LOCAL LANGLANDS CORRESPONDENCE FOR
CLASSICAL GROUPS AND AFFINE HECKE ALGEBRAS

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Abstract. Using the results of J. Arthur on the representation theory of classical
groups with additional work by Colette Moeglin and its relation with representations
of affine Hecke algebras established by the author, we show that the category of
smooth complex representations of a split $p$-adic classical group and its pure inner
forms is naturally decomposed into subcategories which are equivalent to a tensor
product of categories of unipotent representations of classical groups (in the sense of
G. Lusztig). All classical groups (general linear, orthogonal, symplectic and unitary
groups) appear in this context. We get also parameterizations of representations of
affine Hecke algebras, which seem not all to be in the literature yet.

Let $F$ be a non-Archimedean local field of characteristic $0$ - this assumption on
the characteristic is used in [A, M] but not in [H3] -, and $n \geq 1$ an integer. The
symbol $G$ will denote the group of $F$-rational points of a split classical group of
semi-simple rank $n$ defined over $F$. We will mean by that either a symplectic group
or a (full, i.e. non connected) orthogonal group. (The case of unitary groups will
be treated in the annex.) If $G$ is orthogonal, we will denote by $G^-$ its unique pure
inner form [V, GGP]. If $G$ is symplectic, we will leave $G^-$ undefined (there is no
pure inner form $\neq G$). We will often write $G^+$ for $G$ and denote by $\text{Rep}(G^\pm)$ the
category of smooth complex representations of $G^\pm$.

J. Arthur has determined in [A] the parameters of discrete and tempered $L$-
packets of $G$, including the description of the $R$-groups. (The parametrization for
the inner forms of $G$ is forthcoming.) C. Moeglin has deduced from this in [M] the
Langlands-Deligne parameters which correspond to supercuspidal representations
(for both $G$ and $G^-$), including information on reducibility points. The author has
used this information in [H4] to deduce the parameters of the affine Hecke algebras
that have been shown in [H3] to correspond to the Bernstein components of the
category $\text{Rep}(G^\pm)$.

The aim of the present work is to show that, putting together different Bernstein
components, one obtains a natural decomposition of $\text{Rep}(G)$ into subcategories
\( \mathcal{R}_F^{\nu_0}(G) \) which are equivalent to a tensor product of categories of unipotent representations of classical groups (in the sense of [L2]). All classical groups (general linear, orthogonal, symplectic and unitary groups) appear in this context.

Taking into account the local Langlands correspondence (which is in good shape now after the above mentioned results of Arthur and also of Moeglin - however no final account has been written yet, additional results on the preservation of local constants related to non standard \( L \)-functions are in [He, CST]), we get from this parameterizations of representations of affine Hecke algebras, which seem not all to be in the literature yet. In addition, we explain, how to get a direct correspondence for the irreducible representations in \( \mathcal{R}_F^{\nu_0}(G) \) by parameters \((s, u, \Xi)\) in a given complex reductive group.

The plan of this paper is the following: in section 1., we summarize the results of Moeglin (based on Arthur’s work) on the Langlands correspondence for supercuspidal representations of \( G \). We recall the author’s results relating the Bernstein components of \( \text{Rep}(G^\pm) \) to representations of affine Hecke algebras and give the definition of the categories \( \mathcal{R}_F^{\nu_0}(G) \). In section 2., we explain how to get a direct correspondence for the irreducible representations in \( \mathcal{R}_F^{\nu_0}(G) \) by parameters \((s, u, \Xi)\) in a given complex reductive group. The last section 3. is devoted to the parametrization of representations of affine Hecke algebras, taking into account the local Langlands correspondence. At the end, corollary 3.5, we give the final decomposition result (which does not depend on a final written account of the local Langlands correspondence). There are three annexes A, B and C. In annex A, it is explained how results in [H3] generalize to the full orthogonal group which is not connected. In the annex B, we give an account of the notion of tensor product in the context of linear abelian categories and show that it applies to the categories that we are considering. Unitary groups are treated in annex C, although the results are used progressively in the main body of the paper.

Remark that only those results of this paper which apply to non quasi-split inner forms of \( G \) are conditional: for orthogonal groups, some generalization of [A] to inner forms may be required for [M1, M3], but this is forthcoming, and for, unitary groups, the case of the non split inner form has not been treated in [M2, M3], but the result is expected to be true and the proof not to be a major problem. Remark that [A] is based on the stabilization of the twisted trace formula (or at least results which take part of this stabilization), but this has been accomplished recently thanks to a long series of papers by J.-L. Waldspurger.

One may expect that a similar pattern holds for a general \( p \)-adic reductive group. The method of Arthur based on the trace formula does not apply to the general case, but there are technics on the Hecke algebra side developed by E. Opdam [O1, O2] which may be a guide, once the results in [H3] generalized to an arbitrary reductive \( p \)-adic group, to get a conjectural description of local \( L \)-packets for a general reductive \( p \)-adic group. A generalization of [L2, L3] to groups which are not adjoint would also be helpful.
The author thanks S. Riche for many helpful discussions on Lusztig's work [L2, L3].

1. We will denote by $\hat{G}$ the "dual" group of $G$, which means that its connected component is the Langlands dual group of the connected component of $G$ and that it is either a symplectic or a full (disconnected) orthogonal group. We will write $Z_{\hat{G}}$ for the center of the connected component of $\hat{G}$ (which is trivial if and only if $G$ is symplectic and of order two otherwise) and denote by $\iota: \hat{G} \to GL_N(\mathbb{C})$ the canonical embedding, i.e. $N$ equals $2n$ if $G$ is orthogonal and $N$ equals $2n + 1$ if $G$ is symplectic. If $l$ is an integer between 1 and $n$, $H_l$ will denote (the group of $F$-rational points) of a split classical group of semi-simple rank $l$ of the same type (symplectic, even or odd orthogonal) than $G$. The symbols $H_l^+$ and $H_l^-$ will have the appropriate meaning. We will also denote by $\iota$ the canonical embedding $\hat{H}_l \to GL_L(\mathbb{C})$, hoping that this will not be a source of confusion.

Let $W_F$ be the Weil group of $F$. It’s the semi-direct product of the inertial subgroup $I_F$ with the cyclic subgroup generated by a Frobenius automorphism $Fr$, $W_F = \langle Fr \rangle \rtimes I_F$. A character $\chi$ of $W_F$ is called unramified, if it is trivial on $I_F$. By local class field theory, such a character is identified with a character of $F^\times$, trivial on the units of its ring of integers $O_F$, the character sending $Fr - 1$ to $q$ being identified with the absolute value $| \cdot |_F$.

We will call Langlands parameter for $\hat{G}$ a continuous homomorphism of $W_F$ into $\hat{G}$ which sends $Fr$ to a semi-simple element. (It follows from the continuity that the image of $I_F$ is finite.) A homomorphism $\rho: W_F \times SL_2(\mathbb{C}) \to \hat{G}$ will be called a Langlands-Deligne parameter, if its restriction to the first factor is a Langlands parameter and the restriction to the second factor a morphism of algebraic groups. A Langlands or Langlands-Deligne parameter for $\hat{G}$ will be called discrete, if its image is not included in a proper Levi subgroup. Two Langlands or Langlands-Deligne parameters are said equivalent if they are conjugated by an element of $\hat{G}$.

If $\rho$ is an irreducible representation of $W_F$, the set of equivalence classes of representations of the form $\rho^s := \rho | \cdot |_F^s$, $s \in \mathbb{C}$, will be called the inertial class of $\rho$. The group of unramified characters of $W_F$ acts on the inertial class of $\rho$ by torsion. We will denote by $t_\rho$ the order of the stabilizer of the equivalence class of $\rho$. If $\rho$ and $\rho'$ are in the same inertial class, then $t_\rho = t_{\rho'}$, and the definition of $t_\rho$ does not depend neither on the choice of $Fr$.

If $\rho$ is a self-dual representation, we will say that it is of type $\hat{G}$, if it factors through a group of type $\hat{G}$ (meaning that the image of $\rho$ is contained in an orthogonal group if $\hat{G}$ is orthogonal and in a symplectic group if $\hat{G}$ is symplectic). Otherwise, we will say that $\rho$ is not of type $\hat{G}$. We stretch that the use of either of these notions will presume that $\rho$ is self-dual.

If $a$ is an integer $\geq 1$, $sp(a)$ will denote the unique irreducible representation of
$SL_2(\mathbb{C})$ of dimension $a$.

1.1 Theorem: [M1, 1.5.1] 1) A Langlands-Deligne parameter $\varphi : W_F \times SL_2(\mathbb{C}) \to \hat{G}$ corresponds to a supercuspidal representation of $G^+$ or $G^-$, if and only if

$$\iota \circ \varphi = \bigoplus_{\rho \text{ not of type } \hat{G}} a_{\rho} (\hat{\bigotimes}(\rho \otimes \text{sp}(2k))) \oplus \bigoplus_{\rho \text{ of type } \hat{G}} a_{\rho} (\hat{\bigotimes}(\rho \otimes \text{sp}(2k-1))).$$

2) Given $\varphi$ as in 1), denote by $z_{\varphi,\rho,k}$ the diagonal matrix in $\hat{G}$ that acts by $-1$ on the space of the direct summand $\rho \otimes \text{sp}(2k)$ (resp. $\rho \otimes \text{sp}(2k-1)$) of $\iota \circ \varphi$ and by 1 elsewhere. Put $S_{\varphi} = C_{\hat{G}}(\text{Im}(\varphi))/C_{\hat{G}}(\text{Im}(\varphi))^c$. The elements $z_{\varphi,\rho,k}$ lie in $C_{\hat{G}}(\text{Im}(\varphi))$ and their images $\tau_{\varphi,\rho,k}$ generate the commutative group $S_{\varphi}$.

A pair $(\varphi, \epsilon)$ formed by a discrete Langlands-Deligne parameter as in 1) and a character $\epsilon$ of $S_{\varphi}$ corresponds to a supercuspidal representation of either $G^+$ or $G^-$, if and only if $\epsilon(z_{\varphi,\rho,1}) = (-1)^{-1} \epsilon(z_{\varphi,\rho,1})$ with $\epsilon(z_{\varphi,\rho,1}) = -1$ for $\rho$ not of type $\hat{G}$ and $\epsilon(z_{\varphi,\rho,1}) \in \{1,-1\}$ for $\rho$ of type $\hat{G}$. It corresponds to a supercuspidal representation of $G^+$ if $\epsilon(\zeta_{\hat{G}}) = 1$ and to a supercuspidal representation of $G^-$ otherwise.

3) Suppose that $\varphi$ satisfies the property in 1). Let $t_o$ be the number of $\rho$ of type $\hat{G}$ with $a_{\rho}$ odd, put $t_o = 1$ if there are none of them, and let $t_e$ be the number of the remaining $\rho$ of type $\hat{G}$ for which $a_{\rho}$ is even.

If $G$ is symplectic, there are $2^{t_o-1}2^{t_e}$ non isomorphic supercuspidal representations of $G^+$ with Langlands-Deligne parameter $\varphi$.

If $G$ is orthogonal, put $t_o = (-1)^{a_{\rho}+1}$, if $\rho$ is not of type $\hat{G}$, and put $t_o = (-1)^{a_{\rho}}$, if $\rho$ is of type $\hat{G}$ and $a_{\rho}$ even. There exists a supercuspidal representation of $G^+$ with Langlands-Deligne parameter $\varphi$ if and only if either there is a $\rho$ of type $\hat{G}$ with $a_{\rho}$ odd or $\prod_{\rho} \epsilon_{\varphi,\rho} = 1$.

If the above existence condition is satisfied, the number of supercuspidal representations with Langlands-Deligne parameter $\varphi$ equals $2^{t_o-1}2^{t_e}$ and all these representations of $G^+$ are non isomorphic.

The remaining alternating characters correspond to representations of $G^-$, remarking that there are $2^{t_o+t_e}$ alternating characters for orthogonal $G$.

Proof: 1) and 2) are stated as this in the paper of Moeglin. Concerning 3), if $G$ is an orthogonal group, the theorem in the paper of Moeglin says that there is a supercuspidal representation of $G^+$ associated to $\varphi$, if and only if there exists an alternating character $\epsilon_\varphi$ corresponding to $\varphi$ which takes value 1 on $-1$. The number of non isomorphic supercuspidal representations corresponding to $\varphi$ equals the number of alternating characters with this property.

For $\rho$ not of type $\hat{G}$, there is a unique choice of an alternating character and its value on $-1$ is $\prod_{k=1}^{a_{\rho}} (-1)^k$. For $\rho$ of type $\hat{G}$, there are always two choices of an
alternating character. If \( a_\rho \) is even and not divisible by 4 the value taken on \(-1\) is always \(-1\). If \( a_\rho \) is divisible by 4, the value taken on \(-1\) is always 1. If \( a_\rho \) is odd, there is one alternating character which takes value 1 on \(-1\) and another one which takes value \(-1\) on \(-1\).

One concludes by remarking that, if there are alternating characters attached to a \( \rho \) which take respectively value 1 and \(-1\) on \(-1\), then one can of course always find an alternating character for \( \varphi \) with value 1 on \(-1\).

If \( G \) is symplectic, one can conclude as above, after having observed that there is always a \( \rho \) of type \( \hat{G} \) with \( a_\rho \) odd.

\( \blacksquare \)

1.2. Definition: We will fix for the rest of the paper in each inertial class \( O \) of an irreducible representation of \( W_F \) a base point \( \rho_0 \). It will always be assumed to be the equivalence class of a unitary representation, which is in addition self-dual if \( O \) contains such an element. In this last case, we take \( \rho \) of the same type as \( \hat{G} \) if there is such a representation in \( O \). This base point will be called in the sequel a normed representation (w.r.t. \( \hat{G} \)).

A discrete Langlands parameter \( \tau : W_F \to \hat{G} \) will be called normed, if \( \iota \circ \tau \) is the direct sum of inequivalent normed representations of \( W_F \). If \( \hat{M} \simeq GL_{k_1}(\mathbb{C}) \times \cdots \times GL_{k_r}(\mathbb{C}) \times \hat{H}_l \) is a Levi subgroup of \( \hat{G} \), then a discrete Langlands parameter \( \varphi : W_F \to \hat{M} \) is called normed if it is of the form \( \gamma \mapsto (\rho_1(\gamma), \ldots, \rho_r(\gamma), \tau(\gamma)) \), where the \( \rho_i \) are irreducible normed representations and \( \tau \) is a discrete normed Langlands parameter. If \( s \) is in the center of \( \hat{M} \), then \( \chi_s \) will denote the unramified character of \( W_F \), such that \( \chi_s(Fr) = s \), and \( \varphi_s \) will denote the Langlands parameter \( W_F \to \hat{M} \) that satisfies \( \rho_s(\gamma) = \rho(\gamma)\chi_s(\gamma) \). The set of equivalence classes of Langlands parameters of the form \( \varphi_s \) with \( s \) in the center of \( \hat{M} \) will be called the inertial orbit of \( \varphi \).

If \( \rho \) is an irreducible representation of \( W_F \), \( m(\rho'; \varphi) \) will denote the multiplicity of \( \rho' \) in the representation \( \rho_1 \oplus \cdots \oplus \rho_s \oplus \rho_s^\vee \oplus \cdots \oplus \rho_1^\vee \oplus (\iota \circ \tau) \).

A Langlands parameter \( \varphi : W_F \to \hat{G} \) will be called normed, if there is a minimal Levi subgroup \( \hat{M} \) of \( \hat{G} \) containing the image of \( \varphi \), such that \( \varphi \) is a discrete normed Langlands parameter with respect to \( \hat{M} \).

Remark: Normed Langlands parameters will play a similar role than the trivial Langlands parameter for the set of unipotent representations. Unfortunately, it seems not possible to fix in general a ”canonical” base point (see also remark after the next proposition).

1.3 Proposition: Let \( O \) be the inertial orbit of an irreducible representation of \( W_F \) and \( \rho_0 \) the normed representation in this inertial orbit. The map \( \mathcal{O} \to \mathbb{C} \), defined by \( \rho \mapsto f_\rho := \rho(Fr^{**})\rho_0(Fr^{**})^{-1} \), is a bijection. If \( \rho_0 \) is self-dual, then \( \rho \) is self-dual, if and only if \( f_\rho \in \{\pm 1\} \).
Proof: By definition, there is a complex number $s$ such that $\rho = \rho_0 \otimes | \cdot |_F^s$. It follows from this that the map $\rho \mapsto f_\rho$ is well defined and that, for $\rho$ as above, $f_\rho = q^{-st_\rho}$. One sees that the map is surjective. Put $\rho' = \rho_0 \otimes | \cdot |_F^{-s}$. Then $f_\rho = f_{\rho'}$ is equivalent to $q^{(s-s')t_\rho} = 1$. This implies that $| \cdot |_F^{s-s'}$ stabilizes the equivalence class $\rho$. Consequently, $\rho' = \rho \otimes | \cdot |_F^{s-s} = \rho$. So, the map is also injective.

Assume now $\rho_0$ self-dual and that $\chi$ is an unramified character such that $\rho := \rho_0 \otimes \chi$ is also self-dual. This implies that $\chi^2$ stabilizes $\rho_0$ and consequently one has $f_\rho^2 = 1$. One sees that it is enough to twist $\rho_0$ by $| \cdot |_F^{s_{\rho_0}/2}$ to get a representation $\rho'$ that satisfies $f_{\rho'} = -1$.

Remark: It seems in general not possible to distinguish an element $\rho$ of $\mathcal{O}$, such that $f_\rho = 1$, even if $\rho$ is self-dual: although $\rho_0$ is induced from an unramified extension of degree $t_\rho$, there is no reason why $\rho(F^{s_\rho})$ should be a scalar. That is the reason, why we had to make a choice in our definition of a normed representation.

1.4 Definition: If $\rho$ is self-dual, we will denote by $\rho_-$ the unique element in its inertial orbit such that $f_{\rho_-} = -1$.

Remark: If $\rho$ is self-dual, $\rho$ is either orthogonal or symplectic. However, it happens that $\rho_-$ is not of the same type (i.e. symplectic or orthogonal) than $\rho$ (cf. [Mo]). By our convention of a normed representation, this only happens if $\rho_0$ is of type $\widehat{G}$.

1.5 If $\varphi_0 : W_F \to \widehat{G}$ is a normed Langlands parameter, denote by $\text{supp}(\varphi_0)$ the set of irreducible representations $\rho$ of $W_F$ with $m(\rho; \varphi_0) \neq 0$ and by $\text{supp}'(\varphi_0)$ the subset formed by those representations which are self-dual. We will put an equivalence $\sim$ on $\text{supp}(\varphi_0)$ defined by $\rho \sim \rho'$.

Denote by $\mathcal{S}(\varphi_0)$ the set of families of pairs $(a_{\rho,+}, a_{\rho,-})_\rho$ indexed by $\text{supp}'(\varphi_0)$, such that

$$m(\rho; \varphi_0) \geq \begin{cases} a_{\rho,+}(a_{\rho,+} + 1) + a_{\rho,-}(a_{\rho,-} + 1), & \text{if } \rho \text{ and } \rho_- \text{ not of type } \widehat{G}, \\ a_{\rho,+}^2 + a_{\rho,-}(a_{\rho,-} + 1), & \text{if } \rho \text{ of type } \widehat{G}, \text{ but not } \rho_- , \\ a_{\rho,+}^2 + a_{\rho,-}^2, & \text{if } \rho \text{ and } \rho_- \text{ of type } \widehat{G}, \end{cases}$$

with the additional condition that the terms of both sides in the above inequalities have same parity (if $\rho$ and $\rho_-$ are both not of type $\widehat{G}$, this is always satisfied).

Put $\kappa_\rho' = 1$ if $\rho$ is of type $\widehat{G}$ and $\kappa_\rho' = 0$ otherwise. If $S = (a_{\rho,+}, a_{\rho,-})_\rho$ lies in $\mathcal{S}(\varphi_0)$, then the dimension $L_S$ of the representation

$$\bigoplus_{\rho \in \text{supp}'(\varphi_0)} \bigoplus_{k=1}^{a_{\rho,+}} (\rho \otimes sp(2k - \kappa_\rho')) \oplus \bigoplus_{k=1}^{a_{\rho,-}} (\rho_- \otimes sp(2k - \kappa_\rho_-))$$
has the same parity than \( N \). If we denote by \( l_S \) the rank of the group \( H_{l_S} \) whose Langlands dual embeds canonically into \( GL_{l_S}(\mathbb{C}) \), then there is, up to equivalence, a unique discrete Langlands-Deligne parameter \( \varphi^S : W_F \to \widetilde{H}_{l_S} \) [GGP, 8.1.ii] (as we consider the full orthogonal group, the restriction for the even orthogonal group does not apply), such that the above representation is equivalent to \( \iota \circ \varphi^S \). Denote by \( \mathcal{S} \) the set of alternating characters of \( S_{\varphi^S} \) (see theorem 1.1, 2) for the definition of this group) and, for \( \epsilon \in \mathcal{S} \), by \( \epsilon_Z \) its restriction to \( Z_G \) (which can be 1 or \(-1\)). Write \( \tau_{S,\epsilon} \) the irreducible supercuspidal representation of \( H^t_{l_S} \) which corresponds to the Langlands-Deligne parameter \( \varphi^S \) and the alternating character \( \epsilon \) of \( S_{\varphi^S} \).

Denote by \( k_\rho \) the dimension of \( \rho \) and put \( m_\pm(\rho; \varphi^S) = m(\rho; \varphi^S) + m(\rho; \varphi^S) \).

Define \( \widetilde{M}_S \) to be the Levi subgroup of \( \hat{G} \) that is isomorphic to

\[
\prod_{\rho \in (\text{supp}(\varphi_0) - \text{supp}(\varphi_0))/\sim} GL_{k_\rho}(\mathbb{C})^{m(\rho; \varphi_0)} \times \prod_{\rho \in \text{supp}(\varphi_0)} GL_{k_\rho}(\mathbb{C})^{\lfloor m(\rho; \varphi_0) - m_\pm(\rho; \varphi^S) \rfloor/2} \times \widetilde{H}_{l_S}.
\]

(Here \( / \sim \) stands for the equivalence classes w.r.t. the relation \( \sim \) defined above.) Let \( \varphi_S \) be the discrete Langlands-Deligne parameter \( W_F \times SL_2(\mathbb{C}) \to \widetilde{M}_S \) such that

\[
\iota \circ \varphi_S = \oplus_{\rho \in \text{supp}(\varphi_0)} [m(\rho; \varphi_0) - m(\rho; \varphi^S)]/2 \oplus (\iota \circ \varphi^S).
\]

Denote by \( M^\pm_S \) the standard Levi subgroup of \( G^\pm \) which corresponds to \( \widetilde{M}_S \) (\( M^\pm_S \) exists only if \( \widetilde{M}_S \) is “relevant”, i.e. \( l_S \geq 2 \)). It is isomorphic to a product of general linear groups with one factor isomorphic to \( H^t_{l_S} \). For \( S \in \mathcal{S}(\varphi_0) \), \( \epsilon \in \mathcal{S} \), denote by \( \sigma_{S,\epsilon} \) the supercuspidal representation of \( M^t_S \) which corresponds to \( \varphi_S \) and \( \epsilon \) (i.e. the factor \( H^t_{l_S} \) acts by \( \tau_{S,\epsilon} \)) and by \( O_{S,\epsilon} \) the corresponding inertial orbit, i.e. \( O_{S,\epsilon} \) is the set of equivalence classes of representations of \( M^t_S \) which are unramified twists of \( \sigma_{S,\epsilon} \).

In general, if \( \sigma' \) is an irreducible supercuspidal representation of \( M' \), we will denote by \( \varphi_{\sigma'} \) the Langlands-Deligne parameter of \( \sigma' \) obtained by applying respectively 1.1 and the local Langlands correspondence to the \( GL_k \).

**Theorem:** The family \( (M_S, O_{\sigma_{S,\epsilon}})_{S,\epsilon} \) exhausts the set of inertial orbits of supercuspidal representations \( \sigma' \) of standard Levi subgroups \( M' \) of \( G^+ \) and \( G^- \) with \( M' \succeq M \) and such that \( (\iota \circ \varphi_{\sigma'})_{|W_F} \) is an unramified twist of \( (\iota \circ \varphi_0)_{|W_F} \).

One has \( (M_S, O_{\sigma_{S,\epsilon}}) = (M_{S'}, O_{\sigma_{S',\epsilon'}}) \), if and only if \( (S, \epsilon) = (S', \epsilon') \).

**Proof:** The first part follows directly from the constructions and theorem 1.1. For the second part: to have \( (M_S, O_{\sigma_{S,\epsilon}}) = (M_{S'}, O_{\sigma_{S',\epsilon'}}) \), one needs \( l_S = l_{S'} \) and \( \tau_{S,\epsilon} = \tau_{S',\epsilon'} \), but then the other factors of \( \sigma_{S,\epsilon} \) and \( \sigma_{S',\epsilon'} \) must be unramified twists of each other. \( \square \)
1.6 We summarize below the (partly expected) properties of the local Langlands correspondence for \( G \), which is in quite good shape now (see remarks below). If \( \mathcal{O} \) is the inertial orbit of a supercuspidal representation of a Levi subgroup of \( G^\pm \), we will denote by \( \text{Rep}_{\mathcal{O}}(G^\pm) \) the corresponding Bernstein component of \( \text{Rep}(G^\pm) \) [B].

**Local Langlands Correspondence.**

If \( \varphi_0 : W_F \to \hat{G} \) is a normed Langlands parameter, put, for \( S \in \mathcal{S}(\varphi_0), \hat{S}^\pm = \{ \epsilon \in \hat{S} | \epsilon_{\mathbb{Z}} = \pm 1 \}, \mathcal{R}_{\mathcal{E}}^{\varphi_0}(G) = \prod_{S \in \mathcal{S}(\varphi_0), \epsilon \in \hat{S}^\pm} \text{Rep}_{\mathcal{O}_S}(G^\pm) \) and denote by \( \mathcal{R}_{\mathcal{E}}^{\varphi_0}(G) \) the disjoint union of \( \mathcal{R}_{\mathcal{E}}^{\varphi_0,+}(G) \) and \( \mathcal{R}_{\mathcal{E}}^{\varphi_0,-}(G) \).

The set of equivalence classes of pairs \((\varphi, \Xi)\) with \( \varphi : W_F \times SL_2(\mathbb{C}) \to \hat{G} \) a Deligne-Langlands parameter such that \( \varphi|_{W_F} \) is in the inertial orbit of \( \varphi_0 \), and \( \Xi \) an irreducible representation of \( C_{\hat{G}}(\text{Im}(\varphi))/(C_{\hat{G}}(\text{Im}(\varphi))^0 \), is in natural bijection with \( \mathcal{R}_{\mathcal{E}}^{\varphi_0}(G) \).

All smooth irreducible representations of \( G^+ \) and \( G^- \) are obtained in this way. Pairs \((\varphi, \Xi)\) with \( \Xi|_{Z_{\hat{G}}} = 1 \) (resp. \( \Xi|_{Z_{\hat{G}}} = -1 \)) correspond to representations of \( G^+ \) (resp. \( G^- \)), those with \( \varphi \) discrete to square integrable representations and those with \( \varphi(W_F) \) bounded to tempered representations.

In addition, the following equalities of local constants hold: if \( \hat{M} \) is the standard Levi subgroup of a maximal standard parabolic subgroup \( \hat{P} \) of \( \hat{G} \), denote by \( r_1, r_2 \) the irreducible components of the regular representation of \( \hat{M} \) on the Lie algebra of the unipotent radical of \( \hat{P} \). Let \( \pi \) be an irreducible smooth representation of the corresponding maximal Levi subgroup \( M \) of \( G \) and \( \varphi_\pi : W_F \times SL_2(\mathbb{C}) \to \hat{M} \) its Langlands-Shahidi method parameter. Then, the local factors defined by the Langlands-Shahidi method satisfy, for \( i = 1, 2 \),

\[
\gamma(r_i \circ \varphi_\pi, s) = \gamma(\pi, r_i, s), \quad \epsilon(r_i \circ \varphi_\pi, s) = \epsilon(\pi, r_i, s) \quad \text{et} \quad L(r_i \circ \varphi_\pi, s) = L(\pi, r_i, s).
\]

**Remark:**

(i) It is explained in [Sh, 8] how to define the local factors for non generic representations and also for representations of inner forms (see also [H2, section 4]).

(ii) If \( G \) is split and \( \hat{M} = GL_k(\mathbb{C}) \times \hat{H}_i \), then \( r_1 \) is the standard representation \( id_{GL_k(\mathbb{C})} \oplus 1 \) and \( r_2 = \text{Sym}^2 \circ id_{GL_k(\mathbb{C})} \) or \( \wedge^2 \circ id_{GL_k(\mathbb{C})} \), depending if \( \hat{G} \) is symplectic or orthogonal.

(iii) The local Langlands correspondence for classical groups is in quite good shape now in consequence of the work of J. Arthur who describes the discrete series and tempered representations with their \( R \)-groups (see [A] for the split case, the case of inner forms is forthcoming) and the work of C. Moeglin [M1, M3]. Results on the preservation of local factors for symplectic and orthogonal Galois representations have been established by Cogdell-Shahidi-Tsai [CST], completing
work of G. Henniart [He]. However, no final account has been written on all this yet.

Remark that for $G = \text{SO}_{2n+1}(F)$, the case of $\varphi_0 = 1$ has been solved in [L2] with additional work in [W]. The question of local factors seems not to have been addressed.

(iv) For the group $\text{Sp}_4(F)$, W.-T. Gan and S. Takeda gave in [GT] properties for the local Langlands correspondence, which makes it unique.

1.7 The following result of [H4] is obtained by linking the results of C. Mœglin to [H3] (see the remark after theorem A.7 in the annex for the case of the non connected orthogonal group). Recall that it can well happen that $\varphi$ is orthogonal and $\varphi_{-}$ symplectic or vice versa [Mo]. The terminology for affine Hecke algebras with parameters used below is the one from [L1], after evaluation in $q^{1/2}$ as done in [L2, L3].

Recall the equivalence relation on $\text{supp}(\varphi_0)$ given by $\rho \sim \rho^{\vee}$ introduced in 1.5.

**Theorem:** [H4, Theorem 5.2 and remark thereafter] Let $\varphi_0$ be a normed Langlands parameter, $S \in \mathcal{S}(\varphi_0)$, $S = (a_{\rho,+}, a_{\rho,-})_\rho$ and $\epsilon \in \hat{S}$.

The category $\text{Rep}_{\text{O}, S, \epsilon}(G_{\rho})$ is equivalent to the category of right modules over the tensor product $\otimes_{\rho \in \text{supp}(\varphi_0)/\sim} \mathcal{H}_{\varphi_0, S, \rho}$ where $\mathcal{H}_{\varphi_0, S, \rho}$ are extended affine Hecke algebras of the following type:

- if $\rho$ is not self-dual, $\mathcal{H}_{\varphi_0, S, \rho}$ is an affine Hecke algebra with root datum equal to the one of $\text{GL}_m(\rho; \varphi_0)$ and equal parameters $q^{t_\rho}$;

- if $\rho$ and $\rho_{-}$ are both of the same type than $\hat{G}$ and $a_{\rho,+} = a_{\rho,-} = 0$, then $\mathcal{H}_{\varphi_0, S, \rho}$ is the semi-direct product of an affine Hecke algebra with root datum equal to the one of $\text{SO}_m(\rho; \varphi_0)$ and equal parameters $q^{t_\rho}$ by the group algebra of a finite cyclic group of order 2, which acts by the outer automorphism of the root system;

- otherwise, putting $\kappa_{\rho, \pm} = 0$ (resp. $= 1$) if $\rho_{\pm}$ is of type $\hat{G}$ (resp. not of type $\hat{G}$), $\mathcal{H}_{\varphi_0, S, \rho}$ is an affine Hecke algebra with root datum equal to the one of $\text{SO}_m(\rho; \varphi_0) - m_{\pm}(\rho; \varphi_0) + 1$ and unequal parameters $q^{t_\rho}, \ldots, q^{t_\rho}, q^{t_\rho}(a_{\rho,+} + a_{\rho,-} + (\kappa_{\rho,+} + \kappa_{\rho,-} 2))$, remarking that $m(\rho; \varphi_0) - m_{\pm}(\rho; \varphi_0) + 1$ is necessarily an odd number.

Remark: (i) If $\rho$ is not of type $\hat{G}$ and $a_{\rho,+} = a_{\rho,-} = 0$, then it is well known that the affine Hecke algebra $\mathcal{H}_\rho$ expressed above is isomorphic to an affine Hecke algebra with root datum equal to the one of $\text{Sp}_m(\rho; \varphi_0)$ and equal parameter $q^{t_\rho}$.

(ii) The $\kappa_{\rho, -}$ in the above theorem is related to the $\kappa_{\rho, +}$ in 1.5 by the relation $\kappa_{\rho, -}' = 1 - \kappa_{\rho, +}$ and $\kappa_{\rho, +}' = 1 - \kappa_{\rho, -}$.

1.8 The notion of a tensor product of linear abelian categories is treated in [D] and recalled in the annex B. Recall the equivalence relation on $\text{supp}(\varphi_0)$ given by
\( \rho \sim \rho^\vee \) introduced in \textbf{1.5}.

**Corollary:** The category \( \mathcal{R}_F^{\varphi_0}(G) \) is equivalent to the category

\[
\bigoplus_{S \in S(\varphi_0), \epsilon \in \hat{S}^\pm} \left( \bigotimes_{\rho \in \text{supp}(\varphi_0)/\sim} \text{right} - \mathcal{H}_{\varphi_0, \rho} - \text{modules} \right).
\]

**Proof:** It follows from \textbf{B.2} and \textbf{B.3} that the tensor product exists and can be applied to \textbf{1.7} and \textbf{1.6} to give the statement of the corollary. \( \square \)

1.9 The above results generalize to pure inner forms of quasi-split unitary groups over \( F \), as remarked in annex C, C.1 - C.5.

2. The object of this section is to relate the parameters for \( \mathcal{R}_F^{\varphi_0}(G) \) for a given normed Langlands parameter \( \varphi_0 \) to data formed by semi-simple and unipotent elements in a given complex group.

**2.1 Proposition:** ([GGP, section 4])

Let \( \hat{M} = GL_{k_1}(\mathbb{C}) \times \cdots \times GL_{k_d}(\mathbb{C}) \times \hat{H}_l \) be a Levi subgroup of \( \hat{G} \), \( \varphi : W_F \to \hat{M} \) a discrete normed Langlands parameter, \( \iota \circ \varphi = \rho_1 \oplus \cdots \oplus \rho_r \oplus (\iota \circ \tau) \).

Then, one has \( C_\hat{G}(\text{Im}(\varphi)) = \prod_{\rho} H_{\rho; \varphi}(m(\rho; \varphi)) \), the product going over representatives of the equivalence classes of irreducible normed representations \( \rho \) of \( W_F \), while the \( H_{\rho; \varphi}(m) \) are complex classical groups with \( H_{\rho; \varphi}(m) \) isomorphic to \( GL_m(\mathbb{C}) \) if \( \rho \) is not self-dual, to \( Sp_m(\mathbb{C}) \) if \( \rho \) is not of type \( \hat{G} \) and to \( O_m(\mathbb{C}) \) if \( \rho \) is of type \( \hat{G} \) (with the convention \( O_1(\mathbb{C}) = \{ \pm 1 \} \) if \( m = 1 \)).

Finally, \( C_{GL_N(\mathbb{C})}(\text{Im}(\varphi)) = \prod_{\rho} G_{\rho; \varphi}(m(\rho; \varphi)) \), the product going over representatives of the equivalence classes of irreducible representations \( \rho \) of \( W_F \), while \( G_{\rho; \varphi}(m) \) is isomorphic to \( GL_m(\mathbb{C}) \), if \( \rho \) is self-dual, to \( GL_{\frac{m}{2}}(\mathbb{C}) \times GL_{\frac{m}{2}}(\mathbb{C}) \) if \( \rho \) is not self-dual, and the group \( G_{\rho; \varphi}(m) \) contains \( H_{\rho; \varphi}(m) \) in each case.

On the other side, \( C_{GL_N(\mathbb{C})}(\text{Im}(\varphi_s)) \subseteq \prod_{\rho} G_{\rho; \varphi}(m(\rho; \varphi)) \) for every unramified twist \( \varphi_s \) of \( \varphi \) with \( s \) in the center of \( \hat{M} \).

**2.2 Lemma:** Let \( \hat{M} \simeq GL_{k_1}(\mathbb{C}) \times \cdots \times GL_{k_d}(\mathbb{C}) \times \hat{H}_l \) be a standard Levi subgroup of \( \hat{G} \) and let \( \varphi : W_F \to \hat{M} \) be a discrete normed Langlands parameter, \( \gamma \mapsto (\rho_1(\gamma), \ldots, \rho_d(\gamma), \tau(\gamma)) \). Denote by \( \varphi_0 \) the normed Langlands parameter associated to \( \varphi \). Write \( \iota \circ \tau = \tau_1 \oplus \cdots \oplus \tau_r \) for the decomposition of \( \iota \circ \tau \) into irreducible representations. Denote by \( s_\tau \) the element of \( C_{\hat{G}_l}(\text{Im}(\tau)) \) which corresponds to the diagonal matrix \( (f_{\tau_1}, \ldots, f_{\tau_r}) \) and by \( s_\varphi \) the element of \( C_{\hat{G}_l}(\text{Im}(\varphi)) \) that corresponds to \( (f_{\rho_1}, \ldots, f_{\rho_d}, s_\tau) \).

The element \( s_\varphi \) lies in \( C_{\hat{G}_l}(\text{Im}(\varphi_0)) \) and in \( C_{\hat{G}_l}(\text{Im}(\varphi)) \).
Suppose: if $\rho_i$ is self-dual, then it is of the same type than the normed representation in its inertial orbit. Then, $C_{\hat{G}}(\text{Im}(\varphi))$ and $C_{C_{\hat{G}}(\text{Im}(\varphi))}(s_\varphi)$ are canonically isomorphic.

Remark: As $\varphi$ and consequently $\tau$ are discrete, the representations $\tau_i$ are all of type $\hat{G}$ and non isomorphic. In addition, $f_{\tau_i} \in \{\pm 1\}$ for $i = 1, \ldots, r$.

Proof: By the proposition 2.1, one has $C_{\hat{G}}(\text{Im}(\varphi_0)) = \prod_i GL_{l_i}(\mathbb{C}) \times \prod_j Sp_{m_j}(\mathbb{C}) \times \prod_k O_{n_k}(\mathbb{C})$, where the first product goes over the $\rho_i$ which are not self-dual, the second one over the $\rho_j$ which are not of the same type than $\hat{G}$ and the third one over the $\rho_k$ which are of the same type than $\hat{G}$. The centralizer of $\text{Im}(\varphi_0)$ is determined by the partition of the summands of $\iota \circ \varphi_0$ obtained by putting together representations which are either isomorphic or isomorphic to the dual of the other one. The different parts of this partition of the summands of $\iota \circ \varphi_0$ give then rise to factors which are respectively isomorphic to $GL_{l_i}(\mathbb{C})$, $Sp_{m_j}(\mathbb{C})$ or $O_{n_k}(\mathbb{C})$ depending if the representations in the part are not self-dual, orthogonal or symplectic, where $l_i$ denotes half the number of elements in the corresponding part and $m_j$ and $n_k$ the total number of elements in the part. The analog result holds for the centralizer of $\text{Im}(\varphi)$. As $\varphi_0$ is normed, it is clear that the partition of $\iota \circ \varphi$ is finer than the partition of $\iota \circ \varphi_0$.

Writing $C_{\hat{G}}(\text{Im}(\varphi_0))$ as above as a product, one sees that the centralizer of $s_\varphi$ in $C_{\hat{G}}(\text{Im}(\varphi_0))$ is the product of the centralizers of the components of $s_\varphi$ in the different factors. So, one can reduce to consider the following three cases:

(i) All summands of $\iota \circ \varphi$ are in the same inertial orbit and the normed representation in this orbit is not self-dual. In particular, $\tau$ is trivial.

(ii) All summands of $\iota \circ \varphi$ are in the same inertial orbit and the normed representation in this orbit is orthogonal. In particular, $\tau$ is trivial.

(iii) All summands of $\iota \circ \varphi$ are in the same inertial orbit and the normed representation in this orbit is symplectic. In particular, $\iota \circ \tau$ is either trivial or equal to this symplectic representation or an unramified twist of it.

In all three cases the centralizer of $s_\varphi$ is determined by the partition of the coefficients of $s_\varphi$, obtained by putting equal coefficients in the same part. By proposition 1.3, equal coefficients correspond to equal summands of $\iota \circ \varphi$, so that the two partitions correspond canonically to each other and have the same number of elements. The factors of the centralizer of $s_\varphi$ which correspond to the different parts of the partition of the coefficients of $s_\varphi$ are all general linear groups of order equal to the length of the partition in case (i). In case (ii) and (iii) they are general linear groups if the coefficients are $\neq \pm 1$ and groups of the same type than the group they are in in the other cases. As by our assumption we have excluded the appearance of factors of another type in the centralizer of $\text{Im}(\varphi)$, this proves the proposition.
2.3 Lemma: With the same notations as in 2.2, assume that $\rho$ is an irreducible representation of $W_F$ of type $\hat{G}$, such that $\rho_-$ is not of type $\tilde{G}$ and $\iota \circ \varphi_0 \simeq m\rho$.

Then, $C_{GL_N(C)}(Im(\varphi_0))$ is canonically isomorphic to $GL_m(C)$, while $C_{\tilde{G}}(Im(\varphi_0))$ is isomorphic to $O_m(C)$.

Define $s_\varphi$ as in 2.2. The element $s_\varphi$ lies in $C_{\tilde{G}}(Im(\varphi_0))$ and in $C_{\tilde{G}}(Im(\varphi))$.

Write $s_\varphi = \text{diag}(x_1, \ldots, x_{[\frac{d}{2}]}, 1, x_{[\frac{d}{2}]}^{-1}, \ldots, x_1^{-1}) \in GL_m(C)$ (with 1 appearing only when $m$ is odd and $[\frac{d}{2}]$ denoting the integer part of $\frac{d}{2}$). For $x \in \{x_1, \ldots, x_{[\frac{d}{2}]}\}$, denote by $m(x, s_\varphi)$ the multiplicity of $x$ in $s_\varphi$ and put

$$G_x = \begin{cases} GL_m(x_{s_\varphi}) \times GL_m(x^{-1}, s_\varphi), & \text{if } x \not\in \{\pm 1\}, \\ GL_m(\pm 1, s_\varphi), & \text{if } x = \pm 1. \end{cases}$$

The group $C_{GL_N(C)}(s_\varphi)$ is canonically isomorphic to $\prod_x G_x$, the product going over equivalence classes of elements in the set $\{x_1, \ldots, x_{[\frac{d}{2}]}\}$ with respect to the relation $x \sim x^{-1}$, and to $C_{GL_N(C)}(Im(\varphi))$.

Denote by $H_x$ (resp. $H'_x$) the subgroup of $G_x$ defined (with $J$ an appropriate matrix which needs not to be made more precise here) by

$$\begin{align*}
\{(h, Jh^{-1}J) | h \in GL_m(x_{s_\varphi})\} & \text{ if } x \not\in \{\pm 1\}, \\
O_m(1, s_\varphi) & \text{ if } x = 1, \\
Sp_m(-1, s_\varphi) & \text{ (resp. } O_m(-1, s_\varphi), \text{ if } x = -1, \text{ and by } H \text{ (resp. } H') \text{ the image of } \prod_x H_x \text{ (resp. } \prod_x H'_x \text{) in } C_{GL_N(C)}(s_\varphi) \text{ by the above isomorphism.}
\end{align*}$$

Then, $C_{\tilde{G}}(Im(\varphi))$ is isomorphic to $H$ and $C_{\tilde{G}}(s_\varphi)$ to $H'$.

In particular, $C_{C_{\tilde{G}}(Im(\varphi_0))}(s_\varphi)$ and $C_{C_{\tilde{G}}(Im(\varphi))}$ are only isomorphic if $m(-1, s_\varphi) = 0$.

2.4 Remark that at the end of the following definition, we will use results from C.6 and C.7. However, at a first reading, one may avoid to look in the annex C.

Definition: Let $\bar{M} \simeq GL_{k_1}(C) \times \cdots \times GL_{k_d}(C) \times \tilde{H}_l$ be a standard Levi subgroup of $\tilde{G}$ and let $\varphi_0 : W_F \to \bar{M}$ be a discrete normed Langlands parameter, $\gamma \mapsto (\rho_1(\gamma), \ldots, \rho_d(\gamma), \tau(\gamma))$.

Recall that by 2.1, $C_{\tilde{G}}(Im(\varphi_0)) = \prod_{\rho} H_{\rho, \varphi}(m(\rho, \varphi))$, the product going over representatives of the equivalence classes of irreducible normed representations $\rho$ of $W_F$, while the $H_{\rho, \varphi}(m)$ are complex classical groups with $H_{\rho, \varphi}(m)$ isomorphic to $GL_m(C)$ if $\rho$ is not self-dual, to $Sp_m(C)$ if $\rho$ is not of type $\tilde{G}$, and to $O_m(C)$ if $\rho$ is of type $\tilde{G}$ (with the convention $O_1(C) = \{\pm 1\}$ if $m = 1$).

Let $s_\rho$ be a semi-simple element in $C_{\tilde{G}}(Im(\varphi_0))$ and denote by $s_\rho$ the projection of $s$ on $H_{\rho, \varphi}(m(\rho, \varphi))$. Define $C'_{H_{\rho, \varphi}(m(\rho, \varphi))}(s_\rho) = C_{H_{\rho, \varphi}(m(\rho, \varphi))}(s_\rho)$ except if $\rho$
and $\rho_-$ are not of the same type. In that case, denote by $H'_{\rho,\varphi}(m(\rho;\varphi))$ the L-group of the unramified quasi-split unitary group $U_m$ and define $C'_{\varphi}(s) = C'_{\rho_+,\varphi}(m(\rho;\varphi))((-1)^{n_\rho})$ (where $(-1)^{n_\rho}$ is the Langlands parameter for $U_m$ defined in C.7).

Put $C'_{\varphi}(s) = \prod_{\rho} C'_{\rho,\varphi}(m(\rho;\varphi))^{-1}(s)$. For a subset $I$ of $C'_{\varphi}(s)$, denote its centralizer by $C'_{\varphi}(s, I)$. (There is some subtlety if $\rho$ and $\rho_-$ are self-dual, but not of the same type.)

2.5 Theorem: Let $\hat{M} \simeq GL_{k_1}(\mathbb{C}) \times \cdots \times GL_{k_d}(\mathbb{C}) \times \widehat{H}_{\mathfrak{l}}$ be a standard Levi subgroup of $\hat{G}$ and let $\varphi_0 : W_F \to \hat{M}$ be a discrete normed Langlands parameter, $\gamma \mapsto (\rho_1(\gamma), \ldots, \rho_d(\gamma), \tau(\gamma))$.

The set of equivalence classes of Langlands-Deligne parameters $\varphi : W_F \times SL_2(\mathbb{C}) \to \hat{G}$ with $\varphi_{|W_F}$ in the inertial orbit of $\varphi_0$ is in bijection with the set of equivalence classes of pairs $(s, \varphi|SL_2(\mathbb{C}))$ consisting of a semisimple element $s$ and an algebraic homomorphism $SL_2(\mathbb{C}) \to C'_{\varphi}(s)$ by mapping $\varphi$ to $(s, \varphi|SL_2(\mathbb{C}))$, so that $C'_{\varphi}(s, \varphi|SL_2(\mathbb{C}))$ is canonically isomorphic to

$$C'_{\varphi}(s, \varphi|SL_2(\mathbb{C}))/C'_{\varphi}(s, \varphi|SL_2(\mathbb{C}))^0.$$ 

Proof: This is straightforward by the definitions, the above lemmas and C.7, C.8.

2.6 Corollary: Let $\hat{M} \simeq GL_{k_1}(\mathbb{C}) \times \cdots \times GL_{k_d}(\mathbb{C}) \times \widehat{H}_{\mathfrak{l}}$ be a standard Levi subgroup of $\hat{G}$ and let $\varphi_0 : W_F \to \hat{M}$ be a discrete normed Langlands parameter, $\gamma \mapsto (\rho_1(\gamma), \ldots, \rho_d(\gamma), \tau(\gamma))$. Assume, for all $i$, that $\rho_{i-}$ is of type $\hat{G}_i$, if $\rho_i$ is of type $\hat{G}$.

The set of equivalence classes of pairs $(s, u)$ consisting of a semisimple element $s$ and a unipotent element $u$ in $C'_{\varphi}(s)$ such that $s^{-1}us = u^\varphi$ is in bijection with the set equivalence classes of Langlands-Deligne parameters $\varphi : W_F \times SL_2(\mathbb{C}) \to \hat{G}$ with $\varphi_{|W_F}$ in the inertial orbit of $\varphi_0$, so that one has a canonical isomorphism between the group of connected components of the centralizers of the images,

$$C'_{\varphi}(s, \varphi|SL_2(\mathbb{C}))/C'_{\varphi}(s, \varphi|SL_2(\mathbb{C}))^0 \simeq C_{\varphi}(s) / C_{\varphi}(s)^0.$$ 

Proof: If, for all $i$, $\rho_{i-}$ is of the same type than $\rho_i$, then $C'_{\varphi}(s, \varphi|SL_2(\mathbb{C})) = C_{\varphi}(s, \varphi|SL_2(\mathbb{C}))$. But, in this case the set of equivalence classes of pairs $(s, \varphi|SL_2(\mathbb{C}))$ is in bijection with the set of conjugacy classes of pairs $(s, u)$ [BV, 2.1] (it is remarked in [H1, 3.5] that this result is valid for general connected reductive groups, while the image of $SL_2(\mathbb{C})$ lies necessarily in $\hat{G}$).
3. In this section we will show at the end that the previous results allow to relate the category $R_{F}^{\varphi}(G)$ to categories of unipotent representations of $p$-adic classical groups. However, before that, we will state some parameterizations of representations of collections of (possibly extended) affine Hecke algebras that follow from section 1. and the additional remarks about unitary groups in annex C.

3.1 The following is the special case of 1.6 for $G = SO_{2d+1}$, $M = T$ and $\rho$ the trivial representation that is treated in [L2] with modifications in [W]. It follows also by combining 1.6 and 1.7.

**Theorem:** Fix an integer $d \geq 1$. If $(d_+, d_-) \neq 0$ is a pair of integers which are each one products of two consecutive integers, $d_+ = a_+(a_+ + 1)$ and $d_- = a_-(a_- + 1)$, with $d_+ + d_- \leq 2d + 1$, denote by $H(d_+, d_-)$ the affine Hecke algebra with root datum equal to the one of $SO_{2d+1} - d_+ - d_-$ and unequal parameters $q_1, \ldots, q, q^{a_+ + a_- + 1}, q^{a_+ - a_-}$. Denote by $H(0, 0)$ the affine Hecke algebra with root datum equal to the one of $Sp_{2d}(\mathbb{C})$ and equal parameters $q, \ldots, q$.

Then, the set of triples $(s, u, \Xi)$ associated to the group $Sp_{2d}(\mathbb{C})$ with $\Xi(-1) = 1$ is in natural bijection with the set

$$\bigcup_{(d_+, d_-), \frac{d_++d_-}{2} \text{ even}} (\text{irreducible right } - H(d_+, d_-) - \text{modules})$$

and the one with $\Xi(-1) = -1$ is in natural bijection with the set

$$\bigcup_{(d_+, d_-), \frac{d_++d_-}{2} \text{ odd}} (\text{irreducible right } - H(d_+, d_-) - \text{modules}).$$

**Remark:** By natural bijection, we mean what is implied by the conditions for the local Langlands correspondence explained in 1.6. We will not explain this here more, except that compact $s$ should correspond to tempered representations and discrete $s$ (i.e. those which do not lie in a proper parabolic subgroup) to discrete series representations.

3.2 The following follows by combining 1.6 and 1.7 to the special case $G = Sp_{2d}(F)$ (for (i)) and $G = O_{2d}(F)$ (for (ii)) with $\rho$ the trivial representation and $M$ the maximal split torus. Remark that these groups are not of adjoint type, so that [L2] does not apply to these cases, but a slight generalization should do. However, this has not been written yet.

**Theorem:** Fix an integer $d \geq 1$. 
If \((d_+, d_-) \neq 0\) is a pair of integers, which are squares, such that \(d_+ + d_- \leq 2d + 1\), denote by \(H(d_+, d_-)\) the affine Hecke algebra with root datum equal to the one of \(SO_{2d+1-2d-}\), if \(d_+ + d_-\) is even, and equal to the one of \(SO_{2d+2-2d-}\), if \(d_+ + d_-\) is odd, and unequal parameters \(q, \ldots, q, q\sqrt{d_+ + d_-}; q|\sqrt{d_+ - d_-}|\).

In addition, denote by \(H(0, 0)\) the semi-direct product of an affine Hecke algebra with equal parameter \(q\) and root datum equal to the one of \(SO_{2d}\) with the group algebra of a finite cyclic group of order \(2\), which acts by the outer automorphism of the root system. Define \(\epsilon^+_{d_+, d_-} = \begin{cases} 4 & \text{if } d_+\text{ even}, \ d_+ + d_- \in 8\mathbb{Z}, \ d_+ \cdot d_- \neq 0, \\ 1 & \text{if } d_+ = d_- = 0, \\ 0 & \text{if } d_+\text{ even}, \ d_+ + d_- \in 4\mathbb{Z} \setminus 8\mathbb{Z}, \\ 2 & \text{otherwise}. \end{cases}\)

and put \(e_{d_+, d_-} = 4 - \epsilon^+_{d_+, d_-}\) if \(d_+ \cdot d_- \neq 0\), \(e_{d_+, d_-} = 2 - \epsilon^+_{d_+, d_-}\) if exactly one of \(d_+\) and \(d_-\) is 0, and \(e_{0, 0} = 0\).

i) Denote by \(S_0\) the set of pairs of integers \((d_+, d_-)\) such that \(d_+\) and \(d_-\) are squares, \(d_+ + d_-\) is odd and \(\leq 2d + 1\). (Consequently \(e_{d_+, d_-} = 2\). The set of triples \((s, u, \Xi)\) associated to the group \(SO_{2d+1}(\mathbb{C})\) is in natural bijection with the multiset

\[
\bigcup_{(d_+, d_-) \in S_0} \begin{cases} 2 & \text{irreducible right } H(d_+, d_-) - \text{modules}. \end{cases}
\]

ii) Denote by \(S_e\) the set of pairs of integers \((d_+, d_-)\) such that \(d_+\) and \(d_-\) are squares, \(d_+ + d_-\) is even and \(\leq 2d + 1\). The set of triples \((s, u, \Xi)\) associated to the group \(O_{2d}(\mathbb{C})\) with \(\Xi|Z^2_G = \pm 1\) is in natural bijection with the multiset

\[
\bigcup_{(d_+, d_-) \in S_e} e_{d_+, d_-} \begin{cases} \text{irreducible right } H(d_+, d_-) - \text{modules}. \end{cases}
\]

3.3 Assuming the appropriate parts of the Langlands correspondence 1.7 established for the non split pure inner form of the even unramified quasi-split unitary group, the following follows by combining 1.6 and 1.7 as generalized to quasi-split unitary groups in annex C.. Here \(U_m\) will denote an unramified quasi-split unitary group.

Fix \(m\) and consider the set of quadruplets \((s, s_1, u, \Xi)\) associated to \(L^*U_m\) such that

(i) \(s = diag(x_1, \ldots, x_{\frac{m}{2}}; \hat{1}, x_{\frac{m}{2}}^{-1}, \ldots, x_1^{-1}) \in GL_m(\mathbb{C})\) with 1 appearing only when \(m\) is odd;

(ii) Denoting \(1_s\), the element of the inertial class of the trivial Langlands parameter for \(L^*U_m\) deduced from \(s\) (cf. C.7), there is an algebraic homomorphism \(SL_2(\mathbb{C}) \to C_{GL_m(\mathbb{C})}(Im(1_s))\) which sends \(\begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}\) to \(s_1\) and \(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\) to \(u\).
Theorem: If \( d_+ \) is a square integer and \( d_- \) the product of two consecutive integers, \( d_- = a_-(a_- + 1) \), denote by \( \mathcal{H}(d_+, d_-) \) the affine Hecke algebra with root datum equal to the one of \( SO_{2d+1} \). Then, the set of quadruplets \((q, d, s, s)\) associated to \( \mathcal{H}(d_+, d_-) \) is in natural bijection with the multiset

\[
\{q, \ldots, q, q \sqrt{d_+ + a_- + \frac{1}{2}}, q \sqrt{d_- - a_- - \frac{1}{2}}\}.
\]

In addition, denote by \( \mathcal{H}(0,0) \) the affine Hecke algebra with root datum equal to the one of \( SO_{2d+1} \) and unequal parameters \( q, q, \ldots, q, q^{1/2} \).

In addition, denote by \( \mathcal{H}(0,0) \) the affine Hecke algebra with root datum equal to the one of \( SO_{2d+1} \) and unequal parameters \( q, q, \ldots, q, q^{1/2} \).

Denote by \( S_\epsilon \) the set of pairs \((d_+, d_-)\) with \( d_+ \) an even square, \( d_- \) the product of two consecutive integers, \( d_+ + d_- \leq 2d + 1 \), and by \( S_\rho \) the set of pairs \((d_+, d_-)\) with \( d_+ \) an odd integer, \( d_- \) the product of two consecutive integers, \( d_+ + d_- \leq 2d + 1 \).

(i) If \( m \) is an odd integer, \( m = 2d + 1 \), then the set of quadruplets \((s, s, u, \Xi)\) associated to \( L^U_m \) with \( \Xi(\pm 1) \) fixed is in natural bijection with the multiset

\[
\bigcup_{(d_+, d_-) \in S_\rho} (\text{irreducible right} - \mathcal{H}(d_+, d_-) - \text{modules}).
\]

(ii) If \( m \) is an even integer, \( m = 2d \), put \( \epsilon_{d_+, d_-} = 2 \), if \( d_+ \neq 0 \), and \( \epsilon_{d_+, d_-} = 1 \) otherwise. Then the set of quadruplets \((s, s, u, \Xi)\) associated to \( L^U_m \) with \( \Xi(-1) = 1 \) is in natural bijection with the multiset

\[
\bigcup_{(d_+, d_-) \in S_{\epsilon_{d_+, d_-}}} (\text{irreducible right} - \mathcal{H}(d_+, d_-) - \text{modules}),
\]

and the one with \( \Xi(-1) = -1 \) is in natural bijection with the multiset

\[
\bigcup_{(d_+, d_-) \in S_{\epsilon_{d_+, d_-}}} 2 (\text{irreducible right} - \mathcal{H}(d_+, d_-) - \text{modules}).
\]

3.4 If \( t_\varphi \) is an integer \( \geq 1 \), denote by \( F_{t_\varphi} \) the unramified extension of degree \( t_\varphi \), which is unique in a given algebraic closure of \( F \). If \( \varphi \) is Langlands parameter which is not normed and \( \varphi_0 \) is the normed Langlands parameter in its orbit, we put \( \mathcal{R}^\varphi = \mathcal{R}^{\varphi_0} \).

Theorem: Assume \( \iota \circ \varphi_0 = m \rho \). Then,

(i) if \( \rho \) is not self-dual, then the category \( \mathcal{R}^{\varphi_0}(G) \) is equivalent to \( \mathcal{R}^1_{F_{t_\varphi}}(GL_m) \).

(ii) if \( \rho \) is self-dual and not of type \( \hat{G} \), then the category \( \mathcal{R}^{\varphi_0}(G) \) is equivalent to \( \mathcal{R}^1_{F_{t_\varphi}}(Sp_m) \).

(iii) if \( \rho \) and \( \rho_- \) are both of type \( \hat{G} \), then the category \( \mathcal{R}^{\varphi_0}(G) \) is equivalent to \( \mathcal{R}^1_{F_{t_\varphi}}(SO_m) \).
(iv) if $\rho$ and $\rho_-$ are self-dual but not of the same type, then, with $U_m$ equal to the unramified quasi-split unitary group, the category $\mathcal{R}_{F, \rho}^\varphi(G)$ is equivalent to $\mathcal{R}_{F, \rho}^1(U_m)$.

The same holds, if one replaces $\mathcal{R}$ by $\mathcal{R}^{\cdot +}$ or $\mathcal{R}^{\cdot -}$.

**Proof:** This follows from theorem 1.7 (and its generalization to unitary groups in C.5 together with proposition C.6) applied to the special cases when $\iota \circ \varphi_0 = m\rho$ and when $\varphi_0$ is in the orbit of the trivial representation. 

3.5 Recall the equivalence relation on $\text{supp}(\varphi_0)$ given by $\rho \sim \rho^\vee$ introduced in 1.5.

**Corollary:** The category $\mathcal{R}_{F}^{\varphi_0}(G)$ is equivalent to

$$\bigotimes_{\rho \in \text{supp}(\varphi_0)/\sim} \mathcal{R}_{F, \rho}^1(H_\rho(m(\rho; \varphi_0)))$$

with $H_\rho(m)$ equal to $GL_m$, $Sp_m$, $SO_m$ or the unramified quasi-split unitary group $U_m$, if respectively $\rho$ is not self-dual, not of type $\hat{G}$, $\rho$ and $\rho_-$ are both of type $\hat{G}$, or $\rho$ and $\rho_-$ are self-dual but not of the same type.

The same holds, if one replaces $\mathcal{R}$ by $\mathcal{R}^{\cdot +}$ or $\mathcal{R}^{\cdot -}$.

**Proof:** This follows by B.3 and B.4 from 3.4 and 1.8.

A. **Annex:** Equivalence of categories for the full orthogonal group

A.1 The aim of this annex is to show how the results of [H3, H4] generalize to the full orthogonal group, which is not connected. So, in this annex, $H$ will denote a pure inner form of a full split orthogonal group, either split or not. The case when its connected component $H^0$ is an odd orthogonal group is quite easy. Then $H$ is isomorphic to a direct product $H^0 \times \{\pm 1\}$. The Levi subgroups of $H$ are of the form $M = M^0 \times \{\pm 1\}$, where $M^0$ is a Levi subgroup of $H^0$, so that the supercuspidal representations of $M$ are of the form $\sigma^0 \eta$, where $\sigma^0$ is a supercuspidal representation of $M^0$ and $\eta$ a character of $\{\pm 1\}$. One sees immediately that the restriction to $\{\pm 1\}$ of a representation in the supercuspidal support of an irreducible representation $\pi$ of $H$ is determined by the restriction of $\pi$ to this group. So, one may decompose $\text{Rep}(H)$ as a direct sum of subcategories $\text{Rep}_{M^0, \sigma}(H^0) \oplus \text{Rep}_{M^0, \sigma, -1}(H^0)$, where the $\text{Rep}_{M^0, \sigma}(H^0)$ denote the Bernstein components for $H^0$ and the $\text{Rep}_{M^0, \sigma, -1}(H^0)$ denote the part with non-trivial restriction to $\{\pm 1\}$. As the results of [H3,H4] apply to $\text{Rep}(H^0)$, we are done.
A.2 Assume now for the rest of this annex that \( n \) is even. Then, \( H \) is isomorphic to a semi-direct product \( H^0 \rtimes \{1, r_0\} \), where \( H^0 \) is an even orthogonal group and \( r_0 \) is of order 2 and acts on \( H^0 \) by the outer isomorphism. We refer to [GH] for results for the representation theory of a non connected reductive group. We consider only Levi subgroups which are cuspidal in the terminology of [GH]. In particular, one deduces from this paper that the Bernstein decomposition is still valid and that, if \( M \) is a Levi subgroup of \( H \) and \( O \) denotes the inertial orbit of an irreducible supercuspidal representation of \( M \), then \( i_H^M \) is a projective generator of \( \text{Rep}(M, O) \), which implies that the category \( \text{Rep}(M, O) \) is equivalent to the category of right-modules over \( \text{End}_H(i_H^M) \) by Morita theory.

The aim of this annex is to show that \( \text{End}_G(i_H^M) \) has the form given in theorem 1.7.

A.3 Denote by \( \mathcal{W}_0 \) the Weyl group of \( H^0 \) and define \( \mathcal{W} := \mathcal{W}_0 \rtimes \{1, r_0\} \), where \( \mathcal{W}_M \) denotes the Weyl group of \( M \), and similarly for the Levi subgroups of \( H \). If \( M \) is a Levi subgroup of \( H \), define \( W(M) = W^M \setminus \{w \in W| w^{-1} Mw = M\} \) and similarly \( W^0(M^0) \), which will also be denoted (abusively) \( W^0(M) \).

Lemma: One has \( W(M) = W^0(M) \), except if \( M \) is isomorphic to a product of general linear groups and at least one of them has odd rank. In particular, \( W(M) = W^0(M) \) if \( H \) is not quasi-split.

Proof: If \( M \) has a factor \( H_l \) with \( l \geq 2 \), then \( r \in M \). So, \( M \) has to be a product of linear groups if \( W(M) \neq W^0(M) \). If \( M \) is a product of general linear groups of even rank, then every element of \( w \) that satisfies \( w^{-1} Mw = M \) must have an even number of sign changes \( x \mapsto x^{-1} \) on the maximal torus. This means that it lies in \( W^0 \). If \( M \) is a product of general linear groups, one of them being of odd rank \( k \), one sees that there is an element in \( W \) which induces the outer automorphism on \( GL_k \) and which does not lie in \( W^0 \).

A.4 Let \( O \) be the inertial orbit of a supercuspidal representation of a Levi subgroup \( M \) of \( H \). Its restriction to \( M^0 \) decomposes into one or two inertial orbits. Fix an orbit \( \mathcal{O}^0 \) in the restriction. Denote by \( W(M, O) \) (resp. \( W^0(M, O) \)) the subset of elements of \( W(M) \) (resp. \( W^0(M) \)) which stabilize \( O \) (resp. \( O^0 \)).

Lemma: One has \( W(M, O) = W^0(M, O) \) except if \( M \) is a product of general linear groups and at least one factor of \( O^0 \) is the inertial orbit of a self-dual representation of a general linear group of odd rank.

Proof: The group \( W(M, O) \) is a subgroup of \( W(M) \). It follows that the equality \( W(M, O) = W^0(M, O) \) can only fail if \( M \) is a product of general linear groups and at least one of them has odd rank \( k \). In addition, one sees that at least one
factor of \( \mathcal{O} \) corresponding to a \( GL_k(F) \) with \( k \) odd must be the orbit of a self-dual representation. \( \square \)

**A.5** Denote by \( R(\mathcal{O}) \) the subgroup of elements \( r \) of \( W(M, \mathcal{O}) \) that send positive roots for \( M \) to positive roots. Define \( R^0(\mathcal{O}) = R(\mathcal{O}) \cap W^0(M, \mathcal{O}) \). Recall [H3] that \( W^0(M, \mathcal{O}) \) is a semi-direct product \( W^0_0 \times R^0(\mathcal{O}) \), so that one has \( W(M, \mathcal{O}) = W^0_0 \rtimes R(\mathcal{O}) \). As \( \text{ind}_{M_0}^M E_{B_\mathcal{O}} \) is either equal to \( E_{B_\mathcal{O}} \) or a direct sum \( E_{B_\mathcal{O}} \oplus E_{B_{\mathcal{O}'}}, \) one can define, for \( w \in W^0(\mathcal{O}^0) \) and \( r \in R^0(\mathcal{O}) \), operators \( T_w \) and \( J_r \) in \( \text{End}_H(i_H^0 E_{B_{\mathcal{O}}}) \) from the one for \( \text{End}_{G^0}(i_H^0 E_{B_{\mathcal{O}}}) \) by induction. If \( r \in R(\mathcal{O}) \setminus R^0(\mathcal{O}) \), note \( \lambda(r) \) the action of \( r \) on \( i_H^0 E_B \) by left-translation and by \( \tau_r \) the one of \( r \) on \( B_\mathcal{O} \) by right translation [H3], and put \( J_r = \tau_r \lambda(r) \). These operators \( J_r \) commute obviously with the other operators \( J_{r'}, r' \in R(\mathcal{O}) \), and satisfy the commuting relation \( T_w J_r = J_r T_{w^{-1} wr} \) for \( w \in W^0_0 \).

**Lemma:** The operators \( s_p \chi J_r T_w, r \in R(\mathcal{O}), w \in W(M, \mathcal{O}) \), are linearly independent for all \( \chi \in \mathfrak{X}^w(M) \).

**Proof:** The proof of [H3, 5.9] generalizes, as the commuting relations for the operators \( J_r, r \in R(\mathcal{O}) \) are still the same. \( \square \)

**A.6 Lemma:** One has \( \text{Hom}_H(i_H^0 E_{B_{\mathcal{O}}}, i_H^0 E_{K(\mathcal{O})}) = \bigoplus_{w, r} K(\mathcal{O}) J_r T_w \).

**Proof:** This follows from the linear independence and the computation of the Jacquet module with help of the geometric lemma in the non connected case [C, 4.1], taking into account lemma A.4. \( \square \)

**A.7 Theorem:** One has \( \text{End}_H(i_H^0 E_{B_{\mathcal{O}}}) = \bigoplus_{w, r} B_{\mathcal{O}} J_r T_w \).

**Proof:** The proof of [H3, 5.10] generalizes, as the commuting relations for the operators \( J_r, r \in R(\mathcal{O}) \) are still the same. \( \square \)

**Remark:** As the \( T_w \) satisfy the same relations as their restrictions to the space of the representation of the connected component, it follows that \( \text{End}_H(i_H^0 E_{B_{\mathcal{O}}}) \) is an (possibly extended) affine Hecke algebra isomorphic to \( \text{End}_{H^0}(i_H^{00} E_{B_{\mathcal{O}^0}}) \), except if \( M \) is a product of general linear groups and at least one factor of \( \mathcal{O} \) is the inertial orbit of a self-dual representation of a general linear group of odd rank. In this case, one has additional operators \( J_r \) with \( r \in R(\mathcal{O}) \setminus R^0(\mathcal{O}) \).

We also have to make an erratum to [H3, H4]: if \( M = M^0 \) (i.e. \( M \) is a product of general linear groups), \( \text{End}_{H^0}(i_H^{00} E_{B_{\mathcal{O}^0}}) \) is in general isomorphic to a tensor product \( \bigotimes \rho \mathcal{H}_\rho \otimes (( \otimes \rho' \mathcal{H}_\rho') \rtimes C[\mathbb{R}_n]) \), the first product going over elements \( \rho \) in the support of the normed Langlands parameter \( \varphi_0 \) associated to \( \mathcal{O} \) which are not
odd orthogonal and the second product over the odd orthogonal ones, \( R_{nq} \) being generated by Weyl group elements that send positive roots in \( \Sigma_O \) (in the notations of \([H3]\)) to positive roots and have sign changes \( x \mapsto x^{-1} \) on two factors \( GL_k(F) \) and \( GL_{k'}(G) \), on which odd orthogonal representations with distinct inertial orbits are defined. Here the \( \mathcal{H}_p \) denote the (possibly extended) affine Hecke algebras from 1.7 and \( \mathcal{H}_p^0 \) the affine Hecke algebra part (i.e. omitting the finite group algebra part, if there is any). One remarks that the above semi-direct product is with a tensor product of affine Hecke algebras associated to odd orthogonal representations in the support, but does not decompose into a tensor product of semi-direct products of the different affine Hecke algebras with a group algebra.

**Annex B: Tensor product of abelian categories**

**B.1 Definition** [D, 5.] Let \( k \) be a commutative ring and \((A_i)_{i \in I}\) a finite family of \( k \)-linear abelian categories. A \( k \)-linear abelian category \( A \) equipped with a \( k \)-multilinear functor right exact in each variable

\[ \otimes : \prod A_i \to A \]

is called tensor product over \( k \) of the categories \( A_i \) if and only if the following condition is satisfied: denote for a \( k \)-linear abelian category \( C \) by \( \text{Hom}_{k,e \Rightarrow d}(A, C) \) the category of right exact functors from \( A \) to \( C \) and by \( \text{Hom}_{k,e \Rightarrow d}((A_i)_{i \in I}, C) \) the category of right exact functors multilinear in each variable from the product of the \( A_i \) to \( C \).

One asks then that for every category \( C \) the composed functor with the above

\[ \text{Hom}_{k,e \Rightarrow d}(A, C) \to \text{Hom}_{k,e \Rightarrow d}((A_i)_{i \in I}, C) \]

is an equivalence of categories.

**B.2 Proposition:** [D, 5.3] Let \((A_i)_{i \in I}\) be a finite family of coherent \( k \)-algebras that have a coherent tensor product over \( k \). Denote by \((A_i)_{coh}\) (resp. \((\otimes A_i)_{coh}\)) the corresponding abelian category of right modules of finite presentation. The tensor product over \( k \)

\[ \otimes : \prod A_i \to \otimes A_i \]

defines \((\otimes A_i)_{coh}\) as tensor product over \( k \) of the \((A_i)_{coh}\).

**B.3 Proposition:** (i) An extended affine Hecke algebra with unequal parameters is a coherent \( C \)-algebra.

(ii) Any finitely generated right module over an extended affine Hecke algebra with unequal parameters is coherent.
Proof: (i) An extended affine Hecke algebra with unequal parameters is a free module of finite rank over the group ring of a finitely generated lattice. As the group ring of a finitely generated lattice is noetherian as quotient of a polynomial ring, the extended affine Hecke algebra is noetherian as module. But every ideal of this algebra is a submodule. So, it is finitely generated. One concludes that an extended affine Hecke algebra is noetherian and in particular coherent.

(ii) A finitely generated right-module over a noetherian \( C \)-algebra is coherent. ✷

B.4 Proposition: Let \( k \) be a commutative ring and \( (A_i)_{i \in I} \) a finite family of \( k \)-linear abelian categories. Assume that each \( A_i \) is a direct sum of \( k \)-linear categories \( A_{i,j} \), \( j = 1, \ldots, l_i \). Suppose that for each family of integers \( J = (j_i)_{i \in I}, 1 \leq j_i \leq l_i \), the family \( (A_{i,j_i})_{i \in I} \) has a tensor product \( A_J \). Then, the tensor product of the categories \( A_i \) is isomorphic to the direct sum of the categories \( A_J \).

Proof: One has an equivalence of categories between \( \prod_i (\bigoplus_{j=1}^{l_i} A_{j,i}) \) and \( \bigoplus_{J \subseteq J_1 \times \cdots \times J_k} \prod_i A_{j,i} \), and consequently between \( \text{Hom}_{k,e \text{-ad}}((\bigoplus_{j=1}^{l_i} A_{j,i})_{i \in I}, C) \) and \( \bigoplus_{J \subseteq J_1 \times \cdots \times J_k} \text{Hom}_{k,e \text{-ad}}((A_{j,i})_{i \in I}, C) \). Denote by \( A_J \) the tensor product of \( (A_{j,i})_{i \in I} \). One sees immediately that \( \bigoplus_{J \subseteq J_1 \times \cdots \times J_k} A_J \) satisfies the universal property for the tensor product of \( (\bigoplus_{j=1}^{l_i} A_{j,i})_{i \in I} \). ✷

B.5 Proposition: Let \( (H_i)_{i \in I} \) be a finite family of extended affine Hecke algebras with parameters. Let \( B_i \) be a finite family of \( k \)-linear abelian categories with each \( B_i \) equivalent to the category \( (H_i)_f \) of finitely generated modules over \( H_i \). Then, the tensor product of the \( k \)-linear abelian categories \( B_i \) exists and is equivalent to the tensor product of the categories \( (H_i)_f \).

Proof: The equivalence of categories \( B_i \to (H_i)_f \) give equivalences of categories \( \prod B_i \to \prod (H_i)_f \) and \( \text{Hom}_{k,e \text{-ad}}((H_i)_f)_{i \in I}, C) \to \text{Hom}_{k,e \text{-ad}}((B_i)_{i \in I}, C) \). With this, it is immediate that the \( (B_i)_{i \in I} \) satisfy the universal property with respect to the tensor product of the \( H_i \). ✷

Annex C: The case of the unitary group

C.1 In this annex, we will show that the result of the section 1. and 2. generalize to quasi-split unitary groups. To obtain this, we will give a few remarks, justifying that [H3] generalizes to pure inner forms of unitary groups. The reference to [M1] will be replaced by [M2] (see also [M3]), which allows the generalization of [H4].
(Remark that Arthur’s work has been generalized to quasi-split unitary groups in [Mk]. Based on that, inner forms of unitary groups are treated in [TMSW].)

For section 2., one observes that the reference [GGP] contains the appropriate results on unitary groups.

C.2 In this section, $H$ will denote the group of $F$-rational points of a quasi-split unitary group $H$ with respect to a quadratic extension $E/F$. As $H$ is not split, one has to use the $L$-group of $H$ which is a semi-direct product $GL_n(\mathbb{C}) \rtimes Gal(E/F)$, where $GL_n(\mathbb{C})$ is the Langlands dual group of $H$.

According to the parity of $n$, we will say that $H$ is an even or odd unitary group. We will denote by $W_E$ the Weil group of $E$. The notion of a conjugate-orthogonal and a conjugate-symplectic representation of $W_E$ is defined in [GGP]. A conjugate-dual representations $\rho$ of $W_E$ will be said of type $^LH$ if either $n$ is even and $\rho$ is conjugate-symplectic, or $n$ is odd and $\rho$ is conjugate-orthogonal. Otherwise, we will say that $\rho$ is not of type $^LH$. We stretch that the use of either these notions will presume that $\rho$ is conjugate-dual. The same terminology will also be used when $W_E$ is replaced by the Weil-Deligne group $W_E \times SL_2(\mathbb{C})$.

There is a unique pure inner form of $H$ which we will denote by $H^-$. If $n$ is odd, then $H^-$ is isomorphic to $H$ and if $n$ is even then $H^-$ is not quasi-split. We will write again sometimes $H^+$ for $H$.

C.3 A Langlands parameter for $H$ is a morphism $W_F \to ^LH$ such that the projection to the first factor is a Langlands parameter (as defined in 1.) and the projection to the second factor is the projection $W_F \to Gal(E/F)$. The definition of a Langlands-Deligne parameter is straightforward.

It is explained in [GGP, section 8] that Langlands and Langlands-Deligne parameters for $H$ are in bijective correspondence with conjugate-dual representations of type $^LH$ of $W_E$ or $W_E \times SL_2(\mathbb{C})$ respectively. If $\varphi$ is a Langlands or a Langlands-Deligne parameter for $H$ we will denote by $\varphi_E$ the corresponding conjugate dual representation of type $^LH$.

With this terminology, replacing $\iota \circ \varphi$ by $\varphi_E$, it is shown in [M2, 8.4.4] (see also [M3]) that the part of theorem 1.1 that applies to $H^+$ generalizes, the case of an orthogonal group applying to $H$ of even dimension and the one of a symplectic group applying to $H$ of odd dimension. As $H$ is isomorphic to $H^-$ in the odd case, one sees easily that this implies the whole theorem 1.1 in the odd case. As [M2, M3] do not treat the pure inner form of the even quasi-split unitary group, this case has to be left as a conjecture, but it is certainly true.

C.4 The definition in 1.2 has to be modified to choose in each inertial class of an irreducible representation of $W_E$ a base point that is conjugate-dual if there is such a representation in the inertial class and, if possible, even conjugate dual of the same type than $^LH$. 
A standard Levi subgroup $M$ of $H$ has the form $GL_{k_1}(E) \times \cdots \times GL_{k_r}(E) \times H_1$, where $H_1$ is a unitary group of the same type (even or odd) than $H$. One has the equality $n = 2(k_1 + \cdots + k_r) + L$, where $L$ is defined by $\hat{H}_1 = GL_L(\mathbb{C})$. One has then $L^M = GL_{k_1}(\mathbb{C}) \times \cdots \times GL_{k_r}(\mathbb{C}) \times L^{H_1}$. If $\varphi = (\rho_1, \ldots, \rho_k, \tau) : WF \to L^M$ is a discrete Langlands parameter, we will denote by $\rho_{i,E}$ the corresponding irreducible representation $W_E \to GL_{k_i}(\mathbb{C})$. If $s$ is an element in the center of $L^M$ such that the representation $\varphi_{E,s}$ in the inertial class of $\varphi_E$ is conjugated dual of the same type than $\varphi_E$, then we will denote by $\varphi_s$ the corresponding Langlands parameter for $L^H$. The set of the $\varphi_s$ will be the inertial orbit of $\varphi$. One defines the multiplicity $m(\rho; \varphi)$ to be the multiplicity of $\rho$ in $\varphi_E$. The proposition 1.3 generalizes obviously to representations of $W_E$, replacing self-dual by dual-conjugate, remarking that $|.|_E$ is self-conjugate. One defines than for a conjugate-dual representation $\rho$ the representation $\rho^-$ accordingly.

Replacing self-dual by dual-conjugate, the generalizations of the notions defined in 1.5 is straightforward and the theorem at the end remains valid.

In the statement of the local Langlands correspondence 1.6, one has to replace $\hat{G}$ by $L^H$ in the arrival space for the Deligne-Langlands parameter, but the centralizers are taken in $\hat{H}$. The definition of the category $Rep_{E,\varphi}(\hat{H})$, for $\varphi_0$ a normed Langlands parameter for $L^H$, and its subcategories $Rep_{E,\varphi_0,\pm}(\hat{H})$ is then clear. The $L$-function and local factors which have to be used here are those coming from the Asai representation.

C.5 The theorem 1.7 is based on [H3] and [H4], which do not explicitly include unitary groups. However, [H3] generalizes with only minor changes to pure inner forms of quasi-split unitary groups: as the Levi subgroups of $H$ are of the form $GL_{k_1}(E) \times \cdots \times GL_{k_r}(E) \times H_1$, the assumptions made in [H3, 1.13 - 17] and the results therein remain valid. One remarks that the relative reduced roots for $H$ form a root system of type $B$ in the odd case and of type $C$ in the even case. From this, the generalization of [H3, 1.13] is straightforward. The same applies to section 6. and 7. of [H3].

The Plancherel measure of a representation of type $\sigma_{S,\epsilon}$ of a Levi subgroup $M_S$ can be computed as in [H3] according to the results in [M2, M3] (especially [M2, 8.4.4] already mentioned above in C.3), the relation with reducibility points remaining the same as in the orthogonal or symplectic case, using $|.|_E$ instead of $|.|_F$. Replacing self-dual by conjugate-dual, the generalization of 1.7 is straightforward. The corollary 1.8 is then a direct consequence.

C.6 Proposition: [GGP, 3.4] The trivial character of $E^\times$ is always a conjugate-
orthogonal representation. The nontrivial unramified quadratic character of $E^\times$ is
congjugate-symplectic, if and only if $E/F$ is unramified. Otherwise, it is
conjugate-orthogonal.

C.7 The unitary group $H$ is called unramified if $E/F$ is an unramified extension.
Denote by $1$ the Langlands parameter of $H$ such that $1_E$ is $n$ times the trivial
representation of $W_E$. We will write $-1$ for the Langlands parameter $1_{-1}$ for $H$
in the above notations. From C.6 and the definitions, it is immediate that the
normed representation in the inertial class of $1$ is $(-1)^{n-1}$, if $H$ is unramified.

**Proposition:** Assume that $H$ is unramified. Denote by $\hat{T}$ the Langlands dual
of the maximal torus of $H$. Let $s$ be in $\hat{T}$ such that $(-1)^{n-1}$ is a conjugate-dual
representation of type $L^H$.

Write $s = \text{diag}(x_1, \ldots, x_{[n/2]}, \sqrt{x_1}, \ldots, x_{[n/2]}^{-1}) \in GL_n(\mathbb{C})$ (with $1$ appearing only
when $n$ is odd and $[\frac{n}{2}]$ denoting the integer part of $\frac{n}{2}$). For $x \in \{x_1, \ldots, x_{[n/2]}\}$,
denote by $m(x, s)$ the multiplicity of $x$ in $s$ and put

$$C_x = \begin{cases} GL_{m(x, s)}, & \text{if } x \notin \{\pm 1\} \\ O_{m(1, s)}, & \text{if } x = 1, \\ Sp_{m(-1, s)}, & \text{if } x = -1. \end{cases}$$

Then, $C_{\hat{H}}(\text{Im}((-1)^n))$ is isomorphic to $\prod_x C_x$, the product going over equiv-
ance classes of elements in the set $\{x_1, \ldots, x_{[n/2]}\}$ with respect to the relation
$x \sim x^{-1}$.

**Proof:** This follows from the generalization of 1.7 given in C.5, taking into
account C.6. \qed

C.8 Proposition 2.1 remains valid, after replacing the Langlands parameter $\varphi$
by $\varphi_E$ and self-dual by conjugate-dual [GGP, sections 4 and 8]. In the same spirit,
one gets the generalization of 2.2 - 2.6.

C.9 With all these changes, the corollary is 3.5 valid for the quasi-split unitary
group.

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