Geometrical dependence in Casimir-Polder repulsion

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Repulsion, induced from quantum vacuum fluctuations, for an anisotropically polarizable atom on the symmetry axis of an anisotropically polarizable annular disc is studied. There exists two torsion free points on each side of the annular disc, where the interaction energy is orientation independent. The position of second of the two torsion free points, on either side of the disc, is shown to determine the orientation dependence of the atom at distances far from the plate, revealing a geometrical dependence. In the ring limit of the annular disc, new repulsion emerges.

The Casimir effect is an umbrella term referring to interactions induced by quantum fluctuations in the electromagnetic field. The associated forces become dominant in the nanoscale, for neutral configurations. The related van der Waals and London dispersion forces have to do with the domain in which the materials interact weakly at short distances [1–4]. Casimir and Polder [5] extended London dispersion force to include retardation effects at large distances using fourth order perturbation theory. Casimir then computed the attraction between two neutral conducting plates, which was inaccessible to perturbation theory, by evaluating the electromagnetic zero point energy outside and inside the plates with only the difference contributing to a finite inward pressure [6]. Thus, Casimir’s result suggested that the zero point energy could have measurable consequences.

Repulsive forces from the Casimir effect counter the intuition given for its usual attractive nature. Nevertheless, the first repulsive forces were found early on in three-body van der Waals interactions by Axilrod and Teller [7] and Muto [8], and by Craig and Power [9, 10] in Casimir-Polder interactions between anisotropically polarizable atoms. The counterintuitive nature of repulsion induced from quantum fluctuations explains why it was not until a decade ago that a repulsive effect was realized between conductors in the (strong and retarded) nonperturbative Casimir regime. This came by in Ref. [11], where it was argued on physical grounds that the interaction energy between an elongated conductor and a perfectly conducting metal sheet with a circular aperture could have a local minima. They showed that the Casimir force could become repulsive when the needle got sufficiently close to the aperture. It was suggested that the anisotropy in the geometry of a highly conducting object corresponds to an effective anisotropic permittivity and the interplay of these anisotropic permittivities leads to non-monotonic interaction energies that causes repulsion [12, 13]. Only few attempts have been made to understand the result in Ref. [11] analytically. In Ref. [14], using the method of inversion from electrostatics, it was shown that even in the van der Waals (weak and non-retarded) limit, similar configurations lead to repulsion. In Ref. [15], the method of inversion was again used to show that in the (weak and non-retarded) van der Waals regime an anisotropically polarizable atom placed along the symmetry axis of a toroid could experience repulsion. An advance was made in the retarded regime when an analytic formula for the (weak and retarded) Casimir-Polder energy between dielectric bodies was derived in Refs. [13] and [16], demonstrating that repulsion is possible in configurations with anisotropically polarizable atoms and anisotropic dielectric materials. In spite of the above successes in the weak scenario, analytic derivation in the strong regime of a closed-form expression for the interaction energy between a polarizable atom and a highly conducting plate with an aperture still remains elusive [17–19].

Repulsion in Casimir effect also appears if the interacting bodies possess certain electric and magnetic properties [20], or if the medium’s conductivity is greater than that of the materials [21]. However, in the current level of our understanding, apparently, these repulsions have different origins from those discussed in this paper: that of interaction energies between anisotropically polarizable atoms and anisotropically polarizable dielectrics.

In this Letter, we analyze the interaction, including the orientation dependence, of an anisotropically polarizable atom above the center of an anisotropically polarizable ring depicted in Fig. 1 using the methods of Refs. [13, 16]. A full exposition of the interaction for each polarization basis of the ring and the atom is available in Ref. [22], which serves as supplementary material for this article.

FIG. 1. Point atom of polarizability $\alpha$ above a dielectric ring of polarizability $\sigma$. The atom is on the symmetry axis of the ring.
Casimir-Polder repulsion for an atom interacting with the ring demonstrates a richer dependence on polarizability than the corresponding interaction with a plate with a circular aperture in Ref. [13]. In Refs. [13, 16] it was shown that the atom gets repelled at short distances only when its orientation deviates no more than a critical angle away from the axis perpendicular to the plate. An orientation independent point, where the energy is independent of the orientation of the atom and thus feels no torsion there, was found on both sides of the plate. It was established that the energy is minimized when its orientation is parallel to that of the plate for distances less than the torsion free point, and perpendicular to the plate for larger distances.

We show that, for certain polarizabilities of the ring in Fig. 1, see Ref. [22] for details, there exist two torsion-free heights on each side of the ring, in contrast to a single torsion-free point on each side for a plate with circular aperture. We show that for specific orientations the atom could feel a repulsive force when it is near the center of the ring, and, in addition, at intermediate distances from the center of the ring, for example see Fig. 2. To understand this additional repulsive region, and the additional torsion-free point, we analyze the configuration involving an atom interacting with an anisotropically polarizable annular disc. Keeping the inner radius of the annular disc fixed and varying the outer radius, we can smoothly deform the disc into a ring, or a plate of infinite extent with a circular aperture. We find upper bounds on the outer radius of the disc that allows for repulsion at intermediate distances. The second torsion-free height moves farther away as the outer radius of the disc increases. Interestingly, even though one torsion-free point moves to infinity in the plate limit for certain polarizabilities, it does not move afar indefinitely for all the polarizabilities.

In this Letter we focus our attention to the case when the ring is polarizable in the direction of the symmetry axis of the ring. For an extensive analysis we refer to Ref. [22]. The discussion in this article and repulsive Casimir effect in general is expected to be of considerable interest in nanotechnology, because it could assist in reducing ‘stiction’ among the components within nano devices. As a simplenminded application of a configuration studied here, we entertain the plausibility that the interaction between a ring and atom could serve as a prototype for a rotaxane. We discuss an idealized cycle with a circular aperture. We find upper bounds on the interaction energy of Eq. (1) for various orientations $\theta$. The energy is minimized at $h = 0$ for $\theta = 0^\circ$, suggesting that for $h = 0$ the atom tries to align its direction of polarizability with that of the ring. The heights for which the interaction energy is orientation independent and torsion-free is determined when the coefficient of $\cos(2\theta)$ in Eq. (3) is zero, which is at $h = \pm 0.48 a$ and $h = \pm 1.69 a$. Upon crossing a torsion-free point, the orientation that minimizes the energy changes by $90^\circ$ as shown in the inset of Fig. 2. Repulsive forces on the atom requires the parameters to satisfy the inequality

$$\frac{h}{a} < \pm \sqrt{\frac{(19 + 181 \cos 2\theta) \pm D}{80(1 + \cos 2\theta)}}$$

where

$$D = -6039 - 11682 \cos 2\theta + 20601 \cos^2 2\theta.$$  

Repulsion occurs when $60.88^\circ < \theta < 119.12^\circ$ for short distances and has the maximum extent in height for $|h| < 0.47 a$ at $\theta = 90^\circ$. When $|\theta| < 13.27^\circ$, repulsion emerges for a range of intermediate distances. This range is of maximum extent when $\theta = 0^\circ$, for $1.24 a < |h| < 1.41 a$. The repulsive force at these intermediate distances, though small in magnitude, might lead to interesting applications.
FIG. 2. The interaction energy between an atom with polarizability \( \alpha = \alpha_1 \hat{e}_1 \hat{e}_1 \) and a ring of polarizability \( \sigma = \sigma_z \hat{z} \hat{z} \) is illustrated in bottom left corner of figure. The energy in Eq. (3) is plotted as function of height \( h \) for different orientations of \( \hat{e}_1 \) with respect to the symmetry axis of the ring. The first region of repulsion occurs for \( |h| < 0.47 a \) and \( 60.88^\circ < \theta < 119.12^\circ \). A second region of repulsion is not visible, however, the inset provides a zoomed in view of this non-monotonicity, where the switch from attraction to repulsion occurs.

Reulsive force experienced by the atom in the direction of the symmetry axis will be a manifestation of negative slopes in these plots.

We consider an annular disk to understand the differences between the ring and an apertured plate. The annular disc is described by \( \chi_2(\textbf{r}) = \lambda(\omega) \theta(b - \rho) \theta(\rho - a) \delta(z) \), where \( \lambda(\omega) \) is the polarizability per unit area of the annular disc. Here \( b \) is the outer radius of the disc and \( a \) is the inner radius. We consider the case \( \lambda(0) = \lambda_0 \hat{z} \hat{z} \). The limit as \( b \to \infty \) corresponds to a plate with a circular aperture of radius \( a \). Taking the delicate limit of \( b \to a \) with the surface polarizability \( \lambda_0 \to \infty \), such that \( \sigma_z = \lambda(b - a) \) is kept fixed, constructs a ring [22]. The energy of the annular disc is computed to be [22]

\[
E = -\frac{\hbar c \alpha_1 \lambda_2}{64\pi} \frac{1}{5} \left( \frac{(-1)}{\rho^2 + \rho_z^2} \right)^{\frac{1}{2}} \left[ (26\rho^4 + 17h^2\rho^2 + 26h^4) \right. \\
+ \left. (26\rho^4 - 73h^2\rho^2 + 6h^4) \cos 2\theta \right] \bigg|_{\rho = b}^{\rho = a}.
\]

Unlike the case of ring, in Eq. (4), the repulsive regions can not be captured in a closed-form expression for the case of a disc, but can be easily determined numerically. The intermediate region of repulsion, as described in the zoomed in view of Fig. 2 for a ring, vanishes when the width of the annulus is above the critical values given in Table I for the specific orientations of the atom. The four orientation independent heights, \( \pm h_1 \) and \( \pm h_2 \), for the case of disc are determined by

\[
\frac{(26b^4 - 73b^2h^2 + 6h^4)}{(b^2 + h^2)^2} - \frac{(26a^4 - 73a^2h^2 + 6h^4)}{(a^2 + h^2)^2} = 0,
\]

with the solutions above the disc satisfying

\[
0.48a \xrightarrow{\text{ring}} |h_1| \xrightarrow{\text{plate}} 0.60a \quad (8a)
\]

and

\[
1.69a \xrightarrow{\text{ring}} |h_2| \xrightarrow{\text{plate}} 3.44a. \quad (8b)
\]

A difference in the systematics of the features in the energy occurs between axial versus radial polarizability of the disc [22]. The four orientation independent heights for the case of radial polarizability of the disc are given by

\[
0.36a \xrightarrow{\text{ring}} |h_1| \xrightarrow{\text{plate}} 0.44a \quad (9a)
\]

and

\[
3.45a \xrightarrow{\text{ring}} |h_2| \xrightarrow{\text{plate}} \infty. \quad (9b)
\]

Unlike the case of radial polarizability above, in the case of axial polarizability the two orientation independent points, two on each side of the disc, exist even in the limit when the disc becomes a plate. In this scenario, the second orientation independent height increases to a finite value in the plate limit. This implies that the orientation preference of the atom at \( h = 0 \) and \( h \to \infty \) is the same for axial polarizability. This should be contrasted with that of radially polarizable disc in which case the orientation preference of the polarizability of the atom at \( h = 0 \) is orthogonal to that at \( h \to \infty \).

We summarize our results in Fig. 3, which highlights the orientation independent torsion free points and the domains of repulsion. For large distances from the aperture the atom tends to orient its polarizability perpendicular to the polarizability of the plate when the plate is polarizable in the radial direction, while it tends to orient parallel when the plate is polarizable in the axial direction. This feature at \( h \to \infty \) is completely determined.

| Orientation | Criteria for second region of repulsion |
|-------------|----------------------------------------|
| \( \theta = 0^\circ \) | \( a < b < 1.6905a \) |
| \( \theta = 6^\circ \) | \( a < b < 1.559a \) |
| \( \theta = 9^\circ \) | \( a < b < 1.4379a \) |
| \( \theta = 12^\circ \) | \( a < b < 1.2323a \) |

TABLE I. This table lists criteria for the region of repulsion for intermediate distances repulsion to occur between an anisotropically polarizable atom and an annular disc with polarizability in the \( z \)-direction. The second column lists the upper bound on the outer radius \( b \) of the disc for orientation angles \( \theta \) of the atom.
by the existence of the second torsion free point. The repulsion felt at intermediate distances is about six orders of magnitude weaker than the repulsion at short distances which still lacks experimental evidence. It is not clear if the repulsions found for intermediate region of heights will persist when the conductivity of the plate is larger.

As a simple-minded application of the configuration discussed here, we propose a prototype of Casimir machine along the lines of a rotaxane [23], rendering its applicability to molecular transportation, molecular switching, and nanorecording. By requiring the atom to be fixed on an axis and allowing the ring to move with center on the axis, we consider a cyclic path involving the points $A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$ in Fig. 4. The atom starts at the center of the ring while orientated parallel to the polarizability of the ring, denoted by $A$. We put work into the system to rotate the atom by $90^\circ$ so that the energy is maximized at point $B$. The ring is then repelled from the atom at $h = 0.47a$. By expending a small amount of energy the ring adjusts its position to the torsion free point at $h_e = 0.48a$ labeled in the diagram by $C$. Position $D$ represents the ring at $h_e$ but with the atom rotated, without cost of internal energy, to $\theta = 0$. It is energetically favorable for the atom to then return to the center of ring and complete the cycle.

In summary, the interaction between an anisotropically polarizable dielectric annulus and an anisotropically polarizable atom reveals tight constraints on their relative geometrical orientations in their directions of polarizability for repulsion. We have shown that the interaction energies are characterized by torsion free points and domains of repulsion. The intuition gained from this study will guide us in finding a complete understanding of Casimir repulsion.

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