Multi-anyons in the magnetic field

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Abstract

We consider the external magnetic field effects on the two types of anyon with fractional statistical parameters $p/q$ with coprimes $p$ and $q$, one with fractional charge $e/q$ and flux $p\phi_0 (= \hbar c/e)$ (type I), the other with fractional flux $p\phi_0/q$ and fundamental charge $e$ (type II). These two-types of anyons show different behaviors in the presence of the external magnetic field. We also considered the geometry in which a two-dimensional plane contains an island of anyons with different statistical parameter in their equilibrium. The equilibrium inside an island is shown to be periodic with respect to the flux through the island. The period for the type I anyon equals to the integer multiple of the fundamental flux quantum. In the case of type II anyon the period is found to be the fractional multiple of the fundamental flux quantum.

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In two spatial dimension it is possible to interpolate the statistics between the bosonic and fermionic cases [1,2]. F. Wilczek called the particle with this exotic statistics, "anyon" [3]. Anyons play a crucial role in understanding the certain two-dimensional electron systems, most notably the fractional quantum Hall effect [4,5]. Anyons can be considered as bosons carrying a fictitious charges $Q$ and a fictitious flux tube with impenetrable flux $\phi$, mutually interacting via Aharonov-Bohm-type couplings, and such that the fictitious charge and flux are related to the statistical angle $\theta (= \phi/\phi_0^*, \text{where } \phi_0^* = hc/Q)$. The exchange of two anyons gives the phase $\exp[i\theta\pi]$. We will consider the two types of anyons with the same statistical parameter $\theta (= p/q, p$ and $q$ are coprimes), one with the charge $e^*(= e/q)$ and the fictitious flux $p\phi_0$, the other with the charge $e$ and the fractional flux $p\phi_0/q$. Where $\phi_0$ is the fundamental flux $hc/e$.

We will call the former as type I and the latter as type II in the following. The type I anyon is believed to be the elementary excitation in the $\nu (= p'/q)$ fractional quantum Hall liquid [4,6], where $pp' = 1 \mod q$ [7]. In the previous work [8] we considered the persistent currents for the type II anyons of the same statistical angle with the charge $e$ and fractional flux $p\phi_0/q$. This is the quasiparticle around the chiral spin liquids [9]. These two types are indistinguishable when there is no interaction with external electromagnetic (EM) fields. However, we show that, since the external EM-field is minimally coupled to charged particles with a coupling constant proportional to their charge, the two types display different behaviors under the external EM-field.

We first consider the Hamiltonian of $N$ anyons of type I (charge $e^*(> 0)$, flux $p\phi_0$) in an external magnetic field given by

$$H = \frac{1}{2m} \sum_{n=1}^{N} \left[ (p_n - e^* A_n(r_1, \ldots, r_N))^2 - \frac{e^* \hbar}{2mc} \sigma_n B_n(r_1, \ldots, r_N) \right].$$

(1)

where $\sigma_n(\pm)$ represents the spin of $n$th particle and $B_n(r_1, \ldots, r_N)$ is the magnetic field seen by the $n$th particle. The gauge potential $A_n$ corresponding to $B_n$ is assumed to be of the form $A_n(r_1, \ldots, r_N) = A_{ex}(r_n) + \sum_{m(\neq n)} A_{nm}(r_n - r_m)$. The gauge potential $A_{nm}(r_n - r_m)$ which gives the statistical interaction

$$A^i_{nm}(r_n - r_m) = -\theta \frac{\epsilon_{ij}(r^j_n - r^j_m)}{|r_n - r_m|^2}.$$
And the external gauge potential $A_{ex}$ is composed of $A_c + A_s$. Where the $A_c$ corresponds to a uniform magnetic field $B(>0)$ and the $A_s$ corresponds to the singular magnetic flux $\phi_s$ at the origin. They are defined as

$$A_i^s(r_n) = -\phi_s \frac{\epsilon_{ij} r_j^s}{2\pi |r_n|^2}, \quad A_i^c(r_n) = -\frac{\epsilon_{ij}}{4} \frac{\partial}{\partial r_j^s} B |r_n|^2$$

The singular flux at the origin gives rise to the AB geometry. The general arguments about the exact $N$-body ground state of anyonlike objects was discussed in Ref. [10]. There, anyons have been considered as fermions with generic two-body flux-tube interactions. In Ref. [10] they considered the anyonlike particles of charge $-e$ with arbitrary flux. In our case we consider the anyons of fractional charge with the integer multiple of fundamental flux. So the resulting wavefunction becomes slightly different from the ones in Ref. [8]. We shall briefly review some known results of the exact $N$-body ground state for our purpose. Using the singular gauge transformation the multivalued wavefunction $\Psi_s$ is related to the original wave function $\Psi_o$ by the transformation

$$\Psi_s(r_1, \cdots, r_N) = \Omega \Psi_o(r_1, \cdots, r_N),$$

$$\Omega = \prod_n e^{i\alpha_s \Theta_n} \prod_{n<m} e^{i\Theta_{nm}},$$

$$\tan \Theta_n \equiv \frac{y_n}{x_n}, \quad \text{and} \quad \tan \Theta_{nm} \equiv \frac{y_n - y_m}{x_n - x_m},$$

where $\alpha_s = \phi_s/\phi_0^*(= \phi_s/(q\phi_0))$. Since the flux is impenetrable, all the particles cannot be in the same position and moreover cannot be located at the origin. That is to say, the following boundary condition is satisfied

$$\lim_{r_i \to 0} \Psi_s(r_1, \cdots, r_N) = 0, \quad \lim_{r_i \to r_j} \Psi_s(r_1, \cdots, r_N) = 0,$$

for arbitrary $1 < i, j < N$ and $i \neq j$. Hence only the uniform magnetic field contributes to the Zeeman term in the Hamiltonian. And $\Psi_s$ satisfies the eigenequation due to $H_s$ ($k$ is the unit vector perpendicular to the plane $r_{ij} = r_i - r_j$).

$$H_s = \Omega H \Omega^{-1} = \frac{1}{2m} \sum_{n=1}^N \left[ \left( \frac{p_n - e^* B}{2} \mathbf{k} \times \mathbf{r}_n \right)^2 - \frac{e^* \hbar}{2mc} \sigma_n B \right].$$
We note that the total (canonical) angular momentum operator $J_t = -\sum_n \hbar \partial_{\Theta_n}$ commutes with $H_s$ so that the total angular momentum of this system is a good quantum number.

This Hamiltonian can be factored into \[10,11\]

\[H_s = -\sum_{n=1}^{N} \hbar^2 / 2m \left( D_n^x - i\sigma_n D_n^y \right) \left( D_n^x + i\sigma_n D_n^y \right), \tag{5}\]

where $D_n^{x,y} \equiv \partial_{n}^{x,y} - i \hbar c A_{c}^{x,y}(r_n)$, and $A_{c}^{x,y}(r_n)$. Then the ground state is defined by

\[(D_n^x + i\sigma_n D_n^y)\Psi_s(r_1, \cdots, r_N) = 0, \quad n = 1, \cdots, N, \tag{6}\]

since $H_s$ is non-negative definite. With the substitution

\[
\Psi_s(r_1, \cdots, r_N) = f(r_1, \cdots, r_N) \exp \left[ -\frac{\sigma_n \pi B}{2\phi_0} \sum_n |r_n|^2 \right]. \tag{7}\]

Equation (6) becomes

\[
\left[ \frac{\partial}{\partial x_n} + i\sigma_n \frac{\partial}{\partial y_n} \right] f(r_1, \cdots, r_N) = 0, \quad n = 1, \cdots, N,
\]

implying that $f(r_1, \cdots, r_N)$ is an entire function of $z_n \equiv x_n + i\sigma_n y_n$ except the points $r_n = r_m$ for all $n \neq m$ and $r_n = 0$ for all $n$. These points are excluded by the boundary condition (4). For the wavefunction to be physically meaningful, we must take into account the normalizability of $\Psi_s$. In Eq. (7) the normalization of $\Psi_s$ is possible only for $\sigma_n = +1$, which we will assume in the remaining part of this paper. Accounting for the multivaluedness, the most general form of the ground-state wave function becomes the following entire functions,

\[
\Psi_s(z_1, \cdots, z_N) = \prod_i z_i^{\sigma_i} \prod_{i<j}(z_i - z_j)^{\theta} \prod_i z_i^{\nu_i} \prod_{i<j}(z_i - z_j)^{k_{ij}} \exp \left[ -\frac{2\pi B}{4\phi_0} \sum_i |z_i|^2 \right], \tag{8}\]

only in the region where the boundary condition is satisfied, where $S$ is the symmetrizer and $k_{ij}$ is an even integer. The charge of the type I anyon is $e/q$ so that the simple minded gauge invariance gives the multiple period $q\phi_0$. But it seems to be incompatible with the very general argument of Byers and Yang \[12\]. The real constituents are electrons, so the period must always be the fundamental period $\phi_0 = \hbar c / e$, regardless of the nature of the electron-electron interaction. The simple and reasonable interpretation is that the original
wave function must be multivalued in any true AB geometry, following Kivelson and Rocek [13]. The integer $n'_i$ becomes fractional number $n_i/q$ for the multivalued wave function, since it returns to the original state for $q$ turns around the impenetrable flux tube even for $\phi_s = 0$. The total angular momentum of the wave function is $N\alpha_s + \theta N(N - 1)/2 + \sum_i n'_i + \sum_{i<j} k_{ij}$. This total angular momentum returns to the same value for the shift $n'_i \rightarrow n'_i - 1/q$, when the singular AB flux at the origin increased by one flux quantum $\phi_0$, i.e., $\alpha_c \rightarrow \alpha_c + 1/q$. Since the eigenvalues of both the energy and the total angular momentum is periodic with the period of one flux quantum, the system displays the fundamental periodicity with respect to the AB flux, as we expected by the gauge invariance argument.

The terms on the left of $S$ represents the multivaluedness of $\Psi_s$ which comes from the singular gauge transformation Eq. (2). The boundary condition (3) gives the constraint to $\Psi_s$ such that the exponent of $z_i$ and $z_i - z_j$ must be positive. The normalizability of the wave function is automatically satisfied by the $\exp\left[-\frac{2\pi B}{4\phi_0} \sum_i |z_i|^2\right]$ term when the magnetic field exists at the infinity. But in the case the uniform magnetic field is confined in the finite region, this finite region looks like the local solenoid if one sees at infinity. That is, $B|z_i|^2/4 \rightarrow \phi_c \ln |z_i|/2\pi$ as $|z| \rightarrow \infty$, where $\phi_c$ is the total magnetic flux confined in this finite region induced by the uniform magnetic field. Because of the symmetrizer $S$, without loss of generality we can get one term in the symmetric form as the prototype for requiring the normalizability of $\Psi_s$. We choose the term in an increasing order $n'_1 \leq \cdots \leq n'_N$. Requiring that $\Psi_s$ be normalizable as one variable, say $z_i$, gives the condition

\[ n'_N + \alpha_s < \alpha_c - \theta(N - 1) - \sum_{j(\neq i)} k_{ij} - 1, \]  

where $\alpha_c$ is defined by $\phi_c/\phi_0^*$. As two variables $z_i$ and $z_j$ approach the spatial infinity, the condition of the normalizability becomes

\[ n'_{N-1} + n'_N + 2\alpha_s < 2\alpha_c - \theta(N - 1) - \theta(N - 2) - \sum_{j(\neq i)} k_{ij} - \sum_{l(\neq i,j)} k_{jl} - 2. \]  

On the other hand, applying the increasing order to Eq. (9) yields

\[ n'_{N-1} + n'_N + 2\alpha \leq 2(n'_N + \alpha_s) < 2\alpha_c - 2\theta(N - 1) - 2 \sum_{j(\neq i)} k_{ij} - 2. \]  

5
The boundary condition (3) implies $\theta + k_{ij} > 0$. Comparing this with Eq. (11), it is easy to draw a conclusion that for $\theta + k_{ij} > 0$ Eq. (9) alone is sufficient. Combining this with $n'_N + \alpha_s > 0$ from the boundary condition gives the upper bound to the total number $N$ of anyons. For this $N$ to be the greatest integer, $0 < \theta + k_{ij} < 1$ for all $i$ and $j$. Since $\theta$ is the fixed statistical parameter, all the $k_{ij}$’s must have the same value to satisfy $0 < \theta + k_{ij} < 1$. Let us denote this $k_{ij}$ as $k_m$, then the total number $N$ is in the range

$$
0 < N < \frac{\alpha_c - (\alpha_s + n'_N) + \theta + k_m - 1}{\theta + k_m}.
$$

(12)

This equation implies that the zero energy ground state is possible only for $\alpha_c > 1$. Then the maximum number of anyons in the ground state is determined by

$$
N_{max} = \left\lfloor \frac{\alpha_c - (\alpha_s + n'_{N_{max}}) + \theta + k_m - 1}{\theta + k_m} \right\rfloor,
$$

(13)

where $[\alpha]$ means the largest integer less than $\alpha$. From the nature of multivalued wave-functions (2), the statistics is invariant under the parameter change $\theta \rightarrow \theta + \text{integer}$. This periodic behavior agrees with the well-known periodicity of the AB effect. And the statistics is also invariant under the non-zero parameter change $\theta \rightarrow -\theta$ for $\theta = \text{integer}$. It is easily seen that $N_{max}$ is also invariant under the above parameter changes, since we can change the integer $k_m$ so that $\theta + k_m$ does not change. In the region $0 < \theta < 1$, $N_{max}$ becomes the greatest number for $k_m = 0$. With this $N_{max}$ anyons the corresponding wave function for $\alpha_s = 0$ becomes

$$
\prod_{i<j}(z_i - z_j)^{\theta} \exp \left[ -\frac{2\pi B}{4\phi_0} \sum_i |z_i|^2 \right].
$$

(14)

This wave function represents the anyons completely filling the first Landau level. For very large external flux $\alpha_c$, the maximum density in the unit of flux density is $N_{max}/(\phi_c/\phi_0) \approx e^*/(\theta e)$. This agrees with the exact result commented in Ref. [14]. When the external flux $\phi_c$ increases by one flux quantum $\phi_0$, $\alpha_c$ will increase by $\alpha_c \rightarrow \alpha_c + 1/q$. This implies one more quasiparticle to be available in the ground state when $\phi_c$ is increased by $\phi_0$ for $\theta = 1/q$. For general $\theta = p/q$, approximately $1/p$ anyons can be added to the ground state for the increasement of the external flux by one flux quantum.
For the type II anyons, only change in the Hamiltonian is $e^* \rightarrow e$. Hence following the procedure similar to the type I anyons, the ground-state wavefunction becomes

$$\tilde{\Psi}_s(z_1, \cdots, z_N) = \prod_i z_i^{\tilde{\alpha}_s} \prod_{i<j} (z_i - z_j)^\theta S \prod_i z_i^{\tilde{n}_i} \prod_{i<j} (z_i - z_j)^{\tilde{k}_{ij}} \exp \left[ -\frac{2\pi B}{4\phi_0} \sum_i |z_i|^2 \right], \quad (15)$$

where $\tilde{n}_i$ and $\tilde{k}_{ij}$ are integers respectively. This ensures that the original wavefunction $\tilde{\Psi}_0$ is single-valued. And $\tilde{\alpha}_s$ is defined as $\phi_s/\phi_0$. The AB periodicity displays also the fundamental period. From the normalizability of this wavefunction, the maximum number is found to be

$$\tilde{N}_{\text{max}} = \left[ \tilde{\alpha}_c - (\tilde{\alpha}_s + \tilde{n}_{\text{max}}) + \theta + \tilde{k}_m - 1 \right], \quad (16)$$

where $\tilde{\alpha}_c$ is defined as $\phi_c/\phi_0$. When $\phi_c$ is increased by 1, the number of anyons may increase approximately by $1/\theta$ for the type II anyon. For $0 < \theta < 1$ the corresponding wavefunction of the $\tilde{N}_{\text{max}}$ anyons becomes the anyons completely filling the first Landau level,

$$\prod_{i<j} (z_i - z_j)^\theta \exp \left[ -\frac{2\pi B}{4\phi_0} \sum_i |z_i|^2 \right]. \quad (17)$$

compare to Eq. (14), $\phi^*_0$ is replaced by $\phi_0$.

Next we consider that the two kinds of anyons of the same type with different statistical parameters $\theta' (= p'/q')$ and $\theta(= p/q)$ are together in the presence of the external magnetic field. We consider the geometry, in which the two dimensional space contains an island of area $S$. We assume that the $\theta$ anyons reside only inside the island and that the $\theta'$ anyons are outside only. For simplicity, we also assume the area of the island is large enough so that the above argument for the ground state remained valid. And there is no additional impenetrable flux inside the island except the one by the anyon itself. The external uniform magnetic flux through the island is $\phi_c (= SB)$. The ground state wavefunction inside the island with the maximum number of anyons will be (14) and (17) for the type I and type II anyons respectively. Since these wavefunctions are the same as the case of completely filling the first LL, we can consider the situation represented by these wavefunctions as the equilibrium situation. The geometry is similar to what Jain et. al. [15] have considered, where a channel of $\nu' = p'/q'$ FQHE liquid contains an island of the $\nu = p/q$ FQHE liquid.
We will consider the periodicity of the equilibrium with respect to the external flux through the island. When the magnetic flux inside the island is increased by one flux quantum, \( \phi_c \rightarrow \phi_c + \phi_0 \), \( 1/p \) more anyons with \( \theta \) is required to restore the equilibrium for the case of type I. Since we assume that there exist only two kinds of anyons in the system, the charge must be supplied by anyons with \( \theta' \) outside the island. For the type I case we must have

\[
\frac{j}{pq} e^{\frac{1}{p} \phi_0} = j' \frac{e^{\phi_0}}{q'}
\]

to recover the equilibrium. It is easily found the smallest \( j \) is \( pq/s \), where \( s \) is the greatest common factor of \( pq \) and \( q' \). It implies that the equilibrium will be periodic with \( (pq/s)\phi_0 \) period. We note the periodicity found in Jain et. al. can be obtained based on the above argument. Jain et. al. considered the situation when the channel liquid is \( \nu_n \) FQHE liquid and the island liquid is \( \nu_{n-1} \) FQHE liquid, when \( \nu_n(\equiv n/(2n+1)) \) is the filling factor of the principle FQHE liquid. The liquid inside the island is considered in such a way that the quasiholes of the channel \( \nu_n \) FQHE liquid completely fill the lowest LL of their own. That is, the wavefunction of the quasiholes in Ref. [15] is the same as (14). In our consideration this situation may be interpreted as follows. The anyons with \( \theta_n \) are in the equilibrium inside the island. And there are also the anyons with the same statistical parameter outside the island, since the charge is supplied by the quasiholes of \( \nu_n \) FQHE liquid. In this case \( q \) and \( q' \) are the same and equal to be \( 2n+1 \). And \( p \) becomes \( 2n-1 \). Hence this system displays periodicity with the period \( (2n-1)\phi_0 \). It coincide with the result of Jain et. al. deduced from the quasiclassical quantization of the anyon.

For the type II anyons, we need \( q/p \) more anyons with \( \theta \) in the island to recover the equilibrium. In this case the charges are the same for different statistical angles. Hence there is no constraint of the charge conservation. Then the periodicity of the system is solely determined by the change of the maximum available number of anyons inside the island. The system will recover its equilibrium when one additional anyon is available. If \( \phi_c \rightarrow \phi_c + (p/q)\phi_0 \), then \( \tilde{N}_{\text{max}} \) becomes \( \tilde{N}_{\text{max}} + 1 \). Therefore the periodicity of the system becomes \( (p/q)\phi_0 \). Then the system of type II anyons displays a fractional periodicity.
In conclusion, we have considered the exact ground state of two different types of anyons in the presence of external magnetic fields. In case of the local impenetrable external flux, which gives the true AB geometry, AB period is found to be the fundamental period in agreement with the general gauge invariance argument. The maximum number of anyons in the ground state could be determined by the normalizability of the wavefunction. When the total magnetic flux increases by one flux quantum $h\epsilon/e$, one more anyon is available to the ground state for the type I anyon. On the other hand, for the type II anyon the maximum number will increase by $1/\theta$. We have also considered the geometry similar to that of Jain et. al., in which a two-dimensional plane of $\theta' (= p/q)$ anyons contains the island of $\theta (= p/q)$ anyons. The equilibrium inside the island is shown to be periodic with respect to the flux through the island. For the type I anyon the period equals the integer multiple of the fundamental flux, $(pq/s)\phi_0$, where $s$ is the greatest common factor of $pq$ and $q'$. The period for the type II anyon equals the fractional multiple, $(p/q)\phi_0$. We also could reproduce the result of Jain et. al., by considering the normalizability of the exact ground state wavefunction.

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