QUIVERS WITH RELATIONS FOR SYMMETRIZABLE CARTAN MATRICES II: CHANGE OF SYMMETRIZERS

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Abstract. For \( k \geq 1 \) we consider the \( K \)-algebra \( H(k) := H(C, kD, \Omega) \) associated to a symmetrizable Cartan matrix \( C \), a symmetrizer \( D \) and an orientation \( \Omega \) of \( C \), which was defined in [GLS1]. We construct and analyse a reduction functor from \( \text{rep}(H(k)) \) to \( \text{rep}(H(k-1)) \). As a consequence we show that the canonical decomposition of rank vectors for \( H(k) \) does not depend on \( k \), and that the rigid locally free \( H(k) \)-modules are up to isomorphism in bijection with the rigid locally free \( H(k-1) \)-modules. Finally, we show that for a rigid locally free \( H(k) \)-module of a given rank vector the Euler characteristic of the variety of flags of locally free submodules with fixed ranks of the subfactors does not depend on the choice of \( k \).

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1. Introduction

1.1. Overview. In [GLS1] we have started to investigate the representation theory of a new class of quiver algebras associated with symmetrizable generalized Cartan matrices. One of the motivations is to obtain new geometric constructions of the positive part of a symmetrizable Kac-Moody algebra in terms of varieties of representations of these quiver algebras.

An interesting feature of our construction is that for a given generalized Cartan matrix we obtain not just one, but an infinite series \( H(k) \) \((k \geq 1)\) of quiver algebras (the definition of \( H(k) \) is recalled below). In this paper we study the dependence on \( k \) of the representation theory of \( H(k) \). More precisely, for \( k \geq 2 \) we introduce a reduction functor from the module category \( \text{rep}(H(k)) \) to the module category \( \text{rep}(H(k-1)) \). Although \( \text{rep}(H(k)) \) has more indecomposable objects than \( \text{rep}(H(k-1)) \), we show that this functor induces a bijection between isomorphism classes of rigid locally free \( H(k) \)-modules and isomorphism classes of rigid locally free \( H(k-1) \)-modules. Recall that a module \( M \) over an algebra \( A \) is rigid if \( \text{Ext}^1_A(M, M) = 0 \). Moreover, we introduce a notion of canonical decomposition of rank vectors for locally free \( H(k) \)-modules, and we show that it does not depend on \( k \).
Let $M$ be a rigid locally free $H(k)$-module. Given a sequence of rank vectors $\underline{r} = (r_1, \ldots, r_l)$, we define the quasi-projective variety $\text{Flf}_{\underline{r}}(M)$ of flags of locally free submodules of $M$ with prescribed rank vectors $r_1, r_1 + r_2, \ldots, r_1 + \cdots + r_l$. We show that $\text{Flf}_{\underline{r}}(M)$ is smooth and irreducible. Moreover, if $\overline{M}$ is the unique rigid locally free $H(k - 1)$-module with the same rank vector as $M$, we prove that there is a canonical surjective morphism $\text{Flf}_{\underline{r}}(M) \to \text{Flf}_{\underline{r}}(M)$ with all fibers being affine spaces. In particular, choosing $\mathbb{C}$ as base field, we get that the topological Euler characteristics $\chi\left(\text{Flf}_{\underline{r}}(M)\right)$ and $\chi\left(\text{Flf}_{\underline{r}}(M)\right)$ are equal.

This last result will be used in [GLS2] to realize the enveloping algebra of the positive part of a simple complex Lie algebra as a convolution algebra of constructible functions over varieties of representations of $H(k)$. In particular this convolution algebra is independent of $k$. In another work in preparation we plan to use it also for the construction of new cluster characters for skew-symmetrizable acyclic cluster algebras.

The study of canonical decompositions of dimension vectors was started by Kac [K2] for representations of quivers over an algebraically closed field (i.e. for symmetric Cartan matrices and $k = 1$). Schofield [S4] and Crawley-Boevey [CB] showed that this decomposition is independent of the base field. The quiver Grassmannians of rigid representations of acyclic quivers were studied by Caldero and Reineke [CR], who proved their smoothness. Later Wolf [W] proved the smoothness and irreducibility of quiver flag varieties of rigid quiver representations.

We will now give more precise statements of our results.

1.2. Definition of $H(C, D, \Omega)$. We use the notation of [GLS1]. Let $C = (c_{ij}) \in M_n(\mathbb{Z})$ be a symmetrizable generalized Cartan matrix, and let $D = \text{diag}(c_1, \ldots, c_n)$ be a symmetrizer of $C$. This means that $c_i \in \mathbb{Z}_{>0}$, and

\[
c_{ii} = 2, \quad c_{ij} \leq 0 \quad \text{for} \quad i \neq j, \quad c_ic_j = c_jc_i.
\]

When $c_{ij} < 0$ define

\[
g_{ij} := |\gcd(c_{ij}, c_{ji})|, \quad f_{ij} := |c_{ij}|/g_{ij}.
\]

An orientation of $C$ is a subset $\Omega \subset \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ such that the following hold:

(i) $\{(i, j), (j, i)\} \cap \Omega \neq \emptyset$ if and only if $c_{ij} < 0$;

(ii) For each sequence $((i_1, i_2), (i_2, i_3), \ldots, (i_t, i_{t+1}))$ with $t \geq 1$ and $(i_s, i_{s+1}) \in \Omega$ for all $1 \leq s \leq t$ we have $i_1 \neq i_{t+1}$.

For an orientation $\Omega$ of $C$ let $Q := Q(C, \Omega) := (Q_0, Q_1)$ be the quiver with vertex set $Q_0 := \{1, \ldots, n\}$ and with arrow set

\[
Q_1 := \{\alpha_{ij}^{(g)} : j \to i \mid (i, j) \in \Omega, 1 \leq g \leq g_{ij}\} \cup \{\varepsilon_i : i \to i \mid 1 \leq i \leq n\}.
\]

Throughout let $K$ be an algebraically closed field. Let

\[
H := H(C, D, \Omega) := KQ/I
\]

where $KQ$ is the path algebra of $Q$, and $I$ is the ideal of $KQ$ defined by the following relations:

(H1) For each $i$ we have

\[
\varepsilon_i^c_i = 0;
\]
(H2) For each \((i, j) \in \Omega\) and each \(1 \leq g \leq g_{ij}\) we have
\[
\varepsilon_i^{f_{ij}g} \alpha_{ij} = \alpha_{ij} \varepsilon_i^{f_{ij}}.
\]
This definition is illustrated by many examples in [GLS1, Section 13].

Let \(H_i\) be the subalgebra of \(H\) generated by \(\varepsilon_i\). Thus \(H_i\) is isomorphic to the truncated polynomial ring \(K[x]/(x^{c_i})\). In fact, \(\varepsilon_i^{c_i} = 0 \in H\) by (H1). On the other hand the only oriented cycles in \(Q\) consist of the loops \(\varepsilon_i\) by the definition of \(\Omega\). Thus, since the relations (H2) involve arrows which are not loops, \(\varepsilon_i^{c_i-1}\) does not belong to the ideal generated by the relations.

Let \(M = (M_i, M(\varepsilon_i), M(\alpha_{ij}^{(g)}))_{i\in Q_0, (i,j)\in \Omega}\) be a representation of \(H\). Thus \(M_i\) is a finite-dimensional \(K\)-vector space, and \(M(\varepsilon_i): M_i \to M_i\) and \(M(\alpha_{ij}^{(g)}): M_j \to M_i\) are \(K\)-linear maps satisfying the relations (H1) and (H2). Each \(M_i\) can obviously be seen as an \(H_i\)-module. We call \(M\) \emph{locally free} if \(M_i\) is a free \(H_i\)-module for all \(i\). In this case, \(\text{rank}(M) = \text{rank}_{H_i}(M) := (\text{rank}(M_1), \ldots, \text{rank}(M_n))\) is the \emph{rank vector} of \(M\), where \(\text{rank}(M_i)\) denotes the rank of the free \(H_i\)-module \(M_i\). Recall from [GLS1] that an \(H(k)\)-module \(M\) is locally free if and only if \(\text{proj.dim}(M) \leq 1\).

Let \(\langle-, -\rangle_H: \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}\) be the bilinear form defined by
\[
\langle a, b \rangle_H := \sum_{i=1}^{n} c_i a_i b_i + \sum_{(i,j) \in \Omega} c_i c_{ij} a_j b_i
\]
where \(a = (a_1, \ldots, a_n)\) and \(b = (b_1, \ldots, b_n)\). For locally free \(H\)-modules \(M\) and \(N\) we have
\[
\langle \text{rank}(M), \text{rank}(N) \rangle_H = \dim \text{Hom}_H(M, N) - \dim \text{Ext}^1_H(M, N),
\]
see [GLS1, Section 4].

For \(k \geq 1\) and \(H = H(C, D, \Omega)\) we define \(H(k) := H(C, kD, \Omega)\). Observe that
\[
\varepsilon := \sum_{i=1}^{n} \varepsilon_i^{c_i}
\]
belongs to the center of \(H(k)\), and \(H(k)/(\varepsilon^{k-1}H(k)) \cong H(k-1)\) for \(k \geq 2\).

1.3. Canonical decompositions. We study the affine varieties \(\text{rep}_{l.f.}(H(k), \mathbf{r})\) of locally free \(H(k)\)-modules with rank vector \(\mathbf{r}\).

We show that \(\text{rep}_{l.f.}(H(k), \mathbf{r})\) is smooth and irreducible and we compute its dimension. A tuple \((\mathbf{r}_1, \ldots, \mathbf{r}_t)\) of rank vectors with \(\mathbf{r} = \mathbf{r}_1 + \cdots + \mathbf{r}_t\) is the \emph{\(H(k)\)-canonical decomposition} of \(\mathbf{r}\) if there is a dense open subset \(U\) of \(\text{rep}_{l.f.}(H(k), \mathbf{r})\) such that each \(M \in U\) is of the form \(M = M_1 \oplus \cdots \oplus M_t\) with \(M_i\) an indecomposable locally free \(H(k)\)-module with rank vector \(\mathbf{r}_i\) for \(1 \leq i \leq t\). Using [CBS], we show that such an \(H(k)\)-canonical decomposition exists and is unique up to permutation of its entries (see Section 3.3).

For \(k \geq 2\) we construct and study the reduction functor
\[
R: \text{rep}(H(k)) \to \text{rep}(H(k-1)), \quad M \mapsto M/\varepsilon^{k-1}M
\]
and its restriction to locally free modules (see Proposition 3.2). We show that \(R\) induces a natural bijection between isomorphism classes of locally free rigid \(H(k)\)-modules and locally free rigid \(H(k-1)\)-modules. The functor \(R\) also allows us to compare \(H(k)\)-canonical decompositions with \(H(k-1)\)-canonical decompositions. We obtain the following result (see Section 3.4):
Theorem 1.1. For $H(k) = H(C, kD, \Omega)$ the $H(k)$-canonical decompositions of rank vectors do not depend on $k$.

1.4. Flag varieties. We fix for some $l \geq 2$ a sequence of rank vectors $\underline{r} = (r_1, \ldots, r_l) \in \mathbb{N}^{l \times \{1, 2, \ldots, l\}}$ and set $\underline{m} := r_1 + \cdots + r_l$. For a locally free $H(k)$-module $M$ the quiver flag variety of locally free submodules $\text{Flf}_{\underline{r}}^{H(k)}(M)$ is the quasi-projective variety of flags

$$(0 = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_{l-1} \subset U_l = M)$$

of locally free submodules with $\text{rank}_{H(k)}(U_j/U_{j-1}) = r_j$ for $j = 1, 2, \ldots, l$.

There exists up to isomorphism at most one rigid locally free $H(k)$-module $M_{\underline{m}, k}$ with rank vector $\underline{m}$. If there is no such module, we set $M_{\underline{m}, k} = 0$. Thus, we may define $\text{Flf}_{\underline{r}}^{H(k)} := \text{Flf}_{\underline{r}}^{H(k)}(M_{\underline{m}, k})$. We have $R(M_{\underline{m}, k}) \cong M_{\underline{m}, k-1}$.

Theorem 1.2. For all $k \geq 1$ and all rank vector sequences $\underline{r}$ we have $\text{Flf}_{\underline{r}}^{H(k)} \neq \emptyset$ if and only if $\text{Flf}_{\underline{r}}^{H(1)} \neq \emptyset$. In this case, the following hold:

(a) The variety $\text{Flf}_{\underline{r}}^{H(k)}$ is smooth and irreducible of dimension $k \cdot d(\underline{r})$ where

$$d(\underline{r}) := \sum_{a < b} (r_a, r_b)_{H(1)}.$$

(b) For $k \geq 2$ the canonical morphism

$$\pi_k : \text{Flf}_{\underline{r}}^{H(k)} \to \text{Flf}_{\underline{r}}^{H(k-1)}$$

induced by the projection $M_{\underline{m}, k} \to R(M_{\underline{m}, k})$ is a fiber bundle with all fibers being affine spaces of dimension $d(\underline{r})$.

Part (a) will be proved in Proposition 4.7. Though we cannot apply directly Wolf’s results [W], our argument here follows his ideas closely. Part (b), which is based on Part (a), is proved after some preparation, in Section 4.7.

For a complex algebraic variety $X$ let $\chi(X)$ denote the topological Euler characteristic of $X$.

Corollary 1.3. Assume that $K = \mathbb{C}$, and let $H(k) = H(C, kD, \Omega)$ for some $k \geq 1$. Then for all sequences $\underline{r}$ the Euler characteristic $\chi(\text{Flf}_{\underline{r}}^{H(k)})$ does not depend on $k$.

Proof. Recall, that if $\pi : V \to W$ is a surjective morphism between complex algebraic varieties with $\chi(\pi^{-1}(w)) = c$ for all $w \in W$, then $\chi(W) = c \chi(V)$. This is a special case of [Di] Proposition 4.1.31. Thus, Theorem 1.2(b) implies that $\chi(\text{Flf}_{\underline{r}}^{H(k+1)}) = \chi(\text{Flf}_{\underline{r}}^{H(k)})$ for all $k > 0$ since $\chi(A) = 1$ for each finite-dimensional complex vector space $A$. □

Corollary 1.3 has the following application. In [GLS2] we study a convolution algebra $\mathcal{M}(H)$ associated with $H$. For $H$ a path algebra of a quiver (i.e. if $C$ is symmetric and $D$ the identity), Schofield proved that $\mathcal{M}(H)$ is isomorphic to the enveloping algebra $U(n)$ of the positive part $n$ of the Kac-Moody Lie algebra $g(C)$ associated with $C$. We prove in [GLS2] that for $C$ of Dynkin type the same result holds. We first show this for minimal symmetrizers $D$ and then use Corollary 1.3 for the generalization to arbitrary symmetrizers.
1.5. **Notation.** Unless stated otherwise, by a *module* we mean a finite-dimensional left module. We fix a field $K$. Except for the basic results in Section 2, we always assume that $K$ is algebraically closed. Let $A$ be a $K$-algebra. Let mod$(A)$ be the category of $A$-modules, and let proj$(A)$ be the subcategory of projective $A$-modules. We write $\mathbb{N}$ for the natural numbers, including 0.

2. **Reduction functors**

2.1. **Analogy with representations of modulated graphs.** It was shown in [GLS1] Section 5] that $H = H(C, D, \Omega)$ gives rise to a generalized modulated graph, and that the category of $H$-modules is isomorphic to the category of representations of this generalized modulated graph.

As before, let $H_i$ be the subalgebra of $H$ generated by $\varepsilon_i$. For $(i, j) \in \Omega$ we define

$$\iH_j := H_i \text{ Span}_K (\alpha_{ij}^{(g)} | 1 \leq g \leq g_{ij}) H_j.$$ 

It is shown in [GLS1] that $\iH_j$ is an $H_i$-$H_j$-bimodule, which is free as a left $H_i$-module and free as a right $H_j$-module. An $H_i$-basis of $\iH_j$ is given by

$$\{ \alpha_{ij}^{(g)} \varepsilon_j, \ldots, \alpha_{ij}^{(g)} \varepsilon_j^{g_{ij} - 1} | 1 \leq g \leq g_{ij} \}.$$ 

In particular, we have an isomorphism $\iH_j \cong H_i^{[c_{ij}]}$ of left $H_i$-modules, and we have an isomorphism $\iH_j \cong H_j^{[c_{ij}]}$ of right $H_j$-modules.

The tuple $(H_i, \iH_j)$ with $1 \leq i \leq n$ and $(i, j) \in \Omega$ is called a *generalized modulation* associated with the datum $(C, D, \Omega)$. A *representation* $(M_i, M_{ij})$ of this generalized modulation consists of a finite-dimensional $H_i$-module $M_i$ for each $1 \leq i \leq n$, and of an $H_i$-linear map

$$M_{ij} : \iH_j \otimes H_j M_j \to M_i$$

for each $(i, j) \in \Omega$. The representations of this generalized modulation form an abelian category rep$(C, D, \Omega)$ isomorphic to the category of $H$-modules [GLS1] Proposition 5.1]. (Here we identify the category mod$(H)$ of $H$-modules with the category rep$(H)$ of representations of the quiver $Q(C, \Omega)$ satisfying the relations (H1) and (H2).) Given a representation $(M_i, M_{ij})$ in rep$(C, D, \Omega)$ the corresponding $H$-module $(M_i, M(\alpha_{ij}^{(g)}), M(\varepsilon_i))$ is obtained as follows: the $K$-linear map $M(\varepsilon_i) : M_i \to M_i$ is given by

$$M(\varepsilon_i)(m) := \varepsilon_i m.$$ 

(here we use that $M_i$ is an $H_i$-module), and for $(i, j) \in \Omega$, the $K$-linear map $M(\alpha_{ij}^{(g)}) : M_j \to M_i$ is defined by

$$M(\alpha_{ij}^{(g)})(m) := M_{ij}(\alpha_{ij}^{(g)} \otimes m).$$

The maps $M(\alpha_{ij}^{(g)})$ and $M(\varepsilon_i)$ satisfy the defining relations (H1) and (H2) of $H$ because the maps $M_{ij}$ are $H_i$-linear.

2.2. **The central subalgebra $Z(k)$.** Let $H(k) = H(C, kD, \Omega)$. Recall that

$$\varepsilon := \sum_{i=1}^n \varepsilon_i^c$$

belongs to the center of $H(k)$. We denote by $Z(k)$ the subalgebra which is generated by $\varepsilon$. It is easy to see that $Z(k) \cong K[X]/(X^n)$. Obviously, $H(k)$ is a $Z(k)$-algebra, and consequently rep$(H(k))$ is a $Z(k)$-linear category. Moreover, if $C$ is connected, and $D$...
Lemma 2.1. \( H_i(k) \) is the subalgebra of \( H(k) \) generated by \( \varepsilon_i \). Thus \( H_i(k) \cong K[\varepsilon_i]/(\varepsilon_i^{k+1}) \).

Since \( H_i(k) \) is a free \( Z(k) \)-module of rank \( c_i \), one easily checks that locally free \( H(k) \)-modules are free as \( Z(k) \)-modules. In particular, \( H(k) \) is free as a \( Z(k) \)-module, since all projective \( H(k) \)-modules are locally free.

2.3. The reduction functor. We start with the following well-known lemma:

Lemma 2.2. Let \( (A, m) \) be a commutative, local ring with residue field \( \kappa \). Suppose that \( f: A^a \to A^b \) is an \( A \)-linear map. Then \( f \) is surjective, if and only if the induced, \( \kappa \)-linear map \( f: \kappa^a \to \kappa^b \) is surjective. Moreover, in this case \( \text{Ker}(f) \) is a direct summand of \( A^a \).

We fix \( k \geq 2 \). For typographical reasons we abbreviate \( H := H(k), H_i = H_i(k) \) and \( \overline{H} := H/(\varepsilon^{k-1}H) \cong H(k-1) \) and \( \overline{H}_i = H_i(k-1) \). There is an obvious reduction functor \( R = R(k): \text{rep}(H) \to \text{rep}(\overline{H}) \) defined by \( M \mapsto M/(\varepsilon^{k-1}M) \). For \( M \in \text{rep}(H) \) let \( \overline{M} := R(M) = M/(\varepsilon^{k-1}M) \). The functor \( R \) is isomorphic to the tensor functor \( \overline{H} \otimes_H - \) and is therefore right exact. We collect several basic properties of the restriction of the functor \( R \) to locally free modules.

Proposition 2.2. For \( k \geq 2 \) the following hold:

(a) If \( M \) is a locally free \( H \)-module with \( \text{rank}_H(M) = m \), then \( \overline{M} \) is a locally free \( \overline{H} \)-module with \( \text{rank}_{\overline{H}}(\overline{M}) = m \). In particular, this defines an exact functor \( R_{\text{lf}} = R_{\text{lf}}(k): \text{rep}_{\text{lf}}(H) \to \text{rep}_{\text{lf}}(\overline{H}) \).

Moreover, \( R_{\text{lf}} \) induces a full and dense functor \( \text{proj}(H) \to \text{proj}(\overline{H}) \).

(b) To any chain \( X_1 \subset X_2 \subset \cdots \subset X_l \) of locally free \( \overline{H} \)-modules, there exists a chain \( Y_1 \subset Y_2 \subset \cdots \subset Y_l \) of locally free \( H \)-modules with \( \overline{Y}_j \cong X_j \) for \( 1 \leq j \leq l \). In particular, \( R_{\text{lf}} \) is dense.

(c) For \( M, N \in \text{rep}_{\text{lf}}(H) \) the natural map \( R_{\text{lf}}^1(M, N): \text{Ext}^1_H(M, N) \to \text{Ext}^1_{\overline{H}}(\overline{M}, \overline{N}) \) is surjective.

(d) For \( M, N \in \text{rep}_{\text{lf}}(H) \) with \( \text{Ext}^1_{\overline{H}}(\overline{M}, \overline{N}) = 0 \) we have \( \text{Ext}^1_H(M, N) = 0 \). In this case, \( \text{Hom}_H(M, N) \) is a free \( Z(k) \)-module, and the natural map \( R_{\text{lf}}^0(M, N): \text{Hom}_H(M, N) \to \text{Hom}_{\overline{H}}(\overline{M}, \overline{N}) \) is surjective. In particular, if \( M \in \text{rep}_{\text{lf}}(H) \) is indecomposable and rigid, then \( \overline{M} \) is indecomposable and rigid.

Proof. (a) If \( M \in \text{rep}_{\text{lf}}(H) \) and \( 1 \leq i \leq n \), then by definition \( e_i M \) is a free \( H_i \)-module of rank \( m_i \), and \( \varepsilon^{k-1}e_i M = \varepsilon^{(k-1)c_i} M \). Thus \( e_i M/(\varepsilon^{k-1}e_i M) \) is a free \( \overline{H}_i \)-module, also of rank \( m_i \). Since \( R \) is right exact, and since its restriction \( R_{\text{lf}} \) preserves rank vectors, we get that \( R_{\text{lf}} \) is exact. The last claim follows, since projective \( H \)-modules are locally free, and \( \overline{H} = H/(\varepsilon^{k-1}H) \).

(b) We show the case \( l = 2 \). The general case can be done similarly, though with heavier notation.
Thus, let $U = X_1$, $u := \text{rank}_\mathbb{H}(U)$ and $M = X_2$, $m := \text{rank}_\mathbb{H}(M)$. After choosing for each $(i, j) \in \Omega$ left bases $iL_j$ of $iH_j$ the elements $l = l + \varepsilon^{k-1}_jH_j$ with $l \in iL_j$ are identified with a left basis $\bar{L}_{ij}$ of $\mathbb{H}_i$. After choosing $\mathbb{H}_i$-bases $\bar{x}_{i,1}, \ldots, \bar{x}_{i,m_i}$ of $M_i$ such that $\bar{x}_{i,1}, \ldots, \bar{x}_{i,u_i}$ spans $U_i$ as an $\mathbb{H}_i$-module for all $1 \leq i \leq n$, with respect to these bases, the structure maps $M_{ij} : \mathbb{H}_i \otimes \mathbb{H}_j M_j \to M_i$ of $M$ become matrices of the block shape

$$M_{ij} = \begin{pmatrix} U_{ij} & \ast \\ 0 & \ast \end{pmatrix} \in \mathbb{H}_{ij}^{m_i \times (|c_{ij}|m_j)} \text{ and } U_{ij} \in \mathbb{H}_{ij}^{c_{ij} \times (|c_{ij}|u_j)}.$$ 

Since we can see an element of $\mathbb{H}_i$ or $H_i$ just as a truncated polynomial in $\varepsilon_i$, we can interpret $M_{ij}$ and $U_{ij}$ also as matrices with entries in $H_i$. Thus we can take these matrices to define structure maps for locally free $H$-modules $V$ and $N$ with the requested properties that $V \subseteq N$, $\nabla = U$ and $\nabla = M$.

(c) and (d) Observe, that in case $P$ is a projective $H$-module, the space $\text{Hom}_H(P, M)$ is for each locally free $H$-module $M$ naturally a free $Z(k)$-module. By the same token $\text{Hom}_\mathbb{H}(P, M)$ is a free $Z(k-1)$-module of the same rank. It follows that the natural map $R_{l.f.}(P, M) : \text{Hom}_H(P, M) \to \text{Hom}_\mathbb{H}(P, M)$ is surjective. Now, since locally free $H$-modules have projective dimension at most $1$, we can find a projective resolution

$$0 \to P_1 \xrightarrow{p_M} P_0 \to M \to 0,$$

which yields by (a) also a projective resolution of $\mathbb{H}$-modules

$$0 \to \mathbb{P}_1 \xrightarrow{\mathbb{p}_M} \mathbb{P}_0 \to \mathbb{M} \to 0.$$

We obtain a commutative diagram with exact rows:

$$
\begin{array}{cc}
0 & \text{Hom}_H(M, N) \\
R_{l.f.}(M,N) & \text{Hom}_H(P_0, N) \xrightarrow{\partial_M} \text{Hom}_H(P_1, N) \xrightarrow{\partial_M} \text{Ext}^1_H(M, N) \\
0 & \text{Hom}_\mathbb{H}(M, N) \\
R_{l.f.}(P_0,N) & \text{Hom}_\mathbb{H}(P_0, N) \xrightarrow{\partial_M} \text{Hom}_\mathbb{H}(P_1, N) \xrightarrow{\partial_M} \text{Ext}^1_\mathbb{H}(M, N) \\
\end{array}
$$

Now, by the above remark $R_{l.f.}(P_0,N)$ and $R_{l.f.}(P_1,N)$ are surjective. Thus, the $K$-linear map $R_{l.f.}^1(M,N)$ is clearly surjective.

Next, observe that in the above diagram $\partial \circ \partial_M$ is a $Z(k-1)$-linear map between free $Z(k-1)$-modules, say of rank $a$ resp. $b$. Similarly, $\partial \circ p_M$ is a $Z(k)$-linear map between free $Z(k)$-modules, also of rank $a$ resp. $b$. Since $R_{l.f.}(P_0,N)$ and $R_{l.f.}(P_1,N)$ are just the reductions modulo $\varepsilon^{k-1}$ and $Z(k)$ is a commutative local ring, we conclude by Lemma 2.21 that $\partial \circ p_M$ is surjective if and only if $\partial \circ \partial_M$ is surjective. Moreover, in this case $\partial \circ p_M$ splits as $Z(k)$-linear map and $\partial \circ \partial_M$ splits as $Z(k-1)$-linear map. This implies that $R_{l.f.}(M,N)$ is surjective. Finally, if $M$ is rigid, then $\nabla$ is also rigid by (c). If moreover $M$ is indecomposable, then $\text{End}_H(M)$ is local, and $\varepsilon^{k-1} \text{End}_H(M)$ consists of nilpotent elements, so it is contained in the radical. Since $R_{l.f.}(M,M)$ is surjective, this shows that $\text{End}_H(M)$ is local, so $\nabla$ is indecomposable.

2.4. Example. The following example shows that the reduction functor $R_{l.f.}$ is usually not full. It also shows that in general $R_{l.f.}$ does not preserve indecomposables. Let $C$ be a Cartan matrix of Dynkin type $A_2$ with symmetrizer $D = \text{diag}(1, 1)$ and orientation $\Omega = \{(1, 2)\}$. Thus, $H(k)$ is defined by the quiver

$$
\varepsilon_1 \circlearrowleft_1 \alpha \xrightarrow{2} \varepsilon_2
$$
with relations $\varepsilon_1 \alpha - \alpha \varepsilon_2 = 0$ and $\varepsilon_i^k = 0$ for $i = 1, 2$. Consider the locally free $H(2)$-modules

$$E_2: \begin{array}{c} 2 \\ \downarrow \varepsilon_2 \\ 2' \end{array} \quad \text{and} \quad M: \begin{array}{c} 1 \\ \downarrow \alpha \\ 1' \end{array} \rightarrow \begin{array}{c} 2 \\ \downarrow \varepsilon_2 \\ 2' \end{array}$$

In the left picture, 2 and $2'$ denote two basis vectors of $E_2$, regarded as a $K$-vector space, and the arrow means that $M(\varepsilon_2)(2) = 2'$. Similarly, in the right picture, 1 and $1'$ (resp. 2 and $2'$) denote two basis vectors in the space $M_1$ (resp. in the space $M_2$), and the arrows represent the linear maps $M(\varepsilon_i)$ and $M(\alpha)$. Thus, $\text{rank}_{H(2)}(E_2) = (0, 1)$ and $\text{rank}_{H(2)}(M) = (1, 1)$. We get $\overline{M} \cong S_1 \oplus S_2$ and $\overline{E}_2 = S_2$. Here $S_1$ and $S_2$ denote the simple $H(1)$-modules.

Both homomorphism spaces $\text{Hom}_{H(2)}(E_2, M)$ and $\text{Hom}_{H(1)}(\overline{E}_2, \overline{M})$ are 1-dimensional. The space $\text{Hom}_{H(2)}(E_2, M)$ has a basis vector $f: E_2 \to M$ which maps the basis vector 2 of $E_2$ to the basis vector $2'$ of $M$. However, we have $\varepsilon f = 0$ which implies $\overline{f} = 0$.

### 3. Canonical decompositions of rank vectors

#### 3.1. Varieties of locally free $H$-modules

Let $d = (d_1, \ldots, d_n) \in \mathbb{N}^n$ be a dimension vector. Let $\text{rep}(H, d)$ denote the affine variety of representations of $H$ with dimension vector $d$. This is acted upon by the group

$$G_d := \prod_{i=1}^n \text{GL}_{d_i}(K).$$

The $G_d$-orbits in $\text{rep}(H, d)$ are naturally in bijection with the isomorphism classes of $H$-modules with dimension vector $d$. If $M \in \text{rep}(H, d)$ is locally free, its rank vector is $r = (r_1, \ldots, r_n)$ where $r_i := d_i/c_i$. Hence locally free modules can only exist if $d_i$ is divisible by $c_i$ for every $i$. In this case, we say that $d$ is $D$-divisible. Let $\text{rep}_f(H, r)$ be the union of all $G_d$-orbits $\mathcal{O}_M$ of locally free modules $M$ of rank vector $r$. Consider the natural projection

$$\pi: \text{rep}_f(H, r) \to \text{rep}(H_1, d_1) \times \cdots \times \text{rep}(H_n, d_n)$$

defined by

$$(M(\alpha))_{\alpha \in Q_1} \mapsto (M(\varepsilon_1), \ldots, M(\varepsilon_n)).$$

The image of $\pi$ is $\mathcal{O}_{H_1^{r_1}} \times \cdots \times \mathcal{O}_{H_n^{r_n}}$, where $\mathcal{O}_{H_i^{r_i}}$ is the $G_{d_i}$-orbit of the free $H_i$-module $H_i^{r_i}$ of rank $r_i$. (Note that $\text{rep}(H_i, d_i)$ is just a point if $c_i = 1$.)

The free $H_i$-module $H_i$ can also be seen as an $H$-module which we denote by $E_i$. We identify $\text{Im}(\pi)$ with the $G_d$-orbit $\mathcal{O}_{E^r}$ of the locally free $H$-module

$$E^r := \bigoplus_{i=1}^n E_i^{r_i}.$$ 

In particular, $\mathcal{O}_{E^r}$ is smooth and irreducible of dimension

$$\sum_{i=1}^n c_i^2 r_i^2 - \sum_{i=1}^n c_i r_i^2.$$
Here, the summands of the first sum are the dimensions of the groups $G_d$, while the summands of the second sum are the dimensions of the endomorphism rings $\text{End}_H(E^r) \cong \prod_{i=1}^n \text{End}_{H_i}(H_i^{r_i})$.

By Section 2.1, the fibre $\pi^{-1}(E^r_1, \ldots, E^r_n)$ can be identified with

$$\text{rep}_{\text{fib}}(H, r) := \prod_{(i,j) \in \Omega} \text{Hom}_{H_i}(iH_j \otimes_{H_j} E^{r_j}_i, E^{r_i}_i) \cong \prod_{(i,j) \in \Omega} \text{Hom}_{H_i}(E_i^{[c_{ij}]r_j}, E^{r_i}_i).$$

The group

$$G(H, r) := \prod_{i=1}^n \text{GL}_{r_i}(H_i)$$

acts on $\text{rep}_{\text{fib}}(H, r)$ via

$$(g \cdot M)_{ij} = g_i M_{ij} (iH_j \otimes g_j^{-1}).$$

The $G(H, r)$-orbits in $\text{rep}_{\text{fib}}(H, r)$ are naturally in bijection with the isomorphism classes of locally free $H$-modules with rank vector $r$. We have

$$\dim \text{rep}_{\text{fib}}(H, r) = \dim_K G(H, r) - q_H(r) = \sum_{(i,j) \in \Omega} c_i |c_{ij}| r_i r_j.$$

Here and for later use we abbreviate $q_H(r) := (r, r)_H$, see Section 1.2.

**Proposition 3.1.** Let $d = (d_1, \ldots, d_n)$ be $D$-divisible as above. Set $r_i := d_i/c_i$ and $r = (r_1, \ldots, r_n)$. Then $\text{rep}_{\text{lf}}(H, r)$ is a non-empty open subset of $\text{rep}(H, d)$. Moreover we have:

(i) The restriction $\bar{\pi}: \text{rep}_{\text{lf}}(H, r) \to \mathcal{O}_{E^r}$ of $\pi$ to its image defines a vector bundle of rank $\sum_{(i,j) \in \Omega} c_i |c_{ij}| r_i r_j$. In particular, $\text{rep}_{\text{lf}}(H, r)$ is smooth and irreducible of dimension

$$\sum_{i=1}^n c_i (c_i - 1)r_i^2 + \sum_{(i,j) \in \Omega} c_i |c_{ij}| r_i r_j = \dim(G_d) - q_H(r).$$

(ii) If $q_H(r) \leq 0$ then $\text{rep}_{\text{lf}}(H, r)$ has infinitely many $G_d$-orbits.

**Proof.** The function

$$l_r: \text{rep}(H, d) \to \mathbb{N}, \quad M \mapsto \sum_{i=1}^n \text{rank}(M(\varepsilon_i))$$

is lower semicontinuous. Now, $M \in \text{rep}(H, d)$ is locally free if and only if $l_r(M)$ takes the maximum $\sum_{i=1}^n (d_i - d_i/c_i)$. This shows, that the locally free modules form an open subset of $\text{rep}(H, d)$.

Next, notice that $\bar{\pi}$ is by construction $G_d$-equivariant. Since $\mathcal{O}_{E^r}$ is a single $G_d$-orbit, all fibers of $\bar{\pi}$ are isomorphic, and, in particular, are vector spaces of the same dimension.

Consider the trivial vector bundle

$$X := \left( \prod_{\alpha \in Q_+^r} \text{Hom}_K(K^{d_\alpha}, K^{d_\alpha}) \right) \times \mathcal{O}_{E^r}.$$
over $\mathcal{O}_{E'}$. A point of $X$ is given by a tuple $M = \left( (M(\alpha_{ij}^g))_{(i,j)\in\Omega; 1\leq g \leq g_{ij}}, (M(\varepsilon_i))_{1\leq i \leq n} \right)$ of $K$-linear maps. Obviously, the map $\mu : X \to X$ defined by

$$\mu(M) := \left( (M(\varepsilon_i)^{f_j} M(\alpha_{ij}^g) - M(\alpha_{ij}^{g_j}) M(\varepsilon_j)^{f_i}), (M(\varepsilon_i)) \right)$$

is an endomorphism of the vector bundle $X$, and by construction $\text{Ker}(\mu) = \text{rep}_{l.f.}(H, r)$. Since by the above consideration, the fibre

$$\text{Ker}(\mu)(M(\varepsilon_i)) = \tilde{\pi}^{-1}(M(\varepsilon_i))$$

is of constant dimension for all $(M(\varepsilon_i)) \in \mathcal{O}_{E'}$, we have that $\text{Ker}(\mu)$ is a vector bundle of the claimed rank over $\mathcal{O}_{E'}$. This proves (i).

The 1-dimensional torus $\{ (\lambda \text{id}_{d_1}, \ldots, \lambda \text{id}_{d_n}) \mid \lambda \in K^* \} \subset G_d$ acts trivially on the variety $\text{rep}_{l.f.}(H, r)$. So the maximal dimension of a $G_d$-orbit is $\dim(G_d) - 1$. Hence, by (i), if $q_H(r) \leq 0$, every $G_d$-orbit has dimension at most $\dim(\text{rep}_{l.f.}(H, r)) - 1$. This proves (ii).

The vector bundle structure of Proposition 3.1 is inspired by [B, Section 2].

### 3.2. Rigid modules.

**Proposition 3.2.** For $k \geq 2$ the reduction functor

$$R_{l.f.} : \text{rep}_{l.f.}(H) \to \text{rep}_{l.f.}(\overline{H})$$

induces a bijection between the isomorphism classes of rigid locally free $H$-modules and rigid locally free $\overline{H}$-modules.

**Proof.** By Proposition 3.1 we know that $\text{rep}_{l.f.}(H, r)$ is irreducible for all rank vectors $r$. If $M$ is a rigid locally free $H$-module with $\text{rank}(M) = r$, then its orbit in $\text{rep}_{l.f.}(H, r)$ is open and dense. In particular, for a given rank vector $r$ there is at most one rigid module with rank vector $r$ up to isomorphism. Now the result follows from Proposition 2.2.2(d).

### 3.3. Canonical decompositions. Let $r = (r_1, \ldots, r_t)$ be a rank vector and let $d = (d_1, r_1, \ldots, d_n r_n)$ be the corresponding dimension vector. Let $\text{ind}_{l.f.}(H, r)$ be the constructible subset of $\text{rep}_{l.f.}(H, r)$ consisting of all indecomposable locally free $H$-modules with rank vector $r$. The rank vector $r$ is an $H$-Schur root if $\text{ind}_{l.f.}(H, r)$ is dense in $\text{rep}_{l.f.}(H, r)$.

For any tuple $(r_1, \ldots, r_t)$ of rank vectors such that $r_1 + \cdots + r_t = r$ there is a morphism of quasi-projective varieties

$$G_d \times \text{rep}_{l.f.}(H, r_1) \times \cdots \times \text{rep}_{l.f.}(H, r_t) \to \text{rep}_{l.f.}(H, r)$$

defined by $(g, M_1, \ldots, M_t) \mapsto g \cdot (M_1 \oplus \cdots \oplus M_t)$. Let

$$\eta_{(r_1, \ldots, r_t)} : G_d \times \text{ind}_{l.f.}(H, r_1) \times \cdots \times \text{ind}_{l.f.}(H, r_t) \to \text{rep}_{l.f.}(H, r)$$

be its restriction to $G_d \times \text{ind}_{l.f.}(H, r_1) \times \cdots \times \text{ind}_{l.f.}(H, r_t)$.

By Proposition 3.1 we know that the variety $\text{rep}_{l.f.}(H, r)$ is irreducible. Thus, up to permutation of its entries, there is a unique tuple $(r_1, \ldots, r_t)$ of rank vectors such that the image of $\eta_{(r_1, \ldots, r_t)}$ is dense in $\text{rep}_{l.f.}(H, r)$. In this case, it follows that the rank vectors $r_1, \ldots, r_t$ are $H$-Schur roots, compare the proof of [CBS, Theorem 1.1]. We call $(r_1, \ldots, r_t)$ the $H$-canonical decomposition of $r$. 
In contrast to the special case with $C$ symmetric and $D$ the identity matrix, one can in general not expect that for an $H$-Schur root $r$ there exists a module $M \in \text{ind}_{1,f}(H, r)$ with $\text{End}_H(M) = K$.

For rank vector $r$ and $s$ for $H$ set

$$\text{ext}_H(r, s) := \min\{\dim \text{Ext}^1_H(M, N) \mid (M, N) \in \text{rep}_{1,f}(H, r) \times \text{rep}_{1,f}(H, s)\}.$$  

The function $\dim \text{Ext}^1_H(-, ?)$ is upper semicontinuous. Therefore the set of all $(M, N) \in \text{rep}_{1,f}(H, r) \times \text{rep}_{1,f}(H, s)$ with $\dim \text{Ext}^1_H(M, N) = \text{ext}_H(r, s)$ is open in $\text{rep}_{1,f}(H, r) \times \text{rep}_{1,f}(H, s)$.

Let $d$ and $d_1, \ldots, d_t$ be dimension vector for $H$ such that $d = d_1 + \cdots + d_t$. For constructible subsets $C_i \subseteq \text{rep}(H, d_i)$ with $1 \leq i \leq t$

$$C_1 \oplus \cdots \oplus C_t := \{M \in \text{rep}(H, d) \mid M \cong U_1 \oplus \cdots \oplus U_t \text{ with } U_i \in C_i \text{ for all } 1 \leq i \leq t\}.$$  

Let $\overline{C_1} \oplus \cdots \oplus \overline{C_t}$ be the Zariski closure of $C_1 \oplus \cdots \oplus C_t$ in $\text{rep}(H, d)$. The following result generalizes [K2, Proposition 3(a)].

**Theorem 3.3.** For $H = H(C, D, \Omega)$, a tuple $(r_1, \ldots, r_t)$ of rank vector is the $H$-canonical decomposition of the rank vector $r := r_1 + \cdots + r_t$ if and only if the following hold:

(i) $r_i$ is an $H$-Schur root for all $1 \leq i \leq t$.
(ii) $\text{ext}_H(r_i, r_j) = 0$ for all $i \neq j$.

In this case, we have

$$\text{ind}_{1,f}(H, r_1) \oplus \cdots \oplus \text{ind}_{1,f}(H, r_t) = \text{rep}_{1,f}(H, r_1) \oplus \cdots \oplus \text{rep}_{1,f}(H, r_t) = \text{rep}_{1,f}(H, r).$$

**Proof.** We know that $\text{rep}_{1,f}(H, r)$ is an open and irreducible subset of $\text{rep}(H, d)$, where $r = (r_1, \ldots, r_n)$ and $d = (c_1d_1, \ldots, c_n r_n)$. Now the theorem follows directly from [CBS, Theorem 1.2].

### 3.4. Independence of $k$.

**Lemma 3.4.** Let $H = H(k)$ and $\overline{H} = H(k - 1)$. For rank vectors $r$ and $s$ we have $\text{ext}_H(r, s) = 0$ if and only if $\text{ext}_{\overline{H}}(r, s) = 0$.

**Proof.** This follows from the definitions and from Proposition 2.2(c) and (d). 

**Lemma 3.5.** Let $H = H(k)$ and $\overline{H} = H(k - 1)$. A rank vector $r$ is an $H$-Schur root if and only if $r$ is an $\overline{H}$-Schur root.

**Proof.** Assume that $\text{ind}_{1,f}(H, r)$ is dense in $\text{rep}_{1,f}(H, r)$. Let $(r_1, \ldots, r_t)$ be the $\overline{H}$-canonical decomposition of $r$. To get a contradiction, we assume that $t \geq 2$. By induction we get that the $r_i$ are $H$-Schur roots. From Theorem 3.3 we know that $\text{ext}_{\overline{H}}(r_i, r_j) = 0$ for all $i \neq j$. Now Lemma 3.3 implies that $\text{ext}_H(r_i, r_j) = 0$ for all $i \neq j$. Again from Theorem 3.3 we get that

$$\text{rep}_{1,f}(H, r_1) \oplus \cdots \oplus \text{rep}_{1,f}(H, r_t) = \text{rep}_{1,f}(H, r).$$

This is a contradiction, since we assumed that $r$ is an $H$-Schur root.

To show the other direction, let us now assume that $r$ is an $\overline{H}$-Schur root. Let $(r_1, \ldots, r_t)$ be the $H$-canonical decomposition of $r$. To get a contradiction, we assume that $t \geq 2$. By induction we get that the $r_i$ are $\overline{H}$-Schur roots. From Theorem 3.3 we know that
ext\(_H(\mathbf{r}_i, \mathbf{r}_j) = 0\) for all \(i \neq j\). Thus Lemma 3.4 implies that \(\text{ext}_H(\mathbf{r}_i, \mathbf{r}_j) = 0\) for all \(i \neq j\). Again from Theorem 3.3 we get that
\[
\text{rep}_{H,\mathbf{r}}(\mathbf{H}, \mathbf{r}) = \bigoplus_{i=1}^\infty \text{rep}_{H,\mathbf{r}}(\mathbf{H}, \mathbf{r}_i).
\]
This is a contradiction, since we assumed that \(\mathbf{r}\) is an \(\overline{H}\)-Schur root. \(\square\)

As a direct consequence of Lemmas 3.4 and 3.5 we get the following result.

**Theorem 3.6.** Let \(H = H(C, D, \Omega)\), and let \(\mathbf{r}\) be a rank vector for \(H\). Then the \(H\)-canonical decomposition of \(\mathbf{r}\) does not depend on the symmetrizer \(D\).

**3.5. The symmetric case.** Let \(C \in M_n(\mathbb{Z})\) be a symmetric Cartan matrix, and let \(D\) be its minimal symmetrizer. Thus \(D\) is just the identity matrix. Furthermore, let \(\Omega\) be an orientation of \(C\). As in [GLS1] let \(Q = Q(C, \Omega)\) and \(Q^0 = Q \setminus \{\varepsilon_1, \ldots, \varepsilon_n\}\). Set
\[
H(k) = H(C, kD, \Omega) \quad \text{for some} \ k \geq 1.
\]
It follows from [CB] that this assumption can be dropped.

\[
H(k) \cong KQ^0 \otimes_K K[X]/(X^k).
\]
In particular, we have \(H(1) \cong KQ^0\). Now for \(K\) an algebraically closed field one can combine Schofield’s algorithm [Sc] with Theorem 3.6 to compute the \(H(k)\)-canonical decomposition of each rank vector. (Schofield [Sc] is using the extra assumption that the characteristic of \(K\) is zero. It follows from [CB] that this assumption can be dropped.)

4. Varieties of flags of locally free modules

**4.1. Quiver flag varieties.** Recall that for a locally free \(H(k)\)-module \(M\) the quiver flag variety of locally free submodules \(\text{Flf}_H(M)\) is the quasi-projective variety of flags
\[
(0 = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_{l-1} \subset U_l = M)
\]
of locally free submodules with \(\text{rank}_{H(k)}(U_j/U_{j-1}) = r_j\) for \(j = 1, 2, \ldots, l\). Note that in case \(U\) is a submodule of the locally free module \(M\), then \(U\) is locally free if and only if \(M/U\) is locally free by [GLS1] Lemma 3.8]. In particular, we could have defined \(\text{Flf}_H(M)\) alternatively via the rank vectors of the submodules \(U_j\).

In the special case \(l = 2\) we write
\[
\text{Grf}_{\mathbf{e}}(H(k))(M) := \text{Flf}_{(\mathbf{e}, \mathbf{m} - \mathbf{e})}(M).
\]
This is called the quiver Grassmannian of locally free submodules of rank vector \(\mathbf{e}\) of \(M\).

The aim of this section is to prove Theorem 4.2. For Part (a), we closely follow the ideas of Wolf’s Thesis [W] Section 5, see Proposition 4.7.

**4.2. The algebra \(H(k, l)\).** For \(k \geq 1\) and \(l \geq 2\) let \(A_l\) be the path algebra of the linearly oriented quiver \(1 \rightarrow 2 \rightarrow \cdots \rightarrow (l-1)\), and let
\[
H(k, l) := H(k) \otimes_K A_l.
\]
The description of a tensor product of paths algebras of quivers with relations can be found in [L] Section 1.

Thus, we can think of an \(H(k, l)\)-module \(M\) as a tuple
\[
(M_1, \ldots, M_l; \mu_1, \ldots, \mu_{l-2})
\]
\[
\in \text{rep}(H(k))^{l-1} \times \text{Hom}_{H(k)}(M_1, M_2) \times \cdots \times \text{Hom}_{H(k)}(M_{l-2}, M_{l-1}).
\]
In this language,
\[ \text{Hom}_{H(k,l)}(M, M') = \{(f_1, \ldots, f_{l-1}) \in \text{Hom}_{H(k)}(M_1, M'_1) \times \cdots \times \text{Hom}_{H(k)}(M_{l-1}, M'_{l-1}) \mid \mu'_i f_i = f_{i+1} \mu_i \text{ for all } 1 \leq i \leq l - 2\}. \]
We call \( M \) locally free, if all \( M_1, \ldots, M_{l-1} \) are locally free \( H(k) \)-modules. In this case we write
\[ \text{rank}_{H(k,l)}(M) = (\text{rank}_{H(k)}(M_1), \ldots, \text{rank}_{H(k)}(M_{l-1})). \]
The following proposition is inspired from [W] Proof of Theorem 5.27, Appendix B3.

**Proposition 4.1.** Suppose
\[ M = (M_1, \ldots, M_{l-1}; \mu_1, \ldots, \mu_{l-2}) \quad \text{and} \quad M' = (M'_1, \ldots, M'_{l-1}; \mu'_1, \ldots, \mu'_{l-2}) \]
belong to \( \text{rep}_{\text{l.f.}}(H(k,l)) \).

(a) We have \( \text{proj}. \dim(M) \leq 2 \) and \( \text{inj}. \dim(M) \leq 2. \)
(b) If all \( \mu_1, \ldots, \mu_{l-2} \) are injective, then \( \text{proj}. \dim(M) \leq 1. \)
(c) If all \( \mu_1, \ldots, \mu_{l-2} \) are surjective, then \( \text{inj}. \dim(M) \leq 1. \)
(d) The value of the homological bilinear form depends for locally free \( H(k,l) \)-modules only on the rank vector. More precisely, we have
\[ \langle M, M' \rangle_{H(k,l)} := \sum_{i=0}^{2} (-1)^i \text{dim}_K \text{Ext}^i_{H(k,l)}(M, M') \]
\[ = \sum_{j=1}^{l-1} \langle M_j, M'_j \rangle_{H(k)} - \sum_{j=1}^{l-2} \langle M_j, M'_{j+1} \rangle_{H(k)}. \]

**Proof.** The indecomposable projective \( H(k,l) \)-modules are the tensor products of the indecomposable projective \( H(k) \)-modules and the indecomposable projective \( A_l \)-modules. More precisely, these are the modules of the form
\[ P_{(i,j)} = (0, \ldots, 0, P_i, \ldots, P_i; 0, \ldots, 0, \text{id}_{P_i}, \ldots, \text{id}_{P_i}), \quad (1 \leq i \leq n, \ 1 \leq j \leq l - 1), \]
where we have \( l - j \) copies of the indecomposable projective \( H(k) \)-module \( P_i \). Since every locally free \( H(k,l) \)-module has a filtration with successive subquotients of the form
\[ E_{(i,j)} = (0, \ldots, 0, E_i, 0, \ldots, 0; 0, \ldots, 0), \quad (1 \leq i \leq n, \ 1 \leq j \leq l - 1), \]
(where \( E_i \) is in position \( j \)), it is enough to show that \( \text{proj}. \dim(E_{i,j}) \leq 2. \) Clearly \( \text{proj}. \dim(E_{i,l-1}) \leq 1. \) For \( 1 \leq j \leq l - 2 \) we have a projective resolution of the form
\[ 0 \to \bigoplus_{h \in \Omega(-i)} P_{h,j+1}^{[c_{h,i}]} \to P_{i,j+1} \oplus \bigoplus_{h \in \Omega(-i)} P_{h,j}^{[c_{h,i}]} \to P_{i,j} \to E_{i,j} \to 0, \]
compare [GLST] Proposition 3.1. The proof is similar for the injective dimension. This proves (a).

For an \( H(k) \)-module \( M \) and \( 2 \leq j \leq l \), define
\[ M^{(j)} := (0, \ldots, 0, M, \ldots, M; 0, \ldots, 0, \text{id}_M, \ldots, \text{id}_M), \]
where we have \( j - 1 \) copies of \( M \) and \( j - 2 \) identity maps. The \( H(k,l) \)-modules described in (b) are precisely the modules which admit a filtration whose successive subquotients are isomorphic to modules of the form \( M^{(j)} \). If \( 0 \to P' \to P \to M \to 0 \) is a projective resolution of \( M \), then \( 0 \to P'^{(j)} \to P^{(j)} \to M^{(j)} \to 0 \) is a projective resolution of \( M^{(j)} \), hence \( \text{proj}. \dim(M^{(j)}) \leq 1. \) This proves (b). The proof of (c) is dual.
By (a) the form \( \langle -, - \rangle_{H(k,l)} \) descends to the Grothendieck group of the category of locally free \( H(k,l) \)-modules. So it is enough to verify the formula for modules of the form \( E_{i,j} \). We leave this as an exercise. \( \square \)

For \( M \in \text{rep}_{1,1}(H(k)) \) let
\[
\overline{M}^{(l)} := (M, \ldots, M; \text{id}_M, \ldots, \text{id}_M) \in \text{rep}_{1,1}(H(k,l))
\]
denote the repetitive module associated to \( M \).

**Lemma 4.2.** Let \( M \in \text{rep}_{1,1}(H(k)) \) be rigid. Then for any locally free submodule \( U \) of \( M^{(l)} \) and any locally free factor module \( F \) of \( M^{(l)} \) we have
\[
\text{Ext}_{H(k,l)}^{i}(U, F) = 0 \text{ for } i = 1, 2.
\]
In particular,
\[
\dim \text{Hom}_{H(k,l)}(U, M^{(l)}/U) = \sum_{a<b} \langle r_a, r_b \rangle_{H(k)}
\]
with \( r_i := \text{rank}_{H(k)}(U_i/U_{i-1}) \) for \( i = 1, 2, \ldots, l \), where we set \( U_0 = 0 \) and \( U_l = M \).

**Proof.** Clearly we have \( \text{End}_{H(k,l)}(\overline{M}^{(l)}) \cong \text{End}_{H(k)}(M) \). By Proposition 4.1(b) we have \( \text{Ext}_{H(k,l)}^{2}(M^{(l)}, M^{(l)}) = 0 \). By Proposition 4.1(d), \( \langle M^{(l)}, M^{(l)} \rangle_{H(k,l)} = \langle M, M \rangle_{H(k)} \). Thus, all together, again by Proposition 4.1(d),
\[
\text{Ext}_{H(k,l)}^{1}(M^{(l)}, M^{(l)}) \cong \text{Ext}_{H(k)}^{1}(M, M) = 0.
\]
Similarly, by Proposition 4.1 we have \( \text{Ext}_{H(k,l)}^{2}(U, -) = 0 = \text{Ext}_{H(k,l)}^{2}(-, F) \). Thus, we have surjections
\[
0 = \text{Ext}_{H(k,l)}^{1}(M^{(l)}, M^{(l)}) \to \text{Ext}_{H(k,l)}^{1}(U, M^{(l)}) \to \text{Ext}_{H(k,l)}^{1}(U, F),
\]
which shows our first claim.

It follows, that we have in particular
\[
\dim \text{Hom}_{H(k,l)}(U, M^{(l)}/U) = \langle U, M^{(l)}/U \rangle_{H(k,l)}.
\]
Now, the second claim is a straightforward computation with the formula from Proposition 4.1(d). \( \square \)

4.3. **Tangent space.**

**Proposition 4.3.** Let \( \underline{r} = (r_1, \ldots, r_l) \) be a sequence of rank vectors for \( H(k) \), and \( \underline{e} = e(\underline{r}) := (r_1, r_1 + r_2, \ldots, r_1 + r_2 + \cdots + r_{l-1}) \) the corresponding rank vector for \( H(k,l) \). Moreover, let \( M \in \text{rep}_{1,1}(H(k)) \) with \( \text{rank}_{H(k)}(M) = \underline{m} = \sum_{i=1}^{l} r_i \). With this notation we have a natural isomorphism of schemes
\[
\iota : \text{Flf}_{\underline{r}}^{H(k)}(M) \to \text{Grlf}_{\underline{e}}^{H(k,l)}(\overline{M}^{(l)})
\]
defined by
\[
U_\bullet = (U_0, U_1, \ldots, U_l) \mapsto (U_1, \ldots, U_{l-1}; \text{inc}_{1,2}, \ldots, \text{inc}_{l-2,l-1}),
\]
where \( \text{inc}_{i+1} : U_i \hookrightarrow U_{i+1} \) is the natural inclusion. Furthermore, the tangent space at \( U_\bullet \) can be described as \( T_{U_\bullet} \text{Flf}_{\underline{r}}^{H(k)}(M) = \text{Hom}_{H(k,l)}(\iota(U_\bullet), \overline{M}^{(l)}/\iota(U_\bullet)) \).
Proof. Recall that $D$ is the symmetrizer of the Cartan matrix $C$. For a sequence of dimension vectors $\underline{r} = (r_1, \ldots, r_l)$, we will use the shorthand notation $D\underline{r} := (Dr_1, \ldots, Dr_l)$. As observed in the proof of [W, Lemma 5.23] the scheme $Fl^H(k)(M)$ of flags of all submodules of $M$ of type $D\underline{r}$ and the scheme $Gr_{D\underline{r}}^H(M(l))$ of all submodules of $M(l)$ of dimension vector $D\underline{r}$ are isomorphic. Note that Wolf’s easy argument for an arbitrary base field $K$ is valid for quivers $Q$ possibly with loops and any nilpotent representation $M$ of $Q$.

Now, in both schemes the (flags of) locally free submodules form open subschemes whose $K$-rational points are in bijection under $\iota$. In fact, let $U \in Gr_{D\underline{r}}^H(M(l))$ be locally free. Then we can choose for each $U_{i,j}$ a complement $V_{i,j}$ such that $U_{i,j} \oplus V_{i,j} = M_{i,j}(l)$ as an $H_i$-module. Now,

$$U := \{U' \in Gr_{D\underline{r}}^H(M(l)) \mid U'_{i,j} \cap V_{i,j} = 0 \text{ for all } (i,j)\}$$

is an open neighbourhood of $U$ which consists of locally free modules. This proves the first claim.

For the second claim we use that $Gr_{D\underline{r}}^H(M(l))$ is an open subset of the usual quiver Grassmannian $Gr_{D\underline{r}}^H(M(l))$. Thus, we can apply Schofield’s formula for the tangent space of a quiver Grassmannian [Se, Lemma 3.2].

\begin{corollary}
Let $M$ be a rigid locally free $H(k)$-module and $\underline{r} = (r_1, \ldots, r_l)$ a sequence of rank vectors with $\sum_{i=1}^l r_i = \text{rank}_{H(k)}(M)$. Then for each $U_\bullet \in Fl^H(k)(M)$ we have

$$\dim T_{U_\bullet} Fl^H(k)(M) = \sum_{a<b} \langle r_a, r_b \rangle_{H(k)}.$$

Proof. By Proposition 4.3 we have

$$T_{U_\bullet} Fl^H(k)(M) = \text{Hom}_{H(k,l)}(\iota(U_\bullet), M(l)/\iota(U_\bullet)).$$

Since $M$ is rigid, the dimension of the latter space is $\sum_{a<b} \langle r_a, r_b \rangle_{H(k)}$ by Lemma 4.2. \end{corollary}

### 4.4 Dimension and irreducibility.

\begin{lemma}
Let $\underline{r} = (r_1, \ldots, r_l)$ be a sequence of ranks, and set $m := r_1 + \cdots + r_l$. Then $Fl^k_{\underline{x}}(K[x]/(x^k), ((K[x]/(x^k))^m)_{K[x]/(x^k)})$ is a smooth irreducible quasi-projective variety of dimension $k \cdot d(\underline{r})$ for $d(\underline{r}) := (\sum_{a<b} r_a r_b)$. More precisely, we have a fiber bundle

$$\pi: Fl^k_{\underline{x}}(K[x]/(x^k), ((K[x]/(x^k))^m) \to Fl^k_{\underline{x}}(K^m)$$

defined by $U \mapsto U/xU$ with fibers isomorphic to $K^{(k-1)d(\underline{r})}$. \end{lemma}

We leave the proof as an exercise. Note however, that $\pi$ is in general not a vector bundle.

Recall that $S(k) = \prod_{i=1}^n H_i(k)$. Let $\underline{r} = (r_1, \ldots, r_l)$ be a sequence of rank vectors for $S(k)$, and set $\underline{m} := r_1 + \cdots + r_l$. Define

$$S(k)^\underline{m} := \bigoplus_{i=1}^n H_i(k)^{r_i},$$

and let $Fl^S(k) := Fl^S(k)(S(k)^\underline{m})$. 
Corollary 4.6. Let \( \underline{r} = (r_1, \ldots, r_l) \) be a sequence of rank vectors for \( S(k) \). Then \( \text{Flf}_{\underline{r}}^{S(k)} \) is a smooth irreducible quasi-projective variety of dimension

\[
\sum_{a < b} \langle r_a, r_b \rangle_{S(k)} = \sum_{i=1}^{n} \sum_{a < b} k_{c_i} r_{a_i} r_{b_i},
\]

where \( \langle -, - \rangle_{S(k)} \) is the symmetric bilinear form defined by the matrix \( \text{diag}(kc_1, \ldots, kc_n) \).

Proof. The quasi-projective variety \( \text{Flf}_{\underline{r}}^{S(k)} \) is a product of flag varieties of type \( \underline{r}_i \) as considered in Lemma 4.5. This yields the result. \( \square \)

Let \( \underline{r} = (r_1, \ldots, r_l) \) be a sequence of rank vectors for \( H(k) \), and let \( \underline{m} = (m_1, \ldots, m_n) = r_1 + \cdots + r_l \). Set \( d = (kc_1 m_1, \ldots, kc_n m_n) \). We consider

\[
\text{repFlf}_{\underline{r}}^{H(k)}(\underline{m}) := \{ (M, U_\bullet) \in \text{repFlf}_{\underline{r}}^{H(k)}(\underline{m}) \times \text{Flf}_{\underline{r}}^{S(k)} | M_{i,j}(H(k) \otimes_j U_{h,j}) \subseteq U_{h,i} \text{ for all } (i, j) \in \Omega, 1 \leq h \leq l \}
\]

which is a subbundle of rank

\[
\dim_K \text{repFlf}_{\underline{r}}^{H(k)}(H(k), \underline{m}) - \sum_{a < b} \left( \langle r_a, r_b \rangle_{S(k)} - \langle r_a, r_b \rangle_{H(k)} \right)
\]

of the trivial vector bundle \( \text{repFlf}_{\underline{r}}^{H(k)}(H(k), \underline{m}) \rightarrow \text{Flf}_{\underline{r}}^{S(k)} \rightarrow \text{Flf}_{\underline{r}}^{S(k)} \). Now we can use Corollary 4.6 and get that \( \text{repFlf}_{\underline{r}}^{H(k)}(\underline{m}) \) is smooth and irreducible of dimension

\[
\dim_K \text{repFlf}_{\underline{r}}^{H(k)}(H(k), \underline{m}) + \sum_{a < b} \langle r_a, r_b \rangle_{H(k)}.
\]

Let

\[
q: \text{repFlf}_{\underline{r}}^{H(k)}(\underline{m}) \rightarrow \text{repFlf}_{\underline{r}}^{H(k)}(H(k), \underline{m}), \quad (M, U_\bullet) \mapsto M
\]

be the restriction of the projection to the first component. Then obviously \( q^{-1}(M) = \text{Flf}_{\underline{r}}^{H(k)}(M) \).

Finally, with the notation from Section 1 we have the following result, whose proof is directly inspired by [W] Theorem 5.34.

Proposition 4.7. If \( \text{Flf}_{\underline{r}}^{H(k)} \) is non-empty, then it is smooth and irreducible of dimension

\[
\sum_{a < b} \langle r_a, r_b \rangle_{H(k)}.
\]

Proof. Since \( M := M_{\underline{m}, k} \) is rigid, the \( G(H(k), \underline{m}) \)-orbit \( O_M \) of \( M \) in \( \text{repFlf}_{\underline{r}}^{H(k)}(H(k), \underline{m}) \) is open and dense. Thus, with \( q \) defined as above and by our hypothesis, \( q \) is dominant. By Chevalley’s Theorem (see for example [H] II, Ex. 3.22) it follows, that each irreducible component of \( q^{-1}(M) = \text{Flf}_{\underline{r}}^{H(k)} \) has at least dimension

\[
\sum_{a < b} \langle r_a, r_b \rangle_{H(k)} = \dim \text{repFlf}_{\underline{r}}^{H(k)}(\underline{m}) - \dim \text{repFlf}_{\underline{r}}^{H(k)}(H(k), \underline{m}).
\]

By the same token, \( q^{-1}(O_M) \) is dense and open in the irreducible variety \( \text{repFlf}_{\underline{r}}^{H(k)}(\underline{m}) \). Moreover, \( q^{-1}(M') \cong q^{-1}(M) \) for all \( M' \in O_M \). Thus, again by Chevalley’s Theorem \( q^{-1}(M) \) is equidimensional.

To prove the last claim, namely the irreducibility and smoothness of \( \text{Flf}_{\underline{r}}^{H(k)}(M) \cong \text{Grflf}_{\underline{r}}^{H(k,l)}(M^{(l)}) \), we proceed as follows: Recall firstly, that we defined a rank vector \( \underline{e} := \)
Next, let $\text{Mono}$ be connected and smooth group. We start in the SW corner.

The latter applies in particular to vector bundles. We start in the SW corner.

$$\text{HomRep}_{l.f.}^{\text{Mon}}(e, M)_{\text{min}} \overset{\text{open}}{\rightarrow} \text{HomRep}_{l.f.}^{\text{Mon}}(e, M)_{\text{min}}$$

$$\text{rep}_{l.f.}^{\text{Mon}}(e) \overset{\text{open}}{\rightarrow} \text{rep}_{l.f.}^{\text{Mon}}(e, M)_{\text{min}}$$

$$\text{Gr}^{H(k,l),e}-\text{princ.}$$

First, let $\text{Hom}_{S(k)}(e) := \{\text{pt}\}$ if $l = 2$. Otherwise, define

$$\text{Hom}_{S(k)}(e) := \prod_{i=1}^{l-2} \text{Hom}_{S(k)}(S(k)^{e_i}, S(k)^{e_{i+1}}).$$

Next, let $\text{Mono}_{S(k)}(e) := \{\text{pt}\}$ for $l = 2$, and

$$\text{Mono}_{S(k)}(e) := \{\nu := (\nu_1, \ldots, \nu_{l-2}) \in \text{Hom}_{S(k)}(e) \mid \nu_i \text{ is a monomorphism for } i = 1, 2, \ldots, l-2\}.$$

if $l \geq 3$. Thus, $\text{Mono}_{S(k)}(e)$ is smooth and irreducible as an open subset of the vector space $\text{Hom}_{S(k)}(e)$. Define

$$\text{rep}_{l.f.}^{\text{fib}}(H(k,l), e) := \left(\prod_{i=1}^{l-1} \text{rep}_{l.f.}^{\text{fib}}(H(k), e_i)\right) \times \text{Hom}_{S(k)}(e),$$

and let

$$\text{rep}_{l.f.}^{\text{Mon}}(e) := \{(U_1, \ldots, U_{l-1}; \nu) \in \left(\prod_{i=1}^{l-1} \text{rep}_{l.f.}^{\text{fib}}(H(k), e_i)\right) \times \text{Mono}_{S(k)}(e) \mid \nu_i \in \text{Hom}_{H(k)}(U_i, U_{i+1}) \text{ for } i = 1, 2, \ldots, l-2\}.$$

This is, with the projection to the last component, a vector bundle over $\text{Mono}_{S(k)}(e)$. Thus the irreducibility and smoothness of $\text{rep}_{l.f.}^{\text{Mon}}(e)$ follows from the corresponding properties of $\text{Mono}_{S(k)}(e)$. Clearly $\text{rep}_{l.f.}^{\text{Mon}}(e)$ is an open subset of $\text{rep}_{l.f.}^{\text{fib}}(H(k,l), e)$. The function

$$h_M : \text{rep}_{l.f.}^{\text{Mon}}(e) \rightarrow \mathbb{N}$$

defined by

$$(U_\bullet, \nu) \mapsto \dim \text{Hom}_{H(k,l)}((U_\bullet, \nu), M^{(l)})$$

is upper semicontinuous, see for example [3] Section 2.1. Since $\text{Ext}^2_{H(k,l)}(\cdot, M^{(l)}) = 0$, a lower bound for $h_M$ on $\text{rep}_{l.f.}^{\text{Mon}}(e)$ is given by

$$\langle e, \text{rank}_{H(k,l)}(M^{(l)}) \rangle_{H(k,l)} = \langle e_{l-1}, \text{rank}(M) \rangle_{H(k)},$$
see Proposition 11.1. This minimum is achieved for any \((U, \nu) \in \text{Grf}_e^{H(k,l)}(M^{(l)}) \neq \emptyset\) by Lemma 11.2. Thus,
\[
\text{rep}^{\text{Mon}}_{\text{Pl.f.}}(e, M)_{\text{min}} := \{(U, \nu) \in \text{rep}^{\text{Mon}}_{\text{Pl.f.}}(e) | \\
\dim \text{Hom}_{H(k,l)}((U, \nu), M^{(l)}) = \langle e, \text{rank}_{H(k,l)}(M^{(l)}) \rangle_{H(k,l)} \}
\]
is a non-empty open subset of the smooth irreducible quasi-projective variety \(\text{rep}^{\text{Mon}}_{\text{Pl.f.}}(e, M)\). Thus, \(\text{rep}^{\text{Mon}}_{\text{Pl.f.}}(e, M)_{\text{min}}\) is smooth and irreducible.

Let
\[
\text{HomRe}^{\text{Mon}}_{\text{Pl.f.}}(e, M)_{\text{min}} := \{(f, (U, \nu)) \in \text{Hom}_K((U, \nu), M^{(l)}) \times \text{rep}^{\text{Mon}}_{\text{Pl.f.}}(e, M)_{\text{min}} | \\
f \in \text{Hom}_{H(k,l)}((U, \nu), M^{(l)}) \}.
\]
Together with the projection to the second component, this is a vector bundle over \(\text{rep}^{\text{Mon}}_{\text{Pl.f.}}(e, M)_{\text{min}},\) see for example [H Section 2.2]. Thus, the variety \(\text{HomRe}^{\text{Mon}}_{\text{Pl.f.}}(e, M)_{\text{min}}\) is smooth and irreducible because \(\text{rep}^{\text{Mon}}_{\text{Pl.f.}}(e, M)_{\text{min}}\) is so. Define
\[
\text{IHomRe}^{\text{Mon}}_{\text{Pl.f.}}(e, M)_{\text{min}} := \{(f, (U, \nu)) \in \text{HomRe}^{\text{Mon}}_{\text{Pl.f.}}(e, M)_{\text{min}} | f \text{ is injective} \}.
\]
By the discussion above, this is a non-empty open subset of \(\text{HomRe}^{\text{Mon}}_{\text{Pl.f.}}(e, M)_{\text{min}}\). In particular, it is smooth and irreducible because \(\text{HomRe}^{\text{Mon}}_{\text{Pl.f.}}(e, M)_{\text{min}}\) is so.

Finally, we have the canonical morphism
\[
\text{IHomRe}^{\text{Mon}}_{\text{Pl.f.}}(e, M)_{\text{min}} \to \text{Grf}_e^{H(k,l)}(M^{(l)}), (f, (U, \nu)) \mapsto \text{Im}(f)
\]
It is not hard to see, that this is a principal \(G(H(k,l), e)\)-bundle. Thus, \(\text{Grf}_e^{H(k,l)}(M^{(l)}) \cong \text{Flf}^{H(k)}(M)\) is smooth and irreducible because \(\text{IHomRe}^{\text{Mon}}_{\text{Pl.f.}}(e, M)_{\text{min}}\) is so. □

Alternatively, we can prove the smoothness statement in Proposition 11.1 with a short tangent space argument. In fact, since \(M\) is rigid, by Corollary 11.4 Equation (11.1) is the dimension of the tangent space at each point \(U, \nu\) of \(\text{Flf}^{H(k)}(M)\). This implies smoothness (see for example [H III.10.0.3] together with [H II.2.8] and the basic commutative algebra fact [H I.5.2A]).

Given this, it would be nice to have a similar short argument for the connectedness of \(\text{Flf}^{H(k)}\), perhaps by some variant of [H III.11.3], or by the first Bertini Theorem [Sh II.6.1].

4.5. **A reduction map for Grassmannians.** For \(k \geq 2\) let \(B\) be a finite-dimensional \(Z(k)\)-algebra which is free as a \(Z(k)\)-module. We consider \(Z(k)\) as a subalgebra of \(B\). We set \(\overline{B} := B/(e^{k-1}B),\) this is a \(Z(k-1)\)-algebra which is free as a \(Z(k-1)\)-module. Moreover, we set \(\overline{B} := B/(eB)\). We have the following useful result, which is easy to prove.

**Lemma 4.8.** Let \(M\) be a \(B\)-module which is free as a \(Z(k)\)-module, then for \(1 \leq j \leq k-1\) multiplication by \(e\) induces an isomorphism of \(\overline{B}\)-modules
\[
e^{j-1}M/(e^{j-1}M) \to e^jM/(e^{j+1}M).
\]
In particular, \(M/(eM) \cong e^{k-1}M\) as \(\overline{B}\)-modules.
Thus, the action of Lemma 4.9.

Fix a

Proof. The reduction \( M \mapsto \overline{M} \) with respect to our chosen basis \( \{ \overline{M} \} \) can be identified with \( \operatorname{Hom}_B(\pi(U) \to \pi(\overline{M})) \).

Then we have that \( \operatorname{Im}(\pi) \cap \mathcal{U} \) is closed in \( \mathcal{U} \).

Proof. Fix a \( Z(k) \)-basis \( B \) of \( Z(k) \)-basis \( \{ m_1, \ldots, m_l \} \) of \( M \), which we identify with the standard basis of \( Z(k)^l \). We may suppose that \( U \) is the \( Z(k) \)-span of \( \{ m_1, \ldots, m_e \} \).

Thus, the action of \( B \) on \( M \) is described by the matrices

\[
M(b) = \sum_{j=0}^{k-1} \begin{pmatrix} U_j(b) & E_j(b) \\ 0 & F_j(b) \end{pmatrix} e^j \in Z(k)^{m \times m}
\]

for all \( b \in B \) and \( j = 1, 2, \ldots, k-1 \).

(a) With this setup the free \( Z(k) \)-submodule \( \tilde{U} \) of \( Z(k)M = Z(k)^m \) with \( \tilde{U}/\varepsilon^{k-1}\tilde{U} = U/\varepsilon^{k-1}U \) are precisely those which are spanned by the columns of a matrix of the form

\[
\begin{pmatrix} \mathbf{1}_e \\ \varepsilon^{-k-1} S \end{pmatrix} \in Z(k)^{m \times e}
\]

for some \( S \in K^{(m-e) \times e} \). Now, such a subspace \( \tilde{U}_S \) is a \( B \)-submodule of \( M \) if and only if for all \( b \in B \) the left lower \( (m-e) \times e \)-block of

\[
\begin{pmatrix} \mathbf{1}_e & 0 \\ \varepsilon^{-k-1} S & 1_{m-e} \end{pmatrix}^{-1} M(b) \begin{pmatrix} \mathbf{1}_e & 0 \\ \varepsilon^{-k-1} S & 1_{m-e} \end{pmatrix} = \begin{pmatrix} \mathbf{1}_e & 0 \\ -\varepsilon^{-k-1} S & 1_{m-e} \end{pmatrix} M(b) \begin{pmatrix} \mathbf{1}_e & 0 \\ \varepsilon^{-k-1} S & 1_{m-e} \end{pmatrix} \in Z(k)^{m \times m}
\]

vanishes. In other words, if \( F_0(b) \cdot S - S \cdot U_0(b) = 0 \) for all \( b \in B \). This means that by Lemma 4.8, \( \pi^{-1}(\pi(U)) \) can be identified with \( \operatorname{Hom}_B(U/\varepsilon U, F/\varepsilon F) \) for \( F = M/U \).

(b) We may assume that \( \mathcal{U} \) is contained in the standard open neighbourhood of \( \pi(U) \) with respect to our chosen basis \( \{ \overline{m}_i \mid i = 1, \ldots, l \} \) for \( \overline{m}_i := m_i + \varepsilon^{k-1}M \). The elements of
this neighbourhood are submodules $U' = U_\mathcal{S}$ which are $Z(k - 1)$-spanned by the columns of a matrix of the form
\[
\begin{pmatrix} 1_e \\ S \end{pmatrix}
\]
with $S = \sum_{j=0}^{k-2} S_j e^j \in Z(k - 1)^{e \times (m-e)}$
subject to the condition that the left lower $(m - e) \times e$-block of the matrices
\[
\begin{pmatrix} 1_e \\ -S \end{pmatrix} M(b) \begin{pmatrix} 1_e \\ 1_{m-e} \end{pmatrix} \in Z(k)^{m \times m}
\]
vanishes for all $b \in B$. More explicitly, this means
\[
0 = F(b) \cdot \mathcal{S} - \mathcal{S} \cdot \mathcal{U}(b) - \mathcal{S} \cdot \mathcal{E}(b) \cdot \mathcal{S} \in Z(k - 1)^{(m-e) \times e}
\]
for all $b \in B$.

Now, $U_\mathcal{S}$ is in the image of $\pi$ if and only if there exists $S_{k-1} \in K^{(m-e) \times e}$ such that for
\[
\tilde{S} := \sum_{i=0}^{k-1} S_i E_i \in Z(k)^{(m-e) \times e}
\]
the $Z(k)$-span of the columns of the matrix
\[
\begin{pmatrix} 1_e \\ S \end{pmatrix}
\]
is a $B$-submodule of $M$. This is the case if and only if the following, possibly non-homogeneous system of linear equations in the components of $S_{k-1}$ has a solution:
\[
\begin{aligned}
(4.2) \quad (F_0(b) - S_0 \cdot E_0(b)) \cdot S_{k-1} - S_{k-1} \cdot (U_0(b) + E_0(b) \cdot S_0) &= \sum_{0 \leq i,j \leq k-2} S_i \cdot E_{k-1-i-j}(b) \cdot S_j \quad \text{for all } b \in B,
\end{aligned}
\]
where we have set $E_h(b) = 0$ for $h < 0$. Now observe that the family of matrices $(F_0(b) - S_0 \cdot E_0(b))_{b \in B}$ describe the (reduced) factor module $(M/U_\mathcal{S})/(\mathcal{E}(M/U_\mathcal{S}))$, and the family $(U_0(b) + E_0(b) \cdot S_0)_{b \in B}$ describes the (reduced) submodule $U_\mathcal{S}/(\mathcal{E}U_\mathcal{S})$. Thus, by hypothesis the system of linear equations (4.2) has constant rank for all points $U_\mathcal{S} \in \mathcal{U}$. It follows, that the subset where this system has a solution, is closed.

Example 4.8 illustrates the calculations in the above proof.

### 4.6. Closed image.

We show that Lemma 4.9(b) can be used to conclude that the image of the reduction morphism is closed.

**Lemma 4.10.** Let $k \geq 2$, and let $M$ be a rigid locally free $H(k)$-module. For a sequence of rank vectors $\underline{r} = (r_1, \ldots, r_l)$ with $r_1 + \cdots + r_l = \text{rank}_{H(k)}(M)$ we consider the natural reduction morphism
\[
\pi_k: \text{Flf}_{\underline{r}}^H(M) \to \text{Flf}_{\underline{r}}^{H(k-1)}(\overline{M})
\]
defined by $U_\bullet \mapsto U_\bullet/\mathcal{E}^{k-1}U_\bullet$. Then the image of $\pi_k$ is closed in $\text{Flf}_{\underline{r}}^{H(k-1)}(\overline{M})$.

**Proof.** Writing $\underline{r} := (r_1, r_1 + r_2, \ldots, r_1 + \cdots + r_{l-1})$, it is sufficient to show the following equivalent claim: the corresponding reduction map
\[
\pi'_k: \text{Grlf}_{\underline{r}}^{H(k,l)}(\overline{M}^{(l)}) \to \text{Grlf}_{\underline{r}}^{H(k-1,l)}(\overline{M}^{(l)})
\]
has closed image, see Proposition 4.3.
To this end we observe that $B := H(k, l)$ is a $Z(k)$-algebra which is free as $Z(k)$-module, and $\overline{B} := H(k - 1, l) \cong H(k, l)/\langle \varepsilon^{k-1} H(k, l) \rangle$. Now, we show that $\text{Grf}_{\overline{B}}^{B}(M^{(l)})$ is an open subset of $\text{Grf}_{\overline{Z(k-1),e}}^{B}(M^{(l)})$ for an adequate $e \in \mathbb{N}$. This subset fulfills the hypothesis of Lemma 4.9(b) as we shall now see. Indeed, $\text{Grf}_{\overline{B}}^{B}(M^{(l)})$ is an open subset of $\text{Grf}_{\overline{Z(k-1),e}}^{B}(M^{(l)})$, since being locally free is an open condition by the argument at the end of the proof of Proposition 4.3. Moreover, each $U \in \text{Grf}_{\overline{B}}^{B}(M^{(l)})$ yields a short exact sequence of locally free $H(1, l)$-modules

$$0 \to U/(\varepsilon U) \to M^{(l)}/(\varepsilon M^{(l)}) \to (M^{(l)}/U)/(\varepsilon M^{(l)}/U) \to 0.$$ 

Now, by Proposition 2.2 and since $M$ is rigid, $M/\varepsilon M$ is a rigid $H(1)$-module. Thus, since

$$M^{(l)}/(\varepsilon M^{(l)}) \cong M/\varepsilon M$$ 

as $H(1,l)$-module, we can apply Lemma 4.2 to the above exact sequence to see that

$$U \mapsto \dim \text{Hom}_{H(1,l)}(U/(\varepsilon U), (M^{(l)}/U)/(\varepsilon M^{(l)}/U))$$

is a constant function on $\text{Grf}_{\overline{B}}^{B}(M^{(l)})$. It then follows from Lemma 4.9 (with $M$ replaced by $M^{(l)}$ and $U = \text{Grf}_{\overline{B}}^{B}(M^{(l)})$) that $\text{Im}(\pi) \cap U$ is closed in $U$.

Finally, for rank reasons, no locally free submodule $U' \in \text{Grf}_{\overline{B}}^{B}(M^{(l)})$ can be the reduction of some $U \in \text{Grf}_{\overline{B}}^{B}(M^{(l)})$ which is not itself a locally free $B$-module. $\square$

4.7. Conclusion of the proof of Theorem 1.2. We know by Proposition 2.2(b) that for any $k \geq 2$ the quiver flag variety $\text{Fl}_{m}^{H(k)}$ is non-empty if and only if $\text{Fl}_{m}^{H(1)}$ is non-empty. Using this and Proposition 1.7 it remains only to show that $\pi_{k}$ is surjective with all fibers being isomorphic to an affine space of dimension $d(\mathfrak{r})$. Indeed, by Lemma 4.9(a) and Lemma 4.2 we have that $\pi_{k}^{-1}(\pi_{k}(U_{\bullet}))$ is an affine space of dimension $d(\mathfrak{r})$ for all $U_{\bullet} \in \text{Fl}_{m}^{H(k)}$. By Chevalley’s theorem and Lemma 4.10 $\text{Im}(\pi_{k})$ is a closed subset of dimension $\dim \text{Fl}_{m}^{H(k)} - d(\mathfrak{r})$ in $\text{Fl}_{m}^{H(k-1)}$. Now, by Proposition 4.7 we have

$$\dim \text{Fl}_{m}^{H(k-1)} = \dim \text{Fl}_{m}^{H(k)} - d(\mathfrak{r}).$$

Since, again by Proposition 4.7, $\text{Fl}_{m}^{H(k-1)}$ is irreducible, and $\text{Im}(\pi_{k})$ is closed we conclude that $\pi_{k}$ is indeed surjective. This finishes the proof of Theorem 1.2.

Note that the surjectivity of the morphism $\pi_{k} : \text{Fl}_{m}^{H}(M) \to \text{Fl}_{m}^{H(k-1)}(M)$ does not follow from Proposition 2.2(b). Using the notation from Proposition 2.2(b), for a fixed locally free module $X_{l}$ we might need different modules $Y_{l}$ to lift all flags of locally free submodules of $X_{l}$.

4.8. Example. We study the same family of algebras $H(k) = H(C, kD, \Omega)$ as in Example 2.3. Thus $C$ is symmetric and $D$ is the identity matrix. It follows that an $H(k)$-module is locally free if and only if it is free as a $Z(k)$-module.

We consider a locally free $H(k)$-module $N^{(k)}$ with $\text{rank}_{H(k)}(N^{(k)}) = (2, 2)$. It is determined by the $Z(k)$-linear map

$$N_{12}^{(k)} : H_{2}(k) \otimes H_{2}(k) \to N_{1}^{(k)}.$$
We choose this map such that with respect to the standard bases of $N_1(k) := H_1(k)^2$ and $N_2(k) := H_2(k)^2$ it is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. (Note that $H_2(k) \cong H_2(k)$ as $H_2(k)$-modules in this case, so that $H_2(k) \otimes H_2(k) N_2(k) \cong N_2(k)$. Thus the three isomorphism classes of rigid locally free indecomposable $H(k)$-modules appear precisely with multiplicity one. Namely we have

$$N(k) \cong E_1(k) \oplus E_2(k) \oplus F_2(k).$$

In particular, $N(k)$ itself is not rigid. It is known (see for example [W, Example 6.3]) that

$$\text{Gr}^{H(1)}(N^{(1)}) \subset \mathbb{P}^1(K) \times \mathbb{P}^1(K)$$

consists of two irreducible components, each of them isomorphic to $\mathbb{P}^1(K)$, which intersect in exactly one point $U = ([1 : 0], [1 : 0])$. A similar example was studied in [DWZ, Example 3.7]. Note that $U \cong S_1 \oplus S_2 \cong N^{(1)}/U$ and consequently we find for the tangent space

$$T_U \text{Gr}^{H(1)}(N^{(1)}) \cong \text{Hom}_{H(1)}(U, N^{(1)}/U) \cong K^2,$$

see [Sc, Lemma 3.2].

Now consider $\mathbb{P}^1(Z(k)) := \text{Gr}^{H(k)}(Z(k), 1)(H_i(k))^2$, i.e. the space of rank 1 free submodules of $H_i(k)^2 \cong Z(k)^2$. We use a similar notation for the points of this space as the usual notation of points in $\mathbb{P}^1(K)$, namely $[a : b]$ with $a, b \in Z(k)$ and at least one of $a$ and $b$ must be invertible in $Z(k)$; moreover $[a : b] \sim [a' : b']$ if and only if there exists an invertible element $u \in Z(k)^\times$ such that $(a', b') = (ua, ub)$. It follows that $\mathbb{P}^1(Z(k))$ is the disjoint union:

$$\mathbb{P}^1(Z(k)) = \{[1 : a] \mid a \in Z(k)\} \cup \{[\varepsilon a' : 1] \mid a' \in Z(k)\}.$$

With this notation we have

$$\text{Gr}^{H(k)}(N^{(k)}) \subset \mathbb{P}^1(Z(k)) \times \mathbb{P}^1(Z(k)).$$

Consider now elements of the form

$$U_{a,b} = ([1 : a], [1 : b]) \in \mathbb{P}^1(Z(k)) \times \mathbb{P}^1(Z(k)) \text{ with } a, b \in Z(k).$$

It is easy to see that

$$\mathcal{U}^{(k)} := \{U_{a,b} \in \mathbb{P}^1(Z(k)) \times \mathbb{P}^1(Z(k)) \mid 0 = a \cdot b \in Z(k)\}$$

forms a dense open subset of $\text{Gr}^{H(k)}(N^{(k)})$. We leave it as an exercise to show that $\mathcal{U}^{(k)}$ has precisely $k + 1$ irreducible components, each of them isomorphic to an affine space of dimension $k$.

Now, for $U = ([1 : 0], [1 : 0])$ as above, we have that $\pi_2^{-1}(U) = \{U_{\varepsilon a, \varepsilon b} \mid a, b \in K\}$ is precisely one of the three irreducible components of $\mathcal{U}^{(2)}$. Clearly, $\pi_3^{-1}(\pi_2^{-1}(U)) \subset \mathcal{U}^{(3)}$. Due to the defining equations of $\mathcal{U}^{(3)}$ we see that

$$\pi_3^{-1}(U_{\varepsilon a, \varepsilon b}) = \begin{cases} \{U_{\varepsilon a', \varepsilon b + \varepsilon 2b'} \mid a', b' \in K\} & \text{if } a = 0, \\ \{U_{\varepsilon a + \varepsilon 2a', \varepsilon 2b'} \mid a', b' \in K\} & \text{if } b = 0, \\ \emptyset & \text{if } ab \neq 0. \end{cases}$$

In particular, in this situation

$$\pi_3: \text{Gr}^{H(3)}(N^{(3)}) \to \text{Gr}^{H(2)}(N^{(2)})$$

is not surjective, and $\text{Gr}^{H(k)}(N^{(k)})$ is neither smooth nor irreducible.
5. Conjectures

5.1. Irreducibility. Let $H = H(C, D, \Omega)$. In general it is easy to find examples of dimension vectors $d$ such that the variety $\text{rep}(H, d)$ is not irreducible.

**Conjecture 5.1.** Assume that $d$ is a $D$-divisible dimension vector for $H$. Then $\text{rep}(H, d)$ is irreducible. In other words, $\text{rep}_{l.f.}(H, r)$ is dense in $\text{rep}(H, d)$.

5.2. Number of parameters. For $H(k) = H(C, kD, \Omega)$ let $r$ be a rank vector. The number of parameters of $r$ is defined as

$$\mu_{H(k)}(r) := \dim \text{rep}_{l.f.}(H(k), r) - \max\{\dim\mathcal{O}_M \mid M \in \text{rep}_{l.f.}(H(k), r)\}.$$ 

For example, if there exists some rigid $M \in \text{rep}_{l.f.}(H(k), r)$, then $\mu_{H(k)}(r) = 0$, since $\mathcal{O}_M$ is open in this case.

**Conjecture 5.2.** For $k \geq 2$ and all rank vectors $r$ we have $\mu_{H(k)}(r) = k \cdot \mu_{H(1)}(r)$.

5.3. A generalization of Kac’s Theorem. Again let $H = H(C, D, \Omega)$. Recall from [GLS1] that an $H$-module $M$ is $\tau$-locally free provided $\tau^k_H(M)$ is locally free for all $k \in \mathbb{Z}$. Here $\tau_H(-)$ denotes the Auslander-Reiten translation of $H$.

**Conjecture 5.3.** There is a bijection between the set of positive roots of the Kac-Moody Lie algebra $\mathfrak{g}(C)$ associated with $C$ and the set of rank vectors of indecomposable $\tau$-locally free $H$-modules.

For $C$ symmetric and $D$ the identity matrix, Conjecture 5.3 is true and was proved by Kac [K1, K2]. For $C$ of Dynkin type and $D$ arbitrary, Conjecture 5.3 is true, see [GLS1].

5.4. Schur roots. For $H = H(C, D, \Omega)$ let $r$ be an $H$-Schur root. Thus $\text{ind}_{l.f.}(H, r)$ is dense in $\text{rep}_{l.f.}(H, r)$. We conjecture that for a dense open subset $U \subseteq \text{ind}_{l.f.}(H, r)$ there is some $1 \leq i \leq n$ such that $\text{End}_H(M) \cong H_i$ for all $M \in U$. This would be a generalization of [K2 Proposition 1(a)].

5.5. Schofield’s algorithm. Let $H = H(C, D, \Omega)$. We conjecture that Schofield’s algorithm [Sc] (see also [DW]) for the computation of canonical decompositions of dimension vectors for path algebras can be generalized to an algorithm which computes the $H$-canonical decomposition of a given rank vector $r$.

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