On the character table of non-split extension $2^6 \cdot S_8$

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Problem Statement & Objective: Character tables of maximal subgroups of finite simple groups provide considerable amount of information about the groups. In the present article, our objective is to compute the character table of one maximal subgroup of the orthogonal group $PSO_6^+(3)$. Approach: The projective special orthogonal group $PSO_6^+(3) \cong O_6^+(3), 2$ is obtained from the special orthogonal group $SO_6^+(3)$ on factoring by the group of scalar matrices it contains. The group $O_6^+(3), 2$ has a maximal subgroup of the form $2^6 \cdot S_8$ with index 3838185. The group $Q \cong 2^6 \cdot S_8$ is a non-split group extension of an elementary abelian 2-group of order 64 by the symmetric group $S_8$. We apply the Fischer-Clifford theory to compute the irreducible characters of the extension $2^6 \cdot S_8$. Results and Conclusion: We produce 64 conjugacy classes of elements as well as 64 irreducible character of the non-split group extension $2^6 \cdot S_8$ corresponding to the three inertia factors $H_1 = S_8, H_2 = S_6 \times 2$ and $H_3 = (S_4 \times S_2):2$.

1. Introduction

Character table of a group provides considerable amount of information about the group itself, and hence is of great importance in mathematics as well as the physical sciences. Character table of finite groups can be constructed by using various techniques. For example, Todd-Coxeter coset enumeration method, the Schreier-Sims algorithm, the Burnside-Dixon algorithm and various other techniques. However Bernd Fischer developed a very powerful and interesting technique for calculating the character tables of group extensions. This technique, which is known as the technique of the Fischer-Clifford matrices, derives its fundamentals from the Clifford theory. If $Q$ is an appropriate extension of $N$ by $G$, the method involves the construction of a non-singular matrix for each conjugacy class of $G$. In this paper, we apply the technique of Fischer-Clifford matrices (also called Fischer matrices) to compute the character table of group extension of type $2^6 \cdot S_8$, which is a non-split group extension of an elementary abelian group of order $2^6$ by the symmetric group $S_8$.

We computed the character table of a split group extension of the form $2^6:S_8$ that sits inside the orthogonal group $O_6^+(3), 2$ and also in the smallest Fischer group $Fi_{22}$ (see [5]).
In the present article, we aim to study a more complicated situation, in which above mentioned action is of non-split type. We construct the non-split extension inside the orthogonal group $PSO_6^+(3)$ and compute its character table completely. Note that character table of non-split group extension $2^6 \cdot S_8$ is not yet known. So, we compute here character table of non-split extension $Q$.

Let $Q \cong 2^6 \cdot S_8$ be a non-split group extension of $N \cong 2^6$ by $G \cong S_8$, then the technique of Fischer-Clifford matrices involves the construction of a non-singular matrix for each conjugacy class of $Q/N \cong G$. The character table of $Q$ can then be constructed from these Fischer-Clifford matrices and the partial character tables of certain subgroups of $G$, called the inertia factor groups. The conjugacy classes of $Q$ are computed using the coset analysis method. For details on the use of Fischer-Clifford matrices, the coset analysis techniques and other related material, the reader is referred to [1, 2, 3, 4, 5, 6].

In the present article, we construct the group $PSO_6^+(3)$ and compute the character table of one of its maximal subgroup of the form $2^6 \cdot S_8$. The group $Q$ is a non-split group extension of an elementary abelian 2-group of order 64 by the symmetric group $S_8$. We apply the Fischer-Clifford theory to compute the irreducible characters of the non-split extension $Q$.

2. Methodology

We first construct the projective special orthogonal group $PSO_6^+(3)$ inside the special orthogonal group $SO_6^+(3)$. Then with the aid of computer algebra system GAP [7], we can easily determine the elementary abelian group of order 64 and apply the technique of Fischer-Clifford matrices to compute the relevant inertia factor groups and corresponding Fischer matrices.

2.1. Construction of the non-split extension $Q \cong 2^6 \cdot S_8$

The projective special orthogonal group $PSO_6^+(3)$ is a group obtained from the special orthogonal group $SO_6^+(3)$ on factoring by the group of scalar matrices it contains. The group $PSO_6^+(3)$ has a maximal subgroup of the form $2^6 \cdot S_8$ with index 3838185. The group $Q$ is a non-split group extension of an elementary abelian group of order 64 by the symmetric group $S_8$. With the help of GAP [7], we construct non-split extension $Q$ inside the group $PSO_6^+(3)$ as permutation group by the following two generators $x$ and $y$ of orders such that $o(x) = 10, o(y) = 8$ and $o(xy) = 4$ such that $Q = \langle x, y \rangle$. Note that there are three subgroups of index 2 inside $O_6^+(3), O_6^+(3).2_2$ and $O_6^+(3).2_2$ respectively having the structures $2^6 \cdot S_8, 2^6 \cdot S_8$ and $2^6 \cdot (2 \times A_8) = 2^7 \cdot A_8$, respectively.

3. Character Table of non-split group extension $Q \cong 2^6 \cdot S_8$

The use of sections to divide the text of the paper is optional and left as a decision for the author. First, we compute the conjugacy classes of $Q$ using the technique of coset analysis and obtained that there are exactly 64 conjugacy classes of elements of $Q$. Thus, we have 64 irreducible character of $Q$ as well. Corresponding to each conjugacy class of $g \in S_8$, we determine the partial character table of $Q$. Now the partial character table can be determined by multiplying the rows of the Fischer matrix $M(g)$ with the relevant columns of the inertia factor groups of $G$. The complete details on the computation of conjugacy classes and Fischer-Clifford matrices for non-split extension $Q$ will appear in another forthcoming article. The set of irreducible characters of $Q$ will be partitioned into blocks $O_1, O_2$ and $O_3$ corresponding to the inertia factors $S_8, S_6 \times 2$ and $(S_4 \times S_4):2$. In fact $O_1 = \{I_i \mid 1 \leq i \leq 22\}, O_2 = \{I_i \mid 23 \leq i \leq 44\}$ and $O_3 = \{I_i \mid 45 \leq i \leq 64\}$, where $\text{Irr}(Q) = \bigcup_i O_i^\circ$, where $\text{Irr}(Q) = \bigcup_i O_i$. The full character table of $2^6 \cdot S_8$ is displayed in Table 1.

| Table 1. | The character table of $2^6 \cdot S_8$ |
| [\chi]_{i_d} | 2A | 3A | 5A | 15A | 2A |
|-------|---|---|---|----|---|
| 1A | 1 | 1 | 1 | 1 | 1 |
| 2A | 1 | 1 | 1 | 1 | 1 |
| 3A | 4 | 4 | 4 | 4 | 4 |
| 5A | 2 | 2 | 2 | 2 | 2 |
| 1A | 1 | 1 | 1 | 1 | 1 |
| 2A | 1 | 1 | 1 | 1 | 1 |
| 3A | 4 | 4 | 4 | 4 | 4 |
| 5A | 2 | 2 | 2 | 2 | 2 |
| 1A | 1 | 1 | 1 | 1 | 1 |
| 2A | 1 | 1 | 1 | 1 | 1 |
| 3A | 4 | 4 | 4 | 4 | 4 |
| 5A | 2 | 2 | 2 | 2 | 2 |
| 1A | 1 | 1 | 1 | 1 | 1 |
| 2A | 1 | 1 | 1 | 1 | 1 |
| 3A | 4 | 4 | 4 | 4 | 4 |
| 5A | 2 | 2 | 2 | 2 | 2 |
| 1A | 1 | 1 | 1 | 1 | 1 |
| 2A | 1 | 1 | 1 | 1 | 1 |
| 3A | 4 | 4 | 4 | 4 | 4 |
| 5A | 2 | 2 | 2 | 2 | 2 |
| 1A | 1 | 1 | 1 | 1 | 1 |
| 2A | 1 | 1 | 1 | 1 | 1 |
| 3A | 4 | 4 | 4 | 4 | 4 |
| 5A | 2 | 2 | 2 | 2 | 2 |
| 1A | 1 | 1 | 1 | 1 | 1 |
| 2A | 1 | 1 | 1 | 1 | 1 |
| 3A | 4 | 4 | 4 | 4 | 4 |
| 5A | 2 | 2 | 2 | 2 | 2 |
| 1A | 1 | 1 | 1 | 1 | 1 |
| 2A | 1 | 1 | 1 | 1 | 1 |
| 3A | 4 | 4 | 4 | 4 | 4 |
| 5A | 2 | 2 | 2 | 2 | 2 |
| 1A | 1 | 1 | 1 | 1 | 1 |
| 2A | 1 | 1 | 1 | 1 | 1 |
| 3A | 4 | 4 | 4 | 4 | 4 |
| 5A | 2 | 2 | 2 | 2 | 2 |
| 1A | 1 | 1 | 1 | 1 | 1 |
| 2A | 1 | 1 | 1 | 1 | 1 |
| 3A | 4 | 4 | 4 | 4 | 4 |
| 5A | 2 | 2 | 2 | 2 | 2 |
| 1A | 1 | 1 | 1 | 1 | 1 |
| 2A | 1 | 1 | 1 | 1 | 1 |
| 3A | 4 | 4 | 4 | 4 | 4 |
| 5A | 2 | 2 | 2 | 2 | 2 |

**Table 1** The character table of $2^6 \cdot S_6$ (continued)
| $z_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
|-----|---|---|---|---|---|---|---|---|---|---|---|---|
| $z_2$ | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| $z_3$ | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 |
| $z_4$ | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 |
| $z_5$ | 2 | 2 | 2 | 2 | 0 | 0 | 0 | 0 | 4 | 4 | 4 | 4 |
| $z_6$ | 2 | 2 | 2 | 2 | 0 | 0 | 0 | 0 | -4 | -4 | -4 | -4 |
| $z_7$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 10 | 10 | 10 | 10 |
| $z_8$ | 2 | 2 | 2 | 2 | -2 | -2 | -2 | -2 | -10 | -10 | -10 | -10 |
| $z_9$ | 1 | 1 | 1 | 1 | 3 | 3 | 3 | 3 | 5 | 5 | 5 | 5 |
| $z_{10}$ | -1 | -1 | -1 | -1 | -3 | -3 | -3 | -3 | 9 | 9 | 9 | 9 |
| $z_{11}$ | 2 | 2 | 2 | 2 | -2 | -2 | -2 | -2 | -10 | -10 | -10 | -10 |
| $z_{12}$ | -2 | -2 | -2 | -2 | 2 | 2 | 2 | 2 | 10 | 10 | 10 | 10 |
| $z_{13}$ | -1 | -1 | -1 | -1 | 3 | 3 | 3 | 3 | -5 | -5 | -5 | -5 |
| $z_{14}$ | 1 | 1 | 1 | 1 | -3 | -3 | -3 | -3 | 5 | 5 | 5 | 5 |
| $z_{15}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $z_{16}$ | 0 | 0 | 0 | 0 | -4 | -4 | -4 | -4 | -1 | -1 | -1 | -1 |
| $z_{17}$ | 0 | 0 | 0 | 0 | 4 | 4 | 4 | 4 | -1 | -1 | -1 | -1 |
| $z_{18}$ | 0 | 0 | 0 | 0 | -16 | -16 | -16 | -16 | 2 | 2 | 2 | 2 |
| $z_{19}$ | 0 | 0 | 0 | 0 | 16 | 16 | 16 | 16 | -2 | -2 | -2 | -2 |
| $z_{20}$ | 0 | 0 | 0 | 0 | -10 | -10 | -10 | -10 | 1 | 1 | 1 | 1 |
| $z_{21}$ | 0 | 0 | 0 | 0 | 2 | 2 | 2 | 2 | -1 | -1 | -1 | -1 |
| $z_{22}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $z_{23}$ | 0 | 2 | -2 | 2 | 0 | 0 | -4 | -4 | 4 | 4 | -16 | -16 |
| $z_{24}$ | 0 | 2 | 2 | 2 | -6 | 6 | -4 | -4 | 0 | 0 | 2 | 2 |
| $z_{25}$ | 0 | 2 | 2 | 2 | -2 | -2 | -2 | -2 | 6 | 6 | 6 | 6 |
| $z_{26}$ | 0 | 2 | 2 | 2 | 2 | 2 | -2 | -2 | -6 | -6 | -6 | -6 |
| $z_{27}$ | 0 | 2 | 2 | 2 | -2 | -2 | -2 | -2 | 10 | 10 | 10 | 10 |
| $z_{28}$ | 0 | 2 | 2 | 2 | -2 | -2 | -2 | -2 | -10 | -10 | -10 | -10 |
| $z_{29}$ | 0 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 10 | 10 | 10 | 10 |
| $z_{30}$ | 0 | 2 | 2 | 2 | 2 | 2 | -2 | -2 | 4 | 4 | 4 | 4 |
| $z_{31}$ | 0 | 2 | 2 | 2 | -2 | -2 | -2 | -2 | 6 | 6 | 6 | 6 |
| $z_{32}$ | 0 | 2 | 2 | 2 | -2 | -2 | -2 | -2 | -6 | -6 | -6 | -6 |
| $z_{33}$ | 0 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 4 | 4 | 4 | 4 |
| $z_{34}$ | 0 | 0 | 0 | 0 | -4 | -4 | -4 | -4 | 0 | 0 | 0 | 0 |
| $z_{35}$ | 0 | 0 | 0 | 0 | 4 | 4 | 4 | 4 | 0 | 0 | 0 | 0 |
| $z_{36}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -2 | -2 | -2 | -2 |
| $z_{37}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 16 | 16 | 16 | 16 |

Table 1. The character table of $2^5 \cdot S_8$ (continued)
Table 1. The character table of $2^5 \cdot S_8$ (continued)

| $g_{10}$ | 6C | 12A | 7A | 6D | 6E | 4D | 8A |
|----------|----|-----|----|----|----|----|----|
| $x^1$ | 1 | 1 | 1 | 1 | 4 | 4 | 4 | 4 | 1 | 0 | 0 | 0 | 2 | 2 | 2 | 0 | 0 |
| $x^2$ | -1 | -1 | -1 | -1 | 4 | 4 | 4 | 4 | 0 | 0 | 0 | 0 | -2 | -2 | -2 | 0 | 0 |
| $x^3$ | 0 | 0 | 0 | 0 | -3 | -3 | -3 | -3 | 1 | 1 | 1 | 1 | -3 | -3 | -3 | 1 | 1 |
| $x^4$ | 0 | 0 | 0 | 0 | -3 | -3 | -3 | -3 | 1 | 1 | 1 | 1 | 3 | 3 | 3 | 1 | -1 |
| $x^5$ | -1 | -1 | -1 | -1 | -4 | -4 | -4 | -4 | 0 | 0 | 0 | 0 | 2 | 2 | 2 | 0 | 0 |
| $x^6$ | 1 | 1 | 1 | 1 | 3 | 3 | 3 | 3 | 1 | 1 | 1 | -1 | 1 | 3 | 3 | 3 | 1 | -1 |
| $x^7$ | 1 | 1 | 1 | 1 | 3 | 3 | 3 | 3 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 0 | 0 |
| $x^8$ | -1 | -1 | -1 | -1 | 3 | 3 | 3 | 3 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 | 0 | 0 |
| $x^9$ | 0 | 0 | 0 | 0 | -6 | -6 | -6 | -6 | 2 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| $x^{10}$ | 2 | 2 | 2 | 2 | 8 | 8 | 8 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| $x^{11}$ | -2 | -2 | -2 | -2 | 8 | 8 | 8 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | -1 |
| $x^{12}$ | -2 | -2 | -2 | -2 | 8 | 8 | 8 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | -1 |
| $x^{13}$ | 2 | 2 | 2 | 2 | 8 | 8 | 8 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| $x^{14}$ | 1 | 1 | 1 | 1 | -2 | -2 | -2 | -2 | -2 | -2 | -2 | -2 | -2 | -2 | 4 | 4 | 4 | 0 | 0 |
| $x^{15}$ | -1 | -1 | -1 | -1 | -2 | -2 | -2 | -2 | -2 | -2 | -2 | -2 | -2 | -2 | 4 | 4 | 4 | 0 | 0 |
| $z$ | 0 | 0 | 0 | 0 | -6 | -6 | -6 | -6 | 2 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |

For $g_{10}$, $x^1$ through $x^{15}$ are the irreducible characters, and $z$ is the trivial character.
### Table 1. The character table of $2^5 \cdot S_8$ (continued)

| $[\rho]_{15}$ | 1A | 2A | 2B | 3A | 6A | 6B | 5A | 10A | 15A | 12C | 12D | 2E | 12F | 2G |
|---------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $[\lambda]_0$ | 1A | 2A | 2B | 3A | 6A | 6B | 5A | 10A | 15A | 12C | 12D | 2E | 12F | 2G |
| $\lambda_{s}$ | 35 | -5 | 3 | 5 | 1 | -3 | 0 | 0 | 0 | -5 | -1 | 3 | 7 | -1 |
| $\lambda_{6}$ | 35 | -5 | 3 | 5 | 1 | -3 | 0 | 0 | 0 | -5 | -1 | 3 | 7 | -1 |
| $\lambda_{7}$ | 35 | -5 | 3 | 5 | 1 | -3 | 0 | 0 | 0 | -5 | -1 | 3 | 7 | -1 |
| $\lambda_{8}$ | 35 | -5 | 3 | 5 | 1 | -3 | 0 | 0 | 0 | -5 | -1 | 3 | 7 | -1 |
| $\lambda_{9}$ | 70 | -10 | 6 | 10 | 2 | -6 | 0 | 0 | 0 | 14 | -2 | -2 | -2 | 10 |
| $\lambda_{10}$ | 140 | -20 | 12 | 5 | 1 | -3 | 0 | 0 | 0 | 4 | -4 | 4 | 4 | 4 |
| $\lambda_{11}$ | 140 | -20 | 12 | 5 | 1 | -3 | 0 | 0 | 0 | 4 | -4 | 4 | 4 | 4 |
| $\lambda_{12}$ | 140 | -20 | 12 | 5 | 1 | -3 | 0 | 0 | 0 | 4 | -4 | 4 | 4 | 4 |
| $\lambda_{13}$ | 210 | -30 | 18 | 15 | 3 | -9 | 0 | 0 | 0 | -10 | -2 | 6 | 14 | -2 |
| $\lambda_{14}$ | 210 | -30 | 18 | 15 | 3 | -9 | 0 | 0 | 0 | -10 | -2 | 6 | 14 | -2 |
Table 1. The character table of $2^5 \cdot S_8$ (continued)

| [s]_{Sl} | 6B | 2D | 4B | 4C | 10B | 20A |
|----------|-----|-----|-----|-----|-----|-----|
| [s]_{lg} | 6F  | 12B | 6G  | 12C | 4I  | 2H  | 2I  | 4J  | 4K  | 4L  | 4M  | 4N  | 4D  | 8E  | 10A |
| Ze8      | 1   | -1  | -3  | 3   | -1  | 3   | 11  | -5  | 3   | -1  | -1  | 1   | 1   | 0   | 0   |
| Ze8      | 1   | -1  | -3  | 3   | -1  | 3   | -5  | 11  | -1  | -1  | 3   | 1   | 1   | 0   | 0   |
| Ze7      | -1  | 1   | 3   | -3  | -1  | 3   | -5  | 11  | -1  | -1  | 3   | 1   | -1  | -1  | 0   |
| Ze7      | -1  | 1   | 3   | -3  | -1  | 3   | -5  | 11  | -1  | -1  | 3   | 1   | -1  | -1  | 0   |
| Ze6      | 0   | 0   | 0   | 0   | -2  | 6   | 6   | 6   | -2  | 2   | 2   | 0   | 0   | 0   |
| Ze6      | 0   | 0   | 0   | 0   | -2  | 6   | 6   | 6   | -2  | 2   | 2   | 0   | 0   | 0   |
| Ze5      | 1   | -1  | -3  | 3   | -4  | 12  | 12  | 12  | 0   | 0   | 0   | 2   | 2   | 2   | 0   |
| Ze5      | 1   | -1  | -3  | 3   | -4  | 12  | 12  | 12  | 0   | 0   | 0   | 2   | 2   | 2   | 0   |
| Ze4      | 0   | 0   | 0   | 0   | -4  | 12  | -4  | 28  | -4  | 0   | 0   | 0   | 0   | 0   |
| Ze4      | 0   | 0   | 0   | 0   | -4  | 12  | -4  | 28  | -4  | 0   | 0   | 0   | 0   | 0   |
| Ze3      | 1   | -1  | -3  | 3   | 2   | -6  | -6  | -6  | 2   | -2  | 2   | -2  | 2   | 0   | 0   |
| Ze3      | 1   | -1  | -3  | 3   | 2   | -6  | -6  | -6  | 2   | -2  | 2   | -2  | 2   | 0   | 0   |
| Ze2      | 0   | 0   | 0   | 0   | -1  | 3   | 27  | -21 | -1  | -1  | 3   | -3  | 3   | 0   | 0   |
| Ze2      | 0   | 0   | 0   | 0   | -1  | 3   | 27  | -21 | -1  | -1  | 3   | -3  | 3   | 0   | 0   |

(continued)
4. Conclusion
We construct the ordinary character table of non-split group extension of type $2^6 \cdot S_6$ which sits maximally inside the group $PSO^+_6(3)$. We produce 64 conjugacy classes of elements as well as 64 irreducible character of the non-split group extension $2^6 \cdot S_6$ corresponding to the three inertia factors $H_1 = S_6, H_2 = S_6 \times 2$ and $H_3 = (S_4 \times S_4) : 2$. The irreducible characters of $Q$ are in three blocks $O_1, O_2$ and $O_3$ corresponding to the inertia factors $S_6, S_6 \times 2$ and $(S_4 \times S_4) : 2$, respectively.

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