Thermodynamics of a photon gas and deformed dispersion relations

Abel Camacho∗ and Alfredo Macías†
Departamento de Física, Universidad Autónoma Metropolitana–Iztapalapa
Apartado Postal 55–534, C.P. 09340, México, D.F., México.
(Dated: March 24, 2022)

We resort to the methods of statistical mechanics in order to determine the effects that a deformed dispersion relation has upon the thermodynamics of a photon gas. The ensuing modifications to the density of states, partition function, pressure, internal energy, entropy, and specific heat are calculated. It will be shown that the breakdown of Lorentz invariance can be interpreted as a repulsive interaction, among the photons. Additionally, it will be proved that the presence of a deformed dispersion relation entails an increase in the entropy of the system. In other words, as a consequence of the loss of the aforementioned symmetry the number of microstates available to the corresponding equilibrium state grows.

I. INTRODUCTION

Due to its fundamental role in modern science Lorentz symmetry has been subjected to some of the highest precision tests [1, 2, 3] that physics has performed. In spite of this severe experimental scrutiny several quantum gravity models predict the breakdown of Lorentz symmetry [4, 5, 6]. Additionally, it has been suggested an energy dependent speed of light as a possible solution to the so–called GZK paradox [7], namely, the observation ultra high energy cosmic ray above the expected GZK threshold for interaction of such cosmic rays with the cosmic microwave background [7, 8].

The possible detection of these corrections has already been analyzed in several contexts. Indeed, in the realm of interferometry it has been analyzed resorting to first–order coherence proposals [9, 10], or even higher–order coherence experiments, like the so–called Hanbury–Brown–Twiss experiment [11], the modifications upon the Standard Model that a Lorentz invariance violation could have [12, 13, 14].

In the present work the consequences of a Lorentz invariance violation will be considered in a very different approach. The idea here is to introduce a deformed dispersion relation as a fundamental fact for the dynamics of photons. Afterwards we analyze the effects of this assumption upon the thermodynamics of a photon gas. It will be shown that the breakdown of Lorentz symmetry entails an increase in the number of microstates, and in consequence a growth of the entropy, with respect to the case in which Lorentz symmetry exists. Resorting to the case of a non–ideal gas (the interactions among the particles cannot be neglected) it will be proved that the presence of a deformed dispersion relation could be interpreted as a repulsive interaction. The specific heat, the entropy, and other thermodynamic parameters are also calculated, and the consequences of the breakdown of Lorentz symmetry are evaluated.

II. THERMODYNAMICAL CONSEQUENCES OF A DEFORMED DISPERSION RELATION

A. Density of states and partition function

As mentioned above several quantum–gravity models predict a modified dispersion relation [4], the one can be characterized, phenomenologically, through corrections hinging upon Planck’s length, \( l_p \),

\[
E^2 = p^2 \left[ 1 - \alpha \left( E l_p \right)^n \right].
\]  

(1)
Here $\alpha$ is a coefficient, whose precise value depends upon the considered quantum–gravity model, while $n$, the lowest power in Planck’s length leading to a non–vanishing contribution, is also model dependent.

Consider now a photon gas in a container of volume $V$. Though the momentum, as a consequence of the boundary conditions, is quantized, we may assume, as is usual under these conditions $^{13}$, a continuous momentum spectrum. At this point it is noteworthy that this assumption has nothing to do with the relation between energy and momentum, and in consequence the error here introduced is the same that appears in the usual model for a photon gas. Then the number of states in a certain volume of the phase space is given by

$$
\Sigma = \frac{1}{(2\pi \hbar)^3} \int \int d\vec{r}d\vec{p}.
$$

(2)

Our deformed dispersion relation entails that (here $E_p = \sqrt{G/(c^2\hbar)}$ denotes the so–called Planck’s energy)

$$
p^2 = \frac{E_p^2}{c^2} \left[ 1 - \alpha \left( \frac{E}{E_p} \right)^n \right]^{-1}.
$$

(3)

At this point we remember that Statistical Mechanics tells us that the relation between energy and momentum, of the particles comprising a gas, has a very important role in the evaluation of the dependence of the pressure as a function of the energy density $^{13}$. Indeed, assume that the momentum is a power, say $p = \epsilon^n$, (here $p$ and $\epsilon$ are momentum and energy, respectively), then the pressure $P$ and the energy density, $u$, of the corresponding gas are related by the expression $P = su/l$. The last expression entails that we must now have a different state equation, i.e., a deformed dispersion relation must change the thermodynamical properties of a photon gas, since now the relation between momentum and energy is not given by $p \sim \epsilon$.

$$
\frac{dp}{d\epsilon} = \frac{1}{c} \left[ 1 - \alpha \left( \frac{E}{E_p} \right)^n \right]^{-3/2} \left[ 1 + \alpha(n - 1) \left( \frac{E}{E_p} \right)^n \right].
$$

(4)

Then, if the gas is in a container with volume $V$, we may cast the number of states in the following form

$$
\Sigma = \frac{4\pi V}{(2\pi \hbar)^3} \int_0^\infty E^2 \left[ 1 - \alpha \left( \frac{E}{E_p} \right)^n \right]^{-5/2} \left[ 1 + \alpha(n - 1) \left( \frac{E}{E_p} \right)^n \right] dE.
$$

(5)

At this point it is noteworthy to comment that (assuming $\alpha > 0$) the number of available states grows, with respect to the case in which Lorentz symmetry is preserved, when a deformed dispersion relation appears. In other words, the absence of the symmetry implies that with respect to the corresponding equilibrium state now more microstates are available. This last remark implies also that we should not be surprised if the corresponding entropy grows, clearly, here we mean a larger entropy compared against the entropy of a gas under the same conditions, but with the Lorentz symmetry untouched.

Let us now proceed to calculate the thermodynamical properties of such a photon gas, and in order to do this let us remember that the connection between the microscopic world and the thermodynamic behavior is done by the partition function $^{13}$, i.e., the knowledge of this function allows us to deduce all the thermodynamics of the corresponding system.

$$
\ln Z = -\frac{4\pi V}{(2\pi \hbar)^3} \int_0^\infty E^2 \left[ 1 + \alpha(n + 3/2) \left( \frac{E}{E_p} \right)^n \right] \ln \left[ 1 - e^{-E/\xi} \right] dE.
$$

(6)

Explicitly, the logarithm of the partition function reads

$$
\ln Z = \frac{8\pi^5 V(\xi^3)^3}{(2\pi \hbar)^3} \left[ 1 + \frac{\alpha}{2} \left( n + 3/2 \right) \left( n + 2 \right) \right] \frac{\xi(4 + n)}{\xi(4)} (T/T_p)^n.
$$

(7)

In this last expression $K$ is Boltzmann’s constant, $T_p = E_p/K$ Planck’s temperature, and $\xi(x)$ the so–called Riemann’s zeta function $^{10}$. Setting $\alpha = 0$ everything reduces to the usual case. Notice that the new contribution depends upon term $(T/T_p)^n$, a fact that complicates the possible detection of these effects, due to the value of $T_p$. 
B. Thermodynamical parameters

Recalling that the connection between internal energy and partition function is given by [15]

\[ U = K T^2 \left( \frac{\partial \ln Z}{\partial T} \right)_V. \] (8)

\[ U = \frac{24\pi^5 V (KT)^4}{(2\pi\hbar)^3} \left[ 1 + \alpha (n + 3/2) \left( \frac{n + 3}{3!} \right) \xi(4 + n) \right] \xi(4) \left( \frac{T}{T_p} \right)^n. \] (9)

If in (9) we set \( \alpha = 0 \), then we recover the usual internal energy for a photon gas [15]. Also notice that the breakdown of Lorentz symmetry entails an increase in the internal energy of our photon gas. Though the internal energy is not directly a detectable parameter, there are other variables, as the specific heat, that may be measured and which are calculated from the internal energy.

We now proceed to calculate the state equation. This is easily done if we remember that it can be deduced from the characteristic function of the corresponding ensemble [15]. For the present case we have that in the Grand Canonical Ensemble the characteristic function is given by

\[ P V = K T \ln Z. \] (10)

This expression clearly show us that if we have \( \ln Z \) then immediately we obtain the state equation. Our previous results imply that the pressure is given by

\[ P = \frac{8\pi^5 (KT)^4}{(2\pi\hbar)^3} \left[ 1 + \frac{\alpha}{2} (n + 3/2) \left( \frac{n + 3}{3!} \right) \xi(4 + n) \right] \xi(4) \left( \frac{T}{T_p} \right)^n. \] (11)

It is important to mention that the pressure grows, with respect to the case in which Lorentz symmetry is present, the condition \( \alpha = 0 \) takes us back to the usual situation in which the pressure behaves as \( P \sim T^4 \).

This last remark allows us to interpret the breakdown of Lorentz symmetry as a repulsive interaction. Indeed, the presence of a repulsive interaction (among the particles of a gas) entails the increase of the pressure, compared against the corresponding value for an ideal gas.

Let us explain this point deeper. A fleeting glimpse at the cluster expansion and its relation to the virial coefficients [17] clearly shows that the first correction to the ideal gas state equation expressed in terms of the so–called virial state equation \( (P V / (NKT) = \sum_{l=1}^{\infty} a_l(T) (N\lambda^3/V)^{l-1} \) , \( N \) denotes the number of particles) corresponds to a virial coefficient that can be written, as a function of the potential energy of interaction between the i–th and the j–th particle \( v_{ij} \) as

\[ a_2 = -\frac{1}{\lambda} \int f_{12} d^3 r_{12}. \] (12)

Where \( \exp \{-v_{12}/KT\} = 1 + f_{12} \), here \( f_{12} \) is the two particle function (which is a function of the potential \( v_{12} \) between two particles), and \( \lambda = \sqrt{2\pi\hbar^2/mKT} \) is the so–called thermal wavelength [17].

A repulsive interaction means that \( v_{12} > 0 \), and in consequence \( f_{12} < 0 \), and therefore, \( a_2 > 0 \). If we introduce this condition into the virial expression we obtain a pressure larger than that corresponding to an ideal gas. In other words, the introduction of a repulsive interaction among the particles comprising the gas entails an increase of the pressure, compared to the pressure of an ideal gas.

It is in this sense that we say that the loss of the symmetry appears, at the bulk level, as the emergence of a repulsive interaction, and in consequence, at least in principle, we could detect some effects stemming, either from loop quantum gravity, non–commutative geometry, etc.

One of the consequences of Lorentz symmetry can be seen casting the pressure in terms of the energy density \( (u = U/V) \), i.e. \( P = u/3 \). Indeed, if the relation between energy and momentum for a bosonic particle in l–spacelike dimensions is \( \epsilon \sim p^l \), then the pressure becomes \( P = \frac{4}{3} u \) [15]. Notice that for our case the aforementioned relation between energy and momentum does not hold anymore, and in consequence the expression for the pressure in terms of
the energy density can not have this form. In other words, the breakdown of Lorentz symmetry can be, in principle, detected looking at the relation between pressure and energy density, i.e., this kind of effects do appear at the macroscopic level. For our case we have that

\[ P = \frac{1}{3} u \left[ 1 - \alpha (n + 3/2)(n + 2)! \frac{n \xi(4 + n)}{3! \xi(4)} (T/T_p)^n \right]. \tag{13} \]

Setting \( \alpha = 0 \) we obtain the usual result. As expected, the corrections to the usual behavior are a function of the term \((T/T_p)^n\).

Another interesting thermodynamical quantity is the entropy, \( S \). Since \( S = PV/T + U/T \), and with all the results we may write

\[ S = \frac{8\pi^5 V (KT)^3}{90(2\pi \hbar)^3} K \left[ 4 + \alpha (n + 3/2)(n + 4) \frac{(n + 2)! \xi(4 + n)}{2! \xi(4)} (T/T_p)^n \right]. \tag{14} \]

A consequence of (14) is that a reversible adiabatic process is not anymore given by the condition \( VT^3 = \text{const.} \), as is usual \[15, 17\]. Additionally, for a photon gas \( 3S = T \frac{\partial S}{\partial T} V \). This is also lost if \( \alpha \neq 0 \). Notice that the breakdown of Lorentz symmetry increases the number of microstates available to the macrostate, see (5), and in consequence, our entropy grows, as (14) clearly displays, with \( \alpha > 0 \).

Since the internal energy suffers a change in its dependence upon the temperature \( \frac{\partial U}{\partial T} \) then the specific heat, in this case at constant volume, must also show modifications.

Recalling that

\[ C_V = \left( \frac{\partial U}{\partial T} \right)_V, \tag{15} \]

\[ C_V = \frac{24\pi^5 V (KT)^4}{90T(2\pi \hbar)^3} \left[ 4 + \alpha (n + 3/2)(n + 4) \frac{(n + 3)! \xi(4 + n)}{3! \xi(4)} (T/T_p)^n \right]. \tag{16} \]

It shall be no surprise that the value \( \alpha = 0 \) reduces the specific heat to its usual value. This is a measurable quantity and in consequence we have obtained another parameter, which in principle, could be employed in the experimental quest for violations of Lorentz symmetry. The calculated specific heat is larger than the corresponding parameter for an ideal gas, and this result is compatible with our previous interpretation of the breakdown of Lorentz symmetry as the appearance of a repulsive interaction. Indeed, the specific heat at constant volume, in regions where the repulsive interaction among the particles plays the predominant role, is larger than the value for an ideal gas, see table 3.1 on page 77 of \[18\]. This can also be understood if in the van der Waals state equation (in which \( v \) denotes the volume per particle, \( a \) is a parameter related to an attractive interaction among the particles, whereas \( b \) is related to a repulsive one \[15, 17\])

\[ P = \frac{RT}{v - b} - \frac{a}{v^2}, \tag{17} \]

we impose the condition \( a = 0 \), and calculate the corresponding specific heat with \( b > 0 \), a condition that can be rephrased in terms of the repulsive interaction that appears among the particles due to the fact that they are not point–like particles, but do have a non–vanishing volume.

At this point it is noteworthy to mention that, as can be seen from the previous expressions, the modifications, due to this kind of breakdown, behave like \((T/T - p)^n\), and, in consequence, a terrestrial experiment in this context is not feasible.
III. CONCLUSIONS

We have assumed from the onset the modification of the dispersion relation, for photons, a condition stemming in many models that try to quantize gravity. With this new relation the density of states has been evaluated. It has been shown that the number of microstates available to the corresponding equilibrium state grows, compared to the case in which Lorentz symmetry is present. As expected, the entropy becomes larger as an unavoidable consequence of this kind of Lorentz violation (11), nevertheless, in the limit $T \to 0$ the entropy goes to zero, as happens in the case in which Lorentz symmetry is present. In other words, Nernst postulate is not violated if the symmetry is broken.

Additionally, since the breakdown of Lorentz symmetry entails a larger pressure (compared to the case of a photon gas in which Lorentz symmetry is present), see (13), then we may interpret the breakdown of Lorentz symmetry, at least in the bulk realm, as equivalent to the emergence of a repulsive interaction among the photons. This interpretation is corroborated by the ensuing form for the specific heat at constant volume. Indeed, even for a simple case, for instance, van der Waals state equation (in which the fact that the molecules do have a non–vanishing volume can be considered as a repulsive interaction the one appears when the distance between two particles is smaller than the radius of the corresponding molecules) the specific heat may become larger than $3/2$, the value of an ideal classical gas, see [18], table 3.1 on page 77.

Concerning the detection of these effects let us mention that it could be done, at least in principle, resorting to the specific heat at constant volume (clearly, the specific heat at constant pressure is also modified), or the pressure. Unfortunately, the modifications to the usual case appear also as a function of $T/T_p$, where $T_p \sim 10^{32}$ Kelvin is the so–called Planck’s temperature, and this is a very stringent condition, experimentally.

Since the modification of the thermodynamical behavior, as a consequence of a deformed dispersion relation, can be explained as a result of the fact that the state equation of the gas depends upon the relation between momentum and energy of the particles, then it is readily seen that for a bosonic gas comprising massive particles the thermodynamics will be also changed. In this context, since the so–called Bose–Einstein condensation is a purely bosonic effect, then we may wonder what happens to this feature if we introduce the generalization to (11) for massive particles. The consequence is a modification of the condensation temperature. In a quite similar spirit we may analyze the situation of fermions, and for instance, the Fermi temperature is modified. This last case yields an interesting situation, since the Chandrasekhar mass-radius relation for white dwarfs is a direct consequence of the fermionic statistics [15], and hence we expect a modification of this relation due to the breakdown of Lorentz. This last remark opens a new window in this context, an astrophysical one connected to the observation of white dwarfs. The corresponding results of this analysis will be published elsewhere.

Finally, let us comment the coincidences and divergences of the present work with some previous results in this direction, for instance, [19, 20, 21]. Though these aforementioned papers do analyze the implications in the realm of statistical mechanics of quantum–$\kappa$–Poincaré algebra [19], or of a Generalized Uncertainty Principle [20, 21], they do not consider the interpretation of the breaking of Lorentz symmetry, at the bulk level, as the emergence of a pseudo–interaction among the particles of the gas. This fact, which is present even for massive particles [22], allows us define two different classes of Lorentz symmetry breaking, i.e., those related to an attractive pseudo–interaction, and those connected with a repulsive pseudo–interaction, see (11). This kind of connection between thermodynamics and the breaking of Lorentz symmetry has not been carried out [19, 20, 21]. This division into two different classes could be of relevance since the appearance of interactions among the particles is a necessary, though not sufficient condition, for the emergence of critical point in thermodynamical systems [15]. In other words, our approach could allow us to calculate, by means of the cluster expansion, an expression for the pseudo–interaction, see expression (12), associated to a breaking of Lorentz symmetry, and in this form we could look for critical points, a fact that has not been considered previously.

A careful analysis of the link between Lorentz symmetry breaking and thermodynamics has already shed light upon some of the characteristics of black–holes [22]. This work has proved that the second law of thermodynamics is strong enough to cope with the introduction of some of the kind of effects that a breaking of Lorentz symmetry implies. In this context our work pursues this analysis for other kind of thermodynamical systems. The goal behind it is the quest for additional systems that could be considered amplifiers [22].

Finally, as mentioned at the end of the previous section, a terrestrial experiment, as a tool to test this kind of violations, is not a feasible idea. Nevertheless, this last remark does not imply the uselessness of this approach. Indeed, we may wonder if this modification could impinge, for instance, in the evolution of early stages of the universe. As an example of this possibility let us remember that the theory of nucleosynthesis is inconsistent with the cluster baryon fraction in a critical density universe [24]. A possible solution to this problem appears if the theory of nucleosynthesis is flawed, and requires the intervention of new physics [25]. Clearly, a question in this context is the role that a new thermodynamics could play as part of this new physics.
Acknowledgments

This research was partially supported by CONACYT Grants 48404–F and 47000–F. A.C. would like to thank A.A. Cuevas–Sosa for useful discussions and literature hints.

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