Two Eigenvectors for the Price of One

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The 2 × 2 case. It all started with a student’s mistake. In a quiz, students were asked to find eigenvalues and eigenvectors of the matrix

\[ A = \begin{pmatrix} 4 & 1 \\ 2 & 5 \end{pmatrix} \] .

One student correctly found the two eigenvalues \( \lambda_1 = 3 \) and \( \lambda_2 = 6 \). She also correctly computed the matrix \( B_1 = A - \lambda_1 I = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \) (where \( I \) is the identity matrix), only instead of solving the homogeneous system \( B_1 v_1 = 0 \), to find the corresponding eigenvector \( v_1 = \begin{pmatrix} x \\ y \end{pmatrix} \) for \( \lambda_1 = 3 \), she picked a column of \( B_1 \) and (wrongly) declared that this was the required eigenvector, instead of picking, say, \( v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \).

To my surprise, I realized that the chosen vector, \( v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \), is actually an eigenvector for the second eigenvalue \( \lambda_2 = 6! \)

At first I decided this might be a coincidence, but it turns out to be true in general if \( A \) has distinct eigenvalues. A direct check is a good exercise for a motivated student.

We can, however, seek a deeper understanding of the situation and, in doing so, discover a connection with an important result in linear algebra. Namely, assuming \( A \) has distinct eigenvalues \( \lambda_1 \neq \lambda_2 \), if we call \( B_2 = A - \lambda_2 I \) the matrix we use to find the second eigenvector, then the fact that the columns of \( B_1 \) turn out to be solutions \( v_2 \) of \( B_2 v_2 = 0 \) is equivalent to saying that the matrix product \( B_2 B_1 \) is equal to zero, that is, the zero matrix. But if we think a little, we realize that

\[ B_2 B_1 = (A - \lambda_2 I)(A - \lambda_1 I) = A^2 - (\lambda_1 + \lambda_2)A + \lambda_1 \lambda_2 I \]

is equal to \( p(A) \), where \( p(\lambda) \) is the characteristic polynomial of \( A \), \( p(\lambda) = \lambda^2 - (\text{trace } A)\lambda + (\text{determinant } A) \). And the fact that \( p(A) \) is indeed zero is the celebrated Cayley–Hamilton theorem, stating that every square matrix satisfies its own characteristic equation; see, for example, [1] or [2].

Thus, for a 2 × 2 matrix \( A \) with distinct eigenvalues, the computations to find the first eigenvector \( \lambda_1 \) will also provide the second eigenvector for free! We just need to pick any nonzero column vector of \( B_1 = A - \lambda_1 I \).

If \( A \) has a double eigenvalue \( \lambda_1 = \lambda_2 \), then by Cayley–Hamilton \( p(A) = B_2^2 = 0 \), and either \( B_1 = 0 \), and \( A \) is a multiple of the identity, or \( B_1 \neq 0 \), in which case the eigenspace has dimension one, and any nonzero column of \( B_1 \) will provide an eigenvector. To also get a generalized eigenvector (see [1, p. 66]), since \( B_2^2 = 0 \), we just pick any vector \( w \) not in the kernel of \( B_1 \) and compute \( v = B_1 w \). Then \( v \) will be an eigenvector of \( A \), with generalized eigenvector \( w \): \( Av = \lambda v \) and \( Aw = \lambda w + v \).
For example, for the matrix $A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$, we have $\lambda_1 = \lambda_2 = 3$, and $B = A - 3I = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$; pick $w = (1, 0)$, and $v = Bw = (-1, -1)$.

**What about not solving any systems?** It turns out one can use this idea to find the eigenvectors of a $2 \times 2$-matrix $A$ without solving any systems! Indeed, if the eigenvalues $\lambda_1, \lambda_2$ are distinct, compute $B_1 = A - \lambda_i I$; any nonzero column of $B_1$ is an eigenvector for $\lambda_i$, and vice-versa. If $\lambda_1 = \lambda_2$ and $B_1 = 0$, then then $A$ is a multiple of the identity; pick any two linearly independent vectors. If $\lambda_1 = \lambda_2$ and $B_1 \neq 0$, then $B_1^2 = 0$, and any nonzero column of $B_1$ will generate the eigenspace.

**The $3 \times 3$ case.** Let us first assume that a $3 \times 3$ matrix $A$ has three distinct eigenvalues $\lambda_i$ and compute the corresponding matrices $B_i = A - \lambda_i I$, $i = 1, 2, 3$.

Then the Cayley–Hamilton theorem will imply that $B_1B_2B_3 = 0$, so the nonzero columns of the product $B_2B_3$ will be eigenvectors of $\lambda_1$. For example, the matrix

$$A = \begin{pmatrix} 7 & -4 & -5 \\ 3 & -2 & -3 \\ 6 & -4 & -4 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = -2$. Here

$$B_2 = \begin{pmatrix} 5 & -4 & -5 \\ 3 & -4 & -3 \\ 6 & -4 & -6 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 9 & -4 & -5 \\ 3 & 0 & -3 \\ 6 & -4 & -2 \end{pmatrix}, \quad B_2B_3 = \begin{pmatrix} 3 & 0 & -3 \\ -3 & 0 & 3 \\ 6 & 0 & -6 \end{pmatrix}.$$  

The first and the last columns of $B_2B_3$ provide an eigenvector for $\lambda_1 = 1$.

Assume now $A$ has a simple eigenvalue $\lambda_1$ and a double eigenvalue $\lambda_2$. Then we have two sub-cases.

1. **The eigenspace of $\lambda_2$ has dimension two.** Then the minimal polynomial of $A$ is $(\lambda - \lambda_1)(\lambda - \lambda_2)$, so that not only $B_2^2B_1 = 0$, but also $B_2B_1 = 0$; therefore, the nonzero columns of $B_1$ are eigenvectors of $\lambda_2$. Moreover, since $\lambda_1$ is simple, $\dim \ker(B_1) = 1$, and therefore the columns of $B$ generate the entire eigenspace. So here you get three eigenvectors for the price of one! For example,

$$A = \begin{pmatrix} 4 & -9 & -6 \\ -6 & 7 & 6 \\ 12 & -18 & -14 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = -2$. Here

$$B_1 = \begin{pmatrix} 3 & -9 & -6 \\ -6 & 6 & 6 \\ 12 & -18 & -15 \end{pmatrix}$$

can be used to find the eigenvector $v_1 = (1, -1, 2)$, corresponding to $\lambda_1 = 1$. Moreover, a direct check shows that all columns of $B_1$ are eigenvectors for $\lambda_2 = \lambda_3 = -2$, and the first two columns of $B$ (suitably divided by 3) provide the eigenvectors $v_2 = (1, -2, 4)$ and $v_3 = (-3, 2, -6)$ for $\lambda_2$.

2. **The eigenspace of $\lambda_2$ has dimension one.** Then the characteristic polynomial $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)^2$ is also the minimal polynomial. Hence, $B_2^2B_1 = 0$ but $B_2B_1 \neq 0$. As in the first case, the column space of $B_1$ has dimension 2. Moreover, in this case $\dim \ker(B_2) = 1$ and $\dim \ker(B_2^2) = 2$. Consequently, the column space of $B_1$ generates the entire set of eigenvectors and generalized eigenvectors of $\lambda_2$. To find an eigenvector and generalized eigenvector for $\lambda_2$ we proceed as follows:

- Pick a nonzero column $w$ of $B_1$ that is not an eigenvector of $B_2$, that is, such that $B_2w$ is nonzero.
• Compute \( v = B_2 w \). Then \( B_2 v = B_2^2 w = 0 \), since \( w \in \text{col}(B_1) = \ker(B_2^2) \).

We conclude that \( v \) is an eigenvector, and \( w \) is the corresponding generalized eigenvector, for \( \lambda_2 \).

As an example, consider

\[
A = \begin{pmatrix} 5 & -10 & -7 \\ -6 & 7 & 6 \\ 13 & -19 & -15 \end{pmatrix},
\]

with eigenvalues \( \lambda_1 = 1 \) and \( \lambda_2 = \lambda_3 = -2 \). Here

\[
B_1 = A - \lambda_1 I = \begin{pmatrix} 4 & -10 & -7 \\ -6 & 6 & 6 \\ 13 & -19 & -16 \end{pmatrix}.
\]

As usual, we use \( B_1 \) to find an eigenvector for \( \lambda_1 \), say, \( v_1 = (1, -1, 2) \).

Next, a direct check shows that not all columns of \( B_1 \) are eigenvectors for \( \lambda_2 \) (in fact, neither column of \( B_1 \) is). This tells us that the eigenspace for \( \lambda_2 \) has dimension 1 (since otherwise all columns of \( B_1 \) would be eigenvectors). To proceed, we can therefore pick any nonzero column of \( B_1 \), say, the first column: \( w = \begin{pmatrix} 4 \\ -6 \\ 13 \end{pmatrix} \), and calculate \( v_2 = B_2 w = \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} \). We conclude that \( v_2 \) is an eigenvector for \( \lambda_2 \), with corresponding generalized eigenvector \( w \). In the basis \( \{ v_1, v_2, w \} \), the matrix \( A \) will have (Jordan) canonical form.

**A triple eigenvalue.** If \( A \) is a \( 3 \times 3 \)-matrix with a triple eigenvalue \( \lambda \), we have three possibilities, according to the dimension \( k \) of the eigenspace of \( \lambda \) (let us again call \( B = A - \lambda I \)):

1. \( k = 3 \). Then \( A \) is a multiple of the identity.
2. \( k = 1 \). Then \( B^2 \neq 0 \) and has rank one; any nonzero column of \( B^2 \) will provide an eigenvector. It is not hard to find the generalized eigenvectors.
3. \( k = 2 \). Then \( B \neq 0, B^2 = 0 \), and \( B \) has rank one. Any nonzero column of \( B \) will be an eigenvector of \( A \). Unfortunately, here we still need a second eigenvector, and there seems to be no other choice but to choose a second nonzero vector in the kernel of \( B \) that is not in the column space of \( B \).

**Can we avoid solving any systems for a \( 3 \times 3 \)-matrix \( A \)?** It turns out we can, in all cases but one. Namely, If eigenvalues \( \lambda_i \) are distinct, compute \( B_i = A - \lambda_i I, i = 1, 2, 3 \). Any nonzero column of \( B_i B_j \) is an eigenvector for the third \( \lambda_k \). If \( \lambda_1 \neq \lambda_2 = \lambda_3 \), then any nonzero column of \( B_2^2 \) spans the eigenspace for \( \lambda_1 \). Moreover, If \( B_2 B_1 = 0 \), then the eigenspace of \( \lambda_2 \) has dimension two, and is spanned by any two linearly independent columns of \( B_1 \). If \( B_2 B_1 \neq 0 \), then the eigenspace of \( \lambda_2 \) is one-dimensional, and is spanned by any nonzero column of \( B_2 B_1 \). The ideas sketched earlier for a triple eigenvalue show how to obtain the eigenvectors in all cases but one.

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**Summary.** Starting from a mistake done by a student, we discover an unexpected method of finding both eigenvectors for a \( 2 \times 2 \) matrix with distinct eigenvalues in a single computation. We discuss a connection with the Cayley-Hamilton theorem, and show the corresponding generalization for a \( 3 \times 3 \) matrix.

**References**

[1] Axler, S. (1997). *Linear Algebra Done Right*. New York: Springer; p. 173 and p. 207.
[2] Crilly, T. (1992). A Gemstone in Matrix Algebra *Math. Gazette*. 76(475): 182–188. doi.org/10.2307/3620391.