ν-Analysis: A New Notion of Robustness for Large Systems with Structured Uncertainties

Olle Kjellqvist and John C. Doyle

Abstract—We present a new, scalable alternative to the structured singular value, which we call ν, provide a convex upper bound, study their properties and compare them to ϵ1 robust control. The analysis relies on a novel result on the relationship between robust control of dynamical systems and non-negative constant matrices.

I. INTRODUCTION

We consider a system to be robust if it is unlikely to fail. The usual setting to analyze the robustness of a system is to study how it interacts with uncertainty. Standard approaches impose structure on the uncertainty and certify robustness against its size. However, the way we currently measure the size of uncertainty is unsuitable for large-scale networks.

To see this, consider the standard robust control set-up in Fig. 1. G is a stable causal linear system with n inputs and outputs. ∆ is unknown but belongs to the set 𝒟 consisting of diagonal linear time-varying (LTV) systems that are strictly causal, stable, and have n inputs and outputs. We want to determine which of the following two systems are most likely to fail:

\[ P_1 : \begin{cases} x_1(t+1) = \delta_1 x_1(t) \\ x_2(t+1) = \delta_2 x_2(t) \\ \vdots \\ x_n(t+1) = \delta_n x_n(t) \end{cases}, \quad P_2 : \begin{cases} x_1(t+1) = \delta_1 x_2(t) \\ x_2(t+1) = \delta_2 x_3(t) \\ \vdots \\ x_n(t+1) = \delta_n x_1(t) \end{cases} \]

\( P_1 \) is a set of decoupled first-order systems with uncertain time constants, and \( P_2 \) is a delayed ring with uncertain weights. Robustness measures based on structured singular values \([1, 2]\) or \( \epsilon_1 \) robust control methods \([3]\) agree that both systems are robust against diagonal uncertainties whose largest\(^1\) diagonal element is bounded by one. It is tempting to conclude that \( P_1 \) and \( P_2 \) are equally likely to fail. A more careful study reveals that destabilizing \( P_1 \) is easy; a constant gain \( |\delta_k| > 1 \) for any \( k \) will render the closed-loop unstable. However, all of the uncertainties must simultaneously be large \( (\|\delta_1\| \|\delta_2\| \cdots \|\delta_n\| \geq 1) \) to destabilize \( P_2 \). In plain words, destabilizing \( P_2 \) requires large globally coordinated perturbations directly affecting every node.

This article proposes a new robustness measure \( \nu^2 \) that captures sparsity in the uncertainty. \( \nu \) is large for systems that are easily destabilized by sparse perturbations and small for systems that can withstand sparse perturbations. For example, \( \nu(P_1) = 1 \) and \( \nu(P_2) = 1/n \). We focus on diagonal linear time-varying and nonlinear uncertainty in discrete time.

This work is primarily motivated by recent progress to distributed and localized controller design for large-scale networks \([4]\), modeling and analysis of the feedback in neuroanatomy \([5]-[7]\) and the need for better control methods for emerging large-scale systems such as smart-grids and intelligent transportation systems. It is similar in spirit to \([8]\) where the authors considered a sparse \( H_{\infty} \) analysis, but differ in that we consider systems in input/output form. Another approach to reduce conservativeness is to consider stochastic formulations for multiplicative uncertainty as in \([9]\).

A. Outline

Section II introduces notation and gives some background on robust stability for static and dynamic matrices. We introduce and analyze the new robustness measure in Section III and provide a convex upper bound. Section IV describes the properties of the upper bound and in Section V we show how to compute it and characterize the optimal solution. Concluding remarks and directions for future research are contained in Section VI.

II. PRELIMINARIES AND NOTATION

This section contains a brief mathematical background, the reader is referred to the excellent textbook \([10]\). Latin letters denote real-valued vectors and matrices like \( x \in \mathbb{R}^n \) and \( A \in \mathbb{R}^{m \times n} \). For a matrix \( A \in \mathbb{R}^{m \times m} \), \( A_{ij} \) means the element on the \( i \)th row and \( j \)th column, and we refer to the \( i \)th element of a vector \( x \in \mathbb{R}^n \) by \( x_i \). The \( p \)-norm of a vector \( x \) is the largest\(^1\) diagonal element of \( A \).

\[ x_p = \max_{i,j} |a_{ij}| \]

\( 1 \)The robustness measure \( \nu \) is unrelated to Vinnicombe’s \( \nu \)-gap metric. We apologize for the confusion caused by overloading \( \nu \) and highlight the need for further research into new Greek letters.

\( 2 \)In \( H_{\infty} \) and \( \epsilon_1 \)-norm respectively
Given upper bounds $\delta_{ij}$ for $i=1,\ldots,n$ and a stable, causal $n \times n$-dimensional system $G$, the following are logically equivalent:

A. Matrix induced norms and stability of static systems

B. Robust stability with diagonal uncertainty

Before diving into induced norms for dynamical systems, we require insight into the relationship between that of destabilizing the dynamical system $G$ and its magnitude matrix $M_G$. From Theorem 1 we know that if there exists a $\Delta$ that destabilizes Fig. 1, then there exists a constant matrix $M_\Delta$ with the same $\ell_1$ norm so that $I - M_\Delta^{-1} = \mu(\Delta)$.

To study the properties of the new robustness measure and its relationship to $\mu$, we require insight into the relationship between that of destabilizing the dynamical system $G$ and its magnitude matrix $M_G$. From Theorem 1 we know that if there exists a $\Delta$ that destabilizes Fig. 1, then there exists a constant matrix $M_\Delta$ with the same $\ell_1$ norm so that $I - M_\Delta^{-1} = \mu(\Delta)$.

The following Theorem characterizes robust stability of Fig. 1 as conditions on $M_G$.

**Theorem 1** (Theorem 2 in [3]). For $\Delta \in \mathcal{D}$ with $\|\Delta\|_{\infty,\infty} \leq 1$, the following are logically equivalent:

1. The system in Fig. 1 is robustly stable.
2. $\rho(M_G) < 1$, where $\rho()$ denotes the spectral radius.
3. $x = 0$ is stable if $\Delta = 0$ and $x = 0$ imply that $x = 0$.
4. $\inf_{D \in \mathcal{D}} |DM_GD^{-1}|_{\infty,\infty} < 1$
5. $\mu(\Delta) < 1$.

### III. $\nu$: The New $\mu$

Inspired by the role of LASSO [12] in favoring sparse solutions to regression problems, we propose using the sum of $\ell_1$ norms, $\sum_{i=1}^n \|\hat{\delta}_i\|_1$. One could go one step further and study robustness in the $\ell_0$ (number of nonzero $\hat{\delta}_i$) setting. However, any system that has a nonzero diagonal element can be destabilized by a local (possibly very large) perturbation and the corresponding robustness measure will be 1 for almost all systems and not really informative. The new robustness metric, $\nu$, hits the sweet spot and is defined as follows:

**Definition 1 ($\nu$).** Let $\mathcal{D}$ be the set of $\ell_\infty$-stable causal linear time-varying operators with $n$ inputs and outputs, whose off-diagonal elements are zero. Given a causal linear system $G$ with $n$ inputs and outputs

$$
\nu(\Delta) := \frac{1}{\inf_{\Delta \in \mathcal{D}} \|\Delta\|_{\infty,\infty} : \Delta \in \mathcal{D}, (I - G\Delta)^{-1} \text{ unstable}}.
$$

To study the properties of the new robustness measure and its relationship to $\mu$, we require insight into the relationship between that of destabilizing the dynamical system $G$ and its magnitude matrix $M_G$. From Theorem 1 we know that if there exists a $\Delta$ that destabilizes Fig. 1, then there exists a constant matrix $M_\Delta$ with the same $\ell_1$ norm so that $I - M_\Delta^{-1} = \mu(\Delta)$.

**Theorem 2.** Let $\mathcal{D}$ be the set of $\ell_\infty$-stable causal linear time-varying operators with $n$ inputs and outputs, whose off-diagonal elements are zero. Further, let $\mathcal{E} \subset \mathbb{R}^{n \times n}$ be the set of non-negative diagonal matrices. Given upper bounds $\hat{\delta}_{ii}$ for $i=1,\ldots,n$ and a stable, causal $n \times n$-dimensional system $G$, the following are logically equivalent:

| $\mathcal{D}(\Delta)$ | $\mathcal{D}(\Delta)$ | $\mathcal{D}(\Delta)$ |
|-----------------------|-----------------------|-----------------------|
| $1 \cdot 1$           | $\max \sum_{i=1}^n |A_{ij}|$ | NP-HARD               |
| $1 \cdot 2$           | $\sqrt{\max \sum_{i=1}^n |A_{ij}|^2}$ | NP-HARD               |
| $\nu(\Delta)$        | $\max \sum_{i=1}^n |A_{ij}|^2$ | $\max \sum_{i=1}^n |A_{ij}|$ |
1) There exists a $\Delta \in \mathcal{D}$, where each diagonal element is bounded from above, $\|\delta_{ii}\|_{\infty,\infty} \leq \tilde{\delta}_{ii}$, such that the system in Fig. 1 is unstable.

2) There exists a matrix $M_{\Delta} \in \mathcal{C}$, where each diagonal element is bounded from above, $\delta_{ii} \leq \tilde{\delta}_{ii}$, such that $I - \Delta M_{\Delta}$ is singular.

**Proof.** We start by showing that the first claim implies the second. Let $R_{\Delta} = \text{diag}(\delta_{11}, \ldots, \delta_{nn})$, then $\Delta = \hat{\Delta} R_{\Delta}$ for some $\Delta \in \mathcal{D}$, $\|\Delta\|_1 \leq 1$. As Fig. 1 with $\Delta M = \hat{\Delta} R_{\Delta} \Gamma$ is unstable, we conclude the existence of a diagonal non-negative matrix $\hat{M}_{\Delta}$ with $\|\hat{M}_{\Delta}\|_{\infty,\infty} \leq 1$ so that $I - \hat{\Delta} R_{\Delta} M_{\Delta}$ is singular. Taking $M_{\Delta} = \hat{\Delta} R_{\Delta}$ completes the first part of the proof.

The proof of the converse is identical but starts with $M_{\Delta}$.

Theorem 2 implies that we can replace the $\ell_1$ norm in (2) with any norm on the magnitude matrix of $\Delta$ and get 

$$\mu_{\ell_2}(G) = \mu_{\ell_2}(M_{\Delta})$$

Although we do not yet know how to compute $\nu$, from Fig. 1 and Theorem 2 we know that $\nu$ must be absolutely homogeneous and invariant to similarity transforms with matrices that commute with $\mathcal{D}$. Furthermore, we can translate the equivalence relationship between $\|\cdot\|_1$ and $\|\cdot\|_\infty$ into a corresponding relationship between $\nu$ and $\mu$. We summarize the above discussion with the following proposition:

**Proposition 1.** With $G$, $\mathcal{D}$, $\mathcal{C}$ as in Theorem 2, let $D \subset \mathbb{R}^{n \times n}$ be the set of non-negative diagonal matrices, then the following statements are true:

1) $\nu_{\mathcal{D}}(G) = \nu_{\mathcal{C}}(M_{\Delta})$

2) $\nu_{\mathcal{D}}(aG) = |a|\nu_{\mathcal{D}}(G)$ for $a \in \mathbb{R}$.

3) $\nu_{\mathcal{D}}(DG D^{-1}) = \nu_{\mathcal{D}}(G)$ for $D \in \mathbb{D}$.

4) $\mu_{\mathcal{D}}(G)/n \leq \nu_{\mathcal{D}}(G) \leq \mu_{\mathcal{D}}(G)$.

The following theorem tightens the lower bound in 4) by zeroing out different diagonal elements. This result agrees with intuition because we can study how a system interacts with sparse uncertainty by testing the different sparsity patterns separately.

**Theorem 3.** Given $G$ and $\mathcal{D}$ as in Definition 1. Let $I = (i_1, i_2, \ldots, i_m)$ with $m \leq N$ and $i_k \neq i_l$ for $k \neq l$ be an index tuple, and consider the sub-matrix of $M_{\Gamma}$:

$$M_I = \begin{bmatrix}
M_{1i_1} & \cdots & M_{1i_m} \\
\vdots & \ddots & \vdots \\
M_{ni_1} & \cdots & M_{ni_m}
\end{bmatrix}.$$  \hspace{1cm} (3)

Then $\nu_{\mathcal{D}}(G) \geq \frac{\nu(M_I)}{m}$.

**Proof.** Assume without loss of generality that $i_k = k$ for $k = 1, \ldots, m$. This assumption can always be enforced by renaming the signals. Restrict $\Delta$ by setting $\delta_{kk} = 0$ for $k > m$. By Proposition 1 $\nu_{\mathcal{D}}(G) = \nu_{\mathcal{D}}(M_{\Delta})$, so it is sufficient to give the proof in the constant matrix case. Let $\Delta = \text{diag}(\delta_{11}, \ldots, \delta_{nn})$ be the submatrix of $\Delta$ that is nonzero and partition $M_{\Delta}$ into

$$M_{\Delta} = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}.$$  \hspace{1cm} where $M_{11} \in \mathbb{R}^{n \times n}$. Thus $I - \Delta M_{\Delta}$ is invertible if and only if $(I - \Delta I M_{11})$ is invertible, which is equivalent to $\|\Delta I\|_{\infty,\infty} \leq 1/\mu(M_{11})$. From the fourth property of Proposition 1 we conclude that $\nu_{\mathcal{D}}(G) \geq \mu(M_{11})/(m)$.

A. An upper bound of $\nu$

If the norm on the magnitude matrix of $\Delta$ is one in the upper triangle of Table I, then we can use the corresponding dual norm in the lower triangle to construct an upper bound.

Although the induced norm from $\infty$ to 1, in general, is NP-hard to compute, it coincides with the absolute sum for diagonal matrices. To see this, consider

$$\|\Delta\|_{\infty, 1} = \sum_{i=1}^{n} |\delta_{ii}| = \sum_{i=1}^{n} |\delta_{ii}|.$$  \hspace{1cm} (4)

Thus, if $|M_{\Gamma}|_{1,\infty} < 1/|\Delta|_{\infty,1}$ then $I - \Delta M_{\Gamma}$ is non-singular. As $\nu_{\mathcal{D}}$ is invariant under similarity transformations with $D \in \mathbb{D}$, we suggest the following upper bound:

$$\nu_{\mathcal{D}}(G) := \inf_{D \in \mathbb{D}} |DM_{\Delta} D^{-1}|_{1,\infty}.$$  \hspace{1cm} (5)

The 1 to $\infty$ norm is the maximum absolute element of a matrix, see Table I, and can be computed for large-scale connected systems by local evaluation and communication with the closest neighbors.

We end this section by noting that for positive systems, the $H_{\infty}$-norm is achieved by a stationary input [13], [14], so robustness analysis can be done entirely on positive matrices in that case too. We suspect one can derive similar results for positive systems as those in this article.

**Conjecture 1.** For positive systems, there exists a similar convex upper bound for a robustness measure against a causal, diagonal, linear time-varying uncertainty $\Delta$ bounded in the following norm $\|\Delta\|_{\infty, \infty} = \|\delta_{11}\|_{\infty} + \|\delta_{22}\|_{\infty} + \cdots + \|\delta_{nn}\|_{\infty}$.

IV. PROPERTIES OF $\nu$

The lower bound in Theorem 3 shows that if the maximum absolute value is achieved on the diagonal of $M_{\Gamma}$, then the upper bound coincides with the lower bound and is exact. These types of systems are called diagonally maximal and merit a formal definition.

**Definition 2** (Diagonally Maximal). A Matrix $A \in \mathbb{R}^{n \times n}$ is diagonally maximal if the maximum absolute element of $A$ appears on the diagonal. A dynamical system $G$ is diagonally maximal if its magnitude matrix $M_{\Gamma}$ is diagonally maximal.

The following important corollary follows from applying Theorem 3 to each diagonal element.

**Corollary 3.1.** If the matrix $M_{\Gamma} D^{-1}$ is diagonally maximal for some $D \in \mathbb{D}$, then $\nu_{\mathcal{D}}(G) = \nu_{\mathcal{D}}(G)$.

Going back to the systems $P_1$ and $P_2$ in the introduction, we see that $P_1$ is diagonal and hence diagonally maximal and $\nu_{\mathcal{D}}(P_1) = \nu_{\mathcal{D}}(P_1) = 1$. However, for $P_2$ the upper bound is conservative. Indeed, $1/m = \nu_{\mathcal{D}}(P_2) \leq \nu_{\mathcal{D}}(P_2) = 1$. The following theorem describes the gap between $\nu$ and $\nu$.
**Theorem 4.** With \( \nu, G \) and \( \mathcal{D} \) as in Definition 1 and \( \nu \) as in (5), it is true that \( 1 \leq \frac{\nu_G(G)}{\nu_D(G)} \leq n \). Furthermore, the lower bound is achieved by systems \( G \) that are **diagonally maximal** under some similarity transform \( D \) that commutes with \( \mathcal{D} \). Pure rings achieve the upper bound.

**Proof.** By construction \( \nu_G(G) \geq \nu_D(G) \), and by Corollary 3.1 the upper bound is exact for systems that are diagonally maximal under some similarity transform that commutes with \( \mathcal{D} \).

By \( |M_G|_{1,\infty} \leq |M_G|_{\infty,\infty} \) and Proposition 1 we have that \( \nu_G(G) \leq \nu_D(G) \leq n \nu_G(G) \). It remains to show that the upper bound is achieved for pure ring systems. After scaling, balancing, and relabeling the signals, a pure ring system is of the form

\[
x_1(t + 1) = \delta_{11} x_2(t), \quad \ldots, \quad x_n(t + 1) = \delta_{nn} x_1(t).
\]

By Proposition 1, \( \nu_D(G) = \nu_D(M_G) \), so we will study the null space of \( I - M_G \Delta \). \( I - M_G \Delta \) has a nontrivial null space if for some non-zero \( w \in \mathbb{R}^n \),

\[
(I - M_G \Delta)w = 0 \iff \begin{bmatrix} w_1 - \delta_{22} w_2 \\ w_2 - \delta_{33} w_3 \\ \vdots \\ w_n - \delta_{11} w_1 \end{bmatrix} = 0.
\]

If \( w_1 = 0 \), then by substitution we must have \( w = 0 \). So assume without loss of generality that \( w_1 = 1 \). Then we have that \( I - M_G \Delta \) has a nontrivial null space if and only if

\[
\delta_{11} \cdots \delta_{nn} = 1. \quad (6)
\]

We proceed to lower bound \( \sum_{i=1}^{n} \delta_{ii} \) by minimizing it subject to (6). Substitute \( \delta_{nn} = 1/\prod_{i=1}^{n-1} \delta_{ii} \) into the sum to transform the constrained optimization problem into a convex optimization problem over \( \delta_{ii} > 0 \) with the solution \( \min_{\delta_{ii}} \sum_{i=1}^{n} \delta_{ii} = n \). Substitute the lower bound on a destabilizing \( \Delta \) into Definition 1 to get \( \bar{\nu}_D(G) \geq n \nu_G(G) \) as \( \bar{\nu}_G(G) = 1 \). Since the upper bound is equal lower bound, we conclude that the bound is achieved.

By the discussion in this section it is clear that even though \( \bar{\nu} \) bounds \( \nu \), the gap can be pretty significant. It stands to reason that \( \bar{\nu} \) is exact for only a class of disturbances.

**Conjecture 2.** \( \bar{\nu} \) is exact for some class of norm-bounded disturbances.

We conclude this section by studying \( 2 \times 2 \) matrices.

**A. A closed-form formula for \( 2 \times 2 \) matrices.**

Consider without loss of generality, matrices \( M \in \mathbb{R}^{2 \times 2} \) of the form

\[
M = \begin{bmatrix} x & 1 \\ 1 & y \end{bmatrix}.
\]

If \( x > 1 \) or \( y > 1 \) we know that \( \nu_D(M) = \max\{x, y\} \) so only the case \( 0 < x, y < 1 \) remains. We begin by parameterizing all destabilizing \( \Delta \) in \( \delta_{22} \). Setting the determinant to zero we get

\[
\frac{1}{\det(M)} \left( y + \frac{-1}{x - \det(M) \delta_{22}} \right) = \delta_{11}.
\]

Thus \( \nu_D(M) = \delta_{11} (\delta_{22} + \delta_{23}) \) is convex on the domain \([0, 1]\) and the minimum is achieved either on the boundary or at a stationary point. For \( 0 < x, y < 1 \) we have that

\[
\delta_{11} = \frac{y - 1}{\det(M)}, \quad \delta_{22} = \frac{x - 1}{\det(M)}, \quad \nu_D(M) = \frac{\det(M)}{x + y - 2}. \quad (7)
\]

In Fig. 2 we compare the new robustness metric \( \nu \), the upper bound \( \bar{\nu} \) for matrices of the form \( M = \begin{bmatrix} x & w \\ w & y \end{bmatrix} \) for \( x, w, y \in [0, 1] \).

The matrices along the \( \bar{\nu}/\nu = 1 \) line are the diagonally maximal matrices. In the bottom left corner we have the identity matrix, in the top left corner we have the matrix \( \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \) and in the top right corner we have \( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).

Thus \( \nu_D(M) = \delta_{11} (\delta_{22} + \delta_{23}) \) is convex on the domain \([0, 1]\) and the minimum is achieved either on the boundary or at a stationary point. For \( 0 < x, y < 1 \) we have that

\[
\delta_{11} = \frac{y - 1}{\det(M)}, \quad \delta_{22} = \frac{x - 1}{\det(M)}, \quad \nu_D(M) = \frac{\det(M)}{x + y - 2}. \quad (7)
\]

In Fig. 2 we compare the new robustness metric \( \nu \), the upper bound \( \bar{\nu} \) and \( \mu \) for \( 2 \times 2 \)-matrices. We see that \( \bar{\nu} \) is exact for and only for matrices that are diagonally maximal under some \( D \in D \) and conclude that even for diagonally maximal systems, \( \nu \) and \( \mu \) can be very different. As the closed-loop maps generated by system-level synthesis often seem to be diagonally maximal, we conclude that for a large class of relevant systems, computing both \( \bar{\nu} \) and \( \mu \) gives additional information into the nature of destabilizing disturbances even for this class of systems. Based on this observation we state the following conjecture.

**Conjecture 3.** \( \bar{\nu}_D(M) = \nu_D(M) \) only if \( DMD^{-1} \) is diagonally maximal for some \( D \in \mathcal{D} \).

V. COMPUTING \( \bar{\nu} \)

**A. The convex approach**

This section explains how to formulate \( \bar{\nu} \) as a linear program. Let \( M \in \mathbb{R}^{n \times n} \) be a positive matrix. We want to compute

\[
\inf_{D \in D} \max_{ij} \left\{ M_{ij} d_i / d_j \right\}. \quad (8)
\]

As the logarithm is strictly increasing, (8) is equivalent to

\[
\min_{D \in D} \max_{ij} \{ \log(M_{ij}) + \log(d_i) - \log(d_j) \},
\]

where we use the convention that \( \log(0) = -\infty \). Let \( \beta_i = \log(d_i) \), then (8) is equivalent to the following linear program.
that can be solved efficiently using simplex or interior-point methods [15]:
\[
\begin{align*}
\minimize_{\beta \in \mathbb{R}^n, \gamma} & \quad \gamma \\
\text{subject to:} & \quad \log(M_{ij}) + \beta_i - \beta_j \leq \gamma.
\end{align*}
\] (9)

B. Characterizing the solutions of the upper bound

We will relax the positivity assumption of \(d_1, \ldots, d_n\) (8) to allow \(d_i\)s to be zero. Consider the function
\[
\phi_d(M, i, j) = \begin{cases} 
M_{ij} \frac{d_i}{d_j} & \text{if } M_{ij} > 0 \\
0 & \text{if } M_{ij} = 0.
\end{cases}
\] (10)

Then (8) is equivalent to
\[
\inf_{d_1, \ldots, d_n \geq 0} \max_i \phi_d(M, i, j). 
\] (11)

The following theorem shows that if for some \(D \in D\), the maximizing indices of \(DMD^{-1}\) only consists of loops, then \(D\) minimizes (11).

**Theorem 5** (Sufficient condition for optimality). Given a non-negative, non-zero matrix \(M \in \mathbb{R}^{m \times n}\) and non-negative constants \(d_1, \ldots, d_n\). With \(\phi\) as in (10), let \(I\) be the set of maximizing indices of (8), i.e.
\[
I = \left\{(k, l) : \phi_d(M, k, l) = \max_i \phi_d(M, i, j)\right\}.
\]

If for all \((k, l) \in I\) it holds that
\[
\phi_d(M, k, l) = \max_j \phi_d(M, j, l).
\] (12)

Then \(d_1, \ldots, d_n\) is an optimal solution to (8).

**Proof.** First, we show that \(I\) must contain at least one loop. Let \((j_0, j_1) \in I\), and let \(j_{k+1}\) be the smallest integer such that \(\phi_d(M, j_k, j_{k+1}) = \max_i \phi_d(M, j_k, j)\). By induction \((j_k, j_{k+1}) \in I\). Furthermore, as \(n\) is finite, and the selection rule for \(j_{k+1}\) is unique given \(j_k\), there is a \(K \geq 0\) and a \(T \geq 1\) so that \(j_{k+T} = j_k\) for all \(k \geq K\). We denote the limit set containing such points by \(I_* = \{j_k : k \geq K\}\).

Assume towards a contradiction that there are \(d_1', \ldots, d_n'\) so that \(\max_{i} \phi_{d'}(M, i, j) < \max_{i} \phi_{d}(M, i, j)\), and let \((j_0, j_1) \in I_*\). Assume without loss of generality that \(d_{j_1}' > d_{j_1}\), otherwise multiply every \(d_{j_i}'\) by a positive constant so that the assumption holds true. Let \(j_2 = \arg \max_{i} \phi_{d'}(M, j_1, i)\). By assumption, it must hold that \(d_{j_2}' > d_{j_2}d_{j_1}'/d_{j_1}\). Continuing, we have that
\[
d_{j_1}' > \frac{d_{j_{k+T}}'}{d_{j_{k+T-1}}'} > \frac{d_{j_{k+T}}}{d_{j_{k+T-1}}}.
\]

However, since \(j_{k+T} = j_k\) we have that \(d_{j_k}' > d_{j_k}\) which is a contradiction. \(\square\)

By the above theorem, we know that if the maximum is achieved on a loop, then the solution is optimal. It turns out that an optimal solution must contain a loop. This is because if the maximum is achieved on a chain, we can perturb the scales at the end of the chain to make that value smaller, making the chain shorter. Repeating this process reduces all the elements in the maximal chain. We formalize this statement in the following Lemma:

**Lemma 5.1.** Let \(d_1^*, \ldots, d_n^*\) be an optimal solution to (11), and let \(I\) be the set of maximizing indices as in Theorem 5. Then \(I\) contains at least one loop.

**Proof.** If the optimal value is zero, all diagonal elements must be zero, and \((i, i) \in I\) implies that \(I\) contains a loop. Assume towards a contradiction that \(I\) does not contain a loop and that the optimal value is greater than zero. Let \((j_0, j_1) \in I\), and let \(j_{k+1}\) be the smallest integer such that \(\phi_d(M, j_k, j_{k+1}) = \max_j \phi_d(M, j_k, j)\). By assumption there is a \(k\) such that
\[
\phi_d(M, j_k, j_{k+1}) < \max_i \phi_d(M, i, j_k) 
\] (13)

This means that there is a \(d_{j_k}' > 0\) that decreases the right hand side of (13) so that the inequality still holds for \(j_k\), but also holds for \(j_{k-1}\). By induction, this must hold for \(1, \ldots, k\). Repeating for any other chain in \(I\), we conclude that \(\max_j \phi_d(M, i, j) > \max_j \phi_d(M, i, j)\), contradicting optimality. \(\square\)

**Theorem 5 and Lemma 5.1 indicate a relationship between solving (11) and balancing the matrix \(M\) in the maximum absolute element norm. The following theorem strengthens that connection and shows that we can always find a solution to (11) by balancing \(M\).**

**Theorem 6.** For any non-negative matrix \(M \in \mathbb{R}^{m \times n}\), there exists a non-negative solution \(d_1, \ldots, d_n\) to (11) such that
\[
\max_r \phi_d(M, r, k) = \max_r \phi_d(M, k, r), \quad \forall k = 1, \ldots, n. 
\] (14)

**Proof.** We begin by proving the existence of a solution. Assume there is a sequence \(i_1, \ldots, i_m\) such that \(M_{i_ji_{j+1}}M_{ul} \neq 0\) for \(k = 1, \ldots, m\). Then (8) is bounded below by \(\min[M_{ij}, M_{ji}]\) and (8) is equivalent to a linear program with a bounded solution and the minimum is achieved by some \(d_1, \ldots, d_n\). If the assumption is false, we can take \(d = 0\) and the optimal value is zero. If \(M\) is a diagonal matrix, then the claim holds trivially. Assume \(M\) is non-diagonal and let \(\hat{M}\) be the matrix where \(\hat{M}_{ij} = M_{ij}\) for \(i \neq j\) and \(\hat{M}_{jj} = 0.\) Then \(d_1, \ldots, d_n\) are optimal for \(\hat{M}\) if and only if they are optimal for \(\hat{M}\). Note that (14) holds for a maximizing loop of \(\hat{M}\). Let \(d_1, \ldots, d_n\) be an optimal solution to (8) for \(\hat{M}\). By Lemma 5.1, the set of maximizing indices \(I\) contains at least one loop. Remove the rows and columns pertaining the loop from \(\hat{M}\) to get the smaller matrix \(\hat{M}_I\). By recursion on \(\hat{M}_I\) we end up with a new set \(d_1', \ldots, d_n'\) so that (14) is true. \(\square\)

C. An algorithm for balancing the magnitude matrix

We end this section with a simple heuristic algorithm for computing (8) that results from enforcing (14) coordinate-wise in Algorithm 1. The algorithm is similar to Osborne’s algorithm for balancing matrices in the Frobenius norm [16], but balances a matrix in the maximum absolute-element norm and can be computed using local information and
Algorithm 1 Heuristic algorithm for solving (8)

Require: Non-negative $M \in \mathbb{R}^{N \times N}$, $\theta \in (0, 1)$, $T$.

\[
\begin{align*}
& d_1[1] \leftarrow 1 \quad \text{for each } k = 1, \ldots, n \\
& \text{for } t = 1, \ldots, T \text{ do} \\
& \quad \text{for } k = 1, \ldots, n \text{ do} \\
& \quad \quad d_k[t + 1] \leftarrow (1 - \theta) d_k[t] + \theta \sqrt{\max_{\nu \leq k} M_{kk} d_k[t]} / \sqrt{\max_{\nu \leq k} M_{kk} d_k[0]} \\
& \quad \text{end for} \\
& \text{end for}
\end{align*}
\]

that naively taking $\theta = 1$ may cause the algorithm to fail to converge. Consider the matrix,

\[
M = \begin{bmatrix} 0 & 1 \\ x^2 & 0 \end{bmatrix}.
\]

Then $d_1(2) = x$ and $d_2(2) = 1/x$, leading to $D(2)M D^{-1}(2) = M^T$ and the iteration will continue to oscillate back and forth. This is because we are updating each coordinate simultaneously, which is desirable for localized computation. Introducing the interpolation $\theta \in (0, 1)$ seems to solve this issue. Based on the numerical results we conjecture that our algorithm is guaranteed to converge.

Conjecture 4. Algorithm 1 always converges. Moreover the number of iterations required to reach a given tolerance is of $O(n)$ for a fixed $\epsilon$, and $O(\sqrt{c^{-1}})$ for fixed $n$.

VI. CONCLUSIONS

This work introduced and analyzed a new robustness measure $\nu$ that reasonably handles sparsity. We provided a convex upper bound $\mathcal{V}$, characterized its sub-optimality, and gave simple ways to compute it in a distributed way. The companion paper, [17] shows how to compute robust controllers for large-scale systems using $\mu$ and $\nu$. Throughout this article, we gave four conjectures representing important research topics. We conclude with a final conjecture on the computation of $\nu$.

3A Julia implementation of Algorithm 1 can be found at https://github.com/kjellqvist/NuSynthesis.jl.

REFERENCES

[1] K. Zhou and J. C. Doyle, Essentials of Robust Control. Prentice-Hall, 1998.
[2] G. E. Dullerud and F. Paganini, A Course in Robust Control Theory. Springer New York, 2010.
[3] M. A. Dahleh and M. H. Khammash, “Controller design for plants with structured uncertainty,” Autom., vol. 29, pp. 37–56, 1993.
[4] J. Anderson, J. C. Doyle, S. H. Low, and N. Matni, “System level synthesis,” Annual Reviews in Control, vol. 47, pp. 364–393, 2019.
[5] J. Stenberg, J. S. Li, A. A. Sarma, and J. C. Doyle, “Internal feedback in biological control: Diversity, delays, and standard theory,” in 2022 American Control Conference (ACC), 2022, to appear.
[6] J. S. Li, “Internal feedback in biological control: Locality and system level synthesis,” in 2022 American Control Conference (ACC), 2022, to appear.
[7] A. A. Sarma, J. S. Li, J. Stenberg, G. Card, E. S. Heckscher, N. Kasthuri, T. Sejnowski, and J. C. Doyle, “Internal feedback in biological control: Diversity, delays, and standard theory,” in 2022 American Control Conference (ACC), 2022, to appear.
[8] S. You and N. Matni, “A convex approach to sparse $H_\infty$ analysis & synthesis,” in 54th IEEE Conference on Decision and Control (CDC), 2015, pp. 6635–6642.
[9] B. Bamieh and M. Filo, “An input–output approach to structured stochastic uncertainty,” IEEE Transactions on Automatic Control, vol. 65, no. 12, pp. 5012–5027, 2020.
[10] C. Desoer and M. Vidyasagar, Feedback Systems: Input–Output Properties. Academic Press, 1975.
[11] J. Tropp, “Topics in sparse approximation,” 01 2004.
[12] R. Tibshirani, “Regression shrinkage and selection via the lasso,” Journal of the Royal Statistical Society (Series B), vol. 58, pp. 267–288, 1996.
[13] A. Rantzer, “Scalable control of positive systems,” European Journal of Control, vol. 4, pp. 72–80, 2015, st. ECC15.
[14] M. Colombo and R. S. Smith, “A convex characterization of robust stability for positive and positively dominated linear systems,” IEEE Transactions on Automatic Control, vol. 61, no. 7, pp. 1965–1971, 2016.
[15] M. Todd, “The many facets of linear programming,” Mathematical Programming, vol. 91, 04 2002.
[16] E. E. Osborne, “On pre-conditioning of matrices,” J. ACM, vol. 7, pp. 338–345, 1960.
[17] J. S. Li and J. C. Doyle, “Distributed robust control for systems with structured uncertainties,” in 61st IEEE Conference on Decision and Control (CDC), 2022, to appear.