VINBERG’S $\theta$-GROUPS AND RIGID CONNECTION

TSAO-HSIEN CHEN

Abstract. Let $G$ be a simple complex group of adjoint type. In his unpublished work, Z. Yun associated to each $\theta$-group $(G_0, g_1)$ and a vector $X \in g_1$ a flat $G$-connection $\nabla^X$ on $\mathbb{P}^1 - \{0, \infty\}$, generalizing the construction of Frenkel and Gross in [FG]. In this paper we study the local monodromy of those flat $G$-connections and compute the de Rham cohomology of $\nabla^X$ with values in the adjoint representations of $G$. In particular, we show that in many cases the connection $\nabla^X$ is cohomologically rigid.

1. Introduction

1.1. The goal. Let $G$ be a simple complex algebraic group of adjoint type. Motivated by Langlands correspondence, Frenkel and Gross [FG] constructed a flat $G$-connection $\nabla$ on the trivial $G$-bundle on $\mathbb{P}^1 - \{0, \infty\}$ with following remarkable properties:

1. $\nabla$ has regular singularity at 0, and the residue is a regular nilpotent element in the Lie algebra $g$ of $G$.
2. $\nabla$ has irregular singularity at $\infty$ with slope $1/h$, where $h$ is the Coxeter number of $G$ (see [FG, §5] or [CK, §2.2] for the definition of slope).
3. $\nabla$ is cohomologically rigid, i.e., we have $H^*(\mathbb{P}^1, j_! \nabla^{Ad}) = 0$, here $\nabla^{Ad}$ is the $D$-module defined by the connection $\nabla$ with values in the adjoint representation of $G$ and $j_! \nabla^{Ad}$ is the intermediate extension of the $D$-module $\nabla^{Ad}$ to $\mathbb{P}^1$.

The construction used the $\theta$-group $(G_0, g_1)$ studied by Vinberg and his school, which comes from a $\mathbb{Z}/h\mathbb{Z}$-grading on $g$.

In his unpublished work [Yun], Yun generalized the Frenkel-Gross’s construction to all $\theta$-groups. More precisely, starting with a $\theta$-group $(G_0, g_0)$, he constructed a family of flat $G$-connections $\nabla^X$ on $\mathbb{P}^1 - \{0, \infty\}$ parametrizing by vectors $X \in g_0$. We called $\nabla^X$ the $\theta$-connection associated to $X \in g_1$.

The goal of this paper is to study properties of the $\theta$-connections $\nabla^X$. In more detail, recall that each $\theta$-group $(G_0, g_1)$ corresponds to a torsion automorphism of $g = \text{Lie}G$, which we also denote by $\theta \in \text{Aut}(g)$. Let $\sigma$ be the image of $\theta$ in $\text{Out}(g)$. We establish the following properties of $\nabla^X$, parallel to the properties (1), (2) and (3) above:

1. $\nabla^X$ has regular singularity at 0, and the residue $\text{Res}(\nabla^X)$ is a nilpotent element in the Lie algebra $g^\sigma$. Moreover, for generic $X \in g_1$ the residue $\text{Res}(\nabla^X)$ lies in a single nilpotent orbits in $g^\sigma$.
2. $\nabla^X$ has irregular singularity at $\infty$ with slope $e/m$ for any semi-simple $X \in g_1$. Here $m$ (resp. $e$) is the order of $\theta \in \text{Aut}(g)$ (resp. $\sigma \in \text{Out}(g)$).
(3) Assume \( \theta \) is stable with normalized Kac coordinates \( s_0 = 1 \) (see \([21]\) for the definition of stable automorphism and normalized Kac coordinates). Then for any stable element \( X \in g_1 \) the connection \( \nabla^X \) is cohomologically rigid.

Properties (1) and (2) essentially follows from the construction. The proof of property (3) is a variation of the proof in [FG] which uses Kac’s Theorem (or rather its generalization) on Heisenberg subalgebras of affine Kac-Moody algebras (see Proposition 3.3).

1.2. Relation with \([Yun1]\). In \([Yun1]\), starting with a stable torsion automorphism \( \theta \in \text{Aut}(\hat{g}) \) for the Langlands dual of \( g \) and a stable functional \( \phi \in \hat{g}_1^{*\times} \), the author construct a remarkable \( l \)-adic \( G \)-local system \( \text{KL}_G(\phi) \) on \( \mathbb{P}^1 - \{0, \infty\} \), which generalized his early work in [HNY] with Heinloth and Ngô. This \( l \)-adic local system is tamely ramified at 0 and ramified at \( \infty \). He furthermore describe the monodromy of \( \text{KL}_G(\phi) \) at 0 and conditionally deduce the cohomologically rigid of \( \text{KL}_G(\phi) \) (\([Yun1, \text{Theorem 4.7 and Proposition 5.2}]\)). The construction can carry out over the complex number with \( \text{KL}_G(\phi) \) at 0 and ramified at \( \infty \). He furthermore describe the monodromy of \( \text{KL}_G(\phi) \) at 0 and conditionally deduce the cohomologically rigid of \( \text{KL}_G(\phi) \) (\([Yun1, \text{Theorem 4.7 and Proposition 5.2}]\)). The construction can carry out over the complex number with \( \text{KL}_G(\phi) \) at 0 and ramified at \( \infty \). He furthermore describe the monodromy of \( \text{KL}_G(\phi) \) at 0 and conditionally deduce the cohomologically rigid of \( \text{KL}_G(\phi) \) (\([Yun1, \text{Theorem 4.7 and Proposition 5.2}]\)). The construction can carry out over the complex number with \( \text{KL}_G(\phi) \) at 0 and ramified at \( \infty \). He furthermore describe the monodromy of \( \text{KL}_G(\phi) \) at 0 and conditionally deduce the cohomologically rigid of \( \text{KL}_G(\phi) \) (\([Yun1, \text{Theorem 4.7 and Proposition 5.2}]\)). The construction can carry out over the complex number with \( \text{KL}_G(\phi) \) at 0 and ramified at \( \infty \). He furthermore describe the monodromy of \( \text{KL}_G(\phi) \) at 0 and conditionally deduce the cohomologically rigid of \( \text{KL}_G(\phi) \) (\([Yun1, \text{Theorem 4.7 and Proposition 5.2}]\)). The construction can carry out over the complex number with \( \text{KL}_G(\phi) \) at 0 and ramified at \( \infty \). He furthermore describe the monodromy of \( \text{KL}_G(\phi) \) at 0 and conditionally deduce the cohomologically rigid of \( \text{KL}_G(\phi) \) (\([Yun1, \text{Theorem 4.7 and Proposition 5.2}]\)). The construction can carry out over the complex number with \( \text{KL}_G(\phi) \) at 0 and ramified at \( \infty \). He furthermore describe the monodromy of \( \text{KL}_G(\phi) \) at 0 and conditionally deduce the cohomologically rigid of \( \text{KL}_G(\phi) \) (\([Yun1, \text{Theorem 4.7 and Proposition 5.2}]\)). The construction can carry out over the complex number with \( \text{KL}_G(\phi) \) at 0 and ramified at \( \infty \). He furthermore describe the monodromy of \( \text{KL}_G(\phi) \) at 0 and conditionally deduce the cohomologically rigid of \( \text{KL}_G(\phi) \) (\([Yun1, \text{Theorem 4.7 and Proposition 5.2}]\)). The construction can carry out over the complex number with \( \text{KL}_G(\phi) \) at 0 and ramified at \( \infty \). He furthermore describe the monodromy of \( \text{KL}_G(\phi) \) at 0 and conditionally deduce the cohomologically rigid of \( \text{KL}_G(\phi) \) (\([Yun1, \text{Theorem 4.7 and Proposition 5.2}]\)). The construction can carry out over the complex number with \( \text{KL}_G(\phi) \) at 0 and ramified at \( \infty \). He furthermore describe the monodromy of \( \text{KL}_G(\phi) \) at 0 and conditionally deduce the cohomologically rigid of \( \text{KL}_G(\phi) \) (\([Yun1, \text{Theorem 4.7 and Proposition 5.2}]\)). The construction can carry out over the complex number with \( \text{KL}_G(\phi) \) at 0 and ramified at \( \infty \). He furthermore describe the monodromy of \( \text{KL}_G(\phi) \) at 0 and conditionally deduce the cohomologically rigid of \( \text{KL}_G(\phi) \) (\([Yun1, \text{Theorem 4.7 and Proposition 5.2}]\)).

1.3. The paper is organized as follows. In \([2]\) we give a review of Vinberg’s theory of \( \theta \)-groups following [GLRY]. In \([3]\) we recall the Kac-Moyn-Prasad grading for loop algebras and Kac’s theories on automorphism of loop algebras and Heisenberg subalgebras of affine Kac-Moody algebras. In \([4]\) we recall Yun’s construction of \( \theta \)-connections \( \nabla^X \). We compute the residue of \( \nabla^X \) at 0 and the slope and irregularity at \( \infty \). In \([5]\) we prove the main results of this paper: In Theorem \([5.1]\) assume the \( \theta \)-group \( (G_0, g_1) \) is regular, i.e., \( g_1 \) contains regular semi-simple elements, we compute the de Rham cohomology of the intermediate extension to \( \mathbb{P}^1 \) of the \( D \)-module on \( \mathbb{P}^1 - \{0, \infty\} \) defined by the \( \theta \)-connection \( \nabla^X \) associated to a regular semi-simple \( X \in g_1 \) with values in the adjoint representation of \( G \). In Theorem \([5.2]\) assume the \( \theta \)-group is stable and with normalized Kac coordinates \( s_0 = 1 \), we establish the cohomological rigidity of \( \nabla^X \). In \([6]\) we give several examples of \( \theta \)-connections. Finally, in \([8]\) we discuss the relation with \([Yun1]\).

Acknowledgement. The author is grateful to Z. Yun for allowing him to use his unpublished result and for many helpful discussions. An ongoing project with M. Kamgarpour and X. Zhu is the main motivation to write this paper. The author thanks them cordially for the collaboration. The main part of this work was accomplished when the author was visiting the Max Planck Institute for Mathematics in Bonn. He thank the institution for the wonderful working atmosphere.

2. Gradings on simple Lie algebras

2.1. Let \( g \) be the Lie algebra of a simple complex algebraic group \( G \) of adjoint type. Let \( B \) be a Borel subgroup of \( G \) and let \( T \subset B \) be maximal torus. Let \( X \) (resp. \( \bar{X} \)) be the weight lattices (resp. coweight lattices) of \( T \), and \( R \) (resp. \( \bar{R} \)) be the set of roots (resp. co-roots) of \( T \) in \( G \). We fixed a pinning \((X, R, \bar{X}, \bar{R}, \{E_i\})\), where \( E_i \in g \) is a root vector for the simple roots \( \alpha_i \in \Delta \). The choice of pinning
induced an isomorphism $\text{Aut}(g) = G \rtimes \text{Aut}(R, \Delta)$. Let $\exp: V := \mathbb{X} \otimes \mathbb{R} \to T$ be the exponential map given by $\alpha(\exp(x)) = e^{2 \pi i n(x)}$, for all $\alpha \in X$.

Let $g = \bigoplus_{k \in \mathbb{Z}/m \mathbb{Z}} g_k$ be a grading of $g$. The grading on $g$ corresponds to a torsion automorphism $\theta = \theta' \times \sigma \in \text{Aut}(g)$ such that $\theta(v) = \xi_m v$, where $\xi_m$ is a $m$-th root of unity and $v \in g$. The automorphism $\theta$ is $G$-conjugate to one of the form $t \times \sigma$ with $t \in T^\sigma$. Thus without loss of generality, we can and will assume the automorphism $\theta$ has this form.

Let $\Delta^\sigma_{\text{aff}} = \{\alpha_0, \ldots, \alpha_l\}$ be the set simple affine roots associated to $(g, \sigma)$, here $l_\sigma$ is the number of $\sigma$-orbits on the set of simple roots. We denote by $\Phi^\sigma_{\text{aff}}$ the set of affine roots. We identify affine roots with affine functions on $V^\sigma$. For any $\alpha \in \Phi^\sigma_{\text{aff}}$ we denote by $\bar{\alpha}$ the linear part of $\alpha$. Let $C^\sigma = \{x \in V^\sigma | \alpha_i(x) \geq 0, \ i = 0, \ldots, l_\sigma\}$ be the fundamental alcove.

According to [Kac, Theorem 3.8] (see also [OV, §3]), there exists $x \in C^\sigma$ in the fundamental alcove such that $\theta = \exp(x) \times \sigma$. Since $\theta$ has order $m$, we have

$$\alpha_i(x) = \frac{s_i}{m}.$$

The integers $(s_i)_{i=0,\ldots,l_\sigma}$ are the normalized Kac coordinates of $\theta$. These coordinates satisfy

$$e \sum_{i=0}^{l_\sigma} b_i s_i = m,$$

here $e$ is the order of $\sigma$ and the $b_i$ are the labels of Kac’s twisted affine diagram.

Let $\lambda = mx \in \mathbb{X}^\sigma$. The action of $G_m$ on $g$ via $\lambda$ give a grading $g = \bigoplus_{k \in \mathbb{Z}} g(k)$ and each $g_i$ decomposes as

$$(1) \quad g_i = \bigoplus_{k \in \mathbb{Z}} g_i(k).$$

We have $g_i(k) = 0$ unless $k \equiv i \mod \frac{m}{e}$ and $-m + es_0 \leq k \leq m - es_0$.

Let $G_0$ be the reductive subgroup of $G$ with Lie algebra $g_0$. There are natural actions of $G_0 \times \sigma$ on $g_i$. The pair $(G_0, g_1)$ is called $\theta$-group in the terminology of the Vinberg school.

A grading $g = \bigoplus_{k \in \mathbb{Z}/m \mathbb{Z}} g_i$ (resp. a torsion automorphism $\theta \in \text{Aut}(g)$, resp. a $\theta$-group $(G_0, g_1)$) is called regular, if $g_1$ contains a regular semi-simple element; stable if $g_1$ contains a stable element (recall that an element $v$ is called stable if $G_0$-orbit of $v$ is closed and the stabilizer in $G_0$ is finite). According to [GLRY, §5.3], a vector $v \in g_1$ is stable if only if $v$ is a regular semi-simple elements of $g$ and the action of $\theta$ on the Cartan sub-algebra centralizing $v$ is elliptic, i.e. $Z_{g_0}(v) = 0$. We denote by $g^r_1$ (resp. $g^s_1$) the open set of regular semi-simple (resp. stable) elements.

For future reference, we include a lemma about structure of $g_0$:

**Lemma 2.1.**

1) $g_0$ is the reductive subalgebra with Cartan subalgebra $t^\sigma$ and the system of simple roots $\Delta^\sigma_0 = \{\tilde{\alpha}_i | \alpha_i \in \Delta^\sigma_{\text{aff}}, s_i = 0\}$. 2) If $s_0 \neq 0$, we have $g_0 = g_0 \cap g(0) = g(0)^\sigma$.

**Proof.** Part 1) is proved in [Kac, Proposition 8.6] (see also [OV, §3.11]). Part 2) follows from part 1) together with the fact $g(0)^\sigma$ is the Levi subalgebra of $g^\sigma$ with system of simple roots $\{\tilde{\alpha}_i | \alpha_i \in \Delta^\sigma_{\text{aff}}, i \neq 0, s_i = 0\}$. □
3. Kac-Moy-Prasad grading for loop algebras

3.1. Let $\sigma \in Aut(R, \Delta)$ and $\sigma \tilde{g} = g[t, t^{-1}]^\sigma$ be the twisted loop algebra associated to $(g, \sigma)$, where $\sigma$ acts on $t$ by the formula $\sigma(t) = \xi^e t$. Let $\sigma \tilde{g} = \bigoplus_{\alpha \in \Phi^\sigma \cup \{0\}} \sigma \tilde{g}_\alpha$ be the roots space decomposition of the loop algebra, where $\sigma \tilde{g}_\alpha$ is the affine root spaces corresponding to $\alpha \in \Phi^\sigma \cup \{0\}$ (here by definition $\sigma \tilde{g}_0 = t^\sigma$).

A point $x \in V^\sigma$ defines a $\mathbb{Z}$-grading

$$\sigma \tilde{g} = \bigoplus_{x \in \mathbb{Z}} \sigma \tilde{g}_{x,i}$$

called the Kac-Moy-Prasad grading. Explicitly, we have

$$\sigma \tilde{g}_{x,i} = \bigoplus_{\alpha \in \Phi^\sigma \cup \{0\}, \alpha(x) = \frac{i}{m}} \sigma \tilde{g}_\alpha.$$

3.2. We preserve the setup in §2.1. Let $g = \bigoplus_{i \in \mathbb{Z}/m} g_i$ a grading on $g$ and let $\theta = \exp(x) \times \sigma \in Aut(g)$ be the corresponding automorphism. Let $u = t^\pi$, and consider the following $\mathbb{Z}$-graded Lie algebra

$$g(\theta, m) := \bigoplus_{i \in \mathbb{Z}} u^i g_i \subset g[u, u^{-1}].$$

In his book [Kac], Kac proved the following.

**Theorem 3.1.**

1. Let $\lambda = mx \in \mathbb{R}^\sigma$. The automorphism $Ad(\lambda(u^{-1})) : g[u, u^{-1}] \simeq g[u, u^{-1}]$ induces an isomorphism

$$\Phi : g(\theta, m) \simeq \sigma \tilde{g}.$$

2. Under the isomorphism $\Phi$, the $\mathbb{Z}$-grading of $g(\theta, m)$ becomes to the Kac-Moy-Prasad grading of $\sigma \tilde{g}$.

3. Under the isomorphism $\Phi$, the derivation $u \partial_u$ on $g(\theta, m)$ becomes the derivation $D = \frac{m}{\pi} \partial \tau + \text{ad} \lambda$ on $\sigma \tilde{g}$.

4. The invariant form $\langle u^i x, w^j y \rangle = \delta_{i,-j}(x, y)\text{Kil}^t$ on $g[u, u^{-1}]$ induced an invariant from $\langle \cdot, \cdot \rangle_\theta$ (resp. $\langle \cdot, \cdot \rangle_\sigma$) on $g(\theta, m)$ (resp. $\sigma \tilde{g}$) which is compatible with the grading, i.e., we have $\langle v, w \rangle_\theta = 0$ (resp. $\langle v, w \rangle_\sigma = 0$) for $v \in g(\theta, m)_i$, $w \in g(\theta, m)_j$, $i + j \neq 0$ (resp $v \in \sigma \tilde{g}_{x,i}, w \in \sigma \tilde{g}_{x,j}, i + j \neq 0$).

**Corollary 3.2 ([RY, Kac]).** 1) For each $i = 0, 1, \ldots, m - 1$, there is a canonical isomorphism

$$\sigma \tilde{g}_{x,i} = \bigoplus_k t^{\frac{e(i-k)}{m}} g_i(k) \simeq g_i,$$

where the sum is over $-m + es_0 \leq k \leq m - es_0$, $k \equiv i \text{ mod } \frac{m}{e}$.

2) If $i > 0$, all powers $t^{\frac{e(i-k)}{m}}$ appearing in the above sum are positive, i.e., for $0 < i < m$, $g_i(k) = 0$ unless $-m + es_0 \leq k \leq i$, $k \equiv i \text{ mod } \frac{m}{e}$.

**Proof.** Since $g(\theta, m)_i = u^i g_i$ for $i = 0, 1, \ldots, m - 1$, result in §2.1 and above Theorem implies

$$\sigma \tilde{g}_{x,i} = \Phi(u^i g_i) = Ad(\lambda(u^{-1}))(u^i g_i) = \bigoplus_k t^{\frac{e(i-k)}{m}} g_i(k),$$
here the sum is over \(-m + es_0 \leq k \leq m - es_0, \ k \equiv i \mod \frac{m}{c}\). Part 1) follows. Since \(x\) is in the fundamental alcove \(\tilde{C}^\sigma\), direct calculation shows that, for \(i > 0\), \(\sigma \tilde{g}_{x,i}\) is contained in \(\sigma \tilde{g} \cap g[t]\). Part 2) follows.

By [Kac1, Proposition 3.8]

\[\Phi : g(\theta, m) \simeq \sigma \tilde{g}\]

is compatible with the invariant forms on both side, it is enough to prove the corresponding statement for \(g(\theta, m)\).

Let \(s = \bigoplus_{i \in \mathbb{Z}/m} s_i\) be the centralizer of \(X\) in \(g\) and \(b = \bigoplus_{i \in \mathbb{Z}/m} b_i\) be its orthogonal complement with respect to the Killing form \(\langle , \rangle_{\text{Killing}}\). Consider \(a' = \text{Ker}(\text{ad}(uX))\) and its orthogonal complement \(c'\) in \(g(\theta, m)\) with respect to the invariant form \(\langle , \rangle_{\theta}\). We have

\[a' = \bigoplus_{i \in \mathbb{Z}} a'_i, \quad c' = \bigoplus_{i \in \mathbb{Z}} c'_i,\]

where \(a'_i = u^i s_i, \ c'_i = u^i b_i\). Now part (1) and (2) follows from i) \(b_{i+1} = [X, b_i]\) and ii) \(s_i\) and \(s_j\) are orthogonal (resp. non degenerately paired) with respect to the invariant form on \(\sigma \tilde{g}\) if \(i + j \neq 0\) (resp. \(i + j = 0\)).

For part (3), we need to show that the Kac-Moody central extension does not split over \(a\). For this, it is enough to show that for any \(z \neq 0 \in a_n\) there exists \(z' \in a_{-n}\) such that \(\langle td(z), z'\rangle_{\theta} \neq 0\). Now observe that

\[\langle td(z), z'\rangle_{\theta} = \langle \frac{e}{m} (D - \text{ad}\lambda)(z), z'\rangle_{\theta} = \langle \frac{ne}{m} z, z'\rangle_{\theta} - \langle \frac{e}{m} [\lambda, z], z'\rangle_{\theta},\]

where \(D\) is the derivation of \(\sigma \tilde{g}\) in Theorem 3.1. Since \(a\) is commutative by part (2) we have \(\langle \frac{e}{m} [\lambda, z], z'\rangle_{\theta} = 0\) hence \(\langle td(z), z'\rangle_{\theta} = 0\) and the desired claim follows again from part (2).

\[\square\]

4. Yun’s \(\theta\)-connection

In [Yun], Z. Yun associated to each torsion automorphism \(\theta \in \text{Aut}(g)\) and a nonzero vector \(X \in g_1\), a twisted flat \(G\)-connection \(\nabla^X\) on the trivial \(G\)-bundle on \(\mathbb{G}_m = \mathbb{P}^1 - \{0, \infty\}\), called the \(\theta\)-connection associated to \(X\). In this section recall his construction of \(\nabla^X\).
4.1. Twisted flat $G$-connection and cohomological rigidity. In this subsection we recall the definition of twisted flat $G$-connection and cohomological rigidity (see [Yun2] in the setting of $l$-adic sheaves). Let $C$ be a smooth curve and $\tilde{C} \to C$ be a finite etale Galois cover with Galois group $\Gamma$. Let $\eta : \Gamma \to \text{Aut}(G)$ be a homomorphism. We define a $\eta$-twisted flat $G$-connection on $C$ to be a triple $(\mathcal{F}, \nabla, \delta)$ where $\mathcal{F}$ is $G$-bundle on $\tilde{C}$, $\nabla$ is a flat $G$-connection on it, and $\delta$ is a collection of isomorphisms $\delta_{\gamma} : (\mathcal{F}, \nabla) \simeq \gamma^*(\mathcal{F}, \nabla)$ satisfying the usual cocycle relations with respect to the multiplication on $\Gamma$.

When $\mathcal{F}$ is the trivial $G$-bundle, then a $\eta$-twisted flat connection on it may be described as an operator
\[
\nabla = d + A(t)dt,
\]
where $A(t)dt$ is a $\Gamma$-invariant one form on $\tilde{C}$, with $\Gamma$ acting by deck transformation and by the map $\Gamma \xrightarrow{\eta} \text{Aut}(G) \to \text{Aut}(\mathfrak{g})$ on $\mathfrak{g}$.

Let $(\mathcal{F}, \nabla, \delta)$ be a $\eta$-twisted flat $G$-connection on a smooth curve $C$. Then the corresponding flat vector bundle $\nabla^{Ad}$ on $\tilde{C}$ associated to the adjoint representation descends to $C$ by $\Gamma$-equivariance. Let $\nabla^{Ad}$ be the flat vector bundle on $C$ after descent.

**Definition 4.1.** A $\eta$-twisted flat $G$-connection $(\mathcal{F}, \nabla, \delta)$ over an open subset $C$ of a complete smooth curve $\tilde{C}$ is called cohomologically rigid if
\[
H^*(\tilde{C}, j_! \nabla^{Ad}) = 0,
\]
where $j : C \hookrightarrow \tilde{C}$ is the inclusion.

4.2. Construction of $\nabla^X$. We preserve the setup in [2.1]. Let $\theta = \exp(x) \times \sigma \in \text{Aut}(\mathfrak{g}) = G \times \text{Aut}(\mathfrak{r}, \Delta)$ be a torsion automorphism of $\mathfrak{g}$. Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}/m} \mathfrak{g}_i$ be the corresponding grading. Let $X \in \mathfrak{g}_1$ and let us write $X = \sum X_k$, $X_k \in \mathfrak{g}_1(k)$ according to (1). By Corollary [2.1] we have $X_k = 0$ unless $-m + e \sigma_0 \leq k \leq 1$ and $k \equiv 1 \mod \frac{m}{e}$. Define
\[
p_1 = \Phi(uX) = \sum X_k t^{\frac{e(i-k)}{m}} \in \sigma \mathfrak{g},
\]
here $\Phi$ is the isomorphism in Theorem [3.1]. Then the $\theta$-connection associated to $X$ is the following flat $G$-connection on the trivial $G$-bundle on $\mathbb{G}_m = \text{Spec} \mathbb{C}[t, t^{-1}]$
\[
(2) \quad \nabla^X = d + \sum_{-m + e \sigma_0 \leq k \leq 1, \ k \equiv 1 \mod \frac{m}{e}} X_k t^{\frac{e(i-k)}{m}} \frac{dt}{t}.
\]
Note that $\frac{e(i-k)}{m} \in \mathbb{Z}$.

The $\mathfrak{g}$-valued one form $\sum X_k t^{\frac{e(i-k)}{m}} \frac{dt}{t}$ is $\sigma$-invariant, where $\sigma$ acts on $\mathbb{G}_m$ by the formula $t \to \xi_e^{-1} t$ and by the pinned automorphism on $\mathfrak{g}$. Therefore, by the discussion in [4.1] we can regard $\nabla^X$ as a $\sigma$-twisted flat $G$-connection on the trivial $G$-bundle on $\mathbb{G}_m$, where we regard $\sigma$ as a map $\sigma : \mu_e = \langle \xi_e \rangle \to \text{Aut}(G)$ sending $\xi_e \to \sigma$, and $\Gamma = \mu_e$ is the Galois group of the finite etale Galois cover $[e] : \mathbb{G}_m \to \mathbb{G}_m$ given by the $e$-th power map.
4.3. **Residue at 0.** Notice that $\frac{e(1-k)}{m} > 0$ for $k < 1$, thus the connection $\nabla^X$ has regular singularity at 0 with residue $\text{Res}(\nabla^X) = X_1 \in \mathfrak{g}_1(1)^\sigma$. Moreover, since $\mathfrak{g}(1)^\sigma$ consists of nilpotent elements of $\mathfrak{g}^\sigma$, the residue is nilpotent.

There is a dense open subset of $\mathfrak{g}(1)^\sigma$ which lies in a single nilpotent orbit $\langle Y_\theta \rangle$ of $\mathfrak{g}^\sigma$. Thus, for generic $X \in \mathfrak{g}_1$ the residue $\text{Res}(\nabla^X)$ lies in $\langle Y_\theta \rangle$. The assignment $\theta \to \langle Y_\theta \rangle$ gives a well defined map $\theta \to \langle Y_\theta \rangle$.

\{torsion automorphism of $\mathfrak{g}$ whose image in $\text{Aut}(R, \Delta)$ is $\sigma$\} $\to$ \{nilpotent orbits in $\mathfrak{g}^\sigma$\}.

We now assume $\theta$ is stable. Consider the normalized Kac coordinates $\{s_0, s_1, \ldots, s_{\ell_{\sigma}}\}$ of $\theta$. If we omit $s_0$ and double the remaining Kac coordinates we obtain the weighted Dynkin diagram for the nilpotent orbit $\langle Y_\theta \rangle$ of $\mathfrak{g}^\sigma$. Then for $Y$ in $\langle Y_\theta \rangle$, we have $\dim \mathfrak{g}^{\sigma Y} = \dim \mathfrak{g}(0)^\sigma$. The nilpotent class $\langle Y_\theta \rangle$ is distinguished if and only if $\dim \mathfrak{g}(0)^\sigma = \dim \mathfrak{g}(1)^\sigma$.

Recall that stable torsion automorphisms $\theta$ are classified by regular elliptic $W$-conjugacy classes in the coset $W\sigma$ (cf. [GLRY, Corollary 15]). We therefore get a map

$$\begin{aligned}
\{\text{regular elliptic classes in } W\sigma\} &\to \{\text{nilpotent orbits in } \mathfrak{g}^\sigma\}.
\end{aligned}$$

In the case $\sigma = \text{id}$ and the normalized Kac coordinates satisfies $s_0 = 1$, this map is studied in [S] and [GLRY, §7.3] (see §7.3 for more details). The relation between this map and Kazhán-Lusztig map [KL] is discussed in [GLRY, §8.3, Remark 2].

We expect that for any stable vector $X \in \mathfrak{g}_1^\sigma$ we have $X_1 \in \langle Y_\theta \rangle$. In other words, we expect the conjugacy class of the residue $\text{Res}(\nabla^X)$ depends only on $\theta$ and is given by the map (3). We will verify this expectation in some examples in §7.

**Remark 4.2.** Recall that Heisenberg algebras of $\sigma \mathfrak{g}$ are parametrized, up to conjugacy, by $W$-conjugacy classes of the coset $W\sigma$ (see, e.g., [KP] for the case $\sigma = \text{id}$). Given $w \in W\sigma$, let $\hat{\mathfrak{a}}_w$ denote the associated Heisenberg subalgebra of $\hat{\mathfrak{g}}$. One can show that, when $\theta$ is stable torsion automorphism, the algebra $\hat{\mathfrak{a}}$ in Proposition 3.2 is conjugate to the Heisenberg sub-algebra $\hat{\mathfrak{a}}_w$ where $w$ is an element in the regular elliptic conjugacy class of $W\sigma$ corresponding to $\theta$.

4.4. **Slopes and Irregularity at $\infty$.** Let us compute the slopes of $\nabla^X$ at $\infty$. Consider the covering given by $t = a^{-\frac{m}{e}}$. Then the connection $\nabla^X$ becomes

$$d - \frac{m}{e} \sum_k X_k a^{k-1} \frac{da}{a}.$$ 

Taking the gauge transform with $\lambda^{-1}(a)$, then $\text{Ad}(\lambda^{-1}(a))X_i = a^{-k} X_k$, hence the connection becomes

$$d - \frac{m}{e} \lambda \frac{da}{a} + \lambda \frac{da}{a}. $$

When $X$ is semi-simple, using the definition of slopes in [FG, §5] (or [CK, §2.2]), we see that the slopes of the connection at $\infty$ are either 0 or $e/m$.

Recall that any representation $V$ of $G$ gives rise to a flat vector bundle $\nabla^{X,V}$ on $\mathbb{G}_m$. We compute the irregularity of the connection $\text{Irr}_\infty(\nabla^{X,V})$ at infinity when $X$ is semi-simple following [FG, §13]. Since $X$ is semi-simple the leading term of the connection in (4) is diagonalizable in any representation $V$ of $G$. It implies the slopes of the connection $\nabla^{X,V}$ is either 0 or $e/m$, the former occurring at the

---

1 This map and the map in (3) are due to Z. Yun.
zero eigenspaces of $X$ on $V$ and the later occurring at the non-zero eigenspaces. According to [Katz, §1 and §2.3], the irregularity $\text{Irr}_\infty(\nabla^X, V)$ is equal to the sum of the slopes of the connection at $\infty$. This implies

$$\text{Irr}_\infty(\nabla^X, V) = \frac{e}{m}(\dim V - \dim V^X).$$

Assume $\nabla^X, V$ descends to a flat vector bundle $\nabla^X, V$ via the $e$-th power map $[e] : \mathbb{G}_m \to \mathbb{G}_m$. Then we have

$$(5) \quad \text{Irr}_\infty(\nabla^X, V) = \frac{1}{m}(\dim V - \dim V^X).$$

5. MAIN RESULT

We preserve the setup of §4.1. Let $g = \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} g_i$ be a grading of $g$ and let $\theta = \theta' \circ \sigma \in \text{Aut}(g)$ be the corresponding automorphism. Let $X \in g_1$ be a nonzero vector and $\nabla^X$ be the corresponding $\theta$-connection, which is a $\sigma$-twisted flat $G$-connection on $\mathbb{G}_m$.

Consider the adjoint representation $\text{Ad}$ of $G$ on its Lie algebra $g$. The corresponding flat vector bundle $\nabla^{X, \text{Ad}}$ descends via the $e$-th power map $[e] : \mathbb{G}_m \to \mathbb{G}_m$ by $\sigma$-equivariance. Let $\nabla^{X, \text{Ad}}$ be the connection after descent.

Here are the main results of this note:

**Theorem 5.1.** Assume $\theta$ is regular. Then for any regular semi-simple vector $X \in g^r_1$, we have

$$H^0(\mathbb{P}^1, j_* \nabla^{X, \text{Ad}}) = H^2(\mathbb{P}^1, j_* \nabla^{X, \text{Ad}}) = 0$$

and

$$\dim H^1(\mathbb{P}^1, j_* \nabla^{X, \text{Ad}}) = \frac{\# R}{m} - \dim g^{\sigma, X_1}.$$

Here $j : \mathbb{G}_m \hookrightarrow \mathbb{P}^1$ is the canonical embedding and $X_1 \in g_1(1)^\sigma$ is the residue of the connection $\nabla^X$ at $0$ (see §4.3).

**Theorem 5.2.** Assume $\theta$ is stable and its normalized Kac coordinates satisfies $s_0 = 1$. Then for any stable vector $X \in g^s_1$, we have

$$H^i(\mathbb{P}^1, j_* \nabla^{X, \text{Ad}}) = 0$$

for all $i$. I.e., $\nabla^X$ is cohomologically rigid (see Definition 4.1).

Our proof is a variation of the proof in [FG, §7] which uses results of affine Kac-Moody algebras in [3.2].

6. PROOFS

6.1. The key step leading the proofs of Theorem 5.1 and Theorem 5.2 is the computation of the cohomology groups $H^0(D^X_0, \nabla^{X, \text{Ad}})$ and $H^0(D^X_\infty, \nabla^{X, \text{Ad}})$. Here $D^X_0 = \text{Spec} \mathbb{C}((t))$ (resp. $D^X_\infty = \text{Spec} \mathbb{C}((t^{-1}))$) is the formal punctured disc around 0 (resp. $\infty$).

We first recall some notation from [4.1]. Let $\nabla^X = d + p_1 \frac{dt}{t}$ be the $\theta$-connection associated to $X \in g^s_1$. Here $p_1 = \sum X_k \frac{e^{(1-k)}}{m} \in \sigma \mathfrak{g}$. The connection $\nabla^X$ gives a $\mathbb{C}$-linear map

$$\nabla^{X, \text{Ad}} : g[[t, t^{-1}]] \to g[[t, t^{-1}]] \frac{dt}{t}.$$
Let \( f = \sum v_n t^n \in g[[t, t^{-1}]] \) be a solution to \( \nabla^{X, \text{Ad}}(f) = 0 \). The components \( v_n \) satisfy
\[
\begin{align*}
  nv_n + [X_1, v_n] + \sum_{-m+\epsilon_0 \leq i \leq 0, i \equiv 1 \mod \frac{n}{2}} [X_i, v_{ni}] &= 0,
\end{align*}
\]
where \( a_i = n - \frac{e(1-i)}{m} \in \mathbb{Z} \). Notice that \( a_i < n \) for all \( i \).

### 6.1.1. We first compute \( H^0(D_0^\times, \nabla^{X, \text{Ad}}) \).

Recall \( \nabla^{X, \text{Ad}} \) is the descent of \( \nabla^{X, \text{Ad}} \) along the \( e \)-th power map \( [e] : G_m \to G_m \). Thus \( H^0(D_0^\times, \nabla^{X, \text{Ad}}) = H^0(D_0^\times, [e]_* \nabla^{X, \text{Ad}})^\sigma = H^0(D_0^\times, \nabla^{X, \text{Ad}})^\sigma = \text{Ker}(\nabla^{X, \text{Ad}}; g((t))^\sigma \to g(t))^\sigma \frac{dt}{t} \). Let \( f = \sum v_n t^n \in g((t))^\sigma \) be a solution to \( \nabla^{X, \text{Ad}}(f) = 0 \). Then there exists \( b \in \mathbb{Z} \) such that \( v_b \neq 0 \) and \( v_s = 0 \) for \( s < b \). We claim that \( b \geq 0 \). Indeed, if \( b < 0 \), then equation (6) implies
\[
  bv_b + [X_1, v_b] = 0,
\]
which is impossible since the operator \( b \cdot \text{Id} + X_1 \) is invertible (recall \( X_1 \in g(1)^\sigma \) is nilpotent). Thus \( f = \sum v_n t^n \in g[[t]]^\sigma \) and \( v_0 \in g^\sigma \) lies in the kernel of \( \text{ad}(X_1) \). The equation (6) also implies there is an unique solution \( f = \sum v_n t^n \in g[[t]]^\sigma \) for each \( v_0 \in g^\sigma g_{x, n} \). Above discussion shows that
\[
  H^0(D_0^\times, \nabla^{X, \text{Ad}}) = \text{Ker}(\nabla^{X, \text{Ad}}; g((t))^\sigma \to g((t))^\sigma \frac{dt}{t}) = g^\sigma X_1.
\]

### 6.1.2. We show that \( H^0(D_\infty^\times, \nabla^{X, \text{Ad}}) \) is zero. For this, we need some preliminary results about solutions \( f \in g[[t, t^{-1}]]^\sigma \) to \( \nabla^{X, \text{Ad}}(f) = 0 \). Let \( f \) be such a solution. If we write \( f \) in its components for the Kac-Moy-Prasad grading: \( f = \sum y_n \) where \( y_n \in g^\sigma g_{x, n} \), then we have
\[
  \left( \frac{m}{e} t \nabla^{X, \text{Ad}} \right)(f) = \sum_n \left( \frac{m}{e} t \partial_t + \text{ad}\lambda \right)y_n + \frac{m}{e} [p_1, y_n] - [\lambda, y_n] dt = 0.
\]

Recall \( p_1 = \sum X_k t^{\frac{e(1-k)}{m}} \in g^\sigma \).

Notice that the operator \( \frac{m}{e} t \partial_t + \text{ad}\lambda \) is exactly the derivation \( D \) of \( g^\sigma \) in Theorem 3.1 which defines the Kac-Moy-Prasad grading. Thus we have \( \left( \frac{m}{e} t \partial_t + \text{ad}\lambda \right)y_n = ny_n \) and above equation give rise to the identity
\[
  ny_n - [\lambda, y_n] + \frac{m}{e} [p_1, y_n-1] = 0
\]
for all \( n \in \mathbb{Z} \).

We have the following lemma

**Lemma 6.1** ([FG], Lemma 6). Suppose that \( y_n \) satisfying (7) and \( y_n \in a_n \) for some \( n \). Then \( y_m = 0 \) for all \( m \leq n \).

**Proof.** Assume that \( y_n \neq 0 \). Then by part (3) of Corollary 3.3 there exists \( z \in a_{-n} \) such that \( \langle t \partial_t(y_n), z \rangle \neq 0 \). On the other hand, since \( y_n \) satisfies (7), we have
\[
  t \partial_t(y_n) = \frac{m}{e} (D - \text{ad}\lambda)(y_n) = \frac{m}{e} (ny_n - [\lambda, y_n]) = -[p_1, y_n-1] \in c
\]
and it implies \( \langle t \partial_t(y_n), z \rangle = 0 \). We get a contradiction. Hence \( y_n \) must be zero.

Now, the equation (7) shows that if \( y_n = 0 \) then \( y_n-1 \in a_{n-1} \), hence by induction that \( y_m = 0 \) for all \( m \leq n \). \( \square \)
Above lemma implies $H^0(D_{\infty}^\times, \nabla^{X,\text{Ad}}) = 0$. To see this, observe that
\[
H^0(D_{\infty}^\times, \nabla^{X,\text{Ad}}) = \text{Ker}(\nabla^{X,\text{Ad}} : \mathfrak{g}((t^{-1}))^\sigma \rightarrow \mathfrak{g}((t^{-1}))^\sigma dt/t).
\]
Let $f \in \text{Ker}(\nabla^{X,\text{Ad}} : \mathfrak{g}((t^{-1}))^\sigma \rightarrow \mathfrak{g}((t^{-1}))^\sigma)$. Then we have $v_n = 0$ for $n \gg 0$. This implies $y_n = 0$ for $n \gg 0$ (recall that $y_n$ are the components of $f$ for the Kac-Moody-Prasad grading), hence by above lemma $y_n = 0$ for all $n$. So we must have $f = 0$.

6.2. **Proof of Theorem 5.1**

According to [FG] §8, we have
\[
H^0(\mathbb{P}^1, j_* \nabla^{X,\text{Ad}}) = H^0(G_m, \nabla^{X,\text{Ad}}),
\]
\[
H^2(\mathbb{P}^1, j_* \nabla^{X,\text{Ad}}) = H^2_c(G_m, \nabla^{X,\text{Ad}}),
\]
and the exact sequence
\[
0 \rightarrow H^0(G_m, \nabla^{X,\text{Ad}}) \rightarrow H^0(D_{\infty}^\times, \nabla^{X,\text{Ad}}) \oplus H^0(D_{0}^\times, \nabla^{X,\text{Ad}}) \rightarrow H^1_c(G_m, \nabla^{X,\text{Ad}}) \rightarrow H^1(\mathbb{P}^1, j_* \nabla^{X,\text{Ad}}) \rightarrow 0.
\]

We first prove $H^0(\mathbb{P}^1, j_* \nabla^{X,\text{Ad}}) = H^2(\mathbb{P}^1, j_* \nabla^{X,\text{Ad}}) = 0$. Since $H^0(D_{\infty}^\times, \nabla^{X,\text{Ad}}) = 0$ by the result in §6.1.2, $\nabla^{X,\text{Ad}}$ admits no global sections, i.e., $H^0(\mathbb{P}^1, j_* \nabla^{X,\text{Ad}}) = H^0(G_m, \nabla^{X,\text{Ad}}) = 0$. Dually, $H^2(\mathbb{P}^1, j_* \nabla^{X,\text{Ad}}) = H^2_c(G_m, \nabla^{X,\text{Ad}}) = H^2(G_m, \nabla^{X,\text{Ad}})^* = 0$. Here we used the fact adjoint representation $\text{Ad}$ is self-dual, hence $(\nabla^{X,\text{Ad}})^* \simeq \nabla^{X,\text{Ad}}$.

Now we prove $\dim H^1(\mathbb{P}^1, j_* \nabla^{X,\text{Ad}}) = \# R/m - \dim \mathfrak{g}^{\sigma,X_1}$. Results from §6.1.1 §6.1.2 and above exact sequence imply
\[
0 \rightarrow \mathfrak{g}^{\sigma,X_1} \rightarrow H^1_c(G_m, \nabla^{X,\text{Ad}}) \rightarrow H^1(\mathbb{P}^1, j_* \nabla^{X,\text{Ad}}) \rightarrow 0.
\]

Thus it suffices to prove that $\dim H^1_c(G_m, \nabla^{X,\text{Ad}}) = \# R/m$.

Recall the Deligne’s formula in [D §62.11] for the Euler characteristic
\[
\chi_c(G_m, \nabla^{X,\text{Ad}}) := \sum_i (-1)^i \dim H^i_c(G_m, \nabla^{X,\text{Ad}}) = \chi_c(G_m) \text{rank} (\nabla^{X,\text{Ad}}) - \sum_{\alpha = 0, \infty} \text{Irr}_\alpha (\nabla^{X,\text{Ad}}).
\]

Since $\chi_c(G_m) = 0$ and $\nabla^{X,\text{Ad}}$ is regular at 0, it implies
\[
\chi_c(G_m, \nabla^{X,\text{Ad}}) = -\text{Irr}_\infty (\nabla^{X,\text{Ad}}).
\]

Using the vanishing of $H^0_c$, $H^2_c$ and the formula in line (5), we get
\[
\dim H^1_c(G_m, \nabla^{X,\text{Ad}}) = \text{Irr}_\infty (\nabla^{X,\text{Ad}}) = \frac{1}{m} (\dim \mathfrak{g} - \dim \mathfrak{g}^X).
\]

Since $X$ is regular semi-simple, we have $\frac{1}{m} (\dim \mathfrak{g} - \dim \mathfrak{g}^X) = \# R/m$, hence
\[
\dim H^1_c(G_m, \nabla^{X,\text{Ad}}) = \# R/m.
\]

This finished the proof of Theorem 5.1.
6.3. Proof of Theorem 5.2. It is enough to show that $H^1(\mathbb{P}^1, j_* \nabla^{X, \text{Ad}}) = 0$. We begin with the following lemma:

**Lemma 6.2.** For any solution $f = \sum v_n t^n$ of $\nabla^{X, \text{Ad}}(f) = 0$ in $\mathfrak{g}[[t, t^{-1}]]^\sigma$ we have $v_n = 0$ for all $n < 0$.

**Proof.** Write $f = \sum y_n$ in the components for the Kac-Moody-Prasad grading. When $n = 0$, the equation (7) becomes

$$-[\lambda, y_0] + m[p_1, y_{-1}] = 0.$$  

Since $s_0 = 1$ by assumption, Lemma 2.1 implies $y_0 \in \mathfrak{g}_0 = \mathfrak{g}_0 \cap \mathfrak{g}(0) \subset \ker(\text{ad}\lambda)$. Therefore above equation implies $y_{-1} \in \mathfrak{a}_{-1}$ and by Lemma 7.1 we have $y_n = 0$ for $n < 0$, or equivalently $f = \sum_{n \geq 0} y_n$. On the other hand, Corollary 3.2 and the fact $\mathfrak{g}_0 = \mathfrak{g}_0 \cap \mathfrak{g}(0)$ imply $y_n \in \mathfrak{g}[t]$ for $n \geq 0$. The Lemma follows. \hfill \Box

By [FG, §9], we have $H^1_c(\mathbb{G}_m, \nabla^{X, \text{Ad}}) \simeq \text{Ker}(\nabla^{X, \text{Ad}} : \mathfrak{g}[[t, t^{-1}]]^\sigma \to \mathfrak{g}[[t, t^{-1}]]^\sigma \frac{dt}{t})$, which is equal to $\text{Ker}(\nabla^{X, \text{Ad}} : \mathfrak{g}[[t]]^\sigma \to \mathfrak{g}[[t]]^\sigma \frac{dt}{t})$ by above Lemma. The same argument in §6.1.1 shows that $\text{Ker}(\nabla^{X, \text{Ad}} : \mathfrak{g}[[t]]^\sigma \to \mathfrak{g}[[t]]^\sigma \frac{dt}{t}) = \mathfrak{g}^\sigma \cdot X_1$. Therefore the first two terms in the short exact sequence (8) both have dimension $\dim \mathfrak{g}^\sigma \cdot X_1$. This proves the vanishing of $H^1(\mathbb{P}^1, j_* \nabla^{X, \text{Ad}})$, hence finished the proof of Theorem 5.2.

7. Examples

In this section we give several examples of $\theta$-connections. References for this section are [FG, GLRY, RY].

7.1. $S$-distinguished nilpotent case. Recall that a nilpotent element $N$ in $\mathfrak{g}$ is called distinguished if $\mathfrak{g}^N$ consists of nilpotent elements. Let $N \in \mathfrak{g}$ be a distinguished nilpotent element. There is a co-character $\check{\lambda}$ such that $\text{Ad}^\check{\lambda}(t)N = tN$ for all $t \in \mathbb{C}^\times$. This gives a grading $\mathfrak{g} = \bigoplus_{k=-a}^a \mathfrak{g}(k)$ where $\mathfrak{g}(k) = \{x \in \mathfrak{g}| \text{Ad}^\check{\lambda}(t)x = tkx\}$. Set $m = a + 1$ and consider the inner automorphism $\theta_N := \check{\lambda}(\xi_m) \in \text{Aut}(\mathfrak{g})$, here $\xi_m$ is a $m$-th primitive root of unity in $\mathbb{C}$. We have $\mathfrak{g}_0 = \mathfrak{g}(0)$ and $\mathfrak{g}_1 = \mathfrak{g}(1) \oplus \mathfrak{g}(-a)$. Following [RLYG, §7.3], we say that a distinguished nilpotent element $N \in \mathfrak{g}$ is $S$-distinguished if the automorphism $\theta_N$ is stable. According to loc. cit., a nilpotent element $N$ is $S$-distinguished if and only if there exits $E \in \mathfrak{g}(-a)$ such that $N + E \in \mathfrak{g}_1$ is stable. Moreover, assume $\mathfrak{g}$ is of exceptional type, the map $N \to \theta_N$ defines a bijection between the set of $S$-distinguished nilpotent orbits in $\mathfrak{g}$ to the set of stable inner automorphism on $\mathfrak{g}$ with $s_0 = 1$.

Let $\theta_N$ be the stable automorphism of $\mathfrak{g}$ corresponding to a $S$-distinguished nilpotent element $N \in \mathfrak{g}$. Let $X = N + E \in \mathfrak{g}_1 = \mathfrak{g}(1) \oplus \mathfrak{g}(-a)$ be a stable vector. Then by the formula in (2), the corresponding $\theta$-connection $\nabla^X$ takes the form

$$\nabla^X = d + \frac{N}{t} dt + Edt.$$  

Note that $N$ is the residue of $\nabla^X$ at zero.

Let us show that $\nabla^X$ is cohomologically rigid, i.e., $\dim H^*(\mathbb{P}^1, j_* \nabla^{X, \text{Ad}}) = 0$ By Theorem 5.1 we have $H^0 = H^2 = 0$. Thus it remains to show $H^1(\mathbb{P}^1, j_* \nabla^{X, \text{Ad}}) = \#R - \dim \mathfrak{g}^N = 0$. To see this recall that $N$ is distinguished, thus we have $\dim \mathfrak{g}^N = \dim \mathfrak{g}(0) = \dim \mathfrak{g}_0$. On the other hand, we have $\dim \mathfrak{g}_0 = \frac{\#R}{m}$ ([P, Theorem 4.2]). Result follows.
7.1.1. Frenkel-Gross case. We preserve the setup in [2.1] Consider the regular nilpotent element \( N = \sum_{i=1}^{l} E_i \) in \( \frak{g} \). By [Kos, Corollary 6.4], it is \( S \)-distinguished. We can take the co-character to be \( \lambda = \hat{\rho} \) and \( \theta_N = \hat{\rho}(\xi_h) \in \text{Aut}(\frak{g}) \), where \( \xi_h \) is a \( h \)-th primitive root of unity. We have \( g_0 = t \) and \( g_1 = g(1) \oplus g(-h+1) \), \( g(1) = \bigoplus_{i=1}^{l} g_{\alpha_i} \), \( g(-h+1) = g_{-\beta} \). Here \( \beta \) is the highest root. Choosing a generator \( E_0 \) for \( g_{-\beta} \) and identifying \( g_1 \) with \( \bigoplus_{i=0}^{l} \mathbb{C} E_i \), the open subset \( g_1^\circ \) of stable vectors can be identified with \( g_1^\circ = \{ \sum c_i E_i | c_i \neq 0 \text{ for } i = 0, ..., l \} \). For any \( X = \sum c_i E_i \in g_1^\circ \), the corresponding \( \theta \)-connection takes the form
\[
\nabla^X = d + \sum_{i=1}^{l} \frac{c_i}{t} dt + c_0 E_0 dt.
\]
This is the rigid connections constructed in [FG]. Note that the residue of \( \nabla^X \) at 0 is \( N' = \sum_{i=1}^{l} c_i X_i \), which is regular nilpotent.

7.1.2. Type \( G_2 \). Let \( \frak{g} \) is the simple Lie algebra of type \( G_2 \). Let \( \alpha_1, \alpha_2 \) be the simple root of \( \frak{g} \), where \( \alpha_2 \) is the short root. Consider the automorphism \( \theta = \lambda(\xi_3) \), where \( \lambda = \omega_1 \) is the fundamental co-weight dual to \( \alpha_1 \) and \( \xi_3 \) is a 3-th primitive root of unity. According to [RLYG], \( \theta \) is a stable inner automorphism of order 3 with normalized Kac coordinates
\[
1 1 \Rightarrow 0.
\]
Observe that if we omit \( s_0 \) and double remaining the Kac coordinates we obtain
\[
2 \Rightarrow 0,
\]
which is the weighted Dynkin diagram for the nilpotent orbit \( G_2(2) \). This implies \( \theta \) is equal to \( \theta_N \) in \([2.1]\) for some \( N \in G_2(2) \).

We have \( G_0 = \text{GL}_2(\mathbb{C}) \) and \( \frak{g}_1 = \frak{g}(1) \oplus \frak{g}(-2) \), \( \frak{g}(1) = \bigoplus_{k=0}^{3} \frak{g}_{\alpha_1+\kappa \alpha_2} \), \( \frak{g}(-2) = \frak{g}_{-2\delta_1-3\delta_2} \). As a representation of \( G_0 = \text{GL}_2(\mathbb{C}) \), we have
\[
(9) \quad \frak{g}(1) \simeq \text{det}^2 \otimes P_3, \quad \frak{g}(-2) \simeq \text{det}^{-1} \otimes P_0,
\]
where \( P_d \) is the space of homogeneous polynomials of degree \( d \) on \( \mathbb{C}^2 \), with the natural action of \( G_0 = \text{GL}_2(\mathbb{C}) \). Choosing coordinates, we regard a vector \( X \in \frak{g}_1 \) as a pair \( (f, z) \), where \( f = f(x, y) \) is a binary cubic polynomials over \( \mathbb{C} \) and \( a \in \mathbb{C} \). According to [RY, §7.5], we have \( (f, z) \in \frak{g}_1^\circ \) if only if \( z \neq 0 \) and \( f \) has three distinct roots in the projective line. For any \( X \in \frak{g}_1^\circ \), let us write \( X = X_1 + X_{-2} \) according to the decomposition \( \frak{g}_1 = \frak{g}(1) \oplus \frak{g}(-2) \). The corresponding \( \theta \)-connection takes the form
\[
\nabla^X = d + \frac{X_1}{t} dt + X_{-2} dt.
\]
We claim the residue \( X_1 \) is in the subregular nilpotent orbit \( G_2(2) \). In particular, the conjugacy classes of the residue \( \text{Res} (\nabla^X) \) is independent of the choice \( X \in \frak{g}_1^\circ \). To prove the claim, observe that the intersection of \( G_2(2) \) with \( \frak{g}(1) \) is open dense. Thus to show that \( X_1 \) is in \( G_2(2) \) it is enough to show that \( \dim \text{Ad} G_0(X_1) = \dim \frak{g}(1) = 4 \). But it follows from the fact that the centralizer \( Z_{G_0}(X_1) \) of \( X_1 \) in \( G_0 \) is the symmetric group \( S_3 \) (permuting the roots of \( f \), where \( f \) is the binary cubic polynomial corresponding to \( X_1 \) under the isomorphism \([9]\)), hence \( \dim \text{Ad} G_0(X_1) = \dim G_0 = 4 \).
7.2. **Type** $A_{2n}^{(2)}$. Let $\mathfrak{g} = \mathfrak{sl}_{2n+1}(\mathbb{C})$ ($n \geq 1$). Let $\sigma$ be a pinned automorphism of $\mathfrak{g}$. We define $\theta = \hat{\rho}(-1) \rtimes \sigma \in \text{Aut}(\mathfrak{g})$. According to [GLRY], it is a stable involution with Kac coordinates

$$1 \Rightarrow 0 \cdots 0 \Rightarrow 0.$$ 

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the corresponding grading. We give a description of $\mathfrak{g}_0$ and $\mathfrak{g}_1$. Let $V$ be a vector space over $\mathbb{C}$ of dimension $2n+1$ with basis $\{x_{-n}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n\}$. We define an inner product $(,)$ on $V$ by the formula $(\sum a_i x_i, \sum b_i x_i) = \sum_{i=1}^{n} a_i b_{-i}$. For any $X \in \mathfrak{gl}(V)$, let $X^*$ be the adjoint of $X$ with respect to this inner product. Then under the canonical isomorphism $\mathfrak{sl}(V) \simeq \mathfrak{g}$, we have $\theta(X) = -X^*$ for any $X \in \mathfrak{g}$. Thus $\mathfrak{g}_0 \simeq \mathfrak{so}(V) = \{X \in \mathfrak{sl}(V) | X = -X^*\}$, $\mathfrak{g}_1 = \{X \in \mathfrak{sl}(V) | X = X^*\}$. Moreover, we have $\mathfrak{g}_1^* = \mathfrak{g}_1 \cap \mathfrak{g}^{rs}$, here $\mathfrak{g}^{rs}$ is the open subset of regular semi-simple elements in $\mathfrak{g}$.

Since $m = e = 2$ (recall $m$ and $e$ are the order of $\theta$ and $\sigma$), Corollary 3.2 implies $\mathfrak{g}_1 = \mathfrak{g}_1(0)$. Thus for any $X \in \mathfrak{g}_1$, the $\theta$-connection $\nabla^X$ has the form

$$\nabla^X = d + Xdt.$$

In particular, it is unramified at zero.

Finally, since $\mathfrak{g}^\sigma$ is a simple lie algebra of type $B_n$, we have $\dim \mathfrak{g}^\sigma = n(2n + 1)$. Thus for $X \in \mathfrak{g}_1^*$ we have

$$\dim H^1(P^1, j^*_* \nabla^X, \text{Ad}) = \frac{# R}{2} - \dim \mathfrak{g}^\sigma = \frac{2n(2n + 1)}{2} - n(2n + 1) = 0.$$

8. **Relation with [Yun1] and ramified geometric Langlands**

Let $G$ be a split simply-connected almost simple group over a finite field $k$. In [Yun1], starting with a stable torsion automorphism $\theta = \theta \rtimes \sigma \in \text{Aut}(\mathfrak{g})$ and a stable linear function $X \in \mathfrak{g}_1^{*s}$, the author construct an $\sigma$-twisted l-adic $G$-local system on $\mathbb{G}_m$. One can carry out above construction over the complex number with l-adic sheaves replaced by $D$-modules. Starting with a stable torsion automorphism $\theta$ of $\mathfrak{g}$ over $\mathbb{C}$ and a stable linear function $\phi \in \mathfrak{g}_1^{*s}$, we get a $\sigma$-twisted flat $G$-connection $KL_G(\phi)$ on $\mathbb{G}_m$. Fixing $\sigma \in \text{Aut}(R, \Delta)$, the stable torsion automorphism $\theta$ is completely determined by a number $m$, so that $\theta$ is $G$-conjugate to $\hat{\rho}(\xi_m) \rtimes \sigma$. Consider the stable automorphism $\theta$ for the dual Lie algebra $\tilde{\mathfrak{g}}$ determined by $\sigma$ and $m$.

**Conjecture 8.1** (Z. Yun). There is a bijection between the set of stable linear functions $\phi \in \mathfrak{g}_1^{*s}$ and the set of stable vectors $X \in \tilde{\mathfrak{g}}_1^s$, such that whenever $\phi$ corresponds to $X$ under this bijection, there is a natural isomorphism between $\sigma$-twisted flat $\tilde{G}$-connection on $\mathbb{G}_m$

$$KL_{\tilde{G}}(\phi) \simeq \nabla^X.$$

When $m = h$ is the Coxeter number, i.e., in the Frenkel-Gross case (see §7A.1), above conjecture was proved in [Zhu] using a ramified version of Beilinson-Drinfeld’s work on quantization of Hitchin’s integrable systems. With X. Zhu and M. Kamgarpour [CK1, CKZ], we plan to extend the methods in [Zhu] to more general stable automorphisms.
References

[CK] T.H. Chen, M. Kamgarpour.: Preservation of depths in local geometric Langlands, arXiv:1404.0598
[CK1] T.H. Chen, M. Kamgarpour.: Vinberg’s θ-groups and ramified geometric Langlands correspondence on \P^1, in preparation.
[CKZ] T.H. Chen, M. Kamgarpour, X. Zhu.: Note on ramified Hitchin fibration, in preparation.
[D] P. Deligne.: Equations differentielles a points singuliers reguliers, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin, 1970.
[FG] E. Frenkel, D. Gross.: Rigid irregular connection on the projective line, Ann. of Math. (2) 170 (2009), no. 3, 1469-1512.
[GLRY] B. Gross, P. Levey, M. Reeker, J.K. Yu.: Gradings of positive rank on simple Lie algebras, Transformation Groups, 17, No. 4, (2012), 1123-1190.
[HNY] J.Heinloth, B-C. Ngô, Z.Yun.: Kloosterman sheaves for reductive groups, Ann. of Math. (2) 177 (2013), no. 1, 241-310.
[KL] D. Kazhdan, G. Lusztig.: Fixed point varieties on affine flag manifolds, Israel Journal of Mathematics 1988, Volume 62, Issue 2, 29-168
[Kac] V. Kac.: Infinite dimensional Lie algebras, 3rd Edition, Cambridge University Press, 1990.
[Kac1] V. Kac.: Infinite-dimensional algebras, Dedekind’s η-function, classical Möbius function and the very strange formula, Adv. Math. 30 (1978) 85-136.
[KP] V. Kac, D. Peterson.: 112 construction of the basic representation of the loop group of Es, Symposium on anomalies, geometry, topology (Chicago, Ill., 1985), 276-298, World Sci. Publishing, Singapore, 1985.
[Katz] N. Katz.: On the calculation of some differential Galois groups, Invent. Math. 87 (1987), no. 1, 13-61.
[OV] A.L. Onishchik, E.B. Vinberg.: Lie Groups and Lie Algebra III, Encyclopaedia of Mathematical Sciences, Vol. 41
[P] P. Panyushev.: On invariant theory of θ-groups, Jour. Algebra, 283 (2005), pp. 655-670.
[RY] M. Reeder, J.-K. Yu.: Epipelagic representations and invariant theory, Journal of the AMS (electronically published on August 5, 2013).
[S] T.A. Springer.: Regular elements of finite reflection groups, Invent. Math. 25 (1974), 159-198.
[Yun] Z. Yun.: Note on θ-groups and connections, unpublished note.
[Yun1] Z. Yun.: Epipelagic representation and rigid local systems, arXiv:1401.7647
[Yun2] Z. Yun.: Rigidity in automorphic representations and local systems, arXiv:1405.3035
[Zhu] X. Zhu.: Frenkel-Gross’s irregular connection and Heinloth- Ngo–Yun’s are the same, arXiv:1210.2680

E-mail address: chenth@math.northwestern.edu