QUASICONVEXITY VERSUS GROUP INVARIANCE

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Abstract. The lower invariance under a given arbitrary group of diffeomorphisms extends the notion of quasiconvexity. The non-commutativity of the group operation (the function composition) modifies the classical equivalence between lower semicontinuity and quasiconvexity.

In this context null lagrangians are particular cases of integral invariants of the group.

Keywords: quasiconvexity, diffeomorphism groups, invariants, null lagrangians, lower semicontinuity.

1. Introduction

1.1. First notations. In this paper $\Omega \subset \mathbb{R}^n$ is an open bounded set, with smooth boundary and $id$ is the identity map of $\mathbb{R}^n$ $id(x) = x$. $A \subset \subset B$ means that $A$ is compactly included in $B$.

$Diff^\infty_0 = Diff^\infty_0(\mathbb{R}^n)$ is the group of all $C^\infty$ diffeomorphisms with compact support in $\mathbb{R}^n$: $Diff^\infty_0(\mathbb{R}^n) = \{ \phi \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) : \exists \phi^{-1} \in C^\infty(\mathbb{R}^n, \mathbb{R}^n), supp(\phi - id) \subset \subset \mathbb{R}^n \}$.

For any open set $\Omega \subset \subset \mathbb{R}^n$ and any subgroup $G \subset Diff^\infty_0$ we define $G(\Omega) = \{ \phi \in Diff^\infty_0 : supp(\phi - id) \subset \Omega \}$.

Notice that $G(\Omega)$ is a group under the operation "." of functions composition.

The main goal of this paper is to study the lower semicontinuity of an integral functional having the form:

$$I(\mathbf{u}, \Omega) = \int_\Omega W(x, \mathbf{u}(x), \nabla \mathbf{u}(x)) \, dx$$

under inner (or left) and outer (or right) variations in a group of diffeomorphisms $G(\Omega)$. An inner (or left) variation of $I$ consists in the replacement of the argument $\mathbf{u}$ by $\mathbf{u}.\phi$ with $\phi \in G(\Omega)$. Similarly, an outer (or right) variation consists in the replacement of $\mathbf{u}$ by $\phi.\mathbf{u}$. Off course, the argument

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$\mathbf{u}$ belongs to a space of mappings $X$ which is stable under inner or outer variations.

Lower semicontinuity results are important tools in the study of existence and regularity of critical points for (integral) functionals.

1.2. Some problems involving groups of diffeomorphisms. In this section we shall give three examples of problems involving groups of diffeomorphisms and critical points of integral functionals.

1.2.1. Least action principles in nonlinear mechanics. The standard space of configuration of a fluid lying in a vessel $\Omega$, under adherence conditions on the vessel’s wall $\partial \Omega$, is $\text{Diff}_0^\infty(\Omega)$. Moreover, if the fluid is incompressible, then the space of states is $\text{Diff}_0^\infty(dx)(\Omega)$, namely the group of volume-preserving diffeomorphisms with support in $\Omega$.

Arnold [2] first showed that hydrodynamics of an ideal fluid can be formulated in the frame of volume preserving diffeomorphisms: the evolution of an ideal fluid is a geodesic (i.e. minimizer of an integral action functional) in the group of volume preserving diffeomorphisms. This work has been developed in papers like Arnold & Khesin [3], Ebin & Marsden [8] or Shnirelman [16].

Of a related nature is the problem of evolution of an elastic body. Generally, we seek for critical points of an action functional

$$A(\mathbf{u}_t) = \int_{\Omega \times [0,T]} L(x, t, \mathbf{u}(x, t), \nabla \mathbf{u}(x, t)) \ dx \ dt$$

under admissible variations of the form

$$(t, \mathbf{u}(\cdot, t)) \in [0, T] \times X(\mathbf{u}^0(t)) \mapsto (t, \mathbf{v}(\cdot, t)) \in [0, T] \times X(\mathbf{u}^0(t))$$

such that $\mathbf{v}(\cdot, 0) = \mathbf{u}(\cdot, 0)$, $\mathbf{v}(\cdot, T) = \mathbf{u}(\cdot, T)$, $\dot{\mathbf{v}}(\cdot, 0) = \dot{\mathbf{u}}(\cdot, 0)$, $\dot{\mathbf{v}}(\cdot, T) = \dot{\mathbf{u}}(\cdot, T)$. This problem can be reformulated by considering variations of the form

$$(t, \mathbf{u}(\cdot, t)) \in [0, T] \times X(\mathbf{u}^0(t)) \mapsto (t, \mathbf{u}(\cdot, \phi(\cdot, t))) \in [0, T] \times X(\mathbf{u}^0(t))$$

for all

$$\phi \in \text{Diff}_0^\infty(\Omega \times [0, T]), \phi(x, t) = (\phi_t(x), t), \phi_t \in \text{Diff}_0^\infty(\Omega).$$

This set of diffeomorphisms of $\Omega \times [0, T]$ is a subgroup of $\text{Diff}_0^\infty(\Omega \times [0, T])$.

1.2.2. Critical points turned into local minima. Another interesting problem is to find all critical points of an integral functional by variational methods. The direct method gives access only to global minimizers. Zhang [18], [19], Sivaloganathan [17] show that for strictly quasiconvex potential the critical points of the associated functional are local (the word has various precise meanings) minimizers.

The connection of the critical point problem with diffeomorphisms groups is made by the following proposition:
Proposition 1.1. Suppose that \( W : GL_n(R) \to R \) is a \( C^2 \) potential and \( u \in C^3(\Omega, R^n) \) is a local minimizer of the functional

\[
I(v) = \int_\Omega W(\nabla v) \, dx
\]

in the class

\[
uDiff^\infty(\Omega) = \{ u.\phi : \phi \in Diff^\infty(\Omega) \}.
\]

Then \( u \) is a critical point of \( I \).

Proof. Let us consider, for a given but arbitrary \( \eta \in C^\infty_c(\Omega, R^n) \), the one parameter flow \( t \mapsto \phi_t \) generated by \( \eta \). We have then \( I(u) \leq I(u.\phi_t^{-1}) \) for small \( |t| \). Therefore:

\[
\frac{\partial I}{\partial t}(u.\phi_t^{-1})_{|t=0} = 0.
\]

The latter equality means that for any \( \eta \in C^\infty_c(\Omega, R^n) \) we have:

\[
\int_\Omega \left\{ W(\nabla u) \, \text{div} \, \eta - \frac{\partial W}{\partial F_{lk}}(\nabla u)_{i,l} \eta_{i,k} \right\} \, dx = 0.
\]

This is equivalent to:

\[
\int_\Omega \left\{ W(\nabla u)_{i,l} \eta_i - \frac{\partial W}{\partial F_{lk}}(\nabla u)_{i,l} \eta_{i,k} \right\} \, dx = 0.
\]

By the Divergence theorem we have:

\[
\int_\Omega \left\{ \frac{\partial W}{\partial F_{lk}}(\nabla u)_{i,l} \right\} \, dx = 0.
\]

We conclude that

\[
(\nabla u)^T \, \text{div} \left( \frac{\partial W}{\partial F_{lk}}(\nabla u) \right) = 0.
\]

Because \( \nabla u(x) \in GL_n(R) \) it follows that

\[
\text{div} \left( \frac{\partial W}{\partial F_{lk}}(\nabla u) \right) = 0.
\]

\( \square \)

We see from proposition 1.1 that the class of all (sufficiently regular) \( u \) with the property:

\[
(1) \quad I(u) \leq I(u.\phi) \text{ for all } \phi \in Diff^\infty_0(\Omega)
\]

is included in the class of critical points of \( I \). From the proof of the same proposition we notice that any critical point \( u \) of \( I \) is critical in the class
$\mathbf{u}.\text{Diff}^\infty_0(\Omega)$ in the sense that for any $\eta \in C^\infty_c(\Omega, R^n)$ which generates the one parameter flow $\phi_t$ we have:

(2) \quad \frac{d}{dt} I(\mathbf{u}, \phi_t^{-1})_{t=0} = 0

A natural question is: does any critical point $\mathbf{u}$ have the property (1)?

Consider, as an example, the functional

$$I(\theta) = \int_0^L \frac{1}{2} k |\theta'|^2 + \lambda \mathbf{F}(\theta) \, ds,$$

where $\mathbf{F}$ is an arbitrary $C^1$ function.

The group $\text{Diff}^\infty_0(0, L)$ is nothing but:

$$\text{Diff}^\infty_0(0, L) = \{ \phi \in C^\infty((0, L), (0, L)) : \phi' > 0 , \supp (\phi - \text{id}) \subset \subset (0, L) \}.$$

We introduce now the functional:

$$I(\phi; \theta) = I(\theta, \phi^{-1}) = \int_0^L \frac{1}{2} k |\theta'|^2 \frac{1}{\phi'} + \lambda \mathbf{F}(\theta) \phi' \, ds.$$

It is easy to see that $I(\cdot; \theta)$ is convex on $\text{Diff}^\infty_0(0, L)$.

Consider now $\theta$ a critical point of $I$. The function $\text{id}$ is then a critical point of $I(\cdot; \theta)$ in $\text{Diff}^\infty_0(0, L)$. We deduce from the convexity of $I(\cdot; \theta)$ that $\text{id}$ is a global minimum, therefore it satisfies the relation (1).

1.2.3. The invariance problem. Let $X$ be a class of functions from $R^n$ to $R^m$ and $G$ a topological semigroup of diffeomorphisms of $R^n$ such that $X . G = X$.

**Definition 1.1.** The functional $I : X \to R$ is $G$ left invariant if for any $\mathbf{u} \in X$ and $\phi \in G$ we have the equality $I(\mathbf{u}, \phi) = I(\mathbf{u})$.

Such invariants play a central role in continuum mechanics. Indeed, let us consider $X = C^3(\Omega, R^k)$, $G = \text{Diff}^\infty_0(\Omega)$ and

$$I(\mathbf{u}; \Omega) = \int_\Omega L(\nabla \mathbf{u}(x)) \, dx.$$

We prove in this paper the following

**Proposition 1.2.** Let $L : M^{n,n} \to R$ be a $C^2$ map. We have equivalence between the statements:

(i) $L$ is a $\text{Diff}^\infty_0(\Omega)$ invariant,
(ii) $L$ is a null lagrangian.
2. Outline

We address in this paper the problem of proving lower semicontinuity of integral functionals defined over groups of diffeomorphisms.

After a section of preliminaries we introduce the notion of left and right lower invariance of a mapping with respect to a group and prove that for mappings defined on $GL_n(R)$, quasi-convexity in the sense of Morrey is equivalent to the left lower invariance under the group of diffeomorphisms with compact support.

The lower invariance under a given arbitrary group $G$ of diffeomorphisms extends the notion of quasi-convexity. The non-commutativity of the group operation (the function composition) modifies the classical equivalence between lower semicontinuity and quasi-convexity. We introduce a new notion of semicontinuity, named $G$-left (or right) lower semicontinuity.

The main results of this paper (theorems 4.2 and 4.3) show that if the integral functional $I$ is $G$-left lower semicontinuous then the potential $W$ is $G$-left lower invariant; also if $W$ is $G$ right lower invariant then $I$ is $G$ right lower semicontinuous.

In this context null lagrangians are particular cases of integral invariants of the group. We generally find that the only homogeneous integral functionals which are weak * continuous under inner variations in a group of diffeomorphisms are the constant ones.

3. Preliminaries

3.1. Notations. The set $W^{1,\infty}(\Omega)$ is the Sobolev space of $L^\infty(\Omega, R^n)$ functions with the first derivative essentially bounded. $\| \cdot \|_1,\infty$ is the usual norm in $W^{1,\infty}(\Omega, R^n)$. For any $u \in W^{1,\infty}(\Omega, R^n)$ the gradient $\nabla u$ is identified with the approximate gradient of $u$. Therefore the equalities involving gradients are almost everywhere (abbreviated "a.e.").

$A$ is the group of affine homothety-translations. Any element of $A$ has the form:

$$\alpha(x_0,y_0,\epsilon)(x) = f(x) = x_1 + \epsilon(x - x_0) \ , \ x_0, x_1 \in R^n \ , \ \epsilon > 0 \ .$$

We consider on $A$ the punctual convergence of functions defined on $R^n$ with values in $R^n$. $GL_n(R) \subset R^{n \times n}$ is the multiplicative group of all invertible, orientation preserving, matrices, i.e the set of all $F$ such that $\det F > 0$.

We shall use in the paper the affine space

$$W^{1,\infty}_{id}(R^n) = \{ \phi \in W^{1,\infty}_{loc}(R^n, R^n) : \text{supp } (\phi - id) \subset \subset R^n \} \ .$$

3.2. Basic definitions and properties. $G$ is a set of functions from $R^n$ to $R^n$, which satisfies the following axioms:

$A1/ (G,.)$ is a semigroup with the function composition operation ".";

$A2/ G \subset W^{1,\infty}_{id}(R^n) \cap C^1(R^n,R^n) \ ;$
A3/ the following action is well defined:

\[ A : \mathcal{A} \times G \to G , \quad A(f, \phi) = f \cdot \phi \cdot f^{-1} . \]

Further on we shall suppose that \( G \) acts transitively on \( \mathbb{R}^n \). We did not included this statement among the axioms because all the results from the paper hold without the transitivity assumption, but in a more involved form. The same remark is true if we suppose only that for any \( x \in \Omega \) the orbit

\[ G(\Omega)(x) = \{ \phi(x) : \phi \in G(\Omega) \} \]

is dense in \( \Omega \). For any open set \( E \subset \subset \mathbb{R}^n \) the set \( G(E) \) is defined further (definition 3.1).

**Definition 3.1.** For any open set \( E \subset \subset \mathbb{R}^n \) we define

\[ G(E) = \{ \phi \in G : \text{supp} \ (\phi - \text{id}) \subset \subset E \} \ . \]

For any \( x_0 \in \mathbb{R}^n \) the first order jet of \( G \) in \( x_0 \) is:

\[ J^1(x_0, G) = \{ \nabla \phi(x_0) : \phi \in G \} \ . \]

**Definition 3.2.** Let \((\phi_h)_h \in G(\Omega)\) be a sequence and \( \phi \) an element of \( W^{1,\infty}_{id}(\Omega, \mathbb{R}^n) \). We say that \( \phi_h \) converges to \( \phi \) if the sequence \( \phi_h \) converges \( W^{1,\infty} \) weak* to \( \phi \).

\( G^{1,\infty}(\Omega) \) is the closure of \( G \) in \( W^{1,\infty}_{id}(\Omega, \mathbb{R}^n) \) with respect the strong convergence. That is \( G^{1,\infty}(\Omega) \) is the space of all \( u \) which can be obtained as limit points of strong convergent sequences \( \phi_h, \phi_h \in G(\Omega) \).

**Remark 3.1.** The action \( A \) is continuous. The operation "." is continuous in each argument.

In the following lemmas we collect some elementary facts connected to the convergence or to the algebraic structure previously introduced.

**Lemma 3.1.**

1/ Let \( \Omega \subset B(0, R) \). Then \( G(\Omega) \subset G(B(0, R)) \).

2/ \( G^{1,\infty}(\Omega) \subset C^{0,1}(B(0, R)) \). For any sequence \( \phi_h \in G^{1,\infty}(\Omega) \) such that \( \phi_h \rightharpoonup \phi \) there exists a subsequence which converges uniformly to \( \phi \).

3/ \( G^{1,\infty}(\Omega) \) is a semigroup. The composition operation "." is continuous in each argument.

**Lemma 3.2.** Let \( A, B \) be non empty open subsets of \( \mathbb{R}^n \). If \( A \) is bounded then there exists \( f \in \mathcal{A} \) such that the application \( A(f, \cdot) : G(A) \to G(B) \) is well defined, injective and continuous.

**Lemma 3.3.** If \( A \) and \( B \) are two open disjoint sets then for any \( \phi \in G(A), \psi \in G(B) \) we have \( \phi \cdot \psi = \psi \cdot \phi \in G(A \cup B) \).

The following proposition shows that the first order jet associated to the set \( G \) and an arbitrary point \( x \in \mathbb{R}^n \) is a (semi) group which does not depend on \( x \).
Proposition 3.1. There exists $J(G)$ sub semigroup of the multiplicative group $GL_n(R)$ such that for any $x_0 \in \mathbb{R}^n$ we have $J^1(x_0, G) = J(G)$. If $G$ is a group then $J(G)$ is a group.

Proof. We first prove that $J^1(x_0, G)$ is semigroup. We have $id \in G$, hence $I$, the identity matrix, belongs to $J^1(x_0, G)$. Let us consider $R, S \in J^1(x_0, G)$ and $\phi, \psi \in G$ such that $R = \nabla \phi(x_0)$, $S = \nabla \psi(x_0)$. We define the translation $f \in A$: $f(x) = x + \psi(x_0) - x_0$. From $A3/\,$ we have $\tilde{\phi} = f.\phi.f^{-1} \in G$, hence

$$RS = \nabla \phi(x_0) \nabla \psi(x_0) = \nabla \tilde{\phi}(\psi(x_0)) \nabla \psi(x_0) = \nabla(\tilde{\phi} \cdot \psi)(x_0) \in J^1(x_0, G).$$

A simple argument based on $A3/\,$ shows that $J^1(x_0, G)$ does not depend on $x_0 \in \mathbb{R}^n$. For a fixed, arbitrarily chosen, $x_0$ we define $J(G) = J^1(x_0, G)$.

The proof of the fact that if $G$ is a group and $F \in J(G)$ then $F^{-1}$ exists and $F^{-1} \in J(G)$ is similar. □

Definition 3.3. $W^{1,\infty}(G, \Omega)$ is the class of all $u \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ such that we have $\nabla u(x) \in J(G)$ a.e. in $\Omega$.

We describe now two easy procedures of construction of groups satisfying the axioms $A1/\,$, $A2/\,$, $A3/\,$.

Definition 3.4. For any subgroup $M$ of $GL_n(R)$ we define the local group generated by $M$:

$$[M] = \{ \phi \in Diff_1^\infty : \forall x \in \mathbb{R}^n \ \nabla \phi(x) \in M \}.$$

It is obvious that $[M]$ satisfies the axioms and $J([M]) = M$. We notice that the groups $G$ constructed in this way are determined by $J(G)$, that is: if $J(G_1) = J(G_2)$ then $G_1 = G_2$. This property justifies the name "local group".

Definition 3.5. For any semigroup (group) $G$ which satisfies the axioms the completion of $G$ is defined by:

$$G^c = \{ F.\phi.F^{-1} : F, F^{-1} \in J(G), \ \phi \in G \}.$$

Generally $G^c$ is larger than $G$, but not always a semigroup (group).

Example 3.1. We obviously have $[GL_n(R)] = Diff_1^\infty$, therefore $J(Diff_1^\infty) = GL_n(R)$. We have also $Diff_1^{\infty, c} = Diff_1^\infty$.

Example 3.2. Let us consider $Diff_1^\infty(dx)$, the subgroup of $Diff_1^\infty$ containing all volume preserving smooth diffeomorphisms with compact support. We have $Diff_1^{\infty, c}(dx) = [SL_n(R)] = Diff_1^{\infty, c}(dx)$. 
Example 3.3. For any \( u : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) and \( \omega \), the canonical symplectic 2-form on \( \mathbb{R}^{2n} \), we denote by \( u^*(\omega) \) the transport of \( \omega \). Let us define \( \text{Diff}^\infty_0(\omega) \):

\[
\text{Diff}^\infty_0(\omega) = \{ \phi \in \text{Diff}^\infty_0 : \phi^*(\omega) = \omega \}.
\]

The axioms A1/, A2/ A3/ are satisfied. We have the equalities:

\[
J(\text{Diff}^\infty_0(\omega)) = \text{Sp}_n(\mathbb{R}) = \{ F \in \mathbb{R}^{2n \times 2n} : F\omega F^T = \omega \}
\]

and \( \text{Diff}^\infty_0\text{c}(\omega) = \text{Diff}^\infty_0(\omega) = [\text{Sp}_n(\mathbb{R})] \).

Example 3.4. Let us take a group \( G \) which satisfies the axioms. The space of smooth loops \( t \in S^1 \to \phi_t \in G \) can be embedded in the following group:

\[
LG = \{ \phi \in \text{Diff}^\infty_0(\mathbb{R}^{n+1}) : \phi(t, x) = (t, \phi_t(x)), \phi_t \in G \}.
\]

Notice that \( LG \) does not satisfy the transitivity assumption. However, the results from this paper are true in this case, but with minor modifications which are left to the interested reader.

Example 3.5. Consider the class \( \mathcal{H} \) of hamiltonian diffeomorphisms with compact support of \( \mathbb{R}^{2n} \) (see Hofer, Zehnder [11]). This is a group which satisfies the axioms, but it is not local. However, it is complete.

For all the transitivity results needed in these example we refer to Michor & Vizman [13].

4. Lower invariance and semicontinuity

The lower semicontinuity of functionals \( I(\cdot; \Omega) \) defined over Sobolev spaces was systematically studied. Morrey [14] introduced the notion of quasiconvexity and proved that \( W^{1,\infty} \) weak * lower semicontinuity of \( I(\cdot; \Omega) \) is equivalent to the quasiconvexity of the integrand \( W \) in it’s third variable, provided that \( W \) is continuous. Acerbi & Fusco [1], Ball & Murat [7] improved this result and introduced the notion of \( W^{1,p} \) quasiconvexity. Ball [4], [5], considered a new condition, called polyconvexity, which implies quasiconvexity, with important applications in nonlinear elasticity.

4.1. Lower invariance and quasiconvexity. There are several slightly different definitions of quasiconvexity. We prefer the one from Ball [5]:

**Definition 4.1.** \( W \) is quasiconvex in \((x_0, y_0, F) \in \mathbb{R}^n \times \mathbb{R}^n \times GL_n(\mathbb{R})\) if for any open bounded set \( E \subset \mathbb{R}^n \) and any \( \eta \in C^\infty(E, \mathbb{R}^n) \) such that:

i) \( \text{supp} \ \eta \subset \subset E \),

ii) for any \( x \in E \) we have \( F + \nabla \eta(x) \in GL_n(\mathbb{R}) \) (i.e. \( \det (F + \nabla \eta(x)) > 0 \)),

we have the inequality:

\[
\int_E W(x_0, y_0, F + \nabla \eta(y)) \ dy \geq |E|^\infty W(x_0, y_0, F).
\]
We could work in the followings with a topological space \( X \) of continuous functions from \( \mathbb{R}^n \) to \( \mathbb{R}^m \), such that \( X.G = X \) and for any homothety-translation \( h : \mathbb{R}^n \to \mathbb{R}^n \) and any \( u \in X \) we have \( u.h \in X \). Our model space will be \( W^{1,\infty}(G, \Omega) \) (see definition \[\ref{W1infty} \]); all results from this paper can be reformulated for a wide variety of spaces \( X \) in an obvious way. We leave this for further applications.

The non-commutativity of the function composition forces us to consider a "left" and "right" variant of any further definition. For the notions involving the word "right" we shall suppose that \( G.X = X \) and \( X \subset W^{1,\infty}_{loc}(\mathbb{R}^m, \mathbb{R}^n) \).

We introduce the following definition of lower (upper respectively) invariance. In the next definition we shall denote by \( J(X) \) the first order jet of \( X \) (supposing that it does not depend on \( x \)). From the condition \( X.G = X \) we derive that \( J(X).J(G) = J(X) \).

\begin{definition}
Let us consider \( x_0, y_0 \in \mathbb{R}^n \) and \( F \in J(X) \).

The function \( W \) or the functional \( I \) are \( G \) left lower invariant in \((x_0, y_0, F)\), and we shall write "\( G \ L.LI \)", or even "\( L.LI \)" if no confusion arises, if for any bounded open set \( E \subset \mathbb{R}^n \) and any \( \phi \in G(E) \) we have the inequality:
\[
\int_E W(x_0, y_0, F \nabla \phi(y)) \, dy \geq |E| \cdot W(x_0, y_0, F) \ .
\]

\( W \) (or \( I \)) is \( G \) right lower invariant (\( G \ R.LI \)) in \((x_0, y_0, F)\) if for any bounded open set \( E \subset \mathbb{R}^n \) and any \( \phi \in G(E) \) we have:
\[
\int_E W(x_0, y_0, \nabla \phi(y)F) \, dy \geq |E| \cdot W(x_0, y_0, F) \ .
\]

If in the relations \[\ref{4}, \[\ref{5} \] we change "\( \geq \)" by "\( \leq \)" then we obtain the definitions of \( G \) left upper invariance (\( G \ L.UI \)), respectively \( G \) right upper invariance (\( G \ R.UI \)). If \( W \) is right and left LI then we call it \( G \ LI \); also if \( W \) is right (or left) lower and upper invariant we call it right (or left) invariant.

A key observation consists in the following proposition, which shows that quasiconvexity is a particular case of lower invariance.

\begin{proposition}
Let us consider \((x_0, y_0, F) \in \mathbb{R}^n \times \mathbb{R}^n \times GL_n(\mathbb{R}) \). Then \( W \) is \( Diff_0^{\infty} \ L.LI \) in \((x_0, y_0, F)\) if and only if it is quasiconvex in the same triplet.

\[\text{Proof.} \text{ Let } E \subset \mathbb{R}^n \text{ be an open bounded set and } \phi \in Diff_0^{\infty}(E). \text{ The vector field } \eta = F(\phi - id) \text{ verifies i) and ii) from definition} \[\ref{QConv}\text{. Therefore, if } W \text{ is quasiconvex in } (x_0, y_0, F), \text{ we derive from} \[\ref{QConv} \text{ the inequality:}
\]
\[
\int_E W(x_0, y_0, F \nabla \phi(y)) \, dy \geq |E| \cdot W(x_0, y_0, F) \ .
\]

We implicitly used the chain of equalities \( F + \nabla \eta(y) = F + F \nabla \phi(y) - F = F \nabla \phi(y) \). We have proved that quasiconvexity implies \( Diff_0^{\infty} \ L.LI \).

In order to prove the inverse implication we shall suppose that \( E \) is also simply connected. This supposition is not restrictive according to corollary
3.1.1 from Ball \[4\] (see also the references therein and the twin result contained in proposition \[1,3\] from this paper). Let us consider \( \eta \) which satisfies i) and ii) from definition \[4,1\] 

From the hypothesis upon \( E \) the function

\[
\psi(x) = \begin{cases} 
Fx + \eta(x) & \text{if } x \in E \\
Fx & \text{otherwise} 
\end{cases}
\]

is \( C^\infty \) and invertible on \( \mathbb{R}^n \). We have therefore \( \phi = F^{-1} \psi \in Diff_0^\infty(E) \) and \( F \nabla \phi = F + \nabla \eta \). If \( W \) is \( Diff_0^\infty L.LI \) in \((x_0, y_0, F)\) then we use \[4\] with the previously defined \( \phi \) in order to obtain \[3\].

**Remark 4.1.** In definition \[4,2\] \( G(E) \) can be replaced by \( G^{1,\infty}(E) \). This follows from the definition of \( G^{1,\infty}(E) \) and the continuity of \( W \).

**Proposition 4.2.** If \( G \) is a group and \( G^c = G \) then \( W \) is \( G L.LI \) in \((x_0, y_0, F) \subset \mathbb{R}^n \times \mathbb{R}^n \times J(G) \) if and only if \( W \) is \( G R.LI \) in the same triplet.

**Proof.** Let us suppose that \( W \) is \( G R.LI \) in \((x_0, y_0, F)\). We make the change of variable \( x = F^{-1} y \) and we rewrite the hypothesis in the following way: for any open bounded set \( E \subset \mathbb{R}^n \) and any \( \phi \in G(E) \) we have

\[
\int_{F^{-1}(E)} W(x_0, y_0, \nabla(\phi,F)(x)) \, dx \geq |F^{-1}(E)| \, W(x_0, y_0, F) 
\]

The hypothesis of the proposition implies that the application \( \phi \in G(E) \mapsto F^{-1} \phi, F \in G(F^{-1}(E)) \) is well defined and bijective. Therefore \( W \) is \( G R.LI \) in \((x_0, y_0, F)\) if and only if for any bounded open set \( E \subset \mathbb{R}^n \) and for any \( \psi \in G(F^{-1}(E)) \) we have

\[
\int_{F^{-1}(E)} W(x_0, y_0, F \nabla \psi(x)) \, dx \geq |F^{-1}(E)| \, W(x_0, y_0, F) 
\]

The last statement is equivalent to the fact that \( W \) is \( G L.LI \) in \((x_0, y_0, F)\). \( \square \)

The proposition remains true if we change lower invariance with upper invariance.

The following theorem shows that \( G \) lower invariance of \( W \) is a necessary condition for the existence of a minimum of \( I(\cdot; \Omega) \) over \( C^1(\Omega, \mathbb{R}^n) \cap W^{1,\infty}(G, \Omega) \).

**Theorem 4.1.** Let us suppose that there exists \( u \in C^1(\Omega, \mathbb{R}^n) \cap W^{1,\infty}(G, \Omega) \) such that for any \( \phi \in G(\Omega) \), \( \| \phi - id \|_{C(\Omega)} < \epsilon \), we have:

\[
I(u, \phi; \Omega) \geq I(u; \Omega) 
\]

\( W \) is then \( G L.LI \) in \((x_0, u(x_0), \nabla u(x_0))\) for any \( x_0 \in \Omega \).

**Proof.** Let \( x_0 \in \Omega \) and \( E \subset \mathbb{R}^n \) be an open bounded set. We can find then \( \epsilon_0 > 0 \) such that the followings are true:

1. \( x_0 + \epsilon_0 \, E \subset \subset \Omega \);
(2) for any $\psi \in G(\Omega)$, $\|\psi - id\|_{C(\Omega)} < \epsilon_0$, we have:

$$I(u;\psi;\Omega) \geq I(u;\Omega).$$

Let us consider $\phi \in G(E)$, $\epsilon \leq \epsilon_0$, $f_\epsilon(x) = x_0 + \epsilon x$ and $\phi^\epsilon = A(f^\epsilon, \phi)$. From lemma 3.2 it follows that $\phi^\epsilon \in G(\Omega)$. We can choose a sufficiently small $\epsilon$ such that $\|\phi^\epsilon - id\|_{C(\Omega)} < \epsilon_0$. We apply (6) with $\phi^\epsilon$ and we obtain the inequality:

$$\int_{f^\epsilon(E)} W(x, u(\phi^\epsilon(x)), \nabla u(\phi^\epsilon(x)) \nabla \phi^\epsilon(x)) \, dx \geq I(u; f^\epsilon(E)).$$

After the change of variable $f^\epsilon(y) = x$ the inequality becomes:

$$\int_{E} W(f^\epsilon(y), u(f^\epsilon(y)), \nabla u(f^\epsilon(y)) \nabla \phi(y)) \, dy \geq \int_{E} W(f^\epsilon(y), u(f^\epsilon(y)), \nabla u(f^\epsilon(y))) \epsilon^n \, dy .$$

We reduce $\epsilon^n$ from the both members of the inequality. The continuity of $W$ and regularity of $u$ imply that when $\epsilon$ converges to 0 we have the inequality:

$$\int_{E} W(x_0, u(x_0), \nabla u(x_0) \nabla \phi(y)) \, dy \geq |E| W(x_0, u(x_0), \nabla u(x_0)) .$$

The following proposition shows that in the definition the text “for any bounded open set $E$ and any $\phi \in G(E)$ ...” can be replaced by “there is a bounded open set $E$ such that for any $\phi \in G(E)$ ...”.

**Proposition 4.3.** Let us consider $x_0, y_0 \in \mathbb{R}^n$ and $F \in J(G)$. If there exists $E \subset \mathbb{R}^n$, bounded and open, such that for any $\phi \in G(E)$, $\|\phi - id\|_{C(E)} < \epsilon$, we have

$$\int_{E} W(x_0, y_0, F \nabla \phi(y)) \, dy \geq |E| W(x_0, y_0, F)$$

then $W$ is $G LLI$ in $(x_0, y_0, F)$.

**Proof.** Let us take $\Omega = E$ and

$$I(u; E) = \int_{E} W(x, u(x), F \nabla u(x)) \, dx .$$

We apply theorem 4.1 and conclude the proof.

Any quasiconvex function $W$ is both $G LLI$ and $R LI$. This follows from propositions 4.1, 4.2 and the simple remark that if $G \subset G'$ then $G'$ LI implies $G$ LI.

**Open Problem 1.** Find a group $G$ and a function $W$ which is $G$ RLI but not $G$ LLI.
4.2. Semicontinuity and invariance.

**Definition 4.3.** A functional \( I : X(\Omega) \to R \) is left sequentially weak* lower semicontinuous (G L LSC) in \( u \in X(\Omega) \) if for any sequence \( \phi_h \in G^{1,\infty}(\Omega) \) convergent to id we have:

\[
I(u) \leq \liminf_{h \to \infty} I(u, \phi_h)
\]

The functional \( I \) is right sw*lsc (G R LSC) in \( u \) if for any sequence \( \phi_h \in G^{1,\infty}(\Omega) \) convergent to id we have:

\[
I(u) \leq \liminf_{h \to \infty} I(\phi_h, u)
\]

The purpose of this section is to explore the connections between the \( G \) lower invariance of \( W \) and the lower semicontinuity (in the sense of definition 4.3) of the functional \( I(\cdot; \Omega) \). Our results generalize the ones from Morrey [14], Meyers [12], which show that quasiconvexity of \( W \) is equivalent to lower semicontinuity (in the classical sense) of \( I(\cdot; \Omega) \), if the functional is defined over a Sobolev vector space.

**Theorem 4.2.** Let \( \Omega \subset \mathbb{R}^n \) be an open bounded set and \( W : \mathbb{R}^n \times \mathbb{R}^n \times J(G) \to R \) continuous. If for any \( \psi \in G \) and for any sequence \( \phi_h \in G(\Omega) \) convergent to id we have the inequality:

\[
I(\psi; \Omega) \leq \liminf_{h \to \infty} I(\psi, \phi_h; \Omega)
\]

then \( W \) is G L LI in any triplet of the form \( (x, \psi(x), \nabla \psi(x)) \subset \Omega \times \Omega \times J(G), \phi \in G(\Omega) \). If \( W = W(x, F) \) then the conclusion of the theorem is: \( W \) is G L LI in any pair of the form \( (x, F) \subset \Omega \times J(G) \).

**Proof.** Let us consider \( x_1 \in \Omega \) and \( h > 0 \). \( Q_h \) is the cube \( x_1^j < x^j < x^j + 1/h \).

We take \( \phi \in G(Q_1) \) and \( k \in N \). The extension of \( \phi \) by periodicity over \( \mathbb{R}^n \) is denoted by \( \tilde{\phi} \). We define then:

\[
\phi_{h,k}(x) = \begin{cases} (hk)^{-1} \left( \tilde{\phi}(hk(x - x_1) + x_1) - x_1 \right) + x_1 & \text{if } x \in Q_h \\ x & \text{otherwise} \end{cases}
\]

From proposition 3.3 and A3/ we infer that \( \phi_{h,k} \in G \). Any set \( Q_h \) decomposes in \( k^n \) cubes which will be denoted by \( Q_{hk,j}, j = 1, ..., k^n \), such that \( Q_{hk,1} = Q_{hk} \). The corner of \( Q_{hk,j} \) with least distance from \( x_1 \) is denoted by \( x_j \).

Let us now consider \( \psi \in G(\Omega) \cap C^2(\Omega, \mathbb{R}^n) \) and \( y_1 = \psi(x_1), F = \nabla \psi(x_1) \). For a sufficiently large \( h \) we have \( Q_h \subset \Omega \), hence \( I(\psi, \phi_{h,k}; \Omega) \) makes sense. We decompose this integral in two parts:

\[
I(\psi, \phi_{h,k}; \Omega) = I(\psi, \phi_{h,k}; Q_h) + I(\psi; \Omega \setminus Q_h)
\]

\[
I(\psi, \phi_{h,k}; Q_h) = \sum_{j=1}^{k^n} \int_{Q_{hk,j}} [W(x, \psi, \phi_{h,k}(x), \nabla \psi(x)](x)]
\]
\begin{equation}
-W(x_j; \psi \cdot \phi_{h,k}(x_j), \nabla \psi(\phi_{h,k}(x_j)) \nabla \phi_{h,k}(x)) \, dx +
\end{equation}

\sum_{j=1}^{k^n} \int_{Q_{h,k,j}} W(x_j, \psi \cdot \phi_{h,k}(x_j), \nabla \psi(\phi_{h,k}(x_j)) \nabla \phi_{h,k}(x)) \, dx . 

\text{Notice that } \phi_{h,k} \text{ converges weak* to } id. \text{ Because } W \text{ and } \nabla \psi \text{ are continuous and } \phi_{h,k} \text{ converges uniformly to } id, \text{ it follows that the first sum from the right-handed member of the equality (9) converges to zero.}

By the change of variable \( y = hk(x - x_j) + x_1 \) we obtain:

\begin{equation}
\int_{Q_{h,k,j}} W(x_j, \psi \cdot \phi_{h,k}(x_j), \nabla \psi(\phi_{h,k}(x_j)) \nabla \phi_{h,k}(x)) \, dx =
\end{equation}

\( (hk)^{-n} \int_{Q_1} W(x_j, \psi(x_j), \nabla \psi(x_j) \nabla \phi(y)) \, dy . \)

We deduce from here that the second sum of the right-handed member is a Cauchy sum. By a passage to the limit as \( k \to \infty \) we get the equality:

\begin{equation}
\lim_{k \to \infty} I(\psi, \phi_{h,k}; Q_h) = \int_{Q_h} \int_{Q_1} W(x, \psi(x), \nabla \psi(x) \nabla \phi(y)) \, dy \, dx .
\end{equation}

From (7) we have:

\[ \liminf_{k \to \infty} I(\psi, \phi_{h,k}; \Omega) = \liminf_{k \to \infty} I(\psi, \phi_{h,k}; \Omega \setminus Q_h) + I(\psi; \Omega \setminus Q_h) \]

\begin{equation}
\geq I(\psi; Q_h) + I(\psi; \Omega \setminus Q_h) ,
\end{equation}

therefore (11) implies that:

\begin{equation}
\int_{Q_h} \int_{Q_1} W(x, \psi(x), \nabla \psi(x) \nabla \phi(y)) \, dy \, dx \geq
\end{equation}

\[ \int_{Q_h} W(x, \psi(x), \nabla \psi(x)) \, dx . \]

We multiply the relation (13) with \( h^n \) and pass to the limit as \( h \to \infty \). The result is:

\[ \int_{Q_1} W(x_1, \psi(x_1), \nabla \psi(x_1) \nabla \phi(y)) \, dy \geq W(x_1, \psi(x_1), \nabla \psi(x_1)) \]

which concludes the first part of the proof.

Now, if \( W = W(x, F) \) then let us notice that for any \( x_1 \in \Omega \) and \( F \in J(G) \) there exists \( \psi \in G(\Omega) \) such that \( \nabla \psi(x_1) = F \), therefore we can apply what we have already proved in order to obtain the second conclusion of the theorem. \( \square \)
**Theorem 4.3.** Let $W : \mathbb{R}^n \times \mathbb{R}^n \times J(G) \to \mathbb{R}$ be a continuous function. If $W$ is $G$ R.LI in any triplet of the form $(x, y, F) \subset \Omega \times \Omega \times J(G)$, then for any $\psi \in W^{1,\infty}(G, \Omega)$ and any sequence convergent to $\phi_k \in G(\Omega^\prime)$, where $\psi(\Omega) \subset \Omega^\prime$, we have the inequality:

\[(14) \quad I(\psi; \Omega) \leq \liminf_{k \to \infty} I(\phi_k.\psi; \Omega) .\]

**Proof.** For the proof is not restrictive to consider that $\psi(\Omega) \subset \Omega$. Let $G_\nu$ be the cubic lattice constructed from the cube $0 \leq x^i \leq 2^{-\nu}$ and let $\Gamma_\nu$ be the reunion of all cubes of $G_\nu$ included in $\Omega$. Let us consider $\psi \in W^{1,\infty}(G, \Omega)$ and a sequence $\phi_k \in G(\Omega^\prime)$ convergent to $\text{id}$.

Fix $\epsilon > 0$; there exists $\nu^\prime$, sufficiently large such that:

\[(15) \quad | I(\phi_k.\psi; \Omega \setminus \Gamma_{\nu^\prime}) | < \epsilon \quad \forall \ k \in \mathbb{N} ,\]

\[(16) \quad | I(\psi; \Omega \setminus \Gamma_{\nu^\prime}) | < \epsilon .\]

For any $\nu > \nu^\prime$, with the notations from the proof of the theorem, we write $\Gamma_{\nu^\prime}$ in the following way: $\Gamma_{\nu^\prime} = \bigcup_{h=1}^{N_{\nu^\prime}} Q_h$. The integral $I(\phi_k.\psi; \Gamma_{\nu})$ can be regarded as a sum of two terms:

\[(17) \quad I(\phi_k.\psi; \Gamma_{\nu}) = \int_{\Gamma_{\nu}} [W(x, \phi_k.\psi(x), \nabla(\phi_k.\psi(x))) - W(x, \psi(x), \nabla(\phi_k.\psi(x))) \nabla \psi(x)] \ dx + \int_{\Gamma_{\nu}} W(x, \psi(x), \nabla(\phi_k.\psi(x))) \ dx .\]

The first integral of the right-handed member of (17) converges to zero as $k \to \infty$.

We can take in any cube $Q_h \in \Gamma_{\nu}$ a point $x_{h,\nu}$ such that it is a Lebesgue point for all $\nabla \phi_k$. For any $v \in L^1_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ such that all $x_{h,\nu}$ are Lebesgue points we make the notation:

$$\nabla(x) = \begin{cases} v(x_{h,\nu}) & \text{if } x \in Q_h \\ x & \text{otherwise} \end{cases} .$$

The second integral from the right-handed member of (17) can be written as a sum $J_1 + J_2 + J_3$, with:

\[(18) \quad J_1 = \int_{\Gamma_{\nu}} [W(x, \psi(x), \nabla \phi_k(\psi(x))\nabla \psi(x)) - W(x, \psi(x), \nabla \phi_k(\psi(x))\nabla \psi(x))] \ dx ,\]

\[(19) \quad J_2 = \int_{\Gamma_{\nu}} [W(x, \psi(x), \nabla \phi_k(\psi(x))\nabla \psi(x)) -\]

\[(19) \quad J_3 = \int_{\Gamma_{\nu}} [W(x, \psi(x), \nabla \phi_k(\psi(x))\nabla \psi(x)) -\]
\[-W(x, \nabla \psi(x)) \quad dx \,.\]

\[J_3 = \int_{\Gamma} W(x, \nabla \psi(x)) \quad dx \,.
\]

From the continuity of \(W\), the boundedness of \(\nabla \psi\) and the uniform boundedness of \(\nabla \phi_k\) we deduce that:

1. \(J_1\) converges to zero uniformly with respect to \(k\),
2. \(J_3\) converges to \(I(\psi; \Gamma, \nu')\)
as \(\nu \to \infty\).

If \(W\) is \(G\) R.LI in any triplet \((x, y, F) \subset \Omega \times \Omega \times J(G)\) then:

\[\liminf_{k \to \infty} J_2 \geq 0 \,.
\]

From the convergences mentioned at 1., 2. and from the relations (17), (21) we obtain:

\[\liminf_{k \to \infty} I(\phi_k, \psi; \Gamma, \nu') \geq I(\psi; \Gamma, \nu') \,.
\]

The latter relation, together with (15), (16), lead us to the inequality:

\[\liminf_{k \to \infty} I(\phi_k, \psi; \Omega) \geq I(\psi; \Omega) - 2\epsilon \, , \text{ q.e.d.}
\]

**Remark 4.2.** If \(G\) is a group then \(I(\cdot; \Omega)\) is \(G\) L.LSC over \(G\) if and only if it is \(G\) R.LSC. Moreover, the left (or right) lower semicontinuity over \(G\) are equivalent with classical lower semicontinuity.

The theorem 4.2 essentially says that if \(I(\cdot; \Omega)\) is \(G\) L.LSC then \(W\) is \(G\) L.LI The theorem 4.3 asserts that if \(W\) is \(G\) R.LI then \(I(\cdot; \Omega)\) is \(G\) R.LSC.

**Open Problem 2.** Are the inverse implications true?

Our guess is that they are not generally true. Notice that in the proof of the theorem 4.2 there is a key equality (10); all the proof but this equality could be rewritten with the hypothesis that \(I\) is \(G\) R.LSC and the conclusion would be that \(W\) is \(G\) R.LI Analogous remarks can be made for the theorem 4.3 The key step in the proof of this theorem is the uniform convergence to zero of the term \(J_1\), defined in (18).

With the results from this section the formulation of the open problem 1 becomes clear. Indeed, we are interested to find a function \(W : J(G) \to R\) such that:

i) the integral functional which is generated by \(W\) is (left or right) lower semicontinuous;

ii) there is no quasiconvex function \(W^*\) with the property:

\[\forall u \in G^{1,\infty}(\Omega), \int_{\Omega} W(\nabla u(x)) \quad dx = \int_{\Omega} W^*(\nabla u(x)) \quad dx \,.
\]
Suppose that we have found a group $G$ and a function $W$ which bare the open problem 1. Then, according to proposition 4.2 $G \neq G^c$. If item ii) were true, then the integral functional $I$ generated by $W$ would be lower semi-continuous, therefore by theorem 4.2 $W$ would be $G$ L.LI, which contradicts the hypothesis.

**Open Problem 3.** Suppose that $G = G^c$. By remark 4.2 and proposition 4.2 if the integral functional $I$ is $G$ L.LSC then it is $G$ R.LSC. Find a potential $W$ and a group $G = G^c$ such that the functional $I$ generated by $W$ is $G$ R.LSC but not $G$ L.LSC.

If there is a function $W$ which responds to the open problem 3 then the open problem 2 would have a negative answer.

Ball introduced in 4, definition 3.2 and theorem 3.3, the notion of rank one convexity. In order to generalize this notion we introduce first the following definition.

**Definition 4.4.** $TG(\Omega)$ is the class of all vector fields $\eta \in C^\infty(\Omega, \mathbb{R}^n)$ such that the one parameter flow $\phi_t$, $t \in (a,b)$ with $a < 0 < b$, defined by

$$\dot{\phi}_t(x) = \eta(\phi_t(x)) \quad , \quad \phi_0 = \text{id}$$

lies in $G$, that is

$$\phi_t \in G \quad \forall \ t \in (a,b) \ .$$

**Theorem 4.4.** Let $W = W(F)$ be a $G$ L.LI, defined on an open neighbourhood of $G$. Then for any $F \in J(G)$ and for any $\eta \in TG$ we have the inequality:

$$\tag{22} \frac{\partial^2 W}{\partial F_{ij} \partial F_{mr}}(F)F_{ik}F_{mp} \int_\Omega \eta_{k,j} \eta_{p,r} \ dx \geq 0 \ .$$

**Proof.** Consider the function

$$I(t) = \int_\Omega W(F \nabla \phi_t(x)) \ dx \ .$$

This is a $C^2$ function which has a minimum at $t = 0$, according to hypothesis upon $W$. This fact implies that

$$\frac{\partial I}{\partial t}(0) = 0 \ , \quad \frac{\partial^2 I}{\partial t^2}(0) \geq 0$$

The first variation of $I$ has the form:

$$\frac{\partial I}{\partial t}(t) = \int_\Omega \frac{\partial W}{\partial F_{ij}}(F \nabla \phi_t)F_{ik}\phi_{tk,j} \ dx$$
hence for \( t = 0 \) we obtain
\[
\frac{\partial I}{\partial t}(0) = \frac{\partial W}{\partial F_{ij}}(F)F_{ik}\int_{\Omega} \eta_{k,j} \, dx .
\]

The integral from the right-handed member is obviously null because \( \eta \) has compact support in \( \Omega \), therefore we obtain a trivial identity.

The second variation of \( I \) is a sum of three terms:
\[
\frac{\partial^2 I}{\partial t^2}(0) = A + B + C ,
\]

\[(23)\]
\[
A = \int_{\Omega} \frac{\partial W}{\partial F_{ij}}(F)F_{ik}\eta_{k,l}\eta_{l,j} \, dx ,
\]

\[(24)\]
\[
B = \int_{\Omega} \frac{\partial W}{\partial F_{ij}}(F)F_{ik}\eta_{k,jl}\eta_{l} \, dx ,
\]

\[(25)\]
\[
C = \frac{\partial^2 W}{\partial F_{ij}\partial F_{mr}}(F)F_{ik}F_{mp} \int_{\Omega} \eta_{k,j}\eta_{p,r} \, dx .
\]

An integration by parts argument shows that \( A + B = 0 \) therefore we obtain \( C \geq 0 \), q.e.d.

The generalized rank-one convexity is defined further.

**Definition 4.5.** A \( C^2 \) function \( W = W(F) \) is \( G \) (left) rank one convex at \( F \in J(G) \) if for any \( \eta \in TG \) the relation \( (22) \) is true.

**Remark 4.3.** If we take \( G = Diff_0^\infty(\mathbb{R}^n) \) then the relation \( (22) \) becomes the Hadamard-Legendre inequality (see Hadamard \[10\], Ball \[5\] and the references therein). Indeed, for this group we have \( TG(\Omega) = C_0^\infty(\Omega, \mathbb{R}^n) \), hence for any \( \eta \in TG(\Omega) \) and \( F \in J(G) = GL_n(\mathbb{R}) \), the vector field \( F\eta \) belongs to \( TG(\Omega) \). Therefore the relation \( (22) \) can be written as:

\[(26)\]
\[
\frac{\partial^2 W}{\partial F_{ij}\partial F_{kl}}(F) \int_{\Omega} \eta_{i,j}\eta_{k,l} \, dx \geq 0
\]

for any \( \eta \in C_0^\infty(\Omega) \). An argument from Ball \[5\], proof of Theorem 3.4, allow us to consider piecewise affine vector fields \( \eta \). It can be shown that \( (26) \) implies the Legendre-Hadamard inequality:

\[(27)\]
\[
\frac{\partial^2 W}{\partial F_{ij}\partial F_{kl}}(F) a_i a_k b_j b_l \geq 0 ,
\]

for any vectors \( a, b \in \mathbb{R}^n \) (see also remark 6.2).

**Remark 4.4.** Same arguments as in the previous remark, but for the group of volume-preserving diffeomorphisms \( Diff_0^\infty(dx) \) show that the relation \( (24) \) implies the following inequality:

\[(28)\]
\[
\frac{\partial^2 W}{\partial F_{ij}\partial F_{mr}}(F)F_{ik}F_{mp} a_i a_p b_j b_r \geq 0 ,
\]
for any orthogonal vectors $a, b \in \mathbb{R}^n$, $a \cdot b = 0$.

Theorem 4.4 has a correspondent for the case of $G$ R.LI functions. We leave this theorem to the reader.

As a corollary we have:

**Proposition 4.4.** Any $G$ L.LI function $W = W(F)$ is $G$ rank one convex.

**Proof.** The result is obtained from definition 4.5 and theorem 4.4. □

5. Null lagrangians and group invariants

Let us consider the pair $(X,G)$ such that $X = X.G$ ($X = G.X$ respectively) and a functional $I$ representable in integral form:

$$I(u) = \int_{\Omega} L(x, u(x), \nabla u(x)) \, dx$$

for any $u \in X$.

**Definition 5.1.** A $C^2$ function $L : R^n \times R^n \times J(X) \rightarrow R$ is a $(X,G)$ invariant at left (abbreviated i.l.) if for any $u \in X$, $\phi \in G(\Omega)$ we have $I(u.\phi) = I(u)$. Right invariants are defined in a similar way. If $L = L(F)$ then we call it a homogeneous $(X,G)$ left invariant.

**Definition 5.2.** A $C^2$ function $L : R^n \times R^n \times J(G) \rightarrow R$ is a $G$ null lagrangian if for any $F \in J(G)$ and $\phi \in G(\Omega)$ we have $I(F.\phi) = I(F)$. If $L$ depends only on its third variable then is called a homogeneous $G$ null lagrangian.

**Remark 5.1.** Suppose that $G \subset X$ (equivalently $id \in X$) and $J(G) \subset X$. Then any $G$ invariant is a $G$ null lagrangian.

**Remark 5.2.** If we take $X = J(G).G$ then $L$ is a homogeneous $(X,G)$ invariant at left if and only if it is a $G$ null lagrangian at left.

**Proposition 5.1.** If $L = L(x, y, F)$ is a $(W^{1,\infty}(G, \Omega), G)$ i.l. then for any $x, y \in R^n$ the mapping $F \in J(G) \mapsto L(x, y, F)$ is a $G$ n.l.l.

**Proof.** Direct consequence of theorem 4.2. Indeed, from left continuity follows that $L(x, y, \cdot)$ and $-L(x, y, \cdot)$ are both GLLI □

5.1. **Examples.** Let us consider the case $X = W^{1,\infty}(GL_n(R), R^n)$ and $G = Diff_0^\infty(R^n)$. By proposition 4.4 any null lagrangian at left is a classical null lagrangian. The class of null lagrangians is known (see Ball, Currie & Olver [6] or Olver & Sivaloganathan [15]). For $n = 3$ for example, any homogeneous null lagrangian is a linear combination of $F_{ij}$, $adj \ F_{ij}$ and $det \ F$.

In the particular case that we have chosen the homogeneous null lagrangians are also invariants at left. Indeed, take $\Omega$ simply connected and
smooth, \( u \in X \) and compute \( I(u; \Omega) \), where \( I \) is generated by a null lagrangian. We obtain:

\[
\int_\Omega \det \nabla u \, dx = |u(\Omega)|,
\]

\[
\int_\Omega (\nabla u)_{ij} \, dx = \int_{\partial \Omega} u_{,i} n_j \, ds,
\]

\[
\int_\Omega \text{adj} (\nabla u)_{ij} \, dx = \int_{\partial \Omega} (u \wedge n)_{ij} \, ds.
\]

We see that generally \( I(u; \Omega) \) depends only on \( \Omega \) and the value of \( u \) on \( \partial \Omega \). These rest the same under a composition at left with any \( \phi \in \text{Diff}^\infty_0(\Omega) \).

These considerations prove proposition 1.2 from the introduction.

Let us choose now \( X = \text{Diff}^\infty_0(\mathbb{R}^n) \) and \( G = [SL_n(\mathbb{R})] \). The (integral) invariants given by the classical null lagrangians are trivial in this case. An easy \([SL_n(\mathbb{R})]\) invariant at left turns to be \( L(F) = \log \det F \). Indeed, consider \( u \in X \) and \( \phi \in G(\Omega) \). We have then

\[
I(u, \phi^{-1}; \Omega) = \int_\Omega (\log \det \nabla u(x) - \log \det \nabla \phi(x)) \det \nabla \phi(x) \, dx = I(u; \Omega).
\]

Unfortunately the restriction of \( L \) to \( SL_n(\mathbb{R}) \) equals 0, the most trivial null lagrangian.

The classical null lagrangians are all derived from the determinant. The determinant function \( f \mapsto \det F \) transforms matrix multiplication into number multiplication and basically this is the reason which makes determinant to be a null lagrangian.

The following theorem shows an interesting change of behaviour in the case of \( G \) null lagrangians with \( J(G) \subset SL_n \). In this situation we find a class of non-classical null lagrangians which transform matrix multiplication into number addition.

Unfortunately we have not been able to find a group \( M \subset SL_n \) which has non-trivial characters and \([M]\) acts transitively on \( \mathbb{R}^n \). However, the following theorem might offer an illustration of a general phenomenon concerning non-classical null lagrangians.

**Theorem 5.1.** Let \( M \subset SL_n(\mathbb{R}) \) be a Lie subgroup of the linear group of \( n \times n \) matrices with real coefficients and positive determinant. By a character of \( M \) we mean any homeomorphism \( \chi : M \to (0, +\infty) \) from \( M \) to the multiplicative group of \( \mathbb{R} \). For any character \( \chi \) of \( M \) the function:

\[
W : M \to \mathbb{R}, \quad W(F) = \log \chi(F)
\]

is a \([M]\) null lagrangian at left.

**Proof.** Let us consider \( \phi \in [M](E), F \in M \) and \( \eta \in T[M], \text{supp} \eta \subset E \). We denote by \( \phi_t \) the one parameter group generated by \( \eta \) and we introduce
the function:
\[ g(t) = I(\phi_t; E) = \int_E W(\nabla \phi_t(x)) \, dx \]

We have then:
\[ g(t_1 + t_2) = \int_E W(\nabla \phi_{t_2}(\phi_{t_1}(x))) \, dx + \int_E W(\nabla \phi_{t_1}(x)) \, dx \]

By the change of variables \( y = \phi_{t_1}(x) \) in the first integral and taking account of the equality \( \det \nabla \phi_t(y) = 1 \) we obtain:
\[ g(t_1 + t_2) = g(t_1) + g(t_2) \]

therefore \( g \) is linear.

The function \( g \) is also differentiable. We compute the first derivative of \( g \) at \( t = 0 \) and we obtain:
\[ \frac{\partial g}{\partial t}(0) = \int_E \frac{\partial}{\partial F} \log \chi(1) \nabla \eta(x) \, dx = 0 \]

therefore \( g \) is a constant function. From here we derive that \( g \) is constant (provided that the exponential map covers a neighbourhood of the identity, which is obvious). \( \square \)

**Remark 5.3.** Because \([M]\) is a local group, it follows in particular that \([M]^c = [M]\). Therefore, by proposition 4.2 any \([M]\) n.l.l. is a \([M]\) n.l.r.

5.2. **Properties of null lagrangians.**

**Theorem 5.2.** The following statements are true:

(i) If \( I(\cdot; \Omega) \) is left continuous then \( W \) is a \( G \) n.l.l.

(ii) Suppose that \( W \) is a \( C^2 \) function defined over an open neighbourhood of \( J(G) \). If \( W \) is a \( G \) n.l.l. then for any \( F \in J(G) \) and \( \eta \in TG(\Omega) \) we have the equality:

\[ \frac{\partial^2 W}{\partial F_{ij} \partial F_{mr}}(F) F_{ik} F_{mp} \int_{\Omega} \eta_{k,j} \eta_{p,r} \, dx = 0 \]

**Proof.** The first statement comes from theorem 4.2 applied for \( I(\cdot; \Omega) \) and \(-I(\cdot; \Omega)\). The second statement is a consequence of the theorem 4.4. \( \square \)

**Remark 5.4.** We particularly see that if \( I(\cdot; \Omega) \) is left invariant then \( W \) is a \( G \) n.l.l.

**Remark 5.5.** The equality (30) becomes the Hadamard-Legendre equality if \( G = Diff_0^\infty(\mathbb{R}^n) \). In this case (30) is equivalent to

\[ \frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}}(F) = -\frac{\partial^2 W}{\partial F_{il} \partial F_{kj}}(F) \]
Generally, if for any \( F \in J(G) \) we have \( F^T \in J(G) \) (where \( F^T \) is the transpose of \( F \)), then (31) implies the classical Hadamard-Legendre equality:

\[
\frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}}(F)_{ab} = 0
\]

for any \( a, b \in \mathbb{R}^n \).

Let us denote by \( \text{Exp}_G(\Omega) \) the class of all \( \phi \in G(\Omega) \) for which there exist \( \eta \in T_G(\Omega) \) and \( \tau \in \mathbb{R} \) such that \( \phi = \phi_{\tau, \eta} \), where \( \phi_{s, \eta} \) is the one parameter flow generated by \( \eta \).

**Theorem 5.3.** Suppose that \( W \) is a \( C^2 \) function defined over an open neighbourhood of \( J(G) \). Then the following statements are true:

(i) If \( W \) satisfies (31) for any \( F \in J(G) \) then for any \( u \in W^{1,\infty}(G, \Omega) \) and \( \eta \in T_G(\Omega) \) we have the equality:

\[
\int_\Omega \frac{\partial}{\partial x_j} \left( \frac{\partial W}{\partial F_{ij}}(\nabla u(x)) \right) u_{i,p}(x) \eta_p(x) \, dx = 0 .
\]

(ii) If \( W \) satisfies (32) for any \( u \in W^{1,\infty}(G, \Omega) \) and \( \eta \in T_G(\Omega) \) then we have

\[
I(u, \phi; \Omega) = I(u; \Omega)
\]

for any \( u \in W^{1,\infty}(G, \Omega) \) and \( \phi \in \text{Exp}_G(\Omega) \). In this case we say that \( I(\cdot; \Omega) \) is exponentially \( G \) invariant at left.

(iii) If \( I(\cdot; \Omega) \) is exponentially \( G \) invariant at left then for any \( F \in J(G) \) and \( \phi \in \text{Exp}_G(\Omega) \) we have:

\[
\int_\Omega W(F \nabla \phi(x)) \, dx = |\Omega| W(F) .
\]

In this case we say that \( W \) is exponentially \( G \) left invariant.

(iv) If \( W \) is exponentially \( G \) left invariant then \( W \) satisfies the equality (34) for any \( F \in J(G) \) and \( \eta \in T_G(\Omega) \).

**Proof.** In order to prove (i) let us denote by \( A \) the integral from (32). We have the equality:

\[
A = \int_\Omega \frac{\partial^2 W}{\partial F_{ij} \partial F_{kl}}(\nabla u(x)) u_{i,p}(x) \eta_p(x) u_{k,jl}(x) \, dx .
\]

From (31) we see that \( A = 0 \).

For (ii) let us take \( \eta \in W^{1,\infty}(G, \Omega) \) and denote by \( \phi_t \) the one parameter flow generated by \( \eta \). Consider the function \( g(t) = I(u, \phi_t^{-1}; \Omega) \). After some calculations based on integration by parts we obtain the equality:

\[
\frac{\partial g}{\partial t}(t) = \int_\Omega \frac{\partial}{\partial x_j} \left( \frac{\partial W}{\partial F_{ij}}(\nabla (u, \phi_t^{-1}))(x) \right) (u, \phi_t^{-1})_{i,p}(x) \eta_p(x) \, dx .
\]

From the hypothesis the right-handed member of the previous equality equals 0, therefore \( g \) is a constant function and (32) is proven.
For (iii) take \( u(x) = F x \) in (33).

For (iv) remark that the function \( g \), previously defined, is constant. Therefore it’s second variation at \( t = 0 \) is null. We proceed as in the proof of the theorem 4.4 in order to compute this second variation and we finally obtain (30).

\[ \square \]

**Open Problem 4.** Find the class of (homogeneous) \( G \) null lagrangians.

In particular, are there any \([SL_n]\) non-classical null lagrangians?

### 5.3. Polyconvex functions.

**Definition 5.3.** A function \( W = W((x, y, F)) \) is called \( G \) (left or right) polyconvex if there is a continuous function \( g : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \), convex in the third argument and functions \( w_1, \ldots, w_m \) \( G \) null lagrangians (at left or right) such that for any \( F \in G \) we have

\[
W(F) = g(w_1(F), \ldots, w_m(F)).
\]

**Theorem 5.4.** Any \( G \) (left or right) polyconvex function is \( G \) LLI (or r.LI). Therefore, if \( W \) is \( G \) right polyconvex, then \( I(\cdot; \Omega) \) is R.LSC.

**Proof.** For the first part of the theorem the proof is the same in the "right" or "left" cases. We apply the Jensen inequality to the function \( g \):

\[
\frac{1}{|E|} \int_{E} W(x_0, y_0, F \nabla \phi(x)) \, dx \geq g(x_0, y_0, \frac{1}{|E|} \int_{E} w(F \nabla \phi(x)) \, dx) .
\]

The functions \( w_j \) are null lagrangians, therefore we have:

\[
g(x_0, y_0, \frac{1}{|E|} \int_{E} w(F \nabla \phi(x)) \, dx) = g(x_0, y_0, w(F)) = W(x_0, y_0, F).
\]

The second part of the theorem is a consequence of the theorem 4.3. \( \square \)

Polyconvexity, in the classical sense, is of major interest because it is a local condition. Indeed, the class of (classical) null lagrangians was determined by Ericksen [9] and it corresponds to the class of \( Diff_0^\infty \) n.l.l. Therefore \( W \) is polyconvex if and only if it has the form from definition 5.3, where \( w_i(\cdot) \) are known functions (for example, if \( n = 3 \) then any \( w_i(F) \) is a linear combination of \( F_{kl}, (ad F)_{kl} \) and \( \det F \)).

If one solves the open problem 4, the next problem to solve is the following:

**Open Problem 5.** Find the class of all \( G \) polyconvex functions. Give sufficient local or global conditions for a function \( W \) to be \( G \) polyconvex.

Indeed, the knowledge of the class of \( G \) n.l.l. functions would transform the \( G \) polyconvexity condition into a local one. In the case \( G = G^e \) the (right) lower semicontinuity of \( I(\cdot; \Omega) \) can be proved from the polyconvexity of \( W \), which becomes easy to check if the \( G \) n.l.l. functions are known.
6. Conclusions

We have introduced in this paper two notions: (left or right) lower invariance (abbreviated LI) and (left or right) lower semicontinuity (abbreviated LSC). These notions describe the behaviour of integral functionals of the form:

\[ I(u; \Omega) = \int_{\Omega} W(x, u(x), \nabla u(x)) \, dx \]

under inner or outer variations in a group of diffeomorphisms \( G(\Omega) \).

The first notion is a generalization of quasiconvexity in the sense of Morrey whilst the second one is weaker than the classical lower semicontinuity.

For a given group \( G \), the difference between \( G \) LI and quasiconvexity of the potential \( W \) is the same as the difference between \( G \) LSC and the classical LSC. For example, let us consider the case \( G = [SL_n(R)] \), of the group of volume preserving diffeomorphisms. According to proposition 4.2, in this case \( G \) left and right LI are equivalent. Theorem 4.2 shows that \( G \) left LSC implies left LI hence right LI; by theorem 4.3 right LI implies right LSC. We conclude that for general complete groups \( G \) right LSC is weaker than \( G \) left LSC.

If \( G \) right LSC would be equivalent to classical LSC then (at least for our example) \( G \) LI would be equivalent to quasiconvexity. Indeed, as earlier we use theorem 4.3 to deduce that right LI implies right LSC; by hypothesis right LSC implies classical LSC; classical LSC implies quasiconvexity. Therefore right LI implies quasiconvexity. The inverse implication is always true, by proposition 4.4.

\( G \) left or right LSC is in fact classical LSC when we restrict the class of admissible sequences to sequences obtained by inner or outer variations. Therefore any potential which generates integral functionals which are left or right but not classical lsc gives an indications about the local behaviour of the group \( G \).

In this paper we address several open problems. For reader’s convenience we rewrite them here:

**OP1:** Find a group \( G \) and a function \( W \) which is \( G \) right LI but not \( G \) left LI

**OP2:** Does \( G \) left LI generally imply \( G \) left LSC ? Does \( G \) right LSC generally imply \( G \) right LI ?

**OP3:** Find a potential \( W \) and a group \( G = G^c \) such that the functional \( I \) generated by \( W \) is \( G \) right LSC but not \( G \) left LSC

**OP4:** Find the class of (homogeneous) \( G \) null lagrangians. In particular, are there any \( [SL_n] \) non-classical null lagrangians?

**OP5:** Find the class of all \( G \) polyconvex functions. Give sufficient local or global conditions for a function \( W \) to be \( G \) polyconvex.
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