Symmetric coupling of four spin-1/2 systems

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Abstract
We address the non-binary coupling of identical angular momenta based upon the representation theory for the symmetric group. A correspondence is pointed out between the complete set of commuting operators and the reference-frame-free subsystems. We provide a detailed analysis of the coupling of three and four spin-1/2 systems and discuss a symmetric coupling of four spin-1/2 systems.

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1. Introduction

The coupling of several angular momenta is one of the important technical problems in quantum mechanics. Every standard textbook of quantum mechanics discusses at least the coupling of two angular momenta. This problem is of great importance not only for practical purposes when dealing with several particles such as in atomic physics, nuclear physics and so on, but is also of mathematical interest. To be specific, the problem can be rephrased as a well-known problem in group representation theory, namely to obtain a direct sum of irreducible representations of the rotation group. There is a large body of literature in which this problem is studied both analytically and numerically; see, for example, [1, 2].

The complete analytical formula is known for the coupling of any two angular momenta in terms of the Clebsh–Gordan (CG) coefficients. These coefficients immediately allow us to express the bases in irreducible representations as superpositions of the bases in direct product representations. It is a well-established strategy to add more than two angular momenta by successive applications of the addition of two with the aid of CG coefficients. The addition of many angular momenta can be carried out by using the \(n j\) symbols, the Racah coefficients and so on, which are generally studied within the recoupling theory. Indeed, the study of many angular momenta turns out to be a beautiful mathematical physics problem in its own right.
Recently, the study of irreducible representations of many identical spin-$j$ systems, or many $d$-dimensional quantum systems (qudits), has been revived in the context of quantum information and quantum computing. When coupling many spin-$j$ systems, the structure of the Hilbert space can be described by the representation theory of the symmetric group. This fact is known as the Weyl–Schur duality which can be utilized to solve many problems [3]. Interesting examples of this kind include an estimation for a spectrum of an unknown quantum state [4], quantum communication without sharing a reference frame [5], universal coding for a classical-quantum channel [6] and others [4]. One of the important applications is to realize decoherence-free subspaces and subsystems in order to implement useful quantum algorithms [7]. Another interesting feature among these studies is the proposal for an efficient quantum circuit to obtain the irreducible representation of the $N$-fold tensor product of a $d$-dimensional Hilbert space, which requires only a total number of gates of order $N \log(d, \log N, \log 1/\epsilon)$ up to accuracy $\epsilon$ [8–10].

Let us look at the coupling of several angular momenta using the CG coefficients. The first step is to add two angular momenta which are conveniently chosen from all angular momenta. The next step is then to add each of the obtained angular momenta and another one chosen from the yet-uncoupled angular momenta. By repeating this binary coupling many times, one can arrive at the desired result. It is a rather straightforward task to perform each step, but the final result cannot be obtained in a simple manner. The major obstacle is that the computational complexity of such a coupling of several angular momenta grows rather rapidly with the total number of angular momenta and the dimension of each angular momentum.

As the simplest case, we consider the addition of $N$ spin-1/2 systems which can be decomposed into a direct sum of irreducible representations labeled with angular momentum $j$ as

$$D^{\otimes N}_{1/2} = \bigoplus_{j \in J} c_j D_j,$$

where the index set $J = \{N/2, N/2 - 1, \ldots\}$ has $(N + 1)/2$ or $N/2 + 1$ elements if $N$ is odd or even, respectively. The multiplicity of each irreducible space is

$$c_j = \frac{N!(2j + 1)}{(N/2 + j + 1)!(N/2 - j)!}. \quad (2)$$

When another spin-1/2 system is to be added to the obtained result, we need to couple this new spin-1/2 state to $\sum_j c_j \sim (2N)!/(N!)^2 \sim 4^N$ different angular momenta, which number grows exponentially for large $N$.

Another disadvantage of the binary coupling is that the various constituent angular momenta are not treated on equal footing. In other words, the resulting angular momentum states depend on the way one chooses the pairing in the intermediate steps. The binary coupling might not be a wise choice when dealing with many identical angular momenta. To overcome this problem, a novel coupling scheme was proposed more than four decades ago independently by Chakrabarti [11] and Lévy-Leblond and Lévy-Nahas [12]. They studied the non-binary couplings of three angular momenta without employing the binary coupling. Their coupling scheme is generally referred to as the symmetric coupling or the democratic coupling, which reflects the fact that their choice of the complete set of commuting operators (CSCO) contains all three angular momenta with equal weights. To our knowledge, there has been no generalization of their non-binary coupling to the case of more than three angular momenta.

It is our main motivation here to clarify the meaning of the symmetric coupling for four identical spin systems and then to provide a possible solution for the coupling of four
spin-1/2 systems. While the main result was already reported in [13]\(^5\), we have not clarified the meaning of the symmetric coupling as yet. We hope that our construction paves the way toward establishing a non-binary coupling of many angular momenta.

This paper is organized as follows. In section 2 we give the mathematical background of the Weyl–Schur duality, a possible definition of the symmetric coupling of many identical angular momentum systems and discuss the relation to the reference-frame-free (RFF) subsystems. We then study the symmetric coupling of many spin-1/2 systems in the second-largest angular momentum subspace in section 3. The detailed analysis in the case of four spin-1/2 systems is shown in section 4. We close with a summary and discussion in section 5.

2. Symmetric coupling of \(N\) spin-1/2 systems

We briefly summarize some of relevant mathematical facts. Readers are referred to [3] for more concise discussions. In the rest of this paper, we mainly consider \(N\) spin-1/2 systems (two-dimensional systems) unless stated explicitly.

2.1. Weyl–Schur duality

The \(N\)-fold tensor product of the two-dimensional Hilbert space can be decomposed into the following direct sum structure:

\[
(C^2)^{\otimes N} = \bigoplus_{\nu \in \text{Par}(N,2)} S_{\nu} \otimes R_{\nu},
\]

(3)

where \(\text{Par}(N, 2)\) stands for the partition of \(N\) into two non-negative and non-increasing integers, i.e., \(\text{Par}(N, 2) = \{(v_1, v_2) \in \mathbb{Z}^2 | v_1 + v_2 = N, v_1 \geq v_2 \geq 0\}\). In the decomposition (3), known as the Wedderburn decomposition, the subspaces \(R_{\nu}\) are the representation spaces for the general matrix group over the complex field with the dimension

\[
r(\nu) = v_1 - v_2 + 1,
\]

(4)

and \(S_{\nu}\) are the representation spaces for the symmetric group \(S_N\) with the dimension

\[
s(\nu) = \frac{N!}{(v_1 + 1)!v_2!}.
\]

(5)

This dimension \(s(\nu)\) is the same as the multiplicity \(e_j\) in (2) for \(j = (v_1 - v_2)/2\). In other words, we can also label the subspaces with a single quantum number \(j\) in accordance with

\[
v_1 = \frac{N}{2} + j, \quad v_2 = \frac{N}{2} - j,
\]

(6)

with \(j \in J\). Note that \(r(\nu) = 2j + 1\) is the dimension of the subspace \(D_j\) with angular momentum \(j\). As an example, consider the coupling of three spin-1/2 systems, in which case the partition is \(\text{Par}(3, 2) = \{(3, 0), (2, 1)\}\). The dimensions of the corresponding subspaces are \(r(3, 0) = 4, s(3, 0) = 1, r(2, 1) = 2\) and \(s(2, 1) = 2\).

Because of the decomposition (3), the \(N\)-fold tensor product of the two-dimensional non-singular matrices \(A \in \text{GL}(2, \mathbb{C})\), i.e., \(A^{\otimes N}\) acts irreducibly on the subspaces \(R_{\nu}\), and the unitary representations of the permutation operators \(P_{i_1i_2\ldots i_N}\) act irreducibly on the subspaces \(S_{\nu}\). Throughout this paper, we denote the permutation from 1, 2, \ldots, \(N\) to \(i_1i_2\ldots i_N\) by \(P_{i_1i_2\ldots i_N}\), that is,

\[
P_{i_1i_2\ldots i_N} = \begin{pmatrix}
1 & 2 & \cdots & N \\
i_1 & i_2 & \cdots & i_N
\end{pmatrix}.
\]

(7)

\(^5\) For the record, we note that an inadvertent interchange between \(\lambda = 1\) and \(\lambda = 2\) happened in the transition from section 3 to section 4 A in this paper.
Therefore, $A^\otimes N$ and $P_{i_1z_1...i_N}$ are decomposed as follows:

$$A^\otimes N = \bigoplus_{\nu \in \text{Par}(N,2)} I_{\nu} \otimes R_{\nu},$$

$$P_{i_1z_1...i_N} = \bigoplus_{\nu \in \text{Par}(N,2)} S_{\nu} \otimes I_{\nu}.$$  \hfill (8)

These decompositions are the essence of the Weyl–Schur duality which states that operators commuting with all elements of $A^\otimes N$ are expressed as linear combinations of the unitary permutation operators $P_{i_1z_1...i_N}$ with complex coefficients. Moreover, its inverse also holds, that is, if operators commute with all elements of $P_{i_1z_1...i_N}$, they are a linear combination of $A^\otimes N$ with complex coefficients. The Weyl–Schur duality holds for the general case of $N$-fold tensor products of $d$-dimensional systems [3].

### 2.2. CSCO and missing label operators (MLOs)

As a mathematical problem, to calculate the completely reducible representation is equivalent to finding the CSCO whose joint eigenstates define the representation uniquely up to arbitrary phase factors. A simple counting argument shows that the number of elements of the CSCO is $N$ for the case of addition of $N$ arbitrary angular momenta\(^\text{6}\). This follows from the fact that the direct product representation is given by the joint eigenvalues of the $z$-components of the individual angular momentum. To be precise, the states are labeled by $2N$ quantum numbers in the direct product representation. Upon denoting the individual spin operator by $J_\ell = \vec{\sigma}_\ell/2$ ($\ell = 1, 2, \ldots, N$), where $\vec{\sigma}_\ell$ are the Pauli spin operators constituting the Lie algebra $su(2)$, the squares are $J_\ell^2 = J_\ell^x J_\ell^z$ and the $z$-components of the individual angular momenta are $J_\ell^z$.\(^\text{7}\) When considering systems in which each angular momentum is fixed, we will not write the eigenvalues of the Casimir operator $J_\ell^2$ explicitly. This paper deals with such a system and the total number $N$ of constituents thus determines the representation.

From the general theory of quantum angular momentum, two of the commuting operators in the CSCO are immediate. The first one is the Casimir operator of the rotational group, which labels the total angular momentum quantum number, and the second is the $z$-component of the total angular momentum. Therefore, the minimal number of elements in the CSCO that are still to be constructed is $N - 2$. These remaining $N - 2$ operators are usually referred to as the MLOs, and finding the MLOs has been a standard but rather difficult problem in the representation theory [14–16]. Note that the CSCO is not unique in general and to list all possible families of CSCO seems an untrackable problem except for some special cases.

With the total angular momentum $J = \sum_{i=1}^N J_i$, the above-mentioned Casimir operator and the $z$-component of the total angular momentum are $J^2 = J_1^2 + \ldots + J_N^2$ and $J_z = \sum_i J_i^z$, respectively. The former operator $J^2$ specifies the angular momentum space $j$ in (1) and the partition $\nu$ in (3). The latter operator determines the representation $D_j$ in (1) and the representation $R_\nu$ in (3). By definition, the MLOs commute with $J^2$ and $J_z$, and their eigenvalues are non-degenerate within each subspace specified by the common eigenvalues of $J^2$ and $J_z$. Without loss of generality, these non-degenerate eigenvalues can be chosen as real, and thus the MLOs can be given by Hermitian operators. It is not difficult to show that the MLOs live only in the subspace $S_j$, and they are expressed as the superpositions of the unitary permutation operators. This leads

\(^6\) The general theorem guarantees that the minimal number of elements of CSCO for finite-dimensional representations can always be reduced to 1. In this paper, however, we adopt the direct sum decomposition (3) and wish to find the CSCO according to this decomposition.

\(^7\) The spin operators act on the full Hilbert space, as illustrated by $\vec{\sigma}_2 = I_2 \otimes \vec{\sigma} \otimes I_2 \otimes \ldots \otimes I_2$. 
to the key observation that the MLO problem for \(N\) spin-\(j\) systems can be solved by finding a suitable unitary representation of the symmetric group \(S_N\).

2.3. Symmetric coupling of three spin-1/2 systems

The representation theory of the symmetric group has been much studied. In the standard treatment, the irreducible representations can be constructed in real matrix forms by employing the canonical subgroup chain

\[
S_N \supset S_{N-1} \supset \ldots \supset S_2.
\]

In the simplest case of \(S_3\), for example, there exist three different irreducible representations corresponding to three possible Young tableaux. Besides trivial one-dimensional representations for the totally symmetric and anti-symmetric subspaces, the remaining non-trivial one is the two-dimensional subspace. In the above choice of subgroup chain (9), one can choose three possible proper subgroups \(S_3(12), S_2(23)\) and \(S_2(31)\) where \(S_2(i_1, i_2)\) is the transposition subgroup between two indices \((i_1, i_2), \text{e.g., } S_2(12) = \{P_{213}, P_{231}\}\). When the reduction \(S_3 \supset S_2(12)\) is adopted, the representations corresponding to the decomposition (8) are

\[
P_{213} \cong I_1 \otimes I_4 \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I_2,
\]

\[
P_{132} \cong I_1 \otimes I_4 \oplus \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \otimes I_2,
\]

and the other representations for the elements of \(S_3\) are generated by the combinations of these two transpositions. Since a particular proper subgroup \(S_2(12)\) is diagonalized, the symmetry of the subspace representation labeled by the partition \(v = (v_1, v_2)\) is determined by the subgroup \(S_2(12)\), i.e. we have invariance under the transposition between the two indices \((1, 2)\).

With these observations, the coupling of three spin-1/2 systems is solved by specifying the MLOs. As discussed, the number of MLOs is 1 for the \(N = 3\) case. This operator is identified with the transposition operator \(P_{213}\) by adopting the representation (10). Upon noting that the transposition operator between two given systems \((k, \ell)\) is expressed in terms of the Pauli operators \(\sigma_k\) and \(\sigma_\ell\) (\(k \neq \ell\)) by

\[
P_{k\ell} = \frac{1}{2}(I_k + \sigma_k \cdot \sigma_\ell) = P_{1k},
\]

the MLO is written as \(P_{213} = (I_8 + \sigma_1 \cdot \sigma_2)/2\). It is straightforward to see the correspondence with the standard binary coupling, in which the MLO is given by the square of the intermediate coupled angular momenta. In the above choice of MLO \(P_{213}\), the corresponding operator in terms of angular momenta is

\[
J_{12}^2 = \vec{J}_{12} \cdot \vec{J}_{12} = \frac{1}{2}(3I_8 + \sigma_1 \cdot \sigma_2) = P_{123} + P_{213},
\]

where \(\vec{J}_{12} = \vec{J}_k + \vec{J}_\ell\) is the intermediate angular momentum. Because the identity \(P_{123} = I_8\) is irrelevant as far as MLOs are concerned, \(P_{213}\) and \(J_{12}^2\) are essentially the same MLO. This illustrates the claim that the coupling of identical angular momenta is obtained by the representation theory of the symmetric group. The other possible MLOs are also found to be \(P_{23} = P_{213} + P_{132}\) and \(J_{23}^2 = P_{213} + P_{231}\). Importantly, all three representations are related through the action of unitary transformations, and the subject matter of recoupling theory is to study the relationships among these different representations.

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8 For example, see [10, 19, 25, 26].
We now show a different coupling scheme by reconsidering the subgroup chain \((9)\). It is well known that the cyclic permutation is also a proper subgroup of \(S_n\). Denoting the \(N\)-cyclic permutation group by \(C_N\), the alternative subgroup chain is
\[
S_N \supset C_N \supset C_{N-1} \supset \ldots \supset C_2 = S_2.
\] (13)
In the case of three spin-1/2 constituents, the cyclic group of order 3 is \(C_3 = \{P_{123}, P_{231}, P_{312}\}\), and the natural representation of \(C_3\) is the diagonal matrix where the elements are powers of the basic third root of unity. Thus, two-dimensional representations for the elements of \(S_3\) can be generated by the combination of the three-cycle permutation \(P_{231}\) and two-cycle permutation \(P_{213}\). They are
\[
P_{231} \equiv I_1 \otimes I_2 \otimes \left( \frac{\omega_3}{\omega_2^3} \right) \otimes I_2,
\]
\[
P_{213} \equiv I_1 \otimes I_2 \otimes \left( \frac{0}{1} \frac{1}{0} \right) \otimes I_2,
\] (14)
where \(\omega_d = \exp(2\pi i/d)\) is the basic \(d\)th root of unity. One can, of course, check that the two representations in \((10)\) and \((14)\) are related by the unitary matrix
\[
I_1 \otimes I_2 \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ -i & i & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes I_2,
\] (15)
but this representation has higher symmetry than the previous one. This is because the cyclic permutation subgroup \(C_3\) has order 3 rather than 2 for the transposition subgroup. We remark that this choice of irreducible representation is only possible with complex numbers.

The MLOs are readily found through the relation \(P_{231} = P_{132}P_{213}\) and the Pauli operator representation of the transition operator \((11)\),
\[
P_{231} = \frac{1}{2}[I_3 + \vec{\sigma}_1 \cdot \vec{\sigma}_2 + \vec{\sigma}_2 \cdot \vec{\sigma}_3 + i\vec{\sigma}_1 \times (\vec{\sigma}_2 \times \vec{\sigma}_3)].
\] (16)
Since the other element of the cyclic permutation operator \(P_{312}\) is equally good for the MLOs, the two cyclic permutations can be combined to give the real one of the MLOs as
\[
K \equiv \frac{-i}{\sqrt{3}} (P_{231} - P_{312}) = \frac{1}{\sqrt{12}} \vec{\sigma}_1 \cdot (\vec{\sigma}_2 \times \vec{\sigma}_3)
\]
\[
\equiv 0_1 \otimes I_2 \oplus \left( \frac{1}{0} \frac{0}{-1} \right) \otimes I_2.
\] (17)
This form of the MLO is more convenient for us for two reasons. First, the eigenvalues of \(K\) are \(\pm 1\) and, second, \(K\) is projected onto the two-dimensional subspace \(v = (2, 1)\). The corresponding angular momentum states in the \(j = 1/2\) subspace are
\[
|1/2, 1/2; \lambda\rangle = \frac{1}{\sqrt{3}} \{(100)\sigma_2^3 + (010)\sigma_3^2 + (001)\},
\]
\[
|1/2, -1/2; \lambda\rangle = -\frac{1}{\sqrt{3}} \{(011)\sigma_2^2 + (101)\sigma_3^2 + (110)\},
\] (18)
where \(\lambda = 1, 2\) and we denote by \(|0\rangle\) and \(|1\rangle\) the kets with \(m = 1/2\) and \(m = -1/2\), respectively, for the states of the single spin-1/2 constituents.

Note that this CSCO \([J^2, J_z, K]\) is indeed identical with that of \([11, 12]\), where they proved that this set is the CSCO for the general coupling of three angular momenta \(J_\ell\) (\(\ell = 1, 2, 3\)) with the MLO \(K = J_1 \cdot (J_2 \times J_3)\). Although this novel coupling scheme was discovered almost half a century ago, the generalization to the case of more than three angular momenta is—to our knowledge—still an open problem. The reason seems that there exist infinitely many different MLOs, which are in general equivalent. It is then rather hard to conclude that a particular choice of MLOs provides the symmetric coupling without a precise definition of the symmetric coupling. In the next subsection, we attempt to give a possible definition of the symmetric coupling in the case of identical spin systems.
2.4. A proposal for symmetric coupling

We define the symmetric coupling of identical angular momenta, not necessarily for the case of spin-1/2 systems, as follows:

If the MLO projected onto a subspace labeled by the partition $\nu$ can be chosen as a linear combination of cyclic permutation operators with the possible maximal order in this subspace, then identical angular momenta are said to be coupled symmetrically within the subspace.

The word ‘symmetric coupling’ refers to the fact that this coupling respects the cyclic permutation symmetry within each subspace. In this definition, the possible maximal order of the cyclic permutation operator still needs to be stated explicitly. In this paper, we set the order of the cyclic permutation subgroup to be $s(\nu) + 1$ for the partition $\nu$. If the order is less than $s(\nu) + 1$, we do not have a symmetric coupling in the subspace. Note that the cyclic permutation group is Abelian and hence all linear combinations are equally good as the MLOs as long as they have non-degenerate eigenvalues. Another important consequence of the Abelian property is that the obtained irreducible basis is invariant under the same cyclic permutation.

We emphasize that this definition is rather limited since it only applies to the case of addition of identical angular momenta for a few angular momentum systems. In particular, the number $s(\nu) + 1$ becomes greater than $N$ for more than four spin-1/2 systems and we need to refine the meaning of the possible maximal order of the cyclic permutation operator properly. According to the definition above, for example, it follows that five spin-1/2 systems cannot be coupled symmetrically in the subspace $\nu = (3, 2) (j = 1/2)$ whose degeneracy is 5. Nevertheless, we will show that a symmetric coupling of four spin-1/2 systems is possible. The coupling of more spin-1/2 systems and other more general cases is left to future studies.

2.5. Relation to RFF subsystems

Owing to the Weyl–Schur duality, the role of the permutation symmetry is clear in the tensor-product Hilbert space. This leads to the idea of constructing a RFF subsystem, or a RFF qudit, in the context of quantum information theory. A RFF qudit is a $d$-dimensional subsystem of a composite system, which remains invariant under the same unitary transformation on the constituents of the composite system. It is called a rotationally invariant qudit when considering the invariant subsystems under the collective rotation rather than the general unitary transformations. The RFF subsystems have many applications in quantum information and quantum computing [5]. The Werner state for two parties is the simplest example [17], and the generalization to more than two parties has been studied in this context [18–20].

When dealing with spin-1/2 systems, any unitary transformation is equivalent to a rotation ($SU(2) \cong SO(3)$). Then, RFF subsystems can be described by a non-negative unit-trace density operator in the subsystems, which commutes with the $N$-fold tensor product of the rotation $u_j = \exp(i\vec{n} \cdot \vec{J}_j)$. Denoting the density operator by $\rho (\rho \geq 0, \text{tr}[\rho] = 1)$, the condition for the RFF subsystem reads

$$\left[ \rho, \prod_{j=1}^{N} u_j \right] = [\rho, e^{i\vec{n} \cdot \vec{J}}] = 0,$$

(19)
where \( \tilde{J} \) is the total angular momentum operator. From the properties of the Weyl–Schur duality, it immediately follows that the only possible form of the density operator for the RFF subsystem is

\[
\rho_{\text{RFF}} = \bigoplus_{\nu \in \text{Part}(N,2)} \rho_{\nu(v)} \otimes I_{r(v)},
\]

(20)

which is a linear combination of permutation operators \( P_{0i_1...i_2} \). Therefore, the construction of RFF subsystems is essentially a problem of revealing the algebraic relationship between the permutation operators in the various subspaces, that is, to analyze the different choices of MLOs. In fact, the following stronger statement holds:

All possible MLOs for the subspace \( \nu \) are linear combinations of a non-degenerate RFF density operator and the projector onto the subspace \( \nu \),

\[
\alpha \rho_{\nu} + \beta I_{r(v)},
\]

where \( \alpha \) and \( \beta \) are coefficients.

In the Lie algebra theory, the algebra formed by RFF states is known as the (universal) enveloping algebra of \( \text{su}(2) \) [21].

There are many ways to describe a quantum state defined in the \( d \)-dimensional Hilbert space \( \mathbb{C}^d \). The simplest one is to use the \( d \) orthonormal kets \( |k\rangle \) (\( k = 1, 2, \ldots, d \)) that are orthonormal and complete,

\[
(\langle k' | k \rangle = \delta_{kk'}, \quad \sum_{k=1}^d |k\rangle \langle k| = I_d,
\]

(21)

and thus form a basis in \( \mathbb{C}^d \). Correspondingly, the \( d^2 \) operators

\[
Q_{\ell\ell} \equiv |k\rangle \langle \ell| \quad (k, \ell = 1, 2, \ldots, d)
\]

(22)

that satisfy the closure relation \( Q_{\ell\ell} Q_{\ell'\ell'} = \delta_{\ell\ell'} Q_{\ell'\ell'} \) form a basis for the \( d \)-dimensional operator-algebra space, i.e. the \( d \)-dimensional matrix ring over \( \mathbb{C} \). We call these \( d^2 \) operators the RFF basis operators. The state of a \( d \)-dimensional quantum system can be represented as a unit-trace \( d \times d \) matrix that is semi-definite positive. Another possible way is to expand the state in terms of the generators for a \( SU(d) \) Lie group with real coefficients. The standard Gell-Mann matrices together with the semi-definite positivity requirement provide a proper \( d \)-level quantum state [22, 23]. The third option is to use the unitary Heisenberg–Weyl–Schwinger operator basis [24]. The complete set of unitary operators is given by \( U^d V^\ell \) (\( k, \ell = 1, 2, \ldots, d \)), where the unitary operators \( U \) and \( V \) have the period \( d \), i.e. \( U^d = V^d = 1 \), and satisfy the Weyl commutation relation \( U^d V^\ell = \omega_d^{-\ell} V^\ell U^d \) with \( \omega_d = \exp(2\pi i/d) \) as above. From the mathematical point of view, all three operator bases and any other ones are equivalent in the sense that we can convert one description into the others through a bijective mapping, and an advantage over others may show up depending upon the problem of interest. In the following, the first method is mainly considered, and the basis operators \( Q_{\ell\ell} \) are to be constructed.

The Weyl–Schur duality and the decomposition in (8) already provide the relation between any permutation operator and the basis operators \( Q_{\ell\ell}^\nu \) in the subspace labeled by a partition \( \nu \). First, define the RFF basis operators in the subspace \( \nu \) by

\[
Q_{\ell\ell}^\nu = |k\rangle \langle \ell| \otimes I_{r(v)},
\]

(23)

where the operator \( |k\rangle \langle \ell| \) lives in the subspace \( S_\nu \). Then, they form the RFF basis operators satisfying \( Q_{\ell\ell}^\nu Q_{\ell'\ell''}^\nu = \delta_{\nu\nu'} \delta_{\ell\ell'} Q_{\ell''\ell'}^\nu \), and the permutation operator can be written as

\[
P_{0i_1...i_2} = \sum_{\nu \in \text{Part}(N,2)} \sum_{k,\ell=1}^d p_{\ell\ell}^\nu Q_{\ell\ell}^\nu.
\]

(24)
Here, the coefficients \( p^v \) are elements of the matrix representation of the symmetric group within the subspace \( v \). Using the RFF basis operators, the MLO \( M_v \) can be expressed as a linear combination of diagonal elements of them with different coefficients

\[
M_v = \sum_{k=1}^{n(v)} q^v_k Q_{k,k}. 
\]  

(25)

In the example of three spin-1/2 systems, the relation (24) for the subgroup chain (13) reads

\[
P_{123} = Q^{(3,0)}_{111} + Q^{(2,1)}_{111} + Q^{(2,1)}_{222},
\]

\[
P_{213} = Q^{(3,0)}_{111} + \alpha_3 Q^{(2,1)}_{111} + \alpha_2^2 Q^{(2,1)}_{222} = (P_{312})^\dagger,
\]

\[
P_{212} = Q^{(3,0)}_{111} + Q^{(2,1)}_{111} + Q^{(2,1)}_{222},
\]

\[
P_{132} = Q^{(3,0)}_{111} + \omega_3 Q^{(2,1)}_{111} + \omega_2 Q^{(2,1)}_{222},
\]

\[
P_{231} = Q^{(3,0)}_{111} + \alpha_2 Q^{(2,1)}_{111} + \alpha_3 Q^{(2,1)}_{222}.
\]

(26)

Although these equations seem to be over-determined at first sight, i.e. six equations for five variables \( Q_{\ell \ell} \), only five equations are actually linearly independent. Note that in general, there are \( N! \) linear equations for the RFF basis operators of which the total number is \( \sum_{v \in \text{Par}(N,N)} s(v)^2 \).

This number is always less than \( N! \) and, therefore, the occurrence of linearly dependent equations is generic. This is due to the fact that the entire representation space is not exhausted when considering a problem of \( N \) spin-1/2 systems. The complete set of representations can be obtained for the case of \( N \) spin-\( j \) (\( j = N/2 + 1 \)) systems, where the following relation holds:

\[
\sum_{v \in \text{Par}(N,N)} s(v)^2 = N!.
\]  

(27)

By converting equations (26), we obtain the operators \( Q^v_{\ell k} \) in terms of the permutation operators:

\[
\begin{pmatrix}
Q^{(3,0)}_{111} \\
Q^{(2,1)}_{111} \\
Q^{(2,1)}_{222}
\end{pmatrix}_{\text{Par}(3,3)} = \frac{1}{3} \begin{pmatrix}
1 & 1 & 1 \\
1 & \alpha_2 & \alpha_3 \\
1 & \alpha_3 & \alpha_2
\end{pmatrix}_{\text{Par}(3,3)} \begin{pmatrix}
P_{123} \\
P_{213} \\
P_{312}
\end{pmatrix}.
\]

(28)

In terms of the Pauli spin operators, they read

\[
Q^{(3,0)}_{111} = \frac{1}{2} I_8 + \frac{1}{6} (\vec{\sigma}_1 \cdot \vec{\sigma}_2 + \vec{\sigma}_2 \cdot \vec{\sigma}_3 + \vec{\sigma}_3 \cdot \vec{\sigma}_1),
\]

\[
Q^{(2,1)}_{111} = -\frac{1}{12} (\vec{\sigma}_1 \cdot \vec{\sigma}_2 + \vec{\sigma}_2 \cdot \vec{\sigma}_3 + \vec{\sigma}_3 \cdot \vec{\sigma}_1) \pm \frac{1}{\sqrt{48}} \vec{\sigma}_1 \cdot (\vec{\sigma}_2 \times \vec{\sigma}_3),
\]

(29)

\[
Q^{(2,1)}_{222} = (Q^{(2,1)}_{211})^\dagger = \frac{1}{6} (\vec{\sigma}_1 \cdot \vec{\sigma}_2 + \omega_3^2 \vec{\sigma}_2 \cdot \vec{\sigma}_3 + \omega_3 \vec{\sigma}_3 \cdot \vec{\sigma}_1).
\]

With these basis operators \( Q^{(3,0)}_{111} \) and \( Q^{(2,1)}_{\ell k} \), we can express any RFF state as a linear combination of them, and this completes the analysis of three spin-1/2 systems.
3. Symmetric coupling for the second-largest angular momentum

In this section, we investigate a symmetric coupling of \( N \) spin-1/2 systems within the second-largest angular momentum subspace \([13]\) (see also footnote 5). Since the second-largest angular momentum has \( N - 1 \) components and its multiplicity is \( c_{N/2-1} = N - 1 \), there are \((N - 1)^2\) states in the subsystem in total. The basic ingredient is the grading of the \( N \) spin-1/2 constituents \( \mathbf{J}_\ell = \mathbf{\sigma}_\ell/2 \ (\ell = 1, 2, \ldots, N) \),

\[
\vec{\Sigma}(\lambda) = \sum_{\ell=1}^{N}\omega_{N,\ell}^{\lambda}\mathbf{J}_\ell.
\]  

The parameter \( \lambda \) takes the values \( \lambda = 1, 2, \ldots, N \) with modulo \( N \) and the \( \lambda = N \) case reduces to the usual total angular momentum operator \( \Sigma(N) = \mathbf{J} = \sum_{\ell=1}^{N}\mathbf{\sigma}_\ell/2 \). They satisfy \( \vec{\Sigma}(\lambda)^\dagger = \vec{\Sigma}(N-\lambda) \) and the commutation relation

\[
[a \cdot \vec{\Sigma}(\lambda), b \cdot \vec{\Sigma}(\lambda')] = i(a \times b) \cdot \vec{\Sigma}(\lambda + \lambda')
\]  

for all numerical vectors \( a \) and \( b \). With this commutation relation, \( \vec{\Sigma}(\lambda) \) form a so-called graded Lie algebra labeled by the integers \( \lambda \) (mod \( N \)).

Denoting the kets for the single spin-1/2 states with \( m = 1/2 \) and \( m = -1/2 \) by \( |0\rangle \) and \( |1\rangle \) as before, the ket for the state with maximal values of both \( j \) and \( m \), i.e. \( j_1 = m_1 = N/2 \) is

\[
|0_N\rangle = |0\rangle \otimes |0\rangle, \quad (32)
\]

and successive applications of the lowering operator \( J_- = J_x - i J_y \) yield all states for the maximal angular momentum space

\[
|j_1, m_1\rangle = \frac{(j_1 + m_1)!}{j_1!(j_1 - m_1)!} \langle j_1 \rangle^{j_1 - m_1}|0_N\rangle. \quad (33)
\]

The highest states with \( j_2 = m_2 = N/2 - 1 \) for the second-largest angular momentum space are given by the action of \( N - 1 \) lowering operators

\[
\Sigma_-(\lambda) = \Sigma_i(\lambda) - i \Sigma_y(\lambda) = \sum_{\ell=1}^{N}\omega_{N,\ell}^{\lambda}\mathbf{J}_\ell
\]  

onto the highest state \( |0_N\rangle \) in the largest angular momentum space as

\[
|j_2, m_2; \lambda\rangle = \frac{1}{\sqrt{N}}\Sigma_-(\lambda)|0_N\rangle \quad (\lambda = 1, 2, \ldots, N - 1), \quad (35)
\]

and successive applications of \( J_- \) give all the remaining states \( |j_2, m_2; \lambda\rangle \) with \( m_2 = -j_2, -j_2 + 1, \ldots, j_2 \). Since \( \Sigma_- \) and \( J_- \) commute with each other, the resulting states are

\[
|j_2, m_2; \lambda\rangle = \frac{(j_2 + m_2)!}{N(2j_2)!(j_2 - m_2)!} \Sigma_-(\lambda)\mathbf{J}_{-m_2}^{j_2-m_2}|0_N\rangle
\]

\[
= \frac{2j_1 - 1}{(j_1 + m_2 + 1)(j_1 + m_2)} \Sigma_-(\lambda)|j_1, m_2 + 1\rangle, \quad (36)
\]

which are orthonormal in the \( j_2 \) subsystem,

\[
\langle j_2, m_2; \lambda'|j_2, m_2; \lambda\rangle = \delta_{m_2,m_2'}\delta_{\lambda\lambda'}, \quad (37)
\]

\[
\sum_{\lambda=1}^{N-1} \sum_{m_2=-j_2}^{j_2} |j_2, m_2; \lambda\rangle\langle j_2, m_2; \lambda| = I_j_n.
\]
Here, $I_{j_2}$ is the projector onto the angular momentum $j_2$ subsystem. This projector is a polynomial of the Casimir operator $J^2 = \vec{J} \cdot \vec{J}$,

$$I_{j_2} = \prod_{j \neq j_2, j \neq j_2} \frac{J^2 - j(j + 1)}{j_2(j_2 + 1) - j(j + 1)}. \tag{38}$$

Their orthogonality is more transparent when the states (36) are written as

$$|j_2, m_2; \lambda\rangle = \frac{1}{\sqrt{N}} \sum_{\ell = 1}^{N} |\ell; j_2, m_2\rangle \omega^\ell_N, \tag{39}$$

where the $N$ states $|\ell; j_2, m_2\rangle$ form a pyramid,

$$\langle \ell; j_2, m_2 | \ell'; j_2, m_2' \rangle = \delta_{m_2 m_2'} \left( \delta_{\ell \ell'} + \frac{j_2 - m_2}{j_2 + m_2 + 1} \right). \tag{40}$$

From the Weyl–Schur duality, the second-largest angular momentum states $|j_2, m_2; \lambda\rangle$ correspond to the partition $\nu = (N - 1, 1)$ in the decomposition (3), and they are $(N - 1)$-dimensional irreducible representations of the symmetric group $S_N$.

We note that the discrete Fourier transformation that we chose in (34) is just one of many possibilities for defining the $\Sigma_\nu(\lambda)$ and thus the kets $|j_2, m_2; \lambda\rangle$. More generally, any unitary $(N - 1) \times (N - 1)$ matrix, with $N$th-row matrix elements $U_{N1} = 1$, can serve in $\Sigma_\nu(\lambda) = \sum_{i=1}^{N} U_{ij} \sigma_i^{(j)}$. For the specific choice of the discrete Fourier matrix, the projectors $|j_2, m_2; \lambda\rangle \langle j_2, m_2; \lambda|\rangle$ are invariant under the cyclic permutation subgroup of order $N$ generated by $P_{23 \ldots N}$.

Following the construction for the $N = 3$ case, it is natural to look for the representation of $S_N$ that possesses a cyclic permutation symmetry of order $N$. In fact, without exploring the representation theory, we can immediately construct the $(N - 1)^2$ RFF basis operators from the states (36) by tracing over the quantum number $m_2$,

$$Q^{(N-1,1)}_{\lambda'} = \sum_{m_2 = -j_2}^{j_2} |j_2, m_2, \lambda\rangle \langle j_2, m_2, \lambda'| = \left( Q^{(N-1,1)}_{\lambda} \right)^\dagger. \tag{41}$$

Indeed, we can check the properties

$$[\vec{J}, Q^{(N-1,1)}_{\lambda'}] = 0,$$

$$Q^{(N-1,1)}_{\lambda'} Q^{(N-1,1)}_{\lambda''} = \delta_{\lambda' \lambda''} Q^{(N-1,1)}_{\lambda'}. \tag{42}$$

The explicit construction in (36) of the angular momentum states in the subspace with $j_2 = N/2 - 1$ enables us to express the MLO for the second-largest angular momentum subspace as a linear combination of diagonal elements of the RFF basis operators with different coefficients. It is then straightforward but rather tedious to rewrite it in terms of individual Pauli spin operators. In the next subsection, we provide an explicit construction of the MLO for the second-largest angular momentum subspace in terms of cyclic permutation operators.

### 3.1. MLO for the second-largest angular momentum subspace

In the construction of the second-largest angular momentum states in (36), the cyclic permutation of order $N$ is respected according to the subgroup chain of the permutation group, i.e. $S_N \supseteq C_N$. Utilizing this fact we now construct the MLO for the second-largest angular momentum subspace in terms of the Pauli operators. In the following, we restrict ourselves to representations of the permutation group elements within the second-largest angular momentum subspace whose dimension is $N - 1$. 


The cyclic permutation operator $C$ that transforms the index $(1, 2, \ldots, N - 1, N)$ to $(2, 3, \ldots, N, 1)$ is given by

$$C = P_{23\ldots N1} = P_{N-1}P_{N-2} \ldots P_{2}P_{21}, \quad (43)$$

where, as before, $P_{ij}$ denotes the transposition operator between the two indices $(i, j)$,

$$P_{ij} = \frac{1}{2} (I_{2}^{N} + \vec{\sigma}_{i} \cdot \vec{\sigma}_{j}). \quad (44)$$

As stated, we choose a diagonal representation of the above cyclic permutation within the second-largest angular momentum subspace as

$$C \equiv_{j_{2}} \begin{pmatrix} \omega_{N} & \omega_{N}^{2} & 0 & \cdots & \omega_{N}^{N-1} \\ 0 & \omega_{N} & \cdots & \omega_{N}^{N-2} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{N}^{N-1} & \omega_{N}^{N-2} & \cdots & \omega_{N} & \omega_{N} \end{pmatrix} \otimes I_{N-1}, \quad (45)$$

where '$\equiv_{j_{2}}'$ indicates the restriction to the $j_{2}$ subspace. The symmetrically constructed second-largest angular momentum states are the eigenstates of cyclic permutation operator $C_{j_{2}, m_{2}; \lambda} = | j_{2}, m_{2}; \lambda \rangle \omega_{N}^{\lambda}$.

This is a consequence of the permutation invariance for the largest angular momentum states and the commutation relation $C \Sigma_{-}(\lambda) = \omega_{N}^{\lambda} \Sigma_{-}(\lambda) C$.

It is straightforward to construct $N$ orthogonal projectors by the inverse Fourier transform of the permutation operators $\{C, C^{2}, \ldots, C^{N-1}, C^{N} = I_{2N}\}$,

$$P_{c}(\lambda) = \frac{1}{N} \sum_{k=1}^{N} \omega_{N}^{-k} C_{k}^{\lambda}. \quad (47)$$

They have the following representation within the second-largest angular momentum subspace

$$P_{c}(\lambda) \equiv_{j_{2}} \begin{pmatrix} \delta_{\lambda, 1} & 0 & \cdots & 0 \\ 0 & \delta_{\lambda, 2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \delta_{\lambda, N-1} \end{pmatrix} \otimes I_{N-1}, \quad (48)$$

and satisfy the orthogonal relation

$$P_{c}(\lambda) P_{c}(\lambda') = \delta_{\lambda, \lambda'} P_{c}(\lambda). \quad (49)$$

We thus obtain

$$P_{c}(\lambda)| j_{2}, m_{2}; \lambda \rangle = | j_{2}, m_{2}; \lambda \rangle \delta_{\lambda, \lambda'}. \quad (50)$$

These relations are obtained directly from the commutation relation $P_{c}(\lambda) \Sigma_{-}(\lambda') = \Sigma_{-}(\lambda') P_{c}(\lambda - \lambda')$ and the observation that $P_{c}(\lambda)| j_{1}, m_{1} \rangle = | j_{1}, m_{1} \rangle \delta_{\lambda, N}$. We remark that the rank of the projectors $P_{c}(\lambda) (\lambda = 1, 2, \ldots, N - 1)$ is greater than $N - 1$ in general. It follows that they project onto not only the second-largest angular momentum subspace but other lower angular momentum subspaces as well.

With the above result, we can express the MLO for the second-largest angular momentum subspace as

$$M_{j_{2}} = \sum_{\lambda=1}^{N-1} \bar{q}_{\lambda}^{(N-1,1)} P_{c}(\lambda). \quad (51)$$
with \( N - 1 \) different coefficients \( q_k^{(N-1,1)} \). The choice \( q_k^{(N-1,1)} = j_2 + 1 - \lambda \) reads

\[
M_{j_2} = \sum_{k=1}^{N-1} \gamma_k C^k,
\]

\[
\gamma_k = \frac{1}{N} \sum_{\lambda=1}^{N-1} \left( \frac{N}{2} - \lambda \right) \alpha_N^{k \lambda}.
\]

The coefficients \( \gamma_k \) have the properties \( \gamma_N = 0 \) and \( \gamma_{N-k} = -\gamma_k \), and the MLO is stated as the summation of \( C^k - C^{N-k} \) with certain coefficients.

### 4. Symmetric coupling for four spin-1/2 systems

In this section the symmetric coupling for four spin-1/2 systems is accomplished along the ideas described in section 2. The standard binary coupling of four angular momenta is known under names such as the L–S coupling and the j–j coupling [1, 2]. In the first step, two angular momenta \( J_1 \) and \( J_2 \) are coupled and similarly the remaining two \( J_3 \) and \( J_4 \) are coupled. Then, the newly coupled angular momenta \( J_{12} \) and \( J_{34} \) are coupled in the last stage. With this particular choice of intermediate angular momentum states, the CSCO are

\[
\text{CSCO}_{1234} = \{ J^2, J_z, J^z_{12}, J^z_{34} \}.
\]

Since there are three inequivalent ways of pairing four angular momenta as the intermediate states, there are two more choices for CSCO, namely \( \text{CSCO}_{1324} \) and \( \text{CSCO}_{1423} \).

From the Weyl–Schur duality, the partition of 4 into 2 is \( \text{Par}(4, 2) = \{(4, 0), (3, 1), (2, 2)\} \), and the corresponding dimensions of the subspaces for the symmetric group are \( s(\nu) = 1, 3, 2 \), respectively. The total number of basis operators to span the RFF subsystems is thus \( 1^2 + 3^2 + 2^2 = 14 \). As noted earlier, this number is smaller than the number of elements in \( S_4 \), namely \( 14 < 4! = 24 \). The cyclic permutation subgroup of order 4,

\[
C_4 = \{ P_{1234}, P_{2341}, P_{3412}, P_{4123} \},
\]

is used to diagonalize the subspace \((3, 1) \) (\( j = 1 \)), and similarly the cyclic permutation subgroup of order 3,

\[
C_3(123) = \{ P_{1234}, P_{2314}, P_{3124} \},
\]

is used for the subspace \((2, 2) \) (\( j = 0 \)). We remark that there are two other cyclic permutation subgroups of order 4 and three others of order 3. However, all other choices lead essentially to the same result. Following the subgroup chain (13) together with the last transposition permutation subgroup \( S_2(12) \), we obtain the following unitary representations for the permutation operators:

\[
P_{2341} \equiv I_1 \otimes I_5 \oplus \begin{pmatrix} \omega_4 & 0 & 0 \\ 0 & \omega_3^2 & 0 \\ 0 & 0 & \omega_4 \end{pmatrix} \otimes I_3 \oplus \begin{pmatrix} 0 & \omega_3^2 \\ \omega_3 & 0 \\ 0 \end{pmatrix} \otimes I_1,
\]

\[
P_{2314} \equiv I_1 \otimes I_5 \oplus \frac{1}{2} \begin{pmatrix} i & 1 & -i \\ 1 & 0 & 1+i \\ 1 & 1+i & -i \end{pmatrix} \otimes I_3 \oplus \begin{pmatrix} \omega_3 & 0 \\ 0 & \omega_3^2 \\ 0 \end{pmatrix} \otimes I_1,
\]

\[
P_{2134} \equiv I_1 \otimes I_5 \oplus \frac{1}{2} \begin{pmatrix} 1 & 1 & -i \\ 1+i & 0 & 1-i \\ -i & 1+i & 1 \end{pmatrix} \otimes I_3 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes I_1.
\]
and all other elements of the permutation group $S_4$ can be generated with these three elements. Using the above choice of the representations, the basis operators $Q_i^j$ can be expressed in terms of the permutation operators in a straightforward, if somewhat tedious, manner. For completeness, we list all relations using the notations where $I = P_{1234}, P_j$ for the transposition permutations, and

\[
C_1 = P_{341}, C_1^2 = P_{3412}, C_1^3 = P_{4123},
\]

\[
C_2 = P_{2413}, C_2^2 = P_{4321}, C_2^3 = P_{3142},
\]

\[
C_3 = P_{4231}, C_3^2 = P_{2143}, C_3^3 = P_{3412},
\]

\[
D_1 = P_{1342}, D_1^2 = P_{4123},
\]

\[
D_2 = P_{3241}, D_2^2 = P_{4213},
\]

\[
D_3 = P_{3421}, D_3^2 = P_{4312},
\]

\[
D_4 = P_{2314}, D_4^2 = P_{3214}.
\]

(57)

for cyclic permutations as follows. For the subspaces with $\nu = (4, 0)$ and $\nu = (3, 1)$, we have two different representations in terms of odd and even permutation operators as

\[
Q^{(4,0)}_{11} = \frac{1}{12} \left[ I + \sum_{k=1}^{3} C_k^2 + \sum_{l=1}^{4} (D_l + D_l^2) \right]
\]

\[
= \frac{1}{12} \left[ \sum_{k=1}^{3} (C_k + C_k^3) + \sum_{i>j} P_{ij} \right],
\]

\[
Q^{(3,1)}_{11} = \frac{1}{4} (I - C_i^2) + i \sum_{l=1}^{4} (D_l - D_l^2)
\]

\[
= \frac{1}{8} (P_{12} + P_{23} + P_{34} + P_{41}) - \frac{1}{8} (C_2 + C_2^3 + C_3 + C_3^3) + \frac{i}{4} (C_1 - C_1^3),
\]

\[
Q^{(3,1)}_{22} = \frac{1}{4} (I + C_i^2 - C_i^2 - C_i^3)
\]

\[
= \frac{1}{4} (P_{13} + P_{24} - C_i - C_i^3),
\]

\[
Q^{(3,1)}_{13} = (Q^{(2,1)}_{31})^\dagger = \frac{1}{8} \sum_{l=1}^{4} (-1)^l (D_l + D_l^2) + i \frac{3}{4} (C_2 - C_2^3)
\]

\[
= \frac{1}{4} (P_{13} - P_{24}) - \frac{i}{16} (P_{12} - P_{23} + P_{34} - P_{41}) - \frac{i}{16} (C_2 + C_2^3 - C_3 - C_3^3),
\]

\[
Q^{(1,2)}_{12} = (Q^{(1,2)}_{31})^\dagger = \frac{1}{4} (1 \pm i) \sum_{l=1}^{3} \omega_l^0 D_l + \frac{1}{4} (1 \mp i) \sum_{l=1}^{3} \omega_l^2 D_l^2
\]

\[
= \frac{1}{4} (1 + i)(P_{12} - P_{34}) + (1 - i)(P_{23} - P_{41})
\]

\[
\mp \frac{1}{4} [(1 - i)(C_2 - C_2^3) + (1 + i)(C_3 - C_3^3)].
\]

(58)

For the $\nu = (2, 2)$ subspace, we have
Q_{(2,2)}^{(2,2)} = \frac{1}{12} \left( I + \sum_{k=1}^{3} C_{k}^{2} \right) + \frac{\omega \lambda}{12} (D_{1} + D_{2}^{2} + D_{3} + D_{4}^{2}) + \frac{\omega_{2}}{12} (D_{2}^{2} + D_{4} + D_{5}^{2} + D_{6}) \right), \quad (\lambda = 1, 2)

Q_{12}^{(2,2)} = \frac{1}{12} \left( \frac{1}{2} (C_{3} + C_{1}^{3} + P_{12} + P_{34}) + \frac{\omega_{3}}{12} (C_{4} + C_{1}^{3} + P_{13} + P_{24}) \right)

As we see there are many ways of representing the Q_{12}^{(2,2)} by linear combinations of permutation operators, and here we choose the following representation to see representations for the RFF basis operators in terms of the Pauli spin operators.

Define the Hermitian operators $A_{j}^{\pm}$, $K_{j}$ and $L_{j}$ by

$A_{j}^{\pm} = P_{2134} \pm P_{1234}, \quad A_{2}^{\pm} = P_{3142} \pm P_{1324}, \quad A_{3}^{\pm} = P_{3421} \pm P_{1423},$

$K_{1} = i(P_{342} - P_{243}), \quad K_{2} = i(P_{241} - P_{214}),$

$K_{3} = i(P_{431} - P_{413}), \quad K_{4} = i(P_{312} - P_{321}),$

$L_{1} = P_{2143}, \quad L_{2} = P_{3412}, \quad L_{3} = P_{3421},$

where $A_{j}^{\pm}$ are combinations of elements of the transposition permutation subgroup, $K_{j}$ are those of the cyclic permutation subgroup of order 3 and $L_{j}$ are those of the two-cycle permutation operators. Their Pauli spin operator representations are

$K_{1} = -\frac{1}{2} \sigma_{2} \cdot (\sigma_{3} \times \sigma_{4}), \quad K_{2} = -\frac{1}{2} \sigma_{3} \cdot (\sigma_{4} \times \sigma_{1}),$

$K_{3} = -\frac{1}{2} \sigma_{4} \cdot (\sigma_{1} \times \sigma_{2}), \quad K_{4} = -\frac{1}{2} \sigma_{1} \cdot (\sigma_{2} \times \sigma_{3}),$

$L_{1} = \frac{1}{4}(I_{16} + \sigma_{1} \cdot \sigma_{2})(I_{16} + \sigma_{3} \cdot \sigma_{4}),$

$L_{2} = \frac{1}{4}(I_{16} + \sigma_{1} \cdot \sigma_{3})(I_{16} + \sigma_{2} \cdot \sigma_{4}),$

$L_{3} = \frac{1}{4}(I_{16} + \sigma_{2} \cdot \sigma_{3})(I_{16} + \sigma_{4} \cdot \sigma_{1}),$

besides the trivial ones for $A_{j}^{\pm}$. With these notations, we have

$Q_{11}^{(4,0)} = -\frac{1}{4} I_{16} + \frac{1}{6} (A_{1}^{+} + A_{2}^{+} + A_{3}^{+}) + \frac{1}{12} (L_{1} + L_{2} + L_{3}),$

$Q_{11}^{(3,1)} = \frac{1}{4} I_{16} + \frac{1}{8} (K_{1} + K_{2} + K_{3} + K_{4}) - \frac{1}{4} L_{2},$

$Q_{22}^{(2,2)} = \frac{1}{4} I_{16} - \frac{1}{4} (L_{1} + L_{2} + L_{3}),$

$Q_{12}^{(3,1)} = (Q_{21}^{(3,1)})^+, \quad Q_{23}^{(3,1)} = (Q_{12}^{(3,1)})^+$

This agrees with the result presented in section IV D of [13]. We remark that the RFF basis operators for the subspace (2, 2) ($j = 0$) can be expressed in terms of spin-0 singlet states.
The MLOs are constructed from the diagonal elements with the proper choice of coefficients. The most natural form among many different constructions is

\[ 1 \times Q_{11}^{(1,1)} + 0 \times Q_{22}^{(3,1)} + (-1) \times Q_{33}^{(3,1)} = -\frac{1}{4} (K_1 + K_2 + K_3 + K_4) = M_{j=1}. \]

\[ 1 \times Q_{11}^{(2,2)} + (-1) \times Q_{22}^{(2,2)} = \frac{1}{\sqrt{12}} (K_1 - K_2 + K_3 - K_4) = M_{j=0}, \]

and hence the CSCO, up to trivial multiplicative factors, is obtained as

\[ \text{CSCO}_{\text{sym}} = \left\{ J^2, J_z, M_{j=1}, M_{j=0} \right\}. \]

Note that this construction of the MLO for the \( j = 1 \) subspace agrees with (52) of section 3.1.

The corresponding angular momentum states in the \( j = 1 \) subspace are

\[ |1, 1; \lambda \rangle = \frac{1}{2} \left( |0000\rangle \omega_4^2 + |0100\rangle \omega_2^3 + |0011\rangle \omega_4^2 + |0001\rangle \right), \]

\[ |1, 0; \lambda \rangle = \frac{1}{\sqrt{8}} \left[ \left( |1001\rangle - |0110\rangle \right) \omega_2^2 + 1 \right] + \left( |0101\rangle - |1010\rangle \right) \omega_2^3 + 1 \right], \]

\[ |1, -1; \lambda \rangle = -\frac{1}{2} \left( |1011\rangle \omega_2^3 + |1101\rangle \omega_4^2 + |1110\rangle \right), \]

with \( \lambda = 1, 2, 3 \), and those in the \( j = 0 \) subspace are

\[ |0, 0; \lambda \rangle = \frac{1}{\sqrt{6}} \left[ \left( |1001\rangle + |0110\rangle \right) \omega_4^3 + \left( |1010\rangle + |0101\rangle \right) \omega_2^2 + \left( |0011\rangle + |1100\rangle \right) \right] \]

with \( \lambda = 1, 2, 3 \).

From our construction, the symmetry of \( \text{CSCO}_{\text{sym}} \) is clear. To this end, it is more convenient to introduce the following five conjugacy classes of the permutation group \( S_4 \) based upon the disjoint cycles of the subgroups by

\[ Cl(1^4) = \{ P_{1234} \}, \]

\[ Cl(21^2) = \{ P_{1234}, P_{1243}, P_{3214}, P_{1432}, P_{1324}, P_{3241} \}, \]

\[ Cl(31) = \{ P_{1342}, P_{1423}, P_{2431}, P_{1231}, P_{2132}, P_{2314}, P_{1324} \}, \]

\[ Cl(4) = \{ P_{1342}, P_{1423}, P_{2431}, P_{1231}, P_{2132}, P_{3412} \}, \]

\[ Cl(2^2) = \{ P_{1243}, P_{3214}, P_{2134}, P_{2341} \}. \]

Then, besides a trivial symmetry under the identity \( P_{1234} \), \( M_{j=1} \) is invariant under the cyclic permutation subgroup of order 4 in (54),

\[ c_3 M_{j=1} c_4^{-1} = M_{j=1} \quad \text{with} \quad c_4 \in C_4, \]

and \( M_{j=0} \) is symmetric under the three-cycle conjugacy class and the two two-cycle conjugacy classes,

\[ c_3 M_{j=0} c_3^{-1} = M_{j=0} \quad \text{with} \quad c_3 \in Cl(31), \]

\[ c_2^2 M_{j=0} c_2^{-1} = M_{j=0} \quad \text{with} \quad c_2 \in Cl(2^2). \]

We remark that \( M_{j=1} \) is antisymmetric under \( P_{3214}, P_{1423} \in Cl(21^2) \) and \( P_{3241}, P_{2134} \in Cl(2^2) \), and \( M_{j=0} \) is antisymmetric under the conjugacy classes \( Cl(21^2) \) and \( Cl(4) \). These symmetries should be contrasted with the standard binary coupling scheme that gives \( \text{CSCO}_{1234} \) or the like. For identical spins, the MLOs of the kind \( J_{12}^2, J_{34}^2 \) are then only invariant under two of the classes in \( Cl(21^2) \) and one of the classes in \( Cl(2^2) \). This is our main conclusion: the higher symmetry in the MLOs distinguishes this coupling scheme from all other coupling schemes.
5. Summary and discussion

In this paper we have given a detailed analysis of the symmetric coupling of four spin-1/2 systems by revealing the symmetry of the MLOs. The result is based on our proposal for the symmetric coupling of many identical angular momenta in which the Weyl–Schur duality plays a central role. The relation between the MLOs and the RFF subsystems is also clarified through the analysis.

An immediate extension is to provide the MLOs for the coupling of four arbitrary angular momenta without using the binary coupling. This long-standing open problem can be tackled by first solving the case of four identical angular momentum systems and then analyzing the eigenvalues of the obtained MLOs in the non-identical case.

Another interesting problem is to study the possible coupling scheme for $N$ spin-1/2 systems in all subspaces along the ideas presented in this paper.

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