ANTI-YETTER-DRINFELD MODULES FOR QUASI-HOPF ALGEBRAS.

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Abstract. We apply categorical machinery to the problem of defining anti-Yetter-Drinfeld modules for quasi-Hopf algebras. While a definition of Yetter-Drinfeld modules in this setting, extracted from their categorical interpretation as the center of the monoidal category of modules has been given, none was available for the anti-Yetter-Drinfeld modules that serve as coefficients for a Hopf cyclic type cohomology theory for quasi-Hopf algebras. This is a followup paper to the authors’ previous effort that addressed the somewhat different case of anti-Yetter-Drinfeld contramodule coefficients in this and Hopf algebroid setting.

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1. Introduction

It is an interesting fact that the theory of coefficients in Hopf cyclic theories began with what is now known as anti-Yetter-Drinfeld modules in [9, 6, 7] that followed [2, 3]. It was not until [1] that anti-Yetter-Drinfeld contramodules were introduced. The latter, in retrospect, seem a lot more natural, though they involve notions that are less so.

In this followup paper to [10] we make the modifications necessary to deal with the definitions of anti-Yetter-Drinfeld modules for quasi-Hopf algebras which generalize the Hopf algebras by relaxing the coassociativity condition to coassociativity up to a specified isomorphism. This isomorphism complicated matters sufficiently that a direct generalization of the formulaic approach used for Hopf algebras is not possible.

Similar to what we do here, but less laborious calculations, have been performed in [11] where the categorical notion of the center of a monoidal category of modules over a quasi-Hopf algebra has been unwound into formulas. Our task is complicated by two factors: we deal with a certain bimodule category over the category of modules over a quasi-Hopf algebra and we allow not only finite dimensional representations.

The main theme of [10] was exploiting the fact that the category of modules is biclosed, i.e., it possesses internal Homs. This allows for a definition of a natural bimodule category over it, the center of which is what we are looking for. The justification for the importance of the center is the observation that its elements (or rather the ones satisfying an additional stability condition) can be used to quickly manufacture a functor called a symmetric 2-contratrace [8] and thus define a cyclic cohomology theory.

As mentioned above, historically anti-Yetter-Drinfeld modules appeared before their contramodule versions, but in this paper we rely on the conceptual definition of generalized anti-Yetter-Drinfeld contramodules as in [10] to obtain the module version. In particular
the question of stability is explicitly reduced to the contramodule case, though it is immediate that it is equivalent to the stability in [8], though not completely analogous to what is possible to do in the Hopf algebra case; see Remark 2.7.

The paper is organized as follows. In Section 2 we review and augment some generalities from [10] that we use subsequently. Section 3 is a review of the basics of quasi-Hopf algebras and recalls the definition of internal Homs, from [10], for them. Finally in Section 4 we unravel the conceptual definitions of Section 2 into formulas. An interesting observation is the appearance of two distinct ways of writing down the formulas; these are identical for Hopf algebras but very different here. This mirrors a similar phenomenon that occurs in the contramodule case.

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2. Generalities

In this section we will extend the general formalism of [10] from generalized anti-Yetter-Drinfeld contramodule coefficients to their module variant. Recall from [8] that the main ingredient in constructing Hopf-cyclic cohomology is a symmetric 2-contratrace. It is with a view towards this goal that we undertake the following.

Let \( M \) be a biclosed monoidal category, i.e., it possesses internal Homs. More precisely, the property of being biclosed implies in particular the existence of the following adjunctions for \( M, V, W \in M \):

\[
\text{Hom}_M(W \otimes V, M) \cong \text{Hom}_M(W, \text{Hom}^l(V, M)),
\]

and

\[
\text{Hom}_M(V \otimes W, M) \cong \text{Hom}_M(W, \text{Hom}^r(V, M)),
\]

where \( \text{Hom}^l(V, M) \) and \( \text{Hom}^r(V, M) \) are left and right internal homomorphisms respectively.

As in [12], we can introduce the contragradient \( M \)-bimodule category \( M^{op} \). Specifically, for \( M \in M^{op} \) and \( V \in M \), the actions are given by:

\[
M \blacktriangleleft V := \text{Hom}^r(V, M), \quad V \blacktriangleright M := \text{Hom}^l(V, M).
\]

A natural object to consider in this situation is the center of a bimodule category \( N \); roughly speaking a category with a left and a right action of our monoidal category \( M \). However, here it becomes too restrictive. If in the definition of a (strong) center element \( N \in N \) we relax the condition that the maps \( \tau : N' \triangleright N \to N \blacktriangleleft N' \) are isomorphisms, we get a weak center. More formally:

**Definition 2.1.** The weak center \( w\mathbb{Z}_M(N) \) of a \( M \)-bimodule category \( N \) consists of objects that are pairs \( (N, \tau_{N,\cdot}) \), where \( N \in N \) and \( \tau_{N,\cdot} \) is a family of natural morphisms such that for \( V \in M \) we have: \( \tau_{N,V} : V \triangleright N \to N \blacktriangleleft V \), satisfying the hexagon axiom as in the usual definition of center (see [5]).

We need one more definition.
Definition 2.2. Let \( \mathcal{M} \) be a biclosed monoidal category, \( \mathcal{M}^{op} \) a contragradient category as above. For \( M \in w-\mathcal{Z}_\mathcal{M}(\mathcal{M}^{op}) \) let

\[ \sigma_M : M \to M \]

denote the image of \( \text{Id}_M \) under

\[ \text{Hom}_\mathcal{M}(M, M) \simeq \text{Hom}_\mathcal{M}(1, M \triangleright M) \simeq \text{Hom}_\mathcal{M}(1, M \triangleleft M) \simeq \text{Hom}_\mathcal{M}(M, M) \]

where the map in the middle is postcomposition with \( \tau \) and the isomorphisms come from definitions of actions in \( \mathcal{M}^{op} \).

We have from \([10]\):

Lemma 2.3. If \( M \in w-\mathcal{Z}_\mathcal{M}(\mathcal{M}^{op}) \) and \( \sigma_M = \text{Id} \) then \( M \in \mathcal{Z}_\mathcal{M}(\mathcal{M}^{op}) \).

We called such \( M \) stable and denoted the full subcategory containing them by \( \mathcal{Z}'_\mathcal{M}(\mathcal{M}^{op}) \). These are exactly the generalized stable anti-Yetter-Drinfeld contramodules and the functor \( \text{Hom}_\mathcal{M}(-, M) \) is a symmetric 2-contratrace. Recall that for an algebra \( A \in \mathcal{M} \) the collection \( \text{Hom}_\mathcal{M}(A^{\otimes \bullet + 1}, M) \) is naturally a cocyclic vector space.

2.1. The module variant modifications. Assume as above that \( \mathcal{M} \) is biclosed and suppose further that there exists a tensor auto-equivalence (\( - \)\( ^\# \)) of \( \mathcal{M} \) together with natural identifications:

\[ W \triangleright 1 \simeq 1 \triangleleft W^\#. \]

Observe that should such a functor exist we would immediately have natural identifications:

\[ \iota_{V,W} : \text{Hom}_\mathcal{M}(V \otimes W, 1) \simeq \text{Hom}_\mathcal{M}(W^\# \otimes V, 1) \]

for all \( V, W \in \mathcal{M} \).

Consider \( ^\# \mathcal{M} \), an \( \mathcal{M} \)-bimodule category with the right and left \( \mathcal{M} \)-module structures given by:

\[ M \triangleleft V = M \otimes V \quad \text{and} \quad V \triangleright M = V^\# \otimes M. \]

Lemma 2.4. We have a functor between weak centers:

\[ D : (w-\mathcal{Z}_\mathcal{M}(^\# \mathcal{M}))^{op} \to w-\mathcal{Z}_\mathcal{M}(\mathcal{M}^{op}) \]

that sends \( M \) to \( 1 \triangleleft M \).

Proof. The weak center structure on \( 1 \triangleleft M \) is obtained as follows: \( V \triangleright (1 \triangleleft M) \simeq (V \triangleright 1) \triangleleft M \simeq (1 \triangleleft V^\#) \triangleleft M \simeq 1 \triangleleft (V^\# \otimes M) \to 1 \triangleleft (M \otimes V) \simeq (1 \triangleleft M) \triangleright V. \]

Recall a definition of stability for generalized anti-Yetter-Drinfeld modules from \([8]\): For \( M \in w-\mathcal{Z}_\mathcal{M}(^\# \mathcal{M}) \) if

\[ \text{Hom}_\mathcal{M}(M \otimes V, 1) \xrightarrow{-\circ \tau} \text{Hom}_\mathcal{M}(V^\# \otimes M, 1) \xrightarrow{\iota_{M,V}^{-1}} \text{Hom}_\mathcal{M}(M \otimes V, 1) \]

is identity then \( M \) is called stable. It is immediate that the following definition is equivalent to this one.

Definition 2.5. Let \( M \in w-\mathcal{Z}_\mathcal{M}(^\# \mathcal{M}) \), we say that \( M \) is stable if \( DM \in w-\mathcal{Z}_\mathcal{M}(\mathcal{M}^{op}) \) is stable, i.e., if \( \sigma_{1\triangleleft M} = \text{Id} \).
We have the following analogue of Lemma 2.3:

**Lemma 2.6.** Suppose that $M \in w\mathcal{Z}_M(\#M)$ is stable. Assume that $D$ reflects isomorphisms (when considered as a functor from $(\#M)^{op}$ to $M^{op}$), then $M \in \mathcal{Z}_M(\#M)$. We will denote the full subcategory of such $M$ by $\mathcal{Z}'_M(\#M)$.

**Proof.** By definitions if $M$ is stable then $\sigma_1 \triangleright M = Id$ and so by Lemma 2.3 and the proof of Lemma 2.4 the centrality map $\tau : V^\# \otimes M \to M \otimes V$ is such that its right dual is an isomorphism, i.e., $D(\tau)$ is an isomorphism. If $D$ reflects isomorphisms then $\tau$ is an isomorphism as well. □

The $M$ of the lemma above are exactly the generalized stable anti-Yetter-Drinfeld modules. We note that the functor $Hom_\mathcal{M}(M \otimes -, 1)$ is a symmetric 2-contratrace in this case. This follows immediately from its isomorphism to $Hom_\mathcal{M}(-, DM)$ as $DM \in \mathcal{Z}'_M(M^{op})$.

**Remark 2.7.** The current approach to general stability of aYD modules, via stability of aYD contramodules (implicitly as in [8] or explicitly as in Definition 2.5), may seem unsatisfactory but it is the only way in general. In particular situations one may do better. More precisely, it may happen that for $M \in w\mathcal{Z}_M(\#M)$ the map $\sigma_{DM}$ may have a predual, i.e., a $\sigma_M$ such that $Id \triangleright \sigma_M = \sigma_1 \triangleright M$. This is the case for Hopf algebras where $\sigma_M(m) = m_1m_0$, but this question remains open in the quasi-Hopf algebra case.

3. Recalling Quasi-Hopf algebras

Let us remind the reader of all the necessary definitions following [4]. In this section $k$ is a field.

**Definition 3.1.** A quasi-bialgebra is a collection $(A, \Delta, \varepsilon, \Phi)$, where $A$ is an associative $k$-algebra with unity, $\Delta : A \to A \otimes A$ and $\varepsilon : A \to k$ are homomorphisms of algebras, $\Phi \in A \otimes A \otimes A$ is an invertible elements, such that the following equalities hold:

(3.1) $(id \otimes \Delta)(\Delta(a)) = \Phi \cdot ((\Delta \otimes id)(\Delta(a))) \cdot \Phi^{-1}$ \hspace{1cm} \forall a \in A

(3.2) $[(id \otimes id \otimes \Delta)(\Phi)] \cdot [(\Delta \otimes id \otimes id)(\Phi)] = (1 \otimes \Phi) \cdot [(id \otimes \Delta \otimes id)(\Phi)] \cdot (\Phi \otimes 1)$

(3.3) $(\varepsilon \otimes id)(\Delta(a)) = a$, \hspace{1cm} $(id \otimes \varepsilon)(\Delta(a)) = a$ \hspace{1cm} \forall a \in A

(3.4) $id \otimes \varepsilon \otimes id(\Phi) = 1 \otimes 1$

**Remark 3.2.** In this paper we will use the Sweedler notation. Let’s denote

(3.5) $\Phi = X \otimes Y \otimes Z$,

(3.6) $\Phi^{-1} = P \otimes Q \otimes R$,

here we mean the summation. In particular, the equality (3.1) can be written as:

$a^1 \otimes a^{21} \otimes a^{22} = Xa^{11}P \otimes Ya^{12}Q \otimes Za^{2}R$. 
We are interested in the category of left $A$-modules $\mathcal{A}\mathcal{M}$. It was proved in [4] that this category is monoidal if a tensor product of two left $A$-modules $M$ and $N$ is defined by the same formula as in the case of a bialgebra:

\begin{equation}
M \otimes N = M \otimes_k N, \quad a \cdot (m \otimes n) = a^1 m \otimes a^2 n.
\end{equation}

The associativity morphism is no longer trivial as it was in the case of a bialgebra. If one sets the associativity morphism $(M \otimes N) \otimes L \to M \otimes (N \otimes L)$ to be the image of $\Phi$ in $\text{End}_k(M \otimes N \otimes L)$, then it becomes an isomorphism of left $A$-modules by (3.1). Consider $k$ as an $A$-module by $a \cdot 1 = \varepsilon(a)1$ as in the bialgebra case. Then one defines a morphism $\lambda_M: k \otimes M \to M$ as the usual morphism of $k$-modules. Then $\lambda_M$ becomes an $A$-module morphism by (3.3). Similarly one can define a morphism $\rho_M: M \otimes k \to M$. So $k$ is a unit of the monoidal category $\mathcal{A}\mathcal{M}$.

**Remark 3.3.** The equality (3.2) is equivalent to the pentagon axiom. Associativity and unit in the category respect each other by (3.4).

Recall the definition of a quasi-Hopf algebra from [4].

**Definition 3.4.** Let $(H, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra. Then it is called a quasi-Hopf algebra if there exist $\alpha, \beta \in H$ and anti-automorphism $S: H \to H$, such that

\begin{align}
S(h^1)\alpha h^2 &= \varepsilon(h)\alpha, \\
h^1 \beta S(h^2) &= \varepsilon(h)\beta.
\end{align}

If one keeps notation as in Remark 3.2, then there should be equalities

\begin{align}
X \beta S(Y)\alpha Z &= 1, \\
S(P)\alpha Q \beta R &= 1.
\end{align}

**Remark 3.5.** We want to emphasize that the antipode $S$ in the definition above is assumed to be invertible.

It was shown in [4] that the category of $H$-modules for a quasi-Hopf algebra $H$ that are finite dimensional as $k$-vector spaces is rigid. It was shown in [10] that if we consider all modules, i.e., $\mathcal{H}\mathcal{M}$ then it is biclosed. More precisely, for any $M, N \in \mathcal{H}\mathcal{M}$, we can define the left internal Hom by

\begin{equation}
\text{Hom}^l(M, N) = \text{Hom}_k(M, N), \quad h \cdot \varphi = h^1 \varphi(S(h^2)\cdot).\end{equation}

Furthermore, we can define the right internal Hom by

\begin{equation}
\text{Hom}^r(M, N) = \text{Hom}_k(M, N), \quad h \cdot \varphi = h^2 \varphi(S^{-1}(h^1)\cdot).\end{equation}

4. **Anti-Yetter-Drinfeld modules for a quasi-Hopf algebra**

Yetter-Drinfeld modules for a quasi-Hopf algebra were described in [11]. The consideration of $YD$ modules as centers of a monoidal category $\mathcal{H}\mathcal{M}$ was crucial to write down the complicated formulas. In this section we are going to define the anti-Yetter-Drinfeld modules using the categorical approach from [8]. Unlike the Hopf-case, there are two ways to define $aYD$ modules.
As in [8] consider a monoidal functor $\#: H\mathcal{M} \to H\mathcal{M}$, taking a left $H$-module $M$ to $M^\#$, where $M^\#$ is the same as $M$ as a $k$-vector space but the module structure is modified by $S^2$:

$$h \cdot m = S^2(h)m \quad \text{for} \quad m \in M^\#.$$  

Define the $H\mathcal{M}$-bimodule category $H^\#\mathcal{M}$ such that it has the same objects as $H\mathcal{M}$. For $M \in H^\#\mathcal{M}$ and $V \in H\mathcal{M}$, the right and left $H\mathcal{M}$-module structures are given by:

$$M \triangleleft V = M \otimes V, \quad \text{and} \quad V \triangleright M = V^\# \otimes M.$$  

**Remark 4.1.** Observe that as required according to Section 2 we have a very trivial identification $V \triangleright k \simeq k \triangleright V^\#$, indeed the left hand side is $\text{Hom}_k(V, k)$ with $h \cdot \varphi = \varphi(S(h) - \lambda)$ whereas the right hand side is $\text{Hom}_k(V^\#, k)$ with $h \cdot \varphi = \varphi(S^{-1}(h) \cdot \lambda) = \varphi(S(h) - \lambda)$. Furthermore, the $D$ functor being essentially a vector space duality functor reflects isomorphisms and so as long as we insist on stability in the sense of Definition 2.5 we do not need to worry about the difference between the weak and the strong center.

If $H$ is a Hopf algebra it was proved in [8] that the center $Z_{H\mathcal{M}}(H^\#\mathcal{M})$ is the same as anti-Yetter-Drinfeld modules. We are going to use this fact as a guide and give a description of $aYD$ modules in the quasi-Hopf case.

### 4.1. Anti-Yetter-Drinfeld modules I.

We organize the material similar to [11].

**Lemma 4.2.** Let $M \in H^\#\mathcal{M}$. Natural transformations $\tau_\bullet \in \text{Nat}(id \triangleright M, M \triangleleft id)$ are in 1-1 correspondence with $k$-linear maps $\rho : M \to M \otimes H$, denoted by $m \mapsto m^{(0)} \otimes m^{(1)}$, such that

$$h^1m^{(0)} \otimes h^2m^{(1)} = (h^2m)^{(0)} \otimes (h^2m)^{(1)}S^2(h^1).$$

**Proof.** Consider the morphism $\tau_H : H \triangleright M \to M \triangleleft H$. Define $\rho(m)$ as $\tau_H(1 \triangleright m) \in M \otimes H$. Because $\tau_H$ is a morphism in the category (of left $H$-modules), we should have:

$h \cdot \tau_H(1 \triangleright m) = \tau_H(h \cdot (1 \triangleright m))$ for any $h \in H$. The left hand side gives us $h^1m^{(0)} \otimes h^2m^{(1)}$.

Using the definition of $H^\#\mathcal{M}$ we can see that $h \cdot (1 \triangleright m) = S^2(h^1) \triangleright h^2m$. The right action of $H$ on itself is a morphism in $H\mathcal{M}$, so because $\tau_\bullet$ is a natural transformation, one can rewrite the right hand side as $(h^2m)^{(0)} \otimes (h^2m)^{(1)}S^2(h^1)$.

Conversely, given the map $\rho$, for any $V \in H\mathcal{M}$ one can define the map $\tau_V : V \triangleright M \to M \triangleleft V$ by the rule $v \triangleright m \mapsto m^{(0)} \triangleright (m^{(1)}v)$. It is a morphism of $H$-modules by [4,3]. Clearly $\tau_\bullet$ is a natural transformation.

These correspondences are mutually inverse. For a given $v \in V$ consider a morphism $H \to V$ by the rule $h \mapsto h \cdot v$. So, if we want $\tau$ to be a natural transformation, it is uniquely defined from $\rho$. 

In the Hopf case the $k$-linear map $\rho : M \to M \otimes H$ was a part of a right comodule structure on the left $H$-module $M$. But in the quasi-Hopf case the comodule condition must be replaced.

Let $(M, \tau_\bullet) \in Z_{H\mathcal{M}}(H^\#\mathcal{M})$. By the hexagon axiom of the center, the following diagram is commutative:
Consider this diagram in the case $V = W = H$. Start with the element $1 \triangleright m$ in the upper left corner. The definition of the $k$-linear map $\rho : M \rightarrow M \otimes H$ from Lemma 4.2 and the definition of the category $\#_H^M$ imply the following equality:

$$Qm_{(0)} \otimes (Qm_{(0)})^{(1)} S^2(P) \otimes Rm_{(1)} = P(Rm_{(0)}) \otimes Q((Rm_{(1)})^1 S^2(P) \otimes R((Rm_{(1)})^2 S^2(Q).$$

(4.4)

**Remark 4.3.** Notice that if $H$ is Hopf algebra, $\Phi$ is trivial and the condition (4.4) comes down to the definition of a right $H$-comodule.

Consider the following diagram:

$$
\begin{array}{cccccc}
H \triangleright M & \xrightarrow{\rho \otimes id} & k \triangleright M & \xrightarrow{\cong} & M \\
\downarrow{\tau_H} & & \downarrow{\tau_k} & & \downarrow{id} \\
M \triangleleft H & \xrightarrow{id \otimes \tau} & M \triangleleft k & \xrightarrow{\cong} & M.
\end{array}
$$

The left square commutes by the naturality of $\tau$ and the right square commutes by the definition of the center. If we start with $1 \triangleright m$ in the upper left corner we will get the equality:

$$m = \varepsilon(m_{(1)})m_{(0)}.$$  

(4.5)

This condition is exactly the same as in the Hopf case.

**Definition 4.4.** Let $H$ be a quasi-Hopf algebra. A pair $(M, \rho)$, where $M$ is a left $H$-module and $\rho : M \rightarrow M \otimes H$ is a $k$-linear map, written as $\rho(m) = m_{(0)} \otimes m_{(1)}$, is called a left-right anti-Yetter-Drinfeld module of type I, if it satisfies the equalities (4.3), (4.4) and (4.5).

A morphism of two aYD modules $(M, \rho) \rightarrow (M', \rho')$ is an $H$-morphism $f : M \rightarrow M'$, such that $\rho' \circ f = (f \otimes id) \circ \rho$.

**Theorem 4.5.** The category of aYD-modules of type I for a quasi-Hopf algebra $H$ is equivalent to $w-Z_{H,M}(\#_H^M)$, a weak center.

**Proof.** We have seen that an object in the center $(M, \tau)$ gives us an aYD module $(M, \rho)$. Consider a morphism of central objects $f : (M, \tau) \rightarrow (M', \tau')$. In particular the following diagram must commute:

$$
\begin{array}{cccccc}
H \triangleright M & \xrightarrow{\tau_H} & M \triangleleft H \\
\downarrow{Id} & & \downarrow{f \triangleleft id} \\
H \triangleright M' & \xrightarrow{\tau'_H} & M' \triangleleft H.
\end{array}
$$
Conversely, take an aYD module $(M, \rho)$. By Lemma 4.2, there is a natural transformation $\tau : id_M M \to M \triangleleft id$. Formula (4.1) guarantees that $\tau$ satisfies the hexagon axiom. Equality (4.5) gives that $\tau_k = id$. \hfill \Box

4.2. Anti-Yetter-Drinfeld modules II. There is the second way to introduce aYD modules. Let us again consider the central element $(M, \tau) \in Z_H M(H, M)$. So for any $V \in H M$ we have a natural isomorphism: $\tau_V : V \otimes M \to M \otimes V$. Using internal Hom it gives a morphism:

$$\hat{\tau}_V : M \to \text{Hom}^r(V^\#, M \otimes V).$$

Now we want to introduce a new $H$-module $M \otimes^r H$ which is the same as $M \otimes_k H$ as a vector space, but the $H$-action is different:

$$x \cdot (m \otimes h) = x^{21} m \otimes x^{22} h S(x^1).$$

For any $V \in H M$ define a map $r_V : M \otimes^r H \to \text{Hom}^r(V^\#, M \otimes V)$ by the rule: $m \otimes h \mapsto (v \mapsto m \otimes hv)$. This is a morphism in the category by construction of $M \otimes^r H$ and $V^\#$. We can formulate a Lemma similar to Lemma 4.2.

Lemma 4.6. Let $M \in \# H M$. Natural transformations $\tau \in \text{Nat}(id_M M, M \triangleleft id)$ are in 1-1 correspondence with $k$-linear maps $\lambda : M \to M \otimes H$, written as $m \mapsto m_{[0]} \otimes m_{[1]}$, such that

$$(4.6) \quad (hm)_{[0]} \otimes (hm)_{[1]} = h^{21} m_{[0]} \otimes h^{22} m_{[1]} S(h^1).$$

Proof. First assume that $\tau$ is given. Then for $m \in M$ we define $\lambda(m) := (\tau_H(m))(1)$. From the fact that $(\hat{\tau}_H(m))$ is a right internal homomorphism we get (4.6). Conversely, given a $k$-linear map $\lambda : M \to M \otimes H$ we can consider it as an $H$-homomorphism $M \to M \otimes^r H$ by (4.6). Now for any $V \in H M$ we define $\hat{\tau}_V$ by:

$$M \xrightarrow{\hat{\tau}_V} \text{Hom}^r(V^\#, M \otimes V)$$

$$\lambda \downarrow \quad r_V$$

$$M \otimes^r H$$

Everything is constructed naturally and the one-to-one correspondence is clear. \hfill \Box

Remark 4.7. It will be useful to explicitly write the reconstruction formula. Given $\lambda : M \to M \otimes H$, satisfying (4.6), $\tau_V : V \otimes M \to V \otimes M$ is built by the formula:

$$(4.7) \quad v \otimes m \mapsto R^1 m_{[0]} \otimes R^2 m_{[1]} S(Q)S(\alpha)^{S(2)(P)} v.$$

As above to write the replacement of the comodule condition consider the hexagon axiom. Then, using formula (4.7), we get the following equality:

$$(4.8) \quad PR^1 (Rm)_{[0]} \otimes Q(R^2 (Rm)_{[1]} \kappa^1) S^2(P) \otimes R(R^2 (Rm)_{[1]} \kappa^2) S^2(Q)$$

$$= R^1 (QR^1 m_{[0]})_{[0]} \otimes R^2 (QR^1 m_{[0]})_{[1]} \kappa S^2(P) \otimes RR^2 m_{[1]} \kappa,$$

where we set $\kappa = S(Q)S(\alpha)^{S(2)(P)}$. 

By the construction of $\rho$ one gets the condition $\rho \circ f = (f \otimes id) \circ \rho$. Conversely, take an aYD module $(M, \rho)$. By Lemma 4.2, there is a natural transformation $\tau : id_M M \to M \triangleleft id$. Formula (4.1) guarantees that $\tau$ satisfies the hexagon axiom. Equality (4.5) gives that $\tau_k = id$. \hfill \Box
The unital condition that \( \tau_k : k \triangleright M \to M \triangleright k \) is identity gives:
\[
m = R^1 m^{[0]} \cdot \varepsilon(R^2 m^{[1]} \kappa) = \varepsilon(m^{[1]}) R m^{[0]} \cdot (\kappa).
\]
Here we used that \( \varepsilon \) is algebra map and the formula (3.3). For a quasi-Hopf algebra the equality \( \varepsilon \circ S = \varepsilon \) holds (for the proof see [4]). So we can simplify \( \varepsilon(\kappa) = \varepsilon(\alpha) \varepsilon(Q) \varepsilon(P) \).
Using (3.4), we get the final equality:
(4.9)
\[
m = \varepsilon(m^{[1]}) m^{[0]} \varepsilon(\alpha).
\]

**Definition 4.8.** Let \( H \) be a quasi-Hopf algebra. A pair \( (M, \lambda) \), where \( M \) is a left \( H \)-module and \( \rho : M \to M \otimes H \) is a \( k \)-linear map, written as \( \rho(m) = m^{[0]} \otimes m^{[1]} \), is called a left-right anti-Yetter-Drinfeld module of type II, if it satisfies the equalities (4.6), (4.8) and (4.5).

And as before we have the following Theorem with a proof that is very similar to the type I case and so is omitted.

**Theorem 4.9.** The category of \( aYD \)-modules of type II for a quasi-Hopf algebra \( H \) is equivalent to \( \mathcal{W}_H^\# \mathcal{M} \).

**Remark 4.10.** Type I and type II \( aYD \) modules are different, though of course are equivalent as categories. The difference between them is like the difference between two maps: \( H \otimes \text{Hom}^r(H, V) \to V \), where the first map is \( h \otimes f \mapsto f(h) \) (naive evaluation) and the second one is \( \text{ev}^r(h \otimes f) \) (actual evaluation). In the Hopf case these two evaluations are the same, but it is no longer true for a quasi-Hopf algebra.

The reader is also invited to read the proof that in the Hopf case formula (4.3) is equivalent to (4.6) ([8, Lemma 2.2]).

**References**

[1] T. Brzezinski, *Hopf-cyclic homology with contramodule coefficients*, Quantum groups and noncommutative spaces, 1–8, Aspects Math., E41, Vieweg + Teubner, Wiesbaden, 2011.

[2] A. Connes, H. Moscovici, *Hopf algebras, cyclic cohomology and the transverse index theorem*, Commun. Math. Phys. 198 (1998), 199–246.

[3] A. Connes, H. Moscovici, *Cyclic cohomology and Hopf algebras*, Lett. Math. Phys. 48 (1999), 97–108.

[4] V. Drinfeld, *Quasi-Hopf algebras*, Leningrad Mathematical Journal, 1990, 1:6, 1419–1457.

[5] P. Etingof, D. Nikshych and V. Ostrik, *Fusion categories and homotopy theory*, Quantum Topology, 1 (2010), 209–273.

[6] P. M. Hajac, M. Khalkhali, B. Rangipour, and Y. Sommerhauser, *Hopf-cyclic homology and cohomology with coefficients*, C. R. Math. Acad. Sci.Paris 338 (2004), no. 9, 667–672.

[7] P. M. Hajac, M. Khalkhali, B. Rangipour, and Y. Sommerhäuser, *Stable anti-Yetter-Drinfeld modules*, C. R. Acad. Sci. Paris, Ser. I, 338 (2004) 587–590.

[8] M. Hassanzadeh, M. Khalkhali, and I. Shapiro, *Monoidal Categories, 2-Traces, and Cyclic Cohomology*, preprint [arXiv:1602.05441].

[9] P. Jara, D. Stefan, *Hopf-cyclic homology and relative cyclic homology of Hopf-Galois extensions*. Proc. London Math. Soc. (3) 93 (2006), no. 1, 138–174.

[10] I. Kobyzev, I. Shapiro, *A categorical approach to cyclic cohomology of quasi-Hopf algebras and Hopf algebroids*, preprint [arXiv:1803.09194].

[11] S. Majid, *Quantum Double for Quasi-Hopf Algebras*, Letters in Mathematical Physics (1998), 45:1, 1–9.

[12] I. Shapiro, *Some invariance properties of cyclic cohomology with coefficients*, preprint [arXiv:1611.01425].
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