Subspace preserving completely positive maps

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Abstract. A class of quantum channels and completely positive maps (CPMs) are introduced and investigated. These, which we call subspace preserving (SP) CPMs have, in the case of trace preserving CPMs, a simple interpretation as those which preserve probability weights on a given orthogonal sum decomposition of the Hilbert space of a quantum system. Several equivalent characterizations of SP CPMs are proved and an explicit construction of all SP CPMs, is provided. For a subclass of the SP channels a construction in terms of joint unitary evolution with an ancilla system, is presented.

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1. Introduction

Completely positive maps (CPMs) and trace preserving completely positive maps [1] have important use as models of operations on quantum systems. In this article a special type of CPMs, the subspace preserving CPMs, are defined and investigated. To some extent the material presented here should be regarded as a toolbox, to be used in future investigations, like in [2] where subspace local trace preserving CPMs are introduced, and in [3] where the concept of gluings of CPMs is developed. Nevertheless, trace preserving SP CPMs do have a simple conceptual interpretation.

Imagine some kind of box with impenetrable walls. It is assumed that if a particle is put into such a box, it stays there; it neither ‘leaks out’ from the box, nor is it annihilated. Suppose we have two such boxes and one single particle. This particle can be put in an arbitrary state in this box-pair. It may be localized in one of the boxes, in a superposition, or any mixture of localized or delocalized states. The question is: given the restriction that the boxes are impenetrable to the particle, what kind of operations can we, in principle, perform on the state of this particle? Put differently, if the only restriction on the evolution is that there should be no transfer of the particle between the boxes, what kind of evolution is allowed, else allowing any type of interaction with environment or between the boxes? We search for the family of trace preserving CPMs which obey the restriction of no ‘particle transfer’ between the two boxes.

The Hilbert space of the two-box system can be decomposed into an orthogonal sum of two subspaces. One of these subspaces represents the set of pure states localized in one of the boxes, the other subspace represents the pure states localized in the other box. If $P_1$ is the projector onto the subspace of localized pure states of box 1, and if $\Phi$ is the trace preserving CPM of the two-box system, then the condition that the
particle stays in box 1 when put there, can be formulated as $\text{Tr}(P_1\Phi(\rho)) = \text{Tr}(P_1\rho)$, where $\rho$ denotes the initial density operator. This can be interpreted as conservation of probability; it is the same probability to find the particle in box 1, after the operation has been performed, as it was before. This definition, or rather a wider definition which includes more general types of situations, is used to derive some equivalent characterizations of these types of CPMs and also to derive an explicit expression for all such CPMs. In the above example, the SP channels can be characterized as those which preserve certain 2-valued observables, which is related to $[4]$. 

The proofs presented here are all made under the limiting assumption that all involved Hilbert spaces are finite-dimensional. This assumption is made primarily to avoid mathematical technicalities. Much of the material is likely to have analogies in case of separable $[5]$ Hilbert spaces, with some technical modifications. This is not treated here however.

The structure of this article is the following. In section 2 the concept of subspace preserving CPMs is introduced and some equivalent characterizations of this class of CPMs are proved. In section 3 a special type of matrix representation of CPMs is described. In section 4 the matrix representation of the previous section is applied to SP CPMs. Expressions which makes it possible generate all SP CPMs, are deduced. In section 5 we turn to the special case of SP CPMs with identical source and target spaces, and moreover identical decompositions of the source and target spaces, to show a unitary representation for these CPMs. In section 6 a summary is presented.

2. Subspace preserving CPMs

We begin by establish some notation, terminology, and basic concepts used throughout this article. $\mathcal{H}$ denotes a finite-dimensional complex Hilbert space. $\mathcal{H}$ with various subscripts denotes the same. The set of linear operators on $\mathcal{H}$ is denoted $\mathcal{L}(\mathcal{H})$. Moreover, $\mathcal{L}(\mathcal{H}_S, \mathcal{H}_T)$ denotes the set of linear operators from $\mathcal{H}_S$ to $\mathcal{H}_T$. For two Hermitian operators $A, B \in \mathcal{L}(\mathcal{H})$ we let $B \geq A$ denote $\langle \psi | B - A | \psi \rangle \geq 0$ for all $|\psi\rangle \in \mathcal{H}$.

Given a linear map $\phi : \mathcal{L}(\mathcal{H}_S) \to \mathcal{L}(\mathcal{H}_T)$, we say that $\mathcal{H}_S$ is the source-space of $\phi$ and that $\mathcal{H}_T$ is the target space of $\phi$ (or just source and target for short). The source and target space should not be confused with the domain and the range of $\phi$. The domain of $\phi$ is $\mathcal{L}(\mathcal{H}_S)$ and the range is a subspace of $\mathcal{L}(\mathcal{H}_T)$. In this investigation we are concerned with special linear maps $\phi$, the completely positive maps (CPM) $[1]$. It has been shown $[1]$ that if the source and target space of a linear map $\phi$ are separable, then $\phi$ is a CPM if and only if there exists a sequence (finite or countable) of operators $\{V_k\}_k \subset \mathcal{L}(\mathcal{H}_S, \mathcal{H}_T)$, such that $\phi(Q) = \sum_k V_k Q V_k^\dagger$ for all $Q \in \mathcal{L}(\mathcal{H}_S)$ and fulfilling the condition $\sum_k V_k^\dagger V_k \leq a I$, where $0 \leq a < +\infty$. We say that $\{V_k\}_k$ is a Kraus representation of $\phi$. The Kraus representation $\{V_k\}_k$ potentially contains an infinite number of elements (also in the finite-dimensional case) so that we strictly speaking have to define what type of convergence we are considering in the sum $\sum_k V_k Q V_k^\dagger$. However, since the involved spaces are assumed to be finite-dimensional, this is not such an involved question. Moreover, if the target and sources are finite dimensional, it is always possible to find finite Kraus representations (as will be seen).

A CPM is called trace preserving if $\text{Tr}(\Phi(Q)) = \text{Tr}(Q)$ for every trace class operator $Q$. Since the present analysis is restricted to finite-dimensional Hilbert spaces, the set of trace class operators coincide with the set of linear operators. To emphasize
that a CPM is trace preserving, we denote it with a Greek capital letter, while Greek small letters denote general CPMs. The word ‘channel’ is here used as synonymous with trace preserving CPM.

In the following, when discussing CPMs, \( \mathcal{H}_S \) denotes the source space and \( \mathcal{H}_T \) the target space of the CPM in question, unless otherwise stated. These spaces are assumed to be finite-dimensional. Moreover, the source and target spaces are decomposed into orthogonal sums of subspaces as:

\[
\mathcal{H}_S = \mathcal{H}_{s1} \oplus \mathcal{H}_{s2}, \quad \mathcal{H}_T = \mathcal{H}_{t1} \oplus \mathcal{H}_{t2},
\]

where \( \mathcal{H}_{s1}, \mathcal{H}_{s2}, \mathcal{H}_{t1}, \) and \( \mathcal{H}_{t2} \) are assumed to be at least one-dimensional. Furthermore, \( P_{s1} \) denotes the projection operator onto \( \mathcal{H}_{s1} \), \( P_{s2} \) the projection operator onto \( \mathcal{H}_{s2} \), and similarly for \( P_{t1} \) and \( P_{t2} \).

Now we are in position to define subspace preserving CPMs. The word ‘preserving’ refers to preservation of probability weight on selected subspaces. Strictly speaking this terminology is a misnomer for CPMs which are not trace preserving.

**Definition 1** Let \( \phi \) be a CPM with source space \( \mathcal{H}_S \) and target space \( \mathcal{H}_T \). If \( \phi \) fulfills both the conditions

\[
\text{Tr}(P_{t1}\phi(|\psi\rangle\langle\psi|)) = 0, \quad \forall |\psi\rangle \in \mathcal{H}_{s2}, \quad \text{Tr}(P_{t2}\phi(|\psi\rangle\langle\psi|)) = 0, \quad \forall |\psi\rangle \in \mathcal{H}_{s1},
\]

then \( \phi \) is subspace preserving (SP) from \( (\mathcal{H}_{s1}, \mathcal{H}_{s2}) \) to \( (\mathcal{H}_{t1}, \mathcal{H}_{t2}) \).

In proposition 1 it is shown that in case of trace preserving CPMs, this definition is equivalent to a characterization in line with the discussion in the introduction.

In this definition “\(|\psi\rangle\langle\psi|\)” and “\(\forall |\psi\rangle \in \mathcal{H}_{s_j}\)” can be replaced with “\(Q\)” and “\(Q \in \mathcal{L}(\mathcal{H}_{s_j})\)” or with “\(\rho\)” and “for all density operators \(\rho\) on \(\mathcal{H}_{s_j}\).” This because of linearity of \(\phi\) and the fact that any element \(Q\) in \(\mathcal{L}(\mathcal{H}_{s_j})\) can be written as a (complex) linear combination of four density operators on \(\mathcal{H}_{s_j}\), which in turn can be written as a sum of outer products of elements in \(\mathcal{H}_{s_j}\).

**Lemma 1** Let \( \phi \) be a CPM. If \( \text{Tr}(P_{t1}\phi(|\psi\rangle\langle\psi|)) = 0 \) for all \(|\psi\rangle \in \mathcal{H}_{s_j}\) and if \( \{V_k\}_k \) is any Kraus representation of \( \phi \), then \( P_{t1}V_kP_{s_j} = 0 \), \( \forall k \).

**proof.** Let \(|\psi\rangle \in \mathcal{H}_{s_j}\) be arbitrary. Let \(|\chi\rangle \in \mathcal{H}_{t1}\) be an arbitrary normalized vector. By \(|\chi\rangle\langle\chi| \leq P_{t1}\) and by the positivity of \(\phi(|\psi\rangle\langle\psi|)\) follows

\[
0 \leq \langle\chi|\phi(|\psi\rangle\langle\psi|)|\chi\rangle \leq \text{Tr}(P_{t1}\phi(|\psi\rangle\langle\psi|)) = 0.
\]

Let \( \{V_k\}_k \) be an arbitrary Kraus representation of \( \phi \). From 1 one obtains

\[
\sum_k |\langle\chi|V_k|\psi\rangle|^2 = 0, \quad \forall k.
\]

Since this is true for arbitrary \(|\psi\rangle \in \mathcal{H}_{s_j}\) and arbitrary normalized \(|\chi\rangle \in \mathcal{H}_{t1}\) it follows that \( P_{t1}V_kP_{s_j} = 0 \).

**Proposition 1** Let \( \phi \) be a CPM. If \( \phi \) is SP from \( (\mathcal{H}_{s1}, \mathcal{H}_{s2}) \) to \( (\mathcal{H}_{t1}, \mathcal{H}_{t2}) \), then for every Kraus representation \( \{V_k\}_k \) of \( \phi \), there exists operators \( V_{1,k} \) and \( V_{2,k} \), such that

\[
V_k = V_{1,k} + V_{2,k}, \quad P_{t1}V_{1,k}P_{s1} = V_{1,k}, \quad P_{t2}V_{2,k}P_{s2} = V_{2,k}, \quad \forall k.
\]

Conversely if \( \phi \) has a Kraus representation on the form 4, then \( \phi \) is SP from \( (\mathcal{H}_{s1}, \mathcal{H}_{s2}) \) to \( (\mathcal{H}_{t1}, \mathcal{H}_{t2}) \).

**proof.** Let \( \{V_k\}_k \) be any Kraus representation of \( \phi \). If \( \phi \) is assumed to be SP \( (\mathcal{H}_{s1}, \mathcal{H}_{s2}) \) to \( (\mathcal{H}_{t1}, \mathcal{H}_{t2}) \), then by combining lemma 1 with the two conditions 2 it is found that \( P_{t2}V_kP_{s1} = 0 \) and \( P_{t1}V_kP_{s2} = 0 \). Define \( V_{1,k} \) and \( V_{2,k} \) by \( V_{1,k} = P_{t1}V_kP_{s1} \) and \( V_{2,k} = P_{t2}V_kP_{s2} \). These fulfill the conditions stated in the proposition.
The second statement follows since \( \phi \), written on Kraus representation with \( \{V_{1,k} + V_{2,k}\}_k \), fulfills the conditions \( \Box \).

An arbitrary CPM \( \phi \) with source \( \mathcal{H}_S \) and target \( \mathcal{H}_T \), can be written

\[
\phi(Q) = \sum_{i=1,2} \sum_{j=1,2} \sum_{k=1,2} \sum_{l=1,2} P_{ij}\phi(P_{sk}QP_{sl})P_{lj}, \quad \forall Q \in \mathcal{L}(\mathcal{H}_S). \tag{5}
\]

This should be compared with the fourth statement of the following proposition, which gives an analogous expression in case \( \phi \) is SP from \( (\mathcal{H}_{s}, \mathcal{H}_{s}) \) to \( (\mathcal{H}_{t}, \mathcal{H}_{t}) \).

**Proposition 2** Let \( \phi \) be a CPM with source \( \mathcal{H}_S \) and target \( \mathcal{H}_T \). The following are equivalent

\begin{enumerate}[(i)]
  \item \( \phi \) is SP from \( (\mathcal{H}_{s}, \mathcal{H}_{s}) \) to \( (\mathcal{H}_{t}, \mathcal{H}_{t}) \).
  \item \( P_{ij}\phi(Q)P_{lj} = \phi(P_{sk}QP_{sl}) \quad i = 1,2 \quad j = 1,2, \quad \forall Q \in \mathcal{L}(\mathcal{H}_S). \)
  \item \( P_{ij}\phi(P_{sk}QP_{sl})P_{lj} = \delta_{ik}\delta_{lj}P_{ii}\phi(P_{sk}QP_{sl})P_{ij} \quad i, j, k, l = 1,2, \quad \forall Q \in \mathcal{L}(\mathcal{H}_S). \)
  \item \( \phi(Q) = \sum_{i=1,2} \sum_{j=1,2} P_{ij}\phi(P_{sk}QP_{sl})P_{lj}, \quad \forall Q \in \mathcal{L}(\mathcal{H}_S). \)
\end{enumerate}

**proof.** \( \Box \) \( \Rightarrow \) \( \Box \): From the existence of a Kraus representation \( \{V_{1,k} + V_{2,k}\}_k \) given by proposition \( \Box \) follows \( \Box \), since \( P_{ij}\phi(Q)P_{lj} = \sum_{k} V_{ik}V_{kj}^\dagger = \phi(P_{sk}QP_{sl}). \)

\( \Box \Rightarrow \Box \): By \( \Box \) follows \( P_{ii}\phi(P_{sk}QP_{sl})P_{ij} = \phi(P_{sk}QP_{sl}P_{sj}), \) from which follows.

\( \Box \Rightarrow \Box \): By inserting the condition \( \Box \) into equation \( \Box \), \( \Box \) follows.

\( \Box \Rightarrow \Box \): Assuming \( \phi \) fulfills the condition of \( \Box \) then \( \phi \) fulfills conditions \( \Box \) and hence, \( \phi \) is SP from \( (\mathcal{H}_{s}, \mathcal{H}_{s}) \) to \( (\mathcal{H}_{t}, \mathcal{H}_{t}) \). \( \Box \)

**Proposition 3** If a CPM \( \phi_a \) is SP from \( (\mathcal{H}_{s}, \mathcal{H}_{s}) \) to \( (\mathcal{H}_{t}, \mathcal{H}_{t}) \) and if a CPM \( \phi_b \) is SP from \( (\mathcal{H}_{s}, \mathcal{H}_{s}) \) to \( (\mathcal{H}_{r}, \mathcal{H}_{r}) \) then \( \phi_b \circ \phi_a \) is SP from \( (\mathcal{H}_{s}, \mathcal{H}_{s}) \) to \( (\mathcal{H}_{r}, \mathcal{H}_{r}) \).

The spaces \( \mathcal{H}_{t} \) and \( \mathcal{H}_{r} \) are assumed to be finite-dimensional, at least one-dimensional and being orthogonal complements of each other.

**proof.** Using proposition \( \Box \) follows

\[
P_{ij}\phi_a(Q)P_{lj} = \phi_b(P_{ii}\phi_a(Q)P_{ij}) = \phi_b(\phi_a(P_{sk}QP_{sl})), \quad \forall Q \in \mathcal{L}(\mathcal{H}_S). \tag{6}
\]

Hence, by proposition \( \Box \), \( \phi_b \circ \phi_a \) is SP. \( \Box \)

**Proposition 4** Let \( \Phi \) be a trace preserving CPM. The following are equivalent

\begin{enumerate}[(i)]
  \item \( \Phi \) is SP from \( (\mathcal{H}_{s}, \mathcal{H}_{s}) \) to \( (\mathcal{H}_{t}, \mathcal{H}_{t}) \).
  \item \( \text{Tr}(P_{1}\Phi(Q)) = \text{Tr}(P_{s1}Q), \quad \forall Q \in \mathcal{L}(\mathcal{H}_S). \)
  \item \( \text{Tr}(P_{2}\Phi(Q)) = \text{Tr}(P_{s2}Q), \quad \forall Q \in \mathcal{L}(\mathcal{H}_S). \)
\end{enumerate}

**proof.** \( \Box \) \( \Leftrightarrow \) \( \Box \): Using the fact that \( P_{s1} + P_{s2} = \hat{1}_S \) and similarly for the target space, one can write

\[
\text{Tr}(\Phi(Q)) - \text{Tr}(Q) = \text{Tr}(P_{1}\Phi(Q)) - \text{Tr}(P_{s1}Q) + \text{Tr}(P_{2}\Phi(Q)) - \text{Tr}(P_{s2}Q).
\]

Since \( \Phi \) is trace preserving, it follows that \( \text{Tr}(P_{1}\Phi(Q)) - \text{Tr}(P_{1}Q) = - \text{Tr}(P_{2}\Phi(Q)) + \text{Tr}(P_{2}Q), \) from which the equivalence of \( \Box \) and \( \Box \) is obtained.

\( \Box \Rightarrow \Box \): With \( Q = |\psi\rangle\langle\psi|, |\psi\rangle \in \mathcal{H}_S \) into \( \Box \), the first of the conditions \( \Box \) is seen to hold. We know \( \Box \) \( \Rightarrow \) \( \Box \). Hence, with \( \psi \in \mathcal{H}_s \) into \( \Box \), the second of the conditions \( \Box \) is seen to hold. Hence, \( \Phi \) is SP.

\( \Box \Rightarrow \Box \): From proposition \( \Box \) it follows that \( P_{1}\Phi(Q)P_{1} = \Phi(P_{s1}QP_{s1}). \) Hence, \( \text{Tr}(P_{1}\Phi(Q)) = \text{Tr}(P_{1}\Phi(Q)P_{1}) = \text{Tr}(\Phi(P_{s1}QP_{s1})) = \text{Tr}(P_{s1}Q), \) where the last equality follows from \( \Phi \) being trace preserving. \( \Box \)
3. Matrix representation of CPMs

In this section some material is presented which will be useful in the analysis of SP and SL CPMs, as well as for the gluing concept. We here discuss a type of matrix representation of CPMs, where CPMs are represented by positive semi-definite matrices. One can construct many different types of matrix representations, and the choice is a matter of convenience, depending on the application. The type used here has appeared before in the literature (for examples see [8, 9]). Another form can be found in [10]. In order to establish the properties of the representation used here, we state and prove the following proposition.

**Proposition 5** Let \( \{V_m\}_{m=1}^M \subset \mathcal{L}(\mathcal{H}_S, \mathcal{H}_T) \) with \( M = \dim(\mathcal{H}_S) \dim(\mathcal{H}_T) \) be a basis of \( \mathcal{L}(\mathcal{H}_S, \mathcal{H}_T) \). Let the elements of the set \( \{\phi_{m,m'}\}_{m,m'=1}^M \) be defined as

\[
\phi_{m,m'}(Q) = V_m Q V_{m'}^\dagger, \quad \forall Q \in \mathcal{L}(\mathcal{H}_S).
\]

The set \( \{\phi_{m,m'}\}_{m,m'=1}^M \) forms a basis of \( \mathcal{L}(\mathcal{L}(\mathcal{H}_S), \mathcal{L}(\mathcal{H}_T)) \) and hence the equation

\[
\phi(Q) = \sum_{m,m'=1}^M F_{m,m'} V_m Q V_{m'}^\dagger, \quad \forall Q \in \mathcal{L}(\mathcal{H}_S),
\]

defines a linear bijection between the set of complex \( M \times M \) matrices \( F = [F_{m,m'}]_{m,m'} \) and \( \mathcal{L}(\mathcal{L}(\mathcal{H}_S), \mathcal{L}(\mathcal{H}_T)) \).

Moreover, equation (8) defines a bijection between the set of all positive semi-definite matrices \( F \) and the set of all CPMs with source space \( \mathcal{H}_S \) and target space \( \mathcal{H}_T \).

In this type of representation we regard \( \phi \) as a vector in \( \mathcal{L}(\mathcal{L}(\mathcal{H}_S), \mathcal{L}(\mathcal{H}_T)) \), and represent it via a basis in this space. This basis we construct using an arbitrary basis of \( \mathcal{L}(\mathcal{H}_S, \mathcal{H}_T) \). This representation may be compared with more ‘classical’ matrix representations of linear maps. In such a representation we would use a basis of \( \mathcal{L}(\mathcal{H}_S) \) and a basis of \( \mathcal{L}(\mathcal{H}_T) \) to represent linear maps from \( \mathcal{L}(\mathcal{H}_S) \) to \( \mathcal{L}(\mathcal{H}_T) \) as matrices.

**Proof.** Most of the proposition follows immediately if it can be shown that \( \{\phi_{m,m'}\}_{m,m'} \) is a basis of \( \mathcal{L}(\mathcal{L}(\mathcal{H}_S), \mathcal{L}(\mathcal{H}_T)) \). Clearly each \( \phi_{m,m'} \) belongs to this set. The set \( \{\phi_{m,m'}\}_{m,m'} \) contains \( M^2 \) elements, hence it suffices to show that it is linearly independent (since \( \dim(\mathcal{L}(\mathcal{L}(\mathcal{H}_S), \mathcal{L}(\mathcal{H}_T))) = M^2 \)). Define \( \alpha = \sum_{m,m'} C_{m,m'} \phi_{m,m'} \), for some arbitrary complex numbers \( C_{m,m'} \). It has to be shown that \( \alpha = 0 \) implies \( C_{m,m'} = 0 \) for all \( m, m' \). Let \( |\psi\rangle, |\chi\rangle \in \mathcal{H}_S, |\eta\rangle \in \mathcal{H}_T \) all be arbitrary. The condition \( \alpha = 0 \) implies \( \alpha(|\psi\rangle \langle \chi|)\langle \eta| = 0 \). This can be rewritten (by inserting the definition of \( \phi_{m,m'} \) and rearrange) as \( \sum_m (\sum_{m'} C_{m,m'} \langle \chi|V_m^\dagger |\eta\rangle) V_m |\psi\rangle = 0 \). Since \( |\psi\rangle \) is arbitrary, it follows that \( \sum_m (\sum_{m'} C_{m,m'} \langle \chi|V_m^\dagger |\eta\rangle) V_m = 0 \). Since \( \{V_m\}_m \) is a linearly independent set, the last equation implies \( \sum_{m'} C_{m,m'} \langle \chi|V_{m'}^\dagger |\eta\rangle = 0 \), for each \( m \). Since \( |\chi\rangle \) and \( |\eta\rangle \) are arbitrary, it further follows that \( \sum_m C_{m,m'} V_{m'}^\dagger = 0, \quad \forall m \). If \( \{V_m\}_m \) is a linearly independent set, then so is \( \{V_m^\dagger\}_m \) and hence, \( C_{m,m'} = 0, \quad \forall m, m' \). Hence, \( \{\phi_{m,m'}\}_{m,m'} \) is a basis of \( \mathcal{L}(\mathcal{L}(\mathcal{H}_S), \mathcal{L}(\mathcal{H}_T)) \). From \( \{\phi_{m,m'}\}_{m,m'} \) being a basis it follows that the complex numbers \( F_{m,m'} \) in (8) are the expansion coefficients of \( \phi \) with respect to this basis. Hence, the bijectivity stated in the proposition follows.

It remains to show the bijectivity between the set of positive semi-definite matrices \( F \) and the set of CPMs. Assuming \( F \) is positive semi-definite, there exists some unitary matrix \( U \), such that \( U^\dagger D U = F \), where \( D \) is a diagonal matrix \( D = [d_{m,m'}]_{m,m'} \) with \( d_{m,m'} \geq 0 \). Define \( \{W_n\}_n \) by \( W_n = \sqrt{d_n} \sum_m U_{n,m} V_m \) for \( n \) for which \( d_n \neq 0 \).
(For this proof \(d_n \neq 0\) is not needed, but will be useful in a later proof.) One can check that \(\{W_n\}_n\) so defined is a Kraus representation of \(\phi\). Since any element in \(\mathcal{L}(\mathcal{H}_S, \mathcal{H}_T)\) which has a Kraus representation, is a CPM, it follows that \(\phi\) is a CPM.

It remains to show that for any CPM \(\phi\) in \(\mathcal{L}(\mathcal{H}_S, \mathcal{H}_T)\), the corresponding matrix \(F\) is positive semi-definite. Let \(\{|s_n\}_{n=1}^{\infty}\) be an arbitrary orthonormal basis of \(\mathcal{H}_S\). Let |\(\psi\rangle = \sum_n |s_n\rangle |s_n\rangle\). (Hence, |\(\psi\rangle\) is an element of \(\mathcal{H}_S \otimes \mathcal{H}_S\).) Let \(I_N\) denote the identity CPM with source and target \(\mathcal{H}_S\). Since \(\phi\) is a CPM it follows, by definition \(\mathcal{H}_S\), that \(\phi \otimes I_N\) maps positive semi-definite operators to positive semi-definite operators. Let \(A \in \mathcal{L}(\mathcal{H}_S, \mathcal{H}_T)\) be arbitrary. Clearly \((A^\dagger \otimes 1_S)(\phi \otimes I_N)(Q)(A \otimes 1_S)\) is a positive semi-definite operator for any positive semi-definite \(Q\). One may verify that

\[
\langle \psi | (A^\dagger \otimes 1_S)(\phi \otimes I_N)(|\psi\rangle \langle \psi|)(A \otimes 1_S)|\psi\rangle = \sum_{m,m'} F_{mm'} \text{Tr}(A^\dagger V_m) \text{Tr}(V_{m'}^\dagger A)
\]

and hence

\[
\sum_{m,m'} F_{mm'} \text{Tr}(A^\dagger V_m) \text{Tr}(V_{m'}^\dagger A) \geq 0, \quad \forall A \in \mathcal{L}(\mathcal{H}_S, \mathcal{H}_T).
\]  

Since \(\{V_m\}_{m=1}^M\) is a linearly independent set, it follows that the matrix \(F\) has to be positive semi-definite. To see this, one can use that \((A, B) = \text{Tr}(A B^\dagger)\) is an inner product (the Hilbert-Schmidt inner product \(\mathcal{H}_S\), \(\mathcal{H}_T\)). With respect to this inner product we may form a Gram-matrix with elements \(G_{mm'} = \text{Tr}(V_m^\dagger V_{m'})\). Since \(\{V_m\}_{m=1}^M\) is a linearly independent set, this matrix is positive definite and hence invertible. Hence, if \(c = (c_m)_{m=1}^M \in \mathbb{C}\) is arbitrary, we may construct

\[
A = \sum_{m=1}^M c_m V_m.
\]

By inserting this into equation \(\mathcal{H}_S\), one finds that \(c^\dagger G F c \geq 0\) for any \(c \in \mathbb{C}^M\). Hence, \(G F G^\dagger \geq 0\). Since \(G\) is invertible it follows that \(F \geq 0\), which shows the proposition.

**Proposition 6** To every CPM \(\phi\) there exists a linearly independent Kraus representation. The number of elements \(K(\phi)\) in a linearly independent Kraus representation only depends on the CPM and not on the choice of linearly independent Kraus representation. The number \(K(\phi)\), to be called the Kraus number, have the following properties:

(i) \(K(\phi) \leq \dim(\mathcal{H}_S) \dim(\mathcal{H}_T)\).

(ii) \(K(\phi)\) is the minimal number of operators needed in any Kraus representation of \(\phi\).

(iii) \(K(\phi)\) is the number of non-zero eigenvalues, counted with multiplicity of the matrix \(F\) given by proposition \(\mathcal{H}_S\), for any choice of basis of \(\mathcal{L}(\mathcal{H}_S, \mathcal{H}_T)\).

**Proof.** First it has to be proved that for every CPM it is possible to find a linearly independent Kraus representation. Actually we have already constructed such a set in the proof of proposition \(\mathcal{H}_S\). The set \(\{W_n\}_n\), defined in that proof, is such a set. To see why this is the case one can note how \(\{W_n\}_n\) was constructed. By assumption \(\{V_m\}_{m=1}^M\) forms a basis. Hence, the set \(\{\tilde{W}_n\}_{n=1}^M\) defined by \(\tilde{W}_n = \sum_{m=1}^M U^*_n V_m\) for the unitary matrix \(U\), must also be a basis and hence linearly independent. Since the elements \(W_n\) were defined as \(W_n = \sqrt{d_n} \tilde{W}_n\), for the non-zero \(d_n\), it follows that \(\{W_n\}_n\) must also be a linearly independent set. Hence, there exists a linearly independent Kraus representation.

The number of elements in \(\{W_n\}_n\) is equal to the number of non-zero eigenvalues \(d_n\) (counted with multiplicity) of the matrix \(F\). Moreover, there cannot be more than \(\dim(\mathcal{H}_S) \dim(\mathcal{H}_T)\) non-zero eigenvalues.
Consider the same CPM \( \phi \), but represented with respect to some other choice of basis \( \{ \tilde{V}_m \}_{m=1}^M \) of \( \mathcal{L}(\mathcal{H}_S, \mathcal{H}_T) \). As with any change of basis, the new basis and the original are related via an invertible matrix \( A \) as \( V_m = \sum_{m'} \tilde{V}_{m'} A_{m'm} \). When inserting this into (5), one finds that the new matrix \( \tilde{F} \) is related to the old as \( \tilde{F} = AFA^\dagger \). Since \( A \) is invertible, the number of non-zero eigenvalues of \( F \) and \( \tilde{F} \) are the same. The eigenvalues per se may change, but not the number of non-zero eigenvalues. This follows from “Sylvester’s law of inertia” [7], since \( F \) and \( \tilde{F} \) are congruent \( (F = AFA^\dagger \) for some non-singular \( A \)) and Hermitian [4]. Hence, the number of non-zero eigenvalues of the matrix \( F \) is independent of the choice of basis. From this follows directly that the number of operators in a linearly independent Kraus representation is independent of the choice of linearly independent representation. Hence, it is possible to define \( K(\phi) \) as the number of operators in a linearly independent Kraus representation. It also follows that if a Kraus representation has \( K(\phi) \) elements, it has to be a linearly independent Kraus representation. Moreover, there cannot be any Kraus representation with less than \( K(\phi) \) elements.

One may wonder about the nature of the set of all linearly independent Kraus representations of a CPM.

**Proposition 7** Let \( \phi \) be a CPM with Kraus number \( K = K(\phi) \). Let \( \{ V_k \}_{k=1}^K \) be an arbitrary but fixed linearly independent Kraus representation of \( \phi \). Let \( \{ V'_k \}_{k=1}^K \) be defined by

\[
V'_k = \sum_{k'=1}^K U_{kk'} V_{k'}, \quad k = 1, \ldots, K.
\] (10)

Equation (10) defines a bijection between the set of all linearly independent Kraus representations of \( \phi \) and the set of unitary \( K \times K \) matrices \( U = [U_{kk'}]_{k,k'=1}^K \).

Note that in this proposition, two linearly independent Kraus representations \( \{ V_k \}_{k=1}^K \) and \( \{ V'_k \}_{k=1}^K \) are regarded as ‘equal’, if and only if \( V_k = V'_k \) for all \( k = 1, \ldots, K \). Hence, two linearly independent Kraus representations differing only in the ordering of its elements are regarded as ‘different’ in this proposition.

**proof.** It is a well known result [11], [12] that any two Kraus representations of the same CPM can be connected via a unitary matrix (where a possibly smaller set of Kraus operators is padded with zero operators, such that the two sets have the same number of elements), and moreover that any unitary matrix \( U \) creates a new Kraus representation. By combining these facts with proposition 6 it follows that every unitary \( K \times K \) matrix \( U \) gives a linearly independent Kraus representation via (10) and moreover that every linearly independent Kraus representation can be reached in this manner. Hence, (10) is surjective from the set of unitary \( K \times K \) matrices \( U \) to the set of linearly independent Kraus representations. Hence, it remains to show that the mapping is injective. Suppose \( U \) and \( U' \) both give the same linearly independent Kraus representation. Then \( \sum_{k=1}^K U'_{k',k} V_k = \sum_{k=1}^K U_{k',k} V_k \) for all \( k' = 1, \ldots, K \). Since \( \{ V_k \}_{k=1}^K \) is a linearly independent set, it follows that \( U_{k',k} = U'_{k',k} \) for all \( k, k' \). Hence, no two distinct unitary matrices are mapped to the same linearly independent Kraus representation. Hence, the mapping is injective.

We conclude this section by proving that it is always possible to find a linearly independent Kraus representation based on an operator set which is orthonormal with respect to the Hilbert-Schmidt inner product \( (A, B) = \text{Tr}(A^\dagger B) \).
Proposition 8 Let $\phi$ be a CPM with Kraus number $K = K(\phi)$. There exists a linearly independent Kraus representation on the form $\{\sqrt{r_n}Y_n\}_{n=1}^K$ where each $r_n$ is a positive real number $r_n > 0$ and where the set $\{Y_n\}_{n=1}^K$ is orthonormal with respect to the Hilbert-Schmidt inner product.

**proof.** Let $\{V_m\}_m$ be a linearly independent Kraus representation of $\phi$. Form the Gram-Matrix $R = [R_{m,m'}]_{m,m'}$, with elements $R_{m,m'} = \text{Tr}(V_m^d V_{m'})$. Since $\{V_m\}_m$ is a linearly independent set, the Gram matrix is positive definite \[.\] Hence, there exists a unitary $M \times M$ matrix $U$ and numbers $r_n > 0$, such that $\sum_{m,m'} U_{m,n}^* R_{m,m'} U_{m,n'} = r_n \delta_{n,n'}$. Let $\{\hat{Y}_n\}_n$ be the linearly independent Kraus representation defined by $\hat{Y}_n = \sum_m V_m U_{m,n}$. By construction $\text{Tr}(\hat{Y}_n^d \hat{Y}_{n'}) = r_n \delta_{n,n'}$. Since $r_n > 0$ one may define $Y_n = \frac{1}{\sqrt{r_n}} \hat{Y}_n$. The set $\{\sqrt{r_n}Y_n\}_{n=1}^N$ so defined, fulfills the statements of the proposition. \(\square\)

4. Matrix representation of SP CPMs

Here the matrix representation presented in the previous section is applied to the case of SP CPMs.

**Proposition 9** Let $\{V_k\}_k$ be a basis of $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and let $\{W_l\}_l$ be a basis of $\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$. A CPM $\phi$ is SP from $(\mathcal{H}_1, \mathcal{H}_2)$ to $(\mathcal{H}_1, \mathcal{H}_2)$ if and only if it can be written

$$\phi(Q) = \sum_{k,k'} F_{k,k'} V_k Q V_{k'}^\dagger + \sum_{k,l} F_{k,l} V_k Q W_l^\dagger + \sum_{l,l'} F_{l,l'} W_l Q W_{l'}^\dagger, \quad \forall Q \in \mathcal{L}(\mathcal{H}_S), \quad (11)$$

where the matrix $F$ is defined by

$$F = \left[ \begin{array}{cc} [F_{k,k'}]_{k,k'} & [F_{k,l}]_{k,l} \\ [F_{l,k'}]_{l,k'} & [F_{l,l'}]_{l,l'} \end{array} \right]. \quad (12)$$

is positive semi-definite. Moreover, $(11)$ defines a bijection between the set of such positive semi-definite matrices $F$ and the set of CPMs that are SP from $(\mathcal{H}_1, \mathcal{H}_2)$ to $(\mathcal{H}_1, \mathcal{H}_2)$.

**proof.** We begin with the “if” part of the proposition. The set $\{V_k\}_k \cup \{W_l\}_l$ is not a basis of $\mathcal{L}(\mathcal{H}_S, \mathcal{H}_T)$. In order to use proposition 5 we have to complete it with a basis of $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$: $\{Y^{(1)}_n\}_n$, and a basis of $\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$: $\{Y^{(2)}_m\}_m$. The set $B = \{V_k\}_k \cup \{W_l\}_l \cup \{Y^{(1)}_n\}_n \cup \{Y^{(2)}_m\}_m$ forms a basis of $\mathcal{L}(\mathcal{H}_S, \mathcal{H}_T)$. The matrix $(12)$ is a sub-matrix of the matrix $F$ given by proposition 8 with respect to the basis $B$. Moreover, $(12)$ forms the only non-zero part of the matrix. Hence, the matrix $F$ of proposition 5 is positive semi-definite if and only if the sub-matrix $(12)$ is. Hence, $\phi$ defined by $(11)$, is a CPM if and only if $(12)$ is positive semi-definite. By inserting $\phi$ as defined by $(11)$, into the conditions $(2)$, it follows that $\phi$ is an SP CPM.

For the “only if” part, assume $\phi$ is any CPM which is SP from $(\mathcal{H}_1, \mathcal{H}_2)$ to $(\mathcal{H}_1, \mathcal{H}_2)$. Let $F$ be the matrix (given by proposition 4) which represents $\phi$ with respect to the basis $B$: $\phi(Q) = \sum_{s,s'} F_{s,s'} X_s Q X_{s'}^\dagger$, where each $X_s$ is an element of type $V$, $W$, $Y^{(1)}$, or $Y^{(2)}$. Using proposition 2 one can show that $F_{s,s'} = 0$ if $X_s$ or $X_{s'}$ is an element of type $Y^{(1)}$ or $Y^{(2)}$. Hence, if $\phi$ is an SP CPM, then the only (potentially) non-zero part of the sum $\sum_{s,s'} F_{s,s'} X_s Q X_{s'}^\dagger$ is $(12)$, where the sub-matrix
consists of those elements $F_{k,s'}$, for which $X_s$ and $X_{s'}$ are both of the type $V$ or $W$. Hence, we have shown that (11) defines a surjective map from the set of positive semi-definite matrices (12) to the set of SP CPMs. That this map is also injective follows from the bijectivity stated in proposition 9.

Corollary 1 If $\phi$ is SP CPM from $(\mathcal{H}_1, \mathcal{H}_2)$ to $(\mathcal{H}_1, \mathcal{H}_2)$ then

$$K(\phi) \leq \dim(\mathcal{H}_1) \dim(\mathcal{H}_1) + \dim(\mathcal{H}_2) \dim(\mathcal{H}_2).$$

proof. The corollary follows directly from the characterization of the Kraus number $K(\phi)$ as the number of non-zero eigenvalues of the representation matrix $F$ of proposition 9. By proposition 9 only the sub-matrix (12) is non-zero. Since this sub-matrix is a $[\dim(\mathcal{H}_1) \dim(\mathcal{H}_1) + \dim(\mathcal{H}_2) \dim(\mathcal{H}_2)] \times [\dim(\mathcal{H}_1) \dim(\mathcal{H}_1) + \dim(\mathcal{H}_2) \dim(\mathcal{H}_2)]$ matrix, the corollary follows.

For the rest of this section we develop means to rewrite proposition 9. This is done by characterizing positive semi-definite matrices in terms of sub-matrices.

Lemma 2 If $D$ is a complex positive semi-definite $N \times N$ matrix, then

$$|a^\dagger Db|^2 \leq a^\dagger Da b^\dagger Db, \quad \forall a, b \in \mathbb{C}^N.$$  

Note that $a$ and $b$ are regarded as being column vectors.

proof. Consider the $2 \times 2$ matrix $F$

$$F = \begin{bmatrix} a^\dagger Da & a^\dagger Db \\ b^\dagger Da & b^\dagger Db \end{bmatrix}.$$  

Let $c \in \mathbb{C}^2$ be arbitrary. ($c$ is regarded as a column vector with elements $c_1$ and $c_2$.)

$$c^\dagger Fc = (c_1 a^\dagger + c_2 b^\dagger )D(c_1 a + c_2 b) \geq 0,$$

where the last inequality follows since $D$ is positive semi-definite. Since $c$ is arbitrary it follows that $F$ is positive semi-definite. By the fact that the determinant of a positive semi-definite matrix is non-negative, the statement of the lemma follows, since $\det(F) = a^\dagger Da b^\dagger Db - |a^\dagger Db|^2.$

Lemma 3 Let $A$ and $B$ be $N \times N$ and $M \times M$ positive semi-definite matrices, respectively. Let $C$ be a complex $N \times M$ matrix and let $F$ be the $(N + M) \times (N + M)$ matrix

$$F = \begin{bmatrix} A & C \\ C^\dagger & B \end{bmatrix}.$$  

Then $F$ is positive semi-definite if and only if

$$P_{A,0}C = 0, \quad CP_{B,0} = 0, \quad A \geq CB^\oplus C^\dagger,$$

(13)

where $B^\oplus$ denotes the Moore-Penrose pseudo inverse of $B$. $P_{A,0}$ denotes the orthogonal projector onto the zero eigenspace of $A$ and analogously for $P_{B,0}$. In (13), the condition $A \geq CB^\ominus C^\dagger$ can be replaced with the condition $B \geq C^\dagger A^\ominus C$.

A comment on the Moore-Penrose pseudo inverse (MP-inverse) is perhaps suitable here. If a matrix is invertible, its MP-inverse reduces to the ordinary inverse. If a (finite) square matrix $B$ is Hermitian, and if its non-zero eigenvalues are $\lambda_k$, with corresponding orthonormal eigenvectors $b_k$, such that $B = \sum_k \lambda_k b_k b_k^\dagger$ (regard $b_k$ as column vectors) then $B^\ominus = \sum_k \lambda_k^{-1} b_k b_k^\dagger$. The operator $BB^\ominus = B^\ominus B$ is the projector onto the range of $B$. Moreover, $P_{A,0} = I_N - AA^\ominus$ and $P_{B,0} = I_M - BB^\ominus$. In the following the MP-inverse will be used without further comments.

proof. Throughout this proof, elements of $\mathbb{C}^N$, $\mathbb{C}^M$, $\mathbb{C}^{N+M}$ are all regarded as column vectors. We begin by proving that if $F$ is positive semi-definite then $P_{A,0}C = 0,$
\[CP_{B,0} = 0 \text{ and } A \geq CB^\ominus C^\dagger.\] For any \(q_A\) in the zero eigenspace of \(A\), it follows by lemma \(\text{[2]}\) applied to the matrix \(F\), that \(q_A^\dagger Cq_B = 0\) for every \(q_B \in \mathbb{C}^M\). From this follows \(P_{A,0}C = 0\). Similarly one can derive \(CP_{B,0} = 0\). Let \(q \in \mathbb{C}^{N+M}\) be arbitrary and let \(q_A\) denote the projection of \(q\) onto the first \(N\) components of \(q\) and let \(q_B\) denote the projection of \(q\) onto the rest of the \(M\) components (\(q = q_A \oplus q_B\)). Then

\[
q^\dagger Fq = q_A^\dagger Aq_A + q_B^\dagger Cq_B + q_B^\dagger C^\dagger q_A + q_B^\dagger Bq_B. \quad (14)
\]

Let \(q\) be such that \(q_B = -B^\ominus C^\dagger q_A\) and \(q_A \in \mathbb{C}^N\) being arbitrary. By inserting this \(q\) into \((14)\) one finds \(q^\dagger Fq = q_A^\dagger (A - CB^\ominus C^\dagger)q_A\). Since \(F\) is assumed to be positive semi-definite, it follows that \(q_A^\dagger (A - CB^\ominus C^\dagger)q_A \geq 0\). Since \(q_A\) is arbitrary one obtains \(A - CB^\ominus C^\dagger \geq 0\). By an analogous reasoning it can be shown that \(B \geq C^\dagger A^\ominus C\). (Let \(q_B \in \mathbb{C}^M\) be arbitrary and \(q_A = -A^\ominus Cq_B\).

Next we have to prove that if \(P_{A,0}C = 0\), \(CP_{B,0} = 0\) and \(A \geq CB^\ominus C^\dagger\), then \(F\) is positive semi-definite. Let \(q \in \mathbb{C}^{N+M}\) be arbitrary and let \(q = q_A \oplus q_B\). From \(A \geq CB^\ominus C^\dagger\) follows that \(q_A^\dagger Aq_A \geq q_A^\dagger CB^\ominus C^\dagger q_A\). By inserting the last expression into equation \((14)\) one obtains

\[
q^\dagger Fq = q_A^\dagger C\sqrt{B^\ominus} \sqrt{B^\ominus} C^\dagger q_A + q_A^\dagger C\sqrt{B} \sqrt{B} q_B + q_B^\dagger C^\dagger q_A + q_B^\dagger \sqrt{B} \sqrt{B} q_B.
\]

In the last equality above, it has been used that \(\sqrt{B^\ominus} \sqrt{B} = \sqrt{B^\ominus B}\) is the projector onto the range of \(B\). Hence, \(C = C\sqrt{B^\ominus} \sqrt{B}\), since \(CP_{B,0} = 0\). Let \(q_D = \sqrt{B^\ominus} C^\dagger q_A\) and \(q_E = \sqrt{B} q_B\). By inserting the last expression into \((16)\), one obtains \(q^\dagger Fq \geq q_A^\dagger q_D + q_E^\dagger q_D + q_D^\dagger q_D + q_E^\dagger q_E = \|q_D + q_E\|^2 \geq 0\). Since \(q\) is arbitrary, it follows that \(F\) is positive semi-definite. A similar derivation can be done for \(B \geq C^\dagger A^\ominus C\).

Now we are in position to state the reformulation of proposition \(\text{[8]}\) by which we end this section. One may wonder why this reformulation is useful. Loosely speaking, proposition \(\text{[3]}\) is to prefer when we ‘search’ the whole set of SP CPMS (with respect to some choice of source and target decomposition) without any additional assumptions. Proposition \(\text{[10]}\) is more useful when considering, not the whole set of SP CPMS, but subsets for which the sub-matrices \(A\) and \(B\) are fixed. Proposition \(\text{[10]}\) is particularly useful if \(A\) and \(B\) can be chosen to have a simple form, like for example identity matrices. (Indeed such conditions, or similar, do occur \(\text{[3]}\).) Hence, depending on the specific problem at hand, one of propositions \(\text{[3]}\) or \(\text{[10]}\) may be the preferable tool.

By combination of proposition \(\text{[10]}\) and lemma \(\text{[3]}\) the following is obtained.

**Proposition 10** Let \(\{V_k\}_{k=1}^K\) be a basis of \(L(H_{s1}, H_{t1})\), \(K = \dim H_{s1} \dim H_{t1}\), and let \(\{W_l\}_{l=1}^L\) be a basis of \(L(H_{s2}, H_{t2})\), \(L = \dim H_{s2} \dim H_{t2}\). Then \(\phi\) defined by

\[
\phi(Q) = \sum_{k,k'} A_{k,k'} V_k Q V_{k'}^\dagger + \sum_{l,l'} B_{l,l'} W_l Q W_{l'}^\dagger + \sum_{k,l} C_{k,l} V_k Q W_{l'}^\dagger + \sum_{k,l} C_{k,l}^* W_l Q V_{k'}^\dagger. \quad (16)
\]

for all \(Q \in L(H_{s})\), is an SP CPMS from \((H_{s1}, H_{s2})\) to \((H_{t1}, H_{t2})\), if and only if the matrices \(A = [A_{k,k'}]_{k,k'}, B = [B_{l,l'}]_{l,l'}\) and \(C = [C_{k,l}]_{k,l}\) fulfill the relations

\[
A \geq 0, \quad B \geq 0, \quad P_{A,0}C = 0, \quad CP_{B,0} = 0, \quad A \geq CB^\ominus C^\dagger, \quad (17) \quad (18)
\]

with \(P_{A,0}\) and \(P_{B,0}\) defined as in lemma \(\text{[3]}\).
Moreover, \( (16) \) defines a bijection between the set of all CPMs which are SP from \((\mathcal{H}_{s1}, \mathcal{H}_{s2})\) to \((\mathcal{H}_{t1}, \mathcal{H}_{t2})\), and the set of all triples of matrices \((A, B, C)\) fulfilling the conditions \((14)\) and \((15)\).

5. Unitary representation of a subclass of the SP channels

In this section a special case of SP channels is considered, the case of identical source and target spaces and moreover with the orthogonal decomposition of the target and source space being equal. \((\mathcal{H}_T = \mathcal{H}_S, \mathcal{H}_{t1} = \mathcal{H}_{s1}, \mathcal{H}_{t2} = \mathcal{H}_{s2})\) The ‘two-box system’ described in the introduction is one example of such a system. To have a more comfortable terminology; if a CPM \(\phi\) is SP from \((\mathcal{H}_{s1}, \mathcal{H}_{s2})\) to \((\mathcal{H}_{t1}, \mathcal{H}_{t2})\), then we say that \(\phi\) is SP on \((\mathcal{H}_{s1}, \mathcal{H}_{s2})\).

The representation presented here is of the form of a unitary evolution of the system and an ancillary system. Given a channel with identical source and target spaces there always exists a representation in terms of a unitary evolution on the system and an ancilla system \(\mathcal{H}_a\). As such, the representation presented here is nothing new. The important aspect is rather the special form of this unitary representation. The meaning of this form of representation perhaps become a bit more clear in \(\mathcal{H}_a\). In \(\mathcal{H}_a\) a unitary representation, for a class of channels called local subspace preserving channels, is developed. These form a subset of the set of SP channels. One may compare the unitary representation for local subspace preserving channels, with the representation presented in proposition \(11\). This comparison suggests that the local subspace preserving channels consists precisely of the locally acting SP channels, in the sense of \(\mathcal{H}_a\). For more details the reader is referred to \(\mathcal{H}_a\) and \(\mathcal{H}_a\).

**Proposition 11** Let \(\Phi\) be a trace preserving CPM. \(\Phi\) is SP on \((\mathcal{H}_{s1}, \mathcal{H}_{s2})\) if and only if there exists an ancilla space \(\mathcal{H}_a\), a normalized state \(|a\rangle \in \mathcal{H}_a\), and operators \(V_1\) and \(V_2\) on \(\mathcal{H}_S \otimes \mathcal{H}_a\), such that

\[
V_1 V_1^\dagger = V_1^\dagger V_1 = P_{s1} \otimes \hat{1}_a, \quad V_2 V_2^\dagger = V_2^\dagger V_2 = P_{s2} \otimes \hat{1}_a, \quad (19)
\]

\[
\Phi(Q) = \text{Tr}_a(UQ \otimes |a\rangle\langle a|U^\dagger), \quad \forall Q \in \mathcal{L}(\mathcal{H}_S), \quad (20)
\]

where \(U\) is the unitary operator \(U = V_1 + V_2\).

**Proof.** First it has to be shown that \(U\) is a unitary operator. The conditions \((19)\) imply

\[
(P_{s1} \otimes \hat{1}_a)V_1(P_{s1} \otimes \hat{1}_a) = V_1, \quad (P_{s2} \otimes \hat{1}_a)V_2(P_{s2} \otimes \hat{1}_a) = V_2. \quad (21)
\]

This can be shown by using a singular value decomposition \((7)\) of \(V_1\). \(V_1 = \sum_k r_k |\psi_k\rangle \langle \eta_k|\), where \(|\psi_k\rangle\} \)k and \(|\eta_k\rangle\} \)k both form orthonormal sets of vectors, and \(r_k > 0\). By inserting this decomposition into \((19)\), it follows that \(|\psi_k\rangle, |\eta_k\rangle \in \mathcal{H}_{s1} \otimes \mathcal{H}_a\). Hence, \(V_1\) fulfills \((21)\). An analogous reasoning holds for \(V_2\). Using \((21)\) and \((19)\) the unitarity of \(U\) follows.

To prove the “if” part we first note that \(\Phi\) is trace preserving by the form of equation \((20)\). Moreover,

\[
\text{Tr}(P_{s1} \Phi(Q)) = \text{Tr} \left( (P_{s1} \otimes \hat{1}_a)UQ \otimes |a\rangle\langle a|U^\dagger \right)
\]

\[
= \text{Tr}(V_1 Q \otimes |a\rangle\langle a| V_1^\dagger) = \text{Tr}(P_{s1} Q). \quad (22)
\]

According to proposition \(4\) this implies that \(\Phi\) is SP on \((\mathcal{H}_{s1}, \mathcal{H}_{s2})\).
Now we turn to the proof of the “only if” part of the proposition. By proposition 5 there exists a Kraus representation of \( \Phi \) on the form \( \{V_{1,k} + V_{2,k}\}_k \) where

\[
P_{s_1}V_{1,k}P_{s_1} = V_{1,k}, \quad P_{s_2}V_{2,k}P_{s_2} = V_{2,k}.
\]

Since \( \Phi \) is trace preserving, it follows that \( \sum_k (V_{1,k} + V_{2,k})^\dagger (V_{1,k} + V_{2,k}) = \sum_k V_{1,k}^\dagger V_{1,k} + \sum_k V_{2,k}^\dagger V_{2,k} = 1 \), which together with (23) imply

\[
\sum_k V_{1,k}^\dagger V_{1,k} = P_{s_1}, \quad \sum_k V_{2,k}^\dagger V_{2,k} = P_{s_2}.
\]

From proposition 6 it follows that \( \{V_{1,k} + V_{2,k}\}_k \) can be chosen such that it has finitely many elements. Let this number be \( K \). Let \( \{|a_0\rangle\} \cup \{U\}_{k=1}^K \) be an orthonormal basis of an ancilla space \( \mathcal{H}_a \). (If \( K \) is the number of Kraus operators, the ancilla space is of dimension \( K + 1 \).) For \( i = 1, 2 \) define the following operators:

\[
V_i = P_{s_i} \otimes \hat{1}_a - P_{s_i} \otimes |a_0\rangle\langle a_0| - \sum_{kk'} V_{i,k} V_{i,k'}^\dagger \otimes |a_k\rangle\langle a_{k'}| + \sum_k V_{i,k}^\dagger \otimes |a_0\rangle\langle a_k|.
\]

Using equations (24) one may verify that \( V_1 \) and \( V_2 \) fulfill conditions (14). Moreover, one can check, using \( V_{1,k} V_{1,k'}^\dagger + V_{2,k} V_{2,k'}^\dagger = (V_{1,k} + V_{2,k})(V_{1,k'} + V_{2,k'})^\dagger \), that

\[
U = V_1 + V_2 = \hat{1} \otimes \hat{1}_a - \hat{1} \otimes |a_0\rangle\langle a_0| - \sum_{kk'} (V_{1,k} + V_{2,k})(V_{1,k'} + V_{2,k'})^\dagger \otimes |a_k\rangle\langle a_{k'}| + \sum_k (V_{1,k} + V_{2,k}) \otimes |a_k\rangle\langle a_0| + \sum_k (V_{1,k} + V_{2,k})^\dagger \otimes |a_0\rangle\langle a_k|
\]

and that

\[
\text{Tr}_a(UQ \otimes |a_0\rangle\langle a_0|U^\dagger) = \sum_k (V_{1,k} + V_{2,k})Q(V_{1,k} + V_{2,k})^\dagger = \Phi(Q),
\]

which proves the proposition.

\[ \square \]

6. Summary

A special family of completely positive maps (CPMs), named subspace preserving (SP), is introduced. In case of trace preserving CPMs (channels) these can be characterized as those which preserve probability weight on chosen subspaces. More specifically, let \( \Phi \) be a trace preserving CPM which maps density operators on the finite-dimensional Hilbert space \( \mathcal{H}_S \) to density operators on the finite-dimensional Hilbert space \( \mathcal{H}_T \). (We say that \( \mathcal{H}_S \) is the source space of \( \Phi \), and \( \mathcal{H}_T \) the target space of \( \Phi \).) Let \( P_{s_1} \) be the projection operator onto some, at least one-dimensional, subspace \( \mathcal{H}_{s_1} \) of \( \mathcal{H}_S \) and let \( P_{t_1} \) be the projection operator onto some, at least one-dimensional, subspace \( \mathcal{H}_{t_1} \) of \( \mathcal{H}_T \). The channel \( \Phi \) is said to be subspace preserving from \( (\mathcal{H}_{s_1}, \mathcal{H}_{t_2}) \) to \( (\mathcal{H}_{s_2}, \mathcal{H}_{t_2}) \) if \( \text{Tr}(P_{t_1}\Phi(\rho)) = \text{Tr}(P_{s_1}\rho) \) for all density operators \( \rho \) on \( \mathcal{H}_S \). In the actual analysis a more general definition is used which covers also the case of not trace preserving CPMs.

Under the limiting assumption of finite-dimensional source and target spaces, several equivalent characterizations of these types of channels are deduced. Moreover, an expression which makes it possible to generate all CPMs which are SP with respect
to some orthogonal decompositions of source and target spaces, is proved (proposition 9 and 10). In the special case of identical source and target spaces $\mathcal{H}_T = \mathcal{H}_S$ and moreover identical orthogonal decomposition of the source and target $\mathcal{H}_{t1} = \mathcal{H}_{s1}$, $\mathcal{H}_{t2} = \mathcal{H}_{s2}$, a representation in terms of a special form of joint unitary evolution with an ancilla system, is deduced. As a part of the analysis, the concepts of linearly independent Kraus representations and the Kraus number of CPMs are introduced. This investigation is the first in a family of articles investigating properties of channels with respect to orthogonal decompositions of source and target spaces. The subsequent members in this family is [2] and [3], where some of the material presented here is used.

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