Any function of $X_{FS}$ can be written as the sum of a function of $X_{LG}$ and a function of $\Phi_h$. This representation can be made unique by specifying one degree of freedom.

Notice that $\Phi_h \cap X_{LG} = \text{vect}(v_\phi)$, where for all $\ell \in \mathcal{I}_K$, $v_\phi|_{K_\ell} = \phi_{K_\ell}$. Then, counting the degrees of freedom, one can show that $\dim(X_{FS}) = \dim(X_{LG}) + \dim(\Phi_h) + 1$. For problems involving Dirichlet boundary conditions we can prove thus that for $X_{0,FS}$ the representation is unique and $X_{0,FS} = X_{0,\text{LG}} \oplus \Phi_h$. We have $X_{LG} = \text{vect}\left((\phi_{S_i})_{i \in \mathcal{I}_S},(\phi_{F_i})_{f \in \mathcal{I}_F}\right)$ where the basis functions are such that: \quad \forall \ell \in \mathcal{I}_K,

\[
\forall i \in \mathcal{I}_S, \quad \phi_{S_i|K_\ell} = \left\{ \begin{array}{ll}
\lambda_{i,\ell} (2\lambda_{j_\ell} - 1) & \text{if } i \in \mathcal{I}_{S,\ell} \\
0 & \text{if } j \notin \mathcal{I}_{S,\ell}
\end{array} \right.
\]

(7.3)

\[
\forall f \in \mathcal{I}_F, \quad \phi_{F_i|K_\ell} = \left\{ \begin{array}{ll}
4\lambda_{i,\ell} \lambda_{j,\ell} & \text{if } f \in \mathcal{I}_{F,\ell}, \text{ and } F_f = S_i S_j \\
0 & \text{if } f \notin \mathcal{I}_{F,\ell}
\end{array} \right.
\]

For all $\ell \in \mathcal{I}_K$, we will denote by $(\phi_{\ell,j})_{j=1}^6$ the local nodal basis such that:

\[
(\phi_{\ell,j})_{j=1}^3 = (\phi_{S_i|K_\ell})_{i \in \mathcal{I}_{S,\ell}} \quad \text{and} \quad (\phi_{\ell,j})_{j=4}^6 = (\phi_{F_i|K_\ell})_{f \in \mathcal{I}_{F,\ell}}.
\]

The spaces $X_{FS}$ and $X_{0,FS}$ are such that:

\[
X_{FS} = \text{vect}\left((\phi_{S_i})_{i \in \mathcal{I}_S},(\phi_{F_i})_{f \in \mathcal{I}_F},(\phi_{K_\ell})_{\ell \in \mathcal{I}_K}\right),
\]

(7.4)

\[
X_{0,FS} = \text{vect}\left((\phi_{S_i})_{i \in \mathcal{I}_S},(\phi_{F_i})_{f \in \mathcal{I}_F},(\phi_{K_\ell})_{\ell \in \mathcal{I}_K}\right).
\]

We propose here an alternative definition of the Fortin interpolation operator proposed in [7]. Let us first recall the Scott-Zhang interpolation operator [21, 22]. For all $i \in \mathcal{I}_S$, we choose some $\ell_i \in \mathcal{I}_{K_i}$, and we build the $L^2(K_i)$-dual basis $(\phi_{\ell_i,j})_{j=1}^6$ of the local nodal basis such that:

\[
\forall j, j' \in \{1, \cdots, 6\}, \quad \int_{K_{\ell_i}} \phi_{\ell_i,j} \phi_{\ell_i,j'} = \delta_{j,j'}.
\]

Let us define the Fortin-Soulie interpolation operator for scalar functions by:

\[
\pi_{FS} : \quad \mathcal{P}_h H^1 \rightarrow X_{FS} \quad \quad v \mapsto \tilde{\pi} v + \sum_{\ell \in \mathcal{I}_K} v_{K_\ell} \phi_{K_\ell},
\]

with \[
\tilde{\pi} v = \sum_{i \in \mathcal{I}_S} v_{S_i} \phi_{S_i} + \sum_{f \in \mathcal{I}_F} \tilde{v}_f \phi_{F_f},
\]

(7.5)
• The coefficients \((v_{S_i})_{i \in I_S}\) are fixed so that: \(\forall i \in I_S, v_{S_i} = \int_{K_{i,\ell}} v \hat{\phi}_{i,j_i},\)

where \(j_i\) is the index such that \(\int_{K_{i,\ell}} \hat{\phi}_{i,j_i} \phi_{S_i|K_{i,\ell}} = 1\). Using Cauchy-Schwarz inequality and inequality \((4.2)\), we have:

\[
|v_{S_i}| \leq \left( \int_{K_{i,\ell}} \frac{\phi^2_{i,j_i}}{\phi^2_{i,j_i}} \right)^{1/2} \|v\|_{L^2(K_{i,\ell})} \lesssim |K_{i,\ell}|^{-1/2} \|v\|_{L^2(K_{i,\ell})} \lesssim \|\hat{v}\|_{L^2(K_{i,\ell})}.
\]

• The coefficients \((\hat{v}_F)_{f \in I_F}\) are fixed so that: \(\forall f \in I_F, \int_{F_f} \hat{v} = \int_{F_f} \{v\}.\)

We then have:

\[
(7.7) \quad \hat{v}_F = \frac{3}{2} v_{F_f} - \frac{1}{4} \sum_{i \in I_{S,f}} v_{S_i}, \text{ where } v_{F_f} := \frac{1}{|F_f|} \int_{F_f} \{v\}.
\]

For all \(\ell \in I_K\), the coefficient \(v_{K_{\ell}}\) is fixed so that:

\[
\int_{K_{\ell}} \Pi_{FS} v = \int_{K_{\ell}} v.
\]

The definition \((7.3)\) is more general than the one given in \([7]\), which holds for \(v \in H^2(\Omega)\).

We set \(v_{S_i} := (\hat{\pi}v_1(S_i), \hat{\pi}v_2(S_i))^T\) and \(\hat{v}_{F_f} := (\hat{\pi}v_1(F_f), \hat{\pi}v_2(F_f))^T\).

We now define the Fortin-Soulie interpolation operator for vector functions by:

\[
\Pi_{FS} : \begin{cases} 
\mathbf{H}^1(\Omega) & \to \mathbf{X}_{FS} \\
v & \mapsto \sum_{i \in I_S} v_{S_i} \hat{\phi}_{S_i} + \sum_{f \in I_F} \hat{v}_{F_f} \phi_{F_f} + \sum_{\ell \in I_K} v_{K_{\ell}} \phi_{K_{\ell}}.
\end{cases}
\]

For all \(\ell \in I_K\), the vector coefficient \(v_{K_{\ell}} \in \mathbb{R}^2\) is now fixed so that condition \((6.7)\) is satisfied. We can impose for example that the projection \(\Pi_{FS} v\) satisfies:

\[
(7.8) \quad \int_{K_{\ell}} T_{-1}(x) \text{ div } \Pi_{FS} v = \int_{K_{\ell}} T_{-1}(x) \text{ div } v.
\]

Notice that due to \((7.2)\), the patch-test condition is still satisfied. Moreover, one can check that for all \(v \in \mathbf{P}^2(\Omega), \Pi_{FS} v = v\). In particular, if \(v \in \mathbf{P}^1(\Omega)\), we obtain that for all \(\ell \in I_K, v_{K_{\ell}} = 0\). Using definitions \((7.3)\) and \((7.7)\), we obtain for all \(\ell \in I_K:\)

\[
(7.9) \quad (\Pi_{FS} v)|_{K_{\ell}} = \sum_{i \in I_{S,\ell}} v_{S_i} \psi_{S_i} + \frac{3}{2} \sum_{f \in I_{F,\ell}} v_{F_f} \phi_{F_f} + v_{K_{\ell}} \phi_{K_{\ell}},
\]

where \(\psi_{S_i|K_{\ell}} = 3 \lambda_{i,\ell}^2 - 2 \lambda_{i,\ell}\).

Let us estimate \(v_{K_{\ell}}\). By changing the variable, setting \(\hat{v}_{\ell}(\hat{x}) = v \circ T_{\ell}(\hat{x})\), the linear system \((7.8)\) is written as follows, for \(d' \in \{1, 2\}:\)

\[
(\mathbb{B}_{\ell}^{-1} v_{K_{\ell}}) \cdot \int_{K} \hat{x}_{d'} \text{ grad}_{\hat{x}} \hat{\phi}_{K_{\ell}} = \int_{K} \hat{x}_{d'} \text{ div}_{\hat{x}} (\mathbb{B}_{\ell}^{-1} \hat{v}_{\ell}) - \sum_{i \in I_{S,\ell}} (\mathbb{B}_{\ell}^{-1} v_{S_i}) \cdot \int_{K} \hat{x}_{d'} \text{ grad}_{\hat{x}} \hat{\psi}_{S_i} - \frac{3}{2} \sum_{f \in I_{F,\ell}} (\mathbb{B}_{\ell}^{-1} v_{F_f}) \cdot \int_{K} \hat{x}_{d'} \text{ grad}_{\hat{x}} \hat{\phi}_{F_f}.
\]
Noticing that \( \int_K \hat{x} \cdot \mathbf{e}_d = -\frac{1}{4} \mathbf{e}_d \), we have:

\[
\frac{1}{4} \mathbb{B}_{\ell}^{-1} \mathbf{v}_K = \sum_{i \in I_{S_{\ell}}} \int_K \hat{x} (\mathbb{B}_{\ell}^{-1} \mathbf{v}_{S_i}) \cdot \nabla_{K} \hat{\psi}_{S_i}
\]

\[+ \frac{3}{2} \sum_{f \in I_{F_{\ell}}} \int_K \hat{x} (\mathbb{B}_{\ell}^{-1} \mathbf{v}_f) \cdot \nabla_{K} \hat{\phi}_f
\]

\[- \int_K \hat{x} \text{div}_{K} (\mathbb{B}_{\ell}^{-1} \hat{\mathbf{v}}_{\ell}).
\]

(7.10)

Using integration formula (4.1), and Cauchy-Schwarz inequality to bound the last term of (7.10), we have:

\[
|\mathbf{v}_K|^2 \lesssim \sigma^2 \left( \sum_{i \in I_{S_{\ell}}} |\mathbf{v}_{S_i}|^2 + \sum_{f \in I_{F_{\ell}}} |\mathbf{v}_f|^2 + \|\nabla_{K} \hat{\mathbf{v}}_{\ell}\|_{L^2(K)}^2 \right).
\]

(7.11)

**Proposition 6.** The Fortin-Soulie interpolation operator \( \Pi_{FS} \) is such that:

\[
\exists C_{FS} > 0, \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \quad \|\Pi_{FS} \mathbf{v}\|_{h} \leq C_{FS} \|\nabla \mathbf{v}\|_{L^2(\Omega)}.
\]

(7.12)

**Proof.** Let \( \mathbf{v} \in \mathbf{H}^1(\Omega) \). Let us set \( \mathbf{v} \in \mathcal{P}_h \mathbf{H}^1 \) such that \( \forall \ell \in I_K, \mathbf{v}_{\ell} := \mathbf{v} - \int_{K} \mathbf{v} / |K|. \) We have:

\[
\|\Pi_{FS} \mathbf{v}\|_{h}^2 = \sum_{\ell \in I_K} \|\nabla \Pi_{FS} \mathbf{v}\|_{L^2(K_{\ell})}^2 = \sum_{\ell \in I_K} \|\nabla \Pi_{FS} \mathbf{v}\|_{L^2(K_{\ell})}^2.
\]

For \( a, b \in \mathbb{R}^2 \), we set: \( a \otimes b := (a_i b_j)_{i,j=1} \in \mathbb{R}^{2 \times 2} \). According to equation (7.9), we have:

\[
\nabla \Pi_{FS} \mathbf{v} = \sum_{i \in I_{S_{\ell}}} \mathbf{v}_{S_i} \otimes \nabla \psi_{S_i} + \frac{3}{2} \sum_{f \in I_{F_{\ell}}} \mathbf{v}_f \otimes \nabla \phi_f + \mathbf{v}_K \otimes \nabla \phi_K.
\]

We can then make the estimate:

\[
\|\nabla \Pi_{FS} \mathbf{v}\|_{L^2(K_{\ell})}^2 \lesssim \sum_{i \in I_{S_{\ell}}} |\mathbf{v}_{S_i}|^2 \|\nabla \psi_{S_i}\|_{L^2(K_{\ell})}^2
\]

\[
+ \sum_{f \in I_{F_{\ell}}} |\mathbf{v}_f|^2 \|\nabla \phi_f\|_{L^2(K_{\ell})}^2
\]

\[
+ |\mathbf{v}_K|^2 \|\nabla \phi_K\|_{L^2(K_{\ell})}^2.
\]

\[
\lesssim \|\mathbb{B}_{\ell}^{-1}\|_{L^2(\Omega)} \left( \sum_{i \in I_{S_{\ell}}} |\mathbf{v}_{S_i}|^2 + \sum_{f \in I_{F_{\ell}}} |\mathbf{v}_f|^2 + |\mathbf{v}_K|^2 \right).
\]

(7.13)

Thus, using estimate (7.11), and noticing that \( \|\mathbb{B}_{\ell}^{-1}\|^2 |J_{\ell}| \lesssim \sigma^2 \), we have:

\[
\|\nabla \Pi_{FS} \mathbf{v}\|_{L^2(K_{\ell})}^2 \lesssim \sigma^4 \left( \sum_{i \in I_{S_{\ell}}} |\mathbf{v}_{S_i}|^2 + \sum_{f \in I_{F_{\ell}}} |\mathbf{v}_f|^2 + \|\nabla \mathbf{v}_{\ell}\|_{L^2(K)}^2 \right).
\]
Using inequality (7.6) and Poincaré-Steklov inequality (4.4) in cell, component by component, we have:

\[(7.14) \quad |\mathbf{v}_{S,i}|^2 \lesssim \|\text{Grad}_x \mathbf{v}_\ell\|_{L^2(K)}^2.\]

Since the triangulation \(T_h\) is supposed to be shape-regular, there exists a constant \(N_0 \lesssim \sigma^2\) such that for all \(i \in I_{K,i}, N_i \leq N_0\) [16 Rmk. 11.5]. We then obtain that:

\[
\sum_{\ell \in I_K} \sum_{i \in I_{S,i}} \|\text{Grad}_x \mathbf{v}_\ell\|_{L^2(K)}^2 \lesssim N_0 \sum_{\ell \in I_K} \|\text{Grad}_x \mathbf{v}_\ell\|_{L^2(K)}^2.
\]

Thus, summing (7.14), we get:

\[(7.15) \quad \sum_{\ell \in I_K} \sum_{i \in I_{S,i}} |\mathbf{v}_{S,i}|^2 \lesssim N_0 \sum_{\ell \in I_K} \|\text{Grad}_x \mathbf{v}_\ell\|_{L^2(K)}^2.
\]

Using the triangular inequality, equality (4.2), and inequality (4.7), we obtain:

\[
|\mathbf{v}_F|^2 \lesssim |F|^{-1} \sum_{\ell' \in I_{K,f}} \|\mathbf{v}_{F'}|_{F}\|_{L^2(F')}^2 \lesssim \sum_{\ell' \in I_{K,f}} \|\text{Grad}_x \mathbf{v}_{F'}\|_{L^2(F')}^2.
\]

Notice that:

\[
\sum_{\ell \in I_K} \sum_{f \in I_{F,\ell}} \sum_{\ell' \in I_{K,f}} \|\text{Grad}_x \mathbf{v}_{F'}\|_{L^2(F')}^2 \lesssim 6 \sum_{\ell \in I_K} \|\text{Grad}_x \mathbf{v}_\ell\|_{L^2(K)}^2,
\]

so that:

\[(7.16) \quad \sum_{\ell \in I_K} \sum_{f \in I_{F,\ell}} |F|^{-1} \|\mathbf{v}_{F'}|_{F}\|_{L^2(F')}^2 \lesssim \sum_{\ell \in I_K} \|\text{Grad}_x \mathbf{v}_\ell\|_{L^2(K)}^2.
\]

Summing (7.13) over \(\ell \in I_K\), using (7.15) and (7.16), we get that:

\[(7.17) \quad \|\Pi_{FS}\mathbf{v}\|_{K}^2 \lesssim \sigma^2 N_0 \sum_{\ell \in I_K} \|\text{Grad}_x \mathbf{v}_\ell\|_{L^2(K)}^2.
\]

Considering (4.3) for each component, and noticing that \(\|\mathbf{B}\|^2 |J_\ell|^{-1} \lesssim \sigma^2\), we obtain that:

\[(7.18) \quad \|\Pi_{FS}\mathbf{v}\|_{K}^2 \lesssim \sigma^4 N_0 \sum_{\ell \in I_K} \|\text{Grad} \mathbf{v}_\ell\|_{L^2(K)}^2.
\]

We obtain then (7.12) with \(C_{FS} \approx \sigma^2 (N_0)^{1/2}\).

We recall that the discrete Poincaré–Steklov inequality (5.3) holds.

**Theorem 6.** Let \(X_h = X_{FS}\). Then the continuous bilinear form \(a_{S,h}(\cdot, \cdot)\) is \(T\)-coercive and Problem (5.5) is well-posed.

**Proof.** Apply the proof of Theorem 4.

Notice that in the recent paper [23], the inf-sup condition of the mixed Fortin-Soulie finite element is proven directly on a triangle and then using the macroelement technique [24], but it seems difficult to use this technique to build a Fortin operator, which is needed to compute error estimates.

The study can be extended to higher orders for \(d = 2\) using the following papers: [25] for \(k \geq 4, k\) even, [26] for \(k = 3\) and [20] for \(k \geq 5, k\) odd. In [27], the authors propose a local Fortin operator for the lowest order Taylor-Hood finite element [14] for \(d = 3\) which could be used to prove the T-coercivity.
8. Numerical results

Consider Problem (3.1) with data \( f = - \nabla \phi \), where \( \phi \in H^1(\Omega) \cap L^2(\Omega) \). The unique solution is then \( (u, p) := (0, \phi) \). By integrating by parts, the source term in (3.6) reads:

\[
\forall v \in H^1_0(\Omega), \quad \int_\Omega f \cdot v = \int_\Omega \nabla \phi \cdot v.
\]

Recall that the nonconforming space \( X_h \) defined in (3.1) is a subset of \( P_h H^1 \): using a nonconforming finite element method, the integration by parts must be done on each element of the triangulation, and we have:

\[
\forall v \in P_h H^1, \quad \int_\Omega f \cdot v = (\nabla_h v, \phi) + \sum_{f \in I_p} \int_{\partial f} [v \cdot n_f] \phi.
\]

When we apply this result to the right-hand-side of (5.5), we notice that the term with the jumps acts as a numerical source, which numerical influence is proportional to \( 1/\nu \). Thus, we cannot obtain exactly \( u_h = 0 \) (see also (5.9)). Linke proposed in [28] to project the test function \( v_h \in X_h \) on a discrete subspace of \( H(\text{div}; \Omega) \), like Raviart-Thomas or Brezzi-Douglas-Marini finite elements (see [29, 30], or the monograph [15]). Let \( \Pi_{\text{div}} : X_{0,h} \to P_{d-1} \cap H_0(\text{div}; \Omega) \) be some interpolation operator built so that for all \( v_h \in X_{0,h} \), for all \( \ell \in I_K \), \( (\text{div} \Pi_{\text{div}} v_h)_{|K_\ell} = \text{div} v_h_{|K_\ell} \).

Integrating by parts, we have for all \( v_h \in X_{0,h} \):

\[
\int_\Omega f \cdot \Pi_{\text{div}} v_h = \int_\Omega \phi \text{ div} \Pi_{\text{div}} v_h = \sum_{\ell \in I_K} \int_{K_\ell} \phi \text{ div} \Pi_{\text{div}} v_h,
\]

The projection \( \Pi_{\text{div}} \) allows to eliminate the terms of the integrals of the jumps in (8.2).

Let us write Problem (5.5) as:

Find \( (u_h, p_h) \in X_h \) such that

\[
\forall v_h \in X_{0,h}, (u_h, p_h), (v_h, q_h) = \ell_f \left( (\Pi_{\text{div}} v_h, q_h) \right) \quad \forall (v_h, q_h) \in X_h.
\]

In the case of \( X_h = X_{CR} \) and a projection on Brezzi-Douglas-Marini finite elements, the following error estimate holds if \((u, p) \in H^2(\Omega) \times H^1(\Omega)):

\[
||u - u_h||_{L^2(\Omega)} \leq C(h^2 ||u||_{H^2(\Omega)}),
\]

where the constant \( C \) depends on \( h \). The proof is detailed in [31] for shape-regular meshes and [32] for anisotropic meshes. We remark that the error doesn’t depend on the norm of the pressure nor on the \( \nu \) parameter. We will provide some numerical results to illustrate the effectiveness of this formulation, even with a projection on the Raviart-Thomas finite elements, which, for a fixed polynomial order, are less precise than the Brezzi-Douglas-Marini finite elements.

For all \( \ell \in I_K \), we let \( P_h^k(K_\ell) \) be the set of homogeneous polynomials of order \( k \) on \( K_\ell \).

For \( k \in \mathbb{N}^* \), the space of Raviart-Thomas finite elements can be defined as:

\[
X_{RT_k} := \{ v \in H(\text{div}; \Omega); \forall \ell \in I_K, v_{|K_\ell} = a_\ell + b_\ell x \mid (a_\ell, b_\ell) \in P_k(K_\ell)^d \times P_k(K_\ell) \}.
\]
Let $k \leq 1$. The Raviart–Thomas interpolation operator $\Pi_{RT_k} : H^1(\Omega) \cup X_h \to X_{RT_k}$ is defined by: $\forall \mathbf{v} \in H^1(\Omega) \cup X_h$,

\begin{equation}
\begin{cases}
∀ f \in I_F, \quad \int_{F_f} \Pi_{RT_k} \mathbf{v} \cdot \mathbf{n}_f q = \int_{F_f} \mathbf{v} \cdot \mathbf{n}_f q, \quad \forall q \in P^k(F_f) \\
\text{for } k = 1, \forall \ell \in I_K, \quad \int_{K_\ell} \Pi_{RT_k} \mathbf{v} = \int_{K_\ell} \mathbf{v}
\end{cases}
\end{equation}

(8.5)

Note that the Raviart–Thomas interpolation operator preserves the constants. Let $\mathbf{v}_h \in X_h$. In order to compute the left-hand-side of (8.2), we must evaluate $(\Pi_{RT_k} \mathbf{v}_h)_{|K_\ell}$ for all $\ell \in I_K$. Calculations are performed using the proposition below, which corresponds to [33 Lemma 3.11]:

**Proposition 7.** Let $k \leq 1$. Let $\hat{\Pi}_{RT_k} : H^1(\hat{K}) \to P^k(\hat{K})$ be the Raviart–Thomas interpolation operator restricted to the reference element, so that: $\forall \mathbf{v} \in H^1(\hat{K})$,

\begin{equation}
\begin{cases}
∀ \hat{\mathbf{F}} \in \partial \hat{K}, \quad \int_{\hat{F}} \hat{\Pi}_{RT_k} \hat{\mathbf{v}} \cdot \mathbf{n}_{\hat{F}} \hat{q} = \int_{\hat{F}} \hat{\mathbf{v}} \cdot \mathbf{n}_{\hat{F}} \hat{q}, \quad \forall \hat{q} \in P^k(\hat{F}) \\
\text{for } k = 1, \forall \ell \in \hat{I}_K, \quad \int_{\hat{K}} \hat{\Pi}_{RT_k} \hat{\mathbf{v}} = \int_{\hat{K}} \hat{\mathbf{v}}
\end{cases}
\end{equation}

(8.6)

We then have: $\forall \mathbf{v} \in I_K$,

\begin{equation}
(\Pi_{RT_k} \mathbf{v})_{|K_\ell}(x) = \mathbb{B}_\ell \left( \hat{\Pi}_{RT_k} \mathbb{B}_\ell^{-1} \hat{\mathbf{v}}_\ell \right) \circ T_\ell^{-1}(x) \quad \text{where } \hat{\mathbf{v}}_\ell = \mathbf{v} \circ T_\ell(\hat{x}).
\end{equation}

(8.7)

The proof is based on the equality of the $\hat{F}$ and $\hat{K}$-moments of $(\Pi_{RT_k} \mathbf{v})_{|K_\ell} \circ T_\ell(\hat{x})$ and $\mathbb{B}_\ell \left( \hat{\Pi}_{RT_k} \mathbb{B}_\ell^{-1} \hat{\mathbf{v}}_\ell \right) (\hat{x})$. For $k = 0$, setting for $d' \in \{1, \ldots, d\}$: $\psi_{f,d'} := \psi_f e_{d'}$, we obtain that:

\begin{equation}
∀ \ell \in I_K, \forall f \in I_{F,\ell}, \quad (\Pi_{RT_k} \psi_{f,d'})_{|K_\ell} = (d_{|K_\ell})^{-1} \left( x - \hat{O}_S f,\ell \right) S_{f,\ell} \cdot e_d,
\end{equation}

(8.8)

where $S_{f,\ell}$ is the vertex opposite to $F_f$ in $K_\ell$.

For $k = 1$, the vector $(\Pi_{RT_k} \mathbf{v}_h)_{|K_\ell}$ is described by eight unknowns:

$$(\Pi_{RT_k} \mathbf{v}_h)_{|K_\ell} = \mathbf{A}_\ell \mathbf{x} + (\mathbf{b}_\ell \cdot \mathbf{x}) \mathbf{x} + \mathbf{d}_\ell,$$

where $\mathbf{A}_\ell \in \mathbb{R}^{2 \times 2}$, $\mathbf{b}_\ell \in \mathbb{R}^2$, $\mathbf{d}_\ell \in \mathbb{R}^2$.

We compute only once the inverse of the matrix of the linear system (8.6), in $\mathbb{R}^{8 \times 8}$. In the Table [3] (resp. Tables [2] and [3]), we call $\varepsilon_0(\mathbf{u}) = \|u_h\|_{L^2(\Omega)}$ (resp. $\|u - u_h\|_{L^2(\Omega)}$) the velocity error in $L^2(\Omega)$, where $u_h$ is the solution to Problem [5.5] (columns $X_{CR}$ and $X_{FS}$) or [8.3] (columns $X_{CR} + \Pi_{RT_0}$ and $X_{FS} + \Pi_{RT_1}$) and $h$ is the mesh step.

We first consider Stokes Problem [3.1] in $\Omega = (0,1)^2$ with $\mathbf{u} = 0$, $p = (x_1)^3 + (x_2)^3 - 0.5$, $\mathbf{f} = \text{grad } p = 3 \cdot ((x_1)^2, (x_2)^2)^T$. We report in Table 1\textsuperscript{[1]} the values of $\varepsilon_0(\mathbf{u})$ for $h = 5.00e - 2$ and for different values of $\nu$.

| $\nu$   | $X_{CR}$ | $X_{CR} + \Pi_{RT_0}$ | $X_{FS}$ | $X_{FS} + \Pi_{RT_1}$ |
|---------|---------|------------------------|---------|------------------------|
| $1.00e + 0$ | $3.19e - 4$ | $1.34e - 18$ | $3.53e - 7$ | $9.09e - 19$ |
| $1.00e - 3$ | $3.19e - 1$ | $1.34e - 15$ | $3.53e - 4$ | $9.09e - 16$ |
| $1.00e - 4$ | $3.19e + 0$ | $1.34e - 14$ | $3.53e - 3$ | $9.09e - 15$ |

| Table 1. Values of $\varepsilon_0(\mathbf{u})$ for $h = 5.00e - 2$ | | |

When there is no projection, the error is inversely proportional to the $\nu$ parameter,
whereas using the projection, we obtain \( u_h = 0 \) up to machine precision. We now consider Stokes Problem (3.1) in \( \Omega = (0,1)^2 \) with:

\[
\begin{align*}
\mathbf{u} = \begin{pmatrix}
(1 - \cos(2\pi x_1)) \sin(2\pi x_2) \\
(\cos(2\pi x_2) - 1) \sin(2\pi x_1)
\end{pmatrix}, & \quad \begin{cases} p = \sin(2\pi x_1) \sin(2\pi x_2), \\
f = -\nu \Delta \mathbf{u} + \nabla p.
\end{cases}
\end{align*}
\]

We report in Table 2 (resp. 3) the values of \( \varepsilon_0(\mathbf{u}) \) in the case \( \nu = 1.00 e - 3 \) (resp. \( \nu = 1.00 e - 4 \)) for different level of mesh refinement. When there is no projection, \( \varepsilon_0(\mathbf{u}) \) is inversely proportional to \( \nu \), whereas using the projection, \( \varepsilon_0(\mathbf{u}) \) is independent of \( \nu \).

| \( h \) | \( X_{CR} \) | \( X_{CR} + \Pi_{RT_0} \) | \( X_{FS} \) | \( X_{FS} + \Pi_{RT_1} \) |
|---|---|---|---|---|
| \( 5.00 e - 2 \) | \( 5.66 e - 1 \) | \( 1.13 e - 2 \) | \( 2.35 e - 3 \) | \( 2.06 e - 4 \) |
| \( 2.50 e - 2 \) | \( 1.33 e - 1 \) | \( 2.89 e - 3 \) | \( 3.21 e - 4 \) | \( 2.59 e - 5 \) |
| \( 1.25 e - 2 \) | \( 3.88 e - 2 \) | \( 5.40 e - 4 \) | \( 4.20 e - 5 \) | \( 3.40 e - 6 \) |
| \( 6.25 e - 3 \) | \( 8.40 e - 3 \) | \( 1.79 e - 4 \) | \( 5.04 e - 6 \) | \( 4.15 e - 7 \) |
| Rate | \( h^{2.05} \) | \( h^{2.07} \) | \( h^{2.96} \) | \( h^{2.98} \) |

**Table 2. Values of \( \varepsilon_0(\mathbf{u}) \) for \( \nu = 1.00 e - 3 \)**

| \( h \) | \( X_{CR} \) | \( X_{CR} + \Pi_{RT_0} \) | \( X_{FS} \) | \( X_{FS} + \Pi_{RT_1} \) |
|---|---|---|---|---|
| \( 5.00 e - 2 \) | \( 5.66 e - 0 \) | \( 1.13 e - 2 \) | \( 2.35 e - 2 \) | \( 2.06 e - 3 \) |
| \( 2.50 e - 2 \) | \( 1.33 e - 0 \) | \( 2.89 e - 3 \) | \( 3.20 e - 3 \) | \( 2.59 e - 5 \) |
| \( 1.25 e - 2 \) | \( 3.38 e - 1 \) | \( 5.40 e - 4 \) | \( 4.20 e - 4 \) | \( 3.40 e - 6 \) |
| \( 6.25 e - 3 \) | \( 8.40 e - 2 \) | \( 1.79 e - 4 \) | \( 5.04 e - 5 \) | \( 4.15 e - 7 \) |
| Rate | \( h^{2.05} \) | \( h^{2.07} \) | \( h^{2.96} \) | \( h^{2.98} \) |

**Table 3. Values of \( \varepsilon_0(\mathbf{u}) \) for \( \nu = 1.00 e - 4 \)**

Let \( u_{FS} \) (resp. \( u_{FS+RT_1} \)) the solution to Problem (5.5) (resp. (8.3)) computed with Fortin-Soulie finite elements. We represent on Figure 2 the values of the approximations that are exactly divergence free. Notice that using conforming finite elements, the Scott-Vogelius finite elements \([38,39]\) produce velocity approximations that are exactly divergence free.

The code used to get the numerical results can be downloaded on GitHub \([40]\).

9. Conclusion

We analysed the discretization of Stokes problem with nonconforming finite elements in light of the T-coercivity theory, we computed stability coefficients for \( k = 1, d = 2 \) or 3 without regularity assumption; and for \( k = 2, d = 2 \) in the case of a shape-regular simplicial triangulation sequence. For \( k = 2 \), we used an alternative definition of the Fortin-Soulie interpolation operator. We then provided numerical results to illustrate the importance of using \( H(\text{div}) \)-conforming projection. Further, we intend to extend the study to other mixed finite element methods.
Figure 2. Values of \((\mathbf{u}_{FS} - \mathbf{u}_{FS+RT})\). Left: \(x_1\)-component, right: \(x_2\)-component.

Acknowledgements

The author acknowledges Mahran Rihani and Albéric Lefort.

References

[1] P. Ciarlet Jr. T-coercivity: Application to the discretization of Helmholtz-like problems. *Computers & Mathematics with Applications*, 64(1):22–24, 2012.

[2] E. Jamelot and P. Ciarlet, Jr. Fast non-overlapping Schwarz domain decomposition methods for solving the neutron diffusion equation. *Journal of Computational Physics*, 241:445–463, 2013.

[3] P. Ciarlet Jr., E. Jamelot, and F. D. Kpadonou. Domain decomposition methods for the diffusion equation with low-regularity solution. *Computers & Mathematics with Applications*, 74(10):2369–2384, 2017.

[4] L. Giret. *Non-Conforming Domain Decomposition for the Multigroup Neutron SPN Equation*. PhD thesis, Université Paris-Saclay, 2018.

[5] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations*. Springer-Verlag, 1986.

[6] M. Crouzeix and P.-A. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations. *RAIRO, Sér. Anal. Numér.*, 7(3):33–75, 1973.

[7] M. Fortin and M. Soulie. A non-conforming piecewise quadratic finite element on triangles. *International Journal for Numerical Methods in Engineering*, 19(4):505–520, 1983.

[8] A. Ern and J.-L. Guermond. *Finite elements II*, volume 73 of *Texts in Applied Mathematics*. Springer, 2021.

[9] C. Bernardi, M. Costabel, M. Dauge, and V. Girault. Continuity properties of the inf-sup constant for the divergence. *SIAM J. Math. Anal.*, 48(2):1250–1271, 2016.

[10] D. Gallistl. Rayleigh-Ritz approximation of the inf-sup constant for the divergence. *Mathematics of Computation*, 88(315):73–89, 2019.

[11] M. Costabel and M. Dauge. On the inequalities of Babuška-Aziz, Friedrichs and Horgan-Payne. *Arch. Rational Mech. Anal.*, 217:873–898, 2015.

[12] M. Barré and P. Ciarlet Jr. T-coercivité et problèmes mixtes. working paper or preprint, October 2022.

[13] P. Ciarlet Jr. Méthodes variationnelles pour l’analyse de problèmes non coercifs, 2021. M.Sc. AMS lecture notes (ENSTA-IPP).

[14] C. Taylor and T. Hood. Numerical solution of the Navier-Stokes equations using the finite element technique. *Computers & Fluids*, 1:73–100, 1973.

[15] D. Boffi, F. Brezzi, and M. Fortin. *Mixed and hybrid finite element methods and applications*. Springer-Verlag, 2013.
[16] A. Ern and J.-L. Guermond. *Finite elements I*, volume 72 of *Texts in Applied Mathematics*. Springer, 2021.

[17] P. G. Ciarlet. The Effect of Numerical integration for Second-Order Problems. In *Finite Element Methods (Part 1)*, volume II of *Handbook of Numerical Analysis*. Elsevier, 1991.

[18] S. Sauter. The inf-sup constant for Crouzeix-Raviart triangular elements of any polynomial order, 2022.

[19] T. Apel, S. Nicaise, and J. Schöberl. Crouzeix-Raviart type finite elements on anisotropic meshes. *Numerische Mathematik*, 89(2):193–223, 2001.

[20] C. Carstensen and S. Sauter. Critical functions and inf-sup stability of Crouzeix-Raviart elements. *Computers and Mathematics with Applications*, 108:12–23, 2022.

[21] L. R. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54:483–493, 1990.

[22] P. Ciarlet Jr. Analysis of the Scott–Zhang interpolation in the fractional order Sobolev spaces. *J. Numer. Math.*, 21(3):173—180, 2013.

[23] S. Sauter and C. Torres. On the Inf-Sup Stability of Crouzeix-Raviart Stokes Elements in 3D, 2022.

[24] R. Stenberg. Error analysis of some finite element methods for the stokes problem. *Math. Comp.*, 54:495–508, 1990.

[25] A. Baran and G. Stoyan. Gauss-Legendre elements: a stable, higher order non-conforming finite element family. *Computing*, 79(1):1–21, 2007.

[26] C. Carstensen and S. Sauter. Crouzeix-Raviart triangular elements are inf-sup stable, 2021. Preprint.

[27] L. Diening, J. Storn, and T. Tscherpel. Fortin operator for Taylor-Hood element. *Numerische Mathematik*, 150(2):671–689, 2022.

[28] A. Linke. On the Role of the Helmholtz-Decomposition in Mixed Methods for Incompressible Flows and a New Variational Crime. *Comput. Methods Appl. Mech. Engrg.*, 268:782–800, 2014.

[29] P.-A. Raviart and J.-M. Thomas. A mixed finite element method for second order elliptic problems. In *Mathematical aspects of finite element methods*, volume 606 of *Lecture Notes in Mathematics*, pages 292–315. Springer, 1977.

[30] F. Brezzi, J. Douglas, and L. D. Marini. Two families of mixed finite elements for second order elliptic problems. *Numerische Mathematik*, 47(2):217–235, 1985.

[31] C. Brennecke, A. Linke, C. Merdon, and J. Schöberl. Optimal and pressure independent $L^2$ velocity error estimates for a modified Crouzeix-Raviart Stokes element with BDM reconstructions. *Journal of Computational Mathematics*, 33(2):191–208, 2015.

[32] T. Apel, V. Kempf, A. Linke, and C. Merdon. A nonconforming pressure-robust finite element method for the Stokes equations on anisotropic meshes. *IMA Journal of Numerical Analysis*, 42(1):392–416, 2022.

[33] G. N. Gatica. *A Simple Introduction to the Mixed Finite Element Method: Theory and Applications*. SpringerBriefs in Mathematics. Springer, 2014.

[34] E. Dari, R. Durán, and C. Padra. Error estimators for nonconforming finite element approximations of the Stokes problem. *Mathematics of Computation*, 64(211):1017–1033, 1995.

[35] W. Dörfler and M. Ainsworth. Reliable a posteriori error control for nonconforming finite element approximation of Stokes flow. *Mathematics of Computation*, 74(252):1599–1619, 2005.

[36] M. Ainsworth, A. Allendes, G. R. Barrenechea, and R. Rankin. Computable error bounds for nonconforming Fortin–Soulie finite element approximation of the Stokes problem. *IMA Journal of Numerical Analysis*, 32(2):417–447, 2011.

[37] F. Hecht. Construction d’une base de fonctions $P_1$ non conforme à divergence nulle dans $\mathbb{R}^3$. *RAIRO, Sér. Anal. Numér.*, 15(2):119–150, 1981.

[38] L. R. Scott and M. Vogelius. Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials. *RAIRO, Sér. Anal. Numér.*, 19(1):111–143, 1985.

[39] S. Zhang. A new family of stable mixed finite elements for the 3D Stokes equations. *Mathematics of Computation*, 74:543–554, 2005.

[40] E. Jamelot. [https://github.com/cea-trust-platform/Stokes_NCFEM](https://github.com/cea-trust-platform/Stokes_NCFEM)