ON STRONG ARENS IRREGULARITY OF PROJECTIVE TENSOR
PRODUCT OF HILBERT-SCHMIDT SPACE

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Abstract. It was shown in [16] that the Banach algebra $A := S_2(\ell^2) \otimes \gamma S_2(\ell^2)$ is not Arens regular, where $S_2(\ell^2)$ denotes the Banach algebra of the Hilbert-Schmidt operators on $\ell^2$. In this article, employing the notion of limits along ultrafilters, we prove that the irregularity of $S_2(\ell^2) \otimes \gamma S_2(\ell^2)$ is not strong. Along the way, we provide a class of functionals in $A^{**}$ which lie in the topological center but are not in $A$; and, as a consequence, we deduce that $A^{**}$ is not an annihilator Banach algebra with respect to any of the two Arens products.

1. Introduction

For any Banach algebra $A$, Richard Arens (in [1]) defined two products $\Box$ and $\Diamond$ on its bidual space $A^{**}$ such that each product makes $A^{**}$ into a Banach algebra and the canonical isometric inclusion $J : A \to A^{**}$ becomes a homomorphism with respect to both the products. A Banach algebra $A$ is said to be Arens regular if the two products $\Box$ and $\Diamond$ agree on $A^{**}$, i.e. $f \Box g = f \Diamond g$ for all $f, g \in A^{**}$. Soon, people realized that when a Banach algebra is not Arens regular, then, roughly speaking, it can exhibit different “levels” of irregularity. Two such well explored notions are known as strong Arens irregularity (SAI) due to Dales and Lau (see [3, Definition 2.18]) and extreme non-Arens regularity (ENAR) due to Granirer. In this article, we shall concentrate only on strong Arens irregularity.

Briefly speaking, a Banach algebra $A$ is said to be left (resp., right) strongly Arens irregular if its so called left topological center $Z_l^l(A^{**})$ (resp., right topological center $Z_r^r(A^{**})$) equals $A$. And, $A$ is said to be strongly Arens irregular if it is both left and right strongly Arens irregular. Interestingly, on the other extreme, it is known that $A$ is Arens regular if and only if $Z_l^l(A^{**}) = A^{**} = Z_r^r(A^{**})$ - see [5, Page 1]. It thus follows that a Banach algebra is Arens regular as well as strongly Arens irregular if and only if it is reflexive. In particular, the Banach algebra $S_2(\mathcal{H})$ consisting of Hilbert-Schmidt operators on a Hilbert space $\mathcal{H}$ is Arens regular as well as strongly Arens irregular. However, in general, it is a difficult task to realize whether an Arens irregular Banach algebra is strongly Arens irregular or not. Yet, over the years, many known (non-reflexive) Banach algebras have been identified which are strongly Arens irregular and the quest is still on. For instance:

- For a locally compact group $G$, it was proved by Lau and Losert [7] that $L^1(G)$ is always strongly Arens irregular.
- The Fourier algebra $A(G)$ is strongly Arens irregular if $G$ is discrete and amenable - see [8].
- $A(F_2)$ and $A(SU(3))$ are not strongly Arens irregular - see [10].

2020 Mathematics Subject Classification. 46M05, 46B28, 47B10.

Key words and phrases. Banach algebras, Arens regularity, strong Arens irregularity, annihilator, topological centers, Schatten class operators, projective tensor product.

The second named author was supported by the Council of Scientific and Industrial Research (Government of India) through a Senior Research Fellowship with No. 09/263(1133)/2017-EMR-I.
Our interest in the subject grew from the natural question of analyzing the Arens regularity of the projective tensor product of some standard Banach algebras. This project was, in fact, initiated in the 80s by Ülger (see [17]) and over the years it has attracted some good minds and has proved to be quite significant with a good number of quality work already devoted to it - see, for instance, [17, 4, 11, 6] and the references therein. One highlight of this project has been a recent work of Neufang [11], wherein he settled a four-decade-old open problem by proving that the Varopoulos algebra $C(X) \otimes \gamma C(Y)$, for (infinite) compact Hausdorff spaces $X$ and $Y$, is neither Arens regular nor strongly Arens irregular. In fact, Neufang goes on to prove that for arbitrary $C^*$-algebras $A$ and $B$, their projective tensor product $A \otimes \gamma B$ is Arens regular if and only if either $A$ or $B$ has the Phillips property, thereby illustrating a deep relationship that exists between Arens regularity of the projective tensor product and some intrinsic geometric properties of the constituent Banach algebras.

An important class of Banach algebras consists of the Schatten $p$-class operators, $S_p(H)$, on a Hilbert space $H$. They are known to be Arens regular for all $1 \leq p < \infty$ (with respect to multiplication as operator composition). Like the Varopoulos algebras, a natural question that arises is to analyze the Arens regularity of their projective tensor products $S_p(H) \otimes \gamma S_q(H)$ for $1 \leq p, q < \infty$. In [16], the second named author had shown that the Banach algebras $S_p(\ell^2) \otimes \gamma S_q(\ell^2)$ for $1 \leq p, q \leq 2$ are not Arens regular. However, the question whether $S_p(\ell^2) \otimes \gamma S_q(\ell^2)$ is strongly Arens irregular or not remained unanswered. Continuing the work initiated in [16], we now answer this question in the negative for $p = q = 2$. Thus, analogous to the Varopoulos algebra, we now know that the Banach algebra $S_2(\ell^2) \otimes \gamma S_2(\ell^2)$ is neither Arens regular nor strongly Arens regular. Further, via some natural identifications, essentially on similar lines, we observe that the predual $S_1(\ell^2)$ of $B(\ell^2)$ inherits a Banach algebra structure which is neither Arens regular nor strongly Arens regular. (It must be mentioned here that such structures on $S_1(\ell^2)$ are neither new nor unique, as has been illustrated, for instance, in Remark 3.3, 11).

Here is a brief overview of the flow of this article. After a quick section on preliminaries, we first focus in Section 3 on establishing that the Banach algebra $A := S_2(\ell^2) \otimes \gamma S_2(\ell^2)$ is not strongly Arens irregular. The novelty of this paper lies in a judicious exploitation of the notion of a limit along a non-principal ultrafilter, which allows us to provide a class of functionals in $A^{**}$ which lie in the topological center but are not in $A$; thereby establishing that $A$ is not strongly Arens irregular. Also, as another consequence of the technique of limit along a non-principal ultrafilter, we deduce that the bidual space $A^{**}$ is not an annihilator Banach algebra with respect to any of the two Arens products.

2. Preliminaries

2.1. Arens regularity and strong Arens irregularity. For the sake of convenience, we quickly recall the definitions of the two products $\square$ and $\diamond$ mentioned in the Introduction. Let $A$ be a Banach algebra. For $a \in A$, $\omega \in A^*$, $f \in A^{**}$, consider the functionals $\omega a, a \omega \in A^*$, $\omega f, f \omega \in A^{**}$ given by $w_a = (L_a)^*(\omega)$, $\omega w = (R_a)^*(\omega)$; $\omega f(a) = f(\omega a)$ and $f \omega(a) = f(\omega a)$. Then, for $f, g \in A^{**}$ the operations $\square$ and $\diamond$ are given by $(f \square g)(\omega) = f(\omega g)$ and $(f \diamond g)(\omega) = g(\omega f)$ for all $\omega \in A^*$. As recalled in the Introduction, $A$ is said to be Arens regular if the products $\square$ and $\diamond$ are same on $A^{**}$.

Further, the left and the right topological centers of $A$ are defined, respectively, as

\[ Z_t^{(l)}(A^{**}) = \{ \varphi \in A^{**} : \varphi \square \psi = \psi \diamond \psi \text{ for all } \psi \in A^{**} \}; \text{ and} \]

\[ Z_t^{(r)}(A^{**}) = \{ \varphi \in A^{**} : \psi \square \varphi = \psi \diamond \varphi \text{ for all } \psi \in A^{**} \}. \]

It (is known and) can be seen easily that $A \subseteq Z_t^{(l)}(A^{**}) \cap Z_t^{(r)}(A^{**})$.

A Banach algebra $A$ is said to be left (resp., right) strongly Arens irregular if $Z_t^{(l)}(A^{**}) = A$ (resp., $Z_t^{(r)}(A^{**}) = A$). And, $A$ is said to be strongly Arens irregular if it is both left and right
strongly Arens irregular. As mentioned earlier, $A$ is Arens regular if and only if $Z_i^{(l)}(A^{**}) = Z_i^{(r)}(A^{**})$. As a consequence of Goldstine’s Theorem, it is easy to see that any norm dense subset of $A$ is weak*-dense in both $Z_i^{(r)}(A^{**})$ and $Z_i^{(l)}(A^{**})$. We refer the reader to $\textbf{3}$ for further discussion on topological centers.

2.2. von Neumann Schatten classes. For Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and $1 \leq p < \infty$, the $p^{th}$-Schatten class $S_p(\mathcal{H}_1, \mathcal{H}_2)$ denotes the collection
\begin{equation}
S_p(\mathcal{H}_1, \mathcal{H}_2) := \{ T \in B(\mathcal{H}_1, \mathcal{H}_2) : \| T \|_p < \infty \},
\end{equation}
where $\| T \|_p := (\text{Tr}(|T|^p))^{1/p}$ and $\text{Tr}$ denotes the canonical semi-finite positive trace on $B(\mathcal{H}_1)$. It is well known that $S_p(\mathcal{H}_1, \mathcal{H}_2)$ is a subspace of $B(\mathcal{H}_1, \mathcal{H}_2)$, the space of compact operators from $\mathcal{H}_1$ into $\mathcal{H}_2$, and is a Banach space with respect to the Schatten $p$-norm $\| \cdot \|_p$. Further, the finite rank operators are dense in $(S_p(\mathcal{H}_1, \mathcal{H}_2), \| \cdot \|_p)$. For more on Schatten classes, we refer the reader to [12, 19].

Proposition 2.1. Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ be Hilbert spaces. If $R \in B(\mathcal{H}_2)$, $S \in S_p(\mathcal{H}_1, \mathcal{H}_2)$ and $T \in B(\mathcal{H}_1)$, then
\[ \| RST \|_p \leq \| R \| \| S \|_p \| T \|. \]

Remark 2.2. Let $\mathcal{H}, \mathcal{H}_i, i = 1, 2$ be Hilbert spaces.

1. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable, then the Schatten $p$-norm of an operator $T$ in $S_p(\mathcal{H}_1, \mathcal{H}_2)$ can also be expressed as
\[ \| T \|_p = \left( \sum_{i=1}^{\infty} s_i(T)^p \right)^{1/p}, \]
where $\{ s_i(T) : i = 1, 2, \ldots \}$ are the singular values of $T$, i.e., the square roots of the eigenvalues of $|T|$. 

2. $(S_p(\mathcal{H}), \| \cdot \|_p)$ is a Banach $*$-algebra with respect to the composition of operators as multiplication.

3. $(S_2(\mathcal{H}), \| \cdot \|_2)$ consists of the Hilbert-Schmidt operators on $\mathcal{H}$ and forms a Hilbert space with respect to the inner product $\langle T, S \rangle := \text{Tr}(S^* T)$ for $T, S \in S_2(\mathcal{H})$.

For more on Schatten classes, we refer the reader to [12, 19].

2.3. Projective tensor product. Let $A$ and $B$ be Banach algebras. Then, there is a natural multiplication structure on their algebraic tensor product $A \otimes B$ satisfying $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2$ for all $a_i \in A$ and $b_i \in B$, $i = 1, 2$. And, there are various ways to impose a normed algebra structure on $A \otimes B$. For instance, for each $u \in A \otimes B$, its projective norm is given by
\[ \| u \|_\gamma = \inf \left\{ \sum_{i=1}^{n} \| a_i \| \| b_i \| : u = \sum_{i=1}^{n} a_i \otimes b_i \right\}. \]
This norm turns out to be a cross norm, i.e., $\| a \otimes b \|_\gamma = \| a \| \| b \|$ for all $(a, b) \in A \times B$; and, the completion of the normed algebra $A \otimes B$ with respect to this norm is a Banach algebra and is denoted by $A \otimes^\gamma B$. As per our requirements related to the projective tensor product, we shall borrow freely some results from $\textbf{15}$.

2.4. Limits along ultrafilters. We now provide a brief overview of the last (and the most important) ingredient of this paper, namely, the notion of “limits along filters”. Recall that a filter on a set $X$ is a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ such that

1. $X \in \mathcal{F}$;
2. $\emptyset \notin \mathcal{F}$;
(3) whenever $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$; and
(4) $A \cap B \in \mathcal{F}$ for every pair $A, B$ in $\mathcal{F}$.

The cofinite filter on $X$ is the collection of all cofinite subsets of $X$. A filter $\mathcal{F}$ is said to be a principal filter generated by an element $x$, if $\mathcal{F} = \{ A \subseteq X : x \in A \}$. And, a filter $\mathcal{F}$ on a set $X$ is said to be an ultrafilter if for any $A \subseteq X$, either $A$ or $A^c$ is in $\mathcal{F}$. Clearly, every principal filter is an ultrafilter; and, it is known, by Zorn’s Lemma, that a non-principal ultrafilter exists on every infinite set.

Further, given a filter $\mathcal{F}$ on a topological space $Y$, it is said to converge to an element $y$ in $Y$ if every open set $U$ containing $y$ is in $\mathcal{F}$. And, given a filter $\mathcal{F}$ on a set $X$ and a map $f$ from $X$ into a topological space $Y$, $f$ is said to converge along $\mathcal{F}$ to a point $y$ in $Y$ (and denoted as $\lim \mathcal{F} f(x) = y$) if the filter $f_*\mathcal{F} := \{ A \subseteq Y : f^{-1}(A) \in \mathcal{F} \}$ on $Y$ converges to $y$. Note that $f_*\mathcal{F}$ is an ultrafilter on $Y$ if $\mathcal{F}$ is so on $X$. The following well known elementary observations are quite useful:

**Lemma 2.3.**

1. Every non-principal ultrafilter on a set $X$ contains the cofinite filter on $X$.
2. The limit of an ultrafilter on a Hausdorff space, if there exists one, is unique.
3. Every ultrafilter on a compact Hausdorff space converges to a unique point.

3. The Projective Tensor Product of Hilbert-Schmidt Space

Throughout this and the following section, $\mathcal{H}$ will denote an infinite dimensional separable Hilbert space, i.e., $\mathcal{H} \cong \ell^2$; and, $\mathcal{K}$ will denote the Banach algebra $S_2(\mathcal{H})$ consisting of the Hilbert-Schmidt operators on $\mathcal{H}$. Recall that, given any orthonormal basis $\{ e_i : i \in \mathbb{N} \}$ of $\mathcal{H}$, the rank one operators $\{ e_{ij} := e_i \otimes e_j : i, j \in \mathbb{N} \}$ on $\mathcal{H}$ given by $(e_i \otimes e_j)(\xi) = \langle \xi, e_j \rangle e_i$ for $\xi \in \mathcal{H}$, form an orthonormal basis of $\mathcal{K}$. In particular, $\mathcal{K}$ can be identified with $\mathcal{H} \otimes \mathcal{H}$ (Hilbert space tensor product) as Hilbert spaces. Also, in the Banach algebra $S_2(\mathcal{H})$, we have

\[(3.1)\quad e_{ij}e_{kl} = (e_i \otimes e_j) \circ (e_k \otimes e_l) = \delta_{j,k}e_i \otimes e_l = \delta_{j,k}e_{il}\]

for all $i, j, k, l \in \mathbb{N}$.

3.1. Some useful identifications. We shall frequently use some natural identifications that we mention in this subsection.

The identification in the following lemma is a direct adaptation of [15] Corollary 4.11 as per our requirements.

**Lemma 3.1.** There is a surjective conjugate-linear isometry $\theta : \mathcal{K} \otimes \gamma \mathcal{K} \to S_1(\mathcal{K}, \mathcal{K}^*)$ such that $\theta(x \otimes y)(z) = \langle z, x \rangle \langle \cdot, y \rangle$ for all $x, y, z \in \mathcal{K}$.

**Proof.** For each $x, y \in \mathcal{K}$, define $\varphi_{x,y} : \mathcal{K} \to \mathcal{K}^*$ by $\varphi_{x,y}(z) = \langle x, z \rangle \langle \cdot, y \rangle$ for $z \in \mathcal{K}$. Clearly, $\varphi_{x,y} \in S_1(\mathcal{K}, \mathcal{K}^*)$ and $\| \varphi_{x,y} \|_1 = \| x \| \| y \| = \| x \otimes y \|_\gamma$.

Note that, $\mathcal{K}$ and $\mathcal{K}^*$ being Hilbert spaces, it is easily seen that the space of trace-class operators $S_1(\mathcal{K}, \mathcal{K}^*)$ is the same as the space of nuclear operators $\mathcal{N}(\mathcal{K}, \mathcal{K}^*)$ and that the trace-class norm is the same as the nuclear norm. Consider the conjugate Banach space $S_1(\mathcal{K}, \mathcal{K}^*)$. Then, on the lines of [15] Corollary 4.11, it can be shown that the linear mapping

$$\mathcal{K} \otimes \mathcal{K} \ni \sum_i x_i \otimes y_i \mapsto \sum_i \varphi_{x_i,y_i} \in S_1(\mathcal{K}, \mathcal{K}^*)$$

extends to a surjective linear isometric mapping $\overline{\theta} : \mathcal{K} \otimes \gamma \mathcal{K} \to S_1(\mathcal{K}, \mathcal{K}^*)$ such that $\overline{\theta}(x \otimes y)(z) = \varphi_{x,y}(z)$ for all $x, y, z \in \mathcal{K}$. In particular, we obtain a surjective conjugate-linear isometry $\theta : \mathcal{K} \otimes \gamma \mathcal{K} \to S_1(\mathcal{K}, \mathcal{K}^*)$ such that $\theta(x \otimes y)(z) = \langle z, x \rangle \langle \cdot, y \rangle$ for all $x, y, z \in \mathcal{K}$. $\square$
Remark 3.2.  (1) In view of Lemma 3.1, to each \( u \in \mathcal{K} \otimes \mathcal{K} \), we can associate an infinite matrix \([u_{ij,kl}]_{i,j,k,l \in \mathbb{N}}\), namely, the matrix of \( \theta(u) \) with respect to the orthonormal basis \( \{ e_{ij} : i, j \in \mathbb{N} \} \) of \( \mathcal{K} \). Thus, \( u_{ij,kl} := \theta(u)(e_{kl})(e_{ij}) \) for all \( i, j, k, l \in \mathbb{N} \).

Whenever convenient, we shall interchangeably use the matrix \( u = [u_{ij,kl}] \) and the element \( u \in \mathcal{K} \otimes \mathcal{K} \).

(2) There is natural surjective isometry

\[ \vartheta : B(\mathcal{K}, \mathcal{K}^*) \to (\mathcal{K} \otimes \mathcal{K})^* \]

satisfying

\[ \vartheta(f)(x \otimes y) = f(x)(y) \]

for all \( f \in B(\mathcal{K}, \mathcal{K}^*)\), \( x, y \in \mathcal{K} \) - see \([15\text{ pg 24}]\).

(Notice that these identifications are not algebra isomorphisms.)

Lemma 3.3. With notations as in the preceding remark, for each \( u = [u_{ij,kl}]_{i,j,k,l \in \mathbb{N}} \) in \( \mathcal{K} \otimes \mathcal{K} \) and \( m, n \in \mathbb{N} \), the matrix corresponding to the product \( (e_{mn} \otimes e_{mn}) \cdot u \) is given by \([v_{ij,kl}]_{i,j,k,l \in \mathbb{N}}\), where \( v_{ij,kl} = 0 \) if \( i \neq m \) or \( k \neq m \), and \( v_{mj,ml} = u_{nj,nl} \) for all \( l, j \in \mathbb{N} \).

Proof. Recall, from \([15\text{ §2.1}]\), that \( u \) can be expressed as a sum \( u = \sum_{i=1}^{\infty} \beta_i \langle x_i, y_i \rangle \) for some pair of null sequences \( \{ x_i \} \) and \( \{ y_i \} \) in \( \mathcal{K} \), and a sequence of scalars \( \{ \beta_i \} \) satisfying \( \sum_{i=1}^{\infty} |\beta_i| < \infty \). Then, we have

\[ u_{ij,kl} = \theta(u)(e_{kl})(e_{ij}) = \sum_{r=1}^{\infty} \beta_r \langle e_{kl}, x_r \rangle \langle e_{ij}, y_r \rangle \]  

(by Remark 3.2(1))

for all \( i, j, k, l \in \mathbb{N} \). Since, \( (e_{mn} \otimes e_{mn}) \cdot u = \sum_{i=1}^{\infty} \beta_i (e_{mn}x_i) \otimes (e_{mn}y_i) \), we obtain

\[ v_{ij,kl} = \theta((e_{mn} \otimes e_{mn}) \cdot u)(e_{ij}) = \sum_{r=1}^{\infty} \beta_r \langle e_{kl}, e_{mn}x_r \rangle \langle e_{ij}, e_{mn}y_r \rangle \]

for all \( i, j, k, l \in \mathbb{N} \). Notice that, by Equation (3.1), \( \langle e_{kl}, e_{mn}x_r \rangle = \langle e_{mn}e_{kl}, x_r \rangle = 0 \) if \( k \neq m \) and \( \langle e_{ij}, e_{mn}y_r \rangle = \langle e_{mn}e_{ij}, y_r \rangle = 0 \) if \( i \neq m \). Hence, \( v_{ij,kl} = 0 \) if either \( i \neq m \) or \( k \neq m \). Further, it can be easily checked from the above equations that

\[ v_{mj,ml} = \sum_{r=1}^{\infty} \beta_r \langle e_{nl}, x_r \rangle \langle e_{nj}, y_r \rangle = u_{nj,nl} \]

for all \( j, l \in \mathbb{N} \). \( \square \)

3.2. Functionals induced by non-principal ultrafilters. The following observation is the crux of this section.

Proposition 3.4. Let \( A := \mathcal{K} \otimes \mathcal{K} \), \( \mathcal{U} \) be a non-principal ultrafilter on \( \mathbb{N} \) and \( J : A \to A^{**} \) denote the canonical isometry. Then, the following hold:

(1) The sequence \( \{ J(e_{mn} \otimes e_{mn}) \} \) converges along the ultrafilter \( \mathcal{U} \), with respect to the weak*-topology, to a unique element (denoted as) \( \phi_{\mathcal{U}} \in A^{**} \setminus J(A) \), which satisfies

\[ \phi_{\mathcal{U}}(f) = \lim_{f \uparrow \mathcal{U}} \langle f(e_{mn}), e_{nn} \rangle \]

for all \( f \in A^* \).

(2) For each \( f \in A^* \), the function

\[ \mathbb{N} \ni n \mapsto f_{e_{mn} \otimes e_{nn}} \in A^* \]

converges along the ultrafilter \( \mathcal{U} \) to a unique element in the \( w^* \)-compact set \( \{ g \in A^* : \|g\| \leq \|f\| \} \) in \( A^* \) with respect to the \( w^* \)-topology.
(3) We have

\[ (\phi_U \diamond \psi)(f) = \psi \left( \lim_{U} f_{\epsilon_{nn} \otimes \epsilon_{nn}} \right) \text{ and } (\phi_U \Box \psi)(f) = \lim_{U} \psi(f_{\epsilon_{nn} \otimes \epsilon_{nn}}) \]

for all \( \psi \in A^{**} \) and \( f \in A^* \).

Proof. (1): Clearly, the sequence \( \{J(e_{nn} \otimes e_{nn})\} \) is contained in the closed unit ball \( B_1(A^{**}) \), a \( w^* \)-compact set. Hence, by Lemma 2.5 it converges to a unique element along the ultrafilter \( U \), say, \( \phi_U \in B_1(A^{**}) \), with respect to the weak*-topology. Also, we clearly see, via Remark 3.2(2),

\[ \phi_U(f) = \lim_{U} J(e_{nn} \otimes e_{nn})(f) = \lim_{U} f(e_{nn}, e_{nn}) \] for all \( f \in A^* \).

Notice that \( \phi_U \neq 0 \) because \( \phi_{nn}(\eta) = 1 \), where \( \eta : K \to K^* \) is defined as \( \eta(\epsilon_{ij}) = \langle \cdot, \epsilon_{ij} \rangle \) for each \( i, j \in \mathbb{N} \). Further, since \( (K \otimes K)^{**} \cong B(K, K^*) \), in order to show that \( \phi_U \notin J(K \otimes K) \), it suffices to show that \( \phi_U \) vanishes on the space of finite rank operators \( B_{00}(K, K^*) \) whereas \( J(u) \) does not vanish on \( B_{00}(K, K^*) \) for any non-zero \( u \in K \otimes K \).

To see this, let \( f : K \to K^* \) be a finite rank operator defined as \( f(z) = \sum_{i=1}^{k} \lambda_i \langle z, x_i \rangle \langle \cdot, x_i \rangle \), where \( \{x_i\}_{i=1}^{k} \) is an orthonormal set in \( K \) and \( \{\lambda_i\}_{i=1}^{k} \) are some scalars. Note that, \( e_{nn} \to 0 \) weakly in \( K \); so, the function \( \mathbb{N} \ni n \mapsto (x_i, e_{nn}) \in \mathbb{C} \) converges to 0 along the non-principal ultrafilter \( U \) (as \( U \) contains the cofinite filter on \( \mathbb{N} \)). Thus, via the identification \( B(K, K^*)^* \cong (K \otimes K)^{**} \), we obtain

\[ \phi_U(f) = \lim_{U} \sum_{i=1}^{k} \lambda_i |\langle x_i, e_{nn} \rangle|^2 = 0. \]

In particular, \( \phi_U \) vanishes on \( B_{00}(K) \).

Let \( 0 \neq u \in K \otimes K \). Then, the (possibly finite) sequence \( \{\mu_i\} \) consisting of the singular values of \( \theta(u) \in S_1(K, K^*) \) is absolutely summable. Also, by the singular value decomposition of \( \theta(u) \), there exist (possibly finite) orthonormal sequences \( \{x_i\} \) and \( \{y_i\} \) in \( K \) such that \( \theta(u)(x) = \sum_i \mu_i \langle x_i, \cdot \rangle \langle \cdot, y_i \rangle \) for all \( x \) in \( K \), i.e., the sequence of operators

\[ K \ni z \mapsto \sum_{i=1}^{n} \mu_i z_i \otimes y_i \in K^* \]

converges strongly to \( \theta(u) \). In fact, since \( \sum_i \mu_i < \infty \) and since \( \| \cdot \|_{\gamma} \) is a cross-norm, it follows that \( u = \sum_{i=1}^{\infty} \mu_i x_i \otimes y_i \in K \otimes K \). Then, we observe that \( J(u)(x_i \otimes y_i) = \mu_i \neq 0 \) for all \( i \). Hence, \( J(u) \) does not vanish on \( B_{00}(K) \).

(2): Let \( f \in A^* \). Then, we have

\[ \|f_{\epsilon_{nn} \otimes \epsilon_{nn}}\| \leq \|f\| \|e_{nn} \otimes e_{nn}\|_{\gamma} \leq \|f\| \]

for all \( n \in \mathbb{N} \). Since the closed unit ball of \( A^* \) is weak*-compact, it follows from Lemma 4.3 that the function \( \mathbb{N} \ni n \mapsto f_{\epsilon_{nn} \otimes \epsilon_{nn}} \) converges along the ultrafilter \( U \) to a unique element in the weak*-compact set \( \{g \in A^*: \|g\| \leq \|f\|\} \), with respect to the weak*-topology on \( A^* \).

(3): Let \( \phi, \psi \in A^{**} \) and \( f \in A^* \). Notice that, due to double limit criteria

\[ \phi \circ \psi = \lim_{\beta} \lim_{\alpha} f_{\phi_{\alpha} \psi_{\beta}} = \lim_{\alpha} \left( \lim_{\beta} f_{\phi_{\alpha}} \right) \psi_{\beta} = \psi \left( \lim_{\alpha} f_{\phi_{\alpha}} \right) \]

for any pair of nets \( \{J(\phi_{\alpha})\} \) and \( \{J(\psi_{\beta})\} \) in \( J(A) \) converging to \( \phi \) and \( \psi \), respectively, in the weak* topology of \( A^{**} \). Since \( \phi_U = \lim_{U} J(e_{nn} \otimes e_{nn}) \), using (2), we deduce that

\[ (\phi_U \circ \psi)(f) = \psi \left( \lim_{U} f_{\epsilon_{nn} \otimes \epsilon_{nn}} \right). \]
And, for the box product, we have
\[
(\phi_U \Box \psi)(f) = \phi_U(\psi f)
\]
\[
= \lim_{\mathcal{U}} \langle \psi f(e_{nn}), e_{nn} \rangle
\]
\[
= \lim_{\mathcal{U}} \psi f(e_{nn} \otimes e_{nn}) \quad \text{(by Remark 3.2(2))}
\]
\[
= \lim_{\mathcal{U}} \psi(f e_{nn} \otimes e_{nn}).
\]

\[\square\]

3.3. **The main theorem.** We are now all set to establish the main result of this article that the Banach algebra \( A := \mathcal{K} \otimes \mathcal{K} \) is not strongly Arens irregular. We first analyze the annihilating properties of its bidual space \((\mathcal{K} \otimes \mathcal{K})^{**}\) with respect to the two Arens products \(\Box\) and \(\diamond\).

Recall that, for any subset \( S \) of a Banach algebra \( B \), the left and right annihilator ideals of \( S \) in \( B \) are defined, respectively, as
\[
A^r_B(S) = \{ x \in B : xs = 0 \text{ for all } s \in S \}; \quad \text{and}
\]
\[
A^l_B(S) = \{ x \in B : sx = 0 \text{ for all } s \in S \}.
\]

\( B \) is said to be an annihilator Banach algebra if \( A^r_B(B) = (0) = A^l_B(B) \) and for every proper closed left (resp., right) ideal \( I \) (resp., \( J \)), \( A^r_B(I) \neq (0) \neq A^l_B(J) \). Most of the common Banach algebras (such as function spaces and operator algebras) are annihilator algebras. For more, see [12, 13].

An element \( T \in S_1(\mathcal{K}, \mathcal{K}^*) \) is said to have finite support if its matrix \([T]_{ij,kl}(as \text{ in Remark 3.2(1)}) \) has only finitely many non-zero entries.

**Theorem 3.5.** For any non-principal ultrafilter \( \mathcal{U} \) on \( \mathbb{N} \),
\[
A^r_{(\mathcal{A}^r, \circ)}(\{ \phi_U \}) = A^{**}.
\]

**Proof.** Let \( \psi \in A^{**} \) and \( f \in A^* \). We first assert that \( f e_{nn} \otimes e_{nn} \xrightarrow{w} 0 \) as \( n \to \infty \).

To prove this, it is sufficient to show that \( \lim_{n \to \infty} f e_{nn} \otimes e_{nn}(u) = 0 \) for all \( u \in \mathcal{K} \otimes \mathcal{K} \). Notice that \( (e_{nn} \otimes e_{nn}) \cdot u \to 0 \) in norm for all \( u \in \mathcal{K} \otimes \mathcal{K} \) due to Lemma 3.3. Hence, it converges to 0 in norm for all \( u \in \mathcal{K} \otimes \mathcal{K} \), because \( \mathcal{K} \otimes \mathcal{K} \) is dense in \( \mathcal{K} \otimes \mathcal{K} \). Thus, \( f e_{nn} \otimes e_{nn}(u) = f((e_{nn} \otimes e_{nn}) \cdot u) \to 0 \) for all \( u \in \mathcal{K} \otimes \mathcal{K} \); thereby, implying that \( f e_{nn} \otimes e_{nn} \xrightarrow{w} 0 \).

Next, since \( \mathcal{U} \) contains the cofinite filter - see Lemma 2.3, the set \( \{ n : f e_{nn} \otimes e_{nn} \in U \} \), being cofinite, is in \( \mathcal{U} \), for each weak* open neighborhood \( U \) of 0. Hence, 0 is a weak* limit of the function \( \mathbb{N} \ni n \mapsto f e_{nn} \otimes e_{nn} \in A^* \) along the ultrafilter \( \mathcal{U} \). And, since the limit along an ultrafilter in a Hausdorff space is unique, it follows from Equation (3.2) that \( (\phi_U \circ \psi)(f) = 0 \). Since \( \psi \) and \( f \) were arbitrary, this implies that
\[
A^r_{(\mathcal{A}^r, \circ)}(\{ \phi_U \}) = A^{**}.
\]

\[\square\]

Now we turn to the first Arens product \(\Box\) on \((\mathcal{K} \otimes \mathcal{K})^{**}\).

**Theorem 3.6.** For any non-principal ultrafilter \( \mathcal{U} \) on \( \mathbb{N} \),
\[
A^r_{(\mathcal{A}^r, \Box)}(\{ \phi_U \}) = A^{**}.
\]

**Proof.** Let \( \psi \in (\mathcal{K} \otimes \mathcal{K})^{**} \) and \( f \in (\mathcal{K} \otimes \mathcal{K})^* \). Then, from Equation (3.2), we know that
\[
(\phi_U \Box \psi)(f) = \lim_{\mathcal{U}} \psi(f e_{nn} \otimes e_{nn}).
\]
Since \( \mathcal{U} \) contains the co-finite ultrafilter, it can be easily seen that \( \lim_{\mathcal{U}} \psi(f e_{nn} \otimes e_{nn}) \) equals \( \lim_{n \to \infty} \psi(f e_{nn} \otimes e_{nn}) \) if the latter exists. So, in order to show that \( (\phi_U \Box \psi)(f) = 0 \), it suffices to show that \( \lim_{n \to \infty} \psi(f e_{nn} \otimes e_{nn}) = 0 \).
For each \( g \) we define an increasing sequence \( u \) for all \( t \) tends to infinity. Then, there exists an \( \epsilon > 0 \) for which we can choose an increasing sequence \( \{n_t\}_{t=1}^{\infty} \) of natural numbers such that \( |\psi(f_{e_{n_{t+1}}})| > \epsilon \) for all \( t \geq 1 \). Let us define \( g_t \in A^* \) for each natural number \( t \) as \( g_t = c_t \cdot f_{e_{n_{t+1}}}, \) where \( c_t = \frac{\psi(f_{e_{n_{t+1}}})}{|\psi(f_{e_{n_{t+1}}})|} \).

Now, for each \( N \in \mathbb{N} \), let \( h_N := \sum_{t=1}^{N} g_t \). Then,

\[
|h_N(u)| = |f \left( \sum_{t=1}^{N} c_t(e_{n_{t+1}} \otimes e_{n_{t+1}}) \cdot u \right) | \leq ||f|| \left| \sum_{t=1}^{N} (c_t u)_{t_{t+1}} \right|_\gamma
\]

for all \( u \in K \otimes \gamma K \) and \( N \in \mathbb{N} \), where \( u_{n_t} = (e_{n_{t+1}} \otimes e_{n_{t+1}}) \cdot u \).

For each \( N \in \mathbb{N} \), consider the linear operator \( Q_N : K \to K \) satisfying \( Q_N(e_{n_{t+1}}) = c_t e_{n_{t+1}} \) for \( t = 1, 2, ..., N \), \( j \in \mathbb{N} \) and \( Q_N(e_{1}) = 0 \) otherwise. Clearly, \( ||Q_N|| = 1 \). Then, one can easily verify, through the actions on the orthonormal basis \( \{e_{1}\} \), that

\[
\theta \left( \sum_{t=1}^{N} c_t u_{n_t} \right) = Q_N(u) Q_N
\]

for all \( u \in K \otimes \gamma K \) and \( N \in \mathbb{N} \). Thus, from Proposition 2.1 we obtain

\[
\left| h_N(u) \right| \leq |f||u|_\gamma
\]

for all \( u \in K \otimes \gamma K \) and \( N \in \mathbb{N} \). In particular,

\[
\left| h_N(u) \right| \leq |f||u|_\gamma
\]

for all \( u \in K \otimes \gamma K \) and \( N \in \mathbb{N} \). Thus, \( ||h_N|| \leq ||f|| \) for each natural number \( N \).

On the other hand,

\[
\psi(h_N) = \sum_{t=1}^{N} \psi(g_t) = \sum_{t=1}^{N} |\psi(f_{e_{n_{t+1}}})| \quad \text{for all } N \in \mathbb{N}.
\]

Thus, \( \{\psi(h_N) : N \in \mathbb{N} \} \) is an unbounded sequence. But this is absurd because

\[
|\psi(h_N)| \leq ||\psi|| ||h_N|| \leq ||\psi|| ||f||
\]

for all \( N \in \mathbb{N} \). Hence, our assumption was wrong and we must have \( \lim_{n \to \infty} \psi(f_{e_{n_n}}) = 0 \).

Since \( \psi \) and \( f \) were arbitrary, it follows that \( A^* \{\phi_\mathcal{U}\} = A^* \).

It is worth noting that \( K \otimes \gamma K \) does not possess annihilating elements. However, we can deduce from Theorem 3.8 and Theorem 3.9 that its bidual behaves differently with respect to both Arens products.

**Corollary 3.7.** \( (K \otimes \gamma K)^{**} \) is not an annihilator Banach algebra with respect to any of the two Arens products.

**Remark 3.8.** From the preceding discussions, we observe that

\[
A^* \{\phi_\mathcal{U}\} = A^* = A^* \{\phi_\mathcal{U}\}.
\]

This is not very common to see in usual Banach algebras. For example, \( C^* \)-algebras, group algebras \( L^1(G) \), Schatten spaces \( S_p(\mathcal{H}) \) and many other standard Banach algebras do not possess elements which can annihilate the whole algebra.

**Theorem 3.9.** The Banach algebra \( A := K \otimes \gamma K \) is neither Arens regular nor strongly Arens irregular.
Recall that, up to isometric isomorphism, the predual of the von Neumann algebra $S$ given by the space $H$ for all $a$. The predual of $S$ is not Arens regular is known from [16]. To exhibit the failure of strong irregularity, we assert that

$$J(A) + \{ \phi_U : U \text{ is a non-principal ultrafilter on } \mathbb{N} \} \subseteq Z^1(A^{**}).$$

For any non-principal ultrafilter $U$ on $\mathbb{N}$, it follows from Theorem 3.5 and Theorem 3.6 that

$$(\phi_U + J_a) \circ \psi = \phi_U \circ \psi + J_a \circ \psi = 0 + J_a \square \psi = (\phi_U + J_a) \square \psi$$

for all $a \in A$ and $\psi \in A^{**}$. Since non-principal ultrafilters exist on every infinite set, it follows that the left topological center of $A$ contains more than $J(A)$; so, $A$ is not left strongly Arens irregular. In particular, $A$ is not strongly Arens irregular. \hfill $\square$

**Remark 3.10.** Computations as in Proposition 3.4 and Theorem 3.6 yield

$$(\psi \circ \phi_U)(f) = \lim_U \psi(\epsilon_{nn} \otimes e_{nn} f)$$

and

$$(\psi \square \phi_U)(f) = \psi \left( \lim_U \epsilon_{nn} \otimes e_{nn} f \right)$$

for all $\psi \in A^{**}$ and $f \in A^*$. Now, applying symmetrical arguments as in Theorem 3.6 and Theorem 3.3, we can easily conclude that $\phi_U$ is in the left annihilator ideal of $A$ and hence in the right topological center. Hence, $A$ is not right strongly Arens irregular.

### 3.3.1. The predual of $B(\ell^2)$. There is a natural Banach algebra structure on the dual space $K^*$, where the multiplication is given by

$$\langle \cdot, T \rangle \langle \cdot, S \rangle := \langle \cdot, T \circ S \rangle$$

for $T, S \in K$.

Recall that, up to isometric isomorphism, the predual of the von Neumann algebra $B(\mathcal{H})$ is given by the space $S_1(\mathcal{H})$ consisting of the trace-class operators on $\mathcal{H}$; and, it is known that $S_1(\mathcal{H})$ is an Arens regular Banach algebra with respect to the multiplication given by the usual composition of operators.

On the other hand, there exists a natural isometric identification between $S_1(\mathcal{H})$ and the projective tensor product $\mathcal{H}^* \otimes \gamma \mathcal{H}$. Since $\mathcal{H} \cong K$ as Hilbert spaces, from the fact that $K$ is a Banach algebra with respect to the operator composition and the Schatten 2-norm (Remark 2.2), it follows that $S_1(\mathcal{H})$ inherits a canonical Banach algebra structure from that of $K^* \otimes \gamma K$. Using the isometric isomorphism of $K^* \otimes \gamma K$ with $K \otimes \gamma K$, one can conclude that the predual of $B(\mathcal{H})$ is neither Arens regular nor strongly irregular with respect to this new induced multiplication.

**Remark 3.11.**

1. The preceding theorem gives a Banach algebra structure on the predual $S_1(\ell^2)$ of $B(\ell^2)$, which is neither Arens regular nor strongly Arens irregular. This is in contrast to the fact that $S_1(\ell^2)$ with usual operator composition is Arens regular.

2. However, such structures on $S_1(\ell^2)$ are already known to the experts and are not unique. An anonymous expert brought the following example to our notice:

   We know that $\ell^1$ sits as a complemented subspace (known as tensor diagonal) in $X := \ell^2 \otimes \gamma \ell^2$. Thus, $X = \ell^1 \oplus Y$ for some closed space $Y$. We now define a product $\circ$ on $X$ as $(a + y) \circ (b + z) = a \ast b$ for $a, b \in \ell^1$ and $y, z \in Y$, where $\ast$ is the convolution on the semigroup Banach algebra $\ell^1 = \ell^1(\mathbb{N})$. Clearly, $\circ$ is well defined and $X$ is a commutative Banach algebra with respect to this new product.

   Note that, $X$ is not Arens regular, because $\ell^1$ is a non-regular subalgebra of $X$. And, since $Y^{**} \hookrightarrow X^{**}$ and $X \bullet Y = \{0\}$, we observe that $X$ fails to be strongly Arens irregular as well.
Some questions. We conclude by listing a few unresolved natural questions:

1. Is a complete characterization of the topological centers of $K \otimes \gamma K$ (or $K^* \otimes \gamma K$) possible?
2. Is $K^* \otimes \gamma K$ (equivalently, $K \otimes \gamma K$) extremely non-Arens regular?
3. In [10], it was shown that $S_p(\ell^2) \otimes \gamma S_q(\ell^2)$ is not Arens regular for all $1 \leq p, q \leq 2$. So, it still remains to answer whether $S_p(\ell^2) \otimes \gamma S_q(\ell^2)$ is strongly Arens irregular for each pair $1 \leq p, q \leq 2$ with $(p, q) \neq (2, 2)$.
4. Does the predual of an arbitrary von Neumann algebra admit any natural Banach algebra structure? If yes, then what can be said about its Arens regularity?

Acknowledgements. The authors would like to thank the anonymous referee for suggesting some simplifications of some of the proofs.

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