ON A CLASS OF $\alpha$-PARA KENMOTSU MANIFOLDS

K. SRIVASTAVA* AND S. K. SRIVASTAVA**

Abstract. The purpose of this paper is to classify $\alpha$-para Kenmotsu manifolds $M^3$ such that the projection of the image of concircular curvature tensor $L$ in one-dimensional linear subspace of $T_p(M^3)$ generated by $\xi_p$ is zero.

1. Introduction

The geometry of concircular transformations is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism [22]. An interesting invariant of a concircular transformation is the concircular curvature tensor.

Let $(M^{2n+1}, g)$ be a $(2n+1)$-dimensional connected pseudo-Riemannian manifold. The concircular curvature tensor $L$ [27] of $M^{2n+1}$ is defined by

$$L(X,Y)Z = R(X,Y)Z - \frac{\tau}{2n(2n+1)}(g(Y,Z)X - g(X,Z)Y)$$

(1.1)

where $R$ is the curvature tensor, $\tau$ is the scalar curvature and $X, Y, Z \in \chi(M^{2n+1})$, $\chi(M^{2n+1})$ being the Lie algebra of vector fields of $M^{2n+1}$.

We observe immediately from the form of the concircular curvature tensor that pseudo-manifolds with vanishing concircular curvature tensor are of constant curvature. Thus one can think of the concircular curvature tensor as a measure of the failure of a pseudo-Riemannian manifold to be of constant curvature (see also [4]). This paper is organized as follows: In §2 The basic information about almost paracontact metric manifolds, normal almost paracontact metric manifolds and the curvature tensor of the manifolds are given. In §3 we have obtained the relation between second order parallel tensor and the associated metric on $\alpha$-para Kenmotsu manifold. In §4 we found the necessary and sufficient condition for an $\alpha$-para Kenmotsu manifold to be $\xi$-concircularly flat. Finally, we cited of an $\alpha$-para Kenmotsu manifold in §5.

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2. Preliminaries

2.1. Almost paracontact metric manifolds. A $C^\infty$ smooth manifold $M^{2n+1}$ of dimension $(2n+1)$, is said to have triplet $(\phi, \xi, \eta)$-structure, if it admits an endomorphism $\phi$, a unique vector field $\xi$ and a contact form $\eta$ satisfying:

$$\phi^2 = I - \eta \otimes \xi \quad \text{and} \quad \eta(\xi) = 1 \quad (2.1)$$

where $I$ is the identity transformation; and the endomorphism $\phi$ induces an almost paracomplex structure on each fibre of $D = \ker(\eta)$, the contact subbundle, i.e., eigen distributions $D^{\pm1}$ corresponding to the characteristic values $\pm1$ of $\phi$ have equal dimension $n$.

From the equation $(2.1)$, it can be easily deduce that

$$\phi \xi = 0, \quad \eta \circ \phi = 0 \quad \text{and} \quad \text{rank}(\phi) = 2n. \quad (2.2)$$

This triplet structure $-(\phi, \xi, \eta)$ is called an almost paracontact structure and the manifold $M^{2n+1}$ equipped with the $(\phi, \xi, \eta)$-structure is called an almost paracontact manifold [12]. If an almost paracontact manifold admits a pseudo-Riemannian metric [28], $g$ satisfying:

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y) \quad (2.3)$$

where signature of $g$ is necessarily $(n+1, n)$ for any vector fields $X$ and $Y$; then the quadruple $-(\phi, \xi, \eta, g)$ is called an almost paracontact metric structure and the manifold $M^{2n+1}$ equipped with paracontact metric structure is called an almost paracontact metric manifold. With respect to $g$, $\eta$ is metrically dual to $\xi$, that is

$$g(X, \xi) = \eta(X) \quad (2.4)$$

Also, equation $(2.3)$ implies that

$$g(\phi X, Y) = -g(X, \phi Y). \quad (2.5)$$

Further, in addition to the above properties, if the structure $-(\phi, \xi, \eta, g)$ satisfies:

$$d\eta(X, Y) = g(X, \phi Y),$$

for all vector fields $X$, $Y$ on $M^{2n+1}$, then the manifold is called a paracontact metric manifold and the corresponding structure $-(\phi, \xi, \eta, g)$ is called a paracontact structure with the associated metric $g$ [28]. For an almost paracontact metric manifold, there always exists a special kind of local pseudo-orthonormal basis $\{X_i, X^r, \xi\}$; where $X^r = \phi X_i$; $\xi$ and $X_i$’s are space-like vector fields and $X^r$’s are time-like. Such a basis is called $\phi$-basis. Hence, an almost paracontact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is an odd dimensional manifold with a structure group $\mathbb{U}(n, \mathbb{R}) \times Id$, where $\mathbb{U}(n, \mathbb{R})$ is the para-unitary group isomorphic to $\mathbb{C}L(n, \mathbb{R})$. 

2.2. Normal almost paracontact metric manifolds. On an almost paracontact manifold, one defines the \((2,1)\)-tensor field \(N_\phi\) by
\[
N_\phi := [\phi, \phi] - 2d\eta \otimes \xi,
\]
where \([\phi, \phi]\) is the Nijenhuis torsion of \(\phi\). If \(N_\phi\) vanishes identically, then we say that the manifold \(M^{2n+1}\) is a normal almost paracontact metric manifold \([13, 28]\). The normality condition implies that the almost paracomplex structure \(J\) defined on \(M^{2n+1} \times \mathbb{R}\) by
\[
J\left(X, \lambda \frac{d}{dt}\right) = \left(\phi X + \lambda \xi, \eta(X) \frac{d}{dt}\right)
\]
is integrable. Here \(X\) is tangent to \(M^{2n+1}\), \(t\) is the coordinate on \(\mathbb{R}\) and \(\lambda\) is \(C^\infty\) function on \(M^{2n+1} \times \mathbb{R}\). Now we recall the following proposition which characterized the normality of almost paracontact metric 3-manifolds:

**Proposition 2.1.** \([23]\) For almost paracontact metric 3-manifold \(M^3\), the following three conditions are mutually equivalent

(i) \(M^3\) is normal,
(ii) there exist smooth functions \(\alpha, \beta\) on \(M^3\) such that
\[
(\nabla_X \phi)Y = \beta (g(X,Y)\xi - \eta(Y)X) + \alpha (g(\phi X, Y)\xi - \eta(Y)\phi X),
\]
(iii) there exist smooth functions \(\alpha, \beta\) on \(M^3\) such that
\[
\nabla_X \xi = \alpha (X - \eta(X)\xi) + \beta \phi X
\]
where \(\nabla\) is the Levi-Civita connection of the pseudo-Riemannian metric \(g\).

The functions \(\alpha, \beta\) appearing in (2.7) and (2.8) are given by
\[
2\alpha = \text{trace} \{X \to \nabla_X \xi\}, \quad 2\beta = \text{trace} \{X \to \phi \nabla_X \xi\}.
\]

**Definition 2.1.** A normal almost paracontact metric 3-manifold is called

- paracosymplectic if \(\alpha = \beta = 0\) \([6]\),
- quasi-para Sasakian if and only if \(\alpha = 0\) and \(\beta \neq 0\) \([9]\),
- \(\beta\)-para Sasakian if and only if \(\alpha = 0\) and \(\beta\) is non-zero constant, in particular para Sasakian if \(\beta = -1\) \([28]\),
- \(\alpha\)-para Kenmotsu if \(\alpha\) is non-zero constant and \(\beta = 0\) \([25]\).

2.3. Curvature properties of normal almost paracontact metric 3-manifolds.
In \((2n+1)\)-dimensional connected pseudo-Riemannian manifold \((M^{2n+1}, g)\), the curvature tensor \(R\) \([17]\) and the projective curvature tensor \(P\) \([20]\) are defined by
\[
R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,
\]
\[
P(X,Y)Z = R(X,Y)Z - \frac{1}{2n} (g(SY,Z)X - g(SX,Z)Y).
\]
In 3-dimensional pseudo-Riemannian manifold the curvature tensor satisfies [5]:
\[
\tilde{R}(X, Y, Z, W) = g(X, W)g(SY, Z) - g(X, Z)g(SY, W) + g(Y, W)g(X, Z)
- \frac{\tau}{2} \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}
\]
(2.12)

where \(\tilde{R}(X, Y, Z, W) = g(R(X, Y)Z, W)\), \(\tau = \text{trace}(S)\) is the scalar curvature of the manifold and Ricci operator \(S\) is defined by
\[
g(SX, Y) = \text{Ric}(X, Y).
\]
(2.13)

Using (2.1), (2.2), (2.3), (2.5), (2.8), (2.10) and (2.12) it is easy to prove the following lemma:

**Lemma 2.1.** Let \(M^3(\phi, \xi, \eta, g)\) be a normal almost paracontact metric manifold, then we have
\[
R(X, Y)\xi = \{((Y\alpha) + (\alpha^2 + \beta^2)\eta(Y))\phi^2 X - ((X\alpha) + (\alpha^2 + \beta^2)\eta(X))\phi^2 Y
+ \{(Y\beta) + 2\alpha\beta\eta(Y)\}\phi X - \{(X\beta) + 2\alpha\beta\eta(X)\}\phi Y
\]
(2.14)
\[
\text{Ric}(X, Y) = \{Y\eta(Y)(X\alpha) + \eta(\xi\alpha)\}\phi X - \{(Y\alpha) + 3(\alpha^2 + \beta^2) - \frac{\tau}{2}\}\eta(Y)
+ \{(Y\beta) + 2(\alpha^2 + \beta^2)\}\eta(Y).
\]
(2.15)
\[
\xi\beta + 2\alpha\beta = 0.
\]
(2.16)

**Remark 1.** From (2.17), it follows that if \(\alpha, \beta = \text{constant}\), then \(M^3\) is either \(\beta\)-para Sasakian or \(\alpha\)-para Kenmotsu or paracosymplectic.

**Proposition 2.2.** Let \(M^3(\phi, \xi, \eta, g)\) be an \(\alpha\)-para Kenmotsu manifold, then we have
\[
R(X, Y)Z = \{(\tau/2 - 2\alpha^2)(g(Y, Z)X - g(X, Z)Y)
- (\tau/2 - 3\alpha^2)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi
+ (\tau/2 - 3\alpha^2)\{Y\eta(X) - X\eta(Y)\}\eta(Z).
\]
(2.18)

**Proof.** In view of (2.12) and (2.14), we have (2.18). \(\square\)

3. Second order parallel tensor field

**Definition 3.1.** A tensor \(T\) of second order is said to be a second order parallel tensor if \(\nabla T = 0\), where \(\nabla\) denotes the operator of covariant differentiation with respect to the associated metric \(g\). [8]

Now we give the following result which established the relation between second order parallel tensor and the associated metric on \(\alpha\)-para Kenmotsu manifold:
Theorem 3.1. On an $\alpha$-para Kenmotsu manifold $M^3$ a second order parallel tensor is a constant multiple of the associated tensor.

Proof. Let $h$ denotes symmetric $(0, 2)$-tensor field on $M^3$ such that $\nabla h = 0$. Then it follows that
\[ h(R(Z, X)Y, W) + h(Y, R(Z, X)W) = 0 \tag{3.1} \]
for any $X, Y, Z, W \in \Gamma(TM^3)$. Substituting $Y = Z = W = \xi$ in (3.1), we obtain
\[ h(R(\xi, X)\xi, \xi) = 0 \]
which gives by virtue of (2.18) that
\[ h(X, \xi) = \eta(X)h(\xi, \xi). \tag{3.2} \]
Differentiating (3.2) along $Y$ and using (2.8) and (3.2), we have
\[ h(X, Y) = h(\xi, \xi)g(X, Y). \tag{3.3} \]
Again differentiating (3.3) covariantly along any vector field on $M^3$ it can be easily seen that $h(\xi, \xi)$ is constant. This completes the proof. □

Remark 2. If the Ricci tensor field is parallel in an $\alpha$-para Kenmotsu manifold $M^3$, then it is an Einstein manifold.

Let us suppose that $h$ is a parallel 2-form on $M^3$, that is,
\[ h(X, Y) = -h(Y, X) \text{ and } \nabla h = 0. \tag{3.4} \]
Then we prove the following result:

Theorem 3.2. Let $M^3(\phi, \xi, \eta, g)$ be an $\alpha$-para Kenmotsu manifold. Then non-zero parallel 2-form cannot occur on $M^3$.

Proof. For the parallel 2-form, we have from (3.3)
\[ h(\xi, \xi) = 0. \tag{3.5} \]
Differentiating (3.5) covariantly along $X$ and applying (2.8) and (3.5), we have
\[ h(X, \xi) = 0. \tag{3.6} \]
Further differentiating above covariantly with respect to $Y$ and using (2.8) and (3.6) yields
\[ h(X, Y) = 0. \]
This completes the proof. □
4. Main results

ξ-conformally flat K-contact manifolds have been studied by Zhen et al. [29]. Since at each point \( p \in M^{2n+1} \) the tangent space \( T_p(M^{2n+1}) \) can be decomposed into the direct sum \( T_p(M^{2n+1}) = \phi(T_p(M^{2n+1})) \oplus \{ \xi_p \} \), where \( \{ \xi_p \} \) is the one-dimensional linear subspace of \( T_p(M^{2n+1}) \) generated by \( \xi_p \), the conformal curvature tensor \( C \) is a map

\[
C : T_p(M^{2n+1}) \times T_p(M^{2n+1}) \times T_p(M^{2n+1}) \to \phi(T_p(M^{2n+1})) \oplus \{ \xi_p \}.
\]

An almost contact metric manifold \( M^{2n+1} \) is called ξ-conformally flat if the projection of the image of \( C \) in \( \{ \xi_p \} \) is zero [29].

Analogous to the definition of ξ-conformally flat almost contact metric manifold we define ξ-concirurally flat normal almost paracontact metric manifold.

**Definition 4.1.** A normal almost paracontact metric manifold \( M^{2n+1} \) is called ξ-concirurally flat if the condition \( L(X, Y)\xi = 0 \) holds on \( M^{2n+1} \), where concirular curvature tensor \( L \) is defined by (1.1).

**Theorem 4.1.** Let \( M^3(\phi, \xi, \eta, g) \) be an \( \alpha \)-para Kenmotsu manifold. Then \( M^3 \) is ξ-concirurally flat if and only if the scalar curvature \( \tau = 6\alpha^2 \).

**Proof.** Putting \( Z = \xi \) in (1.1) and using (2.14) and (2.16), we have

\[
L(X, Y)\xi = (\alpha^2 - \tau/6)\{\eta(Y)X - \eta(X)Y\}.
\]

This implies that \( L(X, Y)\xi = 0 \) if and only if \( \tau = 6\alpha^2 \).

□

As a corollary of the above theorem we have the following result:

**Corollary 4.1.1.** Let \( M^3(\phi, \xi, \eta, g) \) be a ξ-concirurally flat \( \alpha \)-para Kenmotsu manifold. Then

(i) Ricci is parallel.

(ii) \( M^3 \) is Einstein.

**Remark 3.** An \( \alpha \)-para Kenmotsu manifold \( M^3(\phi, \xi, \eta, g) \) is ξ-projectively flat.

5. Example

We consider the 3-dimensional manifold \( M^3 = \mathbb{R}^2 \times \mathbb{R}^- \subset \mathbb{R}^3 \) with the standard cartesian coordinates \((x, y, z)\). Define the almost paracontact structure \((\phi, \xi, \eta)\) on \( M^3 \) by

\[
\phi e_1 = e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0, \quad \xi = e_3, \quad \eta = dz,
\]

where \( e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} \) and \( e_3 = \frac{\partial}{\partial z} \). By straightforward calculations, one verifies that

\[
[\phi, \phi](e_i, e_j) - 2d\eta(e_i, e_j) = 0, \quad 1 \leq i < j \leq 3,
\]
which implies that the structure is normal. Let $g$ be the pseudo-Riemannian metric defined by

$$[g(e_i, e_j)] = \begin{bmatrix} exp(2z) & 0 & 0 \\ 0 & -exp(2z) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.2}$$

For the Levi-Civita connection, we obtain

$$\nabla_{e_1} e_1 = -exp(2z)e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = e_1,$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = exp(2z)e_3, \quad \nabla_{e_2} e_3 = e_2,$$

$$\nabla_{e_3} e_1 = e_1, \quad \nabla_{e_3} e_2 = e_2, \quad \nabla_{e_3} e_3 = 0. \tag{5.3}$$

Using the above expressions and (2.8), we find $\alpha = 1, \beta = 0$. Hence the manifold is an $\alpha$-para Kenmotsu manifold. With the help of (5.3) and (2.10), we have

$$R(e_1, e_2)e_3 = R(e_1, e_3)e_1 = -exp(2z)e_3, \quad R(e_1, e_3)e_2 = -exp(2z)e_1,$$

$$R(e_1, e_2)e_2 = exp(2z)e_3, \quad R(e_2, e_3)e_3 = e_2,$$

$$R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e_2 = 0. \tag{5.4}$$

From (5.4) it is not hard to see that the scalar curvature $\tau = 6$. Therefore $M^3$ is $\xi$-concircularly flat. Thus theorem 4.1 is verified.

REFERENCES

[1] BEJAN, C. L.—CRASMAREANU, M.: Second order parallel tensors and Ricci solitons in 3-dimensional normal paracontact geometry, Ann. Glob. Anal. Geom., DOI 10.1007/s10455-014-9414-4, 2014.

[2] BLAIR, D. E.: Riemannian geometry of contact and symplectic manifolds, Progress in Mathematics 203, Birkhauser Boston, Inc., Boston, MA, 2002.

[3] BLAIR, D. E.: Two remarks on contact metric structures, Tôhoku Math. J., 29(1977), 319-324.

[4] BLAIR, D. E.—KIM, J. S.—TRIPATHI, M. M.: On the concircular curvature tensor of a contact metric manifold, J. Korean Math. Soc., 42(2005), No. 5, pp. 883-892.

[5] BLAIR, D. E.—KOUGIOGROS, T.—SHARMA, R.: A classification of 3-dimensional contact metric manifolds with $Q\phi = \phi Q$, Kodai Math. J., 13(1990), 391-401.

[6] DACKO, P.: On almost para-cosymplectic manifolds, Tsukuba J. Math., 28(1)(2004), 193-213.

[7] DE, U. C.—MONDAL, A. K.: On 3-dimensional normal almost contact metric manifolds satisfying certain curvature conditions, Commun. Korean Math. Soc. 24(2009), No. 2, pp. 265-275.

[8] DE, U. C.—MONDAL, A. K., ÖZGÜR, C.: Second order parallel tensor on $(k, \mu)$-contact metric manifolds, An. St. Univ. Ovidius Constanța, 18(1)(2010), 229-238.

[9] ERDEM, S.: On almost (para)contact (hyperbolic) metric manifolds and harmonicity of $(\phi, \phi')$-holomorphic maps between them, Houston J. Math., 28(2002), 21-45.

[10] ERKEN, I. K.—DACKO, P.—MURATHAN, C.: Almost $\alpha$-Paracosymplectic manifolds, arXiv:1402.6930, 2014.
[11] JIN, J. E.—PARK, J. H.—SEKIGAWA, K.: Notes on some classes of 3-dimensional contact metric manifolds, Balkan Journal of Geometry and Its Applications, 17(2012), No. 2, pp. 54-65.

[12] KANEYUKI, S.—KONZAI, M.: Paracomplex structure and affine symmetric spaces, Tokyo J. Math., 8(1985), 301-318.

[13] KANEYUKI, S.—WILLAMS, F. L.: Almost paracontact and parahodge structures on manifolds, Nagoya Math. J., 99(1985), 173-187.

[14] LEVI, H.: Symmetric tensors of the second order whose covariant derivatives vanish, Annals of Math., 27(1926), 91-98.

[15] MANEV, M.—STAIKOVA, M.: On almost paracontact Riemannian manifolds of type $(n, n)$, J. Geom., 72(2001), 108-114.

[16] NAKOVA, G.—ZAMKOVOY, S: Almost paracontact manifolds, arXiv:0806.3859v2, 2007.

[17] NEILL, B.O.: Semi Riemannian geometry with applications to Relativity, Academic press, New York, 1983.

[18] ÖZGÜR, C.: $\phi$-conformally flat Lorentzian para-Sasakian manifolds, Radovi Mat. 12(2003), 1-10.

[19] SHARMA, R.: Second order parallel tensor in real and complex space forms, International J. Math. and Math. Sci., 12(1989), 787-790.

[20] SRIVASTAVA, K.—SRIVASTAVA, S. K.: On a class of paracontact metric 3-manifolds, arXiv:1402.6137v1, 2014.

[21] TANNO, S.: Ricci curvatures of contact Riemannian manifolds, Tôhoku Math. J., 40(1988), 441-448.

[22] TRIPATHI, M. M.,—KIM, J. S.: On the concircular curvature tensor of a $(\kappa, \mu)$-manifold, Balkan Journal of Geometry and Its Applications, 9(2004), No. 2, pp. 104-114.

[23] WELYCZKO, J.: On Legendre curves in 3-dimensional normal almost paracontact metric manifolds, Result. Math., 54(2009), 377-387.

[24] WELYCZKO, J.: Para-CR structures on almost paracontact metric manifolds, arXiv:12026383v2, 2012.

[25] WELYCZKO, J.: Slant curves in 3-dimensional normal almost paracontact metric manifolds, Mediterr. J. Math., DOI 10.1007/s00009-013-0361-2, 2013.

[26] YANO, K.—KON, M.: Structures on Manifolds, Series in Pure Math, 3, World Sci, 1984.

[27] YANO, K.: Concircular geometry IV, Proc. Imp. Acad. Tokyo, 16: 195-200, 354-360, 442-448, 505-511, 1940.

[28] ZAMKOVOY, S.: Canonical connections on paracontact manifolds, Ann. Glob. Anal. Geom., 36(2009), 37-60.

[29] ZHEN, G.—CABRERIZO, J. L.—FERNÁNDEZ, L. M.—FERNÁNDEZ M.: On $\xi$-conformally flat contact metric manifolds, Indian J. Pure Appl. Math. 28(1997), No. 6, 723-734.
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**Department of Mathematics**
Central University of Himachal Pradesh
Dharamshala-176215
Himachal Pradesh
India
E-mail address: sachink.ddumath@gmail.com