An Improved Deterministic Rescaling for Linear Programming Algorithms

Rebecca Hoberg* and Thomas Rothvoss†
University of Washington, Seattle
December 15, 2016

Abstract

The perceptron algorithm for linear programming, arising from machine learning, has been around since the 1950s. While not a polynomial-time algorithm, it is useful in practice due to its simplicity and robustness. In 2004, Dunagan and Vempala showed that a randomized rescaling turns the perceptron method into a polynomial time algorithm, and later Peña and Soheili gave a deterministic rescaling. In this paper, we give a deterministic rescaling for the perceptron algorithm that improves upon the previous rescaling methods by making it possible to rescale much earlier. This results in a faster running time for the rescaled perceptron algorithm. We will also demonstrate that the same rescaling methods yield a polynomial time algorithm based on the multiplicative weights update method. This draws a connection to an area that has received a lot of recent attention in theoretical computer science.

1 Introduction

One of the central algorithmic problems in theoretical computer science as well as in more practical areas like operations research is finding the solution to a linear program

\[ \max \{c^T x \mid Ax \geq b\} \]  

where \( A \in \mathbb{R}^{m \times n}, c \in \mathbb{R}^n \) and \( b \in \mathbb{R}^m \). On the theoretical side, linear programming relaxations are the backbone for many approximation algorithms [WS11, Vaz01]. On the practical side, many real-world problems can either be modeled as linear programs or they can be modeled at least as integer linear programs; the latter ones are then solved using Branch & Bound or Branch & Cut methods. Both of these methods rely on repeatedly computing solutions to linear programs [CCZ14].

The first algorithm for solving linear programs was the simplex method due to Dantzig [Dan51]. While the method performs well in practice — and is still the method of choice today — for almost any popular pivoting rule one can construct instances where the algorithm takes exponential time [KM72]. In 1979, Khachiyan [Hac79, Sch86] developed the first polynomial-time algorithm. However, despite

---

*Email: rahoberg@uw.edu
†Email: rothvoss@uw.edu. Supported by an Alfred P. Sloan Research Fellowship. Both authors supported by NSF grant 1420180 with title “Limitations of convex relaxations in combinatorial optimization”. File compiled on December 15, 2016, 01:23.
the desirable theoretical properties, Khachiyan’s ellipsoid method turned out to be too slow for practical applications.

In the 1980s, interior point methods were developed which were efficient in theory and in practice. Karmarkar’s algorithm has a running time of \(O(n^{3.5}L)\), where \(L\) is the number of bits in the input \(\text{[Kar84]}\). Since then, there have been many further improvements in interior point methods. As recently as 2015, it was shown that there is an interior-point method using only \(\tilde{O}(\sqrt{\text{rank}(A)} \cdot L)\) many iterations; this upper bound essentially matches known lower bound barriers \(\text{[LS15]}\).

A common way to find a polynomial-time linear programming algorithm is with a greedy type procedure along with periodic rescaling \(\text{[DVZ16a]}\). One famous example of this is the perceptron algorithm \(\text{[Agr54]}\), which we will focus on in this paper. Instead of solving \(\text{(1)}\) directly, this method finds a feasible point in the open polyhedral cone

\[
P = \{ x \in \mathbb{R}^n \mid Ax > 0 \}
\]

where \(A \in \mathbb{R}^{m \times n}\) – using standard reductions one can interchange the representations \(\text{(1)}\) and \(\text{(2)}\) with at most a linear overhead. The classical perceptron algorithm starts at the origin and iteratively walks in the direction of any violated constraint. In the worst case this method is not polynomial time, but it is still useful due to its simplicity and robustness \(\text{[Agr54]}\). In 2004, Dunagan and Vempala \(\text{[DV06]}\) showed that using a randomized rescaling procedure, the algorithm can be modified to find a point in \(\text{(2)}\) in polynomial time. Explicitly, their algorithm runs in time \(\tilde{O}(mn^4 \log \frac{1}{\rho})\), where \(\rho > 0\) is the radius of the largest ball in the intersection of \(P\) with the unit ball \(B := B(0, 1)\). A deterministic rescaling procedure was provided by Peña and Sohieii in \(\text{[PS16]}\). Their algorithm uses an improved convergence of the perceptron algorithm based on Nesterov’s smoothing technique \(\text{[Nes05, PS12]}\). Overall, their algorithm takes time \(\tilde{O}(m^2n^{2.5} \log \frac{1}{\rho})\).

Another classical LP algorithm that we will discuss in this paper is based on a very general algorithmic framework called the multiplicative weights update (MWU) method. In its general form one imagines having \(m\) experts who each incur some cost in a sequence of iterations. In each iteration we have to select a convex combination of experts so that the expected cost is minimized, where we only have information on the past costs. The MWU method initially gives all experts the same weight and in each iteration the weight of expert \(i\) is multiplied by \(\exp(-\varepsilon \cdot \text{cost incurred by expert } i)\) where \(\varepsilon\) is some parameter. Then on average, the convex combination given by the weights will be nearly as good as the cost incurred by the best expert. MWU is an online algorithm that does not need to know the costs in advance, and it has numerous applications in machine learning, economics and theoretical computer science. In fact, MWU has been reinvented many times under different names in the literature. Recent applications in theoretical computer science include finding fast approximations to maximum flows \(\text{[CKM+11]}\), multicommodity flows \(\text{[GK07, Mad10]}\), solving LPs \(\text{[PST95]}\), and solving semidefinite programs \(\text{[AHK05]}\). We refer to the survey of Arora, Hazan and Kale \(\text{[AHK12]}\) for a detailed overview.

When we apply the MWU framework to linear programming, the experts correspond to the linear constraints. Suppose we use this method to find a valid point in \(P = \{ x : Ax > 0 \}\) where \(\|A_i\|_2 = 1\) for every row \(A_i\). At iteration \(t\), the cost associated with expert \(i\) will be \((A_i, p(t))\) for some vector \(p(t)\). Therefore the weight of expert \(i\) at time \(T\) will be \(e^{-\langle A_i, x \rangle}\) where \(x = \sum_{t=1}^{T} e^{\langle t \rangle} p(t)\). The analysis of MWU consists of bounding the sum of the weights, which in this case is given by the potential function \(\Phi(x) = \sum_{i=1}^{m} e^{-\langle A_i, x \rangle}\). If we choose the update vector \(p(t)\) to be a weighted sum of constraints at every iteration, notice that the resulting walk in \(\mathbb{R}^n\) corresponds to gradient descent on \(\Phi\) – in this case MWU terminates in \(\tilde{O}(\frac{1}{\rho^2})\)

\(^1\)The \(\tilde{O}\)-notation suppresses any \(\text{polylog}(m, n)\) terms.

\(^2\)Notice that normalizing the rows does not affect the feasible region.
iterations. However, \( \rho \) need not be polynomial in the input size, and in fact this method is not polynomial time in the worst case.

### 1.1 Our contribution

For reference, the general form for the rescaled LP algorithms we will present in this paper is given in Algorithm 1.

**Algorithm 1**

FOR \( \tilde{O}(n \log \frac{1}{\rho}) \) phases DO:

1. **Initial phase:** Either find \( x \in P \) or provide a \( \lambda \in \mathbb{R}_\geq 0^m \), \( \|\lambda\|_1 = 1 \) with \( \|\lambda A\|_2 \leq \Delta \).

2. **Rescaling phase:** Find an invertible linear transformation \( F \) so that \( \text{vol}(FU \cap B) \) is a constant fraction larger than \( \text{vol}(P \cap B) \). Replace \( P \) by \( F(P) \).

Our technical and conceptual contributions are as follows:

1. **Improved rescaling:** We design a rescaling method that applies for a parameter of \( \Delta = \Theta(\frac{1}{n}) \), which improves over the threshold \( \Delta = \Theta(\frac{1}{m\sqrt{n}}) \) required by [PS16]. This results in a smaller number of iterations that are needed per phase until one can rescale the system.

2. **Rescaling the MWU method:** We show that in \( \tilde{O}(1/\Delta^2) \) iterations the MWU method can be made to implement the initial phase of Algorithm 1. The idea is that if gradient descent is making insufficient progress then the gradient must have small norm, and from this we can extract an appropriate \( \lambda \). In particular, combining this with our rescaling method, we obtain a polynomial time LP algorithm based on MWU.

3. **Faster gradient descent:** The standard gradient descent approach terminates in at most \( \tilde{O}(1/\Delta^2) \) iterations, which matches the first approach in [PS16]. The more recent work of Peña and Soheili [PS12] uses Nesterov’s smoothing technique to bring the number of iterations down to a linear term of \( \tilde{O}(1/\Delta) \). We prove that essentially the same speedup can be obtained without modifying the objective function by projecting the gradient on a significant eigenspace of the Hessian.

4. **Computing an approximate John ellipsoid:** For a general convex body \( K \), computing a John ellipsoid is equivalent to finding a linear transformation so that \( F(K) \) is well rounded. For our unbounded region \( P \), our improved rescaling algorithm gives a linear transformation \( F \) so that \( F(P) \cap B \) is well-rounded.

### 2 Rescaling of the Perceptron Algorithm

In this section we fix an initial phase for Algorithm 1—in particular, the paper of Peña and Soheili gives a smooth variant of the perceptron algorithm that achieves the following guarantee:

**Lemma 1** ([PS16]). In time \( \tilde{O}(\frac{mw}{\Delta}) \), either the smooth perceptron phase outputs \( x \in P \) or it gives \( \lambda \in \mathbb{R}_\geq 0^m \) with \( \|\lambda\|_1 = 1 \) and \( \|\lambda A\|_2 \leq \Delta \).
We then focus on the rescaling phase of the algorithm. Our main result is that we are able to rescale with $\Delta = O\left(\frac{1}{n}\right)$.

**Lemma 2.** Suppose $\lambda \in \mathbb{R}^m_{\geq 0}$ with $\|\lambda\|_1 = 1$ and $\|\lambda A\|_2 \leq O\left(\frac{1}{n}\right)$. Then in time $O(mn^2)$ we can rescale $P$ so that $\text{vol}(P \cap B)$ increases by a constant factor.

We introduce two new rescaling methods that achieve the guarantee of Lemma 2. First we show that we can extract a thin direction by sampling rows of $A$ using a random hyperplane. The linear transformation that scales $P$ in that direction, corresponding to a rank-1 update, will increase $\text{vol}(P \cap B)$ by a constant factor.

Next we give an alternate rescaling which is no longer a rank-1 update but which has the potential to increase $\text{vol}(P \cap B)$ by up to an exponential factor under certain conditions. In addition, if we take an alternate view where the cone $P$ is left invariant and instead update the underlying norm, we see that this rescaling consists of adding a scalar multiple of a particular Hessian matrix to the matrix defining the norm. We also believe that this view is the right one to make potential use of the sparsity of the underlying matrix $A$, which would be a necessity for any practically relevant LP optimization method.

Combining Lemmas 1 and 2 gives us the following theorem:

**Theorem 3.** There is an algorithm based on the perceptron algorithm that finds a point in $P$ in time $\tilde{O}(mn^3 \log(\frac{1}{\rho}))$.

### 2.1 Rescaling Using a Thin Direction

In this section we will show how we can rescale by finding a direction in which the cone is thin – see Figure 1 for a visualization. First we give the formal definition of width.

**Definition 1.** Define the width of the cone $P$ in the direction $c \in \mathbb{R}^n \setminus \{0\}$ as

$$\text{WIDTH}(P, c) = \frac{1}{\|c\|_2} \max_{x \in P \cap B} |\langle c, x \rangle|.$$

As described in [PS16], we will now show that stretching $P$ in a thin enough direction increases the volume of $P \cap B$ by a constant factor. We reproduce the argument of [PS16] here for the sake of completeness:
Lemma 4 ([PS16]). Suppose that there is a direction $c \in \mathbb{R}^n \setminus \{0\}$ with $\text{WIDTH}(P, c) \leq \frac{1}{3\sqrt{n}}$. Define $F : \mathbb{R}^n \to \mathbb{R}^n$ as the linear map with $F(c) = 2c$ and $F(x) = x$ for all $x \perp c$. Then

$$\text{vol}(F(P) \cap B) \geq \frac{3}{2} \cdot \text{vol}(P \cap B).$$

Proof. We may assume that $\|c\|_2 = 1$. Since $\det(F) = 2$, we know that $\text{vol}(F(P \cap B)) = 2\text{vol}(P \cap B)$. Now suppose that $x \in P \cap B$ and write it as $x = x' + \langle c, x \rangle \cdot c$ where $x' \perp c$. Then $\|F(x)\|_2^2 = \|x' + 2 \langle c, x \rangle \cdot c\|_2^2 = \|x'\|_2^2 + 4 \langle c, x \rangle^2 \leq \|x\|_2^2 + 3 \cdot \text{WIDTH}(P, c)^2 \leq 1 + \frac{1}{6n}$ and taking square roots gives $\|F(x)\|_2 \leq 1 + \frac{1}{6n} \leq e^{1/6n}$. In particular, we know that $F(P \cap B) \subseteq e^{1/6n} \cdot F(P) \cap B$, and so we have

$$\text{vol}(F(P) \cap B) \geq (e^{1/6n})^{-n} \cdot \text{vol}(F(P \cap B)) \geq \frac{3}{4} \text{vol}(F(P \cap B)) = \frac{3}{2} \text{vol}(P \cap B).$$

Explicitly, assuming $\|c\|_2 = 1$, Lemma 4 updates our constraint matrix to $A(I - \frac{1}{2}cc^T)$. In particular, we apply a rank-1 update to the constraint matrix. Given a solution $x$ to these new constraints, a solution to the original problem can be easily recovered as $(I - \frac{1}{2}cc^T)x$.

It remains to argue how one can extract a thin direction for $P$, given a convex combination $\lambda$ so that $\|\lambda A\|_2$ is small. Here we will significantly improve over the bounds of [PS16] which require $\|\lambda A\|_2 \leq O \left( \frac{1}{\sqrt{n}} \right)$. We begin by a new generic argument to obtain a thin direction:

Lemma 5. For any non-empty subset $J \subseteq [m]$ of constraints one has

$$\text{WIDTH} \left( P \sum_{i \in J} \lambda_i A_i \right) \leq \frac{\| \sum_{i=1}^m \lambda_i A_i \|_2}{\| \sum_{i \in J} \lambda_i A_i \|_2}.$$

Proof. First, note that by the full-dimensionality of $P$, we always have $\| \sum_{i \in J} \lambda_i A_i \|_2 > 0$. By definition of width, we can write

$$\text{WIDTH} \left( P \sum_{i \in J} \lambda_i A_i \right) = \frac{1}{\| \sum_{i \in J} \lambda_i A_i \|_2} \max_{x \in P \cap B} \langle \sum_{i \in J} \lambda_i A_i, x \rangle.$$

Now, we know that $\langle A_i, x \rangle \geq 0$ for all $x \in P$ and so

$$\max_{x \in P \cap B} \langle \sum_{i \in J} \lambda_i A_i, x \rangle \leq \max_{i = 1}^{m} \langle \sum_{i \in J} \lambda_i A_i, x \rangle = \| \lambda A \|_2$$

and the claim is proven.

So in order to find a direction of small width, it suffices to find a subset $J \subseteq [m]$ with $\| \sum_{i \in J} \lambda_i A_i \|_2$ large. Implicitly, the choice that Peña and Soheili [PS16] make is to select $J = \{i_0\}$ for $i_0 \in [m]$ maximizing $\lambda_i$. This approach gives a bound of $\| \sum_{i \in J} \lambda_i A_i \|_2 \geq \frac{1}{m}$. We will now prove the asymptotically optimal bound\footnote{It suffices here to consider the trivial example with $\lambda_1 = \ldots = \lambda_n = \frac{1}{n}$ and $A_i = e_i$ being the standard basis. Then $\| \sum_{i \in J} \lambda_i A_i \|_2 \leq \frac{1}{\sqrt{n}}$ for any subset $J$. The optimality of our rescaling can also be seen since the cone in the last iteration is $\tilde{O}(n)$-well rounded, which is optimal up to $\tilde{O}$-terms.} using a random hyperplane:

Lemma 6. Let $\lambda \in \mathbb{R}_{\geq 0}^m$ be any convex combination and $A \in \mathbb{R}^{m \times n}$ with $\|A_i\|_2 = 1$ for all $i$. Take a random Gaussian $g$ and set $J := \{i \in [m] \mid \langle A_i, g \rangle \geq 0\}$. Then with constant probability $\| \sum_{i \in J} \lambda_i A_i \|_2 \geq \frac{1}{4\sqrt{\pi n}}$.
Proof. We set \( v := \frac{g}{\| g \|_2} \). Since \( v \) is unit vector we can lower bound the length of \( \| \sum_{i \in J} \lambda_i A_i \|_2 \) by measuring the projection on \( v \) and obtain \( \| \sum_{i \in J} \lambda_i A_i \|_2 \geq \| \sum_{j \not\in J} \lambda_i \langle A_i, v \rangle \). By symmetry of the Gaussian it then suffices to argue that \( \sum_{i=1}^{m} \lambda_i |\langle A_i, v \rangle| \geq \frac{1}{2\sqrt{\pi n}} \). First we will show that for an appropriate constant \( \alpha \in (0, 1) \),

(1) \( \Pr(\| g \|_2 \geq \sqrt{2n}) \leq \frac{2}{n} \)

(2) \( \Pr(\Sigma_{i=1}^{n} \lambda_i |\langle A_i, g \rangle| < \sqrt{\frac{2}{\pi}}) \leq \alpha. \)

Then, with probability at least \( \gamma = \frac{1-\alpha}{2} \), we have \( \Sigma_{i=1}^{n} \lambda_i |\langle A_i, v \rangle| \geq \frac{1}{2\sqrt{\pi n}} \).

For (1), notice that \( \| g \|_2 \) is just the chi-squared distribution with \( n \) degrees of freedom, and so it has variance \( 2n \) and mean \( n \). Therefore Chebyshev’s inequality tells us that \( \Pr[\| g \|_2^2 \geq 2n] \leq \frac{2}{n} \). Now, for all \( i, \langle A_i, g \rangle \) is a normal random variable with mean 0 and variance 1, and so the expectation of its absolute value is \( \sqrt{\frac{2}{\pi}} \). Summing these up gives \( \sum_{i=1}^{m} |\langle A_i, g \rangle| = \sqrt{\frac{2}{\pi}} \). Moreover, \( \sum_{i=1}^{m} |\langle A_i, g \rangle| \) is Lipschitz in \( g \) with Lipschitz constant 1, and so

\[
\Pr\left( \sum_{i=1}^{m} |\langle A_i, g \rangle| < \sqrt{\frac{2}{\pi}} - t \right) \leq e^{-t^2/\pi^2}.
\]

Letting \( t = \sqrt{\frac{1}{2\pi}} \) gives (2). By a union bound, the probability either of these events happens is at most \( \alpha + \frac{2}{n} \), and so with probability at least \( \frac{1-\alpha}{2} \) neither occurs, which gives us the claim. \( \square \)

While the proof is probabilistic, one can use the method of conditional expectation to derandomize the sampling \[\text{ASE92}\]. More concretely, consider the function \( F(g) := \sum_{i=1}^{m} \lambda_i |\langle A_i, g \rangle| - \frac{1}{\gamma \sqrt{n}} \| g \|_2 \). The proof of Lemma\[\text{6}\] implies that the expectation of this function is at least \( \Omega(1) \). Then we can find a desired vector \( g = (g_1, \ldots, g_n) \) by choosing the coordinates one after the other so that the conditional expectation does not decrease. We are now ready to prove Lemma\[\text{2}\] which we restate here with explicit constants.

**Lemma 7.** Suppose \( \lambda \in \mathbb{R}^{m}_{\geq 0} \) with \( \| \lambda \|_1 = 1 \) and \( \| \lambda A \|_2 \leq \frac{1}{12n\sqrt{\pi}} \). Then in time \( O(mn^2) \) we can rescale \( P \) so that \( \text{vol}(P \cap B) \) increases by a constant factor.

**Proof.** Computing a random Gaussian and checking if it satisfies the conditions of Lemma\[\text{6}\] takes time \( O(mn) \). Since the conditions will be satisfied with constant probability, the expected number of times we do must is constant. Once the conditions are satisfied, finding a thin direction and rescaling can be done in time \( O(n^3) \). Lemmas\[\text{4}\]and\[\text{5}\]guarantee we get a constant increase in the volume. \( \square \)

### 2.2 Deterministic Multi-rank Rescaling

We now introduce an alternate linear transformation we can use to rescale. This is no longer a rank-1 update, but it is inherently deterministic along with other nice properties. For one thing, although we only guarantee constant improvement in the volume, under certain circumstances the rescaling can improve the volume by an exponential factor. This transformation will also take a nice form when we change the view to consider rescaling the unit ball rather than the feasible region.

\[\text{Recall that a function } F: \mathbb{R}^n \to \mathbb{R} \text{ is Lipschitz with Lipschitz constant } 1 \text{ if } |F(x) - F(y)| \leq \|x - y\|_2 \text{ for all } x, y \in \mathbb{R}^n. \text{ A famous concentration inequality by Sudakov, Tsirelson, Borell states that } \Pr[|F(g) - \mu| \geq t] \leq e^{-t^2/\pi^2}, \text{ where } g \text{ is a random Gaussian and } \mu \text{ is the mean of } F \text{ under } g. \]
Lemma 8. Suppose $\lambda \in \mathbb{R}^{m}_{>0}$, $\|\lambda\|_1 = 1$ and $\|\lambda A\|_2 \leq \frac{1}{10n}$. Let $M$ denote the matrix $\sum_{i=1}^{m} \lambda_i A_i A_i^T$ and suppose $0 \leq \alpha \leq \frac{1}{\delta_{\max}}$, where $\delta_{\max} = \|M\|_{\text{op}}$ denotes the maximal eigenvalue of $M$. Define $F(x) = (I + \alpha M)^{1/2}x$. Then $\text{vol}(F(P \cap B)) \geq e^{\alpha/5} \cdot \text{vol}(P \cap B)$.

Proof. First notice that $0 + \text{ball}$ will not lose more volume than shrinking by a factor of $1 + \alpha \delta_i$, where $\delta_{\max} = \|M\|_{\text{op}}$. The point is that every element of $H$ also assume the rows are normalized so that $\|\cdot\|_H$ is the dual norm of $\|\cdot\|_H$. In this view we assume the rows $A_i$ of $A$ are normalized so that $\|A_i\|_{H^{-1}} = 1$.

Note that one always has $\delta_{\max} \leq 1$ and hence in any case one can choose $\alpha \geq 1$. Therefore if $\|\lambda A\|_2 \leq \frac{1}{10n}$, we get constant improvement in $\text{vol}(P \cap B)$. In fact, if the eigenvalues of $M$ happen to be small, we could get up to exponential improvement. This computation can be carried out in time $O(mn^2)$ and so Lemma 8 proves Lemma 2 and hence Theorem 3.

2.3 An Alternate View of Rescaling

Obviously instead of applying a linear transformation to the cone $P$ itself, there is an equivalent view where instead one applies a linear transformation to the unit ball. We will now switch the view in the sense that we fix the cone $P$, but we update the norm in each rescaling step so that the unit ball becomes more representative of $P$.

Recall that a symmetric positive definite matrix $H \in \mathbb{R}^{n \times n}$ induces a norm $\|x\|_H := \sqrt{x^T H x}$. Note that also $H^{-1}$ is a symmetric positive definite matrix and $\|\cdot\|_{H^{-1}}$ is the dual norm of $\|\cdot\|_H$. In this view we assume the rows $A_i$ of $A$ are normalized so that $\|A_i\|_{H^{-1}} = 1$.

5 An easy way to see this is to write $H = \sum_{j=1}^{n} \mu_j u_j u_j^T$ as the eigendecomposition of $H$. Then $H^{-1} = \sum_{j=1}^{n} \frac{1}{\mu_j} u_j u_j^T$ is the inverse; clearly all eigenvalues are positive and the inverse has the same spectrum as $H$. 

7
Let \( B_H := \{ x \in \mathbb{R}^n \mid \| x \|_H \leq 1 \} \) be the unit ball for the norm \( \| \cdot \|_H \). Note that \( B_H \) is always an ellipsoid. We will measure progress in terms of the fraction of the ellipsoid \( B_H \) that lies in the cone \( P \), namely
\[
\mu(H) := \frac{\text{vol}(B_H \cap P)}{\text{vol}(B_H)}. \]
The goal of the rescaling step will then be to increase \( \mu(H) \) by a constant factor. Note that we initially have \( \mu(H) = \mu(I) \geq \rho^n \), and at any time \( 0 \leq \mu(H) \leq 1 \), so we can rescale at most \( O(n \log \frac{1}{\rho}) \) times.

In this view, Lemma 9 takes the following form:

**Lemma 9.** Let \( H \in \mathbb{R}^{n \times n} \) be symmetric with \( H \succ 0 \). Suppose \( \lambda \in \mathbb{R}^m_{\geq 0} \) with \( \| \lambda \|_1 = 1 \) and \( \| \lambda A \|_{H^{-1}} \leq \frac{1}{10n} \), and let \( M := \sum_{i=1}^m \lambda_i A_i A_i^T \). Let \( 0 \leq \alpha \leq \frac{1}{\delta_{\max}} \), where \( \delta_{\max} := \| H^{-1} M \|_{op} \). Then for \( \tilde{H} := H + \alpha M \) one has
\[
\mu(\tilde{H}) \geq e^{\alpha/5} \cdot \mu(H). \]

Algorithm 2 illustrates what the multi-rank rescaling looks like under the alternate view. Notice that the algorithm updates the norm matrix by adding a scalar multiple of the Hessian matrix of the MWU potential function discussed in Section 3. Moreover, throughout the algorithm our matrix \( H \) will have the form \( I + \sum_{i=1}^m h_i A_i A_i^T \) for some \( h_i \geq 0 \). Note that this allows fairly compact representation as we only need \( O(m) \) space to encode the coefficients \( h_i \) that define the norm matrix.

**Algorithm 2**

FOR \( O(n \log \frac{1}{\rho}) \) phases DO:

1. **Initial phase:** Either find \( x \in P \) or give \( \lambda \geq 0, \| \lambda \|_1 = 1 \) with \( \| \lambda A \|_{H^{-1}} \leq O(\frac{1}{n}) \).

2. **Rescaling phase:** Update \( H := H + \alpha M \), where \( M = \sum_{i=1}^m \lambda_i A_i A_i^T \).

### 3 Rescaling for the MWU algorithm

In this section we show that the same rescaling methods can be used to make the MWU method into a polynomial time algorithm for linear programming.

Recall that the MWU algorithm corresponds to gradient descent on a particular potential function. First we show how we can apply rescaling to the standard gradient descent approach. We then introduce a modified gradient descent, which speeds up the MWU phase. Combining this with our rescaling step above gives us the following result:

**Theorem 10.** There is an algorithm based on the MWU algorithm that finds a point in \( P \) in time \( \tilde{O}(mn^{(\omega+1)} \log(\frac{1}{\rho})) \), where \( \omega \approx 2.373 \) is the exponent of matrix multiplication.

#### 3.1 Standard Gradient Descent

Consider the potential function \( \Phi(x) = \sum_{i=1}^m e^{-\langle A_i, x \rangle} \), where \( \| A_i \|_2 = 1 \) for all rows \( A_i \). Notice that \( \Phi(0) = m \) and that if \( \Phi(x) < 1 \) then \( \langle A_i, x \rangle > 0 \) for all \( i \), and hence \( x \in P \). In this section we analyze standard gradient descent on \( \Phi \), starting at the origin. Notice that the gradient takes the form
\[
\nabla \Phi(x) = -\sum_{i=1}^m e^{-\langle A_i, x \rangle} A_i. \]
If we let $λ_i = \frac{1}{y_i} e^{-\langle A_i, x \rangle}$, we see that $\|λ\|_1 = 1$ and $λA = -\frac{\nabla \Phi(x)}{\Phi(x)}$. In particular, if at any iteration this vector has small Euclidean norm, then we will be able to rescale. It remains to show, therefore, that if this vector has large Euclidean norm, then we get sufficient decrease in the potential function.

**Lemma 11.** Suppose $x ∈ \mathbb{R}^n$ and abbreviate $y = -\frac{\nabla \Phi(x)}{\Phi(x)}$. Then

$$\Phi(x + \frac{1}{2} y) ≤ \Phi(x) · e^{-\|y\|_2^2 / 4}.$$

**Proof.** First note that since $\|λ\|_1 = 1$ and $\|A_i\|_2 = 1$, we know that $|⟨A_i, y⟩| ≤ 1$ for all $i$. In our analysis we will also use the fact that for any $z ∈ \mathbb{R}$ with $|z| ≤ 1$ one has $e^z ≤ 1 + z + z^2$. We obtain the following.

$$\Phi(x + \frac{1}{2} y) = \sum_{i=1}^{m} e^{-⟨A_i, x + \frac{1}{2} y⟩} = \sum_{i=1}^{m} e^{-⟨A_i, x⟩} e^{-\frac{1}{2}⟨A_i, y⟩}$$

$$≤ \sum_{i=1}^{m} e^{-⟨A_i, x⟩} (1 - \frac{1}{2}⟨A_i, y⟩ + \frac{1}{4}⟨A_i, y⟩^2)$$

$$= \Phi(x) · \sum_{i=1}^{m} λ_i (1 - \frac{1}{2}⟨A_i, y⟩ + \frac{1}{4} y^T A_i A_i^T y)$$

$$≤ \Phi(x) · \frac{1}{4} \|y\|_2^2.$$

Thus as long as $\|y\|_2 ≥ Ω(\frac{1}{n})$, gradient descent will decrease the potential function by a factor of $e^{-Θ(1/n^2)}$ in each iteration, and so in at most $O(n^2 \ln(m))$ iterations we arrive at a point $x$ with $\Phi(x) < 1$.

### 3.2 Modified Gradient Descent

With $Δ = Θ(\frac{1}{n})$, the standard gradient descent approach implements the initial phase of Algorithm 1 in $O(n^2)$ iterations. It turns out we can get the same guarantee in $O(n)$ iterations by choosing a more sophisticated update direction. While we do not know how to guarantee an update direction that decreases $\Phi(x)$ by factor of more than $e^{-Θ(1/n^2)}$, we are able to find a direction so that the product of $\Phi(x)$ and $\|\nabla \Phi(x)\|_2$ decreases a lot faster. Note that in the following we will work with a general norm so that the results can be applied directly to either Algorithm 1 (with $H = 1$) or Algorithm 2. We assume now that $\|A_i\|_{H^{-1}} = 1$ for all $i$.

**Theorem 12.** Suppose $H > 0$ and $\|λA\|_{H^{-1}} ≥ \frac{β}{n}$, where $β > 0$ is an arbitrary constant. Then in time $O(mn^{ω^{-1}})$, we can find $ε > 0$ and $p ∈ \mathbb{R}^n$ so that

$$\|\nabla \Phi(x + εp)\|_{H^{-1}} · \Phi(x + εp) ≤ \|\nabla \Phi(x)\|_{H^{-1}} · \Phi(x) · e^{-Θ(1/n)}.$$

Before going through the proof, we note that the update step of Theorem 12 yields a MWU phase that runs in time $O(mn^{ω})$. In particular, this gives the running time guarantee of Theorem 10.

---

6 Consider a single phase of the algorithm without rescaling. There exists $x^* ∈ P$ with $\|x^*\|_2 = 1$ so that $B(x^*, ρ) ⊆ P$, where $B(x^*, ρ) := \{x ∈ \mathbb{R}^n \mid \|x - x^*\|_2 ≤ ρ\}$. Then $\|λA\|_2 · \|x^*\|_2 ≥ \langle λA, x^* \rangle = \sum_{i=1}^{m} λ_i (A_i, x^*) ≥ ρ$, since $⟨A_i, x^*⟩ ≥ ρ$ for all $i$. Therefore the algorithm is guaranteed to find a feasible point in $O(\frac{ln(m) + m^ω}{ρ^2})$ iterations without rescaling. This argument is closely related to the classical analysis of the perceptron.
Lemma 13. Suppose $H > 0$, and let $\beta$ be an arbitrary constant. Then in time $O(mn^\omega)$ we can run a MWU phase, which either finds $x \in P$ or gives $\lambda \in \mathbb{R}^n_{\geq 0}$ with $\|\lambda\|_1 = 1$ and $\|\lambda A\|_{H^{-1}} \leq \frac{\beta}{n}$.

Proof. Let $\lambda \geq 0$ be such that $\lambda A = -\frac{\nabla \Phi(x)}{\Phi(x)}$. Then as long as $\|\lambda A\|_{H^{-1}} \geq \frac{\beta}{n}$, Theorem 12 says that the quantity $\|\lambda A\|_{H^{-1}} \cdot \Phi(x)^2$ decreases by a factor of $e^{-\Theta(1/n)}$. Then in $O(n)$ iterations we will have $\|\lambda A\|_{H^{-1}} \cdot \Phi(x)^2 \leq \frac{\beta}{n}$, which implies that either $\Phi(x) < 1$ or $\|\lambda A\|_{H^{-1}} \leq \frac{\beta}{n}$.

The remainder of this section will be devoted to the proof of Theorem 12. We begin by establishing some useful notation. For any symmetric positive definite matrix $H > 0$ we define the inner product $\langle x, y \rangle_H := x^T H y$. Without any subscript $(x, y) = x^T y$ will continue to denote the canonical inner product.

Given $x \in \mathbb{R}^n$, define $\lambda_i = \frac{1}{\Phi(x)} e^{-\langle A_i, x \rangle}$, $y = -\frac{\nabla \Phi(x)}{\Phi(x)} = \sum_{i=1}^m \lambda_i A_i$, and $M = \frac{\nabla^2 \Phi(x)}{\Phi(x)} = \sum_{i=1}^m \lambda_i A_i^T$. Even though all three depend on $x$, we will not denote that here to keep the notation clean.

To prove Theorem 12 we first show how $\Phi(x)$ decreases as we take steps in an arbitrary direction $p$.

Lemma 14. For any $0 < \epsilon \leq 1$ and $p \in \mathbb{R}^n$ with $\|p\|_H \leq 1$, we have

$$\Phi(x + \epsilon p) \leq \Phi(x) \cdot (1 - \epsilon \langle y, p \rangle + \epsilon^2 p^T M p).$$

Proof. Notice that since $\|p\|_H \leq 1$ and $\|A_i\|_{H^{-1}} = 1$ we have $|\langle A_i, \epsilon p \rangle| \leq 1$ by the generalized Cauchy-Schwarz inequality. Writing out the definitions we obtain

$$\Phi(x + \epsilon p) = \sum_{i=1}^m e^{-\langle A_i, x + \epsilon p \rangle} = \sum_{i=1}^m e^{-\langle A_i, x \rangle} e^{-\epsilon \langle A_i, p \rangle} \leq \sum_{i=1}^m e^{-\langle A_i, x \rangle} (1 - \epsilon \langle A_i, p \rangle + \epsilon^2 \langle A_i, p \rangle^2) = \Phi(x) \cdot \sum_{i=1}^m \lambda_i (1 - \epsilon \langle A_i, p \rangle + \epsilon^2 p^T A_i A_i^T p) = \Phi(x) \cdot (1 - \epsilon \langle y, p \rangle + \epsilon^2 p^T M p).$$

In (*) we use the estimate that for any $z \in \mathbb{R}$ with $|z| \leq 1$ one has $e^z \leq 1 + z + z^2$. 

In a similar way, we bound $\|\nabla \Phi(x)\|_{H^{-1}}$ after an update step in an arbitrary direction $p$.

Lemma 15. Suppose $p \in \mathbb{R}^n$ with $\|p\|_H \leq 1$, and $0 < \epsilon \leq 1$ we have

$$\|\nabla \Phi(x + \epsilon p)\|_{H^{-1}} \leq \|\nabla \Phi(x)\|_{H^{-1}} \cdot \left(1 + \frac{\epsilon^2 \langle p, M p \rangle}{\|y\|_{H^{-1}}} + \frac{1}{\|y\|_{H^{-1}}^2} (\epsilon \langle y, M p \rangle_{H^{-1}} + \epsilon^2 \|M p\|_{H^{-1}}^2)\right)$$

Proof. For any $z$ with $|z| \leq 1$, we have $e^z = 1 + z + \eta z^2$ for some $\eta \in \mathbb{R}$ with $|\eta| \leq 1$. In particular, since
Recalling that $\|A_i\|_{H^{-1}} = 1$ and $\|p\|_H \leq 1$, we have $|\langle A_i, \epsilon p \rangle| \leq 1$ and so we have such an $\eta_i$ for each $i$.

$$
\frac{\|\nabla \Phi(x + \epsilon p)\|_{H^{-1}}}{\Phi(x)} = \frac{1}{\Phi(x)} \left\| \sum_{i=1}^{m} (-A_i) \cdot e^{-\langle A_i, x + \epsilon p \rangle} \right\|_{H^{-1}}
$$

$$
= \frac{1}{\Phi(x)} \left\| \sum_{i=1}^{m} (-A_i) e^{-\langle A_i, x \rangle} e^{-\epsilon \langle A_i, p \rangle} \right\|_{H^{-1}}
$$

$$
= \left\| \sum_{i=1}^{m} (-A_i) \cdot \lambda_i \cdot \left(1 - \epsilon \langle A_i, p \rangle + \epsilon^2 \cdot \eta_i \langle A_i, p \rangle^2 \right) \right\|_{H^{-1}}
$$

$$
\leq \left\| \sum_{i=1}^{m} (-A_i) \cdot \lambda_i \cdot (1 - \epsilon \langle A_i, p \rangle) \right\|_{H^{-1}} + \epsilon^2 \cdot \left\| \sum_{i=1}^{m} (-A_i) \cdot \lambda_i \eta_i \langle A_i, p \rangle^2 \right\|_{H^{-1}}
$$

$$
\leq \left\| y - \epsilon \cdot M p \right\|_{H^{-1}} + \epsilon^2 \cdot \sum_{i=1}^{m} \lambda_i \langle A_i, p \rangle^2
$$

$$
= \left( \|y\|_{H^{-1}} - 2\epsilon y^T H^{-1} M p + \epsilon^2 \|M p\|_{H^{-1}}^2 \right)^{1/2} + \epsilon^2 p^T M p
$$

$$
\leq \|y\|_{H^{-1}} \cdot \left( 1 + 2 \frac{1}{\|y\|_{H^{-1}}^2} \left( -\epsilon \langle y, M p \rangle_{H^{-1}} + \epsilon^2 \|M p\|_{H^{-1}}^2 \right) \right)^{1/2} + \epsilon^2 p^T M p
$$

$$
\leq \|y\|_{H^{-1}} \left( 1 + \frac{1}{\|y\|_{H^{-1}}^2} \left( -\epsilon \langle y, M p \rangle_{H^{-1}} + \epsilon^2 \|M p\|_{H^{-1}}^2 \right) \right) + \epsilon^2 p^T M p
$$

Recalling that $\nabla \Phi(x) = -\Phi(x)$ $y$ finishes the proof.

Using Lemmas 14 and 15 we can show a sufficient condition for $p$ to satisfy Theorem 12.

**Lemma 16.** Suppose $p \in \mathbb{R}^n$ with $\|p\|_H \leq 1$ and constant $a > 0$ is such that either

1. $\langle y, p \rangle \geq \frac{\|y\|_{H^{-1}}}{\log n}$ and $\langle y, M p \rangle_{H^{-1}} \geq \frac{\|M p\|_{H^{-1}} \cdot \|y\|_{H^{-1}}}{\log n}$ or

2. $\langle y, p \rangle \geq \frac{\|y\|_{H^{-1}}}{\log n}$ and $\|M p\|_{H^{-1}} \leq O\left( \frac{1}{\log n} \right)$.

Then as long as $\|y\|_{H^{-1}} \geq \frac{\beta}{n}$, choosing $\epsilon = \min \left\{ \frac{\|y\|_{H^{-1}}}{4 \cdot \log n \cdot \|M p\|_{H^{-1}}}, \frac{1}{2 \cdot \log n} \right\}$ gives

$$
\|\nabla \Phi(x + \epsilon p)\|_{H^{-1}} \cdot \Phi(x + \epsilon p) \leq \|\nabla \Phi(x)\|_{H^{-1}} \cdot \Phi(x) e^{-\Theta(1/n)}.
$$

**Proof.** Let $\epsilon = \min \left\{ \frac{\|y\|_{H^{-1}}}{4 \cdot \log n \cdot \|M p\|_{H^{-1}}}, \frac{1}{2 \cdot \log n} \right\}$. Then by Lemma 14 we have

$$
\Phi(x + \epsilon p) \leq \Phi(x) \cdot (1 - \epsilon \langle y, p \rangle + \epsilon^2 \|M p\|_{H^{-1}}) \leq \Phi(x) \cdot (1 - \epsilon \cdot \frac{\|y\|_{H^{-1}}}{\log n} + \epsilon^2 \|M p\|_{H^{-1}}).
$$

Assume first that we are in Case 1. By Lemma 15 since we know $\epsilon \leq \frac{1}{2 \cdot \log n}$, we have

$$
\|\nabla \Phi(x + \epsilon p)\|_{H^{-1}} \leq \|\nabla \Phi(x)\|_{H^{-1}} \cdot \left( 1 - \frac{\epsilon \cdot \|M p\|_{H^{-1}}}{2 \cdot \|y\|_{H^{-1}} \cdot \log n} + \epsilon^2 \cdot \frac{\|M p\|_{H^{-1}}^2}{\|y\|_{H^{-1}}^2} \right).
$$
If \( \epsilon = \frac{1}{2(\log n)^2} \), then \( \| Mp \|_{H^{-1}} \leq \frac{||y||_{H^{-1}}}{2(\log n)^2} \). Using this, \( \Phi(x) \) will decrease by \( e^{-\Theta(||y||_{H^{-1}})} \), and \( \| \nabla \Phi(x) \|_{H^{-1}} \) will decrease.

On the other hand, if \( \epsilon = \frac{1}{4(\log n)^2 \|Mp\|_{H^{-1}}} \), then \( \Phi(x) \) will decrease, and \( \| \nabla \Phi(x) \|_{H^{-1}} \) decreases by \( e^{-\Theta(1)} \). Together, these show that the product decreases by a factor of \( e^{-\Theta(||y||_{H^{-1}})} \).

If we are in Case 2, the only thing that might change is that when \( \epsilon = \frac{1}{2(\log n)^2} \) we might have \( \| \nabla \Phi(x) \|_{H^{-1}} \) increase by up to \( e^{O(1/poly(n))} \). Since \( \| y \|_{H^{-1}} \geq \frac{n}{M} \) this is the dominating term, and so we will still get the appropriate decrease.

Notice that the conditions of Lemma 16 essentially say that both \( p \) and \( Mp \) are close in angle with the vector \( y \). In particular, if the gradient happened to be an eigenvector of the Hessian (and hence \( y \) an eigenvector of \( M \)) then Lemma 16 would be satisfied with \( p = \frac{y}{\|y\|_{H^{-1}}} \). With this in mind, the idea for computing such a direction \( p \) is to project \( y \) onto an appropriate eigenspace of \( M \). To this end, we first prove the following general statement about computing approximate eigenvectors of matrices.

**Lemma 17.** Suppose \( z \in \mathbb{R}^n \) is a unit vector, \( N \) a PSD matrix with no eigenvalue bigger than \( 1 \) and \( K > 0 \) a given parameter. For \( k = 1, ..., K \), define \( z_k = (I - N)^{2^k} z \). Then \( \| z_k \|_2 \leq 1 \), and for an appropriate constant \( C > 0 \) at least one of the following must hold:

1. There exists \( k \leq K \) with
   \[
   \langle z, z_k \rangle \geq \frac{C}{K} \text{ and } \langle z, Nz_k \rangle \geq \frac{C\|Nz_k\|_2}{K^2}
   \]

2. For \( k = K \), we have
   \[
   \langle z, z_K \rangle \geq \frac{C}{K} \text{ and } \| Nz_K \|_2 \leq \frac{K}{2^k}
   \]

**Proof.** First note that we may assume in fact that no eigenvalue of \( N \) is bigger than \( \frac{1}{2} \) since we can always replace \( N \) with \( \frac{1}{2} N \) and only lose a factor of 2. Suppose now the eigenvectors of \( N \) are unit vectors \( v_1, ..., v_n \) with eigenvalues \( \alpha_1, ..., \alpha_n \).

We see that
\[
z_k = \sum_{j=1}^{n} (1 - \alpha_j)^{2^k} \langle z, v_j \rangle v_j.
\]
\[
Nz_k = \sum_{j=1}^{n} \alpha_j (1 - \alpha_j)^{2^k} \langle z, v_j \rangle v_j.
\]
\[
\langle z_k, z \rangle = \sum_{j=1}^{n} (1 - \alpha_j)^{2^k} \langle z, v_j \rangle^2
\]

For any \( k \), we can get the following bounds.

- For any \( \alpha_j \), we either have \( \alpha_j \leq \frac{k}{2^k} \) or \( e^{-\alpha_j 2^k} \leq e^{-k} \leq 2^{-k} \). In either case we have the bound \( \alpha_j (1 - \alpha_j)^{2^k} \leq \alpha_j e^{-\alpha_j 2^k} \leq \frac{k}{2^k} \). Therefore we can conclude that \( \| Nz_k \| \leq \frac{k}{2^k} \).

- Whenever \( \alpha_j \leq \frac{1}{2^k} \) we have \( (1 - \alpha_j)^{2^k} \geq e^{-2 \alpha_j 2^k} \geq \frac{1}{e^2} \) and all other coefficients will be nonnegative, and therefore
  \[
  \langle z, z_k \rangle \geq \frac{1}{e^2} \sum_{j, \alpha_j \leq 2^{-k}} \langle z, v_j \rangle^2
  \]
Now, notice that since \( \sum_{j=1}^{n} \langle z, v_j \rangle^2 = \| z \|^2 \), either there exists \( k \leq K \) so that
\[
\sum_{a_j \in \left[ \frac{1}{2^{k+1}}, \frac{1}{2^k} \right]} \langle z, v_j \rangle^2 \geq \frac{1}{2K},
\]
or else we must have
\[
\sum_{a_j \leq 2^{-k}} \langle z, v_j \rangle^2 \geq \frac{1}{2} \implies \langle z, z_K \rangle \geq \frac{1}{2e^2}
\]
In the latter case we are done, since we already showed \( \| N z_k \| \leq \frac{K}{2e} \).

Otherwise, choose this \( k \), and notice that whenever \( a_j \in \left[ \frac{1}{2^{k+1}}, \frac{1}{2^k} \right] \), we have \( \alpha_j(1 - \alpha_j)^{2^k} \geq \alpha_j e^{-2\alpha_j 2^k} \geq \frac{1}{e^{2^{k+1}K}} \), and all other coefficients will be nonnegative. Therefore \( \langle z, N z_k \rangle = \sum_{j=1}^{n} \alpha_j(1 - \alpha_j)^{2^k} \langle z, v_j \rangle^2 \geq \frac{1}{e^{2^{k+1}K}} \) and so in particular \( \langle z, N z_k \rangle \geq \frac{\| N z_k \|}{2e^2} \), as desired.

Finally, using Lemma 17 we show that we can efficiently compute a vector \( p \) satisfying Lemma 16 and hence complete the proof of Theorem 12.

**Lemma 18.** In time \( \tilde{O}(mn^{\omega-1}) \) we can find \( p \in \mathbb{R}^n \) satisfying the hypotheses of Lemma 16.

**Proof.** Set \( K = 10 \log n \), \( z = H^{-1/2} y \|y\|_{H^{-1}} \) and \( N = H^{-1/2} M H^{-1/2} \) in Lemma 17 to get output of \( z_k \). Let \( p = H^{-1/2} z_k \) and notice that
1. \( \| p \|_H = \| z_k \|_2 \),
2. \( \| M p \|_{H^{-1}} = \| N z_k \|_2 \),
3. \( \langle \frac{y}{\| y \|_{H^{-1}}}, p \rangle = \langle z, z_k \rangle \),
4. \( \langle \frac{y}{\| y \|_{H^{-1}}}, M p \rangle_{H^{-1}} = \langle z, N z_k \rangle \).

In particular, rearranging the statement of Lemma 17 this \( p \) satisfies the hypotheses of Lemma 16. Finally, note that computing \( M \) takes time \( O(n^{\omega-1}) \) and all other matrix operations can be computed in time \( O(n^{\omega}) \). Since we perform at most \( O(\log n) \) iterations, the running time is \( \tilde{O}(mn^{\omega-1}) \), as desired.

## 4 Computing an Approximate John Ellipsoid

It turns out that our algorithm implicitly computes an approximate John ellipsoid for the considered cone \( P \), which gives us geometric insight into \( P \). Recall that a classical theorem of John [Joh48] shows that for any closed, convex set \( Q \subseteq \mathbb{R}^n \), there is an ellipsoid \( E \) and a center \( z \) so that \( z + E \subseteq Q \subseteq z + nE \). The bound of \( n \) is tight in general — for example for a simplex — but it can be improved to \( \sqrt{n} \) for symmetric sets. This is equivalent to saying that for each convex body, there is a linear transformation that makes it well \( n \)-well rounded. Here, a body \( Q \) is \( \alpha \)-well rounded if \( z + r \cdot B \subseteq Q \subseteq z + \alpha \cdot r \cdot B \) for some center \( z \in \mathbb{R} \) and radius \( r > 0 \). See the excellent survey of Ball [Bal97] on this topic.

A summary of our full MWU algorithm with rescaling is given in Algorithm 3. We will prove here that after a minor modification of the algorithm, the set \( P \cap B \) will be well rounded when the algorithm terminates.
Algorithm 3

FOR $O(n \log(\frac{1}{\rho}))$ phases DO

- **MWU phase:**
  1. Normalize $\|A_i\|_2 = 1$ for all $i$, and set $x^{(0)} := 0$
  2. FOR $t := 0$ TO $T$ DO
     3. Set $\lambda^{(t)} := \frac{1}{\Phi(x^{(t)})} \exp(-\langle A_i, x^{(t)} \rangle)$
     4. If $\Phi(x^{(t)}) < 1$ THEN RETURN $x^{(t)} \in P$
     5. IF $\|\lambda^{(t)} A_i\|_2 \leq \frac{\rho}{n}$ THEN GOTO Rescaling phase
     6. Select an update vector $p^{(t)} \in \mathbb{R}^n$ with $\|p^{(t)}\|_2 \leq 1$
     7. Select a step size $0 < \epsilon_t \leq 1$
     8. Update $x^{(t+1)} := x^{(t)} + \epsilon_t p^{(t)}$

- **Rescaling phase:**
  1. Compute an invertible linear transformation $F$ so that $\text{vol}(F(P) \cap B)$ is a constant factor larger than $\text{vol}(P \cap B)$. Replace $P$ by $F(P)$.

**Lemma 19.** Consider Algorithm 3 with the modification that the MWU phase terminates in step (4) only if $\Phi(x) < \frac{1}{e}$. Then in the final iteration $P \cap B$ is $\tilde{O}(n)$-well rounded.

**Proof.** Let us consider the last phase of the algorithm and let $x^{(0)}, \ldots, x^{(T)}$ be the computed sequence of points with $\Phi(x^{(T)}) < \frac{1}{e}$ and $T \leq \tilde{O}(n)$. Then $e^{-\langle A_i, x^{(T)} \rangle} < \frac{1}{e}$ and hence $\langle A_i, x^{(T)} \rangle \geq 1$ for all $i$. The step size of the algorithm is always bounded by 1/2, hence $\|x^{(T)}\|_2 \leq \frac{1}{2}$. Now define $z := \frac{1}{T} \cdot x^{(T)}$ as center. Then $\|z\|_2 \leq \frac{1}{2}$ and $\langle A_i, z \rangle \geq \frac{1}{T}$. Hence $B(z, \frac{1}{T}) \subseteq P \cap B \subseteq B(z, 1)$, which shows that $P \cap B$ is $T$-well rounded. \[\square\]

Note that running the algorithm until $\Phi(x) < \frac{1}{e}$ only increases the worst case running times by a constant factor. Alternatively one can run the algorithm with standard gradient descent and a fixed step size of $\epsilon := \Theta(\frac{1}{n})$ and only terminate when $\Phi(x) < \frac{1}{m}$. This increases the running time by up to a factor of $n$, but the final set $P \cap B$ will be $O(n)$-well rounded, thus removing the logarithmic terms suppressed by the $\tilde{O}$ notation. On the other hand, no linear transformation can make the conic hull of a simplex $o(n)$-well rounded, hence our obtained bound is asymptotically optimal. Note that to obtain the tight factor for well-roundedness it was crucial to have the optimal rescaling threshold of $\Delta = \Theta(\frac{1}{n})$.

**Independent publication.** The multi-rank rescaling was also discovered in a parallel and independent work by Dadush, Végh and Zambelli [DVZ16b] (see their Algorithm 5).

**References**

[Agm54] S. Agmon. The relaxation method for linear inequalities. *Canadian Journal of Mathematics*, 6:382–392, 1954.

[AHK05] S. Arora, E. Hazan, and S. Kale. Fast algorithms for approximate semidefinite programming using the multiplicative weights update method. In *46th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2005)*, 23-25 October 2005, Pittsburgh, PA, USA, Proceedings, pages 339–348, 2005.
[AHK12] S. Arora, E. Hazan, and S. Kale. The multiplicative weights update method: a meta-algorithm and applications. *Theory Comput.*, 8:121–164, 2012.

[ASE92] Noga Alon, Joel H. Spencer, and Pál Commentateur de texte écrit. Erdős. *The Probabilistic method*. Wiley-Interscience series in discrete mathematics and optimization. J. Wiley & sons, New York, Chichester, Brisbane, 1992.

[Bal97] Keith Ball. An elementary introduction to modern convex geometry. In *Flavors of Geometry*, pages 1–58. Univ. Press, 1997.

[CCZ14] M. Conforti, G. Cornuejols, and G. Zambelli. *Integer Programming*. Springer Publishing Company, Incorporated, 2014.

[CKM+11] P. Christiano, J. A. Kelner, A. Madry, D. A. Spielman, and S. Teng. Electrical flows, laplacian systems, and faster approximation of maximum flow in undirected graphs. In *Proceedings of the Forty-third Annual ACM Symposium on Theory of Computing*, STOC ’11, pages 273–282, New York, NY, USA, 2011. ACM.

[Dan51] G.B. Dantzig. Maximization of a linear function of variables subject to linear inequalities. In *Activity Analysis of Production and Allocation*, Cowles Commission Monograph No. 13, pages 339–347. John Wiley & Sons, Inc., New York, N. Y.; Chapman & Hall, Ltd., London, 1951.

[DV06] J. Dunagan and S. Vempala. A simple polynomial-time rescaling algorithm for solving linear programs. *Math. Program.*, 114(1):101–114, 2006.

[DVZ16a] D. Dadush, L. A. Végh, and G. Zambelli. Rescaled coordinate descent methods for linear programming. In *Proceedings of the 18th International Conference on Integer Programming and Combinatorial Optimization - Volume 9682*, IPCO 2016, pages 26–37, New York, NY, USA, 2016. Springer-Verlag New York, Inc.

[DVZ16b] Daniel Dadush, László A. Végh, and Giacomo Zambelli. Rescaling algorithms for linear programming - part I: conic feasibility. *CoRR*, abs/1611.06427, 2016.

[GK07] N. Garg and J. Könemann. Faster and simpler algorithms for multicommodity flow and other fractional packing problems. *SIAM J. Comput.*, 37(2):630–652, 2007.

[Hać79] L.G. Hačijan. A polynomial algorithm in linear programming. *Dokl. Akad. Nauk SSSR*, 244(5):1093–1096, 1979.

[Joh48] F. John. Extremum problems with inequalities as subsidiary conditions. In *Studies and Essays presentsed to R. Courant on his 60th Birthday*, pages 187–204. Interscience Publishers, 1948.

[Kar84] N. Karmarkar. A new polynomial-time algorithm for linear programming. *Combinatorica*, 4(4):373–395, 1984.

[KM72] V. Klee and G. Minty. How good is the simplex algorithm? In *Inequalities, III (Proc. Third Sympos., Univ. California, Los Angeles, Calif., 1969; dedicated to the memory of Theodore S. Motzkin)*, pages 159–175. Academic Press, New York, 1972.

[LS15] Y. Lee and A. Sinford. A new polynomial-time algorithm for linear programming. 2015. https://arxiv.org/abs/1312.6677.
[Mad10] A. Madry. Faster approximation schemes for fractional multicommodity flow problems via
dynamic graph algorithms. In *Proceedings of the Forty-second ACM Symposium on Theory of
Computing*, STOC ’10, pages 121–130, New York, NY, USA, 2010. ACM.

[Nes05] Y. Nesterov. Excessive gap technique in nonsmooth convex minimization. *SIAM Journal on
Optimization*, 16(1):235–249, 2005.

[PS12] J. Peña and N. Soheili. A smooth perceptron algorithm. *SIAM J. Optim.*, 22(2):728–737, 2012.

[PS16] J. Peña and N. Soheili. A deterministic rescaled perceptron algorithm. *Math. Program.*, 155(1-
2):497–510, 2016.

[PST95] S.A. Plotkin, D.B. Shmoys, and E. Tardos. Fast approximation algorithms for fractional pack-
ing and covering problems. *Math. Oper. Res.*, 20(2):257–301, 1995.

[Sch86] A. Schrijver. *Theory of linear and integer programming*. Wiley-Interscience Series in Discrete
Mathematics. John Wiley & Sons, Ltd., Chichester, 1986. A Wiley-Interscience Publication.

[Vaz01] V. Vazirani. *Approximation algorithms*. Springer, 2001.

[WS11] D.P. Williamson and D.B. Shmoys. *The Design of Approximation Algorithms*. Cambridge Uni-
versity Press, 2011.