Extension of Fröhlich’s method to 4-fermion interactions

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Higher order terms of the transformed electron-phonon Hamiltonian $H_{e-ph}$, obtained by performing the Fröhlich’s transformation, are investigated. The influence of terms discarded by Fröhlich (in particular those proportional to the third power of electron-phonon coupling) on the effective Hamiltonian is examined. To this end a second Fröhlich-type transformation is performed, which yields, among others, an effective 4-electron interaction. This interaction is reduced to a form admitting solution of thermodynamics. The form of the coupling of the 4-electron interaction is found. By applying standard approximations, it is shown that this interaction is attractive with interaction $V_{eff}$ between electrons dressed in the phonon field. $V_{eff}$ proved to be attractive for 1-electron energies close to $\varepsilon_F$. A reduced form of $V_{eff}$ was subsequently used by Bardeen, Cooper and Schrieffer in their theory.

The form of higher order terms of the original Fröhlich-transformed $H_{e-ph}$ are also found, up to terms proportional to the 6-th power of the coupling, that is up to those, which yield the effective 4-electron interactions.

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I. INTRODUCTION

The BCS theory proved extremely effective as a theory of superconductivity. The key idea of this theory—the attractive interaction which binds electrons into Cooper pairs—has its essential origins in earlier conclusions drawn by Fröhlich, who performed a unitary transformation $U$ of the electron-phonon Hamiltonian $H_{e-ph}$. $U$ was adjusted so as to eliminate the electron-phonon interaction as far as possible and replace it by an effective interaction $V_{eff}$ between electrons dressed in the phonon field. $V_{eff}$ proved to be attractive for 1-electron energies close to $\varepsilon_F$. A reduced form of $V_{eff}$ was subsequently used by Bardeen, Cooper and Schrieffer in their theory.

Unfortunately, BCS theory proved incapable of explaining superconductivity in type-II superconductors, heavy fermions and high-$T_c$ superconductors (HTSC). The search for an alternative theory of superconductivity proceeds in various directions and one of them exploits the idea of extending BCS theory by adding to the BCS Hamiltonian $H_{BCS}$ further interactions. Rickayzen suggested incorporation of a 4-fermion interaction, motivating such choice by analogies between theory of superconductivity and nuclear theory, where such interactions had been considered. This idea was also mentioned by Volovik, but remained in the realm of theoretical concepts until the late nineties, when Mackowiak and one of authors (PT) proposed a Hamiltonian $H_{MT} = H_{BCS} + W + V_{MT}$, where

$$H_{BCS} = T + V_{BCS} = \sum_{k\sigma} \xi_k n_{k\sigma} - |\Lambda|^{-1} \sum_{kk'} G_{kk'} a^+_k a^+_k a^-_{-k'-a},$$

$$W = \sum_k \gamma_k n_{k+} n_{k-},$$

$$V_{MT} = |\Lambda|^{-1} \sum_{kk'} g_{kk'} a^+_k a^+_k a^+_k a^-_{-k-a}. a^+_{-k'-a} a^-_{-k'a},$$

$|\Lambda|$ in Eqs. 1–3 denotes the system’s volume, whereas $g_{kk'}$, $G_{kk'}$ are bounded functions and $\gamma_k$, $g_{kk'}$, $G_{kk'}$ are invariant under time reversal $k \rightarrow -k$ or $k' \rightarrow -k'$.

Considerations which led to the introduction of $W$ were founded on the analysis of the HTSC normal state. There are grounds to believe that this is not a normal Fermi liquid state. The interaction $W$, first added to $H_{BCS}$
by Czerwonko, guaranteed normal-state behaviour characteristic of the so-called statistical spin liquid, considered earlier by Spałek and Wójcik.

$W$ is a 2-electron interaction. This raises the question, whether it can be obtained by a reduction procedure (different than the BCS one) of the interaction derived by Fröhlich. It can be easily verified that this is impossible. More precisely, for the unique possible reduction of momenta, the coupling vanishes, meaning that the nature of $W$ is not phononic.

Introduction of the 4-fermion interaction $V_{MT}$ was justified in Ref. by its possible role as an attraction between pairs in HTSC, mediated by phonons or other quanta. A conjecture put forward in Ref. suggested also that $V_{MT}$ could be expected to arise as one of the higher-order terms of Fröhlich’s expansion of $UH_{e-ph}U^{-1}$. An alternative justification was given in Ref. where $V_{MT}$ was viewed as a BCS-type interaction between quasi-particles of a free gas represented by $W$ written in the form $\sum_{k} \gamma_{k}\epsilon_{k}^{\sigma}c_{k}^{\sigma}$, with $\alpha_{k} = a_{k,+}a_{k,-}$. Both of these ideas essentially exploit the concept of phonon-type mediation of interactions. The significance of this mediation in HTSC was stressed by Wysokiński.

The question of the form of higher-order terms of Fröhlich’s expansion is interesting itself, not only as providing a possible explanation of the MT extension, but, first of all, because these terms could throw some light on further possible extensions of $H_{BCS}$ related to effects of electron-lattice interaction. Since Fermi-liquid theory will remain the foundation of our formalism, we shall focus our interest on effective electron interactions.

It is worth noting that the possible presence of fermion quadruples in superconductors and superfluids was considered in a number of papers. Schneider and Keller measured the various characteristics of some cuprates and Chevrel-phase superconductors, especially concentrating on the relation between the critical temperature and zero temperature condensate density. They noticed that the experimental data for e.g., YBa$_2$Cu$_3$O$_{6.602}$ point to similarities with the behaviour of a dilute Bose gas. As a result they suggested Bose condensation of weakly interacting fermion pairs as a mechanism of transition from normal to superconducting state. Bunkov et al. pointed to presence of fermion quadruples in $^3$He. Their work was devoted to the problem of influence of spatial disorder on the order parameter in superfluid $^4$He. By resorting to the work of Volovik, they suggested that unusual spectra of $^4$He in aerogel could be explained by a process in which impurities tend to destroy the anisotropic correlations of the order parameter, while correlations of higher symmetry can survive (e.g., four-particle correlations). Recently Schneider et al. discovered of half-$h/2e$ magnetic flux in SQUIDs fabricated of bicrystalline YBa$_2$Cu$_3$O$_{7-\delta}$ films. This situation corresponds to the presence of fermion quadruples in the system. Based on this observation, Aligia et al. investigated a model of interface between two superconductors, based on a one dimensional boson lattice model and proposed formation of quartets of electrons.

In Section higher order terms of the expansion of Fröhlich’s transformation $UH_{e-ph}U^{-1}$ are discussed qualitatively in order to exhibit the emerging structure. Owing to the complexity of this procedure, only terms proportional to the third power of the electron-phonon coupling are found in Section. The resulting extended Fröhlich Hamiltonian is transformed in the next section by a second Fröhlich-type transformation, which produces 4-electron terms. These are discussed in detail in Section. In particular, a reduction similar to the BCS one is performed, which yields an interaction of the form $V_{MT}$ in Eq. The expression for $g_{kk'}$ is derived. Detailed analysis of this expression is performed in the next section, in particular, we showed, by applying some approximations, that this expression is negative i.e., $V_{MT}$ is attractive. The original Fröhlich’s transformation and the resulting terms, up to 6th power of the coupling, are commented in Section. The final section contains a discussion, summary and open questions.

II. HIGHER ORDER TERMS OF THE FRÖHLICH’S TRANSFORMATION

Following Fröhlich (see Appendix A for details), let us consider the electron-phonon Hamiltonian:

$$H_{e-ph} = H_0 + H_{int} = \sum_{k\sigma} \varepsilon_{k} a_{k\sigma}^{\dagger} a_{k\sigma} + \sum_{w} \omega_{w} b_{w}^{\dagger} b_{w} + i \sum_{w} D_{w} (b_{w}^{\dagger} \rho_{w}^{*} - b_{w} \rho_{w}),$$

(4)

where

$$\rho_{w} = \sum_{k\sigma} a_{k\sigma}^{\dagger} a_{-w\sigma},$$

(5)

and $a_{k\sigma}$ ($b_{k}$) are fermion (boson) operators. The coupling $D_{w}$ will be assumed small and $\hbar = 1$.

Since the interaction is spin-independent, the spin index will be suppressed. Summation over electron momenta will include summation over spins.
Fröhlich performed a unitary transformation of $H_{e-ph}$ in order to eliminate (as far as possible) the interaction term. The transformed Hamiltonian is

$$H = e^{S^*} H_{e-ph} e^S = H_{e-ph} - [S, H_{e-ph}] + \frac{1}{2} [S, [S, H_{e-ph}]] + \ldots ,$$

(6)

where

$$S = \sum_q S_q = \sum_q (\gamma^*_q b^*_q - \gamma_q b_q) = -S^*,$$

(7)

$$\gamma_q = \sum_k \phi(k, q) a^*_k a_{k-q},$$

(8)

and the unknown function $\phi(k, q) : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{C}$ is adjusted to achieve the cancellation.

Subsequently, a term which is a combination of products, each with $f$ fermion operators and $b$ boson operators will be written as $(f, b)$. Clearly, $f$ will always be even. For example, $H_0$ consists of terms $(2,0)$ and $(0,2)$.

The rhs of Eq. (6) expresses in terms of commutators $[(f_1, b_1), (f_2, b_2)]$. One easily finds that

$$[(f_1, b_1), (f_2, b_2)] = [f_1, f_2] b_1 b_2 + f_2 f_1 [b_1, b_2] = [f_1, f_2] b_2 b_1 + f_1 f_2 [b_1, b_2].$$

(9)

The necessary commutators $[f_1, f_2], [b_1, b_2]$ are given in Appendix A.

According to Eq. (6), the transformation can be performed, given commutators of the form occurring in Eq. (9) with the first argument equal $S$. The latter is a $(2,1)$ expression, hence

$$[S, (f, b)] = [(2,1), (f, b)] = (f, b+1) + (f+2, b-1),$$

(10)

by virtue of Eqs. (9), (11). Clearly, $(f, b-1) = 0$ for $b = 0$.

Based on this ground, Fröhlich obtained the transformed Hamiltonian (see Appendix A):

$$H_F = \sum_k \varepsilon_k n_k - \frac{1}{2} \sum_{kq} \frac{D^2_w (1 + \Delta(k,w)) (1 - \Delta(q,w))}{\varepsilon_{k-q} - \varepsilon_q + \omega_w} (a^*_k a_{k-w} a^*_q a_{q-w} + c.c.).$$

The second term represents an effective interaction between electrons dressed in the phonon field. If $\varepsilon_{k-w} - \varepsilon_q + \omega_w > 0$, this interaction is attractive.

Omission of higher terms in Eq. (6) results in violation of unitarity. The question thus arises whether partial inclusion of these terms (first of all those of 3rd order in the coupling) could improve agreement between theory and experiment.

Following the general rule for the action of $S$ in consecutive orders, expressed by Eq. (11), one easily finds the form of subsequent terms:

$$\begin{align*}
\begin{cases}
(2,0) \\
(0,2)
\end{cases} & \xrightarrow{S^*} \begin{cases}
(2,1) \\
(2,2)
\end{cases} \xrightarrow{S^*} \begin{cases}
(4,0) \\
(2,2)
\end{cases} \xrightarrow{S^*} \begin{cases}
(4,1) \\
(4,2)
\end{cases} \xrightarrow{S^*} \begin{cases}
(6,0) \\
(4,2)
\end{cases} \xrightarrow{S^*} \begin{cases}
(6,1) \\
(6,2)
\end{cases} \xrightarrow{S^*} \begin{cases}
(8,0) \\
(6,2)
\end{cases}.
\end{align*}$$

(11)

In each consecutive step one obtains terms proportional to the next power of the coupling.

From the view-point of Fermi liquid theory, the terms representing effective inter-electron interactions are most interesting. According to Eq. (11), new terms $(6,0)$ (proportional to $D^2_w$), describing 3-electron interactions and $(8,0)$ (proportional to $D^2_w$), representing 4-electron interactions, appear. Attractive interactions of this type leading to the formation of fermion triplets and quadruples, could affect the behaviour of a superconductor. However, the total spin of an electron triplet is nonzero, so such clusters are unstable, as they are not invariant under time inversion. So far, there has been no experimental evidence of such objects.

Most electrons in a superconductor below $T_c$ are paired, so the 4-electron interaction between Cooper pairs can be expected to prevail. Furthermore, quadruples with a total spin equal zero and appropriate 1-electron momenta are stable under time inversion. On the other hand, under Fröhlich’s conditions for convergence of series (6), the effect of the terms $(6,0), (8,0)$ is weaker.

Evaluation of these terms is a complicated procedure. Before doing this, let us first examine the 3rd order corrections.
III. THIRD ORDER OF THE TRANSFORMATION

Let us consider the effect of the first higher orders discarded by Fröhlich, i.e., terms proportional to $D_0^2$. Then the corrected Hamiltonian takes the form

\[ H' = H_0 - (S, H_0) - H_{\text{int}} + \left( \frac{1}{2} S, [S, H_0] - [S, H_{\text{int}}] \right) - \left( \frac{1}{6} S, [S, [S, H_0]] - \frac{1}{2} S, [S, H_{\text{int}}] \right) + \ldots \]  

(12)

The additional two terms in the last bracket are equal, explicitly,

\[ \frac{1}{2} S, [S, H_{\text{int}}] - \frac{1}{6} S, [S, [S, H_0]] = \sum_{qk} A_{qk} b_q^* n_k a_k^* + \sum_{qk} b_q^* B_{qk} a_k^* a_{-q}^* a_{-k} a_{-q} + C_{qk} b_q^* a_k^* a_{-q}^* a_{-k} a_{-q} + c.c. := H_c, \]  

(13)

where

\[ 2A_{qk} = iD_q \phi^*(k, q) + iD_q \phi(k + q, q) + \frac{1}{3} (\varepsilon_{k-q} - \varepsilon_k + \omega_q) |\phi(k, q)|^2 - \frac{1}{3} (\varepsilon_k - \varepsilon_{k+q} + \omega_q) |\phi(k + q, q)|^2 + c.c., \]  

(14)

\[ 2B_{qk} = iD_q \left\{ \phi^*(k', q) \phi^*(k - q, w) - \phi^*(k', q) \phi^*(k, w) + \phi^*(k, w) \phi(k, q) - \phi^*(k - q, w) \phi(k, q) \right\} \]  

\[ + \frac{1}{3} \phi^*(k', q) \left\{ \phi(k, q) \phi^*(k, w) (\varepsilon_k - \varepsilon_{k-q} + \varepsilon_{k'-q} - \varepsilon_{k'-q} - \varepsilon_{k'-q} + \varepsilon_{k'-q}) \right\}, \]  

(15)

\[ 2C_{qk} = iD_q \left\{ \phi(k', q) \phi^*(k + w - q, w) - \phi(k', q) \phi^*(k + w, w) \right\} \]  

\[ - \phi^*(k + w - q, w) \phi^*(k + w, q) + \phi^*(k + w, w) \phi^*(k, q) \right\} \]  

\[ + \frac{1}{3} \phi(k', q) \left\{ \phi^*(k + w, q) \phi^*(k + w - q, w) (\varepsilon_{k'-q} - \varepsilon_k - \varepsilon_{k+q} - \varepsilon_{k+q}) \right\} \]  

\[ + \phi^*(k, q) \phi^*(k + w + q) \varepsilon_k - \varepsilon_{k-q} + \varepsilon_{k'-q} + \varepsilon_{k'-q} \right\}, \]  

(16)

It can be seen that a new phonon index $w$ has appeared in Eq. (13). It results from the commutator of $S$ with the terms $(4, 0)$, so the harmonic approximation has not been violated.

Substitution of the expression (13) into $H'$ yields

\[ H' = H_a + H_b + H_c + H_d + H_e, \]

where

\[ H_a = \frac{1}{2} \sum_k \varepsilon_k n_k + \frac{1}{2} \sum_q \omega_q b_q^* b_q + c.c., \]

\[ H_b = \sum_{qk} b_q^* b_q n_k \left\{ iD_q (\phi(k + q, q) - \phi(k, q)) - \frac{1}{2} (\varepsilon_k - \varepsilon_{k+q} + \omega_q) |\phi(k + q, q)|^2 + \frac{1}{2} (\varepsilon_{k-q} - \varepsilon_k + \omega_q) |\phi(k, q)|^2 \right\} + c.c., \]

(17)

\[ H_c = \sum_{qk} b_q \left( iD_q + (\varepsilon_{k-q} - \varepsilon_k + \omega_q) \phi(k, q) \right) a_k^* a_{k-q} + c.c., \]

(18)

\[ H_d = \sum_{qkk'} \left( iD_q \phi(k, q) + \frac{1}{2} (\varepsilon_{k-q} - \varepsilon_k + \omega_q) \phi(k, q) \phi^*(k', q) \right) a_k^* a_{k-q} a_{k'-q} a_{k'q} + c.c., \]

(19)

$H_b$ is linear in $b_q^* b_q$ but differs from $H_a$, $H_c$ is part of $H_{\text{int}}$ which had been excluded from the transformation to avoid convergence problems, $H_d$ is the 2-electron interaction obtained, whereas $H_e$ contains terms of the form $(4, 1)$ which represent the obtained correction to Fröhlich's Hamiltonian.

The function $\phi(k, q)$ remains in the form (A7), which guarantees minimization of the contribution $H_c$. 


The form of $H'$ bears similarities to that of $H_{e-ph}$. Both $H'$ and $H_{e-ph}$ contain terms describing the interaction of electrons with phonons: the counterpart of $H_{int}$ in $H_{e-ph}$ is $H_e$ in $H'$. To estimate the effect of $H_e$, let us repeatedly apply Fröhlich’s method and perform a second unitary transformation adjusted to eliminate $H_e$ as far as possible. The form of $H_e$ suggests to take

$$S' = \sum_u (\eta_u^* b_u^* - \eta_u b_u + \xi_u^* b_u^* - \xi_u b_u + \zeta_u^* b_u^* - \zeta_u b_u),$$

where

$$\eta_u^* = \sum_{lm} \psi^*(l, m, u) a_l^* a_{m-u} a_m,$$

$$(20)$$

$$\xi_u^* = \sum_{lmt} \chi^*(l, m, t, u) a_{l-u}^* a_{l-t} a_{m-t} a_m,$$

$$(21)$$

$$\zeta_u^* = \sum_{lmt} \varphi^*(l, m, t, u) a_{l-t} a_{m-t} a_{m+u}.$$

$$(22)$$

The explicit form of the functions $\psi$, $\chi$, $\varphi$ will be found below.

To evaluate the rhs, one needs the commutators

$$[S', H_a] = \sum_{ukl} b_k^* \psi^*(l, k, u) (\varepsilon_k - \varepsilon_{k-u} - \omega_u) a_l^* a_{k-u} a_k$$

$$+ \sum_{uklt} b_k^* \chi^*(l, k, t, u) (\varepsilon_k - \varepsilon_{k-t} + \varepsilon_{l-t} - \varepsilon_{l-u} - \omega_u) a_{l-u}^* a_{l-t} a_{k-t} a_k$$

$$+ \sum_{uklt} b_k^* \varphi^*(l, k, t, u) (\varepsilon_{k+u} - \varepsilon_{k-t} + \varepsilon_{l-t} - \varepsilon_{l-u} - \omega_u) a_{l-t} a_{k-t} a_{k+u} + c.c.,$$

where $D_{kq}$ is defined by $H_b = \sum_q b_q^* b_q n_k D_{kq} + c.c.$ in Eq. 17,

$$[S', H_b] = - \sum_{qk} b_k^* n_k (\eta_q^* + \xi_q^* + \zeta_q^*) (D_{kq} + D_{kq}^*) + c.c.,$$

where $D_{kq}$ is defined by $H_b = \sum_q b_q^* b_q n_k D_{kq} + c.c.$ in Eq. 17,

$$[S', H_c] = - \sum_{qk} E_{k,q,a}^* a_{k-q} (\eta_q^* + \xi_q^* + \zeta_q^*)$$

$$+ \sum_{qkl} E_{k,q,b}^* b_q b_k (\psi^*(l, k, q) n_1 (n_{k-q} - n_k) + (\psi^*(k, l, q) - \psi^*(k - q, l, q)) a_l^* a_{k-q} a_{k-q-l} a_l)$$

$$+ \sum_{klt} E_{k,q,c}^* b_q b_k (E_{k-q,t} \chi^*(l, k, t, q) - E_{k-q,t} \chi^*(l, k - q, t, q)) a_{l-t}^* a_{k-t} a_{k-q}$$

$$+ (E_{k-q,t} \chi^*(k + q, l, t, q) - E_{k-q,t} \chi^*(k, l, t, q)) a_{l-t} a_{k-t} a_{k-q}$$

$$+ (E_{k+q-t, q} \varphi^*(l, k + t, q) - E_{k+q-t, q} \varphi^*(k, l + t, q)) a_l^* a_{k-t} a_{k+t}$$

$$+ (E_{k+q-t, q} \varphi^*(k + t, l, q) - E_{k+q-t, q} \varphi^*(k + t - q, l, q)) a_l^* a_{k-t} a_{k+q} + c.c.,$$

IV. THE SECOND TRANSFORMATION
where \( E_{k,q} \) is defined by \( H_e = \sum_{qk} b_q a^*_k a_{k-q} E_{k,q} + c.c. \) in Eq. (13),

\[
[S', H_d] = \sum_{qkk'} b^*_q \left( F_{k', q} + F^*_k, q \right) \left\{ a^*_k a_{k-q} - a^*_q a_{k-q} + \sum_{qk} \left[ \sum_{q'k'} b^*_q \left( F_{k', q} + F^*_k, q \right) \left\{ a^*_k a_{k-q} - a^*_q a_{k-q} \right\} \right] \right\} + c.c.,
\]

where \( F_{k, k'} \) is defined by \( H_d = \sum_{qkk} F_{k, k'} a_{k-q} a_{k'-q} + c.c. \) in Eq. (13),

\[
[S', H_a] = - \sum_{kq} A_{k,q} a^*_k b^*_q \left\{ \left[ \sum_{qk} \left[ \sum_{q'k'} b^*_q \left( F_{k', q} + F^*_k, q \right) \left\{ a^*_k a_{k-q} - a^*_q a_{k-q} \right\} \right] \right] \right\} + c.c.,
\]

where \( A_{k,q}, B_{k,q'}, C_{k,q,k'} \) are given, respectively by Eqs. (14), (15) and (16),

\[
[S', S', H_a] = - \sum_{qkk'} G_{k,k', q} \left\{ \left[ \sum_{qk} \left[ \sum_{q'k'} b^*_q \left( F_{k', q} + F^*_k, q \right) \left\{ a^*_k a_{k-q} - a^*_q a_{k-q} \right\} \right] \right] \right\} + c.c.,
\]

where

\[
G_{k,k', q} = \psi^*(k', k, q) \left( \varepsilon_{k} - \varepsilon_{k-q} - \omega_q \right),
\]

\[
H_{k,k', t,q} = \chi^*(k', k, t, q) \left( \varepsilon_{k} - \varepsilon_{k-t} - \varepsilon_{k'-t} - \omega_q \right),
\]

\[
I_{k,k', t,q} = \varphi^*(k', k, t, q) \left( \varepsilon_{k+t} - \varepsilon_{k-q} - \omega_q \right).
\]

The commutators in the terms containing phonon operators have not been evaluated, as we are interested first of all in expressions containing exclusively electron operators.

Given the transformed Hamiltonian \( \hat{H} \), we are now in position to minimize the effect of \( H_e \) by imposing, similarly as Fröhlich, the condition \( H_e - [S', H_a] = 0 \). This leads to equations for \( \psi, \chi, \varphi \), which determine these functions uniquely, viz,

\[
\psi^*(k', k, q) = \frac{\phi^*(k, q) A_{k, k'}}{\varepsilon_k - \varepsilon_{k-q} - \omega_q}.
\]
After substituting $\phi$ and $A$ given by Eqs. (A7), (13), one obtains

$$
\psi^\ast(k', k, q) = \frac{2iD_q^3}{3(\varepsilon_{k-q} - \varepsilon_k + \omega_q)^2} \left( \frac{1 - \Delta(k, q)}{\varepsilon_{k'-q} - \varepsilon_{k'-q} + \omega_q} - \frac{1 - \Delta(0, q, q)}{\varepsilon_{k'} - \varepsilon_{k'} + \omega_q} \right). 
$$

(29)

Introduction of additional functions in order to preserve convergence is not necessary here, as $\Delta$ already guarantees this property.

As for $\chi$ and $\varphi$, additional functions preserving convergence are indispensable, viz,

$$
\chi^\ast(k', k, t, q) = \frac{B_{k', t, q, k}}{\varepsilon_{k-k} - \varepsilon_{k'-t} - \varepsilon_{k'-q} - \omega_q} \left( 1 - \tilde{\Delta}(k, k', t, q) \right),
$$

(30)

where

$$
\tilde{\Delta}(k, k', t, q) = \begin{cases} 
1, & \text{if } |\varepsilon_{k-k} - \varepsilon_{k'-t} - \varepsilon_{k'-q} - \omega_q| < \tilde{\Gamma}_q \\
0, & \text{if } |\varepsilon_{k-k} - \varepsilon_{k'-t} - \varepsilon_{k'-q} - \omega_q| \geq \tilde{\Gamma}_q.
\end{cases}
$$

(31)

Taking into account Eqs. (A7), (15), one obtains

$$
\chi^\ast(k', k, t, q) = \frac{iD_q^2D_k}{6(\varepsilon_{k-k} - \varepsilon_{k'-t} - \varepsilon_{k'-q} - \omega_q)} \left( \frac{1 - \Delta(k, t))}{\varepsilon_{k-k} + \omega_k} \left( \frac{1 - \Delta(k, t))}{\varepsilon_{k-k} + \omega_k} \right) \frac{1 - \Delta(k', t))}{\varepsilon_{k'-t} - \varepsilon_{k'-t} - \varepsilon_{k'-q} - \omega_q} \left( 1 - \tilde{\Delta}(k', t, q) \right) \right) \left( 2 + \Delta(k', t, q) \right) 
$$

(32)

$$
\varphi^\ast(k', k, t, q) = \frac{C_{k, t, q, k'}}{\varepsilon_{k-k} - \varepsilon_{k'-t} - \varepsilon_{k'} - \omega_q} \left( 1 - \tilde{\Delta}(k, k', t, q) \right),
$$

(33)

where

$$
\tilde{\Delta}(k, k', t, q) = \begin{cases} 
1, & \text{if } |\varepsilon_{k+q} - \varepsilon_{k-k} - \varepsilon_{k'-q} - \omega_q| < \tilde{\Gamma}_q \\
0, & \text{if } |\varepsilon_{k+q} - \varepsilon_{k-k} - \varepsilon_{k'-q} - \omega_q| \geq \tilde{\Gamma}_q,
\end{cases}
$$

(34)

and substitution of $\psi$ and $C_{k, t, q, k'}$ from Eqs. (A7), (15) yields

$$
\varphi^\ast(k', k, t, q) = \frac{iD_q^2D_k}{6(\varepsilon_{k-k} - \varepsilon_{k-k} - \varepsilon_{k'-q} - \omega_q)} \left( \frac{1 - \Delta(k', t))}{\varepsilon_{k-k} + \omega_k} \left( \frac{1 - \Delta(k, t))}{\varepsilon_{k-k} + \omega_k} \right) \frac{1 - \Delta(k+q, t))}{\varepsilon_{k-k} + \omega_k} \left( 2 + \Delta(k+q, t) \right) 
$$

(35)

Having established $\psi$, $\chi$, $\varphi$, let us use average $\hat{H}$ over the phonon vacuum:

$$
\hat{H}_{av} = \frac{1}{2} \sum_k \varepsilon_{k} n_k + \sum_{qkkk'} F_{k,k',q} a_{k-q}^* a_{k-q} a_{k'-q} a_{k'-q} + \sum_{kq} E_{k,q} a_{k-q}^* a_{k-q} \left( \eta_q^* + \xi_q^* + \zeta_q^* \right) + \frac{1}{2} \sum_{qkkk'} \left( A_{k', q} \phi^*_{k', q} - \frac{1}{2} G_{k,k', q} \right) \left\{ \eta_q n_{k-q} a_{k-q}^* a_{k-q} + \xi_q a_{k-q}^* a_{k-q} a_{k-q} + \zeta_q n_{k-q} a_{k-q}^* a_{k-q} \right\} + \frac{1}{2} \sum_{qkkk'} \left( B_{k', t, q, k} - \frac{1}{2} H_{k,k', t, q} \right) \left\{ \eta_q a_{k-q}^* a_{k-q} a_{k-q} + \xi_q a_{k-q}^* a_{k-q} a_{k-q} + \zeta_q n_{k-q} a_{k-q}^* a_{k-q} \right\} + \frac{1}{2} \sum_{qkkk'} \left( C_{k, t, q, k'} - \frac{1}{2} J_{k,k', t, q} \right) \left\{ \eta_q a_{k-q}^* a_{k-q} a_{k-q} + \xi_q a_{k-q}^* a_{k-q} a_{k-q} + \zeta_q n_{k-q} a_{k-q}^* a_{k-q} \right\} + c.c.,
$$

(36)
\( \hat{H}_{av} \) contains the free-electron term, the 2-electron Fröhlich interaction and terms representing 3-electron and 4-electron interactions. The 3-electron terms were generated by the second transformation of \( H_c \). The source of these terms is thus the non-transformed part of the original interaction, discarded by BCS. If higher-order corrections to BCS theory are of interest, all terms arising from that part, i.e., the 3-electron ones, can be therefore neglected. The same conclusion was drawn above on the grounds of instability.

V. 4-FERMIION INTERACTIONS

The Hamiltonian \( \hat{H}_{av} \) contains several terms representing 4-fermion interactions. Using Eqs. (25), (28), (30), (33), one finds

\[
A_{k',q} \phi^\dagger(k,q) = \frac{1}{2} G_{k,k',q} = \frac{1}{2} \phi^\dagger(k,q) A_{k',q}.
\]

Additionally, taking into account Eqs. (26), (27), we get

\[
B_{k',t,q,k} - \frac{1}{2} H_{k,k',t,q} = \frac{1}{2} B_{k',t,q,k} \left( \hat{\Delta}(k,k',t,q) \right),
\]

\[
C_{k,t,q,k'} - \frac{1}{2} I_{k,k',t,q} = \frac{1}{2} C_{k,t,q,k'} \left( \hat{\Delta}(k,k',t,q) \right).
\]

In terms of \( G_{k,k',q}, H_{k,k',t,q}, I_{k,k',t,q} \) the 4-fermion interactions present in \( \hat{H}_{av} \) express as

\[
H_4^1 = \sum_{qqk'lm} \frac{1}{2} G_{k,k',q} \psi(l,m,q) a_m^* a_m - q n_k a_k^* a_k + c.c.,
\]

\[
H_4^2 = \sum_{qqk'lm} \frac{1}{2} G_{k,k',q} \chi(l,m,t,q) a_m a_m - t a_t^* a_t - q n_k a_k^* a_k + c.c.,
\]

\[
H_4^3 = \sum_{qqk'lm} \frac{1}{2} G_{k,k',q} \varphi(l,m,t,q) a_{m+q} a_{m-t} a_t^* a_t^* - q a_k^* a_k + c.c.,
\]

\[
H_4^4 = \sum_{qqk'lm} \frac{1}{2} H_{k,k',t,q} \psi(l,m,q) a_m^* a_m - q n_k a_k^* a_k - t a_t^* a_k + c.c.,
\]

\[
H_4^5 = \sum_{qqk'lmw} \frac{1}{2} H_{k,k',t,q} \chi(l,m,w,q) a_m^* a_m - w a_w^* a_w^* - q a_k^* a_k - t a_t^* a_k + c.c.,
\]

\[
H_4^6 = \sum_{qqk'lmw} \frac{1}{2} H_{k,k',t,q} \varphi(l,m,w,q) a_{m+w} a_{m-w} a_w^* a_w^* - q a_k^* a_k - t a_t^* a_k + c.c.,
\]

\[
H_4^7 = \sum_{qqk'lm} \frac{1}{2} I_{k,k',t,q} \psi(l,m,q) a_m^* a_m - q n_k a_k^* a_k + c.c.,
\]

\[
H_4^8 = \sum_{qqk'lmw} \frac{1}{2} I_{k,k',t,q} \chi(l,m,w,q) a_m^* a_m - w a_w^* a_w^* - q a_k^* a_k + c.c.,
\]

\[
H_4^9 = \sum_{qqk'lmw} \frac{1}{2} I_{k,k',t,q} \varphi(l,m,w,q) a_{m+w} a_{m-w} a_w^* a_w^* - q a_k^* a_k + c.c.,
\]
Five of these interactions, viz, $H^1_4$, $H^2_4$, $H^3_4$, $H^4_4$, $H^5_4$, contain the operator $n_k$, therefore they are not reducible to the 4-fermion MT potential. This potential is particularly interesting not only for its possible relevance to the physics of superconductors, but also because the thermodynamics of the Hamiltonian $H_{BCS} + W + V_{MT}$ is exactly solvable (Brankov et al).

Let us consider the 8-fold product of operators in $H^5_4$ with explicit spin indices:

$$a^{+}_{m \sigma \tau} a^{+}_{m \sigma' \tau'} a^{-}_{1 \sigma''} a^{-}_{1 \sigma'''},$$

With respect to 1-fermion momenta these are nine possible reductions to the form (3), and each of them allows two or four possibilities related to spin reduction. The values assumed by each momentum index are collected in Table I. In four cases $q = 0$ and the coupling vanishes, since $\chi(l, m, w, 0) = 0$, for all $l$, $m$, $w$. As a consequence, $H^5_4$ assumes the reduced form

$$H^5_{4(red)} = \sum_{k \neq m} \left\{ 2H_k, 2k+m, k+m, 2k+2m \chi(2m + k, m, k + m, 2k + 2m) 
- H_k, m, k + 2m(\chi(2m + k, m, k + m, 2m) + \chi(2m - k, m, k - m, 2m)) 
- \chi(k, m, k + m, 2k) (H_k, 2k + m, k + m, 2k + H_k, 2k - m, k - m, 2k) \right\} a^{+}_{m - k} a^{+}_{m - k} a^{+}_{m + k} a^{+}_{m + k} a^{+}_{k} a^{+}_{k} a^{+}_{k} a^{+}_{k} + c.c..$$

(46)

Terms with $k = m$ have been excluded, similarly as in $V_{BCS}$. Their contribution is accounted for by a shift of 1-fermion energies. The coefficients on the rhs result after performing summation over spins.

This procedure has been also applied to $H^5_4$, $H^5_4$ and $H^5_4$. The corresponding values of 1-fermion momenta are given in Tables II and III. The additional remark $k \rightarrow k - 2m$ (or similar), indicates that a translation of one momentum index is necessary after reduction.

### Table I: Reductions of $H^5_4$

| $H^5_4$ | l | k' | q | t | w | spins |
|--------|---|----|---|---|---|-------|
| 1      | $-k$ | $-m$ | $-2k - 2m$ | $-k - m$ | $k + m$ | $\sigma'' = -\sigma''' = \sigma = -\sigma'$ |
| 2      | $k$ | $-m$ | $-2m$ | $k - m$ | $k + m$ | $\sigma'' = -\sigma''' = -\sigma = \sigma'$ |
| 3      | $-k$ | $4k + m$ | $2k$ | $3k + m$ | $k + m$ | $\sigma'' = -\sigma = -\sigma'' = \sigma'$ |
| 4      | $k$ | $m$ | $0$ | $k + m$ | $k + m$ | $\sigma'' = -\sigma'' = \sigma'' = \sigma''$ |
| 5      | $-k$ | $m$ | $0$ | $k + m$ | $m - k$ | $\sigma'' = -\sigma = \sigma'' = -\sigma$ |
| 6      | $-k$ | $-m$ | $2k$ | $k - m$ | $k + m$ | $\sigma'' = -\sigma = -\sigma = -\sigma''$ |
| 7      | $k$ | $-m$ | $0$ | $k - m$ | $k + m$ | $\sigma'' = -\sigma = -\sigma = \sigma''$ |
| 8      | $-k$ | $-m$ | $0$ | $k - m$ | $m - k$ | $\sigma'' = -\sigma'' = \sigma'' = -\sigma''$ |

### Table II: Reductions of $H^6_4$

| $H^6_4$ | l | k' | q | t | w | spins |
|--------|---|----|---|---|---|-------|
| 1      | $-k$ | $-m$ | $-2k - 2m$ | $-k - m$ | $k + m$ | $\sigma'' = -\sigma''' = \sigma = -\sigma'$ |
| 2      | $k$ | $-m$ | $-2m$ | $k - m$ | $k + m$ | $\sigma'' = -\sigma''' = -\sigma = \sigma'$ |
| 3      | $-k$ | $4k + m$ | $2k$ | $3k + m$ | $k + m$ | $\sigma'' = -\sigma = -\sigma'' = \sigma'$ |
| 4      | $k$ | $m$ | $0$ | $k + m$ | $k + m$ | $\sigma'' = -\sigma'' = \sigma'' = \sigma''$ |
| 5      | $-k$ | $m$ | $0$ | $k + m$ | $m - k$ | $\sigma'' = -\sigma = \sigma'' = -\sigma$ |
| 6      | $-k$ | $-m$ | $2k$ | $k - m$ | $k + m$ | $\sigma'' = -\sigma = -\sigma = -\sigma''$ |
| 7      | $k$ | $-m$ | $0$ | $k - m$ | $k + m$ | $\sigma'' = -\sigma = -\sigma = \sigma''$ |
| 8      | $-k$ | $-m$ | $0$ | $k - m$ | $m - k$ | $\sigma'' = -\sigma'' = \sigma'' = -\sigma''$ |
Collecting all terms in Eqs. (46)–(49), one obtains a $V_{MT}$ interaction as in Eq. (3) with

$$g_{mn} = -4\omega_{2k+2m}A_{km}A_{mk}^* + 2\Theta_{km} + 2\Theta_{mk}^*,$$

After reduction, $H^6_{4}$, $H^8_{4}$, $H^9_{4}$ take the following forms:

$$H^6_{4(\text{red})} = \sum_{k \neq m} \left\{ 2H_{k,k+m,k+m,2k+2m} \varphi(-k, -2k - m, -k - m, 2k + 2m) 
- H_{k,m,k+m,2m}(\varphi(k, -m, k - m, 2m) + \varphi(-k, -m, -m - k, 2m)) 
- \varphi(-k, m - 2k, -k + m, 2k) (H_{k,2k+m,k+m,2k} + H_{k,2k-m,k-m,2k}) \right\} a_m^* a_m + a_m^* a_{-m} + a_{-m}^* a_{-m} - a_{k+k} a_{-k-k} + c.c.,$$

(47)

$$H^8_{4(\text{red})} = \sum_{k \neq m} \left\{ 2I_{-k-2m,-m,-k-m,2k+2m} \chi(k + 2m, m, k + m, 2k + 2m) 
- I_{k-2m,-m,-k-m,2m} \chi(k + 2m, m, k + m, 2m) + \chi(-k + 2m, m, m - k, 2m) 
- \chi(k, m, k + m, 2k) (I_{-k,m,-k,m,2k} + I_{-k,-m,-k,m,2k}) \right\} a_m^* a_{m} + a_m^* a_{-m} + a_{-m}^* a_{-m} - a_{k-k} a_{-k-k} + c.c.,$$

(48)

$$H^9_{4(\text{red})} = \sum_{k \neq m} \left\{ 2I_{-k-2m,-m,-k-m,2k+2m} \varphi(-k, -2k - m, -k - m, 2k + 2m) 
- I_{k-2m,-m,-k-m,2m} \varphi(k, -m, k - m, 2m) + \varphi(-k, -m, -m - k, 2m) 
- \varphi(-k, -2k + m, -k + m, 2k) (I_{-k,m,-k,m,2k} + I_{-k,-m,-k,m,2k}) \right\} a_m^* a_m + a_m^* a_{-m} + a_{-m}^* a_{-m} - a_{k+k} a_{-k-k} + c.c.\cdot$$

(49)

Collecting all terms in Eqs. (46)–(49), one obtains a $V_{MT}$ interaction as in Eq. (3) with

$$g_{mn} = -4\omega_{2k+2m}A_{km}A_{mk}^* + 2\Theta_{km} + 2\Theta_{mk}^*,$$

(50)
where
\[ \Lambda_{km} = \chi^*(2k + m, k, k + m, 2k + 2m) + \varphi^*(-m, -k - 2m, -k - m, 2k + 2m), \] (51)
\[ \Theta_{km} = \omega_m \left( \chi^*(m, k + m, 2m) + \varphi^*(-m, k - 2m - k, m, 2m) \right) \]
\[ \times \left( \chi(2m + k, m, k + m, 2m) + \chi(2m - k, m, m - k, 2m) + \varphi(k, -m - k, 2m) + \varphi(-k, -m - k, 2m) \right). \] (52)

VI. 4-FERMION INTERACTION COUPLING

The most important question about the 4-fermion interaction is positive- or negative-valuedness of \( g_{mk} \). In particular, it would be of interest to determine the domain of \( g_{mk} \) in momentum space where it is attractive. Unfortunately, the form of \( g_{mk} \) given by Eq. (51) is extremely complicated, so this problem cannot be resolved in general.

First, let us specify the quantities occurring in Eq. (50). We assume that \( \varepsilon_k = ak^2, \omega_k = bk, \tilde{\Gamma}_k = ck, \Gamma_k = dk \) and \( \tilde{\Gamma}_k = ek \), where \( a, b, c, d, e \) are real, positive constants.

Following Davydov, we apply some approximations in order to estimate \( g_{mk} \). We assume that the most significant contribution to the 4-fermion interaction (analogously as in the BCS theory) comes from 1-electron momenta \( m, k \) which satisfy the condition \( m \approx k \).

The case of exact equality is, of course, excluded (in accord with the restriction on summation in Eqs. (46) – (49)), so we use \( g_{kk} \) only as an abbreviation for \( g_{mk} \) under our approximation \( m \approx k \).

Thus, the interaction coupling \( g_{kk} \) is given by
\[ g_{kk} = -4\omega_{kk} \Lambda_{kk}^* + 2\Theta_{kk} + 2\Theta_{kk}^*. \] (53)
We can thus rewrite Eq. (52) in the form:
\[ g_{kk} = -16\omega_{kk} |\Lambda_{kk}|^2 + 4\Re \Theta_{kk}, \] (54)
where the first term is always nonpositive and \( \Re \) means real part. The second term requires detailed analysis. Detailed calculations are performed in Appendix A. We have found \( g_{kk} \) in all cases for different values of the constants \( a, b, c, d, e \). Most interesting is the case \( b > c, b > d \) and \( b > d \), because the essential part of initial Hamiltonian is transformed under these conditions. Under a further approximation, we find the form of \( g_{kk} \) for \( k \approx k_F \), where \( k_F \) is Fermi momentum. Additionally we put \( c = d = e \) and \( \varepsilon_{k_F} \gg \omega_{k_F} \) (this is at least true for metals). Under these assumptions
\[ \Theta_{k_F} \approx -\frac{D^6_{k_F}}{4\omega_{k_F}^5}, \quad \Lambda_{k_F} \approx \frac{iD^3_{k_F}}{12\omega_{k_F}^3 \varepsilon_{k_F}}, \]
which implies
\[ g_{kk} \approx \frac{-D^6_{k_F}}{\omega_{k_F}^5} < 0. \]

Now we can compare the magnitudes of the coupling constants of 4-fermion interaction and 2-fermion interaction (under the same assumptions). It is shown in Appendix A that \( G_{kk} \approx -D^6_{k_F}/\omega_{k_F}^5 \). Thus the interaction coupling for 4- and 2-fermion (BCS) interactions fulfills the relation \( g_{kk} \approx G^6_{kk} /\omega_{k_F}^5 \). As a consequence, for a strong pairing, there is a significant contribution from 4-fermion interactions.

Moreover, the 4-fermion interaction coupling \( g_{kk} \) is also negative in most cases without imposing the approximations \( c = d = e \) and \( k \approx k_F \). This is shown in Appendix B.

VII. 4-FERMION INTERACTIONS IN FRÖHLICH'S EXPANSION

As demonstrated in Section III, 4-fermion interaction terms appear in sixth order of the expansion of Fröhlich’s original transformation. These terms are derived in Appendix C. The various resulting 4-fermion terms considerably
outnumber those obtained by applying a second transformation in Section\textsuperscript{12} Again a reduction procedure to $V_{MT}$ is possible for all 4-fermion expressions (\ref{eq:4fermion}), which do not contain the particle number operator and possess three phonon indices. However, the resulting couplings are also complicated functions and, therefore, will not be examined in detail.

The disadvantage of this method is appearance of 3-fermion interactions, which in the double transformation method gave a small contribution. Here additional arguments must be used to discard these terms.

VIII. CONCLUDING REMARKS

We have extended Fröhlich’s transformation of $H_{e-ph}$ to higher order terms. This has been done by performing a second transform of the first terms discarded by Fröhlich. The resulting interactions are of 3- and 4-fermion type. The 3-fermion terms can be expected to be inessential in superconductivity because of their instability under time inversion. The 4-fermion terms are, in general, reducible to $V_{MT}$, a BCS-type 4-fermion interaction. The resulting 4-fermion coupling is extremely complicated, but under reasonable approximations it is negative-valued. Moreover, for 1-fermion momenta in the neighbourhood of Fermi momentum, it has the simple form $-D_{k,k}^6/\omega_{k,k}^2 = G_{k,k}^2/\omega_{k,k}^2$, where $G_{k,k}$ is the 2-fermion coupling. This fact implies that 4-fermion interactions are significant for systems with strong pairing and allows to estimate (relative to 2-fermion coupling and phonon energy at Fermi momentum) their magnitude.

As in BCS theory, where the relation between the gap parameter $\Delta$ and coupling constant $G$ of the BCS interaction allows to estimate the magnitude of $G$, it can be expected that detailed thermodynamics of $H_{MT}$ will provide a relation between $|g_{mk}|$ and other parameters of the theory, thereby allowing to estimate the magnitude of $g_{mk}$. Another emerging question is Cooper’s problem\textsuperscript{12} for a bound quadruple in the presence of $V_{MT}$ or $V_{BCS} + V_{MT}$. Kamei and Miyake deal with this question in a recent work\textsuperscript{25}. The double transformation has unveiled the structure of 3-, 4-, 5-fermion interactions. Since they are proportional to 4th, 6th, 8th power of $D_w$, they are relatively weak, so inclusion of these terms, apart from 4-fermion ones, appears unjustified at present, although Schneider et al.\textsuperscript{13} suggest existence of both quadruples and sextets in some HTSC.

The higher-order expansion terms of the transformed $H_{e-ph}$, including quadruplet, sextet etc. interactions can be expected to reveal themselves in materials with extremely strong pairing correlations between spin 1/2 fermions, since the presence of strongly correlated pairs implies at least some kind of weak interaction between them.

On the other hand, our extension of Fröhlich’s transformation shows that if phonon mediation exists in a superconductor, the 4-fermion and 6-fermion interactions are always present as supplementary to the BCS one. Unfortunately, the Hamiltonian $H_{BCS} + V_{MT}$, although exactly solvable, leads to intricate mean-field equations.

Our results obtained can be generalized in many respects. First of all, the process of averaging over phonon vacuum could be replaced by averaging over the phonon equilibrium state, which could be justified at higher temperatures. This would already lead to an additional 1-fermion term in Fröhlich’s Hamiltonian and modification of 2-fermion and 3-fermion terms in our method. Another extension would result by going beyond the harmonic approximation and including all products of phonon operators. Method of Bogoliubov\textsuperscript{26} could be applied to “dangerous terms” (divergent), omitted in Fröhlich’s method. These questions will be dealt with in further investigations.

APPENDIX A: THE FRÖHLICH’S TRANSFORMATION

Let us recall Fröhlich’s method\textsuperscript{8}, using in some details the more elegant approach due to Davydov\textsuperscript{22}. Consider the electron-phonon Hamiltonian:

$$H_{e-ph} = H_0 + H_{\text{int}} = \sum_{k \sigma} \varepsilon_k a_{k \sigma}^* a_{k \sigma} + \sum_{w} \omega_w b_w^* b_w + i \sum_w D_w (b_w \rho_w^* - b_w^* \rho_w),$$

where

$$\rho_w = \sum_{k \sigma} a_{k-w, \sigma}^* a_{k \sigma},$$

and $a_{k \sigma}$ ($b_k$) are fermion (boson) operators. The coupling $D_w$ will be assumed small and $\hbar \equiv 1$.

The form of the interaction term of $H_{e-ph}$ arises under a number of assumptions: the ions of the lattice move collectively, the coupling depends only on $w$ and electrons interact only with longitudinal phonons for which $\omega_w = ws$, $s$ denoting the velocity of sound. Our interest is focused on the behaviour of electrons, therefore variations of the phonon spectrum will be accounted for only through $s$.\textsuperscript{12}
Fröhlich performed a unitary transformation of $H_{e-ph}$ in order to eliminate (as far as possible) the interaction term. The transformed Hamiltonian is

$$H = e^{S^*} H_{e-ph} e^S = H_{e-ph} - [S, H_{e-ph}] + \frac{1}{2} [S, [S, H_{e-ph}]] + \ldots,$$

where

$$S = \sum_q S_q = \sum_q \left( \gamma_q b_q^* - \gamma_q b_q \right) = -S^*, \quad \gamma_q = \sum_k \phi(k, q) a_k^* a_{k-q},$$

and the unknown function $\phi(k, q) : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{C}^+$ is adjusted so as to minimize the contribution of terms (2, 0). Clearly, $f, b$ terms containing $F_r$ Fröhlich additionally assumed that introduction of experimentally measured velocity of sound $s$ allows to discard all terms containing $b_w^2$, $(b_w^*)^2$, $b_w b_v$, $b_v b_w$ if $w \neq v$ etc. In other words, all $(f, b)$ terms for $b \geq 3$ and part of $(f, 2)$ terms are neglected. As a consequence, the expansion (A1) up to 2nd order reduces to:

$$H = H_0 - ([S, H_0] - H_{int}) + \frac{1}{2} [S, [S, H_0]] - [S, H_{int}] + \ldots \tag{A2}$$

Subsequently, a term which is a combination of products, each with $f$ fermion operators and $b$ boson operators will be written as $(f, b)$. Clearly, $f$ will always be even. For example, $H_0$ consists of terms $(2, 0)$ and $(0, 2)$.

The rhs of Eq. (A2) expresses in terms of commutators $[(f_1, b_1), (f_2, b_2)]$. One easily finds that

$$[(f_1, b_1), (f_2, b_2)] = [f_1, f_2] b_1 b_2 + f_2 f_1 [b_1, b_2] = [f_1, f_2] b_2 b_1 + f_1 f_2 [b_1, b_2]. \tag{A3}$$

The necessary commutators $[f_1, f_2], [b_1, b_2]$ are given in Appendix C.

According to Eq. (A1), the transformation can be performed, given commutators of the form occurring in Eq. (A3) with the first argument equal $S$. The latter is a $(2, 1)$ expression, hence

$$[S, (f, b)] = [(2, 1), (f, b)] = (f, b + 1) + (f + 2, b - 1),$$

by virtue of Eqs. (A3), (C1). Clearly $(f, b - 1) = 0$ for $b = 0$.

One finds

$$[S, H_0] = [(2, 1), (2, 0)] + [(2, 1), (0, 2)] = (2, 1),$$

so the term arising from $H_0$ in first order, has the same form as the one arising from $H_{int}$ in zeroth order. These terms are used to eliminate the interaction in $H_{e-ph}$.

Similarly, for $H_{int}$

$$[S, H_{int}] = [(2, 1), (2, 1)] = (2, 2) + (4, 0).$$

Fröhlich additionally assumed that introduction of experimentally measured velocity of sound $s$ allows to discard all terms containing $b_w^2$, $(b_w^*)^2$, $b_w b_v$, $b_v b_w$ if $w \neq v$ etc. In other words, all $(f, b)$ terms for $b \geq 3$ and part of $(f, 2)$ terms are neglected. As a consequence, the expansion (A1) up to 2nd order reduces to:

$$[S, H_0] = \sum_{kq} b_k (\varepsilon_k - \varepsilon_{k-q} - \omega_q) \phi(k, q) a_k^* a_{k-q} + c.c., \tag{A4}$$

$$[S, H_{int}] = \sum_{q} iD_q \gamma_q \rho_q + \sum_{q} \sum_{kq} iD_q b_q^* b_k \phi(k, q) (n_k - n_{k-q}) + c.c., \tag{A5}$$

$$[S, [S, H_0]] = \sum_{kq} (\varepsilon_{k-q} - \varepsilon_k + \omega_q) b_q^* b_k \phi(k, q) |^2 (n_k - n_{k-q}) + \sum_{kq} (\varepsilon_{k-q} - \varepsilon_k + \omega_q) \phi(k, q) a_k^* a_{k-q} \gamma_q^* + c.c. \tag{A6}$$

$\phi$ is now adjusted so as to minimize the contribution of terms $(2, 1)$. This is achieved by the choice

$$\phi(k, q) = \frac{-iD_q}{\varepsilon_{k-q} - \varepsilon_k + \omega_q} (1 - \Delta(k, q)), \tag{A7}$$

where

$$\Delta(k, q) = \begin{cases} 1, & \text{if } |\varepsilon_{k-q} - \varepsilon_k + \omega_q| < \Gamma_q, \\ 0, & \text{if } |\varepsilon_{k-q} - \varepsilon_k + \omega_q| \geq \Gamma_q, \end{cases} \tag{A8}$$
\( \Delta(k, q) \) is introduced to avoid divergence of the series \([A1]\) and \( \Gamma_q \) is positive energy choosing for convergence. Eqs. \([A4] - [A7]\) yield

\[
H = \sum_k \varepsilon_k a_k^* a_k + \sum_w \omega_w b_w^* b_w + \sum_{wk} b_w^* b_w (n_k - n_{k-\omega}) \left( \frac{1}{2} (\varepsilon_{k-w} - \varepsilon_k + \omega_w |\phi(k, w)|^2 - iD_w \phi(k, w) + c.c. \right) \\
+ i \sum_{kw} D_w \left( b_w a_{k-w}^* - b_w^* a_{k-w} \right) \Delta(k, w) - \frac{1}{2} \sum_{kqw} \frac{D_w^2 (1 + \Delta(k, w)) (1 - \Delta(q, w))}{\varepsilon_{q-w} - \varepsilon_q + \omega_w} (a_{k-w}^* a_{k-w} a_{q-w}^* a_q + c.c.) . \tag{A9}
\]

Discarding the non-transformed part of the initial interaction (4th sum on the rhs) and taking the average in phonon vacuum, one obtains the Fröhlich’s Hamiltonian:

\[
H_F = \sum_k \varepsilon_k n_k - \frac{1}{2} \sum_{kqw} \frac{D_w^2 (1 + \Delta(k, w)) (1 - \Delta(q, w))}{\varepsilon_{q-w} - \varepsilon_q + \omega_w} (a_{k-w}^* a_{k-w} a_{q-w}^* a_q + c.c.) . \tag{A10}
\]

The second term represents an effective interaction between electrons dressed in the phonon field. If \( \varepsilon_{q-w} - \varepsilon_q + \omega_w > 0 \), this interaction is attractive.

**APPENDIX B: CALCULATION OF \( g_{kk} \)**

Our objective is to find the explicit form of the following expression:

\[
g_{kk} = -16 \omega_k |\Lambda_{kk}|^2 + 4 \Re \Theta_{kk} . \tag{B1}
\]

Taking into account Eq. \([B2]\), we have

\[
\Theta_{kk} = \omega_k (\chi^*(k, k, 2k, 2k) + \varphi^*(-k, -k, 0, 2k)) \\
\times \left( \chi(3k, k, 2k, 2k) + \chi(k, k, 0, 2k) + \chi(-k, -k, 0, 2k) + \varphi(-k, -k, -2k, 2k) \right)
\]

\[
= \omega_k \chi^*(k, k, 2k, 2k)(\chi(3k, k, 2k, 2k) + \varphi(-k, -k, -2k, 2k)) ,
\tag{B2}
\]

because \( \varphi^*(-k, -k, 0, 2k) = \chi(k, k, 0, 2k) = \varphi(-k, -k, 0, 2k) = 0 \), as can be seen from the explicit form e.g., of the first of these functions (see Eq. \([B3]\)):

\[
\varphi^*(-k, -k, 0, 2k) = -i \frac{D_{2k}^2}{6 \omega_k \omega_0} \left( 1 - \tilde{\Delta}(-k, -k, 0, 2k) \right) (1 - \Delta(k, 2k)) \\
\times \left\{ \left( 1 - \Delta(-k, 0) \right) \left( -4 + \Delta(k, 0) \right) + \left( 1 - \Delta(k, 0) \right) \left( 4 - \Delta(-k, 0) \right) \right\} = 0 ,
\]

because \( \Delta(-k, 0) = \Delta(k, 0) = 1 \), for all \( k \) (see Eq. \([A5]\)). Similarly for \( \chi(k, k, 0, 2k) \) and \( \varphi(-k, -k, 0, 2k) \).

Similarly, from Eq. \([B1]\), we get

\[
\Lambda_{kk} = \chi^*(3k, k, 2k, 4k) + \varphi^*(-k, -3k, -2k, 4k) . \tag{B3}
\]

On the grounds of Eq. \([B3]\), we obtain

\[
\chi^*(k, k, 2k, 2k) = -i \frac{\sqrt{2} D_{2k}^3}{12 \omega_k} \left( 1 - \tilde{\Delta}(k, k, 2k, 2k) \right) \left\{ \frac{3 (1 - \Delta(k, 2k))}{\omega_k^2} \\
- \frac{(1 - \Delta(k, 2k)) (1 - \Delta(-k, 2k))}{2 \omega_k (4 \varepsilon_k + \omega_k)} (2 + \Delta(-k, 2k)) - \frac{(1 - \Delta(-k, 2k))}{2 (4 \varepsilon_k + \omega_k)^2} (4 - \Delta(k, 2k)) \right\} . \tag{B4}
\]

The functions \( \tilde{\Delta} \) and \( \Delta \) are equal 0 or 1, depending on their argument (see Eqs. \([B1], [A5]\) ), so we obtain several conditions. The most important one is

\[
\tilde{\Delta}(k, k, 2k, 2k) = 0 \iff \omega_k > \tilde{\Gamma}_k \iff b > c , \tag{B5}
\]
and \( \chi^*(k, k, 2k, 2k) = 0 \) otherwise. Other conditions are not so strong, because they do not destroy all of parts of Eq. (B4). We have

\[
\Delta(k, 2k) = 0 \iff \omega_k > \Gamma_k \iff b > d, \tag{B6}
\]

\[
\Delta(-k, 2k) = 0 \iff |4\varepsilon_k + \omega_k| > \Gamma_k \iff k > \frac{d - b}{4a}. \tag{B7}
\]

Taking into account Eqs. (B5), (B6) and (B7) we get

- If \( b \leq c \), then \( \chi^*(k, k, 2k, 2k) = 0 \).
- If \( b > c \) and \( b > d \), then

\[
\chi^*(k, k, 2k, 2k) = \frac{-i\sqrt{2}D_k^3\varepsilon_k (5\omega_k + 12\varepsilon_k)}{3\omega_k^2 (\omega_k + 4\varepsilon_k)^2}.
\]

- If \( b > c \), \( b \leq d \) and \( k \in (0, \frac{d-b}{4a}] \), then \( \chi^*(k, k, 2k, 2k) = 0 \).
- If \( b > c \), \( b \leq d \) and \( k \in (\frac{d-b}{4a}, \infty) \), then

\[
\chi^*(k, k, 2k, 2k) = \frac{i\sqrt{2}D_k^3}{8\omega_k (\omega_k + 4\varepsilon_k)^2}.
\]

Proceeding similarly, we get the form of other functions occurring in Eqs. (B2) and (B3). For \( \chi(3k, k, 2k, 2k) \), we have:

- If \( b \leq c \), then \( \chi(3k, k, 2k, 2k) = 0 \).
- If \( b > c \), \( b > d \) and \( k \in (0, \frac{b-d}{4a}) \cup (\frac{b+d}{4a}, \infty) \), then

\[
\chi(3k, k, 2k, 2k) = \frac{i\sqrt{2}D_k^3 (5\omega_k - 12\varepsilon_k)}{3\omega_k^2 (\omega_k - 4\varepsilon_k)^2}.
\]

- If \( b > c \), \( b > d \) and \( k \in [\frac{b-d}{4a}, \frac{b+d}{4a}] \), then

\[
\chi(3k, k, 2k, 2k) = -\frac{i\sqrt{2}D_k^3}{4\omega_k^3}.
\]

- If \( b > c \), \( b \leq d \) and \( k \in (\frac{b+d}{4a}, \infty) \), then \( \chi(3k, k, 2k, 2k) = 0 \).
- If \( b > c \), \( b \leq d \) and \( k \in (\frac{b+d}{4a}, \infty) \), then

\[
\chi(3k, k, 2k, 2k) = \frac{i\sqrt{2}D_k^3}{8\omega_k (\omega_k - 4\varepsilon_k)^2}.
\]

For \( \varphi(-k, -k, -2k, 2k) \):

- If \( b \leq c \), then \( \varphi(-k, -k, -2k, 2k) = 0 \).
- If \( b > c \), \( b > d \) and \( k \in (0, \frac{b-d}{4a}) \cup (\frac{b+d}{4a}, \infty) \), then

\[
\varphi(-k, -k, -2k, 2k) = \frac{i\sqrt{2}D_k^3\varepsilon_k (\omega_k + 12\varepsilon_k)}{3\omega_k^2 (\omega_k - 16\varepsilon_k^2)}.
\]

- If \( b > c \), \( b > d \) and \( k \in [\frac{b-d}{4a}, \frac{b+d}{4a}] \), then

\[
\varphi(-k, -k, -2k, 2k) = -\frac{i\sqrt{2}D_k^3}{4\omega_k^3}.
\]
For $\chi(3k, k, 2k, 4k)$:

- If $b \leq c$, then $\chi(3k, k, 2k, 4k) = 0$.
- If $b > c$, $b < d/3$ and $k \in \left(0, \frac{b-d}{2a}\right]$, then $\chi(3k, k, 2k, 4k) = 0$.
- If $b > c$, $b < d/3$ and $k \in \left(\frac{d-b}{2a}, \frac{d+b}{2a}\right)$, then
  \[
  \chi(3k, k, 2k, 4k) = \frac{-3iD_k^3\varepsilon_k}{4(\omega_k^2 - 16\varepsilon_k^2)(\omega_k^2 - 4\varepsilon_k^2)} := \chi^{(2)}.
  \]

- If $b > d/3 \leq b \leq d$ and $k \in \left(0, \frac{d-b}{2a}\right]$, then $\chi(3k, k, 2k, 4k) = 0$.
- If $b > c$, $d/3 \leq b \leq d$ and $k \in \left(\frac{d-b}{2a}, \frac{d+b}{2a}\right]$, then $\chi(3k, k, 2k, 4k) = \chi^{(1)}$.
- If $b > c$, $d/3 \leq b \leq d$ and $k \in \left(\frac{d+b}{2a}, \infty\right)$, then $\chi(3k, k, 2k, 4k) = \chi^{(2)}$.
- If $b > c$, $d/3 \leq b \leq d$ and $k \in \left(\frac{d-b}{2a}, \frac{b-d}{2a}\right]$, then
  \[
  \chi(3k, k, 2k, 4k) = \frac{iD_k^3\varepsilon_k}{6(\omega_k^2 - 16\varepsilon_k^2)(\omega_k^2 - 4\varepsilon_k^2)} := \chi^{(3)}.
  \]

- If $b > c$, $d/3 < b < 3d$ and $k \in \left[\frac{b-d}{2a}, \frac{b+d}{2a}\right]$, then
  \[
  \chi(3k, k, 2k, 4k) = \frac{iD_k^3(3\omega_k^2 + 4\varepsilon_k)}{24\omega_k^2(\omega_k^2 + 2\varepsilon_k)(\omega_k + 4\varepsilon_k)} := \chi^{(5)}.
  \]

- If $b > c$, $d/3 < b < 3d$ and $k \in \left(0, \frac{b-d}{2a}\right] \cup \left(\frac{b-d}{2a}, \frac{b+d}{2a}\right) \cup \left(\frac{b+d}{2a}, \infty\right)$, then $\chi(3k, k, 2k, 4k) = \chi^{(3)}$.
- If $b > c$, $d/3 < b < 3d$ and $k \in \left[\frac{b-d}{2a}, \frac{b+d}{2a}\right]$, then $\chi(3k, k, 2k, 4k) = \chi^{(4)}$.
- If $b > c$, $d/3 < b < 3d$ and $k \in \left[\frac{b-d}{2a}, \frac{b+d}{2a}\right]$, then $\chi(3k, k, 2k, 4k) = \chi^{(5)}$.

For $\varphi^{*}(-k, -3k, -2k, 4k)$:

- If $b \leq c$, then $\varphi^{*}(-k, -3k, -2k, 4k) = 0$.
- If $b > c$, $b < d/3$ and $k \in \left(0, \frac{d-b}{2a}\right]$, then $\varphi^{*}(-k, -3k, -2k, 4k) = 0$.
- If $b > c$, $b < d/3$ and $k \in \left(\frac{d-b}{2a}, \frac{d+b}{2a}\right]$, then
  \[
  \varphi^{*}(-k, -3k, -2k, 4k) = \frac{iD_k^3}{16\omega_k(\omega_k + 2\varepsilon_k)(\omega_k - 4\varepsilon_k)} := \varphi^{(1)}.
  \]
\begin{itemize}
  \item If \( b > e, b < d/3 \) and \( k \in \left( \frac{d-b}{2a}, \infty \right) \), then
    \[
    \varphi^*(-k, -3k, -2k, 4k) = \frac{iD_k^3 \varepsilon_k}{4(\omega_k^2 - 16\varepsilon_k^2)(\omega_k^2 - 4\varepsilon_k^2)} := \varphi^{(2)}.
    \]
  \item If \( b > e, d/3 \leq b \leq d \) and \( k \in \left( 0, \frac{d+b}{2a} \right] \), then \( \varphi^*(-k, -3k, -2k, 4k) = 0 \).
  \item If \( b > e, d/3 \leq b \leq d \) and \( k \in \left( \frac{d+b}{2a}, \frac{d-b}{2a} \right] \), then \( \varphi^*(-k, -3k, -2k, 4k) = \varphi^{(3)} \).
  \item If \( b > e, d/3 \leq b \leq d \) and \( k \in \left( \frac{d+b}{2a}, \infty \right) \), then \( \varphi^*(-k, -3k, -2k, 4k) = \varphi^{(2)} \).
  \item If \( b > e, d < b \leq 3d \) and \( k \in \left( 0, \frac{b-d}{2a} \right) \cup \left[ \frac{b-d}{2a}, \frac{b+d}{2a} \right) \), then
    \[
    \varphi^*(-k, -3k, -2k, 4k) = \frac{iD_k^3 \varepsilon_k (\omega_k^2 + 16\varepsilon_k^2)}{6\omega_k^2 (\omega_k^2 - 4\varepsilon_k^2)(\omega_k^2 - 16\varepsilon_k^2)} := \varphi^{(3)}.
    \]
  \item If \( b > e, d < b \leq 3d \) and \( k \in \left[ \frac{b-d}{2a}, \frac{b+d}{2a} \right) \), then
    \[
    \varphi^*(-k, -3k, -2k, 4k) = \frac{-iD_k^3 (3\omega_k^2 + 14\varepsilon_k \omega_k + 40\varepsilon_k^2)}{48\omega_k^2 (\omega_k^2 - 4\varepsilon_k^2)(\omega_k + 4\varepsilon_k)} := \varphi^{(4)}.
    \]
  \item If \( b > e, d < b \leq 3d \) and \( k \in \left[ \frac{b-d}{2a}, \frac{b+d}{2a} \right] \), then
    \[
    \varphi^*(-k, -3k, -2k, 4k) = \frac{iD_k^3}{16\omega_k^2 (\omega_k + 2\varepsilon_k)}.
    \]
  \item If \( b > e, d < b \leq 3d \) and \( k \in \left( \frac{b+d}{2a}, \frac{b-d}{2a} \right) \), then
    \[
    \varphi^*(-k, -3k, -2k, 4k) = \frac{iD_k^3 (3\omega_k - 4\varepsilon_k)}{24\omega_k^2 (\omega_k + 2\varepsilon_k)(\omega_k - 4\varepsilon_k)} := \varphi^{(3)}.
    \]
  \item If \( b > e, 3d < b \) and \( k \in \left( 0, \frac{b-d}{2a} \right) \cup \left( \frac{b-d}{2a}, \frac{b+d}{2a} \right) \cup \left( \frac{b+d}{2a}, \infty \right) \), then \( \varphi^*(-k, -3k, -2k, 4k) = \varphi^{(3)} \).
  \item If \( b > e, 3d < b \) and \( k \in \left( \frac{b-d}{2a}, \frac{b+d}{2a} \right] \), then \( \varphi^*(-k, -3k, -2k, 4k) = \varphi^{(4)} \).
  \item If \( b > e, 3d < b \) and \( k \in \left[ \frac{b-d}{2a}, \frac{b+d}{2a} \right) \), then \( \varphi^*(-k, -3k, -2k, 4k) = \varphi^{(5)} \).
\end{itemize}

On the basis of this results and Eq. (B1), we get the 4-fermion interaction coupling \( g_{kk} \):
\begin{itemize}
  \item If \( b \leq c \), then \( g_{kk} = 0 \).
  \item If \( b \leq c, b > e, b < d/3 \) and \( k \in \left( 0, \frac{d-b}{2a} \right] \), then \( g_{kk} = 0 \).
  \item If \( b \leq c, b > e, b < d/3 \) and \( k \in \left( \frac{d-b}{2a}, \frac{d+b}{2a} \right] \), then
    \[
    g_{kk} = \frac{-D_k^6}{16\omega_k^2 (\omega_k + 2\varepsilon_k)^2 (\omega_k - 4\varepsilon_k)^2} := g^{(1)}_{kk} < 0.
    \]
  \item If \( b \leq c, b > e, b < d/3 \) and \( k \in \left( \frac{d+b}{2a}, \infty \right) \), then
    \[
    g_{kk} = \frac{-D_k^6 \omega_k^2 \varepsilon_k^2}{(\omega_k^2 - 4\varepsilon_k^2)^2 (\omega_k^2 - 16\varepsilon_k^2)^2} := g^{(2)}_{kk} < 0.
    \]
  \item If \( b \leq c, b > e, d/3 \leq b \leq d \) and \( k \in \left( 0, \frac{d+b}{4a} \right] \), then \( g_{kk} = 0 \).
  \item If \( b \leq c, b > e, d/3 \leq b \leq d \) and \( k \in \left( \frac{d+b}{4a}, \frac{d+b}{2a} \right] \), then \( g_{kk} = g^{(1)}_{kk} \).
  \item If \( b \leq c, b > e, d/3 \leq b \leq d \) and \( k \in \left( \frac{d+b}{2a}, \infty \right) \), then \( g_{kk} = g^{(2)}_{kk} \).
\end{itemize}
• If \( b \leq c, b > e, d < b \leq 3d \) and \( k \in \left(0, \frac{b-d}{2a}\right) \cup \left[\frac{b+d}{2a}, \infty\right) \), then
\[
g_{kk} = -\frac{4D_k^6 \varepsilon_k^2 \left(\omega_k^2 + 16\varepsilon_k^2\right)^2}{9\omega_k^6 \left(\omega_k^2 - 4\varepsilon_k^2\right)^2 \left(\omega_k^2 - 16\varepsilon_k^2\right)^2} := g_{kk}^{(3)} < 0.
\]

• If \( b \leq c, b > e, d < b \leq 3d \) and \( k \in \left[\frac{b-d}{2a}, \frac{b+d}{2a}\right) \), then
\[
g_{kk} = -\frac{D_k^6 \left(3\omega_k^4 + 14\varepsilon_k \omega_k + 40\varepsilon_k^2\right)^2}{144\omega_k^6 \left(\omega_k^2 - 4\varepsilon_k^2\right)^2 \left(\omega_k + 4\varepsilon_k\right)^2} := g_{kk}^{(4)} < 0.
\]

• If \( b \leq c, b > e, d < b \leq 3d \) and \( k \in \left[\frac{b-d}{2a}, \frac{b+d}{2a}\right) \), then
\[
g_{kk} = -\frac{D_k^6 \left(3\omega_k^2 - 4\varepsilon_k\right)^2}{36\omega_k^6 \left(\omega_k + 2\varepsilon_k\right)^2 \left(\omega_k - 4\varepsilon_k\right)^2} := g_{kk}^{(5)} < 0.
\]

• If \( b \leq c, b > e, 3d < b \) and \( k \in \left(0, \frac{b-d}{2a}\right) \cup \left(\frac{b+d}{2a}, \frac{b-d}{2a}\right) \cup \left(\frac{b+d}{2a}, \infty\right) \), then \( g_{kk} = g_{kk}^{(3)} \).

• If \( b \leq c, b > e, 3d < b \) and \( k \in \left[\frac{b-d}{2a}, \frac{b+d}{2a}\right) \), then \( g_{kk} = g_{kk}^{(4)} \).

• If \( b \leq c, b > e, 3d < b \) and \( k \in \left[\frac{b-d}{2a}, \frac{b+d}{2a}\right) \), then \( g_{kk} = g_{kk}^{(5)} \).

• If \( b > c, b \leq e, b < d/3 \) and \( k \in \left(0, \frac{b+d}{2a}\right) \), then \( g_{kk} = 0 \).

• If \( b > c, b \leq e, b < d/3 \) and \( k \in \left(\frac{b+d}{2a}, \frac{d-b}{2a}\right) \), then
\[
g_{kk} = -\frac{D_k^6 \left(\omega_k^2 + 8\varepsilon_k^2\right)}{16\omega_k \left(\omega_k^2 + 2\varepsilon_k\right)^2 \left(\omega_k^2 - 16\varepsilon_k^2\right)^2} := g_{kk}^{(6)} < 0.
\]

• If \( b > c, b \leq e, b < d/3 \) and \( k \in \left(\frac{d+b}{2a}, \infty\right) \), then
\[
g_{kk} = -\frac{D_k^6 \left(\omega_k^4 + 64\omega_k^2 \varepsilon_k^2 + 16\varepsilon_k^4\right)}{8\omega_k \left(\omega_k^2 - 16\varepsilon_k^2\right)^2 \left(\omega_k^2 - 4\varepsilon_k^2\right)^2} := g_{kk}^{(7)} < 0.
\]

• If \( b > c, b \leq e, d/3 \leq b \leq d \) and \( k \in \left(0, \frac{d-b}{2a}\right) \), then \( g_{kk} = 0 \).

• If \( b > c, b \leq e, d/3 \leq b \leq d \) and \( k \in \left(\frac{d-b}{2a}, \frac{d+b}{4a}\right] \), then
\[
g_{kk} = -\frac{D_k^6 \left(\omega_k^2 + 8\varepsilon_k^2\right)}{16\omega_k \left(\omega_k^2 + 2\varepsilon_k\right)^2 \left(\omega_k^2 - 16\varepsilon_k^2\right)^2} := g_{kk}^{(6)} < 0.
\]

• If \( b > c, b \leq e, d/3 \leq b \leq d \) and \( k \in \left(\frac{d+b}{4a}, \frac{d+b}{2a}\right) \), then \( g_{kk} = g_{kk}^{(6)} \).

• If \( b > c, b \leq e, d/3 \leq b \leq d \) and \( k \in \left(\frac{d+b}{2a}, \infty\right) \), then \( g_{kk} = g_{kk}^{(7)} \).
- If $b > c$, $b \leq e$, $d < b \leq 3d$ and $k \in (0, \frac{b-d}{2a}) \cup (\frac{k+d}{2a}, \infty)$, then
  \[
g_{kk} = -\frac{4D^6_k \varepsilon_k^2 \left(-\omega_k^2 + 464\omega_k^4 \varepsilon_k^2 - 2848\omega_k^2 \varepsilon_k^4 + 4608\varepsilon_k^6\right)}{9\omega_k^6 \left(\omega_k^2 - 4\varepsilon_k^2\right)^2 \left(\omega_k^2 - 16\varepsilon_k^2\right)^2} := g_{kk}^{(9)}.
\]
  We see that $g_{kk}^{(9)} < 0$ iff
  \[\omega_k^2 + 464\omega_k^4 \varepsilon_k^2 - 2848\omega_k^2 \varepsilon_k^4 + 4608\varepsilon_k^6 > 0.
\]
  Taking into account $\omega_k = bk$ and $\varepsilon_k = ak^2$, we obtain a 3rd order algebraic inequality
  \[4608l^3 - 2848xl^2 + 464x^2l - x^3 > 0,
\]
  for $l = a^2k^2$, $x = b^2$. This can be easily solved—there exist only one real root $t_0^{(9)} \approx 0.002x$, so
  \[g_{kk}^{(9)} < 0 \quad \text{iff} \quad k > k_0^{(9)} \approx 0.047b/a < b/4a.
\]
- If $b > c$, $b \leq e$, $d < b \leq 3d$ and $k \in \left[\frac{b-d}{2a}, \frac{b-d}{2a}\right]$, then
  \[
g_{kk} = -\frac{D^6_k \left(9\omega_k^4 + 144\omega_k^3 \varepsilon_k + 784\omega_k^2 \varepsilon_k^2 + 1632\omega_k \varepsilon_k^3 + 1152\varepsilon_k^4\right)}{36\omega_k^5 \left(\omega_k + 2\varepsilon_k\right)^2 \left(\omega_k + 4\varepsilon_k\right)^2} < 0.
\]
- If $b > c$, $b \leq e$, $d < b \leq 3d$ and $k \in \left(\frac{b+d}{4a}, \frac{b+d}{4a}\right]$, then
  \[
g_{kk} = -\frac{D^6_k \left(9\omega_k^4 - 48\omega_k^3 \varepsilon_k - 832\omega_k^2 \varepsilon_k^2 - 2944\omega_k \varepsilon_k^3 + 1664\omega_k^2 \varepsilon_k^4 + 18432\omega_k \varepsilon_k^5 + 18432\varepsilon_k^6\right)}{36\omega_k^5 \left(\omega_k + 2\varepsilon_k\right)^2 \left(\omega_k + 4\varepsilon_k\right)^2} := g_{kk}^{(11)}.
\]
  $g_{kk}^{(11)} < 0$ iff $k \in (0, 0.073b/a) \cup (0.412b/a, \infty)$.
- If $b > c$, $b \leq e$, $b > 3d$ and $k \in \left(0, \frac{b-d}{4a}\right) \cup \left(\frac{k+d}{4a}, \frac{b+d}{2a}\right) \cup (\frac{b+d}{2a}, \infty)$, then $g_{kk} = g_{kk}^{(9)}$.
- If $b > c$, $b \leq e$, $b > 3d$ and $k \in \left[\frac{b-d}{4a}, \frac{b+d}{4a}\right]$, then $g_{kk} = g_{kk}^{(10)}$.
- If $b > c$, $b \leq e$, $b > 3d$ and $k \in \left[\frac{b-d}{2a}, \frac{b+d}{2a}\right]$, then $g_{kk} = g_{kk}^{(11)}$.
- If $b > c$, $b > e$, $b < d/3$ and $k \in (0, \frac{b+d}{4a})$, then $g_{kk} = 0$.
- If $b > c$, $b > e$, $b < d/3$ and $k \in \left(\frac{b+d}{4a}, \frac{b+d}{2a}\right)$, then
  \[g_{kk} = -\frac{D^6_k}{4 \left(\omega_k + 4\varepsilon_k\right)^2 \left(\omega_k + 4\varepsilon_k\right)^2} < 0.
\]
• If \( b > c, b > e, b < d/3 \) and \( k \in \left[ \frac{d-b}{2a}, \frac{b-d}{2a} \right] \), then

\[
g_{kk} = -\frac{D_k^6 \left( \omega_k^2 + 4\omega_k \varepsilon_k + 2\varepsilon_k^2 \right)}{2 \left( \omega_k^2 - 16\varepsilon_k^2 \right) \left( \omega_k + 2\varepsilon_k \right)^2 \left( \omega_k + 4\varepsilon_k \right)} =: g_{kk}^{(12)} < 0.
\]

• If \( b > c, b > e, b < d/3 \) and \( k \in \left( \frac{d-b}{2a}, \infty \right) \), then

\[
g_{kk} = -\frac{D_k^6 \left( \omega_k^2 - 4\omega_k \varepsilon_k^2 + 16\omega_k \varepsilon_k^3 + 16\varepsilon_k^4 \right)}{4 \left( \omega_k^2 - 16\varepsilon_k^2 \right) \left( \omega_k^2 - 4\varepsilon_k^2 \right) \left( \omega_k + 4\varepsilon_k \right)} =: g_{kk}^{(13)}.
\]

\( g_{kk}^{(13)} < 0 \) for all \( k > 0 \).

• If \( b > c, b > e, d/3 \leq b \leq d \) and \( k \in \left( 0, \frac{d-b}{2a} \right] \), then \( g_{kk} = 0 \).

• If \( b > c, b > e, d/3 \leq b \leq d \) and \( k \in \left( \frac{d-b}{2a}, \frac{d+b}{2a} \right] \), then \( g_{kk} = g_{kk}^{(8)} \).

• If \( b > c, b > e, d/3 \leq b \leq d \) and \( k \in \left( \frac{d+b}{2a}, \frac{d-k}{2a} \right] \), then \( g_{kk} = g_{kk}^{(12)} \).

• If \( b > c, b > e, d/3 \leq b \leq d \) and \( k \in \left( \frac{d+b}{2a}, \infty \right) \), then \( g_{kk} = g_{kk}^{(13)} \).

• If \( b > c, b > e, d < b \leq 3d \) and \( k \in \left( 0, \frac{b-d}{2a} \right) \), then

\[
g_{kk} = -\frac{32D_k^6 \varepsilon_k^2}{9\omega_k^5 \left( \omega_k^2 - 16\varepsilon_k^2 \right)^2 \left( \omega_k^2 - 4\varepsilon_k^2 \right)^2 \left( \omega_k + 4\varepsilon_k \right)} \times \left( -3\omega_k^7 - 20\omega_k \varepsilon_k^2 + 84\omega_k \varepsilon_k^3 + 400\omega_k \varepsilon_k^4 - 508\omega_k \varepsilon_k^5 - 2400\omega_k \varepsilon_k^6 + 1152\omega_k \varepsilon_k^7 + 4608\varepsilon_k^2 \right) := g_{kk}^{(14)}.
\]

\( g_{kk}^{(14)} < 0 \) iff \( k \in (0.268b/a, \infty) \).

• If \( b > c, b > e, d < b \leq 3d \) and \( k \in \left[ \frac{b-d}{2a}, \frac{b-d}{2a} \right] \), then

\[
g_{kk} = -\frac{16D_k^6 \varepsilon_k^2 \left( 60\omega_k^5 + 225\omega_k \varepsilon_k^2 - 120\omega_k \varepsilon_k^3 - 752\omega_k \varepsilon_k^4 - 960\omega_k \varepsilon_k^5 + 2304\varepsilon_k^6 \right)}{9\omega_k^5 \left( \omega_k - 2\varepsilon_k \right)^2 \left( \omega_k + 2\varepsilon_k \right)^2} := g_{kk}^{(15)}.
\]

\( g_{kk}^{(15)} < 0 \) for all \( k > 0 \).

• If \( b > c, b > e, d < b \leq 3d \) and \( k \in \left[ \frac{b-d}{2a}, \frac{b-d}{2a} \right] \), then

\[
g_{kk} = -\frac{D_k^6 \left( 81\omega_k^4 + 1320\omega_k \varepsilon_k^3 + 6544\omega_k \varepsilon_k^6 + 13056\omega_k \varepsilon_k^9 + 29216\varepsilon_k^{12} \right)}{9\omega_k^5 \left( \omega_k + 2\varepsilon_k \right)^2 \left( \omega_k + 4\varepsilon_k \right)} < 0.
\]

• If \( b > c, b > e, d < b \leq 3d \) and \( k \in \left( \frac{b+d}{2a}, \frac{b+d}{2a} \right) \), then

\[
g_{kk} = -\frac{D_k^6 \left( 36\omega_k^5 - 336\omega_k \varepsilon_k^2 - 2560\omega_k \varepsilon_k^3 - 3264\omega_k \varepsilon_k^4 + 14592\omega_k \varepsilon_k^5 + 4608\omega_k \varepsilon_k^6 + 36864\varepsilon_k^7 \right)}{9\omega_k^5 \left( \omega_k + 2\varepsilon_k \right)^2 \left( \omega_k + 4\varepsilon_k \right)^2} := g_{kk}^{(16)}.
\]

\( g_{kk}^{(16)} < 0 \) iff \( k \in (0, 0.142b/a) \cup (0.345b/a, \infty) \).

• If \( b > c, b > e, d < b \leq 3d \) and \( k \in \left( \frac{b+d}{2a}, \infty \right) \), then \( g_{kk} = g_{kk}^{(14)} \).

• If \( b > c, b > e, d > 3d \) and \( k \in \left( 0, \frac{b-d}{4a} \right) \cup \left( \frac{b+d}{4a}, \frac{b-d}{2a} \right) \cup \left( \frac{b+d}{2a}, \infty \right) \), then \( g_{kk} = g_{kk}^{(14)} \).

• If \( b > c, b > e, d > 3d \) and \( k \in \left[ \frac{b-d}{4a}, \frac{b+d}{4a} \right] \), then \( g_{kk} = g_{kk}^{(15)} \).

• If \( b > c, b > e, d > 3d \) and \( k \in \left[ \frac{b-d}{2a}, \frac{b+d}{2a} \right] \), then \( g_{kk} = g_{kk}^{(16)} \).
APPENDIX C: HIGHER ORDER TERMS OF FRÖHLICH’S TRANSFORMATION

Evaluation of successive commutators of the expansion \( A^1 \) is a time-consuming task. Below the final results are given for the relevant orders.

Since \( S = B - B^* \) in Eq. 7, all expansion terms have the form \( g + g^* \). We focus on the first summand.

The following formulae for boson operators

\[
[b_1, b_2 b_3] = [b_1, b_2] b_3 + b_2 [b_1, b_3],
\]

\[
[b_1 b_2, b_3 b_4] = b_1 b_3 [b_2, b_4] + b_1 [b_2, b_3] b_4 + b_3 [b_1, b_4] b_3 + [b_1, b_3] b_4 b_2,
\]

and fermion operators

\[
[f_1, f_2 f_3] = \{f_1, f_2\} f_3 - f_2 \{f_1, f_3\},
\]

\[
[f_1 f_2, f_3 f_4] = \{f_1, f_3\} f_2 f_3 - \{f_1, f_4\} f_2 f_1 + f_2 \{f_2, f_3\} f_4 - f_1 f_3 \{f_2, f_4\},
\]

will be used. In particular, for fermion creation and annihilation operators,

\[
[a^*_k a_l, a^*_q a_r] = \delta_{lq} a^*_l a_r - \delta_{kr} a^*_q a_l. \tag{C1}
\]

1. Fourth order

To calculate 4th order terms, let us rewrite the 3rd order expression \( H^{(3)}_{\text{int}} \) as a sum of \( H^{(3)}_{\text{int}} = \frac{1}{2} [S, [S, H_{\text{int}}]] \) and \( H^{(3)}_{\text{int}} = -\frac{1}{6} [S, [S, [S, H_0]]] \):

\[
H^{(3)}_{\text{int},0} = \sum_{\text{qk}} A^{\text{int},0}_{\text{qk}} b^*_q n_k \gamma_q + \sum_{\text{qkwk'}} b^*_w \left\{ B^{\text{int},0}_{\text{kwkw'}} a^*_k \gamma_{k-w} a^*_q \gamma_{k'-q} a^*_k \gamma_{k'w} + C^{\text{int},0}_{\text{kwkw'}} a^*_k \gamma_{k-q} a^*_k \gamma_{k-q} a^*_k \gamma_{k+w} \right\} + \text{c.c.},
\]

with

\[
A^{\text{int}}_{\text{qk}} = \frac{1}{2} i D_q \left( \phi^*(k, q) + \phi(k + q, q) \right) + \text{c.c.},
\]

\[
A^0_{\text{qk}} = \frac{1}{6} (\varepsilon_{k-q} - \varepsilon_k + \omega_q) |\phi(k, q)|^2 - \frac{1}{6} (\varepsilon_k - \varepsilon_{k+q} + \omega_q) |\phi(k + q, q)|^2 + \text{c.c.},
\]

\[
B^{\text{int}}_{\text{kwkw'}} = \frac{1}{2} i D_q \left( \phi^*(k', q) \phi^*(k - q, w) - \phi^*(k', q) \phi^*(k, w) + \phi^*(k, w) \phi(k, q) - \phi^*(k - q, w) \phi(k - w, q) \right),
\]

\[
B^0_{\text{kwkw'}} = \frac{1}{6} \phi^*(k', q) \left\{ \phi(k, q) \phi^*(k, w) (\varepsilon_k - \varepsilon_{k-q} + \varepsilon_{k'-q} - \varepsilon_{k'}) + \phi(k - w, q) \phi^*(k - q, w) (\varepsilon_{k-w-q} - \varepsilon_{k-w} - \varepsilon_{k'-q} + \varepsilon_{k'}) \right\},
\]

\[
C^{\text{int}}_{\text{kwkw'}} = \frac{1}{2} i D_q \left( \phi(k', q) \phi^*(k + w - q, w) - \phi(k', q) \phi^*(k + w, w) - \phi^*(k + w - q, w) \phi(k + w, q) + \phi^*(k + w, w) \phi(k, q) \right),
\]

\[
C^0_{\text{kwkw'}} = \frac{1}{6} \phi(k', q) \left\{ \phi^*(k + w, q) \phi^*(k + w - q, w) (\varepsilon_{k'-q} - \varepsilon_{k'} - \varepsilon_{k+w-q} + \varepsilon_{k+w}) + \phi^*(k, q) \phi^*(k + w, w) (\varepsilon_{k-q} - \varepsilon_k - \varepsilon_{k'-q} + \varepsilon_{k'}) \right\}.
\]
The 4th order term in Eq. (6) equals

\[ H^{(4)} = \frac{1}{6} \left[ S, [S, [S, H_{\text{int}}]] \right] + \frac{1}{24} \left[ S, [S, [S, H_0]] \right] = -\frac{1}{3} \left[ S, H_{\text{int}}^{(3)} \right] - \frac{1}{4} \left[ S, H_0^{(3)} \right] := H^{(4)}_{\text{int}} + H^{(4)}_0. \]

Explicitly,

\[ H^{(4)}_{\text{int},0} = \sum_{kq} A^{(4)}_{\text{int},0} \gamma q a_k a_q^* + \sum_{kqk'} B^{(4)}_{\text{int},0} \gamma w a_k a_w a_k a_{k'} a_{k'} + \sum_{kq} C^{(4)}_{\text{int},0} \gamma w a_k a_{k'} a_q a_{k' + w} + \sum_{kqk'} D^{(4)}_{\text{int},0} b^*_w a_k a_q a_{k'} + \sum_{kq} E^{(4)}_{\text{int},0} b^*_w a_k a_q + \sum_{kqk'} F^{(4)}_{\text{int},0} b^*_w a_k a_{k'} a_q + \sum_{kqk'} G^{(4)}_{\text{int},0} b^*_w a_k a_{k'} a_q + \sum_{kqk'} I^{(4)}_{\text{int},0} b^*_w a_k a_{k'} a_q + c.c., \]

where

\[ A^{(4)} = \frac{1}{3} A^{\text{int}}_{\text{kq}}, \quad A^{(4)}_0 = \frac{1}{4} A^{\text{0}}_{\text{kq}}, \quad B^{(4)}_{\text{int}} = \frac{1}{3} B^{\text{int}}_{\text{kqkwk'}}, \quad B^{(4)}_0 = \frac{1}{4} B^{\text{0}}_{\text{kqkwk'}}. \]

Other coefficients in Eq. (C2) arise according to the same scheme, viz,

\[ C^{(4)}_{\text{int}} = \frac{1}{3} C^{\text{int}}_{\text{kqkwk'}}, \quad D^{(4)}_{\text{int}} = \frac{1}{3} A^{\text{int}}_{\text{kq}} \left( |\phi(k', q)|^2 - |\phi(k' + q, q)|^2 \right), \]

\[ E^{(4)}_{\text{int}} = \frac{1}{3} \left( A^{\text{int}}_{k - q, q} - A^{\text{int}}_{k q} \right) \phi(k, q), \quad F^{(4)}_{\text{int}} = \frac{1}{3} \left( B^{\text{int}}_{k q, w, k'} - \phi(k' - q, w) - B^{\text{int}}_{k q, w, k'} \phi(k', w) \right), \]

\[ G^{(4)}_{\text{int}} = \frac{1}{3} \left( C^{\text{int}}_{k q, w, k'} \phi(k' + w, w) - C^{\text{int}}_{k q, w, k' + w} \phi(k' + w - q, w) \right), \]

\[ I^{(4)}_{\text{int}} = \frac{1}{3} \left( B^{\text{int}}_{k q, w, k'} \phi(k, w) - B^{\text{int}}_{k q, w, k'} \phi(k + w - q, w) + C^{\text{int}}_{k q, w, k'} \phi(k' - q, w) - C^{\text{int}}_{k q, w, k'} \phi(k' + w, w) \right), \]

and similarly for \( C^{(4)}_0, D^{(4)}_0, \) etc.

2. Fifth order

Proceeding analogously as with 4th order, one obtains

\[ H^{(5)} = \frac{1}{24} \left[ S, [S, [S, [S, H_{\text{int}}]]] \right] - \frac{1}{120} \left[ S, [S, [S, [S, H_0]]] \right] = \frac{1}{4} \left[ S, H^{(4)}_{\text{int}} \right] - \frac{1}{5} \left[ S, H^{(4)}_0 \right] := H^{(5)}_{\text{int}} + H^{(5)}_0. \]
\[ H_{\text{int,0}}^{(5)} = \sum_{qkwk'} A_{\text{int,0}}^{(5)} b_{w}^{*} q k a_{qk'} - w a_{qk'} + \sum_{qkw} D_{\text{int,0}}^{(5)} b_{w}^{*} q a_{qk'} - w a_{qk'} + \sum_{qkwk'} C_{\text{int,0}}^{(5)} b_{w}^{*} q k - w a_{qk'} - q k' a_{qk'} \]

\[ + \sum_{qkwk'u} D_{\text{int,0}}^{(5)} b_{w}^{*} q k a_{w} - q k - q k' - q k' + u + \sum_{qkwk'u} E_{\text{int,0}}^{(5)} b_{w}^{*} q k - w a_{w} - q k - q k' - q k' + u \]

\[ + \sum_{qkwk'u} F_{\text{int,0}}^{(5)} b_{w}^{*} q k a_{w} - q k - q k' - q k' + u + \sum_{qkwk'u} G_{\text{int,0}}^{(5)} b_{w}^{*} q k - w a_{qk'} - q k' - q k' + u \]

\[ + \sum_{qkwk'u} H_{\text{int,0}}^{(5)} b_{w}^{*} q k a_{w} - q k - q k' - q k' + u + \sum_{qkwk'u} J_{\text{int,0}}^{(5)} b_{w}^{*} q k a_{w} - q k - q k' - q k' + u \]

\[ + \sum_{qkwk'u} K_{\text{int,0}}^{(5)} b_{w}^{*} q k a_{w} - q k - q k' - q k' + u + \sum_{qkwk'u} L_{\text{int,0}}^{(5)} b_{w}^{*} q k a_{w} - q k - q k' - q k' + u \]

\[ + \sum_{qkwk'u} M_{\text{int,0}}^{(5)} b_{w}^{*} q k a_{w} - q k - q k' - q k' + u + \sum_{qkwk'u} N_{\text{int,0}}^{(5)} b_{w}^{*} q k a_{w} - q k - q k' - q k' + u \]

\[ + \sum_{qkwk'u} O_{\text{int,0}}^{(5)} b_{w}^{*} q k a_{w} - q k - q k' - q k' + u + \sum_{qkwk'u} P_{\text{int,0}}^{(5)} b_{w}^{*} q k a_{w} - q k - q k' - q k' + u \]

\[ + \sum_{qkwk'u} Q_{\text{int,0}}^{(5)} b_{w}^{*} q k a_{w} - q k - q k' - q k' + u + \sum_{qkwk'u} R_{\text{int,0}}^{(5)} b_{w}^{*} q k a_{w} - q k - q k' - q k' + u \]

\[ + \sum_{qkwk'u} S_{\text{int,0}}^{(5)} b_{w}^{*} q k a_{w} - q k - q k' - q k' + u + \sum_{qkwk'u} T_{\text{int,0}}^{(5)} b_{w}^{*} q k a_{w} - q k - q k' - q k' + u \]

\[ + \sum_{qkwk'u} U_{\text{int,0}}^{(5)} b_{w}^{*} q k a_{w} - q k - q k' - q k' + u + \sum_{qkwk'u} W_{\text{int,0}}^{(5)} b_{w}^{*} q k a_{w} - q k - q k' - q k' + u + c.c., \]

where

\[ A_{\text{int,0}}^{(5)} = -\left\{ \frac{1}{4} \right\} A_{\text{int,0}}^{(4)} \times \left( \phi^{*}(k', w) \phi^{*}(k' + q, q) - \phi^{*}(k' + q, w) \phi^{*}(k' + q - w, q) \right), \]

\[ B_{\text{int,0}}^{(5)} = -\left\{ \frac{1}{4} \right\} \left( A_{\text{int,0}}^{(4)} + A_{\text{int,0}}^{(4)*} - A_{\text{int,0}}^{(4)*} - A_{\text{int,0}}^{(4)*} \right) \phi^{*}(k, w), \]

\[ C_{\text{int,0}}^{(5)} = -\left\{ \frac{1}{4} \right\} \left( A_{\text{int,0}}^{(4)} + A_{\text{int,0}}^{(4)*} \right) \phi^{*}(k', w) \phi^{*}(k', q) - \phi^{*}(k' - q, w) \phi^{*}(k' - q, w), \]

\[ D_{\text{int,0}}^{(5)} = -\left\{ \frac{1}{4} \right\} \left( B_{\text{int,0}}^{(4)} \phi^{*}(k + u, u) - B_{\text{int,0}}^{(4)*} \phi^{*}(k + u, u) \right), \]

\[ E_{\text{int,0}}^{(5)} = -\left\{ \frac{1}{4} \right\} \left( B_{\text{int,0}}^{(4)} \phi^{*}(k - w, u) - B_{\text{int,0}}^{(4)*} \phi^{*}(k - w, u) \right), \]

\[ F_{\text{int,0}}^{(5)} = -\left\{ \frac{1}{4} \right\} \left( B_{\text{int,0}}^{(4)} \phi^{*}(l, u) \phi(l, w) - \phi^{*}(l - w, u) \phi(l - u, w) \right), \]

\[ G_{\text{int,0}}^{(5)} = \left\{ \frac{1}{4} \right\} \left( B_{\text{int,0}}^{(4)} \phi^{*}(k' - q, u) - B_{\text{int,0}}^{(4)*} \phi^{*}(k', u) \right), \]

\[ H_{\text{int,0}}^{(5)} = \left\{ \frac{1}{4} \right\} \left( B_{\text{int,0}}^{(4)} \phi^{*}(k - q, u) - B_{\text{int,0}}^{(4)*} \phi^{*}(k - q, u) \right), \]

\[ I_{\text{int,0}}^{(5)} = \left\{ \frac{1}{4} \right\} \left( B_{\text{int,0}}^{(4)} \phi(l + w, u) \phi(l + u, w) - \phi(l, u) \phi(l + u, w) \right). \]
\[ K_{\text{int},0}^{(5)} = - \left\{ \frac{1}{4}, \frac{1}{5} \right\} \left( C_{\text{int},0}^{(4)} \phi^*(k - q, u) - C_{\text{int},0}^{(4)k \rightarrow k-u} \phi^*(k + w, u) \right), \]

\[ L_{\text{int},0}^{(5)} = - \left\{ \frac{1}{4}, \frac{1}{5} \right\} \left( C_{\text{int},0}^{(4)} \phi^*(k', u) - C_{\text{int},0}^{(4)k' \rightarrow k'-u} \phi^*(k' - q, u) \right), \]

\[ M_{\text{int},0}^{(5)} = - \left\{ \frac{1}{4}, \frac{1}{5} \right\} C_{\text{int},0}^{(4)} \left( \phi^*(l, u) \phi(l, w) - \phi^*(l - w, u) \phi(l - u, w) \right), \]

\[ N_{\text{int},0}^{(5)} = \left\{ \frac{1}{4}, \frac{1}{5} \right\} \left( C_{\text{int},0}^{(4)} \phi(k - q, u) - C_{\text{int},0}^{(4)k' \rightarrow k'+u} \phi(k' + u - q, u) \right), \]

\[ O_{\text{int},0}^{(5)} = \left\{ \frac{1}{4}, \frac{1}{5} \right\} \left( C_{\text{int},0}^{(4)} \phi(k' + u, u) - C_{\text{int},0}^{(4)k' \rightarrow k'+u} \phi(k' + u - q, u) \right), \]

\[ P_{\text{int},0}^{(5)} = \left\{ \frac{1}{4}, \frac{1}{5} \right\} C_{\text{int},0}^{(4)} \phi(l + w, u) \phi(l - u + w, w) - \phi(l, u) \phi(l + w, w). \]

\[ R_{\text{int},0}^{(5)} = \left\{ \frac{1}{4}, \frac{1}{5} \right\} \left( D_{\text{int},0}^{(4)} + D_{\text{int},0}^{(4)*} \right), \quad S_{\text{int},0}^{(5)} = \left\{ \frac{1}{4}, \frac{1}{5} \right\} F_{\text{int},0}^{(4)} \phi, \quad T_{\text{int},0}^{(5)} = \left\{ \frac{1}{4}, \frac{1}{5} \right\} \left( F_{\text{int},0}^{(4)} + F_{\text{int},0}^{(4)*k \rightarrow k'} \right), \]

\[ U_{\text{int},0}^{(5)} = \left\{ \frac{1}{4}, \frac{1}{5} \right\} \left( G_{\text{int},0}^{(4)} + G_{\text{int},0}^{(4)*k \rightarrow k'} \right), \quad W_{\text{int},0}^{(5)} = \left\{ \frac{1}{4}, \frac{1}{5} \right\} \left( I_{\text{int},0}^{(4)} + I_{\text{int},0}^{(4)*k \rightarrow k'} \right). \]

1/4 in the curly brackets refering to int terms and 1/5 to 0 terms.

3. Sixth order

\[ H^{(6)} = -\frac{1}{120} [S, [S, [S, [S, [S, [S, H_{\text{int}}]]]]]] + \frac{1}{720} [S, [S, [S, [S, [S, [S, H_0]]]]]] = -\frac{1}{2} [S, H_{\text{int}}^{(5)}] - \frac{1}{6} [S, H_0^{(5)}] := H_{\text{int}}^{(6)} + H_0^{(6)}. \]
\[
H_{\text{int},0}^{(6)} = \left\{ \begin{array}{l}
\frac{1}{5} \frac{1}{6} \right\} \sum_{qkwk'} A_{\text{int},0}^{(5)} \gamma \omega q k q k' \sum_{-w q k w q +} + \sum_{qkwk'} B_{\text{int},0}^{(5)} \gamma \omega q q k q k' + \sum_{qkwk'} C_{\text{int},0}^{(5)} \gamma \omega q k q k' + \sum_{qkwk'} D_{\text{int},0}^{(5)} \gamma \omega q k q k' + \sum_{qkwk'} E_{\text{int},0}^{(5)} \gamma \omega q k q k' + \sum_{qkwk'} F_{\text{int},0}^{(5)} \gamma \omega q k q k' + \sum_{qkwk'} G_{\text{int},0}^{(5)} \gamma \omega q k q k' \end{array} \right. \\
+ \sum_{qkwk'} H_{\text{int},0}^{(5)} \gamma \omega q k q k' + \sum_{qkwk'} J_{\text{int},0}^{(5)} \gamma \omega q k q k' + \sum_{qkwk'} K_{\text{int},0}^{(5)} \gamma \omega q k q k' + \sum_{qkwk'} L_{\text{int},0}^{(5)} \gamma \omega q k q k' + \sum_{qkwk'} M_{\text{int},0}^{(5)} \gamma \omega q k q k' + \sum_{qkwk'} N_{\text{int},0}^{(5)} \gamma \omega q k q k' + \sum_{qkwk'} O_{\text{int},0}^{(5)} \gamma \omega q k q k' + \sum_{qkwk'} P_{\text{int},0}^{(5)} \gamma \omega q k q k' + \sum_{qkwk'} Q_{\text{int},0}^{(5)} \gamma \omega q k q k' + \sum_{qkwk'} R_{\text{int},0}^{(5)} \gamma \omega q k q k' + \sum_{qkwk'} S_{\text{int},0}^{(5)} \gamma \omega q k q k' + \sum_{qkwk'} T_{\text{int},0}^{(5)} \gamma \omega q k q k' + \sum_{qkwk'} U_{\text{int},0}^{(5)} \gamma \omega q k q k' + \sum_{qkwk'} V_{\text{int},0}^{(5)} \gamma \omega q k q k' \right. \\
\left. \sum_{qkwk'} W_{\text{int},0}^{(5)} \gamma \omega q k q k' + \sum_{qkwk'} X_{\text{int},0}^{(5)} \gamma \omega q k q k' + \sum_{qkwk'} Y_{\text{int},0}^{(5)} \gamma \omega q k q k' + \sum_{qkwk'} Z_{\text{int},0}^{(5)} \gamma \omega q k q k' \right. \\
\left. \right\} \\
(\text{C5})
\]

The last term denoting all (6, 2) expressions which are irrelevant after averaging over phonon vacuum. The coefficients are the same as in the 5th order terms, except for the factors 1/5, 1/6.

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this and following inequalities are solved by *Mathematica* package