ASYMPTOTIC PROFILES OF SOLUTIONS FOR THE GENERALIZED FORNBERG–WHITHAM EQUATION WITH DISSIPATION

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Abstract

We consider the Cauchy problem for the generalized Fornberg–Whitham equation with dissipation. This is one of the nonlinear, nonlocal and dispersive-dissipative equations. The main topic of this paper is an asymptotic analysis for the solutions to this problem. We prove that the solution to this problem converges to the modified heat kernel. Moreover, we construct the second term of asymptotics for the solutions depending on the degree of the nonlinearity. In view of those second asymptotic profiles, we investigate the effects of the dispersion, dissipation and nonlinear terms on the asymptotic behavior of the solutions.

Keywords: Fornberg–Whitham equation with dissipation; Cauchy problem; Asymptotic profiles.

1 Introduction

We consider the Cauchy problem for the following integro-differential equation:
\[
\begin{align*}
\frac{u_t}{u_t} + (|u|^{p-1}u)_x + \int_{\mathbb{R}} Be^{-(x-y)|u(y,t)|} u_y(y,t)dy &= \mu u_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \\
u(x,0) &= u_0(x), \quad x \in \mathbb{R},
\end{align*}
\]
(1.1)

where \( u = u(x,t) \) is a real-valued unknown function, \( u_0(x) \) is a given initial data, \( p > 2 \) and \( B, b, \mu > 0 \). The subscripts \( t \) and \( x \) denote the partial derivatives with respect to \( t \) and \( x \), respectively. This equation is one of a model for nonlinear waves taking into account the dispersive-dissipative processes as well as the convection effects. The purpose of this study is to analyze the large time asymptotic behavior of the solutions to (1.1). Based on that analysis, we would like to investigate the effects of the dispersion, dissipation and nonlinear terms on the asymptotic profiles of the solutions.

First of all, let us explain about the original problem and background of (1.1). If we take \( \mu = 0 \) and replace \( (|u|^{p-1}u)_x \) with \( \beta uu_x \) (\( \beta \neq 0 \)) in (1.1), we obtain the Fornberg–Whitham equation:
\[
\begin{align*}
\frac{u_t}{u_t} + \beta uu_x + \int_{\mathbb{R}} Be^{-(x-y)|u(y,t)|} u_y(y,t)dy &= 0, \quad x \in \mathbb{R}, \quad t > 0, \\
u(x,0) &= u_0(x), \quad x \in \mathbb{R},
\end{align*}
\]
(1.2)

The above equation (1.2) was derived by Whitham [23] and Fornberg–Whitham [2] in the late 1900s, as a mathematical model for so-called “breaking waves”. Here, roughly speaking, wave-breaking means blow-up of the spatial derivative of the solution, i.e. \( \limsup_{t \uparrow T_0} \|u_x(\cdot,t)\|_{L^\infty} = \infty \) for some \( T_0 > 0 \). Wave-breaking phenomena for equations with the nonlocal dispersion term, was first studied by Seliger [20]. He studied the following more general equation called the Whitham equation:
\[
\begin{align*}
\frac{u_t}{u_t} + \beta uu_x + \int_{\mathbb{R}} K(x-y)u_y(y,t)dy &= 0, \quad x \in \mathbb{R}, \quad t > 0, \\
u(x,0) &= u_0(x), \quad x \in \mathbb{R},
\end{align*}
\]
(1.3)

where \( K(x) \) is a given real even function. In [20], he presented that wave-breaking is possible for (1.3), by a formal argument. Also, the wave-breaking phenomena for solutions to (1.3) with a regular kernel \( K(x) = Be^{-b|x|} \) like (1.2) was studied in [22]. For another perspective on (1.3), see e.g. [5]. In addition, for the related results about more general nonlocal dispersive equations, we can also refer to [18].

A mathematically rigorous analysis of the wave-breaking phenomena for equations with the nonlocal dispersion term such as (1.2) was first performed by Constantin–Escher [1]. They gave a sufficient condition for the blow-up of solutions to (1.3). After that, their result was improved by Ma–Liu–Qu in
for any $2 \leq q \leq \infty$, where $C_0 = C_0(B, b, \beta, \mu)$ is a certain positive constant.

Moreover, in [4], the second and the third order asymptotic profiles of the solutions to (1.4) also have been obtained. Here, we remark that the similar results of them were also obtained for other Burgers type equations such as the KdV–Burgers equation in [9, 13] and the BBM–Burgers equation in [7]. In addition, the author and Itasaka also mentioned in [4] that the effect of the nonlocal dispersion term on the more higher-order asymptotic profiles. Furthermore, they compared the results for (1.4) with the ones of the following KdV–Burgers equation:

$$u_t + \frac{2B}{b}u_x + \beta uu_x + \frac{2B}{b^2}u_{xxx} = \mu u_{xx}, \quad x \in \mathbb{R}, \ t > 0,$$

(1.5)
Roughly speaking, the results given in [4] suggest that the difference between (1.4) and (1.5) appears from the third order asymptotic profiles of the solutions (for details see [4]).

On the other hand, compared with (1.4), our target equation (1.1) has the more general nonlinearity \((|u|^{p-1}u)_x\). In particular, for \(p > 2\), the nonlinearity seems weak because \((|u|^{p-1}u)_x\) decays faster than \(uu_x\) as the solution decays. For this reason, the asymptotic profile of the solution to (1.1) is expected to be different from that given in [4]. Therefore, it is worthwhile to study (1.1) to investigate the structure of the solution and some interaction between nonlinear and dispersion effects. In order to obtain the asymptotic profile for the solution to (1.1), we develop the method used in [15] for the Cauchy problem of some dispersive-dissipative equations, such as the following generalized KdV–Burgers equation:

\[
\frac{\partial u}{\partial t} - \mu u_{xx} + u_{xxx} + (|u|^{p-1}u)_x = 0, \quad x \in \mathbb{R}, \quad t > 0,
\]

\[
u(x, 0) = u_0(x), \quad x \in \mathbb{R}.
\]

**Main Result.** Now, let us state our main result:

**Theorem 1.1.** Let \(p > 2\). Assume that \(u_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})\) and \(E_0 := \|u_0\|_{H^1} + \|u_0\|_{L^1}\) is sufficiently small. Then, (1.1) has a unique global mild solution \(u \in C([0, \infty); H^1(\mathbb{R}))\) satisfying

\[
\|\partial_x^l u(t, :t)\|_{L^2} \leq CE_0(1 + t)^{-\frac{l}{2} - \frac{1}{p}} , \quad t \geq 0, \quad l = 0, 1.
\]

Moreover, the solution \(u(x, t)\) satisfies the following estimate:

\[
\|u(\cdot, t)\|_{L^q} \leq CE_0(1 + t)^{-\frac{q}{2} - \frac{1}{p} - \frac{1}{4}}, \quad t \geq 0, \quad 2 \leq q \leq \infty.
\]

Furthermore, if \(xu_0 \in L^1(\mathbb{R})\), then the solution \(u(x, t)\) satisfies the following asymptotics:

\[
\lim_{t \to \infty} t^{\frac{1}{4} - \frac{1}{p} + \frac{1}{2} - \frac{1}{q}} \|u(\cdot, t) - MG_0(\cdot, t) + (|M|^{p-1}M) W_\rho(\cdot, t)\|_{L^q} = 0, \quad 2 < p < 3,
\]

\[
\lim_{t \to \infty} t^{\frac{1}{4} - \frac{1}{p} + \frac{1}{q} \log t} \|u(\cdot, t) - MG_0(\cdot, t) + \frac{M^3}{4\sqrt{3}\mu} (\log t) \partial_x G_0(\cdot, t)\|_{L^q} = 0, \quad p = 3,
\]

\[
\lim_{t \to \infty} t^{\frac{1}{4} - \frac{1}{p} + \frac{1}{q} + \frac{1}{2} \log t} \|u(\cdot, t) - MG_0(\cdot, t) + (m + M) \partial_x G_0(\cdot, t) + \frac{2BM}{\rho^3} \partial_x^2 G_0(\cdot, t)\|_{L^q} = 0, \quad p > 3,
\]

for any \(2 \leq q \leq \infty\). Here, \(G_0(x, t)\) and \(W_\rho(x, t)\) are defined as follows:

\[
G_0(x, t) := G\left(x - \frac{2B}{\rho} l, t\right), \quad W_\rho(x, t) := t^{-\frac{1}{2}} \rho^{\frac{1}{2}} w_\rho\left(\frac{x - \frac{2B}{\rho} l}{\sqrt{t}}\right), \quad x \in \mathbb{R}, \quad t > 0,
\]

where \(G(x, t)\) and \(w_\rho(x)\) are defined by

\[
G(x, t) := \frac{1}{\sqrt{4\pi\mu t}} \exp\left(\frac{x^2}{4\mu t}\right), \quad w_\rho(x) := \frac{d}{dx} \left(\int_0^t (G(1 - s) * G^p(s))(x)ds\right).
\]

Also, we define the constants \(M\), \(m\) and \(M\) as follows:

\[
M := \int_\mathbb{R} u_0(x)dx, \quad m := \int_\mathbb{R} xu_0(x)dx, \quad \mathcal{M} := \int_0^\infty \int_\mathbb{R} (|u|^{p-1}u)(y, \tau)d\tau dy \quad \text{for} \quad p > 3.
\]

**Remark 1.2.** From the above result, we see that the first asymptotic profile of the solution to (1.1) is given by \(MG_0(x, t)\) for all \(p > 2\). As we can see from the shape of \(MG_0(x, t)\), the strongest effect is the dissipation effect and the dispersion term acts as a convection effect. This means that the dispersion effect is not so strong in the first term of the asymptotics. A result analogous to this conclusion has been obtained for the more general dispersive-dissipative nonlinear equation in Theorem 4.13 of [8]. On the other hand, the present result provides the more specific asymptotic formula for the solution, including the second asymptotic profiles.

**Remark 1.3.** The second asymptotic profiles of the solution are divided into three cases depending on \(p\). In particular, note that for \(2 < p \leq 3\), the effect of the dispersion term does not appear strongly in the second asymptotic profiles as well as in the first asymptotic profile. This can be said to indicate that the effect of the nonlinearity is stronger than the effect of the dispersion term. On the other hand, note that for \(p > 3\), the second asymptotic profile contains all of the dissipation, dispersion, and nonlinear effects.
Remark 1.4. Some similar results for (1.8), (1.9) and (1.10) and other related results are also obtained for several dissipative type equations, such as the generalized KdV–Burgers equation [14, 15], the generalized BBM–Burgers equation [15, 19] and also the convection-diffusion equation [24].

Remark 1.5. In the case of \( p = 3 \), we can actually prove the more stronger result than (1.9). More precisely, the more improved asymptotic rate can be obtained. For details, see Theorem 4.3 below.

Moreover, in view of the second asymptotic profiles, we are able to obtain the optimal asymptotic rates to the modified heat kernel \( G_0(x, t) \) as follows:

**Corollary 1.6.** Under the same assumptions in Theorem 1.1, we have the following estimate:

\[
\|u(\cdot, t) - MG_0(\cdot, t)\|_{L^p} = \begin{cases} 
\left( |M|^p \|u_0\|_{L^q} + o(1) \right) t^{-\frac{1}{2}(1-\frac{1}{q})-\frac{p-3}{2}}, & 2 < p < 3, \\
\left( \frac{|M|^3}{4\sqrt{3\pi} \mu} \|\partial_x G(\cdot, 1)\|_{L^q} + o(1) \right) t^{-\frac{1}{2}(1-\frac{1}{q})-\frac{1}{2} \log t}, & p = 3, \\
\left( \left\| (m + M)\partial_x G(\cdot, 1) + \frac{2BM}{b^3} \partial_x^3 G(\cdot, 1) \right\|_{L^q} + o(1) \right) t^{-\frac{1}{2}(1-\frac{1}{q})-\frac{1}{2}}, & p > 3,
\end{cases}
\]

as \( t \to \infty \), for any \( 2 \leq q \leq \infty \).

The rest of this paper is organized as follows. First, we prove the global existence and the time decay estimates for the solutions to (1.1) in Section 2. Next, in Section 3, we introduce some auxiliary lemmas and propositions to prove the main result. Finally, we give the proof of our main result Theorem 1.1 in Section 4. This section is divided into three subsections. Subsection 4.1 is for \( 2 < p < 3 \), Subsection 4.2 is for \( p = 3 \) and Subsection 4.3 is for \( p > 3 \). The main difficulty of the proof of Theorem 1.1, especially the proof of (1.10), is how to treat the nonlocal dispersion term \( \int \mathbb{R} B e^{-b|y|/u_0(y, t)} dy \). To overcome this difficulty, we transform this term to \( \frac{2BM}{b^3} \partial_x^3 u + \frac{2BM}{b^3} \partial_x u + \frac{2BM}{b^3} (b^2 - \partial_x^2)^{-1} \partial_x^3 u \) via the Fourier transform, and apply the idea of the asymptotic analysis for the generalized KdV–Burgers equation used in [15].

**Notations.** For \( 1 \leq p \leq \infty \), \( L^p(\mathbb{R}) \) denotes the usual Lebesgue spaces. Then, for \( f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \), we denote the Fourier transform of \( f \) and the inverse Fourier transform of \( g \) as follows:

\[
\hat{f}(\xi) := \mathcal{F}[f](\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx, \quad \mathcal{F}^{-1}[g](x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} g(\xi) d\xi.
\]

Also, for \( s \geq 0 \), we define the Sobolev spaces by

\[
H^s(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}); \|f\|_{H^s} : = \left( \int_{\mathbb{R}} (1 + \xi^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty \right\}.
\]

Moreover, we denote \( C([0, \infty); H^s(\mathbb{R})) \) as the space of \( H^s \)-valued continuous functions on \([0, \infty)\).

Throughout this paper, \( C \) denotes various positive constants, which may vary from line to line during computations. Also, it may depend on the norm of the initial data. However, we note that it does not depend on the space variable \( x \) and the time variable \( t \).

## 2 Global Existence and Decay Estimates

In this section, we would like to show the global existence and the decay estimates (1.6) and (1.7) of the solutions to (1.1). To discuss them, we consider the following integral equation associated with (1.1):

\[
u(t) = T(t) * u_0 - \int_0^t \partial_x T(t - \tau) * (|u|^p - 1) u(\tau) d\tau,
\]

where the integral kernel \( T(t, x) \) is defined by

\[
T(x, t) := \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1} \left[ \exp \left( -\mu t \xi^2 - i2Bbt \xi \right) \right](x).
\]

For this function, we note that the following estimate holds. The proof is the same as Lemma 2.2 in [3].
Lemma 2.1. Let $s$ be a non-negative integer. Suppose $f \in H^s(\mathbb{R}) \cap L^1(\mathbb{R})$. Then, the estimate
\[
\| \partial_x^l (T(t) * f) \|_{L^2} \leq C (1 + t)^{-\frac{d}{2} - \frac{1}{2}} \| f \|_{L^1} + C e^{-\mu t} \| \partial_x^l f \|_{L^2}, \quad t \geq 0
\] (2.3)
holds for any integer $l$ satisfying $0 \leq l \leq s$.

Now, let us prove the global existence and the decay estimates (1.6) and (1.7) of the solutions to (1.1). We give the proof of them by slightly modifying the method used in Theorem 2.2 of [4].

Proposition 2.2. Let $p \geq 2$. Assume that $u_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$ and $E_0 = \| u_0 \|_{H^1} + \| u_0 \|_{L^1}$ is sufficiently small. Then, (1.1) has a unique global mild solution $u \in C([0, \infty); H^1(\mathbb{R}))$ satisfying
\[
\| \partial_x^l u(\cdot, t) \|_{L^2} \leq C E_0 (1 + t)^{-\frac{d}{2} - \frac{1}{2}}, \quad t \geq 0, \ l = 0, 1.
\] (1.6)
Moreover, the solution satisfies the following estimate:
\[
\| u(\cdot, t) \|_{L^p} \leq C E_0 (1 + t)^{-\frac{d}{p} + \frac{1}{2}}, \quad t \geq 0, \ 2 \leq q \leq \infty.
\] (1.7)

Proof. We solve the integral equation (2.1) by using the contraction mapping principle for the mapping
\[
N[u] := T(t) * u_0 - \int_0^t \partial_x T(t - \tau) \cdot (|u|^{p-1} u)(\tau) d\tau.
\] (2.4)

Let us introduce the Banach space $X$ as follows:
\[
X := \left\{ u \in C([0, \infty); H^1(\mathbb{R})); \ \| u \|_X := \sup_{t \geq 0} (1 + t)^{\frac{1}{2}} \| u(\cdot, t) \|_{L^2} + \sup_{t \geq 0} (1 + t)^{\frac{d}{2}} \| u_x(\cdot, t) \|_{L^2} < \infty \right\}.
\] (2.5)

Now, we set $N_0 := T(t) * u_0$. Then, it follows from Lemma 2.1 that
\[
\exists C_0 > 0 \ s.t. \ \| N_0 \|_X \leq C_0 E_0.
\] (2.6)

In what follows, we apply the contraction mapping principle to (2.4) on the closed subset $Y$ of $X$ below:
\[
Y := \left\{ u \in X; \ \| u \|_X \leq 2 C_0 E_0 \right\}.
\]

In order to complete the proof, it is sufficient to show the following estimates:
\[
\| N[u] \|_X \leq 2 C_0 E_0, \ u \in Y,
\] (2.7)
\[
\| N[u] - N[v] \|_X \leq \frac{1}{2} \| u - v \|_X, \ u, v \in Y.
\] (2.8)

If we have shown (2.7) and (2.8), by using the Banach fixed point theorem, we can see that (1.1) has a unique global mild solution in $Y$ satisfying the $L^2$-decay estimate (1.6).

In the following, $E_0$ is assumed to be sufficiently small. First, from the Sobolev inequality
\[
\| f \|_{L^\infty} \leq \sqrt{2} \| f \|_{L^2}^{\frac{2}{p}} \| f' \|_{L^2}^{\frac{2}{p}}, \ f \in H^1(\mathbb{R}),
\] (2.9)
we have
\[
\| u(\cdot, t) \|_{L^\infty} \leq \| u \|_X (1 + t)^{-\frac{d}{2}}.
\] (2.10)

In addition to (2.10), we need to prepare the following estimates:
\[
\| (|u|^{p-1} u - |v|^{p-1} v)(\cdot, t) \|_{L^1} \leq C (\| u \|_X + \| v \|_X) \| u - v \|_X (1 + t)^{-\frac{d}{2} - \frac{1}{2}}, \ u, v \in Y,
\] (2.11)
\[
\| (|u|^{p-1} u - |v|^{p-1} v)(\cdot, t) \|_{L^2} \leq C (\| u \|_X + \| v \|_X) \| u - v \|_X (1 + t)^{-\frac{d}{2} - \frac{1}{2}}, \ u, v \in Y,
\] (2.12)
\[
\| \partial_x (|u|^{p-1} u - |v|^{p-1} v)(\cdot, t) \|_{L^1} \leq C (\| u \|_X + \| v \|_X) \| u - v \|_X (1 + t)^{-\frac{d}{2}}, \ u, v \in Y,
\] (2.13)
\[
\| \partial_x (|u|^{p-1} u - |v|^{p-1} v)(\cdot, t) \|_{L^2} \leq C (\| u \|_X + \| v \|_X) \| u - v \|_X (1 + t)^{-\frac{d}{2} - \frac{1}{2}}, \ u, v \in Y.
\] (2.14)

We shall prove only $L^1$-decay estimates (2.11) and (2.13), since we can prove (2.12) and (2.14) in the same way. First, we recall the following basic inequality:
\[
\| u^{p-1} u - |v|^{p-1} v \| \leq C (|u|^{p-1} + |v|^{p-1}) \| u - v \|.
\] (2.15)
For (2.11), by using the above inequality (2.15), the Schwarz inequality, (2.5) and (2.10), we have
\[
\| (|p|^{-1}u - |v|^{-1}v) (\cdot, t) \|_{L^1} \\
\leq C \| (|p|^{-1}u - |v|^{-1}v) (\cdot, t) \|_{L^1} + C \| (|p|^{-1}u - |v|^{-1}v) (\cdot, t) \|_{L^1} \\
\leq C \| u (\cdot, t) \|^2_{L^2} \| v (\cdot, t) \|_{L^2} + C \| v (\cdot, t) \|^2_{L^2} \| u (\cdot, t) \|_{L^2} \\
\leq C \left( \left\| u \right\|_X + \left\| v \right\|_X \right) \| u - v \|_X (1 + t)^{-\frac{p-3}{2}} (1 + t)^{-\frac{p-1}{2}}.
\]

For (2.13), noticing \( \frac{\partial L}{\partial L} \) using the Plancherel theorem and splitting the
\[
\int_{\mathbb{R}} \left\{ \right\} \sum_{\xi \in \mathbb{R}} |\xi|^l \exp \left( -2(t - \tau) \xi^2 \right) d\xi \leq C(1 + t - \tau)^{-\frac{p}{2} - \frac{1}{2}}, \quad j \geq 0,
\]
it follows from (2.11) and (2.13) that
\[
I_1(t) \leq \int_0^t \left( |\xi|^{l+1} \exp \left( -\mu(t - \tau) \xi^2 - \frac{12B \mu(t - \tau)}{b^2 + \xi^2} \right) \mathcal{F} \left[ |u|^{-1}u - |v|^{-1}v \right] (\xi, \tau) \right) d\tau \\
\leq \int_0^{t/2} \sup_{|\xi| \leq 1} \mathcal{F} \left[ |u|^{-1}u - |v|^{-1}v \right] (\xi, \tau) \left( \int_{|\xi| \leq 1} |\xi|^{2(l+1)} \exp \left( -2\mu(t - \tau) \xi^2 \right) d\xi \right) \frac{1}{\xi^2} d\tau \\
+ \int_{t/2}^t \sup_{|\xi| \leq 1} \left| (i\xi)^l \mathcal{F} \left[ |u|^{-1}u - |v|^{-1}v \right] (\xi, \tau) \right| \left( \int_{|\xi| \leq 1} |\xi|^2 \exp \left( -2\mu(t - \tau) \xi^2 \right) d\xi \right) \frac{1}{\xi^2} d\tau \\
\leq C \int_0^{t/2} (1 + t - \tau)^{-\frac{p}{2} - \frac{1}{2}} \| (|u|^{-1}u - |v|^{-1}v) (\cdot, \tau) \|_{L^1} d\tau \\
+ C \int_{t/2}^t (1 + t - \tau)^{-\frac{p}{2}} \| \frac{\partial L}{\partial L} (|u|^{-1}u - |v|^{-1}v) (\cdot, \tau) \|_{L^1} d\tau \\
\leq C \left( \left\| u \right\|_X + \left\| v \right\|_X \right) \| u - v \|_X \\
\times \left( \int_0^{t/2} (1 + t - \tau)^{-\frac{p}{2} - \frac{1}{2}} (1 + \tau)^{-\frac{p-3}{2}} d\tau + \int_{t/2}^t (1 + t - \tau)^{-\frac{p}{2} - \frac{1}{2}} (1 + \tau)^{-\frac{p-1}{2}} d\tau \right) \\
\leq C \left( \left\| u \right\|_X + \left\| v \right\|_X \right) \| u - v \|_X \\
\times \begin{cases} 
(1 + t)^{-\frac{p-3}{2}}, & 2 \leq p < 3, \\
(1 + t)^{-\frac{p}{2} - \frac{1}{2}} \log(2 + t), & p = 3, \\
(1 + t)^{-\frac{p}{2} - \frac{1}{2}}, & p > 3,
\end{cases}
\leq C \left( \left\| u \right\|_X + \left\| v \right\|_X \right) \| u - v \|_X (1 + t)^{-\frac{p-3}{2}}, \quad p \geq 2,
\]
for any $t \geq 0$ and $l = 0, 1$. Next, for $|\xi| \geq 1$, by using the Schwarz inequality, we have

$$
\left| (i\xi)^l \hat{I}(\xi, t) \right| = \left| (i\xi)^{l+1} \int_0^t \exp \left( -\mu(t - \tau)\xi^2 - \frac{12Bb(t - \tau)\xi}{b^2 + \xi^2} \right) \mathcal{F} \left[ |u|^{p-1}u - |v|^{p-1}v \right] (\xi, \tau) d\tau \right|
\leq \int_0^t |\xi| \exp \left( -\mu(t - \tau)\xi^2 \right) \left| (i\xi)^l \mathcal{F} \left[ |u|^{p-1}u - |v|^{p-1}v \right] (\xi, \tau) \right| d\tau
\leq \left( \int_0^t \xi^2 \exp \left( -\mu(t - \tau)\xi^2 \right) d\tau \right)^{\frac{1}{2}}
\times \left( \int_0^t \exp \left( -\mu(t - \tau)\xi^2 \right) \left| (i\xi)^l \mathcal{F} \left[ |u|^{p-1}u - |v|^{p-1}v \right] (\xi, \tau) \right|^2 d\tau \right)^{\frac{1}{2}}
\leq C \left( \int_0^t \exp \left( -\mu(t - \tau)\xi^2 \right) \left| (i\xi)^l \mathcal{F} \left[ |u|^{p-1}u - |v|^{p-1}v \right] (\xi, \tau) \right|^2 d\tau \right)^{\frac{1}{2}}.
$$

Therefore, it follows from (2.12) and (2.14) that

$$
I_2(t) \leq C \left( \int_0^t \exp \left( -\mu(t - \tau)\xi^2 \right) \left| (i\xi)^l \mathcal{F} \left[ |u|^{p-1}u - |v|^{p-1}v \right] (\xi, \tau) \right|^2 d\tau d\xi \right)^{\frac{1}{2}}
\leq C \left( \int_0^t \exp \left( -\mu(t - \tau) \right) \int_{|\xi| \geq 1} \left| (i\xi)^l \mathcal{F} \left[ |u|^{p-1}u - |v|^{p-1}v \right] (\xi, \tau) \right|^2 d\tau d\xi \right)^{\frac{1}{2}}
\leq C \left( \int_0^t \exp \left( -\mu(t - \tau) \right) \|\partial_x^l \left( |u|^{p-1}u - |v|^{p-1}v \right) (\cdot, \tau) \|^2_{L^2} d\tau \right)^{\frac{1}{2}}
\leq C \left( \left\| N[u] \right\|_X + \left\| v \right\|_X \right) \left\| u - v \right\|_X \left( \int_0^t \exp \left( -\mu(t - \tau) \right) (1 + \tau)^{-\frac{2p-1}{p} - \frac{l}{2}} d\tau \right)^{\frac{1}{2}}
\leq C \left( \left\| N[u] \right\|_X + \left\| v \right\|_X \right) \left\| u - v \right\|_X \left( 1 + t \right)^{-\frac{2p-1}{p} - \frac{l}{2}} p \geq 2, \right.

\text{(2.19)}
$$

for any $t \geq 0$ and $l = 0, 1$. Combining (2.16) through (2.19), we obtain

$$
\left\| \partial_x^l \left( N[u] - N[v] \right) (t) \right\|_{L^2} \leq C \left( \left\| N[u] \right\|_X + \left\| v \right\|_X \right) \left\| u - v \right\|_X \left( 1 + t \right)^{-\frac{2p-1}{p} - \frac{l}{2}} t \geq 0, \ l = 0, 1.
$$

Hence, there exists a positive constant $C_1 > 0$ such that

$$
\left\| N[u] - N[v] \right\|_X \leq C_1 \left( \left\| N[u] \right\|_X + \left\| v \right\|_X \right) \left\| u - v \right\|_X \leq 4C_0C_1E_0 \left\| u - v \right\|_X, \ u, v \in Y.
$$

Here, we choose $E_0$ which satisfies $4C_0C_1E_0 \leq 1/2$, then we have (2.8). Moreover, we can see that (2.7) holds from (2.8). Actually, taking $v = 0$ in (2.8), it follows that

$$
\left\| N[u] - N[0] \right\|_X \leq C_0E_0, \ u \in Y.
$$

Therefore, since $N[0] = N_0$, we obtain from (2.8) that

$$
\left\| N[u] \right\|_X \leq \left\| N[0] \right\|_X + \left\| N[u] - N[0] \right\|_X \leq 2C_0E_0, \ u \in Y.
$$

Thus, we get (2.7). This completes the proof of the global existence and of the $L^2$-decay estimate (1.6).

Finally, we shall prove (1.7). From the Sobolev inequality (2.9), we immediately obtain

$$
\left\| u(\cdot, t) \right\|_{L^\infty} \leq E_0 \left( 1 + t \right)^{-\frac{1}{2}}, \ t \geq 0.
$$

(2.20)

The estimate (1.7) for $2 < q < \infty$ can be obtained by (1.6), (2.20) and an interpolation inequality

$$
\left\| f \right\|_{L^q} \leq \left\| f \right\|_{L^\infty}^{\frac{2}{q}} \left\| f \right\|_{L^2}^{\frac{q}{2}}, \ 2 < q < \infty,
$$

as follows:

$$
\left\| u(\cdot, t) \right\|_{L^q} \leq \left\| u(\cdot, t) \right\|_{L^\infty}^{\frac{2}{q}} \left\| u(\cdot, t) \right\|_{L^2}^{\frac{q}{2}}
\leq CE_0 \left( 1 + t \right)^{-\frac{1}{2} (1 - \frac{1}{q})} (1 + t)^{-\frac{1}{2} \frac{1}{q}} \leq CE_0 \left( 1 + t \right)^{-\frac{1}{2} (1 - \frac{1}{q})}, \ t \geq 0.
$$

(2.22)

This completes the proof. \qed
3 Auxiliary Lemmas and Propositions

In this section, we prepare some auxiliary lemmas and propositions to prove the main result. First, we introduce the asymptotic formula for the integral kernel \( T(x, t) \). Now, we remark that

\[
\int_{\mathbb{R}} Be^{-b|x-y|} u(y, t) dy = \mathcal{F}^{-1} \left[ \frac{i2Bb}{b^2 + \xi^2} b(\xi) \right] (x)
\]

\[
= 2Bb(b^2 - \partial_x^2)^{-1} \partial_x u = \frac{2B}{b} \partial_x u + \frac{2B}{b}(b^2 - \partial_x^2)^{-1} \partial_x^3 u.
\]

(3.1)

Therefore, the integral kernel \( T(x, t) \) is defined by (2.2) can be rewritten by

\[
T(x, t) = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1} \left[ \exp \left( -\mu \xi^2 - \frac{i2Bt \xi}{b} + \frac{i2Bt \xi^3}{b(b^2 + \xi^2)} \right) \right] (x).
\]

By using the above expression, we can show the following estimates (for the proof, see Lemma 4.1 in [4]):

**Lemma 3.1.** Let \( l \) be a non-negative integer and \( 2 \leq q \leq \infty \). Then, we have

\[
\| \partial_x^l T(\cdot, t) \|_{L^q} \leq Ct^{-\frac{1}{2}(1-\frac{1}{q})-\frac{l}{2}}, \quad t > 0,
\]

(3.2)

\[
\| \partial_x^l (T(\cdot, t) - G_0(\cdot, t)) \|_{L^q} \leq Ct^{-\frac{1}{2}(1-\frac{1}{q})-\frac{l}{2}}, \quad t > 0,
\]

(3.3)

where \( T(x, t) \) and \( G_0(x, t) \) are defined by (2.2) and (1.11), respectively.

By virtue of the above lemma, we can prove the following approximation formula for the Duhamel term in the integral equation (2.1). The following result plays an important role of the proof of Theorem 1.1.

**Proposition 3.2.** Let \( p > 2 \). Assume that \( u_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R}) \) and \( E_0 = \| u_0 \|_{L^1} + \| u_0 \|_{L^1} \) is sufficiently small. Then, the solution \( u(x, t) \) to (1.1) satisfies

\[
\lim_{t \to \infty} t^{\frac{1}{2}(1-\frac{1}{q})+\frac{1}{2}} \left\| \int_0^t \partial_x (T - G_0) (t - \tau) * (|u|^{p-1} u) (\tau) d\tau \right\|_{L^q} = 0,
\]

(3.4)

for any \( 2 \leq q \leq \infty \), where \( T(x, t) \) and \( G_0(x, t) \) are defined by (2.2) and (1.11), respectively.

**Proof.** First, we split the integral as follows:

\[
\int_0^t \partial_x (T - G_0) (t - \tau) * (|u|^{p-1} u) (\tau) d\tau = \left( \int_0^{t/2} + \int_{t/2}^t \right) \partial_x (T - G_0) (t - \tau) * (|u|^{p-1} u) (\tau) d\tau =: D_1(x, t) + D_2(x, t).
\]

(3.5)

In order to evaluate \( D_1(x, t) \) and \( D_2(x, t) \), we need to prepare some decay estimates for \((|u|^{p-1} u)(x, t)\). It follows from the decay estimate (1.7) that

\[
\| (|u|^{p-1} u)(\cdot, t) \|_{L^q} \leq \left\| u(\cdot, t) \right\|_{L^\infty}^{p-2} \| u(\cdot, t) \|_{L^q}^{2} \leq C E_0 (1 + t)^{-\frac{p}{2}} (1 + t)^{-(1-\frac{1}{q})-\frac{l}{2}}, \quad t \geq 0, \quad 1 \leq q \leq \infty.
\]

(3.6)

Moreover, we can see that the following estimate holds:

\[
\| \partial_x (|u|^{p-1} u)(\cdot, t) \|_{L^r} \leq C E_0 (1 + t)^{-\frac{p}{2}(1-\frac{1}{q})-\frac{r}{2}}, \quad t \geq 0, \quad 1 \leq r \leq 2.
\]

(3.7)

Actually, we get from (1.7) and (1.6) that

\[
\| \partial_x (|u|^{p-1} u)(\cdot, t) \|_{L^2} = p \| (|u|^{p-1} u_x)(\cdot, t) \|_{L^2} \leq p \| u(\cdot, t) \|_{L^\infty}^{p-1} \| u_x(\cdot, t) \|_{L^2} \leq C E_0 (1 + t)^{-\frac{p}{2}} (1 + t)^{-\frac{1}{2}} \leq C E_0 (1 + t)^{-\frac{1}{2}}, \quad t \geq 0.
\]

(3.8)

Moreover, using the Schwarz inequality, similarly we have

\[
\| \partial_x (|u|^{p-1} u)(\cdot, t) \|_{L^1} = p \| (|u|^{p-1} u_x)(\cdot, t) \|_{L^1} \leq p \| u(\cdot, t) \|_{L^\infty}^{p-2} \| u(\cdot, t) \|_{L^2} \| u_x(\cdot, t) \|_{L^2}
\]
Then, we have $t > \frac{1}{2}$ for any inequality, (3.3) and (3.6) that

Let $D_1(x, t)$ and $D_2(x, t)$ in (3.5). First, for $D_1(x, t)$, it follows from Young’s inequality, (3.3) and (3.6) that

$$
\|D_1(\cdot, t)\|_{L^q} \leq \int_0^{t/2} \|\partial_x((T - G_0)(\cdot, t - \tau))\|_{L^q} \|(|u|^{p-1} u)(\cdot, \tau)\|_{L^r} d\tau
\leq CE_0 \int_0^{t/2} (t - \tau)^{-\frac{3}{2} + \frac{1}{1 - \frac{1}{q}} - \frac{1}{r}} (1 + \tau)^{-\frac{2}{q - 1}} d\tau
\leq CE_0 \left\{ \begin{array}{ll}
t^{\frac{1}{2} - \frac{3}{2} \frac{1}{r}} & 2 < p < 3, \\
t^{\frac{1}{2} - \frac{3}{2} \frac{1}{r}} \log(2 + t), & p = 3, \\
t^{\frac{1}{2} - \frac{1}{r}} & p > 3, \end{array} \right.
$$

for any $t > 0$ and $2 \leq q < \infty$. On the other hand, from Young’s inequality, (3.3) and (3.7), we get

$$
\|D_2(\cdot, t)\|_{L^q} \leq \int_0^{t} \|((T - G_0)(\cdot, t - \tau))\|_{L^q} \|\partial_x(|u|^{p-1} u)(\cdot, \tau)\|_{L^r} d\tau
\leq CE_0 \int_0^{t} (t - \tau)^{-\frac{3}{2} + \frac{1}{1 - \frac{1}{q}} - \frac{1}{r}} (1 + \tau)^{-\frac{2}{q - 1}} d\tau
\leq CE_0 t^{\frac{1}{2} - \frac{3}{2} \frac{1}{r}} + CE_0 \lim_{t \to \infty} \left\{ \begin{array}{ll}
t^{\frac{1}{2} - \frac{3}{2} \frac{1}{r}} & 2 < p < 3, \\
t^{\frac{1}{2} \log(2 + t)}, & p = 3, \\
t^{\frac{1}{2}} & p > 3, \end{array} \right.
$$

Finally, combining (3.5), (3.10) and (3.11), we arrive at

$$
\limsup_{t \to \infty} t^{\frac{1}{2} + \frac{1}{r}} \int_0^t \partial_x((T - G_0)(t - \tau) * (|u|^{p-1} u)(\tau)) d\tau \leq C \left( \frac{1}{q} + 1 = \frac{1}{2} + \frac{1}{r} \right)
$$

It means that the asymptotic formula (3.4) holds. This completes the proof.

Next, let us introduce several properties of the modified heat kernel $G_0(x, t)$ defined by (1.11). This function satisfies the following decay estimate (3.12). The proof is the same as the one for the usual heat kernel $K(x, t)$ defined by (1.12) (for details, see e.g. [6]).

**Lemma 3.3.** Let $k$ and $l$ be non-negative integers. Then, for $1 \leq q < \infty$, we have

$$
\|\partial^k x^l G_0(\cdot, t)\|_{L^q} \leq Ct^{-\frac{3}{2} + \frac{1}{1 - \frac{1}{q}}}, \quad t > 0.
$$

Moreover, we can prove the following asymptotic formula for $G_0(x, t)$:

**Proposition 3.4.** Let $l$ be a non-negative integer and $1 \leq q < \infty$. Suppose $u_0 \in L^1(\mathbb{R})$ and $xu_0 \in L^1(\mathbb{R})$. Then, we have

$$
\|\partial^l (G_0(t) * xu_0 - MG_0(t), t)\|_{L^q} \leq C\|xu_0\|_{L^1(\mathbb{R})} t^{-\frac{1}{2} + \frac{1}{q}} \quad t > 0,
$$

$$
\lim_{t \to \infty} t^{\frac{1}{2} + \frac{1}{q}} \|\partial^l x (G_0(t) * xu_0 - MG_0(t), t + M\partial x G_0(\cdot, t))\|_{L^q} = 0,
$$

where $G_0(x, t)$, $M$ and $m$ are defined by (1.11) and (1.13), respectively.
Proof. We shall only prove (3.14) because (3.13) can be derived by a standard way (see e.g. [6]). First, it follows from the definition of $M$ given by (1.13) that

$$G_0(t) * u_0 - MG_0(x, t) = \int_{\mathbb{R}} (G_0(x - y, t) - G_0(x, t)) u_0(y)dy. \tag{3.15}$$

Now, recalling Taylor’s theorem, we have

$$f(1) = f(0) + \int_0^1 f'(\theta)d\theta = f(0) + \int_0^1 f''(\theta)(1 - \theta)d\theta.$$ 

Then, applying the above formula for $f(\theta) := G_0(x - \theta y, t)$ for $\theta \in \mathbb{R}$, we obtain

$$G_0(x - y, t) = G_0(x, t) - y \left( \int_0^1 \partial_\theta G_0(x - \theta y, t)d\theta \right), \tag{3.16}$$

$$G_0(x - y, t) = G_0(x, t) - y\partial_\theta G_0(x, t) + y^2 \left( \int_0^1 \partial_\theta^2 G_0(x - \theta y, t)(1 - \theta)d\theta \right). \tag{3.17}$$

Since $xu_0 \in L^1(\mathbb{R})$, for any $\varepsilon_0 > 0$, there exists a constant $L = L(\varepsilon_0) > 0$ such that $\int_{|y| \geq L} |y\partial_\theta u_0(y)|dy < \varepsilon_0$. Now, we split the $y$-integral in (3.15) as follows:

$$G_0(t) * u_0 - MG_0(x, t) + m\partial_\theta G_0(x, t)$$

$$= \int_{|y| \leq L} \left( \int_0^1 \partial_\theta^2 G_0(x - \theta y, t)(1 - \theta)d\theta \right) y^2 u_0(y)dy$$

$$+ \int_{|y| \geq L} \left\{ \partial_\theta G_0(x, t) - \left( \int_0^1 \partial_\theta G_0(x - \theta y, t)d\theta \right) \right\} yu_0(y)dy. \tag{3.18}$$

where we have used the facts (3.16) and (3.17). Then, from (3.18) and Lemma 3.3, we can see that

$$\| \partial_\theta (G_0(t) * u_0 - MG_0(\cdot, t) + m\partial_\theta G_0(\cdot, t)) \|_{L^s}$$

$$\leq \int_{|y| \leq L} \left( \int_0^1 \|\partial_\theta^2 G_0(\cdot - \theta y, t)(1 - \theta)\|_{L^s} d\theta \right) |y|^2 |u_0(y)|dy$$

$$+ \int_{|y| \geq L} \left\{ \|\partial_\theta^2 G_0(\cdot, t)\|_{L^s} + \left( \int_0^1 \|\partial_\theta^{p+1} G_0(\cdot - \theta y, t)\|_{L^s} d\theta \right) \right\} |yu_0(y)|dy$$

$$\leq C L^2 \|u_0\|_{L^1} t^{-\frac{2}{3}(1-\frac{1}{q})} + \varepsilon_0 C t^{-\frac{2}{3}(1-\frac{1}{q})} + \varepsilon_0 C, \quad t > 0.$$ 

Thus, we finally arrive at

$$\limsup_{t \to \infty} t^\frac{2}{3}(1-\frac{1}{q}) + \varepsilon_0 \|G_0(t) * u_0 - MG_0(\cdot, t) + m\partial_\theta G_0(\cdot, t)\|_{L^s} \leq \varepsilon_0 C.$$ 

Therefore, we get (3.14), because $\varepsilon_0 > 0$ can be chosen arbitrarily small.

Finally, in this section, we would like to introduce some useful lemmas to prove the main theorem. To doing that, we first show that the solution $u(x, t)$ to (1.1) can be approximated by $MG_0(x, t)$.

**Proposition 3.5.** Let $p > 2$. Assume that $u_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$, $xu_0 \in L^1(\mathbb{R})$ and $E_0 = \|u_0\|_{H^1} + \|u_0\|_{L^1}$ is sufficiently small. Then, the solution $u(x, t)$ to (1.1) satisfies

$$\|u(\cdot, t) - MG_0(\cdot, t)\|_{L^s} \leq CE_1 \begin{cases} t^{-\frac{2}{3}(1-\frac{1}{q})} \frac{1}{\varepsilon_0^2}, & 2 < p < 3, \\ t^{-\frac{2}{3}(1-\frac{1}{q})} - \frac{1}{2} \log(2 + t), & p = 3, \\ t^{-\frac{2}{3}(1-\frac{1}{q}) - \frac{1}{4}}, & p > 3, \end{cases} \tag{3.19}$$

for any $t \geq 1$ and $2 \leq q \leq \infty$, where $G_0(x, t)$ and $M$ are defined by (1.11) and (1.13), respectively. Also, the above constant $E_1$ is defined by $E_1 := E_0 + \|xu_0\|_{L^1}$. 

Proof. We rewrite the integral equation (2.1) as follows:

\[
u(x, t) - MG_0(x, t) = (T - G_0)(t) \ast u_0 + G_0(t) \ast u_0 - MG_0(x, t) - \left( \int_0^{t/2} + \int_{t/2}^t \right) \partial_x T(t - \tau) \ast (|u|^{p-1} u)(\tau) d\tau
\]

\[= (T - G_0)(t) \ast u_0 + \{G_0(t) \ast u_0 - MG_0(x, t)\} + N_1(x, t) + N_2(x, t). \tag{3.20}\]

Then, from Lemma 3.1 and Young’s inequality, we can easily show

\[\|G_0(t) \ast u_0 - MG_0(x, t)\|_{L^q} \leq C\|u_0\|_{L^1} t^{-\frac{q}{2}}(1 + q - \frac{2}{p}) \leq \infty, \quad t > 0, \ 2 \leq q \leq \infty. \tag{3.31}\]

For the second term in the right hand side of (3.20), we just recall (3.13):

\[\|N_1(t)\|_{L^q} \leq C\int_0^{t/2} \|\partial_x T(\cdot, t - \tau)\|_{L^q} \|(|u|^{p-1} u)(\cdot, \tau)\|_{L^1} d\tau \leq CE_0 \int_0^{t/2} (t - \tau)^{-\frac{1}{2}}(1 + \tau)^{-\frac{2}{p} - 1} \ d\tau \leq CE_0 \int_0^{t/2} \left\{ \begin{array}{ll}
\frac{1}{2} \log(2 + 1) & , \quad 2 < p < 3, \\
\frac{1}{2} \log(2 + 1) & , \quad p = 3, \\
\frac{1}{2} \log(2 + 1) & , \quad p > 3,
\end{array} \right. \tag{3.22}\]

for any \(t > 0\) and \(2 \leq q \leq \infty\). On the other hand, for \(N_2(x, t)\), in the same way to get (3.11), we have from Young’s inequality, (3.2) and (3.7) that

\[\|N_2(t)\|_{L^q} \leq CE_0 \int_0^{t/2} (t - \tau)^{-\frac{1}{2}}(1 + \tau)^{-\frac{2}{p} - 1} \ d\tau \leq CE_0 t^{-\frac{1}{2}}(1 + p - \frac{2}{3}) \leq \infty, \quad t > 0, \ 2 \leq q \leq \infty. \tag{3.23}\]

Combining (3.20) through (3.23), we arrive at (3.19). This completes the proof. \(\square\)

By virtue of Proposition 3.5, we can get the following two lemmas, which will be used in the proofs of (1.8) and (1.9). The methods of the proofs of these lemmas are based on the techniques used for the generalized KdV–Burgers equation and the generalized BBM–Burgers equation (see Lemma 5.3 in [15]).

**Lemma 3.6.** Let \(p > 2\). Assume that \(u_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R}), x u_0 \in L^1(\mathbb{R})\) and \(E_0 = \|u_0\|_{H^1} + \|u_0\|_{L^1}\) is sufficiently small. Then, the solution \(u(x, t)\) to (1.1) satisfies

\[\|(|u|^{p-1} u)(\cdot, t) - (|MG_0|^{p-1} MG_0)(\cdot, t)\|_{L^q} \leq CE_1 \left\{ \begin{array}{ll}
t^{-\frac{1}{2}} \left(1 + \frac{2}{3}\right) \log(2 + t) & , \quad 2 < p < 3, \\
t^{-\frac{1}{2}} \left(1 + \frac{2}{3}\right) \log(2 + t) & , \quad p = 3, \\
t^{-\frac{1}{2}} \left(1 + \frac{2}{3}\right) & , \quad p > 3, \end{array} \right. \tag{3.24}\]

for any \(t \geq 1\) and \(2 \leq q \leq \infty\), where \(G_0(x, t)\) and \(M\) are defined by (1.11) and (1.13), respectively. Also, the constant \(E_1\) is defined by \(E_1 = E_0 + \|x u_0\|_{L^1}\).

**Proof.** By using the inequality (2.15), (1.7), Lemma 3.3 and Proposition 3.5, we obtain

\[\|(|u|^{p-1} u)(\cdot, t) - (|MG_0|^{p-1} MG_0)(\cdot, t)\|_{L^q} \leq C \left( \|u(\cdot, t)\|^{p-1}_{L^\infty} + \|MG_0(\cdot, t)\|^{p-1}_{L^\infty} \right) \|u(\cdot, t) - MG_0(\cdot, t)\|_{L^q} \]
for any $t \geq 1$ and $2 \leq q \leq \infty$. This completes the proof.

\begin{lemma}
Assume that $u_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$, $xu_0 \in L^1(\mathbb{R})$ and $E_0 = \|u_0\|_{H^1} + \|u_0\|_{L^1}$ is sufficiently small. Then, there exists a positive function $\eta \in L^\infty(0, \infty)$ satisfying $\lim_{t \to \infty} \eta(t) = 0$ such that the solution $u(x, t)$ to (1.1) satisfies

\begin{equation}
t^{\frac{p-1}{2}} \|(|u|^{p-1}u)(\cdot, t) - (|MG_0|^{p-1}MG_0)(\cdot, t)\|_{L^1} \leq \eta(t), \quad t > 0,
\end{equation}

where $G_0(x, t)$ and $M$ are defined by (1.11) and (1.13), respectively.

\end{lemma}

\begin{proof}
It follows from the inequality (2.15), (1.7), Lemma 3.3 and Proposition 3.5 that

\begin{equation}
t^{\frac{p-1}{2}} \|(|u|^{p-1}u)(\cdot, t) - (|MG_0|^{p-1}MG_0)(\cdot, t)\|_{L^1}
\leq C t^{\frac{p-1}{2}} \left( \|u\|_{L^p} + \|M\|_{L^p} \right) \|u(\cdot, t) - MG_0(\cdot, t)\|_{L^2}
\leq C t^{\frac{p-1}{2}} \left( \|u(\cdot, t)\|_{L^p} + \|M\|_{L^p} \|G_0(\cdot, t)\|_{L^p} \right) \|u(\cdot, t) - MG_0(\cdot, t)\|_{L^2}
\leq C t^{\frac{p-1}{2}} \left( E^{p-1}(1 + t)^{- \frac{p-2}{2}} (1 + t)^{- \frac{p}{2}} + \|M\|_{L^p} t^{\frac{p-2}{2}} t^{- \frac{p}{4}} \right) \cdot CE_1 \begin{cases}
\frac{t^{- \frac{p-2}{2}}}{t-1}, & 2 \leq p < 3,
\frac{t^{- \frac{p}{2}} \log(2 + t)}{t^{- \frac{p}{4}}}, & p = 3,
\frac{t^{- \frac{p}{4}}}{t^{- \frac{p}{4}}}, & p > 3
\end{cases}
\leq CE_1 \begin{cases}
\frac{t^{- \frac{p-2}{2}}}{t-1}, & 2 \leq p < 3,
\frac{t^{- \frac{p}{2}} \log(2 + t)}{t^{- \frac{p}{4}}}, & p = 3,
\frac{t^{- \frac{p}{4}}}{t^{- \frac{p}{4}}}, & p > 3
\end{cases}
=: g(t), \quad t \geq 1.
\end{equation}

Moreover, we have from (3.6) and Lemma 3.3 that

\begin{equation}
t^{\frac{p-1}{2}} \|(|u|^{p-1}u)(\cdot, t) - (|MG_0|^{p-1}MG_0)(\cdot, t)\|_{L^1}
\leq t^{\frac{p-1}{2}} \left\{ \|(|u|^{p-1}u)(\cdot, t)\|_{L^1} + \frac{t^{- \frac{p}{4}}}{t-1} \|(|MG_0|^{p-1}MG_0)(\cdot, t)\|_{L^1} \right\}
\leq t^{\frac{p-1}{2}} \left\{ CE_0 (1 + t)^{- \frac{p}{4}} + \|M\|_{L^p} \|G_0(\cdot, t)\|_{L^p} \right\}
\leq t^{\frac{p-1}{2}} \left\{ CE_0 t^{- \frac{p-2}{2}} + C|M|t^{- \frac{p}{4}} \right\} \leq CE_0 =: C_0, \quad t > 0.
\end{equation}

Therefore, combining (3.26) and (3.27), we can conclude that (3.25) is true with the following $\eta(t)$:

\begin{equation}
\eta(t) := \begin{cases}
g(t), & t \geq 1,
C_0, & 0 < t < 1.
\end{cases}
\end{equation}

This completes the proof.
\end{proof}

\section{Proof of the Main Result}

In this section, we shall prove our main result Theorem 1.1, i.e. we establish the asymptotic formulas (1.8), (1.9) and (1.10). This section is divided into three subsections below.
4.1 Proof of Theorem 1.1 for $2 < p < 3$

First in this subsection, we would like to prove Theorem 1.1 in the case of $2 < p < 3$, i.e. we shall prove (1.8). To doing that, we prepare the following approximation formula for the Duhamel term of (2.1).

**Proposition 4.1.** Let $2 < p < 3$. Assume that $u_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$, $xu_0 \in L^1(\mathbb{R})$ and $E_0 = \|u_0\|_{H^1}$ is sufficiently small. Then, the solution $u(x,t)$ to (1.1) satisfies

$$\lim_{t \to \infty} t^{\frac{1}{2}(1 - \frac{1}{p}) + \frac{p-2}{2p}} \left\| \int_0^t \partial_x T(t - \tau) \ast (|u|^{p-1} u) (\tau) d\tau - (|M|^{p-1} M) W_p (\cdot, t) \right\|_{L^q} = 0, \quad (4.1)$$

for any $2 \leq q \leq \infty$, where $T(x,t)$, $W_p(x,t)$ and $M$ are defined by (2.2), (1.11) and (1.13), respectively.

**Proof.** First of all, for simplicity, we set

$$K(x,t) := \int_0^t \partial_x G_0(t - \tau) \ast (|u|^{p-1} u - |MG_0|^{p-1} MG_0) (\tau) d\tau. \quad (4.2)$$

In what follows, we shall prove

$$\lim_{t \to \infty} t^{\frac{1}{2}(1 - \frac{1}{p}) + \frac{p-2}{2p}} \|K(\cdot, t)\|_{L^q} = 0, \quad 2 \leq q \leq \infty. \quad (4.3)$$

We start with the evaluation for the $L^q$-norm of $K(x,t)$ in the case of $2 \leq q < \infty$. From Young’s inequality, Lemmas 3.3 and 3.7 and the change of valuable, we obtain

$$\|K(\cdot, t)\|_{L^q} \leq \int_0^t \| \partial_x G_0(\cdot, t - \tau) \|_{L^q} \left\| (|u|^{p-1} u) (\cdot, \tau) - (|MG_0|^{p-1} MG_0) (\cdot, \tau) \right\|_{L^1} d\tau \leq C \int_0^t (t - \tau)^{\frac{1}{2}(1 - \frac{1}{p}) - \frac{1}{2} - \frac{p-1}{2p} = 0, \quad 2 \leq q < \infty. \quad (4.4)$$

To handle the $L^\infty$-norm, splitting the $\tau$-integral and using Young’s inequality, Lemmas 3.3, 3.7 and 3.6 and the change of valuable, we have

$$\|K(\cdot, t)\|_{L^\infty} \leq \int_0^{t/2} \| \partial_x G_0(\cdot, t - \tau) \|_{L^\infty} \left\| (|u|^{p-1} u) (\cdot, \tau) - (|MG_0|^{p-1} MG_0) (\cdot, \tau) \right\|_{L^1} d\tau + \int_{t/2}^t \| \partial_x G_0(\cdot, t - \tau) \|_{L^1} \left\| (|u|^{p-1} u) (\cdot, \tau) - (|MG_0|^{p-1} MG_0) (\cdot, \tau) \right\|_{L^\infty} d\tau \leq C \int_0^{t/2} (t - \tau)^{-\frac{1}{2} - \frac{p-1}{2p} = 0, \quad t \geq 2.$$}

Thus, we can get the following result similarly as (4.4):

$$\limsup_{t \to \infty} t^{\frac{1}{2}(1 - \frac{1}{p}) + \frac{p-2}{2p}} \|K(\cdot, t)\|_{L^\infty} \leq C \lim_{t \to \infty} \int_0^{1/2} (1 - s)^{-\frac{1}{2} - \frac{p-1}{2p} = 0, \quad t \geq 2.$$}

Combining (4.4) and (4.5), we can say that (4.3) is true.
Next, let us derive $W_p(x, t)$ defined by (1.11). To simplify the calculation, we set

$$\alpha := \frac{2B}{b}. \quad (4.6)$$

Recalling the definitions of $G_0(x, t)$ and $G(x, t)$ (i.e. (1.11) and (1.12), respectively) and using the change of valuable several times, we are able to see that

$$
\int_0^t G_0(t - \tau) * G_0^p(\tau)d\tau = \int_0^t \int_\mathbb{R} G_0(x - y, t - \tau)G_0^p(y, \tau)dyd\tau
$$

$$
= \int_0^t \int_\mathbb{R} G(x - y - \alpha(t - \tau), t - \tau)G^p(y - \alpha\tau, \tau)dyd\tau
$$

$$
= \int_0^t \int_\mathbb{R} G(x - \alpha t - z, t - \tau)G^p(z, \tau)dzd\tau \quad (z = y - \alpha\tau)
$$

$$
= t \int_0^t \int_\mathbb{R} G(x - \alpha t - z, t(1 - s))G^p(z, ts)dzds \quad (\tau = ts)
$$

$$
= t^2 \int_0^t \int_\mathbb{R} G \left( x - \alpha t - \sqrt{t}v, t(1 - s) \right)G^p \left( \sqrt{t}v, ts \right)dvds \quad (z = \sqrt{t}v)
$$

$$
= t^2 \int_0^t \int_\mathbb{R} t^{-\frac{\alpha}{\sqrt{t}}}G \left( \frac{x - \alpha t}{\sqrt{t}} - v, 1 - s \right) t^{-\frac{\alpha}{\sqrt{t}}}G^p(v, s)dvds
$$

$$
= t^{-\frac{\alpha}{\sqrt{t}}} \int_0^t (G(1 - s) * G^p(s)) \left( \frac{x - \alpha t}{\sqrt{t}} \right) ds. \quad (4.7)
$$

Therefore, from (4.7), (4.6) and (1.11), we can derive $W_p(x, t)$ as follows:

$$
\int_0^t \partial_x G_0(t - \tau) * (|MG_0|^{p-1}MG_0)(\tau)d\tau
$$

$$
= (|M|^{p-1}M) t^{-\frac{\alpha}{\sqrt{t}}} \partial_x \left( \int_0^1 (G(1 - s) * G^p(s)) \left( \frac{x - \alpha t}{\sqrt{t}} \right) ds \right)
$$

$$
= (|M|^{p-1}M) t^{-\frac{\alpha}{\sqrt{t}}} \frac{d}{dx} \left( \int_0^1 (G(1 - s) * G^p(s))(x) ds \right) \bigg|_{x=\frac{x-\alpha t}{\sqrt{t}}}
$$

$$
= (|M|^{p-1}M) t^{-\frac{\alpha}{\sqrt{t}}} \partial_x \left( \frac{x - \alpha t}{\sqrt{t}} \right) = (|M|^{p-1}M) W_p(x, t). \quad (4.8)
$$

Finally, we shall prove (4.1). From, (4.8), we note that the following relation holds:

$$
\int_0^t \partial_x T(t - \tau) * (|u|^{p-1}u)(\tau)d\tau - (|M|^{p-1}M) W_p(x, t)
$$

$$
= \int_0^t \partial_x T(t - \tau) * (|u|^{p-1}u)(\tau)d\tau - \int_0^t \partial_x G_0(t - \tau) * (|MG_0|^{p-1}MG_0)(\tau)d\tau
$$

$$
= \int_0^t \partial_x (T - G_0)(t - \tau) * (|u|^{p-1}u)(\tau)d\tau
$$

$$
+ \int_0^t \partial_x G_0(t - \tau) * (|u|^{p-1}u - |MG_0|^{p-1}MG_0)(\tau)d\tau. \quad (4.9)
$$

Therefore, by virtue of (4.9), Proposition 3.2, (4.2) and (4.3), we can conclude that (4.1) is true.

**End of the Proof of Theorem 1.1 for $2 < p < 3$.** We note that the following relation holds:

$$
u(x, t) - MG_0(x, t) + (|M|^{p-1}M) W_p(x, t)
$$

$$= \{T(t) * u_0 - G_0(t) * u_0\} + \{G_0(t) * u_0 - MG_0(x, t)\}
$$

$$- \left( \int_0^t \partial_x T(t - \tau) * (|u|^{p-1}u)(\tau)d\tau - (|M|^{p-1}M) W_p(x, t) \right). \quad (4.10)
$$

Therefore, from (4.10), Young’s inequality, Lemma 3.1, Propositions 3.4 and 4.1, we can conclude that the asymptotic formula (1.8) is true. This completes the proof of Theorem 1.1 for $2 < p < 3$. \qed
Next, in this subsection, let us treat the case of $p = 3$, i.e. we shall prove (1.9). In order to prove it, we derive the following key asymptotic formula. The methods used in the proof of the proposition below are based on the techniques used for Proposition 4.3 in [3, 16].

**Proposition 4.2.** Let $p = 3$. Assume that $u_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$, $xu_0 \in L^1(\mathbb{R})$ and $E_0 = \|u_0\|_{H^1} + \|u_0\|_{L^1}$ is sufficiently small. Then, the solution $u(x, t)$ to (1.1) satisfies

\[
\left\| \int_0^t \partial_x T(t - \tau) \ast u^3(\tau) \, d\tau - \frac{M^3}{4\sqrt{3\pi} \mu} (\log t) \partial_x G_0(\cdot, t) \right\|_{L^q} \leq CE_1 t^{-\frac{3}{2}(1 - \frac{1}{q}) - \frac{1}{2}}, \quad t \geq 2,
\]

(4.11)

for any $2 \leq q \leq \infty$, where $T(x, t)$, $G_0(x, t)$ and $M$ are defined by (2.2), (1.11) and (1.13), respectively. Also, the constant $E_1$ is defined by $E_1 = E_0 + \|xu_0\|_{L^1}$.

**Proof.** First, we split the integral in the Duhamel term of (2.1) as follows:

\[
\int_0^t \int_0^{t/2} \partial_x T(t - \tau) \ast u^3(\tau) \, d\tau \leq \int_0^t \partial_x T(t - \tau) \ast u^3(\tau) \, d\tau
\]

\[
= \int_0^1 \partial_x T(t - \tau) \ast u^3(\tau) \, d\tau
\]

\[
+ \int_0^1 \partial_x (T - G_0)(t - \tau) \ast u^3(\tau) \, d\tau
\]

\[
\leq CE_0 t^{-\frac{3}{2}(1 - \frac{1}{q}) - \frac{1}{2}}, \quad t \geq 2, \quad 2 \leq q \leq \infty.
\]

(4.12)

(4.13)

In what follows, let us evaluate $R_i(x, t)$ for all $i = 1, 2, 3, 4$. We start with evaluation for $R_1(x, t)$. Modifying the way to get (3.22) and using Young’s inequality, (3.2) and (3.6), we have

\[
\|R_1(\cdot, t)\|_{L^q} \leq \int_0^1 \|\partial_x T(\cdot, t - \tau)\|_{L^q} \|u^3(\cdot, \tau)\|_{L^1} \, d\tau \leq CE_0 \int_0^1 (t - \tau)^{-\frac{3}{2}(1 - \frac{1}{q}) - \frac{1}{2}} (1 + \tau)^{-1} \, d\tau
\]

\[
\leq CE_0 (t - 1)^{-\frac{3}{2}(1 - \frac{1}{q}) - \frac{1}{2}} \int_0^1 (1 + \tau)^{-1} \, d\tau \leq CE_0 t^{-\frac{3}{2}(1 - \frac{1}{q}) - \frac{1}{2}}, \quad t \geq 2, \quad 2 \leq q \leq \infty.
\]

(4.14)

For $R_2(x, t)$, we can use the estimate (3.23) because $R_2(x, t) \equiv N_2(x, t)$. Therefore, we obtain

\[
\|R_2(\cdot, t)\|_{L^q} \leq CE_0 t^{-\frac{3}{2}(1 - \frac{1}{q}) - \frac{1}{2}}, \quad t > 0, \quad 2 \leq q \leq \infty.
\]

(4.15)

Next, we deal with $R_3(x, t)$. In a similar way to get (3.10) for $p = 3$, by using Young’s inequality, (3.3) and (3.6), we are able to see that

\[
\|R_3(\cdot, t)\|_{L^q} \leq \int_0^1 \|\partial_x (T - G_0)(\cdot, t - \tau)\|_{L^q} \|u^3(\cdot, \tau)\|_{L^1} \, d\tau
\]

\[
\leq CE_0 \int_0^1 (t - \tau)^{-\frac{3}{2}(1 - \frac{1}{q}) - \frac{1}{2}} (1 + \tau)^{-1} \, d\tau
\]

\[
\leq CE_0 t^{-\frac{3}{2}(1 - \frac{1}{q}) - \frac{1}{2}} \log(1 + t), \quad t \geq 2, \quad 2 \leq q \leq \infty.
\]

(4.16)

Finally, let us treat $R_4(x, t)$, it follows from Young’s inequality, Lemma 3.3 and (3.26) for $p = 3$ that

\[
\|R_4(\cdot, t)\|_{L^q} \leq \int_0^1 \|\partial_x G_0(\cdot, t - \tau)\|_{L^q} \|(u^3 - (MG_0)^3)(\cdot, \tau)\|_{L^1} \, d\tau
\]

\[
\leq CE_1 \int_0^1 (t - \tau)^{-\frac{3}{2}(1 - \frac{1}{q}) - \frac{1}{2}} \log(2 + \tau) \, d\tau
\]

\[
\leq CE_1 t^{-\frac{3}{2}(1 - \frac{1}{q}) - \frac{1}{2}}, \quad t \geq 2, \quad 1 \leq q \leq \infty.
\]
Therefore, it follows from (4.18) that

$$L(x, t) = M^3 \int_1^{t/2} \int_1^R \partial_x G(x - y - \alpha(t - \tau), t - \tau) G^3(y - \alpha \tau, \tau) dy d\tau$$

$$= M^3 \int_1^{t/2} \int_1^R \partial_x G(x - \alpha t - z, t - \tau) G^3(z, \tau) dz d\tau \quad (z = y - \alpha \tau)$$

$$= M^3 \int_1^{t/2} \partial_x G(x - \alpha t, t - \tau) \int_R G^3(\eta, \tau) d\eta d\tau$$

$$- M^3 \int_1^{t/2} \int_0^{t/2} \int_0^\infty \partial_x^2 G(x - \alpha t - z, t - \tau) \int_z^\infty G^3(\eta, \tau) d\eta dz d\tau$$

$$+ M^3 \int_1^{t/2} \int_{-\infty}^0 \partial_x^2 G(x - \alpha t - z, t - \tau) \int_{-\infty}^z G^3(\eta, \tau) d\eta dz d\tau$$

$$=: L_0(x, t) + L_1(x, t) + L_2(x, t). \quad (4.17)$$

Now, let us evaluate $L_1(x, t)$ and $L_2(x, t)$. For the heat kernel $G(x, t)$ defined by (1.12), we recall the following well known estimate (for the proof, see e.g. [6]):

$$\|\partial_t^q \partial_x^k G(t)\|_{L^\alpha} \leq C t^{-\frac{\alpha}{2} \left(1 - \frac{k}{2}\right)}, \quad t > 0, \quad 1 \leq q \leq \infty. \quad (4.18)$$

Therefore, it follows from (4.18) that

$$\|L_1(\cdot, t)\|_{L^\alpha} \leq |M|^3 \int_1^{t/2} \int_0^\infty \|\partial_x^k G(\cdot - \alpha t - z, t - \tau)\|_{L^\alpha} \int_z^\infty G^3(\eta, \tau) d\eta dz d\tau$$

$$\leq C |M|^3 t^{-\frac{\alpha}{2} \left(1 - \frac{k}{2}\right)} \int_1^{t/2} \int_0^\infty \int_0^\infty G^3(\eta, \tau) d\eta dz d\tau$$

$$= C |M|^3 t^{-\frac{\alpha}{2} \left(1 - \frac{k}{2}\right)} \int_1^{t/2} \int_0^\infty \int_0^\eta G^3(\eta, \tau) d\eta dz d\tau$$

$$\leq C |M|^3 t^{-\frac{\alpha}{2} \left(1 - \frac{k}{2}\right)} \int_1^{t/2} \int_0^\eta G^3(\eta, \tau) d\eta dz d\tau$$

$$\leq C |M|^3 t^{-\frac{\alpha}{2} \left(1 - \frac{k}{2}\right)} \int_1^{t/2} \tau^{-\frac{k}{2}} d\tau$$

$$\leq C |M|^3 t^{-\frac{\alpha}{2} \left(1 - \frac{k}{2}\right)}, \quad t \geq 2, \quad 1 \leq q \leq \infty, \quad (4.19)$$

where we have used the following fact:

$$\int_0^\infty \eta G^3(\eta, \tau) d\eta = \int_0^\infty \frac{\eta}{(4\pi \mu \tau)^{\frac{3}{2}}} \exp \left( -\frac{3\eta^2}{4\mu \tau} \right) d\eta \leq C \tau^{-\frac{1}{2}}.$$

Analogously, we can obtain the same estimate for $L_2(x, t)$ as follows:

$$\|L_2(\cdot, t)\|_{L^\alpha} \leq C |M|^3 t^{-\frac{\alpha}{2} \left(1 - \frac{k}{2}\right)}, \quad t \geq 2, \quad 1 \leq q \leq \infty. \quad (4.20)$$

Finally, we would like to treat $L_0(x, t)$. First, we note that

$$\int_\mathbb{R} G^3(\eta, \tau) d\eta = \int_\mathbb{R} \frac{1}{(4\pi \mu \tau)^{\frac{3}{2}}} \exp \left( -\frac{3\eta^2}{4\mu \tau} \right) d\eta = \frac{\tau^{-\frac{1}{2}}}{4\sqrt{3\pi} \mu}.$$ 

Therefore, we can see that

$$L_0(x, t) = M^3 \int_1^{t/2} \partial_x G(x - \alpha t, t - \tau) \int_\mathbb{R} G^3(\eta, \tau) d\eta d\tau = \frac{M^3}{4\sqrt{3\pi} \mu} \int_1^{t/2} \partial_x G(x - \alpha t, t - \tau) \tau^{-1} d\tau$$

$$= \frac{M^3}{4\sqrt{3\pi} \mu} \int_1^{t/2} \partial_x (G(x - \alpha t, t - \tau) - G(x - \alpha t, t)) \tau^{-1} d\tau + \frac{M^3}{4\sqrt{3\pi} \mu} \partial_x G(x - \alpha t, t) \log \frac{t}{2}$$
On the other hand, it directly follows from Lemma 3.3 that

\[
L(u) \leq M^3 \frac{\log 2}{4\sqrt{3\pi\mu}} \partial_x G_0(x,t) + \frac{M^3}{4\sqrt{3\pi\mu}} \partial_x G_0(x,t)
\]

\[
=: L_{0.1}(x,t) + L_{0.2}(x,t) + \frac{M^3}{4\sqrt{3\pi\mu}} \partial_x G_0(x,t).
\]

To evaluate \(L_{0.1}(x,t)\), we use the following fact:

\[
\|\partial_x (G(\cdot - \alpha t, t - \tau) - G(\cdot - \alpha t, t))\|_{L^q} \leq C\tau(t - \tau)^{-\frac{1}{2} + \frac{1}{q}} - \frac{1}{2} - \frac{1}{q}, \quad t > \tau, \quad 1 \leq q \leq \infty,
\]

which comes from the mean value theorem

\[
\partial_x G(x - \alpha t, t - \tau) - \partial_x G(x - \alpha t, t) = -\tau \int_0^1 (\partial_x G)(x - \alpha t, t - \tau) d\theta
\]

and (4.18). Therefore, it follows from (4.22) that

\[
\|L_{0.1}(\cdot, t)\|_{L^q} \leq C|M|^3 \int_1^{t/2} \tau(t - \tau)^{-\frac{1}{2} + \frac{1}{q}} - \frac{1}{2} - 1 d\tau
\]

\[
\leq C|M|^3 t^{\frac{1}{2} + \frac{1}{q}} - \frac{1}{2} - \frac{1}{q}, \quad t \geq 2, \quad 1 \leq q \leq \infty.
\]

On the other hand, it directly follows from Lemma 3.3 that

\[
\|L_{0.2}(\cdot, t)\|_{L^q} \leq C|M|^3 t^{\frac{1}{2} + \frac{1}{q}} - \frac{1}{2} - \frac{1}{q}, \quad t \geq 2, \quad 1 \leq q \leq \infty.
\]

Eventually, combining (4.12) through (4.17), (4.19) through (4.21), (4.23) and (4.24), we arrive at

\[
\left\| \int_0^t \partial_x T(t - \tau) * u(\tau) \right\|_{L^q} \leq CE_1 t^{\frac{1}{2} + \frac{1}{q}} - \frac{1}{2} - \frac{1}{q}, \quad t \geq 2, \quad 2 \leq q \leq \infty.
\]

Therefore, we obtain the desired estimate (4.11). This completes the proof. \(\square\)

Now, we note that the following relation holds:

\[
u(x,t) - M G_0(x,t) + \frac{M^3}{4\sqrt{3\pi\mu}} \partial_x G_0(x,t)
\]

\[
= \{T(t) * u_0 - G_0(t) * u_0\} + \{G_0(t) * u_0 - M G_0(x,t)\}
\]

\[
- \left\{ \int_0^t \partial_x T(t - \tau) * (|u|^{p-1} u) \right\} - \frac{M^3}{4\sqrt{3\pi\mu}} \partial_x G_0(x,t)).
\]

Therefore, by virtue of (4.25), Young’s inequality, Lemma 3.1, Propositions 3.4 and 4.2, we can immediately conclude that the following asymptotic formula is true:

**Theorem 4.3.** Let \(p = 3\). Assume that \(u_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})\), \(xu_0 \in L^1(\mathbb{R})\) and \(E_0 = \|u_0\|_{H^1} + \|u_0\|_{L^1}\) is sufficiently small. Then, the solution \(u(x,t)\) to (1.1) satisfies

\[
\left\| u(\cdot, t) - M G_0(\cdot, t) + \frac{M^3}{4\sqrt{3\pi\mu}} \partial_x G_0(\cdot, t) \right\|_{L^q} \leq CE_1 t^{\frac{1}{2} + \frac{1}{q}} - \frac{1}{2} - \frac{1}{q}, \quad t \geq 2,
\]

for any \(2 \leq q \leq \infty\), where \(G_0(x,t)\) and \(M\) are defined by (1.11) and (1.13), respectively. Also, the constant \(E_1\) is defined by (1.11) and (1.13), respectively. Also, the constant \(E_1\) is defined by \(E_1 = E_0 + \|xu_0\|_{L^1}\).

**End of the Proof of Theorem 1.1 for \(p = 3\).** The desired result (1.9) can be easily obtained from the above Theorem 4.3. This completes the proof of Theorem 1.1 for \(p = 3\). \(\square\)
4.3 Proof of Theorem 1.1 for $p > 3$

Finally in this subsection, we complete the proof of Theorem 1.1 for $p > 3$, i.e. we shall prove (1.10). In order to show it, we need to analyze the linear part of the solution to (1.1) in more details. To do that, let us further transform the dispersion term in (1.1). Now, recalling (3.1) and noticing that

$$\frac{2B}{b} \partial_x^2 u + \frac{2B}{b^3} \partial_x^2 \partial_x^3 u = \frac{2B}{b} \partial_x^2 u + \frac{2B}{b^3} \partial_x^2 (\partial_x^2 - \partial_x^3 \partial_x^3 u).$$  \hfill (4.26)

Therefore, the integral kernel $T(x, t)$ is defined by (2.2) can be rewritten by

$$T(x, t) = \frac{1}{\sqrt{2\pi}} F^{-1} \left[ \exp \left( \frac{-\mu \xi^2 - \frac{i2Bt\xi}{b} + \frac{i2Bt\xi^3}{b^3}}{b^3(\xi^2 + \xi^2)} \right) \cdot \exp \left( -i\theta_1 2Bt\xi^5 \right) \right] (x).$$ \hfill (4.27)

To prove (1.10), we need to find the asymptotic profile of both the linear part and the Duhamel part of (2.1). First, we shall explain about the asymptotic analysis for the linear part. The following proposition is a key to derive the leading term of (2.1). For some related results to this formula, see e.g. [4, 15].

**Proposition 4.4.** Let $l$ be a non-negative integer and $2 \leq q \leq \infty$. Then, we have

$$\left\| \partial_x^l \left( T(\cdot, t) - G_0(\cdot, t) + \frac{2B}{b^3} \partial_x^3 G_0(\cdot, t) \right) \right\|_{L^q} \leq C t^{-\frac{l}{2}(1 - \frac{1}{q})} \left( 1 + t^{-\frac{1}{2}} \right), \quad t > 0,$$ \hfill (4.28)

where $T(x, t)$ and $G_0(x, t)$ are defined by (2.2) and (1.11), respectively.

**Proof.** First, applying the Fourier transform to $T(x, t)$, then from (4.27) and Taylor’s theorem, there exist $\theta_0, \theta_1 \in (0, 1)$ such that the following relation holds:

$$\hat{T}(\xi, t) = \frac{1}{\sqrt{2\pi}} \exp \left( \frac{-\mu \xi^2 - \frac{i2Bt\xi}{b} + \frac{i2Bt\xi^3}{b^3}}{b^3(\xi^2 + \xi^2)} \right) \cdot \exp \left( -i\theta_1 2Bt\xi^5 \right) = \frac{1}{\sqrt{2\pi}} \exp \left( \frac{-\mu \xi^2 - \frac{i2Bt\xi}{b} + \frac{i2Bt\xi^3}{b^3}}{b^3(\xi^2 + \xi^2)} \right) \cdot \exp \left( -i\theta_1 2Bt\xi^5 \right)$$

$$= \frac{1}{\sqrt{2\pi}} \exp \left( \frac{-\mu \xi^2 - \frac{i2Bt\xi}{b} + \frac{i2Bt\xi^3}{b^3}}{b^3(\xi^2 + \xi^2)} \right), \quad \xi \in \mathbb{R}, \ t > 0,$$ \hfill (4.29)

where the remainder term $R(\xi, t)$ is defined by

$$R(\xi, t) := -\frac{2B^2\xi^5}{\sqrt{2\pi}b^6} \exp \left( \frac{-\mu \xi^2 - \frac{i2Bt\xi}{b} + \frac{i\theta_1 2Bt\xi^3}{b^3}}{b^3(\xi^2 + \xi^2)} \right).$$

Here, we note that the following estimate holds:

$$|R(\xi, t)| \leq C \left( t^2 \xi^6 + t^6 \xi^5 \right) e^{-\mu \xi^2}, \quad \xi \in \mathbb{R}, \ t > 0.$$ \hfill (4.30)

Therefore, by using the Plancherel theorem, from (4.29) and (4.30), we have

$$\left\| \partial_x^l \left( T(\cdot, t) - G_0(\cdot, t) + \frac{2B}{b^3} \partial_x^3 G_0(\cdot, t) \right) \right\|_{L^2}^2 \leq C \int_{\mathbb{R}} \xi^l \left( t^2 \xi^6 + t^6 \xi^5 \right) e^{-2\mu \xi^2} d\xi \leq C \int_{\mathbb{R}} \left( t^2 \xi^2(\xi^6 + t^2 \xi^2(\xi^5)) e^{-2\mu \xi^2} d\xi \right) = C \left( t^{\frac{3}{2}(-l) + \frac{3}{2}(-l)} \right) = Ct^{\frac{3}{2}(-l) + \frac{3}{2}(-l)}(1 + t^{-l}), \quad t > 0.$$
Thus, we have the $L^2$-decay estimate:
\[
\left\| \partial_x^2 \left( T(\cdot, t) - G_0(\cdot, t) + \frac{2B}{b^4} l \partial_x^2 G_0(\cdot, t) \right) \right\|_{L^2} \leq Ct^{-\frac{3}{p}} \left( 1 + \frac{t}{q} \right), \quad t > 0.
\]  
(4.31)

For the $L^\infty$-decay estimate, from the Sobolev inequality (2.9), we can see that
\[
\left\| \partial_x^2 \left( T(\cdot, t) - G_0(\cdot, t) + \frac{2B}{b^4} l \partial_x^2 G_0(\cdot, t) \right) \right\|_{L^\infty} \leq Ct^{-\frac{3}{p}} \left( 1 + \frac{t}{q} \right), \quad t > 0.
\]  
(4.32)

In addition, the desired estimate (4.28) for $2 < q < \infty$ can be easily obtained by using (4.31), (4.32) and the interpolation inequality (2.21), in the same way to get (2.22). This completes the proof. □

By using Young’s inequality and Proposition 4.4, we immediately have the following formula:

**Corollary 4.5.** Let $l$ be a non-negative integer and $2 \leq q \leq \infty$. Suppose $u_0 \in L^1(\mathbb{R})$. Then, we have
\[
\left\| \partial_x^2 \left( T(t) * u_0 - G_0(\cdot, t) + \frac{2B}{b^4} l \partial_x^2 G_0(t) * u_0 \right) \right\|_{L^q} \leq C \|u_0\|_{L^1} t^{-\frac{3}{p} - 1} \left( 1 + t^{-\frac{1}{q}} \right), \quad t > 0,
\]  
where $T(x, t)$ and $G_0(x, t)$ are defined by (2.2) and (1.11), respectively.

Next, we shall introduce a key proposition to derive the leading term of the Duhamel term in (2.1), in the case of $p > 3$. The method for the proof of the following proposition is based on the technique used in the proof of Lemma 6.2 in [4].

**Proposition 4.6.** Let $p > 3$. Assume that $u_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$ and $E_0 = \|u_0\|_{H^1} + \|u_0\|_{L^1}$ is sufficiently small. Then, the solution $u(x, t)$ to (1.1) satisfies
\[
\lim_{t \to \infty} t^{\frac{2}{p} - 1} \left\| \int_0^t \partial_x T(t - \tau) \left( |u|^{p-1} u \right)(\tau)d\tau - M \partial_x G_0(x, t) \right\|_{L^q} = 0,
\]  
(4.34)

for any $2 \leq q \leq \infty$, where $T(x, t)$, $G_0(x, t)$ and $M$ are defined by (2.2), (1.11) and (1.13), respectively.

**Proof.** By virtue of Proposition 3.2, it is sufficient to show the following formula:
\[
\lim_{t \to \infty} t^{\frac{2}{p} - 1} \left\| \int_0^t \partial_x^2 G_0(t - \tau) \left( |u|^{p-1} u \right)(\tau)d\tau - M \partial_x G_0(x, t) \right\|_{L^q} = 0, \quad 2 \leq q \leq \infty.
\]  
(4.35)

In what follows, we shall prove (4.35). First, from the definition of (1.13), we have

\[
\int_0^t \partial_x G_0(t - \tau) \left( |u|^{p-1} u \right)(\tau)d\tau - M \partial_x G_0(x, t)
\]
\[
= \int_0^t \int_R \partial_x G_0(x - y, t - \tau) \left( |u|^{p-1} u \right)(y, \tau)d\tau - \left( \int_0^\infty \int_R \left( |u|^{p-1} u \right)(y, \tau)d\tau \right) \partial_x G_0(x, t)
\]
\[
= \int_0^t \int_R \partial_x G_0(x - y, t - \tau) \left( |u|^{p-1} u \right)(y, \tau)d\tau - \partial_x G_0(x, t) \left( |u|^{p-1} u \right)(y, \tau)d\tau
\]
\[
- \left( \int_0^t \int_R \left( |u|^{p-1} u \right)(y, \tau)d\tau \right) \partial_x G_0(x, t) =: X(x, t) + Y(x, t).
\]  
(4.36)

Next, let us evaluate $X(x, t)$. Before doing that, for the latter sake, we shall rewrite $G_0(x, t)$. Now, recalling (1.11) and (4.6), we can see that
\[
G_0(x - y, t - \tau) = G(x - y - \alpha(t - \tau), t - \tau), \quad G_0(x, t) = G(x - \alpha t, t), \quad \alpha = \frac{2B}{b}.
\]

Here, we take small $\varepsilon > 0$. By using the change of variable and splitting the integral, we get
\[
X(x, t) = \int_0^t \int_R \partial_x G_0(x - y, t - \tau) - \partial_x G_0(x, t) \left( |u|^{p-1} u \right)(y, \tau)d\tau
\]
\[
= \int_0^t \int_R \partial_x G(x - y - \alpha(t - \tau), t - \tau) - \partial_x G(x - \alpha t, t) \left( |u|^{p-1} u \right)(y, \tau)d\tau
\]
\[
= \int_0^t \int_{\mathbb{R}} (\partial_x G(x - at - z, t - \tau) - \partial_x G(x - at, t)) ([u]^{p-1}u) (z + \alpha t, \tau) \, dz \, d\tau \\
= \int_{\mathbb{R}} \int_{t/2}^t (\partial_x G(x - at - z, t - \tau) - \partial_x G(x - at, t)) ([u]^{p-1}u) (z + \alpha t, \tau) \, dz \, d\tau \\
+ \int_{0}^{t/2} \int_{|z| > \epsilon \sqrt{T}} (\partial_x G(x - at - z, t - \tau) - \partial_x G(x - at, t)) ([u]^{p-1}u) (z + \alpha t, \tau) \, dz \, d\tau \\
+ \int_{0}^{t/2} \int_{|z| < \epsilon \sqrt{T}} (\partial_x G(x - at - z, t - \tau) - \partial_x G(x - at, t)) ([u]^{p-1}u) (z + \alpha t, \tau) \, dz \, d\tau \\
=: X_1(x, t) + X_2(x, t) + X_3(x, t).
\]

Now, let us evaluate \(X_1(x, t)\) to \(X_3(x, t)\). First for \(X_1(x, t)\), from Young’s inequality, (4.18), (3.6) and (3.7), we obtain

\[
\|X_1(\cdot, t)\|_{L^q} = \left\| \int_{t/2}^t \int_{\mathbb{R}} (\partial_x G(\cdot - at - z, \cdot - t - \tau) - \partial_x G(\cdot - at, \cdot)) ([u]^{p-1}u) (z + \alpha \cdot, \tau) \, dz \, d\tau \right\|_{L^q} \\
\leq \left\| \int_{t/2}^t \int_{\mathbb{R}} \partial_x G(\cdot - at - z, \cdot - t - \tau) ([u]^{p-1}u) (z + \alpha \cdot, \tau) \, dz \, d\tau \right\|_{L^q} \\
+ \|\partial_x G(\cdot, \cdot)\|_{L^q} \int_{t/2}^t \int_{\mathbb{R}} ([u]^{p-1}u) (z + \alpha \cdot, \tau) \, dz \, d\tau \\
\leq \int_{t/2}^t \|G(\cdot - t - \tau)\|_{L^q} \int_{\mathbb{R}} \partial_x ([u]^{p-1}u) (\cdot + \alpha \cdot, \tau) \, d\tau \cdot \left(\frac{1}{q} + 1 = \frac{1}{2} + \frac{1}{r}\right) \\
+ \|\partial_x G(\cdot, \cdot)\|_{L^q} \int_{t/2}^t \left\| ([u]^{p-1}u) (\cdot + \alpha \cdot, \tau) \right\|_{L^r} \, d\tau \\
\leq CE_0 \int_{t/2}^t (t - \tau)^{-\frac{1}{2}} (1 + \tau)^{-\frac{3}{4}(1 + \frac{1}{r}) - \frac{5}{2} + \frac{6}{2}} \, d\tau + CE_0 \int_{t/2}^t (1 + \tau)^{-\frac{1}{2} + \frac{6}{2}} \, d\tau \\
\leq CE_0 \int_{t/2}^t \left(\frac{t}{2} + \frac{6}{2} \right)^{-\frac{3}{4}(1 + \frac{1}{r}) + \frac{6}{2}} \, d\tau \\
+ CE_0 \int_{t/2}^t \left(\frac{t}{2} + \frac{6}{2} \right)^{-\frac{3}{4}(1 + \frac{1}{r}) + \frac{6}{2}} \, d\tau, \quad t > 0, \ 2 \leq q \leq \infty.
\]

Next, let us treat \(X_2(x, t)\). Similarly as (4.38), we can easily have

\[
\|X_2(\cdot, t)\|_{L^q} \leq \int_{0}^{t/2} \int_{|z| > \epsilon \sqrt{T}} \|\partial_x G(\cdot - at - z, \cdot - t - \tau)\|_{L^q} + \|\partial_x G(\cdot - at, \cdot)\|_{L^q} \, dz \, d\tau \\
\times \left\| ([u]^{p-1}u) (z + \alpha \cdot, \tau) \right\|_{L^r} \, d\tau \leq Ct^{-\frac{1}{4}} (1 + \frac{1}{r}) - \frac{5}{2} Z(t), \quad t > 0, \ 2 \leq q \leq \infty,
\]

where \(Z(t)\) is defined by

\[
Z(t) := \int_{0}^{t/2} \int_{|z| > \epsilon \sqrt{T}} \left\| ([u]^{p-1}u) (z + \alpha \cdot, \tau) \right\|_{L^r} \, dz \, d\tau.
\]

In addition, applying Lebesgue’s dominated convergence theorem, we are able to see

\[
\lim_{t \to \infty} Z(t) = 0,
\]

because it follows from (3.6) that

\[
|\mathcal{M}| \leq \int_0^\infty \int_{\mathbb{R}} \left\| ([u]^{p-1}u) (y, \tau) \right\|_{L^q} \, dy \, d\tau = \int_0^\infty \left\| ([u]^{p-1}u) (\cdot, \tau) \right\|_{L^q} \, d\tau \\
\leq CE_0 \int_0^\infty (1 + \tau)^{-\frac{6}{2}} \, d\tau \leq CE_0 < \infty, \quad p > 3.
\]

Finally, we shall treat \(X_3(x, t)\). By using the mean value theorem and (4.18), we have

\[
\|\partial_x G(\cdot - at - z, \cdot - t - \tau) - \partial_x G(\cdot - at, \cdot)\|_{L^q} = \|\partial_x G(\cdot - z, \cdot - \tau) - \partial_x G(\cdot, \cdot)\|_{L^q}.
\]
Thus, combining the fact (4.41) and the above estimate, we obtain
\[
\|X_\delta(t)\|_{L^q} \leq C t^{\frac{1}{2}\left(1 - \frac{1}{q}\right) + \frac{1}{2}} \int_0^t \int_{|z| \leq \sqrt{t}} \|\partial_x G(-z, t - \tau) - \partial_x G(t - \tau)\|_{L^q} d\tau \leq C t^{\frac{1}{2}\left(1 - \frac{1}{q}\right) + \frac{1}{2}} \int_0^t \int_{|z| \leq \sqrt{t}} \|\partial_x G(-z, t - \tau) - \partial_x G(t - \tau)\|_{L^q} d\tau \leq C t^{\frac{1}{2}\left(1 - \frac{1}{q}\right) + \frac{1}{2}}, \quad t > 0, \; 2 \leq q \leq \infty.
\]
(4.42)

On the other hand, from (3.6), we can easily have
\[
\|Y(\cdot, t)\|_{L^q} \leq \|\partial_x G_0(\cdot, t)\|_{L^q} \int_0^t \int_{\mathbb{R}} \|u|^{p-1} u\| (y, \tau) dy d\tau = \|\partial_x G_0(\cdot, t)\|_{L^q} \int_0^t \int_{\mathbb{R}} \|u|^{p-1} u\| (\cdot, \tau) d\tau \leq C t^{\frac{1}{2}\left(1 - \frac{1}{q}\right) + \frac{1}{2}} \int_0^t \int_{|z| \leq \sqrt{t}} \|\partial_x G(-z, t - \tau) - \partial_x G(t - \tau)\|_{L^q} d\tau \leq C t^{\frac{1}{2}\left(1 - \frac{1}{q}\right) + \frac{1}{2}}, \quad t > 0, \; 2 \leq q \leq \infty.
\]
(4.43)

Summarizing up (4.37), (4.38) through (4.43), for \( p > 3 \), we eventually arrive at
\[
\lim_{t \to \infty} t^{\frac{1}{2}\left(1 - \frac{1}{q}\right) + \frac{1}{2}} \left\| \int_0^t \partial_x G_0(t - \tau) \left( |u|^{p-1} u \right) (\cdot, \tau) d\tau - M \partial_x G_0(\cdot, t) \right\|_{L^q} \leq C \varepsilon, \quad 2 \leq q \leq \infty.
\]
Thus, we obtain (4.35) because \( \varepsilon > 0 \) can be chosen arbitrarily small. Therefore, combining Proposition 3.2 and (4.35), we arrive at the desired result (4.34). This completes the proof.

End of the Proof of Theorem 1.1 for \( p > 3 \). We note that the following relation holds:
\[
u(x, t) - MG_0(x, t) + (m + \mathcal{M}) \partial_x G_0(x, t) + \frac{2BM}{b^3} t \partial_x^3 G_0(x, t) \]
\[
= \left\{ T(t) \ast u_0 - G_0(t) \ast u_0 + \frac{2B}{b^3} t \partial_x^3 G_0(t) \ast u_0 \right\}
\]
\[
+ \left\{ \partial_x G_0(t) \ast u_0 - MG_0(x, t) + m \partial_x G_0(x, t) \right\} - \frac{2B}{b^3} t \partial_x^3 G_0(t) \ast u_0 - MG_0(x, t)
\]
\[
- \left\{ \int_0^t \partial_x T(t - \tau) \ast \left( |u|^{p-1} u \right) (\cdot, \tau) d\tau - M \partial_x G_0(x, t) \right\}.
\]
(4.44)

Therefore, from (4.44), Corollary 4.5, Propositions 3.4 and 4.6, we can conclude that the asymptotic formula (1.10) is true. This completes the proof of Theorem 1.1 for \( p > 3 \).

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