THE MINIMAL NUMBER OF SINGULAR FIBERS
OF A SEMISTABLE CURVE OVER $\mathbb{P}^1$

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1. Introduction

The purpose of this paper is to try to answer the following

Szpiro’s Question. ([Sz], [B1]) Let $f : S \to \mathbb{P}_C^1$ be a family of semistable curves of genus $g$, which is not trivial. Then, what is the minimal number of the singular fibers of $f$?

Beauville gives a lower bound for the number of singular fibers.

Beauville’s Theorem. [B1] With the notations as above, if $g \geq 1$, then

1) $f$ admits at least 4 singular fibers.

2) If $f$ admits 4 singular fibers, then $S$ is algebraically simply connected with $p_g(S) = 0$, and the irreducible components of the 4 singular fibers are rational curves (may be singular), which generate a hyperplane of the $\mathbb{Q}$-vector space $\text{Pic}(S) \otimes \mathbb{Q}$.

Furthermore, Beauville [B1] gives an example of semistable elliptic fibration over $\mathbb{P}^1$ with 4 singular fibers, and one example of genus 3 with 5 singular fibers, and he gives also a series of such examples with 6 singular fibers for all $g > 1$. In fact, Beauville conjectured that for $g \geq 2$, there is no such fibrations with 4 singular fibers. In [B2], Beauville classified all semistable elliptic fibrations over $\mathbb{P}^1$ with 4 singular fibers.

Szpiro [Sz] considered that problem over a field with characteristic $p > 0$, and he proved that the minimal number of singular fibers is at least 3, and if the surface is of general type, then the number is at least 4.

The main result of this paper is...
**Theorem 1 (Beauville’s conjecture).** If \( f : S \to \mathbb{P}_C^1 \) is a non-trivial semi-stable fibration of genus \( g \geq 2 \), then \( f \) admits at least 5 singular fibers.

Theorem 1 is an immediate consequence of Beauville’s theorem and the following “strict canonical class inequality” (cf. Sect. 2.1).

**Theorem 2.** Let \( f : S \to C \) be a locally non-trivial semistable fibration of genus \( g \geq 2 \) with \( s \) singular fibers. Then we have
\[
\deg f^* \omega_{S/C} < \frac{g}{2}(2g(C) - 2 + s).
\]

The validity of these two theorems is heavily dependent on Miyaoka-Yau inequality. In Sect. 4, we shall give an example \( f : S \to \mathbb{P}_1 \) of genus 2 with 5 singular fibers. Note that Beauville has also given such an example for \( g = 3 \).

In the first version of this paper, I proved Beauville’s conjecture for \( g \geq 6 \) by using Miyaoka’s inequality. In the second version (MPI preprint 94-45), the two theorems of this paper are proved by considering Kodaira-Parshin’s construction. After the second version submitted, I simplified the proof and used the method to obtain a linear (in \( g \)) and effective height inequality for algebraic points on curves over functional fields [Ta]. Later, the editor informed me that, based on the first version of this paper, Nguyen Khac Viet had also independently given the simple proof of Beauville’s conjecture.

I would like to thank Prof. A. Beauville, Prof. F. Hirzebruch, and Prof. G. Xiao for their helps and encouragements. Prof. Xiao kindly informed me that he had independently obtained most of the steps in original version except for the proof of the main theorems. Special thanks are due to the referee for the suggestion for correction of the original version.

**2. Preliminaries**

**2.1. Invariants of fibrations.** Let \( f : S \to C \) be a relatively minimal fibration of genus \( g \geq 2 \), i.e., \( S \) contains no \((-1)\)-curves in a fiber of \( f \). Let
\[
\chi_f = \deg f^* \omega_{S/C} = \chi(O_S) - (g-1)(g(C)-1),
\]
\[
K^2_S/C = K^2_S - 8(g-1)(g(C)-1),
\]
\[
e_f = \chi_{top}(S) - 4(g-1)(g(C)-1).
\]
They are the basic invariants of \( f \). If \( K^2_{S/C} \) is the relative canonical divisor of \( f \), then \( \chi_f = \deg f^* \omega_{S/C} \), where \( \omega_{S/C} = \mathcal{O}(K^2_{S/C}/C) \). If \( f \) is not locally trivial, by the well-known Arakelov-Parshin Theorem ([Ar], [Pa]), we know \( \chi_f > 0 \) and \( K^2_{S/C} > 0 \). Then we can define the slope of \( f \) as \( \lambda_f = K^2_{S/C}/\chi_f \), which is an important invariant of \( f \).
of \( f \). In [Xi], Xiao shows that if \( f \) is a locally non-trivial fibration of genus \( g \geq 2 \), then we have
\[
\lambda_f \geq 4 - \frac{4}{g}.
\]
Furthermore, if the slope of \( f \) is \( 4 - 4/g \), then \( e_f > 0 \), hence \( f \) admits at least one singular fiber. Cornalba and Harris [CH] have also obtained (1) for semistable fibrations.

2.2. Miyaoka’s inequality and Vojta’s inequality. We refer to [Hi] for the details of the following Miyaoka’s inequality.

**Lemma 2.3.** [Mi] If \( S \) is a smooth surface such that the canonical divisor \( K_S \) is nef (numerically effective), and \( E_1, \cdots, E_n \) are disjoint ADE curves on \( S \), then we have
\[
\sum_{i=1}^{n} m(E_i) \leq 3c_2(S) - c_1^2(S),
\]
where \( m(E) \) is defined as follows,
\[
m(A_r) = 3(r+1) - \frac{3}{r+1},
\]
\[
m(D_r) = 3(r+1) - \frac{3}{4(r-2)}, \quad \text{for } r \geq 4,
\]
\[
m(E_6) = 21 - \frac{1}{8},
\]
\[
m(E_7) = 24 - \frac{1}{16},
\]
\[
m(E_8) = 27 - \frac{1}{40}.
\]

Finally, we should mention Vojta’s interesting inequality for semistable fibrations \( f \) with \( s \) singular fibers, i.e., the “canonical class inequality” [Vo]:
\[
K_{S/C}^2 \leq (2g - 2)(2g(C) - 2 + s).
\]

Our proof of Theorem 2 is based on these inequalities and Beauville’s theorem.

3. The proof of Theorem 2

First of all, we give some notations. Let \( f : S \longrightarrow C \) be a relatively minimal semistable fibration with \( s \) singular fibers. We denote by \( f^\# : S^\# \longrightarrow C \) the corresponding stable model, and by \( q \) a singular point of \( S^\# \). Then \( q \) is a rational double point of type \( A_n \). Let \( \mu_q = n \) be the Mihir number of \((S^\#, q)\), i.e., the number of \((-2)\)-curves in the exceptional set \( E_q \) of the minimal resolution of \( q \). We also denote by \( q \) a singular point of a fiber on the smooth part of \( S^\# \), in this case \( \mu_q = 0 \).
Lemma 3.1. If $s > 0$, then

$$K^2_{S/C} < (2g - 2)(2b - 2 + s).$$  \hfill (3)

Proof. First we consider the base changes $\pi : \tilde{C} \to C$ of degree $de$ ramified exactly over the $s$ critical points, where $d$ and $e > 1$ are natural numbers. We assume that the fiber of $\pi$ over each critical point consists of $d$ points with ramification index $e$. By Kodaira-Parshin construction such a base change exists for all $e$ if $b > 0$ and for odd $e$ if $b = 0$. (cf. [Vo]). Let $\tilde{f} : \tilde{S} \to \tilde{C}$ be the pullback fibration of $f$ under $\pi$, then it is easy to know that the inequality (3) holds if and only if it holds for $\tilde{f}$. By Beauville’s theorem, if $b = 0$, then $s \geq 4$, so $g(\tilde{C}) > 0$. Hence we can assume that $b > 0$. Now we know that

$$K_S \sim K_{S/C} + (2b - 2)F$$

is nef, where $F$ is a fiber of $f$. Since $f$ is semistable, we know that $e_f$ is the number of singular points of the fibers, which implies $e_f = \sum_q (\mu_q + 1)$. Then by using Miyaoka’s inequality, we can obtain easily that

$$K^2_{S/C} \leq \sum_q \frac{3}{\mu_q + 1} + (2g - 2)(2b - 2).$$ \hfill (4)

(cf. Vojta’s proof. [Vo]).

Consider the pullback fibration $\tilde{f}$ mentioned above, and denote by $\tilde{\cdot}$ the corresponding objects of $\tilde{f}$, we have

$$\tilde{s} = ds, \quad K^2_{\tilde{S}/\tilde{C}} = deK^2_{S/C}, \quad \mu_{\tilde{q}} + 1 = e(\mu_q + 1),$$

$$2\tilde{b} - 2 = de(2b - 2) + d(e - 1)s.$$  

Applying (4) to $\tilde{f}$, we have

$$deK^2_{S/C} \leq \frac{d}{e} \sum_q \frac{3}{\mu_q + 1} + (2g - 2)((2b - 2)de + d(e - 1)s),$$

i.e.,

$$K^2_{S/C} - (2g - 2)(2b - 2 + s) \leq -\frac{(2g - 2)s}{e} + \frac{1}{e^2} \sum_q \frac{3}{\mu_q + 1},$$ \hfill (5)

let $e$ be large enough we can see that the right hand side of (5) is negative. This completes the proof of the lemma. \hfill Q.E.D.

Proof of Theorem 2. From (1) and (2), we have

$$\chi_f \leq \frac{g}{2}(2b - 2 + s).$$
If the above equality holds, then the equalities in (1) and (2) hold. On the other hand, a locally non-trivial fibration with $\lambda_f = 4 - 4/g$ admits at least one singular fiber, hence the equality in (2) contradicts Lemma 3.1. This completes the proof of Theorem 2. Q.E.D.

4. An example of genus $g = 2$ with $s = 5$

In this section, we shall construct a semistable fibration $f : S \to \mathbb{P}^1$ of genus 2 with 5 singular fibers.

Let $\phi$ and $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ be two morphisms with $\deg \phi + \deg \psi = 2g + 2$. We assume that there exists a subset $R = \{p_1, \cdots, p_5\} \subset \mathbb{P}^1$ satisfying

i) the branched points of $\phi$ and $\psi$ are contained in $R$, and the ramification points of them are of index 2.

ii) $\phi^{-1}(p_i) \cap \psi^{-1}(p_i)$ consists of non-ramified points of $\phi$ and $\psi$, and if $p \notin R$, then $\phi^{-1}(p) \cap \psi^{-1}(p)$ is empty.

In $\mathbb{P}^1 \times \mathbb{P}^1$, we consider the divisors $\Gamma_\phi$ and $\Gamma_\psi$, graphes of $\phi$ and $\psi$ respectively. Let $B = \Gamma_\phi + \Gamma_\psi$. Then $B$ is an even divisor of type $(2g+2, 2)$. Let $\pi : \Sigma \to \mathbb{P}^1 \times \mathbb{P}^1$ be a double cover branched along $B$, and let $S$ be the canonical resolution of the singularities of $\Sigma$. Then the second projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ induces a semistable fibration of genus $g$ with 5 singular fibers.

Now we give an example of genus 2 with 5 singular fibers. Let $a$ and $b$ be two nonzero complex numbers such that the discriminant of the polynomial

$$p(x) = x^3 + (2a - b^2)x^2 + (a^2 + 2ab^2)x - a^2b^2,$$

is zero. Hence $p(x)$ has (at most) two zeros $x_1$ and $x_2$. Let $\phi$ and $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ be two morphisms defined by

$$\phi(t) = t^2 + \frac{a^2}{t^2}, \quad \psi(t) = -2a - \frac{t^2 + b^2}{t^2 - b^2}.$$

Note that if $\phi(t) = \psi(t)$, then $p(t^2) = 0$. Take $R = \{\infty, 2a, -2a, x_1 + a^2/x_1, x_2 + a^2/x_2\}$. It is easy to check that $\phi$ and $\psi$ satisfy i) and ii). This completes the construction.

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