A COHERENT STATE ASSOCIATED WITH SHAPE-ININVARIANT POTENTIALS

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Abstract

An algebraic treatment of shape-invariant potentials is discussed. By introducing an operator which reparametrizes wave functions, the shape-invariance condition can be related to a generalized Heisenberg-Weyl algebra. It is shown that this makes it possible to define a coherent state associated with the shape-invariant potentials.

1. Introduction

Coherent states play an important role in physics today. The original coherent state is closely related to Heisenberg-Weyl group and this property has been extended to a number of Lie groups, and the term coherent states is now applied to many objects. Recently, a “coherent state” is proposed from a quite different point of view. It is based on the idea of the reparametrization invariance property of exactly solved potentials, called shape-invariance. This property is introduced by the use of supersymmetry in quantum mechanics. The key idea of constructing a coherent state is to introduce an operator denoting the reparametrization, which makes it possible to express the shape-invariance condition as a commutation relation. In this note, we construct a coherent state associated with shape-invariant potentials based on this commutation relation.

2. Shape-Invariance Revisited

Let us first consider the following operators

\[ D_n \equiv \frac{1}{\sqrt{2}} \left( W_n(x) + \frac{d}{dx} \right) \]  

(1)
with the property
\[ D_n D_n^\dagger = D_{n+1} D_{n+1}^\dagger + R_{n+1} \quad (n = 0, 1, 2, \cdots) \tag{2} \]
where \( W_n(x) \) is so-called superpotential and \( R_{n+1} \) is a constant independent of \( x \). They are, in a sense, a generalization of the boson creation and annihilation operators. In terms of the superpotentials this relation is given by
\[ W_n^2 + W_n' = W_{n+1}^2 - W_{n+1}' + 2R_{n+1} \quad (n = 0, 1, 2, \cdots) \tag{3} \]
where dash denotes the derivative with respect to \( x \). This infinite chain of differential equations reduce to only one equation if we adopt an anzats
\[ W_n(x) = W(x, a_n), \quad a_n \equiv f(f(\cdots f(a_0)\cdots)) \tag{n \text{ times}} \tag{4} \]
In fact, substituting this into Eq.3, we can easily confirm the following statement: If the relation
\[ W^2(x, a) + W'(x, a) = W^2(x, f(a)) - W'(x, f(a)) + 2R(f(a)) \tag{5} \]
holds identically with respect to \( a \), the set of Eq.3 do not depend on \( n \). Now let us denote \( D(a_n) \equiv D_n \). Then we have the relation
\[ D(a) D^\dagger(a) = D^\dagger(f(a)) D(f(a)) + R(f(a)) \tag{6} \]
as the shape-invariance condition. Note that this resembles that of the harmonic oscillator commutation relation. However, because of the difference in parameter dependence among operators, the above relation cannot be expressed naively as a form a commutation relation. Typical solutions of Eq.5 are as follows: 1) \( f(a) = a - 1 \), \( W(x, a) \) consists of finite power series of \( a \). In this case, we have six types of potentials classified by Infeld and Hull.\(^3\) 2) \( f(a) = qa \), \( W(x, a) = aW(ax) \). This corresponds to the self-similar potential discovered by Shabat and Spiridonov.\(^4\) For such shape-invariant potentials we can calculate the eigenvalues and corresponding eigenstates in an algebraic way. To see this, let us consider the following sequence of hamiltonians
\[ H_0 = D^\dagger(a_0) D(a_0) + R(a_0) \]
\[ H_1 = D(a_0) D^\dagger(a_0) + R(a_0) = D^\dagger(a_1) D(a_1) + R(a_0) + R(a_1) \]
\[ \vdots \]
\[ H_{n+1} = D(a_n) D^\dagger(a_n) + \sum_{k=0}^{n} R(a_k) = D^\dagger(a_{n+1}) D(a_{n+1}) + \sum_{k=0}^{n+1} R(a_k) \]
\[ \vdots \]
For these hamiltonians we can see the following two properties provided that there exist normalized 0-eigenvalue states satisfying \( D(a_n)|\psi_0(a_n)\rangle = 0 \) \((n = 0, 1, 2, ...):\)

1) \( D^\dagger(a_n)D(a_n) \) and \( D(a_n)D^\dagger(a_n) \) are superpartners each other, i.e., they have the same eigenvalues except for the 0-eigenvalue of the former. 2) The lowest eigenvalue of \( H_{n+1} \) is \( \sum_{k=0}^{n+1} R(a_k) \) since \( D^\dagger(a_{n+1})D(a_{n+1}) \) has a 0-eigenvalue. Combining these two properties, we can conclude that the \( n \)th eigenvalue of \( H_0 \) and corresponding eigenstate are given by

\[
E_n(a_0) = \sum_{k=0}^{n} R(a_k),
\]

\[
|\psi_n(a_0)\rangle \propto D^\dagger(a_0)D^\dagger(a_1) \cdots D^\dagger(a_{n-1})|\psi_0(a_n)\rangle.
\]

(7)

Therefore, once we know two important functions of \( a \), i.e., \( R(a) \) and \( f(a) \), we can calculate eigenvalues and eigenstates in an algebraic way.

3. A Coherent State Associated with Shape-Invariant Potentials

It should be noted again that the shape-invariance condition Eq.6 can be considered as a generalization of the oscillator commutation relation. To stress the analogy between them, we introduce formally the operator \( T \) defined by

\[
T|\phi(x, a)\rangle = |\phi(x, a_1)\rangle, \quad a_1 = f(a),
\]

(8)

which denotes the reparametrization of the parameter \( a \). Using this operator, we define the following operators

\[
A_+(a) \equiv D^\dagger(a)T, \quad A_-(a) \equiv T^{-1}D(a).
\]

(9)

Then we achieve the following expression of the shape-invariance condition in the form of a commutation relation

\[
[A_-(a), A_+(a)] = R(a).
\]

(10)

However, it is not closed in general because of the existence of \( T \) in \( A \) operators. From now on, we assume that \( |\psi_0(a_n)\rangle \) \( (n = 0, 1, ...) \) are all normalizable eigenstates, i.e., \( H_0 \) has infinite number of bound states. After some calculation we get the expression of the normalized eigenstate of Eq.7

\[
|\psi_n(a_0)\rangle = \frac{1}{\sqrt{[n]_0!}} (A_+(a_0))^n|\psi_0(a_0)\rangle,
\]

(11)

where

\[
[n]_k \equiv R(a_{k+1}) + R(a_{k+2}) + \cdots + R(a_{k+n}), \quad [\widetilde{n}]_k \equiv [n]_k T,
\]

\[
[n]_k! \equiv [\widetilde{n}]_k! [n-1]_k ! \cdots [1]_k ! T^{-n}.
\]

(12)
The appearance of $T$ in $[n]_k$ reflects the non-commutative character between $R(a)$ and $A_\pm(a)$. This expression can be considered as a generalization of the expression of the eigenstate of the usual harmonic oscillator, i.e., if we replace $[n]_0$ and $A_+$ by the natural number and boson creation operator, respectively, Eq.11 reduces to the well-known formula.

Now let us define a “coherent state” associated with the commutation relation Eq.10. Here coherent state means the eigenstate of the “annihilation” operator $A_-(a)$. For this purpose, we first define the generalized exponential function

$$
\exp_k(x) \equiv \sum_{n=0}^{\infty} \frac{1}{[n]_k!} x^n, \quad (13)
$$

using Eq.12. Next, we define the state

$$
|z, a_0\rangle \equiv \exp_0\{z A_+(a_0)\}|\psi_0(a_0)\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{[n]_0!}} z^n |\psi_n(a_0)\rangle, \quad (14)
$$

which is, of course, a generalized definition of the usual boson coherent state. From the direct calculation, we can easily confirm the relation $A_-(a_0)|z, a_0\rangle = z|z, a_0\rangle$.

4. Examples with $f(a) = a - 1$

Let us concretely calculate the newly defined coherent state Eq.14 for typical shape-invariant potentials. We follow the classification of Infeld and Hull in Ref.3 and confine our attention to systems with only bound states. Constants $a, b, c, d$ in this reference correspond to $\alpha, \beta, \gamma, \delta$, respectively.

(I) Types (C) and (D). These are the simplest cases among the shape-invariant potentials. $W$ and $R$ in Eq.5 are given by

$$
W(x, a) = \begin{cases} 
\frac{(a + \delta)}{x} + \frac{\beta x}{2}, & \text{for (C)} \\
\beta x + \delta, & \text{for (D)}
\end{cases}
$$

$$
R(a) = \beta, \quad (15)
$$

where $\beta$ and $\delta$ are some real constants. In these cases we can set $R(a) = 1$ without loss of generality. Then

$$
[n]_k = n, \quad [n]_k! = n!
$$

are independent of $k$, namely, they are the usual natural numbers and factorial, respectively. The coherent state becomes, therefore,

$$
|z, a_0\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |\psi_n(a_0)\rangle, \quad (16)
$$

$$
\mathcal{N}_z(a_0) = \exp(|z|^2),
$$
where $\mathcal{N}_z(a_0) = \langle z, a_0 | z, a_0 \rangle$ is a normalization of the coherent state, and from now on we denote the normalized coherent state as $| z, a_0 \rangle = \mathcal{N}_z^{-1/2} | z, a_0 \rangle$. For the measure

$$d\mu(z) = \frac{1}{\pi} d\text{Re} z d\text{Im} z,$$

we have the following completeness relation

$$\int d\mu(z) | z, a_0 \rangle \langle z, a_0 | = \sum_{m,n=0}^{\infty} \frac{1}{\sqrt{m!n!}} | \psi_m(a_0) \rangle \langle \psi_n(a_0) | \frac{1}{\pi} \int d\text{Re} z d\text{Im} z \exp(-|z|^2) z^m z^n$$

$$= \sum_{n=0}^{\infty} | \psi_n(a_k) \rangle \langle \psi_n(a_k) |.$$

Next we briefly mention the coordinate representation of the coherent state for these simplest cases. Since the potential of type (D) is that of the harmonic oscillator and it is almost trivial, we restrict ourselves to type (C). Since the coherent state is an eigenstate of $A_-$, the relation

$$\{ W(x, a_0) + d/dx \} \langle x | z, a_0 \rangle = \sqrt{2} z T \langle x | z, a_0 \rangle$$

holds. The solution of this equation is given by

$$\langle x | z, a_0 \rangle = x^{-(a_0+\delta)} e^{-(\beta/4)x^2} e^{zz^T/\sqrt{2}} C(z, a_0),$$

where $C(z, a_0)$ is independent of $x$. On the other hand, by noting $D(a_0) | \psi(a_0) \rangle = 0$, the ground state in $x$-representation is given by

$$\langle x | \psi_0(a_0) \rangle = \mathcal{N}^{-1/2}(a_0) x^{-(a_0+\delta)} e^{-(\beta/4)x^2},$$

$$\mathcal{N}(a_0) = \int_{-\infty}^{\infty} dx x^{-2(a_0+\delta)} e^{-(\beta/2)x^2}.$$  

By using the property $\langle \psi_0(a_0) | z, a_0 \rangle = 1$, we have

$$1 = \int_{-\infty}^{\infty} dx \mathcal{N}^{-1/2}(a_0) x^{-(a_0+\delta)} e^{-(\beta/4)x^2} \cdot x^{-(a_0+\delta)} e^{-(\beta/4)x^2} e^{zz^T/\sqrt{2}} C(z, a_0)$$

$$= \mathcal{N}^{-1/2}(a_0) \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{z}{\sqrt{2}} \right)^n C(z, a_n) \mathcal{N}(a_n)$$

$$= \mathcal{N}^{-1/2}(a_0) e^{(zT/\sqrt{2})} C(z, a_0) \mathcal{N}(a_0).$$

Solving this with respect to $C(z, a_0)$, we end up with

$$\langle x | z, a_0 \rangle = x^{-(a_0+\delta)} e^{-(\beta/4)x^2} e^{zz^T/\sqrt{2}} \mathcal{N}^{-1}(a_0) e^{zT/\sqrt{2}} \mathcal{N}^{1/2}(a_0).$$
(II) Types (A) and (B). \( W \) and \( R \) are given by

\[
W(x, a) = \begin{cases} 
\alpha(a + \gamma) \cot \alpha(x + p) + \delta / \sin \alpha(x + p), & \text{for (A)} \\
\iota \alpha(a + \gamma) + \delta \exp(-\iota \alpha x), & \text{for (B)} 
\end{cases}
\]

\[R(a) = -\alpha^2(a + \gamma + \frac{1}{2}), \quad (22)\]

where \( \alpha \) is in general a real or a pure imaginary constant for type (A), while it is pure imaginary for type (B), and \( \gamma, \delta \) and \( p \) are real constants. However, as we assume that the hamiltonian under consideration has only bound states, we confine our attention to the type (A) with real \( \alpha \). Note that in \( W \) and \( R \), \( \gamma \) always appears as a form \( a + \gamma \). Therefore, we can put \( \gamma = 0 \) without loss of generality. Furthermore, \( R \) can reduce to \( R(a) = -(a + 1/2) \) by replacing \( A_\pm \to A_\mp / \alpha \). Finally, we can fix the various parameters by choosing \( a_0 \) properly. Among them, for example, if we choose \( a_0 = -1/2 \), we have a simple expression for the coherent state Eq.14 as follows; first we have

\[
[n]_k = \frac{1}{2} n(2k + n + 1), \quad [n]_k! = \frac{1}{2^n} n!(2k + 2n)! / (2k + n)!
\]

It should be noted the factorial depends on \( k \), namely, on \( a \). For \( k = 0 \), we have \( [n]_0! = (2n)!/2^n \) and

\[
|z, a_0\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{(2n)!}} z^n \psi_n(a_0) \rangle,
\]

\[
\mathcal{N}_z(a_0) = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{(2n)!} = \cosh(|z|), \quad (23)
\]

where we replace \( \sqrt{2}z \to z \). For the measure

\[
d\mu_0(z) = \frac{1}{2\pi} \mathcal{N}_z(a_0) \exp(-|z|) |z| d Re z d Im z, \quad (24)
\]

we have a similar completeness relation in example (I).

For other examples, see Ref.2.

5. An Example with \( f(a) = qa \)

Before calculating the coherent state Eq.14, we must confirm that there really appears the q-oscillator algebra since the shape-invariance condition Eq.10 is here represented by the usual oscillator-like commutation relation. By substituting \( f(a) = qa \) and \( W(x, a) = aW(ax) \) into Eq.5, we have

\[
W^2(x) + \frac{dW(x)}{dx} = q^2 W^2(qx) - q \frac{dW(qx)}{dx} + \frac{2R(f(a))}{a^2}. \quad (25)
\]
As previously mentioned, this equation should hold identically with respect to $a$. The last term should be, therefore, a constant, denoted here by $\gamma(q)(>0)$:

$$ R(a) = \frac{\gamma(q)}{2q^2}a^2. \quad (26) $$

Hereafter, we set $\gamma(q) = 2$ for simplicity. The commutation relation denoting the shape-invariance Eq.10 is then given by $[A_-(a), A_+(a)] = a^2/q^2$. Though this seems oscillator-like algebra, it is not closed since $A$ operators and $a$ are not commutative. It is possible, however, to get a closed relation as follows: Let us introduce modified $A$-operators

$$ A_q^+(a) \equiv \frac{1}{a}A_+(a), \quad A_q^-(a) \equiv A_-(a)\frac{1}{a}. \quad (27) $$

Then, the above commutation relation is rewritten as

$$ A_q^-(a)A_q^+(a) - q^2A_q^+(a)A_q^-(a) = 1, \quad (28) $$

which is essentially equivalent to the one derived by Spiridonov.\(^4\)

Next, let us calculate $[n]_0$ and $[n]_0!$ in Eq.12. By definition, we have $a_k = q^k a_0$ and therefore $R(a_k) = q^{2(k-1)}a_0^2$. Then we have

$$ [n]_0 = [n]_q \cdot a_0^2, \quad [n]_0! = [n]_q! \cdot a_0^2 a_1^2 \cdots a_{n-1}^2, \quad (29) $$

where $[n]_q \equiv (1 - q^{2n})/(1 - q^2)$ is a $q$-deformed $n$ and $[n]_q! \equiv [n]_q[n-1]_q \cdots [1]_q$ is a $q$-deformed factorial.

Before we calculate the coherent state Eq.14, recall that we define this state based on the commutation relation Eq.10. What is important here is that this relation is invariant under the transformation

$$ A_g^+(a) = g(a)A_+(a), \quad A_g^-(a) = A_-(a)\frac{1}{g(a)}. \quad (30) $$

where $g(a)$ is an arbitrary function of $a$. Using this property, we can immediately define a coherent state which is the eigenstate of $A_g^-$ as follows;

$$ A_g^-|z, a_0\>_g = z|z, a_0\>_g \quad (31) $$

with

$$ |z, a_0\>_g = \exp_0\{z A_q^+(a_0)\}|\psi_0(a_0)\>. \quad (32) $$

Now let us choose $g(a) = a$. Then we have

$$ A_g^-(a) = A_q^-(a), \quad A_g^+(a) = a^2 A_q^+(a). \quad (33) $$

By the use of Eq.29, the exponent in Eq.32 is calculated as follows;

$$ \exp_0\{z A_q^+(a_0)\} = \sum_{n=0}^{\infty} \frac{1}{[n]_q! \cdot a_0^2 a_1^2 \cdots a_{n-1}^2} \{za_0^2 A_q^+(a_0)\}^n = \exp_q\{z A_q^+(a_0)\}. \quad (34) $$
where \( \exp_q(x) \equiv \sum_{n=0}^{\infty} x^n/[n]_q! \) is a q-deformed exponential function. Therefore, we conclude that a coherent state associated with the shape-invariant potentials naturally leads to the q-coherent state in case of the self-similar potentials.

6. Summary

By introducing an operator \( T \) denoting a reparametrization of \( a \), we have represented the shape-invariance condition as a form of a commutation relation, and based on it we have defined a coherent state associated with shape-invariant potentials. It is shown that in cases of the usual harmonic oscillator and recently found self-similar potentials, it reduces to the usual and the q-deformed coherent state, respectively. We expect this state should play a similar role as the usual and generalized coherent states have been playing in various fields in modern physics.

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8. References

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