ON TRUNCATED LOGARITHMS OF FLOWS ON A RIEMANNIAN MANIFOLD

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Abstract. This paper gives quantitative global estimates between a time dependent flow on a Riemannian manifold $(M)$ and the flow of a vector field constructed by truncating the formal Magnus expansion for the logarithm of the flow. As a corollary, we also find quantitative estimates between the composition of the flows of two given time independent vector fields on $M$ and the flow of a truncated version of the Baker-Cambel-Hausdorff-Dynkin expansion associated to the two given vector fields.

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1. Introduction

For the purposes of this paper, let $M$ be a connected manifold without boundary, $\Gamma(TM)$ be the space of smooth vector fields on $M$, $g$ be a Riemannian metric on $M$, $d = d_g$ be the induced length metric on $M$ (see Notation 2.1), $\nabla = \nabla^g$ be the associated Levi-Civita covariant derivative, and $R = R^g$ be the curvature tensor of $\nabla$ (see Definition 2.5).

**Definition 1.1** (Complete vector fields). Let $J = [0,T]$ (or possibly some other interval) and $J \ni t \to Y_t \in \Gamma(TM)$ be a smoothly varying time dependent vector field. We say that $Y$ is **complete** provided for every $s \in J$ and $m \in M$, there exists a solution, $\sigma : J \to M$ solving the ordinary differential equation (ODE for short),

\[
\dot{\sigma}(t) = Y_t(\sigma(t)) \quad \text{with} \quad \sigma(s) = m,
\]

where $\dot{\sigma}(0)$ and $\dot{\sigma}(T)$ are to be interpreted as the appropriate one-sided derivative. [See Corollary 2.12 below for some necessary conditions on $(M,g)$ and $Y$ which imply that $Y$ is complete.]

**Definition 1.2** (Flows). If $J = [0,T] \ni t \to Y_t \in \Gamma(TM)$ is a smoothly varying time dependent complete vector field, let $\mu_{Y,t,s}(m) := \sigma(t)$ where $\sigma(\cdot)$ is the solution to Eq. (1.1). Thus the **flow associated to $Y$, $\mu^Y_{t,s} : M \to M** for $s,t \in J$, satisfies,

\[
\frac{d}{dt} \mu^Y_{t,s}(m) = Y_t \circ \mu^Y_{t,s} \quad \text{with} \quad \mu^Y_{s,s} = Id_M.
\]

When $Y_t = Y$ is independent of $t$, we denote $\mu^Y_{t,0}$ by $e^{tY}$ so that $\mu^Y_{t,s} = e^{(t-s)Y}$ for all $s,t \in \mathbb{R}$.

The **logarithm problem** in this context is the question of finding vector fields, $Z_t \in \Gamma(TM)$, so that $\mu^Y_{t,0} = e^{tZ_t}$. This problem seems to have first been formally studied by Magnus [27] in the context of linear differential equations although the special case encoded in the Baker-Cambel-Hausdorff-Dynkin formula is much older. For a short derivation of Magnus result see [1], and for an extensive review of the Magnus’ expansion and its many generalizations and applications see the survey article, [6]. Although it is not within the author’s ability to give a systematic review of the many uses of Magnus’ idea, in order to understand the breadth of applications let me give a sampling references involving quantum physics, control theory, geometric numerical integration, and stochastic analysis. An early reference
in quantum physics is [37]. An example in control theory is [23] who is discussing taking logarithms of the Neumann-Dyson series and their non-linear extensions described by K.T. Chen [12,13] and Fliess [16]. The following references, [9,15,17,19,25,26,30] along with the survey articles [8,22] and the monographs [7,18], give only a sporadic sampling of the extensive literature in geometric numerical integration theory. For references from stochastic analysis, see [2,3,10,11,20,21,35,36] which pertain to approximating stochastic flows and see [33,34] for a couple of example involving stochastic control theory. Hopefully the reader sees from this sampling of references how ubiquitous the “logarithm problem” has become.

A key starting point for this paper and many of the references above is Strichartz’s [32] Eq. (G.C-B-H-D) (and also see [5,28]) formal series solution,

\[ Z_t \sim \sum_{m=1}^{\infty} \sum_{\sigma \in P_m} \left( \frac{(-1)^{e(\sigma)}}{m^2 (m-1) e(\sigma)} \right) \int_{\Delta_m(t)} \left( -1 \right)^m \text{ad}_{Y_{\tau_m}} \cdots \text{ad}_{Y_{\tau_2}} Y_{\tau_1} d\tau, \]

to the logarithm problem, i.e. \( Z_t \in \Gamma(TM) \) “represented” by the series above formally solves \( \mu^Y_{t,0} = e^{Z_t} \). In Eq. (1.2), \( P_m \) is the set of permutations of \( \{1,2,\ldots,m\} \),

\[ \Delta_m (t) = \{ 0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_m \leq t \} \text{, and} \]

\[ e(\sigma) := \# \{ j < m : \sigma(j) > \sigma(j+1) \} \]

is the number of “errors” in the ordering of \( \sigma(1),\ldots,\sigma(m) \).

If \( M \) is a Lie group and \( Y_t \) is a family of left invariant vector fields on \( M \) then the expansion in Eq. (1.2) will converge when \( t \) is sufficiently close to 0 and the resulting sum will solve the logarithm problem in this context. There is a rather vast literature exploring when such series expansions actually converge, see for example [4,14,24,29] just to give a very thin sample. In the general context of arbitrary time dependent vector fields, the expansion in Eq. (1.2) will typically not converge. In this paper, our goal is not to discuss convergence of the series but rather to estimate the errors made by truncating the expansion in Eq. (1.2).

For \( n \in \mathbb{N} \), let \( Z^{(n)}_t \) denote the series expansion in Eq. (1.2) where the first sum, \( \sum_{m=1}^{\infty} \), is truncated to \( \sum_{m=1}^{n} \). Roughly speaking, the main object of this paper is (under certain added hypothesis on \( Y_t \)) to estimate the distance between \( \mu^Y_{t,0} \) and \( e^{Z^{(n)}_t} \). The rest of this introduction will be devoted to summarizing the main results of this paper and in particular to stating Theorems 1.30 and 1.32 below.

1.1. Basic flow estimates.

Notation 1.3. If \( \rho_m \geq 0 \) for all \( m \in M \), we let

\[ \rho_M := \sup_{m \in M} \rho_m. \]
If $J$ is a compact subinterval of $\mathbb{R}$ and $M \times J \ni (m,t) \to \rho^*_m(t) \geq 0$ is a continuous function we let,

$$|\rho^*_M| := \int_J \rho_M(t) \, dt = \int_J \sup_{m \in M} \rho^*_m(t) \, dt.$$  

When $J = [0,t]$ for some $t > 0$ we will simply write $|\rho^*_t|$ for $|\rho^*_M|$. Note that if $J \ni t \to \rho(t) \geq 0$ does not depend on $m \in M$, then

$$\rho^*_t = \int_J \rho(t) \, dt = \|\rho\|_{L^1(J,M)}$$

where $m$ is Lebesgue measure on $\mathbb{R}$.

**Notation 1.4.** For $X \in \Gamma(TM)$, $m \in M$, and $v, w \in T_m M$, let

$$\nabla^2_{v \otimes w} X := \nabla_v (\nabla_w X) - \nabla_{v \otimes w} X$$

where $W \in \Gamma(TM)$ is chosen so that $W(m) = w$.

See Definition 2.3 and Remark 2.6 below for an alternative but equivalent definition of $\nabla^2 X$ as well as the verification that $\nabla^2_{v \otimes w} X$ is well defined bilinear form on $T_M M \times T_M M$.

**Notation 1.5** (Tensor Norms). If $X \in \Gamma(TM)$ and $m \in M$, let

$$|X_m| := |X(m)|_g,$$

$$|\nabla X_m| := \sup_{|v_m| = 1} |\nabla_{v_m} X|_g,$$

$$|\nabla^2 X_m| := \sup_{|v_m| = 1 = |w_m|} |\nabla^2_{v_m \otimes w_m} X|_g,$$

$$|R(X, \cdot)|_m := \sup_{|v_m| = 1 = |w_m|} |R(X(m), v_m) w_m|_g,$$

and

$$H_m(X) := |\nabla^2 X|_m + |R(X, \cdot)|_m.$$ 

Let us give a few examples of how this notation will be used.

**Example 1.6.** Suppose that $J \ni t \to X_t \in \Gamma(TM)$ is a continuously varying time dependent vector field, then

$$|X_t|_M := \sup_{m \in M} |X_t|_m,$$

$$|\nabla^2 X_t|_M := \sup_{m \in M} |\nabla^2 X_t|_m,$$

$$|\nabla^2 X_t|_M^* := \int_J \sup_{m \in M} |\nabla^2 X_t|_m \, dt,$$

$$H_M(X_t) := \sup_{m \in M} \left( |\nabla^2 X_t|_m + |R(X_t, \cdot)|_m \right) \leq |\nabla^2 X_t|_M + |R(X_t, \cdot)|_M,$$

and

$$H(X_\cdot)^*_J = \int_J \sup_{m \in M} H_m(X_t) \, dt \leq |\nabla^2 X_\cdot|_J^* + |R(X_\cdot, \cdot)|^*_J.$$

The next theorem is a combination of Theorem 2.29 and Corollary 2.30 below.
Theorem 1.7. Let \( J = [0, T] \ni t \to X_t, Y_t \in \Gamma(TM) \) be two smooth complete time dependent vector fields on \( M \) and \( \mu^X \) and \( \mu^Y \) be their corresponding flows. Then for \( m \in M \) and \( t > 0 \) (for notational simplicity) we have the following estimates,

\[
d\left(\mu^X_{t,0}(m), \mu^Y_{t,0}(m)\right) \leq \int_0^t e^{\int_0^s |\nabla X_{\sigma}|_{\mu^X_{\mu_{\mu,0}(m)}}} \cdot |Y_s - X_s|_{\mu^Y_{\mu_{\mu,0}(m)}} \ ds
\]

\[
\leq e^{\int_t^0 |\nabla X|^*_{t}} |Y - X|^*_t,
\]

\[
d\left(\mu^Y_{t,0}(m), m\right) \leq \int_0^t |Y_s|_{\mu^Y_{\mu_{\mu,0}(m)}} \ ds \leq |Y|^*_t, \quad \text{and}
\]

\[
d\left(\mu^Y_{t,0}(m), m\right) \leq \int_0^t e^{\int_0^s |\nabla Y_{\sigma}|_{\mu^Y_{\mu_{\mu,0}(m)}}} \cdot |Y_s|_{m} \ ds \leq e^{\int_t^0 |\nabla Y|^*_t} \int_0^t |Y_s(m)| ds.
\]

We are also interested in estimating the distance between the differentials, \((\mu^X_{t,0})_*\) and \((\mu^Y_{t,0})_*\), of \( \mu^X \) and \( \mu^Y \). To do so we endow \( TM \) with its “natural” Riemannian metric induced from the Riemannian metric, \( g \), on \( M \) (see Definition 5.1 of Section 5 below) and let \( d^{TM} \) be the induced length metric on \( TM \). The next theorem is a combination of Theorem 7.2 and Corollary 7.3 below.

Theorem 1.8. If \( J = [0, T] \ni t \to X_t, Y_t \in \Gamma(TM) \) are smooth complete (see Definition 1.1) time dependent vector fields on \( M \) and \( \mu^X \) and \( \mu^Y \) be their corresponding flows, then

\[
\sup_{v \in TM: |v| = 1} d^{TM}\left((\mu^X_{t,0})_* v, (\mu^Y_{t,0})_* v\right)
\]

\[
\leq e^{2|\nabla X|^*_t + |\nabla Y|^*_t} \cdot (1 + H (X)_t^* + |\nabla (Y - X)|^*_t),
\]

and

\[
\sup_{v \in TM: |v| = 1} d^{TM}\left((\mu^Y_{t,0})_* v, v\right) \leq e^{\int_t^0 |\nabla Y|^*_t} \cdot (|Y|^*_t + |\nabla Y|^*_t).
\]

The next proposition starts to indicate how the two previous theorems fit into the logarithm approximation theorem.

Proposition 1.9. Suppose that \( J = [0, T] \ni t \to X_t \in \Gamma(TM) \) is a complete time dependent vector field and \( J = [0, T] \ni t \to Z_t \in \Gamma(TM) \) is another time dependent vector field such that \( Z_t \) is complete for each fixed \( t \) and \( Z_0 \equiv 0 \). Then

\[
d\left(\mu^X_{t,0}(m), e^{Z_t}(m)\right) \leq \int_0^t e^{\int_0^s |\nabla X_{\sigma}|_{\mu^X_{\mu_{\mu,0}(m)}} d\sigma} \cdot |W_s - X_s|_{e^{Z_s}(m)} \ ds
\]

where

\[
W^Z_t := \int_0^1 e^{sZ_t} \dot{Z}_t o e^{-sZ_t} ds = \int_0^1 \text{Ad}_{e^{sZ_t}} \dot{Z}_t \ ds \in \Gamma(TM).
\]
Proof. By Corollary \[2.23\] below, which states,
\[
\frac{d}{dt} e^{Z_t} = W_t Z_t \circ e^{Z_t} \quad \text{with} \quad e^{Z_0} = Id
\]
and so \( \mu_{t,0}^{W_Z} = e^{Z_t} \) for all \( t \in [0,T] \). Thus the estimate in Eq. \[1.3\] follows by applying Theorem \[1.7\] with \( Y_t = W_t Z_t \).
\[\square\]

Because of Proposition \[1.9\], in order to find good approximate logarithms for the flow, \( \mu^X \), we should choose \( Z_t \in \Gamma(TM) \) so that \( Z_0 = 0 \) and \( |W_s^Z - X_s|_{e^{Z_s(m)}} \) is small. Ideally we would like to choose \( Z \) so that \( W_s^Z = X_s \) but this is not possible in general. However, formally solving the equation \( W_s^Z = X_s \) would lead to the expansion in Eq. \[1.2\]. In order to get precise estimates we are now going to make more assumptions (in the spirit of control theory) on what we allow for our choice of \( X_t \). These additional assumptions and necessary notations will be explained in the next subsection.

1.2. Free nilpotent Lie groups and dynamical systems.

**Definition 1.10 (Tensor Algebras).** Let \( T(R^d) := \bigoplus_{k=0}^{\infty} [R^d] \otimes^k \) be the tensor algebra over \( R^d \) so the general element of \( \omega \in T(R^d) \) is of the form
\[
\omega = \sum_{k=0}^{\infty} \omega_k \quad \text{with} \quad \omega_k \in (R^d) \otimes^k, \quad \text{for} \quad k \in N_0
\]
where we assume \( \omega_k = 0 \) for all but finitely many \( k \). Multiplication is the tensor product and associated to this multiplication is the Lie bracket,
\[
[A,B]_\otimes := A \otimes B - B \otimes A \quad \text{for all} \quad A,B \in T(R^d).
\]

**Definition 1.11 (Free Lie Algebra).** The free Lie algebra over \( R^d \) will be taken to be the Lie-subalgebra, \( F(R^d) \), of \( (T(R^d), [,], \otimes) \) generated by \( R^d \).

**Remark 1.12.** If \((g, [\cdot, \cdot])\) is a Lie algebra and \( V \subset g \) is a subspace, then using Jacobi’s identity one easily shows that Lie sub-algebra (Lie \((V)\)) of \( g \) generated by \( V \) may be described as:
\[
\text{Lie} \,(V) = \text{span} \cup_{k=1}^{\infty} \{ \text{ad}_{v_1} \ldots \text{ad}_{v_{k-1}} v_k : v_1, \ldots, v_k \in V \},
\]
where \( \text{ad}_A B := [A,B] \) for all \( A,B \in g \). As a consequence of this remark it follows that \( F(R^d) \) is a \( \mathbb{N}_0 \)-graded algebra with
\[
F(R^d) = \bigoplus_{k=0}^{\infty} F_k(R^d) \quad \text{where} \quad F_k(R^d) = F(R^d) \cap [R^d] \otimes^k \subset F(R^d).
\]
According to this grading, if \( A \in F(R^d) \) we let \( A_k \in F_k(R^d) \) denote the projection of \( A \) into \( F_k(R^d) \).
See [31], for general background information on free Lie algebras. The spaces $T(\mathbb{R}^d)$ and $F(\mathbb{R}^d)$ are infinite dimensional. We are going to be most interested in the finite dimensional truncated versions of these algebras.

**Definition 1.13** (Truncated Tensor Algebras). Given $\kappa \in \mathbb{N}$, let

$$T^{(\kappa)}(\mathbb{R}^d) := \bigoplus_{k=0}^{\kappa} \mathbb{R}^d \otimes_k \subset T(\mathbb{R}^d)$$

which is algebra under the multiplication rule,

$$AB = \sum_{k=0}^{\kappa} (AB)_k = \sum_{k=0}^{\kappa} \sum_{j=0}^{k} A_j \otimes B_{k-j} \quad \forall \ A, B \in T^{(\kappa)}(\mathbb{R}^d)$$

and a Lie algebra under the bracket operation, $[A, B] := AB - BA$ for all $A, B \in T^{(\kappa)}(\mathbb{R}^d)$.

**Notation 1.14.** Let $\pi_{\leq \kappa} : T(\mathbb{R}^d) \rightarrow T^{(\kappa)}(\mathbb{R}^d)$ and $\pi_{> \kappa} := I_{T(\mathbb{R}^d)} - \pi_{\leq \kappa} : T(\mathbb{R}^d) \rightarrow \bigoplus_{k=\kappa+1}^{\infty} [\mathbb{R}^d]^\otimes k$ be the projections associated to the direct sum decomposition,

$$T(\mathbb{R}^d) = T^{(\kappa)}(\mathbb{R}^d) \oplus \left( \bigoplus_{k=\kappa+1}^{\infty} [\mathbb{R}^d]^\otimes k \right).$$

Further let

$$\mathfrak{g}^{(\kappa)} = \bigoplus_{k=1}^{\kappa} [\mathbb{R}^d]^\otimes k$$

which is a two sided ideal as well as a Lie sub-algebra of $T^{(\kappa)}(\mathbb{R}^d)$.

With this notation the multiplication and Lie bracket on $T^{(\kappa)}(\mathbb{R}^d)$ may be described as,

$$AB = \pi_{\leq \kappa} (A \otimes B) \quad \text{and} \quad [A, B] = \pi_{\leq \kappa} [A, B]_\otimes.$$

**Notation 1.15** (Induced Inner product). The usual dot product on $\mathbb{R}^d$ induces an inner product, $\langle \cdot , \cdot \rangle$ on $T^{(\kappa)}(\mathbb{R}^d)$ uniquely determined by requiring $T^{(\kappa)}(\mathbb{R}^d) := \bigoplus_{k=0}^{\kappa} [\mathbb{R}^d]^\otimes k$ to be an orthogonal direct sum decomposition, $\langle 1, 1 \rangle = 1$ for $1 \in [\mathbb{R}^d]^\otimes 0$, and

$$\langle v_1 v_2 \ldots v_k, w_1 w_2 \ldots w_k \rangle = \langle v_1, w_1 \rangle \langle v_2, w_2 \rangle \ldots \langle v_k, w_k \rangle$$

for any $v_j, w_j \in \mathbb{R}^d$ and $1 \leq k \leq \kappa$. We let $|A| := \sqrt{\langle A, A \rangle}$ denote the associated Hilbertian norm of $A \in T^{(\kappa)}(\mathbb{R}^d)$.

Often, it turns out to be more convenient (see Proposition 3.24 below) to measure the size of $A \in \mathfrak{g}^{(\kappa)}$ using the following “homogeneous norms.”

**Definition 1.16** (Homogeneous norms). For $A \in \mathfrak{g}^{(\kappa)} \subset T^{(\kappa)}(\mathbb{R}^d)$, let

$$N(A) := \max_{1 \leq k \leq \kappa} |A_k|^{1/k}$$
and for \( f \in C ([0, t], \mathfrak{g}^{(\kappa)}) \) let

\[
N_t^* (f) := \max_{1 \leq k \leq \kappa} |f_k|^1 = \max_{1 \leq k \leq \kappa} \left( \int_0^t |f_k (\tau)| \, d\tau \right)^{1/k}
\]

be the homogeneous \( L^1 \)-norm of \( f \). [Note that \( N (A) \) is the best constant such that \( |A_k| \leq N (A)^k \) for \( 1 \leq k \leq \kappa \).]

Let us observe that for \( t \in \mathbb{R} \),

\[
(1.6) \quad N (tA) = \max_{1 \leq k \leq \kappa} \left[ |t|^{1/k} |A_k|^{1/k} \right] \leq \max_{1 \leq k \leq \kappa} \left[ |t|^{1/k} \right] \cdot N (A) \leq (1 \lor |t|) \cdot N (A)
\]

and if \( \delta_t : T^{(\kappa)} \left( \mathbb{R}^d \right) \rightarrow T^{(\kappa)} \left( \mathbb{R}^d \right) \) is the dilation operator defined by \( \delta_t (A) = \sum_{k=0}^{\kappa} t^k A_k \), then

\[
(1.7) \quad N (\delta_t A) = \max_{1 \leq k \leq \kappa} \left[ |t^k A_k|^{1/k} \right] = |t| \cdot N (A).
\]

**Definition 1.17** (Free Nilpotent Lie Algebra). *The step \( \kappa \) free Nilpotent Lie algebra on \( \mathbb{R}^d \) may then be realized as the Lie sub-algebra, \( F^{(\kappa)} \left( \mathbb{R}^d \right) \), of \( \mathfrak{g}^{(\kappa)} \) generated by \( \mathbb{R}^d \subset T^{(\kappa)} \left( \mathbb{R}^d \right) \).*

Again, a simple consequence of Remark [1.12] is that, as vector spaces, \( F^{(\kappa)} \left( \mathbb{R}^d \right) = \pi_{\leq \kappa} \left( F \left( \mathbb{R}^d \right) \right) \) and \( F^{(\kappa)} \left( \mathbb{R}^d \right) \) is graded as

\[
F^{(\kappa)} \left( \mathbb{R}^d \right) = \bigoplus_{k=0}^{\kappa} F_k^{(\kappa)} \left( \mathbb{R}^d \right)
\]

where

\[
F_k^{(\kappa)} \left( \mathbb{R}^d \right) := F^{(\kappa)} \left( \mathbb{R}^d \right) \cap \left[ \mathbb{R}^d \right] ^{\otimes k} \subset F^{(\kappa)} \left( \mathbb{R}^d \right) \quad \text{for} \quad 1 \leq k \leq \kappa.
\]

The set,

\[
(1.8) \quad G^{(\kappa)} \left( \mathbb{R}^d \right) := 1 + \mathfrak{g}^{(\kappa)} \subset T^{(\kappa)} \left( \mathbb{R}^d \right),
\]

forms a group under the multiplication rule of \( T^{(\kappa)} \left( \mathbb{R}^d \right) \) which is a Lie group with Lie algebra, \( \text{Lie} \left( G^{(\kappa)} \right) = \mathfrak{g}^{(\kappa)} \). Moreover, the exponential map,

\[
\mathfrak{g}^{(\kappa)} \ni \xi \rightarrow e^\xi = \sum_{k=0}^{\kappa} \frac{\xi^k}{k!} \in G^{(\kappa)} \left( \mathbb{R}^d \right),
\]

is a diffeomorphism whose inverse is given by

\[
(1.9) \quad \log \left( 1 + \xi \right) = \sum_{k=1}^{\kappa} \frac{(-1)^{k+1}}{k} \xi^k.
\]

[See Section [3] for more details.] We will mostly only use the following subgroup of \( G^{(\kappa)} \left( \mathbb{R}^d \right) \).
Definition 1.18 (Free Nilpotent Lie Groups). For \( \kappa \in \mathbb{N} \), let \( G^{(\kappa)}_{\text{geo}}(\mathbb{R}^d) \subset G^{(\kappa)}(\mathbb{R}^d) \) be the simply connected Lie subgroup of \( G^{(\kappa)}(\mathbb{R}^d) = 1 \oplus_{k=1}^{\kappa} \mathbb{R}^d \otimes_k \) whose Lie algebra is \( F^{(\kappa)}(\mathbb{R}^d) \). This subgroup is a step-\( \kappa \) (free) nilpotent Lie group which we refer to as the geometric sub-group of \( G^{(\kappa)} \).

It is well known as a consequence of the Baker-Campel-Dynken-Hausdorff formula (see Proposition 3.12 of Section 3) that the exponential map restricted to \( F^{(\kappa)}(\mathbb{R}^d) \),

\[
F^{(\kappa)}(\mathbb{R}^d) \ni \xi \mapsto e^\xi = \sum_{k=0}^{\kappa} \frac{\xi^k}{k!} \in G^{(\kappa)}_{\text{geo}}(\mathbb{R}^d),
\]

is again diffeomorphism.

Notation 1.19. Let \( \text{LD} \left( C^\infty (M, \mathbb{R}) \right) \) denote the algebra of smooth linear differential operators from \( C^\infty (M, \mathbb{R}) \).

As usual we view the smooth vector fields, \( \Gamma \left( TM \right) \), on \( M \) as a subspace of \( \text{LD} \left( C^\infty (M, \mathbb{R}) \right) \).

Definition 1.20 (Dynamical systems). A \( d \)-dimensional dynamical system on \( M \) is a linear map, \( \mathbb{R}^d \ni w \to V_w \in \Gamma \left( TM \right) \).

A \( d \)-dimensional dynamical system on \( M \) is completely determined by knowing \( \{V_{e_j}\}_{j=1}^d \subset \Gamma (TM) \) where \( \{e_j\}_{j=1}^d \) is the standard basis for \( \mathbb{R}^d \). The tensor algebra, \( T \left( \mathbb{R}^d \right) \), of Definition 1.10 satisfies the following universal property; if \( V : \mathbb{R}^d \to \mathcal{A} \) is a linear map, \( \mathcal{A} \) is another associative algebra with identity, then \( V \) extends uniquely to an algebra homomorphism from \( T \left( \mathbb{R}^d \right) \) to \( \mathcal{A} \) which we still denote by \( V \). The extension is uniquely determined by \( V_1 = 1_{\mathcal{A}} \) and \( V_{e_1 \cdots e_k} = V_{e_1} \cdots V_{e_k} \) for all \( e_i \in \mathbb{R}^d \) and \( k \in \mathbb{N} \). The following example is of primary importance to this paper.

Example 1.21. Every \( d \)-dimensional dynamical system on \( M \), \( \mathbb{R}^d \ni w \to V_w \in \Gamma (TM) \subset \text{LD} (C^\infty (M, \mathbb{R})) \), extends to an algebra homomorphism from \( T \left( \mathbb{R}^d \right) \) to \( \text{LD} (C^\infty (M, \mathbb{R})) \). We will still denote this extension by \( V \). Because of Remark 1.12, it is easy to see that \( V \left( F \left( \mathbb{R}^d \right) \right) \subset \Gamma (TM) \) and \( V|_{F(\mathbb{R}^d)} : F \left( \mathbb{R}^d \right) \to \Gamma (TM) \) is a Lie algebra homomorphism.

Notation 1.22 (Extension of \( V \) to \( F^{(\kappa)}(\mathbb{R}^d) \)). The restriction, \( V|_{F^{(\kappa)}(\mathbb{R}^d)} \), of \( V \) to the subspace \( F^{(\kappa)}(\mathbb{R}^d) \) of \( F \left( \mathbb{R}^d \right) \) will be denoted by \( V^{(\kappa)} : F^{(\kappa)}(\mathbb{R}^d) \to \Gamma (TM) \).

Remark 1.23. It is not generally true that \( V^{(\kappa)} := V|_{F^{(\kappa)}(\mathbb{R}^d)} : F^{(\kappa)}(\mathbb{R}^d) \to \Gamma (TM) \) is a Lie algebra homomorphism. In order for this to be true we must require that \( \text{ad}_V^{a_1} \cdots \text{ad}_V^{a_j} V_{a_0} = 0 \) for all \( \{a_j\}_{j=0}^\kappa \subset \mathbb{R}^d \), i.e. \( \{V_a : a \in \mathbb{R}^d \} \) should generate a step-\( \kappa \) nilpotent Lie sub-algebra of \( \Gamma (TM) \).
Definition 1.24 (Dynamical System Norms). If $V$ is a dynamical system and $\kappa \in \mathbb{N}$, we let

\begin{align}
(1.10) & \quad |V^{\kappa}|_M := \left\{|V_A|_M : A \in F^{(\kappa)} \left(\mathbb{R}^d\right) \text{ with } |A| = 1\right\}, \\
(1.11) & \quad \nabla V^{(\kappa)} |_M := \left\{\nabla V_A |_M : A \in F^{(\kappa)} \left(\mathbb{R}^d\right) \text{ with } |A| = 1\right\}, \\
(1.12) & \quad \nabla^2 V^{(\kappa)} |_M := \left\{\nabla^2 V_A |_M : A \in F^{(\kappa)} \left(\mathbb{R}^d\right) \text{ with } |A| = 1\right\}, \text{ and} \\
(1.13) & \quad H_M \left(V^{(\kappa)}\right) := \sup \left\{H_M (V_A) : A \in F^{(\kappa)} \left(\mathbb{R}^d\right) \text{ with } |A| = 1\right\}
\end{align}

where we allow for the possibility that any of these expressions might be infinite. [Recall that $H_M (V_A)$ is defined in Notation 1.5 and Example 1.6.]

1.3. Approximate logarithm theorems.

Definition 1.25 (See Definition 3.6). For $\xi \in C^1 \left([0,T], F^{(\kappa)} \left(\mathbb{R}^d\right)\right)$, let $g^\xi \in C^1 \left([0,T], G_{geo}\right)$ denote the solution to the ODE,

\begin{equation}
(1.14) \quad \dot{g}^\xi (t) = g^\xi (t) \xi (t) \text{ with } g^\xi (0) = 1
\end{equation}

and

\begin{equation}
(1.15) \quad C^\xi (t) := \log \left(g^\xi (t)\right) = \sum_{k=1}^{\kappa} \frac{(-1)^{k+1}}{k} \left(g^\xi (t) - 1\right)^k \in F^{(\kappa)} \left(\mathbb{R}^d\right),
\end{equation}

Notation 1.26. For $f, g \in C^1 \left(M, M\right)$, let

\begin{align}
&\quad d_M (f, g) := \sup_{m \in M} d \left(f (m), g (m)\right) \text{ and} \\
&\quad d^TM_M (f_x, g_x) := \sup_{v \in TM, |v|=1} d^TM \left(f_x v, g_x v\right)
\end{align}

where again $d^TM$ is defined in Section 7 below.

Definition 1.27 ($\kappa$-complete). We say that a dynamical system, $\mathbb{R}^d \ni w \rightarrow V_w \in \Gamma (TM)$, is $\kappa$-complete if for any $\xi \in C^1 \left([0,T], F^{(\kappa)} \left(\mathbb{R}^d\right)\right)$ the time dependent vector-field, $[0,T] \ni t \rightarrow V_{\xi (t)} \in \Gamma (TM)$, is complete as defined in Definition 1.1.

Assumption 1. Unless otherwise stated, the dynamical system $V : \mathbb{R}^d \rightarrow \Gamma (TM)$ is assumed to be $\kappa$-complete.

The next two theorems are the main theorems of this paper. The first theorem is a combination of Theorem 4.11, Eq. (4.18), and Corollary 4.15. To simplify the statements we first introduce the following notation.

Notation 1.28. For $\lambda \geq 0$ and $m, n \in \mathbb{N}$ with $m < n$, let

\begin{align}
&\quad Q_{[m,n]} (\lambda) := \max \left\{\lambda^k : k \in \mathbb{N} \cap [m,n]\right\} = \max \left\{\lambda^m, \lambda^n\right\} \text{ and} \\
&\quad Q_{(m,n)} (\lambda) = Q_{[m+1,n]} (\lambda) := \max \left\{\lambda^k : k \in \mathbb{N} \cap (m,n]\right\} = \max \left\{\lambda^{m+1}, \lambda^n\right\}.
\end{align}
Notation 1.29. Given two functions, $f(x)$ and $g(x)$, depending on some parameters indicated by $x$, we write $f(x) \lesssim g(x)$ if there exists a constant, $C(\kappa)$, only possibly depending on $\kappa$ so that $f(x) \leq C(\kappa)g(x)$ for the allowed values of $x$. Similarly we write $f(x) \asymp g(x)$ if both $f(x) \lesssim g(x)$ and $g(x) \lesssim f(x)$ hold.

Theorem 1.30. There is a constant $c(\kappa) < \infty$ such that

$$d_M \left( \frac{V_{\xi}}{\mu_{T,0}}, e^{V_{\log(\delta^c(T))}} \right)$$

$$\lesssim \left| V^{(\kappa)} \right|_M \left| \nabla V^{(\kappa)} \right|_M e^{c(\kappa)} \left| \nabla V^{(\kappa)} \right|_M Q_{[1,\kappa]} \left( N_T^*(\hat{\xi}) \right) Q_{[\kappa,\kappa+1]} \left( N_T^*(\hat{\xi}) \right)$$

for every $\xi \in C^1([0,T], F^{(\kappa)}(\mathbb{R}^d))$. Moreover, if $A, B \in F^{(\kappa)}(\mathbb{R}^d)$, then

$$d_M \left( e^{V_B}, Id_M \right) \leq \left| V^{(\kappa)} \right| \left| B \right| \leq \left| V^{(\kappa)} \right| Q_{[1,\kappa]} \left( N(\kappa) \right)$$

and

$$d_M \left( e^{V_B} \circ e^{V_A}, e^{V_{\log(A^cB)}} \right)$$

$$\lesssim K_0 N(A) N(B) Q_{[\kappa,2\kappa-2]} \left( N(A) + N(B) \right)$$

where

$$K_0 := \left| V^{(\kappa)} \right|_M \left| \nabla V^{(\kappa)} \right|_M e^{c(\kappa)} \left| \nabla V^{(\kappa)} \right|_M Q_{[1,\kappa]} \left( N(A) + N(B) \right).$$

Remark 1.31 (Dialating Theorem 1.30). If we define the dilation homomorphism, $\delta_\lambda : T^{(\kappa)}(\mathbb{R}^d) \to T^{(\kappa)}(\mathbb{R}^d)$, where $\delta_\lambda A = \sum_{k=0}^\kappa \lambda^k A_k$ for $\lambda > 0$ and $A \in T^{(\kappa)}(\mathbb{R}^d)$, then $N_T^* \left( \lambda \hat{\xi} \right) = N_T^* \left( \hat{\xi} \right)$ and hence it follows from Theorem 1.30 that

$$d_M \left( \frac{V_{\delta_\lambda \xi}}{\mu_{T,0}}, e^{V_{\log(\delta_\lambda^c(T))}} \right) = O \left( \lambda^{\kappa+1} \right) \text{ and } \lambda \to 0.$$

If is also easy to verify, 1) $N(\delta_\lambda A) = \lambda N(A)$ for all $A \in F^{(\kappa)}(\mathbb{R}^d)$, 2) $g^{\delta_\lambda \xi} = \delta_\lambda \left( g^\xi \right)$,

$$3) \log \left( g^{\delta_\lambda \xi} \right) = \log \left( \delta_\lambda \left( g^\xi \right) \right) = \delta_\lambda \log \left( g^\xi \right),$$

and 4) $\delta_\lambda \hat{\xi} (t) = \lambda \hat{\xi} (t)$ in the special case where $\xi (t) \in \mathbb{R}^d \subset F^{(\kappa)}(\mathbb{R}^d)$.

The next theorem (which is a combination of Theorem 8.4, Eq. (4.19), and Corollary 8.5) is an analogue of Theorem 1.30 for the differentials of $\mu_{T,0}$ of $e^{V_{\log(\delta^c(T))}}$.

Theorem 1.32. If $\xi \in C^1([0,T], F^{(\kappa)}(\mathbb{R}^d))$, then

$$d_M \left( \frac{V_{\xi}}{\mu_{T,0}}, e^{V_{\log(\delta^c(T))}} \right) \leq K \cdot Q_{[\kappa,2\kappa]} \left( N_T^* \left( \hat{\xi} \right) \right),$$

where

$$K := \left| V^{(\kappa)} \right|_M \left| \nabla V^{(\kappa)} \right|_M e^{c(\kappa)} \left| \nabla V^{(\kappa)} \right|_M Q_{[1,\kappa]} \left( N(\kappa) \right) Q_{[\kappa,\kappa+1]} \left( N(\kappa) \right).$$
where
\[ \mathcal{K} = \mathcal{K} \left( T, \left| V^{(\kappa)} \right|_{M}, \left| \nabla V^{(\kappa)} \right|_{M}, \left| \nabla^2 V^{(\kappa)} \right|_{M}, |R \langle V, \cdot \rangle|_{M}, N^*_{T} (\xi) \right) \]
is a (fairly complicated) increasing function of each of its arguments. Moreover, if \( A, B \in F^{(\kappa)} (\mathbb{R}^d) \), then
\[ d^T_M \left( e^{V_B}, Id_M \right) \leq \left| V^{(\kappa)} \right| |B| \leq \left| V^{(\kappa)} \right| Q_{[1, \kappa]}(N(A) + N(B)) \]
and
\[ d^T_M \left( [e^{V_B} \circ e^{V_A}]_{*}, e_*^{V_{\log(g^{(T)})}} \right) \leq \mathcal{K}_1 \cdot N(A) \cdot N(B) \cdot Q_{(\kappa_1, 2(\kappa - 1))} \left( N(A) + N(B) \right). \]

This paper separates into two parts. The first part consisting of Sections 2 – 4 which develops the results needed to prove Theorem 1.30 estimating error between the flow \( \mu^T_{\xi} \) and \( e^{V_{\log(g^{(T)})}} \). The second part of the paper consists of Sections 5 – 8 where the tools are developed to estimate the error between the differentials of \( \mu^T_{\xi} \) and \( e^{V_{\log(g^{(T)})}} \) given in Theorem 1.32. The computations in the second part are necessarily more complicated and this is where curvature of \( M \) enters the scene. Lastly, the Appendix 9 gathers some basic Gronwall type estimates used in the body of this paper.

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2. **Geometric notation and background**

2.1. **Riemannian distance.** Given \(-\infty < a < b < \infty\), a path, \( \sigma \in C ([a, b] \to M) \) is said to be **absolutely continuous** provided for any chart \( x \) on \( M \) and closed positive length subinterval, \( J \subset [a, b] \) such that \( \sigma (J) \subset D(x) \) (\( D(x) \) is the domain of \( x \)) we have then \( x \circ \sigma|_{J} : J \to \mathbb{R}^d \) is absolutely continuous.

**Notation 2.1.** For \(-\infty < a < b < \infty\), let \( AC ([a, b] \to M) \) denote the **absolutely continuous paths** from \([a, b] \) to \( M \). Moreover, if \( p, q \in M \), let
\[ AC_{p,q} ([a, b] \to M) := \{ \sigma \in AC ([a, b] \to M) : \sigma (a) = p \text { and } \sigma (b) = q \}. \]
The **length**, \( \ell_M (\sigma) \), of a path in \( \sigma \in AC ([a, b] \to M) \) is defined by
\[ \ell_M (\sigma) := \int_{a}^{b} |\dot{\sigma} (t)| \, dt \]
and (as usual) the distance between \( m, m' \in M \) is defined by
\[
d(m, m') := \inf \{ \ell_M(\sigma) : \sigma \in AC_{m,m'}([0,1] \to M) \}.
\]

Given \( v \in TM \), let \( \sigma_v(t) \) be the geodesic in \( M \) such that \( \sigma_v(0) = v \), \( \exp(v) = \sigma_v(1) \in M \) for those \( v \in TM \) such that \( \sigma_v(1) \) exists, and for \( m \in M \) we let \( \exp_m := \exp|_{T_mM} : T_mM \to M \).

Throughout this paper we will use the following geometric notations.

**Notation 2.2** (Metric vector bundles and connections). Let \( (M,g) \) be a Riemannian manifold, \( \pi : E \to M \) be a real Hermitian vector bundle over \( M \) (with fiber dimension, \( D \)) with the fiber metric denoted by, \( \langle \cdot, \cdot \rangle_E \). We further assume that \( E \) is equipped with a metric compatible covariant derivative, \( \nabla = \nabla^E \). Typically we are interested in the setting where \( E = TM \) in which case we always take \( \nabla = \nabla^{TM} \) to be the Levi-Civita covariant derivative on \( TM \). Further,

1. Let \( E_m := \pi^{-1}(\{m\}) \) be the fiber over \( m \) which is isomorphic to \( \mathbb{R}^D \),
2. Let \( \nabla_t^E(\sigma) : E_{\sigma(a)} \to E_{\sigma(t)} \) denote parallel translation along a curve \( \sigma \in C^1([a,b] \to M) \) or more generally along \( \sigma \in AC([a,b] \to M) \) – the space of \( M \)-valued absolutely continuous paths on \([a,b]\), and
3. If \( \xi(t) \in E_{\sigma(t)} \) for \( t \in [a,b] \), let
\[
\nabla_t \xi(t) = \frac{\nabla \xi}{dt}(t) := \nabla_t^E(\sigma) \frac{d}{dt} [\nabla_t^{-1} \xi(t)].
\]

By assumption, for every \( m \in M \), there exists an open neighborhood \((W)\) of \( m \) and a smooth function \( W \times \mathbb{R}^D \ni (m, \alpha) \to u(m) \alpha \in E \) such that \( u(m) : \mathbb{R}^D \to E_m \) is an isometric isomorphism of inner product spaces. We refer to \( (u,W) \) as a (local) orthogonal frame of \( E \). We also let \( SO(\mathbb{R}^D) \) be the group of \( D \times D \) real orthogonal matrices with determinant equal to \( 1 \) and let \( so(\mathbb{R}^D) \) be its Lie algebra of \( D \times D \) real skew-symmetric matrices.

**Remark 2.3** (Local model for \( E \)). As described just above, after choosing a local orthogonal frame, we may identify (locally) \( E \) with the trivial bundle \( W \times \mathbb{R}^D \) where \( W \) is an open subset of \( W \). In this local model we have:

1. \( \pi(m, \alpha) = m \) for all \( m \in W \) and \( \alpha \in \mathbb{R}^d \).
2. \( \langle (m, \alpha), (m, \beta) \rangle = \alpha \cdot \beta \) for all \( m \in W \) and \( \alpha, \beta \in \mathbb{R}^d \).
3. There exists and so \( (\mathbb{R}^D) \)-valued one form, \( \Gamma \), such that if \( S(m) = (m, \alpha(m)) \) is a section of \( E \) and \( v \in T_mW \), then
\[
\nabla_v S = (m, d\alpha(v_m) + \Gamma(v_m) \alpha(m)).
\]
4. If \( \sigma \in C^1([\alpha, \beta] \to W) \), then \( \nabla_t^E(\sigma(a), \alpha) = (\sigma(t), g(t) \alpha) \) where
\[
g(t) \in SO(\mathbb{R}^D) \text{ is the solution to the ordinary differential equation,}
\]
\[
\dot{g}(t) + \Gamma(\dot{\sigma}(t)) g(t) = 0 \quad \text{with} \quad g(a) = I_{\mathbb{R}^D}.
\]
5. If \( \xi(t) = (\sigma(t), \alpha(t)) \) is a \( C^1 \)-path in \( E \), then
\[
\nabla_t \xi(t) = (\sigma(t), \dot{\alpha}(t) + \Gamma(\dot{\sigma}(t)) \alpha(t)).
\]
For completeness, here is the verification of Eq. \((2.1)\):

\[
\frac{d}{dt} \left[ \left\| \tau (t) \right\| \right] = \frac{d}{dt} \left[ \left( \tau (a) , g (t) \right) \right] = \left( \tau (a) , g (t) \right)
\]

where \(\tau (a) = g (t) \cdot \dot{\tau} (t) - g (t) \cdot \dot{g} (t) g (t) \cdot \alpha (t) = \left( \tau (a) , g (t) \right) + \Gamma (\dot{\tau} (t)) \alpha (t) \cdot \alpha (t) \).

The next elementary lemma illustrates how the structures in Notation 2.2 fit together.

**Lemma 2.4.** If \(\xi : [a, b] \to E\) is a \(C^1\)-curve and \(\sigma := \pi \circ \xi \in C^1 ([a, b], M)\), then

\[
\left\| \xi (b) - \xi (a) \right\| \leq \left\| \nabla \sigma^{-1} \xi (b) - \xi (a) \right\| \leq \int_a^b \left\| \nabla \xi (t) \right\| dt.
\]

**Proof.** By the metric compatibility of \(\nabla\), \(\left\| \xi (b) \right\| = \left\| \nabla \sigma^{-1} \xi (b) \right\|\) and therefore

\[
\left\| \xi (b) - \xi (a) \right\| \leq \left\| \nabla \sigma^{-1} \xi (b) - \xi (a) \right\| \leq \left\| \nabla \sigma^{-1} \xi (b) - \xi (a) \right\|
\]

which proves the first inequality in Eq. \((2.2)\). By the fundamental theorem of calculus and the definition of \(\frac{d}{dt}\),

\[
\left\| \nabla \sigma^{-1} \xi (b) - \xi (a) \right\| = \int_a^b \frac{d}{dt} \left[ \left\| \tau (t) \right\| \right] dt = \frac{d}{dt} \left[ \left\| \tau (t) \right\| \right] = \int_a^b \frac{d}{dt} \left[ \left\| \tau (t) \right\| \right] dt.
\]

The second inequality in Eq. \((2.2)\) now follows from this identity and the triangle inequality for vector valued integrals,

\[
\left\| \int_a^b \frac{d}{dt} \left[ \left\| \tau (t) \right\| \right] dt \right\| \leq \int_a^b \left\| \nabla \tau (t) \right\| dt = \int_a^b \frac{d}{dt} \left[ \left\| \tau (t) \right\| \right] dt.
\]

\(\Box\)

**Definition 2.5.** If \(X \in \Gamma (TM)\) and \(v_m, w_m \in T_m M\), let

\[
\nabla^2_{v_m \otimes w_m} X := \frac{d}{dt} \left[ \left\| \tau (t) \right\| \left( \nabla \tau (t) \right) \right]_{v_m \otimes w_m} X
\]

where \(\sigma (t) \in M\) is chosen so that \(\dot{\sigma} (0) = v_m\). In this notation the curvature tensor may be defined by

\[
R (v_m, w_m) \xi_m = \nabla^2_{v_m \otimes w_m} X - \nabla^2_{w_m \otimes v_m} X,
\]

where \(X \in \Gamma (TM)\) is any vector field such that \(X (m) = \xi_m \in T_m M\).
Remark 2.6. If \( W, X \in \Gamma (TM) \) and \( v_m = \dot{\sigma} (0) \in T_m M \), then

\[
\nabla_{v_m} \nabla_W X = \frac{d}{dt} |_{\sigma(0)} \left( \nabla_W X \right)(\sigma(t)) \\
= \frac{d}{dt} \left[ |_{\sigma(t)} \right] \left( \nabla_W X \right)(\sigma(t)) \\
= \frac{d}{dt} \left[ |_{\sigma(t)} \right] \left( \nabla_{\dot{\sigma}(t)} W(\sigma(t))) X \right) \\
= \frac{d}{dt} \left|_{\sigma(t)} \right| \left( \nabla_{\dot{\sigma}(t)} W(\sigma(t))) X \right) + \frac{d}{dt} \left| \nabla W(\sigma(t))) X \right|
\]

(2.3)

\[
= \nabla_{v_m \otimes W(m)} X + \nabla_{\nabla v_m W} X.
\]

This shows two things: 1) that \( \nabla^2_{v_m \otimes w_m} X \) is independent of the choice of curve, \( \sigma(t) \) such that \( \dot{\sigma}(0) = v_m \) since

\[
\nabla^2_{v_m \otimes W(m)} X = \nabla_{v_m \nabla W} X - \nabla_{\nabla v_m W} X,
\]

and 2) that with this definition of \( \nabla^2 X \) the natural product rule derived in Eq. (2.3) holds.

Definition 2.7. For \( f \in C^1 (M, M) \) let \( f_* : TM \to TM \) be the differential of \( f \),

\[
|f_*|_m := \sup_{v \in T_M \; |v| = 1} |f_* v| \quad \text{for each } m \in M, \quad \text{and}
\]

\[
|f_*|_M := \sup_{m \in M} |f_*|_m = \sup_{v \in T_M \; |v| = 1} |f_* v|.
\]

Definition 2.8. We say \( f \in C^1 (M, M) \) is Lipschitz if there exists \( K = K(f) < \infty \) such that

\[
d (f (m), f (m')) \leq K d (m, m') \quad \forall m, m' \in M.
\]

The best smallest \( K \in [0, \infty] \) such that Eq. (2.4) holds is denoted by \( \text{Lip} (f) \), i.e.

\[
\text{Lip} (f) := \sup_{m \neq m'} \frac{d (f (m), f (m'))}{d (m, m')},
\]

We will write \( \text{Lip} (f) = \infty \) if \( f \) is not Lipschitz.

Lemma 2.9. If \( f \in C^1 (M, M) \) then \( \text{Lip} (f) = |f_*|_M \).

Proof. Let \( m, m' \in M \) and \( \sigma \in AC ([0,1], M) \) such that \( \sigma (0) = m \) and \( \sigma (1) = m' \). Then \( f \circ \sigma \in AC ([0,1], M) \) and \( \frac{d}{dt} f (\sigma (t)) = f_* \dot{\sigma} (t) \) for a.e. \( t \) and therefore,

\[
d (f (m), f (m')) \leq \ell (f \circ \sigma) = \int_0^1 |f_* \dot{\sigma} (t)| dt
\]

\[
\leq \int_0^1 |f_*|_M |\dot{\sigma} (t)| dt = |f_*|_M \ell_M (\sigma).
\]

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Taking the infimum of this inequality over all $\sigma \in AC_{m,m'}([0,1], M)$ then shows

$$d\left( f(m), f(m') \right) \leq |f_*|_M d(m, m')$$

which implies $\text{Lip}(f) \leq |f_*|_M$.

For the opposite inequality let $m \in M$, $v \in T_m M$ with $|v| = 1$, and let $\sigma_v(t) := \exp_m(tv)$ for $t$ near 0. Further let $\gamma(t)$ be the smooth curve in $T_{f(m)} M$ satisfying $\gamma(0) = 0_{f(m)}$ and $f(\sigma_v(t)) = \exp_{f(m)}(\gamma(t))$.

Then it follows that

$$\dot{\gamma}(0) = \left( \exp_{f(m)} \right)_* \dot{\gamma}(0) = f_* \dot{\sigma}_v(0) = f_* v$$

and for $t$ sufficiently close to 0, that

$$|\gamma(t)| = d(\gamma(0), f(\sigma_v(t))) \leq \text{Lip}(f) d(m, \sigma_v(t)) = \text{Lip}(f) |v| |t| = |f_*|_M t.$$

Since

$$\lim_{t \to 0} \frac{1}{t} \gamma(t) = \lim_{t \to 0} \frac{1}{t} [\gamma(t) - \gamma(0)] = \dot{\gamma}(0) = f_* v$$

we may conclude that

$$|f_* v| = \lim_{t \to 0} \frac{1}{t} |\gamma(t)| \leq \text{Lip}(f).$$

As $v \in TM$ was arbitrary, it follows that $|f_*|_M \leq \text{Lip}(f)$. \hfill \Box

**Lemma 2.10.** If $X \in \Gamma(TM)$ satisfies, $|\nabla X|_M < \infty$, then

\begin{equation}
||X(p)| - |X(m)|| \leq |\nabla X|_M \cdot d(p, m) \quad \forall \ m, p \in M,
\end{equation}

i.e. $\text{Lip}(|X(\cdot)|) \leq |\nabla X|_M$.

**Proof.** Let $\sigma \in C^1([0,1], M)$ satisfy $\sigma(0) = m$ and $\sigma(1) = p$ and define $\xi(t) = X(\sigma(t))$ and note that

$$\left| \nabla_{d\xi(t)} t \right| = |\nabla_{\dot{\sigma}(t)} X| \leq |\nabla X|_M |\dot{\sigma}(t)|.$$ 

Therefore by Lemma 2.3

\begin{equation}
\int_0^1 \left| \nabla_{d\xi(t)} t \right| dt \leq \int_0^1 |\nabla X|_M |\dot{\sigma}(t)| dt = |\nabla X|_M \cdot \ell(\sigma).
\end{equation}

Taking the infimum of the last term in this inequality over all paths joining $m$ to $p$ gives Eq. (2.5). \hfill \Box

**Theorem 2.11** (Distance estimates). Suppose that $[0,T] \ni t \to Y_t \in \Gamma(TM)$ smoothly varying time dependent vector field and $(a,b) \subset [0,T]$ and $\sigma : (a,b) \to M$ solves

$$\dot{\sigma}(t) = Y_t(\sigma(t)) \quad \text{for all } t \in (a,b)$$

Then $\text{Lip}(|X(\cdot)|) \leq |\nabla X|_M$.
where $\dot{\sigma}(a)$ and $\dot{\sigma}(b)$ are interpreted as appropriate one sided derivatives.

Then for any $s, t \in (a, b)$,

$$
    d(\sigma(t), \sigma(s)) \leq |Y|^*_{J(s,t)} \leq |Y|^*_{T} \quad \text{and}
$$

$$
    d(\sigma(t), \sigma(s)) \leq e^{\|\nabla Y\|_{J(s,t)}} |Y(s,m)|^*_{J(s,t)} \leq e^{\|\nabla Y\|_{T}} : |Y(m)|^*_{T}.
$$

Proof. Without loss of generality we may assume that $s \leq t$. Since $d(\sigma(t), \sigma(s))$ is no more than the length of $|\sigma|_{[s,t]}$ we immediately find,

$$
    d(\sigma(t), \sigma(s)) \leq \int_s^t |\dot{\sigma}(\tau)| d\tau = \int_s^t |Y_\tau(\sigma(\tau))| d\tau |Y|^*_{J(s,t)} \leq |Y|^*_{T}
$$

which gives the first inequality. To prove the second inequality we use the estimate in Eq. (2.6) with $X = Y_t$ to find,

$$
    |\dot{\sigma}(t)| = |Y_t(\sigma(t))| \leq |Y_t(\sigma(s))| + |\nabla Y_t|_M \ell(\sigma|_{[s,t]})
$$

$$
    = |Y_t(\sigma(s))| + |\nabla Y_t|_M \int_s^t |\dot{\sigma}(r)| dr.
$$

If we define

$$
    \psi(\tau) := \int_s^{s+\tau} |\dot{\sigma}(r)| dr \quad \text{for } 0 \leq \tau \leq b - s,
$$

then the inequality in Eq. (2.7) may be rewritten as,

$$
    \dot{\psi}(\tau) = |\dot{\sigma}(s+\tau)| \leq |Y_{s+\tau}(m)| + |\nabla Y_{s+\tau}|_M \psi(\tau) \quad \text{with } \psi(0) = 0.
$$

By Gronwall’s inequality (see Proposition 9.1) and a simple change of variables we find,

$$
    \int_s^{s+\tau} |\dot{\sigma}(r)| dr = \psi(\tau) \leq \int_0^\tau e^{\int_r^\tau |\nabla Y_{s+\tau}|_M d\sigma} \cdot |Y_{s+\tau}(m)| dr
$$

$$
    = \int_0^\tau e^{\int_0^\tau |\nabla Y|_M d\sigma} \cdot |Y_{s+\tau}(m)| dr.
$$

Choosing $\tau$ so that $s + \tau = t$ and making another translational change of variables yields

$$
    d(\sigma(t), \sigma(s)) \leq \int_s^t |\dot{\sigma}(r)| dr \leq \int_0^{t-s} e^{\int_s^{s+\tau} |\nabla Y|_M d\sigma} \cdot |Y_{s+\tau}(m)| dr
$$

$$
    = \int_s^t e^{\int_s^{\tau} |\nabla Y|_M ds} \cdot |Y_\tau(m)| d\tau \leq e^{\|\nabla Y\|_{J(s,t)}} |Y(m)|^*_{J(s,t)}
$$

which gives the second inequality. □

Corollary 2.12. If $(M, g)$ is a complete Riemannian manifold and either $|Y|^*_{T} < \infty$ or $|\nabla Y|^*_{T} < \infty$, then $Y$ is complete.

Proof. Suppose that $s \in [0, T]$ and $m \in M$ are given and that $\sigma : (a, b) \to M$ is a maximal solution to the ODE,

$$
    \dot{\sigma}(t) = Y_t(\sigma(t)) \quad \text{with } \sigma(s) = m.
$$
In order to handle both cases at once, let $R := |Y|^*_T$ or $R = e^{\left|\nabla Y\right|^*_T} |Y(m)|^*_T$ so that according to Theorem \[2.11\] $\sigma(t) \in K := B(0,R)$ for all $t \in (a,b)$.

Since $R < \infty$ and $M$ is complete we know that $K$ is compact and hence
\[
|\sigma'(t)| = |Y_t(\sigma(t))| \leq C_K := \max_{0 \leq s \leq T \& m \in K} |Y_s(m)| < \infty.
\]

Thus it follows that
\[
d(\sigma(t), \sigma(s)) \leq C_K |t-s| \text{ for } s, t \in (a,b).
\]

From this we conclude that $\lim_{t \uparrow b} \sigma(t)$ exists as $\{\sigma(t) : t \uparrow \tau\}$ is a Cauchy and $(M,g)$ is complete and similarly, $\lim_{t \downarrow a} \sigma(t)$ exists and we may extend $\sigma$ to a continuous function on $[a,b]$.

We now claim that the one sided derivatives of $\sigma(t)$ at $t = a$ and $t = b$ exist and are given by $Y_a(\sigma(a))$ and $Y_b(\sigma(b))$ respectively. Indeed, if $\lim_{t \uparrow b} \sigma(t) = p := \sigma(b)$ and $f \in C^\infty(M)$, then for $a < t < b$
\[
f(\sigma(b)) - f(\sigma(t)) = \lim_{\tau \uparrow b} f(\sigma(\tau)) - f(\sigma(t))
= \lim_{\tau \uparrow b} \int_t^\tau df(\sigma'(r)) \, dr
= \lim_{\tau \uparrow b} \int_t^\tau (Y_r f)(\sigma(r)) \, dr = \int_a^b (Y_t f)(\sigma(r)) \, dr
\]

and hence
\[
\lim_{t \uparrow b} \frac{f(\sigma(b)) - f(\sigma(t))}{b-t} = \lim_{t \uparrow b} \frac{1}{b-t} \int_t^b (Y_r f)(\sigma(r)) \, dr = (Y_b f)(\sigma(b)).
\]

Since this holds for all $f \in C^\infty(M)$, it follows that $\sigma$ has a left derivative at $b$ given by $Y_b(\sigma(b))$. Similarly, one shows the right derivative of $\sigma(t)$ exists at $t = a$ and is given by $Y_a(\sigma(a))$.

To complete the proof, for the sake of contradiction, suppose that $b < T$. By local existence of ODEs we may find $\gamma : (b-\varepsilon,b+\varepsilon) \rightarrow M$ such that
\[
\dot{\gamma}(t) = Y_t(\gamma(t)) \text{ with } \gamma(b) = p = \sigma(b).
\]

The path
\[
\tilde{\sigma}(t) := \begin{cases} 
\sigma(t) & \text{if } 0 \leq t \leq b \\
\gamma(t) & \text{if } b \leq t < b + \varepsilon
\end{cases}
\]
then satisfies the ODE on a longer time interval which violates the maximality of the solution and so we in fact must have $b = T$. Similarly, one shows that $a$ must be $0$ as well and hence $Y$ is complete.

\[\Box\]

**Corollary 2.13** ($|\nabla X|_M < \infty$ growth implications). If $X \in \Gamma(TM)$ is a complete time independent vector field, then for all $m \in M$,
\[
d(e^X(m), m) \leq |X|_M, \text{ and }
\]
\[
d(e^{-X}(m), m) \leq |X(m)| \cdot e^{|\nabla X|_M}.
\]
Proof. This follows immediately from Theorem 2.11 with \( Y_t = X \) for all \( t \) and \( \sigma (t) = e^{tX} (m) \). The inequalities in the theorem are applied with \( t = 1 \) and \( s = 0 \). \( \Box \)

2.2. Flows. The next theorem recalls some basic properties of flows associated to complete time dependent vector fields.

**Theorem 2.14.** Suppose that \( J = [0, T] \ni t \rightarrow Y_t \in \Gamma (TM) \) is a smoothly varying complete vector field on \( M \) and for fixed \( s \in J \) and \( m \in M \), \( J \ni t \rightarrow \mu_{t,s} (m) \) is the unique solution to the ODE,

\[
\frac{d}{dt} \mu_{t,s} (m) = Y_t (\mu_{t,s} (m)) \quad \text{with} \quad \mu_{s,s} (m) = m.
\]

Then,

(1) \( J \times J \times M \ni (t,s,m) \rightarrow \mu_{t,s} (m) \in M \) is smooth.

(2) For all \( r,s,t \in J \),

\[
\mu_{t,s} \circ \mu_{s,r} = \mu_{t,r}.
\]

(3) For all \( s,t \in J \), \( \mu_{t,s} \in \text{Diff} (M) \), \( \mu_{t,s}^{-1} = \mu_{s,t} \), the map \( (s,t,m) \rightarrow \mu_{t,s}^{-1} (m) \) is smooth and

\[
\frac{d}{dt} \mu_{t,s}^{-1} = \mu_{s,t} = -(\mu_{s,t})_* Y_t = -(\mu_{t,s}^{-1})_* Y_t.
\]

(4) For all \( s,t \in J \),

\[
\mu_{t,s} = \mu_{t,0} \circ \mu_{s,0}^{-1}.
\]

**Proof.** We take each item in turn.

(1) The smoothness of \( (t,s,m) \rightarrow \mu_{t,s} (m) \) is a basic consequence of the fact that ODE’s depend smoothly on parameters and initial conditions. For example \( \sigma (\tau) = \mu_{T(\tau),s} (m) \) and \( T(\tau) = \tau + s \), then

\[
\frac{d}{d\tau} \left( \begin{array}{c} \sigma (\tau) \\ T(\tau) \end{array} \right) = \left( \begin{array}{c} Y_T (\sigma (\tau)) \\ 1 \end{array} \right) \quad \text{with} \quad \left( \begin{array}{c} \sigma (0) \\ T(0) \end{array} \right) = \left( \begin{array}{c} m \\ T(0) = s \end{array} \right)
\]

and hence \( \sigma (\tau, m, s) = \mu_{\tau+s,s} (m) \) depends smoothly on \( (\tau, m, s) \) and hence \( \mu \) depends smoothly on all of its variables.

(2) To prove Eq. (2.10) simply notice that \( t \rightarrow \mu_{t,s} \circ \mu_{s,r} \) and \( t \rightarrow \mu_{t,r} \) both satisfy the differential equation,

\[
\frac{d}{dt} \nu_t = Y_t \circ \nu_t \quad \text{with} \quad \nu_s = \mu_{s,r}
\]

(3) Taking \( r = t \) in Eq. (2.10) gives,

\[
\mu_{t,s} \circ \mu_{s,t} = \mu_{t,t} = \text{Id}_M \quad \text{for all} \quad s,t \in J.
\]

By interchanging \( s \) and \( t \) in the above equation may also be written as

\[
\mu_{s,t} \circ \mu_{t,s} = \text{Id}_M \quad \text{for all} \quad s,t \in J.
\]
From these last two equations we see that $\mu_{t,s} \in \text{Diff} (M)$ for all $s, t \in J$ and moreover that $\mu_{t,s}^{-1} = \mu_{s,t}$ which also shows the map $(s, t, m) \rightarrow \mu_{t,s}^{-1}(m)$ is smooth. To prove Eq. (2.11) we differentiate Eq. (2.13) with respect to $t$ to find,

$$0 = \dot{\mu}_{t,s} \circ \mu_{s,t} + (\mu_{t,s}) \_ \dot{s}_{s,t} = Y_t \circ \mu_{t,s} \circ \mu_{s,t} + (\mu_{t,s}) \_ \dot{\mu}_{s,t}$$

and hence

$$\dot{s}_{s,t} = - (\mu_{t,s})^{-1} Y_t = - (\mu_{s,t}) \_ Y_t.$$

(4) This one is easily deduced by what has already been proved;

$$\mu_{t,s} = \mu_{t,0} \circ \mu_{0,s} = \mu_{t,0} \circ \mu_{s,0}^{-1}.$$

2.3. $\text{Diff} (M)$-Adjoint Action.

**Definition 2.15** (Adjoint actions and Lie derivatives). If $f \in \text{Diff} (M)$ and $Y \in \Gamma (TM)$, let $\text{Ad}_f Y = f_* Y \circ f^{-1} \in \Gamma (TM)$, i.e. $\text{Ad}_f Y$ is the vector field defined by

$$(2.15) \quad (\text{Ad}_f Y) (m) = f_* Y (f^{-1} (m)) \in T_m M \forall \ m \in M.$$  

If $X, Y \in \Gamma (TM)$, then the Lie derivative of $Y$ with respect to $X$ is

$$(2.16) \quad L_X Y := \frac{d}{dt} \bigg|_0 \text{Ad}_{e^{-tX}} Y = \frac{d}{dt} \bigg|_0 \text{e}^{-tX} Y \circ e^{tX}.$$  

We further let

$$(2.17) \quad \text{ad}_X := \frac{d}{dt} \bigg|_0 \text{Ad}_{e^{tX}} = -L_X.$$  

**Remark 2.16.** The following identities are well known and easy to prove.

1. If $f, g \in \text{Diff} (M)$ then $\text{Ad}_{f \circ g} = \text{Ad}_f \text{Ad}_g$.
2. The Lie derivative, $L_X Y$, is again a vector field on $M$ which may also be computed using

$$L_X Y = [X, Y] = XY - YX.$$  

3. If $X, Y \in \Gamma (TM)$ and $f \in \text{Diff} (M)$, then

$$(2.18) \quad \text{Ad}_f [X, Y] = [\text{Ad}_f X, \text{Ad}_f Y].$$  

For example, to verify Eq. (2.18), observe that $\text{Ad}_f Y$ is the unique vector field on $M$ such that

$$(2.19) \quad f_* Y = (\text{Ad}_f Y) \circ f,$$

i.e. such that $Y$ and $\text{Ad}_f Y$ are “$f$-related.” Since the commutator of two $f$-related vector fields are $f$-related, it follows that $[X, Y]$ and $[\text{Ad}_f X, \text{Ad}_f Y]$ are $f$-related, i.e.

$$f_* [X, Y] = [\text{Ad}_f X, \text{Ad}_f Y] \circ f \implies \text{Ad}_f [X, Y] = f_* [X, Y] \circ f^{-1} = [\text{Ad}_f X, \text{Ad}_f Y].$$
Proposition 2.17 (Adjoint flow equations). Suppose $Y \in \Gamma(TM)$ and $\nu_t \in \text{Diff}(M)$ is smoothly varying in $t$. If we define

$$W_t := \dot{\nu}_t \circ \nu_t^{-1} \in \Gamma(TM) \quad \text{and} \quad \tilde{W}_t = (\nu_t^{-1})_* \dot{\nu}_t \in \Gamma(TM),$$

then the adjoint flows of $Y$, $\text{Ad}_{\nu_t} Y$ and $\text{Ad}_{\nu_t^{-1}} Y$, satisfy

\begin{equation}
\frac{d}{dt} \text{Ad}_{\nu_t} Y = [\text{Ad}_{\nu_t} Y, W_t] = \text{Ad}_{\nu_t} \left[ Y, \tilde{W}_t \right]. \tag{2.20}
\end{equation}

and

\begin{equation}
\frac{d}{dt} \text{Ad}_{\nu_t^{-1}} Y = \left[ \tilde{W}_t, \text{Ad}_{\nu_t^{-1}} Y \right] = \text{Ad}_{\nu_t^{-1}} \left[ W_t, Y \right]. \tag{2.21}
\end{equation}

Proof. Let $Y_t := \text{Ad}_{\nu_t} Y$ for $t \in \mathbb{R}$ (as in Eq. (2.19)) so that,

$$Y_t \circ \nu_t = (\text{Ad}_{\nu_t} Y) \circ \nu_t = \nu_t \ast Y.$$

Hence, if $\varphi \in C^\infty(M, \mathbb{R})$, then

$$(Y_t \varphi) \circ \nu_t = (\nu_t \ast Y_t) \varphi = Y (\varphi \circ \nu_t)$$

and differentiating this equation in $t$ gives

$$\dot{Y_t \varphi} \circ \nu_t + (W_t Y_t \varphi) \circ \nu_t = Y ((W_t \varphi) \circ \nu_t) = (\nu_t \ast Y) (W_t \varphi) = (Y_t W_t \varphi) \circ \nu_t.$$

The last equation is equivalent to the first equality in Eq. (2.20). Since $\text{Ad}_{\nu_t} \tilde{W}_t = W_t$,

$$[Y_t, W_t] = \left[ \text{Ad}_{\nu_t} Y, \text{Ad}_{\nu_t} \tilde{W}_t \right] = \text{Ad}_{\nu_t} \left[ Y, \tilde{W}_t \right]$$

which gives the second equality in Eq. (2.20).

As, by Theorem 2.14,

$$\frac{d}{dt} \nu_t^{-1} = -\nu_t^{-1} W_t = -\nu_t^{-1} W_t \circ \nu_t = -W_t \circ \nu_t^{-1},$$

it follows from Eq. (2.20) that

$$\frac{d}{dt} \left( \text{Ad}_{\nu_t^{-1}} Y \right) = \left[ \text{Ad}_{\nu_t^{-1}} Y, -\tilde{W}_t \right] = \left[ \tilde{W}_t, \text{Ad}_{\nu_t^{-1}} Y \right] \quad \text{and}$$

$$\frac{d}{dt} \left( \text{Ad}_{\nu_t^{-1}} Y \right) = \text{Ad}_{\nu_t^{-1}} \left[ Y, -W_t \right] = \text{Ad}_{\nu_t^{-1}} \left[ W_t, Y \right],$$

which proves both equalities Eq. (2.21). \hfill \Box

Corollary 2.18. If $Y, X \in \Gamma(TM)$ with $X$ being complete, then

$$\frac{d}{dt} \text{Ad}_{e^{tX}} Y = \text{Ad}_{e^{tX}} \text{ad}_X Y = \text{ad}_X \text{Ad}_{e^{tX}} Y$$

where $\text{ad}_X = -L_X$.

Proof. The result follows directly from Eq. (2.20) by taking $\nu_t = e^{tX}$ and noting that $W_t = X = \tilde{W}_t$ in this case as $\text{Ad}_{e^{tX}} X = X$ for all $t \in \mathbb{R}$. \hfill \Box
2.4. Vector field differentiation of flows.

**Definition 2.19** (Differentiating \( \mu^X \) in \( X \)). Let \( X_t, Y_t \in \Gamma (TM) \) be smoothly varying time dependent vector fields on \( M \). We say \( \mu^X \) is **differentiable relative to** \( Y \) if there exists \( \{ X^\varepsilon_t \}_{\varepsilon, t} \subset \Gamma (TM) \) such that \( (t, \varepsilon, m) \rightarrow X^\varepsilon_t (m) \in TM \) is smooth and \( X^\varepsilon_t \) is complete for \( \varepsilon \) near 0, \( X^0_t = X_t \), and \( \frac{d}{d\varepsilon}|_{0}X^\varepsilon_t = Y_t \).

If all of this holds we let

\[
(2.22) \quad \partial_Y \mu^X_{t,s} := \frac{d}{d\varepsilon}|_{0}\mu^X_{t,s}.
\]

**Theorem 2.20.** If \( X_t, Y_t \in \Gamma (TM) \) are as in Definition 2.19 so that \( \partial_Y \mu^X_{t,s} = \frac{d}{d\varepsilon}|_{0}\mu^X_{t,s} \) exists, then

\[
(2.23) \quad \partial_Y \mu^X_{t,s} = \int_{s}^{t} \mu^X_{t,\tau, s} [Y_{\tau} \circ \mu^X_{\tau, s}] d\tau
\]

\[
(2.24) = \left( \int_{s}^{t} \Ad_{\mu^X_{\tau, s}} Y_{\tau} d\tau \right) \circ \mu^X_{t, s}
\]

\[
(2.25) = (\mu^X_{t, s})_{*} \int_{s}^{t} \Ad_{\mu^X_{\tau, s}} Y_{\tau} d\tau.
\]

**Proof.** Let \( V_{t,s} := (\mu^X_{t,s})_{*} \partial_Y \mu^X_{t,s} \in \Gamma (TM) \) so that

\[
(2.26) \quad \partial_Y \mu^X_{t,s} = \frac{d}{d\varepsilon}|_{0}\mu^X_{t,s} = (\mu^X_{t,s})_{*} V_{t,s}
\]

Notice that Eq. (2.26) is equivalent to, for all \( f \in \mathcal{C}^\infty (M) \),

\[
\frac{d}{d\varepsilon}|_{0}[f \circ \mu^X_{t,s}] = df \left( \frac{d}{d\varepsilon}|_{0}\mu^X_{t,s} \right) = df \left( \partial_Y \mu^X_{t,s} \right)
\]

\[
(2.27) = df \left( (\mu^X_{t,s})_{*} V_{t,s} \right) = V_{t,s} [f \circ \mu^X_{t,s}].
\]

So, on one hand,

\[
\frac{d}{d\varepsilon}|_{0} [f \circ \mu^X_{t,s}] = \frac{d}{dt} \frac{d}{d\varepsilon}|_{0} [f \circ \mu^X_{t,s}] = \frac{d}{dt} V_{t,s} [f \circ \mu^X_{t,s}]
\]

\[
(2.28) = V_{t,s} [f \circ \mu^X_{t,s}] + V_{t,s} [X_t f \circ \mu^X_{t,s}].
\]

On the other hand,

\[
\frac{d}{dt} [f \circ \mu^X_{t,s}] = (X_t f) \circ \mu^X_{t,s}
\]

and differentiating this equation in \( \varepsilon \) implies while using Eq. (2.27) with \( f \) replaced by \( X_t f \) implies,

\[
\frac{d}{d\varepsilon}|_{0} \frac{d}{dt} [f \circ \mu^X_{t,s}] = Y_t f \circ \mu^X_{t,s} + \frac{d}{d\varepsilon}|_{0} [(X_t f) \circ \mu^X_{t,s}]
\]

\[
(2.29) = Y_t f \circ \mu^X_{t,s} + V_{t,s} [X_t f \circ \mu^X_{t,s}].
\]

Comparing Eqs. (2.28) and (2.29) shows,

\[
(\mu^X_{t,s} \dot{V}_{t,s}) f = \dot{V}_{t,s} [f \circ \mu^X_{t,s}] = Y_t f \circ \mu^X_{t,s} = (Y_t \circ \mu^X_{t,s}) f \quad \forall \ f \in \mathcal{C}^\infty (M)
\]
which implies,

\[(2.30) \quad V_{t,s} = \mu_{s,t}^X Y_t \circ \mu_{t,s}^X.\]

Since \(\mu_{s,s}^X = Id_M\) we know that \(\partial_Y \mu_{s,s}^X = 0\) and hence \(V_{s,s} = 0\) and so integrating Eq. \((2.30)\) implies,

\[
V_{t,s} = \int_s^t \mu_{s,\tau}^X Y_\tau \circ \mu_{\tau,s}^X d\tau. 
\]

This equality along with Eq. \((2.26)\) proves Eq. \((2.25)\). The proofs of Eqs. \((2.23)\) and \((2.24)\) now easily follows since

\[
(\mu_{t,s}^X)_* V_{t,s} = \int_s^t (\mu_{\tau,s}^X)_* \mu_{s,\tau}^X Y_\tau \circ \mu_{\tau,s}^X d\tau 
\]

\[
= \int_s^t \mu_{t,\tau}^X Y_\tau \circ \mu_{\tau,s}^X d\tau 
\]

\[
= \int_s^t \mu_{t,\tau}^X Y_\tau \circ \mu_{\tau,s}^X \circ \mu_{\tau,\tau}^X d\tau = \left( \int_s^t \text{Ad}_{\mu_{t,\tau}^X} Y_\tau d\tau \right) \circ \mu_{t,s}^X.
\]

The following theorem is an important special case of Theorem \(2.20\).

**Theorem 2.21** (Differential of \(e^{tX} \) in \(X\)). Suppose that \(M\) is a smooth manifold and for each \(\sigma \in \mathbb{R}, \{X_\sigma\} \subset \Gamma(TM)\) is a smooth varying one parameter family of complete vector fields on \(M\) and let \(X := X_0\) and \(Y := \frac{d}{d\sigma}|_0 X_\sigma\). Then

\[
(2.31) \quad \partial_Y e^{tX} = \frac{d}{d\sigma}|_0 e^{tX_\sigma} = e^{tX} \int_0^t e_{\tau}^{-\tau X} Y \circ e^{\tau X} d\tau
\]

\[
(2.32) \quad = \int_0^t e^{(t-\tau)X} Y \circ e^{\tau X} d\tau
\]

\[
(2.33) \quad = \left[ \int_0^t \text{Ad}_{e^{\tau X}} Y d\tau \right] \circ e^{tX}.
\]

**Notation 2.22.** To each smooth path, \(t \to Z_t \in \Gamma(TM)\), of complete vector fields, let

\[
(2.34) \quad W^Z_t := \int_0^1 e^{sZ_t} \dot{Z}_t \circ e^{-sZ_t} ds = \int_0^1 \text{Ad}_{e^{sZ_t}} \dot{Z}_t ds
\]

**Corollary 2.23.** If \(t \to Z_t \in \Gamma(TM)\) is a smooth path of complete vector fields, then

\[
(2.35) \quad \frac{d}{dt} e^{Z_t} = W^Z_t \circ e^{Z_t}.
\]

**Proof.** Theorem \(2.21\) with \(t = 1\) and \(X_\sigma = Z_{t+s} \), gives

\[
\frac{d}{dt} e^{Z_t} = \left[ \int_0^1 \text{Ad}_{e^{\tau X_0}} X_\sigma d\tau \right] \circ e^{X_0} = \left[ \int_0^1 \text{Ad}_{e^{\tau Z_t}} \dot{Z}_t d\tau \right] \circ e^{Z_t}.
\]
2.5. Jacobian formulas and estimates for flows.

**Notation 2.24.** Let \( \mathbb{R} \ni t \to W_t \in \Gamma (TM) \) be a smoothly varying time dependent vector field and suppose that \( \mathbb{R} \times M \ni (t,m) \to \nu_t (m) \in M \) is in \( C^\infty (\mathbb{R} \times M, M) \) satisfies the ordinary differential equation,

\[
(2.36) \quad \dot{\nu}_t = W_t \circ \nu_t.
\]

Notice that if \( W_t (\cdot) \) is complete, then \( \nu_t = \mu_{t,s} \circ \nu_s \) for any \( s \in \mathbb{R} \). The general goal of this section is to find estimates on \( \nu_t, \nu_t^*, \) and \( \nabla \nu_t^* \) (see Definition 5.25 below) expressed in terms of the geometry of \( M \) and \( W_t \).

**Proposition 2.25.** If \( W_t \in \Gamma (TM) \) and \( \nu_t \in C^\infty (M, M) \) are as in Notation 2.24, then

\[
(2.37) \quad \nabla_{dt} \nu_t^* v = \nabla_{\nu_t^* v} W_t \quad \forall v \in TM.
\]

**Proof.** If \( \sigma (s) \) is a curve in \( M \) so that \( \sigma' (0) = \frac{d}{ds} |_{0} \sigma (s) = v \), then

\[
\nabla_{dt} \nu_t^* v = \nabla_{d} \frac{d}{ds} |_{0} \nu_t (\sigma (s)) = \nabla_{d} \frac{d}{dt} \nu_t (\sigma (s)) = \nabla_{\nu_t^* v} W_t.
\]

**Corollary 2.26.** If \( W_t \in \Gamma (TM) \) and \( \nu_t \in C^\infty (M, M) \) are as in Notation 2.24, then

\[
(2.38) \quad |\nu_t^*|_m \leq |\nu_s^*|_m \cdot e^{\int_{s,t} |\nabla W_t|_{\nu_t} (m) d\tau} \leq |\nu_s^*|_m \cdot e^{\int_{s,t} |\nabla W_t|}\]

and in particular,

\[
(2.39) \quad \text{Lip} (\nu_t) = |\nu_t^*|_M \leq |\nu_s^*|_M \cdot e^{\int_{s,t} |\nabla W_t|}.
\]

We also have the following time derivative estimates,

\[
(2.40) \quad \left| \frac{\nabla}{dt} \nu_t^* \right|_m \leq |\nu_s^*|_m \cdot |\nabla W_t|_{\nu_t} (m) \cdot e^{\int_{s,t} |\nabla W_t|} d\tau
\]

and

\[
(2.41) \quad \left| \frac{\nabla}{dt} \nu_t^* \right|_M \leq |\nu_s^*|_M \cdot |\nabla W_t| \cdot e^{\int_{s,t} |\nabla W_t|}.
\]

**Proof.** If we define \( H_t v := \nabla_t W_t \) for all \( v \in TM \), then Proposition 2.25 states, for any \( v \in TM \), that

\[
(2.42) \quad \frac{\nabla}{dt} \nu_t^* v = H_t \nu_t^* v.
\]

Therefore by the geometric Bellman-Gronwall’s inequality in Corollary 9.3 (with \( G \equiv 0 \)),

\[
|\nu_t^* v| \leq e^{\int_{s,t} \|H_t\|_{op} d\tau} |\nu_s^* v| = e^{\int_{s,t} |\nabla W_t|_{\nu_t} (m) d\tau} |\nu_s^* v|
\]
which proves Eqs. (2.38) and (2.39). By Eq. (2.42),

\[ \left| \frac{\nabla}{dt} \nu_t v \right| \leq \|H_t\|_{op} \cdot |\nu_t v| = |\nabla W_t|_{\nu_t(m)} \cdot |\nu_t v| \leq |\nabla W_t|_{\nu_t(m)} |\nu_{s*} v| = e^{f_{J(s,t)} |\nabla W_r|_{\nu_r(m)} dr} \leq |\nabla W_t|_{\nu_t(m)} |\nu_{s*} v| \cdot e^{f_{J(s,t)} |\nabla W_r|_{\nu_r(m)} dr} . \]

Taking the supremum of this inequality over \( v \in T_m M \) with \( |v| = 1 \) gives the estimate in Eq. (2.40) which then easily implies Eq. (2.41). □

The following corollary records the results in Proposition 2.25 and Corollary 2.26 when \( W_t = X \in \Gamma(TM) \) is a complete vector field and \( \nu_t = e^{tX} \).

**Corollary 2.27.** If \( X \in \Gamma(TM) \) is a complete vector field and \( t \in \mathbb{R} \), then

\[
\frac{\nabla}{dt} e^{tX} v_m = \nabla e^{tX} v_m X \text{ with } e^{tX} v_m = v_m ,
\]

\[
\left| e^{tX} \right|_{m} \leq e^{\int_{J(0,t)} |\nabla X| e^{rX} dr} \leq e^{t |\nabla X|} ,
\]

\[
\left| \frac{\nabla}{dt} e^{tX} \right|_{m} \leq |\nabla X| e^{tX} \cdot e^{\int_{J(0,t)} |\nabla X| e^{rX} dr} ,
\]

\[
\text{Lip} (e^{tX}) = |e^{tX}|_{m} \leq e^{t |\nabla X|} , \text{ and}
\]

\[
\left| \frac{\nabla}{dt} e^{tX} \right|_{M} \leq |\nabla X| e^{t |\nabla X|} .
\]

**Corollary 2.28.** If \( \mathbb{R} \ni t \mapsto W_t \in \Gamma(TM) \) is a complete time dependent vector field and \( Z \in \Gamma(TM) \), then

\[
\left| \text{Ad}_{\nu_t}^W Z \right|_{m} \leq e^{\int_{J(0,t)} |\nabla W_r|_{\nu_r(m)} dr} \cdot |Z|_{\nu_t}^{W} (m) \leq e^{\int_{J(0,t)} |\nabla W_r| dr} \cdot |Z|_{\nu_t} (m) .
\]

As a special case, if \( X \in \Gamma(TM) \) is complete, then

\[
|\text{Ad}_{e^{tX}} Z|_{m} \leq e^{\int_{0}^{t} |\nabla X| e^{-rX} dr} \cdot |Z|_{e^{-X}} (m) \leq e^{t |\nabla X|} \cdot |Z|_{M} .
\]

**Proof.** Let \( \nu_t := \mu^W_{t,s} \), then \( \nu_t^{-1} = \mu^W_{s,t}, \nu_s = Id_M \), and

\[
\left( \text{Ad}_{\nu_t^{-1}}^W Z \right) (m) = |(\nu_t)_{s} Z (\nu_t^{-1} (m))| \leq |(\nu_t)_{s} | (\nu_t^{-1} (m)) \cdot |Z|_{\nu_t^{-1} (m)} \leq |\nu_{s*} | (\nu_t^{-1} (m)) \cdot e^{\int_{J(s,t)} |\nabla W_r|_{\nu_r} (m) dr} \cdot |Z|_{\nu_t^{-1} (m)} \leq e^{\int_{J(s,t)} |\nabla W_r|_{\nu_t} (m) dr} \cdot |Z|_{\nu_t} (m) .
\]

For the second assertion we take Eq. (2.44) with \( s = 0, t = 1 \), and \( W_t = X \) for all \( t \) to find,

\[
|\text{Ad}_{e^{tX}} Z|_{m} \leq e^{\int_{0}^{t} |\nabla X| e^{-rX} dr} \cdot |Z|_{e^{-X}} (m) = e^{\int_{0}^{t} |\nabla X| e^{-rX} dr} \cdot |Z|_{e^{-X}} (m) .
\]

□
2.6. **Distance estimates for flows.** This subsection is devoted to estimating the distance between two flows, \(\mu^X_t\) and \(\mu^Y_t\). A key observation in the proofs to follow is, given \(t \in [0, T]\), that
\[
(2.46) \quad [0, t] \ni s \to \Theta_s(m) := \mu^X_{t,s} \circ \mu^Y_{s,0}(m)
\]
is a natural path in \(M\) which interpolates between \(\Theta_0(m) = \mu^X_{t,0}(m)\) at \(s = 0\) and \(\Theta_t(m) = \mu^Y_{t,0}(m)\) at \(s = t\).

**Theorem 2.29.** Let \(J = [0, T] \ni t \to X_t, Y_t \in \Gamma(TM)\) be two smooth complete time dependent vector fields on \(M\) and \(\mu^X_t\) and \(\mu^Y_t\) be their corresponding flows. Then for \(t > 0\) (for notational simplicity)
\[
(2.47) \quad d(\mu^X_{t,0}(m), \mu^Y_{t,0}(m)) \leq \int_0^t e^{\int_0^t |\nabla s|_{\mu^X_{t,s}(m)} d\sigma} \cdot |Y_s - X_s|_{\mu^Y_{s,0}(m)} \, ds
\]
and in particular,
\[
(2.48) \quad d_M(\mu^X_{t,0}, \mu^Y_{t,0}) \leq e^{|\nabla X|^*} \cdot |Y - X|^*.
\]

**Proof.** Fix \(t \in [0, T]\). If \(\Theta_s(m)\) is as in Eq. (2.46), then
\[
(2.49) \quad d(\mu^X_{t,0}(m), \mu^Y_{t,0}(m)) \leq \int_0^t |\Theta_s'(m)| \, ds.
\]
Making use of Theorem 2.14 we find,
\[
\Theta_s'(m) = \left( \frac{d}{ds} \mu^X_{t,s} \right) \circ \mu^Y_{s,0}(m) + \mu^X_{t,s} \left( \frac{d}{ds} \mu^Y_{s,0}(m) \right) = \left( \mu^X_{t,s} \right)_* [-X_s + Y_s] \circ \mu^Y_{s,0}(m).
\]
The Jacobian estimate in Corollary 2.26 with \(nu = \mu^X_{t,s}\) states that,
\[
(2.51) \quad |\mu^X_{t,s}|_{m} \leq e^{\int_0^t |\nabla X|_{\mu^X_{t,s}(m)} d\sigma} \leq e^{\int_0^t |\nabla X|_{M} d\sigma}.
\]
By this Jacobian estimate and Eq. (2.50), we find that
\[
|\Theta_s'(m)| \leq \left( \mu^X_{t,s} \right)_* \left| Y_s - X_s \right|_{\mu^Y_{s,0}(m)} \leq e^{\int_0^t |\nabla X|_{M} d\sigma} \left| Y_s - X_s \right|_{\mu^Y_{s,0}(m)}
\]
which then substituted back into Eq. (2.49) completes the proof. \(\square\)

**Corollary 2.30.** Let \(J = [0, T] \ni t \to Y_t \in \Gamma(TM)\) be a smooth complete time dependent vector field on \(M\) and \(\mu^Y_t\) be the corresponding flow. Then for \(t > 0\)
\[
(2.53) \quad d(m, \mu^Y_{t,0}(m)) \leq \int_0^t |Y_s|_{\mu^Y_{s,0}(m)} \, ds \leq |Y|^*
\]
and
\[
(2.54) \quad d(\mu^Y_{t,0}(m), m) \leq \int_0^t e^{\int_0^t |\nabla Y|_{\mu^Y_{t,s}(m)} d\sigma} \cdot |Y_s|_{m} \, ds \leq e^{|\nabla Y|^*} \int_0^t |Y_s(m)| \, ds.
\]
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Proof. This corollary easily follows from Theorem 2.11. Alternatively, taking $X \equiv 0$ in Theorem 2.29 gives Eq. (2.53) while taking $Y \equiv 0$ shows

$$d (\mu_{X,0} (m) , m) \leq \int_0^t e^{\int_s^t |\nabla X| u |\mu_{X,s} (m) | ds} |X_s| m \ ds \leq e^{\int_0^t |X_s (m) | ds}.$$  
Equation (2.54) now follows by relabeling $X$ to $Y$. □

3. Nilpotent Lie Algebras (Group) Results

Suppose that $A$ is a non-commutative associative algebra with unit, 1, over $\mathbb{R}$ such that: 1) $\dim_{\mathbb{R}} A < \infty$, 2) $A = \mathbb{R} \cdot 1 \oplus g$ where $g$ is a sub-algebra of $A$ without unit, and 3) there exists $\kappa \in \mathbb{N}$ such that $\xi_1 \ldots \xi_{\kappa+1} = 0$ whenever $\xi_1, \ldots, \xi_{\kappa+1} \in g$. We make $A$ into a Lie algebra using the commutator, $[\xi, \eta] := \xi \eta - \eta \xi$ for all $\xi, \eta \in A$, as the Lie bracket. Note that $g$ is a Lie-subalgebra of $A$ and as usual we let $\text{ad}_\xi : A \to A$ be the linear operator defined by $\text{ad}_\xi \eta = [\xi, \eta]$. See Example 3.2 below for the key example of this setup that is used in the bulk of this paper.

3.1. Calculus and functional calculus on $A$.

Definition 3.1. Let $H_0$ denote the germs of functions which are analytic in a neighborhood of $0 \in \mathbb{C}$ and for $f (z) = \sum_{k=0}^{\infty} a_k z^k \in H_0$ and $\xi \in g$, let

$$f (\xi) := \sum_{k=0}^{\infty} a_k \xi^k = \sum_{k=0}^{\kappa} a_k \xi^k \in A$$

and

$$f (\text{ad}_\xi) := \sum_{k=0}^{\infty} a_k \text{ad}_{\xi}^k = \sum_{k=0}^{\kappa-1} a_k \text{ad}_{\xi}^k : A \to A.$$

In most of the results below, we describe properties of $f (\xi)$ for $f \in H_0$ with the understanding that similar results hold equally as well for $f (\text{ad}_\xi)$.

Proposition 3.2. For each fixed $\xi \in g$, the map

$$H_0 \ni f \to f (\xi) \in A$$

is an algebra homomorphism and for each fixed $f \in A_0$, the map

$$g \ni \xi \to f (\xi) \in A$$

is smooth, i.e. it is infinitely continuously differentiable. Moreover, if $J := (a,b) \ni t \to \xi (t) \in g$ is differentiable with $[\xi (t) , \xi (s)] = 0$ for $s, t \in J$, then

$$\frac{d}{dt} f (\xi (t) ) = f' (\xi (t) ) \hat{\xi} (t) = \dot{\xi} (t) f' (\xi (t) ) \forall \ t \in J.$$

Proof. The standard fact that the map in Eq. (3.1) is an algebra homomorphism is easily seen to be a direct consequence of the multiplication rules for power series. The smoothness of $f : g \to A$ is a consequence of the fact
that \( f(\xi) \) is a finite linear combination of the smooth multi-linear maps, \( \mathfrak{g} \ni \xi \to \xi^k \in \mathcal{A} \), for each \( k \in \{0, 1, 2, \ldots, \kappa \} \). For arbitrary \( \xi, \eta \in \mathfrak{g} \) we have
\[
\partial_\eta \xi^k = \sum_{j=0}^{k-1} \xi^j \eta \xi^{k-j}
\]
which simplifies to
\[
\partial_\eta \xi^k = k \eta \xi^{k-1} = k \xi^{k-1} \eta \text{ when } [\xi, \eta] = 0.
\]

With these observations the proof of Eq. (3.2) is a consequence of the following simple computation,
\[
\frac{d}{dt} f(\xi(t)) = \sum_{k=0}^{\kappa} a_k \frac{d}{dt} \xi(t)^k = \sum_{k=0}^{\kappa} a_k k \xi(t) \xi(t)^{k-1} = \left( \sum_{k=0}^{\infty} a_k k \xi(t) \xi(t)^{k-1} \right) = f'(\xi(t)) \dot{\xi}(t) = \dot{\xi}(t) f'(\xi(t)).
\]

□

For our purposes, the functions, \( e^z, (1+z)^{-1}, \log (1+z) \),

(3.3) \[ \psi(z) := \frac{e^z - 1}{z} = \sum_{k=1}^{\infty} \frac{z^{k-1}}{k!} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!}, \]

(3.4) \[ \psi_-(z) := \frac{1}{\psi(-z)} = \frac{z}{1-e^{-z}} = \frac{e^z \cdot z}{e^z - 1}, \text{ and} \]

(3.5) \[ \mathcal{L}(z) = \psi_-(\log (1+z)) := \frac{(1+z) \log (1+z)}{z} = 1 + \sum_{j=2}^{\infty} \frac{(-1)^j}{j \cdot (j-1)} z^{j-1} \]

are the most important functions in \( \mathcal{H}_0 \).

**Lemma 3.3.** The subset,
\[ G := 1 + \mathfrak{g} = \{1 + \xi : \xi \in \mathfrak{g}\}, \]

equipped with the algebra multiplication law forms a group where the inverse operation is given by
\[ (1 + \xi)^{-1} = \frac{1}{1+\xi}. \]

The following corollary follows directly from Proposition 3.2.

**Corollary 3.4.** The three maps, \( \mathfrak{g} \ni \xi \to e^\xi \in G \), \( \mathfrak{g} \ni \xi \to (1+\xi)^{-1} \in G \), and \( \mathfrak{g} \ni \xi \to \log (1+\xi) \in \mathfrak{g} \) are smooth and these maps satisfy the following natural identities.
For all $\xi \in g$,
\[
\frac{d}{dt} e^{t\xi} = \xi e^{t\xi}, \quad \text{and} \quad \frac{d}{dt} \log (1 + t\xi) = \frac{1}{1 + \xi}.
\]
Moreover generally, if $t \to \xi(t) \in g$ is differentiable near $t_0 \in \mathbb{R}$ and $[\xi(t), \xi(s)] = 0$ for $s$ and $t$ near $t_0$, then
\[
\frac{d}{dt} \xi(t) = \dot{\xi}(t) e^{\xi(t)} \quad \text{and} \quad \frac{d}{dt} \log (1 + \xi(t)) = \frac{\dot{\xi}(t)}{1 + \xi(t)} = \dot{\xi}(t) (1 + \xi(t))^{-1}.
\]
(2) For all $s, t \in \mathbb{R}$, $e^{t\xi} e^{s\xi} = e^{(t+s)\xi}$ and $e^{-t\xi} = [e^{t\xi}]^{-1}$.

**Proposition 3.5.** The map,
\[
g \ni \xi \to e^\xi \in G,
\]
is a diffeomorphism and the map,
\[
G \ni g = 1 + \xi \to \log (g) = \log (1 + \xi) \in g,
\]
is its inverse map.

**Proof.** By Corollary 3.4,
\[
\frac{d}{dt} \log (e^{t\xi}) = \left[ e^{t\xi} \right]^{-1} \frac{d}{dt} e^{t\xi} = e^{-t\xi} \frac{d}{dt} e^{t\xi} = \xi,
\]
from which it follows that $\log (e^{t\xi}) = \log (1) + t\xi = t\xi$. Taking $t = 1$ in this identity shows $\log (e^\xi) = \xi$.

Similarly, by Corollary 3.4 if we let $g(t) := e^{\log(1+t\xi)} \in G$, then
\[
\dot{g}(t) = \frac{d}{dt} e^{\log(1+t\xi)} = e^{\log(1+t\xi)} \cdot \frac{d}{dt} \log (1 + t\xi) = g(t) \frac{\xi}{1 + t\xi} \quad \text{with} \quad g(0) = 1.
\]
Since $t \to (1 + t\xi)$ satisfies the same equation as $g(t)$, by uniqueness of solutions we conclude that $g(t) = 1 + t\xi$ for all $t \in \mathbb{R}$ and in particular taking $t = 1$ shows
\[
e^{\log(1+\xi)} = g(1) = 1 + \xi.
\]

For $k \in G$, let $L_k \in \text{Diff}(G)$ be defined by $L_k g = kg$ for all $g \in G$. For $\xi \in g$, let $G \ni g \to \xi(g) := L_{g*} \xi \in T_g G$ be the left invariant vector field on $G$ associated to $\xi \in g$. If $f : G + 1 + g \cong g \to \mathbb{R}$ is a smooth function, then
\[
\left( \dot{\xi} f \right)(g) = \frac{d}{dt} \left( e^{t\xi} \right) \frac{d}{dt} f \left( g e^{t\xi} \right) = \left( \partial_{g*} \xi \right)(g) f'(g) g\xi.
\]
Thus if $\xi, \eta \in g$,
\[
\left( \dot{\eta} \xi f \right)(g) = f''(g) [g\eta \otimes g\xi] + f'(g) g\eta \xi.
\]
and since \( f''(g) \) is symmetric,
\[
\left[ \tilde{\eta}, \tilde{\xi} \right] f (g) = f'(g) g(\eta \xi - \xi \eta) = ((\eta \xi - \xi \eta) \sim f)(g).
\]

Therefore the standard left invariant vector-field Lie algebra associated to \( G \) has bracket,
\[
[\eta, \xi] = \left[ \tilde{\eta}, \tilde{\xi} \right](1) = \eta \xi - \xi \eta
\]
which is the same as the Lie algebra associated to the algebra multiplication law.

**Definition 3.6.** For \( \xi \in C^1([0,T], \mathfrak{g}) \), let \( g^\xi(t) \in G \) denote the unique solution to the linear differential equation,
\[
(3.7) \quad \dot{g}^\xi(t) = g^\xi(t) \dot{\xi}(t) = \tilde{\xi}(t)(g(t)) \quad \text{with} \quad g^\xi(0) = 1 \in G
\]
and further let
\[
(3.8) \quad C^\xi(t) = \log \left( g^\xi(t) \right).
\]

**Remark 3.7.** If \( \xi \in C^1([0,T], \mathfrak{g}) \), then \( t \rightarrow \tilde{\xi}(t) \in \Gamma(TG) \) is a \( C^0 \)-varying vector field on \( G \) with associated flow, \( \tilde{\mu}_{t,s}^\xi \), which satisfies \( L_k \circ \tilde{\mu}_{t,s}^\xi = \tilde{\mu}_{t,s}^\xi \circ L_k \).

Applying this equation to \( 1 \in G \) shows
\[
\tilde{\mu}_{t,s}^\xi(k) = k \cdot \mu_{t,s}^\xi(1) \in G \quad \text{for all} \quad k \in G
\]
where \( \mu_{t,s}^\xi(1) \in G \) satisfies the ODE,
\[
\frac{d}{dt} \mu_{t,s}^\xi(1) = \tilde{\xi}(t) \circ \mu_{t,s}^\xi(1) = \tilde{\mu}_{t,s}^\xi(1) \xi(t) \quad \text{with} \quad \mu_{s,s}^\xi(1) = 1.
\]

As \( g^\xi(s)^{-1} g^\xi(t) \) satisfies this same differential equation, it follows that
\[
\tilde{\mu}_{t,s}^\xi(k) = kg^\xi(s)^{-1} g^\xi(t) = R_{g^\xi(s)^{-1} g^\xi(t)} k \quad \forall \quad s, t \in [0,T].
\]

In particular if \( \xi \in \mathfrak{g} \) is constant, then \( e\tilde{\xi} = R_{e\xi} \), i.e.
\[
e_{t \xi}(k) = ke_{t \xi} \quad \text{for all} \quad k \in G
\]

**Proposition 3.8.** For \( \xi, \eta \in \mathfrak{g} \),
\[
\partial_\xi e^\eta = e^\eta \int_0^1 \text{Ad}_{e^{-t \eta}} \eta dt = \left[ \int_0^1 \text{Ad}_{e^{-t \eta}} \xi dt \right] e^\eta.
\]

Consequently if \( C(t) \in \mathfrak{g} \) is a smooth curve then
\[
\frac{d}{dt} e^{C(t)} = \left[ \int_0^1 \text{Ad}_{e^{-sC(t)}} \dot{C}(t) ds \right] e^{C(t)}
\]
\[
= e^{C(t)} \left[ \int_0^1 \text{Ad}_{e^{-sC(t)}} \dot{C}(t) ds \right]
\]
Proof. First proof. Differentiating the identity,
\[ \frac{d}{dt} e^{t(\eta + s\xi)} = (\eta + s\xi) e^{t(\eta + s\xi)}, \]
in \( s \) shows
\[ \frac{d}{dt} \frac{d}{ds} |_{s=0} e^{t(\eta + s\xi)} = \frac{d}{ds} |_{s=0} \left[ (\eta + s\xi) e^{t(\eta + s\xi)} \right] \]
\[ = \xi e^{tn} + \eta \frac{d}{ds} |_{s=0} e^{t(\eta + s\xi)} \text{ with } \frac{d}{ds} |_{s=0} e^{0(\eta + s\xi)} = 0. \]
Solving this equation by Duhamel's principle gives,
\[ \frac{d}{ds} |_{s=0} e^{t(\eta + s\xi)} = \int_0^1 e^{(1-t)\eta} \xi e^{t\eta} dt, \]
i.e.
\[ \partial_s e^{t\eta} = e^{t\eta} \int_0^1 \text{Ad}_{e^{-t\eta}} \xi dt = \left[ \int_0^1 \text{Ad}_{e^{t\eta}} \xi dt \right] e^{t\eta}. \]

Second proof. This proof relies on the fact that the statement of this proposition is in fact a special case of Theorem 2.21. Indeed using this theorem along with Remark 3.7 shows,
\[ \partial_\eta e^{t\xi} = \text{Ad}_{e^{t\eta}} \xi = \text{Ad}_{e^{t\eta}} \xi (1) = \left[ \int_0^t \text{Ad}_{e^{\tau\xi}} \eta d\tau \right] \circ e^{t\xi} (1) \]
\[ = \left[ \int_0^t \text{Ad}_{e^{\tau\xi}} \eta d\tau \right] \left( e^{t\xi} \right) \]
where
\[ (\text{Ad}_{e^{\tau\xi}} \eta) (k) = \left( e^{\tau\xi} \eta \circ e^{-(\tau\xi)} \right) (k) = e^{\tau\xi} \left( L_{e^{-(\tau\xi)}(k)} \right)_* \eta \]
\[ = e^{\tau\xi} (L_{ke^{-(\tau\xi)}})_* \eta = (R_{e^{\tau\xi}})_* (L_{ke^{-(\tau\xi)}})_* \eta \]
\[ = ke^{-(\tau\xi)} \eta e^{\tau\xi}. \]

Since
\[ \left[ \int_0^t \text{Ad}_{e^{\tau\xi}} \eta d\tau \right] \left( e^{t\xi} \right) = e^{t\xi} \int_0^t e^{-(\tau\xi)} \eta e^{\tau\xi} d\tau = \int_0^t e^{(t-\tau)\xi} \eta e^{\tau\xi} d\tau, \]
the result is again proved. \( \square \)

Lemma 3.9. For \( \eta \in \mathfrak{g} \) and \( t \in \mathbb{R}, \text{Ad}_{e^{t\eta}} = e^{t \text{ad}_\eta} \) and
\[ (3.9) \int_0^1 \text{Ad}_{e^{t\eta}} dt = \psi (\text{ad}_\eta) \]
where \( \psi \) is as in Eq. (3.3).

Proof. Let \( \xi \in \mathfrak{g} \). Since
\[ \frac{d}{dt} [\text{Ad}_{e^{t\eta}} \xi] = \frac{d}{dt} \left[ e^{t\eta} \xi e^{-t\eta} \right] = \eta e^{t\eta} \xi e^{-t\eta} - e^{t\eta} \xi e^{-t\eta} \eta \]
\[ = \text{ad}_\eta \text{Ad}_{e^{t\eta}} \xi \]
and $e^{t \text{ad}_\eta} \xi$ solves the same equation with the same initial condition of $\xi$ at $t = 0$, we conclude that $\text{Ad}_{e^{t \eta}} \xi = e^{t \text{ad}_\eta} \xi$. As this is true for all $\xi \in \mathfrak{g}$, it follows that $\text{Ad}_{e^{t \eta}} = e^{t \text{ad}_\eta}$. The last equality is now proved by integrating the series expansion for $e^{t \text{ad}_\eta}$:

$$
\int_0^1 \text{Ad}_{e^{t \eta}} dt = \int_0^1 e^{t \text{ad}_\eta} dt = \int_0^1 \sum_{n=0}^{\infty} \frac{t^n}{n!} \text{ad}_\eta^n dt
$$

$$
= \sum_{n=0}^{\infty} \int_0^1 \frac{t^n}{n!} \text{ad}_\eta^n dt = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \text{ad}_\eta^n = \psi(\text{ad}_\eta).
$$

Corollary 3.10. Let $g(t)$ be a smooth curve in $G$,

$$
\dot{\xi}(t) := L_{g(t)^{-1}} \dot{g}(t) = g(t)^{-1} \dot{g}(t) \in \mathfrak{g} \text{ and } C(t) := \log(g(t)) \in \mathfrak{g}.
$$

Then $C(t) := \log(g(t)) \in \mathfrak{g}$ is the unique solution to the ODE,

(3.10) \quad \dot{\psi}(-\text{ad}_{C(t)}) \dot{C}(t) = \int_0^1 \text{Ad}_{e^{-sC(t)}} \dot{C}(t) ds = \dot{\xi}(t) \text{ with } C(0) = \log(g(0))

or equivalently (in more standard form) $C(t)$ satisfies,

(3.11) \quad \dot{C}(t) = \psi_-(\text{ad}_{C(t)}) \dot{\xi}(t) = \frac{\text{ad}_{C(t)}}{1 - e^{-\text{ad}_{C(t)}}} \dot{\xi}(t) \text{ with } C(0) = \log(g(0)),

where $\psi_-(z) = 1/\psi(-z)$ as in Eq. (3.4).

Proof. Since $g(t) = e^{C(t)}$, it follows from Proposition 3.8 and Lemmas 3.9 that

$$
g(t) \dot{\xi}(t) = \dot{g}(t) = \frac{d}{dt} e^{C(t)} = e^{C(t)} \int_0^1 \text{Ad}_{e^{-sC(t)}} \dot{C}(t) ds
$$

$$
= g(t) \int_0^1 \text{Ad}_{e^{-sC(t)}} \dot{C}(t) ds = g(t) \psi(-\text{ad}_{C(t)}) \dot{C}(t).
$$

Multiplying this identity on the left by $g(t)^{-1}$ gives the Eq. (3.10) while Eq. (3.11) then follows by multiplying Eq. (3.10) on the left by $\psi_-(\text{ad}_{C(t)})$. \qed

Definition 3.11. Let $\Gamma : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be the function defined by

$$
\Gamma(\xi, \eta) := \log\left(e^\xi e^\eta\right) \in \mathfrak{g} \text{ for all } \xi, \eta \in \mathfrak{g}.
$$

The next proposition deals with Lie sub-algebras of $\mathfrak{g}$ and simply connected Lie subgroups of $G$.

Proposition 3.12. Let $\mathfrak{g}_0$ be a Lie subalgebra of $\mathfrak{g}$ and $G_0 \subset G$ be the unique connected Lie subgroup of $G$ which has $\mathfrak{g}_0$ as its Lie algebra. If $g(t) \in G_0$ is a smooth curve connecting 1 to $g \in G_0$ and $\xi(t) := g(t)^{-1} \dot{g}(t) \in \mathfrak{g}_0$, then $C(t) := \log(g(t)) \in \mathfrak{g}_0$. 
Proof. We know that $C(t)$ may be characterized as the solution to the ODE
\[
\dot{C}(t) = F_\xi(t, C(t)) \quad \text{with } C(0) = 0,
\]
where
\[
F_\xi(t, \eta) = \frac{\text{ad}_\eta}{1 - e^{-\text{ad}_\eta}} \dot{\xi}(t) = \frac{1}{\psi} (-\text{ad}_\eta) \dot{\xi}(t). 
\]
As $F_\xi(t, \cdot) : \mathfrak{g}_0 \to \mathfrak{g}_0$, it follows that $C(t) \in \mathfrak{g}_0$ as required. \qed

Corollary 3.13. If we continue the assumptions and notation in Proposition 3.12, then $\log (G_0) = \mathfrak{g}_0$ and $\log |G_0 : G_0 \to \mathfrak{g}_0$ is a diffeomorphism with inverse given by
\[
\mathfrak{g}_0 \ni A \to e^A \in G_0.
\]
Proof. These assertions follow directly using $e^A \in G_0$ for $A \in \mathfrak{g}_0$ along with Proposition 3.5 and Proposition 3.12. \qed

For later purposes it is useful to record an “explicit” formula for $g^{\xi}(t)$ as defined in Definition 3.6.

Proposition 3.14. The path, $g^{\xi}(t) \in G$, as in Definition 3.6 may be expressed as
\begin{equation}
(3.12) \quad g^{\xi}(t) = 1 + \sum_{k=1}^{\kappa} \int_{0 \leq s_1 \leq s_2 \leq \ldots \leq s_k \leq t} \dot{\xi}(s_1) \ldots \dot{\xi}(s_k) \, ds_1 \ldots ds_k.
\end{equation}

Proof. From Definition 3.6 and the fundamental theorem of calculus,
\[
g^{\xi}(t) = 1 + \int_0^t g^{\xi}(\tau) \dot{\xi}(\tau) \, d\tau.
\]
Feeding this equation back into itself then shows,
\[
g^{\xi}(t) = 1 + \int_0^t \left[ 1 + \int_0^\tau g^{\xi}(s) \dot{\xi}(s) \, ds \right] \dot{\xi}(\tau) \, d\tau = 1 + \int_0^t \dot{\xi}(\tau) \, d\tau + \int_0^t d\tau \int_0^\tau ds g^{\xi}(s) \dot{\xi}(s) \dot{\xi}(\tau).
\]
Continuing this way inductively shows for any $m \in \mathbb{N}$ that,
\begin{equation}
(3.13) \quad g^{\xi}(t) = 1 + \sum_{k=1}^{m-1} \int_{0 \leq s_1 \leq s_2 \leq \ldots \leq s_k \leq t} \dot{\xi}(s_1) \ldots \dot{\xi}(s_k) \, ds + R_m(t)
\end{equation}
where $\sum_{k=1}^{m-1} [\ldots] = 0$ when $m = 1$ and
\[
R_m(t) := \int_{0 \leq s_1 \leq s_2 \leq \ldots \leq s_m \leq t} g^{\xi}(s_1) \dot{\xi}(s_1) \ldots \dot{\xi}(s_m) \, ds
\]
where $ds$ is short hand for $ds_1 \ldots ds_m$ in the above formula. Since
\[
\dot{\xi}(s_1) \ldots \dot{\xi}(s_{\kappa+1}) = 0 \in \mathcal{A},
\]
it follows that $R_{\kappa+1}(t) = 0$ and so Eq. (3.13) with $m = \kappa + 1$ gives Eq. (3.12). \qed
Corollary 3.15. If \( g^\xi (t) \in G \) is as in Definition 3.6, then

\[
\text{Ad}_{g^\xi (t)} = I + \sum_{k=1}^{\kappa} \int_{0 \leq s_1 \leq s_2 \leq \ldots \leq s_k \leq t} \text{ad}_{\xi(s_1)} \cdots \text{ad}_{\xi(s_k)} \, ds_1 \cdots ds_k.
\]

Proof. Since

\[
\frac{d}{dt} \text{Ad}_{g^\xi (t)} = \text{Ad}_{g^\xi (t)} \text{ad}_{\xi(t)} \quad \text{with} \quad \text{Ad}_{g^\xi (t)} = \text{Id}_\mathfrak{g},
\]

the proof of Eq. (3.14) is exactly the same as the proof of Eq. (3.12) provided the reader changes \( g^\xi \) to \( \text{Ad}_{g^\xi} \) and \( \xi \) to \( \text{ad}_{\xi} \) everywhere. \( \square \)

Notation 3.16. For \( j \in \mathbb{N}, a : [0, \infty)^j \to \mathbb{R} \) is a bounded measurable function, \( t \in [0, T] \), and \( \xi \in L^1 ([0, T], \mathfrak{g}) \), let

\[
\hat{a}_t (\xi) = \int_{[0, t]^j} a (s_1, \ldots, s_j) \xi (s_1) \cdots \xi (s_j) \, ds_1 \cdots ds_j \in \mathfrak{g}
\]

and \( \hat{a}_t (\text{ad}_\xi) : \mathfrak{g} \to \mathfrak{g} \) be the linear transformation defined by

\[
\hat{a}_t (\text{ad}_\xi) := \int_{[0, t]^j} a (s_1, \ldots, s_j) \text{ad}_{\xi(s_1)} \cdots \text{ad}_{\xi(s_j)} \, ds_1 \cdots ds_j.
\]

Note that \( \hat{a}_t (\xi) = 0 \) if \( j > \kappa \) and \( \hat{a}_t (\text{ad}_\xi) = 0 \) if \( j \geq \kappa \).

The proof of the following lemma is elementary and is left to the reader.

Lemma 3.17. If \( a : [0, \infty)^j \to \mathbb{R} \) and \( b : [0, \infty)^k \to \mathbb{R} \) are bounded and measurable functions, \( t \in [0, T] \), and \( \xi \in L^1 ([0, T], \mathfrak{g}) \), then

\[
\hat{a}_t (\xi) \hat{b}_t (\xi) = [a \otimes b]_t (\xi) \quad \text{and} \quad \hat{a}_t (\text{ad}_\xi) \hat{b}_t (\text{ad}_\xi) = [a \otimes b]_t (\text{ad}_\xi)
\]

where \( a \otimes b : [0, \infty)^{j+k} \to \mathbb{R} \) is the bounded measurable function defined by

\[
a \otimes b (s_1, \ldots, s_j, t_1, \ldots, t_k) = a (s_1, \ldots, s_j) b (t_1, \ldots, t_k).
\]

Proposition 3.18. If \( g (t) = g^\xi (t) \in G \) and \( C (t) = C^\xi (t) = \log (g^\xi (t)) \in \mathfrak{g} \) are as in Definition 3.6 and \( f \in \mathcal{H}_0 \), then for each \( j \in \mathbb{N} \cap [1, \kappa - 1] \), there exists bounded measurable functions, \( f^j : [0, \infty)^j \to \mathbb{R} \) such that

\[
f (\text{ad}_{C(t)}) = f (0) \, \text{Id}_\mathfrak{g} + \sum_{j=1}^{\kappa-1} \hat{f}^j_t (\text{ad}_{\xi})
\]

Moreover, each function \( f^j \) depends linearly on \( f (0), \ldots, f^{(\kappa-1)} (0) \). \( \square \)

\[1\]The fact that the \( f^j \) depend linearly on first \((\kappa-1)\)-derivatives of \( f \) is easily understood from the identity, \( f (\text{ad}_{C(t)}) = \sum_{j=0}^{\kappa-1} \left( f^{(j)} (0) / j! \right) \text{ad}_{C(t)} \).
Proof. For $\lambda \in \mathbb{R}$, let $u(w) := f(\log(1 + w))$ and observe by a simple exercise in differentiation shows there exists $\alpha_{n,k} \in \mathbb{Z}$ such that $u(n)(0) = \sum_{k=0}^{n} \alpha_{n,k} f^{(k)}(0)$. [For example, one has $u(0) = f(0)$, $u'(0) = f'(0)$, $u''(0) = f''(0) - f'(0)$, and $u'''(0) = f'''(0) - 3f''(0) + 2f'(0).$]

Since $g(t) = e^{C(t)}$, it follows that $\text{Ad}_{g(t)} = e^{\text{ad}_{C(t)}}$ and therefore

$$\text{ad}_{C(t)} = \log(\text{Ad}_{g(t)}) = \log(Id_g + [\text{Ad}_{g(t)} - Id_g])$$

and hence,

$$f(\text{ad}_{C(t)}) = f(0)I + \sum_{j=1}^{\kappa-1} \frac{u^{(j)}(0)}{j!} \left(\text{Ad}_{g(t)} - I\right)^j.$$

By Corollary 3.15,

$$\text{Ad}_{g(t)} - Id_g = \sum_{j=2}^{\kappa} \hat{\Delta}^j \left(\text{ad}_\xi\right)$$

where

$$b^k(s_1, \ldots, s_k) := 1_{0 \leq s_1 \leq s_2 \leq \cdots \leq s_k}$$

and so it follows that

$$f(\text{ad}_{C(t)}) = f(0)I + \sum_{j=1}^{\kappa-1} \frac{u^{(j)}(0)}{j!} \sum_{k_1, \ldots, k_j=1}^{\kappa-1} \hat{b}^{k_1}_{1} \left(\text{ad}_\xi\right) \cdots \hat{b}^{k_j}_{j} \left(\text{ad}_\xi\right).$$

By repeated use of Lemma 3.17, the last identity may be written in the form described in Eq. (3.15).

The formula for $C^\xi(t)$ now follows directly from Eqs. (3.11) and (3.16).

□

Corollary 3.19. If $g(t) = g^\xi(t) \in G$ and $C(t) = C^\xi(t) = \log(g(t)) \in \mathfrak{g}$ are as in Definition 3.6, then there exists bounded measurable functions, $\Delta^j : [0, \infty)^{j-1} \to \mathbb{R}$ for $j \in \mathbb{N} \cap [2, \kappa]$ such that

(3.16) $\psi_-(\text{ad}_{C(t)}) = Id_g + \sum_{j=2}^{\kappa} \hat{\Delta}^j \left(\text{ad}_\xi\right)$

and

(3.17) $\dot{C}^\xi(t) = \dot{\xi}(t) + \sum_{j=2}^{\kappa} \hat{\Delta}^j \left(\text{ad}_\xi\right) \dot{\xi}(t)$

Proof. Applying Proposition 3.18 with $f = \psi_-$ and $\lambda = 1$ gives Eq. (3.16). Equation (3.17) then follows from Eq. (3.16) and Eq. (3.11). □
Remark 3.20. It is possible, see for example [32], to work out explicit formula for the functions $\Delta^j$ in Corollary 3.19 and this would lead to a proof of Eq. (1.2). For our purposes, these explicit formula are not needed.

Corollary 3.21. If $g(t) = G^\xi(t) \in G$ and $C(t) = C^\xi(t) = \log (g(t)) \in g$ are as in Definition 3.6, there exists bounded measurable functions, $c^j : [0, \infty)^j \to \mathbb{R}$ for $j \in \mathbb{N} \cap [2, \kappa]$ such that

$$C^\xi(t) = C(0) + \xi(t) + \sum_{j=2}^\kappa \hat{c}^j \left( \dot{\xi} \right).$$

Proof. Integrating Eq. (3.17) shows,

$$(3.18) \quad C^\xi(t) = C(0) + \xi(t) + \sum_{j=2}^\kappa \int_0^t \hat{\Delta}^j \left( \text{ad} \dot{\xi} \right) \dot{\xi}(\tau) d\tau$$

where

$$\int_0^t \hat{\Delta}^j \left( \text{ad} \dot{\xi} \right) \dot{\xi}(\tau) d\tau = \int_0^t \int_{[0, \tau]^{j-1}} ds_1 \ldots ds_{j-1} \Delta^j(s_1, \ldots, s_{j-1}) \text{ad} \dot{\xi}(s_1) \ldots \text{ad} \dot{\xi}(s_{j-1}) \dot{\xi}(\tau)$$

$$= \int_{[0, t]^j} \tilde{\Delta}^j(s_1, \ldots, s_j) \text{ad} \dot{\xi}(s_1) \ldots \text{ad} \dot{\xi}(s_{j-1}) \dot{\xi}(s_j) ds_1 \ldots ds_j$$

and

$$\tilde{\Delta}^j(s_1, \ldots, s_j) := \Delta^j(s_1, \ldots, s_{j-1}) 1_{\max\{s_1, \ldots, s_{j-1}\} \leq s_j}.$$  

By expanding out all of the commutators and permuting the variables of integration in each of the resulting terms we may rewrite the previous expression in the form $\hat{c}^j \left( \xi \right)$ for some bounded measurable function, $c^j : [0, \infty)^j \to \mathbb{R}$.

Alternatively: simply apply log to Eq. (3.12) and then repeatedly use Lemma 3.17 to arrive at the stated assertion. □

3.2. Truncated tensor algebra estimates. We now apply the above results with $g = g^{(\kappa)} \subset \mathcal{A} = T^{(\kappa)}(\mathbb{R}^d)$ as in Notation 1.14. In what follows we will make use of the simple estimates in the following remark without further mention.

Remark 3.22. For any $m, n \in \mathbb{N} \cap [1, 2\kappa]$ with $m < n$, it is easy to show, for $\mu, \lambda \geq 0$, that

$$Q_{(m,n)}(\lambda) \asymp \sum_{k=m+1}^n \lambda^k,$$

$$Q_{[m,n]}(\lambda) \asymp \sum_{k=m}^n \lambda^k,$$

and

$$(3.19) \quad Q_{(m,n)}(\lambda + \mu) \asymp Q_{(m,n)}(\lambda) + Q_{(m,n)}(\mu).$$
For example, the first estimate follows from the more precise estimate,

\[ Q_{(m,n)}(\lambda) \leq \sum_{k=m+1}^{n} \lambda^k \leq (n - m) Q_{(m,n)}(\lambda). \]

Recalling from Definition 1.16 that

\[ N(A) := \max_{1 \leq k \leq \kappa} |A_k|^{1/k} \text{ for } A \in g^{(\kappa)}, \]

we find

\[ (3.20) \quad |A| \leq \sum_{k=1}^{\kappa} |A_k| \leq \sum_{k=1}^{\kappa} N(A)^{k} \leq \kappa Q_{[1,\kappa]}(N(A)). \]

Similarly if \( f \in C([0,t],g^{(\kappa)}) \), then

\[ (3.21) \quad |f|^* \leq \sum_{k=1}^{\kappa} |f_k|^* \leq \sum_{k=1}^{\kappa} N^*_k(f)^{k} \leq \kappa Q_{[1,\kappa]}(N^*_k(f)). \]

Let us also recall that if \( a = (a_j)_{j=1}^{N} \) is a sequence \( N = \infty \) allowed, then

\[ \|a\|_p := \left( \sum_{j=1}^{N} |a_j|^p \right)^{1/p} \]

is a decreasing function of \( p \in [1, \infty) \). In particular using \( \|a\|_p \leq \|a\|_1 \) with \( a_j \) replaced by \( a_j^{1/p} \) it follows (as is easily proved directly) that

\[ (3.22) \quad \left( \sum_{i=1}^{m} a_j \right)^{1/p} \leq \sum_{j=1}^{N} a_j^{1/p} \text{ when } a_j \geq 0 \text{ and } p \geq 1. \]

**Lemma 3.23.** If \( \{A(j)\}_{j=1}^{r} \subset g^{(\kappa)} \), then

\[ (3.23) \quad N\left( \sum_{j=1}^{r} A(j) \right) \leq \sum_{j=1}^{r} N(A(j)). \]

If \( A, B \in g^{(\kappa)} \) and \( 2 \leq k \leq 2\kappa \), then

\[ (3.24) \quad |[A \otimes B]|_k \leq N(A)N(B) \cdot (N(A) + N(B))^{k-2}. \]

**Proof.** For \( 1 \leq k \leq \kappa \),

\[ \left| \sum_{j=1}^{r} A(j) \right|^{1/k} \leq \left( \sum_{j=1}^{r} |A(j)|_k \right)^{1/k} \leq \sum_{j=1}^{r} |A(j)|^{1/k} \leq \sum_{j=1}^{r} N(A(j)) \]

wherein we have used Eq. (3.22) with \( p = k \) for the second inequality. Since this is true for all \( 1 \leq k \leq \kappa \), Eq. (3.23) is proved. The proof of the second...
inequality follows by the simple estimates;

\[
\| [A \otimes B]_k \| = \left| \sum_{m,n=1}^{\kappa} 1_{m+n=k} \cdot A_m \otimes B_n \right| \leq \sum_{m,n=1}^{\kappa} 1_{m+n=k} \cdot |A_m \otimes B_n| \\
\leq \sum_{m,n=1}^{\kappa} 1_{m+n=k} \cdot |A_m| |B_n| \leq \sum_{m,n=1}^{\kappa} 1_{m+n=k} \cdot N(A)^m N(B)^n \\
= N(A) N(B) \sum_{m,n=0}^{\kappa-1} 1_{m+n=k-2} \cdot N(A)^m N(B)^n \\
\leq N(A) N(B) (N(A) + N(B))^{k-2},
\]

wherein we have used all the coefficients in the binomial formula are greater than or equal to 1 for the last inequality.

Recall from Notation 3.16 with \( g = g^{(\kappa)} \) that if \( 1 \leq \ell \leq \kappa \), \( \Delta : [0,T]^\ell \to \mathbb{R} \) is a bounded measurable function, and \( \xi \in L^1([0,T], g^{(\kappa)}) \), then we let

\[
(3.25) \quad \hat{\Delta}_t (\xi) := \int_{[0,t]^{\ell}} \Delta (s_1, \ldots, s_\ell) \xi (s_1) \ldots \xi (s_\ell) \, ds \in g^{(\kappa)} \forall \, t \in [0,T],
\]

where \( ds := ds_1 \ldots ds_\ell \).

**Proposition 3.24.** Suppose that \( 1 \leq \ell \leq \kappa \), \( \Delta : [0,T]^\ell \to \mathbb{R} \), and \( \xi \in L^1([0,T], g^{(\kappa)}) \), and

\[
\hat{\Delta}_t (\xi) = \sum_{k=1}^{\kappa} \left[ \hat{\Delta}_t (\xi) \right]_k \in \oplus_{k=\ell}^{\kappa} \left[ \mathbb{R}^d \right]^{\otimes k}
\]

are as above. Then

\[
(3.26) \quad \left| \left[ \hat{\Delta}_t (\xi) \right]_k \right| \leq \# (\Lambda_{k,\ell}) \cdot \| \Delta \|_\infty \cdot N_t^* (\xi)^k
\]

\[
(3.27) \quad N \left( \hat{\Delta}_t (\xi) \right) \leq C(\kappa) \max \left( \| \Delta \|_{1/\ell}^{1/\ell}, \| \Delta \|_{1/\kappa}^{1/\kappa} \right) \cdot N_t^* (\xi) \text{ for all } 0 \leq t \leq T
\]

where \( \| \Delta \|_\infty \) is the essential supremum of \( \Delta \) on \([0,T]^\ell\) ,

\[
\Lambda_{k,\ell} := \left\{ (j_1, \ldots, j_\ell) \in \mathbb{N}^\ell : \sum_{i=1}^{k} j_i = k \right\},
\]

and

\[
C(\kappa) := \max_{1 \leq \ell \leq \kappa} \max_{\ell \leq k \leq \kappa} \left[ \left( \# (\Lambda_{k,\ell}) \right)^{1/k} \right].
\]
Proof. For $k \in [\ell, \kappa] \cap \mathbb{N}$,

$$
\left| \left[ \hat{\Delta}_t (\xi) \right]_k \right| = \left| \sum_{(j_0, \ldots, j_\ell) \in \Lambda_{k, \ell}} \int_{[0,t]^{\ell}} \Delta (s_1, \ldots, s_\ell) \xi_{j_1} (s_1) \ldots \xi_{j_\ell} (s_\ell) \, ds \right|
$$

$$
\leq \| \Delta \|_\infty \sum_{(j_0, \ldots, j_\ell) \in \Lambda_{k, \ell}} \int_{[0,t]^{\ell}} |\xi_{j_1} (s_1) \ldots \xi_{j_\ell} (s_\ell)| \, ds
$$

$$
= \| \Delta \|_\infty \prod_{i=1}^{\ell} |\xi_{j_i}|_t \leq \| \Delta \|_\infty \sum_{(j_0, \ldots, j_\ell) \in \Lambda_{k, \ell}} \prod_{i=1}^{\ell} N_T^* (\xi)^{j_i}
$$

$$
\leq \| \Delta \|_\infty \cdot \# (\Lambda_{k, \ell}) \cdot N_T^* (\xi)^k
$$

which proves Eq. (3.26). Equation (3.27) is an easy consequence of Eq. (3.26) and the observation that

$$
\left[ \# (\Lambda_{k, \ell}) \right]^{1/k} \cdot \| \Delta \|^{1/k}_\infty \leq C (\kappa) \max \left( \| \Delta \|^{1/\ell}_\infty, \| \Delta \|^{1/\kappa}_\infty \right).
$$

$$\square$$

**Proposition 3.25.** Suppose that $1 \leq \ell \leq \kappa$, $\Delta : [0, T]^\ell \to \mathbb{R}$, and $\xi \in L^1 ([0, T], g^{(\kappa)})$, then

$$
\int_0^T \left| \left[ \hat{\Delta}_t (\text{ad}_\xi) \xi (t) \right]_k \right| \, dt \lesssim \| \Delta \|_\infty N_T^* (\xi)^k.
$$

**Proof.** For $k \in (\ell, \kappa] \cap \mathbb{N}$, let

$$
\Lambda_{k, \ell} := \left\{ (j_0, j_1, \ldots, j_\ell) \in \mathbb{N}^{\ell+1} : \sum_{i=0}^k j_i = k \right\}.
$$

We then have

$$
\left| \left[ \hat{\Delta}_t (\text{ad}_\xi) \xi (t) \right]_k \right|
$$

$$
= \left| \sum_{(j_0, j_1, \ldots, j_\ell) \in \Lambda_{k, \ell}} \int_{[0,t]^{\ell}} \Delta (s_1, \ldots, s_\ell) \text{ad}_{\xi_{j_1} (s_1)} \ldots \text{ad}_{\xi_{j_\ell} (s_\ell)} \xi_{j_0} (t) \, ds \right|
$$

$$
\leq 2^\ell \| \Delta \|_\infty \sum_{(j_0, j_1, \ldots, j_\ell) \in \Lambda_{k, \ell}} \int_{[0,t]^{\ell}} |\xi_{j_1} (s_1)| \ldots |\xi_{j_\ell} (s_\ell)| |\xi_{j_0} (t)| \, ds
$$

$$
\leq 2^\ell \| \Delta \|_\infty \sum_{(j_0, j_1, \ldots, j_\ell) \in \Lambda_{k, \ell}} \int_{[0,T]^{\ell}} |\xi_{j_1} (s_1)| \ldots |\xi_{j_\ell} (s_\ell)| |\xi_{j_0} (t)| \, ds
$$
Integrating this estimate on $t \in [0, T]$ shows,

$$
\int_0^T \left| \left[\hat{\Delta}_t \left( \text{ad}_\xi \right) \xi(t) \right]_k \right| dt \\
\leq 2^\ell \|\Delta\|_\infty \sum_{(j_0, j_1, \ldots, j_\ell) \in \Lambda_{k, \ell}} \int_0^T dt \int_{[0, T]^\ell} |\xi_{j_1}(s_1)| \ldots |\xi_{j_\ell}(s_\ell)| |\xi_{j_0}(t)| ds
\\
= 2^\ell \|\Delta\|_\infty \sum_{(j_0, j_1, \ldots, j_\ell) \in \Lambda_{k, \ell}} \prod_{i=0}^\ell |\xi_{j_i}|_T
\\
\leq 2^\ell \|\Delta\|_\infty \sum_{(j_0, j_1, \ldots, j_\ell) \in \Lambda_{k, \ell}} N^*_\ell(\xi)^k = 2^\ell \#(\Lambda_{k, \ell}) \|\Delta\|_\infty \cdot N^*_\ell(\xi)^k.
$$

We end this section with a few key estimates that we will need in the remainder of the paper. In each of the next three results we assume that $\xi \in C^1([0, T], F^{(\kappa)}(\mathbb{R}^d))$ and $C(t) = C^\xi(t) = \log(g^\xi(t)) \in F^{(\kappa)}(\mathbb{R}^d)$ are as in Definition 1.25.

**Proposition 3.26.** To each $f \in H_0$ there exists $K(f) > 0$ depending linearly on $(|f(0)|, \ldots, |f^{(\kappa-1)}(0)|)$ and independent of $\xi$ such that

$$
\int_0^T \left| \left( f \left( \text{ad}_{C(t)} \right) \dot{\xi}(t) \right)_n \right| dt \leq K(f) N^*_T(\dot{\xi})^n
$$

**Proof.** Recall that Proposition 3.18 asserts that $f \left( \text{ad}_{C(t)} \right) \dot{\xi}(t) = f(0) \dot{\xi}(t) + \sum_{j=1}^{\kappa-1} \hat{\mathcal{F}}_t \left( \text{ad}_\xi \right) \dot{\xi}(t)$ where the functions $\mathcal{F}^j$ depend linearly on $(f(0), \ldots, f^{(\kappa-1)}(0))$. So by repeated application of Proposition 3.25 with $\xi$ replaced by $\dot{\xi}$ shows,

$$
\int_0^T \left| \left( f \left( \text{ad}_{C(t)} \right) \dot{\xi}(t) \right)_n \right| dt \\
\leq |f(0)| \int_0^T |\dot{\xi}_n(t)| dt + \sum_{j=1}^{\kappa-1} \int_0^T \left| \left[ \hat{\mathcal{F}}_t \left( \text{ad}_\xi \right) \dot{\xi}(t) \right]_n \right| dt
\\
\leq |f(0)| N^*_T(\dot{\xi})^n + \sum_{j=1}^{\kappa-1} \|\mathcal{F}^j\|_\infty N^*_T(\dot{\xi})^n \leq K(f) N^*_T(\dot{\xi})^n
$$

where $K(f) > 0$ may be chosen to depend linearly on $(|f(0)|, \ldots, |f^{(\kappa-1)}(0)|)$. $\square$
Corollary 3.27. If $\xi \in C^1([0,T], F^{(\kappa)}(\mathbb{R}^d))$ and $C^\xi (t) = \log (g^\xi (t)) \in F^{(\kappa)}(\mathbb{R}^d)$ are as above, then

\begin{equation}
N_T^+ (C^\xi) \preceq N_T^+ (\dot{\xi}),
\end{equation}

\begin{equation}
|C^\xi (\cdot)|_{\infty, T} \preceq N_T^+ (\dot{\xi})^n \quad \forall \ n \in [1, \kappa] \cap \mathbb{N}, \quad \text{and}
\end{equation}

\begin{equation}
|C^\xi (\cdot)|_{\infty, T} \preceq Q_{[1, \kappa]} \left( N_T^+ (\dot{\xi}) \right)^n.
\end{equation}

Proof. By Corollary 3.10, $\dot{C}^\xi (t) = f (\text{ad}_{C(t)}) \dot{\xi} (t)$ where $f (z) = 1/\psi (-z)$ and so by Proposition 3.26,

\[ |\dot{C}^\xi (\cdot)|_{1, T} = \int_0^T \left| \left( f (\text{ad}_{C(t)}) \dot{\xi} (t) \right)_n \right| \, dt \leq K (f) N_T^+ (\dot{\xi})^n \quad \text{for } 1 \leq n \leq \kappa. \]

This proves Eq. (3.29) and also Eqs. (3.30) and (3.31) since (as $C^\xi (0) = 0$),

\[ |C^\xi (\cdot)|_{\infty, T} \leq |\dot{C}^\xi (\cdot)|_{1, T} \leq K (f) N_T^+ (\dot{\xi})^n \]

and hence

\[ |C^\xi (\cdot)|_{\infty, T} \leq \sum_{n=1}^\kappa |C^\xi (\cdot)|_{\infty, T} \leq \sum_{n=1}^\kappa N_T^+ (\dot{\xi})^n \preceq Q_{[1, \kappa]} \left( N_T^+ (\dot{\xi}) \right)^n. \]

\[ \square \]

Corollary 3.28. Suppose $\xi \in C^1([0,T], F^{(\kappa)}(\mathbb{R}^d))$ and $C (t) = C^\xi (t) = \log (g^\xi (t)) \in F^{(\kappa)}(\mathbb{R}^d)$ are as in Definition 1.25 and $f \in \mathcal{H}_0$. Then there exists $K (f) > 0$ such that $K (f)$ depends linearly on $(|f (0)|, \ldots, |f^{(\kappa-1)} (0)|)$ and on $\kappa$ such that

\begin{equation}
\int_0^T \left| C^\xi (t) \right| \left| \left( f (\text{ad}_{C(t)}) \dot{\xi} (t) \right)_n \right| \, dt \leq K (f) N_T^+ (\dot{\xi})^{m+n} \quad \forall \ m, n \in [1, \kappa] \cap \mathbb{N}.
\end{equation}

Proof. Making use of the estimates in Proposition 3.26 and Corollary 3.28, we find,

\[ \int_0^T \left| C^\xi (t) \right| \left| \left( f (\text{ad}_{C(t)}) \dot{\xi} (t) \right)_n \right| \, dt \]

\[ \leq \left| C^\xi (\cdot) \right|_{\infty, T} \cdot \int_0^T \left| \left( f (\text{ad}_{C(t)}) \dot{\xi} (t) \right)_n \right| \, dt \]

\[ \preceq N_T^+ (\dot{\xi})^m \cdot K (f) N_T^+ (\dot{\xi})^n = K (f) N_T^+ (\dot{\xi})^{m+n}. \]

\[ \square \]
4. Logarithm Approximation Problem

Recall from Definition 1.20 that a \textit{d-dimensional dynamical system} on \(M\) is a linear map, \(\mathbb{R}^d \ni w \rightarrow V_w \in \Gamma(TM)\). For \(A \in F^{(\kappa)}(\mathbb{R}^d)\) we know that \(V_A \in \Gamma(TM)\) by Example 1.21. Let us again emphasize that we assume Assumption 1 is in force, i.e. \(V\) is \(\kappa\)-complete.

To help motivate the next key theorem, let \(A\) and \(B\) be in the full free Lie algebra, \(F(\mathbb{R}^d)\). Working heuristically (using \(\sim\) to indicate equality of formal series), we should have

\[
\text{Ad}_{e^{sV_A}} V_B = e^{s\text{ad}_V A} V_B = e^{-sL_{V_A} V_B}
\]

\[
\sim \sum_{k=0}^{\infty} \frac{(-1)^k s^k}{k!} L_{V_A} V_B \sim \sum_{k=0}^{\infty} \frac{(-1)^k s^k}{k!} V_{\text{ad}_V A}^k V_B
\]

\[
\sim V \sum_{k=0}^{\infty} \frac{(-1)^k s^k}{k!} \text{ad}_V^k B = V e^{-s\text{ad}_V A} B.
\]

Integrating this formal identity then suggests,

\[
\int_0^1 \text{Ad}_{e^{sV_A}} V_B ds \sim \int_0^1 e^{-sL_{V_A} V_B} ds \sim \int_0^1 V e^{-s\text{ad}_V A} B ds \sim \int_0^1 e^{-s\text{ad}_V A} B ds.
\]

Although the above series need not converge, this computation is suggestive of the following key Taylor type approximation theorem for \(\int_0^1 \text{Ad}_{e^{sV_A}} V_B ds \in \Gamma(TM)\) when \(A, B \in F^{(\kappa)}(\mathbb{R}^d)\).

\textbf{Theorem 4.1.} Let \(\psi(z)\) be as in Eq. (3.3) and \(V: \mathbb{R}^d \rightarrow \Gamma(TM)\) be a dynamical system satisfying Assumption \(\square\) so that in particular, \(V_A \in \Gamma(TM)\) is complete for all \(A \in g_0 = F^{(\kappa)}(\mathbb{R}^d)\). Then for all \(A, B \in g_0\),

\[
\int_0^1 \text{Ad}_{e^{sV_A}} V_B ds
\]

\[
= V \left( \int_0^1 \text{Ad}_{e^{s-1A}} B \ ds \right) + \int_0^1 \text{Ad}_{e^{sV_A}} V_{\pi > \kappa [A, \psi((s-1) \text{ad}_V A) B]} (s-1) \ ds
\]

\[
= V \psi(-\text{ad}_V A) B + \int_0^1 \text{Ad}_{e^{sV_A}} V_{\pi > \kappa [A, \psi((s-1) \text{ad}_V A) B]} (s-1) \ ds.
\]

\textbf{Proof.} The heart of the proof is to show, for \(0 \leq l \leq \kappa\), that

\[
\int_0^1 \text{Ad}_{e^{sV_A}} V_B ds = V \sum_{k=1}^l (-1)^{k+1} \frac{\text{ad}_A^{k-1} B}{k!} + \frac{1}{l!} \int_0^1 \text{Ad}_{e^{sV_A}} V_{\text{ad}_A^l B} (s-1)^l ds
\]

\[
+ \int_0^1 \text{Ad}_{e^{sV_A}} V_{\pi > \kappa [A, \sum_{k=1}^l (-1)^{k-1} \frac{\text{ad}_A^{k-1} B}{k!}]} ds,
\]

where \(\sum_{k=1}^l \cdots = 0\) and \(l! = 1\) when \(l = 0\). The proof of these identities will be by induction on \(l\). In the proof of this identity we will use Corollary 2.18 which in this context implies,

\[
\frac{d}{ds} \text{Ad}_{e^{sV_A}} V_C = \text{Ad}_{e^{sV_A}} \text{ad}_V A = -\text{Ad}_{e^{sV_A}} [V_A, V_C] = -\text{Ad}_{e^{sV_A}} V_{[A, C]}.
\]
for all $A, C \in F(\kappa) (\mathbb{R}^d)$. In the proof to follow, $C = \text{ad}_A^l B$ for some $l$.

When $l = 0$, there is nothing to prove. For the induction step, we integrate by parts the middle term on the right side of Eq. (4.3),

$$
\frac{1}{l!} \int_0^1 \text{Ad}_{e^s \lambda} V_{\text{ad}_A^l B} (s - 1)^l ds
$$

$$
= \frac{1}{(l + 1)!} \int_0^1 \text{Ad}_{e^s \lambda} V_{\text{ad}_A^l B} d (s - 1)^{l+1}
$$

$$
= \frac{1}{(l + 1)!} \text{Ad}_{e^s \lambda} V_{\text{ad}_A^l B} (s - 1)^{l+1} |_0^1
$$

$$
- \frac{1}{(l + 1)!} \int_0^1 \left( \frac{d}{ds} \text{Ad}_{e^s \lambda} V_{\text{ad}_A^l B} \right) (s - 1)^{l+1} ds
$$

$$
= \frac{(-1)^l}{(l + 1)!} \text{Ad}_{e^s \lambda} V_{\text{ad}_A^l B}
$$

$$
+ \frac{1}{(l + 1)!} \int_0^1 \text{Ad}_{e^s \lambda} [V_A, V_{\text{ad}_A^l B}] (s - 1)^{l+1} ds
$$

$$
= \frac{(-1)^l}{(l + 1)!} \text{Ad}_{e^s \lambda} V_{\text{ad}_A^l B}
$$

$$
+ \frac{1}{(l + 1)!} \int_0^1 \text{Ad}_{e^s \lambda} V_{[A, \text{ad}_A^{l+1} B]} (s - 1)^{l+1} ds.
$$

Combining this result with Eq. (4.3) and the fact that

$$
[A, \text{ad}_A^l B] \otimes = [A, \text{ad}_A^l B] + \pi_{\kappa} [A, \text{ad}_A^l B] \otimes
$$

completes the inductive step.

To finish the proof observe that

$$(s - 1) \psi ((s - 1) \text{ad}_A) = \sum_{k=1}^\kappa \frac{(s - 1)^k}{k!} \text{ad}_A^{k-1}
$$

and taking $s = 0$ in this equation also shows,

$$
\sum_{k=1}^\kappa \frac{(-1)^{k+1}}{k!} \text{ad}_A^{k-1} = \psi (- \text{ad}_A) = \int_0^1 \text{Ad}_{e^{-s \lambda}} ds,
$$

where the last equality comes from Eq. (4.1). So from the last two displayed equations and Eq. (4.3) with $l = \kappa$ and the fact that $\text{ad}_A^\kappa B = 0$ so that

$$
\frac{1}{\kappa!} \int_0^1 \text{Ad}_{e^s \lambda} V_{\text{ad}_A^\kappa B} (s - 1)^\kappa ds = 0,
$$
Remark 4.2. The expression, $\text{Ad}_{e^{t\mathcal{L}_C}} V_B$, appears on the left side of Eq. (4.1) while on the right side we have the expression, $\text{Ad}_{e^{-s\mathcal{L}_C}} B$ which involves a change of $s$ to $-s$. This change of sign is a simple consequence of the fact that vector-fields on $M$ may naturally be identified with right invariant vector fields on $\text{Diff}(M)$ while on the other hand we have chosen to view $A, B \in F^{(s)}(\mathbb{R}^d)$ as left invariant vector fields on $G_0 = G^{(s)}_{\text{geo}}(\mathbb{R}^d)$. This left right interchange is the reason for the sign changes in Eq. (4.1).

Notation 4.3. To each $C \in C^1([0, T], F^{(s)}(\mathbb{R}^d))$, let

$$W_C^t := \int_0^1 \text{Ad}_{e^{s\mathcal{L}_C(t)}} V_{C(t)} \, ds \in \Gamma(TM).$$

Notation 4.4. For $\xi \in C^1([0, T], F^{(s)}(\mathbb{R}^d))$, let $g(t) = g_\xi(t) \in G_0$ be as in Definition 3.6, $C_\xi(t) := \log(g_\xi(t)) \in F^{(s)}(\mathbb{R}^d)$, and $\mu_\xi^t := \mu_\xi^0 \in \text{Diff}(M)$ denote the flow defined by

$$\dot{\mu}_\xi^t = V_{C(t)} \circ \mu_\xi^t \quad \text{with} \quad \mu_\xi^0 = \text{Id}_M.$$

Our goal is now to estimate the distance between $\mu_\xi^0$ and $e^{V_\log(s\mathcal{L}(t))} = e^{V_{C(t)}}$. Since (by Corollary 2.23 with $Z_t = V_{C(t)} \in \Gamma(TM)$)

$$\frac{d}{dt} e^{V_{C(t)}} = W_t \circ e^{V_{C(t)}},$$

the desired distance estimates will be a consequence of applying Theorem 2.29 with $X_t = V_{C(t)}$ and $Y_t = W_t^{C_\xi}$. Before carrying out the details we need to develop a few auxiliary results first.

Notation 4.5. For $0 \leq s \leq 1$, let $u(s, \cdot) \in \mathcal{H}_0$ be defined by $u(s, z) := \psi((s - 1)z) / \psi(-z)$.

Lemma 4.6. Let $\xi$ and $C = C_\xi$ be as in Notation 4.4 and $W^{C_\xi}$ be as in Eq. (4.1). Then the difference vector field,

$$U_\xi^t := Y_t - X_t = W_t^{C_\xi} - V_{C(t)} \in \Gamma(TM),$$

may be expresses as

$$U_\xi^t = \int_0^1 \text{Ad}_{e^{s\mathcal{L}_C} C(t)} V_{\mathcal{L}_C [C(t), u(s, \mathcal{L}_C(t))\xi(t)]} \, ds.$$
Proof. By Corollary 3.10,

\[ \psi \left( -\text{ad}_{C(t)} \right) \hat{C}(t) = \int_0^1 \text{Ad}_{e^{-sC(t)}} \hat{C}(t) ds = \dot{\xi}(t) \]

which combined with Theorem 4.1 with \( A = C(t) \) and \( B = \hat{C}(t) \) implies

\[ W_t^C = V_{\xi(t)} + \int_0^1 \text{Ad}_{e^{sC(t)}} V_{\pi_{>\kappa}}[C(t), \psi((s-1)\text{ad}_{C(t)})\hat{C}(t)] (s-1) ds. \]

Since \( \psi((s-1)z) = u(s, z) \psi(-z) \), it follows (with the aid of Eq. (4.7))

\[ \psi((s-1)\text{ad}_{C(t)}) \hat{C}(t) = u(s, \text{ad}_{C(t)}) \psi(-\text{ad}_{C(t)}) \hat{C}(t) = u(s, \text{ad}_{C(t)}) \dot{\xi}(t) \]

which combined with Eq. (4.8) gives Eq. (4.6).

\[ \square \]

Corollary 4.7. If \( \{ V_a : a \in \mathbb{R}^d \} \) generates a step-\( \kappa \) nilpotent Lie subalgebra of \( \Gamma(TM) \), then

\[ \mu_{t,0}^V = e^{V_{\xi(t)}} \quad \text{for all } \xi \in C^1 \left( [0, T], F^{(\kappa)} \left( \mathbb{R}^d \right) \right). \]

Moreover, for any \( A, B \in F^{(\kappa)} \left( \mathbb{R}^d \right) \), we have

\[ e^{V_B} \circ e^{V_A} = e^{V_{\log(e^{A}B)}}. \]

Proof. The given assumption implies \( V_{\pi_{>\kappa}}[C(t), \psi((s-1)\text{ad}_{C(t)})\hat{C}(t)] \equiv 0 \) and hence \( U^\xi \equiv 0 \) and the Eq. (4.9) now follows from Theorem 2.29. To prove the second assertion \( \xi : [0, \infty) \to F^{(\kappa)} \left( \mathbb{R}^d \right) \) be defined by

\[ \xi(t) := \begin{cases} tA & \text{if } 0 \leq t \leq 1 \\ A + (t-1)B & \text{if } 1 \leq t < \infty \end{cases}. \]

With this choice of \( \xi \) we have; \( \dot{\xi}(t) = 1_{t \leq 1} A + 1_{t > 1} B \) (for \( t \neq 1 \),

\[ g^\xi(t) = \begin{cases} e^{tA} & \text{if } 0 \leq t \leq 1 \\ e^Ae^{(t-1)B} & \text{if } 1 \leq t < \infty \end{cases}, \]

\[ \mu_{t,0}^\xi = \begin{cases} e^{tV_A} & \text{if } 0 \leq t \leq 1 \\ e^{(t-1)V_B} \circ e^{V_A} & \text{if } 1 \leq t < \infty \end{cases}, \]

all of which is valid where \( V \) is step-\( \kappa \) nilpotent or not. If \( V \) is step-\( \kappa \) nilpotent we “apply” Eq. (4.9) at \( t = 2 \), to find,

\[ e^{V_B} \circ e^{V_A} = \mu_{2,0}^\xi = e^{V_{\xi(2)}} = e^{V_{\log(e^{A}B)}}. \]

The slight flaw in this argument is that \( \xi(\cdot) \) is not continuously differentiable at \( t = 1 \). To correct this flaw, choose \( \varphi \in C^\infty_{[0, 1]}(\mathbb{R}, [0, \infty)) \) which is supported in \( (0, 1) \) and satisfies \( \int_0^1 \varphi(t) dt = 1 \). We then run the above argument with \( \xi \in C^\infty_{\infty}([0, \infty), F^{(\kappa)} \left( \mathbb{R}^d \right) \) defined so that

\[ \dot{\xi}(t) = \varphi(t) A + \varphi(t-1) B \]
In more detail, if we let
\[(4.13) \quad \tilde{\varphi}(t) := \int_{-\infty}^{t} \varphi(\tau) \, d\tau,\]
then
\[(4.14) \quad \xi(t) = \tilde{\varphi}(t) A + \tilde{\varphi}(t-1) B,\]
\[(4.15) \quad g^\xi(t) = e^{\varphi(t-1)} A e^{\varphi(t-1)} B, \quad \text{and}\]
\[(4.16) \quad \mu_{t,0}^\xi = e^{\varphi(t-1)} V_B \circ e^{\varphi(t)} V_A\]
and in particular at \(t = 2\) we again have,
\[(4.17) \quad e^{V_B} \circ e^{V_A} = \mu_{2,0}^\xi \quad \text{and} \quad C(2) = \log\left(g^\xi(2)\right) = \log\left(e^{A} e^{B}\right).\]
Thus when \(V\) is step-\(\kappa\) nilpotent we are now justified in applying Eq. \((4.9)\) at \(t = 2\) to arrive at Eq. \((4.10)\).

**Notation 4.8** (Commutator bounds). If \(V : \mathbb{R}^d \to \Gamma(TM)\) is a dynamical system and \(m, n \in \mathbb{N}\) with \(\kappa < m + n \leq 2\kappa\), let
\[S_{m,n} := \left\{(A, B) \in F_m^{(\kappa)}(\mathbb{R}^d) \times F_n^{(\kappa)}(\mathbb{R}^d) : |A| = 1 = |B|\right\},\]
\[C_{m,n}^0 \left(V^{(\kappa)}\right) := \sup \{ |[V_A, V_B]|_M : (A, B) \in S_{m,n} \},\]
\[C_{m,n}^1 \left(V^{(\kappa)}\right) := \sup \{ |\nabla [V_A, V_B]|_M : (A, B) \in S_{m,n} \},\]
and
\[C_j \left(V^{(\kappa)}\right) := \sum_{m+n>\kappa} 1_{m,n>\kappa} C_{m,n}^j \left(V^{(\kappa)}\right) \quad \text{for} \quad j = 0, 1.\]

Since
\[[V_A, V_B] = \nabla_{V_A} V_B - \nabla_{V_B} V_A \quad \text{and}\]
\[\nabla_v [V_A, V_B] = \nabla^2_{v \otimes V_A} V_B + \nabla_{vV_A} V_B - (A \leftrightarrow B)\]
it follows that
\[C_{m,n}^0 \left(V^{(\kappa)}\right) \leq 2 \left| V^{(\kappa)} \right|_M \left| \nabla V^{(\kappa)} \right|_M \quad \text{and}\]
\[C_{m,n}^1 \left(V^{(\kappa)}\right) \leq 2 \left( \left| \nabla^2 V^{(\kappa)} \right|_M \cdot \left| V^{(\kappa)} \right|_M + \left| \nabla V^{(\kappa)} \right|_M^2 \right)\]
and therefore
\[(4.18) \quad C^0 \left(V^{(\kappa)}\right) \leq \kappa (\kappa + 1) \left| V^{(\kappa)} \right|_M \left| \nabla V^{(\kappa)} \right|_M \quad \text{and}\]
\[(4.19) \quad C^1 \left(V^{(\kappa)}\right) \leq \kappa (\kappa + 1) \left( \left| \nabla^2 V^{(\kappa)} \right|_M \cdot \left| V^{(\kappa)} \right|_M + \left| \nabla V^{(\kappa)} \right|_M^2 \right).\]
The previous estimates are in general not sharp. For example if \(V\) is \(\kappa\)-nilpotent, then \(C^0 \left(V^{(\kappa)}\right) \equiv 0\) while \(2 \left| V^{(\kappa)} \right|_M \left| \nabla V^{(\kappa)} \right|_M \) will typically be positive.
Lemma 4.9. If $\xi \in C^1([0, T], F^{(\kappa)}(\mathbb{R}^d))$ and $C^\xi(t) = \log (g^{\xi}(t)) \in F^{(\kappa)}(\mathbb{R}^d)$ are as in Definition 4.25 or Notation 4.4 and $u(s, z)$ is as in Notation 4.5 then

\begin{equation}
\int_0^1 ds \int_0^T dt \left| \nabla_{\varphi^{\kappa}[C(t), u(s, ad_{C(t)})]} \xi(t) \right|_M \lesssim C^0(V^{(\kappa)}) Q_{(\kappa, 2\kappa)} \left( N_T^* (\hat{\xi}) \right) .
\end{equation}

and

\begin{equation}
\int_0^1 ds \int_0^T dt \left| V_{\varphi^{\kappa}[C(t), u(s, ad_{C(t)})]} \xi(t) \right|_M \lesssim C^1(V^{(\kappa)}) Q_{(\kappa, 2\kappa)} \left( N_T^* (\hat{\xi}) \right) .
\end{equation}

Proof. Applying the triangle inequality to the identity,

\begin{equation}
V_{\varphi^{\kappa}[C(t), u(s, ad_{C(t)})]} \xi(t) = \sum_{m, n=1}^\kappa 1_{m+n>\kappa} V_{[C(t)_m, (u(s, ad_{C(t)})\xi(t))_n]} \right|_M
\end{equation}

while using Corollaries 3.27 and 3.28 and the definition of $C^0(V^{(\kappa)})$ shows,

\begin{align}
&\int_0^T \left| V_{\varphi^{\kappa}[C(t), \psi((s-1) ad_{C(t)})C(t)]} \right|_M dt \\
&\leq \sum_{m, n=1}^\kappa 1_{m+n>\kappa} \int_0^T \left| V_{[C(t)_m, (u(s, ad_{C(t)})\xi(t))_n]} \right|_M dt \\
&\leq \sum_{m, n=1}^\kappa 1_{m+n>\kappa} C^0_{m, n} \left( V^{(\kappa)} \right) \int_0^T \left| C_m \right|_{\infty, T} \left| (u(s, ad_{C(t)}) \xi(t))_n \right|_T dt \\
&\lesssim K(u(s, \cdot)) \sum_{m, n=1}^\kappa 1_{m+n>\kappa} C^0_{m, n} \left( V^{(\kappa)} \right) N_T^* (\hat{\xi})^{m+n} \\
&\leq K(u(s, \cdot)) \sum_{m, n=1}^\kappa 1_{m+n>\kappa} C^0_{m, n} \left( V^{(\kappa)} \right) Q_{(\kappa, 2\kappa)} \left( N_T^* (\hat{\xi}) \right)
\end{align}

A simple differentiation exercise shows $p_n(s) := \left( \frac{d}{ds} \right)^n u(s, z) |_{z=0}$ is a degree $n$-polynomial function of $s$ with $p_0(s) = 1$. As $K(u(s, \cdot))$ depends linearly on $\left\{ \left( \frac{d}{ds} \right)^j u(s, z) |_{z=0} \right\}_{j=0}^{\kappa-1}$ it follows that $K(u(s, \cdot))$ is bounded by a polynomial function of $s$ and in particular,

$$\int_0^1 K(u(s, \cdot)) (1-s) ds < \infty.$$
Thus multiplying Eq. (4.23) by $(1 - s)$ and then integrating on $s \in [0, 1]$ completes the proof of Eq. (4.20). The proof of Eq. (4.21) is very similar. Simply apply $\nabla$ to both sides of Eq. (4.22) and then continue the estimates as above with $C^0_m, n(V^{(k)})$ and $C^0_m, n(V^{(k)})$ replaced by $C^1_m, n(V^{(k)})$ and $C^1_m, n(V^{(k)})$ respectively.

\textbf{Theorem 4.10.} If $\xi \in C^1([0, T), F^{(k)}(\mathbb{R}^d))$ and $C^\xi(t) = \log(g^{\xi}(t)) \in F^{(k)}(\mathbb{R}^d)$ be as in Definition 1.25 or Notation 4.4 and $U^\xi_t \in \Gamma(TM)$ as in Eq. (4.6) of Lemma 4.6, then

\begin{equation}
|U^\xi|_{T} \lesssim C^0(V^{(k)}) e^{\nabla V^{(k)}}|_{M} C^\xi|_{\infty, T} Q_{[0, n+1]} \left(N^s_T(\hat{\xi})\right)
\end{equation}

which combined with Eq. (3.31) shows there exists $C(\kappa) < \infty$ such that

\begin{equation}
|U^\xi|_{T} \lesssim C^0(V^{(k)}) e^{C(\kappa)}|\nabla V^{(k)}|_{M} Q_{[0, n]}(N^s_T(\hat{\xi})) Q_{[n+1]} \left(N^s_T(\hat{\xi})\right).
\end{equation}

\textbf{Proof.} By Corollary 2.27 if $Y \in \Gamma(TM)$, then

\begin{equation}
\left|\text{Ad}_{e^{sV_{C(r)}}} Y\right|_{M} = \left|e^{sV_{C(r)}} Y \circ e^{-sV_{C(r)}}\right|_{M} = \left|e^{sV_{C(r)}} Y\right|_{M} \leq e^{s|\nabla V_{C(r)}|_{M}|Y|_{M}} \leq e^{s|\nabla V^{(k)}|_{M}|C(r)|}|Y|_{M}
\end{equation}

and so (see Eq. (4.6)),

\begin{equation}
\left|U^\xi_t\right|_{M} \leq \int_{0}^{1} \left|\text{Ad}_{e^{sV_{t}}} V_{\pi_{>\kappa}[C(t), u(s, \text{ad}_{C(t)})\hat{\xi}(t)]}\right|_{M} (s - 1) \, ds
\end{equation}

and so

\begin{equation}
\left|U^\xi|_{T} \lesssim e^{s|\nabla V^{(k)}|_{M}|C^{(r)}|_{\infty, T} \int_{0}^{1} ds (1 - s) \int_{0}^{T} \left|V_{\pi_{>\kappa}[C(t), u(s, \text{ad}_{C(t)})\hat{\xi}(t)]}\right|_{M} dt\right|_{M} (s - 1) \, ds.
\end{equation}

which combined with Lemma 4.9 proves Eq. (4.24). \hfill \square

\textbf{Theorem 4.11 (Approximate log-estimate).} If $\xi \in C^1([0, T), F^{(k)}(\mathbb{R}^d))$, then

\begin{equation}
d_M \left(\frac{V^{\xi}}{\mu_{T, 0}}, e^{\log(g^{(T)})}\right) \lesssim C^0(V^{(k)}) e^{C(\kappa)}|\nabla V^{(k)}|_{M} Q_{[0, n]}(N^s_T(\hat{\xi})) Q_{[n+1]} \left(N^s_T(\hat{\xi})\right).
\end{equation}

\textbf{Proof.} By Theorem 2.29 with $X_t = V^{\xi}(t)$ and $Y_t = W^{C}_{t}$, we know that

\begin{equation}
d_M \left(\frac{V^{\xi}}{\mu_{T, 0}}, e^{V_{C(T)}}\right) \leq e^{s|\nabla V^{\xi}|_{M}} \left|U^\xi\right|_{T} \lesssim e^{s|\nabla V^{(k)}|_{M} |\hat{\xi}|_{T.}} \left|U^\xi\right|_{T}.
\end{equation}

\textsuperscript{2}We will see in Theorem 8.4 below that a similar estimate holds for the distance between the differentials of $\mu_{T, 0}^{V^{\xi}}$ and $e^{\log(g^{(T)})}$.\hfill \square
Combining this estimate with the estimate for $|U^\xi|^T_T$ in Theorem 4.10 and the estimate for $|\dot{\xi}|^*_T$ in Eq. (3.21) gives Eq. (4.27). □

For the rest of this section we assume that $\xi \in C^\infty ([0, \infty), F^{(\kappa)} (\mathbb{R}^d))$ is defined as in Eq. (4.12) of the proof of Corollary 4.7, i.e.

\begin{equation}
\xi(t) = \bar{\varphi}(t) A + \bar{\varphi}(t-1) B \in F^{(\kappa)} (\mathbb{R}^d),
\end{equation}

where

\begin{equation}
\bar{\varphi}(t) = \int_{-\infty}^t \varphi(\tau) d\tau
\end{equation}

and $\varphi \in C^\infty_c (\mathbb{R}, [0, \infty))$ with $\bar{\varphi}(1) = \bar{\varphi}(\infty) = 1$.

**Corollary 4.12.** If $A, B \in F^{(\kappa)} (\mathbb{R}^d)$, then

\begin{equation}
d_M \left( e^{VB} \circ e^{VA}, e^{V \log(e^{A+B})} \right)
\end{equation}

\begin{equation}
\lesssim C^0 \left( V^{(\kappa)} \right) e^{C(\kappa)|\nabla V^{(\kappa)}|_M Q_{[1,\kappa]}^N (N(A) + N(B)) Q_{(\kappa,\kappa+1)}^N \left( N(A) + N(B) \right)}.
\end{equation}

**Proof.** From the definition of $\xi$ in Eq. (4.29), we find

\begin{equation}
|\dot{\xi}|^*_2 = |A_k| + |B_k| \quad \text{for } 1 \leq k \leq \kappa
\end{equation}

and hence with the aid of Eq. (3.22),

\begin{equation}
N_2^* (\dot{\xi}) \leq N(A) + N(B).
\end{equation}

Moreover, by the identities in Eq. (4.17) we know that

\begin{equation}
d_M \left( e^{VB} \circ e^{VA}, e^{V \log(e^{A+B})} \right) = d_M \left( V_k, e^{V \log(C^0)} \right).
\end{equation}

So an application of Theorem 4.11 for this $\xi$ and taking $T = 2$ gives Eq. (4.30). □

The estimate in Eq. (4.30) is not as sharp as we would like. For example the right side of Eq. (4.30) is only 0 when $A = 0 = B$ while the left side is 0 when either $A = 0$ or $B = 0$. To improve upon the estimate in Eq. (4.30) (see Corollary 4.15) we need to examine the form of the difference vector field, $U^\xi_t$, for $\xi$ in Eq. (4.29). We begin with a couple of lemmas.

**Lemma 4.13.** If $f \in \mathcal{H}_0$ satisfies, $f(0) = 0$, then $[f \circ \text{ad}_{C(t)}] B]_1 = 0$ and for $2 \leq k \leq \kappa$,

\begin{equation}
\max_{1 \leq t \leq 2} \left| \left[ f \circ \text{ad}_{C(t)} \right] B \right|_k \leq K(f) N(A) N(B) (N(A) + N(B))^{k-2}
\end{equation}

where $K(f) < \infty$ is a constant which depends linearly on $\left\{ |f^{(j)}(0)| \right\}_{j=1}^{\kappa-1}$. 
Proof. By Proposition 3.18 there exists bounded measurable functions, \( f^j : [0, \infty)^j \to \mathbb{R} \) depending linearly on \( (f(0), \ldots, f^{(\kappa-1)}(0)) \) such that

\[
\int f\left(\text{ad}_{C(t)}\right) = \sum_{j=1}^{\kappa-1} \hat{f}_t \left(\text{ad}_{\xi^j}\right).
\]

As \( \left[\hat{f}_t \left(\text{ad}_{\xi}\right) B\right]_k = 0 \) if \( j \geq k \), to finish the proof it suffices to show for each \( 1 \leq j < k \) that

\[
(4.34) \quad \max_{1 \leq t \leq 2} \left\| \hat{f}_t \left(\text{ad}_{\xi}\right) B\right\|_k \leq \|f\|_\infty N(A) N(B) (N(A) + N(B))^{k-2}.
\]

Let us now fix \( 1 \leq j < k \).

For \( 1 \leq t \leq 2 \),

\[
\hat{f}_t \left(\text{ad}_{\xi}\right) B = \int_{[0,t]} f^j(t_1, \ldots, t_j) \text{ad}_{\xi(t_1)} \ldots \text{ad}_{\xi(t_{j-1})} \text{ad}_{\xi(t_j)} B dt_1 \ldots dt_j
\]

\[
= \int_{[0,t]^{j-1}} f^j(t_1, \ldots, t_j) \varphi(t_j) \text{ad}_{\xi(t_1)} \ldots \text{ad}_{\xi(t_{j-1})} [A, B] dt_1 \ldots dt_{j-1} dt_j,
\]

wherein we have used \( \text{ad}_{\xi(t_j)} B = \varphi(t_j) [A, B] \) for all \( t \geq 0 \). Since \( \int \varphi(t) dt = 1 \), it is simple to verify that

\[
(4.35) \quad \left| \left(\hat{f}_t \left(\text{ad}_{\xi}\right) B\right)_k \right| \leq \|f\|_\infty \int_{[0,t]^{j-1}} \left| \left(\text{ad}_{\xi(t_1)} \ldots \text{ad}_{\xi(t_{j-1})} [A, B]\right)_k \right| dt
\]

where \( dt := dt_1 \ldots dt_{j-1} \). We now estimate the integral in the usual way, namely;

\[
(4.36) \quad \int_{[0,t]^{j-1}} \left| \left(\text{ad}_{\xi(t_1)} \ldots \text{ad}_{\xi(t_{j-1})} [A, B]\right)_k \right| dt \leq \sum \int_{[0,t]^{j-1}} \left| \text{ad}_{\xi_{k_1}(t_1)} \ldots \text{ad}_{\xi_{k_{j-1}}(t_{j-1})} [A_{m,B_n}] \right| dt
\]

where the sum is over \((m, n, k_1, \ldots, k_{j-1}) \in \mathbb{N}^{j+1} \) such that \( \sum_{i=1}^{j-1} k_i + m + n = k \). Using \( \|A, B\| \leq 2 |A| |B| \) for all \( A, B \in F^{(\kappa)}(\mathbb{R}^d) \), each term on the right side of Eq. (4.36) may be estimated by

\[
2^j \int_{[0,t]^{j-1}} \left| \xi_{k_1}(t_1) \ldots \xi_{k_{j-1}}(t_{j-1}) \right| |A_m| |B_n| dt \leq 2^j \prod_{i=1}^{j-1} |\xi_{k_1}| |A_m| |B_n| \leq 2^j \prod_{i=1}^{j-1} N_i^{k_i} (\xi)^{k_i} N(A)^m N(B)^n
\]

\[
\leq 2^j (N(A) + N(B))^{k-m-n} N(A)^m N(B)^n
\]

\[
(4.37) \quad \leq 2^j N(A) N(B) (N(A) + N(B))^{k-2}.
\]

Combining the estimates in Eqs. (4.35), (4.37) completes the proof of Eq. (4.34) and hence the proof of the lemma. \( \square \)
Proposition 4.14. If \( A, B \in F(\kappa) (\mathbb{R}^d) \) and \( \xi \in C^\infty ([0, \infty), F(\kappa) (\mathbb{R}^d)) \) is as in Eq. (4.29), then
\[
\left[ C^\xi (t), u \left( s, \text{ad}_{C^\xi(t)} \right) \hat{\xi} (t) \right]_\otimes
\]
(4.38)
\[
= \varphi (t - 1) \left( [A, B]_\otimes + \left[ \bar{C}^\xi (t), B \right]_\otimes + \left[ C^\xi (t), \bar{u} \left( s, \text{ad}_{C^\xi(t)} \right) B \right]_\otimes \right)
\]
where
\[
\bar{C}^\xi (t) := C^\xi (t) - \xi (t) \quad \text{and} \quad \bar{u} (s, z) := u (s, z) - u (s, 0) = u (s, z) - 1.
\]
Moreover for \( k \geq 2 \), the following estimates hold:
\[
(4.39) \quad \max_{0 \leq t \leq 2} \left| \bar{C}^\xi _k (t) \right| \lesssim N (A) N (B) (N (A) + N (B))^{k-2}
\]
(4.40) \quad \max \max_{0 \leq t \leq 2 0 \leq s \leq 1} \left| \bar{u} \left( s, \text{ad}_{C^\xi(t)} \right) B \right|_k \lesssim N (A) N (B) (N (A) + N (B))^{k-2}.

Proof. From Eq. (4.15), \( C^\xi (t) = \bar{\varphi} (t) A = \xi (t) \) when \( t \leq 1 \) and therefore
\[
\left[ C^\xi (t), u \left( s, \text{ad}_{C^\xi(t)} \right) \hat{\xi} (t) \right]_\otimes = \bar{\varphi} (t) \varphi (t) \left[ A, u \left( s, \text{ad}_{\bar{\varphi}(t)A} \right) A \right]_\otimes
\]
= \( \varphi (t) \varphi (t) \left[ A, u \left( s, 0 \right) A \right]_\otimes = 0 \).

which proves Eq. (4.38) for \( t \leq 1 \). When \( t \geq 1 \), \( \xi (t) = A + \bar{\varphi} (t - 1) B \), \( \hat{\xi} (t) = \varphi (t - 1) B \), and
\[
u \left( s, \text{ad}_{C^\xi(t)} \right) \hat{\xi} (t) = u \left( s, \text{ad}_{C^\xi(t)} \right) B = B + \bar{u} \left( s, \text{ad}_{C^\xi(t)} \right) B
\]
and hence
\[
\left[ C^\xi (t), u \left( s, \text{ad}_{C^\xi(t)} \right) \hat{\xi} (t) \right]_\otimes
\]
= \( \left[ C^\xi (t), \hat{\xi} (t) + \bar{u} \left( s, \text{ad}_{C^\xi(t)} \right) \hat{\xi} (t) \right]_\otimes
\]
= \( \varphi (t - 1) \left( \left[ A + \bar{\varphi} (t - 1) B + \bar{C}^\xi (t), B \right]_\otimes + \left[ C^\xi (t), \bar{u} \left( s, \text{ad}_{C^\xi(t)} \right) B \right]_\otimes \right)
\]
which easily gives Eq. (4.38) for \( t \geq 1 \).

By Eq. (3.10),
\[
\hat{C}^\xi (t) = \frac{1}{\psi} (- \text{ad}_{C(t)}) \hat{\xi} (t) = \hat{\xi} (t) + g \left( \text{ad}_{C(t)} \right) \hat{\xi} (t)
\]
wherein the last equality we used \( 1/\psi (0) = 1 \) and have set
\[
g (z) := \frac{1}{\psi (-z)} - \frac{1}{\psi (0)} = \frac{1}{\psi (-z)} - 1.
\]
Thus it follows that
\[ C^z(t) = \int_0^t g \left( \text{ad}_{C^z(\tau)} \right) \xi(\tau) d\tau = \int_0^t \varphi(\tau - 1) g \left( \text{ad}_{C^z(\tau)} \right) B d\tau. \]
By Lemma 4.13,
\[ \max_{1 \leq \tau \leq 2} \left| g \left( \text{ad}_{C^z(\tau)} \right) B \right| \leq K(g) N(A) N(B) (N(A) + N(B))^{k-2} \]
and so it now easily follows that
\[ \left| \xi(t) \right| \leq K(g) N(A) N(B) (N(A) + N(B))^{k-2} \text{ for all } 0 \leq t \leq 2. \]
By another application of Lemma 4.13
\[ \max_{0 \leq \tau \leq 2} \left| \xi(t) \right| \leq K(g) N(A) N(B) (N(A) + N(B))^{k-2} \]
where \( K(\xi) \) is bounded in \( s \in [0,1] \) as the derivatives of \( \xi \) as \( z = 0 \) are polynomial functions in \( s \). These last two inequalities verify Eqs. (4.39) and (4.40) and hence complete the proof. \( \square \)

**Corollary 4.15.** If \( A, B \in F^{(\kappa)}(\mathbb{R}^d) \), then
\[ d_M \left( e^{B}, \text{Id}_M \right) \leq \left| V^{(\kappa)} \right| \left| B \right| \leq V^{(\kappa)} Q_{[1,\kappa]}(N(B)) \]
and there exists \( C(\kappa) < \infty \) such that
\[ d_M \left( e^{B} \circ e_{A}, e^{V_{\log(sAeB)}} \right) \leq K_0 N(A) N(B) Q_{[\kappa,2\kappa-2]}(N(A) + N(B)) \]
where
\[ K_0 := C^0 \left( V^{(\kappa)} \right) e^{C(\kappa) |\nabla V^{(\kappa)}|_{M} Q_{[1,\kappa]}(N(A)+N(B))} \]
\[ \leq 2 \left| V^{(\kappa)} \right|_{M} \left| \nabla V^{(\kappa)} \right|_{M} e^{C(\kappa) |\nabla V^{(\kappa)}|_{M} Q_{[1,\kappa]}(N(A)+N(B))}. \]

**Proof.** The first inequality follows as an application of Corollary 2.30 with \( Y_{\xi} := V_{B} \) using
\[ \left| Y_{\xi} \right| = \left| V_{B} \right| \leq \left| V^{(\kappa)} \right| \left| B \right|. \]
To prove the second inequality we let \( \xi(t) \) be as in Proposition 4.14. By Eq. (4.28) in the proof of Theorem 4.11 we then have, to find,
\[ d_M \left( e^{V_{B} \circ e_{A}, \xi}, e^{V_{\log(sAeB)}} \right) = d_M \left( \mu_{2,0}, e^{V_{\xi}} \right) \]
\[ \leq e^{\left| \nabla V^{(\kappa)} \right| |\xi|^2_{2}}, \left| U_{\xi} \right|^2_{2} = e^{\left| \nabla V^{(\kappa)} \right| |\xi|^2_{2}}, \left| U_{\xi} \right|^2_{2}. \]
where, by Eq. (4.26) of the proof of Theorem 4.10
\[ \left| U_{\xi} \right|^2_{2} \leq e^{\left| \nabla V^{(\kappa)} \right| |\xi|^2_{2}}, \int_0^1 ds (1-s) \int_0^2 dt \left| V_{\mu(s,C(t),\xi)} \right|_{M}. \]
From Proposition 4.14

\[
\left| V\left( [C(t), u(s, \text{ad}_{C(t)}) \xi(t)]_{\otimes} \right) \right|_M \\
\leq \varphi (t-1) \left( \left| V\left( [A,B]_{\otimes} \right) \right|_M + \left| V\left( [\tilde{C} \xi(t), B]_{\otimes} \right) \right|_M \right) \\
+ \left| V\left( [\tilde{C} \xi(t), u(s, \text{ad}_{\tilde{C} \xi(t)}) B]_{\otimes} \right) \right|_M 
\]

(4.47)

We now estimate each of the three terms appearing on the right side of Eq. (4.47).

(1) Since, for \( m, n \in [1, \kappa] \) with \( m + n = k \),

\[ |A_m| |B_n| \leq N(A)^m N(B)^n \leq N(A) N(B) (N(A) + N(B))^{k-2}, \]

we find

\[
\left| V\left( [A,B]_{\otimes} \right) \right|_M \leq \sum_{m,n=1}^{\kappa} 1_{m+n=k} \left| [V_{A_m}, V_{B_n}] \right|
\leq \sum_{m,n=1}^{\kappa} 1_{m+n=k} \mathcal{C}^0_{m,n} (V^{(\kappa)}) \left| A_m \right| \left| B_n \right|
\leq \mathcal{C}^0 (V^{(\kappa)}) N(A) N(B) (N(A) + N(B))^{k-2}.
\]

(2) Using Eq. (4.39) and (by definition) \( \tilde{C}_1^\xi = 0 \), it follows that

\[
\left| V\left( [\tilde{C} \xi(t), B]_{\otimes} \right) \right|_M \\
\leq \sum_{m,n=1}^{\kappa} 1_{m+n=k} \mathcal{C}^0_{m,n} (V^{(\kappa)}) \left| \tilde{C}_m^\xi (t) \right| \left| B_n \right|
\leq \sum_{n=1}^{\kappa} \sum_{m=2}^{\kappa} 1_{m+n=k} \mathcal{C}^0_{m,n} (V^{(\kappa)}) N(A) N(B) (N(A) + N(B))^{m-2} N(B)^{n-1}
\leq \mathcal{C}^0 (V^{(\kappa)}) N(A) N(B) (N(A) + N(B))^{k-2}.
\]
(3) Similarly using Eqs. (3.30) and (4.40),
\[
\left| \left[ C^\xi(t), \bar{u}(s, \text{ad}_C^\xi(t)) B \right] \right|_{\otimes k}^{M \otimes k}
\]
\[
\leq \sum_{m,n=1}^{\kappa} 1 + n = k C^{0}_{m,n} \left( V^{(\kappa)} \right) \left| C^\xi_m (t) \right| \left| (\bar{u} \left( s, \text{ad}_C^\xi(t) \right)) B \right|_{n}
\]
\[
\lesssim \sum_{n=2}^{\kappa} \sum_{m=1}^{\kappa} 1 + n = k C^{0}_{m,n} \left( V^{(\kappa)} \right) \left| C^\xi_m (t) \right| \cdot N (A) N (B) (N (A) + N (B))^{n-2}
\]
\[
\lesssim \sum_{n=2}^{\kappa} \sum_{m=1}^{\kappa} 1 + n = k C^{0}_{m,n} \left( V^{(\kappa)} \right) \cdot N (A) N (B) (N (A) + N (B))^{m+n-2}
\]
\[
\lesssim C^{0} \left( V^{(\kappa)} \right) N (A) N (B) (N (A) + N (B))^{k-2}.
\]

Combining the last three estimates with Eqs. (4.47) and (4.46) shows (with \( \mathcal{K} \) having the form as in Eq. (4.43)),
\[
\left| U^\xi \right|_{L^2} \leq e^{[\nabla V^{(\kappa)}]_{M|C}_{\infty,2}} \sum_{k=\kappa+1}^{2\kappa} \int_{0}^{1} ds (1 - s) \int_{0}^{2} dt \left| \left[ C(t), u(s, \text{ad}_C^\xi(t)) \right] \right|_{k}^{M}
\]
\[
\lesssim e^{[\nabla V^{(\kappa)}]_{M|C}_{\infty,2}} \sum_{k=\kappa+1}^{2\kappa} C^{0} \left( V^{(\kappa)} \right) N (A) N (B) (N (A) + N (B))^{k-2}
\]
\[
\lesssim C^{0} \left( V^{(\kappa)} \right) e^{[\nabla V^{(\kappa)}]_{M|C}_{\infty,2}} N (A) N (B) Q_{[\kappa-1,2\kappa-2]} (N (A) + N (B))
\]
\[
(4.48)
\]
\[
\lesssim \mathcal{K} N (A) N (B) Q_{[\kappa-1,2\kappa-2]} (N (A) + N (B)),
\]

wherein the last inequality we have also used the estimate in Eq. (3.31). This estimate combined with Eq. (4.45), while using Eqs. (3.20) and (3.19) in order to show \( |A| + |B| \lesssim Q_{[1,k]} (N (A) + N (B)) \), completes the proof. \( \square \)

This completes part I. of the paper. The second remaining part of the paper is devoted to developing estimates for the distance between the differentials of \( \mu_{T,0}^{V^\xi} \) and \( e^{V^\log (\xi (T))} \). In order to formulate our results we must first define a distance between \( f_* \) and \( g_* \) for \( f, g \in C^1 (M, M) \). To do so we will use the metric on \( M \) to endow \( TM \) with a Riemannian metric and then make use of this metric to construct the desired distance. It will also be necessary to develop some of the basic properties of the induced distance function on \( TM \) which is the topic of the next section.

5. Riemannian Distances on \( TM \)

5.1. Riemannian distances on vector bundles. For clarity of exposition (and since it is no harder), it is convenient to carry out these constructions in the more general context of an arbitrary Hermitian vector bundle, \( \pi : \)
Later we will specialize to the case of interest where \( E = TM \).

**Definition 5.1** (Riemannian metric on \( E \)). Continuing the setup in Notation 2.2, we define a Riemannian metric on \( TE \) by defining
\[
\langle \dot{\xi}(0), \dot{\eta}(0) \rangle_{TE} := \langle \pi_* \dot{\xi}(0), \pi_* \dot{\eta}(0) \rangle_{g} + \langle \nabla_{\xi} dt(0), \nabla_{\eta} dt(0) \rangle_{E}
\]
whenever \( \xi(t) \) and \( \eta(t) \) are two smooth curves in \( E \) such that \( \pi(\xi(0)) = \pi(\eta(0)) \).

**Remark 5.2.** Let \( \sigma(t) \) and \( \gamma(t) \) be two smooth paths in \( M \) so that \( \sigma(0) = \gamma(0) \) and suppose that \( \alpha(t) \) and \( \beta(t) \) are two smooth paths in \( \mathbb{R}^D \).

Then in the local model described in Remark 2.3 we have,
\[
\pi_* \dot{\xi}(0) = \pi_* \left( \dot{\sigma}(0), \dot{\alpha}(0)_{\alpha(0)} \right) = \dot{\sigma}(0),
\pi_* \dot{\eta}(0) = \pi_* \left( \dot{\gamma}(0), \dot{\beta}(0)_{\beta(0)} \right) = \dot{\gamma}(0),
\nabla_{\xi} dt(0) = (m, \dot{\alpha}(0) + \Gamma(\dot{\sigma}(0))\alpha(0)),
\nabla_{\eta} dt(0) = (m, \dot{\beta}(0) + \Gamma(\dot{\gamma}(0))\beta(0)),
\]
and
\[
\langle \dot{\xi}(0), \dot{\eta}(0) \rangle_{TE} = \langle \dot{\sigma}(0), \dot{\gamma}(0) \rangle_{g} + (\dot{\alpha}(0) + \Gamma(\dot{\sigma}(0))\alpha(0)) \cdot (\dot{\beta}(0) + \Gamma(\dot{\gamma}(0))\beta(0)).
\]

From this expression we see that \( \langle \cdot, \cdot \rangle_{TE} \) is indeed a Riemannian metric on \( E \). For example, \( |\dot{\xi}(0)|_{TE}^2 = 0 \) implies
\[
0 = |\dot{\sigma}(0)|_{g}^2 + |\dot{\alpha}(0) + \Gamma(\dot{\sigma}(0))\alpha(0)|_{\mathbb{R}^D}^2
\]
from which it follows that \( \dot{\sigma}(0) = 0 \) and then \( |\dot{\alpha}(0)|_{\mathbb{R}^D} = 0 \) so that \( \dot{\alpha}(0) = 0 \), i.e. \( \dot{\xi}(0) = 0 \in T_{\xi(0)}E \).

**Definition 5.3.** As usual, the length of a smooth path, \( t \to \xi(t) \in E \), is defined by
\[
\ell_E(\xi) = \int_{0}^{1} |\dot{\xi}(t)| dt = \int_{0}^{1} \sqrt{\left| \pi_* \dot{\xi}(t) \right|^2 + \left| \nabla_{\xi} dt(t) \right|^2_{E}} dt
\]
and the distance, \( d^E \), is then the distance associated to this length.

Our first goal is to give a more practical way (see Eq. (5.5) of Corollary 5.7 below) of computing \( d^E(e, e') \) for \( e, e' \in E \).

**Notation 5.4.** Given a path \( \sigma : [0, 1] \to M \), let
\[
L_{\sigma}(e, e') := \sqrt{\ell_{M}(\sigma)^2 + \|L_{1}(\sigma)^{-1} e' - e\|_{E}} \quad \forall \, e \in E_{\sigma(0)} \text{ and } e' \in E_{\sigma(1)}
\]
with the convention that \( L_{\sigma}(e, e') = \infty \) if \( \sigma \) is not absolutely continuous.
**Theorem 5.5.** If \( \sigma \in AC([0, 1], M) \), \( \xi \in AC_{\sigma}([0, 1], E) \), and

\[
s(t) := \int_0^t |\dot{\sigma}(\tau)| d\tau - \text{arc-length of } |\sigma|_{[0,t]},
\]

then

\[
L_\sigma(\xi(0), \xi(1)) \leq \sqrt{\ell_M(\sigma)^2 + \left[\int_0^1 |\nabla_t \xi(t)| \, dt\right]^2} \leq \ell_E(\xi) \leq \int_0^1 [|\dot{\sigma}(t)| + |\nabla_t \xi(t)|] \, dt.
\]  

(5.1)

and moreover

\[
L_\sigma(\xi(0), \xi(1)) = \sqrt{\ell_M(\sigma)^2 + \left[\int_0^1 |\nabla_t \xi(t)| \, dt\right]^2} = \ell_E(\xi)
\]  

(5.2)

when

\[
\xi(t) = \xi(0) + t(\xi(1) - \xi(0)) \quad \text{if } s(1) = 0 \quad \text{or}
\]

(5.3)

\[
\xi(t) = /\!/_t(\sigma) \left[ e_m + \frac{s(t)}{s(1)} (//_1(\sigma)^{-1} \xi(1) - \xi(0)) \right] \quad \text{if } s(1) > 0.
\]

(5.4)

Proof. If we let \( w(t) := /\!/_t(\sigma)^{-1} \xi(t) \in E_{\sigma(0)} \), then \( |\nabla_t \xi(t)| = |\dot{w}(t)| \) and so

\[
\int_0^1 |\nabla_t \xi(t)| \, dt = \int_0^1 |\dot{w}(t)| \, dt \geq |w(1) - w(0)| = \left| //_1(\sigma)^{-1} \xi(1) - \xi(0) \right|,
\]

wherein we have used the length of \( w \) is greater than or equal to \( |w(1) - w(0)| \).

The last inequality is equivalent to the first inequality in Eq. (5.1).

If we let

\[
u(t) := \int_0^t |\nabla_t \xi(\tau)| \, d\tau,
\]

then \( t \to (s(t), u(t)) \in \mathbb{R}^2 \) is an absolutely continuous path in \( \mathbb{R}^2 \) from \((0, 0)\) and so the length of this path,

\[
\int_0^1 \sqrt{\dot{s}(t)^2 + \dot{u}(t)^2} \, dt = \int_0^1 |\dot{\sigma}(t)|^2 + |\nabla_t \xi(t)|^2 \, dt = \ell_E(\xi),
\]

is greater than or equal to

\[
\| (s(1), u(1)) \|_{\mathbb{R}^2} = \sqrt{\ell_M(\sigma)^2 + \left[\int_0^1 |\nabla_t \xi(t)| \, dt\right]^2}.
\]

This proves the second inequality in Eq. (5.1). To prove the last inequality in Eq. (5.1) simply observe (see Eq. (3.22) with \( p = 2 \)) that

\[
\sqrt{|\dot{\sigma}(t)|^2 + |\nabla_t \xi(t)|^2} \leq |\dot{\sigma}(t)| + |\nabla_t \xi(t)|.
\]

\[\text{If } 0 = s(1) = \ell(\sigma), \text{ then necessarily } m = \sigma(0) = \sigma(1) = p.\]
If \( s(1) > 0 \) and \( \xi \) is given as in Eq. \((5.4)\), then
\[
|\nabla \xi(t)| = \left| \frac{1}{\ell_M(\sigma)} \left( \left/ \frac{1}{\sigma} e_p - e_m \right. \right) \right| = \frac{|\dot{\sigma}(t)|}{\ell_M(\sigma)} \left| /\left/ \frac{1}{\sigma} e_p - e_m \right. \right|
\]
and hence
\[
\ell_E(\xi) = \int_0^1 \sqrt{\left| \dot{\sigma}(t) \right|^2 + \left( \left/ \frac{1}{\sigma} \right. \right)^2} \left| /\left/ \frac{1}{\sigma} \xi(t) - \xi(0) \right. \right|^2 \, dt
\]
\[
= \int_0^1 \frac{|\dot{\sigma}(t)|}{\ell_M(\sigma)} \sqrt{\ell_M^2(\sigma) + \left| /\left/ \frac{1}{\sigma} e_p - e_m \right. \right|^2} \, dt = L_\sigma(\xi(0), \xi(1))
\]
which verifies Eq. \((5.2)\) in this case. Similarly by a simple calculation, Eq. \((5.2)\) holds when \( \ell_M(\sigma) = 0 \) and \( \xi \) is given as in Eq. \((5.3)\).

**Notation 5.6.** To each \( \sigma \in AC([0, 1], M) \), let \( AC_\sigma([0, 1], E) \) denote those \( \xi \in AC([0, 1], E) \) such that \( \xi(t) \in T_{\sigma(t)}M \) for all \( 0 \leq t \leq 1 \).

**Corollary 5.7.** If \( e_m \in E_m \), and \( e_p \in E_p \), then
\[
d^E(e_m, e_p) = \inf \{ L_\sigma(e_m, e_p) : \sigma \in AC([0, 1], M), \sigma(0) = m, \sigma(1) = p \}
\]
where for \( \sigma \in AC([0, 1], M) \) with \( \sigma(0) = m \) and \( \sigma(1) = p \),
\[
L_\sigma(e_m, e_p) = \min \{ \ell_E(\xi) : \xi \in AC_\sigma([0, 1], E), \xi(0) = e_m, \ell_E(\xi(1)) = e_p \}.
\]

**Proof.** The first equation is an easy consequence of the second. For the second equation, if \( \xi \in AC_\sigma([0, 1], E) \) with \( \xi(0) = e_m \), and \( \xi(1) = e_p \), then by Theorem 5.5 \( L_\sigma(e_m, e_p) \leq \ell_E(\xi) \) with equality occurring when \( \xi \) is given by Eq. \((5.3)\) if \( \ell_M(\sigma) = 0 \) or by Eq. \((5.4)\) if \( \ell_M(\sigma) > 0 \).

**Remark 5.8.** One might suspect that if \( e, e' \in E_m \), then \( d^E(e, e') = |e - e'| \). However this is not necessarily the case unless the holonomy group of \( \nabla^E \) at \( m \) is trivial (in particular this implies the curvature of \( \nabla^E = 0 \).) For example of \( |e| = |e'| = 1 \), there may be a very short loop, \( \sigma \), starting and ending at \( m \), so that \( /\left/ \sigma^{-1} e' = e \right. \) in which case it would follow that \( d^E(e, e') \leq \ell_M(\sigma) \) which can easily be smaller than \( |e - e'| \) which could be as large as \( \sqrt{2} \). If \( \sigma \) is the constant loop sitting at \( m \), then \( L_\sigma(e, e') = |e' - e| \) and hence \( d^E(e, e') \leq |e' - e| \) whenever \( e, e' \in E_m \) for some \( m \in M \).

**Proposition 5.9.** \(|\cdot|_E \) is Lipschitz. If \( e, e' \in E \), then
\[
|e|_E - |e'|_E \leq d^E(e, e'),
\]
i.e. fiber metric on \( E, |\cdot|_E \), is 1-Lipschitz relative to \( d^E \).

**Proof.** Let \( e_m \) and \( e'_p \) in \( E \) and \( \sigma \) be an absolutely continuous path joining \( m \) to \( p \). Then by Lemma 2.4
\[
|e_m|_E - |e'_p|_E \leq \left| e_m - /\left/ \sigma e'_p \right. \right|_{E_m} \leq L_\sigma(e_m, e'_p)
\]
and therefore by Corollary 5.7

\[ \left| e_m \right|_E - \left| e'_p \right|_E \leq \inf_{\sigma} L_{\sigma} (e_m, e'_p) = d^E (e_m, e'_p), \]

where the infimum is over all paths, \( \sigma \), joining \( m \) to \( p \).

**Proposition 5.10** (Completeness of \( E \)). If \((M, g)\) is a complete Riemannian manifold then the vector bundle, \( E \), with the Riemannian structure in Definition 5.1 is again a complete Riemannian manifold.

**Proof.** Let \( \pi : E \to M \) be the natural projection map and observe that

\[ \left| \pi_* \dot{\xi} (0) \right|_M \leq \left| \dot{\xi} (0) \right|_E \quad \forall \dot{\xi} (0) \in T E. \]

If \( e_0, e_1 \in E \) and \( e (\cdot) \in AC ([0, 1], E) \) is a path joining \( e_0 \) to \( e_1 \), then \( \pi \circ e \in AC ([0, 1], M) \) is path joining \( \pi (e_0) \) to \( \pi (e_1) \) and

\[ d_M (\pi (e_0), \pi (e_1)) \leq \ell_M (\pi \circ e) = \int_0^1 \left| \pi_* \dot{u} (t) \right|_M dt \]
\[ \leq \int_0^1 \left| \dot{u} (t) \right|_E dt = \ell_E (e). \]

Minimizing this inequality over \( e \) as described above shows

\[ d_M (\pi (e_0), \pi (e_1)) \leq d^E (e_0, e_1). \]

Hence if \( \{ e_n \}_{n=1}^{\infty} \) is a Cauchy sequence in \( E \), then \( \{ p_n = \pi (e_n) \}_{n=1}^{\infty} \) is a Cauchy sequence in \( M \). As \( M \) is complete we know that \( p = \lim_{n \to \infty} p_n \) exists in \( M \). Let \( W \) be an open neighborhood of \( p \) in \( M \) over which \( M \) is trivial and let \( U \) be a local orthonormal frame (as described after Notation 2.2) of \( E \) defined over \( W \) and, for large enough \( n \), let \( v_n := U (p_n)^{-1} e_n \in \mathbb{R}^N \) where \( N \) is the fiber dimension of \( E \). From Proposition 5.9 we know \( \{ |e_n|_E = |v_n|_{\mathbb{R}^N} \}_{n=1}^{\infty} \) is a Cauchy sequence in \( \mathbb{R} \) and hence bounded and hence there exists a subsequence, \( \{ v_{n_k} \}_{k=1}^{\infty} \) of \( \{ v_n \} \) so that \( v := \lim_{k \to \infty} v_{n_k} \) exists in \( \mathbb{R}^N \). It then follows that

\[ \lim_{k \to \infty} e_{n_k} = \lim_{k \to \infty} U (p_{n_k}) v_{n_k} = U (p) v \] exists.

As \( \{ e_n \}_{n=1}^{\infty} \) was Cauchy in \( E \) and has a convergent subsequence, it follows that \( \lim_{n \to \infty} e_n = U (p) v \) exists in \( E \) and hence \( E \) is complete. \( \square \)

**Theorem 5.11.** Let \( \pi : E \to M \) be a vector bundle equipped with a fiber metric and metric compatible covariant derivative as above. If \( \lambda \geq 0 \) and \( e_m, e'_p \in E \), then

\[ d^E (\lambda e_m, \lambda e'_p) \leq (\lambda \vee 1) d^E (e_m, e'_p). \]
Proof. Let $\sigma$ be a curve joining $m$ to $p$, then
\[
d^E (\lambda e_m, \lambda e'_p) \leq L_\sigma (\lambda e_m, \lambda e'_p) = \sqrt{\ell_M (\sigma)^2 + \left| /_{/1 (\sigma)}^{-1} e' - \lambda e \right|^2} = \sqrt{\ell_M (\sigma)^2 + \left| /_{/1 (\sigma)}^{-1} e' - \lambda e \right|^2} \leq \lambda \vee 1 \cdot L_\sigma (e_m, e'_p)
\]
and the result now follows from Corollary 5.7 as $\sigma$ was arbitrary. \qed

**Definition 5.12** (Bundle maps). A smooth function, $F : E \to E$, is a **bundle map** provided there exist a smooth map, $f : M \to M$ such that $F(E_m) \subset E_{f(m)}$ for all $m \in M$ and $F|_{E_m} : E_m \to E_{f(m)}$ is linear. We will refer to such an $F$ as a **bundle map covering** $f$.

We are interested in measuring the distance between two bundle maps, $F, G : E \to E$. For such maps we can no longer define $d^E_\infty (F, G) = \sup_{\lambda > 0} d^E (F\lambda e, G\lambda e)$ since $\sup_{\lambda > 0} d^E (F\lambda e, G\lambda e) = \infty$ if $|F e| \neq |G e|$.

Indeed if $\sigma \in C^1 ([0, 1], M)$ is any path such that $F e \in E_{\sigma(0)}$ and $G e \in E_{\sigma(1)}$, then
\[
L_\sigma (\lambda F e, \lambda G e) = \sqrt{\ell_M (\sigma)^2 + \left| /_{/1 (\sigma)}^{-1} \lambda G e - \lambda F e \right|^2} \geq \left| /_{/1 (\sigma)}^{-1} \lambda G e \right| - |\lambda F e| = |\lambda| ||G e| - |F e||
\]
and hence by Corollary 5.7
\[
d^E (F\lambda e, G\lambda e) \geq |\lambda| ||G e| - |F e|| \to \infty \text{ as } \lambda \uparrow \infty.
\]
On the other hand, as bundle maps are fiber linear they are determined by their values, $\{F e : e \in E \text{ with } |e| = 1\}$. With these comments in mind we make the following definition.

**Definition 5.13** (Bundle map norms and distances). Given a bundle maps, $F : E \to E$, $m \in M$, and $\sigma \in C ([0, 1], M)$, let
\[
|F|_m := \sup_{e \in E_m, |e| = 1} |F e|, \quad |F|_\sigma := \sup_{t \in [0, 1]} |F|_{\sigma(t)}, \text{ and } |F|_M := \sup_{m \in M} |F|_m = \sup_{m \in M} |F|_m = \sup_{e \in E_m, |e| = 1} |F e|.
\]
If $G : E \to E$ is another bundle map, let
\[
d^E_\infty (F, G) := \sup_{e \in E, |e| = 1} d^E (F e, G e).
\]

**Remark 5.14.** Let us note that $d^E_\infty (F, G) = 0$ iff $F e = G e$ for all $|e| = 1$ which suffices to show $F \equiv G$ since both $F$ and $G$ are fiber linear.
Lemma 5.15. If $F, G : E \to E$ are bundle maps, then
\begin{equation}
||F||_M - |G|_M| \leq d_\infty^E (F,G).
\end{equation}

Proof. If $e \in E$ with $|e| = 1$, then (by Proposition 5.9)
\begin{equation}
||Fe| - |Ge|| \leq d^E (Fe, Ge) \leq d_\infty^E (F,G)
\end{equation}
and therefore
\begin{equation}
|Fe| \leq |Ge| + d_\infty^E (F,G) \leq |G|_M + d_\infty^E (F,G).
\end{equation}
As this is true for all $e \in E$ with $|e| = 1$ we may further conclude that
\begin{equation}
|F|_M \leq |G|_M + d_\infty^E (F,G).
\end{equation}
Reversing the roles of $F$ and $G$ also shows
\begin{equation}
|G|_M \leq |F|_M + d_\infty^E (F,G)
\end{equation}
and together the last two displayed equations proves Eq. (5.9). \qed

The next proposition contains the typical mechanism we will use for estimating $d_\infty^E (F,G)$.

Proposition 5.16. If $\{F_t\}_{0 \leq t \leq 1}$ is a smoothly varying one parameter family of bundle maps from $E$ to $E$ covering $\{f_t\}_{0 \leq t \leq 1} \subset C^\infty (M, M)$, then for any $e \in E_m$,
\begin{equation}
d^E (F_0e, F_1e) \leq \sqrt{\ell_M^2 (f_\cdot (m)) + \left[ \int_0^1 \| \nabla F_t \|_M dt \right]^2},
\end{equation}
and
\begin{equation}
d_\infty^E (F_0, F_1) \leq \sqrt{\sup_{m \in M} \ell_M^2 (f_\cdot (m)) + \left[ \int_0^1 \| \nabla F_t \|_M dt \right]^2}.
\end{equation}
(5.12)
\begin{equation}
\leq \sup_{m \in M} \ell_M (f_\cdot (m)) + \int_0^1 \| \nabla F_t \|_M dt
\end{equation}
(5.13)
\begin{equation}
\leq \int_0^1 \left[ \| \dot{f}_t \|_M + \| \nabla F_t \|_M \right] dt.
\end{equation}

Proof. If $e \in E_m$, then, by Corollary 5.7 with $\sigma (t) = f_t (m)$,
\begin{equation}
d^E (F_0e, F_1e) \leq L_\sigma (F_0e, F_1e)
\end{equation}
(5.14)
\begin{equation}
= \sqrt{\ell_M^2 (t \to f_t (m)) + \| f_\cdot (m) \|^{-1}_1 (f_t \cdot (m)) F_1e - F_0e \|^2}.
\end{equation}
This inequality along with Lemma 2.4 applied with $\xi (t) = F_t e$ then gives Eq. (5.10) and Eq. (5.10) along with Eq. (3.22) with $p = 2$ then show,
\begin{equation}
d^E (F_0e, F_1e) \leq \ell_M (f_\cdot (m)) + \int_0^1 \| \nabla F_t \|_M dt.
\end{equation}
Taking the supremum of these estimates over \(|e| = 1\) then give the remaining stated estimates since, Definition 5.13.

\[
(5.16) \quad \left\| \frac{\nabla F_t}{dt} \right\|_M := \sup \left\{ \left\| \frac{\nabla F_t}{dt} e \right\| : e \in E \text{ with } |e| = 1 \right\}.
\]

\[
\square
\]

Remark 5.17. A more elementary way to arrive at Eq. (5.15) is again to let \(\sigma (t) = f_t (m)\) and \(\xi (t) = F_t e\) and then observe that

\[
d^E (F_0 e, F_1 e) \leq \ell_E (\xi) = \int_0^1 \sqrt{\left( \frac{d}{dt} \right)^2 (\dot{\sigma} (t))^2 + \left( \frac{d}{dt} \dot{\xi} (t) \right)^2} dt
\]

\[
\leq \int_0^1 \left( |\dot{\sigma} (t)| + \left| \frac{d}{dt} \dot{\xi} (t) \right| \right) dt = \ell_M (f_{\xi} (m)) + \int_0^1 \left| \frac{d}{dt} \dot{F}_t e \right| dt
\]

wherein we have used Eq. (3.22) with \(p = 2\) for the last inequality.

Lastly we turn our attention to estimating \(d^E (Fe, Fe')\), where \(e, e' \in E\) and \(F : E \to E\) is a bundle map covering \(f : M \to M\). As a warm up let us begin with the following flat special case.

Lemma 5.18. Suppose \(M = \mathbb{R}^n\) with the standard metric, \((W, \langle \cdot, \cdot \rangle)\) is a finite dimensional inner product space, and \(E = M \times W\) which is equipped with flat covariant derivative, i.e. \(\Gamma \equiv 0\) in this trivialization. \([\text{We denote } e = (m, w) \in E = M \times W \text{ by } w_m.]\) If \(f \in C^\infty (M, M)\) and \(F \in C^\infty (M, \text{End } (W))\), then \(Fw := \left[ F (m) w \right]_{f(m)}\) is a bundle map covering \(f\) and this map satisfies,

\[
(5.17) \quad d^E (Fw, Fw') \leq \left( \max \left\{ \text{Lip} (f), \left\| \dot{F} (p) \right\| \right\} + \left\| \dot{F}' \right\|_M \left\| w \right\| \right) d^E (w, w') .
\]

Proof. Written in this form we find

\[
(d^E)^2 (Fw, Fw') = \| f (m) - f (p) \|^2 + \| \dot{F} (m) w - \dot{F} (p) w' \|^2
\]

\[
\leq \text{Lip}^2 (f) \| m - p \|^2 + \left( \| \dot{F} (m) w - \dot{F} (p) w \| + \| \dot{F} (p) (w - w') \| \right)^2
\]

\[
\leq \text{Lip}^2 (f) \| m - p \|^2 + \left( \left\| \dot{F}' \right\|_M \| w \| \| m - p \| + \| \dot{F} (p) \| \| w - w' \| \right)^2
\]

\[
= \text{Lip}^2 (f) \| m - p \|^2 + \left( \left\| \dot{F} (p) \right\| \| w - w' \|^2 + \left\| \dot{F}' \right\|_M \| w \| \| m - p \| ^2 + \left\| \dot{F} (p) \right\| \| w - w' \|^2 \right)
\]

\[
+ 2 \left\| \dot{F}' \right\|_M \| w \| \left\| \dot{F} (p) \right\| \| m - p \| \| w - w' \| .
\]
Using $\rho = \max \left\{ \text{Lip}(f), \| \hat{F}(p) \| \right\}$ the above estimate implies,

$$
(dE)^2 (Fw_m, Fw'_p) \leq \left[ \rho^2 + \left\| \hat{F}' \right\|_M \|w\| + 2 \left\| \hat{F}' \right\|_M \|w\| \rho \right] (dE)^2 (w_m, w'_p)
$$

which gives the estimate in Eq. (5.17). \(\square\)

By swapping $w_m$ with $w'_p$ in Eq. (5.17) we of course also have

$$
(dE)^2 (Fw_m, Fw'_p) \leq \max \left\{ \text{Lip}(f), \left\| \hat{F}(m) \right\| \right\} + \left\| \hat{F}' \right\|_M \|w'\| \rho (dE)^2 (w_m, w'_p)
$$

In Theorem 5.22 below, we will show that the analogue of Eq. (5.18) holds in full generality. The following notation will be used in the statement of this theorem.

**Definition 5.19.** Suppose that $f \in C^\infty(M, M)$ and $F : E \to E$ is a bundle map covering $f$. For $v \in T_m M$, let $\nabla_v F \in \text{Hom}(E_m, E_{f(m)})$ be defined by;

$$
\nabla_v F := \frac{d}{dt} \bigg|_0 \left( f \circ \sigma \right)^{-1} F_{\sigma(t)} \left( \left. S \right|_{\sigma(t)} \right)
$$

where $\sigma$ is any $C^1$-curve in $M$ such that $\dot{\sigma}(0) = v$ and $F_{\sigma(t)} := F\big|_{E_{\sigma(t)}}$.

**Lemma 5.20** (Product rule). If $F : E \to E$ is a bundle map covering $f$, $S \in \Gamma(E)$ and $\sigma(t) \in M$, then

$$
\nabla_v F \big|_0 (FS)(\sigma(t)) = (\nabla_{\dot{\sigma}(0)} F) S(m) + F_{\sigma(0)} \nabla_{\dot{\sigma}(0)} S.
$$

**Proof.** This result is easily reduced to the standard product rule matrices and vectors as follows;

$$
\nabla_v F \big|_0 (FS)(\sigma(t)) = \frac{d}{dt} \bigg|_0 \left( \left. (f \circ \sigma)^{-1} \left[ F_{\sigma(t)} S(\sigma(t)) \right] \right) \right)
$$

$$
= \frac{d}{dt} \bigg|_0 \left( \left. (f \circ \sigma)^{-1} \left[ F_{\sigma(t)} \left( f \circ \sigma \right)^{-1} S(\sigma(t)) \right] \right) \right)
$$

$$
= \frac{d}{dt} \bigg|_0 \left( \left. (f \circ \sigma)^{-1} \left[ F_{\sigma(t)} \left( f \circ \sigma \right)^{-1} S(\sigma(t)) \right] \right) \right)
$$

$$
= \frac{d}{dt} \bigg|_0 \left( \left. \left[ F_{\sigma(t)} \left( f \circ \sigma \right)^{-1} S(\sigma(t)) \right] \right) \right)
$$

which is equivalent to Eq. (5.19). \(\square\)
Notation 5.21. Given \( m \in M, \sigma \in C ([0, 1], M) \), \( f \in C^\infty (M, M) \), and a bundle map, \( F : E \to E \), covering \( f \), let
\[
|\nabla F|_m := \sup_{v \in T_m M; |v| = 1} |\nabla_v F|_{op} := \sup_{v \in T_m M; |v| = 1} \sup_{e \in E_m; |e| = 1} |(\nabla_v F) e|,
\]
\[
|\nabla F|_\sigma := \sup_{t \in [0, 1]} |\nabla F|_{\sigma(t)}, \text{ and}
\]
\[
|\nabla F|_M := \sup_{m \in M} |\nabla F|_m.
\]

Theorem 5.22. Let \( F : E \to E \) is a bundle map covering \( f : M \to M \), \( e \in E_m, e' \in E_p \), and \( \sigma \in AC ([0, 1], M) \) be a curve such that \( \sigma (0) = m \) and \( \sigma (1) = p \). Then
\[
d^E (Fe, Fe') \leq (\max (|f_*|_{\sigma}, |F_m|) + |\nabla F|_{\sigma} \cdot |e'\,|) L_\sigma (e, e'),
\]
and in particular,
\[
d^E (Fe, Fe') \leq (\max (\text{Lip} (f), |F_m|) + |\nabla F|_M |e'\,|) d^E (e, e').
\]

Proof. To simplify notation in the proof below let
\[
\rho := \max (|f_*|_{\sigma}, |F_m|),
\]
\[
\tilde{e} := //1 (\sigma)^{-1} e' \in E_m, \text{ and}
\]
\[
A_t := //t (f \circ \sigma)^{-1} F_{\sigma(t)}///t (\sigma) : E_m \to E_{f(m)}.
\]
By Corollary 5.7, it follows that
\[
d^E (Fe, Fe') \leq L_{f \circ \sigma} (Fe, Fe') = \sqrt{\ell^2_M (f \circ \sigma) + //1 (f \circ \sigma)^{-1} Fe' - Fe}^2.
\]
\[
= \sqrt{\ell^2_M (f \circ \sigma) + |A_1 \tilde{e} - A_0 e|^2}.
\]

The first term in the square root is estimated by,
\[
\ell_M (f \circ \sigma) = \int_0^1 |f_* \dot{\sigma} (t)| dt \leq |f_*|_{\sigma} \ell_M (\sigma).
\]
For the second term, we note that
\[
\left| \frac{d}{dt} A_t \right| = //t (f \circ \sigma)^{-1} (\nabla_{\dot{\sigma}(t)} F) //t (\sigma) = |\nabla_{\dot{\sigma}(t)} F| \leq |\nabla F|_{\sigma} |\dot{\sigma} (t)|
\]
and hence
\[
|A_1 - A_0|_{op} = \int_0^1 \left| \frac{d}{dt} A_t \right| dt \leq |\nabla F|_{\sigma} \ell_M (\sigma).
\]

Thus we conclude that
\[
|A_1 \tilde{e} - A_0 e| \leq |A_1 \tilde{e} - A_0 e| + |A_0 (\tilde{e} - e)|
\]
\[
\leq |\nabla F|_{\sigma} \ell_M (\sigma) |\tilde{e}| + |F_m| |\tilde{e} - e|,
\]
\[
= |\nabla F|_{\sigma} \ell_M (\sigma) |e'| + |F_m| //1 (\sigma)^{-1} e' - e|.
\]
Combining the previous estimates then shows,

\[(d^E)^2 (F_e, F_e')\]

\[\leq |f_\ast|^2 L_M^2 (\sigma) + \left[ |\nabla F|_\sigma \ell_M (\sigma) |e'| + |F_m| \right] \left( / 1 (\sigma)^{-1} e' - e \right)^2\]

\[= |f_\ast|^2 L_M^2 (\sigma) + \text{Lip}_\sigma (F) |e'| L_M^2 (\sigma) + |F_m| \left( / 1 (\sigma)^{-1} e' - e \right)^2\]

\[+ 2 |F_m| \left( / 1 (\sigma)^{-1} e' - e \right) \cdot |\nabla F|_\sigma \ell_M (\sigma) |e'|\]

\[\leq \rho L_M^2 (e, e') + \text{Lip}_\sigma (F) |e'| L_M^2 (e, e') + 2 |F_m| |\nabla F|_\sigma |e'| L_M^2 (e, e')\]

\[\leq \rho L_M^2 (e, e') + \text{Lip}_\sigma (F) |e'| L_M^2 (e, e') + 2 \rho |\nabla F|_\sigma |e'| L_M^2 (e, e')\]

\[= (\rho + |\nabla F|_\sigma |e'|)^2 L_M^2 (e, e')\]

which proves Eq. (5.20). Moreover, Eq. (5.20) implies

\[d^E (F_e, F_e') \leq \left( \max (\text{Lip}_f), |F_m| + |\nabla F|_M : |e'| \right) L_M (e, e')\]

and so taking the infimum of this last inequality over \(\sigma \in AC ([0,1], M)\) such that \(\sigma (0) = m\) and \(\sigma (1) = p\) gives (see Corollary 5.7, Eq. (5.21)). \(\square\)

5.2. Metrics on \(TM\). From now we are going to restrict our attention to the case of interest where \(E = TM\) and \(F = f\star\) where \(f \in C^2 (M, M)\).

Before stating the main result in Theorem 5.29 below, let us record that relevant notions of covariant differentiation in this context.

**Definition 5.23 (Vector-fields along \(f\)).** For \(f \in C^\infty (M, M)\), let \(\Gamma_f (TM)\) denote the **vector fields along** \(f\), i.e. \(U \in \Gamma_f (TM)\) iff \(U : M \to TM\) is a smooth function such that \(U (m) \in T_{f(m)} M\) for all \(m \in M\).

**Example 5.24.** If \(Z \in \Gamma (TM)\) and \(f \in C^\infty (M, M)\), then \(f_\ast Z\) and \(Z \circ f\) are both vector fields along \(f\).

**Definition 5.25.** For \(f \in C^\infty (M, M)\), \(U \in \Gamma_f (TM)\), and \(v = v_m \in T_m M\), let \(\nabla_v U \in T_{f(m)} M\) and \(\nabla_v f\star\) be the linear map from \(T_m M\) to \(T_{f(m)} M\) be defined by,

\[\nabla_v U = \left. \frac{d}{dt} \right|_{t = 0} U (\sigma (t)) = \left. \frac{d}{dt} \right|_{t = 0} \left( / t (f \circ \sigma)^{-1} U (\sigma (t)) \right)\]

and

\[\nabla_v f\star = \left. \frac{d}{dt} \right|_{t = 0} \left( / t (f \circ \sigma)^{-1} f_\ast \sigma(t) / t (\sigma) \right)\]

where is any \(C^1\)-curve in \(M\) such that \(\dot{\sigma} (0) = v_m\). [It is easily verified by working in local trivializations of \(TM\) that \(\nabla_v U\) and \(\nabla_v f\star\) are well defined independent of the choice of \(\sigma\) such that \(\dot{\sigma} (0) = v_m\).

**Proposition 5.26 (Chain and product rules).** If \(f \in C^\infty (M, M)\), \(Z \in \Gamma (TM)\), and \(v \in T_m M\), then

\[\nabla_v [Z \circ f] = \nabla_{f_\ast v} Z\]

and

\[\nabla_v [f_\ast Z] = (\nabla_v f\star) Z (m) + f_\ast \nabla_v Z.\]
More generally if \( U \in \Gamma_f (TM) \) and \( g \in C^\infty (M, M) \), then \( U \circ g \in \Gamma_{fof} (M) \), \( g_* U \in \Gamma_{gof} (M) \),

\[
\nabla_v [U \circ g] = \nabla_{g_* v} U, \quad \text{and} \\
\nabla_v [g_* U] = (\nabla_{f_* v} g_*) U (m) + g_{sm} \nabla_v U.
\]

Proof. If \( \sigma (t) \in M \) is chosen so that \( \dot{\sigma} (0) = v_m \), then

\[
\nabla_v [Z \circ f] = \frac{d}{dt}|_0 \left[ \frac{1}{f \circ \sigma}^{-1} (Z \circ f) (\sigma (t)) \right] = \nabla_{f_* v} Z
\]

and

\[
\nabla_v [f_* Z] = \frac{d}{dt}|_0 \left[ \frac{1}{f \circ \sigma}^{-1} f_{\sigma(t)} Z (\sigma (t)) \right]
\]

\[
= \frac{d}{dt}|_0 \left[ \frac{1}{f \circ \sigma}^{-1} f_{\sigma(t)} \frac{1}{f (\sigma)} (Z (\sigma (t))) \right]
\]

\[
= \frac{d}{dt}|_0 \left[ \frac{1}{f \circ \sigma}^{-1} f_{\sigma(t)} \frac{1}{f (\sigma)} Z (m) \right]
\]

\[
+ f_{sm} \frac{d}{dt}|_0 \left[ \frac{1}{f (\sigma)} Z (\sigma (t)) \right]
\]

\[
= (\nabla_{f_* v} Z) (m) + f_* \nabla_v Z.
\]

The more general cases are proved similarly;

\[

abla_v [U \circ g] = \frac{d}{dt}|_0 \left[ \frac{1}{f \circ g \circ \sigma}^{-1} (U \circ g) (\sigma (t)) \right]
\]

\[
= \frac{d}{dt}|_0 \left[ \frac{1}{f \circ g \circ \sigma}^{-1} U (g \circ \sigma) (t) \right]
\]

\[
= \nabla_{g_* v} U
\]

and

\[

abla_v [g_* U] = \frac{d}{dt}|_0 \left[ \frac{1}{f \circ g \circ \sigma}^{-1} g_* U (\sigma (t)) \right]
\]

\[
= \frac{d}{dt}|_0 \left[ \frac{1}{f \circ g \circ \sigma}^{-1} g_* \frac{1}{f (\sigma)} (U (\sigma (t))) \right]
\]

\[
= \frac{d}{dt}|_0 \left[ \frac{1}{f \circ g \circ \sigma}^{-1} g_* \frac{1}{f (\sigma)} U (m) \right]
\]

\[
+ \frac{d}{dt}|_0 [g_{sm} \frac{1}{f (\sigma)} U (\sigma (t))]
\]

\[
= (\nabla_{f_* v} g_*) U (m) + g_{sm} \nabla_v U.
\]

\[
\]

Corollary 5.27. If \( f \in \text{Diff} (M) \), \( Z \in \Gamma (TM) \), and \( v \in T_m M \), then

\[
\nabla_v [\text{Ad}_f Z] = \left( \nabla_{f^{-1}_v f_*} \right) Z (f^{-1} (m)) + f_* \nabla_{f^{-1}_v Z}.
\]

Proof. Since \( \text{Ad}_f Z = (f_* Z) \circ f^{-1} \) with \( f_* Z \in \Gamma_f (TM) \), it follows by first applying Eq. (5.24) and then Eq. (5.25) that

\[
\nabla_v [\text{Ad}_f Z] = \nabla_{f^{-1}_v f_*} (f_* Z) = \left( \nabla_{f^{-1}_v f_*} \right) Z (f^{-1} (m)) + f_* \nabla_{f^{-1}_v Z}.
\]
Definition 5.28. Let $d^{TM}: TM \times TM \to [0, \infty)$ be the metric on $TM$ associated to the Riemannian metric on $E = TM$ with the given fiber Riemannian metric $g$.

In this setting,

$$\max (\text{Lip}(f) \cdot |F_m|) = \max (\text{Lip}(f) \cdot |f_m|) = \text{Lip}(f)$$

and hence the next theorem is an immediate consequence of Theorem 5.22.

Theorem 5.29 ($d^{TM}(f_*v_m, f_*w_p)$ estimates). Let $v_m, w_p \in TM$ and $f \in C^2(M, M)$ and for any path $\sigma \in AC([0, 1], M)$ with $\sigma(0) = v_m$ and $\sigma(1) = w_p$, let

$$L_{\sigma}(v_m, w_p) := \sqrt{\ell_M(\sigma)^2 + |\nu_1(\sigma)^{-1} w_p - v_m|^2}.$$  \hfill (5.26)

Then

$$d^{TM}(f_*v_m, f_*w_p) \leq |f_*|_\sigma + ||\nabla f_*|_{\sigma} \cdot |w_p|| \cdot L_{\sigma}(v_m, w_p)$$  \hfill (5.27)

and consequently, \hfill (5.28)

$$d^{TM}(f_*v_m, f_*w_p) \leq (\text{Lip}(f) + |\nabla f_*| M \cdot |w_p|) \cdot d^{TM}(v_m, w_p).$$

6. First order derivative estimates

6.1. $\nabla \nu_{t_*} -$ estimates. Suppose that $W_t \in \Gamma(TM)$ and $\nu_t \in C^\infty(M, M)$ are as in Notation 2.24. Our next goal is to estimate the local Lipschitz-norm of $\nu_{t_*}$. We will do this using Theorem 5.29 which requires us to estimate $\nabla \nu_{t_*}$. We begin by finding the differential equation solved by $\nabla \nu_{t_*}$.

Proposition 6.1. If $W_t \in \Gamma(TM)$ and $\nu_t \in C^\infty(M, M)$ are as in Notation 2.24, $m \in M$, and $v_m, \xi_m \in T_m M$, then $(\nabla_{v_m} \nu_{t_*}) \xi_m$ satisfies the covariant differential equation:

$$\nabla_t (\nabla_{v_m} \nu_{t_*}) \xi_m = \langle \nabla W_t \rangle \left[ (\nabla_{v_m} \nu_{t_*}) \xi_m \right] + \langle \nabla^2 W_t \rangle [\nu_{t_*} v_m \otimes \nu_{t_*} \xi_m] + R(W_t(\nu_t(m)), \nu_{t_*}v_m) \nu_{t_*} \xi_m.$$  \hfill (6.1)

Proof. Let $\sigma(s)$ be a smooth curve in $M$ such that $v_m := \sigma'(0)$ and define $\xi(s) := ||s(\sigma)\cdot \nu_{t_*} \xi_m\cdot ||$ with this notation we have

$$\nabla = \frac{d}{ds} \left| \nu_{t_*} \xi(s) \right| = \frac{d}{ds} \left| \nu_{t_*} \xi(s) \right| \left( ||s(\sigma) \cdot \nu_{t_*} \xi(s) \right)$$

$$= \frac{d}{ds} \left| \nu_{t_*} \xi(s) \right| \left( ||s(\sigma) \cdot \nu_{t_*} \xi(s) \right) = (\nabla_{v_m} \nu_{t_*}) \xi_m.$$  \hfill (6.2)

The next inequality may be localized if necessary. The point is we may assume that $\ell(\sigma) \leq d_{TM}(v_m, w_p)$ and so we need to compute $\text{Lip}(f)$ and $\text{Lip}(f_s)$ over the ball, $B(m, d_{TM}(v_m, w_p))$. 
Using the relationship of curvature to the commutator of covariant derivatives,
\[
[\nabla_t, \nabla_s] = R \left( \frac{d}{dt} \nu_t (\sigma (s)), \frac{d}{ds} \nu_t (\sigma (s)) \right) = R \left( W_t (\nu_t (\sigma (s))), \nu_t \sigma' (s) \right),
\]
it follows that
\[(6.3)\]
\[
\nabla_t \nabla_s [\nu_t \xi (s)] = \nabla_s \nabla_t [\nu_t \xi (s)] + R \left( W_t (\nu_t (\sigma (s))), \nu_t \sigma' (s) \right) \nu_t \xi (s).
\]
By Proposition 2.25 and the product rule for covariant derivatives the first term in Eq. (6.3) may be written as
\[\nabla_s \nabla_t [\nu_t \xi (s)] = \nabla_s [\nabla_{\nu_t \xi (s)} W_t] = (\nabla^2 W_t) [\nu_t \sigma' (s) \otimes \nu_t \xi (s)] + (\nabla W_t) \nabla_s \nu_t \xi (s).\]
Combining Eqs. (6.2)–(6.4) gives,
\[\nabla_t (\nabla_{\nu_m} \nu_t) \xi_m = \nabla_t \nabla_s [\nu_t \xi (s)] = \nabla_s [\nabla_{\nu_t \xi (s)} W_t] = (\nabla^2 W_t) [\nu_t \sigma' (s) \otimes \nu_t \xi (s)] + (\nabla W_t) \nabla_s \nu_t \xi (s),\]
which is the same as Eq. (6.1). \(\Box\)

Recall from Notations 1.3 and 1.5 (also see Example 1.6) that
\[(6.5)\]
\[
H_m (W_t) = |\nabla^2 W_t|_m + |R (W_t, \bullet)|_m
\]
and for a closed interval, \(J \subset [0, T]\), that
\[(6.6)\]
\[
H^* (W_t)_J = \int_J H_M (W_t) \, dt = \int_J \sup_{m \in M} H_m (W_t) \, dt.
\]

**Corollary 6.2** (\(|\nabla \nu_t|_m\) -estimate). If \(W_t \in \Gamma (TM)\) and \(\nu_t \in C^\infty (M, M)\) are as in Notation 2.2, and we let
\[(6.7)\]
\[
k_J (m) := \int_J |\nabla W|_{\nu^*_t (m)} \, d\tau \leq |\nabla W|_J^*, \text{ and}
\]
\[(6.8)\]
\[
K_J (m) := \int_J H_{\nu^*_t (m)} (W_\tau) d\tau \leq H (W_\tau)_J^*,
\]
then
\[(6.9)\]
\[
|\nabla \nu_t|_m \leq e^{k_J (s, t) (m)} \left[ |\nabla \nu_s|_m + |\nu_{ss}|_m^2 \int_{J(s, t)} H_{\nu^*_t (m)} (W_\tau) e^{k_J (s, \tau) (m)} d\tau \right]
\]
\[(6.10)\]
\[
|\nabla \nu_{ss}|_m \leq e^{k_J (s, t) (m)} |\nabla \nu_s|_m + e^{2k_J (s, t) (m)} K_{J(s, t)} (m) |\nu_{ss}|_m^2.
\]
Proof. By Theorem 5.29 with and so Eq. (6.10) reduces to Eq. (6.11).

Lastly if \( \nu_s = Id_M \), then the above estimate reduces to

\[
|\nabla \nu_t|_m \leq e^{k_{J(s,t)}(m)} \cdot \int_{J(s,t)} H_{\nu_t} W_t \cdot e^{k_{J(s,r)}(m)} d\tau
\]

(6.11)

\[
\leq e^{2k_{J(s,t)}(m)} \cdot K_{J(s,t)}(m)
\]

and in particular,

(6.13)

\[
|\nabla \nu_t|_m \leq e^{2|W|_{J(s,t)}^*} H (W)^* J(s,t).
\]

Proof. To shorten notation in this proof, let

\[
(TM)\nu_t = H_{\nu_t}(W_t) := |\nabla^2 W_t|_{\nu_t(m)} + |R(W_t, \bullet)|_{\nu_t(m)}.
\]

Starting with Eq. (6.11) while using the estimate in Eq. (2.38) allows us to easily conclude that

\[
|\nabla^2 (\nabla W_m \nu_t)| \leq |\nabla^2 W_t|_{\nu_t(m)} |\nabla W_m|_{\nu_t(m)} |\nu_t|_m
\]

\[
+ |R(W_t, \nu_t(m), \nu_t v_m)| |\nu_t|_m.
\]

\[
\leq |\nabla W_t|_{\nu_t(m)} |\nabla W_m|_{\nu_t(m)} + e^{k_{J(s,t)}(m)} h_t |\nu_t|_m^2 \cdot |v_m|.
\]

It follows by the Bellman-Gronwall inequality in Corollary 9.3 of the appendix that

\[
|\nabla \nu_t|_m \leq e^{k_{J(s,t)}(m)} |\nabla W|_{\nu_t(m)}^* d\tau |\nabla \nu_t|_m
\]

\[
+ \int_{J(s,t)} e^{k_{J(s,r)}(m)} H_{\nu_t} W_t \cdot e^{k_{J(s,r)}(m)} h_t |\nu_t|_m^2 d\tau
\]

\[
= e^{k_{J(s,t)}(m)} |\nabla \nu_t|_m + \int_{J(s,t)} e^{k_{J(s,r)}(m)} H_{\nu_t} W_t \cdot e^{k_{J(s,r)}(m)} h_t |\nu_t|_m^2 d\tau
\]

\[
\leq e^{k_{J(s,t)}(m)} |\nabla \nu_t|_m + e^{2k_{J(s,t)}(m)} \int_{J(s,t)} h_t |\nu_t|_m^2 d\tau.
\]

Lastly if \( \nu_s = Id_M \) then \( \nu_s = Id_{TM} \) in which case \( |\nu_s|_m^2 = 1 \) and \( \nabla \nu_s = 0 \) and so Eq. (6.10) reduces to Eq. (6.11).

\[
\square
\]

\textbf{Corollary 6.3.} If \( W_t \in \Gamma (TM) \) and \( \nu_t \in C^\infty (M, M) \) are as in Notation 2.27 and further assuming \( \nu_0 = Id_M \), then

\[
d^TM(\nu_t v_m, \nu_t w_p) \leq e^{2|W|_t^*} (1 + H (W_t)^* \cdot |w_p|) d^TM (v_m, w_p)
\]

Proof. By Theorem 5.29 with \( f = \nu_t \) along with Corollaries 2.26 and Corollary 6.2 we find,

\[
d^TM(\nu_t v_m, \nu_t w_p) \leq (\text{Lip} (\nu_t) + |\nabla \nu_t|_M \cdot |w_p|) d^TM (v_m, w_p)
\]

\[
\leq (e^{2|W|_t^*} + e^{2|W|_t^*} H (W_t)^* \cdot |w_p|) d^TM (v_m, w_p)
\]

\[
\leq e^{2|W|_t^*} (1 + H (W_t)^* \cdot |w_p|) d^TM (v_m, w_p).
\]
The next corollary is the special case of Corollaries 6.2 and 6.3 when \( W_t = X \) is a time independent vector field.

**Corollary 6.4** \((\| \nabla e^tX \|_M \)-estimate). If \( X \) is a complete vector field and

\[
\begin{align*}
\kappa_t (X, m) &:= \int_0^t |\nabla X|_{e^{\tau}X(m)} \, d\tau, \\
\| \nabla e^tX \|_m &\leq e^{\kappa_t (X, m)} \cdot \int_0^t H_{e^{\tau}X(m)} (X) \, d\tau \\
&\leq e^{2\kappa_t (X, m)} \cdot \int_0^t H_{e^{\tau}X(m)} (X) \, d\tau
\end{align*}
\]

and, for \( v_m, w_p \in TM \),

\[
dTM (e^X v_m, e^X w_p) \leq e^{2|\nabla X|_M [1 + H_M (X) |w_p|]} dTM (v_m, w_p).
\]

**Notation 6.5.** For \( X \in \Gamma (TM) \) and \( m \in M \), let

\[
\bar{H}_m (X) := \int_1^0 \| \nabla X \|_{e^{-\tau}X(m)} (X) \, d\tau \leq H_M (X).
\]

**Proposition 6.6.** If \( X, Z \in \Gamma (TM) \) and \( X \) is complete, then

\[
\begin{align*}
|\nabla [\text{Ad}_e X Z]|_m &\leq e^{2\kappa_1 (-X, m)} |\nabla Z|_{e^{-X(m)}} + \bar{H}_m (X) e^{3\kappa_1 (-X, m)} |Z|_{e^{-X(m)}} \\
&\leq e^{3\kappa_1 (-X, m)} \left[ |\nabla Z|_{e^{-X(m)}} + \bar{H}_m (X) |Z|_{e^{-X(m)}} \right]
\end{align*}
\]

where, from Eq. \((6.14)\),

\[
\kappa_1 (-X, m) = \int_0^1 |\nabla X|_{e^{-\tau}X(m)} \, d\tau.
\]

[It is possible, using “transport methods,” to replace \( e^{3\kappa_1 (-X, m)} \) by \( e^{2\kappa_1 (-X, m)} \) in the previous inequalities but we do not bother doing so in this paper.]

**Proof.** As a consequence of the flow property of \( e^{tX} \) and a simple change of variables, it is useful to record;

\[
k_{1-s} (X, e^{-X}(m)) = \int_0^{1-s} |\nabla X|_{e^{-(1-s)}X(m)} \, d\tau = \int_s^1 |\nabla X|_{e^{-uX(m)}} \, du
\]

for any \( s \in [0, 1] \). When \( \sigma = 0 \) this identity may be stated as

\[
k_1 (X, e^{-X}(m)) = k_1 (-X, m).
\]

With this preparation in hand, we now go to the proof of the proposition.
By Corollary 5.27 with \( f = e^X \),
\[
\nabla_{v_m} [\text{Ad}_{e^X} Z] = \nabla_{v_m} [e^X [Z \circ e^{-X}]]
\]
\[
= \left( \nabla_{e^X \cdot v_m} e^X \right) [Z \circ e^{-X} (m)] + e^X \left[ \nabla_{e^X \cdot v_m} Z \right]
\]
and so
\[
|\nabla_{v_m} [\text{Ad}_{e^X} Z]| \leq \left( |\nabla e^X|_{e^{-X}(m)} |Z|_{e^{-X}(m)} + |e^X|_{e^{-X}(m)} |\nabla Z|_{e^{-X}(m)} \right) \cdot |e^{-X}|_{m} |v_m|.
\]
By Corollary 2.27,
\[
|e^{-X}|_{m} \leq e^{\int_0^1 |\nabla X|_{e^{-X}(m)} d\tau} = e^{k_1(-X,m)}
\]
and from this inequality with \( X \) replaced by \(-X\) and \( m \) by \( e^{-X}(m) \) we also have (using Eq. (6.22) with \( t = 1 \)) that
\[
|e^X|_{e^{-X}(m)} \leq e^{k_1(X, e^{-X}(m))} = e^{k_1(-X,m)}.
\]
Similarly from Corollary 6.4 with \( m \) replaced by \( e^{-X}(m) \),
\[
|\nabla e^X|_{e^{-X}(m)} \leq e^{k_1(X, e^{-X}(m))} \cdot \int_0^1 H_{e^{-X}(m)} e^{k_1(X, e^{-X}(m))} d\tau
\]
\[
= e^{k_1(-X,m)} \int_0^1 H_{e^{-X}(m)} e^{k_1(X, e^{-X}(m))} d\tau
\]
\[
= e^{k_1(-X,m)} \int_0^1 H_{e^{-X}(m)} e^{k_1-s(X, e^{-X}(m))} ds
\]
\[
= e^{k_1(-X,m)} \int_0^1 H_{e^{-X}(m)} e^{\int_s^1 |\nabla X|_{e^{-X}(m)} du} ds
\]
\[
\leq e^{2k_1(-X,m)} \int_0^1 H_{e^{-X}(m)} e^{\int_s^1 |\nabla Z|_{e^{-X}(m)} du} ds
\]
Combining these inequalities shows,
\[
|\nabla_{v_m} [\text{Ad}_{e^X} Z]| \leq \left( e^{3k_1(-X,m)} \cdot H_m(X) |Z|_{e^{-X}(m)} + e^{2k_1(-X,m)} |\nabla Z|_{e^{-X}(m)} \right) |v_m|
\]
from which Eq. (6.18) immediately follows. \( \square \)

### 7. First order distance estimates

The main goal of this section is to estimate (see Theorem 7.2) the distance between the differentials of \( \mu^X_{t,0} \) and \( \mu^Y_{t,0} \). To do so we will again need to estimate the time derivative of the interpolator defined Eq. (2.46) above.

**Proposition 7.1.** Let \([0, T] \ni t \to X_t, Y_t \in \Gamma(TM)\) be smooth complete time dependent vector fields on \( M \) and \( \mu^X \) and \( \mu^Y \) be their corresponding
flows. If \( 0 < t \leq T, [0, t] \ni s \to \Theta_s := \mu_{t,s}^X \circ \mu_s^Y \) is the interpolator defined in Eq. (2.46), and \( v_m \in T_m M \), then

\[
\frac{\nabla_m \Theta_{ss}}{ds} \leq e^{2|\nabla X|_t^2 + |\nabla Y|_t^2} \cdot (H(X)_t^* |Y_s - X_s|_M + |\nabla [Y_s - X_s]|_M),
\]

where (as in Eq. (5.16) with \( f = \Theta_s \) and \( F = \Theta \))

\[
\left| \frac{\nabla_m \Theta_{ss}}{ds} \right| = \sup \left\{ \left| \frac{\nabla_m ((\Theta_s)_s v_m)}{ds} \right| : v \in TM \text{ with } |v| = 1 \right\}
\]

and (as in Notation 1.3) \( H_m (X_t) = |\nabla^2 X|_m + |R(X_t, \bullet)|_m \).

**Proof.** Choose \( \sigma (\tau) \in M \) so that \( \dot{\sigma} (0) = v_m \). Then by the properties of the Levi-Civita covariant derivatives, the formula for \( \Theta_s'(m) \) in Eq. (2.50), along with the product and chain rule in Proposition 5.26, it follows that

\[
\nabla_m [(\Theta_s)_s v_m] = \frac{d}{ds} \nabla_m [0 \Theta_s (\sigma (\tau))] = \frac{d}{d\tau} \nabla_m [0 \Theta_s (\sigma (\tau))]
\]

\[
= \nabla_m \left[ (\mu_{t,s})_s (Y_s - X_s) \circ \mu_{s,0}^Y (\sigma (\tau)) \right]
\]

\[
= \nabla_m \left[ (\mu_{t,s})_s (\mu_s^Y (Y_s - X_s)) \right]
\]

\[
= \nabla_m \left[ (\mu_{t,s})_s (Y_s - X_s) \right]
\]

\[
+ (\mu_{t,s})_s \nabla_m \left[ (\mu_{t,s})_s (Y_s - X_s) \right]
\]

and consequently,

\[
\left| \frac{\nabla_m [(\Theta_s)_s v_m]}{ds} \right| \leq \left( \left| \nabla_m \mu_{t,s}^X \right|_M |Y_s - X_s|_M + \left| \mu_{t,s}^X \right|_M |\nabla_m (Y_s - X_s)|_M \right) \left| \mu_{s,0}^Y \right|_M |v_m|.
\]

By Eq. (6.13) of Corollary 6.2 with \( v_t = \mu_{t,s}^X \),

\[
\left| \nabla_m \mu_{t,s}^X \right|_M \leq e^{2\int_0^t |\nabla X_s| \mu_{t,s}^X d\tau} \left| H(X)_t^* \right|_M \leq e^{2\int_0^t |\nabla X_s| \mu_{t,s}^X d\tau} \left| H(X)_t^* \right|_M.
\]

Using the estimate in Eq. (2.51) twice shows,

\[
\left| \nabla_m \mu_{t,s}^X \right|_M \leq e^{\int_0^t |\nabla X_s| \mu_{t,s}^X d\tau} \leq e^{2\int_0^t |\nabla X_s| \mu_{t,s}^X d\tau}, \text{ and }
\]

\[
\left| \mu_{s,0}^Y \right|_M \leq e^{\int_0^t |\nabla Y_s| \mu_{s,0}^Y d\tau} \leq e^{2\int_0^t |\nabla Y_s| \mu_{s,0}^Y d\tau}.
\]

Combining the last four inequalities yields and taking the supremum of the result over \( v_m \in TM \) with \( |v_m| = 1 \) yields Eq. (7.1). \( \square \)

**Theorem 7.2.** If \( [0, T] \ni t \to X_t, Y_t \in \Gamma(TM) \) are smooth complete time dependent vector fields on \( M \) and \( \mu^X \) and \( \mu^Y \) be their corresponding flows,
Corollary 8.2. If
\[ d \left( \mu^Y_{t,0} \right) \cdot \left( \mu^Y_{t,0} \right) \leq e^{2|\nabla X|_t^* + |\nabla Y|_t^*} \cdot (1 + H (X)_t^*) |Y - X|_t^* + |\nabla [Y - X]|_t^* , \]

Proof. Integrating the estimate in Eq. (7.1) shows
\[ \int_0^t |\Theta'_{s}|_M ds \leq e^{2|\nabla X|_t^* |Y - X|_t^*} \leq e^{2|\nabla X|_t^* + |\nabla Y|_t^*} |Y - X|_t^* \]
and integrating the estimate in Eq. (7.1) shows
\[ \int_0^t \frac{\nabla |\Theta'_s|_M}{ds} ds \leq e^{2|\nabla X|_t^* + |\nabla Y|_t^*} \cdot (H (X)_t^* |Y - X|_t^* + |\nabla [Y - X]|_t^* ) . \]

Adding these estimates while making use of an appropriately time scaled version of Eq. (5.13) of Proposition 5.16 with \( E = TM, f_s = \Theta_s, \) and \( F_s = \Theta_{ss} \) completes the proof of Eq. (7.2).

Corollary 7.3. Let \( J = [0, T] \ni t \to Y_t \in \Gamma (TM) \) be a smooth complete time dependent vector field on \( M \) and \( \mu^Y \) be the corresponding flow. Then for \( t > 0 \) (for notational simplicity)
\[ d \left( \mu^Y_{t,0} \right) \cdot Id_{TM} \leq e^{2|\nabla Y|_t^*} \cdot (|Y|_t^* + |\nabla Y|_t^* ) . \]

Proof. Applying Theorem 7.2 with \( X \equiv 0 \) gives Eq. (7.3).

8. First order logarithm estimates

The main purpose of this section is to give a first order version (see Theorem 8.4 below) of the logarithm control estimate in Theorem 4.11. Before doing so we will first need to develop a few more auxiliary estimates.

Proposition 8.1. If \( C (\cdot) \in C ([0, T], F^{(\kappa)} (\mathbb{R}^d)) \) and \( Z \in \Gamma (TM) \), then for \( 0 \leq s \leq 1, \)
\[ \left| \nabla \left[ \text{Ad}_{e^{sV_C(0)}} Z \right] \right|_M \leq e^{3|\nabla V^{(s)}|_M |C(t)|} \cdot \left| H_M \left( V^{(s)} \right) \right|_M |Z|_M |C(t)| + |\nabla Z|_M , \]
where \( H_M (V^{(s)}) \) was defined in Eq. (1.13) of Definition 1.24.

Proof. This result follows directly as an application of Proposition 6.6 with \( X_t = -sV_C(t) \).

Corollary 8.2. If \( C \in C^1 ([0, T], F^{(\kappa)} (\mathbb{R}^d)) \) and \( W_t^C \) is given as in Eq. (4.4), then
\[ \left| \nabla W^C_t \right|_t^* \leq c^3 \left| \nabla V^{(s)} \right|_M |C|_{\infty, t} \cdot \left( H_M \left( V^{(s)} \right) \right) \left| \nabla V^{(s)} \right|_M |C|_{\infty, t} + \left| \nabla V^{(s)} \right|_M \right| C^* t . \]
Moreover, there exists \( c(\kappa) < \infty \) such that, whenever \( \xi \in C^1 ([0, T], F^{(\kappa)} (\mathbb{R}^d)) \) and \( C^\xi (t) = \log \left( g^\xi (t) \right) , \)
\[ \left| \nabla W^C_{\xi t} \right|_t^* \leq K_t \cdot \left( H_M \left( V^{(s)} \right) \right) \left| \nabla V^{(s)} \right|_M |Q|_{t, \xi} \left( N_t^\xi \left( \xi \right) \right) + \left| \nabla V^{(s)} \right|_M \]
where
\[ K_t := e^{c(\xi)\| V^{(\xi)} \|_{M} Q_{[1,\kappa]}(N^*_T(\hat{\xi})) Q_{[1,\kappa]}(N^*_T(\hat{\xi}))}. \]

**Proof.** Let \( \tau \in [0, t] \) and \( s \in [0, 1] \). Applying the estimate in Eq. (8.1) with \( Z = V_{C(\tau)} \) implies,
\[
\left| \nabla \left[ \text{Ad}_{e^\tau V_{C(\tau)}} V_{C(\tau)} \right] \right|_M \leq e^{3\| V^{(\xi)} \|_{M} |C(\tau)|} \left( H_M \left( V^{(\xi)} \right) \left| V^{(\xi)} \right|_M \left| C(\tau) \right| + \left| \nabla V^{(\xi)} \right|_M \left| \dot{C}(\tau) \right| \right) \leq e^{3\| V^{(\xi)} \|_{M} |C|_{\infty,t}} \left( H_M \left( V^{(\xi)} \right) \left| V^{(\xi)} \right|_M \left| C \right|_{\infty,t} + \left| \nabla V^{(\xi)} \right|_M \left| \dot{C}(\tau) \right| \right).
\]
Integrating this inequality on \( s \in [0, 1] \) and \( \tau \in [0, t] \), while using
\[
\left| \nabla W^* \right|_t^s = \int_0^T \left| \nabla W^* \right|_t \, d\tau \leq \int_0^T \left[ \int_0^1 \left| \nabla \left[ \text{Ad}_{e^\tau V_{C(\tau)}} V_{C(\tau)} \right] \right| ds \right] \, d\tau,
\]
gives Eq. (8.2).

Now suppose that \( \xi \in C^1([0, T], F^{(\kappa)}(\mathbb{R}^d)) \) and \( C^\xi(t) = \log(g^\xi(t)) \). Then from Eq. (3.31),
\[
\left| C^\xi(\cdot) \right|_{\infty,t} \lesssim Q_{[1,\kappa]}(N^*_T(\hat{\xi}))
\]
and by the estimates in Eqs. (3.29) and (3.21),
\[
\left| \dot{C} \right|_t^s \lesssim Q_{[1,\kappa]}(N^*_T(\hat{\xi})).
\]
Using the previous two estimates in Eq. (8.2) proves Eq. (8.3). \( \square \)

**Corollary 8.3.** If \( \xi \in C^1([0, T], F^{(\kappa)}(\mathbb{R}^d)) \), \( C^\xi(t) = \log(g^\xi(t)) \), and \( U_1^\xi \in \Gamma(TM) \) is the difference vector field in Eq. (4.6), then there exist \( c(\kappa) < \infty \) such that
\[
\left| \nabla U^* \right|^s_T \lesssim \left[ \left( c^0 \left( V^{(\xi)} \right) H_M \left( V^{(\xi)} \right) Q_{[1,\kappa]}(N^*_T(\hat{\xi})) + C^1 \left( V^{(\xi)} \right) \right) \cdot e^{c(\kappa)|\nabla V^{(\xi)}|_M Q_{[1,\kappa]}(N^*_T(\hat{\xi}))} \cdot Q_{[1,\kappa]}(N^*_T(\hat{\xi})) \right].
\]

**Proof.** Let \( t \in [0, T] \) and \( s \in [0, 1] \). The estimate in Eq. (8.1) with
\[
Z = V_{\pi^*_{\kappa} \left[ C^\xi(t) \cdot [s, \text{ad}_{C^\xi(t)} \hat{\xi}(t)] \right]} \]
becomes
\[
\left| \nabla \left[ \text{Ad}_{e^\tau V_{C^\xi(t)}} V_{\pi^*_{\kappa} \left[ C^\xi(t) \cdot [s, \text{ad}_{C^\xi(t)} \hat{\xi}(t)] \right]} \right] \right|_M \leq e^{3\| V^{(\xi)} \|_{M} |C^\xi(t)| \cdot [\alpha(s, t) + \beta(s, t)]} \leq e^{3\| V^{(\xi)} \|_{M} |C^\xi|_{\infty,T} \cdot [\alpha(s, t) + \beta(s, t)].}
Recalling from Eq. (3.21) that

\[ \alpha(s, t) = H_M \left( V^{(\kappa)} \right) \left| C_\xi \right|_{\infty, T} \cdot \left| V_{\pi > \kappa} \left[ C_\xi(t), u \left( s, a \right) \varphi_{C_\xi(t)} \right] \right| \cdot \]

and

\[ \beta(s, t) = \left| \nabla V_{\pi > \kappa} \left[ C_\xi(t), u \left( s, a \right) \varphi_{C_\xi(t)} \right] \right| \cdot \]

In this notation we have

\[ |\nabla U|_T = e^{3|V^{(\kappa)}|_{\infty, T}} \int_0^T dt \int_0^1 ds (1 - s) [\alpha(s, t) + \beta(s, t)] \]

where according to Lemma 4.9

\[ \int_0^T dt \int_0^1 ds (1 - s) \alpha(s, t) \lesssim H_M \left( V^{(\kappa)} \right) C_\xi \left( V^{(\kappa)} \right) \left| C_\xi \right|_{\infty, T} \cdot Q(\kappa, 2\kappa) \left( N^*_T \left( \hat{\xi} \right) \right) \]

and

\[ \int_0^T dt \int_0^1 ds (1 - s) \beta(s, t) \lesssim C_\xi \left( V^{(\kappa)} \right) Q(\kappa, 2\kappa) \left( N^*_T \left( \hat{\xi} \right) \right) \]

The proof is now completed by combining these estimate with the estimate for \( C_\xi(\cdot) \) in Corollary 8.2 into the previous inequality gives the stated estimate in Eq. (8.7).

\[ \square \]

**Theorem 8.4 (Comparing differentials).** If \( \xi \in C^1 \left( [0, T], F^{(\kappa)} \left( \mathbb{R}^d \right) \right) \), then

\[ d^T_M \left( \left( V^\xi \right)_t, e^{V_{\log \left( \theta \left( T \right) \right)}} \right) \leq K \cdot Q(\kappa, 2\kappa) \left( N^*_T \left( \hat{\xi} \right) \right) \]

where

\[ K = K \left( T, \left| V^{(\kappa)} \right|_{\infty, T}, \left| \nabla V^{(\kappa)} \right|_{\infty, T}, H_M \left( V^{(\kappa)} \right), N^*_T \left( \hat{\xi} \right) \right) \]

is a (fairly complicated) increasing function of each of its arguments.

**Proof.** Our proof of this result is similar to the proof of Theorem 4.11 except that we will being using Theorem 7.2 in place of Theorem 2.29. Applying Theorem 7.2 with \( X_t = V_{\xi(t)} \) and \( Y_t = W^{C_\xi}_{\xi(t)} \) shows

\[ d^T_M \left( \left( V^\xi \right)_t, e^{V_{\log \left( \theta \left( T \right) \right)}} \right) \leq e^{2|\nabla V_{\xi(t)}|_t^* + |\nabla W_{C_\xi}|_t^*} \left( 1 + H \left( V^\xi \right)_t^* \right) \left| U^\xi \right|_t^* + |\nabla U^\xi|_t^* \]

\[ \leq e^{2|\nabla V^{(\kappa)}|_{\infty, T} \left| \xi \right|_t^* + |\nabla W^{C_\xi}|_t^*} \left( 1 + H_M \left( V^{(\kappa)} \right) \right) \left| \xi \right|_t^* \cdot \left| U^\xi \right|_t^* + |\nabla U^\xi|_t^* \cdot \]

Recalling from Eq. (3.21) that \( \left| \xi \right|_t^* \lesssim Q[1, \kappa] \left( N^*_T \left( \hat{\xi} \right) \right) \) and substituting the estimates for \( |\nabla W^{C_\xi}|_t^* \) in Corollary 8.2, \( |U^\xi|_t^* \) in Theorem 4.10, and \( |\nabla U|_T^* \) in Corollary 8.3 into the previous inequality gives the stated estimate in Eq. (8.7). \( \square \)
Corollary 8.5. If $A, B \in F^{(\kappa)}(\mathbb{R}^d)$, then

$$d_M^{TM}(e_{V_B}^*, Id_{TM}) \leq \left[ V^{(\kappa)}_{\kappa} \bigg|_M + \left| \nabla V^{(\kappa)} \right|_{TM} e^{(\nabla V^{(\kappa)}_{\kappa})_{Mj}} \right] |B|,$$

and there exists $K_1$ such that

$$d_M^{TM} \left( e_{V_B}^* \circ e_{V_A}^*, e_{*}^{V_{\log(e_A e_B)}} \right) \leq K_1 \cdot N(A) N(B) Q_{(\kappa-1,2(\kappa-1))} (N(A) + N(B)).$$

where

$$K_1 = K_1 \left( \left| V^{(\kappa)}_{\kappa} \right|_{M} + \left| \nabla V^{(\kappa)}_{\kappa} \right|_{TM} H_M \left( V^{(\kappa)}_{\kappa} \right), N(A) \lor N(B) \right).$$

Proof. From Corollary 7.3 with $Y_t = V_B$ we find

$$d_M^{TM}(e_{V_B}^*, Id_{TM}) \leq e^{\left| \nabla V^{(\kappa)}_{\kappa} \right|_{TM} \cdot (|V_B|^* + \left| \nabla V_B \right|^*)} \leq \left[ V^{(\kappa)}_{\kappa} \bigg|_M + \left| \nabla V^{(\kappa)} \right|_{TM} e^{(\nabla V^{(\kappa)}_{\kappa})_{Mj}} \right] |B|.$$ 

The proof of the second inequality is completely analogous to the proof of the second inequality in Corollary 4.15 with the exception that we now use Theorem 8.4 in place of Theorem 4.11 and we must replace $V_{[A,B]_{\kappa}} = [V_A, V_B]$ by $\nabla V_{[A,B]_{\kappa}} = \nabla [V_A, V_B]$ and $C^0(V^{(\kappa)})$ by $C^1(V^{(\kappa)})$ appropriately.

\[ \square \]

9. Appendix: Gronwall Inequalities

This appendix gathers a few rather standard differential inequalities that are used in the body of the paper.

9.1. Flat space Gronwall inequalities.

Proposition 9.1 (A Gronwall Inequality). Suppose that $\psi : [0, T] \rightarrow \mathbb{R}$ is absolutely continuous, $u \in C([0, T], \mathbb{R})$, and $h \in L^1([0, T])$. If

$$\dot{\psi}(t) \leq u(t) + h(t) \psi(t) \text{ for a.e. } t, \tag{9.1}$$

then

$$\psi(t) \leq \psi(0) e^{\int_0^t h(s) \, ds} + \int_0^t e^{\int_\tau^t h(s) \, ds} \cdot u(\tau) \, d\tau.$$

Proof. Here is the short proof of this standard result for the reader’s convenience. Let $H(t) := \int_0^t h(s) \, ds$ so that $H$ is absolutely continuous with $H(t) = h(t)$ for a.e. $t$. We then have, for a.e. $t$, that

$$\frac{d}{dt} \left[ e^{-H(t)} \psi(t) \right] = e^{-H(t)} \left[ \dot{\psi}(t) - h(t) \psi(t) \right] \leq e^{-H(t)} u(t).$$

Integrating this equation gives

$$e^{-H(t)} \psi(t) - \psi(0) \leq \int_0^t e^{-H(\tau)} u(\tau) \, d\tau.$$
Multiplying this inequality by $e^{H(t)}$ completes the proof as $H(t) = \int_\tau^t h(s) \, ds$.

The following corollary is the form of Gronwall’s inequality which is most useful to us.

**Corollary 9.2.** Let $(V, |\cdot|)$ be a normed space, $-\infty < a < b < \infty$, and $[a, b] \ni t \rightarrow C(t) \in V$ be a $C^1$-function of $t$. If there exists continuous functions, $h(t)$ and $g(t)$, such that
\begin{equation}
(9.2) \quad |\dot{C}(t)| \leq h(t) |C(t)| + g(t) \quad \forall \ t \in [a, b],
\end{equation}
then for any $s, t \in [a, b]$,
\begin{equation}
(9.3) \quad |C(t)| \leq |C(s)| e^{\int_{s}^{t} h(\sigma) \, d\sigma} + \int_{s}^{t} g(\sigma) e^{\int_{\sigma}^{t} h(\sigma') \, d\sigma'} \, d\sigma \quad \forall \ t \in [0, T].
\end{equation}

where
\[J(s, t) := \begin{cases} [s, t] & \text{if } s \leq t, \\ [t, s] & \text{if } t \leq s.\end{cases}\]

**Proof.** If $K := \max_{a \leq t \leq b} |\dot{C}(t)| < \infty$, then for $a \leq s \leq t \leq b$,
\[\|C(t) - C(s)\| \leq |C(t) - C(s)| = \left| \int_{s}^{t} \dot{C}(\tau) \, d\tau \right| \leq \int_{s}^{t} |\dot{C}(\tau)| \, d\tau \leq K |t - s|\]
which shows $|C(t)|$ is Lipschitz and hence absolutely continuous. Moreover, at $t$, where $|C(t)|$ is differentiable, we have
\[
\left| \frac{d}{dt} \left| C(t) \right| \right| = \lim_{s \to t} \left| \frac{|C(t)| - |C(s)|}{t - s} \right| \leq \lim_{s \to t} \left| \frac{\int_{s}^{t} \dot{C}(\tau) \, d\tau}{t - s} \right| = |\dot{C}(t)|
\]
which combined with Eq. (9.2) implies,
\begin{equation}
(9.4) \quad \left| \frac{d}{dt} \left| C(t) \right| \right| \leq h(t) |C(t)| + g(t) \quad \text{for a.e. } t \in [a, b].
\end{equation}

If $s \leq t$ in Eq. (9.3) let $\varepsilon = +1$ while if $t \leq s$ in Eq. (9.3) let $\varepsilon = -1$ and in either case let $\psi_\varepsilon(\tau) := |C(s + \varepsilon \tau)|$ for $\tau \geq 0$ so that $s + \varepsilon \tau \in [a, b]$. Then $\psi_\varepsilon(\tau)$ is still absolutely continuous and satisfies,
\[\psi_\varepsilon(\tau) = \varepsilon \frac{d}{dt} |C(t)| \big|_{t=s+\varepsilon \tau} \leq \left| \frac{d}{dt} \left| C(t) \right| \big|_{t=s+\varepsilon \tau} \right| \leq h(s + \varepsilon \tau) |C(s + \varepsilon \tau)| + g(s + \varepsilon \tau) = h(s + \varepsilon \tau) \psi_\varepsilon(\tau) + g(s + \varepsilon \tau).
\]
Thus Propositions 9.1
\[|C(s + \varepsilon \tau)| = \psi_\varepsilon(\tau) \leq e^{\int_{0}^{\tau} h(s + \varepsilon \rho) \, d\rho} \psi_\varepsilon(0) + \int_{0}^{\tau} e^{\int_{\rho}^{\tau} h(s + \varepsilon \tau) \, d\tau} g(s + \varepsilon \rho) \, d\rho.
\]
We now choose $\tau$ so that $s + \varepsilon \tau = t$ (i.e. $\tau := \varepsilon (t - s) = |t - s|$) to conclude,
\[|C(t)| \leq \int_{0}^{\varepsilon (t-s)} h(s + \varepsilon \rho) \, d\rho \big| C(s) \big| + \int_{0}^{\varepsilon (t-s)} e^{\int_{\rho}^{\varepsilon (t-s)} h(s + \varepsilon \tau) \, d\tau} g(s + \varepsilon \rho) \, d\rho.
\]
which after affine change of variables gives Eq. (9.3).

9.2. A geometric form of Gronwall’s inequality. Recall that $\nabla$ denotes the Levi-Civita covariant derivative on $TM$. We also use $\nabla$ to denote the Levi-Civita covariant derivative extended (by the product rule) to act on any associated vector bundle, let $\Lambda^k(TM)$, $\Lambda^k(T^*M)$, $TM^{\otimes l} \otimes (T^*M)^{\otimes k}$, etc.

The following geometric version of the classic Bellman-Gronwall inequality will be used frequently in this section.

Corollary 9.3 (Covariant Bellman/Gronwall). Let $E := TM^{\otimes k} \otimes T^*M^{\otimes l}$ for some $k, l \in \mathbb{N}_0$, $\sigma \in C^1((a, b), M)$, and suppose that $T_t, G_t \in E_{\sigma(t)}$ and $H_t \in \text{End}(E_{\sigma(t)})$ are given continuously differentiable functions of $t$. If $T_t, H_t,$ and $G_t$ satisfy the differential equation,

\[ \nabla_t T_t = H_t T_t + G_t, \]

then, for all $s, t \in (a, b)$,

\[ |T_t| \leq e^{\int_{J(s, t)} \|H_r\|_{op} dr} |T_s| + \int_{J(s, t)} e^{\int_{J(r, t)} \|H_r\|_{op} dr} |G_r| d\rho. \]

Proof. The point is that, writing $//_t$ for $//_t(\sigma)$, we have from Eq. (9.5) that

\[ \frac{d}{dt} [//_t^{-1} T_t] = //_t^{-1} \nabla_t T_t = //_t^{-1} H_t T_t + //_t^{-1} G_t \]

and therefore

\[ \left| \frac{d}{dt} [//_t^{-1} T_t] \right| \leq \| //_t^{-1} H_t //_t \|_{op} \| //_t^{-1} T_t \| + \| //_t^{-1} G_t \| \]

and Eq. (9.6) now follows directly from Corollary 9.2 above with $C(t) : = //_t^{-1} T_t$ and the observation that $|C(t)| = |T_t|$ for all $t$. \qed

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