BUMP CONDITIONS AND TWO-WEIGHT INEQUALITIES FOR
COMMUTATORS OF FRACTIONAL INTEGRALS

YONGMING WEN AND HUOXIONG WU∗

Abstract. This paper gives new two-weight bump conditions for the sparse operators related to iterated commutators of fractional integrals. As applications, the two-weight bounds for iterated commutators of fractional integrals under more general bump conditions are obtained. Meanwhile, the necessity of two-weight bump conditions as well as the converse of Bloom type estimates for iterated commutators of fractional integrals are also given.

1. Introduction and main results

Let \(0 < \alpha < n, m \in \mathbb{Z}^+\) and \(b \in L^1_{\text{loc}}(\mathbb{R}^n)\). The fractional integral operator \(I_{\alpha}f\) and its higher order commutator \(I_{\alpha}^b,m\) are defined by

\[
I_{\alpha}f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}dy, \quad I_{\alpha}^b,m f(x) = \int_{\mathbb{R}^n} \frac{(b(x) - b(y))^m f(y)}{|x-y|^{n-\alpha}}dy.
\]

In this paper, we consider two weight estimates for \(I_{\alpha}^b,m\)

\[
\left( \int_{\mathbb{R}^n} |I_{\alpha}^b,m f(x)|^q \mu(x)dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p \nu(x)dx \right)^{1/p},
\]

where \((\mu, \nu)\) is a pair of weights. Before this, we recall some backgrounds.

Let \(1 < p < n/\alpha\) and \(1/p - 1/q = \alpha/n\). It is well known that \(I_{\alpha}\) is bounded from \(L^p(\mathbb{R}^n)\) to \(L^q(\mathbb{R}^n)\). Given a function \(b \in L^1_{\text{loc}}(\mathbb{R}^n)\), we say that \(b \in BMO(\mathbb{R}^n)\) if

\[
||b||_{BMO(\mathbb{R}^n)} = \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q|dx < \infty,
\]

where \(b_Q = |Q|^{-1} \int_Q b(x)dx\). In 1982, Chanillo [4] proved that if \(1 < p < n/\alpha, 1/p - 1/q = \alpha/n\) and \(b \in BMO(\mathbb{R}^n)\), then \(I_{\alpha}^b,1\) is bounded from \(L^p(\mathbb{R}^n)\) to \(L^q(\mathbb{R}^n)\). By a weight \(\omega\), we mean a nonnegative locally integrable function on \(\mathbb{R}^n\). We say that \(\omega \in A_{p,q}\) if

\[
[\omega]_{A_{p,q}} = \sup_Q \left( \frac{1}{|Q|} \int_Q \omega(x)^q dx \right) \left( \frac{1}{|Q|} \int_Q \omega(x)^{q'} dx \right)^{q'/q} < \infty, \quad 1 < p < q < \infty.
\]

Muckenhoupt and Wheeden [21] proved that \(I_{\alpha}\) is bounded from \(L^p(\omega^p)\) to \(L^q(\omega^q)\), where \(0 < \alpha < n, 1 < p < n/\alpha, 1/p - 1/q = \alpha/n\) and \(\omega \in A_{p,q}\). Under the same conditions as [21]

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*Corresponding author.

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with $b \in BMO(\mathbb{R}^n)$, Segovia and Torrea $[29]$ obtained the weighted $L^p \rightarrow L^q$ boundedness for commutators of fractional integral operators.

Though the forms of two weight inequalities for singular integral operators and related operators are the generalization of one weight inequalities, two weight estimates for operators are more difficult. For instance, it is well known that the $A_p$ condition
\[
\sup_Q \left( \frac{1}{|Q|} \int_Q \omega(x) \frac{dx}{|Q|} \right) \left( \frac{1}{|Q|} \int_Q \omega(x) x^{1-p'} \frac{dx}{x^{1-p'}} \right)^{p-1} < \infty
\]
is the sufficient condition for singular integral operators and related operators to be bounded on $L^p(\omega)$. However, in general, the $A_p$ condition for a pair of weights $(\mu, \nu)$
\[
\sup_Q \left( \frac{1}{|Q|} \int_Q \mu(x) \frac{dx}{|Q|} \right) \left( \frac{1}{|Q|} \int_Q \nu(x) x^{1-p'} \frac{dx}{x^{1-p'}} \right)^{p-1} < \infty
\]
is necessary but never sufficient for operators to be bounded from $L^p(\nu)$ to $L^p(\mu)$, see $[6]$. To solve this problem, Sawyer $[28]$ first introduced the following test condition: there is a positive constant $C$ such that for any cubes $Q$
\[
\int_Q M(\nu^{1-p'} \chi_Q)(x)^p \mu(x) \frac{dx}{|Q|} \leq C \int_Q \nu(x) x^{1-p'} \frac{dx}{x^{1-p'}} < \infty,
\]
where $M$ is the Hardy-Littlewood maximal operator, he proved that this test condition is necessary and sufficient for $M$ to be bounded from $L^p(\nu)$ to $L^p(\mu)$. However, this condition is very difficult to verify due to the operator $M$ involves in it. This drawback appeals researchers to searching for some simpler sufficient conditions, which are close to (1.1) in some sense. Neugebaur $[23]$ first proved that for some $r > 1$, if a pair of weights $(\mu, \nu)$ satisfies the following power bump condition:
\[
\sup_Q \left( \frac{1}{|Q|} \int_Q \mu(x)^r \frac{dx}{|Q|} \right)^{1/r} \left( \frac{1}{|Q|} \int_Q \nu(x)^{(1-p')r} \frac{dx}{x^{(1-p')r}} \right)^{(p-1)/r} < \infty,
\]
then
\[
\int_{\mathbb{R}^n} (Mf(x))^p \mu(x) \frac{dx}{|Q|} \leq C \int_{\mathbb{R}^n} |f(x)|^p \nu(x) \frac{dx}{|Q|}.
\]

To formulate the following works of seeking for appropriate bump conditions which are sufficient for the two weight inequalities of singular integral operators and related operators. We recall some facts about Orlicz spaces. We say $A(t) : [0, \infty) \rightarrow [0, \infty)$ is a Young function if it is increasing, convex, $A(0) = 0$ and $A(t)/t \rightarrow \infty$ as $t \rightarrow \infty$. Given a Young function $A$, the associated complementary function $\tilde{A}$ is defined by
\[
\tilde{A}(t) = \sup_{s > 0} \{ st - A(s) \}.
\]
Let $1 < p < \infty$ and $A$ be a Young function, we say that $A \in B_p$ if
\[
\int_1^{\infty} \frac{A(t)}{t^{p-1}} \frac{dt}{t} < \infty.
\]
Given a Young function $A$, the Orlicz average on a cube $Q$ of a function $f$ is defined by
\[
\|f\|_{A,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q A(\frac{|f(x)|}{\lambda}) \frac{dx}{|Q|} \leq 1 \right\}.
\]
In 1995, Pérez \cite{26} improved Neugebauer’s result by eliminating the power bump on the left-hand weight \( \mu \) and replacing the right-hand weight \( \nu \) by the “Orlicz bump”. Precisely, he proved that if a pair of weights \((\mu, \nu)\) satisfies
\[
\sup_Q \| \mu^{1/p} \|_{p,Q} \| \nu^{-1/p} \|_{\Phi,Q} < \infty, \quad 1 < p < \infty,
\]
and \( \Phi \in B_p \), then \( M : L^p(\nu) \to L^p(\mu) \). While for Calderón-Zygmund operator \( T \), Cruz-Uribe and Pérez \cite{12} conjectured that if both terms in \( \|T\|_{p,p} \) were bumped, then \( T : L^p(\nu) \to L^p(\mu) \). This conjecture was partially solved in \cite{22} and completely solved by Lerner in \cite{17}. Lerner proved that if a pair of weights \((\mu, \nu)\) satisfies
\[
(1.2) \quad \sup_Q \| \mu^{1/p} \|_{p,Q} \| \nu^{-1/p} \|_{\Phi,Q} < \infty, \quad 1 < p < \infty,
\]
and \( \Phi \in B_p, \tilde{\Psi} \in B_{q'} \), then \( T : L^p(\nu) \to L^p(\mu) \). The separated bump conjecture, which arises from the work of Cruz-Uribe et al. \cite{11}, who asserted that \( T : L^p(\nu) \to L^p(\mu) \) provided that \((1.2)\) is replaced by
\[
\sup_Q \| \mu^{1/p} \|_{p,Q} \| \nu^{-1/p} \|_{\Phi,Q} < \infty \quad \text{and} \quad \| \mu^{1/p} \|_{\Psi,Q} \| \nu^{-1/p} \|_{\Phi' Q} < \infty.
\]
In \cite{13}, Cruz-Uribe et al. only proved this conjecture is true for \( \Phi(t) = t^{p'}(\log(e+t))^{p'-1+\delta} \) and \( \Psi(t) = t^{q'}(\log(e+t))^{q'-1+\delta} \) for some \( \delta > 0 \). This conjecture is still open, and we refer readers to see \cite{16} \cite{19} \cite{20} for more recent works about it. Analogue to the case of singular integral operators, Pérez \cite{25} gave the following sufficient condition:
\[
\sup_Q |Q|^{\alpha+n+1/q-1/p} \| \mu^{1/q} \|_{A,Q} \| \nu^{-1/p} \|_{B,Q} < \infty, \quad \tilde{A} \in B_p, \tilde{B} \in B_{q'},
\]
such that \( I_\alpha : L^p(\nu) \to L^q(\mu) \). The conditions \( \tilde{A} \in B_p, \tilde{B} \in B_{q'} \) were improved to \( \tilde{A} \in B_{p,q}, \tilde{B} \in B_{q',q'} \) in \cite{8}. Here, we say that \( A \in B_{p,q} \) if
\[
\int_1^\infty \frac{A(t)^{q/p} dt}{t^q} < \infty.
\]
Recently, Rahm \cite{27} used “entropy bumps” and “direct comparison bumps” to get the two weight boundedness for fractional sparse operators.

On the other hand, Cruz-Uribe and Moen \cite{7} showed that if \( b \in BMO(\mathbb{R}^n) \) and a pair of weights \((\mu, \nu)\) satisfies
\[
\sup_Q \| \mu^{1/p} \|_{L_p(\log L)^{2p-1+\delta},Q} \| \nu^{-1/p} \|_{L_{p'}(\log L)^{2p'-1+\delta},Q} < \infty,
\]
then the commutator of Calderón-Zygmund operator \( T_b \) is bounded from \( L^p(\nu) \) to \( L^p(\mu) \). This result was recently improved by Lerner et al. \cite{19}, who provided a wider class of weights \((\mu, \nu)\):
\[
\sup_Q \| \mu^{1/p} \|_{L_p(\log L)^{(m+1)p-1+\delta},Q} \| \nu^{-1/p} \|_{B,Q} + \sup_Q \| \mu^{1/p} \|_{A,Q} \| \nu^{-1/p} \|_{L_p(\log L)^{(m+1)p-1+\delta},Q} < \infty,
\]
for which \( \| T_b^m \|_{L^p(\nu) \to L^p(\mu)} < \infty \), where \( b \in BMO(\mathbb{R}^n) \) and \( \tilde{A} \in B_{p'}, \tilde{B} \in B_p \). Very recently, Cruz-Uribe et al. \cite{9} generalized the work in \cite{19} by assuming the Young functions \( A, C \in B_{p'}, B, \tilde{D} \in B_p \) and \((\mu, \nu)\) satisfies
\[
\sup_Q \| \mu^{1/p} \|_{A,Q} \| (b-b_Q)^m \nu^{-1/p} \|_{B,Q} + \sup_Q \| (b-b_Q)^m \mu^{1/p} \|_{C,Q} \| \nu^{-1/p} \|_{B,Q} < \infty.
\]
We also refer readers to see [15] for the result in the matrix setting. For the commutators of fractional integral operators, Cruz-Uribe [5] showed that if a pair of weights \((\mu, \nu)\) satisfies
\[(1.3) \quad \sup_Q |Q|^{\alpha/n+1/q-1/p} \|\mu^{1/q}\|_{A,Q} \|\nu^{-1/p}\|_{B,Q} < \infty,\]
with \(A(t) = t^q (\log(e + t))^{2q - 1 + \delta}, B(t) = t^{\delta'} (\log(e + t))^{2\delta' - 1 + \delta}\), then \(I_{\alpha}^{b,m}\) is bounded from \(L^p(\nu)\) to \(L^p(\mu)\). Recently, Cardenas and Isralowitz [3] established two weight inequality for \(I_{\alpha}^{b,m}\) in the matrix setting.

Inspired by the works in [3, 9, 19], in this paper, we mainly consider two weight inequalities for \(I_{\alpha}^{b,m}\). Our first main result can be formulated as follows.

**Theorem 1.1.** Let \(1 < p \leq q < \infty, 0 < \alpha < n, m \in \mathbb{Z}^+, b \in L^m_{loc}(\mathbb{R}^n)\) and \(S\) be a sparse family.

1. Suppose that \(A, B, C, D\) are Young functions which satisfy \(\tilde{A}, \tilde{C} \in B_q^p\) and \(\tilde{B}, \tilde{D} \in B_{p,q}\). If a pair of weights \((\mu, \nu)\) satisfies
\[
\sup_{Q \in S} |Q|^{\alpha/n+1/q-1/p} \|\mu^{1/q}\|_{A,Q} \|\nu^{-1/p}\|_{B,Q} + \sup_{Q \in S} |Q|^{\alpha/n+1/q-1/p} \|\nu^{-1/p}\|_{D,Q} < \infty,
\]
then
\[(1.4) \quad \|T_{\alpha}^{S,b,m} f\|_{L^q(\mu)} + \|(T_{\alpha}^{S,b,m})^* f\|_{L^q(\mu)} \lesssim \|f\|_{L^p(\nu)}.
\]

2. Conversely, if \((1.4)\) holds, then
\[
\sup_{Q \in S} |Q|^{\alpha/n+1/q-1/p} \|\mu^{1/q}\|_{q,Q} \|\nu^{-1/p}\|_{p',Q} + \sup_{Q \in S} |Q|^{\alpha/n+1/q-1/p} \|\nu^{-1/p}\|_{p',Q} < \infty.
\]

Here
\[
T_{S,\alpha}^{b,m} f(x) = \sup_{Q \in S} |Q|^{\alpha/n} \left(|Q|^{-1} \int_Q |b(x) - b_Q|^m f(x) |dx|\right) \chi_Q(x),
\]
and
\[
(T_{S,\alpha}^{b,m})^* f(x) = \sup_{Q \in S} |Q|^{\alpha/n} |b(x) - b_Q|^m \left(|Q|^{-1} \int_Q |f(x) |dx|\right) \chi_Q(x).
\]

As an application, we can obtain the following two weight bump conditions for iterated commutators \(I_{\alpha}^{b,m}\).

**Theorem 1.2.** Let \(1 < p \leq q < \infty, 0 < \alpha < n, m \in \mathbb{Z}^+, b \in L^m_{loc}(\mathbb{R}^n)\) and \(I_{\alpha}^{b,m}\) be commutators of fractional integral operators. Suppose that \(A, B, C, D\) are Young functions which satisfy \(\tilde{A}, \tilde{C} \in B_q^p\) and \(\tilde{B}, \tilde{D} \in B_{p,q}\). If a pair of weights \((\mu, \nu)\) satisfies
\[
\sup_{Q \in S} |Q|^{\alpha/n+1/q-1/p} \|\mu^{1/q}\|_{A,Q} \|\nu^{-1/p}\|_{B,Q} + \sup_{Q \in S} |Q|^{\alpha/n+1/q-1/p} \|\nu^{-1/p}\|_{D,Q} < \infty,
\]
then
\[
\|I_{\alpha}^{b,m} f\|_{L^q(\mu)} \lesssim \|f\|_{L^p(\nu)}.
\]

Furthermore, as a consequence of Theorem 1.2 we can obtain the more traditional bump conditions by assuming that the multiplier \(b\) lies in an oscillation class related to \(BMO(\mathbb{R}^n)\).
Theorem 1.3. Let $1 < p \leq q < \infty$, $0 < \alpha < n$, $m \in \mathbb{Z}^+$ and $I_{b,m}^{b,m}$ be commutators of fractional integral operators. Assume that $A, B, C, D, X, Y$ are Young functions which satisfy $\hat{A}, \hat{C} \in B_{q'}, \hat{B}, \hat{D} \in B_{p,q}$ and $X, Y$ satisfy

$$X^{-1}(t) \lesssim \frac{B^{-1}(t)}{\Phi^{-1}(t)m} \text{ and } Y^{-1}(t) \lesssim \frac{C^{-1}(t)}{\Phi^{-1}(t)m}$$

for large $t$. If $b \in \text{Osc}(\Phi)$ and a pair of weights $(\mu, \nu)$ satisfies

$$\sup_Q |Q|^{\alpha/n+1/q-1/p} \mu^{1/q} \|A_Q\|_{\mu^{-1/p}, Q} + \sup_Q |Q|^{\alpha/n+1/q-1/p} \mu^{1/q} \|A_Q\|_{\mu^{-1/p}, \nu^{-1/p}, D, Q} < \infty,$$

then $\|I_{b,m}^{b,m} f\|_{L^q(\mu)} \lesssim \|b\|_{\text{Osc}(\Phi)} \|f\|_{L^p(\nu)}$, where $\text{Osc}(\Phi)$ is the space of functions $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ with

$$\|b\|_{\text{Osc}(\Phi)} = \sup_Q \|b - b_Q\|_{\Phi, Q} < \infty.$$

When $b \in \text{BMO}(\mathbb{R}^n)$, we may take $\Phi(t) = \exp t - 1$ in Theorem 1.3. Then we have the following result.

Corollary 1.4. Let $1 < p \leq q < \infty$, $0 < \alpha < n$, $m \in \mathbb{Z}^+$ and $I_{b,m}^{b,m}$ be commutators of fractional integral operators. Suppose that $A, D$ are Young functions which satisfy $\hat{A} \in B_{q'}, \hat{D} \in B_{p,q}$. If $b \in \text{BMO}(\mathbb{R}^n)$ and a pair of weights $(\mu, \nu)$ satisfy

$$\sup_Q |Q|^{\alpha/n+1/q-1/p} \mu^{1/q} \|A_Q\|_{\mu^{-1/p}, \nu^{-1/p}, \nu^{-1/p}, D, Q} < \infty,$$

then $\|I_{b,m}^{b,m} f\|_{L^q(\mu)} \lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)} \|f\|_{L^p(\nu)}$.

In particular, if we take $A(t) = t^\alpha \log(e + t)^{q-1+\delta}$ and $D(t) = t^{\alpha'} \log(e + t)^{\nu-1+\delta}$, then we can get the following more general two-weight bump conditions than 1.3 for $I_{b,m}^{b,m}$.

Corollary 1.5. Let $1 < p \leq q < \infty$, $0 < \alpha < n$, $m \in \mathbb{Z}^+$ and $I_{b,m}^{b,m}$ be commutators of fractional integral operators. If $b \in \text{BMO}(\mathbb{R}^n)$ and a pair of weights $(\mu, \nu)$ satisfies

\begin{equation}
\sup_Q |Q|^{\alpha/n+1/q-1/p} \mu^{1/q} \|A_Q\|_{L^q(\log L)^{q-1+\delta}, Q} \|\nu^{-1/p}\|_{L^{q'}(\log L)^{q'-1+\delta}, Q} \|\nu^{-1/p}\|_{L^{q'}(\log L)^{q'-1+\delta}, Q} < \infty
\end{equation}

for some $\delta > 0$, then $\|I_{b,m}^{b,m} f\|_{L^q(\mu)} \lesssim \|b\|_{\text{BMO}(\mathbb{R}^n)} \|f\|_{L^p(\nu)}$.

Remark 1.6. It is clear that the bump condition in (1.5) for $m = 1$ is more general than one in (1.3). Therefore, Corollary 1.5 is an essential improvement and extension to the corresponding result in [5].

Next, we turn to the necessity of bump conditions for the two-weight boundedness of $I_{a,m}^{b,m}$, which is addressed by the following theorem.
**Theorem 1.7.** Let $1 < p \leq q < \infty$, $0 < \alpha < n$, $m \in \mathbb{Z}^+$ and $I_{\alpha}^{b,m}$ be commutators of fractional integral operators. Suppose that $\mu$ is a doubling weight and for any $b \in BMO(\mathbb{R}^n)$, \[ \|I_{\alpha}^{b,m}f\|_{L^{q,\infty}(\mu)} \lesssim \|b\|_{BMO(\mathbb{R}^n)}\|f\|_{L^{p}(\nu)}. \] Then \[ \sup_Q |Q|^\frac{\alpha}{n} \mu^\frac{1}{p} \|\mu^{1/q}\|_{L^q(Q)} \|\nu^{-1/p}\|_{L^{p'}((\log L)^{m'},Q)} < \infty. \]

Finally, we consider the inverse result related to Bloom type estimate for $I_{\alpha}^{b,m}$. We first recall the relevant definition and backgrounds. Let $\eta$ be a weight, we say that $b \in BMO_\eta$ if \[ \|b\|_{BMO_\eta} := \sup_Q \frac{1}{\eta(Q)} \int_Q |b(x) - b_Q|dx < \infty. \] Bloom [2] first characterized $BMO_\eta$ via the two weight estimate of commutator of Hilbert transform $H$. For the commutator of fractional integral operator, Accomazzo et al. [1] proved that if $\lambda, \mu \in A_{p,q}$ and $\eta = (\mu \lambda^{-1})^{1/m}$, then \[ b \in BMO_\eta \Rightarrow \|I_{\alpha}^{b,m}\|_{L^q(\lambda \nu)} \lesssim \|b\|_{BMO_\eta}\|f\|_{L^p(\mu \nu)} \] and \[ \|I_{\alpha}^{b,m}\|_{L^q(\lambda \nu)} \lesssim \|f\|_{L^p(\mu \nu)} \Rightarrow b \in BMO_\eta. \] The corresponding result for $m = 1$ was obtained by Holmes et al. in [14]. Our next theorem can be regarded as the converse of the above Bloom type estimate for $I_{\alpha}^{b,m}$.

**Theorem 1.8.** Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/p - 1/q = \alpha/n$, $m \in \mathbb{Z}^+$, $\lambda, \mu \in A_{p,q}$ and $I_{\alpha}^{b,m}$ be commutator of fractional integral operators. If $\eta$ is an arbitrary weight which satisfies
\begin{equation}
(1.6) \quad b \in BMO_\eta \Rightarrow \|I_{\alpha}^{b,m}\|_{L^q(\lambda \nu)} \lesssim \|b\|_{BMO_\eta}\|f\|_{L^p(\mu \nu)} \end{equation}
and
\begin{equation}
(1.7) \quad \|I_{\alpha}^{b,m}\|_{L^q(\lambda \nu)} \lesssim \|f\|_{L^p(\mu \nu)} \Rightarrow b \in BMO_\eta, \end{equation}
then $\eta \sim (\mu \lambda^{-1})^{1/m}$ almost where.

We organize the rest of the paper as follows. Section 2 is devoted to the proofs of Theorems 1.1-1.3, Corollaries 1.4 and 1.5. In Section 3, we will show Theorem 1.7 and Theorem 1.8 will be given in Section 4.

We end this section by making some conventions. We denote $f \lesssim g$, $f \sim g$ if $f \leq Cg$ and $f \lesssim g \lesssim f$, respectively. For any ball $B := B(x_0, r) \subset \mathbb{R}^n$, $x_0$ and $r$ are the center and the radius of $B$, respectively, and $\chi_B$ represents the characteristic function of $B$. For any cube $Q \subset \mathbb{R}^n$, the diameter of $Q$ is denoted by $\text{diam } Q$. $C^\infty_c(\mathbb{R}^n)$ is the space of all smooth functions with compact support.

2. Two-weight boundedness for $I_{\alpha}^{b,m}$

In this section, we will prove Theorems 1.1-1.3 and Corollary 1.4-1.5. To begin with recalling some notation, definitions and facts related to sparse families (see [18] [24] for more details). Given a cube $Q \subset \mathbb{R}^n$, let $D(Q)$ be the set of cubes obtained by repeatedly subdividing $Q$ and its descendants into $2^n$ congruent subcubes.
Definition 2.1. A collection of cubes $D$ is called a dyadic lattice if it satisfies the following properties:

1. If $Q \in D$, then every child of $Q$ is also in $D$;
2. For every two cubes $Q_1, Q_2 \in D$, there is a common ancestor $Q \in D$ such that $Q_1, Q_2 \in D(Q)$;
3. For any compact set $K \subset \mathbb{R}^n$, there is a cube $Q \in D$ such that $K \subset Q$.

Definition 2.2. A subset $S \subset D$ is called an $\eta$-sparse family with $\eta \in (0, 1)$ if for every cube $Q \in S$, there is a measurable subset $E_Q \subset Q$ such that $\eta |Q| \leq |E_Q|$, and the sets $\{E_Q\}_{Q \in S}$ are mutually disjoint.

In [1], Accomazzo et al. proved the following sparse domination for commutators of fractional integral operators.

Lemma 2.3. (cf. [1]) Let $0 < \alpha < n$ and $m \in \mathbb{Z}^+$. For every $f \in C^\infty_c(\mathbb{R}^n)$ and $b \in L^m_{loc}(\mathbb{R}^n)$, there exist a family $\{D_j\}_{j=1}^{3^n}$ of dyadic lattices and a family $\{S_j\}_{j=1}^{3^n}$ of sparse families such that $S_j \subset D_j$, for each $j$, and

$$|I^{b,m}_\alpha f(x)| \lesssim \sum_{j=1}^{3^n} \sum_{Q \in S_j} \sum_{k=0}^m |b(x) - b_Q|^{m-k}|Q|^{\alpha/n} \left( \frac{1}{|Q|} \int_Q |b(x) - b_Q|^k |f(x)| \, dx \right) \chi_Q(x).$$

Based on Lemma 2.3, we can prove the following lemma.

Lemma 2.4. Let $0 < \alpha < n$, $m \in \mathbb{Z}^+$, $b \in L^m_{loc}(\mathbb{R}^n)$ and $I^{b,m}_\alpha$ be commutators of fractional integral operators. Then for $f \in C^\infty_c(\mathbb{R}^n)$, there exist $3^n$ sparse families $S_j \subset D_j$, $j = 1, \ldots, 3^n$, such that

$$|I^{b,m}_\alpha f(x)| \lesssim \sum_{j=1}^{3^n} (T^{b,m}_{S_j,\alpha} f(x) + (T^{b,m}_{S_j,\alpha})^* f(x)),$$

where $T^{b,m}_{S,\alpha}$ and $(T^{b,m}_{S,\alpha})^*$ are defined in Theorem 1.1.

Proof. Fix a sparse family $S$, let $Q \in S$ and $x \in Q$, then

$$|Q|^{\alpha/n} \sum_{k=0}^m |b(x) - b_Q|^{m-k} \frac{1}{|Q|} \int_Q |b(y) - b_Q|^k |f(y)| \, dy$$

$$\leq \frac{1}{|Q|} \int_Q \left( \sum_{k=0}^m \max\{|b(x) - b_Q|, |b(y) - b_Q|\}^m \right) |f(y)| \, dy |Q|^{\alpha/n}$$

$$= (m + 1) \frac{1}{|Q|} \int_Q \max\{|b(x) - b_Q|^m, |b(y) - b_Q|^m\} |f(y)| \, dy |Q|^{\alpha/n}$$

$$\lesssim |b(x) - b_Q|^m \frac{1}{|Q|} \int_Q |f(y)| \, dy |Q|^{\alpha/n} + \frac{1}{|Q|} \int_Q |b(y) - b_Q|^m |f(y)| \, dy |Q|^{\alpha/n}.$$ 

This, together with Lemma 2.3, leads to the desired conclusion and completes the proof of Lemma 2.4.

The proof of Theorem 1.1 is converted into the following two propositions.

Proposition 2.5. Let $0 < \alpha < n$, $m \in \mathbb{Z}^+$, $b \in L^m_{loc}(\mathbb{R}^n)$ and $S$ be a sparse family. Assume that $1 < p \leq q < \infty$ and $A, B$ are Young functions that satisfy $A \in B_q$, $B \in B_{p,q}$. If $(\mu, \nu)$
is a pair of weights that satisfies
\[
\sup_{Q \in S} |Q|^{\alpha/n + 1/q - 1/p} \|\mu^{1/q}\|_{A,Q} \| (b - b_Q)^m \nu^{1/p}\|_{B,Q} < \infty,
\]
then
\[
(2.1) \quad \|T_{S,\alpha}^{b,m} f\|_{L^q(\mu)} \leq C \|f\|_{L^p(\nu)}.
\]
Conversely, if \(T_{S,\alpha}^{b,m}\) satisfies (2.1), then the pair of weights \((\mu, \nu)\) satisfies
\[
\sup_{Q \in S} |Q|^{\alpha/n + 1/q - 1/p} \|\mu^{1/q}\|_{\nu,Q} \| (b - b_Q)^m \nu^{1/p}\|_{\nu',Q} < \infty.
\]

Proof. By duality, there exists nonnegative measurable function \(g \in L^{q'}(\mu)\) with \(\|g\|_{L^{q'}(\mu)} = 1\) such that
\[
(2.2) \quad \|T_{S,\alpha}^{b,m}\|_{L^q(\mu)} = \int_{\mathbb{R}^n} T_{S,\alpha}^{b,m} f(x) g(x) \mu(x) dx
\]

\[
\leq \sum_{Q \in S} |Q|^{\alpha/n + 1} \left( \frac{1}{|Q|} \int_Q |b(x) - b_Q|^m |f(x)| dx \right) \left( \frac{1}{|Q|} \int_Q |g(x)| \mu(x) dx \right).
\]

Let \(1/p - 1/q = \beta/n\), it was proved in [8] that
\[
M_{\beta,B} : L^p(\mathbb{R}^n) \to L^{q'}(\mathbb{R}^n).
\]

From this, by (2.2), the generalized Holder inequality and our assumptions yield that
\[
\|T_{S,\alpha}^{b,m}\|_{L^q(\mu)} \leq \sum_{Q \in S} \| (b - b_Q)^m \nu^{1/p}\|_{B,Q} \| f^{1/p}\|_{B,Q} \|\mu^{1/q}\|_{A,Q} \| g\mu^{1/q'}\|_{\hat{A},Q} |Q|^{\frac{\alpha}{n} + \frac{1}{q} - \frac{1}{p} + \frac{1}{q'} - \frac{\beta}{n}}
\]

\[
\leq |E_Q| |Q|^{\beta/n} \| f^{1/p}\|_{B,Q} \| g\mu^{1/q'}\|_{\hat{A},Q}
\]

\[
\leq \int_{\mathbb{R}^n} M_{A}(g\mu^{1/q'}) (x) M_{\beta,B}(f^{1/p}) (x) dx
\]

\[
\leq \|M_{\beta,B}(f^{1/p})\|_{L^{q'}} \|M_{A}(g\mu^{1/q'})\|_{L^{q'}} \lesssim \|f\|_{L^p(\nu)}.
\]

Next, we turn to prove necessity. Fix \(Q \in S\), let \(f = |b - b_Q|^{(p'-1)/p} \nu^{1/p} \chi_Q\). For \(x \in Q\), it is easy to see that
\[
T_{S,\alpha}^{b,m} f(x) \geq |Q|^{\alpha/n - 1} \int_Q |b(x) - b_Q|^{mp'} \nu(x)^{-p'/p} dx,
\]
which implies that
\[
\left( \int_Q T_{S,\alpha}^{b,m} f(x)^q \mu(x) dx \right)^{1/q} \geq |Q|^{\alpha/n - 1} \int_Q |b(x) - b_Q|^{mp'} \nu(x)^{-p'/p} dx \left( \int_Q \mu(x) dx \right)^{1/q}.
\]

On the other hand,
\[
\left( \int_Q T_{S,\alpha}^{b,m} f(x)^q \mu(x) dx \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^{p} \nu(x) dx \right)^{1/p}
\]

\[
= C \left( \int_Q |b(x) - b_Q|^{mp'} \nu(x)^{-p'/p} dx \right)^{1/p}.
\]

Hence, we conclude that
\[
|Q|^{\alpha/n - 1} \int_Q |b(x) - b_Q|^{mp'} \nu(x)^{-p'/p} dx \left( \int_Q \mu(x) dx \right)^{1/q}
\]
Similarly, we have

\[ |b(x) - b_Q|^{mp'} \nu(x)^{-p'/p} dx \]

The desired result follows by rearranging the above terms.

Similarly, we can obtain the following proposition, and we leave the details for the interested readers.

**Proposition 2.6.** Let \( 0 < \alpha < n, m \in \mathbb{Z}^+ \), \( b \in L_{loc}^m(\mathbb{R}^n) \) and \( S \) be a sparse family. Assume that \( 1 < p \leq q < \infty \) and \( C, D \) are Young functions that satisfy \( C \in B_q', D \in B_{p,q} \). If \( (\mu, \nu) \) is a pair of weights that satisfies

\[ \sup_{Q \in S} |Q|^{\alpha/n + 1/q - 1/p} \|(b - b_Q)^m \mu^1/q\nu^{-1/p}\|_{C, Q} < \infty, \]

then

\[ \| (T_{S,a}^{b,m}) f \|_{L^q(\mu)} \leq C \| f \|_{L^p(\nu)}. \]

Conversely, if \((T_{S,a}^{b,m})^*\) satisfies (2.3), then the pair of weights \((\mu, \nu)\) satisfies

\[ \sup_{Q \in S} |Q|^{\alpha/n + 1/q - 1/p} \|(b - b_Q)^m \mu^1/q\nu^{-1/p}\|_{C, Q} < \infty. \]

**Proofs of Theorems 1.1 and 1.2.** Theorem 1.1 follows from Propositions 2.5 and 2.6, and Theorem 1.2 follows from Lemma 2.4 and Theorem 1.1.

Next, we prove Theorem 1.3. We first recall the following lemma.

**Lemma 2.7.** (cf. [6]) Let \( A, B \) be continuous and strictly increasing functions on \([0, \infty)\) and \( C \) be Young function that satisfies \( A^{-1}(t)B^{-1}(t) \lesssim C^{-1}(t) \) for \( t \) large. Then

\[ \|fg\|_{C, Q} \lesssim \|f\|_{A, Q}\|g\|_{B, Q}. \]

**Proof of Theorem 1.3.** Denote \( \Phi_m(t) = \Phi(t^{1/m}) \). Since \( B, X, \Phi \) satisfy

\[ \Phi^{-1}(t)mX^{-1}(t) \lesssim B^{-1}(t) \]

for \( t \) large, by Lemma 2.7, we have that

\[ \|(b - b_Q)^m \nu^{-1/p}\|_{B, Q} \lesssim \|(b - b_Q)^m \Phi_{m, Q}\|_{\nu^{-1/p}, X, Q} \]

\[ = \|(b - b_Q)^m \Phi_{m, Q}\|_{\nu^{-1/p}, X, Q}. \]

Therefore,

\[ \sup_{Q} |Q|^{\alpha/n + 1/q - 1/p} \|\mu^{1/q}\|_{A, Q}\|(b - b_Q)^m \nu^{-1/p}\|_{B, Q} \]

\[ \lesssim \|b\|_{\partial sc(\Phi)} \sup_{Q} |Q|^{\alpha/n + 1/q - 1/p} \|\mu^{1/q}\|_{A, Q}\nu^{-1/p}\|_{X, Q} \lesssim \infty. \]

By Proposition 2.5 we get that

\[ \|T_{S,a}^{b,m} f \|_{L^q(\mu)} \leq C \| f \|_{L^p(\nu)}. \]

Similarly, we have

\[ \sup_{Q} |Q|^{\alpha/n + 1/q - 1/p} \|(b - b_Q)^m \mu^{1/q}\|_{C, Q}\nu^{-1/p}\|_{D, Q} \]

\[ \lesssim \|b\|_{\partial sc(\Phi)} \sup_{Q} |Q|^{\alpha/n + 1/q - 1/p} \|\mu^{1/q}\|_{Y, Q}\nu^{-1/p}\|_{D, Q} \lesssim \infty. \]
This, together with Proposition 2.6 deduces that
\[ \| (T_{S_t,a}^b)^* f \|_{L^q(\mu)} \leq C \| f \|_{L^p(\nu)}. \]

Summing up the above estimates with Lemma 2.4, we obtain the conclusion of Theorem 1.3. \( \square \)

To prove Corollary 1.4, we need to recall the following fact. For \( \varphi(t) = t^p (\log(e + t))^q \) with \( p > 1 \) and \( q \in \mathbb{R} \), Cruz-Uribe et al. [10] showed that
\[ (2.4) \quad \varphi^{-1}(t) \sim \frac{t^{1/p}}{\log(e + t)^{q/p}}, \quad \tilde{\varphi}(t) = \frac{t^p}{\log(e + t)^{q/q}}, \]

Now, we give the proof of Corollary 1.4.

Proof of Corollary 1.4. To prove this corollary, we need only to choose some Young functions that satisfy the conditions of Theorem 1.3. For some \( \delta > 0 \), choose
\[ X(t) = t^{p'} [\log(e + t)]^{(m+1)p' - 1 + \delta}, \quad Y(t) = t^q [\log(e + t)]^{(m+1)q-1 + \delta}, \]
\[ B(t) = t^{p'} [\log(e + t)]^{q' - 1 + \delta}, \quad C(t) = t^{q} [\log(e + t)]^{q-1 + \delta}, \quad \Phi(t) = e^t - 1. \]

It is not hard to check that
\[ \tilde{B}(t) \sim \frac{t^{p'}}{\log(e + t)^{1 + p'\delta}} \in B_p \subset B_{p,q}, \quad \tilde{C}(t) \sim \frac{t^q}{\log(e + t)^{1 + q'\delta}} \in B_{q'} \]
and \( \Phi^{-1}(t) = \log(e + t) \). By (2.4), we have that
\[ X^{-1}(t) \sim \frac{t^{1/p'}}{\log(e + t)^{m+1/p' + \delta}}, \quad Y^{-1}(t) \sim \frac{t^{1/q}}{\log(e + t)^{m + 1/q' + \delta}}, \]
\[ B^{-1}(t) \sim \frac{t^{1/p'}}{\log(e + t)^{1 + p'\delta}}, \quad C^{-1}(t) \sim \frac{t^{1/q}}{\log(e + t)^{1 + q'\delta}}. \]

Then
\[ X^{-1}(t) \Phi^{-1}(t)^m \sim \frac{t^{1/p'}}{\log(e + t)^{m+1/p' + \delta}} [\log(e + t)]^m \sim B^{-1}(t), \]
\[ Y^{-1}(t) \Phi^{-1}(t)^m \sim \frac{t^{1/q}}{\log(e + t)^{m + 1/q' + \delta}} [\log(e + t)]^m \sim C^{-1}(t). \]

Finally, by the John-Nirenberg inequality and \( t \lesssim \Phi(t) \), we get \( \| b \|_{BMO(\mathbb{R}^n)} \sim \| b \|_{Osc(\Phi)}. \)

Thus, Theorem 1.3 implies Corollary 1.4. \( \square \)

Proof of Corollary 1.5. Choosing \( A(t) = t^q [\log(e + t)]^{q-1 + \delta}, \quad D(t) = t^{p'} [\log(e + t)]^{p' - 1 + \delta} \) in Corollary 1.4. Then
\[ \tilde{A}(t) \sim \frac{t^{p'}}{\log(e + t)^{1 + q'\delta}}, \quad \tilde{D}(t) \sim \frac{t^p}{\log(e + t)^{1 + p'\delta}} \in B_p \subset B_{p,q}. \]

Hence, Corollary 1.5 directly follows from Corollary 1.4. \( \square \)
3. Necessity of two weight inequalities for $I_{a}^{b,m}$

In this section, we give the proof of Theorem 1.7. To prove Theorem 1.7 we need the following two lemmas.

**Lemma 3.1.** Let $K_{\alpha}(x, y) = \frac{1}{|x-y|^{n-\alpha}}$. Then for each $A \geq 4$ and each ball $B := B(y_{0}, r)$, there exists a disjoint ball $\hat{B} := B(x_{0}, r)$ with $\text{dist}(B, \hat{B}) \sim Ar$ satisfies $|K_{\alpha}(x, y_{0})| = \frac{1}{A^{n-\alpha}}$, and for any $y \in \hat{B}$ and $x \in B$, there holds

$$|K_{\alpha}(x, y) - K_{\alpha}(x, y_{0})| \lesssim \frac{\epsilon_{A}}{A^{n-\alpha}}$$

where $\epsilon_{A} \to 0$ as $A \to \infty$.

**Proof.** Fix a ball $B = B(y_{0}, r)$ and $A \geq 4$, take $x_{0} = y_{0} + Ar\theta_{0}$, where $\theta_{0} \in \mathbb{S}^{n-1}$. Let $\hat{B} := B(x_{0}, r)$, it is easy to see that $\text{dist}(B, \hat{B}) \sim Ar$ and $K_{\alpha}(x, y_{0}) = \frac{1}{|x_{0} - y_{0}|^{n-\alpha}} = \frac{1}{A^{n-\alpha}}$

For any $y \in \hat{B}$ and $x \in B$, by the mean value theorem, we have

$$|K_{\alpha}(x, y) - K_{\alpha}(x, y_{0})| \leq |K_{\alpha}(x, y) - K_{\alpha}(x, y_{0})| + |K_{\alpha}(x, y) - K_{\alpha}(x, y_{0})|$$

$$\lesssim \frac{|x - x_{0}|}{|x_{0} - y|^{n-\alpha+1}} \leq \frac{1}{(Ar)^{n-\alpha}} =: \frac{\epsilon_{A}}{(Ar)^{n-\alpha}}$$

□

**Lemma 3.2.** (cf. [19]) Assume that $f \in BMO(\mathbb{R}^{n})$, and let $Q$ be a cube such that $f_{Q} = 0$. Then there exists a function $\varphi$ such that $\varphi = f$ on $Q$, $\varphi = 0$ on $\mathbb{R}^{n}\setminus 2Q$ and $\|\varphi\|_{BMO(\mathbb{R}^{n})} \lesssim \|f\|_{BMO(\mathbb{R}^{n})}$.

**Proof of Theorem 1.7.** For any cube $Q \subset \mathbb{R}^{n}$, we define

$$g(x) = \log^{+}\left(\frac{M(\nu^{1-p'}Q)(x)}{(\nu^{1-p'})_{Q}}\right)$$

It is well known that $g \in BMO(\mathbb{R}^{n})$. Moreover, the Kolmogorov inequality yields that

$$\int_{Q}(M(fQ))^{\delta} \lesssim \left(\frac{1}{|Q|}\int_{Q}|f|\right)^{\delta}|Q|, \quad 0 < \delta < 1$$

We then have $g_{Q} \lesssim 1$. According to Lemma 3.2, there is a function $\varphi$ satisfying $\varphi = g - g_{Q}$ on $Q$, $\varphi = 0$ outside $2Q$ and $\|\varphi\|_{BMO(\mathbb{R}^{n})} \lesssim 1$. Choosing a ball $B$ such that the centre of $B$ is the same as cube $Q$ and $r = \text{diam} Q$. Then by Lemma 3.1 there is a ball $\hat{B}$ such that the centre of $\hat{B}$ is the same as cube $B$ and $\text{dist}(B, \hat{B}) = Ar$, where $A \geq 4$ will be determined later.

Now, we return to prove our theorem. By duality, we find that the condition

$$\|I_{a}^{b,m}f\|_{L^{p,\infty}(\mu)} \lesssim \|b\|_{BMO(\mathbb{R}^{n})}^{m}\|f\|_{L^{p}(\mu)}$$

is equivalent to the condition

$$\|(I_{a}^{b,m})^{*}f\|_{L^{p'}(\nu^{1-p'})} \lesssim \|b\|_{BMO(\mathbb{R}^{n})}^{m}\|f/\mu\|_{L^{p',1}(\mu)}$$

One can check that $(I_{a}^{b,m})^{*} = (-1)^{m}I_{a}^{b,m}$, hence, we can still deal with 3.1 by considering $I_{a}^{b,m}$. Let $b = \varphi$, then for $x \in B$ and a non-negative function $f$,

$$I_{a}^{b,m}(f_{X\hat{B}})(x) = \int_{B}(b(x) - b(y))^{m}\frac{f(y)}{|x - y|^{n-\alpha}}dy = \varphi(x)^{m}\int_{\hat{B}}\frac{f(y)}{|x - y|^{n-\alpha}}dy.$$
By (3.1), we immediately get that
\[
\left( \int_B \left( \int_B \frac{f(y)}{|x-y|^{n-\alpha}} dy \right)^{p'} \right)^{1/p'} |\varphi(x)|^{mp'} \nu(x)^{1-p'} dx \right)^{1/p} \lesssim \|f \chi_B / \mu\|_{L^{q',1}(\mu)}.
\]
This, combining with Lemma 3.1, yields that
\[
\frac{1}{A^{n-\alpha}} \left( \int_B |\varphi(x)|^{mp'} \nu(x)^{1-p'} dx \right)^{1/p'} f_B
\]
\[
= \frac{r^{n-\alpha}}{(Ax)^{n-\alpha}} \left( \int_B |\varphi(x)|^{mp'} \nu(x)^{1-p'} dx \right)^{1/p'} f_B
\]
\[
= r^{-\alpha} \left( \int_B \left( \int_B \frac{f(y)}{|x-y|^{n-\alpha}} dy \right)^{p'} |\varphi(x)|^{mp'} \nu(x)^{1-p'} dx \right)^{1/p'}
\]
\[
\leq r^{-\alpha} \left( \int_B \left( \int_B \frac{1}{|x-y|^{n-\alpha}} |f(y)dy|^{p'} |\varphi(x)|^{mp'} \nu(x)^{1-p'} dx \right)^{1/p'}
\]
\[
+ r^{-\alpha} \left( \int_B \left( \int_B \frac{1}{|x-y|^{n-\alpha}} f(y)dy \right)^{p'} |\varphi(x)|^{mp'} \nu(x)^{1-p'} dx \right)^{1/p'}
\]
\[
\lesssim \frac{c_A}{A^{n-\alpha}} \left( \int_B |\varphi(x)|^{mp'} \nu(x)^{1-p'} dx \right)^{1/p'} f_B + r^{-\alpha}\|f \chi_B / \mu\|_{L^{q',1}(\mu)}.
\]
Choosing $A$ large enough, we have
\[
\left( \int_B |\varphi(x)|^{mp'} \nu(x)^{1-p'} dx \right)^{1/p'} f_B \lesssim r^{-\alpha}\|f \chi_B / \mu\|_{L^{q',1}(\mu)}.
\]
Taking $f = \mu$ and using the fact that $\|\chi_B\|_{L^{q',1}(\mu)} \sim \left( \int_B \mu \right)^{1/q}$, we obtain
\[
(3.2) \quad r^\alpha |\tilde{B}|^{-1} \left( \int_B |\varphi(x)|^{mp'} \nu(x)^{1-p'} dx \right)^{1/p'} \left( \int_B \mu(x) dx \right)^{1/q} \lesssim 1.
\]
Similarly, let $b = \chi_B$, following the arguments as (3.2), we get that
\[
(3.3) \quad r^\alpha |\tilde{B}|^{-1} \left( \int_B \nu(x)^{1-p'} dx \right)^{1/p'} \left( \int_B \mu(x) dx \right)^{1/q} \lesssim 1.
\]
Observe that $|\tilde{B}| \sim |Q|$ and $Q \subset \theta \tilde{B}$, where $\theta$ depends only on $A$ and $n$. Combing with these facts and the doubling property of $\mu$, we can replace (3.2) and (3.3) by
\[
r^\alpha |Q|^{-1} \left( \int_Q |g(x) - g_Q|^{mp'} \nu(x)^{1-p'} dx \right)^{1/p'} \left( \int_Q \mu(x) dx \right)^{1/q} \lesssim 1
\]
and
\[
r^\alpha |Q|^{-1} \left( \int_Q \nu(x)^{1-p'} dx \right)^{1/p'} \left( \int_Q \mu(x) dx \right)^{1/q} \lesssim 1,
\]
respectively. Keeping in mind that $g_Q \lesssim 1$, we finally have
\[
r^\alpha |Q|^{-1} \left( \int_Q |g(x)|^{mp'} \nu(x)^{1-p'} dx \right)^{1/p'} \left( \int_Q \mu(x) dx \right)^{1/q}
\]
\[
\lesssim r^\alpha |Q|^{-1} \left( \int_Q |g(x) - g_Q|^{mp'} \nu(x)^{1-p'} dx \right)^{1/p'} \left( \int_Q \mu(x) dx \right)^{1/q}
\]
\[
+ r^\alpha |Q|^{-1} \left( \int_Q \nu(x)^{1-p'} dx \right)^{1/p'} \left( \int_Q \mu(x) dx \right)^{1/q} \lesssim 1,
\]
which implies that
\[
\sup_Q |Q|^\frac{2}{n} + \frac{1}{p'} \left( \frac{1}{|Q|} \int_Q \mu(x) dx \right)^{1/q} \\
\times \left( \frac{1}{|Q|} \int_Q \nu(x)^{1-p'} \left[ \log \left( \frac{\nu(x)^{1-p'}}{\nu(x)^{1-p'}} \right) + e \right]^{mp'} dx \right)^{1/p'} < \infty.
\]

Therefore, using the following fact proved in [30],
\[
\|f\|_{L^\infty Q} \lesssim \frac{1}{|Q|} \int_Q |f(x)| \|\log(|f(x)|/|f + e|)\|^\alpha dx,
\]
we get the desired result. Theorem 1.7 is proved. \(\square\)

4. Converse to Bloom type estimate for \(I_{b,m}^b\)

This section is concerning with the proof of Theorem 1.8. First, we recall and establish some lemmas, which are the keys in our arguments.

**Lemma 4.1.** (cf. [19]) Let \(\eta_1, \eta_2\) be the weights such that \(\eta_1/\eta_2 \notin L^\infty\). Then there exists \(b \in BMO_{\eta_1} \setminus BMO_{\eta_2}\).

**Lemma 4.2.** Let \(\lambda, \mu\) be arbitrary weights satisfy \((1.6)\) and \(p, q, m, \alpha\) be given in Theorem 1.6. Then for each ball \(B := B(y_0, r)\), there exists a disjoint ball \(\tilde{B} := B(x_0, r)\) with \(\text{dist}(B, \tilde{B}) \sim Ar\) such that for any non-negative measurable function \(f\),
\[
\left( \int_B \eta(x)^{mq} \lambda(x)^q dx \right)^{1/q} f_B \lesssim r^{-\alpha} \left( \int_B f(x)^p \mu(x)^p dx \right)^{1/p}.
\]

**Proof.** Let \(\tilde{B}\) be given in Lemma 3.1 and \(b = \eta x_{\tilde{B}}\). Then for \(x \in \tilde{B}\),
\[
I_{b,m}^b(f \chi_B)(x) = \int_B (b(x) - b(y))^m \frac{f(y)}{|x - y|^{n-\alpha}} dy = \eta(x)^m \int_B \frac{f(y)}{|x - y|^{n-\alpha}} dy.
\]
By \((1.6)\), we have
\[
\left( \int_{\tilde{B}} \left( \int_B \frac{f(y)}{|x - y|^{n-\alpha}} dy \right)^q \eta(x)^{mq} \lambda(x)^q dx \right)^{1/q} \lesssim \left( \int_B f(x)^p \mu(x)^p dx \right)^{1/p}.
\]
From this and Lemma 3.1, we deduce that
\[
\frac{1}{A^{n-\alpha}} \left( \int_B \eta(x)^{mq} \lambda(x)^q dx \right)^{1/q} f_B
\]
\[
= \frac{r^{n-\alpha}}{(Ar)^{n-\alpha}} \left[ \int_B \eta(x)^{mq} \lambda(x)^q \left( \frac{1}{|B|} \int_B f(y) dy \right)^q dx \right]^{1/q}
\]
\[
= r^{-\alpha} \left[ \int_B \eta(x)^{mq} \lambda(x)^q \left( \int_B \frac{f(y)}{|x_0 - y_0|^{n-\alpha}} dy \right)^q dx \right]^{1/q}
\]
\[
\leq r^{-\alpha} \left[ \int_B \eta(x)^{mq} \lambda(x)^q \left( \int_B \frac{1}{|x - y|^{n-\alpha}} - \frac{1}{|x_0 - y_0|^{n-\alpha}} \right) f(y) dy \right]^q dx \right]^{1/q}
\]
\[
+ r^{-\alpha} \left[ \int_B \eta(x)^{mq} \lambda(x)^q \left( \int_B \frac{1}{|x - y|^{n-\alpha}} f(y) dy \right)^q dx \right]^{1/q}
\]
\[
\lesssim \frac{\epsilon_A}{A^{n-\alpha}} \left( \int_B \eta(x)^{mq} \lambda(x)^q dx \right)^{1/q} f_B + r^{-\alpha} \left( \int_B f(x)^p \mu(x)^p dx \right)^{1/p}.
\]
Then the desired result directly follows by letting \(A \to \infty\). \(\square\)
Lemma 4.3. Let $\lambda, \mu$ be arbitrary weights satisfy (1.6) and $p, q, m, \alpha$ be given in Theorem 1.8. Then

$$\lambda(x)\eta(x)^m \lesssim \mu(x).$$

Proof. Let $f = 1$ in Lemma 4.2. Keep in mind that $1/p - 1/q = \alpha/n$, then

$$\left( \frac{1}{|B|} \int_B \eta(x)^mq \lambda(x)^q dx \right)^{1/q} \lesssim \left( \frac{1}{|B|} \int_B \mu(x)^p dx \right)^{1/p}.$$

By the Lebesgue differential theorem, we get the desired result. \qed

Now, we are in the position to prove Theorem 1.8.

Proof of Theorem 1.8. By Lemma 4.3, it suffices to prove that

$$\mu \lesssim \lambda\eta^m$$

almost everywhere. Suppose that (4.1) is not true. Denote $\tilde{\eta} = (\mu/\lambda)^{1/m}$. Then $\tilde{\eta}/\eta \not\in L^\infty$. Note that when $\lambda, \mu \in A_{p,q}$, Accomazzo et al. proved that for $b \in BMO_{\tilde{\eta}}$,

$$\|P_{\alpha,m}^b f\|_{L^q(\lambda^q)} \lesssim \|f\|_{L^p(\mu^p)}.$$

This, together with Lemma 4.1, implies that $b \not\in BMO_{\eta}$, which contradicts with (1.7) and completes the proof of Theorem 1.8. \qed

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School of Mathematics and Statistics, Minnan Normal University, Zhangzhou 363000, China
Email address: wenyongmingxmu@163.com

School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
Email address: huoxwu@xmu.edu.cn