Abstract. We prove an $O(\log n)$ bound for the expected value of the logarithm of the componentwise (and, a fortiori, the mixed) condition number of a random sparse $n \times n$ matrix. As a consequence, small bounds on the average loss of accuracy for triangular linear systems follow.

1 Introduction

Triangular systems of linear equations provide one of the few examples in numerical linear algebra where a gap occurs between stability analysis and everyday practice. One could summarize this gap as follows:

Triangular systems of equations are generally solved to high accuracy in spite of being, in general, ill-conditioned.

This state of affairs had been already noted by J.H. Wilkinson in [9, p. 105]: “In practice one almost invariably finds that if $L$ is ill-conditioned, so that $\|L\|\|L^{-1}\| \gg 1$, then the computed solution of $Lx = b$ (or the computed inverse) is far more accurate than [what forward stability analysis] would suggest.”

An explanation to this gap is suggested by N. Higham [4] who notes that the backward error analysis given by Wilkinson for the solution of triangular systems yields (small) componentwise bounds on the perturbation matrix (see Section 6(1) below). Higham then uses this fact to deduce small

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forward error bounds for particular subclasses of triangular systems and to numerically investigate the accuracy of other particular such systems. In doing so, Higham makes use of the mixed condition number introduced by Skeel [5]. This condition number has a natural role in analyzing accuracy of triangular systems since bounds for it, together with the backward analysis of Wilkinson mentioned above, yield forward analysis bounds for the computed solution of the system. Furthermore, the restriction to componentwise perturbations—both in the backward error analysis and in the mixed condition number—forces perturbations to preserve the triangular structure of the data matrices.

A further step in explaining the gap, somehow orthogonal to the work of Higham, was given by D. Viswanath and N. Trefethen in [8] where a precise meaning to the expression “triangular systems are, in general, ill-conditioned” was given. Indeed, if $L_n$ denotes a random triangular $n \times n$ matrix (whose entries are independent standard Gaussian random variables) and $\kappa_n = \|L_n\|\|L_n^{-1}\|$ is its condition number (which is a positive random variable) then, it is shown in [8],

$$\sqrt[n]{\kappa_n} \to 2$$

almost surely as $n \to \infty$. A straightforward consequence of this result is that the expected value of $\log \kappa_n$ satisfies $E(\log \kappa_n) = \Omega(n)$.

The goal of this paper is to close the gap by giving a precise meaning, in the sense of [8], to the other half of the statement above namely, to the expression “triangular systems are generally solved to high accuracy.” More precisely, we consider the mixed condition numbers $m(\dagger)(L_n)$ and $m(L_n, b_n)$ for the problems of matrix inversion and linear equation solving, respectively, for a random triangular $L_n$ as above and a random $b_n \in \mathbb{R}^n$. Then, we show that

$$E(\log m(\dagger)(L_n)), \ E(\log m(L_n, b_n)) = O(\log n).$$

From the bound on $E(\log m(L_n, b_n))$ it follows that the average loss of precision in the solution of random triangular systems is small. From that on $E(\log m(\dagger)(L_n))$, that the one for matrix inversion is small as well. One can therefore replace the summary above by the following:

*Triangular systems of equations are generally solved to high accuracy because their backward error analysis yields small compo-

\footnote{Skeel called it “componentwise.” In this paper, however, following the notation introduced in [3], we will use this word for the condition numbers measuring both data perturbation and computed errors in a componentwise fashion.}
nentwise perturbations and triangular matrices are, in general, well conditioned for these perturbations.

The results showing (1), Theorems 2 and 3 below, are proved in the more general context of sparse matrices (i.e. matrices with a fixed pattern of zeros) and componentwise condition numbers (which ensure high relative accuracy in each component of the computed solution $A^{-1}$ or $x$). Besides triangular matrices, these results apply to other classes of sparse matrices such as, for instance, tridiagonal matrices. In the process of proving them, we found useful to estimate as well the average mixed condition for the computation of the determinant.

2 Preliminaries

Condition numbers measure the worst-case magnification of a small data perturbation in the computed outcome. As originally introduced by Turing [7], they were normwise in the sense that both the data perturbation and the outcome’s error are measured using norms (in the space of data and outcomes respectively). In contrast, mixed condition numbers measure data perturbation componentwise, and componentwise condition numbers measure both data perturbation and outcome’s error in this way.

To define these condition numbers the following form of “distance” function will be useful. For points $u, v \in \mathbb{R}^p$ we define $\mathbb{A}_v = (w_1, \ldots, w_p)$ with

$$w_i = \begin{cases} 
\frac{u_i}{v_i} & \text{if } v_i \neq 0 \\
0 & \text{if } u_i = v_i = 0 \\
\infty & \text{otherwise}.
\end{cases}$$

Then we define

$$d(u, v) = \left\| \frac{u - v}{v} \right\|_\infty.$$

Note that, if $d(u, v) < \infty$,

$$d(u, v) = \min\{\nu \geq 0 \mid |u_i - v_i| \leq \nu|v_i| \text{ for } i = 1, \ldots, p\}.$$

For $\delta > 0$ and $a \in \mathbb{R}^p$ we denote $B(a, \delta) = \{x \in \mathbb{R}^p \mid d(x, a) \leq \delta\}$.

**Definition 1** Let $D \subseteq \mathbb{R}^p$ and $F : D \rightarrow \mathbb{R}^q$ be a continuous mapping. Let $a \in D$ such that $F(a) \neq 0$. 

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(i) The mixed condition number of $F$ at $a$ (with respect to a norm $\| \cdot \|_q$ on $\mathbb{R}^q$) is defined by

$$m(F, a) = \lim_{\delta \to 0} \sup_{x \in B(a, \delta), x \neq a} \frac{\|F(x) - F(a)\|_q}{\|F(a)\|_q} \frac{1}{d(x, a)}.$$

(ii) Suppose $F(a) = (f_1(a), \ldots, f_q(a))$ is such that $f_j(a) \neq 0$ for $j = 1, \ldots, q$. Then the componentwise condition number of $F$ at $a$ is

$$c(F, a) = \lim_{\delta \to 0} \sup_{x \in B(a, \delta), x \neq a} \frac{d(F(x), F(a))}{d(x, a)}.$$

**Proposition 1** For all $a \in D$ and any monotonic norm in $\mathbb{R}^q$, $m(F, a) \leq c(F, a)$.

**Proof.** For all $x \in B(a, \delta)$ and all $i \leq q$, $|F(x)_i - F(a)_i| \leq d(F(x), F(a))|F(a)_i|$. Since $\| \cdot \|$ is monotonic (cf. [1]), this implies $\|F(x) - F(a)\| \leq d(F(x), F(a))\|F(a)\|$ and hence the statement.

In all what follows, for $n \in \mathbb{N}$, we denote the set $\{1, \ldots, n\}$ by $[n]$.

**Definition 2** We denote by $\mathcal{M}$ the set of $n \times n$ real matrices and by $\Sigma$ its subset of singular matrices. Also, for a subset $S \subseteq [n]^2$ we denote

$$\mathcal{M}_S = \{ A \in \mathcal{M} \mid (i, j) \not\in S \text{ then } a_{ij} = 0 \}.$$

We denote by $\mathcal{R}_S$ the space of random $n \times n$ matrices obtained by setting $a_{ij} = 0$ if $(i, j) \in S$ and drawing all other entries independently from the standard Gaussian $N(0, 1)$. As above, if $S = [n]^2$, we write simply $\mathcal{R}$.

In the rest of this paper, for non-singular matrices $A, A'$, we denote their inverses by $\Gamma, \Gamma'$, respectively. Also, we denote by $A_{(ij)}$ the sub-matrix of $A$ obtained by removing from $A$ its $i$th row and its $j$th column.

### 3 Determinant Computation

We consider here the problem of computing the determinant of a matrix $A$ and its componentwise condition number $c_{\text{det}}(A)$. The main result of this section is the following.
Theorem 1  For $S \subseteq [n]^2$ and $t \geq 2|S|$ we have

$$\Pr\{c_{\det}(A) \geq t\} \leq |S|^2 \frac{1}{t}.$$  

Average loss of precision (in a base $b$) is measured by the expected value of the logarithm (in that base) of the condition number. We may use Theorem 1 to obtain one such result for the computation of the determinant. To avoid problems caused by this condition number being less than 1 we consider the function $\log_+(x)$ defined to be $\log(x)$ if $x \geq 1$ and 0 otherwise.

Corollary 1  For a base $b > 1$, $E(\log_+ c_{\det}(A)) \leq 2 \log |S| + \frac{1}{\ln b}$ where $E$ denotes expectation over $A \in \mathcal{R}_S$.

Towards the proof of Theorem 1 we first obtain explicit expressions for $c_{\det}(A)$. We begin by noting that taking $F : \mathcal{M} \to \mathbb{R}$ to be $F(A) = \det(A)$ in Definition 1 we obtain, for $A \in \mathcal{M} \setminus \Sigma$,

$$c_{\det}(A) = \lim_{\delta \to 0} \sup_{A' \in B(A,\delta)} \frac{|\det(A') - \det(A)|}{|\delta| \det(A)}.$$  

Also, for $A \in \Sigma$, we have $c_{\det}(A) = 0$ if

$$\lim_{\delta \to 0} \sup_{A' \in B(A,\delta)} \frac{|\det(A')|}{\delta} = 0$$  

and $c_{\det}(A) = \infty$ otherwise. Note that $c_{\det}(0) = 0$.

Lemma 1  For $A \in \mathcal{M} \setminus \Sigma$,

$$c_{\det}(A) = \sum_{i,j \in [n]} |a_{ij} \gamma_{ji}|.$$  

For $A \in \Sigma$, $c_{\det}(A) = 0$ if

$$\sum_{i,j \in [n]} |a_{ij} \det(A_{(ij)})| = 0$$  

and $c_{\det}(A) = \infty$ otherwise.

Proof.  Let $A \in \mathcal{M}$. For any $i \in [n]$, expanding by the $i$th row,

$$\det(A) = \sum_{j \in [n]} (-1)^{i+j} a_{ij} \det(A_{(ij)}).$$
Hence, for all \( i, j \in [n] \),
\[
\frac{\partial \det(A)}{\partial a_{ij}} = \det(A_{(ij)}).
\]

Using Taylor’s expansion and these equalities we obtain
\[
det(A') = det(A) + \sum_{i,j \in [n]} (a'_{ij} - a_{ij}) \det(A_{(ij)}) + O\left(\|A' - A\|^2\right).
\]

Here, the norm in \( \|A' - A\| \) is not relevant since all norms in \( \mathcal{M} \) are equivalent. By choosing a monotonic norm we have that if \( A' \in B(A, \delta) \) then \( \|A' - A\| = O(\delta) \). It follows that, for \( A \not\in \Sigma \),
\[
c_{\det}(A) = \lim_{\delta \to 0} \sup_{A' \in B(A, \delta)} \frac{|\det(A') - \det(A)|}{\delta |\det(A)|} = \lim_{\delta \to 0} \sup_{A' \in B(A, \delta), i,j \in [n]} \frac{|(a'_{ij} - a_{ij}) \det(A_{(ij)})|}{\delta |\det(A)|} \cdot \frac{|a'_{ij} - a_{ij}|}{|a_{ij}|} \leq \delta.
\]

The supremum above is attained by taking \( a'_{ij} = a_{ij}(1 + \delta) \) and therefore
\[
c_{\det}(A) = \sum_{i,j \in [n]} \left| \frac{a_{ij} \det(A_{ij})}{\det(A)} \right| = \sum_{i,j \in [n]} |a_{ij} \gamma_{ji}|.
\]

If \( A \in \Sigma \) it similarly follows that
\[
\lim_{\delta \to 0} \sup_{A' \in B(A, \delta)} \frac{|\det(A')|}{\delta} = \sum_{i,j \in [n]} |a_{ij} \det(A_{(ij)})|
\]
and hence the statement. \( \square \)

**Lemma 2** Let \( p, q \) be two fixed vectors in \( \mathbb{R}^n \) such that \( \|p\| \leq \|q\| \). If \( x \sim N(0, \text{Id}_n) \) then, for all \( t \geq 2 \),
\[
\text{Prob}\left\{ \left| \frac{x^T p}{x^T q} \right| \geq t \right\} \leq \frac{1}{t}.
\]
Proof. Let $\nu = \|q\|$. By the orthogonal invariance of $N(0, \text{Id}_n)$ we may assume $q = (\nu, 0, \ldots, 0)$. Also, by appropriately scaling, we may assume that $\nu = 1$. Note that then, $\|p\| \leq 1$. We therefore have

$$\text{Prob}\left\{ \left| \frac{x^T p}{x^T q} \right| \geq t \right\} = \text{Prob}\left\{ \left| p_1 + \sum_{i \in \{2, \ldots, n\}} \frac{x_i p_i}{x_1} \right| \geq t \right\}$$

$$= \text{Prob}\left\{ \left| p_1 + \frac{1}{x_1} \alpha Z \right| \geq t \right\}$$

$$= \text{Prob}\left\{ \frac{Z}{x_1} \geq \frac{t - p_1}{\alpha} \right\} + \text{Prob}\left\{ \frac{Z}{x_1} \leq \frac{-t - p_1}{\alpha} \right\}$$

\[(2)\]

where $Z = N(0,1)$ independent of $x_1$ and $\alpha = \sqrt{p_2^2 + \ldots + p_n^2} \leq 1$. Here we used that a sum of independent centered Gaussians is a centered Gaussian whose variance is the sum of the terms variances. Note that in case $\alpha = 0$ the statement is trivially true.

Since the $x_1$ and $Z$ are independent $N(0,1)$, the angle $\theta = \arctan \left( \frac{Z}{x_1} \right)$ is uniformly distributed in $[-\pi/2, \pi/2]$ and we have, for $\gamma \in \mathbb{R}$,

$$\text{Prob}\left\{ \frac{Z}{x_1} \geq \gamma \right\} = \text{Prob}\left\{ \theta \geq \arctan \gamma \right\} = \frac{1}{\pi} \left( \frac{\pi}{2} - \arctan \gamma \right)$$

$$= \frac{1}{\pi} \int_{\gamma}^{\infty} \frac{1}{1 + t^2} dt \leq \frac{1}{\pi} \int_{\gamma}^{\infty} \frac{1}{t^2} dt = \frac{1}{\pi \gamma}.$$  

Similarly, for $\sigma \in \mathbb{R}$,

$$\text{Prob}\left\{ \frac{Z}{x_1} \leq \sigma \right\} = 1 - \text{Prob}\left\{ \theta \geq \arctan \sigma \right\} = 1 - \frac{1}{\pi} \left( \frac{\pi}{2} - \arctan \sigma \right)$$

$$= \frac{1}{\pi} \left( \frac{\pi}{2} - \arctan(\sigma) \right) \leq \frac{1}{\pi (1 + \sigma)}.$$

Using these bounds in (2) with $\gamma = \frac{t - p_1}{\alpha}$ and $\sigma = \frac{-t - p_1}{\alpha}$ we obtain

$$\text{Prob}\left\{ \left| \frac{x^T p}{x^T q} \right| \geq t \right\} \leq \frac{1}{\pi} \left( \frac{\alpha}{t - p_1} + \frac{\alpha}{t + p_1} \right) = \frac{2t}{\pi t^2 - p_1^2} \leq \frac{2t}{\pi t^2 - 1} \leq \frac{1}{t}$$

the last since $t \geq 2$. \hfill \Box

Lemma 3 Let $S \subseteq [n]^2$ be such that $\mathcal{M}_S \subseteq \Sigma$. Then, for all $A \in \mathcal{M}_S$, $c_{\text{det}}(A) = 0$. 

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Proof. Since $M_S \subseteq \Sigma$ and $A \in M_S$, we have $B(A, \delta) \subseteq \Sigma$ for all $\delta > 0$. The result now follows.

Lemma 4 Let $S \subset [n]^2$ such that $M_S \not\subseteq \Sigma$. Then

$$\Prob_{A \in M_S}(A \text{ is singular}) = 0.$$ 

Proof. The set of singular matrices in $M_S$ is the zero set of the restriction of the determinant to $M_S$. This restriction is a polynomial in $\mathbb{R}^{|S|}$ whose zero set, if different from $\mathbb{R}^{|S|}$, has dimension smaller than $|S|$.

Proof of Theorem 1. Case (i): $M_S \subseteq \Sigma$. In this case, the desired inequality is trivial by Lemma 3.

Case (ii): $M_S \not\subseteq \Sigma$. By Lemma 4, with probability 1, $A$ is non-singular. So, by Lemma 1,

$$\Prob\left\{ c_{\det}(A) \geq t \right\} = \Prob \left\{ \sum_{i,j \in [n]} |a_{ij} \gamma_{ji}| \geq t \right\} = \Prob \left\{ \sum_{(i,j) \in S} \left| \frac{a_{ij} \det(A_{ij})}{\det(A)} \right| \geq t \right\}. \quad (3)$$

Assume $(1,1) \in S$ and let $x = a_1$ be the first column of $A$. Also, let $I = \{ i \in [n] \mid (i,1) \in S \}$ and $x_I$ be the vector obtained by removing entries $x_i$ with $i \not\in I$. Then,

$$x_S \sim N(0, \Id_{|I|}). \quad (4)$$

Let $q = (\det(A_{11}), \ldots, \det(A_{n1}))^T$ and $q_I$ be the vector obtained by removing entries $q_i$ with $i \not\in I$. Clearly, $q_I$ is independent of $x_I$. Using this notation, the expansion by the first column yields

$$\det(A) = \sum_{i \in [n]} (-1)^{i+1} a_{i1} \det(A_{i1}) = x_I^T q_I.$$

In addition, $a_{11} \det(A_{11}) = x_I^T (q_1 e_1)$ where $e_1$ is the vector with the first entry equal to 1 and all others equal to 0. Hence,

$$\frac{a_{11} \det(A_{11})}{\det(A)} = \frac{x_I^T (q_1 e_1)}{x_I^T q_I}.$$
Using (4) and Lemma 2 (with \( p = (q_1e_1) \) and \( q = q_I \)) we obtain, for \( z \geq 2 \),
\[
\text{Prob} \left\{ \left| \frac{a_{11} \det(A_{11})}{\det(A)} \right| \geq z \right\} \leq \frac{1}{z}.
\]
The same bound can be proven for all \((i,j) \in S\). Using these bounds with \( z = \frac{1}{|S|} \) and (3) we obtain,
\[
\text{Prob}\{c_{\det}(A) \geq t\} \leq \sum_{(i,j) \in S} \text{Prob} \left\{ \left| \frac{a_{ij} \det(A_{ij})}{\det(A)} \right| \geq \frac{t}{|S|} \right\} \leq \frac{|S|^2}{t}.
\]

The proof of Corollary 1 follows from the following result by taking \( Z = c_{\det}(A) \) and \( t_0 = |S|^2 \).

**Proposition 2** Let \( t_0 > 0 \) and \( Z \geq 1 \) be a random variable satisfying that \( \text{Prob}\{Z \geq t\} \leq t_0 t^{-1} \) for all \( t \geq t_0 \). Then \( E(\log Z) \leq \log t_0 + \frac{1}{\ln b} \) where \( b > 1 \) is the base of the logarithm.

**Proof.** We have \( \text{Prob}\{\log Z \geq t\} \leq t_0 b^{-t} = b^{-(t-\log t_0)} \) for all \( t > \log t_0 \). Therefore,
\[
E(\log Z) = \int_{0}^{\infty} \text{Prob}\{\log Z \geq s\} ds \leq \log t_0 + \int_{\log t_0}^{\infty} b^{-(t-\log t_0)} dt = \log t_0 + \frac{1}{\ln b}.
\]

### 4 Matrix inversion

We now focus on the problem of inverting a matrix \( A \) and its componentwise condition number \( c^{\dagger}(A) \). Our main results in this section are the following.

**Theorem 2** Let \( S \subset [n]^2 \) be such that \( \mathcal{M}_S \subset \Sigma \). Then, for all \( t \geq 2|S| \),
\[
\text{Prob}\{c^{\dagger}(A) \geq t\} \leq 4|S|^2 n^2 \frac{1}{t}
\]
where \( \text{Prob} \) denotes probability over \( A \in \mathcal{R}_S \).

**Corollary 2** Let \( S \subset [n]^2 \) be such that \( \mathcal{M}_S \subset \Sigma \). Then,
\[
E(\log_{+}(c^{\dagger}(A))) \leq 2 \log n + 2 \log |S| + \log 4 + \frac{1}{\ln b}
\]
where \( E \) denotes expectation over \( A \in \mathcal{R}_S \).  

\[\Box\]
Remark 1  Note that, for all monotonic norm on \( M \), the bound above also holds for \( m(A) \) by Proposition 1. This is in contrast with the lower bound linear in \( n \) for the expected value of the logarithm of normwise condition numbers, etc.

Definition 1 yields expressions for the (mixed and componentwise) condition numbers of a matrix \( A \) by taking \( D = M \setminus \Sigma \) and \( F : M \setminus \Sigma \to M \) given by \( F(A) = A^{-1} \). For \( k, \ell \in [n] \) such that \( \gamma_{k\ell} \neq 0 \), we let

\[
c_{k\ell}^\dagger(A) = \lim_{\delta \to 0} \sup_{A' \in B(A, \delta)} \frac{|\gamma'_{k\ell} - \gamma_{k\ell}|}{\delta |\gamma_{k\ell}|},
\]

and for \( k, \ell \in [n] \) such that \( \gamma_{k\ell} = 0 \), we let \( c_{k\ell}^\dagger(A) = 0 \) if

\[
\lim_{\delta \to 0} \sup_{A' \in B(A, \delta)} \frac{|\gamma'_{k\ell} - \gamma_{k\ell}|}{\delta} = 0
\]

and \( c_{k\ell}^\dagger(A) = \infty \) otherwise. Then

\[
c^\dagger(A) = \max_{k, \ell \in [n]} c_{k\ell}^\dagger(A).
\]

Similarly, for a norm \( \| \| \) on \( M \),

\[
m(A) = \lim_{\delta \to 0} \sup_{A' \in B(A, \delta)} \frac{\|\Gamma' - \Gamma\|}{\|\Gamma\|}.
\]

Lemma 5  For \( A \in M \setminus \Sigma \) and \( k, \ell \in [n] \),

\[
c_{k\ell}^\dagger(A) \leq c_{\det}^\dagger(A) + c_{\det}^\dagger(A_{\ell k}).
\]

Proof. We divide the proof by cases. Case (i): \( \gamma_{k\ell} \neq 0 \).

Let \( \delta > 0 \) be sufficiently small so that \( B(A, \delta) \cap \Sigma = \emptyset \) and, for all \( A' \in B(A, \delta) \), \( \left| \frac{\det(A') - \det(A)}{\det(A)} \right| < 1 \). Let \( A' \in B(A, \delta) \).
Since $\gamma_{k\ell} = \frac{\det(A_{(\ell k)})}{\det(A)}$,

\[
\frac{\gamma'_{k\ell} - \gamma_{k\ell}}{\gamma_{k\ell}} = \frac{\det(A)}{\det(A_{(\ell k)})} \left( \frac{\det(A'_{(\ell k)})}{\det(A')} - \frac{\det(A_{(\ell k)})}{\det(A)} \right)
\]

\[
= \frac{\det(A)}{\det(A_{(\ell k)})} \frac{\det(A'_{(\ell k)})}{\det(A')} - 1
\]

\[
= 1 + \frac{\det(A'_{(\ell k)}) - \det(A_{(\ell k)})}{\det(A)} - 1
\]

\[
= \frac{\det(A'_{(\ell k)}) - \det(A_{(\ell k)})}{\det(A)} - \frac{\det(A') - \det(A)}{\det(A)}
\]

\[
= \frac{\det(A'_{(\ell k)}) - \det(A_{(\ell k)})}{\det(A)} - \frac{\det(A') - \det(A)}{\det(A)}.
\]

Using that $\left| \frac{\det(A') - \det(A)}{\det(A)} \right| < 1$,

\[
\frac{\gamma'_{k\ell} - \gamma_{k\ell}}{\gamma_{k\ell}} \leq \frac{\left| \frac{\det(A'_{(\ell k)}) - \det(A_{(\ell k)})}{\det(A_{(\ell k)})} \right| + \left| \frac{\det(A') - \det(A)}{\det(A)} \right|}{1 - \left| \frac{\det(A') - \det(A)}{\det(A)} \right|}
\]

and therefore

\[
\sup_{A' \in B(A,\delta)} \left| \frac{\gamma'_{k\ell} - \gamma_{k\ell}}{\delta \gamma_{k\ell}} \right| \leq \sup_{A' \in B(A,\delta)} \left| \frac{\det(A'_{(\ell k)}) - \det(A_{(\ell k)})}{\delta \det(A_{(\ell k)})} \right| + \sup_{A' \in B(A,\delta)} \left| \frac{\det(A') - \det(A)}{\delta \det(A)} \right|
\]

Taking limits for $\delta \to 0$ on both sides we get

\[
c^0_{k\ell}(A) \leq c_{\det}(A) + c_{\det}(A_{(\ell k)}).
\]

Case (ii): $\gamma_{k\ell} = 0$ and

\[
\lim_{\delta \to 0} \sup_{A' \in B(A,\delta)} \frac{|\gamma_{k\ell}|}{\delta} = 0.
\]

In this case, $c^0_{k\ell}(A) = 0$ and the statement holds.

Case (iii): $\gamma_{k\ell} = 0$ and

\[
0 \neq \lim_{\delta \to 0} \sup_{A' \in B(A,\delta)} \frac{|\gamma_{k\ell}|}{\delta} = \lim_{\delta \to 0} \sup_{A' \in B(A,\delta)} \frac{|\det(A'_{(\ell k)})|}{\delta |\det(A')|}.
\]
In this case
\[
\lim_{\delta \to 0} \sup_{A' \in B(A,\delta)} \frac{|\det(A'_{\ell k})|}{\delta} \neq 0
\]
and therefore \(c_{\text{det}}(A'_{\ell k}) = \infty\). The statement holds as well. \(\square\)

**Proof of Theorem 2.** By definition of \(c^\dagger(A)\),
\[
\text{Prob}\{c^\dagger(A) \geq t\} \leq \sum_{k,\ell \in [n]} \text{Prob}\{c_{k\ell}^\dagger(A) \geq t\}
\]
By Lemma 4, with probability 1, \(A\) is non-singular. So, we can apply Lemma 5 to obtain
\[
\text{Prob}\{c_{k\ell}^\dagger(A) \geq t\} \leq 4|S|^2 \frac{1}{t}
\]
the last inequality by Theorem 1. The statement now follows. \(\square\)

## 5 Linear Equations Solving

We finally deal with the problem of solving linear systems of equations.

**Theorem 3** Let \(S \subset [n]^2\) be such that \(\mathcal{M} \not\subset \Sigma\). Then, for all \(t \geq 2(|S|+n)\),
\[
\text{Prob}\{c(A, b) \geq t\} \leq 10|S|^2 n \frac{1}{t}
\]
where \(\text{Prob}\) denotes probability over \((A, b) \in \mathcal{R}_S \times N(0, \text{Id}_n)\).

**Corollary 3** Let \(S \subset [n]^2\) be such that \(\mathcal{M} \not\subset \Sigma\). Then,
\[
\mathbf{E}(\log_+(c(A, b))) \leq \log n + 2\log |S| + \log 10 + \frac{1}{\ln b}.
\] \(\square\)

Definition 1 yields again expressions for the (mixed and componentwise) condition numbers of a pair \((A, b)\) by taking \(\mathcal{D} = (\mathcal{M} \setminus \Sigma) \times \mathbb{R}^n\) and \(F : (\mathcal{M} \setminus \Sigma) \times \mathbb{R}^n \to \mathbb{R}^n\) given by \(F(A, b) = A^{-1}b\).

For \(A \in \mathcal{M} \setminus \Sigma\) and \(b \in \mathbb{R}^n\) we denote \(x = A^{-1}b\). For \(k \in [n]\) such that \(x_k \neq 0\) we let
\[
c_k(A, b) = \lim_{\delta \to 0} \sup_{(A', b') \in B(A, b, \delta)} \frac{|x'_{k} - x_k|}{|\delta x_k|}.
\]
For $k \in [n]$ such that $x_k = 0$ we let $c_k(A, b) = 0$ if
\[
\lim_{\delta \to 0} \sup_{(A', b') \in B((A, b), \delta)} \frac{|x'_k - x_k|}{\delta} = 0
\]
and $c_k(A, b) = \infty$ otherwise. Then
\[
c(A, b) = \max_{k \in [n]} c_k(A, b).
\]
Similarly, for a norm $\| \|$ in $\mathbb{R}^n,$
\[
m(A, b) = \lim_{\delta \to 0} \sup_{(A', b') \in B((A, b), \delta)} \frac{\|x' - x\|}{\delta \|x\|}.
\]
In what follows let $R_k$ be the matrix obtained by replacing the $k$th column of $A$ by $b.$

**Lemma 6** For any non-singular matrix $A$ and $k \in [n],$
\[
c_k(A, b) \leq c_{\det}(A) + c_{\det}(R_k).
\]
**Proof.** By Cramer’s rule,
\[
x_k = \frac{\det(R_k)}{\det(A)}.
\]
The rest of this proof is similar to the proof of Lemma 5. \(\square\)

**Proof of Theorem 3.** It follows the lines of that of Theorem 2. First, we get
\[
\Prob \{ c(A, b) \geq t \} \leq \sum_{k \in [n]} \Prob \{ c_k(A, b) \geq t \}.
\]
Then, we apply Lemma 6 (using that, with probability 1, $A \notin \Sigma$) and the fact that $|S| \geq n$ to get
\[
\Prob \{ c_k(A, b) \geq t \} \leq \Prob \left\{ c_{\det}(A) \geq \frac{t}{2} \right\} + \Prob \left\{ c_{\det}(R_k) \geq \frac{t}{2} \right\}
\leq \frac{2|S|^2}{t} + 2(|S| + n)^2 \frac{1}{t} \leq 10|S|^2 \frac{1}{t}
\]
from which the statement follows. \(\square\)
6 Additional Remarks

(1) To obtain bounds for the average loss of precision of random triangular systems one may combine Theorem 3 with the following result by Wilkinson [9, Ch.3,§19] which we quote as given in [4].

**Theorem 4**  
Let $T \in \mathbb{R}^{n \times n}$ be a nonsingular triangular matrix, and assume $nu < 0.1$ (here $u$ is the round-off unit). Then, the computed solution $\hat{x}$ to the system $Tx = b$ satisfies

$$(T + E)\hat{x} = b,$$

where, for some universal constant $c$,

$$|e_{ij}| \leq (|i - j| + 2)cu|t_{ij}|.$$  

The use of $c(A)$ actually yields average loss of precision with the latter measured componentwise in the computed solution.

(2) The bound in Corollary 2 appears to be worse than what computer simulations suggest $\mathbb{E}(\log c^\dagger(L_n))$ should be. In [2] matrices $L_n$ were generated for various values of $n$ and an experimental mean of $\mathbb{E}(\log c^\dagger(L_n))$ was obtained from these values. A linear regression for these means shows a best fit of $3.065 \log n - 1.1466$. A probable source of (a good part of) the difference of this value with the estimate $6 \log n + O(1)$ following from Corollary 2 is the broad bound $\mathbb{P}\{\max_{k,\ell \in [n]} c^\dagger_{k\ell}(A) \geq t\} \leq \sum_{k,\ell \in [n]} \mathbb{P}\{c^\dagger_{k\ell}(A) \geq t\}$ in the proof of Theorem 2. In addition to this, the bound $\mathbb{E}(m^\dagger(L_n)) \leq \mathbb{E}(c^\dagger(L_n))$ following from Proposition 1 may be coarse as well. Numerical experiments in [2] suggest a best fit of $\mathbb{E}(\log m^\dagger(L_n)) \approx 1.5334 \log n - 0.5723$.

(3) Section 6 of [8] discusses stability of Gaussian elimination. Having shown that, almost surely, $\kappa(L_n) \approx 2^n$ the authors reflect on how this behavior can be reconciled with the fact that “Gaussian elimination is overwhelmingly stable.” They point out that “The reason appears to be statistical: the matrices $A$ for which $\|L^{-1}\|$ is large occupy an exponentially small proportion of the space of matrices” a claim for which experimental evidence is given in [6]. It would thus follow that “the matrices $L$ produced by Gaussian elimination are far from random.”

Our results show that, in addition, Gaussian elimination needs much less than producing matrices $L$ in the vanishingly small set of triangular
matrices with $\kappa(L)$ small. It is enough to produce matrices $L$ outside the vanishingly small set of matrices with $c(L,b)$ large.

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