The stability of the Einstein static state in $f(T)$ gravity

Puxun Wu and Hongwei Yu

Center of Nonlinear Science and Department of Physics,
Ningbo University, Ningbo, Zhejiang, 315211 China

Abstract

The stability of the Einstein static universe against the homogeneous scalar perturbations in $f(T)$ gravity is analyzed. Both the spatial closed and open universes are considered. We find that the stable Einstein static solutions exist in both cases. Considering a concrete $f(T)$ model and assuming that the cosmic energy has a constant equation of state $w$, we obtain that, in the closed case, $w < 1/3$ is required. Thus, $f(T)$ theory gives a larger region of $w$ than that in general relativity ($-1 < w < -1/3$) to have the stable Einstein static solution. For the open universe, $f(T)$ theory allows the stable Einstein static solution, although this kind of solution is forbidden in general relativity. Thus, a modification of gravity can play a crucial role in stabilizing the Einstein static solution.

PACS numbers: 04.50.Kd, 04.20.Jb, 98.80.-k
I. INTRODUCTION

One of the most important and challenging problems in modern cosmology is how to explain the current cosmic acceleration, discovered firstly from the type Ia supernova observations [1] and then further confirmed by the cosmic microwave background radiation [2] and the large scale structures [3]. One popular way to explain this observed phenomenon is to postulate, within the context of general relativity, the existence of an exotic energy component, called dark energy (see [4] for recent review), in our universe. Another is to modify the general relativity. $f(R)$ theory (see [5] for recent review), obtained by generalizing the Ricci scalar $R$ in the Einstein-Hilbert action to an arbitrary function $f$ of $R$, is one of such modified gravity theories.

In 1928, Einstein [6] first introduced the teleparallel gravity (TG) in his endeavor to unify gravity and electromagnetism with the introduction of a tetrad field. Although not succeeding, as is well known, TG can, however, show up as a theory completely equivalent to general relativity [7, 8]. Since TG is built on the teleparallel geometry, and the Weitzenböck connection rather than the Levi-Civita connection is used in this geometry, the Riemann curvature vanishes automatically and the spacetime has only torsion. The torsion scalar $T$ is the Lagrangian density of TG.

Recently, in analogy to $f(R)$ theory, a new modified gravity to account for the accelerating cosmic expansion, named $f(T)$ theory, is proposed by extending the TG action $T$ to an arbitrary function $f$ of $T$. $f(T)$ theory can not only explain the present cosmic acceleration with no need of dark energy [9], but also provide an alternative to inflation without an inflaton [10, 11]. Moreover, further studies have shown that $f(T)$ may avoid the big bang singularity problem in the standard cosmology [12], realize the crossing of phantom divide line for the effective equation of state [13, 14], fit the current type Ia supernova observation very well [15] and yield an usual early cosmic evolution [16]. It, thus, has recently spurred an increasing deal of interest [17–20]. It is worth pointing out here that $f(T)$ gravity also suffers from some problems, such as the violation of local Lorentz invariance [19] and the violation of the first law of black hole thermodynamics [20].

In this paper, we plan to analyze the stability of the Einstein static state in $f(T)$
theory. Both spatially closed and open universes are considered. Our interest in this issue lies in that our universe might have originated from the Einstein static state and then evolved to the inflation, so as to provide a possible way to resolve the big bang singularity problem [21, 22]. The Einstein static universe has attracted a great deal of attention [23–41]. For instance, the Einstein static solutions have been analyzed in braneworld theory, Hoava-Lifshitz gravity and loop quantum cosmology [24, 27, 38, 39]. The stability of the Einstein static state has been studied in $f(R)$ gravity [28–31] and it was found that, in several concrete $f(R)$ models, the stable solutions do exist under the homogeneous perturbations [28]. However, Goswami et al. [29] argued that only one functional form of $f(R)$ admits an Einstein static solution, which seems to be inconsistent with what was obtained in [28]. This contradiction was reconciled in [31] by considering the stability of the Einstein static universe under the homogeneous and inhomogeneous scalar perturbations in a general $f(R)$ theory. It is worth noting that all above studies are done in the case of a spatially closed universe. More recently, it was found that in the frameworks of Loop Quantum Cosmology and Horava-Lifshitz gravity, the Einstein static solution may also exist in an open universe [41].

We examine, in the present paper, the stability of the Einstein static solution against homogeneous perturbations in $f(T)$ gravity. In the following section, we give a brief review on $f(T)$ gravity. In section III, we give the Einstein static solution and then discuss its stability by analyzing the homogeneous scalar perturbations, and we conclude in Section IV.

II. THE $f(T)$ THEORY

In this section, we give a brief review on $f(T)$ gravity. In this theory, the dynamical object is the vierbein $e^\mu_i$ rather than the metric. If $e^\mu_\mu$ is the inverse matrix of vierbein $e^\mu_i$, the relation between them is

$$e^\mu_i e^\mu_\mu = \delta^i_\mu, \quad e^\mu_i e^\nu_\nu = \delta^\mu_\nu,$$

(1)

where $i$ is an index running over 0, 1, 2, 3 for the tangent space of the manifold, and $\mu$, also running over 0, 1, 2, 3, is the coordinate index on the manifold. This vierbein relates
with the metric through

\[ g_{\mu\nu} = \eta_{ij} e^i_\mu e^j_\nu , \] (2)

where \( \eta_{ij} = \text{diag}(1,-1,-1,-1) \). Using Eq. (1), the above expression can be inverted to obtain

\[ \eta_{ij} = g_{\mu\nu} e^i_\mu e^j_\nu , \] (3)

which means that the vierbein is orthonormal.

As already mentioned in the previous section, TG uses the curvatureless Weitzenböck connection, which is defined as

\[ \Gamma^\lambda_{\mu\nu} = e^\lambda_i \partial^\nu e^i_\mu = -e^i_\mu \partial^\nu e^\lambda_i . \] (4)

From this connection, one can introduce a non-null torsion tensor \( T^\sigma_{\mu\nu} \),

\[ T^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu} - \Gamma^\sigma_{\mu\nu} . \] (5)

Defining other two tensors:

\[ S^\mu\nu_{\sigma} \equiv \frac{1}{2}(K^\mu\nu_{\sigma} + \delta^\mu_{\sigma} T^\alpha_{\nu} - \delta^\nu_{\sigma} T^\alpha_{\mu}) , \] (6)

and

\[ K^\mu\nu_{\sigma} = -\frac{1}{2}(T^\mu\nu_{\sigma} - T^\nu\mu_{\sigma} - T^\mu\sigma_{\nu}) , \] (7)

one can obtain the torsion scalar \( T \)

\[ T \equiv S^\mu\nu_{\sigma} T^\sigma_{\mu\nu} , \] (8)

which is the teleparallel Lagrangian. Thus the TG action can be expressed as

\[ I = \frac{1}{16\pi G} \int d^4 x \ e \ T , \] (9)

where \( e = \det(e^i_\mu) = \sqrt{-g} \). Since \( eT \) just differs from the Einstein-Hilbert Lagrangian \( eR \) by a total derivative term

\[ -e \ R = e \ T - 2\partial_\nu(e \ T^\mu_{\nu}) \] (10)
where \( R \) is the scalar curvature for the Levi-Civita connection, TG is completely equivalent to general relativity. As \( f(R) \) gravity, the action of \( f(T) \) theory is obtained by replacing \( T \) in the TG action by a general function of \( T \)

\[
I = \frac{1}{16\pi G} \int d^4x \, e \, f(T) .
\] (11)

Apparently the last term in Eq. (10) can be discarded by converting it to a boundary term in TG. It, however, remains in \( f(T) \) theory, which leads to the violation of local Lorentz invariance. Adding a matter term in the above equation and doing a derivative with respect to vierbein, one can obtain the field equation of \( f(T) \) gravity.

**III. THE EINSTEIN STATIC UNIVERSE IN \( f(T) \) THEORY**

To analyze the Einstein static universe, we consider the FRW universe with non flat spatial sections. Due to the lack of local Lorentz invariance, pairs of vierbein fields connected by local Lorentz transformations are inequivalent. Thus, one should be careful in obtaining the parallelized frames. Here, we follow the procedure given in Ref. [17] to get the vierbein. For a closed universe, the vierbein is

\[
e^0 = dt ; \quad e^1 = a(t)E^1 ; \quad e^2 = a(t)E^2 ; \quad e^3 = a(t)E^3 ,
\] (12)

with

\[
E^1 = - \cos \theta d\psi + \sin \psi \sin \theta (\cos \psi d\theta - \sin \psi \sin \theta d\phi)
\] (13)

\[
E^2 = \sin \theta \cos \phi d\psi - \sin \psi [(\sin \psi \sin \phi - \cos \psi \cos \theta \cos \phi) d\theta
+ (\cos \psi \sin \phi + \sin \psi \cos \theta \cos \phi) \sin \theta d\phi]
\]

\[
E^3 = - \sin \theta \sin \phi d\psi - \sin \psi [(\sin \psi \cos \phi + \cos \psi \cos \theta \sin \phi) d\theta
+ (\cos \psi \cos \phi - \sin \psi \cos \theta \sin \phi) \sin \theta d\phi],
\]

and, for an open one, it is

\[
e^0 = dt ; \quad e^1 = a(t)\bar{E}^1 ; \quad e^2 = a(t)\bar{E}^2 ; \quad e^3 = a(t)\bar{E}^3 ,
\] (14)
with

\[ E^1 = \cos \theta d\psi + \sinh \psi \sin \theta ( - \cosh \psi d\theta + i \sinh \psi \sin \theta d\phi) \]
\[ E^2 = - \sin \theta \cos \phi d\psi + \sinh \psi (i \sinh \psi \sin \phi - \cosh \psi \cos \theta \cos \phi) d\theta \]
\[ + (\cosh \psi \sin \phi + i \sinh \psi \cos \theta \cos \phi \sin \theta d\phi) \]
\[ E^3 = \sin \theta \sin \phi d\psi + \sinh \psi (i \sinh \psi \cos \phi + \cosh \psi \cos \theta \sin \phi) d\theta \]
\[ + (\cosh \psi \cos \phi - i \sinh \psi \cos \theta \sin \phi \sin \theta d\phi) . \]

Here, the angular coordinates range in the intervals \[0 \leq \phi \leq 2 \pi, 0 \leq \theta \leq \pi\] and \[0 \leq \psi \leq \pi\].

Thus, using Eqs. (2, 12, 14), one can obtain the induced metric

\[ ds^2 = dt^2 - k^2 a^2(t)(d(k\psi)^2 + \sin^2(k\psi)(d\theta^2 + \sin^2 \theta d\phi^2)) , \]

where \( k = 1 \) for the closed universe and \( k = i \) for the open universe. The torsion scalar can be expressed as

\[ T = 6(\pm a^{-2} - H^2) , \]

where + and − correspond to the closed and open cases respectively, and \( H = \frac{\dot{a}}{a} \) is the Hubble parameter. In a non flat universe, the modified Friedmann equation can be expressed as

\[ 12H^2 f'(T) + f(T) = 16\pi G \rho \equiv \kappa \rho , \]
\[ 4(\pm a^{-2} + \dot{H})(12H^2 f''(T) + f'(T)) - f(T) - 4f'(T)(2\dot{H} + 3H^2) = \kappa p , \]

where \( f'(T) = df/dT \), \( f''(T) = d^2f/dT^2 \) and we assume that the matter-content of the universe is a perfect fluid with \( \rho \) and \( p \) being the unperturbed energy density and pressure respectively, which satisfy the conservation equation

\[ \dot{\rho} + 3H(\rho + p) = \dot{\rho} + 3H(1 + w)\rho = 0 . \]

Here we let \( w = \frac{p}{\rho} \) be a constant.

For the Einstein static universe, we have

\[ a = a_0 = \text{const}, \quad \dot{a} = H = 0 , \]
\( T_0 = T(a_0) = \frac{6}{a^2} \) for the closed universe, and \( T_0 = -\frac{6}{a^2} \) for the open one. Thus, using Eqs. (18, 19) we obtain the conditions for the existence of an Einstein static universe

\[ f_0 = f(T_0) = \kappa \rho_0 , \quad \pm \frac{4f_0'}{a_0^2} - f_0 = \kappa p_0 , \tag{22} \]

with \( f_0' \equiv \frac{df}{dT}|_{T=T_0} \), \( \rho_0 = \rho(a_0) \) and \( p_0 = p(a_0) \). Combining the above two expressions and using \( T_0 \), one finds

\[ \left( \frac{f f'}{f} \right)_{T=T_0} = \frac{3}{2} (1 + w) , \tag{23} \]

which gives a constraint on \( f(T) \) to obtain the Einstein static solution.

Now we consider the stability of the Einstein static solution against the linear homogeneous scalar perturbations. Thus, the perturbations in the cosmic scale factor and in the energy density depend only on time and can be expressed as

\[ a(t) = a_0 (1 + \delta a(t)) , \quad \rho(t) = \rho_0 (1 + \delta \rho(t)) . \tag{24} \]

Substituting the above equation into Eqs. (18, 19) and linearizing the results, we have

\[ f_0' \delta T = \kappa \rho_0 \delta \rho , \tag{25} \]

\[ 4(\mp 2a_0^{-2} \delta a + \delta \dot{a}) f_0' \pm 4a_0^{-2} f_0'' \delta T - f_0' \delta T - 8f_0' \delta \ddot{a} = \kappa \delta p , \tag{26} \]

where \( \delta f = f_0' \delta T \) and \( \delta f' = f_0'' \delta T \). Using \( \delta T = -2T_0 \delta a \), \( \delta \rho = w \rho_0 \delta \rho \) and \( a_0^{-2} = \pm \frac{6}{4f_0'} (1 + w) \), one can obtain

\[ \delta \ddot{a} = \frac{\kappa \rho_0}{4f_0'^2} (1 + w) \left( (1 + 3w) f_0' - 3(1 + w) \frac{\dot{f}_0 \ddot{f}_0'}{f_0'} \right) \delta a , \tag{27} \]

which admits a solution

\[ \delta a(t) = C_1 e^{\omega t} + C_2 e^{-\omega t} , \tag{28} \]

where \( C_1 \) and \( C_2 \) are integration constants, and \( \omega \) is given by

\[ \omega^2 = \frac{\kappa \rho_0}{4f_0'^2} (1 + w) \left( (1 + 3w) f_0' - 3(1 + w) \frac{\dot{f}_0 \ddot{f}_0'}{f_0'} \right) \tag{29} \]

Obviously, if \( \omega^2 < 0 \) we have oscillating perturbation modes, which correspond to existence of the stable Einstein static universe.

In what follows, we divide our discussion into the closed and open universes respectively, and consider a concrete power law \( f(T) \) model in order to give the detail conditions for the stable Einstein static solution.
A. The closed universe

In this case, \( T_0 = 6a_0^{-2} \) for the Einstein static solution. We consider a model

\[
    f(T) = T + \alpha \frac{T^2}{6} - \Lambda ,
\]

where \( \alpha \) is a constant and \( \Lambda \) is the cosmological constant. Substituting the above expression into the Einstein static solution given in Eq. (22), one can obtain

\[
    \Lambda = \frac{1}{2}(-12\alpha + \kappa \rho_0(1 + 3w)) ,
    \quad \frac{1}{a_0^2} = \frac{1}{2}(-8\alpha + \kappa \rho_0(1 + w)).
\]

Using Eq. (31), it is easy to prove that the model (Eq. (30)) satisfies the constraints given in Eq. (23). A positive \( a_0^2 \) gives a constraint on \( \alpha \):

\[
    \alpha < \frac{\kappa \rho_0}{8}(1 + w).
\]

Substituting Eq. (30) into Eq. (29), we have

\[
    \omega^2 = \frac{\kappa \rho_0}{4f_0'}(1 + w)[-16\alpha + \kappa \rho_0(1 + w)(1 + 3w)] - 8\alpha + \kappa \rho_0(1 + w),
    \quad \frac{1}{a_0^2} = \frac{\kappa \rho_0}{8f_0'^2}(1 + w)[-16\alpha + \kappa \rho_0(1 + w)(1 + 3w)].
\]

As expected, the limit \( \alpha \to 0 \ (f \to T - \Lambda) \) yields

\[
    \omega^2 = \frac{\kappa \rho_0}{4}(1 + w)(1 + 3w),
    \quad \Lambda = \frac{\kappa \rho_0}{2}(1 + 3w),
    \quad \frac{1}{a_0^2} = \frac{\kappa \rho_0}{2}(1 + w),
\]

which is just the general relativistic result. Apparently, within the general relativity context the solution is stable in the region

\[
    -1 < w < -1/3 ,
\]

which violates the strong energy condition and leads to a negative cosmological constant.

From Eq. (33), we find two stable regions in \( f(T) \) gravity:

\[
    w < -1 , \quad \alpha < \frac{\kappa \rho_0}{8}(1 + w),
\]

and

\[
    -1 < w < 1/3 , \quad \frac{\kappa \rho_0}{16}(1 + w)(1 + 3w) < \alpha < \frac{\kappa \rho_0}{8}(1 + w).\]
Apparently, $f(T)$ gravity enlarges the allowed region of $w$ for obtaining the stable Einstein static solution, and the strong energy condition is not necessarily violated. In addition, different from the general relativity case where the cosmological constant is negative at the static point, $\Lambda$ can be either positive or negative in $f(T)$ theory as shown in Tab. (I).

| $w$ | $\alpha$ | $\Lambda$ |
|-----|----------|----------|
| $w < -1$ | $\frac{\kappa \rho_0}{12}(1 + 3w) < \alpha < \frac{\kappa \rho_0}{8}(1 + w)$ | $\Lambda > 0$ |
| $-1 < w < -1/3$ | $\alpha < \frac{\kappa \rho_0}{12}(1 + 3w)$ | $\Lambda < 0$ |
| $-1/3 < w < 1/3$ | $\frac{\kappa \rho_0}{16}(1 + w)(1 + 3w) < \alpha < \frac{\kappa \rho_0}{12}(1 + w)$ | $\Lambda > 0$ |
| $1/3 < w < 1$ | $\alpha < \frac{\kappa \rho_0}{12}(1 + 3w)$ | $\Lambda < 0$ |

### B. The open universe

This case corresponds to $T_0 = -6a_0^{-2}$ in the Einstein static universe. We consider the same model as that in the closed case. Substituting Eq. (30) into the Einstein static solution given in Eq. (22), we have

$$\Lambda = \frac{1}{2}(-12\alpha + \kappa \rho_0(1 + 3w)),$$

$$\frac{1}{a_0^2} = \frac{1}{2}(8\alpha - \kappa \rho_0(1 + w)). \quad (38)$$

A positive $a_0^2$ leads to $\alpha$:

$$\alpha > \frac{\kappa \rho_0}{8}(1 + w). \quad (39)$$

Doing the same calculation as in the closed case, we obtain

$$\omega^2 = \frac{\kappa \rho_0}{4f_0^2} \frac{(1 + w)[-16\alpha + \kappa \rho_0(1 + w)(1 + 3w)]}{-8\alpha + \kappa \rho_0(1 + w)}$$

$$= \frac{\kappa \rho_0 a_0^2}{8f_0^2}(1 + w)[16\alpha - \kappa \rho_0(1 + w)(1 + 3w)]. \quad (40)$$
When $\alpha = 0$, we get
\[
\omega^2 = \frac{\kappa \rho_0}{4}(1 + w)(1 + 3w), \quad \Lambda = \frac{\kappa \rho_0}{2}(1 + 3w), \quad \frac{1}{a_0^2} = -\frac{\kappa \rho_0}{2}(1 + w). \quad (41)
\]
The stable solution requires $-1 < w < -1/3$, but it leads to $a_0^2 < 0$. Thus, in the spatially open case, the general relativity does not allow a stable Einstein static solution.

In $f(T)$ theory, from Eq. (40), we find that the conditions for stability are
\[
w < -1, \quad \alpha > \frac{\kappa \rho_0}{16}(1 + w)(1 + 3w), \quad (42)
\]
and
\[
w > 1/3, \quad \frac{\kappa \rho_0}{8}(1 + w) < \alpha < \frac{\kappa \rho_0}{16}(1 + w)(1 + 3w), \quad (43)
\]
which means that a phantom or a stiff matter is required for obtaining the stable Einstein static universe. In Tab. (II), we show the properties of $\Lambda$.

| $w < -1$ | $\alpha > \frac{\kappa \rho_0}{16}(1 + w)(1 + 3w)$ | $\Lambda < 0$ |
| $w > 1/3$ | $\frac{\kappa \rho_0}{12}(1 + 3w) < \alpha < \frac{\kappa \rho_0}{16}(1 + w)(1 + 3w)$ | $\Lambda < 0$ |
| $\frac{\kappa \rho_0}{8}(1 + w) < \alpha < \frac{\kappa \rho_0}{12}(1 + 3w)$ | $\Lambda > 0$ |

IV. CONCLUSION

The Einstein static universe has been proposed as the asymptotic origin of our universe for avoiding the big bang singularity problem. Thus, in order to assure that the universe can stay at this static state past-eternally, a stable Einstein static universe is essential. In this paper, we have analyzed the stability of the Einstein static universe against the homogeneous scalar perturbations in $f(T)$ gravity, which is a new modified gravity used to account for the present accelerated cosmic expansion and explain the cosmic inflation.
Different from usual discussions considering only the spatially closed universe, both the closed and open cases are studied in the present paper. In particular, we assume that the matter-content of the universe is a perfect fluid and it has a constant equation of state \( w \). By considering a concrete power law \( f(T) \) model: 
\[
f(T) = T + \alpha \frac{T^2}{6} - \Lambda,
\]
we find that, in the closed case, \( w < 1/3 \) is required to obtain a stable Einstein static solution, which means that the strong energy condition violated in general relativity can be satisfied. Our results show a larger allowed region of \( w \) in \( f(T) \) gravity than that in general relativity where the condition for stability is \(-1 < w < -1/3\). As shown in Tab. (I), \( f(T) \) theory allows both negative and positive \( \Lambda \) in contrast to general relativity in which a negative cosmological constant is needed.

For the open universe, we find that there is no stable Einstein static solution in general relativity, but \( f(T) \) theory allows it. The stable Einstein static universe requires that the perfect fluid is a phantom (\( w < -1 \)) or a stiff matter (\( w > 1/3 \)). Therefore, we can conclude that the modification of gravity can play a crucial role in stabilizing the Einstein static solution.

We only consider the power law \( f(T) \) model in the present paper. However, one can show that there also exists the Einstein static solution in the exponential model, but the stability conditions are much more complicated than that in the power law case. So, we do not give the details here. Finally, we must point out that here only the homogeneous perturbations are analyzed. It is of interest to extend our results to the case of the inhomogeneous perturbations, which will be a topic for study in the future.

**Acknowledgments**

This work was supported by the National Natural Science Foundation of China under Grants Nos. 10935013 and 11075083, Zhejiang Provincial Natural Science Foundation of China under Grants Nos. Z6100077 and R6110518, the FANEDD under Grant No. 200922, the National Basic Research Program of China under Grant No. 2010CB832803,
the NCET under Grant No. 09-0144, and K.C. Wong Magna Fund in Ningbo University.

[1] A. G. Riess, et al., Astron. J. 116, 1009 (1998); S. Perlmutter, et al., Astrophys. J. 517, 565 (1999).
[2] D. N. Spergel, et al., ApJS, 148, 175 (2003); D. N. Spergel, et al., ApJS, 170, 377S (2007).
[3] M. Tegmark, et al., Phys. Rev. D 69, 103501 (2004); D. J. Eisenstein, et al., Astrophys. J. 633, 560 (2005).
[4] T. Padmanabhan, AIP Conf. Proc. 861, 179 (2006); E. J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod. Phys. D 15, 1753 (2006); V. Sahni and A. Starobinsky, Int. J. Mod. Phys. D 15, 2105 (2006); L. Perivolaropoulos, astro-ph/0601014; M. Li, X. Li, S. Wang and Y. Wang. arXiv:1103.5870.
[5] S. Nojiri, S. D. Odintsov, arXiv:0807.0685; T. P. Sotiriou, V. Faraoni, Rev. Mod. Phys. 82, 451 (2010); A. De Felice and S. Tsujikawa, Living Rev. Rel. 13, 3 (2010); S. Nojiri, S. D. Odintsov, arXiv:1011.0544.
[6] A. Einstein, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl., 217 (1928).
[7] A. Einstein, Math. Ann. 102, 685 (1930); K. Hayashi and T. Shirafuji, Phys. Rev. D 19, 3524 (1979); 24, 3312 (1981).
[8] C. Moller, K. Dan. Vidensk. Selsk. Math. Fys. Skr. 1 (1961); C. Pellegrini and J. Plebanski, K. Dan. Vidensk. Selsk. Math. Fys. Skr. 2 (1962);
[9] G. R. Bengochea and R. Ferraro, Phys. Rev. D 79, 124019 (2009).
[10] R. Ferraro and F. Fiorini, Phys. Rev. D 75, 084031 (2007).
[11] R. Ferraro and F. Fiorini, Phys. Rev. D 78, 124019 (2008).
[12] R. Ferraro and F. Fiorini, arXiv:1106.6349.
[13] P. Wu and H. Yu, Eur. Phys. J. C 71, 1552 (2011).
[14] K. Bamba, C. Geng, C. Lee and L. Luo, JCAP 1101, 021 (2011).
[15] P. Wu and H. Yu, Phys. Lett. B 693, 415 (2010); G. R. Bengochea, Phys. Lett. B 695, 405 (2011).
[16] P. Wu and H. Yu, Phys. Lett. B 692, 176 (2010); Y. Zhang, H. Li, Y. Gong and Z. Zhu,
[17] R. Ferraro, F. Fiorini, \texttt{arXiv:1103.0719}.

[18] E. V. Linder, Phys. Rev. D \textbf{81}, 127301 (2010); K. K. Yerzhanov, S. R. Myrzakul, I. I. Kulnazarov, R. Myrzakulov, \texttt{arXiv:1006.3879}; R. Yang, \texttt{arXiv:1007.3571}; P. Y. Tsyba, I. I. Kulnazarov, K. K. Yerzhanov, R. Myrzakulov, Int. J. Theor. Phys. \textbf{50}, 1876 (2011); S. Chen, J. B. Dent, S. Dutta, E. N. Saridakis, Phys.Rev.D \textbf{83}, 023508 (2011); K. Bamba, C. Geng, C. Lee, \texttt{arXiv:1008.4036}; R. Myrzakulov, \texttt{arXiv:1008.4486}; K. Karami, A. Abdolmaleki, \texttt{arXiv:1009.2459}; K. Karami, A. Abdolmaleki, \texttt{arXiv:1009.3587}; R. Yang, Europhys. Lett. \textbf{93}, 60001 (2011); J. B. Dent, S. Dutta, E. N. Saridakis, JCAP \textbf{1101}, 009 (2011); T. Wang, \texttt{arXiv:1102.4410}; B. Li, T. P. Sotiriou, J. D. Barrow, Phys. Rev. D \textbf{83}, 104017 (2011); Y. Cai, S. Chen, J. B. Dent, S. Dutta, E. N. Saridakis, \texttt{arXiv:1104.4349}; S. Chattopadhyay, U. Debnath, Int. J. Mod. Phys. D \textbf{20}, 1135 (2011); M. Li, R. Miao, Y. Miao, \texttt{arXiv:1105.5934}; H. Wei, X. Ma, H. Qi, \texttt{arXiv:1106.0102}; X. Meng, Y. Wang, \texttt{arXiv:1107.0629}; C. G. Boehmer, A. Musa, N. Tamanini, \texttt{arXiv:1107.4455}; H. Wei, H. Qi, X. Ma, \texttt{arXiv:1108.0859}; S. Capozziello, V. F. Cardone, H. Farajollahi, A. Ravanpak, \texttt{arXiv:1108.2789}.

[19] B. Li, T. P. Sotiriou and J. D. Barrow, Phys. Rev. D \textbf{83}, 064035 (2011); T. P. Sotiriou, B. Li, J. D. Barrow, Phys. Rev. D \textbf{83}, 104030 (2011).

[20] R. Miao, M. Li, Y. Miao, \texttt{arXiv:1107.0515}.

[21] G. F. R. Ellis and R. Maartens, Class. Quant. Grav. \textbf{21}, 223 (2004).

[22] G. F. R. Ellis, J. Murugan and C. G. Tsagas, Class. Quant. Grav. \textbf{21}, 233 (2004).

[23] S. Carneiro and R. Tavakol, Phys. Rev. D \textbf{80}, 043528 (2009).

[24] D. J. Mulryne, R. Tavakol, J. E. Lidsey and G. F. R. Ellis, Phys. Rev. D \textbf{71}, 123512 (2005).

[25] L. Parisi, M. Bruni, R. Maartens and K. Vandersloot, Class. Quant. Grav. \textbf{24}, 6243 (2007).

[26] P. Wu and H. Yu, J. Cosmol. Astropart. P. \textbf{05}, 007 (2009).

[27] J. E. Lidsey and D. J. Mulryne, Phys. Rev. D \textbf{73}, 083508 (2006).

[28] C. G. Boehmer, L. Hollenstein and F. S. N. Lobo, Phys. Rev. D \textbf{76}, 084005 (2007).

[29] R. Goswami, N. Goheer and P. K. S. Dunsby, Phys. Rev. D \textbf{78}, 044011 (2008).

[30] N. Goheer, R. Goswami and P. K. S. Dunsby, Class. Quant. Grav. \textbf{26}, 105003 (2009); S. del
Campo, R. Herrera and P. Labrana, J. Cosmol. Astropart. P. 0711, 030 (2007); U. Debnath, Class. Quant. Grav. 25, 205019 (2008); B. C. Paul and S. Ghose, arXiv: 0809.4131.

[31] S. S. Seahra and C. G. Bohmer, Phys. Rev. D 79, 064009 (2009).
[32] C. G. Boehmer and F. S. N. Lobo, Phys. Rev. D 79, 067504 (2009).
[33] J. D Barrow, G. Ellis, R. Maartens, C. Tsagas, Class. Quant. Grav. 20, L155 (2003).
[34] T. Clifton, John D. Barrow, Phys. Rev. D 72, 123003 (2005).
[35] J. D Barrow, C. G Tsagas, [arXiv:0904.1340]
[36] C. G. Boehmer, L. Hollenstein, F. S. N. Lobo, and S. S. Seahra, [arXiv:1001.1266]
[37] A. Odrzywolek, Phys. Rev. D 80, 103515 (2009).
[38] C. G. Boehmer, F. S. N. Lobo, Eur. Phys. J. C 70, 1111 (2010).
[39] P. Wu and H. Yu, Phys. Rev. D 81, 103522 (2010).
[40] K. Zhang, P. Wu and H. Yu, Phys. Lett. B 690, 229 (2010).
[41] R. Canonico, L. Parisi, Phys. Rev. D 82, 064005 (2010).