HYPERSURFACES OF NEARLY KÄHLER TWISTOR SPACES

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Abstract. In this article, we show that a hypersurface of the nearly Kähler $\mathbb{CP}^3$ or $\mathbb{F}_{1,2}$ cannot have its shape operator and induced almost contact structure commute together. This settles the question for six-dimensional homogeneous nearly Kähler manifolds, as the cases of $S^6$ and $S^3 \times S^3$ were previously solved, and provides a counterpart to the more classical question for the complex space forms $\mathbb{CP}^n$ and $\mathbb{CH}^n$. The proof relies heavily on the construction of $\mathbb{CP}^3$ and $\mathbb{F}_{1,2}$ as twistor spaces of $S^4$ and $\mathbb{CP}^2$.

1. Introduction

The classical study of real hypersurfaces in complex space forms has led to extensive lists by Takagi \cite{21, 22} (for $\mathbb{CP}^n$) and Montiel \cite{17} (for $\mathbb{CH}^n$).

Driven by the number of principal curvatures and the importance of Hopf hypersurfaces, i.e. when the ambient complex structure maps the normal vector field to a principal direction, hypersurfaces of $\mathbb{CP}^n$ or $\mathbb{CH}^n$ where the shape operator $A$ and the induced almost contact structure $\varphi$ commute constitute a remarkable class, amenable to classification.

Indeed, by \cite{6} Th. 6.19 their principal curvatures must be constant and in twos or threes. Moreover, they must belong to type A of the Takagi-Montiel lists (cf. \cite{6} Th. 8.37 as well)

An almost Hermitian manifold $(Z, \mathbb{I}, g)$ is called nearly Kähler \cite{12} if $\nabla \mathbb{I}$ is antisymmetric. The best-known (non-Kähler) example is the round sphere $S^6$ with its canonical metric and the structure that comes from octonion multiplication.

In view of the classical theory for complex space forms, it is natural to ask which hypersurfaces of nearly Kähler manifolds satisfy $A\varphi = \varphi A$.

Nearly Kähler manifolds enjoy many topological and geometric properties akin to Kähler geometry (cf. \cite{13}) and have known a recent revival of interest with the structure theorem of Nagy \cite{19} in 2002, which shows that six-dimensional nearly Kähler manifolds act as building blocks, and Butruille’s 2005 classification of homogeneous nearly Kähler six-manifolds \cite{5}, namely $S^6$, $S^3 \times S^3$, $\mathbb{CP}^3$ and $\mathbb{F}_{1,2}$.

While the explicit construction of the nearly Kähler structure (and metric) on $S^3 \times S^3$ is rather involved and ad-hoc, the $\mathbb{CP}^3$ and $\mathbb{F}_{1,2}$ examples both have their origin in twistor theory, as twistor spaces of $S^4$ and $\mathbb{CP}^2$.

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In the case of the four-dimensional sphere, as its unitary frame bundle is SO(5), its twistor space is the associated bundle
\[ \text{SO}(5) \times_{\text{SO}(4)} \text{SO}(4)/U(2) \simeq \text{SO}(5)/U(2) \]
which is \( \mathbb{C}P^3 \) and the twistor projection \( \mathbb{C}P^3 \to S^4 \) is \( \text{span}_Cv \mapsto \text{span}_Hv \), where \( S^4 \simeq \mathbb{H}P^1 \) by the Hopf map. When the spaces are equipped with their canonical metrics, this projection is a Riemannian submersion.

For the two-dimensional complex projective space, one considers \( y \in \mathbb{C}P^2 \) and \((x, y, z)\) mutually orthogonal complex lines in \( \mathbb{C}^3 \). Identifying a complex structure in \( T_y \mathbb{C}P^2 \) with a choice of holomorphic and anti-holomorphic bundles, one shows that
\[
(1) \quad T^{1,0}_y \mathbb{C}P^2 = \text{Hom}(y, x) \oplus \text{Hom}(y, z)
\]
and
\[
(2) \quad T^{0,1}_y \mathbb{C}P^2 = \text{Hom}(x, y) \oplus \text{Hom}(z, y).
\]
There is a one-one correspondence between triples \((x, y, z)\) and couples \((l, p)\), where \( l \) is a complex line and \( p \) a complex plane in \( \mathbb{C}^3 \) with \( l \subset p \), i.e. the flag manifold \( F_{1,2} \).

Since \( \mathbb{C}P^2 \) is self-dual \cite{2}, the integrable almost Hermitian structure is defined by taking the standard Hermitian structure on \( T^{1,0}_y \mathbb{C}P^2 \) and its opposite on \( T^{0,1}_y \mathbb{C}P^2 \). Because of this orientation reversal, this identifies \( F_{1,2} \) with \( Z(\mathbb{C}P^2) \), and the twistor projection \((l \subset p) \in F_{1,2} \mapsto l^\perp \cap p \in \mathbb{C}P^2 \) is also a Riemannian submersion.

There is a general procedure, due to \cite{9} and \cite{19}, to produce nearly Kähler manifolds: If \((Z, I_1, g_1)\) is a Kähler manifold with a Riemannian foliation \( F \), which induces an \((I_1\)-invariant) integrable distribution \( \mathcal{V} \) and its orthogonal complement \( \mathcal{H} \), then the Riemannian metric
\[
g_2(X, Y) = \frac{1}{2}g_1(X, Y) \quad \forall X, Y \in \mathcal{V}
\]
and
\[
g_2(X, Y) = g_1(X, Y) \quad \forall X, Y \in \mathcal{H}
\]
together with the almost complex structure
\[
I_2X = -I_1X \quad \forall X \in \mathcal{V} \quad \text{and} \quad I_2X = I_1X \quad \forall X \in \mathcal{H}
\]
make \((Z, I_2, g_2)\) into a nearly Kähler manifold.

According to Hitchin \cite{14}, \( \mathbb{C}P^3 \) and \( F_{1,2} \) are the only compact twistor spaces \((Z, I_1, g_1)\) to be Kähler and, therefore, the only ones to admit a nearly Kähler structure.

Let \((Z, I_2, g)\) be a nearly Kähler manifold and \( H \hookrightarrow Z \) a hypersurface. Call \( N \) the unit normal to \( H \) and then define an almost contact (metric) structure \( \varphi \) on \( H \) by:
\[
\varphi X = I_2X - g(I_2X, N)N, \quad \forall X \in TH.
\]
One easily verifies that 
\[ g(\varphi X, \varphi Y) = g(X, Y) \quad \forall X, Y \in TH \cap (\mathbb{I}_2 N)^\perp, \]
or more generally 
\[ g(\varphi A, B) = g(A, \varphi B) \quad \forall A, B \in TH, \]
as well as 
\[ \varphi(\mathbb{I}_2 N) = 0. \]

The other fundamental tensor is the shape operator \( A \) of \( H \):

\[ AX = -\nabla^Z_X N, \]
so that

\[ \nabla_X^Z Y = \nabla_X^Z Y + g(AX, Y)N. \]

An immediate remark on hypersurfaces of nearly Kähler manifolds which satisfy \( A\varphi = \varphi A \) is that the Hopf vector field \( \mathbb{I}_2 N \) has to be an eigenvector of \( A \) (of eigenvalue \( \mu \)) and the eigenspaces of \( A|_{(\mathbb{I}_2 N)^\perp} \) must be \( \mathbb{I}_2 \)-stable.

In dimension 6, since we have a full classification of homogeneous nearly Kähler [5], the first two cases of the list, \( S^6 \) and \( S^3 \times S^3 \) have already been investigated.

Combining results of [3] and [16], shows that the only hypersurfaces of \( S^6 \) with \( A\varphi = \varphi A \) are (open parts of) geodesic spheres.

For the nearly Kähler \( S^3 \times S^3 \), that is equipped with the right metric and almost complex structure, its hypersurfaces with \( A\varphi = \varphi A \) must be locally given by the canonical immersion of \( S^2 \times S^3 \) in \( S^3 \times S^3 \) ([15]). Note that this classification contains three immersions but, by [18], they turn out to be all isometric one to the other.

There exists an almost contact counterpart to the nearly Kähler condition, coined nearly cosymplectic (and defined by \( \nabla \varphi \) being antisymmetric). By [4], they must satisfy \( A\varphi = \varphi A \) but while \( S^5 \hookrightarrow S^6 \) is well-known to be nearly cosymplectic, the hypersurface \( S^2 \times S^3 \hookrightarrow S^3 \times S^3 \) is not, as a quick inspection of the eigenvalues of its shape operator reveals.

The objective of this article is to extend these results to the remaining two homogeneous nearly Kähler six-manifolds and to prove the following theorem.

**Theorem.** Let \( Z(M) \) be the nearly Kähler manifold \( \mathbb{C}P^3 \) or \( F_{1,2} \). Then there exists no hypersurface \( H \hookrightarrow Z(M) \) such that its shape operator \( A \) and the induced almost contact structure \( \varphi \) commute:

\[ A\varphi = \varphi A. \]

A direct consequence is that this construction produces only one example of nearly cosymplectic almost contact hypersurface.

**Corollary 1.** The only nearly cosymplectic hypersurface of a homogeneous 6-dimensional nearly Kähler manifold is \( S^5 \hookrightarrow S^6 \).
As a byproduct of the Theorem, we obtain information on the eigenvalues of the shape operator $A$.

**Corollary 2.** There is no totally geodesic or totally umbilical hypersurface of the nearly Kähler manifolds $\mathbb{CP}^3$ or $\mathbb{F}_{1,2}$.

2. Curvature properties of nearly Kähler $\mathbb{CP}^3$ and $\mathbb{F}_{1,2}$

Throughout the rest of this article, we specialize to the cases $M = S^4$ and $M = \mathbb{CP}^2$. Let $Z(M)$ be the twistor space of $M$, equipped with the Riemannian metric $[1]$

$$g_t = \pi^* g_M + t g_{\mathbb{CP}^1}, \quad (t > 0).$$

Two almost complex structures can be defined on $Z(M)$: First the Atiyah-Hitchin-Singer structure $I_1$ on $T_{(x_0, I)} Z(M)$, with $x_0 \in M$ and $I$ a complex structure on $T_{x_0} M$, defined by

$$\begin{align*}
I_1 X &= I X & \text{if } X \in \mathcal{H} \\
I_1 Y &= J \mathbb{CP}^1 Y & \text{if } Y \in \mathcal{V}
\end{align*}$$

where we identify vectors tangent to $M$ with their horizontal lifts in $\mathcal{H} \subset T_{(x_0, I)} Z(M)$;

Second the Eells-Salamon structure $[9]$:

$$\begin{align*}
I_2 &= I_1 & \text{on } \mathcal{H} \\
I_2 &= -I_1 & \text{on } \mathcal{V}.
\end{align*}$$

Then, as the cases we consider are anti-self dual, $[2]$ shows that $(Z(M), g_t, I_1)$ is a Kähler manifold for $t = \frac{12}{s}$, ($s = \text{scal}(M, g_M)$), while $[10, 19]$ prove that $(Z(M), g_t, I_2)$ is nearly Kähler for $t = \frac{6}{s}$.

The next proposition relates the curvature tensors of the twistor space and the base manifold, in terms of the nearly Kähler structure. This will lead to crucial curvature properties in Lemma $[3]$

**Proposition 3.** Let $Z(M)$ be the twistor space of $\mathbb{S}^4$ or $\mathbb{CP}^2$. Write $g = g_{\frac{6}{s}}$ so that $(Z(M), I_2, g)$ is nearly Kähler and denote by $R$ and $R^M$ the respective curvature tensors of $(Z(M), g)$ and $(M, g_M)$.

Let $X, Y, Z, T \in T_p Z(M)$ ($p \in Z(M)$) then
\[ R(X, Y, Z, T) = R^M \left( d\pi(X), d\pi(Y), d\pi(Z), d\pi(T) \right) \]
\[ + 2(b + 2a)g^h(\mathbb{I}_2X, Y)g^h(\mathbb{I}_2Z, T) + (b + 2a)g^h(\mathbb{I}_2X, Z)g^h(\mathbb{I}_2Y, T) \]
\[ - (b + 2a)g^h(\mathbb{I}_2X, T)g^h(\mathbb{I}_2Y, Z) + (c - 5b) \left( g^h(X, Z)g^h(Y, T) - g^h(X, T)g^h(Y, Z) \right) \]
\[ - 2ag^h(\mathbb{I}_2X, Y)g^h(\mathbb{I}_2Z, T) - 2ag^h(\mathbb{I}_2X, Y)g^h(\mathbb{I}_2Z, T) - ag^h(\mathbb{I}_2X, Z)g(\mathbb{I}_2Y, T) \]
\[ - ag(\mathbb{I}_2X, Z)g^h(\mathbb{I}_2Y, T) + ag(\mathbb{I}_2X, T)g(\mathbb{I}_2Y, Z) + ag(\mathbb{I}_2X, T)g^h(\mathbb{I}_2Y, Z) \]
\[ + (b - c) \left( g^h(X, Z)g(Y, T) + g(X, Z)g^h(Y, T) - g^h(X, T)g(Y, Z) - g(X, T)g^h(Y, Z) \right) \]
\[ + c \left( g(X, Z)g(Y, T) - g(X, T)g(Y, Z) \right) \]
where
\[ a = \frac{s}{24} - t \left( \frac{s}{24} \right)^2 ; b = t \left( \frac{s}{24} \right)^2 ; c = \frac{1}{t}, \]
with \( t = \frac{s}{24} \)

**Proof.** We rely on the formula of [1]:

\[ R(X, Y, Z, T) = R^M \left( d\pi(X), d\pi(Y), d\pi(Z), d\pi(T) \right) \]
\[ + 2ag^h(\mathbb{I}_1X, Y)g^v(\mathbb{I}_1Z, T) + 2ag^v(\mathbb{I}_1X, Y)g^h(\mathbb{I}_1Z, T) + ag^h(\mathbb{I}_1X, Z)g^v(\mathbb{I}_1Y, T) \]
\[ + ag^v(\mathbb{I}_1X, Z)g^h(\mathbb{I}_1Y, T) - ag^h(\mathbb{I}_1X, T)g^v(\mathbb{I}_1Y, Z) - ag^v(\mathbb{I}_1X, T)g^h(\mathbb{I}_1Y, Z) \]
\[ + 2bg^h(\mathbb{I}_1X, Y)g^h(\mathbb{I}_1Z, T) + bg^h(\mathbb{I}_1X, Z)g^h(\mathbb{I}_1Y, T) - bg^h(\mathbb{I}_1X, T)g^h(\mathbb{I}_1Y, Z) \]
\[ + bg^h(X, Z)g^v(Y, T) + bg^v(X, Z)g^h(Y, T) - bg^h(X, T)g^v(Y, Z) - bg^v(X, T)g^h(Y, Z) \]
\[ - 3bg^h(X, Z)g^h(Y, T) + cg^v(X, Z)g^v(Y, T) + 3bg^h(X, T)g^h(Y, Z) - cg^v(X, T)g^v(Y, Z), \]
where \( \mathbb{I}_1 \) is the Kähler structure on \( Z(M) \).

Since \( \mathbb{I}_1 \) and \( \mathbb{I}_2 \) agree on the horizontal distribution and are opposite on \( V \), we have

\[ R(X, Y, Z, T) = R^M \left( d\pi(X), d\pi(Y), d\pi(Z), d\pi(T) \right) \]
\[ - 2ag^h(\mathbb{I}_2X, Y)g^v(\mathbb{I}_2Z, T) - 2ag^v(\mathbb{I}_2X, Y)g^h(\mathbb{I}_2Z, T) - ag^h(\mathbb{I}_2X, Z)g^v(\mathbb{I}_2Y, T) \]
\[ - ag^v(\mathbb{I}_2X, Z)g^h(\mathbb{I}_2Y, T) + ag^h(\mathbb{I}_2X, T)g^v(\mathbb{I}_2Y, Z) + ag^v(\mathbb{I}_2X, T)g^h(\mathbb{I}_2Y, Z) \]
\[ + 2bg^h(\mathbb{I}_2X, Y)g^h(\mathbb{I}_2Z, T) + bg^h(\mathbb{I}_2X, Z)g^h(\mathbb{I}_2Y, T) - bg^h(\mathbb{I}_2X, T)g^h(\mathbb{I}_2Y, Z) \]
\[ + bg^h(X, Z)g^v(Y, T) + bg^v(X, Z)g^h(Y, T) - bg^h(X, T)g^v(Y, Z) - bg^v(X, T)g^h(Y, Z) \]
\[ - 3bg^h(X, Z)g^h(Y, T) + cg^v(X, Z)g^v(Y, T) + 3bg^h(X, T)g^h(Y, Z) - cg^v(X, T)g^v(Y, Z) \]
We use the shorthand \( g^h(X, Y) = g(X^h, Y^h) \) and \( g^v(X, Y) = g(X^v, Y^v) \) for \( X = X^h + X^v \) its decomposition in the horizontal and vertical distributions. Since \( g(X, Y) = g^h(X, Y) + g^v(X, Y) \), we have

\[
R(X, Y, Z, T) = R^M \left( d\pi(X), d\pi(Y), d\pi(Z), d\pi(T) \right)
\]

\[
+ 4ag^h(\mathbb{I}_2X, Y)g^h(\mathbb{I}_2Z, T) + 2ag^h(\mathbb{I}_2X, Z)g^h(\mathbb{I}_2Y, T) - 2ag^h(\mathbb{I}_2X, T)g^h(\mathbb{I}_2Y, Z)
\]

\[
- 2ag^h(\mathbb{I}_2X, Y)g(\mathbb{I}_2Z, T) - 2ag(\mathbb{I}_2X, Y)g^h(\mathbb{I}_2Z, T) - ag^h(\mathbb{I}_2X, Z)g(\mathbb{I}_2Y, T)
\]

\[
- ag(\mathbb{I}_2X, Z)g^h(\mathbb{I}_2Y, T) + ag^h(\mathbb{I}_2X, T)g(\mathbb{I}_2Y, Z) + ag(\mathbb{I}_2X, T)g^h(\mathbb{I}_2Y, Z)
\]

\[
+ 2bg^h(\mathbb{I}_2X, Y)g^h(\mathbb{I}_2Z, T) + bg(\mathbb{I}_2X, Z)g^h(\mathbb{I}_2Y, T) - bg^h(\mathbb{I}_2X, T)g^h(\mathbb{I}_2Y, Z)
\]

\[
- 2bg^h(\mathbb{I}_2X, Z)g^h(Y, T) + 2bg^h(X, T)g^h(Y, Z) + bg^h(X, Z)g(Y, T) + bg^h(X, Z)g^h(Y, T)
\]

\[
- bg^h(X, T)g(Y, Z) - bg(X, T)g^h(Y, Z) + (c - 3b)g^h(X, Z)g^h(Y, T)
\]

\[
- (c - 3b)g^h(X, T)g^h(Y, Z)
\]

\[
+ c(g(X, Z)g(Y, T) - g(X, Z)g^h(Y, T) - g^h(X, Z)g(Y, T))
\]

\[
- c(g(X, T)g(Y, Z) - g(X, T)g^h(Y, Z) - g^h(X, T)g(Y, Z)),
\]

and reorganising terms yields the proposition. \(\square\)

Let \( H \rightarrow Z(M) \) be a hypersurface of \((Z(M), \mathbb{I}_2, g)\) satisfying

\[
A\varphi = \varphi A.
\]

We call \( N \) the normal to \( H \) and Equation (3) implies that \( \mathbb{I}_2N \) is an eigenvector of \( A \) (of eigenvalue \( \mu \)). We denote by \( \lambda \) an eigenvalue of \( A \) and observe that the eigenspace \( E_\lambda \cap (\mathbb{I}_2N)^\perp \) is \( \mathbb{I}_2 \)-invariant.

**Lemma 4.** Let \( X \in E_\lambda \cap (\mathbb{I}_2N)^\perp \) then

\[
R(\mathbb{I}_2N, X, \mathbb{I}_2X, N) = -R(\mathbb{I}_2N, \mathbb{I}_2X, X, N),
\]

and \( R(\mathbb{I}_2N, X, \mathbb{I}_2X, N) = \lambda(\lambda - \mu)\|X\|^2 \).

**Proof.** Since both \( X \) and \( \mathbb{I}_2X \) belong to \( E_\lambda \), the Codazzi Equation gives

\[
R(\mathbb{I}_2N, X, \mathbb{I}_2X, N) = g((\nabla_{\mathbb{I}_2N}A)(X) - (\nabla_X A)(\mathbb{I}_2N), \mathbb{I}_2X)
\]

\[
= -g((\nabla_X A)(\mathbb{I}_2N), \mathbb{I}_2X)
\]

since \( A|_{E_\lambda} = \lambda \text{id}_{E_\lambda} \).

Therefore

\[
R(\mathbb{I}_2N, X, \mathbb{I}_2X, N) = (\lambda - \mu)g(\nabla_X \mathbb{I}_2N, \mathbb{I}_2X),
\]

which is \( \mathbb{I}_2 \)-invariant since

\[
\lambda\|X\|^2 = g(AX, X) = -g(\nabla_X N, X) = g((\nabla_X \mathbb{I}_2)(\mathbb{I}_2N) + \mathbb{I}_2\nabla_X \mathbb{I}_2N, X)
\]

\[
= -g((\nabla_{\mathbb{I}_2N}\mathbb{I}_2)(X), X) - g(\nabla_X \mathbb{I}_2N, \mathbb{I}_2X)
\]

\[
= -g(\nabla_X \mathbb{I}_2N, \mathbb{I}_2X).
\]
Motivated by the results of Lemma 4, we use Proposition 3 to obtain the following curvature expression.

**Corollary 5.** If $X$ is a vector field in $(N, \mathbb{I}_2 N)^\perp$, we have

$$R(\mathbb{I}_2 N, \mathbb{I}_2 X, X, N) = R^M \left( d\pi(\mathbb{I}_2 N), d\pi(\mathbb{I}_2 X), d\pi(X), d\pi(N) \right)$$

$$+ (b + 2a)g^h(N, X)^2 + (-4a + 3b - c)g^h(N, \mathbb{I}_2 X)^2 + a(\|N^h\|^2\|X\|^2 + \|X^h\|^2)$$

$$- (2a + b)\|N^h\|^2\|X^h\|^2.$$

From this symmetry of the curvature tensor, we can eliminate vertical normal vector fields.

**Proposition 6.** Let $H$ be a hypersurface of $Z(M)$ such that $A\varphi = \varphi A$. Then the normal vector $N$ cannot be vertical.

**Proof.** If $N$ is vertical then so is $\mathbb{I}_2 N$ and all eigenvectors orthogonal to it must be horizontal. But for such an eigenvector $X$ associated to the eigenvalue $\lambda$ and orthogonal to $\mathbb{I}_2 N$

$$\lambda\|X\|^2 = g(A X, X) = g(-\nabla_X N, X) = g(N, \nabla_X X)$$

$$= g(N, \frac{1}{2}[X, X]) = 0,$$

by O’Neill [20] since $X$ is horizontal. However, this implies, by Lemma 4, that $R(\mathbb{I}_2 N, \mathbb{I}_2 X, X, N)$ is zero, that is, by $N^h = 0$ and Corollary 5

$$a\|X^h\|^2 = 0,$$

which contradicts the fact that $X$ is horizontal, since $a \neq 0$. \hfill \Box

### 3. The Case $(M, Z(M)) = (S^4, \mathbb{C}P^3)$

As $\text{scal}_{S^4} = 12$, the constants in Proposition 3 take on the values $a = \frac{3}{8}$, $b = \frac{1}{8}$ and $c = 2$. Since

$$R^{S^4}(U, V, W, T) = g(V, W)g(U, T) - g(U, W)g(V, T) \quad \forall U, V, W, T \in T_pS^4,$$

we have

$$\pi^* R^{S^4}(\mathbb{I}_2 N, \mathbb{I}_2 X, X, N) = g^h(\mathbb{I}_2 X, N)^2.$$

From Corollary 5 we obtain that for any $X \in (N, \mathbb{I}_2 N)^\perp$

$$R(\mathbb{I}_2 N, \mathbb{I}_2 X, X, N) = \frac{7}{8}g^h(X, N)^2 - \frac{17}{8}g^h(\mathbb{I}_2 X, N)^2 + \frac{3}{8}(\|X^h\|^2 + \|N^h\|^2\|X\|^2)$$

$$- \frac{7}{8}\|X^h\|^2\|N^h\|^2.$$
and
\[-R(\mathbb{I}_2 N, X, \mathbb{I}_2 X, N) = -\frac{17}{8} g^h(X, N)^2 + \frac{7}{8} g^h(\mathbb{I}_2 X, N)^2 + \frac{3}{8} \left( \|X^h\|^2 + \|N^h\|^2 \right) \|X\|^2 - \frac{7}{8} \|X^h\|^2 \|N^h\|^2.\]

By Lemma 4 when \( X \in E_\Lambda \cap (\mathbb{I}_2 N)^\perp \)
\[g^h(X, N)^2 = g^h(\mathbb{I}_2 X, N)^2.\]

As this must remain true for the eigenvector \( X + \mathbb{I}_2 X \), we infer that
\[g^h(X, N) = g^h(\mathbb{I}_2 X, N) = 0.\]

We then easily prove that the vertical component of the normal vector field must be zero.

**Proposition 7.** Let \( H \) be a hypersurface of \( \mathbb{CP}^3 \) such that \( A\phi = \varphi A \). Then the normal vector field must be horizontal.

**Proof.** If \( N^v \neq 0 \), then \( (N^v, \mathbb{I}_2 N^v) \) is a basis of the vertical distribution. But Equation (4) forces
\[g^v(X, N) = g^v(\mathbb{I}_2 X, N) = 0\]
as \( X \) is an eigenvector orthogonal to \( N \) and \( \mathbb{I}_2 N \), so \( X \) must be horizontal. Since this applies to all eigenvectors of \( A \) in \( (\mathbb{I}_2 N)^\perp \), they must be horizontal and orthogonal to \( N \), hence \( N^h \) must vanish, and we conclude with Proposition 6.

The complementary contingency is resolved using tools from twistor theory.

**Proposition 8.** Let \( H \) be a hypersurface of \( \mathbb{CP}^3 \) such that \( A\phi = \varphi A \). Then \( N \) cannot be horizontal.

**Proof.** If \( N^v = 0 \), then for any horizontal \( X \), we have by O’Neill
\[(AX)^v = (\nabla_X N)^v = \frac{1}{2}(\mathbb{I}^v (X, N)).\]

Let \( p = (x_0, I) \in H \subset \mathbb{CP}^3 = Z(S^4) \), \( x_0 \in S^4 \) and \( I \) a complex structure on \( T_{x_0}S^4 \).

Take a positive orthonormal frame \((e_1, e_2, e_3, e_4)\) of \( T_{x_0}S^4 \) such that at \( p \):
\[e_1 = d\pi(N), e_2 = I e_1, e_3 \in \langle e_1, e_2 \rangle^\perp, e_4 = I e_3.\]

Let \( \mathcal{V}_p \) be the vertical space at \( p \in \mathbb{CP}^3 \), i.e. the tangent space to the fibre. We identify \( \Lambda^2 T_{x_0}S^4 \) with \( \mathfrak{so}(T_{x_0}S^4) \), then there exists a surjection
\[\mathfrak{so}(T_{x_0}S^4) \twoheadrightarrow \mathcal{V}_{(x_0, I)} \quad A \mapsto \hat{A} := \left[ I, A \right] = IA - AI.\]

Denote by \((I^+, J^+, K^+, I^-, J^-, K^-)\) the basis of \( \Lambda^2 T_{x_0}S^4 \) with
\[
\begin{align*}
I^+ &= e_1 \wedge e_2 + e_3 \wedge e_4 \\
J^+ &= e_1 \wedge e_3 - e_2 \wedge e_4 \\
K^+ &= e_1 \wedge e_4 + e_2 \wedge e_3 \\
I^- &= e_1 \wedge e_2 - e_3 \wedge e_4 \\
J^- &= e_1 \wedge e_3 + e_2 \wedge e_4 \\
K^- &= -e_1 \wedge e_4 + e_2 \wedge e_3
\end{align*}
\]
so that $I^+ = I$. From [S], we know that

$$(AX)^v = \frac{1}{2}[X, N]^v = \frac{1}{2}R^{S^4}(X \wedge N).$$

One can easily check that

$$R^{S^4}(e_3 \wedge N) = R^{S^4}(e_3 \wedge e_4) = e_1 \wedge e_3 = \frac{1}{2}(J^++J^-),$$

so $R^{S^4}(e_3 \wedge N) = K^+$.

Identifying $e_2$, $e_3$ and $e_4$ with their horizontal lifts, we have $(Ae_3)^v = -\frac{1}{2}K^+$. Similarly $(Ae_4)^v = \frac{1}{2}J^+$.

Therefore the block matrix of $A$ in the basis $\{e_2, \{e_3, e_4\}, \{J^+, K^+\}\}$ is

$$A = \begin{pmatrix} \mu & 0 & 0 \\ 0 & E & F \\ 0 & F & G \end{pmatrix} \quad \text{with} \quad F = -\frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

while $\varphi$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{pmatrix} \quad \text{with} \quad I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

As, by hypothesis $A\varphi = \varphi A$, a straightforward computation shows this to be impossible. \(\Box\)

Combining Propositions 7 and 8 shows the $\mathbb{CP}^3$ case of the Theorem.

4. THE CASE $(M, Z(M)) = (\mathbb{CP}^2, \mathbb{F}_{1,2})$

The curvature tensor of $(\mathbb{CP}^2, g_{\mathbb{CP}^2}, \mathbb{J}_{\mathbb{CP}^2})$ is

$$R^{\mathbb{CP}^2}(U, V, W, S) = g(U, S)g(V, W) - g(U, W)g(V, S) - g(U, \mathbb{J}_{\mathbb{CP}^2}W)g(V, \mathbb{J}_{\mathbb{CP}^2}S) + g(U, \mathbb{J}_{\mathbb{CP}^2}S)g(V, \mathbb{J}_{\mathbb{CP}^2}W) - 2g(U, \mathbb{J}_{\mathbb{CP}^2}V)g(W, \mathbb{J}_{\mathbb{CP}^2}S).$$

for $U, V, W$ and $S$ in $T_{x_0}\mathbb{CP}^2$ ($x_0 \in \mathbb{CP}^2$). We still denote by $\mathbb{J}_{\mathbb{CP}^2}$ the almost complex structure induced on the horizontal distribution $\mathcal{H}$, hence,

$$\pi^* R^{\mathbb{CP}^2}(\mathbb{I}_2 N, \mathbb{I}_2 X, X, N) = g^h(N, \mathbb{I}_2 X)^2 + 2g^h(\mathbb{J}_{\mathbb{CP}^2}N, X)^2 - g^h(\mathbb{J}_{\mathbb{CP}^2}N, \mathbb{I}_2 X)^2 + g^h(\mathbb{I}_2 N, \mathbb{J}_{\mathbb{CP}^2}N)g^h(\mathbb{I}_2 X, \mathbb{J}_{\mathbb{CP}^2}X),$$

and, as $\text{scal}_{\mathbb{CP}^2} = 24$ and $t = \frac{1}{4}, \ a = \frac{2}{3}, \ b = \frac{1}{3}$ and $c = 4$. From Proposition 3 we obtain

$$R(\mathbb{I}_2 N, \mathbb{I}_2 X, X, N) = \frac{7}{4}g^h(N, X)^2 - \frac{21}{4}g^h(N, \mathbb{I}_2 X)^2 + 2g^h(\mathbb{J}_{\mathbb{CP}^2}N, X)^2 - g^h(\mathbb{J}_{\mathbb{CP}^2}N, \mathbb{I}_2 X)^2 + \frac{3}{4}(||N^h||^2 + ||X^h||^2) - \frac{7}{4}||N^h||^2||X^h||^2 + g^h(\mathbb{I}_2 N, \mathbb{J}_{\mathbb{CP}^2}N)g^h(\mathbb{I}_2 X, \mathbb{J}_{\mathbb{CP}^2}X).$$

We deduce, by Lemma 3.
Lemma 9. If \( X \in E_\lambda \) then

\[
(5) \ 7 \left( g^h(N, X)^2 - g^h(N, \mathbb{I}_2 X)^2 \right) + 3 \left( g^h(\mathbb{J}_{CP^2} N, X)^2 - g^h(\mathbb{J}_{CP^2} N, \mathbb{I}_2 X)^2 \right) = 0.
\]

The next result is key to our argument since it reduces the type of the vector field normal to \( H \) to just two possibilities.

Proposition 10. Let \( H \) be a hypersurface of \( \mathbb{F}_{1,2} \) such that \( A \varphi = \varphi A \). Then the normal vector \( N \) must be either vertical or horizontal.

Proof. The proof of Proposition 10 consists of a series of lemmas.

Assume that \( N \) is neither vertical nor horizontal. We consider a basis of the \( T_pS^4 \) given by

\[
e_1 = \frac{d\pi(N^h)}{|N^h|}, \quad e_2 = Ie_1; \]

\[
e_3 = \begin{cases} \text{unitary part of } \mathbb{J}_{CP^2} e_1 \text{ that is normal to } (e_1, e_2), \text{if non-zero}, \\ \text{any unit vector in } (e_1, e_2)^\perp \text{, otherwise} \end{cases} \]

\[
e_4 = Ie_3.
\]

Recall that since \( CP^2 \) is self-dual, \( \mathbb{F}_{1,2} = Z(\mathbb{CP}^2) \) and \( \mathbb{J}_{CP^2} \in \wedge^2(\mathbb{CP}^2) \), so (using the same notation as on page 8) we can consider \( \tilde{c}, \tilde{s} \in \mathbb{R} \), with \( \tilde{c}^2 + \tilde{s}^2 = 1 \), such that \( \mathbb{J}_{CP^2} = \tilde{c}I^+ + \tilde{s}J^- \), which, in the basis \( (e_1, e_2, e_3, e_4) \), translates as

\[
I = I^+ = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathbb{J}_{CP^2} = \begin{pmatrix} 0 & -\tilde{c} & -\tilde{s} & 0 \\ \tilde{c} & 0 & 0 & -\tilde{s} \\ \tilde{s} & 0 & 0 & \tilde{c} \\ 0 & \tilde{s} & -\tilde{c} & 0 \end{pmatrix}.
\]

We first describe the solutions to Equation (5) in Lemma 9 in the basis we just constructed.

Lemma 11. If \( \tilde{s} \neq 0 \), \( d\pi(E_\lambda \cap (\mathbb{I}_2 N)^\perp) \) is included in

\[
\text{Vect} \left( \begin{pmatrix} 1 \\ 0 \\ 0 \\ \delta_- \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -\delta_- \\ 0 \end{pmatrix} \right) \bigcup \text{Vect} \left( \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\delta_+ \end{pmatrix} \right).
\]

with \( \delta_\pm = \frac{6\tilde{s} \pm \sqrt{\tilde{s}^2 + 1}}{6\tilde{c}}. \)

For \( \tilde{s} = 0 \), \( d\pi(E_\lambda \cap (\mathbb{I}_2 N)^\perp) \) is included in \( \text{Vect}(e_3, e_4) \).

Proof. Assume \( X \in E_\lambda \cap (\mathbb{I}_2 N)^\perp \), with \( X^h = (x, y, z, t) \) its coordinates in the basis \( (e_1, e_2, e_3, e_4) \) (identifying vectors tangent to the base manifold with their
horizontal lifts). Then by Lemma 10 we have

\[ 0 = 7 \left( g^h(X, N)^2 - g^h(X, \mathbb{I}_2 N)^2 \right) + 3 \left( g(X, \mathbb{I}_{CP^2} N)^2 - g(X, \mathbb{I}_{CP^2} \mathbb{I}_2 N)^2 \right) \]

\[ = 7(x^2 - y^2) + 3 \left( (\bar{c}y + \bar{s}z)^2 - (\bar{c}x + \bar{s}t)^2 \right) \]

\[ = (7 - 3\bar{c}^2)(x^2 - y^2) + 3\bar{s}^2(z^2 - t^2) + 6\bar{c}\bar{s}(xt + yz). \]

Re-writing this system with the eigenvector \( X + \mathbb{I}_2 X \), yields

\[ 0 = 7g^h(X, N)g^h(X, \mathbb{I}_2 N) + 3g^h(X, \mathbb{I}_{CP^2} N)g^h(X, \mathbb{I}_{CP^2} \mathbb{I}_2 N) \]

\[ = 7xy + 3(\bar{c}y + \bar{s}z)(\bar{c}x + \bar{s}t) \]

\[ = (7 - 3\bar{c}^2)xy + 3\bar{s}^2zt - 3\bar{c}\bar{s}(xz - yt), \]

so \( X^h = (x, y, z, t) \) must satisfy the system

\[
\begin{cases}
(7 - 3\bar{c}^2)(x^2 - y^2) + 3\bar{s}^2(z^2 - t^2) + 6\bar{c}\bar{s}(xt + yz) = 0, \\
(7 - 3\bar{c}^2)xy + 3\bar{s}^2zt - 3\bar{c}\bar{s}(xz - yt) = 0.
\end{cases}
\]

We work with complex numbers \( z_1 = x + iy \) and \( z_2 = z + it \) to re-write (6) as a polynomial in \( z_2 \):

\[ 3\bar{s}^2z_2^2 - 6\bar{c}\bar{s}z_1z_2 + (7 - 3\bar{c}^2)z_1^2 = 0. \]

If \( \bar{s} \neq 0 \), its roots are \( z_2 = \frac{6\bar{c}\bar{s}z_1 \mp \sqrt{36\bar{c}^2 - 1}}{6\bar{s}z_1} = i\delta \pm z_1. \)

Note that \( \delta_\pm \delta_\pm = -\frac{7 - 3\bar{c}^2}{3\bar{s}^2} \), so neither \( \delta_- \) nor \( \delta_+ \) can vanish.

If \( \bar{s} = 0 \) then the set of solutions is \( \{ z_1 = 0 \} \). \( \square \)

This description forces the number of eigenvalues of \( A|_{\mathbb{I}_2 N}^\perp \).

**Corollary 12.** The shape operator \( A \) of the hypersurface \( H \), restricted to \( (\mathbb{I}_2 N)^\perp \), admits two distinct eigenvalues \( \lambda_1 \) and \( \lambda_2 \).

**Proof.** Lemma 11 implies that the dimension of \( d\pi(E_\lambda \cap (\mathbb{I}_2 N)^\perp) \) must be at most two, and since it is \( \mathbb{I}_2 \)-invariant and \( \mathbb{I}_2 N \) cannot be neither vertical nor horizontal, the dimension of \( E_\lambda \cap (\mathbb{I}_2 N)^\perp \) is exactly two. \( \square \)

Next we prove that the horizontal parts of the eigenspaces are in direct sum.

**Lemma 13.** If \( N^v \neq 0 \), then

\[ d\pi : T_{(x_0, t)} \mathbb{F}_{1, 2} \cap (N, \mathbb{I}_2 N)^\perp \rightarrow T_{x_0} \mathbb{CP}^2 \]

is an isomorphism. In particular, as \( T_{(x_0, t)} \mathbb{F}_{1, 2} \cap (N, \mathbb{I}_2 N)^\perp \) decomposes into a direct sum of eigenspaces of \( A \), we have

\[ d\pi(E_{\lambda_1}) \oplus d\pi(E_{\lambda_2}) = T_{x_0} \mathbb{CP}^2. \]
Proof. As $T\mathbb{F}_{1,2} = \mathcal{H} \oplus \mathcal{V}$, at a point $z = (x_0, I) \in H \subset \mathbb{F}_{1,2}$, write $N = (N^h, N^v)$ and $\mathbb{I}_2 N = (IN^h, -IN^v)$ in their horizontal and vertical components. If $(X^h, X^v) \in T_{(x_0, I)} \mathbb{F}_{1,2} \cap (N, \mathbb{I}_2 N)^\perp$, we have
\[
\begin{cases}
(N^h, X^h) + (N^v, X^v) = 0 \\
(IN^h, X^h) - (IN^v, X^v) = 0,
\end{cases}
\]
and clearly if $N^v \neq 0$ then $d\pi$ is injective. □

Observe that, when $N^v \neq 0$, for reasons of dimensions, the case $\tilde{s} = 0$ is excluded by Lemma 13.

We fully describe $E_\lambda$ by obtaining its vertical part.

Lemma 14. Assume that neither $N^h$ nor $N^v$ vanishes, then in the basis
\[
\left( (N^h, I^+ N^h, J^+ N^h, K^+ N^h), (N^v, IN^v) \right)
\]
of $T_p \mathbb{F}_{1,2}$, then one of the eigenspaces of $A|_{(\mathbb{I}_2 N)^\perp}$ is
\[
E_\lambda = \text{Vect} \left( \begin{pmatrix}
\|N^v\|^2 & 0 \\
0 & \delta_+ \|N^v\|^2
\end{pmatrix}, \begin{pmatrix}
0 & \|N^v\|^2 \\
-\delta_+ \|N^v\|^2 & 0
\end{pmatrix} \right),
\]
while the other corresponds to $\delta_-$. \[\]
Proof. From Lemmas 13 and 11 without loss of generality, we know that
\[
d\pi(E_\lambda) = \text{Vect} \left( \begin{pmatrix}
1 & 0 \\
0 & \delta_+
\end{pmatrix}, \begin{pmatrix}
0 & 1 \\
-\delta_+ & 0
\end{pmatrix} \right).
\]
so in the basis $\left( (N^h, I^+ N^h, J^+ N^h, K^+ N^h), (N^v, IN^v) \right)$, as $E_\lambda \perp \text{Vect}(N, \mathbb{I}_2 N)$, we necessarily have the above description of $E_\lambda$. □

To conclude the proof of Proposition 10, first recall that a nearly Kähler manifold satisfies [11]
\[
\left( \nabla_{\mathbb{I}_2} N \right)^2 = R(X, N, X, N) + R(\mathbb{I}_2 N, \mathbb{I}_2 X, X, N),
\]
and, moreover, in dimension six, we have [12]
\[
\left( \nabla_{\mathbb{I}_2} N \right)^2 = \alpha \|X\|^2,
\]
where $\alpha = 1$ for $\mathbb{F}_{1,2} = Z(\mathbb{C}P^2)$, since $\text{scal}_{\mathbb{F}_{1,2}} = 24$ (cf. [10]).
If \( X \) is a vector field in \( E_\lambda \), we have by Proposition 3

\[
R(X, N, X, N) = R^{CP^2}(d\pi(X), d\pi(N), d\pi(X), d\pi(N))
\]

\[
- \frac{14}{7} g^h(N, X)^2 + \frac{21}{4} g^h(N, I_2 X)^2 - \frac{12}{7} (||N^h||^2 ||X||^2 + ||X^h||^2)
\]

\[
+ \frac{14}{7} ||N^h||^2 ||X^h||^2 + 4 ||N||^2,
\]

with

\[
\pi^* R^{CP^2}(X, N, X, N) = -||X^h||^2 ||N^h||^2 + g^h(N, X)^2 - 3g^h(I_2 CP^2 N, X)^2
\]

hence

\[
R(X, N, X, N) = -\frac{7}{4} g^h(N, X)^2 + \frac{21}{4} g^h(N, I_2 X)^2
\]

\[
- 3g^h(I_2 CP^2 N, X)^2 - \frac{15}{4} (||N^h||^2 ||X||^2 + ||X^h||^2) + \frac{7}{4} ||N^h||^2 ||X^h||^2 + 4 ||N||^2
\]

Second, observe that from page 8 we have that

\[
\text{This concludes the proof of Proposition 10.} \quad \Box
\]

\[
\text{and this is impossible by the observation at the end of the proof of Lemma 11.}
\]

Proposition 15. Let \( H \) be a hypersurface of \( \mathbb{F}_{1,2} \) such that \( A\varphi = \varphi A \). Then the normal vector \( N \) cannot be horizontal.
We identify vectors tangent to $C$ elements of $\bigwedge$ and substituting $e$ one easily obtains Proof.

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and by $[8]$, $[X, N] = R^{\mathbb{C}P^2}(X \wedge N)$. Let $p \in H \subset \mathbb{F}_{1,2}$, $p = (x_0, I)$, $x_0 \in \mathbb{C}P^2$. We identify vectors tangent to $\mathbb{C}P^2$ at $x_0$ with their horizontal lifts in $T_p \mathbb{F}_{1,2}$. Let $(e_1, e_2, e_3, e_4)$ be an orthonormal basis of $T_{x_0} \mathbb{C}P^2$ adapted to our problem, i.e.

$e_1 = d\pi(N), \ I = I^+ et \ J = \frac{\partial}{\partial \mathcal{N}} = \frac{\partial}{\partial \mathcal{N}}$. Where

$$
\begin{cases}
I^+ = e_1 \wedge e_2 + e_3 \wedge e_4 \\
J^+ = e_1 \wedge e_3 - e_2 \wedge e_4 \\
K^+ = e_1 \wedge e_4 + e_2 \wedge e_3
\end{cases}
\quad \text{and} \quad
\begin{cases}
I^- = e_1 \wedge e_2 - e_3 \wedge e_4 \\
J^- = e_1 \wedge e_3 + e_2 \wedge e_4 \\
K^- = -e_1 \wedge e_4 + e_2 \wedge e_3
\end{cases}
$$

As in the case of $\mathbb{S}^4$, there exists a surjection

$$
\mathfrak{so}(T_{x_0} \mathbb{C}P^2) \rightarrow \mathcal{V}(x_0, I)
\quad A \mapsto \hat{A} := [I, A] = IA - AI.
$$

One easily obtains

$$
\begin{align*}
R^{\mathbb{C}P^2}(e_1, e_3) e_1 &= -(1 + 3s^2) e_3 - 3\tilde{c} \tilde{s} e_2; \\
R^{\mathbb{C}P^2}(e_1, e_3) e_2 &= (1 - 3\tilde{s}^2) e_4 - 3\tilde{c} \tilde{s} e_1; \\
R^{\mathbb{C}P^2}(e_1, e_3) e_3 &= 3\tilde{c} \tilde{s} e_4 + (3s^2 + 1) e_1; \\
R^{\mathbb{C}P^2}(e_1, e_3) e_4 &= -3\tilde{c} \tilde{s} e_3 - (1 - 3\tilde{s}^2) e_2,
\end{align*}
$$

so that

$$
R^{\mathbb{C}P^2}(e_1 \wedge e_3) = -3\tilde{c} \tilde{s} I^+ - (1 + 3s^2) e_1 \wedge e_3 + (1 - 3\tilde{s}^2) e_2 \wedge e_4
$$

and substituting $e_1 \wedge e_3 = \frac{1}{2} (J^+ + J^-)$ and $e_2 \wedge e_4 = \frac{1}{2} (J^- - J^+)$, we obtain

$$
\overline{R^{\mathbb{C}P^2}(e_1, e_3)} = \left[ I^+, R^{\mathbb{C}P^2}(e_1, e_3) \right] = -2K^+.
$$

Similarly

$$
R^{\mathbb{C}P^2}(e_1 \wedge e_4) = -K^+.
$$

As elements of $\bigwedge^+$ and $\bigwedge^-$ commute with each other

$$
\mathcal{V}(A e_3) = -\frac{1}{2} R^{\mathbb{C}P^2}(e_1, e_3) = K^+ \\
\mathcal{V}(A e_4) = -\frac{1}{2} R^{\mathbb{C}P^2}(e_1, e_4) = -J^+.
$$
In the basis \( \{e_2, e_3, e_4, \{J^+, K^+\}\} \) the endomorphisms \( A \) and \( \varphi \) have the following block matrices

\[
A = \begin{pmatrix}
\mu & 0 & 0 \\
0 & E & F \\
0 & F & G
\end{pmatrix}
\quad \text{with} \quad F = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}.
\]

and

\[
\varphi = \begin{pmatrix}
0 & 0 & 0 \\
0 & I & 0 \\
0 & 0 & -I
\end{pmatrix}
\quad \text{with} \quad I = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}.
\]

By hypothesis, \( A \) and \( \varphi \) commute, which contradicts the form of block \( F \). \( \square \)

REFERENCES

[1] V. Apostolov, G. Grantcharov, and S. Ivanov. Hermitian structures on twistor spaces. *Annals of Global Analysis and Geometry*, 16(3):291–308, 1998.

[2] M. F. Atiyah, N. J. Hitchin, and I. M. Singer. Self-duality in four-dimensional Riemannian geometry. *Proc. Roy. Soc. London Ser. A*, 362(1711):425–461, 1978.

[3] J. Berndt, J. Bolton, and L. M. Woodward. Almost complex curves and Hopf hypersurfaces in the nearly Kähler 6-sphere. *Geometriae Dedicata*, 56(3):237–247, 1995.

[4] D. E. Blair. Almost contact manifolds with Killing structure tensors. *Pacific J. Math.*, 39:285–292, 1971.

[5] J.-B. Butruille. Classification des variétés approximativement kähleriennes homogènes. *Annals of Global Analysis and Geometry*, 27(3):201–225, 2005.

[6] Th. E. Cecil and P. J. Ryan. *Geometry of Hypersurfaces*. Springer New York, 2015.

[7] J. Davidov and O. Muskarov. On the Riemannian curvature of a twistor space. *Acta Mathematica Hungarica*, 58(3-4):319–332, 1991.

[8] P. de Bartolomeis and A. Nannicini. Introduction to differential geometry of twistor spaces. In *Geometric theory of singular phenomena in partial differential equations (Cortona, 1995)*, Sympos. Math., XXXVIII, pages 91–160. Cambridge Univ. Press, Cambridge, 1998.

[9] J. Eells and S. Salamon. Twistorial construction of harmonic maps of surfaces into four-manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 12(4):589–640, 1985.

[10] Th. Friedrich and H. Kurke. Compact four-dimensional self-dual Einstein manifolds with positive scalar curvature. *Mathematische Nachrichten*, 106(1):271–299, 1982.

[11] A. Gray. Almost complex submanifolds of the six sphere. *Proceedings of the American Mathematical Society*, 20(1):277–277, 1969.

[12] A. Gray. Nearly Kähler manifolds. *Journal of Differential Geometry*, 4(3):283–309, 1970.

[13] A. Gray. The structure of nearly Kähler manifolds. *Mathematische Annalen*, 223(3):233–248, 1976.

[14] N. J. Hitchin. Kählerian twistor spaces. *Proc. London Math. Soc. (3)*, 43(1):133–150, 1981.

[15] Z. Hu, Z. Yao, and Y. Zhang. On some hypersurfaces of the homogeneous nearly Kähler \( S^3 \times S^3 \). *Mathematische Nachrichten*, 291(2-3):343–373, 2017.

[16] J. Kenedy Martins. Congruence of hypersurfaces in \( S^6 \) and in \( \mathbb{CP}^n \). *Boletim da Sociedade Brasileira de Matematica*, 32(1):83–105, 2001.

[17] S. Montiel. Real hypersurfaces of a complex hyperbolic space. *J. Math. Soc. Japan*, 37(3):515–535, 1985.
[18] M. Moruz and L. Vrancken. Properties of the nearly Kähler $S^3 \times S^3$. *Publications de l’Institut Mathématique (Belgrade)*, 103(117):147–158, 2018.

[19] P.-A. Nagy. Nearly Kähler geometry and Riemannian foliations. *Asian Journal of Mathematics*, 6(3):481–504, 2002.

[20] B. O’Neill. The fundamental equations of a submersion. *Michigan Math. J.*, 13:459–469, 1966.

[21] R. Takagi. Real hypersurfaces in a complex projective space with constant principal curvatures. *J. Math. Soc. Japan*, 27:43–53, 1975.

[22] R. Takagi. Real hypersurfaces in a complex projective space with constant principal curvatures. II. *J. Math. Soc. Japan*, 27(4):507–516, 1975.

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