Nonsingular 2-D Black Holes and Classical String Backgrounds

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We study a string-inspired classical 2-D effective field theory with nonsingular black holes as well as Witten’s black hole among its static solutions. By a dimensional reduction, the static solutions are related to the \( (SL(2, R)_k \otimes U(1))/U(1) \) coset model, or more precisely its \( O((\alpha')^0) \) approximation known as the 3-D charged black string. The 2-D effective action possesses a propagating degree of freedom, and the dynamics are highly nontrivial. A collapsing shell is shown to bounce into another universe without creating a curvature singularity on its path, and the potential instability of the Cauchy horizon is found to be irrelevant in that some of the infalling observers never approach the Cauchy horizon. Finally a \( SL(2, R)_k/U(1) \) nonperturbative coset metric, found and advocated by R. Dijkgraaf et.al., is shown to be nonsingular and to coincide with one of the charged spacetimes found above. Implications of all these geometries are discussed in connection with black hole evaporation.

February 1993

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† Supported in part by U.S. Dept. of Energy under Contract no. DEAC-03-81ER40050
1. Introduction

The longstanding problem of black hole evaporation has been recently revitalized in the context of 2-D dilaton gravity [1], either on its own or as an effective S-wave sector theory of the 4-D black holes [2]. In 2-D, the coupling to conformal matter, hence the Hawking radiation, can be conveniently represented by the Polyakov-Liouville action of appropriate coefficient, enabling a systematic study of the gravitational backreaction [1][3]. But most of these studies are again plagued by curvature singularities, hidden or naked, and the inability to handle the singularities properly clouds any conclusion concerning the fate of the black holes [3][4][5].

The question arises, then, whether the singularity is an essential feature of black hole physics. For example the Hawking radiation is, strictly speaking, a property of event horizons, and singularities appear only as a result of the gravitational field equation. Is there a physically sensible theory, solutions of which are nonsingular spacetimes but with horizons? The answer is yes. In this paper, we present such an 2-D effective field theory, static solutions of which are related to certain WZW coset models.

While WZW coset models provide us with string theories with interesting target spacetimes, it is often very difficult to study the dynamics since we have to solve the full string theory around nontrivial backgrounds. An easy way out is to abandon the exact model for an effective (approximate) local field theory by using the $\alpha'$ expansion of the sigma model [6]. Higher order corrections can be controlled by introducing extended supersymmetry in some cases, but not always. In particular, the $\alpha'$ expansion is also an expansion in the curvature of the target metric and a curvature singularity can be a signal of big higher order and nonperturbative corrections.

Apparently this is exactly what happens with Witten’s black hole as an $O((\alpha')^0)$ approximation to the $SL(2, R)_k/U(1)$ coset model. So far, two independent attempts to find a nonperturbative exact geometry have yielded an identical static metric [7][8], and, quite surprisingly, this new metric has the causal structure of a nonsingular multi-universe spacetime with horizons, as will be shown in the final section of this paper.

Furthermore we have found a string inspired local field theory in 2-D, whose charged static solutions are mostly of the same causal structure as this exact coset metric. The
action is obtained by a dimensional reduction of the 3-D string effective action of the gravity multiplet and the static solutions are dimensionally reduced versions of the 3-D black strings of Horne et.al. (These 3-D black strings are $O((\alpha')^0)$ approximations to the $(SL(2, R)_k \otimes U(1))/U(1)$ coset model.)

In short, we have not only an exact classical string background whose metric is a nonsingular black hole but also a local field theory, with essentially same type of static solutions, which can serve as a dynamical toy model. The form of the field theoretical solutions is identical to that of the $SL(2, R)_k/U(1)$ exact coset background up to an additive shift of one parameter. Not only the causal structures but the behaviour of the dilatons are the same. This presents us with a unique opportunity as well as a motivation to study these new kind of black holes. In this paper we study the classical properties of the 2-D effective field theory for the most part and detailed discussions on the exact background is postponed until at the end.

In section 2, we introduce the 2-D effective field theory and the much advertised nonsingular 2-D spacetimes. We start with a compactified charged black string. The geodesic equation of motion is studied for radial motion, and the Penrose diagrams of the 2-D part of the metric are shown. After the calculation of the Hawking temperature, we elaborate on the relationship of these new spacetimes to Witten’s black hole, with emphasis on the duality transformation used for the construction of the black string from Witten’s black hole.

We devote the next two sections to the dynamics of the 2-D effective theory, in particular, the stability of the static solutions under the process of a gravitational collapse. In section 3, we investigate the reponse of the geometry to a collapsing shell of massless matter. Even though we are unable to solve the full partial differential equations governing the process, the massless nature of the gravitating source allows us to obtain useful information such as the difference of the curvature across the shell. Crucial to the calculation here is the asymptotic behaviour of the gravitational perturbation. We added an appendix at the end of the paper to derive the neccessary information. In section 4, we examine the instability of the Cauchy horizon and the consequences.

Section 5 is devoted to some of the unresolved issues of the gravitational collapse as
well as implications of the results we obtained in the earlier sections. In particular, the exact metric of the $SL(2,R)_k/U(1)$, found in [7] and recently rediscovered in an entirely different approach by A.A. Tseytlin [8], is shown to have the same geometry as one of the nonextremal (hence nonsingular) solutions. Also discussed are nonperturbative black string solutions. Based on this new set of charged spacetimes, we speculate on the real nature of the 2-D black holes.

2. Compactified Black Strings as Geodesically Complete 2-D Spacetimes

We start with the charged black string solution by Horne et al. [9],

$$S^{(3)} = \int \sqrt{-g^{(3)}} e^{-2\phi}(R^{(3)} + 4(\nabla \phi)^2 + 4\lambda^2 - \frac{1}{12}H^2) \quad (2.1)$$

$$g^{(3)} = -(1 - \frac{m}{r}) dt^2 + \frac{1}{4\lambda^2(r-m)(r-q)} dr^2 + (1 - \frac{q}{r})a^2 d\theta^2 \quad (2.2)$$

and consider the case $\theta$ periodic with period $2\pi$. Then $a$ is the asymptotic radius of the internal circle. We define $q \equiv Q^2/m$ where $Q$ is the axionic charge associated with the antisymmetric 2-tensor. The solution (2.2) is actually $O((\alpha')^0)$ approximation to the coset model $(SL(2,R)_k \otimes U(1))/U(1)$ and $q = 0$ limit corresponds to $(SL(2,R)_k/U(1)) \otimes S^1$, the Witten’s coset model multiplied by a circle of fixed radius. For $m > q > 0$, the metric describes singularities at $r = 0$ hidden inside two distinct sets of horizons at $r = m, q$. The leading divergence of the curvature near $r = 0$ is $\sim -(1/r^2)$. The causal structure is somewhat similar to that of the Reissner-Nordström black holes in that a countable number of asymptotically flat universes are connected through compact regions of inner and outer horizons as well as timelike singularities (for more detail, see reference [9]), but the analogy should not be taken too seriously since the nature of the inner horizons is very different. The Penrose diagram of (2.2) cannot be accurately drawn in the $r-t$ plane.

For small $a$, a big mass gap of order $1/a$ develops for the internal degrees of freedom and all the low energy states have circularly symmetric wave functions. For the most of this paper we will consider the case of $a$ comparable to the Planck length so that the only
Available internal state is the ground state. In other words, we will consider the circularly symmetric sector of the theory. In terms of the classical physics of the low energy observers, it means the only relevant worldlines are those without any angular momentum. In this section we want to explore the static geometry above as seen by such observers. To begin with, consider all geodesics of vanishing angular momenta. This is consistent with the geodesic equations of motion since $\theta$ is a Killing coordinate. With the help of two Killing vector fields, the geodesic equations of motion can be reduced to the following form for the $r$-coordinate as a function of an affine parameter $\tau$ [9].

$$\left(\frac{\dot{r}}{2\lambda r}\right)^2 = (1 - \frac{q}{r})(\epsilon^2 + \alpha(1 - \frac{m}{r}))$$

(2.3)

$\epsilon$ is the covariant $t$-component of the 3-velocity and $\alpha = -1, 0, 1$ for timelike, null and spacelike geodesics. Immediately one finds a few revealing properties of $r(\tau)$. For generic values of the energy $\epsilon$, $r$ bounces at $q$ quadratically, so that an initial inequality $r \geq q$ is maintained throughout the history of the worldline. The only exception occurs for spacelike geodesics with $1 + \epsilon^2 = m/q$, in which case it takes infinite amount of time $\tau$ for $r(\tau)$ to reach $q$. Another important fact is that, for $\alpha \neq 1$, $r = q$ is the unique extremal value inside $r = m$. In other words, the time coordinate $r$ is monotonically increasing toward both future and past of $r = q$, while $r < m$. Obviously, no radial geodesics can penetrate the inner horizon and see the curvature singularity at $r = 0$ and the way it works out for timelike or null worldlines is that all radial observers bounce at the inner horizon into the next asymptotic region. In any case, one can conclude the restricted region defined by $r \geq q$ is by itself geodesically complete as far as radial geodesics are concerned. The singular structure is completely hidden inside a barrier at $r = q$. Therefore, the radial part of the metric (2.2) describes a nonsingular 2-D manifold with a countable number of universes as illustrated in figure 1.

This unusual behaviour can be partially traced to the fact $\theta$ becomes a time coordinate inside the inner horizon, where a radially moving observer, freely falling or accelerated, can never satisfy the on-shell condition. Once we understand the peculiar nature of the inner horizons at $r = q$, it is not difficult to draw the Penrose diagram of the 2-D metric, or (2.2) without the $d\theta^2$ term, on the $r$-$t$ plane. The nonextremal case is similar to that of the
Fig. 1: Penrose diagrams of $g^{(2)}$ for various parameter ranges. The bold lines indicate asymptotically far regions with vanishing effective coupling $e^x$. $e^x = \infty$ at $r = q$. $I^+$ and $I^−$ are the future and the past null infinities of a particular universe.

Reissner-Nordström geometry [10] with parts of it containing the inner horizons and the
singularities cut out and the remaining pieces glued along spacelike hypersurfaces. For this case, the inner horizons at $r = q$ become spacelike lines, to be also called the critical lines from now on, contained in the compact region surrounded by outer horizons $r = m$. In figure 1, the dotted lines of the case $m > q > 0$ are these critical lines. For $m = q = 0$ they are null lines infinitely far away. In the extremal limit $q \to m > 0$, half of the universes are decoupled and each remaining universe is successively attached to the next by the outer horizons at $r = m$. On the other hand the charge-zero limit can be easily shown to be Witten’s black hole with the spacelike singularities forming along the critical lines (or the inner horizons) at $r = q$. The change of variable, $r/\lambda = e^{2\lambda x}$, is necessary to go back to a more familiar coordinate in which the dilaton $\phi$ is linear.

The action and the field equations governing the dynamics of these 2-D spacetimes are obtained by a dimensional reduction of 3-D effective string field theory (2.1). Choosing a convenient set of 2-D fields and splitting the metric to $g^{(2)}$ plus the internal part we obtain the following effective action.

$$S^{(2)} = \int \sqrt{-g^{(2)}} e^{-2\chi} (R^{(2)} + 4(\nabla \chi)^2 + 4\lambda^2 - (\nabla f)^2 - e^{2f} K^2)$$  \hspace{1cm} (2.4)

$$g^{(3)} \equiv g^{(2)} + e^{-2f} a^2 d\theta^2$$

$$e^{-2\chi} \equiv e^{-2\phi - f}$$

$$K_{ij} \equiv \frac{1}{2a} H_{ij\theta}.$$  \hspace{1cm} (2.5)

Here $K$ is an exterior derivative of a 1-form, hence a $U(1)$ gauge field strength. The field equations are, after appending charged source on the right-hand sides, the following: $Q$ is the total charge inside the given value of $r$ and is locally constant in the source-free region.

$$R^{(2)} + 4\nabla^2 \chi - 4(\nabla \chi)^2 + 4\lambda^2 - (\nabla f)^2 + 2Q^2 e^{4\phi} = 0$$  \hspace{1cm} (2.6)

$$\nabla(e^{-2\chi} \nabla f) + 2Q^2 e^{4\phi - 2\chi} = 0$$

$$-\nabla_i \nabla_j e^{-2\chi} + g_{ij} \nabla^2 e^{-2\chi} + \cdots = T_{ij}^{\text{matter}}.$$  \hspace{1cm} (2.7)

The 3-D field $\phi = \chi - f/2$ is kept for later conveniences and $\cdots$ of (2.7) indicates terms of lower derivatives. The Bianchi identity of the left-hand side of (2.7) implies the energy
momentum conservation of $T^{\text{matter}}$. It is not too difficult to see that the static solutions of finite mass are all given by the 2-parameter family above. For $q > 0$ we can define a new coordinate $y$ by $r = q \cosh^2 \lambda y$, which shall be useful later on.

$$g^{(2)} = -(1 - \frac{m}{r})dt^2 + \frac{1}{4\lambda^2 (r - m)(r - q)} dr^2 = -F dt^2 + \frac{1}{F} dy^2$$

$$F \equiv (1 - \frac{m}{q \cosh^2 \lambda y})$$

$$e^{-2\chi} = \frac{r}{\lambda} \sqrt{1 - \frac{q}{r}} = \frac{q}{\lambda} \cosh \lambda y \sinh \lambda y$$

$$e^{-f} = \sqrt{1 - \frac{q}{r}} = \tanh \lambda y$$

$$R^{(2)} = 4\lambda m e^{2\phi} - 6mq e^{4\phi}.$$  \hfill (2.9)

Notice the inequality $e^{-2\phi} \equiv e^{-2\chi+f} \geq q/\lambda$. Even though the effective coupling $e^\chi$ is unbounded, the string coupling $e^\phi$ is bounded for charged cases. In the charge-zero limit, $f \equiv 0$, and $\chi \equiv \phi$ becomes the usual dilaton of [1]. The Hawking temperature [11] can be easily computed by requiring the Euclidean version of $g^{(2)}$ to be nonsingular at the horizon $r = m$. The periodicity of the Euclidean time coordinate is the inverse temperature.

$$T_{\text{Hawking}} = \frac{\lambda}{2\pi} \sqrt{\frac{m-q}{m}}$$

(2.10) reduces to the expected value $\lambda/(2\pi)$ [1] as $q \to 0$.

To illustrate the relationship between these new spacetimes and Witten’s, recall the construction of the charged black strings [9]. A neutral black string is a simple product of a Witten’s black hole with a line. After a Lorentz boost along the line, one can make a duality transformation [12] to get a new classical solution to the effective string field theory with torsion, or charge as we call it in this paper. We wrote the solution in (2.2) for a circle instead of a line. Without any boost ($q/m = 0$), the duality transformation is trivial and $g^{(2)}$ is the original Witten’s black hole. The maximal boost corresponds to the extremal case $q/m = 1$. If the Witten’s black hole we started with were an exact string background, the duality symmetry of string theory would imply all nonextremal solutions of (2.8) support one and the same string theory. But as we shall see in the final section this is not the case.
Simple counting reveals that this gravity system possesses one local degree of freedom. Classically, this means that there are numerous time dependent matter free solutions. This poses an essential difficulty in studying the dynamics and the question of stability becomes quite nontrivial. In particular, one needs to solve the full partial differential equations to study the process of gravitational collapses. For a system devoid of a local degree of freedom, such as the S-wave sector of Einstein-Maxwell theory or the dilaton gravity of [1], the solutions are locally static wherever the matter is absent and for thin shells of collapsing matter we can simply glue different static geometries across the history of the shell [10]. This statement is usually referred to as Birkhoff’s theorem in general relativity. But given a local degree of freedom, the geometry will be fluctuating even after the matter part dies out and can be found only by explicitly integrating the nonlinear dynamical equations. For this reason we will investigate the gravitational collapse in a piecewise manner and try to come up with a physically reasonable scenario in sections 3 and 4.

3. Dynamics of a Collapsing Massless Shell

Discussions above raise an important question. Does the gravitational collapse of charged matter leave behind a nonsingular spacetime similar to the charged static solutions? A related question is whether the inflow of charged matter to a Witten’s black hole will lift the curvature singularity just as the inflow of neutral matter in the linear dilaton vacuum creates a singularity. An even more immediate problem is whether the nonsingular nature of the charged spacetime is stable against extra (charged or neutral) matter inflow. A gravitational collapse can be divided into the following three stages.

1. The immediate response of the geometry to the matter.
2. The residual gravitational fluctuation for \((t + \int F^{-1}dy) < \infty\).
3. The residual gravitational fluctuation as \((t + \int F^{-1}dy) \to \infty\).

While the general treatment of even small gravitational perturbation is a fairly complex problem, the physics of (3) is relatively well understood and is the subject of the next section. Such an asymptotic tail of the gravitational fluctuation is known to be responsible for the instability of the Cauchy horizon in the case of charged black holes of the Einstein gravity.
One needs to analyze the effect of (2) to investigate the stability of the event horizons, along the lines of Chandrasekar’s analysis of the Kerr black holes [13]. But knowing that the 2-parameter family of solution (2.8) is the only finite mass static solution of the theory and that the ADM mass and the charge are conserved quantities, it is difficult to imagine a possible cause of instability that could change the structure outside the event horizons. Our static spacetime, however, has another source of instability under (2), namely the infinite effective coupling at the critical lines, which are the inner horizons of the 3-D metric. We will not pursue the matter here other than what is needed in this section concerning certain gravitational shock waves. The difficulty lies in reducing the linearized equations (3 dynamical and 2 constraint) to a single unconstrained dynamical equation in the region of interest. In other words we were not able to solve the constraint equations.

For part (1), consider a thin shell of conformal matter $T_{\text{matter}}$ collapsing in an initially static spacetime specified by the two numbers $(m_s, q_s)$. (It is important to assume that the spacetime is static initially in that we will need the explicit solution before the collapse.) In the limit of infinitesimal thickness, that is for a shock wave, the matter will induce discontinuous changes in normal derivatives of various fields across the shock. Furthermore one can obtain a set of first order differential equations obeyed by these jumps from the set of the nonlinear field equations of section 2. All one has to do is to take the discontinuities of the homogeneous (source free) equations and to integrate the inhomogeneous equations across the shock.

Since we are dealing with massless matter, it is convenient to introduce a light-cone coordinate in the conformal gauge.

$$g^{(2)} = -e^{2\rho} \, dx^+ dx^-$$  \hspace{1cm} (3.1)

Then $R^{(2)} = 8e^{-2\rho} \partial_+ \partial_- \rho$ and $\nabla^2 = -4e^{-2\rho} \partial_+ \partial_-$. Let $x^- = -\infty$ on past null infinity, to be concrete. Let the shock be centered at $x^+ = x_o^+$. The only nonvanishing component of the matter energy momentum tensor is $T_{\text{matter}}^{++}$, which has an infinitesimally thin support.

1 Unlike the case of timelike shell, normal here does not mean orthogonal. Using a null coordinate pair $(x^+, x^-)$, the normal derivative is $\partial_+$ for a shock propagating along a fixed value of $x^+$. 

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Then we can assume that $\chi$ and $f$ are continuous across the shock while the derivatives along $x^+$ may not be. Furthermore we assume the continuity of $\rho$ as well, the consistency of which needs to be checked. First of all the effect of $T^\text{matter}_{++}$ is felt by the dilaton $\chi$ through the $\delta g^{++}$ equation of (2.7). If we define $\sigma$ as the total flux, we have the equation

$$
\sigma(x^-) \equiv \lim_{\delta \to 0} \int_{x_o^- - \delta}^{x_o^+ + \delta} T^\text{matter}_{++} \, dx^+ = \lim \int -\nabla_+ \nabla_+ e^{-2\chi} + \cdots = -[\partial_+ e^{-2\chi}]. \tag{3.2}
$$

Here, we introduced the notation $[A]$ for the difference of $A$ across the shock. Next, the evolution of $\sigma$ is governed by $\delta g^{+-}$ equation, whose difference across the shock is

$$
\partial_- \sigma = -\partial_- [\partial_+ e^{-2\chi}] = -e^{4\phi_o + 2\rho_o - 2\chi_o} [Q^2] \frac{Q^2}{2}.
$$

Here $\chi_o$ denotes $\chi$ restricted to $x^+ = x_o^+$ and similar for the other fields. Note that we already know values of these $\chi_o, \cdots$. They are simply the corresponding static values owing to the continuity. Naturally all the quantities above are functions of $x^-$ only. On the other hand the energy momentum conservation of $T^\text{matter}$ implies

$$
\partial_- \sigma = -[T^\text{matter}++] = 0. \tag{3.3}
$$

Obviously the procedure is inconsistent for a charged shell ($[Q^2] \neq 0$). The assumption of continuous $\rho$, needed when we derived (3.2) ignoring the connection piece $\Gamma_{++}^+ \partial_+ e^{-2\chi}$, is apparently too strong. A priori, there is no reason that $\rho$, a coordinate dependent function, should be continuous. However for the case of massless and neutral shell ($[Q^2] = 0$), we shall find no further inconsistency below. For this reason we will consider neutral conformal matter only, so that $\sigma$ is constant.

We can also take the discontinuity of the equations (2.3) and (2.5) and the result can be written in the form (after substituting $\sigma$ for $-[\partial_+ e^{-2\chi}]$)

$$
\partial_- [\partial_+ f] - \partial_- \chi_o [\partial_+ f] - \frac{e^{2\chi_o}}{2} \sigma \partial_- f_o = 0, \tag{3.4}
$$

$$
e^{-2\chi_o + 2\rho_o} \frac{8}{R^{(2)}} [R^{(2)}] - \sigma \partial_- \chi_o + e^{-2\chi_o} \frac{1}{2} \partial_- f_o [\partial_+ f] = 0. \tag{3.5}
$$
Reduced to the usual dilaton gravity ($f \equiv 0, Q \equiv 0$), with the Kruskal coordinate in which $e^{-2\chi_o} = e^{-2\rho_o} = m/\lambda - \lambda^2 x_o^+ x^-$, these equations have the solution

$$[R^{(2)}] = 4\lambda \sigma x_o^+ e^{2\chi_o}.$$ 

$e^{2\chi_o} = e^{2\phi_o}$ is unbounded and shows a curvature singularity forming at infinite effective coupling $e^\chi = \infty$. It is easy to see the result reproduce what we would find by actually solving the full equations as done in [1]. In this particular case there is no gravitational fluctuation afterwards and the solution outside the shock is completely determined by this jump. All we have to do is to find the right static solution to match. It is worthwhile to notice that, in spite of the singular behaviour, $\sigma$ is constant and therefore $e^{-2\chi}$ is continuous even at the singularity, a necessary behaviour for the self-consistency.

Now what happens if we start with a charged spacetime? Then we have to solve for $[\partial_+ f]$ first. Again for the sake of consistency we will consider a shock of neutral matter in an initially charged spacetime. Solving (3.4),

$$e^{-\chi_o} [\partial_+ f] = (C + \frac{\sigma}{2} \int_{-\infty}^{x^-} e^{\chi_o} \partial_- f_o \, dx). \quad (3.6)$$

Since the matter shock is neutral $\sigma$ is independent of $x^-$. $C$ is an integration constant. The lower bound of the integral is past null infinity where the shock wave originates.

Consider the effect of $C$ on the jump of the curvature. Near the critical line $e^{-2\chi} = 0$ (or $y = 0$ in the coordinate of (2.8)), the corresponding jump scales like $\sim C/\sqrt{y^3}$ and is infinite at $y = 0$. This is rather strange in that not only the strength of the curvature singularity but also its sign are completely independent of the matter you are throwing in. (As to be shown later, the contribution from $\sigma$ is finite at the critical line.) Even more disturbing is the jump of the internal part of the Ricci tensor $R_{\theta\theta} = -e^f \nabla^2 e^{-f}$. Asymptotically it is $\sim e^{-\lambda y}$, whereas the static value is $\sim e^{-2\lambda y}$. This seems to suggest that the gravitational fluctuation with nonzero $C$ would cost an infinite amount of energy. It turns out that the problem is closely related to the asymptotic nature of the gravitational perturbation.

As demonstrated in the appendix, the gravitational perturbation around a static so-
olution is asymptotically described by the following massive Klein-Gordon equation with
\[ \Psi \equiv e^{\chi_s}(f - f_s). \]
\[ \nabla^2 \Psi - \lambda^2 \Psi = 0. \quad (3.7) \]
\( \chi_s \) and \( f_s \) are the static values of \( \chi \) and \( f \). Across any null line \( x^+ = x^+_o \), \([\partial_+ \Psi] = C\) is allowed by (3.7). But it is quite deceptive since the behaviour of field as we move away from the null line is profoundly affected by the massive nature of the evolution equation. After all, we do not expect a massive field to propagate along a null line. As shown in the appendix, the bump of \( \Psi \) itself owing to the discontinuous derivative tends to diminish rapidly with time and becomes completely undetectible after an infinite amount of time. In other words, if we incorporate such a jump to the initial condition on past null infinity, the field configuration of \( \Psi \) in finite region does not show any sign of discontinuous derivatives. (In terms of the homogeneous solution presented in the appendix, this case corresponds to \( u_o \to -\infty \) before taking the normal derivative.) On the other hand, if we insist that the field configuration show such a jump away from past null infinity, the required initial condition on past null infinity has divergent \( \Psi \), and the energy content of the initial flux of the associated gravitational radiation is infinite. It is simply impossible to produce nonzero effective \( C \) in the finite region with a finite amount of energy.\(^2\) \( C \) should be dropped.

While we discussed the problem in the context of a gravitational collapse of massless matter fields, the conclusion must hold in other contexts. After all, \( C \) represents a homogeneous solution to the gravitational field equations. We should not think of it as being generated by the matter shock. If \( C \neq 0 \) were allowed, it would imply the instability of the critical line under gravitational perturbation automatically, for example. In this sense we have eliminated one possible way for the infinite self-coupling to cause instability.

Rewriting (3.7),
\[ [\partial_+ f] = \frac{\sigma}{2} e^{\chi_o} (I(x^-) - I(-\infty)) = \frac{\sigma}{2} e^{\chi_o} I(x^-). \quad (3.8) \]
\( I \) is the indefinite integral of \( A(x^-) \equiv e^{\chi_o} \partial_- f_o \) over \( x^- \) and invariant under coordinate

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\(^2\) See appendix for detailed discussion on the asymptotics of the gravitational perturbation.
transformations \( x^- \to z^-(x^-) \). One can ask whether the massive nature of the graviton affects this inhomogeneous part of the solution. The answer is no. It turns out that the asymptotic behaviour of the source term \( A \) is such that a separation of variables occurs. The form of \( \Psi \) near the shock is given by a product of two factors each of which is function of one of the null coordinates only. As a result, the estimate (3.8) is reliable even though we derived (3.4) without taking into account the massive nature of the gravitational perturbation. Inserting (3.8) to (3.5),

\[
[R^{(2)}] = 2\sigma e^{-2\rho_o}(2\partial_- e^{2\chi_o} - e^{\chi_o} \partial_- f_o I).
\]  

(3.9)

No infinite jump is possible except maybe at the critical point \( e^{\chi_o} = \infty \). But even at this point the leading terms of two parts cancel each other and the true leading term is a constant. For an initially nonextremal spacetime, an easy way to see this is to make a coordinate transformation \( z^- \equiv y(x^+_o, x^-) \) where \( y \) is the coordinate used in (2.8) and recall that \( e^{-2\chi_o} = qy + O(y^3) \), \( e^{-f_o} = y + O(y^3) \).

\[-\infty < R^{(2)} < \infty \quad \text{near} \quad x^+ = x^+_o.\]  

(3.10)

The conclusion holds for all initially charged cases including the extremal case. Therefore the history of a collapsing shell of neutral conformal matter is qualitatively similar to that of a massless observer. It just bounces into the next universe. It is quite a surprising behaviour when compared to what happens in the usual dilaton gravity [1].

To conclude this section consider the case of an initially neutral spacetime. Will the singularity be lifted by the inflow of some charged matter? Can we open up a curvature singularity by a completely classical process? As we have seen earlier, we should be more careful in dealing with charged matter. In principle, one may proceed by considering two conformal factors inside and outside the shock, each of which is smooth but assumes values different from the other along the shock. However, we will postpone this problem of the gravitational collapse of charged matter in a neutral spacetime to a future project for the following reason. A neutral static spacetime has \( e^{-2f} \equiv 1 \), while the charged case has \( e^{-2f} = \tanh^2 \lambda y \) ranging from 1 to 0. In the framework we are using, \( e^{-2f} \) is a scalar
and should be continuous across the shock. Then the resulting immediate response of the initially neutral solution to the collapsing shell of charged matter must have constant $e^{-2f}$ and hardly resembles a static charged spacetime. Even though it might be that the solution settles down to a nonsingular spacetime with varying $e^{-2f}$ after a while, we can not detect that with this type of local analysis. At the singularity the scalar field can actually jump discontinuously to the desired value, but such a behaviour could imply an
inconsistency. The same argument applies to the original 3-D dilaton field $e^{-2\phi}$. In this sense the shock wave analysis seems less promising for initially neutral cases. Even if we carried out similar calculations for the case of a charged shell, we would be unable to decide whether the spacelike singularity was lifted. Probably we need to go beyond the immediate neighborhood of the shock to probe the geometry outside the shock.

A different way of tackling this problem is to try to establish the stability of the critical lines, which would certainly imply *time dependent* charged solutions without a spacelike singularity. In this regard, the results above on the neutral matter shock in an initially charged spacetime are quite encouraging. We found that, as a consequence of the full nonlinear dynamics, no singularity met the shock. If the critical lines are unstable, it is difficult to imagine where the resulting spacelike singularity along the critical line might start. It has to start somewhere away from the shock.

4. Instability of the Cauchy Horizon

In section 3, we found that a collapsing shell in an initially charged static spacetime does not encounter a curvature singularity and bounces into the next universe. However, any gravitational collapse is followed by residual gravitational fluctuations and so far we have not addressed the effect of them on the subsequent evolution of the geometry. In principle these fluctuations could grow indefinitely to produce curvature singularities. The most obvious place to look for the singularity is the Cauchy horizons, which are generically unstable because of the infinite blue shift of the asymptotic residual fluctuation entering the future event horizon at arbitrary late times. However, unlike the more familiar case of Reissner-Nordström black holes the Cauchy horizons of our geometry coincide with the outer horizons rather than the inner horizons. The inner horizons of the nonextremal geometry, also called the critical lines, are effectively spacelike for any radial observer and cannot be the Cauchy horizons. While the behaviour of the critical lines under the gravitational perturbation (as opposed to the matter inflow) still needs to be studied, we will first concentrate on the more obvious potential source of instability.

A convenient way to analyze the instability of the Cauchy horizon is to consider ingoing
null flux of energy stretched all the way to the infinite future \cite{14}. Assuming an initially static geometry (2.8), choose a new set of coordinates \((v, u)\).

\[
g^{(2)} = -F(y) \, dt^2 + \frac{1}{F(y)} \, dy^2 = -F(y) \, dvdu = -F(y) \, dv^2 + 2 \, dvdy \tag{4.1}
\]

\[
dv = dt + \frac{dy}{F}, \quad du = dt - \frac{dy}{F}.
\]

With this coordinate system the Cauchy horizon of our universe is at \(v = +\infty\) inside the future event horizon. In figure 2, we labeled the Cauchy horizon by CH and the future event horizon by EH. Outside EH, \(v = +\infty\) corresponds to future null infinity \(I^+\) and we are interested in asymptotic inflow of the radiation stretched all the way to \(I^+\).

For simplicity, suppose that the energy inflow is due to matter rather than gravitational radiation.\footnote{We are indebted to E. Poisson for suggesting this approach.} This simplification is well justified by the universal nature of the instability. By the energy-momentum conservation, the energy-momentum tensor of a massless pure inflow is of the form

\[
T^{\text{matter}} = \mu(v) \, dv dv.
\tag{4.2}
\]

As \(v \to \infty\) (future null infinity if outside the event horizon) a typical \(\mu(v)\) shows an inverse powerlike behaviour \cite{15} with finite total energy and has little effects on the geometry outside the event horizon. But inside the event horizon, \(v = \infty\) corresponds to CH at finite physical distance and the energy density measured by a freely falling observer could be infinite at CH. To be definite let us choose Kruskal type coordinates \((V, U)\), to be defined later, good for both CH and EH. In addition, let \(V = 0\) at CH,

\[
g^{(2)} = -e^{2\rho} \, dV dU \quad (V = V(v), U = U(u)), \tag{4.3}
\]

\[
T^{\text{matter}} = \mu(v) \, dv dv = \mu(v) \, (dv/dV)^2 \, dV dV \equiv T_{VV} \, dV dV. \tag{4.4}
\]

Notice that once the flux crosses future event horizon (EH), it looks more like a thin shell of massless matter, even though \(\mu(v)\) has an infinite span outside. The energy density
measured by a timelike observer crossing CH is roughly $\sim T_{VV}$ which diverges unless the fall-off of $\mu(v)$ is fast enough to overcome the infinite blueshift factor $(dv/dV)^2 \to \infty$.

A better estimate can be made using the machinery of the previous section. Since only the large $v$ behaviour is important, let $\mu$ be nonzero only for $v > v_{\min}$ with sufficiently big $v_{\min}$, so that near CH the flux is that of an arbitrarily thin shell. Then certainly the formalism of section 3 is applicable inside EH. Using the definition (3.2), after identifying the corresponding null coordinates $x^+ \equiv V$ and $x^- \equiv U$,

$$\sigma = \int_{V(v_{\min})}^{0} T_{VV} dV = \int_{v_{\min}}^{\infty} \mu(v) (dv/dV) \, dv.$$  \hspace{1cm} (4.5)

As long as $\sigma$ is finite, we can estimate the jump of the curvature on CH induced by the neutral inflow using (3.5). For initially static spacetimes,

$$[R^{(2)}]_{V=0} = 0.$$  

The reason is simply that the dilatons $\chi_s$ and $f_s$ are uniform along CH and $\partial U \chi_o = \partial U f_o = 0$. This is a somewhat fictitious situation since we have assumed that before the on-set of the asymptotic inflow at $v = v_{\min}$ the geometry is strictly static. For a realistic model, the asymptotic inflow originates from the collapsing matter and the geometry is non-static even for $v < v_{\min}$. In particular CH does not necessarily coincide with an apparent horizon, the dilatons $\chi, f$ are no longer uniform along CH, and as a result the curvature jump is generically nonzero and finite, proportional to $\sigma$,

$$[R^{(2)}]_{V=0} \sim \sigma \quad \text{up to some finite factor as a function of } U.$$  

A similar mechanism must work for infinite $\sigma$, even though the formalism of section 3 does not apply in this case. After taking into account other residual fluctuations, an infinite $\sigma$ would most probably generate a null singularity along CH. The role of the extra radiation flowing into the Cauchy horizon rather than parallel to it has been first studied in [14] and found crucial for the instability to manifest itself.

For the particular case of the compactified charged black strings we are studying, we can choose the following Kruskal coordinates
\[ V_{\text{nonextremal}} = -\frac{1}{\beta} e^{-\beta v} \quad \beta = \lambda \sqrt{\frac{m-q}{m}}, \quad (4.6) \]

\[ V_{\text{extremal}} = -\frac{\lambda^2}{v}. \quad (4.7) \]

(Obviously, this choice is not unique, but the same conclusion holds for any other coordinate regular on CH.) For any powerlike \( \mu(v) \), \( \sigma_{\text{nonextremal}} \) is infinite, and the Cauchy horizon is inherently unstable. Unless the asymptotic residual fluctuation turns out to be exponentially small (\( \sim e^{-\lambda v} \) for example), a null curvature singularity will form along CH. But provided that \( \mu(v) \sim v^{-n} \) with \( n > 3 \), \( \sigma_{\text{extremal}} \) could be finite. Whether a finite \( \sigma_{\text{extremal}} \) indicates a stable Cauchy horizon of an initially extremal solution is unclear. The extremal geometry can always be changed to a nonextremal one by an infinitesimal increase of the mass, and once that happens we must use the nonextremal estimate of \( \sigma \).

(However, this classical picture might undergo a drastic modification once we include the semiclassical effect of Hawking radiation. Since the temperature is positive except for the extremal spacetime, the geometry outside the event horizon must approach the extremal geometry in the infinite future (\( v \to \infty \)). If the same is true for the geometry inside the event horizon, we must use the extremal estimate of \( \sigma \) and the Cauchy horizon might as well be stable for any initially charged geometry.)

What are the consequences of the null singularity along CH? Note that the Cauchy horizon of our universe (CH) meets the future event horizon of the next universe. But all the infalling observers escape to the next universe through its past horizon (EH'). That is, none of the observers from our universe, entering EH before the on-set of the asymptotic inflow (4.2), cross CH between EH and EH'. The singularity on CH hardly interferes with the observers entering the next universe. While some massive observers without sufficient momenta would be attracted to the singularity, the rest will probably continue to the asymptotic region away from it. This should be compared to what happens in case of 4-D Reissner-Nordström (nonextremal) black holes. For these 4-D black holes the phenomenon is well known as mass inflation [14]. The static geometries of nonextremal Reissner-Nordström black holes allow timelike observers (entering the future event horizon) to escape to the next universe, the static singularity being timelike. Once we include generic
perturbations, however, the Cauchy horizon becomes singular all the way to the point where it meets the original timelike singularity and even the timelike observers eventually experience infinite tidal force.\(^4\)

Of course, the past event horizon of the next universe (EH') overlaps the Cauchy horizon of another universe which is spacelike separated from our universe. An asymptotic inflow from that universe can destabilize EH'. But if we suppose the charged spacetime is made by a gravitational collapse in our universe, the extended static structure before the collapse is fictitious just as the past event horizon of the Schwarzschild geometry is fictitious for a collapsing star. Then EH', the past event horizon of the next universe, would be as stable as EH.

5. Discussion: Nonsingular Exact String Backgrounds

The most curious feature of the charged black string is probably the way the inner horizons conceal the curvature singularities from the radial observers. The resulting 2-D geometry is geodesically complete and the scalar curvature is finite everywhere.

\[-\lambda^2 \frac{2m}{q} \leq R^{(2)} \leq \lambda^2 \frac{2m}{3q}.\]  \((5.1)\)

In particular, as \(q \rightarrow m\), the absolute value of the curvature is bounded by \(2\lambda^2\). On the other hand, for Witten’s black hole seen as the dimensional reduction of the 4-D dilatonic black hole, \(\lambda^{-1}\) corresponds to the total magnetic charge \(2\), or the radius of the long necked extremal solution called the cornucopion \(17\). If we take this parameter \(\lambda\) much less than the Planck mass, then, as the black hole evaporation progresses \(m \rightarrow q\), the geometry is of macroscopic scale everywhere including inside the event horizons. Furthermore the string coupling \(e^\phi\) is bounded by \(\sqrt{\lambda/q}\). It is possible that the quantum fluctuation of geometry does not change the causal structure qualitatively. Taken seriously, this last statement has a profound ramification in the black hole physics: Whatever the real answer

\(^4\) However, it has been shown the total impulse from the tidal force is finite and the would-be Cauchy horizon is transversible \(16\).
is to the information puzzle\cite{15}, it has little to do with the details of the Planck scale physics. Either the information is completely recovered before the collapsing matter enters the future event horizon, or some will be lost forever to the next universe.

Any remnant scenario is likely to assume some nonsingular objects as the final products of the evaporation process. They are assumed to be able to carry macroscopic amounts of the information yet interact very little with the surroundings. All these assumptions may or may not be realized depending on the details of the Planck scale physics. Even in the case of the cornucopions we still need to explain how the information inside the horizon, far down at the tip of the long neck, propagates out to occupy the whole length of the cornucopions, which could happen only near the end of the process where the singular structure inside the horizon must be resolved somehow. The static extremal solution above could obviate all these details.

But we are hardly in a position to take these static solutions seriously. The Reissner-Nordström nonextremal solution, which is also known to allow some of the infalling observers (massive ones) to escape to the next universe, is dynamically unstable and the resulting generic geometry has a curvature singularity blocking the passage though perhaps not completely \cite{14} \cite{15}. In the present case of the compactified black strings, while the instability of the Cauchy horizon due to the asymptotic inflow from our universe does not block the passage, there is a possibility that the critical lines of the infinite effective coupling become singular under the generic gravitational perturbation. As mentioned at the end of section 3, the fact that a collapsing shell does not encounter a singularity even at the critical line seems to suggest no singularity whatsoever along the entire span of the critical line, but this might be an artifact of the initially static geometry.

Another difficulty concerns the creation of the charged spacetimes from vacua. If the linear dilaton background is the true vacuum, it is very difficult to imagine how the extended structures of the charged spacetime could be made from the gravitational collapse of thin shells without any singularity at the infinite effective coupling. More specifically, it is rather difficult to draw a sensible Penrose diagram describing the process. Even a qualitative understanding in this regard would be an important step.

\footnote{For a recent review see the reference \cite{18}.}
At this point one can easily see that all the unanswered questions are in one way or another associated with the true nature of the critical lines (the inner horizons as seen by 3-D observers), which become curvature singularities under the duality transformation (see section 2). The ultimate question is then, between the two geometrical pictures dual to each other, namely the Witten’s causal structure and our nonsingular version, which one of them resembles the reality more closely \[12\].

Of course the effective field theory (2.1) we started with is correct only up to $O(1)$ in the $\alpha'$ expansion of the sigma-model \[3\]. That is, the extended nature of the fundamental string is not properly taken into account. In particular, the static solutions we have found are related to the coset models $(SL(2, R)_{k} \otimes U(1))/U(1)$ for $m \geq q > 0$, or $(SL(2, R)_{k}/U(1)) \otimes S^1$ for $q = 0$, with $(k - 2) = (2\alpha'\lambda^2)^{-1}$. $k$ is larger than 2 provided that the total dimension of the spacetime, compact or not, is less than the critical dimension. A few months after Witten’s derivation of the 2-D black hole geometry from $SL(2, R)_{k}/U(1)$ coset model, R. Dijkgraaf et.al., attempted to find the exact geometry of the coset model by investigating the tachyon spectrum of the theory \[7\] with the assumption that the string on-shell condition $L_0 = 1$ should be interpreted as the tachyon field equation $\nabla^2 T = 0$.

Recently, A.A. Tseytlin rediscovered the same metric in a functional integral approach where he replaced the classical effective action of the coset WZW model by a quantum effective action, which involved modifying the coefficients of the action from $k$ to $k + cG/2$ \[8\]. The change of metric can be easily written in the following form, in a gauge similar to those used in \[20\] \[7\] \[8\].

\[
g^{(2)}_{Witten} = \frac{1}{\lambda^2} dx^2 - \tanh^2 x dt^2 \quad \rightarrow \quad g^{(2)}_{exact} = \frac{1}{\lambda^2} dx^2 - \frac{(1 - 2/k) \tanh^2 x}{1 - (2/k) \tanh^2 x} dt^2 \quad (5.2)
\]

In the gauge of (2.2), this can be rewritten

\[
g^{(2)}_{exact} = -(1 - \frac{m}{r}) dt^2 + \frac{1}{4\lambda^2(r - m)(r - (2/k)m)} dr^2. \quad (5.3)
\]

Amazingly, this looks exactly like a nonextremal metric, (2.8) with $q/m \rightarrow 1 > 2/k > 0$. The $SL(2, R)_k/U(1)$ coset metric has the causal structure of the Penrose diagram labeled
by \( m > q > 0 \) instead of the singular one \((m > q = 0)\) in figure 1. Furthermore the dilaton part is, according to Tseytlin’s estimate,

\[
e^{-2\chi_{\text{exact}}} = \frac{r}{\lambda} \sqrt{1 - \frac{(2/k)m}{r}}.
\]  

(5.4)

Again, it is identical to that of the nonextremal solution (2.8) with \( q/m = 2/k \). The implication is very remarkable. The reality seems to be better described, at least classically, by the nonsingular causal structures we have found than by Witten’s.

Another encouraging evidence for these nonsingular solutions comes from the duality symmetry of the string theory. The \((SL(2,R)_k \otimes U(1))/U(1)\) coset model has been considered by I. Bars and K. Sfetsos \cite{21}, also nonperturbatively. The resulting metric is singular, unfortunately, and the causal structure is again that of timelike singularities hidden behind inner and outer horizons. However, if we consider the 2-D part of the metric as we did in section 2,

\[
g_{\text{BS}}^{(2)} = -(1 - \frac{m}{r}) dt^2 + \frac{1}{4\lambda^2(r-m)(r-q-(2/k)(m-q))} dr^2,
\]  

(5.5)

after an appropriate redefinition of the parameters in \cite{21}. This metric is identical to the \(O(1)\) approximation (2.8) except that \( q \geq 0 \) is replaced by \( q + (2/k)(m-q) > 0 \). The case of \( q = 0 \) coincides with \( g_{\text{exact}}^{(2)} \) as it should, and the extremal limit is achieved by letting \( q \to m \). (We need to perform a coordinate rescaling on the metric presented in \cite{21} to achieve this limit.) Now since the \(O(1)\) approximations to \( g_{\text{exact}}^{(2)} \) and \( g_{\text{BS}}^{(2)} \) are related to each other by a duality transformation, it is not unreasonable to expect these nonperturbative versions are dual to each other.

Unlike the \(O(1)\) approximations, the 2-D part \( g_{\text{BS}}^{(2)} \) of the nonperturbative black string metric is qualitatively self-dual. The causal structure of the 2-D part does not change upon \( g_{\theta\theta} \to (1/g_{\theta\theta}), \cdots \), with the exception of the extremal case. We no longer have to choose between two entirely different causal structures dual to each other. It is true that there are many examples of different geometries (or even different topologies) supporting the same string theory, and a fundamental string must have a unique classical history independent

---

\text{6 We thank G. Horowitz for drawing our attention to this work.}
of the particular geometry chosen. But here, instead of the horrendous task of figuring out the actual behaviour of test strings inside the event horizon, we are presented with a unique nonsingular causal structure invariant under the duality transformation. The naive expectations based on the behaviour of test particles are unambiguous and might as well indicate the actual behaviour of test strings. A radially moving classical fundamental string propagating into the future event horizon does not see a curvature singularity and simply propagates to another universe.

As for the exceptional case of the extremal solution, we can easily see that it is dual to the linear dilaton background. The relationship between masses of a pair of dual solutions is given by (according to the $O(1)$ estimate)

$$m_{after} = m_{before} \cosh^2 \alpha,$$

where $\alpha$ is the Lorentz boost parameter; $\alpha$ is infinite for the maximal boost [9]. Obviously a finite mass extremal solution is obtained by the duality transformation on a maximally boosted zero mass limit of $SL(2, R)_{k}/U(1) \otimes S^1$, the linear dilaton background up to a trivial internal part. The positive mass of the extremal solution must be an artifact of the effective field theory estimate. While this ambiguity prevents us from viewing either of them as the true ground state, it is quite interesting in that we have an alternative description for the end stage of the black holes, which has an event horizon at finite physical distance and another universe beyond it.

Of course we overlooked two important facts in the arguments above. First of all, the effective coupling is again unbounded.

$$e^{-2 \phi_{BS}} = \frac{r}{\lambda} \sqrt{1 - \frac{(2/k)(m - q)}{r}}, \quad (5.6)$$

$$e^{-2 \chi_{BS}} = \frac{r}{\lambda} \sqrt{1 - \frac{q + (2/k)(m - q)}{r}}, \quad (5.7)$$

which we deduced from [21]. The effective coupling $e^{\chi}$ is infinite at $r = q + (2/k)(m - q)$. In section 3, we have seen this infinity was not strong enough to create a curvature singularity out of an collapsing shell of matter. But we have not been able to address the problem of
gravitational perturbation completely, let alone quantum fluctuations. Secondly, the 3-D metrics of the compactified charged black strings \((m > q > 0)\), perturbative or nonperturbative, possess not only timelike singularities but also closed timelike worldlines inside inner horizons. While it is reassuring that the corresponding regions are completely inaccessible to 2-D observers and that they are causally disconnected from us by event horizons, it is quite unclear whether such an background is physically acceptable. It might be that we should be content with the effective field theory \((2.4)\) rather than try to interpret the 2-D solution as a part of a classical string background.

Finally, we would like to mention that similar causal structures have been discussed in different contexts in \([5][22][23]\). After the completion of this work, we received a preprint by M.J. Perry et.al., \([24]\), which discussed the causal structure of \(g_{exact}^{(2)}\) above. They found the same causal structure as we did. Also A. Tseytlin informed us of \([25]\) where he and C. Vafa observed that the nonpertubative coset metric \((5.2)\), of R. Dijkgraaf et.al., remained bounded when continued to imaginary \(x\).

It is my pleasure to thank J. Park, E. Poisson and K. Thorne for sharing their expertise on general relativity. I would like also to thank J.H. Schwarz, S. Trivedi and my advisor J.P. Preskill for the useful conversations as well as their kindly encouragement.

Appendix A. Asymptotics of the Effective Theory

To study the asymptotic behaviour of the gravitational perturbation, it is convenient to start with the ADM mass formula, obtained using the canonical formalism \([26]\) on \((2.4)\). \(\Delta\) denotes the deviation from the linear dilaton vacuum.

\[
M_{ADM} = 2 Ne^{-2\chi} \Delta \frac{\chi'}{\sqrt{g}} \bigg|_{x \to \infty}
\]

\(g^{(2)} = N^2 dt^2 + g(dx + L dt)^2\quad N, g \to 1; L \to 0\quad \text{as}\quad x \to \infty\)

\(\chi_{\text{vacuum}} = -\lambda x\quad f_{\text{vacuum}} = 0.\)  \hspace{1cm} (A.2)

Readers should be warned that \((A.1)\) is derived with certain assumption on the allowed phase space \([24]\). More explicitly we need to restrict the phase space to \(\Delta \chi_s \sim \Delta \sqrt{g_s} \sim\)
$\Delta f_s \sim e^{-2\lambda x}$ at most, which is sensible for the static solutions. Applied to (2.8) we obtain $M_{ADM} = m$.

Now consider a time-dependent solution. The requirement of finite mass restricts the possible asymptotic behaviours (A.1). Obviously the same asymptotic restrictions apply to $\delta \chi$, $\delta \rho$, the corresponding deviations from a static solution. To be definite, write a time dependent solution in the following form.

$$g^{(2)} = e^{2\delta \rho}(-F(y)\, dt^2 + \frac{1}{F(y)} dy^2) \quad \text{(A.3)}$$

$$\chi = \chi_s + \delta \chi, \quad f = f_s + \delta f$$

$$(\delta \chi' + \lambda \delta \rho) \sim e^{-2\lambda y}, \quad \text{at most.}$$

Notice that $\delta f$ could be much larger than the static part $f_s \sim e^{-2\lambda y}$. Expanding the constraint equations $\delta S^{(2)}/\delta g^{01} = 0$ and $\delta S^{(2)}/\delta g^{00} = 0$ in $e^{-2\lambda y}$, and collecting the leading terms,

$$\partial_t (\delta \chi' + \lambda \delta \rho) = \frac{1}{2} \delta (f' \dot{f}), \quad \text{(A.4)}$$

$$(\delta \chi' + \lambda \delta \rho)' + 2\lambda (\delta \chi' + \lambda \delta \rho) = \frac{1}{4} \delta (f' f' + \dot{f} \dot{f}). \quad \text{(A.5)}$$

If $\delta f \sim e^{-\lambda y}$ then $(\delta \chi' + \lambda \delta \rho) \sim e^{-2\lambda y}$. If $\delta f$ is exponentially smaller than $e^{-\lambda y}$, the leading part of $(\delta \chi' + \lambda \delta \rho) \sim e^{-2\lambda y}$ is static and the subleading time-dependent part of it is again exponentially smaller than $\delta f$. Finally, the possibility of $\delta f$ much larger than $\sim e^{-\lambda y}$ is excluded since it, combined with (A.4), will lead to infinite ADM mass (A.1). After redefining the unperturbed fields to include the static parts of $\delta \chi, \cdots$,

$$(\delta \chi' + \lambda \delta \rho) \sim e^{-\lambda y} \delta f \quad \text{at most.} \quad \text{(A.6)}$$

Therefore it is quite reasonable to expect $\delta \chi \delta f \ll \delta f$ as far as the time-dependent perturbations are concerned. The field equation for $f$ (2.6) can be expanded in the same fashion and up to the leading order $\delta f$ is easily shown to be decoupled from the rest.

$$\nabla^2(\delta f) - 2\nabla \chi_s \nabla(\delta f) = 0 \quad \text{(A.7)}$$
After a field redefinition \( \Psi \equiv e^{-\chi_s \delta f} \), (A.7) becomes,

\[
\nabla^2 \Psi - \lambda^2 \Psi = 0.
\]

(A.8)

The gravitational perturbation is asymptotically described by a massive Klein-Gordon equation.

Now consider solutions to this equation with the following characteristic data, in a flat light-cone coordinate, \( g^{(2)} = -dvdu \).

\[
\Psi \bigg|_{v = v_o} = 0
\]

(A.9)

\[
\psi \equiv \Psi \bigg|_{u = u_o} = 0 \quad \text{for} \quad v < v_o, \quad C(v - v_o) \quad \text{for} \quad v_o < v < v_n.
\]

Since the geometry is flat, we can integrate the equation explicitly using the following recursive solution

\[
\Psi(v, u) = -\frac{\lambda^2}{4} \int_{u_o}^u \int_{v_o}^v \Psi(v', u') \, dv' \, du' + \psi(v).
\]

(A.10)

This uniquely determines \( \Psi \) up to \( v < v_n \) in terms of a Bessel function.

\[
\partial_v \Psi = 0 \quad \text{for} \quad v < v_o
\]

(A.11)

\[
\partial_v \Psi = C J_0(\lambda \sqrt{(u-u_o)(v-v_o)}) \quad \text{for} \quad v_o < v < v_n
\]

Notice \( [\partial_v \Psi]_{v_o} = C \). However the massive nature of the field becomes important as we move away from \( v_o \) toward \( v_n \). In the future direction \( (u > u_o) \), \( \Psi \) is attenuated and spread in a powerlike manner and as \( u \to \infty \) it would take an infinite resolution to see the ever so tiny bump of \( \Psi \) owing to the discontinuous derivative. Because of this tendency to spread the initial signal, if we had started with a smooth version of the jump \( C \) at \( u = u_o \), it would have disappeared without trace for \( \lambda (u - u_o) \gg 1 \). The more striking feature is the behaviour in the past \( (u < u_o) \). Then the square root is pure imaginary and

\[
\Psi \sim e^{\lambda \sqrt{(u-u_o)(v-v_o)}}
\]

(A.12)

The asymptotic geometry is flat within the same approximation.
up to some powerlike prefactor. On the past null infinity \( u = -\infty \), \( \Psi \) is infinite in the interval \((v_o, v_n)\).

In the asymptotically far region, the \( f \) field effectively decouples and the action \((2.4)\) can be considered as the usual dilaton gravity coupled nonlinearly to matter fields \( f \) and \( K \). Then the corresponding Einstein equation of the metric can be written with energy-momentum tensor of these matter fields on the right-hand side. \( \delta g^{uv} \) equation is given by the following. We did not write down the left-hand side explicitly.

\[
\frac{\delta S_{\text{grav}}}{\delta g^{uv}} = e^{-2\chi} \partial_v f \partial_u f + \cdots \tag{A.13}
\]

But the leading contribution to the right-hand side is proportional to the square of \((A.11)\) which is infinite on past null infinity in the finite interval \((v_o, v_n)\). Physically this means an infinite amount of energy is sent in from null past infinity and subsequently the Bondi mass must be also infinite in the same interval.\(^8\) (The reason the right hand side of \((A.13)\) is finite for \( u > -\infty \) in spite of the initially infinite value can be understood again by considering the dispersive effect of the massive dynamical equation. After a sufficient amount of time the initial flux of infinite density and infinite total energy becomes a flux of infinite energy yet of finite density spread all the way to infinite future.)

We can conclude that solutions of the type \((A.11)\) should be excluded energetically since it involves an infinite amount of energy. Furthermore the same conclusion holds if we drop the assumption of initially static spacetime. Since the perturbation equation is linear asymptotically, we can subtract a smooth part from \( \Psi \) to reduce the problem to that of an initially static spacetime (i.e., \( \Psi = 0 \) before the shock).

\(^8\) Unfortunately we were not able to derive the mass formula appropriate for the asymptotic behaviour of \( \delta f \sim e^{-\lambda y} \) But if we naively extrapolate \((A.1)\) to this case and evaluate on the past null infinity, the constraint equations \((A.4)(A.5)\) imply infinite Bondi mass.
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