OPTIMAL REGULARITY FOR THE SIGNORINI PROBLEM
AND ITS FREE BOUNDARY.

JOHN ANDERSSON

Abstract. We will show optimal regularity for minimizers of the Signorini problem for the Lame system. In particular if \( u = (u^1, u^2, u^3) \in W^{1,2}(B_1^+ : \mathbb{R}^3) \) minimizes
\[
J(u) = \int_{B_1^+} |\nabla u + \nabla^\perp u|^2 + \lambda \text{div}(u)^2
\]
in the convex set
\[
K = \{ u = (u^1, u^2, u^3) \in W^{1,2}(B_1^+ : \mathbb{R}^3); \ u^3 \geq 0 \text{ on } \Pi, \ u = f \in C^\infty(\partial B_1) \text{ on } (\partial B_1)^+ \},
\]
where \( \lambda \geq 0 \) say.

Then \( u \in C^{1,1/2}(B_1^+ \backslash \Pi) \). Moreover the free boundary, given by
\[
\Gamma_u = \partial \{ x; \ u^3(x) = 0, \ x_3 = 0 \} \cap B_1,
\]
will be a \( C^{4,\alpha} \) graph close to points where \( u \) is not degenerate.

Similar results have been known before for scalar partial differential equations (see for instance [4] and [5]). The novelty of this approach is that it does not rely on maximum principle methods and is therefore applicable to systems of equations.

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1. Introduction

We are interested in minimizers $u = (u^1, u^2, u^3)$ in the set

$$ K = \{ u = (u^1, u^2, u^3) \in W^{1,2}(B_1^+; \mathbb{R}^3) ; \ u^3 \geq 0 \text{ on } \Pi, \ u = f \in C^\infty(\partial B_1) \text{ on } (\partial B_1)^+ \}. $$

of the following functional

$$ J(u) = \int_{B_1^+} |\nabla u + \nabla^T u|^2 + \lambda \text{div}(u)^2. $$

Here $\nabla^T u = (\nabla u)^T$ is the transpose of the gradient matrix. We will always assume that $\lambda \geq 0$. We could relax the condition that $u \in W^{1,2}$ to $\nabla u + \nabla^T u \in L^2$, as is usually done. But due to Korn’s inequality both conditions are equivalent.

If we denote $\Pi = \{ x ; x_3 = 0 \}$ and $\Lambda_u = \{ x \in \Pi ; u^3(x) = 0 \}$ then it is easy to see that the minimizers solve the following Euler-Lagrange equations

$$ Lu \equiv \Delta u + \frac{2+\lambda}{4} \nabla \text{div}(u) = 0 \quad \text{in } \mathbb{R}^3_+ $$

$$ u^3 = 0 \quad \text{on } \Lambda_u $$

$$ \frac{\partial u^3}{\partial x_3} + \frac{\lambda}{4} \text{div}(u) = 0 \quad \text{on } \Pi \setminus \Lambda_u $$

$$ \frac{\partial u^i}{\partial x_3} + \frac{\partial u^3}{\partial x_i} = 0 \quad \text{on } \Pi \text{ for } i = 1, 2 $$

$$ u^3 \geq 0 \quad \text{on } \Pi $$

$$ \frac{\partial u^3}{\partial x_3} + \frac{\lambda}{4} \text{div}(u) \leq 0 \quad \text{on } \Pi. $$

It is important to note that this problem is highly nonlinear since the set $\Lambda_u$ is not a priori known. The major difficulty in analysing the regularity of this problem consists in understanding not only the behavior of the solution but also of the unknown set $\Lambda_u$.

This minimization problem models the deformation of an elastic body, which we here assume for simplicity to be the half ball $B_1^+$ when it is subjected to some deformation $f$ of the curved part of the boundary $\partial B_1^+$ and is required to stay above a certain obstacle, here given by $x_3 = 0$.

This is of course a version of the Signorini problem. This problem was first formulated by Antonio Signorini in 1933 [16]. In the original formulation of the problem Signorini assumed Neumann data on the boundary and he included the influence of gravity. From a mathematical point of view adding gravity to the functional $J(u)$ does not result in any new difficulties. Our analysis is entirely local so the boundary data on $(\partial B_1)^+$ will play no roll in our analysis. Signorini where interested in the existence and uniqueness of solutions. This was solved by G. Fichera [9] in 1963. With the advances in the calculus of variations since the sixties the existence and uniqueness is today considered to be quite standard.

Here we are interested in the regularity of minimizers and in the regularity of the free boundary $\partial \Lambda_u$.

Mathematical Background: It took almost 50 years from Signorini’s formulation of this problem to the first regularity results was proved by D. Kinderlehrer in 1981 [13]. Kinderlehrer proves that the solution is $C^{1,\beta}$ in the case $n = 2$.

Soon after A.A. Arkhipova and N.N. Uraltseva showed $C^{1,\beta}$ regularity for variational inequalities of diagonal systems in $n$ dimensions [3]. The assumption that the system is diagonal excludes the Signorini problem from their result. The first $C^{1,\beta}$ result for the Signorini problem in general dimensions is due to R. Schumann who proved $C^{1,\beta}$-regularity for some $\beta > 0$ in 1989 [15].
There are several other papers relating to free boundary problems for systems of equations; see for instance M. Fuchs [11] for a pleasant proof of regularity and free boundary regularity for a system.

However, all previous proofs of optimal regularity and free boundary regularity results for systems of equations are based on the reduction of the system to a scalar problem. To the authors knowledge, there are no papers that manage to tackle the difficulties of systems without such a reduction. Let us therefore investigate the development of the regularity theory for the scalar versions of the Signorini problem - where much more is known.

There has been significant progress in the understanding of the regularity questions for the scalar Signorini problem, also called the thin obstacle problem, in the last decade, see [4] and [5]. The thin obstacle problem is the minimisation of the Dirichlet energy

\[ \int_{B^+} |\nabla u|^2 \]

in the set

\[ K = \{ u \in W^{1,2}; \ u \geq 0 \text{ on } \Pi, \ u = f \text{ on } (\partial B_1)^+ \}. \]

To show existence of minimizers to the thin obstacle problem is again rather standard. But the regularity theory is quite subtle, both with respect to the regularity of the solution [4] and its free boundary [5].

We have several good reasons to dwell on the technique used in [4] and [5]. First of all, those papers have provided the framework for this paper even though the techniques we use will be very different from theirs. Secondly, to know something about the scalar problem considered by Athanasopoulos, Caffarelli and Salsa will also help us understand the difficulties of the vectoral case. In particular we will be able to understand why Signorini’s problem have not been solved by the techniques developed in the last thirty years.

In [4] the main result is that minimizers of (9) in the set (9) are \( C^{1,1/2} \), which is the optimal regularity. The proof is based on the Bernstein technique, a monotonicity formula and an iteration. The Bernstein technique is basically to apply the maximum principle to the function

\[ g(x) = \eta(x) \frac{\partial^2 u}{\partial e_i^2} - \lambda |\nabla u|^2 \]

where \( \eta \) is a cut off function and \( e_i \in \Pi \). Since the maximum principle is not true for the Lame system we can not replicate this argument for the vectoral Signorini problem. Neither do the structure of the Lame system allow us to derive the monotonicity formula that is essential for the optimal regularity proof.

In [5] the authors use comparison and boundary comparison principles together with some geometrical insight to show that the free boundary \( \partial \{ u > 0 \} \cap \Pi \) is \( C^{1,\alpha} \) in a small neighbourhood around free boundary points \( x^0 \) where \( \sup_{B_r(x^0)} |u| \approx r^{1+\beta} \) for any \( \beta < 1 \). The usage of comparison principles makes it impossible to apply their technique directly to solutions of the vectoral Signorini problem.

The theory in [4] and [5] was later generalized in [7] to more general thin obstacle problems interpreted as obstacle problems for the fractional Laplacian. But the methods in [7] are quite similar to the methods used in [4] and [5]. In particular the methodology in [7] relies heavily on comparison and maximum principles and is therefore not applicable for our problem.

One can say that this article constitutes the author’s attempt to develop a regularity theory for free boundaries that is not dependent on maximum principles.
Instead of maximum principle methods we will rely on the blow-up method, the Liouville Theorem and linearization together with some simple geometric observations and a nice way to control blow-up sequences that we get from [2].

The article naturally divides itself into two parts that depend on different procedures. The first part, after some introductory and standard considerations, constitutes of section 4-7 where we show that close to free boundary points where \( u \) grows like \( r^{1+\alpha} \) for some \( \alpha < 1 \) the free boundary \( \Gamma_u \) is actually flat. That \( \Gamma_u \) is flat just means that in a small ball \( B_r \cap \Pi \) the free boundary is contained in a strip of width \( \sigma(r) \) for some modulus of continuity \( \sigma \).

The idea of the proof is quite straightforward and uses a result by M. Benedicks [6] that states that the set of positive harmonic functions vanishing on part of \( \Pi \) and has zero Neumann data on the rest of \( \Pi \) is one dimensional. Using this result we may deduce that the tangential derivatives of the blow-up of a solution \( u \) are all multiples of each other. That implies in particular that the blow-up of \( u \) depend only on two directions, say \( x_1 \) and \( x_n \). By means of the Liouville Theorem we can classify such solutions and thus calculate the asymptotic profile of solutions close to points where \( \sup_{B_r} |u| \approx r^{1+\alpha} \). The profile in question is explicitly calculated in Lemma 10. In polar coordinates the asymptotic profile is

\[
\begin{align*}
    u(r, \phi) &= r^{3/2} \left( \frac{18+3\lambda}{40} \cos \left( \frac{5}{2} \phi \right) - \frac{2+\lambda}{8} \cos \left( \frac{1}{2} \phi \right) \right) \\
    v(r, \phi) &= r^{3/2} \left( \frac{6-\lambda}{20} \sin \left( \frac{5}{2} \phi \right) + \frac{2-\lambda}{8} \sin \left( \frac{1}{2} \phi \right) \right).
\end{align*}
\]

Since the growth of the asymptotic profile is \( r^{3/2} \) we can directly conclude that \( u \in C^{1,\beta}(B_{1/2}^+) \) for each \( \beta < 1/2 \), see Lemma 12 and Corollary 5.

In order to use Benedicks result we will derive that global solutions with control of the growth at infinity is actually determined by a harmonic function. Later, in appendix 2, we will also use this method to indicate how to make a simple eigenfunction expansion of the linearized problem. It is quite possible that one could derive all the regularity theory for the vectoral Signorini problem by this reduction to harmonic functions. We will however only use this reduction in our proof that the free boundary is flat, see the proof of Corollary 3, and in appendix 2. We believe that the result in the appendix is well known and that it could be proved by other methods such as spectral theory of operator pencils [14]. I could unfortunately not find any good reference to such a result. And the machinery of operator pencils [14] is too heavy to introduce in this paper to prove a supporting lemma. Therefore, for convenience, we use the harmonic reduction again in appendix 2.

The first part of the proof is quite trivial from a technical point of view and we use mostly standard calculus and elementary pde theory. The result is however very important for the linearisation that follows. In in section 4-7 we show that the asymptotics of the solution is uniquely determined at points of lowest regularity. This allows us to make a linearisation at all such points which will imply everywhere regularity. This is quite different, and much stronger, than the standard outcome of a linearisation argument where an \( \epsilon \)-closeness assumption is needed and only partial regularity (which may or may not be optimal) can be deduced.

The second part of the paper is far more technical and, unfortunately, much harder to read. There we prove that the solutions are in fact \( C^{1,1/2} \) which is optimal as the above asymptotic profile demonstrates. We also prove that the free boundary is \( C^{1,\alpha} \) close to points where \( u \) has the above asymptotic profile (this includes all the points \( x^3 \in \Gamma_u \) where \( \sup_{B_r} |u| > r^{1+\beta} \) for some \( \beta < 1 \) and \( r \) small).

The argument is by linearization and flatness improvement. In particular if the origin is a free boundary point of \( u \) with the asymptotic profile \( p \) where \( p = (u, v) \)
as in equation (10). Then, heuristically at least, the limit (we will use a slightly different limit later)

$$\lim_{r \to 0} \frac{u(rx) - p(rx)}{\|u(rx) - p(rx)\|_{L^2(B^+_r)}} = v$$

will contain information about the regularity of $u$ and the free boundary. The problem is that in order to extract any useful information from (11) we need the convergence to be strong in $L^2$ we would also need to show that $v$ is better than $u$. This is a very delicate matter that will be analysed in sections 9-11.

In the final sections we prove the regularity theorem and free boundary regularity. We also show, Lemma 21, that the analysis in the previous sections can be made uniform.

Intuitively there is a gap in the eigenvalues of the operator for the Lame system on the sphere for the linearized problem. Where the next homogeneous solution after $p$ as in (10) is homogeneous of second order. That implies that $\sup_{B^+_r} |u - p|$ will be of order $r^2$ which implies that the difference between $u$ and $p$ decays geometrically in smaller and smaller balls. This is enough to deduce the regularity of the solution and the free boundary.

Throughout this paper we will not use the maximum principle or comparison principles at all.

There is however a deeper connection with the flatness proof of this paper and the methodology used in [4], [5] and [7]. As mentioned before, the flatness proof we use is based on a result by Benedicks [6]. Both Benedicks paper and the monotonicity formula used in [4] is based on a paper by Friedland and Hayman [10]. Friedland and Hayman’s paper also use some structural properties of the Laplacian that goes back at least to Alfred von Huber [12]. The same methods is used to derive a frequency formula in [7].

Even though there are some technical connections between this paper and previous work on thin obstacle problems. Most of the material here is essentially new. It is the authors hope that the linearisation technique will prove useful also for other free boundary problems involving systems of equations.

Our main regularity result is:

**Main Regularity Theorem.** Let $u$ be a solution to the Signorini problem in $B^+_1$ then

$$\|u\|_{C^{1,1/2}(B^+_1)} \leq C\|u\|_{L^2(B^+_1)}$$

The free boundary regularity result is the same as in [5] but we will need to introduce some notation before we can state the Theorem. The precise formulation can be found in Corollary 7 in section 13.

The regularity theorems are, for simplicity, only formulated for solutions in $B^+_1$ with the constraint $u^3 \geq 0$. The more general problem to minimize

$$\int_D |\nabla u + \nabla^\perp u|^2 + \lambda \text{div}(u)^2$$

in the set

$$K = \{ u = (u^1, u^2, u^3) \in W^{1,2}(D : \mathbb{R}^3); \quad u^3(x) \geq \psi(u^1(x), u^2(x)), \quad u \text{ satisfies appropriate boundary conditions} \},$$

where $D$ is some $C^{1,\beta}$ domain and $\psi$ is a $C^{1,\beta}$ function with $\beta > 1/2$, can be handled by a perturbation argument as in for instance [1].
Notation:

At times we will write \( n \) for the dimension. But some proofs in the paper has been written only for \( n = 3 \). This is for simplicity, since the curl operator is more explicit in \( \mathbb{R}^3 \). The pedantic (here used with no negative connotation) reader can always think that \( n = 3 \).

\( \Pi = \{ x; \ x_n = 0 \} \) is the boundary of \( \mathbb{R}_+^n \).

We will use bold face \( \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{p} \) etc. to denote vector valued functions \( \mathbf{u} = (u^1, u^2, u^3, \ldots, u^n), \mathbf{v} = (v^1, v^2, \ldots, v^n) \) etc. For a continuous function \( \mathbf{u} = (u^1, u^2, \ldots, u^n) \) we set \( \Lambda_{\mathbf{u}} = \Lambda = \{ x; \ x_n = 0, \ u^n(x) = 0 \} \).

For a continuous function \( \mathbf{u} = (u^1, u^2, \ldots, u^n) \) we set \( \Omega_{\mathbf{u}} = \Omega = \Pi \setminus \Lambda_{\mathbf{u}} \).

For a continuous function \( \mathbf{u} \) we define its free boundary \( \Gamma_{\mathbf{u}} = \Gamma = \overline{\Lambda_{\mathbf{u}}} \cap \overline{\Omega_{\mathbf{u}}} \).

\( \nabla \mathbf{u} \) is the matrix:

\[
\begin{bmatrix}
\frac{\partial u^1}{\partial x_1} & \frac{\partial u^1}{\partial x_2} & \cdots & \frac{\partial u^1}{\partial x_n} \\
\frac{\partial u^2}{\partial x_1} & \frac{\partial u^2}{\partial x_2} & \cdots & \frac{\partial u^2}{\partial x_n} \\
\vdots & \vdots & & \vdots \\
\frac{\partial u^n}{\partial x_1} & \frac{\partial u^n}{\partial x_2} & \cdots & \frac{\partial u^n}{\partial x_n}
\end{bmatrix}
\]

\( \nabla^\perp \mathbf{u} \) is the transpose of the matrix \( \nabla \mathbf{u} \):

\[
\begin{bmatrix}
\frac{\partial u^1}{\partial x_1} & \frac{\partial u^2}{\partial x_1} & \cdots & \frac{\partial u^n}{\partial x_1} \\
\frac{\partial u^1}{\partial x_2} & \frac{\partial u^2}{\partial x_2} & \cdots & \frac{\partial u^n}{\partial x_2} \\
\vdots & \vdots & & \vdots \\
\frac{\partial u^1}{\partial x_n} & \frac{\partial u^2}{\partial x_n} & \cdots & \frac{\partial u^n}{\partial x_n}
\end{bmatrix}
\]

We will often use a prime to indicate the projection of an \( n \)-dimensional vector into an \((n-1)\)-dimensional vector: \( x' = (x_1, x_2, \ldots, x_{n-1}) \), \( \nabla' = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{n-1}} \right) \) etc. At times we will slightly abuse notation and write \( x' = (x_1, x_2, \ldots, x_{n-1}, 0) \) and \( \nabla' = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_{n-1}}, 0 \right) \). It will always be clear from context what we intend.

We use the notation \( \nabla'' = (0, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \ldots, \frac{\partial}{\partial x_{n-1}}, 0) \). More generally for any vector \( \xi \in \Pi \) we use the notation \( \nabla_{\xi} = \nabla - e_n \frac{\partial}{\partial x_n} - \xi/|\xi|^2 (\xi \cdot \nabla) \) which is just the gradient restricted to the subspace orthogonal to \( e_n \) and \( \xi \).

\( \text{Pr}(\mathbf{u}, r) \) is a projection operator onto affine functions defined in Definition 3.

By \( W^{k,p}(D) \) we mean the normal Sobolev space. We will often be quite informal when assigning vector valued functions to this space and write \( (u^1, u^2, u^3, \ldots, u^n) \in W^{k,p} \) instead of \( (u^1, u^2, u^3, \ldots, u^n) \in (W^{k,p})^n \) etc.

By \( \| \mathbf{u} \|_{L^p(\Omega)} \) we mean the norm:

\[
\| \mathbf{u} \|_{L^p(\Omega)} = \left( \frac{1}{|\Omega|} \int_{\Omega} |\mathbf{u}|^p \right)^{1/p},
\]

defined in Definition 1.

The homogeneous solutions to the Lame system \( p_{1/2}, p_{3/2}, p_{1/2}^\xi, p_{3/2}^\xi, \ldots \) are defined in Definition 2.

2. Some Simplifying Conventions.

Our main goal is to prove that the solutions are \( C^{1,1/2} \). It will however simplify our exposition significantly if we assume that we have \( C^{1,3} \) regularity. The techniques developed in the following pages is certainly strong enough to prove that solutions are \( C^{1,3} \). But to formally prove the \( C^{1,3} \) regularity we would have to work through the same string of Lemmas and Theorems twice and end up with an
article twice as long. We will therefore assume that the solutions are \( C^{1,\beta} \) without proof. But we have indicated in an appendix how to prove the following lemma. The exposition in the appendix is rather terse and we will freely refer to results proved in the main body of the paper. Hopefully there is enough information in the appendix for a thorough reader to reconstruct the proof.

Another \( C^{1,\beta} \) proof was published in [15]. I have not been able to verify that proof due to a lack knowledge of pseudodifferential operators.

**Lemma 1.** Let \( u \) be a solution to the Signorini problem

\[
\begin{align*}
\Delta u + \frac{2+\lambda}{4} \nabla \text{div}(u) &= 0 \quad \text{in } B_1^+ \\
u^3 &= 0 \quad \text{on } \Lambda \\
\frac{\partial u^3}{\partial x_3} + \frac{\lambda}{4} \text{div}(u) &= 0 \quad \text{on } \Pi \setminus \Lambda \\
\frac{\partial u^3}{\partial x_3} + \frac{\partial u^2}{\partial x_i} &= 0 \quad \text{on } \Pi \text{ for } i = 1, 2 \\
u^3 &\geq 0 \quad \text{on } \Pi \\
\frac{\partial u^3}{\partial x_3} + \frac{\lambda}{4} \text{div}(u) &\leq 0 \quad \text{on } \Pi.
\end{align*}
\]

Then \( u \in C^{1,\beta}(B_{1/2}^+) \) for some \( \beta > 0 \) and \( u \) satisfies the following estimate

\[
\|u\|_{C^{1,\beta}(B_{1/2}^+)} \leq C\|u\|_{L^2(B_1^+)}.
\]

One of the advantages to have \( C^{1,\beta} \) regularity in what follows is that it will significantly simplify our exposition. It is easy to verify that if \( u \) is a solution to the Signorini problem then

\[
u + \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},
\]

is also a solution for any constants \( b_1, b_2 \) and \( a_{ij} \) such that

\[
a_{33} + \frac{\lambda}{4}(a_{11} + a_{22} + a_{33}) = 0.
\]

If \( u \in C^{1,\beta} \) then we can make the following, informal, standing assumption.

**Standing Assumption:** Let \( u \) be a solution to the Signorini problem and assume that \( x^0 \) is a free boundary point of interest (such as a point that we make a blow-up at). Then we will assume that \( |u(x^0)| = 0 \) and that \( \nabla u(x^0) = 0 \).

A more formal way of handling this would be to only consider our solutions modulo affine functions and define

\[
\sup_{B_r(x^0)} |u| \leq C_0
\]

if there is a function \( v \) in the same equivalence class as \( u \) such that the estimate holds etc.

In appendix 1 where we indicate how to prove the \( C^{1,\beta} \)-lemma we will not rely on the standard assumption but instead define a projection operator \( Pr \) (see Definition 3) and consider \( u - Pr(u, r) \). In the main body of the paper this extra \( Pr(u, r) \)-term would only clutter down our already complicated expressions too much. Therefore we will rely on the standing assumption.

At times we will refer to the Liouville Theorem without explanation. Whenever that is done we refer to the following simple result.

**Liouville’s Theorem.** Let \( u \) be any function defined in \( \mathbb{R}^n \) that satisfies the following estimates for all \( k \in \mathbb{N} \), \( R > 1 \) and some constant \( C_0 \)

\[
(12) \quad \sup_{B_R} |D^k u| \leq \frac{C_0}{R^{k+n/2}} \|u\|_{L^2(B_{2R})}.
\]
Assume furthermore that \( u \) satisfies the growth condition
\[
\|u\|_{L^2(B_R)} \leq C_1 R^{k_0+n/2+\alpha}
\]
for all \( R > 1 \), some \( \alpha < 1 \), some \( k_0 \in \mathbb{N} \) and some constant \( C_1 \). Then \( u \) is a polynomial of order \( k_0 \).

The proof is trivial. This applies in particular to harmonic functions of polynomial growth for which the estimates (12) are standard.

Something about the dimension \( n \). All the results in this paper are valid in \( \mathbb{R}^n \) for any \( n \geq 2 \). However the main technical difficulties arise in \( \mathbb{R}^3 \). Also, some proofs will rely on the curl operator that is much more explicit in \( \mathbb{R}^3 \). Therefore some of the proofs are written only for \( n = 3 \). This is an attempt to balance the clarity of the exposition without avoiding any of the intrinsic difficulties of the problem which arise in \( \mathbb{R}^3 \).

In \( \mathbb{R}^n \) we can define the curl operator on a vector field \( u \) according to
\[
\text{curl}(u) = [\ast (d^b u)]^\sharp,
\]
where \( \ast \) is the Hodge star, \( ^b \) the flat and \( ^\sharp \) the sharp operator. With this definition we would still have \( \text{curl}(\nabla f) = 0 \) etc. and all the proofs would still work. Hopefully, the assumption that \( n = 3 \) will increase the clarity enough to motivate the loss of generality.

In the later sections of the paper, where we do not use the curl operator, we will write \( n \) instead of \( 3 \). This is to indicate that the technique is not simplified by the assumption \( n = 3 \). The reader should always remember that we, in order to avoid working with \( [\ast (d^b u)]^\sharp \), always assume that \( n = 3 \).

3. Weak Regularity.

In order to prove \( W^{2,2} \) estimates for solutions to the Signorini problem we need the well known Korn’s inequality found for instance in [8].

**Lemma 2.** [Korn’s Inequality] Let \( u : B^+_1 \to \mathbb{R}^3 \) then
\[
\left( \int_{B^+_1} |\nabla u|^2 \right)^{1/2} \leq C \left( \int_{B^+_1} |\nabla u + \nabla^\perp u|^2 \right)^{1/2}
\]
whenever the right hand side is defined. In particular
\[
\left( \int_{B^+_1} |\nabla u + \nabla^\perp u|^2 \right)^{1/2}
\]
is a semi norm on \( W^{1,2}(B^+_1) \).

Next we state a simple Lemma. The proof is standard and therefore omitted.

**Lemma 3.** Let \( u \) be a solution to the Signorini problem, that is \( u \) minimizes (1) in \( K \), then
\[
\|u\|_{W^{1,2}(B^+_{1/2})} \leq C \|u\|_{L^2(B^+_{3/4})}.
\]

**Lemma 4.** Let \( u \) be a solution to the Signorini problem, that is \( u \) minimizes (1) in \( K \), then \( u \in W^{2,2}(B^+_{1/2}) \) and
\[
\|u\|_{W^{2,2}(B^+_{1/2})} \leq C \|u\|_{L^2(B^+_{3/4})}.
\]
By Kohn’s inequality this implies that
\[ \partial K \] is a competitor for minimality in
\[ K \] if \( 0 \leq t \leq h \) and \( i = 1, 2 \). Thus
\[ 0 \leq \int \nabla u \cdot \nabla v + \nabla^\perp u \cdot \nabla v + \nabla u \cdot \nabla^\perp v + \nabla^\perp u \cdot \nabla^\perp v + \lambda \text{div}(u)\text{div}(v). \]

Differentiating at \( t = 0 \) we get
\[ 0 \leq \int_{B_1^+} \xi^2 \left( \nabla u \cdot \nabla u_h + \nabla^\perp u \cdot \nabla u_h + \nabla u \cdot \nabla^\perp u_h + \lambda \text{div}(u)\text{div}(u_h) \right) + \]
\[ 2\xi \left( u_h \cdot \nabla u \cdot \nabla^\perp u_h + \nabla^\perp u_h \cdot \nabla u_h + \nabla u_h \cdot \nabla^\perp u_h + \lambda \text{div}(u)\text{div}(u_h) \right). \]

If we denote \( u^h(x) = u(x + e_i h) \) then \( u^h \) is a minimizer in \( B_1^{+h} \) and
\[ v_{h,t} = v + t\xi^2 v_{h}^\perp \]
is an admissible competitor for minimization. Therefore by differentiation at \( t = 0 \) and using that \( u^h \) is a minimizer we can conclude that
\[ 0 \leq \int_{B_1^{+h}} \xi^2 \left( \nabla u^h \cdot \nabla u_{h}^h + \nabla^\perp u^h \cdot \nabla u_{h}^h + \nabla^\perp u_{h}^h \cdot \nabla^\perp u_{h}^h + \lambda \text{div}(u^h)\text{div}(u_{h}^h) \right) + \]
\[ 2\xi \left( u_{h}^h \cdot \nabla u^h \cdot \nabla^\perp u_{h}^h + \nabla^\perp u_{h}^h \cdot \nabla^\perp u^h + \lambda \text{div}(u^h)\nabla^\perp u_{h}^h \right). \]

Next we notice that
\[ u_{h}^h(x) = \frac{u(x) - u(x + he_i)}{h} = -u_h(x). \]

Adding (13) and (14), rearranging the terms and dividing by \( h \), we may conclude that
\[ \int_{B_1^+} \xi^2 \left( |\nabla u_h + \nabla^\perp u_h|^2 + \lambda \text{div}(u_h)^2 \right) \leq \]
\[ -2 \int_{B_1^+} 2\xi \left( u_h \cdot \nabla u \cdot \nabla^\perp u_h + \nabla^\perp u_h \cdot \nabla u + \lambda \text{div}(u_h)\nabla^\perp u_h \right) \leq \]
\[ \frac{1}{2} \int_{B_1^+} \xi^2 \left( |\nabla u_h + \nabla^\perp u_h|^2 + \lambda \text{div}(u_h)^2 \right) + C \int_{B_1^+} |u_h|^2 |\nabla \xi|^2. \]

In particular it follows, by letting \( h \to 0 \), that
\[ \int_{B_{1/2}^{+}} \left( |\nabla \partial_i u + \nabla^\perp \partial_i u_h|^2 + \lambda \text{div}(\partial_i u)^2 \right) \leq C \int_{B_{1/2}^+} |\partial_i u|^2. \]

By Kohn’s inequality this implies that \( \partial_i u \in W^{1,2}(B_{1/2}^{+}; \mathbb{R}^3) \) for \( i = 1, 2 \). It directly follows that \( u \) solves the Lame system in \( B_{1/2}^{+} \) with boundary data given by the restriction of a \( W^{2,2} \) function to the boundary and \( W^{2,2} \) regularity of \( u \) follows. The estimate given is a consequence of Lemma 3 and the above. \( \square \)
4. Global Solutions, part 1.  
Reduction of the System.  

In this section we make a very useful reduction of solutions, with controlled growth, of the Lame system in the upper half space into a system with two unknown functions $\xi$ and $\tau$. Later we will even be able to express the solution in terms of one harmonic function $\tau$.

We will call a solution $u$ in the upper half space $\mathbb{R}^3_+ = \mathbb{R}^3 \cap \{x_3 > 0\}$ a global solution. In this section we will only consider solutions in $\mathbb{R}^3_+$ for simplicity. In particular we will utilise the curl operator which is much easier to express in $\mathbb{R}^3$ than in higher dimensions. There is however nothing in this section that requires the dimension to be three and the reader may verify that all the proofs work also in $\mathbb{R}^n$.

**Lemma 5.** Let $u$ be a global solution to the Signorini problem and

$$\liminf_{r \to \infty} \frac{\ln \left( \|u\|_{L^2(B^+_r)} \right)}{\ln(r)} < 2.$$  

Then there exist functions $\xi$ and $\tau$ such that

(15)  
$$u = \nabla \xi + e_3 \tau$$

and

(16)  
$$\Delta \xi + \frac{\lambda + 2}{\lambda + 4} \frac{\partial \tau}{\partial x_3} = 0 \quad \text{in } \mathbb{R}^3_+$$

(17)  
$$\Delta \tau = 0 \quad \text{in } \mathbb{R}^3_+$$

(18)  
$$\frac{\partial \xi}{\partial x_3} = \tau = 0 \quad \text{on } \Lambda_u$$

(19)  
$$\tau = -2 \frac{\partial \xi}{\partial x_3} \quad \text{on } \Omega_u$$

(20)  
$$\frac{\partial \tau}{\partial x_3} = \frac{2\lambda + 8}{3\lambda + 8} \frac{\partial^2 \xi}{\partial x_3^2} \quad \text{on } \Omega_u.$$  

**Proof:** Let $u$ be as in the Lemma and define

$$v = \text{div}(u) \in W^{1,2}_{\text{loc}}(\mathbb{R}^3_+),$$

$$w = \text{curl}(u) = [u_2^3 - u_3^2, u_3^1 - u_3^1, u_1^2 - u_2^2] \in W^{1,2}_{\text{loc}}(\mathbb{R}^3_+).$$

It is easy to see that $\Delta w^i = \Delta v = 0$ for $i = 1, 2, 3$. Moreover on $\Lambda_u$ we have

$$\frac{\partial w^3}{\partial x_3} = \frac{\partial^2 u^2}{\partial x_3 \partial x_1} - \frac{\partial^2 u^1}{\partial x_3 \partial x_2} = \frac{\partial}{\partial x_1} \left( - \frac{\partial u^3}{\partial x_2} \right) - \frac{\partial}{\partial x_2} \left( - \frac{\partial u^3}{\partial x_1} \right) = 0.$$  

And on $\Omega_u$

$$\frac{\partial w^3}{\partial x_3} = \frac{\partial^2 u^2}{\partial x_3 \partial x_1} - \frac{\partial^2 u^1}{\partial x_3 \partial x_2} = \frac{\partial}{\partial x_1} \left( - \frac{\partial u^3}{\partial x_2} \right) - \frac{\partial}{\partial x_2} \left( - \frac{\partial u^3}{\partial x_1} \right) = 0,$$

where we have used (5). In particular $w^3$ satisfies

$$\Delta w^3 = 0 \quad \text{in } \mathbb{R}^3_+$$

$$\frac{\partial w^3}{\partial x_3} = 0 \quad \text{on } \Pi$$

$$\sup_{B^+_R} |w^3| \leq CR^{\alpha} \quad \text{for } R \geq 1 \text{ and an } \alpha < 1.$$
By the Liouville Theorem it follows that $w^3 = \text{constant}$. But from Lemma 1 it follows that for $R < 1$ and some $\beta > 0$

$$\sup_{B_R^x} |w^3| \leq CR^\beta$$

which implies that $w^3 = 0$.

We may therefore conclude that there exist a $\xi$ such that

$$\nabla(\Delta \frac{\partial \xi}{\partial x_3} + 2 + \frac{\lambda}{2} \frac{\partial}{\partial x_3} \text{div}(u)) = 0.$$  

Therefore

$$\Delta \frac{\partial \xi}{\partial x_3} + 2 + \frac{\lambda}{2} \frac{\partial}{\partial x_3} \text{div}(u) = f(x_3)$$

for some function $f(x_3)$. But $\xi$ is not determined up to functions in the $x_3$ variable, that is since the only condition on $\xi$ so far is that $\nabla \xi = (u^1, u^2)$, so we may choose $\xi$ so that

$$\Delta \frac{\partial \xi}{\partial x_3} + 2 + \frac{\lambda}{2} \frac{\partial}{\partial x_3} \text{div}(u) = 0.$$

In particular $u^3$ and $\frac{\partial \xi}{\partial x_3}$ satisfies the same partial differential equation, $\Delta \cdot = -\frac{\lambda^2 + 4}{\lambda + 2} \frac{\partial}{\partial x_3} \text{div}(u)$, and differ thus by a harmonic function which we will denote $\tau$.

The equations (15) and (17) follows.

To show (16) we just notice that

$$0 = \Delta u + \frac{\lambda + 2}{2} \nabla \text{div}(u) = \nabla \left( \Delta \xi + \frac{\lambda}{2} \frac{\partial \xi}{\partial x_3} \right).$$

It immediately follows that

$$\Delta \xi + \frac{\lambda}{2} \frac{\partial \xi}{\partial x_3} = c_0,$$

for some constant $c_0$. By making the substitutions

$$\xi \rightarrow \xi + \frac{c_0}{2} \frac{\lambda}{3} \frac{x_3}{2} + \frac{a}{2} x_3^2$$

and

$$\tau \rightarrow \tau - \frac{\lambda}{\lambda + 2} ax_3$$

we may assume that the constant $c_0$ in (22) is zero. Equation (16) follows. We want to point out that the constant $a$ is arbitrary and that $\tau$ is therefore not determined up to linear functions $ax_3$, a fact that we will use later.

On $\Lambda u$, we have

$$0 = u^3 = \frac{\partial \xi}{\partial x_3} + \tau.$$
and
\[ 0 = \frac{\partial u^i}{\partial x_3} = \frac{\partial^2 \xi}{\partial x_i \partial x_3} \]
for \( i = 1, 2 \). It follows that
\[ (24) \quad \frac{\partial \xi}{\partial x_3} = c_i, \]
where the constant \( c_i \) may differ from component to component of \( \Lambda_u \).

The boundary conditions (18) follows from (23) and (24).

On \( \Omega_u \) we have
\[ 0 = \frac{\partial u^i}{\partial x_3} = \frac{\partial u^3}{\partial x_i} = 2 \frac{\partial^2 \xi}{\partial x_i \partial x_3} + \frac{\partial \tau}{\partial x_i}. \]
It follows that
\[ (25) \quad \tau = -2 \frac{\partial \xi}{\partial x_3} + \tilde{c}_i, \]
where the constant \( \tilde{c}_i \) may differ in different components of \( \Omega_u \).

By making the substitution
\[ (26) \quad \xi \rightarrow \xi + c_1 x_3 \]
we see that we may choose \( \xi \) so that the constant \( c_i \) is zero for one component of \( \Lambda_u \). In particular the boundary conditions (18), in that component \( \Lambda_1 \) of \( \Lambda_u \), follows from (23) and (24).

Next we notice that if \( \Omega_i \) is any component of \( \Omega_u \) adjacent to \( \Lambda_1 \) then by \( C^{1, \beta} \) continuity of \( u \) it follows that \( \tilde{c}_i = 0 \). That is (19) holds in \( \Omega_i \). In particular if (18) is true in one component \( \Lambda_i \) of \( \Lambda_u \) then (19) holds for all components \( \Omega_j \) adjacent to \( \Lambda_1 \). That is if \( c_i = 0 \) in (24) for some component \( \Lambda_i \) then \( \tilde{c}_j = 0 \) in (25) for each \( j \) such that \( \Omega_j \) is a component adjacent to \( \Lambda_i \).

Conversely if (19) holds in a component \( \Omega_i \) (that is \( \tilde{c}_i = 0 \) in (25)) then by \( C^{1, \beta} \) regularity (18) is true for each adjacent component \( \Lambda_i \). So if we make the the substitution (26) then it follows that \( c_i = 0 \) and \( \tilde{c}_i = 0 \) for all components of \( \Lambda_u \) and \( \Omega_u \). The boundary conditions (19) and (18) holds on \( \Pi \).

Finally we have on \( \Omega \) that
\[ 0 = \frac{\partial u^3}{\partial x_3} + \frac{\lambda}{4} \text{div} u = \frac{\partial^2 \xi}{\partial x_3} + \frac{\partial \tau}{\partial x_3} + \frac{\lambda}{4} \left( \Delta \xi + \frac{\partial \tau}{\partial x_3} \right) = \frac{\partial^2 \xi}{\partial x_3} + \frac{3\lambda + 8}{2\lambda + 8} \frac{\partial \tau}{\partial x_3}, \]
where we have used (16). This implies (20). \( \square \)

Next we want to show that \( \tau \) is an \( x_3 \)-derivative of a harmonic function.

**Corollary 1.** Let \( u \) be a global solution to the Signorini problem and
\[ \liminf_{r \to \infty} \frac{\| u \|_{L^2(B^r_+)}}{\ln(r)} < 2. \]
Then there exist functions \( \xi \) and \( \tau \) such that
\[ (27) \quad u = \nabla \xi + e_3 \frac{\partial \tau}{\partial x_3} \]
and
\[ (28) \quad \Delta \xi + \frac{\lambda + 2}{\lambda + 4} \frac{\partial^2 \tau}{\partial x_3^2} = 0 \quad \text{in} \quad \mathbb{R}^3_+ \]
\[ (29) \quad \Delta \tau = 0 \quad \text{in} \quad \mathbb{R}^3_+ \]
\[ (30) \quad \frac{\partial \xi}{\partial x_3} = \frac{\partial \tau}{\partial x_3} = 0 \quad \text{on} \quad \Lambda_u \]
\( \frac{\partial \tau}{\partial x_3} = -2 \frac{\partial \xi}{\partial x_3} \) on \( \Omega_u \)

\( \frac{\partial^2 \tau}{\partial x_3^2} = -\frac{2 \lambda + 8}{3 \lambda + 8} \frac{\partial \xi}{\partial x_3^2} \) on \( \Omega_u \).

**Proof:** Let \( \xi \) and \( \tau \) be as in Lemma 5 and let \( \chi \) be the solution of

\[ \Delta \chi = -\frac{\lambda + 2}{\lambda + 4} \frac{\partial \tau}{\partial x_3} \text{ in } \mathbb{R}^3 \]

\[ \frac{\partial \chi}{\partial x_3} = 0 \text{ on } \Pi. \]

Then

\[ \Delta^2 (\chi - \xi) = 0 \text{ in } \mathbb{R}^3 \]

\[ \frac{\partial^2 (\chi - \xi)}{\partial x_3} = 2 \frac{\partial \xi}{\partial x_3} = -\tau \text{ on } \Pi. \]

In particular if we denote

\[ \tilde{\tau} = \tau + 2 \frac{\partial (\chi - \xi)}{\partial x_3} \]

then

\[ \Delta \tilde{\tau} = 0 \text{ in } \mathbb{R}^3 \]

\[ \tilde{\tau} = 0 \text{ on } \Pi. \]

Moreover \( \sup_{B_R^+} |\tilde{\tau}| \leq CR^{1+\alpha} \) so by Liouville’s Theorem \( \tilde{\tau} = ax_3 \) for some constant \( a \). That is

\( \tau = ax_3 - 2 \frac{\partial (\chi - \xi)}{\partial x_3} \).

But as we pointed out in the discussion right after (22) \( \tau \) is only determined up to linear functions \( ax_3 \). We may therefore choose a \( \tau \) so that \( a = 0 \) in (33).

We have thus shown that \( \tau \) in Lemma 5 is expressible as the \( x_3 \)-derivative of a harmonic function. The corollary follows.

**Corollary 2.** For each \( R > 0 \) we have, with \( \tau \) as in Corollary 1,

\[ \frac{\partial \tau}{\partial x_3} \in W^{2,2}(B_R^+). \]

**Proof:** By Corollary 1

\[ \frac{\partial \xi}{\partial x_3} + \frac{\partial \tau}{\partial x_3} = u^3 \in W^{2,2}(B_R^+). \]

It follows that

\[ \frac{\partial^2 \tau}{\partial x_3 \partial x_3} = \frac{\partial u^3}{\partial x_i} \frac{\partial^2 \xi}{\partial x_3 \partial x_3} = \frac{\partial u^3}{\partial x_i} \frac{\partial u^3}{\partial x_3} \in W^{1,2}(B_R^+), \]

for \( i = 1, 2 \), where we have used that

\[ \frac{\partial \xi}{\partial x_i} = u_i. \]

By the trace theorem we therefore have

\[ \frac{\partial^2 \tau}{\partial x_i \partial x_3} \in W^{1/2,2}(\Pi \cap B_R). \]

That in turn results in

\[ \frac{\partial \tau}{\partial x_3} \in W^{3/2,2}(\Pi \cap B_R). \]

But \( \frac{\partial \tau}{\partial x_3} \) is harmonic so we may conclude that

\[ \frac{\partial \tau}{\partial x_3} \in W^{2,2}(B_R^+) \]

\( \square \)
Remark: It is not hard to show that $\tau \in W^{3,2}(B_R)$, but we have no need for that stronger statement in what follows.

Corollary 3. Let $\xi$ and $\tau$ be as in Corollary 1. Then

$$\frac{\partial^2 \tau}{\partial x_3^2} = 0$$
on $\Omega_u$.

Proof: Let

$$v = \xi - \frac{\lambda + 2}{2(\lambda + 4)} \frac{\partial \tau}{\partial x_3} x_3.$$

Then

$$\Delta v = \Delta \xi - \frac{\lambda + 2}{\lambda + 4} \frac{\partial^2 \tau}{\partial x_3^2} = 0$$
in $\mathbb{R}^3_+$. Moreover, on $\Pi$,

$$\frac{\partial v}{\partial x_3} = -\frac{\lambda + 3}{\lambda + 4} \frac{\partial \tau}{\partial x_3}.$$

This implies that

$$v = -\frac{\lambda + 3}{\lambda + 4} \tau$$
or equivalently that

(34)  $$\xi = \frac{\lambda + 2}{2(\lambda + 4)} \frac{\partial \tau}{\partial x_3} x_3 - \frac{\lambda + 3}{\lambda + 4} \tau.$$

Using (32) we conclude that on $\Omega$

$$\frac{3\lambda + 6}{2\lambda + 8} \frac{\partial^2 \tau}{\partial x_3^2} = \frac{\partial^2 \xi}{\partial x_3^2} = \frac{\lambda + 2}{\lambda + 4} \frac{\partial^2 \tau}{\partial x_3^2} - \frac{\lambda + 3}{\lambda + 4} \frac{\partial^2 \tau}{\partial x_3^2} = -\frac{1}{\lambda + 4} \frac{\partial^2 \tau}{\partial x_3^2}.$$

Rearranging terms we may conclude that

$$\frac{3\lambda + 6}{2\lambda + 8} \frac{\partial^2 \tau}{\partial x_3^2} = 0$$
on $\Omega$ and the Corollary follows.

We will need to refer to (34) on several occasions later so it is convenient to formulate that equality as a Corollary.

Corollary 4. Let $\xi$ and $\tau$ be as in Corollary 1. Then

$$\xi = \frac{\lambda + 2}{2(\lambda + 4)} \frac{\partial \tau}{\partial x_3} x_3 - \frac{\lambda + 3}{\lambda + 4} \tau.$$

Lemma 6. Let $u$ be a global solution to the Signorini problem as in Corollary 1 and denote $v = \text{div}(u)$. Then

$$v = \frac{2}{\lambda + 4} \frac{\partial^2 \tau}{\partial x_3^2}$$

where $\tau$ is as in Corollary 1.

In particular, by Corollary 3,

$$v = 0 \quad \text{on } \Omega.$$

Proof: With the notation of Corollary 1 we have

$$v = \text{div}(u) = \Delta \xi + \frac{\partial^2 \tau}{\partial x_3^2} = \frac{2}{\lambda + 4} \frac{\partial^2 \tau}{\partial x_3^2},$$

where we have used (28).
5. A result by Benedicks.

In this section we will remind ourselves of a Theorem due to M. Benedicks [6]. We will formulate the theorem slightly differently from Benedick’s for convenience. We will however not be able to directly apply the theorem. Therefore we give a slightly different version of it, whose proof is a simple consequence of [10]. We will also prove that global solutions of the Signorini problem with a bound at infinity are uniquely determined by the set \( \Lambda_u \). Later we will refine this result somewhat and it will not be used in the rest of the paper.

**Proposition 1.** Let \( \Lambda \subset \Pi \) be a given set in \( \Pi \) and
\[
\mathcal{P}_\Delta(\Lambda) = \{ u \in W^{1,2}(\mathbb{R}^n \setminus \Lambda); \Delta u = 0 \text{ in } \mathbb{R}^n \setminus \Lambda, u \geq 0 \text{ in } \mathbb{R}^n \setminus \Lambda, u = 0 \text{ on } \Lambda \}.
\]
Then the set \( \mathcal{P}_\Delta(\Lambda) \) is a one or two dimensional set.
Moreover if \( u \) is even in \( x_n \) then \( \mathcal{P}_\Delta(\Lambda) \) is one dimensional.

For a proof see [6].

**Lemma 7.** Let \( \Lambda \subset \Pi \) be a given set in \( \Pi \) and
\[
\Delta u = 0 \quad \text{in } \mathbb{R}^3_+ \\
u = 0 \quad \text{on } \Lambda \\
\frac{\partial u}{\partial x_3} = 0 \quad \text{on } \Pi \setminus \Lambda \\
\sup_{B^+_R} |u| \leq CR^\alpha \quad \text{for } R > 0 \text{ and an } \alpha < 1
\]
then \( u \) has a sign. That is \( u \geq 0 \) or \( u \leq 0 \). In particular Proposition 1 applies.

**Proof:** We may extend \( u \) by an even reflection to
\[
\bar{u}(x) = \left\{ \begin{array}{ll}
u(x) & \text{if } x_3 \geq 0 \\
u(x_1, x_2, -x_3) & \text{if } x_3 < 0.
\end{array} \right.
\]
Then \( \bar{u} \) is harmonic in \( \mathbb{R}^3 \setminus \Lambda \). If \( u \) does not have a sign then \( \bar{u}^\pm \neq 0 \). But \( \sup_{B^+_R} |\bar{u}^\pm| \leq CR^\alpha \). However, since the supports of \( \bar{u}^\pm \) are disjoint it follows by Friedland Hayman’s Theorem [10] that at least one of \( \bar{u}^\pm \) must have at least linear growth at infinity. This is a contradiction. Therefore we can conclude that either \( \bar{u}^+ = 0 \) or \( \bar{u}^- = 0 \). \( \square \)

Next we prove a version of Benedicks’ Theorem for the Signorini problem. We will however need a refined version later and we will therefore not need the following Lemma in what follows.

**Lemma 8.** Let \( \mathcal{P} \) be the set of \( W^{2,2} \) solutions to
\[
\begin{align}
\Delta u + \frac{2+\lambda}{2} \nabla \text{div}(u) &= 0 \quad \text{in } \mathbb{R}^3_+ \\
u^3 &= 0 \quad \text{on } \Lambda \\
\frac{\partial u^3}{\partial x_n} + \lambda \text{div}(U) &= 0 \quad \text{on } \Pi \setminus \Lambda \\
\frac{\partial u^i}{\partial x_3} + \frac{\partial u^3}{\partial x_i} &= 0 \quad \text{on } \Pi \text{ for } i = 1, 2 \\
\sup_{B^+_R} |u| &\leq C(1 + R^{1+\alpha}) \quad \text{for some } \alpha < 1 \text{ and all } R \geq 1 \\
\sup_{B^+_R} |u| &\leq Cr^{1+\beta} \quad \text{for some } \beta > 0 \text{ and all } r < 1.
\end{align}
\]
Here \( \Lambda \) is a fixed set in \( \Pi \).

Then \( \mathcal{P} \) is a one dimensional set. That is \( \mathcal{P} = \{ t\mathbf{v}; t \in \mathbb{R} \} \) for some \( \mathbf{v} \in W^{2,2} \).
Proof: Since $u \in \mathcal{P}$ implies that $u \in W^{2,2}$ we know that $v \equiv \text{div}(u) \in W^{1,2}$. Taking the divergence of (35) it follows that $\Delta v = 0$ in $\mathbb{R}^3 \setminus \Lambda$. Using (38) we see that
\[
\frac{\partial^2 u^i}{\partial x_j \partial x_n} + \frac{\partial^2 u^3}{\partial x_j \partial x_i} = 0
\]
on $\Pi$ for $i, j = 1, 2$ and thus
\[
(41) \quad \frac{\partial v}{\partial x_3} = -\frac{\partial^2 u^3}{\partial x_1^2} - \frac{\partial^2 u^3}{\partial x_2^2} + \frac{\partial^2 u^3}{\partial x_3^2}
\]
on $\Pi$. From (36) we see that $\Delta' u^3 = 0$ on $\Lambda$
\[
\frac{\partial v}{\partial x_3} = \frac{\partial^2 u^3}{\partial x_3^2}
\]
on $\Lambda$. Therefore (35) implies that
\[
0 = \Delta u^3 + \frac{2 + \lambda}{2} \frac{\partial v}{\partial x_3} = \frac{4 + \lambda}{2} \frac{\partial^2 u^3}{\partial x_3^2}
\]
on $\Lambda$, where we also used $\Delta' u^3 = 0$ on $\Lambda$. Thus, using (41)
\[
(42) \quad \frac{\partial v}{\partial x_3} = 0 \quad \text{on } \Lambda.
\]
Thus $v$ satisfies the following boundary value problem
\[
(43) \quad \Delta v = 0 \quad \text{in } \mathbb{R}^3_+
\]
\[
\frac{\partial v}{\partial x_3} = 0 \quad \text{on } \Lambda.
\]
Using (39) and (40) we deduce that
\[
(44) \quad \sup_{B^+_r} |v| \leq C(1 + R^\alpha)
\]
where $\alpha < 1$ and
\[
(45) \quad \sup_{B^+_r} |v| \leq Cr^\beta
\]
where $\beta > 0$ and for every $r < 1$.

We make the following claim.

Claim 1: $v = 0$ on $\Omega$.

Proof of claim 1: This was proved in Lemma 6.

For the next claim we notice that we may define a antisymmetric solution in $\mathbb{R}^3 \setminus \Lambda$ by reflecting $v$
\[
v(x) = \begin{cases} v(x) & \text{if } x_3 \geq 0 \\ v(x_1, x_2, -x_3) & \text{if } x_3 < 0. \end{cases}
\]
Then $v$ is harmonic in $\mathbb{R}^3 \setminus \Lambda$ and satisfies the Neumann condition in (43) on both sides of $\Lambda$.

Claim 2. The set of antisymmetric solutions to (43), (44) and (45) is one dimensional.

Proof of claim 2: By antisymmetry it is enough to show that solutions to
\[
\begin{align*}
\Delta v &= 0 \quad \text{in } \mathbb{R}^3_+ \\
v &= 0 \quad \text{on } \Pi \setminus \Lambda \\
\frac{\partial v}{\partial x_3} &= 0 \quad \text{on } \Lambda
\end{align*}
\]
are one dimensional. This is a special case of a Proposition 1. The claim follows.
We are now ready to prove the Lemma. It is clearly enough to show that if \( u \) and \( v \) are two solutions normalized so that \( \text{div}(u) = \text{div}(v) \) then \( u = v \) since by claim 1 and 2 such a normalization always exists for non-vanishing solutions.

Let \( u, v \in P \) and \( \text{div}(u) = \text{div}(v) \). Then \( w = u - v \in P \) and \( \text{div}(w) = 0 \). From (35) we conclude that
\[
\Delta w = 0.
\]

From (38) it follows, as in the beginning of this proof, that \( \frac{\partial w^i}{\partial x^i} = 0 \) on \( \Pi \) for \( i = 1, 2 \). Using (39) and the Liouville Theorem we may conclude that
\[
w^1(x) = a_1 x_1 + b_1 x_2 \\
w^2(x) = a_2 x_1 + b_2 x_2
\]
for some constants \( a_1, a_2, b_1 \) and \( b_2 \). Using (40) we see that \( a_1 = a_2 = b_1 = b_2 = 0 \). From (38) we may deduce that
\[
\frac{\partial w^3}{\partial x^i} = 0 \\
\text{on } \Pi \text{ for } i = 1, 2
\]

Similarly let
\[
\text{curl}(u) = \frac{\partial u^1}{\partial x^2} - \frac{\partial u^2}{\partial x^1}.
\]
Then \( \Delta \text{curl}(u) = 0 \) and
\[
\text{curl}(u) = \frac{\partial u^1}{\partial x^2} - \frac{\partial u^2}{\partial x^1} = \frac{2 + \lambda}{2} \frac{\partial u^3}{\partial x^1} = 0 \text{ on } \Lambda,
\]
where we have used that
\[
\frac{\partial u^1}{\partial x^2} - \frac{\partial u^2}{\partial x^1} = 0 \text{ and } \frac{\partial u^2}{\partial x^1} = 0
\]
on \( \Lambda \). Moreover
\[
\text{curl}\left((1 + (2 + \lambda)/2)\text{div}(u), \text{curl}(u)\right) = \Delta u^2 + \frac{2 + \lambda}{2} \frac{\partial \text{div}(u)}{\partial x^2} = 0.
\]
That implies that there exist a \( v \) such that
\[
\nabla v = (1 + (2 + \lambda)/2)\text{div}(u), \text{curl}(u))
\]
It is easy to see that
\[
\Delta v = \Delta u^1 + \frac{2 + \lambda}{2} \frac{\partial \text{div}(u)}{\partial x^1} = 0.
\]

**Lemma 9.** Let \( u \) be a solution to the Signorini problem in \( \mathbb{R}^2_+ \) and
\[
\limsup_{r \to \infty} \frac{\ln(\|u\|_{L^2(B_r)})}{\ln(r)} < 2
\]
then \( \Gamma \) contains at most one point.

**Proof:** Taking the divergence of \( Lu = 0 \) we see that \( \text{div}(u) \) is harmonic. From claim 1 in the proof of Lemma 8 we also know that \( \text{div}(u) \) is antisymmetric so
\[
\text{div}(u) = 0 \text{ on } \Sigma.
\]
Also as in Lemma 8 \( \text{div}(u) \) has a sign.

Similarly let
\[
\text{curl}(u) = \frac{\partial u^1}{\partial x^2} - \frac{\partial u^2}{\partial x^1}.
\]
Then \( \Delta \text{curl}(u) = 0 \) and
\[
\text{curl}(u) = \frac{\partial u^1}{\partial x^2} - \frac{\partial u^2}{\partial x^1} = -2 \frac{\partial u^3}{\partial x^1} = 0 \text{ on } \Lambda,
\]
where we have used that
\[
\frac{\partial u^1}{\partial x^2} - \frac{\partial u^2}{\partial x^1} = 0 \text{ and } \frac{\partial u^2}{\partial x^1} = 0
\]
on \( \Lambda \). Moreover
\[
\text{curl}\left((1 + (2 + \lambda)/2)\text{div}(u), \text{curl}(u)\right) = \Delta u^2 + \frac{2 + \lambda}{2} \frac{\partial \text{div}(u)}{\partial x^2} = 0.
\]
Moreover
\[
(49) \quad \frac{\partial v}{\partial x_2} = \text{curl}(u) = 0 \quad \text{on } \Lambda \\
\frac{\partial v}{\partial x_1} = \text{div}(u) = 0 \quad \text{on } \Sigma.
\]
Since both \( \text{div}(u) \) and \( \text{curl}(u) \) have a sign, by Lemma 7, so does \( \frac{\partial v}{\partial x_1} \) and \( \frac{\partial v}{\partial x_2} \). We will assume that \( \frac{\partial v}{\partial x_1} \geq 0 \) and, the other case is handled similarly.

Next we assume that \( a,b \in \Gamma \) and \( a \neq b \). If two such points exist then we may chose them so that \( a \) is the left boundary of an interval \((a,a_0) \subset \Sigma \) and \( b \) is the right endpoint in an interval \((b_0,b) \subset \Sigma \). Observe that from (49) it follows that \( v \) is constant in each component of \( \Sigma \) and thus a solution to the thin obstacle problem in \( B^+_i(a) \) and \( B^+_i(b) \) if \( r \) is small enough.

The asymptotic expansion at free boundary points for the thin obstacle problem is known [4] and we may conclude that in polar coordinates

\[
v(x_1-a,x_2) = v(a,0) - \alpha r^{k+1/2} \sin ((k+1/2)\phi) + O(r^{k+1+1/2})
\]
and

\[
v(x_1-b,x_2) = v(b,0) - \beta r^{l+1/2} \cos ((l+1/2)\phi) + O(r^{l+1+1/2})
\]
for some \( \alpha, \beta \in \mathbb{R}_+ \) and integers \( k, l \geq 1 \). This implies that

\[
\frac{\partial v}{\partial x_2} \leq 0
\]
close to \( a \) and

\[
\frac{\partial v}{\partial x_2} \geq 0
\]
close to \( b \), with strict inequality away from \( \{x_2 = 0\} \). Since \( \frac{\partial u}{\partial y} \) has a sign we get a contradiction.

\[
6. \text{ Global Solutions, Part 2.}
\]

In this section we explicitly calculate the global homogeneous solutions \( u \) in \( \mathbb{R}^2_+ \) to (2)-(5).

**Lemma 10.** Let \( u(x,y) = (u(x,y),v(x,y)) \) be a global homogeneous solution of order \( 1+\alpha > 0 \) to the Lame system with \( \lambda \neq -2 \) in \( \mathbb{R}^2_+ \):

\[
\begin{align*}
\Delta u + \frac{\partial \text{div}(u)}{\partial y} &= 0 & \text{in } \mathbb{R}^2_+ \\
\Delta v + \frac{\partial \text{div}(u)}{\partial y} &= 0 & \text{in } \mathbb{R}^2_+ \\
v(x,0) &= 0 & \text{on } \{x > 0\} \\
\frac{\partial u}{\partial y}(x,0) &= 0 & \text{on } \{x > 0\} \\
\frac{\partial u}{\partial y}(x,0) + \frac{\partial v}{\partial x}(x,0) &= 0 & \text{on } \{x < 0\} \\
\frac{\partial v}{\partial y}(x,0) + \frac{\lambda}{2} \text{div}(u)(x,0) &= 0 & \text{on } \{x < 0\} \\
u & \in W^{2,2}_{\text{loc}}(\mathbb{R}^2_+)
\end{align*}
\]
then \( \alpha \in \mathbb{N} \) or \( \alpha \in \mathbb{N} + \frac{1}{2} = \{1/2, 3/2, 5/2, \ldots\} \), and in polar coordinates, \( x = r \cos(\phi) \) and \( y = r \sin(\phi) \), we have for some \( a \in \mathbb{R} \):

\[(i)\]

\[
\begin{align*}
u(r,\phi) &= r^{1+\alpha} \left( \frac{10+3\alpha+\alpha(2+\lambda)}{8(1+\alpha)} \cos (1+\alpha) \phi - \frac{2+\lambda}{8} \cos (1-\alpha) \phi \right) \cdot a \\
v(r,\phi) &= r^{1+\alpha} \left( \frac{2-\lambda-\alpha(2+\lambda)}{8(1+\alpha)} \sin (1+\alpha) \phi - \frac{2+\lambda}{8} \sin (1-\alpha) \phi \right) \cdot a.
\end{align*}
\]
if \( \alpha \in \mathbb{N} \) and
(ii)

\[
\begin{align*}
  u(r, \phi) &= r^{1+\alpha} \left( \frac{6+\lambda+\alpha(2+\lambda)}{8(1+\alpha)} \cos ((1+\alpha)\phi) - \frac{2+\lambda}{8} \cos ((1-\alpha)\phi) \right) a \\
  v(r, \phi) &= r^{1+\alpha} \left( \frac{6+\lambda-\alpha(2+\lambda)}{8(1+\alpha)} \sin ((1+\alpha)\phi) - \frac{2+\lambda}{8} \sin ((1-\alpha)\phi) \right) a.
\end{align*}
\]

If \( \alpha \in \mathbb{N} + \frac{1}{2} \).

Proof: Denoting \( w = \text{div}(u) \in W^{1,2} \) as we have done before we see that

\[
\begin{align*}
  \Delta w &= 0 \quad \text{in } \mathbb{R}^2_+ \\
  \frac{\partial w}{\partial y}(x,0) &= 0 \quad \text{on } \{ x > 0 \}.
\end{align*}
\]

Also \( w \) will be homogeneous of order \( \alpha \). That is, in polar coordinates,

\[
(50) \quad w(r, \phi) = r^{\alpha} \left( a \cos(\alpha \phi) + b \sin(\alpha \phi) \right).
\]

Using that \( \frac{\partial w}{\partial \phi}(r,0) = 0 \) we can deduce that \( b = 0 \).

Next we consider the ordinary differential equation

\[
\text{div}(u(r, \phi)) = w(r, \phi).
\]

Using that \( u \) is homogeneous of order \( 1 + \alpha \) it is easy to see that \( u \) is of the form

\[
(51) \quad u(r, \phi) = r^{1+\alpha} \left( a_u \cos((1+\alpha)\phi) + b_u \sin((1+\alpha)\phi) + c_u \cos((1-\alpha)\phi) + d_u \sin((1-\alpha)\phi) \right)
\]

and that

\[
(52) \quad v(r, \phi) = r^{1+\alpha} \left( a_v \cos((1+\alpha)\phi) + b_v \sin((1+\alpha)\phi) + c_v \cos((1-\alpha)\phi) + d_v \sin((1-\alpha)\phi) \right).
\]

Also \((u, v)\) solves

\[
\begin{align*}
  \Delta u + \frac{2 + \lambda}{2} \frac{\partial w}{\partial x} &= 0 \quad \text{in } \mathbb{R}^2_+ \\
  \Delta v + \frac{2 + \lambda}{2} \frac{\partial w}{\partial y} &= 0 \quad \text{in } \mathbb{R}^2_+ \\
  v(r,0) &= 0 \\
  \frac{\partial u}{\partial \phi}(r,0) &= 0 \\
  \frac{1}{r} \frac{\partial v}{\partial \phi}(r,\pi) - \frac{\lambda}{4} w(r,\pi) &= 0 \\
  \frac{\partial v}{\partial r}(r,\pi) + \frac{1}{r} \frac{\partial u}{\partial \phi}(r,\pi) &= 0
\end{align*}
\]

and

\[
(57) \quad \text{div}(u(v)) = w \quad \text{in } \mathbb{R}^2_+.
\]

From (53) we may deduce that

\[
(58) \quad c_v = -a_v
\]

and from (54) it follows that

\[
(59) \quad b_u = \frac{1 - \alpha}{1 + \alpha} d_u.
\]
Using (51), (50) and \( b = 0 \) we may deduce that
\[
d_u = 0
\]
and
\[
e_u = -\frac{2 + \lambda}{8}a.
\]
Similarly from (52) we deduce that
\[
a_v = 0
\]
and
\[
d_v = -\frac{(2 + \lambda)}{8}a.
\]
Equation (57) implies that
\[
b_v = \frac{6 + \lambda}{4(1 + \alpha)}a - a_u.
\]
Next we use equation (56) which implies that either
\[
e^{2\pi\alpha} = 1
\]
that is \( \alpha \in \mathbb{N} \) or
\[
a_u = \frac{6 + \lambda + \alpha(2 + \lambda)}{8(1 + \alpha)}a.
\]
From equation (55) we deduce that either
\[
e^{2\pi\alpha} = -1
\]
that is \( \alpha \in \mathbb{N} + \frac{1}{2} \) or
\[
a_u = \frac{10 + 3\lambda + \alpha(2 + \lambda)}{8(1 + \alpha)}a.
\]
Both (58) and (59) holds only if \( \lambda = -2 \). Therefore we must have either \( \alpha \in \mathbb{N} \) and (59) holds or \( \alpha \in \mathbb{N} + \frac{1}{2} \) and (58) holds.
In case \( \alpha \in \mathbb{N} + \frac{1}{2} \) then (58) implies that
\[
u(r, \phi) = r^{1+\alpha}\left(\frac{6 + \lambda + \alpha(2 + \lambda)}{8(1 + \alpha)} \cos((1 + \alpha)\phi) - \frac{2 + \lambda}{8} \cos((1 - \alpha)\phi)\right)a
\]
and
\[
u(r, \phi) = r^{1+\alpha}\left(\frac{6 + \lambda - \alpha(2 + \lambda)}{8(1 + \alpha)} \sin((1 + \alpha)\phi) - \frac{2 + \lambda}{8} \cos((1 - \alpha)\phi)\right)a.
\]
And if \( \alpha \in \mathbb{N} \) then
\[
u(r, \phi) = r^{1+\alpha}\left(\frac{10 + 3\lambda + \alpha(2 + \lambda)}{8(1 + \alpha)} \cos((1 + \alpha)\phi) - \frac{2 + \lambda}{8} \cos((1 - \alpha)\phi)\right)a
\]
and
\[
u(r, \phi) = r^{1+\alpha}\left(\frac{2 - \lambda - \alpha(2 + \lambda)}{8(1 + \alpha)} \sin((1 + \alpha)\phi) - \frac{2 + \lambda}{8} \sin((1 - \alpha)\phi)\right)a.
\]
We will also define a normalized \( L^2 \) norm that scales like the \( L^\infty \) norm which will be convenient later.

**Definition 1.** We will use the notation \( \tilde{L}^p(\Omega) \) for the average \( L^p \) space with norm
\[
\|u\|_{\tilde{L}^p(\Omega)} = \left(\frac{1}{|\Omega|} \int_{\Omega} |u|^p \right)^{1/p}
\]
where \( |\Omega| \) is the measure of \( \Omega \).
Inspired by Lemma 10 we make the following definition of the global normalised homogeneous two dimensional solutions.

**Definition 2.** We will denote by \( p_{1/2}(x_1, x_n), \) \( p_1(x_1, x_n), \) \( p_{3/2}(x_1, x_n), \) etc. the homogeneous solution to the Signorini problem in \( \mathbb{R}_+^2 \) with \( \|p_{1+n}\|_{L^2(B_1^+)} = 1 \) of order \( 1/2, 1, 3/2, \) etc. as specified in Lemma 10.

We will also use the notation \( \hat{p}_{k+1/2}(x_1, x_n) = \frac{\partial p_{k+3/2}(x_1, x_n)}{\partial x_n} \).

**Lemma 11.** Let \( w = (w^1, w^2, w^3) \in W^{1,2}(B_1^+) \) be a solution to the following linear problem

\[
\begin{align*}
\Delta w + \frac{2+\lambda}{2} \nabla \text{div}(w) &= 0 & &\text{in } B_1^+
\end{align*}
\]

\[
\begin{align*}
w^3(x_1, x_2, 0) &= 0 & &\text{on } \{x_1 > 0\} \cap \Pi \\
\frac{\partial w^3}{\partial x_2} + \frac{\partial w^3}{\partial x_1} &= 0 & &\text{on } \Pi \text{ for } i = 1, 2 \\
\frac{\partial w^3}{\partial x_2} + \frac{\lambda}{4} \text{div}(w) &= 0 & &\text{on } \{x_1 < 0\} \cap \Pi \\
\|w\|_{L^\infty(B_1^+)} &\leq 1
\end{align*}
\]

then \( w = \sum_{i=1}^\infty a_i q_i \) where \( q_i \) is a homogeneous solution of order \( i/2 \) to the same problem.

Furthermore if \( w \in W^{2,2}(B_1^+) \) then

\[
w = ap_{3/2} + \sum_{i=0}^\infty a_i q_i.
\]

For a brief sketch of a proof see Appendix 2.

7. Flatness of the Free Boundary

In this section we introduce the first fundamental idea in the paper and show that at non-regular points the free boundary is flat.

From here on we will no longer, with the exception of Lemma 16, need any explicit calculations using the curl operator and we will therefore write \( \mathbb{R}^n \) instead of \( \mathbb{R}^3 \). Some of the ideas in this section to control the growth of blow up sequences comes from [2].

**Proposition 2.** Let \( u \) be a solution to the Signorini problem in \( B_1^+ \) and assume that \( 0 \in \Gamma \). Assume furthermore that

\[
\liminf_{r \to 0} \frac{\ln \left( \|u\|_{L^2(B_1^+)} \right)}{\ln(r)} < 2
\]

then there exists a sub-sequence \( r_j \to 0 \) such that

\[
\frac{\|u(r_j, x)\|_{L^2(B_1^+)} \to v,}
\]

where \( v \) is a global solution to the Signorini problem and furthermore, after a rotation, \( \Lambda_v = \{x \in \mathbb{R}^{n-1}; x_1 \geq 0 \} \).

**Proof:** Assume that the limit in (60) is less than two and call the limit \( \gamma < 2 \) and let \( 0 < \alpha < 1 \) and \( 1 + \alpha > \gamma \), we also let \( u' \) be as in the lemma, \( r_j \to 0 \) be a sequence such that

\[
\frac{\|u(r_j, x)\|_{L^2(B_1^+)} \to 1}.
\]

Such sequences exist since
\[ \limsup_{r \to 0} \left\| \frac{u(x)}{r^{1+\alpha}} \right\|_{L^2(B_r^+)} = \infty. \]
We may also choose \( r_j \) maximal in the sense that
\[ \|u^j(x)\|_{L^2(B_r^+)} \leq j r_j^{1+\alpha} \]
for \( r \geq r_j \). We make the blow-up
\[ v^j(x) = \frac{u^j(r_j x)}{j r_j^{1+\alpha}}. \]
Then, for a sub-sequence \( v^j \to v^0 \), locally and weakly in \( W^{2,2} \) and locally strongly in \( W^{1,2} \).
Also
\[ v^{i,j} \equiv \frac{\partial v^j}{\partial x_i} \]
will converge locally in \( C^0 \cap W^{1,2} \) to a solution \( v^{i,0} \) to the following mixed boundary value problem
\[
\begin{align*}
\Delta v^{i,0} + \frac{\lambda + 2}{4}\nabla \div (v^{i,0}) &= 0 &\text{in } \mathbb{R}^n_+ \\
\frac{\partial e_n \cdot v^{i,0}}{\partial x_n} + \frac{\lambda + 4}{8} \div (v^{i,0}) &= 0 &\text{on } \Pi \cap \{ e_n \cdot v^{i,0} > 0 \} \\
\frac{\partial e_n \cdot v^{i,0}}{\partial x_n} &= 0 &\text{on } \Pi \text{ for } k = 1, 2, \ldots, n-1. 
\end{align*}
\]
if \( i = 1, 2, \ldots, n-1 \) and
\[
\begin{align*}
\Delta v^{n,0} + \frac{\lambda + 2}{4}\nabla \div (v^{n,0}) &= 0 &\text{in } \mathbb{R}^n_+ \\
\frac{\partial e_n \cdot v^{n,0}}{\partial x_n} &= 0 &\text{on } \Pi \cap \{ e_n \cdot v^{n,0} = 0 \} \\
\frac{\partial e_n \cdot v^{n,0}}{\partial x_n} &= 0 &\text{on } \Pi \text{ for } k = 1, 2, \ldots, n-1. 
\end{align*}
\]
Using (61) we may also conclude that
\[ \sup_{B_R^+} |v^0| \leq CR^{1+\alpha}. \]
From the Corollary 4 we can conclude that
\[ v^0 = \nabla \left( \frac{\lambda + 2}{2(\lambda + 4)} \frac{\partial \tau}{\partial x_3} x_3 - \frac{\lambda + 3}{\lambda + 4} \right) + e_3 \frac{\partial \tau}{\partial x_3}, \]
where
\[
\begin{align*}
\Delta \tau &= 0 &\text{in } \mathbb{R}^n_+ \\
\frac{\partial \tau}{\partial x_n} &= 0 &\text{on } \Lambda_{\nu^0} \\
\frac{\partial \tau}{\partial x_n} &= 0 &\text{on } \Omega_{\nu^0} \\
\frac{\partial \tau}{\partial x_n} \in W^{2,2}(B_R^+) &\text{for each } R > 0.
\end{align*}
\]
Naturally the function
\[ \zeta(x) = \frac{\partial \tau}{\partial x_n} \]
will solve
\[
\begin{align*}
\Delta \zeta &= 0 &\text{in } \mathbb{R}^n_+ \\
\zeta &= 0 &\text{on } \Lambda_{\nu^0} \\
\frac{\partial \zeta}{\partial x_n} &= 0 &\text{on } \Omega_{\nu^0} \\
\zeta \in W^{2,2}(B_R^+) &\text{for each } R > 0.
\end{align*}
\]
Moreover,
\[ \zeta_i(x) = \frac{\partial \zeta}{\partial x_i} \text{ for } i = 1, 2, \ldots, n-1 \]
will solve
\[ \begin{align*}
\Delta \zeta_i &= 0 & \text{in } \mathbb{R}^3_+ \\
\zeta_i &= 0 & \text{on } \Lambda_{\varphi_0} \\
\frac{\partial \zeta_i}{\partial x_n} &= 0 & \text{on } \Omega_{\varphi_0} \\
\zeta &\in W^{1,2}(B^+_R) & \text{for each } R > 0.
\end{align*} \tag{62} \]

and
\[ \sup_{B^+_R} |\zeta_i| \leq CR^\alpha. \tag{63} \]

By Benedicks’ Theorem we know that the set of solutions to (62) and (63) is a one dimensional set. We may conclude that there are constants \( a_1, a_2, \ldots, a_{n-1} \) such that
\[ a_1 \zeta_1 = a_2 \zeta_2 = \ldots = a_{n-1} \zeta_{n-1} \]
and therefore that
\[ \eta \cdot \nabla' \zeta = 0 \]
for every
\[ \eta \in \{ \eta \in \mathbb{R}^n; \eta \cdot (a_1, a_2, \ldots, a_{n-1}, 0) = 0 \} \cap \Pi. \]

By rotating the coordinate system we may assume that
\[ \zeta(x) = \zeta(x_1, x_n). \]

We can directly conclude that
\[ \tau(x) = \bar{\tau}(x_1, x_n) + \bar{\tau}(x'). \]

Where \( \Delta \bar{\tau} = \Delta \bar{\tau} = 0 \). We thus have that, as in the proof of Corollary 3,
\[ v^0 = \nabla \left( \frac{\lambda + 2}{2} (\lambda + 4) \frac{\partial \tau}{\partial x_n} x_n - \frac{\lambda + 3}{\lambda + 4} \tau + e_n \frac{\partial \tau}{\partial x_n} \right) + e_n \frac{\partial \tau}{\partial x_n}, \]
\[ = \nabla \left( \frac{\lambda + 2}{2} (\lambda + 4) \frac{\partial \bar{\tau}}{\partial x_n} x_n - \frac{\lambda + 3}{\lambda + 4} \bar{\tau} + e_n \frac{\partial \bar{\tau}}{\partial x_n} + \frac{\lambda + 3}{\lambda + 4} \bar{\tau} \right). \]

But \( \bar{\tau} \) is harmonic and
\[ \sup_{B^+_R} |\bar{\tau}| \leq CR^{2+\alpha}. \]

By the Liouville Theorem it follows that \( \nabla \bar{\tau} \) is an affine function. But by our standing assumption the affine part of \( v^0 \) is zero. We may thus conclude that
\[ \tau(x) = \tau(x_1, x_n). \]

Claim: Neither \( \Lambda_{\varphi_0} \) nor \( \Sigma_{\varphi_0} \) are empty.

Proof of the Claim: Let’s assume that \( \Lambda_{\varphi_0} = \emptyset \) then
\[ \Delta v^0 + \frac{\lambda + 4}{2} \nabla \text{div}(v^0) = 0 \quad \text{in } \mathbb{R}^n_+ \]
\[ e_n \cdot v^0 = 0 \quad \text{on } \Pi \]
\[ \frac{\partial v^0}{\partial x_n} = 0 \quad \text{on } \Pi \text{ for } k = 1, 2. \]

Moreover
\[ \sup_{B^+_R} |v^0| \leq R^{1+\alpha}. \]

In particular
\[ w(x) = \begin{cases} 
\begin{bmatrix}
 v^0(x) \\
 e_1 \cdot v^0(x_1, x_2, \ldots, -x_n) \\
 e_2 \cdot v^0(x_1, x_2, \ldots, -x_n) \\
 \vdots \\
 -e_n \cdot v^0(x_1, x_2, \ldots, -x_n)
\end{bmatrix} & \text{if } x_n > 0 \\
\end{cases} \]

if \( x_n < 0 \).
will solve
\[ \Delta w + \frac{\lambda + 2}{2} \nabla \text{div}(w) = 0 \quad \text{in } \mathbb{R}^3 \]
and
\[ \sup_{B_R} |w| \leq R^{1+\alpha}. \]
It follows, from Liouville’s Theorem, that \( w \) is a plane. This contradicts that \( 0 \in \Gamma_u \).

Using (64) we may consider \( v^0 \) as a solution in \( \mathbb{R}^2_+ \) and use Lemma 9. It follows that the free boundary consists of one point. Extending \( v^0 \) to \( \mathbb{R}^n_+ \) again we see that the free boundary is a plane in \( \Pi \). The proposition follows. \( \square \)

8. Almost Optimal Regularity

We can now easily deduce that the solutions are \( C^{1,\beta} \) for each \( \beta < 1/2 \).

**Lemma 12.** Let \( u \) be a solution to the Signorini problem in \( B_1^+ \) and assume that \( \|u\|_{L^2(B_1^+)} = 1 \) and that \( 0 \in \Gamma_u \) then for each \( \alpha < 1/2 \) there exist a constant \( C_\alpha \) such that
\[ \sup_{B_R^+} |u| \leq C_\alpha r^{1+\alpha}. \]

**Proof:** If not then we can find an \( \alpha < 1/2 \) and a sequence of solutions \( u^j \) and \( r_j \to 0 \) such that
\[ \sup_{B_{r_j}^+} |u^j| = j r_j^{1+\alpha}, \]
and
\[ \sup_{B_R^+} |u^j| \leq j R^{1+\alpha} \]
for each \( R \geq r_j \). Make the blow-up
\[ \tilde{u}^j = \frac{u^j(r_j x + x_0)}{j r_j^{1+\alpha}} \]
then, using Lemma 1, for a sub-sequence \( \tilde{u}^j \to u^0 \) in \( C^{1,\beta}_{\text{loc}}(\mathbb{R}^n_+) \). From Proposition 2 we can conclude that
\[ u^0(x) = u^0(x_1, x_n) \]
and that after a rotation and \( \Sigma \) and \( \Lambda \) are complementary half spaces, furthermore we have \( \sup_{B_R^+} |u^0| \leq R^{1+\alpha} \) for \( R > 1 \). It follows, from Lemma 11, that \( u^0 = 0 \) which contradicts \( \sup_{B_1^+} |u^0| = \lim_{j \to \infty} (\sup_{B_{r_j}^+} |\tilde{u}^j|) = 1. \)

**Corollary 5.** Let \( u \) be a solution to the Signorini problem in \( B_1^+ \) then for each \( \alpha < 1/2 \) there exists a constant \( C_\alpha \) such that
\[ \|u\|_{C^{1,\alpha}(B_1^+)} \leq C_\alpha \|u\|_{L^2(B_1^+)}. \]

**Proof:** We may assume that \( \|u\|_{L^2} = 1 \). Let \( d(x^0) = \text{dist}(x^0, \Gamma) \) then
\[ \sup_{B_{d(x^0)}(x^0)} |u| \leq C_\alpha d(x^0)^{1+\alpha}. \]

Thus
\[ u_{d(x^0)} = \frac{u(d(x^0)x + x^0)}{d(x^0)^{1+\alpha}} \]
is a solution in \( B_1 \) and \( \sup_{B_1} |u_{d(x^0)}| \leq C_\alpha \). We might need, and in that case we do, either evenly or oddly reflect \( u \) across \( \Pi \) in order for \( u_{d(x^0)} \) to be defined.
in the entire unit ball. It follows that \(|\nabla u_{d(x)}(0)| \leq CC_\alpha\). Scaling back we get \(|\nabla u(x)| \leq CC_\alpha d(x)^\alpha\) which implies that \(u \in C^{1,\alpha}\).

9. Fundamental and Technical Results.

With this section we start to get a little more technical and we will start to lay the foundation for the flatness improvement results that leads to optimal regularity and free boundary regularity.

Lemma 13. Let \(u\) be a sequence of solutions to the Signorini problem in \(B_1^+\) and assume furthermore that

\[
\inf_{x \in \Pi} \|u - p_{3/2}^\xi\|_{L^2(B_1^+)} = \inf_{x \in \Pi} \|u - p_{3/2}\|_{L^2(B_1^+)} = \delta_j \to 0
\]

then \(v^0\) solves

\[
\begin{align*}
\Delta v^0 + \frac{\lambda_n}{e_n} \nabla v^0 &= 0 & &\text{in } B_1^+ \\
\frac{\partial e_n}{\partial x_n} v^0 &= 0 & &\text{on } \Pi \cap \{x_1 > 0\} \\
\frac{\partial e_n}{\partial x_k} v^0 &= 0 & &\text{on } \Pi \cap \{x_1 > 0\} \\
\text{and } v^0 &\to v^0 & &\text{weakly in } L^2
\end{align*}
\]

\[
\inf_{a \in \mathbb{R}} \|v^0 - ap_{3/2}\|_{L^2(B_1^+)} = \|v^0\|_{L^2(B_1^+)}.
\]

Proof: That \(v^0\) converges weakly to a solution of the system is simple so we will only prove (66).

By (65) we have

\[
\int_{B_1^+} p_{3/2}(u^j - p_{3/2}) = 0
\]

so by weak convergence we have

\[
\int_{B_1^+} p_{3/2}v^0 = 0.
\]

Now \(\|v^0 - ap_{3/2}\|_{L^2(B_1^+)}\) is convex in \(a\) so it has only one minimum. Therefore (67) implies that the minimum is at \(a = 0\).

Lemma 14. Let \(u\) be a global solution to the Signorini problem and

\[
\inf_{x \in \Pi} \|u - p_{3/2}^\xi\|_{L^2(B_1^+)} \leq \mu \|u\|_{L^2(B_1^+)}
\]

for all \(R \geq 1\). Then if \(\mu\) is small enough then \(u = p_{3/2}^\xi\) for some \(x \in \Pi\).

Also, if \(u\) is a solution to the Signorini problem in \(B_1^+\) then for each \(\epsilon > 0\) there exist a \(\mu_\epsilon > 0\) such that if

\[
\inf_{x \in \Pi} \|u - p_{3/2}^\xi\|_{L^2(B_1^+)} \leq \mu_\epsilon \|u\|_{L^2(B_1^+)}
\]

for all \(r \leq 1\). Then

\[
\|u\|_{L^2(B_1^+)} \geq r^{3/2 + \epsilon}.
\]

Proof: The proof of the first and second parts are very similar so we will only prove the first part.

Let \(\gamma > 0\) be the real non-negative solution to \(\gamma^{(n+3)/2} = 1/2\). Then

\[
\mu \|u\|_{L^2(B_R)} \geq \gamma^{n/2} \|u - p_{3/2}^\xi\|_{L^2(B_R^+)} \geq \gamma^{n/2} \|p_{3/2}^\xi - p_{3/2}\|_{L^2(B_R^+)} - \gamma^{n/2} \|u - p_{3/2}\|_{L^2(B_R^+)} \geq \gamma^{n/2} \|p_{3/2}^\xi - p_{3/2}\|_{L^2(B_R^+)} - \gamma^{n/2} \|u - p_{3/2}\|_{L^2(B_R^+)} \geq
\]

(68)
Next we notice that
\[ \gamma^{n/2} \| \mathbf{p}^{\xi_R}_{3/2} - \mathbf{p}^{\xi_{R'}}_{3/2} \|_{L^2(B_{R'}^+)} - \mu \gamma^{n/2} \| \mathbf{u} \|_{L^2(B_{R'}^+)} . \]
Inserting this into (68) results in
\[ (69) \quad \mu \left( \| \mathbf{u} \|_{L^2(B_R^+)} + \gamma^{n/2} \| \mathbf{u} \|_{L^2(B_{R'}^+)} \right) \geq \gamma^{(n+3)/2} |\xi_R - \xi_{R'}| . \]
Next we use the triangle inequality to estimate for any \( T \geq 1 \)
\[ \| \mathbf{u} \|_{L^2(B_T^+)} - \| \mathbf{p}^{\xi_R}_{3/2} \|_{L^2(B_T^+)} \leq \| \mathbf{u} - \mathbf{p}^{\xi_R}_{3/2} \|_{L^2(B_T^+)} \leq \mu \| \mathbf{u} \|_{L^2(B_T^+)} . \]
That is
\[ \| \mathbf{u} \|_{L^2(B_T^+)} \leq \frac{1}{1 - \mu} \| \mathbf{p}^{\xi_R}_{3/2} \|_{L^2(B_T^+)} . \]
If we use (70) in (69) and that \( \gamma^{(n+3)/2} = 1/2 \) we get
\[ \frac{\mu}{1 - \mu} (2 + \sigma) \geq (1 - \sigma) \]
where we have used the notation \( \sigma |\xi_R| = |\xi_{R'}| . \) That is
\[ \sigma \geq 1 - 3 \mu . \]
We have shown that
\[ |\xi_R| \leq \frac{1}{1 - 3 \mu} |\xi_{R'}| . \]
Iterating this relation we get
\[ |\xi_{R^{-k}}| \leq (1 - 3 \mu)^{-k} |\xi_1| . \]
We may normalize so \( |\xi_1| = 1 \) and use (70) to deduce that
\[ (71) \quad \| \mathbf{u} \|_{L^2(B_{R^{-k}}^+)} \leq \frac{1}{1 - \mu} (1 - 3 \mu)^{-k} \gamma^{-\frac{k}{2}} . \]
In particular if \( \mu \) is small enough to satisfy
\[ (72) \quad (n + 3) \frac{\ln \left( \frac{1}{1 - 3 \mu} \right)}{\ln(2)} < 1 \]
Then
\[ (73) \quad \lim_{R \to \infty} \frac{\ln \left( \frac{\| \mathbf{u} \|_{L^2(B_R^+)}}{\ln(R)} \right)}{\ln(R)} < 2 . \]
We may conclude, as in the argument of Proposition 2 that \( \mathbf{u}(x) = \mathbf{u}(x_1, x_n) \) in some coordinate system. From Lemma 11 we conclude that
\[ \mathbf{u} = \alpha \mathbf{p}_{3/2}(x) + \sum_{i=2}^{\infty} \alpha_i \mathbf{q}_i . \]
From (73) it follows that \( \alpha_i = 0 \) for all \( i \) and the first part of the Lemma follows.

The second part is similar.

Lemma 15. Assume that \( \| \mathbf{u} \|_{W^{1,2}(B_T^+)} \leq C_1 \) and
\[ \inf_{\xi \in \Pi} \| \mathbf{u} - \mathbf{p}^{\xi}_{3/2} \|_{L^2(B_T^+)} \leq \delta , \]
where the minimizing \( \xi \) satisfies \( |\xi| = 1 \) assume furthermore that \( \delta \) and \( \delta_1 \) are small enough and that
\[ (74) \quad \| \nabla^m \mathbf{u} \|_{L^2(B_T^+)} \leq \delta_1 . \]
Then $|\xi - e_1| \leq C\delta_1$ and
\[ \|\nabla^2_{\xi} u\|_{L^2(B_1)} \leq (CC_1 + 1)\delta_1, \]
here $\nabla^2_{\xi} = \nabla - e_n \frac{\partial}{\partial x_n} - \xi_w/|\xi_w|^2 (\xi_w \cdot \nabla)$ is the gradient restricted to the subspace orthogonal to $e_n$ and $\xi$.

Proof: We may rotate the coordinates so that $\xi = (\xi_1, \xi_2, 0, 0, \ldots)$ where $\xi_2$ is very small, we may also assume that $\xi_2 \geq 0$. Notice that when $\xi_2$ and $\delta$ are small then

\[ \frac{p_{3/2}^\xi - p_{3/2}}{\delta} = \frac{\xi_2}{\delta} x_2 p_{1/2} + q \]

where
\[ \|q\|_{L^2(B^+_1)} = o(\xi_2/\delta). \]

Let
\[ v(x) = u(x) - \frac{p_{3/2}^\xi(x)}{\delta} \]

and
\[ w(x) = u(x) - \frac{p_{3/2}(x)}{\delta}. \]

Then by the minimizing property of $p_{3/2}^\xi$ it follows that

\[ \|v\|_{L^2(B^+_1)} \leq \|w\|_{L^2(B^+_1)}. \]

From the assumption (74) it follows that $\|\nabla'' w\|_{L^2(B^+_1)} \leq \delta_1/\delta$ which by the Poincare inequality implies that

\[ \|w - \bar{w}\|_{L^2(B^+_1)} \leq C\delta_1^\frac{1}{\delta} \]

where $\bar{w}(y) = \bar{w}(y_1, y_n)$ is the average of $w$ on the $(n-1)$-plane $B^+_1 \cap \{x_1 = y_1, x_n = y_n\}$. Form (75) it follows that

\[ \int_{B^+_1} |v|^2 = \int_{B^+_1} \left( |w|^2 + 2\frac{\xi_2}{\delta} x_2 p_{1/2} w + \frac{(\xi_2)^2}{\delta^2} x_2^2 |p_{1/2}|^2 \right) + o(\xi_2/\delta) \geq \]

\[ \int_{B^+_1} |w|^2 + c\frac{(\xi_2)^2}{\delta^2} - 2\frac{\xi_2}{\delta} \sqrt{\int_{B^+_1} x_2^2 |p_{1/2}|^2} \sqrt{\int_{B^+_1} |w - \bar{w}|^2} \]

where we have used that $\int_{B^+_1} x_2 \bar{w} = 0$. Applying (77) and (76) we may deduce that

$C\delta_1 \geq \xi_2$

which implies the first conclusion in the Lemma.

The second conclusion follows by writing $\nabla'' = \nabla'' + \xi_2 e_2 \frac{\partial}{\partial x_2} + (1 - \sqrt{1 + \xi_2^2}) e_1 \frac{\partial}{\partial x_1}$ and thus

\[ \|\nabla''_\xi u\|_{L^2(B^+_1)} \leq \|\nabla'' u\|_{L^2(B^+_1)} + C|\xi_2| \|\nabla u\|_{L^2(B^+_1)} \leq \delta_1 + CC_1\delta_1. \]

The next Lemma looks more complicated than it is in reality. It just states that a gradient can not be written as something that is not a gradient plus a small perturbation.
Lemma 16. Let $u$ solve the Signorini problem in $B_1^+$ and assume that $u = p_{3/2} + R$
where
\begin{align}
\nabla R = \begin{bmatrix}
a_1 p_{1/2}^1 & a_2 p_{1/2}^1 & a_3 p_{1/2}^1 \\
a_1 p_{1/2}^2 & a_2 p_{1/2}^2 & a_3 p_{1/2}^2 \\
a_1 p_{1/2}^3 & a_2 p_{1/2}^3 & a_3 p_{1/2}^3 \\
\end{bmatrix} + \begin{bmatrix}
m_{11}(x) & m_{12}(x) & m_{13}(x) \\
m_{21}(x) & m_{22}(x) & m_{23}(x) \\
m_{31}(x) & m_{32}(x) & m_{33}(x) \\
\end{bmatrix} &= P + \tilde{M}
\end{align}
and
\begin{align}
\|\tilde{M}\|_{L^2(B_1^+ \cap \{x_3 > \epsilon\})} \leq \epsilon c \sum_{i=1}^{3} (a^i)^2
\end{align}
for any $\epsilon \in [0, 1/4]$ and some small enough, but universal, $c$. Assume furthermore that
\begin{align}
|a^1 - a^3| > \frac{1}{4} (|a^1| + |a^3|).
\end{align}
Then $a^1 = a^2 = a^3 = 0$.

Proof: If we apply the curl operator on both sides of (78) we can deduce that
\begin{align}
0 = \text{curl}(\nabla R) = \text{curl}\left( \begin{bmatrix}
a_1 p_{1/2}^1 & a_2 p_{1/2}^1 & a_3 p_{1/2}^1 \\
a_1 p_{1/2}^2 & a_2 p_{1/2}^2 & a_3 p_{1/2}^2 \\
a_1 p_{1/2}^3 & a_2 p_{1/2}^3 & a_3 p_{1/2}^3 \\
\end{bmatrix} + \begin{bmatrix}
m_{11}(x) & m_{12}(x) & m_{13}(x) \\
m_{21}(x) & m_{22}(x) & m_{23}(x) \\
m_{31}(x) & m_{32}(x) & m_{33}(x) \\
\end{bmatrix} \right)
+ \text{curl}\left( \begin{bmatrix}
m_{11}(x) & m_{12}(x) & m_{13}(x) \\
m_{21}(x) & m_{22}(x) & m_{23}(x) \\
m_{31}(x) & m_{32}(x) & m_{33}(x) \\
\end{bmatrix} \right).
\end{align}
Rearranging terms and taking the $L^2(B_{1-2\epsilon} \cap \{x_3 > 2\epsilon\})$ norm on both sides we may conclude that
\begin{align}
\|\text{curl}\left( \begin{bmatrix}
a_1 p_{1/2}^1 & a_2 p_{1/2}^1 & a_3 p_{1/2}^1 \\
a_1 p_{1/2}^2 & a_2 p_{1/2}^2 & a_3 p_{1/2}^2 \\
a_1 p_{1/2}^3 & a_2 p_{1/2}^3 & a_3 p_{1/2}^3 \\
\end{bmatrix} \right) \|_{L^2(B_{1-2\epsilon} \cap \{x_3 > 2\epsilon\})} = \|\text{curl}(\tilde{M})\|_{L^2(B_{1-2\epsilon} \cap \{x_3 > 2\epsilon\})}.
\end{align}
But
\begin{align}
\|\text{curl}\left( \begin{bmatrix}
a_1 p_{1/2}^1 & a_2 p_{1/2}^1 & a_3 p_{1/2}^1 \\
a_1 p_{1/2}^2 & a_2 p_{1/2}^2 & a_3 p_{1/2}^2 \\
a_1 p_{1/2}^3 & a_2 p_{1/2}^3 & a_3 p_{1/2}^3 \\
\end{bmatrix} \right) \|_{L^2(B_{1-2\epsilon} \cap \{x_3 > 2\epsilon\})} = \left\| \begin{bmatrix}
-a_1 \frac{\partial p_{1/2}^1}{\partial x_3} & -a_2 \frac{\partial p_{1/2}^1}{\partial x_3} & -a_3 \frac{\partial p_{1/2}^1}{\partial x_3} \\
-a_1 \frac{\partial p_{1/2}^2}{\partial x_3} & -a_2 \frac{\partial p_{1/2}^2}{\partial x_3} & -a_3 \frac{\partial p_{1/2}^2}{\partial x_3} \\
-a_1 \frac{\partial p_{1/2}^3}{\partial x_3} & -a_2 \frac{\partial p_{1/2}^3}{\partial x_3} & -a_3 \frac{\partial p_{1/2}^3}{\partial x_3} \\
\end{bmatrix} \right\|_{L^2(B_{1-2\epsilon} \cap \{x_3 > 2\epsilon\})},
\end{align}
where we have used that $p_{1/2}(x) = p_{1/2}(x_1, x_3)$. Next we notice that by definition
\begin{align}
\tilde{p}_{1/2} = \frac{\partial p_{3/2}}{\partial x_3}
\end{align}
and that
\begin{align}
p_{1/2} = \frac{\partial p_{3/2}}{\partial x_1}.
\end{align}
In particular this implies that
\begin{align}
\frac{\partial p_{1/2}^1}{\partial x_3} = \frac{\partial \tilde{p}_{1/2}^1}{\partial x_1}
\end{align}
for $i = 1, 2, 3$. Therefore

$$
\| \text{curl} \left( \begin{bmatrix}
    a_1 p_{1/2}^1 & a_2 p_{1/2}^1 & a_3 p_{1/2}^1 \\
    a_1 p_{1/2}^2 & a_2 p_{1/2}^2 & a_3 p_{1/2}^2 \\
    a_1 p_{1/2}^3 & a_2 p_{1/2}^3 & a_3 p_{1/2}^3 
\end{bmatrix} \right) \|_{L^2(B_{1-\epsilon} \cap \{x_3 > 2\epsilon\})}
$$

$$
\geq c_0 \sqrt{a_1^2 \int_{B_{1-\epsilon} \cap \{x_3 > \epsilon\}} \| \nabla p_{1/2} \|^2 + (a_1 - a_3)^2 \int_{B_{1-\epsilon} \cap \{x_3 > \epsilon\}} \left( \frac{\partial^2 p_{3/2}}{\partial x_1 \partial x_3} \right)^2}
$$

$$
\geq c_1 \sqrt{a_1^2 + (a_1 - a_3)^2} \geq c_2 \sqrt{a_1^2 + a_2^2 + a_3^2},
$$

where we have used (80) in the last inequality. Putting (81) and (82) together we have thus shown that

$$
c_2 \sqrt{a_1^2 + a_2^2 + a_3^2} \leq \| \text{curl}(\tilde{M}) \|_{L^2(B_{1-\epsilon} \cap \{x_3 > 2\epsilon\})}.
$$

Next we use that each column of $\tilde{M}$ solves the Lame system and therefore we have the estimate

$$
\| \text{curl}(\tilde{M}) \|_{L^2(B_{1-\epsilon} \cap \{x_3 > 2\epsilon\})} \leq C \| \nabla \tilde{M} \|_{L^2(B_{1-\epsilon} \cap \{x_3 > 2\epsilon\})} \leq \frac{C}{\epsilon} \| \tilde{M} \|_{L^2(B_{1-\epsilon} \cap \{x_3 > \epsilon\})}
$$

so we have shown that

$$
\sqrt{a_1^2 + a_2^2 + a_3^2} \leq \frac{C}{c_2 \epsilon} \| \tilde{M} \|_{L^2(B_1^+)} \leq \frac{Cc}{c_2} \sqrt{a_1^2 + a_2^2 + a_3^2},
$$

where we have used (79). But if $\epsilon \leq c_2/(2C)$ this implies that

$$
\sqrt{a_1^2 + a_2^2 + a_3^2} \leq \frac{1}{2} \sqrt{a_1^2 + a_2^2 + a_3^2}
$$

which implies that

$$
\sqrt{a_1^2 + a_2^2 + a_3^2} = 0.
$$

\hfill \square

10. Regularity Improvement for the Tangential Gradient.

In the next proposition, and also afterwards, we will denote

$$
\nabla'' = \left(0, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \ldots, \frac{\partial}{\partial x_{n-1}}, 0\right).
$$

Lemma 17. Let $u$ solve the Signorini problem in $B_1^+$ and $0 < \gamma < 1/2$ then there exists a $\delta_\gamma > 0$ such that if

$$
\inf_{\xi \in \mathbb{R}} \| u - p_{3/2}^\xi \|_{L^2(B_1^+)} = \| u - p_{3/2} \|_{L^2(B_1^+)} \leq \delta_\gamma,
$$

and

$$
\| \nabla'' u \|_{L^2(B_1^+)} \leq \delta_\gamma
$$

then there exist an $0 < s_\gamma < 1$ such that

$$
\| \nabla'' u \|_{L^2(B_1^+)} \leq s_\gamma^{1/2 + \gamma} \| \nabla'' u \|_{L^2(B_1^+)}.
$$
Proof: Let \( \mathbf{u}^j \) be a sequence as in the Lemma corresponding to \( \delta_j \to 0 \) and a fixed \( \gamma \in (0, 1) \). Consider

\[
\begin{bmatrix}
\mathbf{v}^{1,j} \\
\mathbf{v}^{2,j} \\
\mathbf{v}^{3,j} \\
\vdots \\
\mathbf{v}^{n-1,j} \\
\mathbf{v}^{n,j}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\|\nabla \mathbf{u}^j - \mathbf{p}_{3/2}\|_{L^2(B_1^+)}^{x_1}} \frac{\partial \mathbf{u}^j - \mathbf{p}_{3/2}}{\partial x_1}
\\
\frac{1}{\|\nabla \mathbf{u}^j\|_{L^2(B_1^+)^n}} \frac{\partial \mathbf{u}^j}{\partial x_2}
\\
\frac{1}{\|\nabla \mathbf{u}^j\|_{L^2(B_1^+)^n}} \frac{\partial \mathbf{u}^j}{\partial x_3}
\\
\vdots
\\
\frac{1}{\|\nabla \mathbf{u}^j\|_{L^2(B_1^+)^n}} \frac{\partial \mathbf{u}^j}{\partial x_n}
\end{bmatrix}
\]

that is \( \mathbf{v}^{i,j} = \frac{\frac{1}{\|\nabla \mathbf{u}^j\|_{L^2(B_1^+)^n}} \frac{\partial \mathbf{u}^j}{\partial x_i}}{\partial x_i} \) for \( i = 2, \ldots, n - 1 \) and as in the displayed formula when \( i = 1 \) or \( i = n \).

Then for \( i = 2, 3, \ldots, n - 1 \) the function \( \mathbf{v}^i \) solves

\[
\begin{align*}
\Delta \mathbf{v}^{i,j} + \frac{\lambda_j}{\mathbf{e}_3} \cdot \mathbf{v}^{i,j} &= 0 & & \text{in } B_1^+ \cap \Lambda_{\mathbf{u}^j} \to \{ x_1 > 0 \} \\
\frac{\partial \mathbf{v}^{i,j}}{\partial x_3} + \frac{\lambda_j}{\mathbf{e}_3} \cdot \mathbf{v}^{i,j} &= 0 & & \text{on } \Lambda_{\mathbf{u}^j} \to \{ x_1 > 0 \} \\
\frac{\partial \mathbf{v}^{i,j}}{\partial x_3} &= 0 & & \text{on } \sum_{\mathbf{u}^j} \to \{ x_1 < 0 \} \text{ for } k = 1, 2
\end{align*}
\]

and \( \| \mathbf{v}^{i,j} \|_{L^\infty(B_{1/\gamma}^+)} \| \mathbf{v}^{i,j} \|_{W^{1,2}(B_{1/\gamma})} \leq C \). We may also assume that \( \lim_{j \to \infty} \mathbf{v}^{i,j} = \mathbf{v}^i \) by taking a sub-sequence if necessary. By Lemma 11 it follows that

\[
\mathbf{v}^i = \sum_{k=0}^{\infty} a_k^i \mathbf{q}_k
\]

where \( \mathbf{q}_k \) are homogeneous of order \( k/2 \).

Also, since \( \| \mathbf{v}^{i,j} \|_{L^2} \leq 1 \), we get that \( \mathbf{v}^{1,j} \) converge weakly in \( L^2 \) to a solution \( \mathbf{v}^1 \) of (84).

Since \( \mathbf{v}^{i,j} \) convergence strongly in \( L^2(B_{1-\epsilon}^+) \) and \( \mathbf{v}^{1,j} \) and \( \mathbf{v}^{n,j} \) converges strongly in \( L^2(B_{1-\epsilon} \cap \{ x_n > \epsilon \}) \) for each \( \epsilon > 0 \), we may, in the set \( B_{1-\epsilon}^+ \cap \{ x_n > \epsilon \} \), write \( \mathbf{u}^j = \mathbf{p}_{3/2} + \nabla R^j \) where

\[
\nabla R^j = \begin{bmatrix}
\| \nabla (\mathbf{u}^j - \mathbf{p}_{3/2})\|_{L^2(B_1)} \sum_{k=0}^{\infty} a_k^j \mathbf{q}_k \\
\| \nabla'' \mathbf{u}^j\|_{L^2(B_1)} \sum_{k=0}^{\infty} a_k^{n-1} \mathbf{q}_k \\
\| \nabla' \mathbf{u}^j\|_{L^2(B_1)} \sum_{k=0}^{\infty} a_k \mathbf{q}_k \\
\| \nabla (\mathbf{u}^j - \mathbf{p}_{3/2})\|_{L^2(B_1^+)} \sum_{k=0}^{\infty} a_k^1 \mathbf{p}_{1/2} \\
\| \nabla'' \mathbf{u}^j\|_{L^2(B_1)} a_1^j \mathbf{p}_{1/2} \\
\| \nabla' \mathbf{u}^j\|_{L^2(B_1)} a_0^j \mathbf{p}_{1/2} \\
\| \nabla (\mathbf{u}^j - \mathbf{p}_{3/2})\|_{L^2(B_1^+)} a_0 \mathbf{p}_{1/2} \\
\| \nabla'' \mathbf{u}^j\|_{L^2(B_1)} a_0^{n-1} \mathbf{p}_{1/2} \\
\| \nabla' \mathbf{u}^j\|_{L^2(B_1^+)} a_0^1 \mathbf{p}_{1/2} \\
\| \nabla (\mathbf{u}^j - \mathbf{p}_{3/2})\|_{L^2(B_1^+) a_0} \mathbf{p}_{1/2}
\end{bmatrix}
\]

where the little-o terms are considered to be little-o in \( L^2 \) norm, that is \( f^j(x) = o(\|g^j\|_{L^2\cap\{x_n>\epsilon\}}) \) if \( \|f^j\|_{L^2(B_1^+ \cap \{ x_n > \epsilon \})}/\|g^j\|_{L^2\cap\{x_n>\epsilon\}} \to 0 \) as \( j \to \infty \).

Now consider \( \mathbf{u}^j_s = \mathbf{u}^j(s x)/\sqrt{s} \) then \( \mathbf{u}^j = s \mathbf{p}_{3/2} + \nabla R_s^j \) where

\[
\nabla R_s^j = \begin{bmatrix}
\| \nabla (\mathbf{u}^j - \mathbf{p}_{3/2})\|_{L^2(B_1)} a_1^1 \mathbf{p}_{1/2} \\
\| \nabla'' \mathbf{u}^j\|_{L^2(B_1)} a_0^j \mathbf{p}_{1/2} \\
\| \nabla' \mathbf{u}^j\|_{L^2(B_1)} a_1^j \mathbf{p}_{1/2} \\
\| \nabla (\mathbf{u}^j - \mathbf{p}_{3/2})\|_{L^2(B_1^+)} a_0 \mathbf{p}_{1/2} \\
\| \nabla'' \mathbf{u}^j\|_{L^2(B_1)} a_0^{n-1} \mathbf{p}_{1/2} \\
\| \nabla' \mathbf{u}^j\|_{L^2(B_1^+)} a_0^1 \mathbf{p}_{1/2} \\
\| \nabla (\mathbf{u}^j - \mathbf{p}_{3/2})\|_{L^2(B_1^+) a_0} \mathbf{p}_{1/2}
\end{bmatrix}
\]
Moreover we may disregard the lower order terms, that is terms of order \( s' \). Therefore (80) holds.

**Proof of the claim:** We may assume that \( j \) is very large and \( s \) very small. Moreover we may disregard the \( s \)-terms since they vanish in the limit. Writing

\[
P^j = \begin{bmatrix}
\|\nabla^0 (u^j - p_{3/2})\|_{L^2(B_1^+)} a_0^j & 0 \\
\|\nabla^0 u^j\|_{L^2(B_1^+)} a_0^j & 0 \\
\|\nabla^0 u^j\|_{L^2(B_1^+)} a_0^j & 0 \\
\vdots & \vdots \\
\|\nabla^0 u^j\|_{L^2(B_1^+)} a_0^j & 0 \\
\|\nabla (u^j - p_{3/2})\|_{L^2(B_1^+)} a_0^j & 0
\end{bmatrix}
\]

and

\[
\tilde{R}_s^j = s \begin{bmatrix}
\|\nabla^0 (u^j - p_{3/2})\|_{L^2(B_1^+)} \sum_{k=1}^{\infty} s^{k-1} a_k^1 q_k \\
\|\nabla^0 u^j\|_{L^2(B_1^+)} \sum_{k=1}^{\infty} s^{k-1} a_k^2 q_k \\
\|\nabla^0 u^j\|_{L^2(B_1^+)} \sum_{k=1}^{\infty} s^{k-1} a_k^3 q_k \\
\vdots \\
\|\nabla^0 u^j\|_{L^2(B_1^+)} \sum_{k=1}^{\infty} s^{k-1} a_k^{n-1} q_k \\
\|\nabla (u^j - p_{3/2})\|_{L^2(B_1^+)} \sum_{k=1}^{\infty} s^{k-1} a_k^n q_k
\end{bmatrix}.
\]

Then

\[
\nabla R_s^j = p^j + \tilde{R}_s^j + l.o.t.
\]

where \( l.o.t. \) indicates lower order terms: that is terms of order \( s^{3/2}, s^2, \ldots \). We hope to be able to use Lemma 16.

First we need to show that the assumption (80) holds. Since \( P^j \) is a gradient modulo lower order terms we may conclude that

\[
\frac{\partial (e_1 \cdot P^j)}{\partial x_n} = \frac{\partial (e_n \cdot P^j)}{\partial x_1}
\]

which implies that \( a_0^1 = a_0^0 \). Therefore (80) holds.

Next we need to verify that (79) holds for some \( \epsilon \in (0, 1/4) \), that is

\[
\|\tilde{R}_s^j\|_{L^2(B_1^+)} \leq c \left( \|\nabla (u^j - p_{3/2})\|_{L^2(B_1^+)}^2 (a_0^1)^2 + \sum_{i=2}^{n-1} \|\nabla^0 u^j\|_{L^2(B_1^+)}^2 (a_0^i)^2 + \|\nabla (u^j - p_{3/2})\|_{L^2(B_1^+)}^2 (a_0^n)^2 \right)^{1/2}.
\]

If (86) holds for some \( s > 0 \) then \( a_0^1 = a_0^2 = \ldots = a_0^n = 0 \) by Lemma 16.

If (86) is not true then

\[
\|\nabla^0 u^j\|_{L^2(B_1^+)} \sqrt{\sum_{i=2}^{n-1} (a_0^i)^2} \leq c s \|\nabla (u^j - p_{3/2})\|_{L^2(B_1)}
\]

We make the following claim.

**Claim:** Either \( \|\nabla^0 u^j\|_{L^2(B_1^+)} = 0 \) or \( a_0^0 = 0 \) for \( i = 1, \ldots, n \) or both.
for some constant \( c \). Since \( s \) is arbitrary so the only two possibilities in (87) are that either \( \| \nabla'' u \|_{L^2(B^+_s)} = 0 \) and the Lemma is trivially true or \( \sqrt{\sum_{i=2}^{n-1} (a_i^0)^2} = 0 \) which is what we claim. We may therefore assume that (86) is true.

We may thus apply Lemma 16 and conclude that \( a_0^0 = a_0^1 = \ldots = a_0^n = 0 \). This finishes the proof of the claim.

We have therefore shown that

\[
v^{i,j} \to v^i = \sum_{k=1}^{\infty} q_k
\]

strongly in \( L^2 \). Since we have no \( q_i \) term in the sum and therefore the lowest homogeneity is 1. It is easy to see that there exist an \( s_\gamma \) such that

\[
\| v^i \|_{L^2(B^+_s, \gamma)} \leq \frac{1}{2} s^{1/2+\gamma}\| v^i \|_{L^2(B^+_1)}
\]

for each \( \gamma \in (0, 1/2) \). The Lemma follows by strong convergence. \( \Box \)

11. Decay of the Solution.

In the next proposition we prove that the difference between \( u \) and \( p_{3/2} \) is small in \( L^2 \) norm then the difference decays geometrically. This is implies regularity of the solutions in a standard way as will be shown later.

**Proposition 3.** Let \( u \) solve the Signorini problem in \( B^+_1 \) and \( 0 < \gamma < 1/2 \) then there exists a \( \delta_\gamma > 0 \) such that if

\[
\inf_{\xi \in \Pi} \| u - p_{3/2}^\xi \|_{L^2(B^+_1)} = \| u - p_{3/2} \|_{L^2(B^+_1)} \leq \delta_\gamma
\]

and

\[
\| \nabla'' u \|_{L^2(B^+_1)} \leq \| u - p_{3/2} \|_{L^2(B^+_1)}
\]

then there exist an \( s_\gamma > 0 \) such that

\[
\inf_{\xi \in \Pi} \| u - p_{3/2}^\xi \|_{L^2(B^+_s, \gamma)} \leq s_\gamma \| u - p_{3/2} \|_{L^2(B^+_1)} \| u \|_{L^2(B^+_s, \gamma)}
\]

and

\[
\| \nabla'' u \|_{L^2(B^+_s, \gamma)} \leq s_\gamma^{1/2+\gamma}\| \nabla'' u \|_{L^2(B^+_1)}.
\]

**Proof:** Let us first point out that (89) was proved in Lemma 17.

The proof is unfortunately quite long so we will divide it into several Lemmas.

**Lemma 18.** Let \( u^j \) be a sequence of solutions as in Proposition 3 corresponding to \( \delta_{\gamma,j} = \delta_j \to 0 \) then there exist a modulus of continuity \( \sigma \) such that

\[
\inf_{\xi \in \Pi} \| u^j - p_{3/2}^\xi \|_{L^2(B^+_s, \gamma)} \leq \sigma(\delta_j) \| u^j \|_{L^2(B^+_s, \gamma)}
\]

for each \( t < 1 \). In particular, from Lemma 14, for each \( u^j \) we have

\[
\limsup_{s \to 0} \inf_{\xi \in \Pi} \| u^j - p_{3/2}^\xi \|_{L^2(B^+_s, \gamma)} = 0.
\]

**Proof:** If (90) is not true then for some small \( \mu > 0 \) there is a sequence \( t_j \) such that

\[
\inf_{\xi \in \Pi} \| u^j - p_{3/2}^\xi \|_{L^2(B^+_s, \gamma)} = \mu.
\]
We may assume that \( t_j \) is the largest such \( t \) corresponding to \( \mathbf{u}^j \). Since \( \delta_j \to 0 \) it is easy to see that \( t_j \to 0 \). We make the blow-up

\[
\mathbf{w}^j(x) = \frac{\mathbf{u}^j(t_j x)}{\|\mathbf{u}\|_{L^2(B_j^+)}^\xi} \to \mathbf{w}^0.
\]

From Lemma 14 we conclude that

\[
\mathbf{w}^0 = p_{3/2}^\xi,
\]

for some \( \xi \in \Pi \). This is a contradiction since

\[
0 = \inf_{\xi \in \Pi} \|\mathbf{w}^0 - p_{3/2}^\xi\|_{L^2(B_j^+)} = \lim_{j \to \infty} \inf_{\xi \in \Pi} \|\mathbf{u}^j(t_j x) - p_{3/2}^\xi\|_{L^2(B_j^+)} = \mu.
\]

The second equality in (91) follows by strong convergence of \( \mathbf{u}^j(t_j x)/\|\mathbf{u}\|_{L^2(B_j^+)} \) (since \( W^{1,2} \) compactly embeds into \( L^2 \)) and of \( p_{3/2}^\xi \) (since these functions are contained in a finite dimensional subspace of \( L^2 \)).

Before we state our next lemma let us describe the general idea of the proof of Proposition 3.

The general idea to prove Proposition 3 is to argue by contradiction and assume that there exist \( \mathbf{u}^j \), \( \delta_j \to 0 \) and \( s_j \to 0 \) such that \( \|\mathbf{u} - p_{3/2}\|_{L^2(B_j^+)} = \delta_j \), \( \|\nabla'' \mathbf{u}\|_{L^2(B_j^+)} \leq \delta_j \) and

\[
\inf_{\xi \in \Pi} \|\mathbf{u}^j - p_{3/2}^\xi\|_{L^2(B_j^+)} = C_j s_j^{\gamma_j}.
\]

Where \( C_j \to \infty \) and \( 0 < \gamma_j < 1/2 \). We will assume that the sequence \( \gamma_j \to \gamma_0 \), for some \( \gamma_0 \) that may be zero. However, as the proof will show, \( \gamma_j \to 1/2 \) or else we get a contradiction.

We may assume, if not the proposition is clearly true, that

\[
\inf_{\xi \in \Pi} \|\mathbf{u}^j - p_{3/2}^\xi\|_{L^2(B_j^+)} \to \infty
\]

for some sequence \( \tilde{s}_j \to 0 \). We also know from Lemma 18 that

\[
\lim_{s \to 0} \inf_{\xi \in \Pi} \|\mathbf{u}^j - p_{3/2}^\xi\|_{L^2(B_j^+)} = 0
\]

for each \( j \). Therefore we can choose \( s_j \to 0 \), \( C_j \to \infty \) such that

\[
\inf_{\xi \in \Pi} \|\mathbf{u}^j - p_{3/2}^\xi\|_{L^2(B_j^+)} \leq C_j \quad \text{if } s < s_j
\]

\[
\inf_{\xi \in \Pi} \|\mathbf{u}^j - p_{3/2}^\xi\|_{L^2(B_j^+)} \leq C_j s_j^{\gamma_j} \quad \text{if } s > s_j.
\]

We make the blow-up

\[
\mathbf{v}^j = \frac{\mathbf{u}^j(s_j x) - s_j^{3/2} p_{3/2}^\xi}{C_j s_j^{\gamma_j} \delta_j \|\mathbf{u}\|_{L^2(B_j^+)}^\xi}.
\]

Then \( \mathbf{v}^j \to \mathbf{v}^0 \) weakly in \( L^2(B_j^+) \) for each \( R > 1 \) and also

\[
\|\mathbf{v}^j\|_{L^2(B_j^+)} = \frac{1}{C_j s_j^{\gamma_j} \delta_j \|\mathbf{u}\|_{L^2(B_j^+)}^\xi} \|\mathbf{u}^j(s_j x) - s_j^{3/2} p_{3/2}^\xi\|_{L^2(B_j^+)} \leq
\]
\[
\frac{1}{C^j s_j^3 / \delta_j \| u' \|_{L^2(B^+_{s_j})}} \left( \| u' (s_j x) - p_{\xi_j}^{\epsilon_j} \|_{L^2(B^+_{s_j})} + r^{3/2} s_j^{3/2} \| \xi_j \|_{L^2(B^+_{s_j})} \right)
\]

\[
\begin{cases}
2 \frac{\| u' \|_{L^2(B^+_{s_j})}}{\| u' \|_{L^2(B^+_{s_j})}} + \frac{r^{3/2} s_j^{3/2} \| \xi_j \|_{L^2(B^+_{s_j})}}{C^j s_j^3 / \delta_j} & \text{if } r \leq 1 \\
\gamma_j \frac{\| u' \|_{L^2(B^+_{s_j})}}{\| u' \|_{L^2(B^+_{s_j})}} + \frac{r^{3/2} s_j^{3/2} \| \xi_j \|_{L^2(B^+_{s_j})}}{C^j s_j^3 / \delta_j} & \text{if } r > 1,
\end{cases}
\]

The proof of Proposition 3 will be finished in three steps. First we will show, in Lemma 19 that (95) implies that \( \| v^0 \|_{L^2(B^+_{s_j})} \leq C (r^{3/2} + \gamma_j) \). This will imply, by using Lemma 11 that either \( \gamma_j \geq 1/2 \) or \( v^0 = 0 \). Then in Lemma 20 we will show that \( v^0 \to v^0 \) strongly which excludes that \( v^0 = 0 \) that will imply that \( \gamma_j \geq 1/2 \) so in particular, for each \( \gamma < 1/2 \) there has to be a \( C_\gamma \) such if \( u \) satisfies the conditions of the proposition with \( \delta \) small enough then

\[
\inf_{\xi \in \mathbb{N}} \frac{\| u - p_{\delta_j}^{\epsilon_j} \|_{L^2(B^+_{s_j})}}{\delta \| u \|_{L^2(B^+_{s_j})}} \leq C_\gamma \| u \|_{L^2(B^+_{s_j})}
\]

for all \( s < 1 \).

**Lemma 19.** Let \( v^0 \) be as in (94), \( s_j, \xi_j, c_j \) and \( \delta_j \) chosen as in the discussion leading up to (94), in particular we assume that (93) holds. Also let \( C > 1 \) then for each \( r \) there exist a \( j_r \) such that

\[
\| v^0 \|_{L^2(B^+_{s_j})} \leq \begin{cases} 
C r^{3/2} & \text{if } r \leq 1 \\
C r^{3/2 + \gamma_j} & \text{if } r > 1,
\end{cases}
\]

if \( j > j_r \).

**Proof:** We need to estimate the two terms in (95). Notice that by Lemma 18 and Lemma 14, in particular the equations (71) and (72) in the proof with \( \mu = \sigma(\delta_j) \) will imply that

\[
\| u' \|_{L^2(B^+_{s_j})} \leq C r^{3/2}
\]

when \( \sigma(\delta_j) \) is small enough.

In order to estimate

\[
\frac{r^{3/2} s_j^{3/2} \| \xi_j \|_{L^2(B^+_{s_j})}}{C^j s_j^3 / \delta_j} \| u' \|_{L^2(B^+_{s_j})}
\]

we let

\[
\bar{u}^j (x) = \frac{u^j (s_j x) - s_j^{3/2} \xi_j \bar{v}^j (x)}{\| u' - p_{\delta_j}^{\epsilon_j} \|_{L^2(B^+_{s_j})}}.
\]

Then

\[
\| \bar{u}^j \|_{L^2(B^+_{s_j})} = 1.
\]

So \( \bar{u}^j \to \bar{u}^0 \) weakly in \( L^2(B_{2s}) \). After a rotation we may assume that \( \xi_{s_j} \bar{v}^j = e_1 \) and conclude that \( \bar{u}^j \to \bar{u}^0 \) strongly in \( W^{1,2}(B_{2s} \setminus \{ \sqrt{|x_1|^2 + |x_n|^2} \leq t \}) \) for any
\( t > 0 \). This is true since \( \bar{u}^j \) is a solution, with bounded \( L^2 \)-norm, of the following Lame system

\[
\begin{align*}
\Delta \bar{u}^j + \frac{1}{2} \nabla \text{div}(\bar{u}^j) &= 0 \quad \text{in } B_1^+ \\
\partial_\nu \bar{u}^j + \frac{1}{2} \text{div}(\bar{u}^j) &= 0 \quad \text{on } \Pi \cap \{ x_1 < -t \} \\
\bar{u}^j \cdot e_n &= 0 \quad \text{on } \Pi \cap \{ x_1 > t \} \\
\partial_x \bar{u}^j + \partial \bar{u}^j \cdot e_n &= 0 \quad \text{on } \Pi \setminus \{ |x_1| < t \} \text{ for } i = 1, 2, ..., n - 1
\end{align*}
\]

and for any \( t > 0 \) if \( j \) is large enough.

In particular, by Lemma 11 we may conclude that

\[
\bar{u}^0 = \sum_{i=0}^{\infty} a_i q_i.
\]

But \( a_1 = 0 \) since we subtracted \( p_{3/2}^\xi \) in the definition of \( \bar{u}^j \) (see Lemma 13).

We therefore have, with the notation

\[
\inf_{\xi \in \Pi} \| \bar{u}^j - p_{3/2}^\xi \|_{L^2(B_1^+)} = \| \bar{u}^j - p_{3/2}^\xi \|_{L^2(B_1^+)},
\]

that

\[
\| p_{3/2}^\xi \|_{L^2(B_1^+)} = o(1)
\]

That is

\[
(97) \quad \| p_{3/2}^\xi - p_{3/2}^\xi \|_{L^2(B_1^+)} = s_j^{-3/2} \| u_j - p_{3/2}^\xi \|_{L^2(B_1^+)},
\]

Using equation (93) then (97) can be estimated, when \( r > 1 \), by

\[
s_j^{-3/2} \| u_j - p_{3/2}^\xi \|_{L^2(B_1^+)} \leq s_j^{-3/2} C_j s_j \delta_j \| u_j \|_{L^2(r,s_j)} \leq C_j s_j^{-3/2} \delta_j s_j^{-3/2} r^{3/2} \| u_j \|_{L^2(B_1^+)}
\]

where we used (96) in the last inequality. And when \( r \leq 1 \) we may estimate (97) according to

\[
s_j^{-3/2} \| u_j - p_{3/2}^\xi \|_{L^2(B_1^+)} \leq s_j^{-3/2} C_j \delta_j \| u_j \|_{L^2(r,s_j)} \leq C_j s_j^{-3/2} \delta_j s_j^{-3/2} \| u_j \|_{L^2(B_1^+)}.
\]

It follows that

\[
\frac{r^{3/2} s_j^{3/2} \| p_{3/2}^\xi - p_{3/2}^\xi \|_{L^2(B_1^+)} \| u_j \|_{L^2(B_1^+)} = o(1)}{C_j s_j \delta_j}
\]

when \( r > 1 \). And

\[
\frac{r^{3/2} s_j^{3/2} \| p_{3/2}^\xi - p_{3/2}^\xi \|_{L^2(B_1^+)} \| u_j \|_{L^2(B_1^+)} = o(1)}{C_j \delta_j}
\]

when \( r \leq 1 \).

Using this and (96) in (95) we get

\[
\| v^j \|_{L^2(B_1^+)} \leq \begin{cases} C_j r^{3/2} & \text{if } r \leq 1 \\
C_j r^{3/2+\gamma_0} & \text{if } r > 1,
\end{cases}
\]

when \( j \) is large enough. \( \square \)

**Lemma 20.** Let \( v^j \) be as in (94) then \( v^j \to v^0 \) strongly for some sub-sequence, in particular \( \| v^0 \|_{L^2} = 1 \).

Moreover in some coordinate system \( v^0(x) = v^0(x_1, x_n) \).
Proof: As before we may rotate the coordinates so that \( \xi_j/|\xi_j| = e_1 \), we may need a different rotation for each \( j \) but that will not affect the proof. Then, as before,

\[
\Delta \nu^j + \frac{2+\lambda}{2} \nabla \text{div}(\nu^j) = 0 \quad \text{in } B^+_1(\gamma_0),
\]

\[
e_n \cdot \nu^j = 0 \quad \text{on } \Pi \cap \{x_1 > t\} \cap B_{1/t},
\]

\[
\frac{\partial \nu^j}{\partial x_n} + \frac{\partial}{\partial x_n} \text{div}(\nu^j) = 0 \quad \text{on } \Pi \cap \{x_n < -t\} \cap B_{1/t},
\]

\[
\frac{\partial \nu^j}{\partial x_n} = 0 \quad \text{on } \Pi \text{ for } i = 1, 2, \ldots, n - 1
\]

for each \( t > 0 \) provided that \( j \) is large enough. So \( \nu^j \rightarrow \nu^0 \) in \( W^{1,2}(B^+_1 \setminus \{\sqrt{|x_1|^2 + |x_n|^2} < t\}) \) for each \( t > 0 \). The claim will follow if we can show that \( \|\nu^j\|_{L^2(\sqrt{(|x_1|^2 + |x_n|^2)} < t) \cap B_1} \) can be chosen arbitrarily small for each \( R \) by considering \( t \) small enough and \( j \) large enough.

Notice that by Lemma 17 and Lemma 18 we have

\[
\|\nabla'' \nu^j\|_{L^2(B^+_1)} \leq s_{1/2+\gamma}^{1/2} \|\nabla'' \nu^j\|_{L^2(B^+_1)}.
\]

By Lemma 14 we have, when \( \delta_j \) is small enough,

\[
\|\nu^j\|_{L^2(B^+_1)} > t^{3/2+\epsilon}
\]

for \( \epsilon > 0 \) being very small. That implies that, after a small rotation of the coordinate system,

\[
w = \frac{\nu^j(s, x)}{\|\nu^j\|_{L^2(B^+_1)}}
\]

satisfies the conditions in Lemma 17. In particular when \( \delta_j \) is small enough we have for \( \gamma > \gamma_j + 1/2 \)

\[
(88) \quad \inf_{\xi \in \Pi} \|w - p_{3/2}\|_{L^2(B_1)} \leq \delta_j.
\]

Since \( \|\nabla'' w\|_{L^2(B^+_1)} \) is \( s_{1/2+\gamma}^{1/2} \delta_j \) and therefore by Lemma 15

\[
\|\nabla'' w\|_{L^2(B^+_1)} \leq C s_{1/2+\gamma}^{1/2} \delta_j
\]

where \( \xi_w \) is the minimizer in (88) and we used the notation \( \nabla'' \nu^j = \nabla - e_n \frac{\partial}{\partial x_n} - \xi_w/|\xi_w|^2(\xi_w \cdot \nabla) \). By possibly decreasing \( s_\gamma \) it follows that

\[
\|\nabla'' w\|_{L^2(B^+_1)} \leq s_{1/2+\gamma}^{1/2} \delta_j
\]

Therefore the conditions of Lemma 17 are satisfied by \( w \).

Iterating this we see that

\[
\|\nabla'' \nu^j\|_{L^2(B_1)} \leq C^{s_j^{k(3/2+\gamma)/2}} \|\nabla'' \nu^j\|_{L^2(B^+_1)} \leq C \frac{s_j^{k(1/2-\gamma-\epsilon)}}{C_j} \rightarrow 0,
\]

where \( k \) is chosen such that \( s_j^{k+1} < s_j \leq s_j^{k} \), \( \epsilon \) is small (depending on \( \delta \)) and is the same constant as in Lemma 14. In particular, since \( \frac{\partial \nu^j}{\partial x_i} \) solves an elliptic problem such as in equation (84), this implies that \( \nabla'' \nu^j \rightarrow 0 \) uniformly in \( B^+_R \). This proves the last statement in the Lemma.

Next we consider any \( \kappa \in (0, R) \) and \( e'' \in \Pi \cap \{x_1 = 0\} \). The estimates on \( \nabla'' \nu^j \) implies that

\[
\|\nu^j\|_{L^2(B_1(\kappa e''))} \leq C \|\nu^j\| + \sup |\nabla'' \nu^j|_{L^2(B^+_1)} \leq C t^{(n+3)} + C \left( \sup |\nabla'' \nu^j| \right)^2 t_{\nu^j}.\]
Therefore, by covering $B_R \cap \{ \sqrt{|x_1|^2 + |x_n|^2} \leq t \}$ by $C R^{n-2}/t^{n-2}$ balls, we may conclude that
\[
\| v' \|_{L^2((\sqrt{|x_1|^2 + |x_n|^2} < t) \cap B_R)} \leq C (t^\delta + t^2) R^3
\]
when $j$ is large enough. Choose $t \leq \epsilon (C R^{n-2})^{-1}$ and it follows that
\[
\| v' \|_{L^2((\sqrt{|x_1|^2 + |x_n|^2} < t) \cap B_R)} \leq \epsilon.
\]
We may conclude that $v^j \to v^0$ strongly in $L^2(B_R^+)$ for every $R$.

Proof: In principle the proof consists of applying Lemma 17 and Proposition 3.

We may therefore conclude that for each $\gamma < 1/2$ there is a $C_\gamma$ such that
\[
\inf_{\xi \in \Pi} \| u - p_{3/2}^\xi \|_{L^2(B_R^+)} \leq C_\gamma \gamma^3
\]
for all $s < 1$. The Proposition follows with slightly smaller $\gamma$ by choosing $s$ small enough.

Corollary 6. Let $u$ solve the Signorini problem in $B^+_R$ and $0 < \gamma < 1/4$ then there exists a $\delta_\gamma > 0$ such that if
\[
\inf_{\xi \in \Pi} \| u - p_{3/2}^\xi \|_{L^2(B^+_R)} = \| u - p_{3/2}^\xi \|_{L^2(B^+_R)} \leq \delta_\gamma,
\]
and
\[
\| \nabla'' u \|_{L^2(B^+_R)} \leq \delta_\gamma
\]
then there exist an $s_\gamma$ such that
\[
\max \left( s_\gamma^{-(1/2+\gamma)} \| \nabla'' u \|_{L^2(B^+_R)}, s_\gamma^{-(3/2+\gamma)} \| u - p_{3/2}^\xi \|_{L^2(B^+_R)} \right) \leq \max \left( \| \nabla'' u \|_{L^2(B^+_R)}, \| u - p_{3/2}^\xi \|_{L^2(B^+_R)} \right)
\]
where $\xi$ is the vector that minimizes $\inf_{\xi \in \Pi} \| u - p_{3/2}^\xi \|_{L^2(B^+_R)}$.

Proof: In principle the proof consists of applying Lemma 17 and Proposition 3. Unfortunately this is not as straightforward as one might hope. We will have to split the proof into four cases.

Properly speaking we only prove that there exist an $\tilde{s}_\gamma \in (s_\gamma^{2n}, s_\gamma)$ such that the Corollary holds where $s_\gamma$ is the constant in Proposition 3. It is easy to see that this is implies the Corollary.

Case 1: If
\[
\| \nabla'' u \|_{L^2(B^+_R)} \leq \| u - p_{3/2} \|_{L^2(B^+_R)}.
\]

Proof of the Corollary in Case 1: Proposition 3 directly implies that
\[
\max \left( s_\gamma^{-(1/2+\gamma)} \| \nabla'' u \|_{L^2(B^+_R)}, s_\gamma^{-(3/2+\gamma)} \inf_{\xi \in \Pi} \| u - p_{3/2}^\xi \|_{L^2(B^+_R)} \right) \leq \max \left( \| \nabla'' u \|_{L^2(B^+_R)}, \| u - p_{3/2} \|_{L^2(B^+_R)} \right)
\]
for $\gamma < 1/2$.
By Lemma 18 the assumptions in Lemma 14 holds and we may thus deduce that
\[
\| u \|_{L^2(B^+_{n/2})} \geq s_\gamma^{3/2+\varepsilon}
\]
so the Corollary follows, with \( \gamma - \varepsilon \) in place of \( \gamma \), if (100) is true. Since \( \varepsilon \) is arbitrarily small the Corollary follows in the situation of case 1.

**Case 2:** If
\[
\| u - p_{3/2} \|_{L^2(B^+_{n/2})} \leq \| \nabla^n u \|_{L^2(B^+_{n/2})}
\]
and
\[
s_\gamma^{-1} \inf_{\xi \in \Pi} \| u - p_{3/2}^\xi \|_{L^2(B^+_{n/2})} \leq \| \nabla'' u \|_{L^2(B^+_{n/2})}.
\]

**Proof of the Corollary in Case 2:** From Lemma 17 we still have for \( \gamma < 1/4 \)
\[
s_\gamma^{-(1/2+2\gamma)} \| \nabla'' u \|_{L^2(B^+_{n/2})} \leq \| \nabla'' u \|_{L^2(B^+_{n/2})}.
\]
Then (102) implies that
\[
s_\gamma^{-(1+\gamma)} \inf_{\xi \in \Pi} \| u - p_{3/2}^\xi \|_{L^2(B^+_{n/2})} \leq \| \nabla'' u \|_{L^2(B^+_{n/2})} \leq s_\gamma^{1/2+\gamma} \| \nabla^n u \|_{L^2(B^+_{n/2})}.
\]
(103) and (104) implies the Corollary.

In order to state the third and the fourth case we need some notation. First we notice that if we are not in case 1 or case 2 then
\[
\| u - p_{3/2} \|_{L^2(B^+_{n/2})} \leq \| \nabla^n u \|_{L^2(B^+_{n/2})}
\]
and
\[
\| \nabla'' u \|_{L^2(B^+_{n/2})} \leq s_\gamma^{-1} \inf_{\xi \in \Pi} \| u - p_{3/2}^\xi \|_{L^2(B^+_{n/2})}.
\]
If (105) and (106) holds then
\[
\max(s_\gamma^{-1/2+\gamma} \inf_{\xi \in \Pi} \| u - p_{3/2}^\xi \|_{L^2(B^+_{n/2})}, s_\gamma^{-(1/2+2\gamma)} \| \nabla'' u \|_{L^2(B^+_{n/2})})
\]
\[= s_\gamma^{-(3/2+\gamma)} \inf_{\xi \in \Pi} \| u - p_{3/2}^\xi \|_{L^2(B^+_{n/2})}.
\]
From Lemma 15 we can deduce that
\[
s_\gamma^{-1/2-\gamma} \| \nabla'' u \|_{L^2(B^+_{n/2})} \leq (CC_1 + 1) \| \nabla'' u \|_{L^2(B^+_{n/2})}.
\]
Or if \( s_\gamma \) is small enough that
\[
s_\gamma^{-1/2-\gamma} \| \nabla^n u \|_{L^2(B^+_{n/2})} \leq \| \nabla'' u \|_{L^2(B^+_{n/2})}.
\]
We thus have that
\[
u^1(x) = \frac{u(s_\gamma,x)}{\sup_{B_1} |u(s_\gamma,x)|}
\]
satisfies the conditions in Case 1, with \( \delta s_\gamma^{n/2} \) in place of \( \delta \).

**Case 3:** Assume (105), (106) and that there exist a \( j_0 \leq 2n \) such that
\[
\| \nabla'' u^j \|_{L^2(B^+_{n/2})} \leq \inf_{\xi \in \Pi} \| u^j - p_{3/2}^\xi \|_{L^2(B^+_{n/2})} = \| u^j - p_{3/2}^\xi \|_{L^2(B^+_{n/2})}
\]
for \( j \leq j_0 \) and

\[
\inf_{\xi \in \Omega} \|u^j - p_{3/2}^\xi\|_{L^2(B^+_1)} = \|u^j - p_{3/2}^{\xi_0}\|_{L^2(B^+_1)} \geq \|\nabla'' u^j\|_{L^2(B^+_1)}.
\]

**Proof of the Corollary in Case 3:** Observe that (107) implies that we may use Lemma 17 and Lemma 15 on \( u^j \) for \( j < j_0 \) and deduce that

\[
\|\nabla'' u^j\|_{L^2(B^+_1)} \leq s^{1/2} \|\nabla'' u^j\|_{L^2(B^+_1)} \leq s^{2(1/2+\gamma)} \|\nabla'' u^j\|_{L^2(B^+_1)} \leq \cdots \leq s^{j_0(1/2+\gamma)} \|\nabla'' u^j\|_{L^2(B^+_1)}
\]

where \( \xi^k \) is the minimizing vector in

\[
\inf_{\xi} \|u^k - p_{3/2}^\xi\|_{L^2(B^+_1)}.
\]

Next we use (108) to conclude that

\[
\|\nabla'' u^k\|_{L^2(B^+_1)} \leq \|u^k - p_{3/2}^\xi\|_{L^2(B^+_1)} = \inf_{\xi \in \Omega} \|u^k - p_{3/2}^\xi\|_{L^2(B^+_1)}.
\]

**Case 4:** Assume (105), (106) and that for each \( k \leq 2n \) the following holds

\[
\|\nabla'' u^k\|_{L^2(B^+_1)} \leq \|u^k - p_{3/2}^\xi\|_{L^2(B^+_1)} = \inf_{\xi \in \Omega} \|u^k - p_{3/2}^\xi\|_{L^2(B^+_1)}.
\]

**Proof of the Corollary in Case 4:** As in case 3 we have that (111) implies that we may apply Lemma 17 to \( u^k \). However \( u^k \) will also satisfy the conditions in case 1. Applying case 1 on \( u^k \) for \( k \leq 2n \) we can conclude that

\[
\|\nabla'' u^k\|_{L^2(B^+_1)} \leq \|u^k - p_{3/2}^\xi\|_{L^2(B^+_1)} = \inf_{\xi \in \Omega} \|u^k - p_{3/2}^\xi\|_{L^2(B^+_1)}.
\]

where \( \tilde{s} = s^{2\gamma} \). But

\[
\|\nabla'' u^k\|_{L^2(B^+_1)} \leq \|u^k - p_{3/2}^\xi\|_{L^2(B^+_1)} = \inf_{\xi \in \Omega} \|u^k - p_{3/2}^\xi\|_{L^2(B^+_1)}.
\]

The inequalities (112) and (113) implies that

\[
\max (\|\nabla'' u^k\|_{L^2(B^+_1)}, \|u - p_{3/2}\|_{L^2(B^+_1)}) \leq s^{-n/2} \max (\|\nabla'' u^k\|_{L^2(B^+_1)}, \|u - p_{3/2}\|_{L^2(B^+_1)}).
\]

But this is true for each \( \gamma < 1/2 \) so

\[
\max (\|\nabla'' u^k\|_{L^2(B^+_1)}, \|u - p_{3/2}\|_{L^2(B^+_1)}) \leq \max (\|\nabla'' u^k\|_{L^2(B^+_1)}, \|u - p_{3/2}\|_{L^2(B^+_1)}).
\]

for each \( \gamma < 1/4 \). This finishes the proof for case 4 and the Corollary. □
12. Regularity of Solutions.

We are now finally ready to prove that solutions are in fact $C^{1,1/2}$ which is the first main result of the paper and the main result of this section.

Before we prove the main theorem we will need one more small Lemma.

**Lemma 21.** Let $u$ solve the Signorini problem in $B^+_1$ and $\|u\|_{L^\infty(B^+_1)} = 1$. Then for every $\delta > 0$ there exist a $C_\delta$ such that if for some $r$ we have

$$\frac{\|u\|_{L^2(B^+_1)}}{C_\delta s^{3/2}} \leq 1$$

for $s \geq r$ and

$$\frac{\|u\|_{L^2(B^+_1)}}{C_\delta r^{3/2}} = 1.$$

Then we have

$$\inf_{\xi \in \Pi} \left\| \frac{u(rx)}{C_\delta s^{3/2}} - p_3^{\xi} \right\|_{L^2(B_1)} < \delta$$

and assuming that the minimizing $\xi = |\xi|e_1$ we also have

$$\left\| \nabla'' \left( \frac{u(rx)}{C_\delta r^{3/2}} \right) \right\|_{L^2(B^+_1)} < \delta.$$

**Proof:** If the Lemma is not true then there exist $u^j$ and $r_j$ such that

$$\frac{\|u^j\|_{L^2(B^+_1)}}{j s^{3/2}} \leq 1$$

for $s \geq r_j$ and

$$\frac{\|u^j\|_{L^2(B^+_1)}}{j r_j^{3/2}} = 1.$$

But

$$\inf_{\xi \in \Pi} \left\| \frac{u^j(r_j x)}{j r_j^{3/2}} - p_3^{\xi} \right\|_{L^2(B_1)} > \delta$$

or

$$\left\| \nabla'' \left( \frac{u^j(r_j x)}{j r_j^{3/2}} \right) \right\|_{L^2(B^+_1)} > \delta$$

for some fixed $\delta > 0$. Since $\|u^j\|_{L^\infty(B^+_1)} = 1$ we may deduce that $r_j \to 0$. We make the blow-up

$$v^j(x) = \frac{u^j(r_j x)}{j r_j^{3/2}}.$$

Then $v^j \to v^0$ strongly in $W^{1,2}$ for some sub-sequence. Next we notice that $v^0 = p_3^{\xi_0}$ for some $\xi_0 \in \Pi$. In particular, for $R > 1$, $\|v^0\|_{L^2(B^{+}_R)} \leq R^{3/2}$ by (116) and $v^0$ is a global solution to the Signorini problem. Arguing as in Proposition 2 one readily deduces that $v^0(x)$ is two dimensional and the assertion that $v^0 = p_3^{\xi_0}$ follows. Rotating the coordinate system we may assume that $\xi_0 = |\xi_0|e_1$ and obviously

$$\inf_{\xi \in \Pi} \left\| v^0 - p_3^{\xi} \right\|_{L^2(B^+_1)} = \left\| \nabla'' v^0 \right\|_{L^2(B^+_1)} = 0$$

this together with strong convergence clearly contradicts (117) and (118) when $j$ is large enough. □
Theorem 1. Let \( \mathbf{u} \) solve the Signorini problem in \( B_1^+ \) then
\[
\| \mathbf{u} \|_{C^{1,1/2}(B_0^+)} \leq C \| \mathbf{u} \|_{L^2(B_1^+)}
\]

Proof: It is enough to show the Theorem for \( \| \mathbf{u} \|_{L^2(B_1^+)} = 1 \), since we may always apply the proof to \( \mathbf{u}/\| \mathbf{u} \|_{L^2(B_1^+)} \). We will therefore assume that \( \| \mathbf{u} \|_{L^\infty(B_1^+)} = 1 \) for the rest of the proof. It is also enough to prove that
\[
(119) \quad \| \mathbf{u} \|_{L^2(B_1^+(x^0))} \leq C r^{3/2}
\]
for all \( r \in (0, 1/2) \) and \( x^0 \in \Gamma \cap B_1/2 \). Once (119) is proved we may argue as in Corollary 5 to show that \( \mathbf{u} \in C^{1,1/2} \). We will therefore assume that \( 0 \in \Gamma \) and show that \( \| \mathbf{u} \|_{L^2(B_1^+)} \leq C r^{3/2} \).

We choose \( \delta < \delta_3 \) where \( \gamma = 1/8 \) and \( \delta_3 \) is as in Corollary 6. Then, by Lemma 21, there exist a \( C_\delta \) with the properties of that Lemma. If \( \mathbf{u} \) is as in the Theorem then either \( \| \mathbf{u} \|_{L^2(B_1^+)} \leq C r^{3/2} \) for each \( r \in (0, 1) \) and we are done. Or there exist a largest \( r \), lets call it \( r_0 \), such that
\[
\| \mathbf{u} \|_{L^2(B_{r_0}^+)} = C r^{3/2}_0.
\]
Consider \( v = \frac{\mathbf{u}(r \mathbf{z})}{\| \mathbf{u}(r \mathbf{z}) \|_{L^2(B_1^+)}} \), which by Lemma 21 satisfies the assumptions of Corollary 6. Using (99) we see that
\[
\inf_{\xi \in \Pi} \| v - p_3^{\xi/2} \|_{L^2(B_{r_0}^+)} \leq s_\gamma^{3/2+\gamma} \delta.
\]
Rescale \( v_{s_\gamma} = \frac{v(s_\gamma \mathbf{x})}{s_\gamma^{3/2}} \) and we get
\[
\inf_{\xi \in \Pi} \| v_{s_\gamma} - p_3^{\xi/2} \|_{L^2(B_{r_0}^+)} \leq s_\gamma^{2} \delta \| v \|_{L^2(B_{r_0}^+)}.
\]
Also, by Corollary 6 and Lemma 15,
\[
\| \nabla \xi v_{s_\gamma} \|_{L^2(B_{r_0}^+)} \leq C s_\gamma^2 \delta \leq s_\gamma^{3/2} \delta,
\]
where we have decreased \( s_\gamma \) so that the last inequality holds. In particular this implies that \( v_{s_\gamma} \) satisfies the conditions in Corollary 6 with \( s_\gamma^{3/2} \delta \) instead of \( \delta \).

Next if we, as usual, let \( \xi_r \) be the minimizer of
\[
\inf_{\xi \in \Pi} \| \mathbf{u} - p_3^{\xi/2} \|_{L^2(B_{r_0}^+)}
\]
then
\[
\| p_3^{\xi_{s_\gamma}} - p_3^{\xi_{s_\gamma}} \|_{L^2(B_{r_0}^+)} \leq \frac{1}{s_\gamma^{3/2}} \| p_3^{\xi_{s_\gamma}} - p_3^{\xi_{s_\gamma}} \|_{L^2(B_{r_0}^+)} \leq \frac{1}{s_\gamma^{3/2}} \| v - p_3^{\xi_{s_\gamma}} \|_{L^2(B_{r_0}^+)} + \frac{1}{s_\gamma^{3/2}} \| v - p_3^{\xi_{s_\gamma}} \|_{L^2(B_{r_0}^+)} \leq s_\gamma^3/2 + \frac{1}{s_\gamma^{n+3/2}} \| v - p_3^{\xi_{s_\gamma}} \|_{L^2(B_{r_0}^+)} \leq \delta \left( s_\gamma^{3/2} + s_\gamma^{-n/2} \right).
\]
Since \( v_{s_\gamma} \) also satisfies the conditions in Corollary 6 with \( s_\gamma^{3/2} \delta \) for \( \delta \) we may iterate this. If
\[
v_{s_\gamma} = \frac{v(s_\gamma \mathbf{x})}{s_\gamma^{3/2}}
\]
then
\[
\| v_{s_\gamma}^{\xi_{s_\gamma}} - p_3^{\xi_{s_\gamma}} \|_{L^2(B_{r_0}^+)} \leq \delta s_\gamma^{3/2},
\]
\[
\| \nabla_s v_{s_\gamma}^{\xi_{s_\gamma}} \|_{L^2(B_{r_0}^+)} \leq \delta s_\gamma^{3/2}
\]
and
\[ \| \xi \|_{L^2(B_i^+)} - \| \xi \|_{L^2(B_i^+)} \leq \sum_{j=1}^{k-1} \| \xi \|_{L^2(B_i^+)} - \| \xi \|_{L^2(B_i^+)} \leq \]
\[ \delta \left( s_{i}^{n/2} + s_{i}^{n/2} \right) \delta \left( s_{i}^{n/2} + s_{i}^{n/2} \right) \leq \tilde{C}(\gamma) \delta. \]

By the triangle inequality we therefore have
\[ \| v \|_{L^2(B_i^+)} \leq \| v \|_{L^2(B_i^+)} + \| p \|_{L^2(B_i^+)} + \| p \|_{L^2(B_i^+)} \leq \delta s_{i}^{n/2} + \tilde{C}(\gamma) \delta + \| p \|_{L^2(B_i^+)} \leq \tilde{C}(\gamma) \delta. \]

Noticing that
\[ \| p \|_{L^2(B_i^+)} \leq 2 \]

since \( \| v \|_{L^2(B_i^+)} \leq 1 \) we may deduce that
\[ \| u \|_{L^2(B_i^+)} \leq C \delta s_{i}^{n/2}. \]

The theorem follows. \( \square \)

13. Free Boundary Regularity

In the previous section we proved that \( u \in C^{1,1/2} \). The proof was based on the fact that if the asymptotic profile of \( u \) at a free boundary point is \( p_{3/2} \) then the blow-up is unique. We can use exactly the same reasoning to show that the free boundary is \( C^{1,\alpha} \) close to a point where the asymptotic profile is \( p_{3/2} \). This is done in this section.

**Theorem 2.** Let \( u \) solve the Signorini problem in \( B_i^+ \) and \( 0 \in \Gamma \) then there exists a \( \delta_0 > 0 \) such that if
\[ \inf_{\xi \in \mathbb{H}} \| u - \xi \|_{L^2(B_i^+)} = \| u - \xi \|_{L^2(B_i^+)} \leq \delta, \]

and
\[ \| \nabla u \|_{L^2(B_i^+)} \leq \delta, \]

and \( \delta \leq \delta_0 \). Then the limit
\[ \lim_{r \to 0} \frac{u(rx)}{r^{3/2}} = u_0 \]

exists is unique and furthermore
\[ \| u_0 - p_{3/2} \|_{L^2(B_i^+)} \leq C \delta. \]

**Proof:** Let \( u \) be as in the Theorem with \( \delta_0 \) small enough then by Corollary 6
\[ \inf_{\xi \in \mathbb{H}} \| u - p_{3/2} \|_{L^2(B_i^+)} \leq \eta \frac{\delta}{\| u \|_{L^2(B_i^+)}} \]

for small \( \eta < < 1 \) depending only on \( \delta_0 \) and \( n \), in particular by choosing \( \delta_0 \) small enough we may make \( \eta \) as small as we need.

Next we notice that
\[ \| p_{3/2} - p_{3/2} \|_{L^2(B_i^+)} \leq \| u - p_{3/2} \|_{L^2(B_i^+)} + \| u - p_{3/2} \|_{L^2(B_i^+)} \leq \]
\[ \eta \delta \| u \|_{L^2(B_i^+)} + \delta s_{i}^{n/2} \leq C \eta \delta (s_{i}^{n/2} + s_{i}^{n/2}). \]

Therefore
\[ \| p_{3/2} - p_{3/2} \|_{L^2(B_i^+)} \leq C \eta \delta (1 + s_{i}^{n/2} + s_{i}^{n/2}). \]
In particular, this together with (89) and Lemma 15 implies that

$$\frac{u(s/2)}{\|u\|_{L^2(B_0^+)}^{s/2}}$$

satisfies the conditions in the Theorem with $\delta_1 \leq C\eta\delta$. If $\delta_0$ is small enough then $\eta < \frac{1}{C}$ and we may conclude that $\delta_1 \leq \delta/2$. We may thus iterate the above and deduce that

$$\|p_{3/2}^k - p_{3/2}\|_{L^2(B_0^+)} \leq \sum_{j=1}^{k} \|p_{3/2}^j - p_{3/2}^{j-1}\|_{L^2(B_0^+)} \leq$$

$$C\delta \left(1 + s^{-\gamma(n+3)/2}\right) \sum_{j=1}^{k} 2^{-j} \leq 4C \left(1 + s^{-\gamma(n+3)/2}\right) \delta.$$

Which implies the Theorem. \qed

**Corollary 7.** Let us solve the Signorini problem in $B_1^+$ and assume that $0 \in \Gamma$ assume furthermore that

$$\lim_{r \to 0} \frac{u(rx)}{r^{3/2}} = p_{3/2}^\xi$$

for some vector $\xi \in \Pi$. Then there exist an $r_0 > 0$ such that $\Gamma \cap B_{r_0}$ is an $(n-2)$-dimensional $C^{1,\alpha}$ manifold.

**Proof:** We may, by normalizing and rotating the coordinates, assume that

$$\lim_{r \to 0} \frac{u(rx)}{r^{3/2}} = p_{3/2}.$$ 

It therefore exist an $s$ such that

$$\|u - p_{3/2}\|_{L^2(B_0^+)} \leq s^{3/2} \delta_\gamma$$

and

$$\|\nabla u\|_{L^2(B_0^+)} \leq s^{1/2} \delta_\gamma$$

where $\delta_\gamma$ is as in Corollary 6.

Using (88) and $\|u\|_{L^2(B_0^+)} \leq C r^{3/2}$, which follows from Theorem 1, we may induce as in Theorem 1 that

$$\inf_{\xi \in \Pi} \|u - p_{3/2}^\xi\|_{L^2(B_0^+)} \leq C r^{3/2+\gamma} \delta_\gamma$$

for $r \leq s$. Also, by Lemma 14 and Proposition 2, if $x^0 \in \Gamma \cap B_{r/2}$ for any $r < s$ then

$$\lim_{t \to 0} \frac{u(tx + x^0)}{\|u\|_{L^2(B_0^+)}} = p_{3/2}^\xi$$

for some $\xi$ and also using Proposition 3 we can deduce that

$$\inf_{\xi \in \Pi} \|u - p_{3/2}^\xi\|_{L^2(B_0^+)} \leq C \inf_{\xi \in \Pi} \|u - p_{3/2}^\xi\|_{L^2(B_{r/2}^+, (x^0))} \leq C r^{3/2+\gamma} \delta_\gamma.$$ 

From this it follows that

$$\|p_{3/2}^{x^0} - p_{3/2}^{\xi}\|_{L^2(B_0^+)} \leq C r^\gamma \delta_\gamma$$

and from Theorem 2 we have

$$\lim_{r \to 0} \frac{u(rx + x^0)}{r^{3/2}} = p_{3/2}^{x^0}$$

where

$$\|p_{3/2} - p_{3/2}^{x^0}\|_{L^2(B_0^+)} \leq C |x^0|^{\gamma} \delta_\gamma.$$ 

We have thus shown that the normal of $\Gamma$ changes in a Hölder continuous fashion in $B_{s/2}$ which implies the Corollary. \qed
14. Appendix 1: Proof of Lemma 1.

In this appendix we will indicate how to prove $C^{1,\beta}$ regularity of the solutions to the Signorini problem. The proof follows the lines of the proof in the main body of the paper, but it is significantly simpler. We will therefore only briefly indicate some main points.

Since we do not know that the solutions are $C^{1,\beta}$ yet we will no longer make the “standing assumption” we did in section 2.

We will also use the curl operator explicitly so we will only, for the sake of simplicity, formulate the proof in $\mathbb{R}^3$.

Lemma 22. Let $u \neq 0$ be a global solution to the Signorini problem and assume that

$$
\liminf_{r \to \infty} \frac{\ln(\|u\|_{L^2(B_r^+)})}{\ln(r)} < \frac{3}{2}
$$

then $u$ is a linear function.

Proof: The proof is almost line for line the same as the proof of Lemma 5. Following that proof we consider $\text{curl}(u) = w$ and deduce that $w^3 = \text{constant}$. Noticing that

$$
\sup_{B_R^+} |w| \leq C(1 + R)^{\alpha - 1}
$$

we may conclude that the constant is zero if $\alpha < 1$. If $\alpha \geq 1$ we may without loss of generality subtract a linear function from $u$ such that $w^3 = 0$. Equation (21) follows. As in Lemma 5 we may conclude that (24) and (25) holds even without the $C^{1,\beta}$ assumption.

In particular we may deduce that

$$
2 \frac{\partial \xi}{\partial x_3} + \tau = \begin{cases} 
\frac{\partial \xi}{\partial x_3} = c_i & \text{in each component of } \Lambda_u \\
c_i & \text{in each component of } \Omega_u.
\end{cases}
$$

It is also easy to see that

$$
2 \frac{\partial \xi}{\partial x_3} + \tau \in W^{2,2}_{\text{loc}}(\mathbb{R}_+^3)
$$

which by the trace Theorem implies that

$$
2 \frac{\partial \xi}{\partial x_3} + \tau \in W^{3/2,2}_{\text{loc}}(\Pi).
$$

But (120) implies that $\nabla w = 0$ almost everywhere on $\Pi$ and we may therefore conclude from (121) that $2 \frac{\partial \xi}{\partial x_3} + \tau$ is constant. In other words $c_i = c_j = \tilde{c}_k = \tilde{c}_l$ for all $i,j,k,l$.

We may conclude, as in the main body of the paper, that $u(x) = u(x_1, x_3)$ and that $\Gamma_u$ contains at most one point. Linearity follows from Lemma 10. $\square$

The following Corollary is an easy consequence of Lemma 22.

Corollary 8. Let $u \neq 0$ be a global solution to the Signorini problem and assume that

$$
\liminf_{r \to \infty} \frac{\ln(\|u\|_{L^2(B_r^+)})}{\ln(r)} \leq 1
$$

then

$$
\liminf_{r \to \infty} \frac{\ln(\|u\|_{L^2(B_r^+)})}{\ln(r)} = 1.
$$

Definition 3. We will denote the $L^2$-projection of $u \in L^2(B^+_r(x^0); \mathbb{R}^3)$ onto the space $P$ by $\text{Pr}(u, r, x^0)$. The space $P$ we is the space of affine functions $l$ satisfying
The proof will progress in several steps.

When $x \to 0$ then there exist an $r$ such that

$$\liminf_{j \to \infty} \frac{|\ln (\|u^j(r_jx) - Pr(u^j,r_j)\|_{L^2(B_R^+)} - 1)}{|\ln(r_j)|} \geq 1 + 2\beta$$

for some $\beta > 0$. Once we have shown (123) the Lemma follows as Corollary 5.

We will assume the contrary that we have sequences $u^j$ satisfying (122) and $r_j \to 0$ such that

$$\lim_{j \to \infty} \frac{|\ln (\|u^j(r_jx) - Pr(u^j,r_j)\|_{L^2(B_R^+)} - 1)}{|\ln(r_j)|} = \alpha \leq 1.$$

The proof will progress in several steps.

**Step 1:** Arguing as we did in Lemma 19 we may find a sub-sequence of $u^j$ and a sequence $r_j \to 0$ such that

$$v^j(x) = \frac{u^j(r_jx) - Pr(u^j,r_j)}{|u^j(r_jx) - Pr(u^j,r_j)|_{L^2(B_R^+)}^2} \to u^0$$

where

$$\sup_{B_R^+} \|u^0\|_{L^2(B_R^+)} \leq C(1 + R)^{1+\epsilon}$$

for some small $\epsilon$. In particular from Corollary 8 it follows that $u^0$ is linear.

**Step 2:** The limit $u^0$ from step 1 satisfies

$$u^0 = \frac{1}{\|\frac{\lambda x_1}{2x} + \frac{\lambda x_2}{2x} + x_3\|_{L^2(B_R^+)}^2} \left[ \frac{\lambda x_1}{2x} \ x_1 \ + \frac{\lambda x_2}{2x} \ x_2 - x_3 \right] = f(x).$$

**Proof of Step 2:** This is a simple consequence of the fact that $u^0$ is a linear solution satisfying $Pr(u^0,1) = 0$. 

(i) $l(x)$ is affine of the following form:

$$l(x) = \begin{bmatrix} a_1 & b_1 \\ b_2 & a_2 \\ c_3 & d_3 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

(ii) $a_{33} + \frac{1}{2}(a_{11} + a_{22} + a_{33}) = 0.$

That is $Pr(u, r, x^0)$ is the element in $P$ that satisfies

$$\|u - Pr(u, r, x^0)\|_{L^2(B_R^+)} = \inf_{p \in B} \|u - p\|_{L^2(B_R^+)}.$$
Step 3: Let \( u^j \) be as in step 2 and \( \mu \) some small constant. Then there exist a sequence \( x^j \in B_{\delta/2} \cap \Pi \) and a sequence of real numbers \( s_j \) such that

\[
\inf_{\gamma \in \mathbb{R}} \frac{\left\| u^j(s_jr_jx + x^j) - \Pr(u^j, r_j, s_j) - \gamma f(x) \right\|_{L^2(B^+_1)}}{\left\| u^j(s_jr_jx + x^j) - \Pr(u^j, r_j, s_j) \right\|_{L^2(B^+_1)}} = \mu.
\]

Proof of Step 3: Notice that since \( 0 \in \Gamma \) we have \( \Lambda \cap B_{\delta/2} \neq \emptyset \) so we may find a small ball \( B_{\delta}(x^j) \) such that \( e_3 \cdot u^j(x^j) > 0 \) in \( B_{\delta} \cap \Pi \).

It is not hard to show that for some sequence \( t_k \to 0 \)

\[
\lim_{t_k \to 0} \frac{u^j(tkr_jx + x^j) - \Pr(u^j, tkr_j, x^j)}{\left\| u^j(tkr_jx + x^j) - \Pr(u^j, tkr_j, x^j) \right\|_{L^2(B^+_1)}} = \tilde{u}^j
\]

where

\[
\Delta \tilde{u}^j + \frac{\lambda + 2}{2} \nabla \text{div}(\tilde{u}^j) = 0 \quad \text{in } B^+_1
\]

\[
\frac{\partial \tilde{u}^j}{\partial x_i} + \frac{\partial \tilde{u}^j}{\partial x_3} = 0 \quad \text{on } \Pi \text{ for } i = 1, 2
\]

\[
\text{Pr}(\tilde{u}^j, 1, 0) = 0.
\]

Standard regularity theory implies that \( \tilde{u}^j \) can be written as a sum of polynomials and \( \Pr(\tilde{u}^j, 1, 0) = 0 \) implies that the zeroth and first order polynomial are identically zero.

It follows that

\[
\inf_{\gamma \in \mathbb{R}} \frac{\left\| u^j(tkr_jx + x^j) - \Pr(u^j, tkr_j, x^j) - \gamma f(x) \right\|_{L^2(B^+_1)}}{\left\| u^j(tkr_jx + x^j) - \Pr(u^j, tkr_j, x^j) \right\|_{L^2(B^+_1)}} \to 1
\]

as \( k \to \infty \). But by step 2 we also have that

\[
\inf_{\gamma \in \mathbb{R}} \frac{\left\| u^j(r_jx + x^j) - \Pr(u^j, r_j, x^j) - \gamma f(x) \right\|_{L^2(B^+_1)}}{\left\| u^j(r_jx + x^j) - \Pr(u^j, r_j, x^j) \right\|_{L^2(B^+_1)}} \to 0
\]

as \( j \to \infty \). An argument of continuity shows that we may choose the \( s_j \) as claimed in the step.

Step 4: Let

\[
w^j = \frac{u^j(r_js_jx + x^j) - \Pr(u^j, r_js_j, x^j)}{\left\| u^j(r_js_jx + x^j) - \Pr(u^j, r_js_j, x^j) \right\|_{L^2(B^+_1)}}.
\]

with \( x^j \) and \( s_j \) as in step 3.

Then there exists a sub-sequence such that \( w^j \to w^0 \) strongly in \( W^{1,2} \) and weakly in \( W^{2,2} \). Moreover if we chose the sequences appropriately \( w^0 \) will be a linear function.

Proof of Step 4: It follows from strong convergence and Step 3 that

\[
\inf_{\gamma \in \mathbb{R}} \frac{\left\| w^0 - \gamma e_3 f(x) \right\|_{L^2(B^+_1)}}{\left\| w^0 \right\|_{L^2(B^+_1)}} = \mu
\]

and, if we chose the \( s_j \) as the largest \( s \in (0, 1) \) such that (124) holds,

\[
\inf_{\gamma \in \mathbb{R}} \frac{\left\| w^0 - \Pr(w^0, R) - \gamma e_3 f(x) \right\|_{L^2(B^+_1)}}{\left\| w^0 \right\|_{L^2(B^+_1)}} \leq \mu,
\]

for each \( R \geq 1 \).

Also from the convergence it follows that

\[
\Pr(w^0, 1) = 0.
\]
Arguing as in Lemma 14 it follows from (126) that

\[
\lim_{R \to \infty} \frac{\ln(\|w^0\|_{L^2(B_R)})}{\ln(R)} \leq 1 + \epsilon.
\]

Using (127) and Lemma 22 we may conclude that \(w^0\) is a linear solution to the Signorini problem, as step 4 claims.

We have thus constructed a solution \(w^0\) such that \(Pr(w^0, 1) = 0\) and such that (125) holds. It follows from \(Pr(w^0, 1) = 0\) and linearity that \(w^0 = \gamma f\) for some \(\gamma\), but that contradicts (125). Our argument of contradiction is therefore complete and we have shown (123).

From (123) and similar argument as in Corollary 5 we may conclude that

\[
\frac{1}{d(x^0)^{n+2+2\beta}} \int_{B_d(x^0) \cap \mathbb{R}^n_+} |u - u(x^0) - (\nabla u)_{r,x^0} \cdot (x - x^0)|^2 \leq C
\]

for each ball \(B_d(x^0)\) with \(|x^0| < 1/2, r \leq 1/4\) and \(d(x^0) = \text{dist}(x^0, \Gamma)\). It is now fairly standard, using the interior regularity for the Lame system, to show that this implies that \(u^0 \in C^{1,\beta}\).

15. APPENDIX 2: SKETCH OF THE PROOF OF LEMMA 11.

In this appendix we will briefly indicate how to prove the eigenfunction expansion in Lemma 11.

We will argue as in section 4. Following the proof of Lemma 5 we denote \(w = \text{curl}(u)\). Then \(\Delta w = 0\). Noticing that a difference quotient argument assures that \(\partial u^2/\partial x_2 \in W^{1,2}\) we can conclude that

\[
\frac{\partial u^2}{\partial x_1} = -\frac{\partial u^3}{\partial x_2} \in H^{1/2}(\Pi)
\]

and \(\partial u^3/\partial x_2 \in H^{1/2}(\Pi)\). Therefore

\[
\frac{\partial w^3}{\partial x_3}
\]

is defined on \(\Pi\). In particular

\[
\Delta w^3 = 0 \quad \text{in } B^+_1,
\]

\[
\frac{\partial w^3}{\partial x_3} = 0 \quad \text{on } \Pi \cap B_1.
\]

By extending \(w^3\) to the lower half ball by an even reflection we see that \(w^3\) may be expressed by a power series

\[
w^3(x) = \sum_{k=0}^{\infty} z_k(x)
\]

where \(z_k\) is a homogeneous polynomial of order \(k\).

We can thus find a \(\xi\) and \((\chi^1, \chi^2)\) such that

\[
(u^1, u^2) = \nabla\xi + (\chi^1, \chi^2)
\]

where

\[
\frac{\partial \chi^2}{\partial x_1} - \frac{\partial \chi^1}{\partial x_2} = \sum_{k=0}^{\infty} z_k(x).
\]

We may thus express \(\chi^1\) and \(\chi^2\) by power series expressions

\[
\chi^i = \sum_{k=0}^{\infty} q_k^i(x)
\]

for \(i = 1, 2\).
If we consider the equations for \( w^1 \) and \( w^2 \) as in Lemma 5 we may conclude that
\[
\frac{\partial}{\partial x_1} \left( \Delta \frac{\partial \xi}{\partial x_3} + \frac{\lambda + 2}{2} \text{div}(u) \right) = -\frac{\partial}{\partial x_3} \Delta \chi^1
\]
and
\[
\frac{\partial}{\partial x_2} \left( \Delta \frac{\partial \xi}{\partial x_3} + \frac{\lambda + 2}{2} \text{div}(u) \right) = -\frac{\partial}{\partial x_3} \Delta \chi^2.
\]
It follows that
\[
\chi^3 = \sum_{k=0}^{\infty} \tilde{z}_k(x)
\]
for some harmonic function \( \tilde{\tau} \) and
\[
\Delta \chi^3 = \sum_{k=0}^{\infty} \tilde{z}_k(x)
\]
where \( \tilde{z}_k \) are homogeneous polynomials of order \( k \). We may therefore express
(129) \[
\chi^3 = \sum_{k=0}^{\infty} q_k^3(x).
\]
That is
(130) \[
u = \nabla \xi + (\chi^1, \chi^2, \chi^3) + \tau e_3
\]
where \( \chi^i \) are analytic functions. Here
\[
\Delta \frac{\partial \xi}{\partial x_3} + \frac{\lambda + 2}{2} \frac{\partial}{\partial x_3} \text{div}(\nabla \chi + \tau e_3) = 0 \quad \text{in } B^+_1 \quad \text{for } i = 1, 2
\]
\[
\Delta \frac{\partial \xi}{\partial x_3} + \frac{\lambda + 2}{2} \frac{\partial}{\partial x_3} \text{div}(\nabla \chi + \tau e_3) = 0 \quad \text{in } B^+_1 \quad \text{in } B^+_1
\]
with the boundary values
(131) \[
\frac{\partial \xi}{\partial x_3} + \tau = \chi^3 = \sum_{k=0}^{\infty} q_k^3(x) \quad \text{on } \{ x_1 > 0 \} \cap \Pi
\]
\[
\frac{\partial^2 \xi}{\partial x_3^2} + \frac{\lambda + 2}{2} \frac{\partial}{\partial x_3} \text{div}(\nabla \chi + \tau e_3) = -\frac{\lambda}{4} \text{div}(\chi^1, \chi^2, \chi^3) - \frac{\partial^2 \chi}{\partial x_3^2} \quad \text{on } \{ x_1 < 0 \} \cap \Pi
\]
\[
2 \frac{\partial^2 \xi}{\partial x_2 \partial x_3} = -\frac{\partial^2 \chi}{\partial x_3^2} - \frac{\partial^2 \chi}{\partial x_3^2} \sum_{k=0}^{\infty} q_k^3 \frac{\partial}{\partial x_3} \sum_{k=0}^{\infty} q_k^3 \quad \text{on } \Pi \quad \text{for } i = 1, 2.
\]
We see that we can split the boundary values into their different homogeneity’s and write
(132) \[
\xi = \sum_{k=0}^{\infty} \xi^k + \xi
\]
(133) \[
\tau = \sum_{k=0}^{\infty} \tau_k + \tilde{\tau},
\]
where \( \xi_k \) and \( \tau_k \) satisfy the boundary conditions
\[
\frac{\partial \xi_{k+1}}{\partial x_3} + \tau_k = \chi^3 = q_k^1(x) \quad \text{on } \{ x_1 > 0 \} \cap \Pi
\]
\[
\frac{\partial^2 \xi_{k+1}}{\partial x_3^2} + \frac{\lambda + 2}{2} \frac{\partial}{\partial x_3} \text{div}(\nabla \xi_{k+1} + \tau_k e_3) = -\frac{\lambda}{4} \text{div}(q_{k+1}^1, q_{k+1}^2, q_{k+1}^3) - \frac{\partial q_{k+1}^3}{\partial x_3} \quad \text{on } \{ x_1 < 0 \} \cap \Pi
\]
\[
2 \frac{\partial^2 \xi_{k+1}}{\partial x_2 \partial x_3} = -\frac{\partial q_{k+1}^3}{\partial x_3} - \frac{\partial q_{k+1}^3}{\partial x_3} \sum_{k=0}^{\infty} q_k^3 \frac{\partial}{\partial x_3} \sum_{k=0}^{\infty} q_k^3 \quad \text{on } \Pi \quad \text{for } i = 1, 2.
\]
It takes some calculation to see that we may chose \( \xi_{k+1} \) and \( \tau_k \) to be polynomials.

The functions \( \xi \) and \( \tilde{\tau} \) will satisfy homogeneous boundary conditions and we can therefore analyse them as in Corollaries 1, 3 and 4 and conclude that there is a harmonic function \( \tilde{\tau} \) solving the boundary value problem
\[
\Delta \tilde{\tau} = 0 \quad \text{in } B^+_1
\]
\[
\frac{\partial \tilde{\tau}}{\partial x_3} = 0 \quad \text{on } \{ x_1 > 0 \} \cap \Pi
\]
\[
\frac{\partial \tilde{\tau}}{\partial x_3} = 0 \quad \text{on } \{ x_1 < 0 \} \cap \Pi
\]
such that
\begin{equation}
\tilde{\tau} = \frac{\partial \bar{\tau}}{\partial x_3}
\end{equation}
and
\begin{equation}
\tilde{\xi} = \frac{\lambda + 2}{2(\lambda + 4)} \frac{\partial \bar{\tau}}{\partial x_3} x_3 - \frac{\lambda + 3}{\lambda + 4} \bar{\tau}.
\end{equation}
Since $\bar{\tau}$ is a harmonic with homogeneous boundary data we can make the expansion of $\bar{\tau}$ into a sum of homogeneous eigenfunctions as
\begin{equation}
\bar{\tau} = \sum_{k=0}^{\infty} E_k \bar{\tau}(x),
\end{equation}
here $E_k \bar{\tau}$ is a homogeneous harmonic function, but not necessarily of order $k$. In general such an eigenfunction expression of a harmonic function may contain generalized eigenfunctions of growth $\ln(|x|)|x|^m$, but since our boundary is so simple no such terms will appear in (136).

Putting (128)-(136) together we have shown that $u$ may be written as a sum of homogeneous functions.
\begin{equation}
 u = \sum_{k=0}^{\infty} H_k(x)
\end{equation}
where $H_k$ is homogeneous, but not necessarily of order $k$. The Lemma follows if we can show that each $H_k$ is homogeneous of order $j/2$ for some $j \in \mathbb{N}$. To that end we notice that a difference quotient argument implies that
\begin{equation}
\frac{\partial^m H_k}{\partial x_2^m} \in W^{1,2} \text{ for each } m \in \mathbb{N}.
\end{equation}
Also $\frac{\partial^m H_k}{\partial x_2^m}$ will be homogeneous of $m$ orders less than $H_k$. But this together with (137) implies that if $m$ is large enough then
\begin{equation}
\frac{\partial^m H_k}{\partial x_2^m} = 0.
\end{equation}
Therefore there exist an $m$ such that
\begin{equation}
\frac{\partial^m H_k}{\partial x_2^m} = L(x_1, x_3).
\end{equation}
By the classification in Lemma 10 it follows that $L$ is homogeneous of order $j$ or $j + 1/2$. The first part of the Lemma follows.

The second part of Lemma 11 is simple to prove. First we notice that in polar coordinates $q_0 = P_{1/2} \notin W^{2,2}$ and therefore $a_0 = 0$ if $w \in W^{2,2}$. Next we notice that $q_1 = ap_{3/2} + (\nu \cdot x)p_{1/2}, \nu \cdot e_1 = \nu \cdot e_2 = 0$, which is only a $W^{2,2}$ function if $\nu = 0$. The second part of the Lemma follows.

REFERENCES

[1] J. Andersson, H. Shahgholian, and G.S. Weiss. Linearization techniques in free boundary problems, with an application to the regularity of free boundary branch points. In preparation.
[2] D. E. Apushkinskaya, H. Shahgholian, and N. N. Uraltseva. Boundary estimates for solutions of a parabolic free boundary problem. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 271(Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funktsii)31):39–55, 313, 2000.
[3] A. A. Arkhipova and N. N. Uraltseva. Regularity of solutions of diagonal elliptic systems under convex constraints on the boundary of the domain. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 152(Kraev. Zadachi Mat. Fiz. i Smezhnye Vopr. Teor. Funktsii18):5–17, 181, 1986.
[4] I. Athanasopoulos and L. A. Caffarelli. Optimal regularity of lower dimensional obstacle problems. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 310(Kraev. Zadachi Mat. Fiz. i Smesh. Vopr. Teor. Funkts. 35 [34]):49–66, 226, 2004.

[5] I. Athanasopoulos, L. A. Caffarelli, and S. Salsa. The structure of the free boundary for lower dimensional obstacle problems. Amer. J. Math., 130(2):485–498, 2008.

[6] Michael Benedicks. Positive harmonic functions vanishing on the boundary of certain domains in \( \mathbb{R}^n \). Ark. Mat., 18(1):53–72, 1980.

[7] Luis A. Caffarelli, Sandro Salsa, and Luis Silvestre. Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian. Invent. Math., 171(2):425–461, 2008.

[8] Philippe G. Ciarlet. Mathematical elasticity. Vol. I, volume 20 of Studies in Mathematics and its Applications. North-Holland Publishing Co., Amsterdam, 1988. Three-dimensional elasticity.

[9] Gaetano Fichera. Sul problema elastostatico di Signorini con ambigue condizioni al contorno. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8), 34:138–142, 1963.

[10] S. Friedland and W. K. Hayman. Eigenvalue inequalities for the Dirichlet problem on spheres and the growth of subharmonic functions. Comment. Math. Helv., 51(2):133–161, 1976.

[11] Martin Fuchs. The smoothness of the free boundary for a class of vector-valued problems. Comm. Partial Differential Equations, 14(8-9):1027–1041, 1989.

[12] Alfred Huber. Über Wachstumseigenschaften gewisser Klassen von subharmonischen Funktionen. Comment. Math. Helv., 26:81–116, 1952.

[13] David Kinderlehrer. Remarks about Signorini’s problem in linear elasticity. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 8(4):605–645, 1981.

[14] V. A. Kozlov, V. G. Maz'ya, and J. Rossmann. Spectral problems associated with corner singularities of solutions to elliptic equations, volume 85 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2001.

[15] Rainer Schumann. Regularity for Signorini’s problem in linear elasticity. Manuscripta Math., 63(3):255–291, 1989.

[16] A. Signorini. Sopra alcune questioni di elasticit. Societ Italiana per il Progresso delle Scienze, 1933.