Persistent Current of a One-Dimensional Wigner Crystal-Ring

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Abstract

We calculate the magnetic moment (‘persistent current’) in a strongly correlated electron system — a Wigner crystal — in a one-dimensional ballistic ring. The flux- and temperature dependence of the persistent current is shown to be essentially the same as for a system of non-interacting electrons. In contrast, by incorporating into the ring geometry a tunnel barrier, that pins the Wigner crystal, the current is suppressed and its temperature dependence is drastically changed. The competition between two temperature effects — a reduced barrier height for macroscopic tunneling and a loss of quantum coherence — results in a sharp peak in the temperature dependence, which for a rigid Wigner crystal appears at $T \sim 0.5 \hbar s/L$, ($s$ is the sound velocity of the Wigner crystal, $L$ is the length of the ring).

PACS numbers: 73.40.Gk, 7340.Jn
In recent experiments \cite{1,2} persistent currents have been observed in the ballistic transport regime of mesoscopic rings formed in the laterally confined two-dimensional electron gas of certain AlGaAs heterostructures. The current $I$ and the associated magnetic moment were found to oscillate as a function of magnetic flux with period $\Phi_0 = hc/e$ — the quantum unit of flux — and amplitude $I_0 \sim ev_F/L$ ($e$ is the electronic charge, $v_F$ the Fermi velocity, and $L$ the length of the circumference of the ring). These results are in excellent agreement with a theory of such Aharonov-Bohm (AB) oscillations based on a free electron model of the ballistic electrons \cite{3,4}. Since electron-electron interactions in the semiconductor ring are not weak, and since electron correlations must play an important role when the density of conducting electrons is low, this agreement is quite surprising. In diffusive metal rings, for example, where the electronic mean free path is short ($\ell \ll L$), it has been suggested that electron correlations significantly enhance the amplitude of the AB oscillations \cite{5}. Thus the question of how Coulomb correlations in a system of ballistic electrons affect the magnitude of the persistent current is of significant interest.

In this Letter we study persistent currents and AB oscillations in systems of spinless interacting electrons confined to a one-dimensional ring; the electrons are assumed to be so strongly correlated that they form a Wigner crystal. In an ideal ring the mechanism of the persistent current is a dissipationless sliding of the crystal as a whole. We demonstrate that the resulting current oscillates as a function of magnetic flux with period $\Phi_0$. Its amplitude at low temperatures is exactly the same, $I_0 = ev_F/L$ ($v_F = \pi \hbar/m a$ is of the Fermi velocity, $a$ is the period of the Wigner crystal, $m$ is the electron mass) as for noninteracting electrons of the same density in accordance with general theorem \cite{6}. If the temperature is raised, the amplitude of the oscillations is exponentially suppressed: $I(T) \sim I_0 \exp(-\pi T/2T_0)$, where $T_0 \equiv \hbar v_F/L$ is the characteristic crossover temperature. Therefore the magnitude as well as the temperature dependence of the persistent current carried by an ideal Wigner crystal looks completely identical to that of a current carried by a free electron gas.

The situation changes drastically if a potential barrier, somewhere along the ring, impedes the motion of the electrons. Charge transport in this case requires that electrons tunnel.
through the barrier - the process which for strongly correlated electrons can be viewed as a macroscopic tunneling of a Wigner crystal-ring. In the case of high enough barrier (strong pinning) it is convenient to think of the motion of the crystal as a two-step process, where first a single electron tunnels through the barrier producing a deformation of a finite portion of the Wigner crystal, which then is relaxed [7,8]. This process necessarily depends on the elastic properties of the crystal, and as a result the magnitude of the persistent current will depend on the sound velocity, $s$, in the Wigner lattice. As our analysis below will show, the temperature dependence of the amplitude of the AB oscillations is also affected in a qualitative way. The presence of the tunnel barrier, which pins the Wigner crystal and makes charge transfer possible only by macroscopic tunneling, strongly decreases the zero temperature value of the persistent current since for a repulsive interaction quantum fluctuations in a strongly correlated electron system renormalize the barrier upward. The finite ring circumference cuts off the divergent renormalization of the barrier height which occurs in the thermodynamic limit of a Luttinger liquid [9] or Wigner crystal [7,8]. Thus the persistent current at zero temperature is greatly reduced but is still finite. The competition between two effects of an increased temperature — a temperature stimulated tunneling and a loss of phase coherence due to the enhancement of destructive interference — leads to a sharp maximum in the temperature dependence of the persistent current. For a rigid crystal this maximum occurs at $T \sim 0.5 T_s$, where $T_s \equiv \hbar s/L \gg T_0$. This effect makes it possible to measure the Wigner crystal sound velocity in a ring with an ‘adjustable barrier’ (height controlled by a gate voltage).

The starting point of our analysis is a model system, where the Wigner crystal is regarded as an elastic chain of spinless electrons forming a ring. In the presence of a potential barrier, smooth on the scale of $a$ but well localized on the scale of $L$, the Lagrangian of such a system in the long wavelength approximation is [8]

$$L = \frac{ma}{8\pi^2} \left\{ \varphi^2 - s^2(\varphi')^2 \right\} - V_0 \delta(x) \cos(\varphi). \quad (1)$$

Here $\varphi = 2\pi u(x)/a$ is the dynamical displacement field of the crystal and $V_0$ is the magnitude
of the pinning potential (without loss of generality placed at the point \( x = 0 \)).

We emphasize that (1) is an effective Lagrangian that describes the long wavelength aspects of the quantum dynamics of the Wigner crystal. The short wavelength fluctuations do not affect the global dynamics of the system, and only result in a renormalization of the magnitude \( V_0 \) of the potential, (already included in (1) but negligible for a stiff Wigner crystal \([8]\)). We assume that the ring circumference is large enough to justify dropping terms from (1) which are irrelevant in an infinite system.

In the presence of a magnetic field, directed normal to the plane of the ring, the one-dimensional Lagrangian (1) acquires an additional term, \( L_{\text{int}} \). This term describes the AB interaction of the Wigner crystal with the vector potential of an electromagnetic field, \( A_\varphi = \Phi/L \) (\( \Phi \) is the magnetic flux through the ring). The AB interaction term, rewritten using the real scalar displacement field \( \varphi \), has the form of a total time derivative,

\[
L_{\text{int}} = \left( \frac{\hbar}{L} \right) \left( \frac{\Phi}{\Phi_0} \right) \dot{\varphi},
\]

and affects, as must be the case, only the quantum dynamics of the crystal.

The flux-induced persistent current \( I(\Phi) = -c\partial F/\partial \Phi, \) is defined in terms of the sensitivity of the free energy of the ring to a magnetic flux. For the following analysis, it is convenient to express the free energy \( F \) as a functional integral over quantum- and thermal fluctuations of the displacement field,

\[
F = -k_B T \ln \left\{ \sum_{n=-\infty}^{\infty} (-1)^{n(N-1)} \int D\varphi_n e^{-S_E[\varphi_n]/\hbar} \right\},
\]

where the action \( S_E \) derives from the Lagrangian (1), (2) in the imaginary time representation. ‘Twisted’ boundary conditions in imaginary time are imposed on the field \( \varphi \) (see, e.g. \([10]\)):

\[
\varphi_n(\tau + \beta, x) = \varphi_n(\tau, x) + 2\pi n.
\]

Here \( n = 0, \pm 1, \pm 2 \ldots \) is the topological (winding) number, classifying homotopically inequivalent trajectories. The physical meaning of the boundary condition (4) follows from
the definition of the field $\varphi = 2\pi u(x)/a$; a uniform shift of the crystal by a distance equal to an integer times the lattice constant $a$ leads, in the ring geometry, to a state identical to the initial state after certain permutations of electrons. For the minimum shift by $1 \times a$ ($\Delta \varphi = 2\pi$), the initial state is recovered after $(N - 1)$ successive permutations of pairs of electrons. The corresponding extra phase $\pi(N - 1)$, that appears in the many-particle wave-function because the electrons obey Fermi statistics, generates the factor $(-1)^{n(N-1)}$ in (3). As we will see below, this factor properly accounts for the parity effects in the response of one-dimensional interacting electrons to a magnetic field [11–13]. We note in passing that the analogous twisted boundary conditions appear when the Luttinger model is applied to a ring geometry [12]. The appearance of the homotopic index $n$ in the boundary condition (4) suggests that the functional integral should first be calculated for trajectories belonging to a definite homotopic class, and then the homotopically non-equivalent classes of trajectories should be summed over.

In every homotopic class we will calculate the functional integral using the saddle point approximation, assuming the saddle point action to be large, $S_n \gg \hbar$, on the extremal trajectory given by the solution of the classical equations of motion in imaginary time. Below we will show that this assumption is justified for a stiff Wigner crystal ($\alpha \ll 1$).

First we calculate the persistent current in the ideal, unpinned crystal ($V_0 = 0$). In a perfect (or weakly pinned) Wigner crystal, long-wave quantum fluctuations are cut off at the wavelength of the order of the crystal size $L$. It is physically evident that we can imagine an ordered crystal structure as long as the mean square fluctuations of the dimensionless field $\varphi$, $\langle \varphi^2 \rangle \sim \alpha \int_{\pi/L}^{\pi/a} \frac{dk}{k} \coth \left( \frac{\hbar}{2k_B T k} \right)$, are small so that $\langle \varphi^2 \rangle \ll 1$ ($T$ is the temperature, $\alpha = \pi \hbar/m s a$ is a dimensionless parameter that characterizes the strength of quantum fluctuations in the Wigner crystal). For $T \to 0$ this restriction imposes an upper bound on the chain length $L \ll a e^{1/\alpha}$; for such samples the thermal fluctuations are suppressed up to a temperature $T \lesssim T_s/\alpha$ ($T_s \equiv \hbar s/L$). The situation is changed drastically for a strongly pinned Wigner crystal where an “intermediate”
One can readily calculate the persistent current of an ideal ring as the problem in the long wavelength limit is described by a quadratic Lagrangian. The extremal trajectory corresponding to the boundary condition (4) is linear in imaginary time and independent of the $x$-coordinate,

$$\varphi_n(\tau) = 2\pi n \left(\frac{\tau}{\hbar\beta}\right).$$

By substituting (6) into (1, 2, 3), it is easy to find an exact solution for the free energy in terms of the Jacobi function $\vartheta_3$ (see e.g. [14]). The asymptotic expressions for the persistent current at high- and low temperatures are

$$I_{WC} \Bigg/ I_0 \simeq \begin{cases} 
\frac{2T_0}{T} e^{-\frac{\pi}{2}\tau_0} (-1)^N \sin \left(2\pi \frac{\Phi}{\Phi_0}\right), & T \gtrsim T_0 \\
1 - 2\{\{\frac{\Phi}{\Phi_0} + \delta_N\}, & T \ll T_0 
\end{cases}$$

Here $\{x\}$ denotes the fractional part of $x$, and the parity dependent term $\delta_N$ is 1/2 (0) for $N$ odd (even). Thus the persistent current carried by an ideal Wigner crystal is a periodic function of flux with period $\Phi_0 = hc/e$ and amplitude $I_0 = ev_F/L$ at low temperatures. The oscillations are exponentially damped at $T \gtrsim T_0 = \hbar v_F/L$. The current has a paramagnetic character when there is an even number of electrons in the ring (i.e. the induced magnetic moment is parallel to the external magnetic field) and diamagnetic for an odd number of electrons. All these properties of the persistent current coincide with those calculated using the model of an ideal Fermi gas. For $T = 0$, this was first shown in Ref. [3] for a general case of arbitrary Coulomb-like interaction. At finite temperatures there are in general contributions due to crystal deformations produced by thermally excited phonons. It is possible to show [15] that, even in a perfect Wigner crystal ring, the contribution of phonon fluctuations to the action results in a correction to the persistent current which is small if the temperature is less than $mv_0\sqrt{s}/2k_B$.

The thermal destruction of the persistent current can be characterized by a crossover temperature $T_c$, where $I \propto \exp(-T/T_c)$. From (4) one has $T_c = (2/\pi)(\hbar v_F/L)$, which is twice as large as the crossover temperature found for a ring of free electrons characterized
by a constant chemical potential \([4,16]\). Rather than with the electron-electron correlations \([12]\) the factor of 2 difference is connected with the fact that in our case the number of electrons — not the chemical potential — is fixed \([17]\).

Let us now consider the persistent current in a Wigner crystal in the presence of a potential barrier. A uniform sliding motion of the crystal is impossible in this case, and charge transport along the ring is connected with macroscopic quantum tunneling (MQT) of the Wigner crystal. The character of the MQT is dictated by the pinning strength. At strong pinning, \(\alpha V_0 \gg T_s\), the mechanism for charge transport around the ring includes tunneling of a finite segment of the Wigner crystal through the barrier, as well as the subsequent relaxation of the associated elastically deformed state of the crystal. In the weak pinning regime, \(\alpha V_0 \ll T_s\), the Wigner crystal as a whole tunnels through the barrier (without essential distortions). The above mechanisms for macroscopic tunneling were first considered in connection with the tunneling of commensurate charge density waves \([9]\) and have also been used to describe the tunneling conductivity of a Wigner crystal \([8]\). In these contexts it was shown \([7,8]\) that in the case of strong pinning the dominating tunneling process is the elastic relaxation of the deformed state arising in the near-barrier region.

In the ring geometry a shift of the crystal as a whole by the lattice period, \(a\), may include one or several ‘tunneling steps’ (by a ‘tunneling step’ we understand the combined processes of tunneling and relaxation of the elastic deformation). In the case of strong pinning, the single-step tunneling is described by the exact solution of the free equation of motion \((V_0 = 0)\) in imaginary time with the twisted boundary condition \((4)\) and \(n = \pm 1\):

\[
\varphi^{(s)}(\sigma) = 2\sigma \arctan \left[ \frac{\coth \left( \frac{\pi |x|}{\hbar s \beta} \right)}{\tan \left( \frac{\pi}{\hbar \beta} - \frac{1}{2} \right)} \right],
\]

\[
\sigma = \pm 1.
\]

A description of the dominating relaxation process in terms of this ‘periodic instanton’-solution \([18]\) is valid in the region outside the interval \([-\ell_0, \ell_0]\) containing the part of the crystal deformed by the initial tunneling process. The length \(\ell_0\), which is inversely proportional to the potential \(V_0\), appears only as a limit of the integration over coordinate \(x\); we
assume that $\ell_0 \ll L/2$, a criterion which one can show to be equivalent to a restriction on temperature, $T \ll \alpha V_0$.

A multi-step solution is a sequence of single steps of type (8), corresponding to all possible intermediate rotations of the crystal

$$\varphi^{(m)}(\tau) = \sum_{\ell} \varphi^{(s)}(\tau - \tau_\ell), \quad \sigma = \sum_{\ell} \sigma_\ell = \pm 1. \quad (9)$$

The set of solutions $\{\varphi^{(m)}\}$, corresponding to different configurations of single steps $\{\sigma_\ell\}$ and time-sequences $\{\tau_\ell\}$ for the tunneling events, form the basis in the well-known dilute instanton gas approximation.

At $T \ll T_s$, the multi-step solution (3) can be used as the extremal trajectory when calculating the partition function that appears in the expression (3) for the free energy. In this manner we get the zero temperature value of the persistent current as

$$I_{WC}(T = 0) \sim (-1)^N \frac{e^{\alpha S}}{L} \left( \frac{T_s}{\alpha V_0} \right)^{1/\alpha} \sin \left( \frac{2\pi \Phi}{\Phi_0} \right).$$

This result for the persistent current of a Wigner crystal in the presence of a pinning potential barrier, clearly shows that the effect of the barrier is simply to suppress the zero temperature amplitude of the current. The net current depends on the elastic properties of the Wigner lattice that reflects the fact that for a strong pinning the charge transport in the ring is due to macroscopic quantum tunneling of the system through a deformed state of the crystal.

Except at very low temperatures, $T \ll T_s$, the only relevant saddle point trajectory is the single-step solution (8). By using it one gets for the normalized current

$$\frac{I_{WC}(T)}{I_{WC}(0)} = \frac{T}{T_s} \exp \left( \frac{1}{\alpha} f \left( \frac{T}{T_s} \right) \right), \quad T \ll \alpha V_0$$

$$f(x) = \frac{\pi}{2} x - \ln \left( \frac{\sinh(\pi x)}{\pi x} \right). \quad (11)$$

This result implies a non-monotonic temperature dependence of the persistent current; for a stiff crystal (small $\alpha$) the current has an exponentially sharp maximum at $T \sim 0.5 T_s$, with a width of the order of $\sqrt{T_0 T_s}$. The physical reason for this non-trivial temperature dependence — shown in Fig. 1 for different values of $\alpha$ — can be explained as follows:
It is easy to see from (8) that as the temperature is increased the picture of the elastic deformation propagating as a ‘sharp’ instanton changes (at $T \sim T_s$) into a picture of a homogeneous sliding of the crystal as a whole. This temperature-induced ‘softening’ of the instanton reduces the contribution to the action from the elastic deformation of the crystal. Hence, the persistent current should increase with temperature. On the other hand this effect competes with a thermal smearing of the phase coherence which — as we showed for the unpinned crystal — tends to reduce the current. The sharp peak in the temperature dependence of the persistent current carried by a strongly pinned Wigner crystal, is a result of this very competition.

At weak pinning a stiff Wigner crystal-ring tunnels through the barrier as a whole and the persistent current does not depend on the elastic properties of the chain. However, if the barrier is high enough $\alpha T_s \lesssim V_p \ll T_s/\alpha$ (moderately weak pinning) the zero temperature current is still greatly suppressed,

$$I_{wp}(T \to 0) \sim (-1)^N I_0 \left( \frac{V_0}{T_0} \right)^{1/4} \exp \left( -\sqrt{2\pi \frac{V_0}{T_0}} \sin \left( 2\pi \frac{\Phi}{\Phi_0} \right) \right). \quad (12)$$

This is in contrast to the case of non-interacting electrons where even a large potential barrier (of the order of the Fermi energy) only leads to a power-law supression of the persistent current [16].

As temperature always makes tunneling easier, one may expect an anomalous temperature behaviour of the persistent current even for a weakly pinned Wigner crystal-ring. This is indeed the case, but unlike in the regime of strong pinning, the maximum of the persistent current (which is now attained at a potential-dependent temperature $T^* \sim \sqrt{V_0 T_0}$) is weakly pronounced. Therefore the only distinctive feature of the temperature dependence of a moderately pinned ($V_0 > T_0$) Wigner crystal-ring, — compared to that of a ring with noninteracting electrons — is the shift of the crossover temperature to higher values, $T_0 \Rightarrow T^* > T_0$. For very weak pinning, $V_p \ll T_0$, the response of a Wigner crystal-ring to a magnetic flux is the same as for free electrons.

Formula (11) is valid in the strong pinning limit, when temperature is much smaller
than $\alpha V_0$. At high temperatures, $T \gtrsim \alpha V_0$, the pinning potential can be treated as a perturbation when calculating the depinning of the Wigner crystal. In this case we find unimportant corrections to the persistent current in an ideal Wigner crystal (the details of this calculation will be published elsewhere [15]).

By measuring the dependence of the persistent current on the barrier height at zero temperature [10] and its temperature dependence [11], one has an opportunity to determine independently the stiffness parameter, $\alpha = \hbar/2msa$, and the sound velocity, $s$, in this system of strongly correlated electrons. This gives us strong reasons to propose an experiment using a gate-controlled barrier in a mesoscopic semiconductor ring in order to study Wigner crystallisation and to measure the parameters of the crystal.

In conclusion we have shown that in an ideal ring with no impurity scattering, the persistent current carried by interacting electrons — so strongly correlated that they form a Wigner crystal — is indistinguishable from the current carried by a non-interacting Fermi gas. By incorporating a potential barrier, $V_0 > \hbar v_F/L$, in the ring structure, a qualitative change of the magnitude and temperature dependence of the persistent current appears. With an adjustable barrier, these differences can be used for detecting and investigating the properties of the Wigner crystal.

We gratefully acknowledge discussions with L. Glazman, A. Nersesyan, A. Sjölander, and A. Zagoskin. This work was supported by the Swedish Royal Academy of Sciences, the Swedish Natural Science Research Council, the Swedish National Board for Industrial and Technical Development, by the NSF through grant DMR-9113911, and by grant PH2-9187-0917 from International Science Foundation. One of us (I.K.) acknowledges the hospitality of the Department of Applied Physics, CTH/GU.
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FIGURES

FIG. 1. Temperature dependence of the normalized persistent current in a strongly pinned Wigner crystal of different stiffness (measured by $\alpha^{-1} = 2msa/h$; $T_s = \hbar s/k_B L$, see text). The sharp peak for stiff crystals is a result of a competition between two effects of temperature: a reduced renormalized tunneling barrier and a loss of quantum coherence.