Riesz Transforms Characterizations of Hardy Spaces $H^1$ for the Rational Dunkl Setting and Multidimensional Bessel Operators

Jacek Dziubański

Received: 21 May 2014 / Published online: 28 September 2015
© Mathematica Josephina, Inc. 2015

Abstract We characterize the Hardy space $H^1$ in the rational Dunkl setting associated with the reflection group $\mathbb{Z}_2^n$ by means of special Riesz transforms. As a corollary we obtain Riesz transforms characterization of $H^1$ for product of Bessel operators in $(0, \infty)^n$.

Keywords Dunkl theory · Riesz transform · Hardy space · Maximal operator · Atomic decomposition

Mathematics Subject Classification Primary: 42B30 · Secondary: 33C52 , 35J05 , 42B25 , 42B35 , 42C05

1 Introduction and Statement of the Result

The theory of Dunkl operators had its origin in a series of seminal works [8–11] and was developed by many mathematicians afterwards. The Dunkl operators form a commuting system of differential-difference operators associated with a finite group of reflections. We refer the reader to the lecture notes [20,21] and references therein for the rational Dunkl theory and to [18] for the trigonometric Dunkl theory.

In the present paper, on the Euclidean space $\mathbb{R}^n$, $n \geq 1$, we consider the Dunkl operators

$$D_j f(x) = \frac{\partial}{\partial x_j} f(x) + \frac{k_j}{x_j} \left[ f(x) - f(\sigma_j x) \right] \quad (j = 1, 2, \ldots, n)$$
associated with the reflections

\[ \sigma_j(x_1, x_2, \ldots, x_j, \ldots, x_n) = (x_1, x_2, \ldots, -x_j, \ldots, x_n) \quad (1.1) \]

and the multiplicities \( k_j \geq 0 \). Their joint eigenfunctions form the Dunkl kernel

\[ E(x, y) = \prod_{j=1}^{n} E_{k_j}(x_j, y_j), \quad (1.2) \]

\[ D_j E(\cdot, y)(x) = y_j E(x, y), \quad (1.3) \]

where

\[ E_k(x, y) = \frac{\Gamma(k + \frac{1}{2})}{\Gamma(k) \Gamma(\frac{1}{2})} \int_{-1}^{+1} (1 - u)^{k-1} (1 + u)^k e^{xyu} \, du \]

\[ = 2^{k-1/2} \Gamma \left( k + \frac{1}{2} \right) |xy|^{1-k} \left( I_{k-1/2}(|xy|) + \text{sgn}(xy)I_{k+1/2}(|xy|) \right) \quad (1.4) \]

(see, for instance, [20, p. 107, Example 2.1]). Here \( I_\nu(x) \) is the modified Bessel function (see, e.g., [17, 27]). Notice that \( E(x, y) = e^{\langle x, y \rangle} \) if all multiplicities \( k_j \) vanish.

The Dunkl Laplacian

\[ L f(x) = \sum_{j=1}^{n} D_j^2 f(x) = \sum_{j=1}^{n} \left\{ \left( \frac{\partial}{\partial x_j} \right)^2 f(x) + \frac{2k_j}{x_j} \frac{\partial}{\partial x_j} f(x) - \frac{k_j}{x_j} \left[ f(x) - f(\sigma_j x) \right] \right\} \]

is the infinitesimal generator of the heat semigroup \( \{e^t L\}_{t \geq 0} \), which acts by linear self-adjoint operators on \( L^2(\mathbb{R}^n, d\mu) \) and by linear contractions on \( L^p(\mathbb{R}^n, d\mu) \), for every \( 1 \leq p \leq \infty \), where

\[ d\mu(x) = d\mu_1(x_1) \ldots d\mu_n(x_n) = |x_1|^{2k_1} \ldots |x_n|^{2k_n} \, dx_1 \ldots dx_n. \quad (1.5) \]

Clearly,

\[ L E(\cdot, y)(x) = |y|^2 E(x, y). \]

The heat semigroup, which is strongly continuous on \( L^p(\mathbb{R}^n, d\mu) \) for \( 1 \leq p < \infty \), consists of integral operators

\[ e^t L f(x) = \int_{\mathbb{R}^n} h_t(x, y) \, f(y) \, d\mu(y) \]

associated with the heat kernel

\[ h_t(x, y) = c_k^{-1} t^{-N/2} e^{-|x|^2 + |y|^2 / 4t} \, E\left( \frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}} \right), \quad (1.6) \]
see, e.g., [19], where
\[
N = n + \sum_{j=1}^{n} 2k_j
\]  
(1.7)
is the homogeneous dimension and
\[
c_k = 2^{N_n} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}} \, d\mu(x) = 2^N \prod_{j=1}^{n} \Gamma(k_j + \frac{1}{2}).
\]
The Dunkl transform is defined by
\[
\mathcal{F} f(\xi) = c_k^{-1} \int_{\mathbb{R}^n} f(x) E(x, -i\xi) \, d\mu(x).
\]  
(1.8)
It is an isometric isomorphism of \(L^2(\mathbb{R}^n, d\mu)\) onto itself with the inversion formula:
\[
f(x) = \mathcal{F}^2 f(-x)
\]
(see, e.g., [7,10]).

The Hardy space \(H^1_{\text{max}, \mathbf{L}}\) associated with \(\mathbf{L}\) is the set of all functions \(f \in L^1(\mathbb{R}^n, d\mu)\) whose maximal heat function
\[
h^*_f(x) = \sup_{t > 0} \left| \int_{\mathbb{R}^n} h_t(x, y) f(y) \, d\mu(y) \right|
\]  
(1.9)
begins to \(L^1(\mathbb{R}^n, d\mu)\) and the norm is given by
\[
\|f\|_{H^1_{\text{max}, \mathbf{L}}} = \|h^*_f\|_{L^1(\mathbb{R}^n, d\mu)}.
\]
Now we turn to the atomic definition of the Hardy space \(H^1\). Notice that \(\mathbb{R}^n\), equipped with the Euclidean distance \(d(x, y) = |x - y|\) and with the measure \(\mu\), is a space of homogeneous type in the sense of Coifman–Weiss [5]. An atom is a measurable function \(a : \mathbb{R}^n \to \mathbb{C}\) such that
\begin{itemize}
  \item \(a\) is supported in a ball \(B\),
  \item \(\|a\|_{L^\infty} \lesssim \mu(B)^{-1}\),
  \item \(\int_{\mathbb{R}^n} a(x) \, d\mu(x) = 0\).
\end{itemize}
By definition, the atomic Hardy space \(H^1_{\text{atom}}\) consists of all functions \(f \in L^1(\mathbb{R}^n, d\mu)\) which can be written as \(f = \sum \lambda_\ell a_\ell\), where the \(a_\ell\)'s are atoms and \(\sum |\lambda_\ell| < +\infty\), and the norm is given by
\[
\|f\|_{H^1_{\text{atom}}} = \inf \sum |\lambda_\ell|,
\]
where the infimum is taken over all atomic decompositions of \(f\).
Hardy spaces on spaces of homogeneous type (see, e.g., [5, 15, 26]) are extensions of the classical real Hardy spaces on $\mathbb{R}^n$. For characterizations and properties of the classical Hardy spaces we refer the reader to the original works [3, 4, 14, 22]. More information are given in the book [24] and references therein.

Hardy spaces associated with the Dunkl operator $L$ were studied in [2]. The following theorem was proved there.

**Theorem 1.1** The spaces $H^1_{\text{max}, L}$ and $H^1_{\text{atom}}$ coincide and the norms $\|f\|_{H^1_{\text{max}, L}}$ and $\|f\|_{H^1_{\text{atom}}}$ are equivalent.

The present paper is a continuation of [2] and deals with the Riesz transforms characterization of $H^1_{\text{max}, L}$. We define the Riesz transforms in the Dunkl setting putting

$$\mathcal{R}_j = D_j (-L)^{-1/2}.$$  

The operators $\mathcal{R}_j$ can be expressed as the Dunkl multiplier operators, namely,

$$\mathcal{R}_j f(x) = D_j (-L)^{-1/2} f(x) = D_j \int_{\mathbb{R}^n} c_k^{-1} \frac{1}{|\xi|} E(x, i\xi) F(\xi) d\mu(\xi)$$

$$= c_k^{-1} \int_{\mathbb{R}^n} i \frac{\xi_j}{|\xi|} E(x, i\xi) F(\xi) d\mu(\xi).$$  \hspace{1cm} (1.10)

Our main result is the following theorem which is an analogue of the result about the characterization of the classical Hardy spaces by the classical Riesz transforms $\partial / \partial x_j (\Delta)^{-1/2}$ (see, e.g., [24, Chapter III, Section 4]).

**Theorem 1.2** Let $f \in L^1(\mathbb{R}^n, d\mu)$. Then $f \in H^1_{\text{max}, L}$ if and only if $\mathcal{R}_j f \in L^1(\mathbb{R}^n, d\mu)$ for $j = 1, 2, \ldots, n$. Moreover, there exists a constant $C > 0$ such that

$$C^{-1} \|f\|_{H^1_{\text{max}, L}} \leq \|f\|_{L^1(\mathbb{R}^n, d\mu)} + \sum_{j=1}^n \|\mathcal{R}_j f\|_{L^1(\mathbb{R}^n, d\mu)} \leq C \|f\|_{H^1_{\text{max}, L}}. \hspace{1cm} (1.11)$$

It is worth pointing out that the boundedness of the Dunkl-type Riesz transforms related to general finite reflection groups on $L^p$ spaces, for $1 < p < \infty$, was proved in [1]. It would be interesting to develop a theory of Hardy spaces for Dunkl operators associated with other than $\mathbb{Z}^n_2$ reflection groups.

Let us emphasize that Theorem 1.2 implies a Riesz transform characterization of the Hardy space $H^1_{\text{max}, L}$ associated with multidimensional Bessel operator $\mathbb{L}$. To be more precise, on $(0, \infty)^n$ equipped with the measure $d\mu$ we consider the Bessel operator

$$\mathbb{L} = \sum_{j=1}^n \left( \frac{\partial^2}{\partial x_j^2} + \frac{2k_j}{x_j} \frac{\partial}{\partial x_j} \right)$$
and the associated semigroup \( \{e^{tL}\}_{t>0} \). The action of the semigroup \( e^{tL} \) on functions is given by integration against the heat kernel \( \mathcal{H}_t(x, y) \), namely,

\[
e^{tL}f(x) = \int_{(0, \infty)^n} \mathcal{H}_t(x, y) f(y) d\mu(y),
\]

(1.12)

where \( \mathcal{H}_t(x, y) = \prod_{j=1}^n h_t^{[j]}(x_j, y_j) \),

\[
h_t^{[j]}(x_j, y_j) = (2t)^{-1} \exp(-(x_j^2 + y_j^2)/4t) I_{kj-1/2} \left( \frac{x_j y_j}{2t} \right) (x_j y_j)^{-k_j+1/2},
\]

(1.13)

(see [27]). We define the Hardy space (see, e.g., [12])

\[
H^1_{max,L} = \left\{ f \in L^1((0, \infty)^n, d\mu) : \sup_{t>0} \|e^{tL}f\|_{L^1((0, \infty)^n, d\mu)} = \|f\|_{H^1_{max,L}} < \infty \right\}.
\]

Let \( R_j = \partial x_j \mathbb{L}^{-1/2} \) denote the Riesz transform associated with \( \mathbb{L} \). Now we state our second main result.

**Theorem 1.3** Assume that \( f \in L^1((0, \infty)^n, d\mu) \). Then \( f \) belongs to \( H^1_{max,L} \) if and only if \( R_j f \in L^1((0, \infty)^n, d\mu) \) for \( j = 1, 2, \ldots, n \). Moreover, there is a constant \( C > 0 \) such that

\[
C^{-1} \|f\|_{H^1_{max,L}} \leq \|f\|_{L^1((0, \infty)^n, d\mu)} + \sum_{j=1}^n \|R_j f\|_{L^1((0, \infty)^n, d\mu)} \leq C \|f\|_{H^1_{max,L}}.
\]

(1.14)

### 2 Poisson Semigroup

The Poisson semigroup \( \{P_t\}_{t>0} \) in the Dunkl setting is defined by:

\[
P_t f(x) = e^{-t\sqrt{-L}} f(x) = \mathcal{F}^{-1}(e^{-t|\xi|} \mathcal{F} f(\xi))(x) = \int_{\mathbb{R}^n} P_t(x, y) f(y) d\mu(y),
\]

where the associated Poisson kernel is given by

\[
P_t(x, y) = c_k^{-2} \int_{\mathbb{R}^n} E(x, i\xi) e^{-t|\xi|} E(y, -i\xi) d\mu(\xi).
\]

By the subordination formula

\[
P_t(x, y) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-u} h_{t/4u}(x, y) \frac{du}{\sqrt{u}},
\]

(2.1)

It easily follows from (2.1), (1.6), (1.2) and (1.4) that \( P_t(x, y) = P_t(y, x) \) is a positive smooth function of the \((t, x, y)\) variables. We shall also show (see Appendix) that for every \( 1 \leq p < \infty \) and \( t > 0 \) there is a constant \( C_{p,t} \) such that
\[
\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} P_t(x, y)^p \, d\mu(y) \leq C_{p, t}.
\]

Let \( \mathcal{L} = \frac{\partial^2}{\partial t^2} + L \). Then
\[
\mathcal{L} P_t f(x) = 0.
\]

Let
\[
P_* f(x) = \sup_{t > 0} |P_t f(x)|
\]
be the maximal operator associated with \( \{P_t\}_{t > 0} \). It is known (see [25]) that the operator \( P_* \) is bounded on \( L^p(\mathbb{R}^n, d\mu) \), that is, for every \( 1 < p \leq \infty \) there is a constant \( C_{k, p} > 0 \) such that
\[
\|P_* f\|_{L^p(\mathbb{R}^n, d\mu)} \leq C_{k, p} \|f\|_{L^p(\mathbb{R}^n, d\mu)}.
\]

For vector valued extensions of maximal functions estimates in the Dunkl setting we refer the reader to [6].

In order to prove Theorem 1.2 we shall use Theorem 2.1 and Proposition 2.2. The proofs of them as well as basic properties of \( P_t(x, y) \) and a short proof of (2.4) are presented in the Appendix.

**Theorem 2.1** Let \( f \in L^1(\mathbb{R}^n, d\mu) \). Then \( f \in H_{\text{max}, L}^1 \) if and only if the maximal function \( P_* f \) belongs to \( L^1(\mathbb{R}^n, d\mu) \). Moreover,
\[
\|P_* f\|_{L^1(\mathbb{R}^n, d\mu)} \sim \|f\|_{H_{\text{max}, L}^1}.
\]

**Proposition 2.2** (a) Assume that \( g \in L^1_{\text{loc}}(\mathbb{R}^n, d\mu) \) and \( \lim_{|x| \to \infty} |g(x)| = 0 \). Then
\[
\lim_{(|x| + t) \to \infty} P_t g(x) = 0.
\]

(b) If \( f \in L^1(\mathbb{R}^n, d\mu) \), then for every \( \varepsilon > 0 \) we have
\[
\lim_{(|x| + t) \to \infty} P_{t+\varepsilon} f(x) = 0.
\]

**3 Key Lemma**

The following lemma, which is perhaps interesting in its own, will play a crucial role in the proof of Theorem 1.2.
Lemma 3.1 For every positive integer \( n \) and every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that for any matrix

\[
B = \begin{bmatrix}
    b_{0,0} & b_{0,1} & \ldots & b_{0,n} \\
    b_{1,0} & b_{1,1} & \ldots & b_{1,n} \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{n,0} & b_{n,1} & \ldots & b_{n,n}
\end{bmatrix}
\]

with real entries we have

\[
\|B\|_2^2 \leq (1 - \delta)\|B\|_{\text{HS}}^2 + \varepsilon \left( (\text{tr } B)^2 + \sum_{i < j} (b_{i,j} - b_{j,i})^2 \right). \tag{3.1}
\]

Here \( \|B\| = \sup_{x \in \mathbb{R}^{n+1}, \|x\| = 1} \|Bx\| \) is the ordinary norm of \( B \) and \( \|B\|_{\text{HS}} = (\sum_{j=0}^n \sum_{\ell=0}^n b_{j,\ell}^2)^{1/2} \) is the Hilbert–Schmidt norm.

Proof Let \( S = (s_{i,j})_{i,j=0,1,...,n} \) and \( A = (a_{i,j})_{i,j=0,1,...,n} \) denote any symmetric and antisymmetric matrix respectively. It is clear that

\[
\|A + S\|_{\text{HS}}^2 = \|A\|_{\text{HS}}^2 + \|S\|_{\text{HS}}^2,
\]

\[
\sum_{i < j} (a_{i,j} - a_{j,i})^2 = 2\|A\|_{\text{HS}}^2. \tag{3.3}
\]

It is known (see, e.g., Section 3.1.2 of Chapter VII of [23]) that (3.1) holds for symmetric trace zero matrices. Observe that it also holds for antisymmetric matrices with \( \delta = 2\varepsilon \). Indeed, from (3.3), we get

\[
\|A\|_2^2 \leq \|A\|_{\text{HS}}^2 = (1 - 2\varepsilon)\|A\|_{\text{HS}}^2 + \varepsilon \sum_{i < j} (a_{i,j} - a_{j,i})^2.
\]

We claim that for any fixed \( \varepsilon > 0 \) and \( A, S \) such that \( \|A\|_{\text{HS}}^2 \geq \frac{1}{\varepsilon}, \|S\|_{\text{HS}}^2 = 1 \), we have

\[
\|A + S\|_2^2 \leq (1 - \varepsilon)\|A\|_{\text{HS}}^2 + 2\varepsilon\|A\|_{\text{HS}}^2. \tag{3.4}
\]

To see (3.4) we utilize (3.2) and obtain

\[
\|A + S\|_2^2 \leq \|A + S\|_{\text{HS}}^2 = \|A\|_{\text{HS}}^2 + \|S\|_{\text{HS}}^2
\]

\[
= (1 - \varepsilon)\|A\|_{\text{HS}}^2 + \varepsilon\|A\|_{\text{HS}}^2 + 1
\]

\[
\leq (1 - \varepsilon)\|A\|_{\text{HS}}^2 + \varepsilon\|A\|_{\text{HS}}^2 + \varepsilon\|A\|_{\text{HS}}^2
\]

\[
\leq (1 - \varepsilon)\|A\|_{\text{HS}}^2 + 2\varepsilon\|A\|_{\text{HS}}^2.
\]
Since (3.1) is homogeneous of degree 2, that is, \( \| tB \|^2 = t^2 \| B \|^2 \), and

\[
(1 - \delta)\| tB \|_{HS}^2 + \varepsilon \left( (\text{tr} tB)^2 + \sum_{i < j} (tb_{i,j} - t b_{j,i})^2 \right) = t^2 \left( (1 - \delta)\| B \|_{HS}^2 + \varepsilon \left( (\text{tr} B)^2 + \sum_{i < j} (b_{i,j} - b_{j,i})^2 \right) \right),
\]

it suffices to prove (3.1) for \( B = A + S \) with \( S \) running over the unit sphere in the Hilbert–Schmidt norm, that is, \( \| S \|^2_{HS} = 1 \). Assume that (3.1) does not hold. Then there is \( \varepsilon > 0 \) such that for every \( \delta_n = \frac{1}{n} \) there are \( A_n \) and \( S_n \), \( \| S_n \|^2_{HS} = 1 \), such that

\[
\| A_n + S_n \|^2 > (1 - \delta_n)\| A_n + S_n \|_{HS}^2 + 2\varepsilon\| A_n \|^2_{HS} + \varepsilon (\text{tr} S_n)^2. \tag{3.5}
\]

It follows from (3.4) that \( \| A_n \|^2_{HS} \leq \frac{1}{\kappa} \) for large \( n \). Thus \( S_n \) and \( A_n \) are in compact sets. There is a subsequence \( n_k \) such that \( S_{n_k} \) and \( A_{n_k} \) converge to \( S \) and \( A \) respectively. Moreover, \( \| S \|^2_{HS} = 1 \). Passing to limit in (3.5) as \( k \to \infty \), we obtain

\[
\| A + S \|^2 \geq \| A + S \|_{HS}^2 + 2\varepsilon\| A \|_{HS}^2 + \varepsilon (\text{tr} S)^2. \tag{3.6}
\]

The inequality (3.6) implies that \( A = 0 \) and \( \text{tr} S = 0 \). Hence \( \| S \|^2 \geq \| S \|^2_{HS} \), which is impossible for a nonzero symmetric matrix \( S \) with \( \text{tr} S = 0 \). \( \square \)

## 4 Riesz Transforms and Cauchy–Riemann Equations

For a function \( f \in L^1(\mathbb{R}^n, d\mu) \) such that \( \mathcal{R} f \in L^1(\mathbb{R}^n, d\mu) \) we define the functions

\[
u_0(t, x) = P_t f(x) = c_k^{-1} \int_{\mathbb{R}^n} e^{-t|\xi|^2} \mathcal{F}_t f(\xi) E(x, i\xi) d\mu(\xi),
\]

\[
u_j(t, x) = -P_t (\mathcal{R}_j f)(x) = -c_k^{-1} \int_{\mathbb{R}^n} i\frac{\xi_j}{|\xi|^2} e^{-t|\xi|^2} \mathcal{F}_t f(\xi) E(x, i\xi) d\mu(\xi).
\tag{4.1}
\]

The functions \( u_j, j = 0, 1, \ldots, n, \) are \( C^\infty \) on \( (0, \infty) \times \mathbb{R}^n \). It is easy to check using (1.3) and (4.1) that they satisfy the following Cauchy–Riemann type equations:

\[
\partial_t u_0(t, x) + \sum_{j=1}^n D_j u_j(t, x) = 0.
\tag{4.2}
\]

From now we shall assume that \( f \) is real-valued, then so are \( u_j, j = 0, 1, \ldots, n \).

Let \( \mathcal{G} \) denote the finite group of reflections in \( \mathbb{R}^n \) generated by \( \sigma_j, j = 1, \ldots, n \) (see (1.1)). For \( \sigma \in \mathcal{G} \) and a function \( f \) defined on \( \mathbb{R}^n \) we denote \( f^\sigma(x) = f(\sigma(x)) \).
Similarly, if \( u(t, x) \) is defined on \((0, \infty) \times \mathbb{R}^n\), then we set \( u^\sigma(t, x) = u(t, \sigma x) \). For \( \sigma \in \mathcal{G} \) and \( j = 1, 2, \ldots, n \), let

\[
\sigma(j) = \begin{cases} 
1 & \text{if } \sigma(x)_j = x_j, \\
-1 & \text{if } \sigma(x)_j = -x_j.
\end{cases}
\]

Additionally define \( \sigma(0) = 1 \). Clearly, \( R_j(f^\sigma) = \sigma(j)(R_j f)^\sigma \) for \( j = 1, 2, \ldots, n \).

If \( u(t, x) = (u_0(t, x), u_1(t, x), u_2(t, x), \ldots, u_n(t, x)) \) satisfies (4.2), then so does

\[
u_\sigma(t, x) = (\sigma(0)u_0^\sigma(t, x), \sigma(1)u_1^\sigma(t, x), \sigma(2)u_2^\sigma(t, x), \ldots, \sigma(n)u_n^\sigma(t, x)).
\]

Moreover, if \( (u_0(t, x), u_1(t, x), \ldots, u_n(t, x)) \) is of the form (4.1), then

\[
u_0^\sigma(t, x) = P_t(f^\sigma)(x), \sigma(j)u_j^\sigma(t, x) = -P_t(R_j(f^\sigma))(x) \text{ for } j = 1, 2, \ldots, n.
\]

For a \( C^2 \) function \( u(t, x) = (u_0(t, x), u_1(t, x), \ldots, u_n(t, x)) \) satisfying (4.2) consider the function \( F: (0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{(n+1)|\mathcal{G}|} \),

\[
F(t, x) = \{u_\sigma(t, x)\}_{\sigma \in \mathcal{G}}.
\]

Observe that \( |F(t, x)| = |F(t, \sigma x)| \) for every \( \sigma \in \mathcal{G} \), where

\[
|F(t, x)|^2 = \sum_{\sigma \in \mathcal{G}} \sum_{\ell = 0}^n |\sigma(\ell)u_\ell^\sigma(t, x)|^2 = \sum_{\sigma \in \mathcal{G}} \sum_{\ell = 0}^n |u_\ell^\sigma(t, x)|^2.
\]

Our main task is to prove the following proposition, which is an analogue of the classical result (see, e.g., [23, Section 3.1 of Chapter VII]).

**Proposition 4.1** There is an exponent \( 0 < q < 1 \) which depends on \( k_1, \ldots, k_n \) such that the function \( |F|q \) is \( L \)-subharmonic, that is, \( L(|F|^q)(t, x) \geq 0 \) on the set where \( |F| > 0 \).

**Proof** Observe that \( |F|^q \) is \( C^2 \) on the set where \( |F| > 0 \). Let \( \cdot \) denote the inner product in \( \mathbb{R}^{(n+1)|\mathcal{G}|} \). In order to unify our notation we denote the variable \( t \) by \( x_0 \). For \( j = 0, 1, \ldots, n \), we have

\[
\partial_{x_j}|F|^q = q|F|^{q-2}(\partial_{x_j}F) \cdot F
\]

\[
\partial_{x_j}^2|F|^q = q(q-2)|F|^{q-4}(\partial_{x_j}F) \cdot F + q|F|^{q-2}(\partial_{x_j}^2F) \cdot F + |\partial_{x_j}F|^2.
\]
Recall that $|F(x_0, x)| = |F(x_0, \sigma x)|$. Hence,

$$\mathcal{L}|F|^q = q(q - 2)|F|^{q-4} \left\{ \left( \partial_{x_0} F \right) \cdot F \right\}^2 + \sum_{j=1}^n \left( \partial_{x_j} F \right) \cdot F \left[ \sum_{j=1}^n \left( \partial_{x_j} F \right) \cdot F \right]^2$$

$$+ q|F|^{q-2} \left\{ \partial_{x_0}^2 F + \sum_{j=1}^n \left( \partial_{x_j}^2 F + \frac{2k_j}{x_j} (\partial_{x_j} F) \right) \right\} \cdot F + |\partial_{x_0} F|^2 + \sum_{j=1}^n |\partial_{x_j} F|^2.$$

(4.3)

Since $D_j D_{\ell} f = D_{\ell} D_j f$ for $f \in C^2(\mathbb{R}^n)$, we conclude from (4.2) applied to $u_{\mathcal{G}}$ that for $\ell = 0, 1, \ldots, n$ we have

$$\partial_{x_0}^2 u_{\ell}^\mathcal{G} + \sum_{j=1}^n \left( \partial_{x_j}^2 u_{\ell}^\mathcal{G} + \frac{2k_j}{x_j} (\partial_{x_j} u_{\ell}^\mathcal{G}) \right) = \sum_{j=1}^n \frac{k_j}{x_j^2} (u_{\ell}^\mathcal{G} - u_{\ell}^{\sigma \mathcal{G}}).$$

Thus,

$$\left( \partial_{x_0}^2 F + \sum_{j=1}^n \left( \partial_{x_j}^2 F + \frac{2k_j}{x_j} (\partial_{x_j} F) \right) \right) \cdot F = \sum_{\sigma \in \mathcal{G}} \sum_{j=1}^n \frac{k_j}{x_j^2} \left( \sigma(\epsilon) u_{\ell}^\sigma - \sigma(\epsilon) u_{\ell}^{\sigma \mathcal{G}} \right) \sigma(\epsilon) u_{\ell}^\sigma$$

$$= \sum_{j=1}^n \sum_{\ell=0}^n \sum_{\sigma \in \mathcal{G}} \frac{k_j}{x_j^2} (u_{\ell}^\mathcal{G} - u_{\ell}^{\sigma \mathcal{G}}) u_{\ell}^\sigma$$

$$= \frac{1}{2} \sum_{j=1}^n \sum_{\ell=0}^n \sum_{\sigma \in \mathcal{G}} \frac{k_j}{x_j^2} (u_{\ell}^\mathcal{G} - u_{\ell}^{\sigma \mathcal{G}})^2$$

$$= \frac{1}{2} \sum_{\sigma \in \mathcal{G}} \sum_{j=1}^n \sum_{\ell=0}^n \frac{k_j}{x_j^2} \left( \sigma(\epsilon) u_{\ell}^\sigma - \sigma(\epsilon) u_{\ell}^{\sigma \mathcal{G}} \right)^2.$$

(4.4)

Thanks to (4.3) and (4.4), it suffices to prove that there is $0 < q < 1$ such that

$$(2-q) \left\{ \left( \partial_{x_0} F \right) \cdot F \right\}^2 + \sum_{j=1}^n \left( \partial_{x_j} F \right) \cdot F \left[ \sum_{j=1}^n \left( \partial_{x_j} F \right) \cdot F \right]^2$$

$$\leq \frac{1}{2} |F|^2 \sum_{\sigma \in \mathcal{G}} \sum_{j=1}^n \sum_{\ell=0}^n \frac{k_j}{x_j^2} \left( \sigma(\epsilon) u_{\ell}^\sigma - \sigma(\epsilon) u_{\ell}^{\sigma \mathcal{G}} \right)^2 + |F|^2 \left( |\partial_{x_0} F|^2 + \sum_{j=1}^n |\partial_{x_j} F|^2 \right).$$

(4.5)

Denote

$$B_\sigma = \begin{bmatrix} \partial_{x_0} \sigma(0) u_0^\sigma & \partial_{x_0} \sigma(1) u_1^\sigma & \ldots & \partial_{x_0} \sigma(n) u_n^\sigma \\ \partial_{x_1} \sigma(0) u_0^\sigma & \partial_{x_1} \sigma(1) u_1^\sigma & \ldots & \partial_{x_1} \sigma(n) u_n^\sigma \\ \partial_{x_n} \sigma(0) u_0^\sigma & \partial_{x_n} \sigma(1) u_1^\sigma & \ldots & \partial_{x_n} \sigma(n) u_n^\sigma \end{bmatrix}.$$
Let $B = \{ B_\sigma \}_{\sigma \in \mathcal{G}}$ be a matrix with $n+1$ rows and $(n+1) \cdot |\mathcal{G}|$ columns. It represents a linear operator from $\mathbb{R}^{(n+1) \cdot |\mathcal{G}|}$ into $\mathbb{R}^{n+1}$.

Observe that

$$(2 - q) \left( (\partial_{x_0} F) \cdot F \right)^2 + \sum_{j=1}^{n} \left( (\partial_{x_j} F) \cdot F \right)^2 \leq (2 - q) |F|^2 \|B\|^2,$$

$$|F|^2 \left( |\partial_{x_0} F|^2 + \sum_{j=1}^{n} |\partial_{x_j} F|^2 \right) = |F|^2 \|B\|^2_{\text{HS}}.$$ 

Clearly,

$$\|B\|^2 \leq \sum_{\sigma \in \mathcal{G}} \|B_\sigma\|^2, \quad \|B\|^2_{\text{HS}} = \sum_{\sigma \in \mathcal{G}} \|B_\sigma\|^2_{\text{HS}}.$$ 

Therefore the inequality (4.5) will be proven if we show that

$$(2 - q) \sum_{\sigma \in \mathcal{G}} \|B_\sigma\|^2 \leq \sum_{\sigma \in \mathcal{G}} \|B_\sigma\|^2_{\text{HS}} + \frac{1}{2} \sum_{\sigma \in \mathcal{G}} \sum_{j=1}^{n} \sum_{\ell=0}^{n} \frac{k_j}{x_j^2} \left( (\sigma(\ell)u_\ell^\sigma - (\sigma(\ell)u_\ell^\sigma_{\sigma j})^2 \right). \quad (4.6)$$

Applying the Cauchy–Riemann type equations (4.2), we obtain

$$(\text{tr}B_\sigma)^2 = \left( - \sum_{j=1}^{n} \frac{k_j}{x_j^2} (\sigma(j)u_j^\sigma - (\sigma(j)u_j^\sigma_{\sigma j}) \right)^2 \leq \left( \sum_{s=1}^{n} k_s \right) \left( \sum_{j=1}^{n} \frac{k_j}{x_j^2} (\sigma(j)u_j^\sigma - (\sigma(j)u_j^\sigma_{\sigma j}))^2 \right), \quad (4.7)$$

$$\sum_{i<j} \left( \partial_{x_i} (\sigma(j)u_j^\sigma - \sigma(j)u_j^\sigma_{\sigma j}) \right)^2 \leq \sum_{j=1}^{n} \frac{k_j}{x_j^2} (u_j^\sigma - u_0^\sigma_{\sigma j})^2 + \sum_{1 \leq i < j} \left( \frac{k_i}{x_i^2} (\sigma(j)u_j^\sigma - (\sigma(j)u_j^\sigma_{\sigma j}) \right)^2 \left( \frac{k_j}{x_j^2} (\sigma(i)u_i^\sigma - (\sigma(i)u_i^\sigma_{\sigma j}) \right)^2 \leq 2 \left( \sum_{s=1}^{n} k_s \right) \left( \sum_{i=0}^{n} \sum_{j=1}^{n} \frac{k_j}{x_j^2} (\sigma(i)u_i^\sigma - (\sigma(i)u_i^\sigma_{\sigma j})^2 \right). \quad (4.8)$$
Using Lemma 3.1 together with (4.7) and (4.8) we have that for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$
\sum_{\sigma \in G} \| B_\sigma \|^2 \leq (1 - \delta) \sum_{\sigma \in G} \| B_\sigma \|_{\text{HS}}^2 + 3\varepsilon \left( \sum_{s=1}^n k_s \right) \sum_{\sigma \in G} \left( \sum_{i=0}^n \sum_{j=1}^n \frac{k_j}{x_j^2} (\sigma(i)u_i^\sigma - \sigma(i)u_i^{\sigma,j})^2 \right).
$$

(4.9)

Taking $\varepsilon > 0$ such that $3\varepsilon \sum_{n \geq 1} k_s \leq \frac{1}{4}$ and utilizing (4.9) we deduce that (4.6) holds with $(1 - \delta) \leq (2 - q)^{-1}$. \hfill \Box

5 Maximum Principles

The following weak maximum principle for $\mathcal{L}$-subharmonic functions was actually proved in Theorem 4.2 of [19].

Theorem 5.1 Let $\Omega \subset \mathbb{R}^{n+1}$ be open, bounded and $\overline{\Omega} \subset (0, \infty) \times \mathbb{R}^n$. Assume that $\Omega$ is $\mathcal{G}$-invariant, that is $(x_0, \sigma(x)) \in \Omega$ for $(x_0, x) \in \Omega$ and all $\sigma \in \mathcal{G}$. Let $f \in C^2(\overline{\Omega}) \cap C(\Omega)$ be real-valued and $\mathcal{L}$-subharmonic. Then

$$
\max_{\Omega} f = \max_{\partial \Omega} f.
$$

It is worth pointing out that for $\mathcal{G}$-invariant functions a strong maximum principle holds. To state the strong maximum principle on $\mathbb{R}^n$ let

$$
\mathcal{L} = \frac{\partial^2}{\partial x_0^2} + \sum_{j=1}^n \left( \frac{\partial^2}{\partial x_j^2} + \frac{2k_j}{x_j} \frac{\partial}{\partial x_j} \right).
$$

Proposition 5.2 Let $f(x_0, x_1, \ldots, x_n)$ be a $C^2$ function defined on an open connected set $\Omega \subset (0, \infty) \times \mathbb{R}^n$. Assume that $f(x_0, x_1, \ldots, -x_j, \ldots, x_n) = f(x_0, x_1, \ldots, x_j, \ldots, x_n)$ whenever $(x_0, x_1, \ldots, -x_j, \ldots, x_n)$ and $(x_0, x_1, \ldots, x_j, \ldots, x_n)$ belong to $\Omega$ and $\mathcal{L}f \geq 0$ on the set $\{(x_0, \ldots, x_n) \in \Omega : x_1 \cdot x_2 \cdot \ldots \cdot x_n \neq 0\}$. Then $f$ cannot attain a local maximum in $\Omega$ unless $f$ is a constant.

Proof The proof takes some ideas from [16], where the one dimensional Bessel operator was considered.

Denote $x = (x_0, x) = (x_0, x_1, \ldots, x_n) \in (0, \infty) \times \mathbb{R}^n$, $U = \{x \in (0, \infty) \times \mathbb{R}^n : x_1 \cdot x_2 \cdot \ldots \cdot x_n \neq 0\}$. Set

$$
v(x_0, x_1, \ldots, x_n) = |x_1|^{2k_1} |x_2|^{2k_2} \cdots |x_n|^{2k_n}.
$$
By the divergence theorem for $C^2$ functions $f$ and $g$ in a smooth region $\bar{D}$ one has
\[
\int_D [g \, \text{div}(v \nabla f) - f \, \text{div}(v \nabla g)] \, dx_0 \, dx_1 \ldots dx_n = \int_{\partial D} v \left( \frac{\partial f}{\partial n} - f \, \frac{\partial g}{\partial n} \right) \, ds, \quad (5.1)
\]
where $n$ is outward normal vector to $D$ at $x \in \partial D$. Observe that $\text{div}(v \nabla f) = v \bar{\Delta} f$ on $D \cap U$. So, if $g$ is additionally $\bar{\Delta}$-harmonic on $D \cap U$ then $\text{div}(v \nabla g) = v \bar{\Delta} g = 0$ on $D \cap U$, and setting $f \equiv 1$ in (5.1) we get
\[
\int_{\partial D} v \, \frac{\partial g}{\partial n} \, ds = 0. \quad (5.2)
\]
Assume that at $a = (a_0, a_1, \ldots, a_n) \in \Omega$ the function $f$ attains a local maximum. By Hopf’s maximum principle (see [13, Section 6.4.2, Theorem 3]) $a$ is not a regular point of $\bar{\Omega}$, that is, $a_1 \cdot a_2 \cdot \ldots \cdot a_n = 0$. There is no loss of generality in assuming that there is $m \in \{1, 2, \ldots, n\}$ such that $a_0 > 0$, $\ldots$, $a_{m-1} > 0$, $a_m = a_{m+1} = \ldots = a_n = 0$. Let
\[
\begin{align*}
    h^{[0]}_\tau(x_0, a_0) &= \frac{1}{\sqrt{4\pi \tau}} \exp(-|x_0 - a_0|^2/4\tau), \\
    h^{[j]}_\tau(x_j, 0) &= \frac{1}{\Gamma(k_j + 1/2)} \tau^{-k_j - 1/2} \exp(-x_j^2/4\tau), \quad j \in \{m, m+1, \ldots, n\}, \\
    h^{[j]}_\tau(x_j, a_j) &= h^{[j]}_\tau(x_j, a_j) \text{ for } j \notin \{0, m, m+1, \ldots, n\}
\end{align*}
\]
(see (1.13)). Put
\[
g_0(x_0, x_1, \ldots, x_n) = \int_0^\infty \prod_{j=0}^n h^{[j]}_\tau(x_j, a_j) \, d\tau.
\]
We have $\bar{\Delta} g_0 = 0$ on $((0, \infty)^m \times \mathbb{R}^{n+1-m}) \cap U$. It is not difficult to check using the asymptotic behavior of the Bessel functions $I_\nu$ (see, e.g., [17]) that there is $r > 0$ such that $\nabla g_0(x) \neq 0$ for every $x \in B(a, 2r) \setminus \{a\} \subset D$. Let $D_R = \{x : g_0(x) > R\} \cup \{a\}$. We take $R$ large enough such that $D_R \subset B(a, r)$. For $\varepsilon > 0$ small enough let $D_{R, \varepsilon} = D_R \setminus B(a, \varepsilon)$. Set $g(x) = g_0(x) - R$. Then $g \equiv 0$ on $\partial D_R$, $g \geq 0$ on $D_{R, \varepsilon}$ and $\frac{\partial g}{\partial n}(x) < 0$ on $\partial D_R$, where $n$ is outward normal vector to $D_R$ at $x \in \partial D_R$. Using (5.2) we have
\[
\int_{\partial D_R} v \, \frac{\partial g}{\partial n} \, ds = \int_{\partial B(a, \varepsilon)} v \, \frac{\partial g}{\partial n} \, ds < 0. \quad (5.3)
\]
Now from (5.1) we conclude that
\[
0 \leq \int_{D_{R,\varepsilon}} g(v L f) \, dx_0 dx_1 \ldots dx_n \\
= \int_{D_{R,\varepsilon}} g \text{ div } (v \nabla f) \, dx_0 dx_1 \ldots dx_n \\
= - \int_{\partial D_R} f v \frac{\partial g}{\partial n} \, ds - \int_{\partial B(a, \varepsilon)} g \frac{\partial f}{\partial n} \, ds + \int_{\partial B(a, \varepsilon)} f v \frac{\partial g}{\partial n} \, ds,
\] (5.4)
where in the last two integrals \(n\) is the outward normal vector to \(B(a, \varepsilon)\). Clearly, the second summand tends to 0 as \(\varepsilon\) tends to 0. On the other hand, by (5.3), the third summand tends to \(f(a) \int_{\partial D_R} v \frac{\partial g}{\partial n} \, ds\). Thus,
\[
0 \leq \int_{\partial D_R} (f(a) - f) v \frac{\partial g}{\partial n} \, ds.
\] (5.5)
Recall that \(f\) attains a local maximum at \(a\) and \(\frac{\partial g}{\partial n} < 0\) on \(\partial D_R\). Hence, from (5.5) we deduce that \(f = f(a)\) on \(\partial D_R\). So \(f\) must be a constant in a neighborhood of \(a\) and, consequently, \(f \equiv f(a)\) on \(\Omega\), since \(\Omega\) is connected. \(\square\)

6 Proof of Theorem 1.2

Proof of Theorem 1.2 The second inequality in (1.11) is a direct consequence of the following multiplier theorem (see [2, Theorem 1.10]).

**Theorem 6.1** Let \(\chi = \chi(\xi)\) be a smooth radial function on \(\mathbb{R}^n\) such that
\[
\chi(\xi) = \begin{cases} 
1 & \text{if } |\xi| \in \left[\frac{1}{2}, 2\right], \\
0 & \text{if } |\xi| \notin \left(\frac{1}{4}, 4\right).
\end{cases}
\]
If a function \(m = m(\xi)\) on \(\mathbb{R}^n\) satisfies
\[
M = \sup_{t > 0} \| \chi m(t \cdot) \|_{W_2^{N/2} + \varepsilon} < +\infty,
\] (6.1)
for some \(\varepsilon > 0\), then the multiplier operator
\[
T_m f = \mathcal{F}^{-1} \{ m(\mathcal{F} f) \}
\]
is bounded on the Hardy space \(H_{\text{max}, L}^1\) and
\[
\| T_m \|_{H_{\text{max}, L}^1 \rightarrow H_{\text{max}, L}^1} \lesssim M.
\]
Here \(W_2^{N/2 + \varepsilon}\) denotes the classical \(L^2\)-Sobolev space on \(\mathbb{R}^n\).
It is not difficult to check that the multiplier $m_j(\xi) = i^{j} \frac{\xi_j}{|\xi|}$, which corresponds to the Riesz transform $\mathcal{R}_j$, satisfies (6.1). Hence $\mathcal{R}_j f$ is bounded from $H^1_{\text{max},L}$ to itself and, consequently, form $H^1_{\text{max},L}$ to $L^1(\mathbb{R}^n, d\mu)$.

Now we turn to prove the first inequality in (1.11). For this purpose we use (2.4), Theorem 2.1, Propositions 2.2, 4.1, and 5.2 combined with the steps of the proof of the characterization of the classical Hardy spaces by the classical Riesz transforms (see, e.g., [24, Chapter III, Section 4]). For the convenience of the reader, we provide the details.

Assume that $f \in L^1(\mathbb{R}^n, d\mu)$ and $\mathcal{R}_j f \in L^1(\mathbb{R}^n, d\mu)$ for $j = 1, 2, \ldots, n$. There is no loss of generality in assuming that $f$ is real valued, and hence so are $\mathcal{R}_j f$. Set $u(x_0, x_1, \ldots, x_n) = (u_0, u_1, \ldots, u_n)$, where $u_j$ are defined in (4.1) (recall that $x_0 = t > 0$). Fix $0 < q < 1$ as in Proposition 4.1 and set $p = 1/q$. Let $F(x_0, x) = \{u^q(x_0, x)\}_{\sigma \in \mathcal{G}}$. Clearly,

$$
\sup_{x_0 > 0} \int_{\mathbb{R}^n} |F(x_0, x_1, \ldots, x_n)| d\mu(x_1, \ldots, x_n) \\
\leq C \left( \|f\|_{L^1(\mathbb{R}^n, d\mu)} + \sum_{j=1}^{n} \|\mathcal{R}_j f\|_{L^1(\mathbb{R}^n, d\mu)} \right). 
$$

(6.2)

Denote $F_\varepsilon(x_1, \ldots, x_n) = F(\varepsilon, x_1, \ldots, x_n), \varepsilon > 0$. Then $|F_\varepsilon| \in C_0(\mathbb{R}^n)$ (see the part (b) of Proposition 2.2) and, by (7.7),

$$
\sup_{\varepsilon > 0} \|F_\varepsilon|^q\|_{L^p(\mathbb{R}^n, d\mu)}^p \leq C \left( \|f\|_{L^1(\mathbb{R}^n, d\mu)} + \sum_{j=1}^{n} \|\mathcal{R}_j f\|_{L^1(\mathbb{R}^n, d\mu)} \right). 
$$

(6.3)

Consider the function $G : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}$,

$$
G(x_0, x_1, \ldots, x_n) = |F(\varepsilon + x_0, x_1, \ldots, x_n)|^q - P_{x_0}(|F_\varepsilon|^q)(x_1, \ldots, x_n).
$$

The function is continuous and vanishes for $x_0 = 0$ and, by Proposition 2.2,

$$
\lim_{(x_0 + |(x_1, \ldots, x_n)|) \rightarrow \infty} G(x_0, x_1, \ldots, x_n) = 0.
$$

Moreover, $G(x_0, x) = G(x_0, \sigma x)$ for every $\sigma \in \mathcal{G}$. We claim that

$$
G(x_0, x) = |F(\varepsilon + x_0, x)|^q - P_{x_0}(|F_\varepsilon|^q)(x) \leq 0.
$$

(6.4)

To prove the claim assume that $G > 0$ at some point. Then it attains a global maximum, say at $a = (a_0, a_1, \ldots, a_n)$. Obviously, $a_0 > 0$ and $|F(a)| > 0$. Take a connected neighborhood $\Omega$ of $a$ such that $G > 0$ on $\Omega$ and $G$ is not constant on $\Omega$. Actually one can take $\Omega$ satisfying additionally $G(x) > G(a)/2$ for $x \in \Omega$ and $G = G(a)/2$ on $\partial \Omega$. Then $G$ is $C^2$ on $\Omega$ and, according to (2.3) and Proposition 4.1, $LG = \nabla G = 0$.  

\( \odot \) Springer
\[ |F|^q \geq 0 \text{ on } \{ (x_0, x_1, \ldots, x_n) \in \Omega : x_1 \cdot x_2 \cdot \ldots \cdot x_n \neq 0 \}. \] This contradicts the maximum principle (see Proposition 5.2). Hence (6.4) is proved.

From (6.4) and (2.4) we conclude that
\[
\| \sup_{x_0 > \varepsilon} |F(x_0, x_1, \ldots, x_n)| \|_{L^1(\mathbb{R}^n, d\mu(x_1, \ldots, x_n))} = \| \sup_{x_0 > \varepsilon} |F(x_0, x_1, \ldots, x_n)|^q \|_{L^p(\mathbb{R}^n, d\mu(x_1, \ldots, x_n))} \leq C_{p} \| |F|^q \|_{L^p(\mathbb{R}^n, d\mu)} \leq C \left( \| f \|_{L^1(\mathbb{R}^n, d\mu)} + \sum_{j=1}^{n} \| \mathcal{R}_j f \|_{L^1(\mathbb{R}^n, d\mu)} \right),
\]
where in the last inequality we have used (6.3). Since \( \sup_{x_0 > \varepsilon} |F(x_0, x_1, \ldots, x_n)| \) converges monotonically to \( \sup_{x_0 > 0} |F(x_0, x_1, \ldots, x_n)| \) as \( \varepsilon \to 0^+ \), we deduce by the Lebesgue monotone convergence theorem that
\[
\| P^*_\varepsilon f \|_{L^1(\mathbb{R}^n, d\mu)} \leq \| \sup_{x_0 > \varepsilon} |F(x_0, x)| \|_{L^1(\mathbb{R}^n, d\mu(x))} \leq C \left( \| f \|_{L^1(\mathbb{R}^n, d\mu)} + \sum_{j=1}^{n} \| \mathcal{R}_j f \|_{L^1(\mathbb{R}^n, d\mu)} \right). \tag{6.5}
\]

Now the first inequality in (1.14) follows from (6.5) and Theorem 2.1.

### 7 Proof of Theorem 1.3

**Proof of Theorem 1.3** Recall that
\[
h_t(x, y) = \prod_{j=1}^{n} h_t^{(j)}(x_j, y_j) \tag{7.1}
\]
(see (1.6), (1.2), and (1.4)), where
\[
h_t^{(j)}(x, y) = (4t)^{-1} \exp(-x^2 + y^2/4t)|xy|^{-k_j+1/2} \left( I_{k_j-1/2} \left( \frac{|xy|}{2t} \right) + \text{sgn}(xy) I_{k_j+1/2} \left( \frac{|xy|}{2t} \right) \right)
\]
is the heat kernel associated with one dimensional Dunkl operator
\[
Lf(x) = f''(x) + \frac{2k_j}{x} f'(x) - \frac{k_j}{x^2} \left( f(x) - f(-x) \right).
\]
Clearly, \( h_{t}^{(j)}(x, y) = h_{t}^{(j)}(y, x) \) is a \( C^{\infty} \) function of \((t, x, y)\). For a function \( f \) defined on \((0, \infty)^n\) let \( \tilde{f} \) denote its extension to the \( \mathcal{G} \) invariant function on \( \mathbb{R}^n \). One can easily check using (1.12), (1.13) that 

\[
(e^{tL} f) = e^{tL}(\tilde{f}).
\]

Hence, \( f \) belongs to the Hardy space \( H_{\text{max}, L}^1 \) if and only if \( \tilde{f} \in H_{\text{max}, L}^1 \). Moreover, 

\[
\|f\|_{H_{\text{max}, L}^1} = c\|\tilde{f}\|_{H_{\text{max}, L}^1}.
\]

Let us note that 

\[
|\mathcal{R}_j \tilde{f}| = |(R_j f)\tilde{|}|
\]

Thus Theorem 1.3 follows from Theorem 1.2.

\[\Box\]

**Acknowledgments** The author wishes to thank Jean-Philippe Anker, Paweł Głowacki and Rysiek Szwarc for their remarks. The author is greatly indebted to Bartosz Trojan for his suggestions which shortened the original proof of Theorem 1.3. Finally, the author wants to thank the referee for her/his careful reading of the manuscript and helpful comments which improved the presentation of the paper. Research supported by the Polish National Science Center (Narodowe Centrum Nauki, grant DEC-2012/05/B/ST1/00672) and by the University of Orléans.

**Appendix**

It is well known that 

\[
\int_{\mathbb{R}} h_{t}^{(j)}(x, y) \, d\mu_j(y) = 1, \quad d\mu_j(y) = |y|^{2k_j} \, dy.
\]  

(7.2)

It was proved in [2] that \( h_{t}^{(j)}(x, y) \) has the following global behavior :

\[
h_{t}^{(j)}(x, y) \asymp \begin{cases} t^{-k_j - \frac{1}{2}} e^{-\frac{x^2 + y^2}{4t}} & \text{if } |xy| \leq t, \\ t^{-\frac{1}{2}} (xy)^{-k_j} e^{-\frac{(x-y)^2}{4t}} & \text{if } xy \geq t, \\ t^\frac{1}{2} (-xy)^{-k_j - 1} e^{-\frac{(x+y)^2}{4t}} & \text{if } -xy \geq t. \end{cases}
\]

(7.3)

From (7.3) we easily conclude that 

\[
0 < h_t(x, y) \leq \frac{C}{\mu(B(x, \sqrt{t}))} \text{ for all } x, y \in \mathbb{R}^n \text{ and } t > 0;
\]  

(7.4)

\[
h_t(x, y) \leq \frac{C}{\mu(B(x, \sqrt{t}))} e^{-c|x|^2/t} \text{ for } |x| > 2n|y|.
\]

(7.5)

We shall need the following inequalities for volumes of the Euclidean balls (see [2])

\[
\left( \frac{R}{r} \right)^n \lesssim \frac{\mu(B(x, R))}{\mu(B(x, r))} \lesssim \left( \frac{R}{r} \right)^N, \quad \forall \, x \in \mathbb{R}^n, \forall \, R, r > 0.
\]

(7.6)
The subordination formula (2.1) combined with (7.1) and (7.2) implies

\[
\int_{\mathbb{R}^n} P_t(x, y) \, d\mu(x) = \int_{\mathbb{R}^n} P_t(x, y) \, d\mu(y) = 1. \tag{7.7}
\]

**Lemma 7.1** There is a constant \( C > 0 \) such that

\[
0 < P_t(x, y) \leq \frac{C}{\mu(B(x, t))}. \tag{7.8}
\]

Moreover, for every \( 0 < \delta < \frac{1}{N} \) there is a constant \( C_\delta \) such that

\[
P_t(x, y) \leq \frac{C_\delta}{\mu(B(x, t))} \left( 1 + \frac{\mu(B(x, |x|))}{\mu(B(x, t))} \right)^{-1-\delta} \quad \text{for } |x| > 2n|y|. \tag{7.9}
\]

**Proof** To see (7.8) we use (2.1) together with (7.4) and (7.6) and obtain

\[
P_t(x, y) \lesssim \int_0^{\frac{1}{4}} \frac{1}{\mu(B(x, t))} \frac{\mu(B(x, t))}{\mu(B(x, \frac{t}{2\sqrt{u}}))} \frac{du}{\sqrt{u}} + \int_{\frac{1}{4}}^{\infty} e^{-u} \frac{\mu(B(x, t))}{\mu(B(x, \frac{t}{2\sqrt{u}}))} \frac{du}{\sqrt{u}}
\]

\[
\leq \frac{1}{\mu(B(x, t))} \left( \int_0^{\frac{1}{4}} u^{n/2} \frac{d}{\sqrt{u}} + \int_{\frac{1}{4}}^{\infty} e^{-u} u^{n/2} \frac{du}{\sqrt{u}} \right) \lesssim \frac{1}{\mu(B(x, t))}.
\]

The proof of the lower bound of \( P_t(x, y) \) is obvious.

In order to prove (7.9) it suffices to consider \( t \leq |x|/2 \). By (7.6), for every \( \delta > 0 \) and \( c > 0 \), we have

\[
\left(1 + \frac{\mu(B(x, |x|))}{\mu(B(x, \sqrt{s}))}\right)^{1+\delta} \leq C_\delta \left(1 + \frac{|x|}{\sqrt{s}}\right)^{(1+\delta)N} \leq C_\delta c e^{c|x|^{2/s}}, \text{ for } s > 0. \tag{7.10}
\]

Utilizing (7.5) together with (7.10) and proceeding similarly to the proof of (7.8) we have

\[
P_t(x, y) \lesssim \int_0^{\infty} \frac{e^{-u}}{\mu(B(x, \frac{t}{2\sqrt{u}}))} \left(1 + \frac{\mu(B(x, |x|))}{\mu(B(x, \frac{t}{2\sqrt{u}}))}\right)^{-1-\delta} \frac{du}{\sqrt{u}}
\]

\[
= \int_0^{\infty} \frac{e^{-u}}{\mu(B(x, t))} \frac{\mu(B(x, t))}{\mu(B(x, \frac{t}{2\sqrt{u}}))} \times \left(1 + \frac{\mu(B(x, |x|))}{\mu(B(x, t))} \frac{\mu(B(x, t))}{\mu(B(x, \frac{t}{2\sqrt{u}}))}\right)^{-1-\delta} \frac{du}{\sqrt{u}}
\]

\[
\lesssim \int_0^{\frac{1}{4}} \frac{1}{\mu(B(x, t))} \frac{\mu(B(x, t))}{\mu(B(x, \frac{t}{2\sqrt{u}}))} \left(1 + \frac{\mu(B(x, |x|))}{\mu(B(x, t))}\right)^{-1-\delta}
\]

\[
\lesssim \frac{C_\delta}{\mu(B(x, t))} \left(1 + \frac{\mu(B(x, |x|))}{\mu(B(x, t))}\right)^{-1-\delta}.
\]
Theorem 7.2
Assume that a set $X$ is equipped with

$$
\times \left( \frac{\mu(B(x, t))}{\mu(B(x, \frac{t}{2\sqrt{u}}))} \right)^{-1-\delta} \frac{d u}{\sqrt{u}}
+ \int_{\frac{1}{4}}^{\infty} e^{-u} \frac{\mu(B(x, t))}{\mu(B(x, \frac{t}{2\sqrt{u}}))} \left( 1 + \frac{\mu(B(x, |x|))}{\mu(B(x, t))} \right)^{-1-\delta} \frac{d u}{\sqrt{u}}.
$$

Fix $0 < \delta < N^{-1}$. Applying (7.6) we obtain

$$
P_t(x, y) \lesssim \int_{0}^{\frac{1}{4}} \frac{1}{\mu(B(x, t))} \left( 1 + \frac{\mu(B(x, |x|))}{\mu(B(x, t))} \right)^{-1-\delta} u^{-\delta N/2} \frac{d u}{\sqrt{u}}
+ \int_{\frac{1}{4}}^{\infty} e^{-u} u^{N/2} \frac{\mu(B(x, t))}{\mu(B(x, \frac{t}{2\sqrt{u}}))} \left( 1 + \frac{\mu(B(x, |x|))}{\mu(B(x, t))} \right)^{-1-\delta} \frac{d u}{\sqrt{u}},
$$

which proves (7.9).

Proof of Theorem 2.1 The proof, which is in its spirit similar to that of the heat kernel characterization of $H^1_{\text{atom}}$ (see [2]), is based on the following result due to Uchiyama [26].

Theorem 7.2 Assume that a set $X$ is equipped with

- a quasi-distance $\tilde{d}$ i.e., a distance except that the triangular inequality is replaced by the weaker condition

$$
\tilde{d}(x, y) \leq A [\tilde{d}(x, z) + \tilde{d}(z, y)], \quad \forall x, y, z \in X;
$$

- a measure $\mu$ whose values on quasi-balls satisfy

$$
\frac{r}{A} \leq \mu(B(x, r)) \leq r, \quad \forall x \in X, \forall r > 0;
$$

- a continuous kernel $K_r(x, y) \geq 0$ such that, for every $r > 0$ and $x, y, y' \in X$,

  - $K_r(x, x) \geq \frac{1}{A}$,
  - $K_r(x, y) \leq r^{-1} \left( 1 + \frac{\tilde{d}(x, y)}{r} \right)^{-1-\delta}$,
  - $\left| K_r(x, y) - K_r(x, y') \right| \leq r^{-1} \left( 1 + \frac{\tilde{d}(x, y)}{r} \right)^{1-2\delta} \frac{\tilde{d}(y, y')}{r^\delta}$ when $\tilde{d}(y, y') \leq \frac{r + \tilde{d}(x, y)}{4A}$.

Here $A \geq 1$ and $\delta > 0$. Then the following definitions of the Hardy space $H^1(X)$ and their corresponding norms are equivalent:

- Maximal definition: $H^1_{\text{max}, K_r}(X)$ consists of all functions $f \in L^1(X, d \mu)$ such that

$$
K^*_f(x) = \sup_{r > 0} \left| \int_X K_r(x, y) f(y) d \mu(y) \right|
$$

belongs to $L^1(X, d \mu)$ and the norm $\| f \|_{H^1_{\text{max}, K_r}(X)} = \| K^*_f \|_{L^1(X, d \mu)}$. 

 Springer
Atomic definition: An atom for $H^1_{\text{atom}}(X, \widetilde{d})$ is a measurable function $a : X \to \mathbb{C}$ such that: $a$ is supported in a quasi-ball $\widetilde{B}$, $\|a\|_{L^\infty} \lesssim \mu(\widetilde{B})^{-1}$ and $\int_X a \, d\mu = 0$ (see [5, 15, 26]). Then $H^1_{\text{atom}}(X, \widetilde{d})$ consists of all functions $f \in L^1(X, d\mu)$ which can be written as $f = \sum a_i b_i$, where the $a_i$’s are atoms and $\sum |\lambda_i| < +\infty$, and the norm $\|f\|_{H^1_{\text{atom}}(X, \widetilde{d})} = \inf \sum |\lambda_i|$, where the infimum is taken over all such representations.

For $X = \mathbb{R}^n$, equipped with the Euclidean distance $d(x, y) = |x - y|$ and the measure $\mu$ (see (1.5)), set

$$\widetilde{d}(x, y) = \inf \mu(B), \quad \forall \ x, y \in \mathbb{R}^n,$$

where the infimum is taken over all closed balls $B$ containing $x$ and $y$. Let $t = t(x, r)$ be defined by $\mu(B(x, \sqrt{t}) = r$. Then

$$\mu(B(x, r)) \sim r$$

and there exists a constant $c > 0$ such that

$$B(x, \sqrt{t}) \subset B(x, r) \subset B(x, c\sqrt{t}), \quad (7.11)$$

where $B(x, r) = \{y \in \mathbb{R}^n : d(x, y) < r\}$ (see, e.g., [2]).

Let us remark that thanks to (7.11) and (7.6) the atomic spaces $H^1_{\text{atom}}(X, \widetilde{d})$ and $H^1_{\text{atom}}$ (defined in Section 1) do coincide and $\|f\|_{H^1_{\text{atom}}(X, \widetilde{d})} \sim \|f\|_{H^1_{\text{atom}}}$. It was proved in [2] that the kernel $h_t$ can be written in the form

$$h_t(x, y) = H_t(x, y) + S_t(x, y),$$

where $H_t(x, y)$ and $S_t(x, y)$ are nonnegative continuous functions such that there are $C_1, C_2, C_4, \delta > 0$ such that

$$H_t(x, x) \geq \frac{C_1}{\mu(B(x, \sqrt{t}))}; \quad (7.12)$$

$$H_t(x, y) \leq \frac{C_2}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{d(x, y)}{\mu(B(x, \sqrt{t}))}\right)^{-1-\delta}; \quad (7.13)$$

$$|H_t(x, y) - H_t(x, y')| \leq \frac{C_4}{\mu(B(x, \sqrt{t}))} \left(1 + \frac{d(x, y)}{\mu(B(x, \sqrt{t}))}\right)^{-1-2\delta} \frac{d(x, y')}{\mu(B(x, \sqrt{t}))} \frac{1}{N} \quad (7.14)$$

for $d(y, y') \leq C_3 \max \{\mu(B(x, \sqrt{t})), d(x, y)\}$, (the kernel $S_t$ is denoted in [2] by $P_t$). Moreover, the maximal function

$$S_a f(x) = \sup_{t > 0} \left| \int S_t(x, y) f(y) d\mu(y) \right|$$

is a bounded operator on $L^1(\mathbb{R}^n, d\mu)$. Springer
Using subordination formula (2.1) we write

\[ P_t(x, y) = U_t(x, y) + W_t(x, y), \]  

(7.15)

where

\[ U_t(x, y) = c_1 \int_0^\infty e^{-u} H_{t/4u}(x, y) \frac{du}{\sqrt{u}}, \quad W_t(x, y) = c_1 \int_0^\infty e^{-u} S_{t/4u}(x, y) \frac{du}{\sqrt{u}}. \]

Clearly, the maximal operator

\[ W_* f(x) = \sup_{t > 0} \left| \int W_t(x, y) f(y) d\mu(y) \right| \]

is bounded on \( L^1(\mathbb{R}^n, d\mu) \), that is,

\[ \| W_* f \|_{L^1(\mathbb{R}^n, d\mu)} \leq C \| f \|_{L^1(\mathbb{R}^n, d\mu)}. \]  

(7.16)

Our task is to prove the following lemma.

**Lemma 7.3** There are constants \( C_1, C_2, C_4, \delta' > 0 \) such that

\[ U_t(x, x) \geq \frac{C_1}{\mu(B(x,t))}; \]  

(7.17)

\[ U_t(x, y) \leq \frac{C_2}{\mu(B(x,t))} \left(1 + \frac{\tilde{d}(x,y)}{\mu(B(x,t))}\right)^{-1-\delta'}; \]  

(7.18)

\[ |U_t(x, y) - U_t(x, y')| \leq \frac{C_3}{\mu(B(x,t))} \left(1 + \frac{\tilde{d}(x,y)}{\mu(B(x,t))}\right)^{-1-2\delta'} \left(\frac{\tilde{d}(y,y')}{\mu(B(x,t))}\right)^{\delta'} \]  

(7.19)

for \( \tilde{d}(y, y') \leq C_3 \max\{\mu(B(x,t)), \tilde{d}(x,y)\} \).

**Proof** Take \( 0 < \delta < N^{-1} \). By (7.12) and the subordination formula we have

\[ U_t(x, x) \geq \int_1^\infty \frac{e^{-u}}{\mu(B(\frac{x}{\sqrt{u}}))} \frac{du}{\sqrt{u}} \geq \int_1^\infty \frac{e^{-u}}{\mu(B(\frac{x}{\sqrt{u}}))} \frac{\mu(B(x,t))}{\mu(B(\frac{x}{\sqrt{u}}))} \frac{du}{\sqrt{u}} \]

\[ \geq \frac{1}{\mu(B(x,t))} \int_1^\infty \frac{e^{-u}}{\sqrt{u}} du \geq \frac{1}{\mu(B(x,t))}, \]

which proves (7.17).
The proof of (7.18) is similar to that of (7.9). Indeed, by (7.13) we have

\[ U_t(x, y) \leq \int_0^{\infty} e^{-u} \frac{\mu(B(x, t))}{\mu(B(x, \frac{t}{2\sqrt{u}}))} \left( 1 + \frac{\tilde{d}(x, y)}{\mu(B(x, t))} \right)^{-1-\delta} \frac{du}{\sqrt{u}} \]

\[ \leq \int_0^{1/4} e^{-u} \frac{\mu(B(x, t))}{\mu(B(x, \frac{t}{2\sqrt{u}}))} \left( 1 + \frac{\tilde{d}(x, y)}{\mu(B(x, t))} \right)^{-1-\delta} \frac{du}{\sqrt{u}} \]

\[ + \int_{1/4}^{\infty} e^{-u} \frac{\mu(B(x, t))}{\mu(B(x, \frac{t}{2\sqrt{u}}))} \left( 1 + \frac{\tilde{d}(x, y)}{\mu(B(x, t))} \right)^{-1-\delta} \frac{du}{\sqrt{u}}. \]

Now using (7.6) we obtain (7.18).

Now we turn to the proof of (7.19). First we show that there is a constant \( C_4 > 0 \) such that

\[ |U_t(x, y) - U_t(x, y')| \leq \frac{C_4}{\mu(B(x,t))} \left( \frac{\tilde{d}(y,y')}{\mu(B(x,t))} \right)^{\frac{1}{N}} \text{ for every } x, y, y' \in \mathbb{R}^n. \]  

(7.20)

Since \( U_t(x, y) \leq C \mu(B(x, t))^{-1} \) (see (7.18)), it suffices to prove (7.20) for \( \tilde{d}(y, y') \leq \mu(B(x,t)) \). Let \( u_0 \leq 1/4 \) be such that \( \mu(B(x, \frac{t}{2\sqrt{u_0}})) = \tilde{d}(y, y') \). Then, using (7.14) and (7.6), we have

\[ \int_0^{u_0} e^{-u} \left| H_{\frac{t}{2\sqrt{u}}} (x, y) - H_{\frac{t}{2\sqrt{u}}} (x, y') \right| \frac{du}{\sqrt{u}} \]

\[ \leq \int_0^{u_0} e^{-u} \frac{\mu(B(x, t))}{\mu(B(x, \frac{t}{2\sqrt{u}}))} \left( \frac{\tilde{d}(y, y')}{\mu(B(x, t))} \right)^{\frac{1}{N}} \frac{du}{\sqrt{u}} \]

\[ = \int_0^{u_0} e^{-u} \frac{\mu(B(x, t))}{\mu(B(x, \frac{t}{2\sqrt{u}}))} \left( \frac{\tilde{d}(y, y')}{\mu(B(x, t))} \right)^{\frac{1}{N}} \left( \mu(B(x, t)) \right)^{1+\frac{1}{N}} \frac{du}{\sqrt{u}} \]

\[ \leq \frac{1}{\mu(B(x, t))} \left( \frac{\tilde{d}(y, y')}{\mu(B(x, t))} \right)^{\frac{1}{N}} \]

\[ \left( \int_0^{1/4} e^{-u} u^{n(1+\frac{1}{N})/2} \frac{du}{\sqrt{u}} + \int_{1/4}^{u_0} e^{-u} u^{N(1+\frac{1}{N})/2} \frac{du}{\sqrt{u}} \right) \]

\[ \leq \frac{1}{\mu(B(x, t))} \left( \frac{\tilde{d}(y, y')}{\mu(B(x, t))} \right)^{\frac{1}{N}}. \]
Similarly, by (7.13), we get
\[ \int_{u_0}^{\infty} \frac{e^{-u}|H_{1/2}(x, y) - H_{1/2}(x, y')|}{\sqrt{u}} \frac{du}{\sqrt{u}} \leq \int_{u_0}^{\infty} \frac{e^{-u}}{\mu(B(x, t))} \left( \frac{\mu(B(x, t))}{\mu(B(x, \frac{t}{\sqrt{u}}))} \right) \frac{du}{\sqrt{u}} \]
\[ \leq \frac{1}{\mu(B(x, t))} \int_{u_0}^{\infty} e^{-u} u^{N/2} \frac{du}{\sqrt{u}} \]
\[ \leq \frac{1}{\mu(B(x, t))} u_0^{-N/2}. \]

Since
\[ \frac{\tilde{d}(y, y')}{\mu(B(x, t))} = \frac{\mu(B(x, \frac{t}{2\sqrt{u_0}}))}{\mu(B(x, t))} \geq u_0^{-N/2}, \]
(see (7.6)), we obtain (7.20).

We are now in a position to continue the proof of (7.19).

If \( \tilde{d}(y, y') \leq \mu(B(x, t)) \) then (7.19) follows from (7.20).

If \( \tilde{d}(y, y') > \mu(B(x, t)) \) and \( \tilde{d}(y, y') < \tilde{d}(x, y)/(2A) \), then \( \tilde{d}(x, y) \leq 2A \tilde{d}(x, y') \).

Hence, from (7.18) we conclude that
\[ \left| U_i(x, y) - U_i(x, y') \right| \leq \frac{C_2}{\mu(B(x, t))} \left( 1 + \frac{\tilde{d}(x, y)}{\mu(B(x, t))} \right)^{-1 - \delta}. \] (7.21)

Consequently, we deduce (7.19) (with perhaps small \( \delta' > 0 \)) from (7.20) and (7.21).

It remains to consider the case when \( \tilde{d}(x, y) > \mu(B(x, t)) \) and \( \tilde{d}(y, y') \geq \tilde{d}(x, y)/(2A) \). Recall that \( \tilde{d}(y, y') \leq \mu(B(x, t)) \). Thus \( \tilde{d}(x, y) \sim \mu(B(x, t)) \). So, finally, using (7.20) we have
\[ \left| U_i(x, y) - U_i(x, y') \right| \leq \frac{C_4}{\mu(B(x, t))} \left( \frac{\tilde{d}(y, y')}{\mu(B(x, t))} \right)^{\frac{1}{2}} \]
\[ \leq \frac{C_4}{\mu(B(x, t))} \left( \frac{\tilde{d}(y, y')}{\mu(B(x, t))} \right)^{\frac{1}{2}} \left( 1 + \frac{\tilde{d}(x, y)}{\mu(B(x, t))} \right)^{-1 - \delta}. \]

This completes the proof of Lemma 7.3. \( \square \)

Set \( K_r(x, y) = U_i(x, y) \), where \( r = \mu(B(x, t)) \). Now Theorem 2.1 follows from (7.15), boundedness of the maximal function \( W_\ast \) on \( L^1(\mathbb{R}^n, \mu) \), and the Uchiyama theorem (see Theorem 7.2) combined with Lemma 7.3.

Now we turn to prove (2.4). Recall that \( P_i(x, y) > 0 \). So, by (7.7), the operator \( P_\ast \) is bounded on \( L^\infty(\mathbb{R}^n, d\mu) \). Thanks to (7.18) and (7.16), it is of weak-type \((1,1)\). Finally, from the Marcinkiewicz interpolation theorem we conclude that \( P_\ast \) is bounded on \( L^p(\mathbb{R}^n, d\mu) \) for \( 1 < p < \infty \). \( \square \)

**Proof of Proposition 2.2** Fix \( \varepsilon > 0 \). There is \( R > 0 \) such that \( |g(x)| < \varepsilon \) for \( |x| > R \). Write
\[ g = g \chi B(0, R) + g \chi B(0, R)^c =: g_1 + g_2. \]
From (7.7) we get $|P_t g_2(x)| < \varepsilon$ for every $t > 0$ and $x \in \mathbb{R}^n$. Now using (7.8) we obtain

$$|P_t g_1(x)| \leq \frac{C}{\mu(B(x,t))} \|g_1\|_{L^1(\mathbb{R}^n, d\mu)} \to 0 \text{ as } t \to \infty.$$ 

On the other hand, if $t$ remains in a bounded interval and $|x| > 2nR$, applying (7.9) we have

$$|P_t g_1(x)| \leq \frac{C}{\mu(B(x,t))} \left(1 + \frac{\mu(B(x,|x|))}{\mu(B(x,t))}\right)^{-1-\delta} \|g_1\|_{L^1(\mathbb{R}^n, d\mu)} \to 0 \text{ as } |x| \to \infty.$$ 

The proof of the first part of Proposition 2.2 is complete.

In order to prove the second part of the proposition we fix $\varepsilon > 0$. We claim that

$$\lim_{|x| \to \infty} P_\varepsilon f(x) = 0.$$ 

To prove the claim let $\varepsilon' > 0$. Take $R > 0$ large enough such that

$$\int_{|y| > R} |f(y)| \, d\mu(y) \leq \varepsilon' \mu(B(0,R)).$$

Write $f = f_1 \chi_{B(0,R)} + f_2 \chi_{B(0,R)^c} =: f_1 + f_2$. Then, by (7.7) and (7.8) we have $|P_\varepsilon f_2| \leq \varepsilon'$. On the other hand from the first part of the proposition we conclude that $\lim_{|x| \to \infty} P_\varepsilon f_1(x) = 0$, which gives the claim. Now (2.5) follows from the first part of Proposition 2.2, since $P_{t+\varepsilon} f = P_t (P_\varepsilon f)$. \(\square\)

References

1. Amri, B., Sifi, M.: Riesz transform for Dunkl transform. Ann. Math. Blaise Pascal 19, 247–262 (2012)
2. Anker, J-Ph, Ben Salem, N., Dziubański, J., Hamda, N.: The Hardy space $H^1$ in the rational Dunkl setting. Constr. Approx. 42, 93–128 (2015)
3. Burkholder, D.L., Gundy, R.F., Silverstein, M.L.: A maximal function characterisation of the class $H^p$. Trans. Am. Math. Soc. 157, 137–153 (1971)
4. Coifman, R.R.: A real variable characterization of $H^p$. Studia Math. 51, 269–274 (1974)
5. Coifman, R.R., Weiss, G.L.: Extensions of Hardy spaces and their use in analysis. Bull. Am. Math. Soc. 83, 569–615 (1977)
6. Deleaval, L.: Fefferman-Stein inequalities for the $Z^d_2$ Dunkl maximal operator. J. Math. Anal. Appl. 360, 711–726 (2009)
7. de Jeu, M.F.E.: The Dunkl transform. Invent. Math. 113, 147–162 (1993)
8. Dunkl, C.F.: Reflection groups and orthogonal polynomials on the sphere. Math. Z. 197, 33–60 (1988)
9. Dunkl, C.F.: Differential-difference operators associated to reflection groups. Trans. Amer. Math. 311, 167–183 (1989)
10. Dunkl, C.F.: Hankel transforms associated to finite reflection groups. In: Proceedings of the Special Session on Hypergeometric Functions on Domains of Positivity, Jack polynomials and applications. Proceedings, Tampa, Contemp. Math., vol. 138, pp. 123–138 (1989)
11. Dunkl, C.F.: Integral kernels with reflection group invariance. Canad. J. Math. 43, 1213–1227 (1991)
12. Dziubański, J., Preisner, M., Wróbel, B.: Multivariate Hörmander-type multiplier theorem for the Hankel transform. J. Fourier Anal. Appl. 19, 417–437 (2013)
13. Evans, L.C.: Partial Differential Equations. Graduate Studies in Mathematics, vol. 19. AMS, Providence, RI (1998)
14. Fefferman, C., Stein, E.M.: $H^p$ spaces of several variables. Acta Math. 129, 137–193 (1972)
15. Macías, R.A., Segovia, C.: A decomposition into atoms of distributions on spaces of homogeneous type. Adv. Math. 33, 271–309 (1979)
16. Muckenhoupt, B., Stein, E.: Classical expansions and their relation to conjugate harmonic functions. Trans. Am. Math. Soc. 118, 17–92 (1965)
17. NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov
18. Opdam, E.M.: Lecture notes on Dunkl operators for real and complex reflection groups. MSJ Memoirs, vol. 8. Mathematical Society of Japan, Tokyo (2000)
19. Rösler, M.: Generalized Hermite polynomials and the heat equation for Dunkl operators. Commun. Math. Phys. 192, 519–542 (1998)
20. Rösler, M.: Dunkl operators: theory and applications. In: Orthogonal polynomials and special functions (Leuven, 2002). Lecture Notes in Mathematics, vol. 1817, pp. 93–135. Springer, Berlin (2003)
21. Rösler, M., Voit, M.: Dunkl theory, convolution algebras, and related Markov processes. In: Harmonic and stochastic analysis of Dunkl processes, Collection Travaux en cours, vol. 71, pp. 1–112. Hermann, Paris (2008)
22. Stein, E.M., Weiss, G.L.: On the theory of harmonic functions of several variables I (the theory of $H^p$ spaces). Acta Math. 103, 25–62 (1960)
23. Stein, E.M.: Singular Integrals and Differentiability of Functions. Princeton University Press, Princeton, NJ (1971)
24. Stein, E.M.: Harmonic Analysis (Real-Variable Methods, Orthogonality, and Oscillatory Integrals). Princeton Mathematics Series, vol. 43. Princeton University Press, Princeton, NJ (1993)
25. Thangavelu, S., Xu, Y.: Convolution operator and maximal function for the Dunkl transform. J. Anal. Math. 97, 25–55 (2005)
26. Uchiyama, A.: A maximal function characterization of $H^p$ on the space of homogeneous type. Trans. Am. Math. Soc. 262(2), 579–592 (1980)
27. Watson, G.N.: A Treatise on the Theory of Bessel Functions. Cambridge University Press, Cambridge, MA (1995)