Supplementary Materials for

Nonlinear elasticity, yielding, and entropy in amorphous solids

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S1 Setup of the effective field theory

The effective field theory (EFT) for solids, and elastic materials in general, is based on the construction of a zero temperature effective action in which the low-energy continuous degrees of freedom are the phonons – the Goldstone modes of translational invariance (44). From a more general perspective (43), condensed matter or soft matter systems can be defined as low-energy phases which break spontaneously the high-energy fundamental Poincaré group. As such, the various phases, i.e., solids, liquids, superfluids, etc., are in 1-to-1 correspondence with the different possible symmetry breaking patterns of the Poincaré group. For the case of solids, as we will see, our description is obviously not complete and in particular it neglects features such as plasticity, thermal effects, and the presence of an underlying lattice breaking rotational invariance (in the case of crystalline solids). Importantly, this theoretical construction is not restricted to the linear elasticity regime but it can be easily extended toward the non-linear region. See (52–55) for a modern treatment of hydrodynamics and viscoelasticity specially in connection to the holographic framework.

Under these assumptions, the fundamental building blocks are given by a set of $d$ (number of spatial dimensions) single-valued scalar fields $\phi^I$ (implying no plasticity nor non-affine dynamics (55, 56)), whose background solution is given by a coordinate dependent vacuum expectation value (VEV):

$$
\langle \phi^I \rangle = \delta^I_j x^j. \quad (S1)
$$

The solution above obviously breaks translational symmetry $x^I \rightarrow x^I + d^I$ spontaneously since it selects a preferred reference frame which can be thought of as the equilibrium position of the atoms or molecules of the medium. The scalars $\phi^I$ serve as a set of co-moving coordinates. The VEVs in Eq. (S1) preserve the rotational invariance in the spatial plane; isotropy is a simplifying assumption which can be easily generalized. While such an assumption might seem odd in the context of periodic crystalline structures, which are obviously incompatible with rotational symmetry at the microscopic level, it is definitely suitable for amorphous systems where no precise ordered lattice is present, specially at length scales larger with respect to the granularity of the system. A second and more fundamental constraint imposed is that of homogeneity at large scales. We indeed assume that at large distances, or equivalently small momenta, the system looks homogeneous. From a technical point of view, this is equivalent to impose a global shift symmetry for the scalars $\phi^I$ which acts on them as $\phi^I \rightarrow \phi^I + d^I$, and it is in some sense reminiscent of their Goldstone nature (shift-symmetric fields).

Following on these lines, the fundamental tensorial object in the theory is the kinetic matrix:

$$
\mathcal{S}^{IJ} \equiv \partial_\mu \phi^I \partial^\mu \phi^J, \quad (S2)
$$

where the Greek index $\mu$ runs on the spacetime coordinates ($t, x$), while the Latin one only on the spatial subset. Notice that this object is Poincaré invariant, a necessary ingredient if we assume our ultraviolet (UV) theory to have such a symmetry. In $d$ dimensions, the scalar objects that can be constructed out of the matrix in Eq. (S2) are given by the traces of its powers:

$$
X_{(n)} = \text{Tr}[\mathcal{S}^n]. \quad (S3)
$$

It is customary and convenient to replace one of the traces with the determinant of the matrix $Z = \text{Det}[\mathcal{S}]$ using the Newton’s identities. For example:

$$
Z = \frac{1}{2} X_{(1)}^2 - \frac{1}{2} X_{(2)} \quad \text{in} \quad d = 2;
$$

$$
Z = \frac{1}{6} X_{(1)}^3 - \frac{1}{2} X_{(2)} X_{(1)} + \frac{1}{3} X_{(3)} \quad \text{in} \quad d = 3;
$$

$$
\ldots
$$

From now on, for simplicity, we will focus on a 2D system. There are no fundamental obstructions in generalizing it to arbitrary dimensions, but several computational complications appear and render the underlying physics blurred. The most general effective action for an isotropic and homogeneous system which breaks spontaneously translational invariance is given by

$$
S_{\text{EFT}} = \int d^3 x \mathcal{V}(X, Z), \quad (S4)
$$

where to avoid clutter we have defined $X = X_{(1)}$. Notice that here the temperature is set to zero and no dissipative effects such as viscosities are considered. We reiterate that the fields $\phi^I$ are taken to be single valued and therefore no plastic effects are considered.

Importantly, the stress tensor of the system can be easily obtained as:

$$
T_{\mu \nu} = - \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{EFT}}}{\delta g^{\mu \nu}} \bigg|_{g=\eta} = - \eta_{\mu \nu} \mathcal{V} + 2 \partial_\mu \phi^I \partial_\nu \phi^J \mathcal{V}_X + 2 \left( \partial_\mu \phi^I \partial_\nu \phi^J X - \partial_\mu \phi^I \partial_\nu \phi^J \mathcal{V}_X \right) \mathcal{V}_Z, \quad (S5)
$$

where we have finally taken the Minkowski background with the metric $g_{\mu \nu} = \eta_{\mu \nu}$. For any time independent scalar field configurations, the stress-energy tensor components read

$$
T_t^t \equiv \rho = \mathcal{V}, \quad (S6)
$$

$$
T_i^t \equiv -p = \mathcal{V} - X \mathcal{V}_X - 2 Z \mathcal{V}_Z, \quad (S7)
$$

$$
T_i^i = 2 \partial_\mu \phi^I \partial_\nu \phi^J \mathcal{V}_X, \quad (S8)
$$

where $\mathcal{V}_X \equiv \partial \mathcal{V} / \partial X$, etc. Here $\rho$ is the energy density and $p$ the mechanical pressure, and the position of the indices is irrelevant since we only consider a flat background metric $g_{\mu \nu} = \eta_{\mu \nu}$. To make a connection with the standard non-relativistic description of elasticity, $T_{ij}$ corresponds to the stress which is usually denoted as $\sigma_{ij}$ and from now on it will be denoted in that way.
Figure S1. Illustration of affine deformations. (A) \(\rightarrow\) (B): a pure bulk deformation corresponding to \(\alpha = 2, \varepsilon = 0\) in Eq. (S10), changing the volume of the system but not the local angles. (C) \(\rightarrow\) (D): a pure shear transformation corresponding to \(\alpha = 1, \varepsilon = 1\) in Eq. (S10), modifying the angles but not the volume.

At this point, the full dynamics of the low energy degrees of freedom, the phonons, can be obtained by considering small fluctuations around the equilibrium positions Eq. (S2):

\[
\phi^l = \langle \phi^l \rangle + \delta \phi^l ,
\]  

and by expanding the action (S4) in terms of the new \(\delta \phi^l\) fields. This will be sufficient to obtain the linear elastic response of the system but not the fully non-linear one. In order to go to the non-linear order, we need to generalize the scalar configuration (S1). In particular, we consider the more generic Ansatz:

\[
\phi^l_{\text{str}} = O_j x^j , \quad O_j = \alpha \left( \sqrt{1 + \varepsilon^2 / 4} \frac{\varepsilon / 2}{\sqrt{1 + \varepsilon^2 / 4}} \right) .
\]  

This solution for the scalars is not an equilibrium solution, but it describes a deformed configuration. In particular, the two parameters \(\alpha\) and \(\varepsilon\) are directly related to a background bulk strain and a background shear strain as shown in Fig. S1. The \(\alpha\) parameter controls the change of volume in the system; on the contrary, the \(\varepsilon\) parameter implements a pure shear deformation where the corresponding strain tensor contribution reads:

\[
\varepsilon_{xy} = \varepsilon ,
\]  

therefore not modifying the volume of the system. Using (S8) combined with our scalar configuration (S10), one can obtain the fully non-linear stress-strain curve:

\[
\sigma(\varepsilon) \equiv T_{xy} = 2 \varepsilon \sqrt{1 + \frac{\varepsilon^2}{4}} \gamma X (2 + \varepsilon^2, 1) .
\]  

Linearizing the expression above and using the standard linear elasticity relation \(\sigma_{xy} = G_0 \varepsilon_{xy} + \ldots\), we can determine the unstrained elastic shear modulus to be:

\[
G_0 = 2 \gamma X (\bar{X}, \bar{Z})
\]  

where \(\bar{X} = 2, \bar{Z} = 1\) are just the values at the equilibrium configuration Eq. (S1). Here, the unstrained shear modulus \(G_0\) is simply given by the zero strain value of the non-linear one \(G \equiv d\sigma / d\varepsilon\). The same procedure can be followed to obtain all the linear elasticity properties of the systems such as the unstrained bulk modulus \(K_0\), the Poisson ratio \(\nu\) and many more in terms of the potential \(\gamma\). To make an instructive analogy, the potential \(\gamma\) plays exactly the same role of the strain energy function utilized in the standard treatment of non-linear elasticity (57,58). Finally, the dispersion relation of the phonon modes on the deformed background Eq. (S10) can be obtained by expanding the original action Eq. (S4) up to quadratic order in the fluctuations \(\delta \phi^l\). The details can be found in (14). Importantly, because of the broken rotational invariance of the background Eq. (S10), transverse and longitudinal phonons are coupled together and the speed of propagation is a function not only of the parameters \(\alpha, \varepsilon\) in Eq. (S10) and the potential \(\gamma\) but also of the propagation angle \(\theta\) (14). This anisotropy is more and more evident for large values of the external shear strain and it can be consistently neglected at small (enough) strain.

S2 Breaking point from effective field theory

Once the dynamics of the EFT is known (as explained in the previous section), one could ask whether and when the non-linear elastic response will display any sort of instability corresponding to breaking points \((\varepsilon_Y, \sigma_Y)\). In a non-relativistic system, one should not worry about any superluminality or causality issue. Therefore, the first and most dangerous instability is the so-called gradient instability, which corresponds to the point at which the (strain dependent) velocity of sound \(v_1\) becomes imaginary and therefore the system dynamically unstable (14). More precisely, stability requires that:

\[
v_1^2(\varepsilon, \alpha, \theta) \geq 0 ,
\]  

and it imposes strong constraints on the maximum strain that the system can support:

\[
v_1^2(\varepsilon_Y) = 0 .
\]  

At least qualitatively, this point of instability can be associated with the yielding point in the non-linear stress-strain response and with the breakdown of the elastic response. Let us consider, for example, a simple linear isotropic system where the propagation speed of the transverse phonons is given as usual by \(v_T^2 = G_0 / \rho_m\) where \(\rho_m\) is the mass density and \(G_0\) the unstrained shear elastic modulus, which can be extracted from \(\sigma = G_0 \varepsilon + \vartheta (\varepsilon^2)\). By promoting this relation
to non-linear level, \( \sigma(\varepsilon) \), one could define a strain dependent velocity \( v^2(\varepsilon) \) and a strain dependent non-linear elastic modulus \( G(\varepsilon) \equiv d\sigma/d\varepsilon \) as follows:\(^1\):

\[
v^2(\varepsilon) = G(\varepsilon)/\rho_m.
\]

Continuing on these lines, one could derive the maximum strain as the point at which:

\[
v^2(\varepsilon_Y) = 0 \quad \rightarrow \quad G(\varepsilon_Y) = 0.
\]

By using the definition of the strain dependent elastic modulus as \( G(\varepsilon) \equiv d\sigma/d\varepsilon \), one can identify the maximum strain and the instability point with the condition:

\[
\frac{d\sigma}{d\varepsilon} \bigg|_{\varepsilon_Y} = 0,
\]

which coincides exactly with the yielding point in the non-linear stress strain curve (see Fig. S2).

Note that the above discussion is rather intuitive since Eq. (S16) is valid only in the linear regime and it indeed receives corrections at the non-linear level. This implies that the two conditions Eqs. (S15) and (S18) are not equivalent for systems with non-linear elasticity. Because of the absence of plasticity (which results in a monotonically increasing stress-strain curve), in the theory, we will always use the precise criterion given in Eq. (S15). On the other hand, to define the breaking point (yielding point) in simulations, we will use for simplicity the criterion in Eq. (S18) (see Sec. S6 for details). This is because that in simulations, only for unaveraged single-sample stress-strain curves, yielding can be defined by a sharp stress drop, which is the signal of a global instability and in this sense closer to the criterion defined in Eq. (S15). Once the average over samples is taken (which is usually necessary in simulations), the average stress-strain curve becomes smooth and the condition Eq. (S18) applies. Nevertheless, as shown explicitly using the numerical simulations in Sec. S6, the two conditions give compatible results: the average yielding point defined based on Eq. (S18) is close to the individual breaking points of each sample. Therefore, at least from a qualitative point of view, the two criteria in Eqs. (S15) and (S18) are interchangeable in simulations.

In order to make concrete predictions, we make use of a non-relativistic potential:

\[
\mathcal{V} = \rho_0 \left( \sqrt{Z} + \gamma_0^2 \left( \frac{X}{2} \right)^A Z^{(B-A)/2} \right),
\]

as done in Section IV of (14). Here, \( \gamma_0 \ll 1 \) is a small parameter that controls the value of the sound speed. Importantly, the first new term in (S19) does not affect the dynamics of the stress-strain curve but only the size of the energy density \( \rho \). Concretely, this is a sensible way of taking the non-relativistic limit for a potential of the type \( \mathcal{V} \sim X^A Z^{(B-A)/2} \) and allow for a speed of sound which is much smaller than the speed of light \( c \), as in realistic systems. The most important effect of this non-relativistic limit is that of enlarging the allowed values of the parameters \( A, B \) to the range \( A > 0, B > 1 \), and avoiding the previous limitation coming from relativistic invariance. We do not repeat the computations of (14) here but only state and discuss the main results related to the non-relativistic potential Eq. (S19).

For this specific potential, by neglecting the effect of anisotropy (see (14)), the strain dependent speed of sound is given by:

\[
v^2(\varepsilon) = 4^{-A}A v^2 (\varepsilon^2 + 2)^{2A-1} \left( A^2 (\varepsilon^2 + 4) \varepsilon^2 - AB (\varepsilon^2 + 4) \varepsilon^2 + 4A + 4(B-1)B \right),
\]

By searching for the roots of the above expression, we can identify the maximum strain

\[
\varepsilon_Y^2 = 2 \sqrt{\frac{A(B-A)+A+(B-1)B}{A(B-A)}} - 2,
\]

and the breaking stress

\[
\sigma_Y = A\varepsilon_Y \sqrt{\varepsilon_Y^2 + 4 (\varepsilon_Y^2 + 2)^{A-1}},
\]

which can be easily re-written in terms of the non-linear exponent \( v \) and the Poisson ratio \( \tau \) using the definitions presented.

\[^1\text{It is not guaranteed that the nonlinear strain dependent velocity will still take the form } v^2 = G/\rho_m \text{ where } G_0 \text{ is simply replaced with the strain dependent non-linear shear modulus } G(\varepsilon). \text{ Indeed, our computations (14) show that it is not the case. Nevertheless, this approximation is sufficient to provide an intuitive relation between the yielding point and the gradient instability.}\]
in the main text. More explicitly, Eq. (S21) becomes
\[
\varepsilon_Y^2 = 2 \sqrt{-\frac{v + M + (v - 1)\tau + 5}{-v + M + (v - 1)\tau + 1} - 2}, \tag{S23}
\]
with \(M = \sqrt{2v - 2v\tau^2 + \tau^2 - 2\tau + 1}\). Expanding \(\varepsilon_Y\) near the incompressible limit, \(\tau = 1\), we obtain
\[
\varepsilon_Y = \frac{2^{3/4}}{(1 - \tau)^{1/8}} v^{-1/8} + O(1 - \tau)^{1/8}. \tag{S24}
\]
This result suggests that, in the incompressible limit (for large values of the Poisson ratio \(\tau\)), the maximum strain follows a power law in terms of the non-linear exponent \(v\) given by
\[
\varepsilon_Y \sim v^{-\kappa} \tag{S25}
\]
where \(\kappa = 1/8\). This analytical scaling is confirmed in Fig. S3.

![Figure S3](image.png)

**Figure S3. Correlation between the yielding strain and the non-linear elasticity exponent from the EFT.** The yielding point strain \(\varepsilon_Y\) in function of the non-linear elasticity exponent \(v\) for different values of the Poisson ratio \(\tau\) close to the incompressible limit \(\tau = 1\). The dashed line indicates the power law scaling \(v^{-1/8}\) of Eq. (S24).

Let us conclude this section with a few comments about these results. Eq. (S21) is a concrete prediction of the EFT formalism. It has to be considered as a qualitative but sharp correlation between the location of the breaking point (which as discussed above can be roughly identified with the yielding point) and the non-linear elastic properties of the material. In particular, the direct prediction of the EFT is that: the yielding strain diminishes with increasing the non-linear elastic exponent \(v\). Despite we are not able to formally and mathematically prove this statement for a generic potential \(\mathcal{V}\), we have substantial evidences that this behaviour, at least from a qualitative point of view, is universal. This behaviour is reversed at very large values of the non-linear exponent \(v\) (e.g. \(v \approx 40\) for \(\tau = 97\%\)) before which nevertheless the approximation scheme of the EFT breaks down and therefore its results cannot be trusted anymore. Moreover, the power-law behaviour in Eq. (S24) is a consequence of the precise power-law form of the EFT potential. In other words, despite the concrete number (in this case \(-1/8\)) is not universal, a power-law form of the stress-strain curve seems to always imply a power-law correlation between the maximum strain and the non-linear parameter \(v\). Unfortunately, our data from the simulations are not enough to prove or disprove a possible power-law correlation between these two quantities. Finally, as discussed in (59), the same qualitative behaviour is observed also in the gravitational model that we will describe in the next section.

### S3 Setup of the holographic gravitational model

The gravitational model makes use of the so-called gauge-gravity duality (or holographic) formalism (17) and it is deeply inspired by the EFT constructions (43) just mentioned. The idea, introduced in (46) and developed in (47) (see also (48) for a recent review), is to embed the EFT structure in Eq. (S4) into an asymptotically anti-de Sitter curved spacetime by using a 4-dimensional bulk action:
\[
S = \int d^4x \sqrt{-g} \left[ R - 2\Lambda - 2m^2 W(\mathcal{V}, \mathcal{Z}) \right], \tag{S26}
\]
where \(g\) is the determinant of the curved metric \(g_{\mu\nu}\), \(R\) the corresponding Ricci scalar, \(\Lambda < 0\) the negative cosmological constant and \(m\) a dimensionful parameter which will be related to the graviton mass. There are two massless scalar bulk fields \(\phi^I\) with \(\mathcal{F}^{IJ} \equiv \partial_\mu \phi^I \partial^\mu \phi^J\) and \(\mathcal{Z} = \frac{1}{2} \text{Tr} [\mathcal{F}^{IJ}]\) and \(\mathcal{F} = \text{Det} [\mathcal{F}^{IJ}]\).

These models are usually called "homogeneous holographic models" and they exhibit several interesting features such as viscoelastic properties and propagating phonon modes (19, 46–48, 59–66). It is important to stress that, a gravitational model with potential \(W(\mathcal{V}, \mathcal{Z})\) is not the dual of an EFT defined by the action Eq. (S4) with the same potential. In other words, the connection between the bulk potential and the dual EFT potential is very non-local and subtle. To avoid any clutter we will always use different symbols for bulk quantities and EFT ones. Additionally, the idea that these gravitational models are the exact duals of the EFTs in (43) is not totally correct since the global symmetries of the EFTs are kept global in the bulk picture. This caveat is emphasized in (67).

The massless scalar fields \(\phi^I\) are now living in the four dimensional curved bulk spacetime whose metric is taken to be:
\[
ds^2 = \frac{1}{u^2} \left( -f(u) e^{-\mathcal{Z}(u)} dt^2 + \frac{du^2}{f(u)} + \gamma_{ij}(u) dx^i dx^j \right), \tag{S27}
\]
where \(u\) is the radial extra dimension spanning from the UV conformal boundary \(u = 0\) to the black hole horizon.
$u = u_b$ at which $f(u)$ vanishes. The temperature of the background (S27) is given by $T = -f'(u_b) e^{-\xi(u_b)}/4\pi$. The spatial metric $\gamma_{ij}$ is generally not invariant under SO(2) rotations because of the background mechanical deformations. The scalar bulk fields $\varphi^i$ are dual, in the holographic sense, to the scalar operators $\phi^a$ used in the EFT description in previous sections and their background solution is taken as,

$$
\left(\varphi^x / \varphi^y\right) = \alpha \left(\cos(\Omega/2) \sin(\Omega/2) / \sin(\Omega/2) \cos(\Omega/2)\right) \left(x / y\right), \quad (S28)
$$

in complete analogy with the EFT description in Eq. (S10). By comparing the two parametrizations, it is easy to identify the shear strain with:

$$
\varepsilon = 2 \sinh(\Omega/2), \quad (S29)
$$

while $\alpha$ corresponds to the bulk deformation. Notice that the strain $\varepsilon$ is here taken as a background parameter, not as an infinitesimal deformation. This is the crucial technical point which allows us to compute the elastic response at a fully non-linear level.

Before proceeding, a very important observation has to be made. In the EFT description, the scalars solution in Eq. (S10) breaks translational invariance spontaneously. In the gravitational setup, the picture is more complicated and it crucially depends on the asymptotics of the scalar fields $\varphi^i$ close to the boundary. For simplicity, let us consider a potential whose asymptotic boundary expansion takes the form:

$$
W(\varphi^x, \varphi^y) = \varphi^x \varphi^y \frac{\sinh(\alpha \varphi^x / \alpha \varphi^y)}{\sinh(\alpha \varphi^x / \alpha \varphi^y)} x / y, \quad (S30)
$$

where the ellipsis indicates subleading term with faster fall-off. Under this assumption, the asymptotic behaviour of the bulk scalars is given by:

$$
\varphi^i = \varphi_0^i (1 + \ldots) + \varphi_1^i u^{2-2\alpha} (1 + \ldots), \quad \text{for} \quad u \to 0. \quad (S31)
$$

Using the so-called standard quantization scheme rooted in the holographic dictionary (68), the leading term in the above expansion must be identified with the external source for the dual field theory operator while the subleading term with its expectation value. The configuration in Eq. (S28) is independent of the radial coordinate $u$ and it has to be identified with the term $\varphi_0^i$ in the expansion Eq. (S31). Following this logic, the solution in Eq. (S28) is a VEV for the dual fields in the field theory side, as in Eq. (S1), only if $5 - 2\alpha < 0$. Therefore, only in that regime, the gravitational theory describes the same physical systems as in the EFT presented in the previous sections. Indeed, the condition above ensures that in the holographic picture the translational invariance of the boundary field theory is broken spontaneously. This guarantees that the dual systems are solids with propagating phonons and well-defined elastic properties as desired. For more details about this point we refer to the vast literature on the topic (19, 61–64, 69–73).

To continue, we define the spatial matrix

$$
\gamma_{ij}(u) = \left(\begin{array}{cc}
e^a(u) \cosh c(u) & \sinh c(u) \\
\sinh c(u) & e^{-a(u)} \cosh c(u)
\end{array}\right), \quad (S32)
$$

which contains two independent functions $c(u), a(u)$ corresponding to the two different polarizations of the graviton. It is easy to show that one can set consistently $a(u) = 0$ reducing this matrix to a single radial dependent function. Having defined our Ansatz, the background equations of motion read:

$$
\begin{align*}
f \left( u^2 c^2 + 12 \right) + 4 \left( m^2 W - u f' - 3 \right) &= 0, \\
c'' + c' \left( \frac{f'}{f} - \frac{2}{u} \right) - \frac{1}{4} u c^3 - \frac{2\alpha^2 m^2 \sinh(\Omega - c) W_x}{f} &= 0, \\
\chi' &= \frac{1}{2} u c^2, \\
\end{align*} \quad (S33)
$$

where we have taken $\Lambda = -3$ and the potential $W$ is evaluated on the background values $\mathcal{F} \equiv \alpha^2 u^2 \cosh(\Omega - c)$ and $\mathcal{F} \equiv \alpha^4 u^4$.

Notice that the $\chi$ function is completely slaved to the $c$ function and therefore the final set of variables can be simply thought as the pair $(f, c)$. The system of (S33) can be solved numerically by imposing the presence of an event horizon at the location $u = u_b$ where $f(u_b) = 0$ and $c(u_b) = c_h$. The UV boundary condition $\chi(u = 0) = 0$ is used to fix the time scale of the time coordinate so that the temperature at the UV boundary is equal to the standard Hawking temperature. Close to the UV boundary $u = 0$, we have $f(u) = 1$ and $\chi(u) = 0$ together with the asymptotic expansion:

$$
c(u) = \mathcal{C}_0 (1 + \ldots) + \mathcal{C}_3 u^3 + \ldots, \quad (S34)
$$

where, within the standard quantization scheme, $\mathcal{C}_0$ is identified with the source for the $T_v$ operator and $\mathcal{C}_3$ with its expectation value $\langle T_v \rangle$. Finally, we will consider only setups with zero source for the stress tensor operator, $\mathcal{C}_0 = 0$. The equations of motion (S33) allow the scaling symmetry

$$
u \to \lambda u, \quad \alpha \to \frac{1}{\lambda} \alpha, \quad (f, c) \to (f, c), \quad (S35)
$$

with $\lambda$ a constant. This is a consequence of scale invariance of the model. One can fix the symmetry defined in Eq. (S35) by setting $\alpha = 1$. This corresponds to considering only pure shear deformations which are volume-preserving.

The full non-linear elastic response can be studied by looking at the expectation value of the stress tensor operator $\langle T_{ij} \rangle \equiv \sigma = \frac{3}{4} \mathcal{C}_3$ in function of the external mechanical strain $\varepsilon = 2 \sinh(\Omega/2)$. Moreover, one could extend the analysis to the out-of-equilibrium time-dependent dynamics as initiated in (71). In this work, we restrict our analysis to the shear sector. The interested reader can find the analysis of the longitudinal sector in (59). Finally, even in this model the maximally allowed strain and stress can be obtained by
looking at the gradient instabilities of the gravitational modes. A partial analysis of this sort has been presented in (59).

Before concluding this short summary of the gravitational model, let us stress the benefits of using this description. (i) This dual formulation permits to introduce a finite temperature $T \neq 0$ without major difficulties. This would be impossible from the standard EFT point of view, in which dissipative and finite temperature effects are notoriously hard to consider (74, 75). (ii) In this scenario, the computation of the entropy of the system is particularly simple and it boils down to the estimation of the black hole entropy given by the famous Bekenstein-Hawking Area law (21, 76).

### S4 Scalings from the gravitational model

Let us consider the gravitational model introduced in the previous section with a bulk potential of the type:

$$W(\mathcal{X}, \mathcal{Z}) = \mathcal{X}^{\nu} \mathcal{Z}^{(b-a)/2},$$  \hspace{1cm} (S36)

where the parameters $a, b$ need to obey the following constraints:

$$a \geq 0, \; b \geq 1, \; \text{no instabilities (14),}$$  \hspace{1cm} (S37)

$$b > 3/2, \; \text{positive shear modulus (19),}$$  \hspace{1cm} (S38)

$$b > 5/2, \; \text{massless phonons (19).}$$  \hspace{1cm} (S39)

Here, we assume always standard quantization for the bulk axion fields $\phi^i$. Importantly, the duality between the EFT presented in the previous sections and the gravitational model is not simply given by the linear map $(A, B) \rightarrow (a, b)$.

The full non-linear stress-strain curve can be extracted numerically for arbitrary values of $a, b$ and analytically in the large temperature limit $T/m \gg 1$ (59). Some benchmark curves are shown in Fig. S4.

At large enough temperatures, after the linear elastic regime $\sigma = G_0 \varepsilon$, two different scaling regimes appear whose powers are given by:

$$v_0 = 2a, \; v = \frac{3a}{b}.$$  \hspace{1cm} (S40)

Going towards low temperature, the first non-linear scaling $v_0$ disappears and the second one remains as the dominant one. These scalings can be analytically derived using the properties of the background strained geometry (see (59) for the detailed derivation). In summary, combining analytical and numerical methods we can predict and observe a non-linear stress-strain scaling law of the form:

$$\sigma(\varepsilon) \sim \varepsilon^v, \; \; \; v = \frac{3a}{b}.$$  \hspace{1cm} (S41)

As a second step, we can compute the thermodynamic entropy $s = 4\pi/\nu^2_h$ in function of the background strain. Our numerical results shown in Fig. S4 present a universal scaling law of the type:

$$s(\varepsilon) \sim \varepsilon^\zeta, \; \; \; \zeta = \frac{2a}{b} \frac{1}{1 + a^2/b^2}.$$  \hspace{1cm} (S42)

In order to understand this result, we need to consider the background Einstein’s equation:

$$f \left( u^2 c^2 + 12 \right) + 4 \left( m^2 W - u f' - 3 \right) = 0,$$  \hspace{1cm} (S43)

and solve it close to the black hole horizon $u = u_h$. Then we have:

$$-4\pi u_h T e^{2\xi(u_h)/2} + 3 - m^2 u_h^{2b} (\cosh (\Omega - c_h))^a = 0.$$  \hspace{1cm} (S44)

One can show that the first term tends to a constant value at large strain (see Fig. S5) and it can be therefore discarded at

![Figure S4. Stress-strain and entropy-strain curves from the gravity theory. (A) Non-linear stress strain curve for different choices of $\nu$. (B) Entropy density as a function of the background strain for different choices of $\zeta$. The dashed lines indicate the scalings in Eqs. (S40) and (S42). The temperature is fixed to $T/m = 0.1$.](image)
least for the scaling analysis. Therefore, we obtain that:

\[ s \sim \left( \cosh(\Omega - c_h) \right)^{a/b}, \quad (S45) \]

which in the limit \( \Omega \gg c_h \) reduces to the simple scaling:

\[ s \sim \varepsilon^{2a/b}. \quad (S46) \]

We numerically verify (see Fig. S5) that \( \Omega \gg c_h \) only when \( a \ll b \), which is exactly the regime in which our analytic formula Eq. (S46) works. More generally, a correction has to be taken into account and the final scaling is given by Eq. (S42) as shown in Fig. S4(B).

Combing the two results Eq. (S41) and Eq. (S42), we can predict a universal relation between the stress and the entropy given by:

\[ s \sim \sigma^{\xi}, \quad \text{with} \quad \xi^{-1} = \frac{3}{2} \left( 1 + \frac{v^2}{9} \right), \quad (S47) \]

where \( v \) is the non-linear shear scaling in Eq. (S41). This prediction is numerically confirmed as displayed in Fig. 2(C). Notice that similar scalings between the non-linear stress and the external strain can be derived within the EFT formalism of (I4). Nevertheless, such a field theory construction does not provide a direct computation for the entropy of the system.

**S5 Low temperature entropy from the gravitational model**

In order to check the behavior of entropy in the gravitational description as the temperature \( T \to 0 \), we plot the entropy with respect to shear strain at low temperatures, as shown in Fig. S6. We find that the entropy at large shear strains is parametrized as \( s = b(T)e^{\xi} \) with \( \xi \) the power defined in Eq. (S42) and the residual zero temperature constant term is negligible in that limit.

By fitting the numerical data at large strain and small temperature, we obtain that \( b(T) \sim 4.3T^{0.05} \). This outcome is consistent with the arguments in (59) about an emergent Lifshitz-like anisotropic geometry in the limit of large strain.

In the case of amorphous solids, the zero-temperature limit corresponds to the regime where the temperature is small compared with any other energy scales, but obviously not exactly zero. Mapping this situation to our holographic setup, we should therefore consider the black hole at sufficiently low but finite temperature. Indeed, as we show in Fig. S6, the entropy for \( T \ll 1 \) remains non-zero, resembling a key feature of amorphous solids. Moreover, the scaling (for large \( \sigma \)),

\[ s \sim \sigma^{\xi}, \quad (S48) \]

is universal at sufficiently low temperatures. Therefore, the entropy increases under shear, according to our gravitational theory.

**S6 Additional simulation data for the frictionless granular model in 3D**

**S1. Discussion on the power-law fitting of stress-strain curves in the shear hardening regime**

To show more clearly the power law behavior of the stress-strain curves in Fig. 3(A), we plot the data on a log-log scale (Fig. S7). As can be seen from the plot, the data in the hardening regime, which begins around \( \varepsilon = 10^{-2} \) and ends around yielding, can be nicely fitted by straight lines on the log-log scale. More precisely, we fit the data to the power-law scaling, \( \sigma \sim \varepsilon^y \), in the window \( \varepsilon \geq 10^{-2} \) and \( \sigma < \lambda \sigma_{\text{max}} \), where \( \sigma_{\text{max}} \) is the maximum stress and \( \lambda \) is a predetermined parameter. The \( \lambda \)-dependence of the exponent \( y \) is shown in the inset of Fig. S7. The exact value of \( y \) depends on the choice of \( \lambda \), but the trend is robust (in Fig. 3(A), we set \( \lambda = 0.9 \)).

**S2. Estimation of the yielding stress and strain**

The yielding point \( \{ \epsilon_Y, \sigma_Y \} \) is estimated at \( \sigma_Y = \sigma_{\text{max}} \). The maximum stress \( \sigma_{\text{max}} \) is determined from a high order polynomial fit of the stress-strain peak. To test the robustness of our results on the parameter \( \lambda \), we fit the estimated \( \Delta \epsilon_Y = \epsilon_Y(\Delta \phi_g) - \epsilon_Y(0) \) as a function of the re-scaled degree of annealing \( \Delta \phi_g \) for several different \( \lambda \) (Fig. S8). Similar to Fig. 3 (where \( \lambda = 0.98 \), \( \epsilon_Y \) decreases with \( \Delta \phi_g \) for \( \phi_0 = 10^{-2} \) and increases (or remains nearly constant) for \( \phi_0 = 10 \). Note that \( \lambda = 1 \) is not ideal due to the ambiguity in determining the peak of the stress-strain curve, for the case of \( \phi_0 = 0.598 \) (see Fig. 3(A)), where the curve is nearly flat after yielding.

**S3. 2D nature of simple shear**

In this subsection, we verify that the effective rheology of our 3D system under simple shear deformations is two-dimensional. Therefore, it is sufficient to consider a 2D theory as presented above. In our 3D simulations, the system is deformed by simple shear in the \( x-y \) plane. We compute the distributions of projected contacting angles between particles in all the three planes, at different shear strains (see Fig. S9). The unstrained systems are isotropic as expected: the contacting angles distribute evenly in all three planes. As the system is strained, the distribution \( P_{\text{th}}(\theta_{xy}) \) in the shear plane becomes anisotropic, while in other two planes the distributions \( P_{\text{th}}(\theta_{xy}) \) and \( P_{\text{th}}(\theta_{yz}) \) remain isotropic. This proves that the dynamics of the system under shear is non-trivial only in the \( x-y \) plane and that the third dimension can be safely neglected in the theory.

**S4. Breaking point in the single stress-strain curve**

The response of amorphous solid to quasi-static external strain is not purely elastic since it contains a non-negligible amount of plasticity (9). The single strain-stress curve of an individual simulated sample is not continuous but presents several local stress drops due to plasticity, as shown in Fig. S10. Furthermore, for deeply annealed samples, yielding can be defined as the breaking point at which the stress presents a sharp drop signaling the onset of a global instability. This instability is
When the specified in this study refer to the averaged values unless otherwise fluctuations are neglected. Note that all simulation data presented
argument presented in Sec. S2, where sample-to-sample fluctuations, we perform averaging over the independent samples.
more as a solid. The definition of breaking point used in our related to the breakdown of the solid elastic behaviour due to plasticity – after yielding the system does not react anymore as a solid. The definition of breaking point used in our theoretical framework is equivalent to this criterion.

However, the yielding point defined in this way varies among samples. To suppress the sample-to-sample fluctuations, we perform averaging over the independent samples. When the average stress-strain curve is considered, yielding can be practically defined as the point at which the derivative of the stress vanishes. As evident from Fig. S10, the location of the global instability in the single samples and the location of the yielding point defined as \(d\sigma/d\varepsilon = 0\) in the average curve are close. This is also consistent with the theoretical argument presented in Sec. S2, where sample-to-sample fluctuations are neglected. Note that all simulation data presented in this study refer to the averaged values unless otherwise specified.

Figure S5. Numerical check of the derivation of the entropy scaling in the gravity theory. (A) The behaviour of the term \(\cosh(\Omega - c_b)\) in Eq. (S44) as the strain \(\varepsilon\) increases for various values of \((a, b)\). The dashed lines indicate the nonlinear scaling \(\varepsilon^2\). (B) The relation between \(4\pi u_0 T e^{\varepsilon(u_0)/2}\) and the strain \(\varepsilon\) for various values of \((a, b)\). We have fixed \(T/m = 0.1\).

Figure S6. Low temperature entropy at finite shear strains in the gravity theory. (A) Entropy density-strain curves for different temperatures. (B) Fit of the entropy density-strain data in the large strain region to \(s = b(T)\varepsilon^5\) (where \(T/m = 0.1\)). (C) The function \(b(T)\) with respect to the temperature \(T\). The best fit gives \(b(T) \sim 4.37^{0.06}\). The potential is fixed to \(W(B^*, Z^*) = B^* \frac{11}{12} Z^* \frac{5}{3}\).

S5. Discussion on the configurational entropy

The configurational entropy is obtained in the following way. A strained configuration at \(\varepsilon\) (for \(\varepsilon < \varepsilon_Y\) is quasi-statically decompressed (keeping the strain unchanged) until it unjams at \(\varphi_j(\varepsilon)\). The decompression step size is \(\delta\varphi = -10^{-4}\) for \(P \geq 1.0\), \(\delta\varphi = -2.5 \times 10^{-5}\) for \(0.3 \leq P < 1.0\), \(\delta\varphi = -5 \times 10^{-6}\) for \(0.03 \leq P < 0.3\), and \(\delta\varphi = -10^{-6}\) for \(P < 0.03\). The configuration entropy data \(s_{\text{conf}}(\varphi_j)\) are collected from Ref. (28), and the relationship \(\varphi_j(\varphi_j)\) from Ref. (27), for the same model. Together with the stress-strain curve \(\sigma(\varepsilon)\), we obtain \(s_{\text{conf}}(\sigma)\) (see Fig. 4(B)).

As suggested by Eq. (3), the entropy data in Fig. 4(B) are fitted to \(s_{\text{conf}} = s + c_1\sigma^5\). The rescaled plot of \((s_{\text{conf}} - s^*)/c_1\) versus \(\sigma\) is presented in Fig. S11, on a log-log scale (same data as in Fig. 4(B)).
Figure S7. Power-law fitting of the shear hardening part on stress-strain curves. Data are obtained from simulations of the 3D frictionless model, for $P_0 = 10^{-2}$ and a few different $\phi_g$. On a log-log scale. The inset shows the dependence of the non-linear exponent $\nu$ on the parameter $x$ and the error bars represent the standard error of the fitting coefficient.

Figure S8. Dependence of the yielding strain on the parameter $c$. The yielding strain difference $\Delta \varepsilon_Y$ is plotted as a function of re-scaled degree of annealing $\Delta \phi_g$, for $c = 1, 0.98, 0.90, 0.80$ (3D frictionless soft sphere (SS) model). Red color for $P_0 = 10^{-2}$ and blue for $P_0 = 10$.

Figure S9. Distributions of projected contact angles under simple shear. We plot the distributions of projected contact angles in $x - y$ (square), $x - z$ (triangle) and $y - z$ (circle) planes, at three different shear strains (A) $\varepsilon = 0$, (B) $\varepsilon = 0.05$ and (C) $\varepsilon = 0.1$. The simple shear is applied in the $x - y$ plane for 3D frictionless systems with $\phi_g = 0.643$.

S6. Edwards entropy

Edwards statistical mechanics of granular matter was introduced by Edwards and co-workers (33, 34). Because energy
without shear, in both simulation (35) and experimental (37) studies, the Lees-Edwards boundary conditions (39) are used. The calculation is based on an analysis of the volume fluctuations of a cluster formed by \( n \) particles surrounding a reference particle, and the variance \( \langle (V_n - \langle V_n \rangle)^2 \rangle \), where \( \langle \cdots \rangle \) represents the average over configurations. We find that for \( n \geq 1500 \), the variance \( \langle (V_n - \langle V_n \rangle)^2 \rangle \) scales linearly with \( n \) as shown in Fig. S12(A), which means that the volume fluctuation is extensive in the large-\( n \) limit and an intensive quantity \( \delta v^2 = \left\langle \frac{(V_n - \langle V_n \rangle)^2}{n^2} \right\rangle \) can be defined. Interestingly, Fig. S12(B) shows that the data of \( \delta v^2 \) versus \( \varphi \), collapse onto a master curve for different \( \varphi \), representing different degrees of annealing. This evidence justifies the application of Edwards theory: even though jammed states are protocol-dependent (\( \varphi \)-dependent), they can be unified in a single volume ensemble for the evaluation of Edwards entropy. The data are then fitted with a quadratic function to obtain \( \delta v^2(\varphi) \) (see Fig. S12(B)).

(ii) Compactivity. The compactivity \( \chi \), which plays the role of “temperature” in the volume ensemble, is defined via the generalized fluctuation relation (we have set the “Boltzmann constant” \( \lambda \) to one),

\[
\langle (V_n - \langle V_n \rangle)^2 \rangle = \frac{\chi^2 d(V_n)}{d\chi},
\]

or via the equivalent integral form,

\[
\frac{1}{\chi(\varphi)} = \int_{\varphi_0}^{\varphi} \frac{1}{\varphi^2 \delta v^2(\varphi)} d\varphi.
\]

Here, \( \varphi = \frac{x_{Ed}}{V_{Ed}} \) and we have set the average volume of grains \( V_{Ed} = 1 \) in Eq. (S50). In addition, we have chosen the lowest jamming density (the J-point density \( \varphi_J \) (30)) as reference, \( \varphi_J = \varphi_0 \approx 0.655 \) (27), whose compactivity has been set to infinity, \( \chi = \infty \). Based on Eq. (S50) and the quadratic fitting of \( \delta v^2(\varphi) \) in Fig. S12(B), we obtain \( \chi(\varphi) \) (see Fig. S12(C)).

(iii) Edwards entropy. Finally, the Edwards entropy \( s_{Ed}(\varphi_J) \) (per grain) is given by

\[
\chi^{-1} = -\varphi^2 \frac{d s_{Ed}}{d \varphi},
\]

or equivalently,

\[
s_{Ed}(\varphi_J) = s_{Ed}(\varphi_J) - \int_{\varphi_0}^{\varphi_J} \frac{1}{\varphi^2 \chi(\varphi)} d\varphi.
\]

The relevant quantity is the entropy change, \( \Delta s_{Ed}(\varphi) = -\int_{\varphi_0}^{\varphi_J} \frac{1}{\varphi^2 \chi(\varphi)} d\varphi \). Here, \( \Delta s_{Ed}(\varphi) \) is negative, which is reasonable since we expect that \( s_{Ed}(\varphi_J) < s_{Ed}(\varphi_0) \) for \( \varphi < \varphi_J \) (the number of possible packings should decrease with the packing density). Because \( s_{Ed}(\varphi_J) \) and \( \sigma(\varphi_J) \) are available (Fig. 4(A) and Fig. 3(A)), \( \Delta s_{Ed}(\varphi) \) can be easily converted to a function of the stress, \( \Delta s_{Ed}(\sigma) \) (see Fig. 4(C)).

As shown in Fig. 4(C), the Edwards entropy \( \Delta s_{Ed}(\sigma) \) increases with the stress \( \sigma \), consistent with the data of configurational entropy in Fig. 4(B). As suggested by Eq. (3), we fit the data to a power law, \( \Delta s_{Ed} = c_{Ed} \sigma^{\xi_{Ed}} \), where \( c_{Ed}, c_{Ed}, \) and \( \xi_{Ed} \) are fitting parameters. The obtained exponent \( \xi_{Ed} \) as a function of stress exponent \( \nu \) is plotted in the inset of Fig. 4(C), together with the exponent \( \xi \) obtained from the configurational entropy in Fig. 4(B). Both \( \xi \) and \( \xi_{Ed} \) decrease with \( \nu \) as predicted by our theory.

Figure S11. Power-law fitting of the configurational entropy obtained in simulations. Rescaled configurational entropy as a function of \( \sigma \), on a log-log scale (3D frictionless model, \( R_0 = 10^{-2} \)). The dashed lines present the fitting, \( s_{conf} = s^* + c_s \sigma^2 \), where \( s^*, c_s \), and \( \xi \) are fitting parameters.

is not conserved in granular matter, the choice of the standard microcanonical or canonical ensembles is not applicable. Edwards et al. (35) proposed an alternative volume ensemble, based on which the Edwards entropy can be defined as the logarithm of the number of micro-states for a given volume. The Edwards entropy in athermal granular matter plays a role analogous to the thermodynamic entropy in thermal systems.

Here, following a previously proposed method (35), we calculate the Edwards entropy of our granular model under simple shear. This method has been used for isotropic systems without shear, in both simulation (35–37) and experimental studies (38). In order to apply the method to sheared configurations, the Lees-Edwards boundary conditions (39) are used. The calculation is based on an analysis of the volume fluctuations through a generalized fluctuation-dissipation relation. The computation is performed for systems at the jamming density \( \varphi_J(\varepsilon) \) (where \( P = 0 \), see Fig. 4(A)), which are decompressed from sheared configurations \( \varphi(\varepsilon) \) (\( P > 0 \)) with the strain \( \varepsilon \) fixed. We describe below the procedure and specify quantities that need to be calculated.

(i) Volume fluctuations. The local Voronoi volume of each particle is defined through the radical Voronoi tessellation, realized using the Voro++ library (77). One then calculates the total volume \( V_n \) of a cluster formed by \( n \) particles surrounding a reference particle, and the variance \( \langle (V_n - \langle V_n \rangle)^2 \rangle \), where \( \langle \cdots \rangle \) represents the average over configurations. We find that for \( n \geq 1500 \), the variance \( \langle (V_n - \langle V_n \rangle)^2 \rangle \) scales linearly with \( n \) as shown in Fig. S12(A), which means that the volume fluctuation is extensive in the large-\( n \) limit and an intensive quantity
The simulation results presented in the main text are obtained from thermally annealed systems (N = 2000), using athermal quasi-static cyclic shear (79). The procedure is as follows: (i) Firstly, random configurations are generated and slowly compressed to a packing fraction \( \phi_0 = 0.66 \), slightly above the I-point density (minimum jamming density) \( \phi_j = 0.655 \) (27). (ii) Next, we apply cyclic shear under constant volume, athermal, quasi-static conditions, with a step size \( \Delta \varepsilon = 10^{-3} \). During one cycle, the shear strain \( \varepsilon \) is changed as \( \{0 \rightarrow \varepsilon_{\text{max}} \rightarrow -\varepsilon_{\text{max}} \rightarrow 0\} \), where \( \varepsilon_{\text{max}} = 0.05 \). (iii) The cycles stop if the system remains unjammed (the energy per particles is less than \( 10^{-13} \)) during one entire cycle. This step increases the jamming density of the system from \( \phi_j = \phi_l = 0.655 \) to \( \phi_j = 0.662 \). (iv)
The unjammed configuration at \( \{ \varphi_0 = 0.66, P_0 = 0 \} \) is slowly compressed to a target pressure \( P_0 = 10^{-2} \).

The above procedure prepares unstrained initial configurations. A stress-strain curve for such a configuration is plotted in Fig. S15, which shows clearly a shear hardening effect. It is known that mechanical annealing is less efficient compared to swap annealing (79). As a result, shear hardening is also weaker in mechanical annealed systems. Nevertheless, we conclude that this effect is independent of the annealing protocol, as demonstrated here.

**S7 Simulations of a frictionless granular model in 2D**

The 2D frictionless model is composed of \( N = 4000 \) soft disks, whose diameters are distributed according to \( P(D) \sim D^{-2} \), with \( D_{\text{min}} \leq D \leq D_{\text{min}}/0.45 \). Two disks are in contact if their separation \( r_{kl} \) is less than their mean diameter \( D_{kl} = (D_k + D_l)/2 \). Two contacting disks interact via a short-range repulsive potential,

\[
V(r_{kl}) = \frac{k_r}{2} \left( 1 - \frac{r_{kl}}{D_{kl}} \right)^2 .
\]  

(S54)

The unit of length is the average diameter of all disks, the unit of energy is \( 10^3 \times k_r \), and all disks have the same unit mass. The stress-strain curves (averaged over 24 independent samples) for \( P_0 = 10^{-2} \) and \( P_0 = 10^2 \) are plotted in Fig. S16, showing similar behavior as the case in 3D.

**S8 Simulations of a frictional granular model in 2D**

We simulate the same frictional 2D model as in Ref. (80). The force between two contacting particles is

\[
\mathbf{F}_{kl} = \mathbf{F}_{kl}^n + \mathbf{F}_{kl}^\tau ,
\]  

(S55)

where the normal component is

\[
\mathbf{F}_{kl}^n = -k^n \left( r_{kl} - \frac{D_k + D_l}{2} \right) \mathbf{n}_{kl} - \xi^n \mathbf{v}_{kl}^n .
\]  

(S56)

Here \( \mathbf{n}_{kl} \) is the unit vector along the contact direction, \( \mathbf{v}_{kl}^n = (\mathbf{v}_{kl} \cdot \mathbf{n}_{kl}) \mathbf{n}_{kl} \) the normal relative velocity, and \( \mathbf{v}_{kl} = \mathbf{v}_k - \mathbf{v}_l \) the relative velocity. The magnitude of the tangential force has to satisfy the Coulomb condition,

\[
|\mathbf{F}_{kl}^\tau| = \min \{ |\tilde{\mathbf{F}}_{kl}|, |\mu_\ell \mathbf{F}_{kl}^\tau| \} ,
\]  

(S57)

where \( \mu_\ell \) is the friction coefficient, and

\[
\tilde{\mathbf{F}}_{kl} = -k^\tau \mathbf{u}_{kl} - \zeta^\tau \mathbf{v}_{kl} ,
\]  

(S58)

with \( \mathbf{u}_{kl} \) and \( \mathbf{v}_{kl} \) being the tangential displacement and velocity. Both translational and rotational motions contribute to the tangential velocity at contact. Thus the tangential velocity is given by

\[
\mathbf{v}_{kl}^\tau = \mathbf{v}_{kl} - \mathbf{v}_{kl}^n - \frac{D_k \xi_k + D_l \xi_l}{2} ,
\]  

(S59)

where \( \omega \) is the angular velocity. The tangential displacement is updated according to

\[
\mathbf{u}_{kl}^\tau = \mathbf{u}_{kl}^\tau ,
\]  

(S60)
Figure S16. Stress-strain curves of 2D systems. Stress-strain curves for (A) $P_0 = 10^{-2}$ and (B) $P_0 = 10^2$ of a 2D frictionless granular model. Yielding points $\{\varepsilon_Y, \sigma_Y\}$ are estimated at $\sigma_Y = c\sigma_{\text{max}}$, where $\sigma_{\text{max}}$ is the maximum stress and $c = 0.98$.

Figure S17. Stress-strain curve of frictional systems. We plot a stress-strain curve obtained from simulations of the 2D frictional granular model ($P_0 = 10^{-2}$).

if $\|\vec{F}_{ij}\| \leq \|\mu \vec{F}_{ij}\|$, otherwise remains unchanged.

Our system contains $N = 4000$ grains, with equal numbers of larger and smaller grains. All grains have the same mass density, and the diameter ratio between big and small particles is 1.4. We set $k' = 0.2k^{\prime}, \zeta' = \zeta'' = \sqrt{m_Lk''}$ and $\mu_k = 1.0$, where $m_L$ is the mass of larger grains. The units of length, mass and energy are $D_L$, $m_L$ and $10^3 \times k''$.

Both compression (decompression) and shear are performed under athermal quasi-static conditions as in the frictionless case. The increment steps are $\delta \varphi = 10^{-4}$ and $\delta \varepsilon = 10^{-4}$. Energy is minimized by following Newton’s law, if $\sum_{k=1}^N \vec{v}_k \cdot \vec{F}_k \geq 0$, where $\vec{F}_k$ is the contacting force of particle $k$ (otherwise, both translational and rotational velocities are abandoned). Mechanical equilibrium is reached when the average force and the average torque per particle are both below $5 \times 10^{-9}$.

We employ the cyclic shear method to obtain annealed systems. Random configurations are generated at an area fraction $\varphi = 0.75$, and then slowly compressed to $\varphi = 0.845$. The pressure becomes non-zero around $\varphi = 0.829$, and thus the system is over-compressed at $\varphi = 0.845$. After 1000 cycles of shear with $\varepsilon_{\text{max}} = 0.04$, the jamming density increases to a larger value about $\varphi = 0.838$. The configurations are then slowly decompressed to $P_0 = 10^{-2}$. The above procedure prepares annealed unstrained initial configurations. The shear hardening effect is not as strong as in Fig. 3(A), but visible (see Fig. S17). As shown in Fig. 3(A), this effect could be further magnified by increasing the degree of annealing.

S9 Proposal for experimental validations

Finally, we expect our numerical observations to be reproducible in experiments: (i) It is not always easy to avoid friction in experimental systems, but our simulation result (Fig. S17) suggests that friction does not destroy the non-linear shear hardening effect. (ii) Our simulation results suggest that shear hardening should be observable in both 2D
disks \((81, 82)\) and 3D spheres \((83)\). (iii) It is possible to implement athermal quasi-static shear using the recently developed multi-ring Couette shear setup \((84)\). (iv) Mechanical annealing can be realized in experiments using cyclic shear \((40, 41)\).

(v) Although direct estimation of the configurational entropy is difficult, one can measure the “Edwards entropy” in experiments \((38)\), based on a thermodynamic framework initially proposed by Edwards \((33)\).
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