Symmetric Boolean Algebras

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Abstract

We define Boolean algebras in the linear context and study its symmetric powers. We give explicit formulae for products in symmetric Boolean algebras of various dimensions. We formulate symmetric forms of the inclusion-exclusion principle.

Introduction

Fix $k$ a field of characteristic zero. A fundamental fact in mathematics is the existence of a functor $< >: \text{Set} \to \text{vect}$ from the category of sets into the category $k$-vector spaces. The functor $< >: \text{Set} \to \text{vect}$ sends a set $x$ into $< x >$ the free $k$-vector space generated by $x$ and sends a map $f: x \to y$ into the linear transformation $\hat{f}: < x > \to < y >$ whose value on $i \in x$ is $f(i)$.

It is important to notice that both $\text{Set}$ and $\text{vect}$ are symmetric monoidal categories with coproduct and that $< >$ is a monoidal functor that respects coproducts. The monoidal structure on $\text{Set}$ is Cartesian product $\times$ and the coproduct is disjoint union $\sqcup$. The monoidal structure on $\text{vect}$ is tensor product $\otimes$ and the coproduct is direct sum $\oplus$. The restricted functor $< >: \text{FinSet} \to \text{vect}$ is such that the dimension $\dim( < x >)$ of $< x >$ is the cardinality $|x|$ of $x$ for each $x$ in $\text{FinSet}$, the category of finite sets.

Using $< >$ we can transform (combinatorial) set theoretical notions into (finite dimensional) linear algebra notions. For example the linear analogue of a monoid is an associative algebra since for any monoid $x$ the vector space $< x >$ carries the structure of an associative algebra. Similarly, the linear analogue of a group is a Hopf algebra since $< x >$ carries a structure of a Hopf algebra for any group $x$.

Boolean algebras has been known since 1854 and are a cornerstone of modern mathematics. Despite its widespread range of applications we believe the right name for Boolean algebras should have been Boolean monoids. For most mathematicians the word algebra implies a linear structure which is certainly not included in the traditional definition of Boolean algebras. For the purposes of this paper we find useful to make the distinction between Boolean monoids and Boolean algebras. The first goal of this paper is to uncover the linear analogue of Boolean monoids, i.e., we answer the question: what algebraic structure does $< x >$ carry for any Boolean monoid $x$? We will see that there are an infinite number of non-isomorphic Boolean
algebras.

The second goal of this paper is to study the symmetric powers of Boolean algebras. We compute the structural constants of such algebras in various dimensions, and show that each symmetric function can be used to formulate a generalization of the inclusion-exclusion principle for the symmetric powers of Boolean algebra. Our final goal is to propose a logical interpretation for Boolean algebras and pose some open problems.

1 Boolean monoids vs Boolean algebras

We recall the definition of Boolean monoids for definiteness and reader convenience, so that he or she may contrast it with the definition of Boolean algebras given below.

**Definition 1.** A Boolean monoid is a set $B$ together with the data

1. Maps $\cup : B \times B \rightarrow B$, $\cap : B \times B \rightarrow B$, $c : B \rightarrow B$ called union, intersection and complement, respectively.
2. Distinguished elements $e, t \in B$ called the empty and total element, respectively.

This data should satisfy the following identities for $a, b, c \in B$

1. $a \cup b = b \cup a$, $a \cap b = b \cap a$.
2. $a \cup (b \cup c) = (a \cup b) \cup c$, $a \cap (b \cap c) = (a \cap b) \cap c$.
3. $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$, $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$.
4. $a \cup e = a$, $a \cap t = a$.
5. $a \cup a^c = t$, $a \cap a^c = e$.
6. $a \cup (a \cap b) = a$, $a \cap (a \cup b) = a$.

To any set $x$ we associate the Boolean monoid $P(x) = \{a \mid a \subseteq x\}$ where

1. $a \cup b = \{i \in x \mid i \in a \text{ or } i \in b\}$.
2. $a \cap b = \{i \in x \mid i \in a \text{ and } i \in b\}$.
3. $a^c = \{i \in x \mid i \notin a\}$.
4. $t$ is $x$ and $e$ is the empty set $\emptyset$.

Let $[n] = \{1, \ldots, n\}$ and $S_n$ be the group of permutations on $n$ letters. We write $P[n]$ instead of $P([n])$ if no confusion arises. Examples of the form $P(x)$ are essentially the unique models of finite Boolean monoids.
**Theorem 2.** Every finite Boolean monoid is isomorphic to \( P(x) \) for a finite set \( x \).

*Proof.* Let \( B \) be a Boolean monoid. Define a partial order \( \leq \) on \( B \) by letting \( a \leq b \) if and only if \( a \cap b = a \). Let \( x \) be the set of primitive elements or atoms in \( B \), i.e.,

\[
x = \{ a \in A \mid a \neq e \text{ and if } b \leq a \text{ then } b = e \}.
\]

The map \( f : B \rightarrow P(x) \) given by \( f(b) = \{ a \in X \mid a \leq b \} \) is an isomorphism between \( B \) and \( P(x) \). \( \square \)

The Boolean monoids \( P(x) \) are described as follows

**Theorem 3.**

- If \( B \) and \( C \) are Boolean monoids then \( B \times C \) is a Boolean monoid.
- \( P(x) \) is isomorphic to \( P[1]^{\lfloor x \rfloor} \).

For a \( k \)-vector space \( V \) we use the symmetry map \( S : V \otimes V \rightarrow V \otimes V \) given by \( S(x \otimes y) = y \otimes x \) for \( x, y \in V \). The identity map \( I : V \rightarrow V \) is given by \( I(x) = x \) for \( x \in V \). We are ready to define the linear analogue of the notion of Boolean monoids.

**Definition 4.** A Boolean algebra is a \( k \)-vector space \( V \) together with the data

1. Linear maps \( \cup : V \otimes V \rightarrow V, \cap : V \otimes V \rightarrow V, c : V \rightarrow V \) called union, intersection and complement, respectively.
2. Linear maps \( T : k \rightarrow V, E : k \rightarrow V \) called the empty map and total map, respectively.
3. Linear map \( \triangle : V \rightarrow V \otimes V \) called coproduct.
4. Linear map \( \text{ev} : V \rightarrow k \) called evaluation.

The axioms below hold

1. \( \cup = \cup \circ S, \cap = \cap \circ S \).
2. \( \cup \circ (\cup \otimes I) = \cup \circ (I \otimes \cup), \cap \circ (\cap \otimes I) = \cap \circ (I \otimes \cap) \).
3. \( \cap \circ (I \otimes \cup) = \cup \circ (\cap \otimes \cap) \circ (I \otimes S \otimes I) \circ (\triangle \otimes I \otimes I), \cup \circ (I \otimes \cap) = \cap \circ (\cup \otimes \cup) \circ (I \otimes S \otimes I) \circ (\triangle \otimes I \otimes I) \).
4. \( \cup \circ (I \otimes E) = I, \cap \circ (I \otimes T) = I \).
5. \( \cap \circ (I \otimes c) \circ \triangle = E \circ \text{ev}, \cup \circ (I \otimes c) \circ \triangle = T \circ \text{ev} \).
6. \( \cap \circ (I \otimes \cup) \circ (\triangle \otimes I) = I \otimes \text{ev}, \cup \circ (I \otimes \cap) \circ (\triangle \otimes I) = I \otimes \text{ev} \).
7. \( (\triangle \otimes I) \circ \triangle = (I \otimes \triangle) \circ \triangle \).
8. \( S \circ \triangle = \triangle \).

Our next result guarantees the existence of infinitely many models of Boolean algebras, namely those naturally associated with Boolean monoids.
Theorem 5. $< P(x) >$ is a Boolean algebra for any set $x$.

Proof. The structural maps are given by

1. $(\Sigma_{a \subseteq x} v_a a) \cup (\Sigma_{b \subseteq x} w_b b) = \Sigma_{a \subseteq x, b \subseteq x} v_a w_b (a \cup b)$.
2. $(\Sigma_{a \subseteq x} v_a a) \cap (\Sigma_{b \subseteq x} w_b b) = \Sigma_{a \subseteq x, b \subseteq x} v_a w_b (a \cap b)$.
3. $c(\Sigma_{a \subseteq x} v_a a) = \Sigma_{a \subseteq x} v_a c(a)$.
4. $\Delta(\Sigma_{a \subseteq x} v_a a) = \Sigma_{a \subseteq x} v_a (a \otimes a)$.
5. $e(s) = s\emptyset$, for $s \in k$.
6. $t(s) = sx$, for $s \in k$.
7. $ev(\Sigma_{a \subseteq x} v_a a) = \Sigma_{a \subseteq x} v_a a$.

Next result characterizes finite dimensional Boolean algebras of the form $< P(x) >$.

Theorem 6.

• If $V$ and $W$ are Boolean algebras then $V \otimes W$ is a Boolean algebra.

• $< P(x) > = < P[1] >^{\otimes |x|}$.

Proof. The Boolean operations on $V \otimes W$ define component wise. \qed

Theorem 6 suggests the following open problems.

Problem 1. Is any Boolean algebra isomorphic to $< B >$ for some Boolean monoid $B$?

Problem 2. Classify all finite dimensional Boolean algebras.

Problem 3. Is any finite dimensional Boolean algebra isomorphic to $< P[n] >$ for some $n \in \mathbb{N}$?

2 Boolean prop

In this section we give a scientific explanation for our choice of axioms for Boolean algebras. We do so by defining the prop in $\text{vect}$ whose algebras are Boolean algebras and showing that this prop actually comes from a prop in $\text{Set}$ whose algebras are Boolean monoids. Discovering the prop that controls a given family of algebras is like unveiling its genetic code [1], [4], [5], [6], [7]. Despite the fact that Boolean algebras have been extensively studied from a myriad of viewpoints its genetic code has not been study so far. Since the theory of props is not widely known we provide an overview using a convenient notation for our purposes. We define props over a symmetric monoidal category $\mathcal{C}$ [1], but the reader should bear in mind that in this work $\mathcal{C}$ is either $\text{Set}$ or $\text{vect}$.

\footnote{For technical reasons we assume that objects of $\mathcal{C}$ are sets and that $\mathcal{C}(1, x) = x$ for $x$ an object of $\mathcal{C}$. We also assume that $\mathcal{C}$ admits finite colimits.}
**Definition 7.** A prop over $\mathcal{C}$ is a symmetric monoidal category $\mathcal{P}$ enriched over $\mathcal{C}$ such that 1) $\text{Ob}(\mathcal{P}) = \mathbb{N}$. 2) The monoidal structure is addition.

Let $\text{Prop}_\mathcal{C}$ be the category whose objects are props over $\mathcal{C}$. Morphisms in $\text{Prop}_\mathcal{C}$ are monoidal functors.

By definition each prop $\mathcal{P}$ is provided with the following data

- For each $n \in \mathbb{N}$, a group morphisms $S_n \rightarrow \mathcal{P}(n, n)$ such that the diagram

$$
\begin{array}{ccc}
S_n \times S_m & \rightarrow & S_{n+m} \\
\downarrow & & \downarrow \\
P(n,n) \otimes \mathcal{C} P(m,m) & \rightarrow & P(n+m,n+m)
\end{array}
$$

commutes. The maps $S_n \rightarrow \mathcal{P}(n,n)$ induce a right action of $S_n$ on $\mathcal{P}(n,m)$ and a left action of $S_m$ on $\mathcal{P}(n,m)$.

- Let $\mathbb{B}$ be the category whose objects are finite sets and whose morphisms are bijections. The actions constructed above are used to define a functor $\mathcal{P} : \mathbb{B}^{\text{op}} \times \mathbb{B} \rightarrow \mathcal{C}$ given by

$$
\mathcal{P}(a,b) = \mathbb{B}(a,|a|) \times_{S_{|b|}} \mathcal{P}(|a|,|b|) \times_{S_{|b|}} \mathbb{B}(|b|,b).
$$

- Morphisms $\mathcal{P}(n,m) \otimes \mathcal{C} P(m,k) \rightarrow \mathcal{P}(n,k)$ for $n,m,k \in \mathbb{N}$.

- Morphisms $\mathcal{P}(n,m) \otimes \mathcal{C} P(k,l) \rightarrow \mathcal{P}(n+k,m+l)$ for $n,m,k,l \in \mathbb{N}$.

In order to define the free prop generated by a functor $G : \mathbb{B}^{\text{op}} \times \mathbb{B} \rightarrow \mathcal{C}$ we need some combinatorial notions.

**Definition 8.** A digraph $\Gamma$ consists of the following data

1. A pair of finite sets $(V_\Gamma, E_\Gamma)$ called the set of vertices and edges of $\Gamma$, respectively.

2. A map $(s, t) : E_\Gamma \rightarrow V_\Gamma \times V_\Gamma$. We call $s(e)$ and $t(e)$ the source and target of $e \in V_\Gamma$, respectively.

We use the notations $\text{in}(v) = \{e \mid t(e) = v\}$, $\text{i}(v) = |\text{in}(v)|$, $\text{out}(v) = \{e \mid s(e) = v\}$, and $\text{o}(v) = |\text{out}(v)|$. The valence of $v \in V_\Gamma$ is $\text{val}(v) = (\text{i}(v), \text{o}(v)) \in \mathbb{N}^2$. Also we introduce the notation $V_{\Gamma,\text{in}} = \{v \in V_\Gamma \mid \text{i}(v) = 0\}$ and $V_{\Gamma,\text{out}} = \{v \in V_\Gamma \mid \text{o}(v) = 0\}$. Digraphs considered in this work do not have oriented cycles. An oriented cycle in $\Gamma$ is a sequence $e_1, \ldots, e_n$ of edges in $\Gamma$ such that $t(e_i) = s(e_{i+1})$ for $1 \leq i \leq n - 1$ and $t(e_n) = s(e_1)$.

**Definition 9.** Let $a$ and $b$ be finite sets. An $(a,b)$-digraph is a triple $(\Gamma, \alpha, \beta)$ such that

1. $\Gamma$ is a digraph.

2. $\alpha : a \rightarrow V_{\Gamma,\text{in}}$ is an injective map.
3. $\beta : b \to V_{\Gamma, \text{out}}$ is an injective map.

Let $\text{Digraph}(a, b)$ be the groupoid of $(a, b)$-digraphs. A functor $G : \mathbb{B}^{\text{op}} \times \mathbb{B} \to C$ induces a functor $G : \text{Digraph}(a, b) \to C$ given by

$$G(\Gamma) = \bigotimes_{v \in V_{\Gamma, \text{int}}} G(\text{in}(v), \text{out}(v)),$$

$t$ an object of $\text{Digraph}(a, b)$ and $V_{\Gamma, \text{int}} = V_{\Gamma} - (\alpha(a) \sqcup \beta(b))$.

**Definition 10.** The prop $P_G$ freely generated by $G : \mathbb{B}^{\text{op}} \times \mathbb{B} \to C$ is given for $n, m \in \mathbb{N}$ by

$$P_G(n, m) := \text{colim} G(\Gamma),$$

the colimit is taken over the groupoid $\text{Digraph}([n], [m])$. Compositions in $P_G$ are given by gluing digraphs.

To define props via generators and relations we need to know what the analogue of an ideal in the prop context is.

**Definition 11.** A subcategory $I$ of $P$ is a prop ideal if

1. $\text{Ob}(I) = \text{Ob}(P)$.
2. $I(n, m) \otimes P(m, k) \to I(n, k)$, $P(n, m) \otimes I(m, k) \to I(n, k)$.
3. $I(n, m) \otimes P(k, l) \to I(n \sqcup k, m \sqcup l)$, $P(n, m) \otimes I(k, l) \to I(n \sqcup k, m \sqcup l)$.

for $n, m, k, l \in \mathbb{N}$.

We are ready to define Boole as an object in $\text{PropSet}$ . The prop Boole is a quotient by a prop ideal $I_B$ of the prop freely generated by vertices

representing union, intersection, complement, coproduct, the empty element, the total element and the valuation, respectively. The prop ideal $I_B$ is generated by the relations given below, each corresponding with an axiom in the definition of Boolean algebras.

1. Commutativity for union and intersection

$$\begin{array}{c}
\bullet \\
\lor \\
\sqcup \\
\boxplus
\end{array} = 
\begin{array}{c}
\bigcirc \\
\land \\
\cap \\
\boxdot
\end{array}$$
2. Associativity for union and intersection

\[ \begin{align*}
\text{ Associativity } & \quad \text{ for union and intersection} \\
\text{ Diagram 1 } & \quad \text{ for union and intersection} \\
\end{align*} \]

3. Distributivity laws

\[ \begin{align*}
\text{ Distributivity laws } & \quad \text{ for union and intersection} \\
\text{ Diagram 2 } & \quad \text{ for union and intersection} \\
\end{align*} \]

4. Properties of the empty and total elements

\[ \begin{align*}
\text{ Properties of the empty and total elements } & \quad \text{ for union and intersection} \\
\text{ Diagram 3 } & \quad \text{ for union and intersection} \\
\end{align*} \]

5. Absorption Laws

\[ \begin{align*}
\text{ Absorption Laws } & \quad \text{ for union and intersection} \\
\text{ Diagram 4 } & \quad \text{ for union and intersection} \\
\end{align*} \]

6. Coassociativity and cocommutativity

\[ \begin{align*}
\text{ Coassociativity and cocommutativity } & \quad \text{ for union and intersection} \\
\text{ Diagram 5 } & \quad \text{ for union and intersection} \\
\end{align*} \]

\[ \text{ Definition 12. For } x \in \text{Ob}(\mathcal{C}) \text{ we let } \text{End}_x^\mathcal{C} \text{ be the prop given by } \text{End}_x^\mathcal{C}(n, m) = \mathcal{C}(x^\otimes n, x^\otimes m), \text{ for } n, m \in \mathbb{N}. \]

\[ \text{ Definition 13. Let } P \text{ be a prop over } \mathcal{C}. \text{ A } P\text{-algebra is a pair } (x, r), \text{ where } x \text{ is an object of } \mathcal{C} \text{ and } r : P \to \text{End}_x^\mathcal{C} \text{ is a prop morphism.} \]

\[ \text{ In practice a } P\text{-algebra } x \text{ is given by a family of maps } r : P(n, m) \to \mathcal{C}(x^\otimes n, x^\otimes m) \text{ satisfying some compatibility conditions.} \]

\[ \text{ Theorem 14. } B \text{ is a Boole-algebra in Set if and only if } x \text{ is a Boolean monoid.} \]

\[ \text{ Proof. Assume that } (B, r) \text{ is a Boole-algebra in Set where } r : \text{Boole} \to \text{End}_B^{\text{Set}} \text{ is a prop morphism. The images under } r \text{ of the generators of Boole give operations } \cup, \cap, (\ ), t, e, \Delta, ev, \text{ respectively. For example } t : \{1\} \to B \text{ and } e : \{1\} \to B \text{ are identified with elements of } B. \]
ev : B → {1} is the constant map and plays no essential part in this story.

We also get a map ∆ : B → B × B which does seem to fit into the definition of Boolean monoids. Assume that ∆ is given by ∆(a) = (f(a), g(a)) for a ∈ B. We use the relations in Boole. The cocommutativity graph implies that f = g. The coassociativity graph implies that f² = f. One of the absorption graphs implies the identity f(a) ∪ (f(a) ∩ b) = a for a, b ∈ B. Thus we obtain

\[ f(a) = f²(a) ∪ (f²(a) ∩ b) = f(a) ∪ (f(a) ∩ b) = a. \]

Thus ∆(a) = (a, a) and it is a simple check that all other relations in Boole turn B into a Boolean monoid.

Assume that B is a Boolean monoid with operations ∪, ∩, ( ), t, and distinguished elements t and e that may be thought as maps from {1} to B. Take ev to be the constant map from B to {1}, and let ∆ be given by ∆(a) = (a, a). Let r be the map assigning to each generator of the Boole prop the corresponding map from the list above. The fact that B is a Boolean monoid guarantees that all the relations defining Boole are satisfied and r extends to a prop morphism r : Boole → End₂^{Set}.

\[ \square \]

Notice that the functor < > : Set → vect induces a functor < > : PROP_{Set} → PROP_{vect} given by < P > (n, m) =< P(n, m) > for n, m ∈ N. The following result follows from the fact that each generator of the Boole prop correspond with an operation on Boolean algebras and each relation in the prop ideal I_B corresponds with an axiom in the definition of Boolean algebras.

**Theorem 15.** V is a < Boole >-algebra in vect if and only if V is a Boolean algebra.

### 3 Symmetric powers of Boolean algebras

The following ideas introduced in [2] are useful for studying the symmetric powers of Boolean algebras. Suppose that a group G acts by automorphisms on the k-algebra A. The space of co-invariants A/G = A/ < ga − a | g ∈ G and a ∈ A > is a k-algebra with product given by

\[ \overline{ab} = \frac{1}{|G|} \sum_{g \in G} a(gb). \]

For each subgroup K ⊂ S_n the Polya functor P_K : k-alg → k-alg from the category of associative k-algebras into itself is defined by: if A is a k-algebra then P_K(A) denotes the k-algebra whose underlying vector space is

\[ P_K(A) = (A \otimes^n)/(a_1 \otimes \cdots \otimes a_n - a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)} : a_i \in A, \sigma \in K). \]

The rule for the product of m elements in P_K(A) is provided by our next result.
Theorem 16. For any \( \{a_{ij}\}_{i=1,j=1}^{m,n} \subseteq A \) the following identity holds in \( P_K(A) \)

\[
|K^{m-1} \prod_{i=1}^{m} \bigotimes_{j=1}^{n} a_{ij} \rangle = \sum_{\sigma \in \{id\} \times K^{m-1}} \prod_{j=1}^{m} a_{i_{\sigma^{-1}(j)}}
\]

In particular for each algebra \( A \) and each positive integer \( m \) the Polya functor \( P_{Sy_n} \) yields an algebra \( P_{Sy_n}(A) \) which we denote by \( Sym^n(A) \). Recall that \( < P[k] > \) denotes the \( k \)-vector space generated by the subsets of \( [k] \). The structural maps \( \cup, \cap, \) and \( (\cdot)^c \) for \( < P[k] > \) are the linear extension of the union, intersection, and complement on \( P[k] \).

Definition 17. \( Sym^m < P[k] > \) is called the symmetric Boolean algebra of type \((m,k)\). It has operation of union, intersection, and complement induced by the corresponding operators in \( < P[k] > \).

\( S_x \) acts by automorphisms on \( < P(x) > \) for any finite set \( x \). The next result gives a characterization of the algebra of co-invariants \( < P(x) > /S_x \).

Proposition 18. 1. \( < P(x) > /S_x \cong < P[1] >^{s_{|x|}} / S_{|x|} = Sym_{|x|} < P[1] > \).

2. \( \dim(< P(x) > /S_x) = |x| + 1 \).

A basis for \( < P[k] > /S_{k} \) is given by \( \hat{0},...\hat{k} \) where \( \hat{i} \) denotes the equivalence class of \( [i] \subseteq [k] \). Now we study in details the operation of union, intersection, and complements on the space \( < P[k] > /S_{k} \). Below we use the notation \( P(x,k) := \{ c \in P(x) : |c| = k \} \) for any set \( x \).

Theorem 19. For \( \hat{a}, \hat{b} \) in the basis of \( < P[k] > /S_{k} \), we have

1. Let \( m = \min(k-a,b) \),

\[
\hat{a} \cup \hat{b} = \frac{1}{k} \sum_{l=0}^{m} \binom{a}{b-l} \binom{k-a}{l} \hat{a} + l,
\]

2. Let \( m = \min(a,b) \),

\[
\hat{a} \cup \hat{b} = \frac{1}{k} \sum_{l=0}^{m} \binom{a}{l} \hat{l}.
\]

3. \( (\hat{a})^c = \hat{k} - \hat{a} \).

Proof. 1. \[
\hat{a} \cup \hat{b} = \frac{1}{k!} \sum_{\sigma \in S_k} \langle a \rangle \cup \sigma[\hat{b}] = \frac{1}{k} \sum_{c \in P([k],b)} \langle a \rangle \cup c = \frac{1}{k} \sum_{c_0 \subseteq P([k]-[a],l)} a \cup c_0 = \frac{1}{k} \sum_{c_1 \subseteq P([a],b-l)} \binom{a}{b-l} \binom{k-a}{l} \hat{a} + l.
\]
2. Follows from the fact that the number of permutations \( \sigma \in S_k \) such that \( |a \cap \sigma([b])| = l \) is given by

\[
\binom{a}{l} \binom{b}{l}! \binom{k - a}{b - l} (b - l)! (k - b)!
\]

3. Obvious.

Let \( \pi = \{b_1, ..., b_k\} \) be a partition of \( x \) and \( S_\pi \subseteq S_x \) the Young subgroup consisting of block preserving permutations of \( x \). Our next result characterizes algebras of the form \( \langle P(x) \rangle / S_\pi \).

**Proposition 20.**

1. \( \langle P(x) \rangle / S_\pi \cong \bigotimes_{i=1}^{k} \langle P[b_i] \rangle / S[b_i] = \bigotimes_{i=1}^{k} \text{Sym}^{[b_i]} < P[1] > \).

2. \( \dim(\langle P(x) \rangle / S_\pi) = \prod_{i=1}^{k} (|b_i| + 1) \).

We close this section by taking a closer look at the symmetric Boolean algebra \( \text{Sym}^2 < P[1] > \) and the cyclic Boolean algebra \( \langle P[1] \rangle \otimes_3 / \mathbb{Z}_3 \).

The space \( \text{Sym}^2 < P[1] > \) has basis \( \hat{0} = ([0], [0]), \hat{1} = ([1], [0]) \) and \( \hat{2} = ([1], [1]) \). The union \( \cup : \text{Sym}^2 < P[1] > \otimes \text{Sym}^2 < P[1] > \rightarrow \text{Sym}^2 < P[1] > \) is given for \( i = 0, 1, 2 \) by

- \( \hat{0} \cup \hat{i} = \hat{i} \).
- \( \hat{1} \cup \hat{1} = \frac{1}{2} \hat{1} + \frac{1}{2} \hat{2} \).
- \( \hat{2} \cup \hat{1} = \hat{2} \).

The intersection \( \cap : \text{Sym}^2 < P[1] > \otimes \text{Sym}^2 < P[1] > \rightarrow \text{Sym}^2 < P[1] > \) is given for \( i = 0, 1, 2 \) by

- \( \hat{0} \cap \hat{i} = \hat{0} \).
- \( \hat{1} \cap \hat{1} = \frac{1}{2} \hat{0} + \frac{1}{2} \hat{1} \).
- \( \hat{2} \cap \hat{1} = \hat{2} \).

The complement \( (\ )^c : \text{Sym}^2 < P[1] > \rightarrow \text{Sym}^2 < P[1] > \) is given by

- \( \hat{0}^c = \hat{2}, \hat{1}^c = \hat{1}, \) and \( \hat{2}^c = \hat{0} \).

Although the algebra \( \text{Sym}^2([1]) \) does not satisfy all axioms required to make it into a Boolean algebra (absorption fails) it does share many of the properties of Boolean algebras, and in any event it is a mathematical object of great interest.

Let us consider in details the third cyclic power of the Boolean algebra \( \langle P[1] \rangle \), namely \( \langle P[1] \rangle \otimes_3 / \mathbb{Z}_3 \). It has basis \( \hat{0} = ([0], [0]), \hat{1} = ([1], [0]), \hat{2} = ([1], [1], [0]) \) and \( \hat{3} = ([1], [1], [1]) \). The union \( \cup : \langle P[1] \rangle \otimes_3 / \mathbb{Z}_3 \langle P[1] \rangle \otimes_3 / \mathbb{Z}_3 \rightarrow \langle P[1] \rangle \otimes_3 / \mathbb{Z}_3 \) is given for \( i = 0, 1, 2, 3 \) by

- \( \hat{0} \cup \hat{i} = \hat{i} \).
\[ \hat{1} \cup \hat{1} = \frac{1}{3} \hat{1} + \frac{2}{3} \hat{2}. \]
\[ \hat{1} \cup \hat{2} = \frac{2}{3} \hat{2} + \frac{1}{3} \hat{3}. \]
\[ \hat{2} \cup \hat{2} = \frac{1}{3} \hat{1} + \frac{2}{3} \hat{3}. \]
\[ \hat{3} \cup \hat{i} = \hat{i}. \]

The intersection \( \cap : \langle P[1] > \otimes^3 / \mathbb{Z}_3 \rangle \rightarrow \langle P[1] > \otimes^3 / \mathbb{Z}_3 \rangle \) is given for \( i = 0, 1, 2, 3 \) by
\[ \begin{align*}
\hat{0} \cap \hat{1} &= \frac{2}{3} \hat{0} + \frac{1}{3} \hat{1}.
\hat{1} \cap \hat{2} &= \frac{2}{3} \hat{0} + \frac{1}{3} \hat{2}.
\hat{1} \cap \hat{3} &= \frac{1}{3} \hat{1} + \frac{2}{3} \hat{3}.
\hat{2} \cap \hat{3} &= \hat{3}.
\end{align*} \]

The complement map \((\quad)^c : P([1]) \otimes^3 / \mathbb{Z}_3 \rightarrow P([1]) \otimes^3 / \mathbb{Z}_3\) is given by
\[ \begin{align*}
[0]^c &= [3],
[1]^c &= [2],
[2]^c &= [1], \text{ and } [3]^c = [0].
\end{align*} \]

### 4 Symmetric inclusion-exclusion principles

In this Section we take \( k = \mathbb{R}. \) We write \( \{a_1, \ldots, a_m\} \) for the basis element \( a_1 \otimes \cdots \otimes a_m \in \langle P[k] > \otimes^m / S_m \rangle. \) The following result follows from Theorem 16.

**Theorem 21.** Let \( \{a_1, \ldots, a_m\} \) be in the basis of \( \langle P[k] > \otimes^m / S_m \rangle \) for \( 1 \leq i \leq n. \) The union in \( \langle P[k] > \otimes^m / S_m \rangle \) is given by
\[
\bigcup_{i=1}^{n} \{a_1^i, \ldots, a_m^i\} = \frac{1}{(m!)^{n-1}} \sum_{\sigma \in \{1\} \times S_{m}^{(n-1)}} \{ \bigcup_{i=1}^{n} a_{\sigma_i(1)}^i, \ldots, \bigcup_{i=1}^{n} a_{\sigma_i(m)}^i \}.
\]

**Example 22.** For \( m, n = 2 \) we get
\[
\{a_1, a_2\} \cup \{a_1^2, a_2^2\} = \frac{1}{2} \{a_1^1 \cup a_2^1, a_2^1 \cup a_2^2\} + \frac{1}{2} \{a_1^1 \cup a_2^2, a_2^1 \cup a_2^1\}.
\]

In a better notation
\[
\{a, b\} \cup \{c, d\} = \frac{1}{2} \{a \cup c, b \cup d\} + \frac{1}{2} \{a \cup d, b \cup c\}.
\]
A measure on a finite set $x$ is a map $\mu : P(x) \to \mathbb{R}$ such that $\mu(a \cup b) = \mu(a) + \mu(b)$ for $a, b \subseteq x$ disjoint. Fix a measure $\mu$ on $[k]$. An element $\{a_1, ..., a_n\}$ in the basis of $< P[k] >^\otimes / S_m$ determines a vector $(\mu(a_1), ..., \mu(a_n)) \in \mathbb{R}^m / S_m$. Functions on $\mathbb{R}^m / S_m$ are known as symmetric functions. There are many interesting examples of polynomial symmetric functions such as the power functions, the elementary symmetric functions, the Schur functions and so on. For example the polynomial $x_1^2 + \cdots + x_m^2$ is $S_m$-invariant. Each symmetric function can be used to obtain a symmetric form of the inclusion-exclusion principle. The reader will find interesting information on the inclusion-exclusion principle and its generalizations in several papers by Rota and his collaborators in [3]. We use the inclusion-exclusion principle in the following form.

**Proposition 23.** Let $a_1, ..., a_n \in P(x)$ then $|\bigcup^n_{i=1} a_i| = \sum_{I \subseteq [n]} (-1)^{|I|+1} |\bigcap_{i \in I} a_i|.$

We consider the symmetric inclusion-exclusion principles derived from the power, elementary, and homogeneous symmetric functions. Other symmetric functions can be used as well but we shall not do so here. The power function $p_l : < P[k] >^\otimes / S_m \to \mathbb{R}$ is given on the basis by $p_l(\{a_1, ..., a_m\}) = \sum_{i=1}^m \mu(a_i)^l$. We use the power functions $p_l$ to get a symmetric form of the inclusion-exclusion principle.

**Theorem 24.** Let $\{a_1^i, ..., a_m^i\}$ be in the basis of $< P[k] >^\otimes / S_m$ for $1 \leq i \leq n$. Then

$$p_l(\bigcup^n_{i=1} a_i^i) = \frac{1}{(m!)^{n-1}} \sum_{\sigma \in \{1\} \times S_{n-1}^m} \mu(\bigcup_{i=1}^n a_i^\sigma) l!$$

Proof.

$$p_l(\bigcup^n_{i=1} a_i^i) = \frac{1}{(m!)^{n-1}} \sum_{\sigma \in \{1\} \times S_{n-1}^m} \mu(\bigcup_{i=1}^n a_i^\sigma) l!$$

$$= \frac{1}{(m!)^{n-1}} \sum_{\sigma \in \{1\} \times S_{n-1}^m} \left( \sum_{I \subseteq [n]} (-1)^{|I|+1} \mu(\bigcap_{i \in I} a_{\sigma_i(j)}) \right) l!$$

$$= \frac{1}{(m!)^{n-1}} \sum_{\sigma \in \{1\} \times S_{n-1}^m} \left( \sum_{\Sigma c_I = l} \left( \prod_{I \subseteq [n]} \left( \bigcap_{i \in I} a_{\sigma_i(j)} \right) \right) \right)$$

$$= \frac{1}{(m!)^{n-1}} \sum_{\sigma \in \{1\} \times S_{n-1}^m} \left( \prod_{\Sigma c_I = l} \left( \bigcap_{i \in I} a_{\sigma_i(j)} \right) \right)$$

$\square$

---

\(^2\) In [3] Gessel uses the name symmetric inclusion-exclusion to refer to a different mathematical gadget.
Corollary 25. For \( l = 1 \) we have
\[
p_1\left(\bigcup_{i=1}^{n}\{a_i^1, \ldots, a_i^m\}\right) = \frac{1}{(m!)^{n-1}} \sum_{\sigma \in \{1\} \times S_{m}^{n-1} \atop j \in \{1, \ldots, m\} \atop I \subseteq [n]} (-1)^{|I|+1} \mu(\bigcap_{i \in I} a_{\sigma(j)}^i).
\]

Corollary 26. For \( l = 1, n = 2 \) we have
\[
p_1(\{a_1^1, \ldots, a_m^1\} \cup \{a_1^2, \ldots, a_m^2\}) = \frac{1}{m!} \sum_{\sigma \in S_m} \mu(a_1^j) + \mu(a_2^j) - \mu(a_1^j \cap a_2^j).
\]

A generalized inclusion-exclusion principle using the elementary symmetric functions
\[
e_l(x_1, \ldots, x_m) = \sum_{1 \leq t_1 < t_2 < \cdots < t_l \leq m} \prod_{j=1}^{l} x_{t_j}.
\]
is given by

Theorem 27. Let \( \{a_1^i, \ldots, a_m^i\} \) be in the basis of \( < P[k] >^\otimes /S_m \) for \( 1 \leq i \leq n \). Then
\[
e_l\left(\bigcup_{i=1}^{n}\{a_i^1, \ldots, a_i^m\}\right) = \frac{1}{(m!)^{n-1}} \sum_{\sigma \in \{1\} \times S_{m}^{n-1} \atop 1 \leq t_1 < t_2 < \cdots < t_l \leq m \atop f:[l] \to P([n])} \prod_{j=1}^{l} (-1)^{|f(j)|+1} \mu(\bigcap_{i \in f(j)} a_i^{\sigma(j)}).
\]

Proof. According to Theorem 21 we get
\[
e_k\left(\bigcup_{i=1}^{n}\{a_i^1, \ldots, a_i^m\}\right) = \frac{1}{(m!)^{n-1}} \sum_{\sigma \in \{1\} \times S_{m}^{n-1} \atop 1 \leq p_1 < p_2 < \cdots < p_l \leq m} \prod_{j=1}^{l} \mu(\bigcap_{i \in I} a_i^{\sigma_i(p_j)}).
\]
\[
= \frac{1}{(m!)^{n-1}} \sum_{\sigma \in \{1\} \times S_{m}^{n-1} \atop 1 \leq p_1 < p_2 < \cdots < p_l \leq m} \prod_{j=1}^{l} \sum_{I \subseteq [n]} (-1)^{|I|+1} \mu(\bigcap_{i \in I} a_i^{\sigma_i(p_j)})
\]
\[
= \frac{1}{(m!)^{n-1}} \sum_{\sigma \in \{1\} \times S_{m}^{n-1} \atop 1 \leq p_1 < p_2 < \cdots < p_l \leq m \atop f:[l] \to P([n])} \prod_{j=1}^{l} (-1)^{|f(j)|+1} \mu(\bigcap_{i \in f(j)} a_i^{\sigma_i(p_j)})
\].
Example 28. Let \( n = 2, m = 2 \) and \( l = 2 \). The map \( e_2 : < P[k] >^2 / S_2 \rightarrow \mathbb{R} \) is given by \( e_2(\{a, b\}) = \mu(a)\mu(b) \) for \( a, b \in P[k] \). Theorem 29 implies that

\[
2e_2(\{a, b\} \cup \{c, d\}) = 2\mu(a)\mu(b) + 2\mu(c)\mu(d) + \mu(a)\mu(d) + \mu(c)\mu(b) + \mu(a)\mu(c) + \mu(d)\mu(c) + \mu(a)\mu(b) + \mu(c)\mu(b) + \mu(a)\mu(b) + \mu(c)\mu(b \cap d) + \mu(a)\mu(b \cap c) + \mu(d)\mu(a \cap c) + \mu(a)\mu(b \cap c).
\]

The generalization of the inclusion-exclusion principle using the homogenous symmetric functions

\[
h_l(x_1, ..., x_m) = \sum_{1 \leq i_1 < i_2 < \cdots < i_l \leq m} \prod_{j=1}^{i_l} x_{i_j}.
\]

is given by

Theorem 29. Let \( \{a^i_1, ..., a^i_m\} \) be in the basis \( < P[k] >^m / S_m \) for \( 1 \leq i \leq n \). Then

\[
h_l(\bigcup_{i=1}^{n} \{a^i_1, ..., a^i_m\}) = \frac{1}{(m!)^{n-1}} \sum_{\sigma \in \{1\} \times S_m^{m-1}} \prod_{j=1}^{l} (-1)^{|f(j)|+1} \mu(\bigcap_{i \in f(j)} a^i_{\sigma(j_i)}).
\]

Example 30. Let \( n = 2, m = 2 \) and \( l = 2 \). The map \( h_2 : < P([k]) >^2 / S_2 \rightarrow \mathbb{R} \) is given by \( h_2(\{a, b\}) = \mu(a)^2 + \mu(a)\mu(b) + \mu(b)^2 \) for \( a, b \in P[k] \). Theorem 29 implies that

\[
2h_2(\{a, b\} \cup \{c, d\}) = [\mu(a) + \mu(c) - \mu(a \cap c)]^2 + [\mu(b) + \mu(d) - \mu(b \cap d)]^2 + [\mu(a) + \mu(d)]^2 + [\mu(b) + \mu(c)]^2 + [\mu(b) + \mu(c) - \mu(b \cap c)]^2 + 2\mu(a)\mu(b) + 2\mu(c)\mu(a) + \mu(a)\mu(d) + \mu(c)\mu(b) + \mu(a)\mu(c) + \mu(d)\mu(a) - \mu(a)\mu(b \cap d) + \mu(c)\mu(b \cap d) + \mu(b)\mu(a \cap c) + \mu(d)\mu(a \cap c) + \mu(a)\mu(b \cap c) + \mu(d)\mu(b \cap c) + \mu(b)\mu(a \cap d) + \mu(c)\mu(a \cap d).
\]

5 Propositional logic and Boolean algebras

It is hard to do any work on Boolean monoids and not to mention at all its relation with propositional logic. Indeed the motivation of Boole himself to introduce Boolean monoids was to describe the mathematical structures that control the laws of though. Propositional logic deals with the relation of deduction among sequences of sets of sentences constructed from a given finite set of propositions connected by a fixed set of connecting symbols. Let us denote the set of given propositions \( C \) and the set of sentences by \( S \). There are many ways to describe a system of propositional logic but in any of them one can imagine that there exists a sort of logical agent capable of performing the following tasks
- Recognize when a grammatical construction is an element of $S$. The agent is able to translate into sentences in $S$ expressions of the form $s \lor t$, $s \land t$, and $-s$ for sentences $s$ and $t$ in $S$.
- Decide whether or not a sequence of sets of sentences $c_1, ..., c_n$ is a deduction.
- Assign a truth value to each sentence in $S$ when provided with an assignment of truth values for propositions in $P$, i.e., an element of $\{0, 1\}^C$.

A sentence $s$ is said to imply a sentence $t$ if there exists a deduction $c_1, ..., c_n$ such that $c_1 = \{s\}$ and $c_n = \{t\}$. The logical operator is said to be sound and complete if the following property holds:

- Sentence $s \in S$ implies sentence $t \in S$ if for any assignment of truth values to propositions in $C$ the truth value of $t$ is 1 if the truth value of $s$ is 1. It is no hard to show the existence of sound a complete logical agents [10].

Boolean monoids appear within the context of propositional logic as follows. Say that sentences $s$ and $t$ in $S$ are equivalent if $s$ implies $t$ and $t$ implies $s$. Let $B(S)$ be the quotient of $S$ by this equivalence relation. $B(S)$ comes equipped with a natural structure of Boolean monoid with operations defined by $[s] \lor [t] = [s \lor t]$, $[s] \land [t] = [s \land t]$, and $[s]^c = [-s]$, for $[s]$ and $[t]$ in $B(S)$.

The total element is $[s \lor -s]$ and the empty element is $[s \land -s]$. The Boolean monoid $B(S)$ is isomorphic to the Boolean monoid $P(\{0, 1\}^C)$ via the map

$$m : B(S) \rightarrow P(\{0, 1\}^C)$$

sending each sentence $[s] \in S$ into the set of its models

$$m([s]) = \{ v \in \{0, 1\}^C \mid \text{the truth value of } s \text{ according to } v \text{ is } 1 \}.$$

Summarizing sentences in $S$ describe subsets of $\{0, 1\}^C$ and two sentences describe the same set if and only if they are equivalent. The power of the logical description of $P(\{0, 1\}^C)$ lies in the possibility of describing the same set in a variety of ways. For example the logical agent may be told that a subset of $\{0, 1\}^C$ is described by a sentence $s$, that another subset of $\{0, 1\}^C$ is described by a sentence $t$, and be asked to provide a sentence which describes the union of those sets. It will readily answer that $s \lor t$ is the sought after sentence.

It is natural to wonder if any logical meaning can be ascribed to the Boolean algebra $< B(S) >$. Although preliminary we venture an answer: assume the logical agent is told that a sentence $s_i$ describes an unknown subset of $\{0, 1\}^C$ with probability $p_i$ for $1 \leq i \leq n$, and that a sentence $t_j$ describes another unknown subset of $\{0, 1\}^C$ with probability $q_j$ for $1 \leq j \leq m$. If asked to find a sentence that describes the union of those subsets the logical agent will answer: the sentence $s_i \lor t_j$ describes the union of the unknown sets with probability $p_i q_j$. This is the only consistent answer with the product rules on $< B(S) >$ which is given by

$$\left( \sum_{i=1}^{n} p_i [s_i] \right) \lor \left( \sum_{j=1}^{m} q_j [t_j] \right) = \sum_{i=1, j=1}^{n, m} p_i q_j [s_i \lor t_j].$$
This probabilistic interpretation applies as well to the Boolean algebra $< P(x) >$. Let $v$ and $w$ be a couple of vectors in $< P(x) >$ given by $v = \sum_{a \subseteq x} v_a a$ and $w = \sum_{b \subseteq x} v_b b$. Assume that the coefficients of $v$ and $w$, respectively, are positive and add to one. This allow us to think that $v_a$ represents the probability that the unknown subset $v$ of $x$ be equal to $a$. Similarly $w_b$ represents the probability that $w$ be equal to $b$. Under this conditions we have that

- The probability that $v \cup w$ be equal to $c$ is given by $(v \cup w)_c = \sum_{a \cup b = c} v_a w_b$.
- The probability that $v \cap w$ be equal to $c$ is given by $(v \cap w)_c = \sum_{a \cap b = c} v_a w_b$.
- The probability that $v^c$ be equal to $a$ is $v_a^c$.

Finally we invite the reader to take another look at the structural coefficients of the algebras $\text{Sym}^2 < P[1] >$ and $< P[1] > \otimes_3 \mathbb{Z}_3$ given in Section 3 and check that they are indeed consistent with the probabilistic interpretation just outlined.

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