ON COMPOSITE AND NON-MONOTONIC GROWTH
FUNCTIONS OF MEALY AUTOMATA

ILLYA I. REZNYKOV

ABSTRACT. We introduce the notion of composite growth function and provide examples that illustrate the primary properties of these growth functions. There are provided examples of Mealy automata that have composite non-monotonic growth functions of the polynomial growth order. There are described examples of Mealy automata that have composite monotonic growth functions of intermediate and exponential growth. The questions concerning the interrelation between the notions “composite” and “non-monotonic” of a Mealy automaton growth function are formulated.

1. INTRODUCTION

The notion of growth was introduced in the middle of last century [10, 14] and was applied to various geometrical, topological and algebraic objects [2, 15]. Mainly, growth functions of studied objects are non-decreasing monotonic functions of a natural argument [2]. For example, the growth function of a semigroup (group) at the point \( n, n \geq 0 \), equals a number of different semigroup elements of length \( n \). Obviously, the growth function of an arbitrary semigroup is a non-decreasing monotonic function.

Growth of Mealy automata have been studied since the 80th of 20th century [3, 6], and it is close interrelated with growth of automatic transformation semigroups (groups), defined by them [6]. However, the growth functions of the Mealy automaton and the corresponding semigroup have different properties; for example, they may have different growth orders [12]. In the paper we consider special type of growth functions of Mealy automata — composite growth functions.

Composite function is a function such that it can be described by different expressions on infinite non-overlapped intervals. There exist Mealy automata that have composite growth functions of various growth orders. Moreover, some of these automata have non-monotonic growth functions. There were not known Mealy automata such that they have non-monotonic growth functions (see survey [7], and [13], [11], etc.).

Preliminaries of the theory of Mealy automata are listed in Section 2. The notion of composite function is introduced in Section 3. In Section 4 we provide several examples of Mealy automata, that have non-monotonic growth functions of the polynomial growth order. In addition, the Mealy automata with composite growth functions such that one of its finite differences consists of doubled values, are provided in Section 5. The theorems concerning the main properties of these
automata are formulated, and we list these theorems without proofs. They can be proved by using the technique, similar to the technique of [11] (see also [12]). We are planning to publish the proofs of these theorems in subsequent papers. For convenience, the propositions, where the normal form of semigroup elements are formulated, are provided for the most complex of the considered automata. Moreover, the questions concerning the composite growth functions are appeared, and some of them are mentioned in Section 6.

I would like to thank Igor F. Reznykov and Alexandr N. Movchan, who helped to find the automata, that are considered in the paper.

2. Preliminaries

The basic notions of the theory of Mealy automata and the theory of semigroups can be found in many books, for example [3, 5, 8]. We use definitions from [12].

2.1. Mealy automata. Let’s denote the set of all finite words over $X_m$, including the empty word $\varepsilon$, by the symbol $X_m^\ast$, and denote the set of all infinite (to right) words by the symbol $X_m^\omega$. We write a function $\phi: X_m \rightarrow X_m$ as

\[
(\phi(x_0) \ \phi(x_1) \ \ldots \ \phi(x_{m-1}))
\]

Moreover, we have in mind $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Let $A = (X_m, Q_n, \pi, \lambda)$ be a non-initial Mealy automaton [9] with the finite set of states $Q_n = \{q_0, q_1, \ldots, q_{n-1}\}$, input and output alphabets are the same and equal $X_m$, $\pi: X_m \times Q_n \rightarrow Q_n$ and $\lambda: X_m \times Q_n \rightarrow X_m$ are its transition and output functions, respectively. The function $\lambda$ can be extended in a natural way to the mapping $\lambda: X_m^\ast \times Q_n \rightarrow X_m^\ast$ or to the mapping $\lambda: X_m^\omega \times Q_n \rightarrow X_m^\omega$. The transformation $f_q: X_m^\ast \rightarrow X_m^\ast$ ($f_q: X_m^\omega \rightarrow X_m^\omega$), defined by the equality

\[
f_q(u) = \lambda(u, q),
\]

where $u \in X_m^\ast$ ($u \in X_m^\omega$), is called the automatic transformation, defined by $A$ at the state $q$. The automaton $A$ defines the set

\[
F_A = \{f_{q_0}, f_{q_1}, \ldots, f_{q_{n-1}}\}
\]

of automatic transformations over $X_m^\omega$. Each automatic transformation defined by the automaton $A$ can be written in the unrolled form:

\[
f_q = (f_{\pi(x_0, q_i)}, f_{\pi(x_1, q_i)}, \ldots, f_{\pi(x_{m-1}, q_i)}) \sigma_{q_i},
\]

where $i = 0, 1, \ldots, n - 1$, and $\sigma_{q_i}$ is the transformation over the alphabet $X_m$ defined by the output function $\lambda$:

\[
\sigma_{q_i} = (\lambda(x_0, q_i) \ \lambda(x_1, q_i) \ \ldots \ \lambda(x_{m-1}, q_i))
\]

Let us define the set of all $n$-state Mealy automata over the $m$-symbol alphabet by the symbol $A_{n \times m}$. The product of Mealy automata is introduced [5] over the set of automata with the same input and output alphabet $X_m$ as their sequential applying. Therefore for the transformations $f_{q_1, A_1}$ and $f_{q_2, A_2}$, $q_1 \in Q_{n_1}, q_2 \in Q_{n_2}$, the unrolled form of the product $f_{q_1, A_1, q_2, A_2}$ is defined by the equality:

\[
f_{(q_1, q_2), A_1 \times A_2} = f_{q_1, A_1} f_{q_2, A_2} = (g_0, g_1, \ldots, g_{m-1}) \sigma_{q_1, A_1} \sigma_{q_2, A_2},
\]

where $g_i = f_{\pi(1, q_2, (x_i), q_1, A_1) f_{\pi(1, q_2, (x_i), q_2, A_2)}$, $i = 0, 1, \ldots, m - 1$, and all transformations are applied from right to left.
The power $A^n$ is defined for any automaton $A$ and any positive integer $n$. Let us denote $A^{(n)}$ the minimal Mealy automaton, equivalent to $A^n$. It follows from definition of a product, that $|Q_{A^{(n)}}| \leq |Q_A|^n$.

**Definition 2.1.** The function $\gamma_A$ of a natural argument, defined by

$$\gamma_A(n) = |Q_{A^{(n)}}|, \quad n \in \mathbb{N},$$

is called the growth function of the Mealy automaton $A$.

2.2. Semigroups.

**Definition 2.2.** Let $A = (X_m, Q_n, \pi, \lambda)$ be a Mealy automaton. The semigroup

$$S_A = sg(f_{q_0}, f_{q_1}, \ldots, f_{q_{n-1}})$$

is called the semigroup of automatic transformations, defined by $A$.

Let $S$ be a semigroup with the finite set of generators $G = \{s_0, s_1, \ldots, s_{k-1}\}$. The elements of the free semigroup $G^+$ are called semigroup words. In the sequel, we identify them with corresponding elements of $S$. Let’s denote the length of a semigroup element $s$ by the symbol $\ell(s)$.

**Definition 2.3.** The function $\gamma_S$ of a natural argument such that

$$\gamma_S(n) = |\{ s \in S \mid \ell(s) \leq n \}|, \quad n \in \mathbb{N},$$

is called the growth function of $S$ relative to the system $G$ of generators.

**Definition 2.4.** The function $\delta_S$ of a natural argument such that

$$\delta_S(n) = |\{ s \in S \mid \ell(s) = n \}|, \quad n \in \mathbb{N},$$

is called the word growth function of $S$ relative to the system $G$ of generators.

From the definitions 2.3, 2.4 and 2.5, the following inequalities hold for $n \in \mathbb{N}$:

$$\delta_S(n) \leq \gamma_S(n) \leq \gamma_S(n) = \sum_{i=0}^{n} \delta_S(i).$$

Similarly, from definition 2.2 it follows that

$$\gamma_A(n) = \gamma_{S_A}(n), \quad n \in \mathbb{N}.$$

2.3. Growth functions. The growth of some object is defined by functions of a natural argument. One of the most used characteristic of these functions is the notion of growth order.

**Definition 2.6.** Let $\gamma_i : \mathbb{N} \to \mathbb{N}$, $i = 1, 2$, are arbitrary functions. The function $\gamma_1$ has no greater growth order (notation $\gamma_1 \preceq \gamma_2$) than the function $\gamma_2$, if there exist numbers $C_1, C_2, N_0 \in \mathbb{N}$ such that

$$\gamma_1(n) \leq C_1 \gamma_2(C_2n)$$

for any $n \geq N_0$. 

Definition 2.7. The growth functions $\gamma_1$ and $\gamma_2$ are equivalent or have the same growth order (notation $\gamma_1 \sim \gamma_2$), if the inequalities $\gamma_1 \preceq \gamma_2$ and $\gamma_2 \preceq \gamma_1$ hold.

The equivalence class of the function $\gamma$ is called the growth order and is denoted by the symbol $[\gamma]$. The growth order $[\gamma]$ is called
1. polynomial, if $[\gamma] = [n^d]$ for some $d > 0$;
2. intermediate, if $[n^d] < [\gamma] < [e^n]$ for all $d > 0$;
3. exponential, if $[\gamma] = [e^n]$.

It is often convenient to encode the growth function of a semigroup in a generating series:

Definition 2.8. Let $S$ be a semigroup generated by a finite set $G$. The growth series of $S$ is the formal power series
$$\Gamma_S(X) = \sum_{n \geq 0} \gamma_S(n)X^n.$$ 

The power series $\Delta_S(X) = \sum_{n \geq 0} \delta_S(n)X^n$ can also be introduced; we then have $\Delta_S(X) = (1 - X)\Gamma_S(X)$. The series $\Delta_S$ is called the word growth series of the semigroup $S$.

The growth series of a Mealy automaton is introduced similarly:

Definition 2.9. Let $A$ be an arbitrary Mealy automaton. The growth series of $A$ is the formal power series
$$\Gamma_A(X) = \sum_{n \geq 0} \gamma_A(n)X^n.$$ 

3. Composite growth functions

Let us introduce the concept of composite growth function in the following way. Let $\gamma : \mathbb{N} \rightarrow \mathbb{N}$ be an arbitrary function, and let $k \geq 1$ be a positive integer. Let us define the functions $\gamma_i : \mathbb{N} \rightarrow \mathbb{N}$, $i = 0, 1, \ldots, k - 1$, by the equalities:
$$\gamma_i(n) = \gamma(k \cdot n + i), n \in \mathbb{N}.$$ 

We say that the function $\gamma$ is composite, if there exists integer $k \geq 2$ such that at least two functions from the set
$$\{\gamma_0, \gamma_1, \ldots, \gamma_{k-1}\}$$
can be defined by different expressions.

Let us fix the notions. Let $A$ be the arbitrary Mealy automaton. Let us denote the semigroup of automatic transformations, defined by $A$, by the symbol $S_A$, and the growth functions of $A$ and $S_A$ by the symbols $\gamma_A$ and $\gamma_{SA}$, respectively. If $\gamma_A$ is a composite function for some integer $k$, then let us denote its “parts” by the symbols $\gamma_{A,i}$, $i = 0, 1, \ldots, k - 1$. Let $\gamma$ be an arbitrary function, and let us denote the $i$-th finite difference of $\gamma$ by the symbols $\gamma^{(i)}$, $i \geq 1$, i.e.
$$\gamma^{(1)}(n) = \gamma(n) - \gamma(n - 1),$$
$$\gamma^{(i)}(n) = \gamma^{(i-1)}(n) - \gamma^{(i-1)}(n - 1),$$
where $i \geq 2, n \geq i + 1$.

Let us consider the example of Mealy automaton with the composite growth function. Let $A_1$ be the 3-state Mealy automaton over the 2-symbol alphabet
whose Moore diagram is shown on Figure 1. Its automatic transformations have the following unrolled forms:

\[ f_0 = (f_0, f_1)(x_1, x_1), \quad f_1 = (f_0, f_2)(x_1, x_1), \quad f_1 = (f_1, f_1)(x_0, x_0). \]

The following theorem holds:

**Theorem 3.1.** (1) The semigroup \( S_{A_1} \) has the following presentation:

\[
S_{A_1} = \left\{ f_0, f_1 \middle| f_1 f_2 = f_0 f_2, f_1^2 f_1 = f_1 f_0 f_2, i = 0, 1; \right. \\
\left. f_0^2 f_1 = f_2 f_0, f_0 f_0 f_1 = f_2^2 f_0, j = 0, 1, 2; \right. \\
\left. f_0^4 = f_0^3, f_0^2 f_2 = f_0^2, f_0 f_2 f_0^2 = f_0 f_1 f_0 f_2; \right. \\
f_1 f_0 f_2 f_0 = f_1 f_0 f_2, f_2 f_0^2 = f_2 f_0^2 \right\}
\]

(2) The growth function \( \gamma_{A_1} \) is a composite function for \( k = 2 \), and is defined by the following equalities:

\[ \gamma_{A_1,0}(n) = 23 \cdot 2^{n-2} - 1, \quad \gamma_{A_1,1}(n) = 32 \cdot 2^{n-2} - 1, \]

where \( n \geq 2 \), \( \gamma_{A_1}(1) = 3, \gamma_{A_1}(2) = 8, \gamma_{A_1}(3) = 14 \).

It follows from Theorem 3.1 that the growth function \( \gamma_{A_1} \) has the exponential growth order and can be written as

\[ \gamma_{A_1}(n) = \begin{cases} 23 \cdot 2^{n-1} - 1, & \text{if } n \text{ is even;} \\ 32 \cdot 2^{n-1} - 1, & \text{if } n \text{ is odd;} \end{cases} \]

where \( n \geq 4 \). The normal form of elements of \( S_{A_1} \) is declared in the following proposition:

**Proposition 3.2.** An arbitrary element \( s \) of \( S_{A_1} \) has the following normal form

\[ s = s' \cdot (f_0 f_2)^{p_1} (f_1 f_0)^{p_2} (f_0 f_2)^{p_3} (f_1 f_0)^{p_4} \ldots (f_0 f_2)^{p_{2k-1}} (f_1 f_0)^{p_{2k}} \cdot s'', \]

where \( s' \in \{1, f_0, f_2\}, s'' \in \{1, f_0, f_1, f_1 f_0, f_0 f_2, f_0 f_2 f_1 f_0 f_2\} \), and \( k \geq 1, p_1, p_{2k} \geq 0, p_i > 0, i = 2, 3, \ldots, 2k - 1, \ell(s) \geq 1. \)

4. Non-monotonic growth functions

The conception of a composite function allows to construct non-monotonic functions easily. For example, let \( k = 2 \) and \( \gamma \) be a function such that \( \gamma_1(n) = 1 \) and \( \gamma_2(n) = 2 \). Obviously, \( \gamma \) is non-monotonic. Below we consider the 2-state Mealy automata over the 4-symbol alphabet, that have non-monotonic growth functions of constant, linear and square growth. There are exist automata that have non-monotonic growth functions of other polynomial growth orders, but their consideration requires more technical details.
4.1. The automaton $A_2$ of constant growth. Let $A_2$ be the Mealy automaton, defined by Moore diagram on Figure 2. Its automatic transformations have the following unrolled forms:

$$f_0 = (f_0, f_0, f_0, f_0) (x_1, x_1, x_0, x_0), \quad f_1 = (f_0, f_1, f_1) (x_0, x_2, x_0, x_1).$$

The automaton $A_2$ has non-monotonic growth function of the constant growth order, and the graph of $\gamma_{A_2}$ is shown on Figure 3. The following theorem holds:

**Theorem 4.1.**

1. The semigroup $S_{A_2}$ has the following presentation:

$$S_{A_2} = \langle f_0, f_1 \mid f_0^2 = f_0 f_1 f_0, f_1^2 = f_0 f_1 f_0^2, f_1 f_0 f_1 f_0^2 = f_1^2, (f_1 f_0)^4 = (f_1 f_0)^2 \rangle.$$

2. The growth function $\gamma_{A_2}$ is a composite function for $k = 2$, and is defined by the following equalities:

$$\gamma_{A_2,0}(n) = 8, \quad \gamma_{A_2,1}(n) = 9,$$

where $n \geq 2$, $\gamma_{A_2}(1) = 2$, $\gamma_{A_2}(2) = 4$, $\gamma_{A_2}(3) = 7$.

4.2. The automaton $A_3$.

The automaton $A_3$, whose Moore diagram is shown on Figure 4, has the following unrolled forms:

$$f_0 = (f_0, f_0, f_0, f_0) (x_1, x_1, x_0, x_0), \quad f_1 = (f_0, f_1, f_1) (x_0, x_2, x_0, x_1).$$

The automaton $A_3$ has non-monotonic growth function of the linear growth order, and the graph of $\gamma_{A_3}$ is shown on Figure 5. The following theorem holds:

**Theorem 4.2.**

The semigroup $S_{A_3}$ has the following presentation:

$$S_{A_3} = \langle f_0, f_1 \mid f_0^2 = f_0 f_1 f_0, f_1^2 = f_0 f_1 f_0^2, f_1 f_0 f_1 f_0^2 = f_1^2, (f_1 f_0)^4 = (f_1 f_0)^2 \rangle.$$

The growth function $\gamma_{A_3}$ is defined by the following equalities:

$$\gamma_{A_3,0}(n) = 8, \quad \gamma_{A_3,1}(n) = 9,$$

where $n \geq 2$, $\gamma_{A_3}(1) = 2$, $\gamma_{A_3}(2) = 4$, $\gamma_{A_3}(3) = 7$. 

Figure 2. The automaton $A_2$

Figure 3. The growth function of $A_2$
Theorem 4.2.  
(1) The semigroup $S_{A_3}$ has the following presentation:

$$S_{A_3} = \langle f_0, f_1 \mid f_0^2 f_1 = f_1 f_0^2 = f_1 f_0^2, \quad i = 0, 1; f_0 f_1 f_0 = f_0, \ f_0 f_1 f_0 = f_0 f_1^2 f_0 = f_0^2, \ f_1 f_0 f_1 = f_1 f_0 f_1^2 \rangle$$

(2) The growth function $\gamma_{A_3}$ is a composite function for $k = 2$, and is defined by the following equalities:

$$\gamma_{A_3,0}(n) = 4n, \quad \gamma_{A_3,1}(n) = 5n + 1,$$

where $n \geq 1$ and $\gamma_{A_3}(1) = 2$.

4.3. The automaton $A_4$ of square growth. Let $A_4$ be the automaton such that its Moore diagram is shown on Figure 6. Its automatic transformations have the following unrolled forms:

$$f_0 = (f_0, f_0, f_0, f_0)(x_0, x_0, x_1, x_1), \quad f_1 = (f_0, f_1, f_1, f_1)(x_2, x_3, x_0, x_1).$$

The automaton $A_4$ has non-monotonic growth function of square growth, and the graph is shown in the following diagram.

Theorem 4.3.  
(1) The semigroup $S_{A_4}$ is infinitely presented, and has the following presentation:

$$S_{A_4} = \langle f_0, f_1 \mid \quad f_0^2 f_1 = f_1 f_0^2 = f_1 f_0^2, \quad i = 0, 1; f_0 f_1 f_0 = f_0, \ f_0 f_1 f_0 = f_0 f_1^2 f_0 = f_0^2, \ f_1 f_0 f_1 = f_1 f_0 f_1^2 f_0 = f_0^2, \ f_1 f_0 f_1 = f_1 f_0 f_1^2 f_0 = f_0^2, \ f_1 f_0 f_1 = f_1 f_0 f_1^2 f_0 = f_0^2, \ f_1 f_0 f_1 = f_1 f_0 f_1^2 f_0 = f_0^2, \ f_1 f_0 f_1 = f_1 f_0 f_1^2 f_0 = f_0^2 \rangle$$

Figure 5. The growth function of $A_3$. 

Figure 4. The automaton $A_3$. 

Figure 5. The growth function of $A_3$. 

The growth function $\gamma_{A_4}$ is a composite non-monotonic function, that is defined by the following equalities:

$$
\gamma_{A_4,0}(n) = 4n^2 - 5n + 6, \quad n \geq 2,
$$

$$
\gamma_{A_4,1}(n) = \frac{7}{2}n^2 + \frac{3}{2}n + 2, \quad n \geq 0
$$

and $\gamma_{A_4}(2) = 4$. The function $\gamma_{A_4}$ has the square growth order.

From the defining relations the proposition follows

**Proposition 4.4.** An arbitrary element $s$ of $S_{A_4}$ admits a unique minimal-length representation as a word of one of the following forms

$$
f_{0}f_{1}^{2p_1} (f_{0}f_{1})^{p_2} \cdot s',
$$

where $p_1 \geq 1$, $p_2 \geq 0$, $s' \in \{1, f_1, f_0, f_2\}$, except the combination $p_1 = 1$, $p_2 = 0$, $s' = 1$, or

$$
f_{1}^{p_1} (f_{0}f_{1})^{p_2} \cdot s',
$$

where $p_1 \geq 0$, $p_2 \geq 1$, $s' \in \{f_1, f_0, f_2\}$, or

$$
f_{0}^{p_1} (f_{0}f_{1})^{p_2} \cdot e,
$$

where $p_1 \geq 0$, $p_2 \geq 0$, $s' \in \{f_1, f_0, f_2\}$.

In this case, the $m \geq 3$ of its finite differences of a Mealy automaton of intermediate and the $e$
5.1. The automata \( \{ B_m, m \geq 3 \} \) of polynomial growth. Let \( B_m, m \geq 3 \), be the 2-state Mealy automaton over the \( m \)-symbol alphabet (Figure 8), and the unrolled forms of the automatic transformations \( f_0 = f_{q_0, B_m} \) and \( f_1 = f_{q_1, B_m} \) are defined in the following way:

\[
\begin{align*}
  f_0 &= (f_0, f_0, f_1, f_0, \ldots, f_0, f_0) (x_1, x_0, x_2, x_3, \ldots, x_{m-2}, x_{m-1}), \\
  f_1 &= (f_0, f_1, f_1, f_1, \ldots, f_1, f_1) (x_1, x_2, x_3, x_4, \ldots, x_{m-1}, x_{m-1}).
\end{align*}
\]

**Theorem 5.1.**

1. For any \( m \geq 3 \) the semigroup \( S_m \) have the following presentation:

\[
S_{B_3} = \langle f_0, f_1 \mid f_1^2 = f_0 f_1^2, f_1 f_0 f_1 = f_0^2 f_1 \rangle,
\]

\[
S_{B_4} = \langle f_0, f_1 \mid f_1^2 = f_1 f_0 f_1, f_1 f_0^p f_1 f_0 f_1 = f_1 f_0^{p+2} f_1, p_1 \geq 0 \rangle,
\]

\[
S_{B_m} = \langle f_0, f_1 \mid \prod_{i=1}^{m-3} (f_1 f_0^{p_i}) f_1^4 = \prod_{i=1}^{m-4} (f_1 f_0^{p_i}) f_1 f_0 f_1^2, \prod_{i=1}^{m-3} (f_1 f_0^{p_i}) f_1 f_0 f_1 = \prod_{i=1}^{m-3} (f_1 f_0^{p_i}) f_0^2 f_1, p_i \geq 0, i = 1, 2, \ldots, m - 3 \rangle.
\]

All semigroups \( S_{B_m} \) for \( m \geq 4 \) are infinitely presented.

2. For \( m \geq 3 \) the growth function \( \gamma_{B_m} \) is defined by the following equalities:

\[
\gamma_{B_m}(n) = \sum_{i=0}^{m-2} \binom{n}{i} + \sum_{i=0}^{\lfloor \frac{m-1}{m-2} \rfloor} \binom{n-2i-1}{m-2} = \sum_{i=0}^{m-2} \binom{n}{i} + \sum_{i=0}^{m-2} \binom{n-2i-1}{m-2},
\]

for all \( n \geq 1 \).

Here \( \lfloor r \rfloor \) denotes the integer part of the real number \( r \), and we assume that \( \binom{n}{k} = 0 \) if \( k \geq n \) or \( n < 0 \).

The following proposition holds in the semigroup \( S_{B_m} \):

**Proposition 5.2.** The normal form of the element \( s \) of \( S_{B_m} \) is one of the following words

\[
f_0^{p_1} f_1 f_0^{p_2} f_1 \ldots f_0^{p_{k-1}} f_1 f_0^{p_k}
\]

where \( 1 \leq k \leq m - 1 \), \( p_i \geq 0 \), \( i = 1, 2, \ldots, k \), \( \ell(s) \geq 1 \), and

\[
f_0^{p_1} f_1 f_0^{p_2} f_1 \ldots f_0^{p_{m-2}} f_1 f_0^{p_{m-1}}
\]
where $p_i \geq 0$, $i = 1, 2, \ldots, m$.

The corollary follows from Theorem 5.1.

**Corollary 5.3.**

1. For all $m \geq 3$ the function $\gamma_{B_m}$ have the growth order $\left\lfloor \frac{n^m - 1}{m} \right\rfloor$.

2. The $(m-2)$-th finite differences of $\gamma_{B_m}$ is defined by the following equality

$$\gamma^{(m-2)}_{B_m}(n) = \left\lfloor \frac{n - m + 1}{2} \right\rfloor + 2,$$

where $n \geq m - 1$.

It follows from (2) that for any $m \geq 4$ the equalities hold

$$\gamma^{(1)}_{B_m}(n) = \gamma_{B_m}(n) - \gamma_{B_m}(n - 1) =$$

$$= \sum_{i=0}^{m-3} \binom{n - i}{i} + \sum_{i \geq 0} \frac{n - 2i - 2}{m - 3} = \gamma_{B_{m-1}}(n - 1),$$

where $n \geq 2$. The growth function $\gamma_{B_3}$ is defined by the equalities

$$\gamma_{B_3}(n) = \begin{cases} \frac{3}{2}n^2 + n + 1, & \text{if } n \text{ is even;} \\ \frac{3}{4}n^2 + n + \frac{3}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

Hence, joining two last equalities, one has

$$\gamma^{(m-2)}_{B_m}(n) = \gamma^{(1)}_{B_3}(n - (m - 3)) = \left\lfloor \frac{n - (m - 3)}{2} \right\rfloor + 1 = \left\lfloor \frac{n - m + 1}{2} \right\rfloor + 2,$$

for any $m \geq 3, n \geq m - 1$. Therefore $(m-2)$-th finite difference of $\gamma_{B_m}$ consists of doubled values, i.e. for any even integer $n$, $n \geq 0$, the equality holds

$$\gamma^{(m-2)}_{B_m}(n + m) = \gamma^{(m-2)}_{B_m}(n + m - 1).$$

### 5.2. The automaton $A_5$ of intermediate growth.

Let $A_5$ be the 3-state Mealy automaton over the 2-symbol alphabet such that its Moore diagram is shown on Figure 9. The following theorem holds:
Theorem 5.4.  (1) The semigroup $S_{A_5}$ is a infinitely presented monoid, and has the following presentation:

\begin{align*}
S_{A_5} = \left\langle e, f_0, f_1 \right| & \quad f_0 f_1^{2^k-1}, f_1 f_0^{2^k+1} f_0 \prod_{i=k}^{1} (f_1^{2^i} f_0) = \\
&= f_1^{2^k+1} f_0 \prod_{i=k}^{1} (f_1^{2^i} f_0), k \geq 0, p = 0, 1.
\end{align*}

(2) The growth series $\Gamma_{A_5}(X)$ of $A_5$ and the growth series $\Gamma_{S_{A_5}}(X)$ of $S_{A_5}$ coincide and are defined by the equality

$$\Gamma_{S_{A_5}}(X) = \Gamma_{A_5}(X) = \frac{1}{(1-X)^{2}} \left( 1 + \frac{X}{1-X} \left( 1 + \frac{X^2}{1-X^2} \left( 1 + \frac{X^4}{1-X^4} \left( 1 + \frac{X^8}{1-X^8} (1 + \ldots) \right) \right) \right) \right).$$

The properties of the growth function $\gamma_{A_5}$ are formulated in the following corollary:

Corollary 5.5.  (1) The growth function $\gamma_{A_5}$ has the intermediate growth order

$$\left[ \frac{\log n}{n^{1/3}} \right].$$

(2) Let us define $\gamma_{A_5}^{(2)}(0) = \gamma_{A_5}^{(2)}(1) = \gamma_{A_5}^{(2)}(2) = 1$. The second finite difference of $\gamma_{A_5}$ is defined by the following equality

$$\gamma_{A_5}^{(2)}(n) = \sum_{i=0}^{[\frac{n-1}{2}]} \gamma_{A_5}^{(2)}(i), n \geq 3.$$

The system of defining relations (3) implies the following normal form:

Proposition 5.6. Each semigroup element $s$ of $S_{A_5}$ can be written in the following normal form

$$f_0^{p_0} f_1^{2^{k_1} p_1 + (2^{k_1}-1)} f_0 \cdots f_1^{2^{k_i} p_i + (2^{k_i}-1)} f_0 \cdots f_1^{2^{k_k} p_k + (2^{k_k}-1)} f_0$$

where $k \geq 0, p_i \geq 0, i = 0, 1, \ldots, k$.

The growth series for the second finite difference $\Delta^{(2)} \Gamma_{A_5}(X)$ can be easily constructed by using the expression for $\Gamma_{A_5}(X)$:

$$\Delta^{(2)} \Gamma_{A_5}(X) = \sum_{n \geq 3} \gamma_{A_5}^{(2)}(n) X^n + \gamma_{A_5}^{(2)}(0) + \gamma_{A_5}^{(2)}(1) X + \gamma_{A_5}^{(2)}(2) X^2 =$$

$$= (1 - X)^2 \Gamma_{A_5}(X) - (1 - X) \gamma_{A_5}(0) - X(\gamma_{A_5}(1) - \gamma_{A_5}(0)) -$$

$$- X^2(\gamma_{A_5}(2) - 2\gamma_{A_5}(1) + \gamma_{A_5}(0)) + 1 + X + X^2 =$$

$$= 1 + \frac{X}{1-X} \left( 1 + \frac{X^2}{1-X^2} \left( 1 + \frac{X^4}{1-X^4} \left( 1 + \frac{X^8}{1-X^8} (1 + \ldots) \right) \right) \right).$$

The right-hand series of the last equality are the formal series for the numbers of partitions of $n$, $n \geq 1$, into "sequential" powers of 2, that is $\gamma_{A_5}^{(2)}(n)$ equals the cardinality of the set

$$\left\{ p_0, p_1, \ldots, p_k \right| k \geq 0, \sum_{i=0}^{k} p_i 2^i = n, p_i \geq 1, i = 0, 1, \ldots, k \right\}.$$
The equality (4) is well-known for these partition numbers \( \mathbf{H} \). Therefore, the second finite difference of \( \gamma_A \) consists of doubled values, i.e. for any even integer \( n \), \( n \geq 2 \), the equality holds

\[
\gamma_A^{(2)}(n) = \gamma_A^{(2)}(n - 1).
\]

5.3. The automaton \( A_6 \) of exponential growth. Let \( A_6 \) be the 3-state Mealy automaton over the 2-symbol alphabet such that its automatic transformations have the following unrolled forms:

\[
f_0 = (f_0, f_0) (x_1, x_0), \quad f_1 = (f_1, f_2) (x_0, x_1), \quad f_2 = (f_1, f_2) (x_0, x_0).
\]

The Moore diagram of \( A_6 \) is shown on Figure 10. The following theorem holds:

**Theorem 5.7.**

1. The semigroup \( S_{A_6} \) has the following presentation:

\[
S_{A_6} = \left\langle f_0, f_1, f_2 \left| \begin{array}{l}
f_0^2 = 1, \quad f_2 f_1 = f_1 f_2 = f_2^2 = f_2 \\
f_1^2 = f_1, \quad f_2 f_0 f_1 f_0 f_2 = f_1 f_0 f_1 f_0 f_2
\end{array} \right. \right\rangle
\]

2. The growth series \( \Gamma_{A_6}(X) \) of \( A_6 \) admits the description

\[
\Gamma_{A_6}(X) = \frac{1}{(1 - X)^2} \left( 2X - 1 + \frac{1 + X + X^3}{1 - X^2 - X^4} \right).
\]

3. The growth series \( \Gamma_{S_{A_6}}(X) \) of \( S_{A_6} \) is defined in the following way

\[
\Gamma_{S_{A_6}}(X) = \frac{1}{(1 - X)^2} \left( X + \frac{1 + X + X^3}{1 - X^2 - X^4} \right).
\]

Let us define the Fibonacci numbers by the symbols \( \Phi_n \), where \( \Phi_n = \Phi_{n-1} + \Phi_{n-2} \), \( n \geq 2 \), and \( \Phi_0 = \Phi_1 = 1 \). It follows from Theorem 5.7 that the growth function \( \gamma_{A_6} \) can be written in close form, and the following corollary holds:

**Corollary 5.8.** The growth function \( \gamma_{A_6} \) is defined by the following equalities:

\[
\gamma_{A_6}(n) = \begin{cases} 
\Phi_{\lfloor \frac{n}{2} \rfloor} + 6 + \Phi_{\lfloor \frac{n}{2} \rfloor} + 4 - 2n - 18, & \text{if } n \text{ is even;} \\
\Phi_{\lfloor \frac{n}{2} \rfloor} + 6 + 2\Phi_{\lfloor \frac{n}{2} \rfloor} + 4 - 2n - 18, & \text{if } n \text{ is odd.}
\end{cases}
\]

The growth function \( \gamma_{A_6} \) has the exponential growth order.

Let \( n \) be any positive integer, and represent \( n = 2k \), when \( n \) is even, and \( n = 2k + 1 \), when \( n \) is odd. It follows from (5), that for any \( k \geq 0 \) the following equalities hold

\[
\gamma_{A_6}^{(1)}(2k + 1) = \Phi_{k+4} - 2, \quad \gamma_{A_6}^{(1)}(2k + 2) = 2\Phi_{k+3} - 2,
\]

and, using the previous equalities, we have

\[
\gamma_{A_6}^{(2)}(2k + 1) = \Phi_{k+1}, \quad \gamma_{A_6}^{(2)}(2k + 2) = \Phi_{k+1}.
\]
Hence, the second finite difference $\gamma_{A_6}$ consists of doubled values, and for all even integer $n$ the equality holds
\[
\gamma_{A_6}^{(2)}(n) = \gamma_{A_6}^{(2)}(n - 1).
\]

6. Final remarks

There are some questions, that concern the composite non-monotonic growth functions of Mealy automata.

(1) Does there exist the Mealy automaton such that its composite growth function includes “parts” of different growth orders?

(2) Does there exist the Mealy automaton which have the non-monotonic growth function of the intermediate or the exponential growth order?

(3) Does there exist the Mealy automaton such that its growth function is non-monotonic, but isn’t a composite function (in the sense of Section 3)?

References

[1] George E. Andrews, *The Theory of Partitions*, Addison-Wesley Publishing Company, London, Amsterdam, Don Mills Ontario, Sydney, Tokio, 1976.

[2] I.K. Babenko, *The problems of the growth and the rationality in algebra and topology*, Uspehi Math. Nauk 41 (1986), 95–142.

[3] Ferenc Gecseg, *Products of automata*, Springer-Verlag, Berlin etc., 1986.

[4] Arthur Gill, *Introduction to the Theory of Finite-State Machines*, McGraw-Hill Book Company, Inc., New York, San Francisco, Toronto, London, 1963.

[5] Victor M. Gluškov, *Abstract theory of automata*, Uspehi Mat. Nauk 16 (1961), 3–62.

[6] Rostislav I. Grigorchuk, *On cancellation semigroups of the degree growth*, Math. Notes 43 (1988), 305–319.

[7] Rostislav I. Grigorchuk, Volodimir V. Nekrashevich, and Vitaliy I. Sushchansky, *Automata, dynamical systems, and groups*, Proceedings of the Steklov Institute of Mathematics 231 (2000), 128–203.

[8] Gérard Lallement, *Semigroups and combinatorial applications*, John Willey & Sons, New York, Chichester, Brisbane, Toronto, 1979, ISBN 0-471-04379-6.

[9] George H. Mealy, *A method for synthesizing sequential circuits*, Bell System Tech. J. 34 (1955), 1045–1079.

[10] John Milnor, *Growth of finitely generated solvable groups*, Journal of Differential Geometry 2 (1968), 447–451.

[11] Illya I. Reznykov, *The growth functions of two-state Mealy automata over the two-symbol alphabet and the semigroups, defined by them*, Kyiv Taras Shevchenko University, Kiev, Ukraine, 2002.

[12] Illya I. Reznykov, *On 2-state Mealy automata of polynomial growth*, Algebra and Discrete Mathematics (2003), 66–85.

[13] Illya I. Reznykov and Vitaliy I. Sushchansky, *2-generated semigroup of automatic transformations, whose growth is defined by Fibonacci series*, Math. Studii 17 (2002), 81–92.

[14] A.S. Svarc, *A volume invariant of coverings*, Dokladi Akademii Nauk SSSR 105 (1955), 32–34.

[15] V.A. Ufnarovskiy, *Combinatorial and asymptotical methods in algebra*, Itogi Nauki i Tekhniki, vol. 57, VINITI, Moscow, 1990, pp. 5–177.

Faculty of Mathematics and Mechanics, Kiev Taras Shevchenko National University, vul. Volodymyrska, 64, Kiev, Ukraine 01033

Current address: Faculty of Mathematics and Mechanics, Kiev Taras Shevchenko National University, vul. Volodymyrska, 64, Kiev, Ukraine 01033

E-mail address: Illya.Reznykov@iuc5.com.ua