TRANSVERSE LS-CATEGORY FOR RIEMANNIAN FOLIATIONS

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Abstract. We study the transverse Lusternik-Schnirelmann category of a Riemannian foliation \( \mathcal{F} \) on a compact manifold \( M \). We obtain necessary and sufficient conditions for when the transverse category \( \text{cat}_\mathcal{F}(M, \mathcal{F}) \) is finite. We also introduce a variation on the concept of transverse LS category, the essential transverse category \( \text{cat}_\mathcal{F}^e(M, \mathcal{F}) \), and show that this is finite for every Riemannian foliation. Also, \( \text{cat}_\mathcal{F}^e(M, \mathcal{F}) = \text{cat}_\mathcal{F}(M, \mathcal{F}) \) if \( \text{cat}_\mathcal{F}(M, \mathcal{F}) \) is finite. A generalization of the Lusternik-Schnirelmann theorem is also given: the essential transverse category \( \text{cat}_\mathcal{F}^e(M, \mathcal{F}) \) is a lower bound for the number of critical leaf closures of a basic \( C^1 \)-function on \( M \).

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1. Introduction

Let $f : M \to \mathbb{R}$ be a $C^1$-function on a closed Riemannian manifold $M$. We say that $x \in M$ is a critical point if the gradient $\nabla f$ vanishes at $x$. A well-known formula of Lusternik-Schnirelmann [46, 39, 40, 18] gives a lower-bound estimate for the number of critical points,

$$\# \{x \mid x \in M \text{ is critical for } f\} \geq \text{cat}(M),$$

where $k = \text{cat}(M)$ is the Lusternik-Schnirelmann category of $M$, which is defined as the least number of open sets $\{U_1, \ldots, U_k\}$ required to cover $M$ such that each $U_\ell$ is contractible in $M$ to a point. The LS category is a measure of the topological complexity of $M$; in the case where $f$ is $C^2$ with only Morse-type singularities, the value $\text{cat}(M)$ gives a lower bound for the number of cells in a cellular decomposition.

Let $G$ be a compact Lie group and suppose there is a smooth action $M \times G \to M$ which we can assume preserves a Riemannian metric on $M$. A $C^1$-function $f : M \to \mathbb{R}$ is $G$-invariant if $f(xA) = f(x)$ for all $A \in G$, hence the set of critical points $\nabla f = 0$ is $G$-invariant. Each $G$-orbit $xG$ is a closed submanifold of $M$, and the number of critical $G$-orbits is estimated by the $G$-category,

$$\# \{xG \mid x \in M \text{ is critical for } f\} \geq \text{cat}_G(M),$$

where now $\text{cat}_G(M)$ is the least number of $G$-invariant open sets $\{U_1, \ldots, U_k\}$ required to cover $M$ such that each $U_\ell$ is $G$-contractible in $M$ to a single orbit (see Marzantowicz [47] for actions of a compact Lie group, and Ayala, Lasheras and Quintero [3] for proper actions of Lie groups.)

A smooth action of a non-compact Lie group $G$ on a compact manifold is never proper, and the study of its LS-category theory in this case is much more difficult, as the analysis must take into account the dynamical properties of the action. In order that the condition that the critical sets $\nabla f = 0$ for the gradient be $G$-invariant, it is useful to assume the action of the group $G$ preserves a Riemannian metric on $M$, and thus the orbits of the action define a singular Riemannian foliation (SRF) of $M$ [30, 52, 53]. If all orbits of the action have the same dimension, then the orbits define a Riemannian foliation of $M$.

In this paper, we study the LS-category theory for Riemannian foliations, and prove a Lusternik-Schnirelmann type estimate for the case of Riemannian foliations.

**Theorem 1.1.** Let $\mathcal{F}$ be a Riemannian foliation for a compact manifold $M$, and let $f : M \to \mathbb{R}$ be a $C^1$-map which is constant along the leaves of $\mathcal{F}$. A leaf $L_x$ of $\mathcal{F}$ through a point $x \in M$ is critical if $\nabla f|_{L_x} = 0$, and hence $\nabla f$ also vanishes on the leaf closure $\overline{L}_x$. Then the number of critical leaf closures has a lower bound estimate by the essential transverse LS category $\text{cat}_\eta^\mathcal{F}(M, \mathcal{F})$ of $\mathcal{F}$,

$$\# \{\overline{L}_x \mid x \in M \text{ is critical for } f\} \geq \text{cat}_\eta^\mathcal{F}(M, \mathcal{F}),$$

In the case where all leaves of $\mathcal{F}$ are compact, Colman [11] proved a lower bound estimate for the number of critical leaves in terms of the transverse LS category $\text{cat}_\eta(M, \mathcal{F})$, which the estimate generalizes to the general case when $\mathcal{F}$ is non-compact. The transverse LS category $\text{cat}_\eta(M, \mathcal{F})$ is infinite when $\mathcal{F}$ has non-compact leaves, while the essential transverse LS category $\text{cat}_\eta^\mathcal{F}(M, \mathcal{F})$ introduced in this paper is always finite.
The basic concept is that of a foliated homotopy: given foliated manifolds $(M, \mathcal{F})$ and $(M', \mathcal{F}')$, a map $f : M \to M'$ is said to be foliated if for each leaf $L \subset M$ of $\mathcal{F}$, there exists a leaf $L' \subset M'$ of $\mathcal{F}'$ such that $f(L) \subset L'$. A $C^r$-map $H : M' \times [0, 1] \to M$, for $r \geq 0$, is said to be a foliated $C^r$-homotopy if $H_t$ is foliated for all $0 \leq t \leq 1$, and $H_0(x) = x$ for all $x \in U$. As usual, $H_t(x) = H(x, t)$.

Unless otherwise specified, we assume that all maps and homotopies are smooth.

Let $U \subset M$ be an open saturated subset. We say that $U$ is transversely categorical if there is a foliated homotopy $H : U \times [0, 1] \to M$ such that $H_1 : U \to M$ has image in a single leaf of $\mathcal{F}$.

**DEFINITION 1.2.** The transverse LS category $\text{cat}_{\text{fr}}(M, \mathcal{F})$ of a foliated manifold $(M, \mathcal{F})$ is the least number of transversely categorical open saturated sets required to cover $M$. If no such covering exists, then set $\text{cat}_{\text{fr}}(M, \mathcal{F}) = \infty$.

The basic properties of transverse LS category are given in [11, 17]. If a foliation $\mathcal{F}$ is defined by a fibration $M \to B$ over a compact manifold $B$, then $\text{cat}_{\text{fr}}(M, \mathcal{F}) = \text{cat}(B) < \infty$, so the LS category of $\mathcal{F}$ agrees with the LS category of the leaf space $M/\mathcal{F}$ in this case. Also, the transverse LS category is an invariant of foliated homotopy. The transverse saturated category $\text{cat}_{\text{fr}}(M, \mathcal{F})$ has been further studied by various authors [13, 14, 16, 36, 38, 43, 65].

The assumption that $\text{cat}_{\text{fr}}(M, \mathcal{F})$ is finite is a strong hypothesis on $\mathcal{F}$, and has consequences for the dynamical properties of $\mathcal{F}$. Let $\{U_1, \ldots, U_k\}$ be a minimal cardinality covering of $M$ by categorical open $\mathcal{F}$-saturated sets, so that $k = \text{cat}_{\text{fr}}(M, \mathcal{F})$. Then each $U_i$ contains a compact minimal set $K_i \subset U_i$ for $\mathcal{F}$. The first author showed in [35] that if $H_t : U_i \times [0, 1] \to M$ is a foliated homotopy, the image $H_{t,1}(K_i)$ is a compact minimal set for all $t$, and in particular $H_{t,1}(K_i)$ must be a compact minimal set. Thus, if $H_t$ is a categorical homotopy, then the image $H_{t,1}$ must be contained in a compact leaf of $L$. If $\mathcal{F}$ has no compact leaves, or simply not enough compact leaves, then a categorical covering of $M$ cannot be found.

Let $U \subset M$ be an open saturated subset. We say that $U$ is essentially transversely categorical if there is a foliated homotopy $H : U \times [0, 1] \to M$ such that $H_1 : U \to M$ has image in a minimal set of $\mathcal{F}$.

**DEFINITION 1.3.** The essential transverse LS category $\text{cat}_{\text{fr}}^e(M, \mathcal{F})$ of a foliated manifold $(M, \mathcal{F})$ is the least number of essentially transversely categorical open saturated sets required to cover $M$. If no such covering exists, then set $\text{cat}_{\text{fr}}^e(M, \mathcal{F}) = \infty$.

With this definition, we obtain the following fundamental result:

**THEOREM 1.4.** Let $\mathcal{F}$ be a Riemannian foliation of a compact smooth manifold $M$. Then the essential transverse category $\text{cat}_{\text{fr}}^e(M, \mathcal{F})$ is finite. If the transverse category $\text{cat}_{\text{fr}}(M, \mathcal{F})$ is finite, then $\text{cat}_{\text{fr}}(M, \mathcal{F}) = \text{cat}_{\text{fr}}^e(M, \mathcal{F})$.

We obtain an exact characterization of which Riemannian foliations have $\text{cat}_{\text{fr}}(M, \mathcal{F})$. Let $L$ be a leaf of $\mathcal{F}$. A foliated isotopy of $L$ is a smooth map $I : L \times [0, 1] \to M$ such that $I_0 : L \to M$ is the inclusion of $L$, and for each $0 \leq t \leq 1$, $I_t : L \to M$ is a diffeomorphism onto its image $L_t$, which is a leaf of $\mathcal{F}$. We say that the image leaf $L_1$ is foliated isotopic to $L$. 
Let $I_L$ denote the set of leaves of $\mathcal{F}$ which are foliated isotopic to $L$. For $x \in M$, we set $I_x = I_{L_x}$.

For an arbitrary foliation, one cannot expect the isotopy classes $I_x$ to have any nice properties at all, and typically one expects that $I_x = L_x$. However, for a Riemannian foliation, each isotopy class $I_x$ is a smooth submanifold of $M$, and the set of isotopy classes of the leaves of $\mathcal{F}$ defines a Whitney stratification of $M$. (This is proven in section 7.) A leaf $L_x$ (as well as its corresponding stratum $I_x$) is said to be locally minimal if $I_x$ is a closed submanifold of $M$.

**Theorem 1.5.** Let $\mathcal{F}$ be a Riemannian foliation of a compact smooth manifold $M$. Then $\text{cat}_{O(q)}(M, \mathcal{F})$ is finite if and only if each locally minimal leaf $L_x$ is compact, and hence the locally minimal set $I_x$ is a union of compact leaves.

The proofs of Theorems 1.1, 1.4 and 1.5 are based on the very special geometric properties of Riemannian foliations, which were developed by Molino in a series of papers [50, 51, 52, 53], and see also Haefliger [28, 29, 30]. We first recall several key facts in order to state our next result; details are given in section 3.

One of the remarkable corollaries of the Molino structure theory is that for a leaf $L$ of a Riemannian foliation, its closure $\overline{L}$ is a minimal set for $\mathcal{F}$. Thus, the condition in Theorem 1.1 that the critical points of a leafwise constant $C^1$-function consists of unions of leaf closures, means that in fact they are unions of minimal sets.

Molino’s analysis of the geometry and structure of a Riemannian foliation is based on the desingularization of $\mathcal{F}$ using the geometry of a foliated $O(q)$-bundle: Let $\hat{M} \to M$ denote the principle $O(q)$-bundle of orthonormal frames for the normal bundle to $\mathcal{F}$. There exists an $O(q)$-invariant foliation $\hat{\mathcal{F}}$ on $\hat{M}$ whose leaves are the holonomy coverings of the leaves of $\mathcal{F}$, and in particular have the same dimension as the leaves of $\mathcal{F}$. The closures of the leaves of $\hat{\mathcal{F}}$ are the fibers of an $O(q)$-equivariant fibration $\hat{\Upsilon} : \hat{M} \to \hat{W}$, where $\hat{W}$ is the quotient space by the leaf closures of $\hat{\mathcal{F}}$, and the $O(q)$-action on $\hat{M}$ induces the smooth action of $\hat{W}$. The quotient space

$$\hat{W}/O(q) \cong W \equiv M/\mathcal{F}$$

is naturally identified with the singular quotient space $W$ of $M$ by the leaf closures of $\mathcal{F}$. Given $w \in \hat{W}$, the inverse image $\hat{\pi}^{-1}(w) = \overline{L}$ is the closure of each leaf $\hat{L}_x \subset \overline{L}$, and the projection of such $\overline{L}$ to $M$ is the closure $L_x$ of a leaf $L_x$ of $\mathcal{F}$.

The smooth action of $O(q)$ on $\hat{W}$ defines the orbit-type stratification of $\hat{W}$ in terms of the stabilizer groups of the action, which is one of the key concepts for the study of smooth actions of $O(q)$ (see [7, 19, 20, 33, 41, 63].) A key idea for this work is the use of the associated Whitney stratification, $\hat{W} = Z_1 \cup \cdots \cup Z_K$, where each set $Z_\ell$ is a closed submanifold of $\hat{W}$ which is $O(q)$-invariant and such that $Z_\ell/O(q)$ is connected (see sections 6, 7 and 8.) A stratum $Z_\ell$ is said to be locally minimal if it is a closed submanifold. Let $\text{cat}_{O(q)}(\hat{W})$ denote the $O(q)$-equivariant category of the space $\hat{W}$, which is finite as $\hat{W}$ is compact. We then have the following interpretation of the essential transverse category:
THEOREM 1.6. Let $\mathcal{F}$ be a Riemannian foliation of a compact manifold $M$. Then
\begin{equation}
\text{cat}_e^\circ (M, \mathcal{F}) = \text{cat}_{O(q)}(\hat{W})
\end{equation}

Note that for a smooth action of a compact group $G$ on a compact manifold $N$, the $G$-equivariant category of $N$ defined using smooth $G$-homotopies is equal to the $G$-category defined using continuous homotopies, as a continuous homotopy can be approximated by a smooth homotopy (Theorem 4.2, Chapter VI of [7]). Thus, the calculation of the equivariant category $\text{cat}_{O(q)}(\hat{W})$ is a purely topological problem.

The proof of Theorem 1.6 introduces one of the main new technical ideas of this paper, which can be called the “synchronous lifting property”. The choice of a projectable Riemannian metric on $M$, which is $\mathcal{F}$-projectable when restricted to the normal bundle to $\mathcal{F}$, defines a Riemannian metric on the principle $O(q)$-frame bundle $\pi: \hat{M} \to M$ which is projectable with respect to the lifted foliation $\hat{\mathcal{F}}$. This projectable Riemannian metric in turn defines a horizontal distribution in $T\hat{M}$ which is transverse to the fibers of $\pi$. The first key idea is that a homotopy $H: U \times [0, 1] \to M$ on $M$ can be lifted to an $O(q)$-equivariant $\hat{\mathcal{F}}$-foliated homotopy of $\hat{U} = \pi^{-1}(U)$, $\hat{H}: \hat{U} \times [0, 1] \to \hat{M}$. This shows that $\mathcal{F}$-categorical open sets on $M$ are equivalent to $\hat{\mathcal{F}}$-categorical, $O(q)$-equivariant categorical sets on $\hat{M}$.

Similarly, the projection $\hat{\Upsilon}: \hat{M} \to \hat{W}$ from the frame bundle to the space of leaf closures is a Riemannian submersion, so has a natural horizontal distribution which is transverse to the projection $\hat{\Upsilon}$. This is used to show that $\hat{\mathcal{F}}$-categorical, $O(q)$-equivariant categorical sets on $\hat{M}$ are equivalent to $O(q)$-equivariant categorical sets on $\hat{W}$.

It is interesting to note that our technique for lifting homotopies to equivariant homotopies, used in both sections 4 and 5, is similar to the method of “averaging isotopies” used in the proof of (Theorem 3.1, Chapter VI of [7]).

The remarkable aspect of these arguments is that the connection data is used to define equivariant lifts of the given homotopy; the horizontal distributions are used to “synchronize” the orthonormal frames along the traces of the homotopies. This is possible, even though the homotopy itself need not transform normal frames to normal frames. This technique also has applications to the theory of secondary characteristic classes of $\mathcal{F}$, especially residue theory [37, 44, 45].

This paper can be viewed as a sequel to the work [16] by the first author with Hellen Colman. The results of this paper also extends the results of [14].

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2. Transverse category

We assume that $M$ is a smooth, compact Riemannian manifold without boundary of dimension $m = p + q$, and $\mathcal{F}$ is a smooth Riemannian foliation of dimension $p$ and codimension $q$. Given $x \in M$ we will denote by $L_x$ the leaf of $\mathcal{F}$ containing $x$.

Let $\mathcal{E}$ denote the singular Riemannian foliation (SRF) of $M$ defined by the closures of the leaves of $\mathcal{F}$. (See Molino \cite{52, 53} for properties of $\mathcal{E}$.) The tangential distribution $E = T\mathcal{E}$ is integrable and satisfies the regularity conditions formulated by Stefan \cite{61, 62}. Note that all leaves of $\mathcal{E}$ are compact.

The notion of foliated homotopy extends naturally to the case of singular foliations, so that one can define the transverse category $\text{cat}_\mathcal{E}(M, \mathcal{E})$ of $\mathcal{E}$. We recall two topological lemmas due to Colman \cite{14} which are used to relate the transverse categories of $\mathcal{F}$ and $\mathcal{E}$.

**Lemma 2.1.** Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be two foliated manifolds and $f : M \to M'$ be a foliated continuous map. Let $\mathcal{E}$ denote the partition of $M$ by the closures of the leaves of $\mathcal{F}$, and $\mathcal{E}'$ the corresponding partition of $M'$. Then $f$ is also $\mathcal{E}$-foliated.

**Proof:** Let $L \subset M$ be a leaf of $\mathcal{F}$, and $L' \subset M'$ the leaf of $\mathcal{F}'$ such that $f(L) \subset L'$. Then $f(L) \subset f(L) \subset L'$.

The second lemma is based on the special property of Riemannian foliations that the closure of every leaf in a Riemannian foliation is a minimal set.

**Lemma 2.2.** Let $\mathcal{F}$ be a Riemannian foliation of a compact manifold $M$. Let $U \subset M$ be an $\mathcal{F}$-saturated open set and $L$ be a leaf of the Riemannian foliation $\mathcal{F}$ such that $L \subset U$. Then $\overline{L} \subset U$.

**Proof:** Let $L \subset U$, and suppose that the closure $\overline{L}$ is not a subset of $U$. Then there exists a leaf $L' \subset \overline{L}$ such that $L' \not\subset U$, and as $U$ is saturated, it follows that $L' \subset M - U$. The complement $M - U$ is a closed saturated set, so $\overline{L} \subset M - U$. But $L' \subset \overline{L}$ and $\overline{L}$ is a minimal set implies that $\overline{L} = \overline{L}$, so $L \subset \overline{L} \subset M - U$, which is a contradiction.

**Proposition 2.3.** Let $U \subset M$ be a saturated open set. If $H : U \times [0,1] \to M$ is an $\mathcal{F}$-homotopy, then $H$ is also an $\mathcal{E}$-homotopy. If $H_1$ has image in a single leaf $L \in \mathcal{F}$, or more generally in a minimal set $K$ of $\mathcal{F}$, then $H_1$ has image in the leaf $K = \overline{L}$ of $\mathcal{E}$.

**Proof:** The open set $U$ is a $\mathcal{E}$-saturated set by Lemma 2.2. The map $H_t$ is $\mathcal{E}$-foliated by Lemma 2.1. Then the map $H_1$ has image in the closure $K = \overline{L}$ by Lemma 2.1.

**Corollary 2.4.** Let $\mathcal{F}$ be a Riemannian foliation of a compact manifold $M$, then

\begin{equation}
\text{cat}_\mathcal{E}(\mathcal{E}) \leq \text{cat}_\mathcal{F}(\mathcal{F}) \leq \text{cat}_\mathcal{E}(\mathcal{F})
\end{equation}
3. Geometry of Riemannian foliations

The Molino structure theory is a remarkable collection of results about the geometry and topology of a Riemannian foliation on a compact manifold. We recall some of the main results, and in doing so establish the notation which will be used in later sections. The reader should consult Molino [51, 52, 53], Haefliger [28, 29, 30], or Moerdijk and Mrčun [54] for further details, noting that our notation is an amalgam of those used by these authors.

Let $\pi : \hat{M} \rightarrow M$ denote a compact, connected smooth manifold without boundary, and $F$ a smooth Riemannian foliation of codimension $q$, with tangential distribution $TF$.

Let $g$ denote a Riemannian metric on $TM$ which is projectable with respect to $F$. Identify the normal bundle $\nu_x$ with the orthogonal space $TF_x^\perp$, and let $Q$ have the restricted Riemannian metric $g_Q = g|Q$. For a vector $X \in T_xM$ let $X^\perp \in Q_x$ denote its orthogonal projection.

Given a leafwise path $\gamma$ between points $x, y$ on a leaf $L$, the transverse holonomy $h_\gamma$ along $\gamma$ induces a linear transformation $dh_\gamma[\gamma] : Q_x \rightarrow Q_y$. The fact that the Riemannian metric $g$ on $TM$ is projectable is equivalent to the fact that the linear holonomy transformation $dh_\gamma[\gamma]$ is an isometry for all such paths.

Let $\{E_1, \ldots, E_q\}$ be an orthogonal basis for $\mathbb{R}^q$. Fix $x \in M$. An orthonormal frame for $Q_x$ is an isometric isomorphism $e : \mathbb{R}^q \rightarrow Q_x$. Let $\text{Fr}(Q_x)$ denote the space of orthogonal frames of $Q_x$. Given $e \in \text{Fr}(Q_x)$ and $A \in O(q)$ we obtain a new frame $R_A(e) = eA = e \circ A$ where $A : \mathbb{R}^q \rightarrow \mathbb{R}^q$ is the map induced by matrix multiplication.

The group $\text{Isom}(Q_x)$ of isometries of $Q_x$ also acts naturally on $\text{Fr}(Q_x)$: for $h_x \in \text{Isom}(Q_x)$ and $e \in \text{Fr}(Q_x)$, we define $h_x e = h_x \circ e : \mathbb{R}^q \rightarrow Q_x$.

Let $\pi : \hat{M} \rightarrow M$ denote the bundle of orthonormal frames for $Q$, where the fiber over $x \in M$ is $\pi^{-1}(x) = \text{Fr}(Q_x)$. By the above remarks, $\hat{M}$ is a principle $O(q)$-bundle. We use the notation $\hat{x} = (x, e) \in \hat{M}$ where $e \in \text{Fr}(Q_x)$.

There is a canonical $\mathbb{R}^q$-valued 1-form, the Solder 1-form $\theta : T\hat{M} \rightarrow \mathbb{R}^q$, defined as follows: for $X \in T_x\hat{M}$, then $d\pi(X) \in T_xM$ and set

$$\theta(X) = e^{-1}(d\pi(X)^\perp) \in \mathbb{R}^q$$

Note that for $A \in O(q)$, $R_A^* \theta = A^{-1} \circ \theta$.

Let $\nabla$ denote the Levi-Civita connection on $Q \rightarrow M$ defined using the Riemannian metric. Let $\omega : T\hat{M} \rightarrow \mathfrak{o}(q)$ denoted the associated Maurer-Cartan 1-form, with values in the Lie algebra $\mathfrak{o}(q)$ of $O(q)$. Recall that $\omega$ is $Ad$-related: for $A \in O(q)$ and $X \in T\hat{M}$, we have

$$R_A^*(\omega) = Ad(A^{-1}) \circ \omega \quad R_A^*(\omega)(X) = \omega(dR_A(X)) = Ad(A^{-1})(\omega(X))$$

Let $\mathcal{H} = \ker(\omega) \subset T\hat{M}$ denote the horizontal distribution for $\omega$. Then $\mathcal{H}$ is invariant under the right action of $O(q)$, and for all $\hat{x} \in \hat{M}$, the differential $d\pi : \mathcal{H}_{\hat{x}} \rightarrow T_xM$ is an isomorphism.
Define an $O(q)$-invariant Riemannian metric $\hat{g}$ on $T\hat{M}$, by requiring that the restriction $d\pi : \mathcal{H} \to TM$ be an isometry, and the fibers of $\pi$ are orthogonal to $\mathcal{H}$. The metric restricted to the fibers is induced by the bi-invariant metric on $O(q)$ which is defined by the inner product on $\mathfrak{o}(q)$, where $\langle A, B \rangle = \frac{1}{2} Tr(AB)$ for matrices $A, B \in \mathfrak{o}(q)$.

The metric on $TM$ is projectable implies the restriction $\nabla^L$ to $Q|L \to L$ is a flat connection for each leaf $L \subset M$, so the horizontal distribution of $\nabla^L$ is integrable. The inverse image $\pi^{-1}(L) \subset \hat{M}$ is an $O(q)$-principal bundle over $L$, and the restricted flat connection $\nabla^L$ defines an $O(q)$-invariant foliation of $\pi^{-1}(L)$, whose leaves cover $L$. The union of all such flat subbundles in $\hat{M}$ defines an $O(q)$-invariant foliation $\hat{\mathcal{F}}$ of $\hat{M}$ whose tangent distribution $T\hat{\mathcal{F}}$ is an integrable subbundle of $\mathcal{H}$.

The metric on $T\hat{M}$ is projectable for $\hat{\mathcal{F}}$, hence $\hat{\mathcal{F}}$ is also a Riemannian foliation.

The direct sum of the Solder and connection 1-forms, $\theta$ and $\omega$, define a 1-form
\[
\tau \equiv \theta \oplus \omega : T\hat{M} \to \mathbb{R}^q \oplus \mathfrak{o}(q) \cong \mathbb{R}^{(q^2 + q)/2}
\]
whose kernel is the distribution $T\hat{\mathcal{F}}$.

Recall that $\{E_1, \ldots, E_q\}$ is an orthonormal basis of $\mathbb{R}^q$, and let $\{E_{ij} \mid 1 \leq i < j \leq q\}$ denote the corresponding orthonormal basis of $\mathfrak{o}(q)$. Define orthonormal vector fields $\{\hat{Z}_1, \ldots, \hat{Z}_q\}$ on $\hat{M}$ by specifying that at $\hat{x} \in \hat{M}$,
\[
\tau(\hat{Z}_i) = (E_i, 0), \quad \hat{Z}_i(\hat{x}) \in T_{\hat{x}}\hat{\mathcal{F}}^\perp
\]
Similarly define vector fields $\{\hat{Z}_{ij} \mid 1 \leq i < j \leq q\}$
\[
\tau(\hat{Z}_{ij}) = (0, E_{ij}), \quad \hat{Z}_{ij}(\hat{x}) \in T_{\hat{x}}\hat{\mathcal{F}}^\perp
\]
The collection of vector fields $\{\hat{Z}_i, \hat{Z}_{ij} \mid 1 \leq i < j \leq q\}$ span $T_{\hat{x}}\hat{\mathcal{F}}^\perp$ for each $\hat{x} \in \hat{M}$.

Recall that a function $f : \hat{M} \to \mathbb{R}$ is $\hat{\mathcal{F}}$-basic if $f$ is constant on the leaves of $\hat{\mathcal{F}}$. Let $\mathcal{A} = \mathcal{A}(M, \mathcal{F}, g)$ denote the vector space consisting of all linear combinations
\[
\tilde{Z} = \sum_{1 \leq i \leq q} a^i \hat{Z}_i + \sum_{1 \leq i < j \leq q} b^{ij} \hat{Z}_{ij}
\]
where $\{a^1, \ldots, a^q\}$ and $\{b^{ij} \mid 1 \leq i < j \leq q\}$ are $\hat{\mathcal{F}}$-basic, smooth functions on $\hat{M}$.

Let $\mathcal{X}(\hat{\mathcal{F}})$ denote the smooth vector fields on $\hat{M}$ that are everywhere tangent to $\hat{\mathcal{F}}$.

One of the fundamental results of the Molino structure theory is that the flows of the vector fields in $\mathcal{A}$ are foliated:

**PROPOSITION 3.1.** Let $\tilde{Z} \in \mathcal{A}$ and $\tilde{X} \in \mathcal{X}(\hat{\mathcal{F}})$, then $L_{\tilde{Z}}(\tilde{X}) \in \mathcal{X}(\hat{\mathcal{F}})$. Hence, for each $t \in \mathbb{R}$, the flow of $\tilde{Z}$, $\Phi^\tilde{Z}_t : \hat{M} \to \hat{M}$, maps leaves of $\hat{\mathcal{F}}$ to leaves of $\hat{\mathcal{F}}$. \hfill $\Box$

As the flows of vector fields in $\mathcal{X}(\hat{\mathcal{F}})$ preserve the leaves of $\hat{\mathcal{F}}$, it follows that the group of foliated diffeomorphisms $\text{Diff}(\hat{M}, \hat{\mathcal{F}})$ for $\hat{\mathcal{F}}$ acts transitively on $\hat{M}$. That is, the foliated manifold $(\hat{M}, \hat{\mathcal{F}})$ is transversally complete (TC), and the collection of vector fields $\{\hat{Z}_i, \hat{Z}_{ij} \mid 1 \leq i < j \leq q\}$ define a transverse parallelism (TP) for $\hat{\mathcal{F}}$. 

Given \( \hat{x} = (x, e) \in \hat{M} \), let \( \hat{L}_\hat{x} \) denote the leaf of \( \hat{F} \) containing \( \hat{x} \), and \( L_x \) be the leaf of \( F \) through \( x \). Given a leafwise closed curve \( \gamma : [0, 1] \to L \) with \( \gamma(0) = \gamma(1) = x \), we have a transverse holonomy map \( h_x[\gamma] \) which depends only on the homotopy class of \( \gamma \). The differential \( dh_x[\gamma] : Q_x \to Q_x \) is an isometry, so induces a map \( dh_x : \pi_1(L_x, x) \to \text{Isom}(Q_x) \). Let \( \mathcal{K}_x \subset \pi_1(L_x, x) \) denote the kernel of \( dh_x \).

The framing \( e : \mathbb{R}^q \to Q_x \) induces an isomorphism \( e^* : \text{Isom}(Q_x) \cong O(q) \), and the composition \( dh_\hat{x} = e^* \circ dh_x : \pi_1(L_x, x) \to O(q) \) is the framed linear holonomy homomorphism at \( \hat{x} \).

Given \( \hat{x} = (x, e) \in \hat{M} \), the leaf \( \hat{L}_\hat{x} \) of \( \hat{F} \) is defined as an integral manifold of the flat connection on the \( O(q) \)-frame bundle over \( L \), so that \( y = (x, f) \in \hat{L}_\hat{x} \) means that there is \( [\gamma] \in \pi_1(L_x, x) \) for which \( f = dh_x[\gamma](e) \). The projection \( \pi : \hat{L}_\hat{x} \to L_x \) is thus the holonomy covering of \( L_x \) associated to the kernel of \( dh_x \).

For simplicity of notation, let \( \mathcal{T}_\hat{x} \) denote the closure \( \overline{\hat{L}_\hat{x}} \) of a leaf \( \hat{L}_\hat{x} \) of \( \hat{F} \). The distinction between \( \mathcal{L}_\hat{x} \subset \hat{M} \) and the leaf closure \( \hat{L}_\hat{x} \subset \hat{M} \) is indicated by the basepoint.

Recall that the foliation \( \hat{F} \) is invariant under the right action of \( O(q) \) on \( \hat{M} \). For \( \hat{x} \in \hat{M} \), define two stabilizer subgroups of \( O(q) \) associated to \( \hat{x} \):

\[
\mathcal{H}_\hat{x} \equiv \{ A \in O(q) \mid \hat{L}_\hat{x} A = \hat{L}_\hat{x} \}
\]

\[
\mathcal{H}_\hat{x} \equiv \{ A \in O(q) \mid \mathcal{T}_\hat{x} A = \mathcal{T}_\hat{x} \}
\]

Clearly, \( \mathcal{H}_\hat{x} \) is the topological closure of \( \mathcal{H}_\hat{x} \) in \( O(q) \).

**Lemma 3.2.** There is a natural identification of \( \mathcal{H}_\hat{x} \) with the image of \( dh_\hat{x} \).

**Proof:** Let \( \hat{x} = (x, e) \) and \( [\gamma] \in \pi_1(L_x, x) \). Set

\[
f \equiv dh_x[\gamma](e) = dh_x[\gamma] \circ e \in \text{Fr}(Q_x), \quad A \equiv e^{-1} \circ f = e^{-1} \circ dh_x[\gamma] \circ e \in O(q)
\]

Then \( \hat{x} A = (x, e A) = (x, f) \). It follows that for \( A = dh_\hat{x}[\gamma] \), we have that \( A \in \mathcal{H}_\hat{x} \).

Conversely, if \( \hat{L}_\hat{x} A = \hat{L}_\hat{x} \) then \( \hat{x} A = (x, e A) = (x, f) \in \hat{L}_\hat{x} \) hence there exists \( [\gamma] \in \pi_1(L_x, x) \) such that \( f = dh_x[\gamma](e) \). Thus, \( A = e^{-1} \circ dh_x[\gamma] \circ e \) is in the image of \( dh_\hat{x} \). \( \square \)

**Corollary 3.3.** The following are equivalent:

1. \( \mathcal{H}_\hat{x} \) is infinite
2. \( \mathcal{H}_\hat{x} \subset \mathcal{H}_\hat{x} \) is a proper inclusion
3. \( \mathcal{K}_x \) has infinite index in \( \pi_1(L_x, x) \). \( \square \)

For \( \epsilon > 0 \), let \( D^\epsilon_x = \{ \hat{X} \in \mathbb{R}^q \mid \|\hat{X}\| < \epsilon \} \). Let \( Q^\epsilon \to M \) denote the unit disk subbundle, so that for each \( \hat{x} = (x, e) \in \hat{M} \), the framing \( e \) restricts to an isometry \( e : D^\epsilon_x \to Q^\epsilon_x \).

The lift of \( \nabla^L \) to \( \hat{Q} = \pi^* Q \to \hat{L}_\hat{x} \) is also flat, and by the definition of \( \hat{M} \) the leaf \( \hat{L}_\hat{x} \) has trivial holonomy. Thus, combining the isometry \( e : D^\epsilon_x \to Q^\epsilon_x \) with lifted Riemannian connection \( \nabla^L \) on \( \hat{Q} \) defines an isometric product decomposition

\[
\xi_\hat{x} : \hat{L}_\hat{x} \times D^\epsilon_x \cong \hat{Q}^\epsilon|\hat{L}_\hat{x}
\]
Let $\exp : TM \to M \times M$ denote the geodesic exponential map, $\pi_2 : M \times M \to M$ the projection onto the second factor, and let $\exp_x = \pi_2 \circ \exp : T_x M \to M$ be the exponential map based at $x \in M$. Choose $\epsilon > 0$ sufficiently small so that for all $x \in M$, the restriction $\exp_x : Q'_x \to M$ is an embedding.

**PROPOSITION 3.4.** The composition

$$\Xi = \pi_2 \circ \exp \circ d\pi \circ \xi : \hat{L}_x \times \mathbb{D}^q \cong \hat{Q} \to M$$

is a foliated immersion. Given any $\hat{y} \in \hat{L}_x$ and unit-vector $\hat{X} \in \mathbb{R}^q$, the path 

$$\gamma_{(\hat{y},\hat{X})}(t) = \Xi(\hat{y}, t \hat{X}), \ -\epsilon < t < \epsilon$$

is a unit speed geodesic in $M$ which is orthogonal to $\mathcal{F}$. □

Note that Proposition 3.4 does not assert that the map $\Xi$ is an isometry, as the metric on $\mathbb{D}^q$ is flat, while the curvature tensor of $M$ transverse to $\mathcal{F}$ need not be zero.

The fundamental group $\pi_1(L_x, x)$ acts on the right on $\hat{L}_x$ via covering deck transformations, and acts on $\mathbb{D}^q$ via the holonomy representation $d\mathcal{H}_x$. The product action on $\hat{L}_x \times \mathbb{D}^q$ preserves the product structure, so we obtain a linear foliation $\mathcal{F}^\omega$ on the quotient

$$Q'|L_x = \left(\hat{L}_x \times \mathbb{D}^q\right)/\pi_1(L_x, x), \ (\hat{y} \cdot \gamma, \hat{X}) \sim (\hat{y}, d\mathcal{H}_x[\gamma](\hat{X}))$$

The map $\Xi$ is constant on the orbits of this action, so we obtain

**COROLLARY 3.5.** The induced map

$$\Xi : Q'|L_x \to M$$

is a foliated immersion. □

Corollary 3.5 implies that a Riemannian foliation has a “linear model” in an open tubular neighborhood of a leaf. This is usually stated for the normal bundle to a compact leaf $L_x$ but is equally valid when formulated in terms of immersed submanifolds. The linear foliation (15) and map (16) yields a precise description of the leaves of $\mathcal{F}$ near to $L_x$.

For $\hat{x} = (x, e) \in \hat{M}$, define the transverse disk to $\mathcal{F}$ as

$$\iota_{\hat{x}} : \mathbb{D}^q \to M, \ \iota_{\hat{x}}(\hat{X}) = \Xi(\hat{x}, \hat{X})$$

The image of $\iota_{\hat{x}}$ will be denoted by $T^x$, which is a local transversal to $\mathcal{F}$ through $x$.

Let $y \in T^x$ and $\hat{X} \in \mathbb{D}^q$ so that $\iota_{\hat{x}}(\hat{X}) = y$. Then $L_y$ is the image under $\Xi$ of the leaf $(\hat{L}_x \times \{\hat{Y}\})/\pi_1(L_x, x) \subset Q'|L_x$.

**PROPOSITION 3.6.** The projection $\pi : Q'|L_x \to L_x$ is a diffeomorphism when restricted to $L_y$ for $y = \iota_{\hat{x}}(\hat{X})$ if and only if $\hat{X}$ is fixed by all elements of $\mathcal{H}_{\hat{x}}$.

**Proof:** $\mathcal{H}_{\hat{x}}$ is the image of the map $d\mathcal{H}_x : \pi_1(L_x, x) \to O(q)$ so that by (16) the covering map $\pi : L_y \to L_x$ has fibers isomorphic to the orbit $\hat{X} \cdot \mathcal{H}_{\hat{x}}$. □
COROLLARY 3.7. Let \( y \in T^*_x \). If \( \pi : L_y \to L_x \) is a diffeomorphism, then there is a 1-parameter family of immersions, \( I : L_x \times [0,1] \to M \), such that \( I_t : L_x \to M \) is a diffeomorphism onto a leaf of \( \mathcal{F} \) for all \( 0 \leq t \leq 1 \), \( I_0 : L_x \to L_x \subset M \) is the inclusion of \( L_x \), and \( I_1 : L_x \to L_y \).

Proof: For \( z \in L_x \), define \( I_t(z) = \Xi(x)(\{z\} \times \{tX\}) \), \( 0 \leq t \leq 1 \). \( \square \)

Finally, we recall the aspects of the Molino structure theory for \( \mathcal{F} \) which give a complete description of the closures of the leaves of \( \mathcal{F} \), and of the foliation induced on them by \( \mathcal{F} \).

THEOREM 3.8. Let \( \mathcal{F} \) be a Riemannian foliation of a closed manifold \( M \).

(1) For each \( \hat{x} \in \hat{M} \), the leaf closure \( \overline{L_{\hat{x}}} \) is a submanifold of \( \hat{M} \), and the set of all such leaf closures defines a foliation \( \hat{\mathcal{E}} \) of \( \hat{M} \) with all leaves compact.

(2) For \( \hat{x}, \hat{y} \in \hat{M} \) there exists a foliated diffeomorphism \( \Phi_{\hat{x}\hat{y}} : \hat{M} \to \hat{M} \) such that \( \Phi_{\hat{x}\hat{y}}(\hat{x}) = \hat{y} \), hence \( \Phi_{\hat{x}\hat{y}}(\overline{L_{\hat{x}}}) = \overline{L_{\hat{y}}} \) and \( \Phi_{\hat{y}\hat{x}}(\overline{L_{\hat{y}}}) = \overline{L_{\hat{x}}} \).

(3) There exists a closed manifold \( \hat{W} \) with a right \( \mathcal{O}(q) \)-action, and an \( \mathcal{O}(q) \)-equivariant fibration \( \hat{\pi} : \hat{M} \to \hat{W} \) whose fibers of \( \hat{\pi} \) are the leaves of \( \hat{\mathcal{E}} \).

(4) The metric on \( T\hat{M} \) defined above is projectable for the foliation \( \hat{\mathcal{E}} \), and for the induced metric on \( T\hat{W} \), the fibration \( \hat{\pi} : \hat{M} \to \hat{W} \) is a Riemannian submersion. \( \square \)

Let \( W = M/\mathcal{F} \) denote the Hausdorff space defined as the quotient of \( M \) by the closures of the leaves of \( \mathcal{F} \), and \( \pi : M \to W \) the quotient map. Then there is an \( \mathcal{O}(q) \)-equivariant commutative diagram:

\[
\begin{array}{ccc}
\mathcal{O}(q) & = & \mathcal{O}(q) \\
\downarrow & & \downarrow \\
\hat{M} & \xrightarrow{\hat{\pi}} & \hat{W} \\
\pi & \downarrow & \downarrow \hat{\pi} \\
M & \xrightarrow{\pi} & W
\end{array}
\]

(18)

Note that given \( \hat{x} \in \hat{M} \) and \( \hat{w} = \hat{\pi}(\hat{x}) \), we have \( \mathcal{T}_{\hat{x}} = \hat{\pi}^{-1}(\hat{w}) \).

COROLLARY 3.9. Let \( \hat{x} = (x,e) \) and \( L_x \) be the leaf of \( \mathcal{F} \) through \( x \). Then

\[
L_x = \pi(\mathcal{T}_{\hat{x}}) = \mathcal{T}_{\hat{x}}/H_{\hat{x}}
\]

(19)

The restriction \( \pi : \mathcal{T}_{\hat{x}} \to L_x \) is a principle \( H_{\hat{x}} \)-fibration, and is a covering map if and only if \( H_{\hat{x}} \) is a finite group. \( \square \)

Let \( \mathcal{W} \) be the horizontal distribution for the Riemannian submersion \( \hat{\pi} : \hat{M} \to \hat{W} \). Then \( \mathcal{W} \subset T\hat{M} \) is the subbundle of vectors orthogonal to the fibers of \( \hat{\pi} : \hat{M} \to \hat{W} \).

Note that \( T\hat{\mathcal{F}} \) is contained in the kernel of \( d\hat{\pi} \), so that \( \mathcal{W} \subset T\hat{\mathcal{F}}^\perp \) and each leaf \( \hat{L}_{\hat{x}} \) is orthogonal to the fibers of \( \pi : \hat{M} \to M \).

The second part of the Molino structure theory concerns the geometry of \( \mathcal{T}_{\hat{x}} \) with the foliation defined by the leaves of \( \hat{\mathcal{F}} \).
THEOREM 3.10. There exists a connected, simply connected Lie group $G$ with Lie algebra $\mathfrak{g}$ such that the restricted foliation $\tilde{\mathcal{F}}$ on $\tilde{L}_x$ is a Lie $G$-foliation with all leaves dense, defined by a $\mathfrak{g}$-valued connection 1-form $\omega : T\mathcal{L}_x \to \mathfrak{g}$.

Moreover, given $\tilde{x} \in \tilde{M}$ and a contractible open neighborhood $\tilde{V} \subset \tilde{W}$ of $\tilde{w} = \tilde{\pi}(\tilde{x})$ there exists an $\tilde{\mathcal{F}}$-foliated diffeomorphism
\begin{equation}
\Phi_\tilde{x} : \tilde{L}_x \times \tilde{V} \to \tilde{U} = \tilde{\pi}^{-1}(\tilde{V}) \subset \tilde{M}
\end{equation}
Hence, $\tilde{\mathcal{F}}|\tilde{U}$ is defined by a $\mathfrak{g}$-valued connection 1-form $\omega^\tilde{U} : T\tilde{U} \to \mathfrak{g}$. \hfill \square

4. Equivariant foliated transverse category

In this section, we introduce the $O(q)$-transverse category of $\tilde{\mathcal{F}}$ and show that this is equal to the transverse category of $\mathcal{F}$. The proof uses the horizontal distribution of a projectable metric on $M$ and Molino theory, and is somewhat analogous to techniques used in the study of foliations with an Ehresmann connection \cite{5, 6}.

Let $\tilde{U} \subset \tilde{M}$ be an $O(q)$-invariant, $\tilde{\mathcal{F}}$-saturated open subset. Let $\tilde{H} : \tilde{U} \times [0, 1] \to \tilde{M}$ be an $O(q)$-equivariant, $\tilde{\mathcal{F}}$-foliated homotopy. Then for $U = \pi(\tilde{U})$, $\tilde{H}$ descends to an $\mathcal{F}$-foliated homotopy $H : U \times [0, 1] \to M$. The following result proves the converse:

PROPOSITION 4.1. Let $H : U \times [0, 1] \to M$ be an $\mathcal{F}$-foliated homotopy. Then there exists an $O(q)$-equivariant, $\tilde{\mathcal{F}}$-foliated homotopy
\begin{equation}
\tilde{H} : \tilde{U} \times [0, 1] \to \tilde{M}
\end{equation}
such that $\pi \circ \tilde{H} = H \circ (\pi \times \text{Id})$. That is, the following diagram commutes:
\begin{equation}
\begin{array}{ccc}
\tilde{U} \times [0, 1] & \xrightarrow{\tilde{H}} & \tilde{M} \\
\downarrow & & \downarrow \\
U \times [0, 1] & \xrightarrow{H} & M
\end{array}
\end{equation}

Proof: Recall that $\mathcal{H} = \ker(\omega) \subset T\tilde{M}$ is the horizontal distribution for the basic connection $\omega$. For each $\tilde{y} = (e, f) \in \tilde{M}$, $d\pi : \mathcal{H}_\tilde{y} \to T_yM$ is an isomorphism.

Given $\tilde{x} = (x, e) \in \tilde{U}$, let $\gamma_x(t) = H(x, t)$ denote the path traced out by its homotopy. Let $\tilde{\gamma}_\tilde{x} : [0, 1] \to \tilde{M}$ denote the horizontal lift of $\gamma_x(t)$ starting at $\tilde{x}$. That is, $\tilde{\gamma}_\tilde{x}(t) \in \mathcal{H}$ and $d\pi(\tilde{\gamma}_\tilde{x}(t)) = \gamma_x(t)$ for all $0 \leq t \leq 1$. Define $\tilde{H}(\tilde{x}, t) = \tilde{\gamma}_\tilde{x}(t)$.

The map $\tilde{H}$ is smooth, as the lift of the path $\gamma_x$ to the solution curve $\tilde{\gamma}_\tilde{x}(t)$ depends smoothly on the initial conditions $\tilde{x}$.

Given $A \in O(q)$ set $\tilde{y} = (x, f) = (x, eA)$, then $R_A(\tilde{\gamma}_\tilde{x}(0)) = \tilde{y}$,
\begin{equation}
R_A(\tilde{\gamma}_\tilde{x}(t))' = dR_A(\tilde{\gamma}_\tilde{x}(t)) \in dR_A\mathcal{H} = \mathcal{H}
\end{equation}
\begin{equation}
d\pi(dR_A(\tilde{\gamma}_\tilde{x}(t))) = (\pi(R_A(\tilde{\gamma}_\tilde{x}(t))))'(t) = (\pi(\gamma_x))'(t) = \gamma_x'(t)
\end{equation}
so by uniqueness we have $R_A(\tilde{\gamma}_\tilde{x}(t)) = \tilde{\gamma}_{\tilde{y}}(t)$. Thus, $\tilde{H}$ is $O(q)$-equivariant.

It remains to show that $\tilde{H}$ is $\tilde{\mathcal{F}}$-foliated.
Let $\mathcal{I}^p = (-1,1)^p$, $\mathcal{I}^q = (-1,1)^q$ and $\mathcal{I}^m = (-1,1)^m$, where $m = p + q$.

Given $x \in U$, let $\varphi : V \to \mathcal{I}^m$ be a foliation chart such that $x \in V \subset U$. The connected components of the leaves of $F|V$ are the plaques

$$\mathcal{P}_\xi = \varphi^{-1}(\mathcal{I}^p \times \{\xi\})$$

$\xi \in \mathcal{I}^q$

Let $\varphi_{\text{tr}} : U \to \mathcal{I}^q$ denote the projection onto the transverse coordinate $\xi$. By assumption, the restriction of $g$ to $\mathcal{Q}|U$ projects to a Riemannian metric $g_U$ on $\mathcal{I}^q$.

Let $\hat{\mathcal{I}}^q \to \mathcal{I}^q$ denote the $O(q)$-bundle of orthogonal frames of $TT^q$, and $\hat{\pi} : \hat{\mathcal{I}}^q \to \mathcal{I}^q$ the projection. Let $\omega_U : \hat{\mathcal{I}}^q \to \mathfrak{o}(q)$ be the Levi-Civita connection 1-form for $g_U$, with horizontal distribution $\mathcal{H}_U \subset T\hat{\mathcal{I}}^q$.

The assumption that $g$ is projectable implies the restriction to $\hat{\mathcal{U}} = \pi^{-1}(U)$ of the connection 1-form $\omega$ for $Q$ satisfies $\omega|\hat{\mathcal{U}} = \varphi_{\text{tr}}(\omega_U)$. In particular, this implies that the horizontal distribution $\mathcal{H}$ contains the tangent vectors to the fibers of the projection map $\varphi_{\text{tr}} : \hat{\mathcal{U}} \to \hat{\mathcal{I}}^q$. Actually, this fact is obvious as the fibers are exactly the plaques of $F|\hat{\mathcal{U}}$. The following technical result is used to show that $\hat{H}$ is a foliated map.

**Lemma 4.2.** Assume given smooth maps $\hat{G} : [0, \delta] \times [0, \epsilon] \to \hat{\mathcal{U}}$ and $G : [0, \delta] \times [0, \epsilon] \to U$ such that $\pi \circ \hat{G} = G$. For each $0 \leq s \leq \delta$, define the smooth curves $\sigma_s(t) = G(s,t)$ and $\hat{\sigma}_s(t) = \hat{G}(s,t)$. Further assume that

1. for each $0 \leq t \leq \epsilon$, the curve $\gamma_t$ defined by $\gamma_t(s) = G(s,t)$ is contained in a leaf of $\mathcal{F}|U$
2. the curve $\hat{\gamma}_0$ defined by $\hat{\gamma}_0(s) = \hat{G}(s,0)$ is contained in a leaf of $\hat{\mathcal{F}}|\hat{\mathcal{U}}$
3. for all $(s, t) \in [0, \delta] \times [0, \epsilon]$, the tangent vector $\hat{\sigma}'_s(t) \in \mathcal{H}|\hat{\mathcal{U}}$

Then the curve $\hat{\gamma}_t$ defined by $\hat{\gamma}_t(s) = \hat{G}(s,t)$ is contained in a leaf of $\hat{\mathcal{F}}|\hat{\mathcal{U}}$.

**Proof:** Consider the diagram

$$
\begin{array}{ccc}
[0, \delta] \times [0, \epsilon] & \xrightarrow{G} & \hat{\mathcal{U}} \\
(\sigma, t) \in [0, \delta] \times [0, \epsilon] & \xrightarrow{G} & U \\
(\hat{\sigma}, \hat{t}) \in [0, \delta] \times [0, \epsilon] & \xrightarrow{G} & \hat{\mathcal{U}}
\end{array}
$$

By (4.2.1) the composition $\tau_s(t) = \varphi_{\text{tr}} \circ G(s,t)$ is constant in $s$, and thus defines a smooth path $\tau : [0, \epsilon] \to \mathcal{I}^q$, $\tau(t) = \tau_s(t)$ for any choice of $0 \leq s \leq \delta$. Let $\hat{\tau} : [0, \epsilon] \to \hat{\mathcal{I}}^q$ be the lift of $\tau$ to a horizontal path with respect to $\omega_U$.

For $0 \leq s \leq \delta$, then $\hat{\sigma}_s' (t) \in \mathcal{H}|\hat{\mathcal{U}}$ by (4.2.3), hence $d\varphi_{\text{tr}}(\hat{\sigma}_s'(t)) = (\varphi_{\text{tr}} \circ \hat{\sigma}_s)'(t)$ is horizontal for $\omega_U$.

The assumption $\pi \circ \hat{G} = G$ implies $d\pi(\hat{\sigma}_s'(t)) = \sigma_s'(t)$ hence $d\pi(\varphi_{\text{tr}} \circ \hat{\sigma}_s)'(t) = \tau_s'(t) = \tau'(t)$. Thus, the curve $\varphi_{\text{tr}} \circ \hat{\sigma}_s(t)$ is a horizontal lift of $\tau(t)$, with initial point $\varphi_{\text{tr}} \circ \hat{\sigma}_s(0)$.

The initial point $\varphi_{\text{tr}} \circ \hat{\sigma}_s(0)$ is independent of $s$ by (4.2.2), thus it follows that the curve $\varphi_{\text{tr}} \circ \hat{\sigma}_s(t)$ is independent of $s$.

That is, for all $0 \leq t \leq \epsilon$, the curve $s \mapsto \hat{\sigma}_s(t)$ is contained in a fixed fiber of $\varphi_{\text{tr}}$, which is a plaque of $\hat{\mathcal{F}}|\hat{\mathcal{U}}$, so contained in a leaf of $\hat{\mathcal{F}}$. $\square$
We now complete the proof of Proposition 4.1. Let \( \hat{x} \in \hat{U} \) and \( \hat{y} \in \hat{L}_x \cap \hat{U} \) be in the plaque containing \( \hat{x} \). We must show that \( \hat{H}_t(\hat{x}) \) and \( \hat{H}_t(\hat{y}) \) lie in the same leaf for all \( 0 \leq t \leq 1 \).

Choose a smooth path \( \tilde{\gamma}_{xy} : [0, 1] \to \hat{L}_x \cap \hat{U} \) with \( \tilde{\gamma}_{xy}(0) = \hat{x} \) and \( \tilde{\gamma}_{xy}(1) = \hat{y} \). Define \( x = \pi(\hat{x}) \), \( y = \pi(\hat{y}) \), and set \( \gamma_{xy}(s) = \pi(\tilde{\gamma}_{xy}(s)) \) so that \( \gamma_{xy} \) is a smooth path in \( L_x \cap U \) from \( x \) to \( y \).

Compose these paths with the given homotopy \( H \) and its lift \( \hat{H} \), to obtain maps
\[
\begin{align*}
\hat{H}(s, t) &= \hat{H}(\tilde{\gamma}_{xy}(s), t) \\
H(s, t) &= H(\gamma_{xy}(s), t)
\end{align*}
\]

The maps \( H_0 \) and \( \hat{H}_0 \) are inclusions, so \( \hat{H}(s, 0) = \tilde{\gamma}_{xy}(s) \) and \( H(s, 0) = \gamma_{xy}(s) \) for \( 0 \leq s \leq 1 \). We are also given that \( \pi \circ \hat{H} = H \), so that \( \pi \circ \hat{H}(s, t) = H(s, t) \) for all \( 0 \leq s \leq 1, 0 \leq t \leq 1 \).

If the image of \( H : [0, 1] \times [0, 1] \to M \) is contained in a foliation chart \( \varphi : V \to T^p \times T^q \), then we can directly apply Proposition 4.1 to obtain the claim.

For the general case, observe that there exists an integer \( N > 0 \) so that for \( s_\mu = \mu/N \) and \( t_\nu = \nu/N \) for each \( 0 \leq \mu, \nu < N \) there is a foliation chart \( \varphi_{\mu\nu} : V_{\mu\nu} \to T^p \times T^q \) such that \( H([s_\mu, s_{\mu+1}] \times [t_\nu, t_{\nu+1}]) \subset V_{\mu\nu} \).

Set \( \delta = \epsilon = 1/N \), and define for \( 0 \leq s, t \leq 1/N \)
\[
\hat{G}_{\mu\nu}(s, t) = \hat{H}(s_\mu + s, t_\nu + t) \quad G_{\mu\nu}(s, t) = H(s_\mu + s, t_\nu + t)
\]

For \( \nu = 0 \) and each \( 0 \leq \mu < N \), the maps \( \hat{G}_{\mu0}, G_{\mu0} \) satisfy the hypotheses of Proposition 4.1. The conclusion of Proposition 4.1 implies that this is again true for \( \nu = 1 \), so that one can apply the Proposition repeated to obtain that the curve \( s \to \hat{H}(s, 1) \) lies in a leaf of \( \hat{F} \) as claimed. \( \square \)

**Definition 4.3.** Let \( \hat{U} \subset \hat{M} \) be an \( O(q) \)-invariant, \( \hat{F} \)-saturated open subset. We say that \( \hat{U} \) is \( O(q) \)-transversely categorical if there exists an \( O(q) \)-equivariant, \( \hat{F} \)-foliated homotopy \( \hat{H} : \hat{U} \times [0, 1] \to \hat{M} \) such that \( \hat{H}_0 \) is the inclusion, and \( \hat{H}_1 \) has image in the orbit \( \hat{L}_x \cdot O(q) \) of the closure \( \hat{L}_x \) of a leaf \( L_x \) of \( \hat{F} \).

The \( O(q) \)-transverse category of \( \hat{F} \), denoted by \( \text{cat}_{O(q)}(\hat{M}, \hat{F}) \), is the least number of \( O(q) \)-invariant, \( \hat{F} \)-saturated open sets required to cover \( \hat{M} \). The results of this section imply:

**Corollary 4.4.** Let \( \mathcal{F} \) be a Riemannian foliation of a compact manifold \( M \), then
\[
\text{cat}_{O(q)}^s(M, \mathcal{F}) = \text{cat}_{O(q)}(\hat{M}, \hat{F})
\]
5. \textbf{O}(q)-equivariant category

In this section, we show that the \textbf{O}(q)-transverse category \text{cat}_{\text{O}(q)}(\hat{M}, \hat{F}) of $\mathcal{F}$ is equal to the \textbf{O}(q)-equivariant category \text{cat}_{\text{O}(q)}(\hat{W}) of $\hat{W}$, which will complete the proof of Theorem 1.6. The proof uses the horizontal distribution of the Riemannian submersion $\hat{\Upsilon}: \hat{M} \to \hat{W}$ defined by the projectable metric for $\hat{\mathcal{F}}$.

Let $\hat{U} \subset \hat{M}$ be an \textbf{O}(q)-invariant, $\hat{\mathcal{F}}$-saturated open subset. Suppose that $\hat{\mathcal{H}}: \hat{U} \times [0, 1] \to \hat{M}$ is an \textbf{O}(q)-equivariant, $\hat{\mathcal{F}}$-foliated homotopy. Then by Proposition 2.3 applied to $\hat{\mathcal{F}}$, we obtain an \textbf{O}(q)-invariant open set $\hat{U} = \hat{\Upsilon}(\hat{U}) \subset \hat{W}$ and an \textbf{O}(q)-equivariant homotopy $\hat{\mathcal{H}}: \hat{U} \times [0, 1] \to \hat{M}$ such that $\hat{\Upsilon} \circ \hat{\mathcal{H}} = \hat{\mathcal{H}} \circ (\hat{\Upsilon} \times \text{Id})$.

**Proof:** For each $(\hat{w}, t) \in \mathcal{U} \times [0, 1]$, set $\hat{w}_t = \hat{\mathcal{H}}(\hat{w}, t)$, and let

$$\hat{\mathcal{H}}'(\hat{w}, t) = \frac{d}{dt} \hat{\mathcal{H}}(\hat{w}, t) \in T_{\hat{w}_t} \hat{W}$$

denote the tangent vector along the time coordinate.

For $A \in \text{O}(q)$, let $R_A : \hat{W} \to \hat{W}$ denote the right action $R_A(\hat{w}) = \hat{w} A$. The assumption that $\hat{\mathcal{H}}$ is \textbf{O}(q)-invariant implies

$$\hat{\mathcal{H}}'(\hat{w} A, t) = \frac{d}{dt} (R_A \hat{\mathcal{H}}(\hat{w}, t)) = dR_A (\hat{\mathcal{H}}'(\hat{w}, t))$$

so that $\hat{\mathcal{H}}'(\hat{w}, t)$ is a \textbf{O}(q)-invariant vector field.

Recall that $\mathcal{W} \subset T\hat{M}$ is the subbundle of vectors orthogonal to the fibers of $\hat{\Upsilon}: \hat{M} \to \hat{W}$. Then $\mathcal{W}$ is $\text{O}(q)$-invariant, as the Riemannian metric $\hat{g}$ on $T\hat{M}$ is $\text{O}(q)$-invariant, and $\hat{\Upsilon}$ is $\text{O}(q)$-equivariant.

For $\hat{x} \in \hat{U}$ with image $\hat{w} = \hat{\pi}(\hat{x})$, define a smooth curve $\hat{x}_t : [0, 1] \to \hat{M}$ by requiring that

$$\hat{x}_0 = \hat{x}, \quad \hat{x}'_t = \frac{d\hat{x}_t}{dt} \in \mathcal{W}, \quad d\hat{\Upsilon}(\hat{x}'_t) = \hat{\mathcal{H}}'(\hat{w}, t) = \hat{w}'_t$$

Thus, $\hat{x}_t$ is an integral curve for the horizontal distribution $\mathcal{W}$. Define $\hat{\mathcal{H}}(\hat{x}, t) = \hat{x}_t$.

It follows from (28) that $\hat{\Upsilon}(\hat{x}_t) = \hat{w}_t$, hence $\hat{\Upsilon} \circ \hat{\mathcal{H}} = \hat{\mathcal{H}} \circ (\hat{\Upsilon} \times \text{Id})$.

The function $\hat{\mathcal{H}}$ is smooth as the integral curves $\hat{x}_t$ depend smoothly on the initial condition $\hat{x}_0 = \hat{x}$. The function $\hat{\mathcal{H}}$ is $\text{O}(q)$-equivariant, as given $A \in \text{O}(q)$

$$\frac{d}{dt} R_A(\hat{x}_t) = dR_A(\hat{x}'_t) \in dR_A(\mathcal{W}) = \mathcal{W}$$

and

$$d\hat{\Upsilon} \left( \frac{d}{dt} R_A(\hat{x}_t) \right) = d\hat{\Upsilon} (dR_A(\hat{x}'_t)) = dR_A \left( d\hat{\Upsilon}(\hat{x}'_t) \right) = \hat{\mathcal{H}}'(\hat{x} A, t)$$
It remains to show that \( \tilde{H}_t : \tilde{U} \to \tilde{M} \) is \( \tilde{F} \)-foliated. For each \( \tilde{x} \in \tilde{M} \), the trace \( t \mapsto \tilde{x}_t = \tilde{H}_t(x) \) is determined by the flow of the non-autonomous vector field \( \tilde{H}' = \tilde{H}'(\tilde{x}, t) \) which is \( \tilde{Y} \)-related to the vector field \( H' = H'(\tilde{x}, t) \).

Given a vector field \( \tilde{X} \in \mathcal{A}(\tilde{F}) \) tangent to \( \tilde{F} \), let \( \Phi_s^{\tilde{X}} : \tilde{M} \to \tilde{M} \) denote its flow. The metric \( \tilde{g} \) on \( T\tilde{M} \) is \( \tilde{F} \)-projectable, so its projection to \( T\tilde{F}^\perp \) is invariant under the flow \( \Phi_s^{\tilde{X}} \).

The flow \( \Phi_s^{\tilde{X}} \) preserves the leaf closures of \( \tilde{F} \), so induces a quotient flow on \( \tilde{W} \) which is constant. Set \( \tilde{x}_{t,s} = \Phi_s^{\tilde{X}}(\tilde{x}_t) \), then \( \tilde{Y}(\tilde{x}_{t,s}) \) is constant as a function of \( s \).

Apply \( \Phi_s^{\tilde{X}} \) to the \( \tilde{Y} \)-related vector field \( \tilde{x}'_t = \tilde{H}'(\tilde{x}, t) \) to obtain
\[
\tilde{Y}_{t,s} = d\Phi_s^{\tilde{X}}(\tilde{H}'(\tilde{x}, t)) = d\Phi_s^{\tilde{X}}(\tilde{x}')
\]
Note that \( \frac{d}{ds}|_{s=0}\tilde{Y}_{t,s} = L_{\tilde{X}}^{\tilde{H}'}(\tilde{x}, t) \). We claim that the Lie bracket
\[
(L_{\tilde{X}}^{\tilde{H}'})(\tilde{x}) = [\tilde{X}, \tilde{H}']|_{\tilde{x}} \in T_{\tilde{x}}\tilde{T}\tilde{F}
\]
which implies that the flow of \( \tilde{H}' \) preserves the foliation \( \tilde{F} \).

At each point \( \tilde{x}_{t,s} \in \tilde{M} \) there is an orthogonal decomposition
\[
T_{\tilde{x}_{t,s}}\tilde{M} = T_{\tilde{x}_{t,s}}^\tilde{F} \tilde{M} \oplus T_{\tilde{x}_{t,s}}^\perp \tilde{M} \oplus W_{\tilde{x}_{t,s}}
\]
where \( T_{\tilde{x}_{t,s}}^\tilde{F} \tilde{M} = T_{\tilde{x}_{t,s}}\tilde{F} \), and \( T_{\tilde{x}_{t,s}}^\perp \tilde{M} = T_{\tilde{x}_{t,s}}T_{\tilde{x}_{t,s}} \cap T_{\tilde{x}_{t,s}}\tilde{F}^\perp \) consists of the vectors orthogonal to \( T_{\tilde{x}_{t,s}}\tilde{F} \) and tangent to the fibers of \( \tilde{Y} \). We decompose \( \tilde{Y}_{t,s} \) into its components,
\[
\tilde{Y}_{t,s} = \tilde{Y}_{t,s}^\tilde{F} + \tilde{Y}_{t,s}^\perp + \tilde{Y}_{t,s}^W
\]
The projection \( d\tilde{Y}(\tilde{Y}_{t,s}) \) is constant as a function of \( s \), so
\[
d\tilde{Y}(\frac{d}{ds}\tilde{Y}_{t,s}^W) = \frac{d}{ds}d\tilde{Y}(\tilde{Y}_{t,s}^W) = \frac{d}{ds}d\tilde{Y}(\tilde{Y}_{t,s}) = 0
\]
and \( d\tilde{Y} : W_{\tilde{x}_{t,s}} \to T_{\tilde{x}_{t}}\tilde{W} \) is an isomorphism, we conclude that \( \frac{d}{ds}|_{s=0}\tilde{Y}_{t,s}^W = 0 \).

The map \( d\Phi_s^{\tilde{X}} \) induces an isometry on \( T\tilde{F}^\perp \), so the length of the vectors \( \tilde{Y}_{t,s}^\tilde{F} \) is a constant function of \( s \). At \( s = 0 \) we have that \( \tilde{Y}_{t,0}^\tilde{F} = \tilde{H}'(\tilde{x}, t)\tilde{Y} = 0 \) as \( \tilde{H}'(\tilde{x}, t) \in W_{\tilde{x}_t} \). Hence, \( \tilde{Y}_{t,s}^\tilde{F} = 0 \) for all \( s \). Thus, we have
\[
L_{\tilde{X}}^{\tilde{H}'}(\tilde{x}, t) = \frac{d}{ds}|_{s=0}\tilde{Y}_{t,s}^\tilde{F} = \frac{d}{ds}|_{s=0}\tilde{Y}_{t,s}^\tilde{F} + \frac{d}{ds}|_{s=0}\tilde{Y}_{t,s}^\perp + \frac{d}{ds}|_{s=0}\tilde{Y}_{t,s}^W = \frac{d}{ds}|_{s=0}\tilde{Y}_{t,s}^\perp \in T_{\tilde{x}_t}\tilde{F}
\]
This completes the proof of Proposition 5.1.2

**Corollary 5.2.** Let \( \mathcal{F} \) be a Riemannian foliation of a compact manifold \( M \), then
\[
\text{cat}_{\mathcal{O}(q)}(\M, \mathcal{F}) = \text{cat}_{\mathcal{O}(q)}(\W)
\]

Note that the proof of Proposition 5.1.1 implies that \( \text{cat}_{\mathcal{O}(q)}(\M, \mathcal{F}) = \text{cat}_{\mathcal{O}(q)}(\M, \mathcal{E}) \).
This is equivalent to (29) as there is a natural equivalence \( \text{cat}_{\mathcal{O}(q)}(\M, \mathcal{E}) = \text{cat}_{\mathcal{O}(q)}(\W) \) because \( \tilde{Y} : \M \to \W \) is a fibration with compact fibers.
6. $G$-equivariant Category and Orbit Type

In this section, we recall some general properties of a smooth action of a compact Lie group $G$ on a compact manifold and their applications to $G$-equivariant category. References for this material are \[ \{3, 4, 17, 20, 32, 47, 63\}\]. These results will be applied to the case of the $O(q)$-action on $\tilde{W}$ in the next section.

Let $G$ be a compact Lie group, and $R : N \times G \to N$ a smooth right action on a closed manifold $N$ such that the quotient $N/G$ is connected. For $A \in G$, denote $u A = R(u, A)$, and let $u G = \{ u A \mid A \in G \}$ denote the orbit of $u$. Define the closed stabilizer subgroup of the action of $G$ on $N$,

\[(30) \quad G_0 = \{ A \in G \mid u A = u \text{ for all } u \in N \}\]

Note that $G_0$ is always normal. The action of $G$ is said to be effective if $G_0$ is the trivial subgroup.

Let $U \subset N$ be a $G$-invariant open set. A map $H : U \times [0,1] \to N$ is said to be a $G$-homotopy if $H$ is $G$-equivariant, and $H_0 : U \to N$ is the inclusion. It is $G$-categorical if, in addition, $H_1 : U \to uG$ has image in a single orbit, for some $u \in N$. A $G$-invariant subset $U \subset N$ is $G$-categorical if there exists a $G$-categorical homotopy $H : U \times [0,1] \to N$. The $G$-category $\text{cat}_G(N)$ of $N$ is the least number of $G$-categorical open sets required to cover $N$.

We recall some basic aspects of the geometry of a smooth $G$-action. We assume that $N$ has a $G$-invariant Riemannian metric $g_N$ on $TN$.

For $u \in N$, $w \in uG$, and $\epsilon > 0$, and define the $\epsilon$-normal bundle

\[
N(w, \epsilon) = \{ \tilde{X} \in T_w(N) \mid \tilde{X} \perp T_w(wG) , \| \tilde{X} \| < \epsilon \}
\]

\[
N(uG, \epsilon) = \{ \tilde{X} \in T_w(N) \mid w \in uG , \tilde{X} \perp T_u(uG) , \| \tilde{X} \| < \epsilon \}
\]

Denote the geodesic exponential map by $\exp : TN \to N$. For $w \in N$, let $\exp_w : T_wN \to N$ be the exponential map based at $w$. Define the $\epsilon$-normal neighborhood to $uG$ by

\[(31) \quad \mathcal{U}(uG, \epsilon) = \{ \exp_w(\tilde{X}) \mid w \in uG , \tilde{X} \in N(w, \epsilon) \}\]

For $w = uA$, $A \in G$, the derivative of the right action $R$ defines an isometric linear representation of the isotropy group,

\[(32) \quad d_wR : G_w \to \text{Isom}(T_w(uG)^\perp)\]

Note that $d_uR_A : T_u(uG)^\perp \to T_w(uG)^\perp$ conjugates $d_wR$ to $d_uR$.

Let $\mathcal{N}_0(w, \epsilon) \subset T_w(uG)^\perp$ denote the fixed vectors for the representation $d_uR$.

**THEOREM 6.1** (Equivariant Tubular Neighborhood). For $u \in N$, there exists $\epsilon > 0$ so that the open neighborhood $U = \mathcal{U}(uG, \epsilon)$ is a $G$-equivariant retract of $uG$. That is, there exists a $G$-equivariant homotopy $H : U \times [0,1] \to N$ such that $H_t|uG$ is the identity for all $0 \leq t \leq 1$, and $H_1(U) = uG$. In particular, $U$ is a $G$-categorical open neighborhood of $uG$. 

**Proof:** For $\epsilon > 0$ sufficiently small, $\exp : \mathcal{N}(uG, \epsilon) \to N$ is a diffeomorphism. The homotopy $\mathcal{H}$ is then defined using the homothety

$$\mathcal{H}(\exp_w(\bar{X}), t) = \exp_w(t \bar{X})$$

of the normal geodesic map, where $\bar{X} \in \mathcal{N}(w, \epsilon)$. □

The next result is a generalization of Theorem 6.1 except that there is no assertion that the action of $G$ on $U$ has a linear model.

**THEOREM 6.2** (Equivariant Borsuk). Let $A \subset N$ be a closed, $G$-invariant subset. Then there exists a $G$-invariant open neighborhood $A \subset U$ and a $G$-equivariant homotopy $\mathcal{H} : U \times [0, 1] \to N$ such that $\mathcal{H}|_A$ is the identity for all $0 \leq t \leq 1$, and $\mathcal{H}|_1(U) = A$.

If $H$ is a closed subgroup of $G$, we denote by $(H)$ the conjugacy class of $H$ in $G$. While for an orbit $uG$ the isotropy group $G_u = \{ A \in G \mid vA = v \}$ depends on the choice of $v \in uG$, the conjugacy class ($G_u$) does not, and is therefore an invariant of $uG$. The conjugacy class ($G_u$) is called the orbit type of $uG$.

We say that $uG$ is a principal orbit if $G_u = G_0$. An orbit $uG$ with dimension less that of a principal orbit is said to be a singular orbit. If $uG$ has the same dimension as a principal orbit, but the inclusion $G_0 \subset G_u$ is proper (and of finite index) then $uG$ is said to be an exceptional orbit.

One of the basic results for a smooth action of a compact group on a compact manifold is that it has a finite set of orbit types, which follows easily from s is a consequence of Theorem 6.4 (See Proposition 1.2, Chapter IV of [7], or Theorem 5.11 of [63]). In the case of the action $R : N \times G \to N$, there exists a finite collection of closed subgroups $\{G_0, \ldots, G_k\}$ of $G$, where $G_0$ is defined by (30), such that for all $u \in N$, there exists $\ell$ such that $(G_u) = (G_\ell)$.

There is a partial order on the set of orbit types of the $G$-space $N$: for $u, v \in N$,

$$v \leq u \iff AG_v A^{-1} \subset G_u$$

We adopt the notation $[u] = (G_u)$, then $[u] \leq [v]$ (resp. $[u] < [v]$) means that the isotropy group of $v$ is conjugate to a (resp. proper) subgroup of the isotropy group of $u$. Thus, the orbit $vG \cong G_v \backslash G$ is a fibration over the orbit $uG \cong G_u \backslash G$ and therefore should be considered "larger". Note that $[u] \leq (G_0)$ for all $u \in N$.

Given a closed subgroup $H$ of $G$, define the $(H)$-orbit type subspaces

$$N_{(H)} = \{ u \in N \mid (G_u) = (H) \}$$
$$N_{\leq (H)} = \{ u \in N \mid (G_u) \leq (H) \}$$

where $N_{(H)}$ is non-empty if and only if $H = G_\ell$ some $0 \leq \ell \leq k$; define $N_\ell = N_{(G_\ell)}$. Here are some standard properties of the orbit type spaces; see [7] [20] [63] for details.

**THEOREM 6.3.** For $0 \leq \ell \leq k$, $N_\ell$ is a $G$-invariant submanifold, the quotient space $N_\ell/G$ is a smooth manifold, and the quotient map

$$\pi_\ell : N_\ell \to N_\ell/G$$

is a fibration. Moreover, $\pi_\ell$ has the $G$-equivariant path lifting property: given a smooth path $\sigma : [0, 1] \to N_\ell/G$ with $\sigma(0) = v$, there exists a $G$-equivariant smooth
map $\Sigma : vG \times [0,1] \to N_\ell$ such that $\Sigma_0 : v \cdot G \to N_\ell$ is the inclusion of the orbit $vG$, and $\pi_\ell \circ \Sigma(vA,t) = \sigma(t)$ for all $A \in G$. □

**COROLLARY 6.4.** Given a smooth path $\sigma : [0,1] \to N_\ell$ with $v = \sigma(0)$, there exists a $G$-equivariant map $\Sigma : vG \times [0,1] \to N_\ell$ such that $\Sigma_0$ is the inclusion, and $\pi_\ell \circ \Sigma(vA,t) = \pi_\ell \circ \sigma(t)$. □

There are various subtleties which arise in the study of the orbit type spaces $N(H)$ for a smooth compact Lie group action. One is that the quotient space $N_\ell/G$ need not be connected, and the connected components of $N_\ell/G$ need not all have the same dimensions. A second issue is that $N(H)$ need not be closed, and the structure of the closure $\overline{N(H)}$ is a fundamental aspect of the study of the action. We consider both of these points in the following. First recall (see, for example, Theorem 5.14 and Proposition 5.15 of [63]):

**PROPOSITION 6.5.** Recall that we assume $N/G$ is connected. Then the principal orbit space $N_0 = N(G_0)$ is an open dense $G$-invariant submanifold of $N$ such that $N_0/G$ is connected. For $1 \leq \ell \leq k$, the submanifold $N_\ell \subset N$ has codimension at least two. If there are no exceptional orbit types, then $N_0$ is connected.

The case where $N_0$ is not connected occurs when there exists $A \in G$ that acts as an involution of $N$ with a codimension-one fixed-point set.

Next, we introduce the $Z$-stratification of $N$ associated to the orbit-type decomposition. For $u \in N$, let $Z_u \subset N$ denote the $G$-orbit of the connected component of $N(G_u)$ containing $u$. Note that for $u,v \in N$, either $Z_u \cap Z_v = \emptyset$ or $Z_u = Z_v$. Note that if $G$ is connected, then $Z_u$ is also connected. The inclusion $uG \subset Z_u$ can be strict. For example, this is always the case when $Z_u$ is not a closed subset of $N$.

**PROPOSITION 6.6.** The collection of sets $Z_u$ for $u \in N$ form a finite stratification of $N$. That is, there exists a finite set of points $\{\eta_0, \ldots, \eta_K\} \subset N$ such that for $Z_i = Z_{\eta_i}$

$$N = Z_0 \cup \cdots \cup Z_K$$

As $N_0/G$ is connected, we can require that $Z_0 = N_0$.

**Proof:** There exists a finite number of orbit types, and for each orbit type $(G_\ell)$ the space $N(G_\ell)$ is a finite union of connected submanifolds. □

The $Z$-stratification of $N$ is the collection of sets

$$\mathcal{M}_G(N) = \{Z_0, Z_1, \ldots, Z_K\}$$

It satisfies the axioms of a Whitney stratification (see Chapter 2 of [20].)

For the study of $G$-category, it is more natural to consider the $Z$-stratification of $N$ than the orbit-type stratification by the manifolds $N_\ell$. This is because a $G$-homotopy preserves connected components of the orbit-type stratification, hence the $Z$-stratification captures more of the $G$-homotopical information about the action.
A fundamental property of the $Z$-stratification is the incident relations between the closures $\overline{Z_i}$ of the strata. This motivates the following definition, which we will subsequently relate to the order-type relations between the strata. Define the \textit{incidence partial order} on the collection of sets $\{Z_1, \ldots, Z_K\}$ by setting

$$Z_i \preceq Z_j \iff Z_i \subset \overline{Z_j}$$

Set $Z_i \approx Z_j$ if $Z_i \subseteq Z_j$ and $Z_j \subseteq Z_i$. Note that $Z_i \approx Z_j$ implies that $Z_i = Z_j$.

We require two fundamental technical lemmas, used to study the properties of the incidence partial order. The first implies that the function $u \mapsto [u]$ is “lower semicontinuous” on $N$.

**LEMMA 6.7.** For $u \in N$, let $U(uG, \epsilon)$ be an $\epsilon$-normal neighborhood as in Theorem 6.4. Then $[u] \leq [v]$ for all $v \in U(uG, \epsilon)$. Moreover, $v \in U(uG, \epsilon)$ satisfies $[u] = [v]$ if and only if $v = \exp_w(\bar{X})$ for some $w \in uG$ and $\bar{X} \in N_0(w, \epsilon)$.

**Proof:** For $v \in U(uG, \epsilon)$ there exists $w \in uA$ and $\bar{X} \in N(w, \epsilon)$ such that $v = \exp_w(\bar{X})$. Then $B \in G_v$ if and only if $w = wB$ and $d_w R_B(\bar{X}) = \bar{X}$. Let $A \in G$ such that $w = uA$, then $A B \in G_v A$ so $G_v \subset A^{-1} G_u A$ and hence $[u] \leq [v]$. If $[u] = [v]$ then for all $B \in A^{-1} G_u A$ we have that $d_w R_B(\bar{X}) = \bar{X}$, hence $\bar{X} \in N_0(w, \epsilon)$. Conversely, $\bar{X} \in N_0(w, \epsilon)$ implies $G_u = A^{-1} G_u A$ hence $[u] = [v]$.

The next result implies that the function $t \mapsto [H(u, t)]$ is “upper semicontinuous” for $G$-equivariant homotopy.

**LEMMA 6.8.** Let $H : U \times [0, 1] \to N$ be a $G$-homotopy. For $u \in U$, set $u_t = H(x, t)$. Then for $0 \leq t \leq 1$, we have $G_u \subset G_{u_t}$ and therefore $[u_t] \leq [u]$.

**Proof:** For $A \in G_u$ then $u_t A = H(u, t) A = H(u, t) = H(u, t) = u_t$.

The following result establishes the relationships between the incidence partial order and the orbit-type partial order.

**PROPOSITION 6.9.** Let $v \in Z_i$ and $u \in Z_j$, for $i \neq j$ (and hence $Z_i \cap Z_j = \emptyset$.) Suppose that $v \in \overline{Z_j} - Z_j$. Then $[v] < [u]$ and $Z_i \subset (\overline{Z_j} - Z_j)$, hence $Z_i \preceq Z_j$.

In particular, $Z_i \preceq Z_j$ and $Z_i \neq Z_j$ implies that $[v] < [u]$.

**Proof:** Given that $v \in \overline{Z_j} - Z_j$ there exists a sequence $\{u_\ell \mid \ell = 1, 2, \ldots\} \subset Z_j$ such that $\lim_{\ell \to \infty} u_\ell = v$. Let $\epsilon > 0$ be such that $U(vG, \epsilon)$ is a $G$-categorical neighborhood of $vG$. Then there exists $\ell$ such that $u_\ell \in U(vG, \epsilon)$. Let $w \in vG$, $\bar{X} \in N(w, \epsilon)$ be such that $\exp_w(\bar{X}) = u_\ell$. Then by Lemma 6.7 $[v] = [u_\ell] = [u]$ unless $\bar{X} \in N_0(w, \epsilon)$. If $\bar{X} \in N_0(w, \epsilon)$, we have that $[v] = [u_\ell] = [u]$, and hence $\exp_w(N_0(w, \epsilon)) \subset Z_j$. Thus, $v \in Z_j$ contrary to assumption, so we must have $[v] < [u]$.

It remains to show that $Z_i \cap (\overline{Z_j} - Z_j) \neq \emptyset$ implies $Z_i \subset (\overline{Z_j} - Z_j)$. This follows from an argument similar to the above.

**COROLLARY 6.10.** A stratum $Z_j$ is minimal for the incidence partial order if and only if $Z_j$ is a closed submanifold.
Proof: If $Z_j$ is not closed, then there exists $v \in \overline{Z_j} - Z_j$, and then $v \in Z_i$ for some $i \neq j$. Then $Z_i \subset Z_j$ by Proposition 6.9 so $Z_j$ is not minimal. The converse is obvious. □

The orbit-type function $u \rightarrow [u]$ is lower-semicontinuous by Lemma 6.7, so it is natural to also consider a notion of minimality based on continuity:

DEFINITION 6.11. A stratum $Z_u$ is said to be locally minimal if there is an open $G$-invariant neighborhood $U$ of the closure $\overline{Z_u}$ such that $[u] \leq [v]$ for all $v \in U$.

PROPOSITION 6.12. $Z_u$ is locally minimal if and only if $Z_u$ is closed. Hence, $Z_u$ is locally minimal if and only if it is a least element for the incidence partial order.

Proof: Assume that $Z_u$ is closed, then by the Equivariant Borsuk Theorem 6.2 there exists a $G$-invariant open neighborhood $U$ of $Z_u$ and a $G$-equivariant homotopy $\mathcal{H} : U \times [0, 1] \rightarrow M$ such that $\mathcal{H}_0(U) = Z_u$. We claim that for all $v \in U$, $[u] \leq [v]$. Let $v \in U$, and set $v_t = \mathcal{H}(v, t)$. Note that $v_1 \in Z_u$ so that $[v_1] = [u]$. By Lemma 6.8 $[v_t] \leq [v]$ for all $t$. In particular, $[u] = [v_1] \leq [v_0] = [v]$, as was to be shown.

Conversely, assume that $Z_u$ is locally minimal, with $G$-invariant open neighborhood $U$ of $\overline{Z_u}$ as in the definition. If there exists $v \in \overline{Z_u} - Z_u \subset U$, then $[v] < [u]$ by Proposition 6.9 This contradicts the assumption that $v \in U$ satisfies $[v] \geq [u]$. □

COROLLARY 6.13. For each $u \in N$, there exists a locally minimal stratum $Z_j \subset \overline{Z_u}$.

For the remainder of this section, we consider the properties of $G$-homotopy with respect to the incidence and orbit-type partial orders, and give applications to $G$-category. The next result implies that $G$-homotopy preserves the $\mathcal{Z}$-stratification.

PROPOSITION 6.14. Let $\mathcal{H} : U \times [0, 1] \rightarrow N$ be a $G$-homotopy. Then for all $u \in U$ and $0 \leq t \leq 1$,

\begin{equation}
\mathcal{H}_t(U \cap Z_u) \subset \overline{Z_u}
\end{equation}

Proof: For $u \in U$, set $u_t = \mathcal{H}(x, t)$. Then $u_0 = u \in U \cap N_{\{G_u\}}$. Lemma 6.8 implies that for all $0 \leq t \leq 1$, $[u_t] \leq [u]$, hence $(G_{u_t}) \leq (G_u)$ so that $u_t \in N_{\{G_u\}}$. Define $s_0 = \sup\{s \mid u_t \in \overline{Z_u} \text{ for all } 0 \leq t \leq s\}$

Note that $u_{s_0} \in \overline{Z_u}$. Suppose that $s_0 < 1$. Then for all $\delta > 0$ there exists $s_0 < t < s_0 + \delta$ such that $u_t \in N_{\{G_u\}}$ but $u_t \notin \overline{Z_u}$. Set $w = u_{s_0}$.

Lemma 6.8 implies that $G_u \subset G_{u_t}$ for all $0 \leq t \leq 1$, so in particular $G_u \subset G_w$. Let $U(w, G, \epsilon)$ be an $\epsilon$-normal neighborhood of $w G$, and $\exp_w : N(w, \epsilon) \rightarrow N$ the $\epsilon$-disk transverse to $w G$.

Let $\mathcal{N}(w, \epsilon, G_u) \subset N(w, \epsilon)$ denote the vectors fixed by the subgroup $G_u$ under the isotropy representation $\alpha_u: G_u \rightarrow \text{Isom}(N(w, \epsilon))$. Then the submanifold $\exp_w(\mathcal{N}(w, \epsilon, G_u))$ contains the intersection $\exp_w(\mathcal{N}(w, \epsilon)) \cap Z_u$, hence

\begin{equation}
\exp_w(\mathcal{N}(w, \epsilon, G_u)) = \exp_w(\mathcal{N}(w, \epsilon)) \cap \overline{Z_u}
\end{equation}
By assumption, there exists $s_0 < t < s_0 + \delta$ such that $u_t \in N_{\leq (G_u)}$ and there exists $\bar{X} \in N(w, \epsilon)$ but $\bar{X} \notin N(w, \epsilon, G_u)$ so that $v = \exp_u(\bar{X}) \in u_t G$. Thus, $G_v \subset G_u$ as the elements of $G_v$ fix the vector $\bar{X}$, so we have the proper inclusions $G_u \subset G_v \subset G_w$. By hypothesis, $u_t \in Z_t \subset N_{\leq (G_u)}$ for some $Z_t \neq Z_u$ hence $G_u$ and $G_v$ are conjugate in $G$. This is impossible, as $G_u$ is a proper subgroup of $G_v$. □

**COROLLARY 6.15.** Let $\mathcal{H} : U \times [0, 1] \to N$ be a $G$-homotopy. Suppose that $Z_u$ is a locally minimal set for $u \in U$. Then for all $0 \leq t \leq 1$,

\[ \mathcal{H}_t(U \cap Z_u) = Z_u \]  

**(Proof):** By Corollary 6.10 the set $Z_u$ is closed, hence $\mathcal{H}_t(U \cap Z_u) \subset Z_u$ by Proposition 6.14. As $Z_u$ is a closed submanifold, and $\mathcal{H}_0$ is the identity, the map $\mathcal{H}_t$ must be surjective for all $0 \leq t \leq 1$. □

Next, we consider the properties of the $G$-category and its relation to the $\mathcal{Z}$-stratification.

**PROPOSITION 6.16.** The $G$-category $\Lambda = \text{cat}_G(N)$ is finite. Moreover, there exists a $G$-categorical covering $\{\mathcal{H}_\ell : U_\ell \times [0, 1] \to N \mid 1 \leq \ell \leq \Lambda\}$ and basepoints $\{w_1, \ldots, w_\Lambda\} \subset N$ such that each $Z_{w_\ell}$ is locally minimal and $\mathcal{H}_\ell(U_\ell)$ is a locally minimal set for $\ell \in [0, 1]$.

**(Proof)**: By Lemma 6.7 every orbit has a $G$-categorical open neighborhood, and $N$ compact implies there is a finite subcovering by $G$-categorical open sets, hence $\text{cat}_G(N)$ is finite.

Let $\{\mathcal{H}_\ell : U_\ell \times [0, 1] \to N \mid 1 \leq \ell \leq \Lambda\}$ be a $G$-categorical open covering, with $\mathcal{H}_\ell(U_\ell) \subset v_\ell G$ for points $\{v_1, \ldots, v_\Lambda\} \subset N$.

By Corollary 6.13 for each $1 \leq \ell \leq \Lambda$, we can choose $u_\ell \in \overline{Z_{w_\ell}}$ such that $Z_{w_\ell}$ is a locally minimal set. Let $U_\ell = \mathcal{H}(u_\ell G, \epsilon)$ be an $\epsilon$-normal neighborhood as in Theorem 6.1 with $G$-homotopy retract $\mathcal{H}''' : U_\ell \times [0, 1] \to N$. (We choose $\epsilon > 0$ sufficiently small so that it works for all $\ell$.) Then there exists $u_\ell \in U_\ell \cap Z_{w_\ell}$ in the same path-component as $v_\ell$.

Let $\sigma_\ell : [0, 1] \to Z_{w_\ell}$ be a smooth path such that $\sigma_\ell(0) = v_\ell$ and $\sigma_\ell(1) = u_\ell$.

By Corollary 6.13 each path $\sigma_\ell$ defines a smooth $G$-equivariant lifting $\Sigma_\ell : v_\ell G \times [0, 1] \to Z_{w_\ell}$ such that $\pi_\ell \circ \Sigma_\ell(v_\ell A, t) = \pi_\ell \circ \sigma_\ell(t)$. Thus, $\Sigma_\ell(v_\ell G, 1) \subset u_\ell G$.

Now, define $\mathcal{H}_\ell : U_\ell \times [0, 1] \to N$ as the concatenation:

\[
\mathcal{H}_\ell(u, t) = \begin{cases} 
\mathcal{H}'(u, t) & \text{for } 0 \leq t \leq 1/3, \\
\Sigma_\ell(\mathcal{H}'(u, 1), 3t - 1) & \text{for } 1/3 \leq t \leq 2/3, \\
\mathcal{H}''''(\Sigma_\ell(\mathcal{H}'(u, 1), 3t - 2)) & \text{for } 2/3 \leq t \leq 1
\end{cases}
\]

This yields a piece-wise smooth $G$-categorical homotopy $\mathcal{H}$ as desired. By adjusting the time parameters, the map $\mathcal{H}_\ell$ can be made smooth. □

One of the main problems for the study of $G$-category is to obtain upper and lower bounds for $\text{cat}_G(N)$ in terms of the geometry and topology of the $G$-action. The homotopy properties of the $\mathcal{Z}$-stratification given above yields a geometric lower bound for $\text{cat}_G(N)$. 


Let $\alpha_G(N)$ denote the number of locally minimum strata in $\mathcal{M}_G(N)$.

**Theorem 6.17.** $\text{cat}_G(N) \geq \alpha_G(N)$.

**Proof:** Let $\{ H_\ell : U_\ell \times [0,1] \to N \mid 1 \leq \ell \leq k \}$ be a $G$-categorical covering of $N$, and let $\{ w_1, \ldots, w_k \} \subset N$ be such that $H_\ell(U_\ell) \subset w_\ell G$.

Let $Z_i$ be a locally minimal set, and suppose that $u \in U_\ell \cap Z_i$. Set $u_{\ell,t} = H_\ell(u,t)$.

Then by Corollary 6.15, $u_{\ell,1} G = w_\ell G \subset Z_i$. Suppose that $Z_i$ and $Z_j$ are disjoint locally minimal sets, such that $u \in U_\ell \cap Z_i$ and $v \in U_k \cap Z_j$. By the above we have that $w_\ell G \subset Z_i$ and $w_k G \subset Z_j$. If $k = \ell$ then $Z_i \cap Z_j \neq \emptyset$, contrary to assumption. It follows that $k \neq \ell$. Thus, for each locally minimal set $Z_i$ we can associate at least one $G$-categorical set $U_{\ell}$ so that distinct locally minimal sets yield distinct indices $\ell$. Hence $\text{cat}_G(N) \geq \alpha_G(N)$.

Proposition 6.14 implies that the restriction of a $G$-categorical open set $U$ to a closed subset $Z_\ell$ is again $G$-categorical. This remark yields another type of lower bound for $\text{cat}_G(N)$:

**Theorem 6.18.** For each $Z_\ell \in \mathcal{M}_G(N)$, $\text{cat}_G(N) \geq \text{cat}_G(Z_\ell)$.

**Proof:** A $G$-categorical covering of $N$ restricts to a $G$-categorical open covering of each closed subspace $Z_\ell$.

In general, neither Theorems 6.17 or 6.18 are optimal lower bounds, and best estimates are obtained by combining the ideas of each estimate for the particular group action in question.

It is also possible to develop very sophisticated lower bound estimates for $\text{cat}_G(N)$ in terms of the cohomology and homotopy theory of the compact group action [15].

There are also several types of upper bound estimates for $\text{cat}_G(N)$ as discussed for example in the works of Marzantowicz [42, 47], Bartsch [4], Ayala, Lasheras and Quintero [34], Colman [13] and the authors [36]. The simplest version is based on the dimension estimate, that for a connected manifold $X$ of dimension $\xi$, there is an upper bound $\text{cat}(X) \leq \xi + 1$. As each stratum $Z_\ell \in \mathcal{M}_G(N) = \{ Z_0, Z_1, \ldots, Z_K \}$ has quotient $Z_\ell \! / \! G$ which is a connected manifold, we can apply this estimate to each stratum to obtain

$$\alpha_G(N) \leq \text{cat}_G(N) \leq \# \mathcal{M}_G(N) + \sum_{0 \leq \ell \leq K} \dim(Z_\ell \! / \! G)$$
In this section, the results of the last section are applied to the case of a Riemannian foliation $F$ and the associated space $O(q)$-manifold $\hat{M}$. We first introduce the *isotopy stratification* of $M$, corresponding to the structure of the leaves of $F$. Then the correspondence between the $O(q)$-orbits on $\hat{W}$ and leaf closures for $F$ is developed: the main result is that the $Z$-stratification of $\hat{W}$ corresponds to the isotopy stratification of $M$. This yields an interpretation of the locally minimal sets $Z_u$ for the $O(q)$-action on $\hat{E}$ in terms of the intrinsic geometry of $F$. References for this section are the works of Molino [51, 52, 53], Haefliger [28, 29] and Salem [59].

Let $L$ be a leaf of $F$. An $F$-*isotopy* of $L$ is a smooth map $I : L \times [0, 1] \to M$ such that $I_0 : L \to M$ is the inclusion of $L$, and for each $0 \leq t \leq 1$, $I_t : L \to M$ is a diffeomorphism onto a leaf $L_t$ of $F$. We then say that the leaf $L_1$ is $F$-*isotopic* to $L_0 = L$, and write $L_0 \sim L_1$. For example, Proposition 5.1 implies that every leaf of $\hat{F}$ is $\hat{F}$-isotopic in $\hat{M}$ to every other leaf of $\hat{F}$. However, for $F$ this need not be true.

Let $\mathcal{I}_L$ denote the set of leaves of $F$ which are $F$-isotopic to $L$. For $L = L_x$ set $\mathcal{I}_x = \mathcal{I}_{L_x}$. The set of $F$-isotopy classes of the leaves of $F$ defines the *isotopy stratification* of $M$.

Next, we use the Molino theory to define the holonomy stratification for $M$. Recall that for $\hat{x} = (x, e) \in \hat{M}$ with $\hat{w} = \hat{\Upsilon}(\hat{x})$, the fiber $\hat{\Upsilon}^{-1}(\hat{w}) = \hat{L}_{\hat{x}}$. The projection $\pi : \hat{M} \to M$ restricts to a covering map $\pi : \hat{L}_{\hat{x}} \to L_x$, and by Corollary 3.9 we have that

$$\mathcal{L}_x = \pi(\mathcal{L}_{\hat{x}}) \cong \mathcal{L}_{\hat{x}}/H_{\hat{x}}$$

where $H_{\hat{x}} \equiv \{ A \in O(q) | \mathcal{L}_{\hat{x}}A = \mathcal{L}_{\hat{x}} \} = H_{\hat{x}}$ by (12).

For $\hat{w} = \hat{\Upsilon}(\hat{x}) \in \hat{W}$, the $O(q)$-orbit $\hat{w}O(q)$ lifts to the $O(q)$-orbit of $\mathcal{L}_{\hat{x}}$, which again projects to the leaf closure $\mathcal{L}_x$. Thus, each orbit $\hat{w}O(q)$ corresponds to exactly one leaf $\mathcal{L}_x$ of $E$. Let $\Upsilon : M \to W$ denote the quotient map to the leaf space of $E$. Then we have a commutative diagram of $O(q)$-equivariant maps:

$$\begin{array}{ccc}
\hat{M} & \xrightarrow{\hat{\Upsilon}} & \hat{W} \\
\downarrow \pi & & \downarrow \hat{\pi} \\
M & \xrightarrow{\Upsilon} & W
\end{array}$$

The set of leaves of $F$ without holonomy form an open dense subset, $M_0 \subset M$. Define open dense subsets $\hat{M}_0 = \pi^{-1}(M_0)$ and $\hat{W}_0 = \hat{\Upsilon}(\hat{M}_0) \subset \hat{W}$. The leaves in $M_0$ and $\hat{M}_0$ are said to be *regular*, and the points of $\hat{W}_0$ are regular orbits.

Let $\hat{w} = \hat{\Upsilon}(\hat{x})$. Then $O(q)\hat{\omega} = H_{\hat{x}}$. It is immediate from the definitions that $\hat{\Upsilon}(\hat{y}) \in \hat{W}_{O(q)\hat{\omega}}$ if and only if $H_{\hat{y}}$ is conjugate in $O(q)$ to a dense subgroup of $H_{\hat{x}}$. 

\[ \text{References for this section are the works of Molino [51, 52, 53], Haefliger [28, 29] and Salem [59].} \]
For a closed subgroup $H \subset O(q)$, $\hat{x} = (x, e) \in \hat{M}$ and $\hat{w} = \hat{\Upsilon}(\hat{x})$, set:

$$
\hat{M}_{(H)} = \hat{\Upsilon}^{-1}(\hat{W}_{(H)}) ; \quad M_{(H)} = \pi(\hat{M}_{(H)})
$$

$$
\hat{M}_{\leq(H)} = \hat{\Upsilon}^{-1}(\hat{W}_{\leq(H)}) ; \quad M_{\leq(H)} = \pi(\hat{M}_{\leq(H)})
$$

$$
\hat{Z}_x = \hat{\Upsilon}^{-1}(\hat{Z}_w) ; \quad Z_x = \pi(\hat{Z}_x)
$$

Note that as $\hat{\Upsilon}$ has connected fibers, $\hat{Z}_x$ can also be described as the $O(q)$-orbit of the connected component of $\hat{M}(O(q)_x)$ containing $\hat{x}$, and $Z_x$ is the connected component of $M(O(q)_x)$ containing $x$.

Let $\mathcal{M}_F(M) = \{Z_0, Z_1, \ldots, Z_K\}$ be the $Z$-stratification of $\hat{M}$ for the action of $O(q)$. Then there exists a finite set of points $\{z_1, \ldots, z_K\} \subset M$ such that $Z_{x_1} = \pi(\hat{\Upsilon}^{-1}(Z_{x_1}))$. Set $Z_i = Z_{x_i}$.

A stratum $Z_i$ is said to be locally minimal if the set $Z_i$ is locally minimal.

**Proposition 7.1.** If $\hat{y} \in \hat{Z}_x$, then $\mathcal{H}_{\hat{y}}$ is conjugate in $O(q)$ to $\mathcal{H}_{\hat{x}}$.

**Proof:** Given $\hat{y} = (y, f) \in \hat{Z}_x$, either there exists a continuous path $\sigma : [0, 1] \to \hat{M}(O(q)_x)$ such that $\sigma(0) = \hat{x}$ and $\sigma(1) = \hat{y}$, or there exists $A \in O(q)$ such this holds for $(y, f A)$. So without loss of generality we can assume that $\hat{y}$ is in the same path component as $\hat{x}$.

Let $\epsilon > 0$ be as in Corollary 3.5. For each $\hat{y}_t = \sigma(t)$ there exists a linear model for $\mathcal{F}$ along the leaf $L_{\hat{y}}$ through the point $y_t = \pi(\hat{y}_t)$. The image $\pi(\sigma(0, 1))$ is compact, so is covered by a finite collection of such linear models. It thus suffices to consider the case where $x$ and $y$ are such that there is a foliated immersion as given in Corollary 3.6.

$$
\Xi_{\hat{z}} : Q^t|L_x = \left(\hat{L}_x \times \mathbb{D}_t^q\right) / \pi_1(L_x, x) \longrightarrow M
$$

and $y \in T_{\hat{y}} \cap Z_x$. Let $\hat{X} \in \mathbb{D}_t^q$ so that $\nu_{\hat{X}}(\hat{X}) = \Xi_{\hat{z}}(\hat{\xi}) = y$, where $\hat{\xi} = ((\hat{x}) \times \{\hat{X}\}) \in Q^t|L_x$. The map $\Xi_{\hat{z}}$ defines an orthonormal framing of $Q_y$ which we denote by $f$, so that $\hat{y} = (y, f) \in \hat{M}$.

By Lemma 3.2 the holonomy group of the foliation $\mathcal{F}^\omega$ on $Q^t|L_x$ at the point $\hat{\xi}$ is given by

$$
\mathcal{H}^\omega_{\hat{\xi}} = \left\{ A \in \mathcal{H}_{\hat{x}} \mid \hat{X}^\ast A = \hat{X} \right\} \subset \mathcal{H}_{\hat{x}} \subset O(q)
$$

Thus $\mathcal{H}^\omega_{\hat{\xi}}$ is conjugate to $\mathcal{H}_{\hat{y}}$ in $O(q)$.

Let $V_\xi \subset \mathbb{D}_t^q$ be the linear subspace of vectors fixed by the subgroup $\mathcal{H}_{\hat{x}}$. Note that $V_\xi$ is also the set of vectors fixed by the closure $H_\hat{x} = \overline{H}_{\hat{x}}$.

Similarly, define $V_\xi \subset \mathbb{D}_t^q$ as the linear subspace of vectors fixed by $\mathcal{H}_{\hat{x}}$. Again, $V_\xi$ is also the set of vectors fixed by the closure $\overline{H}_{\hat{x}} \subset O(q)$ which is conjugate to $H_\hat{y}$.

The key to the proof of Proposition 7.1 is the following result for linear actions.

**Lemma 7.2.** $\mathcal{H}^\omega_{\hat{\xi}} = \mathcal{H}_{\hat{x}} \iff V_\xi = \mathcal{V}_\xi \iff (H_{\hat{x}}) = (H_{\hat{y}})$.
Since \( V \subset H \), we conclude the proof of Proposition \( 7.1 \). The assumption that \( \overline{H \otimes H} = H \) is conjugate to \( H \) implies that \( \overline{H \otimes H} = H \).

PROPOSITION 7.2. For all \( \mu \in M \), of \( \overline{H \otimes H} \).

Proof: We first show that \( H \otimes H \) is finite. For the same reason, for all 0 \( \leq t \leq 1 \), the restriction \( I_t : L_x \to M \) is a diffeomorphism onto the leaf through \( \nu_x(tX) \). As \( \nu_x(X) = y \) and \( \nu_x(0) = x \), this proves that \( y \in I_x \).

Conversely, to show that \( I_x \subset Z_x \), it suffices to consider an isotopy \( I : L_x \times [0, 1] \to M \) with \( I_0(x) = x \) and \( I_1(x) = y \), whose image lies inside a foliated immersion

\[ \Xi_x : Q^1|L_x = \left( \bar{L}_x \times D^2 \right) / \pi_1(L_x, x) \to M \]

where the map is well-defined as \( \bar{X} \in V \) and the action of \( \pi_1(L_x, x) \) on \( V \) is trivial. For the same reason, for all 0 \( \leq t \leq 1 \), the restriction \( I_t : L_x \to M \) is a diffeomorphism onto the leaf through \( \nu_x(tX) \). As \( \nu_x(X) = y \) and \( \nu_x(0) = x \), this proves that \( y \in I_x \).

Conversely, to show that \( I_x \subset Z_x \), it suffices to consider an isotopy \( I : L_x \times [0, 1] \to M \) with \( I_0(x) = x \) and \( I_1(x) = y \), whose image lies inside a foliated immersion

\[ \Xi_x : Q^1|L_x = \left( \bar{L}_x \times D^2 \right) / \pi_1(L_x, x) \to M \]

The image \( L_t = I_t(L_x) \) is isotopic to \( L_x \) so the covering maps \( L_t \to L_x \) are diffeomorphisms, thus the holonomy groups \( H_{\bar{L}_x} \) have constant conjugacy classes. Hence the conjugacy class of the closure \( (H_{\bar{L}_x}) \) is constant, so \( L_t \subset Z_x \) for all 0 \( \leq t \leq 1 \). Thus, \( y \in Z_x \).

COROLLARY 7.4. For \( \hat{\nu} = (x, e) \in \hat{M} \), set \( \hat{w} = \hat{\nu}(\hat{x}) \). Then \( Z_{\hat{w}} \) is locally minimal if and only if \( I_x \) is a closed submanifold of \( M \).
COROLLARY 7.5. For each isotopy class \( L_x \), there is a well-defined conjugacy class \( (\mathcal{H}_x) \) defined as the conjugacy class of the holonomy group \( \mathcal{H}_x \) for any \( \tilde{x} = (x, e) \in \tilde{M} \). □

Corollary 7.5 highlights one of the unique aspects of the study of Riemannian foliations, the duality between intrinsic and extrinsic geometry of leaves. That is, for \( L_y \) near to \( L_x \), we have that \( L_y \sim L_x \) exactly when \( L_y \) is diffeomorphic to \( L_x \) via the orthogonal projection along leaves, and also that \( L_y \sim L_x \) exactly when the holonomy group \( \mathcal{H}_y \) of \( L_y \) is naturally conjugate to that of \( L_x \).

8. Foliations with finite category

Let \( \mathcal{F} \) be a Riemannian foliation \( \mathcal{F} \) of a compact manifold \( M \), then the \( O(q) \)-equivariant LS category \( \operatorname{cat}_{O(q)}(\hat{W}) \) is finite, and by Corollaries 4.4 and 5.2 we have the equalities

\[
\operatorname{cat}^e_{\hat{q}}(\mathcal{M}, \mathcal{F}) = \operatorname{cat}_{O(q)}(\hat{M}, \hat{\mathcal{F}}) = \operatorname{cat}_{O(q)}(\hat{W})
\]

In this section, we prove the promised geometric criteria for when \( \operatorname{cat}_{\hat{q}}(\mathcal{M}, \mathcal{F}) \) is finite, and show that \( \operatorname{cat}^e_{\hat{q}}(\mathcal{M}, \mathcal{F}) = \operatorname{cat}_{\hat{q}}(\mathcal{M}, \mathcal{F}) \) when this criteria is satisfied. Theorem 1.4 follows from Propositions 8.1 and 8.4. As a further consequence, we obtain estimates for \( \operatorname{cat}_{\hat{q}}(\mathcal{M}, \mathcal{F}) \) which extend those proven in [10] for compact Hausdorff foliations.

PROPOSITION 8.1. Let \( Z_x \) be a locally minimal set for \( \mathcal{F} \). Suppose there is given an open saturated set \( U \) such that such that \( U \cap Z_x \neq \emptyset \), and a foliated homotopy \( H : U \times [0,1] \to \mathcal{M} \) such that \( H_1 : U \to \mathcal{M} \) has image in a single leaf of \( \mathcal{F} \). Then every leaf of \( \mathcal{F} \) in \( Z_x \) is compact.

Proof: Let \( y \in U \cap Z_x \). By Proposition 7.3 every leaf of \( \mathcal{F} \) in \( Z_x \) is isotopic, so it suffices to show that there is some \( z \in Z_x \) with \( L_z \) compact.

Let \( z = H_1(y) \), so the image of \( H_1 : U \to \mathcal{M} \) is contained in the leaf \( L_z \) of \( \mathcal{F} \). As noted in section 2, the closure \( \mathcal{T}_y \) is a compact minimal set for \( \mathcal{F} \). By Lemma 2.2, we have that \( \mathcal{T}_y \subset U \), hence by Theorem 1.3 of [25], the image \( H_1(\mathcal{T}_y) \) is contained in a compact leaf of \( \mathcal{F} \), thus \( L_z \) is compact. The proof now follows from the following:

LEMMA 8.2. Let \( Z_x \) be a locally minimal set for \( \mathcal{F} \) and \( U \) an open saturated set such that \( U \cap Z_x \neq \emptyset \). If \( H : U \times [0,1] \to \mathcal{M} \) is a foliated homotopy, then \( H_t(Z_x \cap U) \subset Z_x \) for all \( 0 \leq t \leq 1 \).

Proof: Let \( \tilde{w} \in \hat{W} \) be such that \( Z_x = \pi(\hat{\mathcal{T}}^{-1}(\mathcal{T}_{\tilde{w}})) \), and note that \( \mathcal{T}_{\tilde{w}} \) is a locally minimal set. Let \( \tilde{U} = \pi^{-1}(U) \subset \tilde{M} \), and set \( \tilde{\mathcal{U}} = \hat{\mathcal{T}}(\tilde{U}) \subset \hat{W} \).

Given \( y \in U \cap Z_x \), choose a point \( \tilde{\xi} \in \tilde{\mathcal{U}} \cap \mathcal{T}_{\tilde{w}} \) so that \( \hat{\mathcal{T}}(\pi^{-1}(y)) = \tilde{\xi} \cdot \mathcal{O}(q) \).

By Proposition 4.4, the homotopy \( H \) determines an \( O(q) \)-equivariant homotopy \( \hat{\mathcal{H}} : \tilde{\mathcal{U}} \times [0,1] \to \hat{W} \). By Corollary 6.15, the trace \( \tilde{\xi}_t = \hat{\mathcal{H}}(\tilde{\xi}, t) \in \mathcal{T}_{\tilde{w}} \) for all \( 0 \leq t \leq 1 \), and thus \( \pi(\hat{\mathcal{T}}^{-1}(\tilde{\xi}_t)) \subset Z_x \). In particular, \( L_z = L_z = \pi(\hat{\mathcal{T}}^{-1}(\tilde{\xi}_1)) \subset Z_x \). □
COROLLARY 8.3. Let \( F \) be a Riemannian foliation \( F \) of a compact manifold \( M \) with \( \text{cat}_0(\pi, F) \) finite. Then every local minimal set \( Z_i \) for \( F \) consists of compact leaves.

Proof: Choose \( x \in Z_i \). Then there a categorical open set \( U \) with \( x \in U \), so by Proposition 8.1 every leaf of \( Z_i \) is compact. \( \Box \)

PROPOSITION 8.4. Let \( F \) be a Riemannian foliation \( F \) of a compact manifold \( M \) such that every local minimal set \( Z_i \) for \( F \) consists of compact leaves. Then

\[
\text{cat}(\pi, F) = \text{cat}_e(\pi, F).
\]

Proof: Let \( \{ H_\ell : U_\ell \times [0, 1] \to \tilde{W} \mid 1 \leq \ell \leq k \} \) be an \( O(q) \)-categorical covering of \( \tilde{W} \), for \( k = \text{cat}_0(\pi, \tilde{W}) \). Let \( \{ \tilde{w}_1, \ldots, \tilde{w}_k \} \subset \tilde{W} \) be such that \( H_\ell(U_\ell) \subset \tilde{w}_\ell O(q) \).

By Proposition 6.16 we can assume that \( Z_{\tilde{w}_\ell} \) is locally minimal for each \( 1 \leq \ell \leq k \). Thus, the sets \( Z_{x_\ell} \) are locally minimal for \( F \). By assumption, all leaves in \( Z_{x_\ell} \) are compact.

For each \( 1 \leq \ell \leq k \), let \( \tilde{x}_\ell \in \tilde{Y}^{-1}(\tilde{w}_\ell) \) and set \( x_\ell = \pi(\tilde{x}_\ell) \) and \( L_\ell = L_{x_\ell} \subset M \). Then \( L_{x_\ell} \subset Z_{x_\ell} \) hence is a compact leaf.

Let \( H_\ell : U_\ell \times [0, 1] \to M \) be the \( F \)-foliated homotopy corresponding to \( H_\ell \) for \( U_\ell = \pi(\tilde{Y}^{-1}(U_\ell)) \).

We have that \( H_1 : U_\ell \to L_\ell \), and as each \( L_\ell \) is a compact leaf, we are done. \( \Box \)

More than just characterizing when \( \text{cat}_0(\pi, F) \) is finite, the arguments of the previous sections and the above yield an estimate for the transverse LS category. Proposition 7.3 identifies the isotropy stratification of \( F \) with the \( Z \)-stratification of the \( O(q) \)-action on \( \tilde{W} \). Let \( \{ I_1, \ldots, I_K \} \) be an enumeration of the isotropy strata for \( F \) and assume that \( I_\ell \) is a closed submanifold exactly when \( 1 \leq \ell \leq k \), where \( k \leq K \). Set \( \alpha(M, F) = k \). Then by Lemma 8.2 Theorem 6.18 and equation (45) we obtain

THEOREM 8.5. Let \( F \) be a Riemannian foliation of a compact manifold \( M \). Then

\[
\alpha(M, F) \leq \text{cat}_0(\pi, F) \leq \sum_{1 \leq \ell \leq K} \text{cat}_0(I_\ell, F|I_\ell) \leq K + \sum_{1 \leq \ell \leq K} \dim(I_\ell/F_\ell).
\]

Moreover, if every locally minimal set \( Z_i \) consists of compact leaves, then \( \text{cat}_0(M, F) = \text{cat}_0(M, F) \) so the estimates (45) and (46) also hold for the transverse LS category.

Theorem 8.5 is a complete generalization of the estimate given by Theorem 6.1 in [16] for the transverse LS category of compact Hausdorff foliations.
9. Critical points

One of the applications of the LS-category invariant for a compact manifold \( N \) is to give a lower bound estimate on the number of critical points for a \( C^1 \)-function \( f : N \to \mathbb{R} \) (see \([18, 23, 24, 40, 46, 55, 56]\)). When there is a compact Lie group \( G \) acting on \( N \) and the function \( f \) is invariant for the \( G \)-action, then there is an induced map on the quotient space, \( f : N/G \to \mathbb{R} \), and one can attempt to estimate the number of critical points of \( f \) using \( f \). Unfortunately, the quotient space \( N/G \) need not be a manifold, so the classical theory does not apply. In addition, examples show that the category of \( N/G \) can be much smaller than the category \( \text{cat}_G(N) \).

The solution is to consider the set of critical points for \( f \) as a \( G \)-space, and then the \( G \)-category \( \text{cat}_G(N) \) provides a lower bound for the number of critical \( G \)-orbits \( [4, 23, 24, 47] \).

Let \( f : M \to \mathbb{R} \) be a \( C^1 \)-function on a compact manifold \( M \) with a Riemannian foliation \( \mathcal{F} \). The function \( f \) is said to be \( \mathcal{F} \)-basic if it is constant along leaves of \( \mathcal{F} \). Since \( f \) is continuous, it is constant on the leaves of the SRF \( \mathcal{E} \) of \( M \) defined by the closures of the leaves of \( \mathcal{F} \), thus induces a continuous map on the quotient space \( \phi = f : W = M/\mathcal{E} \to \mathbb{R} \). The differential \( df : TM \to \mathbb{R} \) is a basic 1-form, so if \( L \) is a critical, then \( L \) will consist of critical leaves also. Colman studied in section 5 of \([17]\) the relation between the transverse category \( \text{cat}_\mathcal{F}(M, \mathcal{F}) \) of \( \mathcal{F} \) and the number of critical leaves of \( \mathcal{E} \) for \( f \), in the case where all leaves of \( \mathcal{F} \) are compact, so it is a compact Hausdorff foliation.

An alternate approach is to first lift \( f \) to a smooth function \( \hat{f} = f \circ \pi : \hat{M} \to \mathbb{R} \) which is \( \mathcal{F} \)-basic for \( \hat{\mathcal{F}} \). A leaf \( \hat{L}_x \) of \( \hat{\mathcal{F}} \) will be critical for \( \hat{f} \) if and only if \( L_x \) is critical for \( f \), and so its closure \( \overline{L}_x \) is a critical submanifold for \( df \). Moreover, the function \( \hat{f} \) is \( O(q) \)-invariant, so we can estimate the number of critical leaf closures for \( f \) in terms of the \( O(q) \)-invariant critical submanifolds of \( \hat{M} \). The smooth map \( \hat{f} \) descends to a smooth \( O(q) \)-invariant map \( \hat{\phi} : \hat{M} \to \mathbb{R} \) and the \( O(q) \)-invariant critical sets for \( \hat{\phi} \) correspond exactly to the critical leaf closures of \( f \). Now, the quotient space \( \hat{W} \) is a manifold, so we can apply the usual results of equivariant LS-category theory to estimate the number of critical \( O(q) \)-orbits for \( \hat{\phi} \). This is a great advantage, as the technical estimates required for the theory can be done in the context of a compact group action, instead of the case of a foliation with non-compact leaves. We recall the main result (see \([23, 24, 47, 56]\)):

**THEOREM 9.1.** Let \( G \) be a compact Lie group, and \( R : N \times G \to N \) a smooth right action on a closed manifold \( N \). If \( \hat{\phi} : N \to \mathbb{R} \) is a \( C^1 \)-function which is \( G \)-invariant, then \( \text{cat}_G(N) \) is a lower bound on the number of critical orbits for \( \hat{\phi} \).

Theorem 1.6 and the above discussion then yields the claim of Theorem 1.1 as a consequence:

**COROLLARY 9.2.** If \( f : M \to \mathbb{R} \) be a \( C^1 \)-map which is constant along the leaves of \( \mathcal{F} \), then \( \text{cat}_\mathcal{F}(M, \mathcal{F}) \) is a lower bound for the number of critical leaves of \( \mathcal{E} \).
In this section, we present a collection of examples to illustrate the ideas of the paper. There are three general methods for constructing a Riemannian foliation on a compact manifold: isometric Lie group actions; the group suspension construction applied to an isometric action of a finitely generated group; and the various blow-up constructions for singular Riemannian foliations [1, 2, 52, 53]. Note that for open manifolds, there is an important fourth method, which realizes a Riemannian pseudogroup as a Riemannian foliation of an open manifold [27, 34]. This will not be discussed, as little is known of the transverse LS-category for open manifolds.

**EXAMPLE 10.1.** Compact Hausdorff foliations and finite group actions

Epstein [21] and Millett [49] proved that a foliation $\mathcal{F}$ with all leaves compact and whose leaf space $M/\mathcal{F}$ is Hausdorff with the quotient topology, admits a projectable Riemannian metric, hence is Riemannian. For each $x \in M$, the holonomy $H_x$ of the leaf $L_x$ is always a finite group, hence all leaves of the lifted foliation $\hat{\mathcal{F}}$ of the orthonormal frame bundle $\hat{M}$ are also compact. The lifted foliation $\hat{\mathcal{F}}$ is thus defined by the fibration $\hat{\Upsilon} : \hat{M} \to \hat{W}$.

The quotient space $W = M/\mathcal{F}$ is a Satake manifold [60], or generalized orbifold, as every point is modeled either on $\mathbb{R}^q$, or by a quotient of $\mathbb{R}^q$ by a finite isometry group given by the holonomy of the leaf fiber. The quotient map $\hat{\pi} : \hat{W} \to W$ is an “$O(q)$-desingularization” of $W$, where the regular orbits of $O(q)$ correspond to the leaves of $\mathcal{F}$ without holonomy. A leaf $L_x$ with holonomy for $\mathcal{F}$ corresponds to an orbit $\hat{w} \cdot O(q)$ on $\hat{W}$ with isotropy group $H_x$ that strictly contains the stabilizer group of the action. The foliation $\hat{\mathcal{F}}$ has no singular leaves.

The exceptional set $E_{\mathcal{F}}$ of a compact foliation $\mathcal{F}$ is the union of all leaves with holonomy, and thus corresponds to the union of all strata except for $Z_0$, hence $E_{\mathcal{F}} = Z_1 \cup \cdots \cup Z_K$. The set of leaves without holonomy, $G_{\mathcal{F}} = M - E_{\mathcal{F}}$, is called the good set and corresponds to the set of regular orbits for the $O(q)$-action on $\hat{W}$. The exceptional set $E_{\mathcal{F}}$ admits the Epstein filtration by the holonomy groups of its leaves, and Proposition 7.1 shows that the connected components of the strata in the Epstein filtration correspond to the isotropy stratification of $M$.

Theorem 5.3 of [16] proves that the Epstein filtration is invariant under $\mathcal{F}$-foliated homotopy. This result is a direct consequence of Proposition 6.14 of this paper. Moreover, Theorem 5.6 above implies the estimate of Theorem 6.1 in [16]. In fact, the genesis of this current work was to find a new approach, which would work for all Riemannian foliations, of the results for compact Hausdorff foliations in [16].

**Example 10.1.1:** The papers [13, 15, 16] contain constructions and calculations of the transverse LS category for compact Hausdorff foliations, and the reader is referred to those papers for details. The standard method of construction is to start with a fibration $\tilde{\pi} : \tilde{M} \to \tilde{W}$ of compact manifolds, and assume that $\tilde{\pi}$ is equivariant with respect to a finite group $\Gamma$ which acts freely on $\tilde{M}$. Then the foliation of $\tilde{M}$ by the fibers of $\tilde{\pi}$ descends to a compact Hausdorff foliation of $M$, whose leaf space $W = \tilde{W}/\Gamma$ is thus a good orbifold. Colman has also given examples of compact Hausdorff foliations whose leaf space $M/\mathcal{F}$ is a bad orbifold.
EXAMPLE 10.2. Isometric flows

Let \((M, g)\) be a compact, connected Riemannian manifold, and \(\phi : M \times \mathbb{R}^p \to M\) a non-singular isometric action of \(\mathbb{R}^p\). (The case \(p = 1\) corresponds to an isometric flow on \(M\).) The orbits of \(\phi\) define a Riemannian foliation \(\mathcal{F}_\phi\) for which the metric \(g\) is projectable. The geometric and topological properties of this class of Riemannian foliations had been studied by many authors [10, 22, 26, 31, 32]. The leaves of the foliation \(\mathcal{F}\) are given by a free \(\mathbb{R}^p\) isometric action on \(M\). The closure of the image of \(\mathbb{R}^p\) in \(\text{Isom}(M)\) is a torus \(\mathbb{T}^k\) for some \(k > p\), and the foliation \(\mathcal{F}\) is defined by a free isometric action of \(\mathbb{T}^k\) on \(\tilde{M}\). This class of examples reveals many of the properties of the transverse category theory for Riemannian foliations, and we give several examples to illustrate various phenomena.

Example 10.2.1: The canonical example of an “irrational flow on the torus” is formulated generally as follows. Let \(M = \mathbb{T}^n\) be the \(n\)-torus, considered as the quotient \(\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n\) by the integer lattice. Choose \(1 \leq p < n\) and a real matrix \(A \in \mathbb{R}^{p \times n}\) such that \(AA^T\) is invertible. Then \(A\) defines an injective map \(A : \mathbb{R}^p \to \mathbb{R}^n\), and thus yields an isometric affine action \(\phi_A : \mathbb{R}^p \times \mathbb{T}^n \to \mathbb{T}^n\). The orbits of \(\phi_A\) are the affine planar leaves of \(\mathcal{F}_A\). All leaves of \(\mathcal{F}_A\) are regular, as there is no holonomy.

If all entries of \(A\) are rational numbers, then the leaves of \(\mathcal{F}_A\) are compact tori; otherwise, the leaves have closures which are embedded tori \(\xi_A : \mathbb{T}^k \subset \mathbb{T}^n\) for some \(p < k \leq n\). Thus, \(\text{cat}_0(\mathbb{T}^n, \mathcal{F}_A) = \infty\) unless \(A\) is a rational matrix. On the other hand, the essential transverse category is equal to the category of the foliation \(\mathcal{E}_A\) obtained from the closures of the leaves. The leaf space \(\mathbb{T}^n / \mathcal{E}_A \cong \mathbb{T}^n / \xi_A(\mathbb{T}^k)\) is a torus of dimension \(n - k\), hence \(\text{cat}_0(\mathbb{T}^n, \mathcal{F}_A) = \text{cat}(\mathbb{T}^{n-k}) = n - k + 1\).

Example 10.2.2: The previous examples of isometric flows can be embedded into compact space forms. We begin with the simplest examples of this.

Let \(\vec{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{R}^{n+1}\) and let \(M\) be the unit \(2n + 1\) sphere

\[ M = S^{2n+1} = \{ x = [z_0, z_1, \ldots, z_n] \mid |z_0|^2 + \cdots + |z_n|^2 = 1 \} \]

Define an isometric \(\mathbb{R}\)-action on \(S^{2n+1}\) by

\[ \phi_t([z_0, \ldots, z_n]) = [e^{2\pi \alpha_0 t \sqrt{-1} z_0}, \ldots, e^{2\pi \alpha_n t \sqrt{-1} z_n}] \]

The orbits of \(\phi_t\) define the leaves of the foliation \(\mathcal{F}_{\vec{\alpha}}\). With the assumption that the numbers \(\{1, \alpha_0, \ldots, \alpha_n\}\) are linearly independent over \(\mathbb{Q}\), then the leaves of \(\mathcal{F}_{\vec{\alpha}}\) are defined by the action of the compact abelian group \(\mathbb{T}^{n+1} = S^1 \times \cdots \times S^1\) acting diagonally.

There are precisely \(n + 1\) locally minimal \(\mathcal{F}_{\vec{\alpha}}\)-isotopy classes, corresponding to the orbits of the points \(\vec{e}_i = [0, \ldots, 1, \ldots, 0]\), which are isolated circles. Thus, \(\text{cat}_0(S^{2n+1}, \mathcal{F}_{\vec{\alpha}}) \geq n + 1\). It is also easily seen that for each point \(\vec{e}_i\) there is a flow-equivariant retraction of the open set \(\mathcal{U}_i = \{ [z_0, \ldots, z_n] \in S^{2n+1} \mid z_i \neq 0 \} \) to the orbit of \(\vec{e}_i\). Hence \(\text{cat}_0(S^{2n+1}, \mathcal{F}_{\vec{\alpha}}) = n + 1\).

It is possible to construct a wide variety of variations on this example, based on the general setup where \(G\) is a connected, compact Lie group and \(K \subset G\) is a closed subgroup, and we set \(M = G / K\). Let \(n\) denote the \(\mathbb{R}\)-rank of \(G\), so there is a locally
free action $\Phi_t : \mathbb{R}^n \times G \to G$ whose orbit through the identity $e \in G$ is a maximal torus $T^n \subset G$.

Choose $1 \leq p \leq n$ and a real matrix $A^{p \times n}$ such that $AA^T$ is invertible. Then $A$ defines an injective map $A : \mathbb{R}^p \to \mathbb{R}^n$, and thus yields an isometric affine action $\phi_A : \mathbb{R}^p \times T^n \to T^n$.

For $\vec{v} \in \mathbb{R}^p$ and $x \in G/K$, define the action $\phi_A(x, \vec{v}) = \Phi_t(x, \phi_A(\vec{v}))$ so that we obtain a $p$-dimensional foliation $\mathcal{F}_A$ on the homogenous space $G/K$.

The closure of the orbit of $\phi_A$ through the identity is a compact $k$-torus $T^k_A \subset T^n$ for some $p < k \leq n$. The foliation $\mathcal{E}_A$ - defined by the closures of the leaves of $\mathcal{F}_A$ - is given by the orbits of the closed subgroup $T^k_A$ on $G/K$.

Note that in this generality, there is no assurance that $\mathcal{F}_A$ has any compact leaves, hence generically one has $\text{cat}_{G/K}(G/K, \mathcal{F}_A) = \infty$. On the other hand, $\text{cat}_{G/K}^G(G/K, \mathcal{F}_A)$ equals the equivariant category of $G/K$ for the left action of the compact Lie subgroup $T^k_A \subset G$. The calculation of $\text{cat}_{G/K}^G(G/K, \mathcal{F}_A)$ then follows by methods of the theory of compact Lie groups, and has applications to the residue theory for the secondary classes of $\mathcal{F}_A$ [36, 44, 45, 66, 67].

**EXAMPLE 10.3. Products**

There is a simple remark, that if $M_1, \mathcal{F}_1$) and $(M_2, \mathcal{F}_2)$ are Riemannian foliations of compact manifolds with leaf dimensions $p_1$ and $p_2$ respectively, then the product manifold $M = M_1 \times M_2$ has a Riemannian foliation $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$ whose leaves have dimension $p = p_1 + p_2$. As an example, suppose that $M_2 = T^{n_2}$ has a linear foliation as in Example [10.2] with all leaves dense. Then for any Riemannian foliation $(M_1, \mathcal{F}_1)$ the product foliation $\mathcal{F}$ has no compact leaves, hence $\text{cat}_{G/K}(M, \mathcal{F}) = \infty$. On the other hand, $\text{cat}_{G/K}^G(M, \mathcal{F}) = \text{cat}_{G/K}^G(M_1, \mathcal{F}_1)$.

**EXAMPLE 10.4. Suspension and compact Lie group actions**

Given an action of a finitely generated group on a compact $q$-dimensional manifold, $\alpha : \Gamma \times N \to N$, the suspension construction yields a foliation $\mathcal{F}_\alpha$ of codimension $q$ whose transverse holonomy group is globally defined by the action (for example, see [8, 9]). When the given action is isometric, then this yields a Riemannian foliation, and provides a large class of examples. We use this construction to realize the orbit structure of every compact Lie group action as the transverse geometry of some Riemannian foliation.

**Example 10.4.1:** Let $\{\gamma_1, \ldots, \gamma_d\}$ be a set of generators for $\Gamma$. A left isometric action of $\Gamma$ on $N$ is equivalent to a representation $\alpha : \Gamma \to \text{Isom}(N)$, where each $\alpha(\gamma_i)$ acts isometrically on the left on $N$.

Let $B$ be a compact connected Riemannian manifold with basepoint $b_0 \in B$ such that the fundamental group $\Lambda = \pi_1(B, b_0)$ admits a surjection $\beta : \Lambda \to \Gamma$. For example, one can let $B = \Sigma_d$ be a closed Riemann surface with genus $d$. There is a surjection $\Lambda \to \mathbb{F}^d = \mathbb{Z} \ast \cdots \ast \mathbb{Z}$ onto the non-abelian free group on $d$ generators. The choice of the generators for $\Gamma$ defines a surjection $\mathbb{F}^d \to \Gamma$, and the composition $\beta : \Lambda \to \Gamma$ is then a surjection.
The universal cover $\tilde{B} \to B$ has a right isometric action by $\Lambda$, acting via deck transformations.

Consider the product manifold $\tilde{B} \times N$ with the product foliation $\tilde{F}$ whose leaves are the “horizontal slices” $\tilde{B} \times \{x\}$ for $x \in N$. The Riemannian metrics on $\tilde{B}$ and $N$ define the product metric on $\tilde{B} \times N$.

Define an action of $\Lambda$ on $\tilde{B} \times N$ by specifying, for $\lambda \in \Lambda$, $(b, x) \cdot \lambda = (b \cdot \lambda, \alpha \circ \beta(\lambda^{-1}) \cdot x)$.

Both the product foliation and the product Riemannian metric on $\tilde{B} \times N$ are invariant under this action, so the product foliation descends to a Riemannian foliation $\mathcal{F}_\alpha$ of $M = \tilde{B} \times \Lambda N$. Note that there is an embedding $t_0 : N \to M$ given by $t_0(x) = [b_0, x]$ where $[b, x]$ represents the equivalence class of the pair $(b, x)$ in $M$. Let $N_0 \subset M$ denote the image of this map.

The group of isometries $\text{Isom}(N)$ is compact, so the closure $G = \alpha(\Gamma) \subset \text{Isom}(N)$ is a compact Lie subgroup. Let $\mathcal{E}_\alpha$ denote the singular Riemannian foliation of $M$ by the closures of the leaves of $\mathcal{F}_\alpha$. Then the leaves of $\mathcal{E}_\alpha \cap N_0$ are precisely given by the orbits of $G$.

**Example 10.4.2:**

Let $G$ be a compact connected Lie group, and $\varphi : G \times N \to N$ a smooth isometric action on a compact Riemannian manifold $N$ of dimension $q$. The orbits of $\varphi$ define a singular Riemannian foliation $\mathcal{E}_\varphi$ of $N$ [31, 32, 52, 53]. Ken Richardson [58] showed that there always exists a Riemannian foliation $\mathcal{F}_\varphi$ of a compact manifold $M$ such that the singular Riemannian foliation defined by the closures of the leaves of $\mathcal{F}_\varphi$ is transversally equivalent to $\mathcal{E}_\varphi$. We recall this argument.

Let $\Gamma \subset G$ be a finitely generated dense subgroup; such always exists by a clever argument of Richardson [58]. The restriction of $\varphi$ to the subgroup defines a representation $\alpha : \Gamma \to \text{Isom}(N)$.

Use the suspension construction as in Example 10.4.2, to obtain a Riemannian foliation $(M, \mathcal{F}_\alpha)$ such that the image $\alpha(\Gamma) \subset \text{Isom}(N)$ has closure precisely the compact Lie subgroup $G$. The orbit type stratification of the $G$-action on $N$ equals the stratification of the transversal $N_0$ induced by the closures of the leaves of $\mathcal{F}_\varphi$.

There is a particular case of this construction which yields some very interesting examples. Let $G = \text{SU}(n)$ be the group of $n \times n$ special unitary matrices. Let $N = \text{SU}(n)$ be the group itself, and let the action $\varphi$ be the adjoint, so that $\varphi(A) : \text{SU}(n) \to \text{SU}(n)$ is given by $\varphi(A)(B) = A^{-1}BA$. Let $\mathcal{F}_\alpha$ denote the resulting suspension foliation. Then $\text{cat}_\varphi(M, \mathcal{F}_\alpha) = n$, based on the calculations of [36]. Note that the codimension of $\mathcal{F}_\alpha$ is the dimension of $\text{SU}(n)$ so that $q = n^2 - 1$.

The suspension of the adjoint action of $\text{SU}(n)$ on itself yields a foliation with no exceptional orbits. This is not the case with the groups $\text{SO}(n)$. In fact, for this case, there are isolated exceptional orbits, so by Colman’s results, the transverse category of the suspended foliation will be infinite. However, the essential transverse category will be finite, and its calculation is a very important problem, as it yields estimates for the category of the groups $\text{SO}(n)$ themselves [36].
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