AN INVARIANCE PRINCIPLE FOR THE TAGGED PARTICLE PROCESS IN CONTINUUM WITH SINGULAR INTERACTION POTENTIAL

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Abstract. We consider the dynamics of a tagged particle in an infinite particle environment moving according to a stochastic gradient dynamics. For singular interaction potentials this tagged particle dynamics was constructed first in [FG11], using closures of pre-Dirichlet forms which were already proposed in [GPS77] and [Osa98]. The environment dynamics and the coupled dynamics of the tagged particle and the environment were constructed separately. Here we continue the analysis of these processes: Proving an essential m-dissipativity result for the generator of the coupled dynamics from [FG11], we show that this dynamics does not only contain the environment dynamics (as one component), but is, given the latter, the only possible choice for being the coupled process. Moreover, we identify the uniform motion of the environment as the reversed motion of the tagged particle. (Since the dynamics are constructed as martingale solutions on configuration space, this is not immediate.) Furthermore, we prove ergodicity of the environment dynamics, whenever the underlying reference measure is a pure phase of the system. Finally, we show that these considerations are sufficient to apply [DMFGW89] for proving an invariance principle for the tagged particle process. We remark that such an invariance principle was studied before in [GPS77] for smooth potentials, and shown by abstract Dirichlet form methods in [Osa98] for singular potentials.

Our results apply for a general class of Ruelle measures corresponding to potentials possibly having infinite range, a non-integrable singularity at 0 and a nontrivial negative part, and fulfill merely a weak differentiability condition on $\mathbb{R}^d \setminus \{0\}$.

1. Introduction

In [DMFGW89], de Masi, Goldstein, Ferrari and Wick developed, based on ideas by Kipnis and Varadhan (see [KV86]), a general method for proving an invariance principle for antisymmetric functions of reversible Markov processes and gave, among other examples, a sketch of an application of this method to the tagged particle process corresponding to a stochastic gradient dynamics, which describes a system of stochastically perturbed particles interacting via a pair potential $\phi$ (which they assumed to be $C^2$ with compact support).

The stochastic gradient dynamics of particles at positions $x_i^t \in \mathbb{R}^d$, $i \in \mathbb{N} \cup \{0\}$, $d \in \mathbb{N}$, at time $t \geq 0$, from which one starts is described by the following equations:

$$dx_i^t = -\sum_{\substack{j=1 \\ j \neq i}}^{\infty} \nabla^T \phi(x_i^t - x_j^t) \, dt + \sqrt{2} dB_i^t, \quad i \in \mathbb{N} \cup \{0\},$$

where $(B_i^t)_{t \geq 0}$, $i \in \mathbb{N}$, are independent $\mathbb{R}^d$-valued Brownian motions. It is a non-trivial question whether a solution in the sense of a sequence of random paths $(x_i^t)_{t \geq 0}$ solving the

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above equations can be constructed for a large class of initial configurations. For constructions of solutions in this sense see e.g. \cite{Lan77, Fri87}. However, these constructions require either $d \leq 2$ or rather restrictive assumptions on $\phi$ (including compact support and the absence of singularities). On the other hand, the existence of Dirichlet form solutions or even martingale solutions on the space of locally finite configurations on $\mathbb{R}^d$ has been shown in great generality, see e.g. \cite{Yos96, Osa96, AKR98}. Considering the tagged particle process means to focus on the trajectory of one particular particle and considering the rest of the system as its environment. Ignoring the question of existence of solutions of (1.1), this can be done by denoting the position of the tagged particle at time $t$ by $\xi_t := x^0_t$, and the environment system by $y^i_t := x^i_t - x^0_t$, $i \in \mathbb{N}$. The thereby obtained joint equations of the tagged particle and its environment read as follows:

$$
d\xi_t = \sum_{i=1}^{\infty} \nabla \phi(y^i_t) \, dt + \sqrt{2} \, dB^0_t, \quad \tag{1.2}
$$

$$
dy^i_t = - \sum_{j=1}^{\infty} \nabla \phi(y^j_t - y^i_t) - \nabla \phi(y^i_t) - \sum_{j=1}^{\infty} \nabla \phi(y^j_t) \, dt + \sqrt{2} \, d(B^i_t - B^0_t), \quad \tag{1.3}
$$

$t \geq 0$. Again, the construction schemes from \cite{Lan77, Fri87} may be applied to construct directly solutions to this system, as is mentioned in \cite{GPS7, DMFGW99}. However, of course, the same restrictions apply as in the case of the system (1.1). For a physically realistic setting (singular potentials, e.g. of Lennard-Jones type, arbitrary dimension, e.g. $d = 3$) the first construction of solutions in the sense of the martingale problem on the space of locally finite configurations was given in \cite{FG11}. Let us make the solution concept more precise. The process constructed in \cite{FG11} has values in $\mathbb{R}^d \times \Gamma$, where

$$
\Gamma := \{ \gamma \subset \mathbb{R}^d \mid \#(\gamma \cap \Lambda) < \infty \text{ for all bounded } \Lambda \subset \mathbb{R}^d \},
$$

is the space of simple locally finite configurations in $\mathbb{R}^d$ (for $d = 1$ the state space is slightly larger). Here $\mathbb{R}^d$ represents the position of the tagged particle and the elements of $\Gamma$ replace sequences $(y^i)_{i \in \mathbb{N}}$ of positions of the particles in the environment. That the process solves (1.2), (1.3) in the sense of the martingale problem means that it solves the martingale problem for the generator corresponding to these equations which is (at first) defined on $C_0^\infty(\mathbb{R}^d) \otimes F C_0^\infty(\mathbb{R}^d, \Gamma)$, i.e. functions which are $C_0^\infty$ in the component of the tagged particle and smooth functions depending on the particles from a bounded region in space. (In fact, the martingale problem is solved for all functions in the domain of the Friedrichs extension of this generator in a suitable $L^2$-space. For more details see Section 3.) It is not clear whether these solutions lead to weak solutions in the ordinary sense (it is e.g. not clear whether one can distinguish and enumerate the particles or whether particles enter from infinity etc.). In particular, any result one may obtain for this type of solutions has to be derived without using the equations (1.2), (1.3), but instead the semimartingale decompositions for functions from the domain of the generator. The construction in \cite{FG11} is done with the help of a quasi-regular Dirichlet form defined on $L^2(\mathbb{R}^d \times \Gamma; dx \otimes \mu)$, where $\mu \in \mathcal{G}^{ge}_{ibp}(\Phi_\phi, ze^{-\phi} \, dx)$, the set of grand canonical Gibbs measures for the interaction potential $\phi$ with intensity measure given by $ze^{-\phi} \, dx$ for some $z > 0$, such that the correct integration by parts formulae are valid; roughly spoken the latter means that elements of $\mathcal{G}^{ge}_{ibp}(\Phi_\phi, ze^{-\phi} \, dx)$ are “translation invariant except of the nonconstant intensity” (again see the discussion in Section 3). In addition to this “coupled process” (consisting of the joint dynamics of the tagged particle and the environment) in \cite{FG11} an “environment process” is constructed from another Dirichlet
form on $L^2(\Gamma; \mu)$, which gives a martingale solution only of (1.3) in the above sense. (The
connection between both Dirichlet forms and processes is a priori not clear, but can be
established, see below.)

Our aim is to apply the general result on invariance principles from [DMFGW89] to the
martingale solutions from [FG11] by adapting it to this type of solutions. A detailed under-
standing of the dynamics, in particular, the precise relation between the path of the tagged
particle and its environment, is essential for doing so.

We obtain the applicability of [DMFGW89] by answering a number of natural questions
about solutions constructed in [FG11]. Let us at first explain these questions, which are
interesting in themselves, and then give the connection to what we need for the invariance
principle.

The first question is the following: We already mentioned above that it seems at least
difficult to get from martingale solutions to weak solutions of the above equations. However,
it should be possible to verify at least the validity of the $d$-dimensional equation (1.2). This
consists of two tasks: First, one has to verify that this stochastic equation is indeed fulfilled
with some Brownian motion $(B^0_t)_{t \geq 0}$. Then one has to make sure that this Brownian motion
“fulfills” (1.3), which means here that it should coincide with the stochastic part of the
uniform motion of the particles in the environment. As is mentioned in [DMFGW89], if one
knows that (1.3) and (1.2) are fulfilled, one can recover $B^0_t$ from the environment process
by computing $-\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N (B^i_t - B^0_t)$, showing in particular that the displacement of
the tagged particle is a functional of the environment. In Section 5 we show that this can
also be done when starting with semimartingale decompositions of functions from
$C_0^\infty(\mathbb{R}^d) \otimes D(L_{\text{env}})_b$, where $D(L_{\text{env}})_b$ denotes the set of bounded functions
from the domain of the generator of the environment process. One should mention that this
domain is rather large, since it includes somehow “full” knowledge of the environment part of
the process. Nevertheless, we give another proof of this fact by means of a stronger result: We prove an
$L^q$-uniqueness property (i.e. essential m-dissipativity result) of the generator of the coupled
process on the set $C_0^\infty(\mathbb{R}^d) \otimes D(L_{\text{env}})_b$, where $D(L_{\text{env}})_b$ denotes the set of bounded functions
from the domain of the generator of the environment process. One should mention that this
domain is rather large, since it includes somehow “full” knowledge of the environment part of
the process. Nevertheless, it is not trivial to prove it, because on a technical level (in terms
of generators) one sees that the coupling between the tagged particle and the environment
as well as the drift acting on the tagged particle is not a small perturbation of the stochastic
motion of the environment and the tagged particle. The $L^q$-uniqueness property does not
only give sufficient knowledge of the semigroup corresponding to the coupled process in order
to identify the environment process as the environment component of the coupled process
(and thus essentially reproduces the mentioned result from [Osa98]). It also shows that for
the environment process from \cite{FG11} there exists exactly one possible tagged particle process: it is not possible to construct another “coupled process” having the same environment part and solving the martingale problem for the generator corresponding to \cite{Lq, Lq}. The $L^2$-uniqueness result and the identification of the environment component of the coupled process are contained in Section 4.

The third question is simply the question whether the environment process is ergodic. For the stochastic gradient dynamics an answer is given in \cite{AKR98}: If the grand canonical reference measure $\mu$ for the Dirichlet form is an extremal canonical Gibbs measure, ergodicity is shown there in great generality. Although the approach given in that paper should also work in the present setting, it seems more effective to make use of the fact that contrary to the case of the stochastic gradient dynamics, the environment process contains also a non-degenerate uniform stochastic motion. In fact, we find that under assumptions which (except possibly in very exceptional cases and in the one-dimensional setting with a singular potential) contain the ones from \cite{AKR98}, we obtain ergodicity from this uniform motion, if $\mu$ is extremal in $G_{\text{dir}}(\Phi, z \phi \phi \text{dx})$. In particular, we do not need to check whether this measure is also extremal as a canonical Gibbs measure. Therefore the assumption on $\mu$ is always nonvoid. The result on ergodicity is given in Section 6.

Let us now explain the connection to the approach from \cite{DMFGW89} for proving the invariance principle. One of the main results in that paper states, roughly spoken, that if one has an antisymmetric integrable (plus some more properties) adapted functional of an ergodic reversible Markov process, then under diffusive scaling this functional converges to a Brownian motion scaled by a constant diffusion matrix, if has a semimartingale decomposition (the “standard decomposition”) with a square integrable martingale and a square integrable mean forward velocity.

If one wants to apply this to the tagged particle process from \cite{FG11}, the environment process plays the role of the reversible ergodic (question 3) Markov process, and the displacement of the tagged particle is the antisymmetric functional. Therefore, one needs to prove that indeed the displacement $\xi_t - \xi_0$ of the tagged particle can be described as an adapted functional on the probability space of the environment process. For this one needs to show that $\xi_t - \xi_0$ is a functional of the environment part of the coupled process (question 1) and that this environment part is distributed as the environment process (question 2).

In the case of smooth compactly supported potentials the invariance principle for the tagged particle process was studied in \cite{GPS87}. Another derivation of an invariance principle, in the present setting, i.e. with singular potentials, has been done in \cite{Osa98} by means of a general theorem which is shown by abstract Dirichlet form methods. We have chosen an approach which can be seen complementary: By a deeper analysis of the particular process under consideration (the answers to question 1-3 above), which in our opinion is interesting in itself, we obtain the invariance principle as a by-product using an older and less involved framework, namely the one of \cite{DMFGW89}.

We summarize the core results of this article:

- The standard decomposition is provided for the path of the tagged particle as well as the existence of the corresponding mean forward velocity in the sense of \cite{DMFGW89}, see Theorem 3.3 and Lemma 7.3.
- We represent the displacement of the tagged particle as functional of the environment, making (the use of $B^o_t$ in) equation \cite{Lq} rigorous, see Theorem 5.1.
- An essential m-dissipativity result is shown for the generator of the coupled process, which yields more control of the corresponding semigroup and shows that the coupled
process from [FG11] is the unique realization of the tagged particle dynamics given
the environment process, see Theorem 4.3.
• Ergodicity of the environment process is shown for essentially any reference measure
μ which can be considered as a pure phase of the system (i.e. extremal elements
of the set of grand canonical Gibbs measures which do not break the action of the
group of uniform shifts), see Theorem 6.1.
• An invariance principle for the tagged particle is shown based on the results from
[DMFGW89], see Section 7. This shows that the concepts developed in that paper
are applicable to martingale solutions of the above equations and in particular work
for particle systems with physically realistic pair interactions.

All results apply for a general class of Ruelle measures corresponding to possibly infinite-
range potentials which may have a non-integrable singularity at the origin and a nontrivial
negative part and fulfill merely a weak differentiability condition on \( \mathbb{R}^d \setminus \{0\} \). They are not
restricted to small activity parameter, but instead work for any pure phase (in the sense
explained above) of the system.

2. CONDITIONS ON THE POTENTIAL AND SOME PRELIMINARIES

Throughout this article we assume that we are given a measurable function \( \phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\} \) which fulfills \( \phi(x) = \phi(-x) \), \( x \in \mathbb{R}^d \), and the following assumptions for some
\( p \in (d, \infty) \cap [2, \infty) \). (For some intermediate results we only need \( p \in [2, \infty) \).

(SS) (superstability) There exist \( A > 0 \) and \( 0 \leq B < \infty \) such that for any \( \eta \in \Gamma_0 := \{\gamma \in \Gamma \mid \gamma < \infty \} \) it holds \( \Phi_\phi(\eta) \geq A \sum_{e \in \mathbb{Z}^d} (\eta(Q_e))^2 - B \eta \).

(LR) (lower regularity) There exists a decreasing measurable function \( \psi : [0, \infty) \to [0, \infty) \) such that \( \int_0^\infty \psi(t) t^{d-1} dt < \infty \) and \( -\phi(x) \leq \psi(|x|_2) \) holds for all \( x \in \mathbb{R}^d \).

(I) (integrability) It holds
\[
\int_{\mathbb{R}^d} |1 - e^{-\phi}| dx < \infty.
\]

(DL) (differentiability and \( L^p \)) \( e^{-\phi} \) is weakly differentiable on \( \mathbb{R}^d \) and \( \phi \) is weakly differentiable on \( \mathbb{R}^d \setminus \{0\} \). Moreover, the gradient \( \nabla \phi \), considered as a \( dx \)-a.e. defined
function on \( \mathbb{R}^d \), satisfies
\[
\nabla \phi \in L^1(\mathbb{R}^d; e^{-\phi} dx) \cap L^p(\mathbb{R}^d; e^{-\phi} dx).
\]

Here \( \Phi_\phi(\eta) := \sum_{\{x,y\} \in \eta} \phi(x - y) \), \( \eta \in \Gamma_0 \), and for \( r = (r_1, \ldots, r_d) \in \mathbb{Z}^d \) we set \( Q_r := \{(\cdot) \in \mathbb{R}^d \mid r_1 - \frac{1}{2} \leq x_i < r_i + \frac{1}{2}\} \). \( |\cdot|_2 \) denotes Euclidean norm on \( \mathbb{R}^d \). \( \# M \) denotes the cardinality of a set \( M \).

Remark 2.1. (i) In [FG11] another condition (LS) as in [AKR98] is imposed on \( \phi \). We
prove in the appendix that this condition can be omitted for the results derived in
that paper (due to the fact that a smaller class of reference measures is considered
there than in [AKR98]).
(ii) Condition (LR) is equivalent to the lower regularity condition from [Rue70], see (the
proof of) [KK03, Proposition 2.17]. (Note that we are dealing with a pair interaction
here.)
(iii) (SS) and (I) hold if lower regularity holds and in addition there exist \( r > 0 \) and a
decreasing function \( \theta : [0, r] \to \mathbb{R} \cup \{\infty\} \) such that \( \int_0^r \theta(s) s^{d-1} ds = \infty \) and \( \phi(x) \geq \theta(|x|_2) \) holds whenever \( x \in \mathbb{R}^d \), \( |x|_2 \leq r \), see [Rue70, Proposition 1.4].
(iv) The above assumptions in particular allow the Lennard-Jones potential \( \phi(x) := 4\varepsilon \left( \left( \frac{s}{|x|} \right)^{12} - \left( \frac{s}{|x|} \right)^{6} \right) \), \( x \in \mathbb{R}^3 \), where \( \varepsilon, s \in (0, \infty) \).

For any measurable function \( h : \mathbb{R}^d \to \mathbb{R} \) and \( \gamma \in \Gamma \) we define
\[
\langle h, \gamma \rangle := \sum_{x \in \gamma} h(x),
\]
if this sum is absolutely convergent or \( h \) is nonnegative. We sometimes use the same notation for \( \mathbb{R}^N \)-valued functions \( h : \mathbb{R}^d \to \mathbb{R}^N \), \( N \in \mathbb{N} \).

\( \mathcal{F}C_0^\infty(\mathcal{D}, \Gamma) \) denotes the set of functions of the form \( F = g_F((f_1, \cdot), \cdots, (f_N, \cdot)) \), where \( N \in \mathbb{N} \), \( g_F \in C^\infty(\mathbb{R}^N \to \mathbb{R}) \), i.e. \( g_F \) is smooth and bounded as well as its derivatives, and \( f_1, \cdots, f_N \in \mathcal{D} := C^\infty_0(\mathbb{R}^d) \). For such a function \( F \), the gradient \( \nabla^\Gamma \) on \( \Gamma \) for the natural differentiable structure on this space (see [AKR98]) is given by
\[
\nabla^\Gamma F(\gamma) := \left( \sum_{i=1}^{N} \partial_i g_F((f_1, \gamma), \cdots, (f_N, \gamma))\nabla f_i(x) \right)_{x \in \mathbb{R}^d}
\]
\( \gamma \in \Gamma \). Here \( \gamma \) is considered as a positive Radon measure on \( \mathbb{R}^d \) by identifying it with the sum \( \sum_{x \in \gamma} \delta_x \) of point masses \( \delta_x \), \( x \in \gamma \). Moreover, below we use the generator \( \nabla^\Gamma_\gamma \) of the group of uniform translations on \( \Gamma \), which is given by \( \nabla^\Gamma_\gamma := \sum_{i=1}^{N} \partial_i g_F((f_1, \cdot), \cdots, (f_N, \cdot))\langle \nabla f_i, \cdot \rangle \) for \( F \) as above.

Embedding \( \Gamma \) in the space of Radon measures on \( \mathbb{R}^d \) as mentioned above, it can be equipped with the vague topology. The corresponding Borel \( \sigma \)-field is denoted by \( \mathcal{B}(\Gamma) \). By \( G^g(\Phi_\phi, z\sigma_\phi) \), \( 0 < z < \infty \), we denote the set of all grand canonical Gibbs measures on \( \Gamma \) for \( \phi \) with intensity measure \( z\sigma_\phi = z e^{-\phi} dx \) (see e.g. [KK03] Theorem 3.12), and by \( G^g(\Phi_\phi, z\sigma_\phi) \) we denote the set of elements of \( G^g(\Phi_\phi, z\sigma_\phi) \) which are tempered in the sense of [Rue70]. By the last part of [Rue70] Corollary 5.3 (the proof works also in the case of the intensity measure \( z\sigma_\phi \), which has bounded density w.r.t. Lebesgue measure) any \( \mu \in G^g(\Phi_\phi, z\sigma_\phi) \) fulfills a Ruelle bound w.r.t. \( z\sigma_\phi \) as defined e.g. in [KK03] Definition 4.12 (or in [FG11]).

For \( v \in C_0^\infty(\mathbb{R}^d) \) and \( F \in \mathcal{F}C_0^\infty(\mathcal{D}, \Gamma) \) one defines \( \nabla^\Gamma_\gamma F(\gamma) := \langle v, \nabla^\Gamma_\gamma F(\gamma) \rangle_{\mathcal{T}, \Gamma} \), \( \gamma \in \Gamma \). In the appendix (which is a generalization of [FG11] Theorem 3.3) it is shown that if \( \phi \) fulfills (SS), (LR), (I) and (DL\( ^p \)), any \( \mu \in \mathcal{G}^g(\Phi_\phi, z\sigma_\phi) \) fulfills the following integration by parts formula
\[
\int_{\Gamma} F \cdot \nabla^\Gamma_\gamma G \, d\mu = -\int_{\Gamma} \nabla^\Gamma_\gamma F \cdot G \, d\mu - \int_{\Gamma} B^{\phi,\mu}_v F \cdot G \, d\mu,
\]
(2.1)
for any \( v \in C_0^\infty(\mathbb{R}^d) \), \( F, G \in \mathcal{F}C_0^\infty(\mathcal{D}, \Gamma) \), where
\[
B^{\phi,\mu}_v(\gamma) := \langle \text{div} \ v, \gamma \rangle_{\mathbb{R}^d} - \sum_{x \in \gamma} (\nabla \phi(x), v(x))_{\mathbb{R}^d} - \sum_{\{x,y\} \subseteq \gamma} (\nabla \phi(x-y), v(x)-v(y))_{\mathbb{R}^d},
\]
\( \gamma \in \Gamma \). Note that since \( \mu \) fulfills an improved Ruelle bound (which can be derived from the usual Ruelle bound together with the Meyer-Montroll equation, see e.g. [AKR98 Eq. (4.29)])], by [KK02] Theorem 4.1] the second sum in the previous equation converges absolutely for \( \mu \)-a.e. \( \gamma \in \Gamma \). Similar arguments imply the following lemma:

**Lemma 2.2.** Let \( \phi \) fulfill (SS), (LR), (I), and (DL\( ^p \)) for some \( p \in [1, \infty) \) and let \( \mu \in G^g(\Phi_\phi, z\sigma_\phi) \). Then \( \langle \nabla \phi, \cdot \rangle \in L^1(\Gamma; \mu) \cap L^p(\Gamma; \mu) \) and there exists \( C < \infty \) such that
\[
\|\langle \nabla \phi, \cdot \rangle\|_{L^1(\Gamma; \mu)} + \|\langle \nabla \phi, \cdot \rangle\|_{L^p(\Gamma; \mu)} \leq C(\|\nabla \phi\|_{L^1(\mathbb{R}^d; \sigma_\phi)} + \|\nabla \phi\|_{L^p(\mathbb{R}^d; \sigma_\phi)}).
\]
Proof. It is not difficult to derive the assertion for $p \in \mathbb{N}$ using [KK02, Theorem 4.1] and the Ruelle bound for $\mu$. Using this, for general $p$ it can be derived by interpolating between $L^1 \cap L^k$-spaces.

By $G^{gc}_{ibp}(\Phi_\phi, z \sigma_\phi)$ we denote the set of those $\mu \in G^{gc}_{ibp}(\Phi_\phi, z \sigma_\phi)$ which also fulfill the additional integration by parts formula

$$
\int \Gamma F \cdot \nabla \Gamma G \, d\mu = - \int \Gamma \nabla \Gamma F \cdot G \, d\mu + \int \langle \nabla \phi, \cdot \rangle F G \, d\mu, \quad F, G \in FC^\infty_0(D, \Gamma).
$$

(2.2)

Moreover, by $G^{gc}_{ibp}(\Phi_\phi, z \, dx)$ we denote the set of translation invariant elements of the set $G^{gc}_{ibp}(\Phi_\phi, z \, dx)$ of tempered grand canonical Gibbs measures for $\phi$ with intensity measure $z \, dx$. If $\mathcal{M}$ is a convex subset of a real vector space, we denote the set of its extremal points by $\text{ext} \, \mathcal{M}$.

In [CK11], the following one-to-one correspondences are shown (with the exception of (iii), which is rather immediate).

**Lemma 2.3.** Let $\phi$ fulfill (SS), (LR), (I), (DL$^1$). For $\mu_0 \in G^{gc}_{ibp}(\Phi_\phi, z \, dx)$ we define

$$
\Psi(\mu_0) := \frac{1}{Z_{\mu_0}} e^{-\langle \phi, \cdot \rangle} \mu_0,
$$

where $Z_{\mu_0} = \int e^{-\langle \phi, \cdot \rangle} \, d\mu_0$.

(i) $\Psi : G^{gc}_{ibp}(\Phi_\phi, z \, dx) \to G^{gc}_{ibp}(\Phi_\phi, z \sigma_\phi)$ is a bijection and $\mu_0$ and $\Psi(\mu_0)$ are equivalent for all $\mu_0 \in G^{gc}_{ibp}(\Phi_\phi, z \, dx)$. Moreover, $\Psi$ maps $G^{gc}_{ibp}(\Phi_\phi, z \, dx)$ to $G^{gc}_{ibp}(\Phi_\phi, z \sigma_\phi)$.

(ii) If additionally $d \geq 2$ and $\nabla \phi \in L^1(\mathbb{R}^d \setminus B_1(0))$ then $\Psi$ is also a bijection between $G^{gc}_{ibp}(\Phi_\phi, z \, dx)$ and $G^{gc}_{ibp}(\Phi_\phi, z \sigma_\phi)$.

(iii) For any convex set $\mathcal{M} \subset G^{gc}_{ibp}(\Phi_\phi, z \, dx)$ we have $\Psi(\text{ext} \, \mathcal{M}) = \text{ext} \, \Psi(\mathcal{M})$.

In particular, since $G^{gc}_{ibp}(\Phi_\phi, z \, dx) \neq \emptyset$ (see e.g. [Rue70, Theorem 5.8]), we also have $G^{gc}_{ibp}(\Phi_\phi, z \sigma_\phi) \neq \emptyset$.

**Remark 2.4.** The assertion of Lemma 2.3 (ii) also holds in many cases if $d = 1$, mainly because it is known that for many potentials there are no phase transitions in the one-dimensional setting. We remark that independently of the occurrence of a phase transition Lemma 2.3 (ii) holds in any dimension if $\phi$ fulfills the stronger assumption $\nabla \phi \in L^1(\mathbb{R}^d)$ (in particular if $\phi$ is bounded), as one easily derives from the proof of [CK11] Lemma 3.2.

**Remark 2.5.** Any $\mu \in G^{gc}_{ibp}(\Phi_\phi, z \, dx)$ can be represented as a mixture of elements from $\text{ext} G^{gc}_{ibp}(\Phi_\phi, z \, dx)$, see [Geo75], [Pre76]. (This is well-known, but for the reader’s convenience we give some hints which hopefully simplify a complete step-by-step verification: The mentioned statement is given in Theorem 4.2 and p. 57 in [Pre76]: one has to take into account that all $\mathbb{Q}^d$-shift invariant probability measures on $(\Gamma, \mathcal{B}(\Gamma))$ (which are precisely the translation invariant probability measures on $(\Gamma, \mathcal{B}(\Gamma))$) have a representation as integrals over extremal $\mathbb{Q}^d$-shift invariant, hence $\mathbb{Q}^d$-ergodic probability measures, which is seen as e.g. in [Geo88, Theorem 14.10].) In particular, $\text{ext} G^{gc}_{ibp}(\Phi_\phi, z \, dx)$ is nonempty. Any $\mu \in \text{ext} G^{gc}_{ibp}(\Phi_\phi, z \, dx)$ is trivial on the $\sigma$-algebra of $\mu$-a.s. translation invariant events (see e.g. [Pre76, Theorem 4.1]), i.e. if $A \in \mathcal{B}(\Gamma)$ fulfills $\mu(\{ \gamma + v \mid \gamma \in A \} \Delta A) = 0$ for all $v \in \mathbb{R}^d$, then $\mu(A) \in \{ 0, 1 \}$. Here for $\gamma \in \Gamma$ we set $\gamma + v := \{ x + v \mid x \in \gamma \}$ and $\Delta$ denotes the symmetric set difference. The above lemma shows that these statements also hold with $G^{gc}_{ibp}(\Phi_\phi, z \sigma_\phi)$ replaced with $G^{gc}_{ibp}(\Phi_\phi, z \sigma_\phi)$, if the assumptions from part (ii) of that lemma are fulfilled.
3. The existence of the tagged particle process

In this section we briefly sketch the construction of the environment process and the coupled process. A detailed study of this problem can be found in [FG11].

3.1. The environment process. For $d \geq 2$ the process corresponding to (1.3) is realized on the configuration space $\Gamma$. For $d = 1$ the process exists in general only in the larger space $\tilde{\Gamma}$ of multiple configurations, which can similarly as $\Gamma$ be equipped with the vague topology and the corresponding Borel $\sigma$-field $B(\tilde{\Gamma})$. For any dimension, the process is obtained by analyzing the densely defined, positive definite, symmetric bilinear form $(\mathcal{E}_{env}^{\Gamma,\mu}, F_{C_b}^\infty(D, \Gamma))$ on $L^2(\Gamma; \mu) = L^2(\tilde{\Gamma}; \mu)$, given by

$$
\mathcal{E}_{env}^{\Gamma,\mu}(F, G) = \int_{\Gamma} (\nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma))_{T, \Gamma} \, d\mu(\gamma) + \int_{\Gamma} (\nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma))_{\mathbb{R}^d} \, d\mu(\gamma),
$$

$F, G \in F_{C_b}^\infty(D, \Gamma)$, where $\mu \in G_{dbp}^{ge}(\Phi_\phi, z \sigma_\phi)$. Using (2.1) and (2.2), the following result is derived in [FG11]. (Note that although e.g. for $p \in [1, \infty]$ the spaces $L^p(\Gamma; \mu)$ and $L^p(\tilde{\Gamma}; \mu)$ are the same, we sometimes emphasize the use of $\Gamma$ for example in general quasi-regularity of the Dirichlet forms used below depends on whether one uses $\Gamma$ or $\tilde{\Gamma}$ as state space.) Recall (cf. Remark 2.1(i)) that condition (LS) from [FG11] can be dropped in the statements of the following theorem.

**Proposition 3.1.** Suppose that $\phi$ fulfills (SS), (LR), (I) and (DL$^2$). Let $\mu \in G_{dbp}^{ge}(\Phi_\phi, z \sigma_\phi)$. Then $(\mathcal{E}_{env}^{\Gamma,\mu}, F_{C_b}^\infty(D, \Gamma))$ is closable in $L^2(\Gamma; \mu)$ and its closure $(\mathcal{E}_{env}^{\Gamma,\mu}, D(\mathcal{E}_{env}^{\Gamma,\mu}))$ is a conservative, local, quasi-regular, symmetric Dirichlet form on $L^2(\Gamma; \mu)$. Moreover, for $F, G \in F_{C_b}^\infty(D, \Gamma)$ it holds

$$
\mathcal{E}_{env}^{\Gamma,\mu}(F, G) = \int_{\Gamma} -L_{env}^{\Gamma,\mu} F \cdot G \, d\mu,
$$

where

$$
L_{env}^{\Gamma,\mu} F(\gamma) = \sum_{i,j=1}^N \partial_i \partial_j g_F(\langle f_1, \gamma \rangle, \cdots, \langle f_N, \gamma \rangle) \left( \langle \nabla f_i, \nabla f_j \rangle_{\mathbb{R}^d}, \gamma \right) + \sum_{j=1}^N \partial_j g_F(\langle f_1, \gamma \rangle, \cdots, \langle f_N, \gamma \rangle) \left( 2\Delta f_j, \gamma \right) - \langle \nabla \phi, \nabla f_j \rangle_{\mathbb{R}^d}, \gamma \right) - \langle \nabla \phi, \nabla f_j \rangle_{\mathbb{R}^d} - \sum_{\{x,y\} \subset \gamma} \langle \nabla \phi - y, \nabla f_j(x) - \nabla f_j(y) \rangle_{\mathbb{R}^d}
$$

for $\mu$-a.e. $\gamma \in \Gamma$. The generator $(H_{env}^{\Gamma,\mu}, D(H_{env}^{\Gamma,\mu}))$ of $(\mathcal{E}_{env}^{\Gamma,\mu}, D(\mathcal{E}_{env}^{\Gamma,\mu}))$ is the Friedrichs extension of $(L_{env}^{\Gamma,\mu}, F_{C_b}^\infty(D, \Gamma))$.

Furthermore, by applying the theory of Dirichlet forms the following is shown:

**Theorem 3.2.** Under the assumptions of Proposition 3.1 the following holds:

(i) There exists a (up to $\mu$-equivalence unique, see [MR92] Section IV.6) conservative diffusion process

$$
M_{env}^{\Gamma,\mu} = \left( \Omega_{env}^{\Gamma,\mu}, \mathbf{P}_{env}^{\Gamma,\mu}, \{ \mathbf{F}_{env}^{\Gamma,\mu} \}_{t \geq 0}, \{ \mathbf{X}_{env}^{\Gamma,\mu} \}_{t \geq 0}, \{ \mathbf{P}_{\gamma}^{\env} \}_{\gamma \in \Gamma} \right)
$$
on $\bar{\Gamma}$ which is properly associated with $(F_{env}^{\Gamma,\mu}, D(F_{env}^{\Gamma,\mu}))$, i.e. for all $\mu$-versions of $F \in L^2(\bar{\Gamma}; \mu)$ and all $t > 0$ the function

$$\gamma \mapsto p_t^{env}(\gamma) := E^\mu_{\gamma}[F(X_t^{env})], \quad \gamma \in \bar{\Gamma},$$

is an $F_{env}^{\mu}$-quasi continuous $\mu$-version of $T_{\gamma}^{Tt}F := \exp(tH_{env}^{\Gamma,\mu})F$. (Here $E^{env}_{\gamma}$ denotes the expectation corresponding to $F_{env}^{\mu};$ below we also use analogous notations.) In particular, $M_{env}^{\Gamma,\mu}$ is $\mu$-symmetric, i.e. $\int_{\Gamma} p_t^{env}F \cdot G \, d\mu = \int_{\Gamma} F \cdot p_t^{env}G \, d\mu$ for all nonnegative $B(\bar{\Gamma})$-measurable $F,G : \bar{\Gamma} \to \mathbb{R}$.

(ii) $M_{env}^{\Gamma,\mu}$ has invariant measure $\mu$ and solves the martingale problem for $(H_{env}^{\Gamma,\mu}, D(H_{env}^{\Gamma,\mu}))$ in the following sense: For all $G \in D(H_{env}^{\Gamma,\mu})$, thus in particular for any $G \in FC_0^\infty(D,\Gamma)$, the process defined by

$$\tilde{G}(X_t^{env}) - \tilde{G}(X_0^{env}) - \int_0^t H_{env}^{\Gamma,\mu}(X_s^{env}) \, ds, \quad t \geq 0,$$

is an $(F_{t}^{env})_{t \geq 0}$-martingale under $P_{\gamma}^{env}$ (hence starting at $\gamma$) for $F_{env}^{\Gamma,\mu}$-quasi-all $\gamma \in \bar{\Gamma}$.

Here $\tilde{G}$ denotes an $E_{env}^{\Gamma,\mu}$-quasi-continuous $\mu$-version of $G$.

### 3.2. The coupled process.

For constructing and analyzing the Markov process on $\mathbb{R}^d \times \Gamma$ representing the coupled motion of the tagged particle and its environment, we often consider test functions $\mathcal{G} \in C_0^\infty(\mathbb{R}^d) \otimes FC_0^\infty(D,\Gamma)$, i.e. functions of the form $\mathcal{G} = \sum_{k=1}^k f \otimes F_k$, $f_1, \cdots, f_k \in C_0^\infty(\mathbb{R}^d)$, $F_1, \cdots, F_k \in FC_0^\infty(D,\Gamma)$, $k \in \mathbb{N}$, where for $f \in C_0^\infty(\mathbb{R}^d)$ and $F \in FC_0^\infty(D,\Gamma)$ we set $f \otimes F(\xi,\gamma) := f(\xi)F(\gamma)$, $(\xi,\gamma) \in \mathbb{R}^d \times \Gamma$. Some of the linear operators on $C_0^\infty(\mathbb{R}^d) \otimes FC_0^\infty(D,\Gamma)$ we consider below are the operators $(\nabla_{\gamma}^T - \nabla_{\xi}^T), \nabla_{\gamma}^T, (\nabla_{\gamma}^T, \nabla_{\xi})_{\mathbb{R}^d}$, given by

$$(\nabla_{\gamma}^T - \nabla_{\xi}^T)\mathcal{G}(\xi,\gamma) := \sum_{i=1}^k (f_i(\xi)\nabla_{\gamma}^T F_i - F_i(\gamma)\nabla f_i(\xi))$$

$$\nabla_{\gamma}^T \mathcal{G}(\xi,\gamma) := \sum_{i=1}^k f_i(\xi)\nabla_{\gamma}^T F_i(\gamma)$$

$$(\nabla_{\gamma}^T, \nabla_{\xi})_{\mathbb{R}^d} \mathcal{G}(\xi,\gamma) := \sum_{i=1}^k (\nabla_{\gamma}^T F_i(\gamma), \nabla f_i(\xi))_{\mathbb{R}^d}$$

for $(\xi,\gamma) \in \mathbb{R}^d \times \Gamma$ and $\mathcal{G} = \sum_{k=1}^k f_i \otimes F_i$ as above. As a general rule, the operators $L_{env}^{\Gamma,\mu}, \nabla_{\gamma}^T$ and $\nabla_{\gamma}^T$ are supposed to act only on the $\gamma$-dependent parts of an $\mathcal{G} \in C_0^\infty(\mathbb{R}^d) \otimes FC_0^\infty(D,\Gamma)$, while $\nabla_{\xi}$ and $\Delta_{\xi}$ only act on the $\xi$-dependent parts. We also use similar notations in analogous situations (i.e. for tensor products of other spaces of functions on $\mathbb{R}^d$ and $\Gamma$). Moreover, we sometimes consider a function $F : \Gamma \to \mathbb{R}$ as a function $F : \mathbb{R}^d \times \Gamma \to \mathbb{R}$ by setting $F(\xi,\gamma) := F(\gamma)$ for $(\xi,\gamma) \in \mathbb{R}^d \times \Gamma$.

The coupled process is obtained by analyzing the densely defined, positive definite, symmetric bilinear form $(\mathcal{E}_{\text{coupl}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}}(\mathcal{G}, \mathcal{G}) := \int_{\mathbb{R}^d \times \Gamma} \left( (\nabla_{\gamma}^T \mathcal{G}(\xi,\gamma), \nabla_{\gamma}^T \mathcal{G}(\xi,\gamma))_{\Gamma,\Gamma} + \int_{\mathbb{R}^d \times \Gamma} \left( (\nabla_{\gamma}^T - \nabla_{\xi})\mathcal{G}(\xi,\gamma), (\nabla_{\gamma}^T - \nabla_{\xi})\mathcal{G}(\xi,\gamma) \right)_{\mathbb{R}^d} \, d\tilde{\mu}(\xi,\gamma),$$
Proposition 3.3. Suppose that $\phi$ fulfills (SS), (LR), (I) and (DLP) for $p \geq \text{max}\{2, d\}$. Furthermore, let $\mu$ and $\tilde{\mu}$ be as above. Then $(\mathcal{E}_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}}, D(\mathcal{E}_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}}))$ is closable in $L^2(\mathbb{R}^d \times \Gamma; \tilde{\mu})$ and its closure $(\mathcal{E}_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}}, D(\mathcal{E}_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}}))$ is a conservative, local, quasi-regular, symmetric Dirichlet form on $L^2(\mathbb{R}^d \times \Gamma; \tilde{\mu})$. Moreover, for $\mathfrak{F}, \mathfrak{G} \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{F}_b^\infty(D, \Gamma)$ it holds
\[
\mathcal{E}_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}}(\mathfrak{F}, \mathfrak{G}) = \int_{\mathbb{R}^d \times \Gamma} -L_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}} \mathfrak{F} \cdot \mathfrak{G} \, d\tilde{\mu},
\]
where
\[
L_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}} = L_{\text{env}} - 2(\nabla_{\gamma}, \nabla_{\xi})_{\mathbb{R}^d} + ((\nabla \phi, \cdot), \nabla \xi)_{\mathbb{R}^d} + \Delta_{\xi}.
\]
The generator $(H_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}}, D(H_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}}))$ of $(\mathcal{E}_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}}, D(\mathcal{E}_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}}))$ is the Friedrichs extension of $(H_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}}, C_0^\infty(\mathbb{R}^d) \otimes \mathcal{F}_b^\infty(D, \Gamma))$.

Applying the theory of Dirichlet forms the following is shown in [FG11]:

Theorem 3.4. Under the assumptions of Proposition 3.3 the following holds:

(i) There exists a (up to $\mu$-equivalence unique) conservative diffusion process
\[
M_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}} = \left(\Omega_{\text{coup}}, F_{\text{coup}}(t) \geq 0, (X_{\text{coup}}(t), (X_{\text{coup}}^\xi(\xi, \gamma)(\xi, \gamma) \in \mathbb{R}^d \times \Gamma) \right)
\]
on $\mathbb{R}^d \times \tilde{\Gamma}$ which is properly associated with $(\mathcal{E}_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}}, D(\mathcal{E}_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}}))$. In particular, $M_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}}$ is $\mu$-symmetric.

(ii) $M_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}}$ has $\tilde{\mu}$ as invariant measure and solves the martingale problem for the operator $(H_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}}, D(H_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}}))$ in the following sense: For all $\mathfrak{G} \in D(H_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}})$, in particular for all $\mathfrak{G} \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{F}_b^\infty(D, \Gamma)$, the process defined by
\[
\tilde{\mathfrak{G}}(X^\xi_t(\xi, \gamma)) - \tilde{\mathfrak{G}}(X^\xi_0(\xi, \gamma)) - \int_0^t H_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}} \mathfrak{G}(X^\xi_s) \, ds, \quad t \geq 0,
\]
is an $(F_{\text{coup}}(t) \geq 0)$-martingale under $P_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}}$ quasi-all $(\xi, \gamma) \in \mathbb{R}^d \times \tilde{\Gamma}$. Here $\tilde{\mathfrak{G}}$ denotes an $\mathcal{E}_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}}$-quasi-continuous $\tilde{\mu}$-version of $\mathfrak{G}$.

Below we use the notation $T_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}} := \exp(t H_{\text{coup}}^{\mathbb{R}^d \times \Gamma, \tilde{\mu}}), t \geq 0$.

Remark 3.5. It is important to note that $M_{\text{env}}$ (resp. $M_{\text{coup}}$) is a solution of (1.3) (resp. (1.2) and (1.3)) in the sense that the martingale problem for the corresponding generators is solved as stated in the previous lemmas. There would be at least some work to do in order to show that this gives rise to solutions in a stronger sense (e.g. proving that one can enumerate all the particles from the environment and construct their trajectories, which should not explode). The martingale problem (or the associatedness of the processes with the respective Dirichlet form) is therefore the only possible starting point for deriving results on these processes.

Sometimes we need the statement on the martingale problem also for some functions which are "locally" in $C_0^\infty(\mathbb{R}^d) \otimes \mathcal{F}_b^\infty(D, \Gamma)$. Therefore we prove the following lemma. We denote by $X_{t \text{coup}}^1$ and $X_{t \text{coup}}^2$ the two components of $X_t^\text{coup}$ for $t \geq 0$. 
Theorem 3.6. 
(i) Let $F \in F C^\infty_c(\mathcal{D}, \Gamma)$ and let $c \in \mathbb{R}^d$ (e.g. some standard unit vector or 0). Let $\mathcal{F}(\xi, \gamma) := F(\gamma) + (c, \xi)_{\mathbb{R}^d}$, $(\xi, \gamma) \in \mathbb{R}^d \times \Gamma$ and set $L_{\text{coupl}}^{\mathbb{R}^d \times \Gamma, \mu} \mathcal{F}(\xi, \gamma) := L_{\text{env}}^{\mathbb{R}^d \times \Gamma, \mu} F(\gamma) + (c, \nabla \phi)_{\mathbb{R}^d \times \Gamma}$. Then for $\mathcal{E}_{\text{coupl}}^{\mathbb{R}^d \times \Gamma, \mu}$-quasi every $(\xi, \gamma) \in \mathbb{R}^d \times \Gamma$

\[ M^\mathcal{F}_t := \mathcal{F}(X^\text{coupl}_t) - \mathcal{F}(X^\text{coupl}_0) - \int_0^t L_{\text{coupl}}^{\mathbb{R}^d \times \Gamma, \mu} \mathcal{F}(X^\text{coupl}_s) \, ds, \quad t \geq 0, \]

defines a continuous local $(\mathcal{F}^\text{coupl}_t)_{t \geq 0}$-martingale under $\mathcal{P}^\text{coupl}_{(\xi, \gamma)}$ with quadratic variation process given by

\[ \mathcal{E}_t = 2 \int_0^t (\nabla^2 F(X^\text{coupl}_s, \nabla F(X^\text{coupl}_s))) T_{X^\text{coupl}, \Gamma} + |\nabla F(X^\text{coupl}_s, \xi)|^2 \, ds. \] (3.1)

(ii) For $\mathcal{E}_{\text{coupl}}^{\mathbb{R}^d \times \Gamma, \mu}$-quasi every $(\xi, \gamma) \in \mathbb{R}^d \times \Gamma$ the process

\[ \left( X^\text{coupl,1}_t - X^\text{coupl,1}_0 - \int_0^t \langle \nabla \phi, X^\text{coupl,2}_s \rangle \, ds \right)_{t \geq 0} \]

is distributed under $\mathcal{P}^\text{coupl}_{(\xi, \gamma)}$ like $\sqrt{2}$ times a $d$-dimensional Brownian motion starting in 0.

Proof. For $k \in \mathbb{N}$ choose a function $\chi_k \in C_0^\infty(\mathbb{R}^d)$ such that $1_{[-k, k]^d} \leq \chi_k \leq 1_{[-k-1, k+1]^d}$. Considering $\chi_k$ as a function on $\mathbb{R}^d \times \Gamma$ which is constant in the second argument, we obtain functions $\bar{\mathcal{F}}_k := \chi_k \mathcal{F} \in C_0^\infty(\mathbb{R}^d) \times FC^\infty_c(\mathcal{D}, \Gamma)$. By Lemma 3.3.ii) and the fact that the set of $\mathcal{E}_{\text{coupl}}^{\mathbb{R}^d \times \Gamma, \mu}$-exceptional sets is closed w.r.t. countable unions, there is an $\mathcal{E}_{\text{coupl}}^{\mathbb{R}^d \times \Gamma, \mu}$-exceptional set $N \subset \mathbb{R}^d \times \Gamma$ such that for all $(\xi, \gamma) \in (\mathbb{R}^d \times \Gamma) \setminus N$ all $M^\mathcal{F}_t$, $t \geq 0$, $k \in \mathbb{N}$, are $(\mathcal{F}^\text{coupl}_t)_{t \geq 0}$ martingales under $\mathcal{P}^\text{coupl}_{(\xi, \gamma)}$ with quadratic variation processes given as in (3.1), with $\mathcal{F}$ replaced by $\mathcal{F}_k$. The latter statement follows using [FOT94] Theorems 5.1.3 and 5.2.3 as in [FOT94] Example 5.2.1.

Since $\mathcal{M}^{\text{coupl}}_{(\xi, \gamma)}$ is conservative, we may in addition assume that $\mathcal{P}^\text{coupl}_{(\xi, \gamma)}$ is conservative for all $(\xi, \gamma) \in \mathbb{R}^d \times \Gamma \setminus N$. From this we obtain using the $(\mathcal{F}^\text{coupl}_t)_{t \geq 0}$-stopping times $\tau_k := \inf\{ t \geq 0 \, | \, X^\text{coupl}_t \notin [-k, k]^d \}$, $k \in \mathbb{N}$, that $(M^\mathcal{F}_{t\wedge \tau_k})_{t \geq 0} = (M^{\mathcal{F}_k}_{t\wedge \tau_k})_{t \geq 0}$ is for each $k \in \mathbb{N}$ an $(\mathcal{F}^\text{coupl}_t)_{t \geq 0}$-martingale under $\mathcal{P}^\text{coupl}_{(\xi, \gamma)}$ for all $(\xi, \gamma) \in (\mathbb{R}^d \times \Gamma) \setminus N$ with quadratic variation process given by $(\langle M^\mathcal{F} \rangle_{t\wedge \tau_k})_{t \geq 0} = (\langle M^{\mathcal{F}_k} \rangle_{t\wedge \tau_k})_{t \geq 0}$. (The $\tau_k$ are stopping times by right-continuity of the filtration $(\mathcal{F}^\text{coupl}_t)_{t \geq 0}$.) Since by conservativity we have $\tau_k \to \infty$ as $k \to \infty$ $\mathcal{P}^\text{coupl}_{(\xi, \gamma)}$-a.s., it follows that $(M^\mathcal{F}_t)_{t \geq 0}$ is a local martingale with quadratic variation process given as in (3.1). The second assertion is now a consequence of Lévy’s theorem (see e.g. [EK86]).

Remark 3.7. By the Burkholder-Davis-Gundy inequality and [Dur96] Theorem (2.5) we can drop the “local” in the statement of the above lemma if we can prove that for $\mathcal{E}_{\text{coupl}}^{\mathbb{R}^d \times \Gamma, \mu}$-quasi every $(\xi, \gamma) \in \mathbb{R}^d \times \Gamma$ and all $T \in \mathbb{N}$ it holds

\[ \mathcal{E}^\text{coupl}_{(\xi, \gamma)}(M^\mathcal{F})_T < \infty, \]

which is e.g. true if

\[ \mathcal{E}^\text{coupl}_{(\xi, \gamma)} \int_0^\infty e^{-s} Z(X^\text{coupl}_s) \, ds < \infty, \] (3.2)

where $Z(\xi', \gamma') := \langle \nabla F(\gamma'), \nabla F(\gamma') \rangle{T} + |\nabla F(\gamma')| - c|^{2}$, $(\xi', \gamma') \in \mathbb{R}^d \times \Gamma$. We believe that this is true, but do not further investigate this question, since below we only consider mixtures of the laws $\mathcal{P}^\text{coupl}_{(\xi, \gamma)}$ for which (3.2) is clear.
4. One-particle uniqueness and the environment part of \((T_{t,2}^{\text{comp}})_{t \geq 0}\).

Let \(\phi\) fulfill the assumptions from the beginning of Section 2. Let \(\mu \in \mathcal{C}_{\text{top}}^\infty(\Phi_\infty, \sigma_\infty)\). By symmetry of the sub-Markovian semigroup \((T_{t,2}^{\text{comp}})_{t \geq 0}\) and by the Beurling-Deny theorem (see [LS91, Proposition 1.8]) there exists for each \(q \in [1, \infty]\) a contraction semigroup \((T_{t,q}^{\text{comp}})_{t \geq 0}\) on \(L^q(\mathbb{R}^d \times \Gamma; \hat{\mu}) =: L^q\) extending the restriction of \((T_{t,2}^{\text{comp}})_{t \geq 0}\) to \(L^1 \cap L^\infty\), such that for each \(q < \infty\) the semigroup \((T_{t,q}^{\text{comp}})_{t \geq 0}\) is strongly continuous and \((T_{t,q}^{\text{comp}})_{t \geq 0}\) is the adjoint semigroup of \((T_{t,q}^{\text{comp}})_{t \geq 0}\). For convenience, we sometimes drop the index \(q\) and simply write \((T_{t}^{\text{comp}})_{t \geq 0}\). For identifying the second component of the coupled process as the environment process (see also Lemma 5.2 below), the main step is to prove the identity

\[
T_{t,\infty}(1 \otimes F) = 1 \otimes T_{t,2}^{\text{env}} F, \quad F \in L^\infty(\Gamma; \mu). \tag{4.1}
\]

One proof for this is given in [Osa98, Proof of Lemma 2.3]. However, as explained in the introduction, we give a different proof based on showing a result on essential m-dissipativity of the generator \(H_{\text{comp}}^{\Gamma,\mu}\) in some \(L^q\), \(q \in [1, 2]\), on a rather large set of functions on \(\mathbb{R}^d \times \Gamma\), which is nevertheless small enough for our purposes. Since these functions are of a very simple type only in the first component (and since knowledge of a generator core uniquely determines the semigroup), this result is named one-particle uniqueness. We need some preparations. By \(S(\mathbb{R}^d)\) we denote the Schwartz space of rapidly decreasing smooth functions on \(\mathbb{R}^d\) and by \(D(H_{\text{comp}}^{\Gamma,\mu})\) we denote the set of bounded functions from \(D(H_{\text{comp}}^{\Gamma,\mu})\). Any \(F \in D(H_{\text{comp}}^{\Gamma,\mu})\) is contained in \(D(\mathcal{E}_{\text{comp}}^{\Gamma,\mu})\). In particular, for such \(F\) the object \(\nabla^t \Gamma F \in L^2(\Gamma ; \mathbb{R} ; \mu)\) is reasonably defined as the limit of \((\nabla^t \Gamma F_n)_{n \in \mathbb{N}}\) for a sequence \((F_n)_{n \in \mathbb{N}} \subset \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)\) approximating \(F\) in \(D(\mathcal{E}_{\text{comp}}^{\Gamma,\mu})\) w.r.t. the norm \(\| \cdot \|_{D(\mathcal{E}_{\text{comp}}^{\Gamma,\mu})} := \sqrt{\langle \cdot, \cdot \rangle_{L^2(\Gamma; \mu)} + \mathcal{E}_{\text{comp}}^{\Gamma,\mu}(\cdot, \cdot)}\).

**Lemma 4.1.** Let \(f \in S(\mathbb{R}^d)\) and \(F \in D(H_{\text{comp}}^{\Gamma,\mu})\). Then \(\tilde{g} := f \otimes F \in D(H_{\text{comp}}^{\Gamma,\mu})\) and

\[
H_{\text{comp}}^{\Gamma,\mu} \tilde{g} = H_{\text{comp}}^{\Gamma,\mu} f \otimes F + 2(\nabla^t \Gamma_{\gamma} \nabla \xi_{\mu} \tilde{g})_{\mathbb{R}^d} + \langle (\nabla_{\phi} \cdot, \nabla_{\xi})_{\mathbb{R}^d} \tilde{g} + \Delta \xi \tilde{g}.\rangle
\]

**Proof.** Let at first \(F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)\) and \(f \in C_0^\infty(\mathbb{R}^d)\). Then for any \(g \in C_0^\infty(\mathbb{R}^d)\) and \(G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)\) it holds

\[
\mathcal{E}_{\text{comp}}^{\Gamma,\mu}(f \otimes F, g \otimes G) = \mathcal{E}_{\text{comp}}^{\Gamma,\mu}(F, G)(f, g)_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(\gamma) G(\gamma) (\nabla \xi f(\xi), \nabla \xi g(\xi))_{\mathbb{R}^d} d\xi d\mu(\gamma)
\]

\[= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(\xi) G(\gamma) (\nabla \xi G(\gamma), \nabla \xi f(\xi))_{\mathbb{R}^d} d\xi d\mu(\gamma)
\]

\[= \mathcal{E}_{\text{comp}}^{\Gamma,\mu}(F, G)(f, g)_{L^2(\mathbb{R}^d)} + (\nabla \xi f(\xi), \nabla \xi g(\xi))_{\mathbb{R}^d} d\xi d\mu(\gamma), \tag{4.2}
\]

see also [FG11, Proof of Theorem 4.15]. This immediately extends to \(f, g \in S(\mathbb{R}^d)\). Moreover, it also extends to \(F \in D(\mathcal{E}_{\text{comp}}^{\Gamma,\mu})\), as one sees by approximating \(F\) by \((F_n)_{n \in \mathbb{N}} \subset \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)\).
as mentioned above: Convergence of the right-hand side is shown using the considerations preceding this lemma. To see convergence of the left-hand side we note that for such a sequence we have sup \( L_n \to C \) \( (f \otimes F_n, f \otimes F_n) < \infty \), hence by [MR92, Lemma 1.2.12] it follows \( f \otimes F \in D(\mathcal{E}_{\text{coupl}}^{\Gamma, \mu}) \) and \( \mathcal{E}_{\text{coupl}}^{\Gamma, \mu}(f \otimes F_n, g \otimes G) \to \mathcal{E}_{\text{coupl}}^{\Gamma, \mu}(f \otimes F, g \otimes G) \) as \( n \to \infty \).

For any bounded \( F \in D(\mathcal{E}_{\text{coupl}}^{\Gamma, \mu}) \) we find that the last summand on the right-hand side of Lemma 4.2 can be rewritten as \( L^2 \)-inner product \( ((\nabla f) \otimes ((\nabla \phi, \cdot), F), g \otimes G)_{L^2} \), and if in addition \( F \in D(H_{\text{env}}^{\Gamma, \mu}) \), the first summand equals \( -(f \otimes H_{\text{env}}^{\Gamma, \mu} F, g \otimes G)_{L^2} \). The assertion follows from [MR92, Proposition 1.2.16] and denseness of the linear span of functions \( g \otimes G \) as above in \( D(\mathcal{E}_{\text{coupl}}^{\Gamma, \mu}) \) w.r.t. \( \| \cdot \|_{D(\mathcal{E}_{\text{coupl}}^{\Gamma, \mu})} \).

For any \( H \in L^2(\Gamma \to \mathbb{R}^d; \mu) \) we define a linear operator \( (L_H, \dot{D}) \) on \( L^q(\mathbb{R}^d \times \Gamma; \dot{\mu}) \), \( q \in [1, 2] \), by

\[
L_H \ddot{\mathfrak{g}} := H_{\text{env}}^{\Gamma, \mu} \ddot{\mathfrak{g}} + \Delta_\xi \ddot{\mathfrak{g}} - 2(\nabla_\gamma, \nabla_\xi)_{\mathbb{R}^d} \ddot{\mathfrak{g}} + (H, \nabla_\xi)_{\mathbb{R}^d} \ddot{\mathfrak{g}},
\]

\( \ddot{\mathfrak{g}} \in \dot{D} := S(\mathbb{R}^d) \otimes FC_{c_0}^\infty(D, \Gamma) \).

\( H_{\text{env}}^{\Gamma, \mu} \), considered as an operator acting on \( \dot{D} \), is dissipative in any \( L^q \), \( q \in [1, 2] \), since an extension of it generates in \( L^2 \) the symmetric sub-Markovian (and thus \( L^p \)-contractive) strongly continuous contraction semigroup \((I \otimes T_{\mathbb{R}^2}^{\Gamma, \mu})_{t \geq 0}\). Dissipativity in any \( L^q \), \( q \in [1, 2] \), of \( (\mathcal{E}_{\text{coupl}}^{\Gamma, \mu}, \dot{D}) = (L_{H_\phi}, \dot{D}) \), where \( H_\phi := \langle \nabla \phi, \cdot \rangle \), follows by an analogous argument.

Lemma 4.2. Let \( H \in L^2(\Gamma \to \mathbb{R}^d; \mu) \). Then \( (L_H, \dot{D}) \) is dissipative in any \( L^q \), \( q \in [1, 2] \).

Proof. Let \( \ddot{\mathfrak{g}} \in \dot{D} \). For \( \varepsilon > 0 \) let \( \varphi_\varepsilon \in C^1(\mathbb{R}) \) be such that \( \varphi_\varepsilon(t) = 0 \) for \( t \leq 0 \), \( \varphi_\varepsilon'(t) = t^\varepsilon^{-1} \) for \( t \geq \varepsilon \). Then the function \( \varphi_\varepsilon \circ \ddot{\mathfrak{g}} \) is differentiable w.r.t. \( \xi \) and decreases as well as its derivative quickly at \( \xi = \infty \). By integration by parts in the \( \xi \)-directions we obtain

\[
\int_{\mathbb{R}^d \times \Gamma} ((H - H_\phi), \nabla_\xi (\varphi_\varepsilon \circ \ddot{\mathfrak{g}}))_{\mathbb{R}^d} d\mu_\phi = 0.
\]

Using the chain rule and letting \( \varepsilon \to 0 \) we obtain by Lebesgue’s dominated convergence theorem \( \int_{\mathbb{R}^d \times \Gamma} (L_H - H_{\text{coupl}}^{\Gamma, \mu}) \ddot{\mathfrak{g}} \cdot (\ddot{\mathfrak{g}}^+) q^{-1} d\mu_\phi = 0 \), where \( \ddot{\mathfrak{g}}^+ \) denotes the positive part of \( \ddot{\mathfrak{g}} \). Adding the same equality with \( \ddot{\mathfrak{g}} \) replaced by \( -\ddot{\mathfrak{g}} \), we obtain \( \int_{\mathbb{R}^d \times \Gamma} (L_H - H_{\text{coupl}}^{\Gamma, \mu}) \ddot{\mathfrak{g}} \cdot |\ddot{\mathfrak{g}}|^q q^{-1} \text{sign}(\ddot{\mathfrak{g}}) d\mu_\phi = 0 \). This implies the assertion, since \( H_{\text{coupl}}^{\Gamma, \mu}|_{\dot{D}} \) is the restriction of an \( m \)-dissipative operator in \( L^q \). \( \square \)

We consider the operators \( L_H \) on the domains \( \dot{D}, D := C^\infty_0(\mathbb{R}^d) \otimes D(H_{\text{env}}^{\Gamma, \mu}), \text{ and } \dot{D} := F^{-1}(C^\infty_0(\mathbb{R}^d)) \otimes D(H_{\text{env}}^{\Gamma, \mu}) \). Here \( \mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \) denotes the Fourier transform. \( L_H \) is dissipative on all these domains in all \( L^q \), \( q \in [1, 2] \), and hence closable. Moreover, since \( C^\infty_0(\mathbb{R}^d) \) and \( F^{-1}(C^\infty_0(\mathbb{R}^d)) \) are dense in \( S(\mathbb{R}^d) \) w.r.t. its usual Frechet topology and the latter is continuously and densely embedded in any Sobolev space \( H^{m,p}(\mathbb{R}^d), m \in \mathbb{N}_0, p \in [1, \infty) \), we obtain the following lemma (which is mentioned here rather for completeness and for convenience of the reader).

Lemma 4.3. Let \( H \in L^2(\Gamma \to \mathbb{R}^d; \mu) \). Then the closures of \( (L_H, \dot{D}), (L_H, \dot{D}), \) and \( (L_H, \dot{D}) \) (exist and) coincide in all \( L^q, q \in [1, 2] \).

For a moment let us restrict to the \( L^2 \)-setting. Below we sometimes switch to the complexified function spaces. By \( \mathcal{F}_\xi : L^2(\mathbb{R}^d \times \Gamma; \mu) \to L^2(\mathbb{R}^d \times \Gamma; \mu) \) we denote the Fourier
transform in the first argument, given as the unique unitary operator fulfilling \( \mathcal{F}_\xi(f \otimes F) = (F f) \otimes F, \ F \in L^2(\mathbb{R}^d), \ f \in L^2(\mathbb{R}^d). \) For \( k \in \mathbb{N} \) let \( P_k : L^2 \to L^2 \) be the orthogonal projection given by multiplication with the indicator function \( 1_{B_k(0) \times \Gamma}. \) We define spaces \( L_k^2 := \mathcal{F}_\xi^{-1}(\text{Range}(P_k)), \ k \in \mathbb{N}, \) of \( L^2 \)-functions with compact band in the \( \xi \)-directions. The set \( \hat{D}_k := \mathcal{F}^{-1}_\xi(\mathcal{C}_0^\infty(B_k(0))) \otimes D(H_{\text{env}}^{\Gamma,\mu})_b \) is dense in \( L_k^2. \)

Note that for \( H \in L^2(\Gamma \to \mathbb{R}^d, \mu) \) the unitary operator \( \mathcal{F}_\xi \) transforms \((L_H, \hat{D})\) into the operator \((\hat{L}_H, \hat{D})\), given by

\[
\hat{L}_H \hat{\mathfrak{s}}(\xi, \gamma) := H_{\text{env}}^{\Gamma,\mu} \hat{\mathfrak{s}}(\xi, \gamma) - |\xi|^2_2 \hat{\mathfrak{s}}(\xi, \gamma) + 2i(\xi, \nabla_\gamma^T \hat{\mathfrak{s}}(\xi, \gamma))_{\mathbb{R}^d} - i(H(\gamma), \xi)_{\mathbb{R}^d} \hat{\mathfrak{s}}(\xi, \gamma), \quad (4.3)
\]

\( \hat{\mathfrak{s}} \in \hat{D}, (\xi, \gamma) \in \mathbb{R}^d \times \Gamma. \) This shows that for \( H \in L^2(\Gamma \to \mathbb{R}^d, \mu) \) the operator \((L_H, \hat{D}_k)\) is well-defined as an operator in \( L_k^2, k \in \mathbb{N}. \)

**Lemma 4.4.** Let \( H \in L^\infty(\Gamma \to \mathbb{R}^d, \mu). \) Then for every \( k \in \mathbb{N} \) the operator \((L_H, \hat{D}_k)\) is essentially m-dissipative in \( L_k^2. \) It follows that \((L_H, \hat{D})\) (hence also \((L_H, D)\)) is essentially m-dissipative in \( L^2(\mathbb{R}^d \times \Gamma; \hat{\mu}). \)

(Recall that essential m-dissipativity of an operator \((A, D(A))\) on a Banach space means that \( A \) is dissipative and \( \text{Range}(I - A) \) is dense in \( X. \) The well-known Lumer-Phillips theorem implies that in this case the closure of \( A \) is the unique extension of \( A \) which generates a strongly continuous contraction semigroup.)

**Proof.** The second assertion follows from the first, since the set \( \bigcup_{k \in \mathbb{N}} L_k^2 \) is dense in \( L^2. \) To prove the first assertion, we note that the operator \((H_{\text{env}}^{\Gamma,\mu}, \hat{D}_k)\) is essentially self-adjoint as an operator in \( L_k^2 \) and hence also essentially m-dissipative in this space. We prove that the (antisymmetric, hence dissipative) operator \((L_H - H_{\text{env}}^{\Gamma,\mu}, \hat{D}_k)\) is Kato bounded by \( H_{\text{env}}^{\Gamma,\mu} \) with bound 0. The assertion then follows from standard perturbation theory. The easiest way to show the Kato boundedness is to consider the images of \( L_H \) and \( H_{\text{env}}^{\Gamma,\mu} \) w.r.t. \( \mathcal{F}_\xi, \) which are defined on \( D_k := C_0^\infty(B_k(0)) \otimes D(H_{\text{env}}^{\Gamma,\mu}_b). \) Using (4.3) we find that

\[
\|(L_H - H_{\text{env}}^{\Gamma,\mu}) \hat{\mathfrak{s}}\|_{L^2} \leq k^2 \|\mathfrak{s}\|_{L^2} + 2k \|
abla_\gamma^T \mathfrak{s}\|_{L^2(\mathbb{R}^d \times \Gamma \to \mathbb{R}^d; \hat{\mu})} + k \||H||_{L^\infty(\Gamma; \mu)} \|\hat{\mathfrak{s}}\|_{L^2}
\]

holds for any \( \hat{\mathfrak{s}} \in D_k. \) Since

\[
\|\nabla_\gamma^T \hat{\mathfrak{s}}\|_{L^2(\mathbb{R}^d \times \Gamma \to \mathbb{R}^d; \hat{\mu})} \leq \epsilon_{\text{env}}^{\Gamma,\mu}(\mathfrak{s}, \hat{\mathfrak{s}})
\]

\[
= - (H_{\text{env}}^{\Gamma,\mu} \hat{\mathfrak{s}}, \hat{\mathfrak{s}})_{L^2} \leq \|H_{\text{env}}^{\Gamma,\mu} \hat{\mathfrak{s}}\|_{L^2} \|\hat{\mathfrak{s}}\|_{L^2} \leq \frac{1}{2} \left( \frac{1}{C} \|H_{\text{env}}^{\Gamma,\mu} \mathfrak{s}\|_{L^2} + C \|\hat{\mathfrak{s}}\|_{L^2} \right)^2
\]

for any \( 0 < C < \infty, \) the claimed Kato-boundedness follows. \qed

**Theorem 4.5.** Let \( \phi \) fulfill (SS), (LR), (I) and (DLp) for some \( p \in (d, \infty] \cap [2, \infty) \) and let \( \mu \in \mathcal{G}^{ap}_{dB}(\Phi, \sigma_\phi). \) Then \( H_{\phi} := \langle \nabla \phi, \cdot \rangle \in L^p(\Gamma \to \mathbb{R}^d; \mu) \) and if \( q \in [1, 2] \) is such that \( \frac{1}{q} = \frac{1}{2} + \frac{1}{p}, \) the operator \((H_{\text{coup}}^{\Gamma,\mu}, \hat{D}) = (L_H, \hat{D})\) is essentially m-dissipative in \( L^q(\mathbb{R}^d \times \Gamma; \hat{\mu}). \)

**Proof.** For the first assertion see Lemma 2.2. Let \( \chi \in C_0^\infty(\mathbb{R}^d) \) be such that \( 1_{[-1,1]^d} \leq \chi \leq 1_{[-2,2]^d} \) and define \( \chi_K := \chi(K^{-1}) \) for \( K \in \mathbb{N}. \) Consider each \( \chi_K \) as a function on \( \mathbb{R}^d \times \Gamma \) by setting \( \chi_K(\xi, \gamma) := \chi_K(\xi), (\xi, \gamma) \in \mathbb{R}^d \times \Gamma. \) For \( n \in \mathbb{N}, \) define \( H_n := H_{\phi} \cdot 1_{\{1(\|H_n\|), \leq n\}}. \) Then \( H_n \in L^\infty(\Gamma \to \mathbb{R}^d; \mu). \) By dissipativity of \( L_{H_{\phi}} \) it is sufficient to prove that \((1 - L_{H_{\phi}})D\) is dense in \( L^q, \) and by denseness of \( \hat{D} \) in \( L^q \) this reduces to finding an approximate solution
\[ \mathfrak{F} \in D \text{ of } (1 - L_{H_n}) \mathfrak{F} = \mathfrak{G} \text{ for any } \mathfrak{G} \in \mathcal{D} = \bigcup_{k \in \mathbb{N}} \mathcal{D}_k. \text{ Thus, let } k \in \mathbb{N}, 0 \neq \mathfrak{G} \in \mathcal{D}_k \text{ and } \varepsilon > 0. \text{ For any } \mathfrak{F} \in \mathcal{D}_k \text{ and } K \in \mathbb{N} \text{ it holds}\]

\[
\|((1 - L_{H_0}) \chi_K \mathfrak{F} - \mathfrak{G})\|_{L^q} \\
\leq \|\chi_K ((1 - L_{H_0}) \mathfrak{F} - \mathfrak{G})\|_{L^q} + \delta_K + 2\|\nabla_\xi \chi_K, (\nabla_\gamma^\Gamma - \nabla_\xi) \mathfrak{F}\|_{L^q} \\
+ \|\mathfrak{F}(H_\phi, \nabla_\xi \chi_K)\|_{L^q} + \|\mathfrak{F} \cdot \Delta_\xi \chi_K\|_{L^q},
\]

where \(\delta_K := \|((1 - \chi_K) \mathfrak{G})\|_{L^q}\). Setting \(\theta_K := \|\Delta_\xi \chi_K\|_{L^p(\mathbb{R}^d)} + \|\nabla_\chi K|_2\|_{L^p(\mathbb{R}^d)}\) and using the H"older inequality, we find that the right-hand side can be estimated by

\[
\|\chi_K ((1 - L_{H_0}) \mathfrak{F} - \mathfrak{G})\|_{L^q} + \delta_K \\
+ \theta_K \left(2\|\nabla_\gamma^\Gamma - \nabla_\xi \mathfrak{F}\|_{L^2(\mathbb{R}^d \times \Gamma \rightarrow \mathbb{R}^{d, \hat{\beta}})} + \|\|H_\phi\|_{2\|L^p(\Gamma, \mu)}\|\mathfrak{F}\|_{L^2} + \|\mathfrak{F}\|_{L^2}\right).
\]

Let \(\mathfrak{F} \in L^q\). Note that by dissipativity of \(L_{H_n}\), we have \(\|\mathfrak{F}\|_{L^2} \leq \|((1 - L_{H_n}) \mathfrak{F})\|_{L^2}\) and by antisymmetry of \(L_{H_n} - H_{\text{coup}}^{\Gamma, \hat{\beta}}\) it follows

\[
\|\nabla_\gamma^\Gamma - \nabla_\xi \mathfrak{F}\|_{L^2(\mathbb{R}^d \times \Gamma \rightarrow \mathbb{R}^{d, \hat{\beta}})} \leq C_{\text{coup}}^{\Gamma, \hat{\beta}}(\mathfrak{F}, \mathfrak{F}) \leq ((1 - H_{\text{coup}}^{\Gamma, \hat{\beta}}) \mathfrak{F}, \mathfrak{F})_{L^2} \\
= ((1 - L_{H_n}) \mathfrak{F}, \mathfrak{F})_{L^2} \leq \|((1 - L_{H_n}) \mathfrak{F})\|_{L^2} \|\mathfrak{F}\|_{L^2} \leq \|((1 - L_{H_n}) \mathfrak{F})\|_{L^2}^2.
\]

Setting \(\tilde{\theta}_K := \theta_K (3 + \|\|H_\phi\|_{2\|L^p(\Gamma, \mu)}\|\mathfrak{F}\|_{L^2})\) we obtain

\[
\|((1 - L_{H_0}) \chi_K \mathfrak{F} - \mathfrak{G})\|_{L^q} \\
\leq \|\chi_K ((1 - L_{H_0}) \mathfrak{F} - \mathfrak{G})\|_{L^q} + \delta_K + \tilde{\theta}_K \|((1 - L_{H_n}) \mathfrak{F})\|_{L^2} \\
\leq \|\chi_K ((1 - L_{H_n}) \mathfrak{F} - \mathfrak{G})\|_{L^q} + \|\mathfrak{F} \mathfrak{G} - H_n|_2\|_{L^p} \|\nabla_\xi \mathfrak{F}\|_{L^2} + \delta_K + \tilde{\theta}_K \|((1 - L_{H_n}) \mathfrak{F})\|_{L^2} \\
\leq C_K \|((1 - L_{H_n}) \mathfrak{F} - \mathfrak{G})\|_{L^2} + kC_K \|\|H_\phi - H_n\|_{2\|L^p(\Gamma, \mu)}\|\mathfrak{F}\|_{L^2} \|((1 - L_{H_n}) \mathfrak{F})\|_{L^2} + \delta_K + \tilde{\theta}_K \|((1 - L_{H_n}) \mathfrak{F})\|_{L^2},
\]

where \(C_K := \|\chi_K\|_{L^p(\mathbb{R}^d)}\). For the last estimate above it is crucial to have \(\mathfrak{F} \in \mathcal{D}_k\). Note that \(\tilde{\theta}_K \to 0\) as \(K \to \infty\). Hence, we may fix \(K \in \mathbb{N}\) large enough such that \(\delta_K \leq \varepsilon/4\) and \(\tilde{\theta}_K \leq \frac{\varepsilon}{\delta_K \|\mathfrak{F}\|_{L^2}}\). Then we choose \(n \in \mathbb{N}\) large enough such that \(\|\|H_\phi - H_n\|_{2\|L^p(\Gamma, \mu)}\| \leq \frac{\varepsilon}{2\delta_K \|\mathfrak{F}\|_{L^2}}\). Finally, according to Lemma 4.3 we may choose \(\mathfrak{F} \in \mathcal{D}_k\) such that \(\|((1 - L_{H_n}) \mathfrak{F} - \mathfrak{G})\|_{L^2} \leq \frac{\varepsilon}{4}\) and \(\|((1 - L_{H_n}) \mathfrak{F})\|_{L^2} \leq 2\|\mathfrak{F}\|_{L^2}\). It follows that

\[
\|((1 - L_{H_0}) \chi_K \mathfrak{F} - \mathfrak{G})\|_{L^q} \leq \varepsilon,
\]

and since \(\chi_K \mathfrak{F} \in D\), the theorem is shown.

\[\square\]

**Remark 4.6.** Let \(\mu \in \mathcal{G}^{qc}(\Phi_\phi, \varpi_\phi)\) and consider the situation of Theorem 4.5 above. Let \(M^{\text{env}}, M^{\text{coup}}\) be the processes from Section 3. Assume there is another conservative right process

\[
M = (\Omega, F, (F_t)_{t \geq 0}, (X_t^{(1)}, X_t^{(2)})_{t \geq 0}, (P_{(\xi, \gamma)})(\xi, \gamma) \in \mathbb{R}^d \times \Gamma)
\]

(as e.g. defined in \(\text{[MR92]}\)) with state space \(\mathbb{R}^d \times \Gamma\) and continuous paths, such that for Lebesgue a.e. \(\xi \in \mathbb{R}^d\) we have that \(P_{d_x \otimes \mu} \circ (X^{(2)})^{-1}\), considered as law on \(C([0, \infty), \Gamma)\), coincides with \(P_{\mu^{\text{env}}}\), and moreover \(P_{d_x \otimes \mu}\) solves the martingale problem for \(L^{\text{coup}}\) on \(C^0_0(\mathbb{R}^d) \otimes FC^0_b(D, \Gamma)\). Then with some effort (applying also the proof of Lemma 5.5 to \(M\)) one shows that \(d_\xi \otimes \mu = \hat{\mu}\) is an invariant measure for \(M\) and that e.g. for some bounded probability density \(h \in L^1(\mathbb{R}^d)\) the law \(P_{h \cdot d_x \otimes \mu}\) solves the martingale problem for
\( C_0^\infty(\mathbb{R}^d) \otimes D(H_{conv}^{\iota,\mu})_b \). By invariance the transition semigroup associated with \( M \) gives rise to a strongly continuous contraction semigroup in each \( L^r(\mathbb{R}^d \times \Gamma; \hat{\mu}) \), \( r \in [1, \infty) \), and by the arguments used in the proof of [AR95, Theorem 3.5] and by Theorem 4.4 this semigroup is equal to \( (T_t^{coup})_{t \geq 0} \), hence \( P(\xi, \gamma) = P^{coup}(\xi, \gamma) \) for \( \hat{\mu} \)-a.e. \( (\xi, \gamma) \in \mathbb{R}^d \times \Gamma \). Thus the above uniqueness result shows that, given the environment process is \( M^{conv} \), there is only one possible coupled process. Whether there is also only one possible environment process is another - much more difficult - question, which we do not attack here.

4.2. Proof of (4.1). Let us at first consider a modified setting in which (4.1) is immediate: In this modified setting the tagged particle is only allowed to move in the cube \((-\kappa, \kappa)^d\), \( \kappa \in \mathbb{N} \), with periodic boundary, i.e. in the \( d \)-dimensional torus. (In fact, we do not consider the corresponding stochastic dynamics, but instead stay on the level of functional analytic objects.)

We start from the nonnegative definite bilinear form \((E_{coup,\kappa}, C^\infty_{per}([-\kappa, \kappa]^d) \otimes FC_b^\infty(D, \Gamma))\) on \( L^2([-\kappa, \kappa]^d \times \Gamma; d\xi \otimes \mu) =: L^2_{\kappa}, \) given by

\[
E_{coup,\kappa}(\tilde{\mathfrak{F}}, \mathfrak{G}) := \int_{\Gamma} \int_{[-\kappa, \kappa]^d} (\nabla^\Gamma \tilde{\mathfrak{F}}(\xi, \gamma), \nabla^\Gamma \mathfrak{G}(\xi, \gamma)) \,\mu(\gamma) \,d\xi d\mu(\gamma) + \int_{\Gamma} \int_{[-\kappa, \kappa]^d} (\langle \nabla^\Gamma \mathfrak{G}, \nabla^\Gamma \mathfrak{F} \rangle, (\nabla^\Gamma \mathfrak{F} - \nabla^\Gamma \gamma) \mathfrak{G}(\xi, \gamma))_{\mathbb{R}^d} \,d\xi d\mu(\gamma)
\]

for \( \tilde{\mathfrak{F}}, \mathfrak{G} \in C^\infty_{per}([-\kappa, \kappa]^d) \otimes FC_b^\infty(D, \Gamma) \). Here \( C^\infty_{per}([-\kappa, \kappa]^d) \) denotes the restrictions of \( 2\kappa \)-periodic (in all arguments), infinitely often differentiable functions on \( \mathbb{R}^d \) to the cube \([-\kappa, \kappa]^d\). As in the proof of [FG11, Theorem 4.15] one obtains for any \( f \otimes F \in C^\infty_{per}([-\kappa, \kappa]^d) \otimes FC_b^\infty(D, \Gamma), \kappa \in \mathbb{N} \), that

\[
E_{coup,\kappa}(\tilde{\mathfrak{F}}, \mathfrak{G}) = -(L_{coup,\kappa} \tilde{\mathfrak{F}}, \mathfrak{G})_{L^2_{\kappa}},
\]

where \( L_{coup,\kappa} = L^\Gamma_{coup} + \Delta_{\tau} - 2(\nabla^\Gamma_{\gamma}, \nabla^\Gamma_{\ell})_{\mathbb{R}^d} + (\langle \nabla^\Gamma \phi, \cdot \rangle, \nabla^\Gamma \gamma)_{\mathbb{R}^d} \). (For proving this, periodicity of the functions in the first argument is only needed to ensure that the integrations by parts do not produce boundary terms.) It follows from [MR92, Proposition I.3.3] that the form \( E_{coup,\kappa} \) is closable and its generator \((H_{coup,\kappa}, (H_{coup,\kappa})^\prime))\) is the Friedrichs extension of the operator \((L_{coup,\kappa}, C^\infty_{per}([-\kappa, \kappa]^d) \otimes FC_b^\infty(D, \Gamma))\) defined above. It is not difficult to verify that the closure of \( E_{coup,\kappa} \) is a Dirichlet form, which implies that the associated strongly continuous contraction semigroup \((T_t^{coup})_{t \geq 0}\), given by \( T_t^{coup} := \exp(tH_{coup,\kappa}), \ t \geq 0, \) is sub-Markovian and gives rise to contraction semigroups \((T_t^{\iota,\mu})_{t \geq 0}\) on \( L^\infty_{\kappa} := L^\infty([-\kappa, \kappa]^d \times \Gamma; d\xi \otimes \mu) \) (with analogous properties as the semigroups \((T_t^{\iota,\mu})_{t \geq 0}\), \( q \in [1, \infty) \)). Analogously to the proof of Lemma 4.3 we have \( C^\infty_{per}([-\kappa, \kappa]^d) \otimes D(H_{coup,\kappa})_b \subset D(H_{coup,\kappa}) \) and \( H_{coup,\kappa} \) has on this set the same form as \( L_{coup,\kappa} \). For \( F \in FC_b^\infty(D, \Gamma) \) it is easily seen that \( t \mapsto 1 \otimes T_t^{\iota,\mu} F \) is a solution of the abstract Cauchy problem for \( H_{coup,\kappa} \) with initial value \( 1 \otimes F \) as well as \( t \mapsto T_t^{coup}(1 \otimes F) \). Well-posedness of this abstract Cauchy problem implies the following lemma.

**Lemma 4.7.** For any \( F \in L^2(\Gamma; \mu) \) and \( t \geq 0 \) it holds

\[
T_{t,2}^{coup,\kappa}(1 \otimes F) = 1 \otimes T_{t,2}^{\mu,\kappa} F.
\]

Note that for the above conclusion it is crucial that \( 1 \otimes F \) is square integrable in the modified setting (in \( L^2_{\kappa}\)). In the original setting (in \( L^2 \)) this is not the case, which is why (4.1) is not immediate.

We now use the classical semigroup convergence theorem. For convenience we use the formulation from [Tro58, Kur09]. Let \( q \in [1, 2] \). For \( \kappa \in \mathbb{N} \) define \( P_\kappa : L^q \rightarrow L^{q,\kappa} \) by \( P_\kappa \tilde{\mathfrak{F}} := \mathfrak{F}|_{[-\kappa, \kappa]^d \times \Gamma}, \ \tilde{\mathfrak{F}} \in L^q \). Since \( \|P_\kappa \tilde{\mathfrak{F}}\|_{L^\infty_{\kappa}} \rightarrow \|\tilde{\mathfrak{F}}\|_{L^q} \) as \( \kappa \rightarrow \infty \), we have \( L^{q,\kappa} \rightarrow L^q \) in...
the sense of the abovementioned papers. One says that a sequence \((\mathfrak{F}_n)_{n \in \mathbb{N}}\) with \(\mathfrak{F}_n \in L^q, \kappa\), \(\kappa \in \mathbb{N}\), converges to some \(\mathfrak{F} \in L^q\), if \(\lim_{n \to \infty} \|P_n \mathfrak{F} - \mathfrak{F}_n\|_{L^q, \kappa} = 0\). A uniformly bounded sequence \((A_n)_{n \in \mathbb{N}}\) of bounded linear operators \(A_n\) on \(L^q, \kappa\), \(\kappa \in \mathbb{N}\), is said to converge to a bounded linear operator \(A\) on \(L^q\), if \(A_n f\) converges to \(Af\) as \(n \to \infty\) for all sequences \((f_n)_{n \in \mathbb{N}}\) converging to \(f\) as described above.

**Lemma 4.8.** Assume that \((H^\kappa_{\text{coup}}, C^\infty_0(\mathbb{R}^d) \otimes D(L_{\text{env}})_b)\) is essentially m-dissipative in \(L^q\) for some \(q \in [1,2]\). Then for \(t \geq 0\) we have \(T^\kappa_{\text{coup}} \to T^\kappa_{\text{coup}}\) in the abovedescribed sense as \(\kappa \to \infty\).

**Proof.** For any \(\mathfrak{F} \in C^\infty_0(\mathbb{R}^d) \otimes D(H^\kappa_{\text{env}})_b\) there exists \(\kappa_0 \in \mathbb{N}\) such that \(P_{\kappa_0} \mathfrak{F} \in C^\infty_0((-\kappa, \kappa)^d) \otimes D(H^\kappa_{\text{env}}) \subset D(H^\kappa_{\text{coup}})_b\) and \(H^\kappa_{\text{coup}} P_{\kappa_0} \mathfrak{F} = P_{\kappa_0} H^\kappa_{\text{coup}} \mathfrak{F}\) for all \(\kappa \geq \kappa_0\). Thus \((H^\kappa_{\text{coup}} P_{\kappa_0} \mathfrak{F})_{\kappa \geq \kappa_0}\) converges to \(H^\kappa_{\text{coup}} \mathfrak{F}\) as \(\kappa \to \infty\) along \(L^q, \kappa\) \(\to L^q\). I.e., the extended limit of the sequence of the operators \((H^\kappa_{\text{coup}}, D(H^\kappa_{\text{coup}})), \kappa \in \mathbb{N}\), extends the essentially m-dissipative operator \((H^\kappa_{\text{coup}}), C^\infty_0(\mathbb{R}^d) \otimes D(H^\kappa_{\text{env}})_b\) and the assertion follows from [Kur69, Theorem 2.1] (where the reader also finds the definition of the “extended limit”).

**Theorem 4.9.** Let \(\phi\) fulfill the assumptions (SS), (LR), (I) and (DL\(^p\)) for some \(p \in (d, \infty) \cap [2, \infty)\) and let \(\mu \in G^\infty_0(\Phi, \sigma_\phi)\). Then for any \(F \in L^\infty(\Gamma; \mu)\) and any \(t \geq 0\) equation (4.1) is valid.

**Proof.** We denote by \(\langle \cdot, \cdot \rangle_\kappa\) the dualization between \(L^{1, \kappa}\) and \(L^{\infty, \kappa}\), \(\kappa \in \mathbb{N}\), and by \(\langle \cdot, \cdot \rangle_\infty\) the dualization between \(L^1\) and \(L^\infty\). Let w.l.o.g. \(0 \leq F \leq 1\). Choose \((\chi_n)_{n \in \mathbb{N}} \subset C^\infty_0(\mathbb{R}^d)\) such that \(0 \leq \chi_n \uparrow 1\) as \(n \to \infty\). Then by Lemma 4.8 and Theorem 4.5 we have for some \(q \in [1,2]\)

\[\lim_{n \to \infty} (P_n \mathfrak{F}_n(\chi_n \otimes F) - T^\kappa_{\text{coup}}(\chi_n \otimes F))_{\kappa \to \infty} = 0\]

as \(\kappa \to \infty\) for all \(n \in \mathbb{N}\). Here we extend \(T^\kappa_{\text{coup}}(\chi_n \otimes F)\) to \(\mathbb{R}^d\) by setting it to 0 on \((\mathbb{R}^d \setminus [-\kappa, \kappa]) \times \Gamma\). It follows that for any nonnegative \(\mathfrak{G} \in L^1 \cap L^\infty\) and \(n \in \mathbb{N}\) it holds

\[\langle P_n \mathfrak{G}, T^\kappa_{\text{coup}}(\chi_n \otimes F) \rangle_{\kappa \to \infty} \to \langle \mathfrak{G}, T^\kappa_{\text{coup}}(\chi \otimes F) \rangle_{\infty}\]

as \(\kappa \to \infty\). Thus

\[\lim_{\kappa \to \infty} (P_n \mathfrak{G}, T^\kappa_{\text{coup}}(1 \otimes F))_{\kappa} \geq (\mathfrak{G}, T^\kappa_{\text{coup}}(1 \otimes F))_{\infty}\]

since \((T^\kappa_{\text{coup}})_{\kappa \geq 0}\) is conservative as well as \((T_{t, 2}^\kappa)_{\kappa \geq 0}\), by replacing \(F\) by \(1 - F\) we obtain

\[\limsup_{\kappa \to \infty} (P_n \mathfrak{G}, T^\kappa_{\text{coup}}(1 \otimes F))_{\kappa} \leq (\mathfrak{G}, T^\kappa_{\text{coup}}(1 \otimes F))_{\infty}\]

hence the lim inf and lim sup coincide and we have convergence. Choosing a function \(\mathfrak{G} = g \otimes G\), with nonnegative \(g \in C_0(\mathbb{R}^d)\) and \(G \in L^\infty(\Gamma; \mu)\), and applying Lemma 4.7, we conclude that

\[\langle \mathfrak{G}, T^\kappa_{t, 2}(1 \otimes F) \rangle_{\infty} = \lim_{\kappa \to \infty} \int_{[-\kappa, \kappa]^d} g \, d\xi (G, T^\kappa_{t, 2} F)_{L^2(\Gamma; \mu)} = \langle \mathfrak{G}, 1 \otimes T^\kappa_{t, 2} F \rangle_{\infty}\]

Since the linear span of functions \(\mathfrak{G}\) as above is dense in \(L^1\), the assertion follows. □

5. Representation of the displacement of the tagged particle in terms of the environment process

The aim of this section is to prove (under the assumptions stated below) that all information on the evolution of the tagged particle with the exception of its initial position, can be obtained from the environment process, at least if suitable initial distributions are chosen.
for both processes. We make this precise: Let $0 \leq h \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be a probability density w.r.t. Lebesgue measure. For $\mu \in G^\mu_{ibp}(\Phi, \sigma)$ we define the probability measure $h\mu := (h \, d\xi) \otimes \mu$ on $\mathbb{R}^d \times \Gamma$. Then $P^\mu_{h\mu}(A) := \int_{\mathbb{R}^d \times \Gamma} P^\mu_{\xi,\gamma}(A) d(h\mu)(\xi, \gamma), A \in \Omega^{comp}$, can be considered as a law on $C(0, \infty); \mathbb{R}^d \times \Gamma$. The analogously defined law $P^\mu_{\mu_{env}}$ can be considered as a law on $C(0, \infty); \mathbb{R}^d \times \Gamma$.

Theorem 5.1. Let $\phi$ fulfill (SS), (LR), (I) and (DLP) for some $p \in (d, \infty) \cap [2, \infty)$, let $\mu \in G^{\mu_{ibp}}_{ibp}(\Phi, \sigma)$, and let $P^\mu_{h\mu}$ and $P^\mu_{\mu_{env}}$ be laws on $C(0, \infty); \mathbb{R}^d \times \Gamma$, $C(0, \infty); \Gamma$, respectively, as described above. Then there exists a $C((0, \infty); \mathbb{R}^d)$-valued random variable $(\eta_t)_{t \geq 0}$ on $C((0, \infty); \Gamma)$ with the following properties:

(i) For all $0 \leq a < b$ the random variable $\eta_b - \eta_a$ coincides $P^\mu_{\mu_{env}}$-a.s. with a random variable that is measurable w.r.t. the $\sigma$-field generated by $\{\gamma_s | a \leq s \leq b\}$. In particular, $(\eta_t)_{t \geq 0}$ is adapted to the $P^\mu_{\mu_{env}}$-completion of the filtration $(\sigma(\gamma_s) | 0 \leq s \leq t)_{t \geq 0}$ relative to the Borel $\sigma$-field on $C(0, \infty); \Gamma$.

(ii) The distribution of the paths $(\eta_t)_{t \geq 0}$ under $P^\mu_{\mu_{env}}$ coincides with the distribution of $(\xi_t - \xi_0, \gamma_t)_{t \geq 0}$ under $P^\mu_{h\mu}$.

Proof. This theorem is a consequence of Lemmas 5.2 and 5.3 below. More precisely, Lemma 5.2 gives for each $0 \leq a < b$ an $\mathbb{R}^d$-valued random-variable $X_{[a,b]}$ on $C(0, \infty); \mathbb{R}^d \times \Gamma$ which is a function of $(\gamma_s)^{a \leq s \leq b}$ and coincides $P^\mu_{h\mu}$-a.s. with $\xi_b - \xi_a$. This random variable $X_{[a,b]}$ can therefore be defined as a measurable function $X_{[a,b]} = X_{[a,b]}((\gamma_s)^{a \leq s \leq b})$ on $C((0, \infty); \Gamma)$, and setting $\eta_t := X_{[0,t]}$ we find by Lemma 5.2 that for each $t$ the distribution of $(\eta_t, \gamma_t)$ under $P^\mu_{\mu_{env}}$ coincides with the distribution of $(\xi_t - \xi_0, \gamma_t)$ under $P^\mu_{h\mu}$. In particular $(\eta_t)_{t \geq 0}$ is $P^\mu_{\mu_{env}}$-continuous, so the desired family $(\eta_t)_{t \geq 0}$ can be chosen as a continuous version of $(\hat{\eta}_t)_{t \geq 0}$.

Lemma 5.2. Under the assumptions of Theorem 5.1, the image law $P^\mu_{h\mu}^{comp,2}$ of $P^\mu_{h\mu}$ under the projection $C((0, \infty); \mathbb{R}^d \times \Gamma) \ni (\xi_t, \gamma_t)_{t \geq 0} \mapsto (\gamma_t)_{t \geq 0} \subset C((0, \infty); \Gamma)$ coincides with $P^\mu_{\mu_{env}}$.

Proof. It suffices to compare the finite dimensional distributions of both laws. Let $F_1, \ldots, F_k : \Gamma \to \mathbb{R}$ be bounded $\mathcal{B}(\Gamma)$-measurable functions and let $0 \leq t_1 < \cdots < t_k < \infty, k \in \mathbb{N}$. Then by Theorem 4.2

$E^\mu_{h\mu}^{comp,2}[F_1(\gamma_{t_1}) \cdots F_k(\gamma_{t_k})]$

$= \int_{\mathbb{R}^d \times \Gamma} T_{t_1}^{comp} \left( (1 \otimes F_1) \cdot T_{t_2}^{comp} \left( \cdots (1 \otimes F_{k-1}) \cdot T_{t_k-t_{k-1}}^{comp} (1 \otimes F_k) \cdots \right) \right) d(h\mu)$

$= \int_{\mathbb{R}^d} T_{t_1}^{env} \left( F_1 \cdot T_{t_2-t_1}^{env} \left( \cdots (F_{k-1} \cdot T_{t_k-t_{k-1}}^{env} F_k) \cdots \right) \right) d\mu = E^\mu_{\mu_{env}}[F_1(\gamma_{t_1}) \cdots F_k(\gamma_{t_k})],$

which shows the assertion.

The above lemma allows us to work only with the coupled process for constructing the displacement of the tagged particle from the environment process. For doing so we need some technical preparations: At first we note that the (DLP)-assumption implies that

$q_n := n^{-1} \left( 1 + \int_0^n \int_{\mathbb{R}^d \setminus [-r,c]^d} |\nabla \phi(x)|_2 e^{-\phi(x)} \, dx \, dr \right) \to 0$
as \( n \to \infty \). Thus, setting \( r_n := \sqrt{q_n} \), \( n \in \mathbb{N} \), it follows that \( r_n = q_n/r_n \to 0 \) as \( n \to \infty \). For \( n \in \mathbb{N} \) set \( e_n^0 : [-n,n]^d \to \mathbb{R} \). Then \( \langle e_n^0, \cdot \rangle : \Gamma \to \mathbb{R} \) counts the points in \([-n,n]^d\) of a given configuration.

**Lemma 5.3.** Let \( \mu \in \mathcal{G}_{d;p}(\Phi_\phi, z \sigma_\rho) \). Then

\[
\lim_{n \to \infty} \frac{\langle e_n^0, \gamma \rangle}{r_n n^d} = \infty \quad \text{for } \mu\text{-a.e. } \gamma \in \Gamma. \tag{5.1}
\]

**Proof.** By Remark 2.3 and Lemma 2.4 we only need to show the assertions for measures \( \mu \in \exp \mathcal{G}^c_0(\Phi_\phi, z \, dx) \). For any such measure we have \( \int_\Gamma (1_{[0,1]^d}, \cdot) \, d\mu =: \rho > 0 \). Using the multidimensional ergodic theorem (one can e.g. directly apply [Pit42, Theorem 4]), one finds that

\[
f_n(\gamma) := \frac{1}{2n^d} (1_{[-n,n]^d}, \gamma) \to \hat{f}(\gamma)
\]

as \( n \to \infty \) where \( \hat{f} \) is the conditional expectation of \( f = (1_{[0,1]^d}, \cdot) \) w.r.t. the sub-\( \sigma \)-field of \( \mathcal{B}(\Gamma) \) of the sets which are \( \mu \)-a.e. invariant w.r.t. translations in \( \mathbb{R}^d \). By triviality of \( \mu \) on this \( \sigma \)-field we obtain \( \hat{f}(\gamma) = \rho \mu\text{-a.s.} \), and (5.1) follows. \( \square \)

**Lemma 5.4.** For \( 1 \leq i \leq d \) and \( n \in \mathbb{N} \) we define a function \( \tilde{Y}_i^n : \Gamma \to \mathbb{R} \) by

\[
\tilde{Y}_i^n(\gamma) := \frac{-1}{r_n n^d + \langle e_n^0, \gamma \rangle} \left( \partial_i \phi e_n^0, \gamma \right) + \left( \partial_i \phi, \gamma \right) \langle e_n^0, \gamma \rangle
\]

\[
+ \sum_{(x,y) \in \gamma} \partial_i \phi(x-y)(e_n^0(x) - e_n^0(y)) + \left( \partial_i \phi, \gamma \right),
\]

for \( \gamma \in \Gamma \) such that all sums converge absolutely (which is \( \mu \)-a.e. true). Then \( \tilde{Y}_i^n \to 0 \) in \( L^1(\Gamma; \mu) \) as \( n \to \infty \).

**Proof.** We first note that for \( n \in \mathbb{N} \) it holds \( \mu \)-a.e.

\[
\left| \frac{1}{r_n n^d + \langle e_n^0, \gamma \rangle} \langle \partial_i \phi e_n^0, \gamma \rangle \right| \leq \frac{\langle |\partial_i \phi|, \cdot \rangle}{1 + \langle e_n^0, \cdot \rangle}. \tag{5.2}
\]

Since the numerator on the right-hand side is integrable and by Lemma 5.3 the denominator converges \( \mu \)-a.s. to \( \infty \) as \( n \to \infty \), we obtain convergence of (5.2) to \( 0 \) in \( L^1(\Gamma; \mu) \). Moreover, Lemma 5.3 and \( \mu \)-integrability of \( \langle \partial_i \phi, \cdot \rangle \) imply

\[
\left( 1 - \frac{\langle e_n^0, \cdot \rangle}{r_n n^d + \langle e_n^0, \cdot \rangle} \right) \langle \partial_i \phi, \cdot \rangle \to 0
\]

in \( L^1(\Gamma; \mu) \) as \( n \to \infty \), so we are left to prove convergence to \( 0 \) of

\[
\frac{1}{r_n n^d + \langle e_n^0, \gamma \rangle} \sum_{(x,y) \subseteq \gamma} \partial_i \phi(x-y)(1_{[-n,n]^d}(x) - 1_{[-n,n]^d}(y)) \leq \frac{1}{r_n n^d} \sum_{(x,y) \subseteq \gamma} \left( 1_{[-n,n]^d}(x) 1_{\mathbb{R} e_n \setminus [-n,n]^d}(y) + 1_{[-n,n]^d}(y) 1_{\mathbb{R} e_n \setminus [-n,n]^d}(x) \right) |\partial_i \phi(x-y)|. \tag{5.3}
\]
Since $\mu$ fulfills an improved Ruelle bound (see Section 2), there exists a constant $C < \infty$ such that the $L^1(\Gamma; \mu)$-norm of the right-hand side can be estimated by

$$
C \int_{\mathbb{R}^d \times \mathbb{R}^d} 1_{[-n,n]^d}(x)1_{\mathbb{R}^d \setminus [-n,n]^d}(y) |\partial_i \phi(x-y)| e^{-\phi(x-y)} \, dx \, dy
\leq C \int_{[-n,n]^d} \int_{\mathbb{R}^d \setminus [-n,n]^d} |\partial_i \phi(x-y)| e^{-\phi(x-y)} \, dx \, dy
\leq 2dC(2n)^d-1 \int_0^\infty \int_{\mathbb{R}^d \setminus [-r,r]^d} |\partial_i \phi(y)| e^{-\phi(y)} \, dr.
$$

(Here we again used [KKO02 Theorem 4.1],) It follows that the right-hand side of (5.3) converges to 0 in $L^1(\Gamma; \mu)$. This completes the proof. 

Lemma 5.5. Under the assumptions of Theorem 5.1, for $0 \leq a < b$, $\xi_b - \xi_a$ is $\mathcal{P}_{h\hat{\mu}}^{\text{comp}}$-a.s. $\sigma(\{\gamma_s, a \leq s \leq b\})$-measurable (i.e. coincides except on a null set with a $\sigma(\{\gamma_s, a \leq s \leq b\})$-measurable random variable).

Proof. In [DMFGW89, p. 849] it is proposed to obtain the stochastic part of the uniform motion of the environment from a solution of (1.3) by taking the limit $\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N (B_i - B_i^0)$ (which together with the fact that the drift part is clearly measurable w.r.t. the environment shows the desired adaptedness/measurability property). We emphasize that this idea cannot be applied directly in the present setting, since the dynamics is not obtained as $(\xi_t)_{t \geq 0}$, $(y_i^t)_{i \geq 0}$ fulfilling (1.3), but as a solution of the martingale problem for the generator corresponding formally to the equations (1.2), (1.3). However, the general idea, namely obtaining the uniform motion of the environment by averaging (namely over the particles contained in subsets of $\mathbb{R}^d$) is applicable in this setting, but leads to the following proof, which is a bit more involved than one might expect.

Let $1 \leq i \leq d$. For $n \in \mathbb{N}$ and $\delta > 0$ choose functions $f_n^\delta, c_n^\delta \in C^\infty_0(\mathbb{R}^d)$ such that $f_n^\delta(x) = x_i$ and $c_n^\delta(x) = 1$ for $x = (x_1, \cdots, x_d) \in [-n, n]^d$, $f_n^\delta(x) = c_n^\delta(x) = 0$ for $x \in \mathbb{R}^d \setminus [-n-n, n+n]^d$ and set

$$
F_n^\delta := \frac{(f_n^\delta, \cdots)}{\tau_n \cdot n^d + (c_n^\delta, \cdots)}.
$$

Then $F_n^\delta \in FC^\infty_b(D, \Gamma)$. It follows from Theorem 5.4 that

$$
M_{t,n}^\delta := F_n^\delta(\gamma_t) - F_n^\delta(\gamma_0) + \xi_{i,t} - \xi_{i,0} - \int_0^t I_{\Gamma_n}^{\text{env}} F_n^\delta(\gamma_s) + \langle \partial_i \phi, \gamma_s \rangle \, ds,
$$

t \geq 0, defines a continuous local martingale under $\mathcal{P}_{h\hat{\mu}}^{\text{comp}}$ with quadratic variation process given by

$$
\langle M_{t,n}^\delta \rangle_t = 2 \int_0^t (\nabla F_n^\delta(\gamma_s), \nabla F_n^\delta(\gamma_s))_{\Gamma_n \times \Gamma} + \left( \nabla F_n^\delta(\gamma_s) - e_i, \nabla \gamma_n^\delta(\gamma_s) - e_i \right)_{\mathbb{R}^d} \, ds,
$$

t \geq 0, where $e_i$ denotes the $i$-th unit vector in $\mathbb{R}^d$. Since by Lemma 5.2 and invariance of $\mu$ w.r.t. $\mathcal{M}_h^{\text{env}}$ it holds

$$
E_{h\hat{\mu}}^{\text{comp}} \left[ \int_0^\infty e^{-t} Z(\gamma_s) \, ds \right] = E_{\mu}^{\text{env}} \left[ \int_0^\infty e^{-t} Z(\gamma_s) \, ds \right] \leq \int_0^\infty e^{-t} \| T_{t+1}^{\text{env}} Z \|_{L^1(\Gamma; \mu)} \, ds < \infty
$$

for any $Z \in L^1(\Gamma; \mu)$, we may replace the “local” in the previous statement by square integrable, see also Remark 3.7. We now ignore the increments of $(F_n^\delta(\gamma_t))_{t \geq 0}$ at times $t \geq 0$
where $\gamma_t \cap A^\delta_n \neq \emptyset$ with $A^\delta_n := \{ -n - \delta, n + \delta \} \setminus (-n, n)$: Let $H^\delta_n : \Gamma \to \{0, 1\}$ be the indicator function of the set $\{\gamma \in \Gamma \mid \gamma \cap A^\delta_n = \emptyset\}$ and set

$$\tilde{M}^{n,\delta}_t := \int_0^t H^\delta_n(\gamma_s) dM^s_n.$$

Then

$$\tilde{M}^{n,\delta}_t := \int_0^t H^\delta_n(\gamma_s) dM^s_n$$

$$= \int_0^t H^\delta_n(\gamma_s) d\xi_s - \int_0^t H^\delta_n(\gamma_s) \left( \Gamma_{\text{env}} F^\delta_n(\gamma_s) + (\partial_t \phi, \gamma_s) \right) ds$$

$$= \int_0^t H^\delta_n(\gamma_s) d\xi_s - \int_0^t H^\delta_n(\gamma_s) \tilde{Y}^n_1(\gamma_s) ds, \quad t \geq 0,$$

defines a martingale under $P_{\text{coup}}$ with quadratic variation process given by

$$\langle \tilde{M}^{n,\delta} \rangle_t = \int_0^t (H^\delta_n(\gamma_s))^2 d\langle M^n \rangle_s$$

$$= 2 \int_0^t H^\delta_n(\gamma_s) \left( (\nabla \Gamma F^\delta_n(\gamma_s), \nabla \Gamma F^\delta_n(\gamma_s))_{T_\Gamma} + \left( \nabla \Gamma F^\delta_n(\gamma_s) - e_i, \nabla \Gamma F^\delta_n(\gamma_s) - e_i \right)_{\mathbb{R}^d} \right) ds$$

$$= 2 \int_0^t H^\delta_n(\gamma_s) \left( \frac{(0, \gamma_s)}{(r_n n^d + c_n^d, \gamma_s)^2} + \frac{(0, \gamma_s)}{r_n n^d + c_n^d, \gamma_s} - 1 \right)^2 ds, \quad t \geq 0.$$

Below we use the following (standard) notion of convergence for continuous processes: If $P$ is a probability measure on a measurable space $(\Omega, \mathcal{B})$ and $(Z_t^n)_{t \geq 0}, n \in \mathbb{N}$, and $(Z_t)_{t \geq 0}$ are $C([0, \infty); \mathbb{R})$-valued random variables on $\Omega$, then we write $(Z_t^n)_{t \geq 0} \to (Z_t)_{t \geq 0}$ in $L^r(P)$, $r \in [1, \infty]$, iff for any $T > 0$ it holds $\int_0^T \sup_{0 \leq t \leq T} |Z_t^n - Z_t|^r dP \to 0$ as $n \to \infty$. We consider the limits in this sense of the processes occurring in $(5.4)$. First note that for any $T > 0$ and any $Y \in L^1(\Gamma; \mu)$ it holds by Lemma $5.2$ and invariance of $\mu$ w.r.t. $M^{\text{env}}$

$$E_{h\mu}^{\text{coup}} \left[ \int_0^T \left( H^\delta_n(\gamma_s) - 1 \right) Y(\gamma_s) ds \right] = \int_0^T E_{\mu}^{\text{env}}[\|H^\delta_n(\gamma_s) - 1\| Y(\gamma_s)]$$

$$= T\|H_n^\delta - 1\| Y \|L^1(\Gamma; \mu) \to 0 \tag{5.5}$$

as $\delta \to 0$ by Lebesgue’s dominated convergence theorem, since $1 - H^\delta_n$ decreases to the indicator function of the set $\{\gamma \in \Gamma \mid \gamma \cap [-n, n] \neq \emptyset\}$ which has $\mu$-measure $0$. This already implies that $\left( \int_0^T H^\delta_n(\gamma_s) \tilde{Y}^n_1(\gamma_s) ds \right)_{t \geq 0} \to \left( \int_0^T \tilde{Y}^n_1(\gamma_s) ds \right)_{t \geq 0}$ in $L^1(P_{h\mu}^{\text{coup}})$. Since for $\delta > \delta'$ the martingale $\left( \int_0^T H^\delta_n(\gamma_s) d(M^{n,\delta} - M^{n,\delta'})_s \right)_{t \geq 0}$ is easily verified to be equal to $0$ $P_{h\mu}^{\text{coup}}$-a.s. (by considering its quadratic variation process, which can be computed with the help of Theorem $3.6(1)$), by the Burkholder-Davis-Gundy inequality there is some $C < \infty$ such that for any $0 < T < \infty$ and $\delta > \delta'$

$$E_{h\mu}^{\text{coup}} \left[ \sup_{t \in [0,T]} \left| \int_0^t H^\delta_n(\gamma_s) dM^s_n - \int_0^t H^\delta(\gamma_s) dM^s_n \right|^2 \right]$$

$$= E_{h\mu}^{\text{coup}} \left[ \sup_{t \in [0,T]} \left| \int_0^t \left( H^\delta_n(\gamma_s) - H^\delta(\gamma_s) \right) dM^s_n \right|^2 \right]$$. 

$\mathcal{M}^n_{t} = \tilde{M}^n_t + \xi_{i,t} - \xi_{i,0} - \int_0^t \tilde{Y}^i_n(\gamma_s) ds, \quad t \geq 0$

Letting $n \to \infty$, it is shown analogously as (5.3) using Lemma 5.4 that $\left( \int_0^t Y^i_n(\gamma_s) ds \right)_{t \geq 0}$ converges to 0 in $L^1(\mathcal{P}_{h\mu}^{coup})$ and using Lemma 5.3 we find that the same holds for $\left( \langle \mathcal{M}^n \rangle_t \right)_{t \geq 0}$. By the Burkholder-Davis-Gundy inequality the latter implies that $\left( \mathcal{M}^n \right)_{t \geq 0} \to 0$ in $L^2(\mathcal{P}_{h\mu}^{coup})_{t \geq 0}$, and it follows that in $L^1$-sense $(-\tilde{F}^n_t)_{t \geq 0}$ converges as $n \to \infty$ to $(\xi_{i,t} - \xi_{i,0})_{t \geq 0}$. A subsequence converges a.s. and we denote the limit of such a subsequence by $(\eta_b)_{t \geq 0}$. In order to prove that $\eta_b - \eta_a$ is $\mathcal{P}_{h\mu}^{coup}$-a.s. $\sigma(\gamma_s \mid a \leq s \leq b)$-measurable, we only need to verify this measurability property for $\tilde{F}^{a,b}_{\mu} - \tilde{F}^{a,b}_{\eta}$. To verify this note first that $H^\delta_n$ is the indicator function of a closed set and can be represented as limit of continuous functions dominated by 1, implying that the stochastic integral defining $\tilde{F}^{a,b}_{\mu} - \tilde{F}^{a,b}_{\eta}$ can be approximated in $\mathcal{P}_{h\mu}^{coup}$-measure (and hence for a subsequence also a.s.) by stochastic integrals w.r.t. continuous integrands (see e.g. [RY05, Theorem IV.2.12]), which in turn can be approximated by Riemann-Stieltjes sums depending only on $\gamma_s$ with $s \in [a,b]$ (see e.g. [RY05 Proposition IV.2.13]).

6. Ergodicity of the environment process

As is well-known, in order to prove time-ergodicity (under suitable assumptions on $\mu$) of the law $\mathcal{P}^{exi}_{\mu}$ defined in the previous section, we have to verify that the form $(\mathcal{E}^\Gamma_{env}, D(\mathcal{E}^\Gamma_{env}))$ is irreducible, i.e. that if $F \in D(\mathcal{E}^\Gamma_{env})$ fulfills $\mathcal{E}^\Gamma_{env}(F,F) = 0$, then $F$ is constant. This property can be shown at least in two ways: Firstly, one could prove it for the closure of the bilinear form $(\mathcal{F} \mathcal{C}^\infty_b(D,\Gamma))^2 \ni (F,G) \mapsto \int_F (\nabla^\Gamma F, \nabla^\Gamma G)_{\mathcal{T}\Gamma} d\mu$. This can presumably be done along the lines of [AKR98, Theorem 6.6] by proving at first an integration by parts characterization of the set of canonical Gibbs measures for $\phi$ with intensity measure $\nu$. The disadvantage of this method in the present setting is that one would have to assume that there exists $\mu \in \mathcal{G}^{pe}_{t0}(\Phi_0, z \sigma_0)$ such that $\mu$ is extremal in the set of canonical Gibbs measures for $\phi$ with intensity measure $\sigma_\phi$. Even if all elements of $\mathcal{G}^{pe}_{t0}(\Phi_0, z \sigma_0)$ have this property (which is a natural assumption, see e.g. [AKR98 and Geo79 Theroem 6.14]), we would still need $(\mathcal{G}^{pe}_{t0}(\Phi_0, z \sigma_0)) \cap \mathcal{G}^{pe}_{t0}(\Phi_0, z \sigma_0) \neq \emptyset$, which seems not to be clear in general. The second
way to prove irreducibility of $\mathcal{E}_{\text{env}}^T$, which is the way we use below, is to show this property for the closure of the form $(\mathcal{E}_{\text{fr},o}^T, \mathcal{F}_b^\infty(D, \Gamma))$, defined by

$$
\mathcal{E}_{\text{fr},o}^T(F, G) := \int_{\Gamma} (\nabla_{\gamma}^T F, \nabla_{\gamma}^T G) d\mu,
$$

$F, G \in \mathcal{F}_b^\infty(D, \Gamma)$. It is not difficult to see that for $\mu \in \mathcal{G}_{\text{ibp}}(\Phi_\phi, z \sigma_\phi)$ this form is closable and its closure $(\mathcal{E}_{\text{fr},o}^T, D(\mathcal{E}_{\text{fr},o}^T))$ is a Dirichlet form. It corresponds to the dynamics of a frozen environment seen from a moving particle interacting with the environment via $\phi$. In particular, we cannot expect to obtain irreducibility of $\mathcal{E}_{\text{fr},o}^T$ for the case when $d = 1$ and $\phi$ is singular. (However, one should note that this case is not interesting concerning the derivation of an invariance principle for the tagged particle process using [DMFGWS9], since diffusive scaling seems not appropriate in one-dimensional systems with collisions, see [Har65].)

On the other hand, in the following theorem we will see that in all other cases we obtain a quite natural - and nonvoid - irreducibility statement.

**Theorem 6.1.** Assume that $\phi$ fulfills (SS), (LR), (I) and (DL). Let $d \geq 2$ or $\phi$ be bounded. Assume that $\mu = \frac{1}{z_0} e^{-\phi(x)} d\mu_0$ for some $\mu_0 \in \text{ex} \mathcal{G}_{ibp}^c(\Phi_\phi, z \text{dx})$. Then $(\mathcal{E}_{\text{fr},o}^T, D(\mathcal{E}_{\text{fr},o}^T))$ and hence $(\mathcal{E}_{\text{env}}^T, D(\mathcal{E}_{\text{env}}^T))$ is irreducible. Thus $\text{P}_{\mu}^{\text{env}}$ (as defined at the beginning of Section 2) is time-ergodic.

**Remark 6.2.** If in addition to the assumptions in the above theorem we have $\nabla \phi \in L^1(\mathbb{R}^d \setminus B_r(0); \text{dx})$ for some (hence all) $r > 0$ (or $d = 1$ and $\phi$ bounded), the assumption on $\mu$ can be replaced by the (in this case) equivalent and more natural assumption $\mu \in \text{ex} \mathcal{G}_{ibp}^c(\Phi_\phi, z \sigma_\phi)$, see the discussion at the end of Section 2.

**Proof.** Let $F \in D(\mathcal{E}_{\text{fr},o}^T)$ be such that $\mathcal{E}_{\text{fr},o}^T(F, F) = 0$. We have to prove that $F$ is constant. By the proof of [MR92, Proposition 1.4.17] we may assume that $\mu$ is bounded. By Remark 6.1, we never prove that for any $v \in \mathbb{R}^d$ it holds $F(\gamma + v) = F(\gamma)$ for $\mu$-a.e. $\gamma \in \Gamma$.

We only prove this for $v = e_1$, the first standard unit vector of $\mathbb{R}^d$ and consider only the case $d \geq 2$, since the case $d = 1$ and $\phi$ bounded is easier to treat. Note that by (DL) it is possible to fix a Lebesgue-version $\hat{\phi}^+$ of $\phi^+$ and a Lebesgue nullset $N \subset \mathbb{R}^{d-1}$ such that for all $(x_2, \ldots, x_d) \in \mathbb{R}^{d-1} \setminus N$ the function $\hat{\phi}^+(., x_2, \ldots, x_d)$ is weakly differentiable and continuous. Setting $\hat{\phi}^{\text{sup}}(x_1, \ldots, x_d) := \sup_{s \in [0,1]} \hat{\phi}^+(x_1 + s, x_2, \ldots, x_d)$, $(x_1, \ldots, x_d) \in \mathbb{R}^d$, we therefore obtain a Lebesgue-a.e. finite function $\hat{\phi}^{\text{sup}}$. Moreover, since for any element $g$ of the Sobolev space $W^{1,1}(0, 1)$ it holds $\sup_{s \in [0,1]} |g(s)| \leq \|g\|_{W^{1,1}(0,1)}$, we obtain for any $K \in \mathbb{N}$

$$
\int_{\mathbb{R}^d} \hat{\phi}^{\text{sup}} \wedge K \, dx = \int_{\mathbb{R}^d} \sup_{s \in [0,1]} \left( \hat{\phi}^+ \wedge K \right)(x_1 + s, x_2, \ldots, x_d) \, dx_1 \, dx_2 \cdots dx_d \\
\leq \int_{\mathbb{R}^d} \int_0^1 \left| \left( \hat{\phi}^+ \wedge K \right)(x_1 + s, x_2, \ldots, x_d) \right| + |\partial_{x_1} \left( \hat{\phi}^+ \wedge K \right)(x_1 + s, x_2, \ldots, x_d)| \, ds \, dx_1 \cdots dx_d \\
= \|\hat{\phi}^+ \wedge K\|_{L^1(\mathbb{R}^d)} + \|\partial_{x_1} \left( \hat{\phi}^+ \wedge K \right)\|_{L^1(\mathbb{R}^d)} \leq e^K \|1 - e^\phi\|_{L^1(\mathbb{R}^d)} + e^K \|\partial_{x_1} e^{-\phi}\|_{L^1(\mathbb{R}^d)} < \infty
$$

by (I), (DL) and [GT77, Lemma 7.6], i.e. $\hat{\phi}^{\text{sup}} \wedge K \in L^1(\mathbb{R}^d)$ holds for any $K \in \mathbb{N}$. In particular, by [KK02, Theorem 4.1] we have that $\langle \hat{\phi}^{\text{sup}} \wedge K, \cdot \rangle$ is finite $\mu$-a.s. It follows that for any $K \in \mathbb{N}$ we have
AN INVARIANCE PRINCIPLE

\begin{align*}
\limsup_{N \to \infty} \int_{\Gamma} 1_{\{\hat{\phi}^+_{\text{sup}} \wedge N,>N\}} d\mu(A_K) + \limsup_{N \to \infty} \int_{\Gamma \setminus A_K} 1_{\{\hat{\phi}^+_{\text{sup}} \wedge N,>N\}} d\mu \\
\leq \mu(A_K) + \limsup_{N \to \infty} \int_{\Gamma \setminus A_K} 1_{\{\hat{\phi}^+_{\text{sup}} \wedge N,>N\}} d\mu = \mu(A_K),
\end{align*}

where \(A_K := \{ \gamma \in \Gamma | \gamma \cap \{\hat{\phi}^+ > K\} \neq \emptyset \} \). Since \(\hat{\phi}^+_{\text{sup}}\) is finite Lebesgue-a.e., it follows \(\mu(\bigcap_{K \in \mathbb{N}} A_K) = 0\). Hence \(\mu(A_K) \to 0\) as \(K \to \infty\). We conclude that

\begin{equation}
1_{\{\hat{\phi}^+_{\text{sup}} \wedge N, \leq N\}} \to 1 \quad \text{in} \quad L^1(\Gamma; \mu)
\end{equation}

as \(N \to \infty\).

By the definition of \(E_{\text{f鐭}}^1\), there exists a sequence \((F_n)_{n \in \mathbb{N}} \subset FC^\infty_b(\mathcal{D}, \Gamma)\) such that \(F_n \to F\) w.r.t. the norm \(\sqrt{\langle \cdot, \cdot \rangle_{L^2(\Gamma; \mu)} + E_{\text{f鐭}}^1(\cdot, \cdot)}\). (In particular \(\nabla_{\gamma}^1 F_n \to 0\) in \(L^2(\Gamma; \mu)\).) Moreover, by [MR92, Theorem 1.4.12] we may assume that \((F_n)_{n \in \mathbb{N}}\) is uniformly bounded in \(L^\infty(\Gamma; \mu)\), hence also in \(L^\infty(\Gamma; \mu_0)\), and converges also pointwise \(\mu_0\)-a.s. to \(F\). Since for any \(\gamma \in \Gamma\) and \(n \in \mathbb{N}\) it holds \(F_n(\gamma + e_1) - F_n(\gamma) = \int_0^1 (e_1, \nabla_{\gamma}^1 F_n(\gamma + se_1))_{\mathbb{R}^d} ds\), we find that for any \(n, N \in \mathbb{N}\) and \(G \in FC^\infty_b(\mathcal{D}, \Gamma)\)

\[
\left| \int_{\Gamma} 1_{\{\hat{\phi}^+_{\text{sup}} \wedge N, \leq N\}}(\gamma) G(\gamma)(F_n(\gamma + e_1) - F_n(\gamma)) d\mu_0(\gamma) \right| \\
\leq \int_0^1 \left| \int_{\Gamma} G(\gamma - se_1) 1_{\{\hat{\phi}^+_{\text{sup}} \wedge N, \leq N\}}(\gamma - se_1)(e_1, \nabla_{\gamma}^1 F_n(\gamma))_{\mathbb{R}^d} d\mu_0 \right| ds \\
\leq e^N \int_0^1 \left| \int_{\Gamma} G(\gamma - se_1)(e_1, \nabla_{\gamma}^1 F_n(\gamma))_{\mathbb{R}^d} d\mu_0 \right| ds.
\]

For the first inequality we use the translation invariance of \(\mu_0\), and for the second inequality we use the fact that for \(\gamma \in \Gamma\) fulfilling \(\langle \hat{\phi}^+ \wedge N, \gamma \rangle \leq N\) we automatically have \(\hat{\phi}^+(x) \leq N\) for all \(x \in \gamma\) and hence \(\langle \hat{\phi}^+, \gamma \rangle = \langle \hat{\phi}^+ \wedge N, \gamma \rangle \leq N\). Letting \(n \to \infty\) in (6.2) and using the properties of \((F_n)_{n \in \mathbb{N}}\), we find that

\[
\int_{\Gamma} 1_{\{\hat{\phi}^+_{\text{sup}} \wedge N, \leq N\}}(\gamma)G(\gamma)F(\gamma + e_1) d\mu_0(\gamma) = \int_{\Gamma} 1_{\{\hat{\phi}^+_{\text{sup}} \wedge N, \leq N\}}(\gamma)G(\gamma)F(\gamma) d\mu_0(\gamma).
\]

Finally, we let \(N \to \infty\) and take (6.1) into account to obtain \(\int_{\Gamma} G(\gamma)(F(\gamma + e_1)) d\mu_0(\gamma) = \int_{\Gamma} G(\gamma)F(\gamma) d\mu_0(\gamma)\), which by denseness of \(FC^\infty_b(\mathcal{D}, \Gamma)\) in \(L^1(\Gamma; \mu)\) implies that \(F(\gamma + e_1) = F(\gamma)\) holds \(\mu_0\)-a.s.

7. AN INVARIANCE PRINCIPLE FOR THE TAGGED PARTICLE PROCESS

In this section we apply the main result from [DMFGW89], which (essentially, see Remark 7.2 below) reads as follows:

**Theorem 7.1** (De Masi, Ferrari, Goldstein, Wick). Let \((\Omega, \mathcal{F}, (Y_t)_{t \geq 0}, (P_y)_{y \in \mathcal{X}})\) be a Markov process with (measurable) state space \(\mathcal{X}\) which has an invariant and ergodic measure \(\nu\) and coordinate maps \(\Omega \times [0, \infty) \ni (\omega, t) \mapsto Y_t(\omega) \in \mathcal{X}\) that are jointly measurable. Let \((X_{[a,b]}(\omega))_{\{a,b\} \subset [0, \infty)}\) be a family of random variables on \(\Omega\), indexed by all closed bounded subintervals \([a,b] \subset [0, \infty)\) and having the following properties:

(i) \(X_{[a,b]}(\omega)\) is \(P_{\nu}\)-a.s. \(\sigma(Y_s : a \leq s \leq b)\)-measurable. (I.e. \(X_{[a,b]}\) is measurable w.r.t. the smallest \(\sigma\)-field containing \(\sigma(Y_s : a \leq s \leq b)\) and all \(P_{\nu}\)-null sets from \(\mathcal{F}\).)

(ii) \(X_{[a,b]} \in L^1(P_{\nu})\) for all intervals \([a,b] \subset [0, \infty)\).

(iii) \(X_{[a,b]}\) is antisymmetric w.r.t. time-reversal and covariant w.r.t. time-shift.
Moreover, we already know from Theorem 3.6(ii) that
\( M \) process properties (i)-(v) for the random variables
\((\xi_t)_{t \geq 0} \) is càdlàg.

Moreover, assume that the mean forward velocity
\[
\lim_{\delta \to 0} \frac{1}{\delta} \nu(X_\delta \mid \nu(\gamma_0)) = \varphi
\]
exists as a limit in \( L^1(\mathcal{X}; \nu) \) (the conditional expectation in the above formula is interpreted as a function of the initial value, i.e. a function on \( \mathcal{X} \)), and that the martingale
\[
M_t = X_t - \int_0^t \varphi(Y_s) \, ds, \quad t \geq 0,
\]
is square-integrable. For \( \varepsilon > 0 \) and \( t \geq 0 \) one defines \( X^\varepsilon_t := \varepsilon X_{\varepsilon^{-2}t} \), \( X^\varepsilon := (X^\varepsilon_t)_{t \geq 0} \). Then \( X^\varepsilon \) converges in finite dimensional distribution towards an \( \mathbb{R}^d \)-dimensional Brownian motion scaled by a constant diffusion matrix \( D = (D_{ij})_{1 \leq i,j \leq d} \). If additionally \( \varphi \in L^2(\mathcal{X}; \nu) \), then this convergence holds weakly in \( \nu \)-measure.

**Remark 7.2.** The statement of the above theorem is a simplified version of the theorem from [DMFGW89]. In fact, in that paper also results are derived for the case when the mean forward velocity is only in \( L^1 \). Moreover, in [DMFGW89] it is assumed that \( X_{[a,b]} \) is even \( \sigma(Y_s : a \leq s \leq b) \)-measurable (i.e. measurable w.r.t. the non-completed \( \sigma \)-fields). However, this assumption can be relaxed to \( \nu \)-a.s. measurability (as we do in assumption (i) of the above theorem) without making any changes to the proof of the theorem.

**Remark 7.3.**
(i) Weak convergence in \( \nu \)-measure means in the above theorem that for any bounded continuous function \( F \) on the space \( D([0, \infty); \mathbb{R}^d) \) of càdlàg paths (equipped with the Skorokhod topology) and any \( \delta > 0 \) it holds
\[
\int \left[ \mathbb{E}_y [F(\varepsilon X_{\varepsilon^{-2}t})_{t \geq 0})] - \mathbb{E}[F] \right] > \delta \, d\mu(y) \to 0,
\]
where \( \mathbb{E} \) is the expectation corresponding to the law of the limiting process on \( D([0, \infty); \mathbb{R}^d) \). Note that weak convergence in \( \nu \)-measure implies weak convergence of the laws of \( (\varepsilon X_{\varepsilon^{-2}t})_{t \geq 0} \) under \( \nu \).

(ii) Using an analogon of the portmanteau theorem for weak convergence in measure (see (i)) and recalling that the Skorokhod topology coincides on the space \( C([0, \infty); \mathbb{R}^d) \) with the topology of locally uniform convergence and that this space is closed in \( D([0, \infty); \mathbb{R}^d) \) (for the latter two facts see [EKS86] Exercise 3.25), one finds that if \( (X_t)_{t \geq 0} \) is not only càdlàg but even continuous, then from weak convergence in \( \nu \)-measure it even follows that one has the analogous convergence with \( D([0, \infty); \mathbb{R}^d) \) replaced by \( C([0, \infty); \mathbb{R}^d) \).

It is our aim to apply the above theorem for deriving the limit after diffusive scaling of the displacement \( (\xi_t - \xi_0)_{t \geq 0} \) of the tagged particle distributed according to \( \mathbb{P}^{\text{tag}}_{\eta, \mu} \) for the definition of this law see Section 3], under the assumptions from Section 2 and for \( \mu \in \text{ex} G_{\text{tag}}^\text{top} (\Phi, \sigma) \). As we have seen in Theorem 5.1 we may equivalently consider the limit of the random variables \( (\eta_t)_{t \geq 0} \) on \( C([0, \infty), \Gamma) \). Using also Lemma 6.1 we find that the Markov process \( \mathbb{M}^{\text{tag}} \) fulfills the assumptions of the above theorem, and it is not difficult to prove properties (i)-(v) for \( X_{[a,b]} \) being the version of \( \eta_b - \eta_a \) from Theorem 5.1(ii), \( a,b \in [0, \infty), a < b \). Moreover, we already know from Theorem 5.1(ii) that \( X_t - \int_0^t \langle \nabla \phi, \gamma_s \rangle \, ds, t \geq 0, \) is \( \sqrt{2} \) times a Brownian motion and hence in particular a square-integrable martingale. It remains to prove that the mean forward velocity exists and equals \( \langle \nabla \phi, \cdot \rangle \).
**Lemma 7.4.** (7.1) holds in the above described setting with \( \varphi = \langle \nabla \phi, \cdot \rangle \).

**Proof.** Observe that \( E_{\mu}^{|\nabla \phi, \cdot \rangle} = T_{s}^{|\nabla \phi, \cdot \rangle} \) holds \( \mu \)-a.e. for any \( s \geq 0 \). Using Theorem 3.6(ii) it follows that for \( \varphi = \langle \nabla \phi, \cdot \rangle \)

\[
\left\| \frac{1}{\delta} E_{\mu}^{|\nabla \phi, \cdot \rangle} X_{\delta} \mid \sigma(\gamma_{0}) \right\| = E_{\mu}^{|\nabla \phi, \cdot \rangle} \left[ \left\| \frac{1}{\delta} E_{\mu}^{|\nabla \phi, \cdot \rangle} X_{\delta} \mid \sigma(\gamma_{0}) \right\| - \varphi(\gamma_{0}) \right] \\
= E_{\mu}^{|\nabla \phi, \cdot \rangle} \left[ \int_{0}^{\delta} \varphi(\gamma_{0}) - \varphi(\gamma_{0}) \, ds \mid \sigma(\gamma_{0}) \right] \\
\leq \frac{1}{\delta} \int_{0}^{\delta} E_{\mu}^{|\nabla \phi, \cdot \rangle} \left\| T_{s,1}^{|\nabla \phi, \cdot \rangle} - \varphi(\gamma_{0}) \right\| \, ds \\
= \frac{1}{\delta} \int_{0}^{\delta} \left\| T_{s,1}^{|\nabla \phi, \cdot \rangle} - \varphi \right\| \, ds \to 0
\]

as \( \delta \to 0 \) by the strong continuity of \( (T_{s,1}^{|\nabla \phi, \cdot \rangle})_{s \geq 0} \). \( \square \)

From the above considerations we conclude our final theorem.

**Theorem 7.5.** Suppose that \( \phi \) fulfills the assumptions (SS), (LR), (I) and (DL\(^p\)) for some \( p \in (d, \infty) \cap [2, \infty) \) and that \( \mu \in \mathcal{G}_{\text{ipb}}^{\phi}(\Phi, z, \sigma_{\phi}) \) and assume that \( d \geq 2 \) or \( \phi \) is bounded. Let \( 0 \leq h \in L^{1}(\mathbb{R}^{d}) \cap L^{\infty}(\mathbb{R}^{d}) \) be a probability density w.r.t. Lebesgue measure and define \( h\mu := (h d\xi) \subset \mu \). Denote by \( (X^{\text{tag}})_{t \geq 0} \) the first component of \( (X^{\text{tag}})_{t \geq 0} \). Then, as \( \varepsilon \to 0 \), 

\( \left( \varepsilon X_{\varepsilon t} \right)_{t \geq 0} \) converge weakly in \( \mu \)-measure (in \( C([0, \infty), \mathbb{R}^{d}) \)) to a \( d \)-dimensional Brownian motion scaled with a (constant) diffusion matrix \( D = (D_{ij})_{1 \leq i, j \leq d} \) and starting in 0.

**Appendix**

Here we give a refinement of [FG11] Theorem 3.3 enabling us to drop the condition (LS) on \( \phi \) which is assumed there. In contrast to the situation in [AKR98] this is possible since we are only dealing with grand canonical (in contrast to canonical) Gibbs measures. We remark that an integration by parts formula for grand canonical Gibbs measures (with intensity measure \( z dx \)) has been shown without condition (LS) already in [Yos96]. The present proof has the advantage that it makes use only of the improved Ruelle bound for tempered grand canonical Gibbs measures instead of deriving the necessary estimates by an approximation with finite dimensional systems with empty boundary conditions, and, in our opinion, is much more transparent than the proof given there. (Moreover, it becomes clear that it works for any tempered grand canonical Gibbs measures, not only for those for which the approximation with the mentioned finite dimensional systems holds.) Recall that we defined \( B_{\phi}^{\phi, \cdot} \gamma := \langle \text{div} v, \gamma \rangle - \sum_{x \in \gamma} \langle \nabla \phi(x), v(x) \rangle \mathbb{R}^{d} - \sum_{\{x, y\} \subset \gamma} \langle \nabla \phi(x - y), v(x) - v(y) \rangle \mathbb{R}^{d}, \gamma \in \Gamma \). This expression is well-defined in the sense of \( \mu \)-a.e. absolute convergence of the sums and integrable w.r.t. \( \mu \) by [KK02] Theorem 4.1 and the improved Ruelle bound.

**Theorem A1.** Let \( \phi \) fulfill (SS), (LR), (I), (DL\(^1\)) and let \( z > 0 \). Then for any \( \mu \in \mathcal{G}_{\text{ipb}}^{\phi}(\Phi, z e^{-\phi} dx), F \in FC_{\infty}^{\phi}(\mathcal{D}, \Gamma) \) and \( v \in C_{\infty}(\mathbb{R}^{d}) \) it holds

\[
\int_{\Gamma} \nabla_{v} F \, d\mu = - \int_{\Gamma} F B_{\phi}^{\phi, \cdot} \mu \, d\mu.
\]

**Remark A2.** (i) Note that the above theorem implies (2.1) (replace \( F \) by \( FG \) and use the product rule).
It is also possible to derive analogous results for tempered Gibbs measures \( \mu \in \mathcal{G}_\mu^c(\Phi, z \rho dx) \), when \( \rho \) is a bounded function from \( \mathcal{W}^1,1(\mathbb{R}^d) \), if in the definition of \( B^\phi_\mu \) one replaces \( \sum_{x \in \gamma} (\nabla \phi(x), v(x)) \) by \( \sum_{x \in \gamma} (\frac{-\rho \phi(x)}{\rho(x)}, v(x)) \), where one sets \( \nabla \rho(x) = 0 \) for \( \rho(x) = 0 \). (The proof of the main Lemma A3 below remains exactly the same, since all such measures \( \mu \) fulfill a Ruelle bound).

Condition (DL1) can be weakened to just assuming that \( e^{-\phi} \) is weakly differentiable and \( \nabla e^{-\phi} \in \mathcal{W}^1(\mathbb{R}^d) \). One then sets \( \nabla \phi := e^{\phi} 1_{\{\phi < \infty\}} \nabla e^{-\phi} \) and observes that \( \nabla e^{-\phi} = -e^{-\phi} \nabla \phi \) Lebesgue-a.e.

The main work for proving the above theorem is contained in the following lemma.

Lemma A3. Let \( \phi \) and \( \mu \) be as in the above theorem. There is a set \( \mathcal{M} \subset \Gamma \) with \( \mu \)-measure 0 such that for all \( \gamma \in \Gamma \setminus \mathcal{M} \) the function

\[
W_\phi(\cdot, \gamma) := \sum_{x \in \gamma} \phi(\cdot - x),
\]

exists - in the sense of convergence of the negative parts of the sum - as a locally bounded below function \( \mathbb{R}^d \to \mathbb{R} \cup \{\infty\} \) and is almost everywhere finite, and that \( e^{-W_\phi(\cdot, \gamma)} \in \mathcal{W}^1,1(\mathbb{R}^d) \), and the derivative is given by \( \nabla e^{-W_\phi(\cdot, \gamma)} \sum_{x \in \gamma} \nabla \phi(x - \cdot) \) a.e. on \( \mathbb{R}^d \), where the sum converges absolutely Lebesgue-a.e.

Proof. We compute for any open relatively compact \( \Lambda \subset \mathbb{R}^d \) using the Ruelle bound (which holds for \( \mu \), as mentioned in Section 2)

\[
\int \int_{\Lambda} \sum_{x \in \gamma} |(\phi \wedge 1)(y - x)| \, dy \, d\mu(\gamma) \leq C \int_{\mathbb{R}^d} \int_{\Lambda} |(\phi \wedge 1)|(|y - x|) \, dy \, dx \leq C|\Lambda| \|\phi \wedge 1\|_{\mathcal{W}^1,1(\mathbb{R}^d)} < \infty,
\]

for some \( C < \infty \), where \( |\Lambda| \) denotes the volume of \( \Lambda \). This implies that \( \mu \)-a.e. we have \( dx \)-a.e. finiteness of \( W_{\phi \wedge 1} \). Moreover, by (I) we have

\[
\int \int_{\Lambda} \sum_{x \in \gamma} 1_{\{\phi(y-x) \geq 1\}} \, dy \, d\mu(\gamma) \leq C|\Lambda| \cdot |\{x \in \mathbb{R}^d | \phi(x) \geq 1\}| < \infty,
\]

which implies that for \( \mu \)-a.e. \( \gamma \) we have \( \int_{\Lambda} \#\{x \in \gamma | \phi(y-x) \geq 1\} \, dy < \infty \), so Lebesgue-a.e. we have only finitely many particles in the region which makes the difference between \( W_{\phi \wedge 1} \) and \( W_\phi \). Thus and since \( \mu \)-a.s. we have \( \phi(x) < \infty \) for all \( x \in \gamma \) (because \( \phi < \infty \) Lebesgue-a.e.), it follows that for \( \mu \)-a.e. \( \gamma \) we have Lebesgue-a.e. finiteness of \( W_\phi(\cdot, \gamma) \). Moreover, we have using the Ruelle bound and (LR)

\[
\int_{\Lambda} \sup_{y \in \Lambda} W_{\phi \wedge 1}(y, \gamma) \, dy \leq \int_{\Lambda} \sup_{y \in \Lambda} \psi(|y - \cdot|_2, \gamma) \, d\mu(\gamma) \leq C \int_{\mathbb{R}^d} \sup_{y \in \Lambda} \psi(|y - x|_2) \, dx,
\]

which is finite because \( \psi \) is decreasing. It follows that \( \mu \)-a.s. we have that \( W_{\phi \wedge 1}(\cdot, \gamma) \) is locally bounded and thus \( W_{\phi}(\cdot, \gamma) \) is locally bounded from below.

Now we prove the weak differentiability statement, and we may restrict to \( \Lambda \subset \mathbb{R}^d \) as above. At first let us have a look at what we consider to be the weak derivative. It holds for \( A_M := \{\gamma \in \Gamma | W_{\phi \wedge 1}(\cdot, \gamma) \leq M \} \) on \( \Lambda \)

\[
\int_{A_M} \int_{\Lambda} \sum_{x \in \gamma} |\nabla \phi(y-x)| e^{-W_\phi(y,\gamma)} \, dy \, d\mu(\gamma) \leq \int_{A_M} \int_{\Lambda} \sum_{x \in \gamma} |\nabla e^{-\phi(y-x)}| e^{W_{\phi}(y,\gamma \setminus \{x\})} \, dy \, d\mu(\gamma)
\]
hence by the dominated convergence theorem we obtain
\[ \theta \sum_{n} \int_{\gamma_{B_{n}(0)}} \|\nabla e^{-\gamma(y-x)}\| d\mu(\gamma) \leq C e^{M} \|A\| \int_{\mathbb{R}^{d}} \|\nabla e^{-\gamma}\| dx < \infty, \quad (7.2) \]
which implies existence for \( \mu \)-a.e. \( \gamma \) of what we consider to be the weak derivative in the sense of Lebesgue-a.e. convergence of the sum.

Clearly, the function \( e^{-W_{\phi}(\cdot, y, \gamma_{B_{n}(0)})} \) is weakly differentiable for all \( n \) and \( \gamma \), where \( \gamma_{B_{n}(0)} := \gamma \cap B_{n}(0) \). The weak derivative is given by
\[ \sum_{x \in \gamma_{B_{n}(0)}} -\langle \nabla \phi(\cdot, -x) \rangle e^{-W_{\phi}(\cdot, y, \gamma_{B_{n}(0)})}. \]
We need to prove that \( \mu \)-a.e. this converges to the above expression in \( L^{1}(\Lambda) \), at least for a subsequence; therefore, it suffices to prove convergence in \( L^{1}(\Lambda \times \Gamma; dx \otimes \mu) \). We may restrict to \( A_{M} \):
\[
\int_{A_{M}} \left( \int_{\Lambda} \sum_{x \in \gamma} \nabla \phi(y-x) e^{-W_{\phi}(y, \gamma)} - \sum_{x \in \gamma_{B_{n}(0)}} \nabla \phi(y-x) e^{-W_{\phi}(y, \gamma_{B_{n}(0)})} \right) dy d\mu(\gamma) \\
\leq \int_{A_{M}} \left( \int_{\Lambda} \sum_{x \in \gamma_{B_{n}(0)}} \nabla \phi(y-x) e^{-W_{\phi}(y, \gamma)} \right) dy d\mu(\gamma) \\
+ \int_{A_{M}} \left( \int_{\Lambda} \sum_{x \in \gamma_{B_{n}(0)}} |\nabla(e^{-\phi(y-x)})| e^{-W_{\phi}(y, \gamma) \{x\}} - e^{-W_{\phi}(y, \gamma_{B_{n}(0)}) \{x\}} \right) dy d\mu(\gamma). \quad (7.3)
\]
For the first summand we find by an estimate as in (7.2) that it converges to 0. For the second one we write
\[
\int_{A_{M}} \left( \int_{\Lambda} \sum_{x \in \gamma_{B_{n}(0)}} |\nabla(e^{-\phi(y-x)})| e^{-W_{\phi}(y, \gamma) \{x\}} - e^{-W_{\phi}(y, \gamma_{B_{n}(0)}) \{x\}} \right) dy d\mu(\gamma) = \int \theta_{n} d\mu^{*}(y, x, \gamma),
\]
where \( \mu^{*}(A) := \int A \int_{\gamma} 1_{A}(y, x, \gamma) d\mu(\gamma) dy, A \in \mathcal{B}(\mathbb{R}^{d}) \otimes \mathcal{B}(\mathbb{R}^{d}) \otimes \mathcal{B}(\Gamma) \) and
\[
\theta_{n}(y, x, \gamma) := \|\nabla e^{-\phi(y-x)}\| 1_{\gamma_{B_{n}(0)}(0)}(x) e^{-W_{\phi}(y, \gamma) \{x\}} - e^{-W_{\phi}(y, \gamma_{B_{n}(0)}) \{x\}}} \], \quad y \in \Lambda, \gamma \in \Gamma, x \in \gamma.
\]
Now we know that for a.e. \( \gamma \) and \( y \) we have that \( \phi(y-x) < \infty \) and \( |\nabla e^{-\phi(y-x)}| < \infty \) for all \( x \in \gamma \) and \( W_{\phi}(y, \gamma_{B_{n}(0)}) \{x\}) \rightarrow W_{\phi}(y, \gamma) \{x\} \). Hence we have \( \mu^{*}\)-a.s. convergence to 0 of the \( \theta_{n} \). Moreover, for \( \gamma \in A_{M}, y \in \Lambda \) and \( x \in \gamma \) we have
\[
\theta_{n}(y, x, \gamma) \leq 2|\nabla e^{-\phi(y-x)}| e^{M},
\]
but
\[
\int |\nabla e^{-\phi(y-x)}| d\mu^{*}(y, x, \gamma) = \int_{\Lambda} \int_{\Gamma} \langle |\nabla e^{-\phi(y-x)}|, \gamma \rangle d\mu(\gamma) dy \leq C|\Lambda| \|\nabla e^{-\phi}\|_{L^{1}(\mathbb{R}^{d})} < \infty,
\]
hence by the dominated convergence theorem we obtain \( \theta_{n} \rightarrow 0 \) in \( L^{1}(\Lambda \times \mathbb{R}^{d} \times \Gamma; \mu^{*}) \), and thus convergence of the left-hand side of (7.3) to 0 as \( n \rightarrow \infty \). It follows that for a sequence \( (n_{k})_{k \in \mathbb{N}} \subset \mathbb{N} \) we have for \( \mu \)-a.e. \( \gamma \in A_{M} \) that \( \sum_{x \in \gamma_{B_{n_{k}}(0)}} \nabla \phi(\cdot, -x) e^{-W_{\phi}(\cdot, y, \gamma_{B_{n_{k}}(0)})} \rightarrow \sum_{x \in \gamma} \nabla \phi(\cdot, -x) e^{-W_{\phi}(\cdot, y, \gamma)} \) in \( L^{1}(\Lambda) \). Thus the mentioned weak differentiability holds for \( \mu \)-a.e. \( \gamma \in A_{M} \), and taking the union over all \( M \in \mathbb{N} \), the assertion follows. \( \square \)
Proof of Theorem A1. Choose Λ ⊂ ℝ^d open, relatively compact and large enough such that F does not depend on the configuration outside Λ, and the support of v is contained in Λ. We set

\[ U_\phi^\gamma(\{x_1, \cdots, x_n\}) := \sum_{i<j} \phi(x_i - x_j) + \sum_{i=1}^n W_\phi(x_i, \gamma \setminus \Lambda) + \sum_{i=1}^n \phi(x_i) \]

for \( x_1, \cdots, x_n \in \Lambda \) and \( \gamma \in \Gamma \). The Dobrushin-Lanford-Ruelle equation implies that

\[
\int_{\Gamma} \nabla_v F(\gamma) d\mu = \frac{1}{Z} \int_{\Gamma} \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} \int_{\Lambda^n} v(x_1) \nabla x_1 F(\{x_1, \cdots, x_n\}) e^{-U_\phi^\gamma(\{x_1, \cdots, x_n\})} dx_1 \cdots dx_n d\mu(\gamma),
\]

where \( Z \) is a normalization constant. Due to Lemma A3 we may apply integration by parts for all \( \gamma \in \Gamma \setminus \mathcal{M} \) (and thus for \( \mu \)-a.e. \( \gamma \)) to the inner integral on the right-hand side, which therefore equals

\[
- \sum_{i=1}^n \int_{\Lambda^n} F(\{x_1, \cdots, x_n\}) e^{-U_\phi^\gamma(\{x_1, \cdots, x_n\}, \gamma)} \left( \nabla v(x_i) - v(x_i) \nabla \phi(x_i) \right) dx_1 \cdots dx_n.
\]

Taking the sum inside the brackets in the above expression and summing up the first smmand in the bracket yields \( \langle \nabla v, \{x_1, \cdots, x_n\} \cup \gamma \setminus \Lambda \rangle \), the second gives \( \langle v \nabla \phi, \{x_1, \cdots, x_n\} \cup \Gamma \setminus \Lambda \rangle \) from the last two we obtain \( \sum_{\{z,z'\} \subset \{x_1, \cdots, x_n\} \cup \gamma \setminus \Lambda} \nabla \phi(z - z')(v(z) - v(z')) \). Therefore, the above expression equals

\[
- \int_{\Lambda^n} F(\{x_1, \cdots, x_n\}) e^{-U_\phi^\gamma(\{x_1, \cdots, x_n\}, \gamma)} B_\phi^\gamma(\{x_1, \cdots, x_n\} \cup \gamma \setminus \Lambda) dx_1 \cdots dx_n.
\]

Replacing the inner integral in (7.4) by this expression and noting that the sum over \( n \) exists by integrability of \( B_\phi^\gamma \) w.r.t. \( \mu \), we obtain the assertion. \( \square \)

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