Classical Mechanical Systems with one-and-a-half Degrees of Freedom and Vlasov Kinetic Equation

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to honor of our Teacher Sergey Petrovich Novikov

Abstract

We consider non-stationary dynamical systems with one-and-a-half degrees of freedom. We are interested in algorithmic construction of rich classes of Hamilton’s equations with the Hamiltonian $H = \frac{p^2}{2} + V(x, t)$ which are Liouville integrable. For this purpose we use the method of hydrodynamic reductions of the corresponding one-dimensional Vlasov kinetic equation.

Also we present several examples of such systems with first integrals with non-polynomial dependency w.r.t. momentum.

The constructed in this paper classes of potential functions $V(x, t)$ which give integrable systems with one-and-a-half degrees of freedom are parameterized by arbitrary number of constants.

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1 Introduction

It is well known that many interesting and important classical mechanical systems

\[ \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots, n, \]

determined by Hamiltonians

\[ H(q, p, t) = \frac{1}{2}p^2 + V(q, t), \quad q = (q_1, \ldots, q_n), \quad p = (p_1, \ldots, p_n), \]

are Liouville integrable, i.e. possess \( n \) functions \( F_i(q, p, t) \) such that

\[ \frac{dF_i}{dt} = 0, \quad \{F_i, F_j\}_{p, q} = 0, \]

where we have used the canonical Poisson bracket

\[ \{F, G\}_{p, q} = \sum_{i=1}^{n} \left( \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right). \]

This means in particular that for every \( F = F_i \)

\[ \frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial p_i} \dot{p}_i = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial q_i} = 0, \]
or in a more compact form (Liouville equation)

\[ F_t = \{F, H\}_{p,q}. \]

In this paper we restrict our considerations to the one-dimensional non-autonomous case only. Such systems are usually called systems with one-and-a-half degrees of freedom ([18, 6, 2]). Everywhere below we identify \( q_1 = x, p_1 = p \).

**Definition 1** We call Hamilton’s equations

\[ \dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x} \]  

with the Hamiltonian function

\[ H = \frac{p^2}{2} + V(x, t) \]  

solvable (in hydrodynamic sense) if there exists an additional function \( F(x,t,p) \) satisfying the Vlasov (collisionless Boltzmann) kinetic equation ([40], [11])

\[ F_t - \{ F, H \}_{p,x} = F_t + pF_x - FP_x = 0, \]  

and the potential energy \( V(x, t) \) coincides with the zeroth moment \( A^0(x, t) \) of the asymptotic expansion of the function \( F(x,t,p) \) for \( p \to \infty \):

\[ F(x,t,p) = p + \frac{A^0(x,t)}{p} + \frac{A^1(x,t)}{p^2} + \frac{A^2(x,t)}{p^3} + \ldots, \quad p \to \infty, \quad A^0(x,t) = V(x,t). \]  

Since Hamiltonian systems with one-and-a-half degrees of freedom solvable in hydrodynamic sense have one additional first integral, they are Liouville integrable. Thus we describe in this paper a subclass of Liouville integrable Hamilton’s equations using the method of hydrodynamic reductions. Note that usually (for example in classical mechanics) (3) is interpreted as a first-order linear PDE with the unknown function \( F \) and fixed potential \( V(x,t) \). In our definition we impose a strong ansatz \( V = A^0(x,t) \). In many physical applications of various versions of the Vlasov equation the quantities \( A^k(x,t) \) (called “moments”) are usually introduced as integrals

\[ A^k(x,t) = \int_{-\infty}^{\infty} p^k \Phi(F(x,t,p)) \, dp, \quad k = 0, 1, \ldots, \]  

where \( \Phi(F) \) is an appropriate rapidly decreasing at infinities \( p \to \pm \infty \) function such that the integrals are finite. Certainly we can obtain the coefficients \( A^k(x,t) \) in (4) as residues at infinity \( A^k(x,t) = \frac{1}{2\pi i} \oint_{\gamma} p^k F(x,t,p) \, dp \) or choose another deformation of the contour in this integral. See Appendix A for more detail on possible relations of the coefficients in the asymptotic expansion (4) and the integrals (5).

**Remark.** Below (see, for instance, (7), (8)) we consider some very important solutions \( F(x,t,p) \) which formally speaking do not have the asymptotic behavior (4). However, equation (3) is obviously invariant w.r.t. any point transformations \( F(x,t,p) \mapsto \)
that the variety of potentials $V(x,t)$ corresponding function
sensed (via hydrodynamic reductions with $N \geq 1$, see chronologically: [1, 21, 31, 11, 37, 29, 3, 15, 14, 29]). A powerful method of hydrodynamic reductions for the one-dimensional Vlasov type kinetic equation (including the Vlasov kinetic equation itself) was developed in [15], [31], [27], while corresponding integrable hydrodynamic chains were investigated in [30], [9], [34]. We will describe it in detail below and apply this method to description of a vast class of systems with one-and-a-half degrees of freedom solvable in hydrodynamic sense. Briefly speaking one should impose the following ansatz: $F(x,t,p) = \lambda(u,p)$ where $u = (u^1(x,t), \ldots, u^N(x,t))$ are auxiliary “hydrodynamic” unknown functions. The possible forms of the function $\lambda(u,p)$ are specified in our case explicitly in Section 3. We will describe it in detail below and apply this method to description of a vast class of systems with one-and-a-half degrees of freedom solvable in hydrodynamic sense. Briefly speaking one should impose the following ansatz: $F(x,t,p) = \lambda(u,p)$ where $u = (u^1(x,t), \ldots, u^N(x,t))$ are auxiliary “hydrodynamic” unknown functions. The possible forms of the function $\lambda(u,p)$ are specified in our case explicitly in Section 3. We will describe it in detail below and apply this method to description of a vast class of systems with one-and-a-half degrees of freedom solvable in hydrodynamic sense. Briefly speaking one should impose the following ansatz: $F(x,t,p) = \lambda(u,p)$ where $u = (u^1(x,t), \ldots, u^N(x,t))$ are auxiliary “hydrodynamic” unknown functions. The possible forms of the function $\lambda(u,p)$ are specified in our case explicitly in Section 3. We will describe it in detail below and apply this method to description of a vast class of systems with one-and-a-half degrees of freedom solvable in hydrodynamic sense.

1. The Bogdanov–Konopelchenko–Krichever reduction:

$$\lambda = \frac{p^{N+1}}{N+1} + u^0p^{N-1} + \ldots + u^{N-1} + \sum_{m=1}^{M} \left( \epsilon_m \ln(p - v^m) + \sum_{k=1}^{K_m} \frac{w^{m,k}}{(p - v^m)^k} \right) + \sum_{l=1}^{L} \tilde{\epsilon}_l \ln(p - \tilde{v}^l).$$

(7)

Here $\epsilon_m$ and $\tilde{\epsilon}_l$ are arbitrary constants, $N, M, L, K_s = 0, 1, \ldots$ and $u^n, v^m, w^{i,k}, \tilde{v}^l$ are functions of $x,t$ to be found.

2. The Puiseux type reduction:

$$\lambda = \prod_{n=1}^{N+1} (p - a^n(x,t))^{\epsilon_n}, \quad \sum_{n=1}^{N+1} \epsilon_n a^n(x,t) = 0,$$

(8)
where \( N \) is an integer and \( \epsilon_n \) are arbitrary constants (but \( \sum \epsilon_n \neq 0 \), because the leading term of the expansion of \( \lambda \) for \( p \to \infty \) must be a function of \( p \) only). Only \( a^s(x, t) \) for \( s = 1, \ldots, N \) are independent hydrodynamic variables.

Substitution of these expressions into (6) leads to corresponding hydrodynamic type systems (see [7]) for the hydrodynamic variables \( u^s(x, t), v^s(x, t), w^s(x, t) \) or \( a^s(x, t) \) in the aforementioned reductions. We will use for simplicity the notation \( u = (u^1(x, t), \ldots, u^N(x, t)) \) for the set of all hydrodynamic variables in a given reduction. General solutions of corresponding hydrodynamic type systems are parameterized by \( N \) arbitrary functions of a single variable (see [36, 37]). In the first example (7) we have the following fixed dependence \( V(u) = u^0 \). In the second example (8), \( V(a) = \frac{1}{2} \epsilon_km a^k a^m \) (see (28) below). Now let us suppose that we fix the precise dependence \( \lambda(u, p) \) (like (7) or (8)) so \( V(u) = A^0(u) \) in (4) is fixed as well. In such a case the resulting potential functions \( V(x, t) = V(u(x, t)) \) of systems (1), (2) solvable in hydrodynamic sense are not fixed, they are parameterized by \( N \) arbitrary functions of a single variable for any \( N \geq 1 \), i.e. solutions of the corresponding hydrodynamic type system for the hydrodynamic variables \( u^i(x, t) \) in the aforementioned reductions.

In a particular case (when all rational and logarithmic parts are removed), (7) reduces to the form

\[
\lambda_{(1)} = \frac{p^{N+1}}{N+1} + u^0 p^{N-1} + \ldots + u^{N-1},
\]

which is also equivalent to a particular case of (8) when all constants \( \epsilon_n = 1 \) (the condition \( \sum \epsilon_n a^n = 0 \) in (8) means that the term \( p^N \) vanishes in (9)). In such a case (8) assumes the form

\[
\lambda_{(2)} = \prod_{n=1}^{N+1} (p - a^n(x, t)), \quad \sum_{n=1}^{N+1} a^n(x, t) = 0.
\]

Vlasov kinetic equation is invariant under any point transformation \( F(x, t, p) \to \Phi(F(x, t, p)) \), so \( (N+1)\lambda_{(1)} = \lambda_{(2)} \) give in fact the same reduction, the sets of hydrodynamic variables \( u^i, a^i \) are related by the obvious point transformation given by Vieta’s formulas. This is nothing but the well-known dispersionless limit of the Gelfand–Dikij reduction for the remarkable Kadomtsev–Petviashvili hierarchy ([19]). This unexpected relationship between ansatz (9) for the integrable reductions of the Vlasov kinetic equation, classical mechanical systems with one-and-a-half degrees of freedom and integrable hydrodynamic type systems was implicitly or explicitly observed in a number of publications, in particular [18, 6, 2, 32]. Let us give the following citation from [18]: “It has long been remarked that all the known first integrals of classical mechanical systems are polynomial w.r.t. velocities (or functions of such polynomials). This observation has no complete explanation yet. For this reason the analytical and geometrical nature of polynomial integrals is of big interest”.

In this paper we **constructively** build a rich family of such solvable potentials \( V(x, t) \) with polynomial first integrals (9).

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1. translated from the Russian original by the authors
2. boldface emphasis by the present authors
3. 1989
In our construction the potential function $V(x, t)$ is one of the components of a solution of some $N$ component hydrodynamic type system. In most cases solutions of such systems break down in finite time. That means that the potential function $V(x, t)$ may become singular. The results of the papers [18, 2] on some classes of nonsingular periodic potentials $V(x, t)$ are based exactly on this property of quasilinear systems. Nevertheless (see for instance [4]) a class of (probably piecewise analytic) nonsingular solutions can exist globally for appropriate initial data.

Our second contribution consists in algorithmic construction of potentials $V(x, t)$ solvable in hydrodynamic sense with non-polynomial first integrals $F(x, t, p)$ (cf. for example (7) and (8)), which depend on the momentum $p$ in a nontrivial way.

This paper is organized as follows. In Section 2 we briefly describe the method of hydrodynamic reductions. In Section 3, we consider polynomial and simplest nonpolynomial in $p$ solutions of Vlasov kinetic equation (6). In Section 4 we construct solutions of the waterbag and Puiseux type reductions by the Generalized Hodograph Method. In Section 5 we briefly describe two types of similarity solutions for the Puiseux type reductions and establish a remarkable link to the classical theory of finite-gap potentials [26]. Section 6 presents an alternative method of integration which produced two interesting families of solvable potentials together with explicit formulas for the solution of the corresponding Hamilton’s equations (1) without the need to use Liouville’s theorem on integrability in quadratures. In the Conclusion we remark that the hydrodynamic reduction technique is applicable to a wider class of Hamiltonians $H(V(x, t), p)$. Appendices A, B, C and D contain some technical details important for the method of hydrodynamic reductions.

2 Method of Hydrodynamic Reductions

Substitution of (4) into (3) with the restriction $V(x, t) = A^0(x, t)$ leads to the remarkable Benney hydrodynamic chain (see [1])

$$A^k_t + A^{k+1}_x + kA^{k-1}_x A^0_x = 0, \quad k = 0, 1, \ldots$$  (10)

Substitution of (5) into (10) implies Vlasov kinetic equation (3) again (see Appendix A).

According to the approach established in [15] we suppose that all $A^k(x, t), V(x, t)$ have the form $A^k(x, t) = A^k(r^1(x, t), \ldots, r^N(x, t)), V(x, t) = A^0(x, t) = V(r^1(x, t), \ldots, r^N(x, t))$, where $V(r), A^k(r)$ are some fixed (unknown) functions of the “hydrodynamic variables” $r^i$ and these variables $r^i = r^i(x, t)$ are arbitrary solutions of an $N$ component hydrodynamic type system in diagonalized form (so the corresponding “hydrodynamic field variables” $r^i$ are Riemann invariants of this quasilinear system)

$$r^i_t + \mu^i(r)r^i_x = 0$$  (11)

integrable by the Generalized Hodograph Method (see [36, 37]). In this case the functions $\mu^i(r)$ and $V(r) \equiv A^0(r)$ (see (4) and (6)) satisfy the so-called Gibbons–Tsarev system ([15]) (here $\partial_i \equiv \partial/\partial r^i$)

$$\partial_i \mu^k = \frac{\partial_i V}{\mu^i - \mu^k}, \quad \partial_{ik}^2 V = 2 \frac{\partial_i V \partial_k V}{(\mu^i - \mu^k)^2}, \quad i \neq k,$$  (12)
while the function \( F(x, t, p) = \lambda(r, p) \) satisfies the (generalized) Löwner equations ([15])
\[
\partial_t \lambda = \frac{\partial_i V}{p - \mu^i} \lambda_p, \quad (13)
\]
whose compatibility conditions \( \partial_k (\partial_t \lambda) = \partial_t (\partial_k \lambda) \) lead to the Gibbons–Tsarev system (12). The celebrated Löwner equation initially appeared in 1923 as an ordinary nonlinear differential equation describing deformations of extremal univalent conformal mappings and was used in the solution of the famous Bieberbach Conjecture in 1984 (see an exposition of the history of this Conjecture in [10] and its relation to hydrodynamic reductions of Benney moment equations in [15]). Equations (12) and (13) were recently applied to the equations of Laplacian Growth, Dirichlet Boundary Problem and Hele-Shaw problem (see for instance [25]).

From (11)–(13) we can determine the functional dimension of the variety of potentials \( V(x, t) \) integrable via hydrodynamic reductions with \( N \) hydrodynamic parameters \( r^i(x, t) \). Namely, this variety is parameterized by \( 2N \) functions of a single variable. First, the solutions of the compatible system of equations (12) is parameterized by \( 2N \) functions of a single variable: the values of \( V(r) \) on the coordinate axes \( r^i \) and the values of each \( \mu^i(r) \) on the corresponding coordinate axis \( r^i \) (the Goursat data for the system (12)). The solutions \( r^i(x, t) \) of (11) with fixed \( \mu^i(r) \) are parameterized by \( N \) functions of a single variable and the solutions \( \lambda(r, p) \) of (13) with fixed \( \mu^i(r) \), \( V(r) \) are parameterized by one function of a single variable. However one can see that (13) essentially has only one solution, the others are arbitrary functions of it: \( \lambda \mapsto f(\lambda) \). Also Riemann invariants \( r^i \) for a given diagonalizable hydrodynamic type system are fixed up to the change \( r^i \mapsto f^i(r^i) \) so the variety of integrable potentials \( V(x, t) = V(r(x, t)) \) is parameterized by \( 2N \) functions of a single variable only.

As we have mentioned above we will use the notation \( F(x, t, p) \) for the conservation law only in the nonreduced case and the notation \( \lambda(r, p) \) for the same function in the case when a finite-component hydrodynamic reduction of the Vlasov kinetic equation is considered. According to the symmetric modification of the above method (see [32]), we can consider hydrodynamic type systems (11) written in the special (non-diagonal) conservative form with special hydrodynamic variables \( a^1 = (a^1(x, t), \ldots, a^N(x, t)) \):
\[
a^k_t + \left( \frac{(a^k)^2}{2} + V(a) \right)_x = 0. \quad (14)
\]
Indeed, dividing all elements in Vlasov kinetic equation (6) by \(-F_p\) we get:
\[
-\frac{F_t}{F_p} - p \frac{F_x}{F_p} + V = 0, \quad (15)
\]
so according to the theorem about differentiation of implicit functions, one can conclude that this equation assumes the form
\[
p_t + \left( \frac{p^2}{2} + V \right)_x = 0. \quad (16)
\]
Here and below \( p(x, t, F) \) is the inversion of the function \( F(x, t, p) \) w.r.t. \( p \) (a solution of the implicit equation \( F = F(x, t, p) \)). It is a generating function of conservation laws
for Benney hydrodynamic chain (10) with respect to the parameter $F$. Let’s choose $N$ arbitrary values $\xi_k$ of this parameter $F$ and denote the corresponding functions $p(x, t, \xi_k)$ as $a^k(x, t)$. Then $N$ copies of (16) for distinct values $\xi_k$ yield the hydrodynamic type system (14). In this paper we will suppose that this set $a^k$ of the new hydrodynamic variables is independent. Let us study this problem in more detail. Substitution of the asymptotic series

$$p(x, t, F) = F - \frac{H_0(x, t)}{F} - \frac{H_1(x, t)}{F^2} - \frac{H_2(x, t)}{F^3} - \ldots, \quad F \to \infty$$

(17)

(the inverted asymptotic series (4)) into (16) yields Benney hydrodynamic chain (10) written in the conservative form

$$\partial_t H_k + \partial_x \left( H_{k+1} - \frac{1}{2} \sum_{m=0}^{k-1} H_m H_{k-1-m} \right) = 0, \quad k = 0, 1, \ldots,$$

(18)

where all conservation law densities $H_k$ are polynomials w.r.t. $A^k$. For instance $H_0 = A^0, H_1 = A^1, H_2 = A^2 + (A^0)^2, H_3 = A^3 + 3A^0A^1$. According to the approach presented in [15], all $N$ component hydrodynamic reductions of Benney hydrodynamic chain (18) can be written choosing $N$ physical variables $H_k, k = 0, 1, \ldots, N - 1$ as an independent set of hydrodynamic reduction variables, while all the other $H_{N-1+k}$ must be functions of this basic set $H_0, \ldots, H_{N-1}$ such that all equations in (18) must be consequences of the first $N$ of them. In such a case, we can introduce $N$ formal equalities (see (17))

$$a^k(H_0, \ldots, H_{N-1}) \equiv p(\xi_k) = \frac{H_0}{\xi_k} - \frac{H_1}{\xi_k^2} - \ldots - \frac{H_{N-1}}{\xi_k^N} - \frac{H_N(H_0, \ldots, H_{N-1})}{\xi_k^{N+1}} - \frac{H_{N+1}(H_0, \ldots, H_{N-1})}{\xi_k^{N+2}} - \ldots.
$$

In this paper we consider the generic case: we assume that the point transformation $H \to a(H)$ is invertible, so the Jacobian $\partial a^k/\partial H_j$ is nondegenerate. Nevertheless degenerate cases are also interesting and will be studied elsewhere.

Thus each hydrodynamic reduction (14) has the generating function $p(a, \lambda)$ of conservation laws (cf. (16))

$$p_t + \left( \frac{p^2}{2} + V(a) \right)_x = 0,$$

(19)

producing the infinite series (18) of conservation law densities $H_k(a)$, where (cf. (17))

$$p(a, \lambda) = \lambda - \frac{H_0(a)}{\lambda} - \frac{H_1(a)}{\lambda^2} - \frac{H_2(a)}{\lambda^3} - \ldots, \quad \lambda \to \infty.$$

(20)

The function $V(a)$ satisfies the Gibbons–Tsarev system (cf. (12) in Riemann invariants)

$$(a^i - a^k) \partial_{ik}^2 V = \partial_k V \partial_i \left( \sum_m \partial_m V \right) - \partial_i V \partial_k \left( \sum_m \partial_m V \right), \quad \partial_i \equiv \partial/\partial a^i, \ i \neq k.$$

(21)

System (21) can be easily derived at least by two approaches. First, let us consider the zeroth equation of Benney hydrodynamic chain (18), i.e. $(H_0(a))_t + (H_1(a))_x = 0$ or
\[ \sum \partial_k H_0 a^k + \sum \partial_k H_1 a^k_x = 0. \]

Substituting \( a^k \) from (14), and taking into account that each factor of \( a^k_x \) must vanish independently due to the assumption that \( a^k(x,t) \) are arbitrary solutions of (14), we conclude that \( \partial_k H_1 = (a^k + \sum \partial_m H_0) \partial_k H_0 \). Compatibility conditions \( \partial_i (\partial_k H_1) = \partial_k (\partial_i H_1) \) imply (21), where (as everywhere in this paper) \( V(a) \equiv H_0(a) = A^0(a) \). Also, (19) yields \( \sum \partial_k p(a, \lambda) a^k + \sum \partial_k V(a) a^k_x + \sum \partial_k V(a) a^k_x = 0. \)

Repetition of the above arguments leads to the L"owner equations (cf. (13) in Riemann invariants)

\[ \partial_i \lambda = \frac{\partial_i V}{a^i - p} \left( \sum_m \frac{\partial_m V}{p - a^m} - 1 \right)^{-1} \partial_p \lambda. \]  

whose compatibility conditions \( \partial_k (\partial_i \lambda) = \partial_i (\partial_k \lambda) \) yield again Gibbons–Tsarev system (21).

**Remark.** One can easily obtain L"owner equations for the function \( \lambda(a, p) \):

\[ \partial_i \lambda = \frac{\partial_i V}{a^i - p} \left( \sum_m \frac{\partial_m V}{p - a^m} - 1 \right)^{-1} \partial_p \lambda. \]  

Details of the proof are given in Appendix C. Dividing this equation by \( \partial_p \lambda \) (cf. (15)) and using the theorem about differentiation of implicit functions, one arrives again to the L"owner equations written in the form (22). Nevertheless the L"owner equations written in the form (23) are more suitable for integration. Indeed, introduce the auxiliary function \( \varphi(a, p) \) such that

\[ \partial_p \lambda = \left( \sum_m \frac{\partial_m V}{p - a^m} - 1 \right) \varphi(a, p). \]  

Then L"owner equations (23) reduce to

\[ \partial_i \lambda = \frac{\partial_i V}{a^i - p} \varphi(a, p). \]  

Thus integration of the L"owner equations written in the form (23) is equivalent to computation of the integration factor \( \varphi(a, p) \) subject to compatibility conditions following from (24), (25). We will see in Section 3 that in many cases we are able to integrate (23) completely in such a way.

The second order quasilinear system (21) has a general solution parameterized by \( N \) arbitrary functions of a single variable. Currently we do not have a constructive procedure to find this complete solution. In this paper we will construct a finite-parametric family of solutions depending on \( N \) arbitrary constants for any \( N \geq 1 \) where \( N \) is the number of equations in (14) (see also [29] for solutions with a larger number of constant parameters).

Summarizing the necessary steps of the method of hydrodynamic reductions in application to the problem of constructive classification of mechanical systems with one-and-a-half degrees of freedom solvable in hydrodynamic sense and in order to give formulae for their potentials \( V(x, t) \) and conservation laws \( F(x, t, p) \) we sketch the following algorithm:

1. Once any solution \( V(a) \) of (21) is given, then the corresponding semi-Hamiltonian hydrodynamic type system (14) is fixed.
2. Compute the corresponding solution $\lambda(a, p)$ of Löwner equations (23) (this problem is usually reduced to computation of an integration factor $\varphi(a, p)$ and usually found explicitly).

3. The corresponding system (14) possesses a general solution $a^i(x, t)$ parameterized by $N$ arbitrary functions of a single variable.

4. Thus, taking any given solution $V(a)$ of (21) and a general solution $a^i(x, t)$ of (14), we obtain the potential functions $V(a(x, t))$ as well as the additional functions $F(x, t, p) = \lambda(a(x, t), p)$ parameterized by $N$ arbitrary functions of a single variable and herewith we obtain infinitely many Liouville integrable Hamilton’s equations (1).

Steps 1 and 3 of this algorithm need a close and detailed consideration, since we do not have constructive methods to obtain general solutions of the systems (21) and (14). The following Sections are devoted to a way around this problem which produces in the cases considered in this paper (and in many other cases, cf. [13], [29]) explicit families of solutions.

Namely, in Section 3 we describe a few possible simple solutions of (21) together with corresponding solutions of (23). A method of construction of rich families of solutions of the system (14) is presented in Section 4.

3 Polynomial and Simplest Nonpolynomial Reductions

As we have stated above, at this moment any regular procedure for construction of solutions for the Gibbons–Tsarev system does not exist. Nevertheless, some multi-parametric solutions can be found easily. We give below a few explicit examples of such solutions. In order to simplify the formulas we will give them modulo the obvious point symmetry $a^i \mapsto \lambda a^i + \mu$ ($\lambda, \mu \in \mathbb{R}$) of (21).

I. Substitution of the ansatz $V(a) = \sum f_m(a^m)$, where $f_k(a^k)$ are unknown functions, into (21) yields the following cases:

I.1. a particular $N$ parametric family of solutions (the so called waterbag reduction, see for instance [15], [38])

\[
V(a) = \sum_{m=1}^{N} \epsilon_m a^m, \tag{26}
\]

where all $\epsilon_m$ are arbitrary constants. Then Löwner equations (23) have the following solution

\[
\lambda(a, p) = p - \sum_{m=1}^{N} \epsilon_m \ln(p - a^m). \tag{27}
\]

I.2. A general solution

\[
V(a) = \sum_{m=1}^{N} \epsilon_m e^{\alpha_m},
\]
where all $\epsilon_m$ are arbitrary constants. Then L"owner equations (23) have the following solution

$$\lambda(a, p) = -e^{-p} - \sum_{m=1}^{N} \epsilon_m a^{m-p} \int \frac{e^a dq}{q}.$$ 

II. A broader ansatz $V(a) = f(\Delta)$ (where $\Delta = \sum f_m(a^m)$ and $f_k(a^k)$ are unknown functions) for (21) yields

$$V(a) = \ln(\Delta + \xi), \quad f'_k(a^k) = \epsilon_k \exp\left(-\frac{(a^k)^2}{2}\right),$$

where $\xi$ and $\epsilon_k$ are arbitrary constants. Then L"owner equations (23) have the following solution

$$\lambda(a, p) = \xi \int e^{q^2/2} dp - \sum_{m=1}^{N} \epsilon_m \int \exp\left(-\frac{q^2 + 2pq}{2}\right) dq.$$

III. A quadratic homogeneous polynomial ansatz $V(a) = \frac{1}{2} \epsilon_{km} a^k a^m$ leads to

$$V(a) = \frac{-1}{2(1+\epsilon)} \left[ \sum_{m=1}^{N} \epsilon_m (a^m)^2 + \left( \sum_{m=1}^{N} \epsilon_m a^m \right)^2 \right], \quad \epsilon = \sum_{n=1}^{N} \epsilon_n.$$ (28)

Then L"owner equations (23) have the following solution (cf. (8))

$$\lambda(a, p) = \left( p + \sum_{m=1}^{N} \epsilon_m a^m \right) \prod_{n=1}^{N} (p - a^n)^{\epsilon_n}, \quad \sum_{n=1}^{N} \epsilon_n \neq -1.$$ (29)

This is the so called Puiseux type reduction (see for instance [15]). If all $\epsilon_n = 1$, this is nothing but the dispersionless limit of the Gelfand–Dikij reduction of the Kadomtsev–Petviashvili hierarchy (see [19]). For Hamiltonian systems with one-and-a-half degrees of freedom this class of polynomial integrals was studied in [18, 6, 2] where some results on existence of global nonsingular periodic potentials were given. In this paper our approach is essentially local. If all $\epsilon_n = \pm 1$, this is the so called Zakharov type reduction (see [40]); if $\epsilon_1 = -M$, ($M \neq N$) and all other $\epsilon_n = 1$, this is the so called Kodama reduction (see [32]). These three cases are dispersionless limits of Krichever–Orlov reduction [20, 35] of the Kadomtsev–Petviashvili hierarchy, which can be obtained from (7) if we remove the logarithmic terms.

More complicated reductions can be found also in [29] and in a set of publications [13].

4 Hydrodynamic Reductions. Integrability

In this Section we consider some constructive methods for integration of hydrodynamic reductions (14) of Benney hydrodynamic chain (10). We illustrate in Sections 4.1, 4.2 this construction on two examples: the waterbag reduction (26), (27) and the Puiseux type reduction (28), (29).
According to the Generalized Hodograph Method (see detail in [36], [37]), any semi-Hamiltonian hydrodynamic type system (11), i.e. a system whose characteristic velocities satisfy the integrability (or the semi-Hamiltonian) property
\[
\partial_j \frac{\partial \mu^i}{\mu^k - \mu^j} = \frac{\partial_k \mu^i}{\mu^j - \mu^i}, \quad \partial_i \equiv \partial/\partial r^i, \quad i \neq j \neq k,
\]
possesses infinitely many commuting flows
\[
r^i_\tau = w^i(r)r^i_x,
\]
whose characteristic velocities are solutions of the linear system (again \( \partial_i \equiv \partial/\partial r^i \))
\[
\partial_k w^i = \frac{\partial_k \mu^i}{\mu^k - \mu^i} (w^k - w^i), \quad i \neq k.
\]
The general solution of this compatible system depends on \( N \) arbitrary functions of a single variable. Then a generic solution \( r^i(x,t) \) of hydrodynamic type system (11) in a neighborhood of a generic point is given in an implicit form by the algebraic system for the unknowns \( r^i(x,t) \):
\[
x - \mu^i(r) \cdot t = w^i(r),
\]
where \( w^i(r) \) is a general solution of the compatible linear system (31).

**Remark.** In arbitrary hydrodynamic variables \( u^i(r) \) algebraic system (32) takes the form (see [37])
\[
x \delta_k^i - tv_k^j(u) = w_k^j(u),
\]
where the hydrodynamic type system (11) has the form
\[
u_x^i = \sum_j v_j^i(u)u_x^j, \quad i, j = 1, \ldots, N,
\]
while commuting hydrodynamic type systems (30) have the form
\[
u_x^i = \sum_j w_j^i(u)u_x^j, \quad i, j = 1, \ldots, N.
\]
In order to construct solutions of (14) we first need to prove the Egorov property of Benney hydrodynamic chain (10). This property is very important and many physical systems of hydrodynamic type integrable by the Generalized Hodograph Method possess this property (cf. [37], [33]). We need the following result suitable for investigation of semi-Hamiltonian systems (cf. [33]):

**Lemma 1** Any hydrodynamic reduction of Benney hydrodynamic chain (10) has a Egorov pair of conservation laws
\[
\left( f(u(x,t)) \right)_t = \left( h(u(x,t)) \right)_x, \quad \left( h(u(x,t)) \right)_t = \left( g(u(x,t)) \right)_x.
\]

**Proof.** Indeed, two first conservation laws of (18) are
\[
\partial_t H_0 + \partial_x H_1 = 0, \quad \partial_t H_1 + \left( H_2 - \frac{1}{2}(H_0)^2 \right)_x = 0,
\]
Any hydrodynamic reduction \( H_k = H_k(u) \) of (18) also has these conservation laws so we can take \( f = H_0(u), h = -H_1(u), g = H_2(u) - (H_0(u))^2/2. \)

Using the technique of [33] one easily proves that for arbitrarily chosen conservation law density \( h(r) \) of the original system (in our case (11)) an appropriately chosen commuting flow must have an Egorov pair such that \( f_r = h_x \), where \( f = H_0 \) (see Appendix B for the proof). Algebraic system (32), (or (33) in arbitrary variables) can be written in the form (here \( \partial_i = \partial/\partial r_i \))

\[
x - t \frac{\partial H_1}{\partial_i H_0} = \frac{\partial h}{\partial_i H_0}.
\]

(34)

Indeed, hydrodynamic type system (11) has a conservation law \( \partial_i H_0 + \partial_r H_1 = 0 \), while the commuting hydrodynamic system has the conservation law \( f_r = h_x \). This means that \( \partial H_0 r_i + \partial H_1 r_x = 0 \) and \( \partial H_0 r^i = \partial_i h \cdot r^i_x \). Taking into account (11), (30) and (32), one obtains (34). Multiplying (34) by \( \partial_i H_0 dr^i \) and summing up, one arrives at

\[
xdH_0(r) - tdH_1(r) = dh(r).
\]

Now we rewrite this equation after the invertible point transformation \( (r) \to (a) \) as

\[
x dH_0(a) - tdH_1(a) = dh(a),
\]

so the algebraic system (32) becomes

\[
x \frac{\partial H_0}{\partial a^i} - t \frac{\partial H_1}{\partial a^i} = \frac{\partial h}{\partial a^i}.
\]

(35)

Taking into account that \( \partial H_1 = (a^k + \sum \partial_m V) \partial_i V \) (see (10) and (14), here again \( \partial_i = \partial/\partial a^i \) and we remind that \( V \equiv H_0 = A^p \)) and substituting \( p(a, \lambda) \) instead of \( h(a) \), we obtain the algebraic system (see (22))

\[
x \partial_i V - t \left( a^i + \sum \partial_m V \right) \partial_i V = \frac{\partial_i V}{p - a^i} \left( \sum \frac{\partial_m V}{p - a^m} - 1 \right)^{-1},
\]

which is nothing but the diagonal part of the matrix algebraic system (33). All off-diagonal equations are compatible with the diagonal part ([37]).

So we proved:

**Theorem 1** An arbitrary hydrodynamic reduction (14) of Benney hydrodynamic chain (10) has infinitely many particular solutions \( a^i(x, t) \) in the implicit form (here \( \partial_i \equiv \partial/\partial a^i \))

\[
x - t \left( a^i + \sum_{m=1}^{N} \partial_m V \right) = \frac{1}{p - a^i} \left( \sum_{m=1}^{N} \frac{\partial_m V}{p - a^m} - 1 \right)^{-1},
\]

(36)

where \( p(a, \lambda) \) is the generating function of conservation law densities (see (19)).

Thus, once the function \( V(a) \) is fixed (any solution of Gibbons–Tsarev system (21)), the function \( p(a, \lambda) \) also is found as an inverse function to \( \lambda(a, p) \) solving (23) or computing the integrating factor \( \varphi(a, p) \) in (24), (25). For the particular cases of \( V(a) \) considered in Section 3 respective \( \lambda(a, p) \) are explicitly given. Then the algebraic system (36) determines one parametric family of solutions \( a^i(x, t, \lambda) \) in implicit form and simultaneously
$V(x, t, \lambda) = V(a(x, t, \lambda))$. Thus we found one parametric family of Hamilton’s equations (1), which are Liouville integrable.

In fact, using the Generalized Hodograph Method and the *nonlinear superposition principle* implied by this method (see below) we easily obtain multiparametric families of solvable potentials. Namely expanding the generating function $p(a, \lambda)$ at different points on the Riemannian surface $p = p(a, \lambda)$ with the parameters $(p, \lambda)$ (for example when $p \to \infty$ or $p \to a'$), one can construct infinite multiparametric series of new solutions $V(a(x, t))$. Let us demonstrate this idea in detail.

1. Kruskal series. Substitution of asymptotic expansion (20) into (19) leads to the Kruskal series of particular conservation law densities $p^k_0(a) = H_k(a)$. They can be found in quadratures. Indeed, we have an infinite series of conservation laws (18), where (let us remind) $H_0(a) = A^0(a) = V(a)$:

$$
(H_0(a))_t + (H_1(a))_x = 0, \quad (H_1(a))_t + \left( H_2(a) - \frac{1}{2} H_0^2(a) \right)_x = 0,
$$

$$
(H_2(a))_t + (H_3(a) - H_0(a)H_1(a))_x = 0, \ldots
$$

Taking into account (14), we obtain

$$
\partial_k H_1(a) = (a^k + \delta V) \partial_k V, \quad \partial_k H_2(a) = [(a^k)^2 + a^k \delta V + \sum a^m \partial_m V + (\delta V)^2 + V] \partial_k V, \ldots
$$

where $\delta = \sum \partial / \partial a^m$. Thus, once the potential function $V(a)$ is given, all other higher Kruskal conservation law densities are found by quadratures.

We call this asymptotic expansion Kruskal, because M. Kruskal was first who introduced a similar expansion ($\lambda \to \infty$) for the KdV equation.

From (37) we easily obtain a family of solvable potentials $V(x, t)$ which is written in a compact form using (35) with $h(a) = \sum_s c_s H_s(a)$:

$$
x - t(a^k + \delta V) = c_0 + c_1(a^k + \delta V) + c_2((a^k)^2 + a^k \delta V + \sum a^m \partial_m V + (\delta V)^2 + V) + \ldots
$$

Namely suppose we have any solution $V(a)$ of Gibbons-Tsarev system (21). Then solving algebraic system (38) with a fixed expression $V(a)$, we can find $a^k(x, t)$ and $V(x, t) = V(a(x, t))$. These solvable potentials are parameterized by arbitrary number of constants $c_s$.

2. $N$ principal series. Instead of asymptotic series (20) we can introduce $N$ expansions of $p(a, \lambda)$ at the vicinities of $\lambda = \xi_k$. This means that we consider $N$ series of conservation law densities

$$
p^{(k)}(a, \tilde{\lambda}(k)) = a^k + p_1^k(a)\tilde{\lambda}(k) + p_2^k(a)\tilde{\lambda}^2(k) + p_3^k(a)\tilde{\lambda}^3(k) + \ldots, \quad k = 1, \ldots, N,
$$

so $p^k_m(a)$ are conservation law densities of hydrodynamic type system (14), and $\tilde{\lambda}(k)(\lambda)$ is a corresponding local parameter at vicinity of each point $\lambda = \xi_k$, $p = a^k$. Substitution of each of these series into (19) yields (14) at the first step, while all higher conservation law densities $p^k_m(a)$ can be found at next steps in quadratures as we prove in Appendix D. This approach requires only $V(a)$ to be known explicitly. Another algorithm to find the quantities $p^k_m(a)$ explicitly will be described in Sections 4.1 and 4.2 and requires the
solution $\lambda(a, p)$ of (23). These $N$ series of conservation law densities $p^k_m(a)$ are independent while the Kruskal series is their linear combination. However, in some cases, the Kruskal series has its own interest, because corresponding solutions are symmetric under arbitrary permutation of indices of hydrodynamic variables $a^k$.

Once we found all these conservation law densities $p^k_m(a)$ and the Kruskal series $p^k_0(a)$, we can construct infinitely many particular solutions parameterized by arbitrary number of constants $\sigma^m_k$ in (35):

$$x \frac{\partial H_0}{\partial a^i} - t \frac{\partial H_1}{\partial a^i} = \frac{\partial}{\partial a^i} \left( \sum_{k=1}^N \sum_{m=0}^\infty \sigma^m_k p^k_m(a) \right),$$

or

$$x \frac{\partial H_0}{\partial a^i} - t \frac{\partial H_1}{\partial a^i} = \frac{\partial}{\partial a^i} \oint \varphi(\lambda) p(a, \lambda) d\lambda,$$

where $\varphi(\lambda)$ and the contour can be chosen in many special forms. Formulae (40) and (41) present the nonlinear superposition principle implied by the Generalized Hodograph Method.

Thus, once the function $V(a)$ is fixed (any solution of Gibbons–Tsarev system (21)) and $\lambda(a, p)$ is found from (23) or (24), (25), algebraic system (40) determines multiparametric families of solutions $a^i(x, t)$ in implicit form. By this way we simultaneously found $V(a(x, t))$ and $F(x, t, p) = \lambda(a(x, t), p)$. If the r.h.s. of algebraic system (41) contains $N$ arbitrary functions $\varphi_k(\lambda)$, and the contour consists of $N$ appropriate piecewise smooth curves (see, for instance, [19]), then $a^i(x, t)$ depend on $N$ arbitrary functions of a single variable. Then the potential function $V(a(x, t))$ also depends on $N$ arbitrary functions of a single variable. However, we cannot describe such general solutions explicitly. Below we study in detail some particular cases given in Section 3 and find rich multiparametric families of solvable potentials.

### 4.1 Waterbag Reduction

Waterbag hydrodynamic reduction (see (14) and (26))

$$a^k_i + \frac{(a^k)^2}{2} + \sum_{m=1}^N \epsilon_m a^m = 0$$

has the Kruskal series of conservation laws (18), where Kruskal conservation law densities $H_k(a)$ are nonhomogeneous polynomials w.r.t. $a^k$ (in a generic case, i.e. if $\sum \epsilon_m \neq 0$). These polynomial expressions can be found by substitution of asymptotic series (20) into (cf. (27)) the following equation:

$$\lambda(\infty) - \sum_{m=1}^N \epsilon_m \ln \lambda(\infty) = p - \sum_{m=1}^N \epsilon_m \ln(p - a^m).$$

Here we perform a point transformation for the function $\lambda$: $\lambda = \lambda(\infty) - \sum \epsilon_m \ln \lambda(\infty)$ in order to have the asymptotic series of the form (20). The first few Kruskal conservation
law densities are

\[ H_0(a) = \sum_{m=1}^{N} \epsilon_m a^m, \quad H_1(a) = \frac{1}{2} \sum_{m=1}^{N} \epsilon_m (a^m)^2 + \epsilon H_0(a), \]

\[ H_2(a) = \frac{1}{3} \sum_{m=1}^{N} \epsilon_m (a^m)^3 + \sum_{m=1}^{N} \epsilon_m a^m H_0(a) + \epsilon H_1(a), \]

where \( \epsilon = \sum_{m=1}^{N} \epsilon_m. \)

\( N \) principal series of conservation law densities can be found from

\[ \tilde{\lambda}_{(i)} = (p - a^i) e^{-p/\epsilon_i} \prod_{m \neq i} (p - a^m)^{\epsilon_m/\epsilon_i} \]

for each index \( i \) separately. Below we explicitly describe this procedure. First we choose the corresponding local parameter \( \tilde{\lambda}_{(i)} = e^{-\lambda/\epsilon_i} \), then asymptotic series (39) is applicable. Once a local parameter \( \tilde{\lambda}_{(i)} \) is chosen so that \( \tilde{\lambda}_{(i)} \sim (p - a^i) \), all conservation law densities \( p^k_m(a) \) (see (39)) can be obtained using the Lagrange–Bürmann series (see, for instance, [24]) at the vicinity of each singular point:

**Proposition 1 (Lagrange–Bürmann formula, [24])** The analytic function

\[ y = y_1 \cdot (x - x_0) + y_2 \cdot (x - x_0)^2 + y_3 \cdot (x - x_0)^3 + \ldots \]

can be inverted \( (y(x) \to x(y)) \) as the Lagrange–Bürmann series

\[ x = x_0 + x_1 y + x_2 y^2 + x_3 y^3 + \ldots, \]

whose coefficients are

\[ x_n = \frac{1}{n!} \lim_{x \to x_0} \frac{d^{n-1}}{dx^{n-1}} \left( \frac{x - x_0}{y} \right)^n, \quad n = 1, 2, \ldots \] (42)

This means that the conservation law densities \( p^k_m(a) \) of the waterbag reduction can be obtained with the aid of Lagrange–Bürmann series:

\[ p^i_n = \frac{1}{n! d(a^i)^{n-1}} \left( e^{\alpha^i/\epsilon_i} \prod_{k \neq i} (a^i - a^k)^{-\epsilon_k/\epsilon_i} \right), \quad n = 1, 2, \ldots, \quad i = 1, \ldots, N. \]

For instance, the first conservation law densities are

\[ p^1_1 = e^{\alpha^1/\epsilon_1} \prod_{k \neq i} (a^i - a^k)^{-\epsilon_k/\epsilon_i}, \quad p^2_1 = \frac{e^{\alpha^2/\epsilon_1}}{\epsilon_1} \left( 1 - \sum_{n \neq i} \frac{\epsilon_n}{a^i - a^n} \right) \prod_{k \neq i} (a^i - a^k)^{-2\epsilon_k/\epsilon_i}, \ldots \]

Thus, multiparametric solutions can be found from (40). For example in the simplest case (here \( \kappa_m \) are arbitrary constants)

\[ x \frac{\partial H_0}{\partial a^i} - t \frac{\partial H_1}{\partial a^i} = \frac{\partial}{\partial a^i} \left( \sum_{m=1}^{N} \kappa_m p^m_1 \right), \] (43)
the corresponding algebraic system assumes the form

\[ x - t(a^i + \epsilon) = \frac{\kappa_i p_i}{\epsilon_i} \left( 1 - \sum_{m \neq i} \frac{\epsilon_m}{a^i - a^m} \right) + \sum_{m \neq i} \frac{\kappa_m p_m}{a^m - a^i}, \quad \epsilon = \sum \epsilon_m. \]

### 4.2 Puiseux Type Reduction

Puiseux type hydrodynamic reduction (see (14) and (28), here \( \epsilon = \sum \epsilon_m \neq -1 \))

\[ a_i^k + \left( \frac{(a_i^k)^2}{2} - \frac{1}{2(1 + \epsilon)} \left[ \sum_{m=1}^N \epsilon_m (a_m^i)^2 + \left( \sum_{m=1}^N \epsilon_m a_m^i \right)^2 \right] \right) = 0 \]  (44)

has the Kruskal series of conservation laws (18), where Kruskal conservation law densities \( H_k(a) \) are homogeneous polynomials. These polynomial expressions can be found by substitution of asymptotic series (20) into (see (29))

\[ \lambda = \left( p + \sum_{m=1}^N \epsilon_m a_m^i \right)^{1/\epsilon_i} \prod_{n=1}^N (p - a_n^i)^{\epsilon_i \epsilon_i}. \]

Here we make a point transformation \( \lambda \to \lambda^{1/\epsilon_i} \), in order to obtain the asymptotic series (20).

\( N \) principal series of conservation law densities are found from

\[ \tilde{\lambda}_{(i)} = (p - a_i^i) \prod_{m \neq i} (p - a_m^i)^{\epsilon_m/\epsilon_i} \left( p + \sum_{n=1}^N \epsilon_n a_n^i \right)^{1/\epsilon_i} \]

for each index \( i \) separately. Here we choose the corresponding local parameter \( \tilde{\lambda}_{(i)} = \lambda^{1/\epsilon_i} \), then asymptotic series (39) is applicable. The conservation law densities \( p_n^k(a) \) of the Puiseux type reduction can be obtained using (42)

\[ p_n^i = \frac{1}{n! d(a^i)^{n-1}} \left( \prod_{m \neq i} (a^i - a_m^i)^{-n \epsilon_m/\epsilon_i} \left( a^i + \sum_{k=1}^N \epsilon_k a_k^i \right)^{-n/\epsilon_i} \right), \quad n = 1, 2, \ldots, \quad i = 1, \ldots, N. \]

For instance, the first conservation law densities are

\[ p_1^i = \prod_{m \neq i} (a^i - a_m^i)^{-\epsilon_m/\epsilon_i} \left( a^i + \sum_{k=1}^N \epsilon_k a_k^i \right)^{-1/\epsilon_i}, \]  (45)

\[ p_2^i = -\frac{1}{\epsilon_i} \prod_{m \neq i} (a^i - a_m^i)^{-2 \epsilon_m/\epsilon_i} \left( a^i + \sum_{k=1}^N \epsilon_k a_k^i \right)^{-2/\epsilon_i - 1} \left[ 1 + \epsilon_i + \left( a^i + \sum_{k=1}^N \epsilon_k a_k^i \right) \sum_{n \neq i} \frac{\epsilon_n}{a^i - a_n^i} \right], \ldots \]
1. Polynomial reduction (Dispersionless limit of the Gelfand–Dikij reduction, see, for instance, [19]). If all $\epsilon_m = 1$, then Puiseux type reduction (29) becomes polynomial

$$
\lambda = \left( p + \sum_{m=1}^{N} a^m \right) \prod_{n=1}^{N} (p - a^n).
$$

(46)

As we mentioned before, this polynomial in $p$ case was studied in [18, 6, 2]. Unfortunately constructive results (formulae for the potential $V(x,t)$ and the first integral $F(x,t,p)$) were obtained in the context of classical mechanics only in a few cases. In this paper we present a much wider class of such potentials and their first integrals. For instance, $p_1^i$ in (45) for the Puiseux type reduction have the homogeneity degree $K_i = 1 - \frac{1}{\epsilon_i}(\epsilon + 1)$; in the polynomial case all $K_i = -N$. The densities $p_2^i$ has the homogeneity degree $K_i = 1 - \frac{2}{\epsilon_i}(\epsilon+1)$ for the Puiseux type reduction; in the polynomial case, all $K_i = -2N-1$. Corresponding expressions for conservation law densities are (see (45))

$$
p_1^i = \prod_{m \neq i} (a^i - a^m)^{-1} \left( a^i + \sum_{k=1}^{N} a^k \right)^{-1},
$$

$$
p_2^i = -\prod_{m \neq i} (a^i - a^m)^{-2} \left( a^i + \sum_{k=1}^{N} a^k \right)^{-3} \left[ 2 + \left( a^i + \sum_{k=1}^{N} a^k \right) \sum_{n \neq i} \frac{1}{a^i - a^n} \right], \ldots
$$

2. Zakharov type reduction (see, for instance, [19] and [40]). If all $\epsilon_m = \pm 1$, then Puiseux type reduction (29) assumes rational form with simple poles only, i.e.

$$
\lambda = \left( p + \sum_{m=1}^{N_1} a^m - \sum_{n=1}^{N_2} b^n \right) \frac{\prod_{k=1}^{N_1} (p - a^k)}{\prod_{s=1}^{N_2} (p - b^s)},
$$

where we introduced $N_1$ hydrodynamic variables $a^k(x,t)$ for $\epsilon_k = 1$ and $N_2$ hydrodynamic variables $b^m(x,t)$ for $\epsilon_m = -1$. The Krichever reduction contains multiple poles.

3. Kodama reduction (see, for instance, [19] and [32]). If all $\epsilon_m = 1$ except $\epsilon_1 = -M$ and $M \neq N$, then Puiseux type reduction (29) becomes rational with one multiple pole:

$$
\lambda = \left( p + \sum_{m=2}^{N} a^m - Ma^1 \right) \frac{\prod_{n=2}^{N} (p - a^n)}{(p - a^1)^M}.
$$

5 Similarity Solutions

In this Section we consider a special but a very important sub-class of solutions for Puiseux type reductions (28), (29), (44).

Expansion (4) is invariant under the scaling $F \rightarrow cF$, $p \rightarrow cp$ and $A^k \rightarrow c^{k+2}A^k$, where $c$ is arbitrary constant. Thus, it is easy to see that Benney hydrodynamic chain (10)
admits similarity reductions \( A^k(x, t) = t^{-(\beta+1)(k+2)} B_k(z) \), where \( z = xt^\beta \). Substitution of this ansatz directly into Benney hydrodynamic chain (10) yields a chain of ordinary differential equations

\[
B_{k+1}'(z) + \beta z B'_k(z) - (\beta + 1)(k + 2)B_k(z) + kB_{k-1}(z)B_0'(z) = 0, \quad k = 0, 1, 2, \ldots
\]

Hamilton’s equations (1) are equivalent to a single ordinary differential equation of a second order \( \ddot{x} = -V_x \), which reduces to the form

\[
\frac{d^2z}{d\tau^2} - (2\beta + 1)\frac{dz}{d\tau} + \beta(\beta + 1)z + \tilde{V}'(z) = 0,
\]

where \( t = e^\tau \), \( V(x, t) = A^0(x, t) = t^{-2(\beta+1)}B_0(z) \) and \( \tilde{V}(z) = B_0(z) \). This autonomous equation is equivalent to the first order ordinary differential equation

\[
q\frac{dq}{dz} - (2\beta + 1)q + \beta(\beta + 1)z + \tilde{V}'(z) = 0,
\]

where \( q = dz/d\tau \). Our aim in this paper is to describe all functions \( V(x, t) \) such that the corresponding Hamilton’s equations (1) are solvable in hydrodynamic sense. In the above example, we would like to find such \( \tilde{V}(z) \).

For Puiseux type reductions (28), (29), (44), similarity solutions \( a^i(x, t) = t^{-\beta-1}b^i(z) \) are determined by \( N \) component non-autonomous system of ODEs

\[
\beta z b^k_x - (\beta + 1)b^k_x + \partial_z \left( \frac{(b^k_x)^2}{2} - \frac{1}{2(1+\epsilon)} \left[ \sum_{m=1}^N \epsilon_m (b^m)_x^2 + \left( \sum_{m=1}^N \epsilon_m b^m_x \right)^2 \right] \right) = 0. \quad (48)
\]

If \( \beta = -1/2 \), equation (47) is easily integrable by the method of separation of variables. Simultaneously, system (48) also integrates and yields \( N \) algebraic equations for \( b^k(z) \):

\[
(b^k)^2 - z b^k - \beta_k = \frac{1}{(1+\epsilon)} \left[ \sum_{m=1}^N \epsilon_m (b^m)_x^2 + \left( \sum_{m=1}^N \epsilon_m b^m_x \right)^2 \right],
\]

where \( \beta_k \) are arbitrary constants. Introducing new potential function \( V(b(x,t)) = t \cdot V(a(x,t)) \) (cf. (28)) we find

\[
b^k(z) = \frac{z}{2} \pm \sqrt{\frac{z^2}{4} + \beta_k - 2V(b(z))}. \quad (49)
\]

In fact we may avoid introduction of the new function \( V(b(x,t)) \) and similar functions of the variables \( b(x,t) \) below which differ by a power of \( t \) from \( V(a(x,t)) \) and other original ones, if we will understand \( V(b(x,t)) \) as the result of formal substitution of the variables \( b^i \) instead of \( a^i \) directly into (28) etc. We will follow this understanding everywhere below in this Section, for example for \( A^k(b(x,t)), p^k_s(b(x,t)) \) in (50). The potential \( V(b(z)) \) may be directly found from the algebraic equation

\[
V(b(z)) = -\frac{1}{2(1+\epsilon)} \left[ \sum_{m=1}^N \epsilon_m (b^m)_x^2 + \left( \sum_{m=1}^N \epsilon_m b^m_x \right)^2 \right]
\]
where we should substitute $b^k(z)$ given by (49). Then

$$a^m(x,t) = t^{-1/2} \left( \frac{xt^\beta}{2} \pm \sqrt{\frac{(xt^\beta)^2}{4} + \beta_m - 2V(b(xt^\beta))} \right)$$

and the corresponding first integral is given by (29):

$$F(x,t,p) = \lambda(a(x,t),p) = \left( p + \sum_{m=1}^{N} \epsilon_m a^m(x,t) \right)^N \prod_{n=1}^{N} (p - a^n(x,t))^n.$$  

In the general case (when $\beta \neq -1/2$), integration of the above non-autonomous system (48) is a difficult problem. However, for special values of the similarity exponent $\beta$ similarity solutions are determined according to the Generalized Hodograph Method by an appropriate choice of the commuting flows in (40). For instance, the simplest similarity solution is determined by the algebraic system (see (43) and (45))

$$z \frac{\partial A^0(b)}{\partial b^i} - \frac{\partial A^1(b)}{\partial b^i} = \frac{\partial}{\partial b^i} \left[ \sum_m \kappa_m p_s^m(b) \right], \quad i = 1, \ldots, N$$

(50)

for any indices $k$, $s$. The similarity exponent $\beta$ is determined explicitly by the indices $k$, $s$ and the constants $\epsilon_m$. For example, for $s = 1$, $\beta = -\left( \epsilon_k + \epsilon + 1 \right) / \left( 2\epsilon_k + \epsilon + 1 \right)$.

In the polynomial case (46), instead of algebraic system (50) without free parameters, $N$ parametric similarity solutions can be presented, because all conservation law densities $p_s^k(b)$ with the same $s$ have the same homogeneity, i.e.

$$z \frac{\partial A^0(b)}{\partial b^i} - \frac{\partial A^1(b)}{\partial b^i} = \frac{\partial}{\partial b^i} \left[ \sum_m \kappa_m p_s^m(b) \right], \quad i = 1, \ldots, N,$$

(let us remind that all $\epsilon_k = 1$ in the polynomial case). For example when $s = 1$, $\beta = -\frac{N+2}{N+3}$ and

$$p_1^k(a) = \prod_{m \neq k} (a^i - a^m)^{-1} \left( a^i + \sum_{k=1}^{N} \epsilon_k a^k \right)^{-1}.$$  

Similar computations can be made for arbitrary linear combination $\sum_m \kappa_m p_s^m(a)$ for any $s$.

Let us now consider another similarity reduction $A^k = x^{k+2}B_k(t)$ (which may be obtained from the previous reduction by appropriate limiting procedure with $\beta \to \infty$). Then Benney hydrodynamic chain (10) reduces to the form

$$B_k'(t) + (k+3)B_{k+1}(t) + 2kB_{k-1}(t)B_0(t) = 0,$$

while Hamilton’s equations (1) are equivalent to a single ordinary differential equation of a second order $\ddot{x} = -V_x$, which reduces to the linear ordinary differential equation $\ddot{x} + 2V(t)x = 0$, where $V(x,t) = x^2 V(t)$ and $V(t) = B_0(t)$. Again, integrability of the corresponding Riccati equation in quadratures is an open problem in the general case. By this reason, we also are interested in finding functions $V(t)$ such that the corresponding Hamilton’s equations (1) will be Liouville integrable.
We seek similarity solutions of Puiseux type reductions in the form $a^i(x,t) = b_i(t)x$. Substitution of this ansatz into (44) yields $N$ component system of first order ordinary differential equations

$$b'_k(t) + b_k^2 - \frac{1}{(1 + \epsilon)} \left[ \sum_{m=1}^{N} \epsilon_m b_m^2 + \left( \sum_{n=1}^{N} \epsilon_n b_n \right)^2 \right] = 0. \quad (51)$$

After the potential substitution $b_k(t) = \psi'_k(t)/\psi_k(t)$, system (51) assumes the form

$$\psi''_k + 2V(b)\psi_k = 0, \quad (52)$$

where

$$V(b) = -\frac{1}{2(1 + \epsilon)} \left[ \sum_{m=1}^{N} \epsilon_m (b^m)^2 + \left( \sum_{m=1}^{N} \epsilon_m b^m \right)^2 \right] = -\frac{1}{2(1 + \epsilon)} \left[ \sum_{m=1}^{N} \epsilon_m \psi'^2_m + \left( \sum_{m=1}^{N} \epsilon_m \psi'_m \psi_m \right)^2 \right].$$

Since $\psi_1 \psi''_k - \psi'_k \psi'_1 = 0$ for all $k > 1$ we obtain

$$\psi_k = g_k \psi_1 \int \frac{dt}{\psi_1^2} + e_k \psi_1,$$

where $g_k$ and $e_k$ are arbitrary constants. Then ($k > 1$)

$$b_k = \frac{\psi'_k}{\psi_k} = \frac{\psi'_1}{\psi_1} + \frac{1}{\psi_1^2 \int \frac{dt}{\psi_1^2} + s_k \psi_1^2},$$

where $s_k = e_k/g_k$.

Then (52) reduces to the form

$$-\frac{1}{2}pp'' + \frac{1}{4}p'^2 = \frac{1}{(1 + \epsilon)} \left[ \frac{1}{4} \epsilon (1 + \epsilon)p'^2 - (1 + \epsilon)pp' \sum_{n=2}^{N} \frac{\epsilon_n}{r + s_n} + p^2 \sum_{m=2}^{N} \frac{\epsilon_m}{(r + s_m)^2} + p^2 \left( \sum_{n=2}^{N} \frac{\epsilon_n}{r + s_n} \right)^2 \right],$$

where $\psi_1(t) = p^{-1/2}(r)$ and the new independent variable $r$ is defined by $r'(t) = p(r) = (\psi_1(t))^{-2}$, so

$$b_1 = -\frac{1}{2}p'(r), \quad b_k = -\frac{1}{2}p'(r) + \frac{p}{r + s_k}, \quad \epsilon = \sum_{n=1}^{N} \epsilon_n.$$

Substitutions $p = z^{2(1+\epsilon)^{-1}}$ and

$$z(r) = y(r) \prod_{m=2}^{N} (r + s_m)^{\epsilon_m}$$

lead to the differential equation $y''(r) = 0$. Thus (51) is integrable in quadratures:

$$t = t_0 + \int (Ar + B)^{-2(1+\epsilon)^{-1}} \prod_{m=2}^{N} (r + s_m)^{-2(1+\epsilon)^{-1} \epsilon_m} dr,$$

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and the expressions for $b_k(t)$ via $\psi_1(t)$ and $p(r)$ are given here.

In the polynomial case (i.e. all $\epsilon_m = 1$):

$$t = t_0 + \int (Ar + B)^{-2(1+N)^{-1}} \prod_{m=2}^{N} (r + s_m)^{-2(1+N)^{-1}} dr.$$

If, for instance, $N = 3$, then

$$t = t_0 + \int \frac{dr}{\sqrt{(Ar + B)(r + s_2)(r + s_3)}}.$$

So we see that (51) is solvable in terms of the Weierstrass $\wp$ function. This brings up a remarkable link between the hydrodynamic reduction approach used here to obtain some “solvable” potentials $\tilde{V}(t)$ and the classical theory of finite-gap potentials [26].

6 Multi-Time Generalization. Explicit Solutions

The approach presented in this paper can be extended to higher number of “time” variables. According to [12] the Hamiltonian $\tilde{H} = p^3/3 + pV(x, y, t) + W(x, y, t)$ determines Hamilton’s equations with the new time variable $y$:

$$x_y = \frac{\partial \tilde{H}}{\partial p} = p^2 + V(x, y, t), \quad p_y = -\frac{\partial \tilde{H}}{\partial x} = -p \frac{\partial V(x, y, t)}{\partial x} - \frac{\partial W(x, y, t)}{\partial x}. \quad (53)$$

This Hamiltonian system is compatible with the system (1), (2) (where the potential $V$ also should be considered as a function of three variables $V(x, y, t)$) if and only if $V(x, y, t), W(x, y, t)$ satisfy (55) given below. Theory of such integrable pairs is of obvious interest. We sketch below some aspects of this problem. If we will try to find an appropriate definition of Liouville integrability one should obviously start with the proper generalization of the definition of conservation laws for such a pair of Hamiltonian systems with “potentials” $V(x, y, t), W(x, y, t)$. Then the method of hydrodynamic reductions may be used for construction of explicit formulas for integrable potential pairs $V, W$. Corresponding Vlasov type kinetic equation is

$$F_y + (p^2 + V)F_x - F_u(pV_x + W_x) = 0. \quad (54)$$

Again suppose now that $F(x, t, y, p)$ simultaneously satisfies (54) and (6). Their compatibility conditions $(F_t)_y = (F_y)_t$ are equivalent to the following system of equations for $V(x, y, t), W(x, y, t)$:

$$V_t = -W_x, \quad (W_t - V_y - VV_x)_x = 0. \quad (55)$$

This is a version of the remarkable Lin–Reissner–Tsien equation (see [22]) also known as the Khokhlov–Zabolotskaya equation (see [39]) or a dispersionless limit of the Kadomtsev–Petviashvili equation (see [16])

$$V_t = -W_x, \quad W_t = V_y + VV_x. \quad (56)$$
Substitution of (4) into (54) yields the first commuting flow from the Benney hierarchy (28):
\[ A_y^k + A_x^{k+2} + A^k A^0_x + (k + 1)A^k A^0_x + kA^{k-1} A^1_x = 0, \quad k = 0, 1, \ldots, \] (57)
where \( V = A^0 \) and \( W = A^1 \) (see also (5)). Then all the rest of further computations will be very similar to all that is written here. Functions \( \lambda(a, p) \) are the same as well as all moments \( A^m(a) \). However, a dependence with respect to “time” variable \( y \) is given by different hydrodynamic type system (cf. (14))
\[ a_y^k + \left( \frac{(a^k)^3}{3} + a^k V(a) + W(a) \right)_x = 0, \] (58)
which commutes with (14) if and only if (55) holds. This means that moments \( A^m(a) \) solve hydrodynamic chains (10) and (57), where functions \( a^i(x, t, y) \) solve commuting hydrodynamic type systems (14) and (58).

Thus, we just would like to mention here that the potential function \( V(x, t) \) of Hamilton’s equations (1) can be interpreted as a two-dimensional reduction of the function \( V(x, y, t) = \partial_x S(x, y, t) \), where we introduce \( S(x, y, t) \) to simplify the concept, since \( S(x, y, t) \) is a solution of the Lin–Reissner–Tsien equation written as a single three-dimensional quasilinear equation of a second order
\[ S_{tt} + S_{xy} + S_x S_{xx} = 0 \]
and (see the first equation in (56)) \( W(x, y, t) = -\partial_t S(x, y, t) \), while the first integral \( F(x, y, t, p) \) satisfies two Vlasov type kinetic equations (see (54) and (6))
\[ F_t + pF_x - F_p S_{xx} = 0, \quad F_y + (p^2 + S_x)F_x - F_p(pS_{xx} - S_{xt}) = 0. \]

Let us remind that the both equations are nothing but \( dF/dt = 0 \) and \( dF/dy = 0 \), i.e. \( F(x, y, t, p) = \text{const} \). A method of hydrodynamic reductions for such three dimensional quasilinear equations of the second order was developed in [8].

Nevertheless, obviously, not all solutions of three dimensional quasilinear equation (see (56))
\[ V_{tt} + (V_y +VV_x)_x = 0 \] (59)
can be obtained by the method of hydrodynamic reductions only. However, if such a three dimensional quasilinear equation passes this integrability test (i.e. possesses sufficiently many hydrodynamic reductions), then this equation also can possess particular solutions explicitly parameterized by arbitrary functions of a single variable. Moreover the two examples of solvable potentials \( V(x, y, t) \) given below have a remarkable property: one can obtain explicit formulas for solutions of the corresponding Hamilton’s equations (1), (2) without the need to use Liouville’s theorem on integrability in quadratures.

1. Manakov–Santini solution (see [23]). A particular class of solution \( V(x, y, t) \) of (59) is given in implicit form
\[ x = 2Vy - f(Vy^{1/2}) - \frac{t^2}{4y}, \] (60)
where \( f(U) \) is an arbitrary function. In order to solve the Hamilton’s equations (1), (2), (53) for such potentials, we write them in the differential form

\[
dx = p\,dt + (p^2 + V)\,dy, \quad dp = -V_x\,dt + (-pV_x + V_t)\,dy
\]  
(61)

Under the substitutions \( y = z^2, t = z\tau, zV = U \), (61) read

\[
dx = pz\,d\tau + (2zp^2 + pr + 2U)\,dz, \quad dp = -[2z - f'(U)]^{-1}d\tau - 2[2z - f'(U)]^{-1}pdz,
\]  
(62)

while (60) becomes

\[
x = 2zU - f(U) - \frac{\tau^2}{4}.
\]  
(63)

Substitution (63) into (62) yields

\[
dU = -\frac{q}{2}dp, \quad dq = f'(U)dp,
\]  
(64)

where \( q = \tau + 2pz \), and we consider now the function \( U \) as the function of two variables: \( U(z, \tau) = U(x(z, \tau), z, \tau) \) instead of \( U(x, z, \tau) \) found from (63). Thus, we obtain

\[
q = \pm 2\sqrt{E - f(U)}, \quad p = p_0 \mp \int \frac{dU}{\sqrt{E - f(U)}},
\]  
(65)

where \( p_0 \) and \( E \) are integration constants. Taking into account (64) and \( q = \tau + 2pz \), we obtain \( U(z, \tau) = U(x(z, \tau), z, \tau) \) from the equation

\[
\pm 2\sqrt{E - f(U)} = \tau + 2z \left( p_0 \mp \int \frac{dU}{\sqrt{E - f(U)}} \right).
\]  
(66)

So finally we obtain the solution of the two-time Hamilton’s equations

\[
x(z, \tau) = 2zU(z, \tau) - f(U(z, \tau)) - \frac{\tau^2}{4},
\]

where \( U(z, \tau) \) is given by (66); \( p(z, \tau) \) is found from (65).

2. Simple wave solution. The solution \( V(x, y, t) \) of (59) is determined by

\[
x = Q(V) - F(V)t + (V + F^2(V))y,
\]  
(67)

where \( Q(V) \) and \( F(V) \) are arbitrary functions. Then (61) read

\[
|Q'(V) - F'(V)t + (1 + 2F(V)F'(V))y|\,dV = (p + F(V))[dt + (p - F(V))dy],
\]

\[
-|Q'(V) - F'(V)t + (1 + 2F(V)F'(V))y|\,dp = dt + (p - F(V))dy.
\]  
(68)

These two differentials reduce to

\[
(p + F(V))dp + dV = 0,
\]

\[
-[Q'(V) - F'(V)t + (1 + 2F(V)F'(V))y]|dp = dt + (p - F(V))dy.
\]
The first of them means that \( p = P(V) \), where \( P(V) \) is a solution of Abel equation
\[
(P(V) + F(V))P'(V) = -1. \tag{69}
\]
However, since \( F(V) \) is an arbitrary function, we can express \( F(V) \) via \( P(V) \), i.e.
\[
F(V) = -\frac{1}{P'(V)} - P(V).
\]
Then the differential (68) can be integrated in quadratures (with one effective integration constant, the second one is hidden in (69))
\[
t + (2P(V) + \frac{1}{P'(V)})y + P'(V)e^{-\int P'^2(V)dV} \int Q'(V)e^{\int P'^2(V)dV}dV = 0.
\]
Thus we can find the solution \( V(y, t) = V(x(y, t), y, t) \) of this equation in implicit form. Then (67) yields the solution \( x(y, t) \) of (1), (2), (53):
\[
x(y, t) = Q(V) + \left(\frac{1}{P'(V)} + P(V)\right) t + \left[V + \left(\frac{1}{P'(V)} + P(V)\right)^2\right] y.
\]
The momentum \( p(y, t) \) is found from the relation \( p = P(V) \).

7 Conclusion

We have constructed a few multiparametric families of potentials \( V(x, t) \) with integrals \( F(x, t, p) \) which are either polynomial or non-polynomial in \( p \). There is a strong evidence (cf. \([15]\)) that such families are locally dense in the functional space of all potentials. Unfortunately we do not have a possibility to go into the necessary details here.

In this paper we considered Hamilton’s equations (1), determined by the classical Hamiltonian function (2). They are equivalent to a single equation \( \ddot{x} = -V \). Now we would like to emphasize that our approach is applicable for Hamilton’s equations (1) with Hamiltonian function \( H(x, t, p) \) of much more general form than (2). This is based on the following results.

A complete classification of Vlasov type kinetic equations (cf. (3))
\[
F_t - \{F, H\} = F_t + H_p F_x - F_p H_x = 0
\]
integrable by the method of hydrodynamic reductions was presented in \([27]\) for the Hamiltonian functions \( H(V(x, t), p) \). First three simplest cases (see also \([31]\)) have the form
\[
H = Q_1(p) + V(x, t), \quad H = Q_2(p) + pV(x, t), \quad H = Q_3(p)V(x, t),
\]
where \( Q_i(p) \) are arbitrary solutions of the equations \( Q_1'' = \alpha Q_1' + \beta Q_1 + \gamma, \quad pQ_2'' = \alpha Q_2' + \beta Q_2 + \gamma, \quad Q_3Q_3'' = \alpha Q_3' + \beta Q_3 + \gamma \) and \( \alpha, \beta, \gamma \) are arbitrary constants (if \( Q_1(p) = p^2/2 \), this is nothing but the case considered in this paper). Corresponding analogues of Löwner equation and Gibbons–Tsarev system were derived in \([27]\), \([31]\). Following the approach presented here, one can extract infinitely many particular solutions of Löwner equation and Gibbons–Tsarev system and thus construct infinitely many Hamilton’s equations solvable in hydrodynamic sense.
Appendix A (Asymptotic Expansion and Moments)

As we mentioned in Introduction the distribution function $F(x, t, p)$ in our approach satisfies the Vlasov (collisionless Boltzmann) kinetic equation ([40], [11])

$$F_t + pF_x - F_p V_x = 0,$$

(70)

where the potential energy $V(x, t)$ coincides with the zeroth moment $A^0(x, t)$ of the asymptotic expansion of the function $F(x, t, p)$ for $p \to \infty$:

$$F(x, t, p) = p + \frac{A^0(x, t)}{p} + \frac{A^1(x, t)}{p^2} + \frac{A^2(x, t)}{p^3} + \ldots, \quad p \to \infty. \tag{71}$$

On the other hand in many mechanical and physical applications the moments $A^k$ are defined as

$$A^k(x, t) = \int_{-\infty}^{\infty} p^k \Phi(F(x, t, p)) dp,$$

(72)

where $\Phi(F)$ is an appropriate rapidly decreasing at infinities $p \to \pm \infty$ function such that the integrals are finite.

In this Appendix we study the relation of (70) with the expansion (71) on one hand and the same equation (70) associated with (72) on the other hand.

First, we start with the pair (70)+(71). We will consider an even more general asymptotic behavior at infinity $p \to \infty$

$$F(x, t, p) = a_{-2}(x, t)p + a_{-1}(x, t) + \frac{a_0(x, t)}{p} + \frac{a_1(x, t)}{p^2} + \ldots. \tag{73}$$

Direct substitution of this expansion into (70) leads to an infinite series of equations:

$$a_{-2,x} = 0,$$

(74)

$$a_{-2,t} + a_{-1,x} = 0,$$  \hspace{1cm} (75)

$$a_{-1,t} + a_{0,x} - a_{-2}V_x = 0,$$ \hspace{1cm} (76)

$$a_{k,t} + a_{k+1,x} + ka_{k-1}V_x = 0, \quad k = 0, 1, \ldots$$ \hspace{1cm} (77)

Integration of (74) yields $a_{-2} = a_{-2}(t)$. However, without loss of generality one can set $a_{-2} = 1$. For this we can perform the following point transformation of the independent variables: $(t, x, p) \mapsto (y, z, q)$ with $t = t(y)$, $x = x(y, z) = a(y)z$, $p = p(y, z, q) = \frac{a'(y)}{v(y)}z + \frac{a(y)}{v'(y)}q$. The function $t(y)$ is to be found from the equation $t'(y) = (a_{-2}(t))^2$ and $a(y) = a_{-2}(t(y))$. Indeed, under this transformation Vlasov kinetic equation (70) is transformed into $F_y + qF_z - F_q W_z = 0$ with $W = W(y, z)$ found from $V_x + (t'(y))^{-2}x_{yy} - x_y(t'(y))^{-3}t''(y) = x_z(t'(y))^{-2}W_z$ while asymptotic series (73) becomes

$$F(y, z, q) = q + \tilde{a}_{-1}(y, z) + \frac{\tilde{a}_0(y, z)}{q} + \frac{\tilde{a}_1(y, z)}{q^2} + \ldots$$

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Thus, if we choose \( a_{-2} = 1 \), then (75) yields \( \tilde{a}_{-1} = \tilde{a}_{-1}(y) \). However, we can shift \( q \) by the value \( \tilde{a}_{-1}(y) \). This requires the following transformation for (70): \((y, z, q) \mapsto (t = y, x = z + s(y), p = q + \tilde{a}_{-1}(y))\), \(W(y, z) \mapsto \tilde{V}(x, t) = W(z - s(t), t) + \tilde{a}'_{-1}(t) \cdot z \) with \( s'(t) = a_{-1}(t) \). Then asymptotic series (73) assumes the form

\[
F(x, t, p) = p + \frac{\tilde{a}_0(x, t)}{p} + \frac{\tilde{a}_1(x, t)}{p^2} + \ldots
\]

and (76) reduces to \( \tilde{a}_{0,x} = \tilde{V}_x \). Since the potential function \( V(x, t) \) is involved in Vlasov kinetic equation (70) via its derivative \( V_x \), without loss of generality we can choose

\[
a_0 = V.
\]

Thus, corresponding infinite set of equations (77) together with this condition \( V = a_0 \) implies Benney hydrodynamic chain (10).

Now we study the pair (70)+(72). We will prove here that substitution of (72) into (10) implies Vlasov kinetic equation (70) again. Indeed, at the first step we obtain

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{p^k \Phi' \left(F \right)}{p} (F_t dp + p F_x) dp + k A_x^0 \int_{-\infty}^{\infty} \frac{p^{k-1} \Phi \left(F \right)}{p} dp = 0.
\]

Integrating by parts we get

\[
\int_{-\infty}^{\infty} \frac{p^k \Phi' \left(F \right)}{p} (F_t + p F_x - F_p A_x^0) dp = 0.
\]

Since \( k \) is arbitrary, infinite set of these integrals vanish if the function \( F(x, t, p) \) satisfies the Vlasov kinetic equation

\[
F_t + p F_x - F_p A_x^0 = 0,
\]

where according to this procedure

\[
A_x^0 = \int_{-\infty}^{\infty} \Phi \left(F \right) dp.
\]

As a result of our considerations we conclude that in fact the Benney chain (10) is the pivotal object relating different pairs (70)+(71) and (70)+(72). Certainly we can make a way through Benney chain from one pair to another pair. In this way one obtains an interesting transformation. Namely substitution of (72) into (71) yields a (formal) integral transformation

\[
F(x, t, p) = p + \sum_{m=0}^{\infty} \frac{A_m}{p^{m+1}} = p + \sum_{m=0}^{\infty} \frac{1}{p^{m+1}} \int_{-\infty}^{\infty} q^m \Phi(\tilde{F}(x, t, q)) dq = p + \int_{-\infty}^{\infty} \frac{\Phi(\tilde{F}(x, t, q))}{p - q} dq,
\]

where \( \tilde{F}(x, t, p) \) is a given solution of Vlasov kinetic equation (70), and \( F(x, t, p) \) is a new solution. This transformation was obtained for the Vlasov equation in [15] and
used in hydrodynamics in [4]. One can show directly that this transformation maps a solution $F(x, t, p)$ of Vlasov kinetic equation into another solution of the same equation if $V(x, t) = \int \Phi(F(x, t, q))dq$. Indeed, suppose that some $F(x, t, q)$ satisfies Vlasov kinetic equation (70)

$$F_t + qF_x - F_qV_x = 0.$$  

Then obviously any function $\Phi(F(x, t, q))$ also satisfies the same equation. Let us multiply this equation by $(p - q)^{-1}$ and integrate with respect to $q$ along an arbitrary path $D$:

$$\oint_D \frac{\Phi(F)}{p - q} dq = 0$$

or

$$\left( \oint_D \frac{\Phi(F)}{p - q} dq \right)_t + \left( \oint_D \frac{q\Phi(F)}{p - q} dq \right)_x - V_x \oint_D \frac{\Phi_q(F)}{p - q} dq = 0.$$  

(79)

The second integral can be transformed:

$$\oint_D \frac{q\Phi(F)}{p - q} dq = p \oint_D \frac{\Phi(F)}{p - q} dq - \oint_D \Phi(F) dq,$$

while the third integral reduces to

$$\oint_D \frac{\Phi_q(F)}{p - q} dq = \Phi(F) \bigg|_D \frac{\Phi(F)}{p - q} dq - \oint_D \frac{\Phi(F)}{p - q} dq.$$  

Introducing new function

$$\tilde{F}(x, t, p) = p + \oint_D \frac{\Phi(F)}{p - q} dq,$$

(80)

we can see that (79) becomes

$$\tilde{F}_t + p\tilde{F}_x - \left( \oint_D \frac{\Phi(F)}{p - q} dq \right)_x = V_x \frac{\Phi(F)}{p - q} \bigg|_{\partial D} + \tilde{F}_pV_x - V_x.$$  

(81)

If $\Phi(F)$ vanishes on $\partial D$ (or in the particular case when the integration is performed along the real axis from $-\infty$ up to $+\infty$, then $\Phi(F)$ must be rapidly decreasing function at the infinities), and if we set

$$V(x, t) = \int_D \Phi(F(x, t, q))dq,$$

then (81) is nothing but the same Vlasov kinetic equation (cf. (78)) with the same $V(x, t)$:

$$\tilde{F}_t + p\tilde{F}_x - \tilde{F}_pV_x = 0.$$  

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Expanding (80) for $p \to \infty$ we get the asymptotic expansion (71) where all moments are determined precisely by (72).

**Remark.** In some physical applications (for instance in hydrodynamics, see [5]) Vlasov type kinetic equation (70) derived from some fundamental physical laws contains the potential function $V(x, t)$, which is different from $A^0$. For instance, $V(x, t) = \ln A^0 = \ln \int F dp$. The corresponding Benney-like hydrodynamic chain for $A^k(x, t) = \int p^k \Phi(F(x, t, p)) dp$ is

$$A^k_t + A^{k+1}_x + kA^{k-1} \ln A^0_x = 0, \quad k = 0, 1, \ldots$$

It is non-integrable by the method of hydrodynamic reductions (see [15]). We consider the opposite case in this paper: the integrable (by the method of hydrodynamic reductions) version of Vlasov kinetic equation determined by the restriction $V = A^0$.

**Appendix B (Egorov Pairs of Conservation Laws)**

We will use the techniques of [37, 33] in order to prove the result we need in Section 4 for construction of the basic formulae for solutions (35), (36), namely the statement that for arbitrarily chosen conservation law density $h(r)$ of the original Egorov system (in our case (11)) an appropriately chosen commuting flow must have a Egorov pair such that $f_t = h_x$, where $f = A^0$ (the density $f$ in Lemma 1).

Egorov semi-Hamiltonian hydrodynamic type systems have the following form

$$r^i_t = \frac{\tilde{H}_i}{H_i} r^i_x,$$  \hspace{1cm} (82)

where the (non-flat in general) metric is given by $g_{ii} = \tilde{H}^2_i$ and the rotation coefficients are

$$\beta_{ik} = \frac{\partial_i \tilde{H}_k}{\tilde{H}_i}, \quad i \neq k.$$

The Egorov property for semi-Hamiltonian systems consists in symmetricity of the rotation coefficients: $\beta_{ik} = \beta_{ki}$. Corresponding linear system reads

$$\partial_i H_k = \beta_{ik} H_i, \quad i \neq k.$$  \hspace{1cm} (83)

One particular solution of this system is $\tilde{H}_i$, another particular solution is $\tilde{H}_i$.

We know that Egorov pair $f_t = h_x$, $h_t = g_x$ of conservation laws for (82) are given by the formulae ([33], Theorem 1)

$$\partial_i f = \tilde{H}^2_i, \quad \partial_i h = \tilde{H}_i \tilde{H}_i = \tilde{H}_i \tilde{H}_i, \quad \partial_i g = \tilde{H}^2_i.$$

The conservation law densities for (82) are given by

$$\partial_i \hat{h} = \tilde{H}_i \hat{H}_i,$$  \hspace{1cm} (84)

where $\hat{H}_i$ are arbitrary solutions of (83). Also, we know that all commuting flows have the form

$$r^i_\tau = \frac{\hat{H}_i}{H_i} r^i_x.$$  \hspace{1cm} (85)
where $\tilde{H}_i$ are again arbitrary solutions of (83). The Egorov pair for this commuting flow is given by
\[ \partial_i f = H_i^2, \quad \partial_i \tilde{h} = \tilde{H}_i, \quad \partial_i \tilde{g} = \tilde{H}_i^2. \]
Comparing (84) and (85) we see that we can always choose the same solution $\tilde{H}_i = \check{H}_i$ of (83) and obtain the necessary commuting flow (85) with the required $h$ in its Egorov pair.

**Appendix C (Löwner Equations)**

We start from the Vlasov kinetic equation
\[ F_t + p F_x - F_p V_x = 0 \quad (86) \]
and hydrodynamic type system
\[ a_i^k + \left( \frac{(a_i^k)^2}{2} + V(a) \right)_x = 0, \quad (87) \]
such that $F(x,t,p) = \lambda(a(x,t),p)$ with some fixed $\lambda(a,p)$ satisfies (86) for arbitrary solution $a^i(x,t)$ of (87). Then we obtain
\[ \sum_i (\lambda_i a_i^i + p \lambda_i a_i^x - \lambda_p V_i a_i^i) = 0. \]
Here and everywhere below we use the lower indices to denote partial derivatives w.r.t. $a^i$: $\lambda_i \equiv \partial_i \lambda \equiv \partial \lambda / \partial a^i$. Substituting $a_i^i$ from (87) we get
\[ \sum_i \lambda_i [-a_i^i a_x^i - V_x^i] + p \sum_i \lambda_i a_i^i - \lambda_p V_x = 0, \]
or
\[ p \sum_i \lambda_i a_i^i - \sum_i \lambda_i a_i^i a_x^i = \sum_i \left( \lambda_p + \sum_m \lambda_m \right) V_i a_i^i. \]
Since $a^i$ are arbitrary solutions of (87) we conclude that
\[ \lambda_i = \frac{V_i}{p - a^i} \left( \lambda_p + \sum_m \lambda_m \right). \quad (88) \]
Summing up we obtain
\[ \sum_m \lambda_m = \sum_n \frac{V_n}{p - a^n} \left( \lambda_p + \sum_m \lambda_m \right) \]
or
\[ \sum_m \lambda_m = \sum_n \frac{V_n}{p - a^n} \lambda_p \left( 1 - \sum_n \frac{V_n}{p - a^n} \right)^{-1}. \]
Substituting this into (88) we get

\[ \lambda_i = \frac{V_i}{p - a^i} \left( 1 + \sum_n \frac{V_n}{p - a^n} \left( 1 - \sum_n \frac{V_n}{p - a^n} \right)^{-1} \right) \lambda_p \]

i.e. the required formula

\[ \lambda_i = \frac{V_i}{p - a^i} \left( 1 - \sum_n \frac{V_n}{p - a^n} \right)^{-1} \lambda_p. \]

**Appendix D (Principal Series of Conservation Laws)**

In this Appendix we prove that one can find all principal series \( p_k^i(a) \) of conservation law densities in the expansion

\[ p^{(i)}(a, \tilde{\lambda}(k)) = a^i + p_1^i(a)\tilde{\lambda}(k) + p_2^i(a)\tilde{\lambda}^2(k) + p_3^i(a)\tilde{\lambda}^3(k) + \ldots, \ i = 1, \ldots, N, \]

where (unknown at this point) generating function \( p(a, \lambda) \) of conservation law densities should satisfy

\[ p_t + \left( \frac{p^2}{2} + V(a) \right)_x = 0 \] (90)

and the potential function \( V(a) \) is already found (as a solution of Gibbons-Tsarev equations (21)). Substitution of (89) into (90) yields infinite set of equations

\[ a^i_t + \left( \frac{(a^i)^2}{2} + V(a) \right)_x = 0, \]

\[ (p_1^i(a))_t + (a^i p_1^i(a))_x = 0, \]

\[ (p_2^i(a))_t + \left( a^i p_2^i(a) + \frac{1}{2} (p_1^i(a))^2 \right)_x = 0, \ldots. \]

We will show that all conservation law densities \( p_m^i(a) \) can be found in quadratures in the first case \( p_1^i(a) \). The higher elements of the principal series \( p_m^i(a) \) are found in the same way. First, we observe that (91) coincides with (14). Equations (92) give

\[ \sum_k \partial_k p_1^i(a)a^k_i + p_1^i(a)a^i_x + \sum_k a^i \partial_k p_1^i(a) a^k_x = 0. \]

Substitution of \( a^k_i \) from (91) gives

\[ \sum_k \partial_k p_1^i(a)[a^k a^k_i + V_x] = p_1^i(a)a^i_x + \sum_k a^i \partial_k p_1^i(a) a^k_x, \]

or

\[ \sum_k (a^k \partial_k p_1^i(a)a^k_x + (\delta p_1^i(a)) \partial_k V a^k_x) = p_1^i(a)a^i_x + \sum_k a^i \partial_k p_1^i(a) a^k_x, \]
where \( \delta = \sum_m \partial/\partial a^m \). Since \( a^s(x,t) \) are arbitrary solutions of (91), coefficients at \( a^s \) vanish identically. For \( s = i \) this gives us

\[
a^i \partial_i p_1^i(a) + \delta p_1^i(a) \cdot \partial_i V = p_1^i(a) + a^i \partial_i p_1^i(a).
\]

(94)

If \( s \neq i \) then

\[
a^k \partial_k p_1^i(a) + \delta p_1^i(a) \cdot \partial_k V = a^i \partial_k p_1^i(a).
\]

(95)

Equation (95) simplifies to the form

\[
\partial_k p_1^i(a) = \delta p_1^i(a) \frac{\partial_k V}{a^i - a^k}, \quad k \neq i,
\]

(96)

while (94) is

\[
\delta \ln p_1^i(a) = \frac{1}{\partial_i V}.
\]

(97)

Equation (96) after summation has the form

\[
\delta p_1^i(a) = \delta p_1^i(a) \sum_{m \neq i} \frac{\partial_m V}{a^i - a^m} + \partial_i p_1^i(a)
\]

or

\[
\delta \ln p_1^i(a) \left( 1 - \sum_{m \neq i} \frac{\partial_m V}{a^i - a^m} \right) = \partial_i \ln p_1^i(a).
\]

Then taking into account (97), we obtain

\[
\partial_i \ln p_1^i(a) = \frac{1}{\partial_i V} \left( 1 - \sum_{m \neq i} \frac{\partial_m V}{a^i - a^m} \right).
\]

(98)

Then (96) takes the form

\[
\partial_k \ln p_1^i(a) = \delta \ln p_1^i(a) \frac{\partial_k V}{a^i - a^k}, \quad k \neq i,
\]

and taking into account (97), we obtain

\[
\partial_k \ln p_1^i(a) = \frac{1}{\partial_i V} \frac{\partial_k V}{a^i - a^k}, \quad k \neq i.
\]

So the conservation law densities \( p_1^i(a) \) can be found in quadratures:

\[
d \ln p_1^i(a) = \frac{1}{\partial_i V} \left( 1 - \sum_{m \neq i} \frac{\partial_m V}{a^i - a^m} \right) da^i + \frac{1}{\partial_i V} \sum_{m \neq i} \frac{\partial_m V}{a^i - a^m} da^m.
\]

**Example:** if \( V = \sum_m \epsilon_m a^m \), then

\[
d \ln p_1^i(a) = \frac{1}{\epsilon_i} \left( 1 - \sum_{m \neq i} \frac{\epsilon_m}{a^i - a^m} \right) da^i + \frac{1}{\epsilon_i} \sum_{m \neq i} \frac{\epsilon_m}{a^i - a^m} da^m,
\]

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so

\[ p_1^i(a) = e^\frac{a^i}{m} \prod_{m \neq i} (a^i - a^m)^{-\frac{m}{a^i}}. \]

All higher conservation law densities can be found in the same way. For instance, (93) leads to

\[ \sum_k \partial_k p_2^i(a) a^k_i + p_2^i(a) a^i_x + \sum_k (a^i \partial_k p_2^i(a) a^k_x + p_1^i(a) \partial_k p_1^i(a) a^k_x) = 0. \]

Taking into account (91) again, we obtain

\[ \sum_k \partial_k p_2^i(a) [a^k a^k_x + V_x] = p_2^i(a) a^i_x + \sum_k (a^i \partial_k p_2^i(a) a^k_x + p_1^i(a) \partial_k p_1^i(a) a^k_x). \]

Then

\[ \sum_k (a^k \partial_k p_2^i(a) a^k_x + \delta p_2^i(a) \partial_k V a^k_x) = p_2^i(a) a^i_x + \sum_k (a^i \partial_k p_2^i(a) a^k_x + p_1^i(a) \partial_k p_1^i(a) a^k_x). \]

If \( k \neq i \)

\[ \partial_k p_2^i(a) = \delta p_2^i(a) \frac{\partial_k V}{a^i - a^k} - \frac{p_1^i(a) \partial_k p_1^i(a)}{a^i - a^k}. \quad (99) \]

If \( k = i \)

\[ \delta p_2^i(a) \partial_i V = p_2^i(a) + p_1^i(a) \partial_i p_1^i(a). \]

Then

\[ \delta p_2^i(a) = \frac{1}{\partial_i V} p_2^i(a) + \frac{1}{\partial_i V} p_1^i(a) \partial_i p_1^i(a) \]

(100)

and (99) assumes the form

\[ \partial_k p_2^i(a) = \frac{1}{\partial_i V} \frac{\partial_k V}{a^i - a^k} p_2^i(a) + \frac{1}{\partial_i V} \frac{\partial_k V}{a^i - a^k} p_1^i(a) \partial_i p_1^i(a) - \frac{p_1^i(a) \partial_k p_1^i(a)}{a^i - a^k}. \]

Let us introduce intermediate set of functions \( q_2^i(a) \) such that \( p_2^i(a) = q_2^i(a) p_1^i(a) \). Then (100) by virtue of (97) reduces to the form

\[ \delta q_2^i(a) = \frac{1}{\partial_i V} \partial_i p_1^i(a), \]

(102)

while (101) due to (98) implies:

\[ \partial_k q_2^i(a) = \frac{1}{\partial_i V} \frac{\partial_k V}{a^i - a^k} \partial_i p_1^i(a) - \frac{\partial_k p_1^i(a)}{a^i - a^k}, \quad k \neq i. \]

This equation after summation has the form

\[ \delta q_2^i(a) = \sum_{m \neq i} \frac{1}{\partial_i V} \frac{\partial_m V}{a^i - a^m} \partial_i p_1^i(a) - \sum_{m \neq i} \frac{\partial_m p_1^i(a)}{a^i - a^m} + \partial_i q_2^i(a). \]
Then taking into account (102), we obtain

$$
\partial_i q^i_2(a) = \frac{1}{\partial_i V} \partial_i p^i_1(a) - \sum_{m \neq i} \frac{1}{\partial_i V} \frac{\partial_m V}{a^i - a^m} \partial_i p^i_1(a) + \sum_{m \neq i} \frac{\partial_m p^i_1(a)}{a^i - a^m}.
$$

Thus, $q^i_2(a)$ can be found in quadratures:

$$
dq^i_2(a) = \left( \frac{1}{\partial_i V} \partial_i p^i_1(a) - \sum_{m \neq i} \frac{1}{\partial_i V} \frac{\partial_m V}{a^i - a^m} \partial_i p^i_1(a) + \sum_{m \neq i} \frac{\partial_m p^i_1(a)}{a^i - a^m} \right) da^i +
\sum_{m \neq i} \left( \frac{1}{\partial_i V} \frac{\partial_m V}{a^i - a^m} \partial_i p^i_1(a) - \frac{\partial_m p^i_1(a)}{a^i - a^m} \right) da^m.
$$

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