There is a close relation between duality in $N = 2$ SUSY gauge theories and integrable models. In particular, the quantum moduli space of vacua of $N = 2$ SUSY $SU(3)$ gauge theories coupled to two flavors of massless quarks in the fundamental representation can be related to the spectral curve of the Goryachev-Chaplygin top. Generalizing this to the cases with massive quarks, and $N_f = 0, 1, 2$, we find a corresponding integrable system in seven dimensional phase space where a hyperelliptic curve appears in the Painlevé test. To understand the stringy origin of the integrability of these theories we obtain exact nonperturbative point particle limit of type II string compactified on a Calabi-Yau manifold, which gives the hyperelliptic curve of $SU(2)$ QCD with $N_f = 1$ hypermultiplet.
Last several years we have witnessed very important progress in understanding S-duality of $N = 2$ SUSY gauge theories \cite{1}. The low energy description of these theories can be encoded by Riemann surfaces and the integrals of meromorphic one differentials over the periods of them. Exact effective actions of these theories can be described by holomorphic functions, so-called prepotentials. With these we can probe the strong coupling limit of the theories. There are other type of theories where this structure on Riemann surfaces plays a crucial role. These are the integrable models in two dimensions \cite{2}. Among the methods of solving integrable models, in inverse scattering method we obtain the solitons solutions as potentials of a quantum mechanics problem, given the scattering data. The spectral parameter plays the role of the energy. If we consider the periodic soliton solutions, then the spectral parameter develops forbidden zones, just as we are familiar in solid state physics. Analytic continuation of the spectral parameter with the forbidden zones gives us the Riemann surface with genus $g > 0$. By now there are many works which connect these low-energy effective theories with known integrable systems. To relate effective quantum field theories with integrable systems, one needs averaging over fast oscillations, i.e. Whitham averaging. It was analyzed that the periods of the modulated Whitham solution of periodic Toda lattice give rise to the mass spectrum in the BPS saturated states \cite{3,4}. Furthermore this framework of Whitham dynamics for the Toda lattice was generalized to other gauge groups \cite{5}. For the case of $SU(N_c)$ gauge theory with a single hypermultiplet in the adjoint representation, the corresponding integrable system was found \cite{6} and was recognized to be the elliptic spin Calogero model \cite{7}. This connection was developed in Ref. \cite{8} by identifying the coupling constant of Calogero system with the mass of a hypermultiplet in the adjoint representation, starting from the Lax operator for the Calogero model and calculating the spectral curve explicitly. The integrable system related to gauge theories coupled to massive hypermultiplets in the fundamental representation was discussed \cite{9,10}, and we will be explaining some of the results of Ref. \cite{9}.

Motivated by the works in SUSY gauge theories, the duality really blossomed in the context of string theories \cite{12}. Among these, the $N = 2$ type II/heterotic duality in four dimensions has been proposed in \cite{13} and further studied in many subsequent papers. In fact, it was extended to the $F$-theory/heterotic duality \cite{14} in eight dimensions where the heterotic strings compactified on $T^2$ is dual to $F$-theory compactified on $K3$ which admits an elliptic fibration. Further compactification in six dimensions leads to the duality
between $F$-theories compactified on Calabi Yau (CY) manifolds and heterotic strings on $K3$. Among the many ways to check the consistency on this duality one can consider the point like limit of four dimensional $N = 2$ SUSY compactifications of heterotic strings, and see the resulting gauge theory [13], which reproduces the exact field theory results of Ref. [1]. Additional question would be whether one can get also matter from the point like limit of the string theory compactification. We would like to see whether the $N = 2$ SUSY QCD is embedded in this compactification of string theory [14].

In this talk, we first consider the $N = 2$ SUSY $SU(N_c)$ gauge theories with $N_c$ colors and $N_f$ flavors. The field content of the theories consists, in terms of $N = 1$ superfields, a vector multiplet $W_\alpha$, a chiral multiplet $\Phi$, and two chiral superfields $Q^i_a$ and $\tilde{Q}_{ia}$ where $i = 1, \cdots, N_f$ and $a = 1, \cdots, N_c$. The superpotential reads,

$$W = \sqrt{2} \tilde{Q}_i \Phi Q^i + \sum_{i=1}^{N_f} m_i \tilde{Q}_i Q^i,$$

where $m_i$'s are the bare quark masses and color indices are suppressed. The curve representing the moduli space with $N_f < N_c$ case is as follows [17]:

$$y^2 = (x^{N_c} - \sum_{i=2}^{N_c} u_i x^{N_c-i})^2 - \Lambda_{N_f}^{2N_c-N_f} \prod_{i=1}^{N_f} (x + m_i),$$

where the moduli $u_i$'s are the vacuum expectation values of a scalar field of the $N = 2$ chiral multiplet, and $m_i$'s are the bare quark masses. It turns out that from the point of view of integrable theory, $u_i$'s correspond to the integrals of motion. The second term in Eq.(2) is due to the instanton corrections. For the $N_c \leq N_f < 2N_c$ case, the correction due to matter is different and the curve is given in Ref. [14]. By inspection we see that the case of $N_f = 0$ corresponds to the periodic Toda lattice with $N_c$-particles, after an appropriate rescaling of the variables [3]. In general the following type of hyperelliptic curve appears

$$y^2 = P_{N_c}(x)^2 - Q_m(x),$$

where $P_n(x)$ and $Q_m(x)$ are polynomials of order $n$ and $m$. It is natural to ask which integrable theories have such spectral curves. The form is indicative of Riemann surfaces with punctures as well as genus. We will start with the known case of $y^2 = P_3(x)^2 - ax^2$ ($a$ is a constant) which corresponds to the so called Goryachev-Chaplygin (GC) top. It was noted in Ref. [18] that there exists such a connection.
Let us review the classical mechanics of rotation of a heavy rigid body around a fixed point, which is described by the following Hamiltonian:

\[ H(M, p) = \frac{M_1^2}{2I_1} + \frac{M_2^2}{2I_2} + \frac{M_3^2}{2I_3} + \gamma_1 p_1 + \gamma_2 p_2 + \gamma_3 p_3. \]  

(4)

The phase space of this system is six dimensional: \( M_i \)'s are the components of the angular momentum and \( p_i \)'s are the linear momenta. \( I_i \)'s are the principal moments of inertia of the body and \( \gamma_i \)'s are the coordinates of the center of mass. There are four known integrable cases for the Hamiltonian in Eq.(4). In all these cases there is always one obvious integral of motion, the energy. It is necessary to get one extra integral independent of the energy for complete integrability according to Liouville’s theorem [2].

Apart from the better known cases of Euler’s and Lagrange’s tops, we have following two other cases: i) Kowalewski’s case: \( I_1 = I_2 = 2I_3, \gamma_3 = 0 \). The extra integral can be found by the Painlevé test or the Kowalewski’s asymptotic method. Here the symmetry group is \( SO(3, 2) \). ii) Goryachev-Chaplygin’s case: \( I_1 = I_2 = 4I_3, \gamma_3 = 0 \) We need 

\[ M_1 p_1 + M_2 p_2 + M_3 p_3 = 0, \]

which leads to a new integral of motion together with the Hamiltonian \( H \) and the GC integral \( G \) [19]. The Lax operator for the GC top is given as follows [20]:

\[
L(z) = \begin{pmatrix}
0 & -ip_3/z & M_2 - iM_1 \\
ip_3/z & 2iM_3 & -2iz + (p_2 - ip_1)/z \\
-M_2 - iM_1 & 2iz + (p_2 + ip_1)/z & -2iM_3
\end{pmatrix}.
\]  

(5)

This Lax operator depends on the phase space variables, \( M_i, p_i \) and on the spectral parameter, \( z \). Now it is easy to calculate the spectral curve from the equation \( \text{Det}(L(z) - xI) = 0 \), which gives the spectral curve as follows:

\[ x^3 + 2xH - 2iG - x(4z^2 + \frac{\lambda^2}{z^2}) = 0, \]

(6)

where \( H = \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) - 2p_1 \) is the Hamiltonian, and \( G = M_3(M_1^2 + M_2^2) + 2M_1p_3 \) is the GC integral. We also have the following constraints:

\[ p_1^2 + p_2^2 + p_3^2 = \lambda^2, \quad \text{and} \quad M_1 p_1 + M_2 p_2 + M_3 p_3 = 0. \]

(7)

Now we see that the spectral curve depends on special combinations of \( M_i, p_i \)'s, which are nothing but the integrals of motion. By introducing variable \( y = x(4z^2 - \frac{\lambda^2}{z^2}) \), we thus get
\[ y^2 = (x^3 + 2Hx - 2iG)^2 - 16\lambda^2 x^2, \quad (8) \]

which are the same as the curve for GC top with some rescalings. To relate this to the curve of SUSY gauge theory we make the following simple substitutions:

\[ H \rightarrow -\frac{1}{2}u_2, \quad G \rightarrow -\frac{i}{2}u_3, \quad \lambda^2 \rightarrow \frac{1}{16}\Lambda^4. \quad (9) \]

It is easy to see that Eq.(8) exactly coincides with Eq.(2) for the particular case of \( N_c = 3, N_f = 2 \) and \( m_1 = m_2 = 0! \)

Since we have seen the intimate relation between the GC top and the SUSY SU(3) gauge theory with two flavor massless hypermultiplets, it is natural for us to extend this to the massive case. For this purpose we need an integrable system which has both the GC top and the three body Toda lattice as particular limits, because the latter corresponds to pure gauge theory with no matter. The Hamiltonian system which realizes this is hard to imagine, but there exists a system of coupled seven nonlinear differential equations in mathematical literature [21]. This system has the following nonlinear “equations of motion”:

\[
\begin{align*}
\dot{z}_1 &= -8z_7, \quad \dot{z}_2 = 4z_5, \quad \dot{z}_3 = 2(z_4z_7 - z_5z_6), \quad \dot{z}_4 = 4z_2z_5 - z_7, \\
\dot{z}_5 &= z_6 - 4z_2z_4, \quad \dot{z}_6 = -z_1z_5 + 2z_2z_7, \quad \dot{z}_7 = z_1z_4 - 2z_2z_6 - 4z_3.
\end{align*}
\]

(10)

There are following five constants of motion of the system:

\[
\begin{align*}
6a &= z_1 + 4z_2^2 - 8z_4, \quad 2b = z_1z_2 + 4z_6, \quad c = z_4^2 + z_5^2 + z_3, \\
d &= z_4z_6 + z_5z_7 + z_2z_3, \quad e = z_6^2 + z_7^2 - z_1z_3.
\end{align*}
\]

(11)

Although the Lax operator for this system is not readily available, we can still apply the asymptotic method due to Kowalewski to this system and take \( z_i = t^{-n_i} \sum_{j=0}^\infty A_i^j t^j \) where \( n_i \)'s are positive integers [22,21]. Substituting these Laurent expansions into the system of Eqs.(10) and (11), one finds \( n_i = 1 \) for \( i = 1, 2, 3 \), \( n_i = 2 \) for \( i = 4, 5, 6, 7 \) and a relation between the coefficients of \( A_i^j \)'s. Then we obtain the Laurent solutions for this system with seven parameters, five of which are from the constants of motion, \( a, b, c, d, e \) and two additional ones \( x \) and \( y \) where they satisfy the equation for an hyperelliptic curve [21]:

\[ y^2 = (2x^3 - 3ax + b)^2 - 4(4cx^2 + 4dx + e). \]

(12)

We clearly see that with the following substitution this gives the algebraic curves given in Eq.(3) of \( N = 2 \) SUSY SU(3) gauge theories with massive quarks of two flavors of masses \( m_1 \) and \( m_2 \):
\[ y \rightarrow 2y, \quad a \rightarrow \frac{2}{3}u_2, \quad b \rightarrow -2u_3, \quad c \rightarrow \frac{1}{4}\Lambda_2^4, \quad d \rightarrow \frac{\Lambda_2^4}{4}(m_1 + m_2), \quad e \rightarrow \Lambda_2^4 m_1 m_2. \]

(13)

When we consider the case of \( c = 0 \), then this leads to gauge theory coupled to one massive quark of mass \( m_1 \) or massless one \((N_f = 1)\) after similar substitution. For the case of \( c = d = 0 \), the usual periodic Toda lattice is recovered, and for \( d = e = 0 \) we get back GC top. So clearly we have a unifying model of two seemingly different systems.

Now let us consider the point like limit of string theories, where the \( N = 2 \) SUSY QCD is embedded in a compactification of string theory. We obtain exact nonperturbative point particle limit of a four dimensional \( N = 2 \) SUSY compactification of heterotic strings. Using Heterotic/type II duality, we show how \( N = 2 \) SUSY QCD with one flavor of massless quark arises in type II string compactification on Calabi-Yau manifolds.

Such analyses were performed for the following two cases [13]: First is the case where the \( E_8 \times E_8 \) heterotic string compactified on \( K3 \times T^2 \) is dual to the type IIB(or type IIA) theory compactified on a CY manifold (or its mirror), which is the weighted projective space of weights \( 1,1,2,6,2,6 \) \([23, 24] \). The point like limit of this model was shown to yield the exact results of \([1]\) with pure \( N = 2 \) Yang-Mills theory with gauge group \( SU(2) \). Second case is where the weighted projective space has weights \( 1,1,2,8,12 \) \([24]\), the point like limit is known to be the that of pure \( N = 2 \) \( SU(3) \) Yang-Mills theory. By going to the conifold locus of the CY manifold and blowing it up, one can indeed obtain the algebraic curves for all the cases of \( SU(n) \) gauge groups \([23]\).

In order to relate these gauge theories with \textit{matter} with string compactification scheme on a CY manifold, we look for the known cases where the explicit forms of the discriminant and the Picard-Fuchs operators of the CY manifolds have been worked out. One of the strong candidate is that of the weighted projective space with weights \( 1,1,1,6,9 \) \([24, 23]\). This is because if we look at the discriminant locus in term of the coordinates describing the large moduli parameters, the singularity structure of this is identical to that of \( N = 2 \) \( SU(2) \) gauge theory coupled to single \((N_f = 1)\) flavor in the fundamental representation \([13]\). The dual is the heterotic \( E_8 \times E_8 \) string compactificated on \( K3 \) with \( SU(2) \) bundles with instanton numbers \((13, 11)\) has the hypermultiplet spectrum of \( \frac{9}{2}(56, 1) + \frac{7}{2}(1, 56) + 62(1, 1)\).

Let therefore consider the moduli space of the mirror of the weighted projective space with weights \( 1,1,1,6,9 \) CY manifold with Hodge numbers \( h_{1,1} = 2, \quad h_{2,1} = 272 \) whose
defining polynomial given as follows \[26\]:

\[
p = x_1^{18} + x_2^{18} + x_3^{18} + x_4^3 + x_5^2 - 18 \psi x_1 x_2 x_3 x_4 x_5 - 3 \phi x_1^6 x_2^6 x_3^6 = 0. \tag{14}
\]

This CY manifold has 2 vector multiplets whose scalar expectation values correspond to \( \psi \) and \( \phi \) and 273 hypermultiplets including a dilaton field. It is convenient to introduce the following variables that were used for the complex moduli of the mirror:

\[
x = \frac{3 \phi}{(18 \psi)^6}, \quad y = \frac{1}{(3 \phi)^3}. \tag{15}
\]

The discriminant can be written as \[26,24\]: \( \Delta = (1 - \bar{x})^3 - \bar{x}^3 \bar{y} \), where \( \bar{x} = 2^{4}3^{3}x \), \( \bar{y} = 3^{3}y \). For weak coupling, \( \bar{y} \to 0 \), there exists a triple singularity at \( \bar{x} = 1 \). The locus on which the CY manifold acquires a conifold point is where \( \Delta = 0 \).

In order to go to the point like limit of strings (\( \alpha' \to 0 \)) we would like to identify \( \bar{x} - 1 \) with the vacuum expectation value of 4D gauge theory \( u \) upto leading order of \( \alpha' \). In fact, to be dimensionally correct we need

\[
\bar{x} = 1 + \alpha' u + O(\alpha'^2) = 1 + \frac{\epsilon}{\Lambda_1^2} u + O(\epsilon^2), \tag{16}
\]

where \( \Lambda_1 \) is the renormalization scale parameter of the theory with \( N_f = 1 \). At the conifold locus, we have

\[
\bar{y} = \frac{(1 - \bar{x})^3}{\bar{x}^3} \frac{1}{u^3}. \tag{17}
\]

When we expand for \( \psi \) and \( \phi \) we get

\[
\psi = \frac{1}{18} e^{-\bar{x}} (1 + \epsilon \psi_1 + \cdots), \quad \phi = \frac{1}{3} e^{-1}(1 + \epsilon u + \cdots), \tag{18}
\]

where \( \psi_1 \) is independent of \( u \). With the expressions in the defining polynomial, we can now compare with the the curve of SUSY QCD along the line of Ref. \[25\]. From the requirement that the coefficient of the term linear in \( u \) be order of \( \epsilon \), we immediately see that the product of \( x_1^6 x_2^6 x_3^6 \) should be order of \( \epsilon \). Taking the following expansion,

\[
x_1 = \epsilon^{\frac{1}{18}} a_1 + \cdots, \quad x_2 = \epsilon^{\frac{1}{18}} a_2 + \cdots, \quad x_3 = a_3 (1 + \epsilon b_3 + \cdots),
\]

\[
x_4 = a_4 (1 + \epsilon b_4 + \cdots), \quad x_5 = a_5 (1 + \epsilon b_5 + \cdots), \tag{19}
\]

and by requiring that \( p \) has the following form up to the first power of \( \epsilon \), we recover the hyperelliptic curve for \( SU(2) \ N_f = 1 \) gauge theory:
\[ p = \epsilon \left( 2u - 2x^2 + \frac{\Lambda^3(x + m)}{\hat{z}} + v^2 + w^2 \right) + \mathcal{O}(\epsilon^2), \]  

(20)

once we fix the functions of \( a_i \)’s and \( b_i \)’s as follows:

\[
\begin{align*}
  a_1 &= (\hat{z})^{\frac{1}{18}}, \\
  a_2 &= \left( \frac{2^3 3 x^2}{\Lambda^3(x + m)} \right)^{\frac{1}{18}}, \\
  a_3 &= \left( -\frac{\Lambda^3(x + m)}{3 x^2 \hat{z}} \right)^{\frac{1}{18}}, \\
  2a_5 &= (-2)\frac{x^2}{6}, \\
  b_3 &= -\frac{x^2}{6}, \\
  b_4 &= \frac{y^2}{3a_4^2}, \\
  b_5 &= \frac{w^2}{2a_5^2}, \\
  \psi_1 &= \frac{x^2}{6} - \frac{y^2}{3a_4^2} - b_5.
\end{align*}
\]

(21)

Note that the change of variable \( y = \hat{z} - P_2(x) \) gives rise to the explicit form of the curve given in Eq.(2).

Now we consider the periods. As is the case of pure SUSY Yang-Mills theory [25], \( p = 0 \) differs from (2) by quadratic terms. On the other hand, the holomorphic 3-form [26] is

\[
\Omega = d \left( \ln \frac{\hat{z}}{\sqrt{Q(x)}} \right) \wedge \left[ \frac{dv \wedge dx}{\partial w} \right].
\]

(22)

In order to integrate \( \Omega \) over \( v \) by following similar arguments in [25], we solve for \( w \) from \( p = 0 \). Plugging this value of \( w \) into (22), then the integral over \( v \) becomes trivial. This leads to the following result:

\[
\int_y \Omega = dx d\ln \frac{\hat{z}}{\sqrt{Q(x)}} = d \left( x d\ln \frac{\hat{z}}{\sqrt{Q(x)}} \right)
\]

(23)

Now we see that the integral of \( \Omega \) on a 3-cycle of the CY manifold produces an integral of \( dS \) over the cycle of Riemann surface.

Now we conclude with a list of some further works to be done. i) If one wishes to obtain the prepotentials which are needed for exact effective action in SUSY gauge theory, we should consider quasiclassical \( \tau \) fuctions in the context of integrable theory as in the case of pure gauge theory [4]. It would be interesting to find out intimate relation between them by using the explicit form of Baker-Akiezer function for GC top [20]. ii) There are algebraic curves for higher rank cases with generic \( N_c \) and \( N_f \). The obvious thing to do would be to obtain a ‘higher’ dimensional generalization of GC top, at least for the massless cases. Although there exists multi-dimensional generalization [27] of Kowalewski top, it is not available for GC top. Nevertheless, with all the results from SUSY gauge theories pointing
to the existence of higher dimensional generalizations, it is quite tempting to speculate that there exists higher dimensional GC top. iii) As regards the massive cases, better understanding of the variables, \( z_i \)'s \((i = 1, \cdots, 7)\) in Eq.(10) are needed as well as the symmetry of the system. In this respect, the relation to the quadratic algebra might shed further light in the problem in Refs. [28,29]. Of course it would be nice to find similar integrable theories for other gauge theories coupled with real matter. iv) String duality and integrability of SUSY gauge models are closely related, but still needs further systematic investigation, especially when we have matter [16,25]. Especially it would be nice to compare results from field theory calculations [30,31].

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