On the Fourier transform of the characteristic functions of domains with $C^1$-smooth boundary

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Abstract. We consider domains $D \subseteq \mathbb{R}^n$ with $C^1$-smooth boundary and study the following question: when the Fourier transform $\widehat{1_D}$ of the characteristic function $1_D$ belongs to $L^p(\mathbb{R}^n)$?

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Introduction

Let $D$ be a bounded domain (an open and connected set) in $\mathbb{R}^n$, $n \geq 2$. Consider its characteristic function $1_D$, i.e. the function that takes value $1_D(t) = 1$ for $t \in D$ and value $1_D(t) = 0$ for $t \notin D$. Consider the Fourier transform $\widehat{1_D}$ of this function. In the present work we study the following question: for which domains we have $\widehat{1_D} \in L^p(\mathbb{R}^n)$? Only the case when $1 < p < 2$ is interesting.

It will be convenient for us to deal with the spaces $A_p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, of tempered distributions $f$ on $\mathbb{R}^n$ such that the Fourier transform $\widehat{f}$ belongs to $L^p(\mathbb{R}^n)$. The norm on $A_p(\mathbb{R}^n)$ is defined in the natural way:

$$\|f\|_{A_p(\mathbb{R}^n)} = \|\widehat{f}\|_{L^p(\mathbb{R}^n)}.$$ 

Recall that (see, e.g. [1, Ch. V, § 1]) for $1 \leq p \leq 2$ the Fourier transform (as well as its inverse) is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, $1/p + 1/q = 1$, so each distribution in $A_p(\mathbb{R}^n)$, $1 \leq p \leq 2$, is a function in $L^q(\mathbb{R}^n)$.

Direct calculation shows that if $D$ is a cube in $\mathbb{R}^n$, then $1_D \in A_p(\mathbb{R}^n)$ for all $p > 1$. The same is true in the case when $D$ is a polytope (i.e. a finite union of simplices). On the other hand, using the well known asymptotics for Bessel functions, one can verify that if $D \subseteq \mathbb{R}^n$ is a ball, then $1_D \in A_p(\mathbb{R}^n)$ for $p > 2n/(n+1)$ and $1_D \notin A_p(\mathbb{R}^n)$ for $p \leq 2n/(n+1)$. The same result holds in the general case of bounded domains with twice smooth boundary. (This follows from Theorems 1, 2 of the present work, see Corollary 2.) Thus, for (bounded) domains with $C^2$-smooth boundary $2n/(n+1)$ is the
critical power of integrability for the Fourier transform of the characteristic function.

In the present work we shall obtain a series of results on the behavior of the Fourier transform of the characteristic functions of bounded domains with $C^1$-smooth boundary. Generally speaking, this case is essentially different from the twice smooth case as is shown (see § 3) by an example of a domain $D \subseteq \mathbb{R}^2$ whose boundary is $C^1$-smooth and at the same time $1_D \in A_p(\mathbb{R}^2)$ for all $p > 1$. (The critical value for planar domains with twice smooth boundary is $4/3$.)

We note that various questions on the rate of decrease at infinity for the Fourier transform of characteristic functions of domains and closely related questions on the behavior of the Fourier transform of (smooth) measures supported on surfaces were investigated by many authors and represent a classical topic in harmonic analysis, see Stein’s survey [2], where one can find an ample bibliography, and his book [3] (Ch. VIII). The basic tools to obtain asymptotic estimates in these investigations are the stationary phase method and the van der Corput lemma. The use of this methods requires considerable smoothness of the boundary of a domain. The smoothness should be at least two even in the planar case. The crucial role in this approach is played by the curvature of the surface (of the boundary of a domain). Our approach does not use any arguments related to curvature and allows to consider domains with $C^1$-smooth boundary.

We denote by $\partial D$ the boundary of a domain $D \subseteq \mathbb{R}^n$. Saying that the boundary of $D$ is $C^1$-smooth or $C^2$-smooth we mean that in an appropriate neighborhood of each of its points the boundary $\partial D$ is a graph of a certain (real) function of class $C^1$ or $C^2$ respectively (that is of a function whose all partial derivatives of the first or the second order respectively are continuous).

For each domain $D \subseteq \mathbb{R}^n$ with $C^1$-smooth boundary let $\nu_D(x)$ be the outer unit normal vector to $\partial D$ at a point $x \in \partial D$. The corresponding map $\nu_D : \partial D \to S^{n-1}$ of the boundary of $D$ into the unit sphere $S^{n-1}$ centered at the origin is called normal map. By $\omega(\nu_D, \delta)$ we denote the modulus of continuity of $\nu_D$:

$$\omega(\nu_D, \delta) = \sup_{x, y \in \partial D, \, |x - y| \leq \delta} |\nu_D(x) - \nu_D(y)|, \quad \delta \geq 0,$$

where $|u|$ is the length of a vector $u \in \mathbb{R}^n$. Let then $\omega(\delta)$ be an arbitrary nondecreasing continuous function on $[0, \infty)$, $\omega(0) = 0$. In the case when
\( \omega(\nu_D, \delta) = O(\omega(\delta)), \delta \to +0, \) we say that the boundary \( \partial D \) is \( C^{1,\omega} \)-smooth. For bounded domains this condition is equivalent to the condition that in an appropriate neighborhood of each of its points the boundary of \( D \) is a graph of a certain function of class \( C^{1,\omega} \). In other words, for each point \( x \in \partial D \) one can find a neighborhood \( B \), containing \( x \), and a domain \( V \subseteq \mathbb{R}^{n-1} \) such that \( B \cap \partial D \) is a graph of some (real) function \( \varphi \in C^{1,\omega}(V) \) i.e. of a function with \( \omega(V, \nabla \varphi, \delta) = O(\omega(\delta)), \delta \to +0, \) where

\[
\omega(V, \nabla \varphi, \delta) = \sup_{x, y \in V : |x-y| \leq \delta} |\nabla \varphi(x) - \nabla \varphi(y)|, \quad \delta \geq 0,
\]

is the modulus of continuity of the gradient \( \nabla \varphi \) of \( \varphi \).

If the boundary \( \partial D \) of a domain \( D \) is \( C^1 \), \( C^2 \), or \( C^{1,\omega} \)-smooth, we write \( \partial D \in C^1 \), \( \partial D \in C^2 \), or \( \partial D \in C^{1,\omega} \) respectively.

If \( \omega(\delta) = \delta^\alpha, \, 0 < \alpha \leq 1 \), then we just write \( C^{1,\alpha} \) instead of \( C^{1,\delta^\alpha} \).

In § 1 we give a simple proof of the inclusion \( 1_D \in A_p(\mathbb{R}^n) \), which holds for \( p > 2n/(n+1) \) for all bounded domains \( D \subseteq \mathbb{R}^n \) with \( C^1 \)-smooth boundary (Theorem 1). For convex domains (without smoothness assumptions on the boundary) such assertion was earlier obtained by Herz [4].

In § 2 we obtain the main result of the present work. Namely, we show (Theorem 2) that if \( \partial D \in C^{1,\omega} \) and

\[
\int_0^1 \frac{\delta^{(n-1)p-1}}{\omega(\delta)^{n+p}} d\delta = \infty,
\]

then \( 1_D \not\in A_p(\mathbb{R}^n) \). In particular (Corollary 1), if \( \partial D \in C^{1,\alpha} \), then \( 1_D \not\in A_p(\mathbb{R}^n) \) for

\[
p \leq 1 + \frac{(n-1)\alpha}{n + \alpha}.
\]

Putting \( \alpha = 1 \) here and taking into account the preceding result, we obtain the indicated in the beginning of the introduction assertion on the critical value for domains with twice smooth boundary (Corollary 2).

In § 3 we consider planar domains. According to the result indicated above if for a domain \( D \subseteq \mathbb{R}^2 \) we have \( \partial D \in C^{1,\omega} \) and

\[
\int_0^1 \frac{\delta^{2p-3}}{\omega(\delta)^{2+p}} d\delta = \infty,
\]

then \( 1_D \not\in A_p(\mathbb{R}^2) \). In particular, if \( \partial D \in C^{1,\alpha} \), then \( 1_D \not\in A_p(\mathbb{R}^2) \) for \( p \leq 1 + \alpha/(2 + \alpha) \). We show (Theorem 3) that this result is sharp, namely,
for each class $C^{1,\omega}$ (under certain simple condition imposed on $\omega$) there exists a bounded domain $D \subseteq \mathbb{R}^2$ such that $\partial D \in C^{1,\omega}$ and for all $p > 1$ satisfying

$$\int_0^1 \frac{\delta^{2p-3}}{\omega(\delta)^{2-p}} d\delta < \infty$$

we have $1_D \in A_p(\mathbb{R}^2)$. In particular (Corollary 3), if $0 < \alpha < 1$, then there exists a planar domain $D$ with $C^{1,\alpha}$ -smooth boundary such that $1_D \in A_p(\mathbb{R}^2)$ for all $p > 1 + \alpha/(2 + \alpha)$. It also follows that (Corollary 4) there exists a planar domain $D$ with $C^1$-smooth boundary such that $1_D \in A_p(\mathbb{R}^2)$ for all $p > 1$ (it suffices to take $\omega$ decreasing to zero slower than any power, i.e. so that $\lim_{\delta \to +0} \omega(\delta)/\delta^\varepsilon = \infty$ for all $\varepsilon > 0$).

The results of the present work are essentially based on the results obtained by the author in [5], [6] where for real $C^1$ -smooth functions $\varphi$ we studied the growth of the $A_p$ norms of exponential functions $e^{i\lambda\varphi}$. The question on the growth of the norms of these functions naturally arises in relation with the known Beurling–Helson theorem (see the history of the question in [5]). Simple arguments (Lemma 1 of the present work) allow to reduce the study of the characteristic functions to the study of the behavior of the exponential functions.

We denote by $|E|$ the Lebesgue measure of a (measurable) set $E \subseteq \mathbb{R}^n$ and by $|E|_{S^{n-1}}$ the spherical measure of a set $E \subseteq S^{n-1}$. We use $(x, y)$ to denote the inner product of vectors $x$ and $y$ in $\mathbb{R}^n$. If $E \subseteq \mathbb{R}^n$ and $t \in \mathbb{R}^n$, then we put $E + t = \{x + t : x \in E\}$. Various positive constants are denoted by $c, c_p, c_{p,n}$.

The results of this work were partially presented at 11-th and 14-th Summer St. Petersburg Meetings in Mathematical Analysis [7], [8] and completely at the International Conference “Harmonic Analysis and Approximations, III”, Tsahkadzor, (Armenia) [9].

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§ 1. General case of domains with $C^1$ -smooth boundary

**Theorem 1.** Let $D$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with $\partial D \in C^1$. Then $1_D \in A_p(\mathbb{R}^n)$ for all $p > 2n/(n + 1)$.  

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Proof. For $s > 0$ consider the Sobolev spaces $W_s^2(\mathbb{R}^n)$ of functions $f \in L^2(\mathbb{R}^n)$ satisfying
\[
\|f\|_{W_s^2(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} (|\xi|^{2s} + 1)|\hat{f}(\xi)|^2d\xi \right)^{1/2} < \infty.
\]

It is easy to verify that $W_s^2(\mathbb{R}^n) \subseteq A_p(\mathbb{R}^n)$ for $2n/(n + 2s) < p < 2$. Indeed, put $p^* = 2/p$, $1/p^* + 1/q^* = 1$. We have $spq^* > n$. Using the Hölder inequality, we obtain
\[
\|f\|_{A_p(\mathbb{R}^n)}^p = \left( \int_{\mathbb{R}^n} |\hat{f}(\xi)|^p d\xi \right)^{1/p} = \left( \int_{\mathbb{R}^n} \left( |\hat{f}(\xi)| (|\xi|^s + 1) \right)^p \frac{1}{(|\xi|^s + 1)^p} d\xi \right)^{1/p} \leq c_{p,s} \|f\|_{W_s^2(\mathbb{R}^n)}^p.
\]

To prove the theorem it remains to take into account that for each bounded domain $D \subseteq \mathbb{R}^n$ with $C^1$-smooth boundary we have $1_D \in W_s^2(\mathbb{R}^n)$ for all $s < 1/2$. This is a trivial consequence of the theorem on (pointwise) multipliers of the Sobolev spaces [10, § 5].

We shall give an independent short and simple proof of the inclusion $1_D \in W_s^2$, $s < 1/2$. It is well known that for $0 < s < 1$ the norm $\|f\|_{W_s^2(\mathbb{R}^n)}$ and the norm
\[
\|f\| = \|f\|_{L^2(\mathbb{R}^n)} + \left( \int_{\mathbb{R}^n} \frac{1}{|t|^{n+2s}} \left( \int_{\mathbb{R}^n} |f(x + t) - f(x)|^2dx \right) dt \right)^{1/2}
\]
are equivalent (see, e.g., [11, Ch. V, § 3.5]). Note now that for each $t \in \mathbb{R}^n$ the symmetric difference
\[
((D - t) \setminus D) \cup (D \setminus (D - t))
\]
of the sets $D - t$ and $D$ is contained in the (closed) $|t|$-neighborhood of the boundary $\partial D$ of $D$, so its (Lebesgue) measure is at most $c|t|$. It is also clear that the measure of this symmetric difference is at most $2|D|$. Thus,
\[
\int_{\mathbb{R}^n} |1_D(x + t) - 1_D(x)|^2dx \leq \min(c|t|, 2|D|), \quad t \in \mathbb{R}^n,
\]

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and it remains to use the equivalence of the norm $\| \cdot \|_{W^s_2(\mathbb{R}^n)}$ and the norm defined by (1). The theorem is proved.  

**Remark 1.** The method used in the proof of Theorem 1 can be applied to arbitrary sets (not only domains). Recall [13] that the upper Minkowski dimension $\dim_M F$ of a bounded set $F \subseteq \mathbb{R}^n$ is defined by

$$\dim_M F = \inf \{ 0 \leq \gamma \leq n : |(F)_\delta| = O(\delta^{n-\gamma}), \delta \to +0 \},$$

where $(F)_\delta$ is the $\delta$-neighborhood of $F$. Let $E \subseteq \mathbb{R}^n$, $n \geq 1$, be a bounded set of positive measure. Let $a$ be the upper Minkowski dimension of its boundary $\partial E$. Suppose that $a < n$. Repeating with obvious modifications the arguments used above, we see that then $1_E \in W^s_2(\mathbb{R}^n)$ for all $s < (n-a)/2$. Hence in turn we obtain that $1_E \in A_p(\mathbb{R}^n)$ for all $p > 2n/(2n-a)$. Note that for $s < (n-a)/2$ the usage of the norm (1) is justified since $a \geq n-1$. Indeed, let us verify that if a set $E \subseteq \mathbb{R}^n$ is bounded and has positive measure, then $\dim_M \partial E \geq n-1$. Assuming that $E \setminus \partial E \neq \emptyset$ (otherwise there is nothing to prove) fix a point $x_0 \in E \setminus \partial E$. There exists an open ball $B$ centered at $x_0$ that does not contain points of the boundary $\partial E$ and moreover lies at positive distance from $\partial E$. Denote by $S$ the boundary sphere of the ball $B$. Define a map $\theta : \mathbb{R}^n \setminus B \to S$ as follows. Take a point $x \in \mathbb{R}^n \setminus B$ and consider the ray that passes through $x$ and has its origin at $x_0$. Denote by $\theta(x)$ the point of intersection of this ray with the sphere $S$. Clearly the map $\theta$ is Lipschitz (moreover it is non-expanding, i.e. $|\theta(x_1) - \theta(x_2)| \leq |x_1 - x_2|$ for all $x_1, x_2 \in \mathbb{R}^n \setminus B$). It is easy to see that the image of the boundary of the set $E$ under the map $\theta$ is the whole sphere $S$. At the same time it is known [13, Ch. 7] that Lipschitz maps do not increase the dimension of a set. Thus,

$$n - 1 = \dim_M S = \dim_M \theta(\partial E) \leq \dim_M \partial E.$$

### § 2. Domains with $C^{1,\omega}$-smooth boundary

**Theorem 2.** Let $D$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with $\partial D \in C^{1,\omega}$.

If

$$\int_0^1 \frac{\delta^{n(p-1)-1}}{(\omega(\delta))^{n-p}} d\delta = \infty,$$

\footnote{Note that [12, Corollary 2.2] if $E \subseteq \mathbb{R}^n$, $n \geq 1$, is a set of positive measure, then $1_E \notin W^{1/2}_2(\mathbb{R}^n)$.}
then \(1_D \notin A_p(\mathbb{R}^n)\).

From Theorem 2 we immediately obtain the following corollary.

**Corollary 1.** Let \(0 < \alpha \leq 1\). Let \(D\) be a bounded domain in \(\mathbb{R}^n\), \(n \geq 2\), with \(\partial D \in C^{1,\alpha}\). If

\[
p \leq 1 + \frac{(n-1)\alpha}{n + \alpha},
\]

then \(1_D \notin A_p(\mathbb{R}^n)\).

Note particularly the case of domains with twice smooth boundary and even more general \(C^{1,1}\) case. Namely, using Corollary 1 and Theorem 1, we obtain the following corollary.

**Corollary 2.** Let \(D\) be a bounded domain in \(\mathbb{R}^n\), \(n \geq 2\), with \(\partial D \in C^{1,1}\). Then \(1_D \in A_p(\mathbb{R}^n)\) for \(p > 2n/(n+1)\) and \(1_D \notin A_p(\mathbb{R}^n)\) for \(p \leq 2n/(n+1)\). In particular, this holds for bounded domains with twice smooth boundary.

Before proving the theorem we discuss certain preliminaries and prove a certain lemma.

Recall (see e.g. [11, Ch. IV, § 3.1]) that a function \(m \in L^\infty(\mathbb{R}^n)\) is called an \(L^p\)-Fourier multiplier \((1 \leq p \leq \infty)\) if the operator \(Q\) given by

\[
\hat{Q}f = m\hat{f}, \quad f \in L^p \cap L^2(\mathbb{R}^n),
\]

is a bounded operator from \(L^p(\mathbb{R}^n)\) to itself. The space \(M_p(\mathbb{R}^n)\) of all such multipliers endowed with the norm

\[
\|m\|_{M_p(\mathbb{R}^n)} = \|Q\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)}
\]

is a Banach algebra (with respect to the usual multiplication of functions). It is well known that the characteristic function of an arbitrary parallelepiped is a multiplier for all \(p, 1 < p < \infty\), (see e.g. [11, Ch. IV, § 4.1] and also [14, Ch. I, § 1.3]).

Note that in the case when \(1 < p < 2\) (only this case is interesting to us here) each \(L^p\)-Fourier multiplier is a pointwise multiplier of \(A_p\), that is if \(m \in M_p(\mathbb{R}^n)\), then for every function \(f \in A_p(\mathbb{R}^n)\) we have \(mf \in A_p(\mathbb{R}^n)\) and

\[
\|mf\|_{A_p} \leq \|m\|_{M_p} \|f\|_{A_p}. \quad (2)
\]
This can be easily verified as follows. Note that estimate (2) holds for every function $f \in A_p \cap L^2(\mathbb{R}^n)$ (this is obvious since the Fourier transform and its inverse differ only in sign of variable and normalization factor). It is clear that the set $A_p \cap L^2(\mathbb{R}^n)$ is dense in $A_p(\mathbb{R}^n)$. Let $f \in A_p(\mathbb{R}^n)$ be an arbitrary function. Let $f_k \in A_p \cap L^2(\mathbb{R}^n)$, $k = 1, 2, \ldots$, be a sequence that converges to $f$ in $A_p(\mathbb{R}^n)$. We have

$$
\|m f_j - m f_k\|_{A_p} = \|m \cdot (f_j - f_k)\|_{A_p} \leq \|m\|_{L^p} \|f_j - f_k\|_{A_p} \to 0.
$$

It is obvious that the spaces $A_p(\mathbb{R}^n)$ are Banach spaces, so the sequence $m f_k$, $k = 1, 2, \ldots$, converges in $A_p(\mathbb{R}^n)$ to some function $g \in A_p(\mathbb{R}^n)$. Using the Hausdorff–Young inequality [1, Ch. V, § 1], which in our notation has the form $\|\cdot\|_{L^q} \leq \|\cdot\|_{A_p}$, $1/p + 1/q = 1$, $1 \leq p \leq 2$, we see that the sequences $\{f_k\}$ and $\{m f_k\}$ converge in $L^q(\mathbb{R}^n)$ to $f$ and $g$ respectively. Thus, $m f = g$ and it remains to proceed to the limit in the inequality $\|m f_k\|_{A_p} \leq \|m\|_{L^p} \|f_k\|_{A_p}$.

Let $D_1$ be a domain in $\mathbb{R}^n$ and let $D_2 = l(D_1)$ be its image under a non-degenerate affine map $l : \mathbb{R}^n \to \mathbb{R}^n$. It is easy to see that $1_{D_1} \in A_p(\mathbb{R}^n)$ if and only if $1_{D_2} \in A_p(\mathbb{R}^n)$. It suffices to observe that if $l(x) = Q x + b$, then for each function $f \in L^1(\mathbb{R}^n)$ we have $|\hat{f} \circ l(u)| = |\det Q|^{-1} |\hat{f}((Q^{-1})^* u)|$, where $Q^{-1}$ is the matrix inverse of $Q$ and $(Q^{-1})^*$ is the matrix adjoint to $Q^{-1}$.

Let $E$ be an arbitrary set in $\mathbb{R}^m$, $m \geq 1$. Following [6] we say that a function $f$ defined on $E$ belongs to the space $A_p(\mathbb{R}^m, E)$ if there exists a function $F \in A_p(\mathbb{R}^m)$ such that its restriction $F|_E$ to the set $E$ coincides with $f$. We define the norm on $A_p(\mathbb{R}^m, E)$ by

$$
\|f\|_{A_p(\mathbb{R}^m, E)} = \inf_{F|_E = f} \|F\|_{A_p(\mathbb{R}^m)}.
$$

Note that if $I$ is a parallelepiped in $\mathbb{R}^m$ and $f$ is a function on $I$, then putting

$$
\|f\|_{A_p(\mathbb{R}^m, I)} = \|F\|_{A_p(\mathbb{R}^m)},
$$

where $F$ is the function $f$ extended by zero to the compliment $\mathbb{R}^m \setminus I$ (that is $F = f$ on $I$ and $F = 0$ on $\mathbb{R}^m \setminus I$), we obtain the norm $\|\cdot\|_{A_p(\mathbb{R}^m, I)}$ equivalent to the norm $\|\cdot\|_{A_p(\mathbb{R}^m)}$ for $1 < p < 2$. This follows since for $1 < p < \infty$ the characteristic function of a parallelepiped is an $L^p$-Fourier multiplier.

The following result, obtained by the author in [6, Theorem 1’], is the base of the proof of Theorem 2. Let $1 \leq p < 2$. Let $V$ be a domain in $\mathbb{R}^m$.
and let $\phi \in C^{1,\infty}(V)$ be a real function. Suppose that the gradient $\nabla \phi$ of $\phi$ is non-degenerate on $V$, i.e. the set $\nabla \phi(V)$ is of positive measure. Then for all those $\lambda \in \mathbb{R}$, $|\lambda| \geq 1$, for which $e^{i\lambda \phi} \in A_p(\mathbb{R}^m, V)$, we have

$$\|e^{i\lambda \phi}\|_{A_p(\mathbb{R}^m, V)} \geq c \left( \frac{|\lambda|^{1/p} \chi^{-1} \left( \frac{1}{|\lambda|} \right)}{m} \right)^m,$$

(4)

where $\chi^{-1}$ is the function inverse to $\chi(\delta) = \delta \omega(\delta)$ and $c = c(p, \phi) > 0$ is independent of $\lambda$.

Simple Lemma 1 below (which we shall also use in § 3) allows to reduce the question on inclusion $1_D \in A_p$ to the question on behavior of exponential functions $e^{i\lambda \phi}$ in $A_p$.

For a vector $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ and a number $a \in \mathbb{R}$ let $(x, a)$ denote the vector $(x_1, x_2, \ldots, x_m, a) \in \mathbb{R}^{m+1}$.

Let $I$ be an open parallelepiped in $\mathbb{R}^m$ with edges parallel to coordinate axes. Let $\varphi$ be a continuous bounded function on $I$ such that $\varphi(t) > 0$ for all $t \in I$. Consider the following domain $G$ in $\mathbb{R}^{m+1}$

$$G = \{(t, y) \in \mathbb{R}^m \times \mathbb{R} : t \in I, \ 0 < y < \varphi(t)\}.$$ 

Each domain of this form is called a special domain generated by the pair $(I, \varphi)$.

**Lemma 1.** Let $G \subseteq \mathbb{R}^{m+1}$ be a special domain generated by a pair $(I, \varphi)$. Let $1 < p < 2$. The inclusion $1_G \in A_p(\mathbb{R}^{m+1})$ holds if and only if $e^{i\lambda \phi} \in A_p(\mathbb{R}^m, I)$ for almost all $\lambda \in \mathbb{R}$ and

$$\int_{\mathbb{R}} \frac{1}{|\lambda|^p} \|e^{i\lambda \phi} - 1\|_{A_p(\mathbb{R}^m, I)}^p d\lambda < \infty.$$

**Proof.** For $\lambda \in \mathbb{R} \setminus \{0\}$ define the function $F_\lambda$ on $\mathbb{R}^m$ by

$$F_\lambda(t) = \begin{cases} \frac{1}{-i\lambda}(e^{-i\lambda \varphi(t)} - 1), & \text{if } t \in I, \\ 0, & \text{if } t \in \mathbb{R}^m \setminus I. \end{cases}$$

Note that

$$\hat{1}_G(u, \lambda) = \hat{F}_\lambda(u), \quad (u, \lambda) \in \mathbb{R}^m \times \mathbb{R}, \ \lambda \neq 0.$$
Indeed, direct calculation yields

\[ \hat{1}_G(u, \lambda) = \int \int_{t \in I, 0 < y < \phi(t)} e^{-i(u,t)e^{-i\lambda y}} dy \, dt = \int I \left( \int_0^{\phi(t)} e^{-i\lambda y} dy \right) e^{-i(u,t)} dt \]

\[ = \int I \frac{1}{-i\lambda} (e^{-i\lambda \phi(t)} - 1) e^{-i(u,t)} dt = \hat{F}_\lambda(u). \]

Thus, \( 1_G \in A_p(\mathbb{R}^{m+1}) \) if and only if

\[ \int \| F_\lambda \|_{A_p(\mathbb{R}^m)}^p d\lambda < \infty. \]

It remains only to take into account that

\[ \| F_\lambda \|_{A_p(\mathbb{R}^m)} = \frac{1}{|\lambda|} \| e^{i\lambda \phi} - 1 \|_{A_p(\mathbb{R}^m, I)}, \]

where \( \| \cdot \|_{A_p(\mathbb{R}^m, I)} \) is the equivalent norm on \( A_p(\mathbb{R}^m, I) \) defined above (see (3)). The lemma is proved.

**Proof of Theorem 2.** Assume that contrary to the assertion of the theorem we have \( 1_D \in A_p(\mathbb{R}^n) \).

Each point \( x \) of the boundary \( \partial D \) can be surrounded by an open parallelepiped \( \Pi_x \ni x \) so small that after an appropriate rotation and translation the intersection \( D \cap \Pi_x \) becomes a special domain. Extract a finite subcovering from the covering \( \{ \Pi_x, x \in \partial D \} \) of \( \partial D \). Note that \( \nu_D(\partial D) = S^{n-1} \), so at least for one of the parallelepipeds \( \Pi_x \), which we denote by \( \Pi \), we have

\[ |\nu_D(\partial D \cap \Pi)|_{S^{n-1}} > 0. \quad (5) \]

Consider the domain \( G = D \cap \Pi \). Since the characteristic function of a parallelepiped is an \( L^p \)-multiplier, we have \( 1_G = 1_{\Pi_1} \cdot 1_D \in A_p(\mathbb{R}^n) \).

Replacing, if needed, the domain \( G \) by its copy obtained by rotation and translation, we can assume that \( G \) is a special domain. This domain is generated by a pair \( (I, \phi) \), where \( I \) is a certain parallelepiped in \( \mathbb{R}^m, m+1 = n \), with edges parallel to coordinate axes and \( \phi \) is a certain function in \( C^{1,\omega}(I) \).

It is easy to see that condition (5) implies that the gradient of \( \phi \) is non-degenerate on \( I \), that is, we have \( |\nabla \phi(I)| > 0 \). Indeed (recall that
if \( x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \) and \( a \in \mathbb{R} \), then \((x, a)\) denotes the vector 
\((x_1, x_2, \ldots, x_m, a) \in \mathbb{R}^{m+1}\) consider the map 
\[
\beta(t) = (t, \varphi(t)), \quad t \in I,
\]
\((\beta \text{ maps } I \text{ onto the graph of } \varphi).\) The normal map \( \nu_D \) and the gradient \( \nabla \varphi \) of \( \varphi \) are related by 
\[
\nu_D \circ \beta(t) = \frac{1}{\sqrt{\left|\nabla \varphi(t)\right|^2 + 1}} (-\nabla \varphi(t), 1), \quad t \in I.
\]
Thus, putting 
\[
\gamma(\xi) = \frac{1}{\sqrt{|\xi|^2 + 1}} (-\xi, 1), \quad \xi \in \mathbb{R}^m,
\]
we have \( \nu_D \circ \beta = \gamma \circ \nabla \varphi. \) So for the set \( W = \nabla \varphi(I) \) we obtain 
\[
\gamma(W) = \gamma(\nabla \varphi(I)) = \nu_D \circ \beta(I) = \nu_D(\partial D \cap \Pi),
\]
and relation (5) implies \( |\gamma(W)|_{S^{n-1}} > 0. \) Since \( \gamma \) is a diffeomorphism of \( \mathbb{R}^m \) onto the upper half-sphere 
\[
S^m_+ = \{x = (x_1, x_2, \ldots, x_{m+1}) \in \mathbb{R}^{m+1} : |x| = 1, \ x_{m+1} > 0\}
\]
(where \( m + 1 = n \)), we see that \( |W| > 0. \)
Thus we see that the special domain \( G \) is generated by a pair \((I, \varphi)\) where \( \varphi \in C^{1, \omega}(I) \) is a function with non-degenerate gradient and at the same time we have \( 1_G \in A_p(\mathbb{R}^{m+1}). \)
By Lemma 1 we have 
\[
\int_{\mathbb{R}} \frac{1}{|\lambda|^p} \left| e^{i\lambda \varphi} - 1 \right|_{A_p(\mathbb{R}^m, I)}^p d\lambda < \infty,
\]
so 
\[
\int_{\lambda \geq 1} \frac{1}{\lambda^p} \left| e^{i\lambda \varphi} - 1 \right|_{A_p(\mathbb{R}^m, I)}^p d\lambda < \infty,
\]
and since \( 1 \in A_p(\mathbb{R}^m, I), \ p > 1, \) we see that 
\[
\int_1^\infty \frac{1}{\lambda^p} \left| e^{i\lambda \varphi} \right|_{A_p(\mathbb{R}^m, I)}^p d\lambda < \infty.
\]
Hence, putting $V = I$ in estimate (4), we obtain
\[
\int_1^\infty \lambda^{n-p} \left( \chi^{-1} \left( \frac{1}{\lambda} \right) \right)^{m^p} d\lambda < \infty,
\]
that is (recall that $m = n - 1$)
\[
\int_1^\infty \lambda^{n-1-p} \left( \chi^{-1} \left( \frac{1}{\lambda} \right) \right)^{(n-1)^p} d\lambda < \infty.
\]

The following lemma is of purely technical character. It completes the proof of the theorem.

**Lemma 2.** Let $n \geq 2$, $1 < p < 2$. The following conditions are equivalent:

1) \[
\int_1^\infty \lambda^{n-1-p} \left( \chi^{-1} \left( \frac{1}{\lambda} \right) \right)^{(n-1)^p} d\lambda < \infty;
\]

2) \[
\int_0^1 \frac{\delta^{(p-1)-1}}{(\omega(\delta))^{n-p}} d\delta < \infty.
\]

Certainly, to complete the proof of the theorem it suffices to verify that 1) $\Rightarrow$ 2). The inverse implication for $n = 2$ will be used below in § 3.

**Proof of Lemma 2.** For $0 < \varepsilon < 1$ put
\[
I(\varepsilon) = \int_{1/\chi(1)}^{1/\chi(\varepsilon)} \lambda^{n-1-p} \left( \chi^{-1} \left( \frac{1}{\lambda} \right) \right)^{(n-1)^p} d\lambda, \quad J(\varepsilon) = \int_\varepsilon^1 \frac{\delta^{(p-1)-1}}{(\omega(\delta))^{n-p}} d\delta.
\]

We have
\[
I(\varepsilon) = \frac{1}{n-p} \int_{1/\chi(1)}^{1/\chi(\varepsilon)} \left( \chi^{-1} \left( \frac{1}{\lambda} \right) \right)^{(n-1)^p} d\lambda^{n-p}.
\]
Changing the variable $\lambda = 1/\chi(\delta)$ and integrating by parts we obtain
\[
I(\varepsilon) = \frac{1}{n-p} \left( \varepsilon^{(n-1)(p-1)} (\omega(\varepsilon))^{n-p} - \frac{1}{(\omega(1))^{n-p}} \right) + \frac{(n-1)p}{n-p} J(\varepsilon). \quad (6)
\]
Using this relation we see that
\[ I(\varepsilon) \geq \frac{-1}{(n-p)(\omega(1))^{n-p}} + \frac{(n-1)p}{n-p} J(\varepsilon), \]
so 1) \(\Rightarrow\) 2).

Conversely, assume that condition 2) holds. Then, since
\[ \int_{\varepsilon/2}^{\varepsilon} \frac{\delta^{n(p-1)-1}}{(\omega(\delta))^{n-p}} d\delta \geq \frac{1}{(\omega(\varepsilon))^{n-p}} \int_{\varepsilon/2}^{\varepsilon} \delta^{n(p-1)-1} d\delta \geq c_{n,p} \frac{\varepsilon^{n(p-1)}}{(\omega(\varepsilon))^{n-p}}, \]
we have
\[ \frac{\varepsilon^{n(p-1)}}{(\omega(\varepsilon))^{n-p}} \to 0, \quad \varepsilon \to +0, \]
and using (6) we obtain condition 1). The lemma and thus the theorem are proved.

Remark 2. Theorem 2 (and Corollary 1, which it implies) has local character. The theorem remains true if we assume that only a part of the boundary of a domain \(D\), i.e. the intersection \(B \cap \partial D\), where \(B\) is a certain neighborhood in \(\mathbb{R}^n\), is \(C^{1,\omega}\) smooth and the normal map \(\nu\) defined on \(B \cap \partial D\) is non-degenerate, that is \(|\nu(B \cap \partial D)|_{S^{n-1}} > 0\). The condition that \(D\) is bounded can be replaced by the weaker condition \(|D| < \infty\). (The modification of the proof is obvious.)

For \(n = 2\) the condition of non-degeneracy of the normal map on \(B \cap \partial D\) means that \(B \cap \partial D\) is not a straight line interval.

§ 3. Domains in \(\mathbb{R}^2\)

In this section for each class \(C^{1,\omega}\) (under certain simple condition imposed on \(\omega\)) we shall construct a bounded domain \(D \subseteq \mathbb{R}^2\) with \(C^{1,\omega}\) smooth boundary such that the characteristic function \(1_D\) belongs to \(A_p\) for \(p\) so close to 1 as is allowed by Theorem 2. In addition the domain \(D\) has the property that its boundary does not contain straight line intervals (thus, this domain is essentially different from polygons).

According to Theorem 2 if \(D\) is a bounded domain in \(\mathbb{R}^2\) with \(\partial D \in C^{1,\omega}\) and
\[ \int_0^1 \frac{\delta^{2p-3}}{\omega(\delta)^{2-p}} d\delta = \infty, \]
then \(1_D \notin A_p(\mathbb{R}^2)\). In particular this is the case when \(\partial D \in C^{4,\alpha}\) and 
\(p \leq 1 + \alpha/(2 + \alpha)\). The following theorem shows that this result is sharp.

**Theorem 3.** Suppose that \(\omega(2\delta) < 2\omega(\delta)\) for all sufficiently small \(\delta > 0\). There exists a bounded domain \(D \subseteq \mathbb{R}^2\) with \(\partial D \in C^{1,\omega}\) such that \(1_D \in A_p(\mathbb{R}^2)\) for all \(p, 1 < p < 2\), satisfying

\[
\int_0^1 \frac{\delta^{2p-3}}{\omega(\delta)^{2-p}} \, d\delta < \infty. \tag{7}
\]

In addition the boundary of \(D\) does not contain line intervals.

This theorem immediately implies the following corollaries.

**Corollary 3.** For each \(\alpha, 0 < \alpha < 1\), there exists a bounded domain \(D \subseteq \mathbb{R}^2\) such that its boundary is \(C^{1,\alpha}\)-smooth and \(1_D \in A_p(\mathbb{R}^2)\) for all \(p > 1 + \alpha/(2 + \alpha)\). The boundary of \(D\) does not contain line intervals.

**Corollary 4.** There exists a bounded domain \(D \subseteq \mathbb{R}^2\) such that its boundary is \(C^1\)-smooth and \(1_D \in \bigcap_{p>1} A_p(\mathbb{R}^2)\). The boundary of \(D\) does not contain line intervals.

Note also that from Theorems 2 and 3 it follows that the existence of a domain \(D \subseteq \mathbb{R}^2\) with \(\partial D \in C^{1,\omega}\) and \(1_D \in \bigcap_{p>1} A_p(\mathbb{R}^2)\) is equivalent to the condition that \(\omega(\delta)\) tends to 0 slower than any power, i.e., to the condition that \(\lim_{\delta \to 0^+} \omega(\delta)^{\varepsilon} = \infty\) for all \(\varepsilon > 0\). Theorem 2 implies the necessity of this condition. Theorem 3 implies its sufficiency.

The author does not know whether similar results are true for domains in \(\mathbb{R}^n\) with \(n \geq 3\).

**Proof of Theorem 3.** Let \(A_p(\mathbb{T})\), \(1 \leq p \leq \infty\), be the space of distributions \(f\) on the circle \(\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}\) (where \(\mathbb{Z}\) is the set of integers) such that the

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\(\text{2One should only observe that if this condition holds, then one can find a nondecreasing continuous function } \omega^*(\delta) \text{ on } [0, +\infty) \text{ that tends to 0 slower then any power and satisfies } \omega^*(2\delta) < 2\omega^*(\delta) \text{ for all } \delta > 0 \text{ and } \omega^*(\delta) = O(\omega(\delta)) \text{ as } \delta \to +0. \text{ For instance, one can put } \omega^*(\delta) = \delta/(1 + \delta) + \delta \inf_{0 < x \leq \delta} \omega(x)/x. \text{ It is easy to verify that the second term is a nondecreasing function, see [15, Ch. III, 3.2.5].} \)
sequence of Fourier coefficients \( \hat{f} = \{ \hat{f}(k), \ k \in \mathbb{Z} \} \) belongs to \( l^p \). We put

\[
\| f \|_{A_p(\mathbb{T})} = \| \hat{f} \|_{l^p} = \left( \sum_{k \in \mathbb{Z}} |\hat{f}(k)|^p \right)^{1/p}.
\]

(For \( 1 \leq p \leq 2 \) each distribution in \( A_p(\mathbb{T}) \) is a function in \( L^q(\mathbb{T}) \subseteq L^1(\mathbb{T}) \), \( 1/p + 1/q = 1 \).

For \( p > 1 \) we put

\[
\Theta_p(y) = \left( \int_1^y \left( \chi^{-1}\left( \frac{1}{\tau} \right) \right)^p d\tau \right)^{1/p}, \quad y > 1,
\]

where as above \( \chi^{-1} \) is the function inverse to \( \chi(\delta) = \delta \omega(\delta) \).

In Theorem 2 of the work [5], under assumption that \( \omega(2\delta) < 2\omega(\delta) \) for all sufficiently small \( \delta > 0 \), we constructed a real function \( \varphi \) on the circle \( \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \) such that \( \varphi \in C^{1,\omega}(\mathbb{T}) \) (i.e., \( \varphi \) is a \( 2\pi \)-periodic function of class \( C^{1,\omega}(\mathbb{R}) \)) and for all \( p, \ 1 < p < 2 \), we have

\[
\| e^{i\lambda \varphi} \|_{A_p(\mathbb{T})} \leq c_p \Theta_p(|\lambda|), \quad \lambda \in \mathbb{R}, \quad |\lambda| \geq 2.
\]

(8)

In addition the function \( \varphi \) is nowhere linear, that is it is not linear on any interval.

It is clear that from estimate (8) we have

\[
\| e^{i\lambda \varphi} - 1 \|_{A_p(\mathbb{T})} \leq c_p \Theta_p(|\lambda|), \quad \lambda \in \mathbb{R}, \quad |\lambda| \geq 2.
\]

(9)

It is also clear that for every continuously differentiable function \( f \) on \( \mathbb{T} \) we have \( f \in A_1(\mathbb{T}) \) and

\[
\| f \|_{A_1(\mathbb{T})} \leq c \| f \|_{C^1(\mathbb{T})},
\]

where

\[
\| f \|_{C^1(\mathbb{T})} = \max_{t \in \mathbb{T}} |f(t)| + \max_{t \in \mathbb{T}} |f'(t)|.
\]

So

\[
\| e^{i\lambda \varphi} - 1 \|_{A_p(\mathbb{T})} \leq \| e^{i\lambda \varphi} - 1 \|_{A_1(\mathbb{T})} \leq c \| e^{i\lambda \varphi} - 1 \|_{C^1(\mathbb{T})} \leq c |\lambda|, \quad \lambda \in \mathbb{R}.
\]

(10)

3Theorem 2 of the work [5] contains similar result for \( p = 1 \) as well.
Consider the following set $Q$ on the line $\mathbb{R}$:

$$Q = \{ t \in (0, 2\pi) : \varphi'(t) > 0 \}.$$  

Since $\varphi \neq \text{const}$ and $\varphi(0) = \varphi(2\pi)$, it is clear that $Q \neq \emptyset$ and $Q \neq (0, 2\pi)$. Consider an interval $(a, b)$ which is a connected component of the set $Q$. The derivative $\varphi'$ vanishes at least at one of its endpoints. We can assume that at the right one, that is $\varphi'(b) = 0$, otherwise instead of $\varphi(t)$ and the interval $(a, b)$ we consider the function $-\varphi(-t)$ and the interval $(-b, -a)$. Choose now a point $c$, $a < c < b$. Replacing the function $\varphi(t)$ by $\varphi(t) - \varphi(c)$, we can assume that $\varphi(c) = 0$. We put $I = (c, b)$.

Thus we obtain a nowhere linear function $\varphi \in C^1(\mathbb{T})$ satisfying conditions (9), (10) and an interval $I = (c, b) \subseteq [0, 2\pi]$ such that $\varphi(c) = 0$, the function $\varphi$ is strictly increasing on $I$, and in addition $\varphi'(c) > 0$, $\varphi'(b) = 0$.

Recall the well known relation [16, § 44] between the spaces $A_p(\mathbb{T})$ and $A_p(\mathbb{R})$ for $1 < p \leq 2$. If $f$ is a $2\pi$-periodic function and $f^*$ is its restriction to $[0, 2\pi]$ extended by zero to $\mathbb{R}$, i.e. $f^* = f$ on $[0, 2\pi]$, $f^* = 0$ on $\mathbb{R} \setminus [0, 2\pi]$, then $f \in A_p(\mathbb{T})$ if and only if $f^* \in A_p(\mathbb{R})$. The norms satisfy

$$c_1(p)\|f^*\|_{A_p(\mathbb{R})} \leq \|f\|_{A_p(\mathbb{T})} \leq c_2(p)\|f^*\|_{A_p(\mathbb{R})}.$$

Thus, from estimates (9) and (10) we obtain that for all $p$, $1 < p < 2$,

$$\|e^{i\lambda \varphi} - 1\|_{A_p(\mathbb{R}, I)} \leq c_p \Theta_p(|\lambda|), \quad \lambda \in \mathbb{R}, \quad |\lambda| \geq 2,$$

and correspondingly

$$\|e^{i\lambda \varphi} - 1\|_{A_p(\mathbb{R}, I)} \leq c|\lambda|, \quad \lambda \in \mathbb{R}.$$  

Consider the special domain $G \subseteq \mathbb{R}^2$ generated by the pair $(I, \varphi)$.

**Lemma 3.** For all $p$, $1 < p < 2$, satisfying (7), we have $1_G \in A_p(\mathbb{R}^2)$.

**Proof.** It is easy to verify that condition (7) implies

$$\int_1^\infty \frac{1}{\lambda^p} (\Theta_p(\lambda))^p d\lambda < \infty.$$  

Indeed, for any $a > 1$ integrating by parts we obtain

$$\int_1^a \frac{1}{\lambda^p} (\Theta_p(\lambda))^p d\lambda = \frac{1}{-p+1} \int_1^a (\Theta_p(\lambda))^{p+1} d\lambda - \frac{1}{-p+1} \int_1^a (\Theta_p(\lambda))^{p+1} d\lambda.$$
\[
\begin{align*}
&= \frac{1}{-p+1} \left( (\Theta_p(a))^{p}a^{-p+1} - \int_{1}^{a} \lambda^{-p+1} \left( \chi^{-1} \left( \frac{1}{\lambda} \right) \right)^{p} d\lambda \right) \\
&\leq \frac{1}{p-1} \int_{1}^{a} \lambda^{-p+1} \left( \chi^{-1} \left( \frac{1}{\lambda} \right) \right)^{p} d\lambda,
\end{align*}
\]
and using Lemma 2 with \( n = 2 \) we obtain (13).

Therefore (see (11), (13)),
\[
\int_{|\lambda| \geq 2} \frac{1}{|\lambda|^{p}} \left\| e^{i\lambda \varphi} - 1 \right\|_{A_{p}(\mathbb{R},I)}^{p} d\lambda < \infty.
\]

At the same time (see (12))
\[
\int_{|\lambda| < 2} \frac{1}{|\lambda|^{p}} \left\| e^{i\lambda \varphi} - 1 \right\|_{A_{p}(\mathbb{R},I)}^{p} d\lambda < \infty.
\]
Thus,
\[
\int_{\mathbb{R}} \frac{1}{|\lambda|^{p}} \left\| e^{i\lambda \varphi} - 1 \right\|_{A_{p}(\mathbb{R},I)}^{p} d\lambda < \infty.
\]
It remains to use Lemma 1. The lemma is proved.

Now we shall complete the proof of Theorem 3. The domain \( G \subseteq \mathbb{R}^{2} \), which we have constructed, is of the form
\[
G = \{ (t, y) : c < t < b, \ 0 < y < \varphi(t) \},
\]
where \( \varphi \in C_{1, \infty}(\mathbb{R}) \) is a nowhere linear function. Recall that according to our construction \( \varphi(c) = 0 \) and the function \( \varphi \) strictly increases on the interval \((c, b)\). In addition \( \varphi'(c) > 0 \) and \( \varphi'(b) = 0 \). By Lemma 3 for all \( p \) satisfying (7) we have \( 1_{G} \in A_{p}(\mathbb{R}^{2}) \). Expanding (or contracting) the domain \( G \) in the vertical direction by an appropriate affine map, we can assume that \( \varphi'(c) = 1 \). Let \( G^{*} \) be the domain symmetric to \( G \) with respect to the line \( t = b \). Let \( W = G \cup G^{*} \cup \xi \), where \( \xi \) is the interval with the endpoints \((b, 0)\) and \((b, \varphi(b))\). Take a square \( \Pi \subseteq \mathbb{R}^{2} \) with side length \( 2(b - c) \). We obtain the required domain \( D \) by taking four rigid copies of the domain \( W \) (that is copies obtained by rotation and translation) and gluing them to the sides of the square \( \Pi \) on its outer side. The theorem is proved.

Remark 3. Theorem 3 (and its Corollaries 3, 4) allows the following modification. The property that the boundary of the domain \( D \) does not
contain line intervals can be replaced by the property that $D$ is convex. The author does not know if it is possible to get both this properties simultaneously. The indicated modification follows since there exists a (real) nonconstant function $\varphi \in C^{1,\omega}(\mathbb{T})$ that satisfies (8) and the following condition: the interval $(0, 2\pi)$ is a union of three intervals such that the derivative $\varphi'$ is monotone on each of them. Such a function is constructed by the author in [5] (see the construction before the proof of Theorem 2 of [5]).

It is not clear, even without any assumptions on smoothness of the boundary, if there exists a strictly convex domain $D \subseteq \mathbb{R}^n$, $n \geq 2$, such that $1_D \in \bigcap_{p>1} A_p(\mathbb{R}^n)$ (we call a domain strictly convex if it is convex and its boundary does not contain line intervals).

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