AVERAGED DEHN FUNCTIONS FOR NILPOTENT GROUPS

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Abstract. Gromov proposed an averaged version of the Dehn function and claimed that in many cases it should be subasymptotic to the Dehn function. Using results on random walks in nilpotent groups, we confirm this claim for most nilpotent groups. In particular, if a nilpotent group satisfies the isoperimetric inequality \( \delta(l) < Cl^\alpha \) for \( \alpha > 2 \) then it satisfies the averaged isoperimetric inequality \( \delta_{\text{avg}}(l) < C' l^{\alpha/2} \). In the case of non-abelian free nilpotent groups, the bounds we give are asymptotically sharp.

1. Introduction

Determining the asymptotic behavior of the Dehn function for a group is a much-studied problem in group theory; see [7] for an introduction to the subject. In [10, 5.A_6(c), p. 90], Gromov proposed a variation on this problem: instead of asking for the largest area required to fill a closed curve of a given length, he asked what the average area is, taken over all closed curves. Gromov claimed that in many cases, this averaged Dehn function should be asymptotically smaller than the Dehn function, which was confirmed in the case of finite rank free abelian groups by [13], whose bound was improved by Bogopolski and Ventura [5]. Still, little is known about the averaged Dehn function; some open questions include whether it is invariant under changes of generators or quasi-isometries.

In this paper, we prove upper and lower bounds for the averaged Dehn function of a nilpotent group which show, in particular, that if \( G \) is a nilpotent group which isn’t finite or virtually \( \mathbb{Z} \), then its averaged Dehn function is subasymptotic to its Dehn function. This implies that random walks in nilpotent groups bound much less area on average than the worst-case curves. In addition, our bounds are sharp in many cases.

A related problem appears in statistical mechanics, where the behavior of a charged particle moving randomly in a magnetic field depends on the signed area of its path. The distributions of the signed area and the area of a random loop have thus been considered by mathematical physicists, see for instance, [8] and [4]. In the former, Colomo shows that our bounds break down for infinitely generated groups. He considers the area of a random loop in \( \mathbb{Z}^d \) and shows that if \( n \) is fixed and \( d \to \infty \), \( \delta_{\text{avg}}(n) \to n(n-1)/6 \), in contrast to the finite dimensional case, where \( \delta_{\text{avg}}(n) \) grows strictly subquadratically.

The averaged Dehn function can also be interpreted as reflecting properties of the average-case complexity of the word problem for a group just as the Dehn function reflects its complexity. Kapovich, Myasnikov, Schupp, and Shpilrain studied this average-case complexity by showing that in many groups, it is easy to show that

The author gratefully acknowledges support from a GAANN fellowship during part of the writing of this paper.
most elements are not the identity; the averaged Dehn function represents the complexity of verifying that an element represents the identity. One interesting question, then, might be to find groups where the averaged Dehn function differs substantially from the Dehn function; the results in this paper show that for many nilpotent groups, $\delta_{avg}(n)$ is approximately $\delta_{avg}(\sqrt{n})$.

In section 2, we define Dehn functions and averaged Dehn functions and state the upper bound. In section 3, we define the centralized isoperimetric function of a group, give a method to calculate it for a nilpotent group, and state a lower bound using this function. In sections 4 and 5, we prove these bounds.

The author would like to thank Shmuel Weinberger and Benson Farb for their helpful suggestions and to thank the referee for their detailed suggestions on improving this paper. The work in this paper was done as part of the author’s doctoral thesis at the University of Chicago.

2. Definitions

We will be using big O notation for asymptotic bounds throughout this paper. Recall that $O$ represents an asymptotic upper bound, $\Omega$ represents an asymptotic lower bound, and $\Theta$ represents an asymptotically tight bound. Specifically,

$$f(x) = O(g(x)) \text{iff } \exists M, x_0 \text{ s.t. } f(x) \leq M g(x) \text{ for } x > x_0$$

$$f(x) = \Omega(g(x)) \text{iff } \exists m, x_0 \text{ s.t. } mg(x) \leq f(x) \text{ for } x > x_0$$

$$f(x) = \Theta(g(x)) \text{iff } \exists m, M, x_0 \text{ s.t. } mg(x) \leq f(x) \leq Mg(x) \text{ for } x > x_0.$$  

One can define the Dehn function in a variety of contexts; here, we define it in terms of a presentation of a group. We first define the filling area of a word. Let $G$ be a group with identity $e$, given by a presentation $G = \{e_1, \ldots, e_d | r_1, \ldots, r_s\}$. Let $R = \langle r_1, \ldots, r_s \rangle$ be the normal closure of the relators, and $F$ be the free group on $d$ generators, so that $G = F/R$. If $w$ is a word in the $e_i^{\pm 1}$ that is the identity in $G$, then $w$ lies in $R$ when considered as an element of $F$. We can thus write

$$w = \prod_{i=1}^{k} g_i^{-1} r_i^{\pm 1} g_i,$$

where the equality is taken in $F$. Define $\delta_G(w)$ as the minimal $k$ for which we can write such a decomposition. We will drop the $G$ if context makes it clear which group is meant.

$\delta(w)$ counts the number of applications of relators required to reduce $w$ to the trivial word. We can view this as an area by considering the 2-complex obtained by taking the Cayley graph of $G$ and adding a face for each conjugate of a relator. Then $w$ represents a curve in this complex, and $\delta(w)$ represents the minimal number of faces in a disc with boundary $w$. Alternately, we can view $\delta(w)$ as a metric on $R$; it is the word metric on $R$ corresponding to the (usually infinite) generating set consisting of all conjugates of the $r_i$.

We define the Dehn function of the presentation as the maximum filling area for words shorter than a given length. That is,

$$\delta(n) = \max_{w \in R \cap B(n)} \delta(w),$$
where $B(n)$ denotes the ball of radius $n$ in $F$. This depends on the choice of presentation, but changing the presentation changes the asymptotics only minimally. In particular, if $\delta(n) = \Theta(n^k)$, $k > 1$ for one finite presentation, the same is true for any other finite presentation of $G$. If we view $\delta(w)$ as a distance function on $R$ as above, then the Dehn function of a group measures the distortion of $\delta(w)$ compared to the metric induced on $R$ by inclusion in $F$.

Instead of taking the maximum, however, we can average the filling area over all words shorter than a given length that represent the identity to get an averaged version, $\delta_{avg}$ of the Dehn function. In order to apply results on random walks, we will average over all “lazy” words of exactly a given length, that is, words of the form $a_1 \ldots a_n$, where $a_j \in \{e_1^{-1}, e, e_1^{+1}\}$.

We will define $\delta_{avg}$ using random walks. Let $p \in M(G)$ be the measure

$$p(g) = \begin{cases} \frac{1}{2d+1} & g = e \text{ or } g = e_1^{\pm 1} \\ 0 & \text{otherwise.} \end{cases}$$

(For our purposes, any finitely supported probability measure such that $p(e) > 0$, $p(x) = p(x^{-1})$, and the support of $p$ generates $G$ will suffice.) We use $p$ to construct a random walk where at each step, the probability of moving from $g$ to $gh$ is given by $p(h)$. Then $p^{(n)}(x)$, the $n$th convolution power of $p$, is the probability that an $n$-step random walk starting at $e$ ends at $x$. We also define $p^{(n)}(x, y) \equiv p^{(n)}(x^{-1}y)$, the probability of going from $x$ to $y$ in $n$ steps and $p(x, y) \equiv p^{(1)}(x, y)$.

Define a measure $\rho_n$ on $G^n = G \times \cdots \times G$ by

$$\rho_n(g_1, \ldots, g_n) = \prod_{i=1}^{n-1} p(g_i).$$

Then $\rho_n(g_1, \ldots, g_n)$ represents the probability that an $n$-step random walk starting at $e$ goes to $g_1$, then $g_1g_2$, and so on, ending at $g_1 \ldots g_n$. We consider this as a probability measure on the set of lazy words of length $n$, where $(g_1, g_2, \ldots, g_n) \in G^n$ corresponds to the unreduced lazy word $g_1g_2 \ldots g_n$.

We can then define our averaged Dehn function by considering the measure $\rho_n((g_1, \ldots, g_n)_{g_1 \ldots g_n = e})$. This measure is nonzero, since $\rho_n(e, e, \ldots, e) = p(e)^n > 0$, so we can normalize it to a probability measure $\overline{\rho}_n$ and consider its support as the set of lazy words of length $n$ which are the identity in $G$. We then define the averaged Dehn function

$$\delta_{avg}(n) = E_{\overline{\rho}_n}(\delta(w)) = \sum_w \delta(w)\overline{\rho}_n(w)$$

as the expected area necessary to fill a random closed curve of length $n$. With the choice of measure given above, the probability of any lazy word is the same, and this average is the same as averaging over all lazy words using the counting measure.

Like the Dehn function, the averaged Dehn function depends a priori on the given presentation. For nilpotent groups, we will show the following upper bound, which is independent of the presentation:

**Theorem 1.** If $G$ is a finitely generated nilpotent group with Dehn function $\delta(n) = O(n^k)$, then if $k > 2$, its averaged Dehn function for any presentation satisfies $\delta_{avg}(n) = O(n^{k/2})$, and if $k = 2$, $\delta_{avg}(n) = O(n \log n)$.
By a theorem of Gersten, Holt, and Riley [9], all nilpotent groups have Dehn functions bounded above by a polynomial, so as a consequence, nilpotent groups with Dehn function growing at least quadratically have averaged Dehn function strictly subasymptotic to their Dehn function. On the other hand, if a group has subquadratic Dehn function, it is hyperbolic by a theorem of Gromov, one proof of which can be found in [6]. Since nilpotent groups are amenable, a nilpotent hyperbolic group must be finite or virtually $\mathbb{Z}$, so if $G$ is a nilpotent group which is not finite or virtually $\mathbb{Z}$, its averaged Dehn function is strictly subasymptotic to its Dehn function.

3. Centralized Isoperimetry

The lower bound we will give uses the centralized isoperimetric function defined by Baumslag, Miller and Short [3]; we recall the definition. Let $G = F/R = \{e_1, \ldots, e_d | r_1, \ldots, r_s\}$ as above. If $w$ is a word which is the identity in $G$, we can write

$$w \in \prod_{i=1}^k g_i^{-1} r_i^{\pm 1} g_i[R, F] = \prod_{i=1}^k r_i^{\pm 1} [R, F] = \prod_{i=1}^s b_i [R, F],$$

using the fact that $R[R, F] \subset Z(F/[R, F])$. Define $\delta^\text{cent}_G(w)$ as the minimal $k$ for which we can write such a decomposition (equivalently, the minimal $\sum_{i=1}^s |b_i|$), and for $n \in \mathbb{N}$, define

$$\delta^\text{cent}_G(n) = \max_{w \in R/\{e, w\} \cap B(n)} \delta^\text{cent}_G(w).$$

As before, we will drop the $G$ if the group is clear. This depends a priori on the choice of presentation, but Baumslag, Miller, and Short [3] prove that, like the Dehn function, changing the presentation changes the asymptotics only minimally. In particular, as for the Dehn function, if $\delta^\text{cent}(n) = \Theta(n^k), k > 1$ for one finite presentation, the same is true for any other finite presentation of $G$.

Equivalently, give $F/[R, F]$ the generating set $\{e_1, \ldots, e_d\}$ and give $R/[R, F]$ the generating set $\{r_1, \ldots, r_s\}$. If we denote the distance functions induced by these generators by $d_{F/[R, F]}$ and $d_{R/[R, F]}$, then

$$\delta^\text{cent}(w) = d_{R/[R, F]}(e, w)$$

and

$$\delta^\text{cent}(n) = \max_{w \in R/[R, F] \cap B_{F/[R, F]}(n)} d_{R/[R, F]}(e, w).$$

Thus, in the same way that the Dehn function measures the distortion of the inclusion $R \subset F$ for the metric on $R$ induced by the generating set $\{g^{-1} r_i g\}_{g \in G, 1 \leq i \leq s}$, the centralized isoperimetric function measures the distortion of the inclusion $R/[R, F] \subset F/[R, F]$ for the metric on $R/[R, F]$ induced by the generating set $\{r_i\}_{1 \leq i \leq s}$. Since $R/[R, F]$ is finitely generated and abelian, $\delta^\text{cent}$ is generally easier to calculate than $\delta$ and provides a lower bound for it, since

$$\delta^\text{cent}(w) \leq \delta(w).$$

We can now state our lower bound on the averaged isoperimetric function:

**Theorem 2.** If $G$ is a finitely generated nilpotent group with centralized isoperimetric function $\delta^\text{cent}(n) = \Omega(n^k)$ for $k \geq 2$, then its averaged Dehn function for any presentation satisfies $\delta^\text{avg}(n) = \Omega(n^{k/2})$. 
Many nilpotent groups have $\delta_{\text{cent}}(n)$ and $\delta(n)$ both polynomial of the same degree. If in addition, this degree is $> 2$, our upper and lower bounds are sharp and independent of the presentation of the group.

In the remainder of this section, we will prove some results on centralized isoperimetric functions of nilpotent groups which will be useful in the proof of Theorem [2]. If $G = F/R$ is a finitely generated nilpotent group, $F/[R, F]$ is nilpotent and in fact a central extension of $F/[R, F]$. The asymptotics of $\delta_{\text{cent}}$ are then relatively straightforward to calculate. Since $R/[R, F] \subset F/[R, F]$ is the inclusion of an abelian group into a nilpotent group, we can apply a special case of a theorem of Osin [14]:

**Theorem 3.** Let $G$ be a f.g. nilpotent group, $H$ be an abelian subgroup of $G$, and $H^0$ the set of all elements of infinite order in $H$. If $k$ is maximal such that $H^0 \cap G^{(k)} \neq \{e\}$, then

$$\max_{h \in H \cap B_F(n)} d_H(h, e) = \Theta(n^k).$$

In particular, if $k$ is as above, $w \in H^0 \cap G^{(k)}$ and $w \neq e$, then $d_H(w^{n^k}, e) = \Theta(n)$.

where $G^{(\cdot)}$ is the lower central series of $G$,

$$G^{(1)} = G,$$

$$G^{(n)} = [G, [G^{(n-1)}]].$$

We will call $k$ the degree of distortion of $H$ in $G$. Then

$$\delta_{\text{cent}}(n) = \max_{w \in R/[R, F] \cap B_{F/[R, F]}(n)} d_{R/[R, F]}(e, w).$$

$$= \Theta(n^k).$$

where $k$ is the degree of distortion of $R/[R, F]$ in $F/[R, F]$.

In fact, we will show that any central extension of $G$ provides a lower bound on its centralized isoperimetric function and that in fact $\delta_{\text{cent}}(n)$ can be calculated by considering just central extensions of $G$ by $\mathbb{Z}$. These lower bounds are closely related to the lower bounds for the centralized isometric function found in Theorem 8 of [3] and in [15]. We will prove the following proposition, which we will use in proving a lower bound on the averaged isoperimetric function:

**Proposition 4.** If $G$ is a finitely generated nilpotent group, $A$ is a finitely generated abelian group,

$$0 \to A \to H \to G \to 1$$

is a central extension of $G$, and $k$ is the degree of distortion of $A$ in $H$, then if $k \geq 2$ (in particular, the extension must be nontrivial), then $\delta_{\text{cent}}^G(n) = \Omega(n^k)$.

If $G$ is a finitely generated nilpotent group and $\delta_{\text{cent}}^G(n) = \Theta(n^k)$, then there is a central extension

$$0 \to \mathbb{Z} \to H \to G \to 1$$

such that the degree of distortion of $\mathbb{Z}$ in $H$ is $k$.

We prove the proposition by using the fact that a central extension by $A$ can be described by a map $R/[R, F] \to A$. Recall that any central extension of a nilpotent group is again nilpotent.
Lemma 5. If \( G = F/R = \{ e_1, \ldots, e_d \} \), \( A \) is abelian and
\[
0 \rightarrow A \overset{i}{\rightarrow} H \overset{p}{\rightarrow} G \rightarrow 1
\]
is a central extension of \( G \) by \( A \), then there are maps \( \alpha : R/[R, F] \rightarrow A \) and \( \beta : F/[R, F] \rightarrow H \) such that
\[
0 \rightarrow R/[R, F] \overset{\alpha}{\rightarrow} F/[R, F] \overset{\beta}{\rightarrow} G \rightarrow 1
\]
commutes and image \( \alpha \supseteq A \cap [H, H] \).

Proof of lemma. Choose \( e'_1, \ldots, e'_d \in H \) such that \( p(e'_i) = e_i \). This defines a map \( f : F \rightarrow H \) which sends \( e_i \) to \( e'_i \). We will find \( \alpha \) and \( \beta \) from this map. First, note that \( f(R) \subset \ker p \), so \( f(R) \subset A \). Since \( A \) is in the center of \( H \), \( f([R, F]) \subset [A, H] = \{ 1 \} \) and so we can define \( \beta : F/[R, F] \rightarrow H \) as the quotient of \( f \) by \([R, F]\) and \( \alpha : R/[R, F] \rightarrow A \) as the restriction of \( \beta \) to \([R, F]\). These maps make the diagram commute. Finally, if \( \prod_{i=1}^n [h_{i,1}, h_{i,2}] \in A \cap [H, H] \) for \( h_{i,j} \in H \), choose \( h_{i,j}' \in F \) such that \( p(h_{i,j}')) = p(h_{i,j}) \) (possible because \( p \circ f \) is surjective). Then
\[
p \left( f \left( \prod_{i=1}^n [h_{i,1}', h_{i,2}'] \right) \right) = p \left( \prod_{i=1}^n [h_{i,1}, h_{i,2}] \right) = 1
\]
and so \( \prod_{i=1}^n [h_{i,1}', h_{i,2}'] \in R \). Since \( f(h_{i,j}') = h_{i,j} a_{i,j} \) for some \( a_{i,j} \in A \),
\[
f \left( \prod_{i=1}^n [h_{i,1}', h_{i,2}'] \right) = \prod_{i=1}^n [h_{i,1} a_{i,1}, h_{i,2} a_{i,2}] = \prod_{i=1}^n [h_{i,1}, h_{i,2}]
\]
and so \( \prod_{i=1}^n [h_{i,1}, h_{i,2}] \in \text{image } \alpha \) as desired. \( \square \)

Proof of proposition. For the first part, let \( \widetilde{A} = \text{image } \alpha \). Note that \( \widetilde{A} \supseteq A \cap [H, H] \), so for all \( j \geq 2 \), we have \( \widetilde{A} \cap H^{(j)} = A^{(j)} \cap H^{(j)} \). Since the degree of distortion of \( A \) in \( H \) is \( \geq 2 \), it is the same as that of \( \widetilde{A} \) in \( H \) and so
\[
\max_{a \in \widetilde{A} \cap B_H(n)} d_{\widetilde{A}}(a, e) = \Theta(n^k).
\]
However, \( \alpha \) cannot increase distances by more than a constant multiple, so
\[
d_{\widetilde{A}}(\alpha(w), e) \leq C d_{R/[R, F]}(w, e),
\]
and
\[
\delta_{\text{cent}}(n) = \max_{w \in R/[R, F] \cap B_{F/[R, F]}(n)} d_{R/[R, F]}(e, w).
\]
\[
\geq \frac{1}{C} \max_{a \in \widetilde{A} \cap B_H(n)} d_{\widetilde{A}}(a, e)
\]
\[
= \Omega(n^k).
\]
When applied to extensions by \( \mathbb{Z} \), this bound is a discrete analogue of the lower bound found by Pittet in [13].

For the second part, suppose that \( \delta_{G}^{\text{cent}}(l) = \Theta(l^k) \). We want to find a central extension by \( \mathbb{Z} \) giving this bound. Since the degree of distortion of \( R/[R, F] \) in \( F/[R, F] \) is at least \( k \), there is an element \( z \) of \( R/[R, F] \) of infinite order so that
Let $z \in (F/[R, F])^{(k)}$. Let $\alpha : R/[R, F] \to Z$ be a map such that $\alpha(z) \neq 0$. We can construct a central extension of $G$ by $Z$ such that

$$
\begin{array}{cccccc}
0 & \longrightarrow & R/[R, F] & \longrightarrow & F/[R, F] & \longrightarrow & G & \longrightarrow & 1 \\
\downarrow & & \downarrow \alpha & & \downarrow z & & \approx \\
0 & \longrightarrow & Z & \longrightarrow & H & \longrightarrow & G & \longrightarrow & 1
\end{array}
$$

commutes by letting $H = (F/[R, F])/(\ker \alpha)$. Then $\alpha(z) \in Z$ is of infinite order and $\alpha(z) \in H^{(k)}$, so

$$
\max_{a \in Z \cap B_H(n)} d_Z(a, e) = \Omega(n^k).
$$

Thus, to compute $\delta^\text{cent}$, it suffices to consider central extensions of $G$ by $Z$. □

## 4. Lower Bounds

Here we extend the bounds in Section 3 to the averaged Dehn function. We will be using theorems on the behavior of random walks on nilpotent groups, most notably a theorem of Hebisch and Saloff-Coste:

**Theorem 6 ([11]).** Let $G$ be a finitely generated group with polynomial volume growth of order $D$. Let $p$ be a finitely supported probability measure such that $p(e) > 0$, $p(x) = p(x^{-1})$, and the support of $p$ generates $G$. Then there exist three positive constants $C, C', C''$ such that, for all $x \in G$ and all integers $n$, we have

$$
p^{(n)}(x) \leq C n^{-D/2} \exp \left( -d(e, x)^2 / C'n \right)
$$

$$
p^{(n)}(x) \geq (Cn)^{-D/2} \exp \left( -C'd(e, x)^2 / n \right)
$$

if $x \in B(e, n/C'')$, where $d(e, x)$ denotes distance in the word metric corresponding to the support of $p$ and $B(e, r)$ is the ball of radius $r$ around $e$ in this metric.

We will prove Theorem 2 by applying Theorem 6 to a suitable extension of $G$. We first sketch an outline of the proof, then fill in the calculational details. By Proposition 4, we know that if $G$ is a finitely generated nilpotent group and $\delta^\text{cent}_G(n) = \Theta(n^k)$, then there is a central extension

$$
0 \to Z \to H \to G \to 1
$$

such that

$$
\max_{a \in Z \cap B_H(n)} d_Z(a, e) = \Omega(n^k).
$$

If we choose $a'_i \in H$ that project to a set of generators $a_i \in G$, then lazy words in the $a_i$ that are the identity in $G$ correspond exactly to lazy words in the $a'_i$ representing elements in the image of $Z$. Theorem 6 suggests that a random lazy word of length $n$ in a nilpotent group will represent an element of distance on average $\sim \sqrt{n}$ from the identity. Because of the order $k$ distortion of $Z$ in $H$, this suggests that, on average, those lazy words which are elements of $Z$ will represent elements of distance $\sim n^{k/2}$ from the identity in $Z$. Such a word $w$ will have $\delta^\text{cent}_G(w) \approx n^{k/2}/c$. Formalizing this argument and carefully calculating the probabilities will give the lower bound we want.
Proof of Theorem 2. Construct $H$ and $a'_i$ as above. One minor problem is that the $a'_i$ may not generate $H$, but unless $H = G \times \mathbb{Z}$ (in which case $k = 1$), the subgroup generated by the $a'_i$ is finite index in $H$ and we can replace $H$ by this subgroup.

Lazy words of length $n$ in $H$ will represent powers $z^l$ of $z$; we want to calculate the expected absolute value of $l$. Letting $p_H$ be the measure corresponding to the random walk on $H$ with generating set $\{a'_1, \ldots, a'_d\}$, this is

$$\sum_{l \in \mathbb{Z}} |l| \cdot p_H^{(n)}(e, z^l),$$

$$\sum_{l \in \mathbb{Z}} p_H^{(n)}(e, z^l).$$

Note that since $G$ is nilpotent, it has polynomial volume growth, say of order $D$. We thus apply Theorem 6 to find

$$\sum_{l \in \mathbb{Z}} |l| \cdot p_H^{(n)}(e, z^l),$$

$$\sum_{l \in \mathbb{Z}} p_H^{(n)}(e, z^l) \geq \sum_{z^l \in B(e, n/C'')} |l| \cdot (Cn)^{-D/2} \exp \left( -d(e, z^l)^2 / C' n \right)$$

For clarity, we will replace expressions not depending on $n$ with positive constants $c_i$.

$$\sum_{l \in \mathbb{Z}} |l| \cdot p_H^{(n)}(e, z^l),$$

$$\sum_{l \in \mathbb{Z}} p_H^{(n)}(e, z^l) \geq c_0 \sum_{|l| \leq c_1 n^k} |l| \cdot \exp \left( -c_2 |l|^{1/k} / n \right),$$

$$\sum_{l \in \mathbb{Z}} \exp \left( -|l|^{1/k} / c_2 n \right) \geq c_0 \sum_{l=0}^{c_1 n^k} l \cdot \exp \left( -c_2 l^{2/k} / n \right) \geq c_0 \sum_{l=0}^{c_1 n^k} \exp \left( -l^{2/k} / c_2 n \right) \geq c_0 \int_0^{c_1 n^k} l \cdot \exp \left( -c_2 l^{2/k} / n \right) \, dl + O(n^{k/2})$$

We make the substitutions $x = (c_2/n)^{k/2} l$ and $y = (1/c_2 n)^{k/2} l$.

$$\sum_{l \in \mathbb{Z}} |l| \cdot p_H^{(n)}(e, z^l),$$

$$\sum_{l \in \mathbb{Z}} p_H^{(n)}(e, z^l) \geq c_3 n^k \int_0^{c_4 n^{k/2}} x \cdot \exp \left( -x^{2/k} \right) \, dx + O(n^{k/2})$$

$$= \Omega(n^{k/2})$$

□

5. Upper Bounds

To obtain upper bounds, we will construct discs filling in random loops and bound their expected areas using Theorem 6.
Figure 1. Filling a disc with triangles

Let \( w = a_1 a_2 \ldots a_n = e \) be a word with \( a_i \in \{ e^{\pm 1}, \ldots, e^{\pm d} \} \). We will think of \( w \) as a path in the Cayley graph and write \( w(i) = a_1 \ldots a_i \). Fix shortest paths \( \gamma_{x,y} \) between each pair of elements \( x, y \) of \( G \) so that \( \gamma_{x,y} \) is \( \gamma_{y,x} \) traced backwards.

Let \( w_{i,j} = w(i) \left( \frac{jn}{2^i} \right) \).

for \( i \geq 1, 0 \leq j \leq 2^i \). Note that \( w_{i,j} = w_{i+1,2j} \) and that for any \( i, w_{i,1}, \ldots, w_{i,2^i} \) is an approximation of \( w \). We will inductively build a sequence of fillings such that the boundary of the \( i \)th filling is \( w_{i+1,1}, \ldots, w_{i,2^i+1} \) by gluing triangles as in Figure 1.

We start with two discs filling
\[
\gamma_{w_{1,0},w_{2,1}} \gamma_{w_{2,1},w_{1,1}} \gamma_{w_{1,1},w_{1,0}}
\]
and
\[
\gamma_{w_{1,1},w_{2,3}} \gamma_{w_{2,3},w_{1,2}} \gamma_{w_{1,2},w_{1,1}}
\]
Gluing these discs together, we obtain a filling of
\[
\gamma_{w_{1,0},w_{2,1}} \gamma_{w_{2,1},w_{1,1}} \gamma_{w_{1,1},w_{2,3}} \gamma_{w_{2,3},w_{1,2}} \gamma_{w_{1,2},w_{1,0}} = \gamma_{w_{2,0},w_{2,1}} \gamma_{w_{2,1},w_{2,2}} \gamma_{w_{2,2},w_{2,3}} \gamma_{w_{2,3},w_{2,4}}
\]
After the \( i \)th step, we have a disc filling the geodesic \( 2^{i+1} \)-gon with vertices
\( w_{i+1,1}, \ldots, w_{i+1,2^{i+1}} \).

To refine this to a filling of the next polygon, we add \( 2^{i+1} \) discs, filling the geodesic triangles with vertices
\[
w_{i+1,j} = w_{i+2,2j}, w_{i+2,2j+1}, w_{i+1,j+1} = w_{i+1,2j+2}.
\]
Finally, after the $\lfloor \log_2 n \rfloor$ th step, the boundary is almost $w$. We can apply a number of relators linear in $n$ to get $w$ exactly.

It remains to estimate the total area of the triangles in the $\lfloor \log_2 n \rfloor$ th filling.

**Proof of Theorem 1.** For a random closed path $w$ of length $n$ in $G$, we can construct a disc filling $w$ using the process above. We must find a rigorous estimate of the area. If $x, y, z \in G$, define $\Delta(x, y, z)$ to be the filling area of a geodesic triangle with vertices $x, y, z$, that is,

$$\Delta(x, y, z) = \delta(\gamma_x \gamma_y \gamma_z \gamma_x).$$

The process above gives a bound

$$\delta(w) \leq cn + \sum_{i=1}^{\lfloor \log_2 n \rfloor} \sum_{j=0}^{2^i-1} \Delta(w_{i,j}, w_{i+1,2j+1}, w_{i,j+1})$$

Thus the expected area of a random word of length $n$ is at most

$$\delta_{\text{avg}}(n) = E(\delta(w)) \leq E \left( cn + \sum_{i=1}^{\lfloor \log_2 n \rfloor} \sum_{j=0}^{2^i-1} \Delta(w_{i,j}, w_{i+1,2j+1}, w_{i,j+1}) \right)$$

$$= cn + \sum_{i=1}^{\lfloor \log_2 n \rfloor} \sum_{j=0}^{2^i-1} E(\Delta(w_{i,j}, w_{i+1,2j+1}, w_{i,j+1})),
$$

where the expectations are taken with respect to $\mathbb{P}_n(\delta(w))$. We can bound the averaged Dehn function by bounding the expected area of each triangle. Consider $E(\Delta(w(r), w(s), w(t)))$, the expected area of the triangle with vertices $w(r), w(s)$, and $w(t)$. Since the Dehn function of $G$ is $O(n^k)$,

$$\Delta(w(r), w(s), w(t)) \leq c_3(d(w(r), w(s)) + d(w(s), w(t)) + d(w(t), w(r)))^k,$$

for some constant $c_3$, that is, the expected area is bounded by the $k$th moment of the perimeter. This can be bounded with a classical inequality proved using the triangle inequality for $L^k$ spaces; if we define $d_{s,t}(w) = d(w(s), w(t))$ as a function from the set of words of length $n$ representing the identity to $\mathbb{R}$ and give the set of such functions the $L^k$ norm $|| \cdot ||_{k}$ then $E(d_{s,t}(w)^k) = (||d_{s,t}||_k)^k$ and

$$E(\Delta(w(r), w(s), w(t))) \leq E(c_3(d_{r,s}(w) + d_{s,t}(w) + d_{t,r}(w))^k)$$

$$= (||d_{r,s} + d_{s,t} + d_{t,r}||_k)^k$$

$$\leq (||d_{r,s}||_k + ||d_{s,t}||_k + ||d_{t,r}||_k)^k$$

$$\leq 3^k \max(||d_{r,s}||_k, ||d_{s,t}||_k, ||d_{t,r}||_k)^k$$

$$\leq 3^k \max\{E(d_{r,s}(w)^k), E(d_{s,t}(w)^k), E(d_{t,r}(w)^k)\}$$

Thus we can bound the expectation of $\Delta(w_{i,j}, w_{i+1,2j+1}, w_{i,j+1})$ by considering the $k$th moments of distances $d(w(s), w(t))$.

We first claim that if $w$ is chosen from the unreduced words of length $n$ representing the identity in $G$ according to the probability distribution $\mathbb{P}_n$, then the distribution of $d(w(s), w(t))$ depends only on $|s-t|$ and $n$, that is,

$$P(\{w|d(w(s), w(t)) = x\}) = P(\{w|d(w(0), w(t-s)) = x\})$$

This is true because the family of maps $\tau_i$, taking

$$a_1 \ldots a_n$$
to
\[ a_{i+1} a_{i+2} \ldots a_n a_1 a_2 \ldots a_i \]
preserves \( \mathfrak{p}_n \) and
\[ d(w(s), w(t)) = d(e, r_s(w(t - s))). \]
Then
\[ P(\{w|d(w(s), w(t)) = x\}) = \mathfrak{p}_n(r_s^{-1}(\{w|d(e, w(t - s)) = x\})) = P(\{w|d(e, w(t - s)) = x\}) \]

Thus to estimate the distribution of \( d(w(s), w(t)) \) it suffices to estimate the distribution of \( d(e, w(t - s)) \). We will prove the following lemma:

**Lemma 7.** If \( G \) is as above and \( m \geq 1 \), then there is a constant \( c \) such that for any \( n > 0 \), \( t < n \),
\[ E(d(e, w(t))^m) < ct^{m/2}, \]
where the expectation is taken with respect to \( \mathfrak{p}_n \), i.e., over random closed paths of length \( n \).

*Proof.* Note that
\[ d(e, w(t)) = d(w(n), w(t)), \]
so since the distribution of \( d(w(s), w(t)) \) depends only on \( |s - t| \) and \( n \),
\[ E(d(e, w(t))^m) = E(d(e, w(n - t))^m), \]
and we can assume that \( t \leq n/2 \). Note also that \( d(e, w(t)) \leq t \), so \( E(d(e, w(t))^m) \leq t^m \) for all \( t \). It thus suffices to find bounds on \( E(d(e, w(t))^m) \) for large \( t \).

The probability that a random closed path of length \( n \) is at \( x \) at time \( t \) is:
\[ P(\{w|w(t) = x\}) = \frac{p(t)(e, x)p(n-t)(x, e)}{\sum_{y \in G} p(t)(e, y)p(n-t)(y, e)}. \]
Thus
\[ E(d(e, w(t))^m) = \sum_{x \in G} d(e, x)^m p(t)(e, x)p(n-t)(x, e) \]
\[ \sum_{y \in G} p(t)(e, y)p(n-t)(y, e) \]
Using Theorem 6, we can estimate these sums. For clarity, we’ll replace terms that don’t depend on \( n \) or \( t \) with positive constants \( c_i \).
\[ E(d(e, w(t))^m) \leq c_0 \sum_{x \in G} d(e, x)^m \exp\left(-\frac{m}{t^2}d(e, x)^2/C't - d(e, x)^2/C'(n - t)\right) \]
\[ \sum_{y \in G(C', t)} \exp\left(-\frac{C'd(y)^2}{t} - C'd(e, y)^2/(n - t)\right) \]
Since \( 0 < t \leq n/2 \), \( \frac{1}{t} \leq \frac{1}{t} + \frac{1}{n-t} \leq \frac{2}{t} \), so
\[ E(d(e, w(t))^m) \leq c_0 \sum_{x \in G} d(e, x)^m \exp\left(-\frac{m}{t^2}d(e, x)^2\right) \]
\[ \sum_{y \in G(C', t)} \exp\left(-\frac{C'd(e, y)^2}{t} - \frac{C'd(e, y)^2}{(n - t)}\right) \]
\[ = c_0 \sum_{r=0}^\infty \sum_{d(x) = r} r^m e^{-\frac{m}{t^2}r^2} \]
\[ \sum_{r=0}^\infty \sum_{d(y) = r} e^{-\frac{C'd(e, y)^2}{t}} \]
\[ = c_0 \sum_{r=0}^\infty \#S(e, r) r^m e^{-\frac{m}{t^2}r^2} \]
\[ \sum_{r=0}^\infty \#S(e, r) e^{-\frac{C'd(e, y)^2}{t}} \]
where \( \#S(e, r) \) is the number of \( x \in G \) such that \( d(e, x) = r \). We’d like to make the estimate \( \#S(e, r) \approx r^{D-1} \), where \( D \) is the order of polynomial growth of \( G \).
groups have polynomial volume growth, so we can bound \( B \) to get:

\[
\sum_{i=n}^{m} a_i b_i = a_m B_m - a_{n-1} B_{n-1} - \sum_{i=n}^{m} B_i (a_{i+1} - a_i)
\]

but this remains an open question. Instead, we estimate the sums by a summation by parts argument that uses just the fact that \( \#B(e, r) \approx r^D \).

Recall that Abel’s Formula states that if \( \{a_i\} \) and \( \{b_i\} \) are sequences and \( B_n = \sum_{i=0}^{n} b_i \), then

\[
\sum_{i=n}^{m} a_i b_i = a_m B_m - a_{n-1} B_{n-1} - \sum_{i=n}^{m} B_i a_i + a_i 
\]

Let \( \{d_i\}_{i \in \mathbb{N}} \) a sequence and \( D_n = \sum_{i=1}^{n} d_i \) be its partial sums. If \( B_n \leq D_n \) and \( \{a_i\} \) is decreasing and positive, then

\[
\sum_{i=n}^{m} a_i b_i \leq a_m B_m - a_{n-1} B_{n-1} - \sum_{i=n}^{m} D_i (a_{i+1} - a_i) 
\]

\[
= a_m B_m - a_{n-1} D_n - a_{n-1} (D_n - B_{n-1}) - \sum_{i=n}^{m} D_i (a_{i+1} - a_i) 
\]

\[
= \sum_{i=n}^{m} a_i d_i + a_{n-1} (D_n - B_{n-1}) 
\]

In particular, if \( z n^D \leq B_n \leq Z n^D \) and \( \{a_i\} \) is decreasing and positive, then

\[
\sum_{i=n}^{m} z' a_i i^{D-1} - a_{n-1} (Z - z) (n-1)^D \leq \sum_{i=n}^{m} a_i b_i \leq \sum_{i=n}^{m} z' a_i i^{D-1} + a_{n-1} (Z - z) (n-1)^D 
\]

for some \( z', Z' \).

To use this result, we need to replace \( r^m e^{-\frac{4}{m^2}} \frac{1}{r^2} \) with a decreasing function of \( r \). The function has one extremum, a maximum of \( \left( \frac{m t}{2 c_1} \right)^{m/2} \) at \( \sqrt{\frac{m t}{2 c_1}} \). Let \( \beta_r = r^m e^{-\frac{4}{m^2}} \frac{1}{r^2} \) and let \( \beta'_r = e^{-\frac{4}{m^2}} \frac{1}{r^2} \). Then we replace \( \beta_r \) by \( \left( \frac{m t}{2 c_1} \right)^{m/2} \) when \( r < \sqrt{\frac{m t}{2 c_1}} \) to get:

\[
E \left( d(e, w(t))^m \right) \leq c_0 \frac{\sum_{r=0}^{\infty} \#S(e, r) \beta_r}{\sum_{t'=0}^{t/c''} \#S(e, r) \beta'_r} 
\]

\[
\leq c_0 \frac{\sum_{r=0}^{\infty} \sqrt{\frac{m t}{2 c_1}} \#S(e, r) \left( \frac{m t}{2 c_1} \right)^{m/2}}{\sum_{r=0}^{t/c''} \#S(e, r) \beta'_r} 
\]

\[
\leq c_0 \frac{\#B \left( e, \sqrt{\frac{m t}{2 c_1}} \right) \left( \frac{m t}{2 c_1} \right)^{m/2} + \sum_{r=\sqrt{\frac{m t}{2 c_1}}}^{\infty} \#S(e, r) \beta_r}{\sum_{r=0}^{t/c''} \#S(e, r) \beta'_r} 
\]

Now \( \beta_i \) and \( \beta'_i \) are both decreasing in the intervals of summation. Nilpotent groups have polynomial volume growth, so we can bound \( \#B(e, r) \) by

\[
c_3^{-1} r^D \leq \#B(e, r) \leq c_3 r^D \quad \text{for all } r \geq 1.
\]
Since \( \#B(e, r) = \sum_{i=0}^{r} \#S(e, i) \), we can use the argument above to replace the \( \#S(e, r) \)'s by terms growing like \( r^{D-1} \):

\[
E(d(e, w(t))^{m}) \leq c_0 \left( \sqrt{\frac{mt}{2\pi e}} \right)^D \left( \frac{mt}{2\pi e} \right)^{-m/2} + \sum_{r=\sqrt{\frac{mt}{2\pi e}}}^{\infty} c_4 r^{D-1} e^{-\frac{t}{2}r^2} dr + O(t^{(D+m-1)/2})
\]

\[
\leq c_0 \int_{0}^{t/C''} c_4 r^{D-1} e^{-\frac{t}{2}r^2} dr + O(t^{(D+1)/2})
\]

where in replacing the sums with integrals, we again use that \( r^k e^{-x^2} \) has a maximum of \( \frac{1}{2\pi x} \).

If we let \( \mu = D + m \) and substitute \( x = \sqrt{\frac{t}{2}}r \), we get

\[
E(d(e, w(t))^{m}) \leq c_0 \int_{0}^{t/C''} c_4 x^{D-1} e^{-c_1 x^2} dx + O(t^{(\mu-1)/2})
\]

and thus

\[
E(d(e, w(t))^{m}) < ct^{m/2}
\]

as desired. \( \square \)

One can use the same techniques to show that

\[
E(d(e, w(t))^{m}) = \Theta(t^{m/2}).
\]

We will use this to bound the area of a triangle in the construction. The triangles added in the \( i \)th step of the construction connect points \( w([jn/2^i]), w([(j+1)n/2^{i+1})], \) and \( w([(j+1)n/2^{i+1}]) \). By the lemma above, the \( k \)th moment of the expected distance between any two of these points is at most \( c(n/2^i)^{k/2} \), and thus by \( [3] \),

\[
E(\Delta(w_{i,j}, w_{i+1,2j+1}, w_{i,j+1})) \leq 3^k c_3 (n/2^i)^{k/2} = C_4 n^{k/2} 2^{-ki/2}.
\]

Finally, we find that

\[
\delta_{avg}(n) = cn + \sum_{i=1}^{\log_2 n} \sum_{j=0}^{2^i-1} E(\Delta(w_{i,j}, w_{i+1,2j+1}, w_{i,j+1}))
\]

\[
\leq cn + \sum_{i=1}^{\log_2 n} \sum_{j=0}^{2^i-1} Cn^{k/2} 2^{-ki/2}
\]

\[
\leq cn + \sum_{i=1}^{\log_2 n} Cn^{k/2} 2^{i(1-k/2)}.
\]

If \( k > 2 \), this is a geometric series and

\[
\delta_{avg}(n) \leq cn + \frac{C}{1 - 2^{1-k/2}} n^{k/2} = O(n^{k/2}).
\]

If \( k = 2 \), then

\[
\delta_{avg}(n) \leq cn + \sum_{i=1}^{\log_2 n} Cn = O(n \log n)
\]
as desired.

For many nilpotent groups, $\delta(l)$ and $\delta_{\text{cent}}(l)$ have polynomial growth of the same order; some examples of these are abelian groups, free nilpotent groups\cite{1,3}, and the Heisenberg groups\cite{3}, though there are many more examples. For example, if $G$ is a nilpotent group of nilpotency class $c$ such that $G^{(c)}$ contains elements of infinite order, then $G/G^{(c)}$ has a Dehn function and a central isoperimetric function both with polynomial growth of order $c$. The upper bound on the Dehn function follows from \cite{9}, and the lower bound on the central isoperimetric function follows from Theorem 8 in \cite{3}. When this growth is faster than quadratic, as in non-abelian free nilpotent groups, Theorems 1 and 2 give sharp estimates, independent of the generating set, of the growth of $\delta_{\text{avg}}$.

6. Conclusion

One natural question is how well these results extend to other groups. The general idea that the loop generated by a random walk will have a length scale much smaller than the number of steps taken seems likely to hold in other groups, but the proofs here rely on good upper and lower bounds for the off-diagonal transition probabilities of a random walk, which may not be obtainable in other classes of groups.

One can consider the behavior of random closed paths in general. For any $n$, we can construct a time-dependent random walk $\hat{p}$ describing the behavior of random closed paths of length $n$. Assume that $p$ is symmetric, so that $p(x,y) = p(y,x)$. The probability that a random path of length $n$ from $x$ to $y$ is at $z$ after time $t$ is

$$\frac{p(t)(x,z)p(n-t)(z,y)}{\sum w p(t)(x,w)p(n-t)(w,y)}.$$  

Thus, if a random loop starting at $e$ is at $x$ after $t$ steps, it must return to $e$ after $n-t$ more steps and we can write

$$\hat{p}(x,y;t) = \frac{p(x,y)p(n-(t+1))(y,e)}{\sum w p(x,w)p(n-(t+1))(w,e)}$$

as the probability that its next step will take it to $y$.

We can often write

$$\lim_{n \to \infty} \frac{p(n)(e,x)}{p(n)(e,e)} = f(x);$$

such a theorem is called a ratio limit theorem. In this case,

$$\lim_{n \to \infty} \hat{p}(x,y;t) = \lim_{n \to \infty} \frac{p(x,y)p(n-(t+1))(y,e)}{\sum w p(x,w)p(n-(t+1))(w,e)} = \frac{p(x,y)f(y)}{\sum w p(x,w)f(w)}$$

In amenable groups \cite{2}, for instance, $f(x) = 1$ for all $x$, and thus

$$\lim_{n \to \infty} \hat{p}(x,y;t) = p(x,y).$$

That is, when $n$ is large, a random closed path will look like the standard random walk on small timescales. The solvable Baumslag-Solitar groups $BS(1,n)$ are examples of amenable groups with exponential Dehn function for which the random
walk goes to infinity at a sublinear rate\cite{16}, so it seems likely that its averaged Dehn function is subexponential.

Sublinear growth of distances in random closed paths may be a fairly general phenomenon. In a free group, for example, the map \( w \mapsto d(w(i), e) \) taking random closed paths on the free group to random closed paths on \( N \) is measure-preserving, so, as in the nilpotent case, we find that \( E(d(s), d(t)) = O(\sqrt{|s - t|}) \).

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