Stable Partial Ice Cover Possible for Any Obliquity: Effects of Obliquity, Albedo, and Heat Transport on Ice Cover Dynamics

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The Snowball state refers to when a planet is completely or almost completely covered in ice. The Earth may have passed through several Snowball events in its history which may have been crucial for increasing complexity of life. As we turn our focus to habitable planets outside of our solar system, the question then arises, what planetary characteristics permit a Snowball state and how do they impact the severity of this state? One factor determining planetary ice cover is the distribution of mean annual incoming solar radiation, which in turn depends on the planetary obliquity. In this study, we use an analytical energy balance model with explicit dependence on obliquity to study the probability of a catastrophic transition from partial ice cover to a stable Snowball State. We show that transitions to the Snowball state is more severe but less likely for higher values of the albedo contrast and energy transport across latitudes and that stable partial ice cover is possible at any obliquity. Additionally, this work is general enough to apply to any rapidly rotating planet and could be used to study the likelihood of Snowball transitions on planets within the habitable region of other stars.

INTRODUCTION

The search for habitable exoplanets, perhaps hosting life, is one of the great endeavors of our time. To aid the search, it is important to understand how life developed on Earth and what planetary factors contribute to its continued habitability. In these directions, much work has been conducted. For example, evidence suggests that in its history, the Earth may have passed through several Snowball events—times when Earth was completely covered in ice [8]. Intriguingly, these Snowball events may have been crucial for increasing complexity of life on Earth [9]. The role of partial ice cover in relation to habitability has been studied using high complexity models (e.g. [10], [11]), intermediate complexity models (e.g. [3]) and analytical models (e.g. [1], [4], [18]) for Earth-sized planets. Unfortunately, transitions between partial ice cover and the Snowball state are still not well understood for Earth [6], making it difficult to extrapolate the role that these transitions may play in the habitability and the development of complex life.

In this study we consider partial ice cover and transitions between the Snowball state and partial ice cover in a analytical energy balance model for rapidly rotating rocky planets. An advantage of using analytical models is that they illuminate feedback mechanisms that have a predominant effect on the planet’s climate. While similar effects can also be found using higher complexity models, the computational effort needed to run high complexity models over long time-scales or large parameter sweeps can be prohibitively expensive. Analytical models can be solved computationally (and sometimes explicitly) efficiently over long time scales, admitting the possibility of understanding model behavior in the entire parameter space and identifying and classifying different regions according to the behavior of the system. Here we analyze the system for all potential values of obliquity, albedo contrast, and efficiency of heat transport.

The analytical energy balance model used in this study has an explicit dependence on obliquity, admitting a study on the probability of partial ice cover as it depends on the planet’s obliquity and other physical parameters. [11] have shown that the Earth’s Moon has a stabilizing effect on Earth’s obliquity, and therefore planets without moons or without a large moon may have highly variable obliquities (e.g. Mars as was shown by [12]). [13] have constrained the original range of obliquities for Earth posed by [11]. Among exoplanets, potential obliquities of habitable zone planets Kepler-62f ([17], [21]) and Kepler-186f [21] have been shown to have stable regimes at lower obliquities and variable regimes at higher obliquities. It is therefore important to explore a wide range of obliquities when considering planetary ice cover.

In this work, we model heat transport with a relaxation to the mean annual temperature and compare the results to a diffusive heat transport model already in the literature [15]. In their work, [15] also approximated the mean annual insolation distribution with a second degree polynomial in sine of the latitude. A second degree polynomial is sufficient to capture the qualitative behavior of insolation distributions for planets with very low and very high obliquities, but does not capture the qualitative behavior for planets with obliquities between approximately 45° and 65°. To capture the behavior accurately, one needs at least a sixth degree polynomial in sine of the latitude [15]. Here we show that the higher order approximation is a necessary one; we would not find that stable partial ice cover is possible at any obliquity without it. Further, we find that the mode of heat transport (relaxation of the mean in this study, diffusion in
In the above model, \( y \) symmetry is assumed here so that the latitude non-frozen regions) is denoted by \( \eta \). The line latitude (the boundary between the frozen and the general, stable asymmetrical edges are possible). The ice and the values are reversed for ice belts (note that in from 0 (the equator) to 1 (the north pole) for ice caps, we consider the dynamics of the system as radiative forcing changes in the model. Physically this could be caused by atmospheric effects, such as changing greenhouse gases. In particular, we consider how different parameters in the model affect the Snowball catastrophe, where a small change in radiative forcing causes the planet to quickly become completely ice covered. In the model, this happens due to passage through a saddle node (or fold) bifurcation as radiative forcing is slowly changed. We consider the severity of the Snowball catastrophe bifurcation based on the distance between the bifurcation latitude and the Snowball state. We identify the regions in parameter space where more severe bifurcations occur since a severe Snowball catastrophe may have stronger signals in a planet’s climate record and may be more likely to be observed.

Our paper is laid out as follows. In section 2.1, we present the governing equations and a nondimensionalization of these equations to better quantify the effects of parameter changes on the behavior of the system. In section 2.2, we derive the equations that relate the latitude of the saddle node bifurcation to the corresponding parameter values. In section 3, we calculate the relative likelihood of stable partial ice cover. In section 4, we classify the bifurcation into the Snowball state based on its severity. A discussion of the results follows in section 5 and we conclude in section 6.

GOVERNING EQUATIONS

Annual Average Energy Balance

We consider a one dimensional energy balance model of the form popularized by [14], [20] and [16]. These models describe the time evolution of temperature on a planet depending on the incoming and outgoing radiation and heat transfer across latitudes. We focus on the version of the model described in [24] and used in [14].

\[
\frac{dT(y,t)}{dt} = Q_s(y,\beta)(1-\alpha(y,\eta)) - (A + BT) + C(T-T). \tag{1}
\]

In the above model, \( y \) is the sine of latitude. Hemispheric symmetry is assumed here so that the latitude \( y \) ranges from 0 (the equator) to 1 (the north pole) for ice caps, and the values are reversed for ice belts (note that in general, stable asymmetrical edges are possible). The ice line latitude (the boundary between the frozen and the non-frozen regions) is denoted by \( \eta \). The mean annual amount of incoming solar radiation (insolation) is represented by \( Q \). The insolation distribution function \( s(y,\beta) \) depends on the latitude and on the planetary axial tilt \( \beta \). The co-albedo function \((1 - \alpha(y,\eta))\) determines the proportion of incoming solar radiation absorbed by the planetary surface at each latitude. Outgoing radiation \( A + BT \) is in a linearized form, as in [28]. A temperature dependent heat transfer between latitudes \( y \) is represented as \( C(T-T) \). The horizontal heat transfer is the relaxation to the global mean temperature, where \( T = \int_0^1 T(y,t)dt \). Scientific discussions for the terms in the relaxation to the mean model can be found in [3].

Readers interested in a mathematical discussion should see [27] and North [16]. The parameters and their values for the Earth are specified in Table I.

![FIG. 1: Planets at low obliquity (left) tend to exhibit ice caps, while planets at high obliquity (right) tend to exhibit ice belts. The ice line latitude is marked \( \eta \).](image)

The albedo function is defined piecewise since ice surface and sea surface reflect different amounts of light. Let \( \alpha_p \) be the albedo polarward of the ice line, and \( \alpha_e \) be the albedo equatorward of the ice line. Then the albedo function is defined as follows:

\[
\alpha(y,\eta) = \begin{cases} 
\alpha_e & 0 < y < \eta \\
\frac{\alpha_e + \alpha_p}{2} & y = \eta \\
\alpha_p & \eta < y < 1
\end{cases} \tag{2}
\]

Note that for ice caps, \( \alpha_p > \alpha_e \), and for ice belts, \( \alpha_e > \alpha_p \), since the icy regions are assumed to reflect more incoming solar radiation (Figure 1). In some energy balance models of Earth, the albedo values are \( \alpha_e = \alpha_w = .32 \) and \( \alpha_p = \alpha_i = .62 \) (see Table I), (e.g. in [28]). Ice-line-dependent albedo is used in [14], [28], [27], and [2]. This is in contrast with the temperature-dependent albedo used in [18], [16]. We have chosen the ice-dependent version because we make \( \eta \) a dynamic parameter.

Here we consider the annual mean version of the Budyko equation [1], which models the annual average changes in the temperature profile. Consequently, we use...
the mean annual insolation, which depends only on obliquity $\beta$ and latitude $y$. A fast rotation rate is assumed, so the insolation function has no dependence on longitude. We use a sixth degree approximation for the annual mean insolation function, as in [15]

$$s(y, \beta) \equiv \sigma_6(y, \beta) = 1 - s_2 p_2(\cos \beta)p_2(y) - s_4 p_4(\cos \beta)p_4(y) - s_6 p_6(\cos \beta)p_6(y),$$

where $s_2 = 5/8$, $s_4 = 9/64$, and $s_6 = 65/1024$, and

$$p_2(y) = (3y^2 - 1)/2,$$

$$p_4(y) = (35y^4 - 30y^2 + 3)/8,$$

$$p_6(y) = (231y^6 - 315y^4 + 105y^2 - 5)/16$$

are the Legendre polynomials. Equation (3) is a higher order approximation than the one used in [18]. In [18], $s_4 = 0$ and $s_6 = 0$.

The position of the ice lines, denoted by $\eta$, depends on the mean annual temperature of the ice line. The physical boundaries at the pole and the equator are built into the model, i.e. $\eta$ cannot be greater than 1 or less than 0. A mathematical treatment of the nonsmooth system using a projection rule and a Filippov framework can be found in [21], where the invariance of the physically possible region is shown.

Ice–albedo feedback is incorporated by the dynamic ice line equation that is coupled with equation (1). We use the following dynamic ice line equation, first formulated in [28] for ice caps on Earth as

$$\frac{d\eta}{dt} = \rho(T(\eta, t) - T_c).$$

For ice belts, the right-hand side should be multiplied by negative one.

The mean annual temperature at the ice line is denoted by $T(\eta, t)$. The critical temperature $T_c$ is the highest temperature at which multiyear ice can be present. If the ice line temperature is above $T_c$, then the ice cover shrinks. If the temperature at the ice line is below $T_c$, the ice cover grows. The response constant $\rho$ controls the speed of the ice line response to a change in temperature. We are interested in the equilibrium position of the ice line, which is obtained when the ice line temperature is exactly $T_c$.

We nondimensionalize the system using transformations analogous to those in [18], namely

$$\tau = \omega t = \frac{2\pi}{t_{\text{year}}}, \quad T^* = \frac{A + BT(\eta)}{A + BT_c}.$$  

The nondimensionalized temperature $T^*$ is proportional to temperature and outgoing longwave radiation. At the ice line, $T^*$ is always equal to 1, i.e. $T^*(\eta) = 1$.

The nondimensionalized parameters are summarized in Table I. The parameter transformations are:

| Parameter | Name | Value for Earth | Units |
|-----------|------|----------------|-------|
| $R$ | Surface layer heat capacity | 12.6 | Wm$^{-1}$K$^{-1}$ |
| $Q$ | Annual average insolation | 343 | Wm$^{-2}$ |
| $\beta$ | Obliquity | 23.5 | degrees |
| $\omega$ | Albedo of water | .32 | dimensionless |
| $\alpha_i$ | Albedo of ice | .62 | dimensionless |
| $A$ | Greenhouse gas parameter | 202 | Wm$^{-2}$ |
| $B$ | Outgoing radiation | 1.9 | Wm$^{-2}$K$^{-1}$ |
| $C$ | Heat transport | 3.04 | Wm$^{-2}$K$^{-1}$ |
| $T_c$ | Critical temperature | -10 | °C |
| $\rho$ | Ice line response to temperature change | varies | K$^{-1}$yr$^{-1}$ |

TABLE I: Parameter values used in the standard Budyko-Widiasih Model [24]
### TABLE II: Nondimensionalized parameters

| Parameter | Definition | Name                        | Value for Earth |
|-----------|------------|-----------------------------|-----------------|
| $\zeta$   | $\cos(\beta)$ | Cosine of obliquity         | 0.92            |
| $\gamma$  | $\frac{B_T}{T}$ | Seasonal heat capacity      | 6.13            |
| $\delta$  | $\frac{1}{T}$  | Efficiency of heat transport | 1.6             |
| $q$       | $\frac{1 - \alpha_0}{A + \beta T_c}$ | Radiative forcing       | 1.27            |
| $\sigma$  | $1 - \frac{T_c}{\alpha_0}$ | Albedo contrast             | 0.44            |
| $\lambda$ | $\frac{\alpha_0 (A + \beta T_c)}{B \sigma}$ | Ice line response         | varies          |

These parameters have the following physical interpretations:

- $\gamma$: Seasonal heat capacity of the system relative to the outgoing radiation over one year.

- $\delta$: efficiency of heat transport. A measure of heat transport across latitudes. Note that despite coinciding notation, this parameter does not directly correspond to $\delta$ in [18]. The discrepancy is caused by the fact that the diffusion coefficient in the diffusion model is not a linear scaling of the horizontal heat transfer coefficient $C$ in equation (1).

- $q$: radiative forcing. It is directly proportional to the annual average incoming solar radiation and inversely proportional to the outgoing radiation at critical temperature $T_c$.

- $\sigma$: a measure of albedo contrast that changes the ice—albedo feedback. $\sigma = 0$ means that the ice and the water have the same albedo. $\sigma = 1$ means maximal contrast.

- $\lambda$: a measure of the speed of ice line response to the changes in temperature.

Rewriting the albedo function in accordance with the nondimensionalization, equation (2) for ice caps becomes:

$$\alpha^*(y, \eta) = \begin{cases} 
1, & 0 < y < \eta \\
\frac{2 - \sigma}{2}, & y = \eta \\
1 - \sigma, & \eta < y < 1
\end{cases}.$$ 

(8)

For belts, the positions of 1 and $1 - \sigma$ are swapped.

Thus the nondimensionalized version of the annual mean model [1] has the form:

$$\gamma \frac{\partial T^*}{\partial \tau} = q\sigma_6(y, \zeta)\alpha^*(y, \eta) - T^*(y, \tau) - \delta(T^*(y, \tau) - T^*(\tau)).$$

(9)

The ice—albedo feedback in equation (5) is nondimensionalized as follows:

$$\frac{\partial \eta}{\partial \tau} = \lambda(T^*(\eta) - 1).$$

(10)

### Analysis of Equilibria and Hysteresis Loops in Radiative Forcing

We expect to observe planets that have existed for a long time, and therefore we expect in the simplest case to find the corresponding planets at temperature—ice equilibrium. We focus on the equilibria of the ice line $\eta$ as they undergo bifurcations in radiative forcing $q$ and the associated hysteresis loops in our non-smooth system. Previously, bifurcations in $A$ and $Q$ have been considered for Earth’s range of parameter values ([28]) and bifurcations in $q$ have been studied in [18]. Our analysis sweeps the whole parameter space. In order to compute the parameter values that correspond to the bifurcations, we first need to find an expression for the equilibrium temperature profile $T^*(y, \tau)$ in the nondimensionalized model (5).

Assuming that the mean annual temperature is at equilibrium at each latitude, we can set $\frac{\partial T^*}{\partial \tau} = 0$ and obtain:

$$0 = q\sigma_6(y, \zeta)\alpha^*(y, \eta) - T^*(y, \tau) - \delta(T^*(y, \tau) - T^*(\tau)).$$

(11)

The above equation determines the equilibrium temperature profile $T^*(y)$. In order to solve for $T^*(y)$, first we find $T^*$, the mean equilibrium temperature, by integrating over the latitudes. Since $\int_0^1 T^* = \overline{T^*}$, the last term in (11) is 0 and we can explicitly solve for $\overline{T^*}$, namely

$$\overline{T^*}(\eta) = \int_0^1 q\sigma_6(y, \zeta)(\alpha^*(y, \eta))dy.$$ 

(12)

Since $\alpha^*(y, \eta)$ differs for ice belts and ice caps, $\overline{T^*}$ also differs. Both solutions are polynomials depending on obliquity and ice edge latitude, with lower temperatures in the ice-covered regions.

Let

$$\Sigma_6(\eta) = \int_0^\eta \sigma_6(y, \zeta)dy = \eta - s_2 p_2(\zeta) P_2(\eta) - s_4 p_4(\zeta) P_4(\eta) - s_6 p_6(\zeta) P_6(\eta) = 1 - \int_\eta^1 \sigma_6(y, \zeta)dy,$$

(13)
where \( P_i(y) = \int p_i(y) dy \) are the integrals of the Legendre polynomials. Then for ice caps, the average equilibrium temperature is:

\[
\overline{T^*(\eta)} = q \int_0^\eta \sigma_0(y, \zeta) dy + q \int_\eta^1 \sigma_0(y, \zeta)(1 - \overline{\sigma}) dy \tag{14}
\]

\[
= q \left( \Sigma_0(\eta) - \Sigma_0(\eta)(1 - \overline{\sigma}) + (1 - \overline{\sigma}) \right) \tag{15}
\]

\[
= q \left( (1 - \overline{\sigma}) + \overline{\sigma} \Sigma_0(\eta) \right), \tag{16}
\]

and for ice belts, the average equilibrium temperature is:

\[
\overline{T^*(\eta)} = q \int_0^\eta \sigma_0(y, \zeta)(1 - \overline{\sigma}) dy + q \int_\eta^1 \sigma_0(y, \zeta) dy \tag{17}
\]

\[
= q \left( (1 - \overline{\sigma}) \Sigma_0(\eta) - \Sigma_0(\eta) + 1 \right) \tag{18}
\]

\[
= q \left( 1 - \overline{\sigma} \Sigma_0(\eta) \right). \tag{19}
\]

Note that \( \overline{T^*} \) is proportional to the nondimensionalized radiative forcing \( q \): the more radiative forcing the planet receives, the warmer its mean equilibrium temperature. Using the expression for the mean equilibrium temperature, we proceed to find the expression for the temperature equilibrium \( T^*(\eta) \) evaluated at the ice line \( \eta \). Due to the discontinuity in \( \alpha^*(y, \eta) \), the temperature profile is discontinuous at the ice line. Therefore the temperature equilibrium \( T^*(\eta) \) is assumed to be the average of the left and right limits of \( T^* \):

\[
T^*(\eta) = \frac{\lim_{y \to \eta^-} T^*(y) + \lim_{y \to \eta^+} T^*(y)}{2}, \tag{20}
\]

where for ice caps these limits are:

\[
\lim_{y \to \eta^-} T^*(y) = \frac{q \left( \Sigma_0(\eta) - \Sigma_0(\eta) \right)}{1 + \overline{\sigma}} \tag{21}
\]

\[
\lim_{y \to \eta^+} T^*(\eta) = \frac{q \left( \Sigma_0(\eta) - \Sigma_0(\eta) \right)}{1 + \overline{\sigma}} \tag{22}
\]

and for ice belts, \( \lim_{y \to \eta^-} \) and \( \lim_{y \to \eta^+} \) are swapped. In either case, the temperature at the ice line is given by

\[
T^*(\eta) = \frac{\lim_{y \to \eta^-} T^*(y) + \lim_{y \to \eta^+} T^*(y)}{2} = \frac{q \left( \Sigma_0(\eta) - \Sigma_0(\eta) \right)}{1 + \overline{\sigma}}. \tag{23}
\]

Ice line equilibria occur when \( T^*(\eta) = 1 \). Having derived an equation for the ice line equilibrium, we consider the system’s response to changes in radiative forcing \( q \). We focus finding saddle node bifurcations in \( q \) where hysteresis loops are possible. The hysteresis loops go through a temporary snowball state.

FIG. 2: Plots showing the bifurcation diagrams demonstrating \( q_{\text{free}}, q_{\text{snow}}, \) and the saddle node point \( \frac{\partial q}{\partial \eta} = 0 \) for Earth’s parameter values using the relaxation to the mean model used in this work (left) and the diffusion model used in [18] (right).

For ice caps, a planet is ice-free when \( q = 1 \), and in a snowball state when \( q = 0 \). The relaxation to the mean model allows us to solve exactly for the unique value of radiative forcing \( q \) as a function of a particular ice line equilibrium \( \eta \) with parameters \( \alpha, \delta, \) and \( \zeta \), namely

\[
q_\eta(\zeta, \overline{\sigma}, \delta) = \frac{2(1 + \delta)}{s(\eta, \zeta)(2 - \overline{\sigma}) + 2\delta T^*(\eta, \zeta)}, \tag{24}
\]

where

\[
T^*(\eta, \zeta) = \overline{T^*(\eta, \zeta)} / q = \begin{cases} 
(1 - \overline{\sigma}) + \overline{\sigma} \Sigma_0(\eta) & \text{ice caps} \\
1 - \overline{\sigma} \Sigma_0(\eta) & \text{ice belts} \end{cases}. \tag{25}
\]

Following the conventions from [18], for ice caps, we denote \( q_1 \) as \( q_{\text{free}} \), the lowest value of \( q \) for which the planet admits a stable ice-free state, and \( q_0 \) as \( q_{\text{snow}} \), the highest value of \( q \) for which the planet admits a stable Snowball state. Additionally, the ice free state is stable for \( q > q_{\text{free}} \) (and the Snowball state is stable for \( q < q_{\text{snow}} \)) even though there is no true ice line equilibrium at \( \overline{\sigma} = 0 \) (or \( \eta = 0 \)) for this range of \( q \). In this case we would say \( \eta = 1 \) (or \( \eta = 0 \)) is a stable pseudo-equilibrium in the sense of Filippov [3]. Note that for ice belts, \( q_1 \) is \( q_{\text{snow}} \), and \( q_0 \) is \( q_{\text{free}} \). See Figure 2 for an bifurcation diagram demonstrating \( q_{\text{free}} \) and \( q_{\text{snow}} \).

**LIKELIHOOD OF STABLE PARTIAL ICE COVER**

**Defining the Region of Integration**

Planets with stable partial ice cover are potential candidates for a Snowball bifurcation. We focus on quantifying the likelihood of stable partial ice cover depending on planetary obliquity. Our goal is to compare planets at different obliquity values. Therefore, for each value of \( \zeta \) we find the region in parameter space that admits stable partial ice cover. We compute an estimate of the likelihood of stable partial ice cover based on the size of this
region. We determine the region using the expression for \( q_\eta \) to derive the expression for the critical value of albedo contrast \( \alpha_{\text{crit}} \) that corresponds to the saddle node bifurcations.

A saddle node (or a fold) bifurcation is a local bifurcation where two equilibria collide and annihilate each other. Changes in the bifurcation parameter lead to a hysteresis loop, in which slow changes in equilibria alternate with fast transitions to a different equilibrium state. For a description of saddle node bifurcations, see, for example, [23].

At the saddle node bifurcation, \( \frac{\partial q}{\partial \eta} = 0 \) (Figure 2). We can exploit this fact in order to eliminate the dependence on radiative forcing \( q \) by retaining the information about the corresponding value of \( \eta \). Taking the derivative \( \frac{\partial q}{\partial \eta} \) yields

\[
\frac{\partial q}{\partial \eta} = \frac{-2(1 + \delta)}{[(2 - \pi)\sigma_\eta(\eta, \zeta) + 2\delta T_x(\eta)]^2} \times \left( (2 - \pi) \frac{\partial}{\partial \eta} \sigma_\eta(\eta, \zeta) + 2\delta(\pm \pi \sigma_\eta(\eta, \zeta)) \right)
\]

for caps and belts, where \( \pm \pi \sigma_\eta(\eta, \zeta) \) is the derivative of \( T_x(\eta, \zeta) \) for caps and belts, respectively. Since \( T_x(\eta) \geq 0 \), this definition is well defined. Setting \( \frac{\partial q}{\partial \eta} = 0 \) yields the equation for the corresponding critical value of the albedo contrast at the saddle node bifurcation latitude for given values of obliquity \( \zeta \), efficiency of heat transport \( \delta \), and a given ice line position \( \eta \):

\[
\alpha_{\text{crit}}(\zeta, \delta, \eta) = \begin{cases} 
\frac{2}{\pi} \sigma_\eta(\eta, \zeta) & \text{for caps,} \\
\frac{2}{\pi} \sigma_\eta(\eta, \zeta) & \text{for belts.}
\end{cases}
\]

We call this function \( \alpha_{\text{crit}} \) because stable partial ice cover is possible whenever \( \pi \alpha < \alpha_{\text{crit}}(\zeta, \delta, \eta) \). At \( \pi \alpha = \alpha_{\text{crit}}(\zeta, \delta, \eta) \), a saddle node bifurcation occurs. Note that since \( \delta \) is theoretically unbounded, The function \( \alpha_{\text{crit}} \) can become arbitrarily small. In the next section, we use \( \alpha_{\text{crit}} \) to find the relative likelihood of stable partial ice cover.

**Relative Likelihood of Stable Partial Ice Cover**

In the previous section, we have derived an expression for the parameter region that allows stable partial ice cover at each value of obliquity. In this section, we integrate over the parameter region in order to estimate the relative likelihood of stable partial ice cover. The integration region is defined by \( q, \delta, \alpha \), and \( \alpha_{\text{crit}} \). The integrand is defined by the probability density function of the above parameters.

We assume that for rocky planets with water, there exists a true distribution for each parameter and that the parameters are independent of each other. Since the true probability distribution of \( q, \alpha, \) and \( \delta \) is not known, we consider several candidate probability distributions and verify that they yield qualitatively similar behavior. We integrate the composite probability density function over the region of the domain where stable edges are present to obtain the likelihood of stable partial ice cover for a given value of obliquity \( \zeta \). We normalize our results by the likelihood value for the obliquity of the Earth, \( \zeta = 23.5^\circ \). From the independence assumption, we can write the overall probability density function \( h_{\text{planet}} \) as follows:

\[
h_{\text{planet}}(q, \delta, \pi \alpha) = h_q(q)h_\delta(\delta)h_\alpha(\pi \alpha).
\]

We follow [18] in our choice of candidate probability functions to test in order to facilitate the comparison with their results. Since \( q \) and \( \delta \) are both nonnegative and unbounded, log-normal distributions are used to incorporate the possibility of a logn tail. Since \( \pi \alpha \in [0, 1] \), uniform and beta distributions are used. The beta distribution favors values of \( \pi \alpha \) close to the value for Earth (\( \pi \alpha = 0.44 \)) compared to extreme (\( \pi \alpha = 1 \)) or nonexistent (\( \pi \alpha = 0 \)) albedo contrast.

In all cases, the averages are chosen so that Earth’s parameter values are not unlikely. For PDF0, \( h_\alpha \) is uniform on \([0, 1]\); \( h_\delta \) is log-normal on \([0, \infty]\) with shape parameter 1.0, scale parameter 1.0, and location parameter 0; \( h_q \) is log-normal on \([0, \infty]\) with shape parameter 0.5, scale parameter 1.0, and location parameter 0.

PDF1 is the same as PDF0 except \( h_\delta \) is log-normal on \([0, \infty]\) with shape parameter 2.0, scale parameter \( c \), and location parameter 0. Compared to PDF0 and PDF2, PDF1 makes larger values of \( \delta \) more likely.

PDF2 is the same as PDF0 except \( h_\alpha \) is a parabolic beta-distribution on \([0, 1]\), with mode at 0.5. Compared to PDF0 and PDF1, PDF2 favors medium values of albedo contrast compared to extreme values.

The (non-normalized) likelihood is given by

\[
P_{\text{ice}}(\beta) = \frac{\int_0^1 \int_0^{\alpha_{\text{crit}}} h_{\text{planet}}(q, \delta, \alpha) d\alpha d\delta}{\int_0^\infty \int_0^{\alpha_{\text{crit}}} h_{\text{planet}}(q, \delta, \alpha) d\alpha d\delta},
\]

Note that the denominator in equation (29) is equal to 1 since \( h_{\text{planet}} \) is a probability density function. The integration region is illustrated in Figure 3. For higher values of \( \delta \), the corresponding values of \( \alpha_{\text{crit}} \) becomes vanishingly small for all latitudes \( \eta \). Therefore, even though the region is unbounded, numerical integration converges. The results of the integration are summarized in Figure 3.

The qualitative behavior is similar for PDF0, PDF1, and PDF2. Planets at low obliquity have a higher likelihood of stable partial ice cover than planets at high obliquity, and planets at medium obliquity have the lowest likelihood of stable partial ice cover. Compared to PDF0 and PDF1, PDF2 yields a lower likelihood of stable partial ice cover for high-obliquity planets. The apparent discontinuities in the relative likelihood plots are
due to numerical artifacts. Due to the physical boundaries at the pole and the equator, the integration region is extremely sensitive to changes in $\delta$, resulting in numerical instabilities in Mathematica. In contrast, the shape of the integration region (specifically, $\alpha_{\text{crit}}$) for the diffusion model does not engender numerical artifacts.

At lower obliquities, the diffusion model used by [18] yields a higher likelihood of stable partial ice cover compared to the relaxation to the mean model. We speculate that this effect is due to the fact that the diffusion model tends to exhibit two saddle node bifurcations for some parameter configurations, while the relaxation to the mean model tends to exhibit only one (see Figure 3).

In contrast with [18] the likelihood of stable partial ice cover is never zero in our investigation. This is due to using the sixth degree approximation for the insolation function instead of a second degree approximation. The sixth degree approximation is able to capture the subtle changes in the insolation distribution for mid-obliquity planets. Nevertheless, the likelihood attains its minimum at mid-obliquities for all tested probability distribution functions, in accordance with the results by [18].

**PARTIAL ICE COVER TO THE SNOWBALL STATE**

In addition to assessing the likelihood of partial ice cover, we quantify the severity of the bifurcation that leads to the snowball state. Typically, one is interested in determining whether or not a bifurcation occurs. However, since we are interested in the implication of rapid freezing and melting of the planet, a Snowball catastrophe that occurs far from the Snowball state would have a more drastic impact on the planet than the Snowball catastrophe that occurs close to complete ice cover. A less drastic Snowball catastrophe may also be harder to observe.

The bifurcation parameter $q$ depends on the amount of incoming stellar radiation $Q$, on the atmospheric parameters $A$ and $B$, the critical temperature $T_c$ and the albedo of water.

Changes in the above parameters could result in passage through a hysteresis loop, where for a critical value of $q$, the position of the ice line changes drastically on a shorter time scale compared to changes in $q$ for non-critical values. If $q$ then returns to its previous value, the system does not return to its previous state. Instead, the system follows a different stable equilibrium until another critical value of $q$ is reached.

Due to the physical boundary at the poles and the equator, hysteresis loops are possible where the only accessible stable states are ice-free and Snowball. The most severe Snowball bifurcation occurs when the hysteresis loop is an oscillation directly between snowball and ice-free states, and every intermediate ice line equilibrium is unstable (red curves in Figure 5). The hysteresis loop outlines a rectangular shape due to the Snowball and ice-free states becoming unstable at $q_{\text{free}}$ and $q_{\text{snow}}$. Such a hysteresis loop occurs when $q_{\text{free}} < q_{\text{snow}}$, and for all $0 < \eta < 1$, $q_{\text{free}} \leq q_{\eta} \leq q_{\text{snow}}$. Nonsmooth generalization of supercritical transcritical bifurcation as we pass through $q_{\text{free}}$ or $q_{\text{snow}}$ are described in [22]. Theory is not yet developed for nonsmooth case.

A less severe bifurcation occurs when the ice line continuously transitions from the ice-free state to small stable ice caps and then the ice line drops to the Snowball state (gray curves in Figure 5). Behavior of solutions is the typical passage through a saddle node bifurcation [22].

In particular, Earth goes through a saddle node bifurcation at $\eta = 0.62$, exhibits small stable ice caps for $0.62 < \eta < 1$, undergoes Snowball catastrophe via a saddle node bifurcation at $q = 1.20$. Then, if $q$ increases past $q_{\text{snow}} = 1.41$, the Earth rapidly transitions to an ice-free state. The ice-free state becomes unstable for $q = q_{\text{free}} = 1.31$. The transition from an ice-free state to small ice caps depends continuously on $q$. The corresponding bifurcation diagram is presented in Figure 5.

We also consider the scenario where no saddle node bifurcation occurs, and so the Snowball catastrophe is not present. The stable ice line equilibrium depends continuously on the parameter $q$ (black curves in Figure 5). No Snowball bifurcation occurs when $\eta q_{\text{snow}} < q_{\text{free}}$, and for all $0 < \eta < 1$, $q_{\text{snow}} \leq q_{\eta} \leq q_{\text{free}}$.

We use the relationship between $q_{\text{free}}$, $q_{\text{snow}}$, and the saddle node to compute the regions in Figure 5. For all obliquity values, higher albedo contrast and a higher

\[\alpha_{\text{crit}}\]

\[\beta\]

\[\gamma\]

\[\delta\]

\[\eta\]

\[\eta_{\text{crit}}\]

\[\alpha\]

\[\beta_{\text{crit}}\]

\[\gamma_{\text{crit}}\]

\[\delta_{\text{crit}}\]
FIG. 4: Plots showing the relative likelihood of stable edges for PDF0 (blue), PDF1 (orange), PDF2 (green), for relaxation to the mean model (left) and the diffusion model (right).

FIG. 5: Plots showing the progression from no Snowball bifurcation to a Snowball bifurcation from the ice free state as albedo contrast increases from $\alpha = 0.001$ (left) to $\alpha = 0.2$ (middle) to $\alpha = 0.55$ (right). Left: $\beta = 30^\circ$ and $\delta = 2$. Right: $\beta = 70^\circ$ and $\delta = 1$.

efficiency of heat transport mean a higher likelihood of having a hysteresis loop without stable partial ice cover. For ice caps, the region of hysteresis without stable partial ice cover is larger at higher obliquities. For ice belts, this region is larger at lower obliquities. Since ice caps tend to occur at lower obliquities, and ice belts tend to occur at higher obliquities, we speculate that planets at mid-obliquities are most likely to exhibit such a bifurcation. The no-Snowball window excludes mid-obliquities. The size of the region is controlled by the albedo contrast $\alpha$ and appears to not depend on the heat transport efficiency $\delta$.

In Figure 5, high obliquities are excluded for ice caps, and low obliquities are excluded for ice belts. The excluded obliquity ranges are based on the contribution from the corresponding $\alpha_{\text{crit}}$ function for caps and belts, respectively. While the estimate is based on the saddle node bifurcations, we use it as a proxy for severe bifurcations as well. Both high- and low-obliquity planets could exhibit no-Snowball behavior for small values of $\alpha$. Hysteresis without passage through a saddle node is possible for planets at all obliquity ranges, and for all values of $\delta$. 
obliquity $\beta$, this behavior is favored by high values of albedo contrast $\alpha$ and high values of heat transport efficiency $\delta$. The region where passage though a saddle node bifurcation is possible is larger for low-obliquity planets, and is smallest for mid-obliquity planets. Systems in this parameter window exhibit both a Snowball catastrophe and a severe transition (from ice-free to Snowball state or vice-versa). In particular, Earth is in a gray region with $\alpha = 0.44$ and $\delta = 1.6$.

**DISCUSSION**

In this paper we have analyzed a one dimensional energy balance model with heat transport modeled by relaxation to the global mean temperature. The relaxation to the mean and diffusions versions of the Budyko-Sellers model provide two similar ways to model the multitude of processes involved in energy transfer between latitudes. Both methods ensure that energy is transported from latitudes that are "hot" to ones that are "cold." The diffusive heat transport is a local process that necessitates special treatment at the poles, while relaxation to the mean global temperature is a global process that does not require special boundary conditions [28].

The second degree approximation of the insolation distribution used in [15] does not capture the qualitative distribution of mid-obliquity planets. Planets with obliquities between approximately 45° and 65° have a characteristic ‘W’ shape that requires a degree six (or higher) polynomial approximation to capture [15]. A main result from [15] is that the likelihood of stable partial ice cover goes to zero at 55° obliquity. This is entirely due to the approximation they use, which is constant for 55° obliquity. Notice that in the definition of $\alpha_{\text{crit}}$ (equation [27]), if the insolation approximation were constant then $\alpha_{\text{crit}} = 0$ for any values of the arguments. This means that the integral in the numerator of the likelihood calculation (equation [29]) is zero. Taking a higher degree approximation, as we do here, avoids these problems.

We find that ice caps are more likely to have stable partial ice cover, and planets at middle obliquities are least likely to have stable partial ice cover, which is qualitatively similar to the likelihood computations in [18]. As noted above, we find that stable partial ice cover is possible at all obliquities and that, in particular, the relative likelihood of finding a planet with partial stable ice cover is never less than 20%. This is not an artifact of using the relaxation to the mean version of heat transport; computing the likelihood with diffusive heat transport (as was done in [18]) and a sixth degree polynomial approximation for the insolation approximation yields similar nonzero likelihood for all obliquities as we find here.

Comparing the relaxation to the mean model to the diffusion model, we note that the latter predicts lower likelihood of stable partial ice cover at lower obliquities. We speculate that this is due to the fact that the diffusion model can exhibit a second saddle node bifurcation at high values of $\eta$, close to the poles, while the relaxation to the mean model does not exhibit such behavior.

The relaxation to the mean model also exhibits pronounced differences between PDF0, PDF1 and PDF2 at high obliquities. For high values of $\beta$, the likelihood of stable partial ice cover is lower for PDF2. The difference between PDF2 and other tested probability density functions is due to the differences in $\alpha_{\text{crit}}$ between the relaxation to the mean model and the diffusion model. Since PDF2 changes the distribution of $\alpha$ from a uniform distribution to a parabolic beta distribution, the shape of $\alpha_{\text{crit}}$ results in a more pronounced difference for the relaxation to the mean model than for the diffusion model. The resulting gap between the likelihood curves conveys decreased certainty about the likelihood of stable partial ice cover on high obliquity planets. While the likelihood curves for the relaxation to the mean model appear discontinuous, we suspect that this behavior is an artifact of numerical integration using Mathematica and not an inherent discontinuity in the system. The system exhibits continuous dependence on parameters.

Note also that the likelihood calculation includes the stable edges that are not accessible by a hysteresis loop. However, the general shape follows [15] so we are reasonably confident that their exclusion has a minor effect on the qualitative shape of the graph. In our comparison in Figure 4, we use the inaccessible edges from the system used by [18]. The inclusion of inaccessible edges results in a slight difference in scaling of non-Earth obliquity values.

The bifurcation parameter $q$ depends on the amount of incoming stellar radiation $Q$, on the atmospheric parameters $A$ and $B$, the critical temperature $T_c$ and the
albedo of water. The hysteresis loop that we describe in $q$ could be caused by any one of these physical parameters changing. Changes in $Q$ could be caused by changing amount of atmospheric particles, as discussed in [4] and [20]. Changes in the atmospheric parameters $A$ and $B$ could be caused by changing chemical compositions of the atmosphere, such as increasing greenhouse gases. The dynamics of a changing $A$ in this type of model are explored in [2]. Changes in the critical temperature could model changing deep ocean temperature as was done in [25] where the temperature that ice forms at is gradually increased by $10^\circ$C over the mid-Pleistocene transition to simulate deep ocean cooling.

We quantify the effects of albedo contrast and efficiency of heat transport on the presence of hysteresis loops in radiative forcing. We find that the severity of the snowball bifurcation increases as the albedo contrast $\alpha$ and the efficiency of heat transport $\delta$ increase. For very small values of the albedo contrast $\alpha$ and the efficiency of heat transport $\delta$, sometimes there is no hysteresis in the model at all and the partial stable ice cover will transition smoothly between ice free and complete ice cover as $q$ changes.

The above behavior can be explained by the effect of albedo contrast and efficiency of heat transport on the planetary climate mechanism. When the albedo contrast $\alpha$ is low, ice is not much more reflective than water, resulting in suppressed ice—albedo feedback. When $\delta$ is low, the near-absence of heat transport across latitudes limits the interaction between the ice regions and the water regions of the planet, thus reducing the likelihood of the Snowball catastrophe. When $\alpha$ is high, the reflectivity of ice is much higher than that of water, expediting the ice—albedo processes. When $\alpha$ is high, the heat transport across latitudes makes it difficult to maintain a difference in temperatures between ice regions and water regions, leading to an ice-free or a Snowball planet.

In the range of obliquities where both ice caps and ice belts may be stable, namely between obliquities of 40$^\circ$ and 60$^\circ$, there will always be a hysteresis loop when varying $q$. The hysteresis will either contain a saddle node bifurcation or the most severe snowball bifurcation from ice-free to completely ice covered. The lack of a region without hysteresis in the parameter space is a contributing factor for the decrease in the likelihood of stable partial ice cover for these obliquities in Figure [4].

We find an approximate power law relationship governing the shape of the boundaries between the regions of no hysteresis and hysteresis with partial ice cover as well as between the regions of hysteresis with partial ice cover and most severe hysteresis from ice-free states to snowball. Although these power law relationships may be a by-product of the simplicity of the model, more work is needed to understand whether this is a mathematical phenomenon or if there is some physical explanation for it.

The robustness of the Snowball catastrophe and the parameter regimes where an energy balance model might be applicable has been debated. In the GCM simulations conducted by [7], the Snowball catastrophe occurs only for particular ocean regimes. [20] show that meridional heat transfer may increase ice cover stability. [19] have extended the energy balance models to include ocean heat transport and meridional structure and have found that Snowball catastrophe is possible in those models.

If this work were to be applied to an observed planet, the obliquity $\zeta$ and the albedo contrast $\alpha$ could perhaps be measured, and albedo signatures could be compared to the predictions of our model. The parameter $q$ might be more difficult to estimate. While the mean annual insolation $Q$ could be derived from the information about the star, the dependence of $q$ on the atmospheric parameters $A$ and $B$ would make it more challenging. The parameter $\delta$, efficiency of heat transport, would also be difficult to measure directly because of our limited knowledge of rates of heat transport on different planets. It should be noted that the regime where no Snowball bifurcation is possible appears to exist for all values of $\delta$ within the relevant window of $\zeta$ and $\alpha$, so the observational limitations on $\delta$ should not affect the search for the no-Snowball regime.

**CONCLUSION**

In this paper we have analyzed a one dimensional energy balance model with heat transport modeled by relaxation to the global mean temperature and with explicit dependence on the planet’s obliquity. We have included a dynamic ice line that controls the boundary on the planet between low and high albedo and analyze the stability of ice line equilibria in different regions of parameter space. We pay particular attention to the planet’s obliquity, radiative forcing, albedo contrast, and efficiency of heat transport and find:

1. With an improved approximation to the insolation distribution function, planets at all values of obliquity exhibit at least a 20% likelihood of stable partial ice cover.

2. Low albedo contrast and low efficiency of heat transport favor stable partial ice cover.

3. High albedo contrast and high efficiency of heat transport favor severe Snowball catastrophe in a hysteresis loop caused by changes in radiative forcing.

This work may be interpreted for any rapidly rotating rocky planet with some physical mechanism of heat transport. This work applies to planets where the temperature affects the albedo, in particular, we assume that higher
temperatures decrease the albedo as they do for ice/water on Earth.

A future study may explore a region of interest in the parameter space using a GCM, for example incorporating meridional heat transport. Another extension of this work would be to introduce obliquity variations in time into the energy balance model used in this study, as variations in obliquity are both plausible and likely to change the behavior of planetary ice cover over geological time. Other orbital parameters such as eccentricity could be incorporated into a version of the energy balance model presented here.

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