Fixed Point Actions for Wilson Fermions

U.-J. Wiese

Höchstleistungsrechenzentrum (HLRZ), 5170 Jülich, Germany

December 1, 2021

Abstract

Iterating renormalization group transformations for lattice fermions the Wilson action is driven to fixed points of the renormalization group. A line of fixed points is found and the fixed point actions are computed analytically. They are local and may be used to improve scaling in lattice QCD. The action at the line’s endpoint is chirally invariant and still has no fermion doubling. The Nielsen-Ninomiya theorem is evaded because in this case the fixed point action is nonlocal. The use of this action for a construction of lattice chiral fermions is discussed.
The universal continuum behavior of a lattice field theory is determined by the properties of the corresponding fixed point of the renormalization group \([1]\). Free field theory is one of the few cases where renormalization group transformations can be carried out exactly and the fixed point action can be investigated in detail. For a free lattice scalar field this has been done by Bell and Wilson \([2]\). Here a similar study is presented for free lattice fermions. Recently, Hasenfratz and Niedermayer \([3]\) have used a local fixed point action to improve scaling in asymptotically free lattice scalar field theory. It is natural to apply the same ideas to fermionic systems, e.g. to lattice QCD. It turns out that the fermionic fixed point action is local as long as the renormalization group transformation breaks chiral symmetry. The locality properties of actions obtained from renormalization group transformations of lattice fermions have also been studied by Balaban, O’Carroll and Schor \([4]\).

Lattice fermions suffer from the well-known doubling problem. Wilson has removed the doubler fermions by adding an irrelevant chiral symmetry breaking term to the action \([5]\). This gives the doublers a mass of the order of the cut-off and removes them from the physical spectrum. Then, in the cut-off field theory chiral symmetry is explicitly broken. In fact, the Nielsen-Ninomiya theorem excludes a chirally invariant solution of the doubling problem assuming hermiticity, locality and translation invariance of the fermion action \([6]\). This prevents the lattice formulation of chiral gauge theories (like the Standard Model) because then chiral breaking terms destroy gauge invariance. For free chiral fermions doubling can be avoided in various ways. For example, one may use SLAC fermions \([7]\) which have a nonlocal action. However, when interactions are included the nonlocality causes severe problems already in perturbation theory \([8]\). When a chirally symmetric renormalization group transformation is used the resulting fixed point action is chirally invariant, and it can be used to describe free chiral fermions. This does not (yet) solve the problems of lattice chiral gauge theories, but it suggests an interesting line of thought.

The partition function
\[
Z = \int D\bar{\Psi}D\Psi \exp(-S[\bar{\Psi}, \Psi])
\]
for free Wilson fermions is determined by the action
\[
S[\bar{\Psi}, \Psi] = \frac{1}{2} \sum_{x,\mu}(\bar{\Psi}_x \gamma_\mu \Psi_{x+\hat{\mu}} - \bar{\Psi}_{x+\hat{\mu}} \gamma_\mu \Psi_x) + \sum_x m \bar{\Psi}_x \Psi_x + \frac{r}{2} \sum_{x,\mu}(2 \bar{\Psi}_x \Psi_{x+\hat{\mu}} - \bar{\Psi}_{x+\hat{\mu}} \Psi_x) \Psi_x,
\]
for free Wilson fermions determined by the action
\[
\Delta^{-1}(k) = i \sum_\mu \sin k_\mu \gamma_\mu + m + \frac{r}{2} \sum_\mu \left(2 \sin \frac{k_\mu}{2}\right)^2.
\]
The lattice theory is critical for $m = 0$ where it describes a single massless Dirac fermion.

A renormalization group transformation maps the theory on the original lattice $\Lambda$ to a theory on a blocked lattice $\Lambda'$ with doubled lattice spacing. The points $x' \in \Lambda'$ of the blocked lattice correspond to cubic blocks of $2^d$ points $x \in \Lambda$ of the original lattice, such that each point $x$ belongs to exactly one block $x'$ (this is denoted by $x \in x'$). The geometric situation is illustrated in fig.1. On the blocked lattice one defines new fermion fields $\bar{\Psi}'$ and $\Psi'$. The renormalization group step corresponds to the identity $Z = \int D\bar{\Psi}' D\Psi' \exp(-S'[\bar{\Psi}', \Psi'])$ with

$$\exp(-S'[\bar{\Psi}', \Psi']) = \int D\bar{\Psi}' D\Psi' \exp(-a \sum_{x' \in \Lambda'} (\bar{\Psi}'_{x'} - \frac{b}{2^d} \sum_{x \in x'} \bar{\Psi}_x)(\Psi'_{x'} - \frac{b}{2^d} \sum_{x \in x'} \Psi_x)) \exp(-S[\bar{\Psi}, \Psi]) =$$

$$\int D\bar{\Psi}' D\Psi' D\bar{\eta} D\eta \exp(\sum_{x' \in \Lambda'} ((\bar{\Psi}'_{x'} - \frac{b}{2^d} \sum_{x \in x'} \bar{\Psi}_x)\eta_{x'} + \bar{\eta}_{x'}(\Psi'_{x'} - \frac{b}{2^d} \sum_{x \in x'} \Psi_x) + \frac{1}{a} \eta_{x'} \eta_{x'})) \times \exp(-S[\bar{\Psi}, \Psi]).$$

(3)

In the last step auxiliary fields $\bar{\eta}$, $\eta$ have been introduced. For $a = \infty$ the renormalization group transformation is chirally symmetric, because then the chiral breaking $\eta_{x'} \eta_{x'}$ term disappears. The parameter $b$ renormalizes the fermion field. Later it turns out that $b$ must be set to a special value in order to reach a fixed point of the renormalization group. When iterated starting from a point on the critical surface ($m = 0$), the renormalization group transformation generates a sequence of actions

$$S[\bar{\Psi}, \Psi] \rightarrow S'[\bar{\Psi}', \Psi'] \rightarrow S''[\bar{\Psi}'', \Psi''] \rightarrow \ldots \rightarrow S^*[\bar{\Psi}, \Psi],$$

(4)
that eventually converges to a fixed point action $S^*[\bar{\Psi}, \Psi]$. Of course, also the fixed point is located on the critical surface. To find the fixed point action let us make the ansatz

$$S[\bar{\Psi}, \Psi] = i \sum_{x,y} \rho_\mu(x - y) \bar{\Psi}_x \gamma_\mu \Psi_y + \sum_{x,y} \lambda(x - y) \bar{\Psi}_x \Psi_y.$$  \quad (5)

Charge conjugation invariance requires $\rho_\mu(-z) = -\rho_\mu(z)$ and $\lambda(-z) = \lambda(z)$. Now we go to momentum space (the Brillouin zone $B = [-\pi, \pi]^d$)

$$\rho_\mu(z) = \frac{1}{(2\pi)^d} \int_B d^d k \rho_\mu(k) \exp(ikz), \quad \lambda(z) = \frac{1}{(2\pi)^d} \int_B d^d k \lambda(k) \exp(ikz).$$  \quad (6)

In fig.2 the function $\rho_1(k)$ for the Wilson fermion action is displayed together with the equipotential lines of $\rho(k)^2$ in the $d = 2$ case. After transforming eq.(5) to momentum space we first perform the Gaussian integration over $\bar{\Psi}, \Psi$. Integrating also over $\bar{\eta}, \eta$ one obtains the action $S[\bar{\Psi}, \Psi]$. Iterating this procedure $n$ times leads to the action $S(n)[\bar{\Psi}^n, \Psi^n]$ that is given in terms of $\rho^{(n)}_\mu(k)$ and $\lambda^{(n)}(k)$. Introducing

$$\alpha^{(n)}_\mu(k) = \frac{\rho^{(n)}_\mu(k)}{\rho^{(n)}(k)^2 + \lambda^{(n)}(k)^2}, \quad \beta^{(n)}(k) = \frac{\lambda(k)}{\rho^{(n)}(k)^2 + \lambda^{(n)}(k)^2}$$  \quad (7)

and analogously $\alpha^{(n)}_\mu(k)$, $\beta^{(n)}(k)$ one finds

$$\alpha^{(n)}_\mu(k) = \left(\frac{b^2}{2^d}\right)^n \sum_l \alpha^{(n)}_\mu(k + 2\pi l) \prod_\nu \left(\frac{\sin(k_\nu/2)}{2^n \sin((k_\nu + 2\pi l_\nu)/2^n + 1)}\right)^2,$$

$$\beta^{(n)}(k) = \left(\frac{b^2}{2^d}\right)^n \sum_l \beta^{(n)}_\mu(k + 2\pi l) \prod_\nu \left(\frac{\sin(k_\nu/2)}{2^n \sin((k_\nu + 2\pi l_\nu)/2^n + 1)}\right)^2 + 1 - (b^2/2^d)^n.$$  \quad (8)

The summation extends over vectors $l$ with integer components $l_\mu \in \{1, 2, ..., 2^n\}$. Fig.3 shows the situation after two renormalization group steps for $a = \infty$. The function $\rho^{(2)}_1(k)$ is depicted together with the equipotential lines of $\rho^{(2)}(k)^2$. In the limit $n \to \infty$ we expect to approach a fixed point of the renormalization group. Then only the small $k$ (large distance) behavior of the initial functions $\alpha_\mu(k)$ and $\beta(k)$ is relevant. For the Wilson fermion action one obtains

$$\alpha_\mu(k) \sim \frac{k_\mu}{k^2(1 + mr) + m^2}, \quad \beta(k) \sim \frac{m + k^2 r/2}{k^2(1 + mr) + m^2};$$  \quad (9)

and at the critical point $m = 0$ one has $\alpha_\mu(k) \sim k_\mu/k^2$ and $\beta(k) \sim r/2$. Inserting this in eq.(8) one finds

$$\alpha^{*}_\mu(k) = \lim_{n \to \infty} \left(\frac{b^2}{2^d}\right)^n \sum_l \frac{2^n (k_\mu + 2\pi l_\mu)}{(k + 2\pi l)^2} \prod_\nu \left(\frac{\sin(k_\nu/2)}{k_\nu/2 + \pi l_\nu}\right)^2,$$

$$\beta^{*}(k) = \lim_{n \to \infty} \left(\frac{b^2}{2^d}\right)^n \sum_l \frac{r}{2} \prod_\nu \left(\frac{\sin(k_\nu/2)}{k_\nu/2 + \pi l_\nu}\right)^2 + 1 - (b^2/2^d)^n.$$  \quad (10)
Figure 2: (a) The function $\rho_1(k) = \sin k_1$ for the original Wilson fermion action over the Brillouin zone $B$, and (b) the equipotential lines of $\rho(k)^2$ in $B$. 
Figure 3: (a) The function $\rho^{(2)}_1(k)$ after two renormalization group steps together with (b) the equipotential lines of $\rho^{(2)}(k)^2$. 
The renormalization group transformations converge to a fixed point only if

\[
\left( \frac{b^2}{2^d} \right)^n 2^n = 1 \Rightarrow b = 2^{(d-1)/2}.
\]  

(11)

This is similar to the scalar field case of Bell and Wilson [2], although \( b \) must then be fixed to a different value. Note that \( (d-1)/2 \) is the canonical dimension of a free fermion field in \( d \) dimensions. Hence \( b \) renormalizes the fermion field on the blocked lattice by the appropriate amount. The fixed point action is determined by

\[
\alpha^*_\mu(k) = \sum_{i \in \mathbb{Z}^d} \frac{k_\mu + 2\pi l_\mu}{(k + 2\pi l)^2} \prod_\nu \left( \frac{\sin(k_\nu/2)}{k_\nu/2 + \pi l_\nu} \right)^2, \quad \beta^*(k) = \frac{2}{a}.
\]  

(12)

As in the scalar field case [2] there is a line of fixed points parametrized by \( a \). For finite \( a \) both the renormalization group transformation and the fixed point action break chiral symmetry. As a consequence, the doubler fermions are removed by a Wilson term \( \lambda^*(k) \neq 0 \) and the fixed point action is local (exponentially suppressed at large distances). At the line’s endpoint \( a = \infty \) one obtains \( \rho^*_\mu(k) = \alpha^*_\mu(k) / \rho^*(k)^2 \) and \( \lambda^*(k) = 0 \), i.e. the Wilson term disappears at the fixed point. Strictly speaking, \( \lambda^*(k) \) is singular at the locations of the former doubler fermions. Putting it to zero also at these isolated points still corresponds to a fixed point of the renormalization group. When \( \lambda^*(k) \) vanishes the fixed point action is chirally invariant. Hence, based on the Nielsen-Ninomiya theorem one might suspect that the doubler fermions reappear at the fixed point. Fortunately, this is not the case because the fixed point action is nonlocal and the theorem does not apply. The function \( \rho^*_\mu(k) \) and the equipotential lines of \( \rho^*(k)^2 \) are shown in fig.4 for \( a = \infty \). At the fixed point there is only one zero of \( \rho^*(k)^2 \) and hence only one physical fermion. In fact, due to the nonlocality of the fixed point action the inverse fermion propagator \( \Delta^{-1}(k) = i \sum_\mu \rho^*_\mu(k) \gamma_\mu \) diverges at the locations of the former doubler fermion zeros. Consequently, the topological argument of Nielsen and Ninomiya does not apply because it assumes regularity of the inverse propagator. Fig.4 also shows that for the fixed point action rotation invariance is restored to a larger extent than for the original Wilson action that was depicted in fig.2 (the region of almost circular equipotential lines is much bigger). Unlike chiral symmetry, rotation invariance is not completely restored at the fixed point. Of course, the fixed point action has infinite correlation length, such that the violation of rotation invariance at the scale of the lattice spacing does not affect physical quantities in the continuum limit. The propagator of fixed point fermions is \( \Delta(k) = -i \sum_\mu \alpha^*_\mu(k) \gamma_\mu \). The function \( \alpha^*_1(k) \) shown in fig.5 has a single pole and it is periodic and continuous over the Brillouin zone. In fig.6a the function \( \rho^*_1(z) \) is displayed in coordinate space for \( a = \infty \). It decays rather slowly indicating the nonlocality of the fixed point action. One may optimize the renormalization group transformation by choosing \( a \) such that the fixed point action is as local as possible. For \( d = 2 \) the optimal value is close to \( a = 4 \). The corresponding function \( \rho^*_1(z) \) is depicted in fig.6b. Such an optimized action may be used in the spirit of Hasenfratz and Niedermayer [3] to improve the scaling behavior.
Figure 4: (a) The function $\rho_1^*(k)$ for the fixed point action together with (b) the equipotential lines of $\rho^*(k)^2$. 
of asymptotically free fermionic lattice theories like e.g. QCD or the Gross-Neveu model.

Having obtained the fixed point actions explicitly it is also possible to determine the values of critical exponents. This requires to introduce small perturbations around a fixed point

\[ S[\bar{\Psi}, \Psi] = S^*[\bar{\Psi}, \Psi] + \delta S[\bar{\Psi}, \Psi], \]

and to investigate their behavior under renormalization group transformations. In particular, we are interested in eigenfunctionals \( \delta S[\bar{\Psi}, \Psi] \) that reproduce themselves under renormalization such that

\[ S'[\bar{\Psi}', \Psi'] = S^*[\bar{\Psi}', \Psi'] + \delta S'[\bar{\Psi}', \Psi'] = S^*[\bar{\Psi}', \Psi'] + \gamma \delta S[\bar{\Psi}, \Psi]. \]

Relevant perturbations have eigenvalues \( \gamma > 1 \). They get amplified under renormalization group transformations. In the present case one expects one relevant direction associated with the fermion mass operator \( m \bar{\Psi}_x \Psi_x \). The corresponding critical exponent \( \nu \) describes the divergence of the correlation length \( \xi \) as one approaches the critical point \( m = 0 \), namely \( 1/\xi \propto m^\nu \). For free fermions one has \( \sinh(1/\xi) = m \) such that \( \nu = 1 \). Alternatively, the value of \( \nu \) is related to the relevant eigenvalue \( \gamma \). Because the renormalization group transformation changes the scale by a factor of 2 one has \( \gamma^\nu = 2 \). Using \( \delta \lambda(k) = m \) one obtains \( \beta(k) = 2/a + m/k^2 \) for small \( m \).
Figure 6: (a) The function $i\rho_1^*(z)$ for the fixed point action in coordinate space at $a = \infty$ and (b) $10i\rho_1^*(z)$ at $a = 4$. 
Iterating this according to eq. (8) one finds

\[ \beta(k) = \frac{2}{a} + \sum_{l \in \mathbb{Z}} \frac{m}{(k + 2\pi l)^2} \prod_{\nu} \left( \frac{\sin(k\nu/2)}{k\nu/2 + \pi l\nu} \right)^2 = \beta^*(k) + \delta\beta(k). \]  

(15)

Under a renormalization group step one finds \( \delta\beta'(k) = (b^2/2d)^4 \delta\beta(k) = 2 \delta\beta(k) \). For small \( m \) the contribution to the action is given by \( \delta\lambda(k) = \delta\beta(k)/(\alpha^*(k)^2 + \beta^*(k)^2) \). Hence, under a renormalization group step \( \delta\lambda'(k) = 2 \delta\lambda(k) \) at least for small \( m \).

The corresponding eigenvalue is \( \gamma = 2 \) and indeed \( \nu = 1 \).

The present investigation shows how free Wilson fermions solve their doubling problem at a fixed point of the renormalization group. When a chirally symmetric renormalization group transformation is used the fixed point action is also chirally invariant. Since it still describes a single physical fermion the Nielsen-Ninomiya theorem must be evaded. This is indeed the case because the fixed point action is nonlocal. In practical applications a nonlocal action may cause severe problems because it slows down a numerical simulation. In principle, however, the nonlocality of the chirally invariant fixed point action is acceptable in the framework of local quantum field theory, because it arises naturally due to the integration over the high momentum modes of the field. This is in contrast to SLAC fermions [7] whose nonlocality is put by hand. Still, the nonlocality of the fixed point action prevents the explicit construction of a positive transfer matrix. This is no problem here because the transfer matrix can be constructed for the original Wilson fermion action [9], and the spectrum of low energy states is invariant under the renormalization group.

The fixed point action can be used to put free chiral fermions on the lattice, simply by keeping, for example, the left-handed spinors only. Of course, the question arises if the nonlocality of the action is still acceptable when interactions are included. In particular, the resulting continuum theory should be local and Lorentz-invariant. The existence of a well-behaved continuum limit is related to Reisz’s lattice power counting theorem [10]. The theorem assumes lattice propagators that have a single pole, are periodic over the Brillouin zone, and are differentiable often enough. For example, SLAC fermions are excluded because their propagator is discontinuous at the Brillouin zone boundary. Fixed point fermions, on the other hand, have a propagator \( \Delta(k) = -i \sum_{\mu} \alpha^*_\mu(k) \gamma_\mu \) that is periodic and differentiable infinitely many times (as one sees in fig.5). Still, in its present form the theorem does not apply, because the inverse propagator \( \Delta^{-1}(k) = i \sum_{\mu} \rho^*_\mu(k) \gamma_\mu \) has isolated poles at the boundary of the Brillouin zone. Since the poles correspond to zeros of the propagator they should not cause trouble. It remains to be seen if Reisz’s theorem can be slightly generalized such that it applies to fixed point fermions. Of course, the properties of the free fermion propagator are not sufficient to decide about the interacting theory. Also the vertex functions must be well-behaved. In this respect, an interesting lesson may be learnt from the chiral fermion proposal of ref. [11] and how it failed [12]. To decide if interacting chiral fermions can be constructed with a fixed point action, one should switch on gauge couplings or four-fermion interactions. This is presently under investigation in the context of the Gross-Neveu model.
An alternative formulation of lattice fermions uses staggered fermion fields \[\text{[13]}\]. This has the advantage that the cut-off theory has a remnant chiral symmetry. Blocking transformations consistent with the symmetries of staggered fermions have been discussed in ref. \[\text{[14]}\]. An investigation of the fixed point action for staggered fermions is in progress. A very interesting application of fixed point actions has been suggested by Hasenfratz and Niedermayer to improve scaling in asymptotically free field theories. In QCD, for example, one could use an optimized local fixed point action with finite $a$ which then necessarily breaks chiral symmetry. Of course, this also requires the inclusion of $SU(3)$ gauge fields.

It is a pleasure to thank W. Bietenholz, M. Göckeler, P. Hasenfratz, J. Jersák, F. Niedermayer and T. Reisz for interesting discussions and C. Rebbi for a helpful remark.

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