HETERODIMENSIONAL TANGENCIES ON CYCLES LEADING TO STRANGE ATTRAKTORS

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ABSTRACT. In this paper, we study a two-parameter family \{ϕ_{µ,ν}\} of 3-dimensional diffeomorphisms which have a bifurcation induced by simultaneous generation of a heterodimensional cycle and a heterodimensional tangency associated to two saddle points. We show that such a codimension-2 bifurcation generates a quadratic homoclinic tangency associated to one of the saddle continuations which unfolds generically with respect to some one-parameter subfamily of \{ϕ_{µ,ν}\}. Moreover, from this result together with some well-known facts, we detect some nonhyperbolic phenomena (i.e., the existence of nonhyperbolic strange attractors and the \(C^2\) robust tangencies) arbitrarily close to the codimension-2 bifurcation.

1. INTRODUCTION

When diffeomorphisms act on manifolds of dimension greater than or equal to three, it is well known that nonhyperbolic phenomena are caused by the existence of heteroclinic cycles containing two saddle points with different indexes, called heterodimensional cycles, as well as that of homoclinic tangencies [13, 16, 10, 15, 17, 11]. Heteroclinic cycles presented a new mechanism in dynamics, which has been studied Díaz et al. [7, 5, 8, 2, 9, 3, 4]. As is suggested in [14], these two classes of nonhyperbolic diffeomorphisms seem to occupy a large part in the complement of the hyperbolic diffeomorphisms. In this paper, we study 3-dimensional diffeomorphisms which have heterodimensional cycles and tangencies of certain type simultaneously.

Let \(ϕ\) be a diffeomorphism on a 3-dimensional smooth manifold \(M\) which has two saddle fixed points \(p\) and \(q\) satisfying \(\text{index}(q) = \text{index}(p) + 1\), where \(\text{index}(\cdot)\) denotes the dimension of the unstable manifold of the saddle point. A heteroclinic point \(r\) of the stable manifold \(W^s(p)\) and the unstable manifold \(W^u(q)\) is called a heterodimensional tangency of \(W^s(p)\) and \(W^u(q)\) if \(r\) satisfies

- \(T_rW^s(p) + T_rW^u(q) \neq T_rM\);
- \(\text{dim}(T_rW^s(p)) + \text{dim}(T_rW^u(q)) > \text{dim}(M)\),

where \(\text{dim}(\cdot)\) denotes the dimension of the space.

Our main theorem in this paper is as follows. We will present some definitions and generic conditions used in the statement of Theorem A in Section 2.

Theorem A. Let \(M\) be a 3-dimensional \(C^2\) manifold, and let \(\{ϕ_{µ,ν}\}\) be a two-parameter family of \(C^2\) diffeomorphisms \(ϕ_{µ,ν} : M \to M\) which \(C^2\) depends on \((µ, ν)\)
and has continuations of saddle fixed points \( p_{\mu, \nu} \) and \( q_{\mu, \nu} \) with \( \text{index}(p_{\mu, \nu}) = 1 \) and \( \text{index}(q_{\mu, \nu}) = 2 \). Suppose that the following conditions hold.

- Each \( \varphi_{\mu, \nu} \) is locally \( C^2 \) linearizable in a small neighborhood \( N(q_{\mu, \nu}) \) of \( q_{\mu, \nu} \).
- \( \varphi = \varphi_{0,0} \) has a heterodimensional cycle containing the fixed points \( p = p_{0,0}, q = q_{0,0} \), a nondegenerate heterodimensional tangency \( r \), a quasi-transverse intersection \( s \in W^s(q) \cap W^u(p) \).
- \( \{ \varphi_{\mu, \nu} \} \) satisfies the generic conditions (C1)-(C4) given in Section 2.

Then, for a sufficiently small \( \varepsilon > 0 \) and any \( \mu \) in either \( (0, \varepsilon) \) or \( (-\varepsilon, 0) \), there exist infinitely many \( \nu \) such that \( \varphi_{\mu, \nu} \) has a quadratic homoclinic tangency associated to \( p_{\mu, \nu} \) which unfolds generically with respect to the \( \nu \)-parameter family \( \{ \varphi_{\mu, \nu} \} \).

**Remark 1.1.**

1. The generic conditions (C2) and (C3) imply the setting such that a heterodimensional tangency and a quasi-transverse intersection unfold generically with respect to the parameters \( \mu \) and \( \nu \), respectively. The point to notice is that the newly detected homoclinic tangency in Theorem A can be also controlled by these parameters. It is not hard to get a one parameter family in the infinite dimensional space \( \text{Diff}^2(M) \) with respect to which the homoclinic tangency unfolds generically. However, in our proof, we need to show that the tangency given in Subsection 4.1 unfolds generically with respect to the \( \nu \)-parameter family \( \{ \varphi_{\mu, \nu} \} \) in the two-dimensional subspace \( \{ \varphi_{\mu, \nu} \} \) of \( \text{Diff}^2(M) \).
2. Theorem A holds for a homoclinic tangency associated to \( q_{\mu, \nu} \) instead of \( p_{\mu, \nu} \) if we replace the conditions in (C1) and (C4) by appropriate ones.
3. We need the \( C^2 \) smoothness in the local linearization around \( q_{\mu, \nu} \) to estimate the curvatures of \( W^s(p_{\mu, \nu}) \) and \( W^u(p_{\mu, \nu}) \) at the tangency in Section 3.

Figure 1.1 illustrates an example of heterodimensional cycles containing a nondegenerate heterodimensional tangency of hyperbolic type, see Definition 2.1(3).
The conclusion obtained from Theorem A reminds us of prior works associated with homoclinic tangencies. The one is related to strange attractors and the other $C^2$ robust tangencies.

First, let us discuss the former one. Viana showed the following theorem.

**Theorem 1.2 (Viana [18]).** For a generic subset of one-parameter families $\{\varphi_\mu\}$ of $C^\infty$ diffeomorphisms on any manifolds of the dimension greater than or equal to two that unfolds a homoclinic tangency at parameter value $\mu = 0$ associated to a sectionally dissipative saddle periodic point, there is a subset $S$ of $\mathbb{R}$ such that

- $S \cap (-\epsilon, \epsilon)$ has a positive Lebesgue measure for every $\epsilon > 0$,
- for all $\mu \in S$, $\varphi_\mu$ exhibits nonhyperbolic strange attractors in a $\mu$-dependent neighborhood of the orbit of tangency.

Leal [12] extended this result and showed the existence of infinitely many strange attractors. A saddle periodic point is said to be sectionally dissipative if the product of any two eigenvalues of the derivative at the point has norm less than one. Also, $\Lambda$ is a strange attractor of $\varphi$ if $\Lambda$ is a compact, $\varphi$-invariant, transitive set and the basin $W_s(\Lambda)$ has a nonempty interior and there exists $z_1 \in \Lambda$ such that $\{\varphi^n(z_1) : n \geq 0\}$ is dense in $\Lambda$ and $||d\varphi^n(z_1)|| \geq e^{cn}$ for all $n \geq 0$ and some $c > 0$. Note, Viana [18] assumed that for simplicity $\varphi_\mu$ is $C^3$ linearizable in a neighborhood of the saddle point for $\mu$ is sufficiently close to 0. Combining these extra conditions and our result for the cycle containing the heterodimensional tangency imply the following corollary.

**Corollary B.** Let $\{\varphi_{\mu, \nu}\}$ be the two-parameter family of 3-dimensional $C^\infty$ diffeomorphisms having the heterodimensional cycle with the nondegenerate heterodimensional tangency for $(\mu, \nu) = (0, 0)$ given in Theorem A. Suppose that each $\varphi_{\mu, \nu}$ is locally $C^3$ linearizable in a small neighborhood of $p_{\mu, \nu}$ and $\varphi_{0, 0}$ is sectionally dissipative at $p_{0, 0}$. Then there exists a positive Lebesgue measure subset $A$ of the $\mu \nu$-plane arbitrarily near $(0, 0)$ such that, for any $(\mu, \nu) \in A$, $\varphi_{\mu, \nu}$ exhibits nonhyperbolic strange attractors.

Next, we discuss $C^2$ robust homoclinic tangencies derived from the heterodimensional cycle having a nondegenerate tangency. A homoclinic tangency of a diffeomorphism $\varphi$ associated with a hyperbolic set $\Gamma$ is $C^r$ robust if there is a $C^r$ neighborhood $U$ of $\varphi$ such that every diffeomorphism $\psi \in U$ has a homoclinic tangency associated with the continuations of $\Gamma$ for $\psi$. Newhouse [13] showed, in the $C^2$ topology, a homoclinic tangency of surface diffeomorphisms generates $C^2$ robust homoclinic tangencies. This property yields the so-called $C^2$ Newhouse phenomenon: there is a nonempty open set of $C^2$ diffeomorphisms and its residual subset such that every diffeomorphism in the subset has infinitely many sinks. The result is extended by Palis-Viana [15] to the higher dimensional case. Palis and Viana proved the result under the sectional dissipativeness and linearizing conditions as in [18]. Moreover, Romero [17] proved the following theorem without these conditions.

**Theorem 1.3 (Romero [17]).** Let $\varphi$ be a $C^2$ diffeomorphism on a manifold $M$, $\dim(M) \geq 3$, having a homoclinic tangency associated with a saddle periodic point of $\varphi$ whose index is greater or equal to 2. Then there are diffeomorphisms arbitrarily $C^2$ close to $\varphi$ having robust homoclinic tangencies.

Therefore, we have the following corollary.
Corollary C. Let \{\varphi_{\mu,\nu}\} be the two-parameter family of 3-dimensional \(C^2\) diffeomorphisms having the heterodimensional cycle with the nondegenerate heterodimensional tangency for \((\mu, \nu) = (0, 0)\) given in Theorem A. Then, for a sufficiently small \(\varepsilon > 0\) and any \(\mu\) in either \((0, \varepsilon)\) or \((-\varepsilon, 0)\), there are infinitely many \(\nu\) such that every \(C^2\) neighborhood of \(\varphi_{\mu,\nu}\) contains a diffeomorphism having a \(C^2\) robust homoclinic tangency.

Remark 1.4. If we add a weak dissipative condition (see in [17, Theorem A (1.1)]) to the assumptions of Corollary C then, the \(C^2\) Newhouse phenomenon is obtained from our settings.

Remark 1.5. In the \(C^1\) case, Díaz et al. [6] have already proved that the unfolding of heterodimensional tangencies leads to non-dominated dynamics and therefore (by results of [2] and [3]) to the \(C^1\) Newhouse phenomenon (see also [1] for a different approach to the phenomenon). On the other hand, the theory of strange attractors has not been so far developed in the \(C^l\) category \((l = 1, 2)\).

Outline of the proof of Theorem A. We will finish Introduction by presenting a sketch of the proof of Theorem A.

In Section 2, we give definitions and the generic conditions (C1)-(C4) used in Theorem A. Especially, a nondegenerate heterodimensional tangency, which is one of main ingredients of this paper, is introduced explicitly there. Such tangencies are classified into the elliptic and hyperbolic types, see Definition 2.1-(3).

Figure 1.2. (1) The situation when \((\mu, \nu) = (0, 0)\). (2) \(s_{\mu,0}\) is a quasi-transverse intersection with respect to the new parameter. (3) \(l_m\) converges \(W_{\text{loc}}^{uu}(q_{\mu,0})\) in \(C^2\) topology. (4) \(l_{m,\nu_m}\) is a continuation of \(l_m\) for \(\nu = \nu_m\).

In Section 3 we prove three lemmas. Lemma 3.1 shows the existence of the re-parametrization of \{\varphi_{\mu,\nu}\} such that, for any \((\mu, 0)\) in the new parameter, \(W^s(q_{\mu,0})\)
and $W^u(p_{\mu,0})$ still have a quasi-transverse intersection $s_{\mu,0}$ which unfolds generically with respect to the $\nu$-parameter but the tangency $r$ is annihilated. See Fig.(1.2) (1) and (2). Lemma 3.2 applies the $C^2$ inclination lemma to a shorter curve $l_{m_0}$ in $W^u(p_{\mu,0})$ passing through $s_{\mu,0}$ so that some curve $l_m$ in $\varphi_{\mu,0}^m(l_{m_0})$ containing $\varphi_{\mu,0}^m(s_{\mu,0})$ $C^2$ converges to $W^u_{\text{loc}}(p_{\mu,0})$ as $m \to \infty$. See Fig.(1.2) (3). Lemma 3.3 explains a connection between quadratic tangencies and curvatures, which is used to show that the homoclinic tangencies obtained in Section 3 are quadratic.

Assertion 4.2 and Assertion 4.3 in Section 4 show that the generic unfolding of heterodimensional tangencies introduces the existence of a quadratic homoclinic tangency $\tau_m$ as illustrated in Fig. (1.2) (4). Finally, we prove in Proposition 4.4 that the tangency $\tau_m$ unfolds generically with respect to the $\nu$-parameter. These results assure the proof of Theorem A.

2. Definitions and generic conditions

In this section, we present some definitions needed in later sections and generic conditions adopted as hypotheses in Theorem A.

2.1. Definitions.

Definition 2.1. Suppose that $M$ is a 3-dimensional $C^2$ manifold. Let $\{l_{\nu}\}_{\nu \in J}$, $\{m_{\nu}\}_{\nu \in J}$ be $C^2$ families of regular curves in $M$, and let $\{S_{\nu}\}_{\nu \in J}$, $\{Y_{\nu}\}_{\nu \in J}$ be $C^2$ families of regular surfaces in $M$, where $J$ is an open interval.

(1) Suppose that $l_{\nu_0}$ and $m_{\nu_0}$ intersect at a point $s$ for some $\nu_0 \in J$ and some open neighborhood $U$ of $s$ in $M$ has a $C^2$ change of coordinates with respect to which $m_{\nu} = \{(0,0,z) : z \in U\}$ for any $\nu \in J$ near $\nu_0$. We say that $s$ is a quasi-transverse intersection of $l_{\nu_0}$ and $m_{\nu_0}$ if

$$\dim (T_s(l_{\nu_0}) + T_s(m_{\nu_0})) = 2.$$ 

Moreover, $s$ unfolds generically at $\nu = \nu_0$ with respect to $\{l_{\nu}\}_{\nu \in J}$, $\{m_{\nu}\}_{\nu \in J}$ if there exists a $C^2$ map $s : J \to M$ with $s(\nu) = s_{\nu} \in l_{\nu}$ for any $\nu \in J$ and $s(\nu_0) = s$ and a $C^2$ function $d : J \to \mathbb{R}^+$ with $d(\nu_0) \neq 0$ such that

$$\dim (T_s(l_{\nu}) + T_s(m_{\nu})) = 2,$$

for any $\nu$ near $\nu_0$, where $N$ is the one-dimensional space spanned by the non-zero tangent vector $(ds_{\nu}/d\nu)|_{\nu=\nu_0}$. This property corresponds to the conditions (GU1)–(GU3) in [9] §2.2.1.

(2) Suppose that $l_{\nu_0}$ and $S_{\nu_0}$ intersect at a point $\tau$ for some $\nu_0 \in J$. We say that $\tau$ is a quadratic tangency (or a contact of order 1) of $l_{\nu_0}$ and $S_{\nu_0}$ if there exists some $C^2$ change of coordinates on an open neighborhood $U(\tau)$ of $\tau$ such that $\tau = (0,0,0)$, $S_{\nu} = \{(x,y,z) : z \in U(\tau) \}$ and $l_{\nu}$ has a regular parametrization $l(\nu,t) = (x(\nu,t),y(\nu,t),z(\nu,t))$ with $l(\nu_0,0) = (0,0,0)$ and

$$\frac{\partial z}{\partial t}(\nu_0,0) = 0 \quad \text{and} \quad \frac{\partial^2 z}{\partial t^2}(\nu_0,0) \neq 0,$$

The tangency $\tau$ is said to unfold generically at $\nu = \nu_0$ with respect to $\{l_{\nu}\}_{\nu \in J}$ and $\{S_{\nu}\}_{\nu \in J}$ if

$$\frac{\partial z}{\partial \nu}(\nu_0,0) \neq 0.$$
(3) Suppose that $S_{\nu_0}$ and $Y_{\nu_0}$ intersect at a point $r$ for some $\nu_0 \in J$. We say that $r$ is a nondegenerate heterodimensional tangency of $S_{\nu_0}$ and $Y_{\nu_0}$ if there exists a $C^2$ coordinate on an open set $U$ in $M$ containing $r$ with $r = (u_0, v_0, 0)$ for some $u_0, v_0 \in \mathbb{R}$, $S_{\nu} = \{(x, y, z) \in U : z = 0\}$ and such that $Y_{\nu}$ has a parametrization $(x, y, f_{\nu}(x, y))$ the third entry $f_{\nu}(x, y) = f(\nu, x, y)$ of which is a $C^2$ function satisfying

$$f_{\nu_0}(u_0, v_0) = 0, \quad \frac{\partial f_{\nu_0}}{\partial x}(u_0, v_0) = \frac{\partial f_{\nu_0}}{\partial y}(u_0, v_0) = 0, \quad \det(Hf_{\nu_0}(u_0, v_0)) \neq 0,$$

where $(Hf_{\nu_0})$ is the Hessian matrix of $f_{\nu_0}$ at $(x, y) = (u_0, v_0)$.

The tangency $r$ unfolds generically at $\nu = \nu_0$ if

$$\frac{\partial f}{\partial \nu}(u_0, u_0, v_0) \neq 0.$$

Remark 2.2. It is easy to see that the property (1) does not depend on the coordinates used to set $l_{\nu}$ in the $z$-axis. Similarly, the properties (2) and (3) do not depend on the coordinates used to set $S_{\nu}$ in the $xy$-plane.

When $\det(Hf_{\nu_0}(u_0, v_0)) > 0$ (resp. $< 0$) in Definition 2.1(3), we say that the tangency $r = (u_0, v_0, 0)$ is of elliptic (resp. hyperbolic) type. The Taylor expansion of $f_{\nu_0}$ around $(u_0, v_0)$ is

$$f_{\nu_0}(x, y) = 1 \frac{\partial^2 f_{\nu_0}}{\partial x^2}(u_0, v_0)(x - u_0)^2 + \frac{\partial^2 f_{\nu_0}}{\partial x \partial y}(u_0, v_0)(x - u_0)(y - v_0)$$

$$+ \frac{1}{2} \frac{\partial^2 f_{\nu_0}}{\partial y^2}(u_0, v_0)(y - v_0)^2 + o((|x - u_0| + |y - v_0|)^2).$$

From (2.3) together with the classification of quadratic surfaces in $\mathbb{R}^3$, we know that $Y_{\nu_0}$ has the form near $r = (u_0, v_0, 0)$ as illustrated in Fig. 2.1.

![Figure 2.1](image)

**Figure 2.1.** (1) $r$ is of elliptic type and $\partial^2 f_{\nu_0}(u_0, v_0)/\partial x^2 < 0$. (2) $r$ is of elliptic type and $\partial^2 f_{\nu_0}(u_0, v_0)/\partial x^2 > 0$. (3) $r$ is of hyperbolic type.

2.2. **Generic conditions.** Throughout the remainder of this paper, we suppose that $\varphi$ is a 3-dimensional $C^2$ diffeomorphism with saddle fixed points $p$ of index$(p) = 1$ and $q$ of index$(q) = 2$, and such that $W^s(p)$ and $W^u(q)$ have a nondegenerate
heterodimensional tangency \( r \), \( W^u(p) \) and \( W^s(q) \) have a quasi-transverse intersection \( s \). The \( \varphi \) is locally \( C^2 \) linearizable in a neighborhood \( U(q) \) of \( q \) if there exists a \( C^2 \) linearizing coordinate \((x, y, z)\) on \( U(q) \), that is,

\[
q = (0, 0, 0), \quad \varphi(x, y, z) = (\alpha x, \beta y, \gamma z)
\]

for any \((x, y, z) \in U(q)\) with \( \varphi(x, y, z) \in U(q) \), where \( \alpha, \beta \) and \( \gamma \) are eigenvalues of \((d\varphi)_q\).

One can take a local unstable manifold \( W^u_{\text{loc}}(q) \) so that it is an open disk in the plane \( \{z = 0\} \) centered at \((x, y) = (0, 0)\). We may assume that the both points \( r, s \) are contained in \( U(q) \) if necessary replacing \( r \) (resp. \( s \)) by \( \varphi^{-n}(r) \) (resp. \( \varphi^n(s) \)) with sufficiently large \( n \in \mathbb{N} \). We set

\[
r = (u_0, v_0, 0)
\]

with respect to the linearizing coordinate on \( U(q) \).

We suppose moreover that \( \{\varphi_{\mu, \nu}\} \) is a two-parameter family in \( \text{Diff}^2(M) \) with \( \varphi_{0,0} = \varphi \) and satisfying the conditions of Theorem \( \text{A} \). In particular, \( \varphi_{\mu, \nu} \) is locally \( C^2 \) linearizable in a small neighborhood \( U(q_{\mu, \nu}) \) of \( q_{\mu, \nu} \) in \( M \) and hence \( \varphi_{\mu, \nu} \) has the form as \( \text{(2.4)} \) in \( U(q_{\mu, \nu}) \), where \( \alpha, \beta, \gamma \) are \( C^2 \) functions on \( \mu, \nu \), i.e., \( \alpha = \alpha_{\mu, \nu}, \beta = \beta_{\mu, \nu}, \gamma = \gamma_{\mu, \nu} \).

We will put the following generic conditions (C1)-(C4) as the hypotheses in Theorem \( \text{A} \)

(C1) (Generic condition for \( q \)) The \( \varphi \) is locally \( C^2 \) linearizable at \( q \) given as in \( \text{(2.4)} \). For simplicity, we suppose that every eigenvalues of \((d\varphi)_q\) is positive, that is,

\[
0 < \gamma < 1 < \beta < \alpha.
\]

(C2) (Generic unfolding property for \( r \)) The nondegenerate heterodimensional tangency \( r \) of \( W^u(q) \) and \( W^s(p) \) unfolds generically with respect to the \( \mu \)-parameter families \( \{W^u(q_{\mu,0})\} \) and \( \{W^s(p_{\mu,0})\} \).

(C3) (Generic unfolding property for \( s \)) The quasi-transverse intersection \( s \) of \( W^s(q) \) and \( W^u(p) \) unfolds generically with respect to the \( \nu \)-parameter families \( \{W^s(q_{0,\nu})\} \) and \( \{W^u(p_{0,\nu})\} \).

(C4) (Additional generic conditions) The tangency \( r \) is not on the \( x \)-axis \( W^uu_{\text{loc}}(q) \), that is,

\[
v_0 \neq 0.
\]

There exists a regular parametrization \( l(t) = (x(t), y(t), z(t)) \) \((t \in I)\) of a small curve in \( W^u(p) \cap U(q) \) with respect to the linearizing coordinate \((x, y, z)\) on \( U(q) \) with \( s = l(0) \) and

\[
\frac{dx}{dt}(0) \neq 0,
\]

where \( I \) is an open interval centered at \( 0 \).

There exists a \( C^2 \) function \( f : O \to \mathbb{R} \) defined on an open disk \( O \) in the \( xy \)-plane centered at \( r \) such that \( f(u_0, v_0) = 0 \), \( \{(x, y, f(x, y)) : (x, y) \in O \} \subset W^s(p) \cap U(q) \) and

\[
\frac{\partial^2 f}{\partial x^2}(u_0, v_0) \neq 0.
\]

Note that the condition \( \text{(2.7)} \) is automatically satisfied when \( r \) is of elliptic type.
3. Some lemmas about parametrization and curvatures

The goal of this section is to prove three lemmas needed for the proof of Theorem A. These play important roles in Section 4.

- Lemma 3.1 presents a new parameter \((\hat{\mu}, \hat{\nu})\) such that, for any \(\hat{\mu}\) near 0, there exists a quasi-transverse intersection \(s_{\hat{\mu},0}\) of \(W_s(q_{\hat{\mu},0})\) and \(W^u(p_{\hat{\mu},0})\) which unfolds generically at \(\hat{\nu} = 0\) with respect to the \(\hat{\nu}\)-parameter. After Lemma 3.1 we denote the new parameter \((\hat{\mu}, \hat{\nu})\) again by \((\mu, \nu)\) for simplicity.

- In Lemma 3.2 we show that, for any \(\mu_0\) near 0, there exists a regular curve \(l_m\) in \(W^u(p_{\mu_0},0)\) containing the quasi-transverse intersection \(\varphi_{\mu_0,0}^{-1}(s_{\mu_0,0})\) and arbitrarily \(C^2\)-close to \(W_{uu}\)loc\((q_{\mu_0})\). In particular, this implies that the curvature of \(l_m\) can be taken arbitrarily close to 0 with respect to the linearizing coordinate (2.4) on \(U(q_{\mu_0},0)\).

- Lemma 3.3 gives a connection between the curvature and quadratic tangencies. In fact, we show that a tangency \(\tau\) of a regular curve \(l\) and a regular surface \(S\) in \(\mathbb{R}^3\) is quadratic if the curvature of \(l\) at \(\tau\) is different from the normal curvature of \(S\) at \(\tau\) along the direction tangent to \(l\).

For any \((\mu, \nu)\) near \((0,0)\), we may assume that \(U(q_{\mu,\nu})\) is equal to

\[
D(\delta) := (-\delta, \delta)^3
\]

with respect to the linearizing coordinate given in Subsection 2.2 for some constant \(\delta > 0\). Since \(s\) is a quasi-transverse intersection which unfolds generically with respect to the \(\nu\)-parameter families \(\{W^s(q_{0,\nu})\}\) and \(\{W^u(p_{0,\nu})\}\) by the condition (C3), there exists a \(C^2\) continuation \(\hat{s}_\nu \in W^u(p_{0,\nu}) \cap D(\delta)\) with \(\hat{s}_0 = s\) and such that \(\hat{s}_\nu\) satisfies the conditions same as those for \(s_{\nu}\) in Definition 2.1(1). By (2.5), for any \(\nu\) near 0, the component \(l_{\nu}\) of \(W^u(p_{0,\nu}) \cap D(\delta)\) containing \(\hat{s}_\nu\) meets transversely the \(yz\)-plane at a point \(s_{\nu}\) which defines a \(C^2\) continuation \(\{s_{\nu}\}\) with \(s_0 = s\), see Fig. 3.1. Note that \(d\hat{s}_\nu/d\nu(0) = ds_{\nu}/d\nu(0) + w\) for some \(w \in T_{s}(l_0) = T_s(W^u(p))\), where

\[
\frac{dy}{d\nu}(0) \neq 0.
\]
For any \((\mu, \nu)\) near \((0,0)\), let \(s_{\mu,\nu}\) be a transverse intersection point of \(W^u(p_{\mu,\nu}) \cap D(\delta)\) with the \(yz\)-plane such that \(\{s_{\mu,\nu}\}\) is a \(C^2\) continuation with \(s_{0,\nu} = s_{\nu}\).

**Lemma 3.1.** There exists a constant \(\rho > 0\) and a \(C^2\) function \(\nu : (-\rho, \rho) \to \mathbb{R}\) such that, for any \(\mu \in (-\rho, \rho)\), \(s_{\mu,\nu}(\mu)\) is a quasi-transverse intersection of \(W^s(q_{\mu,\nu}(\mu))\) and \(W^u(p_{\mu,\nu}(\mu))\) which unfolds generically with respect to the \(\nu\)-parameter families \(\{W^s(q_{\mu(\text{fixed}),\nu})\}\) and \(\{W^u(p_{\mu(\text{fixed}),\nu})\}\).

**Proof.** Let \(g(\mu, \nu)\) be the \(y\)-coordinate of \(s_{\mu,\nu}\). By \(\partial g(\mu, \nu) \neq 0\). Hence, by the Implicit Function Theorem, there exists a \(C^2\) function \(\nu : (-\rho, \rho) \to \mathbb{R}\) for some \(\rho > 0\) such that

\[
\nu(0) = 0, \quad g(\mu, \nu(\mu)) = 0, \quad \frac{\partial g}{\partial \nu}(\mu, \nu(\mu)) \neq 0
\]

for any \(\mu \in (-\rho, \rho)\). This implies that \(s_{\mu,\nu}(\mu)\) is a quasi-transverse intersection unfolding generically at \(\nu = \nu(\mu)\) with respect to the \(\nu\)-parameter families \(\{W^s(q_{\mu,\nu})\}\) and \(\{W^u(p_{\mu,\nu})\}\). \(\square\)

**A new parametrization.** Consider the coordinate \((\hat{\mu}, \hat{\nu})\) on the parameter space defined by \(\hat{\mu} = \mu, \hat{\nu} = \nu - \nu(\mu)\). For simplicity, we denote the new coordinate again by \((\mu, \nu)\). Thus, there exists a continuation \(\{s_{\mu,0}\}_{\mu \in (-\rho, \rho)}\) of quasi-transverse intersections of \(W^s(q_{\mu,0})\) and \(W^u(p_{\mu,0})\) such that each \(s_{\mu,0}\) unfolds generically at \(\nu = 0\) with respect to the \(\nu\)-parameter families \(\{W^s(q_{\mu,\nu})\}\) and \(\{W^u(p_{\mu,\nu})\}\).

Fix \(\mu_0\) with sufficiently small \(|\mu_0|\) arbitrarily. By the properties \([2.4] [2.6]\), there exists \(m_0 \in \mathbb{N}\) such that, for any \(m \geq m_0\), one can parameterize the component \(l_m\) of \(W^u(p_{\mu_0,0}) \cap D(\delta)\) containing \(\phi^m_{\mu_0,0}(s_{\mu_0,0})\) so that \(l_m(0) = \phi^m_{\mu_0,0}(s_{\mu_0,0})\) and

\[
l_m(t) = (t, y_m(t), z_m(t)) \quad (t \in (-\delta, \delta)).
\]

**Lemma 3.2.** The sequence \(\{l_m\}\) \(C^2\) converges uniformly to \(W^u_{\text{loc}}(q_{\mu_0,0})\) as \(m \to \infty\).

In particular, for any \(\varepsilon > 0\), there exists \(\hat{m}_0 \geq m_0\) such that the curvature at any point of \(l_m\) is less than \(\varepsilon\) with respect to the standard Euclidean metric on \(U(q_{\mu_0,0}) = D(\delta)\) if \(m \geq \hat{m}_0\).

**Proof.** By \(\{\mu_0\}\), for any \(m \geq m_0\),

\[
l_m(t) = (t, \beta^n y_{\mu_0}(\alpha^{-n} t), \gamma^n z_{\mu_0}(\alpha^{-n} t)),
\]

where \(n = m - m_0\), \(\alpha = \alpha_{\mu_0,0}\), \(\beta = \beta_{\mu_0,0}\), \(\gamma = \gamma_{\mu_0,0}\). Thus we have

\[
\frac{d l_m}{dt}(t) = \left( 1, \frac{\beta^n}{\alpha^n} \frac{dy_{\mu_0}}{dt}(\alpha^{-n} t), \frac{\gamma^n}{\alpha^n} \frac{dz_{\mu_0}}{dt}(\alpha^{-n} t) \right) \quad \text{uniformly} \quad (1, 0, 0),
\]

\[
\frac{d^2 l_m}{dt^2}(t) = \left( 0, \frac{\beta^n}{\alpha^{2n}} \frac{d^2 y_{\mu_0}}{dt^2}(\alpha^{-n} t), \frac{\gamma^n}{\alpha^{2n}} \frac{d^2 z_{\mu_0}}{dt^2}(\alpha^{-n} t) \right) \quad \text{uniformly} \quad (0, 0, 0)
\]

as \(m \to \infty\). Since \(\{l_m(0)\}_{m=m_0}\) converges to \(q_{\mu_0,0} = (0, 0, 0)\), it follows from \([3.2]\) that \(\{l_m\}\) \(C^2\) converges uniformly to the \(x\)-axis in \(D(\delta)\). \(\square\)

**Lemma 3.3.** Let \(S\) be a regular surface in the Euclidean 3-space \(\mathbb{R}^3\) and \(l\) a regular curve tangent to \(S\) at \(\tau\). Suppose that the curvature \(\kappa_l(\tau)\) of \(l\) at \(\tau\) is less than the absolute value of the normal curvature \(\kappa_S(\tau, \mathbf{w})\) of \(S\) at \(\tau\) along a non-zero vector \(\mathbf{w}\) tangent to \(l\). Then tangency of \(S\) and \(l\) at \(\tau\) is quadratic.
Proof. By changing the coordinate \((x, y, z)\) on \(\mathbb{R}^3\) by an isometry, we may assume that \(\tau = (0, 0, 0)\), the tangent space of \(S\) at \(\tau\) is the \(xy\)-plane and \(w/\|w\| = (1, 0, 0)\). Then one can suppose that \(S\) (resp. \(l\)) is parameterized as \((x, y, \psi(x, y))\) (resp. \((x, f_1(x), f_2(x))\)) in a small neighborhood of \((0, 0, 0)\) in \(\mathbb{R}^3\). Since the graph of \(z = \psi(x, 0)\) is the cross section of \(S\) along the \(xz\)-plane,

\[
|\kappa_S(\tau, w)| = \frac{|g''(0)|}{(g'(0)^2 + 1)^{3/2}} = |g''(0)|, 
\]

where \(g(x) = \psi(x, 0)\). Since the graph of \(z = f_2(x)\) coincides with the orthogonal projection \(l\) of \(l\) into the \(xz\)-plane,

\[
\kappa_\tau(\tau) \geq \kappa_S(\tau) = \frac{|f''_2(0)|}{(f'_2(0)^2 + 1)^{3/2}} = |f''_2(0)|. 
\]

It follows from our assumption \(|\kappa_S(\tau, w)| > \kappa_\tau(\tau)|\) that \(g''(0)| > |f''_2(0)|\). This shows that the tangency at \(\tau\) is quadratic. 

\[\square\]

4. Proof of Theorem A

In this section, we give the proof of Theorem A.

- In Subsection 4.1 we show that, for any \(\mu_0\) in either \((-\varepsilon, 0)\) or \((0, \varepsilon)\) and any sufficiently large \(m \in \mathbb{N}\), there exists \(\nu_m\) with \(\lim_{m \to \infty} \nu_m = 0\) such that \(W^u(p_{\mu_0, \nu_m})\) and \(W^s(p_{\mu_0, \nu_m})\) have a quadratic tangency \(\tau_m\) (Assertion 4.2 and Assertion 4.3). Here the sign of \(\mu_0\) is chosen so that \(\mu_0 \cdot b_{\mu_0, 0} < 0\) (resp. \(\mu_0 \cdot b_{\mu_0, 0} > 0\)) if the tangency \(r\) is of elliptic (resp. hyperbolic) type, where \(b_{\mu_0, 0}\) is the coefficient of \((x - \nu_{\mu_0, 0})^2\)-term of the Taylor expansion (1.1). See Fig. 4.1 and Fig. 4.3. As is suggested in Fig. 4.2 the existence of the homoclinic tangency \(\tau_m\) is proved by using the Intermediate Value Theorem. By Lemma 3.2 the curvature of \(W^u(p_{\mu_0, \nu_m})\) at \(\tau_m\) converges to zero as \(m \to \infty\). On the other hand, we will show that the normal curvature of \(W^s(p_{\mu_0, \nu_m})\) at \(\tau_m\) along the direction tangent to \(l_m\) is bounded away from zero. Hence, by Lemma 3.3 the tangency \(\tau_m\) is quadratic.

- In Subsection 4.2 we show that the quadratic homoclinic tangency \(\tau_m\) unfolds generically at \(\nu = \nu_m\) with respect to the \(\nu\)-parameter families \(\{W^s(p_{\mu_0, \nu})\}\) and \(\{W^u(p_{\mu_0, \nu})\}\) by representing a neighborhood of \(\tau_m\) in \(W^s(p_{\mu_0, \nu})\) as the graph of a function of \((x, z)\).

4.1. Existence of quadratic homoclinic tangencies. Let \(\{\varphi_{\mu, \nu}\}\) be the family given in Subsection 2.2. In particular, \(r = (u_0, v_0, 0)\) is a nondegenerate heterodimensional tangency of \(W^u(q)\) and \(W^s(p)\) which unfolds generically with respect to the \(\mu\)-parameter families \(\{W^u(q_{\mu, 0})\}\) and \(\{W^s(p_{\mu, 0})\}\). By our settings in Sections 2 and 3 there exist \(C^2\) functions \(f_{\mu, \nu} : O \subset \mathbb{R}^2 \to \mathbb{R}^2\) depending on \((\mu, \nu)\) with \(f_{0, 0} = f\) and

\[\Sigma(\mu, \nu) := \{(x, y, f_{\mu, \nu}(x, y)) ; (x, y) \in O\} \subset W^s(p_{\mu, \nu}) \cap D(\delta)\]

for any \((\mu, \nu)\) near \((0, 0)\). Since \(\det(Hf)(u_0, v_0) \neq 0\), there exists a uniquely determined \(C^2\) continuation \((u_{\mu, \nu}, v_{\mu, \nu})\) with \((u_{0, 0}, v_{0, 0}) = (u_0, v_0)\) and

\[
\frac{\partial}{\partial x}f_{\mu, \nu}(u_{\mu, \nu}, v_{\mu, \nu}) = \frac{\partial}{\partial y}f_{\mu, \nu}(u_{\mu, \nu}, v_{\mu, \nu}) = 0.
\]
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Proposition 4.1. For a sufficiently small $\varepsilon > 0$ and any $\mu$ in either $(0, \varepsilon)$ or $(-\varepsilon, 0)$, there exists $\nu$ arbitrarily close to 0 such that $\varphi_{\mu, \nu}$ has a quadratic homoclinic tangency associated to $p_{\mu, \nu}$.

By the condition (2.5), $r$ is not in the $x$-axis. One can take the linearizing coordinate on $D(\delta)$ so that $s$ (resp. $r$) is in the upper half space $\{z > 0\}$ (resp. $\{x > 0\}$). The Taylor expansion of $f_{\mu, \nu}$ around $(u_{\mu, \nu}, v_{\mu, \nu})$ has the form:

\[
f_{\mu, \nu}(x, y) = a_{\mu, \nu} + \frac{1}{2} b_{\mu, \nu}(x - u_{\mu, \nu})^2 + c_{\mu, \nu}(x - u_{\mu, \nu})(y - v_{\mu, \nu}) + \frac{1}{2} d_{\mu, \nu}(y - v_{\mu, \nu})^2 + o\left(|x - u_{\mu, \nu}| + |y - v_{\mu, \nu}|\right),
\]

where $a_{0,0} = 0$ and

\[
b_{\mu, \nu} = \frac{\partial^2 f_{\mu, \nu}}{\partial x^2}(u_{\mu, \nu}, v_{\mu, \nu}), \quad c_{\mu, \nu} = \frac{\partial^2 f_{\mu, \nu}}{\partial x \partial y}(u_{\mu, \nu}, v_{\mu, \nu}), \quad d_{\mu, \nu} = \frac{\partial^2 f_{\mu, \nu}}{\partial y^2}(u_{\mu, \nu}, v_{\mu, \nu}).
\]

Since the tangency $r$ unfolds generically with respect to $\varphi = \varphi_{0,0}$ by (C1),

\[
\eta_0 = \frac{\partial a_{\mu, \nu}}{\partial \mu}(\mu, \nu) = (0,0) \neq 0.
\]

If necessary replacing $\mu$ by $-\mu$, we may assume that $\eta_0 > 0$. By the condition (4.1), $b_{0,0} \neq 0$ and hence $b_{\mu, \nu} \neq 0$ for any $(\mu, \nu)$ near $(0, 0)$.

Proposition 4.1 is divided to the following two assertions.

Assertion 4.2 (Elliptic case). If $r$ is of elliptic type, then Proposition 4.1 holds.

Proof. First we consider the case of $b_{\mu, \nu} < 0$ for any $(\mu, \nu)$ near $(0, 0)$. By (4.2), for any sufficiently small $\mu_0 > 0$, the intersection $C_{\mu_0} = \Sigma(\mu_0, 0) \cap \{z = 0\}$ is a circle disjoint from the $x$-axis. For a sufficiently small $h_0 > 0$, $A = \Sigma(\mu_0, 0) \cap \{0 \leq z \leq h_0\}$ is an annulus in $D(\delta)$, see Fig. 4.1(1). Replacing $m_0$ by an integer greater than $m_0$

\[
\text{Figure 4.1. (1) The case of } \mu_0 > 0, \nu = 0, b_{\mu_0, 0} < 0. \text{ (2) The case of } \mu_0 < 0, \nu = 0, b_{\mu_0, 0} > 0. \text{ Each shaded region represents } A.
\]

if necessary, we may assume that $z_m(0) < h_0/2$ for any $m \geq m_0$. By Lemma 3.2, the curve $l_m \subset W^u(p_{\mu_0, 0}) \cap D(\delta)$ given in Section 3 is sufficiently $C^2$ close to the $x$-axis. Thus one can suppose that $\pi_y(l_m) \cap \pi_y(A) = \emptyset$, where $\pi_y : D(\delta) \rightarrow \mathbb{R}$ is
the orthogonal projection defined by $\pi_y(x, y, z) = y$. For any sufficiently small $\nu$, let $l_{m, \nu}$ be the component of $W^u(p_{\mu_0, \nu}) \cap D(\delta)$ such that \{l_{m, \nu}\} is an $\nu$-continuation with $l_{m, 0} = l_m$, and set $A_\nu = \Sigma(\mu_0, \nu) \cap \{0 \leq z \leq h_0\}$. Moreover, one can suppose that $l_{m, \nu}$ is parameterized as $l_{m, \nu}(t) = (t, y_m(\nu, t), z_m(\nu, t)) \ (t \in (-\delta, \delta))$. By the condition (C3), one can take $\bar{\nu} \neq 0$ with arbitrarily small $|\bar{\nu}|$ such that $0 < \pi_y(l_{m_0, \bar{\nu}}(0)) \leq \sup \{\pi_y(l_{m, \bar{\nu}})\} < \min \{\pi_y(A_{\nu_m})\}$.

We may assume that $\bar{\nu} > 0$ if necessary replacing $\nu$ by $-\nu$. For any integer $m$ sufficiently greater than $m_0$, there exists $0 < \nu_m < \bar{\nu}_m$ such that $\{l_{m, \nu_m}\}_{0 \leq \nu \leq \nu_m}$ is well defined and

$$\max \{\pi_y(A_{\nu_m})\} < \inf \{\pi_y(l_{m, \nu_m})\}$$

holds, see Fig. 4.2(1). By the Intermediate Value Theorem, there exists $0 < \nu_m < \bar{\nu}_m$ such that $l_{m, \nu_m}$ and $A_{\nu_m}$ have a tangency $\tau_m$, see Fig. 4.2(2). Since $l_{m, \nu_m} \subset W^u(p_{\mu_0, \nu_m})$ and $A_{\nu_m} \subset W^s(p_{\mu_0, \nu_m})$, $\tau_m$ is a homoclinic tangency associated to $p_{\mu_0, \nu_m}$.

When $b_{\mu, \nu} > 0$ for any $(\mu, \nu)$ near $(0, 0)$, one can prove the existence of a homoclinic tangency $\tau_m$ near $r$ associated to $p_{\mu_0, \nu_m}$ by arguments quite similar to those as above for any $\mu_0$ with $\mu_0 < 0$.

It remains to show that the tangency $\tau_m$ is quadratic. Since $\Sigma(\mu_0, \nu_m)$ is of elliptic type and $\lim_{m \to \infty} \nu_m = 0$, any normal curvature of $\Sigma(\mu_0, \nu_m)$ at $\tau_m$ is greater than some positive constant $\kappa_0$ independent of $m$. On the other hand, by an argument quite similar to that in Lemma 3.2 for any $m$ sufficiently greater than $m_0$, the curvature of $l_{m, \nu_m}$ at $\tau_m$ is less than $\kappa_0$. Thus, by Lemma 3.3, $\tau_m$ is a quadratic tangency.

**Assertion 4.3 (Hyperbolic case).** When $r$ is a tangency of hyperbolic type, Proposition 4.1 holds.

**Proof.** Since $r$ is of hyperbolic type, $\Sigma(0, 0) \cap \{z = 0\}$ consists of two almost straight curves $\alpha_1, \alpha_2$ meeting transversely at $r$, see Fig. 4.3(1). If necessary by reducing the domain $O$ of $f_{\mu, \nu}$ containing $(u_{\mu, \nu}, v_{\mu, \nu})$, we may assume that $\Sigma(\mu, \nu) \cap \{z = 0\}$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4_2.png}
\caption{The cross sections.}
\end{figure}
Figure 4.3. (1) The situation when $(\mu, \nu) = (0, 0)$. (2) The situation when $(\mu, \nu) = (\mu_0, 0)$.

is disjoint from the $x$-axis for any $(\mu, \nu)$ near $(0, 0)$. If $w_i = (\xi_i, \eta_i, 0)$ $(i = 1, 2)$ is a unit vector tangent to $\alpha_i$ at $r$, then $b_{0,0}\xi_i^2 + 2c_{0,0}\xi_i\eta_i + d_{0,0}\eta_i^2 = 0$. This implies that the normal curvature $\kappa_{\Sigma(0,0)}(r, w_i)$ of $\Sigma(0,0)$ at $r$ along $w_i$ is zero. Since $b_{0,0} \neq 0$ by (2.7), both $w_1, w_2$ are not parallel to the unit tangent vector $v_0 = (1, 0, 0)$. Thus we have $\kappa_{\Sigma(0,0)}(r, v_0) \neq 0$. When $b_{0,0} < 0$ (resp. $b_{0,0} > 0$), for any sufficiently small $\mu_0$ with $\mu_0 < 0$ (resp. $\mu_0 > 0$), $\Sigma(\mu_0, 0) \cap \{z = 0\}$ consists of two $C^2$ curves $\beta_1, \beta_2$ separated by a line in the $xy$-plane parallel to the $x$-axis, see Fig. 4.3(2), and

$\kappa_0 := |\kappa_{\Sigma(\mu_0,0)}(\tilde{\tau}, w)| > 0$,

where $\tau$ is a point of $\beta_1 \cup \beta_2$ the tangent line at which is parallel to the $x$-axis. One can take $\bar{\nu} > 0$ and $h_0 > 0$ so that $A_{\nu} = \Sigma(\mu_0, \nu) \cap \{0 \leq z \leq h_0\}$ is a disjoint union of two curvilinear rectangles for any $\nu$ with $0 \leq \nu \leq \bar{\nu}$, see Fig. 4.4. Moreover,

Figure 4.4. (1) The case of $\mu_0 < 0, \nu = 0, b_{\mu_0,0} < 0$. (2) The case of $\mu_0 > 0, \nu = 0, b_{\mu_0,0} > 0$. Each pair of the shaded regions represents $A$.

by (4.3), the $\bar{\nu} > 0$ can be chosen so that $|\kappa_{\Sigma(\mu_0,\nu)}(\tilde{\tau}, w)| > \kappa_0/2$ for any point $\tilde{\tau} \in A_{\mu_0,\nu}$ sufficiently near $\tau$ and any unit vector $w \in T_{\tilde{\tau}}(A_{\mu_0,\nu})$ sufficiently near $v_0$. As in the proof of Assertion 4.2, for any integer $m$ sufficiently greater than
Proof. Recall that $l_{m,\nu} \subset W^s(p_{\mu_0,\nu_m})$ and $A_{\nu_m} \subset W^u(p_{\mu_0,\nu_m})$ have a quadratic tangency $\tau_m$, see Fig. 4.5.

4.2. Generic unfolding of the tangency. For short, set $p_{\mu_0,\nu} = p_\nu, f_{\mu_0,\nu}(x, y) = f_\nu(x, y)$ and $(u_{\mu_0,\nu}, v_{\mu_0,\nu}) = (u_\nu, v_\nu)$.

Let $\tau_m = (\hat{x}_m, \hat{y}_m, f_{\nu_m}(\hat{x}_m, \hat{y}_m))$ be the homoclinic tangency of $W^u(p_{\nu_m})$ and $W^s(p_{\nu_m})$ given in Proposition 4.1. From (4.1), we have

$$
\partial f_{\nu_m}(x, y) = c_m(x - u_{\nu_m}) + d_m(y - v_{\nu_m}) + o_1,
\partial f_{\nu_m}(x, y) = b_m(x - u_{\nu_m}) + c_m(y - v_{\nu_m}) + o_1,
$$

where $b_m = b_{\mu_0,\nu_m}, c_m = c_{\mu_0,\nu_m}, d_m = d_{\mu_0,\nu_m}$ and $o_1 = o(|x - u_{\nu_m}| + |y - v_{\nu_m}|)$. Thus $b_m \partial f_{\nu_m}(x, y)/\partial y - c_m \partial f_{\nu_m}(x, y)/\partial x = (b_m d_m - c_m^2)(y - v_{\nu_m}) + o_1$. On the other hand, since there exists a unit vector tangent to $\Sigma(\mu_0, \nu_m)$ at $\tau_m$ converges to $(1, 0, 0)$ as $m \to \infty$, $\lim_{m \to \infty} \partial f_{\nu_m}(\hat{x}_m, \hat{y}_m)/\partial x = 0$. Since $\lim_{m \to \infty} b_m = b_{\mu_0,0} \neq 0$ and $\lim_{m \to \infty} b_m d_m - c_m^2 = \det(H f_{\mu_0,0})(u_{\mu_0,0}, v_{\mu_0,0}) \neq 0$,

$$
\partial f_{\nu_m}(\hat{x}_m, \hat{y}_m) = \frac{c_m}{b_m} \partial f_{\nu_m}(\hat{x}_m, \hat{y}_m) + \frac{b_m d_m - c_m^2}{b_m}(\hat{y}_m - v_{\nu_m}) + o_1 \neq 0
$$

for all sufficiently large $m$. By the Implicit Function Theorem, there exists a $C^2$ function $y = g_\nu(x, z) = g(\nu, x, z)$ defined in a small neighborhood of $(\nu_m, \hat{x}_m, f_{\nu_m}(\hat{x}_m, \hat{y}_m))$ in the $(\nu, x, z)$-space with

$$(x, y, f_\nu(x, y)) = (x, g_\nu(x, z), z).$$

Proposition 4.4. For all sufficiently large $m$, the quadratic homoclinic tangency $\tau_m$ of $W^s(p_{\nu_m})$ and $W^u(p_{\nu_m})$ unfolds generically at $\nu = \nu_m$ with respect to the $\nu$-parameter families $\{W^s(p_\nu)\}$ and $\{W^u(p_\nu)\}$.

Proof. Recall that $l_{m,\nu}$ has the parametrization $l_{m,\nu}(t) = (t, y_m(\nu, t), z_m(\nu, t))$ with $l_{m,\nu}((\nu_m) = \tau_m$. By Definition 2.1(2), it suffices to show that

$$(4.4) \quad \frac{\partial y_m}{\partial \nu}(\nu_m, \hat{x}_m) \neq \frac{\partial g}{\partial \nu}(\nu_m, \hat{x}_m, z_m(\nu_m, \hat{x}_m))$$
for all sufficiently large \( m \). Note that
\[
\lim_{m \to \infty} \frac{\partial \varphi}{\partial \nu}(\nu_m, \hat{x}_m, z_m(\nu_m, \hat{x}_m)) = \frac{\partial \varphi}{\partial \nu}(0, \hat{x}_\infty, 0),
\]
where \( \hat{x}_\infty \) is the \( x \)-coordinate of a point \( \tau \) in \( \Sigma(\mu_0, 0) \cap \{ z = 0 \} \) the tangent line in \( xy \)-plane at which is parallel to \( (1, 0, 0) \), see Fig. 13(2) in the case that \( r \) is of hyperbolic type. If we set \( \hat{x}_m, \nu = \alpha_\nu^{-n}\hat{x}_m \), then \( \varphi_n^\nu(l_m, \nu)(\hat{x}_m) = l_m(\nu)(\hat{x}_m) \), where \( n = m - m_0 \) and \( \alpha_\nu = \alpha_{\mu_0, \nu} \). As was seen in the proof of Lemma 3.1,
\[
\lim_{m \to \infty} \frac{\partial y_{m_0}}{\partial \nu}(\nu_m, \hat{x}_m, \nu_m) = \frac{\partial y_{m_0}}{\partial \nu}(0, 0) \neq 0.
\]
We denote the \( \nu \)-function \( y_{m_0}(\nu, \hat{x}_m, \nu) \) by \( h_m(\nu) \). Since \( \lim_{m \to \infty} d\hat{x}_m, \nu / d\nu = 0 \), it follows from (4.5) that
\[
\left| \frac{dh_m}{d\nu}(\nu_m) \right| = \left| \frac{\partial y_{m_0}}{\partial \nu}(\nu_m, \hat{x}_m, \nu_m) + \frac{\partial y_{m_0}}{\partial x}(\nu_m, \hat{x}_m, \nu_m) \frac{d\hat{x}_m, \nu}{d\nu}(\nu_m) \right|
\geq \left| \frac{\partial y_{m_0}}{\partial \nu}(\nu_m, \hat{x}_m, \nu_m) \right| - \left| \frac{\partial y_{m_0}}{\partial x}(\nu_m, \hat{x}_m, \nu_m) \frac{d\hat{x}_m, \nu}{d\nu}(\nu_m) \right| > C_0
\]
for some positive constant \( C_0 \) and all \( m \) sufficiently greater than \( m_0 \). Since \( y_m(\nu, \hat{x}_m) = \beta_\nu h_m(\nu) \) for \( \beta_\nu := \beta_{\mu_0, \nu} \),
\[
\frac{\partial y_{m_0}}{\partial \nu}(\nu_m, \hat{x}_m) = \beta_\nu \frac{dh_m}{d\nu}(\nu_m) + n \beta_\nu^{-1} \frac{d\beta_\nu}{d\nu}(\nu_m) h_m(\nu_m)
= \beta_\nu \frac{dh_m}{d\nu}(\nu_m) + n \frac{d\beta_\nu}{d\nu}(\nu_m) y_m(\nu_m, \hat{x}_m).
\]
Since \( \lim_{m \to \infty} \beta_\nu = \beta_0 > 1 \) and \( |y_m(\nu_m, \hat{x}_m)| \leq \delta \), the inequality (4.6) implies \( \lim_{m \to \infty} |\frac{\partial y_m}{\partial \nu}(\nu_m, \hat{x}_m)| / d\nu| = \infty \). This shows (4.4).

\textbf{Proof of Theorem} \[ \text{Propositions 4.1 and 4.4 imply Theorem A.} \]

\textbf{Acknowledgments}

We would like to thank Bau-Sen Du and Yi-Chiuan Chen for their support and hospitality, and Ming-Chia Li and Mikhail Malkin for their discussions during the first draft of this paper was written in Academia Sinica of Taiwan. We also would like to thank the referees for their valuable comments.

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