RELATION ALGEBRAS AND GROUPS

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Abstract. Generalizing results of Jónsson and Tarski, Maddux introduced the notion of a pair-dense relation algebra and proved that every pair-dense relation algebra is representable. The notion of a pair below the identity element is readily definable within the equational framework of relation algebras. The notion of a triple, a quadruple, or more generally, an element of size (or measure) \( n > 2 \) is not definable within this framework, and therefore it seems at first glance that Maddux’s theorem cannot be generalized. It turns out, however, that a very far-reaching generalization of Maddux’s result is possible if one is willing to go outside of the equational framework of relation algebras, and work instead within the framework of the first-order theory. Moreover, this generalization sheds a great deal of light not only on Maddux’s theorem, but on the earlier results of Jónsson and Tarski.

In the present paper, we define the notion of an atom below the identity element in a relation algebra having measure \( n \) for an arbitrary cardinal number \( n > 0 \), and we define a relation algebra to be measurable if its identity element is the sum of atoms each of which has some (finite or infinite) measure. The main purpose of the present paper is to construct a large class of new examples of group relation algebras using systems of groups and corresponding systems of quotient isomorphisms (instead of the classic example of using a single group and forming its complex algebra), and to prove that each of these algebras is an example of a measurable set relation algebra. In a subsequent paper, the class of examples will be greatly expanded by adding a third ingredient to the mix, namely systems of “shifting” cosets. The expanded class of examples—called coset relation algebras—will be large enough to prove a representation theorem saying that every atomic, measurable relation algebra is essentially isomorphic to a coset relation algebra.

1. Introduction

The calculus of relations was created by De Morgan [2], Peirce (see, for example, [14]), and Schröder [15] in the second half of the nineteenth century. It was intended as an algebraic theory of binary relations analogous in spirit to Boole’s algebraic theory of classes, and much of the early work in the theory consisted of a clarification of some of the important operations on and to binary relations and a study of the laws that hold for these operations on binary relations.

It was Peirce [14] who ultimately determined the list of fundamental operations, namely the Boolean operations on and between binary relations (on a base set \( U \)) of forming (binary) unions, intersections, and (unary) complements (with respect

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to the universal binary relation \( U \times U \); and relative operations of forming the (binary) relational composition—or relative product—of two relations \( R \) and \( S \) (a version of functional composition),

\[
R \mid S = \{ (\alpha, \beta) : (\alpha, \gamma) \in R \text{ and } (\gamma, \beta) \in S \text{ for some } \gamma \text{ in } U \};
\]
a dual (binary) operation of relational addition—or forming the relative sum—of \( R \) and \( S \),

\[
R \updownarrow S = \{ (\alpha, \beta) : (\alpha, \gamma) \in R \text{ or } (\gamma, \beta) \in S \text{ for all } \gamma \text{ in } U \};
\]
and a unary operation of relational inverse (a version of functional inversion)—or forming the converse—of \( R \),

\[
R^{-1} = \{ (\beta, \alpha) : (\alpha, \beta) \in R \}.
\]
He also specified some distinguished relations on the set \( U \): the empty relation \( \emptyset \), the universal relation \( U \times U \), the identity relation

\[
id_U = \{ (\alpha, \alpha) : \alpha \in U \},
\]
and its complement the diversity relation

\[
di_U = \{ (\alpha, \beta) : \alpha, \beta \in U \text{ and } \alpha \neq \beta \}.
\]
Tarski, starting with [16], gave an abstract algebraic formulation of the theory. As several of Peirce’s operations are definable in terms of the remaining ones, he reduced the number of primitive operations to the Boolean operations of addition \( + \) and complement \( - \), and the relative operations of relative multiplication \( ; \) and converse \( \sim \), with an identity element \( 1' \) as the unique distinguished constant. Thus, the models for his set of axioms are algebras of the form

\[
\mathfrak{A} = (A, +, -; ; \sim; 1'),
\]
where \( A \) is a non-empty set called the universe of \( \mathfrak{A} \), while \( + \) and \( ; \) are binary operations called addition and relative multiplication, \( - \) and \( \sim \) are unary operations called complement and converse, and \( 1' \) is a distinguished constant called the identity element. He defined a relation algebra to be any algebra of this form in which a set of ten equational axioms is true. These ten axioms are true in any set relation algebra, and the set-theoretic versions of three of them play a small role in this paper, namely the associative law for relational composition, and first and second involution laws for relational converse:

\[
R \mid (S \mid T) = (R \mid S) \mid T, \quad (R^{-1})^{-1} = R, \quad (R \mid S)^{-1} = S^{-1} \mid R^{-1}.
\]
Tarski raised the problem whether all relation algebras—all models of his axioms—are representable in the sense that they are isomorphic to set relation algebras, that is to say, they are isomorphic to subalgebras of (full) set relation algebras

\[
\mathfrak{A}(E) = (Sb(E), \cup, \sim, \mid, \sim^{-1}, id_U)
\]
in which the universe \( Sb(E) \) consists of all subrelations of some equivalence relation \( E \) on a base set \( U \), and the operations are the standard set-theoretic ones defined above, except that complements are formed with respect to \( E \) (which may or may not be the universal relation \( U \times U \)). Tarski and Jónsson [9] proved several positive representation theorems for classes of relation algebras with special properties. However, a negative solution to the general problem was ultimately given by Lyndon [10], who constructed an example of a finite relation algebra that possesses no representation at all. Since then, quite a number of papers have appeared in which
representation theorems for various special classes of relation algebras have been established, or else new examples of non-representable relation algebras have been constructed. In particular, Maddux \cite{11}, generalizing earlier theorems of Jónsson-Tarski \cite{9}, defined the notion of a pair-dense relation algebra, and proved that every pair-dense relation algebra—every relation algebra in which the identity element is a sum of “pairs”, or what might be called singleton and doubleton elements—is representable.

In trying to generalize Maddux’s theorem, a problem arises. The property of being a pair below the identity element 1’ is naturally expressible in the equational language of relation algebras. Generally speaking, however, the size of an element below the identity element—even for small sizes like 3, 4 or 5—is not expressible equationally. To overcome this difficulty another way must be found of expressing size, using the first-order language of relation algebras. This leads to the notion of a measurable atom.

For an element $x$ below the identity element—a subidentity element—the square on $x$ (or the square with side $x$) is defined to be the element $x; 1; x$. In set relation algebras with unit $E = U \times U$, such squares are just Cartesian squares, that is to say, they are relations of the form $X \times X$ for some subset $X$ of the base set $U$. A subidentity atom $x$ is said to be measurable if its square $x; 1; x$ is the sum (or the supremum) of a set of non-zero functional elements, and the number of non-zero functional elements in this set is called the measure, or the size, of the atom $x$. If the set is finite, then the atom is said to have finite measure, or to be finitely measurable. The name comes from the fact that, for set relation algebras in which the unit $E$ is the universal relation $U \times U$, the number of non-zero functional elements beneath the square on a subidentity atom is precisely the same as the number of pairs of elements that belong to that atom. For instance, in such an algebra, a subidentity atom consists of a single ordered pair just in case its square is a function; it consists of two ordered pairs just in case its square is the sum of two non-empty functions; it consists of three ordered pairs just in case its square is the sum of three non-empty functions; and so on.

In fact, the atoms below the square $x; 1; x$ of a measurable subidentity atom $x$ may be thought of as “permutations” of $x$, and they form a group $G_x$ under the restricted operations of relative multiplication and converse, with $x$ as the identity element of the group. Moreover, the set of atoms below an arbitrary rectangle $x; 1; y$ (with $x$ and $y$ measurable atoms) also form a group, one that is isomorphic to a quotient of $G_x$.

A relation algebra is defined to be measurable if its identity element is the sum of a set of measurable atoms. If each of the atoms in this set is in fact finitely measurable, then the algebra is said to be finitely measurable. The pair-dense relation algebras of Maddux are finitely measurable, and in fact each subidentity atom has measure one or two. The purpose of this paper and \cite{1} is to construct two classes of measurable relation algebras: the class of group relation algebras, which is constructed in this paper; and the broader class of coset relation algebras, which is constructed in \cite{1} and whose construction depends on the construction of group relation algebras and the results in this paper. In \cite{5}, an analysis of atomic, measurable relation algebras is carried out, and it is proved that every atomic, measurable relation algebra is essentially isomorphic to a coset relation algebra. If the given algebra is actually finitely measurable, then the assumption of it being
atomic is unnecessary. The results were announced without proofs in [4]. Except for basic facts about groups, the article is intended to be self-contained. For more information about relation algebras, the reader may consult [3], [6], [7], or [12].

2. COMPLEX ALGEBRAS OF GROUPS

In the 1940’s, J. C. C. McKinsey observed that the complex algebra of a group is a relation algebra. Specifically, let \( \langle G, \cdot, ^{-1}, e \rangle \) be a group and \( Sb(G) \) the collection of all subsets, or complexes, of \( G \). The group operations of multiplication (or composition) and inverse can be extended to operations on complexes in the obvious way:

\[
H \cdot K = \{ h \cdot k : h \in H \text{ and } k \in K \}
\]

and

\[
H^{-1} = \{ h^{-1} : h \in H \}.
\]

In order to simplify notation, we shall often identify elements with their singletons, writing, for example, \( g \cdot H \) for \( \{g\} \cdot H \), so that

\[
g \cdot H = \{ g \cdot h : h \in H \}.
\]

The collection \( Sb(G) \) of complexes contains the singleton set \( \{e\} \) and is closed under the Boolean operations of union and complement, as well as under the group operations of complex multiplication and inverse. Thus, it is permissible to form the algebra

\[
\mathcal{Cm}(G) = \langle Sb(G), \cup, \cdot, ^{-1}, \{e\} \rangle,
\]

and it is easy to check that this is a relation algebra. In fact, it is representable via a slight modification of the Cayley representation of the group. In more detail, for each element \( g \) in \( G \), let \( R_g \) be the binary relation on \( G \) defined by

\[
R_g = \{(h, h \cdot g) : h \in G\}.
\]

The correspondence \( g \mapsto R_g \) is a slightly modified version of the Cayley representation of \( G \) as a group of permutations in which the operation of relational composition is used instead of functional composition. In particular,

\[
R_g = id_G \quad \text{if and only if} \quad g = e,
\]

\[
R_g^{-1} = R_k \quad \text{if and only if} \quad g^{-1} = k,
\]

\[
R_f | R_g = R_k \quad \text{if and only if} \quad f \cdot g = k.
\]

For each subset \( X \) of \( G \), write

\[
S_X = \bigcup_{g \in X} R_g,
\]

and take \( A \) to be the set of all relations \( S_X \) for \( X \subseteq G \). Using the properties of the relations \( R_g \) displayed above, and also the complete distributivity of the operations of relational composition and converse over unions, it is a simple matter to check that \( A \) is a subuniverse of the set relation algebra \( \mathfrak{Re}(E) \) with \( E \) the universal relation on the set \( G \), so that the correspondence mapping each set \( X \) to the relation \( S_X \) is an embedding of \( \mathcal{Cm}(G) \) into \( \mathfrak{Re}(E) \). We shall call this mapping the Cayley representation of \( \mathcal{Cm}(G) \).

There is a natural extension of the Cayley representation of a group \( G \) to a representation of a quotient group \( G/H \). If \( \langle H_\gamma : \gamma < \kappa \rangle \) is a coset system for a
normal subgroup $H$ of $G$, then define the representative of a coset $H_\alpha$ to be the binary relation
\[
R_\alpha = \bigcup_{\gamma < \kappa} H_\gamma \times (H_\gamma \cdot H_\alpha).
\]
(To minimize the number of parentheses that are used, we adopt here and everywhere below the standard convention that multiplications—in this case Cartesian products—take precedence over additions—in this case, unions.) Notice that, strictly speaking, $R_\alpha$ is not the Cayley representation of $H_\alpha$, which is the set of ordered pairs
\[
\{(H_\gamma, H_\gamma \cdot H_\alpha) : \gamma < \kappa\}.
\]

The notion of a relation representing a coset can be taken one step further. If $\varphi$ is an isomorphism from a quotient group $G/H$ to another quotient group $F/K$, then $F/K$ is identical with $G/H$ except for the “shape” of its elements, and therefore it makes sense to identify each coset $H_\gamma$ in $G/H$ with its image $\varphi(H_\gamma) = K_\gamma$ in $F/K$. One can then take the representative of a coset $H_\alpha$ to be the relation
\[
R_\alpha = \bigcup_{\gamma < \kappa} H_\gamma \times \varphi(H_\gamma \cdot H_\alpha) = \bigcup_{\gamma < \kappa} H_\gamma \times (K_\gamma \cdot K_\alpha).
\]
Notice that each relation $R_\alpha$ is a union of rectangles, that is to say, it is a union of relations of the form $X \times Y$, and these rectangles are mutually disjoint, because the cosets $H_\gamma$ are mutually disjoint, as are the cosets $K_\gamma \cdot K_\alpha$, for distinct $\gamma < \kappa$.

To illustrate this idea with a concrete example, consider the two groups $\mathbb{Z}_6$ and $\mathbb{Z}_9$ (the integers modulo 6 and the integers modulo 9), and the canonical isomorphism $\varphi$ between the quotients
\[
\mathbb{Z}_6/\{0, 3\} \quad \text{and} \quad \mathbb{Z}_9/\{0, 3, 6\}
\]
that maps the cosets
\[
H_0 = \{0, 3\} \quad \text{to} \quad K_0 = \{0, 3, 6\}, \quad H_1 = \{1, 4\} \quad \text{to} \quad K_1 = \{1, 4, 7\},
\]
\[
H_2 = \{2, 5\} \quad \text{to} \quad K_2 = \{2, 5, 8\}.
\]
Using this correspondence, define three relations as follows:
\[
R_0 = [H_0 \times (K_0 \cdot K_0)] \cup [H_1 \times (K_1 \cdot K_0)] \cup [H_2 \times (K_2 \cdot K_0)]
\]
\[
= [H_0 \times K_0] \cup [H_1 \times K_1] \cup [H_2 \times K_2]
\]
\[
= \{(a, b) : a \in \mathbb{Z}_6, b \in \mathbb{Z}_9 \text{ and } b \equiv a \text{ mod 3}\},
\]
\[
R_1 = [H_0 \times (K_0 \cdot K_1)] \cup [H_1 \times (K_1 \cdot K_1)] \cup [H_2 \times (K_2 \cdot K_1)]
\]
\[
= [H_0 \times K_1] \cup [H_1 \times K_2] \cup [H_2 \times K_0]
\]
\[
= \{(a, b) : a \in \mathbb{Z}_6, b \in \mathbb{Z}_9 \text{ and } b \equiv a + 1 \text{ mod 3}\},
\]
\[
R_2 = [H_0 \times (K_0 \cdot K_2)] \cup [H_1 \times (K_1 \cdot K_2)] \cup [H_2 \times (K_2 \cdot K_2)]
\]
\[
= [H_0 \times K_2] \cup [H_1 \times K_0] \cup [H_2 \times K_1]
\]
\[
= \{(a, b) : a \in \mathbb{Z}_6, b \in \mathbb{Z}_9 \text{ and } b \equiv a + 2 \text{ mod 3}\}.
\]
(See Figure [1]) The relations $R_0$, $R_1$, and $R_2$ are representatives of the cosets $H_0$, $H_1$, and $H_2$ respectively, and together they give a kind of representation of $\mathbb{Z}_3$ that has the flavor of the Cayley representation of $\mathbb{Z}_3$. (Notice, however, that this is not a real representation of $\mathbb{Z}_3$, since we cannot form the composition of these relations.) This is a key idea in the construction of measurable algebras of binary relations from systems of groups and quotient isomorphisms.
3. Systems of groups and quotient isomorphisms

Fix a system

\[ G = \langle G_x : x \in I \rangle \]

of groups \( \langle G_x, \ast, \ast^{-1}, e_x \rangle \) that are pairwise disjoint, and an associated system

\[ \varphi = \langle \varphi_{xy} : (x, y) \in E \rangle \]

of quotient isomorphisms. Specifically, \( E \) is an equivalence relation on the index set \( I \), and for each pair \( (x, y) \) in \( E \), the function \( \varphi_{xy} \) is an isomorphism from a quotient group of \( G_x \) to a quotient group of \( G_y \). We shall call

\[ \mathcal{F} = (G, \varphi) \]

a group pair. The set \( I \) is the group index set, and the equivalence relation \( E \) is the (quotient) isomorphism index set, of \( \mathcal{F} \). The normal subgroups of \( G_x \) and \( G_y \) from which the quotient groups are constructed are uniquely determined by \( \varphi_{xy} \), and will be denoted by \( H_{xy} \) and \( K_{xy} \) respectively, so that \( \varphi_{xy} \) maps \( G_x/H_{xy} \) isomorphically onto \( G_y/K_{xy} \).

For a fixed enumeration \( \langle H_{xy, \gamma} : \gamma < \kappa_{xy} \rangle \) (without repetitions) of the cosets of \( H_{xy} \) in \( G_x \) (indexed by some ordinal number \( \kappa_{xy} \)), the isomorphism \( \varphi_{xy} \) induces a corresponding, or associated, coset system of \( K_{xy} \) in \( G_y \), determined by the rule

\[ K_{xy, \gamma} = \varphi_{xy}(H_{xy, \gamma}) \]

for each \( \gamma < \kappa_{xy} \). In what follows we shall always assume that the given coset systems for \( H_{xy} \) in \( G_x \) and for \( K_{xy} \) in \( G_y \) are associated in this manner. Furthermore, there is no loss of generality in assuming that the first elements in the enumeration of the coset systems are always the normal subgroups themselves, so that

\[ H_{xy, 0} = H_{xy} \quad \text{and} \quad K_{xy, 0} = K_{xy}. \]
**Definition 3.1.** For each pair \((x, y)\) in \(E\) and each \(\alpha < \kappa_{xy}\), define a binary relation \(R_{xy, \alpha}\) by

\[
R_{xy, \alpha} = \bigcup_{\gamma < \kappa_{xy}} H_{xy, \gamma} \times \varphi_{xy}[H_{xy, \gamma} \circ H_{xy, \alpha}] = \bigcup_{\gamma < \kappa_{xy}} H_{xy, \gamma} \times (K_{xy, \gamma} \circ K_{xy, \alpha}).
\]

The index \(\alpha\) enumerating the relations \(R_{xy, \alpha}\) coincides with the index enumerating the coset system for the subgroup \(H_{xy}\), and therefore is dependent upon the particular, often arbitrarily chosen, enumeration of the cosets. It would be much better if the index enumerating the relations were independent of the particular coset system that has been employed. This can be accomplished by using the cosets themselves as indices, writing, for instance, for each coset \(L\) of \(H_{xy}\), that is to say, for each element \(L\) in \(G_x/H_{xy}\),

\[
R_{xy, L} = \bigcup \{H \times \varphi(H \circ L) : H \in G_x/H_{xy}\}
\]

instead of \(R_{xy, \alpha}\). In fact, it is really our intention that the relations be indexed by the cosets and not by the indices of the cosets. However, adopting this notation in practice eventually becomes notationally a bit unwieldy. For that reason, we shall continue to use the coset indices \(\alpha\), but we view these only as convenient abbreviations for the cosets themselves. In places where the distinction is important, we shall point it out.

Notice that the relation \(R_{xy, 0}\) encodes the isomorphism \(\varphi_{xy}\).

In proofs, we shall use repeatedly the fact that operations such as forward and inverse images of sets under functions, Cartesian multiplication of sets, intersection of sets, complex group composition, relational composition, and relational converse are all distributive over arbitrary unions, and we shall usually simply refer to this fact by citing distributivity.

**Lemma 3.2 (Partition Lemma).** The relations \(R_{xy, \alpha}\), for \(\alpha < \kappa_{xy}\), are non-empty and partition the set \(G_x \times G_y\).

**Proof.** Obviously, the relations are non-empty, because the cosets used to construct them are non-empty. The sequence \(\langle H_{xy, \gamma} : \gamma < \kappa_{xy} \rangle\) is a coset system for \(H_{xy}\) in \(G_x\), so these cosets are mutually disjoint and have \(G_x\) as their union. Similarly, the cosets in the corresponding sequence \(\langle K_{xy, \gamma} : \gamma < \kappa_{xy} \rangle\) are mutually disjoint and have \(G_y\) as their union. The sequence obtained by multiplying each \(K_{xy, \gamma}\) on the right by a fixed coset \(K_{xy, \alpha}\) lists the cosets of \(K_{xy}\) in some permuted order. These observations and the distributivity of Cartesian multiplication yield

\[
\bigcup_{\alpha} R_{xy, \alpha} = \bigcup_{\alpha} \bigcup_{\gamma} H_{xy, \gamma} \times (K_{xy, \gamma} \circ K_{xy, \alpha}) = \bigcup_{\gamma} \bigcup_{\alpha} H_{xy, \gamma} \times (K_{xy, \gamma} \circ K_{xy, \alpha}) = \bigcup_{\gamma} H_{xy, \gamma} \times (\bigcup_{\alpha} K_{xy, \gamma} \circ K_{xy, \alpha}) = \bigcup_{\gamma} H_{xy, \gamma} \times G_y = G_x \times G_y.
\]

The cosets \(H_{xy, \gamma}\) and \(H_{xy, \delta}\) are disjoint whenever \(\gamma \neq \delta\), and so are the cosets \(K_{xy, \alpha}\) and \(K_{xy, \beta}\)—and therefore also the cosets \(K_{xy, \gamma} \circ K_{xy, \alpha}\) and \(K_{xy, \gamma} \circ K_{xy, \beta}\)—whenever \(\alpha \neq \beta\). Consequently,

\[
[H_{xy, \gamma} \cap H_{xy, \delta}] \times [(K_{xy, \gamma} \circ K_{xy, \alpha}) \cap (K_{xy, \delta} \circ K_{xy, \beta})] = \emptyset
\]
whenever \( \gamma \neq \delta \) or \( \alpha \neq \beta \). For distinct \( \alpha, \beta \), a simple computation leads to

\[
R_{xy,\beta} = [\bigcup_\gamma H_{xy,\gamma} \times (K_{xy,\gamma} \circ K_{xy,\alpha})] \cap [\bigcup_\delta H_{xy,\delta} \times (K_{xy,\delta} \circ K_{xy,\beta})]
= \bigcup_\gamma [H_{xy,\gamma} \times (K_{xy,\gamma} \circ K_{xy,\alpha})] \cap [H_{xy,\delta} \times (K_{xy,\delta} \circ K_{xy,\beta})]
= \bigcup_\gamma [H_{xy,\gamma} \cap H_{xy,\delta}] \times [(K_{xy,\gamma} \circ K_{xy,\alpha}) \cap (K_{xy,\delta} \circ K_{xy,\beta})]
= \emptyset,
\]
by the definition of \( R_{xy,\alpha} \) and \( R_{xy,\beta} \), the distributivity of intersection and Cartesian multiplication, and (1).

Let \( U \) be the union of the disjoint system of groups, and \( E \) the equivalence relation on \( U \) induced by the isomorphism index set \( \mathcal{E} \),

\[
U = \bigcup \{ G_x : x \in I \} \quad \text{and} \quad E = \bigcup \{ G_x \times G_y : (x, y) \in \mathcal{E} \}.
\]

Write

\[
\mathcal{I} = \{ ((x, y), \alpha) : (x, y) \in \mathcal{E} \text{ and } \alpha < \kappa_{xy} \}
\]

for the relation index set of the group pair \( \mathcal{F} \), that is to say, for the set of indices of the relations \( R_{xy,\alpha} \). For each subset \( \mathcal{X} \) of \( \mathcal{I} \), define

\[
S_{\mathcal{X}} = \{ R_{xy,\alpha} : ((x, y), \alpha) \in \mathcal{X} \},
\]

and let \( A \) be the collection of all of the relations \( S_{\mathcal{X}} \) so defined.

**Theorem 3.3 (Boolean Algebra Theorem).** The set \( A \) is the universe of a complete and atomic Boolean algebra of subsets of \( E \). The distinct elements in \( A \) are the relations \( S_{\mathcal{X}} \) for distinct subsets \( \mathcal{X} \) of \( \mathcal{I} \), and the atoms are the relations \( R_{xy,\alpha} \) for \( ((x, y), \alpha) \) in \( \mathcal{I} \). The unit is the relation \( E = S_{\mathcal{I}} \), and the operations of union, intersection, and complement in \( A \) are determined by

\[
\bigcup_\xi S_{\mathcal{X}_\xi} = S_{\mathcal{Y}}, \quad \bigcap_\xi S_{\mathcal{X}_\xi} = S_{\mathcal{Y}}, \quad S_{\mathcal{I}} \sim S_{\mathcal{X}} = S_{\mathcal{Y}}
\]

where \( \mathcal{Y} = \bigcup_\xi \mathcal{X}_\xi \) in the first case, \( \mathcal{Y} = \bigcap_\xi \mathcal{X}_\xi \) in the second case, and \( \mathcal{Y} = \mathcal{I} \sim \mathcal{X} \) in the third case (for any system \( \{ \mathcal{X}_\xi : \xi < \lambda \} \) of subsets, and any subset \( \mathcal{X} \), of \( \mathcal{I} \)).

**Proof.** The system of rectangles \( \{ G_x \times G_y : (x, y) \in \mathcal{E} \} \) is easily seen to be a partition of \( E \). Combine this with Lemma 3.2, and the definition of the relations \( S_{\mathcal{X}} \) to arrive at the desired result.

Although the set \( A \) is always a complete Boolean set algebra of binary relations, it is not in general closed under the operations of relational composition and converse, nor does it necessarily contain the identity relation \( id_U \) on the set \( U \). Such closure depends on the properties of the quotient isomorphisms. We begin by characterizing when \( A \) contains \( id_U \).

**Theorem 3.4 (Identity Theorem).** For each element \( x \) in \( I \), the following conditions are equivalent.

(i) The identity relation \( id_{G_x} \) on \( G_x \) is in \( A \).
(ii) \( R_{xx,0} = id_{G_x} \).
(iii) \( \varphi_{xx} \) is the identity automorphism of \( G_x/\{e_x\} \).

Consequently, \( A \) contains the identity relation \( id_U \) on the base set \( U \) if and only if (iii) holds for each \( x \) in \( I \).
Proof. Suppose (i) holds, with the intention of deriving (iii). From the assumption in (i), and the definition of the set $A$, it is clear that $id_{G_x}$ must be a (non-empty) union of some of the relations $R_{yz,\alpha}$. Each relation $R_{yz,\alpha}$ in such a union is a subset of the rectangle $G_y \times G_z$, by Partition Lemma 3.2. and it is simultaneously a subset of the square $G_x \times G_x$, because $id_{G_x}$ is a subset of $G_x \times G_x$. The rectangle and the square are disjoint whenever $x \neq y$ or $x \neq z$, so $x = y = z$, and therefore

$$
\bigcup \gamma H_{xx,\gamma} \times (K_{xx,\gamma} \circ K_{xx,\alpha}) = R_{xx,\alpha} \subseteq id_{G_x} = \bigcup \{(g,h) : g \in G_x\},
$$

by the definitions of $R_{xx,\alpha}$ and $id_{G_x}$. This inclusion implies that the cosets $H_{xx,\gamma}$ and $K_{xx,\gamma} \circ K_{xx,\alpha}$ on the left side of (1) contain exactly one element each, and this element is the same for both cosets, for if this were not the case, then the Cartesian product of the two cosets would contain a pair of the form $(g,h)$ with $g \neq h$, in contradiction to (1). Thus, for each $\gamma < \kappa_{xx}$, there is an element $g$ in $G_x$ such that

$$
H_{xx,\gamma} = \{g\}
$$

and

$$
K_{xx,\gamma} \circ K_{xx,\alpha} = \{g\}.
$$

Take $\gamma = 0$ in (2), and apply the convention that $H_{xx,0}$ and $K_{xx,0}$ coincide with the subgroups $H_{xx}$ and $K_{xx}$ respectively; these subgroups are the identity cosets of the quotient groups $G_x/H_{xy}$ and $G_y/K_{xy}$, so

$$
H_{xx} = H_{xx,0} = \{g\}
$$

and

$$
K_{xx,\alpha} = K_{xx,0} \circ K_{xx,\alpha} = \{g\}.
$$

By assumption $H_{xx}$ is a normal subgroup of $G_x$. The only normal subgroup that has exactly one element is the trivial subgroup $\{e_x\}$, so the element $g$ in (3) must coincide with $e_x$. Use the right side of (3) with $g = e_x$ to see that $\alpha$ must be 0.

Invoke (2) one more time to obtain, for each $\gamma < \kappa_{xx}$, an element $g$ in $G_x$ such that

$$
H_{xx,\gamma} = \{g\} = K_{xx,\gamma} \circ K_{xx,\alpha} = K_{xx,\gamma} \circ K_{xx,0} = K_{xx,\gamma}.
$$

The isomorphism $\varphi_{xx}$ is assumed to map $H_{xx,\gamma}$ to $K_{xx,\gamma}$ for each $\gamma$, so (1) shows that $\varphi_{xx}$ maps each singleton $\{g\}$ to itself. It follows that $\varphi_{xx}$ is the identity isomorphism on $G_x/\{e_x\}$. Thus, (iii) holds.

If (iii) holds, then

$$
R_{xx,0} = \bigcup \{(g) \times (\{g\} \circ \{e_x\}) : g \in G_x\} = \{(g,0) : g \in G_x\} = id_{G_x},
$$

by the definition of $R_{xx,0}$, so (ii) holds. On the other hand, if (ii) holds, then (i) obviously holds, by the definition of $A$.

To derive the final assertion of the theorem, assume first that (iii) holds. The identity relation $id_{G_x}$ is then in $A$, by (i). The union, over all $x$, of these identity relations is the identity relation $id_U$. Since $A$ is closed under arbitrary unions, it follows that $id_U$ is in $A$.

Now assume that $id_U$ is in $A$. The squares $G_x \times G_x$ are all in $A$, by Lemma 3.2 and the definition of $A$, so the intersection of each of these squares with $id_U$ is in $A$, by the closure of $A$ under intersection. This intersection is just $id_{G_x}$, so (i) holds, and therefore also (iii), for each $x$.

In order to prove the next two theorems, it is convenient to formulate two lemmas that will be used in both proofs.

Lemma 3.5. Suppose that each of

$$
\langle M_\alpha : \alpha < \kappa \rangle, \quad \langle N_\alpha : \alpha < \kappa \rangle, \quad \langle P_\beta : \beta < \lambda \rangle, \quad \langle Q_\beta : \beta < \lambda \rangle
$$

are sequences of non-empty, pairwise disjoint sets. If
then there is a uniquely determined mapping $\vartheta$ from $\kappa$ into $\lambda$ such that

(ii) $M_\alpha \subseteq P_{\vartheta(\alpha)}$ and $N_\alpha \subseteq Q_{\vartheta(\alpha)}$

for each $\alpha < \kappa$. If equality holds in (i), then equality holds in (ii), and $\vartheta$ is a bijection.

Proof. Consider, first, arbitrary non-empty sets $M$ and $N$. Assume

(1) $M \times N = \bigcup_{\beta < \lambda} P_\beta \times Q_\beta$,

with the intention of proving that $\lambda = 1$ (recall that $\lambda$ is an ordinal), and

(2) $M = P_0$ and $N = Q_0$.

It is obvious from (1) that $P_\beta \times Q_\beta \subseteq M \times N$, and therefore $P_\beta \subseteq M$ and $Q_\beta \subseteq N$, for each $\beta < \lambda$. Consequently,

(3) $\bigcup_{\beta < \lambda} P_\beta \subseteq M$ and $\bigcup_{\beta < \lambda} Q_\beta \subseteq N$.

Use the distributivity of Cartesian multiplication, (3), and (1) to obtain

(4) $\bigcup_{\alpha, \beta < \lambda} P_\alpha \times Q_\beta = (\bigcup_{\beta < \lambda} P_\beta) \times (\bigcup_{\beta < \lambda} Q_\beta) \subseteq M \times N = \bigcup_{\gamma < \lambda} P_\gamma \times Q_\gamma$.

The inclusion of the first union in the last one in (4) implies that every pair $(g, h)$ in a rectangle $P_\beta \times Q_\beta$ must belong to some rectangle $P_\gamma \times Q_\gamma$. This cannot happen if $\alpha \neq \beta$, because in such a case either $\alpha \neq \gamma$ or $\beta \neq \gamma$, and therefore either $P_\alpha$ must be disjoint from $P_\gamma$, or else $Q_\beta$ must be disjoint from $Q_\gamma$. It follows that there is exactly one $\beta$ that is less than $\lambda$. Since $\lambda$ is assumed to be an ordinal, this forces $\lambda = 1$ and $\beta = 0$. Thus, (1) assumes the form

$$M \times N = P_0 \times Q_0,$$

and clearly, (2) holds in this case.

Next, suppose that the equality in (1) is replaced with set-theoretic inclusion, so that

(5) $M \times N \subseteq \bigcup_{\beta < \lambda} P_\beta \times Q_\beta$.

There is then a unique index $\beta < \lambda$ such that

(6) $M \subseteq P_\beta$ and $N \subseteq Q_\beta$.

For the proof, form the intersection of both sides of (5) with $M \times N$, and use (5), the distributivity of intersection, and simple set theory to obtain

(7) $M \times N = (M \times N) \cap (M \times N) = (M \times N) \cap [\bigcup_{\beta < \lambda} (P_\beta \times Q_\beta)]$

$$\cap [\bigcup_{\beta < \lambda} (M \times N) \cap (P_\beta \times Q_\beta)] = \bigcup_{\beta < \lambda} (M \cap P_\beta) \times (N \cap Q_\beta).$$

Drop all terms in the union on the right side of (7) that are empty. The equality of the first and last expressions in (7) shows that (1) holds with $P_\beta$ and $Q_\beta$ replaced by $M \cap P_\beta$ and $N \cap Q_\beta$ respectively. Use the implication from (1) to (2) to conclude that there can only be one index $\beta$ on the right side of (7) for which the intersection is not empty, and for that $\beta$ we have

$$M = M \cap P_\beta$$

and $N = N \cap Q_\beta$,

so that (6) holds.
Turn now to the proof of the implication from (i) to (ii). Fix an arbitrary index \( \alpha < \kappa \). From (i), it follows immediately that
\[
M_\alpha \times N_\alpha \subseteq \bigcup_{\beta < \lambda} P_\beta \times Q_\beta.
\]
Apply the implication from (5) to (6) to obtain a unique \( \beta < \lambda \) such that
\[
M_\alpha \subseteq P_\beta \quad \text{and} \quad N_\alpha \subseteq Q_\beta.
\]
The desired function is the mapping \( \vartheta \) that sends \( \alpha \) to the corresponding \( \beta \), so that (8) holds for each \( \alpha < \kappa \).
Assume finally that equality holds in (i). There are then uniquely determined mappings \( \vartheta \) from \( \kappa \) to \( \lambda \) and \( \psi \) from \( \lambda \) to \( \kappa \) such that
\[
M_\alpha \subseteq P_{\vartheta(\alpha)} \quad \text{and} \quad N_\alpha \subseteq Q_{\psi(\alpha)}
\]
for each \( \alpha < \kappa \), and
\[
P_\beta \subseteq M_{\psi(\beta)} \quad \text{and} \quad Q_\beta \subseteq N_{\psi(\beta)}
\]
for each \( \beta < \lambda \). Combine (9) and (10) to arrive at
\[
M_\alpha \subseteq P_{\vartheta(\alpha)} \subseteq M_{\psi(\psi(\alpha))} \quad \text{and} \quad P_\beta \subseteq M_{\psi(\beta)} \subseteq P_{\vartheta(\psi(\beta))}
\]
for each \( \alpha < \kappa \) and \( \beta < \lambda \). The sets \( M_\alpha \) are pairwise disjoint, as are the sets \( P_\beta \), so the inclusions in (11) force
\[
\psi(\vartheta(\alpha)) = \alpha \quad \text{and} \quad \vartheta(\psi(\beta)) = \beta
\]
for each \( \alpha < \kappa \) and \( \beta < \lambda \). This implies that the mappings \( \vartheta \) and \( \psi \) are bijections and inverses of each other. \( \square \)

**Lemma 3.6.** Suppose \( P \) and \( Q \) are normal subgroups of groups \( G \) and \( \bar{G} \), with coset systems
\[
\langle P_\gamma ; \gamma < \kappa \rangle \quad \text{and} \quad \langle Q_\gamma ; \gamma < \kappa \rangle
\]
respectively. If the mapping \( P_\gamma \mapsto Q_\gamma \) is an isomorphism from \( G/P \) onto \( \bar{G}/Q \), then for all \( \alpha, \beta < \kappa \), we have
\[
\begin{align*}
(i) \quad & \bigcup_{\gamma} P_\gamma \times (Q_\gamma \circ Q_\alpha) = \bigcup_{\gamma} (P_\gamma \circ P_\alpha^{-1}) \times Q_\gamma, \\
(ii) \quad & \bigcup_{\gamma} P_\gamma \times (Q_\gamma \circ Q_\alpha \circ Q_\beta) = \bigcup_{\gamma} (P_\gamma \circ P_\alpha^{-1}) \times (Q_\gamma \circ Q_\beta).
\end{align*}
\]

**Proof.** Fix an index \( \alpha < \kappa \), and observe that \( \langle P_\gamma \circ P_\alpha^{-1} : \gamma < \kappa \rangle \) is also an enumeration of the cosets of \( P \). Consequently, for each \( \gamma < \kappa \) there exists a unique \( \tilde{\gamma} < \kappa \) such that
\[
P_\gamma = P_{\tilde{\gamma}} \circ P_\alpha^{-1}.
\]
The mapping \( P_\gamma \mapsto Q_\gamma \) is assumed to be an isomorphism, so (1) implies that
\[
Q_\gamma = Q_{\tilde{\gamma}} \circ Q_\alpha^{-1}.
\]
Use (1), (2), and the inverse properties of groups to get
\[
\bigcup_{\gamma} P_\gamma \times (Q_\gamma \circ Q_\alpha) = \bigcup_{\gamma} (P_\gamma \circ P_\alpha^{-1}) \times (Q_{\tilde{\gamma}} \circ Q_\alpha^{-1} \circ Q_\alpha) = \bigcup_{\gamma} (P_{\tilde{\gamma}} \circ P_{\alpha}^{-1}) \times Q_{\tilde{\gamma}}.
\]
As \( \gamma \) varies over \( \kappa \), so does \( \tilde{\gamma} \), and vice versa, so the occurrence of \( \tilde{\gamma} \) in the union on the right side of (3) may be replaced by \( \gamma \) to arrive at (i).
Exactly the same reasoning also gives
\[
\bigcup_{\gamma} P_{\gamma} \times (Q_{\gamma} \circ Q_{\alpha} \circ Q_{\beta}) = \bigcup_{\gamma} (P_{\gamma} \circ P_{\alpha}^{-1}) \times (Q_{\gamma} \circ Q_{\alpha}^{-1} \circ Q_{\beta}) \\
= \bigcup_{\gamma} (P_{\gamma} \circ P_{\alpha}^{-1}) \times (Q_{\gamma} \circ Q_{\beta}) \\
= \bigcup_{\gamma} (P_{\gamma} \circ P_{\alpha}^{-1}) \times (Q_{\gamma} \circ Q_{\beta})
\]
which proves (ii).

The next task is to establish necessary and sufficient conditions for the set \(A\) to be closed under converse, and in particular, for \(A\) to contain the converse of every atomic relation. As we shall see in the next theorem, \(A\) will contain the converse of every atomic relation if and only if the isomorphism \(\varphi_{yx}\) for every pair \((x, y)\) in \(E\). Since \(\varphi_{xy}\) maps \(G_x/H_{xy}\) to \(G_y/K_{xy}\), and \(\varphi_{yx}\) maps \(G_y/H_{yx}\) to \(G_x/K_{yx}\), if these two isomorphisms are inverses of one another, then we must have
\[
G_x/H_{xy} = G_x/K_{yx} \quad \text{and} \quad G_y/K_{xy} = G_y/H_{yx},
\]
so that
\[
K_{yx} = H_{xy} \quad \text{and} \quad K_{xy} = H_{yx}.
\]
As mentioned earlier, the enumeration of the cosets of the subgroup \(H_{yx}\) can be chosen freely. Under the given assumption, we can and shall always adopt the following convention regarding the choice of this enumeration.

**Convention 3.7.** If \(\varphi_{xy}\) and \(\varphi_{yx}\) are inverses of one another, then the coset enumeration \(\langle H_{yx, \gamma} : \gamma < \kappa_{yx} \rangle\) is chosen so that \(\kappa_{yx} = \kappa_{xy}\) and \(H_{yx, \gamma} = K_{xy, \gamma}\) for all \(\gamma < \kappa_{yx}\). It then follows that
\[
K_{yx, \gamma} = \varphi_{yx}(H_{yx, \gamma}) = \varphi_{xy}^{-1}(K_{xy, \gamma}) = H_{xy, \gamma}
\]
for all \(\gamma < \kappa_{xy}\).

The next theorem characterizes when \(A\) is closed under converse.

**Theorem 3.8** (Converse Theorem). For each pair \((x, y)\) in \(E\), the following conditions are equivalent.

(i) There are an \(\alpha < \kappa_{xy}\) and a \(\beta < \kappa_{yx}\) such that \(R_{xy, \alpha}^{-1} = R_{yx, \beta}\).

(ii) For every \(\alpha < \kappa_{xy}\) there is a \(\beta < \kappa_{yx}\) such that \(R_{xy, \alpha}^{-1} = R_{yx, \beta}\).

(iii) \(\varphi_{xy}^{-1} = \varphi_{yx}\).

Moreover, if one of these conditions holds, then we may assume that \(\kappa_{yx} = \kappa_{xy}\), and the index \(\beta\) in (i) and (ii) is uniquely determined by \(H_{xy, \alpha}^{-1} = H_{yx, \beta}\). The set \(A\) is closed under converse if and only if (iii) holds for all \((x, y)\) in \(E\).

**Proof.** Observe, first of all, that without using any of the hypotheses in (i)–(iii), only the definition of the relation \(R_{xy, \alpha}\), Lemma 3.6(i), the distributivity of relational converse, and the definition of relational converse, we have
\[
(1) \quad R_{xy, \alpha}^{-1} = \bigcup_{\gamma} H_{xy, \gamma} \times (K_{xy, \gamma} \circ K_{xy, \alpha})^{-1} = \bigcup_{\gamma} (H_{xy, \gamma} \circ H_{xy, \alpha})^{-1} \times K_{xy, \gamma}^{-1} \\
= \bigcup_{\gamma} (H_{xy, \gamma} \circ K_{xy, \gamma}) \times (H_{xy, \gamma} \circ H_{xy, \alpha})^{-1} = \bigcup_{\gamma} K_{xy, \gamma} \times (H_{xy, \gamma} \circ H_{xy, \alpha})^{-1}.
\]
Assume now that (iii) holds, with the intention of deriving (ii). Choose $\beta < \kappa_{xy}$ so that

$$H_{xy,\beta} = H_{xy,\alpha}^{-1}. \tag{2}$$

In view of assumption (iii), Convention 3.7 may be applied to write $\kappa_{yx} = \kappa_{xy}$, and

$$H_{yx,\gamma_1} = K_{xy,\gamma_1}, \quad K_{yx,\gamma_1} = H_{xy,\gamma} \tag{3}$$

for each $\gamma < \kappa_{xy}$. Use the definition of the relation $R_{yx,\beta}$, together with (3), (2), and (1), to conclude that

$$R_{yx,\beta} = \bigcup_{\gamma} H_{xy,\gamma} \times (K_{yx,\gamma} \circ K_{yx,\beta}) = \bigcup_{\gamma} K_{xy,\gamma} \times (H_{xy,\gamma} \circ H_{xy,\beta}) = \bigcup_{\gamma} K_{xy,\gamma} \times (H_{xy,\gamma} \circ H_{xy,\beta}^{-1}) = R_{xy,\alpha}^{-1}. \tag{4}$$

Thus, (ii) holds.

The implication from (ii) to (i) is obvious. Consider now the implication from (i) to (iii). Fix $\alpha < \kappa_{xy}$, and suppose that

$$R_{yx,\beta} = R_{xy,\alpha}^{-1}. \tag{4}$$

Use (4), the definition of $R_{yx,\beta}$, and (1) (with $\gamma$ replaced by another variable, say $\eta$) to obtain

$$\bigcup_{\gamma < \kappa_{yx}} H_{yx,\gamma} \times (K_{yx,\gamma} \circ K_{yx,\beta}) = \bigcup_{\eta < \kappa_{xy}} K_{xy,\eta} \times (H_{xy,\eta} \circ H_{xy,\alpha}^{-1}). \tag{5}$$

Apply Lemma 3.5 to (5) to see that there must be a bijection $\vartheta$ from $\kappa_{xy}$ to $\kappa_{xy}$ such that

$$H_{yx,\gamma} = K_{xy,\vartheta(\gamma)} \quad \text{and} \quad K_{yx,\gamma} \circ K_{yx,\beta} = H_{xy,\vartheta(\gamma)} \circ H_{xy,\beta} \tag{6}$$

for all $\gamma < \kappa_{yx}$.

Take $\gamma = 0$ in (6). It follows from the first equation that $H_{yx,0} = K_{xy,\vartheta(0)}$. Since $H_{yx,0}$ is a subgroup of $G_y$, the same must be true of $K_{xy,\vartheta(0)}$. The only subgroup in the coset enumeration of $K_{xy}$ is $K_{xy,0}$, so $\vartheta(0) = 0$, and therefore

$$H_{yx} = H_{yx,0} = K_{xy,0} = K_{xy}. \tag{7}$$

Apply this observation to the second equation, and use the fact that $\vartheta(0) = 0$, to arrive at

$$K_{yx,\beta} = K_{yx} \ast K_{yx,\beta} = K_{yx,0} \ast K_{yx,\beta} = H_{xy,\vartheta(0)} \ast H_{xy,\beta} = H_{xy,\beta} = H_{xy,\beta} = H_{xy,\beta}. \tag{8}$$

(Recall that $K_{yx}$ and $H_{xy}$ are the identity cosets of their respective coset systems.)

Multiply the left and right sides of the second equation in (6), on the right, by $K_{yx,\beta}^{-1}$, use the inverse law for groups, and use the equality of the first and last cosets in (7), to arrive at

$$K_{yx,\gamma} = K_{yx,\gamma} \ast K_{yx,\beta} \ast K_{yx,\beta}^{-1} = H_{xy,\vartheta(\gamma)} \ast H_{xy,\beta} \ast H_{xy,\beta}^{-1} = H_{xy,\vartheta(\gamma)} \ast H_{xy,\beta} \ast H_{xy,\beta}^{-1} = H_{xy,\vartheta(\gamma)} \ast H_{xy,\beta} \ast H_{xy,\beta}^{-1} = H_{xy,\vartheta(\gamma)} \ast H_{xy,\beta} \ast H_{xy,\beta}^{-1} = H_{xy,\vartheta(\gamma)} \ast H_{xy,\beta} \ast H_{xy,\beta}^{-1} = H_{xy,\vartheta(\gamma)}. \tag{9}$$

for every $\gamma < \kappa_{yx}$. Consequently,

$$\varphi_{yx}(K_{xy,\vartheta(\gamma)}) = \varphi_{yx}(H_{yx,\gamma}) = K_{yx,\gamma} = H_{xy,\vartheta(\gamma)},$$
by (6), the definition of $K_{yx, \gamma}$, and (8). As $\gamma$ runs through the indices less than $\kappa_{yx}$, the image $\vartheta(\gamma)$ runs through the indices less than $\kappa_{xy}$, so the preceding string of equalities shows that

$$\varphi_{yx}(K_{xy, \delta}) = H_{xy, \delta}$$

for every $\delta < \kappa_{xy}$. Since $\varphi_{xy}$ maps each coset $H_{xy, \delta}$ to $K_{xy, \delta}$, it follows from (9) that $\varphi_{yx}$ is the inverse of $\varphi_{xy}$. This completes the proof that conditions (i)–(iii) are equivalent.

If one of the three conditions holds, then all three conditions hold by the equivalence just established. Consequently, using the proof of the implication from (iii) to (ii), we may assume that $\kappa_{yx} = \kappa_{xy}$, and choose $\beta < \kappa_{xy}$ so that (2) holds. This proves the second assertion of the theorem.

Turn to the proof of the final assertion of the theorem. Assume first that (iii) holds for all $(x, y)$ in $E$. The atoms in $A$ are just the relations of the form $R_{xy, \alpha}$, so from the equivalence of (ii) with (iii), it follows that the converse of every atom in $A$ is again an atom in $A$. The elements of $A$ are just the unions of these various atoms, by Theorem 3.3, and the converse of a union of atoms is again a union of atoms, by the preceding observation and the distributivity of converse. Thus, the converse of every element in $A$ belongs to $A$, so $A$ is closed under converse.

Assume now that $A$ is closed under converse, and fix an arbitrary pair $(x, y)$ in $E$. The relation $R_{xy, 0}$ is a subset of $G_x \times G_y$ and belongs to $A$, by Lemma 3.2 and the definition of $A$. It follows that the converse relation $R_{xy, 0}^{-1}$ is a subset of $G_y \times G_x$, and it belongs to $A$ by assumption. Consequently, there must be a non-empty set $\Gamma \subseteq \kappa_{yx}$ such that

$$R_{xy, 0}^{-1} = \bigcup_{\beta \in \Gamma} R_{yx, \beta},$$

by Boolean Algebra Theorem 3.3. The pair $(e_x, e_y)$ belongs to the relation $R_{xy, 0}$, by the definition of $R_{xy, 0}$ (in fact, the pair is in $H_{xy, 0} \times K_{xy, 0}$, which is one of the rectangles that make up $R_{xy, 0}$), so the converse pair $(e_y, e_x)$ belongs to $R_{xy, 0}^{-1}$. For similar reasons, the relation $R_{yx, 0}$ contains the pair $(e_y, e_x)$, and it is the only relation of the form $R_{yx, \beta}$ that contains this pair, because the atomic relations in $A$ are pairwise disjoint. It follows from this observation and (10) that $0$ must be one of the indices in $\Gamma$. In other words,

$$R_{yx, 0} \subseteq R_{xy, 0}^{-1}.$$ 

Reverse the roles of $x$ and $y$ in this argument to obtain

$$R_{xy, 0} \subseteq R_{yx, 0}^{-1}.$$ 

Combine (11) with (12), and use the monotony and first involution laws for converse, to arrive at

$$R_{xy, 0}^{-1} \subseteq (R_{yx, 0}^{-1})^{-1} = R_{yx, 0} \subseteq R_{xy, 0}^{-1}.$$ 

The first and last terms are equal, so equality must hold everywhere. In particular, $R_{xy, 0}^{-1} = R_{yx, 0}$. This shows that condition (i) is satisfied for the pair $(x, y)$ in the case $\alpha = 0$. Invoke the equivalence of (i) with (iii) to conclude that (iii) holds for all pairs $(x, y)$.

It is natural to ask whether, in analogy with Identity Theorem 3.4, one can add another condition to those already listed in Converse Theorem 3.8, for example, the condition that $R_{xy, \alpha}$ be in $A$ for some $\alpha < \kappa_{xy}$. It turns out, however, that in the
absence of additional hypotheses, this condition is not equivalent to the conditions
listed in the lemma. We return to this question at the end of the next section.

Notice that condition (ii) in the preceding theorem, combined with the second
assertion of the theorem, provides a concrete method of computing the converse
of a relation $R_{xy,\alpha}$ in terms of the structure of the quotient group $G_x/H_{xy}$: just
compute the index $\beta$ such that $H_{xy,\alpha}^{-1} = H_{xy,\beta}$, for then we have $R_{xy,\alpha}^{-1} = R_{xy,\beta}$.
This method, in turn, provides a concrete way of computing the converse of any
relation in $A$.

The final and most difficult task is to characterize when the set $A$ is closed under
relational composition, and in particular, when it contains the composition of two
atomic relations. There is one case in which the relative product of two atomic
relations is empty, and therefore automatically in $A$.

**Lemma 3.9.** If $(x, y)$ and $(w, z)$ are in $\mathcal{E}$, and if $y \neq w$, then

$$R_{xy,\alpha} \cap R_{wz,\beta} = \emptyset$$

for all $\alpha < \kappa_{xy}$ and $\beta < \kappa_{wz}$.

Proof. Indeed,

$$R_{xy,\alpha} \subseteq G_x \times G_y \quad \text{and} \quad R_{wz,\beta} \subseteq G_w \times G_z,$$

by Lemma 3.2. Therefore,

$$R_{xy,\alpha} \cap R_{wz,\beta} \subseteq (G_x \times G_y) \cap (G_w \times G_z),$$

by monotony. If $y \neq w$, then the sets $G_y$ and $G_w$ are disjoint, and therefore the
relational composition of $G_x \times G_y$ and $G_w \times G_z$ is empty. □

To clarify the underlying ideas of the remaining case when $y = w$, we again
use cosets as indices of the atomic relations for a few moments. It is natural
to conjecture that (under suitable hypotheses) the composition of the relations
corresponding to cosets $H$ and $\bar{H}$ of $H_{xy}$ and $H_{yz}$ respectively is precisely the
relation corresponding to the group composition of the two cosets,

$$R_{xy,H} \cap R_{yz,\bar{H}} = R_{xz,H \circ \bar{H}}.$$

This form of the conjecture is incorrect. The first difficulty is that the cosets
$H$ and $\bar{H}$ live in disjoint groups, and therefore cannot be composed. To write the
conjecture in a meaningful way, one must first translate the coset $H$ to its copy,
the coset $K = \varphi_{xy}(H)$ of $K_{xy}$ in $G_y$, where $\bar{H}$ “lives”, and then compose this
translation with $\bar{H}$ to arrive at a coset

$$M = K \circ \bar{H}$$

of $K_{xy} \circ H_{yz}$.

The second difficulty is that compositions of subrelations of

$$G_x \times G_y \quad \text{and} \quad G_y \times G_z$$

should be a subrelation of $G_x \times G_z$, and therefore should have $xz$ as part of the
index. The relations indexed with $xz$ are constructed with the help of cosets of
$H_{xz}$, so it is necessary to translate the composite coset $K \circ \bar{H}$ from $G_y$ back to $G_x$
using the mapping $\varphi_{xy}^{-1}$, so that it can be written as a union of cosets of $H_{xz}$. A
more reasonable form of the original conjecture might look like

$$R_{xy,H} \cap R_{yz,\bar{H}} = R_{xz,\varphi_{xy}^{-1}[M]} = R_{xz,\varphi_{xy}^{-1}[K \circ \bar{H}]}.$$
The third difficulty is that the relation on the right side of this last equation has not been defined. At this point, we can only speak in a meaningful way about relations $R_{xz, \hat{H}}$ for single cosets $\hat{H}$ of $H_{xz}$. It therefore is necessary to rewrite the preceding conjecture in the form

$$R_{xy, H} \setminus R_{yz, H} = \bigcup\{ R_{xz, \hat{H}} : \hat{H} \subseteq \varphi^{-1}_{xy}(K \cdot \hat{H}) \}.$$ 

In order for the conjecture to be true, the subgroup $\varphi^{-1}_{xy}(K \cdot H_{yz})$ must include the subgroup $H_{xz}$, so that the coset $\varphi^{-1}_{xy}(K \cdot \hat{H})$ can really be written as a union of cosets $\hat{H}$ of $H_{xz}$. Moreover, it is natural to suspect that some sort of composition of the mappings $\varphi_{xy}$ and $\varphi_{yz}$ should equal the mapping $\varphi_{xz}$,

$$\varphi_{xy} \quad G_x/H_{xy} \rightarrow G_y/K_{xy}, \quad \varphi_{yz} \quad G_y/H_{yz} \rightarrow G_z/K_{yz},$$

$$\varphi_{xz} \quad G_x/H_{xz} \rightarrow G_z/K_{xz}.$$ 

However, the subgroup $K_{xy}$ may not coincide with the subgroup $H_{yz}$ at all, so it is not meaningful to speak about the composition of $\varphi_{xy}$ with $\varphi_{yz}$. In order to be able to compose quotient isomorphisms, one has first to form a common quotient group using the complex product of the subgroups, and then compose the induced isomorphisms $\hat{\varphi}_{xy}$ and $\hat{\varphi}_{yz}$,

$$\hat{\varphi}_{xy} \quad G_x/(H_{xy} \circ H_{xz}) \rightarrow G_y/(K_{xy} \circ H_{yz}) \rightarrow G_z/(K_{xz} \circ K_{yz}).$$

What really should be true is that the composition of the induced mappings $\hat{\varphi}_{xy}$ and $\hat{\varphi}_{yz}$ should equal the induced mapping $\hat{\varphi}_{xz}$. These conditions do indeed prove to be necessary and sufficient for the conjecture to hold. We formulated them in the conventional notation, using the subscripts of the cosets in place of the cosets.

**Theorem 3.10** (Composition Theorem). For all pairs $(x, y)$ and $(y, z)$ in $E$, the following conditions are equivalent.

(i) The relation $R_{xy,0} \setminus R_{yz,0}$ is in $A$.

(ii) For each $\alpha < \kappa_{xy}$ and each $\beta < \kappa_{yz}$, the relation $R_{xy,\alpha} \setminus R_{yz,\beta}$ is in $A$.

(iii) For each $\alpha < \kappa_{xy}$ and each $\beta < \kappa_{yz}$,

$$R_{xy,\alpha} \setminus R_{yz,\beta} = \bigcup\{ R_{xz,\gamma} : H_{xz,\gamma} \subseteq \varphi^{-1}_{xy}(K_{xy,\alpha} \circ H_{yz,\beta}) \}.$$ 

(iv) $H_{xz} \subseteq \varphi^{-1}_{xy}(K_{xy} \circ H_{yz})$ and $\varphi_{xy} \circ \varphi_{yz} = \varphi_{xz}$, where $\varphi_{xy}$ and $\varphi_{xz}$ are the mappings induced by $\varphi_{xy}$ and $\varphi_{xz}$ on the quotient of $G_x$ modulo the normal subgroup $\varphi^{-1}_{xy}(K_{xy} \circ H_{yz})$, while $\varphi_{yz}$ is the isomorphism induced by $\varphi_{yz}$ on the quotient of $G_y$ modulo the normal subgroup $K_{xy} \circ H_{yz}$.

Consequently, the set $A$ is closed under relational composition if and only if (iv) holds for all pairs $(x, y)$ and $(y, z)$ in $E$.

**Proof.** Let $P_0$ be the normal subgroup of $G_y$ generated by $K_{xy}$ and $H_{yz}$,

$$P_0 = K_{xy} \circ H_{yz}.$$
Choose a coset system \( \{P_\xi : \xi < \mu\} \) for \( P_0 \) in \( G_y \), and write
\[
(2) \quad M_\xi = \varphi_{xy}^{-1}[P_\xi]
\]
for \( \xi < \mu \). The isomorphism properties of \( \varphi_{xy} \) imply that
\[
(3) \quad M_0 = \varphi_{xy}^{-1}[P_0] = \varphi_{xy}^{-1}[K_{xy} \cdot H_{yz}]
\]
is a normal subgroup of \( G_z \) that includes \( H_{xy} \) (the inverse image of \( K_{xy} \) under \( \varphi_{xy} \)), and that the sequence \( \{M_\xi : \xi < \mu\} \) is a coset system for \( M_0 \) in \( G_z \). Moreover, the isomorphism \( \varphi_{xy} \) induces a quotient isomorphism \( \hat{\varphi}_{xy} \) from \( G_z/M_0 \) to \( G_y/P_0 \) that maps \( M_\xi \) to \( P_\xi \) for each \( \xi < \mu \). Similarly, write
\[
(4) \quad N_\xi = \varphi_{yz}[P_\xi]
\]
for \( \xi < \mu \), and observe that \( N_0 \) is a normal subgroup of \( G_z \) that includes \( K_{yz} \) (the image of \( H_{yz} \) under \( \varphi_{yz} \)), and that the sequence \( \{N_\xi : \xi < \mu\} \) is a coset system for \( N_0 \) in \( G_z \). Moreover, the isomorphism \( \varphi_{yz} \) induces a quotient isomorphism \( \hat{\varphi}_{yz} \) from \( G_y/N_0 \) to \( G_z/N_0 \) that maps \( P_\xi \) to \( N_\xi \) for each \( \xi < \mu \).

Since \( P_0 \) is a union of cosets of \( K_{xy} \), each coset of \( P_0 \) is a union of cosets of \( K_{xy} \). Thus, there is a partition \( \{\Gamma_\xi : \xi < \mu\} \) of \( \kappa_{xy} \) such that
\[
(5) \quad P_\xi = \bigcup\{K_{xy,\lambda} : \lambda \in \Gamma_\xi\}
\]
for each \( \xi < \mu \). Apply \( \varphi_{xy}^{-1} \) to both sides of \((5)\), and use the distributivity of inverse images, together with \((2)\), to obtain
\[
(6) \quad M_\xi = \bigcup\{H_{xy,\lambda} : \lambda \in \Gamma_\xi\}
\]
for \( \xi < \mu \). Carry out a completely analogous argument with \( H_{yz} \) in place of \( K_{xy} \) to obtain a partition \( \{\Delta_\xi : \xi < \mu\} \) of \( \kappa_{yz} \) such that
\[
(7) \quad P_\xi = \bigcup\{H_{yz,\lambda} : \lambda \in \Delta_\xi\}
\]
for each \( \xi < \mu \). Apply \( \varphi_{yz} \) to both sides of \((7)\), and use the distributivity of forward images, to obtain
\[
(8) \quad N_\xi = \bigcup\{K_{yz,\lambda} : \lambda \in \Delta_\xi\}
\]
for \( \xi < \mu \).

It is well known from group theory that the intersection of the two normal subgroups \( K_{xy} \) and \( H_{yz} \) in \( G_y \) is again a normal subgroup in \( G_y \), and that a coset system for this intersection is just the system of intersecting cosets,
\[
(9) \quad \{K_{xy,\lambda} \cap H_{yz,\chi} : \xi < \mu \text{ and } \lambda \in \Gamma_\xi \text{ and } \chi \in \Delta_\xi\}.
\]
In particular, the cosets in \((9)\) are non-empty and mutually disjoint. Moreover,\[
P_\xi = P_\xi \cap P_\xi = (\bigcup\{K_{xy,\lambda} : \lambda \in \Gamma_\xi\}) \cap (\bigcup\{H_{yz,\lambda} : \lambda \in \Delta_\xi\})
= \bigcup\{K_{xy,\lambda} \cap H_{yz,\chi} : \lambda \in \Gamma_\xi \text{ and } \chi \in \Delta_\xi\}
\]
for each \( \xi < \mu \), by \((5)\), \((7)\), and the distributivity of intersection. The composition of the relations
\[
(10) \quad (H_{xy,\delta} \circ H_{xy,\alpha}^{-1}) \times (K_{xy,\delta} \cap H_{yz,\chi}) \quad \text{and} \quad (K_{xy,\xi} \cap H_{yz,\delta}) \times (K_{yz,\alpha} \circ K_{yz,\beta})
\]
is empty if either $\delta \neq \zeta$ or $\chi \neq \vartheta$, and it is

\begin{equation}
(H_{xy,\delta} \circ H_{xy,\alpha}^{-1}) \times (K_{yz,\chi} \circ K_{yz,\vartheta})
\end{equation}

when $\delta = \zeta$ and $\chi = \vartheta$.

If $\rho < \mu$ and $\delta \in \Gamma_{\rho}$, then

\begin{equation}
K_{xy,\delta} = K_{xy,\delta} \cap P_{\rho} = K_{xy,\delta} \cap (\bigcup_{\chi \in \Delta_{\rho}} H_{yz,\chi}) = \bigcup_{\chi \in \Delta_{\rho}} (K_{xy,\delta} \cap H_{yz,\chi}),
\end{equation}

by (9), (7), and the distributivity of intersection. A completely analogous argument shows that

\begin{equation}
H_{yz,\vartheta} = \bigcup_{\zeta \in \Gamma_{\rho}} K_{xy,\zeta} \cap H_{yz,\vartheta}
\end{equation}

for $\rho < \mu$ and $\vartheta \in \Delta_{\gamma}$.

Without any special assumptions on $A$, we now prove that

\begin{equation}
K_{xy,\alpha} \subseteq P_{\xi} \quad \text{and} \quad H_{yz,\beta} \subseteq P_{\sigma}
\end{equation}

implies

\begin{equation}
R_{xy,\alpha} \mid R_{yz,\beta} = \bigcup_{\rho < \mu} M_{\rho} \times (N_{\rho} \circ N_{\xi} \circ N_{\sigma}).
\end{equation}

Successive transformations of terms lead to the following values for $R_{xy,\alpha} \mid R_{yz,\beta}$.

\begin{equation}
\bigcup_{\delta \in \kappa_{xy}} H_{xy,\delta} \times (K_{xy,\delta} \circ K_{xy,\alpha}) \bigg| \bigg[ \bigcup_{\theta \in \kappa_{yz}} H_{yz,\theta} \times (K_{yz,\theta} \circ K_{yz,\vartheta}) \bigg],
\end{equation}

by the definitions of $R_{xy,\alpha}$ and $R_{yz,\beta}$.

\begin{equation}
\bigcup_{\rho < \mu} \bigcup_{\delta \in \Gamma_{\rho}} H_{xy,\delta} \times (K_{xy,\delta} \circ K_{xy,\alpha}) \bigg| \bigg[ \bigcup_{\gamma < \mu} \bigcup_{\theta \in \Delta_{\gamma}} H_{yz,\theta} \times (K_{yz,\theta} \circ K_{yz,\vartheta}) \bigg],
\end{equation}

because the sets $\Gamma_{\rho}$ (for $\rho < \mu$) and $\Delta_{\gamma}$ (for $\gamma < \mu$) partition $\kappa_{xy}$ and $\kappa_{yz}$ respectively.

\begin{equation}
\bigcup_{\rho,\delta} (H_{xy,\delta} \circ H_{xy,\alpha}^{-1}) \times K_{xy,\delta} \bigg| \bigg[ \bigcup_{\gamma,\vartheta} H_{yz,\theta} \times (K_{yz,\theta} \circ K_{yz,\vartheta}) \bigg],
\end{equation}

by Lemma 3.6(i).

\begin{equation}
\bigcup_{\rho,\delta} (H_{xy,\delta} \circ H_{xy,\alpha}^{-1}) \times (\bigcup_{\chi \in \Delta_{\rho}} K_{xy,\delta} \cap H_{yz,\chi}) \bigg| \bigg[ \bigcup_{\gamma,\vartheta} (\bigcup_{\zeta \in \Gamma_{\gamma}} K_{xy,\zeta} \cap H_{yz,\vartheta}) \times (K_{yz,\theta} \circ K_{yz,\vartheta}) \bigg],
\end{equation}

by (12) and (13).

\begin{equation}
\bigcup_{\rho,\delta,\chi} (H_{xy,\delta} \circ H_{xy,\alpha}^{-1}) \times (K_{xy,\delta} \cap H_{yz,\chi}) \bigg| \bigg[ \bigcup_{\gamma,\vartheta,\zeta} (K_{xy,\zeta} \cap H_{yz,\vartheta}) \times (K_{yz,\theta} \circ K_{yz,\vartheta}) \bigg],
\end{equation}

by the distributivity of Cartesian multiplication.

\begin{equation}
\bigcup_{\rho,\delta,\chi,\gamma,\vartheta,\zeta} ((H_{xy,\delta} \circ H_{xy,\alpha}^{-1}) \times (K_{xy,\delta} \cap H_{yz,\chi})) \bigg| \bigg[ (K_{xy,\zeta} \cap H_{yz,\vartheta}) \times (K_{yz,\theta} \circ K_{yz,\vartheta}) \bigg],
\end{equation}

by (12) and (13).
by the distributivity of relational composition. The relational composition inside the brackets of the preceding union is precisely of the form (10). Apply the conclusions of (10) about this relative product, and in particular use (11), the distributivity of complex group composition, (6), and (8) to obtain

\[
R_{xy,\alpha} \mid R_{yz,\beta} = \bigcup_{\rho,\delta,\chi} (H_{xy,\delta} \circ H_{xy,\alpha}^{-1}) \times (K_{yz,\chi} \circ K_{yz,\beta})
\]

The fourth step is a consequence of the following well-known property of cosets: \(K_{yz}\) is a normal subgroup of \(N_{\rho}\), so if the coset \(K_{yz,\beta}\) is included in the coset \(N_{\sigma}\), then

\[
N_{\rho} \circ K_{yz,\beta} = N_{\rho} \circ N_{\sigma}
\]

There must then be a partition \(\langle \Psi_{\rho} : \rho < \mu \rangle\) of \(\kappa_{xz}\) such that

\[
M_{\rho} = \bigcup_{\delta \in \Psi_{\rho}} H_{xz,\delta}
\]

and the distributivity of forward images, to see that the function \(\tilde{\varphi}_{xz}\) defined by

\[
\tilde{\varphi}_{xz}(M_{\rho}) = \varphi_{xz}[H_{xz,\delta}] = K_{xz,\delta}
\]

is a well-defined isomorphism on the quotient group \(G_x/M_0\). A simple computation using (2) and (4) shows that

\[
(\tilde{\varphi}_{xy} \mid \tilde{\varphi}_{yz})(M_{\rho}) = \varphi_{yz}(\tilde{\varphi}_{xy}(M_{\rho})) = \varphi_{yz}(P_{\rho}) = N_{\rho}
\]

for each \(\rho < \mu\). Combine (19) and (18) to conclude that the following three conditions are equivalent:

\[
\tilde{\varphi}_{xy} \mid \tilde{\varphi}_{yz} = \tilde{\varphi}_{xz}, \quad \tilde{\varphi}_{xz}(M_{\rho}) = N_{\rho} \text{ for all } \rho, \quad N_{\rho} = \bigcup_{\delta \in \Psi_{\rho}} K_{xz,\delta} \text{ for all } \rho.
\]

Turn now to the task of establishing the equivalences in the theorem. Assume (iv), with the goal of deriving (iii). Fix \(\alpha < \kappa_{xy}\) and \(\beta < \kappa_{yz}\), and choose \(\xi, \sigma < \mu\) so that \(K_{xy,\alpha} \subseteq P_{\xi}\) and \(H_{yz,\beta} \subseteq P_{\sigma}\). The multiplication rules for cosets imply that

\[
P_{\xi} \circ P_{\sigma} = K_{xy,\alpha} \circ H_{yz,\beta}.
\]

Choose \(\pi\) so that

\[
P_{\xi} \circ P_{\sigma} = P_{\pi},
\]

and use the isomorphism properties of \(\tilde{\varphi}_{yz}\), together with (14), to obtain

\[
N_{\xi} \circ N_{\sigma} = N_{\pi}.
\]
Compute:

\[ R_{xy,\alpha} \mid R_{yz,\beta} = \bigcup_{\rho < \mu} M_\rho \times (N_\rho \circ N_\pi) \]
\[ = \bigcup_{\rho} (\bigcup_{\delta \in \Psi_\rho} H_{xz,\delta}) \times (N_\rho \circ N_\pi) \]
\[ = \bigcup_{\rho,\delta} H_{xz,\delta} \times (N_\rho \circ N_\pi) \]
\[ = \bigcup_{\rho,\delta} H_{xz,\delta} \times (K_{xz,\delta} \circ N_\pi) \]
\[ = \bigcup_{\rho,\delta} H_{xz,\delta} \times [K_{xz,\delta} \circ (\bigcup_{\gamma \in \Psi_\rho} K_{xz,\gamma})] \]
\[ = \bigcup_{\rho} \bigcup_{\delta < \kappa_{xz}} H_{xz,\delta} \times (K_{xz,\delta} \circ K_{xz,\gamma}) \]
\[ = \bigcup_{\rho} R_{xz,\gamma}. \]

The first equality uses (14) and (23), the second uses (16), the third uses the distributivity of Cartesian multiplication, the fourth uses the multiplication rules \(\pi\) which implies that the last equation in (20) holds with \(\rho \circ \mu\) in place of \(\rho\), the fifth uses the assumption of (iv), which implies that the last equation in (20) holds, the distributivity of complex group composition, the seventh uses the fact that the sets \(\Psi_\rho\) (for \(\rho < \mu\)) partition \(\kappa_{xz}\), and the last uses the definition of \(R_{xz,\gamma}\).

Summarizing,

\[ R_{xy,\alpha} \mid R_{yz,\beta} = \bigcup_{\gamma \in \Psi_\pi} R_{xz,\gamma}. \tag{24} \]

In order to complete the derivation of (iii), it is necessary to characterize the relations \(R_{xz,\gamma}\) such that \(\gamma \in \Psi_\pi\). First,

\[ M_\pi = \varphi_{xy}^{-1} [P_\pi] = \varphi_{xy}^{-1} [P_\rho \circ P_\sigma] = \varphi_{xy}^{-1} [K_{xy,\alpha} \circ H_{yz,\beta}], \tag{25} \]

by (2) (with \(\pi\) in place of \(\rho\)), (22), and (21). Second,

\[ M_\pi = \varphi_{xz}^{-1} [N_\pi] = \varphi_{xz}^{-1} \left( \bigcup_{\gamma \in \Psi_\pi} K_{xz,\delta} \right) = \bigcup_{\gamma \in \Psi_\pi} \varphi_{xz}^{-1} [K_{xz,\delta}] = \bigcup_{\gamma \in \Psi_\pi} H_{xz,\delta}, \tag{26} \]

by the assumption in (iv), which implies that the second and third equations in (20) hold, the distributivity of inverse images, and (14). Combine (26) and (20) to arrive at the equivalence of the three conditions

\[ H_{xz,\gamma} \subseteq \varphi_{xy}^{-1} [K_{xy,\alpha} \circ H_{yz,\beta}], \quad H_{xz,\gamma} \subseteq M_\pi, \quad \gamma \in \Psi_\pi. \]

The equivalence of the first and last formulas permits us to rewrite (24) in the desired form of (iii):

\[ R_{xy,\alpha} \mid R_{yz,\beta} = \bigcup \{ R_{xz,\gamma} : \gamma < \kappa_{xz} \text{ and } H_{xz,\gamma} \subseteq \varphi_{xy}^{-1} [K_{xy,\alpha} \circ H_{yz,\beta}] \}. \]

The implication from (iii) to (ii) follows from the definition of \(A\) as the set of arbitrary unions of atomic relations (see Boolean Algebra Theorem 3.3), and the implication from (ii) to (i) is obvious. Consider finally the implication from (i) to (iv). Under the assumption of (i), the definition of \(A\) implies that \(R_{xy,0} \mid R_{yz,0}\) must be a union of atomic relations of the form \(R_{xz,\zeta}\). In more detail, the inclusions and the equality

\[ K_{xy,0} \subseteq P_0, \quad H_{yz,0} \subseteq P_0, \quad N_\rho \circ N_0 \circ N_0 = N_\rho \]

(which hold by (23), (11), and the fact that \(N_0\) is the identity coset), together with (14), imply (without using the hypothesis of (i)) that

\[ R_{xy,0} \mid R_{yz,0} = \bigcup_{\rho < \mu} M_\rho \times N_\rho, \tag{27} \]
The cosets $M_\rho$ and $N_\rho$ are subsets of $G_x$ and $G_z$ respectively, for each $\rho < \mu$ (see (2) and (4)), so the right-hand side—and therefore also the left-hand side—of (27) must be a subset of the rectangle $G_x \times G_z$. The only atomic relations in $A$ that are not disjoint from this rectangle are those of the form $R_{xz,\zeta}$. Conclusion: there is a subset $\Phi$ of $\kappa_{xz}$ such that

$$R_{xy,0} | R_{yz,0} = \bigcup_{\zeta \in \Phi} R_{xz,\zeta}.$$  

The pair $(e_x, e_z)$ belongs to the rectangle $M_0 \times N_0$, and therefore also to the composition $R_{xy,0} | R_{yz,0}$, by (27). The pair also belongs to the rectangle $H_{xz,0} \times K_{xz,0} = H_{xz} \times K_{xz}$, and therefore to the relation

$$R_{xz,0} = \bigcup_{\gamma \in \kappa_{xz}} H_{xz,\gamma} \times K_{xz,\gamma}.$$  

The relations of the form $R_{xz,\zeta}$ are pairwise disjoint, and $R_{xz,0}$ is the only one that contains the pair $(e_x, e_z)$. In view of (28), the only way this can happen is if 0 is one of the indices in $\Phi$, so that

$$R_{xz,0} \subseteq R_{xy,0} | R_{yz,0}.$$  

Use (29) and (27) to rewrite (30) in the form

$$\bigcup_{\gamma \in \kappa_{xz}} H_{xz,\gamma} \times K_{xz,\gamma} \subseteq \bigcup_{\rho < \mu} M_\rho \times N_\rho.$$  

In view of Lemma 3.5, the inclusion in (31) implies that for every $\gamma < \kappa_{xz}$ there is a $\rho < \mu$ such that

$$H_{xz,\gamma} \subseteq M_\rho \quad \text{and} \quad K_{xz,\gamma} \subseteq N_\rho.$$  

In particular, when $\gamma = 0$, the subgroup $H_{xz,0}$ is included in $M_\rho$ for some $\rho < \mu$. This inclusion forces the group identity element $e_x$ to belong to $M_\rho$, since it belongs to $H_{xz,0}$. The only coset of $M_0$ that includes the group identity element is $M_0$ itself, so $\rho = 0$, that is to say, (15) holds.

On the basis of (15) alone, we saw that a partition $\{\Psi_\rho : \rho < \mu\}$ of $\kappa_{xz}$ exists for which (16) holds. The derivations of (16)–(20) are based only on (19) and (15), so these statements hold in the present situation as well. It is evident from (16) that the coset $H_{xz,\delta}$ is included in $M_\rho$ for each $\delta \in \Psi_\rho$, and therefore the corresponding coset $K_{xz,\delta}$ must be included in $N_\rho$ for each $\delta \in \Psi_\rho$ by (32). Thus,

$$\bigcup_{\delta \in \Psi_\rho} K_{xz,\delta} \subseteq N_\rho$$  

for each $\rho < \mu$. The cosets $K_{xz,\delta}$ (for $\rho < \mu$ and $\delta \in \Psi_\rho$) form a partition of $G_z$, as do the cosets $N_\rho$ (for $\rho < \mu$). These two facts force the inclusion in (33) to be an equality. Use this equality and the equivalence of the first and third formulas in (20) to conclude that $\hat{\phi}_{xy} | \hat{\phi}_{yz} = \hat{\phi}_{xz}$, as desired in (iv).

Turn now to the final assertion of the theorem. If $A$ is closed under relational composition, then (ii) holds for all pairs $(x, z)$ and $(y, z)$ in $E$, so that (iv) must hold by the equivalence of (ii) and (iv) proved above.

On the other hand, if (iv) holds for all pairs $(x, y)$ and $(y, z)$ in $E$, then (ii) holds as well. Combine this with Lemma 3.9 to conclude that $A$ is closed under the composition of any two of its atomic relations. The elements in $A$ are just the various possible unions of atomic relations, and the composition of two such unions is a union of compositions of atomic relations, by the distributivity of relational composition, and hence a union of elements in $A$. Since $A$ is closed under arbitrary
unions, it follows that the composition of any two elements in $A$ is again an element in $A$, as was to be shown. □

Notice that part (iii) of the previous theorem provides a concrete way of computing the composition of any two relations $R_{xy,\alpha}$ and $R_{yz,\beta}$ in terms of the structure of the quotient group $G_y/(K_{xy} \cdot H_{yz})$, the mapping $\varphi_{xy}$, and the cosets $H_{xz,\gamma}$. One first computes the complex product $K_{xy,\alpha} \cdot H_{yz,\beta}$, and then the inverse image of this complex under $\varphi_{xy}^{-1}$. This inverse image is a union of cosets $H_{xz,\gamma}$, and one computes the set $I$ of indices $\gamma$ for which the corresponding cosets are part of this union.

It is natural to ask whether, in condition (i) of the Composition Theorem, one can replace the condition that $R_{xy,0} \mid R_{yz,0}$ be in $A$ by the condition that $R_{xy,\alpha} \mid R_{yz,\beta}$ be in $A$ for some $\alpha < \kappa_{xy}$ and some $\beta < \kappa_{yx}$. It turns out that an additional hypothesis is needed for this to be true. We will return to this matter at the end of the next section.

Lemma 3.11. Suppose $(x, y)$ and $(y, z)$ are in $E$. If $\varphi_{yx} = \varphi_{xy}^{-1}$, then

$$H_{xz} \subseteq \varphi_{xy}^{-1}[K_{xy} \cdot H_{yz}] \quad \text{and} \quad H_{yz} \subseteq \varphi_{yx}^{-1}[K_{yx} \cdot H_{xz}]$$

together imply

$$\varphi_{xy}[H_{xy} \cdot H_{xz}] = K_{xy} \cdot H_{yz} = H_{yx} \cdot H_{yz}.$$  

Proof. The function $\varphi_{xy}$ is an isomorphism from $G_x/H_{xy}$ onto $G_y/K_{xy}$, and the complex product $K_{xy} \cdot H_{yz}$ is a normal subgroup of $G_y$ that includes $K_{xy}$. Therefore, the inverse image

$$\varphi_{xy}^{-1}[K_{xy} \cdot H_{yz}]$$

is a normal subgroup of $G_x$ that includes $\varphi_{xy}^{-1}[K_{xy}] = H_{xy}$. By assumption, (1) also includes $H_{xz}$, so

$$H_{xy} \cdot H_{xz} \subseteq \varphi_{xy}^{-1}[K_{xy} \cdot H_{yz}].$$

Apply $\varphi_{xy}$ to both sides of this inclusion to obtain

$$\varphi_{xy}[H_{xy} \cdot H_{xz}] \subseteq K_{xy} \cdot H_{yz}.$$  

The same argument with the roles of $x$ and $y$ reversed yields

$$\varphi_{yx}[H_{yx} \cdot H_{xz}] \subseteq K_{yx} \cdot H_{xz}.$$  

The assumption $\varphi_{yx} = \varphi_{xy}^{-1}$ implies that

$$H_{yx} = K_{xy} \quad \text{and} \quad K_{yx} = H_{xy}$$

(see Convention 3.7). Use these three equalities to rewrite the inclusion in (3) as

$$\varphi_{xy}^{-1}[K_{xy} \cdot H_{yz}] \subseteq H_{xy} \cdot H_{xz}.$$  

Combine (4) with (2) and (3) to arrive at the desired conclusion. □

Lemma 3.12. If $\varphi_{yx} = \varphi_{xy}^{-1}$ for all pairs $(x, y)$ in $E$, and if

$$\varphi_{xy}[H_{xy} \cdot H_{xz}] = K_{xy} \cdot H_{yz}$$

for all $(x, y)$ and $(y, z)$ in $E$, then

$$\varphi_{yz}[K_{xy} \cdot H_{yz}] = K_{xz} \cdot K_{yz} \quad \text{and} \quad \varphi_{xz}[H_{xy} \cdot H_{xz}] = K_{xz} \cdot K_{yz}.$$
RELATION ALGEBRAS AND GROUPS

Proof. To derive the first equation, observe that if the pairs \((x,y)\) and \((y,z)\) are in \(E\) then so are the pairs \((x,z)\) and \((z,x)\). Use the assumption that \(\varphi_{yx} = \varphi_{xy}^{-1}\), Convention 3.7, the commutativity of normal subgroups, and the hypotheses of the lemma (with \(y, z,\) and \(x\) in place of \(x, y,\) and \(z\) respectively) to arrive at

\[
\varphi_{yz}[K_{xy} \circ H_{yz}] = \varphi_{yz}[H_{yx} \circ H_{yz}] = \varphi_{yz}[H_{yx} \circ H_{yz}] = K_{yz} \circ H_{zx} = K_{yz} \circ K_{xz}.
\]

An entirely analogous argument yields

\[
\varphi_{zx}[H_{xy} \circ H_{xz}] = \varphi_{zx}[H_{xz} \circ H_{xy}] = K_{xz} \circ H_{zy} = K_{xz} \circ K_{yz}.
\]

\(\square\)

Theorem 3.13 (Image Theorem). If \(A\) is closed under converse and composition, then

\[
\varphi_{xy}[H_{xy} \circ H_{xz}] = K_{xy} \circ H_{yz}, \quad \varphi_{yz}[K_{xy} \circ H_{yz}] = K_{zx} \circ K_{yz}, \quad \varphi_{zx}[H_{xy} \circ H_{xz}] = K_{xz} \circ K_{yz}
\]

for all \((x,y)\) and \((y,z)\) in \(E\). In other words,

(i) \(\varphi_{xy}\) maps \(G_x/(H_{xy} \circ H_{xz})\) isomorphically to \(G_y/(K_{xy} \circ H_{yz})\),
(ii) \(\varphi_{yz}\) maps \(G_y/(K_{xy} \circ H_{yz})\) isomorphically to \(G_z/(K_{zx} \circ K_{yz})\),
(iii) \(\varphi_{zx}\) maps \(G_z/(H_{xy} \circ H_{xz})\) isomorphically to \(G_x/(K_{xy} \circ H_{yz})\).

Proof. Assume \(A\) is closed under converse and composition, and consider pairs \((x,y)\) and \((y,z)\) in \(E\). The assumed closure of \(A\) under converse means that part (iii) of the Converse Theorem 3.11 may be applied to obtain

(1) \(\varphi_{yz} = \varphi_{xy}^{-1}\).

The assumed closure of \(A\) under composition means that part (iv) of the Composition Theorem 3.10 may be applied to obtain

\[H_{xz} \subseteq \varphi_{xy}^{-1}[K_{xy} \circ H_{yz}] \quad \text{and} \quad H_{yz} \subseteq \varphi_{yz}^{-1}[K_{xy} \circ H_{xz}].\]

Invoke Lemma 3.12 to obtain

(2) \(\varphi_{xy}[H_{xy} \circ H_{xz}] = K_{xy} \circ H_{yz}\).

This argument establishes the first equation for all pairs \((x,y)\) and \((y,z)\) in \(E\). Use Lemma 3.11 to obtain the second and third equations.

The mappings \(\varphi_{xy}\) and \(\varphi_{xz}\) are defined to be the isomorphisms induced on the quotient group of \(G_x\) modulo \(\varphi_{xy}^{-1}[K_{xy} \circ H_{yz}]\) by the isomorphisms \(\varphi_{xy}\) and \(\varphi_{xz}\) respectively, and \(\varphi_{yz}\) is defined to be the isomorphism induced on the quotient group of \(G_y\) modulo \(K_{xy} \circ H_{yz}\) by the isomorphism \(\varphi_{yz}\), by part (iv) of the Composition Theorem 3.10. In view of the preceding proof, this immediately gives assertions (i)–(iii) of the theorem.

\(\square\)

4. Group frames

In the preceding section, necessary and sufficient conditions are given for a Boolean algebra \(A\) of binary relations constructed from a group pair \(F\) to contain the identity relation and to be closed under the operations of relational converse and composition. In each case, one of these conditions is formulated strictly in terms of the quotient isomorphisms. It is natural to single out the group pairs that satisfy these quotient isomorphism conditions, because precisely these group pairs lead to algebras of binary relations and in fact to measurable relation algebras.
Definition 4.1. A group frame is a group pair
\[ F = ((G_x : x \in I), (\varphi_{xy} : (x, y) \in E)) \]
satisfying the following frame conditions for all pairs \((x, y)\) and \((y, z)\) in \(E\).

(i) \(\varphi_{xx}\) is the identity automorphism of \(G_x/\{e_x\}\) for all \(x\).
(ii) \(\varphi_{yx} = \varphi_{xy}^{-1}\).
(iii) \(\varphi_{xy}[H_{xy} \circ H_{xz}] = K_{xy} \circ H_{yz}\).
(iv) \(\hat{\varphi}_{xy} | \hat{\varphi}_{yz} = \hat{\varphi}_{xz}\).

\[ \Box \]

Given a group frame \(F\), let \(A\) be the collection of all possible unions of relations of the form \(R_{xy,\alpha}\) for \((x, y)\) in \(E\) and \(\alpha < \kappa_{xy}\). Call \(A\) the set of frame relations constructed from \(F\).

Theorem 4.2 (Group Frame Theorem). If \(F\) is a group frame, then the set of frame relations constructed from \(F\) is the universe of a complete, atomic, measurable set relation algebra with base set and unit
\[ U = \bigcup\{G_x : x \in I\} \quad \text{and} \quad E = \bigcup\{G_x \times G_y : (x, y) \in E\} \]
respectively. The atoms in this algebra are the relations of the form \(R_{xy,\alpha}\), and the subidentity atoms are the relations of the form \(R_{xx,0}\). The measure of \(R_{xx,0}\) is just the cardinality of the group \(G_x\).

Proof. Let \(A\) be the set of frame relations constructed from \(F\). This set is the universe of a complete and atomic Boolean algebra of binary relations with base set \(U\) and unit \(E\), and its atoms are the relations of the form \(R_{xy,\alpha}\), by Boolean Algebra Theorem 3.3. The identity relation \(id_U\) is in \(A\), and the subidentity atoms are the relations of the form \(R_{xx,0}\), by Theorem 3.3 Identity Theorem 3.4 and frame condition (i). The closure of \(A\) under the operations of converse and composition follows from Converse Theorem 3.8, Composition Theorem 3.10 and frame conditions (ii)–(iv).

The measure of a subidentity atom \(R_{xx,0}\) is, by definition, the number of non-zero functional atoms below the square
\[ R_{xx,0} | E | R_{xx,0} = G_x \times G_x. \]
These non-zero functional atoms are just the relations \(R_{xx,\alpha}\) for \(\alpha < \kappa_{xx}\), that is to say, they are just the Cayley representations of the elements in \(G_x\), by Partition Lemma 3.2. Consequently, there are as many of them as there are elements in \(G_x\). \(\Box\)

The theorem justifies the following definition.

Definition 4.3. Suppose that \(F\) is a group frame. The set relation algebra constructed from \(F\) in Group Frame Theorem 4.2 is called the (full) group relation algebra on \(F\) and is denoted by \(\mathfrak{G}[F]\) (and its universe by \(G[F]\)). A general group relation algebra is defined to be an algebra that is embeddable into a full group relation algebra.

The task of verifying that a given group pair satisfies the frame conditions, and therefore yields a full group relation algebra, that is to say, it yields an example of a measurable relation algebra, can be complicated and tedious. Fortunately, a few simplifications are possible. To describe them, it is helpful to assume that the group
index set $I$ is linearly ordered, say by a relation $\prec$. Roughly speaking, under
the assumption of condition (i), condition (ii) holds in general just in case it holds for
each pair $(x, y)$ in $E$ with $x < y$. In other words, under the assumption of (i), it is
not necessary to check condition (ii) for the case $x = y$, nor is it necessary to check
both cases $(x, y)$ and $(y, x)$ when $x \neq y$. Also, under the assumption of condition
(i) and the modified form of condition (ii) just described, conditions (iii) and (iv)
will hold in general if they hold for all pairs $(x, y)$ and $(y, x)$ in $E$ with $x < y < z$. In
other words, under the assumption of (i) and the modified (ii), it is not necessary
to check conditions (iii) and (iv) in any case in which at least two of the three
indices $x$, $y$, and $z$ are equal, nor is it necessary to check all six permutations of an
appropriate triple $(x, y, z)$ of distinct indices. Here is the precise formulation of the
theorem.

**Theorem 4.4.** A group pair $F$ is a group frame if and only if the following four
conditions are satisfied.

(i) $\varphi_{xx}$ is the identity automorphism of $G_x/\{e_x\}$ for every $x$ in $I$.

(ii) $\varphi_{yx} = \varphi_{xy}^{-1}$ for every pair $(x, y)$ in $E$ with $x < y$.

(iii) $\varphi_{xy}[H_{xy} \circ H_{xz}] = K_{xy} \circ H_{yz}$ and $\varphi_{yz}[K_{xy} \circ H_{yz}] = K_{xz} \circ K_{yz}$ for all pairs $(x, y)$ and $(y, z)$ in $E$ with $x < y < z$.

(iv) $\varphi_{xy} \mid \varphi_{yz} = \varphi_{xz}$ for all pairs $(x, y)$ and $(y, z)$ in $E$ with $x < y < z$.

**Proof.** By its very definition, a frame must satisfy conditions (i)–(iv) of the theorem.
To establish the reverse implication, suppose that

$$F = ((G_x : x \in I), (\varphi_{xy} : (x, y) \in E))$$

is a group pair satisfying conditions (i)–(iv) of the theorem. It must be shown that
the four frame conditions hold. Obviously, the first frame condition holds, since it
coincides with condition (i) of the theorem. To verify the second frame condition,
assume that $(x, y)$ is a pair in $E$. If $x = y$, then $\varphi_{xy}$ is the identity automorphism,
by condition (i) of the theorem, and therefore

$$\varphi_{yx} = \varphi_{xy}^{-1}.$$  

If $x < y$, then (1) holds, by condition (i) of the theorem. If $y < x$, then $\varphi_{xy} = \varphi_{yx}^{-1}$,
by condition (i) of the theorem (with the roles of $x$ and $y$ reversed), so (1) must
also hold.

Turn now to the task of verifying the last two frame conditions. Assume that
$(x, y)$ and $(y, z)$ are pairs in $E$, and consider first the case when $x = y$. The mapping
$\varphi_{xy}$ is then the identity automorphism of $G_x/\{e_x\}$, by condition (i), so that

$$H_{xy} = H_{xx} = \{e_x\} = K_{xx} = K_{xy}, \quad H_{xz} = H_{yz}, \quad K_{xz} = K_{yz},$$

and therefore

$$H_{xy} \circ H_{xz} = H_{xz}, \quad K_{xy} \circ H_{yz} = H_{yz} = H_{xz}, \quad K_{xz} \circ K_{yz} = K_{yz} \circ K_{yz}.$$  

It follows that

$$\varphi_{xy}[H_{xy} \circ H_{xz}] = \varphi_{xx}[H_{xz}] = H_{xz} = K_{xy} \circ H_{yz}$$

and

$$\varphi_{yz}[K_{xy} \circ H_{yz}] = \varphi_{yz}[H_{yz}] = K_{yz} = K_{xz} \circ K_{yz}.$$
For the same reasons, the isomorphism \( \hat{\varphi}_{xy} \) induced by \( \varphi_{xy} \) on the quotient group \( G_x/(H_{xy} \cdot H_{xz}) \) must coincide with the identity automorphism of \( G_x/H_{xz} \), the isomorphism \( \hat{\varphi}_{yz} \) induced by \( \varphi_{yz} \) on the quotient group \( G_y/(K_{xy} \cdot H_{yz}) \) must coincide with \( \varphi_{yz} \), and the isomorphism \( \hat{\varphi}_{xz} \) induced by \( \varphi_{xz} \) on the quotient group \( G_x/(H_{xy} \cdot H_{xz}) \) must coincide with the isomorphism \( \varphi_{xz} \). Consequently,

\[
\hat{\varphi}_{xy} | \hat{\varphi}_{yz} = \varphi_{yz} = \varphi_{xz} = \hat{\varphi}_{xz}.
\]

The case when \( y = z \) is treated in a completely symmetric fashion. Consider, next, the case when \( x = z \). The mapping \( \varphi_{xz} \) is then the identity automorphism of \( G_x/\{e_x\} \), by condition (i), so that

\[
H_{xz} = H_{xx} = \{e_x\} = K_{xx} = K_{xz}, \quad H_{yz} = H_{yx}, \quad K_{yz} = K_{yx},
\]

and therefore

\[
H_{xy} \cdot H_{xz} = H_{xy}, \quad K_{xy} \cdot H_{yz} = K_{xy} \cdot H_{yz} = K_{xy} \cdot K_{xy} =
\]

\[
K_{xy} = H_{yx} = H_{yz}, \quad K_{xz} \cdot K_{yz} = K_{yz} = K_{yx}.
\]

In the second string of equations, use is being made of the fact that the second frame condition holds, and therefore \( K_{xy} = H_{yx} \) (see the remark preceding Theorem 3.8). It follows that

\[
\varphi_{xy}[H_{xy} \cdot H_{xz}] = \varphi_{xy}[H_{xy}] = K_{xy} = K_{xy} \cdot H_{yz}
\]

and

\[
\varphi_{yz}[K_{xy} \cdot H_{yz}] = \varphi_{xy}[K_{xy}] = \varphi_{yz}[H_{yz}] = K_{yz} = K_{xz} \cdot K_{yz}.
\]

These equations imply that the isomorphism \( \hat{\varphi}_{xy} \) induced by \( \varphi_{xy} \) on the quotient group \( G_x/(H_{xy} \cdot H_{xz}) \) must coincide with \( \varphi_{xy} \), the isomorphism \( \hat{\varphi}_{yz} \) induced by \( \varphi_{yz} \) on the quotient group \( G_y/(K_{xy} \cdot H_{yz}) \) must coincide with \( \varphi_{yz} \), which is the same as \( \varphi_{yz} \) and therefore also the same as \( \varphi_{xy}^{-1} \), by frame condition (ii) (which has been shown to hold by conditions (i) and (ii) of the theorem) and Converse Theorem 3.8, and the isomorphism \( \hat{\varphi}_{xz} \) induced by \( \varphi_{xz} \) on the quotient group \( G_x/(H_{xy} \cdot H_{xz}) \) must coincide with the identity automorphism of \( G_x/H_{xy} \). Consequently,

\[
\hat{\varphi}_{xy} | \hat{\varphi}_{yz} = \varphi_{xy} | \varphi_{yz} = \varphi_{xy} | \varphi_{xz}^{-1} = \hat{\varphi}_{xz}.
\]

Assume now, that the indices \( x, y, \) and \( z \) are all distinct from one another. If \( x < y < z \), then

\[\begin{align*}
(2) & \quad \varphi_{xy}[H_{xy} \cdot H_{xz}] = K_{xy} \cdot H_{yz}, \quad \varphi_{yz}[K_{xy} \cdot H_{yz}] = K_{xz} \cdot K_{yz}, \\
(3) & \quad \hat{\varphi}_{xy} | \varphi_{yz} = \hat{\varphi}_{xz},
\end{align*}\]

by conditions (iii) and (iv) of the theorem, where \( \hat{\varphi}_{xy} \) is the isomorphism from \( G_x/(H_{xy} \cdot H_{xz}) \) to \( G_y/(K_{xy} \cdot H_{yz}) \) that is induced by \( \varphi_{xy} \), while \( \hat{\varphi}_{yz} \) is the isomorphism from \( G_y/(K_{xy} \cdot H_{yz}) \) to \( G_z/(K_{xz} \cdot K_{yz}) \) that is induced by \( \varphi_{yz} \). It follows from (3) that \( \hat{\varphi}_{xz} \) must be the isomorphism from \( G_x/(H_{xy} \cdot H_{xz}) \) to \( G_z/(K_{xz} \cdot K_{yz}) \) that is induced by \( \varphi_{xz} \) (this is not part of the assumption in condition (iii)). Consequently,

\[
\varphi_{xz}[H_{xy} \cdot H_{xz}] = K_{xz} \cdot K_{yz}.
\]

The corresponding equations for the remaining pairs of mappings follow readily from (2), (4), and condition (ii). In more detail,

\[\begin{align*}
(5) & \quad \varphi_{yx} = \varphi_{xy}^{-1}, \quad \varphi_{yz} = \varphi_{yz}^{-1}, \quad \varphi_{xz} = \varphi_{xz}^{-1}.
\end{align*}\]
by (the already verified) frame condition (ii). In particular,

\[ H_{zx} = K_{xz}, \quad K_{zx} = H_{xz}, \quad H_{yz} = K_{yz}, \quad K_{zy} = H_{yz}, \quad H_{yx} = K_{xy}, \quad K_{yx} = H_{xy}. \]

Apply \( \varphi_{zx} \) to both sides of (4), and use (5), to obtain

\[ \varphi_{zx}[K_{xz} \circ K_{yz}] = H_{xy} \circ H_{xz}. \]

With the help of (6), rewrite this last equation as

\[ \varphi_{zx}[H_{zx} \circ H_{zy}] = K_{zx} \circ H_{xy}. \]

Use (6) and (2) repeatedly, together with the fact that the subgroups involved are normal, to obtain

\[ \varphi_{xy}[K_{zx} \circ H_{xy}] = \varphi_{xy}[H_{xy} \circ H_{xz}] = K_{xy} \circ K_{xy} = K_{xy} \circ K_{zy} = K_{zy} \circ K_{xy}. \]

In other words, condition (iii) holds with the variables \( x, y, \) and \( z \) replaced by \( z, x, \) and \( y \) respectively. The other cases of the third frame condition are verified in a similar fashion.

Frame condition (iv) is a simple consequence of the preceding observations, together with (3) and (5). For example, compose both sides of (3) on the left with \( \hat{\varphi}_{yx} \), and use (5), to obtain

\[ \hat{\varphi}_{yx} \mid \hat{\varphi}_{xz} = \hat{\varphi}_{yz}, \]

and compose both sides of (3) on the right with \( \hat{\varphi}_{zy} \), and use (5), to obtain

\[ \hat{\varphi}_{xz} \mid \hat{\varphi}_{zy} = \hat{\varphi}_{xy}. \]

This argument shows that the two permuted versions of (3), the first obtained by transposing the first two indices \( x \) and \( y \) of the triple \((x, y, z)\), and the second by transposing the last two indices \( y \) and \( z \) of the triple, are valid in \( \mathcal{F} \). All permutations of the triple \((x, y, z)\) may be obtained by composing these two transpositions in various ways. For example, if we transpose the first two indices of (3), permuting \((x, y, z)\) to \((y, x, z)\) and arriving at (7), and then transpose the last two indices of (7), permuting \((y, x, z)\) to \((y, z, x)\), we arrive at

\[ \hat{\varphi}_{yz} \mid \hat{\varphi}_{zx} = \hat{\varphi}_{yz}. \]

It follows that frame condition (iv) is valid in \( \mathcal{F} \).

An examination of the preceding proof reveals that only condition (i) is used to verify the second frame condition in the case when \( x = y \), and to verify the last two frame conditions when \( x = z \) or \( y = z \). Also, only conditions (i) and (ii) are used to verify the last two frame conditions when \( x = z \). The following corollary, which will be needed in the construction of coset relation algebras, is a consequence of this observation. In formulating it and the succeeding two corollaries, we use the following simplified notation: if \( f \) is the \( \alpha \)th element in some fixed enumeration of one of the groups \( G_x \) in a group pair, then we write \( H_{xx,f} \) and \( R_{xx,f} \) for \( H_{xx,\alpha} \) and \( R_{xx,\alpha} \) respectively.

**Corollary 4.5.** Let \( \mathcal{F} \) be a group pair satisfying condition (i) of Theorem 4.4. The following conditions hold for all \( x \) in \( I \) and all pairs \((x, y)\) in \( \mathcal{E} \).

(i) \( R_{xx,f}^{-1} = R_{xx,g} \) for \( f \) in \( G_x \) and \( g = f^{-1} \).

(ii) \( R_{xx,f} \mid R_{xy,\beta} = R_{xy,\gamma} \) for \( f \) in \( G_x \) and \( H_{xy,\gamma} = f \circ H_{xy,\beta} \).

(iii) \( R_{xy,\alpha} \mid R_{y,\beta} = R_{xy,\gamma} \) for \( g \) in \( G_y \) and \( K_{xy,\gamma} = K_{xy,\alpha} \circ g \).
(iv) If \( \mathcal{F} \) also satisfies condition (ii) of Theorem 4.4, then
\[
R_{xy, \alpha} \mid R_{yx, \beta} = \bigcup \{ R_{xx, f} : f \in H_{xy, \alpha} \circ H_{xy, \beta} \}
= \{ (g, g \circ f) : g \in G_x \text{ and } f \in H_{xy, \alpha} \circ H_{xy, \beta} \}.
\]
In particular, each of these converses and compositions is in the set of frame relations.

Proof. As an example, we prove (iv). Write \( z = x \) and use condition (i) from Theorem 4.4 to see that
\[
(1) \quad H_{xz} = H_{xx} = \{ e_x \}
\]
and that
\[
(2) \quad \varphi_{xz} = \varphi_{xx}
\]
is the identity automorphism of \( G_x/\{ e_x \} \). Consequently,
\[
(3) \quad H_{xz, f} = H_{xx, f} = \{ f \} \quad \text{and} \quad R_{xz, f} = R_{xx, f} = \{ (g, g \circ f) : g \in G_x \}
\]
for every \( f \) in \( G_x \). The additional assumption of condition (ii) from Theorem 4.4 implies that frame condition (ii) holds, by Theorem 4.4, and therefore
\[
(4) \quad \varphi_{yx} = \varphi_{xy}^{-1}.
\]
Invoke Convention 3.7 to write \( \kappa_{yx} = \kappa_{xy} \) and
\[
(5) \quad H_{yx, \gamma} = K_{xy, \gamma} \quad \text{and} \quad K_{yx, \gamma} = H_{xy, \gamma}
\]
for all indices \( \gamma < \kappa_{xy} \). In particular, taking \( \gamma = 0 \), we obtain
\[
(6) \quad H_{yx} = K_{xy} \quad \text{and} \quad K_{yx} = H_{xy}.
\]
Since
\[
(7) \quad H_{xy} \circ H_{xz} = H_{xy} \circ \{ e_x \} = H_{xy},
\]
by (1), the isomorphism \( \hat{\varphi}_{xz} \) induced by \( \varphi_{xz} \), which coincides with \( \hat{\varphi}_{xx} \), by (2), is the identity automorphism of \( G_x/H_{xy} \), by (2). Similarly, (7) implies that the isomorphism \( \hat{\varphi}_{xy} \) induced by \( \varphi_{xy} \) coincides with \( \hat{\varphi}_{xy} \). Finally, (4) and the preceding observation imply that the isomorphism \( \hat{\varphi}_{yz} \) induced by \( \varphi_{yz} \) coincides with \( \varphi_{xy} \).

Combine these three observations to conclude that
\[
\hat{\varphi}_{xy} \mid \hat{\varphi}_{yz} = \varphi_{xy} \mid \varphi_{xy}^{-1},
\]
which then is the identity automorphism on \( G_x/H_{xy} \), and therefore
\[
(8) \quad \hat{\varphi}_{xy} \mid \hat{\varphi}_{yz} = \hat{\varphi}_{xz}.
\]
The assumption on \( z \), and (5), yield
\[
K_{xy, \alpha} \circ H_{yz, \beta} = K_{xy, \alpha} \circ H_{yx, \beta} = K_{xy, \alpha} \circ K_{xy, \beta},
\]
and therefore, using also the isomorphism properties of \( \varphi_{xy}^{-1} \),
\[
(9) \quad \varphi_{xy}^{-1} [K_{xy, \alpha} \circ H_{yz, \beta}] = \varphi_{xy}^{-1} (K_{xy, \alpha} \circ K_{xy, \beta})
= \varphi_{xy}^{-1} (K_{xy, \alpha}) \circ \varphi_{xy}^{-1} (K_{xy, \beta}) = H_{xy, \alpha} \circ H_{xy, \beta}.
\]
Take \( \alpha = \beta = 0 \) in (9) to arrive at
\[
(10) \quad \varphi_{xy}^{-1} [K_{xy} \circ H_{yz}] = \varphi_{xy}^{-1} [K_{xy, 0} \circ H_{yz, 0}] = H_{xy, 0} \circ H_{xy, 0} = H_{xy} \circ H_{xy} = H_{xy}.
\]
Because $H_{xz}$ coincides with the trivial subgroup $\{e_x\}$, by (11), it may be concluded from (10) that

$$H_{xz} \subseteq \varphi_{xy}^{-1}[K_{xy} \circ H_{yz}].$$

Together, (8) and (11) show that condition (iv) in Composition Theorem 3.10 is satisfied in the case under consideration. Apply the implication from (iv) to (iii) in that theorem, together with (9), (3), and the definition of $R_{xx,f}$, to conclude that

$$R_{xy,\alpha} \mid R_{yz,\beta} = R_{xy,\alpha} \mid R_{yx,\beta} = \bigcup \{R_{xx,f} : H_{xx,f} \subseteq \varphi_{xy}^{-1}[K_{xy,\alpha} \circ H_{yz,\beta}]\} = \bigcup \{\{(g, g \circ f) : g \in G_x \mid f \in H_{xy,\alpha} \circ H_{xy,\beta}\} = \bigcup \{\{(g, g \circ f) : g \in G_x \text{ and } f \in H_{xy,\alpha} \circ H_{xy,\beta}\}.

\square

We return to the question that was posed after the Converse Theorem: can condition (i) in that theorem be replaced by the condition that $R_{xy,\alpha}^{-1}$ be in $A$ for some fixed $\alpha$?

**Corollary 4.6.** If the set $A$ of frame relations contains the identity relation on the base set, then for any pair $(x, y)$ in $E$, the following conditions are equivalent.

(i) $R_{xy,\alpha}^{-1}$ is in $A$ for some $\alpha < \kappa_{xy}$.

(ii) $R_{xy,\alpha}^{-1}$ is in $A$ for all $\alpha < \kappa_{xy}$.

**Proof.** The implication from (ii) to (i) is obvious. To establish the reverse implication, assume that $R_{xy,\xi}^{-1}$ is in $A$ for some $\xi < \kappa_{xy}$, and let $\alpha < \kappa_{xy}$ be an arbitrary index. Choose an element $f$ in $G_x$ such that

$$f \circ H_{xy,\xi} = H_{xy,\alpha}.$$

The assumption on $A$ implies that the group pair $F$ satisfies condition (i) of Theorem 4.4 by the Identity Theorem 3.4. Apply Corollary 4.5(ii) and (11) to obtain

$$R_{xx,f} \mid R_{xy,\xi} = R_{xy,\alpha}.$$

Form the converse of both sides of (2), and use the second involution law for relational composition, to arrive at

$$R_{xy,\xi}^{-1} \mid R_{xx,f} = R_{xy,\alpha}^{-1}.$$

Put

$$f^{-1} = g.$$

Apply Corollary 4.5(i) to (4) to obtain

$$R_{xx,f}^{-1} = R_{xx,g}.$$

Use this equation to rewrite equation (3) in the form

$$R_{xy,\xi}^{-1} \mid R_{xx,g} = R_{xy,\alpha}^{-1}.$$  

The relation $R_{xy,\xi}^{-1}$ is in $A$, by assumption. Corollary 4.5(ii),(iii) and the distributivity of relational composition together imply that the set $A$ is closed under the composition of its elements with relations of the form $R_{xx,g}$. Consequently, the composition on the left side of (5) is in $A$. Use (5) to conclude that $R_{xy,\alpha}^{-1}$ is in $A$. 

\square
Turn next to the question that was posed after the Composition Theorem: can condition (i) in that theorem be replaced by the condition that \( R_{xy,\alpha} \mid R_{yz,\beta} \) be in \( A \) for some fixed \( \alpha \) and \( \beta \)?

**Corollary 4.7.** If the set \( A \) of frame relations contains the identity relation, then for any pairs \((x, y)\) and \((y, z)\) in \( E \), the following conditions are equivalent.

(i) \( R_{xy,\alpha} \mid R_{yz,\beta} \) is in \( A \) for some \( \alpha < \kappa_{xy} \) and some \( \beta < \kappa_{yz} \).

(ii) \( R_{xy,\alpha} \mid R_{yz,\beta} \) is in \( A \) for all \( \alpha < \kappa_{xy} \) and all \( \beta < \kappa_{yz} \).

**Proof.** The implication from (ii) to (i) is obvious. To establish the reverse implication, use an argument similar to the one in the preceding proof. Assume that \( R_{xy,\xi} \mid R_{yz,\eta} \) is in \( A \) for some \( \xi < \kappa_{xy} \) and \( \eta < \kappa_{yz} \). Let \( \alpha < \kappa_{xy} \) and \( \beta < \kappa_{yz} \) be arbitrary, and choose elements \( f \) in \( G_x \) and \( g \) in \( G_z \) so that

\[
\begin{align*}
f \circ H_{xy,\xi} &= H_{xy,\alpha} \\
g \circ K_{yz,\eta} &= K_{yz,\beta}.
\end{align*}
\]

The assumption on \( A \) implies that the group pair \( F \) satisfies condition (i) of Theorem 4.4, by the Identity Theorem 3.4. Apply Corollary 4.5(ii),(ii) and (1) to obtain

\[
R_{xx,f} \mid R_{xy,\xi} = R_{xy,\alpha} \quad \text{and} \quad R_{yz,\eta} \mid R_{zz,g} = R_{yz,\beta}.
\]

These equations and the associative law for relational composition lead immediately to

\[
R_{xy,\alpha} \mid R_{yz,\beta} = (R_{xx,f} \mid R_{xy,\xi}) \mid (R_{yz,\eta} \mid R_{zz,g}) =
\]

\[
R_{xx,f} \mid (R_{xy,\xi} \mid R_{yz,\eta}) \mid R_{zz,g}.
\]

The relation \( R_{xy,\xi} \mid R_{yz,\eta} \) is in \( A \), by assumption. Corollary 4.4 and the distributivity of relational composition over unions together imply that the set \( A \) is closed under the composition of its elements with relations of the form \( R_{xx,f} \) and \( R_{zz,g} \). Consequently, the composition on the right side of (2) is in \( A \). Use (2) to conclude that \( R_{xy,\alpha} \mid R_{yz,\beta} \) is in \( A \). \[\square\]

**5. Examples**

The easiest group frame to construct involves a kind of “power” of a quotient group. Fix a group \( M \) and a normal subgroup \( N \). For each element \( x \) in a given index set \( I \), let \( G_x \) be an isomorphic copy of \( M \) (chosen so that distinct copies are pairwise disjoint) and \( \psi_x \) an isomorphism from the quotient group \( M/N \) to the corresponding quotient group of \( G_x \). Take the mapping \( \varphi_{xy} \) to be the natural isomorphism between the quotient groups of \( G_x \) and of \( G_y \), defined by

\[
\varphi_{xy} = \psi_x^{-1} \mid \psi_y
\]

for distinct \( x \) and \( y \) in \( I \). These mappings are all isomorphisms between copies of the single quotient group \( M/N \). Take \( \varphi_{xx} \) to be the identity automorphism of \( G_x/\{e_x\} \), as required by the definition of a frame.

The resulting pair \( F = (G, \varphi) \) is readily seen to be a group frame, and the corresponding group relation algebra \( \Theta[F] \) is a measurable set relation algebra. If we take the indices \( \alpha \) of the atomic relations in \( \Theta[F] \) to be the corresponding cosets of \( M/N \), then there are especially simple formulas for computing the converse of an atomic relation and the composition of two atomic relations in \( \Theta[F] \):

\[
R_{xy,\alpha}^{-1} = R_{yx,\alpha^{-1}} \quad \text{and} \quad R_{xy,\alpha} \mid R_{yz,\beta} = R_{xz,\alpha \beta}
\]
when \( x, y, \) and \( z \) are distinct elements of \( I \). (Here \( \alpha^{-1} \) denotes the inverse of the coset \( \alpha \), and \( \alpha \circ \beta \) the product of the cosets \( \alpha \) and \( \beta \), in \( M/N \).)

If \( N \) is the trivial (one-element) subgroup of \( M \), then each atomic relation \( R_{xy, \alpha} \) is a function and in fact a bijection from \( G_x \) to \( G_y \). In this case, \( \mathcal{G}[\mathcal{F}] \) is an example of an atomic relation algebra with functional atoms. At the other extreme, if \( N \) coincides with \( M \), then there is only one atomic relation, namely

\[
R_{xy,0} = G_x \times G_y,
\]

for each pair of distinct indices \( x, y \) in \( I \). In general, if the normal subgroup \( N \) has order \( \lambda \) and index \( \kappa \) in \( M \) (that is to say, if \( N \) contains \( \lambda \) elements and has \( \kappa \) cosets in \( M \), then there will be \( \kappa \) distinct atomic relations of the form \( R_{xy, \alpha} \), and each of them will be the union of \( \lambda \) pairwise disjoint bijections from \( G_x \) to \( G_y \).

In the general case of the power construction, \( \mathcal{E} \) is allowed to be an arbitrary equivalence relation on \( I \). Moreover, the group \( M \) and normal subgroup \( N \) are fixed for a given equivalence class of \( \mathcal{E} \), but different equivalence classes may use different groups and normal subgroups.

The most trivial case of the power construction is when the fixed group \( M \) is the one-element group. In this case, \( \mathcal{G}[\mathcal{F}] \) is just the full set relation algebra with base set and unit

\[
U = \bigcup \{ G_x : x \in I \} \quad \text{and} \quad E = \bigcup \{ G_x \times G_y : (x, y) \in \mathcal{E} \}
\]

respectively. Moreover, every full set relation algebra on an equivalence relation can be obtained as a group relation algebra in this fashion, using arbitrary equivalence relations \( \mathcal{E} \) on \( I \). The construction of full set relation algebras may therefore be viewed as the most trivial case of the construction of full group relation algebras, namely the case when all the groups have order one. This class may be characterized abstractly, up to isomorphisms, as the class of complete and atomic singleton-dense relation algebras.

It follows from this observation that the class of algebras embeddable into full group relation algebras coincides with the class of representable relation algebras. In particular, the class is equationally axiomatizable, by the results of Tarski [17]. However, the description of representable relation algebras in terms of group relation algebras seems much more advantageous, because the class of full group relation algebras is substantially more varied and interesting than the class of full set relation algebras.

A second example of the group relation algebra construction that is easy to describe is the one in which all of the groups are cyclic. Suppose \( G = \langle G_x : x \in I \rangle \) is a family of (pairwise disjoint) cyclic groups and \( \mathcal{E} \) an equivalence relation on \( I \). To avoid unnecessary complications in notation, we consider here only the case when the groups are finite. Fix a generator \( g_x \) of each group \( G_x \). Let \( \langle \kappa_{xy} : (x, y) \in \mathcal{E} \rangle \) be a system of positive integers satisfying the following conditions for all appropriate pairs in \( \mathcal{E} \).

(i) \( \kappa_{xy} \) is a common divisor of the orders of \( G_x \) and \( G_y \).
(ii) \( \kappa_{xx} \) is equal to the order of \( G_x \).
(iii) \( \kappa_{yx} = \kappa_{xy} \).
(iv) \( \gcd(\kappa_{xy}, \kappa_{yz}) = \gcd(\kappa_{xy}, \kappa_{xz}) = \gcd(\kappa_{xz}, \kappa_{yz}) \).

Condition (i) ensures that there are (uniquely determined) subgroups \( H_{xy} \) and \( K_{xy} \) of index \( \kappa_{xy} \) in \( G_x \) and \( G_y \) respectively. The quotient groups \( G_x/H_{xy} \) and
are therefore isomorphic, and in fact there is a uniquely determined isomorphism \( \varphi_{xy} \) between them that maps the generator \( g_x/H_{xy} \) of the first quotient to the generator \( g_y/K_{xy} \) of the second. Conditions (ii) and (iii), and the definition of the quotient isomorphisms, ensure that frame conditions (i) and (ii) are satisfied.

The complex product \( H_{xy} \circ H_{xz} \) is a subgroup of \( G_x \) of index \( d = \gcd(\kappa_{xy}, \kappa_{xz}) \). Condition (iv) says that the complex products \( K_{xy} \circ H_{yz} \) and \( K_{xz} \circ K_{yz} \) also have index \( d \). This, together with the definition of the quotient isomorphisms, ensures that frame conditions (iii) and (iv) are satisfied. It follows that the pair \( \mathcal{F} = (G, \varphi) \) is a group frame. This construction using cyclic groups is due jointly to Hajnal Andráska and the author.

If every group in \( \mathcal{F} \) has order one or two, then the group relation algebra \( \mathcal{G}[\mathcal{F}] \) is an example of a pair-dense relation algebra in the sense of Maddux [11]. When \( \kappa_{xy} = 2 \), there are exactly two relations: \( R_{xy,0} \) and \( R_{xy,1} \). Each of them is a function, and in fact a bijection from \( G_x \) to \( G_y \), with exactly two pairs. When \( \kappa_{xy} = 1 \), there is only the one relation \( R_{xy,0} = G_x \times G_y \). It contains either four pairs, two pairs, or one pair, according to whether both groups \( G_x \) and \( G_y \) have order two, exactly one of these groups has order two and the other order one, or both groups have order one. The class of such group relation algebras may be characterized abstractly, up to isomorphisms, as the class of complete and atomic pair-dense relation algebras.

6. A DECOMPOSITION THEOREM

The isomorphism index set \( E \) of a group frame \( \mathcal{F} = (G, \varphi) \) is an equivalence relation on the group index set \( I \), and the unit

\[
E = \bigcup \{(G_x \times G_y : (x, y) \in \mathcal{E})
\]

of the corresponding full group relation algebra \( \mathcal{G}[\mathcal{F}] \) is an equivalence relation on the base set \( U = \bigcup_{x \in I} G_x \). Call a group frame \textit{simple} if the group index set \( I \) is not empty, and the isomorphism index set \( E \) is the universal relation on \( I \). It turns out that the frame \( \mathcal{F} \) is simple if and only if the algebra \( \mathcal{G}[\mathcal{F}] \) is simple in the algebraic sense of the word, namely, it has more than one element, and every non-constant homomorphism on it must be injective; or, equivalently, it has exactly two ideals, the trivial ideal and the improper ideal.

**Theorem 6.1.** Let \( \mathcal{F} \) be a group frame. The group relation algebra \( \mathcal{G}[\mathcal{F}] \) is simple if and only if \( \mathcal{F} \) is simple.

**Proof.** Suppose first that the frame \( \mathcal{F} \) is simple. The isomorphism index set \( \mathcal{E} \) is then the universal relation on the index set \( I \), and consequently the unit \( E \) of \( \mathcal{G}[\mathcal{F}] \) is the universal relation \( U \times U \) on the base set \( U \). Moreover, the base set \( U \) is not empty, because the index set \( I \) is not empty, and the groups indexed by \( I \) are not empty. The algebra \( \mathcal{R}(E) \) therefore consists of all binary relations on a non-empty base set. Such set relation algebras are well known to be simple. Moreover, \( \mathcal{G}[\mathcal{F}] \) is a subalgebra of \( \mathcal{R}(E) \), by Group Frame Theorem 4.2. It is well known that subalgebras of simple relation algebras are simple, so \( \mathcal{G}[\mathcal{F}] \) must also be simple.

We postpone the proof of the reverse implication, that simplicity of \( \mathcal{G}[\mathcal{F}] \) implies that of \( \mathcal{F} \), until after the next theorem.

It turns out that every full group relation algebra can be decomposed into the direct product of simple, full group relation algebras. Here is a sketch of the main
ideas. The details are left to the reader. Given an arbitrary group frame
\[ F = \langle \langle G_x : x \in I \rangle, \langle \varphi_{xy} : (x, y) \in E \rangle \rangle, \]
consider an equivalence class \( J \) of the isomorphism index set \( E \). The universal relation \( J \times J \) on \( J \) is a subrelation of \( E \), and in fact it is a maximal connected component of \( E \) in the graph-theoretic sense of the word. The restriction of \( F \) to \( J \) is defined to be the group pair
\[ F_J = \langle \langle G_x : x \in J \rangle, \langle \varphi_{xy} : (x, y) \in J \times J \rangle \rangle. \]
Each such restriction of \( F \) to an equivalence class of the index set \( E \) inherits the frame properties of \( F \) and is therefore a simple group frame. Call such restrictions the components of \( F \). It is not difficult to check that every frame is the disjoint union of its components in the sense that the group system and the isomorphism system of \( F \) are obtained by respectively combining the group systems and the isomorphism systems of the components of \( F \).

Each component \( F_J \) gives rise to a full group relation algebra \( \mathfrak{G}[F_J] \) that is simple and is in fact a subalgebra of the full set relation algebra with base set and unit
\[ U_J = \bigcup_{x \in J} G_x \quad \text{and} \quad E_J = U_J \times U_J \]
respectively. The group relation algebra \( \mathfrak{G}[F] \) is isomorphic to the direct product of the simple group relation algebras \( \mathfrak{G}[F_J] \) constructed from the components of \( F \) (so \( J \) varies over the equivalence classes of \( E \)). In fact, if internal direct products are used instead of Cartesian direct products, then \( \mathfrak{G}[F] \) is actually equal to the internal direct product of the full group relation algebras constructed from its component frames.

**Theorem 6.2 (Decomposition Theorem).** Every full group relation algebra is isomorphic to a direct product of full group relation algebras on simple frames.

Return now to the proof of the reverse implication in Theorem 6.1. Assume that the frame \( F \) is not simple. If the group index set \( I \) is empty, then the base set \( U \) is also empty, and in this case \( \mathfrak{G}[F] \) is a one-element relation algebra with the empty relation as its only element. In particular, \( \mathfrak{G}[F] \) is not simple. On the other hand, if the group index set \( I \) is not empty, then the isomorphism index set \( E \) has at least two equivalence classes, by the definition of a simple frame. The group relation algebra \( \mathfrak{G}[F] \) is isomorphic to the direct product of the group relation algebras on the component frames of \( F \), by Decomposition Theorem 6.2 and there are at least two such components. Each of these components is a simple frame, so the corresponding group relation algebra is simple, by the first part of the proof of Theorem 6.1. It follows that \( \mathfrak{G}[F] \) is isomorphic to a direct product of at least two simple relation algebras, so \( \mathfrak{G}[F] \) cannot be simple. For example, the projection of \( \mathfrak{G}[F] \) onto one of the factor algebras is a non-constant homomorphism that is not injective.

7. Summary

The present paper generalizes the notion of pair density from Maddux [11] by introducing the notion of a measurable relation algebra. A large class of examples of such algebras has been constructed, namely the class of full group relation algebras. Unfortunately, the class is not large enough to represent all measurable relation algebras: there exist measurable relation algebras that are not essentially...
isomorphic to (full) group relation algebras, and in fact that are not representable as set relation algebras at all. The next paper in this series, [1], greatly extends the class of examples of measurable relation algebras by adding one more ingredient to the mix, namely systems of cosets that are used to modify the operation of relative multiplication. In the group relation algebras constructed in the present paper, the operation of relative multiplication is just relational composition, but in the coset relation algebras to be constructed in the next paper, the operation of relative multiplication is “shifted” by coset multiplication, so that in general it no longer coincides with composition. On the one hand, this shifting leads to examples of measurable relation algebras that are not representable as set relation algebras, see [1] Theorem 5.2. On the other hand, the class of coset relation algebras constructed from systems of group pairs and shifting cosets really is broad enough to include all measurable relation algebras. The task of the third paper in the series, [5], is to prove this assertion, namely that every measurable relation algebra is essentially isomorphic to a coset relation algebra, see [5] Theorem 7.2.

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