ON SOME NEW HADAMARD LIKE INEQUALITIES FOR CO-ORDINATED $s$-CONVEX FUNCTIONS

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Abstract. In this paper, we prove some new inequalities of Hadamard-type for $s$-convex functions on the co-ordinates.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a < b$. The following double inequality;

$$f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_{a}^{b} f(x)dx \leq \frac{f(a) + f(b)}{2}$$

is well known in the literature as Hadamard’s inequality. Both inequalities hold in the reversed direction if $f$ is concave.

In [8], Alomari and Darus defined $s$-convex functions on the co-ordinates as following:

Definition 1. Consider the bidimensional interval $\Delta = [a, b] \times [c, d]$ in $[0, \infty)^2$ with $a < b$ and $c < d$. The mapping $f : \Delta \to \mathbb{R}$ is $s$-convex in the second sense on $\Delta$ if

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda^s f(x, y) + (1-\lambda)^s f(z, w)$$

hold for all $(x, y), (z, w) \in \Delta$ with $\lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$.

A function $f : \Delta \to \mathbb{R}$ is $s$-convex on $\Delta$ if the partial mappings $f_y : [a, b] \to \mathbb{R}, f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}, f_x(v) = f(x, v)$ are $s$-convex for all $y \in [c, d]$ and $x \in [a, b]$ with some fixed $s \in (0, 1]$.

Recall that the mapping $f : \Delta \to \mathbb{R}$ is $s$-convex on $\Delta$ if the following inequality holds.

Moreover, in [8], Alomari and Darus established the following inequalities of Hadamard’s type for co-ordinated $s$-convex functions on a rectangle from the plane $\mathbb{R}^2$.

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Theorem 1. Suppose that \( f : \Delta = [a, b] \times [c, d] \subset [0, \infty)^2 \rightarrow [0, \infty) \) is \( s \)-convex function on the co-ordinates on \( \Delta \). Then one has the inequalities:

\[
\begin{align*}
4^{s-1} f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) & \leq 2^{s-2} \left[ \frac{1}{b - a} \int_a^b f(x, c + d) dx + \frac{1}{d - c} \int_c^d f \left( \frac{a + b}{2}, y \right) dy \right] \\
& \leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dy dx \\
& \leq \frac{1}{2(s + 1)} \left[ \frac{1}{b - a} \int_a^b f(x, c) dx + \frac{1}{b - a} \int_c^b f(x, y) dy \right] \\
& \hspace{1cm} + \frac{1}{d - c} \int_c^d f(a, y) dy + \frac{1}{s + 1} \int_c^d f(b, y) dy \\
& \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{(s + 1)^2}.
\end{align*}
\]

Similar results can be found in [1]-[9].

However, Özdemir et al. established the following lemma for twice partial differentiable mapping on \( \Delta = [a, b] \times [c, d] \).

Lemma 1. Let \( f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R} \) be a twice partial differentiable mapping on \( \Delta = [a, b] \times [c, d] \). If \( \frac{\partial^2 f}{\partial t \partial \lambda} \in L(\Delta) \), then the following equality holds:

\[
\begin{align*}
& \frac{1}{(b - a)(d - c)} \left[ 2 a - (x - a) \int_c^d f(a, v) dv - (b - x) \int_c^d f(b, v) dv \\
& - (d - y) \int_a^b f(u, d) du - (y - c) \int_a^b f(u, c) du + \int_a^b \int_c^d f(u, v) dv du \right] \\
= & \frac{(x - a)^2 (y - c)^2}{(b - a)(d - c)} \int_0^1 \int_0^1 (t - 1)(\lambda - 1) \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1 - t) a, \lambda y + (1 - \lambda) c) d\lambda dt \\
& + \frac{(x - a)^2 (d - y)^2}{(b - a)(d - c)} \int_0^1 \int_0^1 (t - 1)(\lambda - 1) \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1 - t) a, \lambda y + (1 - \lambda) d) d\lambda dt \\
& + \frac{(b - x)^2 (y - c)^2}{(b - a)(d - c)} \int_0^1 \int_0^1 (1 - t)(\lambda - 1) \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1 - t) b, \lambda y + (1 - \lambda) c) d\lambda dt \\
& + \frac{(b - x)^2 (d - y)^2}{(b - a)(d - c)} \int_0^1 \int_0^1 (1 - t)(\lambda - 1) \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1 - t) b, \lambda y + (1 - \lambda) d) d\lambda dt.
\end{align*}
\]
where
\[ A = \frac{(x - a) (y - c) f(a, c) + (x - a) (d - y) f(a, d)}{(b - a) (d - c)} + \frac{(b - x) (y - c) f(b, c) + (b - x) (d - y) f(b, d)}{(b - a) (d - c)}. \]

The main purpose of this paper is to prove some new inequalities of Hadamard-type for \( s \)-convex functions on the co-ordinates.

2. MAIN RESULTS

**Theorem 2.** Let \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta = [a, b] \times [c, d] \) and \( \frac{\partial^2 f}{\partial t \partial \lambda} \in L(\Delta) \). If \( \frac{\partial^2 f}{\partial t \partial \lambda} \) is a \( s \)-convex function on the co-ordinates on \( \Delta \), for some fixed \( s \in (0, 1] \), then the following inequality holds;

\[
(2.1) \quad \leq \frac{1}{(b - a) (d - c) (s + 2)^2} \left[ \left( \frac{(x - a)^2 + (b - x)^2}{(s + 1)^2} \right) \left( \frac{(y - c)^2 + (d - y)^2}{(s + 1)^2} \right) \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, y) \right| \right. \\
+ \left. \left( \frac{(x - a)^2 (y - c)^2 + (d - y)^2}{(s + 1)^2} \right) \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, y) \right| \right. \\
+ \left. \left( \frac{(b - x)^2 (y - c)^2 + (d - y)^2}{(s + 1)^2} \right) \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, y) \right| \right. \\
+ \left. \left( \frac{(y - c)^2 (x - a)^2 + (b - x)^2}{(s + 1)^2} \right) \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, c) \right| \right. \\
+ \left. \left( \frac{(d - y)^2 (x - a)^2 + (b - x)^2}{(s + 1)^2} \right) \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, d) \right| \right. \\
+ \left. \left( \frac{(x - a)^2 (y - c)^2}{(s + 1)^2} \right) \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right| \right. \\
+ \left. \left( \frac{(x - a)^2 (d - y)^2}{(s + 1)^2} \right) \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right| \right. \\
+ \left. \left( \frac{(b - x)^2 (y - c)^2}{(s + 1)^2} \right) \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) \right| \right. \\
+ \left. \left( \frac{(b - x)^2 (d - y)^2}{(s + 1)^2} \right) \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right| \right. \\
\]

where \( A \) is as above.
Proof. From Lemma \[\text{Lemma}^1\] and using the property of modulus, we have

\[
\left| \frac{1}{(b-a)(d-c)} \left[ A - (x-a) \int_c^d f(a,v)\, dv - (b-x) \int_c^d f(b,v)\, dv \\
- (d-y) \int_a^b f(u,d)\, du - (y-c) \int_a^b f(u,c)\, du + \int_a^b f(u,v)\, du\, dv \right] \right| 
\]

\[
\leq \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} 
\times \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t\partial\lambda} (tx + (1-t)a,\lambda y+(1-\lambda)c) \right| d\lambda dt 
\]

\[
+ \frac{(x-a)^2(d-y)^2}{(b-a)(d-c)} 
\times \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t\partial\lambda} (tx + (1-t)b,\lambda y+(1-\lambda)c) \right| d\lambda dt 
\]

\[
+ \frac{(b-x)^2(y-c)^2}{(b-a)(d-c)} 
\times \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t\partial\lambda} (tx + (1-t)b,\lambda y+(1-\lambda)d) \right| d\lambda dt 
\]

\[
+ \frac{(b-x)^2(d-y)^2}{(b-a)(d-c)} 
\times \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t\partial\lambda} (tx + (1-t)b,\lambda y+(1-\lambda)d) \right| d\lambda dt. 
\]

Since \(\left| \frac{\partial^2 f}{\partial t\partial\lambda} \right|\) is co-ordinated s-convex, for some fixed \(s \in (0,1]\), we can write

\[
\left| \frac{1}{(b-a)(d-c)} \left[ A - (x-a) \int_c^d f(a,v)\, dv - (b-x) \int_c^d f(b,v)\, dv \\
- (d-y) \int_a^b f(u,d)\, du - (y-c) \int_a^b f(u,c)\, du + \int_a^b f(u,v)\, du\, dv \right] \right| 
\]

\[
\leq \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \int_0^{\lambda-1} |\lambda-1| 
\times \left[ \int_0^{(t-1)t^s} \left| \frac{\partial^2 f}{\partial t\partial\lambda} (x,\lambda y+(1-\lambda)c) \right| dt + \int_0^{(t-1)(1-t)^s} \left| \frac{\partial^2 f}{\partial t\partial\lambda} (a,\lambda y+(1-\lambda)c) \right| dt \right] d\lambda 
\]
By computing these integrals, we obtain

\[
\begin{align*}
+ \frac{(x-a)^2 (d-y)^2}{(b-a)(d-c)} \int_0^1 |(1-\lambda)| \\
\times \left[ \int_0^1 (t-1) t^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, \lambda y + (1-\lambda) d) \right| dt + \int_0^1 (t-1) (1-t)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, \lambda y + (1-\lambda) d) \right| dt \right] d\lambda \\
+ \frac{(b-x)^2 (y-c)^2}{(b-a)(d-c)} \int_0^1 |(\lambda - 1)| \\
\times \left[ \int_0^1 (1-t) t^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, \lambda y + (1-\lambda) c) \right| dt + \int_0^1 (1-t) (1-t)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, \lambda y + (1-\lambda) c) \right| dt \right] d\lambda \\
+ \frac{(b-x)^2 (d-y)^2}{(b-a)(d-c)} \int_0^1 |(1-\lambda)| \\
\times \left[ \int_0^1 (1-t) t^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, \lambda y + (1-\lambda) d) \right| dt + \int_0^1 (1-t) (1-t)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, \lambda y + (1-\lambda) d) \right| dt \right] d\lambda.
\end{align*}
\]

By computing these integrals, we obtain

\[
\begin{align*}
&\left| \frac{1}{(b-a)(d-c)} \left[ A - (x-a) \int_c^d f(a,v) dv - (b-x) \int_c^d f(b,v) dv \\
&- (d-y) \int_a^b f(u,d) du - (y-c) \int_a^b f(u,c) du + \int_a^b \int_c^d f(u,v) dudv \right] \right|
\leq \frac{(x-a)^2 (d-y)^2}{(b-a)(d-c)} \int_0^1 |\lambda - 1| \left[ \frac{-1}{(s+1)(s+2)} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, \lambda y + (1-\lambda) c) \right| \\
&\left. - \frac{1}{s+2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, \lambda y + (1-\lambda) c) \right| \right] d\lambda \\
&\left. + \frac{(x-a)^2 (d-y)^2}{(b-a)(d-c)} \int_0^1 |1-\lambda| \left[ \frac{-1}{(s+1)(s+2)} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, \lambda y + (1-\lambda) d) \right| \\
&\left. - \frac{1}{s+2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, \lambda y + (1-\lambda) d) \right| \right] d\lambda \\
&\left. + \frac{(b-x)^2 (y-c)^2}{(b-a)(d-c)} \int_0^1 |\lambda - 1| \left[ \frac{-1}{(s+1)(s+2)} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, \lambda y + (1-\lambda) c) \right| \\
&\left. - \frac{1}{s+2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, \lambda y + (1-\lambda) c) \right| \right] d\lambda \right|
\end{align*}
\]
\[
\begin{align*}
&+ \frac{(b - x)^2 (d - y)^2}{(b - a) (d - c)} \int_0^1 |1 - \lambda| \left[ \frac{-1}{(s + 1) (s + 2)} \left| \frac{\partial^2 f}{\partial \lambda^2} (x, \lambda y + (1 - \lambda) d) \right| \right] d\lambda \\
&- \frac{1}{s + 2} \left| \frac{\partial^2 f}{\partial \lambda^2} (b, \lambda y + (1 - \lambda) d) \right| d\lambda.
\end{align*}
\]

Using co-ordinated \(s\)-convexity of \(\frac{\partial^2 f}{\partial \lambda^2}\) again and computing all integrals, we obtain

\[
\begin{align*}
&\left| \frac{1}{(b - a) (d - c)} \left[ A - (x - a) \int_c^d f(a, v) \, dv - (b - x) \int_c^d f(b, v) \, dv \\
&\quad - (d - y) \int_a^b f(u, d) \, du - (y - c) \int_a^b f(u, c) \, du + \int_a^b \int_c^d f(u, v) \, du \, dv \right] \right| \\
\leq
\frac{(x - a)^2 (y - c)^2}{(b - a) (d - c)} \\
\times \left\{ \int_0^1 |\lambda - 1| \left[ \frac{-1}{(s + 1) (s + 2)} \left( \lambda^s \left| \frac{\partial^2 f}{\partial \lambda^2} (x, y) \right| + (1 - \lambda)^s \left| \frac{\partial^2 f}{\partial \lambda^2} (x, c) \right| \right) \right] d\lambda \\
+ \int_0^1 |\lambda - 1| \left[ \frac{-1}{s + 2} \left( \lambda^s \left| \frac{\partial^2 f}{\partial \lambda^2} (a, y) \right| + (1 - \lambda)^s \left| \frac{\partial^2 f}{\partial \lambda^2} (a, c) \right| \right) \right] d\lambda \\
+ \frac{(b - x)^2 (y - c)^2}{(b - a) (d - c)} \\
\times \left\{ \int_0^1 |\lambda - 1| \left[ \frac{-1}{(s + 1) (s + 2)} \left( \lambda^s \left| \frac{\partial^2 f}{\partial \lambda^2} (x, d) \right| + (1 - \lambda)^s \left| \frac{\partial^2 f}{\partial \lambda^2} (x, c) \right| \right) \right] d\lambda \\
+ \int_0^1 |\lambda - 1| \left[ \frac{-1}{s + 2} \left( \lambda^s \left| \frac{\partial^2 f}{\partial \lambda^2} (a, d) \right| + (1 - \lambda)^s \left| \frac{\partial^2 f}{\partial \lambda^2} (a, c) \right| \right) \right] d\lambda \\
+ \frac{(b - x)^2 (y - c)^2}{(b - a) (d - c)} \\
\times \left\{ \int_0^1 |\lambda - 1| \left[ \frac{-1}{(s + 1) (s + 2)} \left( \lambda^s \left| \frac{\partial^2 f}{\partial \lambda^2} (b, y) \right| + (1 - \lambda)^s \left| \frac{\partial^2 f}{\partial \lambda^2} (b, c) \right| \right) \right] d\lambda \\
+ \int_0^1 |\lambda - 1| \left[ \frac{-1}{s + 2} \left( \lambda^s \left| \frac{\partial^2 f}{\partial \lambda^2} (b, c) \right| + (1 - \lambda)^s \left| \frac{\partial^2 f}{\partial \lambda^2} (b, d) \right| \right) \right] d\lambda \right\}.
\end{align*}
\]
\[ \frac{(b - x)^2 (d - y)^2}{(b - a)(d - c)} \]

\[ \times \left\{ \int_0^1 [1 - \lambda] \left[ \frac{-1}{(s + 1)(s + 2)} \left( \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, y) \right| + (1 - \lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, d) \right| \right) \right] d\lambda \right\} \]

\[ + \int_0^1 [1 - \lambda] \left[ \frac{1}{s + 2} \left( \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, y) \right| + (1 - \lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right| \right) \right] \}

\[ \times \left\{ \frac{1}{(b - a)(d - c)} \left[ A - (x - a) \int_a^b f(a, v) dv - (b - x) \int_c^d f(b, v) dv \right. \right. \]

\[ - (d - y) \int_a^b f(u, d) du - (y - c) \int_a^b f(u, c) du + \int_a^b \int_c^d f(u, v) du dv \] \[ \left. \left. \leq \frac{(x - a)^2 (y - c)^2}{(b - a)(d - c)} \right\} \right\} \]

\[ \times \left\{ \frac{1}{(s + 1)^2(s + 2)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, y) \right| + \frac{1}{(s + 1)(s + 2)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, c) \right| \right. \]

\[ + \frac{1}{(s + 1)(s + 2)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, y) \right| + \frac{1}{(s + 2)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right| \right) \}

\[ + \frac{(x - a)^2 (d - y)^2}{(b - a)(d - c)} \]

\[ \times \left\{ \frac{1}{(s + 1)^2(s + 2)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, y) \right| + \frac{1}{(s + 1)(s + 2)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, d) \right| \right. \]

\[ + \frac{1}{(s + 1)(s + 2)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, y) \right| + \frac{1}{(s + 2)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right| \right) \}

\[ + \frac{(b - x)^2 (y - c)^2}{(b - a)(d - c)} \]

\[ \times \left\{ \frac{1}{(s + 1)^2(s + 2)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, y) \right| + \frac{1}{(s + 1)(s + 2)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, c) \right| \right. \]

\[ + \frac{1}{(s + 1)(s + 2)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, y) \right| + \frac{1}{(s + 2)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) \right| \right) \} \]
\[
\frac{(b-x)^2 (d-y)^2}{(b-a)(d-c)} \times \left\{ \frac{1}{(s+1)^2(s+2)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x,y) \right| + \frac{1}{(s+1)(s+2)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x,d) \right| + \frac{1}{(s+1)(s+2)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b,y) \right| + \frac{1}{(s+2)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b,d) \right| \right\}.
\]

so,

\[
\left| \frac{1}{(b-a)(d-c)} \left[ A - (x-a) \int_c^d f(a,v) dv - (b-x) \int_c^d f(b,v) dv \\
- (d-y) \int_a^b f(u,d) du - (y-c) \int_a^b f(u,c) du + \int_a^b \int_c^d f(u,v) dvdu \right] \right|
\]

\[
\leq \left| \left( \frac{(x-a)^2 + (b-x)^2}{(b-a)(d-c)(s+1)(s+2)^2} \right) \int_c^d f(a,v) dv \right| + \left| \left( \frac{(y-c)^2 + (d-y)^2}{(b-a)(d-c)(s+1)(s+2)^2} \right) \int_c^d f(b,v) dv \right|
\]

\[
+ \left| \left( \frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)(s+1)(s+2)^2} \right) \int_a^b f(u,d) du \right| + \left| \left( \frac{(y-c)^2 (d-y)^2}{(b-a)(d-c)(s+1)(s+2)^2} \right) \int_a^b f(u,c) du \right| + \left| \left( \frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)(s+1)(s+2)^2} \right) \int_a^b \int_c^d f(u,v) dvdu \right|
\]

which completes the proof. \(\square\)
Corollary 1. (1) Under the assumptions of Theorem 2, if we choose \( x = a, y = c \), we obtain the following inequality:

\[
\frac{1}{(b - a)(d - c)} \left| f(b, d) - (b - a) \int_c^d f(b, v) \, dv - (d - c) \int_a^b f(u, d) \, du + \int_a^b \int_c^d f(u, v) \, du \, dv \right| 
\]

\[
\leq \frac{1}{(b - a)(d - c)} \left( \frac{1}{(s + 1)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right| + \frac{1}{s + 1} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) \right| + \frac{1}{s + 1} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right| \right). 
\]

(2) Under the assumptions of Theorem 2, if we choose \( x = b, y = d \), we obtain the following inequality:

\[
\frac{1}{(b - a)(d - c)} \left| f(a, c) - (b - a) \int_c^d f(a, v) \, dv - (d - c) \int_a^b f(u, c) \, du + \int_a^b \int_c^d f(u, v) \, du \, dv \right| 
\]

\[
\leq \frac{1}{(b - a)(d - c)} \left( \frac{1}{(s + 1)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right| + \frac{1}{s + 1} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right| + \frac{1}{s + 1} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right| \right). 
\]

(3) Under the assumptions of Theorem 2, if we choose \( x = a, y = d \), we obtain the following inequality:

\[
\frac{1}{(b - a)(d - c)} \left| f(b, c) - (b - a) \int_c^d f(b, v) \, dv - (d - c) \int_a^b f(u, c) \, du + \int_a^b \int_c^d f(u, v) \, du \, dv \right| 
\]

\[
\leq \frac{1}{(b - a)(d - c)} \left( \frac{1}{(s + 1)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right| + \frac{1}{s + 1} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right| + \frac{1}{s + 1} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) \right| \right). 
\]

(4) Under the assumptions of Theorem 2, if we choose \( x = b, y = c \), we obtain the following inequality:

\[
\frac{1}{(b - a)(d - c)} \left| f(a, d) - (b - a) \int_c^d f(a, v) \, dv - (d - c) \int_a^b f(u, d) \, du + \int_a^b \int_c^d f(u, v) \, du \, dv \right| 
\]

\[
\leq \frac{1}{(b - a)(d - c)} \left( \frac{1}{(s + 1)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) \right| + \frac{1}{s + 1} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right| + \frac{1}{s + 1} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right| \right). 
\]
Theorem 3. Let \( f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta = [a, b] \times [c, d] \) and \( \frac{\partial^2 f}{\partial \lambda^2} \in L(\Delta) \). If \( \left| \frac{\partial^2 f}{\partial \lambda^2} \right|^q, q > 1, \) is a \( s \)-convex function in the second sense on the co-ordinates on \( \Delta \), for some fixed \( s \in (0, 1] \), then the following
Proof. From Lemma 1, we have

\[
\frac{1}{(b - a) (d - c)} \left[ A - (x - a) \int_{c}^{d} f(a, v) dv - (b - x) \int_{c}^{d} f(b, v) dv \\
- (d - y) \int_{a}^{b} f(u, d) du - (y - c) \int_{a}^{b} f(u, c) du + \int_{a}^{b} \int_{c}^{d} f(u, v) dv du \right]
\]

\[
\leq \frac{1}{(p + 1)^{\frac{p}{2}} (s + 1)^{\frac{s}{2}}} \times \left\{ \frac{(x - a)^2 (y - c)^2}{(b - a) (d - c)} \left( \frac{\partial^2 f}{\partial x \partial \lambda}(x, y) \right)^q + \left| \frac{\partial^2 f}{\partial y \partial \lambda}(x, c) \right|^q + \left| \frac{\partial^2 f}{\partial y \partial \lambda}(a, y) \right|^q + \left| \frac{\partial^2 f}{\partial y \partial \lambda}(a, c) \right|^q \right\}^{\frac{1}{2}}
\]

\[
+ \frac{(x - a)^2 (d - y)^2}{(b - a) (d - c)} \left( \frac{\partial^2 f}{\partial x \partial \lambda}(x, y) \right)^q + \left| \frac{\partial^2 f}{\partial y \partial \lambda}(x, d) \right|^q + \left| \frac{\partial^2 f}{\partial y \partial \lambda}(a, y) \right|^q + \left| \frac{\partial^2 f}{\partial y \partial \lambda}(a, d) \right|^q \right\}^{\frac{1}{2}}
\]

\[
+ \frac{(b - x)^2 (y - c)^2}{(b - a) (d - c)} \left( \frac{\partial^2 f}{\partial x \partial \lambda}(x, y) \right)^q + \left| \frac{\partial^2 f}{\partial y \partial \lambda}(x, c) \right|^q + \left| \frac{\partial^2 f}{\partial y \partial \lambda}(b, y) \right|^q + \left| \frac{\partial^2 f}{\partial y \partial \lambda}(b, c) \right|^q \right\}^{\frac{1}{2}}
\]

\[
+ \frac{(b - x)^2 (d - y)^2}{(b - a) (d - c)} \left( \frac{\partial^2 f}{\partial x \partial \lambda}(x, y) \right)^q + \left| \frac{\partial^2 f}{\partial y \partial \lambda}(x, d) \right|^q + \left| \frac{\partial^2 f}{\partial y \partial \lambda}(b, y) \right|^q + \left| \frac{\partial^2 f}{\partial y \partial \lambda}(b, d) \right|^q \right\}^{\frac{1}{2}}
\]

where \( p^{-1} + q^{-1} = 1 \).

Proof. From Lemma 1, we have

\[
\frac{1}{(b - a) (d - c)} \left[ A - (x - a) \int_{c}^{d} f(a, v) dv - (b - x) \int_{c}^{d} f(b, v) dv \\
- (d - y) \int_{a}^{b} f(u, d) du - (y - c) \int_{a}^{b} f(u, c) du + \int_{a}^{b} \int_{c}^{d} f(u, v) dv du \right]
\]

\[
\leq \frac{(x - a)^2 (y - c)^2}{(b - a) (d - c)} \int_{0}^{1} \int_{0}^{1} \left| (t - 1) (\lambda - 1) \right| \left| \frac{\partial^2 f}{\partial x \partial \lambda}(tx + (1 - t) a, \lambda y + (1 - \lambda) c) \right| d\lambda dt
\]

\[
+ \frac{(x - a)^2 (d - y)^2}{(b - a) (d - c)} \int_{0}^{1} \int_{0}^{1} \left| (t - 1) (1 - \lambda) \right| \left| \frac{\partial^2 f}{\partial x \partial \lambda}(tx + (1 - t) a, \lambda y + (1 - \lambda) d) \right| d\lambda dt
\]

\[
+ \frac{(b - x)^2 (y - c)^2}{(b - a) (d - c)} \int_{0}^{1} \int_{0}^{1} \left| (1 - t) (\lambda - 1) \right| \left| \frac{\partial^2 f}{\partial x \partial \lambda}(tx + (1 - t) b, \lambda y + (1 - \lambda) c) \right| d\lambda dt
\]

\[
+ \frac{(b - x)^2 (d - y)^2}{(b - a) (d - c)} \int_{0}^{1} \int_{0}^{1} \left| (1 - t) (1 - \lambda) \right| \left| \frac{\partial^2 f}{\partial x \partial \lambda}(tx + (1 - t) b, \lambda y + (1 - \lambda) d) \right| d\lambda dt.
\]
By using the well known Hölder inequality for double integrals, then one has:

\[
(2.4) \quad \left| \frac{1}{(b-a)(d-c)} \left[ A - (x-a) \int_{c}^{d} f(a,v) \, dv - (b-x) \int_{c}^{d} f(b,v) \, dv - (d-y) \int_{a}^{b} f(u,d) \, du - (y-c) \int_{a}^{b} f(u,c) \, du + \int_{a}^{b} \int_{c}^{d} f(u,v) \, dudv \right] \right|
\]

\[
\leq \frac{(x-a)^2 (y-c)^2}{(b-a)(d-c)} \left( \int_{0}^{1} \int_{0}^{1} |(t-1)(\lambda-1)|^p \, d\lambda dt \right)^{\frac{1}{p}}
\]

\[
+ \frac{(x-a)^2 (d-y)^2}{(b-a)(d-c)} \left( \int_{0}^{1} \int_{0}^{1} |(t-1)(\lambda-1)|^p \, d\lambda dt \right)^{\frac{1}{p}}
\]

\[
+ \frac{(b-x)^2 (y-c)^2}{(b-a)(d-c)} \left( \int_{0}^{1} \int_{0}^{1} |(1-t)(\lambda-1)|^p \, d\lambda dt \right)^{\frac{1}{p}}
\]

\[
+ \frac{(b-x)^2 (d-y)^2}{(b-a)(d-c)} \left( \int_{0}^{1} \int_{0}^{1} |(1-t)(1-\lambda)|^p \, d\lambda dt \right)^{\frac{1}{p}}
\]

\[
+ \left( \int_{0}^{1} \int_{0}^{1} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1-t) a, \lambda y + (1 - \lambda) c) \right|^q d\lambda dt \right)^{\frac{1}{q}}
\]

\[
\leq \left( \int_{0}^{1} \int_{0}^{1} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1-t) b, \lambda y + (1 - \lambda) c) \right|^q d\lambda dt \right)^{\frac{1}{q}}
\]

\[
+ \left( \int_{0}^{1} \int_{0}^{1} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1-t) b, \lambda y + (1 - \lambda) d) \right|^q d\lambda dt \right)^{\frac{1}{q}}
\]

\[
\leq \left( \int_{0}^{1} \int_{0}^{1} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1-t) a, \lambda y + (1 - \lambda) d) \right|^q d\lambda dt \right)^{\frac{1}{q}}
\]

\[
+ \left( \int_{0}^{1} \int_{0}^{1} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1-t) a, \lambda y + (1 - \lambda) c) \right|^q d\lambda dt \right)^{\frac{1}{q}}
\]

\[
+ \left( \int_{0}^{1} \int_{0}^{1} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1-t) b, \lambda y + (1 - \lambda) d) \right|^q d\lambda dt \right)^{\frac{1}{q}}
\]

\[
+ \left( \int_{0}^{1} \int_{0}^{1} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1-t) b, \lambda y + (1 - \lambda) c) \right|^q d\lambda dt \right)^{\frac{1}{q}}
\]
Since $|\frac{\partial^2 f}{\partial s \partial \lambda}|^q, q > 1$, is $s$–convex function in the second sense on the co-ordinates on $\Delta$, for some fixed $s \in (0, 1]$, we know that for $t \in [0, 1]$

$$\left| \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1-t) a, \lambda y + (1-\lambda) c) \right|^q \leq t^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, \lambda y + (1-\lambda) c) \right|^q + (1-t)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, \lambda y + (1-\lambda) c) \right|^q$$

$$\leq t^s \left( \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, y) \right|^q + (1-\lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, c) \right|^q \right)$$

$$+ (1-t)^s \left( \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, y) \right|^q + (1-\lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right|^q \right)$$

hence, it follows that

(2.5)

$$\left( \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1-t) a, \lambda y + (1-\lambda) c) \right|^q d\lambda dt \right)^{\frac{1}{q}}$$

$$\leq \left( \int_0^1 \int_0^1 \left( t^s \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, y) \right|^q + t^s (1-\lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, c) \right|^q \right)$$

$$+ (1-t)^s \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, y) \right|^q + (1-t)^s (1-\lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right|^q \right) dtd\lambda \right)^{\frac{1}{q}}$$

$$= \frac{1}{(s+1)^{\frac{2}{q}}} \left( \frac{\partial^2 f}{\partial t \partial \lambda} (x, y) \right)^q + \frac{\partial^2 f}{\partial t \partial \lambda} (x, c) \right|^q + \frac{\partial^2 f}{\partial t \partial \lambda} (a, y) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right|^q \right)^{\frac{1}{q}}$$

A similar way for other integral, since $|\frac{\partial^2 f}{\partial s \partial \lambda}|^q, q > 1$, is co-ordinated $s$–convex function on $\Delta$, we get

(2.6)

$$\left( \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1-t) a, \lambda y + (1-\lambda) d) \right|^q dsdt \right)^{\frac{1}{q}}$$

$$\leq \frac{1}{(s+1)^{\frac{2}{q}}} \left( \frac{\partial^2 f}{\partial t \partial \lambda} (x, y) \right)^q + \frac{\partial^2 f}{\partial t \partial \lambda} (x, d) \right|^q + \frac{\partial^2 f}{\partial t \partial \lambda} (a, y) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right|^q \right)^{\frac{1}{q}},$$

(2.7)

$$\left( \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial s} (tx + (1-t) b, \lambda y + (1-\lambda) c) \right|^q dsdt \right)^{\frac{1}{q}}$$

$$\leq \frac{1}{(s+1)^{\frac{2}{q}}} \left( \frac{\partial^2 f}{\partial t \partial \lambda} (x, y) \right)^q + \frac{\partial^2 f}{\partial t \partial \lambda} (x, c) \right|^q + \frac{\partial^2 f}{\partial t \partial \lambda} (b, y) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) \right|^q \right)^{\frac{1}{q}},$$
By the (2.5)-(2.8), we get the inequality (2.3).

**Corollary 3.** (1) Under the assumptions of Theorem 5 if we choose \( x = a, y = c \), or \( x = b, y = d \), we obtain the following inequality:

\[
1 \leq \frac{1}{(s + 1)^\frac{2}{q}} \left( \left| \frac{\partial^2 f}{\partial \partial \lambda} (a, c) \right|^q + \left| \frac{\partial^2 f}{\partial \partial \lambda} (a, d) \right|^q + \left| \frac{\partial^2 f}{\partial \partial \lambda} (b, c) \right|^q + \left| \frac{\partial^2 f}{\partial \partial \lambda} (b, d) \right|^q \right)^\frac{1}{q}.
\]

(2) Under the assumptions of Theorem 5 if we choose \( x = b, y = d \), we obtain the following inequality:

\[
1 \leq \frac{1}{(s + 1)^\frac{2}{q}} \left( \left| \frac{\partial^2 f}{\partial \partial \lambda} (b, d) \right|^q + \left| \frac{\partial^2 f}{\partial \partial \lambda} (b, c) \right|^q + \left| \frac{\partial^2 f}{\partial \partial \lambda} (a, d) \right|^q + \left| \frac{\partial^2 f}{\partial \partial \lambda} (a, c) \right|^q \right)^\frac{1}{q}.
\]

(3) Under the assumptions of Theorem 5 if we choose \( x = a, y = d \), we obtain the following inequality:

\[
1 \leq \frac{1}{(s + 1)^\frac{2}{q}} \left( \left| \frac{\partial^2 f}{\partial \partial \lambda} (a, d) \right|^q + \left| \frac{\partial^2 f}{\partial \partial \lambda} (a, c) \right|^q + \left| \frac{\partial^2 f}{\partial \partial \lambda} (b, d) \right|^q + \left| \frac{\partial^2 f}{\partial \partial \lambda} (b, c) \right|^q \right)^\frac{1}{q}.
\]
(4) Under the assumptions of Theorem 3, if we choose \( x = b, y = c \), we obtain the following inequality:

\[
\frac{1}{(b-a)(d-c)} \left| f(a, d) - (b-a) \int_c^d f(a, v) dv - (d-c) \int_a^b f(u, d) du + \int_a^b f(u, v) dudv \right| \\
\leq \frac{(b-a)(d-c)}{(p+1)^\frac{q}{2} (s+1)^\frac{q}{2}} \left( \frac{\partial^2 f}{\partial t \partial \lambda} (b, c)^q + \frac{\partial^2 f}{\partial t \partial \lambda} (b, d)^q + \frac{\partial^2 f}{\partial t \partial \lambda} (a, c)^q + \frac{\partial^2 f}{\partial t \partial \lambda} (a, d)^q \right) ^\frac{1}{q}
\]

(5) Under the assumptions of Theorem 3, if we choose \( x = \frac{a+b}{2}, y = \frac{c+d}{2} \), we obtain the following inequality:

\[
\left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4(b-a)(d-c)} - \frac{1}{2(d-c)} \int_c^d f(a, v) dv - \frac{1}{2(b-a)} \int_a^b f(u, v) dudv \right| \\
\leq \frac{(b-a)(d-c)}{16(p+1)^\frac{q}{2} (s+1)^\frac{q}{2} \times \left( \left( \frac{\partial^2 f}{\partial t \partial \lambda} \left( \frac{a+b}{2}, \frac{c+d}{2} \right)^q \right) + \left( \frac{\partial^2 f}{\partial t \partial \lambda} \left( \frac{a+b}{2}, \frac{c+d}{2} \right)^q \right) + \left( \frac{\partial^2 f}{\partial t \partial \lambda} \left( \frac{a+b}{2}, \frac{c+d}{2} \right)^q \right) + \left( \frac{\partial^2 f}{\partial t \partial \lambda} \left( \frac{a+b}{2}, \frac{c+d}{2} \right)^q \right) \right) ^\frac{1}{q}}
\]

\[
+ \left( \frac{\partial^2 f}{\partial t \partial \lambda} \left( \frac{a+b}{2}, \frac{c+d}{2} \right)^q \right) + \left( \frac{\partial^2 f}{\partial t \partial \lambda} \left( \frac{a+b}{2}, \frac{c+d}{2} \right)^q \right) + \left( \frac{\partial^2 f}{\partial t \partial \lambda} \left( \frac{a+b}{2}, \frac{c+d}{2} \right)^q \right) + \left( \frac{\partial^2 f}{\partial t \partial \lambda} \left( \frac{a+b}{2}, \frac{c+d}{2} \right)^q \right) \right) ^\frac{1}{q}
\]

\[
+ \left( \frac{\partial^2 f}{\partial t \partial \lambda} \left( \frac{a+b}{2}, \frac{c+d}{2} \right)^q \right) + \left( \frac{\partial^2 f}{\partial t \partial \lambda} \left( \frac{a+b}{2}, \frac{c+d}{2} \right)^q \right) + \left( \frac{\partial^2 f}{\partial t \partial \lambda} \left( \frac{a+b}{2}, \frac{c+d}{2} \right)^q \right) + \left( \frac{\partial^2 f}{\partial t \partial \lambda} \left( \frac{a+b}{2}, \frac{c+d}{2} \right)^q \right) \right) ^\frac{1}{q}
\]

\[
+ \left( \frac{\partial^2 f}{\partial t \partial \lambda} \left( \frac{a+b}{2}, \frac{c+d}{2} \right)^q \right) + \left( \frac{\partial^2 f}{\partial t \partial \lambda} \left( \frac{a+b}{2}, \frac{c+d}{2} \right)^q \right) + \left( \frac{\partial^2 f}{\partial t \partial \lambda} \left( \frac{a+b}{2}, \frac{c+d}{2} \right)^q \right) + \left( \frac{\partial^2 f}{\partial t \partial \lambda} \left( \frac{a+b}{2}, \frac{c+d}{2} \right)^q \right) \right) ^\frac{1}{q}.\]
Remark 2. From sum of (2.12)–(2.13), we obtain;

\[ (2.13) \]
\[
\begin{align*}
&f(a, c) - (b - a) \int_c^d f(a, v) dv - (d - c) \int_b^a f(u, c) du + \int_b^d f(u, v) dudv \\
&+ f(a, d) - (b - a) \int_c^d f(a, v) dv - (d - c) \int_b^a f(u, d) du + \int_b^d f(u, v) dudv \\
&+ f(b, c) - (b - a) \int_c^d f(b, v) dv - (d - c) \int_b^a f(u, c) du + \int_b^d f(u, v) dudv \\
&+ f(b, d) - (b - a) \int_c^d f(b, v) dv - (d - c) \int_b^a f(u, d) du + \int_b^d f(u, v) dudv \\
&\leq \frac{4(b - a)^2(d - c)^2}{(p + 1)^\frac{3}{2}} \times \\
&\left( \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \right|^q \right)^\frac{1}{q}.
\end{align*}
\]

Theorem 4. Let \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta = [a, b] \times [c, d] \) and \( \frac{\partial^2 f}{\partial x \partial \lambda} \in L(\Delta) \). If \( \left| \frac{\partial^2 f}{\partial x \partial \lambda} \right|^q, q \geq 1, \) is a \( s \)-convex function in the second sense on the co-ordinates on \( \Delta, \) for some fixed \( s \in (0, 1], \) then the following inequality holds;

\[ (2.14) \]
\[
\begin{align*}
&\left| \frac{1}{(b - a)(d - c)} \left[ A - (x - a) \int_c^d f(a, v) dv - (b - x) \int_c^d f(b, v) dv \\
&-(d - y) \int_a^b f(u, d) du - (y - c) \int_a^b f(u, c) du + \int_a^b f(u, v) dudv \right] \\
&\leq \frac{2^2q}{(s + 1)^\frac{3}{2}} \times \\
&\left\{ \frac{(x - a)^2(y - c)^2}{(b - a)(d - c)} \left\{ \left| \frac{\partial^2 f}{\partial t \partial \lambda}(x, y) \right|^q + (s + 1) \left| \frac{\partial^2 f}{\partial t \partial \lambda}(x, c) \right|^q \right\} \\
&+ (s + 1) \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, y) \right|^q + (s + 1)^2 \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \right|^q \right\}^\frac{1}{q}.
\end{align*}
\]
Proof. From Lemma 1, we have

\[+rac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \left\{ \left| \frac{\partial^2 f}{\partial \partial \lambda} (x, y) \right|^q + (s+1) \left| \frac{\partial^2 f}{\partial \partial \lambda} (x, d) \right|^q \right\} + (s+1) \left| \frac{\partial^2 f}{\partial \partial \lambda} (a, y) \right|^q + (s+1)^2 \left| \frac{\partial^2 f}{\partial \partial \lambda} (a, d) \right|^q \left\} \right.\]

+ \frac{(b-x)^2(y-c)^2}{(b-a)(d-c)} \left\{ \left| \frac{\partial^2 f}{\partial \partial \lambda} (x, y) \right|^q + (s+1) \left| \frac{\partial^2 f}{\partial \partial \lambda} (x, c) \right|^q \right\} + (s+1) \left| \frac{\partial^2 f}{\partial \partial \lambda} (b, y) \right|^q + (s+1)^2 \left| \frac{\partial^2 f}{\partial \partial \lambda} (b, c) \right|^q \left\} \right.\]

\[+ \frac{(b-x)^2(y-c)^2}{(b-a)(d-c)} \left\{ \left| \frac{\partial^2 f}{\partial \partial \lambda} (x, y) \right|^q + (s+1) \left| \frac{\partial^2 f}{\partial \partial \lambda} (x, d) \right|^q \right\} + (s+1) \left| \frac{\partial^2 f}{\partial \partial \lambda} (b, y) \right|^q + (s+1)^2 \left| \frac{\partial^2 f}{\partial \partial \lambda} (b, d) \right|^q \left\} \right.\]

\[\leq \frac{1}{(b-a)(d-c)} \left| \frac{A - (x-a) \int_0^d f(a, v) dv - (b-x) \int_0^d f(b, v) dv}{(d-y) \int_a^b f(u, d) du - (y-c) \int_a^b f(u, c) du + \int_a^b \int_c^d f(u, v) dv du} \right| \int_0^1 \left| (t-1)(\lambda-1) \right| \left| \frac{\partial^2 f}{\partial \partial \lambda} (tx + (1-t) a, \lambda y + (1-\lambda)) \right| d\lambda dt \]

\[+ \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 \left| (t-1)(1-\lambda) \right| \left| \frac{\partial^2 f}{\partial \partial \lambda} (tx + (1-t) a, \lambda y + (1-\lambda)) \right| d\lambda dt \]

\[+ \frac{(b-x)^2(y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 \left| (1-t)(\lambda-1) \right| \left| \frac{\partial^2 f}{\partial \partial \lambda} (tx + (1-t) b, \lambda y + (1-\lambda) c) \right| d\lambda dt \]

\[+ \frac{(b-x)^2(y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 \left| (1-t)(1-\lambda) \right| \left| \frac{\partial^2 f}{\partial \partial \lambda} (tx + (1-t) b, \lambda y + (1-\lambda) d) \right| d\lambda dt. \]
By using the well known power mean inequality for double integrals, \( f : \Delta \to \mathbb{R} \) is co-ordinated \( s \)-convex on \( \Delta \), then one has:

\[
(2.15) \quad \left| \frac{1}{(b-a)(d-c)} \left[ A - (x-a) \int_c^b f(a,v)dv - (b-x) \int_c^b f(b,v)dv \right. \right.
\]
\[
\left. - (d-y) \int_a^b f(u,d)du - (y-c) \int_a^b f(u,c)du + \int_a^b \int_c^b f(u,v)du dv \right] \nonumber
\]
\[
\leq \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 |(t-1)(\lambda-1)| d\lambda dt \right)^{1-\frac{q}{q}} \times 
\]
\[
\left( \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1-t)a, \lambda y + (1-\lambda)c) \right|^q \right)^{\frac{1}{q}} 
\]
\[
+ \frac{(x-a)^2(d-y)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 |(t-1)(1-\lambda)| d\lambda dt \right)^{1-\frac{q}{q}} \times 
\]
\[
\left( \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1-t)a, \lambda y + (1-\lambda)c) \right|^q \right)^{\frac{1}{q}} 
\]
\[
+ \frac{(b-x)^2(y-c)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 |(1-t)(\lambda-1)| d\lambda dt \right)^{1-\frac{q}{q}} \times 
\]
\[
\left( \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1-t)b, \lambda y + (1-\lambda)c) \right|^q \right)^{\frac{1}{q}} 
\]
\[
+ \frac{(b-x)^2(d-y)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 |(1-t)(1-\lambda)| d\lambda dt \right)^{1-\frac{q}{q}} \times 
\]
\[
\left( \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1-t)b, \lambda y + (1-\lambda)d) \right|^q \right)^{\frac{1}{q}} 
\]

Since \( \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q \) is \( s \)-convex function in the second sense on the co-ordinates on \( \Delta \), for some fixed \( s \in (0,1] \), we know that for \( t \in [0,1] \)

\[
\left| \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1-t)a, \lambda y + (1-\lambda)c) \right|^q \leq t^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, \lambda y + (1-\lambda)c) \right|^q + (1-t)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, \lambda y + (1-\lambda)c) \right|^q 
\]
and

\[
\left| \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1 - t) a, \lambda y + (1 - \lambda) c) \right|^q \leq t^s \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, y) \right|^q + t^s (1 - \lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, c) \right|^q + (1 - t)^s \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, y) \right|^q + (1 - t)^s (1 - \lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right|^q
\]

hence, it follows that

\[
(2.16) \left( \int_0^1 \int_0^1 |(t - 1) (\lambda - 1)| \left| \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1 - t) a, \lambda y + (1 - \lambda) c) \right|^q \, d\lambda \, dt \right)^{\frac{1}{q}} \leq \left( \int_0^1 \int_0^1 |(t - 1) (\lambda - 1)| t^s \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, y) \right|^q + |(t - 1) (\lambda - 1)| t^s (1 - \lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, c) \right|^q + |(t - 1) (\lambda - 1)| (1 - t)^s \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, y) \right|^q + |(t - 1) (\lambda - 1)| (1 - t)^s (1 - \lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right|^q \right) \, dt \, d\lambda \right)^{\frac{1}{q}} = \left\{ \frac{1}{(s + 1)^2 (s + 2)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, y) \right|^q + \frac{1}{(s + 1)^2 (s + 2)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, c) \right|^q + \frac{1}{(s + 1)^2 (s + 2)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, y) \right|^q + \frac{1}{(s + 1)^2 (s + 2)^2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right|^q \right\}^{\frac{1}{q}} + (s + 1)^2 \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, y) \right|^q + (s + 1)^2 \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right|^q \right\}^{\frac{1}{q}}
\]

A similar way for other integral, since \( \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q \) is co-ordinated \( s \)-convex function in the second sense on \( \Delta \), we get

\[
(2.17) \left( \int_0^1 \int_0^1 |(t - 1) (1 - \lambda)| \left| \frac{\partial^2 f}{\partial t \partial \lambda} (tx + (1 - t) a, \lambda y + (1 - \lambda) d) \right|^q \, ds \, dt \right)^{\frac{1}{q}} \leq \frac{1}{(s + 1)^{\frac{q}{2}} (s + 2)^{\frac{q}{2}}} \left\{ \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, y) \right|^q + (s + 1) \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, d) \right|^q + (s + 1)^2 \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, y) \right|^q + (s + 1)^2 \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right|^q \right\}^{\frac{1}{q}},
\]
Under the assumptions of Theorem 4, if we choose $x = a, y = c$, or $x = b, y = d$, we obtain the following inequality:

\[ \left( \int_0^1 \int_0^1 |(1-t)(\lambda - 1)| \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right| (tx + (1-t)b, \lambda y + (1-\lambda)c) \right] ds dt \]  

\[ \leq \frac{1}{(s + 1)^\frac{\alpha}{2} (s + 2)^\frac{\beta}{2}} \left\{ \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, y) \right|^q + (s + 1) \left| \frac{\partial^2 f}{\partial t \partial \lambda} (x, c) \right|^q \right\}, \]

Corollary 4. (1) Under the assumptions of Theorem 4 if we choose $x = a, y = c$, or $x = b, y = d$, we obtain the following inequality:

\[ \frac{1}{(b-a)(d-c)} \left\{ f(b,d) - (b-a) \int_c^b f(b,v) \, dv - (d-c) \int_a^c f(u,d) \, du + \int_a^b \int_c^d f(u,v) \, dudv \right\} \]

\[ \leq \frac{2^{2-\frac{\alpha}{2}} (b-a)(d-c)}{s+1} \frac{1}{(s + 1)^\frac{\alpha}{2} (s + 2)^\frac{\beta}{2}} \left\{ \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a,c) \right|^q + (s + 1) \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a,d) \right|^q \right\}. \]

(2) Under the assumptions of Theorem 4 if we choose $x = b, y = d$, we obtain the following inequality:

\[ \frac{1}{(b-a)(d-c)} \left\{ f(a,c) - (b-a) \int_c^b f(a,v) \, dv - (d-c) \int_a^c f(u,c) \, du + \int_a^b \int_c^d f(u,v) \, dudv \right\} \]

\[ \leq \frac{2^{2-\frac{\alpha}{2}} (b-a)(d-c)}{s+1} \frac{1}{(s + 1)^\frac{\alpha}{2} (s + 2)^\frac{\beta}{2}} \left\{ \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b,c) \right|^q + (s + 1) \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b,d) \right|^q \right\}. \]
(3) Under the assumptions of Theorem \[4\] if we choose \(x = a, y = d\), we obtain the following inequality:

\[
\begin{align*}
(2.22) & \quad \frac{1}{(b-a)(d-c)} \left| f(b, c) - (b-a) \int_c^d f(b, v) \, dv - (d-c) \int_a^b f(u, c) \, du + \int_a^b \int_c^d f(u, v) \, dudv \right| \\
& \leq 2^{2-\frac{4}{q}} (b-a)(d-c) \left\{ \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \right|^q + (s+1) \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \right|^q \right. \\
& \quad + (s+1) \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \right|^q + (s+1)^2 \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, c) \right|^q \}^{\frac{1}{q}}
\end{align*}
\]

(4) Under the assumptions of Theorem \[4\] if we choose \(x = b, y = c\), we obtain the following inequality:

\[
\begin{align*}
(2.23) & \quad \frac{1}{(b-a)(d-c)} \left| f(a, d) - (b-a) \int_c^d f(a, v) \, dv - (d-c) \int_a^b f(u, d) \, du + \int_a^b \int_c^d f(u, v) \, dudv \right| \\
& \leq 2^{2-\frac{4}{q}} (b-a)(d-c) \left\{ \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \right|^q + (s+1) \left| \frac{\partial^2 f}{\partial t \partial \lambda}(c, d) \right|^q \right. \\
& \quad + (s+1) \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \right|^q + (s+1)^2 \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \right|^q \}^{\frac{1}{q}}
\end{align*}
\]
(5) Under the assumptions of Theorem [4], if we choose \( x = \frac{a + b}{2} \), \( y = \frac{c + d}{2} \), we obtain the following inequality:

\[
\frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4(b-a)(d-c)} - \frac{1}{2(d-c)} \int_c^d f(a,v)\,dv - \frac{1}{2(b-a)} \int_a^b f(u,c)\,du + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v)\,dudv
\]

\[
\leq \frac{(b-a)(d-c)}{4(2(s+1)(s+2))^\frac{s}{2}} \times
\left\{
\left(\left|\frac{\partial^2 f}{\partial \partial \lambda} \left(\frac{a + b}{2}, \frac{c + d}{2}\right)\right|^q + (s+1) \left|\frac{\partial^2 f}{\partial \partial \lambda} \left(\frac{a + b}{2}, c\right)\right|^q\right)
\right. \\
+ (s+1) \left|\frac{\partial^2 f}{\partial \partial \lambda} \left(\frac{a + b}{2}, \frac{c + d}{2}\right)\right|^q + (s+1)^2 \left|\frac{\partial^2 f}{\partial \partial \lambda} \left(\frac{a + b}{2}, \frac{a + d}{2}\right)\right|^q
\left. \\
+ (s+1) \left|\frac{\partial^2 f}{\partial \partial \lambda} \left(\frac{a + b}{2}, \frac{c + d}{2}\right)\right|^q + (s+1) \left|\frac{\partial^2 f}{\partial \partial \lambda} \left(\frac{a + b}{2}, \frac{a + d}{2}\right)\right|^q
\right.
\left. \\
+ (s+1) \left|\frac{\partial^2 f}{\partial \partial \lambda} \left(\frac{a + b}{2}, \frac{c + d}{2}\right)\right|^q + (s+1) \left|\frac{\partial^2 f}{\partial \partial \lambda} \left(\frac{a + b}{2}, \frac{a + d}{2}\right)\right|^q
\right.
\left. \\
+ (s+1) \left|\frac{\partial^2 f}{\partial \partial \lambda} \left(\frac{a + b}{2}, \frac{c + d}{2}\right)\right|^q + (s+1)^2 \left|\frac{\partial^2 f}{\partial \partial \lambda} \left(\frac{a + b}{2}, \frac{a + d}{2}\right)\right|^q
\right.
\left. \\
+ (s+1) \left|\frac{\partial^2 f}{\partial \partial \lambda} \left(\frac{a + b}{2}, \frac{c + d}{2}\right)\right|^q + (s+1)^2 \left|\frac{\partial^2 f}{\partial \partial \lambda} \left(\frac{a + b}{2}, \frac{a + d}{2}\right)\right|^q
\right.
\left. \\
+ (s+1) \left|\frac{\partial^2 f}{\partial \partial \lambda} \left(\frac{a + b}{2}, \frac{c + d}{2}\right)\right|^q + (s+1)^2 \left|\frac{\partial^2 f}{\partial \partial \lambda} \left(\frac{a + b}{2}, \frac{a + d}{2}\right)\right|^q
\right.
\left. \\
\right\}
\]

Remark 3. From sum of [22.20]-[22.20], we get;

\[
|f(a,c) - (b-a)\int_c^d f(a,v)\,dv - (d-c)\int_a^b f(u,c)\,du + \int_a^b \int_c^d f(u,v)\,dudv|
\]

\[
+ |f(a,d) - (b-a)\int_c^d f(a,v)\,dv - (d-c)\int_a^b f(u,d)\,du + \int_a^b \int_c^d f(u,v)\,dudv|
\]

\[
+ |f(b,c) - (b-a)\int_c^d f(b,v)\,dv - (d-c)\int_a^b f(u,c)\,du + \int_a^b \int_c^d f(u,v)\,dudv|
\]

\[
+ |f(b,d) - (b-a)\int_c^d f(b,v)\,dv - (d-c)\int_a^b f(u,d)\,du + \int_a^b \int_c^d f(u,v)\,dudv|
\]
\[
\leq \frac{4(b-a)^2(d-c)^2}{(2(s+1)(s+2))^{\frac{q}{2}}} \times \\
\left\{ \left( \frac{\partial^2 f}{\partial t\partial \lambda}(a, c) \right)^q + (s+1)^2 \left( \frac{\partial^2 f}{\partial t\partial \lambda}(a, d) \right)^q \\
+ \left( \frac{\partial^2 f}{\partial t\partial \lambda}(b, c) \right)^q + (s+1)^2 \left( \frac{\partial^2 f}{\partial t\partial \lambda}(b, d) \right)^q \right\}^{\frac{1}{q}}
\]

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