A regularity criterion for 3D micropolar fluid flows in terms of one partial derivative of the velocity

Sadek Gala
Department of Mathematics, University of Mostaganem
Box 227, Mostaganem, Algeria &
Dipartimento di Mathematica e Informatica
Università di Catania, Viale Andrea Doria, 6, 95125 Catania, Italy
sadek.gala@gmail.com

and

Maria Alessandra Ragusa
Dipartimento di Mathematica e Informatica
Università di Catania, Viale Andrea Doria, 6, 95125 Catania, Italy
maragusa@dmi.unict.it

Abstract

In this work, we prove a regularity criterion for micropolar fluid flows in terms of the one partial derivative of the velocity in Morrey-Campanato space.

Key words: micropolar fluid equations; regularity criterion; weak solutions.

Mathematics Subject Classification(2000): 35Q35, 35B65, 76D05
1 Introduction and the main result

In this paper, we consider the following Cauchy problem for the incompressible micropolar fluid equations in $\mathbb{R}^3$ [9]:

\begin{align}
\begin{aligned}
\partial_t u + (u \cdot \nabla) u - \Delta u + \nabla \pi - \nabla \times \omega &= 0, \\
\partial_t \omega - \Delta \omega - \nabla \text{div} \omega + 2 \omega + u \cdot \nabla \omega - \nabla \times u &= 0, \\
\nabla \cdot u &= 0,
\end{aligned}
\end{align}

where $u$, $\omega$ and $\pi$ denote the unknown velocity vector field, the micro-rotational velocity and the unknown scalar pressure of the fluid at the point $(x, t) \in \mathbb{R}^3 \times (0, T)$, respectively, while $u_0$, $\omega_0$ are given initial data with $\nabla \cdot u_0 = 0$ in the sense of distributions.

When the micro-rotation effects are neglected or $\omega = 0$, the micropolar fluid flows (1.1) reduce to the incompressible Navier-Stokes flows (see, for example, [21, 35]). Much effort has been devoted to establish the global existence and uniqueness of smooth solutions to the Navier–Stokes equations. Different criteria for regularity of the weak solutions have been proposed. The Prodi–Serrin conditions (see [30, 34]) shows that any solution $u$ for the 3D Navier-Stokes equations satisfying

\begin{align}
\begin{aligned}
\frac{2}{p} + \frac{3}{q} \leq 1 \quad \text{and} \quad 3 \leq q \leq \infty,
\end{aligned}
\end{align}

is regular. Notice that the limiting case $u \in L^\infty(0, T; L^3(\mathbb{R}^3))$ was covered by Escauriaza et al. [10] in 2003. Later on, Beirão da Veiga [2] established another regularity criterion by replacing (1.2) with the following condition:

\begin{align}
\begin{aligned}
\nabla u \in L^\beta \left( (0, T; L^\alpha(\mathbb{R}^3)) \right) \quad \text{with} \quad \frac{3}{\alpha} + \frac{2}{\beta} \leq 2 \quad \text{and} \quad \frac{3}{2} < \alpha \leq \infty.
\end{aligned}
\end{align}

In 2004, Penel and Pokorný [29] obtained a different type regularity criterion, which says that if

\begin{align}
\begin{aligned}
\partial_t u \in L^\beta \left( (0, T; L^\alpha(\mathbb{R}^3)) \right) \quad \text{with} \quad \frac{3}{\alpha} + \frac{2}{\beta} \leq 1 \quad \text{and} \quad 2 \leq \alpha \leq \infty,
\end{aligned}
\end{align}

then the solution $u$ to the Navier-Stokes equations is regular. The same result can be found in [41]. Penel and Pokorný’s work has been improved by some other
authors, (see e.g., [5, 20] and the references cited therein). It was already known that if one component of the velocity is bounded in a suitable space, then the solution is smooth (see Penel and Pokorný [29], Zhou [40, 41, 43, 44]). Some of these regularity criteria can be extended to the 3D MHD equations by making the assumptions on both $u$ and $b$ ([4]). Moreover, He and Xin in [11] derived some regularity criteria for the 3D MHD equations only in terms of the velocity field $u$, and they proved that if $u$ satisfies either (1.2) or (1.3), then the solution is regular. Recently, Cao and Wu [7] proved that the condition

$$\partial_3 u \in L^\beta \left( (0, T; L^\alpha (\mathbb{R}^3)) \right) \quad \text{with} \quad \frac{3}{\alpha} + \frac{2}{\beta} \leq \frac{3}{2} \quad \text{and} \quad \alpha > 3,$$

also implies regularity of the solution $(u, b)$ to the 3D MHD equations. Later, Jia and Zhou [36, 37, 39] showed that if

$$\partial_3 u \in L^\beta \left( (0, T; L^\alpha (\mathbb{R}^3)) \right) \quad \text{with} \quad \frac{3}{\alpha} + \frac{2}{\beta} = \frac{3}{4} + \frac{1}{\alpha} \quad \text{and} \quad \alpha > 2,$$

then the solution is regular. For more interesting component reduction results of the regularity criterion, we refer to e.g. [38, 40, 41, 43, 44].

Inspired by the above-mentioned works on regularity criteria of Navier-Stokes and MHD equations, particularly those of Penel and Pokorný [29], Cao and Wu [7] and Jia and Zhou [36, 37, 38, 39], we want to investigate a similar problem for the micropolar fluid flows (1.1). Very recently, Jia et al. [18] proved the following regularity criterion

$$\partial_3 u \in L^\beta \left( (0, T; L^{\alpha, \infty} (\mathbb{R}^3)) \right) \quad \text{with} \quad \frac{3}{\alpha} + \frac{2}{\beta} = 1 \quad \text{and} \quad 3 < \alpha \leq \infty.$$

Here $L^{\alpha, \infty}$ is the Lorentz space.

The purpose of this work is to improve the result in [18] and to prove that if the derivative of the velocity in one direction belongs to $L^{1/r} \left( 0, T, \hat{M}^{2, \gamma}_{2, r} (\mathbb{R}^3) \right)$ with $0 < r < 1$, then the weak solution actually is regular and unique. This work is motivated by the recent results [36]-[44] on the Navier-Stokes equations and MHD equations.

## 2 Preliminaries and main result

Now, we recall the definition and some properties of the space that will be useful in the sequel. These spaces play an important role in studying the regularity of
solutions to partial differential equations; see e.g. [24] and references therein.

**Definition 2.1.** For \(0 \leq r < \frac{3}{2}\), the space \(\dot{X}_r\) is defined as the space of \(f(x) \in L^2_{\text{loc}}(\mathbb{R}^3)\) such that
\[
\|f\|_{\dot{X}_r} = \sup_{\|g\|_{H^r} \leq 1} \|fg\|_{L^2} < \infty.
\]
where we denote by \(H^r(\mathbb{R}^3)\) the completion of the space \(C_0^\infty(\mathbb{R}^3)\) with respect to the norm \(\|u\|_{H^r} = \left\|(-\Delta)^{\frac{r}{2}} u\right\|_{L^2}\).

We have the homogeneity properties: \(\forall x_0 \in \mathbb{R}^3\)
\[
\|f(\cdot + x_0)\|_{\dot{X}_r} = \|f\|_{\dot{X}_r},
\]
\[
\|f(\lambda \cdot)\|_{\dot{X}_r} = \frac{1}{\lambda^r} \|f\|_{\dot{X}_r}, \quad \lambda > 0.
\]

The following imbedding:
\[
L^\frac{3}{r} \subset \dot{X}_r, \quad 0 \leq r < \frac{3}{2}
\]
holds.

Now we recall the definition of Morrey-Campanato spaces (see e.g. [19]):

**Definition 2.2.** For \(1 < p \leq q \leq +\infty\), the Morrey-Campanato space \(\dot{M}_{p,q}\) is defined by:
\[
\dot{M}_{p,q} = \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^3) : \|f\|_{\dot{M}_{p,q}} = \sup_{x \in \mathbb{R}^3} \sup_{R > 0} R^{3/q-3/p} \|f\|_{L^p(B(x,R))} < \infty \right\}.
\]

It is easy to check the following:
\[
\|f(\lambda \cdot)\|_{\dot{M}_{p,q}} = \frac{1}{\lambda^q} \|f\|_{\dot{M}_{p,q}}, \quad \lambda > 0.
\]

We have the following comparison between Lorentz spaces and Morrey-Campanato spaces: for \(p \geq 2\)
\[
L^\frac{3}{r}(\mathbb{R}^3) \subset L^\frac{3}{r,\infty}(\mathbb{R}^3) \subset \dot{M}_{p,q}(\mathbb{R}^3).
\]

Other useful comparison are contained in [32], [31] and [33]. The relation
\[
L^\frac{3}{r,\infty}(\mathbb{R}^3) \subset \dot{M}_{p,q}(\mathbb{R}^3)
\]
is shown as follows. Let \( f \in L^{\frac{2}{r}, \infty}(\mathbb{R}^3) \). Then

\[
\|f\|_{M_{p, r}} \leq \sup_E |E|^{\frac{r}{2} - \frac{1}{p}} \left( \int_E |f(y)|^p \, dy \right)^{\frac{1}{p}}
\]

\[
= \left( \sup_E |E|^{\frac{r}{2} - 1} \int_E |f(y)|^p \, dy \right)^{\frac{1}{p}}
\]

\[
\cong \left( \sup_{R>0} R \left| \left\{ x \in \mathbb{R}^3 : |f(y)|^p > R \right\} \right|^{\frac{r}{p}} \right)^{\frac{1}{p}}
\]

\[
= \sup_{R>0} R \left| \left\{ x \in \mathbb{R}^p : |f(y)| > R \right\} \right|^{\frac{r}{p}}
\]

\[
\cong \|f\|_{L^{\frac{2}{r}, \infty}}.
\]

For \( 0 < r < 1 \), we use the fact that

\[
L^2 \cap \dot{H}^1 \subset \dot{B}^r_{2,1} \subset \dot{H}^r.
\]

Thus we can replace the space \( \dot{X}_r \) by the pointwise multipliers from Besov space \( \dot{B}^r_{2,1} \) to \( L^2 \). Then we have the following lemma given in [25].

**Lemma 2.3.** For \( 0 \leq r < \frac{3}{2} \), the space \( \dot{Z}_r \) is defined as the space of \( f(x) \in L^2_{\text{loc}}(\mathbb{R}^3) \) such that

\[
\|f\|_{\dot{Z}_r} = \sup_{\|g\|_{\dot{B}^r_{2,1}} \leq 1} \|fg\|_{L^2} < \infty.
\]

Then \( f \in \dot{M}_{2, \frac{3}{2}} \) if and only if \( f \in \dot{Z}_r \) with equivalence of norms.

To prove our main result, we need the following lemma due to [28] (see also [42]).

**Lemma 2.4.** For \( 0 < r < 1 \), we have

\[
\|f\|_{\dot{B}^r_{2,1}} \leq C \|f\|_{L^2}^{1-r} \|\nabla f\|_{L^2}^r.
\]

Additionally, for \( 2 < p \leq \frac{3}{r} \) and \( 0 \leq r < \frac{3}{2} \), we have the following inclusion relations ([23], [25]) :

\[
\dot{M}_{p, \frac{3}{p}}(\mathbb{R}^3) \subset \dot{X}_r(\mathbb{R}^3) \subset \dot{M}_{2, \frac{3}{2}}(\mathbb{R}^3) = \dot{Z}_r(\mathbb{R}^3).
\]
The relation
\[ \dot{X}_r(\mathbb{R}^3) \subset \mathcal{M}_{2,\frac{3}{2}}(\mathbb{R}^3) \]
is shown as follows. Let \( f \in \dot{X}_r(\mathbb{R}^3) \), \( 0 < R \leq 1 \), \( x_0 \in \mathbb{R}^3 \) and \( \phi \in C_0^\infty(\mathbb{R}^3) \), \( \phi \equiv 1 \) on \( B\left(\frac{x_0}{R}, 1\right) \). We have
\[
R^{-\frac{3}{2}} \left( \int_{|x-x_0| \leq R} |f(x)|^2 \, dx \right)^{1/2} = R^r \left( \int_{|y-x_0| \leq 1} |f(Ry)|^2 \, dy \right)^{1/2} \\
\leq R^r \|f(R.)\|_{\dot{X}_r} \|\phi\|_{H^r} \\
\leq \|f\|_{\dot{X}_r} \|\phi\|_{H^r} \\
\leq C \|f\|_{\dot{X}_r}.
\]

Before stating our result, let us recall the definition of Leray–Hopf weak solution.

**Definition 2.5** ([27]). Let \((u_0, \omega_0) \in L^2(\mathbb{R}^3) \) and \( \nabla \cdot u_0 = 0 \). A measurable function \((u(x,t), \omega(x,t))\) is called a weak solution to the 3D micropolar flows equations (1.1) on \((0, T)\) if \((u, \omega)\) satisfies the following properties

1. \( u, \omega \in L^\infty((0, T); L^2(\mathbb{R}^3)) \cap L^2((0, T); H^1(\mathbb{R}^3)) \) for all \( T > 0 \);
2. \((u(x,t), \omega(x,t))\) verifies (1.1) in the sense of distribution;
3. The energy inequality
\[
\|u\|_{L^2}^2 + \|w\|_{L^2}^2 + 2 \int_0^t (\|\nabla u\|_{L^2}^2 + \|\nabla w\|_{L^2}^2) \, ds + 2 \int_0^t \|\nabla \cdot w\|_{L^2}^2 \, ds + 2 \int_0^t \|w\|_{L^2}^2 \, ds \\
\leq \|u_0\|_{L^2}^2 + \|w_0\|_{L^2}^2, \text{ for } 0 < t \leq T.
\]

By a strong solution we mean a weak solution \((u, \omega)\) such that
\[
u, \omega \in L^\infty((0, T); H^1(\mathbb{R}^3)) \cap L^2((0, T); H^2(\mathbb{R}^3)) .
\]

It is well known that strong solutions are regular (say, classical) and unique in the class of weak solutions.

More precisely, we will prove
Theorem 2.6. Suppose that \((u_0, \omega_0) \in H^1(\mathbb{R}^3)\) with \(\nabla \cdot u_0 = 0\) in \(\mathbb{R}^3\). If the velocity \(u\) satisfies

\[
\partial_3 u \in L^\frac{2}{1-r} \left(0, T, \mathcal{M}^2_{1,7}(\mathbb{R}^3)\right) \text{ with } 0 < r < 1,
\]
then the solution remains smooth on \((0, T]\). Therefore, \((u, \omega) \in L^\infty (0, T, H^1(\mathbb{R}^3)) \cap L^2 (0, T, H^2(\mathbb{R}^3))\).

For convenience, we will use the following two lemmas will be used in the proofs of our main results (see, e.g., [11, 26, 22]):

Lemma 2.7. Let \(\mu, \lambda\) and \(\gamma\) be three parameters that satisfy

\[
1 \leq \alpha, \lambda < \infty, \quad \frac{1}{\lambda} + \frac{2}{\alpha} > 1 \quad \text{and} \quad 1 + \frac{3}{\gamma} = \frac{1}{\lambda} + \frac{2}{\alpha}.
\]
Assume that \(f \in H^1(\mathbb{R}^3)\), \(\partial_1 f, \partial_2 f \in L^\alpha(\mathbb{R}^3)\) and \(\partial_3 f \in L^\lambda(\mathbb{R}^3)\). Then there exists a constant \(C = C(\alpha, \lambda)\) such that

\[
\|f\|_{L^\gamma} \leq C \|\partial_1 f\|_{L^\alpha}^{\frac{1}{\alpha}} \|\partial_2 f\|_{L^\alpha}^{\frac{1}{\alpha}} \|\partial_3 f\|_{L^\lambda}^{\frac{1}{\lambda}}, \quad 1 \leq \gamma < \infty.
\]

Lemma 2.8. Let \(2 \leq \beta \leq 6\) and assume that \(f \in H^1(\mathbb{R}^3)\). Then there exists a constant \(C = C(\beta)\) such that

\[
\|f\|_{L^\beta} \leq C \|f\|_{L^2}^{\frac{6-\beta}{2\beta}} \|\partial_1 f\|_{L^2}^{\frac{\beta-2}{2} \|\partial_2 f\|_{L^2}^{\frac{\beta-2}{2}} \|\partial_3 f\|_{L^2}^{\frac{\beta-2}{2}}}
\]

\[
\leq C \|f\|_{L^2(\mathbb{R}^3)} \|f\|_{H^1(\mathbb{R}^3)}^{\frac{3(\beta-2)}{2}}.
\]

Now we are in the position to prove Theorem 2.6.

Proof: We differentiate the first and the second equation in (1.1) with respect to \(x_3\), we take the scalar product with \(\partial_3 u\) and \(\partial_3 \omega\), respectively and integrate over \(\mathbb{R}^3\), we get

\[
\frac{1}{2} \frac{d}{dt} \|\partial_3 u\|_{L^2}^2 + \|\nabla \partial_3 u\|_{L^2}^2 = - \int_{\mathbb{R}^3} (\partial_3 u \nabla v) u \partial_3 u dx + \int_{\mathbb{R}^3} \partial_3 (\nabla \times \omega) \cdot \partial_3 u dx.
\]

and

\[
\frac{1}{2} \frac{d}{dt} \|\partial_3 \omega\|_{L^2}^2 + \|\nabla \partial_3 \omega\|_{L^2}^2 + \|\nabla \cdot (\partial_3 \omega)\|_{L^2}^2
\]

\[
\leq - \int_{\mathbb{R}^3} (\partial_3 u \nabla \omega) \cdot \partial_3 \omega dx - 2 \|\partial_3 \omega\|_{L^2}^2 + \int_{\mathbb{R}^3} \partial_3 (\nabla \times u) \cdot \partial_3 \omega dx.
\]
Now, combining (2.4) and (2.5), one has after suitable integration by parts (recall that $\nabla \cdot u = 0$)

$$
\frac{1}{2} \frac{d}{dt} \left[ \|\partial_3 u\|_{L^2}^2 + \|\partial_3 \omega\|_{L^2}^2 \right] + \|\nabla \partial_3 u\|_{L^2}^2 + \|\nabla \partial_3 \omega\|_{L^2}^2 
\leq \int_{\mathbb{R}^3} \partial_3 (\nabla \times \omega) \cdot \partial_3 u dx + \int_{\mathbb{R}^3} \partial_3 (\nabla \times u) \cdot \partial_3 \omega dx - 2 \|\partial_3 \omega\|_{L^2}^2 
- \int_{\mathbb{R}^3} (\partial_3 u \cdot \nabla) u \cdot \partial_3 u dx - \int_{\mathbb{R}^3} (\partial_3 u \cdot \nabla) \omega \cdot \partial_3 \omega dx 
= A_1 + A_2 + A_3 + A_4 + A_5.
$$

(2.6)

Integrating by parts and using Hölder’s inequality and Young’s inequality (as in [15]), we derive the estimation of the first three terms on the right-hand side of (2.6) as

$$
A_1 + A_2 + A_3 = \int_{\mathbb{R}^3} \partial_3 (\nabla \times \omega) \cdot \partial_3 u dx + \int_{\mathbb{R}^3} \partial_3 (\nabla \times u) \cdot \partial_3 \omega dx - 2 \|\partial_3 \omega\|_{L^2}^2 
\leq 2 \|\partial_3 \omega\|_{L^2}^2 + \frac{1}{2} \|\nabla \partial_3 u\|_{L^2}^2 - 2 \|\partial_3 \omega\|_{L^2}^2 = \frac{1}{2} \|\nabla \partial_3 u\|_{L^2}^2.
$$

For $A_4$, using Lemma 2.3 together with the Hölder inequality and Young inequality, we find

$$
|A_4| = \left| \int_{\mathbb{R}^3} (\partial_3 u \cdot \nabla) u \cdot \partial_3 u dx \right|
\leq \|\partial_3 u \cdot \partial_3 u\|_{L^2} \|\nabla u\|_{L^2}
\leq \|\partial_3 u\|_{\mathcal{M}_2} \|\partial_3 u\|_{B^{r}_{2,1}} \|\nabla u\|_{L^2}
\leq \|\partial_3 u\|_{\mathcal{M}_2} \|\nabla \partial_3 u\|_{L^2} \|\partial_3 u\|_{L^2} \|\nabla u\|_{L^2}
$$

(2.7)

by using the following bilinear estimate (see [12, 13, 25]):

$$
\|fg\|_{L^2} \leq C \left\|f\right\|_{\mathcal{M}_2} \left\|g\right\|_{B^{r}_{2,1}}
$$

and the following interpolation inequality [28]:

$$
\|w\|_{B^{r}_{2,1}} \leq C \left\|w\right\|_{L^2}^{1-r} \|\nabla w\|_{L^2}^r.
$$
Similarly, we can bound

\[
|A_5| = \left| \int_{\mathbb{R}^3} (\partial_3 u \cdot \nabla) \omega \partial_\omega dx \right|
\]

(2.8)

\[
\leq \| \partial_3 u \|_{\mathcal{M}_{2, \frac{3}{2}}} \| \partial_3 \omega \|_{B_{2, 1}^r} \| \nabla \omega \|_{L^2} \\
\leq \| \partial_3 u \|_{\mathcal{M}_{2, \frac{3}{2}}} \| \partial_3 \omega \|_{L^2}^{1-r} \| \nabla \partial_3 \omega \|_{L^2}^r \| \nabla \omega \|_{L^2}.
\]

From the above inequalities and (2.6), we obtain

\[
\frac{1}{2} \frac{d}{dt} \left[ \| \partial_3 u \|_{L^2}^2 + \| \partial_3 \omega \|_{L^2}^2 \right] + \frac{1}{2} \| \nabla \partial_3 u \|_{L^2}^2 + \| \nabla \partial_3 \omega \|_{L^2}^2 \\
\leq \| \partial_3 u \|_{\mathcal{M}_{2, \frac{3}{2}}} \| \nabla \partial_3 u \|_{L^2}^r \| \partial_3 \omega \|_{L^2}^{1-r} \| \nabla \omega \|_{L^2} + \| \partial_3 u \|_{\mathcal{M}_{2, \frac{3}{2}}} \| \partial_3 \omega \|_{L^2}^{1-r} \| \nabla \partial_3 \omega \|_{L^2}^r \| \nabla \omega \|_{L^2}
\]

By Young's inequality \((a^{\alpha}b^{1-\alpha} \leq \alpha a + (1 - \alpha)b \leq a + b \) with \(a, b \geq 0\) and \(0 \leq \alpha \leq 1\), we find

\[
\frac{1}{2} \frac{d}{dt} \left[ \| \partial_3 u \|_{L^2}^2 + \| \partial_3 \omega \|_{L^2}^2 \right] + \frac{1}{2} \| \nabla \partial_3 u \|_{L^2}^2 + \| \nabla \partial_3 \omega \|_{L^2}^2 \\
\leq \left( \| \partial_3 u \|_{\mathcal{M}_{2, \frac{3}{2}}}^{\frac{2}{1-r}} \| \partial_3 u \|_{L^2}^{2(\frac{1}{2} + \frac{r}{2})} \| \nabla \omega \|_{L^2}^{\frac{2}{1-r}} \right) \frac{2}{2-r} \left( \| \nabla \partial_3 u \|_{L^2}^{2} \right)^{\frac{r}{2}} \\
+ 3 \left( \| \partial_3 u \|_{\mathcal{M}_{2, \frac{3}{2}}}^{\frac{2}{1-r}} \| \partial_3 \omega \|_{L^2}^{2(\frac{1}{2} + \frac{r}{2})} \| \nabla \omega \|_{L^2}^{\frac{2}{1-r}} \right) \frac{2}{2-r} \left( \| \nabla \partial_3 \omega \|_{L^2}^{2} \right)^{\frac{r}{2}} \\
\leq C \| \partial_3 u \|_{\mathcal{M}_{2, \frac{3}{2}}}^{\frac{2}{1-r}} \| \partial_3 u \|_{L^2}^{2(\frac{1}{2} + \frac{r}{2})} \| \nabla \omega \|_{L^2}^{\frac{2}{1-r}} + C \| \partial_3 u \|_{\mathcal{M}_{2, \frac{3}{2}}}^{\frac{2}{1-r}} \| \partial_3 \omega \|_{L^2}^{2(\frac{1}{2} + \frac{r}{2})} \| \nabla \omega \|_{L^2}^{\frac{2}{1-r}} \\
+ \frac{1}{2} \| \nabla \partial_3 \omega \|_{L^2}^{2} + \frac{1}{2} \| \nabla \partial_3 u \|_{L^2}^{2} \\
= \frac{1}{2} \| \nabla \partial_3 \omega \|_{L^2}^{2} + \frac{1}{2} \| \nabla \partial_3 u \|_{L^2}^{2} + C \| \partial_3 u \|_{L^2}^{2(\frac{1}{2} + \frac{r}{2})} \left( \| \partial_3 u \|_{\mathcal{M}_{2, \frac{3}{2}}}^{\frac{2}{1-r}} \right)^{\frac{1}{2-r}} \left( \| \nabla \omega \|_{L^2}^{2} \right)^{\frac{r}{2-r}} \\
+ C \| \partial_3 \omega \|_{L^2}^{2(\frac{1}{2} + \frac{r}{2})} \left( \| \partial_3 u \|_{\mathcal{M}_{2, \frac{3}{2}}}^{\frac{2}{1-r}} \right)^{\frac{1}{2-r}} \left( \| \nabla \omega \|_{L^2}^{2} \right)^{\frac{r}{2-r}} \\
\leq \frac{1}{2} \| \nabla \partial_3 \omega \|_{L^2}^{2} + \frac{1}{2} \| \nabla \partial_3 u \|_{L^2}^{2} + C \| \partial_3 u \|_{L^2}^{2(\frac{1}{2} + \frac{r}{2})} \left( \| \partial_3 u \|_{\mathcal{M}_{2, \frac{3}{2}}}^{\frac{2}{1-r}} + \| \nabla \omega \|_{L^2}^{2} \right) \\
+ C \| \partial_3 \omega \|_{L^2}^{2(\frac{1}{2} + \frac{r}{2})} \left( \| \partial_3 u \|_{\mathcal{M}_{2, \frac{3}{2}}}^{\frac{2}{1-r}} + \| \nabla \omega \|_{L^2}^{2} \right).
which implies that

\[
\frac{1}{2} \frac{d}{dt} (1 + \| \partial_3 u \|^2_{L^2} + \| \partial_3 \omega \|^2_{L^2}) + \| \nabla \partial_3 u \|^2_{L^2} + \| \nabla \partial_3 \omega \|^2_{L^2} 
\]

\[
\leq C(1 + \| \partial_3 u \|^2_{L^2}) \left( \| \partial_3 u \|_{M^2_{\mathbb{R}^3}}^2 + \| \nabla u \|^2_{L^2} \right) + C(1 + \| \partial_3 \omega \|^2_{L^2}) \left( \| \partial_3 u \|_{M^2_{\mathbb{R}^3}}^2 + \| \nabla \omega \|^2_{L^2} \right) 
\]

\[
\leq C(1 + \| \partial_3 u \|^2_{L^2} + \| \partial_3 \omega \|^2_{L^2}) \left( \| \partial_3 u \|_{M^2_{\mathbb{R}^3}}^2 + \| \nabla u \|^2_{L^2} + \| \nabla \omega \|^2_{L^2} \right),
\]

since \((\frac{1-r}{2-\sigma}) < 1\). It follows from Gronwall’s inequality together with the energy inequality (1.6) that

\[
(1 + \| \partial_3 u(t, \cdot) \|^2_{L^2} + \| \partial_3 \omega(t, \cdot) \|^2_{L^2}) 
\]

\[
\leq (1 + \| \partial_3 u_0 \|^2_{L^2} + \| \partial_3 \omega_0 \|^2_{L^2}) \exp \left( C \int_0^t \| \partial_3 u(s, \cdot) \|_{M^2_{\mathbb{R}^3}}^2 + \| \nabla u(s, \cdot) \|^2_{L^2} + \| \nabla \omega(s, \cdot) \|^2_{L^2} \, ds \right) 
\]

\[
\leq (1 + \| \partial_3 u_0 \|^2_{L^2} + \| \partial_3 \omega_0 \|^2_{L^2}) \exp \left( C \int_0^t \| \partial_3 u(s, \cdot) \|_{M^2_{\mathbb{R}^3}}^2 \, ds + C \| u_0 \|^2_{L^2} + \| \omega_0 \|^2_{L^2} \right) 
\]

\[
= (1 + \| \partial_3 u_0 \|^2_{L^2} + \| \omega_0 \|^2_{L^2}) e^{C(\| u_0 \|^2_{L^2} + \| \omega_0 \|^2_{L^2})} \exp \left( C \int_0^t \| \partial_3 u(s, \cdot) \|_{M^2_{\mathbb{R}^3}}^2 \, ds \right)
\]

and

\[(2.9) \quad \int_0^t (\| \nabla \partial_3 u(s, \cdot) \|^2_{L^2} + \| \nabla \partial_3 \omega(s, \cdot) \|^2_{L^2}) \, ds \leq C.\]

Here \(C\) denotes a constant dependent on the initial data and \(\| \partial_3 u(s, \cdot) \|_{L^{\frac{2}{1-r}}(0,T; M^2_{\mathbb{R}^3})}\).

Now we establish

\[(u, \omega) \in L^\infty(0,T; H^1) \cap L^2(0,T; H^2).\]

Taking the inner product of the equation (1.1) with \(-\Delta u\) and \(-\Delta \omega\) in \(L^2(\mathbb{R}^3)\), respectively, after suitable integration by parts, by the same calculation as that in [3], [12], [18], we obtain for \(t \in (0,T)\),

\[
\frac{1}{2} \frac{d}{dt} \| \nabla u(t) \|^2_{L^2} + \| \Delta u(t) \|^2_{L^2} = \int_{\mathbb{R}^3} (u \nabla) u. \Delta u \, dx - \int_{\mathbb{R}^3} (\nabla \times \omega) \cdot \Delta u \, dx 
\]

\[
= -\sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_k u \cdot (\partial_k u \nabla u) \, dx - \int_{\mathbb{R}^3} (\nabla \times u) \cdot \Delta \omega \, dx,
\]

10
\[
\frac{1}{2} \frac{d}{dt} \| \nabla \omega(t) \|^2_{L^2} + \| \nabla \omega(t) \|^2_{L^2} + \| \Delta \omega(t) \|^2_{L^2} + 2 \| \nabla \omega(t) \|^2_{L^2} \\
\leq \int_{\mathbb{R}^3} (u \cdot \nabla) \omega \Delta \omega \, dx - \int_{\mathbb{R}^3} (\nabla \times u) \nabla \omega \, dx \\
= - \sum_{k=1}^{3} \int_{\mathbb{R}^3} \partial_k \omega \cdot (\partial_k u \cdot \nabla) \omega \, dx - \int_{\mathbb{R}^3} (\nabla \times u) \Delta \omega \, dx,
\]
where we have used
\[
\int_{\mathbb{R}^3} (\nabla \times \omega) \Delta u \, dx = \int_{\mathbb{R}^3} (\nabla \times u) \Delta \omega \, dx.
\]

We sum the above equations to obtain
\[
\frac{1}{2} \frac{d}{dt} \| \nabla \omega(t) \|^2_{L^2} + \| \nabla \omega(t) \|^2_{L^2} + \| \Delta \omega(t) \|^2_{L^2} + 2 \| \nabla \omega(t) \|^2_{L^2} \\
+ \| \nabla \div \omega(t) \|^2_{L^2} + 2 \| \nabla \omega(t) \|^2_{L^2} \\
\leq C \| \nabla u \|^3_{L^2} + \| \nabla u \|_{L^3} \| \nabla \omega \|^3_{L^3} + 2 \| \nabla u \|_{L^2} \| \Delta \omega \|_{L^2} \\
\leq C \| \nabla u \|^3_{L^2} + (\| \nabla u \|^3_{L^3})^{\frac{1}{3}} (\| \nabla \omega \|^3_{L^3})^{\frac{2}{3}} + C \| \nabla u \|^2_{L^2} + \frac{1}{4} \| \Delta \omega \|^2_{L^2} \\
\leq C \| \nabla u \|^3_{L^2} + \| \nabla \omega \|^3_{L^2} + \frac{1}{4} \| \Delta \omega \|^2_{L^2} + C \| \nabla u \|_{L^2} \\
\leq C \| \nabla u \|^3_{L^2} + \| \nabla \omega \|^3_{L^2} + \frac{1}{4} \| \Delta \omega \|^2_{L^2} + C \| \nabla u \|_{L^2} \\
\leq C \| \nabla u \|^3_{L^2} + \| \nabla \omega \|^3_{L^2} + \frac{1}{4} \| \Delta \omega \|^2_{L^2} + C \| \nabla u \|_{L^2} \\
+ \frac{1}{4} \| \Delta \omega \|^2_{L^2} + C \| \nabla u \|^2_{L^2} \\
= \left( \| \nabla u \|^3_{L^2} \right) \frac{1}{2} \left( C \| \nabla u \|^3_{L^2} \| \nabla \partial_3 u \|_{L^2} \right)^{\frac{1}{2}} \\
+ \frac{1}{4} \| \Delta \omega \|^2_{L^2} + C \| \nabla u \|^2_{L^2} \\
\leq \frac{1}{2} \| \nabla u \|^2_{L^2} + C \| \nabla \omega \|^3_{L^2} \| \nabla \partial_3 u \|_{L^2} + \frac{1}{4} \| \Delta \omega \|^2_{L^2} + C \| \nabla \omega \|^3_{L^2} \| \nabla \partial_3 u \|_{L^2} \\
+ \frac{1}{4} \| \Delta \omega \|^2_{L^2} + C \| \nabla u \|^2_{L^2} \\
= \frac{1}{2} \| \nabla u \|^2_{L^2} + \| \Delta \omega \|^2_{L^2} + C \| \nabla \omega \|^2_{L^2} \| \nabla \partial_3 u \|_{L^2} \\
+ C \| \nabla \omega \|^2_{L^2} \| \nabla \partial_3 u \|_{L^2} + \frac{1}{2} \| \Delta \omega \|^2_{L^2} + \| \nabla u \|^2_{L^2} \\
\leq \frac{1}{2} \| \nabla u \|^2_{L^2} + \| \Delta \omega \|^2_{L^2} + C \| \nabla \omega \|^2_{L^2} \| \nabla \partial_3 u \|_{L^2} \\
+ C \| \nabla \omega \|^2_{L^2} \| \nabla \partial_3 u \|_{L^2} + \frac{1}{2} \| \Delta \omega \|^2_{L^2} + \| \nabla u \|^2_{L^2} \\
+ C \| \nabla \omega \|^2_{L^2} \| \nabla \partial_3 u \|_{L^2} ,
\]
by using Hölder inequality and applying (2.3) with $\alpha = \lambda = 2$ and $\gamma = 6$:

$$\| f \|_{L^6} \leq C \| \partial_1 f \|_{L^2}^{\frac{1}{3}} \| \partial_2 f \|_{L^2}^{\frac{1}{3}} \| \partial_3 f \|_{L^2}^{\frac{1}{3}}.$$

Hence

$$\frac{d}{dt}(\| \nabla u(t) \|_{L^2}^2 + \| \nabla \omega(t) \|_{L^2}^2) + \| \Delta u(t) \|_{L^2}^2 + \| \Delta \omega(t) \|_{L^2}^2 \leq C(1 + \| \nabla u \|_{L^2}^2 + \| \nabla u \|_{L^2}^2) \exp \left( C \int_0^t \| \nabla u(s, .) \|_{L^2}^2 + \| \nabla \omega(s, .) \|_{L^2}^2 \, ds \right).$$

Using Gronwall’s inequality, the energy inequality (1.6) and the estimate (2.9), we conclude that

$$\| \nabla u(t, .) \|_{L^2}^2 + \| \nabla \omega(t, .) \|_{L^2}^2 + \int_0^t (\| \Delta u(s, .) \|_{L^2}^2 + \| \Delta \omega(s, .) \|_{L^2}^2) \, ds \leq C.$$

for all $0 \leq t < T$. Hence

$$(u, \omega) \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)),$$

which gives that $u$ and $\omega$ are smooth. This completes the proof of Theorem 2.6.

Remark 2.1. Theorem 2.6 is still true for the Navier-Stokes equation with $\omega \equiv 0$, so we give an extension of Serrin’s regularity criterion for the Navier-Stokes equations [26].

3 Acknowledgements

The authors thank the referees for their invaluable comments and suggestions which helped improve the paper greatly. This work was done, while the first author was visiting Catania University in Italy. He thanks Department of Mathematics and Computer Science at the Catania university for his hospitality.
References

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[2] H. Beirão da Veiga, A new regularity class for the Navier–Stokes equations in $\mathbb{R}^n$, Chin. Ann. Math., Ser. B 16 (1995), 407-412.

[3] S. Benbernou, M.A. Ragusa, M. Terbeche, Z. Zhang, A note on the regularity criterion for the 3D MHD equations in $B_{\infty,\infty}$ space, Applied Mathematics and Computation, 238, (2014) 245–249, doi: 10.1016/j.amc.2014.03.095

[4] R. E. Caflisch, I. Klapper and G. Steele, Remarks on singularities, dimension, and energy dissipation for ideal hydrodynamics and MHD, Commun. Math. Phys. 184, 1847 (1997).

[5] C. Cao, Sufficient conditions for the regularity to the 3D Navier–Stokes equations, Discrete. Contin. Dyn. Syst., Ser. A 26, 1141 (2010).

[6] C. Cao and E. Titi, Global regularity criterion for the 3D Navier-Stokes equations involving one entry of the velocity gradient tensor, Arch. Rational Mech. Anal. 202 (2011), 919-932.

[7] C. Cao and J. Wu, Two regularity criteria for the 3D MHD equations, J. Differential Equations 248 (2010), 2263-2274.

[8] B.-Q. Dong and W. Zhang, On the regularity criterion for three-dimensional micropolar fluid flows in Besov spaces, Nonlinear Analysis 73 (2010), 2334-2341.

[9] A.C. Eringen, Theory of micropolar fluids, J. Math. Mech. 16 (1966), 1-18.

[10] L. Escauriaza, G.A. Serëgin, V. Šverák, $L^{3,\infty}$-solutions of Navier–Stokes equations and backward uniqueness, Russian Math. Surveys 58 (2003), 211-250.

[11] C. He and Z. Xin, On the regularity of weak solutions to the magnetohydrodynamic equations, J. Differential Equations 213 (2005), 235-254.
[12] S. Gala, Regularity criteria for the 3D magneto-micropolar fluid equations in the Morrey–Campanato space, Nonlinear Differential Equations Appl. 17 (2010), 181-194.

[13] S. Gala, On the regularity criteria for the three–dimensional micropolar fluid equations in the critical Morrey–Campanato space, Nonlinear Anal. Real World Appl. 12 (2011), 2142-2150.

[14] S. Gala, A remark on the logarithmically improved regularity criterion for the micropolar fluid equations in terms of pressure, Math. Meth. Appl. Sci. 34 (2011), 1945-1953.

[15] S. Gala, Z. Guo, M.A.Ragusa, A regularity criterion for the three-dimensional MHD equations in terms of one directional derivative of the pressure, Computers and Mathematics with Applications, (2016)

[16] G.P. Galdi, An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Vol. I & II, Springer-Verlag, 1994.

[17] Y. Giga, Solutions for semilinear parabolic equations in Lp and regularity of weak solutions of the Navier-Stokes equations, J. Differential Equations 62 (1986), 186-212.

[18] Y. Jia, X. Zhang, W. Zhang and B. Dong, Remarks on the regularity criteria of weak solutions to the three-dimensional micropolar fluid equations, Acta Math. Appl. Sinica, 29 (2013), 869-880.

[19] T. Kato, Strong $L^p$ solutions of the Navier-Stokes equations in Morrey spaces. Bol. Soc. Bras. Mat. 22 (1992), 127-155.

[20] I. Kukavica and M. Ziane, Navier–Stokes equations with regularity in one direction, J. Math. Phys. 48 (2007), no. 6, 065203, 10 pp.

[21] O. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Fluids, New York: Gorden Brech, 1969.

[22] O.A. Ladyzhenskaya, The Boundary Value Problems of Mathematical Physics, Springer-Verlag, 1985.
[23] P.G. Lemarié-Rieusset, Recent developments in the Navier-Stokes problem. Research Notes in Mathematics, Chapman & Hall, CRC, 2002.

[24] P.G. Lemarie-Rieusset and S. Gala, Multipliers between Sobolev spaces and fractional differentiation. J. Math. Anal. Appl. 322 (2006), 1030-1054.

[25] P.G. Lemarié-Rieusset, The Navier-Stokes equations in the critical Morrey-Campanato space. Rev. Mat. Iberoam. 23 (2007), no. 3, 897–930.

[26] Q. Liu, A regularity criterion for the Navier-Stokes equations in terms of one directional derivative of the velocity, Acta Appl. Math. DOI 10.1007/s10440-014-9975-z.

[27] G. Lukaszewicz, Micropolar Fluids. Theory and Applications, Modeling and Simulation in Science, Engineering and Technology, Birkhäuser, Boston, 1999.

[28] S. Machihara and T. Ozawa, Interpolation inequalities in Besov spaces. Proc. Amer. Math. Soc. 131 (2003), 1553-1556.

[29] P. Penel and M. Pokorný, Some new regularity criteria for the Navier–Stokes equations containing gradient of the velocity, Appl. Math. 49 (2004), 483-493.

[30] G. Prodi, Un teorema di unicità per le equazioni di Navier–Stokes, Ann. Mat. Pura Appl. 48 (1959), 173-182.

[31] M.A.Ragusa, Homogeneous Herz spaces and regularity results, Nonlinear Analysis, 71 (2009) , E1909-E1914, doi: 10.1016/j.na.2009.02.075.

[32] M.A.Ragusa, Embeddings in Morrey-Lotentz spaces, J.Optim. Theory and Appl. 154, (2)(2012), 491–499, doi:10.1007/s10957-012-0012-y.

[33] M.A.Ragusa, Necessary and sufficient condition for a VMO function, Appl. Math. and Comput., 218, (2012) 11952-11958, doi: 10.1016/j.amc.2012.06.005.

[34] J. Serrin, On the interior regularity of weak solutions of the Navier–Stokes equations, Arch. Rational Mech. Anal. 9 (1962), 187-195.
[35] R. Teman, Navier-Stokes Equations, Theory and Numerical Analysis, Amsterdam: North-Holland, 1977.

[36] X. Jia and Y. Zhou, Regularity criteria for the 3D MHD equations via partial derivatives, Kinetic and Related Models, 5 (2012), 505-516.

[37] X. Jia and Y. Zhou, A new regularity criterion for the 3D incompressible MHD equations in terms of one component of the gradient of pressure, J. Math. Anal. Appl. 396 (2012), 345-350.

[38] X. Jia and Y. Zhou, Remarks on regularity criteria for the Navier-Stokes equations via one velocity component. Nonlinear Anal. Real World Appl. 15 (2014), 239-245.

[39] X. Jia and Y. Zhou, Regularity criteria for the 3D MHD equations via partial derivatives, Kinetic and Related Models, 7 (2014), 291-304.

[40] Y. Zhou, A new regularity criterion for the Navier-Stokes equations in terms of the gradient of one velocity component, Methods Appl. Anal. 9 (2002), 563-578.

[41] Y. Zhou, A new regularity criterion for weak solutions to the Navier-Stokes equations, J. Math. Pures Appl. 84 (2005), 1496-1514.

[42] Y. Zhou and S. Gala, Regularity criteria in terms of the pressure for the Navier-Stokes equations in the critical Morrey-Campanato Space, Zeitschrift für Analysis und ihre Anwendungen 30 (2011), 83-93.

[43] Y. Zhou and M. Pokorny, On a regularity criterion for the Navier-Stokes equations involving gradient of one velocity component, Journal of Mathematical Physics 50 (12), 2009/12.

[44] Y. Zhou and M. Pokorny, On the regularity of the solutions of the Navier-Stokes equations via one velocity component, Nonlinearity 23 (2010), 1097-1107.