Linear logic in normed cones: probabilistic coherence spaces and beyond

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Abstract

Ehrhard, Pagani and Tasson [8] proposed a model of probabilistic functional programming in a category of normed positive cones and stable measurable cone maps, which can be seen as a coordinate-free generalization of probabilistic coherence spaces. However, unlike the case of probabilistic coherence spaces, it remained unclear if the model could be refined to a model of classical linear logic.

In this work we consider a somewhat similar category which gives indeed a coordinate-free model of full propositional linear logic with nondegenerate interpretation of additives and sound interpretation of exponentials. Objects are dual pairs of normed cones satisfying certain specific completeness properties, such as existence of norm-bounded monotone weak limits, and morphisms are bounded (adjointable) positive maps. Norms allow us a distinct interpretation of dual additive connectives as product and coproduct. Exponential connectives are modelled using real analytic functions and distributions that have representations as power series with positive coefficients.

Unlike the familiar case of probabilistic coherence spaces, there is no reference or need for a preferred basis; in this sense the model is invariant. Probabilistic coherence spaces form a full subcategory, whose objects, seen as posets, are lattices. Thus we get a model fitting in the tradition of interpreting linear logic in a linear algebraic setting, which arguably is free from the drawbacks of its predecessors.

Relations with constructions of [8] are left for future research.

1 Introduction

From the very beginning of linear logic it has been implicit in notation and terminology that it should be considered as similar to (multi-)linear algebra.

Indeed, the multiplicative operations, i.e. tensor, linear implication and linear negation are intuitively interpreted as, respectively, the tensor product of vector spaces, vector space of linear maps and vector space duality; and
the additive ones are often understood as corresponding to the product and coproduct of normed spaces. As for the exponentials, they resemble a “Fock space” of power series, i.e. some completion of the free symmetric algebra (as discussed in [3], see also [14]).

However, putting this intuition into solid mathematical form turned out to be rather challenging. Different fragments require different structures on vector spaces, and it seems not so easy to put them all together.

On one hand, linear negation requires involutive duality, which is often problematic in infinite dimensions. This leads to considering topological vector spaces and continuous linear maps.

On the other hand, the additive fragment is better interpreted in the setting of normed vector spaces and norm-bounded linear maps of norm not greater than one, i.e. contractions. Indeed, in the category of topological vector spaces and continuous linear maps, finite products and coproducts are isomorphic, and additive connectives get identified. On the other hand, product and coproduct norms are strictly distinct whenever dimensions are greater than zero. Therefore, in the category of normed vector spaces and contractions, the two operations are not isomorphic. Unfortunately, duality of normed vector spaces is usually not involutive.

Finally, for exponentials it is necessary to consider some version of the free symmetric algebra (as in [3]) and this leads to spaces of power series, i.e. analytic functions. Then it is not always clear how to equip such spaces with norms (since analytic functions globally are unbounded).

In the seminal work [14] Girard proposed to interpret dual formulas as dual pairs of Banach spaces, calling them coherent Banach spaces. This gave a perfectly natural, non-degenerate interpretation of the multiplicative-additive fragment, however failed for the exponentials.

For the case of exponentials it was proposed to consider the Banach space of functions analytic in the unit ball, but there were issues with the behaviour at the boundary. Unfortunately, functions arising from linear logic proofs tend to map the interior of the unit ball to its closure, and there is no reasonable way to extend a general analytic function from the open ball to its boundary. In particular, composing these functions was problematic.

Another model was developed in [7] in the context of Koethe sequence spaces. This gives a successful interpretation of full linear logic. However, since Koethe sequence spaces do not have any norms, there is no distinction between (finite) products and coproducts, and dual additive connectives are identified. Thus, the model of additives is degenerate. A similar degeneracy occurs in the category of convenient vector spaces [2], which models differential linear logic (see [9], [10]), where exponential connectives play a crucial role.

A more recent model of [19] is based on a very general class of vector spaces and interprets full linear logic, but remains additively degenerate in a similar way.

Finally it turned out that the category of probabilistic coherence spaces (PCS), initiated in [15] and developed in [5], provides a nondegenerate model of the full system, including the additives. Omitting some technicalities, a PCS
can be described as a dual pair of (real) Banach spaces, which are spaces of sequences on the same index set. In a more algebraic language, these are Banach spaces with a fixed basis. Having a basis allows us speaking about positive (or, rather, non-negative) vectors, represented as non-negative sequences, and positive maps, represented as non-negative matrices. PCS morphisms can be described as bounded positive maps.

In such a setting, formulas are interpreted as sequence spaces, and proofs, as positive elements (sequences and matrices). Exponential connectives correspond to spaces of power series with positive coefficients converging in the unit ball (which still have an interpretation as real analytic functions). Somewhat remarkably, in this case we do not encounter problems with exponentials typical for coherent Banach spaces. A conceptual reason for that is that analytic functions defined in the open unit ball by positive power series have canonical extensions to the boundary (although not necessarily continuous).

A major drawback of the PCS model is that it is very non-invariant. All spaces come with fixed bases. On the other hand, it has eventually become apparent that what makes the model work is not the choice of bases, but the structure of positivity, i.e. of a partial order (see [8], [25]). In fact, the preferred bases of PCS themselves are determined by the partial order. Sequence spaces, seen as posets (with elementwise ordering) are vector lattices, and preferred bases consist of “minimal” positive elements (elements v such that 0 ≤ u ≤ v implies u = λv for some λ ≤ 1).

This suggests that the whole construction can be reformulated in an invariant fashion using only the language of partially ordered vector spaces. Or, since eventually we are interested only in positive elements and maps, in the language of abstract positive cones (in the sense of [22]). However it soon becomes evident that the lattice structure, characteristic for PCS, is, in fact, not necessary.

In particular, in [8] a coordinate-free generalization of PCS was proposed for modeling probabilistic functional programming, where more general positive cones are considered. These cones are equipped with norms, similar to vector space norms, and satisfy a variant of completeness property: every norm-bounded monotone nondecreasing sequence must have a least upper bound. It turns out that such cones can be organized into a cartesian closed category with specific stable cone maps as morphisms. (This gives already a model of intuitionistic logic, but in order to interpret probabilistic computation further refinements were proposed, leading to the category of measurable cones and measurable stable maps). The question of modeling classical linear logic in such a setting remained open.

In this work we propose to consider normed positive cones in dual pairs of partially ordered vector spaces, or, simply, dual pairs of normed cones. Duality alone would allow us to interpret the multiplicative-additive fragment of linear logic along the lines of [14]. But in order to model the exponential fragment we, similarly to [8], introduce certain completeness property. However our version of completeness is different: any norm-bounded monotone sequence has a weak limit. (It turns out, though, that completeness in our sense implies completeness in the sense of [8].) Here weak limit is understood in the following sense. If
\((P, Q)\) is a dual pair of cones, and \(\{v_n\}\) is a sequence in \(P\), then \(v\) is the weak limit of \(\{v_n\}\) iff for any \(\phi \in Q\) it holds that
\[
\langle \phi, v \rangle = \lim_{n \to \infty} \langle \phi, v_n \rangle.
\]
Completeness then gives us control over convergence of power series needed for exponentials.

A certain exercise is to define a tensor product of such cones, which must be complete itself. This leads us to studying possibilities of cone completion.

By the end of the day we obtain a well-defined sound model of the exponential fragment in terms of real analytic functions and distributions, similar to the case of PCS. The category of PCS itself is identified as a proper full subcategory. Relations with stable and measurable maps of [8] remain to be clarified.

To conclude the introduction, let us comment on some amusing analogies with noncommutative geometry (probably not too strong).

A standard example of a non-lattice partially ordered vector space is the space of self-adjoint operators on a Hilbert space. In fact, a classical result [24] states that the self-adjoint part of a \(C^*-\)algebra is a lattice iff the algebra is commutative. Our passage from vector lattices, which have preferred bases, to general partially ordered vector spaces resembles passage from “commutative” spaces to general noncommutative “pointless” spaces, i.e. general algebras. In particular, probabilistic (“commutative”) coherence spaces have natural representations as subspace of sequence spaces, and these are commutative algebras. There are however genuinely non-lattice (“noncommutative”) objects, especially coming from spaces of self-adjoint operators on a Hilbert space. Let us mention that for finite dimensions such a noncommutative construction was hinted by Girard a while ago under the name of quantum coherence spaces [15].

1.1 Plan of the paper

Our main objects are dual pairs either of vector spaces or of partially ordered vector spaces or of vector cones. These also can be described in a more intrinsic way as spaces equipped with weak topology (in spirit of [19]). We usually prefer the latter representation.

In Section 2 we recall dual pairs of vector spaces and weak topologies on vector spaces.

In Section 3 we discuss partially ordered vector spaces and vector cones. Next we introduce dual pairs of partially ordered vector spaces as well as of vector cones. Then we discuss weak topologies in the partially ordered case. For vector spaces, this is a straightforward adaptation from Section 2. For vector cones, some work is needed. We recall the notion of a uniform space and uniform topology and then introduce uniform cones and weak uniform topologies on cones. We show that vector cones equipped with weak uniform topology are equivalent to dual pairs of cones.

In Section 4 we introduce tensor and cotensor products of cones. In Section 5 we discuss normed cone and study cone completion.
Finally, in Section 6 we define coherent cones, which we use to interpret linear logic, and introduce multiplicative-additive operations on them. Section 7 is concerned with the exponential fragment.

1.2 Notation and background

We assume that the reader is familiar with linear logic (see [12], [13] for an introduction), as well as with its categorical interpretation. The standard reference on linear logic and \(*\)-autonomous categories is [21]. For a modern treatment of categorical semantics of linear logic, especially of the exponential fragment, we refer to [17].

We denote the monoidal product in a \(*\)-autonomous category as \(\otimes\) and call it tensor product, and we denote duality as a star (\(\ast\)). The dual of tensor product is called cotensor and denoted as \(\ast\), i.e. \(A \ast B = (A^\ast \otimes B^\ast)^\ast\). Monoidal unit is denoted \(1\), with its dual \(1^\ast\) denoted \(\bot\). Product and coproduct are denoted as, respectively, \(\times\) and \(\oplus\). The internal homs functor is denoted \(\Rightarrow\), as usual.

We also assume that the reader is familiar with basic notions of locally convex vector spaces. We use [23] for reference. All vector spaces in this paper are real.

2 Dual pairs and reflexivity

The first thing for interpreting linear logic in the context of vector spaces is to get involutive duality. In the finite-dimensional case we have the usual duality of vector spaces. However, when we come to infinite dimensions, there are different ways to generalize this duality, which, in general, produce non-involutive operations. One of the most direct ways to deal with this problem is to use dual pairs.

Recall that a dual pair (also called a dual system or, simply, a duality) is a pair \((E, F)\) of vector spaces equipped with a bilinear pairing \(\langle \cdot, \cdot \rangle : F \times E \to \mathbb{R}\), such that for any \(v \in E\) with \(v \neq 0\) there exists \(\phi \in F\) with \(\langle \phi, v \rangle \neq 0\), and for any \(\phi \in F\) with \(\phi \neq 0\) there exists \(v \in E\) with \(\langle \phi, v \rangle \neq 0\) (i.e., the pairing is nondegenerate).

For finite-dimensional \(E\) there is only one (up to a natural isomorphism) potential partner to form a dual pair, namely the dual space \(E^\ast\). In the infinite-dimensional case there are many different choices for the dual space, and a dual pair is a way to fix one such choice.

A map of dual pairs \((E, F), (E', F')\) is a pair of linear maps \(L : E \to E', M : F' \to F\) satisfying for all \(v \in E, \phi \in F'\) the adjointness condition

\[
\langle M \phi, v \rangle = \langle \phi, Lv \rangle.
\]
Obviously, such maps compose, and dual pairs can be organized in a category. The dual \((E, F)^*\) of a dual pair \((E, F)\) is the dual pair \((F, E)\) equipped with the same pairing, written with the opposite order of arguments. Obviously, this duality is involutive \((E, F)^{**} = (E, F)\). It also extends to maps in the obvious way: \((L, M)^* = (M, L)\), which gives us a contravariant functor from dual pairs to dual pairs.

2.1 Weak topologies and reflexivity

Of course, a dual pair is just a vector space with explicitly specified dual. Always keeping this dual explicit may lead to somewhat cumbersome formulations and expressions (at least, from the notational point of view). We can hide duals from notation by using topology [19]. Indeed, any dual pair \((E, F)\) gives rise to particular locally convex Hausdorff topologies on its members \(E\) and \(F\).

The \(\sigma(E, F)\)-weak (or simply weak) topology on the space \(E\) determined by the dual pair \((E, F)\) is the topology of pointwise convergence on elements of \(F\) (where \(E\) is identified with a subspace of functions on \(F\)).

In a greater detail, the \(\sigma(E, F)\)-weak topology is defined by saying that a net \(\{u_\alpha\}\) in \(E\) converges to an element \(v \in E\) iff for any \(\phi \in F\) the net \(\{\langle \phi, u_\alpha \rangle\}\) converges to \(\{\langle \phi, v \rangle\}\).

In terms of open sets, the base of zero neighborhoods for the \(\sigma(E, F)\)-weak topology consists of the sets

\[ U_{\phi_1, \ldots, \phi_k, \epsilon} = \{ v \in E | \forall i = 1, \ldots, k |\langle \phi_i, v \rangle | < \epsilon \}, \]

where \(k \in \mathbb{N}, \phi_1, \ldots, \phi_k \in F\) and \(\epsilon > 0\).

**Note 1.** For a dual pair \((E, F)\), the \(\sigma(E, F)\)-weak topology is locally convex and Hausdorff.

**Proof** The topology is Hausdorff because the pairing between \(E\) and \(F\) is non-degenerate. Local convexity is immediate. \(\square\)

In the sequel, unless this leads to confusion, we will speak loosely about weak topology without mentioning explicitly the dual pair that determines it.

A crucial property of weak topology determined by a dual pair \((E, F)\) is the following (see, for example, [23, IV.1.2].

**Lemma 1.** For a dual pair \((E, F)\) the space of weakly continuous linear functionals on \(E\) is \(F\). \(\square\)

Now let \(V\) be an arbitrary topological vector space (TVS).

We define the **dual TVS** \(V^*\) as the space of continuous linear functionals on \(V\), with the topology of pointwise convergence. We will understand this topology on \(V^*\) as \(\sigma(V^*, V)\)-weak and, accordingly, we will call it weak. (It is often called \(*\)-weak in literature though.)
We say that a TVS $V$ is *weakly reflexive* if the natural inclusion

$$V \to V^{**}$$

is a topological isomorphism. (We add the prefix “weakly” to avoid confusion with other notions of reflexivity considered in literature.)

Weak reflexivity of $V$ means that topology on $V$ is that of pointwise convergence on elements of $V^*$. In particular, if $(E,F)$ is a dual pair, then the weak topology makes $E$ a reflexive TVS.

The natural notion of TVS map is of course that of a continuous linear map.

Now let $V, U$ be TVS.

A linear map $L : V \to U$ is *adjointable* iff there exists an adjoint map $L^* : U^* \to V^*$ defined by the equation

$$\langle L^* \phi, v \rangle = \langle \phi, L v \rangle. \quad (2)$$

Equation (2) extends duality of TVS to morphisms thus making it a contravariant functor. In fact, Lemma 1 implies the following.

**Note 2** A linear map of weakly reflexive TVS is continuous iff it is adjointable.

**Proof** Let $V, U$ be weakly reflexive, and $L : V \to U$ be a linear map.

If $L$ is continuous, then for any continuous linear functional $\psi$ on $V$ we have a continuous linear functional $L^* \psi = \psi \circ L$ on $U$. This gives us the adjoint map $L^* : U^* \to V^*$.

Assume that $L$ is adjointable. Let $\{v_\alpha\}$ be a converging net in $V$ and $v$ be its limit. Then for any $\phi \in U^*$ we have the net $\{\langle \phi, L v_\alpha \rangle\} = \{\langle L^* \phi, v_\alpha \rangle\}$. Since $V$ is weakly reflexive, its topology is that of pointwise convergence on elements of $V^*$. And since $L^* \phi \in V^*$ and the net $\{v_\alpha\}$ converges to $v$, it follows that the net $\{\langle L^* \phi, v_\alpha \rangle\}$ converges to $\langle L^* \phi, v \rangle = \langle \phi, L v \rangle$. But $\phi \in U^*$ was arbitrary, so $\{L v_\alpha\}$ converges pointwise to $L v$ on all elements of $U^*$. Hence $L$ preserves converging net limits, i.e. is continuous. □

The above discussion readily gives the following.

**Note 3** The category of weakly reflexive TVS and continuous linear maps is equivalent to the category of dual pairs and dual pair maps under the correspondence

$$V \mapsto (V, V^*). \quad (3)$$

The equivalence preserves duality.

**Proof** Correspondence (3) is obviously a functor from TVS to dual pairs.

The “inverse” functor takes a dual pair $(E,F)$ to the vector space $E$ equipped with the $\sigma(E,F)$-weak topology, and a dual pair map $(L,M)$ to the linear map $L$. In this setting we have, by Lemma 1 a natural isomorphism $E^* \cong F$, hence $E$ becomes weakly reflexive, and $L$ adjointable, hence continuous, by Note 2.

Now, composing the two functors in the two possible orders, we get, either the identity (if we start with a weakly reflexive TVS), or the assignment $(E,F) \mapsto (E, E^*)$, where $E$ is seen as a TVS equipped with the $\sigma(E,F)$-weak topology. The latter is a functor isomorphic to the identity by Lemma 1. □
3 Adding positivity

Our ultimate goal is to interpret the exponential fragment of linear logic, which eventually involves analytic maps given by power series, and we need to control their convergence, which may be problematic at certain points. A possible solution is to use positive power series, whose behavior is much simpler. This, in turn, leads us to considering positivity and partial order.

3.1 Positivity and cones

3.1.1 Partial order in vector spaces

Let $E$ be a vector space.

Recall that a cone $P$ in the vector space $V$ is any subset satisfying

- $\forall \lambda \in \mathbb{R}^+ \; \lambda P \subseteq P$;
- $P + P \subseteq P$.

A more fancy way to define a cone in a vector space is to note that any vector space $V$ is a module over the semi-ring $\mathbb{R}^+$ and to say that a cone $P$ in $V$ is an $\mathbb{R}^+$-submodule.

A proper cone $P$ in $V$ is a cone such that $P \cap (-P) = \{0\}$.

A proper cone $P$ gives rise to a partial order on $V$.

Recall that a partially ordered vector space (POVS) (see [23, Chapter V]) is a vector space $V$ equipped with a partial order $\geq$, such that if $u \geq v$ then

- $u + w \geq v + w$ for all $w$;
- $\lambda u \geq \lambda v$ for all $\lambda \geq 0$.

Note 4 For any POVS $V$ the set

$$V_+ = \{v \in V \mid v \geq 0\}$$

is a proper cone. Conversely, any proper cone $V_+$ in $V$ determines a POVS structure on $V$ by:

$$u \geq v \text{ iff } u - v \in V_+.$$

Proof Exercise or see [23, Chapter V]. □

The set $V_+$ is called the positive cone of the POVS $V$. Elements of $V_+$ are called positive, and elements that are differences of positive elements are called regular.

A linear map $L$ between POVS is positive, if $u \geq v$ implies $Lu \geq Lv$. The map $L$ is regular, if it is the difference of two positive ones.

We will say that a POVS is positively generated if it is spanned by its positive elements, i.e., if all its elements are regular.

A natural notion of a POVS morphism is that of a positive map.
3.1.2 Cones abstractly

Eventually we will be interested only in positive elements and positive maps. Therefore it may be more convenient sometimes to speak directly about cones without reference to ambient vector spaces (which may be non-unique). We will consider abstract cones in the sense of [22].

We say that an \((\text{abstract})\) cone is a module over the semi-ring \(\mathbb{R}_+\), satisfying the following properties:

(i) \(p + q = p + q'\) iff \(q = q'\);

(ii) \(p + q = 0\) iff \(p = q = 0\).

A cone map is an \(\mathbb{R}_+\)-linear map of \(\mathbb{R}_+\)-modules.

Any abstract cone \(P\) has an intrinsic partial order defined by

\(v \geq u\) if \(\exists u' \in P\) such that \(v = u + u'\). (5)

Indeed, the relation in (5) is obviously reflexive and transitive, and it is easy to deduce from conditions (i), (ii) above that it is antisymmetric in the sense that \(u \geq v, v \geq u\) iff \(u = v\).

Of course, if we see \(P\) as a positive cone in a vector space \(V\), then the induced partial order on \(V\) extends the intrinsic partial order on \(P\).

**Note 5** If \(P, Q\) are cones, then any cone map \(L : P \to Q\) is monotone: \(u \geq v\) implies \(Lu \geq Lv\). □

It is rather obvious that for a cone \(P\) there is a “minimal” embedding in a vector space.

The enveloping vector space \(EP\) of \(P\) is the \(\mathbb{R}_+\)-module \(P \times P\) quotiented by the equivalence

\[(u + p, v + p) \sim (u + p', v + p').\] (6)

We denote the image of \((u, v) \in P \times P\) in \(EP\) as \([u, v]\).

We define on \(EP\) multiplication by \(-1\) by \(-[u, v] = [v, u]\).

This makes \(EP\) an \(\mathbb{R}\)-module, i.e. a vector space. The original cone \(P\) embeds in \(EP\) by the map \(v \mapsto [v, 0]\), which makes \(EP\) a positively generated POVS.

**Note 6** Let \(P, Q\) be abstract cones. Then any cone map \(L : P \to Q\) has unique extension to a positive linear map \(EL : EP \to EQ\).

**Proof** Exercise. □

**Note 7** The category of cones and cone maps is equivalent to the category of positively generated POVS and positive maps. The equivalence is given by the correspondence

\(P \mapsto EP\).

**Proof** Exercise. □
3.1.3 Positivity and dual pairs

As in the case of vector spaces, we need also an involutive duality for cones or for POVS, and the most direct way is to mimic the construction of dual pairs.

A POVS dual pair \((E, F)\) is a dual pair, whose members are positively generated POVS, such that

- for any \(\phi \in F\) it holds that \(\phi \in F_+\) iff \(\forall v \in E_+ \langle \phi, v \rangle \geq 0\);
- for any \(v \in E\) it holds that \(v \in E_+\) iff \(\forall \phi \in F_+ \langle \phi, v \rangle \geq 0\).

If \((E, F), (E', F')\) are POVS dual pairs, we say that a dual pair map \((L, M) : (E, F) \to (E', F')\) is positive if the maps \(L : E \to E', M : F' \to F\) are positive. (In fact it is sufficient to require positivity of any one of the two maps.) We say that \((L, M)\) is regular if \(L\) and \(M\) are regular.

A POVS dual pair map is a dual pair map that is positive.

Since positively generated POVS are essentially just abstract cones it may be reasonable to consider directly dual pairs of cones.

A cone dual pair is a pair \((P, Q)\) of cones together with an \(R_+-\)bilinear pairing
\[
\langle ., . \rangle : Q \times P \to R_+,
\]
such that

- for any \(u, v \in P\) it holds that \(v \geq u\) iff \(\forall \phi \in Q \langle \phi, v \rangle \geq \langle \phi, u \rangle\);
- for any \(\phi, \psi \in Q\) it holds that \(\phi \geq \psi\) iff \(\forall v \in P \langle \phi, v \rangle \geq \langle \psi, v \rangle\).

Note that the above definition implies also that the pairing is nondegenerate, i.e. separates points (since \(v \geq u\) and \(u \geq v\) implies \(u = v\)).

A map of cone dual pairs \((P, Q), (P', Q')\) is a pair of cone maps \(L : P \to P', M : Q \to Q'\) satisfying adjointness condition [1].

Note 8 The category of POVS dual pairs and POVS dual pair maps is equivalent to the category of cone dual pairs cone dual pair maps.

Proof follows from Note[7] □

Now, as in the case of ordinary dual pairs, we will try to hide explicit duals from notations and define reflexive POVS and reflexive cones.

3.2 Topology and partial order

Let \(X\) be a topological space equipped by with a partial order.

We say that the partial order on \(X\) is compatible with topology if the following property holds: whenever \(x \nleq y\), there exists a neighborhood \(U\) of \(x\) such that for all \(x' \in U\) it holds that \(x' \nleq y\).

Now let \(V\) be a TVS partially ordered by a proper cone \(V_+\).
**Note 9** The partial order on $V$ is compatible with topology iff the cone $V_+$ is closed.

**Proof** Exercise. □

This observation motivates the following definition.

A partially ordered topological vector space (POTVS) is a TVS $V$ equipped with a closed proper positive cone $V_+$. Now let $V$ be a general TVS, and $P$ be a cone in $V$.

The *dual cone* $P^*$ of $P$ is the subset of the topological dual space $V^*$ defined by

$$P^* = \{ \phi \mid \forall v \in P \langle \phi,v \rangle \geq 0 \}.$$  

The *bidual cone* $P^{**}$ is the subset of $P$ defined by

$$P^{**} = \{ v \mid \forall \phi \in P^* \langle \phi,v \rangle \geq 0 \}.$$  

**Remark** There is an ambiguity in notation for $P^{**}$, because we could interpret it also as a subset of the double dual space $V^{**}$. In our case this is harmless, because we always consider weakly reflexive $V$.

**Lemma 2** The bidual $P^{**}$ of a cone $P$ in the topological vector space $V$ coincides with the $\sigma(V,V^*)$-weak closure of $P$ in $V$.

**Proof** The dual cone $P^*$ can be described as

$$P^* = \{ \phi \in V^* | \forall v \in -P \langle \phi,v \rangle < 1 \}.$$  

Then the statement follows from the Bipolar theorem (see [23, IV.1.5] for a formulation). □

Note that the dual cone $P^*$ is always closed in the $\sigma(V^*,V)$-weak topology.

We define the *dual POTVS* $V^*$ of a POTVS $V$ as the space $V^*$ equipped with the $\sigma(V^*,V)$-weak topology and the positive cone $V_+^*$.

**Lemma 2** implies the following.

**Note 10** If $V$ is a POTVS which is weakly reflexive as a TVS, then $V^{**}$ is isomorphic to $V$ both as a POVS and as a TVS. □

We will be interested in the more restricted case of weakly reflexive positively generated POTVS.

### 3.2.1 Reflexive POTVS

Let us say that a *weakly reflexive POTVS* is a positively generated POTVS which is weakly reflexive as a TVS and whose dual is positively generated as well.

It follows from the definition that the dual of a weakly reflexive POTVS is itself a weakly reflexive POTVS.

The following properties of weakly reflexive POTVS are direct analogues of the corresponding properties of weakly reflexive TVS.
Note 11 A positive linear map of weakly reflexive POTVS is continuous iff it is adjointable.

Proof immediate from Note 7 and Note 2. □

Note 12 The category of weakly reflexive POTVS and continuous positive maps is equivalent to the category of POVS dual pairs and POVS dual pair maps under the correspondence

\[ V \mapsto (V, V^*) \).

The equivalence preserves duality.

Proof same as Note 3. □

3.3 Uniformity and cones

We want to describe positive cones in POTVS intrinsically. It turns out that embedding in a TVS equips a cone not only with a topology, but also with a structure of a uniform space.

3.3.1 Uniform spaces

Recall that a uniformity \( \mathcal{U} \) (see [18]) on a set \( X \) is a nonempty collection of subsets of \( X \times X \) satisfying the following properties:

(i) if \( U \in \mathcal{U} \) then \( \Delta_X \subseteq U \), where \( \Delta_X \) is the diagonal,

\[ \Delta_X = \{(x, x) | x \in X\}; \]

(ii) if \( U \in \mathcal{U} \) then \( U^{-1} \in \mathcal{U} \);

(iii) if \( U \in \mathcal{U} \) then there exists \( V \in \mathcal{U} \) such that \( V \circ V \subseteq U \);

(iv) if \( U, V \in \mathcal{U} \) then \( U \cap V \in \mathcal{U} \);

(v) if \( U \in \mathcal{U} \) and \( U \subseteq V \subseteq X \times X \) then \( V \in \mathcal{U} \).

Elements of a uniformity are treated in (ii), (iii) as binary relations on \( X \). The inverse in (ii) means the inverse (or opposite) relation

\[ U^{-1} = \{(x, y) | (y, x) \in U\}, \]

and composition in (iii) is composition of relations.

Elements of uniformity are also called entourages.

A base of a uniformity \( \mathcal{U} \) is any subfamily \( \mathcal{B} \) of \( \mathcal{U} \) such that any entourage in \( \mathcal{U} \) contains some element of \( \mathcal{B} \) as a subset. In other words, \( \mathcal{U} \) is the upward closure of \( \mathcal{B} \).

Note 13 A family \( \mathcal{B} \) forms a base for some uniformity if \( \mathcal{B} \) satisfies conditions (i)-(iv) in the definition above.
Proof Exercise or see [18] Theorems 5.2, 5.3. □

A uniform space is a space equipped with a uniformity.

A prototypical example of a uniform space is metric space. For any metric space $X$ the collection of sets $U_\epsilon$ of the form

$$U_\epsilon = \{(x,y) \mid d(x,y) < \epsilon\}$$

for $\epsilon > 0$ forms a base of a uniformity.

In particular, property (iii) of the above definition corresponds to the fact that for $\epsilon' < \frac{\epsilon}{2}$ we have $U_\epsilon \circ U_{\epsilon'} \subseteq U_{\epsilon'}$.

Given a uniform space $X$, for any $x \in X$ and entourage $U$ we use the notation

$$U[x] = \{y \in X \mid (x,y) \in U\}.$$ 

The collection $\{U[x] \mid x \in X\}$ can be thought of as a system of uniform neighborhoods on $X$.

The uniform topology on a uniform space $X$ is defined by saying that a set $A$ is open iff for any element $x \in A$ there is an entourage $U$ such that the whole uniform neighborhood $U[x]$ is contained in $A$.

It is easy to check that uniform topology is indeed a well-defined topology.

Given two uniform spaces $X,Y$, a function $f : X \to Y$ is uniformly continuous iff for any entourage $V$ in $Y \times Y$ the counterimage $(f \times f)^{-1}(V)$ is an entourage in $X \times X$.

Note 14 Uniformly continuous functions are continuous for the corresponding uniform topologies.

Proof Exercise or see [18] Theorem 5.9. □

Uniform spaces are closed under subspaces, Cartesian products and quotient spaces.

Subspaces. If $X$ is a uniform space and $Y \subseteq X$ is a subset, then the subspace or relative uniformity of $Y$ is defined by the collection of sets $U \cap Y \times Y$, where $U \subseteq X \times X$ is an entourage for $X$.

It is immediate that the corresponding uniform topology on $Y$ is the subspace topology induced by the uniform topology of $X$ and that the inclusion $Y \to X$ is uniformly continuous.

Products. If $X, Y$ are uniform spaces, then the product uniform space $X \times Y$ is defined by the product uniformity whose base is the family of all sets of the form

$$U_1 \ast U_2 = \{(x,y,x',y') \mid (x,x') \in U_1, (y,y') \in U_2\},$$

where $U_1$ is some entourage for $X$ and $U_2$ is some entourage for $Y$.

Note 15 The uniform topology of a product uniform space $X \times Y$ is the product of uniform topologies of the factors $X,Y$. 

Proof
The projections

\[ \pi_1 : X \times Y \to X, \quad \pi_2 : X \times Y \to Y \]

are uniformly continuous.

A function \( f : Z \to X \times Y \), where \( Z \) is a uniform space, is uniformly continuous, iff the compositions \( \pi_1 \circ f \), \( \pi_2 \circ f \) are uniformly continuous.

**Proof** Exercise of see [18, Theorem 5.10] □

It follows from the above that the product of uniform spaces is indeed a Cartesian product (in the categorical sense) in the category of uniform spaces and uniformly continuous functions.

**Quotients.** Let \( X \) be a uniform space, \( Y \), a set, and \( \pi : X \to Y \), a surjective function.

**Theorem 1** ([16]) There exists the largest uniformity on \( Y \), called **quotient uniformity**, making \( \pi \) uniformly continuous.

**Quotient uniformity** satisfies the following universal property: for any uniform space \( Z \) and function \( f : Y \to Z \), it holds that \( f \) is uniformly continuous iff \( f \circ \pi : X \to Z \) is uniformly continuous. □

In the setting as above, the space \( Y \) equipped with the quotient uniformity is called the **uniform quotient** (of \( X \)).

It should be noted that the uniform topology induced by a quotient uniformity in general does **not** coincide with the corresponding quotient topology [16].

**Note 16** In the setting as above, assume that the uniformity \( \mathcal{U} \) on \( X \) is such that the collection

\[ \pi(\mathcal{U}) = \{ U \mid U \subseteq Y \times Y \text{ and } (\pi \times \pi)^{-1}(U) \in \mathcal{U} \} \]

also is a uniformity.

Then the quotient uniformity on \( Y \) coincides with \( \pi(\mathcal{U}) \).

**Proof** Exercise. □

### 3.3.2 TVS and uniformity

For our purposes, an important example of a uniform space is a TVS.

Given a TVS \( E \), we define a uniformity by taking as the base of entourages all sets of the form

\[ \hat{U} = \{(u, v) \in E \times E \mid u - v \in U\}, \quad (8) \]

where \( U \) is a neighborhood of 0 in \( E \).

**Note 17** The system defined by (8) is indeed a base for a uniformity.
Proof In view of Note 13 it is sufficient to check properties (i)-(iv) of the definition of a uniformity above.

The only nontrivial case is property (iii).

Let $U$ be a neighborhood of 0 in a TVS $E$.

Since addition is continuous, there exists a neighborhood $V$ of 0 in $E$ such that $V + V \subseteq U$. Then for any entourage $\tilde{U}$ of form (13) we have that $\tilde{V} \circ \tilde{V} \subseteq \tilde{V}$.

□

Subsets of form (8) have an important property of translation invariance.

Let us say that a subset $U$ of $E \times E$, where $E$ is an additive semi-group, is translation invariant if for any $x \in E$ it holds that $(a, b) \in U$ iff $(a + x, b + x) \in U$.

Note 18 With uniformity on a TVS $E$ defined by (13), the uniform topology coincides with original topology.

Addition and scalar multiplication and all continuous linear maps are uniformly continuous.

Translation-invariant entourages form a base of uniformity.

Proof Exercise. □

3.3.3 Uniform cones

We say that a uniform cone $P$ is a cone equipped with a uniformity such that

(i) addition $P \times P \to P$

and scalar multiplication $\mathbb{R}_+ \times P \to P$

are uniformly continuous;

(ii) intrinsic partial order (5) on $P$ is compatible with the uniform topology (see Section 3.2);

(iii) translation invariant entourages form a base of the uniformity.

(In (i), the space $\mathbb{R}_+$ is considered as a uniform space, with the base of uniformity given by sets \{$(a, b) \in \mathbb{R}_+ \times \mathbb{R}_+ | |a - b| < \epsilon$\}

where $\epsilon > 0$.)

Note 19 Given a POTVS $E$, the positive cone $E_+$ becomes a uniform cone under the subspace uniformity induced by the uniformity of $E$.

Proof by Note 13 and Note 9. □

We make a couple of observations on relationship between uniformities of a positively generated POTVS and its positive cone.

For a POTVS $E$ and a subset $U \subseteq E \times E$ let us denote $U_+ = U \cap E_+ \times E_+$.
Let
\[ s : E \times E \to E, \quad (u,v) \mapsto u - v, \]
be the subtraction map.

**Note 20** Let \( E \) be a positively generated \( \text{POTVS} \).

For any translation invariant subset \( U \subseteq E \times E \) it holds that
\[ U = s^{-1}(s(U_+)). \tag{9} \]

**Proof** The set \( U \) is translation invariant \iff \( U = s^{-1}(s(U)) \).

But \( s(U) = s(U+) \), since for any \((u,v) \in U\), writing \( u = u_+ - u_- \) and \( v = v_+ - v_- \), where \( u_+, u_-, v_+, v_- \geq 0 \), we have \( u - v = (u_+ + v_-) - (v_+ + u_-) \).

But \((u_+ + v_-, v_+ + u_-) = (u + u_- + v_+, v + u_+ + v_-) \in U_+ \).

**Note 21** If \( E \) is a positively generated \( \text{POTVS} \), then a subset \( U \subseteq E \times E \) is an
entourage of \( E \) \iff \( U_+ \) is an entourage of \( E_+ \).

**Proof** The only if part is obvious.

Assume that \( U_+ \) is an entourage of \( E_+ \). Then \( U_+ = U_0 \cap E_+ \times E_+ \) for some
entourage \( U_0 \) of \( E \). The set \( U_0 \) contains some translation invariant entourage \( V \) of \( E \), since translation invariant entourages form a base by Note 8. Then \( V_+ \subseteq U_+ \). It follows from the preceding note that \( V \subseteq U \), hence \( U \) is an
entourage. \( \square \)

### 3.3.4 Enveloping space of a uniform cone

Now let \( P \) be a uniform cone. Consider the enveloping \( \text{POVS} \) \( EP \).

We topologize the enveloping space \( EP \) by a uniform topology.

The uniformity is defined by the base consisting of all sets of the form
\[ DU = s^{-1}(\pi(U)), \tag{10} \]
where \( U \) is some translation invariant entourage of \( P \), and
\[ \pi : P \times P \to EP, \quad (u,v) \mapsto [u,v], \]
is the quotient map.

**Lemma 3** Sets defined by (10) form a base of uniformity.

**Proof** By Note 13 we need to check properties (i)-(iv) in the definition of uniformity.

We prove property (iii) and leave the rest as an exercise.

Let \( U \) be a translation invariant entourage of \( P \). By uniform continuity of addition, there exists a translation invariant entourage \( V \) of \( P \) such that \( V + V \subseteq U \). It is easy to check that \( DV \circ DV \subseteq DU \). \( \square \)

**Lemma 4** In the setting as above, the uniformity of \( EP \) is the quotient uniformity induced by \( \pi : P \times P \to EP \).
Proof Denoting the uniformities of $P$ and $EP$, respectively, as $\mathcal{U}$ and $E\mathcal{U}$, we will prove that $E\mathcal{U} = \pi(\mathcal{U})$ and refer to Note 14.

A direct computation shows that, for any two subsets $U_1, U_2 \subseteq P \times P$ we have

$$s(\pi \times \pi(U_1 \times U_2)) = \pi(U_1 + U_2^{-1}).$$

If $U_1, U_2$ are translation invariant, then the image $\pi \times \pi(U_1 \times U_2) \subseteq EP \times EP$ is translation invariant as well, hence

$$\pi \times \pi(U_1 \times U_2) = s^{-1}(s(\pi \times \pi(U_1 \times U_2))),$$
i.e.

$$\pi \times \pi(U_1 \times U_2) = s^{-1}(\pi(U_1 + U_2^{-1})).$$ \hfill (11)

Now let $U \in \pi(\mathcal{U})$. Then $U$ contains a subset of the form $\pi \times \pi(U_1 \times U_2)$, where $U_1, U_2$ are translation invariant elements of $\mathcal{U}$. By (11) we have that $U \supseteq s^{-1}(\pi(U_1 + U_2^{-1})) \supseteq s^{-1}(\pi(U_1 + (0, 0))) = DU_1$.

So $U \in E\mathcal{U}$.

Let $U \in E\mathcal{U}$. Then $U$ contains a subset of the form $DU_0$, where $U_0$ is some translation invariant element of $\mathcal{U}$. By uniform continuity of addition there is $V \in \mathcal{U}$ such that $V + V \subseteq U_0$. By (11) we have: $U \supseteq s^{-1}(\pi(V + V)) = \pi \times \pi(V + V)$. So $V \in \pi(\mathcal{U})$. \hfill □

Corollary 1 Addition and scalar multiplication on $EP$ are uniformly continuous. \hfill □

Corollary 2 For topological cones $P, Q$, any uniformly continuous cone map $L : P \to Q$ extends to a unique continuous positive map $EL : EP \to EQ$, where topologies on $EP, EQ$ are defined as above.

Lemma 5 The image $i(P)$ of $P$ under the inclusion

$$i : P \to EP, \quad v \mapsto [v, 0]$$
is closed.

Proof Let $w = [u, v] \in EP$ be in the complement of $i(P)$. Then $u \not\geq v$.

By definitions of uniform cone and uniform topology, there exists an entourage $U_0$ of $P$ such that for any $u' \in U_0[u]$ we have $u' \not\geq v$. Without loss of generality $U_0$ is translation invariant.

Let $U = DU_0$. This is an entourage of $EP$.

Then $U[w] = \{w' \in EP | w - w' \in \pi(U_0)\}$.

Assume that $i(P)$ in $EP$ has nonempty intersection with $U[w]$. Then there exists $u' \in P$ such that $w' = [u', 0] \in U[w]$, hence $w - w' = [u, v + u'] \in \pi(U_0)$. But $U_0$ is translation invariant, which means precisely that $U_0 = \pi^{-1}(\pi(U_0))$. So $(u, v + u') \in U_0$, and $v + u' \in U_0[u]$. But $v + u' \geq v$, which gives us a contradiction.

Thus $U[w]$ is contained in the complement of $i(P)$, and since $w$ was arbitrary it follows that $i(P)$ is closed. \hfill □

It follows that $EP$ topologized as the uniform quotient of $P \times P$ is a POTVS.

We will say that $EP$ with such a topology is the *enveloping* POTVS of $P$. 

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Note 22 Let $V$ be a positively generated POTVS, and let $V_+$ be its positive cone considered as a uniform cone under the subspace uniformity. Then the enveloping POTVS $EV_+$ and $V$ are isomorphic as POTVS.

Proof The isomorphism is given by the map

$$F: [u,v] \mapsto u - v.$$ 

It follows from Note 21 and Note 20 that $F \times F$ and $F^{-1} \times F^{-1}$ preserve entourages (compare (9) and (10)). □

Note 23 The uniformity of a uniform cone $P$ coincides with the subspace uniformity induced by the inclusion in $EP$.

Proof Let us treat $P$ as a subset of $EP$.

For any subset $A \subseteq EP$, we have $\pi^{-1}(A) = s^{-1}(A) \cap P \times P$. For any translation invariant subset $B$ of $P$, we have $B = \pi^{-1}(\pi(B))$. Thus for any translation invariant entourage $U$ of $P$ we have

$$DU \cap P \times P = s^{-1}(\pi(U)) \cap P \times P = \pi^{-1}(\pi(U)) = U.$$ 

Since translation invariant subsets form a base, the statement is proven. □

Corollary 3 The category of uniform cones and uniformly continuous cone maps is equivalent to the category of positively generated POTVS and positive continuous linear maps.

The equivalence is given by the assignment $P \mapsto EP$.

Proof follows from Note 7, Note 22, Note 23 and Corollary 2. □

3.3.5 Duality and reflexivity

If $P$ is a cone and $Q$ is a collection of cone maps $P \to \mathbb{R}_+$, we define the uniformity of pointwise convergence on elements of $Q$ by the base consisting of subsets of the form

$$U_{\phi_1, \ldots, \phi_k, \epsilon} = \{(u,v) \in P \times P | \forall i = 1, \ldots, k | \langle \phi, u \rangle - \langle \phi, v \rangle | < \epsilon \}, \hspace{1cm} (12)$$

where $k \in \mathbb{N}$, $\phi_1, \ldots, \phi_k \in Q$ and $\epsilon > 0$.

It is easy to check that the above system is indeed a uniformity, and the corresponding uniform topology is the topology of pointwise convergence on elements of $Q$.

If $(P,Q)$ is a cone dual pair, we define the $\sigma(P,Q)$-weak uniformity, on $P$ as the uniformity of pointwise convergence on elements of $Q$, and the $\sigma(P,Q)$-weak topology, as the corresponding uniform topology. As usual, we will often say simply weak uniformity (topology) without specifying the dual pair.

We define the uniform dual $P^*$ of a uniform cone $P$ as the space of uniformly continuous cone maps $P \to \mathbb{R}_+$ equipped with the $\sigma(P^*,P)$-weak uniformity.
Now, if $P$ is a uniform cone, then every element $v \in P$ defined by pairing a $\mathbb{R}_+$-linear functional on $P^*$, and it is immediate from definition of uniformity for $P^*$ that this functional is uniformly continuous.

We say that a uniform cone $P$ is **reflexive** if the natural map

$$ P \rightarrow P^{**} \tag{13} $$

is an isomorphism of uniform spaces.

Reflexive uniform cones correspond to weakly reflexive POTVS.

**Note 24** If $P$ is a reflexive uniform cone then the enveloping POTVS $EP$ is weakly reflexive with

$$ (EP)^* \cong E(P^*). \tag{14} $$

**Proof** The pairing between $P$ and $P^*$ extends in the obvious way to a pairing between $EP$ and $E(P^*)$. It is easy to see that the latter is nondegenerate since the former is nondegenerate.

It follows that $(EP, E(P^*))$ is a dual pair.

Now the uniformity on $P$ is that of pointwise convergence on elements of $P^*$. It follows from formula defining the uniformity on $EP$ that the uniformity on $EP$ is defined by pointwise convergence on elements of $P^*$ as well, hence the topology on $EP$ is that of pointwise convergence on elements of $P^*$. The latter is equivalent to the $\sigma(EP, E(P^*))$-weak topology, since $E(P^*)$ is the algebraic span of $P^*$. The statement follows from Lemma 1. □

**Note 25** If $V$ is a weakly reflexive POTVS, then the cone $V_+$ equipped with the subspace uniformity is reflexive.

**Proof** By Note 22, we have that $V \cong EV_+$. By Corollary 2, we have that the space $V_+^*$ of uniformly continuous cone maps from $V_+$ to $\mathbb{R}_+$ coincides with the space of positive continuous linear maps from $V$ to $\mathbb{R}$. The latter set is just the positive cone of the dual POTVS $V^*$. The same analysis applies to $V^*$ and $V^+_+$. I.e., the cone $V^+_+$ coincides with the positive cone in the double dual $V^{**}$. The latter cone is just $V_+$ since the POTVS $V$ is assumed weakly reflexive. □

The following must be routine by now.

**Corollary 4** The category of reflexive uniform cones and uniformly continuous cone maps is equivalent to the category of weakly reflexive POTVS and positive continuous linear maps. The equivalence is given by the assignment $P \mapsto EP$. The equivalence preserves duality. □

Putting together Note 12 and Corollary 4 we get the summary.

**Note 26** The following categories are equivalent:

- the category of POVS dual pairs and POVS dual pair maps;
- the category of cone dual pairs and cone dual pair maps;
• the category of weakly reflexive POTVS and positive continuous linear maps;
• the category of reflexive uniform cones and uniformly continuous cone maps.

Corollary 5 A map \( L : P \to Q \) of reflexive uniform cones is uniformly continuous iff there exists an adjoint map \( L^* : Q^* \to P^* \) defined by (2).

Proof Exercise. □

Corollary 6 If \((P, Q)\) is a cone dual pair and \(P\) is equipped with the \(\sigma(P, Q)\)-weak uniformity, then \(P^*\) coincides, as a set, with \(Q\).

Proof Exercise. □

4 Positivity and tensor product

Let \((E_i, F_i), i = 1, 2,\) be POVS dual pairs.

Let us say that a bilinear functional

\[ f : E_1 \times E_2 \to \mathbb{R} \]

is separately weakly continuous if for any \(v_i \in E_i, i = 1, 2,\) the functionals

\[ f(v_1,.) : E_2 \to \mathbb{R}, \quad f(.,v_2) : E_1 \to \mathbb{R} \]

are weakly continuous.

We say that a bilinear separately weakly continuous functional \(f\) is positive if for all \(v_1 \in (E_1)_+, v_2 \in (E_2)_+\) we have \(f(v_2, v_2) \geq 0\). We say that \(f\) is regular if it is a difference of two positive functionals.

Consider the space \(F_1 \mathcal{N}_{top} F_2\) of regular bilinear separately weakly continuous functionals on \(E_1 \times E_2\).

The cone \((F_1 \mathcal{N}_{top} F_2)_+\) of positive elements makes \(F_1 \mathcal{N}_{top} F_2\) a positively generated POVS.

We now specify a dual (i.e. a weak topology) for this space.

Consider the algebraic tensor product \(E_1 \otimes_{alg} E_2\).

We have the pairing \(\langle ., . \rangle\) between \(E_1 \otimes_{alg} E_2\) and \(F_1 \mathcal{N}_{top} F_2\), given for any

\[ w = \sum u_i \otimes v_i \in E_1 \otimes_{alg} E_2, \]

where all \(u_i \in E_1, v_i \in E_2,\) and any functional \(f \in F_1 \mathcal{N}_{top} F_2\) by the formula

\[ \langle f, w \rangle = \sum f(u_i, v_i) \tag{15} \]

Let \(E_1 \otimes_{top} E_2\) be the space \(E_1 \otimes_{alg} E_2\) quotiented by the null-space

\[ W = \{ w \in E_1 \otimes_{alg} E_2 | \langle f, w \rangle = 0 \forall f \in F_1 \mathcal{N}_{top} F_2 \}. \]
Then pairing \([15]\) descends to \(E_1 \otimes_{\text{top}} E_2\) and becomes nondegenerate. Thus \((E_1 \otimes_{\text{top}} E_2, F_1 \mathcal{H}_{\text{top}} F_2)\) is a dual pair of vector spaces.

We define the positive cone \((E_1 \otimes_{\text{top}} E_2)_+\) in \(E_1 \otimes_{\text{top}} E_2\) as the dual cone

\[
(E_1 \otimes_{\text{top}} E_2)_+ = (F_1 \mathcal{H}_{\text{top}} F_2)_+^*,
\]

which makes \(E_1 \otimes_{\text{top}} E_2\) a POVS.

Now, the weak topology on \(F_1 \mathcal{H}_{\text{top}} F_2\) (of pointwise convergence on elements of \(E_1 \otimes_{\text{top}} E_2\)) is just the topology of pointwise convergence on elements of the form \(v_1 \otimes v_2, v_i \in E_i, i = 1, 2\), since they span algebraically the whole \(E_1 \otimes_{\text{top}} E_2\).

It is immediate then that the positive cone \((F_1 \mathcal{H}_{\text{top}} F_2)_+^*\) closed in the weak topology, hence \(F_1 \mathcal{H}_{\text{top}} F_2\), under the weak topology, becomes a POTVS. Then \(E_1 \otimes_{\text{top}} E_2\), equipped with the weak topology, is precisely the dual POTVS, and the two POTVS are weakly reflexive.

Or, using correspondence between weakly reflexive POTVS and POVS dual pairs (Note 12), the pair \((E_1 \otimes_{\text{top}} E_2, F_1 \mathcal{H}_{\text{top}} F_2)\) is a POVS dual pair.

In the above setting, we define the topological tensor product \((E_1, F_1) \otimes_{\text{top}} (E_2, F_2)\) of positive dual pairs as the dual pair

\[
(E_1, F_1) \otimes_{\text{top}} (E_2, F_2) = (E_1 \otimes_{\text{top}} E_2, F_1 \mathcal{H}_{\text{top}} F_2),
\]

and the topological cotensor product as the dual

\[
(E_1, F_1) \mathcal{H}_{\text{top}} (E_2, F_2) = (E_1 \mathcal{H}_{\text{top}} E_2, F_1 \otimes_{\text{top}} F_2).
\]

Definition for tensor product of weakly reflexive POTVS is similar. We refer to Note 26 and do not fill in the details.

Now, let \(P_1, P_2\) be reflexive uniform cones. Then the enveloping POTVS \(EP_1, EP_2\) are weakly reflexive and give rise to POVS dual pairs.

We define the topological tensor product \(P_1 \otimes_{\text{top}} P_2\) of cones \(P_1, P_2\) as the cone of positive elements in the topological tensor product of POTVS

\[
P_1 \otimes_{\text{top}} P_2 = (EP_1 \otimes_{\text{top}} EP_2)_+^*
\]
equipped with the subspace uniformity and the topological cotensor product \(P_1 \mathcal{H}_{\text{top}} P_2\), as the uniform dual

\[
P_1 \mathcal{H}_{\text{top}} P_2 = (P_1^* \otimes_{\text{top}} P_2^*)^*.
\]

Adaptation to the case of cone dual pairs is similar.

We leave it to the reader to check that the definitions for cones agree with definitions for POTVS under the correspondence from Note 26.

We also note that topological cotensor product \(P_1 \mathcal{H}_{\text{top}} P_2\) can be equivalently described as the cone of \(\mathbb{R}_+\)-bilinear separately uniformly continuous functionals on \(P_1^* \times P_2^*\).
4.1 Internal homs

Given two POVS dual pairs $A_i = (E_i, F_i)$, $i = 1, 2$, the space $E_1 \rightarrow E_2$ of regular dual pair maps from $A_1$ to $A_2$ is a vector space in the obvious way. The subset $(E_1 \rightarrow E_2)_+$ of positive maps is a proper generating cone making $E_1 \rightarrow E_2$ a positively generated POVS.

Note 27 The POVS $E_1 \rightarrow E_2$ and $F_1 \mathcal{N}_{\text{top}} E_2$ are isomorphic.

Proof The isomorphism sends a dual pair map $(L, M) : A_1 \rightarrow A_2$ to the functional

$$(u, \phi) \mapsto \langle \phi, Lu \rangle.$$ 

In the opposite direction, if $f \in F_1 \mathcal{N}_{\text{top}} E_2$, then for any $u \in E_1$ the functional $f(u, \cdot)$ is weakly continuous on $F_2$, hence it is represented as an element of $E_2$. This gives us a linear map from $E_1$ to $E_2$, $u \mapsto f(u, \cdot)$. We also get a linear map from $F_2$ to $F_1$ given by the similar map $\phi \mapsto f(\cdot, \phi)$. This pair of linear maps constitutes a dual pair map from $A_1$ to $A_2$.

That the isomorphism preserves partial order is immediate □

Using the above isomorphism, we define the internal homs dual pair $A_1 \rightarrow A_2$ as

$$A_1 \rightarrow A_2 = (E_1 \rightarrow E_2, E_1 \otimes_{\text{top}} F_2).$$

An adaptation of this definition for weakly reflexive POTVS or cones is left to the reader.

Theorem 2 The category of POVS dual pairs and POVS dual pair maps is $\ast$-autonomous, i.e. there is a natural bijection

$$\text{Hom}(A, B \rightarrow C) \cong \text{Hom}(A \otimes_{\text{top}} B, C).$$

The same applies to the categories of cone dual pairs, weakly reflexive POTVS and weakly reflexive cones.

Proof follows from Note 27 (and Note 26). □

5 Adding norms

It is well agreed that in the setting of vector spaces the additive connectives of linear logic correspond to norms, see [13].

We are now going to consider normed cones.

Let $P$ be a cone. We say that a functional

$$\|\cdot\| : P \rightarrow \mathbb{R}_+$$

is a cone norm if the following properties hold:

- $\|\lambda v\| = \lambda \|v\|$;
• \|u + v\| \leq \|u\| + \|v\|;

• \|v\| = 0 iff \ v = 0;

• if \ u \geq v \ then \ \|u\| \geq \|v\|.

We define a \textit{normed cone} as a cone equipped with a norm. If \( P \) is a normed cone, we say that an \( \mathbb{R}_+ \)-linear functional \( \phi : P \to \mathbb{R}_+ \) is \textit{norm-bounded}, or simply \textit{bounded}, iff there exists

\[ ||\phi|| = \sup_{\|v\| \leq 1} \phi(v) < \infty. \tag{16} \]

Similarly, a cone map \( L \) is bounded iff

\[ ||L|| = \sup_{\|v\| \leq 1} ||Lv|| < \infty. \tag{17} \]

A bounded map of norm less or equal to 1 is a \textit{contraction}.

The \textit{norm dual cone} \( P' \) of the normed cone \( P \) is the cone of norm-bounded \( \mathbb{R}_+ \)-linear functionals from \( P \) to \( \mathbb{R}_+ \) equipped with dual norm \( \tag{16} \). (We use a prime in the superscript to avoid confusion with the uniform dual.) Note that \( P' \) is a well-defined normed cone itself.

5.1 Normed dual pairs and reflexive norms

We say that a \textit{normed cone dual pair} is a cone dual pair \((P, Q)\), where \( P, Q \) are normed cones, and the pairing is such that for all \( \phi \in Q, v \in P \) it holds that

\[ \|v\| = \sup_{\psi \in Q, \|\psi\| \leq 1} \langle \psi, v \rangle, \quad ||\phi|| = \sup_{u \in P, \|u\| \leq 1} \langle \phi, u \rangle. \tag{18} \]

If \((L, M) : (P, Q) \to (P', Q')\) is a cone dual pair map, then it is immediate that \( L \) is norm-bounded in the sense of \( \tag{17} \) iff \( M \) is, moreover, in this case \( ||L|| = ||M|| \).

We say that \((L, M)\) is a \textit{map} or a \textit{morphism of normed cone dual pairs} if \( L \) (hence \( M \)) is bounded.

We say that it is a \textit{normed cone dual pair contraction} if \( L \) is a contraction. As usual, we define now corresponding reflexive structures.

Let us say that a norm on the uniform cone \( P \) is \textit{reflexive} if

• any uniformly continuous functional in \( P^* \) is norm-bounded on \( P \), i.e. \( P^* \subseteq P' \);

• the \textit{canonical injection} \( P \to (P^*)' \) sending \( v \in P \) to the functional

\[ \phi \mapsto \langle \phi, v \rangle \tag{19} \]

is norm-preserving, i.e. for any \( v \in P \) we have

\[ \|v\| = \sup_{\phi \in P^*, \|\phi\| \leq 1} \langle \phi, v \rangle. \tag{20} \]

23
A reflexive uniform normed cone $P$ is a reflexive uniform cone equipped with a reflexive norm.

**Note 28** If $P$ is a reflexive uniform normed cone, then the uniform dual $P^*$ of $P$ equipped with dual norm (16) is reflexive itself.

**Proof** Exercise. □

The following must be now routine.

**Note 29** The category of reflexive uniform normed cones and uniformly continuous contractions is equivalent to the category of normed cone dual pairs and contractions. The equivalence is given by the correspondence $P \mapsto (P, P^*)$ and preserves duality.

**Proof** Exercise. □

We would also like to have an equivalent category of reflexive normed POTVS. Basically this means an extension of reflexive cone norms to enveloping weakly reflexive spaces that commutes with duality. Unfortunately, we do not know if such an extension is possible in general.

We discuss now certain extension, which indeed commutes with duality in the finite-dimensional case.

### 5.2 Norm extensions

Let $E$ be a POVS equipped with a usual vector space norm.

A norm on $E$ is called regular if the following properties hold:

(i) for $u, v \in E$, if $-u \leq v \leq u$ then $||v|| \leq ||u||$;

(ii) for any $v \in E$ and $\epsilon > 0$ there exists $u \in E$ such that $u \geq \pm v$ and $||u|| < ||v|| + \epsilon$.

Now let us denote the dual normed space as $E'$. Since $E$ is a POVS, the dual $E'$ has natural partial ordering defined by the cone of positive functionals.

**Note 30** Let $E$ be a positively generated POVS with a positive cone $P$. Assume that $P$ is equipped with a cone norm, which is extended to a regular norm on the whole $E$. Then for any norm-bounded $\mathbb{R}_+$-linear functional on $P$, its linear extension to the whole $E$ is norm-bounded as well. Moreover, the dual cone norm on $P'$ given by (16) coincides with the dual vector space norm on $P'$ inherited from $E'$:

$$||\phi|| = \sup_{v \in E, ||v|| \leq 1} |\langle \phi, v \rangle|.$$  \hspace{1cm} (21)

**Proof** The norm on $E$ is regular, so for any $v \in E$ and any $\epsilon > 0$ there exists $u \geq \pm v$ with $||u|| < ||v|| + \epsilon$. But, if $u \geq \pm v$, then $u \geq \frac{1}{2}v - \frac{1}{2}v = 0$, so $u \in P$.

So, if $\phi \in P'$, then

$$\sup_{v \in E, ||v|| \leq 1} |\langle \phi, v \rangle| \leq \sup_{u \in P, ||u|| < 1 + \epsilon} \langle \phi, u \rangle$$
(since $|\langle \phi, v \rangle| = \max(|\langle \phi, v \rangle|, |\langle \phi, -v \rangle|)$ and $\phi$ is a monotone functional.)
Since $\epsilon > 0$ is arbitrary the claim is proven. □

**Theorem 3** ([6]) If $E$ is a Banach POVS with a regular norm, then the norm on $E'$ is also regular.

**Proof** See [6, Lemma 2.4]. □

Now let $P$ be a normed cone.
We define a norm $|||.||_1$ on the enveloping POVS $EP$ by

$$ ||v||_1 = \inf \{ u \geq v, u \in P \} \|u\|. $$

(22)

(This norm is taken from [6].)

**Lemma 6** The norm $|||.||_1$ above is a well-defined regular vector space norm on $EP$ extending the original cone norm on $P$.

**Proof** Any element $v$ of $EP$ has a representation $v = v_+ - v_-$ with $v_+, v_- \in P$. Then $u = v_+ + v_- \in P$ and $u \geq \pm v$. This shows that $||v||_1$ is defined for any $v \in EP$. Also, if $v \in P$ and $u \geq \pm v$, then $u \in P$ and $||u|| \geq ||v||$. This shows that $||v||_1 \geq ||v||$. But $v \geq \pm v$, hence $||v||_1 \leq ||v||$. This shows that $||v||_1 = ||v||$ for $v \in P$.

Obviously $||\lambda v||_1 = ||\lambda|| \cdot ||v||_1$, and $||v||_1 = 0$ iff $v = 0$.

Also, if $u_1 \geq \pm v_1$ and $u_2 \geq \pm v_2$ with $u_1, u_2 \in P$, then $u_1 + u_2 \geq \pm (v_1 + v_2)$ and $||u_1 + u_2|| \leq ||u_1|| + ||u_2||$. This shows that $||v_1 + v_2||_1 \leq ||v_1||_1 + ||v_2||_1$.

This shows that $||.||_1$ is indeed a vector space norm.

Now, if $-u \leq v \leq u$, then $\frac{1}{2}u \geq \pm \frac{1}{2}u$, hence $u = \frac{1}{2}u + \frac{1}{2}u \geq \frac{1}{2}u - \frac{1}{2}u \geq 0$, so $u \in P$. Also $u \geq \pm v$, hence $||v||_1 \leq ||u|| = ||u||_1$. Thus shows that property (i) of regular norm holds.

Property (ii) is obvious from definition. □

**Remark** Norm extension (22) from $P$ to $EP$ is by no means unique. For example, we can define another norm on $EP$ by

$$ ||v||' = \inf_{v_+, v_- \in P, u = v_+ - v_-} (||v_+|| + ||v_-||), $$

which coincides with (22) on $P$, but is different otherwise (as can be seen already from simple two-dimensional examples.)

### 5.2.1 In finite dimensions

**Corollary 7** If a reflexive uniform cone $P$ is finite-dimensional, then any cone norm on $P$ is reflexive.

**Proof** The enveloping space $EP$ is finite-dimensional, hence extension (22) of the norm on $P$ makes $EP$ a Banach space.
Again, because of finite-dimensionality, topology does not matter, so \( P' = P^* \) is just the cone of all positive functionals on \( EP \).

Finally, \((EP)' = E(P')\), since \( P \) is reflexive.

By Theorem 5 and Lemma 6, the dual vector space norm on \( E(P') \) is regular.

By Note 30, the dual vector space norm of \( E(P') \) extends the dual cone norm of \( P' \).

Applying Note 30 once again we get that the vector space norm of \( EP'' \) coincides on \( P \) with the cone norm of \( P'' \).

But \( EP'' \cong EP \) as normed spaces, and the \( EP \) norm restricted to \( P \) is the original cone norm of \( P \). Hence cone norms of \( P \) and \( P'' \) coincide. □

Unfortunately we do not know under which conditions the above arguments can generalize to infinite dimensions and we do not have any classification of reflexive cone norms.

For possible generalizations it might be worth noting that there are some stronger versions of Theorem 5, see [20].

5.3 Completeness

Unlike the case of vector spaces, a normed cone does not have any intrinsic metric; a metric would require extending the norm to the whole enveloping space. Thus there is no analogue of Banach (i.e., metric complete) spaces in the setting of normed cones. Yet there is a specific notion of completeness, which will be crucial for our purposes.

Let \( P \) be a reflexive uniform normed cone. We say that \( P \) is a complete, if any norm-bounded monotone non-decreasing sequence \( v_0 \leq \ldots \leq v_n \leq \ldots \) in \( P \) has a weak limit in \( P \).

5.3.1 Alternative notions of completeness

It would be more accurate, but also more cumbersome, to use some term like sequential topological order-completeness rather than just completeness.

In fact there are other notions of completeness for normed cones.

In [8], an alternative definition is proposed, which does not involve any topology, but only norms and intrinsic partial order. A cone \( P \) is complete in the sense of [8] if any norm-bounded non-decreasing sequence in \( P \) has a least upper bound. (This could be called sequential order-completeness without the word “topological”.)

It is not hard to see that completeness in our sense implies completeness in the sense of [8], but not vice versa.

**Note 31** If a reflexive uniform normed cone \( P \) is complete (in the sense of this work), then for any norm-bounded monotone non-decreasing sequence \( \{v_n\} \subset P \) its weak limit \( \lim_{n \to \infty} v_n \) is also its least upper bound in \( P \).

**Proof** By definition there exists the weak limit \( v = \lim_{n \to \infty} v_n \in P \).
Then for any $\phi \in P^*$ we have $\langle \phi, v \rangle = \lim_{n \to \infty} \langle \phi, v_n \rangle$. But the numerical sequence $\{\langle \phi, v_n \rangle\}$ is monotone non-decreasing (by Note 3), hence its limit equals its supremum, so $\langle \phi, v \rangle = \sup_n \langle \phi, v_n \rangle$. It follows from Lemma 2 that the difference $v - v_n \in EP$ belongs to the closure of $P$ in $EP$. But, by Lemma 23, the image of $P$ in $EP$ is closed. Hence $v \geq v_n$ for all $n$.

Now if $v' \in P$ is some other upper bound of $\{v_n\}$, then for any $\phi \in P^*$ it holds that $\langle \phi, v' \rangle \geq \sup_n \langle \phi, v_n \rangle = \langle \phi, v \rangle$. Hence $v' \geq v$. Thus $v = \sup_n v_n$. □

Let us give an example of a reflexive uniform normed cone which is complete in our sense, but not in the sense of [8].

Consider the Banach space $l^\infty$ of all real sequences $a = \{a_n\}$ bounded in the norm $\|a\|\infty = \sup_n |a_n|$. The space $l^\infty$ is partially ordered by the cone $l^\infty_+$ consisting of non-negative sequences.

Restriction of the norm $\|\|\infty$ makes $l^\infty_+$ a normed cone, and it is immediate that any norm-bounded sequence $\{a^n\}$ of elements of $l^\infty_+$ has the least upper bound $a \in l^\infty_+$ defined by $a_k = \sup_n a^n_k$.

**Theorem 4 ([1])** There exists a norm-bounded linear functional $LIM : l^\infty \to \mathbb{R}$ such that

(i) $LIM(a) = \lim_{n \to \infty} a_n$ whenever the limit in the righthand side exists;

(ii) $\lim\inf_{n \to \infty} a_n \leq LIM(a) \leq \lim\sup_{n \to \infty} a_n$. □

(See [11, 7.2.1] for a modern proof in English.)

Condition (ii) above implies that $LIM$ is positive.

Let $(l^\infty)^\prime$ be the cone of all positive norm-bounded linear functionals on $l^\infty$ equipped with the dual norm.

Then $(l^\infty)^\prime$ contains the cone $l^1_+$ of nonnegative sequences $b = \{b_n\}$ bounded in the norm $\|b\|1 = \sum b_n$,

if we define the pairing by $\langle b, a \rangle = \sum a_n b_n$.

For any $a \in l^\infty_+$ we have $\|a\|\infty = \sup_{b \in l^1_+, \|b\|1 \leq 1} \langle b, a \rangle,$

and it follows that $\|a\|\infty = \sup_{b \in (l^\infty)^\prime, \|b\| \leq 1} \langle b, a \rangle$.

Thus $(l^\infty_+, (l^\infty)^\prime)$ is a normed cone dual pair.
We equip $l^\infty_+$ with the $\sigma(l^\infty_+, (l^\infty_+)'\}$-weak uniformity. This makes the topological normed cone $l^\infty_+$ a reflexive normed cone (since normed cone dual pairs correspond to reflexive normed cones).

Consider the sequence $\{e^{\leq n}\}_{n=0}^\infty \subset l^\infty_+$, where $e^{\leq n}$ is defined by
\[
e^{\leq n}_k = \begin{cases} 1 & \text{if } k \leq n; \\ 0 & \text{if } k > n. \end{cases}
\]
This sequence is norm-bounded and has the element $e = (1, 1, 1, \ldots) \in l^\infty_+$ as the least upper bound.

Assume that $l^\infty_+$ equipped with the $\sigma(l^\infty_+, (l^\infty_+)'\}$-weak topology is complete (in the sense of this work). Then, by Note 31, the limit of $\{e^{\leq n}\}$ in the $\sigma(l^\infty_+, (l^\infty_+)'\}$-weak topology is $e$. Since $\text{LIM} \in (l^\infty_+)'$, it must be that the sequence $\{\text{LIM}(e^{\leq n})\}$ converges to $\text{LIM}(e)$ as $n \to \infty$.

But, for all $n$, we have $\text{LIM}(e^{\leq n}) = 0$, and $\text{LIM}(e) = 1$. We get a contradiction.

It should be noted though that the example above is very non-constructive and relies heavily on the Axiom of choice (for proving existence of $\text{LIM}$).

We would expect that in all “natural”, constructive examples the two notions of completeness coincide.

### 5.3.2 Completing reflexive cones

Our main use of positivity is in dealing with power series when modeling the exponential fragment. Completeness then gives control on their convergence.

However, prior to interpreting exponentials, we need a multiplicative structure; i.e. we will need tensor products. Since topological tensor products, in general, will not be complete, some procedure of cone completion will be crucial.

It turns out that reflexive normed cones have canonical completions.

Below, we will use the following notation.

If $P$ is a uniform cone equipped with a norm, then $\tilde{P} = (P^*)'$ is the cone of all norm-bounded (not necessarily continuous) cone maps $P^* \to \mathbb{R}_+$, where the uniform dual $P^*$ of $P$ is equipped with the dual norm given by (16). The cone $\tilde{P}$, in its norm is equipped with the norm dual to that of $P^*$.

We also use the following terminology.

For a uniform cone $P$, we say that a subspace $P_0$ is a subcone of $P$ if $P_0$ is a $\mathbb{R}_+$-submodule of $P$ closed in the uniform topology.

It is easy to see that a subcone is itself a uniform cone, when equipped with the subspace uniformity.

When the ambient uniform cone $P$ is equipped with a norm, any subcone of $P$ becomes a normed uniform cone under the norm inherited from $P$.

Now let $P$ be a reflexive uniform normed cone.

Let the dual uniform cone $P^*$ be equipped with dual norm (16). Recall that $P^*$ is reflexive as well (Note 28).

Note 32 The normed cones $\tilde{P}$ and $P^*$ form a normed cone dual pair.
Proof Since the injection $P \to (P^*)' = \tilde{P}$ is norm preserving, we have that for any $v \in P^*$

$$||v|| = \sup_{u \in P: ||u|| \leq 1} \langle u, v \rangle \leq \sup_{\tilde{u} \in \tilde{P}: ||\tilde{u}|| \leq 1} \langle \tilde{u}, v \rangle.$$ 

On the other hand, obviously,

$$||v|| \geq \sup_{\tilde{u} \in \tilde{P}: ||\tilde{u}|| \leq 1} \langle \tilde{u}, v \rangle.$$ 

So

$$||v|| = \sup_{\tilde{u} \in \tilde{P}: ||\tilde{u}|| \leq 1} \langle \tilde{u}, v \rangle.$$ 

Analogous formula for $u \in \tilde{P}$ is obvious. □

We equip $\tilde{P}$ with the $(\tilde{P}, P^*)$-weak uniformity and the corresponding weak topology. This makes $\tilde{P}$ a reflexive uniform normed cone by Note 29.

Note 33 The normed uniform cone $\tilde{P}$ is complete.

Proof If $\{u_n\}$ is a monotone non-decreasing sequence of elements of $\tilde{P}$ bounded in norm by some $A \in \mathbb{R}$, then for any $v \in P^*$ the sequence $\{\langle u_n, v \rangle\}$ is monotone non-decreasing (by Note 5) and bounded by $A||v||$, hence it has a limit in $\mathbb{R}$. Then the functional $u$ defined by $\langle u, v \rangle = \lim_{n \to \infty} \langle u_n, v \rangle$ is norm-bounded by $A$, hence $u \in \tilde{P}$. But $u$ is precisely the weak limit of $\{u_n\}$. □

Note 34 If $Q$ is a subcone of $\tilde{P}$ containing $P$, then $Q$, equipped with the subspace uniformity and restriction of the norm, becomes a reflexive uniform normed cone whose dual coincides, as a set, with $P^*$.

Proof Since $Q$ contains $P$, the pairing of $Q$ and $P^*$ is nondegenerate, and $(Q, P^*)$ is a cone dual pair.

By the same reasoning as in the proof of Note 32 we see that $(Q, P^*)$ is also a normed cone dual pair.

On the other hand, the subspace topology of $Q$ induced by inclusion in $\tilde{P}$ is precisely the $\sigma(Q, P^*)$-weak topology. The statement follows from Note 29. □

Now consider the set $\mathcal{Q}$ of complete subcones of $\tilde{P}$ containing $P$. The set $\mathcal{Q}$ is closed under intersection. It follows that there exists the smallest complete subcone $\mathcal{P}$ of $\tilde{P}$ containing $P$. We call $\mathcal{P}$ the completion of $P$.

The preceding note implies that the dual of $\mathcal{P}$ coincides, as a set, with $P^*$.

Theorem 5 With notation as above, the completion $\mathcal{P}$ satisfies the following universal property.

For any uniformly continuous cone map

$$L : P \to X,$$

where $X$ is a complete reflexive uniform normed cone, there is a unique uniformly continuous

$$\mathcal{L} : \mathcal{P} \to X$$

29
with \(|\bar{L}| = |L|\), making the following diagram commute.

\[
P \xrightarrow{L} X \\
\downarrow i \\
P \\
\]

Here \(i : P \to \bar{P}\) is the inclusion map.

**Proof** Let \(L : P \to X\) be as in the formulation.

There is a double adjoint map \(\bar{L} : \bar{P} \to \bar{X}\) defined by

\[
\langle \phi, \bar{L}v \rangle = \langle L^* \phi, v \rangle,
\]

where \(\phi \in X^*, v \in \bar{P}\). The map \(\bar{L}\) is itself adjointable (with the adjoint \(L^*\)), hence uniformly continuous by Note [5]. It is also bounded with \(|\bar{L}| = |L^*| = |L|\).

Since \(X\) is reflexive, it is identified as a subcone of \(\bar{X}\). Let \(R = \bar{L}^{-1}(X)\).

If \(\{a_n\}\) is a norm-bounded non-decreasing sequence in \(R\), then \(\{\bar{L}a_n\}\) is a norm-bounded non-decreasing (by Note [5]) sequence in \(X\). Since \(X\) is complete, we have that \(x = \lim_{n \to \infty} \bar{L}a_n \in X\). Since \(\bar{L}\) is continuous, it takes \(a = \lim_{n \to \infty} a_n\) to \(x\). Hence \(a \in R\).

It follows that the cone \(R\) is complete, hence \(\bar{P} \subseteq R\). Then the restriction of \(\bar{L}\) to \(\bar{P}\) is a well defined map to \(X\). Its restriction to \(P\) coincides with \(L\). □

**Theorem 6** With notation as above, if \(P^*\) is complete in the \(\sigma(P^*,P)\)-weak topology, then it remains complete in the \(\sigma(P^*,\bar{P})\)-weak topology.

**Proof** It is enough to show that \(\sigma(P^*,P)\)-weak convergence and \(\sigma(P^*,\bar{P})\)-weak convergence, when restricted to monotone sequences, coincide on \(P^*\).

Let \(R\) be the subcone of elements \(v \in (P^*)' = \bar{P}\) satisfying the following property:

- for any monotone non-decreasing sequence \(\{\phi_n\}\) in \(P^*\) with \(\sigma(P^*,P)\)-weak limit \(\phi \in P^*\) it holds that
  \[
  \langle \phi, v \rangle = \lim_{n \to \infty} \langle \phi_n, v \rangle.
  \]

We are going to show that \(R\) is a complete subcone of \(\bar{P}\).

So let \(\{v_n\}\) be a norm-bounded monotone non-decreasing sequence in \(R\). Then there exists \(v = \lim_{n \to \infty} v_n \in \bar{P}\).

Let \(\{\phi_m\}\) be a monotone non-decreasing sequence in \(P^*\) with \(\sigma(P^*,P)\)-weak limit \(\phi \in P^*\). Note that all \(\phi_m\) and \(\phi\) are uniformly continuous on \(\bar{P}\), hence on \(R\).

We have

\[
\langle \phi, v \rangle = \lim_{n \to \infty} \langle \phi, v_n \rangle =
\]
Thus $v \in R$, hence $R$ is complete.

But $P \subseteq R$, hence $\overline{P} \subseteq R$ as well. And this is precisely what we needed to show. \qed

The moral of the above is that if $(P, Q)$ is a normed positive dual pair where $Q$ is complete in the corresponding weak topology, then it has a completion $(\overline{P}, Q)$ to a normed positive dual pair, whose both members are complete. The completion is minimal and canonical as is seen from the universal property of Theorem 5.

### 6 Coherent cones.

We are ready to introduce our category for modeling linear logic.

**Definition 1** Coherent cone is defined by one of the following equivalent structures:

- A complete reflexive uniform normed cone $P$ whose dual $P^*$ is also complete;
- A normed cone dual pair $(P, Q)$ such that $P$, $Q$ are complete for respective weak topologies.

Unless otherwise stated, we will use the first representation and understand coherent cones as complete reflexive uniform normed cones.

Morphisms of coherent cones are uniformly continuous contractions.

Since contractions compose, and identities are contractions, it follows that coherent cones can be organized into a category, which we denote as $\mathbf{CCones}$.

It follows from the very definition that the duality $(\cdot)^*$ equips $\mathbf{CCones}$ with a contravariant functorial involution.

Now let us make sure that coherent cones indeed exist and show some examples.

#### 6.1 Examples

#### 6.1.1 Finite dimensions

If a cone $P$ is finite-dimensional then there is unique uniformity making it a topological cone. It corresponds to the unique topology making the enveloping space $EP$ a TVS.
Then, by Corollary 7, any finite-dimensional normed cone is reflexive.

Now, in a finite-dimensional cone, any closed norm-bounded set is compact, hence, any norm-bounded monotone sequence converges. So a finite-dimensional normed cone is complete.

The dual of a finite-dimensional normed cone is also finite-dimensional, hence it is also reflexive and complete. Thus, any finite-dimensional cone is a coherent cone.

6.1.2 Probabilistic coherence spaces

We briefly recall the notion of probabilistic coherence space (PCS). For more details see [5].

Let $I$ be an at most countable index set.

For any $\alpha \in I$, denote as $e_\alpha$ the sequence indexed by $I$, all whose elements with index other than $\alpha$ are zero, and the element with the index $\alpha$ is 1.

Let $P$ be a set of real nonnegative sequences indexed by $I$.

The polar $P^\circ$ of $P$ consists of all real nonnegative sequences $a$ on the index set $I$ such that $\langle a, x \rangle \leq 1$ for any $x \in P$, where the pairing of sequences is defined by

$$\langle a, x \rangle = \sum_{i \in I} a_i x_i. \quad (23)$$

A probabilistic coherence space (PCS) $(P, I)$ is a pair, where $I$ is an at most countable index set, and $P$ is a set of real nonnegative sequences on $I$, satisfying the following properties.

- $P = P^{\circ\circ}$;
- For any $\alpha \in I$ there exist $\lambda, \mu > 0$ such that $\lambda e_\alpha \in P$ and $\mu e_\alpha \notin P$.

The dual PCS of $(P, I)$ is then the pair $(P^\circ, I)$. (It can be shown that this is indeed a PCS.)

If $(P, I)$ is a PCS then the set $CP = \{\lambda x \mid x \in P, \lambda \in \mathbb{R}\}$

is a vector space, and the subset $C_+ P = \{\lambda x \mid x \in P, \lambda \geq 0\}$

is a positive cone, making $CP$ a positively generated POVS.

It is easy to see that vector spaces $CP$, $CP^\circ$ form a POVS dual pair under pairing.

The space $CP$ is equipped with a norm defined as the Minkowski functional

$$\|x\| = \inf\{\lambda > 0 \mid x \in \lambda P\},$$

making $P$ the “positive unit ball”, i.e. the set of positive elements with norm less or equal to 1.
Restriction of the above norm makes $C_+ P$ a normed cone, and $(C_+ P, C_+ P^o)$ is then a normed positive dual pair. It is easy to see that the cones are complete.

Thus a PCS $A = (P, I)$ gives rise to the dual pair of coherent cones $(CP, CP^o)$, which we will loosely denote with the same letter $A$.

**Remark** Coherent cones coming from PCS have a specific property that their constituent cones, seen as posets, are lattices. It can be shown that in the finite-dimensional case this property characterizes PCS in the class of coherent cones completely. In infinite dimensions, however, this property alone apparently is not sufficient. It might be interesting to work out the missing conditions.

Also, the lattice structure determines for PCS preferred bases, and in this sense they are “commutative” spaces. In particular, they can be seen as subspaces of commutative algebras, namely, algebras of sequences with pointwise multiplication.

We now discuss morphisms of PCS.

Let $A = (P, I), B = (R, J)$ be PCS.

A PCS morphism $u : A \to B$ is a double sequence (a matrix) $u = (u_{ij})$ indexed by $I \times J$, such that for any $x \in P$ the sequence $ux \in R$, where $ux$ is defined by

$$ (ux)_j = \sum_{i \in I} u_{ij}x_i. \quad (24) $$

(It is implicit in the definition that the series in (24) converges for all $j \in J$.)

Now, since $P$ and $R$ are just positive unit balls in the corresponding cones, it is immediate from definition that, for a PCS map $u : A \to B$, formula (24) defines, in fact, a contraction of cones $u : C_+ P \to C_+ R$. Furthermore, this contraction has the adjoint $u^* : C_+ R^o \to C_+ P^o$ given by

$$ (u^* \phi)_i = \sum_{j \in J} u_{ij} \phi_j, $$

(note that the series in the above formula is convergent for all $i$), hence it is continuous. Thus, a PCS map $u : A \to B$ induces also the map $u : A \to B$ of the corresponding coherent cones.

We are going to show that the converse is true as well: any coherent cone map $u : A \to B$ induces a map of corresponding PCS.

So let $(u, u^*) : (C_+ P, C_+ P^o) \to (C_+ R, C_+ R^o)$ be a normed positive dual pair map of norm less or equal to 1.

Define the matrix $(u_{ij})_{i \in I, j \in J}$ by

$$ u_{ij} = \langle ue_i, e_j \rangle = \langle e_i, u^* e_j \rangle. \quad (25) $$

Let $x \in P$ and $j \in J$. Then

$$ (ux)_j = \langle ux, e_j \rangle = \langle x, u^* e_j \rangle = \sum_{i \in I} x_i (u^* e_j)_i = $$

33
Thus $ux$ is defined by matrix $(u_{ij})$ in the sense of formula (24). Also $ux \in R$, since $x \in P$, and $u$ is a contraction.

It follows that the coherent cone map $u : A \to B$ induces the map of corresponding PCS $u : A \to B$ by means of matrix (25), and we have the theorem.

**Theorem 7** The category of PCS and PCS maps is a full subcategory of $\text{CCones}$. □

It can be shown that the model of linear logic in $\text{CCones}$ described in this paper induces the model of [5] when restricted to PCS. We will not go into these routine details here.

### 6.1.3 Bounded operators

A genuinely noncommutative, non-lattice example comes from the space $B(H)$ of bounded operators on a Hilbert space $H$. This requires some background, see for example [4].

The space $B(H)$ is equipped with the norm

$$
\|L\| = \sup_{\|v\| \leq 1} \|Lv\| = \sup \{\lambda | \lambda \text{ is in the spectrum of } \|A\|\},
$$

which makes it a (complex) Banach space.

The subspace $L^1(H)$ of trace-class operators consists of operators $M$ with the norm

$$
\|M\|_1 = tr|M| < \infty.
$$

The above norm also makes $L^1(H)$ a complex Banach space, and $B(H)$ becomes the Banach dual of $L^1(H)$ under the Hermitian pairing

$$
\langle L, M \rangle = tr(L^1M).
$$

The space $L^1(H)$ is also a dual; it is the Banach dual of the subspace $K(H) \subseteq B(H)$ of compact operators.

Now, there are real subspaces

$$
B_s(H) \subset B(H), \ L^1_s(H) \subset L^1(H)
$$

of self-adjoint operators, which are real Banach spaces.

Furthermore, for any operator $A \in B(H)$ we have the decomposition $A = \Re A + i \Im A$, where

$$
\Re A = \frac{A + A^\dagger}{2}, \ \Im A = \frac{A - A^\dagger}{2i}
$$

and $\Re A, \Im A \in B_s(H)$. This decomposition provides isomorphisms

$$
B(H) \cong B_s(H) \otimes_{\mathbb{R}} \mathbb{C}, \ L^1(H) \cong L^1_s(H) \otimes_{\mathbb{R}} \mathbb{C},
$$

(29)
and the spaces $B(H)$, $L^1(H)$ become complexifications of the corresponding real spaces of self-adjoint operators.

Any norm-bounded linear functional $\phi : L^1(H) \to \mathbb{R}$ extends to a norm-bounded functional $\phi \otimes 1 : L(H) \to \mathbb{C}$. Conversely, any norm-bounded functional $\phi : L^1(H) \to \mathbb{C}$ sending $L^1(H)$ to $\mathbb{R}$ yields by restriction a norm-bounded functional $L^1_s(H) \to \mathbb{R}$. It follows that the real Banach dual of $L^1_s(H)$ is identified with a real subspace of the complex Banach dual of $L^1_s(H)$, and then it is easy to compute from (29) that the real Banach dual of $L^1_s(H)$ is isomorphic to $B_s(H)$.

Similarly, it is easy to see that $L^1_s(H)$ is the real Banach dual of the self-adjoint part $K_s(H)$ of $K(H)$.

The self-adjoint spaces are partially ordered, with the positive cones consisting of positive operators $L$, i.e. those for which $\langle v, Lv \rangle \geq 0$ for all $v \in H$.

Then it is easy to check that the cones $B_{s+}(H)$, $L^1_{s+}(H)$ with norms (26), (27) form a normed cone dual pair. They are also complete in the corresponding weak topologies.

Indeed, let $\{X_n\}$ be a norm-bounded monotone non-decreasing sequence in $L^1_{s+}(H)$. Since the space $L^1_{s+}(H)$ is the norm-dual of $K_s(H)$, it follows that the sequence $\{X_n\}$ defines in the limit a norm-bounded positive linear functional $X$ on $K_s(H)$, and $X \in L^1_{s+}(H)$. Now, since for any vector $e \in H$ we have $\langle e, Xe \rangle = \sup_n \langle e, X_n e \rangle = \lim_{n \to \infty} \langle e, X_n e \rangle$ and for any orthonormal basis $\{e_i\}$ of $H$ and $Y \in L^1(H)$ we have $tr(Y) = \sum_i \langle e_i, Ye_i \rangle$, it follows that $tr(X) = \sup_n tr(X_n) = \lim_{n \to \infty} tr(X_n)$. Furthermore, all operators $X - X_n$ are positive, hence $\|X - X_n\| = tr(X - X_n)$ for all $n$. Thus the sequence $\{X_n\}$ converges to $X$ also in the norm-topology and, consequently, in the $\sigma(L^1_{s+}(H), K_{s+}(H))$-weak topology. Thus the cone $L^1_{s+}(H)$ is complete.

As for the cone $B_{s+}(H)$, it is complete in the $\sigma(B_{s+}(H), L^1_{s+}(H))$-weak topology simply because $B_s(H)$ is the norm-dual of $L^1_s(H)$.

It might be interesting to try generalizing this example by considering other normed spaces of self-adjoint (or essentially self-adjoint) operators and pairing (28). A version of such a construction for finite-dimensional Hilbert spaces was hinted by Girard a while ago under the name of quantum coherent spaces [15].

It is worth noting that the PCS example also has a ready interpretation as coming from spaces of self-adjoint operators, when the ambient operator algebras are commutative.

### 6.2 Tensor product

We now show that $\mathbf{CCones}$ is a *-autonomous category, that is, we define tensor and cotensor products and internal homs.

Let $A$, $B$ be coherent cones.

We define the internal homs normed cone $A \to B$ as the set of norm-bounded uniformly continuous maps from $A$ to $B$ equipped with standard norm (17).

This cone can also be described as a cone of bilinear functionals.
Let us say that an $\mathbb{R}_+$-bilinear functional $F : A \times B \to \mathbb{R}_+$ is \textit{bounded}, if

$$
\|F\| = \sup_{\|v\|,\|u\| \leq 1} F(v, u) \leq \infty.
$$

We say that $F$ is \textit{uniformly continuous} if for all $v \in A$, $u \in B$, the functionals $F(v, .) : B \to \mathbb{R}_+$, $F(., u) : A \to \mathbb{R}_+$ are uniformly continuous.

Consider the normed cone $A \otimes_B$ of bilinear separately uniformly continuous functionals on $A^* \times B^*$ bounded in the above norm. (This is a subcone of the topological cotensor product $A \hat{\otimes}_{top} B$).

\textbf{Note 35} The normed cones $A \rightarrow B$ and $A^* \hat{\otimes} B$ are isomorphic.

\textbf{Proof} same as for Note 27 \hfill $\square$

\textbf{Note 36} The cones $A \rightarrow B$, $A^* \hat{\otimes} B$ are complete, respectively, in $\sigma(A \rightarrow B, A \otimes_{top} B^*)$-weak and $\sigma(A^* \hat{\otimes} B, A \otimes_{top} B^*)$-weak topology.

\textbf{Proof} We prove the statement for $A \rightarrow B$.

The topology in question is equivalent to the topology of pointwise convergence on elements of $A \times B^*$, because elements of the form $u \otimes \phi$, where $u \in A$, $\phi \in B^*$, span algebraically the whole $A \otimes B^*$.

Now, if $\{L_n\}$ is a norm-bounded non-decreasing sequence in $A \rightarrow B$, then for any $u \in A$ the sequence $\{L_n u\}$ is non-decreasing and bounded in norm by

$$
\|L_n u\| \leq \sup_n \|L_n\| \cdot \|u\|
$$

(since norms on reflexive cones are monotone). Hence it has a weak limit in $B$. Thus we define

$$
Lu = \lim_{n \to \infty} L_n u.
$$

Obviously, this is adjointable with the adjoint $L^*$ given by the similar formula

$$
L^* \phi = \lim_{n \to \infty} L_n^* \phi,
$$

hence it is uniformly continuous by Corollary 5. It is also bounded with norm less or equal to $\sup \|L_n\|$.

So $L \in A \rightarrow n B$. But $L$, seen as a functional on $A \times B^*$, is precisely the pointwise limit of $\{L_n\}$. \hfill $\square$

Now we equip the topological tensor product of cones $A \otimes_{top} B$ with the norm

$$
\|s\| = \sup_{f \in A^* \hat{\otimes} B^*, \|f\| \leq 1} \langle f, s \rangle
$$

(note that the norm is finite). We define the \textit{tensor product} $A \otimes B$ of coherent cones $A$, $B$ as the completion of the normed cone $A \otimes_{top} B$:

$$
A \otimes B = \overline{A \otimes_{top} B}.
$$
Finally we topologize the normed cone $A \otimes B$ with the topology of pointwise convergence on $A^* \otimes B^*$.

It follows from Note 36 and Theorem 6 that $A \otimes B$ remains complete in this topology.

Thus we define the cotensor product $A \otimes B$ of coherent cones $A$, $B$ as the normed cone $A \otimes B$ above equipped with the $\sigma(A \otimes NB, A^* \otimes B^*)$-weak topology.

Finally, the internal hom-space $A \multimap B$ is defined as the normed cone $A \multimap B$ equipped with topology of $A^* \otimes B$.

**Theorem 8** The category $CCones$ is $*$-autonomous, i.e. there is a natural bijection $\text{Hom}(A, B \multimap C) \cong \text{Hom}(A \otimes B, C)$. □

**Proof** follows from Theorem 2 and Theorem 5. □

### 6.3 Additive structure

The main use of norms is that they allow defining (non-degenerate) additive structure.

The product $P \times R$ and, respectively, the coproduct $P \oplus R$ of normed cones $P$ and $R$ are defined as the set-theoretic cartesian product $P \times R$, equipped respectively with norms

$$||(u,v)|| = \max(||u||, ||v||),$$

and

$$||(u,v)|| = ||u|| + ||v||.$$

The trivial coherent cone $0$ comes from the trivial cone (or vector space) $\{0\}$.

It is immediate that the above operations define indeed categorical product and coproduct in $CCones$. Furthermore we have

$$(A \times B)^* = A^* \oplus B^*,$$

and $A \times B$ is not isomorphic to $A \oplus B$ unless $A \cong 0$ or $B \cong 0$.

### 7 Exponentials

Recall that linear logic exponential connectives allow one to recover the expressive power of intuitionistic logic.

In particular the $!$-modality allows multiple use or waste of a formula in a proof, thus making a bridge between “linear” multiplicative conjunction $\otimes$ and “intuitionistic” additive or context-sharing $\&$, which is summarized in the exponential isomorphism

$$!A \otimes !B \cong !(A \& B).$$
Also, the intuitionistic implication $A \Rightarrow B$ inside linear logic defined as
\[ A \Rightarrow B = !A \multimap B \]
provides an embedding of intuitionistic logic.

This is reflected on the semantic side. A corresponding functor $!$ on a *-autonomous category $\mathcal{C}$ produces a model $\mathcal{K}$ of intuitionistic logic, whose maps from $A$ to $B$ come from $\mathcal{C}$-maps from $!A$ to $B$.

An accurate formulation is in terms of a linear-nonlinear adjunction. We briefly describe this structure; however, see [17] for full details.

### 7.1 Abstract categorical model theory

Recall that a linear-nonlinear adjunction consists of the following data:

- a *-autonomous category $\mathcal{C}$ (i.e. a model of multiplicative linear logic);
- a category $\mathcal{K}$ with a cartesian product $\times$ and a terminal object $T$;
- two functors $R : \mathcal{C} \to \mathcal{K}$ and $L : \mathcal{K} \to \mathcal{C}$, which are monoidal, i.e. there are natural transformations
  \[ RA \times RB \to R(A \otimes B), \ T \to R1, \quad (33) \]
  \[ LA \otimes LB \to L(A \times B), \ 1 \to LT, \quad (34) \]
and moreover are lax symmetric monoidal, which means that the above natural transformations commute in a reasonable way with various combinations of associativity and symmetry morphisms for $\otimes$- and $\times$-monoidal structures;
- an adjunction between $L$ and $R$, i.e. a natural bijection
  \[ \text{Hom}(A, RB) \cong \text{Hom}(LA, B), \quad (35) \]
which is a symmetric monoidal adjunction.

The last condition, i.e. that adjunction (35) is symmetric monoidal is equivalent (see [17], Section 5.17) to saying that the monoidal functor $L$ is strong, i.e. that natural transformations (34) are isomorphisms.

**Theorem 9 ([17])** In the above setting the endofunctor $! = L \circ R$ equips the category $\mathcal{C}$ with a model of the linear logic $!$-connective. $\square$

We now go to the concrete case of coherent cones. As expected, the exponential connectives will be interpreted as spaces of power series, which can be understood as analytic functions and analytic distributions. Using these ingredients, we will construct a category of analytic maps between coherent cones. This will be the second member of a linear-nonlinear adjunction providing us with a model of the exponential fragment.
7.2 Symmetric tensor and cotensor

At first we discuss symmetric (co)tensor algebra in CCones.

Let $A$ be a coherent cone.

For any $n \geq 0$ consider the cones $\mathcal{Y}_{\text{top}}^n A$ of $n$-linear separately uniformly continuous functionals on $A^*$.

Let $\widehat{\mathcal{Y}}_{\text{top}}^n A$ be the subcone of symmetric $n$-linear functionals $a$ in $\mathcal{Y}_{\text{top}}^n A$, i.e. those satisfying

$$a(x_1, \ldots, x_n) = a(x_{\pi(1)}, \ldots, x_{\pi(n)})$$

for any permutation $\pi \in S_n$.

Similarly, consider the $n$-th tensor power $\otimes^n_{\text{top}} A$.

It carries an action of the permutation group $S_n$, given on generators by

$$\sigma x_1 \otimes \ldots \otimes x_n = x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}.$$

Let the cone $\widehat{\otimes}_{\text{top}}^n A$ be the submodule of symmetric tensors in $\otimes^n_{\text{top}} A$ that are fixed by the action of $S_n$.

It is easy to see that $(\widehat{\mathcal{Y}}_{\text{top}}^n A, \widehat{\otimes}_{\text{top}}^n A^*)$ is a cone dual pair. There are different ways to equip its members with norms.

On one hand there are norms inherited from $\mathcal{Y}_{\text{top}}^n A$ and $\otimes^n_{\text{alg}} A$.

Let us denote for $a \in \widehat{\mathcal{Y}}_{\text{top}}^n A$

$$||a||_Y = \sup_{||x_1|| \ldots ||x_n|| \leq 1} a(x_1, \ldots, x_n),$$

and let $||.||_{\otimes}$ be the norm on $\widehat{\otimes}_{\text{top}}^n A$ defined by duality with $\widehat{\mathcal{Y}}_{\text{top}}^n A^*$.

Let us call these norms old.

On the other hand, we define the new norm for $a \in \widehat{\mathcal{Y}}_{\text{top}}^n A$, by

$$||a|| = \sup_{||x|| \leq 1} a(x)$$

where $a(x)$ is a shorthand for

$$a(x) = a(x, \ldots, x).$$

The new norm $||.||$ on $\widehat{\otimes}_{\text{top}}^n A$ is the dual of the new norm on $\widehat{\mathcal{Y}}_{\text{top}}^n A^*$.

It turns out however that the new and the old norms are equivalent, in the sense that sets bounded in one norm are bounded in the other, and vice versa. This follows from the following observation.

**Note 37** The cone $\widehat{\otimes}_{\text{top}}^n A$ is generated by elements of the form $\otimes^n x$, $x \in A$.

**Proof** Obviously, the cone $\widehat{\otimes}_{\text{top}}^n A$ is generated by elements of the form

$$\text{sym}_{\otimes}(x_1 \otimes \ldots \otimes x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} a(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \ x_1, \ldots, x_n \in A.$$
But

\[ \text{sym}_\otimes(x_1 \otimes \ldots \otimes x_n) = \sum_{k=1}^{n} \sum_{i_1 < \ldots < i_k} \otimes^n(x_{i_1} + \ldots + x_{i_k})(-1)^{n-k}. \quad \square \] (38)

**Corollary 8** The normed cones \((\hat{\mathcal{Y}}_{\text{top}}^n A, ||.||),\) and \((\hat{\mathcal{Y}}_{\text{top}}^n A, \|.|\|_\gamma)\) have the same norm-bounded sets.

**Proof** Let \(a \in \hat{\mathcal{Y}}_{\text{top}}^n A.\) Then, obviously,

\[ ||a|| \leq ||a||_\gamma. \]

On the other hand, we have from (38) that

\[ ||a||_\gamma \leq \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} ||a|| = 2^n ||a||. \quad \square \]

Let us denote normed cones

\[ \hat{\mathcal{Y}}_{\text{top}}^n A = (\hat{\mathcal{Y}}_{\text{top}}^n A, ||.||), \quad \hat{\mathcal{Y}}_{\text{new}}^n A = (\hat{\mathcal{Y}}_{\text{top}}^n A, ||.||), \]

\[ \hat{\otimes}_{\text{top}}^n A = (\hat{\otimes}_{\text{top}}^n A, ||.||_\otimes), \quad \hat{\otimes}_{\text{new}}^n A = (\hat{\otimes}_{\text{top}}^n A, ||.||). \]

We have that \(\hat{\mathcal{Y}}_{\text{top}}^n A\) is a topologically closed subspace of \(\mathcal{Y}_{\text{top}}^n A\) (in the weak topology), and \(\mathcal{Y}_{\text{top}}^n A\) is a complete reflexive uniform normed cone. So the uniform normed cone \(\hat{\mathcal{Y}}_{\text{old}}^n A\) is complete. But since the new norm is equivalent to the old, it follows that \(\hat{\mathcal{Y}}_{\text{new}}^n A\) is complete as well.

We define the \(n\)-th symmetric tensor power \(\hat{\otimes}^n A\) as the closure of \(\hat{\otimes}_{\text{new}}^n A\) in the \(\sigma(\hat{\otimes}_{\text{new}}^n A, \hat{\mathcal{Y}}_{\text{new}}^n A^*)\)-weak topology.

We define the \(n\)-th symmetric cotensor power \(\hat{\mathcal{Y}}^n A\) as the normed cone \(\hat{\mathcal{Y}}_{\text{new}}^n A\) equipped with the \(\sigma(\hat{\mathcal{Y}}_{\text{new}}^n A, \hat{\otimes}^n A^*)\)-weak topology.

By Theorem 6 these are indeed coherent cones.

### 7.2.1 Interaction with additive operations

The following easy remarks prepare the exponential isomorphism.

We introduce the following notation. For a \(k\)-element subset

\[ I \subseteq \{1, \ldots, n\} \]

let \(i_1, \ldots, i_k\) denote elements of \(I\) in the increasing order, and \(\overline{i_1}, \ldots, \overline{i_n}\) denote elements of the complement of \(I\) in the increasing order.
Note 38 Let $P$, $Q$ be coherent cones.

The elements $a$ of the cone $\hat{\mathcal{N}}^m (P \oplus Q)$ have representation as tuples

$$(a_{0,m}, \ldots, a_{k,m-k}, \ldots, a_{m,0}),$$

where $a_{k,n} \in (\hat{\mathcal{N}}^k P) \cap (\hat{\mathcal{N}}^n Q)$, such that

$$a((x_1, y_1), \ldots, (x_m, y_m)) = \sum_{k+n=m} \sum_{I \subseteq \{1, \ldots, k+n\}, |I|=k} a_{k,n}(x_i, \ldots, x_{k+i}, y_{I_1}, \ldots, y_{I_n}).$$

Conversely, any such tuple determines an element of $\hat{\mathcal{N}}^m (P \oplus Q)$ by the above formula.

Proof For $a \in \hat{\mathcal{N}}^m (P \oplus Q)$ the coefficients $a_{k,n}$ are given by

$$a_{k,n}(x_1, \ldots, x_k, y_1, \ldots, y_n) = a((x_1, 0), \ldots, (x_k, 0), (0, y_1), \ldots, (0, y_n)).$$

The other direction is obvious. □

7.3 Power series space

For a normed cone $P$ we denote the positive unit ball, i.e. the set of elements with norm less or equal to 1, as $B_+(P)$

Let the cone $\hat{\mathcal{A}}$ consist of all sequences

$$a = (a_n), \ a_n \in \hat{\mathcal{N}}^n A, \ n = 0, 1, \ldots,$$

such that

$$\|a\| = \sup_{x \in B_+(A^*)} \sum a_n(x) < \infty. \quad (39)$$

The cone $\hat{\mathcal{A}}$ has an interpretation as a space of positive power series

$$a = \sum a_n$$

with

$$a_n \in \hat{\mathcal{N}}^n A, \ n = 0, 1, \ldots,$$

which define bounded real analytic functions on $B_+(A)$ by

$$a(x) = \sum a_n(x). \quad (40)$$

In general, however, these functions are not continuous (at the positive unit ball boundary) for any reasonable topology compatible with the norm, as can be observed already in the finite-dimensional case (where the topology is unique).

The second member of the dual pair is the cone $!A$ consisting of all sequences

$$x = (x_n), \ x_n \in \hat{\mathcal{N}}^n A, \ n = 1, 2, \ldots,$$
with only finitely many nonzero elements.

The cones $!A, ?(A^*)$ form a cone dual pair under the pairing

$$\langle a, x \rangle = \sum \langle a_n, x_n \rangle.$$  

(41)

We equip the cone $\hat{!}A$ with the norm dual to norm (39) of $\hat{?}(A^*)$, which makes $(\hat{!}A, ?(A^*))$ a normed cone dual pair.

As several times before we will now complete this pair to get a coherent cone.

First, observe that $\hat{?}A$ is complete (for the weak topology).

Indeed, for $a, b \in \hat{?}A$ we have $a \leq b$ iff $a_k \leq b_k$ for all $k$. Hence if a sequence $\{a_n\}$ in $\hat{?}A$ is monotone non-decreasing and norm-bounded, then for any $k$ the coefficient sequence $\{(a_n)_k\}$ is monotone non-decreasing and norm-bounded, and by completeness of $\hat{N}^kA$ it has the limit $a_k \in \hat{N}^kA$. Then, considering the sequence $a = (a_0, a_1, \ldots)$, we observe that for any $x \in B_+(A^*)$ it holds that

$$\sum_k a_k(x) = \sup_N \sum_{k=0}^N a_k(x) = \sup_n \sum_{n \to \infty} (\lim_{n \to \infty} a_n)_k(x) =$$

$$= \sup_N \sum_{k=0}^N (\sup_n a_n)_k(x) = \sup_n \sum_{k=0}^N (a_n)_k(x) = \sup_n a_n(x) \leq \sup_n ||a_n|| < \infty,$$

hence $a \in \hat{?}A$, and it is immediate that $a$ is the weak limit of $\{a_n\}$.

So Theorem 6 guarantees that the following defines coherent cones.

**Definition 2** With notation as above, the cone $!A$ is the completion of $\hat{!}A$, and $?A$ is the cone $\hat{?}A$ equipped with the topology of pointwise convergence on elements of $!A^*$.

The cone $!A$ has then an interpretation as a space of positive real analytic distributions supported in $B_+(A)$.

In particular $!A$ contains all $\delta$-like distributions $\delta_x$, where $x \in B_+(A)$, defined on analytic functions by

$$\langle a, \delta_x \rangle = a(x).$$  

(42)

Indeed, any such distribution $\delta_x$ is the limit of the monotone norm-bounded sequence $\{\sum_{k=0}^n \otimes^k x\}_{n=0}^\infty$.

### 7.4 Analytic maps

Let $P, Q$ be coherent cones.

A map $F : B_+(P) \to Q$ is analytic if it can be represented as a series

$$F(x) = \sum_n F_n(\otimes^n x)$$

42
converging in topology of $Q$ for all $x \in B_+(P)$, where $F_n \in \hat{\otimes}^n P \to Q$.

The norm of an analytic map $F : B_+(P) \to Q$ is defined by

$$||F|| = \sup_{x \in B_+(P)} ||F(x)||.$$ 

Note 39 Let $P$ be a coherent cone.

The map $\delta : B_+(P) \to !P$, sending $x$ to $\delta_x$ is analytic (where $\delta_x$ is defined by (42)). □

Let $An(P, Q)$ be the set of analytic maps from $B_+(P)$ to $Q$.

Note 40 There is a norm-preserving bijection

$$An(P, Q) \cong !P \twoheadrightarrow Q.$$ (43)

Proof Let $F$ be an analytic map as in the formulation.

The series of coefficients $\sum F_n$ induces a linear map $!P \to Q$ in the obvious way. This linear map has the adjoint $Q^* \to \hat{\otimes}(P^*)$ sending $\phi \in Q^*$ to the function $\phi \circ F$, which is analytic being defined on $x \in B_+(P)$ by the convergent power series $\sum \langle \phi, F_n x \rangle$. By Corollary 5 the constructed linear map $!P \to Q$ is uniformly continuous, and by Theorem 5 it extends to a uniformly continuous map $!P \to Q$, i.e. to an element of $!P \twoheadrightarrow Q$.

Conversely, any element $L \in !P \twoheadrightarrow Q$ induces an analytic map on $B_+(P)$ sending $x \in B_+(P)$ to $L \delta_x = \sum L_n \otimes^n x$, where $L_n$ is the restriction of $L$ to $\otimes^n P$ (the series on the righthand side being convergent by continuity of $L$).

That this bijection is norm-preserving is immediate. □

Lemma 7 Analytic maps of norm less or equal to 1 compose.

Proof Let $P, Q, R$ be coherent cones and $F : B_+(P) \to Q$, $G : B_+(Q) \to R$ be analytic maps.

Since $F$ takes $B_+(P)$ to $B_+(Q)$, the composition $H = G \circ F$ is well defined as a set-theoretic map.

Define $G_{\leq n} : Q \to R$ by

$$G_{\leq n}(y) = \sum_{k=0}^n G_k(\otimes^k y)$$

and

$$H_{(\leq n)} = G_{\leq n} \circ F.$$ 

Now each $H_{(\leq n)}$ is analytic as the composition of a polynomial and an analytic map. Also the sequence $\{H_{(\leq n)}\}$ is monotone and bounded by 1 in norm. Hence, under identification (43) it has a limit $H_\infty$ in $!P \twoheadrightarrow R$ with $H_\infty = \sup H_{(\leq n)}$. 43
But \( \sup H_{\leq n} = \sup G_{\leq n} \circ F = G \circ F = H \). So \( H \in !P \circ R \), i.e. \( H \) is analytic.

Thus coherent cones and analytic maps form a category \( \text{ExpCCones} \), where morphisms between coherent cones \( P \) and \( Q \) are analytic maps from \( B_+(P) \) to \( Q \) of norm less or equal to 1.

**Remark** The proof of Lemma 7 is the only place where we explicitly use that our normed cones are complete.

### 7.4.1 Exponential isomorphism

Observe that Cartesian product \( \times \) in \( \text{CCones} \) (which is just set-theoretic Cartesian product) extends from objects to analytic maps, providing the category \( \text{ExpCCones} \) with a Cartesian product as well. Similarly, the space \( T = \{0\} \) is a terminal object in \( \text{ExpCCones} \) as well as in \( \text{CCones} \).

For modeling the exponential fragment of linear logic, it is crucial how does the Cartesian product in \( \text{ExpCCones} \) interact with the monoidal structure \( \otimes \) in \( \text{CCones} \).

**Note 41** In \( \text{CCones} \) there is a natural isomorphism

\[ !(P \times Q) \cong !P \otimes !Q. \]

**Proof** We establish the dual version \( ?(P \oplus Q) \cong ?P \heartsuit ?Q \).

Let \( P, Q \) be coherent cones.

By Note 38 an element \( a \in ?(A \oplus B) \) is represented as a double sequence \((a_{mn})\) with

\[ a_{mn} \in \left(\hat{\bigotimes}^m P\right) \heartsuit \left(\hat{\bigotimes}^m Q\right), \]

such that the double series \( \sum_{mn} a_{mn} (\hat{\bigotimes}^m x, \hat{\bigotimes}^n y) \) converges for any \( x \in B_+(P) \), \( y \in B_+(Q) \).

On the other hand (using identification (43)), an element \( a \in ?P \heartsuit ?Q \) is represented as a double sequence \((a_{mn})\), such that for any \( x \in B_+(P) \), \( y \in B_+(Q) \) the repeated series \( \sum_{m} \sum_{n} a_{mn} (\hat{\bigotimes}^m x, \hat{\bigotimes}^n y) \) converges. But since all terms are nonnegative, convergence of the repeated series is equivalent to convergence of the double series (the sum of a nonnegative series is the supremum of its partial sums). \( \square \)

We now define functors connecting \( \text{CCones} \) and \( \text{ExpCCones} \) yielding a linear-nonlinear adjunction.

### 7.5 The adjunction

There is the obvious functor

\[ R : \text{CCones} \to \text{ExpCCones} \]
sending coherent cones and uniformly continuous contractions to themselves (since a linear map is obviously an analytic map).

Now, if
\[ F : B_+(P) \to Q \]
is analytic of norm less or equal than 1, we get another analytic map
\[ \delta \circ F : B_+(P) \to !Q \]
(by Lemma 7), which corresponds, under identification (∗), to an element
\[ LF \in !P \to !Q. \]

In particular \( LF \) represents a uniformly continuous \( R_+ \)-linear map from \( !P \) to \( !Q \), which is a contraction, hence a morphism in \( \text{CCones} \).

The corresponding analytic map \( B_+(P) \to !Q \) (i.e. \( \delta \circ F \)) sends \( x \in B_+(P) \) to \( \delta_{F(x)} \), and it is immediate that the assignment \( F \mapsto LF \) is functorial. We extend it to a functor by putting \( LP = !P \) for \( P \in \text{ExpCCones} \).

Correspondence (∗) readily extends to the following.

Note 42 There is a natural correspondence
\[ \text{Hom}(LP, Q) \cong \text{Hom}(R, RQ), \tag{44} \]
where the lefthand side represents \( \text{CCones} \)-morphisms, and the righthand side represents \( \text{ExpCCones} \)-morphisms. □

The exponential functor \( ! : \text{CCones} \to \text{CCones} \) is defined as the composition \( ! = L \circ R \). Explicitly, the functor ! sends coherent cone \( A \) to \( !A \) and continuous contraction \( S : A \to B \) to the element \( !S \in !A \to !B \), which, under identification (∗), represents the analytic map \( x \mapsto \delta_{Sx} \).

Now, if we make sure that correspondence (44) is indeed a linear-nonlinear adjunction, then, by Theorem 9, we get indeed a model of full propositional linear logic. It is sufficient to establish the following.

Note 43 Functors \( L, R \) are lax symmetric monoidal, moreover \( L \) is strong.

Proof All statements about \( L \) are obvious after Note 41. It remains to observe that \( R \) is also (lax symmetric) monoidal. The map
\[ RA \times RB \to R(A \otimes B) \]
is given explicitly as the function
\[ (x, y) \mapsto x \otimes y. \]

Corollary 9 The above defined functor \( ! : \text{CCones} \to \text{CCones} \) is a model of linear logic \( ! \)-modality. □
7.6 Interpretation of exponentials

We now briefly describe the interpretation of basic principles of the exponential fragment.

The functor $!$ has been discussed above. The action of the functor $?$ on morphisms is as follows. For the morphism

$$L : A \to B,$$

the morphism

$$?L : ?A \to ?B$$

sends a function $f \in ?A$ to the function

$$?Lf = f \circ L^*,$$

which belongs to $?B$.

For any coherent cone $A$ the object $?A$ is naturally a monoid in $\mathbf{CCones}$. We have the monoid unit

$$u_A : 1 \to ?A,$$

sending a number to the corresponding constant function, and the multiplication

$$m_A : ?A \otimes ?A \to ?A,$$

given by the diagonalization map which sends the function $f$ of two variables to the function

$$x \mapsto f(x, x).$$

By duality, $!A$ is a comonoid in $\mathbf{CCones}$ with the corresponding dual comonoidal maps.

We also have the important monadic maps

$$\eta_A : A \to ?A, \quad \mu_A : ??A \to ?A,$$

used to interpret principles of dereliction and digging.

The map $\eta_A$ sends an element of $A$ to the corresponding linear function on $A^*$. The map $\mu_A$ is defined by

$$(\mu_A f)(x) = f(\delta_x).$$

8 Conclusion and further work

We proposed a nondegenerate model of full linear logic, which apparently fits in the familiar tradition of interpreting linear logic in the language of vector spaces, but, arguably, is free from various drawbacks of its predecessors. In particular the model is invariant in the sense that we do not refer to any bases. It also encompasses probabilistic coherence spaces as a “commutative” submodel.
There are also genuinely noncommutative objects, especially coming from spaces of self-adjoint operators on Hilbert spaces. The latter look as a variation of Girard’s quantum coherence spaces \cite{15} (see Section \ref{6.1}). They might be interesting to study, in particular, it might be interesting to give directly an explicit description of linear logic connectives specialized to such spaces.

A more important problem, as it seems, is to clarify relations with categories of stable and measurable cone maps of \cite{8}. This is left for future research.

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