Homogenisation of Monotone Parabolic Problems with Several Temporal Scales

The Detailed arXiv e-Print Version

Jens Persson*

Department of Engineering and Sustainable Development,
Mid Sweden University, SE-831 25 Östersund, Sweden

Abstract

In this paper we homogenise monotone parabolic problems with two spatial scales and finitely many temporal scales. Under a certain well-separatedness assumption on the spatial and temporal scales as explained in the paper, we show that there is an H-limit defined by at most four distinct sets of local problems corresponding to slow temporal oscillations, slow resonant spatial and temporal oscillations (the “slow” self-similar case), rapid temporal oscillations, and rapid resonant spatial and temporal oscillations (the “rapid” self-similar case), respectively.

Keywords: homogenisation, H-convergence, multiscale convergence, parabolic, monotone, several temporal scales

MSC 2000: 35B27

1 Introduction

We will give here a brief survey—with some important references—of homogenisation theory and two-scale convergence techniques which is followed by a statement of the research objective of the present paper. Finally in this section we give a list of notations employed in the paper.

Homogenisation theory. Homogenisation theory is the study of the convergence of—in some suitable sense—sequences of equations involving sequences of

*E-mail: jens.persson@miun.se
operators and (possibly) source functions and the responding sequences of solutions. The main applications involve the study of the convergence of sequences of partial differential equations described by heterogeneous coefficients which become more and more refined such that the problem tends to a homogenised limit. In the case of parabolic partial differential equations the convergence modes used to achieve homogenised limits are the so called G- and H-convergences, where the former is employed when the coefficients can be arranged as a symmetric matrix (see [37, 38]), and the latter is the generalisation which includes non-symmetric matrices (see [26, 27, 43, 44]) and even non-linear problems (see [42]). “Homogenising” a problem means in this context to find the limit in the G- or H-convergence process.

Two-scale convergence. The theory of homogenisation experienced a quantum leap in the late 1980’s when the two-scale convergence technique was introduced (see [30, 1])—effectively replacing Tartar’s method of oscillating test functions (see [42, 43]) as the main tool to achieve G- or H-convergence—and the technique has subsequently improved since then. Two-scale convergence (with generalisations such as multiscale convergence [2], “generalised” two-scale convergence [16, 19], scale convergence [25], $\lambda$-scale convergence [18, 36], $\Sigma$-convergence [31, 32] etc.) is today an indispensable tool to the modern homogenisation theorist.

Aims in the present paper. The main purpose of this paper is to perform homogenisation of monotone, possibly non-linear, parabolic problems of the type

\[
\begin{aligned}
\frac{\partial}{\partial t} u_\epsilon(x,t) - \nabla \cdot a(x,t, \frac{x}{\epsilon}, \frac{t}{\epsilon}, \frac{s_1}{\epsilon}, \ldots, \frac{s_m}{\epsilon}; \nabla u_\epsilon) &= f(x,t) \quad \text{in } \Omega \times (0,T), \\
u_\epsilon(x,0) &= u_0(x) \quad \text{in } \Omega, \\
u_\epsilon(x,t) &= 0 \quad \text{on } \partial \Omega \times (0,T),
\end{aligned}
\]

i.e., having two spatial and $m+1$ temporal scales, where $\Omega$ is an open bounded set in $\mathbb{R}^N$ and $T > 0$. As $\epsilon$ tends to 0 we get a sequence of equations given by (1) above and the objective is to find the homogenised problem, i.e., to find the homogenised limit $b$ of the flux $a$ which defines a homogenised equation which admits a limit $u$ of the sequence of solutions $\{u_\epsilon\}$. In order to homogenise (1) we impose a certain separatedness restriction on the scale functions $\epsilon, \epsilon'_1, \ldots, \epsilon'_m$. The homogenised limit $b$ will not contain any fast spatial or temporal oscillations and (if considered as a function of $\nabla u$) is given in terms of an integral over the local variables $y, s_1, \ldots, s_m$ involving the flux $a$ and a function $u_1$ which is the unique solution of some local problems depending on the behaviour of the scale functions.

We discern four distinct cases giving different local problems for $u_1$, namely the cases (i) $\epsilon^2/\epsilon'_m \to 0$ as $\epsilon \to 0$, (ii) $\epsilon'_m \sim \epsilon^2$, and (iii) $\epsilon'_i/\epsilon^2 \to 0$ but $\epsilon'_{i-1}/\epsilon^2 \to \infty$ as $\epsilon \to 0$ for some $\epsilon'_i$ tending more rapidly to 0 than $\epsilon$ does, and (iv) $\epsilon'_{i-1} \sim \epsilon^2$ for
some $\varepsilon'_{\ell-1} \neq \varepsilon'_m$ tending more rapidly to 0 than $\varepsilon$ does. Case (i) corresponds to slow temporal oscillations (compared to the spatial one), (ii) is the so-called “slow” self-similar case where the spatial and temporal oscillations are in resonance, (iii) corresponds to rapid temporal oscillations, and (iv) is the “rapid” self-similar case.

Notations and conventions. The following notations and conventions are used in this paper:

Spatial and temporal domains. Throughout the paper, $\Omega$ defining the spatial domain is a non-empty open bounded set in $\mathbb{R}^N$ with Lipschitz boundary, and $T > 0$ is the maximal time defining the temporal domain $(0, T)$.

Sets of positive integers. We define the following convenient subsets of $\mathbb{Z}$: for any $0 < i \leq j$ in $\mathbb{R}$, $[i, j] = [i, j] \cap \mathbb{Z}$ (the integers between $i$ and $j$); in particular, $[j] = [1, j]$ (the positive integers up to at most $j$). Moreover, if $i < j$, we define $[0], [j, i] = \emptyset$ (empty sets of positive integers); note that we employ the convention that statements over the empty set are by default always trivially true. Examples: $[2, 4] = \{2, 3, 4\}$, $[42] = \{1, 2, 3\}$, $[4, 2] = \emptyset$, $[10, 3] = \emptyset$, and $\ell > \ell$ for all $\ell \in \emptyset$.

Functions with mean value zero and periodic functions. Let $F(A)/\mathbb{R}$ denote all functions in $F(A)$ with mean value zero over $A \subset \mathbb{R}^M$, and let $F^\#$ denote all locally $F$ functions over $\mathbb{R}^M$ that are periodical repetitions of some functions in $F(Z)$ where $Z = (0, 1)^M$. In particular, $F^\#/\mathbb{R}$ is the set of locally $F$ functions over $\mathbb{R}^M$ with mean value zero over $Z$ which are periodic repetitions of some functions in $F(Z)$.

Tensor product sets. The subset $F_1(A_1) \otimes \cdots \otimes F_k(A_k)$ of the tensor product $\mathcal{F}_1(A_1) \otimes \cdots \otimes \mathcal{F}_k(A_k)$ of function spaces $\mathcal{F}_1(A_1), \ldots, \mathcal{F}_k(A_k)$ is the set of all functions $f$ that can be written as the tensor product

$$f = f_1 \otimes \cdots \otimes f_k,$$

i.e.,

$$f(z_1, \ldots, z_k) = f_1(z_1) \cdots f_k(z_k) \quad (z_i \in A_i, i \in [k]),$$

for some $f_i \in \mathcal{F}_i, i \in [k]$. We say that $\mathcal{F}_1(A_1) \otimes \cdots \otimes \mathcal{F}_k(A_k)$ is a tensor product set (which we note spans the tensor product space). Example: Any function $\psi$ in the tensor product set $D(\Omega) \otimes D(0, T)$ defined on $\Omega \times (0, T)$ can be written as

$$\psi = v \otimes c,$$

i.e.,

$$\psi(x, t) = v(x) c(t) \quad (x \in \Omega, t \in (0, T)).$$
for some \( v \in D(\Omega), c \in D(0,T) \). (Note that \( D = C_0^\infty \), i.e., infinitely differentiable functions with compact support on the set argument.)

**Placement of \( \varepsilon \)-indices.** When \( \varepsilon \) is an upper index it refers to an explicit construction like, e.g.,

\[
\psi^\varepsilon(x,t) = \psi(x,t, \varepsilon_1, \ldots, \varepsilon_x, \varepsilon_1, \ldots, \varepsilon_m),
\]

for functions \( \psi \) defined on, in this case, \( \Omega \times (0,T) \times (0,1)^n \times (0,1)^m \). A lower index form refers to an implicit construction not based on (2); see, e.g., the solution \( u_\varepsilon \) to (1) where \( \varepsilon \) only indirectly defines the function.

**Partial derivatives.** There are two kinds of partial derivatives. The partial derivatives of the first kind, \( \nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N}) \) and \( \frac{\partial}{\partial t} \), only discern whether one differentiates with respect to the space variable \( x = (x_1, \ldots, x_N) \) or the time variable \( t \), respectively. The partial derivatives of the second kind, \( \nabla_x = (\partial_{x_1}, \ldots, \partial_{x_N}) \) and \( \partial_t \) (i.e., with the variable as a subscript) are proper partial derivatives with respect to space and time, respectively. Note that partial derivatives of the local variables will always be of the proper, second kind. Example: Let \( \psi = \psi(x,t,y,s) \) be a weakly differentiable real-valued function with respect to the global space and time variables \( x \) and \( t \) and the local space and time variables \( y \) and \( s \). Suppose \( y = \eta x \) and \( s = \sigma t \) for some real constants \( \eta \) and \( \sigma \), then the chain rule and the conventions above give

\[
\nabla \psi = \nabla_x \psi + \eta \nabla_y \psi \quad \text{and} \quad \frac{\partial}{\partial t} \psi = \partial_t \psi + \sigma \partial_s \psi;
\]

these differentiation rules will be important to keep in mind later in this paper.

**Hilbert space structure.** We use the convention that we work solely in \( L^2 \) and derivations such as \( H^1, H^1_k/R \) etc. rather than in the more general \( L^p, p \geq 1 \), with derivations \( W^{1,p}, W^{1,p}_k/R \) etc. The reason we work in \( L^2 \) is of course due to the fact that it is a Hilbert space which means that the topological dual is also \( L^2 \). Heuristically speaking, Hilbert spaces such as e.g. \( L^2 \) are more “natural” than non-Hilbert spaces since they are generalisations of finite-dimensional vector spaces. The drawback is that we lose some important examples of non-linear problems such as e.g. the evolution \( p \)-Laplacian equation (with \( p \neq 2 \)) which describes non-linear diffusion phenomena and employed in e.g. image processing [45].

### 2 Multiscale Convergence

The concept of two-scale convergence was introduced in 1989 by Nguetseng (see [30]) and further developed by Allaire in 1992 (see [1]). In words, two-scale convergence is a kind of weak convergence mode for a sequence of functions of a global
variable where the limit is a function of both the global (or macroscopic) and the local (or microscopic) variable. (For an excellent review on two-scale convergence, see [24].) By using the periodic unfolding (or two-scale transform) technique (see [3, 7]) or alternatively the inverse two-scale transform technique (see [29]), this peculiar convergence mode is realised to be equivalent to an ordinary weak convergence for sequences of functions which depends both on the global and the local variable.

The rigorous definition of two-scale convergence is given below. (If nothing else is stated, in this paper we let \( y \in Y \) where \( Y = (0, 1)^N \).

**Definition 1.** A sequence \( \{u_\varepsilon\} \) of functions in \( L^2(\Omega) \) is said to two-scale converge to a limit \( u_0 \in L^2(\Omega \times Y) \) if, as \( \varepsilon \to 0 \) (from above),

\[
\int_\Omega u_\varepsilon(x) v(x, \frac{\varepsilon}{2}) \, dx \to \int_\Omega \int_Y u_0(x, y) v(x, y) \, dy \, dx
\]

for all \( v \in L^2(\Omega; C_\#(Y)) \), and we write \( u_\varepsilon \overset{2}{\rightharpoonup} u_0 \) as \( \varepsilon \to 0 \).

**Remark 2.** Alternatively one can write \( \overset{\sim}{\rightharpoonup} \) instead of \( \overset{2}{\rightharpoonup} \). Note also that instead of using the positive scale parameter \( \varepsilon \) tending to zero it is possible to employ a perhaps more fundamental scale parameter \( h \) tending to positive infinity. (This means that \( \lim_{h \to \infty} \varepsilon = 0 \); in the remainder of the paper this can at any point be achieved by substituting \( \varepsilon = 1/h \). The substitution would, e.g., give \( hx \) instead of \( \frac{x}{\varepsilon} \) everywhere.)

From now on we assume that all limits are taken as \( \varepsilon \to 0 \) (from above) if nothing else is stated.

In Definition 3 below we introduce the notion of scale functions which are functions with respect to the scale parameter.

**Definition 3.** A scale function \( \varepsilon_* : \mathbb{R}_+ \to \mathbb{R} \) is a real-valued function of the scale parameter \( \varepsilon \) for which \( \varepsilon_*(\varepsilon) \to 0 \) (i.e., \( \varepsilon_* \) is microscopic), and for which there exists \( \delta > 0 \) such that \( \varepsilon_*(\varepsilon) > 0 \) for all \( 0 < \varepsilon < \delta \) (i.e., \( \varepsilon_* \) is ultimately positive).

Note that the scale parameter \( \varepsilon \) itself (i.e., \( \varepsilon_*(\varepsilon) = \varepsilon \)) is a trivial example of a scale function. An example of a function \( \varepsilon_* \) of \( \varepsilon \) that is not a scale function is, e.g., \( \varepsilon_*(\varepsilon) = \varepsilon \sin \frac{1}{\varepsilon} \) since \( \varepsilon_* \) in this case—though being microscopic—is not ultimately positive.

The concept of scale functions leads to the notion of multiscale convergence which was introduced in 1996 by Allaire and Briane (see [2]) as a generalisation of two-scale convergence in order to be able to perform homogenisation of problems with multiple scales. This convergence mode is defined below. (If nothing else is stated, in this paper we let \( y_i \in Y_i \), where \( Y_i = (0, 1)^N \), \( i \in [n] \).)
Definition 4. A sequence \( \{u_\varepsilon\} \) of functions in \( L^2(\Omega) \) is said to \((n + 1)\)-scale converge to a limit \( u_0 \in L^2(\Omega \times Y_1 \times \cdots \times Y_n) \) if

\[
\int_{\Omega} u_\varepsilon(x) \nu(x, x_{\varepsilon_1}, \ldots, x_{\varepsilon_n}) \, dx \\
\rightarrow \int_{\Omega} \int_{Y_1} \cdots \int_{Y_n} u_0(x, y_1, \ldots, y_n) \nu(x, y_1, \ldots, y_n) \, dy_n \cdots dy_1 \, dx
\]

for all \( \nu \in L^2(\Omega; C_\#(Y_1 \times \cdots \times Y_n)) \), and we write \( u_\varepsilon \xrightarrow{\#} u_0 \).

In order to simplify the notation, from now on we will write \( y_n = (y_1, \ldots, y_n) \) and \( Y^n = Y_1 \times \cdots \times Y_n \) so that \( y_n \in Y^n \) which collects the local (spatial) variables and local (spatial) sets under one roof. (Naturally, the Lebesgue measure on \( Y^n \) is denoted \( dy_n \).) We also write \( x_{\varepsilon_k} = (x_{\varepsilon_{k1}}, \ldots, x_{\varepsilon_{kn}}) \) in the same spirit where we note that \( x_{\varepsilon_k} \) actually depends on the particular choice of scale functions \( \varepsilon_1, \ldots, \varepsilon_n \). Of course, multiscale convergence is highly dependent on the behaviour of the (spatial) scale functions. For ordered lists of scale functions we have the following definitions:

Definition 5. The list \( \{\varepsilon_k\}_{k=1}^n \) of scale functions is said to be separated if \( \frac{\varepsilon_{k+1}}{\varepsilon_k} \rightarrow 0 \) for all \( k \in [n-1] \).

Definition 6. The list \( \{\varepsilon_k\}_{k=1}^n \) of scale functions is said to be well-separated if there exists a positive integer \( \ell \) such that \( \frac{1}{\varepsilon_k} \left( \frac{\varepsilon_{k+1}}{\varepsilon_k} \right)^\ell \rightarrow 0 \) for all \( k \in [n-1] \).

Remark 7. Note that well-separatedness is a stronger requirement than separatedness.

Homogenisation for linear parabolic problems with several temporal scales using the multiscale convergence technique was first achieved by Flodén and Olsson in 2007 (see [14]). This was a further development of the work by Holmbom in 1996 and 1997 (see [16] and [17], respectively) where two-scale convergence was employed to homogenise linear parabolic problems with both a spatial and a temporal microscale. General \((n + 1, m + 1)\)-scale convergence can be expressed according to the definition below. (If nothing else is stated, in this paper we let \( s_j \in S_j \), where \( S_j = (0, 1) \), \( j \in [m] \).)

Definition 8. A sequence \( \{u_\varepsilon\} \) in \( L^2(\Omega \times (0, T)) \) is said to \((n + 1, m + 1)\)-scale converge to a limit \( u_0 \in L^2(\Omega \times (0, T) \times Y^n \times S_1 \times \cdots \times S_m) \) if

\[
\int_0^T \int_{\Omega} u_\varepsilon(x, t) \nu(x, t, x', x_{\varepsilon_1}, \ldots, x_{\varepsilon_n}, \frac{t}{T}, \ldots, \frac{1}{T}) \, dx \, dt \\
\rightarrow \int_0^T \int_{\Omega} \int_{Y^n} \int_{S_1} \cdots \int_{S_m} u_0(x, t, y_n, s_1, \ldots, s_m) \\
\times \nu(x, t, y_n, s_1, \ldots, s_m) \, ds_m \cdots ds_1 \, dy_n \, dt
\]

for all \( \nu \in L^2(\Omega \times (0, T); C_\#(Y^n \times S_1 \times \cdots \times S_m)) \), and we write \( u_\varepsilon \xrightarrow{(n+1,m+1)} u_0 \).
Trivially, this definition also works for vector valued functions where the product becomes a dot product, or mixed scalar and vector valued functions which would give vector valued integrals above. All results below concerning the notion of \((n+1, m+1)\)-scale convergence can of course be formulated for such functions as well. In particular, gradient functions will later be of interest.

In order to simplify the notation, from now on we will write \(s_m = (s_1, \ldots, s_m)\) and \(S^m = S_1 \times \cdots \times S_m\) so that \(s_m \in S^m\). (The Lebesgue measure on \(S^m\) will of course be denoted \(ds_m\).) Moreover, \(t^e_m = (\frac{x_1}{t^e_1}, \ldots, \frac{x_m}{t^e_m})\) which is noted to depend on the particular choice of temporal scale functions \(\{\epsilon^e_j\}_{j=1}^m\). Furthermore, introduce \(\Omega_T = \Omega \times (0, T)\) so that \((x, t) \in \Omega_T\) and \(\mathcal{Y}^{nm} = Y^n \times S^m\) so that \((y_n, s_m) \in \mathcal{Y}^{nm}\).

It is clear that we need to introduce some convenient restrictions on the spatial and temporal scale functions \(\{\epsilon_i\}_{i=1}^n\) and \(\{\epsilon^e_j\}_{j=1}^m\) in order for them to collaborate in a meritorious manner. In Definition 9 below we define a certain set of pairs of lists of such meritoriously collaborating spatial and temporal scale functions.

**Definition 9.** Suppose we have a list \(\{\epsilon_i\}_{i=1}^n\) of \(n\) spatial scale functions and a list \(\{\epsilon^e_j\}_{j=1}^m\) of \(m\) temporal scale functions. We say that the pair \((\{\epsilon_i\}_{i=1}^n, \{\epsilon^e_j\}_{j=1}^m)\) belongs to the set \(\mathcal{J}_\text{sep}^{nm}\) if \(\{\epsilon_i\}_{i=1}^n\) and \(\{\epsilon^e_j\}_{j=1}^m\) are both separated and that the following two conditions hold:

(i) There exist possibly empty subsets \(A \subset \llbracket n \rrbracket\) and \(A' \subset \llbracket m \rrbracket\) with \(|A| = |A'| = k\) such that there exist bijections \(\beta : \llbracket k \rrbracket \to A\) and \(\beta' : \llbracket k \rrbracket \to A'\), respectively, such that \(\epsilon \beta(i) = \epsilon^e \beta'(i)\) for all \(i \in \llbracket k \rrbracket\). (We have no requirement in the empty case \(k = 0\).)

(ii) There exists a permutation \(\pi\) on the set \(\llbracket n + m - 2k \rrbracket\) such that the permutation \(\{\epsilon^e_{\pi(i)}\}_{i=1}^{n+m-2k}\) of the list

\[
\{\epsilon^e_i\}_{i=1}^{n+m-2k} = \{\epsilon_i\}_{i \notin A}, \{\epsilon^e_j\}_{j \notin A'}
\]

of the remaining \(n + m - 2k\) scale functions is separated. (We have no requirement in the empty case \(n + m - 2k = 0\).)

If we require well-separatedness instead of mere separatedness we can define the corresponding set \(\mathcal{J}_\text{wsep}^{nm}\).

Note that \(\mathcal{J}_\text{wsep}^{nm} \subset \mathcal{J}_\text{sep}^{nm}\). The idea of the definition above is that we can localise all the spatial and temporal scale functions in two disjoint categories, (i) and (ii), where the former category consists of those that are equal and the latter category consists of those that are jointly (well-)separated. Note also that since neither \(n\) nor \(m\) vanishes, it can not be the case that both categories (i) and (ii) of Definition 9 are empty.
Example 10. As examples of pairs of lists that do and do not belong to $\mathcal{F}_{wsep}^{\text{nm}}$, consider $(e_1,e'_1)$, $(e_2,e'_2)$ and $(e_3,e'_3)$ defined by

$$
\begin{align*}
\left\{ e_1 = \{ \varepsilon, \varepsilon^3 \}, \quad e'_1 = \{ \varepsilon^2, \varepsilon^3, \varepsilon^4 \}, \right. \\
\left. e_2 = \{ \varepsilon, \varepsilon^3 \}, \quad e'_2 = \{ \varepsilon^2, \frac{\varepsilon^2}{|\log \varepsilon|}, \varepsilon^3 \}, \right. \\
\left. e_3 = \{ \varepsilon, \varepsilon^3 \}, \quad e'_3 = \{ \varepsilon, \varepsilon^2, \frac{\varepsilon^2}{|\log \varepsilon|} \}. \right.
\end{align*}
$$

Clearly, the first pair $(e_1,e'_1)$ belongs to $\mathcal{F}_{wsep}^{23}$ since both $e_1$ and $e'_1$ are well-separated lists and the combined list $\{ \varepsilon, \varepsilon^2, \varepsilon^4 \}$ where we have removed the common scale function $\varepsilon^3$ is well-separated.

We have that the middle pair $(e_2,e'_2)$ does not belong to $\mathcal{F}_{wsep}^{23}$ since $e'_2$ is not well-separated.

The last pair $(e_3,e'_3)$ does not belong to $\mathcal{F}_{wsep}^{23}$. Indeed, the combined list $\{ \varepsilon^2, \varepsilon^3, \frac{\varepsilon^2}{|\log \varepsilon|} \}$ (with removed common scale function $\varepsilon$) is not well-separated.

In Proposition 11 below we recall that if $q$, $f$ and $g$ are functions of $\varepsilon$ where $f = qg$ and $q \to 1$, then we say that $f \sim g$, i.e., $f$ and $g$ are asymptotically equal.

Proposition 11. Suppose $u_e \overset{(n+1,m+1)}{\longrightarrow} u_0$ and that $r = r(\varepsilon)$ satisfies $r \sim r_0$, $r_0 \in \mathbb{R}$. Then $r(\varepsilon)u_e \overset{(n+1,m+1)}{\longrightarrow} r_0u_0$.

Proof. Clearly,

$$
\begin{align*}
\int_{\Omega_T} (r(\varepsilon)u_e(x,t)) \nu(x,t,x'_n,t'_m) \, dx dt \\
&= r(\varepsilon) \int_{\Omega_T} u_e(x,t) \nu(x,t,x'_n,t'_m) \, dx dt \\
&\to r_0 \int_{\Omega_T} \int_{\mathcal{Y}_{nm}} u_0(x,t,y_n,s_m) \nu(x,t,y_n,s_m) \, ds_m dy_n \, dx dt \\
&= \int_{\Omega_T} \int_{\mathcal{Y}_{nm}} (r_0u_0(x,t,y_n,s_m)) \nu(x,t,y_n,s_m) \, ds_m dy_n \, dx dt
\end{align*}
$$

for all $\nu \in L^2(\Omega_T;C^\infty_{\#}(\mathcal{Y}_{nm}))$, which precisely means that $r(\varepsilon)u_e \overset{(n+1,m+1)}{\longrightarrow} r_0u_0$. 

Under certain restrictions it can be shown that (5) only has to hold for a certain class of smooth functions in order to get $(n+1,m+1)$-scale convergence; see the proposition below.

Proposition 12. Let $\{u_e\}$ be a bounded sequence in $L^2(\Omega_T)$ and let $u_0 \in L^2(\Omega_T \times \mathcal{Y}_{nm})$. Furthermore, suppose (5) holds for all $\nu \in \mathcal{D}(\Omega_T;C^\infty_{\#}(\mathcal{Y}_{nm}))$. Then $u_e \overset{(n+1,m+1)}{\longrightarrow} u_0$.

Proof. Let $w \in L^2(\Omega_T;C^\infty_{\#}(\mathcal{Y}_{nm}))$ be arbitrary. Furthermore, let $\{v_{\mu}\}$ be a sequence in $\mathcal{D}(\Omega_T;C^\infty_{\#}(\mathcal{Y}_{nm}))$ that converges to $w$ in $L^2(\Omega_T;C^\infty_{\#}(\mathcal{Y}_{nm}))$ as $\mu \to \infty$. It is trivial that
\[ \lim_{\varepsilon \to 0} \int_{\Omega_T} u_{\varepsilon}(x,t) \, w(x,t,x_n^{\varepsilon},t_m^{\varepsilon}) \, dx \, dt \]

\[ = \lim_{\mu \to \infty} \lim_{\varepsilon \to 0} \left( \int_{\Omega_T} u_{\varepsilon}(x,t) \, (w - v_{\mu})(x,t,x_n^{\varepsilon},t_m^{\varepsilon}) \, dx \, dt \right. \]

\[ + \int_{\Omega_T} u_{\varepsilon}(x,t) \, v_{\mu}(x,t,x_n^{\varepsilon},t_m^{\varepsilon}) \, dx \, dt \]  
(6)

holds.

By assumption, for the second term in the right-hand side of (6) we have

\[ \lim_{\mu \to \infty} \int_{\Omega_T} u_{\varepsilon}(x,t) \, v_{\mu}(x,t,x_n^{\varepsilon},t_m^{\varepsilon}) \, dx \, dt \]

\[ = \int_{\Omega_T} \int_{Y_{nm}} u_0(x,t,y_n,s_m) \, v_{\mu}(x,t,y_n,s_m) \, ds_m \, dy_n \, dx \, dt \]

\[ = \int_{\Omega_T} \int_{Y_{nm}} u_0(x,t,y_n,s_m) \, w(x,t,y_n,s_m) \, ds_m \, dy_n \, dx \, dt. \]

The second equality comes from the fact that

\[ \left| \int_{\Omega_T} \int_{Y_{nm}} u_0(x,t,y_n,s_m) \, (v_{\mu} - w)(x,t,y_n,s_m) \, ds_m \, dy_n \, dx \, dt \right| \]

\[ \leq \left\| u_0 \, (v_{\mu} - w) \right\|_{L^1(\Omega_T \times Y_{nm})} \leq C_1 \left\| v_{\mu} - w \right\|_{L^2(\Omega_T \times Y_{nm})} \]

\[ \leq C_1 \left\| v_{\mu} - w \right\|_{L^2(\Omega_T C_\#(Y_{nm}))} \to 0 \]

as \( \mu \to \infty \), where we have used Hölder’s inequality in the second inequality.

It remains to treat the first term in the right-hand side of (6); we want it to vanish. Indeed,

\[ \lim_{\mu \to \infty} \lim_{\varepsilon \to 0} \int_{\Omega_T} u_{\varepsilon}(x,t) \, (w - v_{\mu})(x,t,x_n^{\varepsilon},t_m^{\varepsilon}) \, dx \, dt \]

\[ \leq \lim_{\mu \to \infty} \lim_{\varepsilon \to 0} \| u_{\varepsilon} \, (w^\varepsilon - v_{\mu}^\varepsilon) \|_{L^1(\Omega_T)} \leq \lim_{\mu \to \infty} \lim_{\varepsilon \to 0} C_2 \| w^{\varepsilon} - v_{\mu}^\varepsilon \|_{L^2(\Omega_T)} \]

\[ \leq \lim_{\mu \to \infty} \lim_{\varepsilon \to 0} C_2 \| w - v_{\mu} \|_{L^2(\Omega_T C_\#(Y_{nm}))} = 0, \]

where we have used Hölder’s inequality in the second inequality and employed that \( \{u_{\varepsilon}\} \) is bounded in \( L^2(\Omega_T) \). (The last inequality follows from the fact that the \( L^2(\Omega_T; C_\#(Y_{nm})) \)-norm involves a maximum with respect to the local variables.)

To conclude, (6) becomes

\[ \lim_{\varepsilon \to 0} \int_{\Omega_T} u_{\varepsilon}(x,t) \, w(x,t,x_n^{\varepsilon},t_m^{\varepsilon}) \, dx \, dt \]

\[ = \int_{\Omega_T} \int_{Y_{nm}} u_0(x,t,y_n,s_m) \, w(x,t,y_n,s_m) \, ds_m \, dy_n \, dx \, dt \]

for all \( w \in L^2(\Omega_T; C_\#(Y_{nm})) \); we have in fact shown that \( u_{\varepsilon} \overset{(n+1,m+1)}{\longrightarrow} u_0 \).

We have the following important compactness result.
Theorem 13. Suppose that the pair \((\{\varepsilon_i\}_{i=1}^n, \{\varepsilon'_j\}_{j=1}^m)\) of lists of spatial and temporal scale functions belongs to \(\mathcal{J}_{\text{sep}}^{nm}\). Furthermore, let \(\{u_e\}\) be a bounded sequence in \(L^2(\Omega_T)\). Then there is a function \(u_0 \in L^2(\Omega_T \times \mathcal{Y}_{nm})\) such that, up to a subsequence, \(u_e^{(n+m+1)} \rightarrow u_0\).

Proof. (We assume here that both categories (i) and (ii) of Definition 9 are non-empty, i.e., that \(k \in \lfloor \frac{1}{2}(n+m) \rfloor\). The cases when exactly one category is empty would be even more straightforward to analyse and are thus left out from the discussion for brevity.)

Without loss of generality we can assume that the labelling of the indices is such that \(\varepsilon_i = \varepsilon'_i\), \(i \in [k]\). (If not, simply relabel the scale functions.) Let us introduce the \(k\) number of \((N+1)\)-dimensional local variables \(\tilde{y}_i = (y_i, s_i)\) and corresponding scale functions \(\tilde{\varepsilon}_i = \varepsilon_i = \varepsilon'_i\), \(i \in [k]\). In category (ii) there are now \(n+m-2k\) separated scales to take care of. Introduce the \(n+m-2k\) local “ghost” variables \(\{y_i\}_{i=k+1}^{n+m-k}\) and \(\{s_j\}_{j=k+1}^{n+m-k}\) such that one can form the \(n+m-2k\) number of \((N+1)\)-dimensional local variables \(\tilde{y}_i = (y_j, s_j)\) and scale parameters \(\tilde{\varepsilon}_i = \varepsilon_j\) (if \(s_j\) where \(j \in [k+1, m]\) is the “ghost”) or \(\tilde{\varepsilon}_i = \varepsilon'_j\) (if \(y_j\) where \(j \in [k+1, n]\) is the “ghost”) for \(i \in [k+1, n+m-k]\). (Of course, here it is assumed that \(k \in \lfloor \min\{n, m\} - 1 \rfloor\). If this is not true we simply introduce “ghosts” of only spatial type (i.e., if \(k = m < n\) or temporal type (i.e., if \(k = n < m\).)

In total we have introduced a local variable

\[
\tilde{y}_{n+m-k} = (\tilde{y}_1, \ldots, \tilde{y}_k, \tilde{y}_{k+1}, \ldots, \tilde{y}_{n+m-k}).
\]

which belongs to \(\tilde{\mathcal{Y}}^{n+m-k} = (Y_1 \times S_1) \times \cdots \times (Y_{n+m-k} \times S_{n+m-k})\). Define \(\tilde{\mathcal{W}} = (x, t)\) and \(\tilde{\mathcal{W}} = \Omega_T\) such that \(\tilde{\mathcal{W}} \in \tilde{\mathcal{W}}\) for \((x, t) \in \mathcal{W}\), and \(\tilde{\mathcal{W}}_{n+m-k} = (\tilde{x}_1, \ldots, \tilde{x}_{n+m-k})\).

Furthermore, given an arbitrary test function \(\nu \in L^2(\Omega_T; \hat{G}_\#(\mathcal{Y}_{nm}))\), let

\[
\tilde{u}_\nu(\tilde{x}) = u_\nu(x, t) \quad \text{and} \quad \tilde{\nu}(\tilde{x}, \tilde{y}_{n+m-k}) = \nu(x, t, y, s_m)
\]

for all \(\tilde{\mathcal{W}} \ni \tilde{x} = (x, t) \in \mathcal{W}\) and all \(\tilde{\mathcal{W}}^{n+m-k} \ni \tilde{y}_{n+m-k} = (y_n, s_m) \in \mathcal{Y}_{nm}\). We realise that since \(\nu\) is independent of the \(n+m-2k\) local “ghost” variables, \(\tilde{\nu}\) is too, and we equivalently have that \(\tilde{\nu} \in L^2(\Omega_T; \hat{G}_\#(\tilde{\mathcal{Y}}^{n+m-k}))\).

We have by definition

\[
\int_{\Omega_T} u_\nu(x, t) \nu(x, t, x', t') \, dx \, dt = \int_{\tilde{\mathcal{W}}} \tilde{u}_\nu(\tilde{x}) \tilde{\nu}(\tilde{x}, \tilde{y}_{n+m-k}) \, d\tilde{x}.
\]

According to Theorem 2.4 in [2], up to a subsequence, \(\{\tilde{u}_\nu\}\) \((n+m-k+1)\)-converges to a limit \(\tilde{u}_0 \in L^2(\tilde{\mathcal{W}} \times \tilde{\mathcal{Y}}^{n+m-k})\), i.e.,

\[
\int_{\tilde{\mathcal{W}}} \tilde{u}_\nu(\tilde{x}) \tilde{\nu}(\tilde{x}, \tilde{y}_{n+m-k}) \, d\tilde{x} \rightarrow \int_{\tilde{\mathcal{W}}} \tilde{u}_0(\tilde{x}, \tilde{y}_{n+m-k}) \tilde{\nu}(\tilde{x}, \tilde{y}_{n+m-k}) \, d\tilde{y}_{n+m-k} \, d\tilde{x}.
\]
It is clear that $\bar{u}_0$ does not depend on the local "ghost" variables which implies that there exists $u_0 \in L^2(\Omega_T \times \mathcal{Y}_{nm})$ such that

$$
\bar{u}_0(\bar{x}, \bar{y}_{n+m-k}) = u_0(x, t, y_n, s_m)
$$

for all $\bar{\Omega} \ni \bar{x} = (x, t) \in \Omega_T$ and all $\bar{y}^{n+m-k} \ni \bar{y}_{n+m-k} = (y_n, s_m) \in \mathcal{Y}_{nm}$. If $y_{gh}$ collects the local "ghost" variables and $\mathcal{Y}_{gh}$ is the corresponding local set such that $y_{gh} \in \mathcal{Y}_{gh}$,

$$
\int_{\bar{\Omega}} \int_{\mathcal{Y}_{gh}} u_0(x, t, y_n, s_m) d\bar{y}_{n+m-k} d\bar{x} = \int_{\Omega_T} \int_{\mathcal{Y}_{nm}} u_0(x, t, y_n, s_m) d\bar{y}_{n+m-k} d\bar{x}
$$

for all $v \in L^2(\Omega_T; C_{\#}(\mathcal{Y}_{nm}))$ where $u_0 \in L^2(\Omega_T \times \mathcal{Y}_{nm})$. This means precisely that, for the extracted subsequence, $u_\varepsilon \rightharpoonup u_0$, and we are done.

To conclude, we have shown that

$$
\int_{\Omega_T} u_\varepsilon(x, t) v(x, t, x_n^e, t_m^e) \, dx \, dt \to \int_{\Omega_T} \int_{\mathcal{Y}_{nm}} u_0(x, t, y_n, s_m) v(x, t, y_n, s_m) ds_m \, dy_n \, dx \, dt
$$

for all $v \in L^2(\Omega_T; C_{\#}(\mathcal{Y}_{nm}))$. This means precisely that, for the extracted subsequence, $u_\varepsilon \rightharpoonup u_0$, and we are done.

The proposition below states that under certain restrictions for $v$ defined on $\Omega_T \times \mathcal{Y}_{nm}$, the sequence $\{v^\varepsilon\}$ converges weakly in $L^2(\Omega_T)$ to the average over the local variables.

**Proposition 14.** Suppose that the pair $\{(e_1^i)_{i=1}^n, (e_j^m)_{j=1}^m\}$ of lists of spatial and temporal scale functions belongs to $\mathcal{J}_{seq}^{nm}$. Then

$$
v^\varepsilon \rightharpoonup \int_{\mathcal{Y}_{nm}} v(\cdot, y_n, s_m) \, ds_m \, dy_n \quad \text{in} \ L^2(\Omega_T)
$$

for every $v \in C((\bar{\Omega}_T; C_{\#}(\mathcal{Y}_{nm}))$.

**Proof.** Proceed as in the first part of the proof of Theorem 13—i.e., introducing quantities expressed with tilde—but letting $v \in C((\bar{\Omega}_T; C_{\#}(\mathcal{Y}_{nm}))$ instead. Now we have introduced a collection of $n + m - 2k$ local "ghost" variables collected in the variable $y_{gh} \in \mathcal{Y}_{gh}$. For every $\lambda > 0$, let $\{\bar{K}^\lambda_\mu\}_{\mu=1}^M$ be a covering of $\bar{\Omega}$ where $\bar{K}^\lambda_\mu$ are cubes of side length $\frac{\lambda}{N}$ such that $\bar{K}^\lambda_\mu \cap \bar{\Omega} \neq \emptyset$. Moreover, introduce $\bar{x}^\lambda_\mu \in \bar{K}^\lambda_\mu$, $\mu \in [M]$. According to the convergence result of Lemma 4.2.2 in [35], for any given $v \in C((\bar{\Omega}_T; C_{\#}(\mathcal{Y}_{nm}))$ and fixed $\mu \in [M]$, it holds that

$$
\int_{\bar{\Omega}} \bar{v}(\bar{x}^\lambda_\mu, \bar{x}_{n+m-k}) \, d\bar{x} \to \int_{\bar{\Omega}} \int_{\mathcal{Y}_{nm}} \bar{v}(\bar{x}^\lambda_\mu, \bar{y}_{n+m-k}) \, d\bar{y}_{n+m-k} \, d\bar{x}
$$

(8)

for all $\bar{v} \in L^2(\bar{\Omega})$ since $\bar{v}(\bar{x}^\lambda_\mu, \cdot) \in C_{\#}(\mathcal{Y}_{nm}) \subset L^2(\mathcal{Y}_{nm})$. 

11
Now, define the simple function (with respect to $\bar{x} \in \tilde{\Omega}$)

$$\overline{v}^\lambda(\bar{x}, \bar{y}_{n+m-k}) = \sum_{\mu=1}^M \hat{v}(\bar{x}^\mu_n, \bar{y}_{n+m-k}) \chi_{\tilde{\Omega} \setminus \tilde{\Omega}^\lambda(\bar{x})}(\bar{x}) \quad (\bar{x} \in \tilde{\Omega}, \bar{y}_{n+m-k} \in \tilde{\mathcal{Y}}^{n+m-k}),$$

where $\chi_A$ is the characteristic function on $A \subset \mathbb{R}^{N+1}$, and

$$\hat{\delta}^\lambda(\bar{x}) = \sup_{\bar{y}_{n+m-k}} |(\overline{v} - \overline{v}^\lambda)(\bar{x}, \bar{y}_{n+m-k})|.$$ 

Note that for every fixed $\bar{x} \in \tilde{\Omega}$, the difference $(\overline{v} - \overline{v}^\lambda)(\bar{x}, \cdot)$ is uniformly continuous on $\tilde{\mathcal{Y}}^{n+m-k}$. This means in particular that the supremum above can be taken over any countable dense subset of $\tilde{\mathcal{Y}}^{n+m-k}$ like, e.g., $\tilde{\mathcal{Y}}^{n+m-k} \cap Q^{(n+m-k)(N+1)}$. We observe that $\hat{\delta}^\lambda$ is the supremum of a countable family of measurable functions, and in virtue of claim (9a) on p. 1012 in [48] this implies that $\hat{\delta}^\lambda$ itself is measurable as well. The strong regularity of $\overline{v}$ guarantees that

$$\hat{\delta}^\lambda(\bar{x}) \to 0$$

as $\lambda \to \infty$ for every fixed $\bar{x} \in \tilde{\Omega}$. Furthermore, we clearly have a majoriser

$$|\hat{\delta}^\lambda(\bar{x})| \leq \sup_{\tilde{\Omega} \times \tilde{\mathcal{Y}}^{n+m-k}} |\overline{v}(\bar{x}, \bar{y}_{n+m-k})| + \sup_{\tilde{\Omega} \times \tilde{\mathcal{Y}}^{n+m-k}} |\overline{v}^\lambda(\bar{x}, \bar{y}_{n+m-k})| \leq 2 \sup_{\tilde{\Omega} \times \tilde{\mathcal{Y}}^{n+m-k}} |\overline{v}(\bar{x}, \bar{y}_{n+m-k})|$$

(i.e., a constant majoriser). Hence, according to Lebesgue’s dominated convergence theorem, we have shown that

$$\int_{\tilde{\Omega}} \hat{\delta}^\lambda(\bar{x}) \, d\bar{x} \to \int_{\tilde{\Omega}} 0 \, d\bar{x} = 0.$$

We get the estimation

$$\left| \int_{\tilde{\Omega}} \overline{v}(\bar{x}, \bar{x}_{n+m-k}) \phi(\bar{x}) \, d\bar{x} - \int_{\tilde{\Omega}} \int_{\tilde{\mathcal{Y}}^{n+m-k}} \overline{v}(\bar{x}, \bar{y}_{n+m-k}) \phi(\bar{x}) \, d\bar{y}_{n+m-k} \, d\bar{x} \right|$$

$$\leq \left| \int_{\tilde{\Omega}} \overline{v}^\lambda(\bar{x}, \bar{x}_{n+m-k}) \phi(\bar{x}) \, d\bar{x} - \int_{\tilde{\Omega}} \int_{\tilde{\mathcal{Y}}^{n+m-k}} \overline{v}^\lambda(\bar{x}, \bar{y}_{n+m-k}) \phi(\bar{x}) \, d\bar{y}_{n+m-k} \, d\bar{x} \right|$$

$$+ \left| \int_{\tilde{\Omega}} (\overline{v} - \overline{v}^\lambda)(\bar{x}, \bar{x}_{n+m-k}) \phi(\bar{x}) \, d\bar{x} \right| + \left| \int_{\tilde{\Omega}} (\overline{v} - \overline{v}^\lambda)(\bar{x}, \bar{y}_{n+m-k}) \phi(\bar{x}) \, d\bar{y}_{n+m-k} \, d\bar{x} \right|$$

for every $\phi \in \mathcal{D}(\tilde{\Omega})$. The convergence result (8) implies that the first term tends to zero. For any fixed $\varepsilon > 0$, the middle and last terms are both majorised by $\hat{\delta}^\lambda$, which in the limit $\lambda \to \infty$ means that these terms vanish. Thus, we have proven that for every given $v \in C(\overline{\tilde{\Omega}}; C_\#(\mathcal{Y}_{nm}))$

$$\int_{\tilde{\Omega}} \overline{v}(\bar{x}, \bar{x}_{n+m-k}) \phi(\bar{x}) \, d\bar{x} \to \int_{\tilde{\Omega}} \int_{\tilde{\mathcal{Y}}^{n+m-k}} \overline{v}(\bar{x}, \bar{y}_{n+m-k}) \phi(\bar{x}) \, d\bar{y}_{n+m-k} \, d\bar{x} \quad (9)$$

12
for all \( \phi \in \mathcal{D}(\Omega) \). Since \( \tilde{\phi}(\cdot, y_{n+m-k}) \) is a bounded function in \( L^2(\Omega) \) for every \( y_{n+m-k} \in \mathcal{Y}^{n+m-k} \), the convergence (9) also holds for all \( \phi \in L^2(\Omega) \).

Define \( \phi \) by

\[
\phi(x, t) = \tilde{\phi}(\tilde{x}) \quad (\Omega_T \ni (x, t) = \tilde{x} \in \Omega).
\]

Then \( \tilde{\phi} \in L^2(\Omega) \) is equivalent to saying that \( \phi \in L^2(\Omega_T) \). The convergence result (9) is thus realised to mean that for every given \( v \in C(\Omega_T; C_\#(Y_{nm})) \),

\[
\int_{\Omega_T} v(x, t, x_m^\epsilon, t_m^\epsilon) \phi(x, t) \, dx \, dt \to \int_{\Omega_T} \int_{Y_{nm}} v(x, t, y_n, s_m) \phi(x, t) \, ds_m \, dy_n \, dx \, dt
\]

for all \( \phi \in L^2(\Omega_T) \). Hence, we have shown (7), and the proof is complete.

\( \square \)

**Proposition 15.** Suppose that the pair \( \{\{\epsilon_i\}_{i=1}^n, \{\epsilon_j\}_{j=1}^m\} \) of lists of spatial and temporal scale functions belongs to \( \mathcal{Y}_{sep}^{nm} \). Moreover, assume that \( \{u_\epsilon\} \) converges strongly to \( u \) in \( L^2(\Omega_T) \). Then \( u_\epsilon \rightharpoonup u \).

**Proof.** From Proposition 14 we have

\[
v^\epsilon \rightharpoonup \int_{Y_{nm}} v(\cdot, y_n, s_m) \, ds_m \, dy_n \quad \text{in} \quad L^2(\Omega_T)
\]

for every \( v \in C(\Omega_T; C_\#(Y_{nm})) \). This combined with the assumption

\[
u_\epsilon \rightharpoonup u \quad \text{in} \quad L^2(\Omega_T)
\]

implies

\[
\int_{\Omega_T} u_\epsilon(x, t) \, v(x, t, x_m^\epsilon, t_m^\epsilon) \, dx \, dt \to \int_{\Omega_T} \int_{Y_{nm}} u(x, t) \, v(x, t, y_n, s_m) \, ds_m \, dy_n \, dx \, dt
\]

for every \( v \in C(\Omega_T; C_\#(Y_{nm})) \) \( \subset L^2(\Omega_T; C_\#(Y_{nm})) \), where we have used the weak–strong convergence theorem with respect to \( L^2(\Omega_T) \). Due to Proposition 12 this convergence in fact holds for all functions \( v \in L^2(\Omega_T; C_\#(Y_{nm})) \) due to the inclusion \( C(\Omega_T; C_\#(Y_{nm})) \supset D(\Omega_T; C_\#^\infty(Y_{nm})) \). Hence, \( u_\epsilon \rightharpoonup u \), and we are done.

\( \square \)

For the next theorem, Theorem 18 concerning multiscale convergence of gradient sequences, we need the two lemmas below. Note first that we introduce the following notations. We write \( Y^{[i_1, i_2]} = Y_{i_1} \times \cdots \times Y_{i_2} \) and \( S^{[n, b]} = S_{j_1} \times \cdots \times S_{j_2} \). Moreover, \( y_{[i_1, i_2]} \in Y^{[i_1, i_2]} \) and \( s_{[n, b]} \in S^{[n, b]} \) are the corresponding local variables. The Lebesgue measures on the introduced local sets are written accordingly. Furthermore, we define \( W_k = H_\#^1(Y_k) / \mathbb{R} \), \( k \in [n] \), for brevity. It should be emphasised that all derivatives are taken in the weak (or distributional or generalised) sense.
Lemma 16. Let $\mathcal{H}$ be the subspace of generalised divergence-free functions in $L^2(\Omega \times \mathbb{Y}^n)^N$ defined by
\[
\mathcal{H} = \left\{ \psi \in L^2(\Omega \times \mathbb{Y}^n)^N : \nabla y_n \cdot \psi = 0 \text{ and } \int_{\mathbb{Y}^{|k+1,n|}} \nabla y_k \cdot \psi(x, y_n) \, dy_{[k+1,n]} = 0 \right\}.
\]
for all $x \in \Omega$, $y_k \in \mathbb{Y}^k$ and all $k \in \mathbb{[n-1]}$.

Then the subspace $\mathcal{H}$ has the following properties:

(i) The intersection $D(\Omega; C_\#^\infty(\mathbb{Y}^n)^N) \cap \mathcal{H}$ is dense in $\mathcal{H}$;

(ii) The orthogonal complement $\mathcal{H}^\perp$ in $L^2(\Omega \times \mathbb{Y}^n)^N$ of $\mathcal{H}$ is
\[
\mathcal{H}^\perp = \left\{ \sum_{k=1}^n \nabla y_k u_k : u_k \in L^2(\Omega \times \mathbb{Y}^{k-1}; \mathcal{W}_k) \right\}.
\]

Proof. See Lemma 3.7 in [2].

Lemma 17. Let $k \in \mathbb{[n]}$ and suppose that the list $\{\varepsilon_i\}_{i=1}^n$ is well-separated. Furthermore, introduce
\[
\mathcal{E}_k = \left\{ \phi \in D(\Omega; C_\#^\infty(\mathbb{Y}^n)) : \int_{\mathbb{Y}^{|k,n|}} \phi(x, y_n) \, dy_{|k,n|} = 0 \right\}.
\]
for all $x \in \Omega$, $y_{k-1} \in \mathbb{Y}^{k-1}$.

Then, for any function $\phi \in \mathcal{E}_k$, the sequence $\{1/k^\delta \phi^k\}$ is bounded in $H^{-1}(\Omega)$.

Proof. See Corollary 3.4 in [2].

For the $(n+1,m+1)$-scale convergence of sequences of gradients we have the important Theorem 18 below.

Theorem 18. Suppose that the pair $\{(\varepsilon_i\}_{i=1}^n, \{\varepsilon_j\}_{j=1}^m\}$ of lists of spatial and temporal scale functions belongs to $\mathcal{F}_{\text{wsep}}^{nm}$. Moreover, assume that $\{u_\varepsilon\}$ is a bounded sequence in $H^1(0, T; H_0^1(\Omega), H^{-1}(\Omega))$. Then, up to a subsequence, we have
\[
u_\varepsilon \to u \quad \text{in } L^2(\Omega_T),
\]
\[
u_\varepsilon \to u \quad \text{in } L^2(0, T; H_0^1(\Omega)),
\]
and
\[
\nabla u_\varepsilon^{(n+1,m+1)} \to \nabla u + \sum_{k=1}^n \nabla y_k u_k,
\]
where $u \in L^2(0, T; H_0^1(\Omega))$ and $u_k \in L^2(\Omega_T \times \mathcal{Y}_{(k-1)m}; \mathcal{W}_k)$ for all $k \in \mathbb{[n]}$.

Proof. Since $\{u_\varepsilon\}$ is bounded in $H^1(0, T; H_0^1(\Omega), H^{-1}(\Omega))$, (i) $\{u_\varepsilon\}$ is also bounded in $L^2(0, T; H_0^1(\Omega))$, (ii) $\|\nabla u_\varepsilon\|$ is bounded in $L^2(0, T; H^{-1}(\Omega))$ and (iii) $\{\nabla u_\varepsilon\}$ is bounded in $L^2(\Omega_T)^N$. The first statement (i) implies, up to a subsequence,
\[
u_\varepsilon \to u \quad \text{in } L^2(0, T; H_0^1(\Omega))
\]
for some unique $u \in L^2(0, T; H^1_0(\Omega))$. By Lemmas 8.2 and 8.4 in [10], the statements (i) and (ii) imply, up to a subsequence, that

$$u_\varepsilon \rightarrow u \quad \text{in } L^2(\Omega_T).$$

(11)

Hence, we have proven the convergences for $u_\varepsilon$.

From Theorem 13 and (i) and (iii) we know that, up to a subsequence,

$$\nabla u_\varepsilon \rightarrow w_0$$

(12)

for some limit function $w_0 \in L^2(\Omega_T \times \mathcal{Y}_{nm})^N$.

We will now characterise $w_0$ in terms of gradients. Using the vector valued product test function $\psi \in L^2(\Omega_T; C_\#(\mathcal{Y}_{nm}))$ defined by

$$\psi(x, t, y_n, s_m) = v(x, y_n) c(t, s_m)$$

for all $(x, t) \in \Omega_T$ and all $(y_n, s_m) \in \mathcal{Y}_{nm}$, where $v \in \mathcal{D}(\Omega; C_\#^\infty(Y^n)) \cap \mathcal{H}$ and $c \in \mathcal{D}(0, T) \cap C_\#^\infty(S^m)$, in the $(n+1, m+1)$-scale convergence result (12) yields, up to a subsequence,

$$\int_{\Omega_T} \nabla u_\varepsilon(x, t) \cdot v(x, x^\varepsilon_n) c(t, t^\varepsilon_m) \, dx \, dt \rightarrow \int_{\Omega_T} \int_{\mathcal{Y}_{nm}} w_0(x, t, y_n, s_m) \cdot v(x, y_n) c(t, s_m) \, ds_m \, dy_n \, dx \, dt.$$

(13)

Taking a closer look at the left-hand side of (13) we get

$$\int_{\Omega_T} \nabla u_\varepsilon(x, t) \cdot v(x, x^\varepsilon_n) c(t, t^\varepsilon_m) \, dx \, dt$$

$$= -\int_{\Omega_T} u_\varepsilon(x, t) \left( \nabla x + \sum_{k=1}^{n-1} \frac{1}{\varepsilon_k} \nabla y_k \right) \cdot v(x, x^\varepsilon_n) c(t, t^\varepsilon_m) \, dx \, dt$$

$$= -\int_{\Omega_T} u_\varepsilon(x, t) \left( \nabla x + \sum_{k=1}^{n-1} \frac{1}{\varepsilon_k} \nabla y_k \right) \cdot v(x, x^\varepsilon_n) c(t, t^\varepsilon_m) \, dx \, dt,$$

where we in the first equality have used partial integration on $\Omega$, divergence theorem on $\Omega$ and the fact that both (though only one is necessary) $u_\varepsilon$ and $v$ vanish on $\partial \Omega$, and in the second equality used the fact that $v \in \mathcal{H}$ implying $\nabla y_k \cdot v = 0$.

We claim now that $\nabla y_k \cdot v \in \mathcal{E}_{k+1}$, $k \in [n-1]$. Indeed, for any $k \in [n-1]$ we have $\nabla y_k \cdot v \in \mathcal{D}(\Omega; C_\#^\infty(Y^n))$ and

$$\int_{Y_{[k+1,n]\Omega}} \nabla y_k \cdot v(x, y_n) \, dy_{[k+1,n]} = 0, \quad x \in \Omega, \ y_k \in Y_k,$$

where we have simply employed the definition of $v$ being in $\mathcal{H}$ making the multiple integral vanish, so $\nabla y_k \cdot v \in \mathcal{E}_{k+1}$. Thus, by Lemma 17 we have that $\{ \frac{1}{\varepsilon_{k+1}} \nabla y_k \cdot v^\varepsilon \}$ is bounded in $H^{-1}(\Omega)$ for all $k \in [n-1]$. This boundedness yields an estimation

$$\left| \int_{\Omega_T} u_\varepsilon(x, t) \sum_{k=1}^{n-1} \frac{1}{\varepsilon_k} \nabla y_k \cdot v(x, x^\varepsilon_n) c(t, t^\varepsilon_m) \, dx \, dt \right|^2$$

15
\[ T \int_0^T \left| \int_\Omega u_k(x,t) \sum_{k=1}^{n-1} \frac{1}{\epsilon_k} \nabla y_k \cdot v(x, x_n^e) c(t, t_m^e) \, dx \right|^2 dt \]
\[ \leq T \int_0^T \left| \left\langle \sum_{k=1}^{n-1} \frac{1}{\epsilon_k} \nabla y_k \cdot v^f, u_k(t)(c(t, t_m^e) \right\rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \right|^2 dt \]
\[ \leq T \int_0^T \left\| \sum_{k=1}^{n-1} \frac{1}{\epsilon_k} \nabla y_k \cdot v^f \right\|_{H^{-1}(\Omega)}^2 \| u_k(t)(c(t, t_m^e) \right\|_{H^1_0(\Omega)}^2 dt, \]

i.e.,
\[ \left| \int_\Omega u_k(x,t) \sum_{k=1}^{n-1} \frac{1}{\epsilon_k} \nabla y_k \cdot v(x, x_n^e) c(t, t_m^e) \, dx \right|^2 dt \]
\[ \leq C_1 \left( \sum_{k=1}^{n-1} \frac{1}{\epsilon_k} \| \nabla y_k \cdot v^f \|_{H^{-1}(\Omega)} \right)^2 \int_0^T \left\| u_k(t) \right\|_{H^1_0(\Omega)}^2 \| c(t, t_m^e) \| dt \]
\[ \leq C_2 \left( \sum_{k=1}^{n-1} \frac{1}{\epsilon_k} \right)^2 \int_0^T \left\| u_k(t) \right\|_{H^1_0(\Omega)}^2 dt = C_2 \left( \sum_{k=1}^{n-1} \frac{1}{\epsilon_k} \right)^2 \left\| u_k \right\|_{L^2(0,T; H^1_0(\Omega))}^2 \]
\[ \leq C_3 \left( \sum_{k=1}^{n-1} \frac{1}{\epsilon_k} \right)^2 \to 0 \]

since the scale functions are separated. We thus conclude that the left-hand side of (13) converges according to

\[ \int_\Omega \nabla u_k(x,t) \cdot v(x, x_n^e) c(t, t_m^e) \, dx \]
\[ \to -\int_\Omega \int_{Y^{\text{adm}}} u(x,t) \nabla x \cdot v(x, y_n) c(t, s_m) \, ds_m dy_n \, dx \]
\[ = \int_\Omega \int_{Y^{\text{adm}}} \nabla u(x,t) \cdot v(x, y_n) c(t, s_m) \, ds_m dy_n \, dx \]

for all \( v \in D(\Omega; C^2_0(Y^n)) \cap H \) and all \( c \in D(0,T) \circ C^2_0(S^m) \). From the right-hand side of (13) we thus obtain

\[ \int_\Omega \int_{Y^{\text{adm}}} \left( w_0(x,t, y_n, s_m) - \nabla u(x,t) \right) \cdot v(x, y_n) c(t, s_m) \, ds_m dy_n \, dx = 0, \]

or

\[ \int_0^T \int_{S^m} \left( \int_{T^{\text{adm}}} \left( w_0(x,t, y_n, s_m) - \nabla u(x,t) \right) \cdot v(x, y_n) \, dy_n \right) c(t, s_m) \, ds_m \, dt = 0. \]

By the Variational Lemma and utilising density (i.e., (i) in Lemma 16), for every \( v \in H \) it holds that

\[ \int_\Omega \int_{Y^{\text{adm}}} \left( w_0(x,t, y_n, s_m) - \nabla u(x,t) \right) \cdot v(x, y_n) \, dy_n \, dx = 0 \]
a.e. on \((0,T) \times S^m\). Hence,

\[ w_0 - \nabla u \perp v \quad \text{in } L^2(\Omega \times Y^n)^N \text{ a.e. on } (0,T) \times S^m, \]
i.e., \( w_0 - \nabla u \in H^\perp \) a.e. on \((0,T) \times S^m\). Thus, by (ii) in Lemma 16

\[ w_0 - \nabla u = \sum_{k=1}^n \nabla y_k u_k \quad \text{a.e. on } (0,T) \times S^m, \]
where $u_k \in L^2(\Omega \times Y^{k-1}; \mathcal{W}_k)$ a.e. on $(0, T) \times S^m$.

What remains is to prove that $u_k \in L^2(\Omega_T \times \mathcal{Y}_{(k-1)n}; \mathcal{W}_k)$, $k \in \mathbb{N}$. We will perform a proof by induction accomplished in two steps: the Base case followed by the Inductive step.

**Base case.** We must show that $u_1 \in L^2(\Omega_T \times S^m; \mathcal{W}_1)$. We have, a.e. on $\Omega_T \times \mathcal{Y}_{1m}$,

\[
\nabla y_i u_1(x, t, y_1, s_m) = \int_{Y^{[2,n]}} \nabla y_i u_1(x, t, y_1, s_m) \, dy_{[2,n]} \\
= \int_{Y^{[2,n]}} \sum_{i=1}^n \nabla y_i u_i(x, t, y_i, s_m) \, dy_{[2,n]} \\
= \int_{Y^{[2,n]}} (w_0(x, t, y_n, s_m) - \nabla u(x, t)) \, dy_{[2,n]} \\
= \int_{Y^{[2,n]}} w_0(x, t, y_n, s_m) \, dy_{[2,n]} - \nabla u(x, t),
\]

where the second equality follows from the fact that $u_i$ is $Y_i$-periodic. Hence, by (14) and the well-known characterisation of the $\mathcal{W}_1$-norm in terms of an $L^2$-norm of the gradient (see, e.g., Proposition 3.52 in [3]),

\[
\|u_1\|_{L^2(\Omega_T \times S^m; \mathcal{W}_1)} = \|\nabla y_1 u_1\|_{L^2(\Omega_T \times \mathcal{Y}_{1m})} \\
= \|\int_{Y^{[2,n]}} w_0 - \nabla u\|_{L^2(\Omega_T \times \mathcal{Y}_{1m})} \\
\leq \|\int_{Y^{[2,n]}} w_0\|_{L^2(\Omega_T \times \mathcal{Y}_{1m})} + \|\nabla u\|_{L^2(\Omega_T \times \mathcal{Y}_{1m})}. \tag{15}
\]

Since $w_0 \in L^2(\Omega_T \times \mathcal{Y}_{nm})$, we have that $\int_{Y^{[2,n]}} w_0 \in L^2(\Omega_T \times \mathcal{Y}_{1m})$, and since $u \in L^2(0, T; H^1_0(\Omega))$, it holds that $\nabla u \in L^2(\Omega_T) \subseteq L^2(\Omega_T \times \mathcal{Y}_{1m})$. Thus, by (15),

\[
\|u_1\|_{L^2(\Omega_T \times S^m; \mathcal{W}_1)} < \infty,
\]

which means that $u_1 \in L^2(\Omega_T \times S^m; \mathcal{W}_1)$ as desired; the Base case is complete.

**Inductive step.** Assume that $u_j \in L^2(\Omega_T \times \mathcal{Y}_{(j-1)m}; \mathcal{W}_j)$ for all $j \in \mathbb{N}$ where $\ell \in \mathbb{N}$ (requires $n > 1$; the case $n = 1$ is already treated in the Base case above). We must show that this assumption implies $u_{\ell+1} \in L^2(\Omega_T \times \mathcal{Y}_{\ell+1m}; \mathcal{W}_{\ell+1})$. If $\ell \in \mathbb{N}$, we have, a.e. on $\Omega_T \times \mathcal{Y}_{(\ell+1)m}$,

\[
\nabla y_i u_{\ell+1}(x, t, y_{\ell+1}, s_m) = \int_{Y^{[\ell+2,n]}} \nabla y_i u_{\ell+1}(x, t, y_{\ell+1}, s_m) \, dy_{[\ell+2,n]} \\
= \int_{Y^{[\ell+2,n]}} \sum_{i=1}^n \nabla y_i u_i(x, t, y_i, s_m) \, dy_{[\ell+2,n]} - \int_{Y^{[\ell+2,n]}} \sum_{i=1}^{\ell} \nabla y_i u_i(x, t, y_i, s_m) \, dy_{[\ell+2,n]} \\
= \int_{Y^{[\ell+2,n]}} (w_0(x, t, y_n, s_m) - \nabla u(x, t)) \, dy_{[\ell+2,n]} - \sum_{i=1}^\ell \nabla y_i u_i(x, t, y_i, s_m) \\
= \int_{Y^{[\ell+2,n]}} w_0(x, t, y_n, s_m) \, dy_{[\ell+2,n]} - \nabla u(x, t) - \sum_{i=1}^\ell \nabla y_i u_i(x, t, y_i, s_m), \tag{16}
\]

17
where the second equality follows from the fact that \( u_i \) is \( Y_T \)-periodic. If in this proof interpret integration over “\( Y^{[n+1,n]} \)” as performing no integration at all (i.e., \( \int_{Y^{[n+1,n]}} w_0 = w_0 \) by definition), (16) actually works for \( \ell = n - 1 \) as well. We get the norm

\[
\| u_{\ell+1} \|_{L^2(\Omega_T \times Y_{(\ell+1)m}; W_{\ell+1})} = \| \nabla u_{\ell+1} \|_{L^2(\Omega_T \times Y_{(\ell+1)m})} = \| \int_{Y^{[\ell+2,n]}} w_0 - \nabla u - \sum_{i=1}^{\ell} \nabla y_i u_i \|_{L^2(\Omega_T \times Y_{(\ell+1)m})} \\
\leq \| \int_{Y^{[\ell+2,n]}} w_0 \|_{L^2(\Omega_T \times Y_{(\ell+1)m})} + \| \nabla u \|_{L^2(\Omega_T \times Y_{(\ell+1)m})} + \sum_{i=1}^{\ell} \| \nabla y_i u_i \|_{L^2(\Omega_T \times Y_{(\ell+1)m})} \\
= \| \int_{Y^{[\ell+2,n]}} w_0 \|_{L^2(\Omega_T \times Y_{(\ell+1)m})} + \| \nabla u \|_{L^2(\Omega_T \times Y_{(\ell+1)m})} + \sum_{i=1}^{\ell} \| u_i \|_{L^2(\Omega_T \times Y_{(\ell+1)m}; W_{\ell+1})}. \tag{17}
\]

Since \( w_0 \in L^2(\Omega_T \times Y_{nm}) \), we have that \( \int_{Y^{[\ell+2,n]}} w_0 \in L^2(\Omega_T \times Y_{(\ell+1)m}) \), and since \( u \in L^2(0, T; H^1_0(\Omega)) \), it holds that \( \nabla u \in L^2(\Omega_T) \subset L^2(\Omega_T \times Y_{(\ell+1)m}) \). By the inductive assumption, \( u_j \in L^2(\Omega_T \times Y_{(j-1)m}; W_{\ell+1}) \) for all \( j \in \{\ell\} \). Thus, (17) gives

\[
\| u_{\ell+1} \|_{L^2(\Omega_T \times Y_{nm}; W_{\ell+1})} < \infty,
\]

which means that \( u_{\ell+1} \in L^2(\Omega_T \times Y_{nm}; W_{\ell+1}) \) as desired; the Inductive step is complete, and we are done. \( \square \)

When performing the homogenisation later in this paper we will limit ourselves to two spatial scales, \( n = 1 \), where the microscale is described by the single spatial scale function \( \varepsilon_1 \). The scale function \( \varepsilon_1 \) is, without loss of generality, assumed to coincide with the scale parameter, i.e., \( \varepsilon_1(\varepsilon) = \varepsilon \). Note that in what follows, the list \( \{\varepsilon\} \) of the single spatial scale function will be written as \( \varepsilon \) for brevity. In this setting we have Theorem 20 below. We first need a lemma.

**Lemma 19.** Suppose \( \rho \in C_0^\infty(Y)/\mathbb{R} \). Then there exists a unique \( \theta \in C_0^\infty(Y)/\mathbb{R} \) such that \( \rho = \Delta_y \theta \) where \( \Delta_y \) is the Laplace operator with respect to \( y \) (i.e., \( \Delta_y = \nabla_y \cdot \nabla_y \)).

**Proof.** First we note that for any given \( \rho \in L^2_\#(Y)/\mathbb{R} \) there exists a unique function \( \theta \in H^1_\#(Y)/\mathbb{R} \) such that \( \rho = \Delta_y \theta \). Then we consider only smooth source functions \( \rho \in C_0^\infty(Y)/\mathbb{R} \subset L^2_\#(Y)/\mathbb{R} \) and utilise the hypoellipticity property of the Laplace operator to conclude that \( \theta \) must also belong to \( C_0^\infty(Y)/\mathbb{R} \). (For a further discussion see, e.g., Remark 3.2 in [34].) \( \square \)

In the remainder of the paper, let \( W = H^1_\#(Y)/\mathbb{R} \).
Theorem 20. Suppose that the pair \((\epsilon, \{\epsilon_i\}_{i=1}^m)\) of lists of spatial and temporal scale functions belongs to \(J_{wsp}^1\) and assume that \(\{u_\epsilon\}\) is a bounded sequence in the function space \(H^1(0, T; H^1(\Omega), H^{-1}(\Omega))\). Then, up to a subsequence,

\[
\int_{\Omega_T} \frac{1}{\epsilon} u_\epsilon(x, t) \phi(x, t, \frac{x}{\epsilon}, t^\epsilon_m) \, dx dt \to \int_{\Omega_T} \int_{\Omega_{1m}} u_1(x, t, y, s_m) \phi(x, t, y, s_m) \, ds_m dy dx dt
\]

for all \(\phi \in \mathcal{D}(\Omega) \otimes \mathcal{D}(0, T) \otimes (C^\infty_\#(Y)/\mathbb{R}) \otimes (\prod_{i=1}^m C^\infty_\#(S_i))\), where \(u_1 \in L^2(\Omega_T \times S^m; \mathcal{W})\) is as in Theorem 18 (with \(n = 1\)).

Proof. Fix an arbitrary \(\phi \in \mathcal{D}(\Omega) \otimes \mathcal{D}(0, T) \otimes (C^\infty_\#(Y)/\mathbb{R}) \otimes (\prod_{i=1}^m C^\infty_\#(S_i))\). Then there exist unique \(\psi \in \mathcal{D}(\Omega) \otimes \mathcal{D}(0, T) \otimes (\prod_{i=1}^m C^\infty_\#(S_i))\) and \(\rho \in C^\infty_\#(Y)/\mathbb{R}\) such that \(\phi = \psi \rho\). The left-hand side of (18) can then be written

\[
\int_{\Omega_T} \frac{1}{\epsilon} u_\epsilon(x, t) \phi(x, t, \frac{x}{\epsilon}, t^\epsilon_m) \, dx dt = \int_{\Omega_T} \frac{1}{\epsilon} u_\epsilon(x, t) \psi(x, t, t^\epsilon_m) \rho(\frac{x}{\epsilon}) \, dx dt
\]

for some unique \(\theta \in C^\infty_\#(Y)/\mathbb{R}\) due to Lemma 19. By noting that

\[
\nabla \cdot \sigma(\frac{x}{\epsilon}) = \frac{1}{\epsilon} \nabla_y \cdot \sigma(\frac{x}{\epsilon})
\]

for any \(\sigma\) differentiable over \(Y\) (here \(\sigma = \nabla_y \theta\)), we get by partial integration on \(\Omega\) that

\[
\int_{\Omega_T} \frac{1}{\epsilon} u_\epsilon(x, t) \phi(x, t, \frac{x}{\epsilon}, t^\epsilon_m) \, dx dt = \int_{\Omega_T} u_\epsilon(x, t) \psi(x, t, t^\epsilon_m) \nabla \cdot \nabla_y \theta(\frac{x}{\epsilon}) \, dx dt
\]

\[
= \int_{\Omega_T} \left( \nabla \cdot (u_\epsilon(x, t) \psi(x, t, t^\epsilon_m) \nabla_y \theta(\frac{x}{\epsilon}))
\right.
\]

\[
- \nabla u_\epsilon(x, t) \psi(x, t, t^\epsilon_m) \cdot \nabla_y \theta(\frac{x}{\epsilon})
\]

\[
- u_\epsilon(x, t) \nabla \psi(x, t, t^\epsilon_m) \cdot \nabla_y \theta(\frac{x}{\epsilon}) \right) \, dx dt
\]

\[
= -\int_{\Omega_T} \nabla u_\epsilon(x, t) \psi(x, t, t^\epsilon_m) \cdot \nabla_y \theta(\frac{x}{\epsilon}) \, dx dt
\]

\[
- \int_{\Omega_T} u_\epsilon(x, t) \nabla \psi(x, t, t^\epsilon_m) \cdot \nabla_y \theta(\frac{x}{\epsilon}) \, dx dt,
\]

where we in the last equality for the first term in the integrand have employed the divergence theorem on \(\Omega\) and used the fact that both (though only one is necessary) \(u_\epsilon\) and \(\psi\) vanish on \(\partial \Omega\). Furthermore, by utilisation of Theorem 18 with \(n = 1\), we get (with \(u \in L^2(0, T; H^1_0(\Omega))\) and \(u_1 \in L^2(\Omega_T \times S^m; \mathcal{W})\) as in Theorem 18 with \(n = 1\), up to a subsequence,

\[
\int_{\Omega_T} \frac{1}{\epsilon} u_\epsilon(x, t) \phi(x, t, \frac{x}{\epsilon}, t^\epsilon_m) \, dx dt
\]

\[
\to -\int_{\Omega_T} \int_{\Omega_{1m}} (\nabla u(x, t) + \nabla_y u_1(x, t, y, s_m)) \psi(x, t, s_m) \cdot \nabla_y \theta(y) \, ds_m dy dx dt
\]
we have derived (18). Since \( \theta \) is an arbitrary function, the claim of the theorem follows.

Remark 21. Theorem 20 is a mere variety of Lemma 3.1 in [34] in the special case of periodicity but generalised to include several temporal scales. In its turn, the result in [34] is a mere variation of Corollary 3.3 in [17] generalised to the non-periodic setting and with the sequence \( \{ 1/n \} \) (as derived in Theorem 20 above) instead of the slightly more complicated sequence \( \{ \frac{1}{\epsilon} (u_\epsilon - u) \} \) found in [17].

The convergence mode in Theorem 20 can be regarded as a kind of feeble, or “very weak”, \((2, m + 1)\)-scale convergence of \( \{ 1/n \} \) since the heavily restricted set of test functions in question is more permissible compared to the larger set of test functions employed in ordinary \((2, m + 1)\)-scale convergence.

Finally, we remark that a result analogous to Proposition 11 holds for sequences of the type \( \{ \frac{1}{\epsilon} r(\epsilon) u_\epsilon \} \) having a “very weak” limit \( r_0 u_1 \) instead of \( u_1 \) if \( r(\epsilon) \to r_0 \).

### 3 Monotone Parabolic Operators

Consider the operator-form evolution problem

\[
\begin{cases}
\frac{d}{dt} u + A u = f, \\
\quad u(0) = u^0 \in H, \\
\quad u \in H^1(0, T; V, V'),
\end{cases}
\tag{19}
\]

where \( f \in L^2(0, T; V') \) and \( A : L^2(0, T; V) \to L^2(0, T; V') \). Here \( H \) is some Hilbert space and \( V \) is some Banach space with topological dual \( V' \). Note that \( u \in H^1(0, T; V, V') \) means \( u \in L^2(0, T; V) \) and \( \frac{d}{dt} u \in L^2(0, T; V') \), \( \frac{d}{dt} \) being the weak (or distributional or generalised) derivative with respect to the temporal variable.
\[ t \in (0, T) \]. The definition below establishes a convenient relation between \( H, V \) and \( V' \).

**Definition 22.** Suppose \( H \) is a real and separable Hilbert space and that \( V \) is a real, separable and reflexive Banach space such that \( V \) is continuously embedded and dense in \( H \). We then call \((V, H, V')\) an evolution triple.

**Remark 23.** Recall that \( V \) is continuously embedded in \( H \) if \( V \subset H \) and there exists \( C > 0 \) such that \( \|u\|_H \leq C \|u\|_V \) for all \( u \in V \). Also note that by Riesz’s representation theorem, \( H \) can be identified by its dual \( H' \) and that \( H' \) is continuously embedded and dense in \( V' \). Schematically we have

\[
\begin{array}{c|c|c}
V & \text{Cont. emb. \\ & & dense} & H \\
\text{Riesz’s repr. th.} & \sim & H' \\
& \text{Cont. emb.} & \subset \ V'.
\end{array}
\]

Let for every fixed \( t \in (0, T) \) the operator \( A(t) : V \to V' \) be defined by

\[
A(t)u(t) = (Au)(t) \quad \text{(} u \in L^2(0, T; V) \text{).}
\]

In order for the problem (19) to have a unique solution the operator \( A \) should satisfy the following five conditions:

(A1) \( \langle A(t)u - A(t)v, u - v \rangle_{V', V} \geq 0 \) for all \( u, v \in V \) and all \( t \in (0, T) \) (i.e., \( A(t) \) is monotone);

(A2) The \( [0, 1] \to \mathbb{R} \) function \( q \mapsto \langle A(t)(u + qw), v \rangle_{V', V} \) is continuous for all \( u, v, w \in V \) and all \( t \in (0, T) \) (i.e., \( A(t) \) is hemicontinuous);

(A3) There exists \( C_0 > 0 \) such that \( \langle A(t)u, u \rangle_{V', V} \geq C_0 \|u\|_V^2 \) for all \( u \in V \) and all \( t \in (0, T) \) (i.e., \( A(t) \) is coercive);

(A4) There exist a non-negative function \( \beta \in L^2(0, T) \) and a constant \( C_1 > 0 \) such that \( \|A(t)u\|_V \leq \beta(t) + C_1 \|u\|_V \) for all \( u \in V \) and all \( t \in (0, T) \) (i.e., \( A \) satisfies a certain growth condition);

(A5) The \( (0, T) \to \mathbb{R} \) function \( t \mapsto \langle A(t)u, v \rangle_{V', V} \) is measurable on \( (0, T) \) for all \( u, v \in V \) (i.e., \( t \mapsto A(t) \) is weakly measurable on \( (0, T) \)).

We have the following theorem on existence and uniqueness.

**Theorem 24.** Suppose that \( A : L^2(0, T; V) \to L^2(0, T; V') \) satisfies (A1)–(A5) above and assume that \((V, H, V')\) forms an evolution triple. Then, for every \( f \in L^2(0, T; V') \) and \( u^0 \in H \), there exists a unique solution \( u \) to (19).

**Proof.** See Theorem 30.A in [48].
Let $X = L^2(0, T; V)$ and $X' = L^2(0, T; V')$, and consider a sequence $\{A^\varepsilon\}$ of monotone operators. Equivalently to (19) for this sequence of operators, the evolution problem can be formulated as

$$\begin{align*}
\begin{cases}
\langle \frac{d}{dt} u_\varepsilon, v \rangle_{X', X} + \langle A^\varepsilon u_\varepsilon, v \rangle_{X', X} = \langle f, v \rangle_{X', X}, \\
u(0) = u^0 \in H,
\end{cases}
\end{align*}$$

for all $v \in X = L^2(0, T; V)$, where $u^0 \in H$, $f \in X' = L^2(0, T; V')$ and $(V, H, V')$ is an evolution triple.

Fix $H = L^2(\Omega)$ and $V = H^0_0(\Omega)$ with dual $V' = H^{-1}(\Omega)$. Then

$$(H^0_0(\Omega), L^2(\Omega), H^{-1}(\Omega))$$

is an evolution triple. We let the operators $A^\varepsilon : L^2(0, T; H^0_0(\Omega)) \rightarrow L^2(0, T; H^{-1}(\Omega))$ be defined in terms of a flux $a^\varepsilon : \overline{\Omega_T} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ by

$$\langle A^\varepsilon u, v \rangle_{X', X} = \int_{\Omega_t} a^\varepsilon(x, t; \nabla u) \cdot \nabla v(x, t) \, dx dt$$

for $u, v \in L^2(0, T; H^0_0(\Omega))$, which—by the definition (20) of the time dependent operator—is the same as

$$\langle A^\varepsilon(t)u, v \rangle_{H^{-1}(\Omega), H^0_0(\Omega)} = \int_{\Omega} a^\varepsilon(x, t; \nabla u) \cdot \nabla v(x) \, dx$$

for $u, v \in H^0_0(\Omega)$. We recall that $a^\varepsilon$ is given via $a : \overline{\Omega_T} \times \mathbb{R}^{nN+m} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ according to

$$a^\varepsilon(x, t; k) = a(x, t; x_n^\varepsilon, k_m^\varepsilon, k) \quad ((x, t) \in \Omega_T, k \in \mathbb{R}^N).$$

The problem

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} u_\varepsilon(x, t) - \nabla \cdot a^\varepsilon(x, t; \nabla u_\varepsilon) &= f(x, t) \quad \text{in } \Omega_T, \\
u_\varepsilon(x, 0) &= u^0(x) \quad \text{in } \Omega, \\
u_\varepsilon(x, t) &= 0 \quad \text{on } \partial \Omega \times (0, T),
\end{align*}$$

is the same as (1) but generalised to $n + 1$ spatial scales. Clearly, with the conventions above, (21) is the weakly formulated version of (24). To conclude, the weak formulation is that, given $f \in X' = L^2(0, T; H^{-1}(\Omega))$ and $u^0 \in L^2(\Omega)$, we want to find $u_\varepsilon \in H^1(0, T; H^0_0(\Omega), H^{-1}(\Omega))$ such that

$$\langle \frac{d}{dt} u_\varepsilon, v \rangle_{X', X} + \int_{\Omega_T} a(x, t; x_n^\varepsilon, f_n^\varepsilon; \nabla u_\varepsilon) \cdot \nabla v(x, t) \, dx dt = \int_{\Omega_T} f(x, t) v(x, t) \, dx dt$$

for all $v \in X = L^2(0, T; H^0_0(\Omega))$. The function $a$ should satisfy the following five structure conditions:
(B1) \( a(x, t, y_n, s_m; 0) = 0 \) for all \( (x, t) \in \overline{\Omega}_T \) and all \( (y_n, s_m) \in \mathbb{R}^{nN+m} \);

(B2) \( a(x, t, \cdot ; k) \) is \( \mathcal{Y}_{nm} \)-periodic for all \( (x, t) \in \overline{\Omega}_T \) and all \( k \in \mathbb{R}^N \), and \( a(\cdot ; k) \) is continuous for all \( k \in \mathbb{R}^N \);

(B3) \( a(x, t, y_n, s_m; \cdot) \) is continuous for all \( (x, t) \in \overline{\Omega}_T \) and all \( (y_n, s_m) \in \mathbb{R}^{nN+m} \);

(B4) There exists \( C_0 > 0 \) such that

\[
(a(x, t, y_n, s_m; k) - a(x, t, y_n, s_m; k')) \cdot (k - k') \geq C_0 |k - k'|^2
\]

for all \( (x, t) \in \Omega_T \), all \( (y_n, s_m) \in \mathbb{R}^{nN+m} \) and all \( k, k' \in \mathbb{R}^N \);

(B5) There exist \( C_1 > 0 \) and \( 0 < \alpha \leq 1 \) such that

\[
|a(x, t, y_n, s_m; k) - a(x, t, y_n, s_m; k')| \leq C_1 (1 + |k| + |k'|)^{1-\alpha} |k - k'|^\alpha
\]

for all \( (x, t) \in \Omega_T \), all \( (y_n, s_m) \in \mathbb{R}^{nN+m} \) and all \( k, k' \in \mathbb{R}^N \).

We have the following proposition linking the structural conditions (B1)–(B5) for \( a \) to the conditions (A1)–(A5) for \( A^f \).

**Proposition 25.** Suppose that \( a : \overline{\Omega}_T \times \mathbb{R}^{nN+m} \times \mathbb{R}^N \to \mathbb{R}^N \) fulfils the structure conditions (B1)–(B5). Then \( A^f : L^2(0, T; H^1_0(\Omega)) \to L^2(0, T; H^{-1}(\Omega)) \) defined through (22) satisfies the conditions (A1)–(A5).

**Proof.** We first prove that the monotonicity condition (A1) holds. Fix an arbitrary \( t \in (0, T) \). Then, for any \( u, v = H^1_0(\Omega) \),

\[
\left\langle A^f(t)u - A^f(t)v, u - v \right\rangle_{H^{-1}(\Omega), H^1_0(\Omega)} = \int_\Omega (a(x, t, x_t^n, t_{m_i}; \nabla u) - a(x, t, x_t^n, t_{m_i}; \nabla v)) \cdot (\nabla u(x) - \nabla v(x)) \, dx \\
\geq C_0 \int_\Omega |\nabla u(x) - \nabla v(x)|^2 \, dx = C_0 \|u - v\|^2_{H^1_0(\Omega)} \\
\geq 0,
\]

where we have employed the structure condition (B4) to obtain the first inequality.

Next we prove the hemicontinuity condition (A2). Fix arbitrary \( t \in (0, T) \) and \( q_0 \in [0, 1] \), and let \( q \in [0, 1] \). Then, for any \( u, v, w \in H^1_0(\Omega) \),

\[
\left| \left\langle A^f(t)(u + qw), v \right\rangle_{H^{-1}(\Omega), H^1_0(\Omega)} - \left\langle A^f(t)(u + q_0w), v \right\rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \right| \\
= \left| \int_\Omega (a(x, t, x_t^n, t_{m_i}; \nabla u + q \nabla w) - a(x, t, x_t^n, t_{m_i}; \nabla v + q_0 \nabla w)) \cdot \nabla v(x) \, dx \right| \\
\leq \int_\Omega |a(x, t, x_t^n, t_{m_i}; \nabla u + q \nabla w) - a(x, t, x_t^n, t_{m_i}; \nabla u + q_0 \nabla w)| |\nabla v(x)| \, dx \\
\leq C_1 \int_\Omega \left( 1 + |\nabla u(x) + q \nabla w(x)| \right)
\]

23
\begin{align*}
+ |\nabla u(x) + q_0 \nabla w(x)| \right)^{1-\alpha} |(q - q_0) \nabla w(x)|^\alpha |\nabla v(x)| \, dx \\
\leq C_1 |q - q_0|^\alpha \int_\Omega \left( 1 + 2 |\nabla u(x)| + 2 |\nabla w(x)| \right)^{1-\alpha} |\nabla w(x)|^\alpha |\nabla v(x)| \, dx \\
< 2^{-\alpha} C_1 |q - q_0|^\alpha \int_\Omega \left( 1 + 2 |\nabla u(x)| + 2 |\nabla w(x)| \right) |\nabla v(x)| \, dx \\
\leq 2^{-\alpha} C_1 \| (1 + 2 |\nabla u| + 2 |\nabla w|) \| \nabla v\|_L^1(\Omega) |q - q_0|^\alpha \\
\leq 2^{-\alpha} C_1 \left( \| 1 \|_{L^2(\Omega)} + 2 \| |\nabla u| \|_{L^2(\Omega)} + 2 \| |\nabla w| \|_{L^2(\Omega)} \right) \| \nabla v\|_{L^2(\Omega)} |q - q_0|^\alpha \\
= 2^{-\alpha} C_1 \left( |\Omega|^{\frac{1}{2}} + 2 \| u \|_{H^1(\Omega)} + 2 \| w \|_{H^1(\Omega)} \right) \| \nabla v\|_{H^1(\Omega)} |q - q_0|^\alpha \\
\to 0
\end{align*}

as \( q \to q_0 \), where we have utilised (B_5) for the second inequality and H"older’s inequality together with the triangle inequality to obtain the last inequality. Thus, the hemicontinuity holds.

Let us move on to proving the coercivity condition (A_3). Fix \( t \in (0, T) \). Then, for any \( u \in H^1(\Omega) \),

\begin{align*}
\langle A^\varepsilon(t)u, u \rangle_{H^{-1}(\Omega), H^1(\Omega)} &= \int_\Omega a^\varepsilon(x, t; \nabla u) \cdot \nabla u(x) \, dx \\
&= \int_\Omega \left( a^\varepsilon(x, t; \nabla u) - a^\varepsilon(x, t; 0) \right) \cdot (\nabla u(x) - 0) \, dx \\
&\geq C_0 \int_\Omega |\nabla u(x) - 0|^2 \, dx \\
&= C_0 \| u \|^2_{H^1(\Omega)},
\end{align*}

where we have used structure condition (B_1) to obtain the second equality and (B_4) for the inequality.

The growth condition (A_4) is proven in the following manner. We first note that by (B_1) and (B_5),

\begin{align*}
|a(x, t, y_n, s_m; k)| &= |a(x, t, y_n, s_m; k) - a(x, t, y_n, s_m; 0)| \\
&\leq C_1 (1 + |k|)^{1-\alpha} |k|^\alpha \\
&< C_1 (1 + |k|)^{1-\alpha} (1 + |k|)^\alpha \\
&= C_1 (1 + |k|)^\alpha
\end{align*}

(26)

for all \((x, t) \in \Omega_T\), all \((y_n, s_m) \in \mathbb{R}^{n+m}\) and all \( k \in \mathbb{R}^n \). Fix \( t \in (0, T) \) and let \( u \in H^1(\Omega) \) be arbitrary. Then

\begin{align*}
\| A^\varepsilon(t)u \|_{H^{-1}(\Omega)} &= \sup_{\| v \|_{H^1(\Omega)} \leq 1} \left| \langle A^\varepsilon(t)u, v \rangle_{H^{-1}(\Omega), H^1(\Omega)} \right| \\
&= \sup_{\| v \|_{H^1(\Omega)} \leq 1} \left| \int_\Omega a(x, t, x_n, s_m; \nabla u) \cdot \nabla v(x) \, dx \right|
\end{align*}

24
\[
\leq \sup_{\|v\|_{H^1_0(\Omega)} \leq 1} \int_{\Omega} \left| a(x, t, x^e_n, t_n^e; \nabla u) \right| \left| \nabla v(x) \right| \, dx \\
< C_1 \sup_{\|v\|_{H^1_0(\Omega)} \leq 1} \int_{\Omega} \left( 1 + \left| \nabla u(x) \right| \right) \left| \nabla v(x) \right| \, dx \\
= C_1 \sup_{\|\nabla v\|_{L^2(\Omega)} \leq 1} \left\| (1 + |\nabla u|) |\nabla v| \right\|_{L^1(\Omega)},
\]

where in the second inequality we have employed (26). By Hölder’s inequality we obtain

\[
\left\| \mathcal{A}^e(t) u \right\|_{H^{-1}(\Omega)} \leq C_1 \sup_{\|\nabla v\|_{L^2(\Omega)} \leq 1} \left\| (1 + |\nabla u|) |\nabla v| \right\|_{L^2(\Omega)} \\
\leq C_1 \left\| 1 + |\nabla u| \right\|_{L^2(\Omega)} \leq C_1 \left( \|1\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} \right) \\
= \beta + C_1 \|u\|_{H^1_0(\Omega)}.
\]

This growth constraint is even more regular than anticipated since \( \beta = C_1 \sqrt{|\Omega|} \) is independent of \( t \in (0, T) \).

Finally, the weak measurability condition (A5) follows readily from the continuity assumptions on \( a \) and the boundedness property (26).

The following important theorem follows immediately from Proposition 25 above together with Theorem 24.

**Theorem 26.** Suppose that \( a : \overline{\Omega}_T \times \mathbb{R}^{n+m} \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) fulfils the structure conditions (B1)–(B5). Then, for every \( f \in L^2(\Omega_T) \) and \( u^0 \in L^2(\Omega) \), the evolution problem (24) has a unique weak solution \( u_\varepsilon \in H^1(0, T; H^1_0(\Omega), H^{-1}(\Omega)) \).

### 4 H-Convergence of Monotone Parabolic Problems

In 1967 Spagnolo introduced the notion of G-convergence for linear problems governed by symmetric matrices (see [37]; see also [38, 39, 9]). The name “G”-convergence comes from the fact that this convergence mode corresponds to the convergence of the Green functions associated to the sequence of problems. The G-convergence of symmetric matrices is defined via the weak convergence of solutions to the sequence of problems.

The concept of H-convergence—“H” as in “homogenisation”—is a generalisation of Spagnolo’s G-convergence to cover also non-symmetric matrices. It was introduced in 1976 by Tartar (see [43]; see also [44]) and further developed by Murat in 1978 (see [26, 27]; see also [28]), and in 1977 Tartar defined H-convergence for nonlinear monotone problems (see [42]; see also [5, 6]). Early studies of H-convergence
for non-linear monotone parabolic problems were conducted by Kun’čh and Pankov in 1986 (see [22]), Kun’čh in 1988 (see [21]), and Svanstedt in 1992 (see [40]; see also [41] by Svanstedt and [35] by Pankov for further developments).

We introduce a convenient set of flux functions in the following definition.

**Definition 27.** Suppose \( C_0, C_1 > 0 \) and \( 0 < \alpha \leq 1 \). A function \( a : \Omega_T \times \mathbb{R}^N \to \mathbb{R}^N \) is said to belong to \( \mathcal{M}^a_{C_0,C_1}(\Omega_T) \) if the following four structure conditions are satisfied:

1. \((C_1)\) \( a(x,t;0) = 0 \) a.e. \((x,t) \in \Omega_T;\)
2. \((C_2)\) \( a(\cdot; k) \) is (Lebesgue) measurable for every \( k \in \mathbb{R}^N;\)
3. \((C_3)\) \( (a(x,t;k) - a(x,t;k')) \cdot (k - k') \geq C_0 |k - k'|^2 \) a.e. \((x,t) \in \Omega_T \) and for all \( k,k' \in \mathbb{R}^N;\)
4. \((C_4)\) \( |a(x,t;k) - a(x,t;k')| \leq C_1 (1 + |k| + |k'|)^{1-\alpha} |k - k'|^\alpha \) a.e. \((x,t) \in \Omega_T \) and for all \( k,k' \in \mathbb{R}^N.\)

If no values on \( C_0, C_1 > 0 \) and \( 0 < \alpha \leq 1 \) are fixed we simply say that \( a \in \mathcal{M}(\Omega_T) \), i.e.,

\[
\mathcal{M}(\Omega_T) = \bigcup_{C_0,C_1 > 0} \mathcal{M}^a_{C_0,C_1}(\Omega_T).
\]

We may drop \( \Omega_T \) as soon as there is no hazard of confusion, i.e., \( \mathcal{M}^a_{C_0,C_1} = \mathcal{M}^a_{C_0,C_1}(\Omega_T) \) and \( \mathcal{M} = \mathcal{M}(\Omega_T) \).

The important concept of H-convergence of monotone parabolic problems—coined \( H_{MP} \)-convergence in this paper for brevity—is introduced in the definition below.

**Definition 28.** Suppose \( \{a^e\} \) is a sequence of fluxes in \( \mathcal{M} \). We say that \( \{a^e\} \) \( H_{MP} \)-converges to the flux \( b \in \mathcal{M} \) if, for any \( f \in L^2(0,T;H^{-1}(\Omega)) \) and any \( u^0 \in L^2(\Omega) \), the weak solutions \( u^e \in H^1(0,T;H^1_0(\Omega),H^{-1}(\Omega)) \) to the sequence of evolution problems satisfy

\[
\begin{align*}
\frac{\partial}{\partial t} u^e(x,t) - \nabla \cdot a^e(x,t;\nabla u^e) &= f(x,t) \quad \text{in } \Omega_T, \\
u^e(x,0) &= u^0(x) \quad \text{in } \Omega, \\
u^e(x,t) &= 0 \quad \text{on } \partial \Omega \times (0,T)
\end{align*}
\] (27)

of evolution problems satisfy

\[
\begin{align*}
u^e &\rightharpoonup u \quad \text{in } L^2(0,T;H^1_0(\Omega)), \\
a^e(\cdot;\nabla u^e) &\rightharpoonup b(\cdot;\nabla u) \quad \text{in } L^2(\Omega_T)^N,
\end{align*}
\]
where \( u \in H^1(0,T; H^1_0(\Omega), H^{-1}(\Omega)) \) is the weak unique solution to the evolution problem

\[
\begin{aligned}
\frac{\partial}{\partial t} u(x, t) - \nabla \cdot b(x, t; \nabla u) &= f(x, t) \quad \text{in} \; \Omega_T, \\
u(x, 0) &= u^0(x) \quad \text{in} \; \Omega, \\
u(x, t) &= 0 \quad \text{on} \; \partial \Omega \times (0,T).
\end{aligned}
\]

(28)

Moreover, for brevity, we write this convergence \( a^e \overset{\text{HMP}}{\rightharpoonup} b \), and \( b \) is called the \( \text{HMP-limit of} \) \( \{a^e\} \).

The definition above leads to the compactness result below.

**Theorem 29.** Let \( \{a^e\} \) be a sequence of fluxes in \( \mathcal{M}^{\alpha}_{C_0,C_1} \). Then, up to a subsequence, \( a^e \overset{\text{HMP}}{\rightharpoonup} b \) for some \( b \in \mathcal{M}^{\alpha/(2-\alpha)}_{C'_0,C'_1} \) where \( C'_0, C'_1 > 0 \) only depend on the constants \( C_0, C_1, \alpha \).

**Proof.** This is just a special case of Theorem 3.1 in [41]. \( \square \)

In the case that \( \{a^e\} \) is given according to (23) we have the following proposition linking the structure conditions (B1)–(B5) for \( a \) to the conditions (C1)–(C4) for the sequence \( \{a^e\} \) to be in \( \mathcal{M}^{\alpha}_{C_0,C_1} \).

**Proposition 30.** Suppose that \( a : \overline{\Omega}_T \times \mathbb{R}^{n+m} \times \mathbb{R}^N \to \mathbb{R}^N \) fulfils the structure conditions (B1)–(B5). Then \( \{a^e\} \) defined through (23) is a sequence in \( \mathcal{M}^{\alpha}_{C_0,C_1} \) where \( C_0, C_1 \) and \( \alpha \) are the constants introduced in (B1)–(B5).

**Proof.** We begin by recalling (23), i.e.,

\[
a^e(x, t; k) = a(x, t, x_n^e, t_m^e; k) \quad ((x, t) \in \Omega_T, k \in \mathbb{R}^N).
\]

For condition (C1) we have that

\[
a^e(x, t; 0) = a(x, t, x_n^e, t_m^e; 0) = 0
\]

for all \( (x, t) \in \Omega_T \) by (B1).

Secondly, the (Lebesgue) measurability condition (C2) follows from the continuity and periodicity properties in condition (B2).

Next we wish to verify (C3). For all \( (x, t) \in \Omega_T \) and all \( k, k' \in \mathbb{R}^N \),

\[
(a^e(x, t; k) - a^e(x, t; k')) \cdot (k - k') = (a(x, t, x_n^e, t_m^e; k) - a(x, t, x_n^e, t_m^e; k')) \cdot (k - k') \\
\geq C_0 |k - k'|^2
\]

according to structure condition (B4).

Finally, (C4) is to be checked. For all \( (x, t) \in \Omega_T \) and all \( k, k' \in \mathbb{R}^N \),

\[
|a^e(x, t; k) - a^e(x, t; k')| = |a(x, t, x_n^e, t_m^e; k) - a(x, t, x_n^e, t_m^e; k')|
\]
\[ \leq C_1 (1 + |k| + |k'|)^{1-\alpha} |k - k'|^{\alpha}. \]

We conclude that \( \{a^e\} \) is in \( \mathcal{M}_{C_0,C_1}^\alpha \) where \( C_0, C_1 \) and \( \alpha \) are precisely the constants introduced in (B1)–(B5), and we are done. \( \square \)

We have the following proposition governing an a priori estimate on the solutions to the sequence of evolution problems.

**Proposition 31.** Suppose that \( a : \Omega_T \times \mathbb{R}^{n+m} \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) fulfils the structure conditions (B1)–(B5). Then the sequence \( \{u_e\} \) of weak solutions to the evolution problem (27) with \( \{a^e\} \) defined through (23) satisfies

\[ \|u_e\|_{H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega))} \leq C \]  

for some \( C > 0 \). In other words, \( \{u_e\} \) is uniformly bounded in \( H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega)) \).

**Proof.** For every fixed \( \varepsilon > 0 \) we know as a matter of fact that we have a unique weak solution \( u_e \in H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega)) \) to (24) by Theorem 26.

Let us now verify the uniform boundedness in \( H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega)) \), i.e., (29). By Proposition 30 we know that \( \{a^e\} \) is in \( \mathcal{M} \). We can then apply Proposition 2.3 and Corollary 2.1 in [41] which in this context say that \( \{u_e\} \) and \( \{\frac{\partial}{\partial t} u_e\} \) are uniformly bounded in \( L^2(0,T;H_0^1(\Omega)) \) and \( L^2(0,T;H^{-1}(\Omega)) \), respectively. Thus, we have uniform boundedness in \( H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega)) \), i.e., (29) holds. The proof is complete. \( \square \)

## 5 Homogenisation

The notion of homogenisation of problems with multiple microscales was introduced in 1978 by Bensoussan, Lions and Papanicolaou (see [4]) who homogenised problems with two microscales characterised by the list \( \{\varepsilon, \varepsilon^2\} \) of scale functions. In 1996, Allaire and Briane (see [2]) succeeded to generalise this to homogenisation of linear elliptic problems with an arbitrary number of microscales—even infinitely many—without even assuming the scale functions to be power functions using the notion of (well-)separatedness instead. This was achieved by introducing the multiscale convergence technique. In 2001, Lions, Lukkassen, Persson and Wall performed homogenisation of non-linear monotone elliptic problems with scale functions \( \{\varepsilon, \varepsilon^2\} \) (see [23]), and in 2005 Holmbom, Svanstedt and Wellander studied homogenisation of linear parabolic problems with pairs \( \{\varepsilon, \varepsilon^2\}, e^k \) of lists of scale functions (see [20]). In 2006, Flodén and Olsson generalised to monotone parabolic problems (see [13]; see also [15] by Flodén, Olsson, Holmbom and Svanstedt for
This paper deals with monotone parabolic problems with an arbitrary number of temporal microscales not necessarily characterised by scale functions in the form of power functions but instead using the concept of (well-)separatedness in spirit of [2]. Furthermore—for simplicity—we only consider two spatial scales of which one is microscopical, i.e., henceforth we fix $n = 1$.

Let $k \in [m]$. Define $\mathcal{J}_\text{wsep}^{m-k}$ to be the set of all pairs $(\epsilon, \{\epsilon'_j\}^m_{j=1})$ in $\mathcal{J}_\text{wsep}^m$ such that $\epsilon'_k \sim \epsilon$. (There is no loss of generality to assume mere asymptotic equality rather than the ostensibly more general asymptotic equality modulo a positive constant, i.e., $\epsilon'_k \sim C\epsilon$, $C \in \mathbb{R}$.) In other words, $\mathcal{J}_\text{wsep}^{m-k}$ consists of pairs $(\epsilon, \{\epsilon'_j\}^m_{j=1})$ for which the temporal scale functions are separated and the $k$-th temporal scale function coincides asymptotically with the spatial scale function. (This clearly explains the convenient notation "$\sim k"" which could be read "the spatial scale is asymptotically equal to the $k$-th temporal scale").

Define the collection \( \mathcal{J}_\text{wsep}^{m-k} \{ \mathcal{J}_\text{wsep,j}^{1+2(m-k)} \}_{j=1} \) of $1 + 2(m-k)$ subsets of $\mathcal{J}_\text{wsep}^{m-k}$ by

- $\mathcal{J}_\text{wsep,1}^{m-k} = \{(\epsilon, \{\epsilon'_j\}^m_{j=1}) \in \mathcal{J}_\text{wsep}^{m-k} : \frac{\epsilon'_2}{\epsilon} \to 0\}$,
- $\mathcal{J}_\text{wsep,2}^{m-k} = \{(\epsilon, \{\epsilon'_j\}^m_{j=1}) \in \mathcal{J}_\text{wsep}^{m-k} : \epsilon'_m \sim \epsilon^2\}$,
- $\mathcal{J}_\text{wsep,2+i-k}^{m-k} = \{(\epsilon, \{\epsilon'_j\}^m_{j=1}) \in \mathcal{J}_\text{wsep}^{m-k} : \epsilon'_2 \to 0 \text{ but } \frac{\epsilon'_{i+1}}{\epsilon^2} \to \infty\} \quad (i \in [k+1, m]),$
- $\mathcal{J}_\text{wsep,1+2m+i-k}^{m-k} = \{(\epsilon, \{\epsilon'_j\}^m_{j=1}) \in \mathcal{J}_\text{wsep}^{m-k} : \epsilon'_{i-1} \sim \epsilon^2\} \quad (i^* \in [k+2, m]).$

(Note that if $k = m$, the collection of subsets of $\mathcal{J}_\text{wsep}^{m-k}$ reduces to merely $\{\mathcal{J}_\text{wsep,1}^{m-k}\}$.)

The subsets $\mathcal{J}_\text{wsep,1}^{m-k}$, $\mathcal{J}_\text{wsep,2}^{m-k}$ and the collections of subsets $\{\mathcal{J}_\text{wsep,2+i-k}^{m-k}\}_{i=k+1}^{m}$ and $\{\mathcal{J}_\text{wsep,1+2m+i-k}^{m-k}\}_{i=k+2}^{m}$ of $\mathcal{J}_\text{wsep}^{m-k}$ correspond to slow temporal oscillations, slow resonance (i.e., "slow" self-similar case), rapid temporal oscillations and rapid resonance (i.e., "rapid" self-similar case), respectively.

**Theorem 32.** The collection $\{\mathcal{J}_\text{wsep,j}^{1+2(m-k)} \}_{j=1}^{1+2(m-k)}$ of $1 + 2(m-k)$ subsets of $\mathcal{J}_\text{wsep}^{m-k}$ is mutually disjoint for every $k \in [m]$.

**Proof.** We must prove

$$\mathcal{J}_\text{wsep,j}^{m-k} \cap \mathcal{J}_\text{wsep,j}^{m-k} = \emptyset.$$
for all $i, j \in [1 + 2(m - k)]$ with $i \neq j$. That

$$\mathcal{J}_{wsep, i}^{m-k} \cap \mathcal{J}_{wsep, j}^{m-k} = \emptyset$$

for all $i, j \in [2] \cup [3 + m - k, 1 + 2(m - k)]$ with $i \neq j$, and that

$$\mathcal{J}_{wsep, i}^{m-k} \cap \mathcal{J}_{wsep, j}^{m-k} = \emptyset$$

for all $i, j \in [3, 2 + m - k]$ with $i \neq j$, are simple observations. It thus remains to show that

$$\mathcal{J}_{wsep, i}^{m-k} \cap \mathcal{J}_{wsep, j}^{m-k} = \emptyset$$

for all $i \in [2] \cup [3 + m - k, 1 + 2(m - k)]$ and all $j \in [3, 2 + m - k]$. This is trivial for $k = m$ so it is understood that $k \in [m - 1]$ in the remainder of the proof.

Let $e \in \mathcal{J}_{wsep, 1}^{m-k}$ be arbitrary. For this pair $e$ we have

$$\frac{e^2}{e_{m}'} \to 0,$$

which can be written on the equivalent form

$$\frac{e_{i}'}{e_{i}'} \to \infty,$$

or

$$\frac{e_{i}'}{e_{i}'} \to \infty$$

for every $i \in [m]$. Furthermore,

$$\frac{e_{i}'}{e_{i}'} \to \infty$$

since $\frac{e_{i}'}{e_{i}'}$ either tends to 0 (if $i \in [m - 1]$) or equals 1 (if $i = m$). In particular this holds for all $i \in [k + 1, m]$, and it is clear that $e \not\in \mathcal{J}_{wsep, 2+i-k}^{m-k}$ for all $i \in [k + 1, m]$.

We have shown that

$$\mathcal{J}_{wsep, 1}^{m-k} \cap \mathcal{J}_{wsep, j}^{m-k} = \emptyset$$

for all $j \in [3, 2 + m - k]$.

Let $e \in \mathcal{J}_{wsep, 2}^{m-k}$ be arbitrary. Then we have $e_{m}' \sim e^2$ for the chosen pair $e$ which gives

$$\frac{e_{i}'}{e_{i}'} \sim \frac{e_{i}'}{e_{m}'}$$

$i \in [m]$, which either tends to infinity (if $i \in [m - 1]$) or equals 1 (if $i = m$). In particular this holds for all $i \in [k + 1, m]$. Thus, for every $i \in [k + 1, m]$, $e \not\in \mathcal{J}_{wsep, 2+i-k'}^{m-k}$ and we have proven that

$$\mathcal{J}_{wsep, 2}^{m-k} \cap \mathcal{J}_{wsep, j}^{m-k} = \emptyset$$

for all $j \in [3, 2 + m - k]$. 30
Let \( e \in J_{wsep,i}^{m-k} \), \( i \in [3 + m - k, 1 + 2(m - k)] \), be arbitrary. The introduced pair \( e \) satisfies \( \epsilon_i^{(i-1)} \sim \epsilon^i_i \), \( i \in [k + 2, m] \), giving

\[
\frac{\epsilon_i^i}{\epsilon^i_i} \to \infty, \quad \frac{\epsilon^{i+1}_i}{\epsilon^i_i} \to \begin{cases} 1 \text{ if } i'' = k + 2, & \epsilon_2^i \to \frac{\epsilon_2^i}{\epsilon^i_i} \to \begin{cases} 0 \text{ if } i'' = k + 2, & 1 \text{ if } i'' = k + 3, & \infty \text{ if } i'' \in [k + 4, m] \\
\infty \text{ if } i'' \in [k + 3, m] \end{cases} \\
0 \text{ if } i'' \in [m - 2] \end{cases},
\]

\[
\ldots, \frac{\epsilon^i_i}{\epsilon_{i-2}^i} \to \begin{cases} 1 \text{ if } i'' = m - 1, & \epsilon^{i-1}_i \to \frac{\epsilon^{i-1}_i}{\epsilon^i_i} \to \begin{cases} 0 \text{ if } i'' \in [m - 1]k + 2, & 1 \text{ if } i'' = m \\
\infty \text{ if } i'' \in [m - 2] \end{cases} \end{cases}, \quad \frac{\epsilon^{i-1}_i}{\epsilon^{i-1}_i} \to 0.
\]

We see that \( e \notin J_{wsep,3}^{m-k} \). Indeed, to be in the subset requires \( \frac{\epsilon^{i+1}_i}{\epsilon^i_i} \to 0 \) but \( \frac{\epsilon^i_i}{\epsilon^i_i} \to \infty \), which is clearly impossible. We also see that \( e \notin J_{wsep,4}^{m-k} \) since being in the subset requires \( \frac{\epsilon^{i+1}_i}{\epsilon^i_i} \to 0 \) but \( \frac{\epsilon^{i+2}_i}{\epsilon^{i+1}_i} \to \infty \); the former limit needs \( i'' = k + 2 \) while the latter needs \( i'' \in [k + 3, m] \). We realise that \( e \notin J_{wsep,j}^{m-k} \) for all \( j \in [3, 2 + m - k] \). Hence,

\[
J_{wsep,i}^{m-k} \cap J_{wsep,j}^{m-k} = \emptyset
\]

for all \( i \in [3 + m - k, 1 + 2(m - k)] \) and all \( j \in [3, 2 + m - k] \). The mutual disjointness property has been verified.

In the proposition below we will experience that the introduced collection of mutually disjoint subsets actually forms a partition in the special but very important "classical" case of temporal scale functions expressed as power functions. For this purpose, define the subset

\[
P^{m-k} = \left\{ (e, \{\epsilon_j^i\}_{j=1}^m) \in J_{wsep}^{m-k} : \text{for every } \ell \in [m] \text{ there exists a } p_\ell > 0 \text{ such that } \epsilon^\ell = e^\ell \right\}
\]

of \( J_{wsep}^{m-k} \). We note that in the definition above for \( P^{m-k} \), \( p_k = 1 \). Moreover, \( 0 < p_\ell < 1 \) if \( \ell \in [k - 1] \) (provided \( k \in [2, m] \)) and \( p_\ell > 1 \) if \( \ell \in [k + 1, m] \) (provided \( k \in [m - 1] \)). Furthermore, for each \( i \in [1 + 2(m - k)] \), define the subsets

\[
P_i^{m-k} = P^{m-k} \cap J_{wsep,i}^{m-k}
\]

of \( P^{m-k} \). By Theorem 32 we already know that the collection \( \{P_i^{m-k}\}_{i=1}^{1+2(m-k)} \) is mutually disjoint. Below we will see that it actually also covers all of \( P^{m-k} \).

**Proposition 33.** The collection \( \{P_i^{m-k}\}_{i=1}^{1+2(m-k)} \) forms a partition of \( P^{m-k} \).
Proof. As already mentioned, the mutual disjointness property follows immediately from Theorem \[32\] It remains to show that

\[ \mathcal{P}^{m \sim k} = \bigcup_{i=1}^{1+2(m-k)} \mathcal{P}_{i}^{m \sim k}, \]  

(30)
i.e., that the collection \( \{ \mathcal{P}_{i}^{m \sim k} \}_{i=1}^{1+2(m-k)} \) of subsets covers all of \( \mathcal{P}^{m \sim k} \).

Suppose that there exists a pair

\[ e \in \mathcal{P}^{m \sim k} \setminus \bigcup_{i=1}^{1+2(m-k)} \mathcal{P}_{i}^{m \sim k}, \]  

(31)
which means that we assume that \( \{ \mathcal{P}_{i}^{m \sim k} \}_{i=1}^{1+2(m-k)} \) does not cover all of \( \mathcal{P}^{m \sim k} \). The introduced pair \( e = (\varepsilon, \{ \varepsilon^{p_i} \}_{i=1}^{m}) \) must by definition satisfy

\[ \frac{\varepsilon^2}{\varepsilon^{p_m}} \not\to 0 \]  

(32)since \( e \not\in \mathcal{P}_{1}^{m \sim k} \),

\[ \varepsilon^{p_m} \not\to \varepsilon^2 \]  

(33)since \( e \not\in \mathcal{P}_{2}^{m \sim k} \),

\[ \frac{\varepsilon^{p_i}}{\varepsilon^2} \not\to 0 \]  

(34)since \( e \not\in \bigcup_{\ell=3}^{2+m-k} \mathcal{P}_{\ell}^{m \sim k} \) or \( \frac{\varepsilon^{p_{i-1}}}{\varepsilon^2} \not\to \infty \) \( \forall i \in [k+1, m] \)

and

\[ \frac{\varepsilon^{p_{i-1}}}{\varepsilon^2} \not\to 1 \]  

(35)since \( e \not\in \bigcup_{\ell=3+m-k}^{1+2(m-k)} \mathcal{P}_{\ell}^{m \sim k} \).

The conditions (33) and (35) may be written

\[ (p_{k+1} \neq 2) \land (p_{k+2} \neq 2) \land \ldots \land (p_{m-1} \neq 2) \land (p_{m} \neq 2), \]  

(36)
and (34) can be expressed as

\[ ((p_{k} \geq 2) \lor (p_{k+1} \leq 2)) \land ((p_{k+1} \geq 2) \lor (p_{k+2} \leq 2)) \land \ldots \land ((p_{m-2} \geq 2) \lor (p_{m-1} \leq 2)) \land ((p_{m-1} \geq 2) \lor (p_{m} \leq 2)), \]  

(37)
where we employ the logic symbols \( \land \) ‘and’ (i.e., logical conjunction) and \( \lor \) ‘or’ (i.e., logical disjunction) for clarity.

We begin by noticing that \( p_k = 1 \) by definition, so (37) implies that \( p_{k+1} \leq 2 \). This together with \( p_{k+1} \neq 2 \) from (36) yields \( p_{k+1} < 2 \). Hence, using (37) again and we conclude that \( p_{k+1} \leq 2 \). Consequently, (36) implies \( p_{k+1} < 2 \). Continuing, we end up with \( p_{m} < 2 \). But this contradicts (32) which states that \( p_{m} \geq 2 \). Thus, no pair \( e \) fulfilling (31) can exist so we indeed have (30), and the proof is complete. \( \square \)
**Example 34.** In [14] one considers pairs of the type \((\varepsilon, \{\varepsilon', \varepsilon_r\})\), \(r \in \mathbb{R}_+ \setminus \{1\}\), in the context of linear parabolic problems. Define the mutually disjoint sets

\[
\mathcal{R}^- = \{(\varepsilon, \{\varepsilon', \varepsilon\}) \in \mathcal{P}^{2-2} : 0 < r < 1\},
\]

and

\[
\mathcal{R}^+ = \{(\varepsilon, \{\varepsilon, \varepsilon'\}) \in \mathcal{P}^{2-1} : r > 1\},
\]

and let \(\mathcal{R} = \mathcal{R}^- \cup \mathcal{R}^+\). Introduce the subsets

\[
\mathcal{R}_1^- = \mathcal{P}_1^{2-2} \cap \mathcal{R}^- = \{(\varepsilon, \{\varepsilon', \varepsilon\}) \in \mathcal{P}^{2-2} : 0 < r < 1\} = \mathcal{R}^- \\
\mathcal{R}_2^- = \mathcal{P}_2^{2-2} \cap \mathcal{R}^- = \emptyset
\]

of \(\mathcal{R}^-\), and the subsets

\[
\mathcal{R}_1^+ = \mathcal{P}_1^{2-1} \cap \mathcal{R}^+ = \{(\varepsilon, \{\varepsilon, \varepsilon'\}) \in \mathcal{P}^{2-1} : 1 < r < 2\}, \\
\mathcal{R}_2^+ = \mathcal{P}_2^{2-1} \cap \mathcal{R}^+ = \{(\varepsilon, \{\varepsilon, \varepsilon'\}) \in \mathcal{P}^{2-1} : r = 2\}
\]

and

\[
\mathcal{R}_3^+ = \mathcal{P}_3^{2-1} \cap \mathcal{R}^+ = \{(\varepsilon, \{\varepsilon, \varepsilon'\}) \in \mathcal{P}^{2-1} : r > 2\}
\]

of \(\mathcal{R}^+\). By Proposition 33, \(\mathcal{R}^-\) and \(\mathcal{R}^+\) are partitioned by the collections \(\{\mathcal{R}_1^-, \mathcal{R}_2^-\}\) and \(\{\mathcal{R}_1^+, \mathcal{R}_2^+, \mathcal{R}_3^+\}\), respectively. Thus, according to the developed theory, \(\mathcal{R}\) should be partitioned by the collection

\[
\{\mathcal{R}_1^-, \mathcal{R}_2^-, \mathcal{R}_1^+, \mathcal{R}_2^+, \mathcal{R}_3^+\},
\]

which is verified by looking at the explicit expressions for the subsets derived above. Defining \(\mathcal{R}_1 = \mathcal{R}_1^- \cup \mathcal{R}_1^+\), the partitioning collection of subsets

\[
\{\mathcal{R}_1, \mathcal{R}_2^+, \mathcal{R}_3^+\}
\]

of \(\mathcal{R}\) is seen to correspond to the cases \(0 < r < 2\) with \(r \neq 1\), \(r = 2\) and \(r > 2\), respectively. This is exactly the partition obtained in the homogenisation result of Theorem 10 in [14] leading to three distinct systems of local problems for \(u_1\) corresponding to the mentioned distinct cases for \(r \in \mathbb{R}_+ \setminus \{1\}\).

Let \(S = (0, 1)\) and define \(H^1_h(S; V, V')\), \(V\) being any Banach space with topological dual \(V'\), as the space of functions \(u\) satisfying \(u \in L^2_h(S; V)\) and \(\frac{d}{ds}u \in L^2_h(S; V')\). In order to prove Theorem 37—our first homogenisation result—we first need the lemmas below.
Lemma 35. The tensor product space \( (C_#^\infty(Y)/\mathbb{R}) \otimes C_#^\infty(S) \) is dense in \( H_1^0(S;\mathcal{W},\mathcal{W}') \).

Proof. This is just Proposition 4.6 in [34] in which \( \mathcal{E} \) and \( \mathcal{V} \) correspond to \( (C_#^\infty(Y)/\mathbb{R}) \otimes C_#^\infty(S) \) and \( H_1^0(S;\mathcal{W},\mathcal{W}') \), respectively, of the present paper. \( \square \)

Lemma 36. Suppose that \( u,v \in H_1^0(S;\mathcal{W},\mathcal{W}') \). Then

\[
\langle \partial_s u,v \rangle_{L_2^2(S;\mathcal{W}'_R),L_2^2(S;\mathcal{W})} + \langle \partial_s v,u \rangle_{L_2^2(S;\mathcal{W}'_R),L_2^2(S;\mathcal{W})} = 0
\]

holds. In particular,

\[
\langle \partial_s u, u \rangle_{L_2^2(S;\mathcal{W}'_R),L_2^2(S;\mathcal{W})} = 0.
\]

Proof. This follows immediately from Corollary 4.1 in [34]. \( \square \)

Theorem 37 below is our first homogenisation result.

Theorem 37. Let \( k \in [m] \). Suppose that the pair \( e = (\epsilon,\{\epsilon_j\}_{j=1}^m) \) of lists of spatial and temporal scale functions belongs to \( \bigcup_{i=1}^{1+2(m-k)} \mathcal{J}_{\text{wsep},i}^{m-k} \). Let \( \{u_j\} \) be the sequence of weak solutions in \( H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega)) \) to the evolution problem \( (1) \) with \( a : \Omega_T \times \mathbb{R}^{N+m} \times \mathbb{R}^N \to \mathbb{R}^N \) satisfying the structure conditions (B1)–(B5). Then

\[
u \epsilon \to u \quad \text{in } L^2(\Omega_T),
\]

\[
u \epsilon \twoheadrightarrow u \quad \text{in } L^2(0,T;H_0^1(\Omega)),
\]

and

\[
\nabla u_{\epsilon} \overset{(2m+1)}{\longrightarrow} \nabla u + \nabla y u_1,
\]

where \( u \in H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega)) \) and \( u_1 \in L^2(\Omega_T \times \mathcal{S}_m;\mathcal{W}) \). Here \( u \) is the unique weak solution to the homogenisation problem \( (26) \) with the homogenised flux \( b : \Omega_T \times \mathbb{R}^N \to \mathbb{R}^N \) given by

\[
b(x,t;\nabla u) = \int_{y_1m} a(x,t,y,s_m;\nabla u + \nabla y u_1) \, ds_m \, dy.
\]

Moreover, we have the following characterisation of \( u_1 \):

- If \( e \in \mathcal{J}_{\text{wsep},1}^{m-k} \) then the function \( u_1 \) is the unique weak solution to the local problem

\[
- \nabla y \cdot a(x,t,y,s_m;\nabla u + \nabla y u_1) = 0.
\]

- If \( e \in \mathcal{J}_{\text{wsep},2}^{m-k} \) assuming \( u_1 \in L^2(\Omega_T \times \mathcal{S}_m;\mathcal{W},\mathcal{W}') \), then the function \( u_1 \) is the unique weak solution to the system of local problems

\[
\partial_{s_m} u_1(x,t,y,s_m) - \nabla y \cdot a(x,t,y,s_m;\nabla u + \nabla y u_1) = 0.
\]

- If \( e \in \mathcal{J}_{\text{wsep},2+k}^{m-k} \) for some \( \bar{\ell} \in [k+1,m] \), provided \( k \in [m-1] \), then the function \( u_1 \) is the unique weak solution to the system of local problems

\[
\begin{cases}
- \nabla y \cdot \int_{y_1m} a(x,t,y,s_m;\nabla u + \nabla y u_1) \, ds_{[\bar{\ell},m]} = 0, \\
\partial_i u_1(x,t,y,s_m) = 0 \quad (i \in \bar{\ell},m).
\end{cases}
\]

34
If \( e \in \mathcal{F}_{m-k}^{\text{wsep},1+m+e-2k} \) for some \( \ell^e \in [k+2, m] \), provided \( k \in [m-2] \) and assuming \( u_1 \in L^2(\Omega_T \times S^{\ell^e-2} \times S[\ell^e,m]; H^1_0(S_{\ell^e-1}; \mathcal{W}, \mathcal{W}')) \), then the function \( u_1 \) is the unique weak solution to the system of local problems

\[
\begin{cases}
\partial_{s_{e-1}} u_1(x, t, y, s_m) - \nabla_y \cdot \int_{S[\ell^e,m]} a(x, t, y, s_m; \nabla u + \nabla_y u_1) \, ds_{\ell^e,m} = 0, \\
\partial_y u_1(x, t, y, s_m) = 0 \quad (i \in [\ell^e, m]).
\end{cases}
\]

**Proof.** Since \( a \) fulfils (B1)–(B3) we can use Proposition 31 for the sequence \( \{u_e\} \) of weak solutions; we have ensured uniform boundedness in \( H^1_0(\Omega) \), i.e., (29) holds. We can then employ Theorem 18 (with \( n = 1 \)) obtaining, up to a subsequence,

\[
\begin{align*}
\text{up to a subsequence,} \\
u_e \rightarrow u & \quad \text{in } L^2(\Omega_T), \\
u_e \rightarrow u & \quad \text{in } L^2(0, T; H^1_0(\Omega)), \\
\nabla u_e & \xrightarrow{(2,m+1)} \nabla u + \nabla_y u_1,
\end{align*}
\]

where \( u \in H^1(0, T; H^1_0(\Omega), H^{-1}(\Omega)) \) and \( u_1 \in L^2(\Omega_T \times S^m; \mathcal{W}) \). Consider the sequence \( \{a_e\} \) defined according to

\[
a_e(x, t) = a^e(x, t; \nabla u_e) = a(x, t, \frac{x}{\tau^e}, t_{m';} \nabla u_e) \quad ((x, t) \in \Omega_T).
\]

We have that \( \{a_e\} \) is uniformly bounded in \( L^2(\Omega_T)^N \). Indeed, using (26), the triangle inequality and (29) we get

\[
\begin{align*}
\|a_e\|_{L^2(\Omega_T)^N}^2 & = \int_{\Omega_T} \left| a(x, t, \frac{x}{\tau^e}, t_{m';} \nabla u_e) \right|^2 \, dx \, dt \\
& < C^2 \int_{\Omega_T} \left( 1 + |\nabla u_e(x, t)| \right)^2 \, dx \, dt \\
& = C^2 \|1 + |\nabla u_e|\|^2_{L^2(\Omega_T)} \\
& \leq C^2 \left( \|1\|_{L^2(\Omega_T)} + \|u_e\|_{L^2(0, T; H^1_0(\Omega))} \right)^2 \\
& \leq C^2 \left( \|T\|_\Omega \right)^{\frac{1}{2}} + \|u_e\|_{H^1(0, T; H^1_0(\Omega), H^{-1}(\Omega))}^2 \\
& \leq C^2 \left( \|T\|_\Omega \right)^{\frac{1}{2}} + C^2.
\end{align*}
\]

By Theorem 13 (with \( n = 1 \)) we then know that, up to a subsequence,

\[
a_e \xrightarrow{(2,m+1)} a_0
\]

for some \( a_0 \in L^2(\Omega_T \times Y_{1m})^N \).
Recall the weak form (25) (with \( n = 1 \)) of the evolution problem, i.e.,

\[
\langle \frac{\partial}{\partial t} u_\varepsilon, \psi \rangle_{X', X} + \int_{\Omega_T} a_\varepsilon(x,t) \cdot \nabla \psi(x,t) \, dx \, dt = \int_{\Omega_T} f(x,t) \psi(x,t) \, dx \, dt
\]

(43)

for every \( \psi \in L^2(0,T; H^1_0(\Omega)) \).

Choose an arbitrary \( \psi \in H^1_0(\Omega) \odot D(0,T) \). Then we can shift the weak temporal derivative \( \frac{\partial}{\partial t} \) in (43) from acting on \( u_\varepsilon \) to acting on \( \psi \) instead, i.e.,

\[
\int_{\Omega_T} (-u_\varepsilon(x,t) \frac{\partial}{\partial t} \psi(x,t) + a_\varepsilon(x,t) \cdot \nabla \psi(x,t)) \, dx \, dt = \int_{\Omega_T} f(x,t) \psi(x,t) \, dx \, dt.
\]

(44)

Passing to the limit—using (40) and (42) on the first and second terms on the left-hand side, respectively—we obtain, up to a subsequence,

\[
\int_{\Omega_T} \int_{\gamma_{1m}} (-u_\varepsilon(x,t) \frac{\partial}{\partial t} \psi(x,t) + a_0(x,t,y,s_m) \cdot \nabla \psi(x,t)) \, ds_m \, dy \, dx \, dt
\]

\[
= \int_{\Omega_T} f(x,t) \psi(x,t) \, dx \, dt,
\]

or, in other words,

\[
\int_{\Omega_T} \left( -u_\varepsilon(x,t) \frac{\partial}{\partial t} \psi(x,t) + \int_{\gamma_{1m}} a_0(x,t,y,s_m) \, ds_m \, dy \cdot \nabla \psi(x,t) \right) \, dx \, dt
\]

\[
= \int_{\Omega_T} f(x,t) \psi(x,t) \, dx \, dt.
\]

(45)

Let again \( \frac{\partial}{\partial t} \) act on \( u \). By density, the obtained equality

\[
\langle \frac{\partial}{\partial t} u, \psi \rangle_{X', X} + \int_{\Omega_T} \int_{\gamma_{1m}} a_0(x,t,y,s_m) \, ds_m \, dy \cdot \nabla \psi(x,t) \, dx \, dt
\]

\[
= \int_{\Omega_T} f(x,t) \psi(x,t) \, dx \, dt
\]

(46)

holds for any \( \psi \in L^2(0,T; H^1_0(\Omega)) \). We have obtained the weak form of the homogenised evolution problem (28) with the limit flux given by

\[
b(x,t; \nabla u) = \int_{\gamma_{1m}} a_0(x,t,y,s_m) \, ds_m \, dy.
\]

What remains is to find the local problems for \( u_1 \) and to give the limit \( \alpha_0 \) in terms of \( a \). We will first extract the pre-local-problems, i.e., the problems expressed in terms of \( a_0 \) which become the local problems once \( a_0 \) is given in terms of \( a \).

Introduce \( \omega_\ell \in D(\Omega) \odot D(0,T) \odot (C_0^\infty(Y)/\mathbb{R}) \odot (\prod_{i=1}^\ell C_0^\infty(S_i)), \ell \in [m] \). For each \( \ell \in [m] \) we define the sequence \( \{\omega_\ell^\varepsilon\} \) in the conventional manner. Let \( \{r_\varepsilon\} \) be a sequence of positive numbers such that \( r_\varepsilon \to 0 \). We will now study sequences of test functions \( \{\psi_\ell^\varepsilon\} \) in (44) such that

\[
\psi_\ell^\varepsilon(x,t) = r_\varepsilon \omega_\ell^\varepsilon(x,t) \quad ((x,t) \in \Omega_T)
\]
with appropriate choices of \( \{r_\ell \} \) and \( \ell \) in order to extract the pre-local-problems. We note here that

\[
\nabla \psi_\ell^\varepsilon = r_\varepsilon(\nabla x + \frac{1}{\varepsilon} \nabla y)\omega_\ell^\varepsilon
\]

and

\[
\frac{\partial}{\partial t} \psi_\ell^\varepsilon = r_\varepsilon \left( \partial_t + \sum_{i=1}^\ell \frac{1}{i} \partial_{s_i} \right) \omega_\ell^\varepsilon.
\]

For the sequence \( \{ \psi_\ell^\varepsilon \} , \ell \in [m] \), of test functions given above, (44) becomes

\[
\int_{\Omega_T} \left[ -u_\varepsilon(x,t) r_\varepsilon \left( \partial_t + \sum_{i=1}^\ell \frac{1}{i} \partial_{s_i} \right) \omega_\ell^\varepsilon(x,t) \\
+ a_\varepsilon(x,t) \cdot r_\varepsilon \left( \nabla x + \frac{1}{\varepsilon} \nabla y \right) \omega_\ell^\varepsilon(x,t) \right] \, dxdt = \int_{\Omega_T} f(x,t) r_\varepsilon \omega_\ell^\varepsilon(x,t) \, dxdt.
\]

The right-hand side and the \( \partial_t \) and \( \nabla_x \) terms in the left-hand side clearly vanish in the limit, and what is left is

\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} \left[ -u_\varepsilon(x,t) \sum_{i=1}^\ell \frac{1}{i} \partial_{s_i} \omega_\ell^\varepsilon(x,t) + a_\varepsilon(x,t) \cdot \frac{\partial}{\partial x} \omega_\ell^\varepsilon(x,t) \right] \, dxdt = 0 \tag{47}
\]

recalling that \( \varepsilon'_\ell = \varepsilon \).

Suppose that the real sequence \( \{ \varepsilon'_\ell \} \) is bounded, then the limit equation becomes

\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} \left[ -u_\varepsilon(x,t) \sum_{i=1}^\ell \frac{1}{i} \partial_{s_i} \omega_\ell^\varepsilon(x,t) + a_\varepsilon(x,t) \cdot \frac{\partial}{\partial x} \omega_\ell^\varepsilon(x,t) \right] \, dxdt = 0. \tag{48}
\]

Choose \( r_\varepsilon = \varepsilon'_k \), which implies that \( \{ \varepsilon'_\ell \} = \{ \varepsilon'_k \} \) is bounded for \( \ell \in [k] \). Then (48) becomes

\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} \left[ -u_\varepsilon(x,t) \sum_{i=1}^\ell \frac{1}{i} \partial_{s_i} \omega_\ell^\varepsilon(x,t) + a_\varepsilon(x,t) \cdot \frac{\partial}{\partial x} \omega_\ell^\varepsilon(x,t) \right] \, dxdt = 0. \tag{49}
\]

If \( \ell \in [k-1] \) (provided \( k \in [2,m] \)) the first term tends to 0, and we get in this case

\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} a_\varepsilon(x,t) \cdot \nabla y \omega_\ell^\varepsilon(x,t) \, dxdt = 0,
\]

which after taking the limit can be written

\[
\int_{\Omega_T} \int_{Y_{[1,m]}} a_0(x,t,y,s_m) \cdot \nabla_y \omega_\ell(x,t,y,s_\ell) \, ds_m dy dx dt = 0, \tag{50}
\]

i.e.,

\[
\int_{\Omega_T} \int_{Y_{[1,m]}} \int_{S_{[\ell+1,m]}} a_0(x,t,y,s_m) ds \cdot \nabla_y \omega_\ell(x,t,y,s_\ell) \, ds dy dx dt = 0.
\]

Suppose \( v_1 \in C_0^\infty(Y) / \mathbb{R} \) is the factor of \( \omega_\ell \) with respect to the \( y \) variable. Then, by the Variational Lemma,

\[
\int_{Y_{[1,m]}} \int_{S_{[\ell+1,m]}} a_0(x,t,y,s_m) ds \cdot \nabla_y v_1(y) \, dy = 0 \tag{51}
\]
a.e. on $\Omega_T \times S^\ell$. If $\ell = k$ the limit equation (49) instead reduces to

$$\lim_{\epsilon \to 0} \int_{\Omega_T} \left( -u(x,t) \partial_{s_k} \omega_k^\epsilon(x,t) + a_\ell(x,t) \cdot \nabla y \omega_k^\epsilon(x,t) \right) \, dx \, dt = 0,$$

which in the limit becomes

$$\int_{\Omega_T} \int_{Y} \left( -u(x,t) \partial_{s_k} \omega_k(x,t,y,s_k) + a_0(x,t,y,s_m) \cdot \nabla y \omega_k(x,t,y,s_k) \right) \, ds \, dy \, dx \, dt = 0.$$

The first term gives no contribution since $\omega_k$ is $S_k$-periodic in the $s_k$ variable. Progressing like in the case $\ell \in [k-1]$ we finally arrive at (51) which now also includes $\ell = k$, i.e., (51) holds for all $\ell \in [k]$. But it is clear that (51) holding for $\ell = k$ implies that it holds also for any $\ell \in [k-1]$ (provided $k \in [2,m]$). Thus, we only have to consider (51) for $\ell = k$, i.e., we have so far obtained

$$\int_Y \int_{S_{[k+1,m]}} a_0(x,t,y,s_m) \, ds_{[k+1,m]} \cdot \nabla y v_1(y) \, dy = 0. \quad (52)$$

It should be emphasised here that this equation is always true for $J_{wsep}^{m-k}$ and is not confined to any particular subset $J_{wsep,j} = \{ 1 + 2(m-k) \}$.

If we study the limit equation (47) extracting a factor $\frac{1}{\epsilon}$ in the first term we obtain

$$\lim_{\epsilon \to 0} \int_{\Omega_T} \left( -\frac{1}{\epsilon} u(x,t) \sum_{i=1}^{\ell} \frac{r_i \epsilon^i}{\epsilon^i} \partial_{s_i} \omega_i^\epsilon(x,t) + a_\ell(x,t) \cdot \frac{\epsilon_i}{\epsilon^i} \nabla y \omega_i^\epsilon(x,t) \right) \, dx \, dt = 0,$$

where we have recalled $\epsilon_i^\prime = \epsilon$. Suppose that $\{ \frac{r_i \epsilon^i}{\epsilon^i} \}$ is bounded (in $\mathbb{R}$), it is then clear that the limit equation above reduces to

$$\lim_{\epsilon \to 0} \int_{\Omega_T} \left( -\frac{1}{\epsilon} u(x,t) \sum_{i=1}^{\ell} \frac{r_i \epsilon^i}{\epsilon^i} \partial_{s_i} \omega_i(x,t) + a_\ell(x,t) \cdot \frac{\epsilon_i}{\epsilon^i} \nabla y \omega_i^\epsilon(x,t) \right) \, dx \, dt = 0. \quad (53)$$

- Suppose $(\epsilon, \{ \epsilon_i^\prime \}_{j=1}^{m}) \in J_{wsep,1}^{m-k}$. By definition this means that $(\epsilon, \{ \epsilon_i^\prime \}) \in J_{wsep}^{m-k}$ and $\frac{\epsilon^2}{\epsilon_m^2} \to 0$. Consider first $\epsilon_m \sim \epsilon_k^\prime$, i.e., $k = m$. We have already extracted (52) which in this case, $k = m$, is merely

$$\int_Y a_0(x,t,y,s_m) \cdot \nabla y v_1(y) \, dy = 0, \quad (54)$$

which is the pre-local-problem.

Consider now the situation $\epsilon_m \sim \epsilon_k^\prime$, i.e., $k \in [m-1]$ requiring $m > 1$. We first note that we have already extracted (52) which at this point carries at least one integral (over $S_m$). We want to employ (53) for $\ell \in [k+1,m]$. Choose $r_\ell = \epsilon_k^\prime$, and we get that

$$\frac{r_i \epsilon^i}{\epsilon^i} = \frac{\epsilon^2}{\epsilon_m^2} \frac{\epsilon_i}{\epsilon^i} \quad \rightarrow 0,$$
so \( \{ \frac{r_\ell \epsilon_\ell}{\epsilon_\ell^2} \} \) is indeed bounded (we even have a vanishing limit). We can now use (53) which yields
\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} \left( -\frac{1}{\varepsilon} u_\ell(x,t) \frac{\varepsilon_\ell^2}{r_\ell} \partial_s \omega_\ell^\epsilon(x,t) + a_\ell(x,t) \cdot \nabla_y \omega_\ell^\epsilon(x,t) \right) \, dx \, dt = 0,
\]
which in the limit becomes (50); this can be realised by utilising Theorem 20, considering the final remark in Remark 21 and using \( \frac{\varepsilon_\ell^2}{r_\ell} \to 0 \) such that the contribution from the first term vanishes in the limit. Hence, we have again (51) but for \( \ell \in [k+1,m] \). Apparently we end up at the pre-local-problem (54) again since (51) in the case \( \ell = m \) implies that (51) holds automatically for any \( \ell \in [m-1] \).

- Suppose \( (\varepsilon, \{ \varepsilon_j^\ell \}_{j=1}^m) \in J_{wsep,2}^{m-k} \). By definition this means that \( (\varepsilon, \{ \varepsilon_j^\ell \}) \in J_{wsep}^{m-k} \) and \( \varepsilon_m^\ell \sim \varepsilon_k^\ell \). Let \( \ell = m \) in (53). Choose \( r_\ell = \varepsilon_k^\ell \) again, giving
\[
\frac{r_\ell \epsilon_\ell}{\epsilon_\ell^2} = \frac{\varepsilon_k^2}{\varepsilon_m} \sim 1,
\]
so \( \{ \frac{r_\ell \epsilon_\ell}{\epsilon_\ell^2} \} \) is bounded. The equation (53) then becomes
\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} \left( -\frac{1}{\varepsilon} u_\ell(x,t) \frac{\varepsilon_\ell^2}{r_m} \partial_s \omega_m^\varepsilon(x,t) + a_\ell(x,t) \cdot \nabla_y \omega_m^\varepsilon(x,t) \right) \, dx \, dt = 0,
\]
and by Theorem 20 the limit is
\[
\int_{\Omega_T} \int_{Y_{sm}} \left( -u_\ell(x,t,y,s_m) \partial_s \omega_m(x,t,y,s_m) + a_\ell(x,t,y,s_m) \cdot \nabla_y \omega_m(x,t,y,s_m) \right) \, ds_m \, dy \, dx \, dt = 0.
\]

Suppose \( v_1 \in C_\#^\infty(Y) / \mathbb{R} \) and \( c_m \in C_\#^\infty(S_m) \) are the factors of \( \omega_m \) with respect to the \( y \) and \( s_m \) variables. Utilising the Variational Lemma we then arrive at
\[
\int_{\Omega_T} \int_{S_m} \left( -u_\ell(x,t,y,s_m) v_1(y) \partial_s c_m(s_m) + a_\ell(x,t,y,s_m) \cdot \nabla_y v_1(y) c_m(s_m) \right) \, ds_m \, dy = 0 \quad (55)
\]
a.e. on \( \Omega_T \times S^{m-1} \), which is our pre-local-problem.

- Suppose \( (\varepsilon, \{ \varepsilon_j^\ell \}_{j=1}^m) \in J_{wsep,2+\ell-k}^{m-k} \) for some \( \ell \in [k+1,m] \) where \( k \in [m-1] \) is required. By definition this means that \( (\varepsilon, \{ \varepsilon_j^\ell \}) \in J_{wsep}^{m-k} \) and \( \varepsilon_\ell^\ell \to 0 \) but \( \varepsilon_{\ell+1}^\ell \to \infty \).

We first note that we have already extracted (52) which at this point carries at least one integral and it happens to be independent of \( \ell \). Choose \( r_\ell = \frac{\varepsilon_i^\ell}{\varepsilon_k^\ell} \) where \( i \in [\ell,m] \). Apparently, \( r_\ell \to 0 \) is guaranteed since \( i \in [k+1,m] \). Trivially, \( \{ \frac{r_\ell \epsilon_\ell}{\epsilon_\ell^2} \} \) is bounded. Finally,
\[
\frac{\varepsilon_i^\ell}{\varepsilon_k^\ell} = \frac{\varepsilon_i^\ell}{\varepsilon_k^\ell} \frac{\varepsilon_k^\ell}{\varepsilon_m} \to 0
\]
by assumption and separatedness. Hence, we can utilise (53) (with \( \ell = i \)) giving

\[
\lim_{\epsilon \to 0} \int_{\Omega} \left( -\frac{1}{\epsilon} u_\epsilon(x, t) \partial_3 \omega_\epsilon^i(x, t) + a_\epsilon(x, t) \cdot \frac{\partial}{\partial x} \nabla \omega_\epsilon^i(x, t) \right) \, dx \, dt = 0,
\]

and taking the limit by using Theorem [20].

\[
-\int_{\Omega_t} \int_{Y_{1_m}} u_1(x, t, y, s_m) \partial_3 \omega_1(x, t, y, s_i) \, ds_m \, dy \, dx \, dt = 0 \quad (i \in [\ell, m]).
\]

Proceeding like before, the equation above leads to the pre-local-problem

\[
-\int_{S_i} u_1(x, t, y, s_m) \partial_3 c_i(s_i) \, ds_i \, dy = 0 \quad \text{for all } c_i \in C^\infty_w(S_i) \quad (i \in [\ell, m]). \tag{56}
\]

Note that this means that \( u_1 \) is essentially independent of the temporal local variables \( s_{[\ell, m]} \in S^]\ell, m]. \) Choose now \( r_\ell = \epsilon'_k \) (which indeed tends to 0) and let \( i \in [k + 1, \ell - 1] \) which requires \( \ell \in [k + 2, m] \) (which, of course, in turn requires \( k \in [m - 2] \)). Then \( \left\{ \frac{\epsilon'_k}{\epsilon_i} \right\} \) is bounded since, by assumption and separatedness,

\[
\frac{r_\ell}{\epsilon_i} = \frac{\epsilon'_k}{\epsilon_i} \to 0.
\]

We have shown that we can employ (53) (with \( \ell = i \), giving

\[
\lim_{\epsilon \to 0} \int_{\Omega_t} \left( -\frac{1}{\epsilon} u_\epsilon(x, t) \frac{\epsilon'_k}{\epsilon_i} \partial_3 \omega_\epsilon^i(x, t) + a_\epsilon(x, t) \cdot \frac{\partial}{\partial x} \nabla \omega_\epsilon^i(x, t) \right) \, dx \, dt = 0
\]

for \( i \in [k + 1, \ell - 1] \). Taking the limit by using Theorem [20] remembering that \( \frac{\epsilon'_k}{\epsilon_i} \to 0 \) and taking into consideration the final remark of Remark [21] we arrive at

\[
\int_{\Omega_t} \int_{Y_{1_m}} a_0(x, t, y, s_m) \cdot \nabla \omega_1(x, t, y, s_i) \, ds_m \, dy \, dx \, dt = 0.
\]

Proceeding in the same way as in the derivation of (51) we get

\[
\int_{Y} \int_{S^{[i+1,m]}} a_0(x, t, y, s_m) \, ds_{[i+1,m]} \cdot \nabla \omega_1(y) \, dy = 0 \quad (i \in [k + 1, \ell - 1]).
\]

We conclude that

\[
\int_{Y} \int_{S^{[m]}} a_0(x, t, y, s_m) \, ds_{[m]} \cdot \nabla \omega_1(y) \, dy = 0 \quad \tag{57}
\]

since the case \( \ell = k + 1 \) is taken care of by (52). The extracted pre-local-problems are (56) and (57) in this case.

- Suppose \( (\epsilon, \{\epsilon'_j\}_{j=1}^m) \in J_{\text{we,sep}}^{m-k} \) for some \( \ell^* \in [k + 2, m] \) where it is required that \( k \in [m - 2] \). By definition this means that \( (\epsilon, \{\epsilon'_j\}) \in J_{\text{we}}^{m-k} \) and that \( \epsilon'_{\ell - 1} \sim \epsilon'_k \). Choose \( r_\ell = \frac{\epsilon'_k}{\epsilon_k} \) and let \( i \in [\ell, m] \). It is clearly guaranteed that \( r_\ell \to 0 \) since \( i \in [k + 2, m] \). Moreover, it is trivial that \( \left\{ \frac{\epsilon'_k}{\epsilon_i} \right\} \) is bounded. Finally,

\[
\frac{r_\ell}{\epsilon_i} = \frac{\epsilon'_k}{\epsilon_i} \to 0.
\]
by assumption and separatedness. Hence, we can utilise (53) with \( \ell = i \) giving
\[
\lim_{\ell \to 0} \int_{\Omega_T} \left( -\frac{1}{\varepsilon} u_\varepsilon(x,t) \partial_s \omega_{\ell}^r(x,t) + a_\varepsilon(x,t) \cdot \frac{\varepsilon_{\ell}^r}{\varepsilon_{\ell}^2} \nabla_y \omega_{\ell}^r(x,t) \right) \, dx \, dt = 0,
\]
and taking the limit by using Theorem (20).

\[
-\int_{\Omega_T} \int_{S^m} u_1(x,t,y,s_m) \partial_s \omega_i(x,t,y,s_i) \, ds_m \, dy \, dt = 0 \quad (i \in [\ell', m]).
\]

Proceeding like before, the equation above leads to the pre-local-problem
\[
-\int_{S^m} u_1(x,t,y,s_m) \partial_s c_i(s_i) \, ds_i = 0 \quad \text{for all } c_i \in C^\infty_0(S_i) \quad (i \in [\ell', m]). \quad (58)
\]

Note that this means that \( u_1 \) is essentially independent of the temporal local variables \( s_{[\ell', m]} \in S^{[\ell', m]} \). In particular, (58) implies that
\[
\int_{S^{[\ell', m]}} u_1(x,t,y,s_m) \, ds_{[\ell', m]} = u_1(x,t,y,s_m) \quad (59)
\]
holds a.e. on \( \Omega_T \times Y \times S^m \). For the second pre-local-problem, choose \( r_\varepsilon = \varepsilon_{\ell}^r \) and let \( i = \ell'' - 1 \). Then \( \left\{ \frac{r_\varepsilon \varepsilon_{\ell}^r}{\varepsilon_{\ell''}^r} \right\} \) is bounded since, by assumption,
\[
\frac{r_\varepsilon \varepsilon_{\ell}^r}{\varepsilon_{\ell''}^r} = \frac{\varepsilon_{\ell''}^r}{\varepsilon_{\ell''}^r} \to 1. \quad (60)
\]

We have shown that we can employ (53), giving
\[
\lim_{\ell \to 0} \int_{\Omega_T} \left( -\frac{1}{\varepsilon} u_\varepsilon(x,t) \frac{\varepsilon_{\ell}^2}{\varepsilon_{\ell''}^2} \partial_{s_{\ell''}} \omega_{\ell''}^r(x,t) + a_\varepsilon(x,t) \cdot \nabla_y \omega_{\ell''}^r(x,t) \right) \, dx \, dt = 0.
\]

Taking the limit by using Theorem (20) and (60), we arrive at
\[
\int_{\Omega_T} \int_{S^{[\ell', m]}} \left( -u_1(x,t,y,s_m) \partial_{s_{\ell''}} \omega_{\ell''}^r(x,t,y,s_{\ell''}) 
+ a_0(x,t,y,s_m) \cdot \nabla_y \omega_{\ell''}^r(x,t,y,s_{\ell''}) \right) \, ds_m \, dy \, dt = 0.
\]

Utilising property (59), this becomes
\[
\int_{\Omega_T} \int_{S^{[\ell', m]}} \left( -u_1(x,t,y,s_m) \partial_{s_{\ell''}} \omega_{\ell''}^r(x,t,y,s_{\ell''}) 
+ \int_{S^{[\ell', m]}} a_0(x,t,y,s_m) \, ds_{[\ell', m]} \cdot \nabla_y \omega_{\ell''}^r(x,t,y,s_{\ell''}) \right) \, ds_{\ell''} \, dy \, dt = 0.
\]

Suppose \( v_1 \in C^\infty_0(Y) / IR \) and \( c_{\ell''} \in C^\infty_0(S_{\ell''}) \) are the factors of \( \omega_{\ell''} \) with respect to the \( y \) and \( s_{\ell''} \) local variables, respectively. Employing the Variational Lemma we then get
\[
\int_{Y} \int_{S^{[\ell', m]}} \left( -u_1(x,t,y,s_m) v_1(y) \partial_{s_{\ell''}} c_{\ell''}(s_{\ell''}) 
+ \int_{S^{[\ell', m]}} a_0(x,t,y,s_m) \, ds_{[\ell', m]} \cdot \nabla_y v_1(y) c_{\ell''}(s_{\ell''}) \right) \, ds_{\ell''} \, dy = 0. \quad (61)
\]
a.e. on \( \Omega_T \times S^{r-2} \times S^{[r;m]} \), which is our second pre-local-problem. Concluding the present case, the extracted pre-local-problems are (58) and (61).

What is left to do is to characterise \( a_0 \) in terms of \( a \) such that the pre-local-problems become true local problems, and for this we introduce a sequence \( \{p^\mu\}_{\mu=1}^\infty \) in \( \mathcal{D}(\Omega_T;C^0_\#(Y_{1m})^N) \) of Evans’s perturbed test functions (see [11] [12]) defined according to

\[
p^\mu = \pi^\mu + \pi_{1,\mu} + \delta c \quad (\mu \in \mathbb{Z}_+),
\]

where \( \delta > 0, \pi^\mu \in \mathcal{D}(\Omega_T)^N \) and \( \pi_{1,\mu}c \in \mathcal{D}(\Omega_T;C^0_\#(Y_{1m})^N) \) for all \( \mu \in \mathbb{Z}_+ \). Let \( \{\pi^\mu\}_{\mu=1}^\infty \) and \( \{\pi_{1,\mu}\}_{\mu=1}^\infty \) be such that

\[
\begin{cases}
\pi^\mu \to \nabla u \\
\pi_{1,\mu}(x,t) \to \nabla u(x,t)
\end{cases}
\quad \text{in} \quad L^2(\Omega_T)^N,
\]

and

\[
\begin{cases}
\pi_{1,\mu} \to \nabla_y u_1 \\
\pi_{1,\mu}(x,t,y,s_m) \to \nabla_y u_1(x,t,y,s_m)
\end{cases}
\quad \text{in} \quad L^2(\Omega_T \times Y_{1m})^N,
\]

as \( \mu \to \infty \). Strictly speaking, the last convergence should hold a.e. on \( \Omega_T \times \mathbb{R}^{n+m} \). By periodicity, this is implied from the given assumption, though. For each fixed \( \mu \in \mathbb{Z}_+ \), introduce the sequence \( \{p^\mu\} \) defined by

\[
p^\mu(x,t) = p^\mu(x,t,\frac{x}{\delta},t_m) \quad ((x,t) \in \Omega_T).
\]

A crucial result for the remainder of the proof is

\[
\int_{\Omega_T} \int_{Y_{1m}} \left( -a_0(x,t,y,s_m) + a(x,t,y,s_m; \nabla u + \nabla_y u_1 + \delta c) \right) \cdot \delta c(x,t,y,s_m) \, ds_m dy dt \geq 0 \quad (62)
\]

for every \( \delta > 0 \) and every \( c \in \mathcal{D}(\Omega_T;C^0_\#(Y_{1m})^N) \). Hence, let us prove (62). The point of departure is property (B4) which implies the inequality

\[
(a(x,t,\frac{x}{\delta},t_m; \nabla u_\varepsilon) - a(x,t,\frac{x}{\delta},t_m; p^\mu_\varepsilon)) \cdot (\nabla u_\varepsilon(x,t) - p^\mu_\varepsilon(x,t)) \geq 0 \quad ((x,t) \in \Omega_T),
\]

which after integration over \( \Omega_T \) and expansion of the scalar product becomes

\[
\int_{\Omega_T} \left( a^\varepsilon(x,t; \nabla u_\varepsilon) \cdot \nabla u_\varepsilon(x,t) - a^\varepsilon(x,t; \nabla u_\varepsilon) \cdot p^\mu_\varepsilon(x,t) \\
- a^\varepsilon(x,t; p^\mu_\varepsilon) \cdot \nabla u_\varepsilon(x,t) + a^\varepsilon(x,t; p^\mu_\varepsilon) \cdot p^\mu_\varepsilon(x,t) \right) \, dx dt \geq 0.
\]

We can rewrite the first term by (43) to obtain
which is realised to tend to, as \( \varepsilon \to 0 \) and up to a subsequence, the inequality

\[
- \left\langle \frac{\partial}{\partial t} u, u \right\rangle_{X', X} + \int_{\Omega_T} f(x, t) u(x, t) \, dx dt \\
+ \int_{\Omega_T} \left( -a^e(x, t; \nabla u) \cdot p^e_\mu(x, t) - a^f(x, t; p^e_\mu(x, t)) \cdot \nabla u(x, t) \\
+ a^f(x, t; p^e_\mu(x, t)) \right) \, dx dt \geq 0
\]

since

\[
\left\langle \frac{\partial}{\partial t} u, u \right\rangle_{X', X} \leq \liminf_{\varepsilon \to 0} \left\langle \frac{\partial}{\partial t} u_\varepsilon, u_\varepsilon \right\rangle_{X', X}
\]

(see, e.g., the end of the proof of Theorem 3.1 in [33]). We will now investigate what happens when we let \( \mu \to \infty \) in (63). Immediately from the assumptions on \( \{p_\mu\}_{\mu=1}^\infty \) we have, as \( \mu \to \infty \),

\[
p_\mu \to \nabla u + \nabla_y u_1 + \delta c \quad \text{in } L^2(\Omega_T \times \mathcal{Y}_{1m})^N \text{ and a.e. on } \Omega_T \times \mathcal{Y}_{1m},
\]

which takes care of the first term of the second integral in (63). Moreover, we clearly have

\[
a(x, t, y, s_m, p_\mu) \to a(x, t, y, s_m; \nabla u + \nabla_y u_1 + \delta c) \quad \text{(a.e. on } \Omega_T \times \mathcal{Y}_{1m})
\]

which takes care of the mid term of the second integral in (63), and for the last term of the second integral in (63),

\[
a(x, t, y, s_m, p_\mu) \cdot p_\mu(x, t, y, s_m) \to a(x, t, y, s_m; \nabla u + \nabla_y u_1 + \delta c) \\
\cdot \left( \nabla u(x, t) + \nabla_y u_1(x, t, y, s_m) + \delta c(x, t, y, s_m) \right)
\]
a.e. on \( \Omega_T \times \mathcal{Y}_{1m} \). The key to come any further is to use Lebesgue’s Generalised Dominated Convergence Theorem (LGDCT) on this last integral term. (See, e.g., Theorem (19a) on p. 1015 in [48] for the formulation of LGDCT.) What remains in order to employ LGDCT is to establish majorising, non-negative sequences of functions. By (26) (with \( n = 1 \)), we have

\[
|a(x, t, y, s_m; p_\mu)| < C_1 (1 + |p_\mu(x, t, y, s_m)|) \quad ((x, t) \in \Omega_T, (y, s_m) \in \mathcal{Y}_{1m}).
\]

Hence, by applying this observation and the Cauchy–Schwarz inequality, we have for the last term of the second integral in (63) the majorisation

\[
|a(x, t, y, s_m; p_\mu) \cdot p_\mu(x, t, y, s_m)| \leq |a(x, t, y, s_m; p_\mu)| \cdot |p_\mu(x, t, y, s_m)|
\]
The inequality above can be written

\[ < C_1 \left( 1 + |p_\mu(x, t, y, s_m)| \right) |p_\mu(x, t, y, s_m)| \]

\[ = C_1 \left( |p_\mu(x, t, y, s_m)| + |p_\mu(x, t, y, s_m)|^2 \right) \]

a.e. on \( \Omega_T \times \mathcal{Y}_{1m} \). Due to (64), the majorising right-hand side fulfils, as \( \mu \to \infty \), both

\[ C_1 \left( |p_\mu(x, t, y, s_m)| + |p_\mu(x, t, y, s_m)|^2 \right) \]

\[ \to C_1 \left( |\nabla u(x, t) + \nabla y u_1(x, t, y, s_m) + \delta c(x, t, y, s_m)| \right) \]

\[ + |\nabla u(x, t) + \nabla y u_1(x, t, y, s_m) + \delta c(x, t, y, s_m)|^2 \),

a.e. on \( \Omega_T \times \mathcal{Y}_{1m} \), and

\[ \int_{\Omega_T} \int_{\mathcal{Y}_{1m}} C_1 \left( |p_\mu(x, t, y, s_m)| + |p_\mu(x, t, y, s_m)|^2 \right) ds_m dy dx dt \]

\[ \to \int_{\Omega_T} \int_{\mathcal{Y}_{1m}} \left( |\nabla u(x, t) + \nabla y u_1(x, t, y, s_m) + \delta c(x, t, y, s_m)| \right) \]

\[ + |\nabla u(x, t) + \nabla y u_1(x, t, y, s_m) + \delta c(x, t, y, s_m)|^2 \) ds_m dy dx dt;

thus, LGDCT is applicable. Hence, by finally utilising LGDCT, (63) converges to the inequality

\[ - \left\langle \frac{\partial}{\partial t} u, u \right\rangle_{X, X} + \int_{\Omega_T} f(x, t) u(x, t) dx dt \]

\[ + \int_{\Omega_T} \int_{\mathcal{Y}_{1m}} \left( -a_0(x, t, y, s_m) \cdot (\nabla u(x, t) + \nabla y u_1(x, t, y, s_m) + \delta c(x, t, y, s_m)) \right) \]

\[ - a(x, t, y, s_m; \nabla u + \nabla y u_1 + \delta c) \cdot (\nabla u(x, t) + \nabla y u_1(x, t, y, s_m)) \]

\[ + a(x, t, y, s_m; \nabla u + \nabla y u_1 + \delta c) \cdot (\nabla u(x, t) + \nabla y u_1(x, t, y, s_m)) \right) ds_m dy dx dt \geq 0. \]

The inequality above can be written

\[ \int_{\Omega_T} \int_{\mathcal{Y}_{1m}} \left( -a_0(x, t, y, s_m) \cdot \nabla y u_1(x, t, y, s_m) - a_0(x, t, y, s_m) \cdot \delta c(x, t, y, s_m) \right) \]

\[ + a(x, t, y, s_m; \nabla u + \nabla y u_1 + \delta c) \cdot \delta c(x, t, y, s_m) \right) ds_m dy dx dt \geq 0, \quad (65) \]

where we have used (46) to lose the \( \langle \frac{\partial}{\partial t} u, u \rangle \) and the \( \int f u \) terms. We want to lose the first term in the integrand, and in order to achieve this we must utilise the pre-local-problems.

• Suppose \( (\varepsilon, \{\varepsilon_j\}_{j=1}^m) \in J_{\text{wsep}, 1}^{m-1} \). By density, the pre-local-problem (54) holds for all \( v_1 \in \mathcal{W} = H^1_w(Y)/\mathbb{R} \). (The density property follows from the fact that \( H^1_w(Y) \) is defined to be the closure of \( C_c^\infty(Y) \) in the \( H^1(Y) \)-norm; see, e.g., Definition 3.48 in [8].) Hence, since \( u_1(x, t, s_m) \in \mathcal{W} = H^1_w(Y)/\mathbb{R} \) a.e. on \( \Omega_T \times S^m \)

\[ - \int_{\Omega_T} \int_{\mathcal{Y}_{1m}} a_0(x, t, y, s_m) \cdot \nabla y u_1(x, t, y, s_m) ds_m dy dx dt = 0, \]
i.e., the first term in the integrand of (65) gives no contribution in this case.

- Suppose $(\varepsilon, \{\varepsilon_j\}_{j=1}^{m}) \in J_{\text{wsep},2}^{m-k}$. The pre-local-problem (55) can be written

$$
\int_Y \int_S a_0(x, t, y, s_m) \cdot \nabla_y \omega(y, s_m) \, ds_m \, dy = 0 \quad \text{(a.e. on } \Omega_T \times S^{m-1}),
$$

i.e.,

$$
- \int_Y \int_S a_0(x, t, y, s_m) \cdot \nabla_y \omega(y, s_m) \, ds_m \, dy
\quad = - \int_Y \int_S a_1(x, t, y, s_m) \cdot \nabla_y \omega(y, s_m) \, ds_m \, dy \quad \text{(a.e. on } \Omega_T \times S^{m-1})
$$

for all $\omega \in (C^\infty(Y) / \mathbb{R}) \odot C^\infty(S_m)$ and hence, by the density result of Lemma 35 and the fact that the tensor product set spans the corresponding tensor product space, for all $\omega \in H^1(S_m; W, W')$. In this case we have by assumption that $u_1 \in L^2(\Omega_T \times S^{m-1}; H^1(S_m; W, W'))$, which implies $u_1(x, t, s_{m-1}) \in H^1(S_m; W, W')$ a.e. on $\Omega_T \times S^{m-1}$. Thus,

$$
- \int_{\Omega_T} \int_{S_m} a_0(x, t, y, s_m) \cdot \nabla_y u_1(x, t, y, s_m) \, ds_m \, dy \, dx \, dt
\quad = \int_{\Omega_T} \int_{S_m-1} \left( - \int_Y \int_S a_0(x, t, y, s_m) \cdot \nabla_y u_1(x, t, y, s_m) \, ds_m \, dy \right) \, ds_m \, dx \, dt
\quad = - \int_{\Omega_T} \int_{S_m-1} \left( \partial_{s_m} u_1(x, t, s_{m-1}), u_1(x, t, s_{m-1}) \right) \, ds_m \, dx \, dt.
$$

By Lemma 36 the duality pairing in the right-hand side vanishes, so also in this case the first term in the integrand of (65) gives no contribution.

- Suppose $(\varepsilon, \{\varepsilon_j\}_{j=1}^{m}) \in J_{\text{wsep},2+m-k}^{m-k}$ for some $\ell \in [k + 1, m]$ where $k \in [m - 1]$ is required. By density, the pre-local-problem (57) becomes

$$(66) \quad - \int_Y \int_{S_{\ell,m}} a_0(x, t, y, s_m) \, ds_{\ell,m} \cdot \nabla_y v_1(y) \, dy = 0 \quad \text{(a.e. on } \Omega_T \times S^{\ell-1})$$

for all $v_1 \in \mathcal{W} = H^1(Y) / \mathbb{R}$. Since $u_1$ is almost everywhere constant with respect to $s_{\ell,m} \in S^{m_{\ell}}$ due to the pre-local-problem (66), and $u_1(x, t, s_m) \in \mathcal{W} = H^1(Y) / \mathbb{R}$ a.e. on $\Omega_T \times S^m$, we have

$$
- \int_{\Omega_T} \int_{S_{\ell,m}} a_0(x, t, y, s_m) \cdot \nabla_y u_1(x, t, y, s_m) \, ds_m \, dy \, dx \, dt
\quad = \int_{\Omega_T} \int_{S_{\ell-1,m}} \left( - \int_Y \int_{S_{\ell,m}} a_0(x, t, y, s_m) \, ds_{\ell,m} \cdot \nabla_y u_1(x, t, y, s_m) \, dy \right) \, ds_{\ell-1,m} \, dx \, dt,
$$

which clearly vanishes due to (66). Again, the first term in the integrand of (65) gives no contribution.

- Suppose $(\varepsilon, \{\varepsilon_j\}_{j=1}^{m}) \in J_{\text{wsep},1+m-\ell_{-2k}}^{m-k}$ for some $\ell \in [k + 2, m]$ where $k \in [m - 2]$ is required. The pre-local-problem (61) may be written as
\[
\int_Y \int_{S_{\ell - 1}} (u_1(x, t, y, s_m) \partial_{s_{\ell - 1}} \omega(y, s_{\ell - 1})
- \int_{[\varepsilon, m]} a_0(x, t, y, s_m) \cdot \nabla_y \omega(y, s_{\ell - 1})) \, ds_{\ell - 1} \, dy = 0
\]
a.e. on \( \Omega_T \times S^{\varepsilon - 2} \times S^{[\varepsilon, m]} \) for all \( \omega \in (C_\#^\infty(Y)/\mathbb{R}) \otimes C_\#^\infty(S_{\ell - 1}) \), i.e.,

\[
\int_Y \int_{S_{\ell - 1}} a_0(x, t, y, s_m) \cdot \nabla_y \omega(y, s_{\ell - 1}) \, ds_{\ell - 1} \, dy
= - \int_Y \int_{S_{\ell - 1}} u_1(x, t, y, s_m) \partial_{s_{\ell - 1}} \omega(y, s_{\ell - 1}) \, ds_{\ell - 1} \, dy
\]
a.e. on \( \Omega_T \times S^{\varepsilon - 2} \times S^{[\varepsilon, m]} \) for all \( \omega \in (C_\#^\infty(Y)/\mathbb{R}) \otimes C_\#^\infty(S_{\ell - 1}) \) and hence, by the density result of Lemma 35 and the fact that the tensor product set spans the corresponding tensor product space, for all \( \omega \in H^1_\#(S_{\ell - 1}; W, W') \). By assumption, \( u_1 \in L^2(\Omega_T \times S^{\varepsilon - 2} \times S^{[\varepsilon, m]}; H^1_\#(S_{\ell - 1}; W, W')) \), implying \( u_1 \in H^1_\#(S_{\ell - 1}; W, W') \) a.e. on \( \Omega_T \times S^{\varepsilon - 2} \times S^{[\varepsilon, m]} \). Thus,

\[
\int_{\Omega_T} \int_{S^{\varepsilon - 2}} (\partial_{s_{\ell - 1}} a_0(x, t, y, s_m) \cdot \nabla_y u_1(x, t, y, s_m) \, ds_{\ell - 1} \, dy)
\]

\[
\int_{\Omega_T} \int_{S^{\varepsilon - 2}} (\partial_{s_{\ell - 1}} u_1) \, ds_{\ell - 2} \, dy
\]

By Lemma 36 the duality pairing in the right-hand side vanishes implying that the first term in the integrand of (65) gives no contribution.

To conclude, we have proven the inequality (62) for all considered cases. Divide (62) by \( \delta \), let \( \delta \to 0 \) and finally use the Variational Lemma. Then we clearly have

\[
a_0(x, t, y, s_m) = a(x, t, y, s_m; \nabla_u + \nabla u_1) \quad (\text{a.e. on } \Omega_T \times \mathcal{Y}_{1m})
\]
as desired. This establishes an \( H_{\text{MP}} \)-limit \( b \) on the form (38). Since \( u \) is the unique solution to the homogenised equation and \( u_1 \) is the unique solution to the local problems, the convergences (39)-(41) hold not only for the extracted subsequence but for the whole sequence as well. The proof is complete. \( \square \)

**Remark 38.** The assumption \( u_1 \in L^2(\Omega_T \times S^{m - 1}; H^1_\#(S_m; W, W')) \) in the slow resonant case \( J_{\text{wep}}^{m - k} \) merely amounts to the supposition \( \partial_{s_{\ell - 1}} u_1 \in L^2(\Omega_T \times S^{m - 1}; L^2_\#(S_m; W')) \) since we already know \( u_1 \in L^2(\Omega_T \times S^{m - 1}; L^2_\#(S_m; W)) \) as a fact due to Theorem 18 (with \( n = 1 \)). Similarly, in the rapid resonant case \( J_{\text{wep}}^{m - k} \), the assumption \( u_1 \in L^2(\Omega_T \times S^{\varepsilon - 2} \times S^{[\varepsilon, m]}; H^1_\#(S_{\ell - 1}; W, W')) \) boils down to requiring \( \partial_{s_{\ell - 1}} u_1 \in L^2(\Omega_T \times S^{\varepsilon - 2} \times S^{[\varepsilon, m]}; L^2_\#(S_{\ell - 1}; W')) \).
Define $[\ell]_0 = [\ell] \cup \{0\} = \{0, 1, \ldots, \ell\}$ for any non-negative integer $\ell$. Let $k \in [m]_0$. Define $J_{wsep}^{m-k}$ to be the set of all pairs $(\varepsilon, \{\varepsilon'_j\}_{j=1}^m)$ of lists in $J_{wsep}^1$ such that

$$
\begin{cases}
\{\varepsilon, \varepsilon'_1, \ldots, \varepsilon'_m\} & \text{if } k = 0, \\
\{\varepsilon'_1, \ldots, \varepsilon'_m, \varepsilon, \varepsilon'_{k+1}, \ldots, \varepsilon'_m\} & \text{if } k \in [m-1], \\
\{\varepsilon'_1, \ldots, \varepsilon'_m, \varepsilon\} & \text{if } k = m
\end{cases}
$$

is a well-separated list of scale functions. (Hence, for small enough $\varepsilon, \varepsilon < \varepsilon'_k$, explaining the notation "$\prec k$". This could be read as "the spatial scale is asymptotically less than the $k$-th temporal scale"). Define the collection $\{J_{wsep,i}^{m-k}\}_{i=1}^{1+2(m-k)}$ of $1 + 2(m-k)$ subsets of $J_{wsep}^{m-k}$ according to

$$
\begin{align*}
J_{wsep,1}^{m-k} &= \left\{ (\varepsilon, \{\varepsilon'_j\}_{j=1}^m) \in J_{wsep}^{m-k} : \varepsilon'_n \to 0 \right\}, \\
J_{wsep,2}^{m-k} &= \left\{ (\varepsilon, \{\varepsilon'_j\}_{j=1}^m) \in J_{wsep}^{m-k} : \varepsilon'_m \sim \varepsilon^2 \right\}, \\
J_{wsep,2+i-k}^{m-k} &= \left\{ (\varepsilon, \{\varepsilon'_j\}_{j=1}^m) \in J_{wsep}^{m-k} : \varepsilon'_j \to 0 \text{ but } \frac{\varepsilon'_{i-1}}{\varepsilon} \to \infty \right\} \quad (i \in [k+1, m], \quad (k, i) \neq (0, 1)), \\
J_{wsep,1+m+i-2k}^{m-k} &= \left\{ (\varepsilon, \{\varepsilon'_j\}_{j=1}^m) \in J_{wsep}^{m-k} : \varepsilon'_{i-1} \sim \varepsilon^2 \right\} \quad (i^* \in [k+2, m]),
\end{align*}
$$

and

$$J_{wsep,3}^{m-k} = \left\{ (\varepsilon, \{\varepsilon'_j\}_{j=1}^m) \in J_{wsep}^{m-k} : \frac{\varepsilon'_i}{\varepsilon^2} \to 0 \right\} \quad (67)$$

for $(k, i) = (0, 1)$. Actually, $J_{wsep,3}^{m-k}$ does not really need the second condition—i.e., the non-convergence to 0—since it is already implied by the fact that we are in $J_{wsep}^{m-k}$ since there does not exist any "$\varepsilon'_0"$, we note that we need to impose a special definition $(67)$ for $J_{wsep,3}^{m-k}$ without the extra condition. The collection $\{J_{wsep,i}^{m-k}\}_{i=1}^{1+2(m-k)}$ of subsets of $J_{wsep}^{m-k}$ is clearly mutually disjoint. (Note that if $k = m$, the introduced collection of subsets of $J_{wsep}^{m-k}$ reduces to merely $\{J_{wsep,1}\}$.)

The theorem below is a modification of Theorem 37 where the spatial scale function is not allowed to coincide with any temporal scale function.

**Theorem 39.** Let $k \in [m]_0$. Suppose that the pair $e = (\varepsilon, \{\varepsilon'_j\}_{j=1}^m)$ of lists of spatial and temporal scale functions belongs to $\bigcup_{i=1}^{1+2(m-k)} J_{wsep,i}^{m-k}$. Let $\{u_\varepsilon\}$ be the sequence of weak solutions in $H^1(0, T; H^1_0(\Omega)) \times H^{-1}(\Omega)$ to the evolution problem (1) with $a : \Omega_T \times \mathbb{R}^{N+m} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying the structure conditions (B1)–(B5). Then

$$
u \to u \quad \text{in } L^2(\Omega_T),$$

$$u_\varepsilon \to u \quad \text{in } L^2(0, T; H^1_0(\Omega)).$$
and

\[ \nabla u^{(2,m+1)}_e \rightarrow \nabla u + \nabla_y u_1, \]

where \( u \in H^1(0,T;H^1_0(\Omega),H^{-1}(\Omega)) \) and \( u_1 \in L^2(\Omega_T \times S^m;\mathcal{W}) \). Here \( u \) is the unique weak solution to the homogenised problem (28) with the homogenised flux \( b : \overline{\Omega}_T \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) given by

\[ b(x,t;\nabla u) = \int_{\gamma_m} a(x,t,y,s_m;\nabla u + \nabla_y u_1) \, ds_m \, dy. \]

Moreover, we have the following characterisation of \( u \):

- If \( e \in \mathcal{J}^{m-k}_{wsep,1} \) then the function \( u_1 \) is the unique weak solution to the local problem

\[ -\nabla_y \cdot a(x,t,y,s_m;\nabla u + \nabla_y u_1) = 0. \]

- If \( e \in \mathcal{J}^{m-k}_{wsep,2} \), assuming \( u_1 \in L^2(\Omega_T \times S^{m-1};H^1_\#(S_m;\mathcal{W},\mathcal{W}')) \), then the function \( u_1 \) is the unique weak solution to the local problem

\[ \partial_{s_m} u_1(x,t,y,s_m) - \nabla_y \cdot a(x,t,y,s_m;\nabla u + \nabla_y u_1) = 0. \]

- If \( e \in \mathcal{J}^{m-k}_{wsep,2+\overline{\ell}-k} \) for some \( \overline{\ell} \in [k+1,m] \), provided \( k \in [m-1] \), then the function \( u_1 \) is the unique weak solution to the system of local problems

\[
\begin{cases}
-\nabla_y \cdot \int_{\gamma_m} a(x,t,y,s_m;\nabla u + \nabla_y u_1) \, ds_m = 0, \\
\partial_{s_i} u_1(x,t,y,s_m) = 0 \quad (i \in [\overline{\ell}, m]).
\end{cases}
\]

- If \( e \in \mathcal{J}^{m-k}_{wsep,2+m-k} \) for some \( \ell^* \in [k+2,m] \), provided \( k \in [m-2] \) and assuming \( u_1 \in L^2(\Omega_T \times S^{\ell^*-2} \times S^{[\ell^*,m]};H^1_\#(S^{\ell^*-1};\mathcal{W},\mathcal{W}')) \), then the function \( u_1 \) is the unique weak solution to the system of local problems

\[
\begin{cases}
\partial_{s_j} u_1(x,t,y,s_m) - \nabla_y \cdot \int_{\gamma_m} a(x,t,y,s_m;\nabla u + \nabla_y u_1) \, ds_m = 0, \\
\partial_{s_i} u_1(x,t,y,s_m) = 0 \quad (i \in [\ell^*, m]).
\end{cases}
\]

**Proof.** Let \( \hat{m} = m + 1 \) and \( \hat{k} = k + 1 \). (Note that \( \hat{k} \in [\hat{m}] \) since \( k \in [m] \).) Introduce the list \( \{\hat{\epsilon}_j\}_{j=1}^{\hat{m}} \) of \( \hat{m} \) new temporal scale functions defined according to

\[
\begin{align*}
\hat{\epsilon}_j = \epsilon, & \quad \hat{\epsilon}_j = \epsilon_{j-1} \quad \text{for } j \in [2, \hat{m}] \quad \text{if } \hat{k} = 1, \\
\hat{\epsilon}_j = \epsilon_j & \quad \text{for } j \in [\hat{k} - 1], \quad \hat{\epsilon}_k = \epsilon, \quad \text{and } \hat{\epsilon}_j = \epsilon_{j-1} \quad \text{for } j \in [\hat{k} + 1, \hat{m}] \quad \text{if } \hat{k} \in [2, \hat{m} - 1], \\
\hat{\epsilon}_j = \epsilon_j & \quad \text{for } j \in [\hat{m} - 1], \quad \text{and } \hat{\epsilon}_\hat{m} = \epsilon \quad \text{if } \hat{k} = \hat{m}.
\end{align*}
\]

Since \( (\epsilon, \{\epsilon_j\}_{j=1}^{m}) \in \mathcal{J}^{m-k}_{wsep} \) it must thus equivalently hold that \( (\epsilon, \{\hat{\epsilon}_j\}_{j=1}^{\hat{m}}) \in \mathcal{J}^{m-\hat{k}}_{wsep}. \)

Define \( \hat{a} : \overline{\Omega}_T \times \mathbb{R}^{N+\hat{m}} \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) according to

\[ \hat{a}(x,t,y,s_m;q) = a(x,t,y,s_m;q) \quad ((x,t) \in \overline{\Omega}_T, (y,s_m) \in \gamma_{1m}, q \in \mathbb{R}^N), \]
where we define (provided \( \hat{k} \in [2, \hat{m} - 1] \))

\[
\hat{s}_{\hat{m}} = (s_{k-1}, \hat{s}_{\hat{k}}, s_{[\hat{k}, \hat{m} - 1]}) \quad (s_{\hat{m} - 1} = s_{\hat{m}} \in S^m = S^{\hat{m} - 1})
\]

for any \( \hat{s}_{\hat{k}} \in \hat{S}_{\hat{k}} = (0, 1) \). (The cases \( \hat{k} = 1 \) and \( \hat{k} = \hat{m} \) require obvious respective modifications of the definition.) This means that \( \hat{a} \) is in fact independent of \( \hat{s}_{\hat{k}} \in \hat{S}_{\hat{k}} \), though not manifestly so. Furthermore, define \( \hat{Y}_{1,\hat{m}} = Y \times \hat{S}_{\hat{m}} \) where (provided \( \hat{k} \in [2, \hat{m} - 1] \))

\[
\hat{S}_{\hat{m}} = \hat{S}_{\hat{k}}^{\hat{k} - 1} \times \hat{S}_{\hat{k}} \times \hat{S}_{[\hat{k}, \hat{m} - 1]}.
\]

(The cases \( \hat{k} = 1 \) and \( \hat{k} = \hat{m} \) require obvious respective modifications of the definition.)

It is clear that since \( a \) satisfies (B1)-(B5), so does \( \hat{a} \). Let \( \{\hat{u}_\varepsilon\} \) be the sequence of weak solutions in \( H^1(0, T; H^1_0(\Omega), H^{-1}(\Omega)) \) to the evolution problem \((\Pi)\) with \( \hat{a} \) instead of \( a \). (Note that \( u_\varepsilon = u_\varepsilon \) since \( \hat{a} = a \).) By Theorem 37 (with “hatted” quantities) we then get

\[
\hat{u}_\varepsilon \to \hat{u} \quad \text{in } L^2(\Omega_T),
\]

\[
\hat{u}_\varepsilon \to \hat{u} \quad \text{in } L^2(0, T; H^1_0(\Omega)),
\]

and

\[
\nabla \hat{u}_\varepsilon \xrightarrow{(2,m+2)} \nabla \hat{u} + \nabla_y \hat{u}_1,
\]

where \( \hat{u} \in H^1(0, T; H^1_0(\Omega), H^{-1}(\Omega)) \) and \( \hat{u}_1 \in L^2(\Omega_T \times \hat{S}_{\hat{m}}; \mathcal{W}) \). Here \( \hat{u} \) is the unique weak solution to the homogenised problem \((\hat{\Pi})\) but with the homogenised flux \( \hat{b} : \Omega_T \times \mathbb{R}^N \to \mathbb{R}^N \) given by

\[
\hat{b}(x, t, \nabla \hat{u}) = \int_{\hat{\mathcal{Y}}_1, \hat{\mathcal{M}}} \hat{a}(x, t, y, \hat{s}_{\hat{m}}; \nabla \hat{u} + \nabla_y \hat{u}_1) \, d\hat{s}_{\hat{m}} \, dy,
\]

and \( \hat{u}_1 \) is the unique weak solution to the local problems

\[
- \nabla_y \cdot \hat{a}(x, t, y, \hat{s}_{\hat{m}}; \nabla \hat{u} + \nabla_y \hat{u}_1) = 0
\]

if \((\varepsilon, \{\hat{\varepsilon}_j\}_{j=1}^\hat{m}) \in \mathcal{J}_{\text{wsep}, 1}^{\hat{m} - \hat{k}}\):

\[
\partial_{\hat{s}_{\hat{m}}} \hat{u}_1(x, t, y, \hat{s}_{\hat{m}}) - \nabla_y \cdot \hat{a}(x, t, y, \hat{s}_{\hat{m}}; \nabla \hat{u} + \nabla_y \hat{u}_1) = 0
\]

if \((\varepsilon, \{\hat{\varepsilon}_j\}_{j=1}^\hat{m}) \in \mathcal{J}_{\text{wsep}, 2}^{\hat{m} - \hat{k}}\) and assuming \( \hat{u}_1 \in L^2(\Omega_T \times \hat{S}_{\hat{m}}^{-1}; H^1_#(\hat{s}_{\hat{m}}; \mathcal{W}, \mathcal{W}')\);

\[
\left\{ - \nabla_y \cdot \int_{\hat{\mathcal{Y}}_{\hat{m}}, \hat{\mathcal{M}}} \hat{a}(x, t, y, \hat{s}_{\hat{m}}; \nabla \hat{u} + \nabla_y \hat{u}_1) \, d\hat{s}_{[\hat{y}, \hat{m}]} \right\} = 0,
\]

\[
\partial_{\hat{s}_i} \hat{u}_1(x, t, y, \hat{s}_{\hat{m}}) = 0 \quad (i \in [\hat{\ell}, \hat{m}])
\]

(68)
assuming \( \hat{\varepsilon} \) for simplicity, we consider the strongly rather than weakly formulated versions of the

\[
\left\{ \begin{array}{l}
\partial_{\hat{t}} \hat{u}_1(x, t, y, \hat{s}_m) - \nabla_y \cdot \int_{\hat{S}[\hat{\varepsilon}, \hat{m}]} \tilde{a}(x, t, y, \hat{s}_m; \nabla u + \nabla_y u_1) \, ds \hat{\varepsilon}[\hat{\varepsilon}, \hat{m}] = 0, \\
\partial_{\hat{t}} \hat{u}_1(x, t, y, \hat{s}_m) = 0 \quad (\hat{t} \in [\hat{\varepsilon}, \hat{m}])
\end{array} \right.
\]

\( (71) \)

if \( (\varepsilon, \{ \varepsilon_j \}_{j=1}^m) \in J_{\text{wsep}, 2}^{\hat{m}-\hat{k}} \) for some \( \hat{k} \in [\hat{\varepsilon}, \hat{m}] \) provided \( \hat{k} \in [\hat{\varepsilon} - \hat{k}] \); and

we can write \( \hat{u}_1 \) as

\[
\hat{u}_1(x, t, y, \hat{s}_m) = \hat{b}(x, t, \nabla \hat{u})
\]

which is the local problem if \( (\varepsilon, \{ \varepsilon_j \}_{j=1}^m) \in J_{\text{wsep}, 1}^{\hat{m}-\hat{k}} \), i.e., \( (\varepsilon, \{ \varepsilon_j \}_{j=1}^m) \in J_{\text{wsep}}^{\hat{m}-\hat{k}} \) and \( \varepsilon \rightarrow 0 \), which is equivalent to \( (\varepsilon, \{ \varepsilon_j \}_{j=1}^m) \in J_{\text{wsep}}^{\hat{m}-\hat{k}} \) and \( \varepsilon_j \rightarrow 0 \), i.e., we have precisely \( (\varepsilon, \{ \varepsilon_j \}_{j=1}^m) \in J_{\text{wsep}, 1}^{\hat{m}-\hat{k}} \). Obviously, \( \hat{u}_1 \) must be independent of \( \hat{s}_m \), i.e., we can write the unique solution as \( u_1 = \hat{u}_1 \in L^2(\Omega_T \times S^m; \mathcal{W}) \) which depends only on \( (x, t) \in \Omega_T \) and \( (y, s_m) \in \mathcal{Y}_{1m} \). We thus conclude that the local problem when \( (\varepsilon, \{ \varepsilon_j \}_{j=1}^m) \in J_{\text{wsep}, 1}^{\hat{m}-\hat{k}} \) is

\[
- \nabla_y \cdot a(x, t, y, s_m; \nabla u + \nabla_y u_1) = 0,
\]

and the homogenised flux \( b : \Omega_T \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) is defined by

\[
b(x, t; \nabla u) = \hat{b}(x, t, \nabla \hat{u}) = \int_{\hat{Y}_{1m}} \hat{a}(x, t, y, \hat{s}_m; \nabla \hat{u} + \nabla_y \hat{u}_1) \, ds \hat{m} \, dy = \int_{\hat{Y}_{1m}} a(x, t, y, s_m; \nabla u + \nabla_y u_1) \, ds_m \, dy.
\]

(72)

(This is because \( a^\varepsilon \rightarrow a^\varepsilon \rightarrow \hat{a} \rightarrow \hat{b} \) in \( L^2(\Omega_T) \); see Definition [28])

\( \bullet \) We can write \( (69) \) as

\[
\partial_{s_m} \hat{u}_1(x, t, y, \hat{s}_m) - \nabla_y \cdot a(x, t, y, s_m; \nabla u + \nabla_y \hat{u}_1) = 0,
\]

which is the local problem if \( (\varepsilon, \{ \varepsilon_j \}_{j=1}^m) \in J_{\text{wsep}, 2}^{\hat{m}-\hat{k}} \), i.e., \( (\varepsilon, \{ \varepsilon_j \}_{j=1}^m) \in J_{\text{wsep}}^{\hat{m}-\hat{k}} \) and \( \varepsilon_j \sim \varepsilon^\varepsilon \), which is equivalent to \( (\varepsilon, \{ \varepsilon_j \}_{j=1}^m) \in J_{\text{wsep}}^{\hat{m}-\hat{k}} \) and \( \varepsilon_j \sim \varepsilon^\varepsilon \), i.e., we have precisely \( (\varepsilon, \{ \varepsilon_j \}_{j=1}^m) \in J_{\text{wsep}, 2}^{\hat{m}-\hat{k}} \). Obviously, \( \hat{u}_1 \) must be independent of \( \hat{s}_m \), i.e., we can write the unique solution as \( u_1 = \hat{u}_1 \in L^2(\Omega_T \times S^m; \mathcal{W}) \) which depends only on \( (x, t) \in \Omega_T \) and \( (y, s_m) \in \mathcal{Y}_{1m} \). The assumption \( \hat{u}_1 \in L^2(\Omega_T \times \hat{S}^{\hat{m}-1}; H^1_0(\hat{\mathcal{S}}_{\hat{m}}; \mathcal{W}, \mathcal{W}')) \) is clearly

\[ 50 \]
equivalent to \( u_1 \in L^2(\Omega_T \times S^{m-1}; H^1_\#(S_m; \mathcal{W}, \mathcal{W}')) \). We thus conclude that the local problem when \((\varepsilon, \{\varepsilon_j\}_{j=1}^m) \in \mathcal{J}^{m-k}_{\text{wsep},2} \) assuming \( u_1 \in L^2(\Omega_T \times S^{m-1}; H^1_\#(S_m; \mathcal{W}, \mathcal{W}')) \) is

\[
\partial_s u_1(x, t, y, s_m) - \nabla_y \cdot a(x, t, y, s_m; \nabla u + \nabla_y u_1) = 0,
\]

and the homogenised flux \( b \) is given by (72) again.

• Let \( \ell \) and \( i \) be defined through \( \ell = \ell + 1 \) and \( i = i + 1 \), respectively; we can then write (70) as

\[
\begin{cases}
-\nabla_y \cdot \int_{\mathcal{G}_{\ell,m}} a(x, t, y, s_m; \nabla u + \nabla_y u_1) \, ds_{[\ell,m]} = 0, \\
\partial_s \tilde{u}_1(x, t, y, s_{\tilde{m}}) = 0 \quad (i \in [\ell, m]),
\end{cases}
\]

which are the local problems if \((\varepsilon, \{\varepsilon_j\}_{j=1}^m) \in \mathcal{J}^{m-k}_{\text{wsep},2} \) and \( \hat{\ell} \rightarrow 0 \) but, only necessary if and only if \( \ell \neq k + 1 \) \( \iff \hat{k}, \hat{\ell} \neq (k, \hat{k} + 1) \), \( \hat{\ell} \rightarrow \infty \). This is in turn equivalent to \((\varepsilon, \{\varepsilon_j\}_{j=1}^m) \in \mathcal{J}^{m-k}_{\text{wsep}} \) and \( \hat{\ell} \rightarrow 0 \) but, only necessary if \((k, \hat{\ell}) \neq (0, 1) \iff \hat{k}, \hat{\ell} \neq (1, 2) \), \( \hat{\ell} \rightarrow \infty \), i.e., we have precisely \((\varepsilon, \{\varepsilon_j\}_{j=1}^m) \in \mathcal{J}^{m-k}_{\text{wsep},2+\hat{\ell} - k} \). We thus conclude that the local problems when \((\varepsilon, \{\varepsilon_j\}_{j=1}^m) \in \mathcal{J}^{m-k}_{\text{wsep},2+\hat{\ell} - k} \) for some \( \ell \in [k + 1, m] \) are

\[
\begin{cases}
-\nabla_y \cdot \int_{\mathcal{G}_{\ell,m}} a(x, t, y, s_m; \nabla u + \nabla_y u_1) \, ds_{[\ell,m]} = 0, \\
\partial_s u_1(x, t, y, s_m) = 0 \quad (i \in [\ell, m]),
\end{cases}
\]

and the homogenised flux \( b \) is given by (72) again.

Let \( \ell^* \) and \( i \) be defined through \( \ell^* = \ell + 1 \) and \( i = i + 1 \), respectively; we can then write (71) as

\[
\begin{cases}
\partial_{s_{\ell-1}} \tilde{u}_1(x, t, y, s_{\tilde{m}}) - \nabla_y \cdot \int_{\mathcal{G}_{\ell^*,m}} a(x, t, y, s_{\tilde{m}}; \nabla u + \nabla_y u_1) \, ds_{[\ell^*,m]} = 0, \\
\partial_s \tilde{u}_1(x, t, y, s_{\tilde{m}}) = 0 \quad (i \in [\ell^*, m]),
\end{cases}
\]

which are the local problems if \((\varepsilon, \{\varepsilon_j\}_{j=1}^m) \in \mathcal{J}^{m-k}_{\text{wsep},1+\hat{\ell} - 2k} \), \( \hat{\ell} \rightarrow \hat{k} + 2 \), i.e., \((\varepsilon, \{\varepsilon_j\}_{j=1}^m) \in \mathcal{J}^{m-k}_{\text{wsep}} \) and \( \hat{\ell} \rightarrow 1 \), which is equivalent to \((\varepsilon, \{\varepsilon_j\}_{j=1}^m) \in \mathcal{J}^{m-k}_{\text{wsep}} \) and \( \hat{\ell} \rightarrow 1 \), i.e., we have precisely \((\varepsilon, \{\varepsilon_j\}_{j=1}^m) \in \mathcal{J}^{m-k}_{\text{wsep},1+m+\ell^* - 2k} \), \( \ell^* \in [k + 2, m] \). Obviously, \( \tilde{u}_1 \) must be independent of \( \hat{s}_{\tilde{k}} \), i.e., we can write the unique solution as \( u_1 = \tilde{u}_1 \in L^2(\Omega_T \times S^m; \mathcal{W}) \) which depends only on \((x, t) \in \Omega_T \) and \((y, s_m) \in \mathcal{Y}_{1m} \). We thus conclude that the
local problems when \((\varepsilon, \{\varepsilon_j\}_{j=1}^m) \in \mathcal{J}_{\text{wsep},1}^{m-k}\) for some \(\varepsilon \in [k+2, m]\) assuming \(u_1 \in L^2(\Omega_T \times S^{\varepsilon-2} \times S_{\varepsilon}^{m}; H^1_0(S_{\varepsilon}^{-1}; \mathcal{W}, \mathcal{W}'))\) are

\[
\begin{cases}
\partial_{s_{\varepsilon^i,m}} u_1(x, t, y, s_m) - \nabla_y \cdot \int_{S_{\varepsilon}^{m}} a(x, t, y, s_m; \nabla u + \nabla_y u_1) \, ds_{\varepsilon^i,m} = 0, \\
\partial_{s_{\varepsilon^i,m}} u_1(x, t, y, s_m) = 0 \quad (i \in [\varepsilon, m]),
\end{cases}
\]

and the homogenised flux \(b\) is given by (72) again. The proof is complete. \(\square\)

Define \(\mathcal{J}_{\text{wsep}}^{m-0} = \emptyset\) and \(\mathcal{J}_{\text{wsep}}^{m-0,j} = \emptyset, j \in [1 + 2m]\). Let \(k \in [m]_0\) and introduce \(\mathcal{J}_{\text{wsep}}^{m-k} = \mathcal{J}_{\text{wsep}}^{m-k} \cup \mathcal{J}_{\text{wsep}}^{m-k, i}\) and \(\mathcal{J}_{\text{wsep}}^{m-k, i} = \mathcal{J}_{\text{wsep}}^{m-k} \cup \mathcal{J}_{\text{wsep}}^{m-k, i}, i \in [1 + 2(m-k)]\). (The notation \(\ll k\) could be read as “the spatial scale is asymptotically equal to or less than the \(k\)-th temporal scale”. The asymptotic equality to the 0-th temporal scale is meaningless which explains why we define the corresponding sets of pairs of lists of scale functions as being empty.) From Theorems 37 and 39 we immediately arrive in the corollary below, which is the main result of this e-print paper.

**Corollary 40.** Let \(k \in [m]_0\). Suppose that the pair \(e = (\varepsilon, \{\varepsilon_j\}_{j=1}^m)\) of lists of spatial and temporal scale functions belongs to \(\bigcup_{i=1}^{1+2(m-k)} \mathcal{J}_{\text{wsep},i}^{m-k}\). Let \(\{u_\varepsilon\}\) be the sequence of weak solutions in \(H^1(0, T; H^1_0(\Omega), H^{-1}(\Omega))\) to the evolution problem (1) with \(a : \overline{\Omega}_T \times \mathbb{R}^{N+m} \times \mathbb{R}^N \to \mathbb{R}^N\) satisfying the structure conditions (B1)–(B3). Then

\[
\begin{align*}
&u_\varepsilon \to u \quad \text{in } L^2(\Omega_T), \\
&u_\varepsilon \rightharpoonup u \quad \text{in } L^2(0, T; H^1_0(\Omega)),
\end{align*}
\]

and

\[
\nabla u_\varepsilon \rightharpoonup \nabla u + \nabla_y u_1,
\]

where \(u \in H^1(0, T; H^1_0(\Omega), H^{-1}(\Omega))\) and \(u_1 \in L^2(\Omega_T \times S^m; \mathcal{W})\). Here \(u\) is the unique weak solution to the homogenised problem (28) with the homogenised flux \(b : \overline{\Omega}_T \times \mathbb{R}^N \to \mathbb{R}^N\) given by

\[
b(x, t; \nabla u) = \int_{S_{\varepsilon}^{m}} a(x, t, y, s_m; \nabla u + \nabla_y u_1) \, ds_m \, dy.
\]

Moreover, we have the following characterisation of \(u_1\):

- If \(e \in \mathcal{J}_{\text{wsep},1}^{m-k}\) then the function \(u_1\) is the unique weak solution to the local problem

\[
- \nabla_y a(x, t, y, s_m; \nabla u + \nabla_y u_1) = 0.
\]

- If \(e \in \mathcal{J}_{\text{wsep},2}^{m-k}\), assuming \(u_1 \in L^2(\Omega_T \times S^{m-1}; H^1_0(S_m; \mathcal{W}, \mathcal{W}'))\), then the function \(u_1\) is the unique weak solution to the system of local problems

\[
\begin{align*}
\partial_{s_m} u_1(x, t, y, s_m) &= - \nabla_y a(x, t, y, s_m; \nabla u + \nabla_y u_1) = 0, \\
- \nabla_y \int_{S_m} a(x, t, y, s_m; \nabla u + \nabla_y u_1) \, ds_m &= 0.
\end{align*}
\]
• If \( e \in \mathcal{J}_{wsep,2}^{m \leq k} \) for some \( \ell \in [k + 1, m] \), provided \( k \in [m - 1]_0 \), then the function \( u_1 \) is the unique weak solution to the system of local problems

\[
-\nabla y \cdot \int_{S[\ell, m]} a(x, t, y, s_m; \nabla u + \nabla y u_1) \, ds_{[\ell, m]} = 0,
\]

\[
\partial_s u_1(x, t, y, s_m) = 0 \quad (i \in [\ell, m]).
\]

• If \( e \in \mathcal{J}_{wsep,1}^{m \leq k} \) for some \( \ell^{*} \in [k + 2, m] \), provided \( k \in [m - 2]_0 \) and assuming \( u_1 \in L^2(\Omega_T \times S^{\ell^{*}-2} \times S^{[\ell^{*}, m]}; H^1_{w}(S^{\ell^{*}-1}; W, W')) \), then the function \( u_1 \) is the unique weak solution to the system of local problems

\[
\partial_{s^{*-1}} u_1(x, t, y, s_m) - \nabla y \cdot \int_{S^{[\ell^{*}, m]}} a(x, t, y, s_m; \nabla u + \nabla y u_1) \, ds_{[\ell^{*}, m]} = 0,
\]

\[
\partial_s u_1(x, t, y, s_m) = 0 \quad (i \in [\ell^{*}, m]).
\]

Remark 41. Corollary 40 can only handle the subset \( \bigcup_{i=1}^{1+2(m-k)} \mathcal{J}_{wsep,i}^{m \leq k} \) of \( \mathcal{J}_{wsep}^{m \leq k} \). The conclusion of Proposition 33 is true also in the setting of Corollary 40 though, i.e., the collection \( \{\mathcal{P}_{i}^{m \leq k}\}_{i=1}^{1+2(m-k)} \) forms a partition of \( \mathcal{P}^{m \leq k} \) where \( \mathcal{P}^{m \leq k} \) is the subset of \( \mathcal{J}_{wsep}^{m \leq k} \) with temporal scale functions expressed as power functions, and \( \mathcal{P}_{i}^{m \leq k} \) is the corresponding subset of \( \mathcal{J}_{wsep,i}^{m \leq k} \) for every \( i \in [1+2(m-k)] \).

References

[1] Allaire, G. “Homogenization and two-scale convergence”, SIAM J. Math. Anal. 23 (1992), no. 6, 1482–1518.

[2] Allaire, G.; Briane, M. “Multiscale convergence and reiterated homogenisation”, Proc. Roy. Soc. Edinburgh Sect. A 126 (1996), no. 2, 297–342.

[3] Arbogast, T.; Douglas, J., Jr.; Hornung, U. “Derivation of the double porosity model of single phase flow via homogenization theory”, SIAM J. Math. Anal. 21 (1990), no. 4, 823–836.

[4] Bensoussan, A.; Lions, J.-L.; Papanicolaou, G. “Asymptotic analysis for periodic structures”, Studies in Mathematics and its Applications, 5. North-Holland Publishing Co., Amsterdam-New York, 1978.

[5] Chiadò Piat, V.; Dal Maso, G.; Defranceschi, A. “G-convergence of monotone operators”, Ann. Inst. H. Poincaré Anal. Non Linéaire 7 (1990), no. 3, 123–160.

[6] Chiadò Piat, V.; Defranceschi, A. “Homogenization of monotone operators”, Nonlinear Anal. 14 (1990), no. 9, 717–732.
[7] Ciorănescu, D.; Damlamian, A.; Griso, G. “Periodic unfolding and homogenization”, C. R. Math. Acad. Sci. Paris 335 (2002), no. 1, 99–104.

[8] Ciorănescu, D.; Donato P. “An introduction to homogenization”, Oxford University Press, New York, 1999.

[9] Colombini, F.; Spagnolo, S. “Sur la convergence de solutions d’équations paraboliques”, J. Math. Pures Appl. (9) 56 (1977), no. 3, 263–305.

[10] Constantin, P.; Foiaş, C. “Navier-Stokes equations”, University of Chigaco Press, Chicago, 1988.

[11] Evans, L. C. “The perturbed test function method for viscosity solutions of nonlinear PDE”, Proc. Roy. Soc. Edinburgh Sect. A 111 (1989), no. 3-4, 359–375.

[12] Evans, L. C. “Periodic homogenization of certain fully nonlinear partial differential equations”, Proc. Roy. Soc. Edinburgh Sect. A 120 (1992), no. 3-4, 245–265.

[13] Flodén, L.; Olsson, M. “Reiterated homogenization of some linear and nonlinear monotone parabolic operators”, Can. Appl. Math. Q. 14 (2006), no. 2, 149–183.

[14] Flodén, L.; Olsson, M. “Homogenization of some parabolic operators with several time scales”, Appl. Math. 52 (2007), no. 5, 431–446.

[15] Flodén, L.; Holmbom, A.; Olsson, M.; Svanstedt, N. “Reiterated homogenization of monotone parabolic problems”, Ann. Univ. Ferrara Sez. VII Sci. Mat. 53 (2007), no. 2, 217–232.

[16] Holmbom, A. “Some modes of convergence and their application to homogenization and optimal composites design”, Doctoral thesis 1996:208 D, Department of mathematics, Luleå university, 1996.

[17] Holmbom, A. “Homogenization of parabolic equations: an alternative approach and some corrector-type results”, Appl. Math. 42 (1997), no. 5, 321–343.

[18] Holmbom, A.; Silfver, J. “On the convergence of some sequences of oscillating functionals”, WSEAS Trans. Math. 5 (2006), no. 8, 951–956.

[19] Holmbom, A.; Silfver, J.; Svanstedt, N.; Wellander, N. “On two-scale convergence and related sequential compactness topics”, Appl. Math. 51 (2006), no. 3, 247–262.

54
[20] Holmbom, A.; Svanstedt, N.; Wellander, N. “Multiscale convergence and reiterated homogenization of parabolic problems”, *Appl. Math.* 50 (2005), no. 2, 131–151.

[21] Kun’ch, R. N. “G-convergence of nonlinear parabolic operators”, Thesis, Donetsk, 1988.

[22] Kun’ch, R. N.; Pankov, A. A. “G-convergence of monotone parabolic operators”, *Dokl. Akad. Nauk Ukrain. SSR Ser. A* 1986, no. 8, 8–10.

[23] Lions, J.-L.; Lukkassen, D.; Persson, L.-E.; Wall, P. “Reiterated homogenization of nonlinear monotone operators”, *Chinese Ann. Math. Ser. B* 22 (2001), no. 1, 1–12.

[24] Lukkassen, D.; Nguetseng, G.; Wall, P. “Two-scale convergence”, *Int. J. Pure Appl. Math.* 2 (2002), no. 1, 35–86.

[25] Mascarenhas, M. L.; Toader, A.-M. “Scale convergence in homogenization”, *Numer. Funct. Anal. Optim.* 22 (2001), no. 1-2, 127–158.

[26] Murat, F. “H-convergence”, Séminaire d’analyse fonctionnelle et numérique de l’Université d’Alger, 1978.

[27] Murat, F. “Compacité par compensation”, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 5 (1978), no. 3, 489–507.

[28] Murat, F. “Compacité par compensation. II”, *Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis* (Rome, 1978), pp. 245–256, Pitagora, Bologna, 1979.

[29] Nechvátal, L. “Alternative approaches to the two-scale convergence”, *Appl. Math.* 49 (2004), no. 2, 97–110.

[30] Nguetseng, G. “A general convergence result for a functional related to the theory of homogenization”, *SIAM J. Math. Anal.* 20 (1989), no. 3, 608–623.

[31] Nguetseng, G. “Homogenization structures and applications. I”, *Z. Anal. Anwendungen* 22 (2003), no. 1, 73–107.

[32] Nguetseng, G. “Homogenization structures and applications. II”, *Z. Anal. Anwendungen* 23 (2004), no. 3, 483–508.

[33] Nguetseng, G.; Woukeng, J. L. “Deterministic homogenization of parabolic monotone operators with time dependent coefficients”, *Electron. J. Differential Equations* 2004, No. 82, 23 pp.
[34] Nguetseng, G.; Woukeng, J. L. “Σ-convergence of nonlinear parabolic operators”, *Nonlinear Anal.* 66 (2007), no. 4, 968–1004.

[35] Pankov, A. “G-convergence and homogenization of nonlinear partial differential operators”, Mathematics and its Applications, 422. *Kluwer Academic Publishers*, Dordrecht, 1997.

[36] Silfver, J. “G-convergence and homogenization involving operators compatible with two-scale convergence”, Mid Sweden University Doctoral Thesis 23, 2007.

[37] Spagnolo, S. “Sul limite delle soluzioni di problemi di Cauchy relativi all’equazione del calore”, *Ann. Scuola Norm. Sup. Pisa* (3) 21 (1967), 657–699.

[38] Spagnolo, S. “Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche”, *Ann. Scuola Norm. Sup. Pisa* (3) 22 (1968), 571–597.

[39] Spagnolo, S. “Convergence in energy for elliptic operators”, *Numerical solution of partial differential equations, III* (Proc. Third Sympos. (SYNSPADE), Univ. Maryland, College Park, Md., 1975), pp. 469–498. *Academic Press, New York*, 1976.

[40] Svanstedt, N. “G-convergence and homogenization of sequences of linear and nonlinear partial differential operators”, Doctoral thesis 1992:105 D, Department of mathematics, Luleå university, 1992.

[41] Svanstedt, N. “G-convergence of parabolic operators”, *Nonlinear Anal.* 36 (1999), no. 7, Ser. A: Theory Methods, 807–842.

[42] Tartar, L. “Cours peccot”, Collège de France, 1977, unpublished, partially written in [26].

[43] Tartar, L. “Quelques remarques sur l’homogénéisation”, In: M. Fujita, Editor, *Functional Analysis and Numerical Analysis, Proc. JapanFrance Seminar 1976*, Japanese Society for the Promotion of Science (1978), 468-482.

[44] Tartar, L. “Compensated compactness and applications to partial differential equations”, *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV*, 136–212, Res. Notes in Math., 39, *Pitman, Boston, Mass.-London*, 1979.

[45] Weickert, J. “Anisotropic diffusion in image processing”, *Teubner Verlag*, Stuttgart, 1998. Available online from: [http://www.mia.uni-saarland.de/weickert/book.html](http://www.mia.uni-saarland.de/weickert/book.html)
[46] Woukeng, J. L. “Periodic homogenization of nonlinear non-monotone parabolic operators with three time scales”, *Ann. Mat. Pura Appl.* (4) 189 (2010), no. 3, 357–379.

[47] Zeidler, E. “Nonlinear functional analysis and its applications. II/A. Linear monotone operators”, *Springer-Verlag*, New York, 1990.

[48] Zeidler, E. “Nonlinear functional analysis and its applications. II/B. Nonlinear monotone operators”, *Springer-Verlag*, New York, 1990.