New Development of the Enclosure Method for Inverse Obstacle Scattering

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1 Introduction

The Enclosure Method introduced in [11, 10, 13] has become a well-known guiding principle in attacking various inverse obstacle problems [33] governed by partial differential equations. It is simpler than the Probe Method which has been introduced in [8, 9, 12].

The Enclosure Method aims at obtaining information about the geometry of unknown discontinuity. The method consists of three steps listed below:

- choosing a special solution $v$ depending on a large parameter $\tau > 0$ and independent of the unknown discontinuity;
- constructing a so-called indicator function of independent variable $\tau$ by using observation data and $v$;
- studying asymptotic behaviour of the indicator function as $\tau \to \infty$.

From the asymptotic behaviour of the indicator function we find a domain that encloses unknown discontinuity. The Enclosure Method is quite flexible and its realization depends on

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the choice of \( v \) in the first step and whether the observation data in the second step depend on \( v \) or not.

Now we have many applications of this flexible method to various inverse obstacle problems governed by elliptic partial differential equations or systems. See [22] for the systematic explanation of the Enclosure Method from the beginning and also [15, 19, 26, 31, 32, 35, 36] and [37] with references therein for further applications. For a nonlinear partial differential equation we cite also a recent remarkable work [3].

It was the paper [14] which opened the door to various possibilities of the Enclosure Method in the time domain inverse obstacle problems governed by the heat or wave equations in one-space dimension. Now we have several papers [28, 16, 29, 18, 27, 30] in which the range of application of the Enclosure Method has been extended to inverse obstacle problems governed by parabolic or hyperbolic equations over a finite time interval in three-space dimensions.

The aim of this chapter is to make a review of the recent results using the Enclosure Method on inverse obstacle problems governed by the wave equation and the Maxwell system in time domain. We also describe some of unsolved problems related to further possibility of the Enclosure Method itself. Those are not mentioned in the expository paper [22] and survey paper [17].

2 The enclosure method for inverse obstacle scattering in time domain

The description of the problem is simple. Send a wave and observe the reflected wave by an unknown obstacle (discontinuity). What information about the obstacle can one extract from the observed wave? This type of problems have their origin in sonar, radar, nondestructive testing, etc..

Recently, using the Enclosure Method as the guiding principle, we considered the problem under the constraints: sending at most finitely many waves at a finite distance from the obstacle; observing a reflected wave over a finite time interval and thus at a finite distance from the obstacle; sending and observing places are same. In particular, we use neither the asymptotic behaviour of the wave as time goes to infinity nor the far field profile. In this section we present some of recent results from [20, 24] and their applications.

Let \( D \) be a non-empty bounded open subset of \( \mathbb{R}^3 \) with \( C^2 \)-boundary such that \( \mathbb{R}^3 \setminus \overline{D} \) is connected. Let \( 0 < T < \infty \). Let \( f \in L^2(\mathbb{R}^3) \) satisfy \( \text{supp} f \cap \overline{D} = \emptyset \).

We denote by \( u_f \) the (weak) solution of the following initial boundary value problem for the wave equation:

\[
\begin{align*}
\partial^2_t u - \Delta u &= 0 \text{ in } (\mathbb{R}^3 \setminus \overline{D}) \times [0, T], \\
\frac{\partial u}{\partial \nu} - \gamma(x)\partial_t u - \beta(x)u &= 0 \text{ on } \partial D \times [0, T], \\
u(x, 0) &= 0 \text{ in } \mathbb{R}^3 \setminus \overline{D}, \quad \partial_t u(x, 0) = f(x) \text{ in } \mathbb{R}^3 \setminus \overline{D},
\end{align*}
\]

where \( \nu \) denotes the unit outward normal to \( D \) on \( \partial D \), \( \beta \in L^\infty(\partial D) \), \( \gamma \in L^\infty(\partial D) \) and \( \gamma(x) \geq 0 \) a.e. \( x \in \partial D \). We omit the description about the solution class taken from [5]. See [20] for the description.

The role of \( \gamma \geq 0 \) can be seen from the formal computation

\[
\mathcal{E}'(t) = -\int_{\partial D} \gamma(x)|\partial_t u|^2 dx \leq 0,
\]
where
\[ E(t) = \frac{1}{2} \int_{\mathbb{R}^3 \setminus \overline{D}} (|\partial_t u|^2 + |\nabla u|^2) \, dx + \frac{1}{2} \int_{\partial D} \beta(x)|u|^2 \, dS, \quad t \in [0, T]. \]

We think that the distribution of the values of \( \gamma \) and \( \beta \) on \( \partial D \) is a mathematical model of the state of the surface of the obstacle.

Let \( B \) be the open ball centred at \( p \) with very small radius \( \eta \) and satisfy \( B \cap \overline{D} = \emptyset \). We denote by \( \chi_B \) the characteristic function of \( B \).

**Problem 2.1.** Generate \( u = u_f \) by the initial data \( f = \chi_B \) and observe \( u \) on \( B \) over time interval \([0, T]\). Extract information about the geometry of \( D \), \( \gamma \) and \( \beta \) from the observed data.

The correspondence \( (D, \gamma, \beta) \mapsto u|_{B \times [0, T]} \) is nonlinear. Therefore, Problem 2.1 becomes a nonlinear problem.

### 2.1 Indicator function

The Enclosure Method in time domain also introduces an indicator function like the Enclosure Method in frequency domain. It starts with introducing a special solution with a large parameter. In what follows we always choose \( f = \chi_B \) unless otherwise specified and \( u_f \) is the solution of (2.1).

Let \( \tau > 0 \) and \( v = v_f(\cdot, \tau) \in H^1(\mathbb{R}^3) \) be the solution of
\[
(\Delta - \tau^2)v + f = 0 \text{ in } \mathbb{R}^3. \tag{2.2}
\]

\( v_f \) has the expression
\[
v_f(x, \tau) = \frac{1}{4\pi} \int_B e^{-\tau|x-y|} \, dy. \tag{2.3}
\]

Define
\[
w_f(x, \tau) = \int_0^T e^{-\tau t} u_f(x, t) \, dt, \quad x \in \mathbb{R}^3 \setminus \overline{D}. \tag{2.4}
\]

Using \( v_f \) and \( w_f \), we define the indicator function of \( \tau \):
\[
I_B(\tau) = \int_B (w_f - v_f) \, dx.
\]

This indicator function looks different from the one in the previous version of the Enclosure Method [10]. So someone may have a question: why should it be called the indicator function?

However, in [20], we have pointed out the asymptotic formula:
\[
I_B(\tau) = \int_{\partial \Omega} \left( \frac{\partial v_f}{\partial \nu} w_f - \frac{\partial w_f}{\partial \nu} v_f \right) \, dS + O(\tau^{-1} e^{-\tau T}),
\]

for an arbitrary fixed \( T < \infty \) and provided \( \overline{B} \cap \overline{\Omega} = \emptyset \) and \( \overline{D} \subset \Omega \).

And from (2.4) we have the space-time expression
\[
\int_{\partial \Omega} \left( \frac{\partial v_f}{\partial \nu} w_f - \frac{\partial w_f}{\partial \nu} v_f \right) \, dS = \int_M \left( \frac{\partial (e^{-\tau t} v_f)}{\partial \nu} u_f - \frac{\partial u_f}{\partial \nu} (e^{-\tau t} v_f) \right) \, dS dt,
\]

where \( M = \partial \Omega \times [0, T] \). Note also that: (2.2) implies that \( e^{-\tau t} v_f \) satisfies the wave equation in a neighbourhood of \( \overline{D} \).

Therefore, one can say that \( I_B(\tau) \) is essentially similar to the indicator function in the previous version of the Enclosure Method. It is a space-time version.
2.2 Qualitative state of the surface, distance and direction

The following result is the starting point of the Enclosure Method in time domain.

**Theorem 2.1**([20]). Let \( T > 2 \text{dist}(D, B) \). Let \( C \) be a positive constant.

We have: if \( \gamma(x) \leq 1 - C \) a.e. \( x \in \partial D \), then there exists \( \tau_0 > 0 \) such that \( I_B(\tau) > 0 \) for all \( \tau \geq \tau_0 \); if \( \gamma(x) \geq 1 + C \) a.e. \( x \in \partial D \), then there exists \( \tau_0 > 0 \) such that \( I_B(\tau) < 0 \) for all \( \tau \geq \tau_0 \).

Moreover, in both cases, the formula

\[
\lim_{\tau \to \infty} \frac{1}{\tau} \log |I_B(\tau)| = -2 \text{dist}(D, B),
\]

is valid.

Define \( d_{\partial D}(p) = \inf_{x \in \partial D} |x - p| \). We see that knowing \( \text{dist}(D, B) \) is equivalent to knowing \( d_{\partial D}(p) \) since \( \text{dist}(D, B) = d_{\partial D}(p) - \eta \). Thus Theorem 2.1 yields the sphere \( |x - p| = d_{\partial D}(p) \) whose exterior contains \( D \) and on which at least one point on \( \partial D \) exists. Thus this should be called an enclosing method by using the exterior of a sphere. And also, roughly speaking, we can know the qualitative state of the surface of the unknown obstacle, that is whether \( \gamma >> 1 \) or \( \gamma << 1 \) by the signature of indicator function \( I_B(\tau) \) for a large \( \tau \) as Theorem 2.1 states.

Finally we present a procedure for making a decision around \( p \) whether given direction \( \omega \in S^2 \) the point \( p + d_{\partial D}(p)\omega \) belongs to \( \partial D \) or not provided \( d_{\partial D}(p) \) is known.

Fix a large \( T \) and small \( s \in ]0, d_{\partial D}(p)[. \) Give direction \( \omega \in S^2 \) choose an open ball \( B' \) centred at \( p + s\omega \) such that \( B' \) is contained in the open ball centred at \( p \) with radius \( d_{\partial D}(p) \).

**Step 1.** Generate \( u_f \) by the initial data \( f = \chi_{B'} \) and observe \( u_f \) on \( B' \) over time interval \( ]0, T[. \)

**Step 2.** Calculate \( d_{\partial D}(p + s\omega) \) from the data obtained in Step 1 via (2.5) in Theorem 2.1 in the case when \( B \) is replaced with \( B' \).

We always have \( d_{\partial D}(p + s\omega) \geq d_{\partial D}(p) - s \). Moreover, it holds that:

- if \( d_{\partial D}(p + s\omega) = d_{\partial D}(p) - s \), then \( p + d_{\partial D}(p)\omega \) is on \( \partial D \);
- if \( d_{\partial D}(p + s\omega) > d_{\partial D}(p) - s \), then \( p + d_{\partial D}(p)\omega \) not on \( \partial D \).

Therefore one can make a decision around \( p \) whether \( p + d_{\partial D}(p)\omega \) is on \( \partial D \) or not.

2.3 A sketch of the proof of Theorem 2.1

The proof of Theorem 2.1 consists of three parts as described below.

**Lemma 2.1.** We have, as \( \tau \to \infty \)

\[
\|R\|_{L^2(R^3 \setminus \overline{D})} = O(e^{-\tau \text{dist}(D, B)} + e^{-\tau T}),
\]

\[
\|\nabla R\|_{L^2(R^3 \setminus \overline{D})} = O(\tau(e^{-T} + e^{-\tau \text{dist}(D, B)}))
\]

and

\[
\|R\|_{L^2(\partial D)} = O(\tau^{1/2}(e^{-\tau \text{dist}(D, B)} + e^{-T})),
\]

where \( R = w_f - v_f \).

A brief outline of the proof of Lemma 2.1 is as follows. It follows from (2.1) and (2.2) that \( R \) satisfies

\[
\left\{
\begin{array}{l}
(\Delta - \tau^2)R = e^{-\tau TF} \text{ in } R^3 \setminus \overline{D}, \\
\frac{\partial R}{\partial \nu} - cR = - \left( \frac{\partial v}{\partial \nu} - cv \right) + e^{-\tau T}G \text{ on } \partial D,
\end{array}
\right.
\]

(2.6)
where
\[ c = c(x, \tau) = \gamma(x)\tau + \beta(x), \]
\[ F = F(x, \tau) = \partial_t u(x, T) + \tau u(x, T), \quad (2.7) \]
\[ G = G(x) = \gamma(x)u(x, T). \]

Equation (2.6) and integration by parts give
\[ \int_{\mathbb{R}^3 \setminus \overline{D}} (|\nabla R|^2 + \tau^2 |R|^2 + e^{-\tau T} F R) \, dx \]
\[ + \int_{\partial D} \left\{ c|R|^2 - \left( \frac{\partial v}{\partial \nu} - cv \right) R + e^{-\tau T} GR \right\} \, dS = 0. \quad (2.8) \]

Using a trace theorem [7] and the assumption \( \gamma(x) \geq 0 \) a.e. \( x \in \partial D \), from (2.8) one can easily deduce the conclusion. See Lemma 2.1 and (2.28) in [20].

Let us continue the sketch of the proof of Theorem 2.1. Since \( v \) satisfies (2.2), we obtain
\[ \int_{\mathbb{R}^3 \setminus \overline{D}} f R \, dx = \int_{\partial D} \frac{\partial v}{\partial \nu} R \, dS + \int_{\mathbb{R}^3 \setminus \overline{D}} (\nabla v \cdot \nabla R + \tau^2 v R) \, dx. \quad (2.9) \]

On the other hand, from (2.6) we obtain
\[ 0 = \int_{\partial D} \left\{ cR - \left( \frac{\partial v}{\partial \nu} - cv \right) \right\} v \, dS + \int_{\mathbb{R}^3 \setminus \overline{D}} (\nabla R \cdot \nabla v + \tau^2 R v) \, dx \]
\[ + e^{-\tau T} \left( \int_{\mathbb{R}^3 \setminus \overline{D}} F v \, dx + \int_{\partial D} G \, v dS \right). \quad (2.10) \]

Taking the difference of (2.9) from (2.10), we obtain
\[ I_B(\tau) = \int_{\partial D} \left( \frac{\partial v}{\partial \nu} - cv \right) R dS + \int_{\partial D} \left( \frac{\partial v}{\partial \nu} - cv \right) v dS \]
\[ - e^{-\tau T} \left( \int_{\mathbb{R}^3 \setminus \overline{D}} F v \, dx + \int_{\partial D} G \, v dS \right). \]

Then, applying (2.8) to the first term on this right-hand side, we obtain another expression
\[ I_B(\tau) = \int_{\mathbb{R}^3 \setminus \overline{D}} (|\nabla R|^2 + \tau^2 |R|^2) \, dx + \int_{\partial D} c|R|^2 \, dS + \int_{\partial D} \left( \frac{\partial v}{\partial \nu} - cv \right) v \, dS \]
\[ + e^{-\tau T} \left( \int_{\mathbb{R}^3 \setminus \overline{D}} F R \, dx + \int_{\partial D} G R \, dS - \int_{\mathbb{R}^3 \setminus \overline{D}} F v \, dx - \int_{\partial D} G v \, dS \right). \quad (2.11) \]

It is easy to see that, as \( \tau \to \infty \), \( \|v\|_{L^2(\mathbb{R}^3 \setminus \overline{D})} = O(\tau^{-2}) \) and \( \|v\|_{L^2(\partial D)} = O(e^{-\tau \text{dist}(D,B)}) \). Thus from this, (2.7) and Lemma 2.1 we see that the last term in the right-hand side on (2.11) has bound \( O(\tau^{-1} e^{-\tau T}) \) as \( \tau \to \infty \). Therefore we have, as \( \tau \to \infty \)
\[ I_B(\tau) = E(\tau) + J(\tau) + O(\tau^{-1} e^{-\tau T}), \quad (2.12) \]
where

\[ E(\tau) = \int_{\mathbb{R}^3 \setminus \overline{D}} (|\nabla R|^2 + \tau^2 |R|^2) dx + \int_{\partial D} c|R|^2 dS \]

and

\[ J(\tau) = \int_{\partial D} \left( \frac{\partial v}{\partial \nu} - cv \right) v dS. \]

A combination of this and Lemma 2.1 yields the following estimates on the bound of the indicator function.

**Lemma 2.2.** We have the following asymptotic estimates:

(i) if \( 0 \leq \gamma(x) \) a.e. \( x \in \partial D \), then as \( \tau \to \infty \) we have

\[ I_B(\tau) \geq J(\tau) + O(\tau^{-1} e^{-\tau T}); \quad (2.13) \]

(ii) if \( \gamma(x) \geq C' \) a.e. \( x \in \partial D \) for a positive constant \( C' \), then as \( \tau \to \infty \) we have

\[ I_B(\tau) \leq J(\tau) + \int_{\partial D} \left| \frac{1}{c} \left( \frac{\partial v}{\partial \nu} - cv \right) \right|^2 dS + O(\tau^{-1} e^{-\tau T}); \quad (2.14) \]

(iii) as \( \tau \to \infty \) we have

\[ |I_B(\tau)| = O(\tau^2 e^{-2\tau \text{dist}(D,B)}) + \tau^{-1} e^{-\tau T}). \quad (2.15) \]

A brief outline of the proof of Lemma 2.2 is as follows. From (2.3), (2.12) and Lemma 2.1 we have (2.15); (2.13) is clear from (2.12) and the positivity of \( E(\tau) \) for \( \tau >> 1 \) which is a consequence of the trace theorem \[7\]. We present here a sketch of the proof of (2.14). See [20] for the full proof. Assume that \( \gamma(x) \geq C' \) a.e. \( x \in \partial D \) for a positive constant \( C' \). Rewrite (2.8) as

\[
\int_{\mathbb{R}^3 \setminus \overline{D}} \left( 2|\nabla R|^2 + 2\tau^2 \left| R + \frac{e^{-\tau T} F}{2 \tau^2} \right|^2 \right) dx
\]

\[
+ \int_{\partial D} 2c \left| R - \frac{1}{2c} \left( \frac{\partial v}{\partial \nu} - cv \right) - e^{-\tau T} G \right|^2 dS
\]

\[
= \frac{e^{-2\tau T}}{2 \tau^2} \int_{\mathbb{R}^3 \setminus \overline{D}} |F|^2 dx + \int_{\partial D} \frac{1}{2c} \left| \left( \frac{\partial v}{\partial \nu} - cv \right) - e^{-\tau T} G \right|^2 dS. \quad (2.16)
\]

Since we have

\[ \tau^2 |R|^2 \leq 2 \tau^2 \left| R + \frac{e^{-\tau T} F}{2 \tau^2} \right|^2 + \frac{e^{-2\tau T} |F|^2}{2 \tau^2} \]

and

\[ c|R|^2 \leq 2c \left| R - \frac{1}{2c} \left( \frac{\partial v}{\partial \nu} - cv \right) - e^{-\tau T} G \right|^2 + \frac{1}{2c} \left| \left( \frac{\partial v}{\partial \nu} - cv \right) - e^{-\tau T} G \right|^2, \]

noting the trivial inequality \( |\nabla R|^2 \leq 2|\nabla R|^2 \), from (2.16) we obtain

\[ E(\tau) \leq \frac{e^{-2\tau T}}{\tau^2} \int_{\mathbb{R}^3 \setminus \overline{D}} |F|^2 dx + \int_{\partial D} \frac{1}{c} \left| \left( \frac{\partial v}{\partial \nu} - cv \right) - e^{-\tau T} G \right|^2 dS. \quad (2.17)
\]

Writing

\[
\left| \left( \frac{\partial v}{\partial \nu} - cv \right) - e^{-\tau T} G \right|^2 = \left| \frac{\partial v}{\partial \nu} - cv \right|^2 - 2 \left( \frac{\partial v}{\partial \nu} - cv \right) e^{-\tau T} G + e^{-2\tau T} |G|^2,
\]

...
we have
\[
\int_{\partial D} \frac{1}{c} \left| \left( \frac{\partial v}{\partial \nu} - cv \right) - e^{-\tau T} G \right|^2 \, dS = \int_{\partial D} \frac{1}{c} \left| \frac{\partial v}{\partial \nu} - cv \right|^2 \, dS + O(\tau e^{-\tau (\text{dist}(D,B)+T)} + e^{-2\tau T}).
\]

Therefore (2.17) yields
\[
E(\tau) \leq \int_{\partial D} \frac{1}{c} \left| \frac{\partial v}{\partial \nu} - cv \right|^2 \, dS + O(\tau e^{-\tau (\text{dist}(D,B)+T)} + e^{-2\tau T}). \tag{2.18}
\]

Now a combination of (2.18) and (2.12) gives (2.14).

By virtue of Lemma 2.2 it suffices to study the asymptotic behaviour of two Laplace type integrals in (i) and (ii) from below and above, respectively. For this we have the following estimates.

**Lemma 2.3.** Let $C$ be a positive constant.

(i) If $0 \leq \gamma(x) \leq 1 - C$ a.e. $x \in \partial D$, then there exist positive numbers $\mu, C'$ and $\tau_0$ such that, for all $\tau \geq \tau_0$
\[
J(\tau) \geq C' \tau^{-\mu} e^{-2\tau \text{dist}(D,B)}.
\]

(ii) If $\gamma(x) \geq 1 + C$ a.e. $x \in \partial D$, then there exist positive numbers $\mu, C'$ and $\tau_0$ such that, for all $\tau \geq \tau_0$
\[
J(\tau) + \int_{\partial D} \frac{1}{c} \left| \frac{\partial v}{\partial \nu} - cv \right|^2 \, dS \leq -C' \tau^{-\mu} e^{-2\tau \text{dist}(D,B)}.
\]

For the proof of Lemma 2.3 we refer the reader to [20]. The proof given therein is based on an argument done in [29] and covers more general $f$. Here we describe roughly why $\gamma(x) = 1$ is exceptional in Lemma 2.3.

Applying the mean value theorem [4] to (2.3), we have
\[
v(x) = \frac{\varphi(\tau \eta) e^{-\tau |x-p|}}{\tau^3 |x-p|}, \quad x \in \mathbb{R}^3 \setminus B,
\]
where $\varphi(\xi) = \xi \cosh \xi - \sinh \xi$. Let $x \in \Lambda_{\partial D}(p)$. Since $\nu_x = (p - x)/|x - p|$, from the expression above we obtain
\[
\frac{\partial v}{\partial \nu} = \tau v \left( 1 + \frac{1}{\tau |x - p|} \right) \sim \tau v
\]
and hence
\[
\frac{\partial v}{\partial \nu} - cv \sim \tau (1 - \gamma(x)) v.
\]

Since all the points in $\Lambda_{\partial D}(p)$ attains the minimum of the function: $\partial D \ni x \mapsto |x - p|$, roughly speaking, one may expect, as $\tau \to \infty$
\[
J(\tau) \sim \tau \int_{\partial D} (1 - \gamma) v^2 \, dS
\]
and
\[
J(\tau) + \int_{\partial D} \frac{1}{c} \left| \frac{\partial v}{\partial \nu} - cv \right|^2 \, dS = \int_{\partial D} \frac{1}{c} \left( \frac{\partial v}{\partial \nu} - cv \right) \frac{\partial v}{\partial \nu} \, dS \sim \tau \int_{\partial D} \frac{1 - \gamma v^2}{\gamma} \, dS.
\]

These suggest (i) and (ii) of Lemma 2.3. Note also that $1 - \gamma(x) \leq (1 - \gamma(x))/\gamma(x)$ if $\gamma(x) > 0$. 7
Now it is easy to see that from Lemmas 2.2 and 2.3 one obtains (2.5) and other statements of Theorem 2.1. In the proof we never make use of the idea of geometrical optics which is classical. Everything can be done in the context of the weak solution of [5] and main tool is just integration by parts. Note that in Theorem 2.1 \( \gamma \) is just essentially bounded on \( \partial D \) and thus may have, for example, a first kind of discontinuity.

### 2.4 Curvatures and counting number

Let \( p \in \mathbb{R}^3 \setminus \overline{D} \). Define \( \Lambda_{\partial D}(p) = \{ q \in \partial D \mid |q - p| = d_{\partial D}(p) \} \). We call \( \Lambda_{\partial D}(p) \) the first reflector from \( p \) to \( \partial D \) and the points in the first reflector are called the first-reflection points, going from \( p \) to \( \partial D \). Let \( z \in \mathbb{R}^3 \) and \( 0 < r \). In what follows we denote by \( B_r(z) \) the open ball centred at \( z \) and with radius \( r \).

Let \( q \in \partial D \). Given \( v \in T_q(\partial D) \) define \( S_q(\partial D)v = -\frac{d}{dt}(\nu_{q(t)})_{|t=0} \), where \( q(t) \in \partial D, q(0) = q \) and \( dq/dt(0) = v \). We have \( S_q(\partial D)v \in T_q(\partial D) \). The operator \( S_q(\partial D) : T_q(\partial D) \rightarrow T_q(\partial D) \) is called the shape operator (or Waiengarten map) of \( \partial D \) at \( q \) derived from \( v \). The shape operator is symmetric with respect to the induced inner product on \( T_q(\partial D) \) and its eigenvalues \( k_1(q) \leq k_2(q) \) are called the principle curvatures at \( q \). \( K_{\partial D}(q) = \lambda_1(q)k_1(q) \) and \( H_{\partial D}(q) = \lambda_1(q) + k_2(q) \) are called the Gaussian and mean curvatures at \( q \), respectively.

Let \( q' \in \partial B_{d_{\partial D}(p)}(p) \) and \( S_{q'}(\partial B_{d_{\partial D}(p)}(p)) \) denote the shape operator of \( \partial B_{d_{\partial D}(p)}(p) \) at \( q' \) derived from the unit inward normal to \( \partial B_{d_{\partial D}(p)}(p) \). If \( q \in \Lambda_{\partial D}(p) \), then we have \( q \in \partial B_{d_{\partial D}(p)}(p) \), \( T_q(\partial D) = T_q(\partial B_{d_{\partial D}(p)}(p)) \) and \( S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D) \geq 0 \).

Since \( S_q(\partial B_{d_{\partial D}(p)}(p)) = (1/d_{\partial D}(p))I \), we have

\[
\det(S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D)) = (\lambda - k_1(q))(\lambda - k_2(q)),
\]

where \( \lambda = 1/d_{\partial D}(p) \).

**Theorem 2.2(24)).** Let \( \gamma = 0 \). Assume that \( \partial D \) is \( C^3 \) and \( \beta \in C^2(\partial D) \); \( \Lambda_{\partial D}(p) \) is finite and satisfies

\[
\det(S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D)) > 0, \forall q \in \Lambda_{\partial D}(p).
\]

If \( T > 2 \text{dist}(D,B) \), then we have

\[
\lim_{\tau \rightarrow \infty} \tau^4 e^{2\tau \text{dist}(D,B)} I_B(\tau) = \frac{\pi}{2} \left( \frac{\eta}{d_{\partial D}(p)} \right)^2 A_{\partial D}(p),
\]

where

\[
A_{\partial D}(p) = \sum_{q \in \Lambda_{\partial D}(p)} \frac{1}{\sqrt{\det(S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D))}}
\]

Using Theorem 2.2, one can give a procedure for extracting the curvatures at a known first reflection point. More precisely, let \( p \in \mathbb{R}^3 \setminus \overline{D} \) and \( q \in \Lambda_{\partial D}(p) \). From (2.19) we have

\[
\det(S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D)) = Q(\lambda) \equiv \lambda^2 - 2H_{\partial D}(q)\lambda + K_{\partial D}(q),
\]

where \( \lambda = 1/d_{\partial D}(p) \). Replace \( p \rightarrow p_j = p - s_j \nu_q, j = 1, 2, 0 < s_1 < s_2 < d_{\partial D}(p) \). Then \( \Lambda_{\partial D}(p_j) = \{ q \} \) and \( \det(S_q(\partial B_{d_{\partial D}(p_j)}(p_j)) - S_q(\partial D)) > 0 \) since \( S_q(\partial B_{d_{\partial D}(p_j)}(p_j)) > S_q(\partial B_{d_{\partial D}(p)}(p)) \) and \( S_q(\partial B_{d_{\partial D}(p)}(p)) - S_q(\partial D) \geq 0 \) (\( q \) attains \( \min_{x \in \partial D} |x - p| \)).

Let \( B_1 \) and \( B_2 \) denote two open balls centred at \( p - s_j \nu_q, j = 1, 2 \), respectively with 0 < \( s_1 < s_2 < d_{\partial D}(p) \) and satisfy \( \overline{B_1} \cup \overline{B_2} \subset \mathbb{R}^3 \setminus \overline{D} \). Let \( T > 2 \max \text{dist}(D,B_j) \) and \( f = \chi_{B_j} \). Applying
(2.21) to this case, we obtain

$$\lim_{\tau \to \infty} \tau^4 e^{2\tau \text{dist}(D, B_j)} I_{B_j}(\tau) = \frac{\pi}{2} \left( \frac{\text{diam } B_j}{2d_{\partial D}(p_j)} \right)^2 \frac{1}{\sqrt{Q(\lambda_j)}},$$

where $$\lambda_j = 1/d_{\partial D}(p_j)$$. Since $$\text{dist}(D, B_j) = d_{\partial D}(p_j) - s_j$$ and $$d_{\partial D}(p_j) = d_{\partial D}(p) - s_j$$, one can know $$Q(\lambda_j)$$ with $$j = 1, 2$$ from $$u_f(x, t)$$ given at all $$(x, t) \in B_j \times \{0\}$$, $$T$$ for $$f = \chi_{B_j}$$ with $$j = 1, 2$$.

Then, solving the system

$$\begin{pmatrix} -2\lambda_1 & 1 \\ -2\lambda_2 & 1 \end{pmatrix} \begin{pmatrix} H_{\partial D}(q) \\ K_{\partial D}(q) \end{pmatrix} = \begin{pmatrix} Q(\lambda_1) - \lambda_1^2 \\ Q(\lambda_2) - \lambda_2^2 \end{pmatrix},$$

(2.23)

we obtain both $$K_{\partial D}(q)$$ and $$H_{\partial D}(q)$$. Thus, one can know an approximate shape of the obstacle in a neighbourhood of $$q$$. Note that if $$d_{\partial D}(p) \to \infty$$, then $$\lambda_j \to 0$$ and thus it will be difficult to extract $$H_{\partial D}(q)$$ from (2.23).

Another simple corollary is a formula for counting the number of unknown spherical obstacles with the same and known radius nearest to the center of the support of $$f$$. Assume that $$D = B_r(x_1) \cup \cdots \cup B_r(x_m)$$, where $$B_r(x_j), j = 1, \ldots, m$$ is the open ball centred at $$x_j$$ with a known radius $$\epsilon > 0$$ and $$B_r(x_i) \cap B_r(x_j) = \emptyset$$ if $$i \neq j$$.

Given $$p \in \mathbb{R}^d \setminus D$$ it is easy to see that: $$\Lambda_{\partial D}(p)$$ consists of finite points; (2.20) is satisfied; there exists at most one first reflection point going from $$p$$ on each $$\partial B_r(x_j)$$. Therefore, one can apply Theorem 2.2 to this case and obtain the formula which enables us to know the counting number of the balls which are closest to the centre of $$B$$, that is,

$$z\Lambda_{\partial D}(p) = \left( \frac{1}{d_{\partial D}(p)} + \frac{1}{\epsilon} \right) \frac{2}{\pi} \left( \frac{\text{diam } B_{d_{\partial D}(p)}(p)}{\text{diam } B} \right)^2 \lim_{\tau \to \infty} \tau^4 e^{2\tau \text{dist}(D, B)} I_B(\tau),$$

where

$$\Lambda_{\partial D}(p) = \left\{ x_i + \epsilon \frac{p - x_i}{|p - x_i|} \mid |p - x_i| = \min_j |p - x_j| \right\}.$$

### 2.5 A sketch of the proof of Theorem 2.2

Let $$\gamma \equiv 0$$. Integration by parts yields

$$J(\tau) = \int_D (|\nabla v|^2 + \tau^2 |v|^2) dx - \int_{\partial D} \beta |v|^2 dS.$$  

Applying a trace theorem [7] to the second integral on this right-hand side, we see that $$J(\tau) > 0$$ for all $$\tau > 1$$. Then, we have, as $$\tau \to \infty$$

$$E(\tau) = J(\tau)(1 + O(\tau^{-1/2}))$$

(2.24)

and thus from (2.12) we obtain

$$I_B(\tau) = 2J(\tau)(1 + O(\tau^{-1/2})) + O(\tau^{1} e^{-\tauT}).$$

(2.25)

Using the Laplace method [2], one can expand $$J(\tau)$$ under the condition (2.20) and we find its leading term which contains information about the geometry of $$\partial D$$ at all the first reflection points, going from the centre of $$B$$ to $$\partial D$$. This yields (2.21).
Thus the crucial point of the proof of Theorem 2.2 is the derivation of (2.24). It is a combination of a modification of the Lax-Phillips reflection argument in [34] and a change of a dependent variable near $\partial D$. Here we describe the idea of the derivation of (2.24) in the simplest case $\gamma = \beta \equiv 0$.

Since $G \equiv 0$, it follows from (2.8)

$$E(\tau) = \int_{\partial D} \frac{\partial v}{\partial \nu} RdS - e^{-\tau T} \int_{\mathbb{R}^3 \setminus \overline{D}} FRdx.$$  

and applying the boundary condition in (2.6) to $J(\tau)$, we obtain

$$E(\tau) - J(\tau) = \int_{\partial D} \left( \frac{\partial v}{\partial \nu} R + \frac{\partial R}{\partial \nu} v \right) dS - e^{-\tau T} \int_{\mathbb{R}^3 \setminus \overline{D}} F Rdx.$$  

Choose $\tilde{v}(x), x \in \mathbb{R}^3 \setminus D$ in such a way that $\tilde{v}$ has a compact support and satisfies

$$\begin{cases}
\tilde{v} = v & \text{on } \partial D, \\
\frac{\partial \tilde{v}}{\partial \nu} = -\frac{\partial v}{\partial \nu} & \text{on } \partial D.
\end{cases}$$  

(2.26)

Then, integration by parts and (2.6) gives

$$\int_{\partial D} \left( \frac{\partial v}{\partial \nu} R + \frac{\partial R}{\partial \nu} v \right) dS = \int_{\mathbb{R}^3 \setminus \overline{D}} (\Delta - \tau^2) \tilde{v} \cdot R dx - e^{-\tau T} \int_{\mathbb{R}^3 \setminus \overline{D}} \tilde{v} F dx.$$  

Hence we obtain

$$E(\tau) - J(\tau) = \int_{\mathbb{R}^3 \setminus \overline{D}} (\Delta - \tau^2) \tilde{v} \cdot R dx - e^{-\tau T} \int_{\mathbb{R}^3 \setminus \overline{D}} F (R + \tilde{v}) dx.$$  

(2.27)

The point is the choice of $\tilde{v}$. Let $x'$ denote the reflection in the tubular neighbourhood $\{ x \in \mathbb{R}^3 \setminus D \mid d_{\partial D}(x) < 2\delta_0 \}$ of $\partial D$ with sufficiently small $\delta_0 > 0$. It is given by $x' = 2q(x) - x$, where $q(x)$ denote the unique point on $\partial D$ such that $d_{\partial D}(x) = |x - q(x)|$. It is known that $q(x)$ is $C^2$ for $x \in \mathbb{R}^3 \setminus D$ with $d_{\partial D}(x) < 2\delta_0$ if $\partial D$ is $C^3$ ([6]). Choose a cutoff function $\phi_{\delta}$ with $0 < \delta < \delta_0$ which satisfies $0 \leq \phi_{\delta}(x) \leq 1$; $\phi(x) = 1$ if $d_{\partial D}(x) < \delta$; $\phi(x) = 0$ if $d_{\partial D}(x) > 2\delta$; $|\nabla \phi_{\delta}(x)| \leq C\delta^{-1}$; $|\nabla^2 \phi_{\delta}(x)| \leq C\delta^{-2}$.

Define

$$\tilde{v}(x) = \phi_{\delta}(x) v(x').$$

Clearly (2.26) is satisfied with this $\tilde{v}$. A direct computation gives

$$(\Delta - \tau^2) \tilde{v}(x) = \phi(x) d_{\partial D}(x) \sum_{i,j} a_{ij}(x)(\partial_i \partial_j v)(x') + \text{(lower order terms)},$$  

(2.28)

where $a_{ij}(x)$ with $i, j = 1, 2, 3$ are $C^1$ for $x \in \mathbb{R}^3 \setminus D$ with $d_{\partial D}(x) < 2\delta_0$ and independent of $\tau$, $\phi$ and $v$. Note that the computation is based on the formula

$$2q'(x) - I = I - 2\nu_{q(x)} \otimes \nu_{q(x)} - 2d_{\partial D}(x)(\nu_{q(x)})',$$

where $x \in \mathbb{R}^3 \setminus \overline{D}$ and $d_{\partial D}(x) \ll 1$; $q'(x)$ denotes the Jacobian matrix of the map: $x \mapsto q(x)$. It is a consequence of the expression $q(x) = x - d_{\partial D}(x)\nu_{q(x)}$ and the formula $\nabla (d_{\partial D}(x)) = \nu_{q(x)}$. 

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The point is $d_{\partial D}(x)$ in the first term on the right-hand side of (2.28). By using the change of variable $x = y^*$ we have

$$
\int_{\mathbb{R}^3 \setminus \overline{D}} (\Delta - \tau^2) \tilde{v} \cdot Rdx
$$

$$
= \sum_{i,j} \int_{D} \phi(y^*) d_{\partial D}(y^*) a_{ij}(y^*) (\partial_i \partial_j v)(y) R(y^*) J(y) dy + \cdots,
$$

where $J(y)$ denote the Jacobian of the map: $y \rightarrow y^*$. Since $d_{\partial D}(y^*) \equiv d_{\partial D}(y) = 0$ on $\partial D$, integration by parts yields

$$
\int_{D} \phi(y^*) d_{\partial D}(y^*) a_{ij}(y^*) (\partial_i \partial_j v)(y) R(y^*) J(y) dy
$$

$$
= - \int_{D} \partial_i \{ \phi \delta(y^*) d_{\partial D}(y) a_{ij}(y^*) R(y^*) J(y) \} \partial_j v(y) dy.
$$

Hereafter simply estimating this right-hand side together with other terms in (2.29), we obtain

$$
\left| \int_{\mathbb{R}^3 \setminus \overline{D}} (\Delta - \tau^2) \tilde{v} \cdot Rdx \right|
$$

$$
\leq C((\| \nabla R^* \|_{L^2(D_\delta)}) + \delta^{-1} \| R^* \|_{L^2(D_\delta)}) \| \nabla v \|_{L^2(D)} + \delta^{-2} \| R^* \|_{L^2(D_\delta)} \| v \|_{L^2(D)},
$$

where $D_\delta = \{ y \in D \mid d_{\partial D}(y) < 2\delta \}$ and $R^*(y) = R(y^*)$.

Here we note that

$$
\| \nabla v \|_{L^2(D)} \leq J(\tau)^{1/2}, \quad \| v \|_{L^2(D)} \leq \tau^{-1} J(\tau)^{1/2}
$$

and

$$
\| \nabla R^* \|_{L^2(D_\delta)} \leq C E(\tau)^{1/2}, \quad \| R^* \|_{L^2(D_\delta)} \leq \tau^{-1} E(\tau)^{1/2}.
$$

Choosing $\delta = \tau^{-1/2}$ with $\tau >> 1$, we finally obtain

$$
\left| \int_{\mathbb{R}^3 \setminus \overline{D}} (\Delta - \tau^2) \tilde{v} \cdot Rdx \right| \leq C \tau^{-1/2} (E(\tau) J(\tau))^{1/2}.
$$

From these together with (2.27) and the estimate $\| \tilde{v} \|_{L^2(\mathbb{R}^3 \setminus \overline{D})} \leq C \| v \|_{L^2(D)}$, we obtain

$$
| E(\tau) - J(\tau) | \leq C \tau^{-1/2} (E(\tau) J(\tau))^{1/2} + e^{-\tau T} E(\tau)^{1/2} + e^{-\tau T} J(\tau)^{1/2}
$$

and hence

$$
(1 - 2C \tau^{-1} - 2e^{-\tau T}) E(\tau) \leq (1 + 2C + 2e^{-\tau T}) J(\tau) + 4e^{-\tau T}.
$$

Therefore, there exist positive constants $C'$ and $\tau_0$ such that, for all $\tau \geq \tau_0$

$$
E(\tau) \leq C' (J(\tau) + e^{-\tau T}).
$$

From the case (i) of Lemma 2.3 one can conclude that $e^{-\tau T} J(\tau)$ is decreasing as $\tau \rightarrow \infty$ provided $T > 2\text{dist}(D, B)$. Thus we have $E(\tau) = O(J(\tau))$ as $\tau \rightarrow \infty$. Applying this together with the trivial estimate $J(\tau) = O(1)$ as $\tau \rightarrow \infty$ to the right-hand side on (2.30), we finally obtain (2.24).
2.6 Quantitative state of the surface

A combination of (2.25) and the second order term of the asymptotic expansion of \( J(\tau) \) as \( \tau \to \infty \) yields information about the value of \( \beta \) at all the first reflection points, going from the centre to \( \partial D \).

**Theorem 2.3([24]).** Let \( \gamma \equiv 0 \). Assume that \( \partial D \) is \( C^5 \) and \( \beta \in C^2(\partial D) \); \( \Lambda_{\partial D}(p) \) is finite and satisfies (2.21). For each \( q \in \Lambda_{\partial D}(p) \) let \( e_1, \ldots, e_5 \) be an orthonormal basis of the tangent space at \( q \) of \( \partial D \) with \( e_1 \times e_2 = \nu_q \). Choose an open ball \( U \) centred at \( q \) with radius \( r_q \) in such a way that there exist a \( h \in C^5_0(\mathbb{R}^2) \) with \( h(0,0) = 0 \) and \( \nabla h(0,0) = 0 \) such that \( U \cap \partial D = \{ q + \sigma_1 e_1 + \sigma_2 e_2 + h(\sigma_1, \sigma_2) \nu_q \mid \sigma_1^2 + \sigma_2^2 + h(\sigma_1, \sigma_2)^2 < r_q^2 \} \).

If \( T > 2 \text{dist}(D, B) \), then we have

\[
\lim_{\tau \to \infty} \tau^{5} \left\{ e^{2\tau \text{dist}(D, B)} I_B(\tau) - \frac{1}{\tau^2} \left( \frac{\eta}{d_{\partial D}(p)} \right)^2 A_{\partial D}(p) \right\} = -\frac{\pi \eta}{d_{\partial D}(p)^2} A_{\partial D}(p) + \frac{\pi}{2} \eta^2 B_{\partial D}(p),
\]

where

\[
B_{\partial D}(p) = \sum_{q \in \Lambda_{\partial D}(p)} C_{\partial D}(q) \sqrt{\det(S_q(\partial B_{\partial D}(p)(p)) - S_q(\partial D))},
\]

\[
C_{\partial D}(q) = -\frac{1}{d_{\partial D}(p)^3} + \frac{11 - 12 d_{\partial D}(p) H_{\partial D}(q)}{8 d_{\partial D}(p)^5 \det(S_q(\partial B_{\partial D}(p)(p)) - S_q(\partial D))} - \frac{1}{4 d_{\partial D}(p)^2} h_{\sigma_1, \sigma_2, \sigma_3}(0) h_{\sigma_4, \sigma_5, \sigma_6}(0) \left( \frac{1}{4} B_{pq} B_{ru} + \frac{1}{6} B_{qs} B_{rt} B_{ru} \right)
\]

\[
+ \frac{1}{16 d_{\partial D}(p)^2} h_{\sigma_1, \sigma_2, \sigma_3}(0) B_{pq} B_{qs} - \frac{\beta(q)}{d_{\partial D}(p)^2}
\]

and

\[
(B_{pq}) = -\left( \frac{1}{d_{\partial D}(p)^2} I_2 - \nabla^2 h(0) \right)^{-1}.
\]

Note that we have used the summation convention where repeated indices are to be summed from 1 to 2. The explicit second-order term of an expansion of Laplace type integral \( J(\tau) \) is essential and it is an application of an expansion formula of the Laplace type integral. For this see [2].

As a corollary of Theorem 2.3 we obtain a procedure for calculating the value of \( \beta \) at a known point on \( \Lambda_{\partial D}(p) \) provided \( \gamma \equiv 0 \). More precisely, we assume that:

(i) we know in advance a point \( q \in \Lambda_{\partial D}(p) \);

(ii) we know that \( \partial D \) near \( q \) is given by making a rotation around the normal at \( q \) of a graph of a function \( h \) defined on the tangent plane at \( q \) of \( \partial D \) and that, in an appropriate orthogonal coordinates on the tangent plane, say \( \sigma = (\sigma_1, \sigma_2) \), the Taylor expansion of the function at \( \sigma = 0 \) has the form \( h(\sigma_1, \sigma_2) = \sum_{2 \leq | \alpha | \leq 4} h_\alpha \sigma^\alpha + \cdots \) with known coefficients \( h_\alpha \) for \( 2 \leq | \alpha | \leq 4 \).

Note that, from (i) we know also \( d_{\partial D}(p) = |p - q|, \nu_q = (p - q)/|p - q| \) and the tangent plane \((x - q) \cdot \nu_q = 0 \) at \( q \) of \( \partial D \).

Fix \( s \in ]0, d_{\partial D}(p)[ \). Choose an open ball \( B' \) centred at \( p - s \nu_q \) and satisfying \( \overline{B'} \subset B_{\partial D}(p)(p) \). Let \( T > 2 \text{dist}(D, B') \). Generate the wave \( u_f \) by \( f = \chi_B' \) and observe the wave on \( B' \) over time.
interval ]0, T[. Since \( \Lambda_{BD}(p - s\nu_q) = \{q\} \) and (2.20) for \( p \) replaced with \( p - s\nu_q \) is satisfied, one gets (2.31) in which ball \( B \) is replaced with \( B' \) and \( p \) replaced with \( p - s\nu_q \). Therefore, we obtain \( C_{BD}(q) \) which yields a linear equation with unknown \( \beta(q) \) and thus solving this, one obtains \( \beta(q) \). Note that, in this procedure we do not assume that \( \Lambda_{BD}(p) \) is finite.

Now it is natural to consider the following problem.

**Open problem 2.1.** Assume that, say, \( \gamma \) is sufficiently smooth on \( \partial D \). Find a formula for calculating \( \gamma \) at a known \( q \in \Lambda_{BD}(p) \) from \( u_f \) on \( B' \times ]0, T[ \) generated by \( f = \chi_{B'} \), where \( B' \) is the same as above.

The point is to find the asymptotic profile of \( E(\tau) \) in (2.12) as \( \tau \to \infty \) in terms of \( v \) on \( D \). For one-space dimensional case we have an explicit formula. See [20].

### 2.7 Other wave equations

In [20] a result analogous to Theorem 2.1 has been established also for the equation

\[
\alpha(x)\partial_t^2 u - \Delta u = 0 \text{ in } \mathbb{R}^3 \times ]0, T[
\]

provided: \( \eta \geq \alpha(x) \geq \eta^{-1} \) a.e. \( x \in \mathbb{R}^3 \) for a positive constant \( \eta \) and \( \alpha(x) = 1 \) a.e. \( x \in \mathbb{R}^3 \setminus D \); there exists a positive constant \( C \) such that \( \alpha(x) \leq 1 - C \) a.e. \( x \in D \) or \( \alpha(x) \geq 1 + C \) a.e. \( x \in D \).

In [16] the equation

\[
\partial_t^2 u - \nabla \cdot A(x)\nabla u = 0 \text{ in } \mathbb{R}^3 \times ]0, T[
\]

with a \( 3 \times 3 \) uniformly positive definite real symmetric matrix-valued function coefficient \( A(x) \) satisfying \( A(x) = I_3 \) a.e. \( x \in \mathbb{R}^3 \setminus D \) has been studied. It is assumed that each component of \( A(x) \) is essentially bounded and there exists a positive constant \( C \) such that \( (A(x) - I_3)\xi \cdot \xi \geq C|\xi|^2 \) a.e. \( x \in D \) and all \( \xi \in \mathbb{R}^3 \) or \( -(A(x) - I_3)\xi \cdot \xi \geq C|\xi|^2 \) a.e. \( x \in D \) and all \( \xi \in \mathbb{R}^3 \). Then, it is clear that Theorem 1.2 in [16] for this equation yields also a result analogous to Theorem 2.1. However, for both equations there is no result corresponding to Theorems 2.2 and 2.3 via the Enclosure Method.

### 2.8 Interior problem in time domain

Let \( D \) be a bounded domain of \( \mathbb{R}^3 \) with \( C^2 \)-boundary. Given \( f \in L^2(D) \) satisfying \( \text{supp } f \subset D \), denote by \( u_f \) the solution of the following initial boundary value problem for the wave equation:

\[
\begin{aligned}
\partial_t^2 u - \Delta u &= 0 \text{ in } D \times ]0, T[, \\
- \frac{\partial u}{\partial \nu} - \gamma(x)\partial_t u - \beta(x)u &= 0 \text{ on } \partial D \times ]0, T[, \\
u(x, 0) &= 0 \text{ in } D, \quad \partial_t u(x, 0) = f(x) \text{ in } D,
\end{aligned}
\]

where \( \nu \) denotes the unit outward normal to \( D \) on \( \partial D \), \( \beta \) and \( \gamma \) are the same as those in (2.1).

**Problem 2.2.** Let \( B \) be an open ball and satisfy \( \overline{B} \subset D \). Generate \( u = u_f \) by the initial data \( f = \chi_B \) and observe \( u \) on \( B \) over time interval \( ]0, T[ \). Extract information about the geometry of \( \partial D \), \( \gamma \) and \( \beta \) from the observed data.

In [21], we considered the case when \( \gamma = \beta = 0 \) in (2.32) and obtained two theorems corresponding to Theorems 2.1 and 2.2. It will be possible to obtain a theorem corresponding to Theorem 2.1 for general \( \gamma \) and \( \beta \); theorems corresponding to Theorems 2.2 and 2.3 for \( \gamma = 0 \) and general \( \beta \). Thus, a real problem to be solved should be the same as Open problem 2.1.
3 Further applications and problems

3.1 Bistatic data, spheroid and simultaneous rotation

Let $0 < T < \infty$. Let $f \in L^2(\mathbb{R}^3)$ satisfy $\text{supp } f \cap \overline{D} = \emptyset$. Let $u = u_f(x, t)$ be the solution of the initial boundary value problem:

$$
\begin{cases}
\partial_t^2 u - \Delta u = 0 & \text{in } (\mathbb{R}^3 \setminus \overline{D}) \times ]0, T[, \\
u = 0 & \text{on } \partial D \times ]0, T[, \\
u(x, 0) = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D}, \partial_t u(x, 0) = f(x) \text{ in } \mathbb{R}^3 \setminus \overline{D}.
\end{cases}
$$

**Problem 3.1.** Let $B$ and $B'$ be two known open balls centred at $p \in \mathbb{R}^3$ and $p' \in \mathbb{R}^3$ with radii $\eta$ and $\eta'$, respectively such that $\overline{B} \cap \overline{D} = \emptyset$ and $\overline{B'} \cap \overline{D} = \emptyset$. Let $\chi_B$ denote the characteristic function of $B$ and set $f = \chi_B$. Assume that $D$ is unknown. Extract information about the location and shape of $D$ from the data $u_f(x, t)$ given at all $x \in B'$ and $t \in ]0, T[$.

Let $\chi_{B'}$ denote the characteristic function of $B'$ and set $g = \chi_{B'}$. The results of this subsection are concerned with the asymptotic behaviour of the bistatic indicator function $I_{B, B'}$ defined by

$$
I_{B, B'}(\tau) = \int_{\mathbb{R}^3 \setminus B'} (f v_g - w_f g) \, dx,
$$

where $w_f$ is the same as (2.4) and $v_g$ is the solution of (2.2) in which $f$ is replaced with $g$. Note that $I_{B, B'}(\tau)$ can be computed from $w_f$ on $B$ and thus from $u_f$ on $B \times ]0, T[$.

Define $\phi(x; y, y') = |y - x| + |x - y'|$, $(x, y, y') \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$. This is the length of the broken path connecting $y$ to $x$ and $x$ to $y'$. We denote the convex hull of the set $F \subset \mathbb{R}^3$ by $[F]$.

**Theorem 3.1([23]).** Let $[\overline{B} \cup \overline{B'}] \cap \partial D = \emptyset$ and $T$ satisfy

$$
T > \min_{x \in \partial D, y \in \partial B, y' \in \partial B'} \phi(x; y, y').
$$

Then, there exists a $\tau_0 > 0$ such that $I_{B, B'}(\tau) > 0$ for all $\tau \geq \tau_0$ and the formula

$$
\lim_{\tau \to +\infty} \frac{1}{\tau} \log I_{B, B'}(\tau) = -\min_{x \in \partial D, y \in \partial B, y' \in \partial B'} \phi(x; y, y'),
$$

is valid.

It is easy to see that

$$
\min_{x \in \partial D, y \in \partial B, y' \in \partial B'} \phi(x; y, y') = \min_{x \in \partial D} \phi(x; p, p') - (\eta + \eta').
$$

Thus formula (3.2) enables us to extract $\min_{x \in \partial D} \phi(x; p, p')$ from $u_f(x, t)$ given at all $x \in B'$ and $t \in ]0, T[$.

The quantity $\min_{x \in \partial D} \phi(x; p, p')$ coincides with the shortest length of the broken paths connecting $p$ to a point $q$ on $\partial D$ and $q$ to $p'$, that is, the first reflection distance between $p$ and $p'$ by $D$. Thus, Theorem 3.1 yields a mathematical method for extracting the first reflection distance from the waveform of the observed wave.

Given $c > |p - p'|$ define $E_c(p, p') = \{x \in \mathbb{R}^3 \mid \phi(x; p, p') = c\}$. This is a spheroid with focal points $p$ and $p'$. Given direction $\omega \in S^2$ at $p'$ let $\zeta(\omega; p, p')$ denote the unique point on $E_c(p, p')$. $\zeta(\omega; p, p')$ has the expression $\zeta(\omega; p, p') = p' + s(\omega; p, p') \omega$ with a unique $s(\omega; p, p') > 0$ and the map $S^2 : \omega \mapsto \zeta(\omega; p, p') \in E_c(p, p')$ is bijective.
Define $\Lambda_{\partial D}(p,p') = \{ q \in \partial D \mid \min_{x \in \partial D} \phi(x;p,p') = \phi(q;p,p') \}$. One can write $\Lambda_{\partial D}(p,p') = \partial D \cap E_c(p,p')$, where $c = \min_{x \in \partial D} \phi(x;p,p')$.

Similarly to Section 2.2, using Theorem 3.1, one can make a decision whether given direction $\omega \in S^2$ at $p' \zeta(\omega;p,p')$ which is a point on $E_c(p,p')$ belongs to $\partial D$ or not. It is based on the following characterization of $\Lambda_{\partial D}(p,p')$.

Lemma 3.1(Proposition 5.1 in [23]). Fix $s \in [0,\eta']$. We have:

(i) if $\zeta(\omega;p,p')$ belongs to $\partial D$, then $\min_{x \in \partial D} \phi(x;p,p' + s\omega) = c - s$;

(ii) if $\zeta(\omega;p,p')$ does not belong to $\partial D$, then $\min_{x \in \partial D} \phi(x;p,p' + s\omega) > c - s$.

Therefore, we obtain the following characterization of $\Lambda_{\partial D}(p,p')$:

$$\Lambda_{\partial D}(p,p') = \{ \zeta(\omega;p,p') \mid \min_{x \in \partial D} \phi(x;p,p' + s\omega) = c - s \}.$$

The procedure for finding $\zeta(\omega;p,p')$ belonging to $\partial D$ from a single set of the bistatic data is the following:

**Step 1.** Generate $u_f$ by the initial data $f = \chi_B$ and observe $u_f$ on $B'$ over time interval $[0, T[.$

**Step 2.** Choose an open ball $B'' \subset B'$ centred at $p' + s\omega$.

**Step 3.** Determine $\min_{x \in \partial D} \phi(x;p,p' + s\omega)$ from the restriction of $u_f$ in the first step onto $B'' \times ]0, T[.$ via Theorem 3.1.

From the computed value $\min_{x \in \partial D} \phi(x;p,p' + s\omega)$ in the third step, one has: if $\min_{x \in \partial D} \phi(x;p,p' + s\omega) = c - s$, then $\zeta(\omega;p,p')$ belongs to $\partial D$; if not, then $\zeta(\omega;p,p')$ does not belong to $\partial D$.

Therefore, in principle, one can determine all the points in $\Lambda_{\partial D}(p,p')$ from $u_f$ on $B'' \times ]0, T[.$ for $f = \chi_B$. This is an advantage of the bistatic data not being seen in the monostatic data.

The next theoretical result is concerned with the obtaining information about shape of $D$. In the following theorem, for simplicity of description, we assume that $D$ is convex. In this case $\Lambda_{\partial D}(p,p')$ consists of a single point $q = q(p,p')$.

**Theorem 3.2([23]).** Assume that $\partial D$ is $C^3$. Let $c = \min_{x \in \partial D} \phi(x;p,p')$. Let $T$ satisfy (3.1). Then, we have

$$\det(S_{q(p,p')}(E_c(p,p')) - S_{q(p,p')}(\partial D)) > 0$$

and the formula

$$\lim_{\tau \to \infty} \tau^4 e^{\tau \min_{x \in \partial D, y \in \partial B, y' \in \partial B'} \phi(x;y,y')} I_{B,B'}(\tau) = \pi \left( \frac{\text{diam } B}{2|q - p|} \right) \cdot \left( \frac{\text{diam } B'}{2|q - p'|} \right) \cdot \frac{1}{\sqrt{\det(S_{q}(E_c(p,p')) - S_{q}(\partial D))}_{q=q(p,p')}}$$

is valid.

**Remark 3.1.** For more general condition on $D$ instead of it’s convexity see [23]. In that case, instead of (3.3) we have to assume that

$$\det(S_{q}(E_c(p,p')) - S_{q}(\partial D)) > 0 \forall q \in \Lambda_{\partial D}(p,p').$$

Note that in that case $\Lambda_{\partial D}(p,p')$ does not necessary consists of a single point and thus its finiteness should be assumed. As a result the right-hand side on (3.4) should be changed.

As a consequence of Theorem 3.2 we obtain two procedures for extracting the curvatures and principle directions.
First we describe a procedure for extracting the curvatures of $\partial D$ at $q(p,p')$ provided $q = q(p,p')$ is known. Formula (3.4) gives the information $\det (S_q(E_c(p,p')) - S_q(\partial D))$ from $u_f$ on $B' \times [0, T]$. Then by restricting $u_f$ on $B'' \times [0, T]$ we obtain also from (3.3) $\det (S_q(E_{c,s}(p,p' + s\omega)) - S_q(\partial D))$ for $\omega = (q-p')/|q-p'|$. Using an analogous equation to (2.23) (Lemma 5.1 in [23]), we see that these two quantities construct a linear system with the Gauss curvature $K_{\partial D}(q)$ and a modification of the mean curvature of $\partial D$ at $q$, that is

$$H_{\partial D}(q) - \frac{S_q(\partial D)(A_q(p) \times A_q(p')) \cdot (A_q(p) \times A_q(p'))}{2(1 + A_q(p) \cdot A_q(p'))},$$

where $A_q(x) = (q-x)/|q-x|$. The system corresponds to (2.24) and always uniquely solvable. Thus, by solving the system we obtain those two curvatures. Therefore, one can obtain an approximate shape of $\partial D$ around $q(p,p')$.

Second we show that it is possible to obtain the principal curvature directions of $\partial D$ at $q(p,p')$ by making a rotation of $B$ and $B'$ at the same time around the normal at $q(p,p')$. We denote by $p(\theta)$ and $p'(\theta)$ the points rotated around the line directed $\nu_q$ at $q = q(p,p')$ counterclockwise with rotation angle $\theta \in [0, 2\pi]$ of $p$ and $p'$. Thus $p(0) = p$ an $p'(0) = p'$. Then, for all $\theta \in [0, 2\pi]$ we know that $\Lambda_{\partial D}(p(\theta), p'(\theta)) = (q(p,p'), A_q(p(\theta))) \cdot A_q(p'(\theta)), |A_q(p(\theta)) \times A_q(p'(\theta))|$ and $|p(\theta) - q| + |q - p'(\theta)|$ at $q = q(p,p')$ are invariant with respect to $\theta$.

Let $B(\theta) = \{x \in \mathbb{R}^3, |x-p(\theta)| < \eta\}$ and $B(0) = B; B'(\theta) = \{x \in \mathbb{R}^3, |x-p'(\theta)| < \eta'\}$ and $B'(0) = B'$. We have $\overline{B(\theta) \cup B'(\theta)} \cap D = \emptyset$ provided $\overline{B \cup B'} \cap D = \emptyset$. Let $f(\theta)$ denote the characteristic function of $B(\theta)$.

Then, from $u_f(\theta)$ on $B'(\theta) \times [0, T]$ we obtain the function of $\theta$:

$$\tilde{H}_{\partial D}(q; p(\theta), p'(\theta)) = H_{\partial D}(q) - \frac{1 - A_q(p) \cdot A_q(p')}{2} S_q(\partial D)(V(\theta)) \cdot V(\theta)$$

where $V(\theta)$ denotes the unit vector directed to $A_q(p(\theta)) \times A_q(p'(\theta))$.

Now assume that $A_q(p) \times A_q(p') \neq 0$. Then, $V(\theta)$ attains all the tangent vector at $q$ of $\partial D$ and thus from the behavior of $\tilde{H}_{\partial D}(q; p(\theta), p'(\theta))$ as a function of $\theta$ one can determine all the directions of principle curvatures say, $V(\theta_1)$ and $V(\theta_2)$ with some $\theta_1$ and $\theta_2$. Since we have

$$S_q(\partial D)(V(\theta_1)) \cdot V(\theta_1) + S_q(\partial D)(V(\theta_2)) \cdot V(\theta_2) = 2H_{\partial D}(q),$$

the arithmetic mean of $\tilde{H}_{\partial D}(q; p(\theta_1), p'(\theta_1))$ and $\tilde{H}_{\partial D}(q; p(\theta_2), p'(\theta_2))$ coincides with

$$\left\{ \frac{1 - A_q(p) \cdot A_q(p')}{2} \right\} H_{\partial D}(q).$$

Thus we obtain $H_{\partial D}(q)$. Therefore, we can extract $S_q(\partial D)$ from $u_f(\theta)$ over $B'(\theta) \times [0, T]$ given at all $\theta \in [0, 2\pi]$. This is an advantage of the data collection using a simultaneous rotation of the emitter and the receiver.

**Open problem 3.1.** Extend the results to other boundary conditions, transmission conditions (see Section 2.7) or the Maxwell system (see Section 3.2).

### 3.2 The Maxwell system

In this section we briefly comment on a recent application [25] of the Enclosure Method to an inverse obstacle problem whose governing equation is given by the Maxwell system in the time domain.
Let $0 < T < \infty$. We denote by $\mathbf{E}$ and $\mathbf{H}$ the electric field and the magnetic field, respectively. Assume that $\mathbf{E}$ and $\mathbf{H}$ are induced only by the current density $\mathbf{J}$ at $t = 0$ and that the obstacle is a perfect conductor placed in the whole space $\mathbb{R}^3$. The governing equations of $\mathbf{E}$ and $\mathbf{H}$ take the form

\[
\begin{cases}
\epsilon \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{H} = \mathbf{J} & \text{in } (\mathbb{R}^3 \setminus \overline{D}) \times ]0, T[, \\
\mu \frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = 0 & \text{in } (\mathbb{R}^3 \setminus \overline{D}) \times ]0, T[, \\
\mathbf{v} \times \mathbf{E} = 0 & \text{on } \partial D \times ]0, T[, \\
E|_{t=0} = 0, \ H|_{t=0} = 0 & \text{in } \mathbb{R}^3 \setminus \overline{D},
\end{cases}
\]

where $\mathbf{v}$ denotes the unit outward normal to $D$ on $\partial D$; $\epsilon$ and $\mu$ denote the electric permittivity and magnetic permeability assumed to be positive constants.

There are several choices of the current density $\mathbf{J}$ as a model of the antenna. In [25] it is assumed that $\mathbf{J}$ takes the form

\[
\mathbf{J}(x, t) = f(t) \chi_B(x) \mathbf{a},
\]

where $\mathbf{a} \neq 0$ is a constant unit vector, $\chi_B$ denote the characteristic function of $B$ and $f \in H^1(0, T)$ with $f(0) = 0$; $B$ is an open ball with very small radius and satisfies $\overline{B} \cap \overline{D} = \emptyset$.

In [25] the author considered the following problem.

**Problem 3.2.** Fix $T$. Generate $\mathbf{E}$ and $\mathbf{H}$ by the source $\mathbf{J}$ given by (3.5) and observe $\mathbf{E}$ on $B$ over time interval $]0, T[$. Extract information about the geometry of $D$ from the observed data.

Two theorems corresponding to Theorems 2.1 and 2.2 have been obtained in [25]. The main difference from the scalar case is the existence of directivity of the source at $t = 0$ and one of two theorems catches the effect of the source directivity.

The boundary condition imposed on the surface of the obstacle is a typical one like the Dirichlet boundary condition for the wave equation. As a next step it is natural to ask: how about the case when the electromagnetic wave satisfies a more general boundary condition like the Leontovich condition on the surface of the obstacle (see, e.g., [1])?

**Open problem 3.2.** Consider the Leontovich boundary condition instead of the perfect conductivity condition:

\[
\mathbf{v} \times \mathbf{H} - \lambda(x) \mathbf{v} \times (\mathbf{E} \times \mathbf{v}) = 0 \text{ on } \partial D \times ]0, T[.
\]

Extract information about the geometry of $D$ and $\lambda$ from the observed data.

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