Generalized sine-Gordon model and baryons in two-dimensional QCD

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Abstract
We consider the sl(3,C) affine Toda model coupled to matter (Dirac spinor) (ATM) and through a gauge fixing procedure we obtain the classical version of the generalized sl(3,C) sine-Gordon model (cGSG) which completely decouples from the Dirac spinors. The GSG models are multifield extensions of the ordinary sine-Gordon model. In the spinor sector we are left with Dirac fields coupled to cGSG fields. Based on the equivalence between the U(1) vector and topological currents, which holds in the theory, it is shown the confinement of the spinors inside the solitons and kinks of the cGSG model providing an extended hadron model for “quark” confinement [JHEP0701(2007)027]. Moreover, the solitons and kinks of the generalized sine-Gordon (GSG) model are shown to describe the normal and exotic baryon spectrum of two-dimensional QCD. The GSG model arises in the low-energy effective action of bosonized QCD2 with unequal quark mass parameters [JHEP0703(2007)055]. The GSG potential for three flavors resembles the potential of the effective chiral lagrangian proposed by Witten to describe low-energy behavior of four dimensional QCD. Among the attractive features of the GSG model are the variety of soliton and kink type solutions for QCD2 unequal quark mass parameters. Exotic baryons in QCD2 [Ellis and Frishman, JHEP0508(2005)081] are discussed in the context of the GSG model. Various semi-classical computations are performed improving previous results and clarifying the role of unequal quark masses. The remarkable double sine Gordon model also arises as a reduced GSG model bearing a kink(K) type solution describing a multi-baryon.

1 Introduction
The sine-Gordon model (SG) has been studied over the decades due to its many properties and mathematical structures such as integrability and soliton solutions. It can be used as a toy model for non-perturbative quantum field theory phenomena. In this context, some extensions and modifications of the SG model deserve attention. An extension taking multi-frequency terms as the potential has been investigated in connection to various physical applications [1, 2, 3, 4].

Besides, an extension defined for multi-fields is the so-called generalized sine-Gordon model (GSG) which has been found in the study of the strong/weak coupling
sectors of the $\text{sl}(N,\mathbb{C})$ affine Toda model coupled to matter fields (ATM) [5, 6]. In connection to these developments, the bosonization process of the multi-flavor massive Thirring model (GMT) provides the quantum version of the (GSG) model [7]. The GSG model provides a framework to obtain (multi-)soliton solutions for unequal mass parameters of the fermions in the GMT sector and study the spectrum and their interactions. The extension of this picture to the NC space-time has been addressed (see [8] and references therein).

In the first part of this chapter we study the spectrum of solitons and kinks of the GSG model proposed in [5, 6, 7] and consider the closely related ATM model from which one gets the classical GSG model (cGSG) through a gauge fixing procedure [9]. Some reductions of the GSG model to one-field theory lead to the usual SG model and to the so-called multi-frequency sine-Gordon models. In particular, the double (two-frequency) sine-Gordon model (DSG) appears in a reduction of the $\text{sl}(3,\mathbb{C})$ GSG model. The DSG theory is a nonintegrable quantum field theory with many physical applications [3, 4].

Once a convenient gauge fixing is performed by setting to constant some spinor bilinears in the ATM model we are left with two sectors: the cGSG model which completely decouples from the spinors and a system of Dirac spinors coupled to the cGSG fields. In refs. [10, 11] the authors have proposed a 1 + 1-dimensional bag model for quark confinement, here we follow their ideas and generalize for multi-flavor Dirac spinors coupled to cGSG solitons and kinks. The first reference considers a model similar to the $\text{sl}(2)$ ATM theory, and in the second one the DSG kink is proposed as an extended hadron model.

Recently, in QCD$_4$ there appeared some puzzles related with unequal quark masses [12] providing an extra motivation to consider QCD$_2$ as a testing ground for non-perturbative methods that might have relevance in the real world. Claims for the existence of exotic baryons - that can not be composed of just three quarks - have inspired intense studies of the theory and phenomenology of QCD in the strong-interaction regime. In particular, it has led to the discovery that the strong coupling regime may contain unexpected correlations among groups of two or three quarks and antiquarks. Results of growing number of experiments at laboratories around the world provide contradictory situation regarding the experimental observation of possible pentaquark states, see e.g. [13]. These experiments have thus opened new lines of theoretical investigation that may survive even if the original inspiration - the exotic $\Theta^+$ pentaquark existence- is not confirmed. After the reports of null results started to accumulate the initial optimism declined, and the experimental situation remains ambiguous to the present. The increase in statistics led to some recent new claims for positive evidence [14], while the null result [15] by CLAS is specially significant because it contradicts their earlier positive result, suggesting that at least in their case the original claim was an artifact due to low statistics. All this experimental activity spurred a great amount of theoretical work in all kinds of models for hadrons and a renewed interest in soliton models. Recently,
there is new strong evidence of an extremely narrow $\Theta^+$ resonance from DIANA collaboration and a very significant new evidence from LEPS. For a recent account of the theoretical and experimental situation see e.g. [16].

On the other hand, Quantum Chromodynamics in two-dimensions (QCD$_2$) (see e.g. [17]) has long been considered a useful theoretical laboratory for understanding non-perturbative strong-interaction problems such as confinement [18], the large-$N_c$ expansion [19], baryon structure [20] and, more recently, the chiral-soliton picture for normal and exotic baryons [21, 22]. Even though there are various differences between QCD$_4$ and QCD$_2$, this theory may provide interesting insights into the physical four-dimensional world. In two dimensions, an exact and complete bosonic description exists and in the strong-coupling limit one can eliminate the color degrees of freedom entirely, thus getting an effective action expressed in terms of flavor degrees of freedom only. In this way various aspects have been studied, such as baryon spectrum and its $\bar{q}q$ content [20]. The constituent quark solitons of baryons were uncovered taking into account both bosonized flavor and color degrees of freedom [23]. In particular, the study of meson-baryon scattering and resonances is a nontrivial task for unequal quark masses even in 2D [24].

It has been conjectured that the low-energy action of QCD$_2$ ($e_c >> M_q, M_q$ quark mass and $e_c$ gauge coupling) might be related to massive two dimensional integrable models, thus leading to the exact solution of the strong coupled QCD$_2$ [20]. As an example of this picture, it has been shown that the so-called $su(2)$ affine Toda model coupled to matter (Dirac) field (ATM) [25] describes the low-energy spectrum of QCD$_2$ (one flavor and $N_C$ colors) [26]. The ATM model allowed the exact computation of the string tension in QCD$_2$ [26], improving the approximate result of [27]. The strong coupling sector of the $su(2)$ ATM model is described by the usual sine-Gordon model [30, 28, 29]. The baryons in QCD may be described as solitons in the bosonized formulation. In the strong-coupling limit the static classical soliton which describes a baryon in QCD$_2$ turns out to be the ordinary sine-Gordon soliton, i.e.

$$\Phi(x) = \frac{4}{\beta_0} \tan^{-1} \left[ \exp \beta_0 \sqrt{2m} x \right] \quad (1)$$

where $\beta_0 = \sqrt{4\pi \over N_C}$ is the coupling constant of the sine-Gordon theory, $8\sqrt{2m}/\beta_0$ is the mass of the soliton, and $m$ is related to the common bare mass of the quarks by a renormalization group relation relevant to two dimensions. The soliton in (1) has non-zero baryon number as well as $Y$ charge. The quantum correction to the soliton mass, obtained by time-dependent rotation in flavor space, is suppressed by a factor of $N_C$ compared to the classical contribution to the baryon mass [20]. The considerations of more complicated mass matrices and higher order corrections to the $M_q/e_c \rightarrow 0$ limit are among the issues that deserve further attention.

In the second part of this chapter we show that various aspects of the low-energy effective QCD$_2$ action with unequal quark masses can be described by the
generalized sine-Gordon model. The GSG model has appeared in the study of the strong coupling sector of the \( sl(n, C) \) model theory\[5, 6, 9\], and in the bosonized multilavor massive Thirring model \[7\]. In particular, the GSG model provides the framework to obtain (multi-)soliton solutions for unequal quark mass parameters. Choosing the normalization such that quarks have baryon number \( Q_B^0 = 1 \) and a one-soliton has baryon number \( N_C \), we classify the configurations in the GSG model with baryon numbers \( N_C, 2N_C, \ldots, 4N_C \). For example, the double sine-Gordon model provides a kink type solution describing a multi-baryon state with baryon number \( 4N_C \). Then, using the GSG model we generalize the results of refs. \[20, 21\] which applied the semi-classical quantization method in order to uncover the normal \[20\] and exotic baryon \[21\] spectrum of QCD\(_2\). One of the main features of the GSG model is that the one-soliton solution requires the QCD quark mass parameters to satisfy certain relationship. In two dimensions there are no spin degrees of freedom, so the lowest-lying baryons are related to the purely symmetric Young tableau, the 10 dimensional representation of flavor SU(3). This is the analogue of the multiplet containing the baryons \( \Delta, \Sigma, \Xi, \Omega^- \) in QCD\(_4\). The next state corresponds to a state with the quantum numbers of four quarks and an antiquark, the so-called pentaquark, which in two dimensions forms a 35 representation of flavor SU(3). This corresponds to the four dimensional multiplet 10, which contain the exotic baryons \( \Theta^+, \Sigma, \Xi^{--} \).

Here we improve the results of refs. \[20, 21\], such as the normal and exotic baryon masses, therelevant mass ratios and the radius parameter of the exotic baryons. The semi-classical computations of the masses get quantum corrections due to the unequal mass term contributions and to the form of the diagonal ansatz taken for the flavor field (related to GSG model) describing the lowest-energy state of the effective action. The corrections to the normal baryon masses are an increase of 3.5% to the earlier value obtained in \[21\], and in the case of the exotic baryon our computations improve the behavior of the quantum correction by decreasing the earlier value in 0.34 units, so making the semi-classical result more reliable. Let us mention that for the first exotic baryon \[21\] the quantum correction was greater than the classical term by a factor of 2.46, so that semi-classical approximation may not be a good approximation. As a curiosity, with the relevant values obtained by us for QCD\(_2\) we computed the ratio between the lowest exotic baryon and the \( R = 10 \) baryon masses \( M_{35}/M_{10} \sim 1.65 \), which is only 1% larger than the analogous four dimensional QCD ratio \( M_{\Theta^+}/M_{\text{nucleon}} \sim 1.63 \). In \[21\] the relevant QCD\(_2\) ratio was 17% larger than this value. The mass formulae for the normal and exotic baryons corresponding, respectively, to the representations 10 and 35, in two dimensions resemble the general chiral-soliton model formula in four dimensions \[31\] except that there is no spin-dependent term \( \sim J(J + 1) \), and an analog term containing the soliton moment of inertia emerges.

In the next section we define the \( sl(3, C) \) classical GSG model and describe some of its symmetries. In section 3 we consider the \( sl(3, C) \) affine Toda model coupled to
matter and obtain the cGSG model through a gauge fixing procedure. We discuss the physical properties of its spectrum. The topological charges are introduced, as well as the idea of baryons as solitons (or kinks), and the quark confinement mechanism is discussed. In section 4 it is summarized the bosonized low-energy effective action of QCD\textsubscript{2} and introduced the lowest-energy state described by the GSG action. The global QCD\textsubscript{2} symmetries are discussed. Section 5 provides the GSG solitons and kink solutions relevant to our QCD\textsubscript{2} discussions. In section 6 the semi-classical method of quantization relevant to a general diagonal ansatz is introduced. In subsection 6.1 we briefly review the ordinary sine-Gordon soliton semi-classical quantization in the context of QCD\textsubscript{2}. In section 7 we discuss the quantum correction to the SU(3) GSG ansatz in the framework of semi-classical quantization. In subsection 7.1 the GSG one-soliton state is rotated in SU(3) flavor space by a time-dependent $A(t)$. In subsection 7.2 the lowest-energy baryon state with baryon number $N_C$ is introduced. The possible vibrational modes are briefly discussed in subsection 7.3. In section 8 it is discussed the first and higher multiplet exotic baryons and provided the relevant quantum corrections their masses, the ratio $M_{35}/M_{10}$, and an estimate for the exotic baryon radius parameter. The last section presents a summary and discussions.

2 The generalized sine-Gordon model (GSG)

The generalized sine-Gordon model (GSG) related to $sl(N,\mathbb{C})$ is defined by \cite{5, 6, 7}

$$S = \int d^2x \sum_{i=1}^{N_f} \left[ \frac{1}{2} (\partial_{\mu} \Phi_i)^2 + \mu_i \left( \cos \beta_i \Phi_i - 1 \right) \right].$$  \hspace{1cm} (2)

The $\Phi_i$ fields in (2) satisfy the constraints

$$\Phi_p = \sum_{i=1}^{N-1} \sigma_{pi} \Phi_i, \quad p = N, N+1, ..., N_f, \quad N_f = \frac{N(N-1)}{2},$$  \hspace{1cm} (3)

where $\sigma_{pi}$ are some constant parameters and $N_f$ is the number of positive roots of the Lie algebra $sl(N,\mathbb{C})$. In the context of the Lie algebraic construction of the GSG system these constraints arise from the relationship between the positive and simple roots of $sl(N,\mathbb{C})$. Thus, in (2) we have $(N-1)$ independent fields.

We will consider the $sl(3,\mathbb{C})$ case with two independent real fields $\varphi_{1,2}$, such that

$$\Phi_1 = 2\varphi_1 - \varphi_2; \quad \Phi_2 = 2\varphi_2 - \varphi_1; \quad \Phi_3 = r \varphi_1 + s \varphi_2, \quad s, r \in \mathbb{R}$$  \hspace{1cm} (4)

which must satisfy the constraint

$$\beta_3 \Phi_3 = \delta_1 \beta_1 \Phi_1 + \delta_2 \beta_2 \Phi_2, \quad \beta_i \equiv \beta_0 \nu_i,$$  \hspace{1cm} (5)
where $\beta_0$, $\nu_1$, $\delta_1, \delta_2$ are some real numbers. Therefore, the $sl(3, \mathbb{C})$ GSG model can be regarded as three usual sine-Gordon models coupled through the linear constraint (5).

Taking into account (4)-(5) and the fact that the fields $\varphi_1$ and $\varphi_2$ are independent we may get the relationships

$$\nu_2 \delta_2 = \rho_0 \nu_1 \delta_1 \quad \nu_3 = \frac{1}{r+s} (\nu_1 \delta_1 + \nu_2 \delta_2); \quad \rho_0 \equiv \frac{2s + r}{2r + s}$$

(6)

The $sl(3, \mathbb{C})$ model has a potential density

$$V[\varphi_i] = \sum_{i=1}^{3} \mu_i \left(1 - \cos \beta \Phi_i \right)$$

(7)

The GSG model has been found in the process of bosonization of the generalized massive Thirring model (GMT) [7]. The GMT model is a multilavor extension of the usual massive Thirring model incorporating massive fermions with current-current interactions between them. In the $sl(3, \mathbb{C})$ construction of [7] the parameters $\delta_i$ depend on the couplings $\beta_i$ and they satisfy certain relationship. This is obtained by assuming $\mu_i > 0$ and the zero of the potential given for $\Phi_i = \frac{2\pi}{\beta_0} n_i$, which substituted into (5) provides

$$n_1 \delta_1 + n_2 \delta_2 = n_3, \quad n_i \in \mathbb{Z}$$

(8)

The last relation combined with (6) gives

$$(2r + s) \frac{n_1}{\nu_1} + (2s + r) \frac{n_2}{\nu_2} = 3 \frac{n_3}{\nu_3}.$$  

(9)

The periodicity of the potential implies an infinitely degenerate ground state and then the theory supports topologically charged excitations. The vacuum configuration is related to the fundamental weights [9]. For a future purpose let us consider the fields $\Phi_1$ and $\Phi_2$ and the vacuum lattice defined by

$$(\Phi_1, \Phi_2) = \frac{2\pi}{\beta_0} \left( \frac{n_1}{\nu_1}, \frac{n_2}{\nu_2} \right), \quad n_a \in \mathbb{Z}.$$  

(10)

It is convenient to write the equations of motion in terms of the independent fields $\varphi_1$ and $\varphi_2$

$$\partial^2 \varphi_1 = -\mu_1 \beta_1 \Delta_{11} \sin[\beta_1(2\varphi_1 - \varphi_2)] - \mu_2 \beta_2 \Delta_{12} \sin[\beta_2(2\varphi_2 - \varphi_1)] + \mu_3 \beta_3 \Delta_{13} \sin[\beta_3(r\varphi_1 + s\varphi_2)]$$

$$\partial^2 \varphi_2 = -\mu_1 \beta_1 \Delta_{21} \sin[\beta_1(2\varphi_1 - \varphi_2)] - \mu_2 \beta_2 \Delta_{22} \sin[\beta_2(2\varphi_2 - \varphi_1)] + \mu_3 \beta_3 \Delta_{23} \sin[\beta_3(r\varphi_1 + s\varphi_2)],$$

(11) (12)
where

\[ A = \beta_0^2 \nu_1^2 (4 + \delta^2 + \delta_1^2 \rho_0^2 r^2), \quad B = \beta_0^2 \nu_1^2 (1 + 4 \delta_1^2 + \delta_1^2 \rho_1^2 s^2), \]

\[ C = \beta_0^2 \nu_1^2 (2 + 2 \delta_2^2 + \delta_1^2 \rho_0^2 r s), \]

\[ \Delta_{11} = (C - 2B)/\Delta, \quad \Delta_{12} = (B - 2C)/\Delta, \quad \Delta_{13} = (rB + sC)/\Delta, \]

\[ \Delta_{21} = (A - 2C)/\Delta, \quad \Delta_{22} = (C - 2A)/\Delta, \quad \Delta_{23} = (rC + sA)/\Delta \]

\[ \Delta = C^2 - AB, \quad \delta = \frac{\delta_1}{\delta_2} \rho_0, \quad \rho_1 = \frac{3}{2r + s} \]

Notice that the eqs. of motion (11)-(12) exhibit the symmetries

\[ \varphi_1 \leftrightarrow \varphi_2, \quad \mu_1 \leftrightarrow \mu_2, \quad \nu_1 \leftrightarrow \nu_2, \quad \delta_1 \leftrightarrow \delta_2, \quad r \leftrightarrow s \] (13)

\[ \varphi_a \leftrightarrow -\varphi_a, \quad a = 1, 2 \] (14)

Some type of coupled sine-Gordon models have been considered in connection to various interesting physical problems [32]. For example a system of two coupled SG models has been proposed in order to describe the dynamics of soliton excitations in deoxyribonucleic acid (DNA) double helices [33]. In general these type of equations have been solved by perturbation methods around decoupled sine-Gordon exact solitons. In section 5 the system (11)-(12) will be shown to possess exact soliton and kink type solutions.

### 3 Classical GSG as a reduced Toda model coupled to matter

In this section we obtain the classical GSG model as a reduced model starting from the $sl(3,C)$ affine Toda model coupled to matter fields (ATM) and closely follows ref. [9]. The previous treatments of the $sl(3,C)$ ATM model used the symplectic and on-shell decoupling methods to unravel the classical GSG and generalized massive Thirring (GMT) dual theories describing the strong/weak coupling sectors of the ATM model [5, 6, 29]. The ATM model describes some scalars coupled to spinor (Dirac) fields in which the system of equations of motion has a local gauge symmetry. In this way one includes the spinor sector in the discussion and conveniently gauge fixing the local symmetry by setting some spinor bilinears to constants we are able to decouple the scalar (Toda) fields from the spinors, the final result is a direct construction of the classical generalized sine-Gordon model (cGSG) involving only the scalar fields. In the spinor sector we are left with a system of equations in which the Dirac fields couple to the cGSG fields.

The conformal version of the ATM model is defined by the following equations of motion [25]

\[ \frac{\partial^2 \theta_a}{4i \epsilon_i} = m_1^1 [e^{-i\phi_a} \bar{\psi}_R^l \psi_L^l + e^{i\phi_a} \bar{\psi}_L^l \psi_R^l] + m_3^3 [e^{-i\phi_3} \bar{\psi}_R^3 \psi_L^3 + e^{i\phi_3} \bar{\psi}_L^3 \psi_R^3]; \]
Generalized sine-Gordon model and baryons in QCD

\[ a = 1, 2 \]

\[ -\frac{\partial^2 \bar{\nu}}{4} = \left[ im_{\psi}^2 e^{2 \eta - \phi_1} \bar{\psi}_R^1 \psi_L^1 + im_{\psi}^2 e^{2 \eta - \phi_2} \bar{\psi}_R^2 \psi_L^2 + im_{\psi}^3 e^{\eta - \phi_3} \bar{\psi}_R^3 \psi_L^3 + m^2 e^{3 \eta} \right] \]

\[ -2 \partial_+ \psi_L^1 = m_{\psi}^1 e^{\eta + \phi_1} \psi_R^1, \quad -2 \partial_+ \psi_L^2 = m_{\psi}^2 e^{\eta + \phi_2} \psi_R^2, \quad -2 \partial_+ \psi_L^3 = m_{\psi}^3 e^{\eta + \phi_3} \psi_R^3, \]

\[ 2 \partial_- \psi_R^1 = m_{\psi}^1 e^{2 \eta - \phi_1} \psi_L^1 + \frac{m_{\psi}^2 m_{\psi}^3}{im_{\psi}^1} e^{\eta} (\bar{\psi}_R^3 \psi_L^1 e^{i \phi_2} - \bar{\psi}_R^2 \psi_L^1 e^{-i \phi_3}), \]

\[ 2 \partial_- \psi_R^2 = m_{\psi}^2 e^{2 \eta - \phi_2} \psi_L^2 + \frac{m_{\psi}^1 m_{\psi}^3}{im_{\psi}^2} e^{\eta} (\bar{\psi}_R^3 \psi_L^2 e^{i \phi_1} + \bar{\psi}_R^1 \psi_L^2 e^{-i \phi_3}), \]

\[ 2 \partial_- \psi_R^3 = m_{\psi}^3 e^{2 \eta - \phi_3} \psi_L^3 + \frac{m_{\psi}^1 m_{\psi}^2}{im_{\psi}^3} e^{\eta} (\bar{\psi}_R^1 \psi_L^3 e^{i \phi_1} + \bar{\psi}_R^2 \psi_L^3 e^{i \phi_2}), \]

\[ \partial^2 \eta = 0, \]

where \( \phi_1 \equiv 2 \theta_1 - \theta_2, \phi_2 \equiv 2 \theta_2 - \theta_1, \phi_3 \equiv \theta_1 + \theta_2 \). Therefore, one has

\[ \phi_3 = \phi_1 + \phi_2 \]  

The \( \theta \) fields are considered to be in general complex fields. In order to define the classical generalized sine-Gordon model we will consider these fields to be real.

Apart from the \textit{conformal invariance} the above equations exhibit the \( (U(1)_L)^2 \otimes (U(1)_R)^2 \) left-right local gauge symmetry

\[ \theta_a \rightarrow \theta_a + \xi_a^i (x_+) \text{ and } \xi_a^i (x_-), \quad a = 1, 2 \]

\[ \bar{\nu} \rightarrow \bar{\nu}; \quad \eta \rightarrow \eta \]

\[ \psi^i \rightarrow e^{(1 + \gamma_5)\Xi_+^i} (x_+) + i (1 - \gamma_5) \Xi_-^i (x_-) \psi^i, \]

\[ \bar{\psi}^i \rightarrow e^{-i (1 + \gamma_5) (\Xi_+^i) (x_+) - i (1 - \gamma_5) (\Xi_-^i) (x_-)} \bar{\psi}^i, \quad i = 1, 2, 3; \]

\[ \Xi_+^1 \equiv \pm \xi_+^2 + 2 \xi_+^3, \quad \Xi_+^2 \equiv \pm \xi_+^1 + 2 \xi_+^3, \quad \Xi_+^3 \equiv \Xi_+^1 + \Xi_+^2. \]

One can get global symmetries for \( \xi_+^2 = \mp \xi_+^3 \) = constants. For a model defined by a Lagrangian these would imply the presence of two vector and two chiral conserved
currents. However, it was found only half of such currents \[34\]. This is a consequence of the lack of a Lagrangian description for the \( sl(3)^{(1)} \) CATM model in terms of the model defining fields. So, the vector current

\[
J^\mu = \sum_{j=1}^{3} m_j \bar{\psi}_j \gamma^\mu \psi_j
\]  

(32)

and the chiral current

\[
J^5^\mu = \sum_{j=1}^{3} m_j \bar{\psi}_j \gamma^\mu \gamma_5 \psi_j + 2 \partial_\mu (m_1^1 \theta_1 + m_2^2 \theta_2)
\]  

(33)

are conserved

\[
\partial_\mu J^\mu = 0, \quad \partial_\mu J^5^\mu = 0
\]  

(34)

The conformal symmetry is gauge fixed by setting

\[
\eta = \text{const}.
\]  

(35)

The off-critical model obtained in this way exhibits the vector and topological currents equivalence \[25, 29\]

\[
\sum_{j=1}^{3} m_j \bar{\psi}_j \gamma^\mu \psi_j \equiv \epsilon^{\mu\nu} \partial_\nu (m_1^1 \theta_1 + m_2^2 \theta_2), \quad m_3^3 = m_1^1 + m_2^2, \quad m_i^i > 0.
\]  

(36)

Moreover, it has been shown that the soliton type solutions are in the orbit of the vacuum \( \eta = 0 \).

In the next steps we implement the reduction process to get the cGSG model through a gauge fixing of the ATM theory. The local symmetries (28)-(31) can be gauge fixed through

\[
i \bar{\psi}_j \gamma^\mu \psi_j = i A_j = \text{const}., \quad \bar{\psi}_j \gamma_5 \psi_j = 0.
\]  

(37)

From the gauge fixing (37) one can write the following bilinears

\[
\bar{\psi}_L^j \psi^j_R = 0, \quad j = 1, 2, 3;
\]  

(38)

so, the eqs. (37) effectively comprises three gauge fixing conditions.

It can be directly verified that the gauge fixing (37) preserves the currents conservation laws (34), i.e. from the equations of motion (15)-(26) and the gauge fixing (37) together with (35) it is possible to obtain the currents conservation laws (34).

Taking into account the constraints (37) in the scalar sector, eqs. (15), we arrive at the following system of equations (set \( \eta = 0 \))

\[
\partial^2 \theta_1 = M_1^1 \sin \phi_1 + M_3^3 \sin \phi_3,
\]  

(39)

\[
\partial^2 \theta_2 = M_2^2 \sin \phi_2 + M_3^3 \sin \phi_3, \quad M_i^i \equiv 4 A_i, \quad m_i^i, \quad i = 1, 2, 3.
\]  

(40)
Define the fields $\varphi_1, \varphi_2$ as

$$
\varphi_1 \equiv a\theta_1 + b\theta_2, \quad a = \frac{4\nu_2 - \nu_1}{3\beta_0\nu_1\nu_2}, \quad d = \frac{4\nu_1 - \nu_2}{3\beta_0\nu_1\nu_2} \tag{41}
$$

$$
\varphi_2 \equiv c\theta_1 + d\theta_2, \quad b = -c = \frac{2(\nu_1 - \nu_2)}{3\beta_0\nu_1\nu_2}, \quad \nu_1, \nu_2 \in \mathbb{R} \tag{42}
$$

Then, the system of equations (39)-(40) written in terms of the fields $\varphi_1, \varphi_2$ becomes

$$
\partial^2 \varphi_1 = aM_1^4 \sin[\beta_0\nu_1(2\varphi_1 - \varphi_2)] + bM_2^4 \sin[\beta_0\nu_2(2\varphi_2 - \varphi_1)] + \frac{(a + b)M_3^4}{3\beta_0\nu_1\nu_2} \sin[\beta_0\nu_1(2\varphi_1 - \varphi_2) + (2\varphi_2 - \varphi_1)\varphi_2], \tag{43}
$$

$$
\partial^2 \varphi_2 = cM_1^4 \sin[\beta_0\nu_1(2\varphi_1 - \varphi_2)] + dM_2^4 \sin[\beta_0\nu_2(2\varphi_2 - \varphi_1)] + \frac{(c + d)M_3^4}{3\beta_0\nu_1\nu_2} \sin[\beta_0\nu_1(2\varphi_1 - \varphi_2) + (2\varphi_2 - \varphi_1)\varphi_2] \tag{44}
$$

The system of equations above considered for real fields $\varphi_1, \varphi_2$ as well as for real parameters $M_1^4, a, b, c, d, \beta_0$ defines the classical generalized sine-Gordon model (cGSG). Notice that this classical version of the GSG model derived from the ATM theory is a submodel of the GSG model (11)-(12), defined in section 2, for the particular parameter values $r = \frac{2\nu_2 - \nu_1}{\nu_2}$, $s = \frac{2\nu_1 - \nu_2}{\nu_1}$ and the convenient identifications of the parameters in the coefficients of the sine functions of the both models.

The spinor sector in view of the gauge fixing (37) can be parameterized conveniently as

$$
\begin{pmatrix}
\psi_R^j \\
\psi_L^j
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} \sqrt{A_j/2} u_j \\
i \frac{1}{2} \sqrt{A_j/2} v_j
\end{pmatrix}, \quad \begin{pmatrix}
\bar{\psi}_R^j \\
\bar{\psi}_L^j
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} \sqrt{A_j/2} v_j \\
i \frac{1}{2} \sqrt{A_j/2} u_j
\end{pmatrix}. \tag{45}
$$

Therefore, in order to find the spinor field solutions one can solve the eqs. (17)-(25) for the fields $u_j, v_j$ for each solution given for the cGSG fields $\varphi_1, \varphi_2$ of the system (43)-(44).

### 3.1 Topological charges, baryons as solitons and confinement

In this section we will examine the vacuum configuration of the cGSG model and the equivalence between the $U(1)$ spinor and the topological currents (36) in the gauge fixed model and verify that the charge associated to the $U(1)$ current gets confined inside the solitons and kinks of the GSG model; the explicit form of these type of solutions will be obtained in section 5.

It is well known that in $1 + 1$ dimensions the topological current is defined as $J^\mu_{\text{top}} \sim e^{\nu\nu} \partial_\nu \Phi$, where $\Phi$ is some scalar field. Therefore, the topological charge is

$$
Q_{\text{top}} = \int J^0_{\text{top}} \, dx \sim \Phi(+\infty) - \Phi(-\infty). \tag{46}
$$

In order to introduce a topological current
we follow the construction adopted in Abelian affine Toda models, so we define the field

$$\theta = \sum_{a=1}^{2} \frac{2 \alpha_a}{\alpha_a^2} \theta_a$$

(46)

where $\alpha_a, a = 1, 2$, are the simple roots of $sl(3, \mathbb{C})$. We then have that $\theta_a = (\theta | \lambda_a)$, where $\lambda_a$ are the fundamental weights of $sl(3, \mathbb{C})$ defined by the relation $|35|

$$2 \frac{(\alpha_a | \lambda_b) (\alpha_a | \alpha_a)}{(\alpha_a | \alpha_a)^2} = \delta_{ab}.$$ 

(47)

The fields $\phi_j$ in the equations (15)-(25) written as the combinations $(\theta | \alpha_j)$, $j = 1, 2, 3$, where the $\alpha_j's$ are the positive roots of $sl(3, \mathbb{C})$, are invariant under the transformation

$$\theta \rightarrow \theta + 2 \pi \mu \quad \text{or} \quad \phi_j \rightarrow \phi_j + 2 \pi (\mu | \alpha_j),$$

(48)

$$\mu = \sum_{n_a \in \mathbb{Z}} n_a \frac{2 \lambda_a}{(\alpha_a | \alpha_a)}.$$ 

(49)

where $\mu$ is a weight vector of $sl(3, \mathbb{C})$, these vectors satisfy $(\mu | \alpha_j) \in \mathbb{Z}$ and form an infinite discrete lattice called the weight lattice [35]. However, this weight lattice does not constitute the vacuum configurations of the ATM model, since in the model described by (15)-(26) for any constants $\theta_a^{(0)}$ and $\eta^{(0)}$

$$\psi_j = \tilde{\psi}_j = 0, \quad \theta_a = \theta_a^{(0)}, \quad \eta = \eta^{(0)}, \quad \tilde{\nu} = -m^2 e^{\eta^{(0)}} x^+ x^-$$

(50)

is a vacuum configuration.

We will see that the topological charges of the physical one-soliton solutions of (15)-(26) which are associated to the new fields $\varphi_a, a = 1, 2$, of the cGSG model (43)-(44) lie on a modified lattice which is related to the weight lattice by re-scaling the weight vectors. In fact, the eqs. of motion (43)-(44) for the field defined by

$$\varphi \equiv \sum_{a=1}^{2} \frac{2 \alpha_a}{\alpha_a^2} \varphi_a, \quad \text{such that} \quad \varphi_a = (\varphi | \lambda_a),$$

are invariant under the transformation

$$\varphi \rightarrow \varphi + 2 \pi \sum_{a=1}^{2} \frac{q_a}{\nu_a} \frac{2 \lambda_a}{(\alpha_a | \alpha_a)}, \quad q_a \in \mathbb{Z}.$$ 

(51)

So, the vacuum configuration is formed by an infinite discrete lattice related to the usual weight lattice by the relevant re-scaling of the fundamental weights $\lambda_a \rightarrow \frac{1}{\nu_a} \lambda_a$. The vacuum lattice can be given by the points in the plane $\varphi_1 \times \varphi_2$

$$\left( \varphi_1, \varphi_2 \right) = \frac{2 \pi}{3 \beta_0} \left( \frac{2 q_1}{\nu_1} \frac{q_2}{\nu_2} + \frac{q_1}{\nu_1} + \frac{2 q_2}{\nu_2} \right), \quad q_a \in \mathbb{Z}.$$ 

(52)
In fact, this lattice is related to the one in eq. (10) through appropriate parameter identifications. We shall define the topological current and charge, respectively, as

\[ J^{\mu}_{\text{top}} = \frac{\beta_0}{2\pi} \epsilon^{\mu \nu} \partial_\nu \varphi, \quad Q_{\text{top}} = \int dx J^0_{\text{top}} = \frac{\beta_0}{2\pi}[\varphi(+\infty) - \varphi(\infty)]. \] (53)

Taking into account the cGSG fields (43)-(44) and the spinor parameterizations (45) the currents equivalence (36) of the ATM model takes the form

\[ \sum_{j=1}^{3} m^j_\psi \bar{\psi}^j \gamma^\mu \psi^j \equiv \epsilon^{\mu \nu} \partial_\nu (\zeta^1_\psi \varphi_1 + \zeta^2_\psi \varphi_2), \] (54)

where \( \zeta^1_\psi \equiv \beta^2_0 \nu_1 \nu_2 (m^1_\psi d + m^2_\psi b) \), \( \zeta^2_\psi \equiv \beta^2_0 \nu_1 \nu_2 (m^2_\psi a - m^1_\psi b) \) and the spinors are understood to be written in terms of the fields \( u_j \) and \( v_j \) of (45).

Notice that the topological current in (54) is the projection of (53) onto the vector \( \frac{2\pi}{\beta_0} (\zeta^1_\psi \lambda_1 + \zeta^2_\psi \lambda_2) \).

As mentioned above the gauge fixing (37) preserves the currents conservation laws (34). Moreover, the cGSG model was defined for the off critical ATM model obtained after setting \( \eta = \text{const.} = 0 \). So, for the gauge fixed model it is expected to hold the currents equivalence relation (36) written for the spinor parameterizations \( u_j, v_j \) and the fields \( \varphi_{1,2} \) as is presented in eq. (54). Therefore, in order to verify the \( U(1) \) current confinement it is not necessary to find the explicit solutions for the spinor fields. In fact, one has that the current components are given by relevant partial derivatives of the linear combinations of the field solutions, \( \varphi_{1,2} \), i.e. \( J^0 = \sum_{j=1}^{3} m^j_\psi \bar{\psi}^j \gamma^0 \psi^j = \partial_x (\zeta^1_\psi \varphi_1 + \zeta^2_\psi \varphi_2) \) and \( J^1 = \sum_{j=1}^{3} m^j_\psi \bar{\psi}^j \gamma^1 \psi^j = -\partial_t (\zeta^1_\psi \varphi_1 + \zeta^2_\psi \varphi_2) \).

It is clear that the charge density related to this \( U(1) \) current can only take significant values on those regions where the \( x \)-derivative of the fields \( \varphi_{1,2} \) are non-vanishing. That is one expects to happen with the bag model like confinement mechanism in quantum chromodynamics (QCD). As we will see below the soliton and kink solutions of the GSG theory are localized in space, in the sense that the scalar fields interpolate between the relevant vacua in a limited region of space with a size determined by the soliton masses. The spinor \( U(1) \) current gets the contributions from all the three spinor flavors. Moreover, from the equations of motion (17)-(25) one can obtain nontrivial spinor solutions different from vacuum (50) for each set of scalar field solutions \( \varphi_1, \varphi_2 \). Therefore, the ATM model can be considered as a multflavor generalization of the two-dimensional hadron model proposed in [10, 11]. In the last reference a scalar field is coupled to a spinor such that the DSG kink arises as a model for hadron and the quark field is confined inside the bag.

In connection to our developments above let us notice that two-dimensional QCD\(_2\) has been used as a laboratory for studying the full four-dimensional theory.
providing an explicit realization of baryons as solitons. In the picture described above a key role has been played by the equivalence between the Noether and topological currents. Moreover, one notices that the SU($n$) ATN theory [5, 6] is a 2D analogue of the chiral quark soliton model proposed to describe solitons in QCD [36], provided that the pseudo-scalars lie in the Abelian subalgebra and certain kinetic terms are supplied for them.

4 The bosonized effective action of QCD$_2$ and the GSG model

The QCD$_2$ action is written in terms of gauge fields $A_\mu$ and fundamental quark fields $\psi$ as

$$ S_F[\psi, A_\mu] = \int d^2 x \left\{ -\frac{1}{2e_c^2} \text{Tr}(F_\mu F^{\mu\nu}) - \bar{\psi}^a [(i\partial + \phi)] \psi_a + \mathcal{M}_{ij} \bar{\psi}^a \psi^a_j \right\}, \quad (55) $$

where $a$ is the color index ($a = 1, 2, ..., N_C$) and $i$ the flavor index ($i = 1, 2, ..., N_f$), $e_c$, with dimension of a mass, is the quark coupling to the gauge fields, the matrix $\mathcal{M}_{ij} = m_i \delta_{ij}$ ($m_i$ being the quark masses) takes into account the quark mass splitting, and $F_\mu^{\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ is the gauge field strength.

The bosonized action in the strong-coupling limit ($e_c \gg \text{all } m_i$) becomes [20, 23]

$$ S_{\text{eff}}[g] = N_c S[g] + m^2 N_m \int d^2 x \text{Tr} f \left[ D(g + g^\dagger) \right], \quad (56) $$

where $g$ is a matrix representing $U(N_f)$, $D = \frac{\mathcal{M}}{m_0}$, $m_0$ is an arbitrary mass parameter, and the effective mass scale $m$ is given by

$$ m = \left[ N_c e_c m_0 \frac{e_c \sqrt{N_F}}{\sqrt{2\pi}} \Delta_c \right]^\frac{1}{1+\Delta_c}, \quad (57) $$

with

$$ \Delta_c = \frac{N_c^2 - 1}{N_c (N_c + N_F)}. \quad (58) $$

In (56) $S[g]$ is the WZNW action and $N_m$ stands for normal ordering with respect to $m$. In the large $N_c$ limit, which we use below to justify the semi-classical approximation, the scale $m$ becomes

$$ m = 0.59 N_F^\frac{1}{2} \sqrt{N_c e_c m_0}, \quad (59) $$

so, $m$ takes the value $0.77 \sqrt{N_c e_c m_0}$ for three flavors. Notice that we first take the strong-coupling limit $e_c \gg \text{all } m_i$, and then take $N_c$ to be large, thus it is different from the 't Hooft limit [19], where $e_c^2 N_c$ is held fixed.
Following the Skyrme model approach it is useful to first ask for classical soliton solutions of the bosonic action which are heavy in the $N_C \to$-large limit. The action (56) is a massive WZNW action and possesses the property that if $g$ is non-diagonal it can not be a classical solution, as after a diagonalization to

$$g_0 = \text{diag}(e^{-i\beta_0 \Phi_1(x)}, e^{-i\beta_0 \Phi_2(x)}, \ldots e^{-i\beta_0 \Phi_{N_f}(x)}), \quad \sum_i \Phi_i(x) = \phi(x), \quad \beta_0 \equiv \sqrt{\frac{4\pi}{N_C}}$$

it will have lower energy [37]. Thus, the minimal energy solutions of the massive WZNW model are necessarily in a diagonal form. The majority of particles given by (60) are not going to be stable, but must decay into others.

Previous works consider the diagonal form (60) such that the action (56) reduces to a sum of $N_f$ independent ordinary sine-Gordon models, each one for the corresponding $\Phi_i$ field and parameters

$$\tilde{m}_i^2 = \frac{m_i}{m_0} m^2_i. \quad (61)$$

In this approach the lowest lying baryon is represented by the minimum-energy configuration for this class of ansatz, i.e.

$$\hat{g}_0(x) = \text{diag}(1, 1, \ldots, e^{-i\sqrt{\frac{4\pi}{N_C}}\Phi_{N_f}}), \quad (62)$$

with $m_{N_f}$ chosen to be the smallest mass.

In this paper we will consider the ansatz (60) for

$$N_f = \frac{n}{2}(n - 1), \quad N_f \equiv \text{number of positive roots of } su(n), \quad (63)$$

such that $\frac{(n-2)(n-1)}{2}$ linear constraints are imposed on the fields $\Phi_i$. This model corresponds to the generalized sine-Gordon model (GSG) recently studied in the context of the bosonization of the so-called generalized massive Thirring model (GMT) with $N_f$ fermion species [5, 6, 7]. The classical GSG model and some of its properties, such as the algebraic construction based on the affine $sl(n, \mathbb{C})$ Kac-Moody algebra and the soliton spectrum has been the subject of a recent paper [9].

The WZ term in (56) vanishes for either static or diagonal solution, so, for the ansatz (60) and after redefining the additive constant term the action becomes

$$S[g_0] = \int d^2x \sum_{i=1}^{N_f} \left[ \frac{1}{2} (\partial_{\mu} \Phi_i)^2 + 2\tilde{m}_i^2 \left( \cos \beta_0 \Phi_i - 1 \right) \right], \quad (64)$$

with coupling $\beta_0$ and mass parameters $\tilde{m}_i$ defined in (60) and (61), respectively.
The $\Phi_i$ fields in (64) satisfy certain constraints of the type

$$\Phi_p = \sum_{i=1}^{n-1} \sigma_{pi} \Phi_i, \quad p = n, n + 1, ..., N_f$$

(65)

where $\sigma_{pi}$ are some constant parameters. From the Lie algebraic construction of the GSG model these parameters arise from the relationship between the positive and simple roots of $su(n)$. Even though our treatment in this section and the section 6 is valid for any $N_f$, we will concentrate on the $N_f = 3$ case.

It is interesting to recognize the similarity between the potential of the model (64)-(65) for the $N_f = 3$ case [in $su(3)$ GSG model one has $n = N_f = 3$ and just one constraint equation in (65)] and the effective chiral Lagrangian proposed by Witten to describe low-energy behavior of four dimensional QCD [38]. In Witten’s approach the potential term reads

$$V^{\text{Witten}}(U) = f_\pi^2 \left[ -\frac{1}{2} \text{Tr} M(U + U^\dagger) + \frac{k}{2N_C} (-i \ln \text{Det} U - \theta)^2 \right],$$

(66)

where $U$ is the pseudoscalar field matrix and $M = \text{diag}(m_u; m_d; m_s)$ is the quark mass matrix. Phenomenologically $m_{\eta'}^2 >> m_\pi^2, m_K^2, m_\eta^2$, implying that $\frac{k}{N_C} >> bm_s, bm_d$ [the parameter $b$ is $O(\Lambda)$, where $\Lambda$ is a hadronic scale]. Because $M$ is diagonal, one can look for a minimum of $V^{\text{Witten}}(U)$ in the form $U = \text{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})$. Since the second term dominates over the first, one has $\sum \phi_j = \theta$ up to the first approximation. So, choosing $\theta = 0$, (66) reduces to a model of type (64)-(65) defined for $N_f = 3$. This is the $sl(3)$ GSG model (11)-(12), which possesses soliton and kink type solutions (see section 5), and will be the main ingredient of our developments in sections 7 and 8.

The potential term in (64) is invariant under

$$\Phi_i \rightarrow \Phi_i + \frac{1}{\beta_0} 2\pi N_i, \quad (N_i \in \mathbb{Z}).$$

(67)

All finite energy configurations, whether static or time-dependent, can be divided into an infinite number of topological sectors, each characterized by a set

$$[n_1, n_2, ..., n_{N_f}] = [(N_1^+ - N_1^-), (N_2^+ - N_2^-), ..., (N_{N_f}^+ - N_{N_f}^-)]$$

(68)

$$\Phi_i(\pm \infty) = \frac{1}{\beta_0} 2\pi N_i^\pm$$

(69)

corresponding to the asymptotic values of the fields at $x = \pm \infty$. The $n_i's$ satisfy certain relationship arising from the constraints (65) and the invariance (67) (some examples are given in section 5 for the soliton and kink type solutions in the $SU(3)$ case).
Conserved charges, corresponding to the vector current $J_{ij}^\mu = \Psi_1^a \gamma^\mu \Psi_2^a$, can be computed as

$$Q^A[g(x)] = \int dx [J_0(T^A_2)],$$  

(70)

where $(T^A_2)$ are the $su(n)$ generators and the $U(1)$ baryon number is obtained using the identity matrix instead of $(T^A_2)$. For $g_0$ given in eq. (60) the baryon number of any given flavor $j$ is given by $Q^B_j = N_c n_j$, so, the total baryon number becomes

$$Q_B = N_C(n_1 + n_2 + ... + n_{N_f}),$$  

(71)

and the “hypercharge” is given by

$$Q_Y = \frac{1}{2} \Tr \int dx \left( J_0 \lambda N^j_{-1} \right) = \frac{1}{2} N_C \left( n_1 + n_2 + ... + n_{N_f-1} - (N_f-1)n_{N_f} \right) \sqrt{\frac{2}{N_f^2 - N_f}}.$$  

(72)

The total baryon number is clearly an integer multiple of $N_C$. In the case of (62) they reduce to $Q_B = N_C$ and $Q_Y = -\frac{1}{2} \sqrt{2(N_f-1)/N_FN_C}$, respectively [for $\sqrt{4\pi/N_C\Phi_{N_f}(+\infty)} = 2\pi$, $\Phi_{N_f}(-\infty) = 0$] [20]. We are choosing the convention in which the quarks have baryon number $Q_B = 1$, so the soliton representing a physical baryon has baryon number $N_C$.

A global $U_V(N_f)$ transformation $\tilde{g}_0 = A g_0(x) A^{-1}$ is expected to turn on the other charges. Let us introduce

$$A = \left( \begin{array}{ccc} z_1^{(1)} & \ldots & z_1^{(N_f)} \\ z_2^{(1)} & \ldots & z_2^{(N_f)} \\ \vdots & \ddots & \vdots \\ z_N^{(1)} & \ldots & z_N^{(N_f)} \end{array} \right),$$  

(73)

$$\sum_{p=1}^{N_f} z_p^{(i)} z_p^{(j)*} = \delta_{ij}.$$  

(74)

Now

$$\tilde{g}_0 = \sum_{j=1}^{N_f} e^{i\beta_j \Phi_j} Z^{(j)}, \quad Z^{(j)} = z_p^{(j)} z_q^{(j)*},$$  

(75)

The charges with $\tilde{g}_0$ are

$$\left( \tilde{Q}^0 \right)^A = \frac{1}{2} N_C \Tr \sum_{i} \left( n_i T^A Z^{(i)} \right).$$  

(76)

The baryon number is unchanged. The $U(n)$ possible representations will be discussed below in the semi-classical quantization approach.
5 \( sl(3,C) \) GSG model, solitons and kinks as baryons

Here we summarize some properties of the \( sl(3,C) \) GSG model written in the form (11)-(12) relevant to our discussions above, such as the soliton and kink spectrum [9, 22]. The discussions make some connection to the QCD\(_2\) developments above, such as (multi-) baryon number of solitons and kinks.

In the following we write the 1-soliton(antisoliton) and 1-kink(antikink) type solutions and compute the relevant (multi-)baryon numbers associated to the \( U(1) \) symmetry in the context of QCD\(_2\).

5.1 One soliton/antisoliton pair associated to \( \varphi_1 \)

The functions
\[
\varphi_1 = \frac{4}{\beta_0} \arctan \{ d \exp[\gamma_1(x - vt)] \}, \quad \varphi_2 = 0, \tag{77}
\]
satisfy the system of equations (11)-(12) for the set of parameters
\[
\nu_1 = 1/2, \quad \delta_1 = 2, \quad \delta_2 = 1, \quad \nu_2 = 1, \quad \nu_3 = 1, \quad r = 1. \tag{78}
\]
provided that
\[
13\mu_3 = 5\mu_2 - 4\mu_1, \quad \gamma_1^2 = \frac{\beta_0^2}{13}(6\mu_2 + 3\mu_1). \tag{79}
\]

This solution is precisely the sine-Gordon 1-soliton associated to the field \( \varphi_1 \) with mass
\[
M_{1}^{\text{sol}} = \frac{8\gamma_1}{\beta_0}. \tag{80}
\]

From (4) and taking into account the parameters (78) one has the relationships between the GSG fields
\[
\Phi_1 = -\Phi_2 = \Phi_3 = \varphi_1 \tag{81}
\]

Moreover, from (8)-(9) and (78) one gets the relationships
\[
n_1 = -n_2 = n_3 \tag{82}
\]

Taking into account the QCD\(_2\) motivated formula (71) and the eq. (82) above one can compute the baryon number of the GSG soliton (77) taking \( n_1 = 1 \)
\[
Q_B^{(1)} = N_C, \tag{83}
\]

where the superindex (1) refers to the associated \( \varphi_1 \) field nontrivial solution.
5.2 One soliton/antisoliton pair associated to $\varphi_2$

The functions

$$\varphi_2 = \frac{4}{\beta_0} \arctan \{d \exp[\gamma_2 (x - vt)]\}, \quad \varphi_1 = 0$$

(84)

solve the system (11)-(12) for the choice of parameters

$$\nu_1 = 1, \quad \delta_1 = 1, \quad \delta_2 = 2, \quad \nu_2 = 1/2, \quad \nu_3 = 1, \quad s = 1$$

(85)

provided that

$$13\mu_3 = 5\mu_1 - 4\mu_2, \quad \gamma_2 = \frac{\beta_0^2}{13} (6\mu_1 + 3\mu_2).$$

(86)

This is the sine-Gordon 1-soliton associated to the field $\varphi_2$ with mass

$$M_2^{sol} = \frac{8\gamma_2}{\beta_0^2}. \quad \text{(87)}$$

As above from (4) and the set of parameters (85) one has the relationships

$$-\Phi_1 = \Phi_2 = \Phi_3 = \varphi_2. \quad \text{(88)}$$

From (8)-(9) and (85) one gets the relationship

$$n_1 = -n_2 = -n_3. \quad \text{(89)}$$

So, taking into account the QCD$_2$ motivated formula (71) and the above eq. (89) one computes the baryon number of this GSG soliton taking $n_2 = 1$

$$Q_B^{(2)} = N_C, \quad \text{(90)}$$

where the superindex (2) refers to the associated $\varphi_2$ field.

5.3 1-soliton/1-antisoliton pairs associated to $\hat{\varphi} \equiv \varphi_1 = \varphi_2$

In the case $\varphi_1 = \varphi_2$ one has the 1-soliton solution $\hat{\varphi}$ of the system (11)-(12) associated to the parameters

$$\nu_1 = 1, \quad \delta_1 = -1/2, \quad \nu_2 = 1, \quad \delta_2 = -1/2, \quad \nu_3 = -1/2, \quad r = s = 1.$$ \quad \text{(91)}

One has the 1-soliton

$$\varphi_1 = \varphi_2 \equiv \hat{\varphi}, \quad \hat{\varphi} = \frac{4}{\beta_0} \arctan \{d \exp[\gamma_3 (x - vt)]\},$$

(92)
which requires
\[ \gamma_3^2 = \beta_0^2 \left( \mu_1 + \frac{1}{2} \mu_3 \right), \quad \mu_1 = \mu_2. \] (93)

This is a sine-Gordon 1-soliton associated to both fields \( \varphi_{1,2} \) in the particular case when they are equal to each other. It possesses a mass
\[ M_3^{sol} = \frac{8 \gamma_3}{\beta_0^2}. \] (94)

In view of the symmetry (13) which are satisfied by the parameters (91) and (93) one can think of this solution as doubly degenerated.

As above, from (4) and the set of parameters (91) one has the following relationships
\[ \Phi_1 = \Phi_2 = -\Phi_3 = \hat{\varphi}. \] (95)

From (8)-(9) and (91) one gets the relationship
\[ -2n_3 = n_1 + n_2. \] (96)

So, taking into account the QCD\(_2\) motivated formula (71) and the eq. above (96) one computes the baryon number of this GSG solution taking \( n_3 = -1 \)
\[ Q_B^{(\hat{\varphi})} = N_C, \] (97)
where the superindex refers to the associated \( \hat{\varphi} \) field.

### 5.3.1 Antisolitons and general N-solitons

The GSG system (11)-(12) reduces to the usual SG equation for each choice of the parameters (78), (85) and (91), respectively. Then, the \( N \)-soliton solutions in each case can be constructed as in the ordinary sine-Gordon model.

Using the symmetry (14) one can be able to construct the 1-antisolitons corresponding to the soliton solutions (77), (84) and (92) simply by changing their signs \( \varphi_a \rightarrow -\varphi_a \).

### 5.4 Mass splitting of solitons

It is interesting to write some relationships among the various soliton masses.

i) For \( \mu_1 \neq \mu_2 \) one has respectively the two 1-solitons, (77) and (84), with masses (80) and (87) related by
\[ (M_1^{sol})^2 - (M_2^{sol})^2 = \frac{48N_C}{\pi} (\mu_2 - \mu_1). \] (98)
ii) For $\mu_1 = \mu_2$, there appears the third soliton solution (92)-(93). Then, taking into account (79), (86), (93), (98) and the third soliton mass (94) we have the relationships

$$M_1^{sol} = M_2^{sol}, \quad M_3^{sol} = \sqrt{3/2} M_1^{sol}, \quad \gamma_1 = \gamma_2 = \sqrt{2/3} \gamma_3, \quad \mu_3 = \frac{1}{13} \mu_1.$$  

(99)

Notice that in this case $M_3^{sol} < M_1^{sol} + M_2^{sol}$, and the third soliton is stable in the sense that energy is required to dissociate it.

5.5 **Kink of the double sine-Gordon model as a multi-baryon**

In the system (11)-(12) we perform the following reduction $\phi \equiv \varphi_1 = \varphi_2$ such that

$$\Phi_1 = \Phi_2, \quad \Phi_3 = q \Phi_1,$$  

(101)

with $q$ being a real number.

Moreover, for consistency of the system of equations (11)-(12) we have

$$\mu_1 = \mu_2, \quad \delta_1 = \delta_2 = q/2, \quad \nu_1 = \nu_2, \quad \nu_3 = \frac{q}{2} \nu_1, \quad r = s = 1.$$  

(102)

Thus the system of Eqs.(11)-(12) reduces to

$$\partial^2 \Phi_{DSG} = -\frac{\mu_1}{\nu_1} \sin(\nu_1 \Phi_{DSG}) - \frac{\mu_3 \delta_1}{\nu_1} \sin(q \nu_1 \Phi_{DSG}), \quad \Phi_{DSG} \equiv \beta \phi.$$  

(103)

This is the so-called **two-frequency sine-Gordon model** (DSG) and it has been the subject of much interest in the last decades, from the mathematical and physical points of view.

If the parameter $q$ satisfies

$$q = \frac{n}{m} \in \mathbb{Q},$$  

(104)

with $m, n$ being two relative prime positive integers, then the potential $\frac{\mu_2}{\nu_1} (1 - \cos(\nu_1 \Phi_{DSG})) + \frac{\mu_3}{2\nu_1} (1 - \cos(q \nu_1 \Phi_{DSG}))$ associated to the model (103) is periodic with period

$$\frac{2\pi}{\nu_1 m} = \frac{2\pi}{q \nu_1 n}.$$  

(105)

Then, as mentioned above the theory (103) possesses topological excitations. From (4) and the set of parameters (102) one has the relationships

$$\Phi_1 = \Phi_2 = \frac{1}{q} \Phi_3 = \nu_1 \phi.$$  

(106)
And from (8)-(9) and (102) one gets the relationship
\[ n_3 = \frac{q}{2}(n_1 + n_2). \] (107)

So, taking into account the QCD$_2$ motivated formula (71) and the eq. (107) above one computes the baryon number of this DSG solution
\[ Q_B^{(DSG)} = N_C(1 + \frac{2}{q})n_3, \quad n_3 \in \mathbb{Z}, \] (108)

where the superindex (DSG) refers to the associated DSG solution.

In the following we will provide some kink solutions for a particular set of parameters. Consider
\[ \nu_1 = 1/2, \quad \delta_1 = \delta_2 = 1, \quad \nu_2 = 1/2, \quad \nu_3 = 1/2 \quad \text{and} \quad q = 2, \quad n = 2, \quad m = 1 \] (109)
which satisfy (102) and (104). This set of parameters provide the so-called double sine-Gordon model (DSG), such that from (106) and (109) the field configurations satisfy
\[ \Phi_1 = \Phi_2 = \frac{1}{2}\Phi_3 = \frac{1}{2}\phi. \] (110)

Its potential \(-[4\mu_1(\cos^{2}_{DSG} - 1) + 2\mu_3(\cos\Phi_{DSG} - 1)]\) has period $4\pi$ and has extrema at $\Phi_{DSG} = 2\pi p_1$, and $\Phi_{DSG} = 4\pi p_2 \pm 2\cos^{-1}[1 - |\mu_1/(2\mu_3)|]$ with $p_1, p_2 \in \mathbb{Z}$; the second extrema exists only if $|\mu_1/(2\mu_3)| < 1$. Depending on the values of the parameters $\beta_0, \mu_1, \mu_3$ the quantum field theory version of the DSG model presents a variety of physical effects, such as the decay of the false vacuum, a phase transition, confinement of the kinks and the resonance phenomenon due to unstable bound states of excited kink-antikink states (see [4] and references therein). The semiclassical spectrum of neutral particles in the DSG theory is investigated in ref. [39]. Let us mention that the DSG model has recently been in the center of some controversy regarding the computation of its semiclassical spectrum, see [4, 40].

A particular solution of (103) for the parameters (109) can be written as
\[ \Phi_{DSG} := 4\arctan\left[\frac{1}{d} \frac{1 + h \exp[2\gamma(x - vt)]}{\exp[\gamma(x - vt)]}\right] \] (111)
provided that
\[ \gamma^2 = \beta_0^2(\mu_1 + 2\mu_3), \quad h = -\frac{\mu_1}{4}, \] (112)
5.5.1 A multi-baryon and the DSG kink \((h < 0, \mu_i > 0)\)

For the choice of parameters \(h < 0, \mu_i > 0\) in (112) the equation (111) provides

\[
\phi = \frac{4}{\beta_0} \arctan \left[ -\frac{2|h|^{1/2}}{d} \sinh[\gamma_K (x - vt) + a_0] \right], \quad \gamma_K = \pm \beta_0 \sqrt{\mu_1 + 2\mu_3}. \tag{113}
\]

\[a_0 = \frac{1}{2} \ln |h|.
\]

This is the DSG 1-kink solution with mass

\[
M_K = \frac{16}{\beta_0^2} \gamma_K \left[ 1 + \frac{\mu_1}{\sqrt{2\mu_3(\mu_1 + 2\mu_3)}} \ln \left( \frac{\sqrt{\mu_1 + 2\mu_3} + \sqrt{2\mu_3}}{\sqrt{\mu_1}} \right) \right]. \tag{114}
\]

Since one must have \(\frac{\mu_1}{m} > \frac{1}{3}\) (see below for the range of possible values of these parameters) the potential supports one type of minima and thus there exists only one type of topological kink [3]. So, the DSG model possesses only the topological excitation (113) relevant to our QCD\(_2\) discussion.

One can relate the parameters \(\mu_j\) in (2) to the mass parameters \(m_i\) in the effective lagrangian of QCD\(_2\) in (64). So, for the “physical values” \(N_f = 3\) and \(e_c = 100\text{MeV}\) for the coupling and taking into account (57), (59) and (61) one has for large \(N_C\)

\[
\mu_j = 2 \frac{m_j}{m_0} m^2 \approx N_C m_j 124(\text{MeV}), \tag{115}
\]

thus, the \(\mu_j's\) have dimension \((\text{MeV})^2\).

For the values of the mass parameters \(\mu_1, \mu_3\) in the range \([10^3, 5 \times 10^4]\)(MeV)\(^2\) (take \(m_1 \approx m_2 \approx 52\ \text{MeV};\) \(m_3 = 4\ \text{MeV}\), notice that these values satisfy the relationship (86)) one can determine the values of the ratio \(\kappa\) between the kink (114) and the third soliton (94) masses

\[
\kappa \equiv \frac{M_K}{M_{3\text{sol}}}, \quad 4 < \kappa < 4.2 \tag{116}
\]

The baryon number of this DSG kink solution is obtained from (108) taking \(q = 2, n_3 = 2\)

\[
Q_B^{(K)} = 4N_C, \tag{117}
\]

where the superindex \((K)\) refers to the associated DSG kink solution.

The above relations (116)-(117) suggest that the decay of the kink to four solitons \(\{M_{j\text{sol}}\} (j = 1, 2, 3)\) is allowed by conservation of energy and charge, however one can see from the kink dynamics that it is a stable object and its fission may require an external trigger. For similar phenomena in soliton dynamics see ref. [41].

Let us emphasize that the baryons with charges \(2n_3N_C\) [set \(q = 2\) in (108)] for \(n_3 = 1, 2, ...\) are assumed to be bound states of 2, 4, ... “basic” baryons, and so,
they would correspond to di-baryon states like deuteron ($\frac{1}{2}H^{+}$) and the “$\alpha$ particle” ($\frac{1}{2}He^{+}$). However, we have not found, for the QCD$_2$ motivated parameter space ($\mu_1, \mu_3$) any kink with baryon number $2N_C$. These 2−baryons are expected to be found in the 2−soliton sectors of the GSG model. Notice that in our formalism the four-baryon appears already for $N_f = 3$ as a DSG kink with topological charge $4N_C$, eq. (117). In the formalism of refs. [20, 42] the multibaryons have baryon number $kN_C$ ($k \leq N_f - 1$), so their ($N_f - 1$)−baryon is the one with the greatest baryon number.

5.6 Configuration with baryon number $3N_C$

These solutions do not form stable configurations, nevertheless we describe them for completeness. Let us take $\varphi_1 = \varphi_2$, so one has two 1-soliton solutions $\hat{\varphi}_A (A = 1, 2)$ of the system (11)-(12) associated to the parameters

$$\nu_1 = 1, \delta_1 = 1/2, \nu_2 = 1, \delta_2 = 1/2, \nu_3 = 1/2, r = s = 1.$$  \hfill (118)

As the first 1-soliton one has

$$\varphi_1 = \varphi_2 \equiv \hat{\varphi}_1, \quad \hat{\varphi}_1 = \frac{4}{\beta_0} \arctan\{d \exp[\gamma_4(x - vt)]\},$$ \hfill (119)\hfill (120)

which requires

$$d^2 = 1, \quad 38\gamma_4^2 = \beta_0^2 (25\mu_1 + 13\mu_2 + 19\mu_3)$$ \hfill (121)

This is a sine-Gordon 1-soliton associated to both fields $\varphi_1, 2$ in the particular case when they are equal to each other. It possesses a mass

$$M_4^{sol} = \frac{8\gamma_4}{\beta_0^2}. \hfill (122)$$

In view of the symmetry (13) we are able to write from (121)

$$d^2 = 1, \quad 38\gamma_5^2 = 25\mu_2 + 13\mu_1 + 19\mu_3,$$ \hfill (123)

and then one has another 1-soliton from (119)-(120)

$$\varphi_1 = \varphi_2 \equiv \hat{\varphi}_2, \quad \hat{\varphi}_2 = \frac{4}{\beta_0} \arctan\{d \exp[\gamma_5(x - vt)]\}. \hfill (124)\hfill (125)

It possesses a mass

$$M_5^{sol} = \frac{8\gamma_5}{\beta_0^2}. \hfill (126)$$
Similarly, from (4) and the set of parameters (118) one has the following relationships

$$\Phi_1 = \Phi_2 = \Phi_3 = \hat{\phi}_A, \quad A = 1, 2.$$  \hfill (127)

From (8)-(9) and (118) one gets the relationship

$$2n_3 = n_1 + n_2.$$  \hfill (128)

So, taking into account the QCD$_2$ motivated formula (71) and the eq. (128) one computes the baryon number of this GSG solution taking $n_3 = 1$

$$Q_B^{(A)} = 3N_C,$$  \hfill (129)

where the superindex $(A)$ refers to the associated $\hat{\phi}_A$ field. Therefore, the both solutions $A = 1, 2$, have the same baryon number in the context of QCD$_2$. The individual soliton solutions (120) and (125) have, respectively, a topological charge $N_C$, since they are sine-Gordon solitons. Then, the configuration (127) with total charge $3N_C$ is composed of three SG solitons. Therefore, by conservation of energy and topological charge arguments one has that the rest mass of the static configurations $A = 1, 2$, with baryon number $3N_C$ will be, respectively

$$M^{\text{config.}}_{4,5} \equiv 3M^{\text{sol}}_{4,5},$$  \hfill (130)

where the masses $M^{\text{sol}}_{4,5}$ are given by (122), (126).

Moreover, one can verify the following relationships

$$i) \quad M^{\text{config.}}_4 > M^{\text{sol}}_1 + M^{\text{sol}}_2; \quad \mu_1 \neq \mu_2,$$  \hfill (131)

$$ii) \quad M^{\text{config.}}_4 = M^{\text{config.}}_5 > M^{\text{sol}}_1 + M^{\text{sol}}_2 + M^{\text{sol}}_3; \quad \mu_1 = \mu_2,$$  \hfill (132)

where the soliton masses $M^{\text{sol}}_j (j = 1, 2, 3)$ are given by (80), (87), (94), respectively. One observes that the configurations $A = 1, 2$, do not form bound states (bound states would be formed if the inequalities (131)-(132) are reversed), and they may decay into the “basic” set $\{M^{\text{sol}}_1, M^{\text{sol}}_2\}$ or $\{M^{\text{sol}}_1, M^{\text{sol}}_2, M^{\text{sol}}_3\}$ of solitons, such that the excess energy is transferred to the kinetic energy of the solitons.

6 Semi-classical quantization and the GSG ansatz

In order to implement the semi-classical quantization let us consider

$$g(x, t) = A(t)g_0(x)A^{-1}(t), \quad A(t) \in U(N_f)$$  \hfill (133)

and derive the effective action for $A(t)$ by substituting $g(x, t)$ into the original action. So, following similar steps to the ones developed in [20] one can get
Generalized sine-Gordon model and baryons in QCD

\[ \tilde{S}(g(x,t)) - \tilde{S}(g_0(x)) = \frac{N_C}{8\pi} \int d^2x \text{Tr} \left( \left[ A^{-1} \dot{A}, g_0 \right] \left[ A^{-1} \dot{A}, g_0^\dagger \right] \right) + \frac{N_C}{2\pi} \int d^2x \text{Tr} \left\{ \left( A^{-1} \dot{A} \right) (g_0^\dagger \partial_x g_0) \right\} + m^2 \int d^2x \left[ (DA g_0) A^\dagger - D g_0 \right] + \text{c.c.} \] (134)

The action above for \( D_{ij} = \delta_{ij} \) (in this case the last integrand after taking the trace operation vanishes identically) is invariant under global \( U(N_f) \) transformation

\[ A \to UA, \] (135)

where \( U \in G = U(N_f) \). This corresponds to the invariance of the original action (with mass of the same magnitude for all flavors) under \( g \to U g U^{-1} \). It is also invariant under the local changes

\[ A(t) \to A(t) V(t), \] (136)

where \( V(t) \in H \). This subgroup \( H \) of \( G \) is nothing but the invariance group of \( g_0 \).

Below we will find some particular cases of \( H \).

We define the Lie algebra valued variables \( q^i, y_a \) through

\[ A^{-1} \dot{A} = i \sum \{ \dot{q}_i E_{\alpha_i} + \dot{y}_a H^a \} \] in the generalized Gell-Mann representation [43]. In terms of these variables the action (134), for a diagonal mass matrix such that \( D_{ij} = \delta_{ij} \), takes the form

\[ S[q, y] = \int dt \sum_{i=1}^{N_f} \frac{1}{2M_i} \dot{q}_i \dot{q}_i - \sum_{a=1}^{N_f-1} \sqrt{\frac{2}{(a+1)^2 - (a+1)}} \times \]

\[ \times (n_1 + n_2 + ... + n_a - an_{a+1}) \dot{y}_a, \] (137)

where \( q_{\pm i} \) are associated to the positive and negative roots, respectively, and

\[ \frac{1}{2M_i} = \frac{N_C}{2\pi} \int_{-\infty}^{+\infty} \left[ 1 - \cos \beta_i \Phi_i \right], \quad \Phi_i \neq 0. \] (138)

In the case of vanishing \( \Phi_j \equiv 0 \) for a given \( j \) one must formally set \( M_j = +\infty \) in the relevant terms throughout.

In the case of \( \tilde{g}_0 = \text{diag}(1, 1, ..., e^{i\beta N_f \Phi_N}) \) the second summation in (137) reduces to the unique term \( -N_c \sqrt{\frac{2(N_f - 1)}{N_f}} \dot{y}_{N_f - 1} \) [20].

When written in terms of the general diagonal field \( g_0(x) \) and the \( U(N_f) \) field \( A(t) \), the charges associated to the global \( U(N_f) \) symmetry, (70), become

\[ Q^B = i \frac{N_C}{8\pi} \int dx \text{Tr} \left\{ T^B A \left[ (g_0^\dagger \partial_x g_0 - g_0 \partial_x g_0^\dagger) + [g_0, [A^{-1} \dot{A}, g_0^\dagger]] A^{-1} \right] \right\} \] (139)

A convenient parameterization, instead of the parameters used in (137), is (73) since in the above expressions, for \( Q^B \) and the action (134), there appear the fields
$A, A^{-1}$, as well as their time derivatives. Now, for a diagonal mass matrix such that $D = \frac{m_q}{m_0} \delta_{ij}$, the expression (134) can be written in terms of the variables $z^{(i)}_p$ subject to the relationships (74)

$$S(g(x,t)) - S(g_0(x)) = S[z^{(i)}_p(t), \Phi_i(x)]$$

(140)

$$S[z^{(i)}_p(t), \Phi_i(x)] = \frac{N_C}{2\pi} \int d^2x \sum_{p,q; i<j} [\cos(\beta_i \Phi_i - \beta_j \Phi_j) - 1][z^{(i)}_p \bar{z}^{(j)}_q * z^{(j)}_p * z^{(i)}_q] -$$

$$\frac{iN_C}{2\pi} \int d^2x \sum_{i,p} \beta_i \partial_x \Phi_i z^{(i)}_p \bar{z}^{(i)}_p * +$$

$$\int dt 2 \sum_{i,p} \cos(\beta_i \Phi_i) \bar{m}^2_\pi z^{(i)}_p \bar{z}^{(i)}_p * - \sum_i \cos(\beta_i \Phi_i) \bar{m}^2_\pi$$

(141)

Let us choose the index $k$ corresponding to the smallest mass $m_k$. So, integrating over $x$ in (141) we may write

$$S[z^{(i)}_p(t)] = -\frac{1}{2} \int dt \sum_{i<j} \sum_{p,q} M_{ij}^{-1} z^{(i)}_p \bar{z}^{(i)}_q * z^{(j)}_p * -$$

$$\frac{iN_C}{2\pi} \int dt \sum_i n_i \left[z^{(i)}_p \bar{z}^{(i)}_p * - z^{(i)}_p \bar{z}^{(i)}_p * \right] -$$

$$\frac{2\pi}{N_c} \int dt \left\{ \sum_{i,p} \frac{\bar{m}^2_\pi}{M_i} - \frac{\bar{m}^2_\pi}{M_k} \right\} z^{(i)}_p \bar{z}^{(i)}_p * + \frac{2\pi}{N_c} \left[ \sum_i \frac{\bar{m}^2_\pi}{M_i} - \frac{\bar{m}^2_\pi}{M_k} \right]$$

(142)

where $N_\phi$ is the number of nonvanishing $\Phi_i$ fields and we have introduced some Lagrange multipliers enforcing the relationships (74). The constants $M_{ij}$ above are defined by

$$\frac{1}{2M_{ij}} = \frac{N_C}{2\pi} \int dx [1 - \cos(\beta_0 \Phi_i - \beta_0 \Phi_j)]; \quad i < j.$$

(143)

If the field solutions are such that $\Phi_i = \Phi_j$, then one must set formally $M_{ij} \rightarrow +\infty$ in place of the corresponding constants.

Likewise, we can write the $U(N_f)$ charges, eq. (139), in terms of the $z^{(i)}_p$ variables

$$Q^A = \frac{1}{2} T^A_{\beta\alpha} Q_{\alpha\beta},$$

$$Q_{\alpha\beta} = N_C \sum_j n_j z^{(j)}_\alpha \bar{z}^{(j)}_\beta * - \frac{i}{2} \sum_{i,j} M_{ij}^{-1} z^{(j)}_\alpha \bar{z}^{(j)}_\gamma * z^{(i)}_\gamma \bar{z}^{(i)}_\beta *$$

(144)
The second $U(N_f)$ Casimir operator is obtained from the charge matrix elements $Q_{\alpha\beta}$

$$Q^A Q_A = \frac{1}{2} Q_{\alpha\beta} Q_{\beta\alpha},$$

$$= \frac{1}{2} N_C^2 \sum_i n_i n_i - \frac{1}{4} \sum_{i<j} \left( M_{ij}^{-1} \right)^2 \bar{z}_i^{(j)} \bar{z}_i^{(j)} \bar{z}_i^{(i)} \bar{z}_i^{(i)} \tag{145}$$

The expressions above greatly simplify in certain particular cases of the ansatz (60), the ansatz (62) has been studied extensively in the literature before. In the next subsection we review this case and in further sections we analyze the semiclassical quantization of the GSG ansatz given for $N_f = 3$ flavors.

6.1 Review of usual sine-Gordon soliton and baryons in QCD$_2$

In this subsection we briefly review the formalism applied to the ansatz (62), which is related to the usual SG one-soliton as the lowest baryon state. In order to calculate the quantum correction it is allowed the sine-Gordon soliton to rotate in $SU(N_f)$ space by a time dependent matrix $A(t)$ as in (133). Let us consider the single baryon state defined for the ansatz (62) for the sine-Gordon soliton solution $\Phi_{N_f} \equiv \Phi_{1-soliton}$ [ $\Phi_{1-soliton}$ is given by eq. (1)]; so, in the relations above one must set

\begin{align*}
n_{N_f} &= 1; \quad n_j = 0 (j \neq N_f); \quad M_{jk}^{-1} \equiv 0 (j < k < N_f); \quad M_{jN_f}^{-1} \equiv M_{N_f}^{-1} (j < N_f) \tag{146}
\end{align*}

where $M_{N_f}^{-1}$ can be computed using eq. (138) for $i = N_f$ for the soliton (1)

$$\frac{1}{2M_{N_f}} = \frac{1}{\sqrt{2 \overline{m}}} \left(\frac{N_C}{\pi}\right)^{3/2} \tag{147}$$

Then, for the ansatz (62), i.e. $\hat{g}_0(x) = \mathrm{diag}(1, 1, ..., e^{-i \sqrt{\overline{m}} N_f \Phi})$, the effective action (142) can be written as

\begin{align*}
S[z_j^{(N_f)}(t)] &= \frac{1}{2M_{N_f}} \int dt [\dot{z}_j^{(N_f)} \dot{z}_j^{(N_f)} - (\dot{z}_i^{(N_f)} \dot{z}_i^{(N_f)}) (\dot{z}_k^{(N_f)} \dot{z}_k^{(N_f)})]

- \frac{2\pi}{M_{N_f} N_C} \int dt \sum_{i=1}^{N_f} \left( \overline{m}_{N_f}^2 - \overline{m}_{N_f}^2 \right) z_i^{(N_f)} \bar{z}_i^{(N_f)}

- \frac{i N_C}{2} \int dt n_{N_f} \left( z_j^{(N_f)} \bar{z}_j^{(N_f)} - \bar{z}_j^{(N_f)} \bar{z}_j^{(N_f)} \right)

+ \int dt [z_p^{(N_f)} \bar{z}_q^{(N_f)} - \delta_{pq} \lambda^{pq}], \tag{148}
\end{align*}

where $n_{N_f} = 1$, $M_{N_f}$ is given by (147) and $m_{N_f}$ entering $\overline{m}_{N_f}$ is chosen to be the smallest quark mass. Notice that for equal quark masses the second line in eq. (148)
vanishes identically. According to (135)-(136), the symmetries of $S[z^{(N_f)}(t)]$ are the global $U(N_f)$ group (for equal quark masses) under which
\[ z^{(N_f)}(N_f) \rightarrow z^{(N_f)}(N_f) = U_{\alpha\beta} z^{(N_f)}_{\alpha} \quad U \in U(N_f), \]
and a local $U(1)$ subgroup of $H$ under which
\[ z^{(N_f)}(N_f) \rightarrow z^{(N_f)}(N_f) = e^{i\delta(t)} z^{(N_f)}_{\alpha} . \]

The action (148) has been considered in order to find the quantum correction to the soliton mass for certain representations $R$ of the flavor symmetry $SU(N_f)$. The case of equal quark masses has been studied in the literature \[20, 21, 24, 44\]. Certain properties in the case of different quark masses have been considered in \[23\] for the ansatz (62).

In this approach the minimum-energy configuration for the class of ansatz (62), with $m_{N_f}$ the smallest mass, corresponds to the state of lowest-lying baryon \[20\] which in the large-$N_C$ limit possesses the classical mass
\[ M^{cl}_{\text{baryon}} = 4\tilde{m}_{N_f} \left( \frac{2N_C}{\pi} \right)^{1/2} \approx 1.90N_f^{1/4}\sqrt{c_e m_{N_f} N_C}, \]
where $\tilde{m}_{N_f}$ has been given in (61) for $i = N_f$.

Moreover, for the Ansatz (62) the $SU(N_f)$ charges become
\[ Q_{\alpha\beta} = N_C n_{N_f} z^{(N_f)}_{\alpha} \bar{z}^{(N_f)}_{\beta} + \frac{i}{2M_{N_f}} \left[ z^{(N_f)}_{\alpha} \bar{z}^{(N_f)}_{\beta} \left( \bar{z}^{(N_f)}_{\delta} \bar{z}^{(N_f)}_{\delta} - z^{(N_f)}_{\delta} \bar{z}^{(N_f)}_{\delta} \right) + \right. \]
\[ \left. z^{(N_f)}_{\beta} \bar{z}^{(N_f)}_{\beta} \left( z^{(N_f)}_{\alpha} \bar{z}^{(N_f)}_{\alpha} \right) \right] \]
\[ (152) \]

The corresponding second Casimir can be obtained from (145)
\[ Q_A Q^A = \frac{1}{2} Q_{\alpha\beta} Q_{\beta\alpha} = \frac{1}{2} N_C^2 n_{N_f}^2 + \frac{1}{4M_{N_f}^2} \left( Dz \right)_{\alpha}^{\dagger} \left( Dz \right)_{\alpha} , \quad Dz \equiv \dot{z} - z \left( \dot{z}^\dagger \dot{z} \right) \]
\[ (153) \]

Moreover, denoting the $SU(N_f)$ second Casimir operator by $C_2(N_f)$ one can write
\[ Q_A Q^A = C_2(N_f) + \frac{1}{2N_f} (Q_B)^2, \]
\[ (154) \]
where $Q_B$ is the baryon number (71), which in this case reduces to $Q_B = N_C$.

In the case of a single baryon given by $\hat{g}_0$, eq. (62), and for unequal quark masses, the hamiltonian is linear in the quadratic Casimir operator. To see this we now derive the hamiltonian corresponding to the action (148). The canonical momenta are given by
\[ p_{\alpha} = \frac{\partial L}{\partial \dot{z}^{(N_f)}_{\alpha}} = \frac{1}{2M_{N_f}} \left[ \dot{z}^{(N_f)}_{\alpha} - \left( \dot{z}^{(N_f)}_{\beta} \bar{z}^{(N_f)}_{\beta} \right) z^{(N_f)}_{\alpha} \right] + \frac{iN_C}{2} N_f \]
\[ (155) \]
and there is a conjugate expression for $p_\alpha$. Therefore, from $H = p_\alpha z_\beta^{(N_f)*} + p_\alpha z_\beta^{(N_f)} - L$, one can get the hamiltonian

$$H = \frac{1}{2M_{N_f}} \left( \frac{Dz}{\alpha} \right)^\dagger \left( Dz \right)_\alpha + \frac{2\pi}{M_{N_f}N_C} \sum_{i=1}^{N_f} \left( \bar{m}_i^2 - \bar{m}_{N_f}^2 \right) z_i^{(N_f)*} z_i^{(N_f)}. \quad (156)$$

However, one must take a proper care of the relevant constraint (74) which was incorporated through the addition of a Lagrange multiplier in the action (148). A proper treatment of a constrained system must be performed at this point [20]. In [20, 44] it was shown that the local $U(1)$ gauge symmetry (150) leads to the constraint

$$Q_{N_f N_f} = 0 \Rightarrow Q_B = \sqrt{2N_f(N_f - 1)} Q_Y \quad (157)$$

which has to be imposed on physical states. This implies that the representation $R$ must contain a state with $Y$ charge

$$\bar{Q}_Y = \sqrt{\frac{1}{2N_f(N_f - 1)}} N_C. \quad (158)$$

The remaining states will be generated through the application of the $SU(N_f)$ transformations to this one. For states with only quarks and no antiquarks, the condition that $Q_B = N_C$ implies that only representations described by Young tableaux with $N_C$ boxes appear. The additional constraint that $Q_Y = \bar{Q}_Y$ implies that all $N_C$ quarks belong to $SU(N_f - 1)$, i.e., this state does not involve the $N_f^{th}$ quark flavor. These constraints are automatically satisfied in the totally symmetric representation of $N_C$ boxes, which is the only representation possible in two dimensions. This is because the state wave functions have to be constructed out of the components of the complex vector $z^{(N_f)}$ as

$$\psi(z^{(N_f)}, z^{(N_f)*}) = (z_1^{(N_f)})^{p_1 \ldots (z_{N_f}^{(N_f)})^{p_{N_f}}} (z_1^{(N_f)*})^{q_1 \ldots (z_{N_f}^{(N_f)*})^{q_{N_f}}} \quad (159)$$

with $\sum_{i=1}^{N_f} (p_i - q_i) = N_C$.

The lowest such multiplet has

$$\sum_{i=1}^{N_f} p_i = N_C \text{ and all } q_i = 0 \quad (160)$$

This multiplet corresponds to the Young tableaux

$$\begin{array}{c}
\circ \ldots \circ \\
N_c \\
\end{array} \quad (161)$$
In QCD for $N_C = 3$, $N_F = 3$ we get only the 10 of $SU(3)$.

Then, taking into account (153), (154) and (61), the expression (156) becomes [20, 23]

$$H = M_{baryon}^{cl} \left\{ 1 + \left( \frac{\pi^2}{2N_C} \right)^2 \left[ C_2(R) - \frac{n_f^2 N_C^2}{2N_f} (N_f - 1) \right] + \sum_{i=1}^{N_f} \frac{m_i - m_{N_f}}{m_{N_f}} |z_i^{(N_f)}|^2 \right\},$$

(162)

where $M_{baryon}^{cl}$ is given by (151) and $C_2(R)$ is the value of the quadratic Casimir for the flavor representation $R$ of the baryon. For a baryon state given by SG 1-soliton solution one must set $n_{N_f} = 1$ in the hamiltonian above. Notice that the Hamiltonian depends on $m_0$ only through $M_{baryon}^{cl}$, so the overall mass scale is undetermined, only the mass ratios are meaningful. The mass term contributions come from quantum fluctuations around the classical soliton, consistency with the semi-classical approximation requires that it be very small compared to one. However, these terms vanish for equal quark masses [20, 21]. The 10 baryon mass becomes

$$M(baryon) = M_{classical} \left[ 1 + \left( \frac{\pi^2}{8} \right) \frac{N_f - 1}{N_C} \right].$$

(163)

Notice that the quantum correction is suppressed by a factor of $N_C$. Moreover, the quantum correction for $N_C = 3$, $N_F = 3$ numerically becomes $\sim 0.82$.

The hamiltonian (162) taken for equal quark masses has been used to compute the energy of the first exotic baryon $E_1$ (a state containing $N_C + 1$ quarks and just one anti-quark) by taking the corresponding Casimir $C_2(E_1)$ for $R = 35$ of flavor relevant to the exotic state [21]. For further analysis we record the mass of this exotic baryon

$$M(E_1) = M(classical) \left[ 1 + \frac{\pi^2}{8} \frac{1}{N_C} \left( 3 + N_f - \frac{6}{N_f} \right) + \frac{3\pi^2}{8} \frac{1}{N_C} \left( N_f - \frac{3}{N_f} \right) \right].$$

(164)

In the interesting case $N_C = 3$, $N_f = 3$ this becomes

$$M(35) = M(classical) \left\{ 1 + \frac{\pi^2}{4} \right\}.$$

(165)

In this case the correction due to quantum fluctuations around the classical solution is larger than the classical term. So, the semi-classical approximation may not be a good approximation. However, observe that the ratio $M(35)/M(10) \sim 1.9$, which is 17% larger than the ratio between the experimental masses of the $\Theta^+$ and the nucleon. See more on this point below. These semi-classical approximations may be improved by introducing different ansatz for $g_0$ and considering unequal quark mass parameters. These points will be tackled in the next sections.
7 The GSG model, the unequal quark masses and baryon states

In the following we will concentrate on the effective action (142) for the particular case $N_f = 3$ and unequal quark mass parameters. So, the $SU(3)$ flavor symmetry is broken explicitly by the mass terms.

The effective Lagrangian in the case of $N_f = 3$ from (142), upon using (74), can be written as

$$S[z^{(i)}_p(t)] = \frac{1}{4} \int dt \left\{ (M_{12}^{-1} + M_{13}^{-1} - M_{23}^{-1}) \left[ \dot{z}^{(i)}_\alpha \dot{z}^{(1)*}_\alpha - \dot{z}^{(i)}_\alpha \dot{z}^{(1)}_\alpha \right] + \left( M_{12}^{-1} - M_{13}^{-1} + M_{23}^{-1} \right) \left[ \dot{z}^{(2)}_\alpha \dot{z}^{(2)*}_\alpha - \dot{z}^{(2)}_\alpha \dot{z}^{(2)}_\alpha \right] + \left( - M_{12}^{-1} + M_{13}^{-1} + M_{23}^{-1} \right) \left[ \dot{z}^{(3)}_\alpha \dot{z}^{(3)*}_\alpha - \dot{z}^{(3)}_\alpha \dot{z}^{(3)}_\alpha \right] \right\} - \frac{iN_C}{2} \int dt \sum_{i,p} n_i \left[ \dot{z}^{(i)}_p \dot{z}^{(i)}_p - \dot{z}^{(i)}_p \dot{z}^{(i)}_p \right] - \int dt \left\{ \frac{2\pi}{N_c} \sum_{i,p} \left[ \tilde{m}_{M_i}^2 - \tilde{m}_{M_k}^2 \right] \dot{z}^{(i)}_p \dot{z}^{(i)}_p + \frac{2\pi}{N_c} \sum_i \left[ \tilde{m}_{M_i}^2 - \tilde{m}_{M_k}^2 \right] \right\} \right\} \quad (166)$$

From (145) and following similar steps the second $U(3)$ Casimir operator can be written as

$$Q\Phi Q = \frac{1}{2} Q_{\alpha \beta} Q_{\beta \alpha} = \frac{1}{2} N_C^2 \sum_j n_j n_j + \frac{1}{8} \left( M_{12}^{-2} + M_{13}^{-2} - M_{23}^{-2} \right) \left[ \dot{z}^{(1)}_\alpha \dot{z}^{(1)*}_\alpha - \dot{z}^{(1)}_\alpha \dot{z}^{(1)}_\alpha \right] + \left( M_{12}^{-2} - M_{13}^{-2} + M_{23}^{-2} \right) \left[ \dot{z}^{(2)}_\alpha \dot{z}^{(2)*}_\alpha - \dot{z}^{(2)}_\alpha \dot{z}^{(2)}_\alpha \right] + \left( - M_{12}^{-2} + M_{13}^{-2} + M_{23}^{-2} \right) \left[ \dot{z}^{(3)}_\alpha \dot{z}^{(3)*}_\alpha - \dot{z}^{(3)}_\alpha \dot{z}^{(3)}_\alpha \right]. \quad (167)$$

As a particular case for the ansatz (62) let us take $N_f = 3$, so $n_1 = n_2 = 0$ in (68). In (143) one can set formally $M_{12} \equiv +\infty$ and in view of (138) the remaining parameters can be written as $M_{13} = M_{23} \equiv M_3$. Thus, taking into account these parameters the expressions for the action (166) and the second Casimir (167) reduce to the well known ones (148) and (153), respectively.

Next, we discuss the action (166) and the second Casimir (167) operator for the soliton and kink type solutions of the GSG model. In subsections 5.1, 5.2 and 5.3 we classify these type of solutions. There are three 1–soliton solutions [see eqs. (77), (84) and (92)] which correspond to baryon number $N_C$ [see eqs. (83), (90) and (97)], because the GSG model possesses the symmetry (13) the third soliton is
doubly degenerated. From the fields relationships (81), (88) and (95) one has the three 1-soliton cases
\[ i) \Phi_1 = -\Phi_2 = \Phi_3 = \varphi_1 \quad \Rightarrow \quad M_{13} = +\infty, \quad M_{12} = M_{23} = M_2; \]
\[ M_1 = M_2 = M_3 = \tilde{M}_2, \quad (168) \]
\[ ii) -\Phi_1 = \Phi_2 = \Phi_3 = \varphi_2 \quad \Rightarrow \quad M_{23} = +\infty, \quad M_{12} = M_{13} = M_1; \]
\[ M_1 = M_2 = M_3 = \tilde{M}_1, \quad (169) \]
\[ iii) \Phi_1 = \Phi_2 = -\Phi_3 = \hat{\varphi} \quad \Rightarrow \quad M_{12} = +\infty, \quad M_{13} = M_{23} = M_3; \]
\[ M_1 = M_2 = M_3 = \tilde{M}_3 \quad (170) \]
where the eqs. (143) and (138) have been used, respectively, to define the parameters \( M_j \) and \( \tilde{M}_j \) in the right hand sides of the relationships above.

In section 5 we record the kink type solution [see eq. (113)] which corresponds to the GSG reduced model called double sine-Gordon theory. This solution corresponds to baryon number \( 4N_C \) [see eq. (117)]. Thus, from (110), (143) and (138) one has
\[ \Phi_1 = \Phi_2 = \frac{1}{2} \Phi_3 = \frac{1}{2} \phi \quad \Rightarrow \quad M_{12} = +\infty, \quad M_{13} = M_{23} = M_K; \]
\[ M_1 = M_2 = M_3 = \tilde{M}_K \quad (171) \]

The solutions with baryon numbers \( 2N_C \) and \( 3N_C \) correspond to composite configurations formed by multi-solitons of the GSG model. These states (i.e. multibaryons) deserve a careful treatment which we hope to undertake in future.

7.1 GSG solitons and the states with baryon number \( N_C \)

For the particular cases (168)-(170) one can rewrite the action (166) such that for each case the terms quadratic in time derivatives reduce to a term depending only on one variable, say \( z_i^{(l)} \), related to the \( l \)th column of the matrix \( A \). The reason is that the symmetries of the quantum mechanical lagrangian and actual manifold on which \( A(t) \) lives depend on the properties of the ansatz \( g_0 \). For the ansatz \( g_0 \) related to the GSG model one can see that the space-time dependent field \( g \) in eq. (133) can be rewritten only in terms of certain columns of \( A \). For example, in the case (170) above the matrix \( g(x,t) \) can be written as
\[ g_{\alpha\beta}(x,t) = [A g_0 A^{-1}]_{\alpha\beta} = \delta_{\alpha\beta} e^{i\beta_0 \hat{\varphi}} - 2i \sin(\beta_0 \hat{\varphi}) z^{(3)}_{\alpha} z^{(3)*}_{\beta}, \quad (172) \]
which clearly depends only on the third column of \( A \). So, we may think that the left hand side of (134), i.e. \( [\tilde{S}(g(x,t)) - \tilde{S}(g_0(x))] \), entering the expression of the semi-classical quantization approach, would in principle be written only in terms of the third column of \( A \). However, in order to envisage certain local symmetries
it is useful to write the terms first order in time derivatives as depending on the full parameters \( z^{(j)}_i \) of the field \( A \). These terms arise from the WZW term and provides the Gauss law type \( N_z \) number conservation law [See eq. (179) below]. An additional \( SU(2) \in H \) (see (173)) local symmetry will be described below. Moreover, this picture is in accordance with the counting of the degrees of freedom. In fact, the effective action (134) possesses the local gauge symmetry (136), where in the case of field configuration (170) the gauge group \( H \) becomes

\[
H = SU(2) \times U(1)_B \times U(1)_Y,
\]

with the last two \( U(1) \) factors related to baryon number and hypercharge, respectively. Thus, the effective action (166) will be an action for the coordination describing the coset space \( G/H = SU(3) \times U(1)_B/SU(2) \times U(1)_B \times U(1)_Y = CP^2 \). The \( \Phi_i \) fields and symmetries of \( g_0 \) also determine the values and relationships between the parameters \( M_{ij} \) in (168)-(170), such that certain coefficients in (166) depending on these parameters vanish identically, thus leaving a subset of \( z_i^{(j)} \) variables which must be consistent with the counting of the degrees of freedom. For example this picture is illustrated in the case (170) where the coefficients \( (M_{12}^{-1} + M_{13}^{-1} - M_{23}^{-1}) \) and \( (M_{12}^{-1} - M_{13}^{-1} + M_{23}^{-1}) \) vanish identically, leaving an action with kinetic term depending only on the variables \( z_i^{(3)} \). However, the mass and WZW terms are conveniently written in terms of the complete \( z_i^{(j)} \) variables.

So, for each case in (168)-(170) labeled by \( \hat{l} \), the action can be written as

\[
S[z^{(i)}_p(t)] = \frac{1}{2} \int dt \; M_{\hat{l}}^{-1} \left[ \dot{z}^{(i)}_\alpha \dot{z}^{(i)}_\alpha - \dot{z}^{(i)}_\alpha \dot{z}^{(i)}_\alpha \ast \dot{z}^{(i)}_\beta \dot{z}^{(i)}_\beta \ast \right] - \frac{i N_c}{2} \int dt \sum_{i,p} n_i \left[ \dot{z}^{(i)}_p \dot{z}^{(i)}_p \ast - \dot{z}^{(i)}_p \dot{z}^{(i)}_p \right] - \frac{2\pi N_c}{N_c M_{\hat{l}}} \int dt \sum_{i,j} \tilde{m}_i^2 |z_i^{(j)}|^2.
\]

In the relation above we must assign the relevant set of values to the indices \( n_i (i = 1, 2, 3) \) for the relevant case in (168)-(170). The first term in (174) is the usual \( CP^2 \) quantum mechanical action, while the terms first order in time-derivatives are modifications due to the WZ term, as arisen from (134) and (142). Notice that the last term was originated from the unequal quark mass terms.

Following similar steps as in the single baryon case (see eqs. (155)-(156)) one can obtain the hamiltonian

\[
H = \frac{1}{2M_{\hat{l}}} \left( D_z^{(i)} \right) \ast \left( D_z^{(i)} \right) \ast + \frac{2\pi}{N_c M_{\hat{l}}} \sum_{i,j} \tilde{m}_i^2 |z_i^{(j)}|^2,
\]

where \( \left( D_z^{(i)} \right) \ast = z^{(i)}_\alpha - z^{(i)}_\alpha \ast \left( z^{(i)}_\beta \ast \dot{z}^{(i)}_\beta \right) \).
Similarly, the corresponding second Casimir becomes

\[ Q^A Q_A = \frac{1}{2} Q_{\alpha\beta} Q_{\beta\alpha}, \]

\[ = \frac{1}{2} N_C^2 \sum_i |n_i|^2 + \frac{1}{4 M_l^2} \left( Dz^{(i)} \right)_\alpha^{\dagger} \left( Dz^{(i)} \right)_\alpha \]

(176)

Then from (175)-(176) and taking into account \( Q_A Q_A = C_2 + \frac{1}{2 N_f} \sum_i (Q_B^i)^2 \) one can get

\[ H = 2 M_l \left( C_2 + \frac{1}{2 N_f} \sum_i (Q_B^i)^2 - \frac{1}{2} N_C^2 \sum_i |n_i|^2 \right) + \frac{2 \pi}{N_c M_l} \sum_{i,j=1}^{3} \tilde{m}_i^2 |z_i^{(j)}|^2 \]

(177)

where \( Q_B^i = n_i N_C \) for a convenient choice of the indices \( n_i \), which in the cases (168)-(170) is simply \( |n_i| = 1 \) [see also eqs. (83), (90) and (97) for 1-soliton configurations]. The parameters \( M_l, \tilde{M}_l \) can be computed for the relevant solitons. They become

\[ \frac{1}{2 M_l} = \frac{1}{\tilde{m}} \frac{2 \sqrt{2}}{3} \left( \frac{N_C}{\pi} \right)^{3/2}, \quad \frac{1}{2 \tilde{M}_l} = \frac{1}{\sqrt{2} \tilde{m}} \left( \frac{N_C}{\pi} \right)^{3/2} \]

(178)

Some comments concerning the two hamiltonians (162) and (177) are in order here. Even though they correspond to one baryon state (baryon number \( N_C \)) they look different. In fact, the hamiltonian (177) incorporates additional terms. First, due to the ansatz (60) related to the GSG model one has some set of field solutions comprising in total three possibilities (168)-(170) with baryon number \( N_C \), each case being characterized by the set of parameters \( M_l, \tilde{M}_l \) and relevant combinations of the indices \( n_j \) which are related to the baryon number of the configuration \( \{ \Phi_j \}, j = 1, 2, 3 \). So, the terms \( -\frac{N_C^2}{2} \) and \( \frac{N_C^2}{2 N_f} \) in (162) translate to \( -\frac{N_C^2}{2} \sum_i n_i^2 \) and \( \frac{1}{2 N_f} \sum_i (Q_B^i)^2 \), respectively, in the new hamiltonian (177). Second, the mass term expression allows an exact summation due to unitarity, thus giving a constant additional term to the hamiltonian (see below). The corresponding term in (162), obtained in [23], does not permit an exact summation.

### 7.2 Lowest lying baryon state and the GSG soliton

So far, the treatment for each case (168)-(170) followed similar steps; however, in order to compute the quantum correction to the soliton mass we choose the one from the classification (168)-(170) with the minimum classical energy solution. Thus, taking into account the “physically” motivated inequalities \( m_3 < m_1 < m_2 \) (or \( \mu_3 < \mu_1 < \mu_2 \)) [eq. (115) relates the \( \mu_j \)'s and the \( m_j \)'s] one observes that the soliton with mass \( M_2^{\text{sol}} \) [see eq. (87)] possesses the smallest mass according to the relationship (98). This corresponds to the second case (169) classified above; so one must set the index \( \hat{l} = 1 \) in the action (174).
The variation of the action (174) under $z^{(j)}_\alpha \to e^{i\delta(t)}z^{(j)}_\alpha$ is due to the WZW term: 
\[ \Delta S = N_c (n_1 + n_2 + n_3) \int dt \delta \]
This implies
\[ N_z = \frac{\Delta S}{\Delta \delta} = N_c \left( n_1 + n_2 + n_3 \right), \] (179)
which is an analog of the Gauss law, and restricts the allowed physical states [45].

For the soliton configuration with baryon number $N_C$, (169), under consideration in this subsection, we have $n_1 = -n_2 = -n_3 = -1 \to n_1 + n_2 + n_3 = 1$ [see eq. (89)] implying
\[ N_z = N_C. \] (180)

Therefore, for any wave function, written as a polynomial in $z$ and $z^\ast$ the number of the $z$ minus the number of the $z^\ast$ must be equal to $N_C$. But due to a larger local symmetry we will have more restrictions. Thus, as commented earlier the (massless part) effective action (174) is invariant under the local $SU(2)$ symmetry. This can be easily seen by defining "local gauge potentials"
\[ \tilde{A}_{\beta\alpha}(t) = - \sum_p z_p^{(\beta)} \ast z_p^{(\alpha)}, \quad \alpha, \beta = 2, 3. \] (181)

Under the local gauge transformation corresponding to $\Lambda(t)$, one has
\[ \tilde{A}(t) \to e^{i\Lambda} \tilde{A} e^{-i\Lambda} + \partial_t e^{i\Lambda} e^{-i\Lambda}. \] (182)

Then we have that the WZW term in (174) for the variables $z_p^\alpha, \alpha = 2, 3$ (take $\hat{l} = 1$, $n_2 = n_3 = 1$) remain invariant under the transformation (182)
\[ iN_C \int dt \text{Tr} \dot{z}_p^{(\alpha)} \ast z_p^{(\beta)} \equiv iN_C \int dt \text{Tr} \tilde{A} \Rightarrow iN_C \int dt \text{Tr} \tilde{A} \] (183)

Remember that the variables $z_p^\alpha$ do not appear in the kinetic term of (174). The local symmetry above imply that the allowed physical states must be singlets under the $SU(2)$ symmetry in flavor space. So, the wave functions for $z$’s only (analogous to quarks only for QCD) must be of the form
\[ \psi_2(z) = \Pi_{\alpha_1, \alpha_2}^{N_C} \left( \epsilon_{\alpha_1, \alpha_2} z_{i_1}^{(\alpha_1)} z_{i_2}^{(\alpha_2)} \right), \quad \alpha_1, \alpha_2 = 2, 3, \] (184)
where $1 \leq i_1, i_2 \leq N_f$.

Then, taking into account the restrictions of the types (180) and (184) the most general state can be written as
\[ \tilde{\psi}(z, z^\ast) = \psi_2(z) \left[ \Pi_{\{p,q\}} \left( z_p^{(\alpha)} \ast z_q^{(\alpha)} \right)^{n_{pq}} \right], \] (185)
and the products are defined for some sets of indices. This wave function generalizes the one given in (159).

Next, let us compute the mass of the state represented for wave functions of the form \( \tilde{\psi}(t) = \psi_2(z) \Pi_i(z_i^{(1)})^{p_i} \) where \( \sum_{i=1}^{N_f} p_i = N_C \).

Combining the hamiltonian (177), the relevant parameters (178) and the classical soliton mass term, for the \( R \) baryon we have

\[
\begin{align*}
M(baryon) &= M_{\text{classical}} \left\{ 1 + \frac{3}{4} \left( \frac{\pi}{2N_C} \right)^2 \left[ C_2(R) - \frac{N_C^2}{2} (N_f - 1) + \frac{1}{2m^2} \sum_i \tilde{m}_i^2 \right] \right\},
\end{align*}
\]

where

\[
M_{\text{classical}} = 4\tilde{m}\left( \frac{2N_C}{\pi} \right)^{1/2}, \quad \tilde{m}^2 = \frac{1}{13} \left( \frac{m^2}{m_0} \right) \left( 6m_1 + 3m_2 \right).
\]

The last term in (177) simplifies to a constant term by unitarity condition of the matrix elements \( z_i^{(j)} \) and the parameter \( \tilde{m} \) corresponds to the one-soliton parameter once the identification \( \gamma_2^2 = 2\beta_0^2 \tilde{m}^2 \) is made in (86) by comparing the SG one-solitons (1) and (84). Even though the computations are explicitly made for \( N_f = 3 \) it is instructive to leave the number of flavors as a variable. In the case of the 10 baryon one has

\[
\begin{align*}
M(baryon) &= M_{\text{classical}} \left\{ 1 + \frac{3}{4} \left( \frac{\pi}{2N_C} \right)^2 \left[ C_2(R) - \frac{N_C^2}{2} (N_f - 1) + \frac{1}{2m^2} \sum_i \tilde{m}_i^2 \right] \right\},
\end{align*}
\]

In the following we discuss the correction terms to the earlier expression (163) for the 10 baryon as compared to the last improved expression (188). The quantum correction of (163) is multiplied by 3/4 and the last two terms in (188) are new contributions due to the GSG ansatz used and the unequal quark mass terms. The last term contribution in (186) was simplified providing a numerical term 3/2 in (188) thanks to unitarity and the relationship between the quark masses (86) which is a condition to get the relevant soliton solution. This term apparently may not be consistent with a quantum correction around the classical solution since consistency with the semi-classical approximation requires it be small compared to one. However, this term must be combined with the third term which gives a negative value contribution and is an additional term independent of \( N_C \), as is the last numerical 3/2 term under discussion. In fact, for \( N_C = 3, N_f = 3 \), numerically these two terms contribute \( \sim 0.27 \), which is acceptable. The \( N_C \) dependent term numerically becomes \( \sim 0.62 \) (the term 0.82 of (163) has been multiplied by 3/4). Adding all the quantum contributions one has 0.89, which increases the earlier numerical value 0.86 of (163) in \( \sim 3.5\% \). In fact, this is a small correction to the already known value which was obtained using the ansatz (62) in [20, 21].
7.3 Possible vibrational modes and the GSG model

The only static soliton configurations with baryon number $N_C$, which emerge in the strong-coupling regime of QCD$_2$, are the ones we have considered above in eqs. (168)-(170). Precisely, these are the one-solitons of the GSG model which, in subsection 7.2, have been the subject of semi-classical treatment. Their quantum corrections by time-dependent rotations in flavor space have been computed, we focused on the one with the lowest classical mass. Since in two dimensions there are no spin degrees of freedom, in order to search for higher excitations we must look for vibrational modes which might in principle exist. These type of excitations in the strong coupling limit can be found as classical time-dependent solutions of the GSG equations of motion (11)-(12). Looking at time-dependent solutions of type (169) [see eq. (84)] one has that the field $\varphi_2$ satisfies ordinary sine-Gordon equation

$$\partial_t \varphi_2 - \partial_{xx} \varphi_2 + 2\tilde{m}^2 \sqrt{\frac{4\pi}{N_C}} \sin\left(\frac{4\pi}{N_C} \varphi_2\right), \quad \varphi_1(x,t) \equiv 0. \quad (189)$$

The time dependent one-soliton solution of (189) for the field $\varphi_2$, determines the configuration $\{\Phi_1, \Phi_2, \Phi_3\}$ in (169) with baryon number $N_C$ in the QCD$_2$ context.

To look for higher excitations, for example, one can search for a coupled state of one-baryon and breather type vibrations (soliton-antisoliton bound states) of the GSG system, which can give a total baryon number $N_C$. We were not be able to find a more general time-dependent mixed single-baryon plus vibrational state with baryon number $N_C$ for the general GSG equation. For example, this type of solution, if it exists, may be useful in order to study meson-baryon scattering as considered in [24]. As it is well known the SG eq. (189) does give vibrational solutions in the form of breather states (meson states), for later use we simply recall that in the large $N_C$ limit the lowest-lying mesons have masses of order $\sqrt{m_q e_c}$ [46] ($m_q$ is defined in eq. (194) below). We refer the reader to ref. [21] for more discussion, such as the various meson couplings to baryons with different degrees of exoticity.

8 The GSG solitons and the exotic baryons

8.1 The first exotic baryon

Here we will follow the analog of the rigid-rotor approach (RRA) to quantize solitons and obtain exotic states. In this method it is assumed that the higher order representation multiplets are different rotational (in spin and isospin) states of the same object (the “classical baryon”, i.e the soliton field) [31]. This assumption has allowed in the past the obtention of some relations between the characteristics of the nonexotic baryon multiplets which are satisfied up to a few percent in nature. However, see refs. [47, 48] for some critiques to this conventional approach for exotic baryons. According to these authors the conventional RRA, in which the collective
Generalized sine-Gordon model and baryons in QCD

rotational approach and vibrational modes of the soliton are assumed to be decoupled, and only the rotational modes are quantized, is only justified at large $N_C$ for nonexotic collective states in $SU(3)$ models. On the other hand, the bound state approach (BSA) to quantize solitons, due to Callan-Klebanov [48], considers broken $SU(3)$ symmetry in which the excitations carrying strangeness are taken as vibrational modes, and should be quantized as harmonic vibrations. However, for exotic states the Callan-Klebanov approach does not reproduce the RRA result; indeed this approach gives no exotic resonant states when applied to the original Skyrme model [48]. There was intensive discussion of connections between the both approaches mentioned above. The rotation-vibration approach (RVA) (see [49] and references therein) includes both rotational (zero modes) and vibrational degrees of freedom of solitons and is a generalization of the both methods above, which therefore appear in some regions of the RVA method when certain degrees of freedom are frozen. A major result of the RVA method is that pentaquark states do indeed emerge in both methods above, i.e. in the RRA and BSA. In order to illustrate the present situation of the theoretical controversy let us mention that the RVA approach was criticized in [50], and the reply to this criticism was given in [51].

Following the analog of the RRA, the expression (186) can be used to compute the energy of the first exotic baryon $E_1$ (a state containing $N_C + 1$ quarks and one antiquark) by taking the corresponding Casimir $C_2(E_1)$ for $R = 35$ of flavor relevant to the exotic state in two-dimensions. This state is an analogue of the $\bar{10}, 27$ and $35$ states in four dimensions. So, following [21], in the conventional RRA one has that the mass of the first exotic state becomes

$$M(E_1) = M(\text{classical})\left\{1 + \frac{3}{4} \left[\frac{\pi^2}{8} \frac{1}{N_C} \left(3 + N_f - \frac{6}{N_f}\right) + \frac{3\pi^2}{8} \frac{1}{N_C} \left(N_f - \frac{3}{N_f}\right)\right] - \frac{3\pi^2}{32} \frac{(N_f - 1)^2}{N_f} + \frac{3}{2}\right\}(190)$$

In the interesting case $N_C = 3$, $N_f = 3$ this becomes

$$M(35) = M(\text{classical})\left\{1 + \frac{3\pi^2}{4} - \frac{\pi^2}{8} + \frac{3}{2}\right\}.$$ (191)

In this case the correction due to quantum fluctuations around the classical solution is still larger than the classical term, as it was in the earlier computation (165). However, numerically in eq. (191) the correction is 2.12, whereas in eq. (165) it was 2.46. In fact, the contribution in (191) decreases in 0.34 units the earlier computation. So, we may claim that the introduction of unequal quark masses and the ansatz given by the GSG model slightly improve the semi-classical approximation.

Moreover, notice that the ratio of the experimental masses of the $\Theta^+(1530)$ and the nucleon is 1.63. On the other hand, the ratio of the first exotic to that of the
lightest baryon in the QCD\textsubscript{2} model becomes
\[
\frac{M_{35}}{M_{10}} = \frac{1 + \frac{3\pi^2}{16} - \frac{\pi^2}{8} + \frac{3}{2}}{1 + \pi^2/16 - \pi^2/8 + \frac{3}{2}} \sim 1.65,
\]
which is only 1\% larger to its 4D analog. This must be compared to the earlier calculation which gave a value 17\% larger [see eq. (165)]. However, the result in (192) could be a numerical coincidence, since in two dimensions we are not considering the spin degrees of freedom that is important in QCD\textsubscript{4}, even though the effects of unequal quark masses \(m_3 < m_1 < m_2\) have been incorporated as an exact (without using perturbation theory) contribution to the hamiltonian.

8.2 Exotic baryon higher multiplets

Let us consider exotic states \(E_p\) containing \(p\) antiquarks and \(N_C + p\) quarks. In the case \(N_C = 3, N_f = 3\), the only allowed \(E_2\) state is a 81 representation of flavor. In the particular case \(N_f = 3\), for general \(N_C\) the mass of the \(E_p\) state is
\[
M(E_p) = M(\text{classical}) \left\{ 1 + \frac{3}{4} \left( \frac{\pi}{2N_C} \right)^2 \left[ N_C(p + 1) + p(p + 2) - \frac{2}{3} N_C^2 \right] \right\} + 3/2, \tag{193}
\]
where the correction is considerably larger than unity. For example for \(N_C = 3\) the mass correction becomes 3.76 units. Even though this correction is one unit less than the one obtained in [21], we would not consider it as a consistent semi-classical approximation for \(N_C = 3\). However, we may consider the spacing \(\Delta\) between \(E_{p+1}\) and \(E_p\) exotic states, which for large \(N_C\) becomes
\[
\Delta \equiv E_{p+1} - E_p = \left( \frac{3}{4} \right) \frac{\pi^2}{N_C} \frac{M_{\text{classical}}}{N_C} \sim 3.8 \sqrt{e_c m_q}; \quad m_q \equiv \frac{2m_1 + m_2}{3}, \tag{194}
\]
so, the constant \(\Delta\) of [21] is decreased by a factor of 3/4. Since \(M_{\text{classical}}\) is \(O(N_C^1)\), then the parameter \(\Delta\) is a constant \(O(N_C^0)\) as the exoticaity \(p\) is increased. Notice that the low-lying mesons masses are \(O(N_C^0)\) in the large \(N_C\) limit [21]. This would mean that the constant \(\Delta\) value is like the addition of a meson to the \(p\)-state, in the form of quark-antiquark pair, in order to progress to the next excitation \(p + 1\) [52]. Remember that the low-lying mesons in the SG theory have masses \(\sim 3.2 \sqrt{m_q e_c}\) [46], which are very close to the spacing \(\Delta\) defined in (194).

8.3 Radius parameter of the QCD\textsubscript{2} exotic baryons

In QCD\textsubscript{2}, as found above, the quantum correction to the mass depends on one analogue of the moment of inertia appearing in four dimensions. Following [21] one considers
\[
I = M(\text{classical}) < r^2 >, \tag{195}
\]
the effective soliton radius can be defined by

$$<r>\equiv \sqrt{<r^2>}. \quad (196)$$

Let us compare the quantum mass formula (193) with the corresponding relation in four dimensions [31] in the large $N_C$ limit ($N_C >> p >> 1$), so one has

$$I = \frac{8N_C^2}{3\pi^2 M_{\text{classical}}}, \quad (197)$$

and then

$$<r> = \sqrt{\frac{I}{M_{\text{classical}}}} = \sqrt{\frac{8}{3\pi} \frac{N}{M_{\text{classical}}}} = \frac{1}{0.96\pi N_f^{1/4} e_c^2 m_q}, \quad (198)$$

where $m_q$ was defined in (194). For $N_f = 3$ flavors, $e_c = 100\, MeV$ for the coupling, and quark masses $m_3 = 4\, MeV$, $m_1 = 54.5\, MeV$, and $m_2 = 55.1\, MeV$ [these values satisfy the relationship $13m_3 = 5m_1 - 4m_2$ relevant in two-dimensions as is obtained from (86) and (115)], we get for the effective baryon radius $\approx 1/(294\, MeV) \sim 0.7$ fm. This is 12.5% less than the radius estimated in [21] for QCD$_2$ exotic baryons. As a curiosity, notice that the radius parameter of $\Theta^+$ has been estimated to be around $1.13$ fm $= 5.65$ GeV$^{-1}$ (see e.g. [53] and references therein).

9 Discussion

The generalized sine-Gordon model GSG (11)-(12) provides a variety of soliton and kink type solutions. The appearance of the non-integrable double sine-Gordon model as a sub-model of the GSG model suggests that this model is a non-integrable theory for the arbitrary set of values of the parameter space. However, a subset of values in parameter space determine some reduced sub-models which are integrable, e.g. the sine-Gordon submodels of subsections 5.1, 5.2 and 5.3.

In connection to the ATM spinors it was suggested that they are confined inside the GSG solitons and kinks since the gauge fixing procedure does not alter the $U(1)$ and topological currents equivalence (36). Then, in order to observe the bag model confinement mechanism it is not necessary to solve for the spinor fields since it naturally arises from the currents equivalence relation. In this way our model presents a bag model like confinement mechanism as is expected in QCD.

Besides, through the bosonization process it has been shown that the (generalized) massive Thirring model (GMT) corresponds to the GSG model [7], therefore, in view of the GSG solitons and kinks found above we expect that the spectrum of the GMT model will contain 4 solitons and their relevant anti-solitons, as well as the kink and antikink excitations. The GMT Lagrangian describes three flavor massive spinors with current-current interactions among themselves. So, the total
number of solitons which appear in the bosonized sector suggests that the additional soliton (fermion) is formed due to the interactions between the currents in the GMT sector. However, in subsection 5.3 the soliton masses $M_3$ and $M_4$ become the same for the case $\mu_1 = \mu_2$, consequently, for this case we have just three solitons in the GSG spectrum, i.e., the ones with masses $M_1$, $M_2$ (subsections 5.1-5.2) and $M_3 = M_4$ (subsection 5.3), which will correspond in this case to each fermion flavor of the GMT model. Moreover, the $sl(3,C)$ GSG model potential (7) has the same structure as the effective Lagrangian of the massive Schwinger model with $N_f = 3$ fermions, for a convenient value of the vacuum angle $\theta$. The multiflavor Schwinger model resembles with four-dimensional QCD in many respects (see e.g. [54] and references therein).

In view of the discussions above the $sl(n,C)$ ATN models may be relevant in the construction of the low-energy effective theories of multflavor QCD with the dynamical fermions in the fundamental and adjoint representations. Notice that in the ATN models the Noether and topological currents and the generalized sine-Gordon/massive Thirring models equivalences take place at the classical [6, 29] and quantum mechanical level [7, 30].

On the other hand, the interest in baryons with exotic quantum numbers has recently been stimulated by various reports of baryons composed by four quarks and an antiquark. The existence of these baryons cannot yet be regarded as confirmed, however, reports of their existence have stimulated new investigations about baryon structure (see e.g. [55, 16] and references therein). Recently, the spectrum of exotic baryons in QCD, with $SU(N_f)$ flavor symmetry, has been discussed providing strong support to the chiral-soliton picture for the structure of normal and exotic baryons in four dimensions [21]. The new puzzles in non-perturbative QCD are related to systems with unequal quark masses, so the QCD calculation must take into account the $SU(N_f)$-breaking mass effects, i.e. for $N_f = 3$ it must be $m_s \neq m_{u,d}$. We have extended the results of refs. [20, 21] concerning several properties of normal and exotic baryons by including unequal quark mass parameters. In the case of $N_f = 3$ flavors, the low-energy hadron states are described by the $su(3)$ generalized sine-Gordon model, providing a framework for the exact computations of the lowest-order quantum corrections of various quantities, such as the masses of the normal and exotic baryons. The semi-classical quantization method we adopted is an analogue of the rigid-rotor approach (RRA) applied in four dimensional QCD to quantize normal and exotic baryons (see e.g. [31]). Even though there is no spin in 2D, we have compared our results to their analogues in 4D; so, obtaining various similarities to the results from the chiral-soliton approaches in QCD. The RRA we have followed, as discussed in section 8, may be justified in our case since there is no mixing between the intrinsic vibrational modes and the collective rotation in flavor space degrees of freedom [47]. It is remarkable that the GSG ansatz (60), with soliton solutions which take into account the unequal quark mass parameters, allowed us to improve the lowest order quantum corrections for various physical quantities,
such as the baryon masses; in this way rendering the semi-classical method more reliable in the large $N_C$ limit. Other properties of the baryons such as a proper treatment of $k-$baryon bound states (extending the results of [42] for GSG type ansatz), including baryon-meson scattering amplitudes, are still to be addressed in the future.

Finally, we have found that the remarkable double sine Gordon model arises as a reduced GSG model bearing a kink($K$) type solution describing a multi-baryon; so, the description of the multiflavor spectrum and some resonances in QCD$_2$ may take advantage of the properties of the DSG semiclassical spectrum and $K\bar{K}$ system which are being considered in the current literature [3, 4, 39, 40].

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