Multi-topological Floquet metals in a photonic lattice

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Topological materials are usually classified according to a single topological invariant. The engineering of synthetic structures characterised by more than one class of topological invariants would open the way to the combination of different topological properties, enlarging the richness of topological phase diagrams. Using a synthetic photonic lattice implemented in a two-coupled ring system we engineer an anomalous Floquet metal that is gapless in the bulk and shows simultaneously two different topological properties. On the one hand, this synthetic lattice presents bands characterised by a winding number that directly relates to the period of Bloch oscillations within its bulk. On the other, the Floquet nature of our implementation results in well-known anomalous insulating phases with topological edge states. Our experiments open the way to study unconventional multi-topological phases in synthetic lattices.

One of the most striking properties of topological phases of matter is the appearance of robust unidirectional interface states between two gapped materials of different topology. The existence and number of edge channels is determined by a topological invariant, which is a property of the Hamiltonian describing the bulk materials[1][2]. This pivotal idea, known as the bulk-edge correspondence, has successfully explained the topological edge transport in the quantum Hall effect and in topological insulators[3]. The existence of topological edge states in anomalous Floquet systems[4] and in non-Hermitian lattices[2]. However, the non-trivial topology of a lattice Hamiltonian does not necessarily result in the emergence of edge states. Bright examples are the anomalous velocity due to non-zero Berry curvature[5][6] and the quantized transport in a Thouless pump[7][8]. Enlarging the palette of topological effects in lattices beyond the bulk-edge correspondence is an important resource that would allow combining different topological properties in a single material.

An example of band topologies with properties beyond the bulk-edge correspondence are periodically-driven (Floquet) Hamiltonians with nontrivial band holonomies[9][11]. The eigenvalues of Floquet Hamiltonians can form bands that are periodic both in momentum and quasienergy. This double periodicity enables the possibility of engineering bands with non-trivial windings. An example is shown in Fig. 1E: two bands never touch each other, but still traverse the whole quasienergy Brillouin zone. Since the system is gapless in the sense that bulk states exist at all energies, its spectrum can be identified with that of a metal[12]. In this report we show the experimental implementation of a such an anomalous Floquet metal. We show that the winding of the bands results in Bloch sub-oscillations of a topological origin. Furthermore, the time-periodic nature of the system can be used to engineer anomalous Floquet topological phases with topological edge states.

To establish the multiple topological properties, we use a two-dimensional synthetic photonic lattice implemented in two coupled fiber rings. Recently, photonic platforms based on fiber rings have permitted the study of unconventional topological effects hardly accessible in other systems[6][13][17].

The propagation of light pulses in two rings (Fig. 1A) can be mapped into a lattice of oriented scatterers (Fig. 1B), whose couplings and onsite energies can be manipulated at will[13][18][19]. The dynamics of a light pulse injected in the system follows a split-step coherent walk described by the equations[13][20]:

\[
\alpha_{n+1}^m = (\cos \theta_m \alpha_{n-1}^m + i \sin \theta_m \beta_{n-1}^m) e^{i \varphi_m},
\]

\[
\beta_{n+1}^m = i \sin \theta_m \alpha_{n+1}^m + \cos \theta_m \beta_{n+1}^m,
\]

where \(\alpha_n^m\) and \(\beta_n^m\) denote the complex amplitudes of a light pulse in the left and right fiber ring. The temporal position of a pulse within a ring corresponds to a lattice site \(n\) while the round trip number is the time step \(m\). The splitting ratio of the beamsplitters at step \(m\) is parametrized by \(\theta_m\) so that the transmission and reflection amplitudes are given by \(\cos \theta_m\) and \(\sin \theta_m\), respectively.

Lastly, an electrooptical phase modulator (PM) applies an extra phase \(\varphi_m\) to all light pulses in one of the rings at a time step \(m\).

We consider a time-periodic version of the model described by Eq. (1) with two steps per period \(T_F\). The coupling between rings alternates between \(\theta_1\) and \(\theta_2\) on odd and even steps. Similarly, \(\varphi_m\) takes the values \(\varphi_1 = c_1 \varphi\) and \(\varphi_2 = c_2 \varphi\), where \(\varphi \in [-\pi, \pi]\), and \(c_{1,2}\) are integer coefficients (Fig. 1B). The periodicity of the system in synthetic space and time allows applying the Floquet-Bloch ansatz to the eigenstates of Eq. (1), \((\alpha_n^m, \beta_n^m) = (A, B)e^{-iEm/2}e^{ikE_n/2}\), with \(E\) being the quasienergy, \(k\) the quasimomentum associated to the real-space position in the lattice, and \(\varphi\) can be
thought of as a generalised momentum (Supplementary Section I). This system has two bands \( E_{\pm}(k, \varphi) \) in the two-dimensional \( k, \varphi \) space (see Fig. 1D-E) \[1\].

The periodicity of the Brillouin zone in \( k \), \( \varphi \) and \( E \) allows for the engineering of bands with nontrivial windings. An example of such peculiar band structure is shown in Fig. 1E. The bands are inclined in quasienergy: when \( \varphi \) is changed, they experience a shift in quasienergy and a lateral displacement along quasimomentum \( k \) (Fig. 1F), the combined effect resulting in their winding.

Insights into the topological character of the winding of the bands can be gained by looking at the evolution operator after one Floquet period (two steps in our model):

\[
U_F(k, \varphi) = e^{iK\varphi}T_2S_2T_1S_1, \tag{2}
\]

where the unitary operators \( S_{1,2} \) and \( T_{1,2} \) represent the action of beamsplitters and phase shifts along the lattice, and \( K \equiv (c_1 + c_2)/2 \) (Supplementary Section II). The non-zero matrix elements of \( U_F \) in the real-space picture are sketched in Fig. 1C. From Eq. \( 2 \) one can see that \( K \neq 0 \) imprints a global phase during one Floquet period and breaks the generalized inversion symmetry \( U_F(k, \varphi) \leftrightarrow U_F(-k, -\varphi) \), leading to the winding of the bands along the quasienergy direction.

The quasienergy winding is a topologically protected property of the bulk of the system. The corresponding invariant can be defined using a homotopic property of \( U_F \) \[21\]:

\[
\nu = \sum_{j=\pm} \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{\partial E_j}{\partial \varphi} = \frac{1}{2\pi i} \int_0^{2\pi} d\varphi \text{Tr} \left[ U_F^{-1} \frac{\partial U_F}{\partial \varphi} \right], \tag{3}
\]

which gives \( \nu = 2K \) (Supplementary Section III). Since our model features two bands, the number \( K \) has a simple meaning: it shows how many times each band winds along the quasienergy axis for one full turn of \( \varphi \) from \(-\pi\) to \(\pi\). Note that Eq. \( 3 \) does not depend of \( k \): the winding is a property of the \( \varphi \) synthetic dimension and it takes the same value for any \( k \).

We experimentally demonstrate the Floquet winding metals by injecting a single \( \approx 1 \) ns long laser pulse at a position \( \alpha_{n=1}^0 \) and following the dynamics of the sys-
FIG. 3. Topological Bloch sub-oscillations. Real-space evolution of a wavepacket injected close to \( k = 0 \) into one of the bands and evolved under an adiabatic increase of \( \varphi \) at a rate of \( d\varphi/dt = 2\pi \times 0.008 \text{ rad/turn} \) for (a) \( K = 1 \) \((c_1 = 2, c_2 = 0)\), (b) \( K = 2 \) \((c_1 = 5, c_2 = -1)\), and (c) \( K = 3 \) \((c_1 = 8, c_2 = -2)\). Dashed orange lines show analytical curves. The number of sub-oscillations in one Bloch period is equal to \( K \). (d) Illustration of the experimental procedure. (e) Split-step coherent walk for \( K = 3 \) \((c_1 = 8, c_2 = -2)\) at a higher rate of \( d\varphi/dt = 2\pi \times 0.012 \text{ rad/turn} \), allowing to access more Bloch cycles and (f) evolution of its center-of-mass (dots). Sub-oscillations of different amplitudes are clearly seen and confirmed by analytical calculations (solid line). Gray areas emphasize the difference between the maximal and the minimal achievable amplitudes of sub-oscillations.

The topological feature we have just described does not present any particular effect on the real-space edges of the lattice. However, it has direct consequences on the wavepacket dynamics of the system: it manifest itself in a new kind of Bloch oscillations. If we adiabatically propagate a wavepacket with quasimomentum \( k \) along the \( \varphi \) dimension as sketched in Fig. 3D, the group velocity \( v_g = \partial E(k, \varphi)/\partial k \) periodically changes its sign, resulting in sub-oscillations of the wavepacket. Analytical inspection of the expression for \( v_g \) shows that within one Bloch period the group velocity changes its sign \( 2K \) times, thus leading to observation of \( K \) sub-oscillations. The frequency of these sub-oscillations is a quantity that is topologically protected by the winding (Supplementary Section VI).

To observe the topological Bloch sub-oscillations, we prepare a wavepacket at \( k = 0 \) in one of the quasienergy bands (Supplementary Section VII) and follow its evolution while \( \varphi \), imprinted by the phase modulator, is adiabatically increased at a constant rate \( \partial\varphi/\partial t \). The observed dynamics for winding metals with \( K = 1, 2, \) and
rather emerge from the generalised Floquet topological invariant related to the micromotion of the system during one driving period \( t_{\text{period}} = 1 \). Such anomalous Floquet phases have been reported in 1D photonic lattices \( 20, 23, 24 \) and in 2D systems \( 26, 27 \), and they can also exist in a Floquet winding metal that features two non-touching bands.

The phase diagram for the anomalous Floquet phases in the topological system is determined by the values of \( \theta_1 \) and \( \theta_2 \) for which the gap between the two bands closes (Fig. 4A). The phase diagram only depends on the values of the couplers and does not depend on the winding \( K \). Different topological phases (orange and blue in Fig. 4A) are confirmed by the presence of interface states in simulations when two semi-infinite lattices belonging to different regions in Fig. 4A are linked together. Edge states appear also for the anomalous phase at the edge of a single semi-infinite lattice (Supplementary Section VIII).

In our experiment we take profit of the full control over the couplings between the lattice sites to engineer interfaces between different topological phases. To demonstrate this, we consider a winding metal with \( K = -1 \) and prepare two topologically different phases with an interface at position \( n = 0 \). For lattice sites \( n < 0 \) we set \( \theta_1 = \pi/4, \theta_2 = \pi/4 - 0.4 \), forming an anomalous phase. For \( n > 0 \) we create a trivial phase with \( \theta_1 = \pi/4 - 0.4, \theta_2 = \pi/4 \) (triangles in Fig. 4A). When exciting the interface with a single pulse, the system demonstrates a localized edge state at \( n = 0 \) (see Fig. 4B for \( \varphi = \pi/2 \) ). Simultaneously, the band structure reveals a flat band in one of the gaps between the two bands (Fig. 4C), which can be associated to the localised interface state. To probe the full dispersion of the edge states in \( k \) and \( \varphi \) we perform the full band tomography (Fig. 4D). The spectral flow between two bands is evident in both gaps, in good agreement with simulations (Fig. 4E). While the topological origin of the edge states is confirmed by the fact that it requires the presence of an interface between two different phases, the access to the topological invariant associated to this two-dimensional split-step Floquet operator is still to be addressed.

We have shown the experimental realisation of an anomalous Floquet metal, which simultaneously hosts two different topological properties. Whereas the first one manifests itself in a novel kind of Bloch oscillations, the second one leads to formation of chiral states in \( \varphi \) at the edges of the lattice. Both of these topological properties arise from the Floquet nature of the system and therefore do not have static counterparts. The flexibility of our platform paves the road to studies of Floquet winding metals with unconventional dispersion in higher dimensions, and open unprecedented perspectives in the search for novel topological phases when including, for instance, non-Hermitian hoppings \( 13, 14, 30 \).

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**FIG. 4.** Topological edge states. (a) Phase diagram of anomalous Floquet phases. (b) Measured dynamics when exciting the lattice at a single site located at the interface between two lattices belonging to two different phases (triangles in (a)), showing a localized interface state. Both lattices are prepared with \( K = -1 \) \((c_1 = 1, c_2 = -3)\) and \( \varphi = \pi/2 \) but different values of \( \theta_1 \) and \( \theta_2 \). (c) Measured dispersion showing a flat band in one gap (shown with the arrow), corresponding to the interface state. (d) Experimental and (e) theoretical band tomography of the winding metal for all values of \( \varphi \) confirming the presence of chiral edge states in each of the gaps.
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I. BAND STRUCTURE OF THE TWO-STEP MODEL

We follow the procedure described in [11] to obtain an analytical expression for the band structure. In the case of a two-step periodic modulation, the time evolution of equation (1) can be decomposed into

\[
\begin{align*}
\alpha_{m+1}^n &= (\cos \theta_1 \alpha_{m-1}^n + i \sin \theta_1 \beta_{m-1}^n) e^{i\varphi_1} \\
\beta_{m+1}^n &= i \sin \theta_1 \alpha_{m-1}^n + \cos \theta_1 \beta_{m-1}^n 
\end{align*}
\]

(S1)

for the odd steps and

\[
\begin{align*}
\alpha_{m+2}^{n+1} &= (\cos \theta_2 \alpha_{m+1}^{n+1} + i \sin \theta_2 \beta_{m+1}^{n+1}) e^{i\varphi_2} \\
\beta_{m+2}^{n+1} &= i \sin \theta_2 \alpha_{m+1}^{n+1} + \cos \theta_2 \beta_{m+1}^{n+1} 
\end{align*}
\]

(S2)

for the even steps. We remind that \(\varphi_1 = c_1 \varphi\) and \(\varphi_2 = c_2 \varphi\). Relying on the translation symmetry and time periodicity, we can use a Floquet-Bloch ansatz

\[
\begin{pmatrix}
\alpha_{m+1}^n \\
\beta_{m+1}^n 
\end{pmatrix} =
\begin{pmatrix}
A \\
B 
\end{pmatrix} e^{-iEm/2} e^{i kn/2}
\]

(S3)

and by substituting it into Eqs. (S1)–(S2) and solving the determinant problem we obtain the solution

\[
E_{\pm}(k, \varphi) = \pm \arccos \left[ \cos \theta_1 \cos \theta_2 \cos (k + K \varphi) - \sin \theta_1 \sin \theta_2 \cos (\Delta \varphi) \right] + K \varphi,
\]

(S4)

where \(\varphi_{1,2} = c_{1,2} \varphi\), \(K \equiv (c_1 + c_2)/2\), \(\Delta \equiv (c_1 - c_2)/2\), and \(E_{\pm}\) denote two quasienergy bands. In this work we consider the case of integer \(K\) and \(\Delta\), which makes the period along the \(\varphi\) direction equal to \(2\pi\). The last term \(K \varphi\) in (S4) emphasizes the fact that each band winds \(K\) times along the quasienergy axis when \(\varphi\) is changed by \(2\pi\).

II. DERIVATION OF THE FLOQUET EVOLUTION OPERATOR

For the Floquet period of 2 steps the evolution of the system in real space can be written as

\[
\Psi(m+2) = U \Psi(m),
\]

(S5)

where

\[
\Psi(m) =
\begin{pmatrix}
\cdots \\
\alpha_m^n \\
\beta_m^n \\
\alpha_{m+2}^n \\
\beta_{m+2}^n \\
\cdots 
\end{pmatrix}
\]

(S6)

is a vector representing the state of the system in real space at time step \(m\), and

\[
U = \sum_{x_i, y_j} U_{x_i \rightarrow y_j} |x_j\rangle \langle x_i|
\]

(S7)

is the real-space Floquet evolution operator. Here \(|x_i\rangle\) and \(|y_j\rangle\), where \(x, y \in \{\alpha, \beta\}\) and \(i, j\) are the site number, represent a vector \(\Psi\) with \(x_i = 1\) (or \(y_j = 1\)) and all the other components equal to zero. Non-zero matrix elements
of $U$ can be found from the equations (S1)–(S2):

\[
\begin{align*}
U_{\alpha_n \to \beta_n} &= i s_1 R e^{i r_1} \\
U_{\beta_n \to \alpha_n} &= i s_1 R e^{-i r_1} \\
U_{\beta_n \to \alpha_{n+2}} &= i s_2 R e^{i r_2} \\
U_{\alpha_{n} \to \beta_{n-2}} &= i s_2 R e^{-i r_2} \\
U_{\alpha_{n} \to \alpha_{n+2}} &= s_3 R e^{i r_3} \\
U_{\beta_{n} \to \beta_{n-2}} &= s_3 R e^{-i r_3} \\
U_{\beta_{n} \to \beta_{n}} &= -s_4 R e^{i r_4} \\
U_{\alpha_{n} \to \alpha_{n}} &= -s_4 R e^{-i r_4}
\end{align*}
\]  

(S8)

where $s_1 = \cos \theta_1 \sin \theta_2$, $s_2 = \sin \theta_1 \cos \theta_2$, $s_3 = \cos \theta_1 \cos \theta_2$, $s_4 = \sin \theta_1 \sin \theta_2$, and $R = e^{i r_2}$. These couplings are schematically shown in Fig. 1C of the main text.

To obtain the Floquet evolution operator in reciprocal space, we can use the Floquet-Bloch ansatz (S3) and substitute it into Eqs. (S1)–(S2). This gives

\[
U_F(k, \varphi) = \left( \begin{array}{cc}
\cos \varphi & \sin \varphi \\
-\sin \varphi & \cos \varphi
\end{array} \right)
\]  

(S9)

It can be seen that the Floquet evolution operator can be factorized in a sequential manner

\[
U_F(k, \varphi) = D_2 B_2(k) S_2 D_1 B_1(k) S_1
\]  

(S10)

where $S_{1,2} = S(\theta_{1,2})$ are scattering matrices representing the action of the beamsplitter,

\[
S(\theta) = \left( \begin{array}{cc}
\cos \theta & i \sin \theta \\
i \sin \theta & \cos \theta
\end{array} \right)
\]  

(S11)

$B_{1,2}(k)$ are translation operators

\[
B_1(k) = \left( \begin{array}{cc}
1 & 0 \\
0 & e^{i k}
\end{array} \right), \quad B_2(k) = \left( \begin{array}{cc}
e^{-i k} & 0 \\
0 & 1
\end{array} \right)
\]  

(S12)

and $D_{1,2}$ correspond to the phase shift on odd and even steps:

\[
D_{1,2} = \left( \begin{array}{cc}
e^{i \varphi_{1,2}} & 0 \\
0 & 1
\end{array} \right)
\]  

(S13)

To study the symmetry properties of the unitary evolution operator we symmetrize the matrices $B_{1,2}(k)$ and $D_{1,2}$:

\[
B(k) = \left( \begin{array}{cc}
e^{-i k/2} & 0 \\
0 & e^{i k/2}
\end{array} \right), \quad D(\varphi) = \left( \begin{array}{cc}
e^{i \varphi/2} & 0 \\
0 & e^{-i \varphi/2}
\end{array} \right)
\]  

(S14)

and write

\[
U_F(k, \varphi) = e^{i(\varphi_{1} + \varphi_{2})/2} D(\varphi_2) B(k) S_2 D(\varphi_1) B(k) S_1
\]

\[
e^{i(\varphi_{1} + \varphi_{2})/2} T_2 S_2 T_1 S_1,
\]  

(S15)

where $T_{1,2} = D(\varphi_{1,2}) B(k)$. We notice that both $B(k)$ and $D(\varphi)$ possess inversion symmetry: $\sigma_x B(k) \sigma_x = B(-k)$, $\sigma_x D(\varphi) \sigma_x = D(-\varphi)$, where $\sigma_x$ is the Pauli matrix. Consequently, for $\varphi_1 + \varphi_2 = 0$ the Floquet evolution operator also has the inversion symmetry $\sigma_x U_F(k, \varphi) \sigma_x = U_F(-k, -\varphi)$. However, introducing a net phase $\varphi_1 + \varphi_2 \neq 0$ over one Floquet period breaks this symmetry and leads to winding of the bands.
III. CALCULATION OF THE TOPOLOGICAL INVARIANT

Given the factorized version of the Floquet evolution operator (S10), we can calculate the topological invariant

\[ \nu = \frac{1}{2\pi i} \int_0^{2\pi} d\varphi \mathrm{Tr} \left[ U_F^{-1} \frac{\partial U_F}{\partial \varphi} \right] \]

\[ = \frac{1}{2\pi i} \int_0^{2\pi} d\varphi \mathrm{Tr} \left[ S^\dagger_1 B^\dagger_1 S^\dagger_2 B^\dagger_2 D^\dagger_1 \left[ D_2 B_2 S_1 B_1 S_1 \right] \right] \]

\[ = \frac{1}{2\pi i} \int_0^{2\pi} d\varphi \mathrm{Tr} \left[ D^\dagger_1 \frac{\partial D_1}{\partial \varphi} + D^\dagger_2 \frac{\partial D_2}{\partial \varphi} \right]. \]  

(S16)

By substituting (S13) we get

\[ \nu = \frac{1}{2\pi i} \int_0^{2\pi} d\varphi \left[ i c_1 + i c_2 \right] = 2K. \]

(S17)

IV. EXPERIMENTAL SETUP

![Experimental setup diagram](image)

FIG. S1. Experimental setup. Main components are presented: AM: amplitude EOM, PM: phase EOM, EDFA: erbium-doped fiber amplifier, PD: photodiode.

The experimental setup is very similar to the one reported in [19]. It consists of two fiber rings coupled by an electronically-controlled high-bandwidth variable beamsplitter (EOSpace AX-2x2-0MSS-20). Each of the rings contains an erbium-doped fiber amplifier (Keopsys CEFA-C-HG) followed by a narrow-band optical filter (EXFO XTM-50), an isolator, a polarizer, a variable attenuator, and an optical switch (Photonwares NSSW). One of the rings contains a phase electro-optic modulator (EOM, iXblue MPZ-LN-10). All the fiber components use polarization-maintaining fibers. Each ring has a length of 40 m, and the length difference between the rings is 0.55 m. The mean length of the two rings sets the round trip period, of 205 ns, between the different time steps \( m \). The length difference sets the temporal size of the lattice sites \( n \) in the synthetic spatial dimension, of 2.7 ns.

For the injection of light, emission of a narrow single-frequency laser (IPG Photonics ELR-5-LP) at a wavelength of 1550 nm is chopped into 1.4 ns-long pulses by an amplitude EOM (iXblue MXER-LN-10). Before entering the fiber rings the light passes through an optical switch, which is closed after the injection. This ensures that no spurious signal from the laser enters the fibers during the experiment. The prepared injection signal is coupled into one of the rings through a 70/30 beamsplitter. The light field in the system is probed via an 80/20 beamsplitter in each of the rings.

To get access to both the amplitude and the phase of each light pulse we use optical heterodyning. For this, a fraction of the laser light is modulated by a phase EOM at a frequency of \( \Omega = 3 \) GHz, thus creating sidebands shifted by \( \pm \Omega \) from the laser frequency. Next, the \( +\Omega \) sideband is filtered out by a home-built fiber
The filtered out light field is used as a local oscillator, and its beating with the signal from each ring is measured by a fast photodiode (Thorlabs DET08CFC). Recording the response of the photodiode with a fast oscilloscope (Tektronix MSO64, bandwidth 4 GHz) allows to see the beating, the amplitude and the phase of which directly correspond to the amplitude and the phase of the light field under study.

V. RECONSTRUCTION OF THE BAND STRUCTURE

Due to the periodicity of the system in both synthetic dimension and time, its band structure can be obtained simply by calculating the two-dimensional Fourier transform (2DFT) of a split-step coherent walk. An important prerequisite for this is that each site of the walk ($\alpha_m^n$ and $\beta_m^n$) is a complex number, which accounts for both the amplitude and the phase of the light field. In our experiment the measured quantity is the beating of the signal with the local oscillator at a constant frequency $\Omega$. This allows us to reconstruct the band structure by performing the 2DFT of the measured signal and offsetting it by the frequency $\Omega$.
the same during the subsequent shot.

VI. GROUP VELOCITY OF A WAVEPACKET DURING BLOCH OSCILLATIONS

The group velocity in the real space dimension can be found as \[19\]:

\[ v_{g}^{\pm}(k, \varphi) = \pm \frac{\partial E_{\pm}(k, \varphi)}{\partial k} = \pm \frac{\cos \theta_1 \cos \theta_2 \sin (k + K \varphi)}{\sqrt{1 - [\cos \theta_1 \cos \theta_2 \cos (k + K \varphi) - \sin \theta_1 \sin \theta_2 \cos (\Delta \varphi)]^2}} \]  

(S19)

Due to the term \( \sin (k + K \varphi) \) in the numerator, the sign of \( v_{g} \) changes \( 2K \) times when \( \varphi \) is changed by \( 2\pi \), forcing a wavepacket to experience \( K \) sub-oscillations during one driving period. Since \( \sin (k + K \varphi) \) becomes zero with periodicity of \( \pi/K \) in \( \varphi \), we can claim that the winding number topologically protects the frequency of Bloch sub-oscillations. At the same time if \( K \neq \Delta \), then the term \( \cos (k + K \varphi) \) in the denominator precesses at a different rate than \( \cos (\Delta \varphi) \). Consequently, the translational symmetry \( v_{g}(k, \varphi) = v_{g}(k, \varphi + 2\pi/K) \) gets broken, leading to sub-oscillations of different amplitudes.

Finally, the center-of-mass motion of the wavepacket can be found by integrating the group velocity:

\[ X(k, t) = \int_{0}^{t} v_{g}(k, \varphi(\tau)) d\tau \]  

(S20)

VII. EXCITATION OF A SINGLE BAND

To excite a wavepacket in one band we use the technique described in Ref. \[31\]. We start with a theoretical description for the case of \( \theta_1 = \theta_2 = \pi/4 \), which has simple and intuitive analytical expressions for the eigenstates. At time step \( m = 0 \) we inject a train of pulses with Gaussian envelope into one ring, i.e.

\[ \alpha_0^n = e^{-n^2/\sigma^2}, \beta_0^n = 0. \]  

(S21)

Such excitation populates the eigenstates with narrow quasimomentum spread around \( k \approx 0 \) in both bands. This can be understood knowing that the eigenvectors of the model corresponding to the eigenvalues \( E_{\pm} \) are \[31\]:

\[ \Psi_{\pm} = \begin{pmatrix} A \\ B \end{pmatrix}_{\pm} = \frac{1}{\sqrt{1 + e^{\pm 2 \sin k/2}}} \left( \frac{1}{1} e^{\mp \sin k/2} \right) \]  

(S22)

For \( k = 0 \)

\[ \Psi_{\pm}(k = 0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mp 1 \end{pmatrix}, \]  

(S23)

and for broad Gaussian wavepackets with \( \sigma \gg 1 \) the excitation \( \text{(S21)} \) excites equal fraction of both bands at \( k = 0 \):

\[ \left( \begin{array}{c} \alpha_0^n \\ \beta_0^n \end{array} \right) \approx \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} (\Psi_+ + \Psi_-) \]  

(S24)

To excite a single band, we program the PM during the turn \( m = 1 \) to apply a phase \( \varphi_1 = \pi/2 \). After the first step, the state of the systems becomes

\[ \left( \begin{array}{c} \alpha_1^{n+1} \\ \beta_1^{n+1} \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \alpha_0^n e^{i \varphi_1} \\ i \beta_0^n \end{array} \right) \approx i \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) = i \Psi_-, \]  

(S25)

and, up to the global phase factor, occupies only one single band \( \Psi_- \). Note that choosing \( \varphi_1 = -\pi/2 \) would occupy the \( \Psi_+ \) band.

For the arbitrary choice of \( \theta_1 \) and \( \theta_2 \) one would need to adjust both the phase and the amplitude of signals in two rings in order excite one single band. However, if chosen values of \( \theta_1 \) and \( \theta_2 \) do not alter significantly the shape of the bands (which is the case of our work), one can still transfer the most part of the signal into one band. In our experiment, we can reproducibly inject more than a 80% of the emission into one of the bands (Fig. \[S3\]). Presence of the remaining excitation in the other band manifests itself as a weak signal in the real-space dynamics (Fig. \[3\] of the main text), which does not hinder any of the observed features.
VIII. IDENTIFICATION OF TRIVIAL AND ANOMALOUS FLOQUET PHASES

To identify the trivial and anomalous Floquet phases we compute the quasienergy spectra $E(k, \varphi)$ for a finite size system containing 50 unit cells along the synthetic dimension with fully reflective boundary conditions. The calculated spectra in the trivial and anomalous case are shown in Fig. S4A and B respectively. The anomalous phase clearly shows spectral features traversing the gaps, which correspond to the states localized at the edges of the lattice as in Fig. S4C.

FIG. S4. Calculated bands for the (a) trivial winding metal with $K = -1$, $\theta_1 = \pi/4 - 0.6$, and $\theta_2 = \pi/4$, (b) anomalous winding metal with $K = -1$, $\theta_1 = \pi/4$, and $\theta_2 = \pi/4 - 0.6$. Lines traversing the gap correspond to states localized at the edges. Both models are comprised of 50 sites along the synthetic dimension. (c) Probability amplitude of the red edge state marked by a black arrow in (b) at $\varphi = 1.24\pi$. 

FIG. S3. Excitation of one band