Research Article
An Approximate Solution for a Class of Ill-Posed Nonhomogeneous Cauchy Problems

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In this paper, we consider a nonhomogeneous differential operator equation of first order
\[ u'(t) + Au(t) = f(t) \]
where the coefficient operator \( A \) is linear unbounded and self-adjoint in a Hilbert space. We assume that the operator does not have a fixed sign. We associate to this equation the initial or final conditions
\[ u(0) = \Phi \quad \text{or} \quad u(T) = \Phi. \]
We note that the Cauchy problem is severely ill-posed in the sense that the solution if it exists does not depend continuously on the given data. Using a quasi-boundary value method, we obtain an approximate nonlocal problem depending on a small parameter. We show that regularized problem is well-posed and has a strongly solution. Finally, some convergence results are provided.

1. Introduction

The terms “inverse problems” and “ill-posed problems” have been steadily and surely gaining popularity in modern science since the middle of the 20th century. A little more than fifty years of studying problems of this kind have shown that a great number of problems from various branches of classical mathematics (computational algebra, differential and integral equations, partial differential equations, and functional analysis) can be classified as inverse or ill-posed, and they are among the most complicated ones (since they are unstable and usually nonlinear). At the same time, inverse and ill-posed problems began to be studied and applied systematically in physics, geophysics, medicine, astronomy, and all other areas of knowledge where mathematical methods are used. The reason is that solutions to inverse problems describe important properties of media under study, such as density and velocity of wave propagation, elasticity parameters, conductivity, dielectric permittivity and magnetic permeability, and properties and location of inhomogeneities in inaccessible areas.

Throughout this paper, \( H \) will denote a Hilbert space endowed with the inner product \( \langle \cdot, \cdot \rangle \) and the norm \( \| \cdot \| \). \( A \) is a linear, unbounded self-adjoint operator which has a continuous spectrum with nonfixed sign on \( H \). We assume that \( A \) admits a bounded inverse \( A^{-1} \) within \( H \) and \( f \) is a given function in \( C([0, T], H) \). Let \( T \) be a positive real number and \( \Phi \) is an element in \( H \). We consider the problem of finding a function \( u : [0, T] \rightarrow H \), such that

\[
\begin{align*}
\frac{du}{dt}(t) + Au(t) &= f(t), \quad t \in [0, T], \\
u(0) &= \Phi \quad \text{or} \quad u(T) = \Phi.
\end{align*}
\]

As is known, the nonhomogeneous problem is severely ill-posed in the sense of Hadamard, i.e., solutions do not always exist, and in the case of existence, these do not depend continuously on the given data. In fact, from small noise contaminated physical measurements, the corresponding solutions have large errors. It becomes difficult to do numerical calculations. Hence, a regularization is in order.

There have been many studies on the homogeneous case with the initial condition \( u(0) = \Phi \) or Cauchy problems with final condition \( u(T) = \Phi \) corresponding to \( A \) being of constant sign, using different approaches such as [1–6]. The unhomogeneous has been treated by many authors (see [7]).

We mention that the same problem for the homogeneous equation is treated by Yurchuk and Ababneh [8] and by Bessila [9] by introducing different nonlocal conditions.
In this paper, we introduce into unhomogeneous differential operator equation a nonlocal boundary condition depending on a small parameter $\varepsilon \in [0, 1]$ as follows:

$$
\begin{cases}
  u'(t) + Au(t) = f(t), & t \in [0, T], \\
  eu(0) + (1 - \varepsilon)u(T) = \Phi, & \Phi \in H.
\end{cases}
$$

(2)

Throughout this work, we denote by $\{v_n\}_{n \geq 1}$ the eigenvectors corresponding to positive eigenvalues $\{\lambda_n\}_{n \geq 1}$ and by $\{w_n\}_{n \geq 1}$ the eigenvectors corresponding to negative eigenvalues $\{|\mu_n|\}_{n \geq 1}$ of the operator $A$. The eigenvectors of $A$ $\{\{v_n\}_{n \geq 1}, \{w_n\}_{n \geq 1}\}$ form an orthogonal system in $H$ with $\|v_n\| = 1$ and $\|w_n\| = 1, \forall n \geq 1$.

2. The Approximate Problem

We approximate problem (1) by the following problem:

$$
\begin{cases}
  u'(t) + Au(t) = f(t), & 0 \leq t \leq T, \\
  eu(0) + (1 - \varepsilon)u(T) = \Phi, & \Phi \in H,
\end{cases}
$$

(3)

$$
\begin{align*}
  u_\varepsilon(t) &= \left( \Phi - (1 - \varepsilon) \int_0^T f(s)e^{-A(T-s)}ds \right) \\
  &\quad + \int_0^T g_n(s)e^{-\lambda_n(t-s)}ds \sum_{n \geq 1} \frac{\varepsilon - \lambda_n}{\varepsilon + (1 - \varepsilon)e^{-\lambda_nT}} v_n \\
  &\quad + \int_0^T \sum_{n \geq 1} \frac{\varepsilon - \mu_n}{\varepsilon + (1 - \varepsilon)e^{-\mu_nT}} \delta_n(t-s) w_n \sum_{n \geq 1} \frac{\varepsilon - \mu_n}{\varepsilon + (1 - \varepsilon)e^{-\mu_nT}} \delta_n(t-s) w_n \\
  &\quad + \int_0^T h_n(s)e^{-\delta_n(t-s)}ds \sum_{n \geq 1} \frac{\varepsilon - \mu_n}{\varepsilon + (1 - \varepsilon)e^{-\mu_nT}} \delta_n(t-s) w_n.
\end{align*}
$$

(8)

2.1. Notation. If we denote

$$
\begin{align*}
  M_{\varepsilon,T} &= \left( \varepsilon + (1 - \varepsilon)e^{-\lambda_{\varepsilon}T} \right)^{-1}, \\
  N_{\varepsilon,T} &= \left( \varepsilon e^{-\mu_{\varepsilon}T} + (1 - \varepsilon) \right)^{-1}, \\
  I_\varepsilon(g_n) &= \int_0^T g_n(s)e^{-\lambda_n(t-s)}ds, \\
  I_\varepsilon(h_n) &= \int_0^T h_n(s)e^{-\mu_n(t-s)}ds,
\end{align*}
$$

(9)

then the function $u_\varepsilon(t)$ can be written as follows:

where $A$ is as above and $f \in C([0, T], H)$ is represented by

$$
   f(t) = \sum_{n \geq 1} g_n(t)v_n + \sum_{n \geq 1} h_n(t)w_n,
$$

(4)

where

$$
   g_n(t) = \langle f(t), v_n \rangle, \\
   h_n(t) = \langle f(t), w_n \rangle, \quad \forall n \geq 1.
$$

(5)

The vector $\Phi \in H$ can be represented as follows:

$$
\Phi = \sum_{n \geq 1} \xi_n(t)v_n + \sum_{n \geq 1} \delta_n(t)w_n,
$$

(6)

where $\xi_n = \langle v_n, \Phi \rangle$ and $\delta_n = \langle w_n, \Phi \rangle$.

The classical solution of problem (2) is the function as follows:

$$
\int_0^T \left[ e^{\Delta T} \frac{e^{-A(T-s)}}{\varepsilon + (1 - \varepsilon)e^{-\lambda_{\varepsilon}T}} + \int_0^T f(s)e^{-A(T-s)}ds \right],
$$

(7)

$$
\begin{align*}
  u_\varepsilon(t) &= \sum_{n \geq 1} \left[ \frac{e^{-\lambda_nT}}{\varepsilon + (1 - \varepsilon)e^{-\lambda_{\varepsilon}T}} \right] M_{\varepsilon,T} (\xi_n - (1 - \varepsilon)I_\varepsilon(g_n)) + I_\varepsilon(g_n) \right] v_n \\
  &\quad + \sum_{n \geq 1} \left[ \frac{e^{-\mu_nT}}{\varepsilon + (1 - \varepsilon)e^{-\mu_{\varepsilon}T}} \right] N_{\varepsilon,T} (\delta_n - (1 - \varepsilon)I_\varepsilon(h_n)) + I_\varepsilon(h_n) \right] w_n.
\end{align*}
$$

(10)

Theorem 1.

Problem (3) is well-posed, and its unique solution is $u_\varepsilon(t)$ given by (8). Furthermore,

$$
\sup_{0 \leq t \leq T} \| u_\varepsilon(t) \|^2 \leq \max \left\{ \frac{2}{\varepsilon^2}, \frac{2}{(1 - \varepsilon)^2} \right\} \| \Phi \|^2 + 2T^2 \| f \|^2.
$$

(11)

Proof. For each $0 \leq t \leq T$, we have

$$
I_\varepsilon(g_n) \leq T \| g_n \|, \\
I_\varepsilon(h_n) \leq T \| h_n \|,
$$

(12)

and by using relations (12) and (13), we obtain

$$
\frac{1}{\varepsilon + (1 - \varepsilon)e^{-\lambda_{\varepsilon}T}} \leq \frac{1}{\varepsilon}
$$

(13)
Proof. (i) The value \( (\epsilon (1 - e^{-\lambda T})/\epsilon + (1 - \epsilon)e^{-\lambda T}) \) is positive and increasing with \( \epsilon \), and its derivative with respect to \( \epsilon \) is as follows:

\[
\frac{e^{-\lambda T}}{\epsilon + (1 - \epsilon)e^{-\lambda T}} \geq 0. 
\]

Proof. (ii) It takes its greatest value when \( \epsilon \longrightarrow 1 \), i.e.,

\[
0 \leq \frac{\epsilon (1 - e^{-\lambda T})}{\epsilon + (1 - \epsilon)e^{-\lambda T}} \leq 1 - e^{-\lambda T} \leq 1. \tag{24}
\]

(ii) We know that for all \( \epsilon \in [0, 1] \), we have

\[
0 \leq (1 - \epsilon)e^{-\lambda T} \leq 1, \tag{25}
\]

and then

\[
\frac{\epsilon}{\epsilon + (1 - \epsilon)e^{-\lambda T}} \leq 1. \tag{26}
\]

(iii) On the other hand, the following value:

\[
\frac{\epsilon (\epsilon e^{\lambda T} - 1)}{\epsilon e^{\lambda T} + (1 - \epsilon)} \tag{27}
\]

is negative and decreasing with \( \epsilon \) as long as its derivative with respect to \( \epsilon \) is as follows:

\[
\frac{\epsilon e^{\lambda T} - 1}{(\epsilon e^{\lambda T} + (1 - \epsilon))^2} \leq 0, \tag{28}
\]

and it takes its lowest value at \( \epsilon = (1/2) \), i.e.,

\[
-1 \leq \frac{\epsilon e^{\lambda T} - 1}{\epsilon e^{\lambda T} + (1 - \epsilon)} \leq 0. \tag{29}
\]

(iv) We have also

\[
0 \leq e^{\lambda T} \leq \epsilon e^{\lambda T} + (1 - \epsilon), \tag{30}
\]

and then we obtain relation (19).

(v) For estimate (20), we have

\[
\frac{\epsilon^2 (1 - e^{-\lambda T})^2}{(\epsilon + (1 - \epsilon)e^{-\lambda T})^2} \leq \epsilon, \tag{31}
\]

and then

\[
\frac{\epsilon^2 e^{2\lambda T}}{(\epsilon + (1 - \epsilon)e^{-\lambda T})^2} \leq \frac{\epsilon^2 e^{2\lambda T}}{(1 - \epsilon)^2} \leq \frac{\epsilon^2 e^{2\lambda T}}{(1 - \epsilon)^2}. \tag{32}
\]
(vi) Furthermore, we have
\[
\varepsilon^2 (e^{\mu n T} - 1)^2 \leq \varepsilon^2, \\
\frac{1}{(e^{\mu n T} + (1 - \varepsilon))^2} \leq \frac{1}{(1 - \varepsilon)^2}, \quad (33)
\]
\[
\frac{\varepsilon e^{\mu n T}}{e^{\mu n T} + (1 - \varepsilon)} \leq \frac{\varepsilon}{1 - \varepsilon} \leq 1.
\]
and then we deduce (21) and (22).

\[\square\]

**Theorem 2.**
For all $\Phi \in H$, we have
\[
\lim_{\varepsilon \to 0} \|\Phi - u_\varepsilon (T)\| = 0. \quad (34)
\]

**Proof.** By definitions (6) and (8), we have
\[
\Phi = \sum_{n=1}^{N} \xi_n v_n + \sum_{n=1}^{+\infty} \delta_n w_n, \\
f = \sum_{n=1}^{N} g_n v_n + \sum_{n=1}^{+\infty} h_n w_n,
\]
\[\forall N < \infty.\] Then, by virtue of the Banach–Steinhaus theorem, we obtain

\[
\lim_{\varepsilon \to 0} \|\Phi - u_\varepsilon (T)\| = 0, \quad \forall \Phi \in M, \quad (37)
\]
where $M$ is some special set dense in the Hilbert space $H$. We take all $\Phi$ and $f$ on $M$ of the form as follows:

\[
\left\|\Phi - u_\varepsilon (T)\right\| \leq 2 \left(\left\|\Phi\right\|^2 + T^2 \left\|f\right\|^2\right), \quad (36)
\]

Now, we demonstrate that

\[
\lim_{\varepsilon \to 0} \left\|\Phi - u_\varepsilon (T)\right\| = 0, \quad \forall \Phi \in M,
\]
where $M$ is some special set dense in the Hilbert space $H$. We take all $\Phi$ and $f$ on $M$ of the form as follows:

\[
\left\|\Phi - u_\varepsilon (T)\right\| \leq \sum_{n=1}^{N} \varepsilon^2 \left(1 - \frac{\varepsilon}{e^{\mu n T}}\right)^2 \left\langle v_n, \Phi\right\rangle^2 + 2 \sum_{n=1}^{N} \varepsilon^2 \left(1 - \frac{\varepsilon}{e^{\mu n T}}\right)^2 T^2 \left\langle g_n\right\rangle^2 \\
+ 2 \sum_{n=1}^{+\infty} \varepsilon^2 \left(1 - \frac{\varepsilon}{e^{\mu n T}}\right)^2 \left\langle w_n, \Phi\right\rangle^2 + 2 \sum_{n=1}^{+\infty} \varepsilon^2 \left(1 - \frac{\varepsilon}{e^{\mu n T}}\right)^2 T^2 \left\langle h_n\right\rangle^2, \quad (39)
\]

Combining relations (12), (17), and (20)–(22) (see lemma 1), we obtain
\[
\left\|\Phi - u_\varepsilon (T)\right\| \leq 2 \left(\left\|\Phi\right\|^2 + T^2 \left\|f\right\|^2\right), \quad (40)
\]
which gives that $\lim_{\varepsilon \to 0} \left\|\Phi - u_\varepsilon (T)\right\| = 0.$

**Lemma 2.**
For all $\varepsilon \in ]0, 1[$, we have the following inequalities:
\[
-1 \leq \frac{e^{-\lambda T} - 1}{e^{-\lambda T} + 1} \leq \frac{(\varepsilon - 1)(1 - e^{-\lambda T})}{e + (1 - \varepsilon)e^{-\lambda T}} \leq 0, \quad (41)
\]
\[
1 - \frac{\varepsilon}{1 + \varepsilon} \leq \frac{1 - \varepsilon}{e + (1 - \varepsilon)e^{-\lambda T}}, \quad (42)
\]
\[
0 \leq \frac{(1 - \varepsilon)(1 - e^{\mu T})}{\varepsilon e^{\mu T} + (1 - \varepsilon)} \leq 1 - e^{\mu T} \leq 1, \quad (43)
\]
\[
\frac{(1 - \varepsilon)e^{\mu T}}{\varepsilon e^{\mu T} + (1 - \varepsilon)} \leq 1, \quad (44)
\]
and then by using (12) and (41)–(44), we obtain

\[
\frac{1 - \varepsilon}{\varepsilon + (1 - \varepsilon)e^{-\lambda T}} \leq \frac{1 - \varepsilon}{\varepsilon}
\]

(45)

\[
\frac{(\varepsilon - 1)(1 - e^{-\lambda T})}{\varepsilon + (1 - \varepsilon)e^{-\lambda T}} \leq \frac{(1 - \varepsilon)^2}{\varepsilon^2},
\]

(46)

\[
\frac{(1 - \varepsilon)(1 - e^{\mu T})}{\varepsilon e^{\mu T} + (1 - \varepsilon)} \leq \frac{1 - \varepsilon}{\varepsilon e^{\mu T}},
\]

(47)

\[
\frac{(1 - \varepsilon)e^{\mu T}}{e e^{\mu T} + (1 - \varepsilon)} \leq \frac{1 - \varepsilon}{ee^{\mu T}}
\]

(48)

where \( \mu_N = \min_{\mathbb{R}} \mu_n \).

**Proof.**

(i) For all \( \varepsilon \in [0, 1] \), the following value:

\[
\frac{(\varepsilon - 1)(1 - e^{-\lambda T})}{\varepsilon + (1 - \varepsilon)e^{-\lambda T}}
\]

(49)

is negative and increasing with \( \varepsilon \) since its derivative with respect to \( \varepsilon \) is as follows:

\[
\frac{1 - e^{-\lambda T}}{\varepsilon + (1 - \varepsilon)e^{-\lambda T}} \geq 0,
\]

(50)

and it takes its lowest value when \( \varepsilon = (1/2) \), and then we deduce (41).

(ii) We have also

\[
\frac{1}{1 + \varepsilon} \leq \frac{1}{\varepsilon + (1 - \varepsilon)e^{-\lambda T}} \leq \frac{1}{\varepsilon}
\]

(51)

and then

\[
-1 \leq \frac{1 - \varepsilon}{\varepsilon + (1 - \varepsilon)e^{-\lambda T}} \leq \frac{1}{\varepsilon - 1 + \varepsilon}
\]

(52)

On the other hand, the following function:

\[
\frac{(1 - \varepsilon)(1 - e^{\mu T})}{ee^{\mu T} + (1 - \varepsilon)}
\]

(53)

is positive and decreasing with \( \varepsilon \) since the derivative with respect to \( \varepsilon \) is as follows:

\[
\frac{\varepsilon e^{\mu T}(e^{\mu T} - 1)}{ee^{\mu T} + (1 - \varepsilon)} \leq 0,
\]

(54)

and it takes its greatest value for \( \varepsilon = 0 \), and then we obtain (43).

(iii) Furthermore, we know that for all \( \varepsilon \in ]0, 1[ \),

\[
\frac{1}{ee^{\mu T} + (1 - \varepsilon)} \leq \frac{1}{1 - \varepsilon}
\]

(55)

and then relation (44) is obtained.

(iv) We have

\[
0 \leq (1 - \varepsilon)e^{-\lambda T} \leq 1 - \varepsilon,
\]

(56)

\[
\varepsilon \leq (1 - \varepsilon)e^{-\lambda T} \leq 1 + \varepsilon,
\]

and then we deduce (45) and (46).

(v) Finally, with simple increases, we can easily have inequalities (47) and (48).

\[\Box\]

**Theorem 3.**

For all \( \Phi \in H \), we have

\[
\lim_{\varepsilon \to 1} \|\Phi - u_\varepsilon(0)\| = 0.
\]

(57)

**Proof.** We have

\[
\Phi - u_\varepsilon(0) = \sum_{n=1}^{\infty} \left[ \frac{(\varepsilon - 1)(1 - e^{-\lambda T})}{\varepsilon + (1 - \varepsilon)e^{-\lambda T}} \langle v_n, \Phi \rangle + \frac{1 - \varepsilon}{\varepsilon + (1 - \varepsilon)e^{-\lambda T}} I_n g_n \right] v_n
\]

\[
+ \sum_{n=1}^{\infty} \left[ \frac{(1 - \varepsilon)(1 - e^{\mu T})}{\varepsilon e^{\mu T} + (1 - \varepsilon)} \langle w_n, \Phi \rangle + \frac{(1 - \varepsilon)e^{\mu T}}{\varepsilon e^{\mu T} + (1 - \varepsilon)} I_n h_n \right] w_n
\]

(58)

and then by using (12) and (41)–(44), we obtain

\[
\|\Phi - u_\varepsilon(0)\|^2 \leq 2 \left( \|\Phi\|^2 + T^2\|f\|^2 \right).
\]

(59)

Now, we prove that

\[
\lim_{\varepsilon \to 1} \|\Phi - u_\varepsilon(0)\| = 0, \quad \forall \Phi \in M,
\]

(60)

where \( M \) is a particular set dense in \( H \). We choose \( \Phi \) and \( f \) on \( M \) having the form:

\[
\Phi = \sum_{n=1}^{\infty} \xi_n v_n + \sum_{n=1}^{N} \delta_n w_n,
\]

(61)

\[
f = \sum_{n=1}^{\infty} g_n v_n + \sum_{n=1}^{N} h_n w_n,
\]
\forall N < \infty. \text{ Then,}

\[ \| \Phi - u_\epsilon (0) \|^2 \leq \sum_{n=1}^{\infty} \left\{ \left( 1 - \epsilon \right) (1 - e^{-l_n T}) \left( \langle \nu_n, \Phi \rangle + \frac{1 - \epsilon}{\epsilon (1 - \epsilon) e^{-l_n T}} I_T (g_n) \right)^2 + \frac{1 - \epsilon}{\epsilon (1 - \epsilon) e^{-l_n T}} I_T (g_n) \right\}^2 + \sum_{n=1}^{N} \left\{ \frac{1 - \epsilon}{\epsilon (1 - \epsilon)} \left( 1 - e^{\mu_n T} \right) \left( \langle w_n, \Phi \rangle + \frac{1 - \epsilon}{\epsilon (1 - \epsilon)} e^{\mu_n T} \left( 1 - \epsilon \right) I_T (h_n) \right)^2 \right\}. \]

Using inequalities (45)–(48) in Lemma 2 and relation (12), we obtain that

\[ \| \Phi - u_\epsilon (0) \|^2 \leq 2 \left\{ \frac{(1 - \epsilon)^2}{\epsilon^2} \sum_{n=1}^{\infty} \left\{ \langle \nu_n, \Phi \rangle + T^2 \sum_{n=1}^{\infty} \| g_n \|^2 \right\}^2 + 2 \left\{ \frac{(1 - \epsilon)^2}{\epsilon^2} e^{-2\mu_n T} \left( \| \Phi \|^2 + T^2 \| f \|^2 \right) \right\}. \]

(63)

Finally, according to the Banach–Steinhaus theorem, (57) results from (59) and (63).

\[ \square \]

\textbf{Conclusion 1.} Note that, in this work, using a quasi-boundary value method, we study the nonhomogeneous differential-operator equation introducing nonlocal conditions. The results given in this paper generalized the results of the work given by Yurchuk and Ababneh [8] where they considered the homogeneous case. [10–16]

\textbf{Data Availability}

The content of our article is devoid of any particular data and therefore does not require any provision in this direction.

\textbf{Conflicts of Interest}

The authors declare that they have no conflicts of interest.

\textbf{References}

[1] K. A. Ames, L. E. Payne, and P. W. Schaefer, “Energy and pointwise bounds in some non-standard parabolic problems,” \textit{Proceedings of the Royal Society of Edinburgh: Section A Mathematics}, vol. 134, no. 1, pp. 1–9, 2004.

[2] M. M. Lavrentiev, \textit{Some Improperly Posed Problems of Mathematical Physics}, p. 72, Springer-Verlag, Berlin, Germany, 1967.

[3] K. Miller, “Stabilized quasi-reversibility and other nearly best possible methods for non-well-posed problems,” in \textit{Proceedings of the Symposium on Non-Well-Posed Problems and Logarithmic Convexity}, pp. 161–176, Springer-Verlag, Edinburgh, UK, March 1972.

[4] L. E. Payne, “Some general remarks on improperly posed problems for partial differential equations,” in \textit{Proceedings of the Symposium on Non-Well-Posed Problems and Logarithmic Convexity}, pp. 1–30, Springer-Verlag, Edinburgh, UK, March 1972.

[5] R. E. Showalter, “The final value problem for evolution equations,” \textit{Journal of Mathematical Analysis and Applications}, vol. 47, no. 3, pp. 563–572, 1974.

[6] R. E. Showalter, “Cauchy problem for hyper-parabolic partial differential equations,” in \textit{Trends in the Theory and Practice of Non-linear Analysis}, Elsevier, Amsterdam, Netherlands, 1983.

[7] N. T. Long and A. P. N. Dinh, “Approximation of a parabolic non-linear evolution equation backwards in time,” \textit{Inverse Problems}, vol. 10, no. 4, pp. 905–914, 1994.

[8] N. I. Yurchuk and A. Moussa, “Regularization by nonlocal conditions of the incorrect problems for differential-operator equations of the first order,” \textit{Mathematical Modelling and Analysis}, vol. 97, no. 2, pp. 160–166, 1997.

[9] K. Bessila, “Regularization by a modified quasi-boundary value method of the ill-posed problems for differential-operator equations of the first order,” \textit{Journal of Mathematical Analysis and Applications}, vol. 409, no. 1, pp. 315–320, 2014.

[10] K. A. Ames and L. E. Payne, “Asymptotic behavior for tow regularizations of the Cauchy problem for the backward heat equation,” \textit{Mathematical Models and Methods in Applied Sciences}, vol. 8, no. 1, pp. 187–202, 2011.

[11] M. Ababna, “Regularization by non-local boundary conditions for a control problem by initial condition of evolution operator differential equation,” \textit{Vestnik Beloruskogo Gosudarstvenogo Universiteta. Seriia 1, Fizika, Matematika, Informatika}, vol. 2, pp. 60–63, 1998, in Russian.

[12] G. W. Clark and S. F. Oppenheimer, “Quasireversibility methods for non-well-posed problems,” \textit{Electronic Journal of Differential Equations}, vol. 8, pp. 1–9, 1994.

[13] R. Lattes and J. L. Lions, \textit{Méthode de Quasi-Reversibilité et Applications}, Dunod, Paris, France, 1967.

[14] I. V. Mel’nikova, “Regularization of ill-posed differential problem Sibirsk,” \textit{Sibirskii Matematicheskii Zhurnal}, vol. 33, pp. 126–134, 1989, in Russian.

[15] P. T. Nam, “An approximate solution for nonlinear backward parabolic equations,” \textit{Journal of Mathematical Analysis and Applications}, vol. 367, pp. 337–349, 2010.

[16] D. P. Nguyen, B. Dumitru, T. P. Tran, and L. Le Dinh, “Recovering the source term for parabolic equation with nonlocal integral condition,” \textit{Mathematical Methods in the Applied Sciences}, vol. 44, no. 11, pp. 9026–9041, 2021.