Interacting $Q$-balls

Yves Brihaye$^1$ and Betti Hartmann$^2$

$^1$ Faculté des Sciences, Université de Mons-Hainaut, 7000 Mons, Belgium
$^2$ School of Engineering and Science, Jacobs University Bremen, 28759 Bremen, Germany

E-mail: yves.brihaye@umh.ac.be and b.hartmann@jacobs-university.de

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Abstract

We study non-topological solitons, so-called $Q$-balls, which carry a non-vanishing Noether charge and arise as lump solutions of self-interacting complex scalar field models. Explicit examples of new axially symmetric non-spinning $Q$-ball solutions that have not been studied so far are constructed numerically. These solutions can be interpreted as angular excitations of the fundamental $Q$-balls and are related to the spherical harmonics. Correspondingly, they have higher energy and their energy densities possess two local maxima on the positive $z$-axis.

We also study two $Q$-balls interacting via a potential term in $3 + 1$ dimensions and construct examples of stationary, solitonic-like objects in $(3+1)$-dimensional flat space–time that consist of two interacting global scalar fields. We concentrate on configurations composed of one spinning and one non-spinning $Q$-ball and study the parameter-dependence of the energy and charges of the configuration.

In addition, we present numerical evidence that for fixed values of the coupling constants two different types of 2-$Q$-ball solutions exist: solutions with defined parity, but also solutions that are asymmetric with respect to reflection through the $x$–$y$ plane.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Solitons play an important role in many areas of physics. As classical solutions of nonlinear field theories, they are localized structures with finite energy, which are globally regular. In general, one can distinguish topological and non-topological solitons. While topological...
solitons [1] possess a conserved quantity, the topological charge that stems (in most cases) from the spontaneous symmetry breaking of the theory, non-topological solitons [2, 3] have a conserved Noether charge that results from a symmetry of the Lagrangian. The standard example of non-topological solitons are $Q$-balls [4], which are solutions of theories with self-interacting complex scalar fields. These objects are stationary with an explicitly time-dependent phase. The conserved Noether charge $Q$ is then related to the global phase invariance of the theory and is directly proportional to the frequency. $Q$ can, for example, be interpreted as the particle number [2].

While in standard scalar field theories, it was shown that a non-renormalizable $\Phi^6$-potential is necessary [5], supersymmetric extensions of the standard model (SM) also possess $Q$-ball solutions [6]. In the latter case, several scalar fields interact via complicated potentials. It was shown that cubic interaction terms that result from Yukawa couplings in the superpotential and supersymmetry breaking terms lead to the existence of $Q$-balls with non-vanishing baryon or lepton number or electric charge. These supersymmetric $Q$-balls have been considered recently as possible candidates for baryonic dark matter [7] and their astrophysical implications have been discussed [8].

Two interacting scalar fields are also interesting from another point of view. Until now, the number of explicit examples of stationary solitonic-like solutions that involve two interacting global scalar fields is small. An important example is superconducting strings, which are axially symmetric in $2 + 1$ dimensions extended trivially into the $z$-direction [9]. Axially symmetric generalizations in $3 + 1$ dimension, so-called vortons, have been constructed in [10]. Note that all these solutions have been constructed in models which have a renormalizable $\Phi^4$-potential.

Here, we study two interacting scalar fields in $3 + 1$ dimensions and construct explicit examples of stationary solitonic-like axially symmetric solutions consisting of two global scalar fields. While vortons possess one scalar field with an unbroken $U(1)$ symmetry (the condensate field) and a scalar field whose $U(1)$ is spontaneously broken (the string field), we here consider two scalar fields with unbroken $U(1)$ symmetries. One can thus see our model as the limit of vanishing vacuum expectation value for the second scalar field. Then, stationary solitonic-like objects can be constructed explicitly. Note that the model in [10] contains a renormalizable $\Phi^4$-potential, while we need a non-renormalizable $\Phi^6$-potential here. However, as stated in [10], the explicit construction of vortons was done also using a non-renormalizable potential which contains an interaction term of the form $\Phi^6 \Phi^2_2$.

$Q$-ball solutions in $3 + 1$ dimensions were first studied in detail in [5]. It was realized that next to non-spinning $Q$-balls, which are spherically symmetric, spinning solutions exist. These are axially symmetric with an energy density of toroidal shape and angular momentum $J = kQ$, where $Q$ is the Noether charge of the solution and $k \in \mathbb{Z}$ corresponds to the winding around the $z$-axis. Approximated solutions of the nonlinear partial differential equations were constructed in [5] by means of a truncated series in the spherical harmonics to describe the angular part of the solutions. The full partial differential equation was solved numerically in [11]. It was also realized in [5] that in each $k$-sector, parity-even ($P = +1$) and parity-odd ($P = -1$) solutions exist. Parity-even and parity-odd refer to the fact that the solution is symmetric and anti-symmetric, respectively, with respect to a reflection through the $x–y$ plane, i.e. under $\theta \to \pi - \theta$.

These two types of solutions are closely related to the fact that the angular part of the solutions constructed in [5, 11] is connected to the spherical harmonic $Y_0^0(\theta, \phi)$ for the spherically symmetric $Q$-ball, to the spherical harmonic $Y_1^1(\theta, \phi)$ for the spinning parity-even ($P = +1$) solution and to the spherical harmonic $Y_1^2(\theta, \phi)$ for the parity-odd ($P = -1$) solution, respectively. Radially excited solutions of the spherically symmetric, non-spinning solution were also obtained. These solutions are still spherically symmetric but the scalar
field develops one or several nodes for \( r \in ]0, \infty[ \). In relation to the apparent connection of the angular part of the known solutions to the spherical harmonics, it is natural to investigate whether \( '\theta\)-angular excitations’ of the \( Q \)-balls exist in correspondence to the whole family of spherical harmonics \( Y^k_L(\theta, \varphi), -L \leq k \leq L \). This can further be motivated by the fact that, in the small field limit where a linear approximation can be used, the field equation describing the \( Q \)-ball becomes a standard harmonic equation that can be solved by separation of variables and whose fundamental solutions are given in terms of spherical harmonics for the angular part. Of course, it has to be checked whether this correspondence, expected from the linear limit, still holds for the full, i.e. nonlinear, equation.

In this paper, we present strong numerical arguments that new angularly excited solutions of the nonlinear field equations exist and that the correspondence between angular excitations of the \( Q \)-balls and spherical harmonics indeed holds. In addition to the solutions corresponding to \( Y^0_0 \) and \( Y^0_{-1} \) presented in [5] we have constructed solutions with the angular dependence and symmetries corresponding to the spherical harmonics \( Y^0_k \) and \( Y^1_k \). These solutions are non-spinning but constitute axially symmetric excitations with respect to the angular coordinate \( \theta \). As expected, these new solutions have higher energies and charges than the spherically symmetric solutions and we would thus expect them to be unstable. These solutions thus complete the already known spectrum of \( Q \)-ball solutions and show that not only radial excitations of fundamental soliton solutions but also angular excitations exist.

We also study two interacting \( Q \)-balls and put emphasis on the interaction between a non-spinning and a spinning \( Q \)-ball. In particular, we investigate the dependence of the energy and the charges of the solution on the interaction parameter and the frequencies, respectively.

Next to parity-even and parity-odd solutions, we also construct solutions that have no defined parity with respect to reflection through the \( x-y \) plane.

The explicit construction of solutions with two interacting complex scalar fields is surely of interest for the astrophysical implications of such objects, especially for the construction of such objects in supersymmetric theories. Moreover, it adds to the spectrum of soliton solutions that, for example, possess no definite parity.

The differential equations describing both excited as well as interacting \( Q \)-balls are nonlinear partial differential equations, which—to our knowledge—cannot be solved analytically. We thus solve these equations numerically using an appropriate PDE solver [12].

Our paper is organized as follows: in section 2, we discuss the model and give the equations and boundary conditions. In section 3, we discuss the new \( Q \)-ball solutions for \( k = 0 \), while in section 4, we present our results for two interacting \( Q \)-balls. Section 5 contains our conclusions.

2. The model

In the following, we study a scalar field model in \( 3 + 1 \) dimensions describing two \( Q \)-balls interacting via a potential term. The Lagrangian reads

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \varphi_1 \partial^\mu \varphi_1^* + \frac{1}{2} \partial_\mu \varphi_2 \partial^\mu \varphi_2^* - V(\varphi_1, \varphi_2),
\]

where both \( \varphi_1 \) and \( \varphi_2 \) are complex scalar fields. The potential reads

\[
U(\varphi_1, \varphi_2) = \sum_{i=1}^2 \left( \alpha_i |\varphi_i|^6 - \beta_i |\varphi_i|^4 + \gamma_i |\varphi_i|^2 \right) + \lambda |\varphi_1|^2 |\varphi_2|^2,
\]

where \( \alpha_i, \beta_i, \gamma_i, i = 1, 2 \) are the standard potential parameters for each \( Q \)-ball, while \( \lambda \) denotes the interaction parameter.

In [5] it was argued that a \( \Phi^6 \)-potential is necessary in order to have classical \( Q \)-ball solutions. This is still necessary for the model we have defined here, since we want \( \Phi_1 = 0 \).
and \( \Phi_2 = 0 \) to be a local minimum of the potential. A pure \( \Phi^4 \)-potential which is bounded from below will not fulfil these criteria.

The Lagrangian (1) is invariant under the two global U(1) transformations

\[
\Phi_1 \rightarrow \Phi_1 e^{i \omega_1}, \quad \Phi_2 \rightarrow \Phi_2 e^{i \omega_2}.
\]

which can be applied separately or together. As such the total conserved Noether current \( j^\mu_{(\text{tot})} \), \( \mu = 0, 1, 2, 3 \), associated with these symmetries is just the sum of the two individually conserved currents \( j^\mu_1 \) and \( j^\mu_2 \) with

\[
j^\mu_{(\text{tot})} = j^\mu_1 + j^\mu_2 = -i \left( \Phi_1^* \partial^\mu \Phi_1 - \Phi_1 \partial^\mu \Phi_1^* \right) - i \left( \Phi_2^* \partial^\mu \Phi_2 - \Phi_2 \partial^\mu \Phi_2^* \right)
\]

with \( \partial^\mu j_1^\mu = 0 \), \( \partial^\mu j_2^\mu = 0 \) and \( \partial^\mu j_{(\text{tot})}^\mu = 0 \).

The total Noether charge \( Q_{(\text{tot})} \) of the system is then the sum of the two individual Noether charges \( Q_1 \) and \( Q_2 \):

\[
Q_{(\text{tot})} = Q_1 + Q_2 = - \int j_1^0 d^3 x - \int j_2^0 d^3 x
\]

Finally, the energy–momentum tensor reads

\[
T_{\mu \nu} = \sum_{i=1}^2 \left( \partial_\mu \Phi_i \partial_\nu \Phi_i^* + \partial_\nu \Phi_i \partial_\mu \Phi_i^* \right) - g_{\mu \nu} \mathcal{L}.
\]

2.1. Ansatz

We choose as ansatz for the fields in spherical coordinates:

\[
\Phi_i(t, r, \theta, \phi) = e^{i \omega_i t + i k_i \phi} \phi_i(r, \theta), \quad i = 1, 2,
\]

where the \( \omega_i \) and the \( k_i \) are constants. Since we require \( \Phi_i(\phi) = \Phi_i(\phi + 2\pi) \), \( i = 1, 2 \), we have that \( k_i \in \mathbb{Z} \). It was moreover demonstrated in [5, 11] that \( Q \)-balls exist only in a specific parameter range \( \omega_{\text{min}} < \omega < \omega_{\text{max}} \) and that the charge \( Q \) tends to infinity when either \( \omega \rightarrow \omega_{\text{min}} \) or \( \omega \rightarrow \omega_{\text{max}} \). We discuss the limits in the 2-\( Q \)-ball system in the following section.

The Noether charges of the solution then read

\[
Q_i = 2 \omega_i \int |\Phi_i|^2 d^3 x = 4 \pi \omega_i \int_0^\pi \int_0^\infty r^2 \sin \theta \, dr \, d\theta \, \phi_i^2, \quad i = 1, 2,
\]

while the energy is given by the volume integral of the \( tt \)-component of the energy–momentum tensor:

\[
E = \int T_{00} d^3 x = 2 \pi \int_0^\pi \int_0^\infty r^2 \sin \theta \, dr \, d\theta \sum_{i=1}^2 \left( \omega_i \phi_i + (\phi_i')^2 + \frac{(\phi_i^2)}{r^2} \right)
\]

\[
+ \frac{k_i^2 \phi_i^2}{r^2 \sin^2 \theta} + \alpha_i|\phi_i|^6 - \beta_i|\phi_i|^4 + \gamma_i|\phi_i|^2 \right) + \lambda_i|\phi_i|^2|\phi_2|^2, \]

where the prime and dot denote the derivative with respect to \( r \) and \( \theta \), respectively.

For \( k_i \neq 0 \), the solutions have a non-vanishing angular momentum that is quantized. The total angular momentum \( J \) is the sum of the angular momenta of the two individual \( Q \)-balls:

\[
J = \int T_{0\phi} d^3 x = J_1 + J_2 = k_1 Q_1 + k_2 Q_2.
\]

We thus in the following refer to solutions with \( k_i = 0 \) as non-spinning and to solutions with \( k_i \neq 0 \) as spinning.
The Euler–Lagrange equations read
\[ \phi_i'' + \frac{2}{r} \phi_i' + \frac{1}{r^2} \phi_i + \frac{1}{r^2 \sin^2 \theta} \cot \theta \phi_i - \frac{k_i^2}{r^2 \sin^2 \theta} \phi_i + \omega_i^2 \phi_i = 3\alpha_i \phi_i^5 - 2\beta_i \phi_i^3 + \gamma_i \phi_i + \lambda \phi_i \phi_i^2 \]  
with \( i = 1, 2 \) and \( k \neq i \).

The boundary conditions, which result from requirements of regularity, finiteness of the energy and the symmetry of the solution, are
\[ \partial_r \phi_i(r = 0, \theta) = 0, \quad \partial_r \phi_i(r = \infty, \theta) = 0, \quad \partial_{\theta} \phi_i(r, \theta = 0, \pi) = 0, \quad i = 1, 2 \]  
for non-spinning solutions with \( k_i = 0 \) and
\[ \phi_i(r = 0, \theta) = 0, \quad \phi_i(r = \infty, \theta) = 0, \quad \phi_i(r, \theta = 0, \pi) = 0, \quad i = 1, 2 \]  
for spinning solutions \( k_i \neq 0 \).

2.2. Bounds on \( \omega_1 \) and \( \omega_2 \) in the 2-\( Q \)-ball system

In [5,11] the bounds on the frequency \( \omega \) have been discussed in the case of one \( Q \)-ball. Here, we note that these bounds have to be modified if one considers two interacting \( Q \)-balls. The set of equations (11) can be interpreted as the mechanical equations describing the frictional motion of a particle in two dimensions. The effective potential in this case reads
\[ V(\phi_1, \phi_2) = \frac{1}{4}(\omega_1^2 \phi_1^2 + \omega_2^2 \phi_2^2) - \frac{1}{2} U(\phi_1, \phi_2). \]  
\( Q \)-ball solutions exist provided the configuration \( (\phi_1 = 0, \phi_2 = 0) \) corresponds to a local maximum of the effective potential and provided the effective potential has positive values in any radial direction from the origin in the \( \phi_1-\phi_2 \) plane. This leads to non-trivial bounds for the parameters \( \omega_1 \) and \( \omega_2 \).

The former condition leads to the requirement that
\[ \omega_1^2 < \omega_{1, \max}^2 = \gamma_1, \quad \omega_2^2 < \omega_{2, \max}^2 = \gamma_2. \]  
(15)

The latter condition leads to a more complicated domain of existence in the \( \omega_1-\omega_2 \) plane. To describe this condition, we introduce the polar decomposition of \( \phi_1 \) and \( \phi_2 \) as follows:
\[ \phi_1 = \rho \cos \chi, \quad \phi_2 = \rho \sin \chi, \]  
(16)
where \( 0 \leq \chi < 2\pi \) and \( 0 \leq \rho < \infty \).

The condition on the frequencies \( \omega_1 \) and \( \omega_2 \) then reads
\[ \omega_1^2 \cos^2 \chi + \omega_2^2 \sin^2 \chi > (\omega_1^2 \cos^2 \chi + \omega_2^2 \sin^2 \chi)_{\min} = \min_{\rho}[U(\rho, \chi)/\rho^2]. \quad \forall \chi. \]  
(17)

In the particular case that we have studied throughout this paper, namely, \( \alpha_1 = \alpha_2 = 1, \beta_1 = \beta_2 = 2 \) and \( \gamma_1 = \gamma_2 = 1.1 \), this inequality takes the form
\[ \omega_1^2 \cos^2 \chi + \omega_2^2 \sin^2 \chi > \left[-5\lambda^2 \cos^4 \chi \sin^4 \chi + 20\lambda \cos^2 \chi \sin^2 \chi (\cos^4 \chi + \sin^4 \chi) \right. 
\]  
\[ + 2(\cos^8 \chi + \sin^8 \chi + 11 \cos^6 \chi \sin^2 \chi + 11 \sin^6 \chi \cos^2 \chi 
\]  
\[ - 20 \cos^4 \chi \sin^4 \chi)]/(\cos^4 \chi + \sin^4 \chi - \cos^2 \chi \sin^2 \chi). \]  
(18)

For \( \chi = n\pi/2, n = 0, 1, 2, \ldots \), we recover the results of the one \( Q \)-ball system discussed in [5,11]. For all other values of \( \chi \), the limiting values for \( \omega_1 \) and \( \omega_2 \) will depend on the value of the interaction coupling \( \lambda \). For example, for \( \phi_1 = \phi_2 \), i.e. \( \chi = \pi/4 \), we find
\[ \omega_1^2 + \omega_2^2 > 1/5 + \lambda - 1/8\lambda^2. \]  
(19)
Thus, for small \( \lambda \), the lower bound on the value of \( \omega_1^2 + \omega_2^2 \) will be larger than in the non-interacting limit.
3. New non-spinning $Q$-ball solutions for $\alpha_2 = \beta_2 = \gamma_2 = \lambda = 0$

In order to be able to understand the structure of a system of two $Q$-balls, we have reconsidered the one $Q$-ball system. We set all quantities with index ‘2’ to zero in the following and omit the index ‘1’ for the remaining quantities.

In this section, we would like to point out that more solutions exist than previously discussed in the literature.

For this, we first consider the equation for one $Q$-ball with vanishing potential. This reads

$$
\phi'' + \frac{2}{r} \phi' + \frac{1}{r^2} \phi + \frac{1}{r^2} \cot \theta \phi - \frac{k^2}{r^2 \sin^2 \theta} \phi + \omega^2 \phi = 0.
$$

(20)

Although the solutions of the above equation are well known, it will be useful for the following to recall their properties. Using the standard separation of variables, the solutions read

$$
\phi(r, \theta, \phi) \propto J_{L+1/2}(\omega r) \sqrt{r} Y^L_k(\theta, \phi),
$$

(21)

where $J$ denotes the Bessel function, while $Y^L_k$ are the standard spherical harmonics with $-L \leq k \leq L$.

One may hope that solutions of the full equations with the discrete symmetries corresponding to the ones of the spherical harmonics will exist. Of course, the nonlinear potential interaction will deform the radial part of the solutions of the linear equation in a highly non-trivial manner.

The solutions of the full equation constructed so far for $k = 0$ have been spherically symmetric. With the above arguments, axially symmetric solutions should equally exist with an angular dependence of the form $Y^0_L$, e.g. for $L = 1$, the angular dependence should be of the form $\cos \theta$. In the following, we will denote the solutions of the full nonlinear equations with angular symmetries corresponding to the symmetries of the spherical harmonic $Y^L_k$ by $\phi^L_k$.

3.1. Numerical results

The partial differential equation has been solved numerically subject to the boundary conditions (12) or (13) using the finite difference solver FIDISOL [12]. We have mapped the infinite interval of the $r$ coordinate [0 : $\infty$] to the finite compact interval [0 : 1] using the new coordinate $z := r/(r+1)$. We have typically used grid sizes of 150 points in the $r$-direction and 50 points in the $\theta$ direction. The solutions have relative errors of $10^{-3}$ or smaller. Throughout this section, we choose $\alpha_1 \equiv \alpha = 1$, $\beta_1 \equiv \beta = 2$, $\gamma_1 \equiv \gamma = 1$.1.

In figure 1 (left), we show the profile of a new solution that we obtained for $k = 0$ and $\omega = 0.8$. This solution looks like a deformation of the spherical harmonic $Y^0_1$ with an appropriate symmetry with respect to $\theta = \pi/2$ and is clearly axially symmetric. In particular it fulfils $\phi^0_1(r, \pi/2) = 0$.

The field $\phi^0_1(r, \theta)$ is maximal at a finite distance from the origin on the positive $z$-axis. Moreover, the configuration is anti-symmetric under reflection through the $x$–$y$ plane, i.e. under $\theta \rightarrow \pi - \theta$. Thus, the solution is parity-odd: $P = -1$. Note that we have only plotted the function for $\theta \in [0 : \pi/2]$, but that we have verified the symmetry of the solution.

We also present the corresponding energy density $T_{00}$ in figure 1 (right). It shows that the density of the solution is mainly concentrated within two small ‘balls’ situated around the positive $z$-axis (at $z \approx 2.4$ and $z \approx 7.6$) and separated by a minimum (at $z \approx 5$). The position of this minimum coincides with the maximum of the scalar field $(\phi^0_1)_{\text{max}} \approx \phi^0_1(5, 0) \approx 1.2$. It can be checked that this value corresponds roughly to a local minimum of the potential while the partial derivatives are evidently small in this region, explaining the occurrence of a minimal
value of the energy density at \((x, y, z) \approx (0, 0, 5)\). Of course, due to the anti-symmetry of the solution this pattern is equally given on the negative z-axis.

The classical energy and charge of this new solution are higher than those of the spherically symmetric \(k = 0\) solution (see tables 1 and 2), however lower than those of the \(k = 1\) spinning \(Q\)-ball.

In order to investigate further our idea of constructing new solutions as deformations of the spherical harmonics, we have also investigated solutions with a higher value of \(L\) and we managed to construct solutions \(\phi_0^0\) and \(\phi_0^1\) corresponding in their angular symmetries to those of the spherical harmonics \(Y_0^0 \propto 3 \cos^3 \theta - 1\) and \(Y_0^1 \propto 5 \cos^3 \theta - 3 \cos \theta\), respectively.

In figure 2, we plot \(\phi_0^0, \phi_0^1, \phi_0^2, \phi_0^3\) as functions of \(\theta\) for a fixed value of \(r\) together with the corresponding spherical harmonics \(Y_0^0, Y_0^1, Y_0^2\). Here, we have chosen \(r \sim 5\) for \(\phi_0^0, r \sim 2\) for \(\phi_0^1\) and \(r \sim 6\) for \(\phi_0^2\). The first thing to notice is that the symmetries of the solutions \(\phi_0^0, \phi_0^1, \phi_0^2\) with respect to the reflection at \(\theta = \pi/2\) are exactly equal to those of the corresponding spherical harmonics. The actual solutions are, of course, deformed with respect
Figure 2. $\phi^0_1$ (solid), $\phi^0_2$ (short dashed) and $\phi^0_3$ (long dashed) are shown as functions of $\theta$ for a fixed value of $r$. We have chosen $r \sim 5$ for $\phi^0_1$, $r \sim 2$ for $\phi^0_2$ and $r \sim 6$ for $\phi^0_3$. The corresponding spherical harmonics $Y^0_1 \propto \cos(\theta)$, $Y^0_2 \propto 3 \cos(\theta)^2 - 1$ and $Y^0_3 \propto 5 \cos(\theta)^3 - 3 \cos \theta$ (with an appropriate normalization) are also shown.

Figure 3. The profile of the function $\phi^0_2$ for $\omega = 0.8$, $\alpha = 1$, $\beta = 2$, $\gamma = 1.1$.

to the spherical harmonics, but the correspondence is apparent. For example, the solution $\phi^0_2$ has $\partial_\theta \phi^0_2(r, \pi/2) = 0$ (in contrast to the solution $\phi^0_1$ which has $\phi^0_1(r, \pi/2) = 0$) (figure 3). We do not show the energy density of $\phi^0_2$ and $\phi^0_3$ here, since it resembles that shown in figure 1.

We believe that the correspondence also holds for higher spherical harmonics.

Since we have presented strong numerical evidence that the correspondence with the spherical harmonics holds, it is justified to label the different solutions of the field equation by means of the quantum numbers of the corresponding spherical harmonic, i.e. by $L$ and $k$ referring to $Y^k_L$, with $L, k$ integers and $-L \leq k \leq L$. Needless to say that the numerical
construction becomes more involved when the difference \( L - |k| \) increases. Adopting this notation and fixing the potential according to \( \alpha = 1, \beta = 2, \gamma = 1.1 \), we find for the solutions corresponding to \( \omega = 0.8 \) and \( \omega = 0.84 \) the values for the energy \( E \) and charge per frequency \( Q \) given in tables 1 and 2, respectively.

The first three solutions \( \phi_L^0 \), \( L = 0, 1, 2 \), in this list are static (i.e. non-spinning), while the last, \( \phi_1^1 \), is stationary (i.e. spinning). For all the solutions we constructed, the energy of the non-spinning solutions is lower than the energy of the spinning ones.

4. Interacting \( Q \)-balls

Since in supersymmetric extensions of the SM, \( Q \)-balls exist that result from the interaction of several scalar fields, we investigate the interaction of two classical \( Q \)-balls as a toy model for these systems.

For two spherically symmetric \( Q \)-balls \( (k_1 = k_2 = 0) \) in interaction, the 2-\( Q \)-ball solution is still spherically symmetric and the domain of existence in the \( \omega_1 - \omega_2 \) plane can be determined by using the reasoning given in section 2.2. Here, we put the emphasis on solutions where the two \( Q \)-balls have different symmetries and study the effect of the direct interaction parametrized by the coupling constant \( \lambda \).

We believe that a particularly interesting case is the interaction between a spherically symmetric, non-spinning \( Q \)-ball \( (k_1 = 0) \) and a spinning \( Q \)-ball \( (k_2 = 1) \). We have thus restricted our analysis to this case and set \( k_1 = 0 \) and \( k_2 = 1 \) in the following.

Note that we will index all quantities related to the spherical \( Q \)-ball in the following with ‘1’, while all quantities related to the axially symmetric \( Q \)-ball will be indexed with ‘2’.

For later use, we define the ‘binding energy’ of the solution according to

\[
\Delta E = E - E_{k_1=0} - E_{k_2=1}.
\]

(22)

It represents the difference between the energy \( E \) of the 2-\( Q \)-ball configuration and the sum of the energies of the two single (i.e. non-interacting) \( Q \)-balls \( E_{k_1=0}, E_{k_2=1} \) with the same frequency. We expect those solutions which have \( \Delta E < 0 \) to be stable, while those with \( \Delta E > 0 \) would be unstable.

4.1. Numerical results

We have solved the two coupled partial differential equations using the solver FIDISOL [12] for several values of \( \omega_1, \omega_2 \) and \( \lambda \) and fixing \( \alpha_1 = \alpha_2 = 1, \beta_1 = \beta_2 = 2 \) and \( \gamma_1 = \gamma_2 = 1.1 \). As starting profiles, we have used the corresponding non-interacting \( Q \)-ball solutions. For \( \lambda = 0 \), these solve the two decoupled partial differential equations. We have then slowly increased the parameter \( \lambda \) to obtain the interacting solutions.

4.1.1. \( \omega_1 = \omega_2 \). In order to understand the influence of the interaction parameter \( \lambda \), we show the energy density \( T_{00} \) for \( \omega_1 = \omega_2 = 0.8 \) and three different values of \( \lambda \) in figure 4. For \( \lambda = 0 \), the two \( Q \)-balls are non-interacting and the energy density is just a simple superposition of the energy densities of the two individual \( Q \)-balls. For \( \lambda \neq 0 \) the \( Q \)-balls interact. For \( \lambda > 0 \), it is energetically favourable to have the two \( Q \)-balls’ cores in different regions of space. As seen in figure 4 for \( \lambda = 1 \), the spinning \( Q \)-ball seems to be ‘pushed away’ from the non-spinning, spherically symmetric one. For \( \lambda < 0 \), it is energetically favourable to have two \( Q \)-balls sitting ‘on top of each other’. This is shown in figure 4 for \( \lambda = -0.5 \), where the two \( Q \)-balls seem to be localized at the same place.
Figure 4. The energy density $T_{00}$ of the 2-$Q$-ball solution consisting of a spherically symmetric, non-spinning $Q$-ball ($k_1 = 0$) and a spinning $Q$-ball ($k_2 = 1$) is shown for $\omega_1 = \omega_2 = 0.8$, $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 2$, $\gamma_1 = \gamma_2 = 1.1$ and for three different values of $\lambda = 0, 1, -0.5$.

We have also studied the dependence of the energy $E$, the binding energy $\Delta E$ and the two charges $Q_1$ and $Q_2$ on the interaction parameter $\lambda$. The results are shown in figure 5 for $\omega_1 = \omega_2 = 0.8$. All quantities increase with the increase in $\lambda$; specifically it is evident that the 2-$Q$-ball configuration is energetically more stable for $\lambda < 0$ than for $\lambda > 0$. Specifically, we would thus expect the solution to be stable for $\lambda < 0$ and unstable for $\lambda > 0$.

Following our discussion in section 2.2, we have also studied the dependence of the energy $E$ and of the charges $Q_1$ and $Q_2$ on the frequencies $\omega_1$ and $\omega_2$. Our results for $\omega_1 = \omega_2$ are shown in figure 6 for $\lambda = -0.5, 0$ and $0.5$, respectively.

As expected, the energy $E$ for a given frequency $\omega_1 = \omega_2$ is higher (respectively, lower) than in the non-interacting case for positive (respectively, negative) values of $\lambda$.

As before, we find that the solutions exist in a given interval of the frequency: $\omega_{1,\text{min}}(\lambda) \leq \omega \leq \omega_{1,\text{max}}(\lambda)$ (and equally for $\omega_2$ since $\omega_1 = \omega_2$). We have determined the bounds on $\omega_1$ and $\omega_2$ in section 2.2 for two spherically symmetric $Q$-balls. Here, we would expect that these values change slightly since we have a system of one spherically symmetric and one axially
Symmetric Q-ball. However, we see that the qualitative results are similar here. We observe that for $\lambda \geq 0$, the values of the energy $E$ and of the charges $Q_1$, $Q_2$ diverge at $\omega_1 = \omega_{1\text{, min}}$ and $\omega_1 = \omega_{1\text{, max}}$. Following the discussion of section 2.2 we find that the maximal value of $\omega_1$ is independent of $\lambda$. This can be clearly seen in figure 6 where the energy $E$ and the charges $Q_1$ and $Q_2$ diverge at $\omega_1 = \omega_{1\text{, max}} \approx 1.035$ for all three values of $\lambda$. Note that this maximal value is only slightly lower than the bound given in section 2.2: $\omega_{1\text{, max}}^2 = 1.1$. The reason why the bound is not equal is that here we are dealing with an axially symmetric solution interacting with a spherically symmetric one. Analytic arguments of the type done in section 2.2 are, however, only possible if the Euler–Lagrange equations are ordinary differential equations, i.e. only in the case where the solutions are spherically symmetric. So, it is not surprising that the analytic values differ from the numerical ones.

On the other hand, the minimal value of $\omega_1$ is $\lambda$ dependent. This can be seen in figure 6. We have given our results only for $\omega \geq 0.6$ in this figure since the construction of solutions becomes increasingly difficult for $\omega < 0.6$. However, it can be clearly seen that the energy $E$
and $Q_2$ diverge at different values of $\omega_1 = \omega_{1,\text{min}}$. In agreement with section 2.2, we find that $\omega_{1,\text{min}}$ is increasing for increasing (and small) $\lambda$.

For $\lambda < 0$ the behaviour at the lower bound of $\omega_1$ changes. We observe that $Q_1$, corresponding to the spherically symmetric field $\phi_1$, decreases when $\omega_1$ decreases. The analysis of the profile of the solution reveals that the field $\phi_1$ deviates only slightly from the spherically symmetric configuration for frequencies close to $\omega_{1,\text{max}}$. However, it gets more and more deformed in the equatorial plane when $\omega_1$ decreases. At the same time, the field $\phi_2$ increases in the equatorial plane. This phenomenon is illustrated in figure 7 for $\lambda = -0.5$, $\omega = 0.6$ and $\omega = \omega_{1,\text{max}}$, respectively. In this figure, the fields $\phi_1$ and $\phi_2$ as well as the energy density $T_{00}$ are shown as a function of $r$ for two angles $\theta = 0$ and $\theta = \pi/2$, respectively.

4.1.2. Solutions with $\omega_1 \neq \omega_2$. We have also constructed 2-$Q$-ball solutions for $\omega_1 \neq \omega_2$. The energy density $T_{00}$ of a 2-$Q$-ball solution corresponding to $\omega_1 = 0.65$ and $\omega_2 = 1$ is
The energy density $T_{00}$ of the 2-$Q$-ball solution consisting of a spherically symmetric, non-spinning $Q$-ball ($k_1 = 0$) and a spinning $Q$-ball ($k_2 = 1$) is shown for $\omega_1 = 0.65$, $\omega_2 = 1$, $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 2$, $\gamma_1 = \gamma_2 = 1.1$ and for four different values of $\lambda = 0.5$, $0$, $-0.2$ and $-0.5$.

Figure 8. The energy density $T_{00}$ of the 2-$Q$-ball solution consisting of a spherically symmetric, non-spinning $Q$-ball ($k_1 = 0$) and a spinning $Q$-ball ($k_2 = 1$) is shown for $\omega_1 = 0.65$, $\omega_2 = 1$, $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = 2$, $\gamma_1 = \gamma_2 = 1.1$ and for four different values of $\lambda = 0.5$, $0$, $-0.2$ and $-0.5$.

We have also studied the dependence of the solution’s conserved quantities on $\omega_2 = \omega_1/0.65$ for $\lambda = \pm 0.5$. The dependence of the energy $E$ and the charges $Q_1$, $Q_2$ is shown in figure 9. These results strongly suggest that for $\lambda < 0$ and in the region of the parameter space chosen, the field $\phi_2$ corresponding to the $k_2 = 1$ $Q$-ball tends uniformly to zero for a critical value of $\omega_2 = \omega_2^{(c)}$ such that $Q_2 \to 0$ for $\omega_2 \to \omega_2^{(c)}$. Only the field $\phi_1$ remains non-trivial when $\omega_2 \leq \omega_2^{(c)}$. This effect can also be observed in figure 8, where the solution for $\lambda = -0.5$ has lost nearly all its axially symmetric character.

We observe the inverse phenomenon for $\omega_1 = c\omega_2$ with a constant $c > 0$. We do not present our detailed results here since they are qualitatively equivalent to the case discussed
above. We find that $Q_1 \to 0$ for $\omega_1 \to \omega_1^{(cr)}$. Thus, the spherically symmetric solution disappears from the system, while $Q_2$ remains non-trivial for $\omega_1 \leq \omega_1^{(cr)}$.

Apparently, while in the case $\omega_1 = \omega_2$ and $\lambda < 0$, the charge $Q_1$ associated with the spherical $Q$-ball tends to zero for $\omega_1 \to \omega_1^{(cr)}$, it is the charge $Q_i$, $i = 1, 2$, of the $Q$-ball with the higher frequency that tends to zero for $\omega_i \to \omega_i^{(cr)}$, $i = 1, 2$, when $\omega_1 \neq \omega_2$ and $\lambda < 0$. Note that nothing similar is observed when $\lambda \geq 0$.

While 1-$Q$-ball solutions known so far are always either parity-even or parity-odd with respect to $\theta \to \pi - \theta$, we have constructed several examples of 2-$Q$-ball solutions that do not have a defined parity. One such solution is shown in figure 10 (lower part) together with a parity-even solution (upper part). These solutions exist for exactly the same values of the coupling constants. Both functions $\phi_1, \phi_2$ are clearly neither parity-even nor parity-odd and the field $\phi_2$ possesses in addition nodes in the radial direction. This solution is thus an asymmetric, radially excited 2-$Q$-ball solution. As expected, we observe that this asymmetric solution has much higher energy and charges than the corresponding parity-even solution.

The investigation of solutions of this type and their eventual bifurcation into branches of solutions with defined parity is currently underway.

5. Concluding remarks

In this paper, we have presented numerical evidence that $Q$-ball solutions admit several types of excitations labelled by integers. So far, it was known that the static, spherically symmetric solution is the ‘ground state’ of a series of radially excited solutions. Families of spinning solutions are also known; they are axially symmetric and can be labelled according to the winding $k$ around the axis of symmetry. Here we present evidence that excitation with respect to $\theta$ can be constructed also. Generally, the previous results and the present analysis suggest that families of elementary solutions of the field equations exist and are labelled by $n, L, k$, where $n$ refers to the number of nodes in the radial direction, while $L, k$ refer to the ‘quantum numbers’ related to the spherical harmonics. At the moment, the only analytic argument we have for this property is its analogy to the linearized version (i.e. the small field limit) of the equation where this result holds true by standard harmonic analysis. It is
likely that the qualitative properties of the solutions exist also in the case of the full nonlinear equations.

We have also studied a system of two interacting $Q$-balls and have constructed several examples of axially symmetric, stationary solutions that carry conserved currents and charges. We observe that the $2$-$Q$-ball solutions exist in a finite range of the frequency $\omega_i,_{\min} \leq \omega_i \leq \omega_i,_{\max}$, $i = 1, 2$, where $\omega_i,_{\max}$ is independent of the interaction coupling, while $\omega_i,_{\min}$ is dependent on the interaction coupling in a highly non-trivial manner. We find that the charges $Q_i$, $i = 1, 2$ of the $2$-$Q$-balls in interaction tend to infinity when $\omega_i \to \omega_i,_{\max}$ or $\omega_i \to \omega_i,_{\min}$ as long as $\lambda \geq 0$. For $\lambda < 0$, however, we observe that the charges $Q_i$ associated with the $Q$-ball with the higher frequency $\omega_i$ tend to zero for $\omega_i \to \omega_i,_{\min}^{(cr)}$ or $\omega_i \to \omega_i,_{\max}$. For $\omega_i,_{\min} \leq \omega_i \leq \omega_i,_{\max}$ only the remaining field $\phi_j$, $j \neq i$, is non-zero.

In a future publication, we intend to construct solutions with the more realistic potential available from supersymmetry [6] and put emphasis on the possibility of constructing $Q$-balls and their excited and/or spinning versions with potentials involving only quartic terms in the scalar fields.

Figure 10. The contour plots for $\phi_1$ and $\phi_2$ of a parity-even $2$-$Q$-ball solution (upper part) and of an asymmetric $2$-$Q$-ball solution (lower part) are shown for $\lambda = -0.5$, $\omega_1 = 0.585$ and $\omega_2 = 0.9$.
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