Remarks on distinguishability of Schwarzschild spacetime and thermal Minkowski spacetime using Resonance Casimir-Polder interaction

Chiranjeeb Singha
Department of Physical Sciences, Indian Institute of Science Education and Research Kolkata, Mohanpur - 741 246, WB, India
cs12ip026@iiserkol.ac.in

September 9, 2019

Abstract

One perceives same response of a single-atom detector when placed at a point outside the horizon in Schwarzschild spacetime to that of a static single-atom detector in thermal Minkowski spacetime. So one cannot distinguish Schwarzschild spacetime from thermal Minkowski spacetime by using a single-atom detector. We show that, for Schwarzschild spacetime, beyond a characteristic length scale which is proportional to the inverse of the surface gravity $\kappa$, the Resonance Casimir-Polder interaction (RCPI) between two entangled atoms is characterized by a $1/L^2$ power-law provided the atoms are located close to the horizon. However, the RCPI between two entangled atoms follows a $1/L$ power-law decay for the thermal Minkowski spacetime. Seemingly, it appears that the spacetimes can be distinguished from each other using the RCPI behavior. But our further exploration leads to the conclusion that the length scale limit beyond a characteristic value is not compatible with the local flatness of the spacetime.

1 Introduction

Casimir-Polder interaction (CPI) is a very interesting phenomenon that arises due to the vacuum fluctuations of quantum field. The effects of the CPI are widely

1
investigated in many branches of physics with fairly accurate experimental corroborations [1]. Successful efforts have been made to probe more complicated contexts like entanglement [2], spacetime curvature [3–5], Unruh effect [6–8], Hawking radiation of a black hole [9] and to check thermal and nonthermal scaling in a black hole spacetime [10]. It is shown that the background spacetime and the relativistic motion of interacting systems can modify the $CPI$ [3–15]. So one can in principle extract the information about the background spacetime and the relativistic motion of interacting systems from the $CPI$ [3].

It is well known that the response of a single-atom detector when placed at a point outside the horizon in Schwarzschild spacetime is the same as the response of a static single-atom detector in thermal Minkowski spacetime [16–27]. So one cannot distinguish these two spacetimes by using a single-atom detector. In this paper, we follow the method as used in [3] where the $RCPI$ occurs when one or more atoms are in their excited states and exchange of real photons is involved between them [28, 29]. Our system is modeled as two entangled atoms which are coupled with a massless scalar field. Here we consider the atoms in Schwarzschild spacetime and in thermal Minkowski spacetime. We show that, for Schwarzschild spacetime, beyond a characteristic length scale which is proportional to the inverse of the surface gravity $\kappa$, the Resonance Casimir-Polder interaction ($RCPI$) between two entangled atoms is characterized by a $1/L^2$ power-law decay provided the atoms are located close to the horizon. However, the $RCPI$ between two entangled atoms follows a $1/L$ power-law decay for the thermal Minkowski spacetime. Seemingly, it appears that the spacetimes can be distinguished from each other using the $RCPI$ behavior. But our further exploration leads to the conclusion that the length scale limit beyond a characteristic value is not valid for Schwarzschild spacetime.

This article is organized as follows. In the section 2, we follow the method as used in [3]. Here we apply the open quantum system approach [30, 31] to get the effective Hamiltonian of the two atoms which interact weakly with a massless scalar field. Using this Hamiltonian, we can compute the shifts of the energy level of the symmetric state and the antisymmetric state of the two-atom system. It is shown that these shifts of the energy level are related to the two-point functions which can be calculated along the trajectories of the atoms and thus they depend on the spacetime background. As a result, these shifts of the energy level are different in different spacetimes.

In the section 3, we compute the two-point functions and the shifts of the energy level of the symmetric state and the antisymmetric state for two static atoms in Schwarzschild spacetime when the atoms are located close to the horizon. We show that beyond a characteristic length scale which is proportional to the inverse of the surface gravity $\kappa$, the $RCPI$ between two entangled atoms is characterized by a $1/L^2$ power-law decay for this spacetime. But the length scale limit beyond...
a characteristic value is not valid for Schwarzschild spacetime.

In the section 5, we compute the RCPI between two entangled atoms for thermal Minkowski spacetime. We show that the RCPI is always characterized by a $1/L$ power-law decay for this spacetime.

2 Time evolution of two atoms

Here we follow the method as used in [3] and throughout the paper, we use natural units where $c = h = 1$. For convenience, we adopt the same notations as in [3]. Now we consider two atoms which are mutually independent and identical and these atoms interact weekly with a massless scalar field. Each atom has two internal energy levels correspond to two eigenstates, $+\frac{1}{2}\omega_0$ for the excited state, $|e\rangle$, and $-\frac{1}{2}\omega_0$ for the ground state, $|g\rangle$, respectively. The total Hamilton of the system is given by [3],

$$H = H_A + H_F + H_{int},$$

(1)

where $H_A = \frac{1}{2}\omega_0\sigma_3^{(1)} + \frac{1}{2}\omega_0\sigma_3^{(2)}$ is the total Hamilton of two isolated atoms where superscript (1 or 2) labels the atom number, $\sigma_j$ with $j \in \{1, 2, 3\}$ are Pauli matrices. Here $|e\rangle$ and $|g\rangle$ are eigenstates of $\sigma_3$ and $H_F = \sum_k \omega_k a_k^+ a_k \frac{d^2}{d\tau^2}$ is the free Hamiltonian of the massless field where $a_k^+$ and $a_k$ are the creation and annihilation operators of the massless scalar field with linear dispersion relation $\omega_k = |k|$, [7] and $H_{int}$ is the field-atom interaction term which is assumed to be [3, 7, 10, 32].

$$H_{int}(\tau) = \lambda \left( \sigma_2^{(1)} \Phi(x_1(\tau)) + \sigma_2^{(2)} \Phi(x_2(\tau)) \right),$$

(2)

where $\lambda$ is the coupling constant which is taken to be small. The final results are presented in the paper will remain identical with $\sigma_1$ instead of $\sigma_2$ in the interaction Hamiltonian.

Initially, we assume that there was no interaction between the atoms and the external field [3] and also there is no correlated state between the atoms and quantum field through other medium so the total density matrix of the system can be expressed as $\rho_s(0) = \rho(0) \otimes |0\rangle\langle 0|$. Here $\rho(0)$ is the initial density matrix of the two-atom system and $|0\rangle$ is the vacuum state of the scalar field. In the frame of atoms which is also considered as a proper frame, the time evolution of the density matrix of the total system obeys von Neumann equation [3] i.e.

$$\frac{\partial \rho_s(\tau)}{\partial \tau} = -i[H(\tau), \rho_s(\tau)],$$

(3)

where $\tau$ is the proper time. Here we assumed that the two atoms should be closely spaced so that they should not experience any relative curvature effect. Otherwise,
the proper time carried by the clocks of the atoms will differ by a gravitational red-
shift factor then equation (3) is not viable. We are interested in the time evolution
of the two-atom system and in order to obtain the reduced dynamics, we trace out
the field degrees of freedom i.e. \( \rho(\tau) = Tr_F[\rho(\tau)] \). In the weak-coupling limit,
the resulting equation then becomes the Kossakowski-Lindblad form \([3, 33, 34]\)
which is given by,

\[
\frac{\partial \rho(\tau)}{\partial \tau} = -i[H_e, \rho(\tau)] + L[\rho(\tau)] , \tag{4}
\]

where \( H_e \) is the effective Hamilton of the two-atom system which is given by \([3]\),

\[
H_e = H_A - \frac{i}{2} \sum_{a,b=1}^{3} \sum_{j,k=1}^{3} H_{jk}^{(ab)} \sigma_j^{(a)} \sigma_k^{(b)} . \tag{5}
\]

Here we do not bother about the form of \( L[\rho(\tau)] \) in equation (4) because it is
not important in our calculation \([3]\). The elements, \( H_{jk}^{(ab)} \), in the effective
Hamilton, \( H_e \) (5), can be determined by the Hilbert transforms of the two-point
functions, \( K^{(ab)}(\Omega) \), which are given by \([3]\),

\[
K^{(ab)}(\Omega) = \mathcal{P} \int_{-\infty}^{\infty} d\omega \frac{\mathcal{G}^{(ab)}(\omega)}{\omega - \Omega} , \tag{6}
\]

where \( \mathcal{P} \) is the principal value of the integral. Here \( a, b \) are dummy variables
which can take the value 1 or 2 and \( \mathcal{G}^{(ab)}(\Omega) \) denote the Fourier transforms of the
two-point functions which are given by,

\[
\mathcal{G}^{(ab)}(\Omega) = \int_{-\infty}^{\infty} d\Delta \tau e^{i\Omega \Delta \tau} G^{(ab)}(\Delta \tau) , \tag{7}
\]

where \( G^{(ab)}(\Delta \tau) \) indicate the two-point functions which are given by,

\[
G^{(ab)}(\Delta \tau) = \langle \Phi(\tau, x_a) \Phi(\tau', x_b) \rangle , \tag{8}
\]

and \( \Delta \tau = (\tau - \tau') \). Then the explicit forms of the elements, \( H_{jk}^{(ab)} \), can be written
as \([3]\),

\[
H_{jk}^{(ab)} = A^{(ab)} \delta_{jk} - iB^{(ab)} E_{jki} \delta_{3l} - A^{(ab)} \delta_{3j} \delta_{3k} , \tag{9}
\]

where the parameters \( A^{(ab)} \) and \( B^{(ab)} \) are given by,

\[
A^{(ab)} = \frac{\lambda^2}{4} \left[ \mathcal{K}^{(ab)}(\omega_0) + \mathcal{K}^{(ab)}(-\omega_0) \right] ,
\]

\[
B^{(ab)} = \frac{\lambda^2}{4} \left[ \mathcal{K}^{(ab)}(\omega_0) - \mathcal{K}^{(ab)}(-\omega_0) \right] . \tag{10}
\]
It has been already mentioned that the first term i.e. $H_A$ of the effective Hamiltonian, $H_e$ \((5)\), is the total Hamilton of two isolated atoms. The last term of the effective Hamiltonian \((5)\) i.e.

$$H_{LS} \equiv -\frac{i}{2} \sum_{a,b=1}^{2} \sum_{j,k=1}^{3} H_{jk}^{(ab)} \sigma_j^{(a)} \sigma_k^{(b)},$$

plays the identical role as the Lamb shift of the two-atom system which arises due to the interaction between the atoms and the external field \([3]\).

2.1 The shifts of the energy level of the two-atom system

It is shown that when both the atoms are in the ground state or the excited state, there exists no interatomic correlation between them. Interatomic correlations only exist in the symmetric and antisymmetric states (entangled states) \([3]\). In this work, we want to investigate the effect of interatomic correlations between two atoms. We achieve this by calculating the shifts in the energy levels of the symmetric and antisymmetric states of the entangled two-atom system. By calculating the average values of $H_{LS}$ on the symmetric state $|S\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle|g_2\rangle + |g_1\rangle|e_2\rangle)$ and the antisymmetric state $|A\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle|g_2\rangle - |g_1\rangle|e_2\rangle)$, one can obtain the shifts of the energy level of the symmetric state and the antisymmetric state as \([3]\),

$$\delta E_{S_{LS}} = \langle S|H_{LS}|S\rangle = -\frac{i}{2} \left[ \sum_{j=1}^{3} (H_{1j}^{12} + H_{2j}^{21} + H_{1j}^{11} + H_{2j}^{22}) - 2(H_{3j}^{12} + H_{3j}^{21}) \right],$$

$$\delta E_{A_{LS}} = \langle A|H_{LS}|A\rangle = \frac{i}{2} \left[ \sum_{j=1}^{3} (H_{jj}^{12} + H_{jj}^{21} - H_{jj}^{11} - H_{jj}^{22}) \right].$$

(12)

Here $\delta E_{S_{LS}}$ stands for the first-order energy level shift of the symmetric state and $\delta E_{A_{LS}}$ stands for corresponding energy level shift of the antisymmetric state. We note that these shifts of the energy level are related to the two-point functions through the equations \((6)-(10)\) which can be computed along the trajectories of the atoms and thus they depend on the spacetime background. As a result, these shifts of the energy level are different in different spacetimes. Here we compute these shifts of the energy level for the Schwarzschild spacetime when the atoms are located close to the horizon and these shifts when the atoms are in a thermal Minkowski spacetime.
3 \textbf{RCPI for the Schwarzschild spacetime when two static entangled atoms are located close to the horizon}

3.1 Schwarzschild metric in near horizon region

Here we consider (3+1) dimensional Schwarzschild spacetime to compute the \textit{RCPI} between two entangled atoms when the atoms are located close to the horizon. The Schwarzschild spacetime is described by the metric

\begin{equation}
\text{ds}^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,
\end{equation}

(13)

where \( f(r) = (1 - r_s/r) \) and \( r_s = 2GM \) is the Schwarzschild radius related to the metric. A proper distance from the horizon to a radial distance \( r \) is defined by the formula \[35],

\begin{equation}
l = \int_{r_s}^r \frac{dr'}{\sqrt{1 - \frac{r}{r'}}} = \sqrt{r(r-r_s)} + r_s \sinh^{-1}(\sqrt{\frac{r}{r_s}} - 1).
\end{equation}

(14)

In terms of \( l \), the Schwarzschild metric (13) becomes

\begin{equation}
\text{ds}^2 = -f(r) dt^2 + dl^2 + r_s^2 d\theta^2 + r_s^2 \sin^2 \theta d\phi^2,
\end{equation}

(15)

where \( f(r) = (1 - r_s/r(l)) \). Now near the horizon, where \( r = r_s + \delta \) and \( l = 2\sqrt{r_s}\delta \), within the leading order approximation, the Schwarzschild metric (15) becomes \[35-38]

\begin{equation}
\text{ds}^2 = -l^2 \frac{dt^2}{4r_s^2} + dl^2 + r_s^2 d\theta^2 + r_s^2 \sin^2 \theta d\phi^2.
\end{equation}

(16)

Here we have considered \( \delta \) is a positive parameter and \( \delta << r_s \). Now we can define new coordinates which are given by,

\begin{equation}
X_1 = l \cosh \frac{l}{2r_s}; \quad T = l \sinh \frac{l}{2r_s}.
\end{equation}

(17)

Using these coordinates (17), the metric (16) becomes

\begin{equation}
\text{ds}^2 = -dT^2 + dX_1^2 + r_s^2 d\theta^2 + r_s^2 \sin^2 \theta d\phi^2.
\end{equation}

(18)
If we only focus on a small angular region near the horizon which is around $\theta = 0$ then we can replace the angular coordinates with the Cartesian coordinates which are given by [35, 38]

$$X_2 = r_s \theta \cos \phi ; \quad X_3 = r_s \theta \sin \phi . \quad (19)$$

Using these coordinates (19), the equation (18) becomes

$$ds^2 = -dT^2 + dX_1^2 + dX_2^2 + dX_3^2 , \quad (20)$$

which expresses the Minkowski spacetime [35].

### 3.2 Resonance Casimir-Polder interaction

In the position space, using the coordinates of the inertial metric (20), the two-point function for a massless scalar field can be expressed as

$$G(x, x') \equiv \langle 0|\hat{\Phi}(x)\hat{\Phi}(x')|0\rangle = \langle 0|\hat{\Phi}(T, x)\hat{\Phi}(T', x')|0\rangle , \quad (21)$$

where $|0\rangle$ indicates corresponding vacuum state. Here we ignore back-reaction of this scalar field on the spacetime metric. It can be shown that in Fock quantization, the two-point function (21) becomes in this usual form [21, 25],

$$G(x, x') = -\frac{1}{4\pi^2} \frac{1}{(\Delta T - i\epsilon)^2 - |\Delta x|^2} , \quad (22)$$

where $-\Delta T^2 + |\Delta x|^2 = -(T - T')^2 + (X_1 - X_1')^2 + (X_2 - X_2')^2 + (X_3 - X_3')^2$ is the Lorentz invariant spacetime interval and $\epsilon$ is a small, positive parameter which is introduced to evaluate two-point function.

Now we assume that two static atoms are located at the position $(r, \theta, \phi)$ and $(r, \theta', \phi)$ which are close to the horizon and the angles $\theta$ and $\theta'$ are taken to be small. Using the equations (8171922), we obtain the two-point functions for these two spacetime points which are given by,

$$G^{(11)}(x, x') = G^{(22)}(x, x')$$

$$= -\frac{1}{4\pi^2} \left[ (l \sinh t / 2r_s - l \sinh t' / 2r_s - i\epsilon)^2 
+ (l \cosh t / 2r_s - l \cosh t' / 2r_s)^2 \right]^{-1}$$

$$= -\frac{1}{16\pi^2 l^2 \sinh^2 (\Delta T / 2l - i\epsilon)} , \quad (23)$$

7
and

\begin{align}
G^{(12)}(x, x') &= G^{(21)}(x, x') \\
&= -\frac{1}{4\pi^2} \left[ (l \sinh t/2r_s - l \sinh t'/2r_s - i\varepsilon)^2 \\
&- (l \cosh t/2r_s - l \cosh t'/2r_s)^2 \\
&- (r_s \theta \cos \phi - r_s \theta' \cos \phi)^2 \\
&- (r_s \sin \phi - r_s \theta' \sin \phi)^2 \right]^{-1} \\
&= -\frac{1}{16\pi^2 l^2} \frac{1}{\sinh^2(\frac{4l}{2\pi} - i\varepsilon) - \frac{r_s^2}{4}(\frac{\Delta \theta}{2})^2},
\end{align}

(24)

where \(\Delta \tau = \frac{1}{2r_s}(t - t') = \frac{1}{2r_s}\Delta t\) with \(\tau\) which is the proper time of the static atoms in the Schwarzschild spacetime and \(\Delta \theta = (\theta - \theta')\). Here we have considered the atoms are located at \((r, \theta, \phi)\) and \((r, \theta', \phi)\) respectively, where \(r\) is close to the horizon. One can consider the other cases where the two atoms located at the same \((\theta, \phi)\) value but on the two sides of the horizon and at \((r, \theta, \phi)\) and \((r, \theta, \phi')\). The final results will be different for both the cases because the two-point functions will be different for those cases. Now using this two-point functions (23,24), we can compute the Fourier transforms (7) of these two-point functions which are given by,

\begin{align}
\mathcal{G}^{(11)}(\Omega) &= G^{(22)}(\Omega) \\
&= \int_{-\infty}^{\infty} \frac{1}{16\pi^2 l^2} \frac{1}{\sinh^2(\frac{4l}{2\pi} - i\varepsilon)} e^{i\Omega \Delta \tau} d\Delta \tau \\
&= \frac{\Omega}{2\pi 1 - e^{-2\pi i \Omega}},
\end{align}

(25)

and

\begin{align}
\mathcal{G}^{(12)}(\Omega) &= G^{(21)}(\Omega) \\
&= \int_{-\infty}^{\infty} \frac{1}{16\pi^2 l^2} \frac{1}{\sinh^2(\frac{4l}{2\pi} - i\varepsilon) - \frac{r_s^2}{4}(\frac{\Delta \theta}{2})^2} e^{i\Omega \Delta \tau} d\Delta \tau \\
&= \frac{\Omega}{2\pi 1 - e^{-2\pi i \Omega}} g(\Omega, L/2),
\end{align}

(26)

where we define \(g(\Omega, z) = \frac{\sin[2/\Omega \sin^{-1}(z/l)]}{2z\Omega \sqrt{1+z^2/l^2}}\). Within the leading order approximation, here \(L = r_s(\Delta \theta)\) denotes the proper distance between the two points \((r, \theta, \phi)\) and \((r, \theta', \phi)\) which are close to the horizon where the angles \(\theta\) and \(\theta'\) are taken to
be small. Now using the Fourier transforms of two-points functions \((25, 26)\), we can compute the Hilbert transforms of the two-point functions \((6)\) which are given by,

\[
K^{(11)}(\omega_0) = K^{(22)}(\omega_0) = \frac{P}{2\pi^2 i} \int_{-\infty}^{\infty} d\omega \frac{1}{\omega - \omega_0} \frac{\omega}{1 - e^{-2\pi i \omega}},
\]

and

\[
K^{(12)}(\omega_0) = K^{(21)}(\omega_0) = \frac{P}{2\pi^2 i} \int_{-\infty}^{\infty} d\omega \frac{1}{\omega - \omega_0} \frac{\omega}{1 - e^{-2\pi i \omega}} g(\omega, L/2).
\]

After putting these Hilbert transforms into the equations \((9)\) and \((10)\), we get

\[
H^{(11)}_{jk} = H^{(22)}_{jk} = A_1 \delta_{jk} - iB_1 \varepsilon_{jkl} \delta_{3l} - A_1 \delta_{3j} \delta_{3k},
\]

\[
H^{(12)}_{jk} = H^{(21)}_{jk} = A_2 \delta_{jk} - iB_2 \varepsilon_{jkl} \delta_{3l} - A_2 \delta_{3j} \delta_{3k},
\]

(27)

where the parameters \(A_1, B_1, A_2\) and \(B_2\) are given by,

\[
A_1 = \frac{\lambda^2 P}{8\pi^2 i} \int_{-\infty}^{\infty} d\omega \left( \frac{\omega}{\omega - \omega_0} + \frac{\omega}{\omega + \omega_0} \right) \frac{1}{1 - e^{-2\pi i \omega}},
\]

\[
B_1 = \frac{\lambda^2 P}{8\pi^2 i} \int_{-\infty}^{\infty} d\omega \left( \frac{\omega}{\omega - \omega_0} - \frac{\omega}{\omega + \omega_0} \right) \frac{1}{1 - e^{-2\pi i \omega}},
\]

\[
A_2 = \frac{\lambda^2 P}{8\pi^2 i} \int_{-\infty}^{\infty} d\omega \left( \frac{\omega}{\omega - \omega_0} + \frac{\omega}{\omega + \omega_0} \right) \frac{1}{1 - e^{-2\pi i \omega}} \times g(\omega, L/2),
\]

\[
B_2 = \frac{\lambda^2 P}{8\pi^2 i} \int_{-\infty}^{\infty} d\omega \left( \frac{\omega}{\omega - \omega_0} - \frac{\omega}{\omega + \omega_0} \right) \frac{1}{1 - e^{-2\pi i \omega}} \times g(\omega, L/2).
\]

(28)

Using the equations \((27)\) and \((28)\), one can calculate the shifts of the energy level of the symmetric state and the antisymmetric state of the two-atom system \((12)\) which are given by,

\[
\delta E_{Sls} = -\frac{\lambda^2}{4\pi^2} \int_0^{\infty} d\omega \left( \frac{\omega}{\omega - \omega_0} + \frac{\omega}{\omega + \omega_0} \right) [g(\omega, L/2) + 1],
\]

\[
\delta E_{Als} = \frac{\lambda^2}{4\pi^2} \int_0^{\infty} d\omega \left( \frac{\omega}{\omega - \omega_0} + \frac{\omega}{\omega + \omega_0} \right) [g(\omega, L/2) - 1].
\]

(29)
From the above equations (29), it is shown that the shifts of the energy level of the symmetric state and the antisymmetric state depend on the proper distance, $L$, between the atoms. So the interatomic interactions exist in the symmetric state and the antisymmetric state of the two-atom system \[3\]. Again for computing Casimir-Polder force between the two atoms, one has to take the derivative with respect to $L$. So one can neglect the terms which do not depend on $L$ from the above equations (29) to rewrite the interatomic interactions \[3\]. The interatomic interactions for the symmetric state and the antisymmetric state of the two-atom system are then given by,

$$
\delta E_S = -\frac{\lambda^2}{4\pi^2} \int_0^\infty d\omega \left( \frac{\omega}{\omega - \omega_0} + \frac{\omega}{\omega + \omega_0} \right) g(\omega, L/2),
$$

$$
\delta E_A = \frac{\lambda^2}{4\pi^2} \int_0^\infty d\omega \left( \frac{\omega}{\omega - \omega_0} + \frac{\omega}{\omega + \omega_0} \right) g(\omega, L/2).
$$

We can evaluate the integral in the above equations (30) analytically and the results are given by,

$$
\delta E_S = -\frac{\lambda^2}{4\pi L \sqrt{1 + (L/2l)^2}} \cos(2\omega_0 l \sinh^{-1}(L/2l)),
$$

$$
\delta E_A = \frac{\lambda^2}{4\pi L \sqrt{1 + (L/2l)^2}} \cos(2\omega_0 l \sinh^{-1}(L/2l)).
$$

These are the resonance interatomic interactions between two entangled atoms for the Schwarzschild spacetime when the atoms are located close to the horizon.

We note that the characteristic length scale $l$ which up to the leading order approximation is equal to $\sqrt{1 - \frac{2}{\kappa}}$, where $\kappa = \frac{1}{2\mathcal{V}}$ is the surface gravity. In order to investigate the detailed behavior of the RCP between two entangled atoms widely, here we consider both the limits of the proper distance between the atoms which are larger and smaller than the characteristic length scale. When the proper distance between two atoms is larger than the characteristic length scale, the metric shows a strong noninertial character where the results should be different with that corresponding to the Minkowski spacetime. Whereas, when the proper distance between two atoms is smaller than the characteristic length scale, it is possible to find a local inertial frame where the results should be the same, as obtained in Minkowski spacetime.

From equation (31), it is shown that in the limit $L \gg l$ (or $\Delta \theta \gg 2\sqrt{\delta r}$), the
RCPI can be expressed as

\[ \delta E_S = -\frac{\lambda^2}{2\pi L^2} \cos(2\omega_0 l \log(L/l)) , \]

\[ \delta E_A = \frac{\lambda^2}{2\pi L^2} \cos(2\omega_0 l \log(L/l)) , \]

and in the limit \( L \ll l \) (or \( \Delta \theta \ll 2\sqrt{\frac{\delta}{r_s}} \)), the RCPI can be expressed as

\[ \delta E_S = -\frac{\lambda^2}{4\pi L} \cos(\omega_0 L) , \]

\[ \delta E_A = \frac{\lambda^2}{4\pi L} \cos(\omega_0 L) . \]

In this limit, the effective spacetime curvature is neglected so the result is same as obtained in thermal Minkowski spacetime.

Here we have shown that beyond a characteristic length scale which is proportional to the inverse of the surface gravity \( \kappa \), the RCPI between two entangled atoms is characterized by a \( 1/L^2 \) power-law decay for the Schwarzschild spacetime when the atoms are located close to the horizon. We have also shown that beyond the characteristic length scale, the RCPI depends on the characteristic length scale which is also related to redshifted Unruh-Davies temperature temperature \( T = 1/2\pi l = \frac{\kappa}{2\pi k_B \sqrt{1 - r_s/r}} \) measured by a static observer near the horizon [16–27]. This temperature is equal to the Hawking temperature [16–27, 39–62] measured by the observer.

4 Comparing the relative proper acceleration between two atoms and RCPI in Schwarzschild spacetime

As Schwarzschild spacetime is curved spacetime so for an atom located with a radial distance \( r \) from the black hole, its proper acceleration is \( a = \frac{M}{r^2\sqrt{1 - r_s/r}} \), then the relative proper acceleration of the two atoms located with the same radial distance \( r \) from the black hole with an azimuthal interval \( \Delta \theta = \theta - \theta' \ll 1 \) follows

\[ \Delta a = 2a\sin(\Delta \theta/2) \approx a\Delta \theta = \frac{M}{r^2\sqrt{1 - r_s/r}}\Delta \theta . \]

Now relative acceleration for the atoms located near the horizon becomes,

\[ \Delta a|_{r\to r_s} \approx \frac{M}{r\sqrt{r(r - r_s)}}\Delta \theta \approx \frac{\Delta \theta}{2\sqrt{r_s} \delta} = \frac{\Delta \theta}{l} . \]
So the interatomic separation \( L \gg l \) (or \( \Delta > \gg 2\sqrt{\frac{\kappa}{r}} \)) is equivalent to \( \Delta a_{\mid r \rightarrow r_s} \gg \frac{1}{r_s} \), indicating that the relative proper acceleration of the two atoms in the case of \( L \gg l \) is not negligible, and thus the dynamics of the two atoms cannot be depicted in the same local inertial frame. So we can take \( L \gg l \) limit as a result Eq. (32) is not valid. But the relative proper acceleration of the two atoms in the case of \( L \ll l \) is negligible, and thus the dynamics of the two atoms can be depicted in the same local inertial frame. So we can take \( L \ll l \) limit as a result Eq. (33) is valid. We have shown that, for Schwarzschild spacetime, beyond a characteristic length scale which is proportional to the inverse of the surface gravity \( \kappa \), the Resonance Casimir-Polder interaction (RCPI) between two entangled atoms is characterized by a \( 1/L^2 \) power-law provided the atoms are located close to the horizon. But here we have shown that the length scale limit beyond a characteristic value is not valid for Schwarzschild spacetime.

5 \textit{RCPI for the thermal Minkowski spacetime}

In order to compare the RCPI behavior for thermal Minkowski spacetime and Schwarzschild spacetime Minkowski spacetime here we have considered two static atoms in Minkowski spacetime. These atoms are coupled to a massless scalar field in a thermal state with the temperature \( T = 1/2\pi l \). The two-point functions for this case are given by [3, 21],

\[
G^{(11)}(x, x') = G^{(22)}(x, x') = -\frac{1}{4\pi^2} \sum_{m=-\infty}^{+\infty} \frac{1}{(\Delta \tau - im/T - i\epsilon)^2}
\]

and

\[
G^{(12)}(x, x') = G^{(21)}(x, x') = -\frac{1}{4\pi^2} \sum_{m=-\infty}^{+\infty} \frac{1}{(\Delta \tau - im/T - i\epsilon)^2 - L^2},
\]

where \( \Delta \tau = t - t' \) with \( t \) which is the proper time of the static atoms in Minkowski spacetime and \( L = 2r\sin(\Delta \theta/2) \) is the Euclidean distance between them when the atoms are located at the position \( (r, \theta, \phi) \) and \( (r', \theta', \phi) \) in this spacetime [3]. From these two-point functions, one can compute the RCPI between two entangled atoms for the thermal Minkowski spacetime using the similar method. The interatomic interactions for the symmetric state and the antisymmetric state for
\[
\delta E_{SM} = -\frac{\mu^2}{4\pi L} \cos(\omega_0 L), \\
\delta E_{AM} = \frac{\mu^2}{4\pi L} \cos(\alpha_0 L).
\] (36)

Here we have shown that the \textit{RCPI} between two entangled atoms is always characterized by a \(1/L\) power-law decay for the Minkowski spacetime [3]. We have also shown that the \textit{RCPI} for this spacetime is always temperature-independent [3] and similar to the \textit{RCPI} between two inertial atoms shown in the equation (33). In Minkowski spacetime the surface gravity \(\kappa\) is zero so always \(L \ll l\) and thus the final result is always independent of the temperature and hence of the associated scale \(l\), no matter whether the field state is thermal or not [3].

6 Discussion

Here we have applied the open quantum system approach to obtain effective Hamiltonian of two atoms. This effective Hamiltonian allow us to compute the \textit{RCPI} between two entangled atoms. Subsequently, we have calculated the \textit{RCPI} for the Schwarzschild spacetime when the atoms are located close to the horizon. Although we have shown that beyond a characteristic length scale which is proportional to the inverse of the surface gravity \(\kappa\), the \textit{RCPI} is characterized by a \(1/L^2\) power-law decay for the Schwarzschild spacetime. We have also shown that beyond the characteristic length scale, the \textit{RCPI} for this spacetime depends on the characteristic length scale which is also related to the temperature measured by a static observer near the horizon. Whereas, the \textit{RCPI} is temperature-independent and is always characterized by a \(1/L\) power-law decay for the thermal Minkowski spacetime. Seemingly, it appears that the spacetimes can be distinguished from each other using the \textit{RCPI} behavior. But our further exploration leads to the conclusion that the length scale limit beyond a characteristic value is not valid for Schwarzschild spacetime. In summary, using the \textit{RCPI} between two entangled atoms, one can not distinguish these two spacetimes.

\textit{Acknowledgments.}

I thank Golam Mortuza Hossain, Sumanta Chakraborty, Arnab Chakrabarti and Gopal Sardar for many useful discussions. I also thank IISER Kolkata for supporting this work through doctoral fellowships.
References

[1] D. Dalvit, P. Milonni, D. Roberts, and F. Rosa, *Casimir Physics*, Lecture Notes in Physics 834 (Springer-Verlag Berlin Heidelberg, 2011), 1st ed.

[2] M.A. Cirone, G. Compagno, G.M. Palma, R. Passante, and F. Persico, EPL 78, 30003 (2007).

[3] Z. Tian, J. Wang, J. Jing, and A. Dragan, Sci. Rep. 6, 35222 (2016), arXiv:1605.07350.

[4] Z. Tian and J. Jing, JHEP 07, 089 (2014), arXiv:1405.7439.

[5] J. Zhang and H. Yu, Phys. Rev. A 88, 064501 (2013).

[6] L. Rizzuto, Phys. Rev. A 76, 062114 (2007).

[7] J. Marino, A. Noto, and R. Passante, Phys. Rev. Lett. 113, 020403 (2014).

[8] L. Rizzuto, M. Lattuca, J. Marino, A. Noto, S. Spagnolo, R. Passante, and W. Zhou, Phys. Rev. A94, 012121 (2016), arXiv:1601.04502.

[9] H. W. Yu, and J. Zhang, Phys. Rev. D77, 024031 (2008), arXiv:0806.3602.

[10] G. Menezes, C. Kiefer, and J. Marino, Phys. Rev. D95, 085014 (2017), arXiv:1703.00193.

[11] W. Zhou and H. Yu, Phys. Rev. D96, 045018 (2017).

[12] X. Liu, Z. Tian, J. Wang, and J. Jing, Phys. Rev. D 97, 105030 (2018).

[13] G. V. Steeg and N. C. Menicucci, Phys. Rev. D 79, 044027 (2009).

[14] J. Hu and H. Yu, Phys. Rev. D 88, 104003 (2013).

[15] G. Salton, R. B. Mann, and N. C. Menicucci, New Journal of Physics 17, 035001 (2015).

[16] D. Singleton and S. Wilburn, Phys. Rev. Lett. 107, 081102 (2011), arXiv:1102.5564.

[17] M. Smerlak and S. Singh, Phys. Rev. D 88, 104023 (2013), arXiv:1304.2858 [gr-qc].

[18] W. Israel, Physics Letters A 57, 107 (1976).
[19] W. Israel, Physica A: Statistical Mechanics and its Applications 106, 204 (1981).

[20] K. J. Hinton, Journal of Physics A: Mathematical and General 16, 1937 (1983).

[21] N. D. Birrell and P. C. W. Davies, Quantum fields in curved space, 7 (Cambridge university press, 1984).

[22] L. Hodgkinson, J. Louko, and A. C. Ottewill, Phys. Rev. D89, 104002 (2014), arXiv:1401.2667.

[23] G. Acquaviva, R. Di Criscienzo, M. Tolotti, L. Vanzo, and S. Zerbini, Int. J. Theor. Phys. 51, 1555 (2012), arXiv:1111.6389.

[24] G. Acquaviva, Springer Proc. Phys. 170, 323 (2016), arXiv:1310.6858.

[25] L. Sriramkumar and T. Padmanabhan, Class. Quant. Grav. 13, 2061 (1996), arXiv:gr-qc/9408037.

[26] J. Louko and D. Marolf, Phys. Rev. D 58, 024007 (1998), gr-qc/9802068.

[27] T. Padmanabhan, Gravitation: Foundations and Frontiers (Cambridge University Press, 2010), 1st ed.

[28] D.P. Craig and T. Thirunamachandran, Molecular quantum electrodynamics (AP, 1984).

[29] A. Salam, Molecular Quantum Electrodynamics: Long-Range Intermolecular Interactions (Wiley, 2009), 1st ed.

[30] F. Benatti and R. Floreanini, Phys. Rev. A 70, 012112 (2004).

[31] E. Davis, Quantum Theory of Open Systems (Academic Press, 1976).

[32] R. H. Dicke, Phys. Rev. 93, 99 (1954).

[33] G. Lindblad, Communications in Mathematical Physics 48, 119 (1976).

[34] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, Journal of Mathematical Physics 17, 821 (1976).

[35] B. Ydri (2017), arXiv:1708.00748.
[36] J. Polchinski, in *Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings (TASI 2015): Boulder, CO, USA, June 1-26, 2015* (2017), pp. 353–397, arXiv:1609.04036.

[37] A. Dabholkar and S. Nampuri, Lect. Notes Phys. **851**, 165 (2012), arXiv:1208.4814.

[38] K. Goto and Y. Kazama (2018), arXiv:1803.01672.

[39] S. W. Hawking, Comm. Math. Phys. **43**, 199 (1975).

[40] S. Barman, G. M. Hossain, and C. Singha, Phys. Rev. **D97**, 025016 (2018), arXiv:1707.03614.

[41] P.-H. Lambert, PoS (Modave, 2013), arXiv:1310.8312.

[42] T. Jacobson, in *Lectures on quantum gravity. Proceedings, School of Quantum Gravity, Valdivia, Chile, January 4-14, 2002* (2003), pp. 39–89, gr-qc/0308048.

[43] C. Kiefer, in *DPG School of Physics (Course 2): Galactic Black Hole 2001 Bad Honnef, Germany, August 26-31, 2001* (2002), astro-ph/0202032.

[44] J. H. Traschen, in *Mathematical methods in physics. Proceedings, Winter School, Londrina, Brazil, August 17-26, 1999* (1999), gr-qc/0010055.

[45] B. S. DeWitt, Physics Reports **19**, 295 (1975).

[46] L. H. Ford, in *Particles and fields. Proceedings, 9th Jorge Andre Swieca Summer School, Campos do Jordao, Brazil, February 16-28, 1997* (1997), pp. 345–388, gr-qc/9707062.

[47] S. Hollands and R. M. Wald, Phys. Rept. **574** (2015), arXiv:1401.2026.

[48] T. Padmanabhan, Rept.Prog.Phys. **73**, 046901 (2010), arXiv:0911.5004.

[49] S. Chakraborty, S. Singh, and T. Padmanabhan, JHEP **06**, 192 (2015), arXiv:1503.01774.

[50] S. Chakraborty and K. Lochan (2017), arXiv:1702.07487.

[51] A. D. Helfer, Rept. Prog. Phys. **66**, 943 (2003), gr-qc/0304042.

[52] S. Carlip, Int. J. Mod. Phys. **D23**, 1430023 (2014), arXiv:1410.1486.
[53] S. Fulling and S. Ruijsenaars, Physics Reports 152, 135 (1987).
[54] M. K. Parikh and F. Wilczek, Phys. Rev. Lett. 85, 5042 (2000), hep-th/9907001.
[55] M. Visser, Int. J. Mod. Phys. D12, 649 (2003), hep-th/0106111.
[56] S. Bhattacharya and A. Lahiri, Eur. Phys. J. C73, 2673 (2013), arXiv:1301.4532.
[57] A. S. Lapedes, J. Math. Phys. 19, 2289 (1978).
[58] P. C. W. Davies, J. Phys. A8, 609 (1975).
[59] R. M. Wald, Communications in Mathematical Physics 45, 9 (1975).
[60] S. Singh and S. Chakraborty, Phys. Rev. D90, 024011 (2014), arXiv:1404.0884.
[61] T. Jacobson, Lect. Notes Phys. 870, 1 (2013), arXiv:1212.6821.
[62] J. B. Hartle and S. W. Hawking, Phys. Rev. D13, 2188 (1976).