NON-LINEAR BI-HARMONIC CHOQUARD EQUATIONS

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ABSTRACT. This note studies the fourth-order Choquard equation

\[ i\dot{u} + \Delta^2 u + \epsilon (I_\alpha \ast |u|^p)|u|^{p-2} u = 0. \]

In the mass super-critical and energy sub-critical regimes, a sharp threshold of global well-posedness and scattering versus finite time blow-up dichotomy is obtained.

1. Introduction. In this manuscript, we investigate the Cauchy problem for a bi-harmonic Choquard equation

\[
\begin{cases}
    i\dot{u} + \Delta^2 u + \epsilon (I_\alpha \ast |u|^p)|u|^{p-2} u = 0; \\
    u(0,.) = u_0,
\end{cases}
\tag{1.1}
\]

where \( u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C} \), for some \( N \geq 1 \), \( \epsilon = \pm 1 \), \( 0 < \alpha < N \) and the Riesz-potential is defined on \( \mathbb{R}^N \) by

\[
I_\alpha := \frac{\Gamma(N-\alpha)}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N-\alpha}{2}}|\cdot|^{N-\alpha}}.
\]

The classical Choquard equation is a model of quantum mechanics [17], non-relativistic quantum and Hartree-Fock theories [19, 9]. The particular case \( p = 2 \) with Laplacian operator (instead of bilaplacian) is called Hartree equation and models the dynamics of boson stars [6, 16].

Fourth-order Schrödinger equations, take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr non-linearity [12, 13].

If \( u \) is a solution to the Choquard problem (1.1), then the following scaled function solves the same problem

\[
u_\lambda = \lambda^{\frac{4+\alpha}{2p-4}} u(\lambda^4, \lambda^2), \quad \lambda > 0.
\]

Using the next equality,

\[
||u_\lambda(t)||_{H^s} = \lambda^{\frac{-N}{2} + \frac{4+\alpha}{2p-4}} ||u(\lambda^4t)||_{H^s},
\]

one obtains the unique invariant Sobolev norm under the previous scaling, called critical exponent

\[
s_c := \frac{N}{2} - \frac{4 + \alpha}{2(p-1)}.
\]

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The exponent $s_c = 0$ is called mass-critical case and corresponds to $p_* := 1 + \frac{\alpha + 4}{N}$. The energy-critical case $s_c = 2$ is equivalent to

$$p^* := \begin{cases} 1 + \frac{\alpha + 4}{N - 4}, & N > 4; \\ \infty, & 2 \leq N \leq 4. \end{cases}$$

The well-posedness issues for the mass-super-critical and energy sub-critical classical Choquard equation were investigated recently by many authors [7, 20, 23]. See also [8, 4, 22], for the fractional Choquard equation.

Recall the conservation laws for the Schrödinger problem (1.1),

**Mass**: $M(u(t)) := \int_{\mathbb{R}^N} |u(t, x)|^2 \, dx = M(u_0)$;

**Energy**: $E(u(t)) := \int_{\mathbb{R}^N} \left( |\Delta u(t)|^2 + \frac{\epsilon}{p} (I_\alpha * |u(t)|^p) |u(t)|^p \right) \, dx = E(u_0)$.

The positive (respectively negative) sign of $\epsilon$ refers to the attractive or defocusing (respectively focusing) case, where a local solution in the energy space is claimed to be global and scatters (respectively blows-up in finite time).

It is the purpose of this manuscript to obtain a sharp dichotomy in the mass super-critical and energy sub-critical cases of global well-posedness and scattering versus finite time blow-up of solutions to the fourth-order Choquard problem (1.1), by use of a sharp Gagliardo-Nirenberg type inequality and the existence of ground states. In the scattering part, one uses the concentration-compactness-rigidity method, due to Kenig and Merle [14], which has a deep influence on asymptotic study of Schrödinger problems [5, 10].

To the author knowledge, this paper is the first one dealing with scattering of bi-harmonic Choquard equations.

The plan of this paper is as follows. Section two contains some classical estimates needed in the sequel. In the third section a sharp Gagliardo-Nirenberg type inequality is given. The existence of ground states is proved in section four. In section five, local well-posedness in the energy space is given. A variance identity is established in section six. The existence of global/non global solutions to (1.1) are discussed in section seven. The goal of the last section is to investigate scattering of global solutions.

Here and hereafter $C$ will denote a constant which may vary from line to line and if $A$ and $B$ are non-negative real numbers, $A \leq B$ means that $A \leq CB$.

Denote the Lebesgue space $L^r := L^r(\mathbb{R}^N)$ with the standard norm $\| \cdot \|_r := \| \cdot \|_{L^r}$ and $\| \cdot \| := \| \cdot \|_2$. Take $H^2 := H^2(\mathbb{R}^N)$ the inhomogeneous Sobolev space endowed with the complete norm

$$\| \cdot \|_{H^2} := \left( \| \cdot \|^2 + \| \Delta \cdot \|^2 \right)^{1/2}.$$

If $X$ is an abstract space $C_T(X) := C([0, T], X)$ stands for the set of continuous functions valued in $X$ and $X_{rd}$ is the set of radial elements in $X$, moreover for an eventual solution to (1.1), $T^* > 0$ denotes it’s lifespan.

2. **Preliminary.** This section contains some estimates needed in the sequel. Let us start with a Hardy-Littlewood-Sobolev inequality [18].
Lemma 2.1. Let $0 < \lambda < N \geq 1$ and $1 < s, r < \infty$ be such that $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2$. Then,

$$
\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^{\lambda}} \, dx \, dy \leq C(N, s, \lambda) \|f\|_r \|g\|_s, \quad \forall f \in L^r(\mathbb{R}^N), \forall g \in L^s(\mathbb{R}^N).
$$

The next consequence will be useful [22].

Corollary 2.2. Let $0 < \alpha < N \geq 1$ and $1 < s, r, q < \infty$ be such that $\frac{1}{q} + \frac{1}{r} + \frac{\alpha}{N} = 1 + \frac{\alpha}{N}$. Assume that $f \in L^s(\mathbb{R}^N)$ and $g \in L^q(\mathbb{R}^N)$. Then,

$$
\|(I_\alpha * f)g\|_{r'} \leq C(s, \alpha)\|f\|_s \|g\|_q.
$$

Sobolev injections [2] give a meaning to several computations done in this note.

Lemma 2.3. Let $N \geq 1$, then

1. $H^2 \hookrightarrow L^q$ for any $q \in [2, \frac{2N}{N-4}]$ if $N \geq 5$ and any $2 \leq q < \infty$ if $N \leq 4$;
2. the following injection $H^2_{rd} \hookrightarrow \hookrightarrow L^q$ is compact for any $q \in (2, \frac{2N}{N-4})$ if $N \geq 5$ and any $2 < q < \infty$ if $2 \leq N \leq 4$;
3. for all $\frac{1}{2} < \mu < \frac{N}{2}$,

$$
\sup_{x \neq 0} |x|^{\frac{\mu}{2} - \mu} |u(x)| \leq C(N, \mu) \|(-\Delta)^{\frac{\mu}{2}} u\|, \quad \forall u \in H^\mu_{rd}(\mathbb{R}^N).
$$

(2.1)

Recall a Gagliardo-Nirenberg inequality [21].

Lemma 2.4. Let $N \geq 1$, $1 \leq p, q, r \leq \infty$ and $0 \leq \frac{m}{m} \leq \theta \leq 1$ satisfying

$$
\frac{1}{p} = \frac{\mu}{N} + \theta \left(\frac{1}{r} - \frac{m}{N}\right) + (1 - \theta) \frac{1}{q}.
$$

Then,

$$
\|(-\Delta)^{\frac{\mu}{2}} \cdot \|_p \lesssim \|(-\Delta)^{\frac{\mu}{2}} \cdot \|_r \|_q^{1 - \theta}.
$$

(2.2)

Recall a fractional chain rule [3].

Lemma 2.5. Let $s \in (0, 1]$ and $1 < p, p_i, q_i < \infty$ satisfying $\frac{1}{p} = \frac{1}{p_i} + \frac{1}{q_i}$. Thus,

1. if $G \in C^1(\mathbb{C})$, then $\|\nabla^s G(u)\|_p \lesssim \|G'(u)\|_{p_i} \|\nabla^s u\|_{q_i}$;
2. $\|\nabla^s (u)\|_p \lesssim \|\nabla^s u\|_{p_i} \|v\|_{q_i} + \|\nabla^s v\|_{p_2} \|u\|_{q_2}$.

Definition 2.6. A couple of real numbers $(q, r)$ is said to be s admissible if

$$
\frac{2N}{N - 2s} \leq r < \frac{2N}{N - 4} \quad \text{and} \quad N \left(\frac{1}{2} - \frac{1}{r}\right) = \frac{4}{q} - s.
$$

Strichartz estimate [11, 24] is a classical tool to control solutions to (1.1).

Proposition 2.7. Let $N \geq 2$, $0 \leq s < 2$, $(q, r)$ be an admissible pair and $(\tilde{q}, \tilde{r})$ be $-s$ admissible pair. Then, there exists $C := C_{N, q, \tilde{q}, s}$ such that if $u_0 \in \dot{H}^s$,

$$
\|u\|_{L_t^q(L_x^r)} \leq C \left(\|u_0\|_{\dot{H}^s} + \|i\dot{u} + \Delta^2 u\|_{L_t^{\tilde{q}}(L_x^{\tilde{r}})}\right).
$$

Let us introduce [11] the linear profile decomposition for bounded radial sequences in $H^s$.

Proposition 2.8. Let $N \geq 2$ and $(u_n)$ be a bounded sequence in $H^2_{rd}$. Then for each integer $M$ there exist a sub-sequence still denoted $(u_n)$ and

1. for every $1 \leq j \leq M$, there exists a profile $\psi_j \in H^2$ and a sequence of time shifts $t_{n,j}^j$,
2. there exists a sequence of remainders $W_n^M \in H^2$, such that

$$u_n = \sum_{j=1}^{M} e^{-it_n^j \Delta} \psi^j + W_n^M.$$ 

The time sequences have the pairwise divergence property: For $1 \leq i \neq j \leq M$, 

$$\lim_{n} |t_n^i - t_n^j| = \infty.$$ 

The remainder sequence has the following asymptotic smallness property

$$\lim_{M \to \infty} \lim_{n \to \infty} \| e^{i \Delta^2} W_n^M \|_{S(R)} = 0.$$ 

For fixed $M$ and any $0 \leq \alpha \leq 2$, the asymptotic Pythagorean expansions hold

$$\| u_n \|^2_{H^\alpha} = \sum_{j=1}^{M} \| \psi^j \|^2_{H^\alpha} + \| W_n^M \|^2_{H^\alpha} + o_n(1);$$

$$E(u_n) = \sum_{j=1}^{M} E(e^{-it_n^j \Delta} \psi^j) + E(W_n^M) + o_n(1).$$

Proof. Taking account of [11], the last equality is the only point to prove. It is sufficient to prove that $Q(u) := \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \, dx$ satisfies

$$Q(u_n) = \sum_{j=1}^{M} Q(e^{-it_n^j \Delta} \psi^j) + Q(W_n^M) + o_n(1).$$

Assume as a first case that there exists some $j$ for which $t_n^j$ converges to a finite number, which is supposed to be zero without loss of generality. From the proof of Lemma 5.3 in [11] and the compact embedding $H^{2}_{r,p} \hookrightarrow L^q$ for $2 < q < \frac{2N}{N-4}$, we get $W_n^{j-1} \to \psi^j$ in $L^q$ for $2 < q < \frac{2N}{N-4}$. Write using Lemma 2.1, for $r := \frac{2N}{N+4}$,

$$|Q(W_n^{j-1}) - Q(\psi^j)| \leq C \| W_n^{j-1} \|^p_r - \| \psi^j \|^p_r \| (\| W_n^{j-1} \|^p_r + \| W_n^j \|^p_r)\|p-1$$

$$\leq C \sum_{k=0}^{p-1} \| \psi^j \|^p_r \| W_n^{j-1} \|^k_r \| \psi^j \|^p(k) \|W_n^j\|_{r_p}^{p-k-1}.$$ 

Since, $p < p^*$, we get $2 < rp < \frac{2N}{N-4}$, which implies that $|Q(W_n^{j-1}) - Q(\psi^j)| \to 0$. Let $k \neq j$. Then, $|t_n^k| \to \infty$. Since $p > p_*$, from Lemma 2.1 and the $L^p$ space-time decay estimates of the linear flow associated to (1.1),

$$|Q(e^{-it_n^k \Delta} \psi^k)| \leq \| e^{-it_n^k \Delta} \psi^k \|^p_r \| \psi^k \|_{r_p}^{p-1} \leq \left( \frac{1}{t_n^k} \right)^{\frac{2N}{N-4}} \| \psi^k \|_{\frac{2N}{N-4} <}.$$ 

With the expansion of $u_n$,

$$u_n = \sum_{k=1}^{j-1} e^{-it_n^k \Delta} \psi^k + W_n^{j-1},$$

one gets $u_n \to \psi^j$ in $L^q$ for $2 < q < \frac{2N}{N-4}$. As previously, it follows that $Q(u_n) \to Q(\psi^j)$. Finally, using the identity

$$W_n^M = W_n^{j-1} - \psi^j - \sum_{k=1+j}^{M} e^{-it_n^k \Delta} \psi^k,$$
one gets $W^M_n \to 0$ and $Q(W^M_n) \to 0$ for $M > j$. Similarly, we get the second case: for all $j$, $t^j_n \to \infty$.

\[ \square \]

3. **Gagliardo-Nirenberg inequality.** Denote the real numbers

$$B := \frac{Np - N - \alpha}{2} \quad \text{and} \quad A := 2p - B.$$ 

The goal of this section is to prove a sharp Gagliardo-Nirenberg inequality related to the Choquard problem (1.1).

**Theorem 3.1.** Let $0 < \alpha < N \geq 2$ and $1 + \frac{\alpha}{N} \leq p \leq p^*$. Then,

1. there exists a positive constant $C(N, p, \alpha)$, such that for any $u \in H^2$,

$$\int_{\mathbb{R}^N} (I_\alpha \ast |u|^p)|u|^p \, dx \leq C(N, p, \alpha) \|u\|^A \|\Delta u\|^B. \quad (3.1)$$

Moreover, if $1 + \frac{\alpha}{N} < p < p^*$, then

2. the minimization problem

$$\frac{1}{C(N, p, \alpha)} = \inf \left\{ J(u) := \frac{\|u\|^A \|\Delta u\|^B}{\int_{\mathbb{R}^N} (I_\alpha \ast |u|^p)|u|^p \, dx}, \quad 0 \neq u \in H^2 \right\}$$

is attained in some $Q \in H^2$ satisfying $C(N, p, \alpha) = \int_{\mathbb{R}^N} (I_\alpha \ast |Q|^p)|Q|^p \, dx$ and

$$B\Delta^2 Q + AQ = \frac{2p}{C(N, p, \alpha)} (I_\alpha \ast |Q|^p)|Q|^{p-2}Q = 0; \quad (3.2)$$

3. furthermore

$$C(N, p, \alpha) = \frac{2p}{A} \left( \frac{A}{B} \right)^{\frac{2}{N}} \|\phi\|^{-2(p-1)}, \quad (3.3)$$

where $\phi$ is a ground state solution to (4.1).

**Proof.** The proof contains three steps.

First, let us start by proving the interpolation inequality (3.1). Taking account of Lemma 2.4 and Corollary 2.2, it follows that

$$\int_{\mathbb{R}^N} (I_\alpha \ast |u|^p)|u|^p \, dx \leq C_{N, p, \alpha} \|u\|^2 \frac{2p}{2N_p} \leq C_{N, p, \alpha} \|\Delta u\|^2 \left( \frac{2p}{2N_p} \right) \|u\|^2 \frac{2p}{2N_p}$$

$$\leq C_{N, p, \alpha} \|\Delta u\|^B \|u\|^A.$$

Second, one proves the equation (3.2). Denote $\beta := \frac{1}{C(N, p, \alpha)}$. Using (3.1), there exists a sequence $(v_n)$ in $H^2$ such that $\beta = \lim_n J(v_n)$. Denoting for $a, b \in \mathbb{R}$, the scaling $u_a^b := au(b)$, we compute

$$\|\Delta u^a_b\|^2 = a^2 b^{4-N} \|\Delta u\|^2; \quad \|u^a_b\|^2 = a^2 b^{4-N} \|u\|^2;$$

$$\int_{\mathbb{R}^N} (I_\alpha \ast |u^a_b|^p)|u^a_b|^p \, dx = a^{2p} b^{4-N} \int_{\mathbb{R}^N} (I_\alpha \ast |u|^p)|u|^p \, dx.$$

It follows that

$$J(u^a_b) = J(u).$$

Now, we choose

$$\mu_n := \left( \frac{\|v_n\|}{\|\Delta v_n\|} \right)^{\frac{2}{N}} \quad \text{and} \quad \lambda_n := \frac{\|v_n\|^\frac{2}{N} - 1}{\|\Delta v_n\|^{\frac{2}{N}}}.$$
Thus, \( \psi_n := v_n^{\lambda_n, \mu_n} \) satisfies
\[
\|\psi_n\| = \|\Delta \psi_n\| = 1 \quad \text{and} \quad \beta = \lim_n J(\psi_n).
\]

Then, \( \psi_n \to \psi \) in \( H^2 \) and using Sobolev injections, one gets for a sub-sequence denoted also \( (\psi_n) \),

\[
\int_{\mathbb{R}^N} (I_{\alpha} \ast |\psi_n|^p)|\psi_n|^p \, dx \to \int_{\mathbb{R}^N} (I_{\alpha} \ast |\psi|^p)|\psi|^p \, dx.
\]

In fact, thanks to Lemma 2.1 and Sobolev embedding,

\[
(I_n) := \int (I_{\alpha} \ast |\psi_n|^p)|\psi_n|^p - (I_{\alpha} \ast |\psi|^p)|\psi|^p \, dx
\]

\[
\leq \int \left( |(I_{\alpha} \ast |\psi_n|^p - |\psi|^p)|\psi|^p \right) \, dx
\]

\[
\leq C\|\psi_n\|^p - |\psi|^p \| \frac{2N}{N+\alpha} \left( \|\psi_n\|^p_{2N} + \|\psi\|^p_{2N} \right)
\]

\[
\leq C\|\psi_n\|^p - |\psi|^p \| \frac{2N}{N+\alpha} \left( \|\psi_n\|^p_{H^2} + \|\psi\|^p_{H^2} \right)
\]

\[
\leq C\|\psi_n - \psi\| \frac{2N}{N+\alpha} \left( \|\psi_n\|^p_{H^{2-1}} + \|\psi\|^p_{H^{2-1}} \right) \to 0.
\]

This implies that, when \( n \) goes to infinity

\[
J(\psi_n) \to \frac{1}{\int_{\mathbb{R}^N} (I_{\alpha} \ast |\psi|^p)|\psi|^p \, dx}.
\]

The semi continuity of \( \|\cdot\|_{H^2} \) gives \( \max\{\|\psi\|, \|\Delta \psi\|\} \leq 1 \). Then,

\[
\|\psi\| = \|\Delta \psi\| = 1,
\]

because otherwise, one gets the absurdity \( J(\psi) < \beta \). Thus,

\[
\psi_n \to \psi \quad \text{in} \quad H^2 \quad \text{and} \quad \beta = J(\psi) = \frac{1}{\int_{\mathbb{R}^N} (I_{\alpha} \ast |\psi|^p)|\psi|^p \, dx}.
\]

\( \psi \) satisfies (3.2) because the minimizer satisfies the Euler equation

\[
\partial_\nu J(\psi + \epsilon \eta)|_{\epsilon = 0} = 0, \quad \forall \eta \in C_0^\infty \cap H^2.
\]

Finally, let us establish the equation (3.3). Write \( C(N, p, \alpha) = \frac{1}{\beta} = \int_{\mathbb{R}^N} (I_{\alpha} \ast |\psi|^p)|\psi|^p \, dx \), where \( \psi \) is given in (3.2). Define, the scaling \( \psi = \phi^{a, b} := a \phi(b) \), for \( a, b \in \mathbb{R} \). Then, the equation

\[
B \Delta^2 \phi + A \psi - 2\beta p(I_{\alpha} \ast |\phi|^p)|\phi|^{p-2} \phi = 0,
\]

implies that

\[
A a \left( \frac{B}{A} b^4 \Delta^2 \phi + \phi - 2\beta \frac{2}{p} a^{2(p-1)} b^{2-\alpha} (I_{\alpha} \ast |\phi|^p)|\phi|^{p-2} \phi \right) = 0.
\]

Choosing

\[
b = \left( \frac{A}{B} \right)^{\frac{1}{4}} \quad \text{and} \quad a = \left( \frac{A}{2p} \right)^{\frac{1}{p-2}} \left( \frac{2}{B} \right)^{\frac{1}{2(p-1)}}
\]

it follows that

\[
\Delta^2 \phi + \phi - (I_{\alpha} \ast |\phi|^p)|\phi|^{p-2} \phi = 0.
\]

Now, since \( \|\phi\| = 1 = ab^{-\frac{N}{p}} \|\phi\| \), we get

\[
\beta = \frac{A}{2p} \left( \frac{A}{B} \right)^{\frac{p}{2}} \|\phi\|^{2(p-1)}.
\]

The proof is closed. \( \square \)
4. Existence of ground states. For \( u \in H^2 \) and \( a, b \in \mathbb{R} \), here and hereafter define the quantities
\[
\mu := \min \{ 2a + (N - 4)b, 2a + Nb \}, \quad \bar{\mu} := \max \{ 2a + (N - 4)b, 2a + Nb \};
\]
\( A := \left\{ (a, b) \in \mathbb{R}^+ \times \mathbb{R} \text{ s.t. } \mu \geq 0 \text{ and } \bar{\mu} > 0 \right\}; \)
\( u_{a,b}^\lambda := \lambda^a u(\lambda^{-b}) \), \quad \mathcal{L}_{a,b}(u) := (\partial_\lambda u_{a,b}^\lambda)_{|\lambda=1}; \)
\( K_{a,b}^Q(u) := (2a + (N - 4)b) \| \Delta u \|^2 + (2a + Nb) \| u \|^2; \)
\( K_{a,b}^N(u) := -\frac{2ap + b(N + \alpha)}{p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \, dx; \)
\( S := M + E, \quad K_{a,b} := \mathcal{L}_{a,b} S = K_{a,b}^Q + K_{a,b}^N, \quad H_{a,b} := (1 - \frac{\mathcal{L}_{a,b}}{\mu})S. \)

**Definition 4.1.** We call ground state of (1.1), any solution to
\[ \phi + \Delta^2 \phi - (I_\alpha * |\phi|^p)|\phi|^{p-2} \phi = 0, \quad 0 \neq \phi \in H^2, \] (4.1)
which minimizes the problem
\[ m_{a,b} := \inf_{0 \neq \phi \in H^2_0} \left\{ S(v) \text{ s.t. } K_{a,b}(v) = 0 \right\}. \] (4.2)

**Remark 4.2.** The standing wave \( e^{-it\phi} \) is a global solution to the Schrödinger problem (1.1) which gives the threshold between global well-posedness and finite time blow-up of solutions as proved in section 7.

The following main result of this section follows with variational methods and ensures the existence of ground states.

**Theorem 4.3.** Take \( N \geq 2 \), a couple of real numbers \( (a, b) \in A \) and \( p_* < p < p^* \). Then,
1. \( m := m_{a,b} \) is nonzero and independent of \( (a, b) \);
2. there is a ground state solution to (4.1)-(4.2).

Let us give some intermediate results.

**Lemma 4.4.** Let \( (a, b) \in A \). Then,
1. \( \min \{ \mathcal{L}_{a,b} H_{a,b}(u), H_{a,b}(u) \} > 0 \) for all \( 0 \neq u \in H^2; \)
2. \( \lambda \mapsto H_{a,b}(a^\lambda) \) is increasing.

**Proof.** Compute
\[ H_{a,b}(u) := (1 - \frac{\mathcal{L}_{a,b}}{\bar{\mu}})S(u) = \frac{1}{\bar{\mu}} \left( \bar{\mu} S(u) - K_{a,b}(u) \right) \]
\[ = \frac{1}{\bar{\mu}} \left[ \left( \bar{\mu} - (2a + (N - 4)b) \right) \| \Delta u \|^2 + \left( \bar{\mu} - (2a + Nb) \right) \| u \|^2 \right. \]
\[ \left. + \frac{1}{p} \left( 2ap + b(N + \alpha) - \bar{\mu} \right) \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p \, dx \right]. \]
Since \( \bar{\mu} \geq 0 \) and \( p > p_* \), one obtains, if \( b < 0 \),
\[ 2ap + b(\alpha + N) - \bar{\mu} = 2a(p - 1) + b(\alpha + 4) \]
\[ > 2a(p - 1) - \frac{2a}{N}(4 + \alpha) > 2a(p - p_*) > 0. \] (4.3)
If $b \geq 0$, then $2ap + b(\alpha + N) - \bar{\mu} = 2ap - \bar{\mu} > 0$. Hence, $H_{a,b}(u) > 0$. Moreover,

$$L_{a,b}H_{a,b}(u) = L_{a,b} \left(1 - \frac{L_{a,b}}{\bar{\mu}} \right) E(u)$$

$$= \frac{1}{\bar{\mu}} \left( L_{a,b} - \bar{\mu} \right) \left( L_{a,b} - \mu \right) E(u) + \mu \left( 1 - \frac{L_{a,b}}{\bar{\mu}} \right) E(u)$$

$$= \frac{1}{\bar{\mu}} \left( L_{a,b} - \bar{\mu} \right) \left( L_{a,b} - \mu \right) E(u) + \mu H_{a,b}(u).$$

Since $(L_{a,b} - \bar{\mu})(L_{a,b} - \mu)||u||^2_{H^2} = 0$, one gets

$$L_{a,b}H_{a,b}(u) \geq \frac{1}{\bar{\mu}} \left( L_{a,b} - \bar{\mu} \right) \left( L_{a,b} - \mu \right) \left( \frac{1}{p} \int_{\mathbb{R}^N} (I_{a} \ast |u|^p)|u|^p \, dx \right)$$

$$\geq \frac{1}{p \bar{\mu}} \left( 2ap + b(N + \alpha) - \bar{\mu} \right) \left( 2ap + b(N + \alpha) - \mu \right) \int_{\mathbb{R}^N} (I_{a} \ast |u|^p)|u|^p \, dx.$$ 

Arguing as previously, it follows that $L_{a,b}H_{a,b}(u) > 0$.

The last point follows using the equality $\partial_\lambda H_{a,b}(u^\lambda) = L_{a,b}H_{a,b}(u^\lambda)$. □

The next intermediate result is the following.

**Lemma 4.5.** Let $(a, b) \in A$ and $0 \neq u_n$ be a bounded sequence of $H^2$ such that

$$\lim_n \left( K_{a,b}^Q(u_n) \right) = 0.$$

Then, there exists $n_0 \in \mathbb{N}$ such that $K_{a,b}(u_n) > 0$ for all $n \geq n_0$.

**Proof.** We have

$$K_{a,b}(u_n) = K_{a,b}^Q(u_n) - \frac{2ap + b(N + \alpha)}{p} \int_{\mathbb{R}^N} (I_{a} \ast |u_n|^p)|u_n|^p \, dx.$$

If $b \leq 0$, then $2a + (N - 4)b = \bar{\mu} > 0$ and if $b > 0$, so, $\bar{\mu} = 2a + Nb > 0$, which implies that $b > -\frac{2a}{N}$. Then, $2a + (N - 4)b > 2a - \frac{2a}{N}(N - 4) = \frac{4a}{N} > 0$. Thus,

$$\|\Delta u_n\|^2 \lesssim K_{a,b}^Q(u_n) \to 0.$$

Now, because $B > 2$, using (3.1), for large $n$,

$$\int_{\mathbb{R}^N} (I_{a} \ast |u_n|^p)|u_n|^p \, dx \leq C ||u_n||^4 \|\Delta u_n\|^B = o \left( \|\Delta u_n\|^2 \right) = o(K_{a,b}^Q(u_n)).$$

Thus, when $n \to \infty$,

$$K_{a,b}(u_n) \simeq K_{a,b}^Q(u_n) > 0.$$ □

One can express the minimizing problem (4.2), with negative constraint.

**Lemma 4.6.** Let $(a, b) \in A$. Then,

$$m_{a,b} = \inf_{0 \neq u \in H^2} \left\{ H_{a,b}(u) \text{ s. t } K_{a,b}(u) \leq 0 \right\}.$$

**Proof.** Denoting by $r$ the right hand side of the previous equality, it is sufficient to prove that $m_{a,b} \leq r$. Take $u \in H^2$ such that $K_{a,b}(u) < 0$. Because $\lim_{\lambda \to 0} K_{a,b}^Q(u^\lambda) = 0$, by the previous Lemma, there exists $\lambda \in (0, 1)$ such that $K_{a,b}(u^\lambda) > 0$. With a continuity argument there exists $\lambda_0 \in (0, 1)$ such that $K_{a,b}(u^{\lambda_0}) = 0$, then since $\lambda \mapsto H_{a,b}(u^\lambda)$ is increasing, we get

$$m_{a,b} \leq H_{a,b}(u^{\lambda_0}) \leq H_{a,b}(u).$$

This closes the proof. □
Proof of Theorem 4.3. Let \((\phi_n)\) be a minimizing sequence, namely
\[
0 \neq \phi_n \in H^2_{rad}, \quad K_{a,b}(\phi_n) = 0 \quad \text{and} \quad \lim_n H_{a,b}(\phi_n) = \lim_n S(\phi_n) = m_{a,b}.
\] (4.4)

• First step: \((\phi_n)\) is bounded in \(H^2\).

First case \(a > 0\) and \(b > 0\). Denoting \(\lambda := \frac{b}{2p}\), yields
\[
\|\phi_n\|_{H^2}^2 - \int_{\mathbb{R}^N} (I_\alpha * |\phi_n|^p)|\phi_n|^p \, dx = \lambda \left( 4\|\Delta \phi_n\|^2 - N\|\phi_n\|_{H^2}^2 + \frac{\alpha + N}{p} \int_{\mathbb{R}^N} (I_\alpha * |\phi_n|^p)|\phi_n|^p \, dx \right)
\]
and
\[
\|\phi_n\|_{H^2}^2 - \frac{1}{p} \int_{\mathbb{R}^N} (I_\alpha * |\phi_n|^p)|\phi_n|^p \, dx \rightarrow m_{a,b}.
\]
So the following sequence is bounded
\[
-4\lambda \|\Delta \phi_n\|^2 + \frac{1}{p} \int_{\mathbb{R}^N} (I_\alpha * |\phi_n|^p)|\phi_n|^p \, dx.
\]
Thus, for any real number \(\beta\), the following sequence is also bounded
\[
4\lambda \|\Delta \phi_n\|^2 + (\beta - 1)\|\phi_n\|_{H^2}^2 + \frac{1}{p} \int_{\mathbb{R}^N} (I_\alpha * |\phi_n|^p)|\phi_n|^p \, dx.
\]
Choosing \(\beta \in (1, p + \lambda\alpha)\), it follows that \((\phi_n)\) is bounded in \(H^2\).

Second case \(a > 0\) and \(-\frac{2N}{N-4} < b \leq 0\). Compute
\[
\left(\bar{\mu} - \mathcal{L}_{a,b}\right)S(\phi_n)
\]
\[
= -4b\|\phi_n\|^2 + (2a(p-1) + (\alpha + 4)b) \frac{1}{p} \int_{\mathbb{R}^N} (I_\alpha * |\phi_n|^p)|\phi_n|^p \, dx
\]
\[
\geq (2a(p-1) + (\alpha + 4)b) \frac{1}{p} \int_{\mathbb{R}^N} (I_\alpha * |\phi_n|^p)|\phi_n|^p \, dx.
\]
Moreover, if \(b < 0\), \(\bar{\mu} = 2a + (N-4)b\). Then, since \(\bar{\mu} \geq 0\) and \(p > p^*\), we obtain \(2a(p-1) + (\alpha + 4)b > 0\). Because \(K_{a,b}(\phi_n) = 0\), this implies that
\[
\left(\bar{\mu} + (2a(p-1) + (\alpha + 4)b)\right)S(\phi_n)
\]
\[
= (\bar{\mu} - \mathcal{L}_{a,b})S(\phi_n) + (2a(p-1) + (\alpha + 4)b)S(\phi_n) + \mathcal{L}_{a,b}S(\phi_n)
\]
\[
\geq (2a(p-1) + (\alpha + 4)b)\|\phi_n\|_{H^2}^2.
\]
Hence, \(\phi_n\) is bounded in \(H^2\).

• Second step: the limit of \((\phi_n)\) is nonzero and \(m > 0\).

Taking account of the compact injection in Lemma 2.3, take
\[
\phi_n \rightharpoonup \phi \quad \text{in} \quad H^2
\]
and for all \(2 < p < \frac{2N}{N-4}\), where \(\frac{2N}{N-4} = \infty\) if \(N \leq 4\),
\[
\phi_n \rightarrow \phi \quad \text{in} \quad L^p.
\]
The equality \(K_{a,b}(\phi_n) = 0\) implies that
\[
(2a + (N-4)b)\|\Delta \phi_n\|^2 + (2a + Nb)\|\phi_n\|^2 = \frac{2ap + b(N + \alpha)}{p} \int_{\mathbb{R}^N} (I_\alpha * |\phi_n|^p)|\phi_n|^p \, dx.
\]
Assume that \(\phi = 0\). Using Corollary 2.2, with the fact that \(1 + \frac{N}{N-4} < p < p^*\), write
\[
\int_{\mathbb{R}^N} (I_\alpha * |\phi_n|^p)|\phi_n|^p \, dx \lesssim \|\phi_n\|_{\frac{2p}{2N}}^{2p} \rightarrow 0.
\]
Now, by Lemma 4.5 yields $K_{a,b}(\phi_n)>0$ for large $n$. This contradiction implies that $\phi \neq 0$. Thanks to Lemma 2.1 and Sobolev embedding,

$$
(J_n) := \int |(I_\alpha * |\phi_n|^p)|\phi_n|^p - (I_\alpha * |\phi|^p)|\phi|^p \, dx
$$

$$
\leq \int |(I_\alpha * |\phi_n|^p - |\phi|^p)|\phi|^p \, dx - \int_{\mathbb{R}^N} (I_\alpha * |\phi_n|^p)|\phi|^p - |\phi_n|^p| \, dx
$$

$$
\leq C\|\phi_n^p - |\phi|^p\|_{\frac{2N}{N+2}}^2 \|\phi_n\|^p_{\frac{2N}{N+2}} + \|\phi\|^p_{\frac{2N}{N+2}}
$$

$$
\leq C\|\phi_n^p - |\phi|^p\|_{\frac{2N}{N+2}}^2 \|\phi_n\|^p_{H^2} + \|\phi\|^p_{H^2}
$$

$$
\leq C\|\phi_n - \phi\|_{\frac{2N}{N+2}}^2 \|\phi_n\|_{H^2}^{p-1} + \|\phi\|_{H^2}^{p-1} \to 0.
$$

So, with lower semi continuity of the $H^2$ norm, we have

$$
0 = \liminf_n K_{a,b}(\phi_n)
$$

$$
\geq \frac{2a + (N-4)b}{2} \liminf_n \|\nabla \phi_n\|^2 + \frac{2a + Nb}{2} \liminf_n \|\phi_n\|^2
$$

$$
- \frac{2a \alpha + b(N + \alpha)}{p} \int_{\mathbb{R}^N} (I_\alpha * |\phi|^p)|\phi|^p \, dx
$$

$$
\geq K_{a,b}(\phi).
$$

Similarly, we have $H_{a,b}(\phi) \leq m$. Moreover, thanks to Lemma 4.6, we assume that $K_{a,b}(\phi) = 0$ and $S(\phi) = H_{a,b}(\phi) \leq m$. So, $\phi$ is a minimizer satisfying (4.4) and using previous computation

$$
m = H_{a,b}(\phi) > 0.
$$

- Third step: the limit $\phi$ is a solution to (4.1).

There is a Lagrange multiplier $\eta \in \mathbb{R}$ such that $S'(\phi) = \eta K'_1(\phi)$. Thus,

$$
0 = K_{a,b}(\phi) = \mathcal{L}_{a,b}S(\phi) = \langle S'(\phi), \mathcal{L}_{a,b}(\phi) \rangle = \eta \langle K'_1(\phi), \mathcal{L}_{a,b}(\phi) \rangle,
$$

$$
\mathcal{L}_{a,b}(\phi) = \eta \mathcal{L}_{a,b}K_{a,b}(\phi) = \eta \mathcal{L}_{a,b}^2(\phi).
$$

Taking account of previous computations,

$$
- \mathcal{L}_{a,b}^2 S(\phi) - \bar{\mu}_\alpha S(\phi) = -(\mathcal{L}_{a,b} - \bar{\mu})(\mathcal{L}_{a,b} - \bar{\mu}) S(\phi) > 0.
$$

Therefore, $\mathcal{L}_{a,b} S(\phi) < 0$. Thus, $\eta = 0$ and $S'(\phi) = 0$. So, $\phi$ is a ground state and $m$ is independent of $(a,b)$.

Let us end this section with the so-called generalized Pohozaev identity [15].

**Lemma 4.7.** $\phi \in H^2$ is solution to (4.1) if and only if $S'(\phi) = 0$. Moreover, in a such case

$$
K_{a,b}(\phi) = 0, \quad \text{for any} \quad (a,b) \in \mathbb{R}^2.
$$

5. **Well-posedness in the energy space.** Using a classical fixed point argument and taking account of Strichartz estimates and Sobolev injections, one can obtain the following result.

**Proposition 5.1.** Let $N \geq 2$, $0 < \alpha < N$ such that $\alpha \geq N - 8$, $2 \leq p \leq p^*$ and $u_0 \in H^2$. Then, there exists $T > 0$ such that (1.1) admits a unique local solution

$$
u \in C_T(H^2).
$$

Moreover,

1. the solution satisfies the mass and energy conservation laws;
Taking account of the equation (1.1), one gets

**Proof.**

6. **Virial type identity.** This section is devoted to prove a Virial type identity, which will be useful in order to obtain finite time blow-up of some solutions to the Choquard problem (1.1). Here and hereafter, denote \( \psi \) \( R \)-localized Virial type identity. \( \psi \) Choquard problem (1.1). Here and hereafter, denote \( \psi_R := R^2 \psi(\frac{x}{R}) \), \( R > 0 \), where \( \psi \in C_0^\infty(\mathbb{R}^N) \) is a radial function satisfying \( \psi'' \leq 1 \) and

\[
\psi(x) = \begin{cases} 
\frac{1}{2}|x|^2, & |x| \leq 1; \\
0, & |x| \geq 2.
\end{cases}
\]

A direct computation gives

\[
\psi_R'' \leq 1, \quad \psi_R'(r) \leq r \quad \text{and} \quad \Delta \psi_R \leq N.
\]

Denote the localized Virial

\[
M_\psi[u(t)] := 23 \int_{\mathbb{R}^N} \bar{u}(t) \nabla \psi \nabla u(t) \, dx.
\]

Define the self-adjoint differential operator \( \Gamma_\psi := -i(\nabla \psi + \nabla \psi \cdot \nabla) \), which acts on functions

\[
\Gamma_\psi f = -i \left( \nabla \cdot (\nabla \psi) f + (\nabla \psi) \cdot (\nabla f) \right).
\]

Then,

\[
M_\psi[u(t)] = < u(t), \Gamma_\psi u(t) >.
\]

The main result of this section reads as follows.

**Theorem 6.1.** Let \( N \geq 2 \), \( 0 < \alpha < N \) such that \( \alpha \geq N - 8 \), \( 2 \leq p \leq p^* \) and \( u \in C_T(H^2_{rad}) \) be a solution of (1.1). Then, on \([0, T)\), for any \( R > 0 \) and \( \frac{1}{2} < \mu < 2 \),

\[
\frac{d}{dt} M_\psi[u(t)] \leq 4BE[u] - 2N(p - p_*) \| \Delta u \|^2 + CR^{-4} \\
+ CR^{-2} \| \nabla u \|^2 + \frac{1}{R(N(\frac{2}{N} - \mu)(p - 1 - \frac{2}{N}))} \| \Delta u \|^p \| \nabla u \|^{p + \frac{2}{N}(p - 1 - \frac{2}{N})}.
\]

**Proof.** Taking account of the equation (1.1), one gets

\[
\frac{d}{dt} M_\psi[u(t)] \leq < u(t), \| \Delta^2 + i \Gamma_\psi \| u(t) > + < u(t), -(I_\alpha \ast |u|^p) |u|^{p-2} + i \Gamma_\psi \| u(t) >,
\]

where \([X, Y] := XY - YX\) denotes the commutator of \( X \) and \( Y \). According to computation done in [1], one has

\[
< u(t), \| \Delta^2 + i \Gamma_\psi \| u(t) > \leq 8 \| \Delta u(t) \|^2 + O(R^{-4} + R^{-2} \| \nabla u(t) \|^{2}).
\]

Using computations in [22], it follows that

\[
(N) \leq < u(t), -(I_\alpha \ast |u|^p) |u|^{p-2} + i \Gamma_\psi \| u(t) >
\]

\[
= - \frac{4B}{p} \int_{\mathbb{R}^N} (I_\alpha \ast |u|^p) |u(x)|^p \, dx + O \left( \int_{|x| > R} (I_\alpha \ast |u|^p) |u|^p \, dx \right).
\]
Thanks to (2.1), one has
\[
(I) := \int_{\{|x| > R\}} (I_\alpha \ast |u|^p)|u|^p \, dx
\leq \|u\|_{L^{2N/(N-2)}(\{|x| > R\})}^{2N/(N-2)} \lesssim \|\Delta u\|^p \|u\|_{L^\infty(\{|x| > R\})}^{p-\frac{2}{N}}. 
\]
Take \(\frac{1}{2} < \mu < \min\{2, \frac{N}{2}\}\). Taking account of (2.1) and (2.2), write
\[
(I) \lesssim \|\Delta u\|^p \|u\|_{L^\infty(\{|x| > R\})}^{p-\frac{2}{N}} \lesssim \|\Delta u\|^p \left(R^{-\frac{N}{2} + \mu} \|(-\Delta)^{\frac{\alpha}{2}} u\|\right)^{p-\frac{2}{N}}
\]
\[
\lesssim \|\Delta u\|^p \frac{1}{R^{(\frac{N}{2} - \mu)(p-1-\frac{2}{N})}} \|(-\Delta)^{\frac{\alpha}{2}} u\|^{p-1-\frac{2}{N}}
\]
\[
\lesssim \|\Delta u\|^p \frac{1}{R^{(\frac{N}{2} - \mu)(p-1-\frac{2}{N})}} \left(\|u\|^{1-\frac{2}{N}} \|\Delta u\|^{\frac{2}{N}}\right)^{p-1-\frac{2}{N}}
\]
\[
\lesssim \frac{1}{R^{(\frac{N}{2} - \mu)(p-1-\frac{2}{N})}} \|\Delta u\|^{p + \frac{2}{N} (p-1-\frac{2}{N})}. 
\]

Finally
\[
\frac{d}{dt} M_{\psi_R}[u] = < u, [\Delta^2, i\Gamma_{\psi_R}]u > + < u, (I_\alpha \ast |u|^p)|u|^{p-2}, i\Gamma_{\psi_R}u >
\]
\[
\leq 8\|\Delta u\|^2 + CR^{-4} + CR^{-2}\|\nabla u\|^2
\]
\[
- \frac{AB}{p} \int_{\mathbb{R}^N} (I_\alpha \ast |u|^p)|u|^p \, dx + O \left( \int_{\{|x| > R\}} (I_\alpha \ast |u|^p)|u|^p \, dx \right)
\]
\[
\leq 4BE - 2N(p-p_*)\|\Delta u\|^2 + CR^{-4}
\]
\[
+ CR^{-2}\|\nabla u\|^2 + \frac{1}{R^{(\frac{N}{2} - \mu)(p-1-\frac{2}{N})}} \|\Delta u\|^{p + \frac{2}{N} (p-1-\frac{2}{N})}. 
\]

7. **Global/non global existence of solutions.** In this section, we prove a sharp criteria of finite time blow-up/global existence of solutions to the Choquard problem (1.1) in the focusing regime. In this section one takes \(\epsilon = -1\). Here and hereafter, denote, for \(u \in H^2\), the scale invariant quantities
\[
\mathcal{M}[u] := \frac{E[u] - E[\phi]}{M[\phi] - M[u]}, \quad \mathcal{G}[u] := \frac{\|\Delta u\|^{s_c}\|u\|^{2-s_c}}{\|\Delta \phi\|^{s_c}\|\phi\|^{2-s_c}}. 
\]

The main result of this section reads.

**Theorem 7.1.** Let \(N \geq 2\), \(0 < \alpha < N\) such that \(\alpha > N - 8\), \(0 < s_c < 2\), \(\phi\) be a ground state solution to (4.1) and a maximal solution \(u \in C_T(H^2_{rad})\) of (1.1).

Suppose that
\[
\mathcal{M}[u] < 1. \tag{7.1}
\]

1. Assume that \(p < 3\) and
\[
\mathcal{G}[u] > 1. \tag{7.2}
\]

Then, \(u\) blows-up in finite time, i.e, \(0 < T^* < \infty\) and
\[
\limsup_{t \to T^*} \|\Delta u(t)\| = +\infty;
\]

2. Assume that \(E(u_0) \geq 0\) and
\[
\mathcal{G}[u] < 1. \tag{7.3}
\]
Then, $T^* = \infty$ and $u$ scatters. Precisely, there exists $\psi \in H^2$ such that
\[
\limsup_{t \to \infty} \|u(t) - e^{it\Delta^2} \psi\|_{H^2} = 0.
\]

**Remark 7.2.** 1. The unnatural condition $p < 3$ which seems to be technical is due to a lack of a Virial identity similar to the NLS case;
2. the radial condition is required for the Virial identity in the first case and is assumed for simplicity in the second case;
3. scattering is proved in the next section;
4. the proof of next auxiliary result is omitted because it follows like in [22].

**Lemma 7.3.** The next conditions are invariant under the flow of the problem (1.1), 1. (7.1) and (7.2); 2. (7.1) and (7.3).

**Remark 7.4.** The global well-posedness part of Theorem 7.1 is a consequence of

\[ \tag{7.3} \]

**Lemma 7.5.** Assume that

\[
\text{the second point in Lemma 7.3.}
\]

Then, $T^* < \infty$.

**Proof.** Using the properties of $\psi$, write

\[
|M_{\psi R}[u(t)]| \leq 2\|\nabla \psi_R\|_\infty \|u(t)\| \|\nabla u(t)\| \leq CR\|u_0\| \|\nabla u(t)\| \leq CR\|u_0\|^3 \|\Delta u(t)\|^2.
\]

Thus,

\[
M_{\psi R}[u(t)] \leq -CR \int_{t_0}^t |M_{\psi R}[u(\tau)]|^4 \, d\tau.
\]

Take $z(t) := \int_{t_0}^t |M_{\psi R}[u(\tau)]|^4 \, d\tau$. Then, $z \geq CR^4 t^4 > 0$ for $t > t_0$. Integrating the previous inequality, one obtains for some $t_* > 0$,

\[
\lim_{t \to t_*} M_{\psi R}[u(t)] = -CR \lim_{t \to t_*} z(t) = -\infty.
\]

Then, $u$ cannot be global. Hence $T^* < \infty$. \hfill \Box

We are ready to prove Theorem 7.1. Assume that (7.1)-(7.2) are satisfied and take $\eta > 0$ satisfying

\[
E(u_0)^{s_c} M(u_0)^{2-s_c} < [(1-\eta)E(\phi)]^{s_c} M(\phi)^{2-s_c}
\]

Then, thanks to (7.2), one gets

\[
(1-\eta)(B-2)\|\Delta u(t)\|^2 > BE(u_0).
\]

With Theorem 6.1, for $O_R(1) \to 0$ uniformly in time, and using Young inequality via the fact that $p < 3$, one gets

\[
\frac{d}{dt} M_{\psi_R}[u(t)] \leq 4BE(u_0) - 2N(p-p_*)\|\Delta u\|^2 + CR^{-4}
\]

\[
+ CR^{-2}\|\nabla u\|^2 + \frac{1}{R(\frac{2}{p} - \eta)(p-1 - \frac{2}{p})}\|\Delta u\|^{p + \frac{2}{p} - 1 - \frac{2}{p}}
\]

\[
\leq 2(2(1-\eta)(B-2) - N(p-p_*) + O_R(1))\|\Delta u\|^2
\]
Thanks to Sobolev injection, one has

\[ \left\| \frac{1}{R^{\frac{1}{2}}}(\frac{1}{2} - \frac{1}{p}) \right\| \Delta u \right\|_{p^+}^2 + O_R(1) \leq (-4\eta(B - 2) + O_R(1)) \left\| \Delta u(t) \right\|^2 + O_R(1) \]

\[ + \frac{1}{R^{\frac{1}{2}}}(\frac{1}{2} - \frac{1}{p}) \right\| \Delta u \right\|_{p^+}^2 + O_R(1) \leq (-4\eta(B - 2) + O_R(1)) \left\| \Delta u(t) \right\|^2 + O_R(1) \leq -2\eta(B - 2) \left\| \Delta u(t) \right\|^2. \]

The proof is a consequence of Lemma 7.5.

8. **Scattering.** This section is devoted to prove scattering of global solutions to (1.1), precisely the second part of Theorem 7.1 is proved. For a slab \( I \subset \mathbb{R} \) and \( p > p_* \), define the spaces

\[ S(I) := L^{2p}(I, L^{\frac{2Np}{Np - 1}}) \quad \text{and} \quad W(I) := L^{2p}(I, L^{\frac{2Np}{Np - 1}}). \]

**Remark 8.1.** Thanks to Sobolev injection, one has

\[ \| \cdot \|_{S(I)} \leq C \| |\nabla|^{p_\ast} \cdot \|_{W(I)}. \]

**Proposition 8.2** (Small data). Let \( u_0 \in H^2 \). Then, there exists \( \delta > 0 \) such that if \( \| e^{i\Delta t}u_0 \|_{S(I)} \leq \delta \), then there exists \( u \in C(I, H^2) \) solving (1.1) satisfying

\[ \| u \|_{S(I)} \leq 2\delta \quad \text{and} \quad \| (1 + \Delta)u \|_{W(I) \cap L^\infty(L, L^2)} < cA. \]

**Proof.** First, let us use a fixed point argument. For \( T > 0 \) and \( I := (0, T) \), take the set

\[ X_{\delta, M} := \{ v \in S(I), \quad \| v \|_{S(I)} \leq 2\delta \quad \text{and} \quad \| (1 + \Delta)v \|_{W(I) \cap L^\infty(L, L^2)} \leq M \} \]

equipped with the complete distance

\[ d(u, v) := \| u - v \|_{W(I)}. \]

Set the function

\[ \tilde{v} := u_0(v) := e^{i\Delta t}u_0 + i \int_0^1 e^{i(s-t)\Delta t} (I \ast |v|^p) |v|^{p-2}v(s)ds. \]

By the Strichartz estimate Hölder and Hardy-Littlewood-Sobolev inequalities, one gets for \( (q,r) := (2p, \frac{2Np}{Np-2}) \) and \( w := u - v \),

\[ d(\tilde{u}, \tilde{v}) \leq C((I \ast |u|^p) |u|^{p-2}u - (I \ast |v|^p) |v|^{p-2}v) \leq (I \ast |u|^p) |u|^{p-2}u - (I \ast |v|^p) |v|^{p-2}v \]

\[ \lesssim \| (I \ast |u|^p) |u|^{p-2}u - (I \ast |v|^p) |v|^{p-2}v \|_{L^q(I, L^r)} \]

\[ \lesssim \| (I \ast |u|^p) |u|^{p-2} + |v|^{p-2}u \|_{L^q(I, L^r)} + \| (I \ast |u|^{p-1} + |v|^{p-1})w \|_{L^q(I, L^r)} \]

\[ \lesssim (\| u \|_{S(I)} + \| v \|_{S(I)}) \| w \|_{L^q(I, L^r)} \leq C\delta^{2(p-1)}d(u, v). \]

Now, by the Strichartz estimate, Hardy-Littlewood-Sobolev inequality and fractional chain rule, one gets for \( cA := \frac{M}{2} \),

\[ (I) := \| (1 + \Delta)\tilde{v} \|_{W(I) \cap L^\infty(L, L^2)} \]

\[ \leq \| u_0 \|_{H^2} + C((1 + \Delta)((I \ast |v|^p) |v|^{p-2}v) \|_{L^q(I, L^r)} \]

\[ \leq \frac{M}{2} + C\| v \|_{S(I)}^{2(p-1)} \| (1 + \Delta)v \|_{W(I)} \leq \frac{M}{2} + C\delta^{2(p-1)}M. \]
Thanks to the Sobolev injection in the previous remark, yields
\[ \|\tilde{v}\|_{S(I)} \leq \delta + C\|(1 + \Delta)(\tilde{v} - e^{i\Delta^2 u_0})\|_{W(I)} \leq \delta + C\delta^{2(p - 1)} M. \]

Taking \(\delta > 0\) small enough, it follows that \(\phi_{u_0}\) is a contraction of \(X_{\delta,M}\). Then, the fixed point principle gives the result.

\[ \square \]

**Proposition 8.3** (Long time perturbation theory). *Let \(0 \in I \subset \mathbb{R}\), a time slab. Take \(u \in C(I,H^2)\) a solution of (1.1). Let \(\tilde{u} \in L^\infty(I,H^2)\) satisfying \(\|\tilde{u}\|_{L^\infty(I,H^2) \cap S(I)} \leq A\) for some constant \(A > 0\). Assume that
\[ i\dot{\tilde{u}} + \Delta \tilde{u} + (I_\alpha * |\tilde{u}|^p)\tilde{u}^p - 2\tilde{u} = e \]
and that for \((q,r) := (2p, \frac{2Np}{Np - 4})\), \(\epsilon > 0\),
\[ \|(1 + \Delta)e\|_{L^{\infty}(I,L^{q,r})} \leq \epsilon, \quad \|e^{i\Delta^2}\|_{S(I)} \leq \epsilon. \]

Then, there exists \(\epsilon_0 := \epsilon_0(A)\) such that for \(0 < \epsilon < \epsilon_0\),
\[ \|u\|_{S(I)} \leq C(A). \]

**Proof.** For \(\delta = \delta(A) > 0\) small enough, split \(I \subset \bigcup_j I_j\) such that \(\|\tilde{u}\|_{S(I)} \leq \delta\). Using Duhamel formula and arguing as previously, one gets for \(1 - C\delta^{2(p - 1)} > 0\),
\[ \|(1 + \Delta)\tilde{u}\|_{W(I_j)} \leq CA + C\|I\|_{S(I_j)}^{2(p - 1)} \|(1 + \Delta)\tilde{u}\|_{W(I_j)} + C\|e\|_{L^{\infty}(I,W^{2,q,r})} \leq C(A + \epsilon). \]

Letting \(I_j := [t - 1 + j, t_j]\), one gets
\[ w(t) := u(t) - \tilde{u}(t) \]
\[ = \int_t^{t_j} e^{i(t - t')\Delta^2}[(I_\alpha * |\tilde{u} + w|^p)|\tilde{u} + w|^p - (I_\alpha * |\tilde{u}|^p)|\tilde{u}^p - 2\tilde{u}] dt' \]
\[ + e^{i(t - t_j)\Delta^2}w(t_j) - \int_{t_j}^t e^{i(t - t')\Delta^2}e(t') ds. \]

With a Picard fixed point argument and arguing as in Proposition 8.2, one solves the previous integral equation in \(I_1 = [t_0, t_1] := [0, t_1]\), precisely
\[ \|u\|_{S(I_1)} \leq 2\epsilon, \quad \|(1 + \Delta)w\|_{W(I_1)} \leq C(\epsilon, A). \]

Now, by taking \(t = t_1\) in the previous integral equality and applying \(e^{i(t - t_1)\Delta^2}\), yields
\[ e^{i(t - t_1)\Delta^2}w(t_1) = \int_{t_0}^{t_1} e^{i(t - t'_1)\Delta^2}[(I_\alpha * |\tilde{u} + w|^p)|\tilde{u} + w|^p - 2|\tilde{u} + w| - (I_\alpha * |\tilde{u}|^p)|\tilde{u}^p - 2\tilde{u}] dt' \]
\[ + e^{i(t - t_0)\Delta^2}w(t_0) + \int_{t_0}^{t_1} e^{i(t - t')\Delta^2}e(t') ds. \]

Then, with similar to previous computation, one obtains
\[ \|e^{i(t - t_1)\Delta^2}w(t_1)\|_{S(I_1)} \leq 2\|e^{i(t - t_0)\Delta^2}w(t_0)\|_{S(I_1)} + C\epsilon. \]

Now, iterate the beginning with \(j = 0\), and we obtain
\[ \|e^{i(-t_j)\Delta^2}w(t_j)\|_{S(I)} \leq 2^j\|e^{i(-t_0)\Delta^2}w(t_0)\|_{S(I)} + C2^{j+1}\epsilon \leq C2^{1+j}\epsilon. \]

This finishes the proof. \(\square\)
Proposition 8.4 (Scattering). Let $u \in C(\mathbb{R}, H^2)$ be a global solution to (1.1) with Strichartz norm
\[
\|u\|_{S(\mathbb{R})} < \infty \quad \text{and} \quad \|u\|_{L^\infty(\mathbb{R}, H^2)} < \infty,
\]
then $u(t)$ scatters in $H^2$ as $t \to \infty$. Precisely, there exists $\phi \in H^2$ such that
\[
\lim_{t \to \infty} \|u(t) - e^{it\Delta^2} \phi\|_{H^2} = 0.
\]

Proof. Write with the integral formula
\[
\begin{aligned}
u &= e^{i\Delta^2} u_0 + i \int_0^t e^{i(t-s)\Delta^2} \left[(I_\alpha * |u|^p)|u|^{p-2}u\right] ds; \\
\phi &= u_0 + i \int_0^\infty e^{-is\Delta^2} \left[(I_\alpha * |u|^p)|u|^{p-2}u\right] ds; \\
u - e^{i\Delta^2} \phi &= -i \int_0^\infty e^{i(t-s)\Delta^2} \left[(I_\alpha * |u|^p)|u|^{p-2}u\right] ds.
\end{aligned}
\]

Using Corollary 2.2, write
\[
\|\Delta \left(u - e^{i\Delta^2} u_0\right)\|_{L_p(R)} \lesssim \left\| \left[(I_\alpha * |\Delta|(|u|^p))u|^{p-2}u + (I_\alpha * |u|^p)\Delta(|u|^p)\right] \right\|_{L^p(R)} + \left\| \left| \nabla \left(|u|^p\right)\right|_{L^p(R)} \right\|_{L^p'(R)} := (A) + (B) + (C).
\]
Thus, using the identity $|\Delta(|u|^p)| \leq C_p \|\Delta u\|_p \|u|^{p-1} + \|\nabla u\|^2 \|u|^{p-2}$, denoting $S(I) := L^2_p(I, L^p)$, $\frac{1}{p} = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{a}\right)$ and taking account of the inequality $\|\nabla \|_p \leq C \|\Delta \|_r \|\cdot\|_a$ via Hardy-Littlewood-Sobolev inequality, one gets
\[
(A) \lesssim \|u\|_{S(\mathbb{R})}^{2(p-1)} \|\Delta u\|_{W(\mathbb{R})} + \|\|\nabla u\|_{L^2}^{2} \|u\|_{L^a}^{2p-3} \|q\| \lesssim \|u\|_{S(\mathbb{R})}^{2(p-1)} \|\Delta u\|_{W(\mathbb{R})}.
\]
With the same way, for $p \geq 3$,
\[
(A) + (B) + (C) \lesssim \|u\|_{S(\mathbb{R})}^{2(p-1)} \|\Delta u\|_{W(\mathbb{R})}.
\]
Now, by previous computation
\[
\|\Delta u\|_{W(t, \infty)} \leq C \|u\|_{L^\infty(\mathbb{R}, H^2)} + C \|u\|_{S(t, \infty)}^{2(p-1)} \|\Delta u\|_{W(t, \infty)}.
\]
Taking $t > 0$ large enough such that $\|u\|_{S(t, \infty)} < 1$, then a partition of $[0, t] \subset \bigcup I_j$ with $\sup_j \|u\|_{S(I_j)} < 1$, this implies that
\[
\|\Delta u\|_{W(\mathbb{R})} < \infty.
\]
Thus, when $t \to \infty$,
\[
\|\Delta \left(u - e^{i\Delta^2} \phi\right)\|_{W(t, \infty) \cap L^\infty((t, \infty), L^2)} \leq C \|u\|_{S(t, \infty)}^{2(p-1)} \|\Delta u\|_{W(t, \infty)} \to 0.
\]
With the same way, we prove that when $t \to \infty$,
\[
\|u - e^{i\Delta^2} \phi\|_{L^\infty((t, \infty), L^2)} \to 0.
\]
Finally when $t \to \infty$,
\[
\|u - e^{i\Delta^2} \phi\|_{L^\infty((t, \infty), H^1)} \to 0.
\]
8.1. Critical solution and compactness. In this section, we prepare the proof of the scattering part of Theorem 7.1. Let $u$ be the solution of (1.1) such that the assumptions of the second part of Theorem 7.1 hold. Then, we know that $u$ is global. Thus, combined with Proposition 8.4, the goal is to show that

$$u \in S(\mathbb{R}).$$

Let us prove the claim: there exists $\delta > 0$ such that if

$$E[u_0] M[u_0] \frac{2}{p} - 1 < \delta \quad \text{and} \quad \|u_0\| \frac{2}{p} - 1 \|\Delta u_0\| < \|\phi\| \frac{2}{p} - 1 \|\Delta \phi\|,$$

then $u \in S(\mathbb{R})$. Indeed, write

$$E(u) = \|\Delta u\| \frac{p}{2} - 1 \int_{\mathbb{R}^N} (I_a * |u|^p)|u|^p \, dx \geq \|\Delta u\| \frac{p}{2} \left(1 - \frac{C_{N,p,\alpha}}{p} \|u\| A^N \|\Delta u\|^{B - 2}\right) \geq \|\Delta u\| \frac{p}{2} \left(1 - \frac{2}{A} \left(\frac{B}{A}\right)^2 \frac{\|u\| A^N \|\Delta u\|^{B - 2}}{\|\phi\|^{2(p - 1)}}\right).$$

Taking account of Pohozaev identity, one gets $\|\Delta \phi\| = \frac{B}{A} \|\phi\|$. Then,

$$E(u) \geq \|\Delta u\| \frac{p}{2} \left(1 - \frac{2}{B} \left(\frac{\|u\| A^N \|\Delta u\|^{B - 2}}{\|\phi\|^{2(p - 1)}}\right) \geq \|\Delta u\| \frac{p}{2} \left(1 - \frac{2}{B} \left(\frac{\|u\| A^N \|\Delta u\|^{B - 2}}{\|\phi\|^{2(p - 1)}}\right) \geq \|\Delta u\| \frac{p}{2} \left(1 - \frac{2}{B} \right) \geq \|\Delta u\| \frac{p}{2} \left(1 - \frac{2}{B} \right). \quad (8.1)$$

Since $p > p_*, B > 2$, $E(u)$ is conserved implies that $\|\Delta u(t)\|$ is bounded. The claim follows by Proposition 8.2.

Now, for each $\delta > 0$, define the set

$$S_\delta := \{u_0 \in H^2, E[u_0] M[u_0] \frac{2}{p} - 1 < \delta \quad \text{and} \quad \|u_0\| \frac{2}{p} - 1 \|\Delta u_0\| < \|\phi\| \frac{2}{p} - 1 \|\Delta \phi\|\}.$$

Define also $(ME)_c := \sup \{\delta > 0 \text{ s. t } u_0 \in S_\delta \Rightarrow u \in S(\mathbb{R})\}$. The goal is to prove that $(ME)_c = M[\phi] \frac{2}{p} - 1 E[\phi]$. By contradiction, assume that

$$(ME)_c < M[\phi] \frac{2}{p} - 1 E[\phi]. \quad (8.2)$$

**Proposition 8.5** (Existence of wave operator). *Let $\phi$ be a ground state solution to (4.1) and $\psi \in H^2$ satisfying

$$\|\psi\| \frac{2(2 - \epsilon)}{2 - \epsilon} \|\Delta \psi\| \leq \|\phi\| \frac{2(2 - \epsilon)}{2 - \epsilon} E[\phi].$$

Then, there exists $v \in C(\mathbb{R}, H^2)$ a solution to (1.1) which satisfies

$$\|v(t)\| \frac{2 - \epsilon}{2 - \epsilon} \|\Delta v(t)\| \leq \|\phi\| \frac{2 - \epsilon}{2 - \epsilon} \|\Delta \phi\|, \quad M(v) = \|\psi\|^2, \quad E(v) = \|\Delta \psi\|^2$$

and

$$\lim_{t \to \infty} \|v(t) - e^{it\Delta} \psi\|_{H^2} = 0.$$*
Proposition 8.6 (Existence of a critical solution). Assume that \((ME)_c < M[\phi]^{\frac{2}{2-r_c}}\) \(E[\phi]\). Then, there exists a global solution \(u_c\) to (1.1) with data \(u_{c,0}\) such that \(\|u_{c,0}\| = 1\),

\[
\|\Delta u_{c,0}\| < \|\phi\|^{\frac{2-r_c}{2-r_c}} \|\Delta \phi\|,
\]

\(E[u_c] = (ME)_c\) and \(\|u_c\|_{S(\mathbb{R})} = \infty\).

Proof. There exists a sequence of solutions \(u_n\) to (1.1) with \(H^2\) data \(u_{n,0}\) (rescaled to satisfy \(\|u_n\| = 1\)) such that \(\|\Delta u_{n,0}\| < \|\phi\|^{\frac{2-r_c}{2-r_c}} \|\Delta \phi\|\), \(E[u_{n,0}] \to (ME)_c\) and for any \(n\), \(\|u_n\|_{S(\mathbb{R})} = \infty\). Using the profile decomposition, one gets

\[
u_{n,0} = \sum_{j=1}^{M} e^{-it_j \Delta^2} \psi_j + W_n^M;
\]

\[
E(u_n) = \sum_{j=1}^{M} E(e^{-it_j \Delta^2} \psi_j) + E(W_n^M) + o_n(1).
\]

Then,

\[
(ME)_c = \sum_{j=1}^{M} \lim_{n} E(e^{-it_j \Delta^2} \psi_j) + \lim_{n} E(W_n^M).
\]

With the profile decomposition,

\[
\|\Delta u_{n,0}\|^2 = \sum_{j=1}^{M} \|\Delta \psi_j\|^2 + \|\Delta W_n^M\|^2 + o_n(1); \quad 1 = \sum_{j=1}^{M} \|\psi_j\|^2 + \|W_n^M\|^2 + o_n(1).
\]
Then, $\sum_{j=1}^{M} \|\Delta \psi^j\|^2 \leq \limsup_n \|\Delta u_n,0\|^2$ and $\sum_{j=1}^{M} \|\psi^j\|^2 \leq 1$. So, $\|\Delta \psi_j\| < \|\phi\| \frac{e^{\gamma t}}{\rho} \|\Delta \phi\|$ and with the same way $\lim_n \|\Delta W_n^M\| < \|\phi\| \frac{e^{\gamma t}}{\rho} \|\Delta \phi\|$. Thus, by (8.1), $E(e^{-i t_n^j \Delta^2 \psi^j}) \geq 0$, $\lim_n E(W_n^M) \geq 0$ and so

$$
\lim_n E(e^{-i t_n^j \Delta^2 \psi^j}) \leq (ME)_c.
$$

Claim: only one $\psi_j \neq 0$.

Assume the contrary of the claim. Then, $M[\psi_j] < 1$ for any $j$ and so for large $n$,

$$
M(e^{-i t_n^j \Delta^2 \psi^j}) \frac{2 - e^{\gamma t_n}}{\rho} E(e^{-i t_n^j \Delta^2 \psi^j}) < (ME)_c.
$$

If $|t_n^j| \to +\infty$, assume that up to a sub-sequence, $t_n^j \to \pm \infty$. In this case, by the decay of the linear flow,

$$
\lim_n Q(e^{-i t_n^j \Delta^2 \psi^j}) = 0, \ \forall k.
$$

Then,

$$
\|\psi^j\| \frac{2(2 - e^{\gamma t_n})}{\rho} \|\Delta \psi^j\|^2 = \|e^{-i t_n^j \Delta^2 \psi^j}\| \frac{2(2 - e^{\gamma t_n})}{\rho} \|\Delta[e^{-i t_n^j \Delta^2 \psi^j}]\|^2 < (ME)_c.
$$

Then, from the existence of wave operators (Proposition 8.5) there exists $\tilde{\psi}^j$ such that $\tilde{v}$ the solution of (1.1) with data $\tilde{\psi}^j$ satisfies

$$
\lim_n \|\tilde{v}(-t_n^j) - e^{-i t_n^j \Delta^2 \psi^j}\|_{H^2} = 0,
$$

$$
\|\tilde{\psi}^j\| \frac{2 - e^{\gamma t_n}}{\rho} \|\Delta \tilde{v}(t)\| < \|\phi\| \frac{2 - e^{\gamma t_n}}{\rho} \|\Delta \phi\|, \ \ M(\tilde{\psi}^j) = M(\psi), \ E(\tilde{v}) = \|\Delta \psi^j\|^2.
$$

Then,

$$
M(\tilde{\psi}^j) \frac{2 - e^{\gamma t_n}}{\rho} E(\tilde{\psi}^j) < (ME)_c, \ \ \forall v \in S(\mathbb{R}).
$$

If, $t_n^j \to t'$ finite, then by the continuity of the linear flow in $H^2$, we have

$$
\lim_n \|e^{-i t_n^j \Delta^2 \psi^j} - e^{-i t' \Delta^2 \psi^j}\|_{H^2} = 0.
$$

Let $\tilde{\psi}^j = BNLS(t')e^{-i t' \Delta^2 \psi^j}$ so that $BNLS(-t')\tilde{\psi}^j = e^{-i t' \Delta^2 \psi^j}$.

In both cases, there is a new profile $\tilde{\psi}^j$ associated to each original profile $\psi^j$ such that

$$
\lim_n \|BNLS(-t_n^j)\tilde{\psi}^j - e^{-i t_n^j \Delta^2 \psi^j}\|_{H^2} = 0.
$$

So, one can replace $e^{-i t_n^j \Delta^2 \psi^j}$ by $BNLS(-t_n^j)\tilde{\psi}^j$ in (8.3) to obtain

$$
u_n,0 = \sum_{j=1}^{M} BNLS(-t_n^j)\tilde{\psi}^j + \tilde{W}_n^M,
$$

where

$$
\lim_{M \to \infty} \lim_{n \to \infty} \|e^{i \Delta^2 \tilde{W}_n^M}\|_{S(\mathbb{R})} = 0.
$$

Denote $\nu^j = BNLS(.)\tilde{\psi}^j$, $u_n = BNLS(.)u_{n,0}$, and $\tilde{u}_n = \sum_{j=1}^{M} \nu^j(-, - t_n^j)$. Then,

$$
i \tilde{u}_n + \Delta^2 \tilde{u}_n - \left(I_{\alpha} \ast |\tilde{u}_n|^p\right) |\tilde{u}_n|^{p-2} \tilde{u}_n = e_n,
$$

where

$$
-e_n = \left(I_{\alpha} \ast |\tilde{u}_n|^p\right) |\tilde{u}_n|^{p-2} \tilde{u}_n - \sum_{j=1}^{M} \left(I_{\alpha} \ast |\nu^j(-, - t_n^j)|^p\right) |\nu^j(-, - t_n^j)|^{p-2} \nu^j(-, - t_n^j).
$$

Using the profile decomposition, write

$$
\|e^{-i \Delta^2 \left(\tilde{u}_n - u_n\right)(0)}\|_{S(\mathbb{R})}
$$
\[
\leq \sum_{j=1}^{M} \|e^{-i \Delta^2} (-t_j^n) - e^{-it_j^0 \Delta^2} \psi_j \|_{S(\mathbb{R})} + \|e^{-i \Delta^2} W_n^M \|_{S(\mathbb{R})}
\]
\[
\leq \sum_{j=1}^{M} \|\psi_j \|_{H^\alpha} + \|e^{-i \Delta^2} W_n^M \|_{S(\mathbb{R})}.
\]

Then,
\[
\lim_{M} \lim_{n} \sup \|e^{-i \Delta^2} (\tilde{u}_n - u_n)(0) \|_{S(\mathbb{R})} = 0.
\]

Let us prove two claims.

Claim 1: There exists a large constant \(A\) such that for any \(M\), there exists \(n_0 := n_0(M)\) such that for \(n > n_0\), \(\|u_n\|_{S(\mathbb{R})} < A\).

Claim 2: For each \(M\) and \(\epsilon > 0\), there exist \(n_1 = n_1(M, \epsilon)\) such that for \(n > n_1\), \(\|(1 + \Delta) e_n\|_{W^\epsilon(\mathbb{R})} < \epsilon\).

Let \(M_0\) be sufficiently large such that \(\|e^{i \Delta^2} \tilde{W}_n^{M_0} \|_{S(\mathbb{R})} < \frac{\delta}{2} \) (defined in Proposition 8.2). Thus, from the definition of \(\tilde{W}_n\) that for any \(j > M_0\), \(\|e^{i \Delta^2} \psi_j (-t_j^n) \|_{S(\mathbb{R})} < \delta\). By Proposition 8.2, one obtains
\[
\|v_j (-t_j^n) \|_{S(\mathbb{R})} < 2 \|e^{i \Delta^2} \psi_j (-t_j^n) \|_{S(\mathbb{R})} < 2\delta,
\]
\[
\|(1 + \Delta) v_j (-t_j^n) \|_{W(\mathbb{R})} < c \|v_j (-t_j^n) \|_{H^2}.
\]

Using the identity \(\lim_n \|v_j (-t_j^n) - e^{-it_j^0 \Delta^2} \psi_j \|_{H^2} = 0\), one gets
\[
\|(1 + \Delta) v_j (-t_j^n) \|_{W(\mathbb{R})} < c \|e^{-it_j^0 \Delta^2} \psi_j \|_{H^2} < c \|\psi_j \|_{H^2}.
\]

Thus, by elementary calculation,
\[
\|(1 + \Delta) \tilde{u}_n \|_{W(\mathbb{R})} \leq \sum_{j=1}^{M_0} \|(1 + \Delta) v_j \|_{W(\mathbb{R})} + \sum_{j=M_0+1}^{M} \|(1 + \Delta) v_j \|_{W(\mathbb{R})}
\]
\[
\leq \sum_{j=1}^{M_0} \|(1 + \Delta) v_j \|_{W(\mathbb{R})} + c \sum_{j=M_0+1}^{M} \|\psi_j \|_{H^2}.
\]

On the other hand, by the profile decomposition,
\[
\|\Delta u_{n,0} \|^2 = \sum_{j=1}^{M_0} \|\Delta \psi_j \|^2 + \sum_{j=M_0+1}^{M} \|\Delta \psi_j \|^2 + \|\Delta W_n^M \|^2 + o_n(1).
\]

Then, \(\sum_{j=M_0+1}^{M} \|\psi_j \|^2_{H^2} \) is bounded independently of \(M\) and so \(\|(1 + \Delta) \tilde{u}_n \|_{W(\mathbb{R})} \) is bounded independently of \(M\), for large \(n\). By Sobolev injection \(\|\tilde{u}_n\|_{S(\mathbb{R})} \) is bounded. Then, Claim 1 holds.

Write the expansion of \(e_n\),
\[
-e_n = (I_\alpha * |\tilde{u}_n|^p) |\tilde{u}_n|^p - \sum_{j=1}^{M} (I_\alpha * |v_j^n|^p) |v_j^n|^p - \sum_{j=1}^{M} (I_\alpha * |v_j^n|^p) |v_j^n|^p - 2v_j^n,
\]
\[
= (I_\alpha * |v_j^n|^p) \sum_{j=1}^{M} |v_j^n|^p - \sum_{j=1}^{M} (I_\alpha * |v_j^n|^p) |v_j^n|^p - 2v_j^n.
\]
Then,
\[-e_n = (I_\alpha \ast |v_{i_n}^j|^p - \sum_{j=1}^{M} |v_{i_n}^j|^p)|v_{i_n}^j|^{p-2} \sum_{j=1}^{M} v_{i_n}^j \]
\[+ \sum_{j=1}^{M} (I_\alpha \ast |v_{i_n}^j|^p)|v_{i_n}^j|^{p-2} \sum_{j=1}^{M} v_{i_n}^j - \sum_{j=1}^{M} (I_\alpha \ast |v_{i_n}^j|^p)|v_{i_n}^j|^{p-2} v_{i_n}^j \]
\[= (I_\alpha \ast |v_{i_n}^j|^p - \sum_{j=1}^{M} |v_{i_n}^j|^p)|v_{i_n}^j|^{p-2} \sum_{j=1}^{M} v_{i_n}^j \]
\[+ \sum_{j=1}^{M} (I_\alpha \ast |v_{i_n}^j|^p)|v_{i_n}^j|^{p-2} \sum_{j \neq k=1}^{M} v_{i_n}^k. \]

Then, taking a cross term and arguing as previously and using the inequality

\[|\sum_{j=1}^{M} a_j r - \sum_{j=1}^{M} a_j^j| \leq C_M \sum_{1 \leq j \neq k \leq M} a_j a_k^{p-1}, \quad a_j \geq 0, \]

one gets as previously

\[(A) := \| (1 + \Delta) \left[ (I_\alpha \ast |v_{i_n}^j|^p |v_{i_n}^m|^p) |v_{i_n}^j|^{p-2} \right] \|_{W(\mathbb{R})} \]
\[= \left\| (1 + \Delta) \left[ (I_\alpha \ast |v^j(\cdot - (t_n^j - t_{i_n}^j))|^{p-1} |v^m(\cdot - (t_n^m - t_{i_n}^m))| \right. \right. \]
\[- \left[ v^j(t)|^{p-2} v^k(\cdot - (t_k^j - t_{i_n}^j)) \right] \right\|_{W(\mathbb{R})} \]
\[\leq \| v^j \|^p_{S_{\mathbb{R}}^1} \| v^m \|^p_{S_{\mathbb{R}}^1} \| v^j \|^p_{S_{\mathbb{R}}^2} \| (1 + \Delta) v^k(\cdot - (t_k^j - t_{i_n}^j)) \right\|_{W(\mathbb{R})}. \]

By the fact that \(|t_n^j - t_{i_n}^j| \to \infty, \) for \(1 \leq k \neq j \leq M, \) the cross terms go to zero as \(n \to \infty\) and Claim 2 is proved.

Claim 1 and Claim 2 give a contradiction with Proposition 8.3. This implies that the profile expansion is reduced to the case \(\tilde{\psi} \neq 0\) and \(\psi_j = 0\) for all \(j > 1.\)

Let us show the existence of a critical solution. By the profile decomposition, \(M(\psi^1) \leq 1\) and with previously, \(\lim_n E(e^{-it_n^1 \Delta} \psi^1) \leq (ME)_c.\) If \(\lim_n t_n^1 = 0,\) take \(\tilde{\psi}^1 = \psi^1\) so that

\[\lim_n \| BNLS(-t_n^1) \tilde{\psi}^1 - e^{-it_n^1 \Delta} \psi^1 \|_{H^2} = 0. \]

If \(t_n^1 \to \infty,\) by the decay of the linear flow associated to (1.1), \(Q(e^{-it_n^1 \Delta} \tilde{\psi}^1) \to 0.\) So

\[\| \Delta \tilde{\psi}^1 \|^2 = \lim_n E(e^{-it_n^1 \Delta} \tilde{\psi}^1) \leq (ME)_c. \]

Therefore, by Proposition 8.5, there exist \(\tilde{\psi}^1\) such that

\[M(\tilde{\psi}^1) = M(\psi^1) \leq 1, \quad E(\tilde{\psi}^1) = \| \Delta \tilde{\psi}^1 \|^2 \leq (ME)_c \]

and

\[\lim_{n \to \infty} \| BNLS(-t_n^1) \tilde{\psi}^1 - e^{-it_n^1 \Delta} \tilde{\psi}^1 \|_{H^2} = 0. \]

Take \(\tilde{W}_n^M = W_n^M - (BNLS(-t_n^1) \tilde{\psi}^1 - e^{-it_n^1 \Delta} \tilde{\psi}^1),\) by Strichartz and Sobolev estimates

\[\| e^{-i \Delta^2} \tilde{W}_n^M \|_{S(\mathbb{R})} \leq \| e^{-i \Delta^2} W_n^M \|_{S(\mathbb{R})} + c \| BNLS(-t_n^1) \tilde{\psi}^1 - e^{-it_n^1 \Delta} \tilde{\psi}^1 \|_{H^2}. \]
So
\[
\lim_n \|e^{-i\Delta^2 W_n^M}\|_{S(\mathbb{R})} = \lim_n \|e^{-i\Delta^2 W_n^M}\|_{S(\mathbb{R})}.
\]
Write
\[
u_n,0 = \text{BNLS}(t_n)\psi^1 + W_n^M,
\]
\[M(\tilde{\psi}) \leq 1, E(\tilde{\psi}^1) \leq (ME)_c\] and \[\lim_n \|\psi_n\|_{S(\mathbb{R})} = 0\].

Let \(u_c\) be the solution to (1.1) with data \(u_{c,0} := \psi^1\). Suppose that
\[
\|\text{BNLS}(\cdot - t_n^1)\psi^1\|_{S(\mathbb{R})} = \|\text{BNLS}(\cdot)\psi^1\|_{S(\mathbb{R})} = \|u_c\|_{S(\mathbb{R})} < \infty.
\]
Taking large \(M, n\) such that \(\|e^{i\Delta^2 W_n^M}\|_{S(\mathbb{R})}\) is small enough, then applying the long-time perturbation theory Proposition 8.3, one obtains \(\|u_n\|_{S(\mathbb{R})} < \infty\). This contradiction gives \(\|u_c\|_{S(\mathbb{R})} = \infty\), which implies that \(M[u_c] = 1\) and \(E[u_c] = (ME)_c\). This finishes the proof. \(\square\)

**Proposition 8.7** (pre-compactness of the flow of the critical solution). Let \(u_c\) be as in the previous Proposition, then, the following set is pre-compact in \(H^2\),
\[
\{u_c(t, \cdot), \quad t \geq 0\}.
\]

**Proof.** Denote \(u := u_c\). By contradiction, suppose that \(\exists \eta > 0\) and a sequence \(t_n \to \infty\) such that for all \(n \neq m\),
\[
\|u(t_n) - u(t_m)\|_{H^2} > \eta.
\]
Take the profile decomposition, \(\phi_n := u(t_n) = \sum_{j=1}^{M} e^{-i\Delta^2 \psi^j} W_n^M\). With the energy Pythagorean expansion, one gets
\[
(M\ E)_c = E(\phi_n) = \sum_{j=1}^{M} \lim_n E(e^{-i\Delta^2 \psi^j}) + \lim_n E(W_n^M).
\]
Since as previously, by (8.1) each energy is positive, for any \(j\),
\[
(M\ E)_c \geq \lim_n E(e^{-i\Delta^2 \psi^j}).
\]
By the profile decomposition expansion properties
\[
1 = M(\phi_n) = \sum_{j=1}^{M} \lim_n M(\psi^j) + \lim_n M(W_n^M).
\]
Following the proof of the previous Proposition, we have \(\psi^1 \neq 0 = \psi^j\), for any \(j \neq 1\). Thus,
\[
\phi_n = e^{-i\Delta^2 \psi^1} W_n^M.
\]
Arguing as in the proof of the previous Proposition, one gets
\[
1 = M(\psi^1), \quad \lim_n E(e^{-i\Delta^2 \psi^1}) = (ME)_c, \quad \lim_n E(W_n^M) = 0.
\]
Suppose that \(t_n^1 \to \infty\) and write
\[
\|e^{i\Delta^2 u(t_n)}\|_{S(\mathbb{R})} \leq \|e^{-i(t_n^1 - \cdot)\Delta^2 \psi^1}\|_{S(\mathbb{R})} + \|e^{i\Delta^2 W_n^M}\|_{S(\mathbb{R})}.
\]
Since for large \(n\), \(\|e^{i\Delta^2 W_n^M}\|_{S(\mathbb{R})} \leq \delta\) and \(\lim_n \|e^{-i(t_n^1 - \cdot)\Delta^2 \psi^1}\|_{S(\mathbb{R})} = 0\), one gets a contradiction with the small data scattering. Then, \(t_n^1 \to t^1\) up to a sub-sequence. In such a case, because \(e^{i(t_n^1 - \cdot)\Delta^2 \psi^1} \to e^{it^1 \Delta^2 \psi^1}\) in \(H^2\), this implies that \(\phi_n\) converges in \(H^2\), which contradicts the beginning and concludes the proof. \(\square\)
Proposition 8.8. Let $u$ be a solution to (1.1) such that $\{u(t), t > 0\}$ is pre-compact in $H^2$. Then, for each $\epsilon > 0$, there exists $R > 0$ such that

$$\int_{|x| > R} (|\Delta u|^2 + |u|^2 + \frac{1}{p} (I_\alpha * |u|^p)|u|^p) \, dx < \epsilon.$$ 

Proof. Otherwise, there exist $\epsilon > 0$ and a real numbers sequence $t_n$ such that for any $R > 0$,

$$\int_{|x| > R} (|\Delta u(t_n)|^2 + |u(t_n)|^2 + \frac{1}{p} (I_\alpha * |u(t_n)|^p)|u(t_n)|^p) \, dx > \epsilon.$$

Since $\{u(t), t > 0\}$ is pre-compact, for a sub-sequence $u(t_n) \to \phi$ in $H^2$. Then, for any $R > 0$,

$$\int_{|x| > R} (|\Delta \phi|^2 + |\phi|^2 + \frac{1}{p} (I_\alpha * |\phi|^p)|\phi|^p) \, dx \geq \epsilon.$$

This contradiction ends the proof. \qed

8.2. Rigidity Theorem. In this section, let us prove a Liouville-type theorem.

Proposition 8.9. Let $N \geq 2$, $0 < \alpha < N$ such that $\alpha > N - 8$, $0 < s_c < 2$, $\phi$ be a ground state solution to (4.1) satisfying (7.1) and (7.3). Let $u \in C(\mathbb{R}, H^2)$ be a global solution of (1.1). If $\{u(t), t > 0\}$ is pre-compact, then $u_0 = 0$.

Proof. With the previous computation via Proposition 8.8 and the previous proposition

$$\frac{d}{dt} M_\psi[u(t)] = 8 \|\Delta u(t)\|^2 - \frac{4B}{p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u(x)|^p \, dx$$

$$+ O(R^{-4} + R^{-2} \|\nabla u(t)\|^2) + O \left( \int_{\{|x| > R\}} (I_\alpha * |u|^p)|u|^p \, dx \right)$$

$$\leq 8 \|\Delta u(t)\|^2 - \frac{4B}{p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u(x)|^p \, dx + O_R(1).$$

Claim: there exists $\delta > 0$ such that for large $R > 0$,

$$4\|\Delta u\|^2 - \frac{2B}{p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u(x)|^p \, dx + o_R(1) > \delta \|\Delta u_0\|^2.$$

This implies that

$$\|M_\psi_R(t) - M_\psi_R(0)\| \geq \delta t \|\Delta u_0\|^2.$$

On the other hand

$$\|M_\psi_R(t) - M_\psi_R(0)\| \leq C_R \|\psi\|^2_{H^2}.$$ 

Then, $u_0 = 0$.

It remains to prove the claim. Indeed, since $u_0$ satisfies (7.1) and (7.3), there exists $\delta > 0$ such that

$$E(u)^{s_c} M(u)^{2-s_c} < (1 - \delta) E(\phi)^{s_c} M(\phi)^{2-s_c}, \quad \|\Delta u_0\|^2 < (1 - \delta)x_1,$$

where we take the notations of the proof of Lemma 7.3. Now, $f((1 - \delta)x_1) = (1 - \frac{4}{4N}(1 - \delta)x_1)^{\frac{p}{2} - 1}(1 - \delta)x_1 > (1 - \delta)f(x_1)$. Then,

$$f(X(t)) \leq E(u) < (1 - \delta)f(x_1) < f((1 - \delta)x_1); \quad X(0) < (1 - \delta)x_1.$$

A continuity argument gives

$$\|\Delta u(t)\|^2 < (1 - \delta)x_1, \quad \text{on} \quad \mathbb{R}.$$
Take the function \( F(x) := x^2 - x^B \) and compute using Theorem 3.1,

\[
F\left( \frac{\|u\|^{\frac{2-\alpha}{\alpha}} \|\Delta u\|}{\|\phi\|^{\frac{2-\alpha}{\alpha}} \|\Delta \phi\|} \right) = \left( \frac{\|u\|^{\frac{2-\alpha}{\alpha}} \|\Delta u\|}{\|\phi\|^{\frac{2-\alpha}{\alpha}} \|\Delta \phi\|} \right)^2 - \frac{B}{2p} \left( \frac{1}{\|\phi\|^{\frac{2-\alpha}{\alpha}} \|\Delta \phi\|} \right)^2 M(u_0)^{\frac{2-\alpha}{\alpha}} \int_{\mathbb{R}^N} (I_{\alpha} * |u|^p)|u|^p \, dx.
\]

Now, since \( B > 2 \), there exists \( C_5 > 0 \) such that \( F(x) > C_5 x^2 \) for \( 0 < x < 1 - \delta \). Then, on \( \mathbb{R} \),

\[
\|\Delta u\|^2 - \frac{B}{2p} \int_{\mathbb{R}^N} (I_{\alpha} * |u|^p)|u|^p \, dx > C_5 \|\Delta u\|^2.
\]

The claim follows by the previous inequality via (8.1).

8.3. **Proof of scattering.** Thanks to Proposition 8.7, the critical solution \( u_c \) constructed in Proposition 8.6 satisfies the hypotheses in Proposition 8.9. Therefore, to complete the proof of Theorem 7.1, we apply Proposition 8.9 to \( u_c \), and find that \( u_{c,0} = 0 \), which contradicts the fact that \( \|u_c\|_{S(\mathbb{R})} = \infty \). This contradiction shows that (8.2) is false. Thus, by Proposition 8.4, \( H^2 \) scattering holds.

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