ON THE ℓ-MODULAR COMPOSITION FACTORS OF THE STEINBERG REPRESENTATION

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ABSTRACT. Let $G$ be a finite group of Lie type and $\text{St}_k$ be the Steinberg representation of $G$, defined over a field $k$. We are interested in the case where $k$ has prime characteristic $\ell$ and $\text{St}_k$ is reducible. Tinberg has shown that the socle of $\text{St}_k$ is always simple. We give a new proof of this result in terms of the Hecke algebra of $G$ with respect to a Borel subgroup and show how to identify the simple socle of $\text{St}_k$ among the principal series representations of $G$. Furthermore, we determine the composition length of $\text{St}_k$ when $G = \text{GL}_n(q)$ or $G$ is a finite classical group and $\ell$ is a so-called linear prime.

1. Introduction

Let $G$ be a finite group of Lie type and $\text{St}_k$ be the Steinberg representation of $G$, defined over a field $k$. Steinberg [30] showed that $\text{St}_k$ is irreducible if and only if $[G : B]_{1_k} \neq 0$ where $B$ is a Borel subgroup of $G$. We shall be concerned here with the case where $\text{St}_k$ is reducible. There is only very little general knowledge about the structure of $\text{St}_k$ in this case. We mention the works of Tinberg [33] (on the socle of $\text{St}_k$), Hiss [18] and Khammash [26] (on trivial composition factors of $\text{St}_k$) and Gow [14] (on the Jantzen filtration of $\text{St}_k$).

One of the most important open questions in this respect seems to be to find a suitable bound on the length of a composition series of $\text{St}_k$. Typically, this problem is related to quite subtle information about decomposition numbers; see, for example, Landrock–Michler [27] and Okuyama–Waki [29] where this is solved for groups with a $BN$-pair of rank 1. For groups of larger $BN$-rank, this problem is completely open.

In this paper, we discuss two aspects of this problem.

Firstly, Tinberg [33] has shown that the socle of $\text{St}_k$ is always simple, using results of Green [15] applied to the endomorphism algebra of the permutation module $k[G/U]$ where $U$ is a maximal unipotent subgroup. After some preparations in Sections 2, we show in Section 3 that a similar argument works with $U$ replaced by $B$. Since the corresponding endomorphism algebra (or “Hecke algebra”) is much easier to describe and its representation theory is quite well understood, this provides new additional information. For example, if $G = \text{GL}_n(q)$, then we can identify the partition of $n$ which labels the socle of $\text{St}_k$ in James’ [23] parametrisation of the unipotent simple modules of $G$; see Example 3.6

Quite remarkably, this involves a particular case of the “Mullineux involution” — and an analogue of this involution for other types of groups.

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In another direction, we consider the partition of the simple $kG$-modules into Harish-Chandra series, as defined by Hiss [19]. Dipper and Gruber [6] have developed a quite general framework for this purpose, in terms of so-called “projective restriction systems”. In Section 4, we shall present a simplified, self-contained version of parts of this framework which is tailored towards applications to $St_k$. This yields, first of all, new proofs of some of the results of Szechtman [32] on $St_k$ for $G = \text{GL}_n(q)$; moreover, in Example 4.9, we obtain an explicit formula for the composition length of $St_k$ in this case. Analogous results are derived for groups of classical type in the so-called “linear prime” case, based on [9], [16], [17]. For example, $St_k$ is seen to be multiplicity-free with a unique simple quotient in these cases — a property which does not hold in general for non-linear primes.

2. The Steinberg module and the Hecke algebra

Let $G$ be a finite group and $B, N \subseteq G$ be subgroups which satisfy the axioms for an “algebraic group with a split BN-pair” in [2, §2.5]. We just recall explicitly those properties of $G$ which will be important for us in the sequel. Firstly, there is a prime number $p$ such that we have a semidirect product decomposition $B = U \rtimes H$ where $H = B \cap N$ is an abelian group of order prime to $p$ and $U$ is a normal $p$-subgroup of $B$. The group $H$ is normal in $N$ and $W = N/H$ is a finite Coxeter group with a canonically defined generating set $S$; let $l: W \to N_0$ be the corresponding length function. For each $w \in W$, let $n_w \in N$ be such that $Hn_w = w$. Then we have the Bruhat decomposition

$$G = \coprod_{w \in W} Bn_w B = \coprod_{w \in W} Bn_w U,$$

where the second equality holds since $B = U \rtimes H$ and $H$ is normal in $N$.

Next, there is a refinement of the above decomposition. Let $w_0 \in W$ be the unique element of maximal length; we have $w_0^2 = 1$. Let $n_0 \in N$ be a representative of $w_0$ and $V := n_0^{-1} Un_0$; then $U \cap V = H$. For $w \in W$, let $U_w := U \cap n_w^{-1} V n_w$. (Note that $V$, $U_w$ do not depend on the choice of $n_0$, $n_w$ since $U$ is normalised by $H$.) Then we have the following sharp form of the Bruhat decomposition:

$$G = \coprod_{w \in W} Bn_w U_w, \quad \text{with uniqueness of expressions,}$$

that is, every $g \in Bn_w B$ can be uniquely written as $g = bn_w u$ where $b \in B$ and $u \in U_w$. It will occasionally be useful to have a version of this where the order of factors is reversed: By inverting elements, we obtain

$$G = \coprod_{w \in W} U_w^{-1} n_w B, \quad \text{with uniqueness of expressions.}$$

Now let $R$ be a commutative ring (with identity $1_R$) and $RG$ be the group algebra of $G$ over $R$. All our $RG$-modules will be left modules and, usually, finitely generated. For any subgroup $X \subseteq G$, we denote by $RX$ the trivial $RX$-module. Let $b := \sum_{b \in B} b \in RG$. Then $RGb$ is an $RG$-module which is canonically isomorphic to the induced module $\text{Ind}_B^G(R_B)$. 
Theorem 2.1 (Steinberg [30]). Consider the $RG$-submodule

$$St_R := RGc \subseteq RGb$$

where $c := \sum_{w \in W} (-1)^{l(w)} n_w b$.

(i) The set $\{ue | u \in U\}$ is an $R$-basis of $St_k$. Thus, $St_R$ is free over $R$ of rank $|U|$.

(ii) Assume that $R$ is a field. Then $St_R$ is an (absolutely) irreducible $RG$-module if and only if $[G : B]1_R \neq 0$.

(Note about the proof: Steinberg uses a list of 14 axioms concerning finite Chevalley groups and their twisted versions; all these axioms are known to hold in the abstract setting of “algebraic groups with a split $BN$-pair”; see [2, §2.5 and Prop. 2.6.1].)

When $R = k$ is a field, Tinberg [33, Theorem 4.10] determined the socle of $St_k$ and showed that this is simple. An essential ingredient in Tinberg’s proof are Green’s results [15] on the Hom functor, applied to the endomorphism algebra of the $kG$-module $kG u_1$, where $u_1 = \sum_{u \in U} u$. There is a description of this algebra in terms of generators and relations, and this is used in order to study the indecomposable direct summands of $kG u_1$. Our aim is to show that an analogous argument works directly with the module $kGb$, whose endomorphism algebra has a much simpler description.

So let again $R$ be any commutative ring (with $1_R$), and consider the Hecke algebra

$$\mathcal{H}_R = \mathcal{H}_R(G, B) := \text{End}_{RG} RGb$$

Following Green [15], a connection between (left) $RG$-modules and (left) $\mathcal{H}_R$-modules is established through the Hom functor

$$\mathfrak{g}_R: RG\text{-modules} \to \mathcal{H}_R\text{-modules}, \quad M \mapsto \mathfrak{g}_R(M) := \text{Hom}_{RG}(RGb, M),$$

where $\mathfrak{g}_R(M)$ is a left $\mathcal{H}_R$-module via $\mathcal{H}_R \times \mathfrak{g}_R(M) \to \mathfrak{g}_R(M)$, $(h, \alpha) \mapsto \alpha \circ h$. (See also [7, §2.C] where this Hom functor is studied in a somewhat more general context.) Note that, by [15] (1.3), we have an isomorphism of $R$-modules

$$\text{Fix}_B(M) := \{m \in M | b.m = m \text{ for all } b \in B\} \xrightarrow{\sim} \mathfrak{g}_R(M),$$

which takes $m \in \text{Fix}_B(M)$ to the map $\theta_m: RGb \to M$, $gb \mapsto gm$ ($g \in G$).

Now, $\mathcal{H}_R$ is free over $R$ with a standard basis $\{T_w | w \in W\}$, where the endomorphism $T_w: RGb \to RGb$ is given by

$$T_w(gb) = \sum_{g' \in G/B \text{ with } g^{-1}g' \in Bn_w B} g'b$$

$$(g \in G).$$

The multiplication is given as follows. Let $w \in W$, $s \in S$ and write $q_s := |U_s|1_R$. Then

$$T_s T_w = \begin{cases} T_sw & \text{if } l(sw) > l(w), \\ q_s T_sw + (q_s - 1)T_w & \text{if } l(sw) < l(w). \end{cases}$$
(See [13, §8.4] for a proof and further details.) The crucial step in our discussion consists of determining the \( \mathcal{H}_R \)-module \( \mathfrak{F}_R(\text{St}_R) \). This will rely on the following basic identity, an analogous version of which was shown by Tinberg [33, 4.9] for the action of the standard basis elements of the endomorphism algebra of \( kG_{\mathfrak{l}_1} \) (where \( k \) is a field).

**Lemma 2.2.** We have \( T_w(e) = (-1)^{l(w)}e \) for all \( w \in W \).

**Proof.** It is sufficient to show that \( T_s(e) = -e \) for \( s \in S \). Now, by definition, we have

\[
T_s(e) = \sum_{w \in W} (-1)^{l(w)}T_s(n_w b) = \sum_{w \in W} (-1)^{l(w)} \sum_{g B} g b
\]

where the second sum runs over all cosets \( g B \in G/B \) such that \( n_w^{-1}g \in Bn_s B \). By the sharp form of the Bruhat decomposition, a set of representatives for these cosets is given by \( \{n_w s \} \cup \{n_w v n_s \mid 1 \neq v \in U_s \} \). This yields

\[
T_s(e) = \sum_{w \in W} (-1)^{l(w)}n_w s b + \sum_{w \in W} \sum_{1 \neq v \in U_s} (-1)^{l(w)} n_w v n_s b.
\]

Since \( l(ws) = l(w) + 1 \) for \( w \in W \), the first sum equals \(-e\). So it suffices to show that

\[
\sum_{w \in W} (-1)^{l(w)} \kappa_w = 0 \quad \text{where} \quad \kappa_w := \sum_{1 \neq v \in U_s} n_w v n_s b.
\]

Let \( 1 \neq v \in U_s \). Since \( P_s = B \cup Bn_s B \) is a parabolic subgroup of \( G \), we have \( n_s^{-1} v n_s \in P_s \). By the sharp form of the Bruhat decomposition, \( n_s^{-1} v n_s \not\in B \) and so \( n_s^{-1} v n_s = v' n_s b_v \) where \( v' \in U_s \) and \( b_v \in B \) are uniquely determined by \( v \). Hence, we have \( n_w v n_s b_v = n_w n_s v' n_s b_v = n_w s n_s v' n_s b \) and so

\[
\kappa_w = \sum_{1 \neq v \in U_s} n_w v n_s b_v = \sum_{1 \neq v \in U_s} n_w s v' n_s b_v = \sum_{1 \neq v \in U_s} n_w s v n_s b = \kappa_{ws},
\]

where the third equality holds since, by [33 2.1], the map \( v \mapsto v' \) is a permutation of \( U_s \setminus \{1\} \). Consequently, we have

\[
\sum_{w \in W} (-1)^{l(w)} \kappa_w = \sum_{w \in W} (-1)^{l(w)} \kappa_{ws} = \sum_{w \in W} (-1)^{l(ws)} \kappa_w = - \sum_{w \in W} (-1)^{l(w)} \kappa_w.
\]

We conclude that the identity \( \sum_{w \in W} (-1)^{l(w)} \kappa_w = 0 \) holds if \( R = \mathbb{Z} \). For \( R \) arbitrary, we apply the canonical map \( \mathbb{Z}G \to RG \) and conclude that this identity remains valid in \( RG \). (Such an argument was already used by Steinberg in the proof of [30, Lemma 2].) \( \square \)

**Corollary 2.3.** We have \( \mathfrak{F}_R(\text{St}_R) = \langle \theta_{\mathfrak{u}_1} \rangle_R \) and the action of \( \mathcal{H}_R \) on this \( R \)-module of rank 1 is given by the algebra homomorphism \( \varepsilon : \mathcal{H}_R \to R \), \( T_w \mapsto (-1)^{l(w)} \).

**Proof.** Since \( \{ue \mid u \in U \} \) is an \( R \)-basis of \( \text{St}_R \) and \( H \) normalises \( U \), we have \( \text{Fix}_B(\text{St}_R) = \langle \mathfrak{u}_1 e \rangle_R \) and so \( \mathfrak{F}_R(\text{St}_R) = \langle \theta_{\mathfrak{u}_1} \rangle_R \). It remains to show that \( T_s \theta_{\mathfrak{u}_1} = -\theta_{\mathfrak{u}_1} \) for all \( s \in S \).
In analogy to Tinberg [33, 4.10], we call this the identity on $\text{St}_R$. Thus, if $F$ is a Frobenius map, then Khammash [26, Cor. 3.1] proved that the first inequality always is an equality.

Remark 2.4. Assume that $R$ is an integral domain and that we have a decomposition $RG_b = M_1 \oplus \cdots \oplus M_r$ where each $M_j$ is an indecomposable $RG$-module. Since $\{T_w \mid w \in W\}$ is an $R$-basis of $\mathcal{H}_R$, Lemma 2.2 implies that every idempotent in $\mathcal{H}_R$ either acts as the identity on $\text{St}_R$ or as 0. It easily follows that there is a unique $i$ such that $\text{St}_R \subseteq M_i$.

In analogy to Timberg [33, 4.10], we call this $M_i$ the Steinberg component of $RG_b$.

As observed by Khammash [25, (3.10)], the above argument actually shows that

$$\text{St}_R \subseteq \{ m \in RG_b \mid T_w(m) = (-1)^{l(w)} m \text{ for all } w \in W \} \subseteq M_i.$$  

Then Khammash [26, Cor. 3.3] proved that the first inequality always is an equality.

Remark 2.5. At some places in the discussion below, it will be convenient or even necessary to assume that $G$ is a true finite group of Lie type, as in [2, 1.18]. Thus, using the notation in [loc. cit.], we have $G = G^F$ where $G$ is a connected reductive algebraic group $G$ over $\overline{\mathbb{F}_p}$ and $F: G \to G$ is an endomorphism such that some power of $F$ is a Frobenius map. Then the ingredients of the $BN$-pair in $G$ will also be derived from $G$: we have $B = B^F$ where $B$ is an $F$-stable Borel subgroup of $G$ and $H = T_0^F$ where $T_0$ is an $F$-stable maximal torus contained in $B$; furthermore, $N = N_G(T_0)^F$ and $U = U^F$ where $U$ is the unipotent radical of $B$. This set-up leads to the following two definitions.

1. We define a real number $q > 0$ by the condition that $|U| = q^{\Phi/2}$ where $\Phi$ is the root system of $G$ with respect to $T_0$. Then there are positive integers $c_s > 0$ such that $|U_s| = q^{c_s}$ for all $s \in S$; see [2, 2.9]. Consequently, the relations in $\mathcal{H}_R$ read:

$$T_sT_w = \begin{cases} T_w & \text{if } l(sw) > l(w), \\ q^{c_s}T_{sw} + (q^{c_s} - 1)T_w & \text{if } l(sw) < l(w). \end{cases}$$
(2) The commutator subgroup $[U, U]$ is an $F$-stable closed connected normal subgroup of $U$. We define the subgroup $U^* := [U, U]^F \subseteq U$. Then $[U, U] \subseteq U^*$. (In most cases, we have $U^* = [U, U]$ but there are exceptions when $q$ is very small; see the remarks in §31 p. 258.) The definition of $U^*$ will be needed in Section 4 where we shall consider group homomorphisms $\sigma : U \to R^\times$ such that $U^* \subseteq \ker(\sigma)$.

3. The socle of the Steinberg module

We keep the general setting of the previous section and assume now that $R = k$ is a field and $\ell := \text{char}(k) \neq p$; thus, the parameters of the endomorphism algebra $\mathcal{H}_k$ satisfy $q_s \neq 0$ for all $s \in S$. With this assumption, we have the following two results:

(A) Every simple submodule of $kG\mathfrak{b}$ is isomorphic to a factor module of $kG\mathfrak{b}$, and vice versa; see Hiss [19 Theorem 5.8] where this is proved much more generally.

(B) $\mathcal{H}_k$ is a quasi-Frobenius algebra. Indeed, since $q_s \neq 0$ for all $s \in S$, $\mathcal{H}_k$ even is a symmetric algebra with respect to the trace form $\tau : \mathcal{H}_k \to k$ defined by $\tau(T_1) = 1$ and $\tau(T_w) = 0$ for $w \neq 1$; see, e.g., [13, 8.1.1].

It was first observed in [9, §2] that, in this situation, the results of Green [15] apply (the original applications of which have been to representations of $G$ over fields of characteristic equal to $p$). Let us denote by $\text{Irr}_k(G)$ the set of all simple $kG$-modules (up to isomorphism) and by $\text{Irr}_k(G | B)$ the set of all $Y \in \text{Irr}_k(G)$ such that $Y$ is isomorphic to a submodule of $kG\mathfrak{b}$. In the general framework of [19], this is the Harish-Chandra series consisting of the unipotent principal series representations of $G$. Furthermore, let $\text{Irr}(\mathcal{H}_k)$ be the set of all simple $\mathcal{H}_k$-modules (up to isomorphism). Then, by [15 Theorem 2], the Hom functor $\mathfrak{F}_k$ induces a bijection

$$(\spadesuit) \quad \text{Irr}_k(G | B) \cong \text{Irr}(\mathcal{H}_k), \quad M \mapsto \mathfrak{F}_k(M) = \text{Hom}_k(G(kG\mathfrak{b}), M);$$

furthermore, by [15 Theorem 1], each indecomposable direct summand of $kG\mathfrak{b}$ has a simple socle and a unique simple quotient. Combined with Remark 2.4, this already shows that $\text{St}_k$ has a simple socle. More precisely, we have:

**Theorem 3.1** (Cf. Tinberg [33, 4.10]). Let $Y \subseteq \text{St}_k$ be a simple submodule. Then $u \mathfrak{e} \in Y$ and, hence, $Y$ is uniquely determined. Furthermore, $\dim \mathfrak{F}_k(Y) = 1$ and the action of $\mathcal{H}_k$ on $\mathfrak{F}_k(Y)$ is given by the algebra homomorphism $\varepsilon : \mathcal{H}_k \to k$, $T_w \mapsto (-1)^{(w)}$.

**Proof.** By composing any map in $\mathfrak{F}_k(Y)$ with the inclusion $Y \subseteq \text{St}_k$, we obtain an embedding $\mathfrak{F}_k(Y) \hookrightarrow \mathfrak{F}_k(\text{St}_k)$ and we identify $\mathfrak{F}_k(Y)$ with a subset of $\mathfrak{F}_k(\text{St}_k)$ in this way. Now $Y \subseteq \text{St}_k \subseteq kG\mathfrak{b}$ and so $\mathfrak{F}_k(Y) \neq \{0\}$ by Property (A) above. Consequently, by Corollary 2.3 we have $\mathfrak{F}_k(Y) = \mathfrak{F}_k(\text{St}_k) = \langle \theta_{u \mathfrak{e}} \rangle_k$ and $\mathcal{H}_k$ acts via $\varepsilon$. Furthermore, by the identification $\mathfrak{F}_k(Y) \subseteq \mathfrak{F}_k(\text{St}_k)$, we must have $\theta_{u \mathfrak{e}}(kG\mathfrak{b}) \subseteq Y$ and so $u \mathfrak{e} \in Y$. \qed

**Proposition 3.2.** Let $Y \subseteq \text{St}_k$ be as in Theorem 3.1. Then $Y$ is absolutely irreducible and occurs only once as a composition factor of $\text{St}_k$. Moreover, $Y$ is the only composition factor of $\text{St}_k$ which belongs to $\text{Irr}_k(G | B)$.
Proof. Recall that $F_k(Y) \neq \{0\}$ and $Y$ corresponds to $\varepsilon : \mathcal{H}_k \to k$ via $(\clubsuit)$. Hence, by \cite[2.13(d)]{7}, we have $\text{End}_{kG}(Y) \cong \text{End}_{M}(\varepsilon) \cong k$ and so $Y$ is absolutely irreducible. Now let $\{0\} = Z_0 \subseteq Z_1 \subseteq \ldots \subseteq Z_r = St_k$ be a composition series and $Y_i := Z_i/Z_{i-1}$ for $i = 1, \ldots, r$ be the corresponding simple factors. Since $\ell \neq p$, the restriction of $St_k$ to the subgroup $U \subseteq B$ is semisimple and, hence, isomorphic to the direct sum of the restrictions of the $Y_i$. Taking fixed points under $U$, we obtain

$$\dim \text{Fix}_U(Y_1) + \ldots + \dim \text{Fix}_U(Y_r) = \dim \text{Fix}_U(St_k) = \dim (\mu_1 e)_k = 1.$$ 

Now, if $Y_i \in \text{Irr}_k(G \mid B)$, then $\text{Fix}_U(Y_i) \supseteq \text{Fix}_B(Y_i) \cong F_k(Y_i) \neq \{0\}$ by Property (A) and so we obtain a non-zero contribution to the sum on the left hand side. Hence, there can be at most one $Y_i$ which belongs to $\text{Irr}_k(G \mid B)$. Since $Y_1 = Y \subseteq kG\mathbf{b}$ does belong to $\text{Irr}_k(G \mid B)$, this proves the remaining assertions. \hfill $\Box$

Example 3.3. It is easily seen that $F_k(k_G)$ is also 1-dimensional (spanned by the function $kG\mathbf{b} \to k$ which takes constant value 1 on all $g\mathbf{b}$ for $g \in G$) and $\mathcal{H}_k$ acts on $F_k(k_G)$ via the algebra homomorphism $\text{ind} : \mathcal{H}_k \to k$ such that $\text{ind}(T_s) = q_s$ for all $s \in S$; see, e.g., \cite[4.3.3]{12}. Let $Y$ be the simple socle of $St_k$, as in Theorem 3.1. Then, by $(\clubsuit)$, we obtain:

$$Y \cong k_G \Leftrightarrow F_k(Y) \cong F_k(k_G) \Leftrightarrow \varepsilon = \text{ind} \Leftrightarrow q_s = -1 \text{ for all } s \in S.$$ 

Thus, we recover a result of Hiss \cite{18} and Khammash \cite{26} in this way. Furthermore, Proposition 3.2 implies that, if $q_s \neq 1$ for some $s \in S$, then $k_G$ is not even a composition factor of $St_G$. (This result is also contained in \cite{18}.)

Lemma 3.4. Let $M'$ be the Steinberg component in a given direct sum decomposition of $kG\mathbf{b}$, as in Remark 2.4. Then $St_k = M'$ if and only if $St_k$ is irreducible.

Proof. Assume first that $M' = St_k$ and let $Z \subseteq St_k$ be a maximal submodule. Now $St_k = M'$ is a factor module of $kG\mathbf{b}$ and so $St_k/Z$ belongs to $\text{Irr}_k(G \mid B)$, by Property (A). Hence, by Proposition 3.2, we must have $Z = \{0\}$. Conversely, assume that $St_k$ is irreducible. Then $\ell \mid [G : B]$ by Theorem 2.1. If $\ell = p$, then $\varepsilon$ is a non-zero scalar multiple of an idempotent in $kG$, by \cite[Lemma 2]{30}. Hence, $St_k$ is projective in this case and so $St_k$ is a direct summand of $kG\mathbf{b}$. If $\ell \neq p$, then the assumption $\ell \mid [G : B]$ implies that $kG\mathbf{b}$ is semisimple; see \cite[Lemma 4.3.2]{12}. Hence, again, $St_k$ is a direct summand of $kG\mathbf{b}$. In both cases, it follows that $St_k = M'$. \hfill $\Box$

Example 3.5. Assume that $G$ has a $BN$-pair of rank 1, that is, $W = \langle s \rangle$ where $s \in W$ has order 2. Then, by the sharp form of the Bruhat decomposition, we have $[G : B] = 1 + |U| = 1 + \dim St_k$. Thus, there are only two cases.

If $q_s \neq -1$, then $kG\mathbf{b} \cong k_G \oplus St_k$ and $St_k$ is irreducible by Theorem 2.1.

If $q_s = -1$, then the socle of $St_k$ is the trivial module $k_G$ by Example 3.3.
In the second case, the structure of $\text{St}_k$ can be quite complicated. For example, let $G = 2G_2(q^2)$ be a Ree group, where $q$ is an odd power of $\sqrt{3}$. If $\ell = 2$, then Landrock–Michler [27, Prop. 3.8(b)] determined socle series for $kG\mathfrak{b}$ and $\text{St}_k$:

\[
\begin{align*}
  kG & : \varphi_2 \quad \varphi_4 \quad \varphi_3 \quad \varphi_5 \\
  kG & : \varphi_2 \\
\end{align*}
\]

where $\varphi_i$ ($i = 1, 2, 3, 4, 5$) are simple $kG$-modules and $\varphi_4$ is the contragredient dual of $\varphi_5$.

It is not true in general that $\text{St}_k$ has a unique simple quotient. For example, let $G = \text{GU}_3(q)$ where $q$ is any prime power. Assume that $\ell$ is a prime such that $\ell | q+1$. Then socle series for $\text{St}_k$ are known by the work of various authors; see Hiss [20, Theorem 4.1] and the references there:

\[
\begin{align*}
  \varphi & \quad \varphi \\
  kG & : \varphi \\
\end{align*}
\]

where $\varphi$ and $\vartheta$ are simple $kG$-modules. See also the examples in Gow [14, §5].

**Example 3.6.** Let $G = \text{GL}_n(q)$ and $\mathcal{U}_k(G)$ be the set of all $Y \in \text{Irr}_k(G)$ such that $Y$ is a composition factor of $kG\mathfrak{b}$. James [23, 16.4] called these the unipotent modules of $G$ and showed that there is a canonical parametrisation

$\mathcal{U}_k(G) = \{D_\mu \mid \mu \vdash n\}$. (See also [24, 7.35].) Here, $D_{(n)} = kG$, as follows immediately from [23, Def. 1.11].

The above parametrisation is characterised as follows. For each partition $\lambda \vdash n$, let $M_\lambda$ be the permutation representation of $G$ on the cosets of the corresponding parabolic subgroup $P_\lambda \subseteq G$ (block triangular matrices with diagonal blocks of sizes given by the parts of $\lambda$). Then $D_\mu$ has composition multiplicity 1 in $M_\mu$ and composition multiplicity 0 in $M_\lambda$ unless $\lambda \models \mu$; see [23, 11.12(iv), 11.13]. This shows, in particular, that the above parametrisation is consistent with other known parametrisations of $\mathcal{U}_k(G)$, e.g., the one in [9, §3] based on properties of the $\ell$-modular decomposition matrix of $G$.

If $\ell = 0$, let us set $e := \infty$; if $\ell$ is a prime ($\neq p$), then let

\[
e := \min\{i \geq 2 \mid 1 + q + q^2 + \cdots + q^{i-1} \equiv 0 \text{ mod } \ell\}.
\]

Then, by James [24, Theorem 8.1(ix), (xi)], the subset $\text{Irr}_k(G \mid B) \subseteq \mathcal{U}_k(G)$ consists precisely of those $D_\lambda$ where $\lambda \vdash n$ is $e$-regular. Now let $Y$ be the socle of $\text{St}_k$, as in Theorem 3.1. Then $Y \in \text{Irr}_k(G \mid B)$ and so $Y \cong D_{\mu_0}$ for a well-defined $e$-regular partition $\mu_0 \vdash n$. This partition $\mu_0$ can be identified as follows. Write $n = (e - 1)m + r$ where
0 \leq r < e - 1. (If e = \infty, then m = 0 and r = n.) We claim that
\[
\mu_0 = (m + 1, m + 1, \ldots, m + 1, m, m, \ldots, m) \vdash n.
\]
Indeed, by Theorem 3.1 and (\textcircled{a}), the $kG$-module $Y$ corresponds to the 1-dimensional representation $\varepsilon : \mathcal{H}_k \to k$. Now $\text{Irr}(\mathcal{H}_k)$ also has a natural parametrisation by the $e$-regular partitions of $n$, a result originally due to Dipper and James; see, e.g., [24] 8.1(i), [12] §3.5 and the references there. By [24] Theorem 8.1(xii) (or the general discussion in 3.7 below), this parametrisation is compatible with the above parametrisation of $\mathcal{H}_k(G)$, in the sense that the partition $\mu_0 \vdash n$ such that $Y \cong D_{\mu_0}$ must also parametrise $\varepsilon$. Now note that $\varepsilon \circ \gamma = \text{ind}$, where $\text{ind} : \mathcal{H}_k \to k$ is defined in Example 3.3 and $\gamma : \mathcal{H}_k \to \mathcal{H}_k$ is the algebra automorphism such that $\gamma(T_s) = -q_s T_s^{-1}$ (see [13] Exc. 8.2). The definitions immediately show that $\text{ind}$ is parametrised by the partition $(n)$. Thus, our problem is a special case of describing the “Mullineux involution” on $e$-regular partitions which, for the particular partition $(n)$, has the solution stated above by Mathas [28, 6.43(iii)]. (I thank Nicolas Jacon for pointing out this reference to me.)

We remark that Ackermann [1] Prop. 3.1 already showed that $\text{St}_k$ has precisely one composition factor $D_{\mu_0}$ where $\mu_0$ is the image of $(1^n)$ under the Mullineux involution; however, he does not locate $D_{\mu_0}$ in a composition series of $\text{St}_k$.

3.7. For general $G$, the definition of unipotent modules is more complicated than for $\text{GL}_n(q)$ (see, e.g., [10] §1), but one can still proceed as follows. Let us assume that $G$ is a true finite group of Lie type, as in Remark 2.5. We shall write $\text{Irr}_G(W) = \{E^\lambda \mid \lambda \in \Lambda\}$ where $\Lambda$ is some finite indexing set. It is a classical fact that, if $k = \mathbb{C}$, then there is a bijection $\text{Irr}_G(W) \leftrightarrow \text{Irr}(\mathcal{H}_G)$, $E^\lambda \leftrightarrow E^\lambda_q$, and a decomposition
\[
\mathbb{C}G^B \cong \bigoplus_{\lambda \in \Lambda} \rho^\lambda \oplus \cdots \oplus \rho^\lambda \quad \text{where} \quad \mathfrak{S}_C(\rho^\lambda) \cong E^\lambda_q \quad \text{for all} \quad \lambda \in \Lambda.
\]
Hence, we have a natural parametrisation $\text{Irr}_G(G \mid B) = \{\rho^\lambda \mid \lambda \in \Lambda\}$ in this case; see, e.g., Carter [2] §10.11, Curtis–Reiner [1] §68B (and also [7] Exp. 2.2), where the Hom functor is linked to the settings in [loc. cit.]). In general, under mild conditions on $k$, it is shown in [8] Theorem 1.1 that there is still a natural parametrisation of $\text{Irr}_k(G \mid B)$, but now by a certain subset $\Lambda^0_k \subseteq \Lambda$. We briefly describe how this is done, where we refer to the exposition in [12] §4.4 for further details and references.

First, to each $E^\lambda$ one can attach a numerical value $a_\lambda \in \mathbb{Z}_{\geq 0}$ (Lusztig’s “$a$-invariant”); note that $\lambda \mapsto a_\lambda$ depends on the exponents $c_s$ such that $|U_s| = q^{c_s}$ for $s \in S$. Then, under some mild conditions on $k$, the algebra $\mathcal{H}_k$ is “cellular” in the sense of Graham–Lehrer, where the corresponding cell modules are parametrized by $\Lambda$, and $\Lambda$ is endowed with the partial order $\preceq$ such that $\mu \preceq \lambda$ if and only if $\mu = \lambda$ or $a_\lambda < a_\mu$. Finally, by the general theory of cellular algebras, there is a canonically defined subset $\Lambda^0_k \subseteq \Lambda$ such that
\[
\text{Irr}(\mathcal{H}_k) = \{L^\mu_k \mid \mu \in \Lambda^0_k\},
\]
where $L_k^\mu$ is the unique simple quotient of the cell module corresponding to $\lambda \in \Lambda_k^\circ$. Hence, via the Hom functor and (♦), we obtain the desired parametrisation

$$\text{Irr}_k(G \mid B) = \{Y^\mu \mid \mu \in \Lambda_k^\circ\} \quad \text{where} \quad \mathfrak{S}_k(Y^\mu) \cong L_k^\mu \quad \text{for} \quad \mu \in \Lambda_k^\circ.$$

Let $M \in \text{Irr}(\mathscr{H}_k)$ and denote by $d_{\lambda,M}$ the multiplicity of $M$ as a composition factor of the cell module indexed by $\lambda \in \Lambda$. Then, by [12] 3.2.7, the unique $\mu \in \Lambda_k^\circ$ such that $M \cong L^\mu$ is characterised by the condition that $\mu$ is the unique element at which the function $\{\lambda \in \Lambda \mid d_{\lambda,M} \neq 0\} \to a_\lambda$ takes its minimum value.

Now recall that the simple socle $Y \subseteq \text{St}_k$ belongs to $\text{Irr}_k(G \mid B)$ and it corresponds, via the Hom functor and (♦), to the 1-dimensional representation $\varepsilon : \mathscr{H}_k \to k$. The unique $\mu_0 \in \Lambda_k^\circ$ such that $Y \cong Y^{\mu_0}$ is found as follows. We order the elements of $\Lambda$ according to increasing $a$-invariant; then $\mu_0$ is the first element in this list for which we have $d_{\mu_0,\varepsilon} \neq 0$. Note also that $\varepsilon$ is afforded by a cell module; the unique $\lambda_0 \in \Lambda$ labelling this cell module is characterised by the condition that $a_{\lambda_0} = \max\{a_\lambda \mid \lambda \in \Lambda\}$ (see, e.g., [12] 1.3.3).

For example, if $G = \text{GL}_n(q)$, then $W = \mathfrak{S}_n$ and $\Lambda$ is the set of partitions of $n$. In this case, we have $\lambda_0 = (1^n)$ and $\mu_0$ is described in Example 3.4.

If tables with the decomposition numbers $d_{\lambda,M}$ for $\mathscr{H}_k$ are known, then $\mu_0$ can be simply read off these tables. Thus, $\mu_0 \in \Lambda_k^\circ$ can be determined for all groups of exceptional type, using the information in [13] App. F, [12] Chap. 7; the results are given in Table 1. (If there is no entry in this table corresponding to a certain value of $e$, then this means that $\ell \mid [G : B]$ and so $\text{St}_k$ is simple.)

| $e$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 14 | 15 | 18 | 20 | 24 | 30 |
|-----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| $G_2(q)$ | $1_W$ | $\sigma_2$ | $\sigma_1$ | | | | | | | | | | | | | |
| $^3D_4(q)$ | $1_W$ | $\sigma_2$ | $\varepsilon_1$ | $\sigma_1$ | | | | | | | | | | | | | |
| $^2F_4(q^2)$ | $\sigma_2$ | $\varepsilon_1$ | $\sigma_2$ | | | | | | | | | | | | | |
| $E_4(q)$ | $1_a$ | $4_a$ | $6_a$ | $12$ | $9_a$ | $4_5$ | | | | | | | | | | |
| $^2E_6(q)$ | $8_a$ | $4_1$ | $1_3$ | $4_a$ | $8_4$ | $8_2$ | $9_4$ | $4_5$ | | | | | | | | |
| $E_6(q)$ | $1_p$ | $15_p$ | $10_s$ | $24_p$ | $60_p$ | $30_p$ | $20_p$ | $p_6$ | | | | | | | | |
| $E_7(q)$ | $1_a$ | $15_a$ | $70_a$ | $84_a$ | $210_b$ | $27_a$ | $105_b$ | $35_b$ | $21_b$ | $56_a$ | $27_a$ | $7_a$ | | | | |
| $E_8(q)$ | $1_x$ | $50_x$ | $175_x$ | $168_y$ | $420_y$ | $300_z$ | $2835_z$ | $50_z$ | $448_z$ | $1400_z$ | $700_z$ | $84_x$ | $210_x$ | $112_x$ | $35_x$ | $8_x$ |

In type $E_8$, $\ell > 5$; otherwise, $\ell > 3$; here, $e := \min\{i > 2 \mid 1 + q_0 + q_0^2 + \cdots + q_0^{i-1} \equiv 0 \text{ mod } \ell\}$, with $q_0 := q$ in all cases except for $^2F_4(q^2)$, where $q$ is an odd power of $\sqrt{2}$ and $q_0 := q^2$.

Just to illustrate the procedure (and to fix some notation), let us consider the case where $G = ^2F_4(q^2)$; here, $q$ is an odd power of $\sqrt{2}$. Setting $q_0 := q^2$, we have

$$|B| = q_0^{12}(q_0 - 1)^2 \quad \text{and} \quad [G : B] = (q_0 + 1)(q_0^2 + 1)(q_0^3 + 1)(q_0^6 + 1).$$
Now, \( W = \langle s_1, s_2 \rangle \) is dihedral of order 16 and we have \( \{q_{s_1}, q_{s_2}\} = \{q_0, q_0^2\} \). We fix the notation such that \( q_{s_1} = q_0^2 \) and \( q_{s_2} = q_0 \). As in [12] Exp. 1.3.7, we have
\[
\text{Irr}_C(W) = \{1_W, \varepsilon, \varepsilon_1, \varepsilon_2, \sigma_1, \sigma_2, \sigma_3\}
\]
where \( \varepsilon_1, \varepsilon_2 \) are 1-dimensional and determined by \( \varepsilon_1(s_1) = \varepsilon_2(s_2) = 1 \) and \( \varepsilon_1(s_2) = \varepsilon_2(s_1) = -1 \); furthermore, each \( \sigma_i \) is 2-dimensional and the labelling is such that the trace of \( \sigma_1 \) on \( s_1 s_2 \) equals \( \sqrt{2} \), that of \( \sigma_2 \) equals 0, and that of \( \sigma_3 \) equals \( -\sqrt{2} \).

The “mild condition” on \( k \) is that \( \ell \) must be a “good” prime for the underlying algebraic group; so, \( \ell > 3 \). Assuming that \( \ell \mid [G : B] \), we have the following cases to consider:
\[
\ell \mid q_0 + 1, \quad \ell \mid q_0^2 + 1, \quad \ell \mid q_0^2 - q_0 + 1, \quad \ell \mid q_0^4 - q_0^2 + 1,
\]
which correspond to \( e = 2, 4, 6, 12 \), respectively. For example, for \( e = 2, 4 \), the decomposition numbers \( d_{\lambda,M} \) are given as follows; see [12] Table 7.6:

|\( e = 2 \) | \( a_{\lambda} \) | \( d_{\lambda,M} \) |
|---|---|---|
|1\( W \) | 0 | 1 . . . |
|\( \varepsilon_1 \) | 1 | 1 . . . |
|\( \sigma_1 \) | 2 | . . 1 . |
|\( \sigma_2 \) | 2 | 1 1 . . |
|\( \sigma_3 \) | 2 | . . 1 |
|\( \varepsilon_2 \) | 5 | 1 . . |
|\( \varepsilon \) | 12 | 1 . . |

|\( e = 4 \) | \( a_{\lambda} \) | \( d_{\lambda,M} \) |
|---|---|---|
|1\( W \) | 0 | 1 . . . |
|\( \varepsilon_1 \) | 1 | 1 . . |
|\( \sigma_1 \) | 2 | . . 1 . |
|\( \sigma_2 \) | 2 | . . 1 |
|\( \sigma_3 \) | 2 | 1 1 . . |
|\( \varepsilon_2 \) | 5 | 1 . . |
|\( \varepsilon \) | 12 | 1 . . |

Those representations which belong to \( \Lambda_0^0 \) are marked by “•”. The above procedure for finding \( \mu_0 \) now yields \( \mu_0 = \sigma_2 \) for \( e = 2 \) and \( \mu_0 = \varepsilon_1 \) for \( e = 4 \).

**Remark 3.8.** For groups of classical type, \( \Lambda \) is a set of certain bipartitions of \( n \) and the subsets \( \Lambda^0_k \subset \Lambda \) are explicitly known in all cases; see [12]. Nicolas Jacon has pointed out to me that then \( \mu_0 \in \Lambda^0_k \) can be described explicitly using the results of Jacon–Lecouvey [22]. This will be discussed elsewhere in more detail.

### 4. The Steinberg module and Harish-Chandra series

We shall assume from now on that \( G = G^F \) is a true finite group of Lie type, as in Remark 2.5. Then \( G \) satisfies the “commutator relations” and so the parabolic subgroups of \( G \) admit Levi decompositions; see Carter [2] §2.6; Curtis–Reiner [4] §70A. For each subset \( J \subset S \), let \( P_J \subset G \) be the corresponding standard parabolic subgroup, with Levi decomposition \( P_J = U_J \rtimes L_J \) where \( U_J \) is the unipotent radical of \( P_J \) and \( L_J \) is the standard Levi complement. The Weyl group of \( L_J \) is \( W_J = \langle J \rangle \) and \( L_J \) itself is a true finite group of Lie type. Let \( A \) be a commutative ring (with \( 1_A \)) such that \( p \) is invertible in \( A \). Then we obtain functors, called Harish-Chandra induction and restriction,
\[
R^S_J: AL_J \text{-modules} \to AG \text{-modules},
\]
\[
\ast R^S_J: AG \text{-modules} \to AL_J \text{-modules},
\]
which satisfy properties analogous to the usual induction and restriction, like transitivity, adjointness and a Mackey formula; we refer to [5, 19] and the survey in [7, §3] for further details. An AG-module $Y$ is called cuspidal if $R^S_j(Y) = \{0\}$ for all $J \subseteq S$. In this general setting, we have the following important result due to Dipper–Du [5] and Howlett–Lehrer [21]. Let $I, J \subseteq S$ be subsets such that $wIw^{-1} = J$ for some $w \in W$; let $n \in N$ be a representative of $w$. Then $nL_in^{-1} = L_J$ and

$$R^S_j(X) \cong R^S_j(nX) \quad \text{for any } AL_I\text{-module } X.$$  

(Here, we denote by $nX$ the usual conjugate module for $\mathcal{O}L_J$; see, e.g., [3, §10B]). In analogy to [21 (70.11)], we will refer to this as the “Strong Conjugacy Theorem”.

Furthermore, we now place ourselves in the usual setting for modular representation theory; see, e.g., [3, §16A]. Thus, we assume that our field $k$ has characteristic $\ell > 0$ (where $\ell \neq p$ as before), and that $k$ is the residue field of a discrete valuation ring $\mathcal{O}$ with field of fractions $K$ of characteristic 0. Both $K$ and $k$ will be assumed to be “large enough”, that is, $K$ and $k$ are splitting fields for $G$ and all its subgroups. An $\mathcal{O}G$-module $M$ which is finitely generated and free over $\mathcal{O}$ will be called an $\mathcal{O}G$-lattice. If $M$ is an $\mathcal{O}G$-lattice, then we may naturally regard $M$ as a subset of the $KG$-module $MK := K \otimes_\mathcal{O} M$; furthermore, by “$\ell$-modular reduction”, we obtain a $kG$-module $\overline{M} := M/\mathfrak{p}M \cong k \otimes_\mathcal{O} M$ where $\mathfrak{p}$ is the unique maximal ideal of $\mathcal{O}$. Finally note that, by [3, Exc. 6.16], idempotents can be lifted from $kG$ to $\mathcal{O}G$, hence, $\mathcal{O}G$ is “semiperfect”. We shall freely use standard notions and properties of projective covers, pure submodules etc.; see [3, §4D, §6].

Harish-Chandra induction and restriction are compatible with this set-up. Indeed, let $J \subseteq S$. If $X$ is an $\mathcal{O}L_J$-lattice and $Y$ is an $\mathcal{O}G$-lattice, then $R^S_j(X)$ is an $\mathcal{O}G$-lattice, $^*R^S_j(Y)$ is an $\mathcal{O}L_J$-lattice, and we have

$$KR^S_j(X) \cong R^S_j(KX) \quad \text{and} \quad K^*R^S_j(Y) \cong ^*R^S_j(KY),$$

$$\overline{R^S_j(X)} \cong \overline{R^S_j(X)} \quad \text{and} \quad \overline{R^S_j(Y)} \cong \overline{R^S_j(Y)}.$$

4.1. By Theorem 2.1, we have the “canonical” Steinberg lattice $St_\mathcal{O} = \mathcal{O}G\mathfrak{c}$. Here. We can naturally identify $KSt_\mathcal{O}$ with $St_K$ and $\overline{St_\mathcal{O}}$ with $St_k$. Since $\text{char}(K) = 0$, the $KG$-module $St_K$ is irreducible. We obtain further $\mathcal{O}G$-lattices affording $St_K$ as follows. Let $\sigma : U \to K^\times$ be a group homomorphism. Since $\ell \nmid |U|$, the values of $\sigma$ lie in $\mathcal{O}^\times$. Then $u_\sigma := \sum_{u \in U} \sigma(u)u \in \mathcal{O}G$ and so $\Gamma_\sigma := \mathcal{O}G u_\sigma$ is an $\mathcal{O}G$-lattice. Since $u_\sigma^2 = |U|u_\sigma$ and $|U|$ is a unit in $\mathcal{O}$, the module $\Gamma_\sigma$ is projective. Furthermore, $\text{Hom}_{\mathcal{O}G}(\Gamma_\sigma, St_\mathcal{O}) \cong u_\sigma St_\mathcal{O} = (u_\sigma, e)_\mathcal{O}$ where the last equality holds since $\{ue \mid u \in U\}$ is an $\mathcal{O}$-basis of $St_\mathcal{O}$ and since $u_\sigma u = \sigma(u)^{-1}u_\sigma$ for all $u \in U$. Thus, $\text{Hom}_{\mathcal{O}G}(\Gamma_\sigma, St_\mathcal{O}) = (\varphi_\sigma)_\mathcal{O}$ where $\varphi_\sigma : \Gamma_\sigma \to St_\mathcal{O}, \gamma \mapsto \gamma u_\sigma e$.

The same computation also works over $K$, hence we obtain $\dim \text{Hom}_{K\mathcal{O}}(K\Gamma_\sigma, St_K) = 1$.

**Proposition 4.2** (Cf. Hiss [19, §6]). For any $\sigma : U \to K^\times$ as above, there is a unique pure $\mathcal{O}G$-sublattice $\Gamma'_\sigma \subseteq \Gamma_\sigma$ such that, if we set $\mathcal{I}_\sigma := \Gamma_\sigma/\Gamma'_\sigma$, then $K\mathcal{I}_\sigma \cong St_K$. Furthermore,
\(\varphi_\sigma\) induces an injective \(\mathcal{O}G\)-module homomorphism \(\rho_\sigma: \mathcal{I}_\sigma \rightarrow \text{St}_\sigma\). The \(KG\)-module \(D_\sigma := \mathcal{F}_\sigma / \text{rad}(\mathcal{F}_\sigma)\) is simple and it occurs exactly once as a composition factor of \(\text{St}_k\).

Proof. First we show that a sublattice \(\Gamma'_\sigma \subseteq \Gamma_\sigma\) with the desired properties exists. Now, since \(KG\) is semisimple and \(\dim \text{Hom}_{KG}(KG, \text{St}_k) = 1\), we can write \(K\Gamma_\sigma = Z_1 \oplus Z_2\) where \(Z_1, Z_2\) are \(KG\)-submodules such that \(Z_1 \cong \text{St}_k\) and \(\text{Hom}_{KG}(Z_2, \text{St}_k) = \{0\}\). Then \(\Gamma'_\sigma := \Gamma_\sigma \cap Z_2\) is a pure submodule of \(\Gamma_\sigma\). Consequently, \(\mathcal{I}_\sigma := \Gamma_\sigma / \Gamma'_\sigma\) is an \(\mathcal{O}G\)-lattice such that \(K\mathcal{I}_\sigma \cong \text{St}_k\). Now consider the map \(\varphi_\sigma: \Gamma_\sigma \rightarrow \text{St}_\sigma\). Since \(\text{Hom}_{KG}(Z_2, \text{St}_k) = \{0\}\), we have \(\Gamma'_\sigma \subseteq \ker(\varphi_\sigma)\) and so we obtain an induced \(\mathcal{O}G\)-module homomorphism \(\rho_\sigma: \mathcal{I}_\sigma \rightarrow \text{St}_\sigma\). Since \(K\mathcal{I}_\sigma\) is irreducible and \(\varphi_\sigma \neq 0\), the map \(\rho_\sigma\) is injective.

Let us further write \(\Gamma_\sigma = P_1 \oplus \ldots \oplus P_r\) where each \(P_i\) is an \(\mathcal{O}G\)-lattice which is projective and indecomposable. Then \(K\Gamma_\sigma = KP_1 \oplus \ldots \oplus KP_r\). Since \(\dim \text{Hom}_{KG}(K\Gamma_\sigma, \text{St}_k) = 1\), there is a unique \(i\) such that \(\text{Hom}_{KG}(KP_i, \text{St}_k) \neq \{0\}\). Then \(Z_1 \subseteq KP_i\) and \(KP_j \subseteq Z_2\) for all \(j \neq i\). Thus, we have \(\mathcal{I}_\sigma \cong P_i / (P_i \cap Z_2)\) and so \(P_i\) is a projective cover of \(\mathcal{I}_\sigma\). This certainly implies that \(D_\sigma = \mathcal{F}_\sigma / \text{rad}(\mathcal{F}_\sigma) \cong \mathcal{P}_i / \text{rad}(\mathcal{P}_i)\) is simple and that \(\mathcal{P}_i\) is a projective cover of \(D_\sigma\). Since \(\mathcal{U}, \mathcal{F} \in \rho_\sigma(\mathcal{I}_\sigma)\), the induced map \(\mathcal{P}_\sigma: \mathcal{F}_\sigma \rightarrow \text{St}_k\) is non-zero and so \(D_\sigma\) is a composition factor of \(\text{St}_k\). On the other hand, by standard properties of projective modules, the composition multiplicity of \(D_\sigma\) in \(\text{St}_k\) is bounded above by

\[\dim \text{Hom}_{KG}(P_i, \text{St}_k) \leq \dim \text{Hom}_{KG}(\mathcal{P}_\sigma, \text{St}_k) = \text{Hom}_{KG}(K\Gamma_\sigma, \text{St}_k) = 1.\]

Once the existence of \(\Gamma'_\sigma\) is shown, the uniqueness automatically follows since the intersection of pure submodules is pure and \(\text{St}_k\) has multiplicity 1 in \(K\Gamma_\sigma\).

\[\square\]

4.3. We assume from now on that the center of \(G\) is connected. Furthermore, we shall fix a group homomorphism \(\sigma: U \rightarrow K^\times\) which is a regular character, that is, we have \(U^s \subseteq \ker(\sigma)\) and the restriction of \(\sigma\) to \(U_s\) is non-trivial for all \(s \in S\). (Such characters always exist.) Then the corresponding module \(\Gamma_\sigma = \mathcal{O}G_{U_s}\) is called a Gelfand-Graev module for \(G\); see [2] §8.1 or [31] p. 258. Since the center of \(G\) is assumed to be connected, all regular characters of \(U\) are conjugate under the action of \(H\) and, hence, the corresponding Gelfand-Graev modules will all be isomorphic; see [2] 8.1.2.

For any \(J \subseteq S\), we have \(L_J = L^J\) where \(L\) is an \(F\)-stable Levi subgroup in \(G\); here, the center of \(L\) will also be connected; see [2], 8.1.4. Our regular character \(\sigma\) determines a regular character of \(L_J\); see, e.g., [2] 8.1.6. Hence, we also have a well-defined Gelfand-Graev module for \(\mathcal{O}L_J\), which we denote by \(\Gamma^J_\sigma\). Applying the construction in Proposition 4.2, we obtain an \(\mathcal{O}L_J\)-lattice \(\mathcal{I}^J_\sigma = \Gamma^J_\sigma / (\Gamma^J_\sigma)\)' and a simple \(kL_J\)-module \(D^J_\sigma := \mathcal{F}^J_\sigma / \text{rad}(\mathcal{F}^J_\sigma)\). We have \(K\mathcal{I}^J_\sigma \cong \text{St}^J_{k}\), the Steinberg module for \(KL_J\).

Lemma 4.4. Let \(J \subseteq S\). Then the following hold.

(i) We have \(^*R^J_\sigma(\Gamma_\sigma) \cong \Gamma^J_\sigma\) and \(^*R^J_\sigma(\mathcal{I}_\sigma) \cong \mathcal{I}^J_\sigma\) (as \(\mathcal{O}L_J\)-modules).

(ii) If \(I \subseteq S\) and \(n \in N\) are such that \(nL_In^{-1} = L_J\), then \(^*\mathcal{I}^I_\sigma \cong \mathcal{I}^J_\sigma\) (as \(\mathcal{O}L_J\)-modules) and \(^*D^I_\sigma \cong D^J_\sigma\) (as \(kL_J\)-modules).

Proof. (i) By a result of Rodier ([2] 8.1.5]), we have \(^*R^J_\sigma(\Gamma_\sigma) \cong KT^J_\sigma\); by [4] (71.6), we also have \(^*R^J_\sigma(\text{St}_K) \cong \text{St}^J_{K}\). So (i) follows by a standard argument; see, e.g., [7], 5.14, 5.15.
(ii) Since \( ^*R^j_S(K\Gamma_\sigma) \cong K\Gamma'_\sigma \), it is straightforward to show that \( ^*K\Gamma'_\sigma \cong K\Gamma'_\sigma \), using the “Strong Conjugacy Theorem”. So we also have \( ^*\Gamma'_\sigma \cong \Gamma'_\sigma \) as \( SL_J \)-modules (since these modules are projective). This then implies (ii) by the construction of \( \mathcal{J}_{\sigma}^J \) and \( D_{\sigma}^J \).

4.5. Let \( \mathcal{P}_{\sigma}^* \) be the set of all subsets \( J \subseteq S \) such that \( D_{\sigma}^J \) is a cuspidal \( kL_J \)-module. For \( J \in \mathcal{P}_{\sigma}^* \), we denote by \( \text{Irr}_k(G \mid J, \sigma) \) the set of all \( Y \in \text{Irr}_k(G) \) such that \( Y \) is isomorphic to a submodule of \( R_S^j(D_{\sigma}^J) \). By the “Strong Conjugacy Theorem”, this is a Harish-Chandra series as defined by Hiss [19]. Hence, by [19, Theorem 5.8] (see also [7, §3]), every simple submodule of \( R_S^j(D_{\sigma}^J) \) is isomorphic to a factor module of \( R_S^j(D_{\sigma}^J) \), and vice versa. Furthermore, using also Lemma 4.4(ii), we have for any \( I, J \in \mathcal{P}_{\sigma}^* \):

\[
\text{Irr}_k(G \mid I, \sigma) = \text{Irr}_k(G \mid J, \sigma) \quad \text{if } J = wIw^{-1} \text{ for some } w \in W,
\]

\[
\text{Irr}_k(G \mid I, \sigma) \cap \text{Irr}_k(G \mid J, \sigma) = \emptyset \quad \text{otherwise}.
\]

Thus, having fixed a regular character \( \sigma : U \to K^\times \) as in 4.3, the above constructions produce composition factors of \( kG_k \) arising from subsets \( J \subseteq S \). The following two results are adaptations of Dipper–Gruber [6, Cor. 2.24 and 2.40] to the present setting.

**Proposition 4.6.** Let \( J \in \mathcal{P}_{\sigma}^* \). Then \( St_k \) has a unique composition factor which belongs to the series \( \text{Irr}_k(G \mid J, \sigma) \).

**Proof.** First note that \( St_K \cong K\mathcal{J}_{\sigma} \) and so, by a classical result of Brauer–Nesbitt (see [3 (16.16)]), \( St_k \) and \( \mathcal{J}_{\sigma} \) have the same composition factors (counting multiplicities). Using Lemma 4.4(i) and adjointness, we obtain

\[
\text{Hom}_{kG}(\mathcal{J}_{\sigma}, R_S^j(D_{\sigma}^J)) \cong \text{Hom}_{kL_J}(R_S^j(\mathcal{J}_{\sigma}), D_{\sigma}^J) \cong \text{Hom}_{kL_J}(\mathcal{J}_{\sigma}, D_{\sigma}^J) \neq \{0\}.
\]

Hence, some simple submodule of \( R_S^j(D_{\sigma}^J) \) will be isomorphic to a composition factor of \( \mathcal{J}_{\sigma} \) and so the latter module has at least one composition factor which belongs to \( \text{Irr}_k(G \mid J, \sigma) \). On the other hand, since \( D_{\sigma}^J \) is a quotient of \( \Gamma_{\sigma}^J \), there exists a surjective \( kG \)-module homomorphism \( R_S^j(\Gamma_{\sigma}^J) \to R_S^j(D_{\sigma}^J) \). Now \( R_S^j(\Gamma_{\sigma}^J) \) is projective (see, e.g., [7, 3.4]) and every simple module in \( \text{Irr}_k(G \mid J, \sigma) \) also is a quotient of \( R_S^j(D_{\sigma}^J) \) (see 4.5). Hence, by standard results on projective modules, the total number of composition factors (counting multiplicities) of \( \mathcal{J}_{\sigma} \) which belong to \( \text{Irr}_k(G \mid J, \sigma) \) is bounded above by

\[
\dim \text{Hom}_{kG}(R_S^j(\Gamma_{\sigma}^J), \mathcal{J}_{\sigma}) = \dim \text{Hom}_{kG}(R_S^j(K\Gamma_{\sigma}^J), K\mathcal{J}_{\sigma}).
\]

Using Lemma 4.4(i) and adjointness, the dimension on the right hand side evaluates to \( \dim \text{Hom}_{kL_J}(K\Gamma_{\sigma}^J, K\mathcal{J}_{\sigma}) \), which is one by 4.4.

**Proposition 4.7.** Assume that every composition factor of \( kG_k \) belongs to \( \text{Irr}_k(G \mid J, \sigma) \) for some \( J \in \mathcal{P}_{\sigma}^* \). Then the following hold.

(i) \( St_k \) is multiplicity-free and the length of a composition series of \( St_k \) is equal to the number of \( J \in \mathcal{P}_{\sigma}^* \) (up to \( W \)-conjugacy).

(ii) The induced map \( p_{\sigma} : \mathcal{J}_{\sigma} \to St_k \) is an isomorphism and so \( St_k/\text{rad}(St_k) \cong D_{\sigma} \).

(iii) All composition factors of \( \text{rad}(St_k) \) are non-cuspidal.
Proof. (i) Since \( S_t \subseteq kG \), the hypothesis applies, in particular, to the composition factors of \( S_t \). It remains to use Proposition 4.6.

(ii) By the proof of Proposition 4.2, we have \( \overline{\rho}_\sigma \neq 0 \). Hence, it is enough to show that \( \overline{\rho}_\sigma \) is injective. By [7, Theorem 5.16], this is equivalent to the following statement.

(†) If \( I \subseteq S \) is such that \( \overline{\mathcal{F}}^I \) has a cuspidal simple submodule, then \( I = \emptyset \).

Now (†) is proved as follows. Let \( X \subseteq \overline{\mathcal{F}}^I \) be a cuspidal simple submodule. Using Lemma 4.4(i) and adjointness, we obtain that

\[
\text{Hom}_{kG}(R^S_I(X), \overline{\mathcal{F}}^\sigma) \cong \text{Hom}_{kLJ}(X, \overline{\mathcal{F}}^I_\sigma) \neq \{0\}.
\]

Thus, \( \overline{\mathcal{F}}^\sigma \) has a composition factor which is a quotient of \( R^S_I(X) \). Since \( \overline{\mathcal{F}}^\sigma \) and \( S_t \subseteq kG \) have the same composition factors, it follows that \( kG \) has a composition factor which is a quotient of \( R^S_I(X) \). By our assumption and the characterisation of Harish-Chandra series in [19, Theorem 5.8], the pair \((I, X)\) is \( N \)-conjugate to a pair \((J, D^I_\sigma)\) where \( J \in \mathcal{P}_\sigma \).

So there exists some \( n \in \mathbb{N} \) such that \( nLJ = L_I \) and \( nX \cong D^I_\sigma \). Using Lemma 4.4(ii), we conclude that \( X \cong D^I_\sigma \). Thus, \( D^I_\sigma \) is both isomorphic to a submodule and to a quotient of \( \overline{\mathcal{F}}^I_\sigma \). Now, having a unique simple quotient, the module \( \overline{\mathcal{F}}^I_\sigma \) is indecomposable. Hence, the multiplicity 1 statement in Proposition 4.2 implies that \( D^I_\sigma \cong \overline{\mathcal{F}}^I_\sigma \) and, consequently, we also have \( D^I_\sigma \cong S_t \subseteq kLJ_\sigma \). Thus, \( kLJ_\sigma \cong R^I_\sigma(k_H) \) has a cuspidal simple submodule. Again, by [19, Theorem 5.8], this can only happen if \( I = \emptyset \).

(iii) By our assumption, the only composition factor of \( S_t \) which can possibly be cuspidal is \( D_\sigma \). By (ii) and Proposition 4.2, \( D_\sigma \) is not a composition factor of \( \text{rad}(S_t) \). \( \square \)

Remark 4.8. In analogy to Example 3.6, we define \( \mathcal{U}_k(G) \) to be the set of all \( Y \in \text{Irr}_k(G) \) which are composition factors of \( kG \). We have \( \text{Irr}_k(G | B) \subseteq \mathcal{U}_k(G) \) but note that, in general, we neither have equality nor is \( \mathcal{U}_k(G) \) the complete set of all unipotent \( kG \)-modules as defined, for example, in [10, §1]. (Over \( K \), we do have at least \( \text{Irr}_K(G | B) = \mathcal{U}_K(G) \).

For \( J \subseteq S \), we define \( \mathcal{U}_k(L_J) \) analogously; the standard Borel subgroup of \( L_J \) is given by \( B_J := B \cap L_J \). Let \( X \in \mathcal{U}_k(L_J) \) and \( Y \in \mathcal{U}_k(G) \). Then we have:

(a) All composition factors of \( R^S_I(X) \) belong to \( \mathcal{U}_k(G) \).

(b) If \( *R^S_J(Y) \neq \{0\} \), then all composition factors of \( *R^S_J(Y) \) belong to \( \mathcal{U}_k(L_J) \).

Proof. (a) By the definitions, we have \( kG \cong R^S_{k_H} \) and, similarly, \( kL_J \cong R^I_{k_H} \), where \( B_J = \sum_{b \in B \cap L_J} b \subseteq kL_J \). Hence, using transitivity, we obtain \( kG \cong R^S_{k_H} \). Since Harish-Chandra induction is exact (see [7, 3.4]), \( R^S_I(X) \) is a subquotient of \( kG \).

(b) Since \( kG \cong R^S_{k_H} \), the Mackey formula immediately shows that \( *R^S_J(kG) \) is a direct sum of a certain number of copies of \( kL_J \). It remains to use the fact that Harish-Chandra restriction is also exact. \( \square \)

Example 4.9. Let \( G = GL_n(q) \), where \( n \geq 1 \) and \( q \) is any prime power. Let \( e \geq 2 \) be defined as in Example 3.6, also recall that \( |\mathcal{U}_k(G)| = \pi(n) \), where \( \pi(n) \) denotes the number of partitions of \( n \). By [10, 7.6], \( D_\sigma \) is cuspidal if and only if \( n = 1 \) or \( n = e\ell^j \) for some \( j \geq 0 \). (Note that, if \( \ell \mid q - 1 \), then our \( e \) equals \( \ell \), while it equals 1 in [loc.
Now, the $W$-conjugacy classes of subsets $J \subseteq S$ are parametrised by the partitions $\lambda \vdash n$ (see [13, 2.3.8]); the Levi subgroup $L_J$ corresponding to $\lambda$ is a direct product of general linear groups corresponding to the parts of $\lambda$. Hence, the subsets $J \in \mathcal{P}_\sigma$ are parametrised by the partitions $\lambda \vdash n$ such that each part of $\lambda$ is equal to 1 or to $\ell \ell^j$ for some $j \geq 0$. So Remark [13(1.8)2(a)] and the counting argument in [11 (2.5)] yield $\pi(n)$ simple modules in $\mathcal{U}_k(G)$ which belong to $\text{Irr}_k(G \mid J, \sigma)$ for some $J \in \mathcal{P}_\sigma$. Thus, the hypothesis of Proposition [4.7] is satisfied in this case. Consequently, $\text{St}_k / \text{rad}(\text{St}_k)$ is simple and $\text{St}_k$ is multiplicity-free. (This was also shown by Szechtman [32], using different techniques.) Furthermore, the composition length of $\text{St}_k$ is the coefficient of $t^n$ in the power series

$$\frac{1}{1 - t} \prod_{j=0}^{\infty} \frac{1}{1 - t^{\ell^j}}.$$ 

Indeed, let $c_n$ denote the composition length of $\text{St}_k$. By Proposition [4.7(1)], $c_n$ equals the number of $J \in \mathcal{P}_\sigma^*$ (up to $W$-conjugacy). By the above discussion (see also [11 (2.5)]), this is equal to the number of sequences $(m_1, m_0, m_1, \ldots)$ of non-negative integers such that $m_1 + em_0 + e\ell m_1 + \cdots = n$ (where $\text{GL}_q(q) = \{1\}$ by convention). We multiply both sides by $t^n$ and sum over all $n \geq 0$. This yields

$$\sum_{n \geq 0} c_t t^n = \sum_{(m_1, m_0, m_1, \ldots)} t^{m_1 + em_0 + e\ell m_1 + \cdots} = \left( \sum_{m_1 \geq 0} t^{em_0} \right) \left( \sum_{m_0 \geq 0} t^{e\ell m_1} \right) \cdots \cdots .$$

Using the identity $1/(1 - t^r) = \sum_{i \geq 0} t^{ri}$ for all $r \geq 1$, we obtain the desired formula.

**Remark 4.10.** In the setting of Szechtman [32], the above expression for $c_n$ means that the formula (4) in [32, p. 605] holds for all $n$. (Previously, this was only known for $n \leq 10$; see the remarks in [loc. cit.].) This formula gives an explicit expression of the layers in the Jantzen filtration of $\text{St}_k$ (as defined by Gow [14]), as direct sums of simple modules. It also shows that the layers in this filtration are not always irreducible and, hence, Gow’s conjecture [14, 6.3] does not hold in general. See also the explicit examples in [32] 69.

**Example 4.11.** Let $G = G_n(q)$, $n \geq 1$, be one of the following finite classical groups:

1. The general unitary group $\text{GU}_n(q)$ for any $n$, any $q$.
2. The special orthogonal group $\text{SO}_n(q)$ where $n = 2m + 1$ is odd and $q$ is odd.
3a) The symplectic group $\text{Sp}_n(q)$ where $n = 2m$ is even and $q$ is a power of 2.
3b) The conformal symplectic group $\text{CSp}_n(q)$ where $n = 2m$ is even and $q$ is odd.
4. The conformal orthogonal group $\text{CSO}_n^\pm(q)$ where $n = 2m$ is even and $q$ is odd.

Each of these groups can be realized as $G = G^F$ where $G$ has a connected center and $G$ is simple modulo its center. By convention, $G_0(q)$ is the trivial group, except for the conformal groups, where it is cyclic of order $q - 1$. We define the parameter $\delta$ to be 2 in case (1) and to be 1 in all the remaining cases.

**Theorem 4.12** ([9], Gruber [16], and Gruber–Hiss [17]). Let $G = G_n(q)$ be as in Example 4.11 and assume that $\ell$ is “linear”, that is, $q^{\delta m - 1} \not\equiv -1 \text{ mod } \ell$ for all $m \geq 1$. 
(i) We have \(|\text{Irr}_G(W)| = |\mathcal{U}_k(G)|\).
(ii) If \(Y \in \mathcal{U}_k(G)\), then \(^*R^S_Y(J) \neq \{0\}\) for some subset \(J \subseteq S\) such that \(L_J\) is a direct product of groups of untwisted type \(A\).

Proof. This follows from [9, §4] in the cases (1), (2), (3a), (3b), and from [16] in case (4). We shall refer to the more general setting in [17] (where all of \(\text{Irr}_G(G)\) is considered).

(i) Note that, by the “classical fact” in characteristic 0 mentioned in [3, §7] we certainly know that \(|\text{Irr}_G(W)| = |\mathcal{U}_k(G)|\). Hence, the assertion immediately follows from the block diagonal shape of the decomposition matrix in [17, Theorem 8.2].

(ii) Let \(Q\) be a projective indecomposable \(\mathcal{O}G\)-lattice such that \(\overline{Q}\) is a projective cover of \(Y\). By [17, Cor. 8.6], \(Q\) is a direct summand of \(^*R^S_J(Q')\), where \(J \subseteq S\) and \(Q'\) is a projective indecomposable \(\mathcal{O}L_J\)-lattice such that the following conditions hold. First, we have \(L_J \cong G_\alpha(q) \times L_\lambda\) where \(n = a + 2m\) (\(a, m \geq 0\)) and \(G_\alpha(q)\) is a group of the same type as \(G\); furthermore, \(\lambda\) is a composition of \(m\) and \(L_\lambda\) is a direct product of general linear groups \(\text{GL}_\lambda(q^e)\) where \(\lambda\) runs over the non-zero parts of \(\lambda\). Finally, under the isomorphism \(L_J \cong G_\alpha(q) \times L_\lambda\), we have \(Q' \cong Q'_\alpha \otimes Q'_\lambda\) where \(Q'_\alpha\) is a projective indecomposable \(\mathcal{O}G_\alpha(q)\)-lattice such that \(Q'_\alpha\) has only cuspidal constituents and \(Q'_\lambda\) is an indecomposable direct summand of the Gelfand-Graev lattice for \(\mathcal{O}L_\lambda\).

Now, since \(\overline{Q}\) is a direct summand of \(^*R^S_J(Q')\), we have \(\text{Hom}_{kL_J}(\overline{Q}, {}^*R^S_J(Y)) \neq \{0\}\) by adjointness. This shows, first of all, that \(^*R^S_J(Y) \neq \{0\}\). Using Remark [4,8](b), we conclude that \(\text{Hom}_{kL_J}(\overline{Q}, kL_J\mathcal{B}_J) \neq 0\). Consequently, since \(Q'\) is projective, we also have \(\text{Hom}_{kL_J}(Q', KL_J\mathcal{B}_J) \neq 0\). So, by the above direct product decomposition of \(L_J\), at least one of the cuspidal composition factors of \(KQ'_\alpha\) belongs to \(\mathcal{U}_K(G_\alpha(q))\). But this can only happen if \(G_\alpha(q)\) has \(BN\)-rank equal to 0. Thus, \(L_J\) has the required form. \(\square\)

4.13. Let \(G = G_n(q)\) be as in Example [4,11] and assume that \(\ell\) is linear. By Theorem [4,12](ii) and the characterisation of Harish-Chandra series in [19, Theorem 5.8], every \(Y \in \mathcal{U}_k(G)\) is a submodule of \(^*R^S_J(X)\) where \(J \subseteq S\) is such that \(L_J\) is isomorphic to a direct product of groups of untwisted type \(A\), and \(X \in \text{Irr}_k(L_J)\) is cuspidal. Then, by adjointness, \(X\) is a composition factor of \(^*R^S_J(Y)\) and, hence, \(X \in \mathcal{U}_k(L_J)\) by Remark [4,8](b). But then the known results on general linear groups imply that \(X \cong D^2_J\); see, e.g., [7, Cor. 6.16]. Thus, the hypothesis of Proposition [4,7] is satisfied. (This is also mentioned in Dipper–Gruber [6, 4.22], with only a sketch proof.)

Thus, in all the cases listed in Example [4,11] \(\text{St}_k\) is multiplicity-free, \(\text{St}_k/\text{rad}(\text{St}_k)\) is simple and the composition length of \(\text{St}_k\) is determined as in Proposition [4,7](i). Consequently, one can derive a generating function for the composition length of \(\text{St}_k\), similar to that for \(\text{GL}_n(q)\) in Example [4,9]. We will only give the details for \(G = \text{GU}_n(q)\). Write \(n = 2m\) (if \(n\) is even) or \(n = 2m + 1\) (if \(n\) is odd); furthermore, since \(\delta = 2\), we set
\[
\hat{c} := \min \{i \geq 2 \mid 1 + q^2 + q^4 + \cdots + q^{2(e-1)} \equiv 0 \mod \ell\}
\]
in this case. We can now use the counting argument in the proof of [11, Theorem 4.11]; see also [9, §4]. This shows that the subsets \(J \in \mathcal{P}_\sigma\) are parametrized (up to \(W\)-conjugacy) by the partitions \(\lambda \vdash m\) such that each part of \(\lambda\) is equal to \(1\) or to \(\hat{c}^j\) for some \(j \geq 0\).
So the number of $J \in \mathcal{P}_\sigma^*$ (up to $W$-conjugacy) is equal to the number of sequences $(m_-, m_0, m_1, \ldots)$ of non-negative integers such that $m_- + \tilde{e}m_0 + \ell m_1 + \cdots = m$. Thus, we find that the composition length of $St_k$ for $G = GU_n(q)$ is the coefficient of $t^m$ (and not $t^n$ as in Example 4.9) in the power series
\[ \frac{1}{1-t} \prod_{j \geq 0} \frac{1}{1-t^j} \] (assuming that $\ell$ is linear for $G$).

Remark 4.14. Within the much more general setting of Dipper–Gruber [6], we have considered here the “projective restriction system” $\mathcal{R}(X_G, Y_L)$ where
\[ X_G := \mathcal{S}_\sigma, \quad L = H \quad \text{and} \quad Y_L = \mathcal{O}H \text{ (regular } \mathcal{O}H\text{-module)}. \]
In this particular case, the arguments in [loc. cit.] drastically simplify, and this is what we have tried to present in this section. We note, however, that these methods only yield quite limited information about $St_k$ when $\ell$ is not a “linear prime”. Only two of the composition factors in the socle series displayed in Example 3.5 are accounted for by these methods (namely, $k_G, \varphi_3$ for $^2G_2(q^2)$ and $k_G, \vartheta$ for $GU_3(q)$); all the remaining composition factors are cuspidal. Also note that, in these examples, $St_k$ is not multiplicity-free.

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References
[1] B. Ackermann, The Loewy series of the Steinberg-PIM of finite general linear groups, Proc. London Math. Soc. 92 (2006), 62–98.
[2] R. W. Carter, Finite groups of Lie type: conjugacy classes and complex characters, Wiley, New York, 1985; reprinted 1993 as Wiley Classics Library Edition.
[3] C. W. Curtis and I. Reiner, Methods of representation theory Vol. I, Wiley, New York, 1981.
[4] C. W. Curtis and I. Reiner, Methods of representation theory Vol. II, Wiley, New York, 1987.
[5] R. Dipper and J. Du, Harish-Chandra vertices, J. reine angew. Math. 437 (1993), 101–130.
[6] R. Dipper and J. Gruber, Generalized $q$-Schur algebras and modular representation theory of finite groups with split $(B, N)$-pairs, J. reine angew. Math. 511 (1999), 145–191.
[7] M. Geck, Modular Harish-Chandra series, Hecke algebras and (generalized) $q$-Schur algebras. In: Modular Representation Theory of Finite Groups (Charlottesville, VA, 1998; eds. M. J. Collins, B. J. Parshall and L. L. Scott), p. 1–66, Walter de Gruyter, Berlin 2001.
[8] M. Geck, Modular principal series representations, Int. Math. Res. Notices, vol. 2006, Article ID 41957, pp. 1–20.
[9] M. Geck and G. Hiss, Modular representations of finite groups of Lie type in non-defining characteristic. In: Finite reductive groups (Luminy, 1994; ed. M. Cabanes), Progress in Math. 141, pp. 195–249, Birkhäuser, Boston, MA, 1997.
[10] M. Geck, G. Hiss and G. Malle, Cuspidal unipotent Brauer characters, J. Algebra 168 (1994), 182–220.
[11] M. Geck, G. Hiss and G. Malle, Towards a classification of the irreducible representations in non-defining characteristic of a finite group of Lie type, Math. Z. 221 (1996), 353–386.
[12] M. Geck and N. Jacon, Representations of Hecke algebras at roots of unity, Algebra and Applications 15, Springer-Verlag, 2011.
ON THE $\ell$-MODULAR COMPOSITION FACTORS OF THE STEINBERG REPRESENTATION

[13] M. GECK AND G. PFEIFFER, Characters of finite Coxeter groups and Iwahori–Hecke algebras, London Math. Soc. Monographs, New Series 21, Oxford University Press, 2000.

[14] R. GOW, The Steinberg lattice of a finite Chevalley group and its modular reduction, J. London Math. Soc. 67 (2003), 593–608.

[15] J. A. GREEN, On a theorem of Sawada, J. London Math. Soc. 18 (1978), 247–252.

[16] J. GRUBER, Cuspidale Untergruppen und Zerlegungszahlen klassischer Gruppen, Dissertation, Universität Heidelberg, 1995.

[17] J. GRUBER AND G. HISS, Decomposition numbers of finite classical groups for linear primes, J. reine angew. Math. 485 (1997), 55–91.

[18] G. HISS, The number of trivial composition factors of the Steinberg module, Arch. Math. 54 (1990), 247–251.

[19] G. HISS, Harish-Chandra series of Brauer characters in a finite group with a split $BN$-pair, J. London Math. Soc. 48 (1993), 219–228.

[20] G. HISS, Hermitian function fields, classical unitals, and representations of 3-dimensional unitary group, Indag. Math. 15 (2004), 223–243.

[21] R. B. HOWLETT AND G. I. LEHRER, On Harish-Chandra induction for modules of Levi subgroups, J. Algebra 165 (1994), 172–183.

[22] N. Jacon and C. Lecouvey, On the Mullineux involution for Ariki–Koike algebras, J. Algebra 321 (2009), 2156–2170.

[23] G. D. JAMES, Representations of general linear groups, London Math. Soc. Lecture Note Series, vol. 94, Cambridge University Press, Cambridge, 1984.

[24] G. D. JAMES, The irreducible representations of the finite general linear groups, Proc. London Math. Soc. 52 (1986), 236–268.

[25] A. A. KHAMMASH, A note on a theorem of Solomon–Tits, J. Algebra 130 (1990), 296–303.

[26] A. A. KHAMMASH, On the homological construction of the Steinberg representation, J. Pure Appl. Algebra 87 (1993), 17–21.

[27] P. LANDROCK AND G. O. MICHLER, Principal 2-blocks of the simple groups of Ree type, Trans. Amer. Math. Soc. 260 (1980), 83–111.

[28] A. MATHAS, Iwahori-Hecke algebras and Schur algebras of the symmetric group, University Lecture Series, 15. Amer. Math. Soc., Providence, RI, 1999.

[29] T. OKUYAMA AND K. WAKI, Decomposition numbers of $SU(3, q^2)$, J. Algebra 255 (2002), 258–270.

[30] R. STEINBERG, Prime power representations of finite linear groups II, Canad. J. Math. 9 (1957), 347–351.

[31] R. STEINBERG, Lectures on Chevalley groups, Mimeographed Notes, Yale University, 1967; available at http://www.math.ucla.edu/~rst/YaleNotes.pdf

[32] F. SZECHTMAN, Steinberg lattice of the general linear group and its modular reduction, J. Group Theory 14 (2011), 603–635.

[33] N. B. TINBERG, The Steinberg component of a finite group with a split $(B, N)$-pair, J. Algebra 104 (1986), 126–134.

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