MATRIX GROUP MONOTONICITY
USING A DOMINANCE NOTION

DEBASISHA MISHRA
Department of Mathematics
National Institute of Technology Raipur
Raipur- 492010, India

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Abstract. A dominance rule for group invertible matrices using proper splitting is proposed, and used this notion to show that a matrix is group monotone. Then some possible applications are discussed.

1. Introduction. Matrix monotonicity plays an important role in many places such as finite difference for partial differential equations, input-output production and growth models in economics, Markov processes in probability and statistics, and linear complementarity problems in operations research, to name a few. A real $n \times n$ matrix $A$ is called monotone if $Ax \geq 0 \Rightarrow x \geq 0$. Here, $x \geq 0$ means $x_i \geq 0$ for $i = 1, 2, \cdots, n$. This notion was introduced by Collatz [7], who showed that $A$ is monotone (also called inverse positive) if and only if $A^{-1}$ exists and $A^{-1} \geq 0$, where the latter denotes that all the entries of $A^{-1}$ are nonnegative. Several characterizations and generalizations of monotone matrices are also available in the literature. Motivated by Collatz’s result, Mangasarian [9] extended the concept of monotone matrices to the rectangular case, and proved that a rectangular matrix is monotone if and only if it has a nonnegative left-inverse. Berman and Plemmons extended the notion of monotonicity along several directions using generalized inverses. (See for instance, [3] and [4], and the book [5].) They studied extension of monotonicity of singular square matrices in [4]. Again this notion was generalized by Pye [11]. The Drazin inverse has various applications in finite Markov chains, singular differential and difference equations, cryptography and iterative methods in numerical analysis.

Szidarovszky and Okuguchi, [12] extended the notion of diagonal dominance for square matrices. They obtained a sufficient condition for a matrix having nonnegative inverse. Then they applied it to characterize $M$-matrix and block-$M$-matrix. (A real square matrix $A$ is a Z-matrix if all off-diagonal entries are non-positive and a real square matrix $A$ is called an M-matrix if $A$ is a monotone Z-matrix.)

The objective of this work is to extend the diagonal dominance notion up to certain extent and then to apply to certain class of singular square matrices to show that these are group monotone (see next section for its definition). Our approach here has been inspired and guided mainly by the recent work of Mishra and Sivakumar, [10]. They have provided a generalization of the above fact for rectangular matrices to obtain a similar result using the Moore-Penrose inverse of $A$.

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The organization of this paper is as follows. In Section 2, we fix our notation, and discuss preliminary notions and results that will be used in the sequel. Section 3 presents few results for the Drazin inverse using index-proper splitting and also the main result using proper splitting. Then some possible applications are discussed.

2. Preliminaries. Let $\mathbb{R}^n$ denote the $n$ dimensional real Euclidean space. Throughout, all our matrices are real square matrices of order $n$. We denote the range space, the null space and the transpose of $A$ by $R(A)$, $N(A)$ and $A^T$, respectively. Let $X, Y$ be complementary subspaces of $\mathbb{R}^n$. Let $P_{X,Y}$ denote the projection of $\mathbb{R}^n$ onto $X$ along $Y$. Then $P_{X,Y} A = A$ if and only if $R(A) \subseteq X$ and $AP_{X,Y} = A$ if and only if $N(A) \subseteq Y$. For $A, B \in \mathbb{R}^{n \times n}$, $A \leq B$ denotes that $B - A \geq 0$. The spectral radius of $A \in \mathbb{R}^{n \times n}$, denoted by $\rho(A)$ is defined by $\rho(A) = \max_{1 \leq i \leq n} |\lambda_i|$ where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$.

The group inverse of a matrix $A \in \mathbb{R}^{n \times n}$ (if it exists), denoted by $A^#$ is the unique matrix $X$ satisfying $A = AXA$, $X = XAX$ and $AXA = X$. Equivalently, $A^#$ is the unique matrix $X$ satisfying $XAx = x$ for all $x \in R(A)$ and $xy = 0$ for all $y \in N(A)$. $A$ is called group invertible if $A^#$ exists. A group invertible matrix $A$ is called group monotone if $A^# \geq 0$. The index of $A$ is the least nonnegative integer $k$ such that $\text{rank}(A^{k+1})=\text{rank}(A^k)$, and we denote it by $\text{ind} \ A$. $A^#$ exists if and only if index of $A$ is 1. Another equivalent condition for its existence is $R(A) \oplus N(A) = \mathbb{R}^n$. Let $A \in \mathbb{R}^{n \times n}$ be of index $k$. Then, the Drazin inverse of $A$ is the unique matrix $A^D \in \mathbb{R}^{n \times n}$ which satisfies the equations $A^{k+1}A^D = A^k$, $A^DAA^D = A^D$ and $AA^D = A^D A$. Equivalently, $A^D$ is the unique matrix $X$ satisfying $XAx = x$ for all $x \in R(A^k)$ and $Xy = 0$ for all $y \in N(A^k)$. Drazin inverse and its connection to Krylov subspace methods for solving singular linear system of equations can be found in [16].

We list some of the well-known properties of $A^D$ [1] which will be frequently used in this paper: $R(A^D) = R(A^k)$; $N(A^D) = N(A^k)$; $AA^D = P_{R(A^k),N(A^k)} = A^D A$. In particular, if $x \in R(A^k)$ then $x = A^D Ax$.

Let $A$ be of index $k$. Then a splitting of the form $A = U - V$ with $R(U^k) = R(A^k)$ and $N(U^k) = N(A^k)$ is called an index-proper splitting [6] of $A$. The authors of [6] and [8] also have showed that $A^D = (I - U^DV)^{-1}U^D$. Index-proper splitting is more general than the index splitting introduced in [14], and more on index splitting can be found in [15]. When $\text{ind} \ U = 1$, then $A = U - V$ is called an index splitting. When $k = 1$, then both the above splittings reduce to proper splitting [2]. A splitting $A = U - V$ is called a proper splitting of $A$ if $R(U) = R(A)$ and $N(U) = N(A)$.

Let us revisit some of the earlier results from [8] and [14] to get an expression for $A^D$ using index-proper splitting and index splitting, respectively. In particular, we will also have an expression for $A^#$ using proper splitting.

**Theorem 2.1.** (Theorem 3.1, [8])

Let $A = U - V$ be an index-proper splitting. Then

(a) $AA^D = UU^D = A^D A$;
(b) $I - U^DV$ is invertible;
(c) $A^D = (I - U^DV)^{-1}U^D$.

When index of $U = 1$, then index-proper splitting coincides with index splitting. The fact $A = U - V$ is an index splitting implies $R(U) = R(A^k)$ and $N(U) = N(A^k)$. 

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Therefore $R(U)$ and $N(U)$ are complementary subspaces. Thus $U^\#$ exists and then the above result coincides with Theorem 3.1, [14].

**Corollary 2.2.** (Theorem 3.1, [14])
Let $A = U - V$ be an index splitting. Then
(a) $U^\#$ exists;
(b) $AA^D = UU^\# = A^D A$;
(c) $I - U^\# V$ is invertible;
(d) $A^D = (I - U^\# V)^{-1} U^\#$.

When index of $A$ and $U$ are 1, then $A = U - V$ is a proper splitting of $A$. We then have the following corollary.

**Corollary 2.3.** Let $A = U - V$ be a proper splitting and ind $A = 1$. Then
(a) $U^\#$ exists;
(b) $AA^\# = UU^\# = A^\# A$
(c) $A = U(I - U^\# V)$;
(d) $I - U^\# V$ is invertible;
(e) $A^\# = (I - U^\# V)^{-1} U^\#$.

Note that (c) holds as $R(U) = R(A)$ implies $R(V) \subseteq R(U)$. However, the expression does not hold good in Corollary 2.2. The reason is explained in [8]. The following result will also be used in next section.

**Theorem 2.4.** (Theorem 3.16, [13])
Let $X \geq 0$. Then $\rho(X) < 1$ if and only if $(I - X)^{-1}$ exists and $(I - X)^{-1} = \sum_{k=0}^{\infty} X^k \geq 0$.

3. **Main Results.** The following result will be used to prove our first main result. The proof is same as proof of $(e \Rightarrow a)$ of Theorem 4.14, [8]. We have included here for the completeness.

**Lemma 3.1.** Let $A = U - V$ be an index-proper splitting. If $U^D \geq 0$, $U^D V \geq 0$ and $\rho(U^D V) < 1$, then $A^D \geq 0$.

**Proof.** Since $\rho(U^D V) < 1$, so $I - U^D V$ is invertible. By (c) of Theorem 2.1, we get $A^D = (I - U^D V)^{-1} U^D$. We then have $(I - U^D V)^{-1} = \sum_{j=0}^{\infty} (U^D V)^j \geq 0$ by Theorem 2.4 using the fact that $\rho(U^D V) < 1$ and $U^D V \geq 0$. Hence $A^D \geq 0$. \qed

For index of $U = 1$, we have the following corollary for index splitting.

**Corollary 3.2.** (Theorem 4.2, [14])
Let $A = U - V$ be an index splitting. If $U^\# \geq 0$, $U^\# V \geq 0$ and $\rho(U^\# V) < 1$, then $A^D \geq 0$.

When index of both matrices $A$ and $U$ are 1, then Lemma 3.1 and above Corollary admit the following corollary.

**Corollary 3.3.** Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{n \times n}$. Suppose that $U^\#$ exists, $U^\# \geq 0$, $U^\# V \geq 0$ and $\rho(U^\# V) < 1$. Then $A^\#$ exists and $A^\# \geq 0$.

Using Lemma 3.1, we obtain the following result.
Lemma 3.4. Let $S$ satisfy $R(S) = R(A^k)$ and $N(S) = N(A^k)$ where $k$ is index of $A$. If $S \geq 0$, $P_{R(A^k),N(A^k)} - SA \geq 0$ and $\rho(P_{R(A^k),N(A^k)} - SA) < 1$, then $A^D \geq 0$. Conversely, if $A^D \geq 0$, then there exists $S$ such that $R(S) = R(A^k)$, $N(S) = N(A^k)$, $S \geq 0$, $P_{R(A^k),N(A^k)} - SA \geq 0$ and $\rho(P_{R(A^k),N(A^k)} - SA) < 1$.

Proof. Since $R(S) = R(A^k)$, $N(S) = N(A^k)$ and $R(A^k) \cap N(A^k) = \{0\}$, so $S^#$ exists. Let $U = S^#$ and $V = S^# - A$. Then $R(U) = R(S) = R(A^k)$ and $N(U) = N(S) = N(A^k)$ so that $A = U - V$ is an index splitting of $A$. Therefore $U^#$ exists (by Corollary 2.2 (a)). Now $U^# = S \geq 0$ and $U^#V = U^#(S^# - A) = U^#U - U^#A = A^DA - U^#A = P_{R(A^k),N(A^k)} - SA$. So $U^#V \geq 0$ and $\rho(U^#V) < 1$. Hence we have $A^D \geq 0$ by Corollary 3.2.

The other way follows by setting $S = A^D$.

Corollary 3.5. Let $S$ satisfy $R(S) = R(A)$ and $N(S) = N(A)$. If $S \geq 0$, $A^#A - SA \geq 0$ and $\rho(A^#A - SA) < 1$, then $A^# \geq 0$. Conversely, if $A^# \geq 0$, then there exists $S$ such that $R(S) = R(A)$, $N(S) = N(A)$, $S \geq 0$, $P_{R(A),N(A)} - SA \geq 0$ and $\rho(P_{R(A),N(A)} - SA) < 1$.

The result given above is the corresponding notion of Lemma 3.4 when $\text{ind } A = 1$ while the result given below is for nonsingular $A$.

Corollary 3.6. (Lemma 1, [12])

$A^{-1}$ exists and is nonnegative if and only if there exists a nonsingular matrix $D$ such that $D \geq 0$, $I - DA \geq 0$ and $\rho(I - DA) < 1$.

We hereby proceed to propose a dominance rule for the non-negativity of group inverse of matrices. Let us recall the notion of a certain regularity for a subset of matrices, introduced in [12]. Let $\varphi : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$ be a matrix-matrix mapping and $E : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^m$ be a matrix-vector mapping where $m \geq 1$ is a given integer). The following assumptions are made for $\varphi$ and $E$:

(a) $\varphi$ is linear and idempotent.

(b) Whenever $\alpha \in \mathbb{R}$ with $|\alpha| \geq 1$, the condition

$$E(\alpha A) \succ E(A)$$

holds where $\succ$ is a transitive relation in $\mathbb{R}^m$.

Definition 3.7. (Definition 1, [12])

A matrix $A$ is called $(\varphi, E, \succ)$-dominant if and only if

$$E(\varphi(A)) \succ E(A - \varphi(A)).$$

Definition 3.8. (Definition 2, [12])

The triplet $(\varphi, E, \succ)$ is called regular with respect to $J \subseteq \mathbb{R}^{n \times n}$ if every $(\varphi, E, \succ)$-dominant matrix in $J$ is nonsingular.

Example 3.9. (Example 1, [12])

Let $A = (a_{ij})$. Define $E(A) = (E_1(A), E_2(A), \cdots, E_n(A))$ where $E_i(A) = \sum_{j=1}^{n} |a_{ji}|$ for $i = 1, 2, \cdots, n$ and the relation $\succ$ on $\mathbb{R}^n$ is defined as $x \succ y$ if and only if $x_i > y_i$ for all $i$. Let $\varphi(A) = \text{diag}(a_{11}, a_{22}, \cdots, a_{nn})$. Then $A$ is $(\varphi, E, \succ)$-dominant if and only if it is strictly diagonally dominant and that $(\varphi, E, \succ)$ is regular with respect to $J = \mathbb{R}^{n \times n}$. 
Example 3.10. (Example 2, [12])
Define \( J \) as the set of all irreducible matrices. Define \( \varphi \) and \( E \) as in Example 3.9, and \( x \succ y \) if and only if \( x_i \geq y_i \) for all \( i \), and for at least one \( i, x_i > y_i \). Then \((\varphi, E, \succ)\) is regular with respect to \( J \).

Using the above concept, we propose an extension of regularity to singular square matrices.

**Definition 3.11.** Let \( J \subseteq \mathbb{R}^{n \times n} \) and \( A \) be a matrix of index \( k \). Then we say that \((\varphi, E, \succ)\) is \( D\)-regular with respect to \( J \) if every \((\varphi, E, \succ)\)-dominant matrix \( C \in J \) satisfies \( N(CA^k) = N(A^k) \).

Hence, if \((\varphi, E, \succ)\) is regular with respect to \( J \subseteq \mathbb{R}^{n \times n} \), then \((\varphi, E, \succ)\) is \( D\)-regular with respect to \( J \) and for \( k = 1 \), i.e., \( \text{ind} \ A = 1 \), it is called \( G\)-regular with respect to \( J \). Next, we provide an example of a class of matrices \( J \) satisfying the above definition which extends Example 3.9.

**Example 3.12.** Let \( A \) be of the form \(
\begin{pmatrix}
B & O \\
O & O
\end{pmatrix}
\), where \( B \) is a real square nonsingular matrix of order \( r \). Then \( \text{ind} \ A = 1 \). Define \( \varphi : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n} \) by \( \varphi(A) = \begin{pmatrix} D & O \\ O & O \end{pmatrix} \) where \( D = \text{diag}(b_{11}, b_{22}, \cdots, b_{rr}) \) with \( A = (a_{ij}) \).

Define \( E : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^n \) by \( E(A) = (E_1(A), E_2(A), \cdots, E_r(A), 0, 0, \cdots, 0) \) where \( E_i(A) = \sum_{j=1}^n |a_{ij}| \) for \( i = 1, 2, \cdots, r \). For \( x, y \in \mathbb{R}^n \), let \( x \succ y \) denote \( x_i > y_i \) for \( i = 1, 2, \cdots, r \) and \( x_i \geq y_i \) for \( i = r + 1, r + 2, \cdots, n \). Then \((\varphi, E, \succ)\) is \( G\)-regular with respect to \( J = \mathbb{R}^{n \times n} \).

Similarly, an extension of Example 3.10 can be given. For matrices \( A \in J \) satisfying certain additional conditions, we have the following result. Henceforth, we shall assume that for any \( \mu \geq 1 \), \( A + \mu \varphi(A) \in J \), whenever \( A \in J \).

**Lemma 3.13.** Let \((\varphi, E, \succ)\) be \( G\)-regular with respect to \( J \). Let \( A \in J \) be \((\varphi, E, \succ)\)-dominant. Suppose that \( R(\varphi(A)) = R(A) \) and \( N(\varphi(A)) = N(A) \). Then \( \rho(P_{R(A), N(A)} - \varphi(A)^#A) < 1 \).

**Proof.** Let \( \lambda = \rho(P_{R(A), N(A)} - \varphi(A)^#A) = \rho(A^#A - \varphi(A)^#A) \). (Existence of \( \varphi(A)^# \) follows from the fact that \( R(\varphi(A)) = R(A) \) and \( N(\varphi(A)) = N(A) \) and \( \text{ind} \ A = 1 \).)

Let \( \lambda \geq 1 \). Then
\[
E(\lambda \varphi(A)) \succ E(\varphi(A)) \succ E(A - \varphi(A)).
\]

Let \( C = A + (\lambda - 1)\varphi(A) \). Therefore \( C \in J \). Also
\[
\varphi(C) = \lambda \varphi(A) \text{ and then } C - \varphi(C) = A - \varphi(A).
\]

We have
\[
E(\varphi(C)) = E(\lambda \varphi(A)) \succ E(A - \varphi(A)) = E(C - \varphi(C)).
\]

Hence \( C \) is \((\varphi, E, \succ)\)-dominant. Since \( C \in J \), we have \( N(CA) = N(A) \). The scalar \( \lambda \) satisfies the equation \((A^#A - \varphi(A)^#A)x = \lambda x \) for some \( x \neq 0 \). Since \( \lambda \neq 0 \), it follows that \( x \in R(A) = R(\varphi(A)) \). Therefore \( x = \varphi(A)^#\varphi(A)x \). Set \( y = Cx = (A + (\lambda - 1)\varphi(A))x \). Then \( y \in R(A) \). Also, \( \varphi(A)^#y = \varphi(A)^#Ax + (\lambda - 1)x = \varphi(A)^#Ax + \lambda x - x = \varphi(A)^#Ax + \lambda x - A^#Ax = 0 \). Thus \( y \in N(\varphi(A)^#) = N(\varphi(A)) = N(A) \), so that \( Cx = y = 0 \). As \( x \in R(A) \), with \( x = Az \), we have \( 0 = CAz \) and so \( x = Az = 0 \), a contradiction. Hence \( \rho(P_{R(A), N(A)} - \varphi(A)^#A) < 1 \). \( \square \)
Now $(\varphi, E, \succ)$ is regular with respect to $J$ if every $(\varphi, E, \succ)$-dominant matrix in $J$ is nonsingular. We then have the following corollary to Lemma 3.13.

**Corollary 3.14.** (Lemma 2, [12])
Let $(\varphi, E, \succ)$ be regular with respect to $J$, $A \in J$ be $(\varphi, E, \succ)$-dominant and $\varphi(A)$ be nonsingular. Then $\rho(I - \varphi(A)^{-1}A) < 1$.

**Proof.** Since $(\varphi, E, \succ)$ is regular with respect to $J$, it follows that $A$ is invertible. Since $\varphi(A)$ is nonsingular, we have $R(\varphi(A)) = R(A) = \mathbb{R}^n$ and $N(\varphi(A)) = N(A) = \{0\}$. The proof is complete by observing that $P_{R(A),N(A)} = I$.

Using Lemma 3.4 and 3.13, we are presenting a theorem below which gives sufficient conditions for $A^\# \geq 0$.

**Theorem 3.15.** Let $(\varphi, E, \succ)$ be $G$-regular with respect to $J$, and $\varphi$ be such that $R(\varphi(A)) = R(A)$ and $N(\varphi(A)) = N(A)$. Suppose that $\varphi(A)^\# \geq 0$ and $P_{R(A),N(A)} - \varphi(A)^\# A \geq 0$. If $A \in J$ is $(\varphi, E, \succ)$-dominant, then $A^\# \geq 0$.

**Proof.** Since $A \in J$ and $A$ is $(\varphi, E, \succ)$-dominant, then $A^\#$ exists and $\rho(P_{R(A),N(A)} - \varphi(A)^\# A) < 1$ (by Lemma 3.13). Let $U = \varphi(A)$ and $V = \varphi(A) - A$. Then $R(U) = R(\varphi(A)) = R(A)$ and $N(U) = N(\varphi(A)) = N(A)$ so that $A = U - V$ is a proper splitting. So $U^\#$ exists. Also, $U^\# = \varphi(A)^\# \geq 0$ and $U^#V = U^#(U - A) = U^#U - U^#A = A^#A - \varphi(A)^\# A = P_{R(A),N(A)} - \varphi(A)^\# A \geq 0$. By Lemma 3.4, (using $\varphi(A)^\#$ for $S$ there) it now follows that $A^\# \geq 0$.

**Corollary 3.16.** (Theorem 1, [12])
Suppose that $(\varphi, E, \succ)$ is regular with respect to $J$, $\varphi(A)^{-1}$ exists, $\varphi(A)^{-1} \geq 0$ and $I - \varphi(A)^{-1}A \geq 0$. If $A \in J$ is $(\varphi, E, \succ)$-dominant, then $A^{-1}$ exists and is nonnegative.

Next, we illustrate some applications of our main result for group invertible matrices.

I. Let $A$ be a $Z$-matrix of index 1 with $R(A) = \text{span}\{e^i : i \text{ satisfies } a_{ii} \neq 0\}$ where $e^i$ is the $n$-vector with $i$th entry is 1 and others 0. Define 

$$
\varphi(A) = \text{diag}(b_{11}, b_{22}, \ldots, b_{rr}, 0, 0, \ldots, 0)
$$

where $B = (b_{ij})$, and 

$$
E(A) = (E_1(B), E_2(B), \ldots, E_r(B), 0, 0, \ldots, 0)
$$

where $E_i(B) = \sum_{j=1}^{r} |b_{ij}|$ and $x \succ y$ if and only if $x_i \geq y_i$ for all $i$. Then, $A$ is $(\varphi, E, \succ)$-dominant if and only if $A$ is diagonally dominant. Therefore $R(\varphi(A)) = R(A)$, $N(\varphi(A)) = N(A)$ and $P_{R(A),N(A)} - \varphi(A)^\# A \geq 0$. So $\varphi(A)^\#$ exists and $\varphi(A)^\# \geq 0$. Hence, by Theorem 3.15, $A^\# \geq 0$.

II. Define $\varphi$ and $\succ$ as in case I, and suppose that $A$ is defined as in case I. Define 

$$
P_i(A) = \sum_{j \neq i} |a_{ij}|, \quad Q_i(A) = \sum_{j \neq i} |a_{ji}|, \quad P_{i,\alpha}(A) = \alpha P_i(A) + (1 - \alpha)Q_i(A).
$$

If anyone of the following conditions hold, then $A^\# \geq 0$.

(a) $a_{ii} \geq P_{i,\alpha}(A)$ for all $i$ and some $\alpha \in [0,1]$;
(b) $a_{ii} \geq P_{i,\alpha}(A)Q_i^{1-\alpha}(A)$ for all $i$ and some $\alpha \in [0,1]$;
(c) $a_{ii} \geq P_i(A)$ for all $i$;
exists and \( \phi \) nonsingular with \( E \) and only if \( n \).

Outline of the proof:

(c) and (d) are particular cases of (a) when

We can prove these only by choosing appropriate \( E \) so that \( A \) becomes \( (\phi, E, \succ) \)-dominant. Then other conditions of Theorem 3.15 will follow similarly as in case I. Hence \( A^\# \geq 0 \).

(a) Given that \( a_{ii} \geq \alpha P_i(A) + (1 - \alpha)Q_i(A) \) for all \( i \) and some \( \alpha \in [0, 1] \). Now, define \( E(A) = (E_1(A), E_2(A), \ldots, E_n(A)) \) with \( E_i(A) = \alpha E_i(A) + (1 - \alpha)\bar{E}_i(A) \) where \( \bar{E}_i(A) = \sum |a_{ij}|, \bar{E}_i(A) = \sum |a_{ji}| \). Thus, it follows that \( A \) is \( (\phi, E, \succ) \)-dominant.

(b) Now \( E(A) = (E_1(A), E_2(A), \ldots, E_n(A)) \) with \( E_i(A) = \bar{E}_i(A) + (1 - \alpha)\bar{E}_i(A) \) where \( E_i(A) \) and \( \bar{E}_i(A) \) are as defined earlier yields \( A \) is \( (\phi, E, \succ) \)-dominant.

(c) and (d) are particular cases of (a) when \( \alpha = 0 \) and \( \alpha = 1 \).

For proving (e), (f), (g), (h) and (i), let us redefine \( E : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^m \) where \( m = n(n - 1)/2 \) such that \( E(A) = \{E_{ij}(A)\} \) for all \( i, j = 1, 2, \ldots, n \) and \( j > i \) such that

(e) \( E_{ij}(A) = \bar{E}_{ij}(A) \).

(f) \( E_{ij}(A) = \bar{E}_{ij}(A) \).

(g) \( E_{ij}(A) = \bar{E}_{ij}(A) + \bar{E}_{ji}(A) \).

(h) \( E_{ij}(A) = \bar{E}_{ij}(A) + \bar{E}_{ji}(A) \).

(i) \( E_{ij}(A) = \bar{E}_{ij}(A) \bar{E}_{ij}^{-\alpha}(A) \bar{E}_{ji}^{-\alpha}(A) \).

Then \( A \) is \( (\phi, E, \succ) \)-dominant in all five conditions.

III. Let \( \text{ind } A = 1 \) and \( A \) be of the form

\[
\begin{pmatrix}
B & O_1 & O_2 \\
O_3 & B & O_4 \\
O_5 & O_6 & O_7
\end{pmatrix}
\]

where \( B \) and \( \overline{B} \) are nonsingular \( M \)-matrices, \( O_i \)'s are zero matrices. Define

\[ \varphi(A) = \text{diag}(b_{11}, b_{22}, \ldots, b_{rr}, \overline{b}_{r+1}, \overline{b}_{r+2}, \ldots, \overline{b}_{pp}, 0, 0, \ldots, 0) \]

where \( B = (b_{ij}), \overline{B} = (\overline{b}_{ij}) \),

\[ E(A) = (E_1(B), E_2(B), \ldots, E_r(B), E_{r+1}(\overline{B}), E_{r+2}(\overline{B}), \ldots, E_p(\overline{B}), \ldots, 0) \]

with \( E_i(B) = \sum_{j=1}^r |b_{ij}| \) for \( i = 1, \cdots r \) and \( E_i(\overline{B}) = \sum_{j=r+1}^p |\overline{b}_{r+i} \overline{b}_{r+j}| \) for \( i = 1, \cdots p \), and \( x \succ y \) if and only if \( x_i \geq y_i \) for all \( i \). Then, \( A \) is \( (\varphi, E, \succ) \)-dominant if and only if \( B \) and \( \overline{B} \) are strictly diagonally dominant. In this case, we also have \( R(\varphi(A)) = R(A), N(\varphi(A)) = N(A) \) and \( P_{R(A), N(A)} - \varphi(A)^\# A \geq 0 \). So \( \varphi(A)^\# \) exists and \( \varphi(A)^\# \geq 0 \). Hence, by Theorem 3.15, \( A^\# \geq 0 \).

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E-mail address: kapamath@gmail.com, dmishra@nitrr.ac.in