Consumption–investment optimization with Epstein–Zin utility in incomplete markets

Hao Xing

Received: 22 February 2015 / Accepted: 12 November 2015 / Published online: 12 April 2016
© Springer-Verlag Berlin Heidelberg 2016

Abstract In a market with stochastic investment opportunities, we study an optimal consumption–investment problem for an agent with recursive utility of Epstein–Zin type. Focusing on the empirically relevant specification where both risk aversion and elasticity of intertemporal substitution are in excess of one, we characterize optimal consumption and investment strategies via backward stochastic differential equations. The superdifferential of indirect utility is also obtained, meeting demands from applications in which Epstein–Zin utilities were used to resolve several asset pricing puzzles. The empirically relevant utility specification introduces difficulties to the optimization problem due to the fact that the Epstein–Zin aggregator is neither Lipschitz nor jointly concave in all its variables.

Keywords Consumption–investment optimization · Epstein–Zin utility · Backward stochastic differential equation

Mathematics Subject Classification (2000) 93E20 · 91G10

JEL Classification G11 · D91

1 Introduction

Risk aversion and elasticity of intertemporal substitution (EIS) are two parameters describing two different aspects of preferences: risk aversion measures agents’ attitude toward risk, while EIS regulates agents’ willingness to substitute consumption over time. However, commonly used time-separable utilities force EIS to be the reciprocal of risk aversion, leading to a rich literature on asset pricing anomalies, such
as the equity premium puzzle, the risk-free rate puzzle, the excess volatility puzzle, the credit spread puzzle, etc.

Recursive utilities of Kreps–Porteus or Epstein–Zin type and their continuous-time analogues disentangle risk aversion and EIS, providing a framework to resolve the aforementioned asset pricing puzzles; cf. [2] and [1] for the equity premium puzzle and the risk-free rate puzzle, [4] for the excess volatility puzzle, and [5] for the credit spread puzzle. All these studies require EIS $\psi$ to be larger than 1 in order to match empirical observations. Bansal and Yaron [2] also empirically estimated $\psi$ to be around 1.5. On the other hand, empirical evidence suggests that risk aversion $\gamma$ is in excess of 1. It then follows from $\gamma > 1$ and $\psi > 1$ that $\gamma \psi > 1$. Hence an agent with such a utility specification prefers early resolution of uncertainty (cf. [30] and [41]), therefore asks a sizeable risk premium to compensate for future uncertainty in the state of the economy.

Other than the aforementioned utility specification, two other ingredients are also important in these asset pricing applications. First, investment opportunities in these models are driven by some state variables, which usually lead to an unbounded market price of risk; for example, the Heston model in [9, 26] and [31], or the Kim and Omberg model in [24] and [43]. Second, the first step in all these applications is to understand the superdifferential of the indirect utility for the representative agent, because it is the source to read out the equilibrium risk-free rate and risk premium; cf. [2, Appendix]. Therefore, it is important to rigorously study the consumption–investment problem simultaneously accounting for these three ingredients: utility specification, models with unbounded market price of risk, and superdifferential of indirect utility. However, the following literature review shows that such a study in a continuous-time setting was still missing from the literature. The present paper fills this gap.

In the seminal paper by Duffie and Epstein [13], stochastic differential utilities (the continuous-time analogue of recursive utilities, cf. [28]) are assumed to have Lipschitz-continuous aggregators. Hence the Epstein–Zin aggregator, which is non-Lipschitz, is excluded. Schroder and Skiadas [38] studied the case where $\theta = \frac{1 - \gamma}{1 - 1/\psi}$ is positive.1 However, the empirically relevant parameter specification $\gamma, \psi > 1$ leads to $\theta < 0$. Kraft, Seifried and Steffensen [29] studied incomplete market models with unbounded market price of risk; however, their assumption on $\gamma$ and $\psi$ (cf. Eq. (H) therein) excludes the case $\gamma > 1$ and $\psi > 1$.

Regarding market models, Schroder and Skiadas [38] studied a complete market with bounded market price of risk. Schroder and Skiadas [39, Sect. 5.6], Chacko and Viceira [9] both considered incomplete markets and Epstein–Zin utility with unit EIS. Chacko and Viceira [9], Kraft, Seifried and Steffensen [29] studied a market model whose investment opportunities are driven by a square root process, leading to an unbounded market price of risk.

Regarding the superdifferential of indirect utility, its form can be obtained by a heuristic calculation using the utility gradient approach; cf. [15]. However, rigorous verification needs the aggregator to satisfy a Lipschitz growth condition (cf. [13] and [15]), or joint concavity in both consumption and utility variables (cf. [16]). As

---

1The parameter $1 + \alpha$ in [38] is $\theta$ here. Hence equation (8c) therein implies $\theta > 0$. 
we shall see later, when $\gamma > 1$ and $\psi > 1$, the Epstein–Zin aggregator is neither Lipschitz-continuous nor jointly concave. On the other hand, for Epstein–Zin utility with $\theta > 0$, Schroder and Skiadas [38] verified the heuristic form of the superdifferential via an integrability condition (cf. [38, Lemma 2]). Then the optimality of the consumption stream is verified using the condition that the sum of the deflated wealth process and the integral of the deflated consumption stream is a supermartingale for an arbitrary admissible strategy, and a martingale for the optimal strategy (cf. [38, Eq. (1)]). This argument is carried out in [38, Theorem 2 and 4] for complete market models with a bounded market price of risk.

In this paper, we analyze a consumption–investment problem for an agent with Epstein–Zin utility with $\gamma, \psi > 1$ and a bequest utility at a finite time horizon. This agent invests in an incomplete market whose investment opportunities are driven by a multivariate state variable. Rather than the Campbell–Shiller approximation, which is widely applied for utilities with non-unit EIS, we study exact solutions. As illustrated in [29, Sect. 6], there can be a sizeable deviation of the Campbell–Shiller approximation from the exact solution, highlighting the importance of an exact solution.

A similar problem has also been studied recently by Kraft, Seiferling and Seifried [27]. In [27], the relation between $\gamma$ and $\psi$ in [29] is removed and all configurations of $\gamma$ and $\psi$ are considered, including the $\gamma, \psi > 1$ case. A verification result is obtained following the utility gradient approach in [15] and [38], complemented by a recent note of Seiferling and Seifried [40] for the $\gamma, \psi > 1$ case. Nevertheless, [27] focuses on models with a bounded market price of risk (cf. Assumptions (A1) and (A2) therein). This excludes models such as the Heston and Kim–Omberg models, which are widely used in the aforementioned asset pricing applications. Comparing to [27] and all other aforementioned existing results, the current paper extends the previous literature in three respects.

First, in contrast to the utility gradient approach, our verification result is proved by comparison results for backward stochastic differential equations (BSDEs). Rather than employing the dynamic programming method as in [29] and [27], optimal consumption and investment strategies are represented by a BSDE solution; cf. Theorem 2.14 below. Extending techniques of Hu, Imkeller and Müller [20] and Cheridito and Hu [11], who studied optimal consumption–investment problems for time-separable utility, we verify the candidate optimal strategies for Epstein–Zin utility.

Second, our method is designed for market models with an unbounded market price of risk. Utilizing Lyapunov functions, borrowed from [42, Chap. 10], we prove in Lemma B.2 below that a certain exponential local martingale is a martingale, which is a key component of our verification argument.

Third, regarding the superdifferential of indirect utility, the integrability condition in [38, Lemma 2] is satisfied when $\gamma, \psi > 1$. It is proved in [38, Theorems 2 and 4] that the sum of the deflated wealth process and the integral of the deflated consumption stream is a supermartingale for any admissible strategy, and a martingale for the

---

2The specification $\gamma, \psi > 1$ is related to [38, Case 3 on page 113], which established the utility gradient inequality. Even though its proof is independent of the market model, it uses the existence and concavity of Epstein–Zin utility, which are established in [38, Appendix A] under the assumption $\theta > 0$. Therefore, one needs to replace [38, Appendix A] by Propositions 2.2 and 2.4 below which confirm the existence and concavity of Epstein–Zin utility when $\theta < 0$. During the revision of this paper, these properties were also confirmed in [40] for a general semimartingale setting.
optimal one, for complete market models with a bounded market price of risk. We obtain this property (see Theorem 2.16 below) as a by-product of our verification result. This result is established for models with an unbounded market price of risk, hence meets demands coming from the aforementioned applications to asset pricing puzzles.

Our general results in Sect. 2 are specialized to two examples in Sect. 3. There numerical results reveal an interesting phenomenon. As the time horizon goes to infinity, convergence of the finite-horizon solution to its stationary long-run limit is very slow when \( \psi > 1 \). Figure 2 shows that this convergence takes at least 60 years in an empirically relevant utility and market setting. Moreover, the convergence is sensitive to the time discounting parameter: it is much slower when the discounting parameter decreases slightly. This is in contrast to the \( \psi < 1 \) case, where the convergence is much faster (around 20 years) and is less sensitive to the time discounting parameter. This observation implies that in the \( \psi > 1 \) setting, the finite-horizon optimal strategy can be far away from its infinite-horizon analogue, even when we consider a lifelong consumption-investment problem.

The remainder of this paper is organized as follows. After Epstein–Zin utility is introduced in Sect. 2.1, the consumption–investment problem is introduced and the main results are presented in Sect. 2.2. Then the main results are specialized in two examples in Sect. 3, where the general assumptions of the main results are verified under explicit parameter restrictions, which include many empirically relevant cases. All proofs are postponed to appendices.

## 2 Main results

### 2.1 Epstein–Zin preferences

We work throughout the paper on a filtered probability space \((\Omega, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathcal{F}, \mathbb{P})\). Here \((\mathcal{F}_t)_{0 \leq t \leq T}\) is the augmented filtration generated by a \((k+n)\)-dimensional Wiener process \(B = (W, W^\perp)\), where \(W\) and \(W^\perp\) are the first \(k\) and the last \(n\) components, respectively, and satisfies the usual hypotheses of right-continuity and completeness.

Let \(\mathcal{C}\) be the class of nonnegative progressively measurable processes on \([0, T]\). For \(c \in \mathcal{C}\) and \(t < T\), \(c_t\) stands for the consumption rate at \(t\) and \(c_T\) represents a lump sum consumption at \(T\). We consider an agent whose preference over \(\mathcal{C}\)-valued consumption streams is described by a continuous-time stochastic differential utility of Kreps–Porteus or Epstein–Zin type. To describe this preference, let \(\delta > 0\) represent the discounting rate, \(0 < \gamma \neq 1\) be the relative risk aversion, and \(0 < \psi \neq 1\) be the EIS. We focus on the \(\gamma > 1\) case. In this case, define the Epstein–Zin aggregator \(f : [0, \infty) \times (-\infty, 0] \to \mathbb{R}\) via

\[
 f(c, v) := \delta \frac{(1 - \gamma)v}{1 - \frac{1}{\psi}} \left( \frac{c}{((1 - \gamma)v)^{1/\psi}} \right)^{1 - \frac{1}{\psi}} - 1.
\]  

(2.1)

This is a standard parametrization used, for instance, in [12]. Given a bequest utility function \(U(c) = c^{1-\gamma} / (1 - \gamma)\), the Epstein–Zin utility over the consumption stream
Consumption–investment optimization with Epstein–Zin utility

$c \in C$ on a finite time horizon $T$ is a process $V^c$ which satisfies

$$V_t^c = \mathbb{E}_t \left[ \int_t^T f(c_s, V^c_s) \, ds + U(c_T) \right], \quad \text{for all } t \in [0, T],$$  \hspace{1cm} (2.2)

where $\mathbb{E}_t$ stands for $\mathbb{E}[\cdot \mid \mathcal{F}_t]$.

Remark 2.1 Epstein–Zin utility generalizes the standard time-separable utility with constant relative risk aversion. Indeed, when $\gamma = 1/\psi$, the aggregator reduces to $f(c, v) = \delta \frac{1 - \gamma}{1 - \psi} - \delta v$. Then (2.2) with $t = 0$ can be represented explicitly as the standard time-separable utility

$$V_0^c = \mathbb{E} \left[ \int_0^T \delta e^{-\delta s} \frac{c_s^{1 - \gamma}}{1 - \gamma} \, ds + e^{-\delta T} U(c_T) \right].$$

As discussed in the introduction, we are interested in the empirically relevant case where $\gamma > 1$ and $\psi > 1$. In this case, $\gamma = 1/\psi$ is violated; hence (2.2) is not time-separable.

When $c$ follows a diffusion, the existence of $V^c$ was established by Duffie and Liou [14] via partial differential equation techniques. We work with a non-Markovian setting and construct $V^c$ via the BSDE

$$V_t^c = U(c_T) + \int_t^T f(c_s, V^c_s) \, ds - \int_t^T Z^c_s \, dB_s, \quad 0 \leq t \leq T. \hspace{1cm} (2.3)$$

Denote

$$\theta := \frac{1 - \gamma}{1 - \frac{1}{\psi}}.$$  \hspace{1cm}

When $\gamma, \psi > 1$, we have $\theta < 0$. The generator in (2.3) is

$$f(c, v) = \delta \frac{c^{1 - \frac{1}{\psi}}}{1 - \frac{1}{\psi}} ((1 - \gamma)v)^{1 - \frac{1}{\theta}} - \delta v.$$

Then $f$ has superlinear growth in $v$ when $\theta < 0$. Hence the generator of (2.3) is not Lipschitz. Nevertheless, consider $(Y_t, Z_t) := e^{-\delta \theta t} (1 - \gamma)(V_t^c, Z_t^c)$ and the transformed BSDE

$$Y_t = e^{-\delta \theta T} c_T^{1 - \gamma} + \int_t^T F(s, c_s, Y_s) \, ds - \int_t^T Z_s \, dB_s,$$  \hspace{1cm} (2.4)

where $F(t, c_t, y) := \delta \theta e^{-\delta t} c_t^{1 - \frac{1}{\psi}} y^{1 - \frac{1}{\theta}}$.

When $\theta < 0$, $F$ in (2.4) satisfies the monotonicity condition, i.e., $y \mapsto F(t, c_t, y)$ is decreasing. This allows us to establish the existence and uniqueness of solutions to (2.3), and hence to define $V^c$ satisfying (2.2).
Let us introduce the set of *admissible* consumption streams as\(^3\)

\[ C_a := \left\{ c \in C : \mathbb{E}\left[ \int_0^T e^{-\delta s} c_s^{1-\frac{1}{\psi}} ds \right] < \infty \text{ and } \mathbb{E}[c_T^{1-\gamma}] < \infty \right\}. \]

**Proposition 2.2** Suppose \( \gamma, \psi > 1 \) and \( c \in C_a \). Then (2.4) admits a unique solution \((Y, Z)\) in which \( Y \) is continuous, strictly positive and of class \( D \), and \( \int_0^T |Z_t|^2 dt < \infty \) a.s. Moreover, \( V^c_t := e^{\delta t} Y_t/(1 - \gamma), \ t \in [0, T], \) satisfies (2.2).

**Remark 2.3** When a BSDE satisfies the monotonicity condition, it is customary to assume its terminal condition to be square-integrable; cf. [34, Theorem 2.2]. However, this imposes unnecessary restrictions for the later described utility maximization problem, in the sense that the bequest utility needs to be square-integrable to define the associated Epstein–Zin utility. Therefore, Proposition 2.2 only asks for the terminal condition to be an integrable random variable.

Having defined \( V^c_0 \), we expect that as a utility functional, \( C_a \ni c \mapsto V^c_0 \) is concave. This would follow from the standard argument when \( f(c, v) \) is jointly concave in \( c \) and \( v \); cf. [13, Proposition 5]. However, calculation shows that \( f \) in (2.1) is *not* jointly concave when \( \gamma > 1 \) and \( \psi > 1.4 \) Nevertheless, utilizing an order equivalent transformation of \( V^c_0 \), introduced in [13, Example 3], the following proposition confirms the concavity of \( c \mapsto V^c_0 \).

Let us define \((Y, Z) := (Y^{1/\theta}, 1/\theta Y^{1/\theta-1} Z)/(1 - 1/\psi)\). Calculation shows that \((Y, Z)\) satisfies

\[ Y_t = e^{-\delta T} c_T^{1-\frac{1}{\psi}} + \int_t^T \left( \delta e^{-\delta s} c_s^{1-\frac{1}{\psi}} + \frac{1}{2} (\theta - 1) \frac{Z_s^2}{Y_s} \right) ds - \int_t^T Z_s dB_s. \quad (2.5) \]

Observe that the generator of (2.5) is now jointly concave in \((c, Y, Z)\) when \( \theta < 1 \).

**Proposition 2.4** Suppose that \( \gamma, \psi > 1 \). For any \( c, \tilde{c} \in C_a \) and \( \alpha \in [0, 1] \) such that \( \alpha c + (1 - \alpha)\tilde{c} \in C_a \), we then have

\[ \alpha V^c_0 + (1 - \alpha)V^\tilde{c}_0 \leq V^{\alpha c + (1 - \alpha)\tilde{c}}_0. \]

**Remark 2.5** The integrability condition in \( C_a \) does not imply the convexity of \( C_a \). Indeed, since \( \psi > 1 \), \( \mathbb{E}[\int_0^T e^{-\delta s} b_s^{1-\frac{1}{\psi}} ds] < \infty \) for both \( b = c \) and \( b = \tilde{c} \) does not imply the same integrability for \( b = \alpha c + (1 - \alpha)\tilde{c} \). However, Proposition 2.4 implies the concavity of \( c \mapsto V^c_0 \) on any convex subset of \( C_a \), for example,

\[ C^1_a = \left\{ c \in C_a : \mathbb{E}\left[ \int_0^T e^{-\delta s} c_s ds \right] < \infty \right\}. \]

\(^3\)This admissible set is similar to its counterpart in [11] for time-separable utilities, but is larger than its analogue in [38, Eq. (8a)], where \( \mathbb{E}[\int_0^T c^\ell ds] < \infty \) for all \( \ell \in \mathbb{R} \) is needed for an admissible consumption stream \( c \).

\(^4\)\( f \) is jointly concave in \( c \) and \( v \) if and only if \( \gamma \psi \leq 1 \).
2.2 Consumption–investment optimization

Having established the existence of Epstein–Zin utility in the previous section, we consider an optimal consumption–investment problem for an agent with such a utility. Consider a model of a financial market with a risk-free asset \( S^0 \) and risky assets \( S = (S^1, \ldots, S^n) \) with dynamics

\[
\begin{align*}
    dS^0_t &= S^0_t r(X_t) dt, \\
    dS_t &= \text{diag}(S_t) \left( (r(X_t)1_n + \mu(X_t)) dt + \sigma(X_t)dW^\rho_t \right),
\end{align*}
\]

where \( \text{diag}(S) \) is a diagonal matrix with the elements of \( S \) on the diagonal and \( 1_n \) is an \( n \)-dimensional vector with every entry 1. Given a correlation function \( \rho : \mathbb{R}^k \to \mathbb{R}^{n \times k} \) and \( \rho^\perp : \mathbb{R}^k \to \mathbb{R}^{n \times n} \), satisfying \( \rho \rho^\prime + \rho^\perp (\rho^\perp)^\prime = 1_{n \times n} \) (the \( n \)-dimensional identity matrix), \( W^\rho := \int \rho(X_s)dW_s + \int \rho^\perp(X_s)dW^\perp_s \) defines an \( n \)-dimensional Brownian motion. In (2.6), \( X \) is an \( E \)-valued state variable satisfying

\[
\begin{align*}
    dX_t &= b(X_t) dt + a(X_t)dW_t, \\
    X_0 &= x \in E.
\end{align*}
\]

Here \( E \) is an open domain in \( \mathbb{R}^k \), and the model coefficients \( r : E \to \mathbb{R}, \mu : E \to \mathbb{R}^n, \sigma : E \to \mathbb{R}^{n \times n}, b : E \to \mathbb{R}^k \) and \( a : E \to \mathbb{R}^{k \times k} \) satisfy

**Assumption 2.6** \( r, \mu, \sigma, b, a \) and \( \rho \) are all locally Lipschitz in \( E \); \( A := aa^\prime \) and \( \Sigma = \sigma \sigma^\prime \) are positive definite on any compact subdomain of \( E \); \( r + \frac{1}{\Sigma^\prime} \Sigma^{-1} \mu \geq 0 \) is bounded from below on \( E \); moreover, the dynamics of (2.7) does not hit the boundary of \( E \) in finite time.

In the previous assumption, local Lipschitz-continuity of the coefficients and the nonexplosion assumption combined imply that (2.7) admits a unique \( E \)-valued strong solution \( X \). When the interest rate \( r \) is bounded from below, due to \( r + \frac{1}{\Sigma^\prime} \Sigma^{-1} \mu \geq 0 \), \( r + \frac{1}{\Sigma^\prime} \Sigma^{-1} \mu \) is bounded from below as well.

An agent, whose preference is described by an Epstein–Zin utility, invests in this financial market. Given an initial wealth \( w \), an investment strategy \( \pi \) and a consumption rate \( c \), the wealth of the agent follows

\[
dW^\pi,c_t = W^\pi,c_t \left( (r_t + \pi'_t \mu_t) dt + \pi'_t \sigma_t dW^\rho_t \right) - c_t dt,
\]

\[
W^\pi,c_0 = w.
\]

Throughout the paper, \( r_t, \mu_t, r_t, \sigma_t \) stand for \( r(X_t), \mu(X_t), \rho(X_t) \) and \( \sigma(X_t) \), respectively, and the superscript \( (\pi, c) \) is sometimes suppressed on \( W \) to simplify notation. A pair \( (\pi, c) \) of investment strategy and consumption stream is admissible if \( c \in C_a \) and its associated wealth process is nonnegative. The agent aims to maximize her utility \( V^c_0 \) in the class of admissible strategies. We further restrict admissible strategies to a permissible set, and solve the agent’s optimization problem on this permissible set.

Let us first characterize the optimal value process via a heuristic argument. By the homothetic property of Epstein–Zin utility, we speculate that the utility evaluated at
the optimal strategy has the decomposition\(^5\)

\[
V_t^* = \frac{W_t^{1-\gamma}}{1-\gamma} e^{Y_t}, \quad t \in [0, T],
\]  

(2.9)

where \(Y\) satisfies the BSDE

\[
Y_t = \int_t^T H(s, Y_s, Z_s, Z_s^\perp) \, ds - \int_t^T Z_s \, dW_s - \int_t^T Z_s^\perp dW_s^\perp.
\]  

(2.10)

Let us determine the generator \(H\) in what follows. Parameterizing \(c\) by \(c = \tilde{c} \mathcal{W}\), the wealth process satisfies

\[
\frac{d\mathcal{W}_t}{\mathcal{W}_t} = (r_t - \tilde{c}_t + \pi'_t \mu_t) \, dt + \pi'_t \sigma_t \, dW_t^\rho.
\]

From the dynamic programming principle, we expect that the process

\[
\frac{\mathcal{W}_t^{1-\gamma}}{1-\gamma} e^{Y_t} + \int_0^t f \left( c_s, \frac{\mathcal{W}_s^{1-\gamma}}{1-\gamma} e^{Y_s} \right) ds
\]

is a supermartingale for an arbitrary strategy, and a martingale for the optimal strategy. Let us calculate the drift of that process. Calculation shows that

\[
d\mathcal{W}_t^{1-\gamma} = \mathcal{W}_t^{1-\gamma} \left( (1-\gamma)(r_t - \tilde{c}_t + \pi'_t \mu_t) - \frac{\gamma(1-\gamma)}{2} \pi'_t \Sigma_t \pi_t \right) dt
\]

\[
+ (1-\gamma) \mathcal{W}_t^{1-\gamma} \pi'_t \sigma_t dW_t^\rho,
\]

\[
de^{Y_t} = e^{Y_t} \left( - H(t, Y_t, Z_t, Z_t^\perp) + \frac{1}{2} Z_t Z_t' + \frac{1}{2} Z_t^\perp (Z_t^\perp)' \right) dt
\]

\[
+ e^{Y_t} (Z_t dW_t + Z_t^\perp dW_t^\perp).
\]

Therefore, the drift of \(\frac{\mathcal{W}_t^{1-\gamma}}{1-\gamma} e^{Y_t} + \int_0^t f \left( c_s, \frac{\mathcal{W}_s^{1-\gamma}}{1-\gamma} e^{Y_s} \right) ds\) reads (the time subscript is omitted to simplify notation)

\[
\frac{\mathcal{W}_t^{1-\gamma}}{1-\gamma} e^{Y_t} \left( (1-\gamma)r - \delta \bar{\theta} + \frac{1}{2} ZZ' + \frac{1}{2} Z^\perp (Z^\perp)' + \left( - (1-\gamma) \tilde{c} + \delta \bar{\theta} e^{-\frac{1}{2} \gamma \bar{c}} \right) + \left( - \frac{\gamma(1-\gamma)}{2} \pi' \Sigma \pi + \frac{1}{2} \gamma \pi' (\mu + \sigma \rho Z + \sigma \rho^Z Z^\perp) \right) \right)
\]

\[
- H(\cdot, Y, Z, Z^\perp).
\]  

(2.11)

We expect that the drift above is nonpositive for arbitrary \((\pi, \tilde{c})\) and zero for the optimal strategy. Therefore, the generator \(H\) for (2.10) can be obtained by formally taking the supremum over \(\pi\) and \(\tilde{c}\) in the previous drift and setting it to be zero. Following this direction, we notice that the randomness in \(H\) comes only from \(X\),

\(^5\)The decomposition (2.9) is widely used for (time-separable) power utilities; cf. e.g. [35].
which is driven by $W$; moreover, the terminal condition of (2.10) is zero. As a result, $Z^t$ is necessarily zero. Therefore, we can reduce (2.10) to

$$Y_t = \int_t^T H(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s,$$

where $H$ is given by

$$H(t, y, z) = (1 - \gamma)r_t - \delta \theta + \frac{1}{2}zz' + \inf_{\tilde{c}} \left( - (1 - \gamma)\tilde{c} + \delta \theta e^{-\frac{1}{2}\gamma \tilde{c}^{1-\frac{1}{\psi}}} \right)$$

$$+ \inf_{\pi} \left( - \frac{\gamma(1 - \gamma)}{2} \pi' \Sigma \pi + (1 - \gamma)\pi'(\mu_t + \sigma_t \rho_t z') \right)$$

$$= \frac{1}{2}z'M_t z' + \frac{1 - \gamma}{\gamma} \mu_t' \Sigma_t^{-1} \sigma_t \rho_t z' + \theta \frac{\delta \psi}{\psi} e^{-\frac{1}{2}\gamma y} + h_t - \delta \theta. \tag{2.13}$$

Here, suppressing the subscript $t$,

$$\Sigma := \sigma \sigma'(X), \quad M := 1_{k \times k} + \frac{1 - \gamma}{\gamma} \rho' \sigma' \Sigma^{-1} \sigma(X),$$

$$h := (1 - \gamma)r(X) + \frac{1 - \gamma}{2\gamma} \mu' \Sigma^{-1} \mu(X),$$

where $1_{k \times k}$ is the $k$-dimensional identity matrix. Recall from Assumption 2.6 that $r + \frac{1 - \gamma}{2\gamma} \mu' \Sigma^{-1} \mu$ is bounded from below. Therefore, $\gamma > 1$ implies that there exists a positive constant $h_{\text{max}}$ such that $h \leq h_{\text{max}}$ on $E$. Due to $\gamma > 1$, the drift in (2.11) being nonpositive leads to the two minimization problems in (2.13), whose minimizers are

$$\pi^*_t = \frac{1}{\gamma} \Sigma_t^{-1} (\mu_t + \sigma_t \rho_t Z'_t) \quad \text{and} \quad \frac{c^*_t}{\mathcal{W}^*_t} = \tilde{c}^*_t = \delta \psi e^{-\frac{1}{2}\gamma y}, \quad t \in [0, T), \tag{2.14}$$

where $\mathcal{W}^*$ is the wealth process associated to the strategy $(\pi^*, c^*)$. Therefore, $\pi^*$ and $c^*$ are candidate optimal strategies.

Coming back to (2.12), even though the generator $H$ has an exponential term in $y$ and a quadratic term in $z$, the parameter specification $\gamma, \psi > 1$ allows us to derive a priori bounds on $Y$. In particular, $Y$ is bounded from above by a constant. Meanwhile, since the quadratic term of $z$ in $H$ will be shown to be nonnegative, a lower bound on $Y$ can be obtained by studying a BSDE whose generator does not contain this quadratic term. As a result, a solution to (2.12) can be constructed under the following mild integrability conditions.

**Assumption 2.7**

(i) $\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} = \mathcal{E}(\int \frac{1 - \gamma}{\gamma} \mu' \Sigma^{-1} \sigma(X_s) dW_s)_T$ defines a probability measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$.

(ii) $\mathbb{E}^{\tilde{\mathbb{P}}}[\int_0^T h(X_s) ds] > -\infty.$

Here $\mathcal{E}(\int \alpha_s dW_s)_T := \exp(-\frac{1}{2} \int_0^T |\alpha_s|^2 ds + \int_0^T \alpha_s dW_s)$ denotes the stochastic exponential at time $T$ for $\int_0^T \alpha_s dW_s$. 
Remark 2.8 Since the generator $H$ contains a linear term in $z$, it is natural to apply Girsanov’s theorem. Assumption 2.7(i) allows us to do this and write (2.12) under $\mathbb{P}$. This assumption can be checked by explosion criteria; see Sect. 3 for examples. In (ii), the standard exponential moment condition in [6] is avoided, due to the special structure of $H$: the quadratic term in $z$ is nonnegative, and $H(\cdot, 0, 0)$ is bounded from above by $h_{\text{max}} - \delta \theta$.

Proposition 2.9 When $\gamma, \psi > 1$, let Assumption 2.7 hold. Then (2.12) admits a solution $(Y, Z)$ such that for any $t \in [0, T]$,

\[
\mathbb{E}_t^\mathbb{P}\left[ \int_t^T h(X_s) \, ds \right] - \delta \theta (T - t) + \theta \frac{\psi}{\psi - \delta} e^{(\frac{\psi}{\psi} - \psi) h_{\text{max}} T} (T - t) \leq Y_t \leq -\delta \theta (T - t) + \log \mathbb{E}_t^\mathbb{P}\left[ \exp \left( \int_t^T h(X_s) \, ds \right) \right],
\]

and $\mathbb{E}_t^\mathbb{P}[\int_0^T |Z_s|^2 \, ds] < \infty$. In particular, since $h \leq h_{\text{max}}$, $Y$ is bounded from above by $(h_{\text{max}} - \delta \theta) + T$.

Having constructed $(Y, Z)$, the strategies $(\pi^*, c^*)$ in (2.14) are well defined. To verify their optimality, we need to further restrict the admissible strategies to a permissible set: $(\pi, c)$ is permissible if $c \in C_a$ and $(\Lambda^{\pi, c})^{1-\gamma} e^{Y}$ is of class $D$ on $[0, T]$.

To verify the optimality for $(\pi^*, c^*)$, we introduce an operator $\tilde{\mathcal{G}}$ by

\[
\tilde{\mathcal{G}}[\phi] := \frac{1}{2} \sum_{i,j=1}^k A_{ij} \partial^2_{x_i x_j} \phi + \left( b + \frac{1 - \gamma}{\gamma} a \rho' \sigma' \Sigma^{-1} \mu \right)' \nabla \phi + \frac{1}{2} \nabla \phi' a M a' \nabla \phi + h
\]

for $\phi \in C^2(E)$. The dependence on $x$ is suppressed on both sides above. The function $\phi$ in the following assumption is called a Lyapunov function. Its existence facilitates proving that a certain exponential local martingale is in fact a martingale, hence verifying optimality of the candidate strategies. This proof strategy has been applied to portfolio optimization problems for time-separable utilities; cf. [18] and [37].

Assumption 2.10 There exists $\phi \in C^2(E)$ such that

(i) $\lim_{n \to \infty} \inf_{x \in E \setminus E_n} \phi(x) = \infty$, where $(E_n)$ is a sequence of open domains in $E$ satisfying $\bigcup_n E_n = E$, $E_n$ compact and $E_n \subset E_{n+1}$, for each $n$;

(ii) $\mathcal{G}[\phi]$ is bounded from above on $E$.

The final assumption before the main results imposes an integrability assumption on the market price of risk $\lambda$. This ensures $\mathbb{E}[\int_0^T e^{-\delta s} (c^*)^{1-1/\psi} \, ds] < \infty$. Together

---

6When $h$ is bounded from below, for example, if both $r$ and $\mu' \Sigma^{-1} \mu$ are bounded, (2.15) implies that $Y$ is bounded from below as well. Then $(\pi, c)$ is permissible if $c \in C_a$ and $(\Lambda^{\pi, c})^{1-\gamma}$ is of class $D$ on $[0, T]$. This is exactly the definition of permissibility used in [11] for the time-separable utilities with $\gamma > 1$. 

\[ \text{Springer} \]
Consumption–investment optimization with Epstein–Zin utility

with the integrability of \((c_T^*)^{1-\gamma}\) implied by Corollary B.3 below, the admissibility for the candidate optimal consumption stream \(c^*\) is verified.

**Assumption 2.11** There exists \(\lambda : E \to \mathbb{R}^n\) satisfying \(\mu = \sigma \lambda\) and defining a local martingale measure \(Q^0\) for discounted asset prices via \(dQ^0/d\mathbb{P} = \mathcal{E}(\int -\lambda_s dW_s^\rho)^T\). Moreover,

\[
\mathbb{E}^{Q^0}\left[e^{(\psi - 1)\int_0^T r^+(X_s)ds} \mathcal{E}\left(\int \lambda' (X_s) dW_s^0\right)^T\right] < \infty, \quad (2.17)
\]

where \(W^0 := W^\rho + \int_0^\cdot \lambda_s ds\) is a \(Q^0\)-Brownian motion and \(r^+ = \max\{r, 0\}\).

**Remark 2.12** Assumption 2.11 is stated under the minimal martingale measure \(Q^0\) (cf. [17]). A careful examination of Lemma B.4 shows that \(Q^0\) can be replaced by any local martingale measure \(Q\) such that \(\mathbb{E}^{Q}[\exp((\psi - 1)\int_0^T r^+(X_s)ds)(d\mathbb{P}/dQ)^\psi]\) is finite.

**Remark 2.13** When \(r\) and \(\lambda\) are bounded, Assumption 2.11 holds automatically and Assumption 2.10 is not needed, even for non-Markovian models. Indeed, Assumption 2.10 is used to prove that the stochastic exponential in Lemma B.2 below is a martingale. When \(r\) and \(\lambda\) are bounded, \(h\) is bounded, hence \(H(\cdot, 0, 0)\) is bounded as well. Therefore, (2.15) implies that \(Y\) is bounded, and \(\int_0^\cdot Z_s dW_s\) is a BMO-martingale; cf. e.g. [33, Lemma 3.1]. Then the stochastic exponential in Lemma B.2 can be proved to be a martingale directly. However, many models do not have a bounded market price of risk. Therefore, we retain Assumptions 2.10 and 2.11 in their general forms. These conditions impose some market conditions. In particular, for Markovian models, these conditions will be specified as explicit parameter restrictions in two examples in Sect. 3 below.

Now we are ready to state our first main result.

**Theorem 2.14** When \(\gamma, \psi > 1\), let Assumptions 2.6, 2.7, 2.10 and 2.11 hold. Then \(\pi^*\) and \(c^*\) in (2.14) maximize the Epstein–Zin utility among all permissible strategies. Moreover, the optimal Epstein–Zin utility is given by

\[
\frac{w^{1-\gamma}}{1-\gamma} e^{Y_0}.
\]

The second main result below focuses on the superdifferential of indirect utility. Let us first define the optimal value process

\[
V^*_t := \frac{(\mathcal{W}^*_t)^{1-\gamma}}{1-\gamma} e^{Y_t}, \quad t \in [0, T], \quad (2.18)
\]

where \(\mathcal{W}^*_t\) is the optimal wealth process and \(Y\) comes from Proposition 2.9. Schroder and Skiadas [38] conjectured in their Assumption C3 that the superdifferential is

\[
D^*_t = w^\gamma e^{-Y_0} \exp\left(\int_0^t \partial_v f(c^*_s, V^*_s)ds\right) \partial_c f(c^*_t, V^*_t), \quad t \in [0, T]. \quad (2.19)
\]
The constant \( w^\gamma e^{-Y_0} \) in (2.19) normalizes \( D_0^* \) to be 1. Indeed, combining (2.1), (2.14) and (2.18), calculation shows that

\[
D_t^* = w^\gamma e^{-Y_0} \exp \left( \int_0^t \delta(\theta - 1)((1 - \gamma)V_s^*)^{\frac{1}{\theta}} \left( \nabla^\gamma c_s^* \right)^{\frac{1}{\psi}} ds - \delta \theta t \right) 
\times \delta(1 - \gamma)^{\frac{1}{\theta}} (c_t^*)^{-\frac{1}{\psi}} 
\times \exp \left( \int_0^t (\theta - 1)\delta\psi e^{-\frac{\psi}{\theta} Y_s} ds - \delta \theta t \right) \frac{(W_T^Y)^{-\gamma} e^{Y_t}}{w^{-\gamma} e^{Y_0}}.
\]

Therefore, the previous identity implies that \( D_0^* = 1 \) and \( D^* \) is nonnegative.

In [38, Theorems 2 and 4], \( D^* \) is confirmed to be the superdifferential when the market is complete with a bounded market price of risk. This is proved using an integrability assumption in [38, Lemma 2], together with the property that \( WD^* + \int_0^T D_s^* c_s ds \) is a supermartingale for an arbitrary strategy, and a martingale for the optimal one. The integrability assumption in [38, Lemma 2] is satisfied in our case. Indeed, (2.20) shows that \( \partial_v f(c^*, V^*) = (\theta - 1)\delta\psi e^{-\frac{\psi}{\theta} Y} - \delta \theta \), which is bounded due to \( \theta < 0 \) and since \( Y \) is bounded from above. Now the following result confirms the aforementioned property for \( WD^* + \int_0^T D_s^* c_s ds \) also in markets with an unbounded market price of risk.

**Lemma 2.15** For \( D^* \) given by (2.20), it satisfies

\[
dD_t^* = -r_t D_t^* dt + D_t^* \left( -\gamma (\pi_t^*)' \sigma_t dW_t + Z_t dW_t \right), \quad D_0^* = 1,
\]

where \( Z \) comes from Proposition 2.9. Therefore, for any admissible strategy \((\pi, c)\), \( WD^* + \int_0^T D_s^* c_s ds \) is a nonnegative local martingale, hence a supermartingale.

Finally, our second main result below confirms that \( WD^* + \int_0^T D_s^* c_s ds \) is in fact a martingale. This result has been proved for recursive utilities with a Lipschitz-continuous aggregator which is also jointly concave in all its variables; cf. [16, Theorems 4.2 and 4.3]. However, as we have seen before, none of these conditions are satisfied when \( \gamma, \psi > 1 \).

**Theorem 2.16** When \( \gamma, \psi > 1 \), let Assumptions 2.6, 2.7, 2.10 and 2.11 hold. Then, for the optimal strategy \((\pi^*, c^*) \) given in (2.14), \( WD^* + \int_0^T D_s^* c_s ds \) is a martingale. Therefore, for any admissible strategy \((\pi, c)\),

\[
\mathbb{E} \left[ W_T^{\pi, c} D_T^* + \int_0^T D_s^* c_s ds \right] \leq w = \mathbb{E} \left[ W_T^{\pi^*, c^*} D_T^* + \int_0^T D_s^* c_s ds \right].
\]

**Remark 2.17** The property of \( D^* \) established in Theorem 2.16, together with the utility superdifferential inequality

\[
V_0^c \leq V_0^{c^*} + \mathbb{E} \left[ \int_0^T D_s^* (c_s - c_s^*) ds + D_T^* (W_T^{\pi, c} - W_T^{\pi^*, c^*}) \right],
\]

would verify the optimality of \( c^* \) among all admissible strategies, not necessarily restricted to permissible ones. However, the utility superdifferential inequality above...
is only established among $c$ satisfying $\mathbb{E}[\int_0^T c_s^\ell ds + c_T^\ell] < \infty$ for all $\ell \in \mathbb{R}$; cf. [38, Lemma 2] for the case $\theta > 0$ and [40, Theorem 3.4] for the case $\gamma, \psi > 1$. The candidate optimal consumption stream $c^\ast$ may fail to satisfy this integrability condition, in particular for models with an unbounded market price of risk.

In an equilibrium setting where the representative agent has an Epstein–Zin utility, given the consumption stream, the equilibrium risk-free rate and risk premium can be read out from $D^\ast$, providing a framework to study various asset pricing puzzles as discussed in the introduction.

3 Examples

This section specifies the general results in the previous section to two extensively studied models, where explicit parameter restrictions are presented so that all assumptions in the previous section are satisfied; hence the statements of Theorems 2.14 and 2.16 hold. These parameter restrictions cover many empirically relevant specifications.

3.1 Stochastic volatility

The following model has a 1-dimensional state variable, following a square-root process as suggested by Heston, which simultaneously affects the interest rate, the excess return of risky assets and their volatility. This model has been studied by [9] for recursive utilities with unit EIS, and by [26, 31] for time-separable utilities. The model is specified as

$$
\begin{align*}
\begin{cases}
\frac{dS_t}{S_t} = \text{diag}(S_t) \left[ (r(X_t) 1_n + \mu(X_t)) dt + \sqrt{X_t} \sigma dW_t^\rho \right], \\
\frac{dX_t}{X_t} = b(\ell - X_t) dt + a \sqrt{X_t} dW_t,
\end{cases}
\end{align*}
$$

where $r(x) = r_0 + r_1 x$, $\mu(x) = \sigma \lambda x$, with $r_0, r_1 \in \mathbb{R}$, $\sigma \in \mathbb{R}^{n \times n}$, $\lambda, \rho \in \mathbb{R}^n$, and $b, \ell, a \in \mathbb{R}$. These parameters satisfy

**Assumption 3.1** $b, \ell, r_1 + \frac{1}{2\gamma} \lambda' \Sigma^{-1} \sigma \lambda \geq 0, a > 0$ and $b \ell > \frac{1}{2} a^2$.

This assumption ensures that $X$ takes values in $(0, \infty)$ and $r + \frac{1}{2\gamma} \mu' \Sigma^{-1} \mu$ is bounded from below; hence Assumption 2.6 is satisfied with $E = (0, \infty)$. The following result provides parameter restrictions such that the statements of Theorems 2.14 and 2.16 hold.

**Proposition 3.2** When $\gamma, \psi > 1$, let Assumption 3.1 and the following parameter restrictions hold:

(i) Either $r_1 > 0$ or $\lambda' \Sigma^{-1} \sigma \lambda > 0$;
(ii) $(\psi - 1) (r_1 + \frac{b\lambda'\rho}{a} + \frac{1}{2} \lambda' (\psi 1_{n \times n} - (\psi - 1) \rho \rho') \lambda) < \frac{b^2}{2a^2}$.

Then the statements of Theorems 2.14 and 2.16 hold.
Fig. 1 Both figures use the parameters in (3.2) and \( r = 0.05, \delta = 0.08, \ell = 0.0225 \). They are both time-0 values for a problem with time horizon \( T = 10 \) years. The left panel takes \( \gamma = 5 \), and the right panel uses \( \psi = 1.5 \).

In item (i), either the interest \( r(x) \) or the excess rate of return \( \mu(x) \) increase linearly in \( x \). In item (ii), the inequality asks that either \( b \), the mean-reversion speed of the state variable, is large, or that the volatility \( a \) is small, or that the EIS is close to 1. In particular, when \( r_1 = 0 \) (i.e., constant interest rate) and \( \psi > 1 \), the condition in item (ii) is satisfied when\(^7\)

\[
b\lambda'\rho \leq -\frac{1}{2}\psi a\lambda'\lambda.
\]  

(3.1)

This condition covers the empirically relevant specification in [32], where the parameter values are

\[
\lambda = 0.47, \quad \sigma = 1, \quad b = 5, \quad a = 0.25 \quad \text{and} \quad \rho = -0.5.
\]  

(3.2)

Taking \( \psi = 1.5 \) from [2], (3.1) is verified by calculation.

Figure 1 shows the optimal consumption–wealth ratio \( c^*/W^* \) and the optimal investment fraction \( \pi^* \) with respect to the volatility \( \sqrt{X} \) for different values of the risk aversion and the EIS. Meanwhile, our numerical results show that the EIS has little impact on the optimal investment fraction, and different risk aversions hardly change the optimal consumption–wealth ratio. Figure 2 compares the optimal consumption–wealth ratio for \( \psi = 0.2 \) (top panel) and \( \psi = 1.5 \) (bottom panel). When \( \psi = 0.2 \), the finite-horizon optimal consumption–wealth ratio converges quickly to its infinite-horizon stationary limit. For the parameter specification in (3.2), when the horizon is longer than 20 years, the time-0 optimal consumption strategy is already close to its stationary limit. However, this convergence is much slower when \( \psi = 1.5 \), requiring at least 60 years for the time discounting parameter \( \delta = 0.08 \). Moreover, in contrast to the \( \psi = 0.2 \) case, the convergence speed is sensitive to \( \delta \) when \( \psi = 1.5 \). In this case, the convergence is much slower for smaller values of \( \delta \). Intuitively, an agent with small discounting parameter is more patient. But she still prefers early consumption

\(^7\)Since \( \psi > 1 \), (3.1) yields \( \frac{b\lambda'\rho}{a} + \frac{1}{2}\psi \lambda'(\psi \lambda'\rho - (\psi - 1)\rho')\lambda \leq \frac{b\lambda'\rho}{a} + \frac{1}{2}\psi \lambda'\lambda \leq 0 \). Hence the left-hand side of the inequality in Proposition 3.2(ii) is negative.
Consumption–investment optimization with Epstein–Zin utility

Fig. 2 Optimal consumption–wealth ratio as a function of time when the volatility is 20%. Both figures use the parameters in (3.2), \( r = 0.05, \ell = 0.0225 \) and \( \gamma = 5 \). The upper panel takes \( \psi = 0.2 \) and \( T = 30 \) years. The lower panel takes \( \psi = 1.5 \) and \( T = 100 \) years.

when \( \psi > 1 \). Therefore these two competing forces delay the convergence. All comparative statics are produced by solving the partial differential equation counterpart of (2.12) numerically using finite difference methods.

3.2 Linear diffusion

Both the interest rate and the excess return of risky assets in the following model are linear functions of a state variable, which follows a 1-dimensional Ornstein–Uhlenbeck process. This model has been studied in [24] and [43] for the time-separable utility setting, and in [7] for recursive utilities in a discrete-time setting.

The model dynamics is given by

\[
\begin{align*}
    dS_t &= \text{diag}(S_t) \left( (r(X_t) 1_n + \mu(X_t))dt + \sigma dW_t \right), \\
    dX_t &= -bX_t dt + adW_t,
\end{align*}
\]

where \( r(x) = r_0 + r_1x \) and \( \mu(x) = \sigma(\lambda_0 + \lambda_1x) \) for some \( r_0, r_1 \in \mathbb{R}, \lambda_0, \lambda_1 \in \mathbb{R}^n, \sigma \in \mathbb{R}^{n \times n} \), \( b, a \in \mathbb{R} \) and \( \rho \in \mathbb{R}^n \). These coefficients satisfy

**Assumption 3.3** \( a, b > 0 \), and either \( r_1 = 0 \) or \( \lambda_1^T \Sigma^{-1} \sigma \lambda_1 > 0 \).

This assumption implies that Assumption 2.6 is satisfied with \( E = \mathbb{R} \). Under the following parameter restrictions, the statements of Theorems 2.14 and 2.16 hold.

**Proposition 3.4** When \( \gamma, \psi > 1 \), let Assumption 3.3 and the following parameter restrictions hold:
Both figures use the parameters in (3.3) and $r = 0.0014$ and $\delta = 0.0052$. They are both time-0 values for a problem with time horizon $T = 12$ months. The left panel takes $\gamma = 5$. The optimal consumption–wealth ratio for the $\psi = 0.2$ case is much larger than those displayed in the left panel. The right panel takes $\psi = 1.5$

(i) Either $-b + \frac{1 - \nu}{\nu} a \lambda_1' \sigma' \Sigma^{-1} \sigma \rho < 0$ or $\lambda_1' \sigma' \Sigma^{-1} \sigma \lambda_1 > 0$;
(ii) $(\psi - 1) (\frac{b \lambda_1' \rho}{a} + \frac{1}{2} \lambda_1' (\psi I_{n \times n} - (\psi - 1) \rho \rho') \lambda_1) < \frac{b^2}{2a^2}$.

Then the statements of Theorems 2.14 and 2.16 hold.

In the above item (i), observe that $(-b + \frac{1 - \nu}{\nu} a \lambda_1' \sigma' \Sigma^{-1} \sigma \rho) X$ is the drift of $X$ under $\mathbb{P}$. Therefore, item (i) assumes that either $X$ is mean-reverting under $\mathbb{P}$ or the excess rate of return $\mu(x)$ increases linearly in $x$. Item (ii) is interpreted similarly as Proposition 3.2(ii). In particular, when $\psi > 1$, the inequality in item (ii) is satisfied when

$$
 b \lambda_1' \rho \leq -\frac{1}{2} \psi a \lambda_1' \lambda_1.
$$

This condition already covers many empirically relevant specifications. For example, in [3] and [43], a single risky asset was considered and parameter values (in monthly units) are

$$
\lambda_1 = 1, \quad \sigma = 0.0436, \quad b = 0.0226, \quad a = 0.0189, \quad \rho = -0.935, \quad \psi = 1.5.
$$

Figure 3 shows the optimal consumption–wealth ratio $c^*/W\pi^*$ and the optimal investment fraction $\pi^*$ with respect to the state variable $X$.

Acknowledgements  The author is grateful to Anis Matoussi for inspiring discussions, to Paolo Guasoni for valuable comments on the draft, and to two anonymous referees and the editor Jaksha Cvitanic for their precise comments which helped me to improve this paper.

---

8The proof is the same as in footnote 7.
Appendix A: Proofs for Sect. 2.1

Let us first introduce several notations which are used throughout the appendices.

- Let $\mathcal{S}^2$ denote the space of 1-dimensional continuous adapted processes $(Y_t)_{0 \leq t \leq T}$ such that $\mathbb{E}[\sup_{0 \leq s \leq T} |Y_s|^2] < \infty$.
- Let $\mathcal{S}^\infty$ be the subspace of those $Y \in \mathcal{S}^2$ such that $\|\sup_{0 \leq s \leq T} |Y_s|\|_\infty < \infty$.
- $\mathcal{T}$ is the set of all $(\mathcal{F}_t)$-stopping times $\tau$ such that $0 \leq \tau \leq T$. The process $Y$ is of class $D$ if the family $\{Y_{\tau}; \tau \in \mathcal{T}\}$ is uniformly integrable.
- Let $\mathcal{M}^2$ denote the class of (multidimensional) predictable processes $(Z_t)_{0 \leq t \leq T}$ such that $\mathbb{E}[\int_0^T |Z_s|^2 \, ds] < \infty$.
- BMO is the class of martingales $M$ with $\sup_{\tau \in \mathcal{T}} \|\mathbb{E}[\langle M \rangle_T - \langle M \rangle_\tau |\mathcal{F}_\tau]\|_\infty < \infty$.

Proof of Proposition 2.2

The proof is split into several steps. First, when the terminal condition is bounded, the solution is constructed by slightly modifying the proof of [34, Theorem 2.2]. For general terminal conditions, the solution is obtained by the localization technique in [6]. Finally, uniqueness is proved and (2.2) is verified. To ease notation, we denote $\xi = e^{-\delta \theta T} c_T^{1-\gamma}$ throughout this proof.

Step 1: Bounded terminal condition. When $\xi^2 \leq C$ for some constant $C$, consider the truncated BSDE

$$Y^n_t = \xi + \int_t^T F^n(s, c_s, Y^n_s) \, ds - \int_t^T Z^n_s \, dB_s, \quad (A.1)$$

where $F^n(t, c_t, y) = \delta \theta e^{-\delta t} (c_t^{1-\frac{1}{\theta}} \wedge n)(|y| \wedge n)^{1-\frac{1}{\theta}}$. Note that $y \mapsto F^n(t, c_t, y)$ is Lipschitz; in particular, it is differentiable at $y = 0$ due to $1 - 1/\theta > 0$. Therefore, (A.1) admits a unique solution $(Y^n, Z^n) \in \mathcal{S}^2 \times \mathcal{M}^2$. The first component of this solution is also nonnegative. Indeed, consider (A.1) with zero as the terminal condition. Then BSDE admits a unique solution $(\tilde{Y}^n, \tilde{Z}^n) \equiv (0, 0)$ in $\mathcal{S}^2 \times \mathcal{M}^2$. Since $\xi \geq 0$, it follows from the comparison theorem for BSDEs with Lipschitz generators that $Y^n \geq \tilde{Y}^n = 0$. On the other hand, since $\theta < 0$, $(F^n)$ is decreasing and the comparison theorem implies that $(Y^n)$ is decreasing. Hence $Y := \downarrow \lim_{n \to \infty} Y^n$ is well defined and nonnegative.

To take the limit of $(Y^n, Z^n)$, let us derive the following uniform estimate. Applying Itô’s formula to $(Y^n)^2$ yields

$$(Y^n_t)^2 + \mathbb{E}_t \left[ \int_t^T |Z^n_s|^2 \, ds \right] = \mathbb{E}_t[\xi^2] + 2 \mathbb{E}_t \left[ \int_t^T Y^n_s F^n(s, c_s, Y^n_s) \, ds \right] \leq \mathbb{E}_t[\xi^2] \leq C$$

for any $t, n$, where the first inequality follows from $Y^n \geq 0$ and $F^n \leq 0$. The previous estimate yields

$$(Y^n)^2 \leq C \quad \text{and} \quad \mathbb{E} \left[ \int_0^T |Z^n_s|^2 \, ds \right] \leq C \quad \text{for any } n. \quad (A.2)$$

Therefore, there exists $Z \in \mathcal{M}^2$ such that $(Z^n)$ converges to $Z$ weakly.
Note that \( \lim_{n \to \infty} F^n(t, c_t, y) = F(t, c_t, y) \), \( \lim_{n \to \infty} Y^n = Y \) and

\[
0 \geq F^n(t, c_t, Y^n) \geq F(t, c_t, Y^n) \geq C_{\frac{1}{2} - \frac{1}{2\theta} - \frac{1}{\psi}} c_t^{1 - \frac{1}{\psi}} \quad \text{for any } n,
\]

where the third inequality holds due to the first estimate in (A.2). The dominated convergence theorem then implies that

\[
\lim_{n \to \infty} \int_T^t |F^n(s, c_s, Y^n_s) - F(s, c_s, Y_s)| ds = 0 \quad \text{for any } t.
\]

Now we prove the convergence of \((Z^n)\) in \(\mathcal{M}^2\). Applying Itô’s formula to \(|Y^n - Y^m|^2\) yields

\[
\mathbb{E}[|Y^n_0 - Y^m_0|^2] + \mathbb{E} \left[ \int_0^T |Z^n_s - Z^m_s|^2 ds \right]
= 2 \mathbb{E} \left[ \int_0^T (Y^n_s - Y^m_s)(F^n(Y^n_s) - F^m(Y^m_s)) ds \right]
= 2 \mathbb{E} \left[ \int_0^T (Y^n_s - Y^m_s)(F^n(Y^n_s) - F^n(Y^m_s)) ds \right]
+ 2 \mathbb{E} \left[ \int_0^T (Y^m_s - Y^n_s)(F^m(Y^m_s) - F^m(Y^n_s)) ds \right]
\leq 4 \delta |\theta| C_{\frac{1}{2} - \frac{1}{2\theta}} \mathbb{E} \left[ \int_0^T e^{-\delta s} \left| c_s^{1 - \frac{1}{\psi}} \wedge n - c_s^{1 - \frac{1}{\psi}} \wedge m \right| ds \right],
\]

where the first inequality holds due to the fact that \( y \mapsto F^n(t, c_t, y) \) is decreasing and the second follows from the first estimate in (A.2). Since \( c \in C_a \), dominated convergence implies that the right-hand side of (A.3) converges to zero as \( n, m \to \infty \). Combining this with the weak convergence of \((Z^n)\), we obtain

\[
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^T |Z^n_s - Z_s|^2 ds \right] = 0.
\]

The Burkholder–Davis–Gundy inequality then implies

\[
\mathbb{P}-\lim_{n \to \infty} \sup_{t \leq T} \left| \int_t^T (Z^n_s - Z_s) dB_s \right| = 0,
\]

where \( \mathbb{P}\)-lim stands for the convergence in probability. Passing to a subsequence, we obtain almost sure convergence. Therefore, sending \( n \to \infty \) in (A.1), we obtain that \((Y, Z) \in \mathcal{S}^\infty \times \mathcal{M}^2\) solves (2.4) and \( Y \) is nonnegative. Moreover, since

\[
|Y^n_t - Y^m_t| \leq \int_t^T |F^n(s, c_s, Y^n_s) - F^m(s, c_s, Y^m_s)| ds + \left| \int_t^T (Z^n_s - Z^m_s) dB_s \right|
\]
after taking limits in \( m \) and the supremum over \( t \), we obtain

\[
\sup_{t \leq T} |Y^n_t - Y_t| \leq \int_0^T \left| F^n(s, c_s, Y^n_s) - F(s, c_s, Y_s) \right| ds + \sup_{t \leq T} \left| \int_t^T (Z^n_s - Z_s) dB_s \right|.
\]

Therefore, \((Y^n)\) converges to \( Y \) uniformly in \( t \), implying the continuity of \( Y \).

**Step 2: Unbounded terminal condition.** Set \( \xi := \xi \wedge n \) and consider

\[
Y^n_t = \xi + \int_t^T F(s, c_s, Y^n_s) ds - \int_t^T Z^n_s dB_s.
\]

The results from the previous step imply that this BSDE admits a solution \((Y^n, Z^n)\) in \( S^\infty \times \mathcal{M}^2 \) with \( Y^n \geq 0 \). Moreover, since \( F \leq 0 \), \( Y^n_t \leq \mathbb{E}_t[\xi] \) for all \( n \) and \( t \in [0, T] \).

This a priori bound allows us to construct a solution to (2.4) via the localization technique in [6]. We outline the construction below.

Consider the stopping time \( \tau_k := \inf\{t \geq 0 : \mathbb{E}_t[\xi] \geq k \} \wedge T \), for any \( k \in \mathbb{N} \), and define \((Y^{n,k}_t, Z^{n,k}_t) := (Y^n_{t \wedge \tau_k}, Z^n_{t \wedge \tau_k})\), which satisfies the BSDE

\[
Y^{n,k}_t = Y^n_{t \wedge \tau_k} + \int_t^{t \wedge \tau_k} F(s, c_s, Y^{n,k}_s) ds - \int_t^{t \wedge \tau_k} Z^{n,k}_s dB_s.
\]

Since \( 0 \leq Y^{n,k}_t \leq \mathbb{E}_{s \wedge \tau_k}[\xi] \leq k \), we have

\[
0 \geq F(s, c_s, Y^{n,k}_s) \geq \delta \theta k^{1-\frac{1}{\psi}} e^{-\delta s} c_s^{1-\frac{1}{\psi}}.
\]

Then \( c \in C_d \) implies \( \mathbb{E}[\int_0^T F(s, c_s, Y^{n,k}_s) ds] < \infty \). On the other hand, since \((\xi^n)\) is increasing and \( y \mapsto F(\cdot, \cdot, y) \) satisfies the monotonicity condition, the comparison result (cf. [34, Theorem 2.4]) implies that \((Y^{n,k})\) is increasing in \( n \). Utilizing the same argument as in Step 1, we obtain \( \tilde{Y}^k := \uparrow \lim_n Y^{n,k} \) and \( \tilde{Z}^k \in \mathcal{M}^2 \) such that \( \lim_n Z^{n,k} = \tilde{Z}^k \) in \( \mathcal{M}^2 \), and \((\tilde{Y}^k, \tilde{Z}^k)\) solves the BSDE

\[
\tilde{Y}^k_t = \tilde{Y}^k_{t \wedge \tau_k} + \int_t^{t \wedge \tau_k} \left[ F(s, c_s, \tilde{Y}^k_s) - \tilde{Z}^k_s \right] dB_s,
\]

where \( \tilde{Y}^k_{t \wedge \tau_k} = \uparrow \lim_n Y^{n,k}_{t \wedge \tau_k} \). From the definition of \((\tilde{Y}^k, \tilde{Z}^k)\), we have that \( \tilde{Y}^{k+1}_{t \wedge \tau_k} = \tilde{Y}^k_t \) and \( \tilde{Z}^{k+1}_{t \wedge \tau_k} = \tilde{Z}^k_t \). Therefore, we define

\[
Y_t := \tilde{Y}^k_t \quad \text{and} \quad Z_t := \tilde{Z}^k_t, \quad \text{when } t \in [0, \tau_k].
\]

This construction implies \( \lim_{k \to \infty} Y_{t \wedge \tau_k} = Y_t = \xi \). Indeed, on \( \{\xi \leq k\} \), we have \( \tau_k = T \) and \( \lim_{t \to T} Y^n_t = \xi \) for any \( n \geq k \). Therefore, when \( n \geq k \),

\[
\lim_{t \to T} Y_t = \lim_{t \to \tau_k} \tilde{Y}^k_t = \lim_{t \to \tau_k} Y^{n,k}_t = \lim_{t \to T} Y^n_t = \xi \quad \text{on } \{\xi \leq k\}.
\]

This implies \( \lim_{t \to \infty} Y_t = \xi \), since \( \uparrow \lim_{k \to \infty} \{\xi \leq k\} = \Omega \). Now sending \( k \to \infty \) on both sides of (A.4), we confirm that \((Y, Z)\) solves (2.4). By this construction, \( Y \) is
continuous and satisfies $0 \leq Y_t \leq \mathbb{E}_t[\xi]$ for $t \in [0, T]$; hence $Y$ is of class $D$. The same argument as in [6, after Eq. (7)] shows $\int_0^T |Z_t|^2 dt < \infty$.

**Step 3: Remaining statements.** For future reference, we prove a comparison result for (2.4). Let $(Y, Z)$ (resp. $(\tilde{Y}, \tilde{Z})$) be a supersolution (resp. subsolution) to (2.4), i.e.,

$$Y + \int_0^t F(s, c_s, Y_s) ds \text{ is a local supermartingale and}$$

$$\tilde{Y} + \int_0^t F(s, c_s, \tilde{Y}_s) ds \text{ is a local submartingale}$$

with $Y_T \geq \xi \geq \tilde{Y}_T$, where $Z$ and $\tilde{Z}$ are determined by the Doob–Meyer decomposition and martingale representation. Assuming that both $Y$ and $\tilde{Y}$ are of class $D$, then $Y \geq \tilde{Y}$. Moreover, if $Y_T > \tilde{Y}_T$, then $Y_t > \tilde{Y}_t$ for any $t \leq T$.

To prove this comparison result, define

$$\alpha_t := \begin{cases} 
F(t, c_t, Y_t) - F(t, c_t, \tilde{Y}_t) & \text{for } Y_t \neq \tilde{Y}_t, \\
0 & \text{for } Y_t = \tilde{Y}_t.
\end{cases}$$

Since $y \mapsto F(\cdot, \cdot, y)$ is decreasing, we have $\alpha \leq 0$. It then follows that $e^{\int_0^t \alpha_s ds} (Y - \tilde{Y})$ is a local supermartingale, hence a supermartingale, since the exponential factor is bounded and both $Y$ and $\tilde{Y}$ are of class $D$. Therefore, $Y_T \geq \tilde{Y}_T$ implies $Y \geq \tilde{Y}$. Moreover, when $Y_T > \tilde{Y}_T$, we obtain the strict comparison $Y_t > \tilde{Y}_t$ for any $t \leq T$. The uniqueness follows directly from the comparison result. Since $\gamma > 1$, then $\xi = e^{-\delta \theta T} c_T^{1-\gamma} > 0$. Therefore, $Y > 0$ follows from the strict comparison.

Finally, we verify that $V^c$ satisfies (2.2). To this end, since $(Y, Z)$ solves (2.4), $(V^c_T, Z^c_T) = e^{\delta \theta t} (Y_t, Z_t)/(1 - \gamma)$ satisfies (2.3), implying that $V^c + \int_0^t f(c_s, V^c_s) ds$ is a local martingale. Taking a localizing sequence $(\sigma_n)$ for $V^c + \int_0^T f(c_s, V^c_s) ds$, we obtain

$$V^c_0 + \delta \theta \mathbb{E}\left[ \int_0^{T \wedge \sigma_n} V^c_s ds \right] = \mathbb{E}\left[ V^c_{T \wedge \sigma_n} + \int_0^{T \wedge \sigma_n} \frac{1-\frac{1}{\psi}}{1 - \frac{1}{\psi}} (1 - \gamma) V^c_s \right] ds.$$

Sending $n \to \infty$ on both sides, note that $V^c \leq 0$ and $\psi > 1$; therefore, the integrand on the left side is negative and the integrand on the right side is positive. The monotone convergence theorem and the class $D$ property of $V^c$ then yield

$$V^c_0 + \delta \theta \mathbb{E}\left[ \int_0^T V^c_s ds \right] = \mathbb{E}\left[ U(c_T) + \int_0^T \frac{1-\frac{1}{\psi}}{1 - \frac{1}{\psi}} (1 - \gamma) V^c_s \right] ds. \quad (A.5)$$

Note that

$$0 \geq \mathbb{E}\left[ \int_0^T V^c_s ds \right] = \frac{1}{1 - \gamma} \mathbb{E}\left[ \int_0^T e^{\delta \theta s} Y_s ds \right] \geq \frac{1}{1 - \gamma} \mathbb{E}\left[ \int_0^T Y_s ds \right] > -\infty.$$
where the second inequality holds since $\gamma > 1$ and $\theta < 0$, and the third follows from $\mathbb{E}[Y_s] \leq \mathbb{E}[\xi]$ and $\gamma > 1$. Subtracting $\delta \theta \mathbb{E}[\int_0^T V^c_s ds]$ on both sides of (A.5), we confirm (2.2). □

Concavity of $c \mapsto V^c$ is proved next. This proof utilizes simultaneously the joint concavity of the generator for (2.5) and the class $D$ property of the solution to (2.4).

**Proof of Proposition 2.4** Denote the generator of (2.5) as

$$F(t,c_t,y,z) = \delta e^{\gamma t} c^{1-\frac{1}{\psi}} + \frac{1}{2} (\gamma - 1) z^2.$$ 

For $c, \tilde{c}$ and $\alpha c + (1 - \alpha) \tilde{c} \in C_\alpha$, denote $\Delta X = \alpha X + (1 - \alpha) \tilde{X}$, for $X = c, Y, Z$ and $\tilde{X} = \tilde{c}, \tilde{Y}, \tilde{Z}$, respectively. It follows from (2.5) that

$$d\Delta Y_t = \left( - \delta e^{\gamma t} \frac{\Delta c_t}{1 - \frac{1}{\psi}} - \frac{1}{2} (\theta - 1) \frac{\Delta Z_t^2}{\Delta Y_t} + A_t \right) dt + \Delta Z_t dB_t,$$

where, due to the concavity of $(c_t, y, z) \mapsto F(t,c_t,y,z)$,

$$A_t = \frac{\delta e^{\gamma t}}{1 - \frac{1}{\psi}} \left( \Delta c_t^{1-\frac{1}{\psi}} - \alpha c_t^{1-\frac{1}{\psi}} - (1 - \alpha) \tilde{c_t}^{1-\frac{1}{\psi}} \right)$$

$$+ \frac{1}{2} (\theta - 1) \left( \frac{\Delta Z_t^2}{\Delta Y_t} - \alpha \frac{Z_t^2}{Y_t} - (1 - \alpha) \tilde{Z}_t^2 \right) \geq 0,$$

and $\Delta Y_T \leq e^{-\delta T} \Delta c_T^{1-\frac{1}{\psi}} / (1 - \frac{1}{\psi})$. Set

$$\Delta Y = \left( (1 - \frac{1}{\psi}) \Delta Y \right)^\theta \quad \text{and} \quad \Delta Z = (1 - \gamma) \left( (1 - \frac{1}{\psi}) \Delta Y \right)^{\theta - 1} \Delta Z.$$

Itô’s formula yields

$$d\Delta Y_t = \left( - \delta \theta e^{\gamma t} \Delta c_t^{1-\frac{1}{\psi}} \Delta Y_t^{1-\frac{1}{\psi}} + (1 - \gamma) \Delta Y_t^{1-\frac{1}{\psi}} A_t \right) dt + \Delta Z_t dB_t,$$

where $(1 - \gamma) \Delta Y_t^{1-\frac{1}{\psi}} A_t \leq 0$. On the other hand, $\Delta Y_T \geq e^{-\delta T} \Delta c_T^{1-\gamma}$. Therefore, $(\Delta Y, \Delta Z)$ is a supersolution to (2.4). On the other hand, $\Delta Y$ is of class $D$. Indeed, since $\theta < 0$,

$$\Delta Y = \left( (1 - \frac{1}{\psi}) \Delta Y \right)^\theta \leq \alpha \left( (1 - \frac{1}{\psi}) Y \right)^\theta + (1 - \alpha) \left( (1 - \frac{1}{\psi}) \tilde{Y} \right)^\theta$$

$$= a Y + (1 - a) \tilde{Y}, \quad (A.6)$$

where $Y$ (resp. $\tilde{Y}$) is the first component of the solution to (2.4) with $c$ (resp. $\tilde{c}$). Therefore, $\Delta Y$ is of class $D$, because both $Y$ and $\tilde{Y}$ are. Now consider $Y^{\Delta c}$ as the
first component of the solution of (2.4), where $c$ is replaced by $\Delta c$. It then follows from (A.6) and the comparison result in Step 3 of the previous proof that

$$\alpha Y_0 + (1 - \alpha) \tilde{Y}_0 \geq \Delta Y_0 \geq Y_0^{\Delta c}.$$ 

Dividing both sides by $(1 - \gamma)$, we confirm that $\alpha V_c^\gamma + (1 - \alpha) \tilde{V}_c^\gamma \leq V_0^{\alpha c + (1 - \alpha) \tilde{c}}$. □

**Appendix B: Proofs for Sect. 2.2**

Even though the generator $H$ in (2.12) has an exponential term in $y$, the parameter specification $\gamma > 1$ and $\psi > 1$ allows us to derive a priori bounds for $Y$. Then a solution to (2.12) is constructed via the localization technique in [6].

**Proof of Proposition 2.9** The BSDE (2.12) can be rewritten under $\mathbb{P}$; therefore, all expectations are taken with respect to $\mathbb{P}$ throughout this proof. Due to Assumption 2.7(i), $\overline{W} := W - \int_0^T \frac{1 - \gamma}{\gamma} \rho' \sigma' \Sigma^{-1} \mu(X_s) ds$ is a $\mathbb{P}$-Brownian motion. On the other hand, recall that $\gamma > 1$ and $r + \frac{1}{2} \mu' \Sigma^{-1} \mu$ is bounded from below. Therefore, there exists a constant $h_{\text{max}}$ such that $h \leq h_{\text{max}}$. However, $\mu' \Sigma^{-1} \mu$, in many widely used models, is an unbounded function of the state variable; hence $h$ and $H(t, 0, 0) = h_t - \delta \theta + \theta \frac{\psi}{\psi}$ are not bounded from below. Therefore, we introduce

$$\mathcal{Y}_t = \xi + \int_t^T \mathcal{H}(s, \mathcal{Y}_s, Z_s) ds - \int_t^T Z_s d\overline{W}_s,$$ 

(B.1)

where $\mathcal{Y}_t = Y_t + \int_0^T (h_s - \delta \theta) ds, \xi = \int_0^T (h_s - \delta \theta) ds$ and

$$\mathcal{H}(t, y, z) = \frac{1}{2} z M_t z' + \theta \frac{\psi}{\psi} \int_0^T h_s - \delta \theta ds e^{-\frac{\psi}{\psi} y}.$$

Consider a truncated version of (B.1), namely

$$\mathcal{Y}_t^n = \xi^n + \int_t^T \mathcal{H}^n(s, \mathcal{Y}_s^n, Z_s^n) ds - \int_t^T Z_s^n d\overline{W}_s,$$ 

(B.2)

where $\xi^n = \int_0^T h_s \vee (-n) - \delta \theta ds$ is bounded and

$$\mathcal{H}^n(t, y, z) = \frac{1}{2} z M_t z' + \theta \frac{\psi}{\psi} \int_0^T h_s \vee (-n) - \delta \theta ds (e^{-\frac{\psi}{\psi} y} \wedge n).$$

This truncated generator $\mathcal{H}^n$ is Lipschitz in $y$ and quadratic in $z$. Indeed, since the eigenvalues of $\sigma' \Sigma^{-1} \sigma$ are either 0 or 1, we have $0 \leq z \rho' \sigma' \Sigma^{-1} \rho z' \leq z \rho' \rho z' \leq |z|^2$. Then $\gamma > 1$ and the definition of $M$ after (2.13) imply

$$0 < \frac{1}{\gamma} |z|^2 \leq z M(X) z' \leq |z|^2.$$ 

(B.3)
Consumption–investment optimization with Epstein–Zin utility

Therefore [25, Theorem 2.3] implies that (B.2) has a solution \((Y^n, Z^n) \in S^\infty \times M^2\). Moreover, due to \(\theta < 0\), \((\mathcal{H}^n)\) is decreasing in \(n\). The construction of \(\mathcal{Y}^n\) in [25, Theorem 2.3] yields \(\mathcal{Y}^n \geq \mathcal{Y}^{n+1}\). In what follows, we derive a priori bounds on \(\mathcal{Y}^n\) uniformly in \(n\). This uniform estimate facilitates the construction of a solution to (B.1).

First \(\theta < 0\) and the third inequality in (B.3) yield \(\mathcal{H}^n(t, y, z) \leq \frac{1}{2} |z|^2\). Consider

\[
\bar{Y}_t^n = \xi^n + \int_t^T \frac{1}{2} |\bar{Z}_s^n|^2 ds - \int_t^T \bar{Z}_s^n d\bar{W}_s,
\]

which has the explicit solution \(\bar{Y}_t^n = \log \mathbb{E}_t[\exp(\int_0^T h_s \vee (n - \delta \theta) ds)]\). Then

\[
\bar{Y}_t^n - \int_0^t h_s \vee (n - \delta \theta) ds = \log \mathbb{E}_t\left[ e^{\int_0^T h_s \vee (n - \delta \theta) ds} \right] \leq (h_{\max} - \delta \theta)(T - t).
\]

On the other hand, when \(y - \int_0^T h_s \vee (n - \delta \theta) ds \leq (h_{\max} - \delta \theta)(T - t)\), the first inequality in (B.3) and \(\theta < 0\) imply \(\mathcal{H}^n(t, y, z) \geq \theta \frac{\delta \psi}{\psi} e^{(\delta \psi - \frac{\psi}{\theta} h_{\max}) T} \). Therefore, consider the BSDE

\[
Y_t = \xi + \theta \frac{\delta \psi}{\psi} e^{(\delta \psi - \frac{\psi}{\theta} h_{\max}) T} (T - t) - \int_t^T Z_s d\bar{W}_s,
\]

whose solution \(Y\) admits the representation

\[
\mathbb{E}_t[\xi] + \theta \frac{\delta \psi}{\psi} e^{(\delta \psi - \frac{\psi}{\theta} h_{\max}) T} (T - t) = Y_t \leq Y_t^n \leq \bar{Y}_t^n = \log \mathbb{E}_t\left[ e^{\int_0^T h_s \vee (n - \delta \theta) ds} \right] \leq (h_{\max} - \delta \theta) T,
\]

for any \(n > 0\). These uniform bounds on \(\mathcal{Y}^n\) allow us to construct a solution \((\mathcal{Y}, Z)\) to (B.1) using the localization technique in [6, Theorem 2]; see also Step 2 in the proof of Proposition 2.2. The resulting \(\mathcal{Y}\) satisfies

\[
\mathbb{E}_t[\xi] + \theta \frac{\delta \psi}{\psi} e^{(\delta \psi - \frac{\psi}{\theta} h_{\max}) T} (T - t) \leq Y_t \leq \log \mathbb{E}_t[\exp(\int_0^T h_s ds) e^{\xi}], \tag{B.4}
\]

The previous inequalities imply that \(\lim_{t \to T} Y_t = \xi\). Hence \(\mathcal{Y}\) satisfies the terminal condition of (B.1). The desired estimates on \(Y\) follows by subtracting \(\int_0^t h_s - \delta \theta ds\) on both sides of the previous inequalities; in particular,

\[
Y_t = \mathcal{Y}_t - \int_0^t h_s - \delta \theta ds \leq \log \mathbb{E}_t\left[ e^{\int_0^T (h_s - \delta \theta) ds} \right] \leq (h_{\max} - \delta \theta)(T - t). \tag{B.5}
\]
To prove the integrability of $Z$, take a localizing sequence $(\sigma_n)$ for $\int_0^T Z_s d\bar{W}_s$; then (B.1) yields

$$\frac{1}{2} \mathbb{E} \left[ \int_0^{\sigma_n} Z_s M_s Z'_s ds \right] = \mathcal{Y}_0 - \mathbb{E}[\mathcal{Y}_{\sigma_n}] - \theta \frac{\delta \psi}{\psi} \mathbb{E} \left[ \int_0^{\sigma_n} e^{\psi} \int_0^s h_u - \delta \theta du e^{-\theta} \mathcal{Y}_u ds \right].$$

Sending $n \to \infty$ on both sides, applying the second inequality in (B.3) to the left-hand side, the first inequality in (B.4) to the second term on the right-hand side, and (B.5) to the third term, we confirm that

$$\mathbb{E} \left[ \int_0^T |Z_s|^2 ds \right] < \infty. \quad \square$$

The following several results prepare the proofs of Theorems 2.14 and 2.16. First we show that $w^{1-\gamma} \frac{1-\gamma}{1-\gamma} e^{Y_0}$ is an upper bound for the optimal value among permissible strategies.

**Lemma B.1** Let Assumption 2.7 hold. For any permissible $(\pi, c)$,

$$\frac{w^{1-\gamma}}{1-\gamma} e^{Y_0} \geq V^c_0,$$  \hspace{1cm} (B.6)

where $V^c$ is defined in Proposition 2.2, $Y$ is constructed in Proposition 2.9, and $c$ is financed by $\pi$ via (2.8).

**Proof** This proof extends the technique in [20] to recursive utilities. For a permissible $(\pi, c)$, define

$$R_{\pi, c}^t := (\mathcal{W}_t)^{1-\gamma} e^{Y_t} + \int_0^t c_s (\mathcal{W}_s)^{1-\gamma} e^{Y_s} ds, \quad t \in [0, T],$$

where $\mathcal{W} = \mathcal{W}^{\pi, c}$. Then (2.11) and (2.13) imply that $R$ is a local supermartingale. Due to the Doob–Meyer decomposition and martingale representation, there exist an increasing process $A$ and $Z^R$ such that $R^{\pi, c} = -A + \int_0^\cdot Z^R dB_s$. Therefore, $(\mathcal{W})^{1-\gamma} e^{Y}, Z^R$ is a supersolution to (2.3) and $(\mathcal{W}_T)^{1-\gamma} / (1 - \gamma) \in \mathbb{L}^1$. Indeed, since $(\mathcal{W})^{1-\gamma} e^{Y}$ is of class $D$ by permissibility and $Y_T = 0$, we have $\mathbb{E}[(\mathcal{W}_T)^{1-\gamma}] < \infty$. On the other hand, consider the utility $V^c_0$ associated to the consumption stream $c$ and the terminal lump sum $\mathcal{W}_T$. The comparison result in the proof of Proposition 2.2 confirms (B.6). \square

In what follows, we show that $(\pi^*, c^*)$ is a permissible strategy and attains the upper bound $w^{1-\gamma} \frac{1-\gamma}{1-\gamma} e^{Y_0}$. First, we establish the important result that a certain exponential local martingale associated to $\pi^*$ is a martingale.

**Lemma B.2** Let Assumptions 2.6, 2.7 and 2.10 hold. Then

$$Q := \mathcal{E} \left( \int (1 - \gamma)(\pi^*_s)' \sigma_s dW^\rho_s + \int Z_s dW_s \right)$$

is a $\mathbb{P}$-martingale on $[0, T]$.

\( \square \) Springer
Proof It follows from (2.14), the definition of \(W^\rho\) and \(M\) that
\[
(1-\gamma)(\pi^*)'\sigma dW^\rho + Z dW
= \left(\frac{1-\gamma}{\gamma} \mu_1' \Sigma_1^{-1}\sigma_1 + ZM\right) dW + \frac{1-\gamma}{\gamma} (\mu_1' + Z\rho'\sigma') \Sigma_1^{-1}\sigma_1 dW^\perp
=: L^{(1)} dW + L^{(2)} dW^\perp.
\]
Here we suppress time subscripts to simplify the notation. We now first claim that if
\[
Q^{(1)} := \mathcal{E}(\int L^{(1)}_s dW_s)
\]
is a martingale, so is \(Q\). Indeed, for any \(t \leq T\),
\[
\mathbb{E}[Q_t] = \mathbb{E}\left[\mathcal{E}\left(\int L^{(1)}_s dW_s\right)\mathcal{E}\left(\int L^{(2)}_s dW^\perp_s\right)\right]
= \mathbb{E}\left[\mathcal{E}\left(\int L^{(1)}_s dW_s\right)\mathbb{E}\left[\mathcal{E}\left(\int L^{(2)}_s dW^\perp_s\right) | \mathcal{F}^W\right]\right]
= \mathbb{E}\left[\mathcal{E}\left(\int L^{(1)}_s dW_s\right)\right]
= 1.
\]
Here \(\mathcal{F}^W = \sigma(W_s; 0 \leq s \leq T)\), the third equality follows from [21, Lemma 4.8] since \(L^{(2)}\) and \(W^\perp\) are independent, and the fourth is due to the martingale assumption on \(Q^{(1)}\). In the rest of the proof, we prove the martingale property of \(Q^{(1)}\).

For \((E_n)\) in Assumption 2.10(i), define \(\tau_n := \inf\{t \geq 0 : X_t \notin E_n\} \land T\). We first prove that \(Y_{\land \tau_n}\) is bounded. Since Proposition 2.9 already proves that \(Y\) is bounded from above, it suffices to show that \(\mathbb{E}_{\land \tau_n} [\int_0^T h_s d\mathcal{F}^W_s]\) is bounded from below. Then (2.15) implies that \(Y_{\land \tau_n}\) is bounded from below as well. Define
\[
y(t, x) := \mathbb{E}_{\mathbb{F}}\left[\int_t^T h(X_s) ds \bigg| X_t = x\right].
\]
The Feynman–Kac formula (see [19] when the equation is not uniformly parabolic) implies that under Assumption 2.6, \(y\) is in \(C^{1,2}((0, T] \times E)\) and the unique solution to
\[
\partial_t y + \mathcal{L}y + h = 0, \quad y(T, x) = 0,
\]
where \(\mathcal{L}\) is the infinitesimal generator of \(X\) under \(\mathbb{P}\). Now since \(E_n\) is compact, the continuity of \(y\) implies that \(y(\cdot \land \tau_n, X_{\land \tau_n})\) is bounded.

As a solution to (2.12), \((Y, Z)\) satisfies
\[
Y_t = Y_{\tau_n} + \int_t^{\tau_n} H(s, Y_s, Z_s) ds - \int_t^{\tau_n} Z_s dW_s, \quad t \in [0, \tau_n].
\]
Since both \(X_{\land \tau_n}\) and \(Y_{\land \tau_n}\) are bounded, it follows from the BMO-estimate for quadratic BSDEs (cf. e.g. [33, Lemma 3.1]) that \(\int_0^{\land \tau_n} Z_s dW_s\) is in BMO. Note that both \(\mu_1' \Sigma^{-1}\sigma_1(X_{\land \tau_n})\) and \(M(X_{\land \tau_n})\) are bounded. Therefore, \(\int_0^{\land \tau_n} L^{(1)}_s dW_s\) is in BMO as well. Then [23, Theorem 2.3] implies that \(\mathcal{E}(\int L^{(1)}_s dW_s)_{\land \tau_n}\) is a martingale. Therefore, \(d\mathbb{Q}^n/d\mathbb{P} := \mathcal{E}(\int L^{(1)}_s dW_s)_{\tau_n}\) defines \(\mathbb{Q}^n\) on \(\mathcal{F}_{\tau_n}\) which is equivalent to \(\mathbb{P}\).
Assuming \( \lim_{n \to \infty} \mathbb{Q}^n[\tau_n < T] = 0 \), by the monotone convergence theorem,

\[
\begin{align*}
\mathbb{E}\left[ \mathcal{E}\left( \int L_s^{(1)} dW_s \right)_T \right] &= \lim_{n \to \infty} \mathbb{E}\left[ \mathcal{E}\left( \int L_s^{(1)} dW_s \right)_{\tau_n} \mathbb{I}_{[\tau_n = T]} \right] \\
&= \lim_{n \to \infty} \mathbb{E}\left[ \mathcal{E}\left( \int L_s^{(1)} dW_s \right)_{\tau_n} \right] - \lim_{n \to \infty} \mathbb{E}\left[ \mathcal{E}\left( \int L_s^{(1)} dW_s \right)_{\tau_n} \mathbb{I}_{[\tau_n < T]} \right] \\
&= 1 - \lim_{n \to \infty} \mathbb{Q}^n[\tau_n < T] \\
&= 1,
\end{align*}
\]

proving the martingale property of \( \mathcal{E}(\int L_s^{(1)} dW_s) \) on \([0, T]\).

It remains to prove that \( \lim_{n \to \infty} \mathbb{Q}^n[\tau_n < T] = 0 \). To this end, (2.12) yields

\[
Y_t = Y_0 - \int_0^t H(s, Y_s, Z_s) ds + \int_0^t Z_s dW_s.
\]

On the other hand, recalling \( \mathfrak{F} \) from (2.16), we have from Itô’s formula that

\[
\phi(X_t) = \phi(x) + \int_0^t b'\nabla \phi(X_s) + \frac{1}{2} \sum_{i,j=1}^k A_{ij} \partial_{x_i x_j} \phi(X_s) ds + \int_0^t \nabla \phi' a(X_s) dW_s
\]

\[
= \phi(x) + \int_0^t \left( \mathfrak{F}[\phi] - \frac{1}{2} \nabla \phi' a \Sigma^{-1} \rho \nabla \phi - \frac{1 - \gamma}{\gamma} \mu \Sigma^{-1} \sigma \nabla \phi \right) ds
\]

\[
+ \int_0^t \nabla \phi' a(X_s) dW_s.
\]

Taking the difference of the previous two identities, we have for \( t \leq \tau_n \) that

\[
\begin{align*}
Y_t - \phi(X_t)
&= Y_0 - \phi(x) + \int_0^t \left( Z_s - \nabla \phi' a(X_s) \right) dW_s - \int_0^t \left( \theta \frac{\delta \psi}{\psi} e^{-\frac{\psi}{\sigma} Y_t} - \delta \theta + \mathfrak{F}[\phi] \right) ds \\
&\quad - \int_0^t \left( \frac{1}{2} Z M Z' - \frac{1}{2} \nabla \phi' a M \nabla \phi + \frac{1 - \gamma}{\gamma} \mu \Sigma^{-1} \sigma \rho Z' \nabla \phi \right) ds \\
&= Y_0 - \phi(x) + \int_0^t \left( Z_s - \nabla \phi' a(X_s) \right) dW_s - \int_0^t \left( \theta \frac{\delta \psi}{\psi} e^{-\frac{\psi}{\sigma} Y_t} - \delta \theta + \mathfrak{F}[\phi] \right) ds \\
&\quad - \int_0^t \left( \frac{1}{2} Z M Z' - \nabla \phi' a M \nabla \phi - (Z - \nabla \phi' a) M Z' \right) ds \\
&= Y_0 - \phi(x) + \int_0^t \left( Z_s - \nabla \phi' a(X_s) \right) dW_s^n
\]

\[
+ \int_0^t \left( \frac{1}{2} (Z - \nabla \phi' a) M (Z' - a \nabla \phi) - \theta \frac{\delta \psi}{\psi} e^{-\frac{\psi}{\sigma} Y_t} + \delta \theta - \mathfrak{F}[\phi] \right) ds,
\]

\( \odot \) Springer
where \( W^n := W - \int_0^{\tau_n} L_s^{(1)} \, ds \) is a \( \mathbb{Q}^n \)-Brownian motion on \([0, \tau_n]\). On the right-hand side, the quadratic term is nonnegative, \(-\theta \frac{\psi}{\bar{\psi}} e^{-\bar{\psi} Y_t} \) is nonnegative since \( \theta < 0 \), and \( \delta\theta - \mathfrak{H}[\phi] \) is also bounded from below due to Assumption 2.10(ii). Therefore, there exists some negative constant \( C \) such that

\[
Y_{\tau_n} - \phi(X_{\tau_n}) \geq Y_0 - \phi(x) + C \tau_n + \int_0^{\tau_n} (Z_s - \nabla \phi' a) \, dW^n_s. \tag{B.8}
\]

The stochastic integral on the right-hand side has zero expectation under \( \mathbb{Q}^n \). Indeed, since \( \int_0^{\tau_n} Z_s \, dW_s \) is in \( \text{BMO}(\mathbb{P}) \) and \( \nabla \phi' a(X_{\tau_n}) \) is bounded, we obtain that \( \int_0^{\tau_n} (Z_s - \nabla \phi' a(X_s)) \, dW_s \) is in \( \text{BMO}(\mathbb{P}) \) as well. Now since \( \int_0^{\tau_n} L_s^{(1)} \, dW_s \) is in \( \text{BMO}(\mathbb{P}) \), [23, Theorem 3.6] implies that \( \int_0^{\tau_n} (Z_s - \nabla \phi' a(X_s)) \, dW^n_s \) is in \( \text{BMO}(\mathbb{Q}^n) \) as well. Therefore, its expectation under \( \mathbb{Q}^n \) is zero. It then follows from (B.8) that

\[
\mathbb{E}^{\mathbb{Q}^n}[Y_{\tau_n} - \phi(X_{\tau_n})] \geq Y_0 - \phi(x) + CT > -\infty \quad \text{for each } n. \tag{B.9}
\]

Now since \( Y \) is bounded from above and \( \phi \) is bounded from below due to Assumption 2.10(i), there exists a constant \( C \) such that

\[
Y_{\tau_n} - \phi(X_{\tau_n}) = (Y_{\tau_n} - \phi(X_{\tau_n})) I_{\{\tau_n < T\}} + (Y_T - \phi(X_T)) I_{\{\tau_n = T\}} \\
\leq C - \inf_{x \in \partial E_n} \phi(x) I_{\{\tau_n < T\}}.
\]

Now sending \( n \to \infty \) in (B.9), Assumption 2.10(i) and the previous inequality confirm that \( \lim_{n \to \infty} \mathbb{Q}^n[\tau_n < T] = 0. \)

The martingale property in the previous result helps to verify the permissibility of \((\pi^*, c^*)\).

**Corollary B.3** Let Assumptions 2.6, 2.7 and 2.10 hold. Then \((\mathcal{W}^*)^{1-\gamma} e^Y\) is of class \(D\) on \([0, T]\), where \(\mathcal{W}^*\) is the wealth process associated to \((\pi^*, c^*)\).

**Proof** The calculation leading to (2.13) yields

\[
d(\mathcal{W}^*_t)^{1-\gamma} e^{Y_t} = - (\mathcal{W}^*_t)^{1-\gamma} e^{Y_t} \left( \delta \theta (c^*_t)^{1-\gamma} e^{-\gamma Y_t} \right) dt \\
+ (\mathcal{W}^*_t)^{1-\gamma} e^{Y_t} (1 - \gamma) (\pi^*_t)' c_t dW^\rho_t + Z_t \, dW_t \\
= - (\mathcal{W}^*_t)^{1-\gamma} e^{Y_t} (\delta \psi e^{-\psi Y_t} - \delta \theta) dt \\
+ (\mathcal{W}^*_t)^{1-\gamma} e^{Y_t} ((1 - \gamma) (\pi^*_t)' c_t dW^\rho_t + Z_t \, dW_t),
\]

where the second identity follows from the form of \( c^* \) in (2.14). Therefore,

\[
(\mathcal{W}^*_t)^{1-\gamma} e^{Y_t} = w^{1-\gamma} e^{Y_0} \exp \left(- \int_0^t (\delta \psi \theta e^{-\psi Y_s} - \delta \theta) \, ds \right) \\
\times \mathcal{E} \left( \int (1 - \gamma) (\pi^*_s)' c_s dW^\rho_s + \int Z_s \, dW_s \right)_t.
\]
Since $\theta < 0$ and $Y$ is bounded from above, the second exponential term on the right is bounded, uniformly in $t$. Meanwhile, due to Lemma B.2, the stochastic exponential on the right is of class $D$ on $[0, T]$. The statement is then confirmed.

Lemma B.4 Let Assumptions 2.6, 2.7, 2.10 and 2.11 hold. Let $c^*$ be as in (2.14) and $c_T = \mathcal{W}_T^*$. Then $c^* \in \mathcal{C}_a$.

Proof Since $Y_T = 0$, the class $D$ property of $(\mathcal{W}_T^*)^{1-\gamma} e^Y$ in Corollary B.3 yields $\mathbb{E}[ (\mathcal{W}_T^*)^{1-\gamma} ] < \infty$. On the other hand, the expression of $\tilde{c}^*$ in (2.14) implies that

$$e^{-\delta s} (c_s^*)^{1-\frac{1}{\psi}} = e^{-\delta s} \delta^{\psi-1} e^{-\frac{Y_s}{\delta}} (\mathcal{W}_s^*)^{1-\frac{1}{\psi}}.$$

Since $\psi > 1$, $\theta < 0$, and $Y$ is bounded from above, the first three terms on the right-hand side are bounded. Therefore, it suffices to prove that

$$\mathbb{E} \left[ \int_0^T (\mathcal{W}_s^*)^{1-\frac{1}{\psi}} ds \right] < \infty.$$

To this end, it follows from Assumption 2.11 that

$$\mathbb{E} \left[ \int_0^T (\mathcal{W}_s^*)^{1-\frac{1}{\psi}} ds \right] = \int_0^T \mathbb{E}^Q \left[ e^{(1-\frac{1}{\psi}) \int_0^T r_u du} \mathcal{E} \left( \int \lambda_u dW_u^0 \right)^T e^{-\frac{1}{\psi} \int_0^T r_u du} (\mathcal{W}_s^*)^{1-\frac{1}{\psi}} \right] ds \leq \int_0^T \mathbb{E}^Q \left[ e^{(1-\frac{1}{\psi}) \int_0^T r_u du} \mathcal{E} \left( \int \lambda_u dW_u^0 \right)^T e^{-\frac{1}{\psi} \int_0^T r_u du} (\mathcal{W}_s^*)^{1-\frac{1}{\psi}} \right] ds \leq \mathbb{E}^Q \left[ e^{(\psi-1) \int_0^T r_u du} \mathcal{E} \left( \int \lambda_u dW_u^0 \right)^T e^{-\frac{1}{\psi} \int_0^T r_u du} (\mathcal{W}_s^*)^{1-\frac{1}{\psi}} \right] ds \leq w^{1-\frac{1}{\psi} T} \mathbb{E}^Q \left[ e^{(\psi-1) \int_0^T r_u du} \mathcal{E} \left( \int \lambda_u dW_u^0 \right)^T e^{-\frac{1}{\psi} \int_0^T r_u du} (\mathcal{W}_s^*)^{1-\frac{1}{\psi}} \right] ds < \infty.$$

Here the first inequality follows from $\psi > 1$, the second holds due to Hölder’s inequality, the third is obtained using the fact that $e^{-\int_0^T r_u du} \mathcal{W}_s^*$ is a nonnegative $\mathbb{Q}^0$-local martingale, hence a $\mathbb{Q}^0$-supermartingale, and the fourth holds thanks to (2.17). □

Now we are ready to prove the first main result.

Proof of Theorem 2.14 Corollary B.3 and Lemma B.4 have already shown that $(\pi^*, c^*)$ is permissible. Choosing $(\pi^*, c^*)$, we have from (2.11), (2.13) and $Y_T = 0$.

© Springer
that
\[
\frac{(W^*_t)^{1-\gamma}}{1-\gamma} e^{Y_t} = \frac{(W^*_T)^{1-\gamma}}{1-\gamma} + \int_t^T f \left( e^s, \frac{(W^*_s)^{1-\gamma}}{1-\gamma} \right) ds - \int_t^T Z_s dB_s
\]
for some $Z$. Then the class $D$ property of $(W^*)^{1-\gamma} e^{Y}$ and Proposition 2.9 combined imply
\[
\frac{w^{1-\gamma}}{1-\gamma} e^{Y_0} = \mathbb{E} \left[ \int_0^T f \left( e^s, \frac{(W^*_s)^{1-\gamma}}{1-\gamma} e^{Y_t} \right) ds + \frac{(W^*_T)^{1-\gamma}}{1-\gamma} \right]. \tag{B.10}
\]
Therefore, the upper bound in Lemma B.1 is attained by $(\pi^*, c^*)$. \hfill \Box

Finally, we prove Lemma 2.15 and Theorem 2.16.

Proof of Lemma 2.15 Calculation using (2.8) and (2.21) confirms the local martingale property of $W^* + \int_0^\cdot D^*_s c_s ds$. It then remains to prove (2.21). To ease notation, suppress all time subscripts. Using (2.12) and (2.14), calculation shows
\[
d\left( \frac{W^*_t}{W^*_T} \right) = \left( -\frac{\gamma}{W^*_t} \right) \mu \cdot \Sigma^{-1} \mu + \frac{1}{\gamma} \mu' \Sigma^{-1} \sigma \rho Z' - \frac{1}{\gamma} \mu' \Sigma^{-1} \sigma \rho Z' \right) dt
\]
\[
d e^{Y} = e^{Y} \left( -H(t, Y, Z) + \frac{1}{2} ZZ' \right) dt + e^{Y} ZdW.
\]
Combining the previous two identities, (2.20) and the expression for $\tilde{c}^*$ in (2.14), we obtain
\[
d D^* = D^* \left( -\gamma (r - \tilde{c}^*) + (\theta - 1) \delta \psi e^{-\frac{\psi}{\tilde{\psi}}} Y - \delta \theta \right)
\]
\[
+ \frac{1-\gamma}{\gamma} \mu' \Sigma^{-1} \mu + \frac{1-\gamma}{\gamma} \mu' \Sigma^{-1} \sigma \rho Z' + \frac{1}{2} Z \Sigma Z' - H(t, Y, Z) \right) dt
\]
\[
+ D^* \left( -\gamma (\pi^*)' \sigma dW^\rho + ZdW \right)
\]
Proof of Theorem 2.16  It follows from (2.14) and (2.20) that

\[
\mathcal{W}_t^* D_t^* + \int_0^t D_s^* c_s^* ds = C_t(\mathcal{W}_t^*)^{1-\gamma} e^{Y_t} + \int_0^t C_s \delta^{Y_s} e^{-\frac{\psi}{\pi} Y_s} (\mathcal{W}_s^*)^{1-\gamma} e^{Y_s} ds. \tag{B.11}
\]

Here \( C_t = \omega^\gamma e^{-Y_0} \exp(\int_0^t (\theta - 1) \delta^{Y_u} e^{-\frac{\psi}{\pi} Y_u} du - \delta^\theta t), \ t \in [0, T]. \) Since \( \theta < 0, \) the process \( C \) is bounded from above by a constant. We have already seen in Lemma 2.15 that \( \mathcal{W}_* D^* + \int_0^T D_s^* c_s^* ds \) is a nonnegative local martingale. It suffices to prove that it is of class \( D. \) To this end, it follows from (B.10) that

\[
\mathbb{E}\left[ \int_0^T \delta (c_s^* \frac{1}{1-\frac{1}{\psi}} ((\mathcal{W}_s^*)^{1-\gamma} e^{Y_s})^{1-\frac{1}{\psi}} ds \right] = \frac{\omega^{1-\gamma} e^{-Y_0}}{1-\frac{1}{\psi}} \mathbb{E}(\mathcal{W}_T^*)^{1-\gamma} + \frac{\delta}{1-\frac{1}{\psi}} \int_0^T \mathbb{E}(\mathcal{W}_s^*)^{1-\gamma} e^{Y_s} ds \]

\(< \infty. \)

Here since \( (\mathcal{W}_s^*)^{1-\gamma} e^Y \) is of class \( D, \) \( \mathbb{E}(\mathcal{W}_s^*)^{1-\gamma} e^{Y_s} \) is bounded uniformly in \( s. \) Therefore, the previous inequality holds. On the other hand, using the expression of \( c^* \) in (2.14),

\[
\mathbb{E}\left[ \int_0^T \delta (c_s^* \frac{1}{1-\frac{1}{\psi}} ((\mathcal{W}_s^*)^{1-\gamma} e^{Y_s})^{1-\frac{1}{\psi}} ds \right] = \frac{\delta^\psi}{1-\frac{1}{\psi}} \mathbb{E}\left[ \int_0^T (\mathcal{W}_s^*)^{1-\gamma} e^{1-(1-\frac{1}{\psi})Y_s} ds \right].
\]

Then \( \psi > 1 \) and the previous two equations combined yield that the second term on the right-hand side of (B.11) is bounded from above by an integrable random variable and hence of class \( D. \) Meanwhile, using the class \( D \) property of \( (\mathcal{W}_s^*)^{1-\gamma} e^Y \) again, the first term on the right of (B.11) is also of class \( D. \) This confirms the class \( D \) property of \( \mathcal{W}_* D^* + \int_0^T D_s^* c_s^* ds. \)

\[\square\]

Appendix C: Proofs for Sect. 3

To prove Proposition 3.2, let us recall the following result on the Laplace transform of an integrated square root process; cf. [36, Eq. (2.k)] or [8, Eq. (3.2)].

Lemma C.1 Consider \( X \) with dynamics

\[
dX_t = (\vartheta - \kappa X_t)dt + a \sqrt{X_t} dW_t,
\]

where \( W \) is a 1-dimensional Brownian motion. When

\[
q < \frac{\kappa^2}{2a^2},
\]

\[\square\] Springer
the Laplace transform
\[
\mathbb{E}\left[ \exp\left( q \int_0^T X_s ds \right) \middle| X_0 = x \right]
\]
is well defined for any \( T \geq 0 \).

**Proof of Proposition 3.2** Assume 2.7, 2.10 and 2.11 are verified in what follows. Note that \( \sigma(x) = \sqrt{x} \sigma \), \( \Sigma(x) = x \Sigma \), \( b(x) = b(\ell - x) \), \( a(x) = a \sqrt{x} \) and \( \Theta = \sigma' \Sigma^{-1} \sigma \).

**Assumption 2.7:** Note that \( \frac{1-\gamma}{\gamma} \mu'(x) \Sigma^{-1}(x)\sigma(x)\rho(x) = \frac{1-\gamma}{\gamma} \lambda' \Theta \rho \sqrt{x} \). Consider the martingale problem associated to \( \mathcal{L} := (b \ell - (b - \frac{1-\gamma}{\gamma} a \lambda' \Theta \rho) x) \partial_x + \frac{1}{2} a^2 x \partial_x^2 \) on \((0, \infty)\). Since \( b \ell > \frac{1}{2} a^2 \), Feller’s test for explosion implies that this martingale problem is well posed. Then [10, Remark 2.6] implies that the stochastic exponential in Assumption 2.7(i) is a \( \mathbb{P} \)-martingale; hence \( \mathbb{P} \) is well defined. For Assumption 2.7(ii), \( h(x) = (1 - \gamma) r_0 + ((1 - \gamma) r_1 + \frac{1-\gamma}{2\gamma} \lambda' \Theta \lambda) x \). Since \( X \) has under \( \mathbb{P} \) the dynamics
\[
dX_t = \left( b \ell - \left( b - \frac{1-\gamma}{\gamma} a \lambda' \Theta \rho \right) \right) X_t + a \sqrt{X_t} d\mathcal{W}_t,
\]
where \( \mathcal{W} \) is a \( \mathbb{P} \)-Brownian motion, \( \mathbb{E}^\mathbb{P}[\int_0^T h(X_s) ds] > -\infty \) follows from the fact that \( \mathbb{E}^\mathbb{P}[X_s] \) is bounded uniformly for \( s \in [0, T] \).

**Assumption 2.10:** The operator \( \mathfrak{A} \) in (2.16) reads
\[
\mathfrak{A}[\phi] = \frac{1}{2} a^2 x \partial_x^2 \phi + \left( b \ell - bx + \frac{1-\gamma}{\gamma} a \lambda' \Theta \rho x \right) \partial_x \phi
+ \frac{1}{2} \tilde{M} a^2 x (\partial_x \phi)^2 + (1 - \gamma)(r_0 + r_1 x) + \frac{1-\gamma}{2\gamma} \lambda' \Theta \lambda x,
\]
where \( \tilde{M} = 1 + \frac{1-\gamma}{\gamma} \rho' \Theta \rho > 0 \). Consider \( \phi(x) = -c \log x + \tilde{c} x \) for two positive constants \( c \) and \( \tilde{c} \) determined later. It is clear that \( \phi(x) \uparrow \infty \) when \( x \downarrow 0 \) or \( x \uparrow \infty \). On the other hand, calculation shows that
\[
\mathfrak{A}[\phi] = C + \left( \frac{1}{2} a^2 c + \frac{1}{2} a^2 \tilde{c}^2 \tilde{M} - b \ell \tilde{c} \right) \frac{1}{x}
+ \left( - \left( b - \frac{1-\gamma}{\gamma} a \lambda' \Theta \rho \right) \tilde{c} + \frac{1}{2} a^2 \tilde{c}^2 \tilde{M} + (1 - \gamma) r_1 + \frac{1-\gamma}{2\gamma} \lambda' \Theta \lambda \right) x,
\]
where \( C \) is a constant. Since \( b \ell > \frac{1}{2} a^2 \), the coefficient of \( 1/x \) is negative for sufficiently small \( c \). When \( r_1 \) or \( \lambda' \Theta \lambda > 0 \), since \( \gamma > 1 \), the coefficient of \( x \) is negative for sufficiently small \( \tilde{c} \). Therefore, these choices of \( c \) and \( \tilde{c} \) imply that \( \mathfrak{A}[\phi](x) \downarrow -\infty \) when \( x \downarrow 0 \) or \( x \uparrow \infty \); hence \( \mathfrak{A}[\phi] \) is bounded from above on \((0, \infty)\), verifying Assumption 2.10.
Assumption 2.11: Let us consider the martingale problem associated to the operator $\mathcal{L}^0 := (b\ell - bx - a\rho'\lambda x)\partial_x + \frac{1}{2}a^2 x\partial_x^2$ on $(0, \infty)$. Since $b\ell > \frac{1}{2}a^2$, Feller’s test for explosion implies that this martingale problem is well posed and its solution, denoted by $Q^\rho$, satisfies $dQ^\rho/dP = E(\int -\lambda'\rho\sqrt{X_s}dW_s)_T$. Define $Q^0$ via

$$
\frac{dQ^0}{dP} := E\left( -\int \lambda'\rho\sqrt{X_s}dW_s \right) = \frac{dQ^\rho}{dP} = E\left( \int -\lambda'\rho\sqrt{X_s}dW^\rho_s \right) .
$$

Here, due to the independence between $X$ and $W^\perp$, a proof similar to (B.7) implies that both stochastic exponentials on the right are $P$-martingales; hence $Q^0$ is well defined, and $\lambda$ in Assumption 2.11 can be chosen as $\lambda\sqrt{X}$.

To verify (2.17), note that

$$
E(\int \lambda'\sqrt{X_s}dW_0^\psi)_T = \exp\left( \frac{1}{2}(\psi^2 - \psi)\lambda'\lambda\int_0^T X_s ds \right) \times E(\int \psi\lambda'\sqrt{X_s}dW_0^\psi)_T ,
$$

where $W_0 := W^\rho + \int_0^\cdot \lambda'\sqrt{X_s}ds$ is a $Q^0$-Brownian motion. Following the construction of $Q^0$, one can similarly show that $E(\int \psi\lambda'\sqrt{X_s}dW_0^\psi)$ is a $Q^0$-martingale. Hence $Q^\psi$ can be defined via

$$
\frac{dQ^\psi}{dQ^0} := E\left( \int \psi\lambda'\sqrt{X_s}dW_0^\psi \right) .
$$

Combining the above two measure changes, the dynamics of $X$ can be rewritten as

$$
dX_t = \left( b\ell - (b - (\psi - 1)a\lambda')X_t \right)dt + a\sqrt{X_t}dW^\psi_t ,
$$

where $W^\psi := W + \int_0^\cdot (1 - \psi)\lambda'\rho\sqrt{X_s}ds$ is a 1-dimensional $Q^\psi$-Brownian motion.

On the other hand, calculation using (C.1) shows that

$$
E^{Q^0}\left[ e^{(\psi - 1)\int_0^T r_s ds} E\left( \int \eta_s d\mathcal{B}_s^{Q^0} \right)_T \right] = e^{(\psi - 1)r_0T} E^{Q^\psi}\left[ \exp\left( (\psi - 1)r_1 + \frac{1}{2}(\psi^2 - \psi)\lambda'\lambda\int_0^T X_s ds \right) \right] .
$$

Then Lemma C.1 implies that the expectation on the right-hand side is finite when

$$
(\psi - 1)r_1 + \frac{1}{2}(\psi^2 - \psi)\lambda'\lambda < \frac{(b - (\psi - 1)a\lambda')^2}{2a^2}.
$$

This is exactly the assumption in Proposition 3.2(ii). \hfill \Box

Proof of Proposition 3.4 Assumptions 2.7, 2.10 and 2.11 are verified. Then the statements of Theorems 2.14 and 2.16 follow. We denote $\Theta = \sigma'\Sigma^{-1}\sigma$ throughout the proof to simplify notation.

$\copyright$ Springer
Assumption 2.7: Note that \( \frac{1 - \gamma}{\gamma} \mu'(x) \Sigma^{-1}(x)\sigma(x)\rho(x) = \frac{1 - \gamma}{\gamma} (\lambda_0 + \lambda_1 x)'\Theta\rho \). Consider the martingale problem associated to the operator

\[
\mathcal{L} := \left( -bx + \frac{1 - \gamma}{\gamma} a(\lambda_0 + \lambda_1 x)'\Theta\rho \right) \partial_x + \frac{1}{2} a^2 \partial_x^2
\]

on \( \mathbb{R} \). This martingale problem is well posed since all coefficients of \( \mathcal{L} \) have at most linear growth. Then [10, Remark 2.6] implies that the stochastic exponential in Assumption 2.7 is a \( \mathbb{P} \)-martingale; hence \( \mathbb{P} \) is well defined. For Assumption 2.7(ii), \( h(x) = (1 - \gamma)(r_0 + r_1 x) + \frac{1 - \gamma}{2\gamma} (\lambda_0 + \lambda_1 x)'\Theta(\lambda_0 + \lambda_1 x) \) is bounded from below when either \( r_1 = 0 \) or \( \lambda_1'\Theta\lambda_1 > 0 \). Since \( X \) is another Ornstein–Uhlenbeck process with modified linear drift under \( \mathbb{P} \), \( X \) has all finite moments; cf. [22, Chap. 5, Eq. (3.17)]. So Assumption 2.7(ii) is satisfied.

Assumption 2.10: The operator \( \mathfrak{H} \) in (2.16) reads

\[
\mathfrak{H}[\phi] = \frac{1}{2} a^2 \partial_x^2 \phi + \left( -bx + \frac{1 - \gamma}{\gamma} a(\lambda_0 + \lambda_1 x)'\Theta\rho \right) \partial_x \phi + \frac{1}{2} a^2 M(\partial_x \phi)^2
\]

\[
+ (1 - \gamma)(r_0 + r_1 x) + \frac{1 - \gamma}{2\gamma} (\lambda_0 + \lambda_1 x)'\Theta(\lambda_0 + \lambda_1 x),
\]

where \( M = 1 + \frac{1 - \gamma}{\gamma} \rho'\Theta\rho > 0 \). Consider \( \phi(x) = cx^2 \) for a positive constant \( c \) determined later. It is clear that \( \phi(x) \uparrow \infty \) as \( |x| \uparrow \infty \). On the other hand, calculation shows

\[
\mathfrak{H}[\phi] = ca^2 + 2c \left( -bx^2 + \frac{1 - \gamma}{\gamma} a(\lambda_0 + \lambda_1 x)'\Theta\rho x \right) + 2c^2 a^2 Mx^2
\]

\[
+ (1 - \gamma)(r_0 + r_1 x) + \frac{1 - \gamma}{2\gamma} (\lambda_0 + \lambda_1 x)'\Theta(\lambda_0 + \lambda_1 x)
\]

\[
= \left( -2cb + 2c \frac{1 - \gamma}{\gamma} a\lambda_1'\Theta\rho + 2c^2 a^2 M + \frac{1 - \gamma}{2\gamma} \lambda_1'\Theta\lambda_1 \right) x^2
\]

\[+ \text{lower order terms in } x.\]

When \(-b + \frac{1 - \gamma}{\gamma} a\lambda_1'\Theta\rho < 0 \), noticing that \( \frac{1 - \gamma}{2\gamma} \lambda_1'\Theta\lambda_1 \leq 0 \), we can choose sufficiently small \( c \) such that \( \mathfrak{H}[\phi] \downarrow -\infty \) as \( |x| \uparrow \infty \). When \( \lambda_1'\Theta\lambda_1 > 0 \), we have \( \frac{1 - \gamma}{2\gamma} \lambda_1'\Theta\lambda_1 < 0 \); hence we can also choose sufficiently small \( c \) such that \( \mathfrak{H}[\phi] \) has the same asymptotic behavior. In both cases, \( \mathfrak{H}[\phi] \) is bounded from above on \( \mathbb{R} \); hence Assumption 2.10 is verified.

Assumption 2.11: Let us consider the martingale problem associated to the operator \( \mathcal{L}_0 := (-bx - a(\lambda_0 + \lambda_1 x)'\rho) \partial_x + \frac{1}{2} a^2 \partial_x^2 \) on \( \mathbb{R} \). Since all coefficients have at most linear growth, this martingale problem is well posed, and its solution, denoted by \( \mathbb{Q}^\rho \), satisfies

\[
\frac{d\mathbb{Q}^\rho}{d\mathbb{P}} = \mathcal{E} \left( \int - (\lambda_0 + \lambda_1 X_s)'\rho dW_s \right). 
\]
Define $Q^0$ via

$$
\frac{dQ^0}{dQ} = \mathcal{E} \left( \int - (\lambda_0 + \lambda_1 X_s)'dW^\rho_s \right)_T.
$$

An argument similar to (B.7) implies that $Q^0$ is well defined. Therefore, $\lambda$ in Assumption 2.11 can be chosen as $\lambda_0 + \lambda_1 X$.

To verify (2.17), note that

$$
\mathcal{E} \left( \int (\lambda_0 + \lambda_1 X_s)'dW^0_s \right)_T^{\psi} = \exp \left( \frac{1}{2} (\psi^2 - \psi) \int_0^T |\lambda_0 + \lambda_1 X_s|^2 ds \right) \mathcal{E} \left( \int \psi (\lambda_0 + \lambda_1 X_s)'dW^0_s \right)_T^{\psi},
$$

where $W^0 := W^\rho + \int_0^t (\lambda_0 + \lambda_1 X_s)ds$ is a $Q^0$-Brownian motion. Following the construction of $Q^0$, a similar argument shows that $\mathcal{E} (\int \psi (\lambda_0 + \lambda_1 X_s)'dW_0^0)$ is a $Q^0$-martingale. Hence $Q^{\psi}$ can be defined via

$$
\frac{dQ^\psi}{dQ^0} := \mathcal{E} \left( \int \psi (\lambda_0 + \lambda_1 X_s)'dW^0_s \right)_T^{\psi}.
$$

Combining the above two measure changes, the dynamics of $X$ can be rewritten as

$$
dX_t = \left( (\psi - 1)a\lambda'_0 \rho - (b - (\psi - 1)a\lambda'_1 \rho) X_t \right) dt + a d\tilde{W}_t^\psi,
$$

where $W^\psi := W + \int_0^t (1 - \psi) (\lambda_0 + \lambda_1 X_s)' \rho ds$ is a 1-dimensional $Q^\psi$-Brownian motion. On the other hand, calculation shows, for any $\epsilon > 0$,

$$
\begin{align*}
\mathbb{E}^{Q^0} \left[ e^{(\psi - 1) \int_0^T r^+(X_s) + (\lambda_0 + \lambda_1 X_s)'dW^0_s} \right]^{\psi} \\
= C \mathbb{E}^{Q^0} \left[ \exp \left( (\psi - 1) \int_0^T (r^+_1 X_s) + \frac{1}{2} (\psi^2 - \psi) \lambda'_1 \lambda_1 \int_0^T X^2_s ds \right) \right] \\
\leq C \mathbb{E}^{Q^\psi} \left[ \exp \left( \frac{1}{2} (\psi^2 - \psi) \lambda'_1 \lambda_1 + \epsilon \right) \int_0^T X^2_s ds \right],
\end{align*}
$$

where $C$ is a constant and $C_\epsilon$ a constant depending on $\epsilon$.

In order to appeal to Lemma C.1 to calculate the expectation on the right-hand side of (C.2), let us introduce another measure $\tilde{Q}^\psi$ via $\frac{d\tilde{Q}^\psi}{dQ^0} = \mathcal{E} (- (\psi - 1) \lambda'_0 \rho W^\psi_T)$. Under this measure, $X$ has dynamics

$$
dX_t = -(b - (\psi - 1)a\lambda'_1 \rho) X_t dt + a d\tilde{W}_t^\psi,
$$
where $\tilde{W}^\psi := W^\psi + \int_0^\cdot (\psi - 1)\lambda_0' \rho \, ds$ is a $\tilde{Q}^\psi$-Brownian motion. Let $Y := X^2$. It then has dynamics

$$dY_t = \left( a^2 - 2(b - (\psi - 1)a\lambda_1' \rho)Y_t \right) dt + 2a\sqrt{Y_t} d\tilde{W}^\psi_t,$$

which is of the same type as $X$ in Lemma C.1.

Coming back to (C.2), Hölder’s inequality implies, for any $\delta > 0$,

$$E^{\tilde{Q}^\psi}\left[ \exp\left( \frac{1}{2} (\psi^2 - \psi) \lambda_1' \lambda_1 + \epsilon \right) \int_0^T X_s^2 \, ds \right]$$

$$= E^{\tilde{Q}^\psi}\left[ \frac{dQ^\psi}{d\tilde{Q}^\psi} e^{\frac{1}{2} (\psi^2 - \psi) \lambda_1' \lambda_1(1+\epsilon)} \int_0^T X_s^2 \, ds \right]$$

$$\leq E^{\tilde{Q}^\psi}\left[ \left( \frac{dQ^\psi}{d\tilde{Q}^\psi} \right)^{\frac{1+\delta}{1+\frac{3}{2}}} \right]^{\frac{1+\frac{3}{2}}{1+\delta}} E^{\tilde{Q}^\psi}\left[ e^{(1+\delta)\left( \frac{1}{2} (\psi^2 - \psi) \lambda_1' \lambda_1(1+\epsilon) \right)} \int_0^T X_s^2 \, ds \right]^{\frac{1+\delta}{1+\frac{3}{2}}}.$$  

Observe that the first expectation on the right-hand side is finite, due to the fact that $dQ^\psi/d\tilde{Q}^\psi = \mathcal{E}(\psi - 1)\lambda_0' \rho \tilde{W}^\psi_T$ has all finite moments. For the second expectation, we can choose sufficiently small $\delta$ and $\epsilon$ such that according to Lemma C.1, when

$$\frac{1}{2} (\psi^2 - \psi) \lambda_1' \lambda_1 < \frac{4(b - (\psi - 1)a\lambda_1' \rho)^2}{8a^2},$$  

(C.3)

the second expectation is finite. Combining the previous estimates and (C.2), we confirm (2.17). Finally, note that (C.3) is exactly the assumption in Proposition 3.4(ii).

References

1. Bansal, R.: Long-run risks and financial markets. Fed. Reserve Bank St. Louis Rev. 89, 1–17 (2007)
2. Bansal, R., Yaron, A.: Risks for the long run: a potential resolution of asset pricing puzzles. J. Finance 59, 1481–1509 (2004)
3. Barberis, N.: Investing for the long run when returns are predictable. J. Finance 55, 1481–1509 (2004)
4. Benzoni, L., Collin-Dufresne, P., Goldstein, R.: Explaining asset pricing puzzles associated with the 1987 market crash. J. Financ. Econ. 101, 552–573 (2011)
5. Bhamra, H., Kuehn, L., Strebulaev, I.: The levered equity risk premium and credit spreads: a unified framework. Rev. Financ. Stud. 23, 645–703 (2010)
6. Briand, P., Hu, Y.: BSDE with quadratic growth and unbounded terminal value. Probab. Theory Relat. Fields 136, 604–618 (2006)
7. Campbell, J., Viceira, L.: Consumption and portfolio decisions when expected returns are time varying. Q. J. Econ. 114, 433–495 (1999)
8. Carr, P., Geman, H., Madan, D., Yor, M.: Stochastic volatility for Lévy processes. Math. Finance 13, 345–382 (2003)
9. Chacko, G., Viceira, L.: Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets. Rev. Financ. Stud. 18, 1369–1402 (2005)
10. Cheridito, P., Filipović, D., Yor, M.: Equivalent and absolutely continuous measure changes for jump-diffusion processes. Ann. Appl. Probab. 15, 1713–1732 (2005)
11. Cheridito, P., Hu, Y.: Optimal consumption and investment in incomplete markets with general constraints. Stoch. Dyn. 11, 283–299 (2011)
12. Duffie, D., Epstein, L.: Asset pricing with stochastic differential utility. Rev. Financ. Stud. 5, 411–436 (1992)
13. Duffie, D., Epstein, L.: Stochastic differential utility. Econometrica 60, 353–394 (1992)
14. Duffie, D., Lions, P.-L.: PDE solutions of stochastic differential utility. J. Math. Econ. 21, 577–606 (1992)
15. Duffie, D., Skiadas, C.: Continuous-time security pricing: a utility gradient approach. J. Math. Econ. 23, 107–131 (1994)
16. El Karoui, N., Peng, S., Quenez, M.C.: A dynamic maximum principle for the optimization of recursive utilities under constraints. Ann. Appl. Probab. 11, 664–693 (2001)
17. Föllmer, H., Schweizer, M.: Hedging of contingent claims under incomplete information. In: Davis, M.H.A., Elliott, R.J. (eds.) Applied Stochastic Analysis. Stochastics Monographs, vol. 5, pp. 389–414. Gordon and Breach, London (1991)
18. Guasoni, P., Robertson, S.: Portfolios and risk premia for the long run. Ann. Appl. Probab. 22, 239–284 (2012)
19. Heath, D., Schweizer, M.: Martingales versus PDEs in finance: an equivalence result with examples. J. Appl. Probab. 37, 947–957 (2000)
20. Hu, Y., Imkeller, P., Müller, M.: Utility maximization in incomplete markets. Ann. Appl. Probab. 15, 1691–1712 (2005)
21. Karatzas, I., Kardaras, C.: The numéraire portfolio in semimartingale financial models. Finance Stoch. 11, 447–493 (2007)
22. Karatzas, I., Shreve, S.: Brownian Motion and Stochastic Calculus, 2nd edn. Springer, New York (1988)
23. Kazamaki, N.: Continuous Exponential Martingales and BMO. Lecture Notes in Mathematics, vol. 1579. Springer, Berlin (1994)
24. Kim, T., Omberg, E.: Dynamic nonmyopic portfolio behavior. Rev. Financ. Stud. 9, 141–161 (1996)
25. Kobylanski, M.: Backward stochastic differential equations and partial differential equations with quadratic growth. Ann. Probab. 28, 558–602 (2000)
26. Kraft, H.: Optimal portfolios and Heston’s stochastic volatility model: an explicit solution for power utility. Quant. Finance 5, 303–313 (2005)
27. Kraft, H., Seiferling, T., Seifried, F.-T.: Asset pricing and consumption-portfolio choice with recursive utility and unspanned risk. Working paper (2014). http://ssrn.com/abstract=2424706
28. Kraft, H., Seifried, F.-T.: Stochastic differential utility as the continuous-time limit of recursive utility. J. Econ. Theory 151, 528–550 (2014)
29. Kraft, H., Seifried, F.-T., Steffensen, M.: Consumption-portfolio optimization with recursive utility in incomplete markets. Finance Stoch. 17, 161–196 (2013)
30. Kreps, D., Porteus, E.: Temporal resolution of uncertainty and dynamic choice theory. Econometrica 46, 185–200 (1978)
31. Liu, J.: Portfolio selection in stochastic environments. Rev. Financ. Stud. 20, 1–39 (2007)
32. Liu, J., Pan, J.: Dynamic derivative strategies. J. Financ. Econ. 69, 401–430 (2003)
33. Morlais, M.-A.: Quadratic BSDEs driven by a continuous martingale and applications to the utility maximization problem. Finance Stoch. 13, 121–150 (2009)
34. Pardoux, É.: BSDEs, weak convergence and homogenization of semilinear PDEs. In: Clarke, F.H., et al. (eds.) Nonlinear Analysis, Differential Equations and Control, Montreal, QC, 1998. NATO Sci. Ser. C Math. Phys. Sci., vol. 528, pp. 503–549. Kluwer Acad. Publ., Dordrecht (1999)
35. Pham, H.: Smooth solutions to optimal investment models with stochastic volatilities and portfolio constraints. Appl. Math. Optim. 46, 55–78 (2002)
36. Pitman, J., Yor, M.: A decomposition of Bessel bridges. Probab. Theory Relat. Fields 59, 425–457 (1982)
37. Robertson, S., Xing, H.: Long term optimal investment in matrix valued factor models. Working paper (2014). arXiv:1408.7010
38. Schroder, M., Skiadas, C.: Optimal consumption and portfolio selection with stochastic differential utility. J. Econ. Theory 89, 68–126 (1999)
39. Schroder, M., Skiadas, C.: Optimal lifetime consumption-portfolio strategies under trading constraints and generalized recursive preferences. Stoch. Process. Appl. 108, 155–202 (2003)
40. Seiferling, T., Seifried, F.-T.: Stochastic differential utility with preference for information: existence, uniqueness, concavity, and utility gradients. Working paper (2015). http://ssrn.com/abstract=2625800
41. Skiadas, C.: Recursive utility and preferences for information. Econ. Theory 12, 293–312 (1998)
42. Stroock, D.W., Varadhan, S.R.S.: Multidimensional Diffusion Processes. Classics in Mathematics. Springer, Berlin (2006). Reprint of the 1997 edition
43. Wachter, J.: Portfolio and consumption decisions under mean-reverting returns: an exact solution for complete markets. J. Financ. Quant. Anal. 37, 63–91 (2002)