Transference theorems for Diophantine approximation with weights. *

Oleg N. German

Abstract

In this paper we prove transference inequalities for regular and uniform Diophantine exponents in the weighted setting. Our results generalise the corresponding inequalities that exist in the ‘non-weighted’ case.

2010 Mathematics Subject Classification: Primary 11H60, 11J13; Secondary 11J25, 11J20

Key words and phrases: Diophantine approximation with weights, Diophantine exponents, transference inequalities

1 Introduction

In 1926 A. Ya. Khintchine in his seminal paper [1] proved the famous transference inequalities connecting two dual problems. The first one concerns simultaneous approximation of given real numbers \( \theta_1, \ldots, \theta_n \) by rationals, the second one concerns approximating zero with the values of the linear form \( \theta_1 x_1 + \ldots + \theta_n x_n + x_{n+1} \) at integer points. Later on, Khintchine’s inequalities were generalised to the case of several linear forms by F. Dyson [2]. Given a matrix

\[
\Theta = \begin{pmatrix}
\theta_{11} & \cdots & \theta_{1m} \\
\vdots & \ddots & \vdots \\
\theta_{n1} & \cdots & \theta_{nm}
\end{pmatrix} \in \mathbb{R}^{n \times m}, \quad n + m \geq 3,
\]

let us consider the system of inequalities

\[
\begin{cases}
|x| \leq t^{1/m} \\
|\Theta x - y| \leq t^{-\gamma/n}
\end{cases},
\]

(1)

where \( x \in \mathbb{R}^m, y \in \mathbb{R}^n \), and \(| \cdot |\) denotes the sup-norm.

Definition 1. The Diophantine exponent \( \omega(\Theta) \) is defined as the supremum of real \( \gamma \) such that the system (1) admits nonzero solutions in \( (x, y) \in \mathbb{Z}^{m+n} \) for some arbitrarily large \( t \).

*This research was supported by RSF grant 18-41-05001
In this setting Dyson’s result reads as follows:

$$\omega(\Theta^\top) \geq \frac{(n-1) + m\omega(\Theta)}{n + (m-1)\omega(\Theta)},$$

(2)

where $\Theta^\top$ denotes the transposed matrix.

Along with the regular Diophantine exponents an important role is played by their uniform analogues.

**Definition 2.** The uniform Diophantine exponent $\hat{\omega}(\Theta)$ is defined as the supremum of real $\gamma$ such that the system (1) admits nonzero solutions in $(x, y) \in \mathbb{Z}^{m+n}$ for every $t$ large enough.

The first transference result concerning uniform exponents belongs to V. Jarník [3]. He showed that in the simplest nontrivial case $n = 1, m = 2$ we have

$$\hat{\omega}(\Theta^\top) + \hat{\omega}(\Theta) - 1 = 2.$$  

(3)

In higher dimension there is no equality any longer, the corresponding inequalities

$$\hat{\omega}(\Theta^\top) \geq \begin{cases} m - \hat{\omega}(\Theta)^{-1} & \text{if } \hat{\omega}(\Theta) \geq n/m \\ m - 1 & \text{if } \hat{\omega}(\Theta) \leq n/m \\ n - 1 \end{cases}$$

(4)

for arbitrary $n, m$ were obtained by the author in [4], [5].

The aim of the current paper is to prove analogues of (2) and (4) for the so-called weighted setting.

The rest of the paper is organised as follows. In Section 2 we formulate our main results; in Section 3 we focus on particular cases $m = 1, n = 1$ and analyse in our context a recent result by A. Marnat; in Section 4 we apply our results to prove transference inequalities in the inhomogeneous setting; in Sections 5, 6 we prove Theorems 1, 2, which are the main result of the paper; and, finally, in Section 7 we analyse why the generalisation of Dyson’s theorem proposed in [6] is not optimal.

## 2 Weighted setting

Let us fix weights $\sigma = (\sigma_1, \ldots, \sigma_m) \in \mathbb{R}_{>0}^m, \rho = (\rho_1, \ldots, \rho_n) \in \mathbb{R}_{>0}^n, \quad \sigma_1 \geq \ldots \geq \sigma_m, \quad \rho_1 \geq \ldots \geq \rho_n, \quad \sum_{j=1}^m \sigma_j = \sum_{i=1}^n \rho_i = 1,$

and define the weighted norms $|\cdot|_{\sigma}$ and $|\cdot|_{\rho}$ by

$$|x|_{\sigma} = \max_{1 \leq j \leq m} |x_j|^{1/\sigma_j} \quad \text{for } x = (x_1, \ldots, x_m),$$

$$|y|_{\rho} = \max_{1 \leq i \leq n} |y_i|^{1/\rho_i} \quad \text{for } y = (y_1, \ldots, y_n).$$

Consider the system of inequalities

$$\begin{cases} |X|_{\sigma} \leq t \\
|\Theta x - y|_{\rho} \leq t^{-\gamma} \end{cases}.$$
Definition 3. The weighted Diophantine exponent $\omega_{\sigma, \rho}(\Theta)$ is defined as the supremum of real $\gamma$ such that the system (5) admits nonzero solutions in $(x, y) \in \mathbb{Z}^{m+n}$ for some arbitrarily large $t$.

Definition 4. The uniform weighted Diophantine exponent $\hat{\omega}_{\sigma, \rho}(\Theta)$ is defined as the supremum of real $\gamma$ such that the system (5) admits nonzero solutions in $(x, y) \in \mathbb{Z}^{m+n}$ for every $t$ large enough.

The following two theorems are the main result of the paper.

Theorem 1. Set $\omega = \omega_{\sigma, \rho}(\Theta)$ and $\omega^\top = \omega_{\rho, \sigma}(\Theta^\top)$. Then

$$\omega^\top \geq \frac{(\rho_n^{-1} - 1) + \sigma_m^{-1}\omega}{\rho_n^{-1} + (\sigma_m^{-1} - 1)\omega}. \quad (6)$$

Theorem 2. Set $\hat{\omega} = \hat{\omega}_{\sigma, \rho}(\Theta)$ and $\hat{\omega}^\top = \hat{\omega}_{\rho, \sigma}(\Theta^\top)$. Then

$$\hat{\omega}^\top \geq \begin{cases} 
1 - \sigma_m\hat{\omega}^{-1} & \text{if } \hat{\omega} \geq \sigma_m/\rho_n \\
1 - \sigma_m & \text{if } \hat{\omega} = \sigma_m/\rho_n \\
1 - \rho_n & \text{if } \hat{\omega} \leq \sigma_m/\rho_n 
\end{cases}. \quad (7)$$

Clearly, in the ‘non-weighted’ case, when all the $\sigma_j$ are equal to $1/m$ and all the $\rho_i$ are equal to $1/n$, (6) turns into (2), and (7) turns into (4).

We cannot avoid mentioning a recent paper [6] by S. Chow, A. Ghosh et al., where they propose another generalisation of Dyson’s inequality, different from (6). Namely, they showed that

$$\omega^\top \geq \frac{(m + n - 1)(\rho_n^{-1} + \sigma_m^{-1}\omega) + \sigma_1^{-1}(\omega - 1)}{(m + n - 1)(\rho_n^{-1} + \sigma_m^{-1}\omega) - \rho_1^{-1}(\omega - 1)}. \quad (8)$$

For $\omega = 1$ (6) and (8) obviously coincide, as well as in the Dyson’s ‘non-weighted’ case. In every other case (6) is strictly stronger than (8). At first glance, this fact seems to be rather surprising, as both (6) and (8) are proved essentially by applying Mahler’s theorem. However, there is a certain freedom of choice, at which moment to apply Mahler’s theorem. Different choices result in different inequalities. We spend some time analysing this phenomenon in Section 7.

Concerning inversion of (6) and (7). Sometimes it is useful to have inverted versions of (6) and (7). For instance, such a need arises in Section 4.

First of all let us notice that Minkowski’s convex body theorem provides the following trivial bound we should always keep in mind:

$$\omega(\Theta) \geq \hat{\omega}(\Theta) \geq 1.$$

The inequality (6) is very simple to invert, as the function

$$f(x) = \frac{(\rho_n^{-1} - 1) + \sigma_m^{-1}x}{\rho_n^{-1} + (\sigma_m^{-1} - 1)x}$$

maps $[1, +\infty)$ monotonously onto $[1, (1 - \sigma_m)^{-1})$. Furthermore, any statement concerning $\Theta$ produces a statement concerning $\Theta^\top$ by just swapping $(m, n, \sigma, \rho, \Theta)$ for $(n, m, \rho, \sigma, \Theta^\top)$. Thus, we get
Corollary 1. Set $\omega = \omega_{\rho, \sigma}(\Theta)$ and $\omega^\top = \omega_{\rho, \sigma}(\Theta^\top)$. Suppose $\omega < (1 - \rho_n)^{-1}$. Then

$$\omega^\top \leq \frac{\rho_n}{\sigma_m} \cdot \frac{\omega - (1 - \sigma_m)}{1 - (1 - \rho_n)\omega}. $$

As for inverting (7), it needs an additional observation due to the splitting into two cases. If $\rho_n \geq \sigma_m$, then only the first case remains, and the argument is very simple. Suppose $\rho_n < \sigma_m$. Let us set

$$f_1(x) = \frac{1 - \rho_n}{1 - \rho_n x}, \quad f_2(x) = \frac{1 - \sigma_m x^{-1}}{1 - \sigma_m}. $$

The mappings

$$f_1 : [1, \sigma_m / \rho_n] \to \left[1, \frac{1 - \rho_n}{1 - \sigma_m}\right], \quad f_2 : [\sigma_m / \rho_n, +\infty) \to \left[\frac{1 - \rho_n}{1 - \sigma_m}, \frac{1}{1 - \sigma_m}\right]$$

are monotonous and one-to-one. Hence, denoting by $f_1^{-1}$ and $f_2^{-1}$ the corresponding inverse functions, we get

$$\begin{align*}
\begin{cases}
1 \leq x \leq \sigma_m / \rho_n \\
f_1(x) \leq y < \frac{1}{1 - \sigma_m}
\end{cases}
\iff
\begin{cases}
1 \leq y \leq \frac{1 - \rho_n}{1 - \sigma_m} \\
1 \leq x \leq f_1^{-1}(y)
\end{cases}
\end{align*}$$

or

$$\begin{align*}
\begin{cases}
x \geq \sigma_m / \rho_n \\
f_2(x) \leq y < \frac{1}{1 - \sigma_m}
\end{cases}
\iff
\begin{cases}
\frac{1 - \rho_n}{1 - \sigma_m} \leq y < \frac{1}{1 - \sigma_m} \\
1 \leq x \leq f_2^{-1}(y)
\end{cases}
\end{align*}$$

(cf. Fig. 1).

![Figure 1: How to invert (7)](image)

Swapping again $(m, n, \sigma, \rho, \Theta)$ for $(n, m, \rho, \sigma, \Theta^\top)$, we get

Corollary 2. Set $\hat{\omega} = \hat{\omega}_{\rho, \sigma}(\Theta)$ and $\hat{\omega}^\top = \hat{\omega}_{\rho, \sigma}(\Theta^\top)$. Suppose $\hat{\omega} < (1 - \rho_n)^{-1}$. Then

$$\hat{\omega}^\top \leq \begin{cases}
\rho_n & \text{if } \hat{\omega} \geq \frac{1 - \sigma_m}{1 - \rho_n} \\
\frac{1 - (1 - \rho_n)\hat{\omega}}{1 - (1 - \sigma_m)\hat{\omega}^{-1}} & \text{if } \hat{\omega} \leq \frac{1 - \sigma_m}{1 - \rho_n}
\end{cases}$$

assuming that $(1 - \rho_n)^{-1} = +\infty$ for $\rho_n = 1$. 

\[4\]
3 Case of one linear form and Marnat’s examples

It is worth singling out the case \( m = 1 \) and the case \( n = 1 \), as transference theorems are more often applied in those particular cases, than in the general one.

Theorem 1 and Corollary 1 provide the following statement for \( n = 1 \).

**Theorem 3.** Set \( \omega = \omega_{\sigma, \rho}(\Theta) \) and \( \omega^\top = \omega_{\rho, \sigma}(\Theta^\top) \). Suppose \( n = 1 \). Then
\[
\frac{\omega}{\sigma_m + (1 - \sigma_m)\omega} \leq \omega^\top \leq \frac{\omega - (1 - \sigma_m)}{\sigma_m}.
\]

As for the uniform exponents, it appears that both for \( m = 1 \) and for \( n = 1 \) exactly one of the inequalities (7) survives. For \( n = 1 \) we have \( \rho_n = 1 \) and \( \hat{\omega}(\Theta) \geq 1 > \sigma_m = \sigma_m/\rho_n \), i.e. the second alternative in (7) is inconsistent.

The case \( m = 1 \) is slightly more difficult. It appears that in this case \( \hat{\omega}(\Theta) \) cannot be greater than \( \rho_n^{-1} \) (unless \( \Theta \) is rational), which eliminates the first alternative in (7). In fact, a stronger statement holds.

**Proposition 1.** Let \( m = 1 \).

(i) If \( \Theta \in \mathbb{Q}^{n\times 1} \), then \( \hat{\omega}(\Theta) = \omega(\Theta) = \hat{\omega}(\Theta)^\top = \omega(\Theta)^\top \). Let us prove statement (ii). The argument is the same as in the ‘non-weighted’ case.

Let \( p_{\nu-1}/q_{\nu-1} \) and \( p_{\nu}/q_{\nu} \) be two consecutive convergents for \( \theta_{k_1} \). Set \( t = q_{\nu} - \varepsilon \) with arbitrary positive \( \varepsilon \). Then for every nonzero \( (x, y_1, \ldots, y_n) \in \mathbb{Z}^{n+1} \) such that \( |x| \leq t \) we have
\[
|\theta_{k_1}x - y_k| \geq |\theta_{k_1}q_{\nu-1} - p_{\nu-1}| \geq \left| \frac{p_{\nu}}{q_{\nu}q_{\nu-1} - p_{\nu-1}} \right| = \frac{1}{q_{\nu}} = (t + \varepsilon)^{-1}.
\]

Thus, given \( \gamma > \rho_k^{-1} \), one can find \( t \), arbitrarily large, for which the system (5) admits no nonzero integer solutions. Hence \( \hat{\omega}(\Theta) \leq \rho_k^{-1} \). \( \square \)

Theorem 2, Corollary 2, and Proposition 1 provide the following statement for \( n = 1 \).

**Theorem 4.** Set \( \hat{\omega} = \hat{\omega}_{\sigma, \rho}(\Theta) \) and \( \hat{\omega}^\top = \hat{\omega}_{\rho, \sigma}(\Theta^\top) \). Suppose \( n = 1 \) and \( \Theta \notin \mathbb{Q}^{1\times m} \). Then
\[
\begin{align*}
(1 - \sigma_m)\hat{\omega}^\top + \sigma_m\hat{\omega}^{-1} & \geq 1, \\
\sigma_m\hat{\omega}^\top + (1 - \sigma_m)\hat{\omega}^{-1} & \leq 1.
\end{align*}
\]

Moreover, we also have
\[
\sigma_k\hat{\omega}^\top \leq 1,
\]
where \( k \) is the minimal index such that \( \theta_{1k} \) is irrational.

For \( m = 2 \) we obviously have \( \sigma_m = \sigma_2 \) and \( 1 - \sigma_m = \sigma_1 \), which makes (10) look even nicer.

**Corollary 3.** Set \( \hat{\omega} = \hat{\omega}_{\sigma, \rho}(\Theta) \) and \( \hat{\omega}^\top = \hat{\omega}_{\rho, \sigma}(\Theta^\top) \). Suppose \( n = 1, m = 2, \) and \( \Theta \notin \mathbb{Q}^{1\times 2} \). Then
\[
\begin{align*}
\sigma_1\hat{\omega}^\top + \sigma_2\hat{\omega}^{-1} & \geq 1, \\
\sigma_2\hat{\omega}^\top + \sigma_1\hat{\omega}^{-1} & \leq 1.
\end{align*}
\]

Moreover, if \( \theta_{11} \) is irrational, we also have
\[
\sigma_1\hat{\omega}^\top \leq 1.
\]
Remark 1. If, within the hypothesis of Corollary 3, $\theta_{11}$ is rational, then $\hat{\omega} = +\infty$, whereas for $\Theta^\top$ the system (5) reduces to a system
\[
\begin{aligned}
|x| & \leq t \\
|\theta x - y| & \leq t^{-\gamma_2}
\end{aligned}
\]
with an irrational $\theta$, which by the argument in the spirit of Proposition 1 implies that $\hat{\omega}^\top = \sigma_2^{-1}$. Thus, in this case we always have $(\hat{\omega}, \hat{\omega}^\top) = (+\infty, \sigma_2^{-1})$.

It is very interesting now to analyse a result by A. Marnat [7], who proved the existence of uncountably many $\Theta$ for $n = 1$, $m = 2$ with prescribed values of $\hat{\omega}$ and $\hat{\omega}^\top$, showing thus that there is no analogue of Jarník’s relation in the weighted case. Namely, he proved that for every positive $a < (3\sigma_1)^{-1}$ and every $b$ satisfying the inequalities
\[
\begin{aligned}
\sigma_1 b + \sigma_2 a & \geq 1, \\
\sigma_2 b + \sigma_1 a & \leq 1, \\
\sigma_1 b & \leq 1,
\end{aligned}
\]
there exist uncountably many $\Theta$ with $\hat{\omega} = a^{-1}$ and $\hat{\omega}^\top = b$.

Particularly, in the case of irrational $\theta_{11}$, it follows from Marnat’s result that for $\hat{\omega} > 3\sigma_1$ the inequalities (11), (12) are sharp. Of course, it would be interesting to prove this fact for every $\hat{\omega} \geq 1$.

4 Application to inhomogeneous approximation

Another important class of Diophantine problems concerns the inhomogeneous setting. Given $\eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$, consider the system
\[
\begin{aligned}
|x| & \leq t \\
|\Theta x - y - \eta| & \leq t^{-\gamma}.
\end{aligned}
\]

Definition 5. The inhomogeneous weighted Diophantine exponent $\omega_{\sigma, \rho}(\Theta, \eta)$ is defined as the supremum of real $\gamma$ such that the system (13) admits nonzero solutions in $(x, y) \in \mathbb{Z}^{m+n}$ for some arbitrarily large $t$.

Definition 6. The inhomogeneous uniform weighted Diophantine exponent $\hat{\omega}_{\sigma, \rho}(\Theta, \eta)$ is defined as the supremum of real $\gamma$ such that the system (13) admits nonzero solutions in $(x, y) \in \mathbb{Z}^{m+n}$ for every $t$ large enough.

In the aforementioned paper [6] S. Chow, A. Ghosh et al. proved the following inequalities, the ‘non-weighted’ version of which belongs to M. Laurent and Y. Bugeaud [8]:
\[
\omega_{\sigma, \rho}(\Theta, \eta) \geq \frac{1}{\hat{\omega}_{\sigma, \rho}(\Theta^\top)}, \quad \hat{\omega}_{\sigma, \rho}(\Theta, \eta) \geq \frac{1}{\omega_{\rho, \sigma}(\Theta^\top)}.
\]
Corollaries 1 and 2 combined with (14) provide the following two results. A similar approach was used in [9] and [10] in the ‘non-weighted’ case.
Theorem 5. Set $\omega = \omega_{\sigma, \rho}(\Theta)$ and $\hat{\omega}_\eta = \hat{\omega}_{\sigma, \rho}(\Theta, \eta)$. Suppose $\omega < (1 - \rho_n)^{-1}$. Then

$$\hat{\omega}_\eta \geq \frac{\sigma_m}{\rho_n \left(1 - (1 - \rho_n)\omega\right)}.$$

(15)

Theorem 6. Set $\hat{\omega} = \hat{\omega}_{\sigma, \rho}(\Theta)$ and $\omega_\eta = \omega_{\sigma, \rho}(\Theta, \eta)$. Suppose $\hat{\omega} < (1 - \rho_n)^{-1}$. Then

$$\omega_\eta \geq \begin{cases} \frac{1 - (1 - \rho_n)\hat{\omega}}{\rho_n} & \text{if } \hat{\omega} \geq \frac{1 - \sigma_m}{1 - \rho_n} \\ \frac{\sigma_m}{1 - (1 - \sigma_m)\hat{\omega}^{-1}} & \text{if } \hat{\omega} \leq \frac{1 - \sigma_m}{1 - \rho_n} \end{cases},$$

(16)

assuming that $(1 - \rho_n)^{-1} = +\infty$ for $\rho_n = 1$.

Notice that due to the trivial inequalities $\omega_\eta \geq \hat{\omega}_\eta, \omega \geq \hat{\omega}$ both (15) and (16) provide lower estimates for $\omega_\eta$ in terms of $\omega$. One can easily check that the one provided by (15) is weaker than the one provided by (16). However, there is a small disadvantage in the latter caused by the condition on $\hat{\omega}$. But in the cases $m = 1$ and $n = 1$ that condition luckily disappears, which turns Theorems 5, 6 into the following symmetric statement.

Theorem 7. Let $\omega, \hat{\omega}, \omega_\eta, \hat{\omega}_\eta$ be as in Theorems 5, 6.

(i) Suppose $n = 1$. Then

$$\hat{\omega}_\eta \geq \frac{\sigma_m}{\omega - (1 - \sigma_m)}, \quad \omega_\eta \geq \frac{\sigma_m}{1 - (1 - \sigma_m)\hat{\omega}^{-1}}.$$

(ii) Suppose $m = 1$ and $\omega < (1 - \rho_n)^{-1}$. Then

$$\hat{\omega}_\eta \geq \frac{\omega^{-1} - (1 - \rho_n)}{\rho_n}, \quad \omega_\eta \geq \frac{1 - (1 - \rho_n)\hat{\omega}}{\rho_n}.$$

5 Dyson’s transference with weights

In this Section we prove Theorem 1.

5.1 Mahler’s theorem in terms of pseudo-compound parallelepipeds

All the Dyson-like transference theorems base upon a phenomenon described in its utmost generality by the classical Mahler theorem on a bilinear form (see [11], [12], [13]). We believe that from the geometric point of view Mahler’s theorem is most vividly formulated in terms of pseudo-compound parallelepipeds and dual lattices. An interested reader can find this interpretation performed in detail in [14] (see also [15] for more information about pseudo-compounds in the context of Mahler’s theory). In this Section we simply formulate the corresponding version of Mahler’s theorem (Theorem 8 below).

Definition 7. Given positive numbers $\lambda_1, \ldots, \lambda_d$, consider the parallelepiped

$$\mathcal{P} = \left\{ z = (z_1, \ldots, z_d) \in \mathbb{R}^d \mid |z_i| \leq \lambda_i, \ i = 1, \ldots, d \right\}.$$
We call the parallelepiped
\[ P^* = \left\{ z = (z_1, \ldots, z_d) \in \mathbb{R}^d \mid |z_i| \leq \frac{1}{\lambda_i} \prod_{j=1}^{d} \lambda_j, \ i = 1, \ldots, d \right\} \]
the pseudo-compound of \( P \).

We remind that, given a full-rank lattice \( \Lambda \) in \( \mathbb{R}^d \), its dual is defined as
\[ \Lambda^* = \left\{ z \in \mathbb{R}^d \mid \langle z, w \rangle \in \mathbb{Z} \text{ for all } w \in \Lambda \right\}, \]
where \( \langle \cdot, \cdot \rangle \) denotes inner product.

**Theorem 8** (Interpretation of Mahler’s theorem). Let \( P \) be as in Definition 7. Let \( \Lambda \) be a full-rank lattice in \( \mathbb{R}^d \), \( d \geq 2 \), \( \det \Lambda = 1 \). Then
\[ P^* \cap \Lambda^* \neq \{0\} \implies cP \cap \Lambda \neq \{0\}, \]
where \( c = d^{\frac{1}{2(d-1)}} \) and \( 0 \) denotes the origin.

With the given value of \( c \) Theorem 8 was proved in [14]. In Mahler’s formulation \( c \) equals \( d - 1 \). However, for our purposes any constant depending only on \( d \) will do, as we are concerned only with exponents.

### 5.2 Dual lattices and two-parametric families of parallelepipeds

Set \( d = m + n \). Then, as assumed in the beginning of the paper, \( d \geq 3 \). Define
\[ \Lambda = \begin{pmatrix} I_m & \Theta \\ -\Theta & I_n \end{pmatrix} \mathbb{Z}^d. \]

Then the dual lattice is given by
\[ \Lambda^* = \begin{pmatrix} I_m & \Theta^\top \\ \Theta & I_n \end{pmatrix} \mathbb{Z}^d. \]

For each \( t > 1, \gamma \geq 1, \ s > 1, \ \delta \geq 1 \) set
\[ P(t, \gamma) = \left\{ z = (z_1, \ldots, z_d) \in \mathbb{R}^d \mid \begin{array}{c} |(z_1, \ldots, z_m)|_{\sigma} \leq t \\ |(z_m+1, \ldots, z_d)|_{\rho} \leq t^{-\gamma} \end{array} \right\}, \]
\[ Q(s, \delta) = \left\{ z = (z_1, \ldots, z_d) \in \mathbb{R}^d \mid \begin{array}{c} |(z_1, \ldots, z_m)|_{\sigma} \leq s^{-\delta} \\ |(z_m+1, \ldots, z_d)|_{\rho} \leq s \end{array} \right\}. \]

We can reformulate Definition 3 in the following way.

**Proposition 2.**
\[ \omega_{\sigma, \rho}(\Theta) = \sup \left\{ \gamma \geq 1 \mid \text{there is } t, \text{ however large, s.t. } P(t, \gamma) \cap \Lambda \neq \{0\} \right\}, \]
\[ \omega_{\rho, \sigma}(\Theta^\top) = \sup \left\{ \delta \geq 1 \mid \text{there is } s, \text{ however large, s.t. } Q(s, \delta) \cap \Lambda^* \neq \{0\} \right\}. \]

Now, the preparations are complete, and we are ready to prove Theorem 1.
5.3 Proof of Theorem 1

For every \( t, \gamma \in \mathbb{R} \) such that \( t > 1, 1 \leq \gamma < (1 - \rho_n)^{-1} \) set
\[
s = t^{\gamma - (\gamma - 1)\rho_n^{-1}}, \quad \delta = \frac{1 + (\gamma - 1)\sigma_m^{-1}}{\gamma - (\gamma - 1)\rho_n^{-1}}.
\]
Then
\[
s^{-\delta \sigma_j} \leq t^{-\sigma_j + 1 - \gamma}, \quad j = 1, \ldots, m,
\]
\[
s^{\rho_i} \leq t^{\rho_i + 1 - \gamma}, \quad i = 1, \ldots, n,
\]
whence
\[
Q(s, \delta) \subseteq P(t, \gamma)^*, \quad (18)
\]
\[
\mathcal{P}(t, \gamma)^* = \left\{ z = (z_1, \ldots, z_d) \in \mathbb{R}^d \mid |z_j| \leq t^{-\sigma_j + 1 - \gamma}, \quad j = 1, \ldots, m \right\} \cap \left\{ z_{m+i} \leq t^{\rho_i + 1 - \gamma}, \quad i = 1, \ldots, n \right\}.
\]

Combining (18) with Theorem 8, we get the key relation
\[
Q(s, \delta) \cap \Lambda^* \neq \{0\} \implies c\mathcal{P}(t, \gamma) \cap \Lambda \neq \{0\}. \quad (19)
\]

The assumption \( 1 \leq \gamma < (1 - \rho_n)^{-1} \) guarantees that the correspondence \( \gamma \to \delta \) given by (17) generates a one-to-one monotonic mapping \([1, (1 - \rho_n)^{-1}) \to [1, +\infty)\). Particularly, \( s \) and \( t \) tend to \(+\infty\) simultaneously, and \( \gamma \) can be expressed as a function of \( \delta \).

Thus, in view of Proposition 2, (19) implies that
\[
\omega_{\sigma, \rho}(\Theta^\top) \geq \delta \implies \omega_{\sigma, \rho}(\Theta) \geq \gamma = \frac{(\sigma_m^{-1} - 1) + \rho_n^{-1}\delta}{\sigma_m^{-1} + (\rho_n^{-1} - 1)\delta}. \quad (20)
\]

Hence
\[
\omega_{\sigma, \rho}(\Theta) \geq \frac{(\sigma_m^{-1} - 1) + \rho_n^{-1}\omega_{\sigma, \rho}(\Theta^\top)}{\sigma_m^{-1} + (\rho_n^{-1} - 1)\omega_{\sigma, \rho}(\Theta^\top)}.
\]

Swapping \((\sigma, \rho, \Theta)\) for \((\rho, \sigma, \Theta^\top)\) gives (6). Theorem 1 is proved.

It is clear that (19) also provides an analogue of (6) for uniform exponents, but there is no need for such an analogue, as we are about to prove a stronger statement, namely, Theorem 2.

6 Uniform transference with weights

In this Section we prove Theorem 2.

6.1 Analogue of Theorem 8 for second pseudo-compounds

As we noticed in the beginning of Section 5.1, Theorem 8 is the core of any transfer-
ence theorem for regular exponents. But if we want to prove something about uniform
exponents, we must use a more delicate tool. In this Section we propose an analogue
of Theorem 8 dealing with pairs of lattice points (Theorem 9 below). The idea of this
approach was used by the author in [5] to prove (4).

Let \( e_1, \ldots, e_d \) be the standard basis of \( \mathbb{R}^d \). Let us associate to each \( Z \in \bigwedge^2 \mathbb{R}^d \) its representation
\[
Z = \sum_{1 \leq i < j \leq d} Z_{ij} e_i \wedge e_j.
\]
**Definition 8.** Given positive numbers \( \lambda_1, \ldots, \lambda_d \), consider the parallelepiped

\[
P = \left\{ z = (z_1, \ldots, z_d) \in \mathbb{R}^d \left| |z_i| \leq \lambda_i, \; i = 1, \ldots, d \right. \right\}.
\]

We call the parallelepiped

\[
P^\circ = \left\{ Z \in \bigwedge^2 \mathbb{R}^d \left| |Z_{ij}| \leq \frac{1}{\lambda_i \lambda_j} \prod_{k=1}^d \lambda_k, \; 1 \leq i < j \leq d \right. \right\}
\]

the second pseudo-compound of \( P \).

**Remark 2.** Our terminology differs a bit from that which W. M. Schmidt uses in his exposition of Mahler’s theory in [15]. Instead of \( P^\ast \) and \( P^\circ \) he actually considers \( \ast P^\ast \) and \( \ast P^\circ \) – the images of \( P^\ast \) and \( P^\circ \) under the action of the Hodge star operator. Respectively, he calls them the \((d - 1)\)-th and the \((d - 2)\)-th pseudo-compounds of \( P \). It agrees well with Mahler’s definition of compound bodies [16], [17], but in our context it seems more appropriate to omit the Hodge star and reverse the numeration order.

Given a full-rank lattice \( \Lambda \) in \( \mathbb{R}^d \) and its dual \( \Lambda^\ast \), let us denote by \( \Lambda^\circ \) the set of decomposable elements of the lattice \( \bigwedge^2 \Lambda^\ast \), i.e.

\[
\Lambda^\circ = \left\{ z_1 \wedge z_2 \left| z_1, z_2 \in \Lambda^\ast \right. \right\}.
\]

**Theorem 9.** Let \( P \) be as in Definition 8. Let \( \Lambda \) be a full-rank lattice in \( \mathbb{R}^d \), \( \det \Lambda = 1 \). Then

\[
P^\circ \cap \Lambda^\circ \neq \{0\} \implies c' P \cap \Lambda \neq \{0\},
\]

where \( c' = \left( \frac{d(d-1)}{2} \right)^{1/(2(d-2))} \) and \( 0 \) denotes the origin.

Proof of Theorem 9 is based on three facts. The first one is Minkowski’s convex body theorem, the second one is Vaaler’s theorem [18], which states that the \( k \)-dimensional volume of any \( k \)-dimensional central section of a unit cube is not less than 1, and the third one is the following observation.

**Proposition 3.** Let \( \Lambda \) be a full-rank lattice in \( \mathbb{R}^d \), \( \det \Lambda = 1 \). Let \( L \) be a \( k \)-dimensional subspace of \( \mathbb{R}^d \). Suppose \( \Gamma = L \cap \Lambda \) has rank \( k \). Consider the orthogonal complement \( L^\perp \) and denote \( \Gamma^\perp = L^\perp \cap \Lambda^\ast \). Then \( \Gamma^\perp \) has rank \( d - k \) and

\[
\det \Gamma^\perp = \det \Gamma.
\]

Since the lattice is assumed to be unimodular, Proposition 3 by linearity reduces to the case \( \Lambda = \mathbb{Z}^d \), which seems to be a rather classical statement. The corresponding proof can be found at least in [19] and [5].

Proof of Theorem 9. Set

\[
v = \frac{1}{2} (\text{vol} \; P)^{1/d} = \prod_{k=1}^d \lambda_k^{1/d}.
\]

Consider the diagonal matrix \( T = \text{diag}(\lambda_1/v, \ldots, \lambda_d/v) \). Then \( T^{-1} P = v B \), where

\[
B = \left\{ z = (z_1, \ldots, z_d) \in \mathbb{R}^d \left| |z_i| \leq 1, \; i = 1, \ldots, d \right. \right\}.
\]
As \((T^{-1}(\Lambda))^* = T^T(\Lambda^*) = T(\Lambda^*)\), the second compound matrix \(T^{(2)}\) is acting on \(\Lambda^2 \mathbb{R}^d\) thought of as the ambient space for \(\Lambda^2 \mathbb{R}^d\). Since
\[
T^{(2)}(\mathcal{P}^*) = v^{d-2} \mathcal{B}^*,
\]
where
\[
\mathcal{B}^* = \left\{ Z \in \Lambda^2 \mathbb{R}^d \left| |Z_{ij}| \leq 1, \ 1 \leq i < j \leq d \right\},
\]
we are to show that
\[
(v^{d-2} \mathcal{B}^*) \cap (T^{-1}(\Lambda))^* \neq \{0\} \implies (c'v \mathcal{B}) \cap (T^{-1}(\Lambda)) \neq \{0\}.
\]
(21)
Now, the left hand side of (21) implies that there is a sublattice \(\Gamma\) in \((T^{-1}(\Lambda))^*\) of rank 2 with
\[
\det \Gamma \leq v^{d-2} \text{diam} \mathcal{B}^* = v^{d-2} \left( \frac{d(d-1)}{2} \right)^{1/2} = (c'v)^{d-2}.
\]
The determinant of \((T^{-1}(\Lambda))^*\) equals 1, so by Proposition 3 there is a sublattice \(\Gamma^\perp\) in \(T^{-1}(\Lambda)\) of rank \(d-2\) with
\[
\det \Gamma^\perp = \det \Gamma \leq (c'v)^{d-2}.
\]
By Vaaler’s theorem the \((d-2)\)-dimensional volume of \(S = \text{span}_{\mathbb{R}}(\Gamma^\perp) \cap (c'v \mathcal{B})\) is not less than \((2c'v)^{d-2}\). Applying Minkowski’s convex body theorem, we get that there is a nonzero point of \(\Gamma^\perp\) in \(S\), which completes the proof.

6.2 ‘Nodes’ and ‘leaves’: main parametric construction

Let us adopt the notation of Section 5.2. Our proof of Theorem 2 bases upon a construction involving parallelepipeds \(Q(\cdot, \cdot)\) defined in Section 5.2. In this Section we describe this construction.

Let us fix arbitrary \(s, \delta, \alpha \in \mathbb{R}\) such that
\[
s > 1, \ \delta \geq 1, \ 1 \leq \alpha \leq \delta,
\]
and denote
\[
S = s^{\delta/\alpha}.
\]
To each \(r > 1\) let us associate the parallelepiped
\[
Q_r = Q(r, \alpha \log(sS/r)/\log r) = \left\{ z \in \mathbb{R}^d \left| (z_1, \ldots, z_m)_{\sigma} \leq (sS/r)^{-\alpha} \right\} \right\},
\]
Consider the following three families of parallelepipeds:
\[
\mathcal{S} = \mathcal{S}(s, \delta, \alpha) = \left\{ Q_r \left| s \leq r \leq \sqrt{sS} \right\}, \right\}
\]
\[
\mathcal{A} = \mathcal{A}(s, \delta, \alpha) = \left\{ Q_r \left| \sqrt{sS} \leq r \leq S \right\}, \right\}
\]
\[
\mathcal{L} = \mathcal{L}(s, \delta, \alpha) = \left\{ Q(r, \alpha) \left| s \leq r \leq S \right\} \right\}.
\]
Let us call \(\mathcal{S}\) the ‘stem’ family, \(\mathcal{A}\) the ‘anti-stem’ family, \(\mathcal{L}\) the ‘leaves’ family. Let us call each element of \(\mathcal{S}\) a ‘node’, each element of \(\mathcal{A}\) an ‘anti-node’, each element of \(\mathcal{L}\) a ‘leaf’.

We say that a ‘node’ or an ‘anti-node’ \(Q_r\) produces a ‘leaf’ \(Q(r', \alpha)\) if
\[
r' = r \quad \text{or} \quad r' = sS/r.
\]
Proposition 4.

(i) If $r < r'$, then $Q_r \subset Q_{r'}$. For the root ‘node’ $Q_s$ we have $Q_s = Q(s, \delta)$.

(ii) For each $r \in \mathbb{R}$, $s \leq r \leq \sqrt{sS}$, the ‘node’ $Q_r$ and the ‘anti-node’ $Q_{sS/r}$ produce exactly two ‘leaves’

$$Q(r, \alpha) \quad \text{and} \quad Q(sS/r, \alpha),$$

whose intersection is the ‘node’ and whose union is contained in the ‘anti-node’.

(iii) Each ‘leaf’ $Q(r, \alpha)$ is produced by exactly one ‘node’ $Q_r$ and one ‘anti-node’ $Q_{sS/r}$ with

$$r' = \begin{cases} 
    r, & \text{if} \ r \leq \sqrt{sS} \\
    sS/r, & \text{if} \ r \geq \sqrt{sS} 
\end{cases}$$

Proof. All three statements follow directly from the definition of $Q_r$ and the definition of producing.

We illustrate Proposition 4 by Figure 2, where we use $u$ and $v$ to denote $|(z_{m+1}, \ldots, z_d)|_\rho$ and $|(z_1, \ldots, z_m)|_\sigma$ respectively.

![Figure 2: A ‘node’, its ‘anti-node’, and their ‘leaves’](image)

Lemma 1. Suppose $\alpha < \delta$. Let $\Sigma$ be an arbitrary discrete subset of $\mathbb{R}^d$ with the following two properties:

(i) every ‘leaf’ in $\Sigma$ contains a point of $\Sigma$;
(ii) the root ‘node’ $Q_s$ contains no points of $\Sigma$.

Then there is a ‘leaf’ containing two distinct points of $\Sigma$, one of which lies in the ‘node’ that produces the ‘leaf’.
Proof. Denote by \( r_0 \) the smallest \( r \) such that the ‘node’ \( Q_r \) contains a point \( v \) of \( \Sigma \). The existence of \( r_0 \) follows from the fact that \( Q_{\sqrt{sS}} \in L \). Then \( r_0 > s \) and \( v \) lies on the boundary of \( Q_{r_0} \). Since, by Proposition 4, this ‘node’ coincides with the intersection of its ‘leaves’, \( v \) lies on the boundary of one of them, say, \( Q(r_1, \alpha) \). Since \( r_1 \) equals either \( r_0 \), or \( sS/r_0 \), we have

\[
  s < r_0 \leq r_1 \leq sS/r_0 < S.
\]

If there are no other points of \( \Sigma \) in \( Q(r_1, \alpha) \), let us perturb this ‘leaf’ by adding a small \( \varepsilon \) to \( r_1 \), so that \( v \) is no longer in \( Q(r_1 + \varepsilon, \alpha) \). Since ‘leaves’ are compact and \( \Sigma \) is discrete, for \( \varepsilon \) small enough no other points of \( \Sigma \) will enter \( Q(r_1 + \varepsilon, \alpha) \). This contradicts property (i), which proves that along with \( v \) there is another point of \( \Sigma \) in \( Q(r_1, \alpha) \), distinct from \( v \). \( \square \)

**Lemma 2.** Within the hypothesis of Lemma 1 there are two distinct points of \( \Sigma \), such that one of them lies in a ‘node’ \( Q_r \) and the other one lies in the corresponding ‘anti-node’ \( Q_{sS/r} \).

*Proof.* Consider the ‘leaf’ provided by Lemma 1. Then the ‘node’ and the ‘anti-node’ that produce this leaf satisfy the statement of the Lemma. \( \square \)

Lemma 2 is the key ingredient provided by the ‘stem’-and-‘leaves’ approach for proving Theorem 2. The only additional statement we need to formulate before we can proceed to the proof itself is the following technical Lemma.

**Lemma 3.** Suppose \( a, b \in \mathbb{R}^2 \) satisfy

\[
  a \in \left\{ (z_1, z_2) \in \mathbb{R}^2 \mid |z_1| \leq a, |z_2| \leq b \right\}, \quad b \in \left\{ (z_1, z_2) \in \mathbb{R}^2 \mid |z_1| \leq A, |z_2| \leq B \right\}.
\]

Then \( |a \wedge b| \leq 2 \max(aB, bA) \).

The proof is elementary and we leave it to the reader.

### 6.3 Proof of Theorem 2

Let us keep on holding to the notation of Section 5.2 and reformulate Definition 4 the same way we reformulated Definition 3.

**Proposition 5.**

\[
  \hat{\omega}_{\sigma, \rho}(\Theta) = \sup \left\{ \gamma \geq 1 \mid \text{for every } t \text{ large enough } \mathcal{P}(t, \gamma) \cap \Lambda = \{0\} \right\},
\]

\[
  \hat{\omega}_{\rho, \sigma}(\Theta^\top) = \sup \left\{ \alpha \geq 1 \mid \text{for every } r \text{ large enough } \mathcal{Q}(r, \alpha) \cap \Lambda^* = \{0\} \right\}.
\]

For every \( \alpha \geq 1 \) set

\[
  \gamma = \begin{cases} 
    \frac{1 - \rho_n \alpha^{-1}}{1 - \rho_n} & \text{if } \alpha \geq \rho_n/\sigma_m \\
    \frac{1 - \sigma_m}{1 - \sigma_m \alpha} & \text{if } \alpha \leq \rho_n/\sigma_m
  \end{cases}
\]

(22)
Then
\[ 1 \leq \gamma < (1 - \rho_n)^{-1}. \]

As in Section 5.3, for every \( t > 1 \) set
\[ s = t^{\gamma - (\gamma - 1)\rho_n^{-1}}, \quad \delta = \frac{1 + (\gamma - 1)\sigma_m^{-1}}{\gamma - (\gamma - 1)\rho_n^{-1}}. \]  
(23)

It is a simple exercise to show that with this choice of parameters we have
\[ \alpha = 1 \iff \gamma = 1 \iff \delta = 1 \quad \text{and} \quad \alpha < \delta \iff \alpha > 1. \]

We will prove Theorem 2 by showing that
\[ \hat{\omega}_{s, \rho}(\Theta^\top) \geq \alpha \implies \hat{\omega}_{\sigma, \rho}(\Theta) \geq \gamma. \]

If \( \hat{\omega}_{s, \rho}(\Theta^\top) = 1 \), we are to take \( \alpha = \gamma = 1 \), which makes both inequalities trivial. So we may assume hereafter that
\[ \hat{\omega}_{s, \rho}(\Theta^\top) > 1. \]  
(24)

Let \( S, \Theta, A, L \) be as in Section 6.2.

**Lemma 4.** Suppose \( \alpha > 1 \). Suppose that
(i) every ‘leaf’ in \( L \) contains a nonzero point of \( \Lambda^* \);
(ii) the root ‘node’ \( Q_s = Q(s, \delta) \) contains no nonzero points of \( \Lambda^* \).
Then
\[ (2P(t, \gamma)^\circ \cap \Lambda^*) \neq \{0\}. \]

**Proof.** By Definition 8
\[ P(t, \gamma)^\circ = \left\{ Z \in \Lambda^* \mathbb{R}^d \left| \begin{array}{ll}
|Z_{ij}| \leq t^{-\sigma_i - \sigma_j + 1 - \gamma}, & 1 \leq i < j \leq m \\
|Z_{m+i,m+j}| \leq t^{\rho_i + \rho_j + 1 - \gamma}, & 1 \leq i < j \leq n \\
|Z_{j,m+i}| \leq t^{-\sigma_j + \rho_i + 1 - \gamma}, & 1 \leq j \leq m, \, 1 \leq i \leq n
\end{array} \right. \right\} \]
with no first or second line of inequalities if \( m = 1 \) or \( n = 1 \) respectively.

Since \( \alpha > 1 \), we have \( \alpha < \delta \). Let us apply Lemma 2 with \( \Sigma = \Lambda^* \setminus \{0\} \). Then there are two distinct nonzero points \( v_1, v_2 \in \Lambda^* \) and an \( r \in \mathbb{R}, s < r < S \), such that
\[ v_1 \in Q_r = \left\{ z \in \mathbb{R}^d \left| \begin{array}{l}
|z_j| \leq (sS/r)^{-\sigma_j}, \quad j = 1, \ldots, m \\
|z_{m+i}| \leq \rho_i, \quad i = 1, \ldots, n
\end{array} \right. \right\} \],
\[ v_2 \in Q_{sS/r} = \left\{ z \in \mathbb{R}^d \left| \begin{array}{l}
|z_j| \leq (sS/r)^{-\rho_i}, \quad j = 1, \ldots, m \\
|z_{m+i}| \leq (sS/r)^{\rho_i}, \quad i = 1, \ldots, n
\end{array} \right. \right\} \].

Let us show that
\[ v_1 \wedge v_2 \in 2P(t, \gamma)^\circ. \]

This will prove the Lemma.

We are to show that the coefficients in the representation
\[ v_1 \wedge v_2 = \sum_{1 \leq i < j \leq d} V_{ij}e_i \wedge e_j \]
satisfy
\[
|V_{ij}| \cdot t^{\sigma_i+\sigma_j-1+\gamma} \leq 2, \quad 1 \leq i < j \leq m, \tag{25}
\]
\[
|V_{m+i+m+j}| \cdot t^{-\rho_i-\rho_j-1+\gamma} \leq 2, \quad 1 \leq i < j \leq n, \tag{26}
\]
\[
|V_{j+m+i}| \cdot t^{\sigma_j-\rho_i-1+\gamma} \leq 2, \quad 1 \leq j \leq m, 1 \leq i \leq n. \tag{27}
\]
We shall make use of the inequalities
\[
\gamma(1 - \rho_n) - (1 - \rho_n \alpha^{-1}) \leq 0,
\]
\[
\gamma(1 - \sigma_m \alpha) - (1 - \sigma_m) \leq 0,
\tag{28}
\]
that, as follows from (22), hold for every $\alpha \geq 1$.

**Checking (25)** By Lemma 3 for $V_{ij}$ with $1 \leq i < j \leq m$ we have
\[
|V_{ij}| \leq 2 \max(r^{-\alpha \sigma_i} (sS/r)^{-\alpha \sigma_j}, r^{-\alpha \sigma_j} (sS/r)^{-\alpha \sigma_i}) \leq 2 \max(s^{-\alpha \sigma_i}, s^{-\alpha \sigma_j}, s^{-\alpha \sigma_i}, s^{-\alpha \sigma_j}) = 2 \max(s^{-\alpha \sigma_i}, s^{-\alpha \sigma_j}, s^{-\alpha \sigma_i}, s^{-\alpha \sigma_j}).
\]
It follows from (23) and (28) that
\[
s^{-\alpha \sigma_i-\delta \sigma_i} t^{\alpha \sigma_i+\sigma_j+\gamma} = t^{(\gamma-(\gamma-1)\rho_n^{-1})\alpha \sigma_i-(1+(\gamma-1)\sigma_m^{-1})\sigma_j} t^{\alpha \sigma_i+\sigma_j-1+\gamma} =
\]
\[
n^{-\rho_i} t^{(\gamma-(\gamma-1)\rho_n^{-1})\alpha \sigma_i+(\gamma-1)(1-\sigma_m^{-1})\sigma_j} \leq
t^{\gamma-(\gamma-1)\rho_n^{-1})\alpha \sigma_i} = n^{-\rho_i} t^{\gamma-(\gamma-1)\rho_n^{-1})\alpha \sigma_i} = 1.
\]
Similarly, interchanging $i$ and $j$, we get $s^{-\alpha \sigma_j-\delta \sigma_j} t^{\alpha \sigma_i+\sigma_j-1+\gamma} \leq 1$. Thus, (25) is fulfilled.

**Checking (26)** By Lemma 3 for $V_{m+i+m+j}$ with $1 \leq i < j \leq n$ we have
\[
|V_{m+i+m+j}| \leq 2 \max(r^{-\rho_i} (sS/r)^{-\rho_j}, r^{-\rho_j} (sS/r)^{-\rho_i}) \leq 2 \max(s^{-\rho_i}, s^{-\rho_j}, s^{-\rho_i}) = 2 \max(s^{-\rho_i}, s^{-\rho_j}, s^{-\rho_i}).
\]
It follows from (23) and (28) that
\[
s^{\rho_i+\delta \rho_i} t^{-\rho_i-\rho_j-1+\gamma} = t^{(\gamma-(\gamma-1)\rho_n^{-1})\rho_i+(\gamma-1)(1-\rho_n^{-1})\rho_j} t^{-\rho_i-\rho_j-1+\gamma} =
\]
\[
n^{-\rho_i} t^{(\gamma-(\gamma-1)\rho_n^{-1})\rho_i+(\gamma-1)(1-\rho_n^{-1})\rho_j} \leq
t^{\gamma-(\gamma-1)\rho_n^{-1})\rho_i} = t^{\rho_i} t^{\gamma-(\gamma-1)\rho_n^{-1})\rho_i} = 1.
\]
Similarly, interchanging $i$ and $j$, we get $s^{\rho_j+\delta \rho_j} t^{-\rho_i-\rho_j-1+\gamma} \leq 1$. Thus, (26) is fulfilled.

**Checking (27)** By Lemma 3 for $V_{j+m+i}$ with $1 \leq j \leq m$, $1 \leq i \leq n$ we have
\[
|V_{j+m+i}| \leq 2 \max(r^{-\alpha \sigma_j+\rho_i}, (sS/r)^{-\alpha \sigma_j+\rho_i}) \leq 2 \max(s^{-\alpha \sigma_j+\rho_i}, s^{-\delta \sigma_j+\rho_i}) = 2 \max(s^{-\alpha \sigma_j+\rho_i}, s^{-\delta \sigma_j+\rho_i}).
\]
It follows from (23) and (28) that
\[ s^{-\alpha_j} - \rho_j t^{\sigma_j - \gamma_j - 1 + \gamma} = t^{-\gamma_j - (\gamma_j - 1)\rho_j - 1 + \gamma} \]
\[ = t^{\sigma_j - (\gamma_j - 1)\rho_j - 1 + \gamma} \leq t^{\sigma_j - (\gamma_j - 1)\rho_j - 1 + \gamma} \leq t^{\sigma_j - (\gamma_j - 1)\rho_j - 1 + \gamma} \leq 1 \]
and
\[ s^{-\sigma_j + \delta_j + \rho_j} - \rho_j t^{\sigma_j - \gamma_j - 1 + \gamma} = t^{1 + (\gamma_j - 1)\sigma_j - \gamma_j - 1 + \gamma} \]
\[ = t^{-\gamma_j + (1 + (\gamma_j - 1)\sigma_j - \gamma_j - 1 + \gamma} \leq t^{-\gamma_j + (1 + (\gamma_j - 1)\sigma_j - \gamma_j - 1 + \gamma} \leq 1 \]
Thus, (27) is also fulfilled.

Hence \( v_1 \land v_2 \in \mathcal{P}(t, \gamma) \), which proves the Lemma. \( \square \)

Having Lemma 4 and Theorem 9, we can prove Theorem 2 in the blink of an eye. As we showed in Section 5.3,
\[ \mathcal{Q}(s, \delta) \cap \Lambda^* \neq \{0\} \implies \mathcal{C}(t, \gamma) \cap \Lambda \neq \{0\}. \] (29)
This observation, Lemma 4, and Theorem 9 give us the key relation
\[ \mathcal{Q}(r, \alpha) \cap \Lambda^* \neq \{0\} \text{ for every } r \in [s, S] \implies \mathcal{C}(t, \gamma) \cap \Lambda \neq \{0\}, \] (30)
where \( c'' = (2d(d-1))^{\frac{1}{2(d-2)}} \). Indeed, if \( \mathcal{Q}(s, \delta) \) contains a nonzero point of \( \Lambda^* \), then we are done by (29), since \( c < c'' \). Otherwise, the hypothesis of Lemma 4 is satisfied, whence \( \mathcal{P}(t, \gamma) \cap \Lambda^* \neq \{0\} \). Since \( 2\mathcal{P}(t, \gamma) = \mathcal{C'(t, \gamma)} \) with \( c''' = 2^{d/(d-2)} \), Theorem 9 gives \( \mathcal{C'(t, \gamma)} \cap \Lambda \neq \{0\} \). This proves (30).

Assuming (24), let us choose an arbitrary \( \alpha \) such that \( 1 < \alpha < \hat{\omega}_{\rho, \sigma}(\Theta) \). Then for every \( s \) large enough \( \mathcal{Q}(s, \alpha) \cap \Lambda^* \neq \{0\} \). As we already noticed in Section 5.3, \( s \) and \( t \) tend to \( +\infty \) simultaneously. Thus, in view of Proposition 5 relation (30) implies that
\[ \hat{\omega}_{\rho, \sigma}(\Theta) > \alpha \implies \hat{\omega}_{\sigma, \rho}(\Theta) > \gamma. \]
Since \( \gamma \) continuously depends on \( \alpha \), we get for every \( \alpha \geq 1 \)
\[ \hat{\omega}_{\rho, \sigma}(\Theta) > \alpha \implies \hat{\omega}_{\sigma, \rho}(\Theta) \geq \gamma. \]
Hence
\[ \hat{\omega}_{\sigma, \rho}(\Theta) \geq \begin{cases} 
\frac{1 - \rho_n \hat{\omega}_{\rho, \sigma}(\Theta)}{\hat{\omega}_{\rho, \sigma}(\Theta)} - 1 & \text{if } \hat{\omega}_{\rho, \sigma}(\Theta) \geq \rho_n/\sigma_m \\
\frac{1 - \rho_n}{\hat{\omega}_{\rho, \sigma}(\Theta)} & \text{if } \hat{\omega}_{\rho, \sigma}(\Theta) \leq \rho_n/\sigma_m \\
\frac{1 - \sigma_m}{\hat{\omega}_{\rho, \sigma}(\Theta)} & \text{if } \hat{\omega}_{\rho, \sigma}(\Theta) \leq \rho_n/\sigma_m 
\end{cases} \]
Swapping \( (\alpha, \rho, \Theta) \) for \( (\rho, \sigma, \Theta) \) gives (7). Theorem 2 is proved.
7 Variety of inequalities generalising Dyson’s theorem

As we noticed in Section 5.1, all the Dyson-like transference theorems actually base upon Theorem 8. In the weighted setting, with the notation of Sections 5.1, 5.2, we can describe the scheme of a possible proof rather generally by the following diagram:

\[ Q(s, \delta) \subseteq \mathcal{P}^* \rightarrow \mathcal{P} \subseteq \mathcal{P}(t, \gamma). \] (31)

This diagram means that we find an appropriate parallelepiped \( \mathcal{P} \), to which we can apply Theorem 8, and then choose \( t, \gamma, s, \delta \) providing the inclusions in (31). Given such \( \mathcal{P}, t, \gamma, s, \delta \), we can claim that, if there is a nonzero point of \( \Lambda^* \) in \( Q(s, \delta) \), then there is a nonzero point of \( \Lambda \) in \( c\mathcal{P}(t, \gamma) \). In our proof of Theorem 1 (see Section 5) we chose \( \mathcal{P} = \mathcal{P}(t, \gamma) \).

However, generally one can try and choose another \( \mathcal{P} \).

Let us consider arbitrary \( t, \gamma, s, \delta \in \mathbb{R} \) such that

\[ t > 1, \quad \gamma \geq 1, \quad s > 1, \quad 1 \leq \gamma < (1 - \rho_n)^{-1}, \]

and arbitrary positive \( \lambda_1, \ldots, \lambda_d \) determining \( \mathcal{P} \) by

\[ \mathcal{P} = \left\{ \mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d \mid |z_i| \leq \lambda_i, \ i = 1, \ldots, d \right\}. \]

Let us define \( a, b, \mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n, \) and \( \Delta \) by

\[
\begin{align*}
    s &= t^\alpha, & \delta &= b/a, \\
    \lambda_j &= t^{\mu_j}, & \lambda_{m+i} &= t^{\nu_i}, & j = 1, \ldots, m, & i = 1, \ldots, n, \\
    \Delta &= \sum_{1 \leq j \leq m} \mu_j + \sum_{1 \leq j \leq n} \nu_i.
\end{align*}
\]

Consider the diagonal matrix \( T = \text{diag}(\lambda_1, \ldots, \lambda_d) \). Then \( T^{-1} \mathcal{P} = \mathcal{B} \) and \( T \mathcal{P}^* = t^\Delta \mathcal{B} \), where

\[ \mathcal{B} = \left\{ \mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{R}^d \mid |z_i| \leq 1, \ i = 1, \ldots, d \right\}. \]

Hence the inclusions in (31) are equivalent to \( t^{-\Delta} T Q(s, \delta) \subseteq \mathcal{B} \subseteq T^{-1} \mathcal{P}(t, \gamma) \). Or, more explicitly,

\[
\begin{align*}
    t^{-\Delta - \mu_j - \delta \sigma_j} &\leq 1, & t^{-\mu_j + \sigma_j} &\geq 1, & j = 1, \ldots, m, \\
    t^{-\Delta + \nu_i + \alpha \rho_i} &\leq 1, & t^{-\nu_i - \gamma \rho_i} &\geq 1, & i = 1, \ldots, n.
\end{align*}
\]

Thus, the inclusions in (31) take place if and only if for every \( i \) and \( j \) we have

\[
\begin{align*}
    \mu_j &\leq \sigma_j, & \nu_i &\leq -\gamma \rho_i, & (32) \\
    b &\geq (\mu_j - \Delta)/\sigma_j, & a &\leq (\Delta - \nu_i)/\rho_i. & (33)
\end{align*}
\]

We are interested in \( \delta = b/a \) to be as small as possible, so for every \( \mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n \) it is best to set

\[
b = \max_{1 \leq j \leq m} \frac{\mu_j - \Delta}{\sigma_j}, \quad a = \min_{1 \leq i \leq n} \frac{\Delta - \nu_i}{\rho_i}. \]

Furthermore, notice that under the condition (32) we have \( \Delta - \mu_j \leq -\sigma_j + 1 - \gamma \) and \( \Delta - \nu_i \leq \gamma \rho_i + 1 - \gamma \), since \( \sigma_1 + \ldots + \sigma_m = \rho_1 + \ldots + \rho_n = 1 \). Hence

\[
b = \max_{1 \leq j \leq m} \frac{\mu_j - \Delta}{\sigma_j} \geq \max_{1 \leq j \leq m} \left( 1 + (\gamma - 1)\sigma_j^{-1} \right) = 1 + (\gamma - 1)\sigma_m^{-1}, \]

17
\[ a = \min_{1 \leq i \leq n} \frac{\Delta - \nu_i}{\rho_i} \leq \min_{1 \leq i \leq n} (\gamma - (\gamma - 1)\rho_i^{-1}) = \gamma - (\gamma - 1)\rho_n^{-1}, \]

with the equalities attained if for every \( i \) and \( j \) we have \( \mu_j = \sigma_j \) and \( \nu_i = -\gamma \rho_i \), i.e. if \( \mathcal{P} = \mathcal{P}(t, \gamma) \). These values of \( b \) and \( a \) provide us with \( \delta \) we used in Section 5.3.

However, if the choice of \( \mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n \) is not optimal, for instance, if at least one of the \( m + n \) inequalities in (32) is strict, then either \( b \) is bounded away from \( 1 + (\gamma - 1)\sigma_m^{-1} \), or \( a \) is bounded away from \( \gamma - (\gamma - 1)\rho_n^{-1} \). Thus, if \( \mathcal{P} \) is chosen as a proper subset of \( \mathcal{P}(t, \gamma) \), then \( \delta \) will be greater than \( (1 + (\gamma - 1)\sigma_m^{-1})/(\gamma - (\gamma - 1)\rho_n^{-1}) \). This is the reason the generalisation of Dyson’s inequality obtained in [6] happens to be weaker than the one provided by Theorem 1.

We leave it to the reader to prove that the weakest possible inequality that can be obtained in such a way corresponds to \( \mathcal{P} \) chosen so that \( \mathcal{P}^* = \mathcal{Q}(s, \delta) \).

We end up with a remark that for the ‘non-twisted’ case no problem of this kind arises, as in that case all the inequalities in (32), (33) can be turned into equalities, providing thus a very nice relation \( \mathcal{Q}(s, \delta) = \mathcal{P}(t, \gamma)^* \).

Acknowledgements. The author is a Young Russian Mathematics award winner and would like to thank its sponsors and jury.

References

[1] A. Ya. Khintchine Über eine Klasse linearer Diophantischer Approximationen. Rend. Sirc. Mat. Palermo, 50 (1926), 170–195.

[2] F. J. Dyson On simultaneous Diophantine approximations. Proc. London Math. Soc., (2) 49 (1947), 409–420.

[3] V. Jarník Zum Khintchineschen “Übertragungssatz”. Trav. Inst. Math. Tbilissi, 3 (1938), 193–212.

[4] O. N. German Intermediate Diophantine exponents and parametric geometry of numbers, Acta Arithmetica, 154:1 (2012), 79–101.

[5] O. N. German On Diophantine exponents and Khintchine’s transference principle, Moscow Journal of Combinatorics and Number Theory, 2:2 (2012), 22–51.

[6] S. Chow, A. Ghosh, L. Guan, A. Marnat, D. Simmons Diophantine transference inequalities: weighted, inhomogeneous, and intermediate exponents. Annali della Scuola Normale Superiore di Pisa – Classe di Scienze, to appear; arXiv:1808.07184.

[7] A. Marnat There is no analogue to Jarník’s relation for twisted Diophantine approximation. Monatsh. Math, 181:3 (2016), 675–688.

[8] Y. Bugeaud, M. Laurent On exponents of homogeneous and inhomogeneous Diophantine approximation. Mosc. Math. J., 5 (2005), 747–766.

[9] V. Beresnevich, S. Velani An inhomogeneous transference principle and Diophantine approximation. Proc. Lond. Math. Soc. (3), 101 (2010), 821–851.
[10] A. Ghosh, A. Marnat *On Diophantine transference principles*. Math. Proc. Camb. Phil. Soc., 166:3 (2019), 415–431.

[11] K. Mahler *Ein Übertragungsprinzip für lineare Ungleichungen*. Čas. Pešť. Mat. Fys., 68 (1939), 85–92.

[12] K. Mahler *Ein Übertragungsprinzip für konvexe Körper*. Čas. Pešť. Mat. Fys., 68 (1939), 93–102.

[13] J. W. S. Cassels *An introduction to Diophantine approximation*. Cambridge University Press (1957).

[14] O. N. German, K. G. Evdokimov *A strengthening of Mahler’s transference theorem*. Izvestiya: Mathematics, 79:1 (2015), 60–73.

[15] W. M. Schmidt *Diophantine Approximation*. Lecture Notes in Math., 785, Springer-Verlag (1980).

[16] K. Mahler *On compound convex bodies, I*. Proc. London Math. Soc. (3), 5 (1955), 358–379.

[17] K. Mahler *On compound convex bodies, II*. Proc. London Math. Soc. (3), 5 (1955), 380–384.

[18] J. D. Vaaler *A geometric inequality with applications to linear forms*. Pacif. J. Math., 83:2 (1979), 543–553.

[19] W. M. Schmidt *Diophantine Approximation and Diophantine Equations*. Lecture Notes in Math., 1467, Springer-Verlag (1991).