Birational geometry of algebraic varieties
fibred into Fano double spaces

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Abstract. We develop a quadratic technique for proving the birational rigidity of Fano–Mori fibre spaces over a higher-dimensional base. As an application, we prove the birational rigidity of generic fibrations into Fano double spaces of dimension $M \geq 4$ and index 1 over a rationally connected base of dimension at most $(M - 2)(M - 1)/2$. We obtain a near-optimal estimate for the codimension of the set of hypersurfaces of a given degree in projective space that have positive-dimensional singular sets.

Keywords: Fano–Mori fibre space, Fano variety, maximal singularity, birational map, linear system.

Introduction

0.1. Statement of main result. In [1] we proved the birational rigidity of two large classes of higher-dimensional Fano–Mori fibre spaces: generic fibrations into double spaces of index 1 and dimension $M \geq 5$ when the dimension of the base does not exceed $(M - 4)(M - 1)/2 - 1$, and generic fibrations into Fano hypersurfaces of index 1 and dimension $M - 1 \geq 9$ when the dimension of the base does not exceed $(M - 7)(M - 6)/2 - 6$ (in both cases assuming the fibration to be sufficiently twisted over the base). The question of the birational rigidity of Fano–Mori fibre spaces over the projective line is well studied (see [2], Ch. 5). However, it should be noted that almost all results on the birational rigidity of Fano–Mori fibre spaces over the line were obtained by a quadratic technique (that is, by analyzing the singularities of the self-intersection of the mobile linear system that determines the birational map), whereas the main result of [1] was obtained by a linear technique (that is, by directly analyzing the singularities of the linear system itself, without using the quadratic operation of taking the self-intersection). The quadratic technique requires fewer restrictions on the variety considered and, therefore, enables one to embrace a considerably larger class of rationally connected varieties. It is more efficient in many respects (at least, at the present stage of the theory of birational rigidity).

The aim of this paper is to develop a quadratic technique for studying the birational rigidity of Fano–Mori fibre spaces over a higher-dimensional base and apply it to fibrations into Fano double spaces of index 1. We considerably improve the result of [1] for this class of varieties by proving the birational rigidity of generic
fibrations into Fano double spaces of dimension $M \geq 4$ and index 1 over a rationally
connected base of dimension at most $(M - 2)(M - 1)/2$. This result considerably
increases the admissible dimension of the base of the fibre space. It is obtained by
means of the quadratic technique of counting multiplicities (see [2], Ch. 5), which
was not used in [1].

We now make precise statements.

Consider a Fano–Mori fibre space $\pi: V \to S$, where the base $S$ is non-singular,
the variety $V$ is factorial and has at most terminal singularities, the anticanonical
class $(-K_V)$ is relatively ample and

$$\text{Pic } V = \mathbb{Z} K_V \oplus \pi^* \text{Pic } S.$$  

We say that a fibre $F = F_s = \pi^{-1}(s)$, $s \in S$, satisfies condition $(h)$ if, for every
irreducible subvariety $Y \subset F$ of codimension 2 and every point $o \in Y$, we have

$$\frac{\text{mult}_o Y}{\deg Y} \leq \frac{4}{\deg F},$$

where the degrees are understood in the sense of the anticanonical class, that is,

$$\deg Y = (Y \cdot (-K_V)^{\dim Y}), \quad \deg F = (F \cdot (-K_V)^{\dim F}),$$

and condition $(hd)$ if, for every mobile linear system $\Delta \subset | -n(K_V|_F)|$ and every
irreducible subvariety $Y \subset F$ of codimension 2, we have

$$\text{mult}_Y \Delta \leq n.$$  

We also say that the fibre $F$ satisfies condition $(v)$ if, for every prime divisor $Y \subset F$
and every point $o \in F$, we have

$$\frac{\text{mult}_o Y}{\deg Y} \leq \frac{2}{\deg F}.$$  

Finally, we say that the fibre space $V/S$ satisfies the $K$-condition if, for every mobile
family $\mathcal{C}$ of curves on $S$ sweeping out $S$ and for a general curve $\overline{C} \in \mathcal{C}$, the class of
the algebraic cycle

$$-N(K_V \cdot \pi^{-1}(\overline{C})) - F$$

of dimension $\dim F$ for any $N \geq 1$ is not effective (that is, is not rationally equivalent
to an effective cycle of dimension $\dim F$), and the $K^2$-condition if, for every mobile
family $\mathcal{C}$ of curves on $S$ sweeping out $S$ and for a general curve $\overline{C} \in \mathcal{C}$, the class of
the algebraic cycle

$$N(K^2_V \cdot \pi^{-1}(\overline{C})) - H_F$$

of dimension $\dim F - 1$ is not effective for any $N \geq 1$, where $H_F = (-K_V \cdot F)$ is the
class of the anticanonical section of the fibre.

The following assertion is the main result of this paper.

**Theorem 0.1.** Assume that $\dim F \geq 4$ and every fibre $F$ of the projection $\pi$
is a variety with at most quadratic singularities of rank at least 4, and moreover $\text{codim} (\text{Sing } F \subset F) \geq 4$. Further assume that every fibre $F$ satisfies conditions $(h)$, $(hd)$ and $(v)$ and the fibre space $V/S$ satisfies the $K$-condition and the $K^2$-condition.
Then the fibre space $V/S$ is birationally rigid: every birational map $\chi: V \rightarrow V'$ onto the total space of a rationally connected fibre space $V'/S'$ is fibrewise, that is, one can find a rational dominant map $\beta: S \rightarrow S'$ such that the following diagram commutes:

$$
\begin{array}{ccc}
V \xrightarrow{\chi} V' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
S \xrightarrow{\beta} S'.
\end{array}
$$

(We recall that a morphism $\pi': V' \rightarrow S'$ of projective algebraic varieties is a rationally connected fibre space if the base $S'$ and the general fibre $\pi'^{-1}(s')$, $s' \in S'$, are rationally connected.)

The following assertion is a direct corollary of Theorem 0.1.

**Corollary 0.1.** Under the assumptions of Theorem 0.1, the variety $V$ has no structures of a rationally connected fibre space over a base of dimension higher than $\dim S$. In particular, $V$ is non-rational. Every birational self-map of $V$ is fibrewise and induces a birational self-map of the base $S$. Hence there is a natural homomorphism of groups $\rho: \text{Bir} V \rightarrow \text{Bir} S$, whose kernel $\text{Ker} \rho$ is the group $\text{Bir} F_\eta = \text{Bir}(V/S)$ of birational self-maps of the generic fibre $F_\eta$ (over a non-closed generic point $\eta$ of the base $S$). The group $\text{Bir} V$ is an extension of the normal subgroup $\text{Bir} F_\eta$ by the group $\Gamma = \rho(\text{Bir} V) \subset \text{Bir} S$:

$$1 \rightarrow \text{Bir} F_\eta \rightarrow \text{Bir} V \rightarrow \Gamma \rightarrow 1.$$  

We recall that the following fact was proved in [1].

**Theorem 0.2.** Let $\pi: V \rightarrow S$ be a Fano–Mori fibre space satisfying the following conditions.

i) Every fibre $F_s = \pi^{-1}(s)$, $s \in S$, is a factorial Fano variety with at most terminal singularities and with Picard group $\text{Pic} F_s = \mathbb{Z}K_{F_s}$. Moreover, $F_s$ has complete intersection singularities and $\text{codim}(\text{Sing} F \subset F) \geq 4$.

ii) For every effective divisor $D \in |-nK_{F_s}|$ on an arbitrary fibre $F_s$, the pair $(F_s, \frac{1}{n}D)$ is log canonical. For every mobile linear system $\Sigma_s \subset |-nK_{F_s}|$, the pair $(F_s, \frac{1}{n}\Sigma_s)$ is canonical (that is, the pair $(F_s, \frac{1}{n}D)$ is canonical for a general divisor $D \in \Sigma_s$).

iii) For every mobile family $C$ of curves on $S$ sweeping out $S$ and for a general curve $C \in \mathcal{C}$, the class of the algebraic cycle

$$-N(K_V \cdot \pi^{-1}(\overline{C})) - F$$

of dimension $\dim F$ (where $F$ is a fibre of the projection $\pi$) is not effective for any $N \geq 1$. In other words, this cycle is not rationally equivalent to an effective cycle of dimension $\dim F$.

Then any birational map $\chi: V \rightarrow V'$ onto the total space of a rationally connected fibre space $V'/S'$ is fibrewise, that is, there is a rational dominant map
\( \beta: S \to S' \) such that the following diagram commutes:

\[
\begin{array}{ccc}
V & \xrightarrow{\chi} & V' \\
\downarrow \pi & & \downarrow \pi' \\
S & \xrightarrow{\beta} & S'.
\end{array}
\]

Let us compare the hypotheses of Theorems 0.1 and 0.2. The canonicity of the pair \((F_s, \frac{1}{n}D)\) in condition ii) of Theorem 0.2 (that is, essentially the birational superrigidity of the fibre \(F_s\)) follows from the conditions \((h)\) and \((hd)\) of Theorem 0.1 (and is actually equivalent to them because the main method of proving the birational superrigidity of primitive Fano varieties is to combine the \(4n^2\)-inequality and the exclusion of maximal subvarieties of codimension 2; see [2], Ch. 2). The log canonicity of the pair \((F_s, \frac{1}{n}D)\) in condition ii) of Theorem 0.2 is replaced in Theorem 0.1 by the condition \((v)\), which is much easier to check for certain classes of Fano varieties. Finally, in Theorem 0.1 we have a new global condition for the Fano–Mori fibre space \(V/S\): the \(K^2\)-condition, which is easy to check.

Theorem 0.1 will be applied to fibrations into double spaces of index 1, when the conditions \((h)\) and \((v)\) hold automatically because \(\deg F = 2\).

### 0.2. Fibrations into double spaces of index 1

We use the notation of §1.2 in [1]: the symbol \(\mathbb{P}\) stands for the projective space \(\mathbb{P}^M, M \geq 4\), and \(W = \mathbb{P}(H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(2M)))\) is the space of hypersurfaces of degree \(2M\) in \(\mathbb{P}\). The following general fact holds.

**Theorem 0.3.** The closed algebraic subset of homogeneous polynomials \(f\) of degree \(d\) in \(N+1\) variables such that the hypersurface \(\{f = 0\} \subset \mathbb{P}^N\) has a singular set of positive dimension, is of codimension at least \((d-2)N\) in the space \(H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))\).

**Proof.** The proof will be given in §3. \(\square\)

The following theorem is an immediate corollary of Theorem 0.3.

**Theorem 0.4.** There is a Zariski-open subset \(W_{reg} \subset W\) such that every hypersurface \(W \in W_{reg}\) has finitely many singular points, each of which is a quadratic of rank at least 3 and, moreover, the following estimate holds:

\[
\text{codim}(W \setminus W_{reg} \subset W) \geq \frac{(M-2)(M-1)}{2} + 1.
\]

**Proof.** Taking \(d = 2M\) and \(N = M\) in Theorem 0.3, we obtain that every hyperplane \(W\) in the complement of a closed subset of codimension \(2M(M-1)\) in \(W\) has finitely many singular points. It is easy to check that the closed set of hypersurfaces \(W\) with a quadratic singular point of rank at most 2 or with a singularity \(o \in W\) of multiplicity \(\text{mult}_o W \geq 3\), is of codimension \((M-2)(M-1)/2 + 1\) in \(W\). \(\square\)

If \(F \to \mathbb{P}\) is a double covering branched over a hypersurface \(W \in W_{reg}\), then \(F\) is a factorial Fano variety with terminal singularities (see [1], §3.1, [3] and Proposition 1.4 below). It satisfies the conditions \((h)\) and \((v)\) because \(\deg F = 2\). The condition \((hd)\) can easily be proved by standard methods (see [2], Ch. 2; it holds trivially for \(M \geq 5\) because we have \(\deg Y \geq 2\) for every irreducible subvariety \(Y \subset F\) of codimension 2). Thus, in order to apply Theorem 0.1, it suffices to require that
every fibre $F_s$, $s \in S$, is branched over a regular hypersurface $W_s \in \mathcal{W}_{\text{reg}}$ and the fibre space $V/S$ satisfies the $K$-condition and the $K^2$-condition.

In the notation of [1], §1.2, let $S$ be a non-singular rationally connected variety of dimension $\dim S \leq (M - 2)(M - 1)/2$. Let $L$ be a locally free sheaf of rank $M + 1$ on $S$ and let $X = \mathbb{P}(L) = \text{Proj} \bigoplus_{i=0}^{\infty} L^\otimes i$ be the corresponding $\mathbb{P}^M$-bundle. We may assume that $L$ is generated by its global sections, whence so is the sheaf $\mathcal{O}_{\mathbb{P}(L)}(1)$. Let $L \in \text{Pic} X$ be the class of this sheaf, so that

$$\text{Pic} X = \mathbb{Z}L \oplus \pi_X^* \text{Pic} S,$$

where $\pi_X: X \to S$ is the natural projection. Take a general divisor $U \in |2(ML + \pi_X^* R)|$, where $R \in \text{Pic} S$ is some class. If this system is sufficiently mobile, then the assumption on the dimension of $S$ and Theorem 0.4 enable us to assume that for every point $s \in S$ the hypersurface $U_s = U \cap \pi_X^{-1}(s)$ belongs to $\mathcal{W}_{\text{reg}}$ and, therefore, the hypotheses of Theorem 0.1 hold for the double space branched over $U_s$. Let $\sigma: V \to X$ be the double covering branched over $U$. Set $\pi = \pi_X \circ \sigma: V \to S$, so that $V$ is a fibration into Fano double spaces of index 1 over $S$. Recall that the divisor $U \in |2(ML + \pi_X^* R)|$ is assumed to be sufficiently general.

**Theorem 0.5.** Assume that the variety $V$ is general in the sense of the construction described above and the divisorial class $(K_S + R)$ is pseudo-effective. Then all the assertions of Theorem 0.1 and Corollary 0.1 hold for the fibre space $\pi: V \to S$. In particular,

$$\text{Bir} V = \text{Aut} V = \mathbb{Z}/2\mathbb{Z}$$

is a cyclic group of order 2.

**Proof.** Since the class $L$ is numerically effective and

$$((\sigma^* L)^M \cdot F) = ((\sigma^* L)^{M-1} \cdot H_F) = 2,$$

it suffices to verify the inequalities

$$((\sigma^* L)^M \cdot K_V \cdot \pi^{-1}(\overline{C})) \geq 0 \quad \text{and} \quad ((\sigma^* L)^{M-1} \cdot K_V^2 \cdot \pi^{-1}(\overline{C})) \leq 0.$$

As mentioned in [1], §1.2, the first inequality coincides (up to a positive factor) with the inequality $((K_S + R) \cdot \overline{C}) \geq 0$, which holds because the class $(K_S + R)$ is pseudo-effective and the family of curves $\overline{C}$ is mobile and sweeps out the base $S$. As for the second inequality, elementary calculations show that it can be written (up to a positive factor) in the form

$$2((K_S + R) \cdot \overline{C}) + ((\det \mathcal{L}) \cdot \overline{C}) \geq 0,$$

which holds a fortiori since the locally free sheaf $\mathcal{L}$ is generated by global sections. □

**Remark 0.1.** For fibrations into double spaces of index 1, the $K^2$-condition follows from the $K$-condition. Theorem 0.5 strengthens Theorem 1.3 of [1] in respect of the genericity conditions for all fibres of $V/S$: these conditions are weaker (and hence the set $\mathcal{W}_{\text{reg}}$ is larger) in Theorem 0.5. This enables us to prove birational rigidity for fibre spaces over bases of higher dimension and, in particular, for fibrations into 4-dimensional double spaces.
0.3. Structure of the paper. This paper is organized in the following way. The content of §1 is mostly the first part of the proof of Theorem 0.1. Here we construct a modification $S^+ \to S$ of the base $S$ such that the centre of each maximal singularity on the pullback $\pi_+: V^+ \to S^+$ of the original fibre space to $S^+$ covers a divisor in $S^+$ (this procedure is often referred to as the flattening of maximal singularities). The arguments are similar to those in §1 of [1]. In contrast to [1], they do not prove the main theorem but only show the existence of a supermaximal singularity (assuming that Theorem 0.1 does not hold). This notion plays an important role in the proof of the birational superrigidity of fibre spaces over $\mathbb{P}^1$; see [2], Ch.5. Here we extend it to the case of fibrations over a base of arbitrary dimension. We complete §1 by studying quadratic singularities whose rank is bounded below (this is used to establish that the modified fibre space $\pi_+: V^+ \to S^+$ is factorial and terminal).

In §2 we complete the proof of Theorem 0.1. This is done by excluding the supermaximal singularity whose existence was proved in §1. The exclusion is performed by the standard technique of counting multiplicities (see [2], Ch. 5), adjusted to the present situation.

In §3 we obtain an estimate for the codimension (in the set of all hypersurfaces of degree $d$ in $\mathbb{P}^N$) of the closed set of those hypersurfaces of degree $d$ in $\mathbb{P}^N$ whose singularity set has positive dimension. The estimate is nearly optimal. This general and quite useful result is proved by elementary (but non-trivial) methods of algebraic geometry. To the best of the author’s knowledge, this estimate is new.

0.4. Historical remarks and acknowledgements. The history of problems related to the birational rigidity of Fano–Mori fibre spaces over a base of positive dimension was reviewed in detail in the introduction to [1], and we shall not consider it here. We note, however, that Sarkisov’s theorem on conic bundles [4], [5] was proved by quadratic methods (using the self-intersection of the mobile linear system that determines the birational map) although the quadratic technique of counting multiplicities is not needed for this class of varieties.

The problem of birational rigidity for del Pezzo fibrations over a base of dimension greater than 1 is entirely open. Clearly, the quadratic technique is precisely what is needed for this class of varieties, although a combination of linear and quadratic methods may also be successful. A lot of work has recently been done in the direction of computing the possible values of log canonical thresholds on del Pezzo surfaces; see [6]–[9].

Finally, we mention the recent work [10], where the results of [11], [12] (see also [2], Ch. 7) were used to answer the question (posed in [13]) of the existence of rationally connected varieties that are not Fano-type varieties.

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§ 1. Maximal and supermaximal singularities

The content of this section is the first part of the proof of Theorem 0.1. In § 1.1 we modify the fibre space $V/S$. This procedure is similar to that in § 2 in [1]. As a result, we obtain a new Fano–Mori fibre space $V^+/S^+$ satisfying the hypotheses of Theorem 0.1 and the following additional condition: the centre on $V^+$ of any maximal singularity covers a divisor in $S^+$. In § 1.2 we consider the self-intersection of the mobile linear system $\Sigma$ related to the birational map $\chi$ and prove the existence of a supermaximal singularity. In § 1.3 we give more precise information on quadratic singularities of bounded rank.

1.1. Modification of the fibre space $V/S$. In the notation of Theorem 0.1, we fix a birational map $\chi: V \dashrightarrow V'$. Repeating the arguments in § 2.1 in [1], we consider an arbitrary very ample linear system $\Sigma'$ on $S'$. Let $\Sigma' = (\pi')^*\Sigma'$ be its pullback to $V'$ so that the divisors $D' \in \Sigma'$ are composed of fibres of $\pi'$ and, therefore, we have $(D' \cdot C) = 0$ for every curve $C \subset V'$ that is contracted by $\pi'$. Clearly, the linear system $\Sigma'$ is mobile. We write

$$\Sigma = (\chi^{-1})_*\Sigma' \subset |-nK_V + \pi^*Y|$$

for its strict transform on $V$, where $n \in \mathbb{Z}_+$. Clearly, the map $\chi$ is fibrewise if and only if $n = 0$. Thus Theorem 0.1 holds for $n = 0$. Therefore we assume that $n \geq 1$ and show that this assumption leads to a contradiction.

As shown in [1], Lemma 2.1, the inequality $(\overline{C} \cdot Y) \geq 0$ holds for every mobile family of curves $\overline{C} \in \overline{\mathcal{C}}$ on $S$ sweeping out $S$.

Following [1], we call a prime divisor $E$ over $V$ a maximal singularity of the birational map $\chi$ if its image in $V'$ is a prime divisor covering the base $S'$ and the Noether–Fano inequality holds:

$$\varepsilon(E) = \text{ord}_E \Sigma - na(E) > 0,$$

where $a(E)$ is the discrepancy of $E$ with respect to $V$. It was proved in [1], Proposition 2.1, that maximal singularities exist. Let $\mathcal{M}$ be the (finite) set of all maximal singularities.

In proving the existence of maximal singularities, an important role is played by a very mobile family $\mathcal{C}'$ of rational curves on $V'$. We recall ([1], § 2.1) that a family $\mathcal{C}'$ of rational curves on $V'$ is very mobile if the curves $\mathcal{C}' \in \mathcal{C}'$ are contracted by the projection $\pi'$, sweep out an open dense subset of $V'$, do not intersect the indeterminacy set of the map $\chi^{-1}: V' \dashrightarrow V$, and a general curve $\mathcal{C}' \in \mathcal{C}'$ intersects the image of each maximal singularity $E \in \mathcal{M}$ transversally at points in general position. We fix a very mobile family of curves on $V'$ and denote its strict transform on $V$ by $\mathcal{C}$, and its projection $\pi(\mathcal{C})$ to $S$ by $\overline{\mathcal{C}}$.

Proposition 1.1. For every maximal singularity $E \subset \mathcal{M}$, its centre

$$\text{centre}(E, V) = \varphi(E)$$

on $V$ does not cover the base: $\pi(\text{centre}(E, V)) \subset S$ is a proper closed subset of $S$. 

Proof. Although the statement of this proposition repeats verbatim that of Proposition 2.2 in [1], a new proof is needed since the assumptions are different. Again, it suffices to show that the restriction $\Sigma|_F$ of the linear system $\Sigma$ to a fibre $F = \pi^{-1}(s)$ in general position has no maximal singularities (in the standard, weaker sense; see [2], Ch.2). This follows immediately from the conditions $(h)$ and $(hd)$, which hold for the variety $V$. □

Following §2.2 in [1], we now construct a modification $\sigma_S: S+ \to S$ of the base and the corresponding modification of the total space

$$\sigma_S: V+ = V \times_S S+ \to V$$

of the fibre space $V/S$ such that the new fibre space $\pi+: V+ \to S$ satisfies the following conditions.

1) The base $S+$ is non-singular.

2) For every singularity $E$ of the birational map $\chi \circ \sigma: V+ \dasharrow V'$ which is realized on $V'$ by a divisor covering the base $S'$, its centre on $V+$ covers a divisor on $S+$, that is,

$$\text{codim}(\pi+(\text{centre}(E, V+)) \subset S+) = 1.$$ 

The modification $\sigma_S$ is constructed as a sequence of blow-ups with non-singular centres. Since the fibre of $V+/S+$ over a point $p \in S+$ is naturally isomorphic to the fibre of the original fibre space $V/S$ over the point $\sigma_S(p) \in S$, and the base $S+$ is non-singular, we see from the assumption about the singularities of the fibres of $V/S$ that the variety $V+$ has at most quadratic (in particular, hypersurface) singularities of rank at least 4 and, moreover, $\text{codim}(\text{Sing} V+ \subset V+) \geq 4$, so that the variety $V+$ is factorial and terminal. Clearly,

$$\text{Pic} V+ = \mathbb{Z}K_+ \oplus \pi_+^* \text{Pic} S+,$$

whence $V+/S+$ is again a Fano–Mori fibre space. Let $\overline{T}$ be the set of all $\sigma_S$-exceptional prime divisors on $S+$, and let $T$ be the set of all $\sigma$-exceptional prime divisors on $V+$. The map

$$T \ni T \mapsto \pi+(T) = T \ni \overline{T}$$

is a bijection between $T$ and $\overline{T}$. The inverse map is

$$\overline{T} \ni \overline{T} \mapsto \pi^{-1}(\overline{T}) = T \in T.$$ 

Clearly, $\text{Pic} S+ = \sigma_S^* \text{Pic} S \bigoplus \bigoplus_{T \in T} \mathbb{Z} \overline{T}$ and a similar equality holds for $\text{Pic} V+$.

**Proposition 1.2.** The $K$-condition and the $K^2$-condition hold for the Fano–Mori fibre space $V+/S+$.

**Proof.** Let $\overline{R}$ be a mobile family of curves on $S+$ sweeping out $S+$, and let $\overline{R} \in \overline{R}$ be a general curve. Then, clearly, $\sigma_S(\overline{R})$ is a mobile family of curves on $S$ sweeping out $S$, and $\sigma_S(\overline{R})$ is a general curve in this family. We have

$$K_{S+} = \sigma_S^* K_S + \sum_{T \in T} a_T \overline{T}$$

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and accordingly
\[ K_+ = \sigma^* K_V + \sum_{T \in \mathcal{T}} a_T T \]
(the discrepancies of the prime divisors \( T \) and \( T = \pi_+^{-1}(T) \) with respect to \( S \) and \( V \) obviously coincide). Moreover, \( a_T > 0 \) for all \( T \in \mathcal{T} \). Consider the class of an algebraic cycle
\[ \sigma_*[-N(K_+ \cdot \pi_+^{-1}(\overline{R})) - F] = -N(K_V \cdot \pi_+^{-1}(\sigma_S(\overline{R}))) - \alpha F, \]
where \( \alpha = N \sum_{T \in \mathcal{T}} a_T (\overline{\mathcal{T}} \cdot \overline{R}) + 1 \geq 1 \). Since the \( K \)-condition holds for \( V/S \), we see from this that it holds for \( V^+/S^+ \). Consider the \( K^2 \)-condition. Writing down \( K^2_+ \) explicitly, we obtain that
\[ (K^2_+ \cdot \pi_+^{-1}(\overline{R})) = (\sigma^* K^2_V \cdot \pi_+^{-1}(\overline{R})) + 2\left(\sigma^* K_V \cdot \left(\sum_{T \in \mathcal{T}} a_T T\right) \cdot \pi_+^{-1}(\overline{R})\right) \]
\[ = (\sigma^* K^2_V \cdot \pi_+^{-1}(\overline{R})) - 2 \sum_{T \in \mathcal{T}} a_T (\overline{\mathcal{T}} \cdot \overline{R}) H_F \]
since \( ((\sum_{T \in \mathcal{T}} a_T T)^2 \cdot \pi_+^{-1}(\overline{R}))) = 0 \). Therefore,
\[ \sigma_*[N(K^2_+ \cdot \pi^{-1}(\overline{R})) - H_F] = N(K^2_V \cdot \pi^{-1}(\sigma_S(\overline{R}))) - \beta F, \]
where \( \beta = 2N \sum_{T \in \mathcal{T}} a_T (\overline{T} \cdot \overline{R}) + 1 \geq 1 \). Since the \( K^2 \)-condition holds for \( V/S \), this implies that it also holds for \( V^+/S^+ \). \( \square \)

Clearly, the map \( \chi \circ \sigma: V^+ \rightarrow V \) is fibrewise with respect to the projections \( \pi_+, \pi' \) if and only if \( \chi \) is fibrewise. Therefore we shall prove Theorem 0.1 for the Fano–Mori fibre space \( V^+/S^+ \). By construction, its fibres are the fibres of the original fibre space \( V/S \). Hence all the hypotheses of Theorem 0.1 hold for \( V^+/S^+ \).

From now on, we simplify the notation by assuming that \( V^+/S^+ \) is our original Fano–Mori fibre space \( V/S \), which now possesses the following new property. Let \( E \) be any singularity of the map \( \chi \) (which is still not fibrewise) whose centre on \( V' \) is divisorial and covers the base \( S' \). Then its centre on \( V \), to be denoted by \( \text{centre}(E, V) \), covers a prime divisor on \( S \). In particular, this holds for every maximal singularity \( E \in \mathcal{M} \). To simplify the notation, we shall use the symbols \( \mathcal{T}, \overline{\mathcal{T}} \) in the following new sense: \( \overline{\mathcal{T}} \) is the set of all prime divisors \( \overline{T} \) on \( S \) such that \( \pi(\text{centre}(E, V)) = \overline{T} \) for some maximal singularity \( E \in \mathcal{M} \), and \( \mathcal{T} \) is the set of pre-images \( T = \pi^{-1}(\overline{T}) \) of these divisors on \( V \). The projection \( \pi \) establishes a one-to-one correspondence between \( \mathcal{T} \) and \( \overline{\mathcal{T}} \). Let
\[ \tau: \mathcal{M} \rightarrow \mathcal{T} \]
be the map sending each maximal singularity \( E \in \mathcal{M} \) to the divisor \( T \in \mathcal{T} \) that contains \( \text{centre}(E, V) \). Put \( \overline{\tau} = \pi \circ \tau: \mathcal{M} \rightarrow \overline{\mathcal{T}} \), that is, \( \overline{\tau}(E) = \pi(\text{centre}(E, V)) \subset S \).

For every \( T \in \mathcal{T} \) we put \( \mathcal{M}_T = \tau^{-1}(T) \), so that
\[ \mathcal{M} = \bigsqcup_{T \in \mathcal{T}} \mathcal{M}_T. \]
Remark 1.1. In the situation considered in [1], the modification of the base completes the proof of the birational rigidity of the fibre space because the assumption on the global log canonical threshold of every fibre implies that there can be no maximal singularity whose centre covers a divisor on the base. Since we make no assumptions on the global log canonical thresholds in this paper, the main part of the proof of Theorem 0.1 begins when the base is modified and the centre of every maximal singularity covers a divisor on the base. In the next subsection we carry out some preparatory work for the subsequent exclusion of maximal singularities.

1.2. Supermaximal singularities. For every maximal singularity $E \in \mathcal{M}$ set

$$t_E = \text{ord}_E T,$$

where $T = \tau(E)$. By construction, $T \supset \text{centre}(E, V)$, whence $t_E \geq 1$. Let $\varphi : \tilde{V} \to V$ be a birational morphism resolving the singularities of the map $\chi$. Every maximal singularity $E \in \mathcal{M}$ is realized on $\tilde{V}$ by a prime divisor, which we will again denote by $E$. By the definition of $t_E$, the divisor

$$\varphi^* T - \sum_{E \in \mathcal{M}_T} t_E E$$

is effective and contains none of the maximal singularities $E \in \mathcal{M}$ as a component. Consider the strict transform $\tilde{C}$ on $\tilde{V}$ of the mobile family of curves $C$ that was fixed in §1.1. For $\tilde{C} \in \tilde{C}$ we have

$$\left( \left( \varphi^* T - \sum_{E \in \mathcal{M}_T} t_E E \right) \cdot \tilde{C} \right) \geq 0.$$ 

Set $\nu_E = \text{ord}_E \Sigma$ and let $a_E \geq 1$ be the discrepancy of $E$ with respect to $V$. Denote the canonical class $K_{\tilde{V}}$ by $\tilde{K}$. Then for the strict transform $\tilde{\Sigma}$ of the linear system $\Sigma$ on $\tilde{V}$ we have

$$\tilde{\Sigma} \subset [-n\tilde{K} + \tilde{Y} + \Xi],$$

where $\tilde{Y} = \varphi^* \pi^* Y - \sum_{E \in \mathcal{M}} \varepsilon(E) E$ (recall that $\varepsilon(E) = \nu_E - na_E$) and $\Xi$ is a linear combination of $\varphi$-exceptional divisors $E'$ such that either the centre of $E'$ on $V'$ is a subvariety of codimension at least 2, or $\varepsilon(E') \leq 0$ and, therefore, $(\tilde{C} \cdot \Xi) \geq 0$. Thus we have

$$\sum_{E \in \mathcal{M}} \varepsilon(E)(E \cdot \tilde{C}) > (\tilde{Y} \cdot \tilde{C}). \quad (1)$$

Recall that $(Y \cdot \bar{C}) \geq 0$ by the $K$-condition. On the other hand, we saw earlier that

$$(\bar{T} \cdot \bar{C}) = (\varphi^* T \cdot \bar{C}) \geq \sum_{E \in \mathcal{M}_T} t_E (E \cdot \bar{C}). \quad (2)$$

We now consider the self-intersection $Z = (D_1 \circ D_2)$ of the mobile linear system $\Sigma$ (where $D_1, D_2 \in \Sigma$ are general divisors having no common components because of the mobility). We write this effective algebraic cycle of codimension 2 in the following form:

$$Z = Z^h + Z^v + Z^\varnothing,$$
where the subcycle $Z^h$ contains all the components of $Z$ that cover the base (the horizontal part of $Z$), the subcycle $Z^v$ contains all the components of $Z$ that are contained in the divisors $T \in \mathcal{T}$ and cover $\mathcal{T}$ (the vertical part of $Z$), and the subcycle $Z^\varnothing$ contains all other components of $Z$ (this part of $Z$ is irrelevant for us). Clearly, we also have a representation

$$Z^v = \sum_{T \in \mathcal{T}} Z^v_T,$$

where the subcycle $Z^v_T$ consists of those components of the vertical part that are contained in $T$ and cover $T$.

Let $F = F_s = \pi^{-1}(s)$ be the fibre over a point $s \in \overline{T}$ in general position. Since $\text{Pic } F = \mathbb{Z}K_F$, we have

$$Z^v_T|_F \sim -\lambda_T K_F$$

for some $\lambda_T \in \mathbb{Z}_+$. Hence,

$$(Z^v_T \cdot \pi^{-1}(\mathcal{C})) = \lambda_T (T \cdot \mathcal{C}) H_F.$$ 

**Definition 1.1.** A maximal singularity $E \in \mathcal{M}_T$ is said to be supermaximal if

$$2n \varepsilon(E) > \lambda_T \text{ ord}_E T.$$  

This definition is modelled on that of a supermaximal singularity for Fano fibre spaces over $\mathbb{P}^1$ (see [2], Ch. 5) and plays the same role.

**Proposition 1.3.** A supermaximal singularity exists.

**Proof.** Since

$$Z \sim n^2 K^2_V + 2n((-K_V) \cdot \pi^*(Y)) + \pi^*(Y^2),$$

we have

$$(Z \cdot \pi^{-1}(\mathcal{C})) = n^2(K^2_V \cdot \pi^{-1}(\mathcal{C})) + 2n(Y \cdot \mathcal{C}) H_F$$

because, clearly, $(\pi^*(Y^2) \cdot \pi^{-1}(\mathcal{C})) = 0$. On the other hand,

$$(Z \cdot \pi^{-1}(\mathcal{C})) = (Z^h \cdot \pi^{-1}(\mathcal{C})) + \left( \sum_{T \in \mathcal{T}} \lambda_T (T \cdot \mathcal{C}) \right) H_F + \lambda_\varnothing H_F$$

for some $\lambda_\varnothing \in \mathbb{Z}_+$. By the $K^2$-condition, we have

$$2n(Y \cdot \mathcal{C}) \geq \sum_{T \in \mathcal{T}} \lambda_T (T \cdot \mathcal{C}).$$  

(4)

Combining (1), (2) and (4), we obtain that

$$2n \sum_{E \in \mathcal{M}} \varepsilon(E)(E \cdot \mathcal{C}) > \sum_{T \in \mathcal{T}} \lambda_T \left( \sum_{E \in \mathcal{M}_T} t_E(E \cdot \mathcal{C}) \right).$$

Since the set $\mathcal{M}$ of maximal singularities is a disjoint union of $\mathcal{M}_T$, $T \in \mathcal{T}$, we see that each maximal singularity occurs only once in the last inequality. Therefore, for some singularity $E \in \mathcal{M}_T$ we have

$$2n \varepsilon(E)(E \cdot \mathcal{C}) > \lambda_T t_E(E \cdot \mathcal{C}).$$

Since $(E \cdot \mathcal{C}) > 0$ for all $E \in \mathcal{M}$, this yields (3). $\square$
1.3. A remark on quadratic singularities. In [14], Theorem 4, and [1], §3.1, it was shown that quadratic singularities of rank at least \( r \geq 1 \) are stable with respect to blow-ups. This fact can be sharpened in the following way.

**Proposition 1.4.** Let \( X \) be an algebraic variety having at most quadratic singularities of rank at least \( r \), and suppose that

\[
\text{codim}(\text{Sing } X \subset X) \geq r.
\]

Then for every irreducible subvariety \( B \subset X \) there is a Zariski-open subset \( U \subset X \) such that \( U \cap B \neq \emptyset \) and the blow-up \( \tilde{U} \to U \) along the subvariety \( B_U = B \cap U \) has at most quadratic singularities of rank at least \( r \) and we have

\[
\text{codim}(\text{Sing } \tilde{U} \subset \tilde{U}) \geq r.
\]  \( (5) \)

**Remark 1.2.** The following obvious fact was used in [1], [14]. If \( X \) has at most quadratic singularities of rank at least \( r \), then \( \text{codim}(\text{Sing } X \subset X) \geq r - 1 \). Hence the codimension of the singular set \( \text{Sing } \tilde{U} \) is at least \( r - 1 \). The proposition just stated sharpens results in [1], [14]: the property of the singular set of a variety \( X \) to have codimension at least \( r \) is also stable with respect to blow-ups.

**Proof of Proposition 1.4.** By [4], Theorem 4, and [1], §3.1, we need only prove (5). Clearly, it may be assumed that \( B \subset \text{Sing } X \). Arguing as in §3.1 of [1], we consider a Zariski-open subset \( U \subset X \) such that \( B_U \) is a non-singular subvariety and the rank of quadratic points \( b \in B_U \) is constant and equal to \( r_1 \geq r \). Let \( E_U \subset \tilde{U} \) be the exceptional divisor of the blow-up \( \varphi_B: \tilde{U} \to U \) of the subvariety \( B_U \). Clearly, \( \varphi_B|_{E_U}: E_U \to B_U \) is a fibration into quadrics of rank \( r_1 \). Clearly, the set \( \text{Sing}(\tilde{U} \setminus E_U) \) of singular points has codimension at least \( r \). However, a quadric of rank \( r_1 \) has a singular set of codimension \( r_1 - 1 \). Therefore

\[
\text{codim}(\text{Sing } E_U \subset E_U) = r_1 - 1,
\]

and, a fortiori,

\[
\text{codim}((\text{Sing } \tilde{U} \cap E_U) \subset E_U) \geq r_1 - 1,
\]

whence

\[
\text{codim}((\text{Sing } \tilde{U} \cap E_U) \subset U) \geq r_1 \geq r. \quad \Box
\]

§ 2. Exclusion of supermaximal singularities

In this section we complete the proof of Theorem 0.1 by showing that a supermaximal singularity cannot exist. To do this, we use the technique of counting multiplicities (§2.1) in a modified form adjusted to varieties with quadratic singularities. We prove that the multiplicities of the self-intersection of the mobile linear system \( \Sigma \) along the centres of the supermaximal singularity satisfy a certain quadratic inequality, which is shown by our calculations in §2.2 to be impossible. This contradiction completes the proof of Theorem 0.1. In §2.3 we correct a minor inaccuracy in [14].
2.1. The technique of counting multiplicities. We fix a supermaximal singularity $E$ and the corresponding divisor $T = \pi^{-1}(\bar{T})$. To simplify the notation, we write $Z^v$ instead of $Z^v_T$ and $\lambda$ instead of $\lambda_T$ since the other singularities and divisors $T' \in T$ do not occur in the subsequent constructions. Let

$$V_K \to \cdots \to V_i \to V_{i-1} \to \cdots \to V_0 = V$$

be a resolution of the singularity $E$, that is, a sequence of blow-ups $\varphi_{i,i-1}: V_i \to V_{i-1}$ of irreducible subvarieties $B_{i-1} = \text{centre}(E, V_{i-1})$ with exceptional divisors $E_i = \varphi_{i,i-1}^{-1}(B_{i-1})$, where the last exceptional divisor $E_K$ is the supermaximal singularity $E$. The set of subscripts $I = \{1, \ldots, K\}$ labelling the blow-ups is a disjoint union

$$I = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_{M-1},$$

where $M = \dim F$ is the dimension of the fibre, and $i \in I_k$ if and only if $\dim B_{i-1} = \dim S - 1 + k$ (clearly, $\pi \circ \varphi_{i-1,0}(B_{i-1}) = T$, whence $\dim B_{i-1} \geq \dim T$; here

$$\varphi_{i,j} = \varphi_{j+1,j} \circ \cdots \circ \varphi_{i,i-1}: V_i \to V_j$$

is a composite of elementary blow-ups). Some sets $I_k$ may be empty. By Proposition 1.4 we have $B_{j-1} \not\subset \text{Sing} V_{j-1}$ for $j \in I_{M-2} \cup I_{M-1}$. Putting

$$\mu_j = \text{mult}_{B_{j-1}} V_{j-1} \in \{1, 2\}$$

for all $j \in I$, we thus have $\mu_j = 1$ for $j \in I_{M-2} \cup I_{M-1}$. We shall add the superscript $j$ to denote the strict transform of a subvariety, an effective divisor or a linear system on $V_j$. Given a general divisor $D \in \Sigma$, we write

$$D^j = \varphi_{j,j-1}^* (D^{j-1}) - \nu_j E_j.$$

Let $Z = (D_1 \circ D_2)$ be the self-intersection of the mobile system $\Sigma$. Writing in the usual way (see [2], Ch. 2)

$$(D_1^i \circ D_2^i) = (D_1^{i-1} \circ D_2^{i-1})^i + Z_i,$$

where $Z_i$ is an effective cycle of codimension 2 supported inside the exceptional divisor $E_i$, we define the degree $d_i$ of the cycle $Z_i$ in the following way. If $B_{i-1} \not\subset \text{Sing} V_{i-1}$, then $\varphi_{i,i-1}^{-1}(p)$ is the projective space $\mathbb{P}^{d_i}$ for every point $p \in B_{i-1}$ in general position and

$$d_i = \deg(Z_i|_{\varphi_{i,i-1}^{-1}(p)})$$

is the degree of an effective divisor in this projective space. If $B_{i-1} \subset \text{Sing} V_{i-1}$, then the fibre $\varphi_{i,i-1}^{-1}(p)$ is an irreducible quadric in the projective space $\mathbb{P}^{d_i+2}$ for a general point $p \in B_{i-1}$, and $d_i$ is the degree of the effective cycle $Z_i|_{\varphi_{i,i-1}^{-1}(p)}$ in this projective space. In both cases, $\delta_i$ stands for the elementary discrepancy $\text{codim}(B_{i-1} \subset V_{i-1}) - \mu_i$. As usual, we break the set $I$ into the lower part

$$I_l = I_0 \sqcup \cdots \sqcup I_{M-2}.$$
and the upper part $I_u = I_{M-1}$, and set

$$L = \max\{i \in I_i\}.$$ 

Finally, for $0 \leq i < j \leq L$ we set

$$m_{i,j} = \text{mult}_{B_{j-1}}(Z_i^{j-1}).$$

When $i = 0$, we write simply $m_j$. The technique of counting multiplicities ([2], Ch. 2) yields a system of equations

$$\begin{align*}
\mu_1 \nu_1^2 + d_1 &= m_1, \\
\mu_2 \nu_2^2 + d_2 &= m_2 + m_{1,2}, \\
\cdots \\
\mu_i \nu_i^2 + d_i &= m_i + m_{1,i} + \cdots + m_{i-1,i}, \\
\cdots \\
\mu_L \nu_L^2 + d_L &= m_L + m_{1,L} + \cdots + m_{L-1,L}.
\end{align*}$$

We also have the estimate

$$d_L \geq K \sum_{i=L+1}^K \nu_i^2.$$ 

Let $\Gamma$ be the directed graph of resolution of the singularity $E$, that is, the graph with vertex set $I$ and an oriented edge (an arrow) connecting vertices $i$ and $j$ (notation: $i \to j$) if and only if $i > j$ and $B_{i-1} \subset E_{j-1}^i$. We recall ([2], Ch. 2, Definition 2.1) that a function $a : I \to \mathbb{R}^+$ is compatible with the structure of the graph $\Gamma$ if

$$a(i) \geq \sum_{I_i \ni j \to i} a(j)$$

for every $i \in I_i$.

**Proposition 2.1.** The function

$$r_i = r(i) = \text{ord}_E \varphi^*_{K,i}E_i$$

is compatible with the structure of the graph $\Gamma$.

**Proof.** The Cartier divisor

$$\varphi^*_{K,i}E_i - \sum_{I_i \ni j \to i} \varphi^*_{K,j}E_j$$

is effective, whence the proposition follows immediately. □

Using Proposition 2.4 in Ch. 2 of [2], we obtain that

$$\sum_{i=1}^L r_i m_i \geq \sum_{i=1}^L r_i \mu_i \nu_i^2 + r_L \sum_{i= L+1}^K \nu_i^2.$$
Extending the definition of the numbers $r_i$ to $i \in I_{M-2}$ and using the obvious observation that $r_i$ is a non-increasing function of $i$, we finally have

$$\sum_{i=1}^{L} r_i m_i \geq \sum_{i=1}^{K} r_i \mu_i \nu_i^2. \quad (6)$$

Remark 2.1. Let $p_{ai}$ be the number of paths from a vertex $a$ to a vertex $i \neq a$ in the directed graph $\Gamma$ (so that $p_{ai} = 0$ for $a < i$). We also put $p_{ii} = 1$ for all $i \in I$. The standard technique of counting multiplicities (see [2], Ch. 2) makes use of the numbers $p_{Ki}$ instead of $r_i$ in inequalities of type (6), and it is easy to see that $r_i = p_{Ki}$ for $\mu_1 = 1$. If $\mu_1 = 2$, then $r_1 \geq p_{K1}$ (see below). The inequality (6) remains valid if we replace $r_i$ by $p_{Ki}$, but this modification is less useful because it is the coefficients $r_i$ that occur both in the explicit form of the Noether–Fano inequality and in the explicit expression for $\ord E \varphi_{K,0}^* T$.

Put $L_{\text{sing}} = \max \{1 \leq i \leq L \mid \mu_i = 2\}$.

**Proposition 2.2.**

i) When $i \geq 1 + L_{\text{sing}}$ we have $r_i = p_{Ki}$.

ii) When $1 \leq i \leq L_{\text{sing}}$ we have

$$p_{Ki} \leq r_i \leq 2p_{Ki}.$$

**Proof.** Part i) is obvious since when $i \geq 1 + L_{\text{sing}}$ the exceptional divisor $E_i$ is non-singular over a generic point of $B_{i-1}$, whence

$$\ord E \varphi_{K,i}^* E_i = \sum_{j \rightarrow i} \ord E \varphi_{K,j}^* E_j$$

and a descending induction yields that $r_i = p_{Ki}$. When $i \leq L_{\text{sing}}$, the fibre of the exceptional divisor $E_i$ over a point in general position on $B_{i-1}$ is a quadric of rank at least 4. If we have $j \rightarrow i$ when $j \leq L_{\text{sing}}$ and $j > i$, then, clearly,

$$\varphi_{j,j-1}^*(E_{i}^{j-1}) = E_{i}^{j} + E_{j} \quad (7)$$

as in the non-singular case. If $j \rightarrow i$ for some $j \geq 1 + L_{\text{sing}}$, then two cases can occur:

1) $B_{j-1} \not\subset \text{Sing } E_{i}^{j-1}$, and then (7) again holds;

2) $B_{j-1} \subset \text{Sing } E_{i}^{j-1}$, and then we have

$$\varphi_{j,j-1}^*(E_{i}^{j-1}) = E_{i}^{j} + 2E_{j} \quad (8)$$

We emphasize that if (8) holds, then $j > L_{\text{sing}}$, whence

$$\ord E \varphi_{K,j}^* E_j = p_{Kj}.$$

Therefore every directed path in $\Gamma$ from the top vertex $K$ to a vertex $i$ contributes either 1 or 2 to the number $r_i$, and the latter occurs if and only if the path is of the form

$$i = j_0 \leftarrow j_1 \leftarrow \cdots \leftarrow j_k \leftarrow j_{k+1} \leftarrow \cdots \leftarrow j_m = K,$$

where $j_k \leq L_{\text{sing}}$, $j_{k+1} > L_{\text{sing}}$ and the arrow $j_{k+1} \rightarrow j_k$ realizes case 2) described above. □
2.2. Completion of the proof of Theorem 0.1. We recall that the elementary discrepancies \( \delta_i = \text{codim}(B_{i-1} \subset V_{i-1}) - \mu_i \) were defined above for \( i = 1, \ldots, K \). Put
\[
L_{\text{fibre}} = \max\{1 \leq i \leq K \mid B_{i-1} \subset T^{i-1}\}.
\]
For \( 1 \leq i \leq L_{\text{fibre}} \) we define numbers \( \gamma_i \in \mathbb{Z} \) by the equalities
\[
\varphi_{i,i-1}^* (T^{i-1}) = T^i + \gamma_i E_i,
\]
so that \( \gamma_i \in \{1, 2\} \).

Proposition 2.3. The following assertions hold.

i) The multiplicity of the linear system \( \Sigma \) with respect to \( E \) satisfies the relation
\[
\text{ord}_E \Sigma = \sum_{i=1}^K r_i \nu_i. \tag{9}
\]

ii) The multiplicity of the divisor \( T \) with respect to \( E \) satisfies the relation
\[
\text{ord}_E T = \sum_{i=1}^K r_i \gamma_i. \tag{10}
\]

iii) The discrepancy of \( E \) satisfies the relation
\[
a(E) = \sum_{i=1}^K r_i \delta_i. \tag{11}
\]

Proof. The proof repeats the arguments in the non-singular case ([2], Ch. 2) verbatim with the number of paths \( p_{K_i} \) replaced by the new coefficients \( r_i \). We shall prove (9) since the arguments in the other cases are similar. The proof is by induction on \( K \geq 1 \). If \( K = 1 \), then (9) is obvious. Suppose that \( K \geq 2 \). For a general divisor \( D \in \Sigma \) we write
\[
\varphi_{1,0}^* D = D^1 + \nu_1 E_1,
\]
so that \( \varphi_{K,0}^* D = \varphi_{K,1}^* D^1 + \nu_1 \varphi_{K,1}^* E_1 \) and, therefore,
\[
\text{ord}_E \Sigma = \text{ord}_E D = \text{ord}_E D^1 + r_1 \nu_1.
\]
Then (9) holds for \( D^1 \) by the induction hypothesis. \( \Box \)

Set \( L^* = \min(L, L_{\text{fibre}}) \) and
\[
m_i^h = \text{mult}_{B_{i-1}} (Z^h)^{i-1} \quad \text{for} \quad i = 1, \ldots, L,
\]
\[
m_i^v = \text{mult}_{B_{i-1}} (Z^v)^{i-1} \quad \text{for} \quad i = 1, \ldots, L^*.
\]
The left-hand side of (6) can now be rewritten in the form
\[
\sum_{i=1}^L r_i m_i^h + \sum_{i=1}^{L^*} r_i m_i^v. \tag{12}
\]
The first summand in this sum does not exceed \(4n^2 \sum_{i=1}^{L} r_i\) since the sequence of multiplicities \(m_i^h\) is non-increasing and
\[
m_i^h = \text{mult}_{B_0} Z^h \leq \text{mult}_{B_0} (Z^h \circ T) \leq 4n^2
\]
by condition (h). The ‘vertical’ summand in (12) does not exceed
\[
2\lambda \sum_{i=1}^{L^*} r_i \leq 2\lambda \text{ord}_E T
\]
by condition (v) (see (10)), and the right-hand side of the last inequality is strictly smaller than \(4ne\) (where \(e = \varepsilon(E)\)) by the definition of a supermaximal singularity (see (3)). Combining these estimates, we obtain that the left-hand side of (6) is strictly smaller than the expression
\[
4n^2 \sum_{i=1}^{L} r_i + 4ne.
\]
We now consider the right-hand side of (6). By the definition of the number \(\varepsilon(E)\) we have
\[
\sum_{i=1}^{K} r_i \nu_i = n \sum_{i=1}^{K} r_i \delta_i + e \tag{13}
\]
(in this notation, the Noether–Fano inequality takes the form \(e > 0\)). It is easy to check by standard methods that the minimum of the right-hand side of (6) on the hyperplane with equation (13) in the space \(\mathbb{R}^K_{(\nu_1,...,\nu_K)}\) is attained at \(\nu_i = \theta/\mu_i\), where \(\theta\) is found from (13). We introduce the following notation:
\[
\Sigma_l = \sum_{i=1}^{L} r_i, \quad \Sigma_u = \sum_{i=L+1}^{K} r_i, \quad \Sigma_{\text{sing}} = \sum_{i=1}^{L_{\text{sing}}} r_i, \quad \Sigma_{\text{non-sing}} = \sum_{i=L_{\text{sing}}+1}^{K} r_i.
\]
In this notation, we obtain from (6) that
\[
4n^2 \Sigma_l + 4ne > 2 \left( \frac{2n \Sigma_l + n \Sigma_u + e}{\Sigma_{\text{sing}} + 2\Sigma_{\text{non-sing}}} \right)^2.
\]
Since \(\Sigma_{\text{sing}} + \Sigma_{\text{non-sing}} = \Sigma_l + \Sigma_u\), easy calculations show that
\[
2n^2 \Sigma_l \Sigma_{\text{non-sing}} + 2ne \Sigma_{\text{non-sing}} > 2n^2 \Sigma_l^2 + 2n^2 \Sigma_l \Sigma_u + n^2 \Sigma_u^2 + 2ne \Sigma_l + e^2.
\]
However, \(\Sigma_{\text{non-sing}} \leq \Sigma_l + \Sigma_u\), so that the previous inequality yields the estimate
\[
2ne \Sigma_u > n^2 \Sigma_u^2 + e^2,
\]
which cannot be true. The resulting contradiction excludes the supermaximal singularity and completes the proof of Theorem 0.1.
2.3. Birationally rigid Fano hypersurfaces. In the context of the constructions performed in this section, we consider the problem of estimating the codimension of the set of non-rigid hypersurfaces of degree \( M \) in \( \mathbb{P}^M \). This problem was posed and solved in [14], but the proof of the \( 4n^2 \)-inequality for Fano hypersurfaces with quadratic singularities of rank at least 5 for \( M \geq 5 \) ([14], §3) contained an inaccuracy, which was found by the author while working over the present paper. In this subsection we explain what was incorrect and how it should be corrected. Note that the main assertion ([14], Proposition 1) and the method of its proof remain valid.

We recall that the following local fact was proved in §3 of [14]. Let \( X \) be an algebraic variety with quadratic (in particular, hypersurface) singularities of rank at least 5 (so that the set of singular points \( \text{Sing} \, X \) has codimension at least 4 and, therefore, the variety \( X \) is factorial), let \( B \subset \text{Sing} \, X \) be an irreducible subvariety, and let \( \Sigma \) be a mobile linear system on \( X \) such that the pair \((X, \frac{1}{n} \Sigma)\) is non-canonical for some \( n \geq 1 \) or, more precisely, this pair has a non-canonical singularity \( E \) with centre at \( B \). Then the self-intersection \( Z = (D_1 \circ D_2) \), where \( D_i \in \Sigma \) are general divisors, satisfies the inequality

\[
\text{mult}_B Z > 4n^2.
\]

(The multiplicity is understood in the usual sense; see [2], Ch. 2.) In fact, the assumptions can be relaxed. The following proposition holds.

**Proposition 2.4.** Let \( X \) be a variety with quadratic singularities of rank at least 4, and assume that \( \text{codim} \, (\text{Sing} \, X \subset X) \geq 4 \). Further assume that a certain divisor \( E \) over \( X \) is a non-canonical singularity of the pair \((X, \frac{1}{n} \Sigma)\) with centre \( B \subset \text{Sing} \, X \), where \( \Sigma \) is a mobile linear system. Then the self-intersection \( Z \) of the system \( \Sigma \) satisfies the inequality

\[
\text{mult}_B Z > 4n^2.
\]

**Proof.** We only point out what should be modified in the arguments in [14], §3. It follows from Proposition 1.4 that the technique of counting multiplicities works without change under weaker assumptions on the rank of the quadratic singularities. Furthermore, it is erroneously claimed in §3 of [14] that the Noether–Fano inequality takes the form

\[
\sum_{i=1}^{K} p_i \nu_i > n \left( \sum_{i=1}^{K} p_i \delta_i \right),
\]

where \( p_i \) is the number of paths in the directed graph of the resolution of the singularity \( E \) from the top vertex to vertex \( i \) (this notation has exactly the same meaning as in §2.1). In fact, one must replace the coefficients \( p_i \) in (14) by the coefficients \( r_i \) introduced in §2.1. After that, all the arguments in §3 of [14] can be repeated verbatim and prove Proposition 2.4. □

§3. Hypersurfaces with non-isolated singularities

In this section we prove Theorem 0.3. The procedure for estimating the codimension of the set of hypersurfaces with a positive-dimensional singular set in the projective space depends on the type of the singular set. In §3.1 we consider
some simple cases (for example, when the singular set is a line) admitting a direct estimation or explicit computation of the codimension of the set of hypersurfaces with singular sets of a given type. In §3.2 we develop a technique that enables us to estimate the codimension of the set of hypersurfaces with at least a finite but sufficiently large set of singular points. An application of this technique in §3.3 completes the proof of Theorem 0.3.

3.1. Sets of singular hypersurfaces. Let $\mathbb{P}^N$ be a projective space with homogeneous coordinates $(x_0 : x_1 : \cdots : x_N)$, $N \geq 3$, and let $\mathcal{P}_{N,d} = H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))$ be the vector space of homogeneous polynomials of degree $d$. For every $f \in \mathcal{P}_{N,d}$ we denote the set of singular points of the hypersurface $\{f = 0\}$ by $\text{Sing}(f)$. Set

$$\mathcal{P}^{(i)}_{N,d} = \{f \in \mathcal{P}_{N,d} \mid \dim \text{Sing}(f) \geq i\}.$$  

These are closed subsets of $\mathcal{P}_{N,d}$ and we have $\mathcal{P}^{(j)}_{N,d} \subset \mathcal{P}^{(i)}_{N,d}$ when $i \geq j$.

Example 3.1. Let $\mathcal{P}^{\text{line}}_{N,d}$ be the closed subset of $\mathcal{P}_{N,d}$ consisting of all polynomials $f$ such that $\text{Sing}(f)$ contains a line in $\mathbb{P}^N$. Fix a line $L \subset \mathbb{P}^N$. We can assume that $L = \{x_2 = \cdots = x_N = 0\}$. Then the condition $L \subset \text{Sing}(f)$ is equivalent to the following set of equalities:

$$\frac{\partial f}{\partial x_0} \bigg|_L \equiv \frac{\partial f}{\partial x_1} \bigg|_L \equiv \cdots \equiv \frac{\partial f}{\partial x_N} \bigg|_L \equiv 0,$$

whence, taking into account the dimension of the Grassmannian of lines in $\mathbb{P}^N$, we obtain that

$$\text{codim}(\mathcal{P}^{\text{line}}_{N,d} \subset \mathcal{P}_{N,d}) = (d - 2)N + 3.$$

Theorem 0.3 is an immediate corollary of the following assertion.

Theorem 3.1. We have

$$\text{codim}(\mathcal{P}^{(1)}_{N,d} \subset \mathcal{P}_{N,d}) \geq (d - 2)N.$$

Remark 3.1. It seems that one can improve the inequality in Theorem 3.1 with its right-hand side replaced by $(d - 2)N + 3$, and then it becomes precise. However, the proof below is insufficient for this purpose. In any case, the assertion in Theorem 3.1 is much stronger than we need in this paper.

Proof of Theorem 3.1. Let $\mathcal{P}^{(i,k)}_{N,d} \subset \mathcal{P}_{N,d}$ be the closure of the set of all polynomials $f$ such that $\text{Sing}(f)$ contains an irreducible component $C$ of dimension $i \geq 1$ whose linear span $\langle C \rangle$ is a $k$-plane in $\mathbb{P}^N$, $k \geq i$. For example, $\mathcal{P}^{(1,1)}_{N,d} = \mathcal{P}^{\text{line}}_{N,d}$. Clearly,

$$\mathcal{P}^{(i)}_{N,d} = \bigcup_{k=i}^{N} \mathcal{P}^{(i,k)}_{N,d}.$$

Hence, to estimate the codimension of $\mathcal{P}^{(i)}_{N,d}$, it suffices to estimate the codimension of each set $\mathcal{P}^{(i,k)}_{N,d}$, $k = i, \ldots, N$. Furthermore, for every $k$-plane $P \subset \mathbb{P}^N$ let $\mathcal{P}^{(i,k)}_{N,d}(P)$ be the closure of the set of all polynomials $f$ such that $\text{Sing} f$ contains an irreducible component $C$ of dimension $i$ with $\langle C \rangle = P$. The following fact is obvious.
Proposition 3.1. We have
\[ \text{codim}(\mathcal{P}_{N,d}^{(i,k)} \subset \mathcal{P}_{N,d}) \geq \text{codim}(\mathcal{P}_{N,d}^{(i,k)}(P) \subset \mathcal{P}_{N,d}) - (k + 1)(N - k). \]

Finally, let \( \mathcal{P}_{N,d}^{(i,k)}(P) \subset \mathcal{P}_{N,d}^{(i,k)}(P) \) be the closure of the set of all \( f \) such that (in terms of the definition of \( \mathcal{P}_{N,d}^{(i,k)}(P) \)) the set of singular points \( \text{Sing}(f|_P) \) contains an irreducible component \( B \) of dimension \( l \) with \( C \subset B \subset P \). In particular, \( l \geq i \) and
\[ \mathcal{P}_{N,d}^{(i,k)}(P) = \bigcup_{l=i}^{k} \mathcal{P}_{N,d}^{(i,k,l)}(P). \]

Therefore, by Proposition 3.1, Theorem 3.1 follows from the system of inequalities
\[ \text{codim}(\mathcal{P}_{N,d}^{(1,k,l)}(P) \subset \mathcal{P}_{N,d}) \geq (d - 2)N + (k + 1)(N - k). \tag{15} \]

We shall prove these inequalities for all \( l, k \) with \( 1 \leq l \leq k \leq N \) and for the fixed \( k \)-plane \( P \subset \mathbb{P}^N \) given by the equations \( \{x_{k+1} = \cdots = x_N = 0\} \).

Example 3.2. Consider the case \( l = k = 2 \). Then \( P \) is a plane, \( P \subset \{f = 0\} \) and the closed set \( \text{Sing}(f) \) contains an irreducible plane curve \( C \subset P \) of degree \( q \geq 2 \). This gives \( (d + 1)(d + 2)/2 \) independent conditions on the coefficients of the polynomial \( f|_P \) (they all vanish) and \( N - 2 \) polynomials
\[ \frac{\partial f}{\partial x_3}|_P, \ldots, \frac{\partial f}{\partial x_N}|_P \]
vanish on the curve \( C \). Note that the coefficients of the polynomials \( f|_P, \partial f/\partial x_i|_P, i = 3, \ldots, N \), up to a non-zero integer factor, are distinct coefficients of \( f \). We can assume that at least one of the polynomials \( \partial f/\partial x_i|_P \) is not identically equal to zero, say, \( \partial f/\partial x_3|_P \neq 0 \). Then the curve \( C \) is an irreducible component of the plane curve \( \{\partial f/\partial x_3|_P = 0\} \). Fixing the polynomial \( \partial f/\partial x_3|_P \), we finally obtain
\[ \frac{(d + 1)(d + 2)}{2} + (N - 3) \left( qd - \frac{q(q - 1)}{2} \right) \]
independent conditions on the coefficients of \( f \), where \( 2 \leq q \leq d - 1 \). It is easy to see that this number satisfies the inequality (15).

Example 3.3. Consider the case \( l = 1, k = 2 \). Then \( \text{Sing}(f) \) contains an irreducible plane curve \( C \subset P \) of degree \( q \geq 2 \), but \( f|_P \neq 0 \). Hence \( \{f|_P = 0\} \) is a reducible plane curve of degree \( d \) containing \( C \) as a double component, so that \( 2q \leq d \). An easy dimension count gives
\[ \frac{1}{2} \left( 5q^2 - (4d + 3)q + d^2 + 3d + 4 \right) \]
independent conditions on the coefficients of the polynomial \( f|_P \). The minimum of the last expression is attained at \( q = 2 \). Since the polynomials \( \partial f/\partial x_i|_P \),
$i = 3, \ldots, N$, vanish on the curve $C$, we obtain at least $(N - 2)(2d + 1)$ additional independent conditions on the coefficients of $f$. As a result, we get

$$\text{codim}(\mathcal{P}_{d,N}^{(1,2;1)}(P)) \geq \frac{d(d - 5)}{2} + (N - 2)(2d + 1) + 9$$

and it is easy to see that (15) holds.

From now on we assume that $k \geq 3$. We recall the following definition.

**Definition 3.1** (see [15], §3 or [2], Ch. 3). A sequence of homogeneous polynomials $g_1, \ldots, g_m$ of arbitrary degrees on the projective space $\mathbb{P}^e$, $e \geq m+1$, is called a *good sequence* (and an irreducible subvariety $W \subset \mathbb{P}^e$ of codimension $m$ is called an *associated subvariety* for this sequence) if there is a sequence of irreducible subvarieties $W_j \subset \mathbb{P}^e$ with $\text{codim} W_j = j$ (in particular, $W_0 = \mathbb{P}^e$) such that

1) $g_{j+1}\big|_{W_j} \not\equiv 0$ for $j = 0, \ldots, m + 1$;
2) $W_{j+1}$ is an irreducible component of the closed algebraic set $g_{j+1}\big|_{W_j} = 0$;
3) $W_m = W$.

A good sequence can have more than one associated subvariety, but their number is bounded above by a constant depending on the degrees of the polynomials $q_j$ (see [15], §3).

Consider two more examples that are similar to Examples 3.2 and 3.3.

**Example 3.4.** Consider the case $l = k$. This is a generalization of Example 3.2. We have $f\big|_P \equiv 0$, which gives $\binom{k+d}{d}$ independent conditions on the coefficients of $f$. Since the polynomials

$$\frac{\partial f}{\partial x_{k+1}}\big|_P, \ldots, \frac{\partial f}{\partial x_N}\big|_P$$

vanish identically on $C$ and the curve $C$ is an irreducible component of the set $\text{Sing}(f)$, we can choose $k - 1$ of these polynomials that form a good sequence with associated subvariety $C$ (in particular, $N - k \geq k - 1$). Fixing these polynomials, we obtain the condition

$$\left(\frac{\partial f}{\partial x_i}\big|_P\right)\big|_C \equiv 0 \quad (16)$$

for each of the remaining $N+1 - 2k$ polynomials, where the curve $C$, being one of the associated subvarieties of a fixed good sequence, can also be assumed fixed. It was shown in §3 of [15] that the condition (16) determines a closed set of codimension at least $(d - 1)k + 1$. Therefore,

$$\text{codim}(\mathcal{P}_{N,d}^{(i,k;k)} \subset \mathcal{P}_{N,d}) \geq \binom{k+d}{d} + (N + 1 - 2k)((d - 1)k + 1)$$

and elementary calculations show that (15) holds.

**Example 3.5.** Consider the case $l = k - 1$. This is a generalization of Example 3.3. Here the hypersurface $\{f\big|_P = 0\}$ has a multiple irreducible non-degenerate component of degree $q$, where $2q \leq d$. Hence the coefficients of the polynomial $f\big|_P$
belong to a closed subset of codimension
\[
\binom{k + d}{k} - \binom{k + d - 2q}{k} - \binom{k + d}{k}
\]
in the space \( \mathcal{P}_{d,k} \). Furthermore, since \( C \) is an irreducible component of \( \text{Sing}(f) \), the set of polynomials
\[
\left. f \right| _{P}, \left. \frac{\partial f}{\partial x_{k+1}} \right| _{P}, \ldots, \left. \frac{\partial f}{\partial x_{N}} \right| _{P}
\]
contains a good sequence (starting with \( \left. f \right| _{P} \)) with associated subvariety \( C \). In particular, \( N + 2 \geq 2k \). Fixing the polynomials in this sequence, we can assume that the curve \( C \) is fixed. Arguing as in Example 3.4, we obtain \((N + 2 - 2k)((d - 1)k + 1)\) independent conditions on the coefficients of \( f \) in addition to the conditions on the coefficients of the polynomial \( \left. f \right| _{P} \). An elementary, although tedious, verification shows that (15) holds.

To prove (15) in the case when \( l \leq k - 2 \), we need a new technique which is developed below.

3.2. Linearly independent points. The following assertion holds.

**Lemma 3.1.** Assume that \( d \geq 3 \). Then, for every set of \( m \) linearly independent points \( p_1, \ldots, p_m \in \mathbb{P}^N \), \( m \leq N + 1 \), the condition
\[
\{p_1, \ldots, p_m\} \subset \text{Sing}(g),
\]
for every \( g \in \mathcal{P}_{N,d} \), determines a vector subspace of codimension \( m(N + 1) \) in \( \mathcal{P}_{N,d} \).

**Proof.** We can assume that the points
\[
p_1 = (1 : 0 : 0 : \cdots : 0), \quad p_2 = (0 : 1 : 0 : \cdots : 0)
\]
and so on correspond to the first \( m \) vectors of the standard basis in the vector space \( \mathbb{C}^{N+1} \). The condition \( p_i \in \text{Sing}(g) \) means that the coefficients at the monomials \( x_{i-1}^d \) and \( x_{i-1}^{d-1}x_j \) vanish for all \( j \neq i - 1 \). When \( d \geq 3 \), all of these \( m(N + 1) \) monomials are distinct. □

We now consider an arbitrary vector subspace \( \Pi \subset \mathbb{P}^N \) of codimension \( r + 1 \), where \( r \geq 1 \), given by a system of \( r + 1 \) equations
\[
l_0(x) = 0, \quad l_1(x) = 0, \quad \ldots, \quad l_r(x) = 0,
\]
where \( l_0, \ldots, l_r \) are linearly independent forms. For every \( i = 1, \ldots, r \) we fix an arbitrary tuple of distinct constants \( \lambda_{i0}, \ldots, \lambda_{i,d-1} \in \mathbb{C} \) such that \( \lambda_{i0} = 0 \) for all \( i = 1, \ldots, r \). Then, given any integer point
\[
e = (e_1, \ldots, e_r) \in \mathbb{Z}_+^r, \quad e_i \leq d - 1,
\]
we write \( \Theta(\ne) \) for the linear subspace
\[
\{l_i(x) - \lambda_{i,e_i}l_0(x) = 0 \mid i = 1, \ldots, r\} \subset \mathbb{P}^N
\]
of codimension $r$. Clearly, $\Theta(e) \supset \Pi$. Set

$$|e| = e_1 + \cdots + e_r \in \mathbb{Z}_+.$$

For each tuple $e \in \mathbb{Z}_+^r$ with $|e| \leq d - 3$ we consider an arbitrary set

$$S(e) = \{p_1(e), \ldots, p_m(e)\} \subset \Theta(e) \setminus \Pi$$

of $m$ linearly independent points (so that $m \leq N - r + 1$).

**Proposition 3.2.** The set of conditions

$$S(e) \subset \text{Sing}(g|_{\Theta(e)}), \quad e \in \mathbb{Z}_+^r, \quad |e| \leq d - 3,$$

determines a vector subspace of codimension $m(N - r + 1)|\Delta|$ in $\mathcal{P}_{N,d}$, where

$$\Delta = \{e_1 \geq 0, \ldots, e_r \geq 0, e_1 + \cdots + e_r \leq d - 3\} \subset \mathbb{R}^r$$

is an integer simplex and $|\Delta|$ is the number of integer points in it: $|\Delta| = \#(\Delta \cap \mathbb{Z}^r)$.

**Proof.** We may assume that $l_0 = x_0$, $l_1 = x_1$, $\ldots$, $l_r = x_r$. To simplify the formulae, we shall prove an affine version of the proposition. Put $v_1 = x_1/x_0$, $\ldots$, $v_r = x_r/x_0$ and $u_i = x_{i+1}/x_0$, $i = 1, \ldots, N - r$. The affine spaces $A(e) = \Theta(e) \setminus \Pi$ are contained in the affine space $A^N \subset \mathbb{P}^N$, $A^N = \mathbb{P}^N \setminus \{x_0 = 0\}$, with coordinates $(u, v) = (u_1, \ldots, u_{N-r}, v_1, \ldots, v_r)$:

$$A(e) = \Theta(e) \cap A^N,$$

so that $S(e) \subset A(e)$ for all $e$. Clearly,

$$A(e) = \{v_1 = \lambda_{1,e_1}, \ldots, v_r = \lambda_{r,e_r}\} \subset A^N$$

is an ($N - r$)-plane parallel to the coordinate ($N - r$)-plane $(u_1, \ldots, u_{N-r}, 0, \ldots, 0)$. We write the polynomial $g$ in terms of the affine coordinates $(u, v)$ as follows:

$$g(u, v) = \sum_{\varepsilon \in \mathbb{Z}_+^r, |\varepsilon| \leq d} g_{e_1,\ldots,e_r}(u) \prod_{i=1}^r \prod_{j=0}^{e_i-1} (v_i - \lambda_{ij})$$

(if $e_i = 0$, then the corresponding product is assumed to be equal to 1). Here $g_{\varepsilon}(u) = g_{e_1,\ldots,e_r}(u)$ is an affine polynomial in $u_1, \ldots, u_{N-r}$ of degree $\deg g_{\varepsilon} \leq d - |\varepsilon|$. This presentation is unique for fixed $\lambda_{ij}$. By Lemma 3.1, the condition

$$S(0) = S(0, \ldots, 0) \subset \text{Sing}(g|_{A(0)})$$

determines a vector subspace of codimension $m(N - r + 1)$ in the space of polynomials $\mathcal{P}_{N-r,d}$. However, it is easy to see that

$$g|_{A(0)} = g_{0,\ldots,0}(u)$$
since for every $e \neq 0$ the product
\[
\prod_{i=1}^{r} \prod_{j=0}^{e_i-1} (v_i - \lambda_{ij})
\]
contains at least one factor $(v_i - \lambda_{i0}) = v_i$, which vanishes when restricted to the $(N - r)$-plane $A(0)$. Therefore the condition $S(0) \subset \text{Sing}(g|_{A(0)})$ imposes precisely $m(N - r + 1)$ independent linear conditions on the coefficients of the polynomial $g_0, ..., g_0(u)$, whereas the polynomials $g_e(u)$ with $e \neq 0$ can be arbitrary.

We now complete the proof of Proposition 3.2 by induction on $|e|$. More precisely, for every $a \in \mathbb{Z}_+$ we set
\[
\Delta_a = \{e \in \mathbb{Z}_{r+}, e_i \geq 0, e_1 + \cdots + e_r \leq a\} \subset \mathbb{R}^r,
\]
so that $\Delta = \Delta_{d-3}$ and we shall prove Proposition 3.2 in the following form for every $a = 0, \ldots, d-3$.

**Assertion (**)a. The set of conditions**
\[
S(e) \subset \text{Sing}(g|_{A(\xi)}), \quad e \in \mathbb{Z}_{r+}, \quad |e| \leq a,
\]
determines a vector space of codimension $m(N - r + 1)|\Delta_a|$ in $P_{N,d}$. The linear restrictions are imposed on the coefficients of the polynomials $g_e(u)$ for $e \in \Delta_a$, whereas the polynomials $g_e(u)$ with $e \notin \Delta_a$ can be arbitrary.

The case $a = 0$ has already been considered. Therefore we assume that $a \leq d-4$ and the assertions (**) have already been proved for all $j = 0, \ldots, a$. Let us prove (**) for $a+1$. Let $\xi \in \mathbb{Z}_{r+}$ be an arbitrary multi-index with $|\xi| = a+1$. Restriction to the affine subspace $A(\xi)$ is performed by the substitution $v_1 = \lambda_{1,e_1}, \ldots, v_r = \lambda_{r,e_r}$. Therefore the polynomial $g_\xi(u)$ occurs in the restriction $g|_{A(\xi)}$ with a non-zero coefficient,
\[
\alpha_\xi = \prod_{i=1}^{r} \prod_{j=0}^{e_i-1} (\lambda_{i,e_i} - \lambda_{ij}).
\]
On the other hand, for every $\xi' \neq \xi$ with $|\xi'| \geq a + 1$, the product
\[
\prod_{i=1}^{r} \prod_{j=0}^{e'_i-1} (\lambda_{i,e_i} - \lambda_{ij})
\]
is equal to zero since we have $e'_i > e_i$ for at least one index $i \in \{1, \ldots, r\}$ and, therefore, the product contains a zero factor. Hence $g|_{A(\xi)}$ is the sum of the polynomial $\alpha_\xi g_\xi$ and a linear combination (with constant coefficients) of the polynomials $g_{\xi'}$, where $|\xi'| \leq a$. Now, fixing the polynomials $g_{\xi'}, |\xi'| \leq a$, we obtain that the condition
\[
S(\xi) \subset \text{Sing}(g|_{A(\xi)})
\]
determines an affine subspace (generally speaking, not a vector subspace) of codimension $m(N - r + 1)$ in the space of polynomials $g_\xi(u_1, \ldots, u_{N-r})$ of degree
at most $d - |e|$. The underlying vector subspace of this affine subspace is given by the condition

$$S(\xi) \subset \text{Sing} g_{\xi}(u).$$

Note that we impose no restrictions on the coefficients of the other polynomials $g_{\xi'}$ with $|\xi'| = a + 1$.

This completes the proof of $(\ast)_a$ for all $a = 0, \ldots, d - 3$. Proposition 3.2 is proved. □

3.3. Completion of the proof of Theorem 3.1. Let

$$\Theta = \Theta[l_0, \ldots, l_r; \lambda_{i,j}, i = 1, \ldots, r, j = 0, \ldots, d - 1] = \{\Theta(\xi) \mid \xi \in \Delta\}$$

be a set of linear subspaces of codimension $r$ in $\mathbb{P}^N$ occurring in Proposition 3.2. We define a subset

$$P_{N,d}(\Theta) \subset P_{N,d}$$

by the following condition. For every subspace $\Theta(\xi)$ with $|\xi| \leq d - 3$ there is a set $S(\xi) \subset \Theta(\xi) \setminus \Pi$ consisting of $m$ linearly independent points such that $S(\xi) \subset \text{Sing}(g|_{\Theta(\xi)})$.

**Proposition 3.3.** We have

$$\text{codim}(P_{N,d}(\Theta) \subset P_{N,d}) \geq m|\Delta|.$$ 

**Proof.** The proof is by an obvious dimension count. The subspaces $\Theta(\xi)$ are fixed, so that every point $p_i(\xi)$ varies over an $(N - r)$-dimensional family. □

We now complete the proof of Theorem 3.1. Consider the set $P_{N,d}^{(1,k;l)}(P)$, where $P \subset \mathbb{P}^N$ is the fixed $k$-plane $\{x_{k+1} = \cdots = x_N = 0\}$ and $l \leq k - 2$. We apply Proposition 3.3 with the space $P$ instead of $\mathbb{P}^N$ and with the space of polynomials $P_{k,d}$ instead of $P_{N,d}$. Given an arbitrary set $\Theta = \{\Theta(\xi) \mid \xi \in \Delta\}$ of vector subspaces of codimension $l$ in $P = \mathbb{P}^k$, we write

$$P_{N,d}^{(1,k;l)}(P, \Theta) \subset P_{N,d}^{(1,k;l)}$$

for the set of polynomials $f \in P_{N,d}^{(1,k;l)}$ such that the set $\text{Sing}(f|_P)$ has an irreducible component $Q$ of dimension $l$ which contains a curve $C \subset \text{Sing}(f)$ and is in general position with respect to the subspaces in $\Theta$: for every $\xi \in \Delta$ the set $\Theta(\xi) \cap Q$ contains $(k - l + 1)$ linearly independent points. Since $\langle Q \rangle = \langle C \rangle = P$, the subset $P_{N,d}^{(1,k;l)}(P, \Theta)$ is a Zariski-open subset of $P_{N,d}^{(1,k;l)}$. Hence the inequality (15) will be proved if we can establish it for $P_{N,d}^{(1,k;l)}(P, \Theta)$ instead of $P_{N,d}^{(1,k;l)}$. By Proposition 3.3 for the space $P$, the condition $f \in P_{N,d}^{(1,k;l)}(P, \Theta)$ imposes at least $(k - l + 1)|\Delta|$ independent conditions on the coefficients of the polynomial $f|_P$. Furthermore, the set of $N + 1$ polynomials

$$\frac{\partial f}{\partial x_0}|_P, \ldots, \frac{\partial f}{\partial x_N}|_P$$
contains a good sequence of $k - 1$ polynomials with associated subvariety $C$, where $C$ is a curve with $\langle C \rangle = P$. This good sequence can be chosen in such a way that its first $k - l$ polynomials are among the polynomials

$$\frac{\partial f}{\partial x_0} \bigg|_P, \ldots, \frac{\partial f}{\partial x_k} \bigg|_P$$

(and some subvariety $Q \supset C, Q \subset P$ of dimension $l$ is an associated subvariety for this subsequence) while the next $l - 1$ polynomials are among the polynomials

$$\frac{\partial f}{\partial x_{k+1}} \bigg|_P, \ldots, \frac{\partial f}{\partial x_N} \bigg|_P.$$

By fixing the polynomial $f \big|_P$ and the other polynomials of the good sequence, we can assume that the curve $C \subset \text{Sing}(f)$ of singular points is also fixed. Then the condition $\frac{\partial f}{\partial x_i} \big|_C \equiv 0$ for all $i \in \{k+1, \ldots, N\}$ such that $\partial f / \partial x_i$ does not belong to the good sequence, gives in total $(N + 1 - k - l)((d - 1)k + 1)$ additional independent conditions on the coefficients of $f$. An elementary, although tedious, verification shows that the inequality

$$(k - l + 1)|\Delta| + (N + 1 - k - l)((d - 1)k + 1) \geq (d - 2)N + (k + 1)(N - k)$$

holds for all the values of $k$, $l$ under consideration. This completes the proof of the inequality (15) and Theorem 3.1 and, therefore, of Theorem 0.3. □

**Remark 3.2.** It is easy to see that the worst estimate for the codimension of $\mathcal{P}_{N,d}^{(1,k;1)}(P)$ corresponds to the case when $k = N$ and $l = 1$, that is, the hypersurface $\{f = 0\}$ has a non-degenerate curve of singular points. In this case, Proposition 3.3 yields that

$$\text{codim} \left( \mathcal{P}_{N,d}^{(1,k;1)} \subset \mathcal{P}_{N,d} \right) \geq (d - 2)N.$$ 

It seems improbable that the presence of a non-degenerate curve of singular points imposes fewer (although slightly fewer) independent conditions on the coefficients of $f$ than the presence of a line consisting of singular points (when the bound for the codimension is precise). And indeed, when we use Proposition 3.3, we are essentially replacing the curve consisting of singular points by a finite (although large) set of singular points. The technique used in the proof of Theorem 3.1 can probably be improved to obtain a more precise estimate in the case of a non-degenerate curve of singular points. This is what was meant in Remark 3.1.

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