On the optical-mechanical analogy in general relativity
Kamal K. Nandi and Anwarul Islam
Department of Mathematics, University of North Bengal, Darjeeling (W.B.)
734430, India

Abstract

We demonstrate that the Evans-Rosenquist formulation of the optical mechanical analogy, so successful in the application to classical problems, also describes the motion of massless particles in the Schwarzschild field of general relativity. It is possible to obtain the well known equations for light orbit and radar echo delay which account for two exclusive tests of Einstein’s field equations. Some remarks including suggestions for future work are also added.

I. INTRODUCTION

The historical optical-mechanical analogy\cite{1,2} has recently been cast into a familiar form by Evans and Rosenquist.\cite{3-5} This new formulation, based on Fermat’s principle, provides an interesting approach that can be profitably utilized in the solution of many classical problems. The approach works either way: The well-known ideas and techniques of classical mechanics can be successfully applied to the problem of classical optics and vice versa. Such success naturally prompts further enquiry as to whether the applicability of the optical-mechanical analogy could be extended even beyond the classical regimes. More specifically, a curious student might ask: Can the Evans-Rosenquist (ER) formulation be used to describe the phenomenon of light propagation and massive particle motion in general relativity (GR)? In order to decide this question, let us note that it addresses two distinct areas of investigation: One has to first examine if the “F = ma” optics of ER does at all describe light propagation in the GR scenario. If it does, then the first hurdle is overcome and the question of applying the optical analogy to massive particle motion becomes meaningful. To what extent this analogy would describe planetary motion in GR is a matter of separate investigation.

In this paper, we shall examine only the first part of the question that relates to the motion of massless particles in GR. We choose Schwarzschild field of gravity primarily because of its experimental importance. Besides, on many occasions, it is used as an ideal example for illustrating fundamental notions of GR.

The exterior Schwarzschild metric, which is a solution of Einstein’s GR field equations, has been astoundingly successful in describing various gravitational phenomena. Famous experimental tests of GR include the bending of light rays, perihelion advance of a planet, radar echo delay, and gravitational red shift in the environment of the Schwarzschild gravity field generated by a static, spherically symmetric material source like the sun.\cite{6}

In the present approach, the only ingredient of GR to be used is the isotropic form of the metric relevant to the gravity field (here the Schwarzschild field). No sophisticated mathematics involving four vectors, tensors, Christoffel symbols, or geodesic equations will be necessary. In addition to being simpler and hence understandable by a wide range of readers, the ensuing exposition offers
a different window through which the GR events can be visualized. From the instructional point of view, it is always useful to look at familiar things from as many different perspectives as possible.

It would be worthwhile to display the relevant formulas of the optical-mechanical analogy that we are going to use in our paper. This is done in Sec.II. Thereafter, as a necessary step, the Schwarzschild gravity field is portrayed as a refractive optical medium in Sec.III. GR equations for the light orbit and the radar echo delay in Schwarzschild gravity are derived in Secs.IV and V respectively. Finally, some relevant remarks including a few suggestions for future work are made in Sec.VI.

II. OPTICAL-MECHANICAL ANALOGY

We need not go into the detailed formulation of the “F = ma” optics of ER. Instead, for our present purpose, we reproduce only a table which is self-explanatory

| Quantity               | Mechanics       | Optics            |
|------------------------|-----------------|-------------------|
| Position               | \( \vec{r}(t) \) | \( \vec{r}(a) \) |
| “Time”                 | \( t \)         | a                 |
| “velocity”             | \( \frac{d\vec{r}}{dt} \equiv \dot{\vec{r}} \) | \( \frac{d\vec{r}}{da} \equiv \ddot{\vec{r}} \) |
| “Potential energy”     | \( U(\vec{r}) \) | \( -\frac{n^2}{2}(\vec{r})/2 \) |
| “Mass”                 | m               | 1                 |
| “Kinetic energy”       | \( T = \frac{m}{2} |\dot{\vec{r}}|^2 \) | \( \frac{1}{2} |\ddot{\vec{r}}|^2 \) |
| “Total energy”         | \( \frac{m}{2} |\dot{\vec{r}}|^2 + U \) | \( \frac{1}{2} |\ddot{\vec{r}}|^2 - \frac{n^2}{2} \) |
| “Equation of motion”   | \( m \ddot{\vec{r}} = -\text{grad}U \) | \( \ddot{\vec{r}} = \text{grad} \left( \frac{n^2}{2} \right) \) |

As can be seen, the role of time \( t \) is played by the ER stepping parameter \( a \) having the dimension of length so that \( \vec{r}' \) is not a velocity. It is a dimensionless quantity. The transition between \( t \) and \( a \) can be accomplished by using the relation

\[
d a = \frac{c_0}{n^2} dt, \tag{1}
\]

where \( n \) denotes the refractive index of the optical medium, \( c_0 \) denotes the vacuum speed of light. Evans\(^4\) has shown that the stepping parameter can be physically interpreted as “optical action”. Just as a mechanical particle progresses in time along its trajectory, light progresses in optical action along its ray.

The identification

\[
U(\vec{r}') = -\frac{n^2}{2} \tag{2}
\]

giving the “potential” \( U \) essentially bestows a mechanical character to photon motions; the only restriction being that the motion corresponds to mechanics at “zero total energy.” We can also imagine the possibility that, at least in the classical regime, optical and mechanical motions can take place under the same
form of "force"/force law. An excellent example is probably the Luneburg lens in optics and the harmonic oscillator in mechanics\(^3\). In order to use the ER formulation in the GR regime, it would be necessary to associate a single scalar function, the optical refractive index \(n\), with the gravity field under consideration. The "force" on the massless particles moving in that gravity field can then be explicitly obtained\(^7\) via the ER expression \(\text{grad}\left(-\frac{n^2}{2}\right)\).

From now on, we shall use primes (\('\)) only to designate a radial coordinate \((r')\) and not differentiation with respect to the stepping parameter \(a\). All the differentiations will be displayed in full.

### III. GRAVITY FIELD AS A REFRACTIVE MEDIUM

One might wonder what relationship could there possibly be between two entities apparently as diverse as a gravity field and a refractive optical medium? But, indeed, there is one! It was guessed by Einstein and Eddington\(^8\), formally developed by Plebanski and others\(^9\) and utilized in the investigation of specific problems by many physicists.\(^8,10\) We shall only quote the results in a form that is easily intelligible. Plebanski\(^9\) has shown that, in a curved spacetime with a metric tensor \(g_{\alpha\beta}\), Maxwell’s electromagnetic equations can be rewritten as if they were valid in a flat spacetime in which there is an optical medium with a constitutive equation. More specifically, with regard to light propagation, the gravity field behaves as a refractive optical medium. For example, the gravitational field exterior to a static, spherically symmetric mass \(M\) is equivalent to an isotropic, non-homogeneous optical medium with a refractive index \(n\) given by\(^10\)

\[
n^2(r) = \left(1 + \frac{m}{2r}\right)^6 \left(1 - \frac{m}{2r}\right)^{-3}, \quad m = GM/c^2
\]  

(3)

where, as usual, \(G\) is the gravitational constant, \(r\) is the Euclidean radial coordinate.

Advanced students who are likely to have a fair grasp on the two field theories, Einstein’s and Maxwell’s, along with the details of algebraic manipulations should have no difficulty in pursuing the analysis of Plebanski and de Felice\(^9,10\). However, there is a simpler alternative derivation of Eq.(3), given below, that demands only the knowledge of the form of the Schwarzschild exterior metric. One may also take Eq.(3) at its face value and proceed by treating this \(n(r)\) as just a given choice of the refractive index in the ER formulation. We shall essentially investigate the consequences of such a choice in the succeeding sections.

Consider the exterior Schwarzschild metric in standard coordinates \((r', \theta, \varphi, t)\)

\[
ds^2 = \left(1 - \frac{2m}{r'}\right)c_0^2 dt^2 - \left(1 - \frac{2m}{r'}\right)^{-1} dr'^2 - r'^2 (d\theta^2 + \sin^2 \theta d\varphi^2)
\]

\[= B(r')c_0^2 dt^2 - A(r') dr'^2 - r'^2 (d\theta^2 + \sin^2 \theta d\varphi^2)
\]  

(4)
where \( r' > 2m \). Redefine the radial coordinate as \( r' = r\phi^{-1}(r) = r \left(1 + \frac{m}{2r} \right)^2 \), so that, if \( r = u^{-1} \) and \( r' = u'^{-1} \), then

\[
\begin{align*}
\phi(u) &= \left(1 + \frac{mu}{2} \right)^{-2}.
\end{align*}
\]

The metric (4) can then be rewritten in the so called isotropic coordinates \((r, \theta, \varphi, t)\) as

\[
\begin{align*}
ds^2 &= \Omega^2(r) \left[ \sum_{k=0}^1 \left( \phi^{-2}(r) \left( d\varphi^2 + \sin^2 \theta d\varphi^2 \right) \right) \right] = \Omega^2(r)^2 c_0^2 dt^2 - \frac{du^2}{u^2} + \frac{dr^2}{r^2} + \frac{r^2 d\theta^2}{\sin^2 \theta} + \sin^2 \theta d\varphi^2,
\end{align*}
\]

where \( \Omega(r) = \left(1 + \frac{m}{2r} \right)^{-1} \left(1 - \frac{m}{2r} \right) \).

The above form of the metric has a conformally Euclidean spatial part. Therefore, the isotropic coordinate speed of light \( c(r) \) at any arbitrary point in the gravitational field, obtained by putting \( ds^2 = 0 \), is

\[
c(r) = \left| \frac{dr}{dt} \right| = c_0 \phi(r) \Omega(r).
\]

Defining the refractive index as \( n(r) = c_0/c(r) \), we have

\[
n(r) = \Omega^{-1} \phi^{-1} = \left(1 + \frac{m}{2r} \right)^3 \left(1 - \frac{m}{2r} \right)^{-1},
\]

which is precisely the same as Eq.(3) above. It is easy to see that

\[
\begin{align*}
\phi^2(u') &= 2^{-4} \left[ 1 + (1 - 2mu')^{1/2} \right]^4, \\
\Omega^2(u') &= 1 - 2mu'
\end{align*}
\]

and

\[
du' = \Omega(u)\phi(u) du.
\]

All the above relationships will be used throughout the paper. It should be mentioned that Sjödin\(^{12}\) has also derived the expression for \( \phi, \Omega \) and \( n \) by a different method based on Rindler’s operational definitions of length and time in a gravitational field.

The refractive index \( n(r) \) above corresponds to a radial “force law” having a magnitude

\[
aF^m = \frac{dU}{dr} = \frac{m(m - 4r)(m + 2r)^5}{8(m - 2r)^3 r^5} = \frac{2m}{r^2} + \frac{15m^2}{2r^3} + \frac{27m^3}{2r^4} + O(m^4).
\]

IV. LIGHT ORBIT EQUATION
Within the ER framework, the optical analog of the mechanical zero total energy or in short “zero total energy” (see the entry in the table in Sec.II) is given by

\[ \frac{1}{2} \left( \frac{d \mathbf{r}}{da} \right)^2 - \frac{n^2}{2} = 0. \] (13)

We straightaway claim that this very equation is the GR light orbit equation in Schwarzschild gravity, provided \( n \) is given by Eq.(9). In order to see that, we proceed as follows. Since there is spherical symmetry in the problem, we can, without loss of generality, fix a plane in which the orbiting has to take place. It is customary\(^\text{13}\) to choose \( \theta = \pi/2 \). Writing out Eq.(13) in plane polar coordinates, we have

\[ \left( \frac{dr}{da} \right)^2 + r^2 \left( \frac{d\varphi}{da} \right)^2 - n^2(r) = 0. \] (14)

Noting that Eq.(13) is the first integral of the “equation of motion” \( \mathbf{r}'' = \text{grad} \left( \frac{n^2}{2} \right) \).

Since \( n \) does not depend on \( \varphi \), the \( \varphi \) component of the equation of motion yields a conserved quantity called by ER as the “angular momentum” \( h \) given by

\[ h = r^2 \frac{d\varphi}{da} = \text{const}. \] (15)

Eliminating the stepping parameter \( a \), and writing \( r = u^{-1} \), we have

\[ u^2 + \left( \frac{du}{d\varphi} \right)^2 - n^2 h^{-2} = 0. \] (16)

Evans and Rosenquist have solved a number of problems in classical optics/mechanics using different choices for \( n(r) \). Below we shall discuss, in the context of GR, some aspects of the Eqs.(13)-(16):

(i) As ER have already discussed, the optical quantity \( h \) is completely different from the corresponding conserved mechanical quantity \( h_0 = r^2 d\varphi/dt \) which is proportional to the areal velocity. In the optical formalism, we have \( h = c_0^2 n^2 r^2 d\varphi/dt \) so that \( h_0 \propto n^{-2} \) and \( d\varphi/dt \propto n^{-2} r^{-2} \). Hence, with our form of \( n(r) \), Eq.(9), neither the areal velocity \( h_0 \) nor the angular velocity \( d\varphi/dt \) remains constant. Of course, for the same force law, the optical and mechanical zero energy orbits must have the same form, since \( a \) or \( t \) are ultimately eliminated. We see that the GR light orbit equation has the form of well known classical optics equation\(^\text{14}\), although this interesting fact is not widely noticed. The reason for this is that in most text books on GR, the light orbit equation is given in a completely different form.

(ii) Let us obtain the familiar text book form. Using Eqs. (5), (11) and (12) in Eq.(16), we get

\[ \phi^{-2}(1 - 2mu')^{-1} \left( \frac{du'}{d\varphi} \right)^2 + u'^2 \phi^{-2} - n^2 h^{-2} = 0. \] (17)
Using Eq.(9) for \( n \), we find

\[
\begin{align*}
\dot{u}'^2 + \left( \frac{d\dot{u}'}{d\varphi} \right)^2 - 2mu'^3 - h^{-2} &= 0. \quad (18)
\end{align*}
\]

Differentiating with respect to \( \varphi \), we get

\[
\begin{align*}
\dot{u}' + \frac{d^2\dot{u}'}{d\varphi^2} - 3mu'^2 &= 0. \quad (19)
\end{align*}
\]

This is precisely light orbit equation in Schwarzschild gravity in standard coordinates. Therefore, one can say that the familiar Eq.(19) represents in disguise just the optical analog of the classical zero total energy mechanics. This conclusion will find a further justification in the development of Sec.V.

(iii) It would be of interest to see how the “angular momentum” \( h \) is related to the conserved GR quantities \( E \) and \( L \), associated with the Killing fields \( \partial/\partial t \) and \( \partial/\partial \varphi \), respectively. This would also provide a relationship between the ER stepping parameter \( a \) and the geodesic affine parameters used by relativists.

Integration of GR null geodesic equation gives

\[
\begin{align*}
\left( \frac{d\vec{r}}{dt} \right)^2 &= c_0^2 \Omega^2 \phi^2, \quad \Omega^2 \frac{dt}{dp} = E, \quad \phi^{-2}r^{-2} \frac{d\varphi}{dp} = L, \quad (20)
\end{align*}
\]

where \( ds^2 = \lambda dp^2 \), \( \lambda \) is a constant and \( p \) is a new geodesic affine parameter such that \( ds^2 = 0 \Rightarrow \lambda = 0, \ dp^2 \neq 0 \). Eliminating \( t \) from Eqs.(20), we get

\[
\begin{align*}
c_0^{-2} E^{-2} \phi^{-4} \left| \frac{d\vec{r}}{dp} \right|^2 - n^2 &= 0. \quad (21)
\end{align*}
\]

If we now define

\[
\begin{align*}
dp = c_0^{-1} E^{-1} \phi^{-2} da
\end{align*}
\]

then Eq.(13) follows immediately from Eq.(21). We also find that \( h = L/c_0E \).

From Eqs.(20 and (22), there also follows the ER relation (1) connecting \( dt \) and \( da \). Further Eq.(22) implies that \( ds^2 = \lambda c_0^{-2} E^{-2} \phi^{-4}da^2 \), giving the connection between \( a \) and the affine parameter \( s \).

(iv) By integrating Eq.(19), one obtains all allowable light orbits\(^{15} \); bound, unbound, or even the so-called boomerang orbits\(^{16} \). There have been many observations confirming the GR predictions of the bending of light rays just grazing the sun, the amount being \( \Delta \varphi \sim 4m/R_0 \), where \( R_0 \) is the solar radius. Interested readers may consult any textbook on GR. We shall only make a relevant remark here. M"oller\(^{17} \) has shown that the bending of light rays is due partly to the geometrical curvature of space and partly to the variation of light speed in a Newtonian potential. In fact, the ratio of the parts is 50 : 50. The GR null trajectory equations can be integrated, once assuming a Euclidean space with a variable light speed and again a curved space with a constant light speed; both contributing just half the amount of the observed bending. In the present approach, on the other hand, we are describing light motion by means
of a scalar function $n(r)$. Thus with regard to the light propagation, it looks as if spatial curvature and Newtonian potential lost their separate identities and merged into an equivalent refractive medium. There is, of course, no point in asking which one has a physical reality and which one has not; all these are mathematical constructs [like the $n(r)$ here] designed only to interpret our physical observations.

(v) Finally, some words of caution. From the similarity of Eqs.(16) and (18), one might be tempted to conclude that in the $u'$ coordinates, $n$ is given by $n(u') = (1 + 2mh^2u'^3)^{1/2}$, but that would be incorrect. The correct expression for $n(u')$ is obtained by using the fact that $n(u)$ transforms as a scalar. Hence, one obtains

$$n(u') = 4(1 - 2mu')^{-1/2}[1 + (1 - 2mu')^{1/2}]^{-2}. \quad (23)$$

Also, it must be understood that, with this $n(u')$, it is not possible to define an isotropic coordinate velocity of light in the $u'$ coordinates. From a direct calculation with the metric (4), it will turn out that the coordinate velocities of light are different in radial and cross radial directions.

V. RADAR ECHO DELAY

Let us now go beyond the geometrical shape of the light orbit and consider the dynamics along its path. In other words, we shall derive the GR equations of motion involving the time $t$ for the propagation of light rays around a static, spherically symmetric gravitating mass $M$.

Once again we start from the ER “zero energy” equation $\frac{1}{2} \left\| \frac{dr}{da} \right\|^2 - \frac{\phi^4}{2} \left( \frac{du'}{da} \right)^2 = 0$ and claim that it is the GR equation we are looking for provided $n$ is given by Eq.(9). To see this, consider Eqs.(13)-(15) and write

$$\phi^4 \left( \frac{du'}{da} \right)^2 + h^2(1 - 2mu')u'^2 - 1 = 0. \quad (25)$$

At the position $r_0' = u_0'^{-1}$ of the closest approach to the gravitating mass $M$, we have $dr'/da = 0 \Rightarrow du'/da \mid_{u_0'} = 0$, giving $h^2 = (1 - 2mu_0')^{-1}u_0'^{-2}$. Putting it in Eq.(25), we find

$$\phi^4 \Omega^{-2} u'^{-4} \left( \frac{du'}{da} \right)^2 + u_0'^{-2}u'^2(1 - 2mu')^{-1} - \Omega^{-2} = 0. \quad (26)$$

Noting from the metric (4) that $B(r') = \Omega^2$ and $A(r') = \Omega^{-2}$, and using Eq.(1), we finally have

$$c_0^{-2} A(r')B^{-2}(r') \left( \frac{dr'}{dt} \right)^2 + r_0'^2 r'^{-2}B^{-1}(r_0') - B^{-1}(r') = 0. \quad (27)$$
This is precisely the textbook form of what is known as the radar echo delay equation in Schwarzschild gravity. And, once again, we see that it is still the same ER “zero energy” mechanics, only buried in the \((r', t)\) language. Equation (27) is integrated to obtain the time \(t\) required by the light signal (in practice, a radar signal) to travel from one point of space to another. It is also evident that the coordinate speed of light \(dr'\,dt\) is less than what it would be if the gravitating mass were absent. In other words, light is slowed down and the travel time is longer. All observers will nonetheless measure a photon’s speed to be \(c_0\) through their positions. In a round trip journey around the mass \(M\), there would be a net GR delay in the radar echo reception. This GR prediction has been confirmed to a great accuracy by sending radar signals from Earth to Mercury at superior conjunction and back.

VI. SOME REMARKS

The contents of the entire paper vividly demonstrate that light propagation in Schwarzschild gravity is indeed describable by the language of “\(F = ma\)” optics, a shorthand for the optical mechanical analogy. It is remarkable that the ER formulation, emerging basically from the investigations of classical problems, works equally well also in the GR regime.

We are concerned with the predictions for light propagation that follow exclusively from Einstein’s field equations. On the other hand, the formula for gravitational redshift can be derived from special relativity and the principle of equivalence alone. The field equations or their solutions need not be used. In that sense, the redshift is not a prediction following exclusively from field equations, albeit the effect does follow also from the latter. In the present approach, the usual principle of equivalence that equates the effects of gravity and artificial forces, is translated into an equivalence of the effects of gravitation and refractive medium. Using this idea, the gravitational redshift formula can be obtained in its familiar form. We shall present the deductions in a separate paper.

It is evident that the considered approach can deal with a reasonably general class of gravity fields where the metric can be cast into an isotropic form. The latter makes it possible to construct a scalar function \(n\) which acts as a representative of the gravity field in the matter of light propagation. However, the whole range of GR solutions corresponding to different field distributions can not be made to correspond to such single scalar functions. At best, a detailed constitutive tensor for the equivalent medium may be developed\(^{10}\). In this case, the challenging task would be to generalize the ER equations appropriately. Nevertheless, we can list some immediate future works:

(i) The motion of a light pulse inside and across the body of a spherical star can be tackled quite comfortably. The interior of a spherical star (like the sun) is described by the Schwarzschild interior metric. It corresponds to a refractive medium with index, say, \(n_1\) while the exterior field corresponds to \(n\), Eq.(9). Therefore the whole problem is reduced to one of light propagation in a composite media with indices \(n\) and \(n_1\). At the interface, the matching...
condition is provided by none other than good old Snell’s law. The result will provide a theoretical idea about the total path of light rays between the sun’s core and the Earth.

(ii) For the sake of completeness, we would expect the “F = ma” optics to describe also the motion of massive particles (planetary motion) in the Schwarzschild gravity field. To that end, efforts are underway to extend the present treatment. The answer to the second part of the question raised in sec.1 would then be available.

Finally, it must be emphasized that there is neither any substitute for nor shortcut to to the beauty, generality, and richness of Einstein’s general relativity theory. One must eventually grasp all the details of the physics, mathematics, and the philosophy of this magnificent never ending edifice. On the other hand, the language of the ER optical-mechanical analogy has the power to stimulate the interests of a wide cross section of readers who do not have a formal training in the sophistications of GR. Those who have the training may, however, regard the preceding developments as providing yet another avenue to the same experimentally verified tests of light motion in GR. The educational importance of such alternative points of view cannot be mistaken.

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2 Historically, the basic idea of the optical mechanical analogy was proposed much earlier by none other than Descartes (1637). This information is due to J. Arnaud, ”Analogy between optical rays and nonrelativistic particle trajectories: A comment”, Am. J. Phys. 44, 1067-1069 (1976). In this paper are also discussed the limitations and significance of the analogy.

3 Essential ingredients of what is referred to as the ER approach are contained in: J. Evans and M. Rosenquist, ””F=ma” optics”, Am. J. Phys. 54, 876-883 (1986).

4 J. Evans, ”Simple forms of equations of rays in gradient-index lenses”, Am. J. Phys. 58, 773-778 (1990). see especially p.774.

5 Newton’s laws of motion are obtained directly from Fermat’s principle, in M. Rosenquist and J. Evans, ”The classical limit of quantum mechanics from Fermat’s principle and the de Broglie relation”, Am. J. Phys. 56, 881-882 (1988).

6 For a general information: Latest confirmations of GR come from the observations of the Hulse-Taylor binary pulsar PSR1913+16. There is a rate of decrease of the orbital period \[P \sim (3.2 \pm 0.6) \times 10^{-12} \text{s s}^{-1}\] as well as precession of the order of 4\(^\circ\) per year. Observations are in excellent agreement with the GR predictions. See, J.H. Taylor, L.A. Fowler and P.M. McCulloch, ”Measure-
ments of general relativistic effects in the binary pulsar PSR1913+16", Nature (London) 277, 437-440 (1979).

7To repeat, the "force" is not the force in the mechanical sense. Nonetheless, a material particle acted on by a mechanical force of the same form will follow the same null track. See Ref.16 below for the citation of an interesting example.

8Sir A.S. Eddington calculated the bending of light rays round a massive object by assuming that the space around the sun is filled with a medium with a refractive index \( n = \left(1 - \frac{2m}{r}\right)^{-1} \approx 1 + \frac{2m}{r} \). Incidentally, he was also the first to experimentally observe such a bending in 1919 during the solar eclipse in Sobral, Brazil. See, A.S. Eddington, Space, Time and Gravitation (Cambridge University, Cambridge, 1920), reissued in the Cambridge Science Classics Series, 1987, p.109. However, F. de Felice (Ref.10) reports that Einstein himself was the first to suggest the idea of equivalent refractive medium.

9Among a number of works in this direction, we list only a few, just to give an idea: J. Plebanski, "Electromagnetic waves in gravitational fields", Phys. Rev. 118, 1396-1408 (1960). B. Bertotti, "The luminosity of distant galaxies", Proc. R. Soc. 294, 195-207 (1966). F. Winterberg, "Deflection of gravitational waves by stellar scintillation in space", Nuovo Cim. B 53, 264-279 (1968). See also Ref.10 for a collection of other references.

10F. de Felice shows that the equivalence of a gravity field with the refractive medium can be successfully employed as a method of investigation: F. de Felice, "On the gravitational field acting as an optical medium", Gen. Rel. Grav. 2, 347-357 (1971).

11Because of the general coordinate covariance of Einstein’s GR, its solutions can be freely expressed in any coordinate system we like. Such a freedom, however, does not affect the unique observable predictions of GR. This important point is further illuminated in some recent papers. See, Ya.B. Zel’dovich and L.P. Grishchuk, "The general theory of relativity is correct", Sov. Phys. Usp. 31, 666-671 (1988), and references therein. T. Ohta and T. Kimura, "Coordinate independence of physical observables in classical tests of general relativity", Nuovo Cim. B 106, 291-305 (1991). A.N. Petrov, "New harmonic coordinates for the Schwarzschild geometry and the field approach", Astron. Astrophys. Trans. 1, 195-205 (1992). We would particularly recommend these papers to the advanced graduate students and researchers in GR.

12T. Sjödin, in Physical Interpretations of Relativity Theory, edited by M.C. Duffy (British Soc. Philos. Sc., London, 1990), pp 515-521. Sjödin also obtains the light orbit equations starting from Rindler’s prescription. See, W. Rindler, Essential Relativity (Springer, New York, 1977), especially secs. 8.3 and 8.4.

13The reason for fixing the \( \theta = \pi /2 \) place is that \( \partial / \partial \varphi \) is a Killing field on the entire manifold while \( \partial / \partial \theta \) is not.

14Eq.(16) is the classical optics equation. See, A. Maréchal, Optique Géométrique Généralé, edited by S. Flügge, Handbuch der Physik, Vol.34 (Springer, Berlin, 1956), p.44. M. Born and E. Wolf, Principles of Optics, 2nd ed. (Pergamon Press, New York, 1964), pp. 121-124.

15A number of light orbits has been plotted in S. Chandrasekhar, The Mathematical Theory of Black Holes (Oxford University, Oxford, 1983). See also the
classic treatise: L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields* (Pergamon, New York, 1975), pp. 313-316.

16W.M. Stuckey, "The Schwarzschild black hole as a gravitational mirror", Am. J. Phys. 61, 448-456 (1993). The possibility of a boomerang shaped light orbit is interesting in its own right. We can say that the path of a real massive boomerang around a central mass is also obtained by the same force law in mechanics. The total energy, however, need not be zero.

17The detailed calculations appear in C. Möller, The Theory of Relativity, 2nd ed. (Oxford University, Oxford, 1972), pp.498-501. The variable speed of light is given by $c^* = c_0(1 + 2\chi/c_0^2)^{1/2}$, where $\chi = -GM/r'$. Purely spatial curvature is represented by the metric $ds^2 = c_0^2dt^2 - (1 - 2m/r')^{-1}dr'^2 - r'^2(d\theta^2 + \sin^2\theta d\phi^2)$. Möller expresses the null trajectory in terms of $c^*$ and the metric tensor $(g_{\alpha\beta})$ to demonstrate his result.

18Notice that at the Schwarzschild black hole radius $r' = 2m$, $n(u') = \infty$. Also from Eq.(9), at $r = m/2$, $n(u) = \infty \Rightarrow c(r) = 0$. This result reflects the fact that, to a distant observer, even light comes to a standstill! To a local observer, however, light always travels at a speed $c_0$.

19I.I. Shapiro et al, "Fourth test of general relativity: New radar result", Phys. Rev. Lett. 26, 1132-1135 (1971). The GR predicted value of time delay for the Earth-Mercury system is $\sim 240 \mu$sec while the observed value is $(245 \pm 12) \mu$sec. A remarkable agreement indeed.