DYNAMICS OF INTERACTING BOSONS:
A COMPACT REVIEW

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The success of the Gross–Pitaevskii and Bogoliubov theories in the description of large systems of interacting bosons led to a substantial effort into rigorously deriving these effective theories. In this work we review the related literature in the context of dynamics of large bosonic systems.

1. Introduction

1.1. Setup

Bose–Einstein Condensation (BEC) is a phenomenon that occurs in systems of bosonic particles where, at sufficiently low temperatures, a macroscopic fraction of the particles starts to occupy a single quantum state. The idea of BEC dates back to the works of Bose and Einstein in 1924 [27, 60] in which they analysed non-interacting systems. Experimentally BEC has been produced only in 1995 by Cornell and Wieman, and Ketterle [11, 55]. Since then, fundamental questions in the rigorous understanding of condensation and fluctuations around the condensate in interacting systems have remained unanswered. Some of them are essential for the understanding of interesting quantum effects that can be observed even on macroscopic scales (such as, for example, superfluidity).

In this work we will review the existing results concerning the dynamics of Bose–Einstein condensates. The physical picture we have in mind is as follows. Let $\Psi_{N,0}$ be the ground state of $H_N^V$ given by

$$H_N^V = \sum_{j=1}^N (-\Delta x_j + V(x_j)) + \frac{1}{N-1} \sum_{1 \leq j < k \leq N} w_N(x_j - x_k).$$

(1.1)
Here \( V \in L^\infty_{\text{loc}}(\mathbb{R}^3, \mathbb{R}) \), satisfying \( V(x) \to \infty \) as \( |x| \to \infty \), is an external potential (which models the trapping of the particles while they are cooled during the experiment) and \( w_N \) is the inter-particle potential that describes their interactions (and could a priori depend on \( N \)). The Hamiltonian acts on the symmetric space \( \mathcal{H}^N = \bigotimes_{\text{sym}}^N L^2(\mathbb{R}^3) \). The underlying physical space, if not specified otherwise, is assumed to be three-dimensional.

When the external potential \( V \) is turned off, \( \Psi_{N,0} \) is no longer a ground state of the Hamiltonian \( H_N \equiv H_N^{V=0} \) and the time evolution

\[
\Psi_{N,t} = e^{-itH_N} \Psi_{N,0}
\]

is observed. Although the Schrödinger evolution is linear, its complexity increases dramatically when \( N \) becomes large. In typical experiments \( N \) can be of order \( 10^5 \). Therefore, for computational purposes, it is important to derive effective descriptions for collective behaviour of the quantum system.

In this contribution we review results concerning the approximation of the time-evolved bosonic many-body quantum system.

1.2. Scaling regimes

From a mathematical point of view, the large number of particles involved will be modeled by taking the limit \( N \to \infty \). Note that the coupling constant \( 1/(N - 1) \) in front of the interaction is to ensure the kinetic energy is of the same order (in \( N \)) as the interaction energy. We could choose \( 1/N \) instead of \( 1/(N - 1) \) as well. The interaction potential \( w_N \),

\[
w_N(x) = N^{3\beta} w(N^{\beta} x)
\]

for some \( w \in L^1(\mathbb{R}^3) \), is chosen to be of \( O(1) \) (as \( N \to \infty \)) in the sense that \( \int w_N = \int w \). Here and thereafter, if not specified differently, the integration sign will correspond to integration over the whole space \( \mathbb{R}^3 \) for each variable.

The parameter \( \beta \) in the definition of \( w_N \) characterises different scaling regimes that correspond to different physical situations. When \( \beta = 0 \) (the so-called Hartree or mean-field scaling) then the Hamiltonian models a situation in which there are many but weak collisions between the particles. This is because the size of the interaction potential is \( O(N^{-1}) \) (because of the prefactor \( 1/(N - 1) \) in front of the interaction term), while the range of the interaction (which is \( O(1) \)) is much larger than the mean inter-particle distance \( N^{-1/3} \). Therefore, heuristically, each particle ‘sees’ all other particles.
When $\beta > 0$, then $w_N$ converges formally to the Dirac-delta interaction

\[ \left( \int w \right) \delta_0. \quad (1.2) \]

As long as $0 \leq \beta < 1/3$, the range of the interaction potential (which is $O(N^{-\beta})$) is much larger than the average distance between the particles and there are many but weak collisions. Therefore, to the leading order, the interaction potential experienced by each particle can be still approximated by the effective mean-field potential $\rho * w_N$ where $\rho$ is the density of the system and $f * g$ denotes the convolution of two function $f$ and $g$.

If $\beta > 1/3$, then the analysis is expected to be more complicated due to strong correlations between particles. Despite the physical difference between the case when $\beta \leq 1/3$ and $\beta > 1/3$, the formal limiting behaviour of the interaction in (1.2) is the same in both cases. Therefore, we will call the regime when $\beta \in (0, 1)$ the NLS regime as the limiting effective description of the condensate will be in that case be given by the nonlinear Schrödinger (NLS) equation.

The case when $\beta = 1$ corresponds to the celebrated Gross–Pitaevskii regime in which strong correlations occur on very short length scales. The macroscopic properties of the system are well captured by the famous Gross–Pitaevskii theory [84, 137]. In this theory, a quantum particle is effectively felt by the others as a hard sphere whose radius is the scattering length of the interaction potential. Recall that the scattering length $a$ of the potential $w$ is defined by the variational formula

\[ 8\pi a = \inf \left\{ \int \left( 2|\nabla f|^2 + w|f|^2 \right), \quad \lim_{|x| \to \infty} f(x) = 1 \right\}. \quad (1.3) \]

When $w$ is sufficiently smooth, (1.3) has a minimizer $0 \leq f \leq 1$ that satisfies

\[ (-2\Delta + w)f = 0. \quad (1.4) \]

The scattering length can then be recovered from the formula

\[ 8\pi a = \int w f. \quad (1.5) \]

By scaling, the scattering length of $w_N = N^2 w(N\cdot)$ is $a N^{-1}$. If we formally replace the interaction potential $w_N(x-y)$ in $H_N$ by $8\pi a N^{-1} \delta_0(x-y)$, we obtain a Hamiltonian with a Delta interaction. Such an object is (in three dimensions) mathematically not well defined. Nevertheless, it is usually taken as a starting point in the physics literature on cold gases.
1.3. **Types of approximation**

Recall that our goal is to understand how the \( N \)-body wave function \( \Psi_{N,t} \) behaves when \( N \) is very large. In the context of dynamics one usually considers three possible effective descriptions. The first one, usually called the **leading order approximation**, considers the approximation of \( \Psi_{N,t} \) in terms of reduced density matrices. Recall, that the (one-body) reduced density matrix of a state \( \Psi_N \in \mathcal{H}^N \) (here we will restrict ourselves to zero temperature) is the positive, trace class operator \( \gamma_{\Psi_N} : \mathcal{H} \rightarrow \mathcal{H} \) with kernel

\[
\gamma_{\Psi_N}(x, y) = N \int dx_2 \cdots dx_N \Psi_N(x, x_2, \ldots, x_N)\bar{\Psi}_N(y, x_2, \ldots, x_N).
\]  

(1.6)

The knowledge of \( \gamma_{\Psi_N} \) allows to determine the expectation values of one-body observables in the state \( \Psi_N \). Indeed, let \( O : \mathcal{H} \rightarrow \mathcal{H} \) be an observable and let \( O_i \) denote the corresponding operator acting on the \( i \)-th particle in the \( N \)-body space. Then

\[
\langle \Psi_N, \left( \sum_{i=1}^N O_i \right) \Psi_N \rangle = \text{Tr}(O \gamma_{\Psi_N}).
\]  

(1.7)

We will say that the full many-body evolution \( \Psi_{N,t} \) is to leading order approximated by \( \Phi_{N,t} \) if

\[
\lim_{N \to \infty} \frac{1}{N} \text{Tr} |\gamma_{\Psi_{N,t}} - \gamma_{\Phi_{N,t}}| = 0.
\]  

(1.8)

Note that, due to (1.7), the trace norm topology is natural in this context. The hope is that \( \Phi_{N,t} \) can be determined in an easier way than \( \Psi_{N,t} \).

The convergence (1.8) is closely related to the definition of BEC [131]. We will say that a system of bosons exhibits BEC in the state \( \Psi_N \in \mathcal{H}^N \) if

\[
\lim_{N \to \infty} \frac{1}{N} \text{Tr} \left| \frac{1}{N} \gamma_{\Psi_N} - |\phi\rangle\langle\phi| \right| = 0
\]  

\[1.9\]

for some \( \phi \in \mathcal{H} \). One then often says that \( \phi \) is the wave function of the condensate. This terminology is related to the fact that if one considers the so-called Hartree or product state, i.e., the uncorrelated \( N \)-body wave function of the form \( \phi^\otimes N := \phi(x_1) \cdots \phi(x_N) \) in which all particles occupy the same one-particle state, then

\[\gamma_{\phi^\otimes N} = N|\phi\rangle\langle\phi|\].

This is why one might sometimes run across the notation

\[\Psi_N \approx \phi^\otimes N \text{ to leading order}\]
which means asymptotic equality in terms of reduced density matrices in
the trace norm topology. Note, that BEC does not mean that all particles
occupy one single-particle state, but only that a macroscopic fraction does.
In fact, while a product state is a ground state of non-interacting system
\( w_N = 0 \), it can’t be one of an interacting system. In particular, if one
considers the state (let us skip the symmetry of the wave function for a
moment)

\[
\Xi_N := \prod_{i=1}^{N-1} \phi(x_i) \phi^\dagger(x_N), \quad \text{with} \quad \phi^\perp \perp \phi
\]

then obviously

\[
\Xi_N \perp \phi \otimes \cdots \otimes N \in L^2(\mathbb{R}^{3N})
\]

but physically, for large \( N \), both states describe a very similar situation.
In particular, both states exhibit BEC with the same condensate wave
function.

Often, the knowledge about correlations in the system is crucial in or-
der to understand some of the physical properties of the system. As seen in
the example above, the leading order approximation is not enough for this
purpose. This is why one considers the most straightforward indicator of
closeness: the \( N \)-particle Hilbert space norm (or, shortly, norm approxima-
tion). More precisely, the goal is to find a \( N \)-body wave function \( \Xi_{N,t} \in \mathcal{F}^N \)
that is easier to compute than \( \Psi_{N,t} \) and such that

\[
\lim_{N \to \infty} \| \Psi_{N,t} - \Xi_{N,t} \|_{\mathcal{F}^N} = 0.
\] (1.10)

Clearly, since

\[
\text{Tr} \left| O(\gamma_{\Psi_N} - \gamma_{\Xi_N}) \right| \leq 2\| O \| \| \Psi_N - \Xi_N \|
\] (1.11)

for any bounded operator \( O \), the norm approximation implies the conver-
gence of reduced one-body density matrices.

Another possible way of approximating the wave function is given by
the Fock space approximation. In this approach, one considers the problem
in the grand-canonical setting, where the number of particles in the system
is not fixed. To this end one introduces the Fock space

\[
\mathcal{F} \equiv \mathcal{F}(\mathcal{F}) = \bigoplus_{n=0}^\infty \mathcal{F}^n = \mathbb{C} \oplus \mathcal{F} \oplus \mathcal{F}^2 \oplus \cdots
\]

The wave function in the Fock space is denoted by \( \Psi \in \mathcal{F}(\mathcal{F}) \) and

\[
\Psi = \{ \Psi^{(0)}, \Psi^{(1)}, \ldots, \Psi^{(j)}, \ldots \}
\]
where $\Psi^{(0)} \in \mathbb{C}$ and $\Psi^{(i)} \in \mathcal{H}^i$ for $i \geq 1$. The inner product on $\mathcal{F}$ is defined as

$$\langle \Psi_1, \Psi_2 \rangle_\mathcal{F} = \sum_{i \geq 0} \langle \Psi_1^{(i)}, \Psi_2^{(i)} \rangle_{\mathcal{H}^i}.$$ 

A state $\Psi_N$ with exactly $N$ particles is described on the Fock space $\mathcal{F}$ by a sequence $\Psi = \{\Psi^{(n)}\}_{n \geq 0}$ where $\Psi^{(n)} = 0$ for all $n \neq N$ and $\Psi^{(N)} = \Psi_N$.

One can lift the many-body evolution to the Fock space. To this end we define the Hamiltonian $\mathcal{H}_N$ on $\mathcal{F}$ by

$$\mathcal{H}_N \Psi^{(n)} = \mathcal{H}^{(n)}_{\Psi} \Psi^{(n)}$$  \hspace{1cm} (1.12)

with the $n$-th sector operator

$$\mathcal{H}^{(n)}_{\Psi} = H_N^{\Psi} = \sum_{j=1}^{n} \left(-\Delta x_j + V(x_j)\right) + \frac{1}{N-1} \sum_{1 \leq j < k \leq n} w_N(x_j - x_k)$$

where now the subscript $N$ is not related to the number of particles, but only reflects the scaling in the interaction potential (of course, in the end, $N$ will also be related with the number of particles in the initial Fock state; otherwise, there would be no relation with the scaling regime).

In particular the $N$-particle evolution can be embedded into the Fock space in the following way

$$e^{-it\mathcal{H}_N} \{0, 0, \ldots, \Psi_N, 0, \ldots\} = \{0, 0, \ldots, e^{-it\mathcal{H}_N} \Psi_N, 0, \ldots\}.$$ 

This follows from the fact that the Hamiltonian $\mathcal{H}_N$ commutes with the particle number operator $N$ given by

$$(N \Psi)^{(n)} = n \Psi^{(n)}.$$  \hspace{1cm} (1.13)

Let $\Psi_0 \in \mathcal{F}$ be a state in the Fock space. We will say that $\Xi_t \in \mathcal{F}$ approximates the many-body evolution of $\Psi_0$ in the Fock space if

$$\lim_{N \to \infty} \|e^{-it\mathcal{H}_N} \Psi_0 - \Xi_t\|_\mathcal{F} = 0.$$  \hspace{1cm} (1.14)

Sometimes, it is possible to get some information on the $N$-particle space convergence from convergence in the Fock space. This approach, however, usually leads to worse estimates than direct methods on $N$-particle space and often requires additional assumptions on the initial states.
1.4. Outline

The paper will be organized as follows. In the next section we will provide a brief overview about the ground state properties of (trapped) bosonic systems. In Section 3 we will review the existing results on the leading order approximation. In Section 4 we will review the literature on the norm approximation. In Section 5 we will mention results on the Fock space approximation.

2. Ground state properties

Recall that we want to consider initial states that are ground states of Hamiltonians of the form (1.1). Therefore it makes sense to briefly review some basic facts concerning this issue. For more details and references we refer to the excellent review [142].

2.1. Leading order approximation for the ground state

It is widely expected that ground states of trapped systems exhibit (complete) BEC. In fact, when $0 \leq \beta < 1$ we have

$$\lim_{N \to \infty} \left( \inf_{\|\Psi_N\|_{1,N} = 1} \frac{\langle \Psi_N, H_N^V \Psi_N \rangle}{N} - \inf_{\|u\|_{1} = 1} E_{H,N}^V(u) \right) = 0$$

(2.1)

where

$$E_{H,N}^V(u) := \frac{1}{N} \langle u^\otimes N, H_N^V u^\otimes N \rangle = \int \left( |\nabla u|^2 + V |u|^2 + \frac{1}{2} |u|^2 (w_N * |u|^2) \right).$$

Moreover, if the Hartree energy functional $E_{H,N}^V(u)$ has a unique minimizer $u_H$, then the ground state $\Psi_N$ of $H_N^V$ condensates on $u_H$ in the sense that

$$\lim_{N \to \infty} \text{Tr} \left( \frac{1}{N} \gamma \Psi_N^\dagger - |u_H\rangle\langle u_H| \right) = 0.$$  (2.2)

The rigorous justifications for (2.1) and (2.2) in various specific cases have been given in [16, 70, 108, 112, 133, 139, 143]. Later, in a series of works [102–104], Lewin, Nam and Rougerie provided proofs in a very general setting. Most recently, the next order term in the expansion (2.2) has been established in [32, 122].

When $\beta = 1$ (the Gross–Pitaevskii regime), the Hartree functional has to be modified to capture the strong correlation between particles. In that case

$$\lim_{N \to \infty} \left( \inf_{\|\Psi_N\|_{1,N} = 1} \frac{\langle \Psi_N, H_N^V \Psi_N \rangle}{N} - \inf_{\|u\|_{1} = 1} E_{\text{GP}}^V(u) \right) = 0$$

(2.3)
where
\[ \mathcal{E}_{GP}(u) := \int \left( |\nabla u|^2 + V|u|^2 + 4\pi \alpha |u|^4 \right) \] (2.4)
is the Gross–Pitaevskii functional. In that case one also has BEC on the Gross–Pitaevskii minimizer
\[ \lim_{N \to \infty} \frac{1}{N} \gamma_{GP} = |u_{GP}\rangle\langle u_{GP}| \] (2.5)
in trace norm. The convergence (2.3) has been first proven by Lieb, Seiringer and Yngvason in [111] while (2.5) has been first proven by Lieb and Seiringer in [109,110] (see also [125]). More recently, the optimal rates of convergence for (2.3) and (2.5) have been given in [21,23] (translation invariant case) and [123] (trapped case with smallness condition on \( \alpha \), see also [85]).

2.2. Second order correction

The next order correction to the lower eigenvalues and eigenfunctions of \( H_N^V \) is predicted by Bogoliubov’s approximation [26]. In the mean-field limit, this has been first derived rigorously by Seiringer in [143], and then extended in various directions in [58,77,106,127]. Bogoliubov theory is formulated in the Fock space \( \mathcal{F} \). At this point, let us briefly recall the notion of second quantization.

We define the creation operator \( a^*(f) \) and the annihilation operator \( a(f) \) that for every \( f \in \mathcal{F} \) is given by
\[
(a^*(f)\Psi)(x_1,\ldots,x_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} f(x_j)\Psi(x_1,\ldots,x_{j-1},x_{j+1},\ldots,x_{n+1}),
\]
\[
(a(f)\Psi)(x_1,\ldots,x_n-1) = \sqrt{n} \int dx_n f(x_n)\Psi(x_1,\ldots,x_n)
\]
for all \( \Psi \in \mathcal{F}^n \) and for all \( n \). These operators satisfy the canonical commutation relations (CCR)
\[ [a(f),a(g)] = [a^*(f),a^*(g)] = 0, \quad [a(f),a^*(g)] = \langle f,g \rangle \] (2.6)
for all \( f, g \in \mathcal{F} \). Creation and annihilation operators are used to represent many-body states and operators on the Fock space. It is well-known (see e.g. [20] or [146]) that for a symmetric operator \( H \) on \( \mathcal{F} \) and an orthonormal basis \( \{f_n\}_{n \geq 1} \subset D(h) \) of \( \mathcal{F} \) one has
\[ d\Gamma(H) := 0 \oplus \bigoplus_{N=1}^{\infty} \sum_{j=1}^{N} H_j = \sum_{m,n \geq 1} \langle f_m,H f_n \rangle a^*(f_m)a(f_n). \] (2.7)
Similarly, for a symmetric operator $W$ on $\mathcal{H} \otimes \mathcal{H}$ such that

$$\langle f_m \otimes f_n, W f_p \otimes f_q \rangle = \langle f_n \otimes f_m, W f_p \otimes f_q \rangle$$

for all $m, n, p, q \geq 1$ we have

$$0 \oplus 0 \oplus \bigoplus_{N=2}^{\infty} \sum_{1 \leq i < j \leq N} W_{ij} = \frac{1}{2} \sum_{m,n,p,q \geq 1} \langle f_m \otimes f_n, W f_p \otimes f_q \rangle_{\mathcal{H}^2} a^*(f_m)a^*(f_n)a(f_p)a(f_q). \quad (2.8)$$

If one does not want to work on a specific orthonormal basis, it is possible to use the operator-valued distributions $a^*_x$ and $ax$, with $x \in \mathbb{R}^3$, defined by

$$a^*_x(f) = \int dx f(x)a^*_x \quad \text{and} \quad a(f) = \int dx \overline{f(x)}a_x$$

for all $f \in \mathcal{H}$. The canonical commutation relations (2.6) then imply that

$$[a^*_x, a^*_y] = [a_x, a_y] = 0 \quad \text{and} \quad [a_x, a_y^*] = \delta(x - y). \quad (2.9)$$

The second quantization formulas (2.7) and (2.8) can be rewritten as

$$d\Gamma(H) = \int dx \, dy \, H(x, y)a^*_x a_y, \quad (2.10)$$

$$0 \oplus 0 \oplus \bigoplus_{N=2}^{\infty} \sum_{1 \leq i < j \leq N} W_{ij} = \frac{1}{2} \int dx \, dy \, dx' \, dy' \, W(x, y; x', y') a^*_x a^*_y a^*_x a_y, \quad (2.11)$$

where $H(x, y)$ and $W(x, y; x', y')$ are the kernels of $H$ and $W$, respectively.

For example, the aforementioned particle number operator can be written as

$$\mathcal{N} := d\Gamma(1) = \bigoplus_{n=0}^{\infty} n1_{\mathcal{H}^n} = \int dx \, a^*_x a_x$$

and the $N$-body Hamiltonian $H_N$ can be extended to an operator on Fock space $\mathcal{F}(\mathcal{H})$ as

$$\mathcal{H}_N = d\Gamma(-\Delta) + \frac{1}{2(N - 1)} \int dx \, dy \, w_N(x - y)a^*_x a^*_y a_x a_y. \quad (2.12)$$

As already mentioned, Bogoliubov theory is formulated in the Fock space $\mathcal{F}$ or, more precisely, the excited Fock space $\mathcal{F}(\mathcal{H}_+) \equiv \mathcal{F}(\{u_H\}^+)$. Let $\{u_m\}_{m \geq 0}$ be an orthonormal basis of $\mathcal{H}$ such that $u_H \equiv u_0$. In the mean-field limit, the condensate is described by the Hartree minimizer $u_H$ and
the excited particles are effectively described by a quadratic Hamiltonian $H^V$ of the form

$$H^V = \sum_{m,n \geq 1} \langle u_m, (h + K_1)u_n \rangle a_m^* a_n + \sum_{m,n \geq 1} \frac{1}{2} \langle u_m \otimes u_n, K_2 \rangle a_m^* a_n^* + \frac{1}{2} \langle K_2, u_m \otimes u_n \rangle a_m a_n$$

acting on $\mathcal{F}(\{u_0\}^\perp)$ and where $K_1 : \mathcal{H} \to \mathcal{H}$ and $K_2 : \mathcal{H} \to \mathcal{H}$ are operators defined by

$$\langle u, K_1 v \rangle = \int dx \, dy \, \overline{u(x)} v(y) u_0(x) u_0(y) w(x - y),$$

$$\langle u, K_2 v \rangle = \int dx \, dy \, \overline{u(x)} v(y) u_0(x) u_0(y) w^2(x - y)$$

for all $u, v \in \mathcal{H}$. Finally, $h$ is the one-body operator given by

$$h = -\Delta + V + |u_0|^2 * w - \mu$$

which comes from the Hartree equation (the Euler–Lagrange equation for the Hartree functional). Here $\mu$ is an appropriate constant to make $hu_0 = 0$.

It has been proven in [106] by Lewin, Nam, Serfaty, Solovej that if the Hartree minimizer $u_H$ is non-degenerate (in the sense that the Hessian of $E^V_{\mathcal{H}}(u)$ at $u_H$ is bigger than a positive constant), then the ground state $\Psi^V_N$ of $H^V_N$ admits the norm approximation

$$\lim_{N \to \infty} \left\| \Psi^V_N - \sum_{n=0}^{N} u_H^{\otimes (N-n)} \otimes \psi_n \right\|_{\mathcal{H}^N} = 0$$

where $\Phi^V = (\psi_n)_{n=0}^\infty \in \mathcal{F}(\{u\}^\perp)$ is the (unique) ground state of $H^V$. Note that the norm convergence (2.15) shows what we mentioned before, that is, the fact that if $w \neq 0$, then $\Phi^V$ is not the vacuum $\Omega := 1 \oplus 0 \oplus 0 \cdots$, and hence $\Psi^V_N$ is never close to $u_H^{\otimes N}$ in norm. For $\beta > 0$, Bogoliubov theory has been justified for translation invariant systems in [24] ($\beta < 1$) and in [22] ($\beta = 1$).

3. Leading order approximation

In this section we shall review the results about the leading order approximation for the Schrödinger evolution of a bosonic many-body wave function in the sense of (1.8). From a physics perspective we want to answer the following question: if the initial state of a trapped system exhibits BEC, does
the condensate endure once the trap is switched off and the system starts evolving in time?

Thus, one would like to show that if the initial many-body wave function $\Psi_{N,0}$ satisfies

$$\lim_{N \to \infty} \text{Tr} \left[ \frac{1}{N} \gamma_{N,0} \Psi_{N,0} - |\phi_0\rangle \langle \phi_0| \right] = 0$$

for some $\phi \in L^2(\mathbb{R}^3)$, then

$$\lim_{N \to \infty} \text{Tr} \left[ \frac{1}{N} \gamma_{N,t} \Psi_{N,t} - |\phi_t\rangle \langle \phi_t| \right] = 0$$

for some $\phi_t$ which can be found via an effective theory.

First results of that type have been obtained in the mean-field regime ($\beta = 0$) by Hepp [86] (for differentiable $w$) and by Spohn in [147] (for $w$ bounded) (although the setup there, especially in the work of Hepp, was a priori quite different than the one presented here). In the more familiar setup explained in the introduction the discussed question has been raised in the literature again in the 2000’s. Since then, a substantial effort of the community led to many interesting results which often differ only slightly. Those differences might be difficult to spot for non-specialists and one of the goals of this work is to clarify some of these issues.

### 3.1. Results for different scaling regimes

#### 3.1.1. Mean-field scaling

As mentioned in the introduction, the simplest regime to consider is the Hartree scaling. In this case, the general (and imprecise) form of the statement describing the leading order approximation is given by the following

**Theorem 3.1:** (Leading order approximation for mean-field dynamics) Let $\Psi_{N,0}$ be an initial state satisfying

$$\lim_{N \to \infty} \text{Tr} \left[ \frac{1}{N} \gamma_{N,0} \Psi_{N,0} - |u_0\rangle \langle u_0| \right] = 0$$

for a normalized wave function $u_0 \in L^2(\mathbb{R}^3)$. Then

$$\lim_{N \to \infty} \text{Tr} \left[ \frac{1}{N} \gamma_{N,t} \Psi_{N,t} - |u_t\rangle \langle u_t| \right] = 0$$
where \( \Psi_{N,t} = e^{-itH_N} \Psi_{N,0} \) is the many-body wave function evolved by the mean-field (with \( \beta = 0 \)) Hamiltonian \( H_N \) (with \( V = 0 \)) and \( u_t \) is the solution of the time-dependent Hartree equation

\[
i\partial_t u_t = (-\Delta + w \ast |u_t|^2 - \mu_t) u_t
\]  

(3.5)

with the initial datum \( u_0 \) and for some appropriate phase \( \mu_t \in \mathbb{R} \).

Note, that for the leading order the phase plays no role as it does not alter the projection \( |u_t \rangle \langle u_t| \).

The first result of the form of Theorem 3.1 was obtained by Bardos, Golse and Mauser in [13] (with the additional condition \( \langle \Psi_{N,0}, H_N \Psi_{N,0}\rangle \leq CN \)). Shortly afterwards Erdös and Yau obtained in [69] the same result for initial states that had to be a product state. Clearly, the assumption (3.3) allows for more general initial states. We refer to [14] for a recap and comparison of the two papers. We note that the work of Erdös and Yau allowed to take \( w(x) = 1/|x| \), i.e., the Coulomb potential. Both these works use the BBGKY (Bogoliubov–Born–Green–Kirkwood–Yvon) hierarchy method (cf. Section 3.2). In particular, the BBGKY method does not give any rates of convergence in (3.4).

The question of the convergence rate has been first answered by Rodnianski and Schlein in [140]. Using the method of coherent states (cf. Section 3.2) they showed that the convergence rate in (3.4) is of the form

\[
\frac{C}{\sqrt{N}} e^{Ct}
\]

for an initial state that is a product state. Their work included the Coulomb interaction. This result has been extended in [95] by Knowles and Pickl to cover more singular potentials (in the sense of the function \( w \) rather than the scaling which was still mean-field) and initial states that are not necessarily product states but satisfy the more general condition (3.3). In their work Knowles and Pickl used a method that was developed by Pickl in [135] (which also provides a relatively simple, quantitative proof of (3.4) for nice potentials in the mean-field setting, see also [3, 8, 72, 107]).

For nicer (bounded and integrable) interaction potentials Erdös and Schlein proved in [62] an optimal convergence rate

\[
\frac{C}{N} e^{Ct}
\]

(again, they assumed factorized initial conditions). This result has been extended in [36] by Chen and Lee to cover more general potentials and then,
together with Schlein, further improved to cover the Coulomb case [37] (see also [96]).

In [61, 116] the convergence (3.4) was established for particles with a relativistic dispersion relation (the kinetic energy \(-\Delta\) is in that case replaced by \(\sqrt{1-\Delta}\)) and with Coulomb type interaction \(w(x) = \pm 1/|x|\) (this situation is physically interesting because it describes systems of gravitating bosons, so called boson stars, and the related phenomenon of stellar collapse). These systems have been further studied in [99] (optimal convergence rate) and [9] (convergence in Sobolev trace norms). A detailed analysis of the differences in various results regarding the leading order convergence of mean-field bosonic systems can be found in [88].

Further developments in the study of the leading order behaviour of bosonic systems include the analysis of the mean-field limit coupled to a semi-classical limit [5–7, 71, 73, 75, 76, 98], compound mean-field systems [10, 56, 100, 114], systems with magnetic fields [113], systems with three-body interactions [101], central limit type theorems for bosonic mean-field dynamics [15, 35, 92]. Finally, let us mention that for bounded potentials a systematic, perturbative way to compute higher order terms in the expansion of (3.4) has been developed recently in [30] (see also [130]).

3.1.2. NLS regime

The analysis of the dynamics becomes more complicated for positive \(\beta\). For \(\beta \in (0, 1)\) the typical result is of the form

**Theorem 3.2:** (Leading order approximation in the NLS regime) Let \(\Psi_{N,0}\) be an initial state satisfying

\[
\lim_{N \to \infty} \text{Tr} \left| \frac{1}{N} \gamma_{\Psi_{N,0}} - |u_0\rangle \langle u_0| \right| = 0
\]

for a normalized wave function \(u_0 \in H^1(\mathbb{R}^3)\). Then

\[
\lim_{N \to \infty} \text{Tr} \left| \frac{1}{N} \gamma_{\Psi_{N,t}} - |u_t\rangle \langle u_t| \right| = 0
\]

where \(\Psi_N(t) = e^{-itH_N} \Psi_{N,0}\) is the many-body wave function evolved by the many-body Hamiltonian \(H_N\) (with \(V = 0\)) in the NLS regime \((\beta \in (0, 1))\) and \(u_t\) is the solution of the time-dependent nonlinear Schrödinger equation

\[
i\partial_t u_t = (-\Delta + b_0|u_t|^2 - \mu_t)u_t
\]

with the initial data \(u_0\). Here \(b_0 = \int w\) and \(\mu_t \in \mathbb{R}\) is an appropriate phase.
The first, complete proof of Theorem 3.2 has been given by Erdös, Schlein and Yau in [65]. It was valid for $\beta < 1/2$ and initial states which are a product state. Later, in [68], Erdös, Schlein and Yau extended this result to all $\beta \in (0, 1)$ and the general initial states (3.6). Both papers used the BBGKY approach (cf. Section 3.2). For $\beta < 1/6$, a similar result (for general initial states and without the assumption on the positivity of the interaction potential) has been obtained by Pickl in [134]. This work provided also explicit bounds on the convergence rate.

In one dimension, for $\beta \in (0, 1)$, the problem has been solved by Adami, Golse and Teta [1] (see also [2, 141]). In two dimensions (on a torus) the problem has been studied (for $\beta < 3/4$) by Kirkpatrick, Schlein and Stafillani in [93] and more recently by Jeblick and Pickl in [90] (without the positivity assumption on the interaction). Other results about the leading order approximation in the NLS regime include lower dimensional systems with attractive interactions [45, 49], systems with three-body interactions [40, 50], derivations of lower dimensional dynamics from the three dimensional problem [28, 44, 48, 91, 144].

3.1.3. The GP regime.

The leading order approximation problem in the Gross–Pitaevski regime has been first solved by Erdös, Schlein and Yau in [65] where they proved the following theorem:

**Theorem 3.3:** (Leading order approximation in the Gross–Pitaevskii regime [67, Thm. 3.1]) Assume $w \geq 0$ is a smooth, even potential that decays sufficiently fast and has scattering length $a$. Let $\Psi_{N,0}$ be a family of initial wave functions such that

$$\langle \Psi_{N,0}, H_N \Psi_{N,0} \rangle \leq CN,$$

which exhibit BEC

$$\lim_{N \to \infty} \text{Tr} \left| \frac{1}{N} \gamma_{\Psi_{N,0}} - |\varphi_0\rangle\langle \varphi_0| \right| = 0 \quad (3.9)$$

for a normalized wave function $\varphi_0 \in H^1(\mathbb{R}^3)$. Then

$$\lim_{N \to \infty} \text{Tr} \left| \frac{1}{N} \gamma_{\Psi_{N,t}} - |\varphi_t\rangle\langle \varphi_t| \right| = 0 \quad (3.10)$$

where $\Psi_{N,t} = e^{-itH_N} \Psi_{N,0}$ is the many-body wave function evolved by many-body Gross–Pitaevskii Hamiltonian $H_N$ (with $V = 0$ and $\beta = 1$) and $\varphi_t$ is
the solution of the time-dependent Gross–Pitaevskii equation

\[ i\partial_t \varphi_t = \left( -\Delta + 8\pi a|\varphi_t|^2 \right) \varphi_t \tag{3.11} \]

with the initial condition \( \varphi_0 \).

Earlier, the same authors proved in [68] the same result under the additional assumption that

\[ \sup_{r \geq 0} \{ r^2 w(r) \} + \int_0^\infty dw(r) \tag{3.12} \]

is small enough. To remove this smallness condition Erdős, Schlein and Yau used an intrinsic characterization of the correlation structure in terms of the two-particle scattering wave operator. Generally, however, both works [67, 68] were based on the BBGKY hierarchy method and did not provide any quantitative estimates on the convergence.

Explicit bounds on the convergence rate in (3.10) have been later obtained by Benedikter, de Oliveira and Schlein in [17], by Pickl in [136], and by Brennecke and Schlein in [33].

In the periodic setting on a unit torus partial results in the spirit of Theorem 3.3 (with a modified many-body Hamiltonian which had a cut-off to prevent pair interactions whenever at least three particles come into a region with diameter much smaller than the typical inter-particle distance) have been obtained by Erdős, Schlein and Yau in [64] and then the problem has been solved by Sohinger in [145].

The two-dimensional problem has been solved by Jehlick, Leopold and Pickl in [89]. At this point let us stress that the Gross–Pitaevskii scaling in two dimensions is characterized by the scaling \( w_N(x) = e^{2N} w(e^N x) \) rather than \( w_N(x) = N^2 w(N^2 x) \). Other results in this regime include the dimensionally reduced dynamics [29, 31], dynamics in magnetic fields [129], dynamics of (pseudo-)spinor systems [115] and central limit type theorems for dynamics [138].

3.2. Methods

In this section we shall very briefly explain the two main approaches to prove the leading order convergence. For a more pedagogical introduction we refer to the excellent lecture notes of Benedikter, Porta and Schlein [18].
3.2.1. The BBGKY hierarchy

The BBGKY approach is based on the idea of investigating the $k$-body reduced density matrices rather than the wave function itself. For a given $k$, the $k$-body reduced density matrix of the state $\Psi_N$ is the generalization of the one-body reduced density matrix and allows to compute expectation values of $k$-body operators. It is defined as the operator $\gamma^{(k)}_\Psi$ on $\mathcal{H}^k$ whose kernel satisfies

$$\binom{N}{k}^{-1} \gamma^{(k)}_\Psi(x_1, \ldots, x_k; y_1, \ldots, y_k) = \int dx_{k+1} \ldots dx_N \Psi_N(x_1, \ldots, x_k, x_{k+1}, \ldots, x_N) \Psi_N(y_1, \ldots, y_k, x_{k+1}, \ldots, x_N).$$

(3.13)

Note that by setting $k = 1$ we recover (1.6). In other words,

$$\gamma^{(k)}_\Psi = \binom{N}{k} \text{Tr}_{k+1} |\Psi_N\rangle\langle\Psi_N|.$$

Using the Schrödinger equation, or, more precisely, the von Neumann equation, one can obtain a hierarchy of equations for $\tilde{\gamma}^{(k)} = (N)\gamma^{(k)}_\Psi$ of the form\footnote{To be precise, the hierarchy (3.14) arises for the $H_N$ with the coupling constant $N^{-1}$ rather than $(N-1)^{-1}$ in front of the interaction term.}

$$i\partial_t \tilde{\gamma}^{(k)}_{\Psi, t} = \sum_{j=1}^k [-\Delta x_j, \tilde{\gamma}^{(k)}_{\Psi, t}] + \frac{1}{N} \sum_{i<j} w_N(x_i - x_j), \tilde{\gamma}^{(k)}_{\Psi, t}$$

$$+ \frac{N - k}{N} \sum_{j=1}^k \text{Tr}_{k+1} \left[ w_N(x_j - x_{k+1}), \tilde{\gamma}^{(k+1)}_{\Psi, t} \right].$$

(3.14)

where we use the convention $\tilde{\gamma}^{(N+1)}_{\Psi, t} = 0$. Consider the mean-field limit, i.e., $w_N = w$. Taking a formal limit $N \to \infty$ one obtains

$$i\partial_t \tilde{\gamma}^{(k)}_{\infty, t} = \sum_{j=1}^k [-\Delta x_j, \tilde{\gamma}^{(k)}_{\infty, t}] + \sum_{j=1}^k \text{Tr}_{k+1} \left[ w(x_j - x_{k+1}), \tilde{\gamma}^{(k+1)}_{\infty, t} \right].$$

(3.15)

In the NLS/GP regime the limiting equation obtained from (3.14) is of the form

$$i\partial_t \tilde{\gamma}^{(k)}_{\infty, t} = \sum_{j=1}^k [-\Delta x_j, \tilde{\gamma}^{(k)}_{\infty, t}] + \sigma \sum_{j=1}^k \text{Tr}_{k+1} \left[ \delta(x_j - x_{k+1}), \tilde{\gamma}^{(k+1)}_{\infty, t} \right].$$

(3.16)

with $\sigma = \int w$ for $\beta \in (0, 1)$ and $\sigma = 8\pi a$ for $\beta = 1.$
One can check that (3.15)/(3.16) has a solution given by the Hartree/NLS(GP) equation, i.e.,

\[ \tilde{\gamma}^{(k)}_{\infty, t} = \langle |u_t\rangle \langle u_t| \rangle \otimes k \]

where \( u_t \) solves the Hartree/NLS(GP) equation. This leads to the following strategy of proving results like Theorems 3.1-3.3 which consists of three main steps:

1. **Compactness**: one needs to prove compactness of the sequence (in \( N \)) of \( \{ \tilde{\gamma}^{(k)}_{\Psi_{N, t}} \}_{k=1}^N \) with respect with an appropriate (weak) topology.
2. **Convergence**: one needs to characterize limit points of the sequence \( \{ \tilde{\gamma}^{(k)}_{\Psi_{N, t}} \}_{k=1}^N \) as solutions of (3.15)/(3.16).
3. **Uniqueness**: one has to prove the uniqueness of the solution of (3.15)/(3.16).

Proofs of all these steps can be accomplished in various ways depending on the details of the model (like the regularity and sign of \( w \), initial conditions etc.). Compactness is usually achieved via a priori estimates. The larger \( \beta \), the more difficult it is to obtain those a priori estimates. The a priori estimates will also determine the functional spaces where the solutions can live in. In general, the most difficult step is to prove uniqueness. In the NLS regime \( (\beta < 1/2) \) Erdös, Schlein and Yau [65] proved uniqueness using Feynman diagrams (for example, in their earlier work with Elgart [63] they were not able to show uniqueness). In [94] Klainerman and Machedon provided an alternative approach based on appropriate (conjectured) space-time bounds on limit points of \( \{ \tilde{\gamma}^{(k)}_{\Psi_{N, t}} \}_{k=1}^N \). This set up a program in which various research groups tried to establish these bounds. That was first successfully done by Kirkpatrick, Schlein and Staffilani on \( \mathbb{T}^d \) in [93]. In \( \mathbb{R}^3 \) the conjecture for \( \beta < 1 \) has been established by X. Chen and Holmer in [47] (see also [46] and the works by Chen and Pavlović [39,41–43]). For \( \beta = 1 \) a new proof of uniqueness of the hierarchy has been given by Chen, Hainzl, Pavlović and Seiringer in [38]. Other recent, related works include [4, 51, 87].

### 3.2.2. Quantitative approaches

As mentioned earlier, the BBGKY hierarchy approach does not, in general, provide any convergence rate in (3.2). In 2009 Rodnianski and Schlein used the coherent states approach to obtain a quantitative version of Theorem 3.1 for the first time. Their method was inspired by the work of Hepp [86] and Ginibre and Velo [74]. The idea is to consider the problem in the Fock
space. The initial state is a coherent state which is obtained by applying Weyl’s unitary operator $W(f) = \exp(a^*(f) - a(f))$ to the vacuum:

$$W(f)\Omega = e^{-\|f\|^2/2} \sum_{n\geq 0} \frac{1}{\sqrt{n!}} f^\otimes n.$$  \hspace{1cm} (3.17)

In particular, each $n$-particle component of this state is a product state. The Fock space evolution is then governed by $H_N$ defined in (1.12). More precisely, the initial state, in order to model a system of $N$ particles has to be scaled and is given by $W(\sqrt{N}\phi)\Omega$ where $\phi$ corresponds to the initial data condition (3.1).

For states $\Psi \in \mathcal{F}$ in the Fock space we define the one-body reduced density matrix of $\Psi$ to be the operator (on $\mathcal{H}$) with the kernel

$$\Gamma_{\Psi}(x; y) := \frac{\langle \Psi, a^* x a y \Psi \rangle}{\langle \Psi, \mathcal{N} \Psi \rangle}.$$  

Clearly, this definition reduces (up to a normalization factor) to (1.6) for states with exactly $N$ particles.

Rodnianski and Schlein proved in [140] that in the mean-field limit, the one-body reduced density matrix $\Gamma_{\Psi_t}$ of the state $\Psi_t = e^{iH_N t} W(\sqrt{N}u_0)\Omega$ satisfies

$$\text{Tr} \left| \Gamma_{\Psi_t} - |u_t\rangle \langle u_t| \right| \leq \frac{C e^{Kt}}{N}$$

for some constants $C$ and $K$. Here $u_t$ is the solution of the Hartree equation with initial condition $u_0$.

Let us stress again, that in this set-up the state $\Gamma_{\psi_t}$ depends on $N$ in the grand-canonical sense: $N$ is the expected number of particles in the initial state. In particular, the result does not, a priori, cover canonical initial conditions (which would be states coming from wave functions in $\mathcal{H}^N$). However, a nice property of coherent states allows to project the result above to the $N$-particle sector. To do this, one uses the following representation of a product state in terms of coherent states:

$$\frac{(a^*(u))^N}{\sqrt{N!}} \Omega = d_N \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta N} W(e^{-i\theta} \sqrt{N} u)\Omega$$

where the constant $d_N = \sqrt{N!} N^{-N/2} e^{-N/2}$ satisfies $d_N \approx N^{1/4}$ for large $N$. For the $N$-particle state

$$\Psi_{N,t} = e^{H_N} \frac{(a^*(u_0))^N}{\sqrt{N!}} \Omega$$
one then gets

\[
\gamma_{\Psi_N,t} = \frac{d^2}{N} \int_0^{2\pi} \frac{d\theta_1}{2\pi} \int_0^{2\pi} \frac{d\theta_2}{2\pi} e^{-i(\theta_1 - \theta_2)N}
\times \langle e^{itH_N} W(e^{-i\theta_1} \sqrt{N} u_0) \Omega, a_{\phi_t}^* a_{\phi_t} e^{itH_N} W(e^{-i\theta_2} \sqrt{N} u_0) \Omega \rangle_F.
\]

For the inner product in the Fock space one can then use the result about \(\Gamma_{\Psi_t}\). We see that the price one pays by projecting the Fock space result onto the \(N\)-particles sector is given by the constant \(d^2_N \approx N^{-1/2}\). This is the reason why the rate of convergence in the canonical ensemble is \(N^{-1/2}\) rather than the \(N^{-1}\) above. In the Gross–Pitaevskii (\(\beta = 1\)) regime a similar approach has been adopted by Benedikter, De Oliveira and Schlein in [17] where they proved (by introducing Bogoliubov transformations to track the correlations) that

\[
\text{Tr} \left| \Gamma_{\Psi_t} - |\phi_t\rangle\langle\phi_t| \right| \leq \frac{C e^{Kt}}{N^{1/2}}
\]

where \(\phi_t\) solves the Gross–Pitaevskii equation. As before, this result was formulated in the Fock space but this time for \(\text{correlated}\) initial states (cf. (5.1)). In this case the extension to initial \(N\)-particle states can only be done under additional assumptions which also make the convergence slower (see [66] for an earlier analysis on how correlations form). Most recently, the method of [17] has been further extended in [33] by Brennecke and Schlein to the case of \(N\)-particle initial states (with convergence rate \(O(N^{-1/2})\)).

A different method has been introduced by Pickl in [134]. It has been used in many works, in particular in [136] where the Gross–Pitaevskii regime was analyzed. The Pickl method, as it now often called, is based on the analysis of a certain functional, usually called \(\alpha_N(\Psi_N, \varphi)\), which counts (in a weighted way) the number of particles of the \(N\)-body state \(\Psi_N\) that are in the one-particle state \(\varphi\). The functional is applied to the Schrödinger time evolved many-body wave function \(\Psi_{N,t}\) and the relevant one-body state \(u_t\) in the mean-field or NLS regime or \(\varphi_t\) in the GP regime. The crucial property of the functional \(\alpha_N\) is that

\[
(\alpha_N(\Psi_N, \varphi) \to 0) \Rightarrow (N^{-1} \gamma_{\Psi_N} \to |\varphi\rangle\langle\varphi|)
\]

as \(N \to \infty\). The analysis then concentrates on deriving an estimate on the time derivative of \(\alpha_N(\Psi_{N,t}, \varphi_t)\) in order to apply Grönwall’s argument. In particular, this method is suited to cover initial conditions of the (general) form (3.1).
4. Norm approximation

We shall now review results concerning the norm approximation of many-boson dynamics. As mentioned in the Introduction (recall (1.11)), this notion of closeness is more precise than the leading order approximation discussed in the previous section. The norm approximation is also well suited for initial states that are $N$-particle states.

In the mean-field regime, this problem has been first analyzed by Lewin, Nam and Schlein in [105]. They considered the $N$-particle initial states of the form

$$\Psi_{N,0} = \sum_{n=0}^{N} u_0 \otimes (N-n) \otimes_s \psi_{n,0}$$

(4.1)

where $\Phi_0 := (\psi_{n,0})_{n=0}^{\infty} \in \mathcal{F}(\{u_0\}^+)$. This form is motivated by the ground state property (2.15) of trapped systems. It was proved in [105] that when $\beta = 0$, the time evolution $\Psi_{N,t} = e^{-iH_N t} \Psi_{N,0}$ satisfies the norm approximation

$$\lim_{N \to \infty} \left\| \Psi_{N,t} - \sum_{n=0}^{N} u_t \otimes (N-n) \otimes_s \psi_{n,t} \right\|_{F^N} = 0$$

(4.2)

where $u_t$ is the Hartree evolution (3.5) with the right phase factor

$$\mu_t = \frac{1}{2} \int dx \int dy |u_t(x)|^2 w(x-y)|u_t(y)|^2$$

and the evolution of $\Phi_t = (\psi_{n,t})_{n=0}^{\infty} \in \mathcal{F}(\{u_t\}^+)$ is generated by a quadratic Bogoliubov Hamiltonian. This approach was later developed by Nam and the author of this article in [118] for $\beta < 1/3$ and in [119] for $\beta < 1/2$ (see also [120]).

The crucial ingredient to pass from the $N$-body Hilbert space to the Fock space (which is natural when describing correlations) is the mapping first introduced in the static case by Lewin, Nam, Serfaty and Solovej in the derivation of Bogoliubov theory [106]. The transformation allows to factor out the condensate from the many-body wave function and is given by

$$U_N(t) : \mathcal{F}^N \to \mathcal{F}_+^\leq N(t) := \bigoplus_{n=0}^{N} \mathcal{F}_+(t)^n,$$

(4.3)

$$\Psi \mapsto \psi_0 \oplus \psi_1 \oplus \cdots \oplus \psi_N.$$

where $\mathcal{F}_+(t) = \{u_t\}^\perp$. The space $\mathcal{F}_+^\leq N$ is often called the truncated (because the $m$-particle sectors for $m > N$ are zero), excited (because it describes
excitations around the condensate) Fock space. The idea is to reformulate the Schrödinger evolution $\Psi_{N,t} = e^{-iH_N} \Psi_{N,0}$ in terms of

$$\Phi_{N,t} := U_N(t) \Psi_{N,t}$$

which belongs to $\mathcal{F}_N \subseteq \mathcal{F}^\leq N(t)$ and satisfies the equation

$$\left\{ \begin{array}{l}
\overline{\Phi}_{N,t} = \overline{H}_N(t) \Phi_{N,t}, \\
\Phi_{N,0} = 1^{\leq N} \Phi_0.
\end{array} \right. \tag{4.4}$$

Here $1^{\leq N}$ is the projection onto $\mathcal{F}^\leq N = \mathbb{C} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H}^N$ and

$$\overline{H}_N(t) = 1^{\leq N} \left[ \mathbb{H}(t) + \frac{1}{2} \sum_{j=0}^{4} (R_j + R_j^*) \right] 1^{\leq N}$$

with

$$\mathbb{H}(t) := d\Gamma(h(t)) + \frac{1}{2} \int dx \, dy \left( K_2(t,x,y) a_x^* a_y^* + \overline{K_2(t,x,y)} a_x a_y \right),$$

$$h(t) = -\Delta + \left| u_t(\cdot) \right|^2 * w_N - \mu_t + Q(t) \widetilde{K}_1(t) Q(t),$$

$$K_2(t, \cdot, \cdot) = Q(t) \otimes Q(t) \widetilde{K}_2(t, \cdot, \cdot).$$

Here $\widetilde{K}_1(t)$ is the operator on $\mathcal{H}$ with kernel $\widetilde{K}_1(t,x,y) = u_t(x) w_N(x - y) \overline{u_t(y)}$, and $\widetilde{K}_2(t,x,y) = u_t(x) w_N(x - y) u_t(y)$.

The state $\Phi_{N,t}$ describes the excitations around the condensate. Bogoliubov theory assumes that the operators $R_j$ (which we did not write out explicitly) are small in an appropriate sense. Thus one may expect that the evolution $\Phi_{N,t}$ in (4.4) is close (in norm) to the solution of the effective Bogoliubov equation

$$\left\{ \begin{array}{l}
\overline{i\partial_t} \Phi_t = \mathbb{H}(t) \Phi_t, \\
\Phi_{t=0} = \Phi_0.
\end{array} \right. \tag{4.5}$$

Heuristically, the final steps consist of the proof that the number of (excited) particles in the state $\Phi_t$ is uniformly bounded. To obtain these kind of bounds one exploits the Bogoliubov equation (4.5). In particular, to do so, one uses that appropriate norms of the solution of the Hartree equation (which for $\beta > 0$ is $N$-dependent) are uniform in $N$. This approach, at least for $\beta < 1/3$, turns out to work also in the case when the interaction is attractive (up to times for which the effective equation is well posed). This has been later exploited by Nam and the author of this note in [121] where they derived the focusing NLS in dimensions one and two.
In regimes with $\beta > 1/2$ the short scale correlation structure developed by the solution of the many-body Schrödinger equation cannot be appropriately described by a time-dependent Bogoliubov transformation satisfying an equation of the form (4.5). A modified approach is needed and this has been done by Brennecke, Nam, Schlein an the author of this note in [34]. To take correlations into account more precisely, it is useful to consider the ground state of the Neumann problem

$$\left[-\Delta + \frac{1}{2N}w_N\right]f_N = \lambda_N f_N$$

(4.6)
on the ball $|x| \leq \ell$, for a fixed $\ell > 0$. One fixes $f_N(x) = 1$, for $|x| = \ell$, and extends $f_N$ to $\mathbb{R}^3$ requiring that $f_N(x) = 1$ for all $|x| \geq \ell$. Because of the scaling of the potential $w_N$, the scattering process takes place in the region $|x| \ll 1$; for this reason, the precise choice of $\ell$ is not very important, as long as $\ell$ is of order one.

The solution of (4.6) can be used, first of all, to give a better approximation of the evolution of the condensate wave function, replacing the solution of the limiting nonlinear Schrödinger equation (3.8) with the solution of the modified, $N$-dependent, Hartree equation

$$i\partial \varphi_{N,t} = -\Delta \varphi_{N,t} + (w_N f_N |\varphi_{N,t}|^2)\varphi_{N,t}$$

(4.7)
with initial data $\varphi_{N,0} = \varphi_0$ describing the condensate at time $t = 0$.

Furthermore, (4.6) can be used to describe correlations among particles. To this end, let

$$T_{N,t} = \exp \left(\frac{i}{2} \int dx \, dy \, [k_{N,t}(x,y) a_x a_y - \text{h.c.}] \right)$$

(4.8)
with the integral kernel

$$k_{N,t}(x,y) = (Q_{N,t} \otimes Q_{N,t}) \left[-N(1 - f_N(x-y)\varphi_{N,t}(x+y)/2)^2 \right]$$

(4.9)
where $Q_{N,t} = 1 - |\varphi_{N,t}| \langle \varphi_{N,t} \rangle$ is the orthogonal projection onto the orthogonal complement of the solution of the modified Hartree equation (4.7). In particular, in this context the operator $U_{N}(t)$ will now project onto the orthogonal compliment of $\varphi_{N,t}$ and will be denoted by $U_{\varphi_{N,t}}$. Since $T_{N,t}$ aims at generating correlations, it is natural to define its kernel $k_{N,t}$ through the solution of (4.6). In particular, the choice (4.8) guarantees a crucial cancellation in the generator of the fluctuation dynamics. The final result can be formulated as follows
Theorem 4.1: (Norm approximation in the NLS regime [34, Theorem 3]) Consider the initial state $\Psi_{N,0} \in L^2_\gamma(\mathbb{R}^{3N})$ with the reduced one-particle density matrix $\gamma_{\Psi_{N,0}}$ such that

$$N - \langle \varphi_0, \gamma_{\Psi_{N,0}} \varphi_0 \rangle \leq C$$

(4.10)

and

$$\left| \frac{1}{N} \langle \Psi_{N,0}, H_{\text{GP}}^N \Psi_{N,0} \rangle - \mathcal{E}_{\text{GP}}^V(\varphi_0) \right| \leq C N^{-1}$$

(4.11)

with $\mathcal{E}_{\text{GP}}^V$ defined in (2.4). Let $\Psi_{N,t}$ be the solution of the Schrödinger equation with initial data $\Psi_{N,0}$. Then, for all $\alpha < \min\{\beta/2, (1 - \beta)/2\}$, there exists a constant $C > 0$ such that

$$\| \Psi_{N,t} - U_{\varphi_{N,t}}^* T_{N,t} e^{-\int_0^t d\tau \eta_N(\tau) U_2(t; 0) T_{N,0} U_{\varphi_{N,0}}^* \Psi_{N,0}} \|$$

$$\leq C N^{-\alpha} \exp(C \exp(C|t|))$$

(4.12)

for all $N$ sufficiently large and all $t \in \mathbb{R}$. Here $\eta_N(t)$ is a phase factor and $U_2(t; 0)$ is a unitary dynamics on $\mathcal{F}$ with an appropriate quadratic generator that can be defined using $\varphi_{N,t}, w_N, k_{N,t}$ (see [34, eq. (41)]).

Notice that the conditions (4.10) and (4.11) have been recently justified in [123]. Other results on the norm approximation (involving a slightly different approach which avoids using second quantization) in the mean-field regime have been obtained by Mitrouskas, Petrat and Pickl in [117]. Petrat, Pickl and Soffer extended this result to a mean-field analysis coupled to a large volume in [132]. A perturbative expansion has been analyzed by Bossmann, Petrat, Pickl and Soffer in [30]. In [126] Nam and Salzmann provided a norm approximation for systems with three-body interactions in the NLS regime.

Let us present an outline of the proof of Theorem 4.1, as the strategy is slightly different than the one for $\beta < 1/2$ and does not involve the analysis of the Bogoliubov equations. The start is similar and involves the action of the map $U_{\varphi_{N,t}}$ on $\Psi_{N,t}$. This allows us to remove the condensate described at time $t$ by $\varphi_{N,t}$ and to focus on the orthogonal fluctuations. We set

$$\Phi_{N,t} = U_{\varphi_{N,t}}^* \Psi_{N,t},$$

(4.13)

and we observe that $\Phi_{N,t} \in \mathcal{F}_{\leq N}$ satisfies the equation

$$i \partial_t \Phi_{N,t} = \mathcal{L}_{N,t} \Phi_{N,t}$$

(4.14)

with the generator

$$\mathcal{L}_{N,t} = (i \partial_t U_{\varphi_{N,t}}) U_{\varphi_{N,t}}^* + U_{\varphi_{N,t}}^* H_N U_{\varphi_{N,t}}.$$
A tedious but straightforward computation shows that one can write

$$L_{N,t} = \sum_{j=0}^{4} L_{N,t}^{(j)}$$

(4.16)

where

$$L_{N,t}^{(0)} = \frac{N+1}{2} \langle \varphi_{N,t}, [w_N(1-2f_N) \ast |\varphi_{N,t}|^2] \varphi_{N,t} \rangle - \mu_N(t),$$

$$L_{N,t}^{(1)} = \frac{1}{2} \langle \varphi_{N,t}, [w_N \ast |\varphi_{N,t}|^2] \varphi_{N,t} \rangle \frac{N(N+1)}{N}$$

$$+ \left[ \sqrt{N} \left[ a^*(Q_{N,t}[w_N(1-f_N)]) \ast |\varphi_{N,t}|^2 |\varphi_{N,t}\rangle \right. \right.$$

$$- a^*(Q_{N,t}[w_N \ast |\varphi_{N,t}|^2] \varphi_{N,t}) \frac{N}{N} \sqrt{\frac{N-N}{N}} + \text{h.c.} \left. \right],$$

$$L_{N,t}^{(2)} = d\Gamma \left( -\Delta + (w_N f_N) \ast |\varphi_{N,t}|^2 + K_{1,N,t} - \mu_{N,t} \right)$$

$$+ d\Gamma \left( Q_{N,t}[w_N(1-f_N)] Q_{N,t} \right)$$

$$- d\Gamma \left( Q_{N,t}[w_N \ast |\varphi_{N,t}|^2] Q_{N,t} + K_{1,N,t} \right) \frac{N}{N}$$

$$+ \left[ \frac{1}{2} \int dx \, dy \, K_{2,N,t}(x,y)a^*_x a^*_y \sqrt{(N-N)(N-N-1)} \frac{N}{N} + \text{h.c.} \right],$$

$$L_{N,t}^{(3)} = \left[ \frac{1}{\sqrt{N}} \int dx \, dy \, dx' \, dy' \, (Q_{N,t} \otimes Q_{N,t} w_N Q_{N,t} \otimes 1)(x,y;x',y') \right.$$  

$$\times \varphi_{N,t}(y')a^*_x a^*_y a_{x'}a_{y'} \sqrt{\frac{N-N}{N}} + \text{h.c.} \left. \right],$$

$$L_{N,t}^{(4)} = \frac{1}{2N} \int dx \, dy \, dx' \, dy' \, (Q_{N,t} \otimes Q_{N,t} w_N Q_{N,t} \otimes Q_{N,t})(x,y;x',y')$$

$$\times a^*_x a^*_y a_{x'}a_{y'}$$

with

$$\mu_N(t) := \langle \varphi_{N,t}, [(w_N(1-f_N)) \ast |\varphi_{N,t}|^2] \varphi_{N,t} \rangle$$

and

$$K_{1,N,t} = Q_{N,t} \bar{K}_{1,N,t} Q_{N,t},$$

$$K_{2,N,t} = Q_{N,t} \otimes Q_{N,t} \bar{K}_{2,N,t}.$$
where $\tilde{K}_{1,N,t}$ is the operator on $L^2(\mathbb{R}^3)$ with integral kernel
\begin{equation}
\tilde{K}_{1,N,t}(x,y) = \varphi_{N,t}(x)w_N(x-y)\varphi_{N,t}(y)
\end{equation}
and $\tilde{K}_{2,N,t}$ is a function in $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$:
\begin{equation}
\tilde{K}_{2,N,t}(x,y) = \varphi_{N,t}(x)w_N(x-y)\varphi_{N,t}(y).
\end{equation}

Next, we have to remove the singular correlation structure from $\Phi_{N,t}$. Since $\Psi_{N,t} = U_{\varphi_{N,t}}^* \Phi_{N,t}$ and since $U_{\varphi_{N,t}}^*$ just adds products of solutions of the nonlinear equation (4.7), it is clear that all correlations developed by $\Psi_{N,t}$ must be contained in $\Phi_{N,t}$. To remove correlations from $\Phi_{N,t}$ we apply the Bogoliubov transformation $T_{N,t}$ defined in (4.8). Unfortunately, $T_{N,t}$ does not preserve the number of particles, and therefore it does not leave the truncated Fock space $\mathcal{F}_{\varphi_{N,t}}$ invariant. Since $T_{N,t}$ only creates few particles, this should not be a serious obstacle. To circumvent it, it seems natural to give up the restriction on the number of particles and consider $\Phi_{N,t}$ as a vector in the untruncated Fock space $\mathcal{F}_{\perp \varphi_{N,t}}$. The drawback of this approach is the fact that the generator $L_{N,t}$ computed in (4.16) is defined only on sectors with at most $N$ particles. So, we proceed as follows; first we approximate $\Phi_{N,t}$ by a new, modified, fluctuation vector $\tilde{\Phi}_{N,t}$, whose dynamics is governed by a modified generator $\tilde{L}_{N,t}$ which, on the one hand, is close to $L_{N,t}$ when acting on vectors with a small number of particles and, on the other hand, is well-defined on the full untruncated Fock space $\mathcal{F}_{\perp \varphi_{N,t}}$.

To define $\tilde{L}_{N,t}$ we proceed as follows. Starting from the expression on the r.h.s. of (4.16), we replace first of all the factor $\sqrt{(N-\mathcal{N})(N-\mathcal{N}-1)}$ by $N-\mathcal{N}$ and then we replace $\sqrt{N-\mathcal{N}}$ by $\sqrt{NG_b(N/N)}$ where $G_b(t)$ is the Taylor series for $\sqrt{1-x}$ around $x = 0$ up to order $b$.

Finally, we add a term of the form $C_b e^{C_b |N(N/N)^{2b}}$ with a sufficiently large constant $C_b$. Since the generators $L_N$ and $\tilde{L}_N$ will act on states with small number of particles, one expects this term to have a negligible effect on the dynamics (on the other hand, it allows for better control the energy). With these changes, one obtains the modified generator
\begin{align*}
\tilde{L}_{N,t} &= \frac{N+1}{2} \langle \varphi_{N,t}, \rangle [w_N (1 - 2f_N)*|\varphi_{N,t}|^2] \varphi_{N,t} - \mu_N(t) \\
&+ \frac{1}{2} \langle \varphi_{N,t}, \rangle [w_N *|\varphi_{N,t}|^2] \varphi_{N,t} \frac{N(N+1)}{N} \\
&+ \sqrt{N} a^* (Q_{N,t} [w_N (1 - f_N)*|\varphi_{N,t}|^2] \varphi_{N,t})G_b(N/N) + h.c. \\
&- \sqrt{N} a^* (Q_{N,t} [w_N *|\varphi_{N,t}|^2] \varphi_{N,t}) \frac{N}{\sqrt{N}} G_b(N/N) + h.c.
\end{align*}
Using this modified generator, we define the modified fluctuation dynamics \( \tilde{\Phi}_{N,t} \) as the solution of the Schrödinger equation

\[
\begin{align*}
\frac{\partial}{\partial t} \tilde{\Phi}_{N,t} &= \tilde{L}_{N,t} \tilde{\Phi}_{N,t},
\end{align*}
\]

with the appropriately transformed initial data. One can then prove that for all \( \alpha < (1 - \beta)/2 \), there exists a constant \( C > 0 \) such that

\[
\begin{align*}
\| \Phi_{N,t} - \tilde{\Phi}_{N,t} \|^2 &\leq CN^{-\alpha} \exp(C\exp(C|t|))
\end{align*}
\]

for all \( t \in \mathbb{R} \).

Finally, one applies the Bogoliubov transformation (4.8) to the modified fluctuation evolution \( \tilde{\Phi}_{N,t} \) defined in (4.20). Let

\[
\xi_{N,t} = T_{N,t} \tilde{\Phi}_{N,t}.
\]

Then \( \xi_{N,t} \in \mathcal{F}_{\perp \Phi_{N,t}} \) (with no restriction on the number of particles) and it solves the Schrödinger equation

\[
\begin{align*}
\frac{\partial}{\partial t} \xi_{N,t} &= \mathcal{G}_{N,t} \xi_{N,t},
\end{align*}
\]

with the generator

\[
\mathcal{G}_{N,t} = (i\partial_t T_{N,t}) T^*_{N,t} + T_{N,t} \tilde{L}_{N,t} T^*_{N,t}.
\]

As explained above, the application of the Bogoliubov transformation \( T_{N,t} \) takes care of correlations and makes it possible for us to approximate the evolution (4.22) with the unitary evolution \( U_{2,N} \), having as generator the quadratic part of (4.23). The generator \( U_0 \) that appears in the statement of Theorem 4.1 is what one obtains from \( U_{2,N} \) in limit \( N \to \infty \).
5. Fock space approximation

The methods used in the proof of Theorem 4.1 were inspired strongly by the result of Boccato, Cenatiempo and Schlein who proved in [25] an analogous (i.e., for $\beta < 1$) result in the Fock space setting (that is in the spirit of (1.14)). In that work the authors also used the fluctuation dynamics approach with the correlations described by the Bogoliubov transformation (4.8) (as in [17]).

Earlier, the program of deriving effective dynamics in the Fock space setting for singular interactions was initiated by Grillakis, Machedon and Margetis in [82]. In this work they considered the mean-field regime and proved a result of the type (1.14) for Coulomb potentials (see [83] for an extension). In [78], Grillakis and Machedon considered the NLS regime with $\beta < 1/3$.

The approach of Grillakis, Machedon and co-authors is in spirit very similar to the one of [25]. However, there is one crucial difference that we would like to point out. To this end let us briefly explain the approach in [78].

As mentioned before, in the NLS regime correlations play an important role and to include them in the analysis, similarly to (4.8), Grillakis and Machedon introduce a Bogoliubov transformation (in fact, they did not use this terminology in [78])

$$T(k_t) = \exp \left( \frac{1}{2} \int dx \, dy \left[ k_t(x,y) a_x a_y - \text{h.c.} \right] \right)$$

for some function $k_t(x,y)$. They considered initial Fock space states of the form

$$\Phi(0) = W^* \left( \sqrt{N} \varphi_0 \right) T^*(k_0) \Omega \quad (5.1)$$

which are, a priori, more general than coherent states. In particular, by choosing $k_0 = 0$ one obtains a coherent state as an initial state. Their idea was to approximate the Fock space many-body evolution

$$\Phi(t) = e^{-i t \mathcal{H}_N} \Phi(0)$$

by an effective quadratic evolution that would capture the creation and evolution of correlations. To this end, Grillakis and Machedon postulate that

$$\Phi(t) \approx e^{i N \xi(t)} W^* \left( \sqrt{N} \varphi_t \right) T^*(k_t) \Omega$$

for some phase $\xi(t)$. Next, they introduce the so-called reduced dynamics

$$\Phi_{\text{red}}(t) = T(k_t) W \left( \sqrt{N} \varphi_t \right) \Phi(t).$$
Note that $\Phi_{\text{red}}(0) = c\Omega$ (for some $c$ such that $|c| = 1$) and the goal is to find such a $k_t$ so that also the evolved reduced state satisfies
$$
\Phi_{\text{red}}(t) \approx \Omega.
$$
Thus, as we can see, so far the general idea — the analysis of the fluctuation dynamics — is the same as in the work Boccato, Cenatiempo and Schlein. Here comes the main difference, however. While Boccato, Cenatiempo and Schlein postulated the kernel of $k_t$ to be of the form (4.9) straight away, Grillakis and Machedon derived an equation for $k_t$ so that (5.2) can be satisfied. More precisely, using the properties of coherent states and Bogoliubov transformations one can compute (similarly to (4.23)) the time evolution
$$
i\partial_t \Phi_{\text{red}}(t) = \mathcal{H}_{\text{red}}(t)\Phi_{\text{red}}(t)
$$
and determine $\mathcal{H}_{\text{red}}(t)$. The goal is to choose such $\varphi_t$ and $k_t$ so that
$$
\mathcal{H}_{\text{red}}(t) = N\mu(t) + \int dx\,dy\, L_t(x,y)a^*_x a_y + N^{-1/2}\mathcal{E}(t) \tag{5.3}
$$
where $\mathcal{E}(t)$ is an error term containing polynomials in $a$ and $a^*$ up to degree four, $L_t(x,y)$ is the kernel of some (self-adjoint) operator and $\mu(t)$ is an appropriate phase. This leads to the following set of equations
$$
i\partial_t \varphi_t = (-\Delta + w_N |\varphi_t|^2)\varphi_t, 
\tag{5.4}$$
$$
i\partial_t \text{sh}(2k_t) = -g_N \circ \text{sh}(2k_t) - \text{sh}(2k_t) \circ g_N + m_N \circ \text{ch}(2k_t) + \text{ch}^T(2k_t) \circ m_N
$$
where $\circ$ denotes the composition of operators and the operators $g_N$, $m_N$ are given by
$$
g_N = -\Delta + |\varphi_t|^2 + w_N + \widetilde{K}_1(t),
$$
$$
m_N = \widetilde{K}_2(t).
$$
Here $\widetilde{K}_1(t)$ and $\widetilde{K}_2(t)$ are the same operators as in (4.17) and (4.18), i.e. the operators with the kernels $\widetilde{K}_1(t,x,y) = \varphi_t(x)w_N(x-y)\overline{\varphi}_t(y)$ and $\widetilde{K}_2(t,x,y) = \varphi_t(x)w_N(x-y)\varphi_t(y)$. Furthermore, for an operator $A$ with kernel $a$ we define $\text{sh}(A)$ and $\text{ch}(A)$ to be the operators with the kernels
$$
\text{sh}(a) := a + \frac{1}{3!}a \circ \overline{a} \circ a + \cdots,
$$
$$
\text{ch}(a) := \delta(x-y) + \frac{1}{2!}a \circ \overline{a} \circ a + \cdots,
$$
respectively.
Thus, as one could expect from earlier results, the dynamics of $\varphi_t$ is governed by the Hartree equation in the NLS regime. What is important, the Hartree equation is uncoupled from the equation for $k_t$. The latter equation is in fact, under appropriate assumptions, equivalent to the Bogoliubov equations (4.5) (see [128] for more details). In [118] this equation has been derived in a different manner. In fact, the derivation of Grillakis and Machedon is closely related to the diagonalization problem of quadratic Hamiltonians (see [124] for more details).

In the Grillakis–Machedon approach one has to derive the properties of $k_t$ using the equation. In particular, in order to prove that $\mathcal{E}(t)$ in (5.3) can be treated as an error term, various norms of $k_t$ need to be estimated uniformly in $N$. Obtaining such estimates is more difficult in the case of attractive interactions and this has been done in [53]. In [97] Kuz extended the analysis to cover the case when $\beta < 1/2$. In that work the equations (5.4) remained unchanged.

To cover the NLS regime with $\beta > 1/2$ it turns out that the uncoupled equations (5.4) are not sufficient. In that case, in [79] Grillakis and Machedon suggested a new set of coupled equations that would allow to treat correlations for larger $\beta$. Briefly, the equations have been derived from the condition that

$$X_1 = 0 \quad \text{and} \quad X_2 = 0$$

where $X_1, X_2$ are one and two-body states in the Fock space given by

$$\mathcal{H}_{\text{red}} \Omega = (X_0, X_1, X_2, X_3, X_4, 0, \ldots).$$

In fact, this condition can be obtained by minimizing the term $X_0$ over $\varphi_t$ and $k_t$ (this can be seen as a time-dependent version of the Beliaev theorem introduced in [59] and used in [57]). The resulting equations are then quite similar to (5.4), but now the operator $m_N$ has to be replaced by the operator $\Theta$ with the kernel

$$\Theta(x, y) = -w_N(x - y)\left(\varphi_t(x)\varphi_t(y) + \frac{1}{2N}\text{sh}(2k_t)(x, y)\right)$$

and a similar $O\left(\frac{1}{N}\right)$ correction appears in the Hartree equation. Under certain smoothness assumptions on $\varphi_0$ and $k_0$, in [80] Grillakis and Machedon were able to show for $\beta \in \left(\frac{1}{3}, \frac{2}{3}\right)$ that if $\varphi_t$ and $k_t$ are solution of the coupled equations, then

$$\left\|e^{it\mathcal{H}_N} W^* (\sqrt{N}\varphi_0) T^* (k_0) \Omega - e^{ix_1 W^* (\sqrt{N}\varphi_0) T^* (k_t) \Omega}\right\|_p \leq \frac{C}{N^{1/6}}.$$
locally in time (i.e., for some $t \in (0, T_0)$) and for an appropriate phase factor $\chi_t$.

The main difficulty in the proof of the result above lies in the analysis of the coupled equations, in particular in establishing uniform in $N$ estimates on the solutions $\varphi_t$ and $k_t$ in certain function spaces. This result has been further extended for all $\beta < 1$ (still locally in time) in [81] and then for all $\beta < 1$ but globally in time in [54] (see also [52] for results in one dimension). Finally, let us mention that equations similar to those coupled equations used by Grillakis and Machedon are often called Hartree–Fock–Bogoliubov equations and have been analyzed also in [12,19].

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**References**

1. R. Adami, F. Golse, and A. Teta, “Rigorous Derivation of the Cubic NLS in Dimension One”, *J. Stat. Phys.* 127, 1193–1220 (2007).
2. Z. Ammari and S. Breteaux, “Propagation of chaos for many-boson systems in one dimension with a point pair interaction”, *Asymptot. Anal.* 76, 123–170 (2012).
3. Z. Ammari, M. Falconi, and B. Pawilowski, “On the rate of convergence for the mean field approximation of many-body quantum dynamics”, *Comm. Math. Sci.* 14, 1417–1442 (2014).
4. Z. Ammari, Q. Liard, and C. Rouffort, “On Well-Posedness for General Hierarchy Equations of Gross-Pitaevskii and Hartree Type”, *Arch. Rational Mech. Anal.* 238, 845–900 (2020).
5. Z. Ammari and F. Nier, “Mean field limit for bosons and infinite dimensional phase-space analysis”, *Ann. Henri Poincaré* 9, 1503–1574 (2008).
6. Z. Ammari and F. Nier, “Mean-field limit for bosons and propagation of Wigner measures”, *J. Math. Phys.* 50, 042107 (2009).
7. Z. Ammari and F. Nier, “Mean field propagation of Wigner measures and BBGKY hierarchies for general bosonic states”, *J. Math. Pures App.* 95, 585–626 (2011).
8. I. Anapolitanos, “Rate of Convergence Towards the Hartree–von Neumann Limit in the Mean-Field Regime”, *Lett. Math. Phys.* 98, 1–31 (2011).
9. I. Anapolitanos and M. Hott, “A simple proof of convergence to the Hartree dynamics in Sobolev trace norms”, *J. Math. Phys.* 57, 122108 (2016).
10. I. Anapolitanos, M. Hott, and D. Hundertmark, “Derivation of the Hartree equation for compound Bose gases in the mean field limit”, Rev. Math. Phys. 29, 1750022 (2017).
11. M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman and E.A. Cornell, “Observation of Bose–Einstein Condensation in a Dilute Atomic Vapor”, Science 269, 198–201 (1995).
12. V. Bach, S. Breteaux, T. Chen, J. Fröhlich, and I. M. Sigal, “The time-dependent Hartree–Fock–Bogoliubov equations for bosons”, e-print arXiv:1602.05171 (2016).
13. C. Bardos, F. Golse, and N.J. Mauser, “Weak coupling limit of the N-particle Schrödinger equation”, Methods Appl. Anal. 7, 275–293 (2000).
14. C. Bardos, L. Erdős, F. Golse, N.J. Mauser, and H.T. Yau, “Derivation of the Schrödinger–Poisson equation from the quantum N-body problem”, Comptes Rendus Mathematique 334, 515–520 (2002).
15. G. Ben Arous, K. Kirkpatrick, and B. Schlein, “A central limit theorem in many-body quantum dynamics”, Commun. Math. Phys. 321, 371–417 (2013).
16. R. Benguria and E. H. Lieb, “Proof of the Stability of Highly Negative Ions in the Absence of the Pauli Principle”, Phys. Rev. Lett. 50, 1771–1774 (1983).
17. N. Benedikter, G. de Oliveira, and B. Schlein, “Quantitative Derivation of the Gross–Pitaevskii Equation”, Comm. Pure App. Math. 68, 1399–1482 (2015).
18. N. Benedikter, M. Porta, and B. Schlein, Effective Evolution Equations from Quantum Dynamics, Springer, Cham (2016).
19. N. Benedikter, J. Sok, and J.P. Solovej, “The Dirac–Frenkel Principle for Reduced Density Matrices, and the Bogoliubov–de Gennes Equations”, Ann. Henri Poincaré 19, 1167–1214 (2018).
20. F. Berezin, The method of second quantization, Pure and applied physics. A series of monographs and textbooks, Academic Press, 1966.
21. C. Boccato, C. Brennecke, S. Cenatiempo, and B. Schlein, “Complete Bose–Einstein condensation in the Gross–Pitaevskii regime”, Commun. Math. Phys. 359, 975–1026 (2018).
22. C. Boccato, C. Brennecke, S. Cenatiempo, and B. Schlein, “Bogoliubov theory in the Gross–Pitaevskii limit”, Acta Mathematica 222, 219–335 (2019).
23. C. Boccato, C. Brennecke, S. Cenatiempo, and B. Schlein, “Optimal rate for Bose–Einstein condensation in the Gross–Pitaevskii regime”, Commun. Math. Phys. 376, 1311–1395 (2020).
24. C. Boccato, C. Brennecke, S. Cenatiempo, and B. Schlein, “The excitation spectrum of Bose gases interacting through singular potentials”, J. Eur. Math. Soc. 22, 2331–2403 (2020).
25. C. Boccato, S. Cenatiempo, and B. Schlein, “Quantum many-body fluctuations around nonlinear Schrödinger dynamics”, Ann. Henri Poincaré 18, 113–191 (2017).
26. N. N. Bogoliubov, “On the theory of superfluidity”, J. Phys. (USSR), 11, 23–32 (1947).
27. S. N. Bose, “Plancks Gesetz und Lichtquantenhypothese”, Z. Phys. 26, 178–181 (1924).
28. L. Bossmann, “Derivation of the 1d nonlinear Schrödinger equation from the 3d quantum many-body dynamics of strongly confined bosons”, J. Math. Phys. 60, 031902 (2019).
29. L. Bossmann, “Derivation of the 2d Gross–Pitaevskii Equation for Strongly Confined 3d Bosons”, Arch. Rational Mech. Anal. 238, 541–606 (2020).
30. L. Bossmann, S. Petrat, P. Pickl, and A. Soffer, “Beyond Bogoliubov Dynamics”, e-print arXiv:1912.11004 (2019).
31. L. Bossmann and S. Teufel, “Derivation of the 1d Gross–Pitaevskii Equation from the 3d Quantum Many-Body Dynamics of Strongly Confined Bosons”, Ann. Henri Poincaré 20, 1003–1049 (2019).
32. L. Bossmann, S. Petrat and R. Seiringer, “Asymptotic expansion of the low-energy excitation spectrum for weakly interacting bosons”, e-print arXiv:2006.09825 (2020).
33. C. Brennecke and B. Schlein, “Gross–Pitaevskii dynamics for Bose–Einstein condensates”, Anal. PDE 12, 1513–1596 (2019).
34. C. Brennecke, P. T. Nam, M. Napiórkowski, and B. Schlein, “Fluctuations of N-particle quantum dynamics around the nonlinear Schrödinger equation”, Annales de l’Institut Henri Poincaré C, Analyse non linéaire 36, 1201–1235 (2019).
35. S. Buchholz, C. Saffirio, and B. Schlein, “Multivariate central limit theorem in quantum dynamics”, J. Stat. Phys. 154, 113–152 (2014).
36. L. Chen and J. O. Lee, “Rate of convergence in nonlinear Hartree dynamics with factorized initial data”, J. Math. Phys. 52, 052108 (2011).
37. L. Chen, J. O. Lee, and B. Schlein, “Rate of Convergence Towards Hartree Dynamics”, J. Stat. Phys. 144, 872–903 (2011).
38. T. Chen, C. Hainzl, N. Pavlović, and R. Seiringer, “Unconditional uniqueness for the cubic Gross–Pitaevskii hierarchy via quantum de Finetti”, Commun. Pure Appl. Math. 68, 1845–1884 (2015).
39. T. Chen and N. Pavlović, “On the Cauchy problem for focusing and defocusing Gross–Pitaevskii hierarchies”, Discr. Contin. Dyn. Syst. 27, 715–739 (2010).
40. T. Chen and N. Pavlović, “The quintic NLS as the mean field limit of a boson gas with three-body interactions”, J. Funct. Anal. 260, 959–997 (2011).
41. T. Chen and N. Pavlović, “A new proof of existence of solutions for focusing and defocusing Gross–Pitaevskii hierarchies”, Proc. Am. Math. Soc. 141, 279–293 (2013).
42. T. Chen and N. Pavlović, “Derivation of the Cubic NLS and Gross–Pitaevskii Hierarchy from Manybody Dynamics in d=3 Based on Spacetime Norms”, Ann. Henri Poincaré 15, 543–588 (2014).
43. T. Chen and N. Pavlović, “Higher Order Energy Conservation and Global Well-Posedness of Solutions for Gross–Pitaevskii Hierarchies”, Comm. PDE 39, 1597–1634 (2014).
44. X. Chen and J. Holmer, “On the rigorous derivation of the 2d cubic nonlinear Schrödinger equation from 3d quantum many-body dynamics”, Arch.
52. J.J.W. Chong, “Uniform in $N$ global well-posedness of the time-dependent Hartree–Fock–Bogoliubov equations in $\mathbb{R}^{1+1}$”, *Lett. Math. Phys.* 108, 2255–2283 (2018).

53. J.J.W. Chong, “Dynamics of Large Boson Systems with Attractive Interaction and a Derivation of the Cubic Focusing NLS in $\mathbb{R}^3$”, e-print arXiv:1608.01615 (2016).

54. J.J.W. Chong and Z. Zhao, “Dynamical Hartree–Fock–Bogoliubov approximation of interacting bosons”, pre-print arXiv:1711.00610 (2020).

55. K.B. Davis, M.-O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, and W. Ketterle, “Bose–Einstein condensation in a gas of sodium atoms”, *Phys. Rev. Lett.* 75, 3969–3973 (1995).

56. G. de Oliveira and A. Michelangeli, “Mean-field dynamics for mixture condensates via Fock space methods”, *Rev. Math. Phys.* 31, 1950027 (2019).

57. J. Dereziński, K.A. Meißner, and M. Napiórkowski, “On the Energy-Momentum Spectrum of a Homogeneous Fermi Gas”, *Ann. Henri Poincaré* 14, 1–36 (2013).

58. J. Dereziński and M. Napiórkowski, “Excitation spectrum of interacting bosons in the mean-field infinite-volume limit”, *Ann. Henri Poincaré* 15, 2409–2439 (2014); erratum: *Ann. Henri Poincaré* 16, 1709–1711 (2015).

59. J. Dereziński, M. Napiórkowski, and J.P. Solovej, “On the minimization of Hamiltonians over pure Gaussian states”, in: *Complex Quantum Systems. Analysis of Large Coulomb Systems*, 151–162 (2013).

60. A. Einstein, “Quantentheorie des einatomigen idealen Gases”, *Sitzber. Kgl. Preuss. Akad. Wiss.* 261–267 (1924).
61. A. Elgart and B. Schlein, “Mean field dynamics of boson stars”, *Comm. Pure App. Math.* **60**, 500–545 (2007).
62. L. Erdös and B. Schlein, “Quantum Dynamics with Mean Field Interactions: a New Approach”, *J. Stat. Phys.* **134**, 859–870 (2009).
63. A. Elgart, L. Erdös, B. Schlein, and H.-T. Yau, “Gross–Pitaevskii Equation as the Mean-Field Limit of Weakly Coupled Bosons”, *Arch. Rational Mech. Anal.* **179**, 265–283 (2006).
64. L. Erdös, B. Schlein, and H.-T. Yau, “Derivation of the Gross–Pitaevskii hierarchy for the dynamics of Bose–Einstein condensate”, *Commun. Pure Appl. Math.* **59**, 1659–1741 (2006).
65. L. Erdös, B. Schlein, and H.-T. Yau, “Derivation of the cubic non-linear Schrödinger equation from quantum dynamics of many-body systems”, *Invent. Math.* **167**, 515–614 (2007).
66. L. Erdös, A. Michelangeli, and B. Schlein, “Dynamical Formation of Correlations in a Bose-Einstein Condensate”, *Commun. Math. Phys.* **289**, 1171–1210 (2009).
67. L. Erdös, B. Schlein, and H.-T. Yau, “Rigorous derivation of the Gross–Pitaevskii equation with a large interaction potential”, *J. Amer. Math. Soc.* **22**, 1099–1156 (2009).
68. L. Erdös, B. Schlein, and H.-T. Yau, “Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate”, *Ann. of Math. (2)* **172**, 291–370 (2010).
69. L. Erdös and H.-T. Yau, “Derivation of the nonlinear Schrödinger equation from a many body Coulomb system”, *Adv. Theor. Math. Phys.* **5**, 1169–1205 (2001).
70. M. Fannes, H. Spohn, and A. Verbeure, “Equilibrium states for mean field models”, *J. Math. Phys.* **21**, 355–358 (1980).
71. J. Fröhlich, S. Graffi, and S. Schwarz, “Mean-Field- and Classical Limit of Many-Body Schrödinger Dynamics for Bosons”, *Commun. Math. Phys.* **271**, 681–697 (2007).
72. J. Fröhlich, A. Knowles, and A. Pizzo, “Atomism and quantization”, *J. Phys. A* **40**, 3033–3045 (2007).
73. J. Fröhlich, A. Knowles, and S. Schwarz, “On the mean-field limit of bosons with Coulomb two-body interaction”, *Commun. Math. Phys.* **288**, 1023–1059 (2009).
74. J. Ginibre and G. Velo, “The classical field limit of scattering theory for nonrelativistic many-boson systems. I”, *Commun. Math. Phys.* **66**, 37–76 (1979).
75. F. Golse, C. Mouhot, and T. Paul, “On the Mean Field and Classical Limits of Quantum Mechanics”, *Commun. Math. Phys.* **343**, 165–205 (2016).
76. F. Golse and T. Paul, “The Schrödinger Equation in the Mean-Field and Semiclassical Regime”, *Arch. Rational Mech. Anal.* **223**, 57–94 (2017).
77. P. Grech and R. Seiringer, “The excitation spectrum for weakly interacting bosons in a trap”, *Comm. Math. Phys.* **322**, 559–591 (2013).
78. M. Grillakis and M. Machedon, “Pair excitations and the mean field ap-
proximation of interacting Bosons, I\textsuperscript{\textprime}, \textit{Commun. Math. Phys.} \textbf{324}, 601–636 (2013).
79. M. Grillakis and M. Machedon, “Beyond mean field: On the role of pair excitations in the evolution of condensates”, \textit{J. Fixed Point Theory and Appl.} \textbf{14}, 91–111 (2013).
80. M. Grillakis and M. Machedon, “Pair excitations and the mean field approximation of interacting Bosons, II\textsuperscript{\textprime}, \textit{Comm. PDE} \textbf{42}, 24–67 (2017).
81. M. Grillakis and M. Machedon, “Uniform in N estimates for a Bosonic system of Hartree–Fock–Bogoliubov type”, \textit{Comm. PDE} \textbf{44}, 1431–1465 (2019).
82. M. G. Grillakis, M. Machedon, and D. Margetis, “Second-order corrections to mean field evolution of weakly interacting bosons. I\textsuperscript{\textprime}, \textit{Commun. Math. Phys.} \textbf{294}, 273–301 (2010).
83. M. G. Grillakis, M. Machedon, and D. Margetis, “Second-order corrections to mean field evolution of weakly interacting bosons. II\textsuperscript{\textprime}, \textit{Adv. Math.} \textbf{228}, 1788–1815 (2011).
84. E.P. Gross, “Structure of a quantized vortex in boson systems”, \textit{Il Nuovo Cimento} \textbf{20}, 454–457 (1961).
85. C. Hainzl, “Another proof of BEC in the GP-limit”, e-print arXiv:2011.09450 (2020).
86. K. Hepp, “The classical limit for quantum mechanical correlation functions”, \textit{Comm. Math. Phys.} \textbf{35}, 265–277 (1974).
87. S. Herr and V. Sohinger, “The Gross-Pitaevskii Hierarchy on General Rectangular Tori”, \textit{Arch. Rational Mech. Anal.} \textbf{220}, 1119–1158 (2016).
88. M. Hott, “Convergence rate towards the fractional Hartree equation with singular potentials in higher Sobolev norms”, e-print: arXiv:1805.01807 (2018).
89. M. Jeblick, N. Leopold, and P. Pickl, “Derivation of the Time Dependent Gross-Pitaevskii Equation in Two Dimensions”, \textit{Commun. Math. Phys.} \textbf{372}, 1–69 (2019).
90. M. Jeblick and P. Pickl, “Derivation of the Time Dependent Two Dimensional Focusing NLS Equation”, \textit{J. Stat. Phys.} \textbf{172}, 1398–1426 (2018).
91. J. von Keler and S. Teufel, “The NLS limit for bosons in a quantum waveguide”, \textit{Ann. Henri Poincaré} \textbf{17}, 3321–3360 (2016).
92. K. Kirkpatrick, S. Rademacher, and B. Schlein, “A large deviation principle in many-body quantum dynamics”, e-print arXiv:2010.13754 (2020).
93. K. Kirkpatrick, B. Schlein, and G. Staffilani, “Derivation of the two-dimensional nonlinear Schrödinger equation from many body quantum dynamics”, \textit{Amer. J. Math.} \textbf{133}, 91–130 (2011).
94. S. Klainerman and M. Machedon, “On the uniqueness of solutions to the Gross–Pitaevskii hierarchy”, \textit{Commun. Math. Phys.} \textbf{279}, 169–185 (2008).
95. A. Knowles and P. Pickl, “Mean-field dynamics: singular potentials and rate of convergence”, \textit{Commun. Math. Phys.} \textbf{298}, 101–138 (2010).
96. E. Kuz, “Rate of Convergence to Mean Field for Interacting Bosons”, \textit{Comm. PDE} \textbf{40}, 1831–1854 (2015).
97. E. Kuz, “Exact evolution versus mean field with second-order correction for
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bosons interacting via short-range two-body potential”, *Diff. Int. Eq.* **30**, 587–630 (2017).

98. L. Lafleche and C. Saffirio, “Strong Semiclassical Limit From Hartree And Hartree-Fock to Vlasov-Poisson Equation”, e-print arXiv:2003.02926 (2020).

99. J.O. Lee, “Rate of Convergence Towards Semi-Relativistic Hartree Dynamics”, *Ann. Henri Poincaré* **14**, 313–346 (2013).

100. J. Lee, “Rate of convergence towards equations of Hartree type for mixture condensates with factorized initial data”, e-print arXiv:1907.03388 (2019).

101. J. Lee, “Rate of convergence towards mean-field evolution for weakly interacting bosons with singular three-body interactions”, e-print arXiv:2006.13040 (2020).

102. M. Lewin, P. Nam, and N. Rougerie, “Derivation of non-linear Gibbs measures from many-body quantum mechanics”, *Journal de l’École polytechnique – Mathématiques* **2**, 65–115 (2015).

103. M. Lewin, P. T. Nam, and N. Rougerie, “Derivation of Hartree’s theory for generic mean-field Bose gases”, *Adv. Math.* **254**, 570–621 (2014).

104. M. Lewin, P. T. Nam, and N. Rougerie, “The mean-field approximation and the non-linear Schrödinger functional for trapped Bose gases”, *Trans. Amer. Math. Soc.* **368**, 6131–6157 (2016).

105. M. Lewin, P. T. Nam, and B. Schlein, “Fluctuations around Hartree states in the mean-field regime”, *Amer. J. Math.* **137**, 1613–1650 (2015).

106. E. H. Lieb and W. Liniger, “Exact analysis of an interacting Bose gas. I. The general solution and the ground state”, *Phys. Rev.* **130**, 1605–1616 (1963).

107. E. H. Lieb and R. Seiringer, “Proof of Bose–Einstein Condensation for Dilute Trapped Gases”, *Phys. Rev. Lett.* **88**, 170409 (2002).

108. E. H. Lieb and R. Seiringer, “Derivation of the Gross-Pitaevskii equation for rotating Bose gases”, *Commun. Math. Phys.* **264**, 505–537 (2006).

109. E. H. Lieb, R. Seiringer, and J. Yngvason, “Bosons in a trap: A rigorous derivation of the Gross-Pitaevskii energy functional”, *Phys. Rev. A* **61**, 043602 (2000).

110. E. H. Lieb and H.-T. Yau, “The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics”, *Commun. Math. Phys.* **112**, 147–174 (1987).

111. J. Lührmann, “Mean-field quantum dynamics with magnetic fields”, *J. Math. Phys.* **53**, 022105 (2012).

112. A. Michelangeli and A. Olgiati, “Mean-field quantum dynamics for a mixture of Bose-Einstein condensates”, *Anal. Math. Phys.* **7**, 377–416 (2017).

113. A. Michelangeli and A. Olgiati, “Gross-Pitaevskii non-linear dynamics for pseudo-spinor condensates”, *J. Nonlinear Math. Phys.* **24**, 426–464 (2017).

114. A. Michelangeli and B. Schlein, “Dynamical Collapse of Boson Stars”, *Commun. Math. Phys.* **311**, 645–687 (2012).

115. D. Mitrouskas, S. Petrat, and P. Pickl, “Bogoliubov Corrections and Trace
Norm Convergence for the Hartree Dynamics”, *Rev. Math. Phys.* **31**, 1950024 (2019).

118. P. T. Nam and M. Napiórkowski, “Bogoliubov correction to the mean-field dynamics of interacting bosons”, *Adv. Theor. Math. Phys.* **21**, 683–738 (2017).

119. P. T. Nam and M. Napiórkowski, “A note on the validity of Bogoliubov correction to mean-field dynamics”, *J. Math. Pures et Appliquées* **108**, 662–688 (2017).

120. P. T. Nam and M. Napiórkowski, “Norm approximation for many-body quantum dynamics and Bogoliubov theory”, *Advances in Quantum Mechanics: contemporary trends and open problems*, Springer (2017).

121. P. T. Nam and M. Napiórkowski, “Norm approximation for many-body quantum dynamics: focusing case in low dimensions”, *Adv. Math.* **350**, 547–587 (2019).

122. P. T. Nam and M. Napiórkowski, “Two-term expansion of the ground state one-body density matrix of a mean-field Bose gas”, *Calc. Var. PDE* **60**, 1–30 (2021).

123. P. T. Nam, M. Napiórkowski, J. Ricaud, and A. Triay, “Optimal rate of condensation for trapped bosons in the Gross–Pitaevskii regime”, e-print arXiv:2001.04364 (2020), to appear in *Anal. PDE*.

124. P. T. Nam, M. Napiórkowski, and J. P. Solovej, “Diagonalization of bosonic quadratic Hamiltonians by Bogoliubov transformations”, *J. Funct. Anal.* **270**, 4340–4368 (2016).

125. M. Napiórkowski, “Recent advances in the theory of Bogoliubov Hamiltonians”, *Macroscopic Limits of Quantum Systems*, Springer PROMS **270**, 101–123 (2018).

126. A. Olgiati, “Remarks on the Derivation of Gross–Pitaevskii Equation with Magnetic Laplacian”, In: Michelangeli A., Dell’Antonio G. (eds) *Advances in Quantum Mechanics*. Springer INdAM Series, vol. 18. Springer, Cham.

127. O. Penrose and L. Onsager, “Bose-Einstein Condensation and Liquid Helium”, *Phys. Rev.* **104**, 576–584 (1956).
134. P. Pickl, “Derivation of the Time Dependent Gross–Pitaevskii Equation Without Positivity Condition on the Interaction”, *J. Stat. Phys.* **140**, 76–89 (2010).

135. P. Pickl, “A simple derivation of mean-field limits for quantum systems”, *Lett. Math. Phys.* **97**, 151–164 (2011).

136. P. Pickl, “Derivation of the time dependent Gross Pitaevskii equation with external fields”, *Rev. Math. Phys.* **27**, 1550003 (2015).

137. L. P. Pitaevskii, “Vortex lines in an imperfect Bose gas” *Sov. Phys. JETP* **13**, 451–454 (1961).

138. S. Rademacher, “Central limit theorem for Bose gases interacting through singular potentials”, *Lett. Math. Phys.* **110**, 2143–2174 (2020).

139. G. A. Raggio and R. F. Werner, “Quantum statistical mechanics of general mean field systems”, *Helv. Phys. Acta* **62**, 980–1003 (1989).

140. I. Rodnianski and B. Schlein, “Quantum fluctuations and rate of convergence towards mean field dynamics”, *Commun. Math. Phys.* **291**, 31–61 (2009).

141. M. Rosenzweig, “The Mean-Field Limit of the Lieb-Liniger Model”, e-print arXiv:1912.07585 (2019).

142. N. Rougerie, “Scaling limits of bosonic ground states, from many-body to nonlinear Schrödinger”, e-print arXiv:2002.02678 (2020).

143. R. Seiringer, “The excitation spectrum for weakly interacting bosons”, *Commun. Math. Phys.* **306**, 565–578 (2011).

144. S. Shen, “The rigorous derivation of the $T^2$ focusing cubic NLS from 3D”, e-print arXiv:2003.09693 (2020).

145. V. Sohinger, “A rigorous derivation of the defocusing cubic nonlinear Schrödinger equation on $T^3$ from the dynamics of many-body quantum systems”, *Annales de l’Institut Henri Poincaré C, Analyse non linéaire* **32**, 1337–1365 (2015).

146. J. P. Solovej, “Many body quantum mechanics”, Lecture notes at the Erwin Schrödinger Institute 2014, available online.

147. H. Spohn, “Kinetic equations from Hamiltonian dynamics: Markovian limits”, *Rev. Mod. Phys.* **52**, 569–615 (1980).