Discontinuous Deformation Analysis in Rock Mechanics and Rock Engineering

Genhua Shi

1 1746 Terrace Drive, Belmont, California, USA

Abstract. Discontinuous Deformation Analysis (DDA) is a Finite Element style discontinuous engineering numerical method. DDA is also an implicit version of DEM. DDA was developed especially for rock stability analysis, which is related to safety of personals and engineering projects. Therefore, DDA is naturally for rock mechanics and rock engineering. This paper in a natural way includes the basic formulation of DDA, the stability related simplex integration, and the discontinuous related contact algorithms and the important applications in rock mechanics and rock engineering.

1. The basic formulation of DDA

The mesh of DDA is real rock blocks and produced by real joints. DDA is a step-by-step FEM style, displacement-unknown, implicit, engineering numerical method.

DDA is for large movement, discontinuous, dynamic or static, forward and backward analyses.

1.1. Block Deformations

In basic DDA version, block has constant stress and strain or linear displacements.

\[
\begin{align*}
    u &= a_1 + a_2 x + a_3 y \\
    v &= b_1 + b_2 x + b_3 y
\end{align*}
\]

where, \((u, v)\): Displacement of point \((x, y)\), and

\[
\begin{pmatrix}
    \bar{u} \\
    \bar{v}
\end{pmatrix} = [T_i(x, y)] [D_i].
\]

\[
[T_i(x, y)] = \begin{pmatrix}
    1 & 0 & - (y - y_0) & (x - x_0) & 0 & (y - y_0)/2 \\
    0 & 1 & (x - x_0) & 0 & (y - y_0) & (x - x_0)/2
\end{pmatrix}
\]

\[
[D_i] = \begin{pmatrix}
    u_0 \\
    v_0 \\
    \tau_0 \\
    \epsilon_x \\
    \epsilon_y \\
    \gamma_{xy}
\end{pmatrix}
\]

1.2. Energy Minimization and Equilibrium Equations

Equilibrium equations of \(n\) blocks are given as:
\[
\begin{pmatrix}
K_{11} & K_{12} & K_{13} & \ldots & K_{1n} \\
K_{21} & K_{22} & K_{23} & \ldots & K_{2n} \\
K_{31} & K_{32} & K_{33} & \ldots & K_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
K_{n1} & K_{n2} & K_{n3} & \ldots & K_{nn}
\end{pmatrix}
\begin{pmatrix}
D_1 \\
D_2 \\
D_3 \\
\vdots \\
D_n
\end{pmatrix} = 
\begin{pmatrix}
F_1 \\
F_2 \\
F_3 \\
\vdots \\
F_n
\end{pmatrix}
\]  

(4)

where: \( K_{ij} \): 6 \times 6 connection submatrix of block \( i,j \); \( D_i \): 6 \times 1 displacement submatrix of block \( i \); \( F_i \): 6 \times 1 force submatrix of block \( i \)

1.3. Parallel transition

\[
\begin{pmatrix}
u \\
v
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\
v_0 \end{pmatrix}
\]

(5)

where \((u_0, v_0)\): Parallel translation (see Figure 1), \((u, v)\): Displacement of point \((x, y)\)

Equilibrium of the forces along \(x\)- and \(y\)-direction are:

\[
\frac{\partial \Pi}{\partial u_0} = 0
\]

(6)

\[
\frac{\partial \Pi}{\partial v_0} = 0
\]

(7)

1.4. Rotation

\[
\begin{pmatrix}
u \\
v
\end{pmatrix} = \begin{pmatrix} -\left(y - y_0\right) \\ \left(x - x_0\right) \end{pmatrix} \begin{pmatrix} r_0 \end{pmatrix}
\]

(8)

where \((u, v)\): Displacement of point \((x, y)\); \(r_0\): Angle of rotation (see Figure 2); \((x_0, y_0)\): Center of rotation.

Moment equilibrium of forces on block \(i\)

\[
\frac{\partial \Pi}{\partial r_0} = 0
\]

(9)

![Figure 1. Parallel translation](image1)

![Figure 2. Rotation](image2)

1.5. Normal strains

\[
\begin{pmatrix}
u \\
v
\end{pmatrix} = \begin{pmatrix} x - x_0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\
y - u_0 \end{pmatrix} \begin{pmatrix} \epsilon_x \\
\epsilon_y \end{pmatrix}
\]

(10)

where \(\epsilon_x, \epsilon_y\): Normal strains (see Figure 3); \((u, v)\): Displacement of point \((x, y)\); Equilibrium of forces and stress.
Along $x$ direction:
\[
\frac{\partial \Pi}{\partial \varepsilon_x} = 0
\]  \hspace{1cm} (11)
Along $y$ direction:
\[
\frac{\partial \Pi}{\partial \varepsilon_y} = 0
\]  \hspace{1cm} (12)

1.6. Shear strain

\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix}
= \begin{pmatrix}
  (y - y_0)/2 \\
  (x - x_0)/2
\end{pmatrix}
(\gamma_{xy})
\]  \hspace{1cm} (13)

where: $\gamma_{xy}$: Shear strain (see Figure 4); $(u, v)$: Displacement of point $(x, y)$.

Equilibrium of shear forces and shear stress on block $i$.
\[
\frac{\partial \Pi}{\partial \gamma_{xy}} = 0
\]  \hspace{1cm} (14)

1.7. Stiffness submatrix of a block

\[
S = \frac{E}{(1-\nu^2)} \begin{pmatrix}
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & \nu & 0 & 0 \\
  0 & \nu & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1-\nu^2/2 & 0
\end{pmatrix}
\rightarrow [K_{ii}]
\]  \hspace{1cm} (15)

where $S$ is area of block $i$.

1.8. Submatrix of initial stress

\[
S \begin{pmatrix}
  \sigma_x^0 \\
  \sigma_y^0 \\
  \tau_{xy}^0
\end{pmatrix}
\rightarrow [F_i]
\]  \hspace{1cm} (16)
1.9. Submatrix of point loading

\[ [T_i(x,y)]^T \begin{bmatrix} F_x \\ F_y \end{bmatrix} \rightarrow [F_i] \]  

(17)

where \((F_x, F_y)\): Loading force and \((x, y)\): Loading point.

1.10. Submatrix of volume loading

\[
\begin{bmatrix} f_x S_x \\ f_y S_y \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow [F_i]
\]  

(18)

where \((f_x, f_y)\): Body force.

1.11. Submatrix of displacement constraints in a direction

Constraint direction \((l_x, l_y)\), and

\[ l_x^2 + l_y^2 = 1 \]  

(19)

\[ p[C_i][C_i]^T \rightarrow [K_{ii}] \]  

(20)

\[ [C_i] = [T_i]^T \begin{bmatrix} l_x \\ l_y \end{bmatrix} \]  

(21)

1.12. Mass Submatrix and inertia submatrix

\[
\frac{2M}{\Delta} \int [T_i]^T[T_i] \, dx \, dy \rightarrow [K_{ii}] 
\]  

(22)

\[
\frac{1}{\Delta} \left( \int [T_i]^T[T_i] \, dx \, dy \right) [V_0] \rightarrow [F_i] 
\]  

(23)

where \(\Delta\): Time interval of this step and \([V_0]\): Starting velocity of this step.

Starting velocity \([V_1]\) of next step:

\[
[V_1] = \Delta \frac{\delta^2 [p(t)]}{\delta t^2} + [V_0] = \left( \frac{2}{\Delta} [D_i] - [V_0] \right) 
\]  

(24)

\[
\int [T_i]^T[T_i] \, dx \, dy = 
\begin{pmatrix}
S & 0 & 0 & 0 & 0 & 0 \\
0 & S & 0 & 0 & 0 & 0 \\
0 & 0 & S_1 + S_2 & -S_3 & S_3 & (S_1 - S_2)/2 \\
0 & 0 & -S_3 & S_1 & 0 & S_3/2 \\
0 & 0 & S_3 & 0 & S_2 & S_3/2 \\
0 & 0 & (S_1 - S_2)/2 & S_3/2 & S_3/2 & (S_1 + S_2)/4 \\
\end{pmatrix} 
\]  

(25)

where: \((x_0, y_0)\): Center of gravity and

\[
S_1 = \int (x^2 - x_0x) \, dx \, dy \\
S_2 = \int (y^2 - y_0y) \, dx \, dy \\
S_3 = \int (xy - x_0y) \, dx \, dy 
\]  

(26)

1.13. Submatrix of contact spring

\[
\Delta = \begin{bmatrix} 1 & x_1 + u_1 & y_1 + v_1 \\ 1 & x_2 + u_2 & y_2 + v_2 \\ 1 & x_3 + u_3 & y_3 + v_3 \end{bmatrix} 
\]  

(27)

\[ d = \frac{\Delta}{r} \]  

(28)
\[ l = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2} \quad (29) \]

where: \( d \): Penetration distance, if \( \Delta < 0 \): Penetration, and if \( \Delta > 0 \): Open.

\[ d = \frac{\Delta}{l} \begin{bmatrix} 1 & x_2 + u_2 & y_2 + v_2 \\ 1 & x_3 + u_3 & y_3 + v_3 \end{bmatrix} \quad (30) \]

\[ l = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2} \quad (31) \]

\[ \Delta = \begin{vmatrix} 1 & x_1 + u_1 & y_1 + v_1 \\ 1 & x_2 + u_2 & y_2 + v_2 \\ 1 & x_3 + u_3 & y_3 + v_3 \end{vmatrix} + \begin{vmatrix} 1 & u_1 & v_1 \\ 1 & u_2 & v_2 \\ 1 & u_3 & v_3 \end{vmatrix} \quad (32) \]

Neglecting second order small, it yields:

\[ \Delta = S_0 + u_1(y_2 - y_3) + v_1(x_3 - x_2) \]
\[ + u_2(y_3 - y_1) + v_2(x_1 - x_3) \]
\[ + u_3(y_1 - y_2) + v_3(x_2 - x_1) \quad (33) \]

\[ S_0 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \quad (34) \]

\[ d = \frac{S_0}{l} + \left( e_1 e_2 e_3 e_4 e_5 e_6 \right) \]

Submatrix of a stiff spring is

\[ p[E_i][E_i]^T \rightarrow [K_{ii}] \]
\[ p[E_i][G_j]^T \rightarrow [K_{ij}] \]
\[ p[G_j][E_i]^T \rightarrow [K_{ji}] \]
\[ p[G_j][G_j]^T \rightarrow [K_{jj}] \]
\[ -\frac{S_0}{l} [E_i] \rightarrow [F_i] \]
\[ -\frac{S_0}{l} [G_j] \rightarrow [F_j] \quad (36) \]

Submatrix of a normal stiff spring is
\[ [E_j] = \frac{1}{I} [T_j(x_1, y_1)]^T (y_2 - y_3) \]
\[ [G_j] = \frac{1}{I} [T_j(x_2, y_2)]^T (y_3 - y_1) \]
\[ + \frac{1}{I} [T_j(x_3, y_3)]^T (y_1 - y_2) \]

where \( p \): Stiffness of the spring.

Submatrix of a shear stiff spring is:
\[ [E_j] = \frac{1}{I} [T_j(x_1, y_1)]^T (x_3 - x_2) \]
\[ [G_j] = \frac{1}{I} [T_j(x_2, y_2)]^T (y_3 - y_2) \]
\[ + \frac{1}{I} [T_j(x_3, y_3)]^T (y_1 - y_2) \]

where \( P_0 = (x_0, y_0) \): Contact point in \( P_2P_3 \).

Submatrix of Friction forces is
\[ - \frac{PS_1}{I} [E_i] \rightarrow [F_i] \]
\[ \frac{PS_1}{I} [G_i] \rightarrow [F_i] \]

where \( pS_1 \) is sliding force and
\[ [E_i] = \frac{1}{I} [T_i(x_1, y_1)]^T (x_3 - x_2) \]
\[ [G_i] = \frac{1}{I} [T_i(x_2, y_2)]^T (y_3 - y_2) \]

2. Simplex integration
At least every engineer has to compute the volume of generally shaped blocks. Is there a formula where the coordinates of boundary vertices precisely represent the volume? If block movements are considered, center of gravity has to be computed. Is there a formula where the coordinates of boundary vertices also represent the center of gravity? The simplex integration developed for DDA computation can also solve these questions. The convergency and accuracy of DDA and other integration methods depend upon mainly the analytical integrations on complex shapes.

2.1. Normal integration on a simplex
Transfer a general simplex to the coordinate simplex as:
\[ (x_0, y_0) \rightarrow (0,0) \]
\[ (x_i, y_i) \rightarrow (1,0) \]
\[ (x_j, y_j) \rightarrow (0,1) \]

\[ \begin{pmatrix} 1 & 1 & 1 \\ x_0 & x_i & x_j \\ y_0 & y_i & y_j \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \]
\[ 1 = \lambda_0 + \lambda_1 + \lambda_2 \]
\[ x = \lambda_0 x_0 + \lambda_1 x_i + \lambda_2 x_j \]
\[ y = \lambda_0 y_0 + \lambda_1 y_i + \lambda_2 y_j \]

\[ J_{0ij} = \frac{d(x,y)}{d(\lambda_1,\lambda_2)} = \begin{vmatrix} x_i - x_0 & x_j - x_0 \\ y_i - y_0 & y_j - y_0 \end{vmatrix} \]
\[ = \begin{vmatrix} 1 & x_0 & y_0 \\ 1 & x_i & y_i \\ 1 & x_j & y_j \end{vmatrix} \]
Normal integral of $f(x,y)$ on triangle $0ij$
\[\int_0^1 \int_0^{1-\lambda_1} f(x_1, x_2) |J_{0ij}| \ d\lambda_1 \ d\lambda_2 \] (45)

2.2. Simplex integration

Simplex integral of $f(x,y)$ on triangle $0ij$

\[S_{0ij} = \int_0^1 \int_0^{1-\lambda_1} f(x,y) |J_{0ij}| \ d\lambda_2 \ d\lambda_1 \] (46)

Simplex integral are additive (see Figure 5)

\[S_{12345} = S_{012} + S_{023} + S_{034} + S_{045} + S_{051} \] (47)

![Figure 5](image)

**Figure 5** Algebraic addition of simplex integration

Application of 2D simplex integrations

\[S_1 = \iint_{123...m} \ dxdy \]
\[= \sum_{i=1}^{m} \int_0^1 \int_0^{1-\lambda_1} J_{0\ i\ i+1} \ d\lambda_1 \ d\lambda_2 \]
\[= \sum_{i=1}^{m} \frac{1}{2} J_{0\ i\ i+1} \]
\[= \sum_{i=1}^{m} \frac{1}{2} \begin{vmatrix} x_0 & y_0 \\ x_i & y_i \\ x_{i+1} & y_{i+1} \end{vmatrix} \] (48)

\[S_x = \iint_{123...m} x \ dxdy \]
\[= \sum_{i=1}^{m} \frac{1}{6} (x_0 + x_i + x_{i+1}) J_{0\ i\ i+1} \] (49)

\[S_y = \iint_{123...m} y \ dxdy \]
\[= \sum_{i=1}^{m} \frac{1}{6} (y_0 + y_i + y_{i+1}) J_{0\ i\ i+1} \] (50)

\[S_{xx} = \iint_{123...m} x^2 \ dxdy \]
\[= \sum_{i=1}^{m} \frac{1}{12} (x_0^2 + x_i^2 + x_{i+1}^2 + x_0 x_i + x_0 x_{i+1} + x_i x_{i+1}) J_{0\ i\ i+1} \] (51)
\[ S_{xy} = \iiint_{123...m} xy \, dx \, dy \]
\[ = \sum_{l=1}^{m} \frac{1}{24} (2x_0y_0 + 2x_ly_l + 2x_{i+1}y_{i+1} + x_0y_l + x_{i+1}y_0 + x_0y_{i+1} + x_0y_0 + x_{i+1}y_l + x_{i+1}y_0) \]

\[ \sum_{l=1}^{m} \frac{1}{24} (2x_0y_0 + 2x_ly_l + 2x_{i+1}y_{i+1} + x_0y_l + x_{i+1}y_0 + x_0y_{i+1} + x_0y_0 + x_{i+1}y_l + x_{i+1}y_0) \]

3. Contact Algorithm

Contact is a most common phenomenon. The difference of continuous and discontinuous computations is “contact”. Since DDA is related with safety, in order to make DDA to be a mathematically proven method, “contact theory and algorithm” were developed and explained intuitively here in this paper. The contact algorithm will insure all the possible rock movements are considered, and then all of the failure modes are computed. Based on “Contact Theory”, the contact algorithms on DDA are satisfying contact inequality equations.

3.1. Basic geometric representations

As presented by Minkowski Sum (1910),
- \( a \ b \) is an edge, which is from \( a \) to \( b \);
- \( (a \ b) = (x_j - x_i, y_j - y_i, z_j - z_i) \) is a vector;
- \( A + B = \bigcup_{a \in A, b \in B} (a + b) \);
- \( A + x \) is a parallel translation of block \( A \) along vector \( x \), and
- \( A \) one-dimensional solid angle \( A \) with a top \((0, 0, 0)\) along \( \vec{e}_1 \) is a ray \( A \vec{e}_1 = \bigcup_{t \geq 0} t \vec{e}_1 \).

A 2D solid angle \( A \vec{e}_1 \vec{e}_2 \) means the swept area from vector \( \vec{e}_1 \) to \( \vec{e}_2 \) along the direction from \( o \vec{x} \) to \( o \vec{y} \) (see Figure 6). A 3D solid angle \( A \) with top \((0, 0, 0)\) is represented as \( A \vec{e}_1 \vec{e}_2 ... \vec{e}_{u-1} \vec{e}_u \). Vectors \( \vec{e}_1, \ldots, \vec{e}_{u-1}, \vec{e}_u \) are the edge vectors (see Figure 7).

![Figure 6. A 2D solid angle](image)

![Figure 7. A 3D solid angle](image)

A plane \( A \) passing \((0, 0, 0)\) is defined as the set of \( x \) satisfying the equation

\[ \perp n_{1r} = \{ x \cdot \vec{n}_{1r} = 0 \} \]

(53)

Half space \( A \) passing \((0, 0, 0)\) can be represented as the set of \( x \) satisfying the homogeneous inequality equation (see Figure 8),

\[ \uparrow n_{1r} = \{ x \cdot \vec{n}_{1r} \geq 0 \} \]

(54)
3.2. Representation of block

A block can be represented by its boundary, i.e.,

- A one-dimensional block is an edge: $a_1a_2$.
- A two-dimensional block is a polygon: $a_1a_2 \cdots a_{p-1}a_p$, rotates upward in the right-hand rule, and
- The boundary of a 3D block is composed by polygons. $a_{1k}a_{2k} \cdots a_{p-1k}a_{pk}$.

The rotation axis points outward to block $A$ (see Figures 9 and 10).

$$A = \cap_{k=1,\cdots,f}(\uparrow m_{1k} + a_k)$$

$$A = \cup_{i} \cap_{k \in L_i}(\uparrow m_{ik} + a_k)$$

Denote a vertex of block $A$ as
Denote a boundary vector or edge as:
\[ A_i(0), A(0) = \bigcup_i A_i(0). \] (58)
Denote a boundary 2D solid angle or polygon as:
\[ A_i(1), A(1) = \bigcup_i A_i(1) \] (59)

If \( d \in \partial A \), it defines
\[ \nabla d = \bigcup_{dA \subset A} \{ \nabla(dA) \backslash dA \subset A \}. \] (61)

3.3. Entrance block
The complicated contact conditions between two general blocks \( A \) and \( B \) can be simplified as the relations of a reference point \( a_0 \) and an entrance block \( E(A, B) \).

Definition of contact \( A:B \) (Figure 11) as:
\[ (A \cap B) \subset (\partial A \cap \partial B) \neq \emptyset. \] (62)

Theorem of common boundary as:
\[ A \cap B \subset (\partial A \cap \partial B) \neq \emptyset \iff A \cap B \neq \emptyset, \quad \text{int}(A) \cap \text{int}(B) = \emptyset. \] (63)

Definition of distance:
\[ |A,B| = \min\{|x| \backslash (A + x) \cap B \neq \emptyset\}. \] (64)

Choosing any point \( a_0 \) from block \( A \), the geometric definition of entrance block \( E(A, B) \) (see Figure 12):
\[ E(A,B) = \{a_0 + x \backslash (A + x) \cap B \neq \emptyset\} \] (65)
Theorem of entrance block:

\[ E(A, B) = B - A + a_0 \]  \hspace{1cm} (66)

3.4. Basic theorems of entrance block

The following theorems will make the computation of entrance blocks more efficient.

- Theorem of entrance:

  \[ A \cap B = \emptyset \iff a_0 \notin E(A, B) \]  \hspace{1cm} (67a)

  \[ A \cap B \neq \emptyset \iff a_0 \in E(A, B) \]  \hspace{1cm} (67b)

- Theory of distance:

  \[ |A, B| = |a_0, E(A, B)| \]  \hspace{1cm} (68)

- Theorem of convex blocks:

  \( A \) and \( B \) are convex, the entrance block \( E(A, B) \) is convex.

- Theorem of finite covers of entrance blocks:

  As shown in Figure 13, \( A \) and \( B \) are \( n \) dimensional blocks, \( n = 2, 3 \), then

  \[ E(A, B) = \bigcup_{k=0}^{n} E(A(n - k), B(k)) \]

  \[ \partial E(A, B) \subseteq \bigcup_{k=0}^{n-1} E(A(n - 1 - k), B(k)) \]  \hspace{1cm} (69)
Figure 14 shows the finite covers of two dimensional $E(A, B)$. This is a simple way to compute entrance block $E(A, B)$. This method is not efficient, it only been used to find the boundary structure of $E(A, B)$.

$$E(a, b) \in \partial E(A, B) \Rightarrow \text{int}(a) \cap \text{int}(b) = \emptyset. \quad (70)$$

$A$ and $B$ are $n$ dimensional convex blocks, $n = 2, 3$.

$$E(a, b) \in \partial E(A, B) \Leftrightarrow \text{int}(a) \cap \text{int}(b) = \emptyset. \quad (71)$$

3.5. Finding contact vector covers on 2D entrance surface

As the discontinuous computation follows time steps, the step displacements can be chosen small enough so that all contacts become independent solid angle to solid angle contacts. (see Figure 15)

![Figure 15](image)

(1) angle to angle  \hspace{1cm} (2) parallel edges

Figure 15 Local contacts of solid angle to solid angle

Considering the entrance of two 2D solid angles

$$A = \mathcal{A}\hat{e}_1\hat{e}_2 + \mathbf{e}, \quad (72a)$$

$$B = \mathcal{A}\hat{h}_1\hat{h}_2 + \mathbf{h}. \quad (72b)$$

Denote

$$C(0,1) = \{E(e, \hat{h}_1 + \hat{h}) \setminus \mathcal{A}\hat{e}_1\hat{e}_2; \uparrow n_{2j}\}. \quad (73a)$$

$$C(1,0) = \{E(\hat{e}_1 + e, h) \setminus \uparrow n_{1j}; \mathcal{A}\hat{h}_1\hat{h}_2\}. \quad (73b)$$

Theorem of contact vectors of 2D solid angle contact (Figure 16):
\[ \partial E(A,B) \subset C(0,1) \cup C(1,0). \]  

(74)

Figure 16 Contact vectors of a 2D convex solid angle and a 2D concave solid angle

Thus, if block \( A \) and block \( B \) are convex. (see Figures 17 and 18).

\[ \partial E(A,B) = C(0,1) \cup C(1,0). \]  

(75)

Figure 17 Examples of contact vectors of two 2D solid angles
Symmetric 2D solid angle contact

Finding contact vectors of symmetric 2D solid angle contact

Figure 18 Contact between two vertically opposite convex angles

3.6. Entrance convex round corner solid angle of round corner convex solid angles

A is the round corner angle of solid angle $A_0$ (see Figure 19). B is the round corner angle of solid angle $B_0$. The normal vector $\vec{n}_{ij}$ point the inside of a half plane.

\[ A_0 = (\uparrow n_{11} + e) \cap (\uparrow n_{12} + e). \quad (76a) \]
\[ B_0 = (\uparrow n_{21} + h) \cap (\uparrow n_{22} + h). \quad (76b) \]

Figure 19 round corner angle is an entrance angle

Assuming the normal vectors are unit vectors, $A_1$ and $B_1$ are discs:

\[ A_1 = \{ |x - a_0| \leq r_1 \}, \]
\[ B_1 = \{ |x - b_0| \leq r_2 \}. \quad (77) \]

Let,
Define two inner angles $A_2$ and $B_2$ by the following equations:

$$A_2 = (\uparrow n_{11} + e_0) \cap (\uparrow n_{12} + e_0),$$

$$B_2 = (\uparrow n_{21} + h_0) \cap (\uparrow n_{22} + h_0).$$

The round corner angles $A$ and $B$ are defined as follows:

$$A = E(A_1, A_2) = A_2 - A_1 + a_0,$$

$$B = E(B_1, B_2) = B_2 - B_1 + b_0,$$

where $b_0$ is the reference point of $B_1$.

Based on Section 3.3, $E(A_1, B_1)$ is disc

$$E(A_1, B_1) = \{ |x - b_0| \leq r_1 + r_2 \}.$$
Figure 20 The entrance block of two 2D round corner convex solid angles is also a round corner convex solid angle.

3.7. Finding contact edge covers on surface of 2D entrance block

The following theorems are for understanding and finding the boundaries of 2D entrance blocks.

If blocks $A$ and $B$ are convex,

\[
A = a_1 a_2 \cdots a_p a_{p+1} = a_1,
\]

\[
B = b_1 b_2 \cdots b_q b_{q+1} = b_1.
\]

Denote

\[
C(0,1) = \{E(a_i, b_j) \setminus \exists a_i: \uparrow m_{2j}\},
\]

\[
C(1,0) = \{E(a_i, a_{i+1}, b_j) \setminus \uparrow m_{1i}: \exists b_j\}.
\]

Theorem of contact edges of 2D blocks:
If blocks $A$ and $B$ are convex,

\[
\partial E(A, B) \subset C(0,1) \cup C(1,0).
\]

where $C(0,1)$ and $C(1,0)$ are fully connected by $C(0,0)$.

In each time step, maximum displacement is $r$. Only a small part of $E(A, B)$ is used:

\[
\{\|x - a_0\| \leq 2r\} \cap E(A, B).
\]

If $A, B$ are concave, $\partial E(A, B) \subset C(0,1) \cup C(1,0)$. $C(0,1)$ and $C(1,0)$ are partly connected by $C(0,0)$, concave connections use cutting. In each time step, maximum displacement is $r$. Only a small part of $E(A, B)$ is used.

Figure 21 shows the contact edges and the covers of $\partial E(A, B)$. Figure 22 shows contact edges of concave blocks.
Figure 21 Contact edges of two dimensional $E(A, B)$ form the cover of the boundary of $E(A, B)$
3.8. Finding contact polygon covers on surface of 3D entrance block

The 3D blocks $A$ and $B$ are the same as defined in Section 3.2. The faces of block $A$ are polygons, which rotate outward. The vertices of face $k$ are:

$$P_k = a_{1k}a_{2k} \cdots a_{p-1k}a_{pk}, \quad a_{p+1k} = a_{1k}. \quad (88)$$

The normal vector $\vec{m}_{1k}$ of face $k$ points inward. The faces of block $B$ are polygons, which rotate the outward. The vertices of face $l$ are:

$$Q_l = b_{1l}b_{2l} \cdots b_{q-1l}b_{ql}, \quad b_{q+1l} = b_{1l}. \quad (89)$$

The normal vector $\vec{m}_{2l}$ of face $l$ points the inside of block $B$. Denote (see Figures 23, 24)

$$C(0,2) = \{E(e, Q_l) \setminus \{e: \uparrow m_{2l}\}\}. \quad (90a)$$

$$C(2,0) = \{E(P_k, h) \uparrow m_{1k}: \{h\}\}. \quad (90b)$$

$$C(1,1) = \{E(e e_r, h h_s) \setminus \{e e_r: \{h h_s\}\}\}. \quad (90c)$$

Theorem of contact polygons of 3D entrance block. If $A$ and $B$ are convex blocks,

$$\partial E(A, B) = C(0,2) \cup C(2,0) \cup C(1,1). \quad (91)$$

![Figure 22](image1.png) Computed $E(A, B)$ and all contact edges

![Figure 23](image2.png) Contact polygon of a 3d vertex and a face polygon
If $A, B$ are convex, $\partial E(A, B) = C(0, 2) \cup C(1, 1) \cup C(2, 0), C(0, 2), C(1, 1)$ and $C(2, 0)$ is fully connected by $C(1, 0)$ and $C(0, 1)$. In each time step, maximum displacement is $r$. Only a small part of $E(A, B)$ is used:

$$\{ |x - a_0| \leq 2r \} \cap E(A, B). \quad (92)$$

Figures 25 and 26 show the computed three dimensional $E(A, B)$ of convex blocks. Figures 27, 28 and 29 show the computed three dimensional $E(A, B)$ and all contact polygons where the entrance blocks of Figure 27 are convex; the entrance blocks of Figures 28, 29 are concave or curved.

Figure 24 A 3D edge to edge contact

Figure 25 Computed convex $E(A, B)$
Figure 26 Computed convex $E(A, A)$

Figure 27 Computed convex $E(A, B)$ with all its contact polygons

If $A, B$ are concave, $\partial E(A, B) \subseteq C(0,2) \cup C(1,1) \cup C(2,0)$. $C(0,2), C(1,1)$ and $C(2,0)$ are partly connected by $C(1,0)$ and $C(0,1)$. Cutting makes the concave connections. In each time step, maximum displacement is $r$. Only a small part of $E(A, B)$ is used:

$$\{|x - a_0| \leq 2r\} \cap E(A, B).$$

(93)
Figure 28 Three-dimensional concave entrance block $E(A, B)$

Figure 29 Computed curved $E(A, B)$ with all its contact polygons

3.9. Open-close iterations to ensue block contact
The contact theory is for contact of block pairs. For the contact of whole block systems, open-close iterations are used to ensure no-tension, no-penetration. The open-close iteration converges due to the analytical mass matrices. This iteration is to transfer system of inequality to subset of inequalities and then to soluble equations. DEM use explicit open-close iteration, DDA use implicit open-close iteration.

4. Rock stability computations
DDA is used for surface and underground rock excavation stability, mining stability, rock reinforcement, rock slope stability, rock toppling stability and dam foundation stability analyses. The way of using DDA to do the rock stability analysis is demonstrated.
The stability analyses require the computation itself is stable. Three convergences are required:

- Equilibrium equation convergence to ensure the equilibrium;
- Open-close iteration convergence to ensure the contact, and
- Time depending convergence to ensure the stability.

Based on simplex integration, analytical mass matrix ensures all 3 convergences.

### 4.1. Backward Analysis

DDA was started from backward analysis, which knows measured displacements to compute the discontinuous movements. Currently the practical stability judgements are based on displacement measurements, as the discontinuous displacements are the true criteria of failure, backward analysis cases also are explained for future applications (see Figures 30 and 31).

![Figure 30](image1.png) Back ward analysis of a rock test

![Figure 31](image2.png) Back ward analysis of a shaft discontinuous deformation

### 4.2. Underground rock stability analysis

The rock stability of tunnel and underground chamber rock excavation is critical for both the personal safety and the structure safety. Figure 32-35 show the collapse, bolt support, bolt tensions and time depending bolt tension forces respectively.
**Figure 32** The collapse without bolt support

**Figure 33** Stable underground chamber with bolts

**Figure 34** The computed tensions of bolts
Figure 35 The computed time-depnding bolt force

4.3. Slope stability analysis
Very likely, major civil engineering projects have rock slope issue. The dimension of landslide can be very large. Slope, landslide stability is important. Figures 36-39 show the rock sliding, cable support, cable tensions and time depending cable tension forces respectively.

Figure 36 Slope blocks slide

Figure 37 Cable reinforced stable slope
4.4. *Toppling stability Analysis*

Rock slope toppling is as often to happen as landslide. The stability analysis of rock toppling requires more accurate discontinuous computation. *Figure 40,41* show the toppling computation and computed time-depending displacements of chosen points respectively.
The computed time depending movements of measured points

4.5. Dam foundation stability Analysis
The importance of dam foundation stability is imaginable. DDA computation even can find the critical friction angle or cohesion, which is the condition from stable to moving. (Figure 42-45)

Figure 41 The computed time depending movements of measured points

Figure 42 A gravity dam failure computation

Figure 43 A RCC gravity dam failure computation
4.6. Arch dam stability Analysis
The arch dam stability includes arch block stability and its abutment stability. DDA can compute both. Figure 46, 47 show the computed arch instability condition.

Figure 44 Stable condition of a buttress dam

Figure 45 The sliding of a buttress dam

Figure 46 An arch dam failure computation

Figure 47 A 3d arch dam failure computation
References

[1]. Shi, G.H. Contact Theory. Science China, Technological Sciences, Vol.58, No 9, 2015 1450-1496.
[2]. Wu W, Zhu H, Zhuang X, Ma G and Cai Y. A Multi-shell cover algorithm for contact detection in
the three-dimensional discontinuous deformation analysis. Theoretical and Applied Fracture
Mechanics, 2014, 72: 136-149.
[3]. He L. Three-dimensional numerical manifold method and rock engineering applications. PhD
thesis, Nanyang Technological University, 2010.
[4]. Shi, G.H. Block system modeling by discontinuous deformation analysis. Computational
Mechanics Publications, Southampton, UK and Boston, USA, 1993.
[5]. Zhong ZH, Nilsson L. A contact-searching algorithm for general contact problems. Computers &
Structures, 1989, 33(1): 197-209.
[6]. Goodman, R.E., Taylor, R.L., and Brekke, T.L. A Model for the Mechanics of Jointed Rock.
Journal of the Soil Mechanics and Foundations Div., ASCE, 1968, 94(3): 637-659.