Edge states and the integer quantum Hall effect of a spin-chiral ferromagnetic kagomé lattice with general spin coupling

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Abstract. The chiral edge states and the quantized Hall conductance (QHC) in the two-dimensional kagomé lattice with spin anisotropies included in a general Hund’s coupling region are studied. This kagomé lattice system is periodic in the $x$-direction but has two edges in the $y$-direction. Numerical results show that the strength of Hund’s coupling, as well as spin chirality, affects the edge states and the corresponding QHC. Within the topological edge theory, we express the QHC with the winding number of the chiral edge states on a Riemann surface. This expression is also compared with that within the topological bulk theory, and they are found to be consistent with each other.

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1. Introduction

The quantum Hall effect (QHE), in which the Hall conductance (HC) $\sigma_{xy}$ is quantized with extremely high accuracy, is a typical realization of topological effects in condensed matter physics [1]. The topological aspect of the QHE with a periodic potential was first discussed by Thouless et al [2]. In their work, HC is given by the Chern number [3] over the magnetic Brillouin zone, which is a topological expression by the bulk state. Later Hatsugai [4] suggested that HC can be given by another topological quantity, i.e. the ‘winding number’ of the edge states on the complex-energy surface that is generally a high-genus Riemann surface (RS). To distinguish them, we call the former ‘bulk theory’ and the latter ‘edge theory’. The two topological expressions for the HC, which look quite different, actually give the same integer number.

A recently established recognition points out that the conventional QHE originates from the nonlocal effect provided by the external magnetic field—more exactly, by the vector potential that describes the magnetic field. That means a nonzero vector potential, for example, which can be provided by spin–orbit interaction or by spin chirality, will induce the QHE even if the external magnetic field vanishes. This type of QHE can be called ‘anomalous QHE’ [5]–[7]. Since Haldane introduced the famous ‘Haldane model’ in his pioneering work [5] in 1988, the anomalous QHE in spin–orbit coupled [8]–[10] or spin-chiral ferromagnetic systems [11]–[14] has been a hot topic in condensed matter physics. Based on the tight-binding two-dimensional (2D) graphite model [15], the Haldane model includes the next-nearest-neighbor interaction and a periodic local magnetic-flux density, which breaks time-reversal invariance and creates a chirality. However, because the introduction of local flux is technically difficult, it is not so easy to realize the Haldane model in real materials. Another typical spin-chiral system is the ferromagnetic system, which is represented by pyrochlore compounds $R_2Mo_2O_7$ ($R = \text{Nd}, \text{Sm}, \text{Gd}$), in which the spin configuration is noncoplanar and spin chirality appears. Ohgushi et al [6] first pointed out that the chiral spin state can be realized by the introduction of spin anisotropy in an ordered spin system on the 2D kagomé lattice, which is the cross section of the pyrochlore lattice perpendicular to the $(1, 1, 1)$ direction [16]. In this case, it has been shown in the topological bulk theory [6, 17] that the presence of a chiral spin state may induce gauge-invariant nonzero Chern number, thus resulting in a QHE in the insulating state. However, as stressed by Halperin [18], this gauge invariance has a relation with the edge states, which are localized near the sample boundaries. Recently, we reinvestigated the kagomé lattice with boundaries. Using the topological edge theory established by Hatsugai in the last decade [4],

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we interpreted the quantized Hall conductance (QHC) in the insulating state with the winding number of the edge states on the complex-energy surface [19].

In the above-mentioned theoretical works [6, 17, 19] on the 2D kagomé lattice, a very important limit has been used. That is, the hopping electron spins are aligned with localized spins at each site of the lattices, which is also called the ‘infinite (strong) Hund’s coupling limit’. However, the latest detailed experiments on the pyrochlores [20] showed that the spin-chiral mechanism alone cannot explain the anomalous transport phenomena in these systems. Hence, in this paper, we study the 2D kagomé lattice with boundaries in a general Hund’s coupling region, especially in the weak coupling region [21]. We find that the edge states and QHC are not only affected by the spin chirality, but also by the strength of Hund’s coupling. Varying the spin chirality and the strength of Hund’s coupling, two different types of phenomena are obtained. In one case, in which the strength of Hund’s coupling is larger than its critical value (see [22] or section 3), there are only two edge-state energies in each bulk energy gap. With the help of the topological edge theory, we obtain that the corresponding HC is quantized as $\sigma_{xy} = \pm e^2/h$ in the insulating state. However, in the other case, in which the strength of Hund’s coupling is smaller than its critical value, there are possibly four edge-state energies in one bulk gap. In this case, within the topological edge theory, we obtain that the corresponding HC is quantized as $\sigma_{xy} = \pm 2e^2/h$ in this insulating state, which cannot occur in the strong Hund’s coupling case. We also make a comparison between these results and those in the bulk theory [22], and find that they are consistent with each other.

This paper is organized as follows. In section 2, we introduce the tight-binding model of the 2D kagomé lattice with boundaries along the $y$-direction and obtain the eigenvalue equations for sites. We also write out the Hamiltonian of the 2D kagomé lattice without boundaries in reciprocal space. Because the analytical derivation is very difficult, in section 3 we numerically calculate the energy spectrum. Using the characters of the edge-state energies in the spectrum, we study the QHE within the topological edge theory. As a comparison, we also recalculate the HC in the infinite system in this section. We give a conclusion in the last section.

2. Model

We consider the double-exchange ferromagnetic kagomé lattice schematically shown in figure 1(a). The triangle is one face of the tetrahedron. Here we consider a pure spin model with anisotropic Dzyaloshinskii–Moriya interactions on a kagomé lattice. It consists of an umbrella of three spins per unit cell of the kagomé lattice. Each umbrella can be described by the spherical coordinates of the three spins ($\pi/6, \theta$), ($5\pi/6, \theta$) and ($-\pi/2, \theta$), as shown in figure 1(b). The angle $\theta$ ranges from $0$ to $\pi$.

The tight-binding model of the 2D kagomé system can be written as follows [22]:

$$H = \sum_{i,j,\sigma} t_{ij} \left( c_{i\sigma}^\dagger c_{j\sigma} + \text{H.c.} \right) - J_0 \sum_{i,\alpha,\beta} c_{i\alpha}^\dagger (\sigma_{\alpha\beta} \cdot \mathbf{n}_i) c_{i\beta},$$

(1)

where $t_{ij}$ is the hopping integral between two neighboring sites $i$ and $j$; $c_{i\sigma}^\dagger$ and $c_{i\sigma}$ are the creation and annihilation operators of an electron with spin $\sigma$ on site $i$. $J_0$ is the effective coupling constant to each local moment $\mathbf{S}_i$, and these moments are treated below as classical variables. $\mathbf{n}_i$ is a unit vector collinear with the local moment $\mathbf{S}_i$. $\sigma_{\alpha\beta}$ are the Pauli matrices. In the following, we change notation $i \rightarrow (lm s)$, where $(lm)$ labels the kagomé unit cell and $s$ denotes sites A, B and C in this cell. The size of the unit cell is set to be unity throughout this paper.
This Hamiltonian has already been discussed in the infinite Hund’s coupling limit $J_0 \to \infty$ in [6, 17, 19]. In this limit the two $\sigma = \uparrow, \downarrow$ bands are infinitely split and the model describes a fully polarized electron subject to modulation of a fictitious magnetic field, corresponding to the molecular field associated with the magnetic texture.

Now, we suppose that the system is periodic in the $x$-direction but has two edges in the $y$-direction (see figure 1(c)). Since the system is periodic in the $x$-direction, we can use a momentum representation of the electron operator

$$c_{(lms)} = \frac{1}{\sqrt{L_x}} \sum_{k_x} e^{i k_x X_{(lm)}} \gamma_{ms}(k_x),$$

(2)

where $\mathbf{R}_{(lm)} = (X_{(lm)}, Y_{(lm)})$ is the coordinate of site $s$ in the unit cell $(lm)$ and $k_x$ is the momentum along the $x$-direction. Let us consider the one-particle state $|\Psi(k_x)\rangle = \sum_{ms} \Psi_{ms}(k_x) \gamma^\dagger_{ms}(k_x)|0\rangle$. Inserting it into the Schrödinger equation $H|\Psi\rangle = E|\Psi\rangle$, we can easily obtain the following three eigenvalue equations for sites A, B and C:

$$E_{\Psi_{mA}} = e^{-ik_x/4} \Psi_{m+1A\sigma} + e^{-ik_x/4} \Psi_{m1B\sigma} + e^{ik_x/4} \Psi_{mC\sigma} + e^{-ik_x/4} \Psi_{m+1C\sigma}$$

$$- \sigma J_0 \cos \theta \Psi_{mA\bar{\sigma}} + i \sigma J_0 \sin \theta \Psi_{mA\bar{\sigma}},$$

$$E_{\Psi_{mB}} = e^{ik_x/4} \Psi_{mA\sigma} + e^{-ik_x/4} \Psi_{m-1A\sigma} + 2 \cos(k_x/2) \Psi_{mC\sigma}$$

$$- \sigma J_0 \cos \theta \Psi_{mB\bar{\sigma}} - J_0 e^{i 5\pi/6} \sin \theta \Psi_{mB\bar{\sigma}},$$

(3)

$$E_{\Psi_{mC}} = e^{-ik_x/4} \Psi_{mA\sigma} + e^{ik_x/4} \Psi_{m-1A\sigma} + 2 \cos(k_x/2) \Psi_{mB\sigma}$$

$$- \sigma J_0 \cos \theta \Psi_{mC\bar{\sigma}} - J_0 e^{i 5\pi/6} \sin \theta \Psi_{mC\bar{\sigma}}.$$

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Here we set $t_{ij} = 1$ as the energy unit, $\sigma = \pm 1$ denotes the spin up and down, respectively. Eliminating the B- and C-sublattice sites, one can obtain a difference equation for $\Psi_{A\sigma}$, which is the famous Harper equation [23]. Because the expression of the Harper equation is too sophisticated to help in analytically resolving the edge states, we make a numerical calculation and analysis from equation (3).

If the system is also periodic in the $y$-direction, the Hamiltonian can be rewritten in reciprocal space. We use the momentum representation of the electron operator $c_{(lms)} = (1/\sqrt{L_x L_y}) \sum_k e^{i k \text{R}_{(lms)}} \gamma_{s\sigma}(k)$. Inserting the one-particle state $|\Psi(k)\rangle = \sum_{s\sigma} \Psi_{s\sigma}(k) \gamma_{s\sigma}^\dagger(k)|0\rangle$ into the Schrödinger equation $H|\Psi\rangle = E|\Psi\rangle$, we can easily obtain the Hamiltonian in the reciprocal space $H(k)$, which is given by

$$
H(k) = \begin{pmatrix}
-J_0 \cos \theta & p_k^1 & p_k^3 & iJ_0 \sin \theta & 0 & 0 \\
p_k^1 & -J_0 \cos \theta & p_k^2 & 0 & -J_0 e^{-i\pi/6} \sin \theta & 0 \\
p_k^3 & p_k^2 & -J_0 \cos \theta & 0 & 0 & -J_0 e^{-i\pi/6} \sin \theta \\
-iJ_0 \sin \theta & 0 & 0 & J_0 \cos \theta & p_k^1 & p_k^3 \\
0 & -J_0 e^{i\pi/6} \sin \theta & 0 & p_k^1 & J_0 \cos \theta & p_k^2 \\
0 & 0 & -J_0 e^{i\pi/6} \sin \theta & p_k^3 & p_k^2 & J_0 \cos \theta
\end{pmatrix},
$$

(4)

where $p_k^1 = 2 \cos(k_x/4 + \sqrt{3}k_y/4)$, $p_k^2 = 2 \cos(k_x/2)$ and $p_k^3 = 2 \cos(-k_x/4 + \sqrt{3}k_y/4)$.

3. Edge states and HC

Before investigating the chiral edge states and QHC of the 2D kagomé lattice with boundaries, we briefly discuss the energy structures of this system with different exchange interaction $J_0$. Firstly, in the absence of the exchange interaction, i.e. $J_0 = 0$, equation (1) becomes independent of both the spin index $\sigma$ and the chiral parameter $\theta$. In this case the two $\sigma = \uparrow, \downarrow$ bands are completely degenerate and there are only three bulk energy bands, between which there are no bulk energy gaps, as shown in figure 2. From figure 2, one can see that the lower energy band becomes dispersionless ($E = -2$), which reflects the fact that the 2D kagomé lattice is a line graph of the honeycomb structure [24]. This flat band touches at $k_x = 0$ with the middle band, whereas the middle band touches at $k_x = 2\pi/3, 4\pi/3$ with the upper band.

In the existence of exchange interaction, i.e. $J_0 \neq 0$, the energy spectrum splits into two parts due to the spin-dependent potential. For very large values of $J_0$, the spectrum is divided into two groups of three bands, which is the infinite Hund’s coupling case studied in [19]. For nonzero but not too large values of $J_0$, we numerically calculate the energy spectrum and find that there are mainly two different phenomena occurring for QHE. The critical value of the exchange interaction $J_c$ depends on the chiral parameter $\theta$. In the topological bulk theory [22], this critical value is analytically obtained, which reads as

$$
J_c(\theta) = \pm 2/\sqrt{1 + 3 \cos^2 \theta}.
$$

(5)
Figure 2. Energy spectra of the 2D kagomé lattice (a) with boundaries along the $y$-direction and (b) without boundaries. In both figures, the exchange interaction $J_0 = 0$. There are only three energy bands and there are no bulk gaps between them. The lower band is flat, which reflects that the 2D kagomé lattice is a line graph of the honeycomb structure. In (a), the blue lines denote the up-edge-state energies.

Although in the present model with boundaries, the critical value $J_c(\theta)$ cannot be analytically obtained, the numerical calculations tell us that the above expression of $J_c(\theta)$ (equation (5)) is also valid for the present model. In the following, we investigate HC in both cases.

3.1. Case I: $J_0 < J_c$

In the following, we study the QHE of the 2D kagomé system with boundaries in both cases. First, we investigate the case $J_0 < J_c$. As an example, we choose the fixed chiral parameter as $\theta = \pi/3$ and the exchange interaction as $J_0 = 1$ ($J_c = 4/\sqrt{7} \approx 1.51$). The number of sites A (or B, C) in the $y$-direction is chosen to be $L_y = 31$. By numerically solving equation (3), we draw in figure 3(a) the energy spectrum in this case. From this figure, one can clearly see that the two groups of three bulk bands are not completely divided (i.e. there is no Mott gap) and there are two bulk gaps appearing. The range of the lower energy gap is between $-2.68$ and $-2.6$, and that of the higher energy gap is between $1.1$ and $1.4$, which are enlarged in figures 3(b) and (c), respectively.

In the topological edge theory [4], when the Fermi energy lies in one energy gap, the HC of the system is given by the winding number of the edge states $I$, $\sigma_{\text{edge}} = (-e^2/h)I$. The winding number is given by the number of intersections between the canonical loop $\alpha_i$ on the RS (the complex-energy surface) and the trace of the edge-state energy $\mu_i$. Although in the present model the RS is more sophisticated than that in the infinite Hund’s coupling case [19] and that in the Haldane model [25], we can also take the same analytical structure to study the properties of the edge states and the corresponding QHE in this general model with finite Hund’s coupling.

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(a) Energy spectrum of the 2D kagomé lattice with edges along the $y$-direction. The parameters are chosen as $J_0 = 1$, $\theta = \pi/3$. The shaded areas are energy bands and the lines are the spectrum of the edge states. In this case, there is no Mott gap and there are two energy gaps, which are enlarged in (b) and (c). In (b) and (c), the red and blue lines correspond to the down- and up-edge-state energies, respectively.

To represent the Harper equation (3) in transfer matrix form, let us introduce a new wave function $\varphi_n$, which is a linear transformation of the original wave function $\Psi_n$. Because it is too sophisticated and does not affect the following analysis, here we do not write out the expression of this new wave function. $\varphi_n$ can be written in transfer matrix form

$$
\begin{pmatrix}
\varphi_n \\
\varphi_{n-1}
\end{pmatrix} =
\begin{pmatrix}
1 & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
\varphi_{n-1} \\
\varphi_{n-2}
\end{pmatrix} \equiv \tilde{M}(\epsilon) \begin{pmatrix}
\varphi_{n-1} \\
\varphi_{n-2}
\end{pmatrix}.
$$

More generally, we take $\varphi_0$ and $\varphi_{L_y}$ as the wave functions at two open edges. Then we get a reduced transfer matrix linking the two edges as follows:

$$
\begin{pmatrix}
\varphi_{L_y+1} \\
\varphi_{L_y}
\end{pmatrix} =
\begin{pmatrix}
M(\epsilon) & \varphi_1 \\
\varphi_0 & \varphi_0
\end{pmatrix}.
$$

where

$$M(\epsilon) = \left[\tilde{M}(\epsilon)\right]^{L_y} =
\begin{pmatrix}
M_{11}(\epsilon) & M_{12}(\epsilon) \\
M_{21}(\epsilon) & M_{22}(\epsilon)
\end{pmatrix}.$$

The general open boundary condition is

$$\varphi_{L_y} = \varphi_0 = 0.$$

With equations (7) and (8), one can easily obtain that the solutions satisfy

$$M_{21}(\epsilon) = 0$$

and

$$\varphi_{L_y-1} = -M_{11}(\epsilon) \varphi_1.$$
If we use a usual normalized wave function, the state is localized at the edges as

\[
\begin{cases}
|M_{11}(\epsilon)| < 1 \text{ localized at } y \approx 1 \text{ (down edge)}, \\
|M_{11}(\epsilon)| \gg 1 \text{ localized at } y \approx L_y - 1 \text{ (up edge)}.
\end{cases}
\]  

(12)

Now let us ignore the open boundary condition and consider the bulk Bloch function at sites with a \(y\)-coordinate of \((L_y - 1)\). For the Bloch function, \(\varphi^{(b)}_0\) and \(\varphi^{(b)}_1\), compose an eigenvector of \(M\) with the eigenvalue \(\rho\),

\[
M(\epsilon) \begin{pmatrix} \varphi^{(b)}_1 \\ \varphi^{(b)}_0 \end{pmatrix} = \rho(\epsilon) \begin{pmatrix} \varphi^{(b)}_1 \\ \varphi^{(b)}_0 \end{pmatrix}.
\]  

(13)

In order to discuss the wave function of the edge state, we extend the energy to a complex energy. In the following, we use a complex variable \(z\) instead of real energy \(\epsilon\). From equation (13) we get

\[
\rho(z) = \frac{1}{2}[(\Delta(z) - \sqrt{\Delta^2(z) - 4}]
\]  

(14)

and

\[
\varphi_{L_y-1} = -\frac{M_{11}(z) + M_{22}(z) - \omega}{-M_{11}(z) + M_{22}(z) + \omega}M_{21}(z)\varphi_1,
\]  

(15)

where \(\Delta(z) = \text{Tr}[M(z)]\) and \(\omega = \sqrt{\Delta^2(z) - 4}\). Clearly,

\[
\det M(\epsilon) = 1
\]  

(16)

since \(\det \tilde{M}(\epsilon) = 1\). From equation (15) one can find that the analytic structure of the wave function is determined by the algebraic function \(\omega = \sqrt{\Delta^2(z) - 4}\). The RS of \(\omega = \sqrt{\Delta^2(z) - 4}\) on the complex-energy plane can be built by the conglutination between different analytic branches. Here, the close complex-energy plane can be obtained from the open complex-energy plane through spheral pole mapping (see figure 4(a)). Now let us discuss the analytic structure of \(\omega = \sqrt{\Delta^2(z) - 4}\) on the open complex-energy plane. If the system has \(q\) energy bands separated by bulk gaps, i.e.

\[
\epsilon \in [\lambda_1, \lambda_2], \ldots, [\lambda_{2j-1}, \lambda_{2j}], \ldots, [\lambda_{2q-1}, \lambda_{2q}],
\]  

(17)

where \(\lambda_j\) denote energies of the band edges and \(\lambda_i < \lambda_j, i < j\), then \(\omega\) can be factorized as

\[
\omega = \sqrt{\Delta^2(z) - 4} = \sqrt{\prod_{j=1}^{2q} (z - \lambda_j)}.
\]  

(18)

The two single-value analytic branches are defined on the same complex-energy plane with \(q\) secants. The difference between the two branches is specified in the following paragraph.

For an up- or down-edge-state energy \(\mu_j\) in the gap \([\lambda_{2j}, \lambda_{2j+1}]\), in order to ensure \(\omega(\mu_j) \geq 0\), we divide the two single-value analytic branches in terms of the parity (evenness or oddness) of \(j\). Let us consider the case where the energy \(z\) lies in the band \([\lambda_{2j-1}, \lambda_{2j}]\) (see figure 4(a)). The left side of this energy band is the \((j - 1)\)th gap, whereas the right side is the \(j\)th gap. When the energy \(z\) moves from the \(j\)th band to the \((j - 1)\)th gap (the \(j\)th gap) along an arbitrary path \(c_1\) \((c_2)\), only the singularities \(\lambda_{2j-1}\) and \(\lambda_{2j}\) have contributions.

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Figure 4. (a) The open complex-energy plane is mapped to the close complex-energy plane through spherical pole projection. (b) Two sheets with two cuts that correspond to the energy bands. The RS of the Bloch function is obtained by gluing the two spheres along the arrows near the cuts.

... to the variance of the principal value of the argument of $\omega$. On the up bank of the secant, we distinguish two branches $R^+$ and $R^-$ as follows: for even values of $j$, if we set $\arg(z - \lambda_{2j-1}) = 0$ and $\arg(z - \lambda_{2j}) = \pi$, which corresponds to $\omega(\mu_{j-1}) > 0$ ($\omega(\mu_j) < 0$) when $z$ moves along $c_1$ ($c_2$), then the branch $R^+$ is defined as a complex plane with $q$ secants. However, if we set $\arg(z - \lambda_{2j-1}) = 2\pi$ and $\arg(z - \lambda_{2j}) = \pi$, which corresponds to $\omega(\mu_{j-1}) < 0$ ($\omega(\mu_j) > 0$) when $z$ moves along $c_1$ ($c_2$), then the branch $R^-$ is defined as a complex plane with $q$ secants. The definitions of $R^+$ and $R^-$ for odd values of $j$ are reverse to those for even values of $j$. Hence if $z$ lies in the $j$th gap from below on the real axis,

$$ \alpha(-1)^j \omega \geq 0, \quad z: \text{real on } R^\alpha (\alpha = +, -), $$

and at energies $\mu_j (\in R^\alpha, \alpha = +, -)$ of the edge states, we have

$$ \omega(\mu_j) = \alpha(-1)^j |M_{11}(\mu_j) - M_{22}(\mu_j)|. \quad (19) $$

In addition, one can easily obtain

$$ \Delta(\epsilon) \begin{cases} 
\leq -2 & \text{for } j \text{ odd}, \\
\geq 2 & \text{for } j \text{ even},
\end{cases} \quad (20) $$

where the energy $\epsilon$ (on $R^\alpha$) is in the $j$th gap.

When the branches $R^+$ and $R^-$ on the open complex-energy plane are mapped to the close complex-energy plane through spherical pole projection, one can obtain two single-value analytic spherical surfaces. The RS is obtained by gluing the two spherical surfaces at these branch cuts by making sure that the $\pm$ banks face the $\mp$ banks of the other sphere (see figure 4(b)). Note that there are two real axes after gluing. The wave function is defined on the RS $\Sigma_\alpha(k_x)$. 

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Figure 5. RSs of the Bloch function for case I (a) and case II (b). In (a) there are two bulk gaps. The corresponding winding numbers are $I = 1$ and $I = 2$ in lower and higher gaps (HGs), respectively. In (b) there are four bulk gaps. The corresponding winding numbers are $I = 1, 1, 0$ and $-1$ in gap-I, gap-II, gap-III (Mott gap) and gap-IV, respectively.

The branch of the Bloch function is specified as $\omega > 0$, which we have discussed above. With equations (20), (19) and (15), and using the fact that $\varphi_{L, -1}(\mu_j) = 0$ for $\mu_j \in R^a$ and $\varphi_{L, -1}(\mu_j) \neq 0$ for $\mu_j \in R^-a$, one can obtain that when the zero point is on the upper sheet of RS ($R^+$), the edge state is localized at the down edge; when the zero point is on the lower sheet of RS ($R^-$), the edge state is localized at the up edge.

Figures 5(a) and (b) schematically show the RSs for the general kagomé model with system parameters belonging to case I ($J_0 < J_c$) and case II ($J_0 > J_c$), respectively. On each RS $\Sigma_g(k_x)$, the energy gap corresponds to the loop around the hole of $\Sigma_g(k_x)$ and the energy bands correspond to the closed paths vertical to the energy gap loop on $\Sigma_g(k_x)$. The Bloch function is defined on this surface. For fixed $k_x$ and $\phi$, there is always a $g$ zero point at the down-edge-state energy $\mu_j^{(\text{down})}$. Since there are two real axes on $\Sigma_g(k_x)$, correspondingly there are $g$ zero points at the up-edge-state energy $\mu_j^{(\text{up})}$.

The above analysis is for fixed $k_x$. Changing $k_x$ in one period, we can consider a family of RSs $\Sigma_g(k_x)$. $\Sigma_g(k_x)$ can be modified by this change. However, all the RSs $\Sigma_g(k_x)$ with different $k_x$ are topologically equivalent if, as happens in the present model, there are stable energy gaps in the 2D energy spectrum. By identifying topologically equivalent RSs $\Sigma_g(k_x)$, the behavior of the track of $\mu_j(k_x)$ (including the up-edge-state energy $\mu_j^{(\text{up})}$ and the down-edge-state energy $\mu_j^{(\text{down})}$) depends on system parameters. From figure 5, one can observe that by varying $k_x$ the down-edge-state energy $\mu_j^{(\text{down})}(k_x)$ moves from the lower band edge to the upper band edge, whereas the up-edge-state energy $\mu_j^{(\text{up})}(k_x)$ moves from the upper band edge to the
Figure 6. Intersection number $I(A, B)$ of two curves $A$ and $B$. Each intersection point contributes by $+1$ or $-1$ according to the direction.

lower band edge. That is to say, the two edge-state energy tracks in the same energy gap move around the hole and form an oriented loop $C(\mu_j)$.

It is known that on a general genus-$g$ RS, all kinds of loops (the first homotopy group) are generated by $2g$ canonical loops (generators), $\alpha_i$ and $\beta_i$, $i = 1, \ldots, g$ (see figure 5). The intersection number of these curves (including directions) [4] is given by (see figure 6)

$$I(\alpha_i, \beta_j) = \delta_{ij}. \tag{21}$$

Any curves on the RS are spanned homotopically by $\alpha_i$ and $\beta_i$. When the edge-state energy loop $\mu_j(k_x)$ moves $p$ times around the $j$th hole with some integer $p$, one has

$$C(\mu_j) \approx \beta_j^p, \tag{22}$$

which means

$$I(\alpha_i, C(\mu_j)) = p\delta_{ij}. \tag{23}$$

When the Fermi energy $\epsilon_f$ of the 2D kagomé lattice lies in the $j$th gap, HC is given by the winding number of the edge state [4], which is given by the number of intersections $I(\alpha_i, C(\mu_j))$ between the canonical loop $\alpha_i$ on the RS and the trace of $\mu_j$ and reads as $\sigma_{xy}^{\text{edge}} = -(e^2/h)I$. Considering the winding direction (see figure 6), one obtains that in figure 5(a) $I = 1$ ($I = 2$) when $\epsilon_f$ lies in the lower gap (LG) (HG).

The above analysis finally can be concluded as follows. The winding number of the edge states is given by the sum of the intersection number between the Fermi energy $\epsilon_f$ and the energies of the down (or up) edge state in one period. The down (or up) edge states refer to the eigenstates that are localized near the down (or) up boundaries of the sample. The intersection number is obtained as follows. In the left neighbor of the intersection, if the down-edge-state energy is connected with the lower energy band, or the up-edge-state energy is connected with the upper energy band, the intersection number is given by ‘$+1$’. If not, it is given by ‘$-1$’.

Now with the above results, we investigate the HC of the system. From figure 3(b), one can see that when the Fermi energy $\epsilon_f$ lies in the LG, there is only one intersection between $\epsilon_f$ and the down-edge-state energy, which is labeled as A in figure 3(b). In the left neighbor of intersection A, the down-edge-state energy is connected with the lower energy band. So the winding number of the edge state is $I_1 = +1$. With the same method, one can find in figure 3(c) that there are four edge-state energies lying in the HG. When the Fermi energy $\epsilon_f$ lies in the HG, there are two intersections (labeled as B and C in figure 3(b)) between $\epsilon_f$ and the down-edge-state energies. Because in the left neighbor of the intersections, the down-edge-state energies
Figure 7. (a) Energy spectrum of the 2D kagomé lattice without edges. The parameters are the same as those in figure 3. (b) HC $\sigma_{xy}$ of this system as a function of the Fermi energy $\epsilon_f$ at the temperature $T = 0$. In both figures, shaded areas are the bulk energy gaps.

are connected with the lower band, we can obtain that the winding number of the edge states is $I_2 = +2$. According to the topological edge theory, we can obtain QHC as

$$
\sigma_{xy}^{\text{edge}} = \begin{cases} 
-\frac{e^2}{h} I_1 = -\frac{e^2}{h}, & \epsilon_f \in \text{the LG}, \\
-\frac{e^2}{h} I_2 = -2\frac{e^2}{h}, & \epsilon_f \in \text{the HG}.
\end{cases}
$$

(24)

To check the above results, we recalculate the energy spectrum and the HC of the 2D kagomé lattice without boundaries within the topological bulk theory. This system was studied by Taillefumier et al [22]. In the bulk theory, when the Fermi energy $\epsilon_f$ lies in the energy gap, the HC of the system in units of $e^2/h$ is given by the sum of the Chern numbers of all the occupied energy bands [3]:

$$
\sigma_{xy}^{\text{bulk}} = \left(\frac{e^2}{h}\right) \sum_{n=1}^{\text{occupied}} C_n.
$$

The Chern number of the $n$th band is defined as $C_n = (1/2\pi) \int \Omega_x^2 d^2k$, where $\Omega_x^2 = \nabla_k \times A_{nk}$, and $A_{nk} = -i \langle u_{nk}|\nabla u_{nk}\rangle$ is the geometric vector potential. With the Hamiltonian in the momentum representation $H(k)$ (equation (4)), after a straightforward numerical calculation, one can obtain that the Chern numbers are $-1, 3, -2, -2, 3, -1$ from the lowest to the topmost band. Hence, when the Fermi energy lies in the LG, $\sigma_{xy}^{\text{bulk}} = \left(\frac{e^2}{h}\right) C_1 = -e^2/h$; when the Fermi energy lies in the HG, $\sigma_{xy}^{\text{bulk}} = \left(\frac{e^2}{h}\right) \sum_{n=1}^{4} C_n = -2e^2/h$. Comparing with equation (24), one obtains that $\sigma_{xy}^{\text{edge}} = \sigma_{xy}^{\text{bulk}}$. In figure 7(b), we plot HC as a function of the Fermi energy $\epsilon_f$ using the finite-temperature formula $\sigma_{xy} = \left(\frac{e^2}{h}\right) \sum_{n=1}^{\text{occupied}} C_n = -2e^2/h$. From this figure, one can see that when the system is in the insulating state, HC is quantized. In the LG, the conductance is $\sigma_{xy} = -e^2/h$, and in the HG, the conductance is $\sigma_{xy} = -2e^2/h$. Figure 7(a) plots the energy spectrum of the system without boundaries.

Note that in the strong Hund’s coupling limit [19], the nonzero QHC only takes two values, $\pm e^2/h$. So this is a novel result that the HC can take the value $\pm 2e^2/h$ in weak Hund’s coupling.
Figure 8. (a) Energy spectrum of the 2D kagomé lattice with edges along the $y$-direction. Parameters are chosen as $J_0 = 2$, $\theta = \pi/3$. The shaded areas are energy bands and the lines are the spectrum of the edge states. In this case, there is a Mott gap. Besides that, there are three other energy gaps, which are enlarged in (b)–(d). In (b)–(d), the red and blue lines correspond to the down- and up-edge-state energies, respectively.

3.2. Case II: $J_0 > J_c$

Now we consider the other case, in which $J_0 > J_c$. As an example, we choose the fixed chiral parameter $\theta = \pi/3$ and the exchange interaction $J_0 = 2$. The number of sites A (or B, C) in the $y$-direction is also chosen to be $L_y = 31$. With equation (3), we plot in figure 8(a) the energy spectrum of the 2D kagomé lattice with boundaries. From figure 8(a), one can find that there are four gaps in this case. We call them G-I, G-II, G-III (which is the Mott gap) and G-IV corresponding to that between $-3.54$ and $-3.36$, between $-2.0$ and $-1.65$, between $-1$ and $0$, and between $1.65$ and $2.0$, respectively. To clearly see these gaps and the edge states, we enlarge G-I, -II and -IV in figures 8(b)–(d), respectively.

From figures 8(b)–(d), one can see that when the Fermi energy $\epsilon_f$ lies in G-I (or -II, -IV), there is only one intersection between $\epsilon_f$ and the down-edge-state energy, which is labeled as point A (B, C) in figure 8(b) (8(c), 8(d)). In the left neighbor of intersection A (B), the down edge-state energy is connected with the lower band. Hence the winding number of the edge state is $I_{III} = +1$. But in the left neighbor of intersection C, the down-edge-state energy is connected with the upper band. The winding number of the edge state is $I_{IV} = -1$. Since there is no edge-state energy lying in the Mott gap, there is no intersection between $\epsilon_f$ and the left edge-state energy when $\epsilon_f$ lies in the Mott gap, and correspondingly the winding number is $I_{III} = 0$. 

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So, with the topological edge theory, one can easily obtain that when the Fermi energy \( \epsilon_f \) lies in the bulk gaps, the QHC is

\[
\sigma_{xy}^{\text{edge}} = \begin{cases} 
-\frac{e^2}{h} I_I = -\frac{e^2}{h}, & \epsilon_f \in \text{G-I}, \\
-\frac{e^2}{h} I_{II} = -\frac{e^2}{h}, & \epsilon_f \in \text{G-II}, \\
-\frac{e^2}{h} I_{III} = 0, & \epsilon_f \in \text{the Mott gap}, \\
-\frac{e^2}{h} I_{IV} = \frac{e^2}{h}, & \epsilon_f \in \text{G-IV}. 
\end{cases}
\]  
(25)

The corresponding RS of the Bloch function is also shown in figure 5(b).

Similar to the discussion in case I, to check the above results, we also recalculate the HC of the corresponding 2D kagomé lattice without boundaries. After a straightforward numerical calculation, one obtains that the Chern numbers are \(-1, 0, 1, 1, 0, -1\) from the lowest to the topmost band. When the Fermi energy \( \epsilon_f \) lies in G-I, only the lowest band is fully filled. Hence the HC is \( \sigma_{xy}^{\text{bulk}} = (e^2/h)C_1 = -e^2/h \). When \( \epsilon_f \) lies in G-II, the lowest two bands are fully occupied, \( \sigma_{xy}^{\text{II,bulk}} = (e^2/h)(C_1 + C_2) = -e^2/h \). When \( \epsilon_f \) lies in the Mott gap, \( \sigma_{xy}^{\text{III,bulk}} = (e^2/h)\sum_{n=1}^{3} C_n = 0 \). When \( \epsilon_f \) lies in G-IV, the lowest four bands are fully occupied, \( \sigma_{xy}^{\text{IV,bulk}} = (e^2/h)\sum_{n=1}^{4} C_n = e^2/h \). Comparing with equation (25), one can also obtain that \( \sigma_{xy}^{\text{edge}} = \sigma_{xy}^{\text{bulk}} \). In figure 9(b) we plot HC as a function of the Fermi energy \( \epsilon_f \). From this figure, one can see that \( \sigma_{xy} \) is similar to that in the strong Hund’s coupling limit. When the strength \( J_0 \) is greater, the results tend to those in the strong Hund’s coupling limit. Figure 9(a) plots the energy spectrum of the system without boundaries.
4. Summary

In summary, in this paper we have studied the chiral edge states and the QHC of the 2D kagomé lattice with spin anisotropies included in a general Hund’s coupling region. This system is periodic in the $x$-direction but has two edges in the $y$-direction. By numerical calculation, we find that both the strength of Hund’s coupling and spin chirality affect the edge states and the corresponding QHC. Upon varying the chirality and the strength of Hund’s coupling, two types of phenomena occur, which are distinguished by the critical relation between these two parameters (equation (5)). A remarkable difference with the infinite Hund’s coupling limit is that there are possibly four edge-state energies in one bulk gap in the case $J_0 < J_c$. If the Fermi energy lies in this gap, the HC is quantized as $\sigma_{xy} = \pm 2e^2/h$. We also give the corresponding expression of the QHC without boundaries within the topological bulk theory. Both topological expressions give the same QHC.

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