Low-temperature expansion and perturbation theory in 2D models with unbroken symmetry: a new approach

O. Borisenko, V. Kushnir, A. Velytsky

N.N.Bogolyubov Institute for Theoretical Physics, National Academy of Sciences of Ukraine, 252143 Kiev, Ukraine

Abstract

A new method of constructing a weak coupling expansion of two dimensional (2D) models with an unbroken continuous symmetry is developed. The method is based on an analogy with the abelian XY model, respects the Mermin-Wagner (MW) theorem and uses a link representation of the partition and correlation functions. An expansion of the free energy and of the correlation functions at small temperatures is performed and first order coefficients are calculated explicitly. They are shown to coincide with the results of the conventional perturbation theory. We discuss an applicability of our method to analysis of uniformity of the low-temperature expansion.

1 Introduction

Since two-dimensional models with a continuous global symmetry were recognized to be asymptotically free [1] they became a famous laboratory for testing many ideas and methods before applying them to more complicated gauge theories. In this paper we follow this common way and present a method, different from the conventional perturbation theory (PT), which allows to investigate these models in the weak coupling region. The conventional PT is one of the main technical tools of the modern physics. In spite of the belief of the most of community that this method gives the correct asymptotic expansion of such theories as 4D QCD or 2D spin systems with continuous global symmetry, the recent discussion of this problem [2]-[6] has shown that it is rather far from unambiguous solution. Indeed, for the PT to be applicable it is necessary that a system under consideration would possess a well ordered ground state. In two dimensional models, like the O(N)-sigma models the MW theorem guarantees the absence of such a state in the thermodynamic limit (TL) however small coupling constant is [7]. Then, the usual argument in support of the PT is that locally the system

1email: oleg@ap3.bitp.kiev.ua
2email: kushnir@ap3.bitp.kiev.ua
3email: vel@ap3.bitp.kiev.ua
is ordered and the PT is not supposed to be used for the calculation of long-distance observables. On the other hand it should reproduce the correct behaviour of the fixed-distance correlations as well as all thermodynamical functions which can be expressed via short-range observables. The example of 1D models shows that even this is not always true [8], so why should one believe in correctness of the conventional PT in 2D?

In fact, the only way to justify the PT is to prove that it gives the correct asymptotic expansion of nonperturbatively defined models in the TL. Now, it was shown in [2] that PT results in 2D nonabelian models depend on the boundary conditions (BC) used to reach the TL. This result could potentially imply that the low-temperature limit and the TL do not commute in nonabelian models. Actually, the main argumentation of [2] regarding the failure of the PT expansion is based on the fact that the conventional PT is an expansion around a broken vacuum, i.e. the state which simply does not exist in the TL of 2D models. According to [2], the ground state of these systems can be described through special configurations – the so-called gas of superinstantons (SI) and the correct expansion should take into account these saddle points. At the present stage it is rather unclear how one could construct an expansion in the SI background. Fortunately, there exists other, more eligible way to construct the low-temperature expansion which respects the MW theorem and is apriori not the expansion around the broken vacuum. We develop this method on an example of the 2D SU(N) × SU(N) principal chiral model whose partition function (PF) is given by

\[ Z = \int \prod_x DU_x \exp \left[ \beta \sum_{x,n} \text{Re} \, \text{Tr} U_x U_x^+_{x+n} \right], \tag{1} \]

where \( U_x \in SU(N) \), \( DU_x \) is the invariant measure and we impose the periodic BC. The basic idea is the following. As was rigorously proven, the conventional PT gives an asymptotic expansion which is uniform in the volume for the abelian XY model [9]. One of the basic theorems which underlies the proof states that the following inequality holds in the 3D XY model

\[ < \exp(\sqrt{\beta A(\phi_x)}) > \leq C, \tag{2} \]

where \( C \) is \( \beta \)-independent and \( A(\phi + 2\pi) = \phi \). \( \phi_x \) is an angle parametrizing the action of the XY model \( S = \sum_{x,n} \cos(\phi_x - \phi_{x+n}) \). It follows that at large \( \beta \) the Gibbs measure is concentrated around \( \phi_x \approx 0 \) providing a possibility to construct an expansion around \( \phi_x = 0 \). This inequality is not true in 2D in the thermodynamic limit because of the MW theorem, however the authors of [2] prove the same inequality for the link angle, i.e.

\[ < \exp(\sqrt{\beta A(\phi_l)}) > \leq C, \quad \phi_l = \phi_x - \phi_{x+n}, \tag{3} \]

where the expectation value refers to infinite volume limit. Thus, in 2D the Gibbs measure at large \( \beta \) is concentrated around \( \phi_l \approx 0 \) and the asymptotic series can be constructed expanding the action in powers of \( \phi_l \). In the abelian case such an expansion is equivalent to the expansion around \( \phi_x = 0 \) because i) the action depends only on the difference \( \phi_x - \phi_{x+n} \) and ii) the integration measure is flat, \( DU_x = d\phi_x \).
In 2D nonabelian models, again because of the MW theorem, one has to expect in the TL something alike to (3), namely
\[
< \exp(\sqrt{\beta} \arg A(\text{Tr}V)) > \leq C, \quad V_i = U_x U_{x+n}^+.
\] (4)

Despite there is not a rigorous proof of (4), that such (or similar) inequality holds in 2D nonabelian models is intuitively clear and should follow from the chessboard estimates \[10\] and from the Dobrushin-Shlosman proof of the MW theorem \[7\] which shows that spin configurations are distributed uniformly in the group space in the TL. Namely, the probability \( p(\xi) \) that \( \text{Tr}(V_I - I) \leq -\xi \) is bounded by
\[
p(\xi) \leq O(e^{-b\beta\xi}), \quad \beta \to \infty,
\] (5)
if the volume is sufficiently large, \( b \) is a constant. Thus, until \( \xi \leq O((\sqrt{\beta})^{-1}) \) is not satisfied, all configurations are exponentially suppressed. This is equivalent to the statement that the Gibbs measure at large \( \beta \) is concentrated around \( V_I \approx I \), therefore (4) or its analog holds. In what follows it is assumed that (4) is correct, hence \( V_I = I \) is the only saddle point for the invariant integrals\[4\]. Thus, the correct asymptotic expansion, if exists, should be given via an expansion around \( V_I = I \), similarly to the abelian model. If the conventional PT gives the correct asymptotics, it must reproduce the series obtained expanding around \( V_I = I \). However, neither i) nor ii) holds in the nonabelian models, therefore it is far from obvious that two expansions indeed coincide.

Let us parametrize \( V_I = \exp(i\omega_I) \) and \( U_x = \exp(i\omega_x) \). Consider the following expansion
\[
V_I = \exp[i\omega_I] \sim I + \sum_{n=1} \frac{1}{(\beta)^{n/2}} \frac{(i\omega_I)^n}{n!}.
\] (6)

The standard PT states that to calculate the asymptotic expansion one has to re-expand this series as
\[
\omega_I = \omega_x - \omega_{x+n} + \sum_{k=1} \frac{1}{(\beta)^{k/2}} \omega^{(k)}_I,
\] (7)
where \( \omega^{(k)}_I \) are to be calculated from the definition \( U_x U_{x+n}^+ = \exp(i\omega_I) \). This is presumably true in a finite volume where one can fix appropriate BC (like the Dirichlet ones), or to break down the global symmetry by fixing the global gauge (on the periodic lattice). Then, making \( \beta \) sufficiently large one forces all the spin matrices to fluctuate around \( U_x \approx I \), therefore the substitution (7) is justified. We do not see how this procedure could be justified when the volume increases and fluctuations of \( U_x \) spread up over the whole group space. In other words, it is not clear why in the series
\[
\omega_I = \omega_x - \omega_{x+n} + \sum_{k=1} \omega^{(k)}_I
\]
\[4\] It follows already from (3). What is important in (4) is a factor \( \sqrt{\beta} \), otherwise the very possibility of the expansion in \( 1/\beta \) becomes problematic.
the term $\omega_{l+1}^{(k+1)}$ is suppressed as $\beta^{-1/2}$ relatively to the term $\omega_l^{(k)}$. It is only which remains correct in the large volume limit and takes into account all the fluctuations contributing at a given order of the low-temperature expansion.

It is a purpose of the present paper to develop an expansion around $V_l = I$ aiming to calculate the asymptotic series for nonabelian models. First of all, one has to give a precise mathematical meaning to the expansion (3). It is done in the next section.

2 Link representation for the partition and correlation functions

To construct an expansion of the Gibbs measure and the correlation functions using (3) we use the so-called link representation for the partition and correlation functions. First, we make a change of variables $V_l = U_x U_{x+n}^+$ in (9). PF becomes

$$Z = \int \prod_l dV_l \exp \left[ \beta \sum_l \text{Re} \, \text{Tr} V_l + \ln J(V) \right],$$

where the Jacobian $J(V)$ is given by

$$J(V) = \int \prod_x dU_x \prod_l \left[ \sum_r d_r \chi_r \left( V_l U_x U_{x+n}^+ \right) \right] = \prod_p \left[ \sum_r d_r \chi_r \left( \prod_{l \in p} V_l \right) \right].$$

$\prod_p$ is a product over all plaquettes of 2D lattice, the sum over $r$ is sum over all representations of $SU(N)$, $d_r = \chi_r(I)$ is the dimension of $r$-th representation. The $SU(N)$ character $\chi_r$ depends on a product of the link matrices $V_l$ along a closed path (plaquette in our case):

$$\prod_{l \in p} V_l = V_n(x)V_m(x+n)V_n^+(x+m)V_m^+(x).$$

The expression $\sum_r d_r \chi_r(\prod_{l \in p} V_l)$ is the $SU(N)$ delta-function which reflects the fact that the product of $U_x U_{x+n}^+$ around plaquette equals $I$ (original model has $L^2$ degrees of freedom, $L^2$ is a number of sites; since a number of links on the 2D periodic lattice is $2L^2$, the Jacobian must generate $L^2$ constraints). The solution of the constraint

$$\prod_{l \in p} V_l = I$$

is a pure gauge $V_l = U_x U_{x+n}^+$, so that two forms of the PF are exactly equivalent.

The corresponding representation for the abelian XY model reads

$$Z_{XY} = \int \prod_l d\phi_l \exp \left[ \beta \sum_l \cos \phi_l \right] \prod_p J_p ,$$

5Strictly speaking, on the periodic lattice one has to constraint two holonomy operators, i.e. closed paths winding around the whole lattice. We do not expect such global constraints to influence the TL in 2D (see Discussion).
where the Jacobian is given by the periodic delta-function

\[
J_p = \sum_{r=-\infty}^{\infty} e^{i r \phi_p}, \quad \phi_p = \phi_n(x) + \phi_m(x+n) - \phi_n(x+m) - \phi_m(x+n). \tag{13}
\]

Consider two-point correlation function

\[
\Gamma(x, y) = \langle \text{Tr} \, U_x U_y^+ \rangle, \tag{14}
\]

where the expectation value refers to the ensemble defined in (1). Let \(C_{xy}\) be some path connecting points \(x\) and \(y\). Inserting the unity \(U_z U_z^+\) in every site \(z \in C_{xy}\) one gets

\[
\Gamma(x, y) = \langle \text{Tr} \prod_{l \in C_{xy}} (U_x U_{x+n}^+) \rangle = \langle \text{Tr} \prod_{l \in C_{xy}} W_l \rangle, \tag{15}
\]

where \(W_l = V_l\) if along the path \(C_{xy}\) the link \(l\) goes in the positive direction and \(W_l = V_l^+\), otherwise. The expectation value in (13) refers now to the ensemble defined in (13). Obviously, it does not depend on the path \(C_{xy}\) which can be deformed for example to the shortest path between sites \(x\) and \(y\).

In this representation the series (13) acquires a well defined meaning, therefore the expansion of the action, of the invariant measure, etc. can be done.

3 \textbf{XY model: Weak coupling expansion of the free energy}

In this section we prove that for the abelian XY model the large-\(\beta\) expansion in the link representation gives the same results as the conventional PT in the thermodynamic limit. We consider only the free energy but the generalization for the correlation functions is straightforward.

The first step is a standard one, i.e. we rescale \(\phi \to \frac{\phi}{\sqrt{\beta}}\) and make an expansion

\[
\exp \left[ \beta \cos \frac{\phi}{\sqrt{\beta}} \right] = \exp \left[ \beta - \frac{1}{2} (\phi)^2 \right] \left[ 1 + \sum_{k=1}^{\infty} (\beta)^{-k} \sum_{l_1, \ldots, l_k} \frac{a_{l_1} \ldots a_{l_k}}{l_1! \ldots l_k!} \right], \tag{16}
\]

where \(l_1 + 2l_2 + \ldots + kl_k = k\) and

\[
a_k = (-1)^{k+1} \frac{\phi^{2(k+1)}}{(2k + 2)!}. \tag{17}
\]

In addition to this perturbation one has to extend the integration region to infinity. We do not treat this second perturbation, as usually supposing that all the corrections from this perturbation go down exponentially with \(\beta\) (in the abelian case it can be proven rigorously \[13\]). It is more convenient now to go to a dual lattice identifying
plaquettes of the original lattice with its center, i.e. \( p \to x \). Let \( l = (x; n) \) be a link on the dual lattice. Introducing sources \( h_l \) for the link field, one then finds

\[
Z_{XY}(\beta >> 1) = e^{\beta DL^2 - L^2 \ln \beta} \prod_l \left[ 1 + \sum_{k=1}^{\infty} \frac{1}{\beta^k} A_k \left( \frac{\partial^2}{\partial h_l^2} \right) \right] M(h_l) .
\] (18)

Coefficients \( A_k \) are defined in (16) and (17). The generating functional \( M(h_l) \) is given by

\[
M(h_l) = \sum_{r_x=-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi_l \exp \left[ -\frac{1}{2} \sum_l \phi_l^2 + i \sum_l \frac{\phi_l}{\sqrt{\beta}} (r_x - r_{x+n}) + \sum_l \phi_l h_l \right] .
\] (19)

Sums over representations are treated by the Poisson resummation formula. The Gaussian ensemble appears in terms of the fluctuations of \( r \)-fields. The integral over zero mode of \( r \)-field is not Gaussian and leads to a delta-function in the Poisson formula. Thus, the zero mode decouples from the expansion. All the corrections to the integrals over representations in the Poisson formula fall down exponentially with \( \beta \) so that the generating functional becomes

\[
M(h_l) = \exp \left[ \frac{1}{4} \sum_{l,l'} h_l G_{ll'} h_{l'} \right] ,
\] (20)

where we have introduced the following function which we term “link” Green function

\[
G_{ll'} = 2\delta_{l,l'} - G_{x,x'} - G_{x+n,x'+n'} + G_{x,x'+n} + G_{x+n,x'} .
\] (21)

\( G_{x,x'} \) is a “standard” Green function on the periodic lattice

\[
G_{x,x'} = \frac{1}{L^2} \sum_{k_n=0}^{L-1} \frac{e^{\frac{2\pi i}{L} k_n (x-x')}}{D - \sum_{n=1}^{D} \cos \frac{2\pi}{L} k_n} , \quad k_n^2 \neq 0 .
\] (22)

Normalization is such that \( G_{ll} = 1 \). As far as we could check, the expression (18) reproduces the well known asymptotics of the free energy of the \( XY \) model in two dimensions. For example, the first order coefficient of the free energy being expressed in \( G_{ll'} \) reads

\[
C_1 = \frac{1}{64L^2} \sum_l G_{ll}^2 = \frac{1}{32} .
\] (23)

Let us add some comments. In the standard expansion to avoid the zero mode problem one has to fix appropriate BC, like Dirichlet ones or to fix a global gauge if one works on the periodic lattice [12]. In the present scheme the zero mode decouples automatically due to using \( U(1) \) delta-function which takes into account the periodicity of the integrand in link angles. More important observation is that it is allowed to take the TL already in the formula (18): since the generating functional depends only on the link Green function which is infrared finite, the uniformity of the expansion in the volume follows immediately. This is a direct consequence of the fact that the Gibbs measure of the \( XY \) model is a function of the gradient \( \phi_l \) only.
4 Weak coupling expansion in the $SU(2)$ model

We turn now to the nonabelian models. As the simplest example we analyze the $SU(2)$ principal chiral model. The method developed here has straightforward generalization to arbitrary $SU(N)$ or $SO(N)$ model and we shall present it elsewhere.

4.1 Representation for the partition function

We take the standard form for the $SU(2)$ link matrix which is the most suitable for the weak coupling expansion

$$V_l = \exp[i\sigma^k\omega_k(l)],$$

where $\sigma^k, k = 1, 2, 3$ are Pauli matrices. Let us define

$$W_l = \left[\sum_k \omega^2_k(l)\right]^{1/2},$$

and similarly

$$W_p = \left[\sum_k \omega^2_k(p)\right]^{1/2},$$

where $\omega_k(p)$ is a plaquette angle defined as

$$V_p = \prod_{l \in p} V_l = \exp\left[i\sigma^k\omega_k(p)\right].$$

The exact relation between link angles $\omega_k(l)$ and the plaquette angle $\omega_k(p)$ is given in the Appendix B. Then, the partition function (28) can be exactly rewritten to the following form appropriate for the weak coupling expansion

$$Z = \int \prod_l \left[\frac{\sin^2 W_l}{W^2_l} \prod_k d\omega_k(l)\right] \exp\left[2\beta \sum_l \cos W_l\right] \prod_x \frac{W_x}{\sin W_x} \prod_{x, m(x) = -\infty} \int \prod_k d\alpha_k(x) \exp\left[-i \sum_k \alpha_k(x)\omega_k(x) + 2\pi i m(x)\alpha(x)\right],$$

where we have introduced auxiliary field $\alpha_k(x)$ and

$$\alpha(x) = \left[\sum_k \alpha^2_k(x)\right]^{1/2}.$$
4.2 General expansion

To perform the weak coupling expansion we proceed in a standard way, i.e. first we make the substitution

$$\omega_k(l) \rightarrow (2\beta)^{-1/2}\omega_k(l), \quad \alpha_k(x) \rightarrow (2\beta)^{1/2}\alpha_k(x)$$

and then expand the integrand of (28) in powers of fluctuations of the link fields. We would like to give here some technical details of the expansion which could be useful for a future use. It is straightforward to get the following power series:

1. Action

$$\exp \left[ 2\beta \cos \frac{W_l}{\sqrt{2\beta}} \right] = \exp \left[ 2\beta - \frac{1}{2}(W_l)^2 \right] \left[ 1 + \sum_{k=1}^{\infty} (2\beta)^{-k} \sum_{l_1,\ldots,l_k} a_{l_1}^{l_1} \cdots a_{l_k}^{l_k} \right]$$

where \( l_1 + 2l_2 + \ldots + kl_k = k \) and

$$a_k = (-1)^{k+1} \frac{W_l^{2(k+1)}}{(2k+2)!}.$$  

2. Invariant measure

$$\frac{\sin^2 W_l}{W_l^2} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2\beta)^k} C_k W_l^{2k}, \quad C_k = \sum_{n=0}^{k} \frac{1}{(2n+1)(2k-2n+1)!}.$$  

3. Contribution from Jacobian I

$$\frac{W_x}{\sin W_x} = 1 + \sum_{k=1}^{\infty} \frac{J_k}{(2\beta)^k} W_x^{2k}, \quad J_k = \frac{2^{2k-1}}{(2k)!} \mid B_{2k} \mid,$$

where \( B_{2k} \) are Bernoulli numbers.

4. Contribution from Jacobian II

$$\alpha_k(x) \omega_k(x) = \alpha_k(x) \left[ \omega_k^{(0)}(x) + \sum_{n=1}^{\infty} \frac{\omega_k^{(n)}(x)}{(2\beta)^{n/2}} \right],$$

$$\exp \left[ -i \sum_k \alpha_k(x) \omega_k(x) \right] = \exp \left[ -i \sum_k \alpha_k(x) \omega_k^{(0)}(x) \right] \left[ 1 + \sum_{q=1}^{\infty} \frac{(-i)^q}{q!} \left( \sum_k \alpha_k(x) \sum_{n=1}^{\infty} \frac{\omega_k^{(n)}(x)}{(2\beta)^{n/2}} \right)^q \right].$$
Using the relations (75)-(80) one can calculate \( \omega^{(n)}_k(x) \) up to an arbitrary order in \( n \). In particular, \( \omega^{(0)}_k(x) \) is given by (see Fig.1 for our notations of dual links)

\[
\omega^{(0)}_k(x) = \omega_k(l_3) + \omega_k(l_4) - \omega_k(l_1) - \omega_k(l_2) .
\]

We shall use an obvious property

\[
\sum_x \alpha_k(x) \omega^{(0)}_k(x) = \sum_l \omega_k(l) [\alpha_k(x + n) - \alpha_k(x)] , \quad l = (x; n) .
\]

Introducing now the external sources \( h_k(l) \) coupled to the link field \( \omega_k(l) \) and \( s_k(x) \) coupled to the auxiliary field \( \alpha_k(x) \) and adjusting the definitions

\[
\omega_k(l) \rightarrow \frac{\partial}{\partial h_k(l)} , \quad \alpha_k(x) \rightarrow \frac{\partial}{\partial s_k(x)} ,
\]

we get finally the following formal weak coupling expansion for the PF (28)

\[
Z = C(2\beta) Z(0, 0) \prod_k \left[ \left( 1 + \sum_{k=1}^{\infty} (2\beta)^{-k} \sum_{l_1, \ldots, l_k} \frac{a_{l_1} \ldots a_{l_k}}{l_1! \ldots l_k!} \right) \left( 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2\beta)^k} C_k W^{2k}_x \right) \right] \prod_k \left[ \left( 1 + \sum_{k=1}^{\infty} \frac{J_k}{(2\beta)^k} W^{2k}_x \right) \left( 1 + \sum_{q=1}^{\infty} \frac{(-i)^q}{q!} \left( \sum_k \omega_k^{(n)}(x) \frac{\omega_k^{(n)}(x)}{(2\beta)^n/2} \right)^{q} \right) \right] M(h, s) ,
\]

where

\[
C(\beta) = \exp \left[ 2\beta L^2 - \frac{3}{2} L^2 \ln \beta \right] .
\]

As usually, one has to put \( h_k = s_k = 0 \) after taking all the derivatives. \( M(h, s) \) is a generating functional which we study in the next subsection. It is obvious that the ground state satisfies

\[
< (\omega^{(0)}_k(x))^p > = 0 , \quad p = 1, 2, \ldots ,
\]

precisely like the abelian model.

### 4.3 Generating functional and zero modes

Here we are going to study the generating functional \( M(h, s) \) given by

\[
M(h, s) = \frac{Z(h, s)}{Z(0, 0)} ,
\]

and

\[
Z(h, s) = \int_{-\infty}^{\infty} \prod_{x, k} d\alpha_k(x) \int_{-\infty}^{\infty} \prod_{l, k} d\omega_k(l) \exp \left[ -\frac{1}{2} \omega_k^2(l) - i\omega_k(l) [\alpha_k(x + n) - \alpha_k(x)] \right] \sum_{m(x) = -\infty}^{\infty} \exp \left[ 2\pi i \sqrt{2\beta} \sum_{x} m(x) \alpha(x) + \sum_{l, k} \omega_k(l) h_k(l) + \sum_{x, k} \alpha_k(x) s_k(x) \right] .
\]

9
As in the abelian case we expect that integrals over zero modes of the auxiliary field are not Gaussian and should lead to some constraint on the sums over \( m_x \). To see this, we put \( h_k = s_k = 0 \) and integrate out the link fields. Partition function becomes

\[
Z(0, 0) = \sum_{m(x) = -\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{x, k} d\alpha_k(x) \exp \left[ -\alpha_k(x)G_{x,x',x}^{-1}\alpha_k(x') + 2\pi i \sqrt{2\beta m(x)\alpha(x)} \right], \quad (44)
\]

with \( G_{x,x'} \) given in (22) and sum over repeating indices is understood here and in what follows. It is clear that in this case the zero modes should be controlled via integration over the radial component of the vector \( \vec{\alpha}(x) \). To see this, we change to the spherical coordinates and treat only a constant mode in the angle variables (since the zero mode problem could only arise from this configuration). One has

\[
Z(0, 0) \sim \sum_{m_x = -\infty}^{\infty} \int_0^{\infty} \prod_x \alpha_x^2 d\alpha_x \exp \left[ -\alpha_x G_{x,x}^{-1}\alpha_{x'} + 2\pi i \sqrt{2\beta m_x\alpha_x} \right]
\]

\[
\sim \sum_{m_x = -\infty}^{\infty} \delta \left( \sum_x m_x \right) \exp \left[ -2\pi^2 \beta \sum_{x,x'} m_x G_{x,x'} m_{x'} \right] + O(m_x^2), \quad (45)
\]

and we used the notation \( \alpha_x \) for the radial component of the vector \( \vec{\alpha}(x) \). Since the zero mode of the radial component in the \( x \) space is

\[
\alpha(p = 0) = \left( \frac{1}{L D} \sum_k (\sum_x \alpha_k(x))^2 \right)^{1/2}
\]

one has to omit the zero mode from the Green function in each term of the sum over \( k \) in the integrand of (44). As in the abelian case the integration over zero modes produces delta-function in (45). Since however only \( m_x = 0 \) for all \( x \) contribute to the asymptotics of the free energy and fixed distance correlation function (other values of \( m_x \) being exponentially suppressed) we have to put \( m_x = 0 \) omitting at the same time all zero modes. Calculating resulting Gaussian integrals we come to

\[
M(h, s) = \exp \left[ \frac{1}{4} s_k(x)G_{x,x'} s_k(x') + \frac{i}{2} s_k(x)D_l(x)h_k(l) + \frac{1}{4} h_k(l)G_{ll'} h_k(l') \right], \quad (46)
\]

where \( G_{ll'} \) was introduced in (21) and

\[
D_l(x') = G_{x,x'} - G_{x+n,x'}, \quad l = (x, n). \quad (47)
\]

From (46) one can deduce the following simple rules

\[
< \omega_k(l)\omega_n(l') > = \frac{\delta_{kn}}{2} G_{ll'}, \quad < \alpha_k(x)\alpha_n(x') > = \frac{\delta_{kn}}{2} G_{xx'},
\]

\[
-\frac{i}{\delta_{kn}} < \omega_k(l)\alpha_n(x') > = \frac{\delta_{kn}}{2} D_l(x'). \quad (48)
\]

We describe some simple properties of the functions \( G_{ll'} \) and \( D_l(x') \) in the Appendix C. The expansion (40), representation (46) for the generating functional and rules (48)
are main formulas of this section which allow to calculate the weak coupling expansion of both the free energy and any short-distance observable. Let us now comment on the infrared finiteness of the expansion. It follows from the representation for the generating functional that all expectation values of the link fields are expressed only via the link Green functions $G_{ll'}$ and $D_l(x)$ which are infrared finite by construction. All combinations of auxiliary fields which contain odd overall powers of the fields are expressed only via $D_l(x)$ and, therefore are also infrared finite. However, even powers include $G_{xx'}$ and the infrared finiteness is not provided automatically. In particular, it means that unlike $XY$ model we are not allowed to take the TL at this stage.

Our last comment concerns the partition function (44). We believe it can be regarded as an analog of the corresponding expression in the $XY$ model, i.e. this is a nonabelian analog of the so-called “spin-wave–vortex” representation for the partition function. One can see that the nonabelian model is not factorized into three abelian components and is periodic in the length of the vector $\vec{\alpha}(x)$ rather than in its components $\alpha_k(x)$.

### 4.4 First order coefficient of the correlation function

As the simplest example we would like to calculate the first order coefficient of the correlation function (15). Expanding (15) in $1/\beta$ one has

$$\Gamma(x, y) = 1 - \frac{1}{4 \beta} < \sum_{k=1}^3 \left( \sum_l \omega_k(l) \right)^2 > + O(\beta^{-2}) = 1 - \frac{3}{8 \beta} \sum_{l, l' \in C_{xy}} G_{ll'} + O(\beta^{-2}), \quad (49)$$

where $C_{x'y'}$ is a path dual to the path $C_{xy}$, i.e. consisting of the dual links which are orthogonal to the original links $l, l' \in C_{xy}$. The form of $G_{ll'}$ ensures independence of $\Gamma(x, y)$ of a choice of the path $C_{xy}$. After some algebra it is easy to get the result

$$\Gamma(x, y) = 1 - \frac{3}{4 \beta} D(x - y), \quad D(x) = \frac{1}{L^2} \sum_{k_n=0}^{L-1} \frac{1 - e^{2\pi i k_n x}}{D - \sum_{n=1}^D \cos \frac{2\pi k_n}{L}}, \quad k_n^2 \neq 0, \quad (50)$$

which coincides with the result of the conventional PT.

### 5 First order coefficient of the free energy

The main result of our study is the first order coefficient of the $SU(2)$ free energy

$$F = \frac{1}{2L^2} \ln Z = 2\beta - \frac{3}{4} \ln \beta + \frac{1}{2\beta L^2} C^1 + O(\beta^{-2}). \quad (51)$$

There are four contributions at this order to $C^1$

$$C^1 = C^1_{ac} + C^1_{meas} + C^1_{J_1} + C^1_{J_2}. \quad (52)$$
Contribution from the action (31) is given by
\[
\frac{1}{2L^2} C_{ac}^1 = \frac{5}{128L^2} \sum_l G^2_{ll} = \frac{5}{64} .
\]
Contribution from the measure (33) is given by
\[
\frac{1}{2L^2} C_{\text{meas}}^1 = -\frac{1}{8L^2} \sum_l G_{ll} = -\frac{1}{4} .
\]

Contribution from third brackets in (31) is proportional to \([\omega_k^{(0)}]^2\) and equals 0, because of (31). There are two contributions from the expansion of the Jacobian (31). The first one is given by the expectation value of the operator \(-i \sum_x \sum_{k=1}^3 \alpha_k(x) \omega_k^{(2)}(x)\). \(\omega_k^{(2)}(x)\) is given in the Appendix B (83). From the form of the generating functional (40) one can see that the expectation value of this operator depends only on link Green functions \(G_{ll}\) and \(D_l(x')\). One gets after long but straightforward algebra\(^6\)

\[
\frac{1}{2L^2} C_{J1}^1 = \frac{1}{4L^2} \sum_l \sum_x D_l(x)
\]

\[
\left( \frac{1}{2} \sum_{i=1}^4 (\delta_{l_i}(G_{l_3l_4} + G_{l_4l_3} - G_{l_3l_3} - G_{l_3l_3}) + G_{l_4l_3}(\delta_{l_1} + \delta_{l_2} - \delta_{l_3} - \delta_{l_4})) + \right.
\]

\[
\left. \delta_{l_1}(G_{l_3l_4} + 2G_{l_3l_2} + 2G_{l_3l_2}) + \delta_{l_2}(G_{l_3l_4} - G_{l_3l_4} - G_{l_3l_4}) + \right.
\]

\[
\left. \delta_{l_3}(G_{l_4l_1} + G_{l_4l_3} - G_{l_4l_3} - G_{l_4l_3}) - \delta_{l_4}(2G_{l_3l_4} + 2G_{l_3l_4} + G_{l_3l_4}) \right) ,
\]

where links \(l_i\) are defined in Appendix C (see Fig.1), \(l = (x, n)\). In terms of standard \(D\)-functions defined in Appendix C the result reads
\[
\frac{1}{2L^2} C_{J1}^1 = \frac{1}{4}[6 - 2D(2, 0) - D(1, 1)] = \frac{1}{2} + \frac{3}{2\pi} .
\]

The second term is given by the operator \(\frac{1}{2} \left( \sum_x \sum_{k=1}^3 \alpha_k(x) \omega_k^{(1)}(x) \right)^2\). \(\omega_k^{(1)}(x)\) is given in the Appendix B (82). In terms of Green functions it reads
\[
\frac{1}{2L^2} C_{J2}^1 = -\frac{3}{16} (Q^{(1)} + Q^{(2)}) ,
\]

\[
Q^{(1)} = \frac{1}{2L^2} \sum_{x,x',i<j} \sum_{j'<j} \sum_{j''<j'} G_{x,x'}(G_{i,i',j'} G_{i,i',j'} - G_{i,i',j'} G_{i,i',j'}),
\]

\[
Q^{(2)} = \frac{1}{2L^2} \sum_{x,x',i<j} \sum_{j'<j} \sum_{j''<j'} (G_{i,i',j'} D_{j'}(x) D_{j}(x') + G_{i,i',j'} D_{j'}(x') D_{j}(x) -
\]

\[
G_{i,i',j'} D_{j'}(x) D_{j}(x') - G_{i,i',j'} D_{j'}(x') D_{j}(x)) .
\]

\(^6\)Our previous version suffered from incorrect sign in this term which led to wrong final result.
Link $l_i (l_j')$ refers to one of four links attached to a given site $x (x')$. We performed both analytical and numerical studies of last expressions. Details are given in the Appendix C. We find

\[ Q^{(1)} = \frac{1}{2} \] (60)

and

\[ Q^{(2)} = 1 + \frac{8}{\pi} . \] (61)

Collecting all coefficients we finally obtain

\[ \frac{1}{2L^2} C^1 = \frac{3}{64} , \] (62)

which coincides with the result of the conventional PT [8].

6 Discussion

In this paper we propose to use an invariant link formulation to investigate some properties of 2D models in the weak coupling region. We argued that this approach is more suitable for calculation of asymptotic expansions of invariant functions in cases when the Mermin-Wagner theorem forbids spontaneous symmetry breaking in the thermodynamic limit. We have found that both in the abelian $XY$ model and in non-abelian $SU(2)$ model our results for the first order coefficients of the free energy and correlation function agree with the standard PT expansion in the TL. In our next paper [13] we show that the second order coefficient of the correlation function also agrees with the conventional PT.

Now we can return to the question raised in the Introduction, namely whether conventional PT gives uniform asymptotic expansion for non-abelian models. It is well known for a long time that it is no so in one-dimensional non-abelian models. In [13] we address this question in the link formulation and show that non-uniformity in this case originates from the expansion of holonomy operator which imposes certain global condition on the configurations of link matrices. While this global condition itself vanishes in true TL, it survives large volume limit if the low-temperature expansion is performed in a finite volume. In 2D there are two such operators which we did not consider in the present paper. The reason being that the contributions from these holonomy operators are suppressed as $O(1/L)$. Since in 2D one could encounter only logarithmic divergences it is rather unlikely that holonomies are relevant for the TL. In this sense there is no similarity between one and two dimensional models. Nevertheless, one cannot exclude the possibility of non-uniformity of the low-temperature expansion arising from the remainder to the PT series [9]. This problem is extremely hard to resolve by means of the standard approaches, see for instance [3]. Contrary, in the link formulation we are able to calculate the exponential remainder at a given order of the low-temperature expansion. Therefore, the problem of the infrared finitness of the remainder can be addressed explicitly. Such investigations are presently in progress.
Acknowledgements.

We are grateful to J. Polonyi who found time to go through many details of the expansion presented here and for many encouraging discussions. We would like to thank V. Miransky and V. Gusynin for interesting discussions and healthy critics on different stages of this work. Our special thanks to B. Rusakov for the explanation of his paper [11] regarding the calculation of the Jacobian $J(V)$.

References

[1] E. Brezin, J. Zinn-Justin, Phys.Rev. B14 (1976) 3110; E. Brezin, J. Zinn-Justin, J.C. LeGuillou, Phys.Rev. D14 (1976) 2615; A.M. Polyakov, Phys.Lett. B59 (1975) 79.

[2] A. Patrascioiu, E. Seiler, Phys.Rev.Lett. 74 (1995) 1920.

[3] A. Patrascioiu, E. Seiler, Phys.Rev.Lett. 74 (1995) 1924.

[4] F. David, Phys.Rev.Lett. 75 (1995) 2626.

[5] F. Niedermayer, M. Niedermaier and P. Weisz, Questionable and Unquestionable in the Perturbation Theory of non-Abelian Models, [hep-lat/9612002].

[6] A. Patrascioiu, E. Seiler, Phys.Rev. D57 (1998) 1394.

[7] N.D. Mermin, H. Wagner, Phys.Rev.Lett. 22 (1966) 1133; R. Dobrushin, S. Shlosman, Com.Math.Phys. 42 (1975) 31.

[8] Y. Brihaye, P. Rossi, Nucl.Phys. B235 (1984) 226.

[9] J. Bricmont, J.-R. Fontaine, J.L. Lebowitz, E.H. Lieb, and T. Spencer, Comm.Math.Phys. 78 (1981) 545.

[10] J. Frohlich, R. Israel, E.H. Lieb, B. Simon, Comm.Math.Phys. 62 (1978) 1.

[11] B. Rusakov, Phys.Lett. B398 (1997) 331.

[12] P. Hasenfratz, Phys.Lett. B141 (1984) 385.

[13] O. Borisenko, V. Kushnir, in preparation.
7 Appendix

7.1 A: Representation for $SU(2)$ partition function

We start from the following partition function in the link representation

$$Z = \int \prod_l dV_l \exp \left[ \beta \sum_l \text{Tr} V_l \right] \prod_p J_p(V). \quad (63)$$

Using the formula for $SU(2)$ characters

$$\chi_r(\phi) = \frac{\sin(2r + 1)\phi}{\sin \phi}, \quad (64)$$

with $r = 0, 1/2, 1, 3/2, \ldots$, we can write down the Jacobian $J(V)$ (6) in the form

$$J_p(V) = \sum_{r_p=-\infty}^{\infty} \frac{r_p \sin r_p \phi_p}{\sin \phi_p}, \quad (65)$$

where $\phi_p$ is some plaquette angle. Now $r_p$ takes only integer values. Let us parametrize the $SU(2)$ link matrices as

$$V_l = \exp \left[ i\sigma^k \omega_k(l) \right] = \cos W_l + i\sigma^k \omega_k(l) \frac{\sin W_l}{W_l}, \quad (66)$$

where we used $W_l$ defined in (29). One gets in this parametrization the following expressions:

1. Action

$$\frac{1}{2} \text{Tr} V_l = \cos W_l. \quad (67)$$

2. Invariant measure

$$dV_l = \frac{\sin^2 W_l}{W_l^2} \prod_k d\omega_k(l). \quad (68)$$

3. Jacobian

$$J_p(V) = \sum_{r_p=-\infty}^{\infty} r_p \frac{\sin r_p W_p}{\sin W_p}, \quad (69)$$

where $W_p$ is defined in (20). Substituting (67)-(69) into (63) we get

$$Z = \int \prod_l \left[ \frac{\sin^2 W_l}{W_l^2} \prod_k d\omega_k(l) \right] \exp \left[ 2\beta \sum_l \cos W_l \right] \prod_p \sum_{r_p=-\infty}^{\infty} \frac{r_p \sin r_p W_p}{\sin W_p}. \quad (70)$$

To get an expression for the Jacobian convenient for the large-$\beta$ expansion we use the equality

$$\sum_{r=-\infty}^{\infty} r \frac{\sin r W}{\sin W} = \frac{W}{\sin W} \sum_{m=-\infty}^{\infty} \int \prod_k d\alpha_k \exp \left[ -i \sum_k \alpha_k \omega_k + 2\pi i m \alpha \right], \quad (71)$$
where we have introduced \( \alpha = [\sum_k \alpha_k^2]^{1/2} \). To prove (71) we write
\[
F(\sum_k \omega_k^2) = \int_{-\infty}^{\infty} \prod_k d\alpha_k e^{-i\alpha_k \omega_k} \int_{-\infty}^{\infty} \prod_k dt_k F(\sum_k t_k^2)e^{i\alpha_t t_k}. \tag{72}
\]
Integrals over \( t_k \) are calculated in the spherical coordinates. Taking then \( F(t^2) = \frac{\sin rt}{t} \) and using the Poisson resummation formula we come to (71). Substituting (71) into the partition function (70) we finally get
\[
Z = \int \prod_l \left[ \frac{\sin^2 W_l}{W_l^2} \prod_k d\omega_k(l) \right] \exp \left[ 2\beta \sum_l \cos W_l \right] \prod_p \frac{W_p}{\sin W_p} \prod_p \sum_{m(p) = -\infty}^{\infty} \int \prod_k d\alpha_k(p) \exp \left[ -i \sum_k \alpha_k(p) \omega_k(p) + 2\pi i m(p) \alpha(p) \right]. \tag{73}
\]

### 7.2 B: Relation between link and plaquette angles

Let us introduce the following notations for links of a given plaquette \( p \)
\[
l_1 = (x; n), \quad l_2 = (x + n; m), \quad l_3 = (x + m; n), \quad l_4 = (x; m). \tag{74}
\]
Then, from the definition (27) and (10) one gets the following formulas relating link and plaquette angles
\[
V_p = \cos W_p + i\sigma^k \omega_k(p) \frac{\sin W_p}{W_p}, \tag{75}
\]
where
\[
\cos W_p = \cos M_1 \cos M_2 + \sum_k \nu_1^k \nu_2^k \sin M_1 \sin M_2, \tag{76}
\]
\[
\omega_k(p) \frac{\sin W_p}{W_p} = \nu_1^k \cos M_2 \frac{\sin M_1}{M_1} - \nu_2^k \cos M_1 \frac{\sin M_2}{M_2} + \epsilon^{kpq} \nu_1^k \nu_2^q \frac{\sin M_1}{M_1} \frac{\sin M_2}{M_2}. \tag{77}
\]
\( M_i \) and \( \nu_i^k \) are given by
\[
\cos M_1 = \cos W(l_1) \cos W(l_2) - \sum_k \omega_k(l_1) \omega_k(l_2) \frac{\sin W(l_1) \sin W(l_2)}{W(l_1)W(l_2)}, \tag{78}
\]
\[
\cos M_2 = \cos W(l_3) \cos W(l_4) - \sum_k \omega_k(l_3) \omega_k(l_4) \frac{\sin W(l_3) \sin W(l_4)}{W(l_3)W(l_4)},
\]
\[
\nu_1^k \sin M_1 = \omega_k(l_1) \cos W(l_2) \frac{\sin W(l_1)}{W(l_1)} + \omega_k(l_2) \cos W(l_1) \frac{\sin W(l_2)}{W(l_2)} - \epsilon^{kpq} \omega_p(l_1) \omega_q(l_2) \frac{\sin W(l_1) \sin W(l_2)}{W(l_1)W(l_2)}, \tag{79}
\]
\[
\nu_2^k \sin M_2 = \omega_k(l_3) \cos W(l_4) \frac{\sin W(l_3)}{W(l_3)} + \omega_k(l_4) \cos W(l_3) \frac{\sin W(l_4)}{W(l_4)} - \epsilon^{kpq} \omega_p(l_3) \omega_q(l_4) \frac{\sin W(l_3) \sin W(l_4)}{W(l_3)W(l_4)}. \]
Now it is straightforward to calculate the following expansion \((p \to x)\)
\[
\omega_k(x) = \omega_k^{(0)}(x) + \omega_k^{(1)}(x) + \omega_k^{(2)}(x) + \ldots.
\] (80)

On a dual lattice (see Fig. 1) the first coefficients can be written down as
\[
\omega_k^{(0)}(x) = \omega_k(l_3) + \omega_k(l_4) - \omega_k(l_1) - \omega_k(l_2),
\] (81)
\[
\omega_k^{(1)}(x) = -\epsilon^{k_p q} \sum_{i<j}^4 \omega_p(l_i)\omega_q(l_j),
\] (82)
\[
\omega_k^{(2)}(x) = \frac{1}{3} \epsilon^{mnrc} \omega_n^{(0)}(x) \sum_{i=1}^4 \omega_r(l_i)\omega_m(l_i) + 
\frac{2}{3} (\epsilon^{mnrc} + \epsilon^{mrcn}) [\omega_r(l_3)\omega_m(l_1)\omega_n(l_2) + \omega_r(l_4)\omega_m(l_1)\omega_n(l_2) - 
\omega_r(l_3)\omega_m(l_4)\omega_n(l_1) - \omega_r(l_3)\omega_m(l_4)\omega_n(l_2)].
\] (83)

### 7.3 C: Coefficients \(Q^{(1)}\) and \(Q^{(2)}\)

We define \(D\)-function as
\[
D(x_1, x_2) = \frac{1}{L^2} \sum_{k_n=0}^{L-1} \frac{1 - \frac{2\pi}{L} (k_1 x_1 + k_2 x_2)}{2 - \sum_{n=1}^2 \cos \frac{2\pi}{L} k_n}, \quad k_n^2 \neq 0,
\] (84)

and \(G_0\) as
\[
G_0 = \frac{1}{L^2} \sum_{k_n=0}^{L-1} \frac{1}{2 - \sum_{n=1}^2 \cos \frac{2\pi}{L} k_n}, \quad k_n^2 \neq 0.
\] (85)

The normalization is chosen such that \(G_{ll} = 1\). All \(D\)-functions can be expressed in terms of \(D(1, 0)\) and \(D(1, 1)\). In the TL one has
\[
D(1, 0) = \frac{1}{2}, \quad D(1, 1) = \frac{2}{\pi},
\]
\[
D(2, 0) = 2(2D(1, 0) - D(1, 1)) = 2 - \frac{4}{\pi}, \quad D(2, 1) = 2D(1, 1) - D(1, 0) = \frac{4}{\pi} - \frac{1}{2},
\]
\[
D(3, 0) = 4(D(2, 0) - D(1, 1)) + D(1, 0) = \frac{1}{2} + 8(1 - \frac{3}{\pi}).
\] (86)

\(G_0\) is known to diverge logarithmically in two dimensions and behaves numerically as
\[
G_0 = a_0 + a_1 \ln L + o(1), \quad a_0 \approx 0.0974006, \quad a_1 \approx 0.318331 \approx \frac{1}{\pi}.
\] (87)

We also need some representation for the function \(G_{ll'}\) in the momentum space. One finds from the definition (21)
\[
G_{ll'} = \frac{2\delta_{nn'} - 1}{L^2} \sum_{k_n=0}^{L-1} \frac{e^{\frac{2\pi i}{L} k_n (x-x')}}{f(k)} F(n, n'),
\]
\[
F(n = n') = 2(1 - \cos \frac{2\pi}{L} k_p), \quad n \neq p,
\]
\[
F(n \neq n') = (1 - e^{\frac{2\pi i}{L} k_n})(1 - e^{-\frac{2\pi i}{L} k_n'}),
\] (88)
Figure 1: Plaquette of original lattice and links of dual lattice as they enter sums for \(Q^{(1)}\) and \(Q^{(2)}\). Link is determined by point \(x\) and a positive direction, e.g. \(l_3 = (x - n_1; n_1)\).

where we have denoted

\[
f(k) = 2 - \sum_{n=1}^{2} \cos \frac{2\pi}{L} k_n .
\]  

(89)

Using this representation it is easy to prove the following “orthogonality” relations for the functions \(G_{ll'}\) and \(D_l(x)\) which are very useful for calculation of lattice sums

\[
\sum_b G_{lb} G_{bl'} = 2G_{ll'} ,
\]  

(90)

\[
\sum_b D_b(x) G_{bl'} = 0 ,
\]  

(91)

\[
\sum_b D_b(x) D_b(x') = 2G_{x,x'} .
\]  

(92)

To visualize the summation over indices \(i, i'\) in (58) and (59) we depicted dual links in Fig.1.

7.3.1 \(Q^{(1)}\)

We divide \(Q^{(1)}\) into two pieces

\[
Q^{(1)} = B_1 + B_2 ,
\]  

(93)

where \(B_1\) includes first and second powers of Green functions \(G_{x,x'}\) whereas \(B_2\) consists of terms with only third powers of \(G_{x,x'}\). We thus have for \(B_1\) from (58)

\[
B_1 = \frac{1}{2L^2} \sum_{x,x'} \sum_{i<j} \sum_{i'<j'} G_{x,x'}(\delta_{i,i'} \delta_{j,j'} G_{i,j} G_{i',j'} + G_{i,i'} \delta_{j,j'} + G_{i,i'} \delta_{j,j'} - G_{i,i'} \delta_{j,j'}) 
\]  

(94)

Performing all summations and using Eq.(86) to express all \(D\) functions appearing in the last equation in terms of \(D(1,0)\) and \(D(1,1)\) we get

\[
B_1 = 4D(1,1) - 4D(1,0) - 4G_0 .
\]  

(95)

For \(B_2\) we adduce an expression in the momentum space

\[
B_2 = \frac{1}{2L^4} \sum_{k_1, k_2} \frac{B(k_1^2, k_2^2)}{f(k_1) f(k_2) f(k_1 + k_2)} ,
\]  

(96)
where

\[ B(k_n^1, k_n^2) = (z_1 + z_2^* + z_1^* + z_2 + z_2 z_1^* + z_1 z_2^*)(z_4 + z_3^* + z_3 + z_4^* + z_4 z_3^* + z_3 z_4^*) + (97) \]
\[ 2(z_1 + z_1^*)(z_4 + z_4^*) - 2z_1 z_2 z_4 z_3^* - 2z_1 z_2(z_3 z_4)^* - (z_1 z_2 + (z_1 z_2)^*)(z_3 z_4 + (z_3 z_4)^*) - \]
\[ z_1^2(z_3)^2 - z_2^2(z_4)^2 - 2z_1^2 z_2^2(z_3^2 + z_4^2) - 2z_1 z_2((z_3^2 + (z_4)^2) + (z_1^2 - (z_1)^2)(z_4^2 - (z_4)^2) \]
and we have introduced

\[ z_1 = 1 - e^{i\nu_1} , \quad z_2 = 1 - e^{i\nu_2} , \quad z_3 = 1 - e^{i\nu_3} , \quad z_4 = 1 - e^{i\nu_4} ; \quad p_i \equiv \frac{2\pi}{L} k_i . \]  

(98)

The result of summation can be expressed in terms of \( D \) functions given in (86)

\[ B_2 = 5D(1,0) - 4D(1,1) + 4G_0 . \]  

(99)

One sees that all divergences exactly cancel. The final result is

\[ Q^{(1)} = B_1 + B_2 = D(1,0) = \frac{1}{2} . \]

(100)

We performed numerical check of our result for \( B_2 \). The function was calculated for lattice sizes \( L \in [10 - 120] \) and fitted to the form

\[ B_2 = a_0 + a_1 \ln L + a_2 \frac{\ln L}{L} + a_3 \frac{L}{L} . \]

(101)

From the result \( Q^{(1)} = \frac{1}{2} \) and from Eq.(87) one concludes that coefficients \( a_0 \) and \( a_1 \) should be equal to

\[ a_0 = 0.34312331 , \quad a_1 = 1.273324 . \]

Our fit gives

\[ a_0 = 0.343476 , \quad a_1 = 1.27327 . \]

### 7.3.2 \( Q^{(2)} \)

As before we divide \( Q^{(2)} \) into two pieces

\[ Q^{(2)} = K_1 + K_2 , \]

(102)

where \( K_1 \) includes second powers of Green functions, \( K_2 \) is cubic in \( G_{x,x'} \). \( K_1 \) is given by

\[ K_1 = \frac{1}{2L^2} \sum_{x,x'} \sum_{i<j}^4 \sum_{i'<j'}^4 (\delta_{i,i'} D_{i,i'}(x) D_{i,i'}(x') + \delta_{i,i'} D_{i,i'}(x) D_{i,i'}(x')) - \]
\[ \delta_{i,i'} D_{i,i'}(x) D_{i,i'}(x') - \delta_{i,i'} D_{i,i'}(x) D_{i,i'}(x') \] .  

(103)
Calculations are rather lengthy nevertheless straightforward. We get

\[ K_1 = 20D(1,0)^2 + 4(D(2,0) - D(1,0))(2D(1,1) - 3D(1,0) + D(2,0)) = 8 - \frac{8}{\pi}. \] (104)

\( K_2 \) in the momentum space can be written as

\[ K_2 = \frac{1}{L^4} \sum_{k_1, k_2} \frac{K(k_1, k_2)}{f(k_1)f(k_2)f(k_1 + k_2)}. \] (105)

where

\[ K(k_1, k_2) = (z_1 + z_2 + z_3 + z_4 - z_1z_3 - z_2z_4)[(z_1 + z_2)^2(z_3^* + z_4^*) - (z_1 + z_1^* + z_2 + z_2^*)(z_3 + z_4)]
- 2z_1z_4^* [2(z_2 + 1)(z_2^* + z_4^* - z_2^*z_4^*) + z_1(z_2 + z_4 - z_2z_4)]. \] (106)

Notations are as in Eq.(98). All sums can be done analytically resulting in

\[ K_2 = \frac{16}{\pi} - 7. \] (107)

Finally we obtain

\[ Q^{(2)} = \frac{8}{\pi} + 1 \approx 3.5464791. \] (108)

\[ ^7 \text{This analytical expression agrees well with our previous numerical result } Q^{(2)} = 3.5466309 \]