SUBFACTORS OF FREE PRODUCTS OF RESCALINGS OF A
\(\text{II}_1\)--FACTOR

KEN DYKEMA

Abstract. Let \(Q\) be any \(\text{II}_1\)--factor. It is shown that any standard lattice \(G\) can be realized as the standard invariant of a free product of (several) rescalings of \(Q\). In particular, if \(Q\) has fundamental group equal to the positive reals and if \(P\) is the free product of infinitely many copies of \(Q\), then \(P\) has subfactors giving rise to all possible standard invariants. Similarly, given a \(\text{II}_1\)--subfactor \(N \subset M\), it is shown there are subfactors \(\bar{N} \subset \bar{M}\) having the same standard invariant as \(N \subset M\) but where \(\bar{M}\), respectively \(\bar{N}\), is the free product of \(M\), respectively \(N\), with rescalings of \(Q\).

1. Introduction

The systems of higher relative commutants of finite index subfactors of \(\text{II}_1\)--factors were classified by S. Popa [7], when he proved certain conditions on a lattice of finite dimensional commuting squares (together with certain other data such as Markov traces) to be equivalent to it arising as the higher relative commutants of a subfactor. Such an abstract lattice is called a standard lattice. The question of whether there is a single \(\text{II}_1\)--factor \(M\) whose subfactors give rise to all standard lattices was answered in the affirmative by Popa and Shlyakhtenko [8], where they proved this universal property for \(M = L(F_\infty)\), the von Neumann algebra of the free group on infinitely many generators. Their proof has two main components. First, given a standard lattice \(G\), they construct a non–degenerate commuting square

\[
\begin{align*}
B & \subset A \\
\cup & \cup \\
D & \subset C
\end{align*}
\]

of type I von Neumann algebras having atomic centers, which encodes the standard lattice. Then, for any \(\text{II}_1\)--factor \(Q\), they consider the resulting inclusion \(N \subset M\) of amalgamated free product von Neumann algebras

\[
\begin{align*}
M &= (Q \otimes B, \tau_Q \otimes \text{id}_B) \ast_B (A, E_B^A) \\
N &= (Q \otimes D, \tau_Q \otimes \text{id}_D) \ast_D (C, E_D^C)
\end{align*}
\]

where \(\tau_Q\) is the tracial state on \(Q\) and \(E_B^A\) and \(E_D^C\) are the conditional expectations from the commuting square (1). By relating it to a construction from [7], they show

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Subfactors in free products

that $N \subset M$ is an inclusion of $\text{II}_\infty$ factors which is an infinite amplification of an inclusion $N \subset M$ of $\text{II}_1$-factors whose system of higher relative commutants is the original standard lattice $G$. The second main part of their proof is to show that if $Q$ is taken to be $L(F_\infty)$, then both $N$ and $M$ are isomorphic to $L(F_\infty)$.

In this paper, we identify $N$ and $M$ for general $Q$ as a free scaled product \cite{4} of rescalings of $Q$. This allows us to generalize Popa and Shlyakhtenko's results. For example, (Theorem 4.2), we show that for every standard lattice $G$ of finite depth and for every $\text{II}_1$-factor $Q$ there is a subfactor $N \subseteq M$ whose standard lattice is $G$ and where

$$M = Q_{s_1} \ast \cdots \ast Q_{s_m}$$

$$N = Q_{t_1} \ast \cdots \ast Q_{t_n} * L(F_b)$$

for some $m, n \in \mathbb{N}$, positive real numbers $s_1, \ldots, s_m$ and $t_1, \ldots, t_n$ and $b > 1 - n$ depending only on $G$. (Here we are using the notation of \cite{3} for $L(F_a)$ with $a$ possibly negative). Taking $Q$ to be the hyperfinite $\text{II}_1$-factor $R$ and using the result \cite{2} that the $n$-fold free product of $R$ is $L(F_n)$, this gives after rescaling if necessary that if $1 < t < \infty$ and if $G$ is a standard lattice with finite depth, then $L(F_t)$ has a subfactor whose standard invariant is $G$; This result was first proved by Rădulescu \cite{9}.

We also show, (Theorem 4.4), that for every standard lattice $G$ and for every $\text{II}_1$-factor $Q$, there is a subfactor $N \subset M$ whose standard lattice is $G$ and such that

$$M = \bigotimes_{i=1}^{\infty} Q_{s_i}$$

$$N = \bigotimes_{i=1}^{\infty} Q_{t_i}$$

for some positive numbers $s_1, s_2, \ldots$ and $t_1, \ldots, t_2$ depending only on $G$. As a consequence of these two results, we obtain (Theorem 4.3) that if $Q$ is a $\text{II}_1$-factor whose fundamental group is equal to the set of all positive reals then the free product $Q * Q * \cdots$ of infinitely many copies of $Q$ has the universal property with respect to subfactors, namely, its subfactors give rise to all standard lattices as their standard invariants.

Popa and Shlyakhtenko \cite{8} also proved that if $N \subset M$ is any subfactor of a $\text{II}_1$-factor then there is a subfactor pair $\hat{N} \subseteq \hat{M}$ having the same standard lattice as $N \subset M$ and such that $\hat{N} \cong N * L(F_\infty)$ and $\hat{M} \cong M * L(F_\infty)$. Their proofs involved amalgamated free products of the form

$$(Q \otimes B, \tau_Q \otimes \text{id}_B) *_B (M \otimes B(H), E),$$

where $E$ is a conditional expectation onto a copy of the type I algebra $B$ embedded in the $\text{II}_\infty$-factor $M \otimes B(H)$, and where they use $Q = L(F_\infty)$.

We will identify the free product (3) for an arbitrary $\text{II}_1$-factor $Q$ in terms of free subproducts, and thereby prove generalizations of the above mentioned result where $L(F_\infty)$ in the formula for $\hat{N}$ and $\hat{M}$ is replaced by free products of rescalings of $Q$. (See Theorems 4.6, 4.7 and 4.8.)
The organization of the rest of the paper is as follows: in §2 and, respectively, §3, isomorphism results are proved for amalgamated free products as in (3) and, respectively (2). (However, the simplifying assumption is made that the algebra $B$ is commutative — an easy trick reduces the general case to this. Moreover, because it entails no extra difficulty and in fact makes the proof more transparent, we consider free products where the algebra $Q \otimes B$ of (3) and (3) is replaced by a direct sum $\bigoplus_{i \in I} Q(i)$ of II$_1$–factors whose center is $B$.) In §4 we prove the existence of subfactors, including those described above.

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2. Amalgamated free products, I

In this section, we consider amalgamated free products related to those of the form (2) when $B$ is taken to be commutative, and we show the result can be expressed in terms of free scaled products of von Neumann algebras. See [1] for the definition of and results about free scaled products.

Take an index set $I$ equal to $\{1, 2, \ldots, m\}$ for some integer $m \geq 2$ or to the natural numbers (in which case we write $m = \infty$) and let $Q = \bigoplus_{i \in I} Q(i)$, where each $Q(i)$ is a II$_1$–factor. Let $B$ be the center of $Q$ and let $p_i$ denote the minimal projection of $B$ that is the support of the $i$th summand $Q(i)$ of $Q$. Let $E_B^Q : Q \to B$ be the conditional expectation that is the center–valued trace on $Q$. Let $M$ be a type II$_1$ or II$_\infty$ factor with fixed faithful (finite or semifinite) trace $\text{Tr}$. Let $N$ denote the II$_1$–factor obtained as $eMe$ if $M$ is type II$_\infty$, where $e \in M$ is a projection of trace 1, and as $M_{\text{Tr}(1)}^{-1}$ if $M$ is type II$_1$; (thus if $M$ is type II$_1$ and $\text{Tr}(1) \geq 1$ then $N \cong eMe$ where $e$ is as above). Suppose $B$ is embedded in $M$ as a unital subalgebra in such a way that $\text{Tr}(p_i) < \infty$ for all $i \in I$ and let $E_B^M : M \to B$ be the $\text{Tr}$–preserving conditional expectation. Finally, let $\beta(i) = \text{Tr}(p_i)$.

In this and the next section, when $X_1$ and $X_2$ are subsets of an algebra, we will use the symbols $\Lambda^\beta(X_1, X_2)$ to denote the set of all alternating words in $X_1$ and $X_2$, i.e. all words of the form $a_1a_2 \cdots a_n$, where $a_j \in X_{i(j)}$ and $i(j) \neq i(j + 1)$. In fact, we will frequently blur the distinction between the set of words per se and the set of elements in the algebra that are equal to such words.

**Proposition 2.1.** Consider the amalgamated free product of von Neumann algebras

$$(\mathcal{M}, E) = (M, E_B^M) *_B (Q, E_B^Q).$$

(4)

If $\sum_{i=1}^m \beta(i) < \infty$ then $\mathcal{M}$ is a II$_1$–factor, while if $\sum_{i=1}^m \beta(i) = \infty$ then $\mathcal{M}$ is a II$_\infty$–factor. In either case, we have

$$p_1 \mathcal{M} p_1 \cong N_{\beta(1)} *_{i \in I} [\beta(i), Q(i)].$$

(5)

**Proof.** We denote also by $\text{Tr}$ the semifinite trace $\text{Tr}|_B \circ E$ on $\mathcal{M}$. The method of [3] shows that $E$ is faithful, hence $\text{Tr}$ is faithful on $\mathcal{M}$; we will show the isomorphism (2), from which it will follow that $\mathcal{M}$ is a factor with faithful trace $\text{Tr}$, which is finite or semifinite depending on $\sum_{i=1}^m \beta(i)$.
Let $p_{[1,n]} = p_1 + p_2 + \cdots + p_n$ and let $\beta[1,n] = \text{Tr}(p_{[1,n]})$. For every $n \in I$, let

$$\mathcal{M}(n) = W^*(p_{[1,n]} \mathcal{M}[1,n] \cup p_{[1,n]} Q).$$

Then $\mathcal{M}(1)$ is generated by $p_1 M p_1 \cong N_{\beta(1)}$ and $p_1 Q \cong Q(1)$, which are free, so

$$\mathcal{M}(1) \cong N_{\beta(1)} * Q(1).$$

Now for $n \in I$, $n \geq 2$, we will find $\mathcal{M}(n)$ in terms of $\mathcal{M}(n-1)$. Let $K \in \mathbb{N}$ be such that $\beta[1,n-1] \geq \beta(n)/K$ and let $(e_{ij})_{1 \leq i,j \leq K}$ be a system of matrix units in $M$ such that $\sum_{i=1}^K e_{ii} = p_n$. Let $A = \text{span} \{e_{ij} \mid 1 \leq i,j \leq K\}$ and let $\mathcal{P} = W^*(A \cup p_n Q)$. Clearly $A$ and $p_n Q$ are free with respect to $\beta(n)^{-1} \text{Tr}|_{p_n M p_n}$, so

$$\mathcal{P} \cong Q(n) * M_K(C) \cong Q(n) * L(F_{1-K^{-2}}).$$

Let $v \in M$ be such that $v^* v = e_{11}$ and $q := vv^* \leq p_{[1,n-1]}$. Then

$$\mathcal{M}(n) = W^*(\mathcal{M}(n-1) \cup \mathcal{P} \cup \{v\}), \quad q \mathcal{M}(n) q = W^*(q \mathcal{M}(n-1) q \cup v \mathcal{P} v^*).$$

We claim that $q \mathcal{M}(n-1) q$ and $v \mathcal{P} v^*$ are free with respect to $K \beta(n)^{-1} \text{Tr}|_{q M q}$. This is equivalent to freeness of $v^* \mathcal{M}(n-1) v$ and $e_{11} \mathcal{P} e_{11}$ with respect to $K \beta(n)^{-1} \text{Tr}|_{e_{11} M e_{11}}$, which is what we will show. Using the trace-preserving conditional expectation $\mathcal{P} \to A$ along with the facts that $e_{11}$ is a minimal projection in $A$ and that $\{p_n \cup \Lambda^o(A \cap \ker Tr, p_n Q \cap \ker Tr)$ spans a dense subspace of $\mathcal{P}$, we find that a dense subspace of $e_{11} \mathcal{P} e_{11} \cap \ker Tr$ is spanned by

$$\Phi := \bigcup_{1 \leq i,j \leq K} e_{ij} \Phi_{ij},$$

where $\Phi$ is the set of all words belonging to $\Lambda^o(A \cap \ker Tr, p_n Q \cap \ker Tr)$ whose first and last letters come from $p_n Q \cap \ker Tr$. On the other hand, since

$$p_{[1,n-1]} B \cup \Lambda^o(p_{[1,n-1]} M p_{[1,n-1]} \cap \ker E, p_{[1,n-1]} Q \cap \ker E)$$

spans a dense subspace of $\mathcal{M}(n-1)$, we see that $v^* \mathcal{M}(n-1) v \cap \ker Tr$ is contained in the closed linear span of

$$\tilde{\Psi} := (e_{11} M e_{11} \cap \ker Tr) \cup \Psi,$$

where $\Psi$ is the set of all words belonging to $\Lambda^o(M \cap \ker E, p_{[1,n-1]} Q \cap \ker E)$ with first letter from $e_{11} M p_{[1,n-1]}$ and with last letter from $p_{[1,n-1]} M e_{11}$. Now to show freeness of $q \mathcal{M}(n-1) q$ and $v \mathcal{P} v^*$, it will suffice to show

$$\Lambda^o(\Phi, \tilde{\Psi}) \subseteq \ker Tr. \quad (6)$$

Let $x \in \Lambda^o(\Phi, \tilde{\Psi})$. If the first letter in $x$ comes from $e_{11} \Phi e_{ij} \subseteq \Phi$ then substitute $e_{11} = (e_{11} - K^{-1} p_n) + K^{-1} p_n$ for this first $e_{11}$, while if the last letter in $x$ comes from $e_{11} \Phi e_{ij} \subseteq \Phi$ then substitute this value for this last $e_{11}$ and then distribute. Using $e_{11} e_{11} M p_{[1,n-1]} \subseteq \ker E$ and $e_{11} e_{11} M e_{11} \cap \ker Tr) e_{ij} \subseteq \ker E$ for all $i,j \in \{1, \ldots, K\}$ and regrouping, we see that $x$ is equal to a linear combination of at most four words from $\Lambda^o(M \cap \ker E, Q \cap \ker E)$, and hence $\text{Tr}(x) = 0$ by freeness. We have shown (6) and thereby freeness of $q \mathcal{M}(n-1) q$ and $v \mathcal{P} v^*$. 

Subfactors in free products
What we have shown above implies there is an isomorphism
\[ p_{[1,n-1]} \mathcal{M}(n)p_{[1,n-1]} \sim \mathcal{M}(n-1) * \left[ \frac{\beta(n)}{K_\beta[1,n-1]}, v^* \right] \] (7)

intertwining the inclusion \( \mathcal{M}(n-1) \hookrightarrow p_{[1,n-1]} \mathcal{M}(n)p_{[1,n-1]} \) and the canonical embedding of \( \mathcal{M}(n-1) \) in the RHS of (7). By results of [3],
\[ v^* p \cong P_{1/K} \cong Q(n)_{1/K} * L(F_{K^2-1}) \]

so using the technology of free scaled products [4], we get isomorphisms
\[ \mathcal{M}(n-1) * \left[ \frac{\beta(n)}{K_\beta[1,n-1]}, Q(n)_{1/K} * L(F_{K^2-1}) \right] \sim \]
\[ \sim \mathcal{M}(n-1) * \left[ \frac{\beta(n)}{K_\beta[1,n-1]}, Q(n)_{1/K} \right] * \left[ \frac{\beta(n)}{K_\beta[1,n-1]}, L(F_{K^2-1}) \right] \]
\[ \sim \mathcal{M}(n-1) * L(F_{(\beta(n)/\beta[1,n-1]^2(1-K^{-2})}) * \left[ \frac{\beta(n)}{K_\beta[1,n-1]}, Q(n)_{1/K} \right] \]
\[ \sim \mathcal{M}(n-1) * \left[ \frac{\beta(n)}{K_\beta[1,n-1]}, Q(n) \right] , \]

all intertwining the canonical embeddings of \( \mathcal{M}(n-1) \) into the free scaled products; (the last isomorphism above is obtained from Theorem 5.5 of [4], and the others follow routinely from the definition of free scaled product). Now compressing by \( p_1 \) and appealing to Theorem 4.9 of [4], we get an isomorphism
\[ p_1 \mathcal{M}(n)p_1 \sim p_1 \mathcal{M}(n-1)p_1 * \left[ \frac{\beta(n)}{\beta(1)}, Q(n) \right] \] (8)

intertwining the inclusion \( p_1 \mathcal{M}(n-1)p_1 \hookrightarrow p_1 \mathcal{M}(n)p_1 \) and the canonical embedding of \( p_1 \mathcal{M}(n-1)p_1 \) in the RHS of (8).

Taking the inductive limit as \( n \) increases gives the desired isomorphism (7). \( \square \)

The following corollary is a direct consequence of Proposition 2.1 and results from [3] and [4].

**Corollary 2.2.** Let
\[ (\mathcal{M}, E) = (M, E^M_B) * (Q, E^Q_B) \]

with accompanying notation be as in Proposition 2.1.

(A) If \( I = \{1, \ldots, m\} \) is finite then
\[ \mathcal{M} \cong M * Q(1)_{1/\beta(1)} * Q(2)_{1/\beta(2)} * \cdots * Q(m)_{1/\beta(m)} * L(F_t) \]

where
\[ t = -m + \sum_{i=1}^{m} \beta(i)^2 . \]

(B) If \( I = \mathbb{N} \) is infinite and if one or more of the following conditions hold:

(i) \( Q(i) \cong Q(i) * L(F_\infty) \) for some \( i \in \mathbb{N} \),
(ii) \( N \cong N * L(F_\infty) \),
(iii) \( \sum_{i=1}^{\infty} \beta(i)^2 = \infty \),
then
\[ p_1 M p_1 \cong N_{\beta(1)} * \left( \bigotimes_{i=1}^{\infty} Q(i)_{\beta(i)} \right). \]

3. Amalgamated free products, II

In this section, we will describe amalgamated free product von Neumann algebra related to those in \([3]\), but with \(B\) commutative, in terms of free scaled products.

We begin by letting \(A\) be a separable, type I von Neumann algebra with atomic center \(Z(A)\) and let \(B\) be a unital commutative subalgebra of \(A\) with the property that \(B \cap Z(A) = \mathbb{C}1\). We assume that there is a normal, faithful, semifinite trace \(\text{Tr}_A\) on \(A\) such that \(\text{Tr}_A(p) < \infty\) for every minimal projection \(p\) in \(B\). Let \(E_B^A : A \to B\) be the \(\text{Tr}_A\)-preserving conditional expectation onto \(B\). Let \((p_i)_{i \in I}\) be the minimal projections in \(B\), where we take either \(I = \{1, \ldots, n\}\), some \(n \in \mathbb{N}\), or \(I = \mathbb{N}\), in which case we set \(n = \infty\). Let \(Q(i)\) be a \(II_1\)-factor \((i \in I)\) and let \(Q = \bigoplus_{i \in I} Q(i)\), with the center of \(Q\) identified with \(B\) in such a way that \(p_i\) is the support projection of the \(i\)th summand \(Q(i)\). Let \(E_B^Q : Q \to B\) be the conditional expectation that is the center valued trace on \(Q\). Our goal in this section is to write the amalgamated free product von Neumann algebra
\[ (\mathcal{P}, E) = (A, E_B^A) *_B (Q, E_B^Q) \]
as a free scaled product of \(II_1\)-factors.

We have implicitly chosen an ordering of the minimal projections in \(B\). Let \((q_j)_{j \in J}\) be the minimal projections in \(Z(A)\). Define \(J_m\) recursively \((m \in I)\) by
\[
J_1 = \{ j \in J \mid q_j p_1 \neq 0 \} \\
J_m = \{ j \in J \mid q_j p_m \neq 0 \} \setminus (J_1 \cup \cdots \cup J_{m-1}) \quad (m \in I \setminus \{1\}).
\]
Let \(\beta(i) = \text{Tr}_A(p_i),\) \((i \in I)\) and if \(f_j\) is a minimal projection of \(A\) that lies under \(q_j\), then let \(\alpha(j) = \text{Tr}_A(f_j),\) \((j \in J)\). For every \(m \in I \setminus \{1\}\), let \(e_m\) be a minimal projection in \(A\) such that \(e_m \leq p_1 + \cdots + p_m - 1\) and \(e_m\) is equivalent to a subprojection of \(p_m\). (Such a projection must exist by the condition \(Z(A) \cap B = \mathbb{C}1\)) Let \(j(m) \in J\) be such that \(e_m \leq q_{j(m)}\) and let \(\gamma(m) = \alpha(j(m)) = \text{Tr}_A(e_m)\).

**Proposition 3.1.** Let
\[ (\mathcal{P}, E) = (A, E_B^A) *_B (Q, E_B^Q) \]
be the free product of von Neumann algebras with amalgamation over \(B\). If \(\sum_{i=1}^{n} \beta(i) < \infty\) then \(\mathcal{P}\) is a \(II_1\)-factor, while if \(\sum_{i=1}^{n} \beta(i) = \infty\) then \(\mathcal{P}\) is a \(II_\infty\)-factor. In either case,
\[ p_1 M p_1 \cong (Q(1) * L(F_r))^n \left[ \frac{\gamma(j)}{\beta(1)}, Q(i)_{\beta(i)} \right], \]
where
\[
r = \frac{1}{\beta(1)^2} \left( \beta(1)^2 - \sum_{j \in J_1} \alpha(j)^2 \right) + \frac{1}{\beta(1)^2} \sum_{m=2}^{n} \left( \beta(m)^2 - \gamma(m)^2 - \sum_{j \in J_m} \alpha(j)^2 \right). \]
The hypothesis $Z(A) \cap B = C$ ensures that $\mathcal{P}$ is a factor. We have the normal, faithful, semifinite trace $\text{Tr} = \text{Tr}_A \circ E$ on $\mathcal{P}$, which is finite if and only if $\sum_{i=1}^{\infty} \beta(i) < \infty$.

It remains to find the isomorphism class of $p_1 \mathcal{P} p_1$. For $m \in I$, let
\[
p_{[1,m]} = p_1 + \cdots + p_m
\]
and let
\[
\beta[1,m] = \beta(1) + \cdots + \beta(m) = \text{Tr}(p_{[1,m]})
\]
and let
\[
\mathcal{N}(m) = W^*(p_m Q \cup p_m A p_m)
\]
\[
\mathcal{P}(m) = W^*(p_{[1,m]} Q \cup p_{[1,m]} A p_{[1,m]}) .
\]
Then
\[
\mathcal{N}(m) \cong Q(m) * L(F_{s(m)})
\]
where $s(m)$ is the free dimension of $p_m A p_m$ with respect to $\beta(m)^{-1} \text{Tr}|_{p_m A p_m}$. We have $\mathcal{P}(1) = \mathcal{N}(1)$ and
\[
s(1) = \frac{1}{\beta(1)^2} \left( \beta(1)^2 - \sum_{j \in J_1} \alpha(j)^2 \right).
\]

Note that $\mathcal{P}$ is the increasing union of $\mathcal{P}(1) \subseteq \mathcal{P}(2) \subseteq \cdots$ and thus $p_1 \mathcal{P} p_1$ is the increasing union of $p_1 \mathcal{P}(1) p_1 \subseteq p_1 \mathcal{P}(2) p_1 \subseteq \cdots$. We will show that each $p_1 \mathcal{P}(m) p_1$ is isomorphic to a free scaled product of $\Pi_1$–factors in such a way that the inclusions $p_1 \mathcal{P}(m) p_1 \subseteq p_1 \mathcal{P}(m+1) p_1$ become canonical embeddings. Taking the inductive limit will allow us to express $p_1 \mathcal{P} p_1$ as a free scaled product of $\Pi_1$–factors.

Fix $m$ such that $m, m + 1 \in I$. Let
\[
K_m = \bigcup_{i=1}^{m} J_i,
\]
\[
K_{m}^{(1)} = \{ k \in K_m \mid q_k p_{m+1} \neq 0 \}.
\]
\[
K_{m}^{(0)} = K_m \setminus K_{m}^{(1)} .
\]
The hypothesis $Z(A) \cap B = C$ ensures that $K_{m}^{(1)}$ is nonempty; in fact, $j(m+1) \in K_{m}^{(1)}$.

Make an ordering $K_{m}^{(1)} = \{ k(i) \mid i \in L \}$ where $L = \{ 0, 1, \ldots, \ell \}$ or $L = \{ 0 \} \cup \mathbb{N}$, such that $k(0) = j(m + 1)$. For every $k \in K_{m}^{(1)}$, let $v_k \in A$ be such that $v_k^* v_k$ is a minimal projection in $A$, $v_k^* v_k \leq p_{[1,m]}$ and $v_k v_k^* \leq p_{m+1}$. Note that $\text{Tr}(v_k^* v_k) = \alpha(k)$; we assume without loss of generality that $v_k^* v_k = e_{m+1}$.

Let
\[
\mathcal{R}(0) = W^*(\mathcal{P}(m) \cup \mathcal{N}(m + 1) \cup \{ v_k(0) \}) .
\]
Then
\[
p_{[1,m]} \mathcal{R}(0) p_{[1,m]} = W^*(\mathcal{P}(m) \cup v_k(0)_{m+1} \mathcal{N}(m + 1) v_k(0)) .
\]

---

1. We are using the “free dimension” that was introduced in [1] — see [4] for further discussion. Since each $p_m A p_m$ is finite dimensional, its free dimension is well defined, and we may use the term “free dimension” without inhibition.
We will show that the pair
\[ e_{m+1} \mathcal{P}(m)e_{m+1}, \quad v^*_k N(m+1)v_k(0) \]
is free with respect to $\gamma(m+1)^{-1} \text{Tr}[e_m \mathcal{P}e_m]$. Using $C^o$ to denote $C \cap \ker E$ for any subalgebra $C$ of $\mathcal{P}$, since $e_m$ is a minimal projection in $A$, $\mathcal{P}(m)^o$ is densely spanned by
\[ \Theta := e_m \left( \Lambda^o \left( (Qp_{[1,m]}^o, \langle p_{[1,m]} A p_{[1,m]} \rangle)^o \rangle \right) \right) e_m \]
and $(v^*_k N(m+1)v_k(0))^o$ is densely spanned by
\[ \Omega := v^*_k \left( \Lambda^o \left( (Qp_{m+1}^o, \langle p_{m+1} A p_{m+1} \rangle)^o \right) \right) v_k(0) . \]
To show freeness of the pair $(\Theta, \Omega)$, it will suffice to show $\Lambda^o(\Theta, \Omega) \subseteq \ker E$. However, since
\[ \langle p_{[1,m]} A p_{[1,m]} \rangle v^*_k \langle p_{m+1} A p_{m+1} \rangle \subseteq \ker E_B^A, \]
given $x \in \Lambda^o(\Theta, \Omega)$, by regrouping we see that $x$ is equal to some $x' \in \Lambda^o(Q^o, A^o)$, which is contained in $\ker E$ by freeness. This shows that the pair $(\Theta, \Omega)$ is free. Therefore, there is an isomorphism
\[ p_{[1,m]} \mathcal{R}(0)p_{[1,m]} \sim \mathcal{P}(m) * \left[ \frac{\gamma(m+1)}{\beta_{[1,m]}}, N(m+1) \frac{\gamma(m+1)}{\beta_{[m+1]}} \right] \]
intertwining the inclusion $\mathcal{P}(m) \hookrightarrow p_{[1,m]} \mathcal{R}(0)p_{[1,m]}$ and the canonical embedding of $\mathcal{P}(m)$ in the RHS of $(\Theta, \Omega)$. By $(\Theta, \Omega)$, $\mathcal{R}(0)$ is a factor.

Define $\mathcal{R}(i)$ recursively for every $i \in L \setminus \{0\}$ by
\[ \mathcal{R}(i) = W^* (\mathcal{R}(i-1) \cup \{ v_{k(i)} \}) . \]
We will show that every $\mathcal{R}(i)$ is a factor that can be written as a free subproduct involving $\mathcal{R}(i-1)$. Fixing $i$, suppose that $\mathcal{R}(i-1)$ is a factor. Let $w_1 \in \mathcal{R}(i-1)$ be such that $w_1 w_1^* = v_{k(i)}^* v_{k(i)}$. Let $f = w_i^* w_i$ and $g = w_i w_i^*$. Note that $f \leq p_{[1,m]} q_{k(i)}$ and $g \leq p_{m+1} q_{k(i)}$. We will show that in $f \mathcal{P} f$ and with respect to $\alpha(k(i))^{-1} \text{Tr}[f df]$, $w_i^* v_{k(i)}$ is a Haar unitary and is $*$-free from $f \mathcal{R}(i-1) f$. Let
\[ C = W^* \left( p_{[1,m]} A p_{[1,m]} \cup p_{m+1} A p_{m+1} \cup \{ v_k(0), \ldots, v_{k(i-1)} \} \right) . \]
Then $fCg = \{0\}$ and $\mathcal{R}(i-1)$ is densely spanned by $\Lambda^o(C^o, (p_{[1,m+1]} Q^o))$. In order to show that $w_i^* v_{k(i)}$ is a Haar unitary and is $*$-free from $f \mathcal{R}(i-1) f$, it will suffice to show
\[ \Lambda^o \left( (f \mathcal{R}(i-1) f)^o \cup f \mathcal{R}(i-1) g \cup g \mathcal{R}(i-1) f \cup (g \mathcal{R}(i-1) g)^o, \{ v_{k(i)}, v_{k(i)}^* \} \right) \subseteq \ker E . \]
Because $f$ and $g$ are minimal projections in $C$ and $fCg = \{0\}$,
\[ (f \mathcal{R}(i-1) f)^o \cup f \mathcal{R}(i-1) g \cup g \mathcal{R}(i-1) f \cup (g \mathcal{R}(i-1) g)^o \subseteq \text{span} \left( \Lambda^o(C^o, (p_{[1,m+1]} Q^o)) \right) . \]
But since $Cv_{k(i)} C \subseteq \ker E$, we have
\[ \Lambda^o \left( \Lambda^o(C^o, (p_{[1,m+1]} Q^o)) \setminus C^o, \{ v_{k(i)}, v_{k(i)}^* \} \right) \subseteq \ker E . \]
Thus \( w^*_i v_{k(i)} \) is a Haar unitary and is \(*\)-free from \( f\mathcal{R}(i-1)f \). This gives a \(*\)-isomorphism

\[
p_{[1,m]}\mathcal{R}(i)p_{[1,m]} \xrightarrow{\sim} p_{[1,m]}\mathcal{R}(i-1)p_{[1,m]} \ast \left[ \frac{\alpha(k(i))}{\beta(i)}, L(\mathbb{Z}) \right] \quad (12)
\]

intertwining the inclusion \( p_{[1,m]}\mathcal{R}(i)p_{[1,m]} \hookrightarrow p_{[1,m]}\mathcal{R}(i-1)p_{[1,m]} \) and the canonical embedding of \( p_{[1,m]}\mathcal{R}(i-1)p_{[1,m]} \) into the RHS of (12).

Since \( \mathcal{P}(m+1) = \bigcup_{i \in I} \mathcal{R}(i) \), by composing the isomorphisms (11) and (12), we get an isomorphism

\[
p_{[1,m]}\mathcal{P}(m+1)p_{[1,m]} \xrightarrow{\sim} \left( \mathcal{P}(m) \ast \left[ \frac{\gamma(m+1)}{\beta(i)}, N(m+1) \frac{\gamma(m+1)}{\beta(m+1)} \right] \right) \ast \left[ \frac{\alpha(k(i))}{\beta(i)}, L(\mathbb{Z}) \right] \quad (13)
\]

intertwining the inclusion \( \mathcal{P}(m) \hookrightarrow p_{[1,m]}\mathcal{P}(m+1)p_{[1,m]} \) and the canonical embedding of \( \mathcal{P}(m) \) in the RHS of (13). Using that \( N(m+1) \cong Q(m+1) \ast L(F_{s(m+1)}) \) with

\[
s(m+1) = 1 - \sum_{j \in K_{(m)} \cup J_m} \left( \frac{\alpha(j)}{\beta(m+1)} \right)^2,
\]

by results from [4] there is an isomorphism

\[
p_1\mathcal{P}(m+1)p_1 \xrightarrow{\sim} (p_1\mathcal{P}(m)p_1 \ast L(F_{r(m+1)})) \ast \left[ \frac{\gamma(m+1)}{\beta(1)}, Q(m+1) \frac{\gamma(m+1)}{\beta(m+1)} \right], \quad (14)
\]

where

\[
r(m+1) = \frac{1}{\beta(1)^2} \left( \beta(m+1)^2 - \gamma(m+1)^2 - \sum_{j \in J_m} \alpha(j)^2 \right),
\]

intertwining the inclusion \( p_1\mathcal{P}(m)p_1 \hookrightarrow p_1\mathcal{P}(m+1)p_1 \) and the canonical embedding of \( p_1\mathcal{P}(m)p_1 \) in the RHS of (13). Taking the inductive limit as \( m \to \infty \), we obtain the isomorphism (2).

The next corollary follows directly from Proposition 3.1 and results of [3] and [4].

**Corollary 3.2.** Let \( A, B, \mathcal{Q} = \bigoplus_{i \in I} \mathcal{Q}(i) \) and

\[
(\mathcal{P}, E) = (\mathcal{Q}, E_B^Q) \ast_B (A, E_B^A)
\]

with accompanying notation be as in Proposition 3.1.

(A) If \( I = \{1, \ldots, n\} \) is finite then

\[
\mathcal{P} \cong Q(1) \left[ \frac{1}{\beta(1)} \right] \ast Q(2) \left[ \frac{1}{\beta(2)} \right] \ast \cdots \ast Q(n) \left[ \frac{1}{\beta(n)} \right] \ast L(F_t)
\]

where

\[
t = -n + 1 + \sum_{i=1}^{n} \beta(i)^2 - \sum_{j \in I} \alpha(j)^2
\]

\[= -n + 1 + \text{fdim}(A) - \text{fdim}(B).
\]
(B) If \( I = \mathbb{N} \) is infinite and if one or more of the following conditions hold:

(i) \( Q(i) \cong Q(i) * L(\mathbf{F}_\infty) \) for some \( i \in \mathbb{N} \),

(ii) \( \sum_{i=1}^{\infty} \gamma(i)^2 = \infty \),

(iii) \( r = \infty \),

then

\[
p_1 \mathcal{P}_1 \cong \prod_{i=1}^{\infty} Q(i) \frac{\omega(i)}{\pi(i)}.
\]

4. Subfactors

In this section, we will recall constructions and results of Popa and Shlyakhtenko \cite{PS2} and we will use them together with the results from \S 2 and \S 3 to prove the existence of subfactors in free subproducts and free products, including those mentioned in the introduction.

Let \( Q \) be any \( \Pi_1 \)-factor. Let \( \mathcal{M}_{-1} \subset \mathcal{M}_0 \) be a \( \Pi_1 \)-subfactor of finite index \( \lambda \), let \( \mathcal{G} = \mathcal{G}_{\mathcal{M}_{-1}, \mathcal{M}_0} \) be the associated standard \( \lambda \)-lattice and let \( \mathcal{F} \) be the standard graph and \( \mathcal{F}' \) the second standard graph of \( \mathcal{M}_{-1} \subset \mathcal{M}_0 \). Popa and Shlyakhtenko \cite{PS2} construct semifinite von Neumann algebras

\[
\begin{align*}
\mathcal{M}_{-1} & \subset \mathcal{M}_0 \\
\cup & \cup \\
\mathcal{A}_{-1}^0 & \subset \mathcal{A}_0^0 \\
\cup & \cup \\
\mathcal{A}_{-1}^{-1} & \subset \mathcal{A}_0^{-1}
\end{align*}
\]

with a specified faithful semifinite trace \( \text{Tr} \) on \( \mathcal{M}_0 \) and they prove the following:

I. The trace preserving semifinite trace \( \text{Tr} \) makes \( \mathcal{M}_i \cong M_i \otimes B(\mathcal{K}) \) a diagram of commuting squares.

II. \( \mathcal{M}_i \cong M_i \otimes B(\mathcal{K}) \), \( i = -1, 0 \).

III. Each algebra \( \mathcal{A}_k^i \) is a type I von Neumann algebra with atomic center, and \( \text{Tr} \) is finite on all minimal projections in \( \mathcal{A}_k^i \).

IV. The graphs of the inclusions \( \mathcal{A}_{-1}^{-1} \subset \mathcal{A}_{0}^{-1} \) and \( \mathcal{A}_{-1}^{-1} \subset \mathcal{A}_{0}^{-1} \) are \( \Gamma \) and \( \Gamma' \), respectively.

V. The commuting square

\[
\begin{align*}
\mathcal{A}_{-1}^0 & \subset \mathcal{A}_0^0 \\
\cup & \cup \\
\mathcal{A}_{-1}^{-1} & \subset \mathcal{A}_0^{-1}
\end{align*}
\]

depends only on the standard \( \lambda \)-lattice \( \mathcal{G} \), and the commuting square \( \mathcal{I} \) is functorial in \( \mathcal{G} \), (see \cite{PS2} Thm. 2.9 for details).

VI. Consider the amalgamated free products of von Neumann algebras

\[
\begin{align*}
\mathcal{P}_0 & = \mathcal{A}_0^0 \star_{\mathcal{A}_{0}^{-1}} (Q \otimes \mathcal{A}_{0}^{-1}) \\
\mathcal{P}_{-1} & = \mathcal{A}_{-1}^{-1} \star_{\mathcal{A}_{-1}^{-1}} (Q \otimes \mathcal{A}_{-1}^{-1})
\end{align*}
\]
where the amalgamation is with respect to the trace preserving conditional expectations. Then \( \mathcal{P}_{-1} \subset \mathcal{P}_0 \) is an inclusion of type \( \text{II}_\infty \) factors whose standard invariant (i.e. the standard invariant of \( p\mathcal{P}_{-1}p \subset p\mathcal{P}_0p \) where \( p \in \mathcal{P}_{-1} \) is any finite projection) is \( \mathcal{G} \).

VII. Consider the amalgamated free products of von Neumann algebras

\[
\hat{\mathcal{M}}_0 = \mathcal{M}_0 *_{\mathcal{A}_0^{-1}} (Q \otimes \mathcal{A}_0^{-1}) \tag{19}
\]

\[
\hat{\mathcal{M}}_{-1} = \mathcal{M}_{-1} *_{\mathcal{A}_{-1}^{-1}} (Q \otimes \mathcal{A}_{-1}^{-1}) \tag{20}
\]

where the amalgamation is with respect to the trace preserving conditional expectations. Then \( \hat{\mathcal{M}}_{-1} \subset \hat{\mathcal{M}}_0 \) is an inclusion of type \( \text{II}_\infty \) factors whose standard invariant (i.e. the standard invariant of \( p\hat{\mathcal{M}}_{-1}p \subset p\hat{\mathcal{M}}_0p \) where \( p \in \hat{\mathcal{M}}_{-1} \) is any finite projection) is \( \mathcal{G} \).

4.1. Consider the commuting square \((\mathfrak{K})\) determined by \( \mathcal{G} \). Let \( \mathcal{P}_0 \) be as in \((17)\) and let \( q(0) \in \mathcal{A}_0^{-1} \) be the sum of a maximal family of mutually orthogonal and nonequivalent minimal projections of \( \mathcal{A}_0^{-1} \). Letting \( B = q(0)\mathcal{A}_0^{-1}q(0) \), we have that \( B \) is commutative and

\[
q(0)\mathcal{P}_0q(0) = q(0)\mathcal{A}_0^{-1}q(0) *_{B} (Q \otimes B),
\]

where the amalgamation is with respect to the trace preserving conditional expectations. Fix any minimal projection \( q_1(0) \) in \( B \). Let \( m(0) \in \mathbb{N} \cup \{ \infty \} \) be the number of minimal projections in \( B \) and let \( (\beta(0,i))_{i=1}^{m(0)} \) be the values of \( \text{Tr} \) on the minimal projections in \( B \), with \( \beta(0,1) = \text{Tr}(q_1(0)) \). Proposition 3.1 shows that for certain \( r(0) \in [0, \infty) \) and \( \gamma(0, i) \leq \beta(0, i) \), we have

\[
q_1(0)\mathcal{P}_0q_1(0) \cong \begin{cases} Q * L(F_{r(0)}) & \text{if } m(0) = 1 \\ (Q * L(F_{r(0)}))_{i=2}^{m(0)} \left[ \frac{\gamma(0,i)}{\beta(0,1)} \right] & \text{if } m(0) > 1. \end{cases} \tag{21}
\]

Analogously, if \( \mathcal{P}_{-1} \) is as in \((18)\) and if \( q_1(-1) \) is any minimal projection in \( \mathcal{A}_{-1}^{-1} \), if \( m(-1) \) is the number of minimal projections in the center of \( \mathcal{A}_{-1}^{-1} \), letting \( (\beta(-1,i))_{i=1}^{m(-1)} \) be \( \text{Tr} \) applied to a maximal family of mutually inequivalent minimal projections in \( \mathcal{A}_{-1}^{-1} \), with \( \beta(-1,i) = \text{Tr}(q_1(-1)) \), we have

\[
q_1(-1)\mathcal{P}_{-1}q_1(-1) \cong \begin{cases} Q * L(F_{r(-1)}) & \text{if } m(-1) = 1 \\ (Q * L(F_{r(-1)}))_{i=2}^{m(-1)} \left[ \frac{\gamma(-1,i)}{\beta(-1,1)} \right] & \text{if } m(-1) > 1. \end{cases} \tag{22}
\]

for certain \( r(-1) \in [0, \infty) \) and \( \gamma(-1,i) \leq \beta(-1,i) \).

**Theorem 4.2.** Let \( \mathcal{G} \) be a standard lattice of finite depth. Then there are \( m,n \in \mathbb{N} \), positive real numbers \( s_1, \ldots, s_m \) and \( t_1, \ldots, t_n \) and there are \( a > 1-m \) and \( b > 1-n \) such that for any \( \text{II}_1 \)-factor \( Q \), there is an inclusion \( \mathcal{P}_{-1} \subset \mathcal{P}_0 \) of \( \text{II}_1 \)-factors whose
standard invariant is $\mathcal{G}$ and such that

\[ P_0 = Q_{s_1} \ast \cdots \ast Q_{s_m} \ast L(F_a) \]
\[ P_{-1} = Q_{t_1} \ast \cdots \ast Q_{t_n} \ast L(F_b). \]

Proof. Since $\mathcal{G}$ is of finite depth, it follows from [8] (see point IV above) that the centers of $\mathcal{A}_0^{-1}$ and $\mathcal{A}_1^{-1}$ are finite dimensional. Thus $m(0)$ and $m(1)$ in (21) and (22) are finite. By results in [4], from (21) and (22) we get

\[ q_1(0) \mathcal{P}_0 q_1(0) \cong Q_{s'_1} \ast \cdots \ast Q_{s'_m(0)} \ast L(F_{a'}) \]
\[ q_1(-1) \mathcal{P}_{-1} q_1(-1) \cong Q_{t'_1} \ast \cdots \ast Q_{t'_m(1)} \ast L(F_{b'}) \]

for some $a' > 1 - m(0)$ and $b' > 1 - m(-1)$, where $s'_i = \beta(0, i)/\beta(0, 1)$ and $t'_i = \beta(-1, i)/\beta(-1, 1)$. By results of [8] (see point VI above), the $\mathcal{II}_1$-subfactor $q \mathcal{P}_{-1} q \subset q \mathcal{P}_0 q$ has standard invariant $\mathcal{G}$, where $q$ is any finite projection in $\mathcal{P}_{-1}$. But $q \mathcal{P}_0 q$ is a rescaling of $q_1(0) \mathcal{P}_0 q_1(0)$ by $\lambda := \text{Tr}(q)/\text{Tr}(q_1(0))$, and thus by results of [8] we have

\[ q \mathcal{P}_0 q \cong Q_{\lambda s'_1} \ast \cdots \ast Q_{\lambda s'_m(0)} \ast L(F_a) \]

where $a = \lambda^{-2}a' + (m(0) - 1)(\lambda^{-2} - 1)$. Rescaling $q_1(-1) \mathcal{P}_{-1} q_1(-1)$ yields

\[ q \mathcal{P}_{-1} q \cong Q_{t'_1} \ast \cdots \ast Q_{t'_m(1)} \ast L(F_b) \]

for appropriate values of $t_i$ and $b$. \qed

Remark 4.3. In the proof of Theorem 4.2, $\lambda$ can be chosen to arrange either $a = 0$, provided $m(0) > 1$, or $b = 0$, provided $m(1) > 1$.

Theorem 4.4. Given a standard lattice $\mathcal{G}$, there are some positive real numbers $s_1, s_2, \ldots$ and $t_1, t_2, \ldots$ such that for any $\mathcal{II}_1$-factor $Q$, there is an inclusion $P_{-1} \subset P_0$ of $\mathcal{II}_1$-factors whose standard invariant is $\mathcal{G}$ and such that

\[ P_0 \cong \bigast_{i=1}^{\infty} Q_{s_i} \]
\[ P_{-1} \cong \bigast_{i=1}^{\infty} Q_{t_i} \]

(23)

Proof. If $\mathcal{G}$ is of finite depth, let $\widetilde{Q} = Q \ast Q \ast \cdots$ be the free products of infinitely many copies of $Q$. By [8], a free product of infinitely many $\mathcal{II}_1$-factors absorbs a free product with $L(F_\infty)$, and the rescaling by $\lambda$ of a free product of infinitely many factors is isomorphic to the free product of the same factors each rescaled by $\lambda$. Applying these facts and Theorem 4.2 to $\widetilde{Q}$, we find a subfactor $P_{-1} \subset P_0$ whose standard invariant is $\mathcal{G}$ and where

\[ P_0 \cong (Q_{s_1} \ast Q_{s_1} \ast \cdots) \ast \cdots \ast (Q_{s_n} \ast Q_{s_n} \ast \cdots) \]
\[ P_{-1} \cong (Q_{t_1} \ast Q_{t_1} \ast \cdots) \ast \cdots \ast (Q_{t_m} \ast Q_{t_m} \ast \cdots). \]

If $\mathcal{G}$ has infinite depth, let $\widetilde{Q} = Q \ast L(F_\infty)$. Let $\mathcal{P}_0$ and $\mathcal{P}_{-1}$ be as in (17) and (18), but with $Q$ replaced by $\widetilde{Q}$. Then we have the isomorphisms as in (21) and (22) but with $Q$ replaced by $\widetilde{Q}$. Since $\mathcal{G}$ has infinite depth, $m(0)$ and $m(-1)$ are infinite.
Since \( \widetilde{Q} \) absorbs a free product with \( L(F_{\infty}) \), using [4, Theorem 3.11] and the above mentioned results about rescalings and absorbtion of \( L(F_{\infty}) \), we have the following isomorphisms:

\[
q_1(0)P_0q_1(0) \cong \widetilde{Q} \ast \left( \bigotimes_{i=2}^{\infty} \widetilde{Q} \beta(0,1)_{i} \right) \cong Q \ast Q \beta(0,1)_{i}
\]

\[
q_1(-1)P_1q_1(-1) \cong \widetilde{Q} \ast \left( \bigotimes_{i=2}^{\infty} \widetilde{Q} \beta(1,0)_{i} \right) \cong Q \ast Q \beta(1,0)_{i}
\]

As in the proof of Theorem 4.2, we get a subfactor \( P_1 = qP_1q \subset P_0 = qP_0q \) with isomorphisms (23) as desired.

The following theorem shows that certain II\(_1\)-factors other than \( L(F_{\infty}) \) have the universal property for subfactors.

**Theorem 4.5.** Let \( Q \) be any II\(_1\)-factor whose fundamental group is equal to the positive real numbers, and let \( P = Q \ast Q \ast \cdots \) be the free product of infinitely many copies of \( Q \). Then for any standard lattice \( \mathcal{G} \) there is a subfactor \( N = N_{G} \) of \( P \) whose standard invariant is \( \mathcal{G} \); also \( N \) is isomorphic to \( P \). Moreover, the map \( \mathcal{G} \mapsto N_{G} \subset P \) is functorial, as described in [8, Theorem 4.3].

**Proof.** The existence of the subfactor \( N_{G} \subset P \) is a direct application of Theorem 4.4. Functoriality follows from results of Popa and Shlyakhtenko as in the proof of [8, Theorem 4.3].

Consider the commuting square (15) determined by the subfactor \( M_{-1} \subset M_{0} \) and let \( \hat{M}_{0} \) be as in (19). With notation as in (11) we have

\[
q(0)\hat{M}_{0}q(0) = q(0)M_{0}q(0) *_{B} (Q \otimes B),
\]

\[
q(-1)\hat{M}_{-1}q(-1) = q(-1)M_{-1}q(-1) *_{B} (Q \otimes B),
\]

where the amalgamated free products are with respect to \( \text{Tr} \)-preserving conditional expectations. From Proposition 27.1 we have

\[
q_1(0)\hat{M}_{0}q_1(0) \cong \begin{cases} (M_{0})_{\beta(0,1)} & \text{if } m(0) = 1 \\ (M_{0})_{\beta(0,1)} & \text{if } m(0) > 1 \end{cases}
\]

and similarly

\[
q_1(-1)\hat{M}_{0}q_1(-1) \cong \begin{cases} (M_{-1})_{\beta(-1,1)} & \text{if } m(-1) = 1 \\ (M_{-1})_{\beta(-1,1)} & \text{if } m(-1) > 1 \end{cases}
\]

From these, and in a manner analogous to the proofs of Theorems 4.2, 4.4 and 4.5, one proves the following results.

**Theorem 4.6.** Let \( Q \) be any II\(_1\)-factor. Then for any finite depth subfactor \( M_{-1} \subset M_{0} \) of II\(_1\)-factors, there are \( m, n \in \mathbb{N} \), positive real numbers \( s_1, \ldots, s_m \) and \( t_1, \ldots, t_n \) and there are \( a > 1 - m \) and \( b > 1 - n \), all depending only on the standard invariant
of $M_{-1} \subset M_0$, and there is a $II_1$-subfactor $\hat{M}_{-1} \subset \hat{M}_0$ having the same standard invariant as $M_{-1} \subset M_0$ and where

$$\hat{M}_0 \cong M_0 \ast Q_{s_1} \ast \cdots \ast Q_{s_m} \ast L(F_a)$$

$$\hat{M}_{-1} = M_{-1} \ast Q_{t_1} \ast \cdots \ast Q_{t_n} \ast L(F_b).$$

**Theorem 4.7.** Let $Q$ be any $II_1$-factor. Then for any finite depth subfactor $M_{-1} \subset M_0$ of $II_1$-factors, there are positive real numbers $s_1, \ldots, s_m$ and $t_1, \ldots, t_n$ depending only on the standard invariant of $M_{-1} \subset M_0$ and there is a $II_1$-subfactor $\hat{M}_{-1} \subset \hat{M}_0$ having the same standard invariant as $M_{-1} \subset M_0$ and where

$$\hat{M}_0 \cong M_0 \ast \left( \bigast_{i=1}^{\infty} Q_{s_i} \right)$$

$$\hat{M}_{-1} = M_{-1} \ast \left( \bigast_{i=1}^{\infty} Q_{t_i} \right).$$

**Theorem 4.8.** Let $Q$ be any $II_1$-factor having fundamental group equal to $\mathbb{R}^*_+$ and let $P = Q \ast Q \ast \cdots$ be the free product of infinitely many copies of $Q$. Then for any subfactor $N \subset M$ of a $II_1$-factor, there is a subfactor $\hat{N} \subset \hat{M}$ whose standard lattice is equal to that of $N \subset M$ and such that $\hat{N} \cong N \ast P$ and $\hat{M} \cong M \ast P$.

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DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION TX 77843–3368, USA

E-mail address: Ken.Dykema@math.tamu.edu