Elastic scattering and spatial inelastic profiles of high energy protons

I.M. Dremin

Lebedev Physics Institute, Moscow 119991, Russia
National Research Nuclear University ”MEPhI”, Moscow 115409, Russia

Abstract

The ratio of elastic to total proton cross sections is related to the darkness of the spatial profile of inelastic interactions by a single parameter in the framework of a simple analytical model. Their critical values at LHC energies are discussed. Two possible variants of their asymptotical behavior are described.

Keywords: elastic scattering, inelastic profile

PACS number: 13.85.Dz

1 Introduction

Studies of elastic scattering of high energy protons lead to several unexpected results reviewed, e.g., in Refs [1, 2]. Among them, the increase of the share of elastic scattering with the energy increase by a factor about 1.5 from ISR-energies to LHC is the most surprising and yet unexplained phenomenon. Up to now it is unclear why inelastic processes become losing their competition with elastic scattering. With the help of the unitarity condition this feature can be formulated in terms of the darkness of the spatial profile of inelastic interactions which also increases.

One of the ways to understanding these results lies through the detailed analysis of the intriguing shape of the elastic differential cross section with respect to the transferred momentum. Its characteristic fast exponential decrease at comparatively small transferred momenta in the so-called diffraction cone and subsequent (dip/bump + slower decrease) structure at higher momenta have been carefully studied by experimentalists in a wide energy interval and, especially, at LHC [3, 4, 5, 6, 7, 8].
Many phenomenological models were proposed (see, e.g., recent papers [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and references therein) for explanation of these peculiarities. Among them, the kfkm-model [17, 18] must be especially highlighted. The analytical expressions for the imaginary and real parts of the elastic scattering amplitude based on some QCD arguments are presented there as functions both of the transferred momentum measured in experiment and of the impact parameter relevant to the spatial view of the process. It describes successfully experimental data in a wide energy interval from 20 GeV to 8 TeV in the center-of-mass system choosing the energy dependence of the adjustable parameters.

In total, there are 8 such parameters each of which contains the energy independent terms and those increasing with energy $s$ as $\log^{\sqrt{s}}$ and $\log^2\sqrt{s}$ (see Eqs (29)-(36) in [17]). Thus 8 coefficients should be determined from comparison with experiment at a given energy and 24 for the description of the energy dependence in a chosen interval. Besides, there is another constant $a_0$ (see Eqs (8)-(10) in [17]) which is proclaimed to be fixed within the factor 1.5 from some theoretical arguments about the correlation length of the gluon vacuum expectation value. All that concerns the nuclear part of the amplitude for transferred momenta $0.05 < |t| < 2$ GeV$^2$. Additional parameters have to be ascribed for outer intervals. They are related to the Coulomb-nuclear interference region at very small transferred momenta and to the three-gluon exchange term assumed to be relevant at $|t| > 2$ GeV$^2$. Other models mentioned above use the comparable number of adjustable parameters. Hard computer work is required to reveal the otherwise hidden impact of a particular coefficient on the quality of the fit. That is why it is desirable to get a simplified model with direct analytical estimates of this impact and smaller number of parameters.

2 The model

Here, such a model aimed on the rather accurate qualitative description of experimental results is presented. Its main outcome contains a single parameter only. The proposed model is strongly inspired by the phenomenological QCD-motivated kfkm-model [17, 18], which describes experimental data quantitatively in a wide energy interval. That is why we review first the main findings of the kfkm-model. They are at the ground of the simplified model.

The crucial assumption of the proposed model is the complete neglect by
the real part of the elastic scattering amplitude $f_R$ at high energies. Such an assumption can be guessed from the results of the kfk-model.

It was well known from the dispersion relations [21, 22, 23] that the real part at high energies is much smaller than the imaginary part $f_I$ for the forward scattering: $f_R(s, t = 0) = (0.1 - 0.14)f_I(s, t = 0)$. It is confirmed by LHC results and satisfied within the kfk-model. Moreover, the real part in the kfk-model becomes even much smaller at low transferred momenta within the diffraction cone compared to the imaginary part (see Fig. 3 of [17]). It possesses zero inside there in accordance with theoretical claims of Refs [24, 25]. The integral contribution of the real part of the amplitude to the elastic cross section amounts to less than 1.5% since the diffraction cone dominates. It is demonstrated in the Table II of [18]. The role of the real part becomes noticeable for the differential cross section only at its dip where the imaginary part vanishes as seen from Fig. 4 of [17]. However the integral contribution from this region is negligible because all the values at the dip are very low. The position of the dip $t_{\text{dip}}$ practically coincides with the position of the zero of the imaginary part $t_0$ (see Fig. 4 in [18]) because the cross section at the dip is much smaller than its values in the diffraction peak. Namely, $d\sigma / dt|_{\text{dip}} / d\sigma / dt|_{t=0} \approx 3 \cdot 10^{-5}$ at 7 TeV, i.e. the real part at the dip $f_R(t = t_{\text{dip}})$ is less than $5 \cdot 10^{-3}$ of the imaginary part in the diffraction peak $f_I(t = 0)$.

These findings validate the neglect by the real part of the amplitude $f_R$ in the simplified approach. One can approximate the differential cross section by the following expression

$$\frac{d\sigma}{dt} \approx f_I^2$$

(1)

neglecting $f_R^2$ compared to $f_I^2$.

The two most typical features of the imaginary part of the amplitude in the kfk-model are its steep exponential decrease at low transferred momenta in the so-called diffraction cone and its single zero at some transferred momentum. These features can be accounted by the following expression for the imaginary part $f_I$ of the nuclear amplitude of the elastic scattering of high energy protons used in our simplified model:

$$f_I(s, t) = \frac{\sigma_{\text{tot}}(s)}{4\sqrt{\pi}}(1 - (t/t_0(s))^2)e^{B(s)t/2}.$$  

(2)

The variables $s$ and $t$ are the squared energy and transferred momentum of colliding protons in the center-of-mass system $s = 4E^2 = 4(p^2 + m^2)$,
\(-t = 2p^2(1 - \cos \theta)\) at the scattering angle \(\theta\). The amplitude is normalized according to the optical theorem. Its two typical features are the exponential factor with the slope \(B\) which governs mainly the behavior of the diffraction cone measured at low transferred momenta \(t\) and the zero at \(t = t_0\) which is crucial for the description of the dip/bump region. Thus there are only two energy-dependent parameters \(t_0\) and \(B\) in the model. We consider the total cross section as fixed by the optical theorem at the normalization point \(t = 0\).

The formula (2) fits quite well the kfk-graph for \(T_I\) in Fig. 4 of [17] in the interval \(0.05 < |t| < 2\) GeV\(^2\). The negative term in the brackets steepens the shape of the diffraction cone that is often approximated by another exponent with a larger slope at the end of the cone.

Thus the differential cross section is given by the following expression

\[
\frac{d\sigma}{dt} \approx f_I^2 = \frac{\sigma_{\text{tot}}^2(s)}{16\pi} \left(1 - \left(\frac{t}{t_0(s)}\right)^2\right)^2 e^{B(s)t}. \tag{3}
\]

It coincides practically with \(d\sigma^I/dt\) shown in Fig. 4 of [17] for \(|t_0| = 0.4757\) GeV\(^2\) and \(B \approx B^I = 19.90\) GeV\(^{-2}\) at 7 TeV given in the Tables I and II of [18]. Therefore we do not reproduce their almost identical shapes here.

Surely, our assumption leads to the zero of the differential cross section at \(t = t_0\) (as in Figs 3, 4 of [17, 18] for \(f_I\) and \(d\sigma^I/dt\)) instead of the dip but rather accurately reproduces its behavior in other \(t\)-regions which are more important for the integral contributions.

The elastic cross section is

\[
\sigma_{\text{el}} = \frac{\sigma_{\text{tot}}^2(s)}{16\pi B} \left(1 - \frac{4}{(Bt_0)^2} + \frac{24}{(Bt_0)^4}\right). \tag{4}
\]

The structure of the obtained expression is very transparent. The main normalization factor \(\sigma_{\text{tot}}^2(s)/16\pi B\) is determined by the height of the diffraction peak \(\sigma_{\text{tot}}\) defined by the unitarity condition at \(t = 0\) and its width \(B\). The terms in the brackets demonstrate the suppression of the diffraction peak at its end. Would the real part of the amplitude be taken into account, this expression is multiplied by the factor \(\approx 1 + (f_R(s,0)/f_I(s,0))^2 \approx 1.02\). To be more precise, the integral contributions of real and imaginary parts inside the diffraction cone should be evaluated that shifts the above factor even closer to 1 for the kfk-model because \(f_R\) decreases there faster than \(f_I\).

According to experimental measurements the ratio of the elastic to total cross section increases from ISR to LHC up to the value about 1/4. In what
follows we use the ratio
\[ r = \frac{4\sigma_{el}}{\sigma_{tot}} = \frac{\sigma_{tot}(s)}{4\pi B} \left( 1 - \frac{4}{(Bt_0)^2} + \frac{24}{(Bt_0)^4} \right). \] (5)

It is close to 1 at LHC with both factors near 1. The energy dependence of the first factor is especially important for \( r \) in view of the smallness of the correction terms in the brackets.

These terms depend on the single dimensionless product \( B|t_0| \). They show how deep the zero position \( t_0 \) penetrates inside the diffraction cone at a given energy. This motion of zero is often approximated in exponential fits of experimental data on the differential cross section by the steeper falling exponent at the end of the diffraction cone. For the present model it is taken into account and mimicked by the negative contribution of the terms in the brackets. They are small at LHC energies because \( B|t_0| \approx 10 \) there. However the position of the dip (close to \( t_0 \)) seems to move to smaller values with energy increase faster than \( B \) increases even in the range of LHC energies. It is confirmed by the kfk-model (see Table I and Table II of [18]). The corrections can become larger at higher energies. Then the shape of the diffraction cone modifies and can not be treated as the exponential one for small enough values of \( |t_0| \).

The same factor determines the dip/bump structure of the differential cross section beyond the diffraction cone. The bump position is defined by the zero of the derivative of \( f_I \). The relative shift of the bump position \( |t_b| \) to the dip is given by
\[ \frac{|t_b| - |t_0|}{|t_0|} \approx \frac{2}{B|t_0|}. \] (6)

This ratio depends on the same product \( B|t_0| \). It is about 0.2 at LHC or \( |t_b| - |t_0| \approx 0.1 \). The estimate is a qualitative one. The distance between the dip and the bump can be somewhat larger because the \( t \)-dependence of the real part is important in this interval of the transferred momenta.

3 The spatial inelastic profile

The ratio \( r \) is very close to another important characteristics of elastic processes
\[ \zeta(s) = \frac{1}{2\sqrt{\pi}} \int_{0}^{\infty} d|t| f_I(s, t). \] (7)
It can be called as the darkness factor because it determines the attenuation in the spatial profiles of interaction regions for elastic and inelastic processes in the impact parameter $b$-presentation. The impact parameter $b$ is determined as the transverse distance between the trajectories of the centers of the colliding protons. The knowledge of the attenuation in inelastic processes at different impact parameters is gained from the unitarity condition which connects elastic and inelastic channels of the reaction. Here we consider only the strength of the attenuation in central ($b = 0$) inelastic collisions referring the readers for the detailed description of the unitarity condition and for all-$b$-view to the reviews cited above.

The unitarity condition for central head-on collisions with $b = 0$ reads

$$G(s, b = 0) = \zeta(s)(2 - \zeta(s)),$$

where $G(s, b)$ is the $b$-profile of inelastic collisions. The darkness of central inelastic collisions is complete ($G(s, b = 0) = 1$) at $\zeta = 1$. Both values are critical ones because any slight decline of $\zeta$ from 1 by $\pm \delta$ results in much smaller and always negative decline of $G(s, 0)$ from 1 by $-\delta^2$. One gets the complete attenuation at $\zeta = 1$. The attenuation becomes weaker for any value of $\zeta$ which differs from 1. Thus the energy behavior of $\zeta$ determines the deformation of the inelastic profile with energy.

For the proposed simple model one gets

$$\zeta = \frac{\sigma_{\text{tot}}(s)}{4\pi B} \left(1 - \frac{8}{(Bt_0)^2}\right).$$

Here, the correction term in the brackets is somewhat different from those in the ratio $r$. However, all of them are small at LHC where $B|t_0| \approx 10$. Our analytical estimates show how severe are the requirements upon the accuracy of experimental measurements in the vicinity of the critical values of $r$ and $\zeta$ close to 1 observed at LHC. The factor $1 + (f_R(s, 0)/f_I(s, 0))^2 \approx 1.02$ should be again kept in mind if the real part is accounted.

The ratio $r$ is always larger than $\zeta$:

$$\frac{r}{\zeta} = \frac{1 - \frac{4}{(Bt_0)^2} + \frac{24}{(Bt_0)^4}}{1 - \frac{8}{(Bt_0)^2}} \approx 1 + \frac{4}{(Bt_0)^2} + \frac{88}{(Bt_0)^4} > 1.$$

---

\[1\] Let us note that the integral from 0 to $|t_0|$ is positive and that from $|t_0|$ to $\infty$ is negative because $f_I$ changes the sign at $t_0$. 
Actually, this relation is the main goal of our model. The energy behavior of the ratio $r/\zeta$ determines the evolution of their relative values. It depends on a single variable $Bt_0$ only but not on $B$ and $t_0$ separately. It is about 10 at LHC with $B \approx 20$ GeV$^{-2}$ and $|t_0| \approx |t_{dip}| \approx 0.5$ GeV$^2$. Thus, $r$ exceeds $\zeta$ by less than 5%. Both of them are close to 1. However the precision of measurements of $r$ is still not high enough.

This single variable $B|t_0|$ can be gained from experimental results where the exponential fit of the low-$t$ region is done and the position of the dip $t_{dip} \approx t_0$ is determined. Using this formula one can easily estimate the accuracy of measurements which is required for obtaining the accurate enough values. It can be used to get the value of $\zeta$ after the ratio of the elastic and total cross section $r$ is measured precisely enough and the parameter $B|t_0|$ is defined.

The increase of the ratio $r$ from 0.67 at ISR energies to about 1 at LHC is directly related to the increase of $\zeta$. Therefore their precise values in that energy interval are very important. The need in better accuracy of experimental results is evident. The above discussion of the attenuation dependence on the darkness factor shows that the values of $r$ about 1 obtained from experimental data at LHC can be considered as critical ones. The accuracy of experimental data at LHC is still not enough to get the variables $r$ and $\zeta$ with high enough precision near 1. The desired accuracy is easily estimated with the help of the formula $r(10)$.

Further behavior of these variables at higher energies is especially crucial. It is reasonable to assume that the values of $r$ will increase following the (yet unexplained!) trend at lower energies. The tendencies of $\zeta$ to saturate at 1 or increase above 1 at higher energies would lead to different predictions about the inelastic profiles with complete darkness or decreased attenuation at the center, correspondingly. In the kfk-model asymptotical values of $r$ and $\zeta$ are equal to 1.416 and 1, correspondingly. The parameter $B|t_0|$ should become at least twice smaller there as follows from Eq. $r(10)$. Thus one predicts that the dip must move deeper inside the cone at higher energies. That is the qualitative feature observable in experiment.

The region of the diffraction cone contributes mostly to both $\zeta$ and $r$ because they are defined as integrals of $f_I$ and $f_2^I$. To be more definite, the role of the region beyond the cone (at transferred momenta larger than the dip position) is estimated by integration from $|t_0|$ to infinity. Its contribution
$\Delta \zeta$ to $\zeta$ happens to be negligibly small and negative:

$$\Delta \zeta \frac{4 \pi B}{\sigma_{tot}} = - \frac{4}{B|t_0|} \left(1 - \frac{2}{B|t_0|}\right) e^{-B|t_0|/2}$$

which turns out to be about $-2 \cdot 10^{-3}$ at LHC. Therefore the role of the tail of the differential cross section is very mild.

The unitarity condition imposes the limit $\zeta \leq 2$. It is required by the positivity of the inelastic profile [8]. Then there are no inelastic processes for central collisions ($G(s, 0) = 0$ according to Eq. (8)). It is strange that this limit was called as the "black disk". We prefer to call it as TEH - the Toroidal Elastic Hollow [1, 2]. The inelastic profile acquires the toroidal shape with a hole at the very center which allows only the elastic scattering in there. In principle, such regime is not excluded asymptotically but it is not realized in the kfk-model where $\zeta$ saturates at 1. It asks for the relation

$$\sigma_{el} = \sigma_{inel} = \sigma_{tot}/2.$$

It is not fulfilled at present energies. The height of the profile of elastic collisions $\zeta^2$ at the center $b = 0$ completely dominates and saturates the total profile $2\zeta$ for $\zeta = 2$.

## 4 Discussion and conclusions

If taken into account, the neglected real part of the elastic amplitude would ask for many new parameters to be introduced. We have restricted the considered range of the transferred momenta to those which provide main contribution to the integral characteristics $r$ and $\zeta$ described above. The integral contribution of the real part can be definitely omitted there.

The special problem to be discussed is the energy behavior of $\zeta$. It is important that the value of $\zeta$ in the kfk-model starts increasing at ISR and approaches 1 at LHC energies being below 1 by several percents only. At the same time, if the steady increase of the share of elastic processes from ISR to LHC persists at higher energies with $r$ becoming larger than 1, one should consider the intriguing possibility that $\zeta$ will also exceed 1. Surely, that can happen only if the ratio of the total cross section to $4\pi B$ becomes noticeably larger than 1. It increased from about 0.67 at ISR-energies to $1.02 \pm 0.04$ at LHC. Actually, the experimental values of $r$ at LHC energies range from
1.01±0.06 \textsuperscript{3} to 1.06±0.06 \textsuperscript{6}. The factor in brackets in Eq.(5) is about 0.96. Thus the situation at LHC energies is critical in the sense that all estimates of $r$ and $\zeta$ are near 1 within the accuracy of experimental data. The further insight into proton collisions can be gained if more precise data on elastic scattering at LHC will be obtained. The accuracy of measurements of $\sigma_{tot}$, $B$ and $t_{dip}$ is decisive.

In the case of the further noticeable increase of $\sigma_{tot}/4\pi B$ at higher energies the attenuation for central inelastic collisions can become incomplete ($G(s, b = 0) < 1$) after passing through its completeness at LHC. However if the value of $\zeta$ tends to 1 asymptotically, no such peculiar behavior will be observed. $G(s, b = 0)$ will tend to 1. The last possibility looks quite probable both from our intuitive expectations and from conclusions of the kfk-model.

In conclusion, we have proposed the simple model of elastic scattering of high energy protons which admits analytic calculations and easy estimates with the help of a single parameter related to experimentally measurable characteristics. The accuracy of experimental measurements is directly translated into the required precision of the estimates of its value. The analytic expressions allow experimentalists to find the necessary demands upon the accuracy of measurements by direct estimation of the product of the cone slope $B$ and the dip position $t_{dip} \approx t_0$ by providing the accurate values of $r$ and $\zeta$. That is especially important in the LHC energy range where both $r$ and $\zeta$ reach their critical values close to 1.

What concerns the further perspective, one can state that there is yet no consensus about possibilities for energy behavior of the share of elastic processes $r$ and of $\zeta$ at higher energies. Their asymptotical approach to 1 or increase above 1 would tell us not only about elastic scattering but reveal interesting features of inelastic collisions as well.

Acknowledgments

I am grateful for support by the RAS-CERN program and the Competitiveness Program of NRNU ”MEPhI” (M.H.U.).

References

[1] I.M. Dremin, Int. J. Mod. Phys. A 19 1650107 (2016).

[2] I.M. Dremin, Physics-Uspekhi 187 353 (2017).
[3] G. Antchev et al. (TOTEM Collab.), *Eur. Phys. Lett.* **96** 21802 (2011).

[4] G. Antchev et al. (TOTEM Collab.), *Eur. Phys. Lett.* **101** 21002, 21004 (2013).

[5] G. Antchev et al. (TOTEM Collab.), *Phys. Rev. Lett.* **111** 012001 (2013).

[6] G. Antchev et al. (TOTEM Collab.), *Nucl. Phys. B* **899** 527 (2015).

[7] G. Aad et al. (ATLAS Collab.), *Nucl. Phys. B* **889** 486 (2014).

[8] M. Aaboud et al. (ATLAS Collab.), *Phys. Lett. B* **761** 158 (2016).

[9] V.A. Khoze, A.D. Martin, M.G. Ryskin, *Int. J. Mod. Phys. A* **30** 1542004 (2015).

[10] E. Gotsman, E. Levin, U. Maor, *Eur. Phys. J. C* **75** 1 (2015).

[11] M.M. Block, L. Durand, P. Ha, F. Halzen, *Phys. Rev. D* **92** 014030 (2015).

[12] F. Nemes, T. Csörgő, M. Csand, *Int. J. Mod. Phys. A* **30** 1550076 (2015).

[13] D.A. Fagundes, A. Grau, G. Pancheri, *et al. Phys. Rev. D* **91** 11 (2015).

[14] I.M. Dremin, *Physics-Uspekhi* **185** 65 (2015).

[15] V.V. Anisovich, *Physics-Uspekhi* **185** 1043 (2015).

[16] O.V. Selyugin, *Nucl. Phys. A* **922** 180 (2014).

[17] A.K. Kohara, E. Ferreira, T. Kodama, *Eur. Phys. J. C* **73** 2326 (2013).

[18] A.K. Kohara, E. Ferreira, T. Kodama, arXiv:1408.1599, hep-ph.

[19] D.A. Fagundes, M.J. Menon, P.V.R.G. Silva, *J. Phys. G* **40** 065005 (2013).

[20] C. Bourrely, J.M. Myers, J. Soffer, T.T. Wu, *Phys. Rev. D* **85** 096099 (2012).

[21] I.M. Dremin, M.T. Nazirov, *JETP Lett.* **37** 198 (1983).
[22] M.M. Block, R.N. Cahn, *Rev. Mod. Phys.* **57** 563 (1985).

[23] M.M. Block, F. Halzen, *Phys. Rev. D* **73** 054022 (2006).

[24] A. Martin, *Lett. Nuovo Cimento* **7** 811 (1973).

[25] A. Martin, *Phys. Lett. B* **404** 137 (1997).