Global small solutions to heat conductive compressible nematic liquid crystal system: smallness on a scaling invariant quantity

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Abstract. In this paper, we consider the Cauchy problem to the three dimensional heat conducting compressible nematic liquid crystal system in the presence of vacuum and with vacuum far fields. Global well-posedness of strong solutions is established under the condition that the scaling invariant quantity

\[
\left(\|\rho_0\|_{\infty} + 1\right)\left(\|\rho_0\|_3 + (\|\rho_0\|_{\infty} + 1)^2\left(\|\sqrt{\rho_0} u_0\|_2 + \|\nabla d_0\|_2^2\right)\right)\left(\|\nabla u_0\|_2^2 + (\|\rho_0\|_{\infty} + 1)(\|\sqrt{\rho_0} E_0\|_2^2 + \|\nabla^2 d_0\|_2^2\right)\right)\]

is sufficiently small with the smallness depending only on the parameter s appeared in the system.

Keywords. Heat conducting compressible nematic liquid crystal system, Global well-posedness, Vacuum, Scaling invariant quantity

1 Introduction

Liquid crystals are intermediate phases between solids and fluids. The continuum theory of liquid crystals was established by Ericksen [5] and Leslie [25] during the period of 1958 through 1968. The present paper concerns a simplified version of the general Ericksen-Leslie system, which roughly speaking is a coupled system of the compressible Navier-Stokes equations and the harmonic heat flow (see [30, 32]). The equations of the heat conducting compressible nematic liquid system read as:

\[
\begin{align*}
\rho_t + \text{div}(\rho u) & = 0, \\
\rho(u_t + u \cdot \nabla u) + \nabla P & = \mu \Delta u + (\mu + \lambda)\text{div}u - \Delta d \cdot \nabla d, \\
c_v\rho(\theta_t + u \cdot \nabla \theta) + P\text{div}u - \kappa \Delta \theta & = Q(\nabla u) + |\Delta d + |\nabla d|^2 d|^2|, \\
d_t + u \cdot \nabla d & = \Delta d + |\nabla d|^2 d,
\end{align*}
\]

where \(\rho \in \mathbb{R}_+\) is the density of the fluid, \(u \in \mathbb{R}^3\) is the velocity field and \(d \in S^2\) represents macroscopic average of the nematic liquid crystal orientation field, with \(S^2\) denoting the unit spherical surface in \(\mathbb{R}^3\). Here, \(P = R\rho \theta\) is the pressure with \(R\) being a positive constant, \(\lambda\) and \(\mu\) are constant viscosity coefficients satisfying the physical conditions \(\mu > 0\) and \(2\mu + 3\lambda \geq 0\), heat capacity \(c_v = \frac{R}{\gamma - 1}\) with \(\gamma > 1\) being the adiabatic constant, \(\kappa > 0\) is the heat conductive constant coefficient, and

\[
Q(\nabla u) = \frac{\mu}{2} |\nabla u + (\nabla u)^t|^2 + \lambda (\text{div}u)^2.
\]

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The additional assumption that $2\mu > \lambda$ will also be used in this paper.

Mathematical analysis of the nematic liquid crystals have attracted a lot of attentions for several decades. For the incompressible case, Lin \[30\] first introduced and studied a simplified Ericksen-Leslie system modeling the incompressible liquid crystal flows. From then on, the solvability and stability of the incompressible liquid crystal flows have been substantially developed. The global existence and partial regularity of weak solutions to the Ginzburg-Landau type approximation system were obtained by Lin and Liu \[32, 33\], while the global existence of weak solutions to the original system was proved by Lin et al. \[31\], Hong \[12\], and Hong and Xin \[13\] for two-dimensional case, and by Lin and Wang \[35\] for three-dimensional case under some geometric assumptions on the initial director field $d_0$. Uniqueness of weak solutions in two dimensions was proved by Lin and Wang \[34\], Li et al. \[27\], and Wang et al. \[40\], while the nonuniqueness of weak solutions and finite time blow up of classic solutions in three dimensions was, respectively, addressed by Gong et al. \[10\] and Huang et al. \[16\]. Feireisl et al. \[7, 8\] considered a non-isothermal Ginzburg-Landau model of nematic liquid crystals and investigated the global existence of weak solutions, while the global existence of weak solutions to the corresponding non-isothermal Ericksen-Leslie system in two dimensions was established by Li and Xin \[28\]. One can refer to De Anna and Liu \[2\] for the derivation of the general non-isothermal Ericksen-Leslie system. Huang and Wang \[17\] established a blow-up criterion for the short time classical solutions to incompressible liquid crystal flows in dimensions two and three. Hong et al. \[14\] established the local well-posedness and blow-up criteria of strong solutions to the liquid crystal system with general Oseen-Frank free energy density.

Concerning the compressible case, the model of the liquid crystals becomes more complicated since the density variation affects the mechanical behavior of the fluid. Global well-posedness of the isentropic compressible nematic liquid crystals in one dimension was proved by Ding et al. \[3, 4\], while the global existence of weak solutions in multi-dimensions was proved by Jiang et al. \[22, 23\] under a smallness condition on the third component of initial orientation field. The local existence of unique strong solution to the initial value or initial-boundary value problem was proved in Huang et al. \[18, 19\], where a series of blow-up criterion of strong solutions were established as well. Li et al. \[29\] obtained the global classical solutions to the Cauchy problem with small initial energy but possibly large oscillations and the initial density may allowed vacuum. The long-time behavior of classical solution was considered in \[9\]. By virtue of the Fourier splitting method, the authors built optimal temporal decay rate of the global solution. For more results on simplified isothermal Ericksen-Leslie system, the readers can refer to \[12, 38, 39, 39\] and references therein.

Inspired by the introduction of non-isothermal models of incompressible nematic liquid crystals by Feireisl et al. in \[7, 8\], the compressible non-isothermal nematic liquid crystal flows is now attracting increasing research attention. Fan et al. \[9\] first investigated the local existence of unique strong solution to the initial boundary value problem. Guo et al. \[11\] obtained the global existence of smooth solutions for the Cauchy problem provided that the initial datum is close to a steady state and gave the algebraic decay rate of the global solution. A blow up criterion was established in \[12\] for the strong solutions to the two-dimensional non-isothermal flows in a bounded domain under a geometric condition introduced by Lei et al. in \[24\]. Recently, Liu and Zhong \[37\] proved the global well-posedness of strong solutions with the initial data can have compact support provided that the quantity $\|\rho_0\|_{L^\infty} + \|\nabla d_0\|_{L^3}$ is suitably small with the smallness depending not only on the parameters involved in the system, but also on some high order norm of the initial data.

The purpose of this paper is to investigate the global well-posedness of strong solutions to
the Cauchy problem of (1.1)-(1.4) along with the initial condition:
\[ \langle \rho, u, \theta, d \rangle |_{t=0} = (\rho_0, u_0, \theta_0, d_0). \] (1.5)
The initial data allows far field vacuum and the smallness assumption, which depends only on the parameters involved in the system, is imposed exclusively on some quantities that are scaling invariant with respect to the following scaling transform:
\[ (\rho_0(\lambda x), u_0(\lambda x), \theta_0(\lambda x), d_0(\lambda x)) = (\rho_0(x), \lambda u_0(\lambda x), \lambda^2 \theta_0(\lambda x), d_0(\lambda x)), \text{ for any } \lambda \neq 0. \] (1.6)
This scaling transform on the initial data is motivated from the natural scaling invariant property of the compressible nematic liquid crystal flow (1.1)-(1.4):
\[ \rho_\lambda(x, t) = \rho(\lambda x, \lambda^2 t), \quad u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), \quad \theta_\lambda(x, t) = \lambda^2 \theta(\lambda x, \lambda^2 t), \quad d_\lambda(x, t) = d(\lambda x, \lambda^2 t). \]
More precisely, if \((\rho, u, \theta, d)\) is a solution with the initial data \((\rho_0, u_0, \theta_0, d_0)\), then, by straightforward calculations, one can find that \((\rho_\lambda, u_\lambda, \theta_\lambda, d_\lambda)\) is also a solution with the transformed initial data \((\rho_0(\lambda x), u_0(\lambda x), \theta_0(\lambda x), d_0(\lambda x))\) for any nonzero \(\lambda\).
As already explained in [26], imposing smallness assumptions on the scaling invariant quantities is necessary for obtaining globally well-posed to system (1.1)-(1.5). In fact, if assuming that \(\mathcal{M}\) is a functional such that
\[ \mathcal{M}(\rho_\lambda, u_\lambda, \theta_\lambda, d_\lambda) = \lambda^\ell (\rho_0, u_0, \theta_0, d_0), \quad \text{for any } \lambda \neq 0 \text{ and some constant } \ell \neq 0, \]
and that the global well-posedness holds for any initial data \((\rho_0, u_0, \theta_0, d_0)\) satisfying
\[ \mathcal{M}(\rho_0, u_0, \theta_0, d_0) \leq \varepsilon_0, \]
then, by suitably choosing the scaling parameter \(\lambda\), one can show that the global well-posedness holds for arbitrary large initial data. However, such global well-posedness for arbitrary large initial data is far from what we have already known.
Throughout this paper, the following notations are needed. For \(1 \leq p \leq \infty\), denote \(L^p = L^p(\mathbb{R}^3)\) as the standard \(L^p\) Lebesgue spaces with the norm \(\| \cdot \|_p\). For \(1 \leq p \leq \infty\) and positive integer \(k\), denote by \(W^{k,p} = W^{k,p}(\mathbb{R}^3)\) the standard Sobolev spaces, whose norm is denoted as \(\| \cdot \|_{W^{k,p}}\) or \(\| \cdot \|_{H^k}\) with \(H^k = W^{k,2}\). To simplify the expressions, the norm \(\sum_{i=1}^K \| f_i \|_X \) or \(\left(\sum_{i=1}^K \| f_i \|_X^p \right)^{\frac{1}{p}}\) are sometimes denoted by \(\| (f_1, f_2, \ldots , f_K) \|_X\). For \(1 \leq r \leq \infty\), \(D^{k,r}\) is the homogeneous Sobolev space, which is defined by
\[ D^{k,r} = \{ u \in L^1_{\text{loc}}(\mathbb{R}^3) \mid \| \nabla^k u \|_r < \infty \}, \quad D^k = D^{k,2}; \]
\[ D^0 = \{ u \in L^6(\mathbb{R}^3) \mid \| \nabla u \|_2 < \infty \}. \]
For simplicity, let
\[ \int f dx = \int_{\mathbb{R}^3} f dx. \]
\(\dot{f} = f_t + u \cdot \nabla f\) denotes the material derivative of \(f\).

**Definition 1.1** Let \(T > 0\), \((\rho, u, \theta, d)\) is called a strong solution to the compressible nematic liquid crystal flow (1.1)-(1.4) in \(\mathbb{R}^3 \times (0, T)\) with initial condition (1.5), if for some \(q \in (3, 6)\),
\[ \rho \in C([0, T]; H^1 \cap W^{1,q}), \quad (u, \theta) \in C([0, T]; D^1 \cap D^2) \cap L^2(0, T; D^{2,q}), \]
\[ \nabla d \in C([0, T]; H^2) \cap L^2(0, T; H^3), \quad \rho_t \in C([0, T]; L^2 \cap L^q), \quad (u_t, \theta_t) \in L^2(0, T; D^0), \]
\[ (\sqrt{\rho} u_t, \sqrt{\rho} \theta_t) \in L^\infty(0, T; L^2), \quad d_t \in C([0, T]; H^1) \cap L^2(0, T; H^2), \quad |d| = 1, \]
and \((\rho, u, \theta, d)\) satisfies (1.1)-(1.4) a.e. in \(\mathbb{R}^3 \times (0, T)\) and fulfills the initial condition (1.5).
Theorem 1.1 Assume that the initial data $\rho_0, u_0, \theta_0$ and $d_0$ satisfy
\[ \rho_0, \theta_0 \geq 0, \quad \rho_0 \in H^1 \cap W^{1, q}, \quad \sqrt{\rho_0} \theta_0 \in L^2, \]
\[ (u_0, \theta_0) \in D^0 \cap D^2, \quad \nabla d_0 \in H^2, \quad \text{and} \quad |d_0| = 1, \]
for some $q \in (3, 6]$. Set $\overline{\rho} := \|\rho_0\|_{\infty} + 1$. In addition, the following compatibility condition
\[ -\mu \Delta u_0 - \left( \mu + \lambda \right) \nabla \mathrm{div} u_0 + \nabla P_0 + \Delta d_0 \cdot \nabla d_0 = \sqrt{\rho_0} g_1, \quad (1.7) \]
\[ \kappa \Delta \theta_0 + \mathcal{Q}(\nabla u_0) + |\Delta d_0| + |\nabla d_0|^2 d_0|^2 = \sqrt{\rho_0} g_2, \quad (1.8) \]
hold with $g_1, g_2 \in L^2$, where $P_0 = R \rho_0 \theta_0$.

Then, there is a positive constant $\varepsilon_0$ depending only on $R, \gamma, \mu, \lambda$, and $\kappa$, such that if
\[ \mathcal{N}_0 := \overline{\rho} \left[ \|\rho_0\|_3 + \overline{\rho}^2 \left( \|\sqrt{\rho_0} u_0\|_2 + \|\nabla d_0\|_2 \right) \right] \left[ \|\nabla u_0\|_2 + \overline{\rho} \left( \|\sqrt{\rho_0} E_0\|_2 + \|\nabla^2 d_0\|_2 \right) \right] \leq \varepsilon_0, \]
where $E_0 = \frac{|u_0|^2}{2} + c_r \theta_0$, the problem (1.1)-(1.2) has a unique global strong solution.

Remark 1.1 It is obvious that if there is an initial data $(\rho_0, u_0, \theta_0, d_0)$ satisfying $\mathcal{N}_0 \leq \varepsilon_0$, then any $(\rho_{0\lambda}, u_{0\lambda}, \theta_{0\lambda}, d_{0\lambda})$ defined by the scalar transform (1.6) with $\lambda \neq 0$ also satisfies $\mathcal{N}_0 \leq \varepsilon_0$.

Remark 1.2 The global well-posedness of strong solutions to the Cauchy problem for compressible non-isothermal nematic liquid crystal flows with vacuum as far field density has recently been proved in [37], which needs the small initial data satisfying
\[ \|\rho_0\|_{L^\infty} + \|\nabla d_0\|_{L^3} \leq \varepsilon_0 = \varepsilon_0(\|\rho_0\|_1, \|\sqrt{\rho_0} u_0\|_2, \|\nabla u_0\|_2, \|\sqrt{\rho_0} E_0\|_2, \|\nabla d_0\|_2, \|\nabla^2 d_0\|_2, \mu, \lambda, R, \gamma, \kappa). \]

Note that the explicit dependence of $\varepsilon_0$ on the initial norms was not derived in [37]. Therefore, the scaling invariant quantities may not be expected there.

Note that the system (1.1)-(1.5) contains the full compressible Navier-Stokes equations as subsystem, it inherits the difficulties of the full compressible Navier-Stokes equations. A typical difficulty is that the basic energy estimate does not yield the desired dissipation estimates $\int_0^T \|\nabla u\|^2 dt$. Another difficulty is that the presence of the liquid crystal director field $d$ brings strong coupling term $u \cdot \nabla d$ and nonlinear terms $\nabla d \cdot \Delta d$ and $|\nabla d|^2 d$. In order to overcome these difficulties, we adopt the idea in [26] to get the $L^\infty(0, T; L^3)$ estimate of $\rho$ and introduce the spatial $L^2$-norm of $\nabla d$ and $\nabla^2 d$. This motivates us to put smallness assumptions on $\|\rho_0\|_\infty, \|\sqrt{\rho_0} u_0\|_2, \|\sqrt{\rho_0} u_0\|_2$ and $\|\nabla d_0\|_2, \|\nabla^2 d_0\|_2$, which are both scaling invariant. As a result, by continuity arguments, some necessary lower order time-independent estimates are obtained. Then, we give higher order estimates and eliminate the impact of vacuum by introducing the effective viscous flux and the material derivative.

The paper is organized as follows. In section 2 and section 3, we derive some lower order and higher order a priori estimates for the solutions to the Cauchy problem (1.1)-(1.5), respectively. Section 4 is devoted to proving the global well-posedness.

2 Time independent lower order a priori estimates

First, the following local well-posedness of strong solutions with initial vacuum can be established in a similar way as in [1] and [6].
Lemma 2.1 (local well-posedness) Assume that the initial data \((\rho_0, u_0, \theta_0, d_0)\) satisfies the conditions in Theorem 1.1. Then, there is a positive time \(T_0\) depending only on \(R, c_v, \mu, \lambda, \kappa,\) and \(\Phi_0,\) such that the system \((1.1), (1.5)\) admits a unique strong solution in \(\mathbb{R}^3 \times (0, T_0),\) where \(\Phi_0\) is a positive constant such that 
\[ \|\rho_0\|_{W^{1,\infty}\cap H^1} + \|(u_0, \theta_0)\|_{D^{1\cap,2}} + \|\nabla d_0\|_{H^2} + \|\sqrt{\rho_0} \theta_0, g_1, g_2\|_2 \leq \Phi_0. \]

In the rest of this section, as well as in the next section, we always assume that \((\rho, u, \theta, d)\) is a strong solution to system \((1.1), (1.5)\) in \(\mathbb{R}^3 \times (0, T)\) for some positive time \(T.\)

Lemma 2.2 Assume that \(2\mu > \lambda.\) It holds that 
\[ \sup_{0 \leq t \leq T} (\|\sqrt{\rho} u\|_2^2 + \|\nabla d\|_2^2) + \int_0^T (\|\nabla u\|_2^2 + \|d_t\|_2^2 + \|\nabla^2 d\|_2^2) dt \]
\[ \leq C(\|\sqrt{\rho_0} u_0\|_2^2 + \|\nabla d_0\|_2^2) + C \int_0^T \|\rho\|_5^2 \|\nabla \theta\|_2^2 dt + C \int_0^T \|\nabla d\|_2^2 (\|\nabla u\|_2^2 + \|\nabla^2 d\|_2^2) dt, \]
for a positive constant \(C\) depending only on \(\mu\) and \(\lambda.\)

**Proof.** Multiplying \((1.2)\) by \(u,\) integrating it over \(\mathbb{R}^3,\) and using integration by parts, one gets 
\[ \frac{1}{2} \frac{d}{dt} \|\sqrt{\rho} u\|_2^2 + \mu \|\nabla u\|_2^2 + (\mu + \lambda) \|\nabla u\|_2^2 \]
\[ = - \int \nabla (R \rho \theta) \cdot u dx - \int \Delta d \cdot \nabla d \cdot u dx \]
\[ \leq R \|\rho\|_5 \|\theta\|_6 \|\nabla u\|_2^2 + \|\nabla u\|_6 \|\Delta d\|_3 \|\nabla^2 d\|_2 \]
\[ \leq C(\|\rho\|_5 \|\theta\|_6 \|\nabla u\|_2^2 + C \|\nabla u\|_2 \|\nabla d\|_3 \|\nabla^2 d\|_2 \]
\[ \leq (\mu + \lambda) \|\nabla u\|_2^2 + C(\|\rho\|_5^2 \|\nabla \theta\|_2^2 + \frac{\mu}{2} \|\nabla u\|_2^2 + C \|\nabla d\|_3 \|\nabla^2 d\|_2^2. \]

Using \((1.4)\) and the Sobolev inequality, it follows 
\[ \frac{d}{dt} \|\nabla d\|_2^2 + \int (|d_t|^2 + |\nabla^2 d|^2) dx = \int |d_t - \Delta d|^2 dx = \int |u \cdot \nabla d - |\nabla d|^2 d|^2 dx \]
\[ \leq C(\|\nabla u\|_2^6 \|\nabla^2 d\|_2 + \|\nabla d\|_4^4) \leq C \|\nabla d\|_3^2 (\|\nabla u\|_2^2 + \|\nabla^2 d\|_2^2), \]
where \(\|\nabla d\|^2 = -\Delta d \cdot d\) guaranteed by \(|d| = 1\) was used. Adding \((2.1)\) and \((2.2)\) yields
\[ \frac{d}{dt} (\|\sqrt{\rho} u\|_2^2 + \|\nabla d\|_2^2) + \mu \|\nabla u\|_2^2 + \|d_t\|_2^2 + \|\nabla^2 d\|_2^2 \]
\[ \leq C(\|\rho\|_5^2 \|\nabla \theta\|_2^2 + C \|\nabla d\|_3^2 (\|\nabla u\|_2^2 + \|\nabla^2 d\|_2^2). \]

The conclusion follows by integrating in \(t\) to the above inequality. \(\square\)

**Lemma 2.3** It holds that 
\[ \sup_{0 \leq t \leq T} (\|\nabla^2 d\|_2^2 + \|\nabla d\|_2^4) + \int_0^T (\|\nabla d_t\|_2^2 + \|\nabla^3 d\|_2^2 + \|\nabla d_t\|\nabla^2 d\|_2^2) dt \]
\[ \leq C(\|\nabla^2 d_0\|_2^2 + \|\nabla d_0\|_2^4) + C \sup_{0 \leq t \leq T} (\|\nabla u\|_2^5 \|\nabla d\|_2^2) \int_0^T \|\nabla u\|_2^5 dt \]
\[ + C \sup_{0 \leq t \leq T} (\|\nabla d\|_2^2 \|\nabla^2 d\|_2^2) \int_0^T \|\nabla^3 d\|_2^2 dt, \]
for an absolute positive constant \(C.\)
Proof. Applying the operator $\nabla$ to (1.4) yields
\[
\nabla d_t - \nabla \Delta d = -\nabla(u \cdot \nabla d) + |\nabla d^2|d, \tag{2.3}
\]
from which one derives
\[
\frac{d}{dt} \|\nabla \Delta d\|_2^2 + \int (|\nabla d_t|^2 + |\nabla^3 d|^2) dx = \int |\nabla d_t - \nabla \Delta d|^2 dx = \int |\nabla(u \cdot \nabla d) - |\nabla d^2|d|^2 dx \\
\leq C \int (|\nabla u|^2 |\nabla d|^2 + |u|^2 |\nabla^2 d|^2 + |\nabla d|^6) dx + 2 \int |\nabla d|^2 |\nabla^2 d|^2 dx. \tag{2.4}
\]
Multiplying (2.3) by $4|\nabla d|^2 \nabla d$, and then integrating it over $\mathbb{R}^3$, one obtains
\[
\frac{1}{2} \frac{d}{dt} \int |\nabla d|^4 dx + 4 \int (|\nabla d| |\nabla^2 d|^2 + 2|\nabla d^2| |\nabla ((|\nabla d|)^2) dx \\
= 4 \int |\nabla d^2| \nabla d(-\nabla(u \cdot \nabla d) + \nabla(|\nabla d|^2)) dx \\
\leq C \int (|\nabla d|^3 |\nabla u| + |\nabla d^3| |\nabla^2 d||u| + |\nabla d|^4 |\nabla^2 d| + |\nabla d|^6) dx \\
\leq \int |\nabla d|^2 |\nabla^2 d|^2 dx + C \int (|\nabla u|^2 |\nabla d|^2 + |u|^2 |\nabla^2 d|^2 + |\nabla d|^6) dx, \tag{2.5}
\]
where $|\nabla d|^2 = -\Delta d \cdot d$ guaranteed by $|d| = 1$ was used. Adding (2.3) and (2.5) yields
\[
\frac{d}{dt} \int (|\nabla d|^2 |\nabla d|^4 dx + \int (|\nabla d_t|^2 + |\nabla^3 d|^2 + |\nabla d|^2 |\nabla^2 d|^2) \\
\leq C \int (|\nabla u|^2 |\nabla d|^2 + |u|^2 |\nabla^2 d|^2 + |\nabla d|^6) dx. \tag{2.6}
\]
It follows from the Gagliardo-Nirenberg and Young inequalities that
\[
\int |\nabla u|^2 |\nabla d|^2 dx \leq \|\nabla d\|^2_{\infty, \infty} \|\nabla u\|^2_2 \leq C \|\nabla d\|^\frac{1}{2} \|\nabla^3 d\|^\frac{3}{2} \|\nabla u\|^2_2 \\
\leq \frac{1}{4} \|\nabla^3 d\|^2_2 + C (\|\nabla d\|^2_2 \|\nabla u\|^2_2) \|\nabla u\|^2_2, \tag{2.7}
\]
\[
\int |\nabla d|^6 dx \leq C\|\nabla d\|^2_6 \|\nabla^2 d|^2_6 \|\nabla d|^2_2 \leq C \|\nabla d\|^2_0 \|\nabla (\nabla d)^2\|^2_2 \|\nabla d\|^2_4 \\
\leq \frac{1}{2} \|\nabla d\|^2_4 \|\nabla^2 d|^2_4 + C \|\nabla^2 d\|^4_2 \|\nabla d\|^4_4 \\
\leq \frac{1}{2} \|\nabla d\|^2_2 \|\nabla^2 d|^2_2 + C \|\nabla d\|^2_2 \|\nabla^3 d|^2_2 \|\nabla d|^2_2 \|\nabla^2 d|^2_2 \\
\leq \frac{1}{2} \|\nabla d\|^2_2 \|\nabla^2 d|^2_2 + C \|\nabla d\|^2_2 \|\nabla^2 d|^2_2 \|\nabla^3 d|^2_2 \|\nabla^2 d|^2_2 \tag{2.8}
\]
and
\[
\int |u|^2 |\nabla^2 d|^2 dx \leq C \|u\|^2_6 \|\nabla^2 d|^2_6 \|\nabla^2 d|^2_2 \leq C \|\nabla u\|^2_2 \|\nabla^3 d|^2_2 \|\nabla d|^2_2 \\
\leq \frac{1}{4} \|\nabla^3 d\|^2_2 + C (\|\nabla d\|^2_2 \|\nabla u\|^2_2) \|\nabla u\|^2_2. \tag{2.9}
\]
Substituting (2.7)-(2.9) into (2.6) yields
\[
\frac{d}{dt} \int (|\nabla^2 d|^2 + |\nabla d|^4) dx + \frac{1}{2} \int (|\nabla d_t|^2 + |\nabla^3 d|^2 + |\nabla^2 d|^2 |\nabla d|^2) dx \\
\leq C \|\nabla d\|^2_2 \|\nabla^2 d\|^2_2 + C (\|\nabla d\|^2_2 \|\nabla u\|^2_2) \|\nabla u\|^2_2,
\]
which implies the conclusion by integrating in \( t \).

**Lemma 2.4** It holds that
\[
\sup_{0 \leq t \leq T} \|\sqrt{\rho} E\|^2_2 + \int_0^T (\|\nabla \theta\|^2_2 + \|u\| |\nabla u\|^2_2) dt \\
\leq C \|\sqrt{\rho} E_0\|^2_2 + C \int_0^T \|\rho\|_\infty \|\rho\|_3^{1/2} \sqrt{\rho} \theta \|\nabla \theta\|^2_2 + \|u\| |\nabla u\|^2_2) dt + C \sup_{0 \leq t \leq T} \|\nabla^2 d\|^4_2 \int_0^T \|\nabla u\|^2_2 dt \\
+ C \sup_{0 \leq t \leq T} (\|\nabla d\|^2_2 \|\nabla^2 d\|^2_2) \int_0^T \|\nabla d\| |\nabla^2 d|^2_2 dt + C \sup_{0 \leq t \leq T} (\|\nabla d\|^2_2 \|\nabla^2 d\|^2_2) \int_0^T \|\nabla^3 d\|^2_2 dt,
\]
for a positive constant \( C \) depending only on \( R, c_v, \mu, \lambda, \) and \( \kappa \), where \( E = \frac{\|u\|^2}{2} + c_v \theta \).

**Proof.** From the energy \( E \), multiplying (1.2) by \( u \) and adding the resultant to (1.3), one has
\[
\rho (E_t + u \cdot \nabla E) + \text{div}(uP) - \kappa \Delta \theta = \text{div}(\mathbf{S} \cdot u) - \Delta d \cdot \nabla d \cdot u + |\Delta d + |\nabla d|^2|^2, \quad (2.10)
\]
where \( \mathbf{S} = \mu (\nabla u + (\nabla u)^T) + \lambda \text{div} u \mathbf{I} \). Then, multiplying (2.10) with \( E \) and integrating it over \( \mathbb{R}^3 \) yield
\[
\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho} E\|^2_2 + \kappa c_v \|\nabla \theta\|^2_2 \leq \frac{\kappa c_v}{2} \|\nabla \theta\|^2_2 + C \|u\| |\nabla u\|^2_2 + C \int \rho^2 \theta^2 |u|^2 dx \\
- \int (\Delta d \cdot \nabla d \cdot u) Edx + \int |\Delta d + |\nabla d|^2|^2 E dx
\]
and, thus,
\[
\frac{d}{dt} \|\sqrt{\rho} E\|^2_2 + \kappa c_v \|\nabla \theta\|^2_2 \leq C \|u\| |\nabla u\|^2_2 + C \int \rho^2 \theta^2 |u|^2 dx \\
- 2 \int (\Delta d \cdot \nabla d \cdot u) Edx + 2 \int |\Delta d + |\nabla d|^2|^2 E dx. \quad (2.11)
\]
One can rewrite the right hand side of (1.2) in divergence form since
\[
-\Delta d \cdot \nabla d = -\text{div} \left( \nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 \mathbb{I}_3 \right),
\]
where \( \nabla d \odot \nabla d \triangleq (d_{x_i} \cdot d_{x_j})_{3 \times 3} \) and \( \mathbb{I}_3 \) denotes the identity matrix of order 3. By the Sobolev
inequality and integration by parts, one deduces

\[- \int (\Delta d \cdot \nabla d \cdot u) E dx \]

\[= - \int \text{div} \left( \nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 I_3 \right) : u E dx \]

\[= \int \left( \nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 I_3 \right) \cdot \nabla (u E) dx \]

\[\leq C \|\nabla d\|_6^2 \|\nabla u\|_2 \|E\|_6 + \frac{K_C}{16} \|\nabla \theta\|_2^2 + C \||u||\nabla u||_2^2 + C \int |\nabla d|^4 |u|^2 dx \]

\[\leq \frac{K_C}{8} \|\nabla \theta\|_2^2 + C \||u||\nabla u||_2^2 + C \|||\nabla d|_2|\|\nabla u||_2^2 + C \|||\nabla d|_6|\|\nabla u||_2^2 + C \||\nabla d|_3|\|\nabla u||_2^2 + C \||\nabla d|_6|\|\nabla u||_2^2 \]

and

\[\int |\Delta d + |\nabla d|^2 d|^2 E \leq 2 \int |\Delta d|^2 E dx + 2 \int |\nabla d|^4 E dx \]

\[\leq 2 \int |\nabla d||\nabla^3 d| E dx + 2 \int |\nabla d||\nabla^2 d||\nabla E| dx + \int |\nabla d|^4 |u|^2 dx + 2c_v \int |\nabla d|^4 d dx \]

\[\leq C \||\nabla d||_3|\|\nabla^3 d|_2 \|E\|_6 + C \||\nabla d||_3|\|\nabla^2 d|_6 \|\nabla E\|_2 + C \||u||\nabla u||_6 + C \||u||\nabla \theta\|_6 \|\nabla d\|_2 \|\nabla d\|_6 \|\nabla d\|_6 \|\nabla d\|_6 \|\nabla d\|_6 \]

\[\leq C \|||\nabla u||_2|\|\nabla u||_2^2 + C \||\nabla d|_3|\|\nabla^3 d|_2 + C \||\nabla d|_6|\|\nabla^2 d|_2 \|\nabla d\|_2 \|\nabla d\|_2 \|\nabla d\|_2 \]

\[\leq \frac{K_C}{8} \|\nabla \theta\|_2^2 + C \||u||\nabla u||_2^2 + C \||\nabla d|_3|\|\nabla^3 d|_2 + C \||\nabla d|_6|\|\nabla^2 d|_2 \|\nabla d\|_2 \|\nabla d\|_2 \|\nabla d\|_2 \]

Putting the above two inequalities into (2.11) leads to

\[\frac{d}{dt} \|\sqrt{\rho} E\|_2^2 + \frac{K_C}{2} \|\nabla \theta\|_2^2 \leq C \||u||\nabla u||_2^2 \| + C \int \rho^2 \theta^2 |u|^2 dx \]

\[+ C \||\nabla d|_2^2 \|\nabla u||_2^2 + C \||\nabla d|_6^2 \|\nabla d||_2^2 + C \||\nabla d|_6^2 \|\nabla d||_2^2 \|\nabla d||_2^2 \|\nabla d||_2^2 \|\nabla d||_2^2 \].

(2.12)

To control the term \||u||\nabla u||_2^2 in (2.12), we need to multiply (1.2) with \|u|^2 u to obtain that

\[\frac{1}{4} \frac{d}{dt} \|\sqrt{\rho}|u|^2\|_2^2 - \int (\mu \Delta u + (\mu + \lambda) \nabla \text{div} u) \cdot |u|^2 u dx \]

\[= - \int P \text{div}(|u|^2 u) dx - \int \left( \nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 I_3 \right) \text{div}(|u|^2 u) dx \]

\[\leq C \int (\rho \theta + |\nabla d|^2)|u|^2 dx \]

\[\leq \frac{1}{2} \left( \mu - \frac{\lambda}{2} \right) \||u||\nabla u||_2^2 + C \int \rho^2 \theta^2 |u|^2 + |\nabla d|^4 |u|^2 dx \]

\[\leq \frac{1}{2} \left( \mu - \frac{\lambda}{2} \right) \||u||\nabla u||_2^2 + C \int \rho^2 \theta^2 |u|^2 dx + C \||u||_6^2 \|\nabla d\|_2 \|\nabla d\|_6 \|\nabla d\|_6 \|\nabla d\|_6 \]

\[\leq \left( \mu - \frac{\lambda}{2} \right) \||u||\nabla u||_2^2 + C \int \rho^2 \theta^2 |u|^2 dx + C \||\nabla d\|_2^2 \|\nabla^2 d\|_2^2 \||\nabla d\|\nabla^2 d||_2^2 \].

(2.13)
By direct computation, one has

$$-\int (\mu \Delta u + (\mu + \lambda) \nabla \text{div } u) \cdot |u|^2 \, u \, dx \geq (2\mu - \lambda) ||u||\nabla u||^2_2.$$  

Hence, it follows from the above inequality and (2.13) that

$$\frac{d}{dt} \|\sqrt{\rho}u\|^2_2 + 2(2\mu - \lambda)||u||\nabla u||^2_2 \leq C \int \rho^2 \theta^2 |u|^2 \, dx + C \|\nabla d\|^2_2 \|\nabla^2 d\|_2 \|\nabla d\||\nabla^2 d||^2_2. \quad (2.14)$$

Now, multiplying (2.14) by $N > 0$, which is a sufficiently large number and depending only on $R, c_\nu, \mu, \lambda$, and $\kappa$, then adding the resultant to (2.12), one gets

$$\frac{d}{dt} (\|\sqrt{\rho}E\|^2_2 + N\|\sqrt{\rho}u\|^2_2) + \frac{kC_\nu}{2} \|\nabla \theta\|^2_2 + (2\mu - \lambda) N ||u||\nabla u||^2_2 \leq C \|\rho\|_\infty \|\rho\|^\frac{1}{3}_3 \|\sqrt{\rho} \theta\|_2 \|\nabla \theta\|_2 ||u||_6 \|\nabla u||_2 + C \|\nabla^2 d\|^2_2 ||\nabla d||_2 + C \|\nabla d\|^2_2 \|\nabla^2 d\|_2 \|\nabla^3 d||^2_2 \quad (2.15)$$

$$+ C \|\nabla d\|^2_2 \|\nabla^2 d\|_2 \|\nabla^2 d\|_3 \|\nabla d\|_2,$$

where we have used the fact

$$\int \rho^2 \theta^2 |u|^2 \, dx \leq C \|\sqrt{\rho} \theta\|_2 ||\rho||_6 \|\nabla u||_6 \|\rho||^\frac{3}{2}_3 \leq C \|\rho\|_\infty \|\rho||^\frac{1}{3}_3 \|\sqrt{\rho} \theta\|_2 \|\nabla \theta\|_2 ||u||\nabla u||_2. \quad (2.16)$$

The proof can be completed by integrating (2.15) over $[0,t]$.

**Lemma 2.5** It holds that

$$\sup_{0 \leq t \leq T} \|\rho\|^3_3 + \int_0^T \int \rho^3 P \, dx \, dt \leq C \|\rho_0\|^3_3 + C \sup_{0 \leq t \leq T} (\|\rho\|^\frac{2}{3}_\infty \|\sqrt{\rho} \theta\|_2 \|\nabla \theta\|_2 ||u||_6 \|\rho||^\frac{3}{2}_3)$$

$$+ C \int_0^T (\|\rho\|^2_\infty \|\rho^2_3 \|\nabla u||^2_2 \, dt) + C \int_0^T (\|\rho\|_\infty \|\rho||^2_3 \|\nabla^2 d\|^3_3 \, dt),$$

for a positive constant $C$ depending only on $R, c_\nu, \mu, \lambda$, and $\kappa$.

**Proof.** If we apply the operator $\Delta^{-1} \text{div}$ to (1.2), it holds that

$$\Delta^{-1} \text{div}(\rho u)_t + \Delta^{-1} \text{div}(\rho u \otimes u) - (2\mu + \lambda) \text{div } u + P = -\Delta^{-1} \text{div } (\nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 \mathbb{I}_3). \quad (2.17)$$

Taking advantage of (1.1), we have

$$(\rho^3)_t + \text{div}(u \rho^3) + 2 \text{div } u \rho^3 = 0. \quad (2.18)$$

Then, multiplying (2.17) by $\rho^3$ and using above equality, we obtain

$$\frac{2\mu + \lambda}{2} ((\rho^3)_t + \text{div}(u \rho^3)) + \rho^3 P + \rho^3 \Delta^{-1} \text{div}(\rho u)_t + \rho^3 \Delta^{-1} \text{div}(\rho u \otimes u)$$

$$= -\rho^3 \Delta^{-1} \text{div } (\nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 \mathbb{I}_3). \quad (2.19)$$
Using (2.18), it follows
\[
\int \rho^3 \Delta^{-1} \text{div}(\rho u) \, dx = \frac{d}{dt} \int \rho^3 \Delta^{-1} \text{div}(\rho u) \, dx + \int [\text{div}(\rho^3 u) + 2 \text{div}^2 \rho^3] \Delta^{-1} \text{div}(\rho u) \, dx
\]
\[
= \int [2 \text{div}^2 \rho^3 \Delta^{-1} \text{div}(\rho u) - \rho^3 u \cdot \nabla \Delta^{-1} \text{div}(\rho u)] \, dx + \frac{d}{dt} \int \rho^3 \Delta^{-1} \text{div}(\rho u) \, dx.
\]
Thanks to this, it follows from integrating (2.19) over \( \mathbb{R}^3 \) that
\[
\frac{d}{dt} \int \left( \frac{2\mu + \lambda}{2} + \Delta^{-1} \text{div}(\rho u) \right) \rho^3 \, dx + \int \rho^3 P \, dx
\]
\[
= \int \left[ \rho^3 (u \cdot \nabla \Delta^{-1} \text{div}(\rho u) - \Delta^{-1} \text{divdiv}(\rho u \otimes u)) - 2 \text{div}^2 \rho^3 \Delta^{-1} \text{div}(\rho u) \right] \, dx
\]
\[
- \int \rho^3 \Delta^{-1} \text{divdiv} \left( \nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 I_3 \right) \, dx.
\]
The conclusion in this lemma then follows from the same estimates as in Proposition 2.4 in [26] and the following bound for the last term in above equality
\[
\int \rho^3 \left| \Delta^{-1} \text{divdiv} \left( \nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 I_3 \right) \right| \, dx
\]
\[
\leq C \|\rho\|_\infty \|\rho\|_3^2 \left\| \Delta^{-1} \text{divdiv} \left( \nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 I_3 \right) \right\|_3
\]
\[
\leq C \|\rho\|_\infty \|\rho\|_3^2 \|\nabla d\|^2 \leq C \|\rho\|_\infty \|\rho\|_3^2 \|\nabla^2 d\|_2^2,
\]
where the elliptic estimates were applied. □

**Lemma 2.6** Assume that
\[
\sup_{0 \leq t \leq T} \|\rho\|_\infty \leq 4 \overline{\rho}.
\]
Then, it holds that
\[
\sup_{0 \leq t \leq T} \|\nabla u\|_2^2 + \int_0^T \left\| \left( \sqrt{\rho u_t}, \frac{\nabla G}{\sqrt{\rho}}, \frac{\nabla \omega}{\sqrt{\rho}} \right) \right\|_2^2 \, dt
\]
\[
\leq C \|\nabla u_0\|_2^2 + C \overline{\rho} \sup_{0 \leq t \leq T} \|\sqrt{\rho} \theta\|_2^2 + C \overline{\rho} \int_0^T \|\nabla u\|_2^4 (\|\nabla u\|_2^2 + \|\nabla \theta\|_2^2) \, dt
\]
\[
+ C \int_0^T (\overline{\rho} + \overline{\rho}^2 \|\rho\|_2^2 \|\sqrt{\rho} \theta\|_2^2 (\|\nabla \theta\|_2^2 + \|\nabla u\|_2^2) \) \, dt
\]
\[
+ C \|\nabla d_0\|_4^4 + C \|\nabla d\|_4^4 + \varepsilon_1 \overline{\rho} \int_0^T \|\nabla \theta\|_2^2 \, dt + C \overline{\rho} \sup_{0 \leq t \leq T} (\|\nabla d\|_2 \|\nabla^2 d\|_2) \int_0^T (\|\nabla d\|_2^2 + \|\nabla^3 d\|_2^2) \, dt
\]
\[
+ C \overline{\rho} \sup_{0 \leq t \leq T} (\|\nabla d\|_2 \|\nabla^2 d\|_2) \int_0^T \|\nabla (|\nabla d|^2)\|_2^2 \, dt,
\]
where \( G = (2\mu + \lambda) \text{div} - p, \) \( \omega = \nabla \times u \) and the constant \( C > 0 \) depending only on \( R, c_v, \mu, \lambda, \) and \( \kappa. \)
**Proof.** Multiplying (1.2) by $u_t$, it follows from integration by parts that

$$
\frac{1}{2} \frac{d}{dt} (\mu \|\nabla u\|_2^2 + (\mu + \lambda) \|\text{div} u\|_2^2) - \int P\text{div}u_t dx + \|\sqrt{\rho} u_t\|_2^2
= -\int \rho(u \cdot \nabla) u \cdot u_t dx - \int \Delta d \cdot \nabla d \cdot u_t dx.
$$

(2.20)

By the definition of effective viscous flux $G$, it is easy to see $\text{div} u = \frac{G + P}{2 \mu + \lambda}$, which implies

$$
-\int P\text{div}u_t dx = -\frac{d}{dt} \int P\text{div}u dx + \int P_t\text{div}u dx
= -\frac{d}{dt} \int P\text{div}u dx + \frac{1}{2(2\mu + \lambda)} \frac{d}{dt} \|P\|_2^2 + \frac{1}{2\mu + \lambda} \int P_t G dx.
$$

(2.21)

On the other hand, it follows from (1.3) that

$$
P_t = (\gamma - 1)(\mathcal{Q}(\nabla u) - P\text{div} u + \kappa \Delta \theta + |\Delta d + |\nabla d|^2 |d|^2) - \text{div}(uP),
$$

which leads to

$$
\int P_t G dx = \int [(\gamma - 1)(\mathcal{Q}(\nabla u) - P\text{div} u + |\Delta d + |\nabla d|^2 |d|^2)G + (uP - \kappa(\gamma - 1) \nabla \theta \cdot \nabla G)] dx.
$$

(2.22)

Due to $\|\nabla u\|_2^2 = \|\omega\|_2^2 + \|\text{div} u\|_2^2$ and combining with (2.20)-(2.22), one can deduce that

$$
\frac{1}{2} \frac{d}{dt} \left( \mu \|\omega\|_2^2 + \frac{\|G\|_2^2}{2\mu + \lambda} \right) + \|\sqrt{\rho} u_t\|_2^2
= -\int \rho(u \cdot \nabla) u \cdot u_t dx - \int |\Delta d \cdot \nabla d \cdot u_t dx + \frac{1}{2\mu + \lambda} \int (\kappa(\gamma - 1) \nabla \theta - uP) \cdot \nabla G dx
- \frac{\gamma - 1}{2\mu + \lambda} \int (\mathcal{Q}(\nabla u) - P\text{div} u + |\Delta d + |\nabla d|^2 |d|^2)G dx.
$$

(2.23)

In order to bound the right hand side of (2.23), we need to reformulate (1.2) in the following form with the help of $\Delta u = \nabla \text{div} u - \nabla \times \nabla \times u$:

$$
\rho(u_t + u \cdot \nabla u) = \nabla G - \mu \nabla \times \omega - \Delta d \cdot \nabla d.
$$

(2.24)

Then, multiplying both sides of (2.24) by $\nabla G$, it follows

$$
\|\nabla G\|_2^2 = \int (\rho(u_t + u \cdot \nabla u) \cdot \nabla G + \Delta d \cdot \nabla d \cdot \nabla G) dx
\leq \int \left( \frac{|\nabla G|^2}{2} + 2\rho |u_t|^2 \right) dx + \int (\rho(u \cdot \nabla) u \cdot \nabla G + \Delta d \cdot \nabla d \cdot \nabla G) dx
$$

where $\int \nabla G \cdot \nabla \times \omega dx = 0$ and $\|\rho\|_{\infty} \leq 4\pi$ were used. This gives that

$$
\frac{\|\nabla G\|_2^2}{16\pi} \leq \frac{1}{4} \|\sqrt{\rho} u_t\|_2^2 + \frac{1}{8\pi} \int (\rho(u \cdot \nabla) u \cdot \nabla G + \Delta d \cdot \nabla d \cdot \nabla G) dx.
$$

(2.25)
Similarly, one has

$$\frac{\mu^2 \| \nabla \omega \|^2}{16\rho} \leq \frac{1}{4} \| \sqrt{\rho \mu} \|^2 + \frac{1}{8\rho} \int (\rho(u \cdot \nabla) u \cdot \nabla \omega + \Delta d \cdot \nabla d \cdot \nabla \omega) dx. \quad (2.26)$$

Putting (2.25) and (2.26) into (2.23), one gets

$$\frac{1}{2} \frac{d}{dt} \left( \mu \| \omega \|^2 + \frac{\| G \|^2}{2\mu + \lambda} \right) + \frac{1}{2} \| \sqrt{\rho \mu t} \|^2 + \frac{1}{16\rho} (\| \nabla G \|^2 + \mu^2 \| \nabla \omega \|^2)$$

$$\leq C \int \rho |u| |\nabla u| \left( \| u_t \|^2 + \frac{1}{\beta} (|\nabla G| + |\nabla \omega|) \right) dx + C \int (|\nabla \theta| + \rho \theta |u|) |\nabla G| dx$$

$$+ C \int (|\nabla u|^2 + \rho \theta |\nabla u|) |G| dx - \int \Delta d \cdot \nabla d \cdot u_t dx + C \int (|\Delta d| + |\nabla d|^2 |d|^2) G dx$$

$$+ \frac{C}{\rho} \int |\Delta d| |\nabla d| (|\nabla G| + |\nabla \omega|) dx =: \sum_{i=1}^{6} I_i. \quad (2.27)$$

Estimate $I_i, i = 1, 2, \ldots, 6$ are given as follows. It follows from the Hölder and Young inequalities that

$$I_1 \leq C \sqrt{\beta} \|u\|_2 \|\nabla u\|_2 \|\sqrt{\rho \mu} \|^2 + C \|u\|_2 \|\nabla u\|_2 (\| \nabla G \|^2 + \| \nabla \omega \|^2)$$

$$\leq \frac{1}{12} \| \sqrt{\rho \mu} \|^2 + \frac{1}{192\beta} (\| \nabla G \|^2 + \mu^2 \| \nabla \omega \|^2) + C \beta \|u\| \|\nabla u\|_2^2,$$

and

$$I_2 \leq C \| \nabla \theta \|^2 \|G\|^2 + \| \rho \theta u \|^2 \| \nabla G \|_2$$

$$\leq C \| \nabla \theta \|^2 \|G\|^2 + C \sqrt{\beta} \| \rho \|^2 \| \sqrt{\rho \mu} \|^2 \| \nabla \theta \|^2 \|u\| \|\nabla u\|_2^2 \|G\|_2$$

$$\leq \frac{1}{192\beta} \| \nabla G \|^2 + C \frac{1}{\beta} (\| \nabla \theta \|^2 \|u\| \|\nabla u\|_2^2),$$

where (2.16) was used in $I_2$. For $I_3$, noticing that

$$\| \nabla u \|_6 \leq C (\| \omega \|_6 + \| \text{div} u \|_6) \leq C (\| \omega \|_6 + \| G \|_6 + \| \rho \theta \|_6) \leq C (\| \nabla \omega \|_2 + \| \nabla G \|_2 + \beta \| \nabla \theta \|_2), \quad (2.28)$$

it follows from the Hölder, Young and Sobolev inequalities that

$$I_3 \leq C \| \nabla u \|_2 \| \nabla u \|_6 \|G\|_3 + \| \nabla u \|_2 \| \rho \theta \|_6 \|G\|_3$$

$$\leq C \| \nabla u \|_2 (\| \nabla G \|^2 + \beta \| \nabla \theta \|^2) \|G\|^2 + C \beta \| \nabla u \|_2 \| \nabla \theta \|^2 \|G\|^2$$

$$\leq \frac{1}{192\beta} (\| \nabla G \|^2 + \mu^2 \| \nabla \omega \|^2) + C \beta \|u\|_2 \| \nabla \theta \|^2 \|G\|^2 + C \beta \| \nabla \theta \|_2^2.$$

Note that

$$I_4 = - \int \text{div} \left( \nabla d \otimes \nabla d - \frac{1}{2} |\nabla d|^2 1_3 \right) u_t dx = \int (\nabla d \otimes \nabla d) : \nabla u_t dx - \frac{1}{2} \int |\nabla d|^2 \text{div} u_t dx.$$
Using the Young and Sobolev inequalities, one obtains
\[
\int (\nabla d \circ \nabla d) : \nabla u dx = \frac{d}{dt} \int (\nabla d \circ \nabla d) : \nabla u dx - \int (\nabla d_t \circ \nabla d) : \nabla u dx - \int (\nabla d \circ \nabla d_t) : \nabla u dx
\]
\[
\leq \frac{d}{dt} \int (\nabla d \circ \nabla d) : \nabla u dx + C \|\nabla d\|_3 \|\nabla d_t\|_2 \|\nabla u\|_6
\]
\[
\leq \frac{d}{dt} \int (\nabla d \circ \nabla d) : \nabla u dx + \varepsilon > \frac{\varepsilon}{4p} \|\nabla u\|_6^2 + C\overline{\rho} \|\nabla d\|_3^2 \|\nabla d_t\|_2^2,
\]
where \(\varepsilon > 0\) is a sufficiently small constant. Similarly,
\[
\int |\nabla d|^2 \text{div} u dx \leq \int |\nabla d|^2 \text{div} u dx + \varepsilon > \frac{\varepsilon}{4p} \|\nabla u\|_6^2 + C\overline{\rho} \|\nabla d\|_3^2 \|\nabla d_t\|_2^2.
\]
Therefore, we get
\[
I_4 \leq \frac{d}{dt} \int (\nabla d \circ \nabla d) : \nabla u + |\nabla d|^2 \text{div} u| dx + \frac{\varepsilon}{2p} \|\nabla u\|_6^2 + C\overline{\rho} \|\nabla d\|_3^2 \|\nabla d_t\|_2^2.
\]
Using (2.28) again, we obtain
\[
I_4 \leq \frac{d}{dt} \int (\nabla d \circ \nabla d) : \nabla u + |\nabla d|^2 \text{div} u| dx + \frac{\varepsilon}{2p} \|\nabla u\|_6^2 + C\overline{\rho} \|\nabla d\|_3^2 \|\nabla d_t\|_2^2
\]
\[+ \frac{1}{192\overline{\rho}} (\|\nabla \omega\|_2^2 + \|\nabla G\|_2^2) + \frac{\varepsilon_1 \overline{\rho}}{2} \|\nabla \omega\|_2^2 + C\overline{\rho} \|\nabla d\|_3^2 \|\nabla d_t\|_2^2,
\]
where \(\varepsilon_1 = \frac{\varepsilon}{2p}\) is sufficiently small.

Now, let us turn to \(I_5\) and \(I_6\). By virtue of the H"{o}lder, Young and Sobolev inequalities, one deduces
\[
I_5 = C \int (|\Delta d + |\nabla d|^2| d|^2 G) dx
\]
\[
\leq C \int (|\nabla d|^3 |\nabla^3 d| |G| + |\nabla d|^2 |\nabla d| |\nabla G|) dx + C \|\nabla d\|_2 \|\nabla d\|_6 \|\nabla d\|_6 \|G\|_6
\]
\[
\leq C \|\nabla d\|_3 \|\nabla^3 d\|_2 \|\nabla G\|_2 + C \|\nabla d\|_2 \|\nabla^2 d\|_2 \|\nabla (|\nabla d|^2)\|_2 \|\nabla G\|_2
\]
\[
\leq \frac{1}{192\overline{\rho}} \|\nabla G\|_2^2 + C \overline{\rho} \|\nabla d\|_3^2 \|\nabla^3 d\|_2^2 + C \overline{\rho} \|\nabla d\|_3^2 \|\nabla^2 d\|_2^2 \|\nabla (|\nabla d|^2)\|_2^2,
\]
where \(|\Delta d + |\nabla d|^2| d|^2 = |\Delta d|^2 + 2\Delta d \cdot d |\nabla d|^2 + |\nabla d|^4 = |\Delta d|^2 - |\nabla d|^2| d|^2 was used, and
\[
I_6 \leq \frac{C}{4p} \|\Delta d\|_6 \|\nabla d\|_3 (|\nabla G|_2 + |\nabla \omega\|_2) \leq \frac{1}{192\overline{\rho}} (\|\nabla G\|_2^2 + \mu^2 \|\nabla \omega\|_2^2) + C \|\nabla d\|_3^2 \|\nabla d\|_2^2
\]
Putting all these estimates for \(I_i, i = 1, 2, \ldots, 6\) into (2.27) leads to
\[
\frac{d}{dt} \left( \frac{\mu \|\omega\|_2^2}{2\mu + \lambda} + \frac{\|G\|_2^2}{2\mu + \lambda} \right) + \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{16\overline{\rho}} (\|\nabla G\|_2^2 + \mu^2 \|\nabla \omega\|_2^2)
\]
\[
\leq \frac{d}{dt} \int (\nabla d \circ \nabla d) : \nabla u + |\nabla d|^2 \text{div} u| dx + C(\overline{\rho} + \overline{\overline{\rho}}) \|\rho\|_{1/3} \|\nabla \theta\|_2 (\|\nabla \omega\|_2^2 + \|\nabla u\|_2^2 + \|\nabla d\|_2^2)
\]
\[
+ C \overline{\rho} \|\nabla d\|_2^2 \|\nabla^2 d\|_2 \|\nabla (|\nabla d|^2)\|_2.
\]
Note that

\[ \left\| \nabla u \right\|_2 \leq C(\left\| \omega \right\|_2 + \left\| G \right\|_2 + \left\| \rho \theta \right\|_2) \leq C(\left\| \omega \right\|_2 + \left\| G \right\|_2 + \sqrt[3]{\left\| \mathcal{P} \right\|} \sqrt[3]{\left\| \theta \right\|_2} ) \]  

(2.30)

and

\[ \int [(\nabla d \odot \nabla d) : \nabla u + |\nabla d|^2 \text{div} u] dx \leq C \left\| \nabla u \right\|_2 \|\nabla d\|^2_2 \leq \varepsilon_2 \left\| \nabla u \right\|_2^2 + C \left\| \nabla d \right\|_4^4. \]  

(2.31)

Integrating (2.29) in t, substituting (2.30) and (2.31) into the resultant and choosing \( \varepsilon_2 \) small enough, one gets the desired result. \( \square \)

**Lemma 2.7** Assume that

\[ \sup_{0 \leq t \leq T} \|\rho\|_\infty \leq 4\mathcal{P}. \]

Then, it holds that

\[ \sup_{0 \leq t \leq T} \|\rho\|_\infty \leq \|\rho_0\|_\infty e^{C_\mathcal{P}^2 \sup_{0 \leq t \leq T} \left\| \sqrt{\rho(u)} \right\|_2^\frac{3}{2} \left\| \sqrt{\rho} u \right\|_2^\frac{1}{2} + C_\mathcal{P} J_0^T \left\| \nabla u \right\|_2 \left\| (\nabla G, \nabla \omega, \nabla \theta) \right\|_2 dt + C J_0^T \left\| \nabla^2 d \right\|_2 dt J_0^T \left\| \nabla^3 d \right\|_2 dt \right)^{\frac{3}{2}}, \]

for a positive constant \( C \) depending only on \( R, c_v, \mu, \lambda, \) and \( \kappa. \)

**Proof.** In view of (2.17), one has

\[ \Delta^{-1} \text{div}(\rho u)_t + u \cdot \nabla \Delta^{-1} \text{div}(\rho u) - (2\mu + \lambda) \text{div} u + P + \Delta^{-1} \text{div} \text{div} \left( \nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2_3 \right) \]

\[ = u \cdot \nabla \Delta^{-1} \text{div}(\rho u) - \Delta^{-1} \text{div} \text{div}(\rho u \odot u) = [u, \mathcal{R} \odot \mathcal{R}](\rho u), \]

where \( \mathcal{R} \) is the Riesz transform on \( \mathbb{R}^3. \) To obtain the estimates of \( \|\rho\|_\infty, \) we adapt the arguments by [26]. Exactly in the same way as in Proposition 2.6 of [26], one can prove Thus, we obtain

\[ \sup_{0 \leq t \leq T} \|\rho\|_\infty \leq \|\rho_0\|_\infty e^{C_\mathcal{P}^2 \sup_{0 \leq t \leq T} \left\| \sqrt{\rho(u)} \right\|_2^\frac{3}{2} \left\| \sqrt{\rho} u \right\|_2^\frac{1}{2} + C_\mathcal{P} J_0^T \left\| \nabla u \right\|_2 \left\| (\nabla G, \nabla \omega, \nabla \theta) \right\|_2 dt + C J_0^T \left\| \nabla^2 d \right\|_2 dt \right)^{\frac{3}{2}}. \]  

(2.32)

Thanks to this and noticing that

\[ \int_0^T \|\nabla d\|_\infty^2 dt \leq C \int_0^T \|\nabla^2 d\|_2 \|\nabla^3 d\|_2 dt \leq C \left( \int_0^T \|\nabla^2 d\|_2 dt \right)^{\frac{1}{2}} \left( \int_0^T \|\nabla^3 d\|_2 dt \right)^{\frac{1}{2}}, \]

the conclusion follows. \( \square \)

**Lemma 2.8** Let

\[ \mathcal{N}_T := \mathcal{P}(\|\rho\|_3 + \sqrt[3]{\mathcal{P}^2}(\|\sqrt{\rho(u)}\|_2^3 + \|\nabla d\|_2^3)) (\|\nabla u\|_2^3 + \mathcal{P}(\|\sqrt{\mathcal{P}}E\|_2^3 + \|\nabla^2 d\|_2^3))(t). \]

Then there is a positive constant \( \eta_0 \) depending only on \( R, c_v, \mu, \lambda, \) and \( \kappa. \) such that if

\[ \eta \leq \eta_0, \quad \sup_{0 \leq t \leq T} \|\rho\|_\infty \leq 4\mathcal{P}, \text{ and } \mathcal{N}_T \leq \sqrt{\eta}, \]

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then it holds that

\[
\sup_{0 \leq t \leq T} \|\rho\|_3 + \left( \int_0^T \int \rho^3 P dxdt \right)^{\frac{1}{3}} \leq C \left( \|\rho_0\|_3 + \overline{\rho}^2 \left( \|\sqrt{\rho_0}u_0\|_2^2 + \|\nabla d_0\|_2^2 \right) \right),
\]

(2.33)

\[
\overline{\rho}^2 \left( \sup_{0 \leq t \leq T} \left( \|\sqrt{\rho}u\|_2^2 + \|\nabla d\|_2^2 \right) + \int_0^T \left( \|\nabla u, d, \nabla^2 d\|_2^2 dt \right) \right)
\leq C \left( \|\rho_0\|_3 + \overline{\rho}^2 \left( \|\sqrt{\rho_0}u_0\|_2^2 + \|\nabla d_0\|_2^2 \right) \right),
\]

(2.34)

\[
\sup_{0 \leq t \leq T} \left[ \overline{p} \left( \|\nabla^2 d\|_2^2 + \|\nabla d\|_2^2 + \|\nabla \theta\|_2 \right) + \|\nabla u\|_2^2 \right] + \int_0^T \left( \|\nabla d_t, \nabla^3 d, |\nabla d||\nabla^2 d|, \nabla \theta, |u||\nabla u|\|_2^2 + \left\| \left( \sqrt{\rho}u, \frac{\nabla G}{\sqrt{\rho}}, \frac{\nabla \omega}{\sqrt{\rho}} \right) \right\|_2^2 dt \right)
\leq C \left( \overline{p} \left( \|\nabla^2 d_0\|_2^2 + \|\sqrt{\rho_0}E_0\|_2^2 + \|\nabla u_0\|_2^2 \right) \right),
\]

(2.35)

\[
\sup_{0 \leq t \leq T} \|\rho\|_{\infty} \leq \overline{p} c \overline{\rho}_0 \overline{\rho}^{\frac{1}{2}} + C \overline{\rho}_0^{\frac{1}{2}},
\]

(2.36)

where the constant \( C > 0 \) depending only on \( R, c_v, \mu, \lambda, \) and \( \kappa \).

**Proof.** In view of Lemma 2.2 and by choosing \( \eta_0 < 1 \) small enough, we have

\[
\sup_{0 \leq t \leq T} \left( \|\sqrt{\rho}u\|_2^2 + \|\nabla d\|_2^2 \right) + \int_0^T \left( \|\nabla u\|_2^2 + \|d_t\|_2^2 + \|\nabla^2 d\|_2^2 \right) dt \leq C \left( \|\sqrt{\rho_0}u_0\|_2^2 + \|\nabla d_0\|_2^2 \right) + C \sup_{0 \leq t \leq T} \|\rho\|_3^2 \int_0^T \|\nabla \theta\|_2^2 dt.
\]

(2.37)

It follows from Lemma 2.3 and the assumptions that

\[
\overline{p} \sup_{0 \leq t \leq T} \left( \|\nabla^2 d\|_2^2 + \|\nabla d\|_2^2 \right) + \overline{p} \int_0^T \left( \|\nabla d_t\|_2^2 + \|\nabla^3 d\|_2^2 + \|\nabla d||\nabla^2 d\|_2^2 \right) dt \leq C \left( \|\nabla^2 d_0\|_2^2 + \|\nabla d_0\|_2^2 \right) + C \eta \int_0^T \|\nabla u\|_2^2 dt + C \eta \int_0^T \|\nabla^3 d\|_2^2 dt.
\]

(2.38)

With the help of (2.37) and since \( \overline{p} = \|\rho_0\|_\infty + 1 \), one deduces from the assumption that

\[
\int_0^T \|\nabla u\|_2^2 dt \leq \sup_{0 \leq t \leq T} \|\nabla u\|_2^2 \int_0^T \|\nabla u\|_2^2 dt \leq C \sup_{0 \leq t \leq T} \|\nabla u\|_2^2 \left( \sup_{0 \leq t \leq T} \left( \|\sqrt{\rho}u\|_2^2 + \|\nabla d\|_2^2 \right) + \sup_{0 \leq t \leq T} \|\rho\|_3^2 \int_0^T \|\nabla \theta\|_2^2 dt \right)
\]

\[
\leq C \eta \sup_{0 \leq t \leq T} \|\nabla u\|_2^2 + C \eta \int_0^T \|\nabla \theta\|_2^2 dt,
\]

which together with (2.38) and by choosing \( \eta_0 \) small enough, implies that

\[
\overline{p} \sup_{0 \leq t \leq T} \left( \|\nabla^2 d\|_2^2 + \|\nabla d\|_2^2 \right) + \overline{p} \int_0^T \left( \|\nabla d_t\|_2^2 + \|\nabla^3 d\|_2^2 + \|\nabla d||\nabla^2 d\|_2^2 \right) dt \leq C \left( \|\nabla^2 d_0\|_2^2 + \|\nabla d_0\|_2^2 \right) + C \eta \sup_{0 \leq t \leq T} \|\nabla u\|_2^2 + C \eta \int_0^T \|\nabla \theta\|_2^2 dt.
\]

(2.39)
Next, applying Lemma 2.4, using the assumptions and (2.37), we obtain

\[
\sup_{0 \leq t \leq T} \| \sqrt{\rho} E \|_2^2 + \int_0^T (\| \nabla \theta \|_2^2 + \| u \| \| \nabla u \|_2^2) \, dt \\
\leq C \| \sqrt{\rho_0} E_0 \|_2^2 + C \eta \frac{1}{2} \int_0^T (\| \nabla \theta \|_2^2 + \| u \| \| \nabla u \|_2^2) \, dt + C \eta \int_0^T \| \nabla^3 d \|_2^2 \, dt + C \eta \frac{1}{2} \int_0^T \| \nabla d \| \| \nabla^2 d \|_2^2 \, dt \\
+ C \sup_{0 \leq t \leq T} \| \nabla^2 d \|_2^2 (\| \sqrt{\rho} u \|_2^2 + \| \nabla d \|_2^2) + C \sup_{0 \leq t \leq T} \| u \| \| \nabla u \|_2 \int_0^T \| \nabla \theta \|_2^2 \, dt \\
\leq C \| \sqrt{\rho_0} E_0 \|_2^2 + C \eta \frac{1}{2} \int_0^T (\| \nabla \theta \|_2^2 + \| u \| \| \nabla u \|_2^2) \, dt + C \eta \int_0^T \| \nabla^3 d \|_2^2 \, dt + C \eta \frac{1}{2} \int_0^T \| \nabla d \| \| \nabla^2 d \|_2^2 \, dt \\
+ C \eta \frac{1}{2} \sup_{0 \leq t \leq T} \| \nabla^2 d \|_2^2 + C \eta \int_0^T \| \nabla \theta \|_2^2 \, dt.
\]

This, combined with the fact \( \eta_0 \) is small enough, implies that

\[
\sup_{0 \leq t \leq T} \| \sqrt{\rho} E \|_2^2 + \int_0^T (\| \nabla \theta \|_2^2 + \| u \| \| \nabla u \|_2^2) \, dt \\
\leq C \| \sqrt{\rho_0} E_0 \|_2^2 + C \eta \int_0^T \| \nabla^3 d \|_2^2 \, dt + C \eta \frac{1}{2} \int_0^T \| \nabla d \| \| \nabla^2 d \|_2^2 \, dt + C \eta \frac{1}{2} \sup_{0 \leq t \leq T} \| \nabla^2 d \|_2^2.
\]  

(2.40)

Then, using the assumptions and Sobolev inequality, it follows from Lemma 2.6 that

\[
\sup_{0 \leq t \leq T} \| \nabla u \|_2^2 + \int_0^T (\sqrt{\rho} u, \frac{\nabla G}{\sqrt{\rho}}, \frac{\nabla \omega}{\sqrt{\rho}})\|_2^2 \, dt \\
\leq C \| \nabla u_0 \|_2^2 + C \bar{\rho} \sup_{0 \leq t \leq T} \| \sqrt{\rho} E \|_2^2 + C \bar{\rho}^3 \int_0^t \| \nabla u \|_2 \left( \| \nabla u \|_2^2 + \bar{\rho} \| \nabla E \|_2^2 \right) \, dt \\
+ C \int_0^T (\bar{\rho} + \bar{\rho}^2) \| \nabla u \|_2 \left( \| \nabla \theta \|_2^2 + \| u \| \| \nabla u \|_2^2 \right) \, dt + \eta \bar{\rho} \int_0^T \| \nabla \theta \|_2^2 \, dt \\
+ C \| \nabla d_0 \|_2^2 + C \| \nabla d \|_2 \| \nabla^2 d \|_2 + C \eta \bar{\rho} \int_0^T (\| \nabla d \|_2^2 + \| \nabla^3 d \|_2^2) \, dt \\
+ C \eta^2 \bar{\rho} \int_0^T \| \nabla (|\nabla d|^2) \|_2^2 \, dt,
\]

where we choose \( \varepsilon_1 \leq \eta \) small enough. By (2.37) and (2.40), we get

\[
\bar{\rho}^3 \int_0^T \| \nabla u \|_2 \left( \| \nabla u \|_2^2 + \bar{\rho} \| \nabla E \|_2^2 \right) \, dt \\
\leq C \bar{\rho}^3 \sup_{0 \leq t \leq T} (\| \nabla u \|_2^2 + \bar{\rho} \| \nabla E \|_2^2) \sup_{0 \leq t \leq T} \| \nabla u \|_2^2 \\
\cdot \left( \sup_{0 \leq t \leq T} (\| \sqrt{\rho} u \|_2^2 + \| \nabla d \|_2^2) + \sup_{0 \leq t \leq T} \| \rho \|_3 \int_0^T \| \nabla \theta \|_2^2 \, dt \right) \\
\leq C \eta^2 \sup_{0 \leq t \leq T} \| \nabla u \|_2^2 + C \eta \int_0^T \| \nabla \theta \|_2^2 \, dt
\]  

(2.42)
and
\[
\bar{p} \sup_{0 \leq t \leq T} ||\sqrt{\rho E}||^2 + \int_0^T (\bar{p} + \bar{p}' ||\rho||^\frac{1}{2} ||\sqrt{\rho} \theta||^2) \left(||\nabla \theta||^2 + ||u||^2 \right) dt
\]
\[
\leq \bar{p} \sup_{0 \leq t \leq T} ||\sqrt{\rho E}||^2 + (\bar{p} + \bar{p}' \sup_{0 \leq t \leq T} (||\rho||^\frac{1}{2} ||\sqrt{\rho} E||^2)) \int_0^T \left(||\nabla \theta||^2 + ||u||^2 \right) dt
\]
\[
\leq C\bar{p} ||\sqrt{\rho_0} E_0||^2 + C\bar{p} \eta \int_0^T ||\nabla d||^2 dt + C\bar{p} \eta^\frac{1}{2} \int_0^T ||\nabla d||^2 ||\nabla u||^2 dt
\]  \hspace{1cm} (2.43)

Substituting (2.42) and (2.43) into (2.41) and using \(\eta_0\) is small enough, one obtains
\[
\sup_{0 \leq t \leq T} ||\nabla u||^2 + \int_0^T ||(\sqrt{\rho} u_t, \nabla G \nabla \omega \sqrt{\rho})||^2 dt
\]
\[
\leq C(||\nabla u_0||^2 + p ||\sqrt{\rho_0} E_0||^2 + ||\nabla d_0||^2) + C\eta \bar{p} \int_0^T ||\nabla \theta||^2 dt
\]
\[
+ C\eta^\frac{1}{2} \int_0^T ||\nabla d||^2 ||\nabla u||^2 dt + C\eta \bar{p} \int_0^T ||\nabla^3 d||^2 dt.
\]  \hspace{1cm} (2.44)

The combination of (2.39), (2.40) and (2.41) yields that
\[
\sup_{0 \leq t \leq T} \left(\bar{p} \left(||\nabla d||^2 + ||\nabla u||^2 + ||\sqrt{\rho} E||^2\right) + \int_0^T \left(||(\sqrt{\rho} u_t, \nabla G \nabla \omega \sqrt{\rho})||^2 dt
\right.
\]
\[
+ \bar{p} \int_0^T \left(||\nabla d||^2 + ||\nabla d||^2 + ||\nabla d||^2 + ||\nabla d||^2\right) dt
\]
\[
\leq C \left(||\nabla d_0||^2 + ||\sqrt{\rho_0} E_0||^2\right) + C\eta \bar{p} \int_0^T ||\nabla \theta||^2 dt
\]
\[
+ C\eta^\frac{1}{2} \int_0^T ||\nabla d||^2 ||\nabla u||^2 dt + C\eta \bar{p} \int_0^T ||\nabla d||^2 dt + C\eta^\frac{1}{2} \int_0^T ||\nabla d||^2 dt
\]
\[
\leq C \left(||\nabla d_0||^2 + ||\sqrt{\rho_0} E_0||^2\right) + ||\nabla u_0||^2 + C\eta \bar{p} \int_0^T ||\nabla \theta||^2 dt
\]
\[
+ C\eta^\frac{1}{2} \int_0^T ||\nabla d||^2 ||\nabla d||^2 dt + C\eta^\frac{1}{2} \eta \int_0^T ||\nabla d||^2 ||\nabla d||^2 dt
\]
\[
+ C\eta \bar{p} \int_0^T ||\nabla d||^2 dt + C\eta^\frac{1}{2} \int_0^T ||\nabla d||^2 dt,
\]
from which, choosing \(\eta_0\) small enough, one gets (2.35) and
\[
\bar{p} \int_0^T ||\nabla \theta||^2 dt \leq C \left(||\nabla d_0||^2 + ||\sqrt{\rho_0} E_0||^2 + ||\nabla u_0||^2\right).
\]  \hspace{1cm} (2.45)
Recalling (2.37) and the assumptions, and using (2.45), we have

\[
\sup_{0 \leq t \leq T} (\|\sqrt{\rho} u\|^2_T + \|\nabla d\|^2_T) + \int_0^T (\|\nabla u\|^2_T + \|d_t\|^2_T + \|\nabla^2 d\|^2_T) dt \\
\leq C(\|\sqrt{\rho_0} u_0\|^2_T + \|\nabla d_0\|^2_T) + C \left( \frac{1}{\rho} \sup_{0 \leq t \leq T} \|\rho\|_3 \left( \|\sqrt{\rho} u_0\|^2_T + \|\nabla d_0\|^2_T + \|\nabla u_0\|^2_T \right) \right) \\
\leq C(\|\sqrt{\rho_0} u_0\|^2_T + \|\nabla d_0\|^2_T) + C \left( \frac{1}{\rho^2} \sup_{0 \leq t \leq T} \|\rho\|_3 \right)
\]

(2.46)

It follows from Lemma 2.5, (2.46), the Young inequality and the assumptions that

\[
\sup_{0 \leq t \leq T} \|\rho\|_3^3 + \int_0^T \int \rho^3 P dx dt \\
\leq C(\|\sqrt{\rho_0} u_0\|^2_T + \|\nabla d_0\|^2_T) + C \left( \frac{1}{\rho} \sup_{0 \leq t \leq T} \|\rho\|_3 \left( \|\sqrt{\rho_0} u_0\|^2_T + \|\nabla d_0\|^2_T + \eta \frac{1}{\rho^2} \sup_{0 \leq t \leq T} \|\rho\|_3 \right) \right) \\
\leq C(\|\sqrt{\rho_0} u_0\|^2_T + \|\nabla d_0\|^2_T) + C \left( \frac{1}{\rho} \sup_{0 \leq t \leq T} \|\rho\|_3 \right) \\
\leq C(\|\rho_0\|_3 + C(\eta^{\frac{1}{2}} + \frac{1}{\rho^4} \sup_{0 \leq t \leq T} \|\rho\|_3) \right) \\
\leq C(\|\rho_0\|_3 + C(\eta^{\frac{1}{2}} + \frac{1}{\rho^4} \sup_{0 \leq t \leq T} \|\rho\|_3) \right)
\]

which implies (2.33) by choosing \(\eta_0\) sufficiently small.

Now, substituting (2.33) into (2.46) yields that

\[
\rho^2 \left( \sup_{0 \leq t \leq T} (\|\sqrt{\rho} u\|^2_T + \|\nabla d\|^2_T) + \int_0^T (\|\nabla u\|^2_T + \|d_t\|^2_T + \|\nabla^2 d\|^2_T) dt \right) \\
\leq C \rho^2 (\|\sqrt{\rho_0} u_0\|^2_T + \|\nabla d_0\|^2_T) + C \eta^{\frac{1}{2}} \sup_{0 \leq t \leq T} \|\rho\|_3 \\
\leq C (\|\rho_0\|_3 + \rho^2 (\|\sqrt{\rho_0} u_0\|^2_T + \|\nabla d_0\|^2_T))
\]

which gives (2.34).

Finally, (2.36) follows immediately from Lemma 2.7, (2.34) and (2.35), and the proof is complete. \(\square\)

**Lemma 2.9** Let \(\eta_0, N_T, \) and \(N_0\) be as in Lemma 2.8. Then, there exists a number \(\varepsilon_0 \in (0, \eta_0)\) such that if

\[
\sup_{0 \leq t \leq T} \|\rho\|_\infty \leq 4\rho, \ N_T \leq \sqrt{\varepsilon_0} \ \text{and} \ N_0 \leq \varepsilon_0, \n\]

then

\[
\sup_{0 \leq t \leq T} \|\rho\|_\infty \leq 2\rho \ \text{and} \ N_T \leq \frac{\sqrt{\varepsilon_0}}{2}, \n\]

where \(\varepsilon_0\) depending only on \(R, c_v, \mu, \lambda, \) and \(\kappa.\)
Proof. If \( \varepsilon_0 \leq \eta_0 \) is sufficiently small, all the conditions in Lemma 2.8 hold. Therefore, we obtain

\[
\mathcal{N}_T \leq C\rho\left(\|\rho_0\|_3 + \overline{p}^2\left(\|\sqrt{\rho_0}u_0\|_2 + \|\nabla d_0\|_2\right)\left(\|\nabla u_0\|_2 + \overline{p}(\|\sqrt{\rho_0}E_0\|_2 + \|\nabla^2 d_0\|_2)\right)\right) \leq C\varepsilon_0 \leq \frac{\sqrt{\varepsilon_0}}{2}.
\]

At the same time,

\[
\sup_{0 \leq t \leq T} \|\rho\|_\infty \leq \overline{p}C\epsilon_0^{\frac{1}{2}} + C\epsilon_0^{\frac{1}{4}} \leq \overline{p}C\epsilon_0^{\frac{1}{2}} + C\epsilon_0^{\frac{1}{4}} \leq 2\overline{p}.
\]

We complete the proof of the lemma.

Thus, based on Lemma 2.1 and Lemma 2.9 together with the standard continuity argument, one can deduce the following proposition:

Proposition 2.1 Assume \( \mathcal{N}_0 \leq \varepsilon_0 \) with \( \varepsilon_0 \) defined as in Lemma 2.9. Then, it holds that

\[
\mathcal{N}_T \leq \frac{\varepsilon_0}{2} \quad \text{and} \quad \sup_{0 \leq t \leq T} \|\rho\|_\infty \leq 2\overline{p}.
\]

Moreover, there is a constant \( C > 0 \) such that the following estimates hold:

\[
\sup_{0 \leq t \leq T} \left(\|\sqrt{\rho}E, \sqrt{\rho}u, \nabla u, d_t, \nabla d, \nabla^2 d\|_2^2 + \|\rho\|_3 + \|\rho\|_\infty + \|\nabla d\|_4 + \|\nabla^2 d\|_4\right) \leq C,
\]

\[
\int_0^T \left(\|\nabla\theta, |u|\nabla u, \sqrt{\rho}u_t, \nabla u, \nabla G, \nabla \omega, d_t, \nabla d_t, \nabla^2 d, \nabla^3 d, |\nabla d||\nabla^2 d|\right)dt \leq C,
\]

\[
+ \int_0^T \rho^3 P dx \ dt \leq C,
\]

where \( C \) depends only on \( R, c_v, \mu, \lambda, \kappa, \overline{p}, \|\rho_0\|_3, \|\sqrt{\rho_0}u_0\|_2, \|\sqrt{\rho_0}E_0\|_2, \|\nabla u_0\|_2, \|\nabla d_0\|_2, \|\nabla^2 d_0\|_2, \text{ and } \|\nabla d_0\|_3.\)

3 Time dependent higher order estimates

Taking advantage of Proposition 2.1 we can obtain the higher order estimates which is sufficient to ensure the existence of global strong solutions. Precisely, the following estimate is proved in this section

\[
\sup_{0 \leq t \leq T} (\|\nabla\theta\|_{H^1} + \|\nabla^2 u, \sqrt{\rho}u, \sqrt{\rho}\theta, \nabla^3 d, \nabla d_t\|_2^2 + \|\rho\|_{H^1 \cap W^{1,4}}) \leq C.
\]

This a priori estimates can be established by modifying the methods of [20, 21, 41] for the compressible Navier-Stokes equations and magnetohydrodynamic equations.

In the rest of this section, we always assume that \( (\rho, u, \theta, d) \) is a strong solution to system (1.1)-(1.5) in \( \mathbb{R}^3 \times (0, T) \), for a positive time \( T \).

Lemma 3.1 Assume \( \mathcal{N}_0 \leq \varepsilon_0 \). It holds that

\[
\sup_{0 \leq t \leq T} (\|\nabla\theta\|_2^2 + \|\sqrt{\rho}u\|_2^2 + \|\nabla d_t\|_2^2) + \int_0^T \|\nabla\theta, \nabla u, d_t, \Delta d_t\|_2^2 dt \leq C_T,
\]

where \( C_T \) depending only on \( R, c_v, \mu, \lambda, \kappa, \Phi_0, \text{ and } T \).
Proof. Applying \( \dot{u}_j (\partial_t + \text{div}(u \cdot \cdot)) \) to (2.1) and integrating over \( \mathbb{R}^3 \), it follows
\[
\frac{1}{2} \frac{d}{dt} \| \sqrt{\rho \dot{u}} \|_2^2 = - \int \dot{u}_j (\partial_j P_t + \text{div}(u\partial_j P)) dx + \mu \int \dot{u}_j (\partial_j \Delta u_j + \text{div}(u \Delta u_j)) dx \\
+ (\mu + \lambda) \int \dot{u}_j (\partial_j \text{div} u_t + \text{div}(u \partial_j \text{div} u)) dx \\
- \int \partial_i (M_{i,j}(d)) \dot{u}_j dx - \int \partial_k (u_k \partial_i (M_{i,j}(d))) \dot{u}_j dx =: \sum_{i=1}^{5} J_i,
\]
where \( M_{i,j}(d) = \partial_i d \cdot \partial_j d - \frac{1}{2} |\nabla d|^2 \delta_{i,j} \). It follows from the Hölder, Young and Sobolev inequalities that
\[
J_1 = - \int \dot{u}_j [\partial_j P_t + \partial_j \text{div}(u P) - \text{div}(\partial_j u P)] dx \\
= \int \text{div} \dot{u} (P_t + u P) dx - \int \nabla \dot{u}_j \cdot \partial_j u P dx \\
= R \int \text{div} \dot{u} \rho \theta dx - R \int \nabla \dot{u}_j \cdot \partial_j u \rho \theta dx \\
\leq \frac{\mu}{8} \| \nabla \dot{u} \|_2^2 + C \| \rho \theta \|_2^2 + C \int \rho^2 \theta^2 \| \nabla u \|_2^2 dx \\
\leq \frac{\mu}{8} \| \nabla \dot{u} \|_2^2 + C (1 + \| \rho \theta \|_2^2 + \| \nabla \theta \|_2^2 + \| \nabla u \|_4^2),
\]
where Proposition 2.1 was used. By virtue of integration by parts, we compute
\[
J_2 = - \mu \int (\partial_i \dot{u}_j (\partial_i u_j))_t + \Delta u_j u \cdot \nabla \dot{u}_j) dx \\
= - \mu \int (|\nabla \dot{u}|^2 - \partial_i \dot{u}_j \partial_k \partial_i u_j - \partial_i \dot{u}_j \partial_k \partial_u k u_j + \Delta u_j u \cdot \nabla \dot{u}_j) dx \\
= - \mu \int (|\nabla \dot{u}|^2 + \partial_i \dot{u}_j \partial_k u_j \text{div} u - \partial_i \dot{u}_j \partial_k \partial_u k u_j - \partial_i u_j \partial_k \partial_k \dot{u}_j) dx \\
\leq - \frac{7\mu}{8} \| \nabla \dot{u} \|_2^2 + C \| \nabla u \|_4^4.
\]
In the same way, one gets
\[
J_3 \leq \frac{7(\mu + \lambda)}{8} \| \text{div} \dot{u} \|_2^2 + C \| \nabla u \|_4^4.
\]
For \( J_4 \) and \( J_5 \), by integration by parts and the Hölder, Young and Sobolev inequalities, one has for \( \eta_1 \in (0, 1] \)
\[
J_4 \leq \int |\nabla d| \| \nabla d_t \| \| \nabla \dot{u} \| dx \leq C \| \nabla d \|_6 \| \nabla d_t \|_3 \| \nabla \dot{u} \|_2 \leq C \| \nabla^2 d \|_2 \| \nabla d_t \|_2 \| \nabla \dot{u} \|_2 \\
\leq \varepsilon \| \nabla \dot{u} \|_2 + \eta_1 \| \nabla^2 d \|_2 + C(\varepsilon, \eta_1) \| \nabla d_t \|_2^2,
\]
where \( \sup_{0 \leq t \leq T} \| \nabla^2 d \|_2 \leq C \) guaranteed by 2.1 was used, and
\[
J_5 \leq \int |u| \| \nabla d \| \| \nabla^2 d \| \| \nabla \dot{u} \| dx \leq C \| u \|_6 \| \nabla d \|_6 \| \nabla^2 d \|_6 \| \nabla \dot{u} \|_2 \leq \varepsilon \| \nabla \dot{u} \|_2 + C(\varepsilon) \| \nabla^3 d \|_2^2,
\]
for some \( \varepsilon \).
Thus, (3.4), (3.6), the Sobolev and Young inequalities yield
\begin{equation}
\frac{d}{dt} \| \sqrt{\rho u} \|_2^2 + \mu \| \nabla u \|_2^2 \\
\leq C \eta_1 \| \nabla^2 d_t \|_2^2 + C(1 + \| \rho \theta \|_2^2 + \| \nabla \theta \|_2^2 + \| \nabla u \|_4^4 + \| \nabla^3 d \|_2^2) + C(\eta_1) | \nabla d_t |_2^2.
\end{equation}
(3.2)

Next, multiplying (1.3) by $\dot{\theta}$ and integrating the resultant over $\mathbb{R}^3$ yield
\begin{equation}
\kappa \frac{d}{dt} \| \nabla \theta \|_2^2 + c_v \| \sqrt{\rho \dot{\theta}} \|_2^2 = -\kappa \int \nabla \theta \cdot \nabla (u \cdot \nabla \theta) dx + \lambda \int | \text{div} u |^2 \dot{\theta} dx + \frac{\mu}{2} \int | \nabla u + (\nabla u)^T |^2 \dot{\theta} dx \\
- R \int \rho \text{div} u \dot{\theta} dx + \int | \Delta d + | \nabla d |^2 |^2 \dot{\theta} dx := \sum_{i=1}^5 K_i.
\end{equation}
(3.3)

It follows from elliptic estimates, Proposition 2.1 and Gagliardo-Nirenberg and Young inequalities that
\begin{align*}
| | \nabla \theta |^2_{H^1} & \leq C + C \| \sqrt{\rho \theta} \|_2^2 + | | \nabla \theta |^2_{L^2} + C \int \rho^2 \theta^2 | \nabla u |^2 dx + C \| | \nabla u |^4_{L^2} + C | | \Delta d + | \nabla d |^2 |^2_{L^2} \\
& \leq C + C \| \sqrt{\rho \theta} \|_2^2 + C \| | \nabla \theta |^2_{L^2} + C \| | \nabla u |^2_{L^\infty} | | \nabla \theta |^2_{L^\infty} + C \| | \nabla u |^4_{L^2} + C | | \Delta d + | \nabla d |^2 |^2_{L^2} \\
& \leq \frac{1}{2} | | \nabla \theta |^2_{H^1} + C(1 + \| \sqrt{\rho \theta} \|_2^2 + | | \theta |^2_{L^2} + \| \nabla u |^4_{L^2} + C | | \Delta d + | \nabla d |^2 |^2_{L^2},
\end{align*}
which implies
\begin{equation}
\| \nabla \theta \|_2^{2}_{H^1} \leq C(1 + \int \sqrt{\rho \theta} + \| \nabla \theta \|_2^2 + \| \nabla u \|_4^4 + \| \Delta d + | \nabla d |^2 |^2_{L^2}).
\end{equation}
(3.4)

Moreover, by (1.4), the Hölder, Young and Sobolev inequalities and Proposition 2.1, one can get by the elliptic estimates that
\begin{align*}
\| \nabla^2 d \|_2^2 & \leq C \| \nabla d_4 \|_2 + \| \nabla u \cdot \nabla d \|_2 + C \| \nabla u \|_2 + C \| | \nabla^2 d |^3 \|_{L^2} + C \| | \nabla d |^2 |^2 \|_{L^2} \\
& \leq C \| \nabla d_4 \|_2 + C \| \nabla u \|_4 \| \nabla d \|_4 + C \| \nabla u \|_6 \| \nabla^2 d \|_3 + C \| \nabla d \|_6 \| \nabla^2 d \|_3 \\
& \leq C \| \nabla d_4 \|_2 + C \| \nabla u \|_4 + C \| \nabla^2 d \|_2 \| \nabla^2 d \|_2 + C \\
& \leq \frac{1}{2} \| \nabla^2 d \|_2 + C(1 + \| \nabla d \|_2 + \| \nabla u \|_4),
\end{align*}
which gives
\begin{equation}
\| \nabla d \|_{H^2} \leq C(1 + \| \nabla d \|_2 + \| \nabla u \|_4).
\end{equation}
(3.5)

According to (3.5), one gets by the Sobolev embedding inequality and Proposition 2.1 that
\begin{equation}
\| \Delta d + | \nabla d |^2 |^2 \|_2 \leq C(\| \Delta d \|_{H^1} + \| \nabla d \|_{L^\infty} \| \nabla d \|_4^2) \leq C \| \nabla d \|_{H^2} \leq C(1 + \| \nabla d \|_2 + \| \nabla u \|_4).
\end{equation}
(3.6)

Thus, (3.4), (3.6), the Sobolev and Young inequalities yield
\begin{equation}
K_1 = -\kappa \int \nabla \theta \cdot (u \cdot \nabla \theta) dx \leq C \int | \nabla \theta |(\| u \| | \nabla^2 \theta | + | \nabla u \| | \nabla \theta |) dx \\
\leq C(\| \nabla \theta \|_3 \| u \|_6 \| \nabla^2 \theta \|_2 + \| \nabla u \|_2 \| \nabla \theta \|_6 \| \nabla \theta \|_3) \\
\leq C \| \nabla u \|_2 \| \nabla \theta \|_2 + \| \nabla \theta \|_2^2 \leq \epsilon \| \nabla \theta \|_2^2 + C(\epsilon) \| \nabla \theta \|_2^2 \\
\leq C \epsilon \| \sqrt{\rho \theta} \|_2^2 + C(\epsilon)(1 + \| \nabla \theta \|_2^2 + \| \nabla u \|_4^4 + | \nabla d \|_2^2).
\end{equation}
By integration by parts, it follows from the H"older, Young and Sobolev inequalities and Proposition 2.1 that

\[ K_2 = \lambda \int (\text{div} u)^2 \theta_t dx + \lambda \int (\text{div} u)^2 u \cdot \nabla \theta dx \]

\[ = \lambda \left( \int (\text{div} u)^2 \theta dx \right)_t - 2\lambda \int \theta \text{div} \text{div}(\dot{u} - u \cdot \nabla u) dx + \lambda \int (\text{div} u)^2 u \cdot \nabla \theta dx \]

\[ = \lambda \left( \int (\text{div} u)^2 \theta dx \right)_t - 2\lambda \int \theta \text{div} \text{div}u \dot{u} dx + 2\lambda \int \theta \text{div} \text{div}u \partial_j u_i dx + \lambda \int u \cdot \nabla (\theta (\text{div} u)^2) dx \]

\[ \leq \lambda \left( \int (\text{div} u)^2 \theta dx \right)_t + C \| \theta \|_6 \| \nabla u \|_2^2 \| \nabla u \|_4^2 (\| \nabla \dot{u} \|_2 + \| \nabla u \|_2^2) \]

\[ \leq \lambda \left( \int (\text{div} u)^2 \theta dx \right)_t + \eta_1 \| \nabla \dot{u} \|_2^2 + C(\eta_1)(1 + \| \nabla u \|_4^2 + \| \nabla \theta \|_2^2). \]

Similarly, one has

\[ K_3 \leq \frac{\mu}{2} \left( \int |\nabla u + (\nabla u)^4|^2 \theta dx \right)_t + \eta_1 \| \nabla \dot{u} \|_2^2 + C(\eta_1)(1 + \| \nabla u \|_4^2 + \| \nabla \theta \|_2^2). \]

Using Proposition 2.1 again, we get

\[ K_4 \leq C \| \sqrt{\rho} \|_{\infty} \| \sqrt{\rho} \|_{\frac{1}{2}} \| \theta \|_{\frac{3}{2}} \| \nabla u \|_4 \leq \| \sqrt{\rho} \|_{\frac{3}{2}} + C(\varepsilon)(1 + \| \nabla \theta \|_2^2 + \| \nabla u \|_4^2). \]

At last, for \( K_5 \), noticing that \(|\Delta d + |\nabla d|^2 d|^2 = |\Delta d|^2 - |\nabla d|^4 \) and \( \Delta d \cdot d = -|\nabla d|^2 \) guaranteed by \(|d| = 1 \), it follows from the H"older, Young and Sobolev inequalities that

\[ K_5 = \int |\Delta d + |\nabla d|^2 d|^2 \theta dx + \int |\Delta d + |\nabla d|^2 d|^2 u \cdot \nabla \theta dx \]

\[ = \left( \int |\Delta d + |\nabla d|^2 d|^2 \theta dx \right)_t - 2 \int (\Delta d \cdot \Delta d_t - 4|\nabla d|^2 \nabla d : \nabla d_t) \theta dx \]

\[ + \int |\Delta d + (\Delta d \cdot d) d|^2 u \cdot \nabla \theta dx \]

\[ = \left( \int |\Delta d + |\nabla d|^2 d|^2 \theta dx \right)_t - 2 \int (\Delta d \cdot \Delta d_t + 4\Delta d \cdot d \nabla d : \nabla d_t) \theta dx \]

\[ + \int |\Delta d + (\Delta d \cdot d) d|^2 u \cdot \nabla \theta dx \]

\[ \leq \left( \int |\Delta d + |\nabla d|^2 d|^2 \theta dx \right)_t + C \| \theta \|_6 \| \Delta d_t \|_2 \| \Delta d \|_3 + C \| \Delta d \|_2 \| \nabla d \|_6 \| \nabla d_t \|_6 \| \theta \|_6 \]

\[ + C \| \nabla d \|_2 \| u \|_6 \| \nabla^2 d \|_6^2 \]

\[ \leq \left( \int |\Delta d + |\nabla d|^2 d|^2 \theta dx \right)_t + C \| \nabla \theta \|_2 \| \Delta d_t \|_2 \| \nabla^2 d \|_2 \| \nabla^2 d \|_2 \cdot \| \nabla^2 d \|_2^2 + C \| \Delta d_t \|_2 \| \nabla d \|_2 \]

\[ + C \| \nabla d \|_2 \| \nabla u \|_2 \| \nabla^3 d \|_2^2 \]

\[ \leq \left( \int |\Delta d + |\nabla d|^2 d|^2 \theta dx \right)_t + \eta_1 \| \Delta d_t \|_2^2 + C(\eta_1)(1 + \| \nabla \theta \|_2^2)(1 + \| \nabla^3 d \|_2^2). \]
Now, substituting the estimates for \(K_i, i = 1, 2, \ldots, 5\) into (3.3) and then choosing \(\varepsilon\) small enough, we deduce that

\[
\frac{d}{dt} \int \Phi dx + c_v \sqrt{\theta} \leq C(1 + ||\nabla \theta||_2^2) (1 + ||\nabla d||_2^2 + ||\nabla \theta||_2^2) + C \eta_1 ||\Delta \tilde{u}||_2^2 + C \eta_1 ||\Delta d_t||_2^2
\]

(3.7)

where

\[
\Phi := \kappa ||\nabla \theta||^2 - 2\theta(\lambda(\text{div} u)^2 + \frac{\mu}{2} ||\nabla u + (\nabla u)^t||^2 + ||\Delta d + ||\nabla d||^2\|d\|^2). \quad (3.8)
\]

On the other hand, applying \(\partial_t\) to (3.11), we have

\[
d_{tt} - \Delta d_t = (-u \cdot \nabla d + |\nabla d|^2) d_t.
\]

It follows from integration by parts, the Sobolev and Young inequalities and Proposition 2.1 that

\[
\frac{d}{dt} \|\nabla d_t\|_2^2 + ||(d_{tt}, \Delta d_t)||_2^2 = \int \left|d_{tt} - \Delta d_t\right|^2 dx
\]

\[
= \int \left|(-u \cdot \nabla d)_t + (|\nabla d|^2) d_t\right|^2 dx
\]

\[
\leq \int \left(|u_t|^2 ||\nabla d||^2 + |u|^2 ||\nabla d_t||^2 + ||\nabla d||^2 |d_t|^2 + ||\nabla d||^2 |d_t|^2\right) dx
\]

\[
\leq \int \left(|\tilde{u}_t|^2 ||\nabla d||^2 + |u|^2 ||\nabla u||^2 ||\nabla d||^2\right) dx + C ||\nabla d||_6^4 ||d_t||_{\mu}^2
\]

\[
+ C ||u||_6^2 + ||\nabla d||_6^2 \|\nabla d_t\|_2 \|\nabla d_t\|_6
\]

\[
\leq \frac{1}{2} \|\Delta d_t\|_2^2 + C ||\tilde{u}||_6^2 \|\nabla d\|_2^2 + C ||u||_6^2 \|\nabla u||_6^2 \|\nabla d\|_6^2 + C \|\nabla d_t\|_2^2
\]

\[
\leq \frac{1}{2} ||\Delta d_t||_2^2 + C ||\tilde{u}||_6^2 + C ||\nabla u||_6^2 + C ||\nabla d_t||_2^2,
\]

which yields

\[
\frac{d}{dt} \|\nabla d_t\|_2^2 + ||(d_{tt}, \Delta d_t)||_2^2 \leq C ||\nabla \tilde{u}||_2^2 + C ||\nabla u||_6^2 + C ||\nabla d_t||_2^2. \quad (3.9)
\]

Thus, multiplying (3.2) and (3.9) by \(\eta_1^\frac{1}{2}\) and \(\eta_1^\frac{1}{2}\), respectively, then adding the result with (3.7) and choosing \(\eta_1\) suitably small, we finally obtain

\[
2 \frac{d}{dt} \int \left(\Phi + \eta_1^\frac{1}{2} ||\nabla d_t||^2 + \eta_1^\frac{1}{2} \rho ||\tilde{u}||^2\right) dx + c_v ||\sqrt{\theta}||_2^2 + \eta_1^\frac{1}{2} \left||d_{tt} + \Delta d_t||_2^2 + \mu \eta_1^\frac{1}{2} ||\nabla \tilde{u}||_2^2\right.
\]

\[
\leq C (1 + ||\nabla \theta||_2^2) (1 + ||\nabla d||_2^2 + ||\nabla \theta||_2^2) + C \left(||\nabla u||_4^4 + ||\nabla d_t||_2^4 + ||\nabla d_t||_2^4 + ||\nabla u||_6^4\right),
\]

(3.10)

where \(\Phi\) is given by (3.8).

Next, one needs to show the estimate of \(||\nabla u||_6\) in order to bound \(||\nabla u||_4\). To this end, decompose \(u = v + w\), where \(v\) satisfies

\[
\mu \Delta v + (\mu + \lambda) \nabla \text{div} v = \nabla P. \quad (3.11)
\]

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According to Lemma 2.3 in [20], there exists a unique \( v(\cdot, t) \in D_o^1 \cap D^{2,2} \cap D^{2,q} \) satisfying (3.11) and the following \( L^p \), \( p \in [2, 6] \) and \( L^\infty \) estimates for \( t \in [0, T] \):

\[
\| \nabla v \|_p \leq C \| \rho \theta \|_p, \tag{3.12}
\]

and

\[
\| \nabla v \|_\infty \leq C \left( 1 + \log(e + \| \nabla (\rho \theta) \|_q) \| \rho \theta \|_\infty + \| \rho \theta \|_2 \right), \quad q \in (3, 6]. \tag{3.13}
\]

While, \( w \) satisfies

\[
\mu \Delta w + (\mu + \lambda) \nabla \text{div} w = \rho \dot{u} + \Delta d \cdot \nabla d. \tag{3.14}
\]

By the elliptic estimates, it holds that

\[
\| \nabla w \|_6 + \| \nabla^2 w \|_2 \leq C \| \rho \dot{u} \|_2 + C \| \Delta d \cdot \nabla d \|_2, \tag{3.15}
\]

and

\[
\| \nabla^2 w \|_6 \leq C \| \rho \dot{u} \|_6 + C \| \Delta d \cdot \nabla d \|_6. \tag{3.16}
\]

Hence, by the Hölder and Sobolev inequalities, it follows from Proposition 2.1 that

\[
\| \nabla u \|_6 \leq C \| \rho \theta \|_6 + C \| \rho \dot{u} \|_2 + C \| \Delta d \cdot \nabla d \|_2
\]

\[
\leq C \| \nabla \theta \|_2 + C \| \rho \dot{u} \|_2 + C \| \nabla d \|_6 \| \nabla^2 d \|_2 \| \nabla^2 d \|_6^\frac{1}{2}
\]

\[
\leq C \| \nabla \theta \|_2 + C \| \rho \dot{u} \|_2 + C \| \nabla^2 d \|_2^\frac{1}{2}. \tag{3.17}
\]

Thus, by the Young and Sobolev inequalities, one has

\[
\| \nabla u \|_4^4 \leq C \| \nabla u \|_2 \| \nabla u \|_6^3 \leq C \| \nabla u \|_4^4 + C \| \nabla u \|_6^3 \leq C (1 + \| \nabla \theta \|_2^\frac{4}{3} + \| \sqrt{\rho} \dot{u} \|_2^4 + \| \nabla^3 d \|_2^\frac{3}{4}). \tag{3.18}
\]

Substituting (3.18) into (3.10), we obtain

\[
\frac{2}{3} \frac{d}{dt} \int (\Phi + \eta_1^\frac{1}{2} \| \nabla dt \|_2^2 + \eta_1^\frac{1}{2} \| \rho \dot{u} \|_2^2) dx + c_v \| \sqrt{\rho} \dot{\theta} \|_2^\frac{4}{3} + \eta_1^\frac{1}{2} \| (dt, \Delta dt) \|_2^2 + \mu \eta_1^\frac{1}{2} \| \nabla \dot{u} \|_2^2
\]

\[
\leq C (1 + \| \nabla \theta \|_2^\frac{2}{3} + \| \nabla^3 d \|_2^3 + \| \nabla \theta \|_2^2) + C (\| \sqrt{\rho} \dot{u} \|_2^4 + \| \nabla dt \|_2^4 + \| \nabla d \|_2^4 + \| \nabla \dot{d} \|_2^2), \tag{3.19}
\]

where \( \Phi \) is given by (3.8).

Now, we want to show the lower bound of \( \Phi \). By the elliptic estimates, it follows from (1.4), the Hölder, Sobolev and Young inequalities that

\[
\| \nabla^3 d \|_2 \leq C (\| \nabla d \|_2 + \| \nabla u \cdot \nabla d \|_2 + \| u \cdot \nabla^2 d \|_2 + \| \nabla d \| \nabla^2 d \|_2 + \| \nabla d \| \nabla^2 d \|_2
\]

\[
\leq C (\| \nabla d \|_2 + \| \nabla u \|_4 \| \nabla d \|_4 + \| u \|_6 \| \nabla^2 d \|_2 \| \nabla^3 d \|_2 + \| \nabla d \|_6 \| \nabla^2 d \|_2 \| \nabla^3 d \|_2^\frac{3}{4} + \| \nabla d \|_6 \| \nabla^2 d \|_2 \| \nabla^3 d \|_2^\frac{1}{4})
\]

\[
\leq \frac{1}{4} \| \nabla^3 d \|_2 + C (1 + \| \nabla d \|_2 + \| \nabla u \|_4),
\]

where we have used Proposition 2.1. By (3.17) and the Cauchy inequality, it holds that

\[
\| \nabla u \|_6 \leq C (1 + \| \nabla \theta \|_2 + \| \rho \dot{u} \|_2) + \frac{1}{4} \| \nabla^3 d \|_2.
\]

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Combining the above two inequalities, together with (3.18), lead to
\[ \|\nabla u\|_6 + \|\nabla^3 d\|_2 \leq C (1 + \|\nabla \theta\|_2 + \|\rho \dot{u}\|_2 + \|\nabla d\|_2). \] (3.20)

Thus, from the definition of $\Phi$, (3.20), the Young and Sobolev inequalities, one deduces by Proposition 2.1 that
\[
2 \int \left( \Phi + \eta_1^2 \rho |\dot{u}|^2 + \eta_1^2 |\nabla d|^2 \right) dx \\
\geq 2 \kappa \|\nabla \theta\|_2^2 - C \|\nabla u\|_6^3 + \|\nabla u\|_6^2 - C \|\nabla\|_6 \|\Delta d\|_2^3 + \|\Delta d\|_2^2 - C \|\nabla\|_6 \|\nabla d\|_{\frac{4}{3}}^4 \\
+ 2 \int \left( \eta_1^2 \rho |\dot{u}|^2 + \eta_1^2 |\nabla d|^2 \right) dx \\
\geq \frac{3}{2} \kappa \|\nabla \theta\|_2^2 - C (1 + \|\nabla u\|_6 + \|\nabla^3 d\|_2) + 2 \int \left( \eta_1^2 \rho |\dot{u}|^2 + \eta_1^2 |\nabla d|^2 \right) dx \\
\geq \frac{3}{2} \kappa \|\nabla \theta\|_2^2 - C (\|\nabla \nabla\|_2 + \|\nabla \nabla\|_2 + \|\nabla d\|_2) + 2 \int \left( \eta_1^2 \rho |\dot{u}|^2 + \eta_1^2 |\nabla d|^2 \right) dx \\
\geq \kappa \|\nabla \theta\|_2^2 - C (\eta_1) + \int \left( \eta_1^2 \rho |\dot{u}|^2 + \eta_1^2 |\nabla d|^2 \right) dx.
\] (3.21)

Finally, integrating (3.19) over $[0, t]$, and then using (3.21) and Grönwall’s inequality, the conclusion follows.

As a straightforward consequence of Lemma 3.1 and using (3.20), we have the following corollary:

**Corollary 3.1** Assume $N_0 \leq \varepsilon_0$. It holds that
\[
\sup_{0 \leq t \leq T} (\|\nabla u\|_6 + \|\nabla^3 d\|_2 + \|\nabla d\|_2) \leq C_T,
\]
where $C_T$ depending only on $R, c_v, \mu, \lambda, T$ and the initial data.

**Lemma 3.2** Assume $N_0 \leq \varepsilon_0$. It holds that
\[
\sup_{0 \leq t \leq T} (\|\sqrt{\rho} \theta\|_2^2 + \|\nabla^2 \theta\|_2^2) + \int_0^T \|\nabla \theta\|_2^2 dt \leq C_T,
\]
where $C_T$ depending only on $R, c_v, \mu, \lambda, \kappa, \Phi_0$, and $T$.

**Proof.** Recalling (3.4), by the Sobolev inequality, Proposition 2.1 and Lemma 3.1 in order to get this result, it remains to bound the term $\sup_{0 \leq t \leq T} \|\sqrt{\rho} \theta\|_2^2$. Applying the operator $\partial_t + \text{div}(u \cdot \nabla \theta)$ to (1.3), by tedious computations developed in Appendix A, it follows
\[
c_v \rho (\dot{\theta} + v \cdot \nabla \theta) \\
= \kappa \Delta \dot{\theta} + \kappa (\text{div} u \Delta \theta - \partial_i (\partial_i u \cdot \nabla \theta) - \partial_i (\nabla \nabla \theta)) + \left( \lambda (\text{div} u)^2 + \frac{\mu}{2} \nabla u + (\nabla u)^2 \right) \text{div} u \\
+ R \rho \partial_k u \cdot \partial_k d - R \rho \partial_k d \text{div} u - R \rho \partial_k d \text{div} u + 2 \lambda (\text{div} v \partial_i u - \partial_i u \partial_k u) \text{div} u \\
+ \mu (\partial_i u_j + \partial_j u_i) (\partial_i \dot{u} + \partial_j \dot{u} - \partial_i u_j - \partial_j u_i) \\
+ \partial_t (|\Delta d + |\nabla d|^2 d|^2) + \text{div} (|\Delta d + |\nabla d|^2 d|^2 u).
\] (3.22)
Recalling that $|\Delta d + |\nabla d|^2|^2 = |\Delta d|^2 - |\nabla d|^4$ and $\Delta d + |\nabla d|^2 d = \Delta d - (d \cdot \Delta d)\Delta d$ guaranteed by $|d| = 1$, one has $\partial_t |\Delta d + |\nabla d|^2|^2 = 2\Delta d \cdot \Delta d_t - 4|\nabla d|^2 \nabla d : \nabla d$. Thanks to this, multiplying (3.22) by $\dot{\theta}$, using integration by parts, Proposition 2.1, Lemma 3.1 and Corollary 3.1, we have
\[
\frac{c_v}{2} \int \frac{d}{dt} \sqrt{\rho} \dot{\theta}^2 + \kappa \|\nabla \dot{\theta}\|^2 \leq C \int |\nabla u|(|\nabla^2 \theta||\dot{\theta}| + |\nabla \theta||\nabla \dot{\theta}|)dx + \int |\nabla u|^2 |\dot{\theta}|(|\nabla u| + \theta)dx + C \int \rho |\dot{\theta}|^2 |\nabla u|dx + C \int \rho |\dot{\theta}| |\nabla \dot{u}|dx + C \int |\nabla u||\nabla \dot{u}|dx
\]
\[
+ C \int (|\Delta d||\Delta d_t||\dot{\theta}| + |\nabla d|^3||\nabla d_t||\dot{\theta}| + |\Delta d - (d \cdot \Delta d)|^2|u||\nabla \dot{\theta}|)dx
\leq C \|\nabla u\|_3 (|\nabla^2 \theta|_2 \|\dot{\theta}\|_6 + |\nabla \theta|_6 \|\nabla \dot{\theta}\|_2) + C \|\nabla u\|_3 \|\dot{\theta}\|_6 (|\nabla u|_6 + \|\theta\|_6)
\]
\[
+ C \|\nabla u\|_3 \|\rho |\dot{\theta}|_2 \|\dot{\theta}\|_6 + C \sqrt{\rho \theta}_{1/2} \|\theta\|_6 \|\nabla u\|_2 \|\dot{\theta}\|_6 + C \|\nabla u\|_3 \|\nabla \dot{u}\|_2 \|\dot{\theta}\|_6
\]
\[
+ C \|\Delta d\|_3 \|\Delta d_t\|_2 \|\dot{\theta}\|_6 + C \|\nabla d\|_3 \|\nabla d_t\|_6 \|\dot{\theta}\|_6 + C \|\Delta d\|_2^2 \|u\|_6 \|\nabla \dot{\theta}\|_2
\leq \frac{\kappa}{2} \|\nabla \dot{\theta}\|^2 + C (1 + |\nabla^2 \theta|^2 + \|\sqrt{\rho \theta}\|_2^2 + \|\nabla \dot{u}\|^2 + \|\Delta d_t\|^2).
\]

Thanks to (1.8), Lemma 3.1 and Corollary 3.1, applying Grönwall’s inequality, we arrive at
\[
\sup_{0 \leq t \leq T} \|\sqrt{\rho \theta}\|^2_2 + \int_0^T \|\nabla \dot{\theta}\|^2_2 dt \leq C_T,
\]
which completes the proof. \qed

**Lemma 3.3** Assume $N_0 \leq \varepsilon_0$. It holds that
\[
\sup_{0 \leq t \leq T} (\|\rho\|_{H^{1/2}W^{1,\infty}} + \|\nabla^2 u\|_2) \leq C_T,
\]
where $C_T$ depending only on $R, c_v, \mu, \lambda, \kappa, \Phi_0$, and $T$.

**Proof.** By (3.13), Lemma 3.1 and Lemma 3.2 it follows
\[
\|\nabla v\|_\infty \leq C_T \log(e + \|\nabla \rho\|_q), \quad q \in (3, 6).
\]

Meanwhile, it follows from (3.15), Lemma 3.1 and Corollary 3.1 that
\[
\|\nabla w\|_6 + \|\nabla^2 w\|_2 \leq C \|\sqrt{\rho} u\|_2 + C \|\nabla^2 d\|_3^3 \|\nabla^3 d\|_2^1 \leq C_T.
\]

Due to (3.16), Proposition 2.1, and Corollary 3.1 one deduces by the Sobolev inequality that
\[
\|\nabla^2 w\|_6 \leq C (\|\nabla \dot{u}\|_2 + \|\nabla^3 d \cdot \nabla d\|_2 + \|\nabla^2 d\|_2^2) \leq C (\|\nabla \dot{u}\|_2 + \|\nabla d\|_{H^2}^2) \leq C (\|\nabla \dot{u}\|_2 + C_T).
\]

Hence, by the Sobolev inequality, (3.24) and (3.25) give us
\[
\|\nabla w\|_\infty \leq C \|\nabla \dot{u}\|_2 + C_T,
\]
which combined with (3.23) implies
\[ \| \nabla u \|_\infty \leq C_T \log(e + \| \nabla \rho \|_q) + C \| \nabla \dot{u} \|_2, \quad q \in (3,6]. \quad (3.26) \]

Applying the elliptic estimates to (1.2), one has for \( 2 \leq p \leq q \)
\[ \| \nabla^2 u \|_p \leq C(\| \rho \dot{u} \|_p + \| \Delta d \cdot \nabla d \|_p + \| \nabla P \|_p) \]
\[ \leq C(\| \rho \dot{u} \|_p + \| \Delta d \|_p \| \nabla d \|_\infty + \| \rho \nabla \theta \|_p + \| \nabla \rho \|_p) \]
\[ \leq C(1 + \| \rho \dot{u} \|_p + \| \nabla \rho \|_p) \]
\[ \leq C_T (1 + \| \nabla \dot{u} \|_2 + \| \nabla \rho \|_p), \quad (3.27) \]

where Proposition 2.1 and Lemma 3.2 were used. On the other hand, some straightforward calculations show that, for \( 2 \leq p \leq q \)
\[ \frac{d}{dt} \| \nabla \rho \|_p \leq C(1 + \| \nabla u \|_\infty) \| \nabla \rho \|_p + C \| \nabla^2 u \|_p, \quad (3.28) \]
which together with (3.26) and (3.27) yields
\[ \frac{d}{dt} \| \nabla \rho \|_p \leq C_T (1 + \log(e + \| \nabla \rho \|_q) + \| \nabla \dot{u} \|_2) \| \nabla \rho \|_p + C_T (1 + \| \nabla \dot{u} \|_2 + \| \nabla \rho \|_p). \]

Set \( f(t) = e + \| \nabla \rho \|_q \) and \( g(t) = 1 + \| \nabla \dot{u} \|_2 \),
then
\[ \frac{d}{dt} f(t) \leq C_T g(t) f(t) \log f(t). \]

By solving the above ordinary differential inequality and using Lemma 3.1 one gets
\[ \sup_{0 \leq t \leq T} \| \nabla \rho \|_q \leq C_T. \quad (3.29) \]

Combing (3.29) with (3.26) yields
\[ \int_0^T \| \nabla u \|_\infty^2 dt \leq C_T. \quad (3.30) \]

Choosing \( p = 2 \) in (3.28), it follows from Lemma 3.1, (3.27), (3.30) and Grönwall’s inequality that
\[ \sup_{0 \leq t \leq T} \| \nabla \rho \|_2 \leq C_T. \]

This together with (3.27) and Lemma 3.1 implies
\[ \sup_{0 \leq t \leq T} \| \nabla^2 u \|_2 \leq C(\| \rho \dot{u} \|_2 + \| \nabla \rho \|_2) + C_T \leq C_T. \]

Thus this lemma is proved. □

As a direct consequence of Lemma 3.1, Lemma 3.2, Lemma 3.3 and Corollary 3.1 we have the following Corollary.
Corollary 3.2 Assume that $N_0 \leq \varepsilon_0$. Then, it holds that
\[
\sup_{0 \leq t \leq T} \left( \|\nabla \theta\|_{H^1}^2 + \|(\nabla^2 u, \sqrt{\rho} \dot{\theta}, \nabla^3 d, \nabla d_t)\|_{H^{1,\eta}}^2 + \|\rho\|_{H^1 \cap W^{1,\eta}} \right) \leq C_T,
\]
\[
\int_0^T \|(\nabla \dot{u}, d_t, \nabla^2 d_t, \nabla \dot{\theta})\|_{L^2}^2 dt \leq C_T,
\]
for a positive constant $C_T$ depending only on $R, c_v, \mu, \kappa, \Phi_0$, and $T$.

4 Proof of Theorem 1.1

Proof of Theorem 1.1 Let $(\rho, u, \theta, d)$ be the unique local solution guaranteed by Lemma 2.1. By applying the local well-posedness, i.e. Lemma 2.1 inductively, one can extend the $(\rho, u, \theta, d)$ uniquely to the maximal time $T_{\text{max}}$ of existence. We claim that $T_{\text{max}} = \infty$ and thus the conclusion holds. Assume by contradiction that $T_{\text{max}} < \infty$. Let $\varepsilon_0$ be as in Lemma 2.1 and assume that $N_0 \leq \varepsilon_0$. Then, it follows from Proposition 2.1 and Corollary 3.2 that
\[
\sup_{0 \leq t \leq T} \left( \|\rho\|_{W^{1,\eta}\cap H^1(t)} + \|(u, \theta)\|_{D^0_\rho \cap D^2(t)} + \|\nabla d\|_{H^2(t)} + \|(\sqrt{\rho} \dot{\theta}, \sqrt{\rho} u, \sqrt{\rho} \dot{\theta})\|_{L^2(t)} \right) \leq C_0,
\]
for any $T \in (0, T_{\text{max}})$, where $C_0$ is a positive constant independent of $T \in (0, T_{\text{max}})$. Take $\delta > 0$ and choose $T_{\text{max}} - \delta$ as the initial time. Thanks to the above estimate, one can check that all the conditions in Lemma 2.1 hold and, in particular, the compatibility conditions hold. Then, by Lemma 2.1 there is a positive time $T_0$, depending only on $R, c_v, \mu, \kappa$, and $C_0$ (thus independent of $\delta$), such that the solution $(\rho, u, \theta, d)$ can be extended to the time $T_{\text{max}} - \delta + T_0$. By choosing $\delta$ sufficiently small, it holds that $T_{\text{max}} - \delta + T_0 > T_{\text{max}}$, which contradicts to the definition of $T_{\text{max}}$. This contradiction implies that $T_{\text{max}} = \infty$. Therefore, we get a global strong solution and the conclusion follows.

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Appendix A

Applying the operator $\partial_t + \text{div}(u \cdot)$ to (1.3), from the definition of material derivative, one has
\[
c_v \left[ \partial_t (\rho \dot{\theta}) + \text{div}(u \rho \dot{\theta}) \right] + \partial_t (P \text{div} u) + \text{div}(u P \text{div} u) - \kappa [\partial_t \Delta \theta + \text{div}(u \Delta \theta)]
= \frac{\mu}{2} \left[ \partial_t (|\partial_j u_i + \partial_i u_j|^2) + \text{div}(u |\partial_j u_i + \partial_i u_j|^2) \right] + \lambda [\partial_t (\text{div} u)^2 + \text{div}(u (\text{div} u)^2)]
+ \partial_t (|\Delta d + |\nabla d|^2 d|^2) + \text{div}(|\Delta d + |\nabla d|^2 d|^2 u).
\]
By (1.1) and some straightforward calculations, it follows that

\[ c_v [\partial_t (\rho \dot{\theta}) + \text{div}(u \rho \dot{\theta})] = c_v \rho (\dot{\theta}_t + u \cdot \nabla \dot{\theta}), \tag{4.2} \]

\[ \partial_t (P \text{div} u) + \text{div}(u P \text{div} u) = R [\partial_t \rho \dot{\theta} \text{div} u + \rho \dot{\theta} \partial_t \text{div} u + \rho \dot{\theta} \text{div} u_t + \rho \theta (\text{div} u)^2 + u \cdot \nabla \rho \dot{\theta} \text{div} u + \rho u \cdot \nabla \dot{\theta} \text{div} u + \rho \theta u \cdot \nabla (\text{div} u)] \]

\[ = R [\rho \dot{\theta} \text{div} u + \rho \dot{\theta} \text{div} u_t + \rho \theta \text{div} (u \cdot \nabla u) + \rho \theta u \cdot \nabla (\text{div} u)] \tag{4.3} \]

\[ = R (\rho \dot{\theta} \text{div} u + \rho \theta \text{div} u) - \rho \theta \partial_k u_l \partial_l u_k, \]

\[- \kappa [\partial_t \Delta \theta + \text{div}(u \Delta \theta)] = - \kappa [\Delta \dot{\theta} - \Delta (u \cdot \nabla \theta) + \text{div}(u \Delta \theta)] \]

\[ = - \kappa [\Delta \dot{\theta} - \Delta (u \cdot \nabla \theta) + \text{div} \Delta \theta + u \cdot \nabla (\Delta \theta)] \tag{4.4} \]

\[ = - \kappa \Delta \dot{\theta} - \kappa [\text{div} \Delta \theta - \partial_i (\partial_i u \cdot \nabla \theta) - \partial_i u \cdot \nabla \partial_i \theta]. \]

Similarly, for the terms on the right hand side of (4.1), we also have

\[ \frac{\mu}{2} [\partial_t (|\partial_j u_i + \partial_i u_j|^2) + \text{div}(u (\partial_j u_i + \partial_i u_j)^2)] \]

\[ = \mu (\partial_j u_i + \partial_i u_j)(\partial_t \partial_j u_i + \partial_t \partial_i u_j) + \frac{\mu}{2} [\nabla u + (\nabla u)^t]^2 \text{div} u + \mu u \cdot (\partial_j u_i + \partial_i u_j) \nabla (\partial_j u_i + \partial_i u_j) \]

\[ = \frac{\mu}{2} [\nabla u + (\nabla u)^t]^2 \text{div} u + \mu (\partial_j u_j + \partial_j u_i)(\partial_t \dot{u}_j + \partial_j \dot{u}_t + \partial_t u_k \partial_k u_j - \partial_j u_k \partial_k u_i), \tag{4.5} \]

and

\[ \lambda [\partial_t (\text{div} u)^2 + \text{div}(u (\text{div} u)^2)] \]

\[ = 2 \lambda (\partial_t \text{div} u) \text{div} u + 2 \lambda (u \cdot \nabla \text{div} u) \text{div} u + \lambda (\text{div} u)^3 \]

\[ = 2 \lambda \text{div} \dot{u} \text{div} u - 2 \lambda \text{div}(u \cdot \nabla u) \text{div} u + 2 \lambda (u \cdot \nabla \text{div} u) \text{div} u + \lambda (\text{div} u)^3 \tag{4.6} \]

\[ = \lambda (\text{div} u)^3 + 2 \lambda (\text{div} u - \partial_k u_l \partial_t u_k) \text{div} u. \]

Thus, (3.22) follows by comibing (4.1)-(4.6) together.

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