$p$-adic singular integral and their commutator in generalized Morrey space

Huixia Mo, Zhe Han, Liu Yang
School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, China

Abstract For a prime number $p$, let $\mathbb{Q}_p$ be the field of $p$-adic numbers. In this paper, we established the boundedness of a class of $p$-adic singular integral operators on the $p$-adic generalized Morrey spaces. The corresponding boundedness for the commutators generalized by the $p$-adic singular integral operators and $p$-adic Lipschitz functions or $p$-adic generalized Campanato functions is also considered.

MSC 42B20, 42B25

Key words $p$-adic field; $p$-adic singular integral operator; commutator; $p$-adic generalized Morrey function; $p$-adic generalized Campanato function; $p$-adic Lipschitz function.

1 Introduction

Let $p$ be a prime number and $x \in \mathbb{Q}$. Then the non-Archimedean $p$-adic normal $|x|_p$ is defined as follows: if $x = 0$, $|0|_p = 0$; if $x \neq 0$ is an arbitrary rational number with the unique representation $x = p^\gamma \frac{m}{n}$, where $m, n$ are not divisible by $p$, $\gamma = \gamma(x) \in \mathbb{Z}$, then $|x|_p = p^{-\gamma}$. This normal satisfies $|xy|_p = |x|_p |y|_p$, $|x + y|_p \leq \max\{|x|_p, |y|_p\}$ and $|x|_p = 0$ if and only if $x = 0$. Moreover, when $|x|_p \neq |y|_p$, we have $|x + y|_p = \max\{|x|_p, |y|_p\}$. Let $\mathbb{Q}_p$ be the field of $p$-adic numbers, which is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean $p$-adic normal $|\cdot|_p$. For $\gamma \in \mathbb{Z}$, we denote the ball $B_\gamma(a)$ with center at $a \in \mathbb{Q}_p^n$ and radius $p^\gamma$ and its boundary $S_\gamma(a)$ by

$$B_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p \leq p^\gamma\}, \quad S_\gamma(a) = \{x \in \mathbb{Q}_p^n : |x - a|_p = p^\gamma\},$$

respectively. It is easy to see that

$$B_\gamma(a) = \bigcup_{k \leq \gamma} S_k(a).$$

For $n \in \mathbb{N}$, the space $\mathbb{Q}_p^n = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p$ consists of all points $x = (x_1, \ldots, x_n)$ where $x_i \in \mathbb{Q}_p$, $i = 1, \ldots, n$, $n \geq 1$. The $p$-adic norm of $\mathbb{Q}_p^n$ is defined by

$$|x|_p = \max_{1 \leq i \leq n} |x_i|_p, \quad x \in \mathbb{Q}_p^n.$$

Thus, it is easy to see that $|x|_p$ is a non-Archimedean norm on $\mathbb{Q}_p^n$. The balls $B_\gamma(a)$ and the sphere $S_\gamma(a)$ in $\mathbb{Q}_p^n$, $\gamma \in \mathbb{Z}$ are defined similar to the case $n = 1$. 

* Corresponding author.
E-mail addresses: huixiamo@bupt.edu.cn.
Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, thus from the standard analysis there exists the Haar measure $dx$ on the additive group $\mathbb{Q}_p^n$ normalized by $\int_{B_0} dx = |B_0|_H = 1$, where $|E|_H$ denotes the Haar measure of a measurable set $E \subset \mathbb{Q}_p^n$. Then by a simple calculation the Haar measures of any balls and spheres can be obtained. From the integral theory, it is easy to see that $|B_\gamma(a)|_H = p^{n\gamma}$ and $|S_\gamma(a)|_H = p^{n\gamma}(1-p^{-n})$ for any $a \in \mathbb{Q}_p^n$. For a more complete introduction to the $p$-adic analysis, one can refer to [1, 2, 3, 4, 5, 6, 7, 8] and the references therein.

The $p$-adic numbers have been applied in string theory, turbulence theory, statistical mechanics, quantum mechanics, and so forth (see [1, 9, 10] for detail). In the past few years, there is an increasing interest in the study of harmonic analysis on $p$-adic field (see [5, 6, 7, 8] for detail).

Let $\Omega \in L^\infty(\mathbb{Q}_p^n)$, $\Omega(p^j x) = \Omega(x)$ for all $j \in \mathbb{Z}$ and $\int_{|x|_p = 1} \Omega(x) dx = 0$. Then the $p$-adic singular integral operator defined by Taibleson [5] is as follows

$$T_k(f)(x) = \int_{|y|_p > p^k} f(x - y) \frac{\Omega(y)}{|y|^n} dy.$$

And the $p$-adic singular integral operator $T$ is defined as the limit of $T_k$ when $k$ goes to $-\infty$.

Moreover, let $\vec{b} = (b_1, b_2, ..., b_m)$, where $b_i \in L_{loc}(\mathbb{Q}_p^n)$ for $1 \leq i \leq m$. Then the higher commutator generated by $\vec{b}$ and $T_k$ can be defined by

$$T_k^{\vec{b}} f(x) = \int_{|y|_p > p^k} \prod_{i=1}^m (b_i(x) - b_i(x - y)) f(x - y) \frac{\Omega(y)}{|y|^n} dy.$$

And the commutator generated by $\vec{b} = (b_1, b_2, ..., b_m)$ and $p$-adic singular integral operator $T$ is defined as the limit of $T_k^{\vec{b}}$, when $k$ goes to $-\infty$.

Under some conditions, the authors in [5, 11], obtained that $T_k$ were of type $(q, q)$, $1 < q < \infty$, and of weak type $(1, 1)$ on local fields. In [12], Wu et al. established the boundedness of $T_k$ on $p$-adic central Morrey spaces. Furthermore, the $\lambda$-central BMO estimates for commutators of these singular integral operators on $p$-adic central Morrey spaces were obtained in [12]. Moreover, in $p$-adic linear space $\mathbb{Q}_p^n$, Volosivets [13] gave the sufficient conditions for the maximal function and Riesz potential in $p$-adic generalized Morrey spaces. Mo et al. [14] established the boundedness of the commutators generated by the $p$-adic Riesz potential and $p$-adic generalized Campanato functions in $p$-adic generalized Morrey spaces.

Motivated by the works of [12, 13, 14], we are going to consider the boundedness of $T_k$ on the $p$-adic generalized Morrey type spaces, as well as the boundedness of the commutators generated by $L_k$ and $p$-adic generalized Campanato functions.

Throughout this paper, the letter $C$ will be used to denote various constants and the various uses of the letter do not, however, denote the same constant. And, $A \lesssim B$ means that $A \leq CB$, with some positive constant $C$ independent of appropriate quantities.
2 Some notations and lemmas

Definition 2.1 [13] Let $1 \leq q < \infty$, and let $\omega(x)$ be a non-negative measurable function in $\mathbb{Q}_p^n$. A function $f \in L^q_{\text{loc}}(\mathbb{Q}_p^n)$ is said to belong to the generalized Morrey space $GM_{q,\omega}(\mathbb{Q}_p^n)$, if
\[
\|f\|_{GM_{q,\omega}} = \sup_{a \in \mathbb{Q}_p^n, \gamma \in \mathbb{Z}} \frac{1}{\omega(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|_H} \int_{B_\gamma(a)} |f(y)|^q dy \right)^{1/q} < \infty,
\]
where $\omega(B_\gamma(a)) = \int_{B_\gamma(a)} \omega(x)dx$.

Let $\lambda \in \mathbb{R}$. If $\omega(B_\gamma(a)) = |B_\gamma(a)|^\lambda$, then $GM_{q,\omega}(\mathbb{Q}_p^n)$ is the classical Morrey spaces $M_{q,\lambda}(\mathbb{Q}_p^n)$. About the generalized Morrey space, see [15], and the classical Morrey spaces, see [16], etc.

Moreover, let $\lambda \in \mathbb{R}$ and $1 \leq q < \infty$. The $p$-adic central Morrey space $CM_{q,\lambda}(\mathbb{Q}_p^n)$ (see [8]) is defined by
\[
\|f\|_{CM_{q,\lambda}} = \sup_{\gamma \in \mathbb{Z}} \left( \frac{1}{|B_\gamma(0)|_H} \int_{B_\gamma(0)} |f(y)|^q dy \right)^{1/q} < \infty.
\]

Definition 2.2 [17] Let $0 < \beta < 1$, then the $p$-adic Lipschitz space $\Lambda_{\beta}(\mathbb{Q}_p^n)$ is defined the set of all functions $f : \mathbb{Q}_p^n \to \mathbb{C}$ such that
\[
\|f\|_{\Lambda_{\beta}(\mathbb{Q}_p^n)} = \sup_{x,h \in \mathbb{Q}_p^n, h \neq 0} \frac{|f(x+h) - f(x)|}{|h|^\beta} < \infty.
\]

Definition 2.3 [13] Let $B$ be a ball in $\mathbb{Q}_p^n$, $1 \leq q < \infty$. And let $\omega(x)$ be a non-negative measurable function in $\mathbb{Q}_p^n$. A function $f \in L^q_{\text{loc}}(\mathbb{Q}_p^n)$ is said to belong to the generalized Campanato space $GC_{q,\omega}(\mathbb{Q}_p^n)$, if
\[
\|f\|_{GC_{q,\omega}} = \sup_{a \in \mathbb{Q}_p^n, \gamma \in \mathbb{Z}} \frac{1}{\omega(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|_H} \int_{B_\gamma(a)} |f(y) - f_{B_\gamma(a)}|^q dy \right)^{1/q} < \infty,
\]
where $f_{B_\gamma(a)} = \frac{1}{|B_\gamma(a)|_H} \int_B f(x)dx$ and $\omega(B_\gamma(a)) = \int_{B_\gamma(a)} \omega(x)dx$.

The classical Campanato spaces can be seen in [18], [19] and etc. The important particular case of $GC_{q,\omega}(\mathbb{Q}_p^n)$ is $CBMO_{q,\lambda}(\mathbb{Q}_p^n)$, where for $1 < q < \infty$ and $0 < \lambda < 1/n$. And the central BMO space $CBMO_{q,\lambda}(\mathbb{Q}_p^n)$ is defined by
\[
\|f\|_{CBMO_{q,\lambda}(\mathbb{Q}_p^n)} = \sup_{\gamma \in \mathbb{Z}} \frac{1}{|B_\gamma(0)|_H} \left( \frac{1}{|B_\gamma(0)|_H} \int_{B_\gamma(0)} |f(y) - f_{B_\gamma(0)}|^q dy \right)^{1/q} < \infty. \tag{2.1}
\]
Lemma 2.1 \[14\] Let $1 \le q < \infty$, and let $\omega$ be a non-negative measurable function. Suppose that $b \in GC_{q,\omega}(\mathbb{Q}_p^n)$, then
\[
|b_{B_k(a)} - b_{B_j(a)}| \le \|b\|_{GC_{q,\omega}} |j - k| \max\{\omega(B_k(a)), \omega(B_j(a))\},
\]
for $j, k \in \mathbb{Z}$ and any fixed $a \in \mathbb{Q}_p^n$.

Thus, for $j > k$, from Lemma 2.1, it deduces that
\[
\left(\int_{B_j(a)} |b(y) - b_{B_k(a)}|^q dy\right)^{1/q} \le (j + 1 - k)|B_j(a)|^{1/q} \omega(B_j(a)) \|b\|_{GC_{q,\omega}}. \tag{2.2}
\]

Lemma 2.2 \[5\] Let $\Omega \in L^\infty(\mathbb{Q}_p^n)$, $\Omega(p^j k x) = \Omega(x)$ for all $j \in \mathbb{Z}$ and $\int_{|x|_p = 1} \Omega(x) dx = 0$. If
\[
\sup_{|y|_p = 1} \sum_{j=1}^\infty \int_{|x|_p = 1} |\Omega(x + p^j y) - \Omega(x)| dx < \infty,
\]
then for $1 < p < \infty$, there is a constant $C > 0$ such that
\[
\|T_k(f)\|_{L^p(\mathbb{Q}_p^n)} \le C \|f\|_{L^p(\mathbb{Q}_p^n)},
\]
for $k \in \mathbb{Z}$, where $C$ is independent of $f$ and $k \in \mathbb{Z}$.

Furthermore, $T(f) = \lim_{k \to -\infty} T_k(f)$ exists in the $L^p$ norm and
\[
\|T(f)\|_{L^p(\mathbb{Q}_p^n)} \le C \|f\|_{L^p(\mathbb{Q}_p^n)}.
\]

Moreover, on the $p$–adic field, the Riesz potential $I_\alpha^p$ is defined by
\[
I_\alpha^p f(x) = \frac{1}{\Gamma_n(\alpha)} \int_{\mathbb{Q}_p^n} \frac{f(y)}{|x - y|_p^{-\alpha}} dy,
\]
where $\Gamma_n(\alpha) = (1 - p^{\alpha - n})/(1 - p^{-\alpha})$, $\alpha \in \mathbb{C}$ and $\alpha \neq 0$.

Lemma 2.3 \[14\] Let $\alpha$ be a complex number with $0 < \text{Re}\alpha < n$, and let $1 < r < \infty$, $1 < q < n/\text{Re}\alpha$, $0 < 1/r = 1/q - \text{Re}\alpha/n$. Suppose that both $\omega$ and $\nu$ are non-negative measurable functions, such that
\[
\sum_{j=\gamma}^\infty p^j \text{Re}\alpha \nu(B_j(a)) \omega(B_j(a)) = C < \infty,
\]
for any $a \in \mathbb{Q}_p^n$ and $\gamma \in \mathbb{Z}$. Then the Riesz potential $I_\alpha^p$ is bounded from $GM_{q,\nu}$ to $GM_{r,\omega}$.
3 Main results

In this section, let us state the main results of the paper.

**Theorem 3.1** Let $1 < q < \infty$, and let $\Omega(p^j x) = \Omega(x)$ for all $j \in \mathbb{Z}$, $\int_{|x|_p = 1} \Omega(x) dx = 0$, and

$$\sup_{|y|_p = 1} \sum_{j = 1}^{\infty} \int_{|x|_p = 1} |\Omega(x + p^j y) - \Omega(x)| dx < \infty.$$ 

Suppose that both $\omega$ and $\nu$ are non-negative measurable functions, such that

$$\sum_{j = \gamma}^{\infty} \nu(B_j(a)) / \omega(B_\gamma(a)) = C < \infty,$$  \hfill (3.1)

for any $\gamma \in \mathbb{Z}$ and $a \in \mathbb{Q}^n_p$. Then the singular integral operators $T_k$ are bounded from $GM_{q,\nu}$ to $GM_{q,\omega}$ for all $k \in \mathbb{Z}$. Moreover, $T(f) = \lim_{k \to -\infty} T_k(f)$ exists in $GM_{q,\omega}$ and the operator $T$ is bounded from $GM_{q,\nu}$ to $GM_{q,\omega}$.

**Corollary 3.1** Let $1 < q < \infty$, $\lambda < 0$, $\Omega \in L^\infty(Q^n_p)$, $\Omega(p^j x) = \Omega(x)$ for all $j \in \mathbb{Z}$, $\int_{|x|_p = 1} \Omega(x) dx = 0$, and

$$\sup_{|y|_p = 1} \sum_{j = 1}^{\infty} \int_{|x|_p = 1} |\Omega(x + p^j y) - \Omega(x)| dx < \infty.$$ 

Then the operators $T_k$ and $T$ are bounded on the space $CM_{q,\lambda}$ for all $k \in \mathbb{Z}$.

In fact, for $\lambda < 0$. Taking $\omega(B) = \nu(B) = |B|^\lambda_H$ in Theorem 3.1, we can obtain the Corollary 3.1. If the Morrey space $M^{q,\lambda}(Q^n_p)$ is replaced by the central Morrey space $CM^{q,\lambda}(Q^n_p)$ in Corollary 3.1, then the conclusion is that of Theorem 4.1 in [12].

**Theorem 3.2** Let $\Omega \in L^\infty(Q^n_p)$, $\Omega(p^j x) = \Omega(x)$ for all $j \in \mathbb{Z}$, $\int_{|x|_p = 1} \Omega(x) dx = 0$, and

$$\sup_{|y|_p = 1} \sum_{j = 1}^{\infty} \int_{|x|_p = 1} |\Omega(x + p^j y) - \Omega(x)| dx < \infty.$$ 

Let $0 < \beta_i < 1$ for $i = 1, 2, \ldots, m$, such that $0 < \beta = \sum_{i = 1}^{m} \beta_i < n$. And, let $1 < r < \infty$, $1 < q < n/\beta$ such that $1/r = 1/q - \beta/n$. Suppose that $b_i \in \Lambda_{\beta_i}, i = 1, 2, \ldots, m$, and both $\omega$ and $\nu$ are non-negative measurable functions, such that

$$\sum_{j = \gamma}^{\infty} p^j \beta \nu(B_j(a)) / \omega(B_\gamma(a)) = C < \infty,$$ \hfill (3.2)

1
for any $\gamma \in \mathbb{Z}$ and $a \in \mathbb{Q}_p^n$. Then the commutators $T_k^\delta$ are bounded from $GM_{q,\nu}$ to $GM_{r,\omega}$, for all $k \in \mathbb{Z}$. Moreover, $T_k^\delta(f) = \lim_{k \to -\infty} T_k^\delta(f)$ exists in the space of $GM_{q,\omega}$, and the commutator $T^\delta$ is bounded from $GM_{q,\nu}$ to $GM_{q,\omega}$.

**Theorem 3.3** Let $\Omega \in L^\infty(\mathbb{Q}_p^n)$, $\Omega(p^j x) = \Omega(x)$ for all $j \in \mathbb{Z}$, $\int_{|x|_p = 1} \Omega(x)dx = 0$, and

$$
\sup_{|y|_p = 1} \sum_{j = 1}^\infty \int_{|x|_p = 1} |\Omega(x + p^j y) - \Omega(x)|dx < \infty.
$$

Let $1 < q, r, q_1, \ldots, q_m < \infty$, such that $1/r = 1/q_1 + 1/q_2 + \cdots + 1/q_m$. Suppose that $\omega, \nu$ and $\nu_i (i = 1, 2, \ldots, m)$ are non-negative measurable functions. If $b_i \in GC_{q_i, \nu_i}(\mathbb{Q}_p^n)$ for $i = 1, 2, \ldots, m$, and the functions $\omega, \nu$ and $\nu_i (i = 1, 2, \ldots, m)$ satisfy the following conditions

(i) $\prod_{i = 1}^m \nu_i(B_1(a))\nu(B_1(a))/\omega(B_1(a)) = C < \infty$,

(ii) $\sum_{j = \gamma + 1}^\infty \prod_{i = 1}^m \nu_i(B_j(a))(j + 1 - \gamma)^m \nu(B_j(a))/\omega(B_1(a)) = C < \infty$,

for any $\gamma \in \mathbb{Z}$ and $a \in \mathbb{Q}_p^n$. Then the commutators $T_k^\delta$ are bounded from $GM_{q,\nu}$ to $GM_{r,\omega}$, for all $k \in \mathbb{Z}$. Moreover, the commutator $T^\delta = \lim_{k \to -\infty} T_k^\delta$ exists in the space of $GM_{q,\omega}$, and $T^\delta$ is bounded from $GM_{q,\nu}$ to $GM_{q,\omega}$.

**Corollary 3.2** Let $\Omega \in L^\infty(\mathbb{Q}_p^n)$, $\Omega(p^j x) = \Omega(x)$ for all $j \in \mathbb{Z}$, $\int_{|x|_p = 1} \Omega(x)dx = 0$, and

$$
\sup_{|y|_p = 1} \sum_{j = 1}^\infty \int_{|x|_p = 1} |\Omega(x + p^j y) - \Omega(x)|dx < \infty.
$$

Let $1 < q, r, q_1, \ldots, q_m < \infty$, such that $1/r = 1/q_1 + 1/q_2 + \cdots + 1/q_m$. Let $0 < \lambda_1, \ldots, \lambda_m < 1/n$, $\lambda < -\sum_{i=1}^m \lambda_i$ and $\tilde{\lambda} = \sum_{i=1}^m \lambda_i + \lambda$. If $b_i \in BMO_{q_i, \lambda_i}(\mathbb{Q}_p^n)$, then the commutators $T_k^\delta$ and $T^\delta$ are bounded from $M_{q,\lambda}$ to $M_{r,\tilde{\lambda}}$.

Moreover, let $1 < r, q, q_1 < \infty$, such that $1/r = 1/q + 1/q_1$. Let $0 < \lambda_1 < 1/n$, $\lambda < -\lambda_1$ and $\tilde{\lambda} = \lambda_1 + \lambda$. If $b \in CBMO_{q_1, \lambda_1}(\mathbb{Q}_p^n)$, then from Corollary 4.1, it follows that the commutator $[T_k, b]$ and $[T, b]$ both are bounded from $CM_{q,\lambda}$ to $CM_{r,\tilde{\lambda}}$. This conclusion is that of Theorem 4.2 in [12].
4 Proof of Theorem 3.1-3.3

Let us give the proof of Theorem 3.1, firstly.

For any fixed \( \gamma \in \mathbb{Z} \) and \( a \in \mathbb{Q}_p^n \), it is easy to see that

\[
\frac{1}{\omega(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|_H} \right) \int_{B_\gamma(a)} |T_k(f)(x)|^q dx \right)^{1/q}
\leq \frac{1}{\omega(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|_H} \right) \int_{B_\gamma(a)} |T_k(f)(f \chi_{B_\gamma(a)})(x)|^q dx \right)^{1/q}
\leq \frac{1}{\omega(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|_H} \right) \int_{B_\gamma(a)} |T_k(f \chi_{B_\gamma(a)})(x)|^q dx \right)^{1/q}
\]

\[
:= I + II,
\]

where \( B_\gamma^c(a) \) is the complement to \( B_\gamma(a) \) in \( \mathbb{Q}_p^n \).

Using Lemma 2.2 and (3.1), it follows that

\[
I = \frac{1}{\omega(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|_H} \right) \int_{B_\gamma(a)} |T_k(f \chi_{B_\gamma(a)})(x)|^q dx \right)^{1/q}
\leq \frac{1}{\omega(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|_H} \right) \left( \int_{B_\gamma(a)} |f(x)|^q dx \right)^{1/q}
\]

\[
= \frac{\nu(B_\gamma(a))}{\omega(B_\gamma(a))} \frac{1}{\nu(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|_H} \right) \left( \int_{B_\gamma(a)} |f(x)|^q dx \right)^{1/q}
\]

\[
\leq \|f\|_{GM_{q, \nu}}.
\]

For \( II \), let us estimate \( |T_k(f \chi_{B_\gamma(a)})(x)| \), firstly.

Since \( x \in B_\gamma(a) \) and \( \Omega \in L^\infty(\mathbb{Q}_p^n) \), then we have
Thus, from (3.1) and (4.3), it follows that

\[ II = \frac{1}{\omega(B_\gamma(a))} \left( \int_{B_\gamma(a)} |T_k(f \chi_{B_\gamma(a)})(x)|^q dx \right)^{1/q} \]

\[ \lesssim \|f\|_{GM_{q,\nu}} \sum_{j=\gamma+1}^{\infty} \nu(B_j(a)) / \omega(B_\gamma(a)) \]  

Combining the estimates of (4.1), (4.2) and (4.4), we have

\[ \frac{1}{\omega(B_\gamma(a))} \left( \int_{B_\gamma(a)} |T_k(f)(x)|^q dx \right)^{1/q} \lesssim \|f\|_{GM_{q,\nu}}, \]

which means that $T_k$ is bounded from $GM_{q,\nu}$ to $GM_{q,\omega}$.

Moreover, from Lemma 2.2 and the definition of $GM_{q,\omega}(Q^n_p)$, it is obvious that $T(f) = \lim_{k \to -\infty} T_k(f)$ exists in $GM_{q,\omega}$ and the operator $T$ is bounded from $GM_{q,\nu}$ to $GM_{q,\omega}$.

Proof of Theorem 3.2.

For any $x \in Q^n_p$, since $\Omega \in L^\infty(Q^n_p)$, and $b_i \in \Lambda_{\beta_i}$, $i = 1, 2, \ldots, m$, then it is easy to see
that
\[
|T_k^f f(x)| \\
\leq \int_{|y| > p^k} \prod_{i=1}^m |b_i(x) - b_i(x - y)||f(x - y)||\Omega(y)| |y|_p^n dy \\
\leq \int_{Q_p} |x - z|^{n - \beta} dz \\
\leq T_p(|f|(x).
\]

Thus, from Lemma 2.2 it is obvious that the commutators $T_k^f$ are bounded from $GM_{q, \nu}$ to $GM_{r, \omega}$, for all $k \in \mathbb{Z}$.

Moreover, from the definition of $GM_{q, \omega}(\mathbb{Q}^n_{p})$, it is obvious that $T_k^f(f) = \lim_{k \to -\infty} T_k^f(f)$ exists in the space of $GM_{q, \omega}$, and the commutator $T_k^f$ is bounded from $GM_{q, \nu}$ to $GM_{q, \omega}$.

**Proof of Theorem 3.3**

Without loss of generality, we need only to show that the conclusion holds for $m = 2$.

For any fixed $\gamma \in \mathbb{Z}$ and $a \in \mathbb{Q}^n$, we write $f^0 = f \chi_{B_1(a)}$ and $f^\infty = f \chi_{B_2(a)}$, then

\[
\frac{1}{\omega(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|} \right) \int_{B_1(a)} |T_k^{(b_1, b_2)}(f)(x)|^r dx \right)^{1/r} \\
\leq \frac{1}{\omega(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|} \right) \int_{B_1(a)} |(b_1(x) - (b_1)_{B_1(a)})(b_2(x) - (b_2)_{B_2(a)})T_k(f^0)(x)|^r dx \right)^{1/r} \\
+ \frac{1}{\omega(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|} \right) \int_{B_1(a)} |(b_1(x) - (b_1)_{B_1(a)})T_k((b_2 - (b_2)_{B_2(a)})f^0)(x)|^r dx \right)^{1/r} \\
+ \frac{1}{\omega(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|} \right) \int_{B_1(a)} |(b_2(x) - (b_2)_{B_2(a)})T_k((b_1 - (b_1)_{B_1(a)})f^0)(x)|^r dx \right)^{1/r} \\
+ \frac{1}{\omega(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|} \right) \int_{B_1(a)} |T_k((b_1 - (b_1)_{B_1(a)})(b_2 - (b_2)_{B_2(a)})f^0)(x)|^r dx \right)^{1/r} \\
+ \frac{1}{\omega(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|} \right) \int_{B_1(a)} |(b_1(x) - (b_1)_{B_1(a)})(b_2(x) - (b_2)_{B_2(a)})T_k(f^\infty)(x)|^r dx \right)^{1/r} \\
+ \frac{1}{\omega(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|} \right) \int_{B_1(a)} |(b_1(x) - (b_1)_{B_1(a)})T_k((b_2 - (b_2)_{B_2(a)})f^\infty)(x)|^r dx \right)^{1/r} \\
+ \frac{1}{\omega(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|} \right) \int_{B_1(a)} |(b_2(x) - (b_2)_{B_2(a)})T_k((b_1 - (b_1)_{B_1(a)})f^\infty)(x)|^r dx \right)^{1/r} \\
+ \frac{1}{\omega(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|} \right) \int_{B_1(a)} |T_k((b_1 - (b_1)_{B_1(a)})(b_2 - (b_2)_{B_2(a)})f^\infty)(x)|^r dx \right)^{1/r} \\
=: E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 + E_8.
\]

In the following, we will estimate every part, respectively.
Since $1/r = 1/q + 1/q_1 + 1/q_2$, then, from Hölder’s inequality, Lemma 2.2 and (i), it follows that

$$E_1 = \frac{1}{\omega(B_{\gamma}(a))} \frac{1}{|B_{\gamma}(a)|_H} \int_{B_{\gamma}(a)} |(b_1(x) - (b_1)_{B_{\gamma}(a)}) (b_2(x) - (b_2)_{B_{\gamma}(a)}) T_k(f^0)(x)|^r \, dx \right)^{1/r}$$

$$\leq \frac{1}{\omega(B_{\gamma}(a))|B_{\gamma}(a)|^{1/r}_H} \prod_{i=1}^{2} \left( \int_{B_{\gamma}(a)} |b_i(x) - (b_i)_{B_{\gamma}(a)}|^{q_i} \, dx \right)^{1/q_i} \left( \int_{B_{\gamma}(a)} |T_k(f^0)(x)|^q \, dx \right)^{1/q}$$

$$\leq \frac{\nu(B_{\gamma}(a))\nu_1(B_{\gamma}(a))\nu_2(B_{\gamma}(a))}{\omega(B_{\gamma}(a))|B_{\gamma}(a)|^{1/q}_H} \prod_{i=1}^{2} \|b_i\|_{GC_{q_i,\nu_i}} \|f\|_{GM_{q,\nu}}$$

$$\lesssim \frac{2}{\prod_{i=1}^{2} ||b_i||_{GC_{q_i,\nu_i}} \|f\|_{GM_{q,\nu}}}.$$

Let $1/\tilde{q} = 1/q + 1/q_2$, then $1/r = 1/q_1 + 1/\tilde{q}$. Thus, from Hölder’s inequality, Lemma 2.2 and (i), we obtain

$$E_2 = \frac{1}{\omega(B_{\gamma}(a))} \frac{1}{|B_{\gamma}(a)|_H} \int_{B_{\gamma}(a)} |(b_1(x) - (b_1)_{B_{\gamma}(a)}) T_k((b_2 - (b_2)_{B_{\gamma}(a)}) f^0)(x)|^r \, dx \right)^{1/r}$$

$$\leq \frac{1}{\omega(B_{\gamma}(a))|B_{\gamma}(a)|^{1/r}_H} \left( \int_{B_{\gamma}(a)} |b_1(x) - (b_1)_{B_{\gamma}(a)}|^{q_i} \, dx \right)^{1/q_i} \left( \int_{B_{\gamma}(a)} |T_k((b_2 - (b_2)_{B_{\gamma}(a)}) f^0)(x)|^q \, dx \right)^{1/q}$$

$$\leq \frac{1}{\omega(B_{\gamma}(a))|B_{\gamma}(a)|^{1/r}_H} \left( \int_{B_{\gamma}(a)} |b_1(x) - (b_1)_{B_{\gamma}(a)}|^{q_i} \, dx \right)^{1/q_i} \left( \int_{B_{\gamma}(a)} |(b_2)_{B_{\gamma}(a)} f(x)|^q \, dx \right)^{1/q}$$

$$\leq \frac{\nu(B_{\gamma}(a))\nu_1(B_{\gamma}(a))\nu_2(B_{\gamma}(a))}{\omega(B_{\gamma}(a))} \prod_{i=1}^{2} \|b_i\|_{GC_{q_i,\nu_i}} \|f\|_{GM_{q,\nu}}$$

$$\lesssim \frac{2}{\prod_{i=1}^{2} ||b_i||_{GC_{q_i,\nu_i}} \|f\|_{GM_{q,\nu}}}.$$

Similarly,

$$E_3 \lesssim \frac{2}{\prod_{i=1}^{2} ||b_i||_{GC_{q_i,\nu_i}} \|f\|_{GM_{q,\nu}}}.$$
For $E_4$, from Lemma 2.2, Hölder’s inequality and (i), we obtain

$$E_4 = \frac{1}{\omega(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|} \int_{B_\gamma(a)} |T_k(b_1 - (b_1)_{B_\gamma(a)})(b_2 - (b_2)_{B_\gamma(a)})f^0(x)|^r dx \right)^{1/r}$$

$$\leq \frac{1}{\omega(B_\gamma(a))|B_\gamma(a)|^{1/r}} \left( \int_{B_\gamma(a)} |(b_1(x) - (b_1)_{B_\gamma(a)})(b_2(x) - (b_2)_{B_\gamma(a)})f(x)|^r dx \right)^{1/r}$$

$$\leq \frac{1}{\omega(B_\gamma(a))|B_\gamma(a)|^{1/r}} \left( \int \prod_{i=1}^2 |b_i(x) - (b_i)_{B_\gamma(a)}|^q\nu_i dx \right)^{1/q} \left( \int \prod_{i=1}^2 |f(x)|^q dx \right)^{1/q}$$

$$\leq \frac{\nu(B_\gamma(a))\nu_1(B_\gamma(a))\nu_2(B_\gamma(a))}{\omega(B_\gamma(a))} \prod_{i=1}^2 \|b_i\|_{GC_{q_i,\nu_i}} \|f\|_{GM_{q,\nu}}$$

$$\lesssim \prod_{i=1}^2 \|b_i\|_{GC_{q_i,\nu_i}} \|f\|_{GM_{q,\nu}}.$$

To estimate $E_5$, we need to consider $|T_k(f^\infty)(x)|$, firstly. In fact from (4.3), it is easy to see that

$$|T_k(f^\infty)(x)| \lesssim \|f\|_{GM_{q,\nu}} \sum_{j=\gamma+1}^\infty \nu(B_j(a)). \tag{4.6}$$

Therefore, from Hölder’s inequality, (4.6) and (ii), we obtain

$$E_5 = \frac{1}{\omega(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|} \int_{B_\gamma(a)} |(b_1(x) - (b_1)_{B_\gamma(a)})(b_2(x) - (b_2)_{B_\gamma(a)})f^\infty(x)|^r dx \right)^{1/r}$$

$$\leq \frac{1}{\omega(B_\gamma(a))|B_\gamma(a)|^{1/r}} \left( \int \prod_{i=1}^2 |b_i(x) - (b_i)_{B_\gamma(a)}|^q\nu_i dx \right)^{1/q} \left( \int \prod_{i=1}^2 |T_k(f^\infty)(x)f(x)|^q dx \right)^{1/q}$$

$$\leq \sum_{j=\gamma+1}^\infty \frac{\nu(B_j(a))\nu_1(B_j(a))\nu_2(B_j(a))}{\omega(B_\gamma(a))} \prod_{i=1}^2 \|b_i\|_{GC_{q_i,\nu_i}} \|f\|_{GM_{q,\nu}}$$

$$\lesssim \prod_{i=1}^2 \|b_i\|_{GC_{q_i,\nu_i}} \|f\|_{GM_{q,\nu}}.$$

It is similar to the estimate of (4.3), for $x \in B_\gamma(a)$, by $\Omega \in L^\infty(Q^*_{Q_p})$ and (2.2), we can
deduce that
\[
|T_k(b_2 - (b_2)_{B,γ(a)}) f^∞(x)| \\
= \left| \int_{|y| > \rho^k} (b_2(x - y) - (b_2)_{B,γ(a)}) f \chi_{B,γ(a)} (x - y) \frac{Ω(y)}{|y|^p} dy \right| \\
\leq \int_{B^c} |b_2(z) - (b_2)_{B,γ(a)}| |f(z)| \frac{Ω(x - z)}{|x - z|^p} dz \\
\lesssim \int_{B^c} |b_2(z) - (b_2)_{B,γ(a)}| |f(z)| \frac{Ω(x - z)}{|x - z|^p} dz \\
\lesssim \sum_{j=\gamma + 1}^{∞} \int_{S_j(a)} p^{-jn} |b_2(z) - (b_2)_{B,γ(a)}| |f(y)| dy \\
\leq \|f\|_{GM_q,ν} \sum_{j=\gamma + 1}^{∞} p^{-jn} |B_j(a)|^{1 - 1/q - 1/p} \left( \int_{S_j(a)} |f(y)|^q dy \right)^{1/q} \left( \int_{S_j(a)} |b_2(y) - (b_2)_{B,γ(a)}|^q dy \right)^{1/q} \\
\lesssim \|b_2\|_{GC_{q_2,v_2}} \|f\|_{GM_q,ν} \sum_{j=\gamma + 1}^{∞} (j + 1 - γ) ν(B_j(a)) ν_2(B_j(a)).
\]

(4.7)

Let 1/\bar{q} = 1/q + 1/q_2, then 1/r = 1/q_1 + 1/\bar{q}. Thus, from Hölder’s inequality, (4.7) and (ii), it follows that
\[
E_6 = \frac{1}{ω(B,γ(a))} \left( \frac{1}{|B,γ(a)|_H} \int_{B,γ(a)} |(b_1(x) - (b_1)_{B,γ(a)}) T_k((b_2 - (b_2)_{B,γ(a)}) f^∞)(x)|^q dx \right)^{1/q} \\
\leq \frac{1}{ω(B,γ(a)) |B,γ(a)|_H^{1/r}} \left( \int_{B,γ(a)} |b_1(x) - (b_1)_{B,γ(a)}|^q dx \right)^{1/q} \left( \int_{B,γ(a)} |T_k((b_2 - (b_2)_{B,γ(a)}) f^∞)(x)|^q dx \right)^{1/q} \\
\leq \prod_{i=1}^{2} \|b_i\|_{GC_{q_i,v_i}} \|f\|_{GM_q,ν} \frac{1}{ω(B,γ(a))} \sum_{j=\gamma + 1}^{∞} (j + 1 - γ) ν(B_j(a)) ν_2(B_j(a)) ν_1(B,γ(a)) \\
\lesssim \prod_{i=1}^{2} \|b_i\|_{GC_{q_i,v_i}} \|f\|_{GM_q,ν}.
\]

It’s analogues to the estimate of E_6, we obtain
\[
E_7 \lesssim \prod_{i=1}^{2} \|b_i\|_{GC_{q_i,v_i}} \|f\|_{GM_q,ν}.
\]
Moreover, since $\Omega \in L^\infty(\mathbb{Q}_p^n)$, then by (2.2), it is easy to see that

$$|T_k((b_1 - (b_1)_{B_\gamma(a)})(b_2 - (b_2)_{B_\gamma(a)})f^\infty)(x)|$$

$$= \int_{|x-z|_p > p^k} (b_1(z) - (b_1)_{B_\gamma(a)})(b_2(z) - (b_2)_{B_\gamma(a)})f \chi_{B_\gamma(a)}(z) \frac{\Omega(x-z)}{|x-z|^n} \, dz$$

$$\leq \int_{B_\gamma(a)} |b_1(z) - (b_1)_{B_\gamma(a)}||b_2(z) - (b_2)_{B_\gamma(a)}||f(z)||\Omega(x-z)|| \, dz$$

$$\lesssim \sum_{j=\gamma+1}^{\infty} \int S_j(a) p^{-jn}|b_1(z) - (b_1)_{B_\gamma(a)}||b_2(z) - (b_2)_{B_\gamma(a)}||f(y)|| dy$$

$$= \sum_{j=\gamma+1}^{\infty} p^{-jn}|B_j(a)|H^{1/q-1/q_1-1/q_2} \left( \int_{S_j(a)} |f(y)|^q \, dy \right)^{1/q} \left( \int_{S_j(a)} |b_1(y) - (b_1)_{B_\gamma(a)}|^q \, dy \right)^{1/q_1}$$

$$\times \left( \int_{S_j(a)} |b_2(y) - (b_2)_{B_\gamma(a)}|^q \, dy \right)^{1/q_2}$$

$$\lesssim \frac{2}{\omega(B_\gamma(a))} \|b_1\|_{GC_{q_1,\nu_1}} \|f\|_{GM_{q,\nu}} \sum_{j=\gamma+1}^{\infty} (j + 1 - \gamma)^2 \nu(B_j(a)) \nu_1(B_j(a)) \nu_2(B_j(a)).$$

(4.8)

Therefore, from (4.8) and (ii), we get that

$$E_8 = \frac{1}{\omega(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|_H} \int_B |T_k((b_1 - (b_1)_{B_\gamma(a)})(b_2 - (b_2)_{B_\gamma(a)})f^\infty)(x)|r \, dx \right)^{1/r}$$

$$\leq \frac{2}{\omega(B_\gamma(a))} \|b_1\|_{GC_{q_1,\nu_1}} \|f\|_{GM_{q,\nu}} \sum_{j=\gamma+1}^{\infty} (j + 1 - \gamma)^2 \nu(B_j(a)) \nu_1(B_j(a)) \nu_2(B_j(a))$$

$$\lesssim \frac{2}{\omega(B_\gamma(a))} \|b_1\|_{GC_{q_1,\nu_1}} \|f\|_{GM_{q,\nu}}.$$

Combining (4.5) and the estimates of $E_1, E_2, \ldots, E_8$, we have

$$\frac{1}{\omega(B_\gamma(a))} \left( \frac{1}{|B_\gamma(a)|_H} \int_{B_\gamma(a)} |T_k(b_1,b_2)(f)(x)|r \, dx \right)^{1/r} \leq \frac{2}{\omega(B_\gamma(a))} \|b_1\|_{GC_{q_1,\nu_1}} \|f\|_{GM_{q,\nu}},$$

which means that the commutator $T_k^{(b_1,b_2)}$ is bounded from $GM_{q,\nu}$ to $GM_{r,\omega}$.

Moreover, from Lemma 2.2 and the definition of $GM_{q,\omega}(\mathbb{Q}_p^n)$, it is obvious that $T_k^\partial(f) = \lim_{k \to -\infty} T_k^\partial(f)$ exists in the space of $GM_{q,\omega}$, and the commutator $T_k^\partial$ is bounded from $GM_{q,\nu}$ to $GM_{q,\omega}$.

Therefore, the proof of Theorem 3.3 is complete.

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