An analytic approach for the study of pulsar spindown

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Abstract
In this work we develop an analytic approach to study pulsar spindown. We use the monopolar spindown model by Alvarez and Carramiñana (2004 Astron. Astrophys. 414 651–8), which assumes an inverse linear law of magnetic field decay of the pulsar, to extract an all-order formula for the spindown parameters using the Taylor series representation of Jaranowski et al (1998 Phys. Rev. D 58 6300). We further extend the analytic model to incorporate the quadrupole term that accounts for the emission of gravitational radiation, and obtain expressions for the period $P$ and frequency $f$ in terms of transcendental equations. We derive the analytic solution for pulsar frequency spindown in the absence of glitches. We examine the different cases that arise in the analysis of the roots in the solution of the non-linear differential equation for pulsar period evolution. We provide expressions for the spin-down parameters and find that the spindown values are in reasonable agreement with observations.

A detection of gravitational waves from pulsars will be the next landmark in the field of multi-messenger gravitational wave astronomy.

Keywords: pulsars, spindown, gravitational waves, detectors, Lambert $W$ function, frequency derivatives, multipole spindown

1. Introduction

Pulsars are highly magnetized neutron stars which are known to spin down due to various physical mechanisms [1]. These mechanisms involve electromagnetic and gravitational emissions which result in the spindown of the GW frequency of the emitted pulsar signal [2].

In their paper on monopolar pulsar spindown, Alvarez and Carramiñana have considered a general multipole spindown to study pulsar evolution [2]. In this model, a monopolar term
was introduced to ensure that the braking indices defined in terms of the frequency and higher
derivatives are in agreement with the trajectories in the $P - \dot{P}$ diagram for pulsar evolution,
where $P$ and $\dot{P}$ denote the pulsar period and its time derivative. Their detailed analysis of the
stationary multipole model ruled out the possibility of a time independent evolutionary equation
for the pulsar frequency. Their time dependent multipole model included dynamics of the
pulsar magnetic moment and thereby the decay of the magnetic field, $B(t)$. They show in their
analysis that an inverse linear decay proposed by Chanmugham and Sang [3], in contrast to an
exponential decay law for pulsar magnetic field [4], did a better fit of the evolutionary trajectories of the four pulsars studied namely the Crab, PSR B1509-58, PSR B0540-69 and Vela.

The evolution of pulsars has been studied in great detail [2–11, 15, 16, 18–22, 24]. Following the approach in [2, 3], we use an inverse linear decay law for $B(t)$. Three parameters are introduced to explain the general multipole spindown which is due to the unipolar interaction between the pulsar and an accretion disc of infalling supernova remains [8], the electromagnetic dipolar radiation and the quadrupolar gravitational radiation. The spindown of pulsars due to the intense magnetic fields that surround them is a phenomenon that will significantly impact on the younger pulsars which can lose a large amount of rotational energy due to physical processes such as electromagnetic and gravitational multipole radiation. Recent studies have shown that young rotating neutron stars undergoing r-mode oscillations are also potentially a strong source of GW, although the detection of the GW could be challenging due to instability of r-modes. Electromagnetic observations of r-modes in targeted neutron stars 4U 1636-536(2001) and XTE J1751-305(2002), which are potential candidates for the detection of GW, were discussed in [50] before the first run of Advanced LIGO took place.

The analytical form of the pulsar frequency evolution is useful particularly in the case of
young pulsars, where the pulsar frequency and only its first derivative [58] are known. Although
there are accurate radio, x-ray and gamma ray timing observations that provide frequency and frequency derivatives sufficient enough to characterize the pulsar frequency evolution during observation runs, an analytic form of the frequency evolution would assist in determining higher order frequency derivatives. This would then assist in refining the parameter space of a narrow band search for continuous gravitational waves from pulsars (Abbott et al [59]).

Advanced LIGO has made at least five observations of gravitational waves, from GW150914
[25], GW151226 [26], GW170104 [27, 28] and so on. Recently, a merger of neutron stars GW170817 has been detected by both GW and electromagnetic observations [29]. The possible observations of post merger events from relatively light neutron stars to form a new neutron star is another challenge to detect GWs in such different scenarios. With increasing detector sensitivity of LIGO, VIRGO and the upcoming KAGRA, and IndIGO detectors, the next wave of optimism and discovery should be on the detection of GW from pulsars. Pulsars are remarkably stable objects and although their GW amplitude is weaker compared to black holes, the fact that they can be tracked over a long period of time and that they emit continuous GW signals will be invaluable in detecting their GW signals.

In earlier works [30–34] we have implemented the Fourier transform (FT) of the Doppler
shifted GW signal from a pulsar with the Plane Wave Expansion in Spherical Harmonics
(PWESH). It turns out that the consequent analysis of the Fourier transform (FT) of the gravita-
tional wave (GW) signal from a pulsar has a very interesting and convenient development
in terms of the resulting spherical Bessel, generalized hypergeometric, Gamma and Legendre
functions. These works considered frequency modulation of a GW signal due to rotational and
circular orbital motions of the detector on the Earth. In later analysis, rotational and orbital eccentric motions of the Earth, as well as perturbations due to Jupiter and the Moon were considered [34]. But a crucial missing component was the consideration of pulsar GW spindown. The present work attempts to partially address this significant problem. The numerical
analysis of this analytic expression including pulsar gravitational wave spindown offers a challenge for efficient and fast numerical and parallel computation. It will be considered in forthcoming studies.

We extend the mathematical analysis of the monopolar pulsar spindown model introduced by Alvarez and Carramiñana [2, 35]. The pulsar spindown parameters representation was defined by Brady et al [36, 37] and Jaranowski et al [38]. We use the spindown convention of Jaranowski et al [38] in the form of a Taylor series for the GW frequency to make a connection to current work on the phase measured at the ground based detectors.

Our paper is organized as follows. In the next section, we present a model for pulsar spindown along with a new solution that includes the quadrupole term due to gravitational radiation emission, which is an extension relevant for younger pulsars. In section 3, we connect this model to the Taylor series form of frequency spindown shifts by Jaranowski et al [38], and determine an analytical form of the spindown parameters. We also compute the first three spindown parameters via this approach for the Crab and PSR 1509-58. Section 4 summarizes our conclusions.

2. A model for pulsar spindown

A simple spin-down model based on the rotational frequency \( f \) and its first few time derivatives would be sufficient to explain the slow-down of pulsars in general. We consider the spindown model derived from a general spindown power-law [2, 8]:

\[
\dot{f} = -F(f, t).
\]  

This law expresses the change of pulsar frequency with respect to time and assumes that the frequency is a positive, antisymmetric and continuous function of time. Also, \( F \) is antisymmetric with respect to frequency. Following these properties, the even powers of \( f \) are excluded and the first three non-zero terms in the Taylor expansion of equation (1) are:

\[
\dot{f} = -s(t)f - r(t)f^3 - g(t)f^5. 
\]  

The first term, namely the monopolar term was introduced by Alvarez and Carramiñana (2004) to take into account particle acceleration mass loss or pulsar winds. This term further enables the braking indices to be in accord with the trajectories in the \( P - \dot{P} \) diagrams for pulsar evolution. The second and third terms incorporate the normal spindown mechanisms of magnetic dipole radiation and gravitational radiation. For the frequencies measured in known, isolated pulsars, higher order terms can be neglected. In contrast, the spindown of millisecond pulsars has shown features strongly in contrast with typical pulsars. Hence, this model does not describe binary pulsars. A thorough analyses of binary pulsar spindown has been done in several excellent works [5, 6], to cite but a few. By an elementary result in wave mechanics, \( f = \frac{1}{P} \) (and correspondingly \( \dot{f} = -\frac{\dot{P}}{P^2} \)), we have that :

\[
\dot{P} = s(t)P + \frac{r(t)}{P} + \frac{g(t)}{P^3}. 
\]  

2.1. \( g(t) = 0 \)

For a simpler analysis for older pulsars, the quadrupole term \( g(t) \) is dropped, as \( g(t) \) does not give an appreciable contribution and firstly the contribution due to the electromagnetic is
considered in detail in this section. We include the formula including GW spin-down in the later half of this section. For the simpler case of \( g(t) = 0 \), we have:

\[
\dot{P} \simeq s(t)P + \frac{r(t)}{P}.
\]  

(4)

Evolutionary tracks on the \( P - \dot{P} \) diagram are studied [2] by writing equation (2) in terms of the period and its derivative:

\[
\dot{P} = \left( \frac{r_0}{P} + s_0P \right) \psi(t/t_c).
\]  

(5)

The assumption is that the frequency/period evolution changes as the magnetic field \( B(t) = B(0) / (1 + t/t_c) \) decays. Moreover, the \( \{s(t), r(t)\} \propto B(0)^2 \psi(\frac{t}{t_c}) \); thereby \( r_0, s_0 \) are obtained from pulsar evolution studies [2], where \( t_c \) is the characteristic time of the magnetic field decay and \( \psi \) is a dimensionless function satisfying \( \psi(\alpha) = 1 \). Evidence of magnetic field decay is provided by anomalous x-ray pulsars (AXPs) interpreted as magnetars. Colpi et al. (2000) required the magnetic field in these neutron star related objects to decay as a power law on timescales of the order of \( 10^4 \) years. In our table 1 we reproduce table 1 of Alvarez and Carramiñana [2, 35] which gives the timing parameters and stationary multi-pole model fitting parameters for the four given pulsars. We compare the obtained values with those of Lyne et al. [8, 15] to check on the validity of the model in [2].

From the use of the inverse linear law in equation (4), we find:

\[
\dot{P} = \left( \frac{1}{1 + t/t_c} \right)^2 \left( \frac{r_0}{P} + s_0P \right),
\]  

(6)

where \( r_0 \geq 0, s_0 \geq 0 \).

If we have the initial condition \( P(t_0) = P_0 \), then the solution of this equation is given below:

\[
P(t) = \frac{1}{s_0} \sqrt{s_0 \exp \left\{ \frac{2t^2s_0(t - t_0)}{(t + t_c)(t_c + t_0)} \right\} (P_0^2s_0 + r_0) - r_0s_0}.
\]  

(7a)

For \( r_0 \ll |s_0| \ll 1, t_c \gg t, t_0 = 0 \), we have:

\[
P(t) = \sqrt{\frac{e^{2s_0t}(P_0^2s_0^2 + r_0s_0)}{s_0} - \frac{r_0}{s_0} = P_0e^{s_0t} \left[ 1 + \frac{r_0}{s_0} \left( 1 - e^{-2s_0t} \right) \right]^{\frac{1}{2}}.}
\]  

(7b)

In the most general form, for \( s_0 > 0, r_0 > 0 \):

\[
f(t) = s_0 \left[ \sqrt{s_0 \exp \left( \frac{2t^2s_0(t - t_0)}{(t + t_c)(t_c + t_0)} \right) \left( \frac{1}{f_0^2s_0^2 + r_0} \right) - r_0s_0} \right]^{-1}
\]

\[
= f_0^{1/2} \left[ (s_0^{1/2} \exp \left( \frac{2t^2s_0(t - t_0)}{(t + t_c)(t_c + t_0)} \right) - \lambda_1 \right]^{-1},
\]  

(8a)

\[
\lambda_0 = f_0^2r_0 + s_0,
\]  

(8b)

\[
\lambda_1 = r_0f_0^2,
\]  

(8c)}
where $f_0$ is the initial frequency at $t = 0$.

For $t_c \gg t$ and $t_0 = 0$:

$$f(t) = \frac{f_0 \sqrt{s_1}}{\lambda_0 \exp(2s_0t)} = \frac{f_0 e^{-s_0t}}{\sqrt{\lambda_0 \lambda_0}} \cdot \frac{1 - \lambda_1 e^{-2s_0t}}{\lambda_0}^{-\frac{1}{2}}. \quad (9a)$$

It should be noted that equation (9a) can also be expressed to account for a possible negative sign in $s_0$ as:

$$f(t) = \frac{f_0 \sqrt{|s_0|}}{\sqrt{\lambda_0 \exp(2s_0t)} - \lambda_1}. \quad (9b)$$

It should be observed that the inclusion of a quadratic term in $f$ for a case in which $f$ is not antisymmetric would not present undue problems in the solution of the nonlinear differential equation (2). Re-arrangement of equation (9b) gives the following expression.

$$f(t) = f_0 \sqrt{\frac{\lambda_0}{s_0}} \exp \left[ -s_0 t \left( 1 + \frac{\lambda_1}{\lambda_0} \exp \left\{ -\frac{2s_0 t}{1 + \frac{\lambda_1}{\lambda_0}} \right\} \right) \right]^{-1/2}. \quad (10)$$

Using the binomial expansion we can rewrite this as $(1 - x)^{-1/2} = 1 + \frac{1}{2} x + \frac{3}{8} x^2 + \ldots$, thus:

$$f_{\text{pulsar}}(t) = f_0 \sqrt{\frac{\lambda_0}{s_0}} \exp \left\{ -s_0 t \right\} \left[ 1 + \frac{\lambda_1}{2\lambda_0} \exp \left\{ -2s_0 t \right\} + \frac{3\lambda_1^2}{8\lambda_0^2} \exp \left\{ -4s_0 t \right\} + \ldots \right]. \quad (11)$$

Higher order terms were dropped. Since $t/t_c \ll 1$, we have:

Table 1. Timing parameters and stationary multipole model fitting parameters for the four pulsars selected [2].

| Pulsar | Crab [15] | 1509-58 [12] | 0540-69 [20] | Vela [16] |
|--------|-----------|--------------|--------------|-----------|
| $P$ (ms) | 33.5 | 150.9 | 50.3 | 89.3 |
| $t_{\text{dyn}}$ (yr) | 2.509 | 2.837 | 2.01 | 1.4 |
| $n$ | ±0.005 | ±0.001 | ±0.02 | ±0.2 |
| $m$ | 10.23 | 14.5 | ... | ... |
| $s_0$ | ±0.03 | ±3.6 | ... | ... |
| $g$ (Hz$^{-3}$) | $3.5 \times 10^{-19}$ | $6.2 \times 10^{-16}$ | $6.6 \times 10^{-18}$ | $3.8 \times 10^{-18}$ |
| $r$ (Hz$^{-1}$) | $1.0 \times 10^{-14}$ | $1.6 \times 10^{-13}$ | $7.1 \times 10^{-15}$ | $1.3 \times 10^{-15}$ |
| $s$ (Hz) | $3.4 \times 10^{-12}$ | $2.0 \times 10^{-12}$ | $5.7 \times 10^{-12}$ | $1.2 \times 10^{-12}$ |
| $P_{\text{birth}}$ (ms)* | 9.9 | 14.2‡ | 17.6 | 50 |
| $\dot{f}$ | $-3.79 \times 10^{-10}$ | $-6.77 \times 10^{-11}$ | $-1.88 \times 10^{-10}$ | $-1.57 \times 10^{-11}$ |
| Calculated $\dot{f}$ from equation (2) | $-3.76 \times 10^{-10}$ | $-6.77 \times 10^{-11}$ | $-1.89 \times 10^{-10}$ | $-1.59 \times 10^{-11}$ |

* $P_{\text{birth}}$ defined as the period one dynamical time ago.
‡ Integration is stopped at $t_{\text{dyn}} = 1$ year.
\[ f_{\text{pulsar}}(t) = f_0 \sqrt{\frac{s_0}{\lambda_0}} \exp\{-s_0t\} \left[ 1 + \frac{\lambda_1}{2\lambda_0} \exp\{-2s_0t\} + \ldots \right]. \quad (12) \]

2.2. \( g(t) \neq 0 \)

In the paper by Abbott et al [59], the authors have discussed the important issue of beating the spin-down limit of the gravitational wave emission of the Crab pulsar. The important question is whether the loss due to electromagnetic radiation can explain for the vast amount of observed rotational energy loss, or if gravitational wave emission has a significant role to play. Abbott et al pointed out that given the uncertainties of the electromagnetic observations of the Crab Nebula, a substantial fraction of the power due to the spin-down emitted as gravitational waves would still be possible.

For pulsars such as PSR B1509-58 [52], the spindown is consistent with the gravitational wave spindown, there could be an initial phase of strong gravitational wave spindown with the quadrupole parameters being at least two orders of magnitude higher than that of the other pulsars (Alvarez and Carramiñana [2]). We extend this model to further include the quadrupole term. Inclusion of the quadrupole term with a similar time dependence for \( g(t) \) assumed as in the electromagnetic case, to consider the effects of spindown to GW emission, gives:

\[ \frac{dP}{dt} = \left( s_0P + \frac{r_0}{P} + \frac{g_0}{P^3} \right) \left( 1 + \frac{t}{t_c} \right)^{-2}. \quad (13) \]

Multiplying by \( P^3 \), and re-arranging the differential equation we obtain:

\[ P^3dP = \left( s_0P^4 + r_0P^2 + g_0 \right) \left( 1 + \frac{t}{t_c} \right)^{-2} dt. \quad (14) \]

Let \( Q = P^2 \), with \( dQ = 2PdP \), so that the above differential equation becomes:

\[ \frac{QdQ}{2(s_0Q^2 + r_0Q + g_0)} = \left( 1 + \frac{t}{t_c} \right)^{-2} dt, \quad (16) \]

\[ \frac{QdQ}{2s_0(Q^2 + \frac{r_0Q}{2} + \frac{g_0}{2})} = \frac{QdQ}{2s_0(Q + a)(Q + b)}. \quad (17) \]

From the data shown in the table 3, it is possible that \( Q + a < 0 \) when \( a < 0 \) for Crab pulsar [17]. The derivation of the second integration is shown on the RHS below in equation (18) in appendix B.

\[ \int \frac{QdQ}{(Q + a)(Q + b)} = \begin{cases} \frac{1}{a-b} \{ a \ln(Q + a) - b \ln(Q + b) \}, & Q + a > 0; \\ \frac{1}{|a|+b} \{ b \ln(Q + b) + |a| \ln(|a| - Q) \}, & Q + a < 0. \end{cases} \quad (18) \]

Thus, we obtain the formulae for two cases:

1. \( Q + a > 0 \):
Equating and integrating the right hand side (RHS) of equation (17) and RHS of equation (16) we obtain:

\[
\frac{1}{a-b} \left\{ a \ln(Q + a) - b \ln(Q + b) \right\} = \frac{-2t_s s_0}{1 + t/t_c} + C. \tag{19}
\]

The initial condition \(P(t_0) = P_0\) can be used to fix \(C\), which can be expressed in the form:

\[
C = \frac{1}{a-b} \left\{ a \ln(P_0^2 + a) - b \ln(P_0^2 + b) \right\} + \frac{2t_s s_0}{1 + t_0/t_c}. \tag{20}
\]

Thereby, the full solution to equation (13) can be expressed as:

\[
\frac{1}{a-b} \left\{ a \ln \left( \frac{P^2 + a}{P_0^2 + a} \right) - b \ln \left( \frac{P^2 + b}{P_0^2 + b} \right) \right\} = 2t_s s_0 \left\{ \frac{1}{1 + t_0/t_c} - \frac{1}{1 + t/t_c} \right\} \tag{21a}
\]

\[
\approx 2t_s s_0 \left\{ 1 - \frac{1}{1 + t/t_c} \right\} \approx 2s_0 t, \tag{21b}
\]

in the approximation, \(t_c \gg t_0, t_c \gg t\).

Equation (21a) suggests a very interesting pattern of the generalized Lambert \(W\) function [39]. We explore the Lambert \(W\) solution in section 2.3.

2. \(Q + a < 0\):

We apply the same procedures from equations (19) to (21) and then get

\[
\frac{1}{|a| + b} \left\{ b \ln \left( \frac{P^2 + b}{P_0^2 + b} \right) + |a| \ln \left( \frac{|a| - P^2}{|a| - P_0^2} \right) \right\} \approx 2s_0 t, \tag{22}
\]

in the approximation, \(t_c \gg t_0, t_c \gg t\).

We can solve for \(a\) and \(b\) by finding the roots of the quadratic expression of \(Q^2 + \frac{r_0}{s_0} Q + \frac{g_0}{s_0}\). Here \(a + b = \frac{r_0}{s_0}\), \(ab = \frac{g_0}{s_0}\), \(a - b = \sqrt{\frac{r_0^2 - 4s_0 g_0}{s_0}}\) and here \(s_0, r_0, g_0\) are given from the formulae in appendix A.

Hence,

\[
a = \frac{1}{2} \left( \frac{r_0 + \sqrt{r_0^2 - 4s_0 g_0}}{s_0} \right), \tag{23a}
\]

\[
b = \frac{1}{2} \left( \frac{r_0 - \sqrt{r_0^2 - 4s_0 g_0}}{s_0} \right). \tag{23b}
\]

We analyze the three possible cases for the roots in equations (23a) and (23b):

1. \(r_0^2 - 4s_0 g_0 > 0, a > b\), e.g. B1509-58(1994).
Table 2. Frequency parameters and stationary multipole model fitting parameters for the two pulsars with real roots.

| Pulsar          | J0007+7303 [53] | J1833-0831 [54] |
|-----------------|-----------------|-----------------|
| \( f (\text{Hz}) \) | 3.166           | 0.132           |
| \( f'(s^{-1}) \) | \(-3.612 \times 10^{-12}\) | \(-6 \times 10^{-14}\) |
| \( f''(s^{-3}) \) | \(4.1 \times 10^{-23}\) | \(-1.3 \times 10^{-20}\) |
| \( \tilde{f}(s^{-4}) \) | \(5.4 \times 10^{-30}\) | \(9 \times 10^{-28}\) |
| \( n \)          | 9.95            | -4.77 \times 10^3 |
| \( m \)          | \(-1.15 \times 10^6\) | \(-7.28 \times 10^9\) |
| \( r (\text{Hz}^{-1}) \) | \(3.269 \times 10^{-8}\) | 1.953 |
| \( g (\text{Hz}^{-1}) \) | \(-1.631 \times 10^{-9}\) | -55.88 |
| \( s (\text{Hz}) \) | \(-1.638 \times 10^{-7}\) | \(-1.706 \times 10^{-2}\) |
| \( \text{sgn}(r_0^2 - 4s_0g_0) \) | +              | +               |
| \( a \)          | \(-0.100\)      | \(-57.235\)     |
| \( b \)          | \(-0.099\)      | \(-57.236\)     |

1 \( f \) of J1833-0831 is negative and \( f \) of J0007+7303, J1833-0831 are positive in contrast to pulsars in tables 3 and 4, which may be due to the presence of glitches.

2. \( r_0^2 - 4s_0g_0 = 0, a = b \). There is little data available except for J1833-0831, but this case can arise from pulsars that have certain values for \( s_0, r_0 \) and \( g_0 \). The two pulsars listed in table 2 satisfy the condition \( a \approx b \).

3. \( r_0^2 - 4s_0g_0 < 0, a \) and \( b \) are complex, e.g. the Vela pulsar. From data in table 1,\( g_0 = 3.8 \times 10^{-18} \text{ Hz}^{-3}, r_0 = 1.3 \times 10^{-15} \text{ Hz}^{-1} \) and \( s_0 = 1.2 \times 10^{-12} \text{ Hz}, \) we have \( r_0^2 - 4s_0g_0 = -1.655 \times 10^{-29} < 0 \).

In [2], they claim that \( s_0 > 0, r_0 > 0 \) and \( g_0 > 0 \) where \( s_0 > r_0 > g_0 \). For the Crab pulsar in table 1 from [2], by letting \( b = ae \) for \( a \approx 2.9057 \times 10^{-3}, b \approx 3.54 \times 10^{-5} \) and \( \epsilon \approx 0.01 \), we obtain:

\[
a \ln \left( \frac{Q + a}{Q_0 + a} \right) + ae \ln \left( \frac{Q_0 + ae}{Q + ae} \right) = 2s_0fa - 2s_0fae,
\]

which simplifies to:

\[
\left[ \ln \left( \frac{Q + a}{Q_0 + a} \right) - 2s_0f \right] + e \left[ \ln \left( \frac{Q_0 + ae}{Q + ae} \right) + 2s_0f \right] = 0.
\]

The same derivation is applied to equation (22) that satisfies \( a < 0 \) and \( |a| > |b| \). For the Crab pulsar(1993) in table 3, whose frequency values are obtained from one of its inter-glitches revolution period near 1988, \( a \approx -4.6 \times 10^{-3} \) and \( b \approx -2.9 \times 10^{-4} \). We let \( b = \epsilon |a| \), where \( \epsilon \approx 0.063 \) and then find:

\[
\left[ \ln \left( \frac{|a| - Q}{|a| - Q_0} \right) - 2s_0f \right] + e \left[ \ln \left( \frac{Q_0 + |a|\epsilon}{Q + |a|\epsilon} \right) + 2s_0f \right] = 0.
\]

For small \( \epsilon \), a perturbation analysis from equations (24) and (25) will be discussed in section 2.4.
Table 3. Timing parameters and stationary multipole model fitting parameters for the two pulsars with real roots.

| Pulsar | Crab(1988) | Crab(1993) | B1509-58(2011) | B1509-58(1994) |
|--------|------------|------------|----------------|----------------|
| $f$ (Hz) | 30.027 | 30.225 | 6.612 | 6.638 |
| $f'$ (s^{-2}) | $-3.780 \times 10^{-10}$ | $-3.86 \times 10^{-10}$ | $-6.694 \times 10^{-11}$ | $-6.770 \times 10^{-11}$ |
| $f''$ (s^{-4}) | $1.208 \times 10^{-20}$ | $1.426 \times 10^{-20}$ | $1.919 \times 10^{-21}$ | $1.959 \times 10^{-21}$ |
| $f'''$ (s^{-6}) | $-6.15 \times 10^{-31}$ | $-6.45 \times 10^{-31}$ | $-9.139 \times 10^{-32}$ | $-1.02 \times 10^{-31}$ |
| $n$ | 2.51 | 2.89 | 2.83 | 2.84 |
| $m$ | 10.1 | 10.23 | 13.3 | 14.49 |
| $r$ (Hz^{-3}) | $1.322 \times 10^{-14}$ | $2.650 \times 10^{-14}$ | $2.228 \times 10^{-13}$ | $1.590 \times 10^{-13}$ |
| $g$ (Hz^{-1}) | $-1.444 \times 10^{-18}$ | $-7.268 \times 10^{-18}$ | $-1.238 \times 10^{-16}$ | $6.083 \times 10^{-16}$ |
| $s$ (Hz) | $1.908 \times 10^{-12}$ | $-5.366 \times 10^{-12}$ | $6.217 \times 10^{-13}$ | $2.012 \times 10^{-12}$ |
| $\text{sgn}(r^2_0 - 4s_0g_0)$ | + | + | + | + |
| $a$ | $7.035 \times 10^{-3}$ | $-4.647 \times 10^{-3}$ | $3.589 \times 10^{-1}$ | $7.501 \times 10^{-2}$ |
| $b$ | $-1.076 \times 10^{-4}$ | $-2.915 \times 10^{-4}$ | $-5.549 \times 10^{-4}$ | $4.031 \times 10^{-3}$ |

2.3. The Lambert $W$ solution

The special case of $a = b$ is a rare situation, when the expressions in the radical sign in equations (23a) and (23b) vanish. This is true for the pulsar J1833-0831 with $a = b \approx -57.23$. This is shown below in table 2, at the bottom of this subsection. The left hand side of equation (18) can be integrated as:

$$\int \frac{Q}{(Q + a)^2} \, dQ = \log(Q + a) + \frac{a}{Q + a} = -\frac{2t_s s_0}{1 + t/t_c} + C. \tag{26}$$

After we insert in the initial condition $\log(Q_0 + a) + \frac{a}{Q_0 + a} = C - 2t_s s_0$ when $t = 0$ and $Q = Q_0$, the solution can be concisely written as:

$$-\log\left(\frac{a}{Q + a}\right) + \frac{a}{Q + a} + \log a = 2s_0 t + \log(Q_0 + a) + \frac{a}{Q_0 + a}. \tag{27}$$

Substituting $\frac{a}{Q + a} = z$ and $\frac{a}{Q_0 + a} = z_0$, we obtain by exponentiation on both sides:

$$e^{-\log z + z} = e^{2s_0 t + \log z_0}.$$  

Rearranging gives:

$$-ze^{-z} = -ze^{-z_0}e^{-2s_0 t}. \tag{28}$$

This transcendental equation can be solved to yield:

$$-z = W\left(-ze^{-z_0}e^{-2s_0 t}\right), \tag{29}$$

where $W$ is the Lambert $W$ function [40]. Thus, again for $t_c \gg t$;

$$-\frac{a}{Q + a} = W\left(-\frac{a}{Q_0 + a}e^{-a/(Q_0 + a)}e^{-2s_0 t}\right), \tag{30}$$

where $Q$ and $Q_0$ as defined previously are $Q = P^2$ and $Q_0 = P_0^2$. $W(z)$ has the series expansion:
\[ W(z) = \sum_{n \geq 1} \frac{(-n)^{n-1}}{n!} z^n \approx z - z^2 + \frac{3z^3}{2} + \ldots. \] (31)

The values of \(a\) and \(b\) for two of pulsars in table 2 show that it is possible to have a Lambert \(W\) solution of equation (2) for \(a \approx b\).

2.4. Spindown as a function of frequency

1. \(Q + a > 0:\)

From equation (24) for \(b = a\epsilon\), we give an equivalent expression in terms of the frequency:

\[ \ln \left( \frac{1}{f^2} + a \right) - \epsilon \ln \left( \frac{1}{f^2} + a\epsilon \right) = (1 - \epsilon)2\sigma t + \epsilon \ln \left( \frac{1}{f^2} + a \right), \] (32)

\[ \ln \left[ \frac{1 + a\epsilon^2}{1 + a\epsilon^2 f^2} \right] = \ln \left[ \frac{1 + a\epsilon^2}{1 + a\epsilon^2 f^2} \right] + (1 - \epsilon)2\sigma t. \] (33)

For \(g_0 \ll \gamma_0 < \gamma_0, a\) given by equation (23\(a\)), with \(b = a\epsilon \approx 10^{-2}\) for the Crab pulsar and \(s_0 = (1 - \epsilon)x_0\), we approximately obtain:

\[ \left( \frac{1 + a\epsilon^2}{1 + a\epsilon^2 f^2} \right) \approx e^{2\sigma t}, \] (34)

which can be written as:

\[ \frac{1}{f^2} = \left( \frac{1 + a\epsilon^2}{f^2} \right) e^{2\sigma t} - a. \] (35)

For the Crab pulsar, for this case of using the inverse magnetic law, \(s_0 = 7.5 \times 10^{-12}\) Hz\(^{-1}\), \(\gamma_0 = 9.4 \times 10^{-15}\) Hz was determined in table 3 of [2], however \(g_0\) was not found as this parameter was assumed to be zero in their model, which did not consider gravitational radiation.

2. \(Q + a < 0:\)

From equation (25) for \(b = |a|\epsilon\) when \(Q + a < 0\), an equivalent expression is given by exponential both sides:

\[ \left( \frac{|a| - Q}{Q^2} \right) \left( 1 + \frac{|a|\epsilon}{Q} \right) - \epsilon = \frac{|a| - Q_0}{(Q_0 + |a|\epsilon)^2} \exp(2\sigma t(1 - \epsilon)). \] (36)

We notice that:

\[ 1 + \frac{|a|\epsilon}{Q} \approx 1 - \frac{|a|\epsilon^2}{Q} + \frac{-\epsilon(-\epsilon - 1)}{2} \left( \frac{|a|\epsilon}{Q} \right)^2 + \ldots \approx 1. \]

The above approximation is guaranteed from table 3 for small \(\epsilon\) comparing to \(|a|Q = |a|f^2\), in which the largest \(\epsilon\) is \(-0.062\) for the Crab(1993) pulsar, and is fairly smaller than \(|b|/Q = 4.245\). From the use of this approximation and \(Q = f^{-2}\), we have:
\[(|af|^2 - 1)f^{2e-2} = Ke^{2s_0(1-e)}. \tag{37}\]

The constant term \( K = \frac{|a|-Q_0}{(Q_0+|a|)e}. \) We will simplify this equation in section 2.5 by perturbation methods.

3. Complex roots of \( a, b:\)

Although Alvarez and Carramiñana [2] considered the coefficients \( s_0, r_0 \), and \( g_0 \) to be \( \geq 0 \), to restrict the study to spindown of pulsars, it is worth considering the cases where one or more of \( s_0, r_0 \), and \( g_0 \) could be \( < 0 \), giving rise to the possibility that \( a \) and/or \( b \) could be \( < 0 \) and/or complex valued. However, \( s_0 > r_0 > g_0 > 0 \) is not always the case. Also, \( r_0^2 - 4s_0g_0 \) can be negative, resulting in the fact that \( a \) and \( b \) now become complex conjugates of each other:

\[a = \frac{1}{2} \left( \frac{r_0 + i\sqrt{4s_0g_0 - r_0^2}}{s_0} \right); \tag{38a}\]
\[b = \frac{1}{2} \left( \frac{r_0 - i\sqrt{4s_0g_0 - r_0^2}}{s_0} \right). \tag{38b}\]

From the ATNF database [24], we obtain other parameters in table 4 for this type of pulsars where \( a \) and \( b \) are complex valued. Further calculation shows that the frequency evolution equation for complex values of \( a \) and \( b \) can be written as:

\[|1 + af|^2 f^2 \exp\left\{ \frac{\Re a}{\Im a} (\theta_1 - \theta_2) \right\} = |1 + af_0^2| f_0^2 \exp(2s_0t). \tag{39}\]

Here, \( \theta_1 = \tan^{-1}\left( \frac{\Im a}{\Re a} \right) \) and \( \theta_2 = \tan^{-1}\left( \frac{\Im a}{\Re a} \right) \). The above equation can be simplified by an approximation that \( C = \frac{\Re a}{\Im a} (\theta_1 - \theta_2) \) is a constant:

\[\left( \frac{|1 + af|^2}{|1 + af_0^2|} \right)(f^2_0) \approx C \exp(2s_0t). \tag{40}\]

Thus, this approximated equation for complex case has a similar form of equation (34).

All parameters related to the case of complex roots are given in the table 4.

It can be seen from table 2 that \( \tilde{f} \) of J1833-0831 is negative and \( \tilde{f} \) of J0007+7303, J1833-0831 are positive in contrast to those of the other pulsars shown in tables 3 and 4. This may be due to the presence of glitches and timing noise. Due to the unknown mechanism of most glitches, numerical evidence provides one possible criterion in distinguishing glitches. Also, it is worthwhile to keep in consideration the possibility that the coefficients \( s_0, r_0, g_0 \) can change sign to indicate the occurrence of glitches in some time segments of pulsar data. Then there is a spinup in those time segments. Glitches are discrete changes in the pulsar rotation rate that are often followed by a relaxation [7, 8]. The cumulative effect of glitches is to reduce the regular long-term spindown rate \( |\tilde{f}| \) of the pulsar. The statistical properties of pulsar glitches and their potential impact on searches for continuous gravitational waves (CGW) has been carefully studied in [60].
2.5. Approximation of the frequency evolution

1. \( Q + a > 0 \):

From equation (35), for \( Q + a > 0, b = ae \ll a, g_0 \neq 0 \), and \( s_c = s_0(1 - \epsilon) \) we approximately obtain:

\[
f(t) = \left[ \left( \frac{1 + af_0^2}{f_0^2} \right) e^{2s_c t} - a \right]^{-\frac{1}{2}}. \tag{41a}
\]

\[
f(t) = f_0 \left[ (1 + af_0^2) e^{2s_c t} - af_0^2 \right]^{-\frac{1}{2}}. \tag{41b}
\]

Comparing the expression of \( f \) here with that for \( g_0 = 0 \):

\[
f(t) = f_0 s_0^\frac{1}{2} (\lambda_0 e^{2s_0 t} - \lambda_1)^{-\frac{1}{2}}, \tag{42}
\]

where \( \lambda_0 = f_0^2 r_0 + s_0, \lambda_1 = r_0 f_0^2 \), and \( a = \frac{a_0}{a_0} \). We find that when \( \epsilon = 0 \), the above two equations are similar in form.

For \( \epsilon \neq 0 \), equation (41b) can be rearranged as:

\[
f(t) = f_0 \left( 1 + af_0^2 \right) e^{2s_c t} \left\{ 1 - \left( \frac{af_0^2}{1 + af_0^2} \right) e^{-2s_c t} \right\}^{-\frac{1}{2}}. \tag{43}
\]

If one ignores the higher order terms in the curly brackets in equation (43), one can simplify further:

\[
f(t) = f_0 \left( 1 + af_0^2 \right)^{-1/2} e^{-s_c t}
\]

\[
= f_0 \left( 1 + af_0^2 \right)^{-1/2} \sum_{i=0}^{\infty} (-s_c t)^i \frac{1}{i!}
\]

\[
= f_0 \left( 1 + af_0^2 \right)^{-1/2} \left( 1 - s_c t + \frac{s_c^2 t^2}{2!} - \frac{s_c^3 t^3}{3!} + \frac{s_c^4 t^4}{4!} + \ldots \right). \tag{44}
\]

---

**Table 4.** Timing parameters and stationary multipole model fitting parameters for the four pulsars under the condition of complex roots.

| Pulsar          | Crab(2015) [8] | J1023-5746 [55] | J1418-6058 [56] | B2334 + 61 [57] |
|-----------------|----------------|-----------------|-----------------|-----------------|
| \( f \)         | 29.947         | 8.971           | 9.044           | 2.019           |
| \( n \)         | 2.34           | 66.8            | 30.2            | 47.0            |
| \( m \)         | 45.5           | 2.98 \times 10^5 | 2.46 \times 10^6 | 3.08 \times 10^6 |
| \( r \) (Hz\(^{-3}\)) | -1.170 \times 10^{-15} | -3.134 \times 10^{-9} | -1.156 \times 10^{-8} | -7.367 \times 10^{-8} |
| \( g \) (Hz\(^{-1}\)) | 7.051 \times 10^{-17} | 1.948 \times 10^{-11} | 7.040 \times 10^{-11} | 9.040 \times 10^{-9} |
| \( s \) (Hz)     | 6.086 \times 10^{-11} | 1.260 \times 10^{-7} | 4.709 \times 10^{-7} | 1.501 \times 10^{-7} |
| \( \text{sgn}(r_0 - 4s_0g_0) \) | —              | —               | —               | —               |
| Phase of \( a \) (Degrees) | 153.3          | 179.7           | 179.9           | 179.9           |
| Magnitude \( |a| (= |b|) \) | 1.076 \times 10^{-3} | 1.243 \times 10^{-2} | 1.223 \times 10^{-2} | 2.454 \times 10^{-1} |
However, the approximation given by equation (44) does not give good numerical values for the frequency spin-down parameters in comparison with observations in table 3, as it ignores the pivotal role of the second term in equations (41a) and (41b).

For the case of \( g_0 \neq 0 \), one can derive better approximation forms for \( a \) and \( 1 + a f_0^2 \):

\[
a = \frac{r_0}{2\lambda_0} + \frac{\sqrt{r_0^2 - 4s_0 g_0}}{2s_0} \approx \frac{r_0}{s_0} - \frac{g_0}{r_0} \quad (45a)
\]

\[
1 + a f_0^2 \approx 1 + f_0^2 \left[ \frac{r_0}{s_0} - \frac{g_0}{r_0} \right] = \frac{\lambda_0}{s_0} - \frac{g_0 f_0^2}{r_0} . \quad (45b)
\]

Thus, a more precise approximation of equation (41b) is:

\[
f(t) = f_0 \left[ \frac{\frac{\lambda_0}{s_0} - \frac{g_0 f_0^2}{r_0}}{\lambda_1} - \frac{\lambda_0}{s_0} + \frac{g_0 f_0^2}{r_0} \right]^{-\frac{1}{2}} . \quad (46a)
\]

\[
f(t) = f_0 \left[ \frac{\frac{\lambda_0}{s_0} e^{2\lambda_1 t} - \frac{\lambda_1}{s_0}}{\lambda_1} - \frac{\lambda_0}{s_0} e^{-2\lambda_1 t} \right]^{-\frac{1}{2}} . \quad (46b)
\]

\( \lambda_0 \) will not be the same as given previously (\( \lambda_0 = s_0 + r_0 f_0^2 \)), but will be instead:

\[
\lambda_c \approx \lambda_0 = \frac{g_0 f_0^2}{r_0} . \quad (47)
\]

Hence:

\[
f(t) = f_0 \sqrt{\frac{s_0}{\lambda_c}} e^{-\lambda_1 t} \left[ 1 - \frac{\lambda_1}{\lambda_c} e^{-2\lambda_1 t} \right]^{-1/2} . \quad (48)
\]

Equation (48) indicates the replacement of \( \lambda \) by \( \lambda_c \) for the more accurate approximation of equation (41b).

For \( g_0 \neq 0 \), simplification of equation (23a) in terms of \( \lambda_0 = r_0 f_0^2 + s_0 \) gives:

\[
a = \frac{r_0}{2\lambda_0} \left\{ 1 + \frac{|r_0|}{r_0} \sqrt{1 - 4g_0 \left( \frac{\lambda_0}{r_0} - \frac{f_0^2}{r_0} \right)} \right\} , \quad (49)
\]

where the \( |r_0| \) addresses the situation where \( r_0 < 0 \).

This expression can be more concisely written as:

\[
a = \frac{r_0}{2\lambda_0} \left\{ 1 + \frac{|r_0|}{r_0} \sqrt{1 - 4g_0 s_0} \right\} . \quad (50)
\]

2. \( Q + a < 0 \):

Since \( Q + a = f^{-2} + a < 0 \), we have \( f^{-2} < -a = |a| \) when \( a < 0 \). Thus, when \( |a| f^2 > 1 \), equation (37) can be written as:

\[
|a| f^2 (1 - \frac{1}{|a| f^2}) = Ke^{2\lambda_0 (1-\lambda)} .
\]
By expansion of the terms in paranthesis in the LHS, we have:

\[
\frac{1}{1 - \frac{1}{|a|f^2}} = 1 + \frac{1}{|a|f^2} + \frac{1}{|a|^2f^4} + \ldots.
\]

When \( |a|f^2 > 5 \), one can just retain the dominant. The approximation of frequency terms in equation (37) is \( (|a|f^2 - 1)f^{2\epsilon - 2} \approx |a|f^{2\epsilon} \). Thus, we obtain:

\[
f = \left( \frac{K}{|a|} \right)^{\frac{1}{2}} \exp \left[ s_0 t \left( \frac{1}{\epsilon} - 1 \right) \right], \quad K = \frac{|a| - f_0^{-2}}{(f_0^{-2} + |a|\epsilon)^\epsilon}.
\]

For smaller \( |a|f^2 \), we retain more terms in expansion to get more accurate approximation of equation (37):

\[
f = \frac{f_0^{-2\epsilon}}{|a|} \left( 1 + \frac{f_0^{-2\epsilon}}{|a|} \right) \approx \frac{1}{K} e^{-2s_0 t(1-\epsilon)}.
\]

From table 3, Crab(1993) with \( |a|f^2 \approx 4.3 \), we have to use a better approximation that includes the second term in equation (52).

3. Complex roots of \( a, b \):

We also discuss the approximation of equation (40). Because the spin-down of pulsars is very slow for small \( \dot{f} \), we are making the approximation that \( \frac{|f_1|}{|a|} \) is independent of time and the ratio is \( r \).

\[
\frac{r f_0^2}{f_0^{2\epsilon}} \approx C \exp(2s_0 t).
\]

A more exact analysis that includes time dependence can be done by using perturbation methods. This will be relevant in cases where \( r \) is substantially different from 1.

From these approximations under different conditions, we are able to estimate spin-down parameters in terms of frequency in the section 3.

3. Gravitational wave signal with spindown corrections

Based on the derivations done in section 2, we have the frequency spin-down evolution \( f_{\text{pulsar}} \) for \( g_0 \neq 0 \) (\( \epsilon \neq 0 \)). We will explore the gravitational wave spin-down parameters below.

At first, the values of the calculated spindown parameters are shown in table 5 using the approximation \( |f_1| \approx \frac{1}{\tau_{\text{min}}} \) [36, 37]. It should be emphasized that while this bound indicates the approximate values of the \( f_1 \) parameters, assuming a spindown decay with a characteristic timescale \( \tau_{\text{min}} \), it does not imply that values of \( f_1 \) outside of these bounds are impossible. In fact, both observations and our values are greater than the values obtained from this approximated relation.

It should be noted that time-dependent \( r(t), s(t) \) are taken from table 3 in Alvarez and Carramiñana [2], assuming an inverse linear magnetic field decay timescale consistent with \( r_0 \geq 0 \) and \( s_0 \geq 0 \). \( g_0 \) is taken from the stationary multipole model in table 1 of [2], including the quadrupole term with a time dependence for \( g(t) \) similar to that of \( r(t) \) and \( s(t) \). Since their values for \( r_0, s_0 \) and \( g_0 \) are obtained to satisfy the assumption of positive spin-down coefficients \( r(t), s(t) \) and \( g(t) \), we give better values listed in tables 2–4 for the verification of the frequency spindown. From observational data in the given tables, the coefficients \( s_0, r_0, g_0 \) do change sign. In particular, for \( r_0 < 0 \), the roots are complex although \( s_0 \) and \( g_0 \) are positive in this context. For
real roots, the signs of \( s_0 \) and \( g_0 \) can both change sign. There are potential corresponding physical variables, such as the time dependent magnetic dipole moment \( M \), moment of inertia \( I \) and inclination angle \( \alpha \) between the magnetic and rotational axes, which could explain such changes.

From [38], \( f_{GW}(t) = 2f_{\text{Pulsar}}(t) \), we are able to give higher order GW signal spin-down parameters from frequency spin-down parameters. Also, we find that the Crab pulsar might be emitting at \( f_{GW} \approx 4f_{\text{Pulsar}}/3 \) through an r-mode if the mode saturates at a small amplitude and thus is long-lived. However, the uncertainty of this frequency is relatively large, of order one part in \( 10^3 \) as was mentioned in the interesting work on LIGO beating the spin-down limit by Abbott et al [41] and again recently in [59]. They have presented direct upper limits on GW emissions from the Crab pulsar. The searches use the known frequency and position of the Crab pulsar. They find that, under the assumption that GW and the electromagnetic signals are phase locked, their single template search results constrain the GW luminosity to be less than 6% of the observed spindown luminosity, and beats the indirect limits obtained from all electromagnetic observations of the Crab pulsar and nebula. Similarly, Abadie et al have given the direct upper limits on GW emissions from the Vela pulsar using data from the VIRGO detector’s second science run [42].

The spindown of GW signal from pulsars has been studied in pioneering works by a parametrized model for the gravitational wave frequency [36, 38]:

\[
f = f_0 \left( 1 + \frac{\vec{v}}{c} \cdot \hat{n} \right) \left( 1 + \sum_{k=1}^{r} f_k \left[ t + \frac{\vec{x}}{c} \cdot \hat{n} \right]^k \right), \tag{54a}
\]

where the terms \( \vec{v}/c \) and \( \vec{x}/c \) account for the Doppler shift. In practice this equation is simplified; \( \vec{x}/c \) is limited to the light travel time between the Earth and Sun, which is \( \approx 500 \) s. So for typical observations \( \gg 500 \) s, \( t \gg \vec{x}/c \) and the second bracketed term becomes \( 1 + \sum f_k t^k \) (equation (18) in [38] and appendix A of [38]). We adopt the widespread convention [38] that the spindowns are the coefficients of a Taylor series, i.e.
where the GW spin-down coefficients \( f_k \) are evaluated at \( t = t_{ref} \) and \( t_{ref} \) is some reference time. By ignoring the Doppler shift (which we investigate in a later paper), we can further express the GW signal in terms of the parametrized series for pulsars as a linear combination of Chebyshev polynomials:

\[
f_{GW}(t) = \sum_{k} f_k \left( \frac{t}{\tau_{min}} \right)^k = \sum_{k} F_k T_k \left( \frac{t}{\tau_{min}} \right),
\]

where \( \tau_{min} \) is the spin-down age of neutron stars [38], or \( \tau_{min} \approx f/f_\text{max} \) is the characteristic time scale over which the frequency might change by a factor \( \sim 1 \). It should be observed that \( F_k \neq f_k \) but is a linear combination of \( f_k \). For young, fast pulsars, \( f \approx 1 \text{ kHz} \) and \( \tau_{min} = 40 \text{ yr} \), and for older and slower pulsars, \( f \approx 200 \text{ Hz} \) and \( \tau_{min} = 1000 \text{ yr} \) [36].

Though this approach is mathematically elegant, tractable and designed to use the orthogonality of the Chebyshev polynomials to derive the \( f_k \), it’s not very practical since the spin-down coefficients appear as a linear combination, in contrast to the original Taylor series, requiring a more detailed analysis that is not necessary. For the limiting case of \( f_k = \frac{1}{\tau_{min}} \) [35, 36], the parameterized series of equation (54a) can be exactly summed to give a closed analytic form in the second expression of equation (55). For such a simplified case, the analytic form of the gravitational wave pulsar signal can be exactly obtained. We use the approach of Jaranowski et al [38] from equations (54a)–(54c), so that we can directly compare the frequency spin-down parameters. Several estimates are given in table 5 from expressions of the frequency evolution in section 2.4 by the following derivations.

1. \( Q + a > 0 \):

A good estimation is given following from equation (46b):

\[
f(t) = f^{(0)} \left[ \frac{\lambda_0 e^{2\pi \epsilon t}}{\lambda_0} \right]^{-\frac{1}{2}},
\]

where \( \lambda_0 = \frac{a \epsilon}{2} \frac{\epsilon_0}{2} (f^{(0)})^2 \), \( \lambda_0 = \frac{a_0}{2} \frac{e_0}{2} (f^{(0)})^2 \), \( s_c = s_0 (1 - \epsilon) \) and \( \epsilon = \frac{\epsilon_0}{2} \). Here we have \( \lambda_0 - \lambda_0 = 1 \).

From the above equation (46b), we apply the Taylor expansion to above rotation frequency evolution function:

\[
f(t) = \frac{f^{(0)}}{\sqrt{\lambda_0} e^{i \epsilon t}} \left[ \frac{1}{\lambda_0} \frac{\lambda_0 e^{2\pi \epsilon t}}{\lambda_0} \right]^{-\frac{1}{2}}
\]

\[
= \frac{f^{(0)}}{\sqrt{\lambda_0}} e^{i \epsilon t} \sum_{n=0}^{\infty} \left[ \frac{\lambda_0 e^{2\pi \epsilon t}}{\lambda_0} \right]^n (-1)^n \left( -\frac{1}{2} \right) \left( \frac{2n+1}{n} \right) e^{-(2n+1) \epsilon t}
\]

\[
= \frac{f^{(0)}}{\sqrt{\lambda_0}} \sum_{n=0}^{\infty} \left( \frac{\lambda_0 e^{2\pi \epsilon t}}{\lambda_0} \right)^n \left( -\frac{1}{2} \right) \left( \frac{2n+1}{n} \right) \sum_{m=0}^{\infty} \left( \frac{(-2n+1) \epsilon t)^m}{m!} \right)
\]

\[
= \frac{f^{(0)}}{\sqrt{\lambda_0}} \sum_{n=0}^{\infty} \left[ \frac{\lambda_0 e^{2\pi \epsilon t}}{\lambda_0} \right]^n \left( -\frac{1}{2} \right) \left( \frac{2n+1}{n} \right) m!
\]

\[
= \sum_{n=0}^{\infty} \left[ \frac{f^{(0)}}{\lambda_0} \right] s_c^m \lambda_0 \sum_{n=0}^{\infty} \left( \frac{\lambda_0 e^{2\pi \epsilon t}}{\lambda_0} \right)^n \left( -\frac{1}{2} \right) \left( \frac{2n+1}{n} \right) m!\epsilon_t^m.
\]
Then the pulsar frequency spin-down parameters \( f^{(k)} \) are:
\[
f^{(k)} = \frac{f_0 s_k}{\pi^{3/2} k!} \sum_{n=0}^{\infty} \left( \frac{\lambda_{1c}}{\lambda_{1c}} \right)^n (-1)^{n+k} \left( -\frac{1}{2} \right) (2n + 1)^k.
\] (56)

The first four frequency spin-down parameters \( f_1 \) to \( f_4 \) are given as follows. They are obtained from the derivatives of the binomial expansion \((1 - x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} x^n (-1)^n \left( -\frac{1}{2} \right) \) and the resulting combinatorial analysis.
\[
\begin{align*}
    f^{(1)} &= -f_0 \delta_0 (\lambda_{1c} + 1); \\
    f^{(2)} &= 2f_0^2 (\lambda_{1c} + 1)(3\lambda_{1c} + 1); \\
    f^{(3)} &= -\frac{1}{6} f_0^3 (\lambda_{1c} + 1)(15\lambda_{1c}^2 + 12\lambda_{1c} + 1); \\
    f^{(4)} &= \frac{1}{24} f_0^4 (\lambda_{1c} + 1)(105\lambda_{1c}^3 + 135\lambda_{1c}^2 + 39\lambda_{1c} + 1).
\end{align*}
\]

Other higher order coefficients can be given by software with symbolic computation, like Maple and Mathematica. The calculated results for the available data from Table 2 are shown in the following Table 5 to check the consistency of the approximated model.

2. \( Q + a < 0 \)

For large value of \(|a|f_0^2\), we expand the exponential part in equation (52) to calculate the frequency spin-down parameters:
\[
f^{(k)} = \frac{2K^{2c}}{k!|a|^2f_0^4} \left( \frac{1}{e} - 1 \right)^k, \quad K = \frac{|a| - f_0^{-2}}{(f_0^{-2} + |a|e)^c}.
\] (57)

However, for \( 1 < |a|f_0^2 < 5 \), we apply the perturbation method. We replace \( f \) in equation (37) by equation (54b) to obtain the approximation of frequency spin-down parameters by comparing with the expansion of \( Ke^{2a(1-e)} \):
\[
(|a|f_0^2 - 1)f_0^{2e-2} = Ke^{2a(1-e)},
\] (58)
\[
|a|(f^{(0)} + f^{(1)} + \ldots)^{2e} - (f^{(0)} + f^{(1)} + \ldots)^{2e-2} = K \sum_{k=0}^{\infty} \frac{[2a/(1-e)]^k}{k!}.
\] (59)

The first three frequency spin-down parameters are listed as follows:
\[
\begin{align*}
    f^{(1)} &= \frac{2K_0 (1 - e)}{2|a|f_0^{(e-1)} - (2e - 2)f_0^{2e-3}}; \\
    f^{(2)} &= \frac{2K_0^2 (1 - e)^2 - \left[ |a| (2c_2f_0^{2e-2} - (2c_2^2 - 2c_2^4)f_0^{2e-4}) \right] f_0^2}{2|a|f_0^{(e-1)} - (2e - 2)f_0^{2e-3}}; \\
    f^{(3)} &= \frac{4K_0^3 (1 - e)^3 - \left[ |a| (2c_2^2 - 2c_2^4)f_0^{2e-2} - (2c_2^2 - 4c_2^4)f_0^{2e-2} \right] f_0^{2e-1}f_0^2 - \left[ |a| (2c_2^3) - (2c_2^3)f_0^{2e-2} - (2c_2^3)f_0^{2e-4} \right] f_0^{2e-3}f_0^3}{2|a|f_0^{(e-1)} - (2e - 2)f_0^{2e-3}}.
\end{align*}
\] (60a, 60b, 60c)

There is only one pulsar, Crab(1993) [17], available for \( Q + a < 0 \), we calculate the estimates listed in Table 5.
18

3. Complex roots of $a, b$

When we approximate $\left| \frac{1+af}{1+af_0} \right| \approx r$, we can get the spin-down parameters of equation (53) by exponential expansion:

$$f_p^{(k)} = \frac{f_0}{k!} \left( \frac{C}{r} \right)^k (2s_0)^k. \tag{61}$$

For a precise approximation of $\left| \frac{1+af}{1+af_0} \right|$ in equation (40), we use a perturbation method.

We will discuss the spin-down parameters in the complex case which involves glitches in future work.

Results for the real roots situation are given in the table 5. Thus, from $f_k \approx 2f_p^{(k)}$, we can obtain the estimates of gravitational wave spin-down parameters, which are listed in table 6.

For $s_0 > 0, g_0 \neq 0, \lambda_c < \lambda_0$, the spin-down coefficient $|g|$ will be either having slightly higher or lower values in comparison to the case when $g_0 = 0$ as seen in table 5. Also when $g_0 < 0$, as can occur for spinups, $C$ will be positive and $|g|$ will be lower during such time segments. For $s_0 < 0$, the expression for $f(t)$ is modified accordingly as demonstrated in the limiting case of $g_0 = 0$ in equation (8b) above, whereby higher values of the spin-down parameters are obtained.

For a relatively old pulsar, such as the Crab pulsar, three spin-down parameters may be adequate. For a young pulsar, $g_0$ could be more significant and more spin-down parameters need to be evaluated. Such a pulsar could potentially be associated with SN 1987A, pending discovery. Santostasi et al [51] suggested that it could be within reach of GW detectors if the strain sensitivity reaches $10^{-27}$, although the claimed observation that underlies that calculation has been called into question [23]. Equations (56) and (59) provide the analytic expression for $f_k$, which could be used for all spin-down parameters.

4. Conclusions

The data analysis for continuous GW, for example from rapidly spinning neutron stars, is an important problem for ground based detectors that demands analytic, computational and experimental ingenuity. The Crab and Vela pulsars are among the iconic sources of GW emission. By using pulsars as cosmic gravitational wave detectors, or timing pulsars which are found to be orbiting black holes, astronomers will be able to examine the limits of general relativity such as the behaviour of spacetime in regions of extremely curved space-time.

In this work, we have presented an analytic formulation for determining spin-down parameters in the Jaranowski et al [38] approach using a pulsar model which assumes an inverse linear law decay of the magnetic field [2]. We were able to extract these parameters using
the exact solution involving the monopolar, dipolar and quadrupolar terms in the model and found these to be in good accord with observation. For the Crab, we obtained the first three frequency spindown parameters which are in reasonable agreement with observations [15, 17] and Jaranowski et al [38]. For B1509-58, which has a more stable spin-down evolution compared to the Crab pulsar, the first three frequency spindown parameters match very well with observations [12, 13]. Both observations and our estimates are higher than the limit $|f_k| \approx \frac{1}{\tau_{\min}}$ given by Brady & Creighton [36].

It is to be hoped that better values of $r_0, s_0$ including the presently undetermined $g_0$ should be available from pulsar data in the coming years. It should also be noted that in the event of a merger of two relatively light neutron stars, there is a possibility of the formation of a new neutron star, though it might be relatively short [48, 49]. With the determination of the quadrupole coefficient $g_0$ from data, our solution can be incorporated with further improvement in accuracy of the spindown parameters. The study of pulsar spindown and evolution of its braking index will lead to further interesting explorations of the anomalies present in the timing structure, not only in connection with gravitational waves, but also in the fundamental aspects of quark deconfinement in pulsar cores [43]. The study of pulsar spindown also implicitly involves the role of spinups. The physics behind glitches is an active ongoing area of research that presents challenging studies such as the interior of neutron stars and the properties of matter at ultra high nuclear densities [44]. We are presently working on utilizing the available data on isolated pulsars towards finding fits for the $P - \dot{P}$ diagram that would also include the quadrupole term $g_0$ using the analytic expressions derived in this work to study gravitational wave data mining for the Crab Pulsar [45] and other pulsars. We will apply statistical methods to obtain more accurate values of the spin-down parameters.

We hope to further improve the accuracy of the spindown parameters for GW signal detection and extend the applicability of our approach to younger pulsars. We intend to conduct a data analysis for $s, r$ and $g$, the three pulsar spindown parameters in an upcoming work [45], which would give us an estimate on their associated fit uncertainties. Young pulsars demonstrate timing noise as Hobbes et al [14], state in their study and our data analysis study will reveal if such variations are accounted for in our model.

Davies et al have an improved method of targeting continuous gravitational wave signals in the LIGO and VIRGO detector data [46]. This method has a higher efficiency than the time-domain Bayesian pipeline that was used in many previous GW searches. In forthcoming work, we plan to develop the analytic Fourier Transform of the pulsar GW signal to include spindown [47]. This scenario is of particular relevance to our work as the detection of GW emission from such relatively young, spinning and rapidly evolving neutron stars [44] makes them important astrophysical sources for GW pulsar spin down.

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Appendix A. The derivation of coefficients

The parameters provided in [2] are obtained from estimates of the second braking index $m$ in order to satisfy their assumption of positive $s_0$, $r_0$ and $g_0$. We note that in applying their values to equation (2), there are differences between the estimates and observations of frequency spin-down parameters. A much more natural way to estimate $s_0$, $r_0$ and $g_0$ is given by solving the following linear system of equations:

\[
\begin{align*}
\dot{f} &= -sf - rf^3 - gf^5 \\
\ddot{f} &= \dot{f}(-s - 3rf^2 - 5gf^4) \\
\end{align*}
\]

Thus, the coefficients of spindown are given as follows:

\[
\begin{align*}
s &= -\frac{\dot{f}}{f} - rf^2 - gf^4; \\
r &= \frac{5\dot{f}}{4f^3} - \frac{5\ddot{f}}{4f^2\dot{f}} + \frac{\dddot{f}}{4f^2\dot{f}^2}; \\
g &= \frac{\ddot{f}^2}{8f^3\dot{f}^3} + \frac{3\dddot{f}}{8f^2\dot{f}^2} - \frac{3\dot{f}}{8f^5} - \frac{\dddot{f}}{8f^2\dot{f}^2}. \\
\end{align*}
\]

From this we obtain the same result comparing with equation (5) in [2]. The reason for a large deviation is due to their estimates on the second braking index, which has no relation to the solution of the above system of linear equations. Although using the braking index simplifies the calculation, the equations we provide are more accurate. It is worthwhile to investigate the uncertainty bounds of pulsar parameters, $s$, $r$ and $g$ for a more accurate description of pulsar spindown.

Appendix B. The derivation of equation (18)

If $Q + a < 0$,

\[
\int \frac{QdQ}{(Q + a)(Q + b)} = -\int \frac{QdQ}{(Q + b)(-a - Q)} = -\int \frac{QdQ}{(Q + b)(|a| - Q)} = \frac{1}{|a| + b} \left( b \int \frac{dQ}{Q + b} - |a| \int \frac{dQ}{|a| - Q} \right).
\]

Then we have the same formula as in equation (18).

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