On the extension and kernels of signed bimeasures and their role in stochastic integration

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Abstract

In this work we provide a necessary and sufficient condition for the extension of signed bimeasures on \( \delta \)-rings and for the existence of relative kernels. This result generalises the construction method of regular conditional probabilities to the more general setting of extended signed measures. Building on this result, we obtain the most general theory of stochastic integrals based on random measures, thus extending and generalising the whole integration theory developed in the celebrated Rajput and Rosinski’s paper (Probab. Theory Relat. Fields, 82 (1989) 451-487) and the recent results by Passeggeri (Stoch. Process. Their Appl., 130, (3), (2020), 1735-1791).

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1 Introduction

Bilinear measure theory has a long history and its origins date back to the work of Fréchet in 1915 [7]. In this work, Fréchet characterized the bounded bilinear functionals on \( C[0,1] \), which are bimeasures when identified as set functions [21]. Across the decades there have been numerous works on bilinear (and multilinear) measure theory. A major example is given by the work [8] by Grothendieck, where he introduced what is now called the Grothendieck’s inequality, namely what he described as “the fundamental theorem in the metric theory of tensor products”. Further relevant works, which include also the efforts of obtaining the Riesz-Markov-Kakutani representation theorem in the bilinear and multilinear case, are [1, 2, 3, 6, 9, 11, 13, 21, 22], among others.

However, all these works focused on algebras or \( \sigma \)-algebras of sets, but not on \( \delta \)-rings. Recall that a non-empty collection of sets is called a \( \delta \)-ring if it is closed under union, relative complementation, and countable intersection. Further, recall that any \( \sigma \)-algebra is a \( \sigma \)-ring and any \( \sigma \)-ring is a \( \delta \)-ring, but the reverse it is not true, namely not every \( \delta \)-ring is a \( \sigma \)-ring.

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and not every $\sigma$-ring is a $\sigma$-algebra. The choice of investigating the $\delta$-rings framework, apart from an intrinsic interest, comes from the fact that this is one of the topological framework mostly used in the field of probability and stochastic analysis.

For example, this is the framework at the core of the theory of random measures (\cite{10}). In the first section of the first chapter of the recent wonderful book of Kallenberg \cite{10}, the author introduces the notion of localising ring, which is a ring closed under countable intersections, thus a $\delta$-ring. The localising ring is then used to build the whole theory of (positive) random measures. Moreover, $\delta$-rings are the topological framework of certain classes of stochastic processes, in particular infinitely divisible (ID) stochastic processes. This class of processes contain some of the most studied processes: Brownian motions, Poisson processes, Lévy processes. The topological framework of the Rajput and Rosinski’s paper \cite{19}, where spectral representations for these processes are obtained, is the $\delta$-ring. Their work is still nowadays one of the main references for ID processes and one of the initial motivations of our work was to generalise their work.

Infinitely divisible (ID) distributions constitute one of the most important classes of probability distributions. Their investigation dates back to the work of Lévy, Kolmogorov and De Finetti. One of their most attractive properties is that their characteristic function have a unique explicit formulation, called the Lévy-Khintchine formulation, in terms of three mathematical objects. These are the drift, which is a real valued constant, the Gaussian component, which is a non-negative constant, and the Lévy measure, which is a measure on $\mathbb{R}$ satisfying an integrability condition and with no mass at $\{0\}$. Gaussian and Poisson distributions are examples of this class.

In 2018, in \cite{13} Sato, Lindner and Pan introduced the class of quasi-infinitely divisible (QID) distributions. A QID random variable is defined as follows: a random variable $X$ is QID if and only if there exist two ID random variables $Y$ and $Z$ s.t. $X+Y \overset{d}{=} Z$ and $Y$ is independent of $X$. Like ID distributions, QID distributions posses a Lévy-Khintchine formulation where the Lévy measure is now allowed to take negative values. Any ID distribution is QID, but the converse is not always true.

In \cite{13}, the authors show that QID distributions are dense in the space of all probability distributions with respect to weak convergence, that distributions with a point mass greater than $1/2$ are QID, and that distributions that are concentrated on the integers are QID if and only if their characteristic functions have no zeros. In \cite{16}, the QID framework is extended to real-valued random measures and stochastic processes. The work \cite{16} also represents the first attempt to extend the celebrated Rajput and Rosinski’s 1989 paper \cite{19} to the QID framework. Moreover, in \cite{17} the author show that QID completely random measures (CRMs) are dense in the space of all CRMs with respect to convergence in distribution (in \cite{17} random measures are defined as in \cite{10}). In other words, any CRM is approximated in distribution by a QID CRM. This result is crucial in fields like Bayesian analysis and corroborates the importance of investigating the properties of QID random measures. It is worth mentioning that QID distributions have already shown to have an impact in various fields: from mathematical physics, see \cite{4} and \cite{5}, to number theory, see \cite{14} and \cite{15}, and from Bayesian analysis \cite{17} to insurance mathematics, see \cite{23}.

The first main contribution of this paper is a general measure theoretical result on the extension of signed bimeasures on $\delta$-rings. In particular, consider a bimeasure on the Cartesian product of $(X, \mathcal{B})$ and $(T, \mathcal{A})$, where the former is a Borel measurable space and the latter is such that $T$ is an arbitrary non-empty set and $\mathcal{A}$ is a $\delta$-ring with the additional condition
that there exists an increasing sequence of sets $T_1, T_2, \cdots \in \mathcal{A}$ s.t. $\bigcup_{n \in \mathbb{N}} T_n = T$. Then, the result states that there exists a necessary and sufficient condition for the existence of a unique signed measure on the $\delta$-ring $\bigcup_{D \in \mathcal{A}} (A \cap D) \otimes \mathcal{B}$, and that its unique Jordan decomposition uniquely extend to measures on the $\sigma$-algebra $\sigma(A) \otimes \mathcal{B}$. Moreover, the result provides unique (sub-Markovian) kernels for all these measures.

In one of the main result of [16], the author provides this result (c.f. Theorem 5.18) only for $\sigma$-algebras even though the topological setting of his work is based on $\delta$-rings as the work [19]. In this paper, we succeed in extending his Theorem 5.18 by obtaining the result for the right and more general topological framework of $\delta$-rings. This result, as Theorem 5.18 in [16], is also an extension of a classical measure theoretical result by Horowitz in [9] and can be seen as the generalisation of the construction method of regular conditional probabilities when the probability measure is an extended (i.e. not necessarily finite) signed measure.

Moreover, this result allows us to completely extend the classical results at the base of the Rajput and Rosinski’s work [19], like the mentioned construction method of regular conditional probabilities, to the signed case. Building on these results, we are able to obtain our second main contribution: we generalise, unify and simplify the theory of stochastic integrals based on QID random measures and so extend the whole integration theory developed in Rajput and Rosinski’s 1989 paper [19] and the results of [16]. In particular, this paper represents the realization of the main goal envisioned and attempted in [16] and it provides the most general integration theory with respect to (real valued) random measures.

In particular Theorem 5.18 in [16] does not allow to obtain a complete extension of the ID results to the QID framework. To circumvent this problem, the author introduces three different sets of assumptions (see Section 5 in [16]). In this paper given our first contribution we are able to obtain the whole QID integration theory under a single assumption which is weaker than all the three alternative sets of assumptions of [16]. More importantly, this assumption is always satisfied in the ID setting and is unavoidable since it comes directly from the necessary and sufficient condition on the extension of signed bimeasures. This allows us to affirm that in this paper we provide a true and complete extension of the results in the Rajput and Rosinski’s 1989 paper [19] and in [16], thus obtaining the most complete integration theory with respect to real-valued random measure.

The paper is structured as follows. Section 2 concerns with the notations and some preliminaries. In Section 3, the mentioned general measure theoretical result is presented (see Theorem 3.4). In Section 4 building on this result we derive the integration theory for QID random measures: Lévy-Khintchine formulations, integrability conditions, and continuity properties for QID stochastic integrals.

## 2 Notation and Preliminaries

By a measure on a measurable space $(X, \mathcal{G})$ we always mean a positive measure on $(X, \mathcal{G})$, i.e. a $[0, \infty]$-valued $\sigma$-additive set function on $\mathcal{G}$ that assigns the value 0 to the empty set. For a non-empty set $X$, by $\mathcal{B}(X)$ we mean the Borel $\sigma$-algebra of $X$, unless stated differently. The law and the characteristic function of a random variable $X$ will be denoted by $\mathcal{L}(X)$ and by $\hat{\mathcal{L}}(X)$, respectively. For two measurable spaces $(X, \mathcal{G})$ and $(Y, \mathcal{F})$, we denote by $\mathcal{G} \otimes \mathcal{F}$ the product $\sigma$-algebra of $\mathcal{G}$ and $\mathcal{F}$, and by $\mathcal{G} \times \mathcal{F}$ their Cartesian product. Let us recall some definitions.
Definition 2.1 (extended signed measure). Given a measurable space \((X, \Sigma)\), that is, a set \(X\) with a \(\sigma\)-algebra \(\Sigma\) on it, an extended signed measure is a function \(\mu : \Sigma \to \mathbb{R} \cup \{\infty, -\infty\}\) s.t. \(\mu(\emptyset) = 0\) and \(\mu\) is \(\sigma\)-additive, that is, it satisfies the equality \(\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)\) where the series on the right must converge in \(\mathbb{R} \cup \{\infty, -\infty\}\) absolutely (namely the value of the series is independent of the order of its elements), for any sequence \(A_1, A_2, \ldots\) of disjoint sets in \(\Sigma\).

As a consequence any extended signed measure can take plus or minus infinity as value, but not both. In this work, we use the term ‘signed measure’ for an extended signed measure. Further, the total variation of a signed measure \(\mu\) is defined as the measure \(|\mu| : \Sigma \to [0, \infty]\) defined by

\[
|\mu|(A) := \sup \sum_{j=1}^{\infty} |\mu(A_j)|,
\]

where the supremum is taken over all the partitions \(\{A_j\}\) of \(A \in \Sigma\). The total variation \(|\mu|\) is finite if and only if \(\mu\) is finite. Let us recall the definition of a signed bimeasure.

Definition 2.2 (Signed bimeasure). Let \((X, \Sigma)\) and \((Y, \Gamma)\) be two measurable spaces. A signed bimeasure is a function \(M : \Sigma \times \Gamma \to [-\infty, \infty]\) such that:

(i) the function \(A \to M(A, B)\) is a signed measure on \(\Sigma\) for every \(B \in \Gamma\),

(ii) the function \(B \to M(A, B)\) is a signed measure on \(\Gamma\) for every \(A \in \Sigma\).

Given a signed bimeasure \(G\) on \(\Sigma \times \Gamma\), we denote by \(G^+\) and \(G^-\) the Jordan decomposition of \(B \mapsto G(A, B)\) for fixed \(A \in \Sigma\), and by \(G_+\) and \(G_-\) the Jordan decomposition of \(A \mapsto M(A, B)\) for fixed \(B \in \Gamma\).

We use the term ‘measure’ and ‘signed measure’ not only in the case of \(\sigma\)-algebra, but also in the case of rings, as follows.

Definition 2.3 (Signed measure on a ring). A set function \(\mu(A)\) defined on the elements of a ring \(\mathcal{R}\) with values in \([-\infty, \infty]\) will be called a signed measure, if \(\mu(\emptyset) = 0\) and if for every sequence \(A_1, A_2, \ldots\) of disjoint sets of \(\mathcal{R}\) for which \(A = \bigcup_{k=1}^{\infty} A_k \in \mathcal{R}\) we have

\[
\mu(A) = \sum_{k=1}^{\infty} \mu(A_k)
\]

and the relation (2) holds absolutely (namely independent of the order of its elements).

Similarly, it is possible to extend the definition of bimeasures on rings. Let us now recall the celebrated Carathéodory’s extension theorem.

Theorem 2.4 (Carathéodory’s extension theorem, see Theorem 1.41 in [12]). Let \(\mu\) be a measure on a ring \(\mathcal{R}\) of subsets of a space \(X\). Assume that \(\mu\) is \(\sigma\)-finite (i.e. that there exists \(S_1, S_2, \ldots\) \(\in \mathcal{R}\) such that \(X = \bigcup_{n=1}^{\infty} S_n\) and that \(\mu(S_n) < \infty\) for every \(n \in \mathbb{N}\)) then there exists a unique \(\sigma\)-finite measure \(\bar{\mu}\) on \(\sigma(\mathcal{R})\) such that \(\mu(A) = \bar{\mu}(A)\) for all \(A \in \mathcal{R}\).

Denote by \(\mathcal{S}\) an arbitrary non-empty set. Let \(\mathcal{S}\) be a \(\delta\)-ring with the additional condition that there exists an increasing sequence of sets \(S_1, S_2, \ldots \in \mathcal{S}\) s.t. \(\bigcup_{n \in \mathbb{N}} S_n = \mathcal{S}\).

Definition 2.5 (random measure). Let \(\Lambda = \{\Lambda(A) : A \in \mathcal{S}\}\) be a real valued stochastic process defined on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We call \(\Lambda\) to be a random measure, if, for every sequence \(\{A_n\}\) of disjoint sets in \(\mathcal{S}\), the random variables \(\Lambda(A_n), n = 1, 2, \ldots\) are independent, and, if \(\bigcup_{n=1}^{\infty} A_n \in \mathcal{S}\), then we have \(\Lambda(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \Lambda(A_n)\) a.s. (where the series is assumed to converge almost surely).
Notice that a more correct, but way more tedious, name for a random measure is ‘independently scattered real-valued completely additive stochastic set function on $S$’. We remark that random measures are called sometimes called random noises, see e.g. [20], to distinguish from the random measures as defined in [10].

We recall the following result from Prékopa’s works.

**Theorem 2.6 (Theorem 2.1 in [18]).** In order that a finitely additive set function $\xi(A)$ defined on the elements of the ring $\mathcal{R}$ should be countably additive it is necessary and sufficient that, for every non-increasing sequence of sets $B_1, B_2, \ldots$ with $B_k \in \mathcal{R}$ ($k = 1, 2, \ldots$) and $B_n \searrow \emptyset$, $\xi(B_n) \xrightarrow{p} 0$ as $n \to \infty$.

Now, we introduce the concept of a quasi-Lévy type measure. We start with the following definition, which we recall from [13]:

**Definition 2.7.** Let $\mathcal{B}_r(\mathbb{R}) := \{B \in \mathcal{B}(\mathbb{R}) \mid B \cap (\mathbb{R} - r, r) = \emptyset\}$ for $r > 0$ and $\mathcal{B}_0(\mathbb{R}) := \bigcup_{r > 0} \mathcal{B}_r(\mathbb{R})$ be the class of all Borel sets that are bounded away from zero. Let $\nu : \mathcal{B}_0(\mathbb{R}) \to \mathbb{R}$ be a function such that $\nu|_{\mathcal{B}_r(\mathbb{R})}$ is a finite signed measure for each $r > 0$ and denote the total variation, positive and negative part of $\nu|_{\mathcal{B}_r(\mathbb{R})}$ by $|\nu|_{\mathcal{B}_r(\mathbb{R})}$, $\nu^+|_{\mathcal{B}_r(\mathbb{R})}$ and $\nu^-|_{\mathcal{B}_r(\mathbb{R})}$ respectively. Then the total variation $|\nu|$, the positive part $\nu^+$ and the negative part $\nu^-$ of $\nu$ are defined to be the unique measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfying

$$|\nu|(\{0\}) = \nu^+ (\{0\}) = \nu^- (\{0\}) = 0$$

and

$$|\nu|(A) = |\nu|_{\mathcal{B}_r(\mathbb{R})}(A), \quad \nu^+(A) = \nu^+|_{\mathcal{B}_r(\mathbb{R})}(A), \quad \nu^-(A) = \nu^-|_{\mathcal{B}_r(\mathbb{R})}(A),$$

for $A \in \mathcal{B}_r(\mathbb{R})$, for some $r > 0$.

As mentioned in [13], $\nu$ is not a a signed measure because it is defined on $\mathcal{B}_0(\mathbb{R})$, which is not a $\sigma$-algebra. In the case it is possible to extend the definition of $\nu$ to $\mathcal{B}(\mathbb{R})$ such that $\nu$ is a signed measure then we will identify $\nu$ with its extension to $\mathcal{B}(\mathbb{R})$ and speak of $\nu$ as a signed measure. Moreover, the uniqueness of $|\nu|$, $\nu^+$ and $\nu^-$ is ensured by an application of the Carathéodory’s extension theorem (see Lemma 2.14 in [16]). Further, notice that $\mathcal{B}_0(\mathbb{R}) = \{B \in \mathcal{B}(\mathbb{R}) : 0 \notin B\} \neq \{B \in \mathcal{B}(\mathbb{R}) : 0 \notin B\}$ (see Remark 2.6 in [16]).

**Definition 2.8 (quasi-Lévy type measure, quasi-Lévy measure, QID distribution, from [13]).** A quasi-Lévy type measure is a function $\nu : \mathcal{B}_0(\mathbb{R}) \to \mathbb{R}$ satisfying the condition in Definition 2.7 and such that its total variation $|\nu|$ satisfies $\int_{\mathbb{R}} (1 \wedge x^2)|\nu|(dx) < \infty$. Let $\mu$ be a probability distribution on $\mathbb{R}$. We say that $\mu$ is quasi-infinitely divisible if its characteristic function has a representation

$$\hat{\mu}(\theta) = \exp \left( i\theta \gamma - \frac{\theta^2}{2} a + \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta x) \nu(dx) \right)$$

where $a, \gamma \in \mathbb{R}$ and $\nu$ is a quasi-Lévy type measure. The characteristic triplet $(\gamma, a, \nu)$ of $\mu$ is unique, and $a$ and $\gamma$ are called the Gaussian variance and the drift of $\mu$, respectively. A quasi-Lévy type measure $\nu$ is called quasi-Lévy measure, if additionally there exist a quasi-infinitely divisible distribution $\mu$ and some $a, \gamma \in \mathbb{R}$ such that $(\gamma, a, \nu)$ is the characteristic triplet of $\mu$. We call $\nu$ the quasi-Lévy measure of $\mu$. 

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The above definition extend to the $\mathbb{R}^d$ case (for $d > 1$) as shown in Remark 2.4 in [13]. As pointed out in Example 2.9 of [13], a quasi-Lévy measure is always a quasi-Lévy type measure, while the converse is not true. Moreover, we say that a function $f$ is integrable with respect to quasi-Lévy type measure $\nu$ if it is integrable with respect to $|\nu|$. Then, we define:

$$\int_B f \, d\nu := \int_B f \, d\nu^+ - \int_B f \, d\nu^-, \quad B \in \mathcal{B}(\mathbb{R}).$$

In this work we always keep the same order for the elements in the characteristic triplet: the first element is the drift, the second one is the Gaussian variance, and the third one is the (quasi) Lévy measure.

**Definition 2.9** (QID random measure). Let $\Lambda$ be a random measure. If $\Lambda(A)$ is a QID random variable, for every $A \in \mathcal{S}$, then we call $\Lambda$ a QID random measure.

**Remark 2.10.** By construction any ID random measure is a QID random measure, but the converse is not always true, namely the set of QID random measures is strictly larger than the set of ID random measures.

Finally, we remark that the practical example one should have in mind when dealing with random measures is the following: a non-negative random measure $\Lambda$ on $\mathcal{B}_b(\mathbb{R})$ (i.e. the set of bounded intervals of $\mathbb{R}$), which has almost surely finite values for any $B \in \mathcal{B}_b(\mathbb{R})$. In this example, using the notation of this work, we have that $S = \mathbb{R}$ and $\mathcal{S} = \mathcal{B}_b(\mathbb{R})$. Then, under certain conditions (i.e. $\Lambda$ is independently scattered) and parametrisations (i.e. $\Lambda([0, t])$) it is possible to associate an additive stochastic process to $\Lambda$. In particular, let $X_t \overset{a.s.}{=} \Lambda([0, t])$ then $(X_t)_{t \in [0, \infty)}$ is a non-negative additive process. Observe that $\mathcal{B}_b(\mathbb{R})$ is not an algebra because $\mathbb{R} \notin \mathcal{B}_b(\mathbb{R})$ and is not a $\sigma$-ring, but a $\delta$-ring, because it is not closed under countable union. From this example, it also appears clear and natural the condition that imposes the existence of an increasing sequence of sets $S_1, S_2, \cdots \in \mathcal{B}_b(\mathbb{R})$ s.t. $\bigcup_{n \in \mathbb{N}} S_n = S$; indeed, think of $S_n$ as concentric balls of radii $n$, namely $(-n, n)$.

### 3 Extension of signed bimeasures and existence of relative kernels

In this section, the necessary and sufficient condition for the extension of signed bimeasures on $\delta$-rings. This result extends the classical results in [9] on the extension of signed bimeasures and Theorem 5.18 in [16] on the Jordan decomposition of such extension. In particular, in [9] the author shows that there exists a signed measure on the product space, under the assumption that the bimeasure is a bimeasure on the Cartesian product of two $\sigma$-algebra and that it satisfies a boundedness conditions. Here, we weaken the first part of the assumption (without strengthening the other one). Indeed, we work with spaces which are not necessarily measurable.

Moreover, in [9] there is no Jordan decomposition of the extended signed measure. Such decomposition has been recently proved in [16] (see Theorem 5.18 in [16]). Here, we prove the same result but under weaker assumptions.

Theorem 3.4 provides a general and complete “signed” version of the fundamental results at the base of Rajput and Rosinski’s work [19] (see Lemma 2.3 and Proposition 2.4 and all the results in their proofs). Theorem 3.4 can also be seen as the extension of the construction method of regular conditional probabilities when the probability measure is a signed measure.
Let us now recall the following results proved in [16].

**Lemma 3.1** (Lemma 2.15 in [16]). Let $X$ an arbitrary non-empty set and let $\mathcal{R}$ be a $\delta$-ring. Then, for every $E \in \mathcal{R}$ we have that $\{E \cap B : B \in \mathcal{S}\}$ is a $\sigma$-algebra.

**Lemma 3.2** (Lemma 2.16 in [16]). Let $X$ an arbitrary non-empty set and let $\mathcal{R}$ a $\delta$-ring. Let $\mu$ be a (possibly infinite) signed measure on $\mathcal{R}$. Then, there exist two unique measures $\mu^+$ and $\mu^-$ on $\mathcal{R}$ such that $\mu = \mu^+ - \mu^-$ and that on any $A \in \mathcal{R}$ they are mutually singular.

**Lemma 3.3** (Lemma 2.18 in [16]). Let $X$ an arbitrary non-empty set and let $\mathcal{R}$ a $\delta$-ring. Let $\mu$ be a $\sigma$-finite signed measure on $\mathcal{R}$ (namely there exists a sequence $S_1, S_2, \ldots \in \mathcal{R}$ s.t. $X = \bigcup_{n \in \mathbb{N}} S_n$ and that $-\infty < \mu(S_n) < \infty$ for every $n \in \mathbb{N}$). Then $\mu^+$ and $\mu^-$ can be uniquely extended to two $\sigma$-finite measures on $(X, \sigma(\mathcal{R}))$.

From the above it is possible to see that $|\mu| = \mu^+ + \mu^-$ is the total variation of $\mu$. Moreover, recall that a measurable space $(X, \mathcal{B})$ is a Lusin measurable space if $X$ is measure-theoretically isomorphic to a Borel subset of a compact metric space, and $\mathcal{B}$ is the induced $\sigma$-algebra. Thus, from definition we have that $\mathcal{B}$ is separable and that any standard Borel space is a Lusin measurable space.

**Theorem 3.4.** Let $(X, \mathcal{B})$ be a Lusin measurable space and let $(T, \mathcal{A})$ be such that $T$ is an arbitrary non-empty set and $A$ is a $\delta$-ring with the additional condition that there exists an increasing sequence of sets $T_1, T_2, \cdots \in \mathcal{A}$ s.t. $\bigcup_{n \in \mathbb{N}} T_n = T$. Let $Q_0(A, B)$ be a (possibly real valued) set function on $\mathcal{A}$, $B \in \mathcal{B}$, satisfying:

(a) for every $A \in \mathcal{A}$, $Q_0(A, \cdot)$ is a signed measure on $(X, \mathcal{B})$,

(b) for every $B \in \mathcal{B}$, $Q_0(\cdot, B)$ is a signed measure on $(T, \mathcal{A})$,

(c) $\sup \sum_{i \in I_A} |Q_0(A_i, B_i)| < \infty$, for every $A \in \mathcal{A}$, where the supremum is taken over all the finite families of disjoints elements of $(A \cap A) \times \mathcal{B}$.

Let $\nu(A) := \sup \sum_{i \in I_A} |Q_0(A_i, B_i)|$, where $A \in \mathcal{A}$. Then, $\nu(\cdot)$ has a unique extension on $(T, \sigma(\mathcal{A}))$. Further, there exist two unique measures $Q^+$ and $Q^-$ on $\sigma(\mathcal{A}) \otimes \mathcal{B}$ s.t.

$$Q^+(C) = \int_X \int_T 1_C(x, t) q^+(t, dx) \nu(dt) \quad \text{and} \quad Q^-(C) = \int_X \int_T 1_C(x, t) q^-(t, dx) \nu(dt), \quad (3)$$

where $C \in \sigma(\mathcal{A}) \otimes \mathcal{B}$. Moreover, there exists a unique finite signed measure on the $\delta$-ring $\bigcup_{D \in \mathcal{A}} (A \cap D) \otimes \mathcal{B}$ s.t.

$$Q(A \times B) = Q_0(A, B) = \int_A q(t, B) \nu(dt), \quad (4)$$

for every $A \in \mathcal{A}$, $B \in \mathcal{B}$, where $q : T \times B \to [-1, 1]$ and $q^+, q^- : T \times B \to [0, 1]$ fulfill the following conditions:

(d) for every $t \in T$, $q(t, \cdot)$ is a signed measure on $\mathcal{B}$,

(e) for every $B \in \mathcal{B}$, $q(\cdot, B)$ is $\sigma(\mathcal{A})$-measurable,

(d$'$) $q^+(t, \cdot)$ and $q^-(t, \cdot)$ are the Jordan decomposition of $q(t, \cdot)$,

(e$'$) $q^+(\cdot, B)$ and $q^-(\cdot, B)$ are $\sigma(\mathcal{A})$-measurable functions.

Further, if $q^+_1(\cdot, \cdot)$ is some other function satisfying (d$'$) and (e$'$), then off a set of $\nu$-measure zero, $q_1(t, \cdot) = q(t, \cdot)$. Similarly, if $q^-_1(\cdot, \cdot)$ and $q^-_1(\cdot, \cdot)$ are some other function satisfying (d$'$) and (e$'$), then off a set of $\nu$-measure zero, $q^+_1(t, \cdot) = q^+(t, \cdot)$ and $q^-_1(t, \cdot) = q^-(t, \cdot)$.

Finally, condition (c) is a necessary and sufficient condition for the existence of $Q$. 

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Proof. Since \( \sup_{i \in I_n} \sum_{i \in I_n} |Q_0(A_i, B_i)| < \infty \) then we have that \( \nu(T_n) < \infty \) for every \( n \in \mathbb{N} \). By Lemma 4.1 we know that \((A, A \cap A)\) is a \( \sigma \)-algebra, and by Theorem 4 in [9] (see also Theorem 5.17 in [16]) we know that \( \nu(\cdot) \) is a finite measure on \((A, A \cap A)\), for every \( A \in A \). Then, \( \nu(\cdot) \) is a measure on \( A \) and by the Carathéodory’s extension theorem we know that there exists a unique \( \sigma \)-finite extension on \( \sigma(A) \), which we still denote it by \( \nu \).

Consider the measurable space \((T_n, A \cap T_n)\). By Theorem 5.18 in [16] we have that there exist two unique finite measures

\[
Q_n^+(C) = \int_{T_n} \int_X 1_C(x, y)q_n^+(x, dy)\nu(dx) \quad \text{and} \quad Q_n^-(C) = \int_{T_n} \int_X 1_C(x, y)q_n^-(x, dy)\nu(dx),
\]

and a unique finite signed measure

\[
Q_n(C) = \int_{T_n} \int_X 1_C(x, y)q_n(x, dy)\nu(dx),
\]

where \( C \in (T_n, A \cap T_n) \otimes (X, B) \), \( q_n^+ \) and \( q_n^- \) are two sub-Markovian kernels such that for every \( x \in T_n \) they are the Jordan decomposition of a finite signed measure \( q_n \). In particular, we have that for every \( A \in A \cap T_n \) and \( B \in B \)

\[
Q_0(A, B) = Q_n(A \times B) = \int_A q_n(x, B)\nu(dx).
\]

Observe that the above holds for any \( n \in \mathbb{N} \). Now, we want to concatenate the sequence of obtained \( q_n^+ \)'s into one measure. For this purpose, let \( q^+(x, B) = q_n^+(x, B) \) when \( x \in T_n \setminus T_{n-1} \) and \( b \in B \). Then, it is possible to see that \( q^+(x, \cdot) \) is a measure for every \( x \in T \). This is because for every \( x \in T \) there exists a \( n \in \mathbb{N} \) large enough such that \( x \in T_n \setminus T_{n-1} \) and so \( q^+(x, \cdot) = q_n^+(x, \cdot) \), and we know that \( q_n^+(x, \cdot) \) is a measure on \((X, B)\). It is also possible to see that \( q^+(\cdot, B) \) is a \( \sigma(A) \)-measurable function, namely that for every \( B \in B \) and every \( A \in B(\mathbb{R}) \) we have that \( (q^+(\cdot, B))^{-1}(A) \in \sigma(A) \). Indeed, consider any \( A \in B(\mathbb{R}) \) and \( B \in B \), then

\[
(q^+(\cdot, B))^{-1}(A) = \{ x \in T \mid q^+(x, B) \in A \} = \bigcup_{n=1}^{\infty} \{ x \in T_n \setminus T_{n-1} \mid q_n^+(x, B) \in A \}
\]

\[
= \bigcup_{n=1}^{\infty} \{ x \in T_n \mid q_n^+(x, B) \in A \} \setminus T_{n-1}.
\]

Further, since \( q_n \) is a \( A \cap T_n \)-measurable functions and \( T_{n-1} \in A \cap T_n \), then \( \{ x \in T_n \mid q_n^+(x, B) \in A \} \setminus T_{n-1} \in A \) and since \( A \cap T_n \subseteq \sigma(A) \) then \( \{ x \in T_n \mid q_n^+(x, B) \in A \} \setminus T_{n-1} \in \sigma(A) \). Therefore, since \( \sigma \)-algebras are closed under countable unions we have that \( \bigcup_{n=1}^{\infty} \{ x \in T_n \mid q_n^+(x, B) \in A \} \setminus T_{n-1} \in \sigma(A) \) and so that \( q^+(\cdot, B) \) is a \( \sigma(A) \)-measurable function, for every \( B \in B \).

Similarly we can define \( q^- \) and by applying the same arguments we have that \( q_n^-(x, \cdot) \) is a measure on \((X, B)\) and \( q^- (\cdot, B) \) is a \( \sigma(A) \)-measurable function. Then, it is possible to define two (possibly infinite) measures \( Q^+ \) and \( Q^- \) on \( \sigma(A) \otimes B \) by

\[
Q^+(C) = \int_T \int_X 1_C(x, y)q^+(x, dy)\nu(dx) \quad \text{and} \quad Q^-(C) = \int_T \int_X 1_C(x, y)q^-(x, dy)\nu(dx),
\]
where $C \in \sigma(A) \otimes B$.

Notice that since $q_n^+(x, B) \leq 1$ and $q_n^-(x, B) \leq 1$ for every $x \in T_n$ and $B \in B$ and since this holds for every $n \in \mathbb{N}$, then $q^+(x, B) \leq 1$ and $q^-(x, B) \leq 1$ for every $x \in T$ and $B \in B$. In other words, $q^+$ and $q^-$ are sub-Markovian kernels. Then we can define $q$ to be $q(x, B) = q^+(x, B) - q^-(x, B)$ and notice that for every $x \in T$ $q^+(x, \cdot)$ and $q^-(x, \cdot)$ are the Jordan decomposition of $q(x, \cdot)$, and that for every $n \in \mathbb{N}$ we have $q(x, B) = q_n(x, B)$ for every $x \in T_n$ and $B \in B$.

Therefore, by putting together the results obtained so far we have, for every $A \in A$ and $B \in B$, that $Q_0(A, B) \leq \infty$ and in particular that

$$Q_0(A, B) = \sum_{n=1}^{\infty} Q_0(A \cap T_n \setminus T_{n-1}, B) = \sum_{n=1}^{\infty} \int_{A \cap T_n \setminus T_{n-1}} q_n(x, B) \nu(dx).$$

Indeed, consider any $D \in A$. For every $A \in A \cap D$ and $B \in B$ we have that

$$Q_0(A, B) = \int_A q(x, B) \nu(dx) = Q(A \times B).$$

Therefore, following Theorem 5.18 in [16] $Q$ is the unique finite signed measure on $(D \cap A) \otimes B$ s.t.

$$Q(C) = \int_T \int_X 1_C(x, y) q(x, dy) \nu(dx), \quad C \in (D \cap A) \otimes B.$$ 

Indeed, it is possible to see this also by the following arguments. For every $D \in A$ consider the measurable space $(D, A \cap D)$ and by applying Theorem 5.18 in [16] we obtain $Q_D$, $Q_D^+$, $Q_D^-$, $Q_D^+$, $Q_D^-$ and $q_D$ (as we have done before when we obtained $Q_n$, $Q_n^+$, $Q_n^-$, $q_n^+$, $q_n^-$ and $q_n$). Then, we would have that for every $A \in A \cap D$ and $B \in B$

$$\int_A q_D(x, B) \nu(dx) = Q_0(A, B) = \int_A q(x, B) \nu(dx)$$

and so $q_D(x, \cdot) = q(x, \cdot)$ off a set of $\nu$-measure zero on $D$. Thus, for every $C \in (D \cap A) \otimes B$

$$Q_D(C) = \int_T \int_X 1_C(x, y) q_D(x, dy) \nu(dx) = \int_T \int_X 1_C(x, y) q(x, dy) \nu(dx) = Q(C).$$

Finally, we prove uniqueness. If $q_1(\cdot, \cdot)$ is some other function satisfying [14], (d) and (e), then, off a set of $\nu$-measure zero, $q_1(t, \cdot) = q(t, \cdot)$ when $t \in T_n$. Thus,

$$\int_{T_n} q(x, B) \nu(dx) - \int_{T_n} q_1(x, B) \nu(dx) = 0.$$
Since this holds for every $n \in \mathbb{N}$, we have
\[
\int_T q(x, B) - q_1(x, B)\nu(dx) = \lim_{n \to \infty} \int_{T_n} q(x, B) - q_1(x, B)\nu(dx) = 0,
\]
for every $B \in \mathcal{B}$. Hence, off a set of $\nu$-measure zero, $q_1(t, \cdot) = q(t, \cdot)$, thus we get the uniqueness of $Q$. Now, from this and from the uniqueness of the Jordan decomposition we deduce that, off a set of $\nu$-measure zero, $q_1^+(t, \cdot) = q^+(t, \cdot)$ and $q_1^-(t, \cdot) = q^-(t, \cdot)$, whence we obtain the uniqueness of $Q^+$ and $Q^-$.

The condition (c) is necessary because $Q$ is a finite signed measure on $\bigcup_{D \in \mathcal{A}} (A \cap D) \otimes \mathcal{B}$, hence for every $A \in \mathcal{A}$ the total variation of $Q$, where $Q$ considered as a signed measure on the $\sigma$-algebra $(A \cap A) \otimes \mathcal{B}$ is larger than $\nu(A)$.

**Remark 3.5.** First, notice that $\bigcup_{D \in \mathcal{A}} (A \cap D) \otimes \mathcal{B} = \{ C \in (A \cap D) \otimes \mathcal{B}, \text{ for some } D \in \mathcal{A} \}$. Second, in many situations $\bigcup_{D \in \mathcal{A}} (A \cap D) \otimes \mathcal{B} = \bigcup_{n \in \mathbb{N}} (A \cap T_n) \otimes \mathcal{B}$. For example, this is the case of measures that takes finite values on bounded set of $\mathbb{R}$. In that case $T_n$’s are the concentric balls around zero and radius $n$, then for every $A \in \mathcal{A}$ there exists an $n \in \mathbb{N}$ large enough such that $A \subseteq T_n$ and so $(A, \mathcal{A} \cap A)$ is contained in $(T_n, A \cap T_n)$. In general, this is the case when $A$ is a localising ring of a measurable space, as it is the case in [10] (see page 15 and 19 in [10]). Third, notice that while
\[
\int_T q(x, B) - q_1(x, B)\nu(dx)
\]
is well defined and finite, the objects $\int_T q(x, B)\nu(dx)$ and $\int_T q_1(x, B)\nu(dx)$ are not well-defined.

**Remark 3.6.** Since $(T, \sigma(A))$ and $(X, \mathcal{B})$ are two measurable spaces, we remark that $q^+$ and $q^-$ are sub-Markovian kernels from $T$ to $X$ (see page 16 in [10]).

**Remark 3.7.** In [11], the authors obtained the result on the extension of signed bimeasures in [9] on $\sigma$-algebras of general measurable spaces, namely without the condition that one of them needs to be a Lusin measurable space. It might be possible to obtain such result also in our setting. However, this result, as the one in [11], will necessary lack the results on Markov kernels which is indeed of fundamental importance for our purpose and in general in probability theory and stochastic analysis, e.g. random measures [10], stochastic processes and stochastic integration [19, 16], regular conditional probabilities, etc...

### 4 Stochastic integration based on random measures

In this section we provide a complete general theory for stochastic integrals based on QID random measures. Therefore, we extend the main results of [16] on QID random measures and all the results in Chapter II and most of the results in Chapter III in [19].

First, let us introduce the topological setting of this section. We denote by $\mathcal{S}$ an arbitrary non-empty set and by $\mathcal{S}$ a $\delta$-ring with the additional condition that there exists an increasing sequence of sets $S_1, S_2, \cdots \in \mathcal{S}$ s.t. $\bigcup_{n \in \mathbb{N}} S_n = S$. In this framework $\mathcal{S}$ does not need to belong to $\mathcal{S}$ (thus $\mathcal{S}$ is not necessarily an algebra) and arbitrary subsets of $\mathcal{S}$ do not need to satisfy the condition $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{S}$ (thus $\mathcal{S}$ is not necessarily a $\sigma$-ring).
The following result extends Lemma 3.1 in [16]. For this result, we use the following notation. Let $\nu_0 : S \mapsto \mathbb{R}$ be a signed measure, $\nu_1 : S \mapsto [0, \infty)$ be a measure, $F_A(\cdot)$ be a quasi-Lévy type measure for every $A \in S$, and $F(B)$ be a finite signed measure for every $B \in B_0(\mathbb{R})$. Observe that such objects are typical for ID and QID random measures, see Section II in [19] and Section 3 and 4 in [16]. Further, we define for every $A \in S$ and $B \in \mathcal{B}(\mathbb{R})$

$$J(A, B) := \int_B (1 \wedge x^2) F_A(dx).$$

Consider the following assumption:

$$\sup_{I_A} \sum_{i \in I_A} |J(A_i, B_i)| < \infty, \quad \forall A \in S,$$

where the supremum is taken over all the finite families of disjoints elements of $(S \cap A) \times \mathcal{B}(\mathbb{R})$. In other words, the supremum is taken over all the finite families of the form $(A_i, B_i)_{i \in I_A}$, where $I_A$ is finite, $A_i \in S \cap A$, $B_i \in \mathcal{B}(\mathbb{R})$, and $(A_i \times B_i) \cap (A_j \times B_j) = \emptyset$ for every $i, j \in I_A$ with $i \neq j$. Since $\int_\mathbb{R} (1 \wedge x^2) |F_A|(dx) < \infty$, then $J(\cdot, B)$ is a finite signed measure on $S$ and $J(A, \cdot)$ is a finite signed measure on $\mathcal{B}(\mathbb{R})$. Further, as in [16] and [19], let

$$\tau(x) := \begin{cases} x & \text{if } |x| \leq 1, \\ \frac{x}{|x|} & \text{if } |x| > 1. \end{cases}$$

**Lemma 4.1.** Let $\nu_0$, $\nu_1$, $F$ be as above and let $F$ satisfy assumption (5). If the triplet $(\nu_0(A), \nu_1(A), F_A)$ is the characteristic triplet of a QID random variable, $\forall A \in S$. Then there exists a unique (in the sense of finite-dimensional distributions) QID random measure $\Lambda$ such that, for every $A \in S$,

$$\mathcal{L}(\Lambda(A))(\theta) = \exp \left( i \theta \nu_0(A) - \frac{\theta^2}{2} \nu_1(A) + \int_{\mathbb{R}} e^{i\theta x} - 1 - i \theta \tau(x) F_A(dx) \right).$$

Moreover, $\int_\mathbb{R} (1 \wedge x^2) F_A(dx)$ is a finite signed bimeasure on $S \times \mathcal{B}(\mathbb{R})$.

**Proof.** The existence of a finitely additive random measure $\Lambda = \{\Lambda(A) : A \in S\}$ follows by a standard application of Kolmogorov extension theorem using the finite additivity of $\nu_0(\cdot)$, $\nu_1(\cdot)$, and of $\int_{\mathbb{R}} e^{i\theta x} - 1 - i \theta \tau(x) F_A(dx)$, $\forall \theta \in \mathbb{R}$. Indeed, for the latter we have that for $A_1, A_2 \in S$ with $A_1 \cap A_2 = \emptyset$

$$\int_{\mathbb{R}} e^{i\theta x} - 1 - i \theta \tau(x) F_{A_1 \cup A_2}(dx) = \int_{|x| \geq \varepsilon} e^{i\theta x} - 1 - i \theta \tau(x) F_{A_1 \cup A_2}(dx) + \int_{|x| < \varepsilon} e^{i\theta x} - 1 - i \theta \tau(x) F_{A_1 \cup A_2}(dx)$$

$$= \int_{|x| \geq \varepsilon} e^{i\theta x} - 1 - i \theta \tau(x) F_{A_1}(dx) + \int_{|x| \geq \varepsilon} e^{i\theta x} - 1 - i \theta \tau(x) F_{A_2}(dx) + \int_{|x| < \varepsilon} e^{i\theta x} - 1 - i \theta \tau(x) F_{A_1 \cup A_2}(dx)$$

$$\xrightarrow{c \to 0} \int_{\mathbb{R}} e^{i\theta x} - 1 - i \theta \tau(x) F_{A_1}(dx) + \int_{\mathbb{R}} e^{i\theta x} - 1 - i \theta \tau(x) F_{A_2}(dx).$$

To prove that $\Lambda$ is countable additive let $A_n \downarrow \emptyset$ with $\{A_n\} \subset S$. Then, by definition of (signed) measures, $\nu_0(A_n) \to 0$ and $\nu_1(A_n) \to 0$. In the following we prove that $\int_{\mathbb{R}} (1 \wedge x^2) |F_{A_n}|(dx) \to 0$ as $n \to \infty$. By the Lévy continuity theorem, this will give us $\Lambda(A_n) \overset{p}{\to} 0$, namely the countable additivity of $\Lambda$. 

\newpage
Let $\epsilon > 0$ and let $J_\epsilon(A, B) := \int_\mathbb{R}(1 \wedge x^2)F_{A_n}(dx)$ for every $A \in \mathcal{S}$ and $B \in \mathcal{B}(\mathbb{R})$ (recall that $\mathcal{B}(\mathbb{R}) = \{B \in \mathcal{B}(\mathbb{R}) : B \cap (-\epsilon, \epsilon) = \emptyset\}$). By the properties of $F$ we have that $J_\epsilon$ is a finite signed bimeasure on $(A_1, S \cap A_1) \times \mathcal{B}(\mathbb{R})$. Notice that we have taken $A_1$ because $A_n \in (A_1, S \cap A_1)$ for every $n \in \mathbb{N}$. Thus, by Theorem 5.18 in [10] we have that there exists a unique finite signed measure on $(A_1, S \cap A_1) \otimes \mathcal{B}(\mathbb{R})$, call it $Q_\epsilon$. Observe that its total variation, which we denote by $|Q_\epsilon|$, is a well-defined finite measure on $(A_1, S \cap A_1) \otimes \mathcal{B}(\mathbb{R})$.

Now, let $E_A^+$ and $E_A^-$ be the Hahn decomposition of $\mathbb{R}$ under the signed measure $F_A(\cdot)$, for every $A \in \mathcal{S}$. Observe that, for every $n \in \mathbb{N}$, we have

$$|Q_\epsilon|(A_n \times (-\infty, -\epsilon] \cup [\epsilon, \infty)) \leq \sup_{I_{A_n} \in I_{A_n}} \sum_{i \in I_{A_n}} |J_i(A_i, B_i)|$$

Since $A_n \to \emptyset$, then $|Q_\epsilon|(A_n \times (-\infty, -\epsilon] \cup [\epsilon, \infty)) \to 0$ as $n \to \infty$. Thus, $\int_{|x| \geq \epsilon}(1 \wedge x^2)|F_{A_n}|(dx) \to 0$ as $n \to \infty$, for every $\epsilon > 0$.

By (5) we have that

$$\lim_{n \to \infty} \int_\mathbb{R}(1 \wedge x^2)|F_{A_n}|(dx) \leq \lim_{n \to \infty} \int_{|x| \geq \epsilon}(1 \wedge x^2)|F_{A_n}|(dx) + \int_{|x| < \epsilon}(1 \wedge x^2)|F_{A_n}|(dx)$$

which goes to zero as $\epsilon \to 0$.

Finally, $B \mapsto \int_B(1 \wedge x^2)F_A(dx)$ is a finite signed measure on $\mathcal{B}(\mathbb{R})$, for every $A \in \mathcal{S}$. Further, since $A \mapsto \int_B(1 \wedge x^2)F_A(dx)$ is finitely additive for every $B \in \mathcal{B}(\mathbb{R})$, and since $\int_B(1 \wedge x^2)|F_{A_n}|(dx) \to 0$ as $n \to \infty$ for every $A_n \downarrow \emptyset$, then $\int_B(1 \wedge x^2)F_A(dx)$ is a finite signed measure on $\mathcal{S}$, for every $B \in \mathcal{B}(\mathbb{R})$.

Notice that in Lemma 4.1 we have to mention the sentence “If $(\nu_0(A), \nu_1(A), F_A)$ is the characteristic triplet of a QID random variable, $\forall A \in \mathcal{S}$”, because, while for every QID distribution there exists a (unique) characteristic triplet, not every characteristic triplet gives rise to a (QID) distribution. Moreover, the reason why Lemma 4.1 is an extension of Lemma 3.1 in [10] is because the assumption on $F$ in the former is weaker than the one in the latter (see Lemma 4.1).

In the following result, we provide a trivial generalisation of Lemma 3.4 in [10].

**Lemma 4.2.** Let $\Lambda$ be random measure. Denote by $(\nu_0(A), \nu_1(A), F_A)$ the characteristic triplet of $\Lambda(A)$, for every $A \in \mathcal{S}$. Assume that (5) hold and that the following hold: for every $\{A_n\} \subset \mathcal{S}$ with $A_n \downarrow \emptyset$,

$$\int_\mathbb{R}e^{i0x} - 1 - i\theta \tau(x)F_{A_n}(dx) \to 0, \quad \forall \theta \in \mathbb{R} \Rightarrow F_{A_n}(B) \to 0, \quad \forall B \in \mathcal{B}_0(\mathbb{R}). \quad (7)$$
Then, \( \nu_0, \nu_1, F \) are as in Lemma 4.1, namely \( \nu_0 : \mathcal{S} \to \mathbb{R} \) is a signed measure, \( \nu_1 : \mathcal{S} \to [0, \infty) \) is a measure, \( F_A(\cdot) \) is a quasi-Lévy measure for every \( A \in \mathcal{S} \), and \( F(B) \) is a signed measure for every \( B \in \mathcal{B}_0(\mathbb{R}) \) and such that \( F \) satisfies (5).

**Proof.** First, since \( \Lambda \) is a QID random measure, it follows that \( F_A \) is a quasi-Lévy measure on \( \mathbb{R} \), for every \( A \in \mathcal{S} \). second, let \( \{A_k\}_{k=1}^n \), \( n \in \mathbb{N} \), be pairwise disjoint sets in \( \mathcal{S} \). By the uniqueness of the Lévy-Khintchine representation of a quasi-ID distribution, it follows, using \( \mathcal{L}(\Lambda(\cup_{k=1}^n A_k)) = \prod_{k=1}^n \mathcal{L}(\Lambda(A_k)) \), that all three set functions \( \nu_0, \nu_1 \) and \( F(B) \) (for every fixed \( B \in \mathcal{B}_0(\mathbb{R}) \)) are finitely additive. Let now \( \{A_n\} \subset \mathcal{S}, A_n \searrow \emptyset \). Since \( \Lambda(A_n) \not\to 0 \) we have that \( \nu_0(A_n) \to 0, \nu_1(A_n) \to 0 \) and \( \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta \tau(x) F_{A_n}(dx) \to 0, \forall \theta \in \mathbb{R} \). Thus, by (7) we obtain the stated result. \( \square \)

**Remark 4.3.** If we restrict to QID random measure satisfying both (5) and (7), then from the above lemmas we conclude that there is a one to one correspondence between a QID random measure satisfying (5) and (7), and a triplet composed by a finite signed measure, a finite measure, and a bi-set function, which is a quasi-Lévy measure for fixed \( A \in \mathcal{S} \), is a finite signed measure for fixed \( B \in \mathcal{B}_0(\mathbb{R}) \), and satisfies (5). We would like to have a one to one correspondence without condition (7), but this appears to be impossible.

We are now ready to present the results on QID stochastic integrals. For the rest of the section we assume the conditions of Lemma 4.1, namely we let: \( \nu_0 \) be a finite signed measure on \( \mathcal{S} \), \( \nu_1 \) be a measure on \( \mathcal{S} \), \( F_A(\cdot) \) be a quasi-Lévy type measure for every \( A \in \mathcal{S} \), \( F(B) \) be a finite signed measure for every \( B \in \mathcal{B}_0(\mathbb{R}) \), and condition (5) be satisfied.

Define the set function \( \nu(A) : \mathcal{S} \to [0, \infty) \) as

\[
\nu(A) := \sup_{I_A} \sum_{i \in I_A} |J(A_i, B_i)|. \tag{8}
\]

Notice that \( \nu(S_n) < \infty \) and that \( \nu \) is a measure on \( (S_n, \mathcal{S} \cap S_n) \). Then, by the Carathéodory’s extension theorem \( \nu \) extends to a \( \sigma \)-finite measure on \( (\mathcal{S}, \sigma(\mathcal{S})) \) (see also Theorem 3.4). To have a better idea of what kind of object \( \nu \) is, compare it with the definition of total variation of a signed measure (see (11)).

Let \( E_A^+ \) and \( E_A^- \) be the Hahn decomposition of \( \mathbb{R} \) under the signed measure \( F_A \). Observe that

\[
\nu(A) = \sup_{I_A} \sum_{i \in I_A} |J(A_i, B_i)| \geq \int_{E_A^+} (1 \wedge x^2) F_A(dx) - \int_{E_A^-} (1 \wedge x^2) F_A(dx) = \int_{\mathbb{R}} (1 \wedge x^2)|F_A|(dx). \tag{9}
\]

Therefore, since \( \nu(A) \) is finite by assumption we have that \( \int_{\mathbb{R}} (1 \wedge x^2)|F_A|(dx) < \infty \).

We show now that the assumption of this setting, namely (5), is weaker than the ones presented in Section 5 in [16]. Indeed, in the rest of this section we both unify and generalise the results on QID stochastic integrals in [16]. In Section 5 in [16], the authors investigate three different sets of assumptions under which the results on QID stochastic integrals are obtained.

In Subsection 5.1 in [16], they assume that the QID random measure is “generated” by two ID random measures. In other words, they assume that there exist two ID random measures \( \Lambda_G \) and \( \Lambda_M \) s.t. for every \( A \in \mathcal{S} \), \( \Lambda(A) + \Lambda_M(A) \overset{d}{=} \Lambda_G(A) \) and \( \Lambda_M(A) \) independent of \( \Lambda(A) \). This case resembles the definition of QID distributions transferred to random measures. In
In this case, \( F \) is given by \( F_A(B) = G_A(B) - M_A(B) \) for every \( A \in \mathcal{S} \) and \( B \in \mathcal{B}_0(\mathbb{R}) \), where \( G \) and \( M \) are the Lévy measure of \( \Lambda_G \) and \( \Lambda_M \). In Subsection 5.2, they assume that \( \mathcal{S} \) is a \( \sigma \)-algebra and that \( F \) is a finite signed bimeasure. In Subsection 5.3, they assume that \( \mathcal{S} \) is a \( \sigma \)-algebra and that \( \int_B (1 \wedge x^2) F_A(dx) \) is a finite signed bimeasure and that (5) holds. It is possible to see that the assumptions of Subsection 5.3 are strictly weaker than the ones in Subsection 5.2.

The assumptions of Subsection 5.3 in [16] are stricter than our assumptions. Indeed, assuming that \( \mathcal{S} \) is a \( \sigma \)-algebra is more restrictive than assuming that \( \mathcal{S} \) is a \( \delta \)-ring with the additional condition that there exists an increasing sequence of sets \( S_1, S_2, \ldots \in \mathcal{S} \) s.t. \( \bigcup_{n \in \mathbb{N}} S_n = \mathcal{S} \), which is the present setting. Moreover, in Lemma 4.4 we are able to show that (5) is enough to ensure that \( \int_B (1 \wedge x^2) F_A(dx) \) is a finite signed bimeasure.

Concerning the assumptions in Section 5.1 in [16], the following result shows that they are stricter than our assumption.

**Lemma 4.4.** Let \( G \) and \( M \) be defined as follow: \( G_A(\cdot) \) is a Lévy measure for every \( A \in \mathcal{S} \) and \( G.(B) \) is a measure for every \( B \in \mathcal{B}_0(\mathbb{R}) \) and similarly for \( M \). Let \( F_A(B) = G_A(B) - M_A(B) \) for every \( A \in \mathcal{S} \) and \( B \in \mathcal{B}_0(\mathbb{R}) \). Define \( \nu \) as in (8). Then,

\[
\int_\mathbb{R} (1 \wedge x^2) G_A(dx) + \int_\mathbb{R} (1 \wedge x^2) M_A(dx) < \infty, \quad \forall A \in \mathcal{S} \quad \Rightarrow \quad \text{Assumption (5).}
\]

**Proof.** Notice that \( \nu(A) < \infty, \forall A \in \mathcal{S} \), is equivalent to Assumption (5) and that

\[
\int_B (1 \wedge x^2) F_A(dx) = \int_B (1 \wedge x^2) G_A(dx) - \int_B (1 \wedge x^2) M_A(dx).
\]

Thus, for every \( A \in \mathcal{S} \), we have that

\[
\nu(A) = \sup_{I_A} \sum_{i \in I_A} |J(A_i, B_i)| \leq \sup_{I_A} \sum_{i \in I_A} \int_{B_i} (1 \wedge x^2) G_{A_i}(dx) + \int_{B_i} (1 \wedge x^2) M_{A_i}(dx).
\]

In the following, we prove that for every family \( (A_i, B_i)_{i \in I_A} \) we have

\[
\sum_{i \in I_A} \int_{B_i} (1 \wedge x^2) G_{A_i}(dx) \leq \int_\mathbb{R} (1 \wedge x^2) G_A(dx).
\]

If the \( A_i \)'s are all disjoints, then the \( B_i \)'s could take any values. In particular, by the (finite) additivity of \( G \) we obtain that

\[
\sum_{i \in I_A} \int_{B_i} (1 \wedge x^2) G_{A_i}(dx) \leq \sum_{i \in I_A} \int_\mathbb{R} (1 \wedge x^2) G_{A_i}(dx) = \int_\mathbb{R} (1 \wedge x^2) G_{\cup_{i \in I_A} A_i}(dx) = \int_\mathbb{R} (1 \wedge x^2) G_A(dx).
\]

Thus, it remains to investigate the case where the \( A_i \)'s have at least one intersection. Let \( (A_i)_{i \in I} \) be any finite family of sets in \( \mathcal{S} \cap A \). It is possible to find a finite set of disjoints elements in \( \mathcal{S} \cap A \), denote it \( (A_i')_{i \in I_A} \), such that \( \bigcup_{i \in I_A} A_i = \bigcup_{i \in I_A} A_i' \). Hence, each \( A_i' \) is a subset of one or more of the \( A_i \)'s. Therefore, the corresponding \( B_i \) of the \( A_i \)'s, whose intersection is \( A_i' \), cannot have intersections, because the rectangles \( (A_i, B_i)'s \) must be disjoint. This implies that for each \( A_i' \) the union of the corresponding \( B_i \)'s is a subset of \( \mathbb{R} \). Hence, we have

\[
\sum_{i \in I_A} \int_{B_i} (1 \wedge x^2) G_{A_i}(dx) \leq \sum_{i \in I_A} \int_\mathbb{R} (1 \wedge x^2) G_{A_i'}(dx) = \int_\mathbb{R} (1 \wedge x^2) G_{\cup_{i \in I_A} A_i'}(dx)
\]
\[ = \int_{\mathbb{R}} (1 \wedge x^2)G_{\cup i \in I_A}A_i(dx) = \int_{\mathbb{R}} (1 \wedge x^2)G_A(dx). \]

Since the same arguments hold for \( M \), we obtain the stated result. \( \square \)

**Remark 4.5.** This remark contains one of the most important take-home messages of this paper. From the proof of Lemma [4.4] and from [9], it is possible to see that for \( G \) we have that for every \( A \in S \)

\[
\sup_{I_A} \sum_{i \in I_A} \int_{B_i} (1 \wedge x^2)G_{A_i}(dx) = \int_{\mathbb{R}} (1 \wedge x^2)G_A(dx).
\]

In the present setting for \( G \), as well as for \( M \) and for the \( \text{levy} \) measure of any \( \text{ID} \) random measure in Rajput and Rosinski [19], we know that \( \int_{\mathbb{R}} (1 \wedge x^2)G_A(dx) < \infty \), because \( G \) is \text{levy} measure. This implies that assumption [5] in these cases is always satisfied. Thus, assumption [5] is always satisfied in the framework of Rajput and Rosinski [19]. Therefore, this assumption does not have to be seen as an artificial restrictive assumption, because in the non-signed case (as in [19]) is not a restrictive at all, and in the signed case it is there to satisfy assumption (c) in Theorem 3.4, which comes from the assumption (5) of Theorem 4 in [9] and represents a necessary and sufficient condition for the extension of signed bimeasures. The extension of signed bimeasures is essential for the development of the whole theory. This is why we also believe that assumption [5] cannot be weakened.

We are now ready to present the results on QID random measures.

**Proposition 4.6.** Let \( \nu_0 : S \mapsto \mathbb{R} \) be a signed measure, \( \nu_1 : S \mapsto \mathbb{R} \) be a measure, \( F_A \) be a quasi-Lévy type measure on \( \mathbb{R} \) for every \( A \in S \), \( S \ni A \mapsto F_A(B) \in (-\infty, \infty) \) be a signed measure for every \( B \in \mathcal{B}(\mathbb{R}) \) such that \( 0 \notin \overline{B} \) and that \( (\nu_0(A), \nu_1(A), F_A) \) is the characteristic triplet of a random variable, call it \( \Lambda(A) \), \( \forall A \in S \). Assume that \( F \) satisfies [5] and let \( \nu \) be defined as in [5]. Moreover, define

\[
\lambda(A) = |\nu_0|(A) + \nu_1(A) + \nu(A). \quad (10)
\]

Then \( \lambda : S \mapsto [0, \infty) \) is a measure s.t. \( \lambda(A_n) \to 0 \) implies \( \Lambda(A_n) \overset{P}{\to} 0 \) for every \( \{A_n\} \subset S \).

**Proof.** \( \lambda(A) \) is a sum of three measures on \( S \), hence it is a measure on \( S \). Further, let \( \lambda(A_n) \to 0 \) for some \( \{A_n\} \subset S \), then we have that \( |\nu_0|, \nu_1 \) and \( \nu \) go to zero. Recall \( \nu \) satisfies [9]. Then, by \( \text{levy} \) continuity theorem \( \Lambda(A_n) \overset{P}{\to} 0 \) as \( n \to \infty \). \( \square \)

**Definition 4.7.** Since \( \lambda(S_n) < \infty \), \( n = 1, 2, \ldots \) we extend \( \lambda \) to a \( \sigma \)-finite measure on \( (S, \sigma(S)) \); we call \( \lambda \) the control measure of \( \Lambda \).

**Lemma 4.8.** Let \( F \) be as in Proposition 4.6. There exists a function \( \rho : S \times \mathcal{B}_0(\mathbb{R}) \mapsto \mathbb{R} \) such that

(i) \( \rho(s, \cdot) \) is a quasi-Lévy type measure on \( \mathcal{B}(\mathbb{R}) \), for every \( s \in S \), with positive and negative parts denoted by \( \rho^+(s, \cdot) \) and \( \rho^-(s, \cdot) \),

(ii) \( \rho^+(\cdot, B) \) and \( \rho^-(\cdot, B) \), are \( \sigma(S) \)-measurable functions, for every \( B \in \mathcal{B}(\mathbb{R}) \).

Moreover, there exist two unique \( \sigma \)-finite measures \( \tilde{F}^+ \) and \( \tilde{F}^- \) on \( \sigma(S) \otimes \mathcal{B}(\mathbb{R}) \) s.t.

\[
\int_{S \times \mathbb{R}} h(s,x)\tilde{F}^+(ds,dx) = \int_S \int_{\mathbb{R}} h(s,x)\rho^-(s,dx)\lambda(ds)
\]
for every $\sigma(S) \otimes \mathcal{B}(\mathbb{R})$-measurable function $h : S \times \mathbb{R} \mapsto [0, \infty]$, and the same holds for $\tilde{F}^-$. This equality can be extended to real and complex-valued functions $h$. Finally, for every $A \in S$ and for every $\mathcal{B}(\mathbb{R})$-measurable real function $g$ s.t. $\int_{A} \int_{\mathbb{R}} |g(x)| \mu(s, dx) \lambda(\text{d}s) < \infty$, we have that

$$\int_{\mathbb{R}} g(x) F_A(\text{d}x) = \int_{A} \int_{\mathbb{R}} g(x) \rho(s, dx) \lambda(\text{d}s),$$

and for every $B \in \mathcal{B}(\mathbb{R})$ s.t. $0 \notin \overline{B}$,

$$\tilde{F}^+(A, B) \geq F^+_A(B) \quad \text{and} \quad \tilde{F}^-(A, B) \geq F^-_A(B).$$

**Proof.** First, notice that $J(A, B)$ satisfies the assumptions of Theorem 3.4 with $(T, \mathcal{A}) = (S, \mathcal{S})$ and $(X, \mathcal{B}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Therefore, there exists a finite real valued set function $Q$ on $\bigcup_{D \in \mathcal{S}}(S \cap D) \otimes \mathcal{B}$ such that

$$Q(A \times B) = J(A, B) = \int_{A} q(s, B) \nu(\text{d}s) = \int_{A} q^+(s, B) \nu(\text{d}s) - \int_{A} q^-(s, B) \nu(\text{d}s),$$

where $q^+$ and $q^-$ satisfy (d)' and (e)', and $q$ satisfies (d) and (e) of Theorem 3.4. Since $J(A, \{0\}) = 0$ for every $A \in S$ and since $q^+(s, \cdot)$ and $q^-(s, \cdot)$ are mutually singular, we deduce that $q^+(s, \{0\}) = 0$ and $q^-(s, \{0\}) = 0$ $\nu$-a.e.

Observe that we can consider $q^+(s, \{0\}) = 0$ and $q^-(s, \{0\}) = 0$ for every $s \in S$. This is because of the following argument. Let $q^+(s, \{0\}) = 0$ $\nu$-a.e. and let $\tilde{q}^+(s, B) = q^+(s, B \setminus \{0\})$ for every $B \in \mathcal{B}(\mathbb{R})$. Then $s \mapsto \tilde{q}^+(s, B)$ is $\sigma(S)$-measurable since $s \mapsto q^+(s, B \setminus \{0\})$ is $\sigma(S)$-measurable, for every $B \in \mathcal{B}(\mathbb{R})$. Moreover, for every sequence of disjoint sets $B_1, B_2, \ldots \in \mathcal{B}(\mathbb{R})$ $\tilde{q}^+(s, \bigcup_{i=1}^{\infty} B_i) = q^+(s, \bigcup_{i=1}^{\infty} B \setminus \{0\}) = \sum_{i=1}^{\infty} q^+(s, B_i \setminus \{0\}) = \sum_{i=1}^{\infty} \tilde{q}^+(s, B_i)$. Therefore, $\tilde{q}^+$ satisfies the same properties of $q^+$, namely (d)' and (e)' of Theorem 3.4 and $\tilde{q}^+(s, \cdot)$, $\tilde{q}^-(s, \cdot)$, off a set of $\nu$-measure zero. The same applies to $q^-$ and it is possible to see that $\tilde{q}^+(s, \cdot)$ and $\tilde{q}^-(s, \cdot)$ are the Jordan decomposition of a signed measure $\tilde{q}(s, \cdot)$, for every $s \in S$, and that $\tilde{q}(s, \cdot) = q(s, \cdot)$, off a set of $\nu$-measure zero. Hence, all the results of Theorem 3.4 applied to the present setting remained unchanged (indeed $\tilde{q}$ can be seen as the ‘$q$’ in the statement of Theorem 3.4). Thus, we consider $q^+(s, \{0\}) = q^-(s, \{0\}) = 0$ for every $s \in S$.

Since $\lambda \gg \nu$, define

$$\rho^+(s, dx) := \frac{d\nu}{d\lambda}(s)(1 \land x^2)^{-1} q^+(s, dx), \quad \text{and} \quad \rho^-(s, dx) := \frac{d\nu}{d\lambda}(s)(1 \land x^2)^{-1} q^-(s, dx).$$

Thus, $\rho^+(\cdot, B)$ and $\rho^-(\cdot, B)$ are Borel measurable (precisely $\sigma(S)$-measurable) functions. Further, notice that

$$\int_{\mathbb{R}} (1 \land x^2) \rho^+(s, dx) = \frac{d\nu}{d\lambda}(s) \int_{\mathbb{R}} q^+(s, dx) \leq 1,$$

where the last inequality comes from the fact that $\frac{d\nu}{d\lambda}(s) \leq 1$ for all $s \in S$. Hence, $\rho^+(s, \cdot)$ is a Lévy measure on $\mathbb{R}$ for all $s \in S$. The same holds for $\rho^-(s, \cdot)$. Further, let

$$\rho(s, B) := \rho^+(s, B) - \rho^-(s, B) \quad \text{for all} \ s \in S, \ B \in \mathcal{B}(\mathbb{R}) \ \text{s.t.} \ 0 \notin \overline{B}.$$
Now, let
\[ \tilde{F}^+(C) = \int_S \int_{\mathbb{R}} 1_C(s, x) \rho^+(s, dx) \lambda(ds), \]
where \( C \in \sigma(S) \otimes \mathcal{B}(\mathbb{R}) \), then \( \tilde{F}^+ \) is a well defined measure that satisfies, for every \( A \in S \) and \( B \in \mathcal{B}(\mathbb{R}) \),
\[ \tilde{F}^+(A \times B) = \int_A \int_B \rho^+(s, dx) \lambda(ds) = \int_A \int_B (1 \wedge x^2)^{-1} q^+(s, dx) \xi(ds) \]
where \( Q^+ \) is the positive extension of \( Q \) (see Theorem [3.4]), thus \( Q^+ \) is a measure on \( \sigma(S) \otimes \mathcal{B}(\mathbb{R}) \). Concerning \( J^+(A, dx) \), recall that the notation \( M^+ \) and \( M^- \) for a bimeasure \( M \) stands for the Jordan decomposition of \( B \mapsto M(A, B) \) for fixed \( A \). The same applies to \( \tilde{F}^- \). Finally, notice that for any \( \mathcal{B}(\mathbb{R}) \)-measurable real function \( g \) s.t. \( \int_A \int_{\mathbb{R}} |g(x)||\rho|(s, dx) \lambda(ds) < \infty \) we have
\[ \int_A \int_{\mathbb{R}} g(x) \rho(s, dx) \lambda(ds) = \int_A \int_{\mathbb{R}} g(x) \rho^+(s, dx) \lambda(ds) - \int_A \int_{\mathbb{R}} g(x) \rho^-(s, dx) \lambda(ds) \]
\[ = \int_{A \times \mathbb{R}} g(x)(1 \wedge x^2)^{-1} Q(ds, dx) = \int_{\mathbb{R}} g(x)(1 \wedge x^2)^{-1} J(A, dx) = \int_{\mathbb{R}} g(x) F_A(dx). \]

\[ \square \]

**Remark 4.9.** The discussion at the beginning of the proof of Lemma [4.8] on the possibility to consider \( q^+(s, \{0\}) = q^- (s, \{0\}) = 0 \), for every \( s \in S \), is implicit in the proofs of Lemma 2.3 in [19], and of Lemmas 5.19 and 5.28 in [16]. We decided to write it explicitly for the sake of clarity and completeness and because our setting requires more attention to detail.

Using the above results, we obtain the following proposition.

**Proposition 4.10.** Under the setting of Proposition [4.6], the characteristic function of \( \Lambda(A) \) can be written in the form:

\[ \mathbb{E}(e^{i\theta \Lambda(A)}) = \exp \left( \int_A K(\theta, s) \lambda(ds) \right), \quad \theta \in \mathbb{R}, A \in S, \]

where

\[ K(\theta, s) = i\theta a(s) - \frac{\theta^2}{2} \sigma^2(s) + \int_{\mathbb{R}} e^{i\theta x} - 1 - i\theta \tau(x) \rho(s, dx), \]

\[ a(s) = \frac{d\nu}{d\lambda}(s), \quad \sigma^2(s) = \frac{d\nu}{d\lambda}(s) \] and \( \rho \) is given by Lemma [4.8] and \( \exp(K(\theta, s)) \) is the characteristic function of a QID random variable if it exists. Moreover, we have

\[ |a(s)| + \sigma^2(s) + \frac{d\nu}{d\lambda}(s) = 1, \quad \lambda-a.e.. \]

**Proof.** The first statement follows from Lemma [4.8] and the Lévy-Khintchine formulation of \( \Lambda(A) \) [4]. The second statement follows from the fact that for every \( A \in S \), we have

\[ \int_A \left( |a(s)| + \sigma^2(s) + \frac{d\nu}{d\lambda}(s) \right) \lambda(ds) = |\nu_0|(A) + \nu_1(A) + \nu(A) + \lambda(A) = \int_A d\lambda(ds). \]

\[ \square \]
Let us recall the definition of $\Lambda$-integrability of a measurable function $f$ (see Definition in [16]).

**Definition 4.11.** Let $f(s) = \sum_{j=1}^{n} x_{j} 1_{A_j}(s)$ be a real simple function on $S$, where $A_j \in \mathcal{S}$ are disjoint. Then, for every $A \in \sigma(S)$, we define

$$\int_{A} f d\Lambda = \sum_{j=1}^{n} x_{j} \Lambda(A \cap A_j).$$

Further, a measurable function $f : (S, \sigma(S)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is said to be $\Lambda$-integrable if there exists a sequence $\{f_n\}$ of simple functions such that

(i) $f_n \to f$, $\lambda$-a.e.,

(ii) for every $A \in \sigma(S)$, the sequence $\{\int_{A} f_{n} d\Lambda\}$ converges in probability as $n \to \infty$.

If $f$ is $\Lambda$-integrable, then we write

$$\int_{A} f d\Lambda = \mathbb{P} - \lim_{n \to \infty} \int_{A} f_{n} d\Lambda$$

where $\{f_n\}$ satisfies (i) and (ii).

As proved in Lemma 5.8 in [16], the integral $\int_{A} f d\Lambda$ is well-defined. In the following result we provide a representation for the characteristic function of $\int_{S} f d\Lambda$. The remaining results of this section follow from the same arguments as the ones used in the proof of their respective results in [16]. This is because, despite the fact that we are considering a larger class of QID random measures (because our assumptions are weaker), the structure of the control measure and of the representations of $F$, $\tilde{F}^+$ and $\tilde{F}^-$ are similar to the ones in [16].

**Proposition 4.12.** Under the setting of Proposition 4.6, if $f$ is $\Lambda$-integrable, then we have that $\int_{S} |K(tf(s), s)| \lambda(ds) < \infty$, where $K$ is given in Proposition 4.10 and that

$$\hat{\mathbb{L}} \left( \int_{S} f d\Lambda \right)(\theta) = \exp \left( \int_{S} K(\theta f(s), s) \lambda(ds) \right), \quad \theta \in \mathbb{R}.$$

**Proof.** It follows from the same arguments as the ones used in the proof of Proposition 5.9 in [16]. \qed

We state an important result on the integrability conditions of $\int_{S} f d\Lambda$.

**Theorem 4.13.** Let $f : S \to \mathbb{R}$ be a $\sigma(S)$-measurable function and consider the setting of Proposition 4.6. Then $f$ is $\Lambda$-integrable if the following three conditions hold:

(i) $\int_{S} |U(f(s), s)| \lambda(ds) < \infty$,

(ii) $\int_{S} |f(s)|^2 \sigma^2(s) \lambda(ds) < \infty$,

(iii) $\int_{S} V_0(f(s), s) \lambda(ds) < \infty$,

where

$$U(u, s) = u a(s) + \int_{\mathbb{R}} \tau(xu) - u \tau(x) \rho(s, dx), \quad \text{and} \quad V_0(u, s) = \int_{\mathbb{R}} (1 \wedge |x u|^2) \rho(s, dx).$$

Further, the characteristic function of $\int_{S} f d\Lambda$ can be written as

(iv) $\hat{\mathbb{L}} \left( \int_{S} f d\Lambda \right)(\theta) = \exp \left( i \theta a_f - \frac{1}{2} \sigma_f^2 + \int_{\mathbb{R}} e^{i \theta x} - 1 - i \theta \tau(x) F_f(dx) \right)$,

where $a_f = \int_{S} U(f(s), s) \lambda(ds)$, $\sigma_f^2 = \int_{S} |f(s)|^2 \sigma(s) \lambda(ds)$, and
\( F_f(B) \) is the unique quasi-Lévy measure determined by the difference of the Lévy measures \( \tilde{F}_f^+ \) and \( \tilde{F}_f^- \), which are defined as: for every \( B \in \mathcal{B}(\mathbb{R}) \)

\[
\tilde{F}_f^+(B) = \tilde{F}^+(\{(s, x) \in S \times \mathbb{R} : f(s) x \in B \setminus \{0\}\}) \quad \text{and}
\tilde{F}_f^-(B) = \tilde{F}^-(\{(s, x) \in S \times \mathbb{R} : f(s) x \in B \setminus \{0\}\}).
\]

**Proof.** It follows from the same arguments as the ones used in the proof of Theorem 5.10 in [16].

We conclude with a result on the continuity of the stochastic integral mapping. Before presenting the result, we need some preliminaries. Define the Musielak-Orlicz space as in [19]:

\[
L_{\Phi^p}(S; \lambda) = \left\{ f \in L_0(S; \lambda) : \int_S \Phi_p(|f(s)|, s) \lambda(ds) < \infty \right\}.
\]

By an equivalent result of Lemma 6.1 in [16] applied here it is possible to see that the space \( L_{\Phi^p}(S; \lambda) \) is a complete linear metric space with the \( F \)-norm defined by

\[
\|f\|_{\Phi^p} = \inf_{c>0} \left\{ \int_S \Phi_p(c^{-1}|f(s)|, s) \lambda(ds) \leq c \right\}.
\]

Simple functions are dense in \( L_{\Phi^p}(S; \lambda) \) and \( L_{\Phi^p}(S; \lambda) \hookrightarrow L_0(S; \lambda) \) is continuous, where in the present case \( L_0(S; \lambda) \) is equipped with the topology of convergence in \( \lambda \)-measure on every set of finite \( \lambda \)-measure. Moreover, we have \( \|f_n\|_{\Phi^p} \rightarrow 0 \Leftrightarrow \int_S \Phi_p(|f(s)|, s) \lambda(ds) \rightarrow 0 \). Now, define, for \( 0 \leq p \leq q \), \( u \in \mathbb{R} \) and \( s \in S \),

\[
\Phi_p(u, s) = U^*(u, s) + u^2 \sigma^2(s) + V_p(u, s),
\]

where

\[
U^*(u, s) = \sup_{|c| \leq 1} |U(cu, s)| \quad \text{and} \quad V_p(u, s) = \int_{\mathbb{R}} |xu|^p 1_{|xu| > 1}(x) + |xu|^2 1_{|xu| \leq 1}(x) |\rho|(s, dx).
\]

**Theorem 4.14.** Let \( 0 \leq p \leq q \) and \( \Phi_p \) defined as in (12). Then

\[
\left\{ f : f \text{ is } \Lambda \text{-integrable and } \mathbb{E} \left[ \int_S |f|^{p} \right] < \infty \right\} \supset L_{\Phi_p}(S; \lambda)
\]

and the linear mapping

\[
L_{\Phi_p}(S; \lambda) \ni f \mapsto \int_S f \lambda \in L_p(\Omega; \mathbb{P})
\]

is continuous.

**Proof.** It follows from the same arguments as the ones used in the proof of Theorem 6.3 in [16].

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