We consider the process of pumping charge through an open quantum system, motivated by the example of a quantum dot with strong repulsive or attractive electron-electron interaction. Using the geometric formulation of adiabatic nonunitary evolution put forward by Sarandy and Lidar, we derive an encompassing approach to ideal charge measurements of time-dependently driven transport, that stays near the familiar approach to closed systems. Following Schaller, Kießlich and Brandes we explicitly account for a meter that registers the transported charge outside the system. The gauge freedom underlying geometric pumping effects in all moments of the transported charge emerges naturally as the calibration of this meter. Remarkably, we find that geometric and physical considerations cannot be applied independently as done in closed systems: physical recalibrations do not form a group due to constraints of positivity (Bochner’s theorem). This complication goes unnoticed when considering only the average charge but it is relevant for understanding the origin of geometric effects in the higher moments of the charge-transport statistics. As an application we derive two prominent existing approaches to pumping, based on full-counting-statistics (FCS) and adiabatic-response (AR), respectively, from our approach in a transparent way. This allows us to reconcile all their apparent incompatibilities, in particular the puzzle how for the average charge the nonadiabatic stationary-state AR result can exactly agree with the adiabatic nonstationary FCS result. We relate this and other difficulties to a single characteristic of geometric approaches to open systems: the system-environment boundary can always be chosen to either include or exclude the ideal charge meter. This leads to a physically motivated relation between the mixed-state Berry phase and the entirely different geometric phase of Landsberg.

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I. INTRODUCTION

A. Geometric phases in open systems

In condensed matter physics geometric and topological quantities are generally appreciated for their robustness against perturbations. In physics, topological quantities often arise from geometric ones and the former have been used to classify phases of isolated solids at zero temperature. Recently, this topological classification has also been addressed for finite temperature and nonequilibrium, which is more challenging since it requires a description in terms of mixed-state density operators rather than pure-state vectors. There is, however, a further complicating factor for such open systems: due to the diversity of kinetic equations that describe their nonunitary dynamics there is a much wider variety of geometric formulation than for closed systems which are all described by the Schrödinger equation. Despite this strong impetus to establish relations between geometric approaches in open systems, little attention has been given to such comparisons so far.

An important issue that is currently being addressed in several of the above works is to find observables that are sensitive to given geometric and topological quantities that one defines for an open system. Some works follow an opposite line of questioning by first focusing on observable transport effects and then expressing these in geometric terms. Identifying geometric quantities this way, even in relatively simple open systems, seems an essential first step towards finding effects indicative of topological order. In closed systems, transport due to slow periodic driving, was found to probe geometry and topology and one might expect the same for open systems. However, as we recently highlighted, the geometric phases that one encounters for pumping through open quantum systems do not relate to the reduced quantum state of the system (obtained by tracing out the environment): instead, external observables – or related generating functions – accumulate these geometric phases. Another issue concerns two prominent open-system approaches that deal with pumping in this way, the adiabatic-response approach (AR) and the full counting statistics (FCS): Despite their completely opposite features they were shown to give identical results for the average pumped charge. Reconciling these features requires a deeper understanding of the distinction between geometric phases in open and closed quantum systems and motivates the present work.
FIG. 1: Equivalent ways of accounting for charge pumping through an open system (circle) with coupling $\Gamma$. (a) Meter outside: the meter is considered as part of the environment, together with the electrodes, all of which are traced out (hatched). (b) Meter inside: the meter is considered as part of a composite open system: only the electrodes are integrated out (hatched). (c) Total system with nothing integrated out.

The key questions can be outlined further, guided by the role of a meter that registers the charge $\hat{N}$ without backaction$^1$: either the meter lies outside (a) or inside (b) the open-system boundary as sketched in Fig. 1.

(a) Meter outside. Steady-state pumping through an open quantum system has been studied in the way sketched in Fig. 1(a) in several works.$^{23,25,26,31–38}$ In Ref. 23 it was shown, that in this approach geometric pumping effects do not result from a geometric phase of the state (reduced density operator). Indeed, when computing this phase in the steady state limit, it turns out to be zero. Instead, the pumped charge per period was shown to equal a single Landsberg-type geometric phase$^{39–43}$ that arises naturally when computing the adiabatic response$^{24}$ (AR) of the current that is linear in the velocity of the driving parameters.

For concreteness, we show two examples for such charge pumping in Fig. 2 which can both be accessed with this formalism. Interestingly, in these examples pumping is generated by strong interaction effects,$^{22,23,25,26,31}$ which in the weak coupling limit can readily be treated using the open-system (density-operator) approach. The geometric origin of the pumping was shown to lie in the nonuniqueness of assigning an operator to an observable measured outside$^2$ the open system, i.e., in the hatched area in Fig. 2(a): One is free to add to the charge operator an offset $g$ that depends on the driving parameters, collected in a vector $\mathbf{R}$,

$$\hat{N}_g(\mathbf{R}) = \hat{N} + g(\mathbf{R}) \mathbb{1},$$

without altering the pumped charge$^{23,25,26}$ Physically, this is a literal gauge, i.e., a recalibration of the scale used for reading out a charge meter. These physical gauge transformations fully explain the emergence of a geometric first moment $\mathcal{M}^{(1)} := \langle \hat{N} \rangle (T) - \langle \hat{N} \rangle (0)$ upon driving the parameters during one period $T$. However, driven transport also involves geometric effects for higher moments $\mathcal{M}^{(k)}$ such as the pumping noise$^{44}$ related to $\langle \hat{N}^2 \rangle$. The calibration freedom $\langle 1 \rangle$ does not explain their occurrence and so far it has remained unclear which further physical gauge freedom is responsible for this.

One should note that the computation of the average transported charge requires a memory-kernel for the observable (in addition to a state-evolution kernel) because the reservoirs and meter have been integrated out. It was shown that through this latter observable kernel the gauge freedom Eq. $\langle 1 \rangle$, responsible for the geometric pumping phase, enters the problem. Also, the nonzero value of this phase could be directly tied to the retarded response (“lag”) of the current through the system to the driving parameters as measured by an outside observable: the pumping current is therefore nonadiabatic.

As highlighted in Ref. 23 the AR approach has the advantage of leading to technically simple calculations, producing such examples as Fig. 2 while at the same time tying geometric quantities to concrete transport physics. For example, the geometric connection equals the physical pumping current. However, even without going into detail, one notices that the features of this approach, listed in Table 1, go against virtually all familiar ideas about the appearance of geometric phases in closed quantum systems (adiabaticity, quantum-state phase accumulation, Berry-Simon type connection, etc.). This paper addresses the important question how and to what extent this familiar picture can be restored by starting from a different, yet equivalent, formulation. In particular, how does the phase-accumulation of the charge observable emerge from the phase of some quantum state?

(b) Meter inside. Another prominent approach to pumping proceeds as sketched in Fig. 1(b): the meter is part of the open system (not integrated out). The FCS approach keeps track of the meter by a formal counting-field parameter $\phi$, a statistical tool.$^{45}$ The charge-transfer statistics is computed from a generating function $Z^\phi$ via a generating-operator $\rho^\phi$ which describes the first cumulant $C^{(1)} := \langle \hat{N} \rangle (T) - \langle \hat{N} \rangle (0)$ but also all higher cumulants $C^{(k)}$. Sinitsyn and coworkers applied this approach to the problem of pumping$^{22,23,25,26,31,33,36,43–47}$ and by an adiabatic decoupling obtained a result for the cumulant generating function $C^{(k)}$, which is nonstationary for fixed parameters. For each continuous value of $\phi$ it has a pumping contribution that is a geometric phase of the Berry-Simon type.$^{51,52}$ The geometric part of the first cumulant $C^{(1)}$ is the measurable pumped charge and it is plotted in Fig. 2 for our example.

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1 Integrating out the meter is equivalent to having no meter at all.

2 Applied to system observables this also covers the Kato-projection approach of Avron, Fraas, and Graf.$^{23,27,34}$
covered by an infinitesimal driving cycle 
result (obtained in either way) for two regimes: (b) Repulsive interaction $U$ discussed in more detail in the paper. The two plots illustrate the $C^{(k)}$ of the transported charge which shows that they all have a geometric contribution. However, their properties have not been discussed much. In particular, as we will see here, the geometric gauge transformations may transform proper generating functions to functions that violate Bochner’s positivity criterion implying their inverse Fourier transforms take negative values and do not correspond to charge-transfer probability distributions anymore, cf. Ref. 55. The nontrivial positivity restriction is typical of the open-system (density-operator) setting and its interplay with geometric considerations requires clarification if gauge transformations affecting the higher moments are to be understood physically.

A further question that received little attention so far relates to the full set of moments / cumulants, i.e., the entire transport process. It is an advantage of the FCS approach that it gives a formal expression for all cumulants $C^{(k)}$ of the transported charge which shows that they all have a geometric contribution. However, their properties have not been discussed much. In particular, as we will see here, the geometric gauge transformations may transform proper generating functions to functions that violate Bochner’s positivity criterion implying their inverse Fourier transforms take negative values and do not correspond to charge-transfer probability distributions anymore, cf. Ref. 55. The nontrivial positivity restriction is typical of the open-system (density-operator) setting and its interplay with geometric considerations requires clarification if gauge transformations affecting the higher moments are to be understood physically.

FIG. 2: (a) Weakly coupled, strongly interacting quantum dot hosting up to two electrons as introduced in Sec. II. The time-dependently driven parameters $R = (V_{\text{gate}}/T, V_{\text{bias}}/T)$ are the gate- and bias-voltage. The charge pumped through such an open system in steady state can be equivalently computed using either of two very different approaches: the geometric part (g) of the transported charge equals $\mathcal{M}^{(1)}_{g} = \int d\mathbf{R} \mathcal{A}$ [AR approach, Eq. (84)] = $\psi^{(1)}_{g}$ $= \int d\mathbf{R} \partial_{\phi} \mathcal{A}^\phi |_{\phi = 0}$ [FCS approach, Eq. (81)], as discussed in more detail in the paper. The two plots illustrate the result (obtained in either way) for two regimes: (b) Repulsive interaction $U = |U| > 0$. Plot (a) is the pumping curvature, the charge pumped through the quantum dot per unit area covered by an infinitesimal driving cycle $\mathcal{C}$, as function of the center $\mathbf{R}$ of the cycle. The pumping signal is nonzero only at points where parametric resonances of the system cross (dashed Coulomb-diamond edges) and provides direct information on spin and coupling asymmetry that is not provided by the stationary transport. (c) Attractive interaction $U = -|U| < 0$, experimentally realized only recently. At zero bias $V_{\text{bias}} = 0$ a single two-lobed pumping response emerges at the electron-pair resonance $V_{\text{gate}} = -|U|/2$ (vertical dashed line). Parameters for (b) and (c): $\Gamma_{L}/\Gamma_{R} = 0.5$, $|U| = 16.67 T \gg \Gamma_{L} + \Gamma_{R}$, and temperature $T$.

The features of the FCS approach listed in Table I are similar to the closed-system geometric approach. However, this similarity is formal: it is not clear to which physical state the adiabaticity and geometric phase refer. This is a serious concern –raised already in Ref. 23– since by its adiabatic decoupling approximation the FCS approach to pumping (correctly) produces the same result as the AR approach, in which the current is undeniably nonadiabatic, see Fig. 2. In fact, all the features of the FCS and AR approach listed in Table I seem to be in conflict. Nevertheless, for the first moment / cumulant they produce identical results within their limits of applicability as shown in detailed analyses. These latter works, however, do not give a simple explanation or, even better, a single systematic approach that rationalizes all their differences in a simple and transparent way.

A further question that received little attention so far relates to the full set of moments / cumulants, i.e., the entire transport process. It is an advantage of the FCS approach that it gives a formal expression for all cumulants $C^{(k)}$ of the transported charge which shows that they all have a geometric contribution. However, their properties have not been discussed much. In particular, as we will see here, the geometric gauge transformations may transform proper generating functions to functions that violate Bochner’s positivity criterion implying their inverse Fourier transforms take negative values and do not correspond to charge-transfer probability distributions anymore, cf. Ref. 55. The nontrivial positivity restriction is typical of the open-system (density-operator) setting and its interplay with geometric considerations requires clarification if gauge transformations affecting the higher moments are to be understood physically.

Motivated by the above, we relate in this paper the charge pumping process to geometric phases of the adiabatically evolving mixed quantum state. We achieve this by including an ideal charge meter in the total system sketched in Fig. 1(c) following Schaller et al. and by applying the adiabatic (mixed-)state evolution (ASE) approach, developed by Sarandy and Lidar. Geometric pumping through an open quantum system can then be understood in a formulation that stays near the established intuition of the familiar closed-system formalism.

We hereby go beyond recent discussions of the technical equivalence of pumping approaches. From our new vantage point both the AR and FCS approach to pumping can be derived and understood transparently: all their opposing features, listed in Table I, can be tied

\[3\] Although in Ref. 23 the ASE approach also played a role, it was not applied to the composite system-plus-meter state which is the crucial advance that we report here.
TABLE I: Comparison of approaches to pumping guiding the discussion in Sections III–VI. Notation: $\hat{n}$ is the reservoir charge operator (used traditionally to account for the transported charge), $\hat{N}$ is the meter charge operator (needle position in Fig. 1) used here to register the transported charge. The meter phase $\phi$ (FCS counting field) is the momentum conjugate to the charge $\hat{N}$ (meter needle position) as discussed in Sec. II. In this table the subscript “g” indicates that only the geometric or pumping part of a quantity is meant, see indicated sections for further notation and details.

| Feature                        | Adiabatic state-evolution (ASE) | Full counting statistics (FCS) | Adiabatic response (AR) |
|-------------------------------|----------------------------------|--------------------------------|-------------------------|
| Object                        | State $\rho$ of system + meter  | Counting operator $\rho^g +$ counting-field $\phi$ | State system $\rho^g +$ reservoir charge $(\hat{n})$ |
| Observables                   | Expectation values: $\langle \hat{N}^k(t) \rangle$ | Cumulants $c^1 = \langle \hat{n}(T) - \langle \hat{n} \rangle(0) \rangle \ldots$ | Moment $M^1 = \int_0^T dt \langle I_n(t) \rangle$ |
| Driving                       | Adiabatic                        | Adiabatic                        | Nonadiabatic            |
| Evolution                     | Nonstationary (parametrically)   | Nonstationary (parametrically)   | Stationary (parametrically) |
| Connection                    | Berry-Simon: $A^g = (w^g | \nabla_R w^g \rangle)$ | Berry-Simon: $A^g = (w^g | \nabla_R w^g \rangle)$ | Landsberg: $A = (\Omega | W_{1N} \nabla_R W_{1N} \rangle$ |
| Geometric phase               | Continuum of phases $dA^g$       | Continuum of phases $dA^g$       | Single phase $dA^g$    |
| Gauge freedom                 | Meter calibration: $| \phi \rangle \rightarrow G^g | \phi \rangle$ | Cumulant currents $\frac{1}{\sqrt{2\pi}} c^g_{k=0} = 0$ for all $k$ | Reservoir charge shift: $\hat{n} \rightarrow \hat{n} + g I$ |
| Parallel transport            | $\Leftrightarrow A^g \dot{R} + \frac{1}{\sqrt{2\pi}} g^g = 0$ for all $\phi$ | $\Leftrightarrow A^g \dot{R} + \frac{\sqrt{2\pi}}{g} = 0$ for all $\phi$ | $\Leftrightarrow AR + \frac{\sqrt{2\pi}}{g} = 0$ |

Most prominently, we directly tie the physical origin of geometric effects on the entire pumping process to the simple fact that a physical meter can be recalibrated. This generalizes an earlier result restricted only to the pumped charge (first moment) to the entire transport process (all moments). We show that this is the origin of geometric pumping effects both in the FCS and AR approach, despite all their differences. In particular, in the FCS this point becomes clear when treating the counting field as the physical momentum conjugate to the meter charge. We furthermore find the announced nontrivial conflict between the physical positivity requirements and geometric considerations that one does not encounter in closed quantum systems: physical meter calibrations do not form a transformation subgroup of the full group of allowed geometric gauges.

On a more general level our findings highlight that taking measurements explicitly into account is important for the geometric and – a fortiori – the topological characterization of open systems. Also, by explicitly accounting for the meter, our approach provides a suitable starting point for consideration of more complicated situations than addressed in this paper where nonideal aspects of the charge measurement process need to be included, such as back-action and spontaneous charge-symmetry breaking in superconducting systems.

C. Outline.

The paper is organized as follows. In Sec. II we introduce a Hamiltonian model which includes an ideal meter as sketched in Fig. 1(c). Integrating out the reservoirs, as in Fig. 1(b), we set up a master equation for the composite quantum state of system plus meter and carefully discuss which momentum (counting-field) is actually conjugate to the charge registered by the meter. Then, in Sec. III we outline the adiabatic state-evolution (ASE) approach of Sarandy and Lidar to open systems and show in Sec. IV how it describes the entire charge-pumping process when applied to the composite system-plus-meter state. This provides the basis for resolving all mentioned concerns regarding AR and FCS approach: in Sec. V we show how our ASE approach to system-plus-meter dynamics through standard, calibrated measurements on the meter naturally includes the FCS approach which focuses on charge-transfer statistics. All features of the FCS approach, in particular, the adiabaticity, can be clearly understood physically by considering the system plus meter. In Sec. VI we show how our approach also naturally leads to the AR approach with its completely opposite feature list– going to a reduced description in which the meter is integrated out as in Fig. 1(a). This allows us to explain in particular how the less familiar Landsberg geometric phase for the observable in the AR approach emerges physically from the more familiar Berry-Simon geometric phase of the composite state of system-plus-meter. We summarize our findings in Sec. VII and indicate how the key ideas presented can be extended beyond the simple limits that we intentionally focus on here for purpose of clarity.

II. MODEL, METER AND MASTER EQUATION

Hamiltonian model. In Fig. 1(c) we sketch a model of an ideal meter that we introduce following Schaller et al. In order to count the net number of electrons entering one of the reservoirs, say the left one, we extend the system plus reservoirs by a suitable Hilbert space to model a meter. This can be any space spanned by an infinite discrete set of orthonormal vectors $\{|N\}, N \in \mathbb{Z} \}$. 
When an electron enters (leaves) the left reservoir one “counts” by changing the meter state from $|N\rangle \rightarrow |N\pm 1\rangle$. The charge operator on the meter thus reads

$$\hat{N} = \sum_{N=-\infty}^{\infty} N|N\rangle\langle N|.$$  \hspace{1cm} (2)

Through this formal trick we are able to explicitly keep track of observable operators outside the system even after integrating out the reservoirs below.

We assume the meter is ideal in the sense that it has no internal dynamics (zero Hamiltonian). As a result, the choice of the meter-space is nonunique: it is possible to shift $|N\rangle \rightarrow |N-\eta\rangle$ by any integer $\eta \in \mathbb{Z}$ without altering the construction (translational invariance). Later we will see that there is a further nontrivial physical gauge freedom at this point in the choice of the meter states, see Eq. (61). The charges are counted instantaneously and without backaction on the rest of the system: when later [Eq. (82a)] on we discard the measurements on the meter in the tunnel-coupling Hamiltonian of the left reservoir by a tensor product

$$e \rightarrow e \otimes e^{i\phi}, \quad e\dagger \rightarrow e\dagger \otimes e^{-i\phi},$$  \hspace{1cm} (3)

with phase operators acting on the meter space ($\eta = \pm$)

$$e^{i\eta\phi} = \sum_{N=-\infty}^{\infty} |N\rangle\langle N+\eta| = \sum_{N=-\infty}^{\infty} |N-\eta\rangle\langle N|.$$  \hspace{1cm} (4)

We note that this construction works for any tunnel coupling model – not just for bilinear ones as considered in Ref. 57. Physically, $\hat{N}$ is the position operator of the needle on an unlimited charge scale of an ideal meter and $\hat{\phi}$ is its conjugate momentum ($\hbar = 1$):

$$[\hat{N}, \hat{\phi}] = i.$$  \hspace{1cm} (5)

Thus, $\hat{N} \rightarrow \partial_{\phi}$ when acting on phase eigenkets $|\phi\rangle := \sum_{N} e^{iN\phi}|N\rangle$. This phase operator is well known in superconductivity and in the $P(E)$-theory of electromagnetic fluctuations of electric circuits coupled to quantum dots. The operator (4) “kicks” the meter by one unit $-\eta = \pm$ for every charge that tunnels to/from the left reservoir (out of/into the system). Charge transport is measured outside the system by projective measurements of the meter observable $\hat{N}$, producing a statistics of outcomes $N$ from which the expectation values of powers $\langle \hat{N} \rangle, \langle \hat{N}^2 \rangle$, etc. can be computed. By measuring these at two different times, all moments and cumulants of the charge transfer (full counting statistics) can be determined, see App. A and App. B.

Calling this a “meter” emphasizes its ideal detection aspects which allow us to incorporate it inside an open system, i.e., even when integrating out the reservoirs. This is sufficient to clarify the issues at the focus of the paper. We stress that it is not intended as a realistic model of measurements, but simply makes explicit what one assumes when using any of the approaches discussed in the paper (ASE, FCS, AR) to compute pumping effects. For concreteness we assume that the model of the total system in Fig. 1(c) takes the form

$$H_{tot} = H \otimes \mathbb{1} + \sum_{r=L,R} H_r \otimes \mathbb{1} + H_{\eta} \otimes e^{i\eta\phi},$$  \hspace{1cm} (6)

where $H$ is the system Hamiltonian. Reservoir $r$ is described by the noninteracting Hamiltonian $H^r$ and has temperature $T$, electrochemical potential $\mu^r$ and density of states $\nu_r$. The last line assumes bilinear coupling and for the left reservoir includes the counting of electrons entering the dot ($\eta = -$) and leaving the dot ($\eta = +$).

Example. For a single-orbital quantum dot one can consider, for example, the Anderson model $H = \epsilon \sum_{r=\uparrow,\downarrow} d^\dagger_r d_r + U \sum_r d^\dagger_r d_r $ with $\sigma = \uparrow, \downarrow$, and bilinear tunnel coupling $H_{\eta} = \sum_{k\sigma} \sqrt{T_r/(2\pi\nu_r)} c_{-\eta r k\sigma} d_{\eta r \sigma}$ where $d_{\eta r \sigma} = d^\dagger_{\eta r \sigma}$, $d_{\eta r \sigma}$ for $\eta = \pm$, and similarly, $c_{\eta r k\sigma} = c_{\eta r k\sigma}^\dagger$, $c_{\eta r k\sigma}$, and reservoirs $H_{r} = \sum_{k\sigma} \epsilon_k c_{k\sigma}^\dagger c_{k\sigma}$. For simplicity we assume wide bands with constant density of states $\nu_r$. This model was used to compute the motivating example of charge pumping shown in Fig. 2 with pumping parameters $V_{bias} = \mu_L - \mu_R$ and $V_{\text{gate}} = -\epsilon$, see also Eq. (f).

Master-equation. By explicitly including the meter, we go back to the way FCS of transport was originally conceived. However, the following is not a mere redescription of FCS: we go beyond Ref. 57 by introducing charge and phase superoperators in order to relate the geometry of the pumping process to meter calibrations. Moreover, we consider driving of the Hamiltonian which is slow in the sense that the velocity of the dimensionless parameters $\hat{T}(t)$ is small relative to the transport rates ($\hat{R}:= |\dot{R}| \ll \Gamma$). For simplicity, we also make the weak-coupling approximation ($\Gamma \ll T$) and the frozen-parameter approximation (memory kernel $K$ below is independent of $R$) which is sufficient for computing the linear-response in $\hat{R}$. See Ref. 23 for a detailed discussion of these steps and also Sec. IV.1. After tracing out only the reservoirs, we obtain a Born-Markov master equation for the density operator $|\rho(t)\rangle$ for system plus meter:

$$\partial_t |\rho(t)\rangle = K[R(t)] |\rho(t)\rangle.$$  \hspace{1cm} (7)
Since it proves useful, we often consider operators as vectors in Liouville space, i.e., we write

\[ |A\rangle := \hat{A}, \quad (A)\hat{\cdot} := \text{Tr} \hat{A} \hat{\cdot} \]  

(8)

Here \( \cdot \) denotes the operator argument of \( (B) \), which is a covector dual to \( |A\rangle \) through the operator scalar product \( (A|B) := \text{Tr} \hat{A} \hat{B} \). The kernel is a superoperator that reflects the form of the Hamiltonian \( \hat{H} \)

\[ K = K^0 \otimes \mathcal{I} + K^R \otimes \mathcal{I} + \sum_{\eta} K^{L,\eta} \otimes e^{i\eta \phi}, \]  

(9)

where \( \mathcal{I} \) is the identity superoperator on the meter. All time dependence is parametric and is not explicitly indicated for simplicity. We now discuss the various terms.

**Example of Fig. 2.** For our concrete example of the Anderson model, the relevant part of the density matrix contains the occupation probabilities of the four charge states. In this case,

\[ K^0 = - \begin{pmatrix}
W_0^{r,0} & 0 & 0 & 0 \\
0 & W_0^{r,1} + W_0^{l,1} & 0 & 0 \\
0 & 0 & W_2^{r,1} + W_0^{l,1} & 0 \\
0 & 0 & 0 & W_1^{r,1} + W_1^{l,1}
\end{pmatrix}, \]  

(10)

is the diagonal part of the rate matrix. Golden-Rule transition rates \( W_{i,j} = \sum_{\sigma} W_{i,j}^{\sigma} \) appear in the matrix elements, where

\[ W_{0,0}^{r} = \Gamma_f f_+^{r}(\epsilon_\sigma), \quad W_{0,0}^{l} = \Gamma_f f_-^{l}(\epsilon_\sigma + U), \quad W_{0,1}^{r} = \Gamma_f f_+^{r}(\epsilon_\sigma + U), \quad W_{0,1}^{l} = \Gamma_f f_-^{l}(\epsilon_\sigma + U), \]  

(11)

with the Fermi functions \( f^{r,l}(\omega) = \left( 1 + e^{\pm(\omega-\mu)/T} \right)^{-1} \) and the tunnel rate \( \Gamma = \sum_{\tau=L,R} \Gamma^\tau \). Next, \( K^R \) contains the transition rates due to the coupling of the system to the right reservoir:

\[ K^R = \begin{pmatrix}
0 & W_1^{r,0} & W_0^{r,0} & 0 \\
W_1^{r,0} & 0 & 0 & W_0^{r,0} \\
W_0^{r,0} & 0 & 0 & W_0^{r,0} \\
W_0^{r,0} & 0 & W_0^{r,0} & 0
\end{pmatrix}. \]  

(12)

Finally, the transitions induced by an electron tunneling from / to \( (\eta = \pm) \) the left reservoir are described by the superoperators \( K^{L,\eta} \), in our example,

\[ K^{L,\eta} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \]  

(13)

**Charge and phase superoperator.** Because the reservoirs which we eliminated are normal metals (charge-diagonal state), in the last term of \( K \) the system superoperators \( K^{L,\eta} \) are combined with the phase superoperator \( \Phi \):

\[ e^{i\phi \Phi} \cdot := e^{i\phi \Phi} \cdot e^{-i\phi \Phi} \]  

(15a)

\[ = \sum_{N=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |N,k)(N+k,\eta| \cdot \]  

(15b)

This superoperator "kicks" the meter by \( -\eta = \pm 1 \) and thereby counts the charge on the reservoir. Due to its central importance below it deserves some comments. We have written (functions of) pure-state projections as super(cod)vectors:

\[ |N\rangle := |N\rangle \langle N|, \quad (N)\hat{\cdot} := \text{Tr} \langle N| \langle N| \hat{\cdot}, \]  

(16a)

\[ (N)\hat{\cdot} := \text{Tr} \langle N| \langle N| \hat{\cdot} = \langle N| \hat{\cdot} |N\rangle, \]  

(16b)

where \( \text{Tr}_M \) denotes the trace of an operator acting on the meter space. (Only for \( |N\rangle \), and below \( |N,k\rangle \), \( \phi \) and \( (\phi,k) \) we deviate from the convention Eq. \( \Phi \), thus, e.g., \( |N\rangle \neq \hat{N}_N \). From these we constructed the \( k \)-offdiagonal versions

\[ |N,k\rangle := |N\rangle \langle N+k|, \quad (N,k)\hat{\cdot} := \text{Tr} |N\rangle \langle N+k| \hat{\cdot}. \]  

(17)

Importantly, the expectation value of any integer power \( (q = 1,2,\ldots) \) of the meter charge operator,

\[ \langle \hat{N}^q \rangle(t) = \text{Tr} (\hat{N}^q | \hat{\rho} = \text{Tr} \hat{N}^q \hat{\rho}(t), \]  

(18)

can be expressed entirely in terms of the following charge superoperator, the anticommutator

\[ \hat{\mathcal{N}} = \frac{1}{2} (\hat{N} \cdot + \hat{\cdot} \hat{N} \cdot) \]  

(19a)

\[ = \sum_{N} \sum_{k} (N+\frac{1}{2}k) \langle N,k| \hat{N},k \rangle. \]  

(19b)

Moreover, writing \( k = N' - N \) in Eq. \( \hat{\mathcal{N}} \) one recognizes the eigenvalue \( N+\frac{1}{2}k = \frac{1}{2} (N+N') \) as the classical symmetrized coordinate of the meter’s needle. This splitting into the sum and difference of \( N \) and \( N' \) is well-known from the Moyal gradient expansion around the semiclassical limit using Wigner \{22\} or Green’s functions \{23\}.

The relevant phase superoperator \( \Phi \) that is conjugate to the charge anticommutator \( \mathcal{N} \propto \hat{\cdot} \hat{N} \cdot + \hat{\cdot} \hat{N} \cdot \) in the sense

\[ [\mathcal{N}, \Phi] = i \]  

(20)

is given by the commutator, the phase difference

\[ \Phi := \hat{\phi} \cdot - \hat{\phi} \cdot \]  

(21)

Thus, \( \Phi \) is the momentum canonically conjugate to the classical charge-meter needle position \( \mathcal{N} \): it implements

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5 We only need the probabilities for an empty, singly \((\sigma = \uparrow, \downarrow)\) or doubly occupied dot due to charge and spin conservation in the total system \{23\} in the eigenbasis of \( \hat{H} \) the diagonal density matrix elements decouple from the off-diagonal ones.
the unitary symmetry transformations \( \{ \mathcal{G}_a \} \) on the meter’s mixed state, i.e., the translations of the charge meter’s needle. As we will see in Sec. [V] the counting field in the density-operator FCS formalism is the eigenvalue \( \phi \) of this phase superoperator \( \Phi \). It should not be confused with the phase operator \( \hat{\phi} \) in the initial Hamiltonian formulation which is conjugate to the charge operator \( \hat{N} \) [Eq. (6)]. (We note that from \( \hat{N} \) and \( \hat{\phi} \) one can construct another pair of conjugate superoperators for charge and phase: \( \hat{N}' = \hat{N} \cdot \hat{\phi} \cdot \hat{\phi} \) and \( \hat{\Phi}' = \frac{1}{2} (\hat{\phi} \cdot \hat{\phi} \cdot \hat{\phi}) \). These do not enter anywhere here: these are required only when considering a phase \( \phi \)-meter–observable \( \Phi \)–whose needle is “kicked” by symmetry generator \( \hat{N}' \) for charge.)

Diagonalization of the kernel in the meter space. In the following [Eq. (33)] we will need the diagonal form of the kernel superoperator [9]. This amounts to going to the eigenbasis of the phase superoperator:

\[
e^{i\phi N} = \sum_{\phi N,k} |N,k\rangle\langle N + \eta,k| \tag{22a}
\]

\[
e^{i\phi N} = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{i\eta \phi} \sum_{k} |\phi,k\rangle\langle \phi,k|. \tag{22b}
\]

Its eigenvectors are Fourier transforms of the pure-state meter operators, the phase superkets\(^6\)

\[
|\phi\rangle := e^{i\hat{N}\phi} = \sum_{N} e^{i\phi N}|N\rangle, \quad \phi \in [-\pi,\pi], \tag{23}
\]

(notating \( |\phi\rangle \neq \hat{\phi} \), the other deviation from convention [8]) and similar operators that are charge-off-diagonal by \( k = 0, \pm 1, \pm 2, \ldots \)

\[
|\phi,k\rangle := e^{i\hat{N}\phi} e^{i\hat{\phi}k} = \sum_{N} e^{i\phi N}|N\rangle\langle N + k|. \tag{24}
\]

In agreement with Eq. (20), \( \hat{N} \rightarrow \partial_\phi \) when acting on the phase superkets\(^7\) \( |\phi\rangle = |\phi,0\rangle \) and \( |\phi,k\rangle \) for \( k \neq 0 \). In Eq. (9) we thus see that after integrating out the reservoirs, the ideal meter is coupled to the transported charge through the meter’s needle momentum superoperator \( \Phi \).

Importantly, inserting Eq. (22) into the kernel \( \mathcal{K} \) we see

that the tensor product structure

\[
K(t) = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \left( K_{0}(t) + K_{\Phi}(t) + \sum_{\eta} K_{\Phi}(t)e^{i\phi} \right) \otimes \sum_{k}|\phi,k\rangle\langle \phi,k| \tag{25a}
\]

\[
= \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} W^\phi(t) \otimes \sum_{k}|\phi,k\rangle\langle \phi,k|. \tag{25b}
\]

reflects the ideality of the meter model \( \mathcal{K} \). The important \( \phi \)-dependent superoperator \( W^\phi \)–acting only on the system space–emerges naturally.

Finally, we note that in the following we can drop all \( k \) sums in the above expressions since we will restrict our attention to physical meter states that are \( \hat{N} \)-diagonal\(^8\), i.e., mixed states of the form \( \sum_{N} G^{N}|N\rangle\langle N| \) with some probability distribution \( G^{N} \). We then only need to deal with pure meter charge-states \( |N\rangle := |N,0\rangle = |N\rangle\langle N| \) and their Fourier transforms, the phase superkets \( |\phi\rangle := |\phi,0\rangle \). However, the \( k \) sums were important above to relate the charge \( \langle \hat{N} \rangle \) and phase operators \( \hat{\phi} \) to the relevant superoperators \( \langle \hat{N}' \rangle \), respectively.

### III. GEOMETRIC PHASES OF ADIABATICALLY EVOLVING MIXED STATES

To keep the paper self-contained, this section collects the key steps of the adiabatic state evolution (ASE) approach to slowly-driven open quantum systems guided by our points of interest in Table I. This approach due to Sarandy and Lidar\(^2\)–in principle not related to transport–generalizes the adiabatic closed-system approach: starting from a time-local master equation of the form \( \mathcal{K} \), it leads to a Berry–Simon type geometric phase for the mixed-state operator. In Sec. [IV] we will apply this approach to the pumping model which explicitly includes the meter. Here we first discuss the less specific problem of the driven state-evolution governed by Eq. \( \mathcal{K} \) [i.e., without assuming Eq. (25)]

**Mixed states, modes and decay rates.** First we consider fixed parameters \( R \) (not indicated), for which the time-evolved state reads

\[
|\rho(t)\rangle = |k_{0}\rangle + \sum_{m=1,2,\ldots} e^{k_{m}t}c_{m}(0)|k_{m}\rangle. \tag{26}
\]

Here, the modes are the right eigenvectors which satisfy \( K|k_{m}\rangle = k_{m}|k_{m}\rangle \). These are operators \( |k_{m}\rangle = |k_{m}\rangle \) labeled \( m = 1, 2, \ldots \). These nonstationary modes account

---

\(^6\) Note that the superkets \( |\phi\rangle := \sum_{N} e^{i\phi N}|N\rangle\langle N| \) are distinct from the projectors \( |\phi\rangle\langle \phi| = \sum_{N,N'} e^{i(N-N')\phi}|N\rangle\langle N'| \) onto the eigenvectors \( |\phi\rangle \) of the phase-operator \( \hat{\phi} \) which play no role here.

\(^7\) Distinguish the action on superkets \( \mathcal{K}|\phi\rangle = \partial_{\phi}|\phi\rangle \) from the action on a mixed state component \( \langle\phi|\mathcal{K}|\psi\rangle = -\partial_{\phi}\langle\phi|\psi\rangle \) which is needed to verify Eq. (20): \( \langle\phi|\mathcal{K}|\phi\rangle = -\partial_{\phi}\phi \langle\phi|\phi\rangle + \phi\partial_{\phi} \langle\phi|\phi\rangle = i\langle\phi|\mathcal{K}|\phi\rangle \).

\(^8\) Acting on \( \hat{N} \)-diagonal meter states the \( k \neq 0 \) terms in the superoperators give zero and can be dropped. They contain no terms that couple the \( k = 0 \) and \( k \neq 0 \) components of the state since the difference \( \hat{N} - \hat{n} \) between the meter and the left reservoir charge is conserved by the construction [3]. (This is no longer true if superconducting electrodes are considered.)
for the dissipative decay since their eigenvalues have \( \text{Re} k_m < 0 \). We label by \( m = 0 \) the zero-mode with eigenvalue \( k_0 = 0 \), i.e., \(|k_0\rangle = \hat{k}_0 \) is the stationary density operator. We assume for simplicity that \( K \) has a completely nondegenerate spectrum for all parameter values, in particular, that the stationary state \(|k_0\rangle\) is unique. This applies to a very large class of practically relevant cases. Because the evolution is non-Hamiltonian, \( K \) is not a normal operator \((K^\dagger K \neq K K^\dagger)\) and the left eigenvectors (covectors) are not the hermitian adjoints of the right ones and need to be determined separately. For each eigenvalue \( k_m \) the corresponding left supereigenvector \((\hat{k}_m)^\dagger\) is thus specified by an operator \( \hat{k}_m \) different from \( \hat{k}_m \), as indicated by the overbar. The covector plays a different role: given the initial state for Eq. (26), \(|\rho(0)\rangle = \sum_m c_m(0)|k_m\rangle\), it determines the amplitude for mode \(|k_m\rangle\) being excited:

\[
c_m(0) = (\hat{k}_m|\rho(0)\rangle) = \text{Tr}(\hat{k}_m|\rho(0)\rangle).
\]

Furthermore, the zero mode \(|k_0\rangle\) always exists since the trace-preservation \( \text{Tr}K \propto (k_0K) = 0 \) requires that there is a corresponding covector with \( \hat{k}_0 \propto 1 \).

**Adiabatic mixed-state decoupling.** When slowly driving the parameters, it can be shown that in the leading approximation the mixed state maintains the form

\[
|\rho(t)\rangle = \sum_{m=0,1,\ldots} e^{z_m(t)} c_m(0) |k_m[R(t)]\rangle
\]

It is therefore meaningful to refer to the \( m = 0 \) \((m \neq 0)\) contributions as its **parametrically (non)stationary components. A key question addressed in this paper is to physically understand how this adiabatic procedure manages to (correctly) capture nonadiabatic effects when applied to a suitably modified open system (including a meter). In the adiabatic decoupling approximation (28), the modes \(|k_m\rangle\) evolve independently since any crosstalk is negligible due to the slow driving. Their coefficients have evolution phases \( z_m(t) = z_{m,0}(t) + z_{m,1}(t) \) which are no longer given by Eq. (26). After one period, at \( t = T \), the modes all return to their initial values since the parameters have traversed a closed curve \( \gamma \) in the parameter space, \( R(T) = R(0) \). However, the state \(|\rho(T)\rangle\) has changed relative to \(|\rho(0)\rangle\) due to all the geometric and nongeometric phases, respectively,

\[
\begin{align*}
z_{m,0}(T) &= -\int_0^T dR \langle \hat{k}_m | \nabla_R k_m \rangle \quad (29a) \\
z_{m,1}(T) &= \int_0^T dt \, k_m[R(t)], \quad (29b)
\end{align*}
\]

9 See Ref. 23 [Eq. (G 10)] for a discussion of the gap condition for adiabatic decoupling in the steady-state limit, cf. Refs. 20, 21.

10 We assume that the \(|k_m[R]\rangle\) are chosen as continuous and smooth functions of \( R \) in all accessed parameter regimes.

one for each mode \( m \) contributing to the state (28).

Thus, the ASE approach formally resembles the Berry-Simon approach to closed systems, except for distinguishing left from right eigenvectors. In closed systems one is essentially only concerned with the right eigenvectors of the time-evolution because the closed system Hamiltonian \( H \) is hermitian (implying the left eigenvectors are simply their adjoints). For such Hamiltonian dynamics, we have \( K(t) = -i[H(t), \bullet] \) and the master equation reduces to the usual Liouville-von Neumann equation. One verifies that in this case one exactly recovers the standard closed-system approach for ket vectors including Berry’s result for a spin in a rotating field.

**Geometric phases and nonstationarity.** However, there is an important difference in the way physical restrictions enter for open systems. In particular, the kernel needs to preserve the trace and hermiticity of the density operator. This requires the adiabatically decoupled state to take the more specific form

\[
|\rho\rangle = c_0(t)|k_0\rangle + \sum_{m \geq 1} \left[ c_m(t)|k_m\rangle + c_m^*(t)|k_m^\dagger\rangle \right],
\]

where the coefficients are \( c_m(t) = (\hat{k}_m|\rho(t)\rangle \) and \(|k_m\rangle \) \(|k_m^\dagger\rangle \) are the eigenoperator for the complex-conjugate eigenvalue \( k_m \) \((k_m^\dagger)\). In view of the following discussion we assume that for \( m \neq 0 \) there all eigenvalues are complex. This implies that steady-state geometric phases arise only if the mixed state has a parametrically nonstationary component, similar to closed systems, see App. C. We need to distinguish the two cases:

First, the parametrically stationary mode \( m = 0 \) in Eq. (30) has a real eigenvalue, so its operator can be chosen hermitian, \( \hat{k}_0 = \hat{k}_0^\dagger \). If we trace-normalize it to be a physical state (stationary state), \( \text{Tr}\hat{k}_0 = 1 \), then \( c_0 = 1 \) is fixed for all times as in Eq. (26). These physical restrictions force both the geometric and the non-geometric phase for the zero mode to vanish: \( z_0 = z_{0,0} = z_{0,1} = 0 \) since \( k_0 = 0 \) in Eq. (25a) and \(|k_0\rangle = |0\rangle\) in Eq. (25a). That \(|k_0\rangle\) can be trace-normalized reflects that it is the only term in the expansion on the right-hand side of Eq. (28) that is a valid physical state [consistent with \(|\rho(t)\rangle \rightarrow |k_0\rangle\) in the stationary limit for fixed parameters]. This should be contrasted with closed systems, where each term in a superposition of pure-state vectors similar to Eq. (28) represents a physical state. Since in the slowly driven steady-state limit the unique zero-mode contribution dominates, the time-dependent steady-state (25) exhibits no geometric phase as a result of trace-normalization.

Second, the parametrically nonstationary modes \( m \geq 1 \) in Eq. (30) are pairs of non-hermitian operators with complex-conjugate coefficients (complex eigenvalue \( k_m \)). The operators for these nonstationary components are necessarily traceless, \( \text{Tr}\hat{k}_m = \text{Tr}\hat{k}_m^\dagger = 0 \), and therefore are not quantum states by themselves or in any combination. They can only be added to the stationary state as in Eq. (30) to form the physical state \(|\rho\rangle\). This
lack of trace-normalization also allows the nonstationary modes \(|k_m\rangle\) —and only these— to accumulate nonzero (non)geometric phases \([29]\).

We note that the positivity of the density operator \(\hat{\rho}(t)\) imposes a further nonlinear constraint on the evolution. Whereas in closed systems this is automatically ensured, in open systems this is not the case. This has received little attention in the context of pumping and geometric phases and we will return to it in Sec. IV [Eq. (61)].

**Gauge freedom and physical restrictions.** The appearance of a geometric phase such as Eq. (29a) in general requires some gauge freedom. Here, similar to closed systems, the gauge freedom resides in the possibility of choosing a different normalization of the eigenvectors in the expansion of the state while leaving the state invariant: each pair of right and left eigenvectors can be modified to gauged eigenvectors

\[
(k_m|G_m = (G_m)^{-1}(k_m), \quad |k_m\rangle_G = G_m|k_m\rangle, \quad (31)
\]

if the coefficients are simultaneously adjusted to \(c_m,G_m = (G_m)^{-1}c_m\) [Eq. (27)]. A gauge for mode \(m\) is thus any choice of a continuous nowhere-vanishing complex function of the parameters, \(G_m[R]\). In contrast to closed systems, in open systems the physical requirements for the gauges are less restrictive: although preservation of the form \([30]\), written compactly as \(|\rho\rangle = \sum_{m \in \mathbb{Z}} c_m(t)|k_m\rangle\) with \(\text{Tr} k_0 = 1, |k_+\rangle := |k_0\rangle\) and \(c_− := c_m^∗\), requires

\[
G_0 = 1, \quad G_m^∗ = G_−, \quad (32)
\]

this still allows the modes \(|k_m\rangle\) to accumulate a normalization factor\(^{11}\) with real phases under non-unitary evolution, in addition to the usual imaginary phase factors.

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\(^{11}\) Note that the gauge transformations do maintain the normalization between left- and right eigenvectors \((k_m|k_m\rangle = 1\) .

**Parallel transport of nonstationary components.** Finally, to see what is precisely geometric about the adiabatic evolution [Eqs. (28), (29)] we can closely follow the Berry-Simon approach to closed quantum systems. In particular, in Eq. (28) the evolution factor \(e^{\varepsilon z_m⟨t⟩}\) for each parametrically-nonstationary mode \((m ≥ 1)\) is equivalent to a geometric parallel transport in the space of parameters \((R)\) and nonzero complex gauge values \((G_m)\). This means for each \(m\) separately that if we try to gauge away this evolution factor we will in general fail as illustrated in Fig. 3. Suppose we look for \(c_m(t)\) such that \(|k_m'(t)⟩ := c_m(t)|k_m(t)⟩\) together with \(\langle\bar{k}_m' | = (\bar{k}_m|/c_m(t))\) satisfy the parallel transport condition

\[
(\bar{k}_m'| \frac{d}{dt}\bar{k}_m') = 0, \quad (33)
\]

for \(t \in [0, T]\), such that the geometric phase \((29a)\) is zero in this new “rotating frame”. Eq. (33) is equivalent to

\[
A_m,c_m\dot{R} := A_m \dot{R} + (c_m)^{-1} \frac{d}{dt}c_m = 0, \quad (34)
\]

where \(A_m = (\bar{k}_m|\nabla_R k_m)\) is the connection appearing in the result \((29a)\). Solving this for \(c_m(t)\) one obtains (only) the nontrivial geometric evolution factor \(c_m(T) = e^{-\int_{t_e}^{t_f} dR A_m c_m(0)} = e^{z_{g,m}(T)c_m(0)}\) in Eq. (28), explaining\(^{12}\) its geometric origin. The geometric phase cannot be gauged away: for a small driving curve ‘\(\varepsilon\)’ around a point \(R\) with nonzero curvature \(B_m = \nabla_R R \times A_m\), the solution \(c_m(t)\) will be discontinuous and fails to be a valid gauge. The phases \(z_{g,m}\) are thus unavoidable and physically relevant. We stress that the adiabatic evolution of a single mixed state involves the parallel transport of

\(^{12}\) The left hand side of Eq. (34) is one of several ways of defining a geometric connection. The definition employed here leads to a clear physical picture later on [Eqs. (63), (76), (87)].
all its nonstationary modes \( m = 1, 2, \ldots \). Only the family of geometric pictures, each as in Fig. 3, is physically meaningful, in contrast to closed systems where often consideration of the geometric phase of a single vector suffices.

**Discussion: difference with Uhlmann’s phase.** This completes our summary of the ASE approach as applied for the following. It is, however, relevant to point out that the parallel transport condition \((33)\) can be formulated also as a Fock-type condition \((34)\):

\[
\left( \frac{d}{dt} \hat{k}' \left| \frac{d}{dt} k' \right. \right) \text{ is stationary} \tag{35}
\]

for all possible curves \( c_m \) parametrized by time. Here the overbar in the covector indicates that the distance measure used depends on the particular nonhermitian evolution kernel \( K \) in Eq. (7), see App. D for details. This should be contrasted with parallel transport in closed systems, where due to the hermicity of the Hamiltonian, the Fock condition “\((d/dt) \psi \left| \frac{d}{dt} \psi \right. \) is stationary” involves the standard hermitian metric, independent of the Hamiltonian. Thereby, Eq. (35) also differs from the definition of the Uhlmann geometric phase\(^{13}\) for density operators which is also defined using the Fock condition for a closed “purified” system\(^3\). This difference is relevant in view of the recent interest\(^{10,13,17}\) in finding measurable quantities that are related to the Uhlmann phase. The inverse route taken in the present paper –starting from physical evolution and measurable quantities and then identifying the relevant geometric structures– leads to a different type of geometric parallel transport condition, one that depends on the particular open-system dynamics from which it was derived. Of course, for topological quantities the connection used to compute them does not matter by the Chern-Weil theorem, but our discussion indicates that considerations of measurements are tied to different types of connections.

**IV. ADIABATIC STATE EVOLUTION OF SYSTEM + METER**

In the following we derive geometric pumping in terms of the established geometric approach to adiabatic mixed states [Sec. III] by exploiting the special structure \((25)\) of the evolution \((7)\) of an open system extended by an ideal meter [Secs. IV]. We purposefully postpone comparison with the FCS and AR approach to Secs. V and VI, respectively, to systematically answer the questions that they raised.

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\(^{13}\) The hermitian inner product is taken between pure states which purify two density operators. In terms of operators, this is the Hilbert-Schmidt scalar product between polar decompositions \(\sqrt{\rho U} U\) with a unitary \(U\).

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\(^{14}\) Crossings of the eigenvalues may actually occur and lead to inter-
Next, we take the steady-state limit of the driven evolution: for each \( \phi \), only the \( l = 0 \) term survives\footnote{In Eq. (30) only the \( l = 0 \) eigenspace contributes in the long time limit. The \( l \neq 0 \) modes decay on the time scale \( \Gamma^{-1} \), see App. D of Ref. \cite{23}.}

\[
|\rho(T)\rangle \approx \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} Z^\phi |w^\phi_0(T)\rangle \otimes |\phi\rangle,
\]

(40)

and we denote the remaining \( l \neq 0 \) coefficient at \( t = T \) by \( Z^\phi = e^{z^\phi} \). We stress that this involves two steps: to extract the steady-state component of \( \rho(t) \) one must \textit{“reset”} the initial condition of the system plus meter, \( c^\phi(0) = 1 \) for all \( \phi \). Importantly, before this resetting the function \( c^\phi(0) \) cannot be properly gauged away (i.e., in a continuous fashion). We discuss this in detail in App. E since this is easily overlooked\footnote{\cite{22}}. After this resetting, we are thus free to fix a reference gauge for the eigenvectors \( |k_i,\phi(t)\rangle \) by requiring

\[
(1|w^\phi_0(t)) = 1 \quad \text{for all } \phi \text{ and } t.
\]

(41)

This gauge ensures that the meter readout statistics is entirely contained in the coefficient \( Z^\phi \) and no additional information remains in the parametrically nonstationary components of the time-dependent state. Indeed, the coefficients

\[
Z^\phi = e^{z^\phi}, \quad z^\phi = z^\phi_n + z^\phi_k.
\]

(43)

of the system-plus-meter steady state pick up a \textit{continuum} of phases for \( \phi \neq 0 \). The nongeometric part,

\[
z^\phi_n := \int_0^T dt \, k_{0,\phi}(t) = \int_0^T dt \, w^\phi_0(t),
\]

(44)

is a “sum of snapshots” of the eigenvalues [Eq. (29a)]. The geometric part,

\[
z^\phi_k := -\oint dR A^\phi(R),
\]

(45)

is determined by the eigenvectors through the geometric connection of the Berry-Simon type [Eq. (29b)]

\[
A^\phi := (k_{0,\phi})^{-1}\nabla_R |0\rangle = (w^\phi_0)\nabla_R |0\rangle.
\]

(46)

Since it relies on the eigenvectors, it arises even when the eigenvalues \( w^\phi_0 \) in Eq. [44] were all zero along the path \( \phi \) of accessed parameters, i.e., if there was no transport at all (including no fluctuations). It thus accounts for \textit{pumping} contributions to the entire, slowly driven transport process.

In our physical model (6), it is clear from the beginning that the transport statistics of charge is exactly registered by the meter incorporated in the quantum state \( \rho(T) \) – and thus in \( Z^\phi \) or \( z^\phi \) because every passing electron “kicks” the meter [Eq. (4) and (15b)]. More precisely, the projective measurements of the reservoir charge at two times – with outcomes \( n \) at \( t = 0 \) and \( n' \) at \( t = T \) – determine the statistics for the change of the reservoir electron number through the 2-point moments \( M^{(k)} \) of order \( k = 1, 2, \ldots \)

\[
M^{(k)} := \sum_{n,n'} (n' - n)^k \rho_{n'n}.
\]

(47)

which is verified in App. A. This is possible because the meter only registers changes of the charge by its ideal coupling to transport through the phase superoperator \( \Phi \) [Eq. (15a)]. Since we include the meter in the open system, we simply compute these averages in the standard way from the meter reduced density operator:

\[
M^{(k)} := \text{Tr}_M \rho M^{(k)}(T) = (-\partial_{\phi})^k Z^\phi|_{\phi=0}.
\]

(49)

Although the last formula is familiar from formal FCS considerations to be discussed in Sec. VII here we can physically understand all of its aspects as shown by the steps of its derivation:

\[
\text{Tr}_M \rho M^{(k)}(T) = \text{Tr}_M \rho M^{(k)}(T)
\]

(50a)

\[
= \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \left(1|w^\phi_0(t)\right) \cdot (\partial_{\phi})^k 2\pi \delta(\phi),
\]

(50b)
The first equality (50a) was discussed in Eq. (18). The second step highlights the physical meaning of the quantity $\phi$, the counting field of the FCS approach discussed in Sec. VI. it is the momentum conjugate to the relevant “classical” charge superoperator $\mathcal{N}$, the position of the needle of the charge meter [Eq. (5) and (20) ff.]:

$$\mathcal{N} \to \partial_{\phi} \quad \text{when acting on } |\phi\rangle. \quad (51)$$

The phase should not be confused with the conjugate to the quantum charge operator $\hat{N}$ [Eq. (21) ff.]. Third, the phase derivative in (50b) is made to act on the system factor $Z^{\phi} |w^{\phi}(T)\rangle$ of the composite state $|\rho(T)\rangle$ [Eq. (40)] by partial integration using the gauge condition $\langle 1 | w^{\phi}(t) | 1 \rangle = 1$ [Eq. (41)]. We see that the extra sign in $-\partial_\phi$ that appears in the formula (49) relative to Eq. (51) simply reflects that the charge added to the system is counted negative by the meter. Finally, this derivative is taken only at $\phi = 0$ due the partial trace which traces out the meter:

$$\text{Tr}_M \mathcal{N} |\phi\rangle = \text{Tr}_M \partial_{\phi} |\phi\rangle = \partial_{\phi} 2\pi \delta(\phi) \quad (52)$$

This shows that setting $\phi = 0$ corresponds to discarding information about further measurement outcomes on the meter. Thus, the formal phase-derivative at zero phase in Eq. (50b) emerges naturally from a measurement of the charge observable on the meter, after which the meter is recalibrated. Thus, when working in some gauge one rather considers ally has a functional dependence on the velocities $\dot{\phi}$.

It remains to clarify why geometric quantities emerge in the transport process as a whole by identifying the physical origin of the gauge freedom. (In the general ASE approach of Sec. III the freedom to re-normalize eigenvectors was merely formal.) This origin is simply the possibility of calibrating the meter and geometric parallel transport is the (failed) attempt to calibrate away any effect of the transport process. To develop both ideas we need to take a few steps.

First, the gauge freedom (31) in the present problem translates to multiplying the system and meter factor in Eq. (38) by opposite gauge functions such that the state $|\rho\rangle$ remains invariant:

$$|\rho^{\phi}\rangle \to |\rho^{\phi}\rangle_G := (G^{\phi})^{-1} |\rho^{\phi}\rangle \quad (53a)$$

$$|\phi\rangle \to |\phi\rangle_G := G^{\phi} |\phi\rangle. \quad (53b)$$

Because there is a continuum of modes, the gauge $G^{\phi}$ is a function of the meter-momentum $\phi$. As function of time it is convenient and without loss of generality to fix the initial condition to $G^{\phi}[\mathbf{R}(0)] = 1$ for all $\phi$. Preservation of the trace and hermiticity of the composite state $\rho$ require that we maintain $|\rho\rangle$:

$$G^0 = 1, \quad (G^{\phi})^* = G^{-\phi}. \quad (54)$$

Analogous to other problems with a gauge structure, the gauge freedom can be exploited as follows: A transformation (53b) to an arbitrary gauge $G^{\phi}$ [relative to the reference (41)] amounts to writing

$$|\rho\rangle = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} S^{\phi}|w^{\phi}\rangle \otimes \left[ G^{\phi} |\phi\rangle \right]. \quad (55)$$

and correspondingly for the meter density operator (42)

$$\rho^M = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} S^{\phi} |\phi\rangle_G \quad (56)$$

with new coefficients $S^{\phi} = Z^{\phi}(G^{\phi})^{-1}$. Expectation values on the meter are gauge-invariant:

$$\mathcal{M}^{(k)} = \text{Tr}_M \hat{N}_k^\dagger \rho^M(T) = (-\partial_{\phi})^k Z^{\phi} G^0 |_{\phi=0}. \quad (57)$$

One thus splits up the problem of computing the $Z^{\phi} = G^{\phi} S^{\phi}$ in Eq. (49) by decomposing it into a simple or convenient gauge function $G^{\phi}$ (absorbed into the meter ket) and an unknown complicated part $S^{\phi}$, the solution to be computed in this gauge. For example, by a choice of $G^{\phi}$ one can eliminate the parametrically time-dependent capacitive screening charges from the calculation of the pumped charge (first moment).

Next, we note that gauge transformations can only be combined with the geometric factor of the solution

$$Z^{\phi} = Z^{\phi}_G Z^{\phi}_n, \quad (58)$$

which is a functional $Z^{\phi}_G = e^{\phi \mathbf{R}_G}$ of the parameters $\mathbf{R}(t)$ only. (The nongeometric factor $Z^{\phi}_n = e^{\phi \mathbf{R}_n}$ additionally has a functional dependence on the velocities $\dot{\mathbf{R}}(t)$.) Thus, when working in some gauge one rather considers the split

$$Z^{\phi} = G^{\phi} S^{\phi}. \quad (59)$$

C. Gauge transformations – meter recalibration

It remains to clarify why geometric quantities emerge in the transport process as a whole by identifying the physical origin of the gauge freedom. (In the general ASE approach of Sec. III the freedom to re-normalize eigenvectors was merely formal.) This origin is simply the possibility of calibrating the meter and geometric parallel

\[16\] Note carefully: in Sec. VI the pumped charge is nevertheless expressed directly as single geometric (Landsberg) phase but this is not a (Berry-Simon) phase of a state, but a (Landsberg) phase of the observable.
rather than splitting up $Z^\phi$.

Finally, to clearly see how physical restrictions affect gauge transformations we need to consider the charge representation (in contrast to geometric considerations which are most evident in the phase representation). The special form of the composite state (68) implies

$$|\phi\rangle = \int_0^\pi \frac{d\phi}{2\pi} |\rho^\phi\rangle \otimes |\phi\rangle = \sum_N |\rho^{-N}\rangle \otimes |N\rangle.$$  \hfill (60)

This reflects that by our construction of the ideal meter model the classical information about the charge exchanged between system and reservoir ($-N$, i.e., lost by system) is “copied” to the ideal meter ($N$) without any backaction. The charge representation in Eq. (60) shows that the gauge transformation (53b) leading to Eq. (55) corresponds to changing the pure meter states to

$$|N\rangle \rightarrow |N\rangle_G$$  \hfill (61)

\begin{equation}
:= \int \frac{d\phi}{2\pi} e^{-i\phi N} G^\phi|\phi\rangle = \sum_{N'} G^{N-N'}|N'\rangle \langle N'|.
\end{equation}

These are physical mixed states provided $G^N \geq 0$ for all $N$ in addition to the restrictions (32) which translate to $\sum_N G^N = 1$ and $G^N \in \mathbb{R}$. In this case, the gauge transformation in the charge representation $G^{N-N'}$ is the classical probability of randomly finding the pure state $|N'\rangle \langle N'|$ when the meter is in the gauged state $|N\rangle_G$. We have thus found that working in such a physical gauge, part of the transport statistics has been absorbed into parametrically evolving mixed meter states $|N\rangle_G$ that are used to count charge: the meter has been recalibrated by making the meter states “noisy” with a probability distribution $G^{N-N'}$ (assumed to be known). This is what the formal freedom of re-normalizing eigen-supervectors of the evolution superoperator $K$ [Eq. (53)] physically amounts to for an ideal pumping process.

D. Bochner’s constraints of positivity.

Before we can develop a physical picture of geometric parallel transport, we need to discuss whether a gauge $G^N$ always has (and should have) positive values and thus make sense as probabilities. Although for closed systems this is always true, for open systems negative values are unavoidable, but, on the other hand, never lead to incorrect results.

We first note that in any case $Z^N$ must be positive because the reduced density operator of the meter (12), $\rho^M = \sum_N Z^{-N}|N\rangle \langle N|$, is a valid quantum state. In the following we simply call $G^\phi$ “positive” if it Fourier transforms to a positive distribution $G^N$. Importantly, this property can be checked with Bochner’s criterion (53) directly on $G^\phi$ without actually performing the Fourier transform, as discussed in App. E. This criterion makes explicit that in the original $\phi$-representation there is a serious constraint on the class of allowed gauge functions $G^\phi$ if they are to have physical meaning as the Fourier transform of the statistical mixing coefficients of meter states [Eq. (61)].

Performing gauge transformations in an open system is thus not an innocent procedure as it is in closed quantum systems: in general, two probability distributions $Z^N$ and $G^N$ cannot be related by a convolution $Z^N = \sum_N S^{N-N'} G^{N'}$ with a third function $S^N$ that is also a probability distribution. Although the function $S^\phi$ in Eq. (56) can be found, given a positive $G^\phi$, Bochner’s criterion imposes nontrivial constraints on $S^\phi$ to be positive as well. We stress, that the correctness of the final result $Z^\phi$ to be computed is never at stake. This should be contrasted with closed systems where physical restrictions and geometric considerations can be cleanly separated. To develop some intuition how such negative values arise we consider some simple examples in Fig. 4.

In Fig. 4(a) three possible scenarios are sketched when assuming for simplicity that the geometric factor dominates, $Z^\phi \approx Z^\phi_g$, and thus is positive. This figure shows that a positive function can be split into two positive functions, a positive and a negative one, or even into two negative ones. In these scenarios, the correct physical, positive $Z^\phi_g$ is obtained even though either the chosen gauge ($G^\phi$) or the computed solution ($S^\phi$) or both may be nonpositive and lack a classical probabilistic interpretation. The final answer is obtained by Eq. (59) corresponding to a charge-convolution $Z^\phi_g = \sum N', Z^{N-N'}_g Z^{N'}_g$.

In Fig. 4(b) we show additional scenarios that arise in the case where $Z^\phi_g$ does not dominate, i.e., the coefficient $Z^\phi = Z^\phi_g Z^\phi_n$ consists of both a nontrivial geometric factor $Z^\phi_g$ and nongeometric factor $Z^\phi_n$. Since it is only required by Eq. (58) that the convolution $Z^\phi = \sum N', Z^{N-N'}_n Z^{N-N'}_g$ is positive, there seems to be no reason that the individual functions $Z^{N-N'}_n$ and $Z^{N-N'}_g$ must be positive. Therefore scenarios arise where $Z^\phi_g$ is negative and can be split into a negative and a positive or even two negative functions. This means that geometric and nongeometric contributions generally do not correspond to well-defined classical processes, i.e., with probability distributions. For example, if nongeometric and pumping effects compete, it should be possible that their contributions partially cancel (e.g., a reduction of noise due to pumping), which requires a nonpositive $Z^\phi_g$ as we illustrated in Fig. 4(b).

Thus, generally, considerations of the geometry / gauge structure and of the probabilistic structure of open quantum systems are nontrivially intertwined. However, in no case one makes a mistake by applying gauge transformation: one is only working in an intermediate picture in which the gauge and/or the solution computed in that gauge may have no physical significance.

E. Parallel transport / horizontal lift.

Aware of the positivity constraints we can now express geometric parallel transport in terms of the pumping pro-
FIG. 4: Positivity restrictions and gauge transformations. Panels (a) and (b) sketch the geometric factor $Z_k^\phi$ of $Z_k^\phi = Z_k^\phi Z_k^\phi$ as a curve in the total space of parameter values $\times$ gauge coordinates. The various possible cases of splitting $Z_k^\phi = G^\phi S^\phi$ into a continuous gauge $G^\phi$ and a discontinuous solution $S^\phi$ (obtained in this gauge) are sketched. Panel (c) shows examples for these cases in the $N$-representation. (a) First consider $Z_k^\phi \approx Z_k^\phi$ implying that $Z_k^\phi$ is positive. (i) The factor $Z_k^\phi$ is obtained in the reference gauge, corresponding to the plane in the sketch. However, one can always choose some other gauge $G^\phi$ to determine the solution $S^\phi$. In (ii)-(iii) we choose $G^\phi$ positive. (ii) If $S^\phi$ satisfies Bochner’s stringent criterion, it is positive. In fact, the only positive gauges for which $S^\phi$ is guaranteed to also be positive are $G^\phi = e^{i\phi_k}$ with integer $k$, corresponding to the trivial shift $G^N = \delta_{N_{-k}}$ of probability distributions. (iii) It may thus be that $S^\phi$ fails to be positive even if $G^\phi$ (by choice) and $Z_k^\phi$ (necessarily) are positive. In this case, the solution $S^N$ is not a probability distribution. (iv) It is even possible to choose a nonpositive gauge $G^\phi$ to compute a solution $S^\phi$, which can again be both positive or nonpositive. (b) In general $Z_k^\phi$ may well be nonpositive, in contrast to (a): only $Z^\phi = Z_k^\phi Z_k^\phi$ is required to be positive. In this case, at least one of $G^\phi$ and $S^\phi$ is nonpositive, i.e., case (ii) cannot occur here. (c) Examples of the splitting $Z_k^\phi = G^\phi S^\phi$ in the $N$ representation. (i) “Target” probability distribution $Z_k^\phi$. (ii) A positive $G^N$ that is shifted and narrowed relative to the target distribution $Z_k^\phi$ leads to a positive solution $S^\phi$. (iii) Broadening the gauge function in (ii) at some point forces the solution $S^\phi$ to take on negative values to narrow down the convolution and produce the target $Z_k^\phi$. (iv) Even when choosing a nonpositive gauge, sharply varying in sign, still allows to achieve the positive target by an also negative solution $S^\phi$.

In Sec. III this corresponds to maintaining the parallel transport condition

$$A^\phi \dot{\mathbf{R}} = A^\phi \mathbf{R} + (G^\phi)^{-1} \frac{d}{dt} G^\phi = 0$$

with the gauged connection [Eq. (41)]

$$A^\phi := G(\mathbf{w}_\phi^\phi | \nabla | w_\phi^\phi) G = A^\phi + (G^\phi)^{-1} \nabla R G^\phi.$$ (64)

Indeed, the geometric factor of the solution for $G^\phi(t) =$...
This holds for the ideal case of “pure pumping” where \( Z^g \) dominates \( Z^φ \) and both are necessarily positive. In this case, the information about the entire transport process can be completely incorporated in a recalibration of the meter, which does not evolve anymore (“rotating frame”). Parallel transport

\[
|N⟩_{Z_g} = \sum_{N'} Z_N^{-N'} |N'⟩⟨N'| \tag{65}
\]

This failure to find a proper gauge (continuous) means that the entire quantum state – system plus meter – exhibits a geometric effect: not just the average charge, but the entire transport process (all fluctuations).

Whenever \( Z^g \) is positive, the geometric parallel transport thus has the physical meaning of a meter recalibration to proper physical mixed states [Eq. (61)]

\[
|N⟩_{Z_g} = \sum_{N'} Z_N^{-N'} |N'⟩⟨N'| \tag{65}
\]

This holds for the ideal case of “pure pumping” where \( Z^g \) dominates \( Z^φ \) and both are necessarily positive. In this case, the information about the entire transport process can be completely incorporated in a recalibration of the meter, which does not evolve anymore (“rotating frame”). Parallel transport [63] is thus the (impossible) attempt to literally calibrate away the pumping effect on all charge transport quantities (current, noise, etc.) in continuous fashion by an adjustment of the ideal meter.

One should, however, keep in mind [Sec. IV D] that the splitting \( Z^φ = Z^g Z^φ \) does not in general guarantee that \( Z^g \) is positive, i.e., pumping is not a separate process. With this proviso, the above arguments can be reversed: because an ideal meter can be literally recalibrated down to the level of its classical fluctuations, slowly driven transport measurements should in general pick up geometric contributions in all moments of the measurement statistics. One overlooks this simple physical origin of geometric effects if one assumes that the states Eq. (2) in the ideal meter are pure, as we did initially [Eq. (2) ff.] following Schaller et. al. 54. Instead, one should in general allow for measurements with “noisy” meter states with known gauged statistics \( G^N \) (i.e., which are part of the meter design).

The equivalence of “trivial charge transport” meter-calibration and geometric parallel transport was demonstrated in the average pumped in Ref. 23. Here we have generalized this natural physical idea to the entire transport process (all moments) [cf. also Sec. V and Sec. VI] noting the nontrivial intertwining with the physical constraint of positivity. In Sec. VI we will show that this complication goes unnoticed when one considers only the first moment (= first cumulant) of pumping as in most studies of pumping. Only in that case one can always consider a given gauge transformation of the average charge as realized by some meter-state calibration.

\[
\mathcal{M}^{(k)} = (-\partial_1)_φ^k Z^φ_{|φ=0}, \quad C^{(k)} = (-\partial_1)_φ^k z^φ_{|φ=0}. \tag{66}
\]

V. FULL COUNTING STATISTICS (FCS)
OF ADIABATIC PUMPING

Having discussed our comprehensive approach [Sec. IV] in terms of mixed quantum-state evolution, we show in the remainder of the paper how two prominent density-operator approaches to pumping, FCS and AR, can be elegantly derived from it, thereby reconciling their seemingly conflicting features in Table I.

In the present section, we first discuss the FCS approach which is most closely related to the ASE approach for an open system with meter inside, as illustrated in Fig. 1(b). The expressions derived in the main part of this paper [Sec. IV] were written such that one obtains the FCS approach by simply dropping the meter ket \(|φ⟩\) as we now show. Importantly, this does not mean that the meter is physically discarded or eliminated – as in the case of the AR approach discussed in the next Sec. VI.

State evolution, generating operator and counting field. In order to compute the statistics of charge transfer to the reservoir, the FCS approach in its usual formulation introduces a generating operator \(ρ^φ\) which is defined only on the system\[8\]. This auxiliary object is constructed such that the generating function \(Z^φ = Tr_S ρ^φ\) produces the desired \(k\)th moment and cumulant, respectively, through\[17\]

\[
\mathcal{M}^{(k)} = (-\partial_1)_φ^k Z^φ_{|φ=0}, \quad C^{(k)} = (-\partial_1)_φ^k z^φ_{|φ=0}. \tag{66}
\]

17 Our sign convention is opposite to the usual one in FCS but is physically motivated. It is fixed by letting the ideal meter indicate the same charge as counted in reservoir, outside the system, and by defining \(φ\) to be the momentum conjugate to the meter-needle position, see Sec. [11]. This moreover ensures that there are no signs in the phase representation \(|φ⟩ = \int \frac{dz}{2π} |φ^z⟩ ⊗ |φ⟩\) in which the adiabatic decoupling is made.
for all \( k = 1, 2, \ldots \). The cumulant generating function \( z^\phi \) is the exponent:

\[
Z^\phi = e^{z^\phi}, \quad z^\phi = \sum_{k=0}^\infty \frac{1}{k!}(-i\phi)^k \mathcal{C}(k).
\] (67)

Here, no reference is made to a composite system-meter state as in Sec. [VI], the meter momentum \( \phi \) is treated as an auxiliary “counting field” variable: taking derivatives and setting \( \phi = 0 \) is just a way of generating the desired expressions for the moments / cumulants [66] based on measurements in the reservoir (see Eq. \([A2]-[A3]\)).

**Derivation of FCS equations from ASE.** The FCS approach thus effectively accounts for the measurements performed outside the system through the \( \phi \) dependence of the quantity \( \rho^\phi \) defined only on the system. In contrast, in our ASE approach to pumping Eq. [66] arises naturally from the ideal meter inside the open system in line with the original motivation of the FCS by a meter model [23], see our discussion of Eq. \([49]\). There we identified the meter phase \( \phi \) as the meter-momentum [eigenvalue of the superoperator \( \Phi \), Eq. \([21]\)] that is conjugate to the classical charge [superoperator \( \mathcal{A} \), Eq. \([195]\)]. Phase derivatives \( \partial_\phi \) of \( \phi \) relate to meter-charge measurements and setting \( \phi = 0 \) discards the meter afterwards. Finally, in our approach the generating function \( Z^\phi = e^{z^\phi} \) naturally appeared when expanding the mixed meter state (after integrating out both system and reservoirs) in the meter-momentum basis \( |\phi\rangle \).

The central equation of the FCS approach describing the evolution of \( \rho^\phi \) is derived quite simply from the adiabatic Berry-Simon phase \((45)\) of a mixed state. Thus, even though Eq. \([68]\) formally resembles the evolution of \( \rho^\phi \) to \( \phi = 0 \) discards the meter afterwards. First, it is not the state of the system, even though Eq. \([68]\) formally resembles the evolution of a quantum state. First, it is not the state of the system, the reduced density operator \( \rho^\phi \); even though it includes that state. This difference between \( \rho^\phi \) and \( \rho^\phi \) is crucial since physical restrictions (normalization, hermiticity) do not quench the gauge freedom for \( \rho^\phi \) as they do for \( \rho^\phi \) [Sec. [VI]], allowing \( \rho^\phi \) to accumulate a nonzero geometric phase for \( \phi \neq 0 \). Second, \( \rho^\phi \) is also not the state of the system plus meter: only the full system-meter expression Eq. \([38]\) has this direct physical meaning.

**Nonstationary FCS and Berry-Simon phase.** For slowly driven parameters, the ASE and FCS approach account for pumping of transport quantities in the same way by performing an adiabatic decoupling approximation. Taking account of the steady-state limit as explained in Sec. [VI], in particular, “resetting” \( \phi^0 = 1 \), and defining the gauge by \( \langle 1 | w_0^\phi \rangle = 1 \) [Eq. \([39]\) ff.] leads to

\[
|\rho^\phi \rangle \approx Z^\phi |w_0^\phi \rangle.
\] (69)

Apart from parametrically following of the eigenvector \( |w_0^\phi \rangle \), the FCS generating operator \( \rho^\phi \) picks up the factor \( Z^\phi = e^{z^\phi} \). In our approach this parametric nonstationarity [19] is seen to correspond to the meter needle “running” (net current) and fluctuating (higher cumulants). The geometric part of the exponent \( z^\phi \), the second term in

\[
z^\phi = \int_0^T dt w_0^\phi - \int_0^T dR A^\phi,
\] (70a)

\[
A^\phi = (w_0^\phi \nabla_R |w_0^\phi \rangle). \tag{70b}
\]

is of the Berry-Simon type due to the formal similarity of Eq. \([68]\) to the closed-system Hamiltonian evolution [Sec. [III]]. Because in our approach we keep \( \phi \), we see that this phase appears in the nonstationary component of the physical state of system plus meter, which is always the case [Sec. [III] Eqs. \([30]\), \([42]\) and App. \([C]\)]. Thus, even though \( \rho^\phi \) itself is not a state, its geometric phase is an adiabatic Berry-Simon phase \((45)\) of a mixed state.

**Gauge freedom.** As discussed in Sec. [IVC] in our ASE approach, the gauge transformations

\[
G^\phi = e^{g^\phi}, \quad g^\phi = \sum_{k=0}^\infty \frac{1}{k!}(-i\phi)^k g^{(k)},
\] (71)

can be naturally regarded as changes of the meter factor \( |\phi\rangle \) in the system-meter state; gauge freedom is thus a consequence of physical recalibration of the meter. If one instead sticks to the FCS equation \([68]\) in which all reference to the meter space remains implicit, gauge transformations are not easily related to meter recalibration. Instead of \([53]\), gauge transformations are then equivalently introduced as a simultaneous rescaling of the eigenvectors and of the generating function

\[
\langle \bar{w}_0^\phi \rangle \to (G^\phi)^{-1} \langle \bar{w}_0^\phi \rangle, \quad |w_0^\phi \rangle \to G^\phi |w_0^\phi \rangle \tag{72a}
\]

\[
Z^\phi \to (G^\phi)^{-1} Z^\phi : = S^\phi \tag{72b}
\]

such that the generating operator \( |\rho^\phi \rangle \) in Eq. \([69]\) stays invariant. As in Sec. [IV] \( S^\phi \) denotes the solution sought working in the gauge \( G^\phi \). This absorption of a factor of the generating function into the eigenvector suggests no relation to physical meter calibrations.

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18 If one does integrate out the meter [Sec. [VI]], we indeed have \( \rho^\phi \big|_{\phi=0} = \rho^\phi \) and from Eq. \([68]\) \( \frac{d}{d\phi} |\rho^\phi \rangle = W |\rho^\phi \rangle \) with \( W := W^\phi \big|_{\phi=0} \).

19 For fixed parameters \( R \), the eigenvalue of \( W^\phi \) with the smallest negative real part determines the cumulant generating function at long times, \( dz^\phi / dt \approx w_0^\phi \), giving a nonstationary \( |\rho^\phi \rangle \approx e^{z^\phi} |w_0^\phi \rangle \) for \( \phi \neq 0 \).
The geometric part of a cumulant is in general not separately measurable (except for the first moment / cumulant). See the corresponding discussion regarding the splitting $Z^\phi = Z_1^\phi Z_2^\phi$ in Sec. IV D.

The geometric phase in Eq. (70b) is generated by a driving-velocity dependent argument in the contribution $-\oint dR A^\phi = -\int_0^T dR A^\phi$ which is well-known from analogous closed-system state-evolution expressions. However, one may equally well consider the $R$-dependence in this expression to indicate a nonadiabatic effect as one finds in response calculations. This point naturally arises when one inquires into the currents defined by Eq. (73) that flow through the system-reservoir boundary, e.g., in the gauge $G^\phi = 1$, $g^\phi = 0$:

$$I^{(k)}_c := \frac{d}{dt} C^{(k)} = w^{(k)}_0 - A^{(k)} \frac{d}{dt} R.$$  (78)

We now compare this with the form one would expect in a response calculation, i.e., the expansion of the exact cumulant currents (with a functional dependence on the driving) to linear order in the driving velocity: $I^{(k)}_c \approx I^{(k)}_c[R] + \delta I^{(k)}_c[R,R] + \ldots$ We see that the adiabatic cumulant currents are just the real-valued coefficients of the parametric eigenvalue [Eq. (75)] $I_j^{(k)}[R] = w^{(k)}_0[R]$. The geometric cumulant current $\delta I^{(k)}_c[R,\dot{R}] = -A^{(k)}[R][\dot{R}]$ is linear in the driving velocity $\dot{R}$ and therefore generates the pumping contribution to the cumulant $C^{(k)} = \int_0^T \delta I^{(k)}_c[A^{(k)}] dt$. The real-valued geometric connection $A^{(k)}$ has the direct physical meaning of an adiabatic-response coefficient:

$$\delta A^{(k)}[R] = \frac{\delta I^{(k)}_c}{\delta \dot{R}} \bigg|_{\dot{R}=0}.$$  (79)

This is the first nonadiabatic correction to the adiabatic cumulant current. The parallel transport (76) is thus equivalent to (trying to find) gauges $g^{(k)}$ that maintain zero nonadiabatic cumulant current $\delta I^{(k)}_c := \frac{d}{dt} C^{(k)} = \delta I^{(k)}_c + \frac{d}{dt} g^{(k)}$ for all $k = 0, 1, 2, \ldots$

$$\delta I^{(k)}_c[A,R,\dot{R}] = 0.$$  (80)

Thus, once one turns to a response formulation, geometry and nonadiabaticity are seen to be closely linked in open systems, even “from within” the FCS/ASE approach based on adiabatic decoupling.

The nonadiabaticity responsible for $\delta I^{(k)}_c$ is due to the finite “time-lag” between system and measurement of the charge outside the system [see Sec. I and Sec. VI]. This is confirmed already by considering the simplest case of the first moment / cumulant $k = 1$ for which Eq. (74) with $G^{(1)} = 1$ gives

$$C^{(1)} = \langle \hat{N}(T) \rangle - \langle \hat{N}(0) \rangle = \int_0^T dt (-\partial_{\phi^k \dot{\phi}}) w^{(k)}_0 |_{\phi=0} + \oint \dot{R} dR \partial_{\phi^k A^\phi} |_{\phi=0}. $$  (81a)

When one actually evaluates the expressions for $A^{(1)} = -\partial_{\phi^k A^\phi} |_{\phi=0}$ within the AR approach, i.e., by really integrating out the meter as discussed in Sec. VI, one finds

Geometric parallel transport = “trivial charge transport". In the FCS approach the parallel transport condition (63) of the system-plus-meter state, a condition holding for all values of $\phi$, can be expressed in terms of geometric parts of the cumulants. To see this, consider the adiabatic evolution equation of the cumulant generating function that leads to Eq. (70a),

$$\frac{d}{dt} \sigma^\phi_G = w^\phi_0 - A^\phi \dot{R} - \frac{d}{dt} \delta^\phi$$  (73)

in an arbitrary gauge $G^\phi = e^{\phi \theta}$ [Eq. (71)]. By Eq. (66) this determines the $\lambda$-th cumulant current

$$\frac{d}{dt} \sigma^{(k)}_G = w^{(k)}_0 - A^{(k)} \dot{R} - \frac{d}{dt} \delta^{(k)}$$  (74)

where, similar to the real-valued cumulants $C^{(k)}$, the terms on the right hand side of Eq. (73) have been expanded in powers of $\phi$,

$$w^{(k)}_0 = \sum_{k=0}^{\infty} \frac{1}{k!} (-i \phi)^k w^{(k)}_0, \quad A^{(k)} = \sum_{k=0}^{\infty} \frac{1}{k!} (-i \phi)^k A^{(k)}.$$  (75)

Geometric parallel transport (63) of the system-meter state due to adiabatic evolution is then equivalent to maintaining zero for the geometric part of the cumulant current for all $k = 0, 1, 2, \ldots$

$$A^{(k)}_G \dot{R} = A^{(k)} \dot{R} + \frac{d}{dt} \delta^{(k)} = 0,$$  (76)

i.e., the second, pumping contribution to

$$C^{(k)} = \int_0^T dt w^{(k)}_0 - \oint dR A^{(k)}.$$  (77)

Geometrically, Eq. (76) defines a horizontal lift curve in the space of parameters $R \times$ cumulant gauges $g^{(k)}$. However, one should note that by our discussion in Sec. IV D maintaining a “trivial transport process” is physically realizable as a recalibration of the meter state only if the geometric factor of $Z^\phi$ has a positive inverse Fourier transform (guaranteed by Bochner’s criterion).

Adiabatic decoupling produces nonadiabatic current? Equation (76) generalizes the parallel transport condition derived in Ref. [23] for the first cumulant / moment ($k = 1$) to arbitrary cumulants ($k \geq 2$). However, in Ref. [23] the considerations were based on the explicitly nonadiabatic AR approach. In contrast, the FCS/ASE result (76) was derived using the adiabatic decoupling procedure [69]. In this last subsection we address this apparent conflict from the FCS side by borrowing a few simple considerations from the AR approach that are further discussed in the next section.
the result explicitly contains a finite “lag” time and thus cannot be an adiabatic quantity as seen by the system only. This second geometric part [81] was plotted for our example system in Fig. 2.

One can thus equally well trace the physical origin of the \( \mathbf{R} \)-dependence that turns \( \int_0^t dR A^0 \) into a geometric curve integral to the “lag” or retardation between the system and the measured observable. In the ASE and FCS approach this explicit expression of the “lag” is never manifest since it accounts for a meter that is “running” inside the open system. The lag is correctly kept track of but in a fragmented and implicit way.\(^{21}\)

Summary. In open quantum systems there is an additional caveat in the term “adiabatic”: the confusion whether the \( \mathbf{R} \)-dependence indicates adiabaticity or nonadiabaticity is not\(^{22}\) a matter of defining “adiabatic” differently (i.e., which energy scales bound the driving frequency). Although in all discussed approaches (ASE, FCS, AR) it is required\(^{23}\) that \( \dot{R} \ll \Gamma \) and the same results are obtained\(^{22,23}\), the FCS is adiabatic relative to the system including the meter, whereas the AR is nonadiabatic relative to the system without the meter. It is revealing that the analogy to closed systems—characteristic of the FCS/ASE approach—breaks down precisely when one inquires in the spirit of the AR approach into quantities \([Eq. (80)]\) characteristic of an open system (currents). Table I outlines how the question of adiabaticity is closely tied to many other difficulties that depend on whether the system boundary includes / excludes the meter. That this is not merely a formal choice becomes even more evident in the next section, where we explicitly place the meter outside the system, changing all the features listed in Table I.

VI. NONADIABATIC PUMPING CURRENT: AR APPROACH

Finally, we show how the AR approach can be derived from the adiabatically evolving state of system plus meter [Sec. IV], thereby clarifying the remaining unsettled questions raised in the introduction. The AR approach is obtained from the ASE approach by physically discarding the meter (measurement outcomes). Similar to the FCS approach, the AR approach thus eliminates \( |\phi\rangle \) from the equations in Sec. IV but this time one additionally sets \( \phi = 0 \), which corresponds to tracing out the meter [Sec. IV B]. This leads to a very different physical and geometric picture.

Derivation of AR equations from ASE. To achieve a description referring only to the system, both for the state and for measurements done outside the system, we need to trace out the meter relative to Sec. IV. We first do this for the state \( \rho(t) \): tracing over the master equation \([7]\), accounting for Eq. \([36]\) and \([38]\), gives a master equation for the reduced state \( \rho^S(t) = Tr_M \rho(t) \):

\[
\frac{d}{dt} \rho^S = \frac{1}{i} [\mathbf{H}^S, \rho^S] + \Gamma \rho^S - \rho^S \mathbf{L}^S - \rho^S \mathbf{L}^S \rho^S + W^\phi |\phi=0\rangle \langle \phi=0| W^\phi \rho^S \langle \phi=0| W^\phi = W^\phi \rho^S = W^\phi \rho^S. \quad (82a)
\]

Here the key step is to use \( Tr_M |\phi\rangle = 2 \pi \delta(\phi) \), i.e., setting \( \phi = 0 \) integrates out the meter [cf. Eq. \([52]\)]. Preservation of hermiticity and normalization for the composite system implies that these are also preserved by the master equation \([82a]\). As we discussed in Sec. III B, these properties restrict the Berry-Simon geometric phase of \( \rho^S \) to be zero in the driven steady state\(^{23}\). Thus, once we physically discard the meter (by setting \( \phi = 0 \)) the geometric phase of the system state is lost.

To recover the pumping effect as a geometric phase we need to consider the transport of an observable outside the system, i.e., keep track of additional information about the reservoirs. Using the reduced state \( \rho^S \) we cannot describe measurements of the charge \( \hat{N} \) registered by the meter. This is reflected by the system kernel \( W \) being the \( \phi = 0 \) component [cf. Eq. \([52]\)] of system-meter kernel \( K \) [Eq. \([36]\)], the sum over all possible charge transfers:

\[
W := W^\phi |\phi=0\rangle = \sum_N W_N. \quad (82b)
\]

To recover the charge as computed in the AR approach we need to consider the transport current into the reservoir as registered by the meter in the ASE approach, \( I_N := \frac{d}{dt} \langle \hat{N} \rangle \) where \( \langle \hat{N} \rangle = Tr_M \hat{N} \rho^M \equiv Tr_M \langle \hat{N} | \rho^M \rangle \) [Eq. \([18]\)]. Using Eq. \([52]\), i.e., tracing out the meter after the measurement, we obtain

\[
\frac{d}{dt} \langle \hat{N} \rangle = Tr_M Tr_S \frac{d}{dt} |\rho\rangle = Tr_M \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} W^\phi |\rho\rangle \otimes Tr_S |\phi\rangle \tag{82c}
\]

\[
= Tr_S \left( -\partial_\phi W^\phi |\rho\rangle \right)_{\phi=0} + Tr_W^\phi \left( -\partial_\phi |\rho\rangle \right)_{\phi=0} \tag{82d}
\]

\[
= (1)|W_N| |\rho\rangle. \tag{82e}
\]

The second term vanishes by the trace preservation of the master equation \([82a]\). \( Tr_S W = Tr_S W^\phi |\phi=0\rangle = 0 \). Thus, by integrating out the ASE approach
approach we obtained\textsuperscript{23} the coupled set of AR equations \textsuperscript{82} in which the observable current is “enslaved” to the driven state.\textsuperscript{25,28,41,48} The current-kernel

\[ W_{I_{N}} = -\partial_{\phi} W^{\phi}|_{\phi=0} \quad (82f) \]

accounts for the additional system-reservoir correlations that are needed to compute the current of charge measured outside the system. We point out the difference between the current \( \frac{d}{dt} \langle N \rangle \), namely the average velocity of the needle of the charge-meter, and the needle’s canonical momentum \( \phi \). Our result \textsuperscript{82c} relates these as \( \frac{d}{dt} \langle N \rangle = \frac{\partial}{\partial \phi}[\langle I W^{\phi}|_{\phi=0}\rangle] \), which interestingly resembles a classical Hamilton equation for the velocity \( \frac{d}{dt} x = \frac{\partial}{\partial \phi} H \), while the phase \( \phi \) plays the role of momentum and the meter is again discarded after the measurement by setting \( \phi = 0 \) [Eq. (82c)].

Nonadiabatic, driven stationary state. We now follow the AR approach and compute the current coupled to the state evolution by Eq. \textsuperscript{82}. Notably, to obtain any pumping contribution at all for the first moment \( M^{(1)} = \langle N(T) \rangle - \langle N(0) \rangle \) we need to treat the parameter driving in equations Eq. \textsuperscript{82} in a way that differs from both the ASE / FCS approach. By putting the meter outside the system in the AR approach we have to deal with the finite physical “lag” between system and meter: we need a nonadiabatic approximation relative to the adiabatic system state which obeys the instantaneous equation (\textsuperscript{“i”}) \( W[R]|_{\phi=0}(R) = 0 \). In the notation of Sec. \textsuperscript{84} this is the parametric stationary state, \( |\rho^{S.i}\rangle = |w_{0}\rangle \). The required nonadiabatic correction or response \( |\rho^{S,i}\rangle \) (\textsuperscript{“i”})

\[ |\rho^{S,i}\rangle \approx |\rho^{S,i}\rangle + |\rho^{S.r}\rangle, \quad (83a) \]

is easily found to depend on \( R \) and linearly on \( \dot{R} \):

\[ |\rho^{S,i}\rangle \approx \frac{1}{W} \dot{R} \nabla_{R}|\rho^{S,i}\rangle. \quad (83b) \]

Since we have discarded the meter the steady system state \( \rho^{S}(t) \) given by Eq. \textsuperscript{83a} does not exhibit a geometric phase, even though it includes nonadiabatic effects\textsuperscript{24}. This closely correlates with the fact that for fixed parameters the system (without the running meter) has a well-defined stationary state. In contrast, the system-meter state \( \rho(t) \) in the ASE approach does have such nonstationary components with geometric phases. Our analysis shows that this opposite state of affairs [Table \textsuperscript{3}]

is a consequence only\textsuperscript{25} of having shifted the boundary defining the open system to exclude the meter \(|\phi = 0\rangle \) as in Fig. \textsuperscript{1}. This is a characteristic of geometric effects in open systems: shifting the boundary can drastically change both the physical and the geometric description without changing any prediction for measurements.

\textit{Landsberg’s geometric phase and transported charge.} The AR equations \textsuperscript{82e} by their “enslaved” structure generate a dissipative geometric phase which was considered in particular by Landsberg\textsuperscript{26,33} cf. also Ref. \textsuperscript{43}. The splitting \textsuperscript{83a} of the geometrically trivial state translates into a nongeometric and geometric contribution to the transported charge

\[ \mathcal{M}^{(1)} = \langle \hat{N}(T) \rangle - \langle \hat{N}(0) \rangle \quad (84a) \]

\[ \approx \int_{0}^{T} dt I_{N}[R(t)] + \int_{0}^{\infty} dR. \quad (84b) \]

The adiabatic part \( |\rho^{S.i}\rangle \) leads to the nongeometric adiabatic current \( I_{N}[R] = \langle I | W_{I_{N}}|\rho^{S.i}\rangle \) that survives even at zero driving velocity (\( \dot{R} = 0 \)). The nonadiabatic response correction \( |\rho^{S,i}\rangle \) gives rise to a nonadiabatic current \( \delta I_{N} \) \( |\rho^{S,i}\rangle \) that naturally defines a geometric connection

\[ \delta I_{N} \bigg|_{\dot{R}=0} = A[R] = \langle I | W_{I_{N}} \frac{1}{W} \nabla_{R}|\rho^{S.i}\rangle. \quad (85) \]

Physically, this Landsberg connection\textsuperscript{23} is just the leading current-response to the driving velocity, a nonadiabatic quantity: it depends on the inverse relaxation kernel \( W^{-1} \), i.e., a finite “lag” or retardation [Eq. (83b)]. The pumped charge contribution is equal to the single geometric phase \( \oint_{\phi} dR \) of Landsberg, in contrast to the ASE and FCS approach where exactly the same result is considered in particular by Landsberg\textsuperscript{26,33}.

\textsuperscript{23} Usually, these equations –including the explicit expressions for the kernels \( W \) and \( W_{I_{N}} \) – are derived more economically\textsuperscript{23} without any reference to FCS, counting fields or the meter model, making it a very practical approach.

\textsuperscript{24} Although \( |\rho^{S,i}\rangle \) and the nonadiabatic correction \( |\rho^{S,r}\rangle \) can both acquire geometric phases, these phases were shown to be zero due to trace preservation in Ref.\textsuperscript{22} [App. G and H], see also Sec.\textsuperscript{III}; the steady state \( |\rho^{S}(t)\rangle \) of the system only is geometrically trivial.

\textsuperscript{25} In our ASE approach we do not put in / leave out anything by hand relative to the FCS and AR approach, as explicitly verified in Refs.\textsuperscript{22,23} See App.\textsuperscript{E} regarding a superfluous term in Ref.\textsuperscript{23}.

\textsuperscript{26} The opposite signs are in a way unavoidable because in the ASE and FCS approach leading to \( A^{(1)} \) the natural convention is to require the expression for the connection [Eq. \textsuperscript{703} to resemble that of Berry [Sec.\textsuperscript{III}]. Landsberg’s connection \( A \), however, is naturally equated [Eq. \textsuperscript{85}] to the physical pumping effect without a sign.
we indicated that higher FCS cumulants \(c^{(k)}\) similarly can be understood as nonadiabatic Landsberg phases.

Practically, the Landsberg curvature \(B = \nabla_R \times A\) obtained from Eq. \(55\) simplifies calculations, and it directly corresponds to the physical pumped charge per unit area in driving-parameter space. Our illustration in Fig. 2 of the charge pumping effect for our example quantum dot of Sec. 1 was computed this way\(^{23}\), coinciding with the FCS expressions.

**Gauge transformations – geometric phase accumulated by observable?** In the AR approach, the gauge freedom that allows the Landsberg geometric phase to accumulate lies in the possibility of recalibrating the meter observable\(^{23,13}\) as \(\hat{N} \rightarrow \hat{N} + g\hat{1}\) [Eq. (1)]. Our considerations in Sec. [IV] readily reveal this freedom: the gauge transformation \(61\) of the meter states \(|N\rangle \rightarrow |N\rangle_G\) defines a new gauged *meter observable* operator\(^{27}\):

\[
\hat{N} = \sum_N N|N\rangle \rightarrow \hat{N}_G = \sum_N N|N\rangle_G = \hat{N} + g\hat{1}, \quad (86)
\]

where \(g = \sum_N NG^N = -\partial_G G^0|_{\phi=0} = -\partial_G g^0|_{\phi=0} = g^{(1)}\) is the real linear coefficient in \(G^0\) or \(g^0\) [Eq. (71)]. This provides a direct physical understanding how this gauge freedom in the observable – uncommon in quantum physics – emerges from the gauge freedom in the system-metric state in the familiar Berry-Simon setting. In the usual formulation of the AR approach\(^{23}\) this requires establishing the gauge covariance of the AR equations\(^{82}\) through a careful derivation of the *current kernel*\(^{23}\)

Another insight going beyond Ref. \(23\) is that for the first moment \(M^{(1)}\) the gauge freedom of the observable can always be understood physically as arising from *some* recalibration of the meter states (mixed instead of pure ones), rather than recalibrating the meter observable \(86\). In App. \(G\) we show that the positivity restriction discussed in Sec. [IVD] does not prohibit the construction of a meter-state gauges corresponding to any given real value of \(g\). Since for pumping of the first moment, the gauge freedom is exhausted by the observable recalibration \(86\), this implies that Bochner’s positivity criterion Sec. [IVD] imposes no constraint on pumping of the average charge unlike the situation for higher moments.

**Parallel transport = zero nonadiabatic current.** Finally, by discarding the meter, the “trivial charge-transport condition” of the AR approach,

\[
\delta I_G(R, \dot{R}) = A_G[R]|\dot{R} = A[R]|\dot{R} + \frac{\partial}{\partial t} g = 0, \quad (87)
\]

is found to be just the first of a family of parallel transport conditions [Eq. \(63\)] that defines the parallel transport of the adiabatically evolving system-meter state (the entire transport process), see Eq. \(76\) for \(k = 1\). The condition \(87\) was related in Ref. \(23\) to attempting to gauge away \(\) by the recalibration \(86\) the nonadiabatic pumping current (as seen by the system), i.e., Landsberg’s connection equals the pumping current.

**Summary.** Our approach developed in Sec. [IV] allowed us to show how the AR approach—with all its unfamiliar aspects listed in Table \(1\) emerges naturally from an adiabatic state-evolution approach similar to closed systems by just a single physical operation: discarding the meter, i.e., shifting the boundary from Fig. 1(b) to Fig. 1(a).

**VII. CONCLUSION AND OUTLOOK**

Starting from an explicit model of system, reservoirs and an ideal meter we have shown that geometric effects in driven transport have their direct origin in the physically obvious freedom to calibrate the meter. In contrast to Ref. \(23\) where only the first moment / cumulant was addressed, this applies to the entire transport process, i.e., all possible charge measurements performed on the meter, or, equivalently, all moments / cumulants of the transported charge. The recalibration allows for a “noisy” meter (with known noise) that counts charges using mixed quantum states instead of the pure ones assumed in Ref. \(57\). Notably, physical recalibrations form only a semigroup in the group of all possible geometric gauges since they must satisfy the nontrivial restrictions of Bochner’s criterion for positive probability distributions. Nevertheless, one can make use of the full geometric gauge group to apply standard considerations of connections on a fiber bundle.

We showed that two widely used approaches to pumping—full counting statistics (FCS) and adiabatic response (AR)—transparently follow from our approach. We showed their many crucial differences in Table \(1\) and all go back to the choice of the open system boundary, either including or excluding the meter. In particular, the central issue of the (non)adiabaticity of pumping transport was fully clarified this way. We also noted that by the computational choice of working either with moments (AR) or cumulants (FCS) one decides where one places the open-system boundary and commits oneself to one of the two very different physical and geometric pictures. If one entirely focuses on the average pumped charge – as often done— this is easily overlooked since the first moment equals the first cumulant.

Finally, we showed that when going beyond the average charge, the probability distributions that appear both in the ASE and FCS approach may turn into nonpositive functions when working in certain gauges. As we explained, this is a generic situation but it is not problematic since the computed results are gauge invariant: unlike closed systems the “intermediate” picture obtained in some gauges need not make physical sense. That the physical constraint of positivity is incompatible with ge-
ometric gauge structure, is important if one wants to use gauge invariance as a principle for constructing models within the open system approach (Liouville space).

Finally, we comment on the few simplifying assumptions that we made:

(i) We limited our attention to the AR approach as formulated for the first moment $\mathcal{M}^{(1)}$. However, an extension of the AR approach to all moments $\mathcal{M}^{(k)}$ is possible by calculating additional observable memory kernels. This leads to additional Landsberg geometric phases, one observable quantities (multi-counting statistics Berry-Simon phases of ASE/ FCS approach.

(ii) Although for clarity we have focused on charge transport, our considerations can be extended to spintronics (spin counting[22] quantum thermodynamics (energy / heat transport[22] and simultaneously measurable quantities (multi-counting statistics[22]).

(iii) For simplicity, we assumed that the eigenvalues $w_\phi^0$ are gapped for all $\phi$ [Eq. (39)] but we noted that crossings of the eigenvalues may in principle occur. This leads to interesting topological effects studied recently[23] and our considerations can be adapted to this.

(iv) Importantly, even when maintaining the strong restrictions of the ideal meter model—underlying all discussed approaches (ASE, FCS, AR)—many of the presented considerations can be extended further, in particular, to general non-Markovian open-system dynamics.

(v) Finally, we raise the interesting question how our relation between meter calibration (gauge freedom) and pumping effects (geometric phases) is modified when extending the model beyond Ref. 57 to non-ideal measurements. It seems promising to combine our approach with related insights from quantum-information and measurement theory (“quantum instruments”).

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Appendix A: Moments and cumulants

In this appendix we discuss the different moments and cumulants used in Sects. [IV V] and verify Eq. (48),

$$\langle \hat{N}^k \rangle (T) = \mathcal{M}^{(k)}. \quad (A1)$$

We distinguish between measurements on the reservoir (charge operator $\hat{n}$) as considered in most works[25] and the measurements on the ideal meter (meter position $\hat{N}$).

Reservoir charge $\hat{n}$: 2-point moments

For measurements of the reservoir charge at two times, with outcome $n'$ at $t = T$ and $n$ at $t = 0$, one can compute the statistical 2-point moments

$$\mathcal{M}^{(k)} := \sum_{n,n'} (n' - n)^k p_{n'n} = \sum_\Delta (\Delta n)^k p_{\Delta n}, \quad (A2)$$

from the distribution $p_{\Delta n} := \sum_{n=0}^{\infty} p_{n+\Delta n,n}$ for changes $\Delta n = n' - n$. The joint distribution $p_{n',n}$ can be computed[26] by the standard quantum rules: expanding $\hat{n} = \sum_n n\hat{O}_n$ with projective measurement operators $\hat{O}_n$, assuming that $[\hat{n}, \hat{p}] = 0$, and tracing out the system (S) and reservoirs (R):

$$p_{n',n} = \text{Tr}_{S,R} \hat{O}_{n'} \hat{U} \hat{O}_n \left( \rho^R \otimes \rho^S \right) \hat{U}^\dagger \hat{O}_{n'}, \quad (A3)$$

where $\hat{U}$ is the evolution operator from time 0 to $T$.

For App. [3] we note that the $k$-th moment is simply the average of the $k$th-power of the change of the Heisenberg charge operator $\hat{n}(t)$,

$$\mathcal{M}^{(k)} = \langle \hat{T} [\hat{n}(T) - \hat{n}(0)]^k \rangle \quad (A4)$$

provided one time-orders the expression by $\hat{T}$. By definition, the generating function has Taylor coefficients proportional to the moments:

$$Z^\phi = \sum_{k=0}^{\infty} \frac{(-i\phi)^k}{k!} \mathcal{M}^{(k)}, \quad \mathcal{M}^{(k)} = (-\partial_{\phi^0})^k Z^\phi \big|_{\phi^0 = 0} \quad (A5)$$

Summing the series with Eq. (A4), one obtains[75 89]

$$Z^\phi = \langle e^{-i\phi \hat{n}(T)} e^{i\phi \hat{n}(0)} \rangle \quad (A6a)$$

$$= \langle \hat{T} e^{-i\phi \hat{n}(T)} \hat{n}(0) \rangle = \langle \hat{T} e^{-i\phi \int_0^T dt \hat{f}_n(t)} \rangle. \quad (A6b)$$

Meter charge $\hat{N}$: 1-point moments

By construction of the ideal meter model, changes of the reservoir charge $\hat{n}$ are exactly copied to the meter position $\hat{N}$: the probabilities for measuring the same change $\Delta n = \Delta N$ between times 0 and $T$ are thus equal:

$$p_{\Delta N} = p_{\Delta N}. \quad (A7)$$

Thus, from the time-propagation of the first $r$ moments $\langle \hat{N} \rangle, \langle \hat{N}^2 \rangle, \ldots, \langle \hat{N}^r \rangle$ of the meter one should be able to extract the first $r$ moments of interest $\mathcal{M}^{(1)}, \ldots, \mathcal{M}^{(r)}$ (or the first $r$ cumulants). Below we establish this recursive relation [Eq. (A10)]. Moreover, when initializing the meter to $|0\rangle \langle 0|$, this simplifies to Eq. (11). Thus the formulas of Sec. [IV] based on 1-point meter measurements—indeed produce the exact 2-point statistics (A1) of charge transport to the left reservoir. There are three steps:

(i) Without the adiabatic / steady-state limit the system-meter state has the form[25]. Tracing out the system,
the meter density operator \[ \rho^N(T) \] at time \( T \) reads\(^\text{28} \)

\[ \rho^N(T) = \sum_N Z^{-N} \rho^M(N) \]. Therefore, the charge-diagonal elements \( P^N(t) = \langle N|\rho^M(t)|N \rangle = (N|\rho^M(t)) \) evolve as

\[ P^N(t) = \sum_N Z^{-(N-N)} P^N(0) \quad (A8) \]

where we restored for clarity a general initial condition, instead of \( P^N(0) = \delta_{N,0} \) used in the main text.

(ii) The conditional distribution defined by \( P^N(T) = \sum_N P^N|N P^N(0) \) is thus translation invariant:

\[ P^N|N = Z^{-(N-N)} = P\Delta N \big|_{\Delta N = N'-N} \quad (A9) \]

Then the expectation value of \( \hat{N}_k \) at time \( T \) reads

\[ \langle \hat{N}_k \rangle(T) = \sum_{N'} (N')^k P^N' \quad (A10a) \]

\[ = \sum_{N,N'} (N' - N + N)^k P^N'|N P^N \quad (A10b) \]

\[ = \sum_{q=0}^k (k)^q \sum_{\Delta N} \Delta N^{k-q} P\Delta N \sum_N N^q P^N \quad (A10c) \]

\[ = \sum_{q=0}^k (k)^q \mathcal{M}^{(k-q)} \langle \hat{N}_q \rangle(0). \quad (A10e) \]

(iii) When in the main text the meter is initialized in a pure state, \( P^N(0) = \langle N|0\rangle(0|N) = \delta_{N,0} \), we have \( \langle \hat{N}_q \rangle(0) = \delta_{q,0} \) in Eq. \((A10)\) which gives the result \((A1)\).

**Reservoir charge \( \hat{n} \): 2-point cumulants.** Finally, in Sec. \([IV]\) we used that the transport process as a whole is characterized either by all cumulants or all moments. This relies on the explicit relation

\[ C^{(k)} := \mathcal{M}^{(k)} - \sum_{l=1}^k \binom{k-1}{l-1} C^{(l)} \mathcal{M}^{(k-l)}, \quad (A11) \]

which is obtained by inserting the series Eq. \((A5)\) and \( z^\phi = \sum_{k=0}^\infty \frac{1}{k!} c^{(k)}(-i\phi)^k \) into \( z^\phi = \ln Z^\phi \). The relation \((A11)\) is nonlinear for \( k \geq 2 \) which complicates the comparison of geometric approaches that compute moments (AR) on the one hand, and cumulants (FCS) on the other. For this reason the present paper focused on comparing \( C^{(1)} = \mathcal{M}^{(1)} \).

---

\(^{28}\) The negative sign in \( Z^{-N} \) indicates that charge \( N \) counted by the meter and entering the reservoir is lost on the system \((-N)\).

**Appendix B: Recalibrating the reservoir charge \( \hat{n} \)?**

Here we point out how gauge transformations enter in formulations that do not use an open-system (density operator) formulation and provide some comments. When expression \((A6)\) is used as a starting point, then the gauge transformations \( \phi(t) = e^{i\phi(t)} = e^{-i\phi(t)} \) discussed in the main text correspond to the formal replacement\(^\text{23} \)

\[ \hat{n}(t) \rightarrow \hat{n}(t) + \gamma \phi \mathbb{1} \quad \text{or} \quad \hat{I}_n(t) \rightarrow \hat{I}_n(t) + \partial_t \gamma \phi \mathbb{1} \quad (B1) \]

with \( \gamma(t) = \gamma(t) \mathbb{R}(t) \). Also, \( \gamma(t)|_{\phi=0} = -\partial_t \omega \mathbb{R}|_{\phi=0} = \omega(t) \) and \( \gamma(t) = -\gamma(t)^* \) since \( \gamma(t)|_{\phi=0} = 0 \) and \( \gamma^* = (\gamma^*)^* \).

For the \( \phi \)-independent case \( \gamma = g \) the gauge transformation \((B1)\) physical corresponds to parametrically changing the charge operator in time, \( \hat{n}(t) \rightarrow \hat{n}(t) + g \mathbb{R}(t) \mathbb{1} \), which will cancel out in the change \( \hat{n}(T) - \hat{n}(0) \). This only captures the gauge freedom of the observable \( \phi \), responsible for geometric pumping of the first moment \( \mathcal{M}^{(1)} \)/cumulant \( C^{(1)} \) as discussed in Sec. \([VI]\). However, the gauge freedom responsible for the geometric higher moments / cumulants discussed in the main text arises when \( \gamma \) is \( \phi \)-dependent, in which case it is less clear what Eq. \((B1)\) physically means.

Another problem is that one cannot really consider the counting field \( \phi \) in Eq. \((B1)\) as the conjugate to the reservoir charge observable \( \hat{n} \). First, the replacement \((B1)\), would amount to shifting the operator \( \hat{n} \) by the eigenvalue of its noncommuting conjugate \( \phi \), which is difficult to understand. Moreover, it is known in quantum measurement and estimation theory\(^\text{23} \) that due to the lower bound \( n = 0 \) of the spectrum there does not exist any phase observable operator\(^\text{29} \) that is conjugate to a Fock-occupation operator (such as \( \hat{n} \)).

None of these complications arise in the ideal meter model \((\phi)\) that we used following Ref. \([57]\) the charge operator \( \hat{n} \) has a well-defined conjugate observable \( \phi \) because \( \hat{n} \) is the position-operator for the meter needle running from \(-\infty \) to \( \infty \). It is an excess charge operator that discretely change \( \hat{n} \) in the reservoirs, similar to the situation in, e.g., superconductivity\(^\text{30,31,32} \) and the \( P(E) \)-theory of electromagnetic circuit\(^\text{13,12} \).

---

\(^{29}\) Such a phase does have description as a POVM-measurement (projection-valued operator measure).
The solution of \( \rho(t) = \sum_{m=1}^{d} |\psi(t)\rangle \langle \psi(t)| \) is stationary

\[
\left( \frac{d}{d\tau} \rho \right) \parallel \left( \frac{d}{d\tau} \rho \right) = 0 \quad (D1)
\]

when varied over all operators \( \rho \). This is equivalent to the parallel transport condition Eq. (D4) as it arises from adiabatic evolution. First, in Eq. (D1) the bar is understood as follows in terms of the left- and right eigenvectors:

\[
(\bar{\rho}) := \sum_{m} e^{-i} \langle \bar{\rho}| k_{m} \rangle, \quad |\rho\rangle := \sum_{m} c_{m} |k_{m}\rangle. \quad (D2)
\]

The inverses of the coefficients \( c_{m} \) in the dual vector \( |\rho\rangle \) guarantee that in the innerproduct

\[
(\bar{\rho}| \rho \rangle = \sum_{m} (\bar{k}_{m}| k_{m} \rangle = \sum_{m} 1 \quad (D3)
\]

The outer product for each term is time-independent: \( (\bar{k}_{m}| k_{m} \rangle = 1 \). Functional variation of the expression (D1) with respect to the coefficients is effected by replacing \( c_{m} \rightarrow c_{m} e^{\epsilon m} \), maintaining the constancy of each term Eq. (D3), expanding to O(\( \epsilon \)), and noting that the gauge exponent \( g_{m}(R(t)) \) is a parametric function of time. Using

\[
\begin{align*}
&c_{m}^{-1} (\bar{k}_{m}| \bar{\rho} \rangle = c_{m}^{-1} (\bar{k}_{m}| \bar{\rho} \rangle = \left[ \frac{d}{d\tau} \bar{\rho} \right] k_{m} \rangle = \bar{c}_{m} \rangle & (D4a) \\
&\left( \frac{d}{d\tau} \bar{\rho} \right) k_{m} \rangle = = \left[ \frac{d}{d\tau} \bar{\rho} \right] k_{m} \rangle = \bar{c}_{m} \rangle & (D4b) \\
&\left( \frac{d}{d\tau} \bar{\rho} \right) (\bar{k}_{m}| k_{m} \rangle = \left[ \frac{d}{d\tau} \bar{\rho} \right] (\bar{k}_{m}| k_{m} \rangle = 0 \quad (D4c)
\end{align*}
\]

we obtain to O(\( \epsilon \)) the stationarity condition

\[
\eta \left( \frac{d}{d\tau} \rho \right) = \left( \frac{d}{d\tau} \rho \right) = 0 \quad (D5a)
\]

\[
\begin{align*}
&= \epsilon \sum_{m} \left[ - \lambda_{m} e^{\epsilon m} (\bar{k}_{m}| \bar{\rho} \rangle + \left( \frac{d}{d\tau} \bar{\rho} \right) k_{m} \rangle = \frac{d}{d\tau} \bar{c}_{m} \rangle & (D5b) \\
&= 2\epsilon \sum_{m} \lambda_{m} e^{-\epsilon m} (\bar{k}_{m}| \bar{\rho} \rangle + \left( \frac{d}{d\tau} \bar{\rho} \right) k_{m} \rangle = \frac{d}{d\tau} A_{m,c} \frac{d}{d\tau} R = 0 \quad (D5d)
\end{align*}
\]

Because the variations \( g_{m} \) are independent we have for each mode \( m \) separately

\[
A_{m,c} = (\bar{k}_{m}| \nabla_{R} k_{m} \rangle + e^{\epsilon m} \nabla_{R} c_{m} = 0, \quad (D6)
\]

This shows Eq. (E15) and Eq. (E14) are equivalent.

Appendix E: Steady-state limit

In this appendix we discuss the transition to the steady state in Eqs. (39) and (40). Our expressions in Secs. IV A and IV B are consistent with those of Ref. [25], and we comment on a different expression obtained in Ref. [22]. To obtain Eq. (40),

\[
|\rho(t)\rangle \approx \sum_{l=0,1,...} \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{i e_{l}^{\phi}(t)} \langle \phi| \epsilon_{l}^{\phi}(0) |w_{l}^{\phi}(t) \rangle \otimes |\phi\rangle \quad (E1a)
\]

\[
\approx \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{i e_{l}^{\phi}(t)} \langle \phi| \epsilon_{l}^{\phi}(0) |w_{l}^{\phi}(t) \rangle \otimes |\phi\rangle \quad (E1b)
\]

we have set \( \epsilon_{l}^{\phi}(0) = 1 \) and identified \( Z^{\phi} := e^{i e_{l}^{\phi}(T)} \) after one driving period. Since the adiabatic decoupling has already been performed in Eq. (E1a), the step (E1b) only concerns the steady-state limit. This can be understood clearly by considering measurable quantities: since we consider slow driving, \( T \gg \Gamma^{-1} \propto \hat{R} \), the cumulant current \( \langle Z^{\phi} \rangle \) reaches a time-dependent steady-state already at the beginning of the first driving period,
at times $t \ll T$ and the charge-transfer statistics becomes steady. As explained after Eq. (67) the steady-state cumulant currents $I_c^{(k)}$ are functions of both $\mathbf{R}$ and $\dot{\mathbf{R}}$ and determine the cumulant-generating current \( I^\phi := \frac{d}{dt} z^\phi = \sum_k \frac{1}{i k} I_c^{(k)} (-i \phi)^k \) [Eq. (67)]. Thus

\[
z^\phi(T) - z^\phi(0) = \int_0^T dt \ I^\phi(\mathbf{R}(t), \dot{\mathbf{R}}(t))
\]

is periodic: $z^\phi(nT) = n \cdot z^\phi(T)$ for integer $n$ since the first term, the sum of “snapshots”, takes the same “snapshots” every period and the second term is geometric and depends on the traversed parameter curve, which is also the same for every period. Thus, in Eq. (E1b) $e^{z^\phi(nT)} I_c^{(\phi)}(0) = (e^{z^\phi(T)} e^{z^\phi(0)} I_c^{(\phi)}(0))$ showing that $Z^\phi = e^{z^\phi(T)}$ is the steady-state generating function for one-period FCS. It is extracted formally by setting $c_0^\phi = 1$ as we did in the main text after which we denoted $z^\phi := z^\phi$. 

**Extra contribution in Ref. [22]** Notably, Ref. [22] does not set $c_0^\phi = 1$ and obtains a modified FCS phase $\int_0^\phi A^\phi + \ln c_0^\phi$ relative to Ref. [26] and the present paper. The authors of Ref. [22] rationalize this difference by noting that the extra term does not contribute to the first cumulant $c^{(1)}$ (pumped charge). Although this last observation is correct[30] provided one assumes that $\rho(0)$ is initialized as the parametric stationary state $|v_0\rangle$, this does not take away the fact that higher moments really do depend on the initial condition through $c_0^\phi$: the modified result does not apply to the steady-state FCS. For the above reason this did not lead to any inconsistencies in Ref. [22] when their FCS result was compared with the steady-state AR result for the first moment only. However, when comparing with other FCS works, Ref. [26] or the present paper one should ignore the extra term in Ref. [22]. In the present paper we consistently treat each of the compared approaches (ASE, FCS, and AR) in the steady-state limit.

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30 The extra term in Ref. [22] in $c_0^\phi$ in the geometric phase $z^\phi$ in principle gives an extra contribution to $M^{(1)} = c^{(1)} = -\partial_{\phi} \exp(z^\phi)|_{\phi=0}$ equal to $-\partial_{\phi} c_0^\phi |_{\phi=0} = 0$ calculated as follows. We already assumed $|\rho(0)\rangle = |\rho^{(0)}(0)\rangle = |\rho^{(1)}(0)\rangle$ and initialized the meter in the pure state $\rho^{(1)}(0) = |0\rangle \langle 0|$ and we now assume also that the system is initially stationary, $|\rho^{(0)}(0)\rangle = |\phi_0\rangle$. This gives $c_0^\phi = (K_{\phi_0} |\phi(0)\rangle) = (w_0^0 |v_0\rangle)$ by Eq. (37b) for $l = 0$ and using $|\phi(0)\rangle = 1$. Using Eq. (120) of Ref. [26] we obtain $c_0^\phi = (1 |v_0\rangle - i \phi |W| W^{-1} |v_0\rangle = 1 + O(\phi^2)$ since the pseudo-inverse $W^{-1}$ projects out $|v_0\rangle$. This gives $-\partial_{\phi} c_0^\phi |_{\phi=0} = 0$.

---

**Appendix F: Bochner’s criterion and gauge transformations**

**Bochner’s criterion.** Bochner’s criterion[31] states that the inverse Fourier transform $G^N$ of a smooth function $G^\phi$ of $\phi \in [-\pi, \pi]$ is a positive (semidefinite) function, $G^N \geq 0$ for all $N \in \mathbb{Z}$, if and only if the associated quadratic form

\[
\sum_{k k'} v_k^* G^{\phi_k - \phi_{k'}} v_{k'} \geq 0
\]

is positive (semidefinite) for an arbitrary finite complex vector with components $v_k$ and a corresponding arbitrary discrete sampling of Fourier frequencies $\phi_k \in [-\pi, \pi]$. The “if” part is easily verified and a constructive proof of the converse can be found in Ref. [33].

**Examples of Fig. 4(c).** The criterion [F1] is easily verified for phase factors $G^\phi = e^{i N \phi}$ with a fixed discrete $N \in \mathbb{Z}$. It can easily be made to fail as illustrated for finite Fourier sums $G^\phi = \sum_N G^N e^{i N \phi}$ in Fig. 4(c) of the main text. These examples indicate given an arbitrary positive function $Z^\phi$ and a resulting target distribution $Z_g^\phi$ as a product $Z_g^\phi := G^\phi_{(ii)} S^\phi_{(ii)}$ of two positive functions

\[
G^\phi_{(ii)} = \frac{1}{6} + \frac{2}{3} e^{i \phi} + \frac{4}{3} e^{2i \phi} \tag{F2}
\]

\[
S^\phi_{(ii)} = \frac{1}{10} + \frac{2}{10} e^{i \phi} + \frac{4}{10} e^{2i \phi} + \frac{2}{10} e^{3i \phi} + \frac{1}{10} e^{4i \phi}. \tag{F3}
\]

We also split the target function in another way, $Z_g^\phi = G^\phi_{(iii)} S^\phi_{(ii)}$ using a different, still positive gauge function

\[
G^\phi_{(iii)} = \frac{1}{3} e^{-i \phi} (0.1 e^{-3i \phi} + 0.2 e^{-2i \phi} + 0.4 e^{-i \phi} + 0.7 e^{0i \phi} + 0.2 e^{2i \phi} + 0.3 e^{3i \phi}) \tag{F4}
\]

whose width exceeds the width of the target distribution $Z_g^\phi$. Considering the convolution graphically in Fig. 4(c) one sees that negative coefficients in $S^\phi_{(iii)}$ are needed to narrow down the chosen $G^N_{(iii)}$ to match the target $Z_g^\phi$. The final example of a negative gauge and a resulting negative solution $S^\phi_{(iv)} = Z_g^\phi / G^\phi_{(iv)}$ is

\[
G^\phi_{(iv)} = \frac{1}{2} e^{0i \phi} - 1 e^{i \phi} + 2 e^{2i \phi} - 1 e^{3i \phi} + \frac{1}{2} e^{4i \phi}. \tag{F5}
\]

These examples indicate given an arbitrary positive gauge $G^N$, the corresponding solution $S^N$ may require negative values to adjust the gauge distribution (e.g., its width) to the target $Z_g^N$.

31 Eq. [F1] is actually Herglotz result, a special case of Bochner’s more general criterion.
Appendix G: Calibrating meter observables and meter states

Here we explain the point made at the end of Sec. IV D that for the first moment \( \mathcal{M}^{(1)} \) the gauge transformation of the observable \( \hat{N} \) can only be understood physically as arising from some recalibration of the meter states \( |\hat{N}\rangle = |\mathcal{N}\rangle |N\rangle \) to mixed states. As shown there, a meter-gauge \( G^0 = \sum_N G^N e^{i\phi_N} \) determines a real-valued observable-gauge \( \hat{N} \to \hat{N} + g \) through \( -\delta_{g0} G^0 |\phi=0\rangle = \sum_N N G^N = g \). Here, the geometric gauge freedom – unconstrained by physics – allows \( G^0 \) to be any function (not necessarily with positive Fourier transform \( G^N \).) We now show conversely that for a given observable gauge \( g \), one can find many state gauges \( \sum_N N G^N = g \). Moreover, these functions always include positive ones which correspond to a physical mixing of the meter states [Sec. IV D]. One simple construction of a class of positive meter gauges is

\[
G^N = (1 - |\frac{g}{N}|) \delta_{N,0} + |\frac{g}{N}| \delta_{N,k}. \tag{G1}
\]

The weighted sum \( \sum_N N G^N \) achieves the value \( g \) by a single contribution at integer value \( k \) with sign \( k = \text{sign } g \). Both nonzero values of \( G^N \) are positive if \( k \) is taken sufficiently large (\( |k| \geq g \) but other than this \( k \) is arbitrary. When \( g \) is parameter dependent, \( k \) can always be chosen large enough such that \( G^N \geq 0 \) for all \( R \) values accessed during the driving cycle. One can thus always consider the observable gauge \( g \) as arising from some physical mixing of meter states during a pumping cycle.

Whereas example (G1) is only piecewise differentiable for \( |g| > 0 \) (which is sufficient) one can also construct a differentiable gauge:

\[
G^N = \frac{1}{2} (1 - \frac{g}{N}) \delta_{N,-k} + \frac{1}{2} (1 + \frac{g}{N}) \delta_{N,k} \tag{G2}
\]

Now the weighted sum achieves the given value of \( g \) by two unequal contributions with positive weights if \( |k| \geq g \). This has the side effect that the trivial observable gauge \( g = 0 \) corresponds to a nontrivial (“noisy”) meter-state \( G^N = (\delta_{N,-k} + \delta_{N,k})/2 \), which is fine if one only considers the first moment.

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