Free Field Representation of Quantum Affine Algebra $U_q(\widehat{\mathfrak{sl}}_2)$

Atsushi MATSUO

Department of Mathematics
Nagoya University, Nagoya 464-01, Japan

Abstract. A Fock representation of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is constructed by three bosonic fields for an arbitrary level with the help of the Drinfeld realization.
1. Introduction

Recently the theory of the q-deformed chiral vertex operators (qVO) is developed by Frenkel and Reshetikhin [1] based on the representation theory of the quantum affine algebra $U_q(\hat{g})$ [2]. They derive a q-difference equation for the n-point correlation function, which is a q-analogue of the Knizhnik-Zamolodchikov equation (KZ) in the Wess-Zumino-Witten (WZW) model [3] and is called the quantum Knizhnik-Zamolodchikov equation (qKZ). The importance of this model is due to the fact that some elliptic solutions of the quantum Yang-Baxter equation (YBE) of face type, including the ABF-solution [4], are obtained as the connection matrices of qKZ\(^1\).

For a detailed study of the model, a free field representation of $U_q(\hat{g})$ seems to be an essential machinery. Jimbo et al. [7] explicitly calculate qVO for level one $U_q(\hat{\mathfrak{sl}}_2)$-modules by making use of the free field representation obtained by Frenkel and Jing [8], which is a q-deformation of the Frenkel-Kac construction [10]. We note that the Drinfeld realization of $U_q(\hat{g})$ [11] is the main tool in their works. However a free field representation for an arbitrary level has not been known even in the $U_q(\hat{\mathfrak{sl}}_2)$ case.

When $q = 1$, the currents and the chiral vertex operators can be constructed by some free fields in general. In fact the Wakimoto representation [12] of the affine Lie algebra $\hat{\mathfrak{sl}}_2$ is quite useful for the purpose. It is realized by one set of a $\beta$-$\gamma$ ghost system and a free bosonic field, and the vertex operators and the screening operators are explicitly written down [13]. This construction allows a generalization to the higher rank case [14], see also [16]. Remarkably, in the $\mathfrak{sl}_2$ case, the materials are also realized by three bosonic fields [15]. For instance, the standard $\mathfrak{sl}_2$ currents of level $k$ are expressed as

$$J^\pm(z) = \frac{1}{\sqrt{2}} \left[ \sqrt{k^2 + 2} \partial \phi_1(z) \pm i \sqrt{k} \partial \phi_2(z) \right] e^{\pm \sqrt{k} [i \phi_2(z) - \phi_0(z)]},$$

(1.1)

$$J^0(z) = -\sqrt{k} \partial \phi_0(z),$$

where $\phi_i(z)$ are independent bosonic fields normalized as $\phi_i(z)\phi_i(w) \sim \log (z - w)$.

\(^1\) Aomoto et al. [5] have independently shown that the ABF-solution is obtained as the connection matrix of a q-difference equation, which is nothing else but a special case of qKZ [6].
In this letter we shall generalize (1.1) to the case of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$. In section 2, we review the Drinfeld realization of $U_q(\hat{\mathfrak{sl}}_2)$, whose generating functions play the role of the $\mathfrak{sl}_2$ currents. In section 3, we express them by means of three bosonic fields with slightly modified normalizations and expressions. Section 4 is devoted to a brief conclusion and discussion.

While completing this work the author learned that J. Shiraishi has also obtained a free field realization of $U_q(\hat{\mathfrak{sl}}_2)$ for an arbitrary level in a different expression [19].
2. Drinfeld realization of $U_q(\widehat{sl}_2)$

Here and after we frequently use the notation

\[(2.1) \quad [m] = \frac{q^m - q^{-m}}{q - q^{-1}}.\]

The quantum affine algebra $U_q(\widehat{sl}_2)$ is isomorphic to the associative algebra generated by the letters $\{x^+_m \mid m \in \mathbb{Z}\}$, $\{a_m \mid m \in \mathbb{Z}_{\neq 0}\}$, $q^{\pm \delta}$ and $q^{\pm a_0}$, satisfying the following defining relations:

\[(2.2) \quad q^{\pm \delta} \in \text{the center of the algebra},\]

\[(2.3) \quad [a_m, a_n] = \delta_{m+n, 0} \frac{[2m][mc]}{m}, \quad [a_m, q^{a_0}] = 0,\]

\[(2.4) \quad q^{a_0} x^m q^{-a_0} = q^{\pm 2} x^m, \quad [a_m, x_n^\pm] = \pm [2m] q^{\mp |m|} c x^\pm_{m+l},\]

\[(2.5) \quad x_{m+1}^+ x_n^\pm - q^{\pm 2} x_n^\pm x_{m+1} = q^{\pm 2} x_m^+ x_{n+1}^\pm - x_{n+1}^\pm x_m^+,\]

\[(2.6) \quad [x_m^+, x_n^-] = \frac{1}{q - q^{-1}} \left( q^{\frac{m-n}{2}} c \psi_{m+n} - q^{\frac{n-m}{2}} c \varphi_{m+n} \right),\]

where $q^{\pm mc}$ for $m > 0$ is understood as $(q^{\pm \delta})^{2m}$ and $\{\psi_r, \varphi_s \mid r \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}_{\leq 0}\}$ are related to $\{a_m \mid m \in \mathbb{Z}_{\neq 0}\}$ by

\[(2.7) \quad \sum_{m=0}^{\infty} \psi_m z^{-m} = q^{a_0} \exp((q - q^{-1}) \sum_{m=1}^{\infty} a_m z^{-m}),\]

\[(2.8) \quad \sum_{m=0}^{\infty} \varphi_{-m} z^{m} = q^{-a_0} \exp(-(q - q^{-1}) \sum_{m=1}^{\infty} a_{-m} z^{m}).\]

Now consider the following generating functions:

\[(2.8) \quad k_+(z) = \sum_{m=0}^{\infty} \psi_m z^{-m}, \quad k_-(z) = \sum_{m=0}^{\infty} \varphi_{-m} z^{m},\]

\[(2.8) \quad x^\pm(z) = \sum_{m \in \mathbb{Z}} x^\pm_m z^{-m}.\]

Compositions of these operators are defined as a formal power series. Suppose that they act on a highest weight module and that $q^{\delta}$ acts by a scalar $q^{\delta}$ for some complex number.
Then we may understand them to be analytically continued outside some locus, and the operators (2.8) are characterized by the following properties:

\[
\begin{align*}
[\mathcal{k}_\pm(z), \mathcal{k}_\pm(z)] &= 0, \quad \frac{q^{k+2}z - w}{q^k z - q^2 w} \mathcal{k}_-(z) \mathcal{k}_+(w) = \mathcal{k}_+(w) \mathcal{k}_-(z) \frac{q^{-k+2}z - w}{q^{-k} z - q^{-2} w}, \\
\mathcal{k}_+(z) x_\pm(w) \mathcal{k}_+(z)^{-1} &= \left( \frac{q^{\frac{k}{2}+2}z - w}{q^{\frac{k}{2}} w - q^2 z} \right)^{\pm1} x_\pm(w) \\
\mathcal{k}_-(z) x_\pm(w) \mathcal{k}_-(z)^{-1} &= \left( \frac{q^{\frac{k}{2}+2}z - w}{q^{\frac{k}{2}} z - q^2 w} \right)^{\pm1} x_\pm(w), \\
(z - q^{\pm2}w) x_\pm(z) x_\pm(w) &= (q^{\pm2}z - w) x_\pm(w) x_\pm(z) \quad \text{and} \\
x_\pm(z) x^-\pm(w) &\sim \frac{1}{q - q^{-1}} \left( \frac{z}{z - q^k w} k_+(q^{\frac{k}{2}} w) - \frac{z}{z - q^{-k} w} k_-(q^{-\frac{k}{2}} w) \right).
\end{align*}
\]

The last formula (operator product expansion) means that \( x_\pm(z) x^-\pm(w) \) is analytically continued with the singular part being the right hand side and it coincides with \( x^-\pm(w) x_\pm(z) \). The defining relations (2.2)–(2.6) are recovered from these properties by a standard argument, see [9].

When \( q \) goes to 1, the algebra \( U_q(\widehat{\mathfrak{s}\mathfrak{l}}_2) \) goes to the enveloping algebra \( U(\widehat{\mathfrak{s}\mathfrak{l}}_2) \) of the affine Lie algebra \( \widehat{\mathfrak{s}\mathfrak{l}}_2 \), and the operators (2.8) go to the standard \( \mathfrak{s}\mathfrak{l}_2 \) currents by the following correspondence:

\[
\begin{align*}
\mathcal{x}_\pm(z) \to z J_\pm(z), \quad k_+(z) - k_-(z) \to 2z J^0(z).
\end{align*}
\]
3. Free field representation of $U_q(\widehat{\mathfrak{sl}}_2)$

Let $k$ be a complex number. Let $\{\alpha_\sigma, \alpha_\sigma(n) \mid \sigma = \pm 1, n \in \mathbb{Z}\}$ be a set of operators satisfying the following relations:

$$(3.1) \quad [\alpha_\sigma(m), \alpha_\tau(n)] = \sigma \delta_{\sigma,\tau} \delta_{m+n,0} \frac{[2m][km]}{m}, \quad [\alpha_\sigma(m), \alpha_\tau] = \sigma \delta_{\sigma,\tau} \delta_{m,0}.$$ 

Here $\delta$ denotes the Kronecker symbol. Let $\{\beta(n) \mid n \in \mathbb{Z} \neq 0\}$ be another set of operators satisfying

$$(3.2) \quad [\beta(m), \beta(n)] = \delta_{m+n,0} \frac{[2m][(k+2)m]}{m}.$$ 

Suppose that they commute with $\alpha_\sigma$ and $\alpha_\sigma(m)$ for any $m$.

The Fock space $\tilde{\mathcal{F}}$ on which these operators act is supposed to be generated by the negative modes $\alpha_1(m), \alpha_{-1}(m), \beta(m)$, for $m < 0$, and by $\alpha_1$ and $\alpha_{-1}$ acting on the vacuum vector $v$ satisfying the following conditions:

$$(3.3) \quad \alpha_1(m)v = \alpha_{-1}(m)v = \beta(m)v = 0 \text{ for any } m > 0, \text{ and}$$

$$(3.4) \quad \alpha_1(0)v \text{ and } \alpha_{-1}(0)v \text{ are scalar multiples of } v.$$

We set

$$K_+(z) = \exp \left\{ (q - q^{-1}) \sum_{m=1}^{\infty} z^{-m} \alpha_1(m) \right\} q^{\alpha_1(0)},$$

$$K_-(z) = \exp \left\{ -(q - q^{-1}) \sum_{m=1}^{\infty} z^m \alpha_1(-m) \right\} q^{-\alpha_1(0)},$$

$$X^+(z) = \frac{1}{q - q^{-1}} \left\{ Y^+(z)Z_+(q^{-\frac{k}{2}z})W_+(q^{-\frac{k}{2}z}) - W_-(q^{\frac{k}{2}z})Z_-(q^{\frac{k}{2}z})Y^+(z) \right\},$$

$$X^-(z) = \frac{-1}{q - q^{-1}} \left\{ Y^-(z)Z_+(q^{\frac{k}{2}z})W_+(q^{\frac{k}{2}z})^{-1} - W_-(q^{-\frac{k}{2}z})^{-1}Z_-(q^{-\frac{k}{2}z})Y^-(z) \right\},$$
where

\[
Y^+(z) = \exp \left\{ \sum_{m=1}^{\infty} q^{-\frac{km}{2}} z^m \left( \alpha_1(-m) + \alpha_{-1}(-m) \right) \right\} e^{2(\alpha_1 + \alpha_{-1}) z^{\frac{1}{2}}(\alpha_1(0) + \alpha_{-1}(0))} \exp \left\{ -\sum_{m=1}^{\infty} q^{\frac{km}{2}} z^{-m} \left( \alpha_1(m) + \alpha_{-1}(m) \right) \right\},
\]

\[
Y^-(z) = \exp \left\{ -\sum_{m=1}^{\infty} q^{\frac{km}{2}} z^m \left( \alpha_1(-m) + \alpha_{-1}(-m) \right) \right\} e^{-2(\alpha_1 + \alpha_{-1}) z^{-\frac{1}{2}}(\alpha_1(0) + \alpha_{-1}(0))} \exp \left\{ \sum_{m=1}^{\infty} q^{\frac{km}{2}} z^{-m} \left( \alpha_1(m) + \alpha_{-1}(m) \right) \right\},
\]

\[
Z^+(z) = \exp \left\{ -(q - q^{-1}) \sum_{m=1}^{\infty} z^{-m} \left[ \frac{m}{2m} \right] \alpha_{-1}(m) \right\} q^{-\frac{1}{2}} \alpha_{-1}(0),
\]

\[
Z^-(z) = \exp \left\{ (q - q^{-1}) \sum_{m=1}^{\infty} z^m \left[ \frac{m}{2m} \right] \alpha_{-1}(-m) \right\} q^{\frac{1}{2}} \alpha_{-1}(0),
\]

\[
W^+(z) = \exp \left\{ -(q - q^{-1}) \sum_{m=1}^{\infty} z^{-m} \left[ \frac{m}{2m} \right] \beta(m) \right\},
\]

\[
W^-(z) = \exp \left\{ (q - q^{-1}) \sum_{m=1}^{\infty} z^m \left[ \frac{m}{2m} \right] \beta(-m) \right\}.
\]

**Proposition.** By analytic continuation, \(X^\pm(z)\) and \(K^\pm(z)\) satisfy the same relations as (2.9)–(2.12).

**Proof.** The relation (2.9) is obvious by the definition. The proofs of (2.10)–(2.11) are straightforward by calculating commutators of the fields. For example, the first relation of (2.10) follows from the following:

\[
K^+_+(z)Y^\pm(w) = q^{\pm 2} \exp \left\{ \sum_{m=1}^{\infty} z^{-m} w^m q^{-\frac{km}{2}} q^{2m} - q^{-2m} \right\} Y^\pm(w) K^+_+(z)
\]

\[
= \left( \frac{q^2 z - q^{\pm \frac{k}{2}} w}{z - q^{\pm \frac{k}{2}} w} \right)^{\pm 1} Y^\pm(w) K^+_+(z).
\]
Here we have used the formula: \( \sum_{m=1}^{\infty} \frac{x^m}{m} = -\log(1-x) \). To prove (2.12) we put \( X^+(z) = \{A(z) - B(z)\}/(q - q^{-1}) \) and \( X^-(z) = -\{C(z) - D(z)\}/(q - q^{-1}) \) where

\[ A(z) = Y^+(z)Z_+(q^{-k+1} z)W_+(q^{-k} z), \quad B(z) = W_-(q^{\frac{k}{2}} z)Z_-(q^{\frac{k}{2}+1} z)Y^+(z), \]

\[ C(z) = Y^-(z)Z_+(q^{\frac{k}{2}+1} z)W_+(q^{\frac{k}{2}} z)^{-1}, \quad D(z) = W_-(q^{-\frac{k}{2}} z)^{-1}Z_-(q^{-\frac{k}{2}+1} z)Y^-(z). \]

Then we have

\[
A(z)C(w) = \frac{q^{-1}z - q^{k+1}w}{z - q^{k}w}Y^+(z)Y^-(w)Z_+(q^{-k+1} z)Z_+(q^{k+1} w)W_+(q^{-k} z)W_+(q^{k} w)^{-1}
\]

\[
\sim -\frac{(q - q^{-1})z}{z - q^{k}w}Y^+(w)Y^-(w)Z_+(q^{k} w)Z_+(q^{k+1} w)
\]

\[
= -\frac{(q - q^{-1})z}{z - q^{k}w}K_+(q^{k} w).
\]

Similarly we have

\[ B(z)D(w) \sim \frac{(q - q^{-1})z}{z - q^{-k}w}K_-(q^{-\frac{k}{2}} w), \]

and the other products \( A(z)D(w) \) and \( B(z)C(w) \) are regular. The proof of the relation \( [X^+(z), X^-(w)] = 0 \) is straightforward. \( Q.E.D. \)

Now suppose that the vacuum vector \( v \) of the Fock space \( \tilde{F} \) satisfies the condition

\[
(3.10) \quad \frac{1}{k}(\alpha_1(0) + \alpha_{-1}(0))v = mv \text{ for some integer } m.
\]

Consider the subspace \( F \) of \( \tilde{F} \) generated by the actions of the negative modes and of \( \alpha_1 + \alpha_{-1} \) on \( v \). Then it is clear by the definition that the mode expansion of \( X^\pm(z) \) and \( K_\pm(z) \) like (2.8) makes sense on \( F \) and that each Fourier component acts there. Thus we obtain a representation of the algebra \( U_q(\hat{sl}_2) \) on \( F \) with the level \( c = k \).
4. Conclusion

In this letter we have constructed the Fock representation of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ for an arbitrary level $k$ in terms of three bosonic fields. In the $q \to 1$ limit our representation goes to the representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ defined by the mode expansion of the currents $(1.1)$. It is equivalent to a bosonization of the Wakimoto representation by a certain transformation [15].

In [16] solutions to KZ are explicitly constructed in the context of the Wakimoto representation, and they give rise to the integral solutions obtained previously [17]. The author expects that the Jackson integral solutions of qKZ [6,17] would be obtained in our formulation.

The present work will contribute to a better understanding of massive deformations of conformal field theory as the Wakimoto representation did in the WZW model. A detailed analysis of the representation and a construction of qVO and the screening operators will be contained in a separate paper.

Acknowledgement.

The author thanks K. Kimura, T. Miwa and J. Shiraishi for discussion.
References.

[1] I.B. Frenkel and N.Yu. Reshetikhin, Commun. Math. Phys. 146 (1992) 1.

[2] V.G. Drinfeld, Sov. Math. Dokl. 32 (1985) 254.
   M. Jimbo, Lett. Math. Phys. 10 (1985) 63.

[3] V.G. Knizhnik and A.B. Zamolodchikov, Nucl. Phys. B247 (1984) 83.
   A. Tsuchiya and Y. Kanie, Adv. Stud. Pure. Math 16 (1988) 297.

[4] G.E. Andrews, R.J. Baxter and P.J. Forrester, J. Stat. Phys. 35 (1984) 193.

[5] K. Aomoto, Y. Kato and K. Mimachi, Int. Math. Res. Notes (Duke Math. J.) 65 (1992) 7.

[6] A. Matsuo, Jackson integrals of Jordan-Pochhammer type and quantum Knizhnik-Zamolodchikov equations, preprint (1991) to appear in Commun. Math. Phys.

[7] M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki, preprint RIMS-873 (1992).

[8] I.B. Frenkel and N.H. Jing, Proc. Nat’l Acad. Sci. USA 85 (1988) 9373.

[9] D. Bernard, Lett. Math. Phys. 17 (1989) 239.

[10] I.B. Frenkel and V.G. Kac, Invent. Math. 62 (1980) 23.

[11] V.G. Drinfeld. Sov. Math. Dokl. 36 (1987) 212.

[12] M. Wakimoto, Commun. Math. Phys. 104 (1986) 605.

[13] D. Bernard and G. Felder, Commun. Math. Phys. 127 (1990) 145.

[14] B. Feigin and E. Frenkel, Commun. Math. Phys. 128 (1990) 161.
   P. Bouwknegt, J. McCarythy and K. Pilch, Prog. Theor. Phys. Suppl. 102 (1990) 67.

[15] D. Nemeschansky, Phys. Lett. B224 (1989) 121.
   K. Ito. Nucl. Phys. B332 (1990) 566.
   T. Jayaraman, K.S. Narain and M.H. Sarmadi, Nucl. Phys. B343 (1990) 418.

[16] G. Kuroki, Commun. Math. Phys. 142 (1991) 511.
   H. Awata, A. Tsuchiya and Y. Yamada, Nucl. Phys. B365 (1991) 680.

[17] E. Date, M. Jimbo, A. Matsuo and T. Miwa, Intern. J. Mod. Phys. B4 (1990) 1049.
   A. Matsuo, Commun. Math. Phys. 134 (1990) 65.
   V.V. Schechtman and A.N. Varchenko, Invent. Math. 106 (1991) 139.

[18] A. Matsuo, Quatum algebra structure of certain Jackson integrals, preprint (1992).
   N.Yu. Reshetikhin, Jackson type integrals, Bethe vectors, and solutions to a difference analog of the Knizhnik-Zamolodchikov system, preprint (1992).
[19] J. Shiraishi. Free boson representation of $U_q(\widehat{sl}_2)$, in preparation.