Abstract

With the prevalence of machine learning services, crowdsourced data containing sensitive information poses substantial privacy challenges. Existing works focusing on protecting against membership inference attacks under the rigorous notion of differential privacy are susceptible to attribute inference attacks. In this paper, we develop a theoretical framework for task-specific privacy under the attack of attribute inference. Under our framework, we propose a minimax optimization formulation with a practical algorithm to protect a given attribute and preserve utility. We also extend our formulation so that multiple attributes could be simultaneously protected. Theoretically, we prove an information-theoretic lower bound to characterize the inherent tradeoff between utility and privacy when they are correlated. Empirically, we conduct experiments with real-world tasks that demonstrate the effectiveness of our method compared with state-of-the-art baseline approaches.

1 Introduction

With the growing demand for machine learning systems provided as services, a massive amount of data containing sensitive information, such as race, income level, age, etc., are generated and collected from local users. This poses a substantial privacy challenge and it has become an imperative object of study in machine learning [Abadi et al., 2016, Gilad-Bachrach et al., 2016], computer vision [Chou et al., 2018, Wu et al., 2018], healthcare [Beaulieu-Jones et al., 2018b,a], security [Shokri et al., 2017], and many other domains. In this paper, we consider a practical scenario where the prediction vendor requests crowdsourced data for a target task, e.g., training a machine learning classifier for product recommendation. The data owner agrees on the data usage for the target task while she does not want her other private information (e.g., age, race) to be leaked. The goal of privacy-preserving in this context is then to protect private attributes of the sanitized data released by data owner from potential attribute inference attacks of a malicious
adversary. For example, in an online advertising scenario, while the user (data owner) may agree to share her historical purchasing events, she also wants to protect her age information so that no malicious adversary can infer her age range from the shared data. Note that simply removing age attribute from the shared data is insufficient for this purpose, due to the redundant encoding in data, i.e., other attributes may have a high correlation with age [Lum and Johndrow, 2016]. Among many other techniques, differential privacy (DP) has been proposed and extensively investigated to protect the privacy of collected data [Dwork and Nissim, 2004, Dwork et al., 2006]. DP embraces formal guarantees for privacy problems such as defending against the membership query attacks [Abadi et al., 2016, Papernot et al., 2016], or ensures the distribution of any two data records statistically indistinguishable [Erlingsson et al., 2014, Duchi et al., 2013, Bassily and Smith, 2015]. However, DP still suffers from attribute inference attacks [Cormode, 2011, Gong and Liu, 2016], as it only prevents an adversary from gaining additional knowledge by inclusion/exclusion of a subject, not from gaining knowledge from the data itself [Dwork et al., 2014]. As a result, an adversary can still accurately infer sensitive attributes of data owners from differentially-private datasets. Such a gap between theory and practice calls for an important and appealing challenge:

Can we find a transformation of the raw data to remove private information related to a sensitive attribute while still maximizing our utility of the target task? If yes, what is the fundamental tradeoff between privacy preservation and utility maximization?

Clearly, under the setting of attribute inference attacks, the notion of privacy preservation should be task-specific: the goal is to protect specific attributes from being inferred by malicious adversaries. Note that this is in sharp contrast with differential privacy, where mechanisms are usually designed to resist worst-case membership query among all the data owners. From this perspective, our relaxed definition of privacy also allows for a more flexible design of algorithms with better utility.

**Our Contributions**  To answer the above questions, in this paper, we first formally define the notion of utility and task-specific privacy and justify why our definitions are particularly suited under the setting of attribute inference attacks. Through the lens of representation learning, we then formulate the problem of utility maximization with privacy constraint as a minimax optimization problem that admits an intuitive game-theoretic interpretation. Our optimization framework is flexible: it can be readily extended to protect multiple sensitive attributes simultaneously, and it can also take advantages of recent advances in deep learning. To investigate the inherent tradeoff between privacy-preservation and utility maximization, we prove an information-theoretic lower bound that characterizes the fundamental limit of the utility of any algorithm under privacy constraint. Empirically, we conduct experiments on two real-world datasets to demonstrate the effectiveness of our method and also corroborate our theoretical findings on the utility lower bound. We believe these insights will be helpful to guide the future design of privacy-preservation algorithms.

2 Preliminary

In this section we introduce our problem setting, the notations used throughout the paper and formally define utility and task-specific privacy.
2.1 Problem Setup and Notation

We focus on the setting where the goal of the adversary is to perform attribute inference. This setting is formally proposed by Dwork et al. [2012] where there are two parties in the system, namely the prediction vendor and the data owner. We consider the practical scenarios where users agree to contribute their data for training a machine learning model for specific purposes but do not want others to infer private attribute in the data, such as health information, race, gender, etc. Note that the prediction vendor will not collect raw user data but processed user data during the model training time. In this case, the attackers can be anyone who can get access to the processed user data to some extent and wants to infer private information. For example, malicious machine learning service providers are motivated to infer more information from users to do user profiling and targeted advertisements. Cybercriminals want to steal the private attribute to launch targeted social engineering attacks [Caputo et al. 2013], or break personal information based authentications [Schechter et al. 2009]. Data brokers can make a profit by selling the user attribute information to others such as advertisers, and insurance industries [Anthes 2015]. The goal of the data owner is to provide as much information as possible to the prediction vendor to maximize the vendor’s own utility, but under the constraint that the data owner should also protect the private information of the data source.

We use $X$, $Y$ and $A$ to denote the input, output and adversary’s output space, respectively. Accordingly, we use $X, Y, A$ to denote the random variables which take values in $X, Y$ and $A$. We note that in our framework the input space $X$ may or may not contain the private attribute $A$. For two random variables $X$ and $Y$, $I(X;Y)$ denotes the mutual information between $X$ and $Y$. Our framework studies the stochastic setting where there is a joint distribution $D$ over $X \times Y \times A$ where data are sampled from. To make our notation consistent, we use $D_X, D_Y$ and $D_A$ to denote the marginal distribution of $D$ over $X, Y$ and $A$.

2.2 Utility and Task-specific Privacy

To simplify the exposition, we mainly discuss the attribute inference setting where $X \subseteq \mathbb{R}^d, Y = A = \{0, 1\}$, but the underlying theory and methodology could easily be extended to the categorical case as well. In what follows, we shall first formally define both the utility of the prediction vendor and the privacy of the data owner. It is worth pointing out that our definition of privacy is task-specific, and this is in contrast with the classic framework of differential privacy where the goal is to preserve privacy in the general and worst-case query scenario. At a colloquial level, we seek to keep the utility of the data while being robust to an adversary on protecting specific information from attack.

A hypothesis is a function $h : \mathcal{X} \to \mathcal{Y}$. The error of a hypothesis $h$ under the distribution $D$ over $\mathcal{X} \times \mathcal{Y}$ is defined as: $\text{Err}(h) := E_D [|Y - h(X)|]$. For binary classification problem, when $h(x) \in \{0, 1\}$, the above loss also reduces to the error rate of classification. Let $\mathcal{H}$ be the Hilbert space of hypotheses. In the context of binary classification, it is then natural to define the utility of a hypothesis $h \in \mathcal{H}$ as the opposite of error:

**Definition 2.1 (Utility).** The utility of a hypothesis $h \in \mathcal{H}$ is $\text{Util}(h) := 1 - E_D [|Y - h(X)|]$.

For binary classification, we always have $0 \leq \text{Util}(h) \leq 1, \forall h \in \mathcal{H}$. Now we proceed to define a measure of privacy in our framework:

**Definition 2.2 (Task-specific Privacy).** The privacy of a hypothesis $h \in \mathcal{H}$ w.r.t. task $A$ is defined as $\text{Priv}_A(h) := 1 - | Pr_D(h(X) = 1 | A = 1) - Pr_D(h(X) = 1 | A = 0) |$. The task-specific privacy
w.r.t. \( \mathcal{H} \) is defined by 
\[
\text{Priv}_A(\mathcal{H}) := \min_{h \in \mathcal{H}} \text{Priv}_A(h).
\]

Again, it is straightforward to verify that \( \forall h \in \mathcal{H}, \ 0 \leq \text{Priv}_A(h) \leq 1 \). Based on our definition, \( \text{Priv}_A(\mathcal{H}) \) then measures the privacy of data under possible attacks from adversaries in \( \mathcal{H} \). Note that the above definition of privacy is task-specific and we have the following proposition hold:

**Proposition 2.1.** Let \( h : \mathcal{X} \to \{0, 1\} \) be a hypothesis, then \( \text{Priv}_A(h) = 1 \) iff \( I(h(X); A) = 0 \) and \( \text{Priv}_A(h) = 0 \) iff \( h(X) = A \) almost surely or \( h(X) = 1 - A \) almost surely.

Proposition 2.1 justifies our definition of task-specific privacy: when \( \text{Priv}_A(h) = 1 \), it means that \( h(X) \) contains no information about the sensitive attribute \( A \). On the other hand, if \( \text{Priv}_A(h) = 0 \), then \( h(X) \) fully predicts \( A \) (or equivalently, \( 1 - A \)) from input \( X \). In the latter case \( h(X) \) also contains perfect information of \( A \) in the sense that \( I(h(X); A) = H(A) \), i.e., the Shannon entropy of \( A \). It is worth pointing out that our definition of privacy is insensitive to the marginal distribution of \( A \), and hence is more robust than other definitions such as the error rate of predicting \( A \). In that case, if \( A \) is extremely imbalanced, even a naive predictor can attain small prediction error by simply outputting constant. We call a hypothesis space \( \mathcal{H} \) symmetric if \( \forall h \in \mathcal{H}, 1 - h \in \mathcal{H} \) as well. Interestingly, when \( \mathcal{H} \) is symmetric, we can also relate the privacy \( \text{Priv}_A(\mathcal{H}) \) to a binary classification problem:

**Proposition 2.2.** If \( \mathcal{H} \) is symmetric, then 
\[
\text{Priv}_A(\mathcal{H}) = \min_{h \in \mathcal{H}} \Pr(h(X) = 0 \mid A = 1) + \Pr(h(X) = 1 \mid A = 0).
\]

**Remark** Consider the following confusion matrix between the actual private attribute \( A \) and its predicted variable \( h(X) \) in Table 1. The false positive rate (equiv. Type-I error) is defined as \( \text{FPR} = \frac{FP}{(FP + TN)} \) and the false negative rate (equiv. Type-II error) is similarly defined as \( \text{FNR} = \frac{FN}{(FN + TP)} \). Using the terminology of confusion matrix, it is then clear that \( \Pr(h(X) = 0 \mid A = 1) = \text{FNR} \) and \( \Pr(h(X) = 1 \mid A = 0) = \text{FPR} \). In other words, Proposition 2.2 says that if \( \mathcal{H} \) is symmetric, then the privacy of a hypothesis space \( \mathcal{H} \) corresponds to the minimum sum of Type-I and Type-II error that is achievable under attacks from \( \mathcal{H} \).

### 3 Minimax Optimization against Attribute Inference Attacks

In this section, we introduce our minimax framework and adversarial algorithms for both single and multi-attributes defense.

#### 3.1 Minimax Formulation

Given a set of samples \( S = \{(x_i, y_i, a_i)\}_{i=1}^n \) drawn i.i.d. from the joint distribution \( \mathcal{D} \), how can the data owner keeps the utility of the data while keeping the sensitive attribute \( A \) private under potential attacks from malicious adversary? Through the lens of representation learning, we seek to find a (non-linear) feature transformation \( f : \mathcal{X} \to \mathcal{Z} \) from input space \( \mathcal{X} \) to feature space \( \mathcal{Z} \) such that \( f \) still preserves relevant information w.r.t. the target task of inferring \( Y \) while hiding sensitive attribute \( A \). Specifically, for \( t \in (0, 1) \), our framework suggests an optimization formulation to...
minimize the empirical error under privacy constraint:

$$\minimize_{h \in \mathcal{H}, f} \hat{\text{Err}}(h \circ f), \quad \text{subject to} \quad \text{Priv}_A(h \circ f) \geq t,$$

where $\mathcal{H} \circ f := \{h \circ f \mid h \in \mathcal{H}\}$. For every $t > 0$, we can solve an equivalent unconstrained Lagrangian with $\lambda > 0$ instead:

$$\minimize_{h \in \mathcal{H}, f} \maximize_{h' \in H} \hat{\text{Err}}(h \circ f) - \lambda_t \left( \Pr\left(h'(f(X)) = 0 \mid A = 1\right) + \Pr\left(h'(f(X)) = 1 \mid A = 0\right) \right)$$

**Game-theoretic Interpretation** It is worth pointing out that the optimization formulation in (1) admits an interesting game-theoretic interpretation, where two agents $f$ and $h'$ play a game whose score is defined by the objective function in (1). Intuitively, $h'$ seeks to minimize the sum of Type-I and Type-II error while $f$ plays against $h'$ by learning transformation to removing information about the sensitive attribute $A$. Algorithmically, for the data owner to achieve the goal of hiding information about the sensitive attribute $A$ from malicious adversary, it suffices to learn representation that is independent of $A$. Formally:

**Proposition 3.1.** Let $f : \mathcal{X} \to \mathcal{Z}$ be a deterministic function and $\mathcal{H} \subseteq 2^\mathcal{Z}$ be a hypothesis class over $\mathcal{Z}$. For any joint distribution $\mathcal{D}$ over $X, A, Y$, if $I(f(X); A) = 0$, then $\text{Priv}_A(\mathcal{H} \circ f) = 1$.

Note that in this sequential game, $f$ is the first-mover and $h'$ is the second. Hence without explicit constraint $f$ possesses a first-mover advantage so that $f$ can dominate the game by simply mapping all the input $X$ to a constant or uniformly random noise. To avoid these degenerate cases, the first term in the objective function of (1) acts as an incentive to encourage $f$ to preserve task-related information. But will this incentive compromise our task-specific privacy? As an extreme case if the target variable $Y$ and the sensitive attribute $A$ are perfectly correlated, then it should be clear that there is a tradeoff in achieving utility and preserving privacy. In Sec. [4] we shall provide an information-theoretic bound to precisely characterize such inherent tradeoff.

### 3.2 Multi-attribute Defense

Although our discussion so far only focuses on the case where there is only one sensitive attribute that the data vendor would like to protect, our optimization framework is flexible enough to extend to the setting where multiple sensitive attributes need to be preserved simultaneously. For instance, in online advertising, the data vendor often needs to keep the personal information about specific users secret, e.g., age range, demographic group, income level, etc. To this end, let $\{A_i\}_{i=1}^K$ be $K$ sensitive attributes that the data vendor would like to protect.

Define $\epsilon_i := \Pr_S(h'_i(f(X)) = 0 \mid A_i = 1) + \Pr_S(h'_i(f(X)) = 1 \mid A_i = 0)$ to simplify the notation. Similar to the optimization formulation in (1), the following general optimization formulation handles the setting of multi-attribute defense:

**Hard version:**

$$\minimize_{h \in \mathcal{H}, f} \maximize_{h'_1, \ldots, h'_K \in \mathcal{H}} \hat{\text{Err}}(h \circ f) + \lambda_t \max_{i \in [K]} (-\epsilon_i)$$

Problem (2) is still a minimax optimization problem. If we initialize all the functions, including $f, h$ and $\{h'_i\}_{i=1}^K$ using deep neural networks, then (2) turns into a nonconvex minimax optimization problem. Inspired by [Ganin et al., 2016], we can use the gradient reversal layer to effectively implement (2) by backpropagation. Essentially, with the gradient reversal layer, we use the
Gradient Descent/Ascent (GDA) [Daskalakis and Panageas, 2018] algorithm to optimize all the model parameters, as opposed to the alternative gradient algorithm, which is known to be unstable in the nonconvex setting [Goodfellow et al., 2014a].

One notable drawback of (2) is that in each iteration, the forward evaluation phase requires computation over all the $K$ adversaries, while in the backward propagation phase only one of them is being utilized due to the hard max operator. This is rather data-inefficient and can waste our computational resources in the forward evaluation phase. To avoid this problem, we propose a smoothed formulation of (2) using the fact that

$$\frac{1}{\gamma} \log \sum_{i \in [K]} \exp(-\gamma \varepsilon_i) \rightarrow \max_{i \in [K]} (-\varepsilon_i) \text{ as } \gamma \rightarrow \infty.$$ 

**Smooth version:**

minimize $h \in H$, maximize $h', \ldots, h'_{K} \in H$ $\hat{\text{Err}}(h \circ f) + \frac{\lambda_t}{\gamma} \log \sum_{i \in [K]} \exp(-\gamma \varepsilon_i)$

(3)

We call the one in (2) as hard version multi-attribute defense and (3) as the smooth version multi-attribute defense. Let $\theta$ denote the model parameters of $f$. Take the derivative w.r.t. $\theta$, we have:

$$\frac{\partial}{\partial \theta} \frac{1}{\gamma} \log \sum_{i \in [K]} \exp(-\gamma \varepsilon_i) = - \sum_{i \in [K]} \frac{\exp(-\gamma \varepsilon_i)}{\sum_{j \in [K]} \exp(-\gamma \varepsilon_j)} \frac{\partial \varepsilon_i}{\partial \theta}.$$ 

Compared with the hard version, the smooth version not only avoids the data-inefficiency problem, but also provides an adaptive way to combine the feedback from all the $K$ adversaries by convex combination. Intuitively, the above formulation suggests that during optimization, the larger the error from one adversary, the smaller the combination weight in the ensemble. This is consistent with Proposition 2.2 where we can see that a larger error essentially corresponds to a better protection of the corresponding sensitive attribute, hence a smaller combination weight.

4 Inherent Tradeoff between Utility and Task-Specific Privacy

As we briefly mentioned in Sec. 3.1 when the protected sensitive attribute $A$ and the target variable $Y$ are perfectly correlated, it is impossible to simultaneously achieve the goal of privacy-preserving and utility-maximizing. But what is the exact tradeoff between utility and privacy when they are correlated? In this section we shall provide an information-theoretic bound to quantitatively characterize the inherent tradeoff between privacy and utility, due to the discrepancy between the conditional distributions of the target variable given the sensitive attribute. Our result is algorithm-independent, hence it applies to a general setting where there is a need to preserve both utility and privacy. To the best of our knowledge, this is the first information-theoretic result to precisely quantify such tradeoff. Due to space limit, we defer all the proofs to appendix.

Before we proceed, we first define several information-theoretic concepts that will be used in our analysis. For two distributions $D$ and $D'$, the Jensen-Shannon (JS) divergence $D_{JS}(D, D')$ is:

$$D_{JS}(D, D') := \frac{1}{2} D_{KL}(D \| D_M) + \frac{1}{2} D_{KL}(D' \| D_M),$$

where $D_{KL}(\cdot \| \cdot)$ is the Kullback–Leibler (KL) divergence and $D_M := (D + D')/2$. The JS divergence can be viewed as a symmetrized and smoothed version of the KL divergence, and it is upper bounded by the $L_1$ distance (total variation) between two distributions through Lin’s Lemma:

**Lemma 4.1** (Lin’s Lemma [Lin 1991]). Let $D$ and $D'$ be two distributions, then $D_{JS}(D, D') \leq \frac{1}{2} ||D - D'||_1.$
Unlike the KL divergence, the JS divergence is bounded: $0 \leq D_{JS}(\mathcal{D}, \mathcal{D}') \leq 1$. Additionally, from the JS divergence, we can define a distance metric between two distributions as well, known as the JS distance \cite{LedendesAndSchindelin2003}:

\[ d_{JS}(\mathcal{D}, \mathcal{D}') := \sqrt{D_{JS}(\mathcal{D}, \mathcal{D}')}. \]

With respect to the JS distance, for any feature space $Z$ and any deterministic mapping $f : \mathcal{X} \rightarrow Z$, we can prove the following lemma via the celebrated data processing inequality:

**Lemma 4.2.** Let $\mathcal{D}_0$ and $\mathcal{D}_1$ be two distributions over $\mathcal{X}$ and let $\mathcal{D}_0^f$ and $\mathcal{D}_1^f$ be the induced distributions of $\mathcal{D}_0$ and $\mathcal{D}_1$ over $Z$ by function $f$, then $d_{JS}(\mathcal{D}_0^f, \mathcal{D}_1^f) \leq d_{JS}(\mathcal{D}_0, \mathcal{D}_1)$. Without loss of generality, any method aiming to predict the target variable

\[ \hat{Y} = h(f(X)) \in \{0, 1\} \]

defines a Markov chain as $X \xrightarrow{f} Z \xrightarrow{h} \hat{Y}$, where $\hat{Y}$ is the predicted target variable given by hypothesis $h$ and $Z$ is the intermediate representation defined by the feature mapping $f$. Hence for any distribution $\mathcal{D}_0(\mathcal{D}_1)$ of $X$, this Markov chain also induces a distribution $\mathcal{D}_0^h(\mathcal{D}_1^h)$ of $\hat{Y}$ and a distribution $\mathcal{D}_0^Y(\mathcal{D}_1^Y)$ of $Z$.

Now let $\mathcal{D}_0^Y(\mathcal{D}_1^Y)$ be the underlying true conditional distribution of $Y$ given $A = 0(A = 1)$. Realize that the JS distance is a metric, the following inequality holds:

\[ d_{JS}(\mathcal{D}_0^h, \mathcal{D}_1^h) \leq d_{JS}(\mathcal{D}_0^Y, \mathcal{D}_1^Y) + d_{JS}(\mathcal{D}_0^h, \mathcal{D}_1^h) + d_{JS}(\mathcal{D}_0^Y, \mathcal{D}_1^Y). \]

Combining the above inequality with Lemma 4.2 to show $d_{JS}(\mathcal{D}_0^h, \mathcal{D}_1^h) \leq d_{JS}(\mathcal{D}_0^Y, \mathcal{D}_1^Y)$, we immediately have:

\[ d_{JS}(\mathcal{D}_0^Y, \mathcal{D}_1^Y) \leq d_{JS}(\mathcal{D}_0^Y, \mathcal{D}_1^Y) + d_{JS}(\mathcal{D}_0^h, \mathcal{D}_1^h) + d_{JS}(\mathcal{D}_0^h, \mathcal{D}_1^h). \]

Intuitively, $d_{JS}(\mathcal{D}_0^Y, \mathcal{D}_1^Y)$ and $d_{JS}(\mathcal{D}_0^h, \mathcal{D}_1^h)$ measure the distance between the predicted and the true target distribution on $A = 0/1$ cases, respectively. Formally, let $\text{Err}_a(h \circ f)$ be the prediction error of function $h \circ f$ conditioned on $A = a$. With the help of Lemma 4.1 the following result establishes a relationship between $d_{JS}(\mathcal{D}_0^Y, \mathcal{D}_1^Y)$ and the utility of the prediction function $h$:

**Lemma 4.3.** Let $\hat{Y} = h(f(X)) \in \{0, 1\}$ be the prediction function, then for $a \in \{0, 1\}$, $d_{JS}(\mathcal{D}_0^Y, \mathcal{D}_1^Y) \leq \sqrt{\text{Err}_a(h \circ f)}$.

Combine Lemma 4.2 and Lemma 4.3 we get the following key lemma that is the backbone for proving the main results in this section:

**Lemma 4.4 (Key lemma).** Let $\mathcal{D}_0, \mathcal{D}_1$ be two distributions over $\mathcal{X} \times \mathcal{Y}$ conditioned on $A = 0$ and $A = 1$ respectively. Assume the Markov chain $X \xrightarrow{f} Z \xrightarrow{h} \hat{Y}$ holds, then $\forall h \in \mathcal{H}$:

\[ d_{JS}(\mathcal{D}_0^Y, \mathcal{D}_1^Y) \leq \sqrt{\text{Err}_0(h \circ f)} + d_{JS}(\mathcal{D}_0^Y, \mathcal{D}_1^Y) + \sqrt{\text{Err}_1(h \circ f)}. \]

We emphasize that for $a \in \{0, 1\}$, the term $\text{Err}_a(h \circ f)$ measures the conditional error of the predicted variable $\hat{Y}$ by composite function $h \circ f$. Similarly, we can define the conditional utility for $a \in \{0, 1\}$:

\[ \text{Util}_a(h \circ f) := 1 - \text{Err}_a(h \circ f). \]

The following main theorem then characterizes a fundamental tradeoff between utility and privacy:

**Theorem 4.1.** Let $\mathcal{H} \subseteq 2^Z$ contains all the measurable functions from $Z$ to $\{0, 1\}$. Given the conditions in Lemma 4.4 $\forall h \in \mathcal{H}$, $\text{Util}_0(h \circ f) + \text{Util}_1(h \circ f) + \text{Priv}_A(h \circ f) \leq 3 - \frac{1}{3}d_{JS}(\mathcal{D}_0^Y, \mathcal{D}_1^Y)$. A few remarks follow. First, note that the maximal value achievable by the sum of the three terms on the L.H.S. is 3. In light of this, the upper bound given in Theorem 4.1 shows that when the marginal distribution of the target variable $Y$ differ between two cases $A = 0$ or $A = 1$, then it is impossible to perfectly maximize utility and privacy. Furthermore, the tradeoff due to the difference in marginal distributions is precisely given by the JS divergence $D_{JS}(\mathcal{D}_0^Y, \mathcal{D}_1^Y)$. Note that in Theorem 4.1 the upper bound holds for any hypothesis $h$ in the richest hypothesis class $\mathcal{H}$ that contains all the possible measurable functions. Put it another way, if we would like to maximally preserve privacy w.r.t. sensitive attribute $A$, then we have to incur a large joint error:
Theorem 4.2. Assume the conditions in Theorem 4.1 hold. If \( \text{Priv}_A(\mathcal{H} \circ f) \geq 1 - D_{\text{JS}}(D_{Y0}^Y, D_{Y1}^Y) \), then \( \forall h \in \mathcal{H}, \text{Err}_0(h \circ f) + \text{Err}_1(h \circ f) \geq \frac{1}{2} \left( d_{\text{JS}}(D_{Y0}^Y, D_{Y1}^Y) - \sqrt{1 - \text{Priv}_A(\mathcal{H} \circ f)} \right)^2 \).

Remark. The above lower bound characterizes a fundamental tradeoff between privacy and joint error. In particular, up to a certain level \( 1 - D_{\text{JS}}(D_{Y0}^Y, D_{Y1}^Y) \), the larger the privacy, the larger the joint error. In light of Proposition 3.1, this means that although the data-owner, or the first-mover \( f \), could try to maximally preserve the privacy via constructing \( f \) such that \( f(X) \) is independent of \( A \), such construction will also inevitably compromise the joint utility of the prediction vendor. It is also worth pointing out that our results in both Theorem 4.1 and Theorem 4.2 are attribute-independent in the sense that neither of the bounds depends on the marginal distribution of \( A \). Instead, all the terms in our results only depend on the conditional distributions given \( A = 0 \) and \( A = 1 \). This is often more desirable than bounds involving mutual information, e.g., \( I(A, Y) \), since \( I(A, Y) \) is close to 0 if the marginal distribution of \( A \) is highly imbalanced.

5 Experiments

To demonstrate the effectiveness of our proposed algorithm, we evaluate it on the UTKFace dataset [Zhang et al., 2017] in both the single and multi-attribute settings. To validate our lower bound (Theorem 4.2), we also conduct experiments on the UCI Adult dataset [Dua and Graff, 2017] where the target attribute and the protected attribute have high correlations. In each experiment, we compare our method with the following baselines: 1). Privacy Partial Least Squares (PPLS) [Enev et al., 2012], 2). Privacy Linear Discriminant Analysis (PLDA) [Whitehill and Movellan, 2012], 3). Minimax filter with alternative update (ALT-UP) [Hamm, 2017], 4). Principal Component Analysis (PCA), and 5). No defense (NO-DEF). Among the five competitors, the first three are state-of-the-art baselines for protecting against attribute inference attacks while the latter two are non-private baselines for comprehensive comparison. Due to space limit, we defer detailed descriptions about experiments to the appendix.

5.1 Datasets and Setup

UTKFace Dataset. The UTKFace [Zhang et al., 2017] dataset is a large-scale face dataset containing more than 20,000 images with annotations of age, gender, and ethnicity. It is one of the benchmark datasets for age estimation, gender and race classifications. In this experiment, we set our target task as gender classification and we use the age and ethnicity as sensitive attributes. For the age and race labels, we preprocess the dataset so that both of them are binary (age: \( \geq 35 \) years old; race: white or not). Since our method could be directly applied with deep neural network (so does the other two baselines NO-DEF and ALT-UP), we instantiate the (non-linear) feature transformation \( f \) in our method as the feature extraction module in the Wide Residual Network [Zagoruyko and Komodakis, 2016]. All the other baselines, including PPLS, PLDA, and PCA, learn a matrix filter as \( f \). The output dimensions of \( f \) in all methods are set to be 2048. Among all methods, we report the one achieving the best performance on the target task. For a fair comparison, we use the same network structure as the the hypothesis among all the methods. The results are shown in Table 2, where we show both the target/private attribute accuracy and the overall tradeoff between them (the difference of these two) in the third row of each chunk.
Table 2: Classification accuracy on predicting Gender and protecting Race/Age on the UTKFace dataset. For Tar. Acc., the larger the better. For Priv. Acc., the smaller the better.

| Private Attribute | NO-DEF  | PPLS     | PLDA     | PCA      | ALT-UP   | Ours      |
|-------------------|---------|----------|----------|----------|----------|-----------|
| Race              | Tar. Acc. 0.898±0.001 | 0.700±0.001 | 0.862±0.001 | 0.850±0.002 | 0.881±0.004 | 0.894±0.004 |
|                   | Priv. Acc. 0.793±0.006 | 0.684±0.002 | 0.573±0.000 | 0.805±0.002 | 0.573±0.000 | 0.573±0.000 |
|                   | Tar. − Priv. 0.100±0.007 | 0.016±0.001 | 0.289±0.003 | 0.044±0.003 | 0.307±0.004 | 0.321±0.004 |
| Age               | Tar. Acc. 0.898±0.001 | 0.710±0.001 | 0.820±0.000 | 0.847±0.002 | 0.848±0.019 | 0.897±0.001 |
|                   | Priv. Acc. 0.799±0.005 | 0.712±0.007 | 0.636±0.000 | 0.792±0.002 | 0.636±0.000 | 0.636±0.000 |
|                   | Tar. − Priv. 0.099±0.005 | -0.002±0.006 | 0.184±0.000 | 0.054±0.002 | 0.212±0.019 | 0.261±0.001 |

To demonstrate that the proposed method can also be used in multi-attribute protection, we also evaluate both the hard and smooth versions of our multi-attribute defense. We compare both versions with no defense for both attributes and defenses for single attribute, as none of the other competitors has a multi-attribute defense extension. The results are shown in Figure 1.

Figure 1: Classification accuracy of multi-attribute defense on the UTKFace dataset. For target accuracy, the larger accuracy the better. For private accuracy, the smaller accuracy the better.

UCI Adult  The UCI adult dataset [Dua and Graff, 2017] is a benchmark dataset for privacy-preservation. The task is to predict whether an individual’s income is greater or less than 50K/year based on census data. The attributes in the dataset includes gender, education, occupation, age, etc. In this experiment we set the target task as income prediction and the private task as inferring gender, age and education, respectively. To show that our lower bound applies to all the methods, here we compute $\operatorname{Err}_0$ and $\operatorname{Err}_1$ of different methods (see Theorem 4.2) and compare it against our information-theoretic lower bound. Again, for fair comparison, we use the same network structure for prediction among all the methods. For non-trivial hypothesis space $\mathcal{H}$, $\operatorname{Priv}_A$ is often intractable to compute, hence we approximate it using all the methods we test. The results are shown in Table 3.
5.2 Results and Analysis

UTKFace  We show both the target task accuracy and private task accuracy of different method when protecting race and age in Table 2. The high private accuracy (around 0.8 in PCA or NO-DEF in both cases) indicates the attacker can perform successful attribute inference attack. As a comparison, all the other methods do a good job in defending such attack, indicated by the low private accuracy. Among all methods, our method achieves the best utility-privacy trade-off: our method achieve the same level of target task accuracy as NO-DEF while keeping the inference accuracy of private attribute to the lowest level in all cases. If we compare the difference of target

Table 3: Experimental results on the UCI Adult dataset. The joint conditional error ($\text{Err}_0 + \text{Err}_1$, the smaller the better) of different methods and the corresponding lower bounds in Theorem 4.2.

| Private Attribute | NO-DEF | PPLS | PLDA | PCA  | ALT-UP | Ours  | Lower Bound |
|-------------------|--------|------|------|------|--------|-------|-------------|
| Gender            | 0.451±0.003 | 0.481±0.001 | 0.545±0.002 | 0.462±0.001 | 0.495±0.012 | **0.455±0.003** | 0.309 |
| Age               | 0.478±0.004 | 0.517±0.006 | 0.546±0.001 | 0.487±0.006 | 0.509±0.013 | **0.476±0.003** | 0.212 |
| Education         | 0.448±0.008 | 0.498±0.006 | 0.586±0.003 | 0.462±0.003 | 0.473±0.009 | **0.450±0.003** | 0.245 |

and private accuracies (the third row in each chunk), it becomes clear that our method dominates all the others in turns of privacy-utility tradeoff.

One unique feature of our method is that it can be readily extended to the setting of multi-attribute defense. From Figure 1, we can see that both the hard and smooth variants help to protect two private attributes, indicated by the low private accuracies in both cases. Notably, the target accuracy does not degrade too much as compared to the one in single-attribute defense. Among the two variants, the smooth variant is slightly more effective, possibly due to the adaptive combination property.

UCI Adult  In this experiment we validate our lower bound by presenting the sum of $\text{Err}_0$ and $\text{Err}_1$ of different methods when protecting gender, age, race and comparing them to the corresponding lower bounds. First, from Table 3 it is clear that in all cases our lower bounds hold. In fact the lower bound is also quite close to the actual joint error in all three cases. Among all the methods trying to protect private attributes, ours achieve the smallest joint error in all the three cases. Again, this demonstrates that our method is effective in maintaining the target utility while preserving privacy.

6 Related Work

Privacy Preservation  Differential privacy has been proposed to bound distinguishability between any two “neighboring” datasets from the released data [Dwork and Nissim, 2004, Dwork et al., 2006] and was used in the training of deep neural network recently [Abadi et al., 2016, Papernot et al., 2016]. It aims to add random noise to the aggregated data to defend against the membership query attacks or add random noise to individual data record (local differential privacy) to ensure any two data records statistically indistinguishable [Erlingsson et al., 2014, Duchi et al., 2013, Bassily and Smith, 2015]. The framework and techniques developed in our paper is orthogonal to the works of differential privacy and can be combined with DP to achieve different privacy
goals [Hamm] [2017]. Various minimax formulations and algorithms have also been proposed to defend against inference attack in different scenarios [Hamm] [2017], [Li et al.] [2018], [Huang et al.] [2017], [Enev et al.] [2012], [Whitehill and Movellan] [2012], [Wu et al.] [2018]. Compared with these work, our framework enjoys an easy extension to multi-attribute defense and we also provide a novel information-theoretic lower bound to characterize the utility-privacy tradeoff.

**Algorithmic Fairness** Although the motivations and applications differ, the task-specific privacy defined in this work has a close connection with learning fair representation in the literature of algorithmic fairness [Dwork et al., 2012, Lum and Johndrow, 2016, Beutel et al., 2017, Madras et al., 2018, Donini et al., 2018]. In fact, our definition of task-specific privacy could be treated as an approximate version of group fairness, also known as the demographic parity, which has been extensively studied under various settings [Zemel et al., 2013, Edwards and Storkey, 2015, Madras et al., 2018, Johndrow et al., 2019]. On the other hand, although impossibility results exist between calibration and fairness [Kleinberg et al., 2016, Pleiss et al., 2017], to the best of our knowledge no prior work has explicitly characterized the tradeoff between fairness and demographic parity quantitatively. The technique developed in this paper could also be easily adapted to show an information-theoretic tradeoff between fairness and utility.

**Adversarial Machine Learning** As discussed in Section 3.1, our minimax formulation against single and multiple attribute inference attacks admits a natural game-theoretic interpretation, which could be understood as an application of the adversarial learning technique in the context of security and privacy. In the general paradigm of adversarial machine learning, two agents compete against each other in order to maximize their own utility functions, and the goal is to achieve a (global or local) Nash equilibrium of the game. In our case, the two agents correspond to the data owner and the prediction vendor respectively, and the goal is to preserve predictive utility under the constraint of removing sensitive information. Besides applications in security [Hamm, 2017, Li et al. 2018, Wu et al. 2018, Chi et al., 2018], other applications of adversarial machine learning include generative modeling [Goodfellow et al., 2014a, Mirza and Osindero, 2014], semi-supervised learning [Salimans et al., 2016], domain adaptation [Ganin et al., 2016, Zhao et al., 2018, 2019], fairness, accountability, transparency [Edwards and Storkey, 2015, Madras et al., 2018], and adversarial examples/robustness [Goodfellow et al., 2014b, Kurakin et al., 2016, Liu et al., 2016].

7 Conclusion

We develop a theoretical framework for task-specific privacy preservation under the setting of attribute inference attacks during the model training time. Under this framework, we propose adversarial algorithms for both single and multi-attribute defense and prove an information-theoretic lower bound to quantify the inherent tradeoff between utility and privacy. Following our formulation and theoretical results, we conduct experiments in two real-world datasets to demonstrate the effectiveness of our method and the validity of our lower bound. We also empirically verify that our algorithm works in the multi-attribute defense setting. We believe our work takes an important step towards better understanding the privacy-utility tradeoff, and also stimulates the future design of privacy-preservation algorithm with adversarial learning techniques.
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Appendix

In this appendix we provide the missing proofs of theorems and claims in our main paper. We also describe detailed experimental settings here.

A Missing Proofs

**Proposition 2.1.** Let \( h : \mathcal{X} \to \{0, 1\} \) be a hypothesis, then \( \text{Priv}_A(h) = 1 \) iff \( I(h(X); A) = 0 \) and \( \text{Priv}_A(h) = 0 \) iff \( h(X) = A \) almost surely or \( h(X) = 1 - A \) almost surely.

*Proof.* We first prove the first part of the proposition. By definition, \( \text{Priv}_A(h) = 1 \) iff \( \Pr_D(h(X) = 1 \mid A = 1) = \Pr_D(h(X) = 1 \mid A = 0) \), which is also equivalent to \( h(X) \perp \perp A \). It then follows that \( h(X) \perp \perp A \Leftrightarrow I(h(X); A) = 0 \).

For the second part of the proposition, again, by definition of \( \text{Priv}_A(h) \), it is clear to see that we either have \( \Pr_D(h(X) = 1 \mid A = 1) = 1 \) and \( \Pr_D(h(X) = 1 \mid A = 0) = 0 \), or \( \Pr_D(h(X) = 1 \mid A = 1) = 0 \) and \( \Pr_D(h(X) = 1 \mid A = 0) = 1 \). Hence we discuss by these two cases. For ease of notation, we omit the subscript \( D \) from \( \Pr_D \) when it is obvious from the context which probability distribution we are referring to.

1. If \( \Pr(h(X) = 1 \mid A = 1) = 1 \) and \( \Pr(h(X) = 1 \mid A = 0) = 0 \), then we know that:

\[
\Pr(h(X) \neq A) = \Pr(A = 0) \Pr(h(X) \neq A \mid A = 0) + \Pr(A = 1) \Pr(h(X) \neq A \mid A = 1)
= \Pr(A = 0) \Pr(h(X) = 1 \mid A = 0) + \Pr(A = 1) \Pr(h(X) = 0 \mid A = 1)
= \Pr(A = 0) \cdot 0 + \Pr(A = 1) \cdot 0
= 0.
\]

2. If \( \Pr(h(X) = 1 \mid A = 1) = 0 \) and \( \Pr(h(X) = 1 \mid A = 0) = 1 \), similarly, we have:

\[
\Pr(h(X) \neq 1 - A) = \Pr(A = 0) \Pr(h(X) \neq 1 - A \mid A = 0) + \Pr(A = 1) \Pr(h(X) \neq 1 - A \mid A = 1)
= \Pr(A = 0) \Pr(h(X) = 0 \mid A = 0) + \Pr(A = 1) \Pr(h(X) = 1 \mid A = 1)
= \Pr(A = 0) \cdot 0 + \Pr(A = 1) \cdot 0
= 0.
\]

Combining the above two parts completes the proof. \( \blacksquare \)

**Proposition 2.2.** If \( \mathcal{H} \) is symmetric, then \( \text{Priv}_A(\mathcal{H}) = \min_{h \in \mathcal{H}} \Pr(h(X) = 0 \mid A = 1) + \Pr(h(X) = 1 \mid A = 0) \).

*Proof.* By definition, we have:

\[
\text{Priv}_A(\mathcal{H}) := \min_{h \in \mathcal{H}} \text{Priv}_A(h)
= \min_{h \in \mathcal{H}} 1 - | \Pr(h(X) = 1 \mid A = 1) - \Pr(h(X) = 1 \mid A = 0) |
= \min_{h \in \mathcal{H}} 1 - ( \Pr(h(X) = 1 \mid A = 1) - \Pr(h(X) = 1 \mid A = 0) )
= \min_{h \in \mathcal{H}} \Pr(h(X) = 0 \mid A = 1) + \Pr(h(X) = 1 \mid A = 0),
\]
where the third equality holds due to the fact that \( \max_{h \in \mathcal{H}} \left| \Pr(h(X) = 1 \mid A = 1) - \Pr(h(X) = 1 \mid A = 0) \right| = \max_{h \in \mathcal{H}} \left( \Pr(h(X) = 1 \mid A = 1) - \Pr(h(X) = 1 \mid A = 0) \right) \). To see this, for any specific \( h \) such that the term inside the absolute value is negative, we can find \( 1 - h \in \mathcal{H} \) such that it becomes positive, due to the assumption that \( \mathcal{H} \) is symmetric.

**Proposition 3.1.** Let \( f : \mathcal{X} \to \mathcal{Z} \) be a deterministic function and \( \mathcal{H} \subseteq 2^\mathcal{Z} \) be a hypothesis class over \( \mathcal{Z} \). For any joint distribution \( \mathcal{D} \) over \( X, A, Y \), if \( I(f(X); A) = 0 \), then \( \text{Priv}_A(\mathcal{H} \circ f) = 1 \).

**Proof.** First, by the celebrated data-processing inequality, \( \forall h \in \mathcal{H} \):

\[
0 \leq I(h(f(X)); A) \leq I(f(X); A) = 0.
\]

By Proposition 2.1, this means that \( \forall h \in \mathcal{H} \), \( \text{Priv}_A(h) = 1 \), which further implies that \( \text{Priv}_A(\mathcal{H} \circ f) = 1 \) by definition.

**Lemma 4.2.** Let \( \mathcal{D}_0 \) and \( \mathcal{D}_1 \) be two distributions over \( \mathcal{X} \) and let \( \mathcal{D}'_0 \) and \( \mathcal{D}'_1 \) be the induced distributions of \( \mathcal{D}_0 \) and \( \mathcal{D}_1 \) over \( \mathcal{Z} \) by function \( f \), then \( d_{JS}(\mathcal{D}'_0, \mathcal{D}'_1) \leq d_{JS}(\mathcal{D}_0, \mathcal{D}_1) \).

**Proof.** Let \( B \) be a uniform random variable taking value in \( \{0, 1\} \) and let the random variable \( Z_B \) with distribution \( \mathcal{D}'_B \) (resp. \( X_B \) with distribution \( \mathcal{D}_B \)) be the mixture of \( \mathcal{D}'_0 \) and \( \mathcal{D}'_1 \) (resp. \( \mathcal{D}_0 \) and \( \mathcal{D}_1 \)) according to \( B \). It is easy to see that \( \mathcal{D}_B = (\mathcal{D}_0 + \mathcal{D}_1)/2 \), and we have:

\[
I(B; X_B) = H(X_B) - H(X_B \mid B)
\]

\[
= - \sum\mathcal{D}_B \log \mathcal{D}_B + \frac{1}{2} \left( \sum\mathcal{D}_0 \log \mathcal{D}_0 + \sum\mathcal{D}_1 \log \mathcal{D}_1 \right)
\]

\[
= - \frac{1}{2} \sum\mathcal{D}_0 \log \mathcal{D}_B - \frac{1}{2} \sum\mathcal{D}_1 \log \mathcal{D}_B + \frac{1}{2} \left( \sum\mathcal{D}_0 \log \mathcal{D}_0 + \sum\mathcal{D}_1 \log \mathcal{D}_1 \right)
\]

\[
= \frac{1}{2} \sum\mathcal{D}_0 \log \frac{\mathcal{D}_0}{\mathcal{D}_B} + \frac{1}{2} \sum\mathcal{D}_1 \log \frac{\mathcal{D}_1}{\mathcal{D}_B}
\]

\[
= \frac{1}{2} D_{KL}(\mathcal{D}_0 || \mathcal{D}_B) + \frac{1}{2} D_{KL}(\mathcal{D}_1 || \mathcal{D}_B)
\]

\[
= D_{JS}(\mathcal{D}_0, \mathcal{D}_1).
\]

Similarly, we have:

\[
D_{JS}(\mathcal{D}'_0, \mathcal{D}'_1) = I(B; Z_B).
\]

Since \( \mathcal{D}'_0 \) (resp. \( \mathcal{D}'_1 \)) is induced by \( f \) from \( \mathcal{D}_0 \) (resp. \( \mathcal{D}_1 \)), by linearity, \( \mathcal{D}'_B \) is also induced by \( f \) from \( \mathcal{D}_B \). Hence \( Z_B = f(X_B) \) and the following Markov chain holds:

\[
B \to X_B \to Z_B.
\]

Apply the data processing inequality, we have

\[
D_{JS}(\mathcal{D}_0, \mathcal{D}_1) = I(B; X_B) \geq I(B; Z_B) = D_{JS}(\mathcal{D}'_0, \mathcal{D}'_1).
\]

Taking square root on both sides of the above inequality completes the proof.

**Lemma 4.3.** Let \( \hat{Y} = h(f(X)) \in \{0, 1\} \) be the prediction function, then for \( a \in \{0, 1\} \), \( d_{JS}(\mathcal{D}^Y_a, \mathcal{D}^h_a) \leq \sqrt{\text{Err}_a(h \circ f)} \).

17
Proof. For $a \in \{0, 1\}$, by definition of the JS distance:

$$d_{JS}^2(D^Y_a, D^h_a) = D_{JS}(D^Y_a, D^h_a)$$

$$\leq ||D^Y_a - D^h_a||_1/2$$

$$= (|\Pr(Y = 0 | A = a) - \Pr(h(f(X)) = 0 | A = a)|$$

$$+ |\Pr(Y = 1 | A = a) - \Pr(h(f(X)) = 1 | A = a)|)/2$$

$$= |\Pr(Y = 1 | A = a) - \Pr(h(f(X)) = 1 | A = a)|$$

$$= |\mathbb{E}[Y | A = a] - \mathbb{E}[h(f(X)) | A = a]|$$

$$\leq \mathbb{E}[|Y - h(f(X))| | A = a]$$

$$= \text{Err}_a(h \circ f),$$

where the expectation is taken over the joint distribution of $X, Y$. Taking square root at both sides then completes the proof.

\[\Box\]

**Theorem 4.1.** Let $\mathcal{H} \subseteq 2^Z$ contains all the measurable functions from $Z$ to $\{0, 1\}$. Given the conditions in Lemma 4.4, $\forall h \in \mathcal{H}$, $\text{Util}_0(h \circ f) + \text{Util}_1(h \circ f) + \text{Priv}_A(\mathcal{H} \circ f) \leq 3 - \frac{1}{2}D_{JS}(D^Y_0, D^Y_1)$.

**Proof.** Before we delve into the details, we first give a high-level sketch of the main idea. The proof could be basically partitioned into two parts. In the first part, we will show that when $\mathcal{H}$ contains all the measurable prediction functions, $1 - \text{Priv}_A(\mathcal{H} \circ f)$ could be used to upper bound $D_{JS}(D^f_0, D^f_1)$. The second part combines Lemma 4.3 and Lemma 4.2 to complete the proof.

In this part we first show that $D_{JS}(D^f_0, D^f_1) \leq 1 - \text{Priv}_A(\mathcal{H} \circ f)$:

$$D_{JS}(D^f_0, D^f_1) \leq \frac{1}{2}||D^f_0 - D^f_1||_1$$

$$= d_{TV}(D^f_0, D^f_1)$$

$$= \sup_{A \in \mathcal{B}} |D^f_0(A) - D^f_1(A)|,$$

where $d_{TV}(\cdot, \cdot)$ denotes the total variation distance and $\mathcal{B}$ is the sigma algebra that contains all the measurable subsets of $Z$. On the other hand, when $\mathcal{H}$ contains all the measurable functions in $2^Z$, we have:

$$1 - \text{Priv}_A(\mathcal{H} \circ f) = 1 - \min_{h \in \mathcal{H}} (1 - |\Pr(h(Z) = 1 | A = 0) - \Pr(h(Z) = 1 | A = 1)|)$$

$$= \max_{h \in \mathcal{H}} |\Pr(h(Z) = 1 | A = 0) - \Pr(h(Z) = 1 | A = 1)|$$

$$= \max_{h \in \mathcal{H}} |D^f_0(h^{-1}(1)) - D^f_1(h^{-1}(1))|$$

$$= \sup_{A \in \mathcal{B}} |D^f_0(A) - D^f_1(A)|,$$

where the last equality follows from the fact that $\mathcal{H}$ is complete and contains all the measurable functions. Combine the above two parts we immediately have $D_{JS}(D^f_0, D^f_1) \leq 1 - \text{Priv}_A(\mathcal{H} \circ f)$.
Now using the key lemma, we have:
\[
d_{JS}(D_{0}^{Y}, D_{1}^{Y}) \leq d_{JS}(D_{0}^{Y}, D_{0}^{h}) + d_{JS}(D_{0}^{h}, D_{1}^{f}) + d_{JS}(D_{1}^{f}, D_{1}^{Y}) \\
\leq \sqrt{\text{Err}_0(h \circ f)} + \sqrt{1 - \text{Priv}_A(\mathcal{H} \circ f)} + \sqrt{\text{Err}_1(h \circ f)} \\
= \sqrt{1 - \text{Util}_0(h \circ f) + 1 - \text{Priv}_A(\mathcal{H} \circ f) + 1 - \text{Util}_1(h \circ f)} \\
\leq \sqrt{3(1 - \text{Util}_0(h \circ f) + 1 - \text{Util}_1(h \circ f) + 1 - \text{Priv}_A(\mathcal{H} \circ f))} \\
= \sqrt{3 - (\text{Util}_0(h \circ f) + \text{Util}_1(h \circ f) + \text{Priv}_A(\mathcal{H} \circ f))}.
\]

Taking square at both sides and then rearrange the terms then completes the proof. ■

**Theorem 4.2.** Assume the conditions in Theorem 4.1 hold. If \( \text{Priv}_A(\mathcal{H} \circ f) \geq 1 - D_{JS}(D_{0}^{Y}, D_{1}^{Y}) \), then \( \forall h \in \mathcal{H}, \text{Err}_0(h \circ f) + \text{Err}_1(h \circ f) \geq \frac{1}{2}(d_{JS}(D_{0}^{Y}, D_{1}^{Y}) - \sqrt{1 - \text{Priv}_A(\mathcal{H} \circ f)})^2 \).

**Proof.** Similarly, using the key lemma, we have:
\[
d_{JS}(D_{0}^{Y}, D_{1}^{Y}) \leq d_{JS}(D_{0}^{Y}, D_{0}^{h}) + d_{JS}(D_{0}^{h}, D_{1}^{f}) + d_{JS}(D_{1}^{f}, D_{1}^{Y}) \\
\leq \sqrt{\text{Err}_0(h \circ f)} + \sqrt{1 - \text{Priv}_A(\mathcal{H} \circ f)} + \sqrt{\text{Err}_1(h \circ f)}
\]

Under the assumption that \( \text{Priv}_A(\mathcal{H} \circ f) \geq 1 - D_{JS}(D_{0}^{Y}, D_{1}^{Y}) \), we have \( d_{JS}(D_{0}^{Y}, D_{1}^{Y}) \geq \sqrt{1 - \text{Priv}_A(\mathcal{H} \circ f)} \), hence by AM-GM inequality:
\[
\sqrt{2(\text{Err}_0(h \circ f) + \text{Err}_1(h \circ f))} \geq \sqrt{\text{Err}_0(h \circ f) + \text{Err}_1(h \circ f)} \geq d_{JS}(D_{0}^{Y}, D_{1}^{Y}) - \sqrt{1 - \text{Priv}_A(\mathcal{H} \circ f)}.
\]

Taking square at both sides then completes the proof. ■

## B Details about Experiments

In this section, we provide more details of the datasets and experiments. We extensively evaluated our proposed method on two datasets: 1). We first evaluate our method on the UTKFace dataset for gender estimation. 2). We then evaluate our method on UCI Adult dataset for income prediction. Due to the differences between these datasets and the corresponding learning tasks, we elaborate more dataset description, model architecture and training parameters in different experiments.

### B.1 Details on UTKFace Dataset Evaluation

UTKFace dataset is a large scale face dataset with annotations of age (range from 0 to 116 years old), gender (male and female), and ethnicity (White, Black, Asian, Indian, and Others). It contains 23,705 \( 64 \times 64 \) aligned and cropped RGB face images and we split the dataset into training set (15171 examples), validation set (3793 examples) and test set (4741 examples), respectively. We further process age label and ethnicity label as binary labels: 0 if the person is not greater than 35 years old for age label (is white for ethnicity label), and 1 if the the person is greater than 35 years old for age label (is non-white for ethnicity label). Table 4 and Table 5 summarize the data distribution of UTKFace dataset for protecting different private attributes.
Table 4: Data distribution of gender \((Y)\) and race \((A)\) in UTKFace dataset.

|       | \(Y = 0\) | \(Y = 1\) |
|-------|-----------|-----------|
| \(A = 0\) | 5477     | 4601     |
| \(A = 1\) | 6914     | 6713     |

Table 5: Data distribution of gender \((Y)\) and age \((A)\) in UTKFace dataset.

|       | \(Y = 0\) | \(Y = 1\) |
|-------|-----------|-----------|
| \(A = 0\) | 6889     | 8218     |
| \(A = 1\) | 5502     | 3096     |

We use the feature extraction module of Wide Residual Network [Zagoruyko and Komodakis, 2016] for the (non-linear) feature transformation \(f\) in NO-DEF, ALT-UP, and our method, while PPLS, PLDA, and PCA learn \(12288 \times 2048\) matrix filter for \(f\). We train all methods using SGD with the initial learning rate 0.01 and momentum 0.9 for 50 epochs. The learning rate is decayed by a factor of 0.1 for every 20 epochs. Note that all tasks are train to minimize the binary cross-entropy loss. Among all methods, we report the one achieving the best performance on the target task in the validation set. Note that we try \(\lambda_t\) to be 1, 5, 10, 15 and 20 for our method and ALT-UP since these two methods can choose different \(\lambda_t\) for controlling the utility and privacy tradeoff. Note that we run the experiments for three times and compute the average.

For the experiment of multi-attribute defense, we choose \(\lambda_t\) to be 3 for both hard and smooth variants. We find that this is a reasonable choice of \(\lambda_t\) since \(\lambda_t\) cannot be too large (otherwise it will cause gradient explosion during training) and cannot be too small (otherwise the private task accuracy is still too high) in this learning task. All other parameter settings are the same as the ones described before. Note that we run the experiments for three times and compute the average.

B.2 Details on UCI Adult Dataset Evaluation

UCI Adult dataset is a benchmark machine learning dataset for income prediction. Each data record contains 14 categorical or numerical attributes, such as occupation, education and gender, to predict whether individual annual income exceeds $50K/year. The dataset is divided into training set (24130 examples), validation (6032 examples), and test set (15060 examples). We choose gender, age, and education as the private attributes, respectively.

We process each private attribute as binary label for each experiment: for age label, 0 if the person is no greater than 35 years old and 1 otherwise; for education label, 0 if the person has not entered college or receive higher education than college, and 1 otherwise. In the mean time, we also remove corresponding private attribute from the input, so the dimension of input data for each experiment is different. The input dimensions for income-gender experiment, income-age experiment, and income-education experiment are 113, 104 and 99, respectively. Table 6, Table 7 and Table 8 summarize the data distribution of UCI Adult dataset for protecting different private attributes.

We use the two-layer ReLU-based neural net for \(f\) and one-layer neural net for \(h\). The output dimensions of \(f\) are 100, 100 and 95 when the private attributes are gender, race and education, respectively. We train all methods using SGD with the initial learning rate 0.001 and momentum 0.9 for 40 epochs. Among all methods, we report the one achieving the best performance on the target task in the validation set. Note that we tune \(\lambda_t\) so that we can achieve the most difference of target task accuracy and private task accuracy.

We summarize the performance of the private task for all methods: for NO-DEF, the private
Table 6: Data distribution of income ($Y$) and gender ($A$) in UCI Adult dataset.

|       | $Y = 0$ | $Y = 1$ |
|-------|---------|---------|
| $A = 0$ | 20988  | 9539   |
| $A = 1$ | 13026  | 1669   |

Table 7: Data distribution of income ($Y$) and age ($A$) in UCI Adult dataset.

|       | $Y = 0$ | $Y = 1$ |
|-------|---------|---------|
| $A = 0$ | 18042  | 2473   |
| $A = 1$ | 15972  | 8735   |

Table 8: Data distribution of income ($Y$) and education ($A$) in UCI Adult dataset.

|       | $Y = 0$ | $Y = 1$ |
|-------|---------|---------|
| $A = 0$ | 20447  | 4248   |
| $A = 1$ | 13567  | 6960   |

task accuracies are 75%, 73% and 72% when the private attributes are gender, age and education, respectively. For all other private methods, we observe the similar experimental results as the UTKFace experiments: PLDA, ALT-UP and our method do well in protecting the private attributes (around 67% for protecting gender, 55% for protecting race and 55% for protecting education).