NAVIER–STOKES EQUATIONS ON THE $\beta$-PLANE

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Abstract. We show that, given a sufficiently regular forcing, the solution of the two-dimensional Navier–Stokes equations on the periodic $\beta$-plane (i.e., with the Coriolis force varying as $f_0 + \beta y$) will become nearly zonal: with the vorticity $\omega(x, y, t) = \bar{\omega}(y, t) + \tilde{\omega}(x, y, t)$, one has $|\tilde{\omega}|^2_{H^s} \leq \beta^{-1} M_s(\cdot \cdot \cdot)$ as $t \to \infty$. We use this to show that, for sufficiently large $\beta$, the global attractor of this system reduces to a point.

1. Introduction

The two-dimensional Navier–Stokes equations (2d NSE) have been the subject of many studies and its basic mathematical properties (existence, uniqueness, regularity, etc.) are now well understood; see, e.g., [5, 11] for reviews. As a tool to understand various geophysical flows, it is often desirable to include the effect of planetary rotation, but a constant rotation rate (the so-called $f$-plane approximation) has no effect on the dynamics when periodic boundary conditions are used. To feel the effect of rotation, we need to go to the so-called $\beta$-plane approximation, in which the rotation is given by $f_0 + \beta y$.

Simple physical arguments and numerical studies [7, 15] suggest that a rotation rate that varies as $\beta y$ tends to force the solution to become more zonal (a zonal flow is one that does not depend on $x$). In this article, we prove that this is indeed the case, by obtaining a bound $|\tilde{\omega}(t)|^2_{H^s} \leq \beta^{-1} M_s(f; \cdot \cdot \cdot)$, valid for large time $t$, on the non-zonal part $\tilde{\omega}$ of the flow in terms of the forcing $f$.

With the further assumption that the forcing is independent of time, it has been shown that the Navier–Stokes equations possess a global attractor $\mathcal{A}$ of finite Hausdorff dimension. The long-known and nearly optimal bound on this dimension [4] also applies to our rotating case, but it does not take into account the effect of the rotation. Using our bounds on $\tilde{\omega}$, we show that the dimension of $\mathcal{A}$ is zero for $\varepsilon$ sufficiently small, reducing the long-time dynamics to a single steady (and stable) flow determined completely by the forcing. This is to be contrasted with the situation for larger (but still small) $\varepsilon$, where the solution, although nearly zonal, evolves in time even though $\partial_t f = 0$.

Among the works similar in spirit to the present article, we mention [6, 13] where weak convergence to zonal flow is proved for the (more difficult) $\beta$-plane shallow-water equations. A related result for the inviscid Euler equation can be found in [10]. The technique of using rapid oscillations to obtain better bounds have been used in different contexts in, e.g., [2, 9].

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In the rest of this section, we describe the problem and set up the notation. In Section 2, we review basic results on 2d NSE which will be needed later. The heart of this article is Section 3, where $L^2$ and $H^s$ bounds are obtained for the non-zonal component of the flow. An application of these bounds to the dimension of the global attractor follows in Section 4. The proof of an $L^\infty$ Agmon inequality is presented in the Appendix.

In dimensional form, the two-dimensional Navier–Stokes equations read
\begin{equation}
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \beta y \mathbf{v}^\perp + \nabla p = \mu \Delta \mathbf{v} + f_v
\end{equation}
where the constant rotation $f_0$ has been dropped since it has no effect (i.e. in 2d NSE, there is no difference between equatorial and mid-latitude $\beta$-planes). Here $\mathbf{v} = (u, v)$ is the velocity with $\mathbf{v}^\perp := (-v, u)$ and $p$ is the pressure obtained by enforcing the incompressibility constraint $\nabla \cdot \mathbf{v} = 0$. In what follows, we will work with the dimensionless form
\begin{equation}
\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{Y}{\varepsilon} \mathbf{v}^\perp + \nabla p = \mu \Delta \mathbf{v} + f_v,
\end{equation}
\begin{equation}
\nabla \cdot \mathbf{v} = 0.
\end{equation}

We work with $x = (x, y) \in \mathcal{M} := [0, L_1] \times [-L_2/2, L_2/2]$, with periodic boundary conditions in both directions. Note that we have replaced $\beta y$ in (1.1) by $Y(y)/\varepsilon$, where $Y(-L_2/2) = L_2/2$ and $Y(y) = y$ for $y \in (-L_2/2, L_2/2)$. Furthermore, we assume the following symmetry on the velocity
\begin{equation}
u(x, -y, t) = u(x, y, t) \quad \text{and} \quad v(x, -y, t) = -v(x, y, t).
\end{equation}
It is readily verified that if the initial data $\mathbf{v}(0)$ and the forcing $f_v(t)$ also satisfy this symmetry, which we henceforth assume, it persists for all $t \geq 0$. Note also that periodicity and (1.3) imply that
\begin{equation}v(x, -L_2/2, t) = v(x, L_2/2, t) = 0.
\end{equation}
With no loss of generality, we require that the integral over $\mathcal{M}$ of $\mathbf{v}$ vanishes.

In two dimensions, it is convenient to work with the vorticity $\omega := \nabla^\perp \cdot \mathbf{v} = \partial_x v - \partial_y u$, whose evolution equation is
\begin{equation}\partial_t \omega + \mathbf{v} \cdot \nabla \omega + \frac{Y'}{\varepsilon} \mathbf{v}^\perp = \mu \Delta \omega + f.
\end{equation}
Here $f := \nabla^\perp \cdot f_v$ and we can recover the velocity using $\mathbf{v} = \nabla^\perp \Delta^{-1} \omega$. By our assumption on $\mathbf{v}$, the integral of $\omega$ over $\mathcal{M}$ is zero; similarly, $\Delta^{-1}$ is defined uniquely by the zero-integral condition. The symmetry (1.3) implies that $\omega(x, -y, t) = -\omega(x, y, t)$ and
\begin{equation}\omega(x, -L_2/2, t) = \omega(x, L_2/2, t) = 0.
\end{equation}
Now $Y'(y) = 1 - L_2 \delta(y - L_2/2)$, where $\delta$ is the Dirac distribution. Using the fact that $v(x, \pm L_2/2, t) = 0$, we replace $vY'$ by $v$ in (1.5) and write
\begin{equation}\partial_t \omega + \mathbf{v} \cdot \nabla \omega + \frac{1}{\varepsilon} \mathbf{v} = \mu \Delta \omega + f.
\end{equation}
This is the form that we will be mostly working with.

It is also convenient to write (1.7) in the usual functional form
\begin{equation}\partial_t \omega + B(\omega, \omega) + \frac{1}{\varepsilon} L \omega + \mu A \omega = f,
\end{equation}
where $B(\omega, \omega^t) := (\nabla^t \Delta^{-1} \omega) \cdot \nabla \omega^t$, $L \omega := \partial_t \Delta^{-1} \omega$ and $A := -\Delta$. The following properties, valid whenever the expressions make sense, are readily verified by integration by parts and the boundary conditions \ref{1.4}–\ref{1.6}

\begin{align}
(B(\omega, \omega^t), \omega^t)_{L^2} &= 0, \\
(L \omega, \omega)_{L^2} &= 0, \\
(A \omega, \omega)_{L^2} &= |\nabla \omega|^2_{L^2}.
\end{align}

### 2. Preliminary Estimates

The estimates derived in this section are standard from the theory of 2d NSE (see, e.g., [12, 5, 8]), with very minor modifications to handle the Coriolis term. We gather them here for later use.

We start by noting that the vanishing of spatial integrals of $v$ and $\omega$ implies the equivalence of the norms $|\omega|_{H^1}$ and $|\nabla^s \omega|_{L^2} := |(-\Delta)^{s/2} \omega|_{L^2}$, which will thus be used interchangeably below. We denote by $c_0$ the constant in Poincaré inequality

\begin{equation}
|\nabla^s \omega|_{L^2} \leq |\nabla^{s+1} \omega|_{L^2}.
\end{equation}

Besides the usual Sobolev and interpolation inequalities for two and one dimensions (for functions depending on $y$ only), we note the one-dimensional Agmon inequality

\begin{equation}
|\bar{w}|_{L^\infty} \leq c |\bar{w}|_{L^2}^{1/2} |\nabla \bar{w}|_{L^2}^{1/2}.
\end{equation}

A version we use for the two-dimensional case is in Appendix A.

The $L^2$ estimate for the velocity is obtained by multiplying \ref{1.7} by $v$ and using Cauchy–Schwarz,

\begin{equation}
\frac{d}{dt} |v|^2_{L^2} + \mu |\nabla v|^2_{L^2} \leq \frac{C}{\mu} |f_v|^2_{L^2}.
\end{equation}

Assuming that $f_v \in L^\infty_t L^2_x$, we thus have $v \in L^\infty_t L^2_x \cap L^2_t H^1_x$. Here and henceforth, $L^p_t L^q_x := L^p((0, \infty); L^q(M))$, and $H^s_x$ and $|\nabla^s \omega|$ below are defined in the usual way. We denote

\begin{equation}
[w] := \sup_{t > 0} |w(t)|_{L^2}.
\end{equation}

Here and elsewhere in this article, $c$ denotes a generic constant depending only on $M$ whose value may not be the same each time it appears, while numbered constants such as $c_1$ have fixed values.

Now let $\varphi \in L^\infty$ be such that $\varphi'(t) = 1$ for $t \in [0, \frac{1}{2}]$ and $\varphi(t) = 1$ for $t > 2$, so $\varphi(t) \simeq \tanh t$. Multiplying \ref{1.7} by $\varphi \omega$ in $L^2$, or equivalently, multiplying \ref{1.2} by $-\varphi \Delta v$ in $L^2$, we obtain

\begin{equation}
\frac{d}{dt} (\varphi |\omega|^2_{L^2}) + \mu \varphi |\nabla \omega|^2_{L^2} \leq \varphi' |\omega|^2 + \frac{C}{\mu} \varphi |f_v|^2_{L^2}.
\end{equation}

Assuming henceforth that $[f_v] < \infty$, we have $\omega \in L^\infty_t L^2_x \cap L^2_t H^1_x$ and, for $t \geq T_0([v(0)]_{L^2}, [f_v]; \mu)$,

\begin{equation}
|\omega(t)|_{L^2} \leq \frac{C}{\mu} [f_v].
\end{equation}

Note that $T_0$ does not depend on $\omega(0)$ and that the requirement $f_v \in L^\infty_t L^2_x$ can be weakened in $t$, but we shall not do so here.
A bound in $H^m$ is obtained as follows. Fix a multi-index $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| := \alpha_1 + \alpha_2 = m$, and multiply (1.7) by $D^{2\alpha} \omega := \partial_x^{2\alpha_1} \partial_y^{2\alpha_2} \omega$ in $L^2$:

$$
\frac{1}{2} \frac{d}{dt} |D^\alpha \omega|^2 + (\mathbf{v} \cdot \nabla \omega, D^{2\alpha} \omega) + \frac{1}{\varepsilon} (\mathbf{v}, D^{2\alpha} \omega) + \mu |\nabla D^\alpha \omega|^2 = (f, D^{2\alpha} \omega)
$$

where here and henceforth $| \cdot |$ and $(\cdot, \cdot)$ denote $L^2$ norm and inner product. The linear term involving $1/\varepsilon$ vanishes, and one then proceeds as usual: Using the fact that $(\mathbf{v} \cdot \nabla D^\alpha \omega, D^\alpha \omega) = 0$, the nonlinear term is bounded as

$$
| (\mathbf{v} \cdot \nabla \omega, D^{2\alpha} \omega) | \leq \sum_{1 \leq |\beta| \leq |\alpha|} | (D^\beta \mathbf{v}) \cdot \nabla D^{\alpha-\beta} \omega, D^\alpha \omega) |
$$

(2.8)

$$
\leq c \sum_{|\beta|} |D^{\beta-1} \omega|_{L^4} |D^{\alpha-\beta+1} \omega|_{L^4} |D^\alpha \omega|_{L^2}
\leq c \sum_{|\beta|} |D^{\beta-1} \omega|_{H^{1/2}} |D^{\alpha-\beta+1} \omega|_{H^{1/2}} |D^\alpha \omega|_{L^2}
\leq c(m) \sum_{|\beta|} |\omega|_{H^{1/2}} |\omega|_{H^{1/2}} |\omega|_{H^m}
$$

where we have used Sobolev inequalities for the second and third line, and where $l := |\beta|$ in the last line. Using the interpolation inequalities

$$
|\omega|_{H^{l/2}} \leq c |\omega|_{L^2}^{(2m-2l+3)/(2m+2)} |\omega|_{H^{l+1}}^{(2l-1)/(2m+2)}
$$

$$
|\omega|_{H^{m-i+3/2}} \leq c |\omega|_{L^2}^{(2l-1)/(2m+2)} |\omega|_{H^{m+i}}^{(2m-2l+3)/(2m+2)}
$$

(2.9)

followed by Cauchy–Schwarz and summing over $\alpha$, we obtain

$$
\frac{d}{dt} |\omega|^2_{H^m} + \frac{3\mu}{2} |\omega|^2_{H^{m+1}} \leq c(m) |\omega|^2_{L^2} |\omega|^2_{H^m} + c'(m) |f|^2_{H^{m-1}}
$$

(2.10)

for $m = 1, 2, \ldots$. Proceeding by Gronwall and induction on (2.10), we have the following uniform bounds independent of the initial data

$$
|\omega(t)|^2_{H^m} \leq \left( \frac{c(m)}{\mu} \| \nabla^{m-1} f \|^2 + 1 \right)^{m+1}
$$

(2.11)

$$
e^{-\nu t} \int_0^{t+\nu} e^{\nu \tau} |\omega(\tau)|^2_{H^{m+1}} \, d\tau \leq \frac{c}{\mu} \left( \frac{c(m)}{\mu} \| \nabla^{m-1} f \|^2 + 1 \right)^{m+1}
$$

valid for all $t \geq T_m(|\mathbf{v}(0)|_{L^2}, \| \nabla^{m-1} f \|; \mu)$. Here $\nu = c_0^2 \mu$ with $c_0$ the constant in Poincaré inequality (2.1). Note that for $T_m$ to depend only on $|\mathbf{v}(0)|_{L^2}$, and for the validity of (2.11) for $\mathbf{v}(0) \notin H^1$, we need to multiply by $\varphi$ as in (2.5), but this was not done explicitly for conciseness.

### 3. Bounds on the Non-zonal Component

Assuming sufficient regularity for $f$, which implies that for $\omega$ for any $t > 0$, we expand them in Fourier series

$$
\omega(x, t) = \sum_k \omega_k(t) e^{ik \cdot x - i\Omega_k t / \varepsilon}
$$

(3.1)

$$
f(x, t) = \sum_k f_k(t) e^{ik \cdot x}
$$

where $k = (k_1, k_2) \in \mathbb{Z}_L := \{(2\pi l_1/L_1, 2\pi l_2/L_2) : (l_1, l_2) \in \mathbb{Z}^2\}$ and $\Omega_k := -k_1/|k|^2$ is $i$ times the eigenvalue of the linear operator $L$ for wavenumber $k$. Since $\omega$ and $f$ have vanishing integrals over $\mathbb{R}$, $\omega_0 = 0$ and $f_0 = 0$. Here and in what follows,
sums over wavenumbers are understood to be taken over $Z_L$. In terms of Fourier components, (1.7) reads

\[
\frac{d\omega_l}{dt} + \sum_{jk} B_{jkl} \omega_j \omega_k e^{i(\Omega_l - \Omega_j - \Omega_k)t/\varepsilon} + \mu |l|^2 \omega_l = f_l e^{i\Omega_l t/\varepsilon}
\]

where the coefficient of the nonlinear term is

\[
B_{jkl} = \langle B(e^{ij \cdot x}, e^{ik \cdot x}), e^{il \cdot x} \rangle = |M| \frac{j \wedge k}{|j|^2} \delta_{j+k-l}
\]

with $j \wedge k := j_1 k_2 - j_2 k_1$. We note that the linear term $\varepsilon^{-1} L \omega$ has been removed from (3.2) by including $\exp(-i\Omega_k t/\varepsilon)$ in (3.1a).

Let us split $\omega$ into a slow part $\bar{\omega}$, for which $\Omega_k = 0$, and the remaining fast part $\tilde{\omega} := \omega - \bar{\omega}$, viz.,

\[
\bar{\omega}(x, t) = \sum_{k_1=0} \omega_k(t) e^{ik \cdot x}
\]

\[
\tilde{\omega}(x, t) = \sum_{k_1 \neq 0} \omega_k(t) e^{ik \cdot x - i\Omega_k t/\varepsilon}.
\]

We note that, also having zero integrals over $M$, $\tilde{\omega}$ and $\bar{\omega}$ are orthogonal in $H^m$ for $m = 0, 1, \cdot \cdot \cdot$. For convenience, we also define

\[
\bar{\omega}_k := \begin{cases} \omega_k & \text{if } k_1 = 0 \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad \tilde{\omega}_k := \begin{cases} \omega_k & \text{if } k_1 \neq 0 \\ 0 & \text{otherwise} \end{cases}.
\]

Our objective in this section is to obtain long-time bounds for $\tilde{\omega}$ that tend to zero as $\varepsilon \to 0$.

### 3.1. Bound in $L^2$

The development in this subsection largely follows that in [13] for the primitive equations, the main difference being the absence of a spectral gap (that is, the eigenvalues of the antisymmetric operator $L$ accumulate at zero in the present case).

We start by multiplying (1.8) by $\tilde{\omega}$ in $L^2$,

\[
(\partial_t \omega, \tilde{\omega}) + (B(\omega, \omega), \tilde{\omega}) + \frac{1}{\varepsilon} (L\omega, \tilde{\omega}) + \mu (A\omega, \tilde{\omega}) = (f, \tilde{\omega}).
\]

Now, using (1.9a) twice and the fact that $B(\bar{\omega}, \tilde{\omega}) = 0$,

\[
B(\omega, \omega), \tilde{\omega}) = (B(\tilde{\omega}, \omega), \tilde{\omega}) + (B(\omega, \tilde{\omega}), \tilde{\omega})
\]

\[
= (B(\tilde{\omega}, \tilde{\omega}), \omega) + (B(\tilde{\omega}, \omega), \tilde{\omega})
\]

\[
= - (B(\tilde{\omega}, \tilde{\omega}), \omega).
\]

Thus (3.6) becomes

\[
\frac{d}{dt} |\omega|^2 + \mu |\nabla \omega|^2 = (B(\tilde{\omega}, \omega), \omega) + (f, \tilde{\omega}).
\]

Dropping the nonlinear term for the moment, the fact that $\tilde{\omega}$ is rapidly varying while $f$ is slow implies that the effective forcing from the rhs becomes weaker for smaller $\varepsilon$. This essentially is the mechanism for the attenuation of the fast part $\tilde{\omega}$; the nonlinear term will be handled in the proof below. Recalling the definition (2.4), we state the result of this subsection.
\textbf{Theorem 1.} Assume that the initial data $\psi(0) \in L^2(M)$ and that the forcing is bounded as $[\nabla^2 f] + [\partial_t f] < \infty$. Then there exist $T_0(\|\psi(0)\|_{L^2}, [\nabla^2 f], [\partial_t f]; \rho)$ and $M_0([\nabla^2 f], [\partial_t f]; \rho)$ such that, for $t \geq T_0$,
\begin{equation}
|\tilde{\omega}(t)|^2_{L^2} \leq \varepsilon M_0,
\end{equation}
\begin{equation}
\mu e^{-\rho(t+t')} \int_t^{t+t'} e^{\rho\tau} |\nabla \tilde{\omega}(\tau)|^2 d\tau \leq \varepsilon M_0.
\end{equation}
\textbf{Proof.} Recalling that $\nu = \mu \rho$, we obtain from (3.11)
\begin{equation}
\frac{d}{dt}(e^{\rho|\tilde{\omega}|^2}) + \mu \rho e^{\rho|\nabla \tilde{\omega}|^2} \leq 2e^{\rho|B(\tilde{\omega}, \tilde{\omega})|} + 2e^{\rho|f, \tilde{\omega}|}.
\end{equation}
We integrate the last term from $0$ to $t$ by parts,
\begin{equation}
\int_0^t e^{\rho\tau}(f, \tilde{\omega}) d\tau = \left|\mathcal{M}\right| \sum_k \int_0^t f_k(\tau) \tilde{\omega}_k(\tau) e^{i\Omega_k \tau/\varepsilon} e^{\rho \Omega_k \tau/\varepsilon} d\tau
\end{equation}
\begin{equation}
= -i\varepsilon |\mathcal{M}| \sum_k \left[ \frac{1}{\Omega_k} \int_0^t \bar{f}_k \tilde{\omega}_k e^{i\Omega_k \tau/\varepsilon} e^{\rho \Omega_k \tau/\varepsilon} d\tau \right]_0^t
\end{equation}
\begin{equation}
+ i\varepsilon |\mathcal{M}| \sum_k \int_0^t \frac{d}{dt} \left[ \bar{f}_k \tilde{\omega}_k e^{i\Omega_k \tau/\varepsilon} \right] e^{i\Omega_k \tau/\varepsilon} d\tau
\end{equation}
where the prime on the sums indicates that the resonant terms (i.e. those with $\Omega_k = 0$) are excluded. Defining the operator $\partial^\ast$ by, for any $w$ for which it makes sense,
\begin{equation}
\partial^\ast w := e^{-tL/\varepsilon} \partial_t (e^{tL/\varepsilon} w)
\end{equation}
\begin{equation}
\Rightarrow \partial^\ast \tilde{\omega} := \partial \tilde{\omega} + \frac{1}{\varepsilon} L \tilde{\omega} = -B(\omega, \omega) - \mu A \tilde{\omega} + \tilde{f},
\end{equation}
and defining the operator $l_{\Omega}$ by
\begin{equation}
l_{\Omega}(x, t) := \sum_k \frac{1}{\Omega_k} f_k(t) e^{ik \cdot x} = i \sum_k \frac{|k|^2}{\Omega_k} f_k(t) e^{ik \cdot x},
\end{equation}
which being the restricted inverse of $L$ is also antisymmetric, we can write
\begin{equation}
\int_0^t e^{\rho\tau}(f, \tilde{\omega}) d\tau = \varepsilon (l_{\Omega}(\tilde{f}, \tilde{\omega}))(t) e^{\rho t} - \varepsilon (l_{\Omega}(\tilde{f}, \tilde{\omega}))(0)
\end{equation}
\begin{equation}
- \varepsilon \int_0^t \left[ \nu(l_{\Omega}(\tilde{f}, \tilde{\omega}) + (\partial_l l_{\Omega}(\tilde{f}, \tilde{\omega}) + (l_{\Omega}(\tilde{f}, \partial^\ast l_{\Omega}(\tilde{\omega})) e^{\rho \tau} d\tau.
\end{equation}
Using (3.13), the endpoint terms can be bounded as
\begin{equation}
|l_{\Omega}(\tilde{f}, \tilde{\omega})| \leq c \varepsilon |\nabla \tilde{\omega}||\nabla \tilde{\omega}|.
\end{equation}
We now bound the terms in the integrand. First,
\begin{equation}
|\partial_l l_{\Omega}(\tilde{f}, \tilde{\omega})| = |\partial_l l_{\Omega}(\tilde{f}, \tilde{\omega})| \leq c |\partial_l \tilde{f}| |\Delta \tilde{\omega}|.
\end{equation}
Next, using (3.12) and noting the fact that $(l_{\Omega}(\tilde{f}, \tilde{\omega}) = 0$, we bound the last term in (3.14) by
\begin{equation}
|l_{\Omega}(\tilde{f}, \mu \Delta \tilde{\omega})| \leq \mu c |\Delta \tilde{f}| |\Delta \tilde{\omega}|;
\end{equation}
and, using Sobolev and interpolation inequalities,
\[
| \langle h_1 \bar{f}, B(\omega, \bar{\omega}) \rangle | \leq c |\nabla \bar{f}|_{L^2} |\nabla B(\omega, \bar{\omega})|_{L^2} \\
\leq c |\nabla \bar{f}|_{L^2} |\omega|_{L^4} |\nabla \omega|_{L^4} + c |\nabla \bar{f}|_{L^2} |\nabla^{-1} \omega|_{L^\infty} |\Delta \omega|_{L^2} \\
\leq c |\nabla \bar{f}|_2 |\nabla \omega|_2 |\Delta \omega|.
\]
(3.18)

Thus the integral in (3.14) is bounded as
\[
\left| \int_0^t \left[ \nu \langle h_1 \bar{f}, \omega \rangle + (\partial_t h_1 \bar{f}, \omega) + \langle h_1 \bar{f}, \partial_\tau \omega \rangle \right] e^{\nu \tau} \, d\tau \right|
\leq c \int_0^t \left\{ |\mu| |\Delta \bar{f}|^2 + |\partial_\tau \bar{f}|^2 + |\nabla \bar{f}|_2 |\nabla \omega|_2 |\Delta \omega| \right\} e^{\nu \tau} \, d\tau.
\]
(3.19)

We now treat the penultimate term in (3.10). First, we write
\[
\langle B(\bar{\omega}, \omega), \omega \rangle = \sum_{jkl} B_{jkl} \bar{\omega}_j \omega_k \omega_l e^{-i(\Omega_j + \Omega_k) t/\varepsilon}
= \frac{1}{2} \sum_{jkl} \left( B_{jkl} + B_{kjl} \right) \bar{\omega}_j \omega_k \omega_l e^{-i(\Omega_j + \Omega_k) t/\varepsilon}
\]
(3.20)

and then note that \( B_{jkl} + B_{kjl} = 0 \) in the resonant case, i.e. when \( \Omega_j + \Omega_k = 0 \) and \( l_1 = 0 \). Furthermore, we have
\[
B_{jkl} + B_{kjl} = \left( \frac{j \wedge k}{|j|^2} + \frac{k \wedge j}{|k|^2} \right) |M| = (j \wedge k) \left( \frac{1}{|j|^2} - \frac{1}{|k|^2} \right) |M|
= j_1 l_2 \left( \frac{1}{|j|^2} - \frac{1}{|k|^2} \right) |M| = -l_2 (\Omega_j + \Omega_k) |M|
\]
(3.21)

whenever \( j + k = l \) and \( l_1 = 0 \). Motivated by (3.20), we introduce the bilinear symmetric operator \( B_\Omega \) by
\[
\langle B_\Omega (\omega^a, \omega^b), \omega \rangle := \frac{1}{2} \sum_{jkl} B_{jkl} + B_{kjl} \bar{\omega}_j \omega_k \omega_l e^{-i(\Omega_j + \Omega_k) t/\varepsilon}
= \frac{|M|}{2} \sum_{jkl} l_2 \bar{\omega}_j \omega_k \omega_l e^{-i(\Omega_j + \Omega_k) t/\varepsilon}
\]
(3.22)

for any \( \omega^a, \omega^b \) and \( \bar{\omega} \), where the prime on the sum again indicates that resonant terms (for which \( \Omega_j + \Omega_k = 0 \)) are omitted. We note that, thanks to (3.21), the resonant terms are also absent in \( B(\bar{\omega}, \omega, \bar{\omega}) \). Integrating by parts, we have
\[
\int_0^t e^{\nu \tau} (B(\bar{\omega}, \omega, \bar{\omega}) \, d\tau = \varepsilon \varepsilon e^{\nu t} (B_\Omega (\bar{\omega}, \omega, \bar{\omega})(t) - \varepsilon (B_\Omega (\bar{\omega}, \omega, \bar{\omega})(0)
+ \varepsilon \int_0^t e^{\nu \tau} \left[ \nu (B_\Omega (\bar{\omega}, \omega, \bar{\omega}) + 2 (B_\Omega (\partial_t \bar{\omega}, \omega, \bar{\omega})
+ (B_\Omega (\bar{\omega}, \omega, \partial_t \omega \rangle) \right) \, d\tau.
\]
(3.23)

For the last term in the integrand, we use the fact that \( \bar{B}(\bar{\omega}, \omega) = \bar{B}(\bar{\omega}, \bar{\omega}) = 0 \) to write
\[
\partial_\tau \omega = \partial_t (\bar{\omega}, \omega) - \mu A \omega + \bar{f}
\]
(3.24)
and estimate, using $H^1 \subset L^\infty$ for $\tilde{f}$ and (A.2) for the $L^\infty$ estimates,
\begin{equation}
\left|\left(B\Omega(\tilde{\omega}, \tilde{\omega}, \partial_{r}\tilde{\omega})\right)\right| \leq c|\tilde{\omega}|_{L^\infty}|\partial_{y}\tilde{\omega}|_{L^2}|\tilde{f}|_{L^\infty} + mc|\tilde{\omega}|_{L^\infty}|\partial_{y}\tilde{\omega}|_{L^1}|\Delta \tilde{\omega}|_{L^2} + c|\tilde{\omega}|_{L^\infty}|\partial_{y}\tilde{\omega}|_{L^2}|\nabla^{-1}\tilde{\omega}|_{L^\infty}|\nabla\tilde{\omega}|_{L^2} \\
\leq c|\tilde{\omega}||\nabla\tilde{\omega}||\tilde{f}| + mc|\tilde{\omega}|^{1/2}|\nabla\tilde{\omega}||\Delta \tilde{\omega}|^{3/2} + c|\nabla\omega|^{3}|\tilde{\omega}| \left(\log \frac{|\Delta \omega|}{c_0|\nabla \omega|} + c'\right).
\end{equation}
\begin{equation}
\text{(25.35)}
\end{equation}
For the term involving $\left(B\Omega(\partial_{x}^{2}\tilde{\omega}, \tilde{\omega}, \tilde{\omega})\right)$, we bound, using $\partial_{x}^{2}\tilde{\omega} + B(\omega, \omega) + mA\tilde{\omega} = \tilde{f}$ and the inequality $|\tilde{\omega}|_{L^\infty} \leq c|\tilde{\omega}|^{1/2}|\tilde{\omega}'|^{1/2}$,
\begin{equation}
|\left(B\Omega(\partial_{x}^{2}\tilde{\omega}, \tilde{\omega}, \tilde{\omega})\right)| \leq c|\partial_{y}\tilde{f}||\tilde{\omega}|_{L^\infty} + c|\tilde{f}||\partial_{y}\tilde{\omega}| \left|\nabla^{-1}\tilde{\omega}\right|_{L^\infty} + c|\nabla^{-1}\omega|_{L^\infty} + c|\nabla\tilde{\omega}|_{L^\infty} \\
\leq c|\nabla \tilde{f}||\tilde{\omega}||\tilde{\omega}'| + c|\tilde{f}||\nabla \tilde{\omega}||\tilde{\omega}'| + mc|\tilde{\omega}||\tilde{\omega}'|^{1/2}|\Delta \tilde{\omega}|^{3/2} + c|\omega|^{2}|\nabla \omega|^{3/2}|\tilde{\omega}'^{1/2} \left(\log \frac{|\Delta \omega|}{c_0|\nabla \omega|} + 1\right)^{1/2}
\end{equation}
\begin{equation}
\text{(26.36)}
\end{equation}
where all unadorned norms are $L^2$. Finally, we bound
\begin{equation}
\left|\left(B\Omega(\tilde{\omega}, \tilde{\omega}, \tilde{\omega})\right)\right| \leq c|\tilde{\omega}| |\partial_{y}\tilde{\omega}| |\tilde{\omega}|_{L^\infty} \\
\leq c|\tilde{\omega}| |\nabla \tilde{\omega}| |\tilde{\omega}'|^{1/2}.
\end{equation}
\begin{equation}
\text{(27.37)}
\end{equation}
Using these also to bound the endpoint terms, the integral in (3.28) is bounded as
\begin{equation}
\begin{aligned}
\int_{0}^{t} e^{\nu \tau} & \left(\left(B\Omega(\tilde{\omega}, \tilde{\omega}, \tilde{\omega})\right)\right) d\tau \\
\leq c e \left[|\tilde{\omega}| |\nabla \tilde{\omega}| |\tilde{\omega}'|^{1/2} |\tilde{\omega}'|^{1/2}\right] (t) e^{\nu t} + c e \left[|\tilde{\omega}| |\nabla \tilde{\omega}| |\tilde{\omega}|^{1/2} |\tilde{\omega}'|^{1/2}\right] (0) \\
+ e \int_{0}^{t} \left\{ c |\nabla \tilde{f}| |\tilde{\omega}| |\nabla \tilde{\omega}| + c |\tilde{f}| |\nabla \tilde{\omega}|^{2} + mc |\omega|^{1/2} |\nabla \omega| |\Delta \omega|^{3/2} + c |\omega| |\nabla \omega|^{5/2} |\Delta \omega|^{1/2} \left(\log \frac{|\Delta \omega|}{c_0|\nabla \omega|} + c'\right)\right\} e^{\nu \tau} d\tau.
\end{aligned}
\end{equation}
\begin{equation}
\text{(3.28)}
\end{equation}
Putting together (3.15), (3.19) and (3.28), we have
\begin{equation}
\begin{aligned}
|\tilde{\omega}(t)|^{2} + \mu & \int_{0}^{t} \left|\nabla \tilde{\omega}\right|^{2} e^{\nu (\tau-t)} d\tau \leq e^{-\nu t} |\tilde{\omega}(0)|^{2} \\
+ e c_{2} (1 + e^{-\nu t}) & \sup_{0 \leq \tau \leq t} \left\{ |\nabla \tilde{f}| |\nabla \tilde{\omega}| + |\omega|^{3/2} |\nabla \omega|^{3/2}\right\} \\
+ e c_{3} (\mu & \int_{0}^{t} \left|\nabla \tilde{f}\right|^{2} + |\partial_{y} \tilde{f}|^{2} + |\nabla \tilde{f}| |\nabla \omega| + |\Delta \omega|^{2} (1 + |\nabla \omega|) \\
+ |\omega| |\nabla \omega|^{5/2} |\Delta \omega|^{1/2} \left(\log \frac{|\Delta \omega|}{c_0|\nabla \omega|} + c'\right)\right\} e^{\nu (\tau-t)} d\tau.
\end{aligned}
\end{equation}
\begin{equation}
\text{(3.29)}
\end{equation}
We now shift the origin of time such that $t = 0$ corresponds to $T_{2}$ in (2.11). The hypothesis that $\left[\nabla^{2} \tilde{f}\right] + \left[\partial_{y} \tilde{f}\right] < \infty$ then implies that both the endpoints and the integral in (3.29) are bounded uniformly for all $t > 0$, independently of the initial data provided that $\nu \in L^{2}$ initially. Rewriting the bound in (3.29) as
\begin{equation}
|\tilde{\omega}(t)|^{2} + \mu \int_{0}^{t} \left|\nabla \tilde{\omega}\right|^{2} e^{\nu (\tau-t)} d\tau \leq e^{-\nu t} |\tilde{\omega}(0)|^{2} + \frac{\epsilon}{2} M_{0} (\left[\nabla^{2} \tilde{f}\right], \left[\partial_{y} \tilde{f}\right]; \mu),
\end{equation}
the proof is complete.
We also note from (3.29) that the hypothesis \( f \in L^\infty_t H^2_x \) and \( \partial_t f \in L^\infty_t L^2_x \) can be weakened to \( f \in L^2_t H^2_x \cap L^\infty_t H^1_x \) and \( \partial_t f \in L^2_t L^2_x \).

3.2. **Bounds in** \( H^s \). With a little extra work, \( H^s \) bounds for \( \tilde{\omega} \) that scales as \( \sqrt{\varepsilon} \) can also be obtained. We do this explicitly for \( |\nabla \tilde{\omega}| \) and sketch the computation for \( s = 2, 3, \cdots \).

For the \( H^1 \) bound, we multiply (3.8) by \( A\tilde{\omega} \) in \( L^2 \) to get

\[
\frac{1}{2} \frac{d}{dt} |\nabla \tilde{\omega}|^2 + \mu |\Delta \tilde{\omega}|^2 + (B(\omega, \omega), A\tilde{\omega}) = (\tilde{f}, A\tilde{\omega}),
\]

which implies [cf. (3.10)]

\[
\frac{d}{dt} (e^{\varepsilon t} |\nabla \tilde{\omega}|^2) + \mu e^{\varepsilon t} |\Delta \tilde{\omega}|^2 \leq 2e^{\varepsilon t} (B(\omega, \omega), \Delta \tilde{\omega}) - 2e^{\varepsilon t} (\tilde{f}, \Delta \tilde{\omega}).
\]

As in the \( L^2 \) case, we integrate from 0 to \( t \),

\[
\begin{align*}
\frac{d}{dt} (e^{\varepsilon t} |\nabla \tilde{\omega}(t)|^2) - |\nabla \tilde{\omega}(0)|^2 + \mu \int_0^t |\Delta \tilde{\omega}|^2 e^{\varepsilon \tau} d\tau
\leq 2 \int_0^t \{(B(\omega, \omega), \Delta \tilde{\omega}) - (\tilde{f}, \Delta \tilde{\omega})\} e^{\varepsilon \tau} d\tau.
\end{align*}
\]

The forcing term gives

\[
\int_0^t e^{\varepsilon \tau} (f, \Delta \tilde{\omega}) d\tau = \varepsilon (I_0 \tilde{f}, \Delta \tilde{\omega})(t)e^{\varepsilon t} - \varepsilon (I_0 \tilde{f}, \Delta \tilde{\omega})(0)
\]

\[
+ \varepsilon \int_0^t [\nu(I_0 \tilde{f}, \Delta \tilde{\omega}) + (\partial_t I_0 \tilde{f}, \Delta \tilde{\omega}) + (I_0 \tilde{f}, \Delta \omega')(\tilde{\omega})] e^{\varepsilon \tau} d\tau,
\]

which can be bounded as in the \( L^2 \) case, giving

\[
-2 \int_0^t (\tilde{f}, \Delta \tilde{\omega}) e^{\varepsilon \tau} d\tau \leq \varepsilon c \{ |\Delta \tilde{\omega}| |\Delta \tilde{\omega}| \} (t) e^{\varepsilon t} + \varepsilon c \{ |\Delta \tilde{\omega}| |\Delta \tilde{\omega}| \} (0)
\]

\[
+ \varepsilon c \int_0^t \{(1 + \mu) |\nabla^3 \omega|^2 + \mu |\nabla \tilde{f}|^2 + |\nabla \tilde{\omega}|^2 + |\nabla \omega'| |\Delta \tilde{\omega}| \} e^{\varepsilon \tau} d\tau.
\]

For the nonlinear term, we use the fact that \( B(\tilde{\omega}, \tilde{\omega}) = 0 \) to write

\[
\begin{align*}
(B(\omega, \omega), A\tilde{\omega}) = (B(\omega, \omega), A\tilde{\omega}) + (B(\tilde{\omega}, \tilde{\omega}), A\tilde{\omega}) + (B(\tilde{\omega}, \tilde{\omega}), A\tilde{\omega}),
\end{align*}
\]

and, using \((B(\omega^2, \omega), A\tilde{\omega}) = (B(\nabla^2 \omega, \tilde{\omega}), \nabla \tilde{\omega}), \) previously used in (2.3), we bound

\[
\begin{align*}
|\langle B(\tilde{\omega}, \tilde{\omega}, A\tilde{\omega}) \rangle| &\leq |\langle B(\tilde{\omega}, \tilde{\omega}, \tilde{\omega}) \rangle| \leq c |\tilde{\omega}|_{L^2} |\nabla \tilde{\omega}|^2_{L^2} \leq c |\tilde{\omega}| |\nabla \tilde{\omega}| |\Delta \tilde{\omega}|
\leq \frac{\mu}{4} |\Delta \tilde{\omega}|^2 + \frac{c}{\mu} |\tilde{\omega}|^2 |\nabla \tilde{\omega}|^2.
\end{align*}
\]

\[
\begin{align*}
|\langle B(\tilde{\omega}, \tilde{\omega}, A\tilde{\omega}) \rangle| &\leq \frac{\mu}{4} |\Delta \tilde{\omega}|^2 + \frac{c}{\mu} |\tilde{\omega}|^2 |\nabla \tilde{\omega}|^2.
\end{align*}
\]
Using Poincaré inequality on the last term in (3.37), we obtain
\begin{equation}
2 \int_0^t (B(\omega, \omega), \Delta \tilde{\omega}) e^{\nu \tau} \, d\tau \\
\leq c_2(\mu) \int_0^t \left\{ |\tilde{\omega}|_{L^\infty} + |\nabla \tilde{\omega}|^2 \right\} e^{\nu \tau} \, d\tau.
\end{equation}
(3.38)

After moving the $|\Delta \tilde{\omega}|^2$ to the left-hand side, a factor of $\varepsilon$ can be obtained by pulling the square bracket outside the integral and using (3.9b). Collecting, we have
\begin{equation}
|\nabla \tilde{\omega}(t)|^2 + \frac{\mu}{2} \int_0^t e^{\nu(t-t)}|\Delta \tilde{\omega}|^2 \, d\tau \leq e^{-\nu t} |\nabla \tilde{\omega}(0)|^2 + c\varepsilon \sup_{t' > 0} |\Delta \tilde{f}(t')||\Delta \omega(t')|
\end{equation}
(3.39)
\begin{equation}
+ \varepsilon c_3(\mu) M_0 \sup_{t' > 0} \left\{ |\tilde{\omega}(t')|_{L^\infty} + |\nabla \omega(t')|^2 \right\}.
\end{equation}

Arguing as in the $L^2$ case, $f \in L^\infty_t H^3_x$ and $\partial_t \tilde{f} \in L^\infty_t H^1_x$ gives us an $O(\sqrt{\varepsilon})$ bound for $\tilde{\omega}(t)$ in $L^\infty_t H^1_x$ uniform for large $t$.

Bounds in $H^s$ can now be obtained inductively. Assuming that Theorem 2 below holds for $s-1$ (we just showed that it does for $s=2$), we multiply (1.3) by $A^s \tilde{\omega}$ and integrate the resulting equation in time as above. We bound the nonlinear term $(B(\omega, \omega), A^s \tilde{\omega}) = (B(\tilde{\omega}, \tilde{\omega}), A^s \tilde{\omega}) + (B(\omega, \tilde{\omega}), A^s \tilde{\omega}) + (B(\tilde{\omega}, \omega), A^s \tilde{\omega})$ as follows. The first term is bounded exactly as in (2.8)–(2.9),
\begin{equation}
|\langle B(\tilde{\omega}, \omega), A^s \tilde{\omega} \rangle| \leq \frac{\mu}{4} |\nabla^{s+1} \tilde{\omega}|^2 + \frac{c(s)}{\mu} |\omega|^2 |\nabla^s \omega|^2.
\end{equation}
(3.40)

We bound the next term by [cf. (2.8)], with $|\alpha| = s$ and $1 \leq |\beta| = r \leq s$,
\begin{equation}
|\langle B(\omega, \omega), A^s \tilde{\omega} \rangle| \leq c \sum_{\alpha, \beta} |D^\beta \omega|_{L^\infty} |D^{s-\beta} \nabla \tilde{\omega}|_{L^2} |D^s \tilde{\omega}|_{L^2}
\end{equation}
(3.41)
\begin{equation}
\leq c(s) \sum_{r=1}^s |\nabla^r \tilde{\omega}| |\nabla^{s-r+1} \tilde{\omega}| |\nabla^s \omega|
\leq c(s) |\nabla^s \tilde{\omega}| |\nabla^s \omega|^2.
\end{equation}

Finally, we bound the last term as, where now $0 \leq |\beta| = r \leq s = |\alpha|$,
\begin{equation}
|\langle B(\tilde{\omega}, \omega), A^s \tilde{\omega} \rangle| \leq c \sum_{\alpha, \beta} |D^\beta \nu|_{L^1} |D^{s-\beta} \nabla \tilde{\omega}|_{L^2} |D^s \tilde{\omega}|_{L^s}
\leq c(s) \sum_{r=0}^s |\tilde{\omega}|_{H^{s-r+1/2}} |\tilde{\omega}|_{H^{s-r+1/2}}
\leq \frac{\mu}{4} |\nabla^{s+1} \omega|^2 + c(s, \mu) \sum_{r=0}^s |\nabla^{s+1} \omega|^2 |\nabla^{s+1} \omega|^2 |\nabla^{s+1} \omega|^2 |\nabla^{s+1} \omega|^2
\leq \frac{\mu}{4} |\nabla^{s+1} \omega|^2 + c(s, \mu) |\nabla^s \omega|^4 |\nabla^{s+1} \omega|^4.
\end{equation}
(3.42)

Moving the $|\nabla^{s+1} \omega|^2$ in (3.40) and (3.42) to the left-hand side of the main inequality, the right-hand side depends at most on $|\nabla^s \omega|^2$, which is of $O(\varepsilon)$ in $L^2_t$ from step $s-1$, and on $|\nabla^{s+1} \omega|^2$. As before, the worst term (i.e. that requires the highest derivative on $f$) in fact comes from bounding $(f, A^s \tilde{\omega})$. 

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We summarise our results as:

**Theorem 2.** Let the initial data \( v(0) \in L^2(M) \) and the forcing be bounded as
\[
K_s(f) := \left[ \nabla^{s+2} f \right] + \left[ \nabla^s \partial_t f \right] < \infty.
\]
Then there exist \( T_s(|v(0)|_{L^2}, K_s; \mu) \) and \( M_s(K_s; \mu) \) such that
\[
|\nabla s \tilde{\omega}(t)|_{L^2}^2 \leq \varepsilon M_s,
\]
(3.44)
\[
\mu e^{-\nu(t+t')} \int_t^{t+t'} e^{\nu \tau} |\nabla s \tilde{\omega}(\tau)|^2 \, d\tau \leq \varepsilon M_s
\]
for all \( t \geq T_s \).

3.3. **Higher-order Bounds.** As in [13], one can obtain bounds that scale as \( \varepsilon^{n/2} \) for \( \tilde{\omega} \) when the force \( f \) is independent of time; see [1].

4. **Stability and the Global Attractor**

When the forcing \( f \) is independent of time, the existence of the global attractor \( A \) follows, just as for the non-rotating 2d Navier–Stokes equations, from the uniform long-time bounds in Section 2, where the planetary rotation does not appear at all. In the non-rotating case, the Hausdorff dimension of \( A \) is bounded by
\[
\dim_H A \leq c G^{2/3}(1 + \log G)^{1/3}
\]
where in our notation the Grashof number is
\[
G := |\nabla^{-1} f|_{L^2}/\mu^2.
\]
The rotation not posing any extra essential difficulty, the usual analysis, e.g. [5 §9.2], carries over essentially line-by-line to our case, giving the bound (4.1) also for the rotating case (1.7).

As discussed in the introduction, and following our results that the flow becomes more zonal ("ordered") as \( \varepsilon \to 0 \), we expect the dimension of the attractor to decrease as \( \varepsilon \to 0 \). In this section, we use a simple computation similar to that used for Theorem 1 to show that \( \dim_H A = 0 \) for \( \varepsilon \) sufficiently small.

**Theorem 3.** Let the forcing \( f \) be time independent, \( \partial_t f = 0 \), and assume the hypotheses of Theorem 2 i.e. \( v(0) \in L^2(M) \) and
\[
|\nabla^2 f|_{L^2} < \infty.
\]
Then there exists an \( \varepsilon_* \left( |\nabla^2 f|; \mu \right) \) such that, for all \( \varepsilon < \varepsilon_* \),
\[
\dim_H A = 0.
\]

Since \( A \) is connected, (4.4) implies that \( A \) consists of a single point, that is, a steady flow \( \omega_\ast \) to which all bounded solutions converge. Following Theorems 1 and 2 this steady flow is nearly, but not exactly, zonal (except in the non-generic case when \( \tilde{f} = 0 \)). Heuristically, an approximation to \( \omega_\ast \) is the steady flow
\[
\omega_\ast^{(1)} = -\mu^{-1} \Delta^{-1} \tilde{f} + \varepsilon L^{-1} \tilde{f},
\]
(4.5)
which satisfies
\[
\frac{1}{\varepsilon} L \omega_\ast^{(1)} + B(\omega_\ast^{(1)}, \omega_\ast^{(1)}) + \mu A \omega_\ast^{(1)} = f
\]
up to \( O(\varepsilon) \). More careful work would be needed to determine \( \omega_\ast \) exactly.
In turbulence parlance, the smallness of $\varepsilon$ demanded by Theorem 3 implies that the Rhines scale [14] is so large that it overhelms the entire spectral range, rendering the dynamics trivial.

A general result related to ours is described in [3, ch. 18], where the trajectory attractor $\mathcal{A}_\varepsilon$ of a dynamical system depending on $t/\varepsilon$ (formally, in our case $\mathcal{A}_\varepsilon$ would simply be the attractor $\mathcal{A}$ for $\varepsilon > 0$) converges weakly to the attractor $\mathcal{A}_0$ of the corresponding averaged system. Formally averaging our equations following this construction (which does not apply directly to our case, in which the oscillations have an infinite number of frequencies which accumulate at zero), we obtain purely zonal NSE, whose dynamics is trivial and whose attractor thus has dimension zero. This is of course consistent with our results: strong convergence at finite $\varepsilon$ of $\mathcal{A}$ to a point (which becomes zonal as $\varepsilon \to 0$).

**Proof.** Fix a solution $\omega(t)$ of (1.8) that lives on $\mathcal{A}$, so the bounds (3.11) hold for all $t$. We consider a nearby solution $\omega(t) + \phi(t)$. The linearised evolution equation for $\phi$ is then

\[
\partial_t \phi = - (\nabla^\perp \Delta^{-1} \omega) \cdot \nabla \phi - (\nabla^\perp \Delta^{-1} \phi) \cdot \nabla \omega(t) - \frac{1}{\varepsilon} \partial_t \Delta^{-1} \phi + \mu \Delta \phi
\]

(4.7)

Multiplying this by $\phi$ in $L^2$ and noting that $(B(\omega, \phi), \phi) = 0$, we obtain

\[
\frac{1}{2} \frac{d}{dt} |\phi|^2 + \mu |\nabla \phi|^2 = (B(\phi, \phi), \omega)
\]

(4.8)

\[
= (B(\phi, \phi), \bar{\omega}) + (B(\phi, \phi), \tilde{\omega}).
\]

For the first term, we split $\phi = \bar{\phi} + \tilde{\phi}$ in analogy with $\omega = \bar{\omega} + \tilde{\omega}$ to get

(4.9)

\[
(B(\phi, \phi), \bar{\omega}) = (B(\bar{\phi}, \bar{\phi}), \bar{\omega})
\]

using the (now familiar) facts that $B(\bar{\phi}, \tilde{\phi}) = 0$ and all tilde-bar-bar terms vanish. Using Poincaré inequality in (4.8) gives us [cf. (3.10)]

\[
\frac{d}{dt} (e^{\nu t} |\phi|^2) + \mu e^{\nu t} |\nabla \phi|^2 \leq 2 e^{\nu t} (B(\bar{\phi}, \bar{\phi}), \bar{\omega}) + 2 e^{\nu t} (B(\phi, \phi), \tilde{\omega}),
\]

(4.10)

which integrates to

\[
|\phi(t)|^2 e^{\nu t} + \mu \int_0^t |\nabla \phi|^2 e^{\nu \tau} \, d\tau \\
\leq |\phi(0)|^2 + 2 \int_0^t \{(B(\bar{\phi}, \bar{\phi}), \bar{\omega}) + (B(\phi, \phi), \tilde{\omega})\} e^{\nu \tau} \, d\tau.
\]

(4.11)

We bound the last term of the integrand using

\[
(B(\phi, \phi), \bar{\omega}) \leq c |\nabla^{-1} \phi|_{L^\infty} |\nabla \phi|_{L^2} |\bar{\omega}|_{L^2}
\]

\[
\leq c_1 |\nabla \phi|^2 |\bar{\omega}|_{L^2}.
\]

(4.12)

The other term needs to be integrated by parts,

\[
\int_0^t (B(\bar{\phi}, \bar{\phi}), \bar{\omega}) e^{\nu \tau} \, d\tau = \varepsilon (B_{11}(\bar{\phi}, \bar{\phi}), \bar{\omega})(t) e^{\nu t} - \varepsilon (B_{11}(\bar{\phi}, \bar{\phi}), \bar{\omega})(0)
\]

\[
- \varepsilon \int_0^t \{\nu (B_{11}(\bar{\phi}, \bar{\phi}), \tilde{\omega}) + (B_{11}(\bar{\phi}, \bar{\phi}), \partial_t \bar{\omega}) + 2 (B_{11}(\partial_t^\varepsilon \bar{\phi}, \bar{\phi}), \tilde{\omega})\} e^{\nu \tau} \, d\tau.
\]

(4.13)
where $\partial_\tau \phi = -B(\omega, \phi) - B(\phi, \omega) - \mu A \phi$. We bound the endpoint terms using

$$2 |(B_\Omega(\tilde{\phi}, \tilde{\phi}), \tilde{\omega})| \leq c_0 |\tilde{\phi}|^2 |\tilde{\omega}|_{L^\infty}.$$  

(4.14) It remains to bound the integrand in (4.13):

$$ |(B_\Omega(\tilde{\phi}, \tilde{\phi}), \tilde{\omega})| \leq c |\tilde{\phi}|^2 |\tilde{\omega}|_{L^4}$$

(4.15) and

$$ |(B_\Omega(\tilde{\phi}, \tilde{\phi}), \tilde{\omega})| \leq c |\tilde{\phi}|^2 |\tilde{\omega}|_{L^5/2},$$

(4.16) Recalling (4.7) for the last term in (4.13), we bound

$$ |(B_\Omega(\tilde{\phi}, \tilde{\phi}), \tilde{\omega})| \leq c |\tilde{\phi}|^2 |\tilde{\omega}|_{L^5/2}$$

(4.17)

Collecting, (4.11) now implies

$$ |\phi(t)|^2 (1 - \varepsilon c_3 |\tilde{\omega}(t)|_{L^\infty}) + \int_0^t \{ \mu - \varepsilon N(\tau) - c_4 |\tilde{\omega}(\tau)|_{L^2} \} |\tilde{\phi}|^2 e^{\mu(\tau-t)} \, d\tau \leq e^{-\mu t} |\phi(0)|^2 \left( 1 + \varepsilon c_4 |\tilde{\omega}(0)|_{L^\infty} \right)$$

(4.18) where

$$ N(t) := c_6 \{ \mu |\tilde{\omega}'|_{L^5/2} + |\tilde{\omega}|_{L^5/2} |\tilde{\omega}|_{L^2} + |\tilde{\omega}'|_{L^{5/2}} + |\tilde{\omega}'|_{L^4} \} (t).$$

(4.19) By (2.11), $f \in H^2$ implies that $\omega \in H^3$ with uniform bound in $t$ since we are already on the attractor, and by Theorem 1 we can find an $\varepsilon_*$ so small that, for $\varepsilon < \varepsilon_*$,

$$ \sup_{t>0} \{ \varepsilon N(t) + c_4 |\tilde{\omega}(t)|_{L^2} \} < \mu.$$  

(4.20) Requiring furthermore that $\varepsilon_*$ also implies, for all $\varepsilon < \varepsilon_*$,

$$ \varepsilon c_5 \sup_{t>0} |\tilde{\omega}'(t)|_{L^\infty} < 1.$$  

(4.21) These and (4.18) then imply that

$$ |\phi(t)|^2 \leq C(\cdots) e^{-\varepsilon t} |\phi(0)|^2,$$  

(4.22) in other words, all phase space volumes contract and thus the global attractor has dimension zero.

It is clear from the above proof that our solution $\omega(t)$ is linearly stable. Since (4.7) only differs by $B(\phi, \phi)$ from the nonlinear system, the fact that $(B(\phi, \phi), \phi) = 0$ implies that stability also holds under the same hypotheses for the full nonlinear system.
Lemma 1. Let \( u \) and \( v \in H^2(M) \) have zero integrals and are \( L^2 \) orthogonal, and let \( w = u + v \). Then the following Agmon inequality holds,

\[
|u|_{L^\infty} \leq c |\nabla w| \left( \log \frac{|\Delta w|}{c_0 |\nabla w|} + 1 \right)^{1/2}. \tag{A.2}
\]

Before the proof, we note that the interpolation inequality

\[
|\nabla w|^2 \leq c_9 |w| |\Delta w| \tag{A.3}
\]

can be written as

\[
2 \log |\nabla w| \leq \log |w| + \log |\Delta w| + \log c_9
\]

\[
\iff
\log |\nabla w| - \log |w| \leq \log |\Delta w| - \log |\nabla w| + \log c_9
\]

\[
\iff
\log |\nabla w| \leq \log \frac{|\Delta w|}{c_9 |\nabla w|} + \log c_9,
\]

which can be used to simplify, e.g., \( |w|_{L^\infty}|\nabla w|_{L^\infty} \) when bounded using \text{(A.2)}.

Proof. For most of this proof, up to \text{(A.7)} below, we follow \cite{3} Lemma 7.1 exactly. For conciseness, we put \( L_1 = L_2 = 1 \) but keep the Poincaré constant \( c_0 \). With \( \kappa > 0 \), we expand \( u \) in Fourier series

\[
u(x) = \sum_{|k| < \kappa} u_k e^{ik \cdot x} + \sum_{|k| \geq \kappa} u_k e^{ik \cdot x} =: u^< (x) + u^> (x), \tag{A.5}
\]

and analogously for \( v \) and \( w \). Then

\[
|u|_{L^\infty} = \sup_x \left| \sum_k u_k e^{ik \cdot x} \right| \leq \sum_{|k| < \kappa} |u_k| + \sum_{|k| \geq \kappa} |u_k|
\]

\[
=: \sum_{|k| < \kappa} |k|^{-1} |u_k| + \sum_{|k| \geq \kappa} |k|^{-2} |k|^2 |u_k|
\]

\[
\leq \left( \sum_{|k| < \kappa} |k|^{-2} \right)^{1/2} \left( \sum_{|k| < \kappa} |k|^2 |u_k|^2 \right)^{1/2}
\]

\[
+ \left( \sum_{|k| \geq \kappa} |k|^{-4} \right)^{1/2} \left( \sum_{|k| \geq \kappa} |k|^4 |u_k|^2 \right)^{1/2}
\]

Now on the right-hand side, \( \sum_{|k| < \kappa} |k|^{-2} \leq c \log \kappa \) and \( \sum_{|k| \geq \kappa} |k|^{-4} \leq c/\kappa^2 \), so fixing

\[
\kappa = |\Delta w|/(c_0 |\nabla w|), \tag{A.7}
\]

the lemma follows from

\[
u|u|_{L^\infty} \leq c |\nabla u^<| \left( \log \frac{|\Delta w|}{c_0 |\nabla w|} \right)^{1/2} + c |\Delta u^>| |\nabla w| |\Delta w| |
\]

\[
\leq c |\nabla w| \left( \log \frac{|\Delta w|}{c_0 |\nabla w|} \right)^{1/2} + c |\nabla w|. \tag{A.8}
\]

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