Flatband Line States in Photonic Super-Honeycomb Lattices

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For the first time, a photonic super-honeycomb lattice (sHCL) is established experimentally by use of a continuous-wave laser writing technique, and thereby two distinct flatband line states that manifest as noncontractible loop states in an infinite flatband lattice are demonstrated. These localized states (“straight” and “zigzag” lines) observed in the sHCL with tailored boundaries cannot be obtained by the superposition of conventional compact localized states because they arise from real-space topological property of certain flatband systems. In fact, the zigzag-line states, unique to the sHCL, are in contradistinction with those previously observed in the Kagome and Lieb lattices. Their momentum-space spectrum emerges in the high-order Brillouin zone where the flatband touches the dispersive bands, revealing the characteristic of topologically protected band-touching. The experimental results are corroborated by numerical simulations. This work may provide insight into Dirac-like 2D materials beyond graphene.

Flatband systems, as represented by lattices hosting at least one completely dispersionless energy band, have attracted enormous interest in different branches of physics ranging from condensed matter to exciton polaritons, and from ultracold atoms to photonics. [1–17] Flatband lattices host localized states commonly referred to as compact localized states (CLSs), [1, 2] typically localized over a few lattice sites in a finite area, while their wavefunction amplitude vanishes on all other sites due to destructive interference. [18, 19] The physical boundary formed by the sites with zero amplitude is related to the “local symmetry partition” that makes the associated electronic wave functions remain localized inside the characteristic trapping “prison.” The frozen amplitude distribution inside the “prison” does not allow any dynamics of the “prisoner” beyond the trapping cell, making the kinetic information solely quenched. [20] Interestingly, the momentum independence of the CLSs leads to divergent effective mass tensor and singularity in the profile of density of states, resulting in anomalous behavior in the transport as well as in the response of the system. [21] The tunability of the CLSs is another important aspect associated with flatband. [22] The robustness of the CLSs to disorder crucially depends on the flatband feature, for instance, whether the flatband touches with the dispersive bands or not. [23]

In optics, photonic lattices composed of evanescently coupled waveguide arrays have provided a promising platform to explore intriguing flatband phenomena associated with the lattice models originally proposed in solid state physics, [24,25] many of which are even difficult to be realized in electronic systems. In particular, the CLSs have been experimentally realized in various (quasi)-1D and 2D flatband photonic lattices [26–33] Moreover, the CLSs arising from the so-called Aharonov–Bohm caging have also been experimentally realized in a photonic platform. [34, 35] These flatband states have been proposed for applications such as light localization, [28, 29] slow light, [22] distortion-free imaging, [31] and even flatband lasing. [36]

In certain flatband lattices hosting a band touching between the flatband and other dispersive bands at discrete points in momentum space, it was found that the CLSs do not form a complete basis for the flatband. [37] The band touching precludes the CLSs from forming a complete set spanning the flatband: a lattice with N unit cells should in principle support N flatband states, but if there exists a band-touching with the dispersive band, the number of CLSs is always less than N. The missing states are the so-called noncontractible loop states (NLSs)
winding around the entire (infinite) lattices, a manifestation of real-space topology.[37,38] NLSs are a new type of flatband eigenstates which cannot be obtained by linear superposition of the conventional CLSs, and they cannot be continuously deformed into the CLSs in a torus geometry representing the periodical boundary conditions. Recently, it was theoretically found that the NLSs are inherent to the singular flatband, where an immovable singularity exists in the band-crossing of the Bloch wavefunctions.[38] Since an infinite lattice or a torus structure (mimicking the periodic boundary conditions) is difficult to establish in experiment, an alternative approach is to observe the “line states” in truncated lattices with appropriate boundaries. The existence of the line states does not rely on the spatial size of the lattice but rather on the boundary termination. Recently, such line states have been experimentally observed in a finite-sized photonic Lieb lattice with bearded edges,[39] where an NLS manifests its existence as a straight line. For Lieb lattices, the flatband touching happens to occur at the corners of the first Brillouin zone (BZ), but it can also appear at other high symmetry points in momentum space such as the BZ centers, as in the Kagome lattice and the super-honeycomb lattice (sHCL).

The sHCL is particularly interesting because it is an example of a lattice where fermionic (spin-½) and bosonic (spin-1) conical band-crossing points coexist. Such lattice structures were initially studied more than two decades ago in electronic systems,[40,41] and were recently revisited by theorists studying their Dirac cones, conical diffraction and massless Kane fermions.[42-45] One wonders if the composite sHCL could also be realized in experiment, and how its qualitatively different band structure would affect the NLSs.

In this work, by employing a cw-laser writing technique, we establish a finite-sized sHCL in the bulk of a nonlinear crystal. More importantly, we experimentally observe two types of line states (the “straight” and “zigzag” lines) in the sHCL with tailored boundaries which cannot be obtained by superposition of conventional flatband CLSs. The straight line states occupy two majority sublattices, while the zigzag line states reside in three majority sublattices. The zigzag states are unique for the sHCL, which have never been predicated or observed before whatsoever. Our theoretical analysis shows that the zigzag line states are related to linear superposition of straight line states. Surprisingly, the k-space spectra of these line states emerge at the high-order BZ (across the centers of the second BZ) where the flatband touches the dispersive bands, in contrast to that of the line states in the Lieb lattices.[39] These results validate further the existence of the NLSs as a direct manifestation of band-touching that arises from real-space topology, which may prove relevant to some 2D materials with flatband-touching spectrum.

The sHCL, also known as edge-centered honeycomb lattice, consists of five lattice sites (A, B, C, D, and E) per unit cell as marked in Figure 1a, which can be considered as introducing additional lattice sites C, D, and E (majority) to the HCL sites.

![Figure 1](https://example.com/figure1.png)

**Figure 1.** Illustration of flatband line states in a photonic sHCL. a) Schematic of sHCL structure consisting of five sites (A, B, C, D, and E) per unit cell shown in dark-dashed square, with a fundamental flatband mode (the CLS) shown in red-dashed square. The right-bottom inset shows the experimentally obtained intensity pattern of the CLS. b) Calculated band structure based on the tight-binding model showing a Lieb-like (pseudospin-1) Dirac cone intersected by a flatband at the first BZ center, and six graphene-like (pseudospin-1/2) Dirac cones located at the corners of the first BZ. The red hexagon depicts the first Brillouin zone (BZ) with high symmetry points. c) The corresponding density of states. The sHCL under two different boundary cuttings that support d) a “straight” line along the x-direction and e) a “zigzag” line along the y-direction. For all figures, sites with zero amplitudes are denoted by gray color, and those with nonzero amplitudes of opposite phase are denoted by red and black colors. f) Illustration of NLSs and CLSs in a torus geometry, representing an infinite sHCL “wrapped” around under periodic boundary conditions.
A and B (minority). A light beam propagating in the 2D sHCL is governed by the Schrödinger-type paraxial wave equation:[46,47]

$$i \frac{\partial \Psi(x,y,z)}{\partial z} = -\frac{1}{2k_0} \nabla^2 \Psi(x,y,z) - \frac{k_0 \Delta n(x,y,z)}{n_0} \Psi(x,y,z) = H_2 \Psi(x,y,z) \quad (1)$$

Here $\Psi$ is the electric field envelope of the probe beam, $\nabla^2 = \partial_x^2 + \partial_y^2$ is the transverse Laplacian operator, $z$ is the longitudinal propagation distance into the photonic lattice, $k_0$ is the wavenumber in the medium, $n_0$ is the background refractive index, and $\Delta n$ is the refractive index change that defines the sHCL. $H_2$ in Equation (1) is the continuum Hamilton for wave propagation in the photonic lattice. If only the nearest-neighbor coupling of the waveguides is considered, the corresponding tight-binding Hamiltonian in the $k$-space can be written as[43]

$$H_T = \begin{bmatrix}
0 & 0 & H_{13} & H_{14} & H_{15} \\
0 & 0 & H_{13} & H_{14} & H_{15} \\
H_{13} & H_{15} & 0 & 0 & 0 \\
H_{14} & H_{15} & 0 & 0 & 0 \\
H_{15} & H_{15} & 0 & 0 & 0
\end{bmatrix} \quad (2)$$

where $H_{13} = \exp(-ik_y a)$, $H_{14} = \exp[i(\sqrt{3}k_y a/2 + k_x a/2)]$, $i$ is the lattice constant, $a$ is the nearest-neighbor coupling constant, and $H_{mn}^*$ denotes the complex conjugate of $H_{mn}$. Diagonalizing $H_T$ yields the band structure $\beta(k)$ as shown in Figure 1b. It consists of five bands, including a completely flatband touching two dispersive conical bands at the center ($I$ point) of the first BZ, which resembles the pseudospin-1 Dirac cone in the Lieb lattice, and the flatband has a delta-function-like density of states as shown in Figure 1c. Interestingly, there are also additional graphene-like pseudospin-1/2 Dirac cones at the BZ corners. In this work, we focus on the flatband states. The Bloch eigenstates of the flatband can be written as $|\beta, k\rangle = (0,0,-\sin(\sqrt{3}k_y),\sin(\sqrt{3}k_y/2-\sqrt{3}k_x/2),\sin(\sqrt{3}k_y/2+\sqrt{3}k_x/2),0,0,0)$, which indicates that the flatband states have zero amplitudes on A and B sites. The fundamental CLS corresponds to filling the A, B sites with zero amplitude and other sites with equal amplitude but alternating opposite phase in each hexagonal plaquette (see red dashed square in Figure 1a), and it is localized due to destructive interference. Typical experimental result of the intensity pattern of such a CLS is shown in the inset of Figure 1a. However, due to the flatband touching with the dispersive bands, there should be flatband line states localized in one direction but extended infinitely in the orthogonal direction to complete the flatband basis in the sHCL. The line states in an infinite lattice can be better visualized as the NLSs in a torus geometry representing the periodic boundary conditions as shown Figure 1f. Clearly, the NLSs are topologically different from the conventional CLSs in real space, as an NLS cannot be continuously deformed into a conventional CLS. As in the Lieb lattice,[36] these line states can exist in a finite lattice with specially tailored boundaries satisfying the requirement of destructive interference. Considering the lattice geometry of the sHCL in real space and its band touching in $k$-space, we propose two types of line states (straight and zigzag) as illustrated in Figure 1d,e. The straight line state is along the $x$-direction (Figure 1d) while the zigzag one is along the $y$-direction (Figure 1e), both of them exist in a finite sHCL terminated with D and E sites.

In experiment, the sophisticated sHCL cannot be readily created by simple optical induction technique based on multi-beam interference. As such, we employ the cw-laser writing technique to establish such a lattice with desired lattice boundaries by site-to-site waveguide “writing” in a non-instantaneous nonlinear strontium barium niobate (SBN) crystal. The experimental setup is quite similar to that used in our recent work of flatband line states in photonic Lieb lattices.[49] An ordinarily polarized quasi-nondiffracting beam from a solid-state laser ($\lambda = 532 \text{ nm}$) propagating through the biased SBN crystal is used as the writing beam to induce the waveguides one by one by conveniently changing the position of the writing beam with a spatial light modulator (SLM). All the waveguides remain intact during the writing process due to the “memory” effect of the nonlinear photorefractive crystal. The written lattice can be visualized by sending a weak extraordinarily polarized quasi-plane wave to probe the index changes. Figure 2a shows an sHCL with edge termination on D and E sites along the $y$-direction corresponding to Figure 1d, which supports the straight line states. The lattice spacing is about 32 $\mu$m, and a single-site excitation of a Gaussian beam leads to discrete diffraction with energy mainly coupling to the nearest-neighbor waveguides after 10 mm propagation through the crystal (see

![Figure 2](image_url)
the inset in Figure 2a), which justifies that the waveguide coupling in the lattice satisfies the tight-binding condition. To observe the straight line states displayed in Figure 1d, the probe beam is shaped into a horizontal stripe pattern, with its phase modulated by an SLM to make sure that the adjacent spots have opposite phase (Figure 2b1). Without the lattice, the input beam profile is not preserved after free-space propagation (Figure 2b2). In contrast, with the sHCL, its overall intensity pattern is well maintained (Figure 2b3). Furthermore, each spot remains localized and the out-of-phase feature persists as verified by the measured interferogram (Figure 2b5). For comparison, corresponding results for an in-phase stripe are presented in Figure 2c1,c2,c3. In this case, the input beam cannot be confined, as the energy couples to zero-amplitude lattice sites A and B (Figure 2c3), although the in-phase feature does not change (Figure 2c5). Due to limited propagation distance in experiment (1 cm), we perform numerical simulations to further corroborate our experimental observations. By taking same parameters used in our experiments but for a much longer propagation distance (4 cm), numerical results based on Equation (1) are presented in Figure 2b4,c4. One can see clearly the difference from output intensity patterns between the out-of-phase and in-phase cases.

Next, we present the experimental results of zigzag line states extended along the y-direction, for which the sHCL edge truncation along the x-direction is shown in Figure 3a. To excite the zigzag line state, the probe beam is shaped into the zigzag pattern and launched into the lattice vertically (see Figure 3a, in which the white dashed rectangle marks the input position). One clearly finds that only the out-of-phase beam can evolve into a flatband zigzag line state, with localized intensity pattern (Figure 3b3) and out-of-phase structure (Figure 3b5) after exiting the lattice. The in-phase beam is not preserved during propagation and its energy couples to the nearest-neighbor minority (A and B) sites (Figure 3c3,c5). The difference is much more evident from long propagation distances, as indicated by the simulations in Figure 3b4,c4. It is worth noting that the zigzag line states realized here do not exist in either the Lieb or the Kagome lattices. In fact, to our knowledge, such flatband zigzag line states have never been theoretically predicted before.

Since the flatband line states are related to the flatband touching in k space, it is instructive to experimentally measure the Fourier spectra of the line states. Typical experimental results are displayed in Figure 4, where the white dashed lines mark the edges of the first and second BZs. From the band structure as shown in Figure 1b, one knows that the flatband touching in sHCL occurs at the first BZ center. Interestingly, the experimentally measured spectra of both the zigzag and straight line states mainly concentrate on higher-order BZ and do not touch the BZ center; the latter follows from the out-of-phase structure of the line states (with zero mean amplitude). For the straight line states the spectra mainly distribute in the third BZ and touch four Γ points in the second BZ center (Figure 4a), while the spectra of the zigzag line states cover all six Γ points in the second BZ, and also additional modes.

![Figure 3. Experimental demonstration of photonic sHCL and a flatband zigzag line state. Figure caption is the same as that for Figure 2, except that the lattice boundaries of interest are at the top and bottom, and the line state has a zigzag shape oriented vertically in the y-direction.](image)

![Figure 4. Momentum-space spectra of line states formed in sHCL. a,b) Experimentally measured and c,d) numerically simulated k-space spectra for the a,c) straight and b,d) zigzag line states. White dashed lines outline the first and second BZs.](image)
locate at higher-order BZ centers (Figure 4b). The experimental results of the spectra agree well with the simulation results (Figure 4c,d). Note that the spectra of the line states are in sharp contrast with those in the Lieb lattices, where the spectra mainly distribute along the first BZ edge. These results clearly support that the line states are related to the flatband touching in k-space, as the touching occurs at the BZ centers in the sHCL.

In order to better understand the difference between the straight and zigzag line states, we first obtain their solutions based on the tight-binding model. If the sHCL is terminated along the x-direction, then $k_x$ is not a good quantum number, so the Bloch eigenstate can be written as $|\beta \rangle = |0,0,0,-1,1\rangle^T$. Clearly, the amplitude is zero on sublattices A, B, and C, and nonzero on sublattices D and E with opposite phase. The analytical result completely agrees with the straight line state shown in Figure 1c. Similarly, if the sHCL lattice is terminated along the y-direction, the Bloch eigenstate reduces to $|\beta, k_y\rangle = |0,0,-2\cos(\sqrt{3}k_x/2),+1,1\rangle^T$. Now the line state occupies C, D, and E sublattices, different from the straight line state that only occupies two sublattices. If the probe is launched into the lattice at normal incidence $k_x = 0$, the Bloch eigenstate is recast into $|\beta \rangle = |0,0,0,2,1\rangle^T$. One finds that the line state has nonzero amplitude only on C, D, and E sublattices, and the site C is out of phase with sites D and E. In fact, this is the unusual characteristics of the zigzag line state, along with that the amplitude on site C is two times larger than that on sites D and E. The reason can be quite intuitive: there are infinite numbers of independent zigzag line states in the horizontally cut sHCL if it is infinitely long along the x-direction and ended on sites D and E; however, every nearest two zigzag line states share and occupy three majority sites (C, D, and E) but the straight line only occupy two sites (D and E). Bear in mind that the sHCL does not change if one rotates it 60° clockwise or anticlockwise due to lattice symmetry. Therefore, in addition to the straight line states along the x-direction, there are also two additional line states along other two translation vector directions with an angle of 120° or 60° relative to the positive x-axis, which can be described by $|\beta \rangle = |0,0,-1,1\rangle^T$ (occupying sites C and D), or $|\beta \rangle = |0,0,1,1\rangle^T$, (occupying sites C and E). Appropriate linear superposition of the two straight line states not only gives rise to $|\beta \rangle = |0,0,0,2,1\rangle^T$, but also $|\beta \rangle = |0,0,0,2,1\rangle^T$. The former one is exactly the solution which describes the zigzag line state, while the latter one is the straight line state. In other words, the zigzag line state is also a reflection of the straight line state, and its appearance is dependent on the spatial geometry of the finite sHCL.

To further confirm the existence of the above observed straight and zigzag flatband line states in finite sHCLs, we numerically solve Equation (1) directly by considering the finite sHCL boundary conditions properly and taking parameters like those used in experiments. We seek for the solution $\Psi(x, y, z) = u(x, y) \exp (i \varepsilon z + i k_y \mu z)$ with $\varepsilon$ being the propagation constant, $u(x, y)$ the envelope of line states, and $\mu$ being $y (x)$ for the straight (zigzag) line state. Plugging this solution ansatz into Equation (1), one obtains an eigenvalue problem, based on which the band structure of the truncated semi-infinite sHCL can be obtained by using the plane-wave expansion method. For convenience, we introduce the dimensionless propagation constant $\beta = k_o x/k$ with $r_o$ being the real probe beam width. In Figure 5, the band structure of the truncated sHCL (one unit cell) shown in Figure 5b1 is displayed. In the band structure, the flatband is quite obvious and the dispersive bands are almost symmetric about the flatband. Therefore, the band structure obtained based on the continuum model can be also utilized to demonstrate the validity of our previous discussions based on the tight-binding model. We choose the flattest case in the flatband and mark it with red color in Figure 5a, on which the amplitude and phase of the flatband

![Figure 5](image-url)

**Figure 5.** Numerical results of line states in sHCL based on continuum model. a,b) The straight line state and c,d) the zigzag line state solutions, where (a) and (c) are band structure with $K$ representing the BZ width, (b1) and (d1) the unit cell of the lattice structure, (b2) and (d2) the amplitudes of the flatband modes with $\beta$ indicated by the red lines in (a) and (c) and $k_y = 0$, and (b3) and (d3) the corresponding phase of (b2) and (d2). Panels (b1)–(b3) are rotated 90° for illustration purpose. Note that the amplitudes are not equal for all sites in (d2) and there are phase differences as indicated in the corresponding areas.
mode with $k_y = 0$ are presented in Figure 5b2,b3, respectively. One finds that there are two independent straight line states extended along $x$ direction per one unit cell in Figure 5b2, and every two nearest-neighbor sites (i.e., D and E sites) are out-of-phase according to the phase distribution in Figure 5b3. The numerical results obtained based on the continuum model are completely in accordance with the experimental results in Figure 2 and the analytical results based on the tight-binding model.

Corresponding to the truncated sHCL (one unit cell) in Figure 5d1, the band structure is displayed in Figure 5c. Similar to Figure 5a, we also choose one flattest case and mark with red color in Figure 5c. The flatband mode with $k_y = 0$ on this flattest band is exhibited in Figure 5d2, and its corresponding phase is shown in Figure 5d3. There are also two independent zigzag line states in one unite cell. Again, one finds that the results agree with the analytical results based on the tight-binding model: there is a $\pi$ phase shift between the amplitudes on sites C, D, and E, and the amplitude on site C is two times larger than that on sites D and E. We note that, since Figure 5c is obtained from the continuous model, the flatband is not perfectly “flat.” In fact, under the continuous model, there is a very small gap between the flatband and dispersion band, but the dispersion relation around is still approximately linear, so the results accordingly will not be affected.

In conclusion, we have presented the first demonstration of finite-size phosphonic sHCLs with desired boundaries by employing the cw-laser writing method in a nonlinear bulk crystal, and more importantly, the demonstration of unconventional (straight and zigzag) flatband line states manifesting the NLSs in infinite lattices. The two line states occupy different sublattice sites in real space, and exhibit different spectra in momentum space. Interestingly, the zigzag line state can be viewed as a linear superposition of straight line states along different directions. Moreover, the spectra of the line states in momentum space distribute in higher-order BZ and occupy the BZ center where the flatband touches the dispersive bands, reflecting unusual characteristic of line states arising from band touching. Our work not only reveals these intriguing flatband states, but also brings about new possibilities to explore both flatband and Dirac physics in one platform, as the sHCL represents an ideal system where fermionic (spin-1/2) and bosonic (spin-1) Dirac points coexist.\[41,42,44]\n
Keywords

flatband states, laser-writing arrays, photonic Dirac materials, real-space topology, super-honeycomb lattices

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Conflict of Interest

The authors declare no conflict of interest.
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