EVIDENCE FOR STABILITY OF EXTREMAL BLACK p-BRANES

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Abstract

We investigate the stability of the extremal black p-brane which contains a n-form and a dilaton. We show that the instability due to the s-mode, which was present in the uncharged and non-extremal p-brane, disappears in the extreme case. This is shown to be consistent with an entropy argument which shows that the zero entropy of the extremal black hole is approached more rapidly than the zero entropy of the black p-brane, which would mean an instability would violate the second law of thermodynamics.

PACS number: 97.60.Lf, 04.20.Jb, 04.60.+n
1. Introductory remarks.

In four dimensions, black holes are characterised by a small set of parameters: charge, mass and angular momentum. The simplest solution, Schwarzschild, representing an uncharged non-rotating black hole has long been known to be classically stable, and in Einstein gravity, the stability of the exterior spacetime of the charged and rotating counterparts (Reissner-Nordstrom, Kerr-Newman) has also been demonstrated. In this latter case, there is an instability of the inner Cauchy horizon, however, this is hidden from us by the event horizon, and is therefore presumably not of physical interest to the exterior observer. Therefore, once a black hole is formed, it remains, forever shielding the details of its constituents from the outside observer, who may only measure the charge, mass, and angular momentum of that matter which formed the black hole.

Quantum mechanically, we have a very different picture. Black holes become black bodies, semi-classically radiating a purely thermal spectrum. In theory, a black hole could radiate away completely, possibly losing forever all the detailed information of its constituents. Such a picture, in which quantum mechanical unitarity is lost, is potentially unsettling, and certainly begs the question as to whether there is not some modification of semi-classical gravity in which information retrieval is in some way possible. An examination of black holes in low energy string theory is a step in this direction. Here, the Einstein action is modified by the addition of a dilaton with a non-trivial coupling to ‘electromagnetism’

\[
S = \int d^4 x \sqrt{-g} [R - 2(\nabla \phi)^2 - e^{-2\phi} F^2].
\]  

(1.1)

Such ‘dilatonic’ black holes have several attractive features. In Einstein gravity, charged black holes have an instability of the inner Cauchy horizon due to matter perturbations in the exterior spacetime\(^1\), however, there is no static charged black hole solution in Einstein gravity with only one horizon and a spacelike singularity. On the other hand, in low
energy string gravity the dilaton greatly changes the causal structure of charged black holes making them like Schwarzschild, having one event horizon and a spacelike singularity. This structure is generic, even if the dilaton has a mass, as is generally believed from, although not necessarily implied by, the principle of equivalence. Additionally, extremal magnetically charged dilaton black holes have an infinite throat, which, unlike the spatially infinite throat of extremal Reissner-Nordström, is a true internal infinity as seen in the so-called string metric

\[ g_{\text{string}} = e^{-2\phi} g_{\text{Einstein}} \] (1.2)

Such an internal infinity allows an effective dimensional reduction down to two-dimensions in the throat and has been a topic of much debate as a simplified model for black hole accretion/radiation.

Since superstring theory is most naturally, though not necessarily, associated with a ten-dimensional background, ideally one should examine string gravity in higher dimensions. In four dimensions, an event horizon must be topologically spherical, but in higher dimensions this need not be the case, for example, in ten dimensions we can have \( S^2 \times \mathbb{R}^6, S^3 \times \mathbb{R}^5 \) etc., topologies for the horizon. In previous work, we have pointed out that such black p-branes are unstable for a large range of (magnetic) charge:mass ratios, however, we were unable to demonstrate an instability for the extremal black p-branes. This was largely because our numerical analysis was tailored towards a general charge:mass ratio, and if “Q=M”, a degeneracy occurs in the perturbation equations reducing the dimensionality of the numerical problem from 5 to 3, this degeneracy causing numerical instability in the solution of the equations. In this paper, we seek to close this loophole in our argument, tailoring our analysis specifically to extremal black p-branes, providing a convincing argument for the stability of such objects.

The layout of the paper is as follows. We first review some heuristic arguments,
particularly a thermodynamic entropy argument, as to the stability (or otherwise) of black p-branes. We then set up the formalism involved in solving the problem and argue that the only possible instability is an s-mode. We then examine this s-mode numerically and show there is no solution to the perturbation equations corresponding to an unstable mode. Finally we summarise our results and discuss their implications.

We would first like to examine several arguments in favour of stability for extremal black p-branes. It is generally believed that, being supersymmetric and satisfying an appropriate Bogomolnyi bound, extremal black p-branes should be stable, although formulating an appropriate argument in low energy string gravity would be necessary to prove this. One of the main problems is that the mass appearing in the Bogomolnyi bound is generally the mass per unit p-area, and this is defined by taking the ADM mass of the D-dimensional black hole. Such a procedure relies crucially on the translational invariance of the metric in the p orthogonal directions. Since our instability, when it exists, involves a dependence on these directions, it technically sidesteps the bound. In order to close this route one would have to define a generalised notion of ‘asymptotically flat’, re-derive the ADM mass and positivity theorems, and finally the Bogomolnyi bound for ‘approximately p-planar’ objects.

A second consideration is that certain extremal p-branes are relevant in the context of S-duality - a generalised strong-weak coupling duality which, loosely speaking, swaps solitons and gauge particles, electric and magnetic charges. If a supersymmetric brane were unstable, it would be rather embarrassing for the dual particle! Although this is not a good reason for stability, there is sufficient circumstantial evidence in favour of a limited form of S-duality that it would be most remarkable if we were to discover that the supersymmetric branes were unstable.

Finally, our main reason for expecting stability is a thermodynamic one. One of the most convincing arguments that we should expect an instability of the uncharged black
p-brane was an entropic inequality. We calculated the entropy of a length \( L \) of black string in five dimensions and compared it to that of a hyperspherical black hole. For the string, the entropy was proportional to \( M^2 L \), for the black hole, \( M^{3/2} L^{3/2} \). Regarding \( M \), the mass per unit length of the string as a constant, this showed that the entropy of the hole is greater for \( L \geq \mathcal{O}(M) \), and hence that a series of black holes is preferred to the string. However, such an argument did not include charge and the non-trivial dilaton field that this entails. It is therefore worth re-investigating this argument and generalising it to include charge. We will perform this analysis for general \( N \), the number of spacetime dimensions, and general \( D = N - p \), so as to be able to show the genericity of the argument.

In order to calculate the entropy of the black p-brane, it is most convenient to work in the “Einstein” frame, i.e., to make a conformal transformation so that the gravitational part of the action is in Einstein form

\[
S = \int d^N x \sqrt{-g} R + .... \quad (1.3)
\]

The appropriate conformal factor for \( N \) dimensions is

\[
g_{\text{Einstein}} = e^{-4\phi/(N-2)} g_{\text{string}} \quad (1.4)
\]

where \( \phi \) is the dilaton field. Applying this conformal factor to the metric of a p-brane yields:

\[
ds_{\text{Ein}}^2 = - \left(1 - \left(\frac{r_+}{r}\right)^{D-3}\right) \frac{dt^2}{\left(1 - \left(\frac{r_+}{r}\right)^{D-3}\right) \left(1 - \left(\frac{r_-}{r}\right)^{D-3}\right) \frac{N-4}{2}} + \frac{dr^2}{\left(1 - \left(\frac{r_+}{r}\right)^{D-3}\right) \left(1 - \left(\frac{r_-}{r}\right)^{D-3}\right) \frac{N-2}{2}} + r^2 \left(1 - \left(\frac{r_-}{r}\right)^{D-3}\right) \frac{2}{N-2} d\Omega^2_{D-2} + \left(1 - \left(\frac{r_+}{r}\right)^{D-3}\right) \frac{2}{N-2} dx_i dx^i \quad (1.5)
\]

From this, we read off the entropy using \( S = \frac{1}{4} A \) as

\[
S = \frac{1}{4} A_{D-2} \left(1 - \left(\frac{r_+}{r}\right)^{D-3}\right) \frac{D-2}{N-2} \quad (1.6)
\]
where \[ A_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \] (1.7)

is the area of a unit n-sphere.

In order to compare the entropies of a p-=(N-D)-brane with an N-dimensional black hole, we need the entropy as a function of mass and charge. To find the mass of a D-dimensional black hole, note that as \( r \to \infty \)

\[ |g_{00}| \sim 1 - \frac{16\pi}{(D-2)A_{D-2}} \frac{M_D}{r^{D-3}} \] (1.8)

Reading off the asymptotic form from (1.5) gives

\[ M_D = \frac{(D-2)A_{D-2}}{16\pi} \left[ r_+^{D-3} - \frac{N-4}{N-2} r_-^{D-3} \right] \] (1.9)

Additionally, \( Q_D^2 = \frac{D-3}{2} (r_+) (r_-)^{D-3} \), which gives

\[ r_+^{D-3} = \frac{8\pi M_D}{(D-2)A_{D-2}} + \sqrt{\frac{64\pi^2 M_D^2}{(D-2)^2 A_{D-2}^2} + \frac{2Q_D^2 (N-4)}{(N-2)(D-3)}} \] (1.10)

\[ r_-^{D-3} = \frac{-8\pi M_D (N-2)}{(D-2)(N-4)A_{D-2}} + \sqrt{\frac{64\pi^2 M_D^2 (N-2)^2}{(D-2)^2 (N-4)^2 A_{D-2}^2} + \frac{2Q_D^2 (N-2)}{(N-4)(D-3)}} \]

Thus we input the appropriate form of \( r_\pm \) into \( S_D \) to find the entropy per unit area of the black p-brane, and setting \( D = N \) gives the relevant quantities for the N-dimensional black hole. We then compare the entropies by taking a ‘cube’ of p-brane, side-length \( L \), and setting

\[ M_N = L^{N-D} M_D \]

\[ Q_N = \sqrt{\frac{(N-3)}{(D-3)} \frac{(D-2)}{(N-2)} \frac{A_{D-2}}{A_{N-2}}} L^{N-D} Q_D = \sqrt{2(N-3)} \frac{4\pi c}{A_{N-2}} L^{N-D} M_D \] (1.11)

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The rather unusual factors appearing in the charge formula are to ensure that the extremal limit of the black p-brane corresponds to the extremal limit of the black hole. If this were not the case, then the p-brane would either become stable before reaching its extremal limit (a scenario we have already eliminated), or would be unstable at all length scales at its extremal limit, and in particular at small length scales close to the extremal limit; this too is ruled out by our previous work which showed that the length scale of the instability was tending to infinity as the extremal limit was approached. We therefore assume that these limits coincide, and introduce the constant $c$ as representing a general charge:mass ratio, so that $c \to 1$ as the extremal limit is approached.

We now set $L^{N-D}S_D = S_N$ to find the critical length, $L_*$, at which it becomes entropically favoured to form an N-dimensional black hole. Taking $2\pi/L_*$ gives $\mu_*$, the critical frequency which, after some algebra, is found to be:

$$
\mu_* = \frac{2\pi}{4^{D-3}D-3} \left( \frac{N-4}{2} \right)^{(D-2)(N-3)} \left( \frac{\Gamma(N+1)}{N-2} \right)^{-1} \left( \frac{\Gamma(D+1)}{D-2} \right)^{-\frac{(N-3)}{(D-3)(N-D)}} \left( \frac{2(D-2)}{(N-2)(N-4)} \right)^{\frac{N-3}{N-D}}
\times \left( 1 + \sqrt{1 + c^2(N-2)(N-4)} \right)^{-\frac{(N(D-2)N-3D+7)}{(N-2)(D-3)}} \left( N-3 - \sqrt{1 + c^2(N-2)(N-4)} \right)^{\frac{N-3}{N-2}}
$$

(1.12)

Clearly

$$
\mu_* \propto (N - 3 - \sqrt{1 + c^2(N-2)(N-4)})^{\frac{N-3}{N-2}}
= (N - 3 - \sqrt{1 - c^2 + c^2(N-3)^2})^{\frac{N-3}{N-2}}
$$

(1.13)

$$\to 0 \quad \text{as} \quad c \to 1.$$

Figure 1. shows the numerical and analytical value for $\mu_{max}$ for which the instability just disappears. It is seen that in both cases $\mu_{max}$ goes to zero in the limit of extreme charge. Hence, as the extremal limit is approached, the scale at which it becomes entropically favourable to form a black hole tends to infinity. Thus the entropy argument, which was so convincing for uncharged black holes, actually reinforces the idea that extremal black
holes should be stable. We now turn to setting up the formalism for the stability analysis. Since this has been done in detail in [5], we will merely paraphrase the process here, quoting only the main formulae.

2. The stability analysis.

The low energy string action used by Horowitz and Strominger\(^6\) is

\[
\int d^{10}x \sqrt{-g} e^{-2\phi} [R + 4(\nabla \phi)^2 - \frac{2}{(D - 2)!} F^2] \tag{2.1}
\]

where \(F\) is a \((D - 2)\)-form field strength, and \(\phi\) the dilaton. This has an extremal magnetically charged black p-brane solution:

\[
ds^2 = -dt^2 + \frac{dr^2}{(1 - (\frac{r_+}{r})^{D-3})^2} + r^2 d\Omega^2_{D-2} + dx^i dx^j \delta_{ij} \\
e^{-2\phi} = (1 - (\frac{r_+}{r})^{D-3}) \\
F = \sqrt{\frac{D-3}{2}} r_+^{D-3} \epsilon_{D-2} \tag{2.2}
\]

where the index \(i\) runs from \(D + 1\) to 10 and \(\epsilon_{D-2}\) is the area form of a unit \((D-2)\)-sphere.

In order to write the perturbation equations, we use the usual notation

\[
\delta g_{ab} = h_{ab} , \tag{2.3}
\]

and a conventional gauge choice of transversality

\[
\nabla_a h^{ab} = \nabla_a (h^{ab} - \frac{1}{2} h g^{ab}) = 0 . \tag{2.4}
\]

To perform the stability analysis, we Fourier decompose the perturbations in terms of the symmetries of the background spacetime. We perform our analysis in Schwarzschild coordinates, and transform to the tortoise coordinate at the future event horizon to check
the regularity of the perturbation equation (see [5] for a more detailed discussion of the issue of boundary conditions and the generalised tortoise coordinate). The Fourier modes in the time and p-brane coordinates are of the form \( e^{\Omega t + i\mu_i x^i} \) for an instability. The spherical harmonic modes will depend on the number of dimensions, \( D \), that the black hole sits in, as well as the tensorial nature of the perturbation we are analysing. However, we note that in the uncharged black p-brane, the higher angular momentum mode equations have no unstable solutions, and also that the s-wave instability shrinks in parameter range as we increase the charge (see figure 1.). In other words, adding charge has a stabilising influence. Thus we do not expect higher angular momentum perturbations of charged black holes to exhibit instabilities. Therefore, for the extremal black p-brane, we focus on the s-mode which caused an instability in the lesser-charged branes. This mode takes the form

\[
\delta \phi = e^{\Omega t + i\mu_i x^i} f(r) \quad (2.5a)
\]

\[
\delta F = 0 \quad (2.5b)
\]

\[
h^{ab} = e^{\Omega t + i\mu_i x^i} \begin{bmatrix}
H^{ij}(r) & H^{it}(r) & H^{ir}(r) & 0 & 0 & \ldots \\
H^{ij}(r) & H^{tt}(r) & H^{tr}(r) & 0 & 0 & \ldots \\
H^{rj}(r) & H^{rt}(r) & H^{rr}(r) & 0 & 0 & \ldots \\
0 & 0 & 0 & K(r) & 0 & 0 \\
0 & 0 & 0 & 0 & K(r)/\sin^2 \theta & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots 
\end{bmatrix} \quad (2.5c)
\]

where we have used the non-trivial result from [5] that \( \delta F = 0 \).

Using the gauge conditions, and (2.5b), the perturbation equations in tensorial notation are

\[
\Box \delta \phi - h^{ab} \nabla_a \nabla_b \phi + 2 h^{ab} \nabla_a \phi \nabla_b \phi - 4 \nabla_a \delta \phi \nabla^a \phi
\]

\[
- \frac{h^{cd}}{(D-4)!} F_{ca_2...a_{D-2}} F_{d}^{a_2...a_{D-2}} = 0 \quad (2.6a)
\]

\[
\Box h_{ab} + 2 R_{cabb} h^{cd} - 2 R_{e(a} h_{b)}^e - 4 \nabla_a \nabla_b \delta \phi - 2 \nabla_c \phi \nabla^c h_{ab} + 4 \nabla_c \phi \nabla_b h_{a}^c
\]

\[
- \frac{4}{(D-4)!} h^{cd} F_{aca_3...a_{D-2}} F_{bd}^{a_3...a_{D-2}} = 0 \quad (2.6b)
\]
Inputting the mode decomposition (2.5) yields the full set of equations which were written down in appendix B of [5]. We therefore set $r_+ = r_-$ in these equations, and without loss of generality we set $D=4$, since we demonstrated that the existence of an instability was not dimensionally dependent. In this case, the equations of [5] simplify neatly. An important feature is that the equations for $f, K, h^{rr}$ become independent of all the other variables. It is in this sense that the extremal limit is degenerate, the number of independent physical variables in the perturbation problem reducing from five to three. We can therefore look first at these functions to see if the instability persists. We should also point out that in the extreme case the behaviour of the modes near $r_+$ is different than in the non-extreme case, this is because of the effect of the equality of $r_-$ and $r_+$. We do the analysis for $D = 4$ in 5 dimensions but it can be generalised for higher dimensions and $D$. The equations for these modes are:

\[
-\frac{(r-r_+)^2}{r^2} f'' - \frac{(2r+r_+)(r-r_+)}{r^3} f' - \frac{(r-2r_+)r_+}{r^2} K = 0 \tag{2.7a}
\]

\[
-\frac{(r-r_+)^2}{2r^2} K'' + \frac{(-3r+2r_+)(r-r_+)}{r^3} K' + \frac{2(r-r_+)^2}{r^5} f' = 0 \tag{2.7b}
\]

\[
-\frac{(r-r_+)^2}{2r^2} h^{rr''} + \frac{2(r-r_+)^4}{r^4} f'' - \frac{(r-3r_+)(r-r_+)}{r^3} h^{rr'} + \frac{2(r-r_+)^3r_+}{r^5} f' = 0 \tag{2.7c}
\]

\[
-\frac{2(r-2r_+)(r-r_+)^3}{r^4} K + \left(\frac{\Lambda^2}{2} + \frac{(2r^2 - 6rr_+ + r_+^2)}{r^4}\right) h^{rr} = 0
\]
Using (2.7b,c) we can obtain

\[
\begin{align*}
-\frac{(r-r_+)^2}{2r^2}h_{rr}'' & \quad - \frac{(r-3r_+)(r-r_+)}{r^3}h_{rr}'' - \frac{4(r-r_+)^3}{r^4}f' \\
-\frac{2(r-2r_+)(r-r_+)^2}{r^3}K + \frac{2\Lambda^2(r-r_+)^2}{r^2}f + \left(\frac{\Lambda^2}{2} + \frac{2(r^2-2rr_+-r_+^2)}{r^4}\right)h_{rr} &= 0
\end{align*}
\]

(2.8)

where \( \Lambda^2 = \Omega^2 + \mu^2 \). We can see that these equations depend only on \( \mu \) through \( \Lambda \). If we would find an unstable mode for a given value of \( \Lambda \) this would imply one for \( \mu = 0 \) and \( \Omega \) equal to \( \Lambda \). This would correspond to an instability for a 4-dimensional dilatonic black hole! However as will be shown below, no such instability was found.

For general \( \Lambda \), we cannot directly determine if there exists a regular unstable solution. We must therefore resort to numerical techniques. These involve finding the regular solution space at infinity, which forms a three dimensional subspace of the total solution space. We then integrate in, adjusting the initial configuration to see if one matches to a regular solution near the horizon. In practise, as described in [5], we will search for evidence of such a match, rather than the match itself.

As \( r \to \infty \) we can expand the regular solution as

\[
\begin{align*}
f & \to f_ie^{\Lambda r} \\
K & \to K_ie^{\Lambda r} \\
h_{rr} & \to H_{rr}^ie^{\Lambda r}
\end{align*}
\]

(2.9)

where \( f_i, K_i, H_{rr}^i \) are arbitrary constants. In fact one of them can be set to 1 due to the linearity of the equation.

In the limit as \( r \to r_+ \) the behaviour of the field is rather different than in the non-
extreme case. We assume the form

\[ f = \bar{f}(r - r_+)^\alpha \]

\[ K = \bar{K}(r - r_+)^\beta \]

\[ h^{rr} = h^{rr}(r - r_+)^\gamma \]

and find that

\[ f = A_+(r - r_+)^{\alpha_1^+} + A_-(r - r_+)^{\alpha_1^-} + B_+(r - r_+)^{\alpha_2^+} \]

\[ B_-(r - r_+)^{\alpha_2^-} + C_+(r - r_+)^{\alpha_3^+} + C_-(r - r_+)^{\alpha_3^-} \]

\[ h^{rr} = -\frac{2\alpha_1^+}{(\alpha_1^+ - 1)r_+^2} A_+(r - r_+)^{\alpha_1^+ + 2} - \frac{2\alpha_1^-}{(\alpha_1^- - 1)r_+^2} A_-(r - r_+)^{\alpha_1^- + 2} - \frac{4\alpha_2^+}{r_+^2} B_+(r - r_+)^{\alpha_2^+ + 2} \]

\[ - \frac{4\alpha_2^-}{r_+^2} B_-(r - r_+)^{\alpha_2^- + 2} - \frac{2\alpha_3^+}{r_+^2} C_+(r - r_+)^{\alpha_3^+ + 2} - \frac{2\alpha_3^-}{r_+^2} C_-(r - r_+)^{\alpha_3^- + 2} \]

\[ K = \frac{(\alpha_1^+ - 1)}{(\alpha_1^+ - 2)} A_+(r - r_+)^{\alpha_1^+} + \frac{(\alpha_1^- - 1)}{(\alpha_1^- - 2)} A_-(r - r_+)^{\alpha_1^-} - \frac{4\alpha_3^+}{r_+^3} B_+(r - r_+)^{\alpha_3^+ + 1} \]

\[ - \frac{4\alpha_3^-}{r_+^3} B_-(r - r_+)^{\alpha_3^- + 1} - \frac{2}{r_+^3} C_+(r - r_+)^{\alpha_3^+ + 1} - \frac{2}{r_+^3} C_-(r - r_+)^{\alpha_3^- + 1} \]

for which

\[ \alpha_1^\pm = \alpha = \beta = \gamma - 2 = -\frac{1}{2} \pm \frac{3}{2} \sqrt{1 + \frac{4\Lambda^2 r_+^2}{9}} \]  \hspace{1cm} (2.12a)

\[ \alpha_2^\pm = \alpha = \beta - 1 = \gamma - 2 = \pm \Lambda r_+ \]  \hspace{1cm} (2.12b)

\[ \alpha_3^\pm = \alpha = \beta - 1 = \gamma - 2 = -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 + 4\Lambda^2 r_+^2} \]  \hspace{1cm} (2.12c)

where \( A_+, A_-, B_+, B_-, C_+, C_- \) are constants determined by the integrated function with large \( r \) limit specified in equation (2.9). Once a solution has been found by integration we
can find $A_+, A_-, B_+, B_-, C_+, C_-$ from the functions $f, K, h^{rr}$ and their derivatives. By varying the initial conditions we vary the ratio $A_-/A_+$ etc, until these ratios go to zero. That would correspond to a regular solution. We have used a Newton-Raphson method to investigate this problem and we were not able to find any regular solution. Therefore we should take $f, K, h^{rr}$ to zero.

The next step is to look at the possibility of an instability due to the other modes. Taking into account the previous result the equation for $h^{tr}$ becomes

$$\frac{2}{r^2}h^{tr''}_r - \frac{5r^2}{r^3}h^{tr'} + \frac{2r^2 - 6rr_+ + 3r^2}{r^4}h^{tr} = 0$$

(2.13)

For $\Lambda r_+ > 1/4$ it is possible to show analytically that there is no regular solution to this equation but for smaller values we have to resort to a numerical analysis similar to the one above. We did not find any regular solution to equation (2.13). The equation for the mode $h^{rz}$ is similar to that for $h^{tr}$. We must therefore take both $h^{rz}$ and $h^{tr}$ to be zero. With this result we can finally show that $h^{tz}, h^{tt}$ and $h^{zz}$ must also be zero as their equations are of the form

$$\frac{2}{r^2}h^{tz''}_r - g(r)h^{tz'} + \Lambda^2 h^{tz}r = 0$$

(2.14)

where $g(r)$ is an unimportant function of $r$ which can be computed from eq (B.11e) of [5].

These equations do not possess regular solutions. Thus we have shown that the irregular modes which were found in [4] and [5] do not have a counterpart for the extreme case.

3. Conclusions.

In this paper we have shown that the s-mode leading to the instability of the uncharged and charged p-branes does not extend to the extreme case. We have made a detailed search for such a mode and we were not able to find one. This absence of an instability confirms our expectations based on an extrapolation of the charged p-branes studied in [5] and on the entropy argument presented earlier. Black p-branes therefore behave in a
thermodynamically consistent manner. We should stress that we have not proved stability, for we have neither studied higher angular momentum modes, nor exhibited completeness of our basis. Nonetheless, the fact that the s-mode which gave an unexpected instability for non- and lesser-charged branes does not give an instability for the extreme case, and also that adding charge and considering higher angular momentum modes has a stabilising effect, we regard as compelling evidence for the overall stability of extremal black p-branes.

Using this result, it is possible to argue that charge might prevent p-branes from breaking. When the instability sets in in the non-extreme case, the apparent horizon expands in some regions and shrinks in others, but the charge per unit length is unchanged, at least to linear order. Thus it seems possible that these regions will evolve towards extremality and would therefore be stabilised. In this case the endpoint would appear to be a rippled brane, and we are investigating the possibility of such static solutions. However, the chargeless case does not appear to have such a mechanism and we still believe that these p-branes will break and reveal, at least temporarily, their singularity.

Acknowledgments.

We would like to thank G.W.Gibbons, J.A.Harvey, G.T.Horowitz and P.K.Townsend for useful conversations. R.G. is supported by P.P.A.R.C. R.L. thanks Los Alamos National Laboratory for support.

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Figure 1. The critical frequency at which the hyperspherical black hole is entropically preferred to the black string. The dotted line corresponds to the numerical analysis done in [5] and the full one to the analytical argument presented here. The apparent crossing of the two curves in the vicinity of $r_- = r_+$ is probably due to numerical sensitivity. In both cases the instability does not seem to extend to the extreme case ($r_- = r_+$).