Abstract

The aim of our work is to provide a simple homogenization and discrete-to-continuum procedure for energy driven problems involving stochastic rapidly-oscillating coefficients. Our intention is to extend the periodic unfolding method to the stochastic setting. Specifically, we recast the notion of stochastic two-scale convergence in the mean by introducing an appropriate stochastic unfolding operator. This operator admits similar properties as the periodic unfolding operator, leading to an uncomplicated method for stochastic homogenization. Secondly, we analyze the discrete-to-continuum (resp. stochastic homogenization) limit for a rate-independent system describing a network of linear elasto-plastic springs with random coefficients.

Keywords: stochastic homogenization, discrete-to-continuum limit, unfolding, spring network models

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1 Introduction

The motivation for this paper is twofold: First, we introduce the method of stochastic unfolding as an analogon to the periodic unfolding method, which in recent years has been successfully applied in the analysis and modeling of multiscale problems with periodic microstructure. Our intention is to provide an easily accessible method for stochastic homogenization and discrete-to-continuum analysis that enjoys many parallels to periodic homogenization via unfolding. Secondly, we analyze the macroscopic behavior (based on stochastic unfolding) of a rate-independent system describing a network of elasto-plastic springs with random coefficients. Our result derives (via stochastic homogenization) a continuum evolutionary rate-independent (linear) plasticity system starting from a discrete model. Discrete spring networks depict solid media as collection of material points that interact via one-dimensional elements with certain constitutive laws. They are widely used in material science and the mechanical engineering community. On the one hand, they are used to model materials with an intrinsic discreteness (on a scale larger than the atomistic scale), such as granular media, truss-like structures, composites. On the other hand, spring network models are used as a numerical approximation scheme for continuum models. We refer to [36, 24, 16] and the references therein. In this introduction we first give a brief overview of the stochastic unfolding method, that we develop in this paper, and then discuss the discrete-to-continuum limits of random spring networks.

In order to give a brief overview of the stochastic unfolding, which we develop in this paper, let us consider for a moment a prototypical example of convex homogenization: Let $O \subset \mathbb{R}^d$ be open and bounded, $p \in (1, \infty)$, $\varepsilon > 0$, and consider the minimization problem

$$
\min_{u \in W^{1,p}(O)} \int_O V_\varepsilon(x, \nabla u(x)) \, dx \quad \text{(subject to suitable boundary conditions).} \tag{1}
$$

Above $V_\varepsilon(x, F)$ denotes a family of energy densities that are convex in $F$, and which we think to rapidly oscillate in $x$ on a scale $\varepsilon$. The objective of homogenization is to derive a simpler minimization problem, say

$$
\min_{u \in W^{1,p}(O)} \int_O V_0(\nabla u(x)) \, dx \quad \text{(subject to suitable boundary conditions)},
$$

with an effective (and simpler) energy density $V_0$ that captures the behavior of (1) for small $\varepsilon$. This is done by an asymptotic analysis for $\varepsilon \to 0$, and a classical way to approach this type of problems is based on two-scale convergence methods. The notion of (periodic) two-scale convergence was introduced and developed by Nguetseng [35] and Allaire [3]. A sequence $u_\varepsilon \in L^p(O)$ is said to two-scale converge to $u \in L^p(O \times \Box)$ if

$$
\int_O u_\varepsilon(x) \varphi \left( x, \frac{x}{\varepsilon} \right) \, dx \to \int_O \int_\Box u(x, y) \varphi(x, y) \, dy \, dx \quad \text{as } \varepsilon \to 0,
$$

for all $\varphi \in L^q(O, C_{\text{per}}(\Box))$. Here, $\Box := [0,1)^d$ is the unit box and $C_{\text{per}}(\Box)$ is the space of continuous, $\Box$-periodic functions. The two-scale limit of a sequence refines its weak limit by capturing oscillations on a prescribed scale $\varepsilon$. It is therefore especially useful in homogenization problems involving linear (or monotone) operators and convex potentials with periodic coefficients. With regard to problem (1), two-scale convergence methods apply, e.g. if $V_\varepsilon(x, F) = V(x, \frac{x}{\varepsilon}, F)$ with $V$ being periodic in its second component and sufficiently regular (e.g. continuous) in its first component.

In [12] the method of periodic unfolding was introduced based on the dilation technique [5]. The idea of unfolding, which is closely related to two-scale convergence, is to introduce an operator (the unfolding operator), which embeds sequences of oscillating functions to a larger
two-scale space, to the effect that two-scale convergence can be characterized by the usual weak convergence in the two-scale space. In some cases, this method facilitates a more straightforward, and operator theory flavored analysis of periodic homogenization problems. In recent years periodic unfolding has been applied to a large variety of multiscale problems, e.g. see [11, 15, 32, 33, 29, 38, 10, 37]. For a systematic investigation of two-scale calculus associated with the use of the periodic unfolding method we refer to [13, 39, 32].

Motivated by periodic two-scale convergence, in [8] a related notion of stochastic two-scale convergence in the mean was introduced. It is tailor-made for the study of stochastic homogenization problems. In particular, it applies to a stochastic version of problem (1): Let $\Omega$ be a probability space with a corresponding measure-preserving dynamical system $\{T_x\}_{x \in \mathbb{R}^d}$ (see Section 2.2). In the context of stochastic homogenization, we might view $\Omega$ as a configuration space, and $\langle \cdot \rangle$ (the associated expected value) as an ensemble average w.r.t. configurations. Then we might consider a stochastic version of (1), namely

$$
\min_{u \in L^p(\Omega) \otimes W^{1,p}(\Omega)} \int_{\Omega} V(T_x \omega, \nabla_x u(\omega, x)) \, dx ,
$$

(2)

where the potential $V(\omega, F)$ is parametrized by $\omega \in \Omega$, and thus minimizers of (2) are random fields, i.e. they depend on $\omega \in \Omega$. The random potential $V_\omega(x, \cdot) = V(T_x \omega, \cdot)$ in (2) is rapidly oscillating and its statistics is homogeneous in space (i.e. for any finite number of points $x_1, \ldots, x_m \in \mathbb{R}^d$ and all $F \in \mathbb{R}^d$, the joint distribution of $(V(T_{x_i} + z, F))_{i=1,\ldots,m}$ is independent of the shift $z \in \mathbb{R}^d$). In contrast to periodic two-scale convergence, stochastic two-scale convergence requires test-functions defined not only on the physical space $\Omega \subset \mathbb{R}^d$, but also on the probability space $\Omega$ (see Remark 3.1, where we recall the definition of stochastic two-scale convergence in the mean of [8, 4] in a discrete version).

**Stochastic unfolding.** In this paper, we introduce a stochastic unfolding method, that (analogously to the periodic case) allows to characterize stochastic two-scale convergence in the mean by mere weak convergence in an extended space. Having discrete-to-continuum problems in mind, we concentrate in this paper on a discrete setting: E.g. in (2) $\Omega$ is replaced by the discrete set $O_\epsilon := O \cap \mathbb{Z}^d$, and instead of the gradient we consider difference quotients. As we shall demonstrate, the stochastic unfolding method features many analogies to the periodic case; as a consequence it allows to lift systematically and easily homogenization results and multiscale models for periodic media to the level of random media. In the following, in particular for readers familiar with periodic unfolding, we briefly summarize the main properties of stochastic unfolding and its analogies to the periodic case:

- We introduce an operator $T_\epsilon : L^p(\Omega \times \epsilon \mathbb{Z}^d) \to L^p(\Omega \times \mathbb{R}^d)$ which is a linear isometry and we call it stochastic unfolding operator (see Section 3.1).

- Two-scale convergence in the mean for $u_\epsilon \in L^p(\Omega \times \epsilon \mathbb{Z}^d)$ reduces to weak convergence of the unfolding $T_\epsilon u_\epsilon$ in $L^p(\Omega \times \mathbb{R}^d)$ (see Remark 3.1).

- We define weak (strong) stochastic two-scale convergence as weak (strong) convergence of the unfolded sequence $T_\epsilon u_\epsilon$ (see Definition 3.1).

- (Compactness). Bounded sequences converge (up to a subsequence) in the weak stochastic two-scale sense (see Lemma 3.1).

- (Compactness for gradients). If $u_\epsilon \in L^p(\Omega \times \epsilon \mathbb{Z}^d)$ is bounded and its (discrete) gradient is bounded, then (up to extracting a subsequence) $u_\epsilon$ weakly two-scale converges to $U_0 \in L^p_{inv}(\Omega) \otimes W^{1,p}(\mathbb{R}^d)$. Moreover, its gradient weakly two-scale converges and the limit has a specific structure: $\nabla U_0 + \chi$ where $\chi \in L^p_{pot}(\Omega) \otimes L^p(\mathbb{R}^d)$. Here, $L^p_{pot}(\Omega) := \{D\varphi : \varphi \in L^p(\Omega)\}$ where the closure is taken in $L^p(\Omega)$, and $L^p_{inv}(\Omega)$ is the space of shift-invariant functions (see Section 2.2). In the ergodic case, the latter reduces to the space of
constant functions, and thus the two-scale limit $U_0$ is deterministic, i.e. it does not depend on $\omega \in \Omega$.

The general structure of this compactness statement for gradients is quite similar to its analogon in the periodic case, which we briefly recall: Up to a subsequence, a bounded sequence $u_\varepsilon \in W^{1,p}(\mathbb{R}^d)$ converges weakly to some $u_0 \in W^{1,p}(\mathbb{R}^d)$, and the gradient $\nabla u_\varepsilon$ weakly two-scale converges to $\nabla u_0(x) + v(x,y)$, where $v \in L^p(\mathbb{R}^d) \otimes L^p_{\text{pot,per}}(\square)$ with $\square$ denoting the reference cell of periodicity, say $\square := [0,1]^d$. In the periodic case, thanks to Poincaré’s inequality on $\square$, any periodic, conservative field $v \in L^p_{\text{per}}(\square)$, can be represented as $v = \nabla_y \varphi$ for some potential $\varphi \in W^{1,p}_{\text{per}}(\square)$. Thus, the weak two-scale limit of $\nabla u_\varepsilon$ takes the form $\nabla u_0(x) + \nabla_y \varphi(x,y)$ with $\varphi \in L^p(\mathbb{R}^d, W^{1,p}_{\text{per}}(\square))$.

In the stochastic case, typically it is not possible to represent $v \in L^p_{\text{pot}}(\Omega)$ with help of a potential defined on $\Omega$. This is one of the main differences between stochastic and periodic homogenization.

• (Recovery sequences). For $U_0$ and $\chi$ as above, we construct a sequence $u_\varepsilon \in L^p(\Omega \times \varepsilon \mathbb{Z}^d)$ which converges as in the previous item and additionally in the strong two-scale sense. The following transformation formula holds

\[
\left\langle \int_{\varepsilon \mathbb{Z}^d} V(T_\varepsilon^d \omega, v(\omega, x)) dm_\varepsilon(x) \right\rangle = \left\langle \int_{\mathbb{R}^d} V(\omega, T_\varepsilon v(\omega, x)) dx \right\rangle.
\]

Using this formula and the previous properties, the $\Gamma$-convergence analysis of the discrete version of (2) becomes straightforward, and relies (as only noteworthy additional ingredient) on the weak lower-semicontinuity of convex integral functionals, see Proposition 3.3 and Theorem 4.1.

We would like to remark that some of the statements above (in particular those that involve only weak two-scale convergence) have already been established in the continuum setting in [8]. Yet, the arguments that we present have a different twist, since they are based on the unfolding operator and apply to the discrete setting.

In contrast to periodic (deterministic) two-scale convergence, in the stochastic case different meaningful notions for two-scale convergence exist, since one may ask for convergence in the $L^p(\Omega)$-sense ($\Omega$ being the probability space) or in a quenched sense, i.e. for a.e. $\omega \in \Omega$. The former corresponds to stochastic two-scale convergence in the mean and the notion of stochastic unfolding that we introduce in this paper. The latter corresponds to a finer notion of stochastic two-scale convergence, as considered in [41, 19].

**Discrete-to-continuum limits of random spring networks.** In the second part of this paper, we study the macroscopic, rate-independent behavior of periodic networks formed of elasto-plastic springs with random material properties. In the following, we briefly summarize our result in the simplest (non-trivial) two-dimensional setting. In Section 4 we shall treat a general, multi-dimensional case. To explain the model, we first consider a single spring that in a natural state has endpoints $x_0, x_1 \in \mathbb{R}^d$, and thus is aligned with $b := x_1 - x_0$. We describe a deformation of the spring with help of a displacement function $u$ that maps an endpoint $x_i$ to its new position $x_i + u(x_i)$. As the measure of relative elongation (resp. compression) of the spring, we consider the Cauchy strain $\frac{b + |\partial_b u|}{|b|}$ with $\partial_b u := \frac{u(x_0 + b) - u(x_0)}{|b|}$. If the displacement is (infinitesimally) small, we arrive (by rescaling and passing to the limit $|\partial_b u| \to 0$) at the linearized strain $\frac{b}{|b|} \cdot \partial_b u$. As it is usual in linear elasto-plasticity (see e.g. [17, Section 3]), we assume that the strain admits an additive decomposition $\frac{b}{|b|} \cdot \partial_b u = e + z$, where $e$ and $z$ are its elastic and plastic parts respectively. The force (its intensity) exerted by the spring is linear in the elastic strain: $\sigma = ae$, $a > 0$ being the spring constant. We define a free energy (describing
materials with linear kinematic hardening)

\[ \varepsilon_b(u, z) := \frac{1}{2} a \left( \frac{b}{|b|} \cdot \partial_b u - z \right)^2 + \frac{1}{2} h z^2, \]

where \( h > 0 \) denotes a hardening parameter. The rate-independent evolution of the elasto-plastic spring under a loading \( l : [0, T] \times \{x_0, x_1\} \to \mathbb{R}^2 \) is determined by \( (i = 0, 1) \)

\[ (-1)^i \left( \frac{b}{|b|} \cdot \partial_b u(t) - z(t) \right) \frac{b}{|b|} + l(t, x_i) = 0, \quad \dot{z}(t) \in \partial I_{[-\sigma_y, \sigma_y]} \left( -\frac{\partial \varepsilon_b}{\partial z} (u(t), z(t)) \right). \]  \( (3) \)

Above, \( \sigma_y \geq 0 \) is the yield stress of the spring, \( \partial I_{[-\sigma_y, \sigma_y]} \) denotes the convex subdifferential of \( I_{[-\sigma_y, \sigma_y]} \), which is the indicator function of the set \( [-\sigma_y, \sigma_y] \). Note that the first two equations are force balance equations (inertial terms are disregarded), reasonable in regimes of small displacements, and the second expression is a flow rule for the variable \( z \).

![Periodic network of springs with heterogeneous coefficients.](image)

We consider a network of springs \( E = \{ e = [x, x + \varepsilon b] : x \in \varepsilon \mathbb{Z}^2, b \in \{ e_1, e_2, e_1 + e_2 \} \} \), where the nodes \( x \in \varepsilon \mathbb{Z}^2 \) represent the reference configuration of particles connected by springs. The displacement of the network is described with help of a map \( u : \varepsilon \mathbb{Z}^2 \to \mathbb{R}^2 \) and the plastic strains of the springs are accounted by an internal variable \( z : \varepsilon \mathbb{Z}^2 \to \mathbb{R}^2 \) (e.g. \( z_1(x) \) is the plastic strain of the spring \( [x, x + \varepsilon e_1] \)). We assume that the particles outside of a set \( O_e = O \cap \varepsilon \mathbb{Z}^d \) are fixed, i.e. \( u = 0 \) in \( \varepsilon \mathbb{Z}^d \setminus O_e \); furthermore we suppose that \( z \) is supported in \( O_e^+ := \{ x \in \varepsilon \mathbb{Z}^2 : (x, x + \varepsilon b) \cap O \neq \emptyset \text{ for some } b \in \{ e_1, e_2, e_1 + e_2 \} \} \). A small external force \( \varepsilon \sigma : [0, T] \times O_e \to \mathbb{R}^2 \) acts on the system. According to the evolution law (3) for single springs, the evolution of the network is determined by (for \( t \in [0, T] \))

\[
\sum_{b \in \{ e_1, e_2, e_1 + e_2 \}} -\partial_{-e_b} \left( |b| a(x, b) \left( \frac{b}{|b|} \cdot \partial_{e_b} u(t, x) - z_b(t, x) \right) \right) \frac{b}{|b|} + l_e(t, x) = 0 \text{ in } O_e,
\]

\[
\dot{z}_b(t, x) \in \partial I_{-\sigma_y(x, b), \sigma_y(x, b)} \left( -\frac{\partial \varepsilon_b}{\partial z_b} (u(t, x), z_b(t, x)) \right) \text{ in } O_e^+, \text{ for } b \in \{ e_1, e_2, e_1 + e_2 \} ,
\]

which is a superposition of (3). We tacitly identify \( b \in \{ e_1, e_2, e_1 + e_2 \} \) with indices \( i = 1, 2, 3 \). The coefficients \( a(x, b), h(x, b), \sigma_y(x, b) \) describe the properties of the spring \( [x, x + \varepsilon b] \).

The above equations may be equivalently recast in the global energetic formulation for rate-independent systems (see Appendix A.1) with the help of energy and dissipation functionals,
resp.: \( \mathcal{E}_\varepsilon : [0, T] \times L^2_0(O_\varepsilon)^2 \times L^2_0(O_\varepsilon^+)^3 \to \mathbb{R}, \Psi_\varepsilon : L^2_0(O_\varepsilon)^2 \times L^2_0(O_\varepsilon^+)^3 \to [0, \infty), \)
\[
\mathcal{E}_\varepsilon(t, u, z) = \int_{O_\varepsilon^+} \frac{1}{2} \mathcal{A}(x) (\nabla^*_s u(x) - z(x)) \cdot (\nabla^*_s u(x) - z(x)) + \frac{1}{2} \mathcal{H}(x) z(x) \cdot z(x) \, \mathrm{dm}_x(z) - \int_{O_\varepsilon} l_c(t, x) \cdot u(x) \, \mathrm{dm}_x(z),
\]
\[
\Psi_\varepsilon(u, z) = \sum_{b \in \{e_1, e_2, e_1 + e_2\}} \sigma_g(x, b) \vert z_b(x) \vert \, \mathrm{dm}_x(z).
\]

Above, \( \nabla^*_s \) stands for the discrete symmetrized gradient \( \nabla^*_s u = \left( \frac{b}{|b|} \cdot \partial_b u \right)_{b \in \{e_1, e_2, e_1 + e_2\}} \). \( \mathcal{A}(x) = \text{diag} \left( a(x, e_1), a(x, e_2), \sqrt{2} a(x, e_1 + e_2) \right) \) and \( \mathcal{H}(x) = \text{diag} \left( h(x, e_1), h(x, e_2), h(x, e_1 + e_2) \right) \). We assume that the coefficients are random fields oscillating on a scale \( \varepsilon \). In particular, the deterministic coefficients \( \mathcal{A}(x) \), \( \mathcal{H}(x) \) and \( \sigma_g(x, b) \) in the above functionals are replaced by realizations of rescaled stationary random fields \( \mathcal{A}(T_\varepsilon \omega), \mathcal{H}(T_\varepsilon \omega) \) and \( \sigma^b_g(T_\varepsilon \omega) \). As a consequence of this, the solutions of the corresponding evolutionary equation at each time instance are not deterministic functions but rather random fields on \( \Omega \times \varepsilon \mathbb{Z}^2 \). Under suitable assumptions (cf. Section 4.3), there exists a unique solution \( (u_\varepsilon, z_\varepsilon) \in C^{Lip}[0, T], (L^2(\Omega) \otimes L^2_0(O_\varepsilon)^2) \times (L^2(\Omega) \otimes L^2_0(O_\varepsilon^+)^3) \) to the above described microscopic evolutionary rate-independent system.

Applying the method of stochastic unfolding, we are able to capture the averaged (w.r.t. the probability measure) behavior of the solution \( (u_\varepsilon, z_\varepsilon) \) in the limit \( \varepsilon \to 0 \). Particularly, we show that, upon assuming suitable strong convergence for the initial data and forces, there exists \( (U, Z, \chi) \in C^{Lip}[0, T], H^0_0(O)^2 \times L^2(\Omega \times \Omega)^3 \times (L^2_\text{pot}(\Omega) \otimes L^2(O))^3 \) which solves an effective rate-independent system on a continuum physical space (see Section 4.3) and for every \( t \in [0, T] \)
\[
(u_\varepsilon(t), z_\varepsilon(t)) \xrightarrow{\text{ct}} (U(t), Z(t), \chi(t)),
\]
where \( \xrightarrow{\text{ct}} \) denotes "cross" two-scale convergence and it is explained in Section 4.3.

In the continuum case, similar results have been obtained, for deterministic periodic materials in [30, 18] (via periodic unfolding), and for random materials recently in [20] (using quenched stochastic two-scale convergence) and [22, 23]. We discuss the literature on problems involving discrete-to-continum transition in more detail in Section 4.

If we consider the constraint \( z_\varepsilon = 0 \) and time-independent force \( l_c(t) = l_c \), the above problem boils down to the homogenization of the functional \( u \mapsto \mathcal{E}_\varepsilon(0, u, 0) \), which corresponds to the discrete-to-continuum limit of the static equilibria of a spring network with (only) elastic interactions.

We remark that the methods in this paper apply as well to systems with different constitutive laws, e.g. one might consider an energy functional with an additional term depending on the gradient of the internal variable \( z_\varepsilon \), as it is the case in gradient plasticity (see Section 4.4). In the ergodic case, we even obtain a deterministic elasto-plastic limiting model. Another interesting extension of our method (which we do not discuss in this paper) is the discrete-to-continuum analysis of random spring networks featuring damage or fracture. The convergence result that we establish can be seen as a justitification of continuum models for microstructural spring networks that feature uncertainty in the constitutive relations on the microscopic scale. In this context, the method could also be applied to prove the consistency of computational schemes based on the lattice method as discussed in the mechanical engineering community (e.g. see [16]).

**Structure of the paper.** In Section 2, we introduce a convenient setting for problems involving homogenization and the passage from discrete to continuum systems. Section 3 is devoted to the introduction of the stochastic unfolding operator and its most important properties. In Section 4, we apply the stochastic unfolding method to an example of a multi-dimensional network of elastic/elasto-plastic springs with random coefficients.
2 General framework

In this section, we introduce the setting for functions on a discrete/continuum physical space suited for problems involving discrete-to-continuum transitions. In addition, we present the standard setting for stochastic homogenization problems.

2.1 Functions and differential calculus on $\varepsilon\mathbb{Z}^d$ and $\mathbb{R}^d$

Throughout the paper we consider $p, q \in (1, \infty)$ exponents of integrability that are dual, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. We denote by $\{e_i\}_{i=1,...,d}$ the standard basis of $\mathbb{R}^d$. For $\varepsilon > 0$, we denote the Banach space of $p$-summable functions by $L^p(\varepsilon \mathbb{Z}^d) := \left\{ u : \varepsilon \mathbb{Z}^d \to \mathbb{R} : (\varepsilon^d \sum_{x \in \varepsilon \mathbb{Z}^d} u^p(x))^\frac{1}{p} < \infty \right\}$. For our purposes it is convenient to view $L^p(\varepsilon \mathbb{Z}^d)$ as the $L^p$-space of $p$-integrable functions on the measure space $(\varepsilon \mathbb{Z}^d, x \mathbb{Z}^d, m_\varepsilon)$ with $m_\varepsilon = \varepsilon^d \sum_{x \in \varepsilon \mathbb{Z}^d} \delta_x$. In particular, we use the notation $\int_{\varepsilon \mathbb{Z}^d} u(x) dm_\varepsilon(x) := \varepsilon^d \sum_{x \in \varepsilon \mathbb{Z}^d} u(x)$. For $u : \varepsilon \mathbb{Z}^d \to \mathbb{R}$ and $g = (g_1,...,g_d) : \varepsilon \mathbb{Z}^d \to \mathbb{R}^d$, we set

$$\nabla_\varepsilon^i u(x) = \frac{u(x + \varepsilon e_i) - u(x)}{\varepsilon}, \quad \nabla_\varepsilon^{i,*} u(x) = \frac{u(x - \varepsilon e_i) - u(x)}{\varepsilon},$$

$$\nabla_\varepsilon u(x) = (\nabla_\varepsilon^1 u(x), \ldots, \nabla_\varepsilon^d u(x)),$$

and we call $\nabla_\varepsilon$ discrete gradient and $\nabla_\varepsilon^{i,*}$ negative discrete divergence. In the case $\varepsilon = 1$, we simply write $\nabla$ and $\nabla_*$ instead of $\nabla_\varepsilon$ and $\nabla_\varepsilon^{,*}$. For $u \in L^p(\varepsilon \mathbb{Z}^d)$, $g \in L^q(\varepsilon \mathbb{Z}^d)^d$, we have the discrete integration by parts formula

$$\int_{\varepsilon \mathbb{Z}^d} \nabla_\varepsilon u(x) \cdot g(x) dm_\varepsilon(x) = \int_{\varepsilon \mathbb{Z}^d} u(x) \nabla_\varepsilon^{,*} g(x) dm_\varepsilon(x).$$

**Definition 2.1 (Weak and strong convergence).** Consider a sequence $u_\varepsilon \in L^p(\varepsilon \mathbb{Z}^d)$ and $U \in L^p(\mathbb{R}^d)$. We say that:

- $u_\varepsilon$ weakly converges to $U$ (denoted by $u_\varepsilon \rightharpoonup U$ in $L^p(\mathbb{R}^d)$) if $\limsup_{\varepsilon \to 0} \|u_\varepsilon\|_{L^p(\varepsilon \mathbb{Z}^d)} < \infty$ and

$$\lim_{\varepsilon \to 0} \int_{\varepsilon \mathbb{Z}^d} u_\varepsilon(x) \eta(x) dm_\varepsilon(x) = \int_{\mathbb{R}^d} U(x) \eta(x) dx \quad \text{for all } \eta \in C^\infty_c(\mathbb{R}^d).$$

- $u_\varepsilon$ strongly converges to $U$ (denoted by $u_\varepsilon \to U$ in $L^p(\mathbb{R}^d)$) if

$$u_\varepsilon \to U \text{ in } L^p(\mathbb{R}^d) \text{ and } \lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^p(\varepsilon \mathbb{Z}^d)} = \|U\|_{L^p(\mathbb{R}^d)}.$$

It is convenient to consider piecewise-constant and piecewise-affine interpolations of functions in $L^p(\varepsilon \mathbb{Z}^d)$.

**Definition 2.2.**

(i) For $u : \varepsilon \mathbb{Z}^d \to \mathbb{R}$, its piecewise-constant interpolation $\Pi : \mathbb{R}^d \to \mathbb{R}$ (subordinate to $\varepsilon \mathbb{Z}^d$) is given by $\Pi(x) = \sum_{y \in \varepsilon \mathbb{Z}^d} 1_{y + \Box} \left( \frac{x}{\varepsilon} \right) u(|x| \varepsilon)$, where $\Box = [-\frac{1}{2}, \frac{1}{2})^d$ is the unit box and $|x| \varepsilon \in \varepsilon \mathbb{Z}^d$ is defined by $x - |x| \varepsilon \in \varepsilon \Box$.

(ii) Consider a triangulation of $\mathbb{R}^d$ into $d$-simplices with nodes in $\varepsilon \mathbb{Z}^d$ (e.g. Freudenthal’s triangulation). For $u : \varepsilon \mathbb{Z}^d \to \mathbb{R}$, we denote its piecewise-affine interpolation (w.r.t. the triangulation) by $\hat{u} : \mathbb{R}^d \to \mathbb{R}$.

(iii) The $\varepsilon \mathbb{Z}^d$-discretization $\pi_\varepsilon : L^1_{loc}(\mathbb{R}^d) \to \mathbb{R}^{\varepsilon \mathbb{Z}^d}$ is defined as

$$(\pi_\varepsilon U)(x) = \int_{x + \varepsilon \Box} U(y) dy.$$
Remark 2.1. Note that \( \pi : L^p(\mathbb{Z}^d) \to L^p(\mathbb{R}^d) \), \( u \mapsto \pi u \) defines a linear isometry. Also, \( \pi_e : L^p(\mathbb{R}^d) \to L^p(\mathbb{Z}^d) \) is linear and bounded with \( \|\pi_e\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{Z}^d)} \leq 1 \). Furthermore, \( \pi_e \circ \pi = \text{Id} \) on \( L^p(\mathbb{Z}^d) \) and we define \( \pi_e := (\pi \circ \pi_e) \), which is a contractive projection, mapping to the subspace of piecewise-constant functions (subordinate to \( \mathbb{Z}^d \)) in \( L^p(\mathbb{R}^d) \).

The proof of the following lemma is an uncomplicated exercise and therefore we omit it.

Lemma 2.1. Let \( u_\varepsilon \in L^p(\mathbb{Z}^d) \) and \( U \in L^p(\mathbb{R}^d) \). The following claims are equivalent.

(i) \( u_\varepsilon \to (\to) U \) in \( L^p(\mathbb{R}^d) \).
(ii) \( \pi_\varepsilon \to U \) weakly (strongly) in \( L^p(\mathbb{R}^d) \).
(iii) \( \hat{u}_\varepsilon \to U \) weakly (strongly) in \( L^p(\mathbb{R}^d) \).

2.2 Description of random media and a differential calculus for random variables

As it is standard in stochastic homogenization, we describe random configurations (e.g. coefficients of a PDE or energy densities that describe properties of some medium with quenched disorder) using a probability space \((\Omega, \mathcal{F}, P)\) together with a measure preserving dynamical system \( T_x : \Omega \to \Omega \) (\( x \in \mathbb{Z}^d \)) such that:

(i) \( T_x : \Omega \to \Omega \) is measurable for all \( x \in \mathbb{Z}^d \),
(ii) \( T_0 = \text{Id} \) and \( T_{x+y} = T_x \circ T_y \) for all \( x, y \in \mathbb{Z}^d \),
(iii) \( P(T_x A) = P(A) \) for all \( A \in \mathcal{F} \) and \( x \in \mathbb{Z}^d \).

We write \( \langle \cdot \rangle \) for the expectation and \( L^p(\Omega) \) for the usual Banach space of \( p \)-integrable random variables. Throughout the paper we assume that \((\Omega, \mathcal{F}, P, T)\) satisfies the properties above, and that \((\Omega, \mathcal{F}, P)\) is a separable measure space; the latter implies the separability of \( L^p(\Omega) \).

The dynamical system \( T \) is called ergodic (we also say \( \langle \cdot \rangle \) is ergodic), if for any \( A \in \mathcal{F} \) the following implication holds:

\[
A \text{ is shift invariant, i.e. } T_x A = A \text{ for all } x \in \mathbb{Z}^d \\
\Rightarrow \quad P(A) \in \{0, 1\}.
\]

Remark 2.2. A multiparameter version of Birkhoff’s ergodic theorem (see [1, Theorem 2.4]) states that if \( \langle \cdot \rangle \) is ergodic and \( \varphi \in L^p(\Omega) \) then

\[
\lim_{R \to \infty} \frac{1}{|B_R|} \sum_{x \in B_R \cap \mathbb{Z}^d} \varphi(T_x \omega) \to \langle \varphi \rangle \quad \text{for } P\text{-a.e. } \omega \in \Omega.
\]

Example 2.1. Let \((\Omega_0, \mathcal{F}_0, P_0)\) denote a separable probability space. Define \((\Omega, \mathcal{F}, P)\) as the \( \mathbb{Z}^d \)-fold product of \((\Omega_0, \mathcal{F}_0, P_0)\), i.e. \( \Omega := \Omega_0^\mathbb{Z}^d, \mathcal{F} = \otimes_{\mathbb{Z}^d} \mathcal{F}_0, P = \otimes_{\mathbb{Z}^d} P_0 \). Note that a configuration \( \omega \in \Omega \) can be seen as a function \( \omega : \mathbb{Z}^d \to \Omega_0 \). We define the shift \( T_x \) (\( x \in \mathbb{Z}^d \)) as

\[
T_x \omega(\cdot) := \omega(\cdot + x).
\]

Then it follows that \((\Omega, \mathcal{F}, P, T)\) satisfies the assumptions above and defines an ergodic dynamical system. With regard to the example in the introduction, (2), we might consider random potentials of the form

\[
V(\omega, F) := a_0(\omega(0)) |F|^2,
\]

where \( a_0 : \Omega_0 \to (\lambda, 1) \) is a random variable, with \( \lambda > 0 \) denoting a positive constant of ellipticity.

We remark that the coefficients appearing in the corresponding energy \( \{a_0(T_x \omega(0))\}_{x \in \mathbb{Z}^d} \) are independent and identically distributed random variables.
Differential calculus for random variables. For \( \varphi : \Omega \to \mathbb{R} \) and \( \psi = (\psi_1, \ldots, \psi_d) : \Omega \to \mathbb{R}^d \) measurable, we introduce the horizontal derivative \( D \) and negative horizontal divergence \( D^*: \)

\[
D_i \varphi(\omega) = \varphi(T_{e_i} \omega) - \varphi(\omega), \\
D \varphi(\omega) = (D_1 \varphi(\omega), \ldots, D_d \varphi(\omega)), \\
D^* \psi(\omega) = \sum_{i=1}^d D_i^* \psi_i(\omega).
\]

Remark 2.3. \( D : L^p(\Omega) \to L^p(\Omega)^d \) and \( D^* : L^p(\Omega)^d \to L^p(\Omega) \) are linear and bounded operators. Furthermore, for any \( \varphi \in L^p(\Omega) \) and \( \psi \in L^q(\Omega)^d \) the integration by parts formula

\[
\langle D \varphi \cdot \psi \rangle = \langle \phi D^* \psi \rangle
\]

holds. Hence, \( D \) (defined on \( L^p(\Omega) \)) and \( D^* \) (defined on \( L^q(\Omega)^d \)) are adjoint operators.

We denote the set of shift-invariant functions in \( L^p(\Omega) \) by

\[
L^p_{\text{inv}}(\Omega) := \{ \varphi \in L^p(\Omega) : \varphi(T_x \omega) = \varphi(\omega) \text{ for all } x \in \mathbb{Z}^d \text{ and a.e. } \omega \in \Omega \},
\]

and note that \( L^p_{\text{inv}}(\Omega) \simeq \mathbb{R} \) if and only if \( \langle \cdot \rangle \) is ergodic. We denote by \( P_{\text{inv}} : L^p(\Omega) \to L^p_{\text{inv}}(\Omega) \) the conditional expectation w.r.t. the \( \sigma \)-algebra generated by the family of shift invariant sets \( \{ A \in \mathcal{F} : T_x A = A \text{ for every } x \in \mathbb{Z}^d \} \). It is a contractive projection and in the ergodic case we simply have \( P_{\text{inv}} f = \langle f \rangle \). The adjoint of \( P_{\text{inv}} \) is denoted by \( P_{\text{inv}}^* : L^q(\Omega) \to L^q(\Omega) \) and a simple computation shows that \( P_{\text{inv}}^* \varphi \in L^p_{\text{inv}}(\Omega) \) for any \( \varphi \in L^q(\Omega) \).

It is easily verified that \( L^p_{\text{inv}}(\Omega) = \ker D \) and by standard arguments (see [9, Section 2.6]) we have the orthogonality relations

\[
\ker D = (\text{ran } D^*)^\perp, \\
L^p_{\text{pot}}(\Omega) := \overline{\text{ran } D^{L^p(\Omega)^d}} = (\ker D^*)^\perp.
\]

The above relations hold in the sense of \( L^p - L^q \) duality (we identify \( L^q(\Omega)' \) with \( L^p(\Omega) \)). Namely, \( D : L^p(\Omega) \to L^p(\Omega)^d \) and \( D^* : L^q(\Omega)^d \to L^q(\Omega) \) and the orthogonal of a set \( A \subset L^q(\Omega) \) is given by

\[
A^\perp = \{ \varphi \in L^q(\Omega)' : \langle \varphi, \psi \rangle_{L^q(\Omega)' \times L^q(\Omega)} = 0 \text{ for all } \psi \in A \}.
\]

Random fields. In this paper measurable functions defined on \( \Omega \times \mathbb{Z}^d \) or on \( \Omega \times \mathbb{R}^d \) are called random fields. We mainly consider the space of \( p \)-integrable random fields \( L^p(\Omega \times \mathbb{Z}^d) \), and frequently use the following notation: If \( X \subset L^p(\Omega) \) and \( Y \subset L^p(\mathbb{Z}^d) \) (resp. \( Y \subset L^p(\mathbb{R}^d) \)) are linear subspaces, then we denote by \( X \otimes Y \) the closure of

\[
X \otimes Y := \overline{\text{span } \left\{ \sum_j \varphi_j \eta_j : \varphi_j \in X, \eta_j \in Y \right\}}
\]

in \( L^p(\Omega \times \mathbb{Z}^d) \) (resp. \( L^p(\Omega \times \mathbb{R}^d) \)). In particular, since \( L^p(\Omega) \) is separable (thanks to our assumption on the underlying measure space), we have \( L^p(\Omega) \otimes \mathcal{L}(\mathbb{Z}^d) = L^p(\Omega \times \mathbb{Z}^d) \) (resp. \( L^p(\Omega) \otimes L^p(\mathbb{R}^d) = L^p(\Omega \times \mathbb{R}^d) \)). Similarly, if above instead we have \( Y \subset W^{1,p}(\mathbb{R}^d) \) is a linear subspace, then \( X \otimes Y \) is defined as the closure of \( X \otimes Y \) in \( L^p(\Omega, W^{1,p}(\mathbb{R}^d)) \). In this respect, we tacitly identify linear and bounded operators on \( X \) (or \( Y \)) by their obvious extension to \( X \otimes Y \). The applications involve problems with homogeneous Dirichlet boundary conditions, and therefore the following space is convenient: For \( O \subset \mathbb{R}^d \) we set

\[
L^p_0(O \cap \mathbb{Z}^d) = \{ u \in L^p(\mathbb{Z}^d) : u = 0 \text{ in } \mathbb{Z}^d \setminus (O \cap \mathbb{Z}^d) \}.
\]
3 Stochastic unfolding

3.1 Definition and properties

For \( u : \Omega \times \varepsilon \mathbb{Z}^d \to \mathbb{R} \) we define the unfolding of \( u \) via

\[(\widetilde{T}_\varepsilon u)(\omega, x) = u(T_{\frac{x}{\varepsilon}} \omega, x).\]

The above expression defines an isometric isomorphism \( \widetilde{T}_\varepsilon : L^p(\Omega \times \varepsilon \mathbb{Z}^d) \to L^p(\Omega \times \varepsilon \mathbb{Z}^d) \). For our purposes, it is convenient to consider \( T_\varepsilon := (\widetilde{T}_\varepsilon)^{-1} \circ \widetilde{T}_\varepsilon : L^p(\Omega \times \varepsilon \mathbb{Z}^d) \to L^p(\Omega \times \mathbb{R}^d) \) which is a linear (non-surjective) isometry. We call both operators \( \widetilde{T}_\varepsilon \) and \( T_\varepsilon \) stochastic unfolding operator. Note that \( \widetilde{T}_\varepsilon \) (defined on \( L^p \)) and \( T_\varepsilon^{-1} \) (given by \( T_\varepsilon^{-1}(\omega, x) = v(T_{\frac{x}{\varepsilon}} \omega, x) \) for \( v \in L^q(\Omega \times \varepsilon \mathbb{Z}^d) \)) are adjoint operators.

**Definition 3.1** (Two-scale convergence). We say that a sequence \( u_\varepsilon \in L^p(\Omega \times \varepsilon \mathbb{Z}^d) \) strongly (weakly) stochastically two-scale converges in the mean to \( U \in L^p(\Omega \times \mathbb{R}^d) \) if

\[ T_\varepsilon u_\varepsilon \to U \quad \text{strongly (weakly)} \quad \text{in} \quad L^p(\Omega \times \mathbb{R}^d), \]

and we use the notation \( u_\varepsilon \overset{\Delta}{\to} u \) \( (u_\varepsilon \overset{\Delta}{\rightharpoonup} u) \) in \( L^p(\Omega \times \mathbb{R}^d) \). For vector valued functions, the convergence is defined componentwise.

**Remark 3.1.** Note that the adaptation of the two-scale convergence in the mean from \([8, 4]\) to the discrete setting reads: \( u_\varepsilon \in L^p(\Omega \times \varepsilon \mathbb{Z}^d) \) stochastically two-scale converges in the mean to \( U \in L^p(\Omega \times \mathbb{R}^d) \) if \( \lim_{\varepsilon \to 0} \langle \int_{\varepsilon \mathbb{Z}^d} (u_\varepsilon(\omega, x))^p dm_\varepsilon(x) \rangle < \infty \) and

\[ \lim_{\varepsilon \to 0} \left\langle \int_{\varepsilon \mathbb{Z}^d} u_\varepsilon(\omega, x) \varphi(T_{\frac{\varepsilon}{\varepsilon}} \omega) \eta(x) dm_\varepsilon(x) \right\rangle = \left\langle \int_{\mathbb{R}^d} U(\omega, x) \varphi(\omega) \eta(x) dx \right\rangle \]

for all \( \varphi \in L^q(\Omega) \) and all \( \eta \in C_c^\infty(\mathbb{R}^d) \). This notion is equivalent to our notion of weak stochastic two-scale convergence.

The following lemma is obtained easily by exploiting the fact that the unfolding is a linear isometry and thanks to the usual properties of weak convergence, and therefore we omit its proof.

**Lemma 3.1** (Basic facts). We consider sequences \( u_\varepsilon \in L^p(\Omega \times \varepsilon \mathbb{Z}^d) \) and \( v_\varepsilon \in L^q(\Omega \times \varepsilon \mathbb{Z}^d) \).

(i) If \( u_\varepsilon \overset{\Delta}{\to} U \) in \( L^p(\Omega \times \mathbb{R}^d) \), then we have \( \|U\|_{L^p(\Omega \times \mathbb{R}^d)} \leq \liminf_{\varepsilon \to 0} \|u_\varepsilon\|_{L^p(\Omega \times \varepsilon \mathbb{Z}^d)} \) and \( \sup_{\varepsilon \in (0, 1)} \|u_\varepsilon\|_{L^p(\Omega \times \varepsilon \mathbb{Z}^d)} < \infty \).

(ii) If \( \limsup_{\varepsilon \to 0} \|u_\varepsilon\|_{L^p(\Omega \times \varepsilon \mathbb{Z}^d)} < \infty \), then there exist \( U \in L^p(\Omega \times \mathbb{R}^d) \) and a subsequence \( \varepsilon' \) such that \( u_{\varepsilon'} \overset{\Delta}{\to} U \) in \( L^p(\Omega \times \mathbb{R}^d) \).

(iii) \( u_\varepsilon \overset{\Delta}{\to} U \) in \( L^p(\Omega \times \mathbb{R}^d) \) if and only if \( \lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^p(\Omega \times \varepsilon \mathbb{Z}^d)} = \|U\|_{L^p(\Omega \times \mathbb{R}^d)} \) and \( u_\varepsilon \overset{\Delta}{\rightharpoonup} U \) in \( L^p(\Omega \times \mathbb{R}^d) \).

(iv) If \( u_\varepsilon \overset{\Delta}{\to} U \) in \( L^p(\Omega \times \mathbb{R}^d) \) and \( v_\varepsilon \overset{\Delta}{\to} V \) in \( L^q(\Omega \times \mathbb{R}^d) \), then

\[ \lim_{\varepsilon \to 0} \left\langle \int_{\varepsilon \mathbb{Z}^d} u_\varepsilon(\omega, x)v_\varepsilon(\omega, x) dm_\varepsilon(x) \right\rangle = \left\langle \int_{\mathbb{R}^d} U(\omega, x)V(\omega, x) dx \right\rangle. \]

As in the periodic setting, it is possible to define a suitable ”inverse” of the unfolding operator - the stochastic folding operator:

\[ \mathcal{F}_\varepsilon : L^p(\Omega \times \mathbb{R}^d) \to L^p(\Omega \times \varepsilon \mathbb{Z}^d), \quad \mathcal{F}_\varepsilon = \widetilde{T}_\varepsilon^{-1} \circ \pi_\varepsilon. \]
Lemma 3.2. \( F_\varepsilon \) is linear and it satisfies:

(i) \( \| F_\varepsilon U \|_{L^p(\Omega \times \varepsilon \mathbb{Z}^d)} \leq \| U \|_{L^p(\Omega \times \mathbb{R}^d)} \) for every \( U \in L^p(\Omega \times \mathbb{R}^d) \).

(ii) \( F_\varepsilon \circ \tau_\varepsilon = \text{Id} \) on \( L^p(\Omega \times \varepsilon \mathbb{Z}^d) \) and \( \tau_\varepsilon \circ F_\varepsilon = \pi_\varepsilon \) on \( L^p(\Omega \times \mathbb{R}^d) \).

(iii) \( F_\varepsilon U \overset{\ast}{\rightharpoonup} U \) in \( L^p(\Omega \times \mathbb{R}^d) \) for every \( U \in L^p(\Omega \times \mathbb{R}^d) \).

The proof of this lemma is omitted since it mostly relies on the definition of the folding operator.

3.2 Two-scale limits of gradients

In this section, we treat two-scale limits of gradients. First we present some compactness results and later we show that weak two-scale limits can be recovered in the strong two-scale sense by convenient linear constructions.

Proposition 3.1 (Compactness). Let \( \gamma \geq 0 \). Let \( u_\varepsilon \in L^p(\Omega \times \varepsilon \mathbb{Z}^d) \) satisfy

\[
\lim_{\varepsilon \to 0} \left\langle \int_{\varepsilon \mathbb{Z}^d} |u_\varepsilon(\omega, x)|^p + \varepsilon^\gamma |\nabla^\varepsilon u_\varepsilon(\omega, x)|^p \, dm_\varepsilon(x) \right\rangle < \infty.
\]  

(i) If \( \gamma = 0 \), there exist \( U \in L^p_{\text{inv}}(\Omega) \otimes W^{1,p}(\mathbb{R}^d) \) and \( \chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(\mathbb{R}^d) \) such that, up to a subsequence,

\[
u_\varepsilon \overset{\ast}{\rightharpoonup} U \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \varepsilon^\gamma \nabla^\varepsilon u_\varepsilon \overset{\ast}{\rightharpoonup} \nabla U + \chi \text{ in } L^p(\Omega \times \mathbb{R}^d)^d.
\]

(ii) If \( \gamma \in (0, 1) \), there exist \( U \in L^p_{\text{inv}}(\Omega) \otimes L^p(\mathbb{R}^d) \) and \( \chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(\mathbb{R}^d) \) such that, up to a subsequence,

\[
u_\varepsilon \overset{\ast}{\rightharpoonup} U \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \varepsilon^\gamma \nabla^\varepsilon u_\varepsilon \overset{\ast}{\rightharpoonup} \chi \text{ in } L^p(\Omega \times \mathbb{R}^d)^d.
\]

(iii) If \( \gamma = 1 \), there exists \( U \in L^p(\Omega \times \mathbb{R}^d) \) such that, up to a subsequence,

\[
u_\varepsilon \overset{\ast}{\rightharpoonup} U \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \varepsilon^\gamma \nabla^\varepsilon u_\varepsilon \overset{\ast}{\rightharpoonup} \chi \text{ in } L^p(\Omega \times \mathbb{R}^d)^d.
\]

(iv) If \( \gamma > 1 \), there exists \( U \in L^p(\Omega \times \mathbb{R}^d) \) such that, up to a subsequence,

\[
u_\varepsilon \overset{\ast}{\rightharpoonup} U \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \varepsilon^\gamma \nabla^\varepsilon u_\varepsilon \overset{\ast}{\rightharpoonup} 0 \text{ in } L^p(\Omega \times \mathbb{R}^d)^d.
\]

See Section 3.4 for the proof. The above statement can be adapted to sequences supported in a domain: Let \( O \subset \mathbb{R}^d \) be open. We denote by \( W^{1,p}(O) \) the closure of \( C_c^\infty(\Omega) \) in \( W^{1,p}(\Omega) \). Since the unfolding operator is naturally defined for functions on \( \mathbb{R}^d \), we tacitly identify functions in \( L^p(O) \) and \( W^{1,p}(O) \) with their trivial extension by \( 0 \) to \( \mathbb{R}^d \). As a corollary of Proposition 3.1 we obtain:

Corollary 3.1. Let \( O \subset \mathbb{R}^d \) be a bounded domain with Lipschitz boundary, and set \( O^{+\varepsilon} := \{ x \in \mathbb{R}^d : \text{dist}(x, O) \leq C\varepsilon \} \) where \( C > 0 \) denotes an arbitrary constant independent of \( \varepsilon > 0 \). Consider a sequence \( u_\varepsilon \in L^p(O^{+\varepsilon} \cap \varepsilon \mathbb{Z}^d) \) satisfying (4). Then additionally to the convergence statements in Proposition 3.1, the two-scale limits (from Proposition 3.1) satisfy:

- If \( \gamma = 0 \), \( U \in L^p_{\text{inv}}(\Omega) \otimes W^{1,p}_{0}(O) \) and \( \chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(O) \).
- If \( \gamma \in (0, 1) \), \( U \in L^p_{\text{inv}}(\Omega) \otimes L^p(O) \) and \( \chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(O) \).
- If \( \gamma \geq 1 \), \( U \in L^p(\Omega) \otimes L^p(O) \).

The proof of the above corollary is left to the reader. We remark that in Proposition 3.1 (i) and (ii) the two-scale limit \( U \) is shift invariant and therefore in the ergodic setting it is deterministic, i.e. \( U = P_{\text{inv}} U = \langle U \rangle \).
Corollary 3.2. Let \( \gamma \in [0,1) \) and \( \langle \cdot \rangle \) be ergodic. Let \( u_\varepsilon \) satisfy the assumptions in Proposition 3.1. Then the claims in Proposition 3.1 (i) and (ii) hold and we have:

(i) If \( \gamma = 0 \), then \( \langle u_\varepsilon \rangle \rightharpoonup U \) in \( L^p(\mathbb{R}^d) \), \( \langle \nabla^\varepsilon u_\varepsilon \rangle \rightharpoonup \nabla U \) in \( L^p(\mathbb{R}^d) \) and \( u_\varepsilon - \langle u_\varepsilon \rangle \overset{2}{\rightharpoonup} 0 \) in \( L^p(\Omega \times \mathbb{R}^d) \).

(ii) If \( \gamma \in (0,1) \), then \( \langle u_\varepsilon \rangle \rightharpoonup U \) in \( L^p(\mathbb{R}^d) \), \( \langle \varepsilon \nabla^\varepsilon u_\varepsilon \rangle \rightharpoonup 0 \) in \( L^p(\mathbb{R}^d) \) and \( u_\varepsilon - \langle u_\varepsilon \rangle \overset{2}{\rightharpoonup} 0 \) in \( L^p(\Omega \times \mathbb{R}^d) \).

(iii) If \( \gamma = 0 \) and \( Q \subset \mathbb{R}^d \) is open, bounded with Lipschitz boundary, then \( \overline{\langle u_\varepsilon \rangle} \rightharpoonup U \) strongly in \( L^p(Q) \).

(iv) If \( \gamma \in [0,1) \) and if additionally \( u_\varepsilon \overset{2}{\rightharpoonup} U \) in \( L^p(\Omega \times \mathbb{R}^d) \), then for any \( \varphi \in L^\infty(\Omega) \) we have \( \langle u_\varepsilon \varphi \rangle \rightharpoonup \langle \varphi \rangle U \) in \( L^p(\mathbb{R}^d) \).

For the proof of this corollary, see Section 3.4.

In the following, we show that weak two-scale accumulation points can be recovered in the strong two-scale sense.

Proposition 3.2. (i) Let \( \gamma \in [0,1) \). For \( \varepsilon > 0 \) there exists a linear and bounded operator \( G^\varepsilon_\gamma : L^p_{\text{pot}}(\Omega) \otimes L^p(\mathbb{R}^d) \to L^p(\Omega \times \varepsilon \mathbb{Z}^d) \) such that

\[
G^\varepsilon_\gamma \chi \overset{2}{\rightharpoonup} 0 \text{ in } L^p(\Omega \times \mathbb{R}^d) \quad \text{and} \quad \varepsilon \nabla^\varepsilon G^\varepsilon_\gamma \chi \overset{2}{\rightharpoonup} \chi \text{ in } L^p(\Omega \times \mathbb{R}^d)
\]

for all \( \chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(\mathbb{R}^d) \). Moreover, the operator norm of \( G^\varepsilon_\gamma \) can be bounded independently of \( 0 < \varepsilon \leq 1 \).

(ii) Let \( U \in L^p_{\text{inv}}(\Omega) \otimes W^{1,p}(\mathbb{R}^d) \), we have

\[
\nabla^\varepsilon F^\varepsilon U \overset{2}{\rightharpoonup} \nabla U \text{ in } L^p(\Omega \times \mathbb{R}^d).
\]

(iii) Let \( \gamma \in (0,1) \). For \( \varepsilon > 0 \) there exists a linear and bounded operator \( F^\varepsilon_\gamma : L^p_{\text{inv}}(\Omega) \otimes L^p(\mathbb{R}^d) \to L^p(\Omega \times \varepsilon \mathbb{Z}^d) \) such that

\[
F^\varepsilon_\gamma U \overset{2}{\rightharpoonup} U \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \varepsilon \nabla^\varepsilon F^\varepsilon_\gamma U \overset{2}{\rightharpoonup} 0 \text{ in } L^p(\Omega \times \mathbb{R}^d)
\]

for all \( U \in L^p_{\text{inv}}(\Omega) \otimes L^p(\mathbb{R}^d) \). Moreover, the operator norm of \( F^\varepsilon_\gamma \) can be bounded independently of \( 0 < \varepsilon \leq 1 \).

(iv) Let \( \gamma \geq 1 \). For any \( U \in L^p(\Omega \times \mathbb{R}^d) \), it holds

\[
\varepsilon \nabla^\varepsilon F^\varepsilon_\gamma U \overset{2}{\rightharpoonup} a_\gamma DU \text{ in } L^p(\Omega \times \mathbb{R}^d),
\]

where \( a_\gamma = \begin{cases} 1 & \text{if } \gamma = 1 \\ 0 & \text{if } \gamma > 1 \end{cases} \).

Corollary 3.3. (i) The mapping

\[
(U, \chi) \mapsto \nabla^\varepsilon u_\varepsilon(U, \chi) =: u_\varepsilon(U, \chi)
\]

is linear and bounded and it holds

\[
u_\varepsilon(U, \chi) \overset{2}{\rightharpoonup} U \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \nabla^\varepsilon u_\varepsilon(U, \chi) \overset{2}{\rightharpoonup} \nabla U + \chi \text{ in } L^p(\Omega \times \mathbb{R}^d).
\]

Moreover, its operator norm is bounded uniformly in \( 0 < \varepsilon \leq 1 \).
(ii) Let $\gamma \in (0, 1)$. The mapping

$$(L^p_{inv}(\Omega) \otimes L^p(\mathbb{R}^d)) \times (L^p_{pot}(\Omega) \otimes L^p(\mathbb{R}^d)) \rightarrow L^p(\Omega \times \varepsilon \mathbb{Z}^d)$$

$$(U, \chi) \mapsto F^\varepsilon U + G^\varepsilon \chi =: u_{\varepsilon}(U, \chi)$$

is linear and bounded and it holds

$$u_{\varepsilon}(U, \chi) \xrightarrow{2} U \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \varepsilon \gamma \nabla^\varepsilon u_{\varepsilon}(U, \chi) \xrightarrow{2} \nabla U + \chi \text{ in } L^p(\Omega \times \mathbb{R}^d)^d.$$  

Moreover, its operator norm is bounded uniformly in $0 < \varepsilon \leq 1$.

Let $O \subset \mathbb{R}^d$ be open, bounded with Lipschitz boundary.

(iii) For any $(U, \chi) \in (L^p_{inv}(\Omega) \otimes W^{1,p}_0(O)) \times (L^p_{pot}(\Omega) \otimes L^p(O))$, we can find a sequence $u_{\varepsilon} \in L^p(\Omega) \otimes L^p_0(O \cap \varepsilon \mathbb{Z}^d)$ such that

$$u_{\varepsilon} \xrightarrow{2} U \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \nabla^\varepsilon u_{\varepsilon} \xrightarrow{2} \nabla U + \chi \text{ in } L^p(\Omega \times \mathbb{R}^d)^d.$$  

(iv) Let $\gamma \in (0, 1)$. There exists a mapping

$$(L^p_{inv}(\Omega) \otimes L^p(O)) \times (L^p_{pot}(\Omega) \otimes L^p(O)) \rightarrow L^p(\Omega) \otimes L^p_0(O \cap \varepsilon \mathbb{Z}^d)$$

$$(U, \chi) \mapsto u_{\varepsilon}(U, \chi),$$

which is linear and bounded and it holds

$$u_{\varepsilon}(U, \chi) \xrightarrow{2} U \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \varepsilon \gamma \nabla^\varepsilon u_{\varepsilon}(U, \chi) \xrightarrow{2} \nabla U + \chi \text{ in } L^p(\Omega \times \mathbb{R}^d)^d.$$  

Moreover, its operator norm is bounded uniformly in $0 < \varepsilon \leq 1$.

For the proof of the above two results, see Section 3.4. We remark that in the case $\gamma \geq 1$, the recovery sequence for $U \in L^p(\Omega \times \mathbb{R}^d)$ is simply given by $F^\varepsilon U$. Moreover, in the case of prescribed boundary data for the recovery sequence, we might consider a cut-off procedure as in (iv) above.

Remark 3.2. Note that the construction of the recovery sequence in the whole-space case (i) and (ii) (and if $\gamma \in (0, 1)$ for a domain (iv)) is linear in the sense that the mapping $(U, \chi) \mapsto u_{\varepsilon}$ is linear. In contrast, the construction for domains (iii) is non-linear, since it relies on a cut-off procedure applied to the whole-space construction. We remark that the cut-off procedure can be avoided in certain cases: For $p = 2$, we can construct the recovery sequence, similarly as in the proof of Proposition 3.2 (i), by defining $u_{\varepsilon}$ as the unique solution of $\nabla^\varepsilon \cdot \nabla^\varepsilon u_{\varepsilon} = \nabla^\varepsilon (\nabla^\varepsilon F^\varepsilon U + F^\varepsilon \chi)$ in the interior of $O \cap \varepsilon \mathbb{Z}^d$ and with prescribed homogeneous Dirichlet boundary data. For $p \neq 2$ the same strategy applies as long as the above discrete elliptic equation satisfies maximal $L^p$-regularity. The latter depends on the regularity of the domain $O$.

3.3 Unfolding and (lower semi-)continuity of convex energies

Let $V : \Omega \times \mathbb{R}^k \rightarrow \mathbb{R}$ be jointly measurable (i.e. w.r.t. $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^k)$) and for $P$-a.e. $\omega \in \Omega$, $V(\omega, \cdot)$ convex. Moreover, we assume that there exists $C > 0$ such that

$$\frac{1}{C}|F|^p - C \leq V(\omega, F) \leq C(|F|^p + 1),$$

for $P$-a.e. $\omega \in \Omega$ and all $F \in \mathbb{R}^k$. Our $\Gamma$-convergence results for convex energies exploit the following result.

Proposition 3.3. Let $V$ be as above. Let $O, O^{+\varepsilon} \subset \mathbb{R}^d$ be bounded domains with Lipschitz boundaries which satisfy $O \subset O^{+\varepsilon} \subset \{x \in \mathbb{R}^d : \text{dist}(x, O) \leq C\varepsilon\}$ for some $C > 0$.  


(i) If \( u_\varepsilon \in L^p(\Omega \times \varepsilon \mathbb{Z}^d)^k \) and \( u_\varepsilon \overset{2}{\rightharpoonup} U \) in \( L^p(\Omega \times \mathbb{R}^d)^k \), then
\[
\liminf_{\varepsilon \to 0} \left\langle \int_{O^+ \cap \varepsilon \mathbb{Z}^d} V(T_\varepsilon \omega, u_\varepsilon(\omega, x)) dm_\varepsilon(x) \right\rangle \geq \left\langle \int_{O} V(\omega, U(\omega, x)) dx \right\rangle.
\]

(ii) If \( u_\varepsilon \in L^p(\Omega \times \varepsilon \mathbb{Z}^d)^k \) and \( u_\varepsilon \overset{2}{\rightharpoonup} U \) in \( L^p(\Omega \times \mathbb{R}^d)^k \), then
\[
\lim_{\varepsilon \to 0} \left\langle \int_{O^+ \cap \varepsilon \mathbb{Z}^d} V(T_\varepsilon \omega, u_\varepsilon(\omega, x)) dm_\varepsilon(x) \right\rangle = \left\langle \int_{O} V(\omega, U(\omega, x)) dx \right\rangle.
\]

The proof of this result is in Section 3.4. We have applications in mind, where such integral functionals are treated and the role of \( u_\varepsilon \) is played by a discrete (symmetrized) gradient (see Section 4).

3.4 Proofs

**Proof of Proposition 3.1.** Before proving the claim, we present a couple of auxiliary lemmas. The following commutator identity for \( u_\varepsilon : \Omega \times \varepsilon \mathbb{Z}^d \to \mathbb{R} \), obtained by direct computation, is practical:
\[
\tilde{T}_\varepsilon \nabla^\varepsilon u_\varepsilon - \nabla^\varepsilon \tilde{T}_\varepsilon u_\varepsilon = \frac{1}{\varepsilon} D^2_\varepsilon u_\varepsilon + (D_1^1 \nabla^\varepsilon_{1}, ..., D_d^d \nabla^\varepsilon_{d}) \tilde{T}_\varepsilon u_\varepsilon. \tag{5}
\]

**Lemma 3.3.** Consider a sequence \( u_\varepsilon \in L^p(\Omega \times \varepsilon \mathbb{Z}^d) \). Suppose that \( u_\varepsilon \overset{2}{\rightharpoonup} U \) in \( L^p(\Omega \times \mathbb{R}^d) \) and \( \varepsilon \nabla^\varepsilon u_\varepsilon \overset{2}{\rightharpoonup} 0 \) in \( L^p(\Omega \times \mathbb{R}^d)^d \). Then \( U \in L^p_{inv}(\Omega) \otimes L^p(\mathbb{R}^d) \).

**Proof.** Since \( L^p_{inv}(\Omega) = (\text{ran} D^\varepsilon)^\perp \), it suffices to show that
\[
\left\langle \int_{\mathbb{R}^d} U(\omega, x) D^\varepsilon_\varphi(\omega) \eta(x) dx \right\rangle = 0 \quad \tag{6}
\]
for any \( \varphi \in L^q(\Omega), \eta \in C^\infty_c(\mathbb{R}^d) \) and \( i \in \{1, ..., d\} \).

We consider the sequence \( v_\varepsilon = F_\varepsilon(\varphi\eta) \in L^q(\Omega \times \varepsilon \mathbb{Z}^d) \) and by Lemma 3.2 (iii), we have \( v_\varepsilon \overset{2}{\to} \varphi\eta \) in \( L^q(\Omega \times \mathbb{R}^d) \). Therefore, using Lemma 3.1 (iv), we obtain
\[
\varepsilon \left\langle \int_{\varepsilon \mathbb{Z}^d} u_\varepsilon(\omega, x) \nabla^\varepsilon_{i} v_\varepsilon(\omega, x) dm_\varepsilon(x) \right\rangle = \left\langle \int_{\varepsilon \mathbb{Z}^d} (\varepsilon \nabla^\varepsilon u_\varepsilon(\omega, x)) v_\varepsilon(\omega, x) dm_\varepsilon(x) \right\rangle \to 0 \quad \text{as} \ \varepsilon \to 0. \tag{7}
\]

Moreover, using the definition of \( F_\varepsilon \),
\[
\varepsilon \nabla^\varepsilon_{i} v_\varepsilon(\omega, x) = \varphi(T_{\varepsilon}^\varphi \omega) \pi_{\varepsilon} \eta(x - \varepsilon e_i) - \varphi(T_{\varepsilon}^\varphi \omega) \pi_{\varepsilon} \eta(x) + \varepsilon \varphi(T_{\varepsilon}^\varphi \omega) \nabla^\varepsilon_{i} \pi_{\varepsilon} \eta(x) + D^\varepsilon_\varphi(T_{\varepsilon}^\varphi \omega) \pi_{\varepsilon} \eta(x), \tag{8}
\]

which implies \( \varepsilon \nabla^\varepsilon_{i} v_\varepsilon \overset{2}{\rightharpoonup} D^\varepsilon_\varphi \eta \) in \( L^q(\Omega \times \mathbb{R}^d) \). Indeed, the first term on the right-hand side of (8) vanishes in the strong two-scale limit since \( \eta \) is compactly supported and smooth. The second term strongly two-scale converges to \( D^\varepsilon_\varphi \eta \). This and Lemma 3.1 (iv) imply
\[
\lim_{\varepsilon \to 0} \varepsilon \left\langle \int_{\varepsilon \mathbb{Z}^d} u_\varepsilon(\omega, x) \nabla^\varepsilon_{i} v_\varepsilon(\omega, x) dm_\varepsilon(x) \right\rangle = \left\langle \int_{\mathbb{R}^d} U(\omega, x) D^\varepsilon_\varphi(\omega) \eta(x) dx \right\rangle,
\]
which, together with (7), yields (6).
Lemma 3.4. Let $u_\varepsilon \in L^p(\Omega \times \varepsilon \mathbb{Z}^d)$ satisfy

$$\limsup_{\varepsilon \to 0} \left( \int_{\varepsilon \mathbb{Z}^d} |u_\varepsilon(\omega, x)|^p + |\nabla^\varepsilon u_\varepsilon(\omega, x)|^p dm_\varepsilon(x) \right) < \infty.$$ 

Then there exists $U \in L^p_{\text{inv}}(\Omega) \otimes W^{1,p}(\mathbb{R}^d)$ such that, up to a subsequence,

$$u_\varepsilon \xrightarrow{\tau} U, \quad P_{\text{inv}} u_\varepsilon \xrightarrow{\tau} U \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \nabla^\varepsilon P_{\text{inv}} u_\varepsilon \xrightarrow{\tau} \nabla U \text{ in } L^p(\Omega \times \mathbb{R}^d)^d.$$

Proof. Step 1. We claim that $\overline{T}_\varepsilon \circ P_{\text{inv}} = P_{\text{inv}} \circ \overline{T}_\varepsilon = P_{\text{inv}}$. By shift invariance, we have $\overline{T}_\varepsilon \circ P_{\text{inv}} = P_{\text{inv}}$. Hence, it suffices to prove $P_{\text{inv}} \circ \overline{T}_\varepsilon = P_{\text{inv}}$. Let $\eta \in L^q(\varepsilon \mathbb{Z}^d)$, $\varphi \in L^q(\Omega)$ and $u_\varepsilon \in L^p(\Omega \times \varepsilon \mathbb{Z}^d)$. We have

$$\left\langle \int_{\varepsilon \mathbb{Z}^d} P_{\text{inv}} \overline{T}_\varepsilon v_\varepsilon(\omega, x) \varphi(\omega) \eta(x) dm_\varepsilon(x) \right\rangle = \left\langle \int_{\varepsilon \mathbb{Z}^d} v_\varepsilon(T_{-\frac{\varepsilon}{2}} \omega, x) P_{\text{inv}}^* \varphi(\omega) \eta(x) dm_\varepsilon(x) \right\rangle$$

$$= \left\langle \int_{\varepsilon \mathbb{Z}^d} v_\varepsilon(\omega, x) P_{\text{inv}}^* \varphi(\omega) \eta(x) dm_\varepsilon(x) \right\rangle$$

$$= \left\langle \int_{\varepsilon \mathbb{Z}^d} P_{\text{inv}} v_\varepsilon(\omega, x) \varphi(\omega) \eta(x) dm_\varepsilon(x) \right\rangle,$$

where we used that $P_{\text{inv}}^* \varphi$ is a shift invariant function. Consequently, by a density argument it follows that $P_{\text{inv}} \circ \overline{T}_\varepsilon = P_{\text{inv}}$.

Step 2. Convergence of $P_{\text{inv}} u_\varepsilon$. Using boundedness of $P_{\text{inv}}$ and the fact that $\nabla^\varepsilon$ and $P_{\text{inv}}$ commute, we obtain

$$\limsup_{\varepsilon \to 0} \left( \int_{\varepsilon \mathbb{Z}^d} |P_{\text{inv}} u_\varepsilon(\omega, x)|^p + |\nabla^\varepsilon P_{\text{inv}} u_\varepsilon(\omega, x)|^p dm_\varepsilon(x) \right) < \infty.$$ 

Applying Lemma 3.1 (ii) and Lemma 3.3, there exists $V \in L^p_{\text{inv}}(\Omega) \otimes L^p(\mathbb{R}^d)$ and $\tilde{V} \in L^p(\Omega \times \mathbb{R}^d)^d$ such that

$$P_{\text{inv}} u_\varepsilon \xrightarrow{\tau} V \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \nabla^\varepsilon P_{\text{inv}} u_\varepsilon \xrightarrow{\tau} \tilde{V} \text{ in } L^p(\Omega \times \mathbb{R}^d)^d,$$

for a (not relabeled) subsequence. Note that, additionally, we have $\tilde{V} \in L^p_{\text{inv}}(\Omega) \otimes L^p(\mathbb{R}^d)^d$.

Let $\varphi \in L^q(\Omega)$ and $\eta \in C_\infty^\varepsilon(\mathbb{R}^d)$ and denote $v_\varepsilon = \overline{T}_\varepsilon(\varphi \eta)$. Since $v_\varepsilon \xrightarrow{\tau} \eta \varphi$ (Lemma 3.2 (iii)), for $i = 1, \ldots, d$, we have

$$\left\langle \int_{\varepsilon \mathbb{Z}^d} \nabla_i^\varepsilon P_{\text{inv}} u_\varepsilon(\omega, x) v_\varepsilon(\omega, x) dm_\varepsilon(x) \right\rangle \to \left\langle \int_{\mathbb{R}^d} \tilde{V}_i(\omega, x) \varphi(\omega) \eta(x) dx \right\rangle \text{ as } \varepsilon \to 0.$$

On the other hand, it holds

$$\left\langle \int_{\varepsilon \mathbb{Z}^d} \nabla_i^\varepsilon P_{\text{inv}} u_\varepsilon(\omega, x) v_\varepsilon(\omega, x) dm_\varepsilon(x) \right\rangle = \left\langle \int_{\varepsilon \mathbb{Z}^d} P_{\text{inv}} u_\varepsilon(\omega, x) \varphi(\omega) \nabla_i^\varepsilon \varphi(\omega) \eta(x) dm_\varepsilon(x) \right\rangle$$

$$\to - \left\langle \int_{\mathbb{R}^d} V(\omega, x) \varphi(\omega) \partial_i \eta(x) dx \right\rangle \text{ as } \varepsilon \to 0.$$

Using this, we conclude that $V \in L^p_{\text{inv}}(\Omega) \otimes W^{1,p}(\mathbb{R}^d)$ and $\nabla V = \tilde{V}$.

Step 3. We show that $u_\varepsilon \xrightarrow{\tau} V$ in $L^p(\Omega \times \mathbb{R}^d)$ (up to another subsequence). Using Lemma 3.1 (ii) and Lemma 3.3, we conclude that there exist another subsequence (not relabeled) and $U \in L^p_{\text{inv}}(\Omega) \otimes L^p(\mathbb{R}^d)$ such that $u_\varepsilon \xrightarrow{\tau} U$ in $L^p(\Omega \times \mathbb{R}^d)$. Since $P_{\text{inv}}$ is a linear and bounded operator, it follows that $P_{\text{inv}}(\overline{T}_\varepsilon u_\varepsilon) \to P_{\text{inv}} U$ in $L^p(\Omega \times \mathbb{R}^d)$, and $P_{\text{inv}} U = U$ by shift invariance of $U$. Furthermore, by Step 1 and Step 2 we have that $P_{\text{inv}} \overline{T}_\varepsilon u_\varepsilon = \overline{T}_\varepsilon P_{\text{inv}} u_\varepsilon \to V$ and therefore $U = V$. This completes the proof. \hfill \Box
Proof of Proposition 3.1. (i) By Lemma 3.4 we deduce that there exists \( U \in L^p_{\text{inv}}(\Omega) \otimes W^{1,p}(\mathbb{R}^d) \) such that \( u_\varepsilon \overset{\ast}{\rightarrow} U \) and by boundedness of \( \nabla^\varepsilon u_\varepsilon \) (Lemma 3.1 (ii)) there exists \( V \in L^p(\Omega \times \mathbb{R}^d)^d \) such that \( \nabla^\varepsilon u_\varepsilon \overset{\ast}{\rightarrow} V \) (up to a subsequence). In order to prove the claim, it suffices to show that

\[
\int_{\mathbb{R}^d} V(\omega, x) \cdot \eta(x) \varphi(\omega) dx = \int_{\mathbb{R}^d} \nabla U(\omega, x) \cdot \eta(x) \varphi(\omega) dx
\]

for any \( \varphi \in L^p(\Omega)^d \) with \( D^* \varphi = 0 \) and \( \eta \in C_0^\infty(\mathbb{R}^d) \). Indeed, this implies that \( \chi := V - \nabla U \in L^p_{\text{pot}}(\Omega) \otimes L^p(\mathbb{R}^d) \) and thus the claim of the proposition.

For the argument consider \( v_\varepsilon = \mathcal{F}_\varepsilon(\eta \varphi) \), the folding acting componentwise. Since \( v_\varepsilon \overset{\ast}{\rightarrow} \eta \varphi \) (Lemma 3.2 (iii)), as \( \varepsilon \to 0 \)

\[
\left\langle \int_{\mathbb{R}^d} \nabla^\varepsilon u_\varepsilon(\omega, x) \cdot v_\varepsilon(\omega, x) dm_\varepsilon(x) \right\rangle \to \left\langle \int_{\mathbb{R}^d} V(\omega, x) \cdot \eta(x) \varphi(\omega) dx \right\rangle.
\]

On the other hand, the commutator identity (5) and the definition of \( \mathcal{F}_\varepsilon \) yield

\[
\left\langle \int_{\mathbb{R}^d} \nabla^\varepsilon u_\varepsilon(\omega, x) \cdot v_\varepsilon(\omega, x) dm_\varepsilon(x) \right\rangle
= \left\langle \int_{\mathbb{R}^d} \left( \nabla^\varepsilon \mathcal{T}_\varepsilon u_\varepsilon(\omega, x) + \frac{1}{\varepsilon} D\mathcal{T}_\varepsilon u_\varepsilon(\omega, x) + (D_1\nabla^\varepsilon_1, \ldots, D_d\nabla^\varepsilon_d)\mathcal{T}_\varepsilon u_\varepsilon(\omega, x) \right) \cdot \pi_\varepsilon \eta(x) \varphi(\omega) dm_\varepsilon(x) \right\rangle.
\]

Since \( D^* \varphi = 0 \), the contribution from the second term on the right-hand side above vanishes. After a discrete integration by parts, the right-hand side reduces to

\[
\sum_{i=1}^d \left\langle \int_{\mathbb{R}^d} \left( \mathcal{T}_\varepsilon u_\varepsilon(\omega, x) + D_i \mathcal{T}_\varepsilon u_\varepsilon(\omega, x) \right) \nabla^\varepsilon_i \pi_\varepsilon \eta(x) \varphi_i(\omega) dm_\varepsilon(x) \right\rangle
- \sum_{i=1}^d \left\langle \int_{\mathbb{R}^d} \left( U(\omega, x) + D_i U(\omega, x) \right) \partial_i \eta(x) \varphi_i(\omega) dx \right\rangle \text{ as } \varepsilon \to 0,
\]

which is concluded by using that \( u_\varepsilon \overset{\ast}{\rightarrow} U \) and that \( \eta \) is smooth and compactly supported. Since \( U \) is shift invariant, the second term on the right-hand side vanishes. After an integration by parts, we are able to infer (9) and conclude the proof of part (i).

(ii) By Lemma 3.1 (ii), there exists \( U \in L^p(\Omega \times \mathbb{R}^d) \) such that \( u_\varepsilon \overset{\ast}{\rightarrow} U \) (up to a subsequence). Since \( \gamma \in (0, 1) \), \( u_\varepsilon \) satisfies the assumptions in Lemma 3.3 and therefore \( U \in L^p_{\text{inv}}(\Omega) \otimes L^p(\mathbb{R}^d) \).

With help of (i), we obtain that for the sequence \( v_\varepsilon := \varepsilon^\gamma u_\varepsilon \), there exist \( V \in L^p_{\text{inv}}(\Omega) \otimes W^{1,p}(\mathbb{R}^d) \) and \( \chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(\mathbb{R}^d) \) such that (up to another subsequence)

\[
v_\varepsilon \overset{\ast}{\rightarrow} V \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \nabla^\varepsilon u_\varepsilon \overset{\ast}{\rightarrow} \nabla V + \chi \text{ in } L^p(\Omega \times \mathbb{R}^d)^d.
\]

However, using that \( u_\varepsilon \overset{\ast}{\rightarrow} U \), we conclude that \( V = 0 \) and the claim is proved.

(iii) Lemma 3.1 (ii) implies that there exist \( U \in L^p(\Omega \times \mathbb{R}^d) \) and \( V \in L^p(\Omega \times \mathbb{R}^d)^d \) such that (up to a subsequence)

\[
u_\varepsilon \overset{\ast}{\rightarrow} U \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \varepsilon \nabla^\varepsilon u_\varepsilon \overset{\ast}{\rightarrow} V \text{ in } L^p(\Omega \times \mathbb{R}^d)^d.
\]

Following the same strategy as in Lemma 3.3 it can be obtained that \( V = DU \).
(iv) Lemma 3.1 (ii) implies that there exist \( U \in L^p(\Omega \times \mathbb{R}^d) \) such that \( u_\varepsilon \rightharpoonup U \) in \( L^p(\Omega \times \mathbb{R}^d) \) (up to a subsequence). Also, using part (iii), for the sequence \( v_\varepsilon := \varepsilon \gamma^{-1} u_\varepsilon \), there exists \( V \in L^p(\Omega \times \mathbb{R}^d) \) such that (up to another subsequence)

\[
v_\varepsilon \rightharpoonup V, \quad \varepsilon \nabla v_\varepsilon \rightharpoonup DV.
\]

The fact that \( u_\varepsilon \rightharpoonup U \) implies that \( V = 0 \) and the proof is complete. \( \square \)

**Proof of Corollary 3.2.**

Proof. (i) The claim follows directly from Lemmas 3.4 and 2.1.

(ii) Exploiting linearity and boundedness of \( \pi \) and Step 1 in the proof of Lemma 3.4, we obtain that

\[
\langle u_\varepsilon \rangle = \pi T_{\varepsilon} u_\varepsilon \rightharpoonup \pi U = U, \quad \langle \varepsilon \nabla \varepsilon u_\varepsilon \rangle = \pi \varepsilon \nabla \varepsilon u_\varepsilon \rightharpoonup \pi \varepsilon \nabla \varepsilon \chi = \langle \chi \rangle.
\]

The above, Lemma 2.1 and the fact that \( \langle \chi \rangle = 0 \) allow us to conclude the proof.

(iii) Lemma 3.4 implies that \( \langle u_\varepsilon \rangle \rightarrow u \) and \( \varepsilon \nabla \langle u_\varepsilon \rangle \rightarrow \nabla u \) weakly in \( L^p(\mathbb{R}^d) \). Lemma 2.1 implies that \( \langle u_\varepsilon \rangle \rightarrow u \) weakly in \( L^p(\mathbb{R}^d) \). Furthermore, for any \( \eta \in L^q(\mathbb{R}^d) \) it holds

\[
\int_{\mathbb{R}^d} \left( \nabla \langle u_\varepsilon \rangle(x) - \varepsilon \nabla \langle u_\varepsilon \rangle(x) \right) \eta(x) dx \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.
\]

As a result of this, \( \langle u_\varepsilon \rangle \rightarrow u \) weakly in \( W^{1,p}(\mathbb{R}^d) \). Rellich’s embedding theorem implies that \( \langle u_\varepsilon \rangle \rightarrow U \) strongly in \( L^p(Q) \). Using Lemma 2.1 we conclude that \( \langle u_\varepsilon \rangle \rightarrow U \) strongly in \( L^p(Q) \).

(iv) We have by Jensen’s inequality and by boundedness of \( \varphi \):

\[
\int_{\mathbb{R}^d} \left( | \langle \varphi \rangle(x) \varphi - \langle \varphi \rangle \rangle(x) \right)^p dx \leq C \left( \int_{\mathbb{R}^d} | \varphi |^p dx \right)^\frac{1}{p}.
\]

The right-hand side of the above inequality equals \( \int_{\mathbb{R}^d} | \varphi \rangle(x) \varphi - \langle \varphi \rangle \rangle(x) dx \) and therefore it vanishes as \( \varepsilon \rightarrow 0 \). \( \square \)

**Proof of Proposition 3.2.** Before presenting the proof, we provide an auxiliary lemma providing a nonlinear approximation for \( \chi \) in the case \( \gamma = 0 \).

**Lemma 3.5** (Nonlinear approximation). For \( \chi \in L^p_\text{pot}(\Omega) \otimes L^p(\mathbb{R}^d) \) and \( \delta > 0 \), there exists a sequence \( g_{\delta,\varepsilon} \in L^p(\Omega \times \varepsilon \mathbb{Z}^d) \) such that

\[
\| g_{\delta,\varepsilon} \|_{L^p(\Omega \times \varepsilon \mathbb{Z}^d)} \leq \varepsilon C(\delta), \quad \lim_{\varepsilon \rightarrow 0} \| T_{\varepsilon} \nabla \varepsilon g_{\delta,\varepsilon} - \chi \|_{L^p(\Omega \times \mathbb{R}^d)} \leq \delta.
\]

Proof. Let \( \chi \in L^p_\text{pot}(\Omega) \otimes L^p(\mathbb{R}^d) \) and \( \delta > 0 \) be fixed. By density, there exists \( V = \sum_{j=1}^{n} \varphi_j \eta_j \) with \( \varphi_j \in L^p(\Omega) \), \( \eta_j \in C^\infty(\mathbb{R}^d) \) and

\[
\| DV - \chi \|_{L^p(\Omega \times \mathbb{R}^d)} \leq \delta.
\]

We define \( g_\varepsilon := \varepsilon F_{\varepsilon} V \) and remark that \( \| g_\varepsilon \|_{L^p(\Omega \times \varepsilon \mathbb{Z}^d)} \leq \varepsilon \| V \|_{L^p(\Omega \times \mathbb{R}^d)} \), which follows from the boundedness of \( F_{\varepsilon} \). This proves the first part.

Note that \( \nabla g_\varepsilon(x) \varphi(x) = D\pi V(T_{\varepsilon} \varphi(x)) + \varepsilon \nabla \pi V(T_{\varepsilon} \varphi(x)) \) and therefore it holds \( T_{\varepsilon} \nabla \varepsilon g_\varepsilon(x) = \pi_{\varepsilon} DV(x) + \varepsilon \nabla \pi_{\varepsilon} V(x) \). Hence

\[
\| T_{\varepsilon} \nabla \varepsilon g_\varepsilon - \chi \|_{L^p(\Omega \times \mathbb{R}^d)} \leq \| \pi_{\varepsilon} DV - DV \|_{L^p(\Omega \times \mathbb{R}^d)} + \| DV - \chi \|_{L^p(\Omega \times \mathbb{R}^d)} + \varepsilon \| \nabla \pi_{\varepsilon} V \|_{L^p(\Omega \times \varepsilon \mathbb{Z}^d)}.
\]
The first and the last term on the right-hand side above vanish as \( \varepsilon \to 0 \) and therefore the claim follows (since we can choose \( \delta \) arbitrarily small). Indeed, for the first term it is sufficient to note that \( DV \) is smooth and has compact support w.r.t. its \( x \)-variable. Also, the last term vanishes thanks to boundedness of \( \pi_{\varepsilon} \) and the boundedness of difference quotients by gradients; specifically (for \( i = 1, \ldots, d \))

\[
\varepsilon^{p} \| \nabla^{i}_{\varepsilon} \pi_{\varepsilon} V \|_{L^{p}(\Omega \times \varepsilon \mathbb{Z}^{d})}^{p} \leq C \varepsilon^{p} \left( \int_{\mathbb{R}^{d}} \left| \frac{V(\omega, x + \varepsilon e_{i}) - V(\omega, x)}{\varepsilon} \right|^{p} \, dx \right) \leq C \varepsilon^{p} \| \nabla V \|_{L^{p}(\Omega \times \mathbb{R}^{d})}^{p}.
\]

\[\square\]

**Proof of Proposition 3.2.** In the following proof we appeal to maximal \( L^{p} \)-regularity for the equation

\[
\lambda u + \nabla^{\varepsilon} \cdot \nabla^{\varepsilon} u = \nabla^{\varepsilon} F + g \quad \text{in} \quad \varepsilon \mathbb{Z}^{d}, \quad \text{(for some} \quad F \in L^{p}(\varepsilon \mathbb{Z}^{d}), \quad g \in L^{p}(\varepsilon \mathbb{Z}^{d})\text{)}
\]

in form of

\[
\lambda^{\frac{1}{2}} \| u \|_{L^{p}(\varepsilon \mathbb{Z}^{d})} + \| \nabla^{\varepsilon} u \|_{L^{p}(\varepsilon \mathbb{Z}^{d})}^{d} \leq C(d, p) \left( \| F \|_{L^{p}(\varepsilon \mathbb{Z}^{d})} + \lambda^{-\frac{1}{2}} \| g \|_{L^{p}(\varepsilon \mathbb{Z}^{d})} \right),
\]

which is uniform in \( \varepsilon \). For \( p = 2 \) this is a standard a priori estimate. For \( 1 < p < \infty \), in the continuum setting this is a classical result (see e.g. [25, Chapter 4, Sec. 4, Theorem 2]), and follows from the Calderón-Zygmund estimate \( \| \partial_{j} u \|_{L^{p}([\mathbb{R}^{d}])} \leq C(d, p) \| \Delta u \|_{L^{p}([\mathbb{R}^{d}])} \). The estimate above follows by the same argument from the Calderón-Zygmund estimate for the discrete laplacian on \( \varepsilon \mathbb{Z}^{d} \), for the latter see e.g. [14, 7].

(i) Let \( 2\gamma < \alpha < 2 \). For a given \( \chi \in L^{p}_{\text{loc}}(\Omega) \otimes L^{p}(\mathbb{R}^{d}) \) we define \( G_{\varepsilon}^{\gamma} \chi := u_{\varepsilon} \) as the unique solution to the following equation in \( L^{p}(\Omega \times \varepsilon \mathbb{Z}^{d}) \) (for \( P \)-a.e. \( \omega \in \Omega \))

\[
\varepsilon^{-\alpha} u_{\varepsilon} + \nabla^{\varepsilon} \cdot \nabla^{\varepsilon} u_{\varepsilon} = \nabla^{\varepsilon} \cdot \varepsilon^{-\gamma} F_{\varepsilon} \chi \quad \text{in} \quad \varepsilon \mathbb{Z}^{d}.
\]

The discrete maximal \( L^{p} \)-regularity theory implies that

\[
\varepsilon^{-\frac{\alpha}{2}} \| u_{\varepsilon} \|_{L^{p}(\Omega \times \varepsilon \mathbb{Z}^{d})} + \| \nabla^{\varepsilon} u_{\varepsilon} \|_{L^{p}(\Omega \times \varepsilon \mathbb{Z}^{d})}^{d} \leq \varepsilon^{-\gamma} C \| F_{\varepsilon} \chi \|_{L^{p}(\Omega \times \varepsilon \mathbb{Z}^{d})}.
\]

As a result of this, we have \( \| u_{\varepsilon} \|_{L^{p}(\Omega \times \varepsilon \mathbb{Z}^{d})} \leq \varepsilon^{-\frac{\gamma}{2}} C \| \chi \|_{L^{p}(\Omega \times \varepsilon \mathbb{Z}^{d})} \) and therefore \( u_{\varepsilon} \xrightarrow[\varepsilon \to 0]{} 0 \) as \( \varepsilon \to 0 \).

We consider the sequence \( g_{\delta, \varepsilon} \) from Lemma 3.5 corresponding to \( \chi \). Note that \( w_{\delta, \varepsilon} := u_{\varepsilon} - \varepsilon^{-\gamma} g_{\delta, \varepsilon} \) is the unique solution in \( L^{p}(\Omega \times \varepsilon \mathbb{Z}^{d}) \) (for \( P \)-a.e. \( \omega \in \Omega \))

\[
\varepsilon^{-\alpha} w_{\delta, \varepsilon} + \nabla^{\varepsilon} \cdot \nabla^{\varepsilon} w_{\delta, \varepsilon} = \nabla^{\varepsilon} \cdot \varepsilon^{-\gamma} (F_{\varepsilon} \chi - \nabla^{\varepsilon} g_{\delta, \varepsilon}) - \varepsilon^{-\alpha-\gamma} g_{\delta, \varepsilon} \quad \text{in} \quad \varepsilon \mathbb{Z}^{d}.
\]

We employ again the discrete maximal \( L^{p} \)-regularity theory to obtain

\[
\| \nabla^{\varepsilon} w_{\delta, \varepsilon} \|_{L^{p}(\Omega \times \varepsilon \mathbb{Z}^{d})} \leq C \left( \varepsilon^{-\gamma} \| F_{\varepsilon} \chi - \nabla^{\varepsilon} g_{\delta, \varepsilon} \|_{L^{p}(\Omega \times \varepsilon \mathbb{Z}^{d})} + \varepsilon^{-\frac{\alpha}{2} - \gamma} \| g_{\delta, \varepsilon} \|_{L^{p}(\Omega \times \varepsilon \mathbb{Z}^{d})} \right).
\]

Multiplication of the above inequality by \( \varepsilon^{\gamma} \) yields

\[
\| \varepsilon^{\gamma} \nabla^{\varepsilon} u_{\varepsilon} - \nabla^{\varepsilon} g_{\delta, \varepsilon} \|_{L^{p}(\Omega \times \varepsilon \mathbb{Z}^{d})} \leq C \left( \| F_{\varepsilon} \chi - \nabla^{\varepsilon} g_{\delta, \varepsilon} \|_{L^{p}(\Omega \times \varepsilon \mathbb{Z}^{d})} + \varepsilon^{-\frac{\alpha}{2}} \| g_{\delta, \varepsilon} \|_{L^{p}(\Omega \times \varepsilon \mathbb{Z}^{d})} \right).
\]

As a result of this and with help of the isometry property of \( T_{\varepsilon} \), we obtain

\[
\| T_{\varepsilon} \varepsilon^{\gamma} \nabla^{\varepsilon} u_{\varepsilon} - \chi \|_{L^{p}(\Omega \times \varepsilon \mathbb{Z}^{d})} \leq C \left( \| F_{\varepsilon} \chi - \nabla^{\varepsilon} g_{\delta, \varepsilon} \|_{L^{p}(\Omega \times \varepsilon \mathbb{Z}^{d})} + \varepsilon^{-\frac{\alpha}{2}} \| g_{\delta, \varepsilon} \|_{L^{p}(\Omega \times \varepsilon \mathbb{Z}^{d})} \right).
\]

Letting first \( \varepsilon \to 0 \) and then \( \delta \to 0 \) the right-hand side of the above inequality vanishes using Lemma 3.5. This completes the proof of (i).
(ii) We consider a sequence \( U_\delta = \sum_{i=1}^{n(\delta)} \varphi_i^\delta \eta_i^\delta \) such that \( \varphi_i^\delta \in L^p_{sc}(\Omega) \), \( \eta_i^\delta \in C_c^{\infty}(\mathbb{R}^d) \), and 
\[
\|U_\delta - U\|_{L^p(\Omega \times \mathbb{R}^d)} + \|
abla U_\delta - \nabla U\|_{L^p(\Omega \times \mathbb{R}^d)^d} \to 0 \text{ as } \delta \to 0.
\]

Using the triangle inequality, it follows that 
\[
\|T_\epsilon \nabla^\epsilon F_\epsilon U - \nabla U\|_{L^p(\Omega \times \mathbb{R}^d)^d} \leq \|T_\epsilon \nabla^\epsilon F_\epsilon U - T_\epsilon \nabla^\epsilon F_\epsilon U_\delta\|_{L^p(\Omega \times \mathbb{R}^d)^d} + \|T_\epsilon \nabla^\epsilon F_\epsilon U_\delta - \nabla U_\delta\|_{L^p(\Omega \times \mathbb{R}^d)^d} + \|
abla U_\delta - \nabla U\|_{L^p(\Omega \times \mathbb{R}^d)^d}.
\]

(10)

First, we treat the first term on the right-hand side. For \( i = 1, ..., d \), by the isometry property of \( T_\epsilon \) and contraction property of \( F_\epsilon \),
\[
\|T_\epsilon \nabla^\epsilon F_\epsilon U - T_\epsilon \nabla^\epsilon F_\epsilon U_\delta\|_{L^p(\Omega \times \mathbb{R}^d)^d}^p \leq \left( \int_{\mathbb{R}^d} \frac{|U(\omega, x + \epsilon e_i) - U_\delta(\omega, x + \epsilon e_i) - U(\omega, x) + U_\delta(\omega, x)|^p}{\epsilon} dx \right) \leq C \left( \int_{\mathbb{R}^d} |\nabla U(\omega, x) - \nabla U_\delta(\omega, x)|^p dx \right).
\]

(11)

In the last inequality we use the fact that difference quotients are bounded by gradients. Second, we compute \((i = 1, ..., d)\)
\[
T_\epsilon \nabla^\epsilon F_\epsilon U_\delta(\omega, x) = \frac{1}{\epsilon} \left( \pi_\epsilon U_\delta(T_\epsilon, \omega, x + \epsilon e_i) - \pi_\epsilon U_\delta(T_\epsilon, \omega, x) \right) + \frac{1}{\epsilon} \left( \pi_\epsilon U_\delta(T_\epsilon, \omega, x) - \pi_\epsilon U_\delta(\omega, x) \right).
\]

(12)

The second part of the right-hand side of the above equality vanishes (for \( P\text{-a.e. } \omega \in \Omega \)) by shift invariance of \( U_\delta \). Further, we have
\[
\|T_\epsilon \nabla^\epsilon F_\epsilon U_\delta - \partial_i U_\delta\|_{L^p(\Omega \times \mathbb{R}^d)} \leq \left( \int_{\mathbb{R}^d} \frac{|\pi_\epsilon U_\delta(\omega, x + \epsilon e_i) - \pi_\epsilon U_\delta(\omega, x)|}{\epsilon} dx \right)^{\frac{1}{p}}.
\]

(13)

For any \( \delta > 0 \) the last expression converges to 0 as \( \epsilon \to 0 \) since \( U_\delta \) is smooth in its \( x \)-variable. Finally, in (10) we first let \( \epsilon \to 0 \) and then \( \delta \to 0 \) to conclude the proof.

(iii) Let \( 0 < \alpha < 2\gamma \). For a given \( U \in L^p_{sc}(\Omega) \otimes L^p(\mathbb{R}^d) \) we define \( F_\epsilon U := u_\epsilon \) as the unique solution to the following equation in \( L^p(\Omega \times \epsilon \mathbb{Z}^d) \) (for \( P\text{-a.e. } \omega \in \Omega \))
\[
\epsilon^{-\alpha} u_\epsilon + \nabla \epsilon^{1-\alpha} \nabla \epsilon u_\epsilon = \epsilon^{-\alpha} F_\epsilon U \quad \text{in } \epsilon \mathbb{Z}^d.
\]

The maximal \( L^p \)-regularity theory and boundedness of \( F_\epsilon \) imply that
\[
\|\nabla \epsilon u_\epsilon\|_{L^p(\Omega \times \epsilon \mathbb{Z}^d)^d} \leq \epsilon^{-\frac{\gamma}{2}} C\|U\|_{L^p(\Omega \times \mathbb{R}^d)}.
\]

As a result of this and the isometry property of \( T_\epsilon \), we obtain that \( \epsilon^{1-\alpha} \nabla \epsilon u_\epsilon \to 0 \). We consider a sequence \( U_\delta = \sum_{i=1}^{n(\delta)} \varphi_i^\delta \eta_i^\delta \) such that \( \varphi_i^\delta \in L^p_{sc}(\Omega) \), \( \eta_i^\delta \in C_c^{\infty}(\mathbb{R}^d) \), and 
\[
\|U_\delta - U\|_{L^p(\Omega \times \mathbb{R}^d)} \to 0 \text{ as } \delta \to 0.
\]

Note that \( w_{\delta, \epsilon} := u_\epsilon - F_\epsilon U_\delta \) is the unique solution in \( L^p(\Omega \times \epsilon \mathbb{Z}^d) \) to (for \( P\text{-a.e. } \omega \in \Omega \))
\[
\epsilon^{-\alpha} w_{\delta, \epsilon} + \nabla \epsilon^{1-\alpha} \nabla \epsilon w_{\delta, \epsilon} = \epsilon^{-\alpha} (F_\epsilon U - F_\epsilon U_\delta) - \nabla \epsilon^{1-\alpha} \nabla \epsilon F_\epsilon U_\delta \quad \text{in } \epsilon \mathbb{Z}^d.
\]

The maximal \( L^p \)-regularity theory implies that
\[
\epsilon^{-\frac{\gamma}{2}} \|w_{\delta, \epsilon}\|_{L^p(\Omega \times \epsilon \mathbb{Z}^d)} \leq C \left( \epsilon^{-\frac{\gamma}{2}} \|F_\epsilon U - F_\epsilon U_\delta\|_{L^p(\Omega \times \epsilon \mathbb{Z}^d)} + \|\nabla \epsilon F_\epsilon U_\delta\|_{L^p(\Omega \times \epsilon \mathbb{Z}^d)^d} \right).
\]
We multiply the above inequality by $\varepsilon^{\frac{2}{3}}$ and use boundedness of $F_{\varepsilon}$, to obtain
\[
\|u_{\varepsilon} - F_{\varepsilon}U_\delta\|_{L^p(\Omega \times \mathbb{R}^d)} \leq C \left( \|U - U_\delta\|_{L^p(\Omega \times \mathbb{R}^d)} + \varepsilon^{\frac{2}{3}} \|\nabla^\varepsilon F_{\varepsilon} U_\delta\|_{L^p(\Omega \times \mathbb{R}^d)} \right) .
\]

Using the above inequality and the isometry property of $T_{\varepsilon}$, we obtain
\[
\left\| T_{\varepsilon} u_{\varepsilon} - U \right\|_{L^p(\Omega \times \mathbb{R}^d)} \leq C \left( \|U - U_\delta\|_{L^p(\Omega \times \mathbb{R}^d)} + \varepsilon^{\frac{2}{3}} \|\nabla^\varepsilon F_{\varepsilon} U_\delta\|_{L^p(\Omega \times \mathbb{R}^d)} + \|T_{\varepsilon} F_{\varepsilon} U_\delta - U_\delta\|_{L^p(\Omega \times \mathbb{R}^d)} \right) .
\]

Letting $\varepsilon \to 0$ the last two terms on the right-hand side of the above inequality vanish. Indeed, the middle term is bounded by $C\varepsilon^{\frac{2}{3}} \|\nabla U_\delta\|_{L^p(\Omega \times \mathbb{R}^d)}$ (cf. part (ii) (11)) and the last term vanishes using Lemma 3.2 (iii). Finally, letting $\delta \to 0$ we conclude that $u_{\varepsilon} \xrightarrow{\ast} U$.

(iv) We consider a sequence $U_\delta = \sum_{i=1}^{n(\delta)} \varphi_i^\delta \eta^\delta_i$ such that $\varphi_i^\delta \in L^p(\Omega)$, $\eta_i^\delta \in C_c^\infty(\mathbb{R}^d)$, and
\[
\|U_\delta - U\|_{L^p(\Omega \times \mathbb{R}^d)} \to 0 \text{ as } \delta \to 0 .
\]

We have
\[
\|T_{\varepsilon} \varepsilon^\gamma \nabla^\varepsilon F_{\varepsilon} U - a_\gamma D U\|_{L^p(\Omega \times \mathbb{R}^d)} \leq \|T_{\varepsilon} \varepsilon^\gamma \nabla^\varepsilon F_{\varepsilon} (U - U_\delta)\|_{L^p(\Omega \times \mathbb{R}^d)} + \|a_\gamma (U_\delta - U)\|_{L^p(\Omega \times \mathbb{R}^d)} + \|T_{\varepsilon} \varepsilon^\gamma \nabla^\varepsilon F_{\varepsilon} U_\delta - a_\gamma D U_\delta\|_{L^p(\Omega \times \mathbb{R}^d)} .
\]

The first term on the right-hand side above is bounded by $\varepsilon^{-1}C\|U - U_\delta\|_{L^p(\Omega \times \mathbb{R}^d)}$ (using boundedness of all of the appearing operators). We compute, as in (12) (part (ii)), for $i = 1, ..., d$
\[
T_{\varepsilon} \varepsilon^\gamma \nabla^\varepsilon F_{\varepsilon} U_\delta(\omega, x) \varepsilon^{-1} \left( \pi_i^\delta U_\delta(T_{\varepsilon, \omega, x} + \varepsilon e_i) - \pi_i^\delta U_\delta(T_{\varepsilon, \omega, x}) \right) + \varepsilon^{-1} \pi_i^\delta D U_\delta(\omega, x).
\]

As a result of this, we obtain
\[
\|T_{\varepsilon} \varepsilon^\gamma \nabla^\varepsilon F_{\varepsilon} U - a_\gamma D U_\delta\|_{L^p(\Omega \times \mathbb{R}^d)} \leq \varepsilon^\gamma \|\pi_i^\delta U_\delta(\omega, x) + \varepsilon e_i - \pi_i^\delta U_\delta(\omega, x)\|_{L^p(\Omega \times \mathbb{R}^d)} + \varepsilon^{-1} \pi_i^\delta D U_\delta - a_\gamma D U_\delta\|_{L^p(\Omega \times \mathbb{R}^d)} .
\]

The first term on the right-hand side above is bounded by $\varepsilon^{-1}C\|\nabla U_\delta\|_{L^p(\Omega \times \mathbb{R}^d)}$ and therefore it vanishes in the limit $\varepsilon \to 0$. The second term vanishes as well in the limit $\varepsilon \to 0$. Collecting the above claims and letting first $\varepsilon \to 0$ and then $\delta \to 0$ in (13), we conclude the proof.

**Proof of Corollary 3.3**

Proof. (i) and (ii) follow directly from Lemma 3.2 (iii) and Proposition 3.2.

(iii) For $\delta > 0$ we consider a cut-off function $\eta_\delta \in C_c^\infty(\mathbb{R}^d)$ such that $0 \leq \eta_\delta \leq 1$, $\eta_\delta = 0$ in $\mathbb{R}^d \setminus O$, $\eta_\delta = 1$ in $O^{-\delta} := \{ x \in O : \text{dist}(x, \partial O) \geq \delta \}$ and $|\nabla \eta_\delta| \leq \frac{C}{\delta}$.

Also, by density we can choose a sequence $U_\delta(\omega, x) = \sum_{i=1}^{n(\delta)} \varphi_i^\delta(\omega) \xi_i^\delta(x)$ such that $\varphi_i^\delta \in L^p_{\text{inv}}(\Omega)$ and $\xi_i^\delta \in C_c^\infty(\mathbb{R}^d)$, $\text{dist}(\text{supp}(U_\delta), \partial O) \geq \mu(\delta)$ (with $\mu(\delta) \to 0$ as $\delta \to 0$) and as $\delta \to 0$
\[
U_\delta \to U \text{ strongly in } L^p(\Omega) \otimes W^{1,p}(\mathbb{R}^d).
\]

Let $u_{\varepsilon, \delta} = F_{\varepsilon} U_\delta + \eta_\delta G_{\varepsilon}^0 \chi$, where $G_{\varepsilon}^0$ denotes the operator given in Proposition 3.2. We have
\[
\|u_{\varepsilon, \delta} - (F_{\varepsilon} U + G_{\varepsilon}^0 \chi)\|_{L^p(\Omega \times \mathbb{R}^d)} + \|\nabla^\varepsilon (u_{\varepsilon, \delta} - (F_{\varepsilon} U + G_{\varepsilon}^0 \chi))\|_{L^p(\Omega \times \mathbb{R}^d)} \leq \|F_{\varepsilon} U_\delta - F_{\varepsilon} U\|_{L^p(\Omega \times \mathbb{R}^d)} + \|\nabla^\varepsilon (\eta_\delta U_\delta - G_{\varepsilon}^0 \chi)\|_{L^p(\Omega \times \mathbb{R}^d)} + \|\nabla^\varepsilon (\eta_\delta G_{\varepsilon}^0 \chi - G_{\varepsilon}^0 \chi)\|_{L^p(\Omega \times \mathbb{R}^d)} .
\]
Above on the right-hand side, the first term can be bounded by \( \|U_\delta - U\|_{L^p(\Omega \times \mathbb{R}^d)} \) (by boundedness of \( \mathcal{F}_\varepsilon \)), the second term is bounded by \( \|T_\varepsilon G^0_\varepsilon \chi\|_{L^p(\Omega \times \mathbb{R}^d \setminus O^{-\delta})} \) (using the properties of \( \eta_\delta \)) and the third term is bounded by \( C \| \nabla U_\delta - \nabla U\|_{L^p(\Omega \times \mathbb{R}^d)} \) (similarly as in (11)). The last term is treated as follows. We take advantage of the following product rule: for \( f, g : \varepsilon \mathbb{Z}^d \to \mathbb{R} \) it holds \( \nabla_\varepsilon^i f(x)g(x) = f(x + \varepsilon e_i)\nabla_\varepsilon^i g(x) + g(x)\nabla_\varepsilon^i f(x) \). Consequently, we obtain

\[
\| \nabla^\varepsilon (\eta_\delta G^0_\varepsilon \chi - G^0_\varepsilon \chi)\|_{L^p(\Omega \times \varepsilon \mathbb{Z}^d)} \leq \| (\eta_\delta - 1) \nabla^\varepsilon G^0_\varepsilon \chi\|_{L^p(\Omega \times \varepsilon \mathbb{Z}^d)} + C \sum_{i=1}^d \left( \int_{\varepsilon \mathbb{Z}^d} |G^0_\varepsilon \chi(x_1 + \varepsilon e_i)\nabla_\varepsilon^i \eta_\delta(x)|^p dm_\varepsilon(x) \right)^{\frac{1}{p}}. \tag{15}
\]

The first term on the right-hand side of (15) is bounded by \( \|T_\varepsilon \nabla^\varepsilon G^0_\varepsilon \chi\|_{L^p(\Omega \times \mathbb{R}^d \setminus O^{-\delta})} \) and for small enough \( \varepsilon \), the second term is bounded by \( \frac{C}{\delta} \|T_\varepsilon G^0_\varepsilon \chi\|_{L^p(\Omega \times \mathbb{R}^d)} \). Note that

\[
\lim_{\delta \to 0} \sup_{\varepsilon \to 0} \left( \|T_\varepsilon \nabla^\varepsilon G^0_\varepsilon \chi\|_{L^p(\Omega \times \mathbb{R}^d \setminus O^{-\delta})} + \frac{C}{\delta} \|T_\varepsilon G^0_\varepsilon \chi\|_{L^p(\Omega \times \mathbb{R}^d)} \right) = 0
\]

since \( T_\varepsilon G^0_\varepsilon \to 0 \) and \( T_\varepsilon \nabla^\varepsilon G^0_\varepsilon \to \chi \) as \( \varepsilon \to 0 \) (Proposition 3.2 (i)).

Collecting all the above bounds for the inequality (14), using the isometry property of \( T_\varepsilon \) and with help of part (i), we obtain that

\[
\lim_{\delta \to 0} \sup_{\varepsilon \to 0} \left( \|T_\varepsilon u_{\varepsilon, \delta} - U\|_{L^p(\Omega \times \mathbb{R}^d)} + \|T_\varepsilon \nabla^\varepsilon u_{\varepsilon, \delta} - (\nabla U + \chi)\|_{L^p(\Omega \times \mathbb{R}^d)} + g(\varepsilon, \delta) \right) = 0,
\]

where \( g(\varepsilon, \delta) = \begin{cases} 0 & \text{if } \varepsilon \leq \frac{\mu(\delta)}{C(d)} \text{ and } C(d) \text{ is the diameter of } \delta \text{. Hence, there exists a diagonal sequence } u_\varepsilon := u_{\varepsilon, \delta(\varepsilon)} \text{ which satisfies the claim of the corollary.} \\
1 & \text{if } \varepsilon > \frac{\mu(\delta)}{C(d)} \end{cases} \text{ and } C(d) \text{ is the diameter of } \delta \text{. Hence, there exists a diagonal sequence } u_\varepsilon := u_{\varepsilon, \delta(\varepsilon)} \text{ which satisfies the claim of the corollary.} \}

(iv) For \( (U, \chi) \in (L^p_{\text{loc}}(\Omega) \otimes L^p(\Omega)) \times (L^p_{\text{loc}}(\Omega) \otimes L^p(\Omega)) \) we define \( u_\varepsilon(U, \chi) := \eta(\varepsilon) (\mathcal{F}_\varepsilon U + G^\varepsilon_\varepsilon \chi) \), where \( \eta(\varepsilon) \) is the cut-off function from part (iii) with \( \delta(\varepsilon) = \varepsilon \mathbb{Z}^d \). For notational convenience, we write \( u_\varepsilon \) instead of \( u_\varepsilon(U, \chi) \). We have

\[
\|T_\varepsilon u_\varepsilon - U\|_{L^p(\Omega \times \mathbb{R}^d)} + \|T_\varepsilon \nabla^\varepsilon u_\varepsilon - \chi\|_{L^p(\Omega \times \mathbb{R}^d)} \leq \|T_\varepsilon u_\varepsilon - T_\varepsilon (F_\varepsilon U + G^\varepsilon_\varepsilon \chi)\|_{L^p(\Omega \times \mathbb{R}^d)} + \|T_\varepsilon (F_\varepsilon U + G^\varepsilon_\varepsilon \chi) - U\|_{L^p(\Omega \times \mathbb{R}^d)} + \|T_\varepsilon \nabla^\varepsilon u_\varepsilon - T_\varepsilon \nabla^\varepsilon (F_\varepsilon U + G^\varepsilon_\varepsilon \chi)\|_{L^p(\Omega \times \mathbb{R}^d)} + \|T_\varepsilon \nabla^\varepsilon (F_\varepsilon U + G^\varepsilon_\varepsilon \chi) - \chi\|_{L^p(\Omega \times \mathbb{R}^d)}.
\]

The second and last terms on the right-hand side above vanish as \( \varepsilon \to 0 \) using the claim of part (ii). The first term is bounded by \( \|T_\varepsilon (F_\varepsilon U + G^\varepsilon_\varepsilon \chi)\|_{L^p(\Omega \times \mathbb{R}^d \setminus O^{-\delta(\varepsilon)})} \) (cf. part (iii)) and therefore it vanishes as \( \varepsilon \to 0 \) using the fact that \( T_\varepsilon (F_\varepsilon U + G^\varepsilon_\varepsilon \chi) \) converges strongly (and therefore it is uniformly integrable). For small enough \( \varepsilon \), the third term is bounded (up to a constant) by \( \|T_\varepsilon \nabla^\varepsilon (F_\varepsilon U + G^\varepsilon_\varepsilon \chi)\|_{L^p(\Omega \times \mathbb{R}^d \setminus O^{-\delta(\varepsilon)})} + \varepsilon^2 \|T_\varepsilon (F_\varepsilon U + G^\varepsilon_\varepsilon \chi)\|_{L^p(\Omega \times \mathbb{R}^d)} \) (cf. (15) in part (iii)). The last expression vanishes in the limit \( \varepsilon \to 0 \) using the properties of \( F_\varepsilon U + G^\varepsilon_\varepsilon \chi \). The proof is complete. \( \square \)

**Proof of Proposition 3.3**

**Proof.** For notational convenience, let \( E_\varepsilon(u_\varepsilon) := \langle \int_{O + \varepsilon \mathbb{Z}^d} V(T_\varepsilon \omega, u_\varepsilon(\omega, x)) dm_\varepsilon(x) \rangle \).

(i) Consider a sequence of open sets \( A_k \subset \subset O \) which satisfy \( A_k \subset A_{k+1} \) and \( |O \setminus A_k| \to 0 \) as
For small enough $\varepsilon$, we have

$$
\mathcal{E}_\varepsilon(u_\varepsilon) = \left\langle \int_{O^+ \cap \mathbb{Z}^d} V(\omega, \mathcal{T}_\varepsilon u_\varepsilon(\omega, x)) \, dm_\varepsilon(x) \right\rangle \\
= \sum_{x \in O^+ \cap \mathbb{Z}^d} \left\langle \int_{x + \varepsilon} V(\omega, \mathcal{T}_\varepsilon u_\varepsilon(\omega, x)) \, dx \right\rangle \\
= \left\langle \int_{A_k} V(\omega, \mathcal{T}_\varepsilon u_\varepsilon(\omega, x)) \, dx \right\rangle + \left\langle \int_{L_{\varepsilon,k}} V(\omega, \mathcal{T}_\varepsilon u_\varepsilon(\omega, x)) \, dx \right\rangle,
$$

for a suitable small boundary layer set $L_{\varepsilon,k} \subset \{ x : \text{dist}(x, O^+ \setminus A_k) < C_1 \varepsilon \}$, where $C_1$ is a fixed constant depending only on the dimension $d$ (see Figure 2). The growth conditions of $V$ yield

$$
\mathcal{E}_\varepsilon(u_\varepsilon) \geq \left\langle \int_{A_k} V(\omega, \mathcal{T}_\varepsilon u_\varepsilon(\omega, x)) \, dx \right\rangle - C|L_{\varepsilon,k}|.
$$

Figure 2: Small boundary layer.

Letting $\varepsilon \to 0$, we obtain

$$
\liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon \to 0} \left\langle \int_{A_k} V(\omega, \mathcal{T}_\varepsilon u_\varepsilon(\omega, x)) \, dx \right\rangle - C|O \setminus A_k| \\
\geq \left\langle \int_{A_k} V(\omega, U(\omega, x)) \, dx \right\rangle - C|O \setminus A_k|.
$$

The last inequality is obtained using the fact that the functional $U \mapsto \left\langle \int_{A_k} V(\omega, U) \right\rangle$ is weakly lower-semicontinuous (since it is convex and continuous w.r.t. the strong $L^p$-topology). Finally, letting $k \to \infty$ we obtain the claimed result.

(ii) Similarly as in part (i), we find suitable boundary layer sets $L_{\varepsilon}^+$ and $L_{\varepsilon}^-$ (with $|L_{\varepsilon}^\pm| \to 0$ as $\varepsilon \to 0$), such that

$$
\mathcal{E}_\varepsilon(u_\varepsilon) = \left\langle \int_{O} V(\omega, \mathcal{T}_\varepsilon u_\varepsilon(\omega, x)) \, dx \right\rangle + \left\langle \int_{L_{\varepsilon}^+} V(\omega, \mathcal{T}_\varepsilon u_\varepsilon(\omega, x)) \, dx \right\rangle - \left\langle \int_{L_{\varepsilon}^-} V(\omega, \mathcal{T}_\varepsilon u_\varepsilon(\omega, x)) \, dx \right\rangle \\
\leq \left\langle \int_{O} V(\omega, \mathcal{T}_\varepsilon u_\varepsilon(\omega, x)) \, dx \right\rangle + C(|L_{\varepsilon}^+| + |L_{\varepsilon}^-|) + C \left\langle \int_{L_{\varepsilon}^+} |\mathcal{T}_\varepsilon u_\varepsilon(\omega, x)|^p \right\rangle,
$$

where we use the growth conditions of the integrand $V$. The terms in the middle on the right-hand side vanish as $\varepsilon \to 0$, as well as the last term (using strong convergence of $\mathcal{T}_\varepsilon u_\varepsilon$). As a result of this and using strong continuity of $U \mapsto \left\langle \int_{O} V(\omega, U) \right\rangle$, we obtain that $\limsup_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \leq \left\langle \int_{O} V(\omega, U(\omega, x)) \, dx \right\rangle$. Using part (i), we conclude the proof. 

\[\square\]
4 Stochastic homogenization of spring networks

In this section, we illustrate the capabilities of the stochastic unfolding operator in homogenization of energy driven problems that invoke convex functionals. We treat a multidimensional analogon of the problem presented in the introduction - a network of springs which exhibit either elastic or elasto-plastic response. The material coefficients are assumed to be rapidly oscillating random fields and we derive effective models in the sense of a discrete-to-continuum transition.

In Section 4.1, we briefly present the setting of periodic lattice graphs and the corresponding differential calculus. If the springs display only elastic behavior and the forces acting on the system do not depend on time, the static equilibrium of the spring network is determined by a convex minimization problem. Accordingly, in Section 4.2 we present homogenization results for convex functionals. On the other hand, in the case of elasto-plastic springs, the evolution of the system is embedded in the framework of evolutionary rate-independent systems (ERIS). A short description of that framework can be found in Appendix A.1, and for a detailed study we refer to [27, 30]. In the limit, as the characteristic size of the springs vanishes, we obtain a homogenized model, which is also described by an ERIS on a continuum physical space (Section 4.3 and 4.4).

We remark that homogenization results concerning minimization problems in the discrete-to-continuum setting are already well-established. Earlier works (e.g. [2]) treat more general problems than we do in this paper. Namely, the considered potentials might be nonconvex and if the media is ergodic, previous works feature quenched homogenization results. This means that for almost every configuration, the energy functional $\Gamma$-converges to a homogenized energy functional (cf. Remark 4.3). In our results for convex minimization problems, we obtain weaker, averaged homogenization results (see Section 4.2). Despite our results being weaker in a general situation, we would like to point out that our strategy relies only on the simple idea of the unfolding operator, namely the compactness properties of the unfolding (which are closely related to compactness statements in usual $L^p$-spaces) and lower semicontinuity of convex functionals play a central role in our analysis. On the other hand, previous works are based on more involved techniques, such as the subadditive ergodic theorem [1] ([2], [34]), or the notion of quenched stochastic two-scale convergence ([21]).

4.1 Functions on lattice graphs

Let $E_0 = \{b_1, ..., b_k\} \in \mathbb{Z}^d \setminus \{0\}$ be the edge generating set and we assume that $E_0$ includes $\{e_i\}_{i=1,...,d}$. We consider the rescaled periodic lattice graph $(\varepsilon \mathbb{Z}^d, \varepsilon E)$, where the set of edges is given by $E = \{[x, x + b_i] : x \in \mathbb{Z}^d, i = 1, ..., k\}$. For $u : \varepsilon \mathbb{Z}^d \to \mathbb{R}^d$ the difference quotient along the edge generated by $b_i$ is

$$\partial_{\varepsilon}^i u : \mathbb{Z}^d \to \mathbb{R}^d, \quad \partial_{\varepsilon}^i u(x) = \frac{u(x + \varepsilon b_i) - u(x)}{\varepsilon |b_i|}.$$ 

Note that for each $b_i$ there exists $B_i : \mathbb{Z}^d \to \mathbb{Z}^d$ such that

$$\partial_{\varepsilon}^i u(x) = \sum_{y \in \mathbb{Z}^d} \nabla^\varepsilon u(x - \varepsilon y) B_i(y),$$

where the discrete gradient $\nabla^\varepsilon u$ is defined as in Section 2.1. We define the discrete symmetrized gradient as $\nabla_{\varepsilon}^s u : \varepsilon \mathbb{Z}^d \to \mathbb{R}^k$

$$\nabla_{\varepsilon}^s u(x) = \left( \frac{b_1}{|b_1|} \cdot \partial_{\varepsilon}^1 u(x), ..., \frac{b_k}{|b_k|} \cdot \partial_{\varepsilon}^k u(x) \right).$$

$B_i$ are not uniquely determined, however we consider one such choice corresponding to a path between 0 and $b_i$. 

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Furthermore, we introduce a suitable symmetrization operator for random fields as follows. For a matrix \( F \in \mathbb{R}^{d \times d} \), we denote by \( F_{+} \in \mathbb{R}^{k} \) the vector with entries \((F_{+})_{i} = \frac{b_{i}}{|b_{i}|} \cdot F \frac{b_{i}}{|b_{i}|} \) \((i = 1, \ldots, k)\). Analogously, for \( F : \Omega \to \mathbb{R}^{d \times d} \) measurable, we set \( F_{+} : \Omega \to \mathbb{R}^{k} \),

\[
(F_{+})_{i}(\omega) = \frac{b_{i}}{|b_{i}|} \cdot \sum_{y \in \mathbb{Z}^{d}} F(T_{y} \omega)B_{i}(y), \quad (i = 1, \ldots, k).
\]

If \( F = \nabla U \) or \( F = DU \), instead of \( F_{+} \) we write \( \nabla_{s} U \) or \( \partial_{s} U \).

Throughout the paper, we assume that \((\mathbb{Z}^{d}, \mathcal{E})\) satisfies a discrete version of Korn’s inequality:

There exists \( C(d, p) > 0 \) such that

\[
\int_{\mathbb{Z}^{d}} |\nabla u(x)|^{p} \, dm(x) \leq C(d, p) \int_{\mathbb{Z}^{d}} |\nabla_{s} u(x)|^{p} \, dm(x) \quad \text{(Korn)}
\]

for all compactly supported \( u : \mathbb{Z}^{d} \to \mathbb{R}^{d} \).

**Remark 4.1.** An example of a lattice satisfying the above Korn’s inequality is \((\mathbb{Z}^{d}, \mathcal{E})\) with

\[
\mathcal{E}_{0} = \left\{ \sum_{i=1}^{d} \delta_{i} e_{i} : \delta \in \{0, 1\}^{d} \setminus \{0\} \right\}.
\]

The assumption \((\text{Korn})\) implies a continuum version of Korn’s inequality. Namely, let \( O \subset \mathbb{R}^{d} \) be open and bounded, there exists \( C(p) > 0 \) such that

\[
\int_{O} |\nabla U|^{p} \, dx \leq C(p) \int_{O} |\nabla_{s} U|^{p} \, dx \quad \text{for any } U \in W_{0}^{1,p}(O). \tag{16}
\]

This inequality is obtained applying \((\text{Korn})\) to \( \pi_{\varepsilon} u_{\delta} \), where \( u_{\delta} \) is a smooth approximation of \( u \), and passing to the limits \( \varepsilon \to 0 \) and \( \delta \to 0 \) (cf. Lemma 4.1 in Section 4.5).

Note that \((\text{Korn})\) implies another, stochastic version of Korn’s inequality - see Lemma 4.2 in Section 4.5.

**4.2 Static problem**

As a preparation for the rate-independent evolutionary problem, we first discuss a homogenization procedure for a static convex minimization problem, and discuss different notions of convergence in the homogenization result. We consider a set of particles with reference positions at \( \varepsilon \mathbb{Z}^{d} \). It is assumed that the edges \( \varepsilon \mathcal{E} \) represent springs with elastic response (cf. Introduction with internal variable \( z = 0 \) and loading \( l(t) = l \)). The equilibrium state of the system is determined by a minimization problem which (in a slightly more general setting) reads:

\[
\min_{u \in \{L^{p}(\Omega) \otimes L_{0}^{p}(O \cap \mathbb{Z}^{d})\}^{d}} \left\{ \int_{O + \varepsilon \cap \mathbb{Z}^{d}} V(T_{z} \omega, \nabla_{s} u(\omega, x)) \, dm_{\varepsilon}(x) - \int_{O \cap \mathbb{Z}^{d}} l_{\varepsilon}(\omega, x) \cdot u(\omega, x) \, dm_{\varepsilon}(x) \right\}. \tag{17}
\]

We assume the following:

(A0) \( O \subset \mathbb{R}^{d} \) is a bounded domain with Lipschitz boundary.

We set \( O^{+\varepsilon} = O \cup \{ x \in \mathbb{R}^{d} : (x, x + \varepsilon b_{i}) \cap O \neq \emptyset \text{ for some } b_{i} \in \mathcal{E}_{0} \} \).

(A1) \( V : \Omega \times \mathbb{R}^{k} \to \mathbb{R} \) is jointly measurable (i.e. w.r.t. the product \( \sigma \)-algebra \( \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^{k}) \)).

(A2) For \( P\text{-a.e. } \omega \in \Omega: V(\omega, \cdot) \) is convex.

(A3) There exists \( C > 0 \) such that

\[
\frac{1}{C} |F|^{p} \leq V(\omega, F) \leq C(|F|^{p} + 1),
\]

for \( P\text{-a.e. } \omega \in \Omega \) and all \( F \in \mathbb{R}^{k} \).
In the case that the loading \( l_\varepsilon \) converges in a sufficiently strong sense (see Remark 4.2), in order to describe the asymptotic behavior of minimizers in (17), it is sufficient to consider the energy functional

\[
\mathcal{E}_\varepsilon : (L^p(\Omega) \otimes L^p_0(O \cap \varepsilon \mathbb{Z}^d))^d \to \mathbb{R}, \quad \mathcal{E}_\varepsilon(u) = \left\langle \int_{O + r \varepsilon \mathbb{Z}^d} V(T_\varepsilon \omega, \nabla u(\omega, x))dm_\varepsilon(x) \right\rangle.
\]

As shown below for \( \varepsilon \to 0 \), we derive the effective two-scale functional

\[
\mathcal{E}_0 : (L^p_{inv}(\Omega) \otimes W^{1,p}_0(O))^d \times (L^p_{pot}(\Omega) \otimes L^p(O))^d \to \mathbb{R},
\]

\[
\mathcal{E}_0(U, \chi) = \left\langle \int_O V(\omega, \nabla U(\omega, x) + \chi(\omega, x))dx \right\rangle.
\]

Moreover, if we assume that \( \langle \cdot \rangle \) is ergodic, the effective energy reduces to a single-scale functional

\[
\mathcal{E}_{\text{hom}} : W^{1,p}_0(O)^d \to \mathbb{R}, \quad \mathcal{E}_{\text{hom}}(U) = \int_O V_{\text{hom}}(\nabla U(x))dx,
\]

where the homogenized energy density \( V_{\text{hom}} : \mathbb{R}^{d \times d} \to \mathbb{R} \) is defined by the corrector problem

\[
V_{\text{hom}}(F) = \inf_{\chi \in L^p_{\text{pot}}(\Omega)^d} \langle V(\omega, F + \chi(\omega)) \rangle.
\]

**Theorem 4.1** (Two-scale homogenization). Assume (A0) – (A3).

(i) (Compactness) For \( u_\varepsilon \in (L^p(\Omega) \otimes L^p_0(O \cap \varepsilon \mathbb{Z}^d))^d \) with \( \limsup_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon) < \infty \), there exist a subsequence (not relabeled), \( U \in (L^p_{\text{inv}}(\Omega) \otimes W^{1,p}_0(O))^d \) and \( \chi \in (L^p_{\text{pot}}(\Omega) \otimes L^p(O))^d \) such that

\[
u_\varepsilon \overset{2}{\rightharpoonup} U \text{ in } L^p(\Omega \times \mathbb{R}^d)^d, \quad \nabla \varepsilon u_\varepsilon \overset{2}{\rightharpoonup} \nabla U + \chi \text{ in } L^p(\Omega \times \mathbb{R}^d)^{d \times d}.
\]

(ii) (Lower bound) Assume that the convergence (19) holds for the whole sequence \( u_\varepsilon \). Then

\[
\liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{E}_0(U, \chi).
\]

(iii) (Upper bound) For any \( U \in (L^p_{\text{inv}}(\Omega) \otimes W^{1,p}_0(O))^d \) and \( \chi \in (L^p_{\text{pot}}(\Omega) \otimes L^p(O))^d \), there exists a sequence \( u_\varepsilon \in (L^p(\Omega) \otimes L^p_0(O \cap \varepsilon \mathbb{Z}^d))^d \) such that

\[
u_\varepsilon \overset{2}{\rightharpoonup} U \text{ in } L^p(\Omega \times \mathbb{R}^d)^d, \quad \nabla \varepsilon u_\varepsilon \overset{2}{\rightharpoonup} \nabla U + \chi \text{ in } L^p(\Omega \times \mathbb{R}^d)^{d \times d}, \quad \lim_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \mathcal{E}_0(U, \chi).
\]

**Remark 4.2** (Convergence of minimizers). Under the assumptions of Theorem 4.1 and if the loadings \( l_\varepsilon \in L^p(\Omega \times \varepsilon \mathbb{Z}^d)^d \) satisfy \( l_\varepsilon \overset{2}{\rightharpoonup} l \) where \( l \in L^p(\Omega \times O)^d \), the above theorem implies (by a standard argument from \( \Gamma \)-convergence) that minimizers \( u_\varepsilon \) in (17) satisfy (up to a subsequence)

\[
u_\varepsilon \overset{2}{\rightharpoonup} U \text{ in } L^p(\Omega \times \mathbb{R}^d)^d \quad \text{and} \quad \nabla \varepsilon u_\varepsilon \overset{2}{\rightharpoonup} \nabla U + \chi \text{ in } L^p(\Omega \times \mathbb{R}^d)^{d \times d},
\]

\[\text{where } (U, \chi) \in (L^p_{\text{inv}}(\Omega) \otimes W^{1,p}_0(O))^d \times (L^p_{\text{pot}}(\Omega) \otimes L^p(O))^d \text{ is a minimizer of the two-scale functional}
\]

\[I_0 : (L^p_{\text{inv}}(\Omega) \otimes W^{1,p}_0(O))^d \times (L^p_{\text{pot}}(\Omega) \otimes L^p(O))^d \to \mathbb{R}, \quad I_0(U) = \mathcal{E}_0(U, \chi) - \left\langle \int_O l \cdot U dx \right\rangle.
\]

In the ergodic case, the limit is deterministic:

**Theorem 4.2** (Ergodic case). Assume (A0) – (A3) and that \( \langle \cdot \rangle \) is ergodic.
(i) (Compactness and lower bound) Let \( u_\varepsilon \in (L^p(\Omega) \otimes L^p_0(O \cap \varepsilon \mathbb{Z}^d))^d \) satisfy
\[
\limsup_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon) < \infty.
\]
There exists \( U \in W^{1,p}_0(O)^d \) such that, up to a subsequence,
\[
\langle u_\varepsilon \rangle \to U \text{ in } L^p(\mathbb{R}^d)^d, \quad \langle \nabla u_\varepsilon \rangle \to \nabla U \text{ in } L^p(\mathbb{R}^d)^{d \times d}, \quad \mathcal{E}_{\text{hom}}(U) \leq \liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon).
\]

(ii) (Upper bound) For any \( U \in W^{1,p}_0(O)^d \), there exists \( u_\varepsilon \in (L^p(\Omega) \otimes L^p_0(O \cap \varepsilon \mathbb{Z}^d f))^d \) such that
\[
\langle u_\varepsilon \rangle \to U \text{ in } L^p(\mathbb{R}^d)^d, \quad \langle \nabla u_\varepsilon \rangle \to \nabla U \text{ in } L^p(\mathbb{R}^d)^{d \times d}, \quad \lim_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \mathcal{E}_{\text{hom}}(U).
\]

Remark 4.3. In the ergodic case, \( \mathcal{E}_\varepsilon \) \( \Gamma \)-converges to the deterministic functional \( \mathcal{E}_{\text{hom}} \). In fact, it is known that for \( P \)-a.e. \( \omega \in \Omega \) the functional
\[
\mathcal{E}_\varepsilon(\omega, \cdot) : u \in L^p_0(O \cap \varepsilon \mathbb{Z}^d)^d \mapsto \int_{O \cap \varepsilon \mathbb{Z}^d} V(T_\varepsilon x, \nabla u(x))dm_{\varepsilon}(x) \in \mathbb{R}
\]
\( \Gamma \)-converges to \( \mathcal{E}_{\text{hom}} \). This quenched convergence result can be found e.g. in [2], where even non-convex integrands are treated. Based on stochastic unfolding, we obtain the weaker “averaged” result of Theorem 4.2 as a corollary of Theorem 4.1 and Corollary 3.2. While our argument is relatively easy, the analysis of the stronger quenched convergence result is based on the subadditive ergodic theorem [1] and it is more involved. We remark that minimizers \( \omega \mapsto u_\varepsilon(\omega) \) of the functionals \( \mathcal{E}_\varepsilon(\omega, \cdot) \) present a random field (under the above assumptions), which minimizes the averaged energy \( \mathcal{E}_\varepsilon(\cdot) \) (and vice versa).

If we, additionally, assume that \( V \geq 0 \) and for \( P \)-a.e. \( \omega \in \Omega \) it is uniformly convex with modulus \((\cdot)_p\), i.e. there exists \( C > 0 \) such that for all \( F, G \in \mathbb{R}^k \) and \( t \in [0, 1] \)
\[
V(\omega, tF + (1 - t)G) \leq tV(\omega, F) + (1 - t)V(\omega, G) - (1 - t)tC|F - G|^p,
\]
then we obtain strong two-scale convergence for minimizers:

Proposition 4.1 (Strong convergence). Let the assumptions of Theorem 4.2 hold, as well as the above assumptions. Let \( l_\varepsilon \in L^p(\Omega \times \varepsilon \mathbb{Z}^d)^d \) satisfy \( l_\varepsilon \overset{L^p}{\to} l \) in \( L^p(\Omega \times \mathbb{R}^d)^d \), where \( l \in L^p(O)^d \). The problem (17) admits a unique minimizer \( u_\varepsilon \in (L^p(\Omega) \otimes L^p_0(O \cap \varepsilon \mathbb{Z}^d))^d \) and it satisfies
\[
u_\varepsilon \overset{2}{\to} U \text{ in } L^p(\Omega \times \mathbb{R}^d)^d,
\]
where \( U \in W^{1,p}_0(O)^d \) is the unique minimizer of
\[
I_{\text{hom}} : W^{1,p}_0(O)^d \to \mathbb{R}, \quad I_{\text{hom}}(U) = \mathcal{E}_{\text{hom}}(U) - \int_O l \cdot Udx.
\]

Remark 4.4 (Periodic homogenization). As mentioned earlier, one knows that \( \mathcal{E}_\varepsilon(\omega, \cdot) \overset{\Gamma}{\to} \mathcal{E}_{\text{hom}} \) in the quenched sense, whereas we obtain convergence for minimizers in an averaged sense (as above in (20)). Yet if we consider the setting for periodic homogenization, using the above convergence in the mean, we recover a standard (pointwise) periodic homogenization result. In particular, for \( N \in \mathbb{N} \) we set \( \Omega = N \square \cap \varepsilon \mathbb{Z}^d \) the discrete \( N \)-torus with a corresponding (rescaled) counting measure. The dynamical system \( (T_x) \) is defined as \( T_x \omega = \omega + x \mod N \). The above example of the probability space with the dynamical system \( (T_x) \) satisfies the assumptions given in Section 2.2 and it is ergodic. We remark that in this case \( \Omega \) is a finite set and therefore (20) implies that \( u_\varepsilon(\omega) \to U \) in \( L^p(\mathbb{R}^d)^d \) for all \( \omega \in \Omega \).
4.3 Rate-independent evolutionary problem

Let us first describe the system we have in mind. A system of particles connected with springs is represented using \((\varepsilon\mathbb{Z}^d,\varepsilon\mathcal{E})\). Namely, \(\mathbb{Z}^d \cap O\) serves as the reference configuration of particles. The edges \(\varepsilon\mathcal{E}\) represent springs with elasto-plastic response (cf. Introduction). Upon an external loading \(l\), the system evolves according to an ERIS (see Appendix A.1). Let \(O, O^{+\varepsilon} \subset \mathbb{R}^d\) be open (see below for specific assumptions). The following model is a random and discrete counterpart of the model considered in \([32]\), where the periodic, continuum case is treated.

- The state space is \(Y_\varepsilon = (L^2(\Omega) \otimes L^2_0(O \cap \varepsilon\mathbb{Z}^d))^d \times (L^2(\Omega) \otimes L^2_0((O^{+\varepsilon} \cap \varepsilon\mathbb{Z}^d))^k\) - the displacement \(u_\varepsilon\) and the internal variable \(z_\varepsilon\) are merged into a joint variable \(y_\varepsilon = (u_\varepsilon, z_\varepsilon)\).

We equip \(Y_\varepsilon\) with the scalar product

\[
(y_1, y_2)_{Y_\varepsilon} = \left( \int_{\varepsilon\mathbb{Z}^d} u_1(\omega, x) \cdot u_2(\omega, x) dm_\varepsilon(x) \right) + \left( \int_{\varepsilon\mathbb{Z}^d} \nabla^\varepsilon u_1(\omega, x) : \nabla^\varepsilon u_2(\omega, x) dm_\varepsilon(x) \right)
\]

\[
+ \left( \int_{\varepsilon\mathbb{Z}^d} z_1(\omega, x) \cdot z_2(\omega, x) dm_\varepsilon(x) \right).
\]

- The total energy functional is \(\mathcal{E}_\varepsilon : [0, T] \times Y_\varepsilon \to \mathbb{R}\),

\[
\mathcal{E}_\varepsilon(t, y_\varepsilon) = \frac{1}{2} \langle A_{\varepsilon} y_\varepsilon, y_\varepsilon \rangle_{Y_\varepsilon} - \left( \int_{O^{+\varepsilon} \cap \varepsilon\mathbb{Z}^d} \pi_{\varepsilon l}(t)(x) \cdot u_\varepsilon(\omega, x) dm_\varepsilon(x) \right),
\]

\[
\langle A_{\varepsilon} y_1, y_2 \rangle_{Y_\varepsilon} = \left( \int_{O^{+\varepsilon} \cap \varepsilon\mathbb{Z}^d} A(T_{\varepsilon}^\omega)(\nabla^\varepsilon u_1(\omega, x)) \cdot (\nabla^\varepsilon u_2(\omega, x)) dm_\varepsilon(x) \right).
\]

- The dissipation potential is \(\Psi_\varepsilon : Y_\varepsilon \to [0, \infty),\)

\[
\Psi_\varepsilon(y_\varepsilon) = \left( \int_{O^{+\varepsilon} \cap \varepsilon\mathbb{Z}^d} \rho(T_{\varepsilon}^\omega, z_\varepsilon(\omega, x)) dm_\varepsilon(x) \right).
\]

We consider the ERIS (see Appendix) associated with \((\mathcal{E}_\varepsilon, \Psi_\varepsilon)\) and we denote the set of stable states at time \(t\) corresponding to these functionals by \(S_\varepsilon(t)\).

We assume the following:

\begin{enumerate}
\item[(B0)] \(O \subset \mathbb{R}^d\) is a bounded domain with Lipschitz boundary.
We set \(O^{+\varepsilon} = O \cup \{x \in \mathbb{R}^d : (x, x + cb_i) \cap O \neq \emptyset\ \text{for some } b_i \in \mathcal{E}_0\}\).
\item[(B1)] \(A \in L^\infty(\Omega, \mathbb{R}^{2m \times 2m})\) and it satisfies: there exists \(C > 0\) such that \(A(\omega)F \cdot F \geq C|F|^2\) for \(\forall F \in \mathbb{R}^d\).
\item[(B2)] \(\rho : \Omega \times \mathbb{R}^k \to [0, \infty)\) is jointly measurable (i.e. w.r.t. \(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^k)\)) and for \(\forall \omega \in \Omega\), \(\rho(\omega, \cdot)\) is convex and positively homogeneous of order 1, i.e. \(\rho(\alpha \omega, F) = \alpha \rho(\omega, F)\) for all \(\alpha \geq 0\) and \(F \in \mathbb{R}^k\) (shorter positively 1-homogeneous).
\end{enumerate}

**Remark 4.5.** If we assume \((B0) - (B2), l \in C^1([0, T], L^2(O)^d)\) and \(y_\varepsilon^0 \in S_\varepsilon(0)\), then Theorem A.1 implies that there exists a unique energetic solution \(y_\varepsilon \in C^{Lip}([0, T], Y_\varepsilon)\) to the ERIS associated with \((\mathcal{E}_\varepsilon, \Psi_\varepsilon)\) with

\[
y_\varepsilon(0) = y_\varepsilon^0, \ \text{and moreover} \ \|y_\varepsilon(t) - y_\varepsilon(s)\|_{Y_\varepsilon} \leq C|t - s| \ \text{for all } t, s \in [0, T].
\]

The passage to the limit model (as \(\varepsilon \to 0\)) is conducted in the setting of evolutionary \(\Gamma\)-convergence [31] and it involves a discrete-to-continuum transition. The homogenized model as well is described by an ERIS:

- The state space is given by \(Y = H_0^1(O)^d \times L^2(\Omega \times O)^k \times (L^2_\text{sym}(\Omega) \otimes L^2(O))^d\) and we denote the state variable by \(y = (U, Z, \chi)\).
The energy functional is
\[
\mathcal{E}_0 : [0, T] \times Y \to \mathbb{R}, \quad \mathcal{E}_0(t, y) = \frac{1}{2} \langle Ay, y \rangle_{Y^*, Y} - \int_O l(t) \cdot U dx,
\]
\[
\langle Ay, y \rangle_{Y^*, Y} = \int_O \left\langle A(\omega) \left( \nabla_s U(x) + \chi_s(\omega, x) \right), \left( \frac{\nabla_s U(x) + \chi_s(\omega, x)}{Z(\omega, x)} \right) \right\rangle dx.
\]

The limit dissipation functional is given by
\[
\Psi_0 : Y \to [0, \infty], \quad \Psi_0(y) = \int_O \langle \rho(\omega, Z(\omega, x)) \rangle dx.
\]

\(S(t)\) denotes the set of stable states w.r.t. \((\mathcal{E}_0, \Psi_0)\) at time \(t\).

**Remark 4.6.** If we assume \((B0)-(B2)\), \(l \in C^1([0, T], L^2(O)^d)\) and \(y^0 \in S(0)\), then the assumptions of Theorem A.1 are satisfied (see Lemma 4.4) and therefore there exists a unique energetic solution \(y \in C^{L^{rp}}([0, T], Y)\) to the ERIS associated with \((\mathcal{E}_0, \Psi_0)\) with
\[
y(0) = y^0, \text{ and moreover } \|y(t) - y(s)\|_Y \leq C|t - s| \quad \text{for all } t, s \in [0, T].
\]

For notational convenience we introduce the following abbreviation: for \(y_\varepsilon \in Y_\varepsilon\) and \(y \in Y\)
\[
y_\varepsilon \overset{c^2}{\to} y \iff u_\varepsilon \overset{2}{\to} U, \nabla^s u_\varepsilon \overset{2}{\to} \nabla U + \chi \text{ and } z_\varepsilon \overset{2}{\to} Z \quad \text{(in the corresponding } L^2\text{-spaces)}.\]

Also, we use \(c^2\) if the above quantities strongly two-scale converge. The "c" in this shorthand refers to "cross" convergence as in the periodic case. The proof of the following homogenization theorem follows closely the strategy developed in [32]. In that paper, the periodic unfolding method is applied to a similar (continuum) problem with periodic coefficients.

**Theorem 4.3.** Assume \((B0)-(B2)\), \(\langle \cdot \rangle\) is ergodic, \(l \in C^1([0, T], L^2(O)^d)\) and \(y^0_\varepsilon \in S_\varepsilon(0)\) with
\[
y^0_\varepsilon \overset{c^2}{\to} y_0 \in Y.
\]

Let \(y_\varepsilon \in C^{L^{rp}}([0, T], Y_\varepsilon)\) be the unique energetic solution associated with \((\mathcal{E}_\varepsilon, \Psi_\varepsilon)\) and \(y_\varepsilon(0) = y^0_\varepsilon\). Then
\[
y_0 \in S(0) \text{ and for every } t \in [0, T] : \quad y_\varepsilon(t) \overset{c^2}{\to} y(t),
\]
where \(y \in C^{L^{rp}}([0, T], Y)\) is the unique energetic solution associated with \((\mathcal{E}_0, \Psi_0)\) and \(y(0) = y^0\).

**Remark 4.7.** We remark that the above result holds true in the case that \(\langle \cdot \rangle\) is not ergodic (with minor changes in the proof) with a modified state space for the continuum model, specifically
\[
Y = (L^2_{inv}(\Omega) \otimes H^1_0(O))^d \times L^2(\Omega \times O)^k \times (L^2_{pot}(\Omega) \otimes L^2(O))^d.
\]

### 4.4 Gradient plasticity

The limit rate-independent system from the previous section cannot be equivalently recast as a rate-independent system with deterministic properties as in the case of convex minimization (Theorem 4.2). The reason for this is that the limiting internal variable \(Z\) is in general not deterministic. The microscopic problem might be regularized by adding a gradient term of the internal variable \(z_\varepsilon\) and in that way homogenization yields a deterministic limit problem. This strategy was demonstrated in [18], where periodic homogenization of gradient plasticity in the continuum setting is discussed. In the following, we show that the same applies in our stochastic, discrete-to-continuum setting.
Let $\gamma (0,1)$. The new microscopic system involves the same dissipation potential $\Psi_\varepsilon$ as before, as well as the same state space $Y_\varepsilon$, yet now equipped with the scalar product

$$
\langle y_1, y_2 \rangle_{Y_\varepsilon} = \int_{\mathbb{R}^d} u_1(\omega, x) \cdot u_2(\omega, x) dm_\varepsilon(x) + \int_{\mathbb{R}^d} \nabla^\varepsilon u_1(\omega, x) : \nabla^\varepsilon u_2(\omega, x) dm_\varepsilon(x)
$$

$$
+ \int_{\mathbb{R}^d} z_1(\omega, x) \cdot z_2(\omega, x) dm_\varepsilon(x) + \int_{\mathbb{R}^d} \varepsilon^2 \nabla^\varepsilon z_1(\omega, x) : \nabla^\varepsilon z_2(\omega, x) dm_\varepsilon(x).
$$

We consider a modified energy functional $E_\varepsilon: [0,T] \times \Omega \to \mathbb{R}$

$$
E_\varepsilon(t, y_\varepsilon) = E_\varepsilon(t, y_\varepsilon) + \int_{\mathbb{R}^d} G(T_\varepsilon^\varepsilon \varepsilon^\gamma \nabla^\varepsilon z_\varepsilon(\omega, x) : \varepsilon^2 \nabla^\varepsilon z_\varepsilon(\omega, x) dm_\varepsilon(x),
$$

where $G: \Omega \to \mathbb{R}^{k \times d}$. We assume the following:

(B3) $G \in L^\infty(\Omega, \text{Lin}(\mathbb{R}^{k \times d}, \mathbb{R}^{k \times d}))$ and it satisfies: there exists $C > 0$ such that $G(\omega) F_1 : F_1 \geq C |F_1|^2$ and $G(\omega) F_1 : F_2 = F_1 : G(\omega) F_2$ for P-a.e. $\omega \in \Omega$ and all $F_1, F_2 \in \mathbb{R}^{k \times d}$.

The set of stable states at time $t$ corresponding to $(E_\varepsilon, \Psi_\varepsilon)$ is denoted by $S_\varepsilon(t)$.

**Remark 4.8.** If we assume (B0) - (B3), $l \in C^1([0,T], L^2(\mathbb{R}^d)^d)$ and $y_0^0 \in S_\varepsilon(0)$, then there exists a unique energetic solution $y_\varepsilon \in C^1([0,T], Y_\varepsilon)$ to the ERIS associated with $(E_\varepsilon, \Psi_\varepsilon)$ with $y_\varepsilon(0) = y_0^0$ and additionally for all $t, s \in [0,T]$

$$
\|y_\varepsilon(t) - y_\varepsilon(s)\|_{Y_\varepsilon} \leq C|t-s|.
$$

In the limit $\varepsilon \to 0$, we obtain a deterministic rate-independent system described as follows.

- The state space is $Q = H^1(\Omega)^d \times L^2(\Omega)^k$ and the state variable is denoted by $q = (U, Z)$.
- The energy functional is $E_{\text{hom}}: [0,T] \times Q \to \mathbb{R}$

$$
E_{\text{hom}}(t, q) = \int_O V_{\text{hom}}(\nabla_s U, Z) dx - \int_O l(t) \cdot U dx,
$$

where $V_{\text{hom}}$ is given by the corrector problem: For $F_1, F_2 \in \mathbb{R}^k$,

$$
V_{\text{hom}}(F_1, F_2) = \inf_{\chi \in \mathbb{L}_{\text{sym}}^2(\Omega)^d} \left\{ A(\omega) \left( \begin{array}{cc} F_1 + \chi_s(\omega) \\ F_2 \\ \end{array} \right) \cdot \left( \begin{array}{cc} F_1 + \chi_s(\omega) \\ F_2 \\ \end{array} \right) \right\}.
$$

In fact, it can be shown that $V_{\text{hom}}$ is quadratic: There exists $A_{\text{hom}} \in \mathbb{R}^{2k \times 2k}_{\text{sym}}$ positive-definite such that $V_{\text{hom}}(F_1, F_2) = A_{\text{hom}}(F_1) \cdot (F_2)$ for all $F_1, F_2 \in \mathbb{R}^k$.
- The dissipation potential is given by $\Psi_{\text{hom}}: Q \to [0, \infty)$

$$
\Psi_{\text{hom}}(q) = \int_O \langle \rho(\omega, Z(x)) \rangle dx.
$$

The set of stable states at time $t$ corresponding to the functionals $(E_{\text{hom}}, \Psi_{\text{hom}})$ is denoted by $S_{\text{hom}}(t)$.

**Remark 4.9.** If we assume (B0) - (B3), $l \in C^1([0,T], L^2(\mathbb{R}^d)^d)$ and $q_0^0 \in S_{\text{hom}}(0)$, then there exists a unique energetic solution $q \in C^1([0,T], Q)$ to the ERIS associated with $(E_{\text{hom}}, \Psi_{\text{hom}})$ with $q(0) = q_0^0$.

**Theorem 4.4.** Assume (B0)-(B3), $(\cdot)$ is ergodic. Let $l \in C^1([0,T], L^2(\mathbb{R}^d)^d)$, $y_0^0 \in S_\varepsilon(0)$, $q_0 \in Q$, $\chi \in (L_{\text{pot}}^2(\Omega) \otimes L^2(\Omega))^d$ satisfy

$$
y_0^0 \xrightarrow{C^2} (q_0, \chi), \; \varepsilon^2 \nabla^\varepsilon z_0^0 \xrightarrow{L^2} 0 \text{ in } L^2(\Omega \times \mathbb{R}^d)^{k \times d}, \; E_{\text{hom}}(0, q_0) = E_0(0, (q_0, \chi)).
$$
Let \( y_e \in C^{\text{Lip}}([0, T], Y_e) \) be the unique energetic solution associated with \((E^\gamma, \Psi_e)\) and \( y_e(0) = y_0 \). Then \( q_0 \in S_{\text{hom}}(0) \) and for every \( t \in [0, T] \):

\[
u_e(t) \xrightarrow{z} U(t) \text{ in } L^2(\Omega \times \mathbb{R}^d)^d, \quad z_e(t) \xrightarrow{z} Z(t) \text{ in } L^2(\Omega \times \mathbb{R}^d)^k,
\]

where \( q = (U, Z) \in C^{\text{Lip}}([0, T], Q) \) is the unique energetic solution associated with \((E_{\text{hom}}, \Psi_{\text{hom}})\) and \( q(0) = q_0 \).

### 4.5 Proofs

Before presenting the proofs, we consider a couple of auxiliary lemmas.

**Lemma 4.1** (Two-scale convergence of symmetrized gradients). Let \( u_e \in L^p(\Omega \times \varepsilon \mathbb{Z}^d)^d \) and \( F \in L^p(\Omega \times \mathbb{R}^d)^d \times d \) satisfy \( \nabla \varepsilon u_e \xrightarrow{\text{Lip}} F \) in \( L^p(\Omega \times \mathbb{R}^d)^d \). Then

\[
\nabla \varepsilon u_e \xrightarrow{\text{Lip}} F_s \text{ in } L^p(\Omega \times \mathbb{R}^d)^k.
\]

If we have strong two-scale convergence for \( \nabla \varepsilon u_e \), strong two-scale convergence for \( \nabla \varepsilon u_e \) follows.

**Proof.** For any \( i \in \{1, \ldots, k\} \) we compute

\[
T_{\varepsilon} (\nabla \varepsilon u_e)_i(\omega, x) = \frac{b_i}{|b_i|} \cdot T_{\varepsilon} \sum_{y \in \mathbb{Z}^d} \nabla \varepsilon u_e(\omega, x - \varepsilon y) B_i(y) = \frac{b_i}{|b_i|} \cdot \sum_{y \in \mathbb{Z}^d} T_{\varepsilon} \nabla \varepsilon u_e(T_{-y} \omega, x - \varepsilon y) B_i(y).
\]

For any fixed \( y \in \mathbb{Z}^d \), the function \( (\omega, x) \mapsto T_{\varepsilon} \nabla \varepsilon u_e(T_{-y} \omega, x - \varepsilon y) B_i(y) \) weakly converges to \( (\omega, x) \mapsto F(T_{-y} \omega, x) B_i(y) \). If we assume strong two-scale convergence for the gradient, the previous quantities converge in the strong sense. Using this and the fact that \( B_i(y) = 0 \) for all but finitely many \( y \in \mathbb{Z}^d \), the claim follows.

**Lemma 4.2** (Stochastic Korn’s inequality). Recall that it is assumed that \((\mathbb{Z}^d, E)\) satisfies (Korn) (see Section 4.1). There exists \( C > 0 \) such that

\[
\langle |\chi|^p \rangle \leq C \langle |\chi_s|^p \rangle \quad \text{for all } \chi \in \mathbf{L}^p_{\text{pot}}(\Omega)^d.
\]

**Proof.** We show the inequality in the case \( \chi = D\varphi \) for \( \varphi \in \mathbf{L}^p(\Omega)^d \). For general \( \chi \in \mathbf{L}^p_{\text{pot}}(\Omega)^d \), it is obtained by an approximation argument. We denote by \( \tilde{\varphi} : \Omega \times \mathbb{Z}^d \to \mathbb{R}^d \) the stationary extension of \( \varphi \), i.e. \( \tilde{\varphi}(\omega, x) = \varphi(T_{\omega} \omega) \).

Let \( R > 0 \) and \( K > 0 \) such that \( K > \sup \{|b_i| : b_i \in E_0\} \). Let \( \eta_R \) be a cut-off function given by \( \eta_R : \mathbb{Z}^d \to \mathbb{R} \) with \( \eta_R = 1 \) in \( B_{R+K} \cap \mathbb{Z}^d \) and \( \eta_R = 0 \) otherwise (\( B_R \subset \mathbb{R}^d \) is a ball of radius \( R \) with center in \( 0 \)). Using the properties of \( \eta_R \) and the discrete Korn’s inequality, we obtain

\[
\left\langle \int_{B_R \cap \mathbb{Z}^d} |D\varphi(T_{\omega} \omega)|^p dm(x) \right\rangle \leq \left\langle \frac{1}{|B_R|} \int_{\mathbb{Z}^d} |D\varphi(T_{\omega} \omega)|^p dm(x) \right\rangle
\]

\[
\leq \left\langle \frac{C}{|B_R|} \int_{\mathbb{Z}^d} |\nabla s(\tilde{\varphi}(\omega, x) \eta_R(x))|^p dm(x) \right\rangle
\]

\[
= \left\langle \frac{C}{|B_R|} \int_{B_R \cap \mathbb{Z}^d} |\nabla s(\tilde{\varphi}(\omega, x))|^p dm(x) \right\rangle
\]

\[
+ \left\langle \frac{C}{|B_R|} \int_{(B_{R+2K} \setminus B_R) \cap \mathbb{Z}^d} |\nabla s(\tilde{\varphi}(\omega, x)) \eta_R(x)|^p dm(x) \right\rangle.
\]

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By invariance of \( P \) the left-hand side of the above inequality equals \( \langle |D\varphi|^p \rangle \) for any \( R \). Moreover, the first term on the right-hand side equals \( C \langle |D_x\varphi|^p \rangle \). Therefore, it is sufficient to show that the second term vanishes in the limit \( R \to \infty \). To obtain that, we estimate (for \( P \)-a.e. \( \omega \in \Omega \))

\[
\frac{1}{|B_R|} \int_{(B_{R+2K} \setminus B_R) \cap \mathbb{Z}^d} |\nabla_s (\tilde{\varphi}(\omega, x) \eta_R(x))|^p dm(x) \\
\leq \frac{C}{|B_R|} \int_{(B_{R+2K} \setminus B_{R-K}) \cap \mathbb{Z}^d} |\tilde{\varphi}(\omega, x)|^p |\eta_R(x)|^p dm(x) \\
\leq \frac{C}{|B_R|} \int_{(B_{R+2K} \setminus B_{R-K}) \cap \mathbb{Z}^d} |\tilde{\varphi}(\omega, x)|^p dm(x) \\
= \frac{C}{|B_R|} \int_{B_{R+2K} \cap \mathbb{Z}^d} |\tilde{\varphi}(\omega, x)|^p dm(x) - \frac{C|B_{R-K}|}{|B_R|} \int_{B_{R-K} \cap \mathbb{Z}^d} |\tilde{\varphi}(\omega, x)|^p dm(x).
\]

(21)

In the first inequality above, we used the fact that \( \nabla_s : L^p(\varepsilon \mathbb{Z}^d)^d \to L^p(\varepsilon \mathbb{Z}^d)^k \) is a bounded operator. An integration of (21) over \( \Omega \) yields

\[
\left\langle \frac{C}{|B_R|} \int_{(B_{R+2K} \setminus B_R) \cap \mathbb{Z}^d} |\nabla_s (\tilde{\varphi}(\omega, x) \eta_R(x))|^p dm(x) \right\rangle \leq \langle |\varphi|^p \rangle \left( \frac{C|B_{R+2K}|}{|B_R|} - \frac{C|B_{R-K}|}{|B_R|} \right) \overset{R \to \infty}{\longrightarrow} 0.
\]

This concludes the proof.

The above result and the direct method of calculus of variations imply the following.

**Lemma 4.3.** For any \( F \in \mathbb{R}^{d \times d} \), there exists \( \chi \in \mathbf{L}^p_{\text{pot}}(\Omega)^d \) which attains the infimum in (18) in Section 4.2.

**Proof of Theorem 4.1.**

(i) The growth conditions of \( V \), Korn property of the lattice and a discrete Poincaré inequality imply

\[
\lim \sup_{\varepsilon \to 0} \left\langle \int_{\varepsilon \mathbb{Z}^d} |u_\varepsilon(x)|^p + |\nabla^\varepsilon u_\varepsilon(x)|^p dm_\varepsilon(x) \right\rangle < \infty.
\]

Therefore, the claim follows from Proposition 3.1 (i) and Corollary 3.1.

(ii) The claim directly follows from Lemma 4.1 and Proposition 3.3 (i).

(iii) The claim follows from Corollary 3.3 (iii), Lemma 4.1 and Proposition 3.3 (ii).

**Proof of Theorem 4.2**

(i) By Theorem 4.1 there exist \( U \in W^{1,p}_0(\Omega)^d \) and \( \chi \in (\mathbf{L}^p_{\text{pot}}(\Omega) \otimes \mathbf{L}^p(\Omega))^d \) and a two-scale convergent subsequence such that

\[
\lim \inf_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{E}_0(U, \chi) \geq \mathcal{E}_{\text{hom}}(U).
\]

The corresponding convergence for \( u_\varepsilon \) follows from Corollary 3.2.

(ii) It is sufficient to show that for \( U \in W^{1,p}_0(\Omega) \), there exists \( \chi \in (\mathbf{L}^p_{\text{pot}}(\Omega) \otimes \mathbf{L}^p(\Omega))^d \) such that

\[
\mathcal{E}_0(U, \chi) = \mathcal{E}_{\text{hom}}(U).
\]
Indeed, if this holds, Theorem 4.1 (iii) implies that there exists \( u_\varepsilon \in \left( L^p(\Omega) \otimes L^p_0(O \cap \varepsilon \mathbb{Z}^d) \right)^d \) such that

\[
\begin{align*}
\varepsilon & \to u \text{ in } L^p(\Omega \times \mathbb{R}^d)^d, \\
\varepsilon \varepsilon u_\varepsilon & \to \nabla U + \chi \text{ in } L^p(\Omega \times \mathbb{R}^d)^{d \times d}, \\
\lim_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon) &= \mathcal{E}_0(U, \chi) = \mathcal{E}_{\text{hom}}(U)
\end{align*}
\]

and the corresponding convergence for \( \langle u_\varepsilon \rangle \) and \( \langle \nabla^\varepsilon u_\varepsilon \rangle \) follows by Corollary 3.2 and Lemma 2.1.

To show the above claim, we note that \( \nabla U \in L^p(O)^{d \times d} \) and consider a sequence of piecewise constant functions \( F_n(x) = \sum_{i=1}^{k(n)} 1_{O_i}(x) F_i \) where \( F_i \in \mathbb{R}^{d \times d} \) and such that

\[
F_n \to \nabla U \text{ strongly in } L^p(O)^{d \times d}.
\]

For each \( F_i \), there exists a \( \chi_i^0 \in L^p_{\text{pot}}(\Omega)^d \) such that \( \mathcal{V}_{\text{hom}}(F_i) = \langle V(\omega, (F_i)_s + (\chi_i^0)_s(\omega)) \rangle \). We define \( \chi_n(x, \omega) = \sum_{i=1}^{k(n)} 1_{O_i}(x) \chi_i^0(\omega) \). Note that \( \chi_n \in (L^p_{\text{pot}}(\Omega) \otimes L^p(O))^d \) and with the help of the growth assumptions (A3) and the stochastic Korn’s inequality (Lemma 4.2), we obtain

\[
\limsup_{n \to \infty} \left( \int_O |\chi_n(x, \omega)|^p \, dx \right) < \infty.
\]

As a result of this, there exist a subsequence \( n' \) and \( \chi \in (L^p_{\text{pot}}(\Omega) \otimes L^p(O))^d \) such that

\[
\chi_{n'} \rightharpoonup \chi \text{ weakly in } (L^p_{\text{pot}}(\Omega) \otimes L^p(O))^d.
\]

Furthermore, we have

\[
\left\langle \int_O V(\omega, \nabla s U(x) + \chi_s(\omega, x)) \, dx \right\rangle \leq \liminf_{n' \to \infty} \int_O \langle V(\omega, (F_{n'})_s(x) + (\chi_{n'})_s(\omega, x)) \rangle \, dx
\]

\[
= \liminf_{n' \to \infty} \int_O \mathcal{V}_{\text{hom}}(F_{n'}(x)) \, dx = \int_O \mathcal{V}_{\text{hom}}(\nabla U(x)) \, dx.
\]

Above, in the first inequality we use weak lower-semicontinuity of the functional \( G \in L^p(\Omega \times O)^{d \times d} \rightarrow \langle \int_O V(\omega, G(\omega, x)) \, dx \rangle \) and the facts that \( F_{n'} \to \nabla U, \, \chi_{n'} \rightharpoonup \chi \) and that \( (\cdot)_s \) is a linear and bounded operator. In order to justify the last equality, we remark that \( \mathcal{V}_{\text{hom}} \) is convex (and therefore continuous) and satisfies the growth assumptions \( -C - C|F|^p \leq \mathcal{V}_{\text{hom}}(F) \leq C|F|^p + C \) which implies that the functional \( G \in L^p(O)^{d \times d} \rightarrow \int_O \mathcal{V}_{\text{hom}}(G(x)) \, dx \) is strongly continuous. On the other hand, it is easy to see that \( \int_O \mathcal{V}_{\text{hom}}(\nabla U(x)) \, dx \leq \langle \int_O V(\omega, \nabla s U(x) + \chi_s(\omega, x)) \rangle \). This concludes the proof.

**Proof of Proposition 4.1**

*Proof.* The uniqueness of the minimizer in (17) follows by uniform convexity assumption on the integrand \( V \). Theorem 4.2 implies that (up to a subsequence) \( u_\varepsilon \rightharpoonup u \) in \( L^p(\Omega \times \mathbb{R}^d)^d \), where \( U \) is a minimizer of \( \mathcal{I}_{\text{hom}} \). As in the proof of Theorem 4.2, we find \( \chi \in (L^p(\Omega) \otimes L^p(O))^d \) with \( \mathcal{I}_{\text{hom}}(U) = \mathcal{E}_0(U, \chi) + \int_O f \cdot U \, dx \). By Theorem 4.1, there exists a strong two-scale recovery sequence \( v_\varepsilon \in (L^p(\Omega) \otimes L^p_0(O \cap \varepsilon \mathbb{Z}^d))^d \) for \( (U, \chi) \). We have

\[
\left\langle \int_{\mathbb{R}^d} |T_\varepsilon u_\varepsilon(\omega, x) - U(x)|^p \, dx \right\rangle 
\leq C \left( \left\langle \int_{\mathbb{R}^d} |T_\varepsilon u_\varepsilon(\omega, x) - T_\varepsilon v_\varepsilon(\omega, x)|^p \, dx \right\rangle + \left\langle \int_{\mathbb{R}^d} |T_\varepsilon v_\varepsilon(\omega, x) - U(x)|^p \, dx \right\rangle \right).
\]
The second term on the right-hand side vanishes in the limit $\varepsilon \to 0$ by the properties of $v_\varepsilon$. The first term is treated as follows.

$$
\left\langle \int_{\mathbb{R}^d} |T_\varepsilon u_\varepsilon(\omega, x) - T_\varepsilon v_\varepsilon(\omega, x)|^p dx \right\rangle \leq C \left\langle \int_{\mathbb{R}^d} |\nabla_\varepsilon^c u_\varepsilon(\omega, x) - \nabla_\varepsilon^c v_\varepsilon(\omega, x)|^p dm_\varepsilon(x) \right\rangle
\leq C (\mathcal{I}_\varepsilon(v_\varepsilon) - \mathcal{I}_\varepsilon(u_\varepsilon)),
$$

where we use the discrete Poincaré-Korn inequality and the convexity assumption on $V$. It follows that

$$
\limsup_{\varepsilon \to 0} \left\langle \int_{\mathbb{R}^d} |T_\varepsilon u_\varepsilon(\omega, x) - T_\varepsilon v_\varepsilon(\omega, x)|^p dx \right\rangle \leq C \left( \limsup_{\varepsilon \to 0} \mathcal{I}_\varepsilon(v_\varepsilon) - \liminf_{\varepsilon \to 0} \mathcal{I}_\varepsilon(u_\varepsilon) \right) \leq 0.
$$

This yields the claim for a subsequence. Convergence for the whole sequence follows by a contradiction argument and using the uniqueness of the minimizer $U$.

**Lemma 4.4.** Let $A$ and $Y$ be as in Section 4.3 (with the assumptions as in Remark 4.6). There exists $C > 0$ such that $\langle Ay, y \rangle_{Y^*, Y} \geq C\|y\|_Y^2$ for all $y \in Y$.

**Proof.** First, we have $\langle Ay, y \rangle_{Y^*, Y} \geq C \left( \int_{O} |\nabla_\varepsilon U(\omega, x) + \chi_\varepsilon(\omega, x)|^2 + |Z(\omega, x)|^2 \right)$. Note that, $\nabla U$ does not depend on $\omega$ and therefore $\int_{O} \langle \nabla_\varepsilon U(x) \cdot \chi_\varepsilon(\omega, x) dx \rangle = 0$. This implies

$$
\left\langle \int_{O} |\nabla_\varepsilon U(x) + \chi_\varepsilon(\omega, x)|^2 dx \right\rangle = \int_{O} |\nabla_\varepsilon U(x)|^2 dx + \int_{O} |\chi_\varepsilon(\omega, x)|^2 \right\rangle dx.
$$

Using the continuum Korn’s inequality (16) (Section 4.1) and the stochastic Korn’s inequality Lemma 4.2, we conclude the proof.

**Proof of Theorem 4.3.** First, we prove as an auxiliary result, the existence of joint recovery sequences, which implies the stability of two-scale limits of solutions.

**Lemma 4.5.** Let $t \in [0, T]$ and $y_\varepsilon \in S_\varepsilon(t)$ such that $y_\varepsilon \overset{c_l}{\to} y \in Y$. For any $\tilde{y} \in Y$ there exists $\tilde{y}_\varepsilon \in Y_\varepsilon$ such that $\tilde{y}_\varepsilon \overset{c_l}{\to} \tilde{y}$ and

$$
\lim_{\varepsilon \to 0} \{ E_\varepsilon(t, \tilde{y}_\varepsilon) + \Psi_\varepsilon(\tilde{y}_\varepsilon - y_\varepsilon) - E_\varepsilon(t, y_\varepsilon) \} = E_0(t, \tilde{y}) + \Psi_0(\tilde{y} - y) - E_0(t, y).
$$

This implies $y \in S(t)$.

**Proof.** Corollary 3.3 (i) implies that there exists a sequence $v_\varepsilon \in (L^2(\Omega) \otimes L^2_0(\Omega^c \cap \varepsilon \mathbb{Z}^d))^d$ with

$$
v_\varepsilon \overset{2}{\to} \tilde{U} - U \text{ in } L^2(\Omega \times \mathbb{R}^d)^d, \quad \nabla_\varepsilon v_\varepsilon \overset{2}{\to} \nabla_\varepsilon \tilde{U} - \nabla U + \tilde{\chi} - \chi \text{ in } L^2(\Omega \times \mathbb{R}^d)^{d \times d}.
$$

The sequence $g_\varepsilon \in (L^2(\Omega) \otimes L^2_0(\Omega^c \cap \varepsilon \mathbb{Z}^d))^k$, given by $g_\varepsilon = 1_{O^c} \mathcal{F}_\varepsilon(\tilde{Z} - Z)$, satisfies

$$
g_\varepsilon \overset{2}{\to} \tilde{Z} - Z \text{ in } L^2(\Omega \times \mathbb{R}^d)^k.
$$

We define $\tilde{y}_\varepsilon$ componentwise: $\tilde{u}_\varepsilon = u_\varepsilon + v_\varepsilon$ and $\tilde{z}_\varepsilon = z_\varepsilon + g_\varepsilon$. By weak two-scale convergence of $y_\varepsilon$, we have that $\tilde{y}_\varepsilon \overset{c_l}{\to} \tilde{y}$, and furthermore $\tilde{y}_\varepsilon - y_\varepsilon \overset{c_l}{\to} \tilde{y} - y$. The energy functional is quadratic and thus it satisfies

$$
E_\varepsilon(t, \tilde{y}_\varepsilon) - E_\varepsilon(t, y_\varepsilon) = \frac{1}{2} \left[ \int_{O^c} A(T^\varepsilon_\omega) \left( \nabla_\varepsilon^c (\tilde{u}_\varepsilon - u_\varepsilon)(\omega, x) \cdot \nabla_\varepsilon^c (\tilde{z}_\varepsilon - z_\varepsilon)(\omega, x) \right) dm_\varepsilon(x) \right]
\leq \int_{O^c} \pi_\varepsilon l(t)(x) \cdot (\tilde{u}_\varepsilon - u_\varepsilon)(\omega, x) dm_\varepsilon(x).
$$

(22)
Moreover, by boundedness of Corollary 3.1, the limit may be identified with an
L
if
\lim_{t \to 0} \left\langle \int_{\Omega} \pi \cdot \nabla \right( y \cdot u + \chi_s \cdot \chi_s \cdot \chi_s \cdot \chi_s \right) \right\rangle dx.

This expression is a scalar product of strongly and weakly convergent sequences (see Lemma 4.1), and therefore it converges to (as \(\varepsilon \to 0\))

\[
\frac{1}{2} \left\langle \int_{\mathbb{R}^d} A(\omega) \left( \nabla \bar{u} - \nabla u + \bar{\chi} - \chi \right) \cdot \left( \nabla \bar{U} + \nabla U + \bar{\chi} + \chi \right) \right\rangle dx.
\]

Similarly, the second term on the right-hand side of (22) converges to \(- \left\langle f_\omega^l(t) \cdot (\bar{U} - U) \right\rangle\).

Furthermore, by Jensen’s inequality we obtain

\[
\Psi_\varepsilon(\bar{y} - y) \leq \left\langle \int_{\mathbb{Z}^d} \rho(x, \pi \cdot (\bar{z} - Z)(x)) dm_\varepsilon(x) \right\rangle 
\leq \left\langle \int_{\mathbb{Z}^d} \int_{x + \varepsilon \mathbb{Z}^d} \rho(\omega, Z(\omega, y)) dy dm_\varepsilon(x) \right\rangle 
= \Psi_0(\bar{y} - y).
\]

On the other hand, using Fatou’s lemma and the fact that \(\rho(\omega, \cdot)\) is continuous, we obtain

\[\lim_{\varepsilon \to 0} \Psi_\varepsilon(\bar{y} - y) \geq \Psi_\varepsilon(\bar{y} - y)\]

This concludes the proof. \(\square\)

**Proof of Theorem 4.3.** We consider the sequence \(v_\varepsilon := (T_\varepsilon u_\varepsilon, T_\varepsilon \nabla u_\varepsilon, T_\varepsilon z_\varepsilon) : [0, T] \to L^2(\Omega \times \mathbb{R}^d, \mathbb{R}^d) \times L^2(\Omega \times \mathbb{R}^d, \mathbb{R}^d) \times L^2(\Omega \times \mathbb{R}^d, \mathbb{R}^d)\): \(H\). By Remark 4.5, \(v_\varepsilon\) is uniformly bounded in \(C^{1,p}(0, T, H)\). Therefore, the Arzelà-Ascoli theorem implies that there exist \(v \in C^{1,p}(0, T, H)\) and a subsequence (not relabeled), such that for every \(t \in [0, T]\)

\[v_\varepsilon(t) \rightharpoonup v(t) \text{ weakly in } H.
\]

Moreover, by boundedness of \(y_\varepsilon(t)\) and the above, we conclude that for every \(t \in [0, T]\), \(v(t) = (U(t), \nabla U(t) + \chi(t), Z(t))\), for some \(y(t) = (U(t), Z(t), \chi(t)) \in Y\). Here we use the fact that if \(z_\varepsilon \in L^2(\Omega) \otimes L^2(\Omega) \cap \varepsilon \mathbb{Z}^d\) converges in the weak two-scale sense, then, similarly as in Corollary 3.1, the limit may be identified with an \(L^2(\Omega \times O)\) function. In other words, we have

\[y_\varepsilon(t) \rightharpoonup y(t)\].\] Lemma 4.5 implies that \(y(t) \in S(t)\) for every \(t \in [0, T]\). We show that \(y(t)\) satisfies

\[
\mathcal{E}_0(t, y(t)) + \int_0^t \Psi_\varepsilon(y(s)) ds \leq \mathcal{E}_0(0, y(0)) - \int_0^t \int_O \pi \cdot U(s) ds.
\]

The other inequality in (EB) can be shown using the stability of \(y\) (see [27, Section 2.3.1]). The (EB) of the discrete system reads

\[
\frac{1}{2} \left\langle \hat{A}_\varepsilon y_\varepsilon(t) - y_\varepsilon(t) \right\rangle_{Y^*} - \left\langle \int_{O \cap \varepsilon \mathbb{Z}^d} \pi \cdot l(t) \cdot u_\varepsilon(t) \right\rangle_{Y^*} + \int_0^t \Psi_\varepsilon(y_\varepsilon(s)) ds
\]
The strong convergence of the initial data implies that the first two terms on the right-hand side converge to $\mathcal{E}(0, y(0))$. The remaining term converges to $-\int_0^t \int_{\Omega} i(s) \cdot \nabla u(s) \, ds$ by the Lebesgue dominated convergence theorem. Moreover, using Proposition 3.3 and the strong convergence of $\pi_\epsilon l(t)$ we obtain

$$\liminf_{\epsilon \to 0} \left( \frac{1}{2} \langle A_\epsilon y_\epsilon(t), y_\epsilon(t) \rangle_{Y^*_\epsilon, Y_\epsilon} - \left( \int_{\Omega \times \mathbb{R}^d} \pi_\epsilon l(t)(x) \cdot u_\epsilon(t)(\omega, x) \, dm_\epsilon(x) \right) \right) \geq \mathcal{E}_0(t, y(t)).$$

To treat the last term on the left-hand side of (24), we consider a partition \( \{t_i\} \) of \([0, t]\). We have

$$\sum_i \Psi_0(y(t_i)) \leq \liminf_{\epsilon \to 0} \sum_i \Psi_\epsilon(y_\epsilon(t_i) - y_\epsilon(t_{i-1})).$$

Taking the supremum over all partitions \( \{t_i\} \) of \([0, t]\) and exploiting the homogeneity of $\Psi_0$, we obtain

$$\int_0^t \Psi_0(\dot{y}(s)) \, ds \leq \liminf_{\epsilon \to 0} \int_0^t \Psi_\epsilon(\dot{y}_\epsilon(s)) \, ds. \quad (25)$$

This proves (23).

To obtain strong two-scale convergence, we construct a strong recovery sequence $\tilde{y}_\epsilon(t) \in Y_\epsilon$ for $y(t) \in Y$ (for every $t \in [0, T]$) in the sense that

$$\tilde{y}_\epsilon(t) \overset{\text{strong}}{\to} y(t),$$

(c.f. proof of Lemma 4.5). For notational convenience, we drop the "t" from the sequences and we denote $v_\epsilon := (T_\epsilon u_\epsilon, T_\epsilon \nabla \tilde{u}_\epsilon, T_\epsilon \tilde{z}_\epsilon), \tilde{v}_\epsilon := (T_\epsilon \tilde{u}_\epsilon, T_\epsilon \nabla \tilde{u}_\epsilon, T_\epsilon \tilde{z}_\epsilon)$ and $V := (U, \nabla U + \chi, Z)$. By the triangle inequality,

$$\|v_\epsilon - V\|_H \leq \|v_\epsilon - \tilde{v}_\epsilon\|_H + \|	ilde{v}_\epsilon - V\|_H. \quad (26)$$

The second term on the right-hand side vanishes in the limit $\epsilon \to 0$. Also, since the energy is quadratic,

$$\|v_\epsilon - \tilde{v}_\epsilon\|^2_H \leq C \left( \mathcal{E}_\epsilon(t, y_\epsilon) - \mathcal{E}(t, \tilde{y}_\epsilon) + \langle A_\epsilon \tilde{y}_\epsilon, \tilde{y}_\epsilon - y_\epsilon \rangle_{Y^*_\epsilon, Y_\epsilon} \right. \left. + \left( \int_{\Omega \times \mathbb{R}^d} \pi_\epsilon l(x) \cdot (u_\epsilon - \tilde{u}_\epsilon)(\omega, x) \, dm_\epsilon(x) \right) \right).$$

The last two terms on the right-hand side vanish as $\epsilon \to 0$ (c.f. proof of Lemma 4.5). The first two terms are treated as follows. By the first part of the proof we obtain

$$\lim_{\epsilon \to 0} \left( \mathcal{E}_\epsilon(t, y_\epsilon) + \int_0^T \Psi_\epsilon(\dot{y}_\epsilon(s)) \, ds \right) = \mathcal{E}_0(t, y) + \int_0^T \Psi_0(\dot{y}(s)) \, ds.$$

As a result of this, $\limsup_{\epsilon \to 0} \mathcal{E}_\epsilon(t, y_\epsilon) + \liminf_{\epsilon \to 0} \int_0^T \Psi_\epsilon(\dot{y}_\epsilon(s)) \, ds = \mathcal{E}_0(t, y) + \int_0^T \Psi_0(\dot{y}(s)) \, ds$ and using (25)

$$\limsup_{\epsilon \to 0} (\mathcal{E}_\epsilon(t, y_\epsilon) - \mathcal{E}_\epsilon(t, \tilde{y}_\epsilon)) \leq \limsup_{\epsilon \to 0} \mathcal{E}_\epsilon(t, y_\epsilon) - \mathcal{E}_0(t, y) \leq 0.$$

This shows that the first two terms on the right-hand side of (26) vanishes in the limit $\epsilon \to 0$ and therefore the claim about strong convergence follows.

To show that the convergence holds for the whole sequence, for a fixed $t \in [0, T]$, we consider $e_\epsilon(t) := \|v_\epsilon(t) - V(t)\|_H$. For any subsequence $\epsilon'$ of $\epsilon$, we can find a further subsequence $\epsilon''$ such that $e_{\epsilon''}(t) \to 0$ by the uniqueness of the solution $y$. From this follows that the whole sequence converges in the sense given in the statement of the theorem. \[ \square \]
**Proof of Theorem 4.4.** The proof of this theorem follows the same strategy and it is very similar to the proof of Theorem 4.3. Therefore, we only sketch the proof, emphasizing the differences to the prior setting.

**Sketch of proof.** Step 1. Compactness.  
To the solution \( y_\varepsilon \) we correspond a sequence \( v_\varepsilon := (T_\varepsilon u_\varepsilon, T_\varepsilon \varepsilon \nabla^e u_\varepsilon, T_\varepsilon \varepsilon \dot{z}_\varepsilon, T_\varepsilon \varepsilon \nabla^e \dot{z}_\varepsilon) : [0,T] \rightarrow L^2(\Omega \times \mathbb{R}^d)_e \times L^2(\Omega \times \mathbb{R}^d)^{d \times d} \times L^2(\Omega \times \mathbb{R}^d)_k \times L^2(\Omega \times \mathbb{R}^d)^{k \times d} =: H. \) With help of Remark 4.8 and Corollary 3.1, analogously as in the proof of Theorem 4.3, we obtain that (up to a subsequence) for every \( t \in [0,T] \)

\[
y_\varepsilon(t) \overset{\varepsilon}{\rightarrow} (q(t),\chi_1(t)), \quad \varepsilon \nabla^e \dot{z}_\varepsilon \overset{\varepsilon}{\rightarrow} \chi_2(t),
\]

where \( q(t) \in Q, \chi_1(t) \in (L^2_{\text{pot}}(\Omega) \otimes L^2(\Omega))^d \) and \( \chi_2(t) \in (L^2_{\text{pot}}(\Omega) \otimes L^2(\Omega))^k \).

Step 2. Stability.  
We fix \( t \in [0,T]. \) For an arbitrary \( \bar{q} \in Q, \) similarly as in the proof of Theorem 4.2 (ii), we can find \( \bar{\chi} \in (L^2_{\text{pot}}(\Omega) \times L^2(\Omega))^d \) such that

\[
E_{\text{hom}}(t,\bar{q}) = E_0(t, (\bar{q}, \bar{\chi})).
\]

Corollary 3.3 (iii) implies that for \((\bar{U} - U(t), \bar{\chi} - \chi_1(t))\) there exists a sequence \( v_\varepsilon \in L^2(\Omega) \times L^2_0(\Omega \cap \varepsilon \mathbb{Z}^d)_e \) such that

\[
v_\varepsilon \overset{\varepsilon}{\rightarrow} \bar{U} - U(t), \quad \nabla^e v_\varepsilon \overset{\varepsilon}{\rightarrow} \nabla \bar{U} - \nabla U(t) + \bar{\chi} - \chi_1(t).
\]

Furthermore, Corollary 3.3 (iv) implies that for \((\bar{Z} - Z(t), -\chi_2(t))\), there exists a sequence \( g_\varepsilon \in L^2(\Omega) \times L^2_0(\Omega \cap \varepsilon \mathbb{Z}^d)_k \) such that

\[
g_\varepsilon \overset{\varepsilon}{\rightarrow} \bar{Z} - Z(t), \quad \varepsilon \nabla^e g_\varepsilon \overset{\varepsilon}{\rightarrow} -\chi_2(t).
\]

We define \( \tilde{y}_\varepsilon \) componentwise: \( \tilde{u}_\varepsilon = u_\varepsilon + v_\varepsilon \) and \( \tilde{z}_\varepsilon = z_\varepsilon + g_\varepsilon. \) Following the steps in the proof of Lemma 4.5 (with the new energy \( E_\varepsilon^\gamma \)), we obtain

\[
\lim_{\varepsilon \rightarrow 0} \{ E_\varepsilon^\gamma(t, \tilde{y}_\varepsilon) + \Psi_\varepsilon(\tilde{y}_\varepsilon - y_\varepsilon) - E_\varepsilon^\gamma(t, y_\varepsilon) \} = E_0(t, (\bar{q}, \bar{\chi})) + \Psi_{\text{hom}}(\bar{q} - q(t)) - E_0(t, (q(t), \chi_1(t))) - \int_O \langle G(\omega) \chi_2(t)(\omega, x) \cdot \chi_2(t)(\omega, x) \rangle \, dx \\
\leq E_{\text{hom}}(t, \bar{q}) + \Psi_{\text{hom}}(\bar{q} - q(t)) - E_{\text{hom}}(t, q(t)).
\]

As a result of this, we obtain \( q(t) \in S_{\text{hom}}(t). \) Another important fact following from this inequality is obtained by setting \( \bar{q} = q(t) \) and using positive 1-homogeneity of \( \Psi_{\text{hom}}. \)

\[
E_0(t, (q(t), \chi_1(t))) + \int_O \langle G(\omega) \chi_2(t)(\omega, x) \cdot \chi_2(t)(\omega, x) \rangle \, dx \leq E_{\text{hom}}(t, q(t)).
\]

As a result of this, we conclude that \( \chi_1(t) \) is the corrector corresponding to \( q(t), \) i.e.

\[
E_{\text{hom}}(t, q(t)) = E_0(t, (q(t), \chi_1(t)))
\]

and moreover we obtain that \( \chi_2 = 0. \)

Step 3. Energy balance.  
The energy balance equality is obtained in the same manner as in the proof of Theorem 4.3 by using the assumptions on the initial data and using that

\[
\liminf_{\varepsilon \rightarrow 0} E_\varepsilon^\gamma(t, y_\varepsilon(t)) \geq E_0(t, (q(t), \chi_1(t))) + \int_O \langle G(\omega) \chi_2(t)(\omega, x) \cdot \chi_2(t)(\omega, x) \rangle \, dx = E_{\text{hom}}(t, q(t)),
\]

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which is obtained with help of Proposition 3.3 (i).

Step 4. Strong convergence.

This part of the proof is obtained in the same way as strong convergence in the proof of Theorem 4.3. We remark that the recovery sequence to be used relies on the construction from Proposition 3.3 for \((U(t),\chi_1(t))\) (for the “\(u_\varepsilon\)”-variable) and for \((Z(t),0)\) (for the “\(z_\varepsilon\)”-variable). Also, the observation (27) is useful in this part. Convergence for the whole sequence is obtained as before by a contradiction argument.

\[\square\]

A Appendix

A.1 Abstract evolutionary rate-independent systems

We consider evolutionary rate-independent systems in the global energetic setting. For a detailed study, we refer the reader to [27, 30]. We consider the Hilbert space case and equations involving quadratic energy functionals. The main ingredients of the theory are:

- State space: \(Y\), Hilbert space (dual \(Y^*\));
- External force: \(l \in C^1([0,T],Y^*)\);
- Energy functional: \(E(t,y) = \frac{1}{2} \langle Ay,y \rangle_{Y,Y^*} - \langle l(t),y \rangle_{Y,Y^*}, A \in \text{Lin}(Y,Y^*)\) is self-adjoint and coercive, i.e. there exists \(\alpha > 0\) such that \(\langle Ay,y \rangle \geq \alpha \|y\|^2\) for all \(y \in Y\);
- Dissipation potential: \(\Psi : Y \to [0, +\infty]\), which is convex, proper, lower semi-continuous and positively homogeneous of order 1, i.e. \(\Psi(\alpha y) = \alpha \Psi(y)\) for all \(\alpha > 0\) and \(y \in Y\) and \(\Psi(0) = 0\).

After a prescribed initial state \(y_0 \in Y\), the system’s current configuration is described by \(y : (0,T) \to Y\). We say that \(y \in AC([0,T],Y)\) is an energetic solution associated with \((E,\Psi)\) if for all \(t \in [0,T]\)

\[
E(t,y(t)) \leq E(t,\tilde{y}) + \Psi(\tilde{y} - y(t)) \quad \text{for all } \tilde{y} \in Y, \quad (S)
\]

\[
E(t,y(t)) + \int_0^t \Psi(\dot{y}(s))ds = E(0,y(0)) - \int_0^t \langle \dot{l}(s), y(s) \rangle ds. \quad (EB)
\]

The stability condition is usually stated equivalently using the set of stable states:

\[
S(t) := \{ y : E(t,y) \leq E(t,\tilde{y}) + \Psi(\tilde{y} - y) \quad \text{for all } \tilde{y} \in Y \}, \quad (S) \text{ is equivalent to } y(t) \in S(t).
\]

For the proof of the following existence result, see [27].

**Theorem A.1.** Let \(l \in C^1([0,T],Y^*)\) and \(y_0 \in S(0)\). There exists a unique energetic solution \(y \in C^{Lip}([0,T],Y)\) associated with \((E,\Psi)\) with \(y(0) = y_0\). Moreover,

\[
\|y(t) - y(s)\|_Y \leq \frac{\text{Lip}(l)}{C}|t - s| \quad \text{for any } t, s \in [0,T].
\]

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