Abstract—Operator splitting techniques have recently gained popularity in convex optimization problems arising in various control fields. Being fixed-point iterations of nonexpansive (NE) operators, such methods suffer many well known downsides, which include high sensitivity to ill conditioning and parameter selection, and consequent low accuracy and robustness. As universal solution we propose SuperMann, a Newton-type algorithm for finding fixed points of NE operators. It generalizes the classical Krasnosielski–Mann scheme, enjoys its favorable global convergence properties and requires exactly the same oracle. It is based on a novel separating hyperplane projection tailored for nonexpansive mappings which makes it possible to include steps along any direction. In particular, when the directions satisfy a Dennis–Moré condition we show that SuperMann converges superlinearly under mild assumptions, which, surprisingly, do not entail nonsingularity of the Jacobian at the solution but merely metric subregularity. As a result, SuperMann enhances and robustifies all operator splitting schemes for structured convex optimization, overcoming their well known sensitivity to ill-conditioning.

Index Terms—Convex functions, optimal control, optimization methods.

I. INTRODUCTION

Operator splitting techniques (also known as proximal algorithms), introduced in the 50’s for solving partial differential equations (PDEs) and optimal control problems, have been successfully used to reduce complex problems into a series of simpler subproblems. The most well known operator splitting methods are the alternating direction method of multipliers (ADMM), forward–backward splitting (FBS) also known as proximal–gradient method in composite convex minimization, Douglas–Rachford splitting (DRS), and the alternating minimization method (AMM) [1]. Operator splitting techniques pose several advantages over traditional optimization methods such as sequential quadratic programming and interior point methods.

1) They can easily handle nonsmooth terms and abstract linear operators.
2) Each iteration requires only simple arithmetic operations.
3) The algorithms scale gracefully as the dimension of the problem increases.
4) They naturally lead to parallel and distributed implementation.

Therefore, operator splitting methods cope well with limited amount of hardware resources making them particularly attractive for (embedded) control [2], signal processing [3], and distributed optimization [4], [5].

The key idea behind these techniques when applied to convex optimization is to reformulate the optimality conditions of the problem at hand into a problem of finding a fixed point of a nonexpansive (NE) operator and then apply relaxed fixed-point iterations. Although sometimes a fast convergence rate can be observed, the norm of the fixed-point residual decreases, at best, with $Q$-linear rate, and due to an inherent sensitivity to ill conditioning oftentimes the $Q$-factor is close to one. Moreover, all operator splitting methods are basically “open-loop”, since the tuning parameters, such as stepsizes and preconditioning, must be set before their execution. In fact, such methods are very sensitive to the choice of parameters and sometimes there is not even a concrete way of selecting them, as it is the case of ADMM. All these are serious obstacles when it comes to using such types of algorithms for real-time applications, such as embedded MPC, or to reliably solve cone programs.

As an attempt to solve the issue, people have considered the employment of variable metrics to reshape the geometry of the problem and enhance convergence rate [6]. However, unless such metrics have a very specific structure, even for simple problems the cost of operating in the new geometry outweighs the benefits.

Another interesting approach that is gaining more and more popularity tries to exploit possible sparsity patterns by means of chordal decomposition techniques [7]. These methods can improve scalability and reduce memory usage, but unless the problem comes with an inherent sparse structure they yield no tangible benefit.

Alternatively, the task of searching fixed points of an operator $T$ can be translated to that of finding zeros of the corresponding residual $R = I - T$. Many methods with fast asymptotic convergence rates such as Newton-type exist that can be
employed for efficiently solving nonlinear equations, see, e.g., [8, Section VII] and [9]. However, such methods converge only when close enough to the solution, and in order to globalize the convergence there comes the need of a merit function to perform a linesearch along candidate directions of descent. The typical choice of the square residual $\|Rx\|^2$ unfortunately is of no use, as in meaningful applications $R$ is nonsmooth.

A. Proposed Methodology

In response to these issues, in this paper, we propose a universal scheme that globalizes Newton-type methods for finding fixed points of any NE operator on real Hilbert spaces. Admittedly with an intended pun, since it exhibits superlinear convergence rates and generalizes the Krasnosel’ski–Mann (KM) iterations we name our algorithm SuperMann. The method is based on a novel hyperplane projection step tailored for nonexpansive mappings.

Furthermore, we consider a modified Broyden’s scheme which was first introduced in [10] and show how it fits into our framework enabling superlinear asymptotic convergence rates. One of the most appealing properties of SuperMann is that, thanks to its quasi-Fejérian behavior, achieving superlinear convergence does not necessitate nonsingularity of the Jacobian at the solution, which is the usual requirement of quasi-Newton schemes, but merely metric subregularity. This property considerably widens the range of problems which can be solved efficiently, in that, for instance, the solutions need not be isolated for superlinear convergence to take place.

B. Contributions

Our contributions can be summarized as follows.

1) In Section IV, we design a universal algorithmic framework (see Algorithm 1) for finding fixed points of NE operators, which generalizes the classical KM scheme and possesses its same global and local convergence properties.

2) In Section V, we introduce a novel separating hyperplane projection tailored for nonexpansive mappings; based on this, in Definition V.3 we then propose a generalized KM (GKM) iteration.

3) We define a linesearch based on the novel projection, suited for any NE operator and update direction (see Theorem V.4).

4) In Section VI, we combine these ideas and derive the SuperMann scheme (see Algorithm 2), an algorithm that performs the following tasks.

a) It globalizes the convergence of Newton-type methods for finding fixed points of NE operators (see Theorem VI.1).

b) It reduces to the local method $x_{k+1} = x_k + d_k$ when the directions $d_k$ are superlinear, as it is the case for a modified Broyden’s scheme (see Theorems VI.4 and VI.8).

c) It has superlinear convergence guarantees even without the usual requirement of nonsingularity of the Jacobian at the limit point, but simply under metric subregularity; in particular, the solution need not be unique!

C. Paper Organization

The paper is organized as follows. Section II serves as an informal introduction to highlight the known limitations of fixed-point iterations and to motivate our interest in Newton-type methods with some toy examples. The formal presentation begins in Section III with the introduction of some basic notation and known facts. In Section IV, we define the problem at hand and propose a general abstract algorithmic framework for solving it. In Section V, we provide a generalization of the classical KM iterations that is key for the global convergence and performance of SuperMann, an algorithm which is presented and analyzed in Section VI. Finally, in Section VII, we show how the theoretical findings are backed up by promising numerical simulations, where SuperMann dramatically improves classical splitting schemes. For the sake of readability some of the proofs are deferred to the Appendix.

II. Motivating Examples

Given an NE operator $T: \mathbb{R}^n \to \mathbb{R}^n$, consider the problem of finding a fixed point, i.e., a point $x_\ast \in \mathbb{R}^n$ such that $x_\ast = Tx_\ast$. The independent works of Krasnosel’ski and Mann [11], [12] provided a very elegant solution, which is simply based on recursive iterations $x^{n+1} = (1 - \alpha)x^n + \alpha Tx^n$ with $\alpha \in (0, \bar{\alpha})$ for some $\bar{\alpha} \geq 1$. The method, known as the KM scheme, has been studied intensively ever since, also because it generalizes a plethora of optimization algorithms. It is well known that the scheme is globally convergent with square-summable and monototonically decreasing residual $R = \text{id} - T$ (in norm), and also locally $Q$-linearly convergent if $R$ is metrically subregular at the limit point $x_\ast$. Metric subregularity basically amounts to requiring the distance from the set of solutions to be upper bounded by a multiple of the norm of $R$ for all points sufficiently close to $x_\ast$; it is quite mild a requirement—for instance, it does not entail $x_\ast$ to be an isolated solution—and as such linear convergence is quite frequent in practice. However, the major drawback of the KM scheme is its high sensitivity to ill conditioning of the problem, and cases where convergence is prohibitively slow in practice despite the theoretical (sub)linear rate are also abundant. Illus- trative examples can be easily constructed for the problem of finding a point in the intersection of two closed convex sets $C_1$ and $C_2$ with $C_1 \cap C_2 \neq \emptyset$. The problem can be solved by means of fixed-point iterations of the (nonexpansive) alternating projections operator $T = \Pi_{C_2} \circ \Pi_{C_1}$.

In Fig. 1(a), we consider the case of two polyhedral cones, namely $C_1 = \{x \in \mathbb{R}^2 \mid 0.1x_1 \leq x_2 \leq 0.2x_1\}$ and $C_2 = \{x \in \mathbb{R}^2 \mid 0.3x_1 \leq x_2 \leq 0.35x_1\}$. Alternating projections is then linearly convergent (to the unique intersection point 0) due to the fact that $R = \text{id} - T$ is piecewise affine and, hence, globally metrically subregular. However, the convergence is extremely slow due to the pathological small angle between the two cones, as it is apparent in Fig. 1(a).

As an attempt to overcome this frequent phenomenon, Giselson et al. [13] proposes a foretracking linesearch heuristic, which is particularly effective when subsequent fixed-point iterations proceed along almost parallel directions. Iterationwise, in such instances the linesearch does yield a considerable improvement upon the plain KM scheme; however, each fore- track prescribes extra evaluations of $T$ and unless $T$ has a specific structure the computational overhead might outweigh the advantages. Moreover, its asymptotic convergence rates do not improve upon the plain KM scheme. Fig. 1(b)
illustrates this fact relative to $C_1 = \{ x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1 \}$ and $C_2 = \{ x \in \mathbb{R}^2 \mid x_1 = 1 \}$. Despite a good performance on early iterations, the linesearch cannot improve the asymptotic sublinear rate of the plain KM scheme due to the fact that the residual $R$ is not metrically subregular at $x$. In particular, it is evident that medium-to-high accuracy cannot be achieved in a reasonable number of iterations with either methods.

In response to this limitation there comes the need to include some “first-order-like information”. Specifically, the problem of finding a fixed point of $T$ can be rephrased in terms of solving the system of nonlinear (monotone) equations $R x = 0$, which could possibly be solved efficiently with Newton-type methods.

In the toy simulations of this section, the purple lines correspond to the semismooth Newton iterations

$$x^+ = x - G^{-1} R x$$

for some $G \in \partial R x$ where $\partial R$ is the Clarke generalized Jacobian of $R$ [8, Definition 7.1.1]. Interestingly, in the proposed simulations, this method exhibits fast convergence even when the limit point is a nonsolved solution, as shown in the case of the second-order cone $C_1 = \{ x \in \mathbb{R}^3 \mid x_3 \geq 0.1 \sqrt{x_1^2 + x_2^2} \}$ and the tangent plane $C_2 = \{ x \in \mathbb{R}^3 \mid x_3 = 0.1 x_2 \}$ considered in Fig. 1(c).

However, computing the generalized Jacobian might be too demanding and require extra information not available in close form. For this reason, we focus on quasi-Newton methods

$$x^+ = x - H R x$$

where the linear operator $H$ is progressively updated with only evaluations of $R$ and direct linear algebra in such a way that the vector $H R x$ is asymptotically a good approximation of a Newton direction $G^{-1} R x$. The yellow lines in the simulations of this section correspond to $H$ being selected with Broyden’s quasi-Newton method.

The crucial issue is convergence itself. Though in these trivial simulations it is not the case, it is well known that Newton-type methods in general converge only when close to a solution, and may even diverge otherwise. In fact, globalizing the convergence of Newton-type methods is a key challenge in optimization, as the dedicated recent book [9] confirms.

In this paper, we provide the SuperMann scheme, a globalization strategy for Newton-type methods (or any local scheme in general) that applies to any (nonsmooth) monotone equation deriving from fixed-point iterations of NE operators. Our method covers almost all splitting schemes in convex optimization, such as FBS also known as proximal-gradient method, DRS, and the ADMM, to name a few. We also provide sufficient conditions at the limit point under which the method reduces to the local scheme and converges superlinearly.

III. NOTATION AND KNOWN RESULTS

With $\text{bdry} A$, we denote the boundary of the set $A$, and given a sequence $(x_k)_{k \in \mathbb{N}}$ we write $(x_k)_{k \in \mathbb{N}} \subset A$ to indicate that $x_k \in A$ for all $k \in \mathbb{N}$. For $p > 0$, we let

$$\ell^p := \left\{ (x_k)_{k \in \mathbb{N}} \subset \mathbb{R} \mid \sum_{k \in \mathbb{N}} |x_k|^p < \infty \right\}$$

denote the set of real-valued sequences with summable $p$th power, and with $\ell^+_{p}$ the subset of the positive-valued ones.

The positive part of $x \in \mathbb{R}$ is $[x]_+ := \max\{x, 0\}$.

A. Hilbert Spaces and Bounded Linear Operators

Throughout the paper, $\mathcal{H}$ is a real separable Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and with induced norm $\| \cdot \|$. The Euclidean norm and scalar product are denoted as $\| \cdot \|_2$ and $\langle \cdot, \cdot \rangle_2$, respectively. For $\bar{x} \in \mathcal{H}$ and $r > 0$, the open ball centered at $\bar{x}$ with radius $r$ is indicated as $B(\bar{x}; r) := \{ x \in \mathcal{H} \mid \|x - \bar{x}\| < r \}$. For a closed and nonempty convex set $C \subset \mathcal{H}$, we let $\Pi_C$ denote the projection operator on $C$.

Given $(x_k)_{k \in \mathbb{N}} \subset \mathcal{H}$ and $x \in \mathcal{H}$, we write $x_k \to x$ and $x_k \rightharpoonup x$ to denote, respectively, strong and weak convergence of $(x_k)_{k \in \mathbb{N}}$ to $x$. The set of weak sequential cluster points of $(x_k)_{k \in \mathbb{N}}$ is indicated as $\mathcal{W}(x_k)_{k \in \mathbb{N}}$. 

Fig. 1 (a) Alternating projections on polyhedral cones. $R = \text{id} \circ \Pi_{C_2} \circ \Pi_{C_1}$ is globally metrically subregular, however, the $Q$-linear convergence of the KM scheme is very slow. (b) Alternating projections on ball and tangent line. With or without linesearch the KM scheme is not linearly convergent due to the fact that the residual $R$ is not metrically subregular at $x$. (c) Alternating projections on second-order cone and tangent plane. In contrast with the slow sublinear rate of KM both with and without linesearch, and despite the nonisolatedness of any solution, Broyden’s scheme exhibits an appealing linear convergence rate.
The set of bounded linear operators $\mathcal{H} \to \mathcal{H}$ is denoted as $\mathcal{B}(\mathcal{H})$. The adjoint operator of $L \in \mathcal{B}(\mathcal{H})$ is indicated as $L^*$, i.e., the unique operator in $\mathcal{B}(\mathcal{H})$ such that $\langle Lx, y \rangle = \langle x, L^*y \rangle$ for all $x, y \in \mathcal{H}$.

### B. NE Operators and Fejér Sequences

We now briefly recap some known definitions and results of NE operator theory that will be used in this paper.

**Definition III.1:** An operator $T : \mathcal{H} \to \mathcal{H}$ is said to be

i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in \mathcal{H}$;

ii) averaged if it is $\alpha$-averaged for some $\alpha \in (0, 1)$, i.e., if there exists an NE operator $S : \mathcal{H} \to \mathcal{H}$ such that $T = (1 - \alpha)id + \alpha S$;

iii) firmly nonexpansive (FNE) if it is $\frac{1}{2}$-averaged.

Clearly, for any NE operator $T$ the residual $R = id - T$ is monotone, in the sense that $(Rx - Ry, x - y) \geq 0$ for all $x, y \in \mathcal{H}$; if $T$ is additionally FNE, then not only is $R$ monotone, but it is FNE as well. For notational convenience, we extend the definition of $\alpha$-averagedness to the case $\alpha = 1$ which reduces to plain nonexpansiveness.

Given an operator $T : \mathcal{H} \to \mathcal{H}$, we let

$$z_T = \{z \in \mathcal{H} \mid Tz = 0\}$$

and

$$\text{fix } T := \{z \in \mathcal{H} \mid Tz = z\}$$

denote the set of its zeros and fixed points, respectively. For $\lambda \in \mathbb{R}$, we define the $\lambda$-averaging of $T$ as

$$T_\lambda := (1 - \lambda)id + \lambda T.$$

Notice that

$$id - T_\lambda = \lambda(id - T)$$

for all $\lambda \in \mathbb{R}$.

And, therefore, $\text{fix } T_\lambda = \text{fix } T$ for all $\lambda \neq 0$. Moreover, if $T$ is $\alpha$-averaged and $\lambda \in (0, \frac{1}{\alpha})$, then

$$T_\lambda$$

is $\alpha\lambda$-averaged.

[14, Corollary 4.28] and in particular $T_{1/2\alpha}$ is FNE.

**Definition III.2:** Relative to a nonempty set $S \subseteq \mathcal{H}$, a sequence $(x_k)_{k \in \mathbb{N}} \subseteq \mathcal{H}$ is

i) Fejér (monotone) if $\|x_{k+1} - s\| \leq \|x_k - s\|$ for all $k \in \mathbb{N}$ and $s \in S$;

ii) quasi-Fejér (monotone) if for all $s \in S$, there exists a sequence $(\varepsilon_k(s))_{k \in \mathbb{N}} \subseteq \ell^1_+$ such that

$$\|x_{k+1} - s\|^2 \leq \|x_k - s\|^2 + \varepsilon_k(s) \quad \forall k \in \mathbb{N}.$$

This definition of quasi-Fejér monotonicity is taken from [15] where it is referred to as of type III, and generalizes the classical definition [16].

**Theorem III.3:** Let $T : \mathcal{H} \to \mathcal{H}$ be an NE operator with $\text{fix } T \neq \emptyset$, and suppose that $(x_k)_{k \in \mathbb{N}} \subseteq \mathcal{H}$ is quasi-Fejér with respect to $\text{fix } T$. If $(x_k - T x_k)_{k \in \mathbb{N}} \to 0$, then there exists $x_\ast \in \text{fix } T$ such that $x_k \to x_\ast$.

**Proof:** From [15, Proposition 3.7(i)], we have $\mathcal{W}(x_k)_{k \in \mathbb{N}} \neq \emptyset$; in turn, from [14, Corollary 4.18], we infer that $\mathcal{W}(x_k)_{k \in \mathbb{N}} \subseteq \text{fix } T$. The claim then follows from [15, Th. 3.8].

### IV. General Abstract Framework

Unless differently specified, in the rest of this paper, we work under the following assumption.

**Assumption 1:** $T : \mathcal{H} \to \mathcal{H}$ is an $\alpha$-averaged operator for some $\alpha \in (0, 1]$ and with $\text{fix } T \neq \emptyset$. With $R := id - T$ we denote its $(2\alpha$-Lipschitz continuous) fixed-point residual.

We also stick to this notation, so that, whenever mentioned, $T$, $R$, and $\alpha$ are as in Assumption 1. Our goal is to find a fixed point of $T$, or, equivalently, a zero of $R$.

$$\text{find } x_\ast \in \text{fix } T = \text{zer } R. \quad (3)$$

In this section, we introduce Algorithm 1, an abstract procedure to solve problem (3). The scheme is not implementable in and of itself, as it gives no hint as to how to compute each of the iterates, but it rather serves as a comprehensive ground framework for a class of algorithms with global convergence guarantees. In Section VI, we will derive the SuperMann scheme, an implementable instance which also enjoys appealing asymptotic properties.

The general framework prescribes three kinds of updates.

$K_0$) **Blind updates:** Inspired from [17], whenever the residual $\|Rx_k\|$ at iteration $k$ has sufficiently decreased with respect to past iterates we allow for an uncontrolled update. For an efficient implementation such guess should be somehow reasonable and not completely a "blind" guess; however, for the sake of global convergence the proposed scheme is robust to any choice.

$K_1$) **Educated updates:** To encourage favorable updates, similarly to what has been proposed in [8, Section VIII-C2] and [9, Section V-C1], an educated guess $x_{k+1}$ is accepted whenever the candidate residual is sufficiently smaller than the current.

$K_2$) **Safeguard (Fejérian) updates:** This last kind of updates is similar to $K_1$ as it is also based on the goodness of $x_{k+1}$ with respect to $x_k$. The difference is that instead of checking the residual, what needs to be sufficiently decreased is the distance from each point in $\text{fix } T$. This is meant in a Fejérian fashion as in Definition III.2.

Blind $K_0$- and educated $K_1$-updates are somehow complementary; the former is enabled when enough progress has been made in the past, whereas the latter when the candidate update yields a sufficient improvement. Progress and improvement are meant in terms of a linear decrease of (the norm of) the residual; at iteration $k$, $K_0$ is enabled if $\|Rx_k\| \leq c_0\|Rx_k\|$ where $c_0 \in (0, 1)$ is a user-defined constant and $k$ is the last blind iteration before $k$; $K_1$ is enabled if $\|Rx_{k+1}\| \leq c_1\|Rx_k\|$ where $c_1 \in (0, 1)$ is another user-defined constant and $x_{k+1}$ is the candidate next iterate. To ensure global convergence, educated updates are authorized only if the current residual $\|Rx_k\|$ is not larger than $\|Rx_{k+1}\|$ (up to a linearly decreasing error $q^k$); here $\tilde{k}$ denotes the last $K_1$-update before $k$.

While blind $K_0$- and educated $K_1$-updates are in charge of the asymptotic behavior, what makes the algorithm converge are safeguard $K_2$-iterations.

### A. Global Weak Convergence

To establish a notation, we partition the set of iteration indices $\mathcal{K} \subseteq \mathbb{N}$ as $K_0 \cup K_1 \cup K_2$. Namely, relative to Algorithm 1, $K_0$, $K_1$, and $K_2$ denote the sets of indices $\tilde{k}$ passing the test at steps 2, 3(a), and 3(b), respectively. Furthermore, we index the sets $K_0$...
Algorithm 1: General Framework for Finding a Fixed Point of the $\alpha$-Averaged Operator $T$ With Residual $R = \text{id} - T$.

**Require** $x_0 \in \mathcal{H}$, $c_0, c_1, q \in [0, 1)$, $\sigma > 0$

**Initialize** $\eta_0 = r_{\text{safe}} = \|Rx_0\|$, $k = 0$

1. If $Rx_k = 0$, THEN STOP.
2. If $\|Rx_k\| \leq c_0\eta_k$, THEN set $\eta_{k+1} = \|Rx_k\|$, proceed with a blind update $x_{k+1}$ and go to step 4.
3. Set $\eta_{k+1} = \eta_k$ and select $x_{k+1}$ such that
   3(a) EITHER the safe condition $\|Rx_k\| \leq r_{\text{safe}}$, holds, and $x_{k+1}$ is educated:
      $\|Rx_{k+1}\| \leq c_1 \|Rx_k\|
   $5$ in which case update $r_{\text{safe}} = \|Rx_{k+1}\| + q^k$;
   3(b) OR it is Féjérien with respect to $\text{fix} T$:
      $\|x_{k+1} - z\|^2 \leq \|x_k - z\|^2 - \sigma \|Rx_k\|^2 \quad \forall z \in \text{fix} T.$

4. Set $k \leftarrow k + 1$ and go to step 1.

and $K_1$ of blind and educated updates as

$$K_0 = \{k_1, k_2, \ldots\}, \quad K_1 = \{k_1', k_2', \ldots\}.$$  \hspace{1cm} (5)

To rule out trivialities, throughout this paper, we work under the assumption that a solution is not found in a finite number of steps, so that the residual of each iterate is always nonzero. As long as it is well defined, the algorithm, therefore, produces an infinite number of iterates.

**Theorem IV.1 (Global convergence of the general framework Algorithm 1):** Consider the iterates generated by Algorithm 1 and suppose further that for all $k$ it is always possible to find a point $x_{k+1}$ complying with the requirements of either step 2, 3(a), or 3(b), and further satisfying

$$\|x_{k+1} - x_k\| \leq D \|Rx_k\| \quad \forall k \in K_0 \cup K_1$$  \hspace{1cm} (6)

for some constant $D \geq 0$. Then,

i) $(x_k)_{k \in \mathbb{N}}$ is quasi-Fejér monotone with respect to $\text{fix} T$;

ii) $Rx_k \to 0$ with $(\|Rx_k\|)_{k \in \mathbb{N}} \in \ell^2$;

iii) $(x_k)_{k \in \mathbb{N}}$ converges weakly to a point $x_\ast \in \text{fix} T$;

iv) if $c_0 > 0$ the number of blind updates at step 2 is infinite.

**Proof:** See Appendix A.

**B. Local Linear Convergence**

More can be said about the convergence rates if the mapping $R$ possesses metric subregularity. Differently from (bounded) linear regularity [18], metric subregularity is a local property and as such it is more general. For a (possibly multivalued) operator $R$, metric subregularity at $x_\ast$ is equivalent to calmness of $R^{-1}$ at $x_\ast$ [19, Th. 3.2], and is a weaker condition than metric regularity and Aubin property. We refer the reader to [20, Section IX] for an extensive discussion.

**Definition IV.2 (Metric subregularity at zeros):** Let $R : \mathcal{H} \to \mathcal{H}$ and $x_\ast \in \text{zer} R$, $R$ is metrically subregular at $x_\ast$ if there exist $\varepsilon, \gamma > 0$ such that

$$\|x - x_\ast\| \leq \varepsilon \|R^\ast\| \quad \forall x \in B(x_\ast; \varepsilon)$$  \hspace{1cm} (7)

where $\gamma$ and $\varepsilon$ are (one) modulus and (one) radius of subregularity of $R$ at $x_\ast$, respectively.

In finite-dimensional spaces, if $R$ is differentiable at $x_\ast$ and $x_\ast$ is isolated in $\text{zer} R$ (e.g., if it is the unique zero), then metric subregularity is equivalent to nonsingularity of $JR_\ast x$. Metric subregularity is, however, a much weaker property than nonsingularity of the Jacobian, first, because it does not assume differentiability, and second, because it can cope with “wide” regions of zeros; for instance, any piecewise linear mapping is globally metrically subregular [21].

If the residual $R = \text{id} - T$ of the $\alpha$-averaged operator $T$ is metrically subregular at $x_\ast \in \text{zer} R = \text{fix} T$ with modulus $\gamma$ and radius $\varepsilon$, then

$$\frac{1}{\gamma} \text{dist}(x, \text{fix} T) \leq \|R^\ast\| \leq 2\alpha \text{dist}(x, \text{fix} T)$$  \hspace{1cm} (8)

for all $x \in B(x_\ast; \varepsilon)$. Consequently, if $\|Rx_k\| \to 0$ for some sequence $(x_k)_{k \in \mathbb{N}} \subset \mathcal{H}$, so does $\text{dist}(x_k, \text{fix} T)$ with the same asymptotic rate of convergence, and vice versa. Metric subregularity is the key property under which the residual in the classical KM scheme achieves linear convergence; in the next result we show that this asymptotic behavior is preserved in the general framework of Algorithm 1.

**Theorem IV.3 (Linear convergence of the general framework Algorithm 1):** Suppose that the hypotheses of Theorem IV.1 hold, and suppose further that $(x_k)_{k \in \mathbb{N}}$ converges strongly to a point $x_\ast$ (this being true if $\mathcal{H}$ is finite dimensional) at which $R$ is metrically subregular.

Then, $(x_k)_{k \in \mathbb{N}}$ and $(Rx_k)_{k \in \mathbb{N}}$ are $R$-linearly convergent.

**Proof:** See Appendix A.

**C. Main Idea**

Being interested in solving the nonlinear equation (3), one could think of implementing one of the many existing fast methods for nonlinear equations that achieve fast asymptotic rates such as Newton-type schemes. At each iteration, such schemes compute an update direction $d_k$ and prescribe steps of the form $x_{k+1} = x_k + \tau_k d_k$, where $\tau_k > 0$ is a stepsize that needs to be sufficiently small in order for the method to enjoy global convergence; on the other hand, fast asymptotic rates are ensured if $\tau_k = 1$ is eventually always accepted. The stepsize is a crucial feature of fast methods, and a feasible $\tau_k$ is usually backtracked with a linesearch on a smooth merit function. Unfortunately, in meaningful applications of the problem at hand arising from fixed-point theory the residual mapping $R$ is nonsmooth, and the typical merit function $x \mapsto \|Rx\|^2$ does not meet the necessary smoothness requirement.

What we propose, in this paper, is a hybrid scheme that allows for the employment of any (fast) method for solving nonlinear equations, with global convergence guarantees that do not require smoothness, and which is based only on the
A classical KM scheme

Starting from a point \(x_0 \in \mathcal{H}\), the classical KM scheme performs the following updates:

\[
x_{k+1} = T_{\lambda_k} x_k = (1 - \lambda_k) x_k + \lambda_k T x_k
\]

and converges weakly to a fixed point of \(T\) provided that \(\lambda_k \in [0, \gamma_0)\) and \((\lambda_k/\gamma_0 - \lambda_k)k \in \mathbb{N} \notin \ell^1\) [14, Th. 5.14]. The key property of KM iterations is Fejér monotonicity

\[
\|x_{k+1} - z\|^2 \leq \|x_k - z\|^2 - \lambda_k(\gamma_0 - \lambda_k)\|Rx_k\|^2 \quad \forall z \in \text{fix } T.
\]

In particular, in Algorithm 1, KM iterations can be used as safeguard updates at step 3(b). The drawback of such a selection is that it completely discards the hypothetical fast update direction \(d_k\) that blind and educated updates try to enforce. This is particularly penalizing when the local method for computing the directions \(d_k\) is a quasi-Newton scheme; such methods are indeed very sensitive to past iterations, and discarding directions is neither theoretically sound nor beneficial in practice.

In this section, we provide alternative safeguard updates that while ensuring the desirable Fejér monotonicity are also amenable to take into account arbitrary directions. The key idea lies in interpreting KM iterations as projections onto suitable half-spaces (see Fig. 2), and then exploiting known properties of these facts shown in the next result. To this end, let us remark that the projection \(\Pi_C\) onto a nonempty closed and convex set \(C\) is FNE [14, Proposition 4.8], and that consequently its \(\alpha\)-averaging \(\Pi_{C,\alpha}\) is \(\gamma_0\)-averaged for any \(\alpha \in (0, 2]\), as it follows from (2).

**Proposition V.1 (KM iterations as projections):** For \(x \in \mathcal{H}\), define

\[
C_x = C^T_{x,\alpha} := \{ z \in \mathcal{H} \mid \|Rx\|^2 - 2\alpha \langle Rx, x - z \rangle \leq 0 \}.
\]

Then,

i) \(x \in C_x\) iff \(x \in \text{fix } T\);

ii) \(\text{fix } T = \bigcap_{x \in \mathcal{H}} C_x\).

For \(\lambda \in [0, \gamma_0]\), let

\[
x^+ := x - \lambda \frac{\rho}{\|Rw\|^2} Rw.
\]

Then, the following conditions hold.

i) \(x^+ = \Pi_{C_{x,2\alpha}} x\) where \(C_{x,\alpha} = C^T_{x,\alpha}\) as in (10).

ii) \(|x^+ - z|^2 \leq \|x - z\|^2 - \lambda(\gamma_0 - \lambda) \frac{\rho^2}{\|Rw\|^2} \quad \forall z \in \text{fix } T.

**Proof:** Proposition V.2(i) easily follows from (11) and (12), since by condition (13) the positive part in the formula may be omitted. In turn, (ii) follows from [14, Proposition 4.25(iii)] by

\[\text{for any } \lambda \in [0, \gamma_0]\text{ it holds that } T_{\lambda} x = \Pi_{C_{x,2\alpha}} x = (1 - 2\alpha\lambda)x + 2\alpha\lambda \Pi_{C_x} x.
\]

**Proposition V.2:** Suppose that \(x, w \in \mathcal{H}\) are such that

\[
\rho := \|Rw\|^2 - 2\alpha \langle Rw, w - x \rangle > 0.
\]

For \(\lambda \in [0, \gamma_0]\), let

\[
x^+ := x - \lambda \frac{\rho}{\|Rw\|^2} Rw.
\]

Then, the following conditions hold.

i) \(x^+ = \Pi_{C_{x,2\alpha}} x\) where \(C_{x,\alpha} = C^T_{x,\alpha}\) as in (10).

ii) \(|x^+ - z|^2 \leq \|x - z\|^2 - \lambda(\gamma_0 - \lambda) \frac{\rho^2}{\|Rw\|^2} \quad \forall z \in \text{fix } T.

**Proof:** Proposition V.2(i) easily follows from (11) and (12), since by condition (13) the positive part in the formula may be omitted. In turn, (ii) follows from [14, Proposition 4.25(iii)] by
observing that $\Pi_{C_w,2\lambda}$ is $\alpha\lambda$-averaged due to [14, Proposition 4.8] and (2), and that $\text{fix} T \subseteq C_w$, as shown in Proposition V.1(ii).

Notice that condition (13) is equivalent to $x \notin C_w$. Therefore, Proposition V.2(ii) states that whenever a point $x$ lies outside the half-space $C_w$ for some $w \in \mathcal{H}$, since $\text{fix} T \subseteq C_w$ (cf. Proposition V.1) the projection onto $C_w$ moves $x$ closer to $\text{fix} T$. This means that after moving from $x$ along a candidate direction $d$ to the point $w = x + d$, even though $w$ might be farther from $\text{fix} T$ the point $x^\gamma = \Pi_x w$ is not. We may then use this projection as a safeguard step to prevent from diverging from the set of fixed points. Based on this, we define a GKM update along a direction $d$.

Definition V.3 (GKM update): A GKM update at $x$ along $d$ for the $\alpha$-averaged operator $T : \mathcal{H} \rightarrow \mathcal{H}$ with relaxation $\lambda \in [0,1]$ is

$$x^\tau := \begin{cases} x, & \text{if } w \in \text{fix } T \\ x - \lambda \frac{\rho}{\|Rw\|^2} Rw, & \text{otherwise} \end{cases}$$

where $w = x + d$ and $\rho := \|Rw\|^2 - 2\alpha\langle Rw, w - x \rangle$. In particular, $d = 0$ yields the classical KM update $x^\tau = T_\lambda x$.

C. Linesearch for GKM

It is evident from Definition V.3 that a GKM update trivializes to $x^\tau = x$ if either $w \in \text{fix } T$ or $\rho \leq 0$. Having $w \in \text{fix } T$ corresponds to having found a solution to problem (3), and the case deserves no further investigation. In this section, we address the remaining case $\rho \leq 0$, showing how it can be avoided by simply introducing a suitable linesearch. In order to recover the same global convergence properties of the classical KM scheme we need something more than simply imposing $\rho > 0$. The next result addresses this requirement showing further that it is achieved for any direction $d$ by sufficiently small step sizes.

Theorem V.4: Let $x, d \in \mathcal{H}$ and $\sigma \in [0,1)$ be fixed, and consider

$$\tilde{\tau} = \begin{cases} 1, & \text{if } d = 0 \\ \frac{1 - \sigma \|Rx\|}{2\alpha}, & \text{otherwise} \end{cases}$$

Then, for all $\tau \in (0, \tilde{\tau}]$ the point $w = x + \tau d$ satisfies

$$\rho := \|Rw\|^2 - 2\alpha\langle Rw, w - x \rangle \geq \sigma\|Rx\|\|Rx\|.$$  \hspace{2cm} (15)

Proof: Let a constant $c \geq 0$ be determined be such that

$$\tau\|d\| = \|w - x\| \leq c\|Rx\|.$$}

Observe that $\rho = 4\alpha^2 \langle w - T_{1/2\alpha} w, x - T_{1/2\alpha} w \rangle$, and recall from (1) and (2) that $T_{1/2\alpha}$ is FNE with residual $\text{id} - T_{1/2\alpha} = \frac{1}{2\alpha} R$. Then,

$$\rho = 4\alpha^2 \langle w - T_{1/2\alpha} w, w - T_{1/2\alpha} w, x - w \rangle$$

using Cauchy–Schwartz inequality

$$\geq 4\alpha^2 \|w - T_{1/2\alpha} w\|^2 \geq 2\alpha\|x - T_{1/2\alpha} x\|$$

the bound on $\|x - w\|$ into

$$\geq 2\alpha\|Rw\|\|w - T_{1/2\alpha} w\| - 2\alpha\|x - T_{1/2\alpha} x\|$$

the (reverse) triangular inequality

$$\geq 2\alpha\|Rw\|(1 - 2\alpha c)\|x - T_{1/2\alpha} x\|$$

the nonexpansiveness of $\text{id} - T_{1/2\alpha}$

$$\geq 2\alpha\|Rw\|\left(\frac{1 - 2\alpha c}{2\alpha}\right)\|Rx\| - \|w - x\|$$

and again the bound on $\|w - x\|$

$$\geq (1 - 4\alpha c)\|Rw\|\|Rx\|$$

equating $\sigma = 1 - 4\alpha c$ the assert follows. □

Notice that if $d = 0$, then $\rho = \|Rx\|^2 \geq \sigma\|Rx\|^2$ for any $\sigma \in [0,1)$, and therefore, the linesearch condition (15) is always satisfied; in particular, the classical KM step $x^\tau = T\lambda x$ is always accepted regardless of the value of $\sigma$.

Let us now observe how a GKM projection extends the classical KM depicted in Fig. 2 and how the linesearch works. In the following, we use the notation of Theorem V.4, and for the sake of simplicity we consider $\sigma = 0$ in (15) and a FNE operator $T$. Suppose that the fixed point $z$ and the points $x, Tx, w$ and $w$ are as in Fig. 3(a); due to firm nonexpansiveness, the image $Tw$ of $w$
Algorithm 2: SuperM ann Scheme for Solving (3), Given an \( \alpha \)-Averaged Operator \( T \) With Residual \( R = \text{id} - T \).

\[
\begin{align*}
\text{REQUIRE} & \quad x_0 \in \mathcal{H}, \ \alpha, c_0, q \in (0, 1), \ \beta, \sigma \in (0, 1), \ \lambda \in (0, 1). \\
\text{INITIALIZE} & \quad \eta_0 = r_{\text{safe}} = \|Rx_0\|, \ k = 0 \\
1. & \quad \text{If } Rx_0 = 0, \text{ then STOP.} \\
2. & \quad \text{Choose an update direction } d_k \in \mathcal{H} \\
3. & \quad (K_0) \text{ If } \|Rx_k\| \leq c_0 \eta_k, \text{ then set } \eta_{k+1} = \|Rx_k\|, \text{ proceed with a blind update } x_{k+1} = w_k := x_k + d_k \text{ and go to step 6.} \\
4. & \quad \text{Set } \eta_{k+1} = \eta_k \text{ and } \tau_k = 1. \\
5. & \quad \text{Let } w_k = x_k + \tau_k d_k. \\
5(a) & \quad (K_1) \text{ If the safe condition } \|Rx_k\| \leq r_{\text{safe}} \text{ holds and } w_k \text{ is educated:} \\
& \quad \|Rw_k\| \leq c_1 \|Rx_k\| \\
& \quad \text{then set } x_{k+1} = w_k, \text{ update } r_{\text{safe}} = \|Rw_k\| + q^k, \text{ and go to step 6.} \\
5(b) & \quad (K_2) \text{ If } \rho_k := \|Rw_k\|^2 - 2\alpha \langle Rw_k, w_k - x_k \rangle \geq \sigma \|Rw_k\| \|Rx_k\| \text{ then set} \\
& \quad x_{k+1} = x_k - \lambda \frac{\rho_k}{\|Rw_k\|^2} Rw_k \\
& \quad \text{otherwise set } \tau_k \leftarrow \beta \tau_k \text{ and go to step 5.} \\
6. & \quad \text{Set } k \leftarrow k + 1 \text{ and go to step 1.}
\end{align*}
\]

is somewhere in the intersection of the orange circles. We want to avoid the unfavorable situation depicted in Fig. 3(b), where the couple \( (w, Tw) \) generates a half-space \( C_w \) that contains \( x \), i.e., such that \( \rho \leq 0 \). In fact, with simple algebra it can be seen that \( \rho \leq 0 \) iff \( Tw \) belongs to the dashed circle of Fig. 3(b)

\[
B_{x,w} := \{ \bar{w} \mid \langle w - \bar{w}, x - \bar{w} \rangle \leq 0 \}. \tag{16}
\]

Since the dashed orange circle (in which \( Tw \) must lie) is simply the translation by a vector \( Tx - x \) of \( B_{x,w} \), both having diameter \( \tau \|d\| \), for sufficiently small \( \tau \) the two have empty intersection, meaning that \( \rho > 0 \) regardless of where \( Tw \) is.

VI. SUPERMANN SCHEME

In this section, we introduce the SuperMann scheme (see Algorithm 2), a special instance of the general framework of Algorithm 1 that employs GKM updates as safeguard \( K_2 \)-steps. While the global worst case convergence properties of SuperMann are the same as for the classical KM scheme, its asymptotic behavior is determined by how blind \( K_0 \)- and educated \( K_1 \)-updates are selected. In Section VI-B, we will characterize the “quality” of update directions and the mild requirements under which superlinear convergence rates are attained; in particular, Section VI-C is dedicated to the analysis of quasi-Newton Broyden’s directions.

The scheme follows the same philosophy of the general abstract framework. The main idea is localizing a method for solving the monotone equation \( Rx = 0 \), in such a way that when the iterates get close enough to a solution the fast convergence of the local method is automatically triggered. Approaching a solution is possible thanks to the GKM updates [step 5(b)], provided enough backtracking is performed, as assured by Proposition V.2(ii) and Theorem V.4. When a basin of fast (i.e., superlinear) attraction for the local method is reached, the (norm of) \( Rx \) will decrease more than linearly, and the condition triggering the educated updates of step 5(a) (which is checked first) will be verified without performing any backtracking.

To discuss its global and local convergence properties we stick to the same notation of the general framework of Algorithm 1, denoting the sets of blind, educated, and safeguard updates as \( K_0, K_1, \) and \( K_2 \), respectively.

A. Global and Linear Convergence

To comply with (6), we impose the following requirement on the magnitude of the directions (see also Remark VI.9).

Assumption 2: There exists a constant \( D \geq 0 \) such that the directions \((d_k)_{k \in \mathbb{N}} \) in the SuperMann scheme (see Algorithm 2) satisfy

\[
\|d_k\| \leq D \|Rx_k\| \quad \forall k \in \mathbb{N}. \tag{17}
\]

Theorem VI.1 (Global and linear convergence of the SuperMann scheme): Consider the iterates generated by the SuperMann scheme (see Algorithm 2) with \((d_k)_{k \in \mathbb{N}}\) selected so as to satisfy Assumption 2. Then,

i) \((x_k)_{k \in \mathbb{N}}\) is quasi-Fejér monotone with respect to \( \text{fix} \ T \); 
ii) \( \tau_k = 1 \) if \( d_k = 0 \), and \( \tau_k \geq \min \{ \beta \frac{1 - \sigma}{\alpha D}, 1 \} \) otherwise; 
iii) \( Rx_k \) converges strongly to a point \( x^* \in \text{fix} \ T \), where \( x^* \) is a fixed point of \( T \) if \( c_0 > 0 \) the number of blind updates at step 3 is infinite. Moreover, if \((x_k)_{k \in \mathbb{N}}\) converges strongly to a point \( x^* \) (this being true if \( \mathcal{H} \) is finite dimensional) at which \( R \) is metrically subregular, then 
vi) \((x_k)_{k \in \mathbb{N}}\) and \((Rx_k)_{k \in \mathbb{N}}\) are \( R \)-linearly convergent.

Proof: See Appendix B.

B. Superlinear Convergence

Though global convergence of the SuperMann scheme is independent of the choice of the directions \( d_k \), its performance and tail convergence surely does. We characterize the quality of the directions \( d_k \) in terms of the following definition.

Definition VI.2 (Superlinear directions for the SuperMann scheme): Relative to the sequence \((x_k)_{k \in \mathbb{N}}\) generated by the SuperMann scheme, we say that \((d_k)_{k \in \mathbb{N}} \subset \mathcal{H}\) are superlinear directions if the following limit holds:

\[
\lim_{k \to \infty} \frac{\|Rx_k + d_k\|}{\|Rx_k\|} = 0.
\]

Remark VI.3: Definition VI.2 makes no mention of a limit point \( x^* \) of the sequence \((x_k)_{k \in \mathbb{N}}\), differently from the definition in [8], which instead requires \( \frac{\|x_k + d_k - x^*\|}{\|x_k - x^*\|} \) to be vanishing with no mention of \( R \). Due to \( 2\alpha \)-Lipschitz continuity of \( R \), whenever
the directions \(d_k\) are bounded as in (17) we have
\[
\frac{\|R(x_k + d_k)\|}{\|Rx_k\|} \leq 2\alpha D \frac{\|x_k + d_k - x_*\|}{\|d_k\|}.
\]

Invoking [8, Lemma 7.5.7] it follows that Definition VI.2 is implied by the one in [8] and is, therefore, more general.

**Theorem VI.4:** Consider the iterates generated by the SuperMann scheme (see Algorithm 2) with either \(c_0 > 0\) or \(c_1 > 0\), and with \((d_k)_{k \in \mathbb{N}}\) being superlinear directions as in Definition VI.2. Then,

i) eventually, stepsize \(\tau_k = 1\) is always accepted and safeguard updates \(K_2\) are deactivated (i.e., the scheme reduces to the local method \(x_{k+1} = x_k + d_k\));

ii) \((Rx_k)_{k \in \mathbb{N}}\) converges Q-superlinearly;

iii) if the directions \(d_k\) satisfy Assumption 2, then \((x_k)_{k \in \mathbb{N}}\) converges R-superlinearly;

iv) if \(c_0 > 0\), then the complement of \(K_0\) is finite.

**Proof:** See Appendix B.

Theorem VI.4 shows that when the directions \(d_k\) are good, then eventually the SuperMann scheme reduces to the local method \(x_{k+1} = x_k + d_k\) and consequently inherits its local convergence properties. The following result specializes to the choice of semismooth Newton directions.

**Corollary VI.5 (Superlinear convergence for semismooth Newton directions):** Suppose that \(\mathcal{H}\) is finite dimensional, and that \(R\) is semismooth. Consider the iterates generated by the SuperMann scheme (see Algorithm 2) with either \(c_0 > 0\) or \(c_1 > 0\) and directions \(d_k\) chosen as solutions of
\[
(G_k + \mu_k id) d_k = -Rx_k \quad \text{for some } G_k \in \partial Rx_k
\]
where \(\partial R\) denotes the Clarke generalized Jacobian of \(R\) and \(0 \leq \mu_k \to 0\). Suppose that the sequence \((x_k)_{k \in \mathbb{N}}\) converges to a point \(x\) at which all the elements in \(\partial R\) are nonsingular.

Then, \((d_k)_{k \in \mathbb{N}}\) are superlinear directions as in Definition VI.2, and in particular all the claims of Theorem VI.4 hold.

**Proof:** Any \(G_k \in \partial R\) is positive semidefinite due to the monotonicity of \(R\) and, therefore, \(d_k\) as in (18) is well defined for any \(\mu_k > 0\). The bound (17) holds due to [8, Th. 7.5.2]. Moreover,
\[
\frac{\|Rx_k + G_k d_k\|}{\|d_k\|} = \mu_k \to 0
\]
as \(k \to \infty\), and the proof follows by invoking [8, Th. 7.5.8(a)] and Remark VI.3.

Notice that since \(\partial R = id - \partial T\), nonsingularity of the elements in \(\partial R(x_k)\) is equivalent to having \(\|G\| < 1\) for all \(G \in \partial T(x_k)\), i.e., that \(T\) is a local contraction around \(x_\star\).

Despite the favorable properties of semismooth Newton methods, in this paper, we are oriented toward choices of directions that are defined for any nonexpansive mapping, regardless of the (generalized) first-order properties, and that require exactly the same oracle information as the original KM scheme. This motivates the investigation of quasi-Newton directions, whose superlinear behavior is based on the classical Dennis–Moré criterion, which we provide next. We first recall the notions of semi- and strict-differentiability.

**Definition VI.6:** We say that \(R : \mathcal{H} \to \mathcal{H}\) is

i) strictly differentiable at \(\bar{x}\) if it is differentiable there with \(JR(\bar{x})\) satisfying
\[
\lim_{(x,y) \to (\bar{x},\bar{x})} \frac{\|Ry - Rx - JR(\bar{x})(y-x)\|}{\|y-x\|} = 0;
\]

ii) semidifferentiable at \(\bar{x}\) if there exists a continuous and positively homogeneous function \(DR(\bar{x}) : \mathcal{H} \to \mathcal{H}\), called the semiderivative of \(R\) at \(\bar{x}\), such that
\[
Rx = R\bar{x} + DR(\bar{x})(x - \bar{x}) + o(\|x - \bar{x}\|);
\]

iii) calmly semidifferentiable at \(\bar{x}\) if there exists a neighborhood \(U_\varepsilon\) of \(\bar{x}\) in which \(R\) is semidifferentiable and such that for all \(v \in \mathcal{H}\) with \(\|v\| = 1\) the function \(U_\varepsilon \ni x \mapsto DR(x)v\) is Lipschitz continuous at \(\bar{x}\).

There is a slight ambiguity in the literature, as strict differentiability is sometimes referred to rather as strong differentiability [22], [23]. We choose to stick the proposed terminology, following [20]. Semidifferentiability is clearly a milder property than differentiability in that the mapping \(DR(\bar{x})\) need not be linear. More precisely, since the residual \(R\) of an NE operator is (globally) Lipschitz continuous, then semidifferentiability is equivalent to directional differentiability [8, Proposition 3.1.3] and the semiderivative is sometimes called \(B\)-derivative [8], [22]. The three concepts in Definition VI.6 are related as (iii) \(\Rightarrow\) (i) \(\Rightarrow\) (ii) [23, Th. 2] and do not require the existence of the (classical) Jacobian around \(\bar{x}\).

**Theorem VI.7** (Dennis–Moré criterion for superlinear convergence): Consider the iterates generated by the SuperMann scheme (see Algorithm 2) and suppose that \((x_k)_{k \in \mathbb{N}}\) converges strongly to a point \(x_\star\) at which \(R\) is strictly differentiable. Suppose further that the update directions \((d_k)_{k \in \mathbb{N}}\) satisfy Assumption 2 and the Dennis–Moré condition
\[
\lim_{k \to \infty} \frac{\|Rx_k + JR(x_k)d_k\|}{\|d_k\|} = 0.
\]

Then, the directions \(d_k\) are superlinear as in Definition VI.2. In particular, all the claims of Theorem VI.4 hold.

**Proof:**

\[
0 \overset{(20)}{=} \lim_{k \to \infty} \frac{\|Rx_k + JR(x_k)d_k + (R(x_k + d_k) - R(x_k + d_k))\|}{\|d_k\|} \\
\overset{(17)}{=} \lim_{k \to \infty} \frac{\|R(x_k + d_k)\|}{\|d_k\|} \geq \frac{1}{D} \lim_{k \to \infty} \frac{\|Rx_k\|}{\|d_k\|}
\]
where in the second equality we used strict differentiability of \(R\) at \(x_\star\).

**C. Modified Broyden’s Direction Scheme**

In practical applications the Hilbert space \(\mathcal{H}\) is finite dimensional, and consequently it can be identified with \(\mathbb{R}^n\). Then, the computation of quasi-Newton directions \(d_k\) in the SuperMann scheme amounts to selecting
\[
d_k = -B_k^{-1}Rx_k
\]
where \(B_k \in \mathbb{R}^{n \times n}\) are recursively defined by low-rank updates satisfying a secant condition, starting from an invertible matrix \(B_0\). The most popular quasi-Newton scheme is the two-rank BFGS formula, which also enforces symmetry. As such,
BFGS is well performing only when the Jacobian at the solution $JRx_s$ possesses this property, a requirement that is not met by the residual $R$ of generic nonexpansive mappings.

For this reason, we consider Broyden’s method as a universal alternative. We adopt Powell’s modification [10] to enforce nonsingularity and make (21a) well defined: For a fixed parameter $\vartheta \in (0,1)$, matrices $B_k$ are recursively defined as

$$
B_{k+1} = B_k + \frac{1}{\vartheta k} \left( y_k - B_k s_k \right) s_k^T
$$

(21b)

where for $\gamma_k := \langle \beta_k y_k, s_k \rangle / \|s_k\|^2$ we have defined

$$
\begin{align*}
    s_k &= w_k - x_k \\
y_k &= R w_k - R x_k \\
    y_k &= (1 - \vartheta_k) B_k s_k + \vartheta_k y_k
\end{align*}
$$

and $\vartheta := \begin{cases} 1, & \gamma_k \geq \vartheta \\ 1 - \frac{\text{sgn}(\gamma_k)}{\vartheta}, & \gamma_k < \vartheta \end{cases}$

(21c)

with the convention $\text{sgn} 0 = 1$. Letting $H_k := B_k^{-1}$ and using the Sherman–Morrison identity, the inverse of $B_k$ is given by

$$
H_{k+1} = H_k + \frac{1}{\langle y_k, s_k \rangle} \left( s_k - H_k y_k \right) \left( s_k^T H_k \right). 
$$

(21d)

Consequently, there is no need to compute and store the matrices $B_k$ and we can directly operate with their inverses $H_k$.

Theorem VI.8 (Superlinear convergence of the SuperMann scheme with Broyden’s directions): Suppose that $H$ is finite dimensional. Consider the sequence $(x_k)_{k \in \mathbb{N}}$ generated by the SuperMann scheme (see Algorithm 2), $(d_k)_{k \in \mathbb{N}}$ being selected with the modified Broyden’s scheme (21) for some $\vartheta \in (0,1)$.

Suppose that $(H_k)_{k \in \mathbb{N}}$ remains bounded, and that $R$ is calmly semidifferentiable and metrically subregular at the limit $x_\ast$ of $(x_k)_{k \in \mathbb{N}}$. Then, $(d_k)_{k \in \mathbb{N}}$ satisfies the Dennis–More condition (20). In particular, all the claims of Theorem VI.7 hold.

Proof: See Appendix B.

Remark VI.9: It follows from Theorem VI.1(iv) that the SuperMann scheme is globally convergent as long as $\|d_{k}\| \leq D \|Rx_k\|$ for some constant $D$. To enforce it, we may select a (large) constant $D > 0$ and as a possible choice truncate $d_k \leftarrow D \frac{\|d_{k}\|}{\|d_{k}\| + 1} d_k$ whenever $d_k$ does not satisfy (17).

Let us observe that in order to achieve superlinear convergence the SuperMann scheme does not require nonsingularity of the Jacobian at the solution. This is the standard requirement for asymptotic properties of quasi-Newton schemes, which is needed to show first that the method converges at least linearly.

Aragón Artacho et al. [24] generalizes this property invoking the concepts of (strong) metric (sub)regularity (see also [19] for an extensive review on these properties). However, if $R$ is strictly differentiable at $x_\ast$, then strong subregularity, regularity, and strong regularity are equivalent to injectivity, surjectivity, and invertibility of $JR(x_\ast)$, respectively, these conditions being all equivalent for mappings $H \rightarrow H$ with $H$ finite dimensional. In particular, contrary to the SuperMann scheme standard approaches require the solution $x_\ast$ at least to be isolated

Restarted (modified) Broyden’s Scheme: Broyden’s scheme requires storing and operating with $n \times n$ matrices, where $n$ is the dimension of the optimization variable, and is consequently feasible in practice only for small problems. Alternatively, one can restrict Broyden’s update rule (21d) to only the most recent pairs of vectors $(s_i, y_i)$. As detailed in Algorithm 3, this can be done by keeping track of the last vectors $s_i$ and some auxiliary vectors $\tilde{s}_i = \frac{y_i - H_i y_i}{\langle s_i, H_i y_i \rangle}$. These are stored in some buffers $S$ and $\tilde{S}$, which are initially empty and can contain up to $m$ vectors. The memory $m$ is a small integer typically between 3 and 20; when the memory is full, the buffers are emptied and Broyden’s scheme is restarted. The choice of a restarted rather than a limited-memory variant obviates the need of a nested for-loop to account for Powell’s modification.

D. Parameters Selection in SuperMann

As shown in Theorem VI.4, the SuperMann scheme makes sense as long as either $c_0 > 0$ or $c_1 > 0$; indeed, safeguard $K_2$-steps are only needed for globalization, while it is blind $K_0$- and educated $K_1$-steps that exploit the quality of the directions $d_k$. Evidently, $K_1$-updates are more reliable than $K_0$-updates in that they take into account the residual of the candidate next point. As such, it is advisable to select $c_1$ close to 1 and use small values of $c_0$ if more conservatism and robustness are desired. To further favor $K_1$-updates, the parameter $q$ used for updating the safeguard $r_{\text{safe}}$, at step 5(a) may be also chosen very close to 1.

As to safeguard $K_2$-steps, a small value of $\sigma$ makes condition (15) easier to satisfy and results in fewer backtracking; the averaging factor $\lambda$ may be chosen equal to 1 whenever possible, i.e., if $\alpha \leq 1$ (which is the typical case when, e.g., $T$ comes from splitting schemes in convex optimization), or any close value otherwise. In the simulations of Section VII, we used $c_0 = c_1 = q = 0.99$, $\sigma = 0.1$, $\lambda = 1$, and $\beta = \frac{1}{2}$. For a matter of scaling, we multiplied the summable terms $q^k / \|R x^k\|$ in updating the parameter $r_{\text{safe}}$ at step 5(a). The directions were computed according to the restarted modified Broyden’s scheme (see Algorithm 3) with memory $m = 20$ and $\vartheta = 0.2$; we applied the truncation rule as in Remark VI.9 with $D = 10^4$. We also imposed a maximum of eight backtracking after which a nominal fixed-point iteration would be executed.

E. Comparisons With Other Methods

1) Hybrid Global and Local Phase Algorithms: Blind $K_0$-updates in the SuperMann scheme are inspired from [17, Algorithm 1], and so is the notation $K_0 = \{k_0, k_1, \ldots\}$. Educated $K_1$- and safeguard $K_2$-updates instead play the role of inner- and outer-phases in the general algorithmic framework described in [9, Section V-C] for finding a zero of a candidate merit function $\varphi$ (e.g., $\varphi(x) = \frac{1}{2} \|R x\|^2$ in our case). Differently from [9, Algorithm 5.16] where all previous inner-phase
iterations are discarded as soon as the required sufficient decrease is not met, the SuperMann scheme allows for an alternation of phases that eventually stabilizes on the fast local one, provided the solution is sufficiently regular. Our scheme is more in the flavor of [9, Algorithm 5.19], although with less conservative requirements for triggering inner $K_1$-updates ($\varphi(x_{k+1})$ is here compared with $\varphi(x_k)$, whereas in the cited scheme with the smallest past past value).

2) Inexact Newton Methods for Monotone Equations:

The GKM updates are closely related to the extra-gradient steps described in [25, Algorithm 2.1]. This work introduces an inexact Newton algorithm for solving systems of continuous monotone equations $Rx = 0$, where id $- R$ need not be nonexpansive. At a given point $x$, first a direction $d$ is computed as (possibly approximate) solution of $Gd = -Rx$, where $G$ is some positive definite matrix; then, an intermediate point $w = x + \tau d$ is retrieved with a linesearch on $\tau$ that ensures the condition

$$
\|Rw\|^2 - \langle Rw, x - Tw \rangle \leq -\sigma \|d\|^2
$$

for some $\sigma > 0$; here, we defined $T := \text{id} - R$ to highlight the symmetry with (10). Finally, the new iterate is given by $x^{+} = \Pi_{H_w} x$, where

$$
H_w := \{ z \in \mathcal{H} \mid \|Rw\|^2 - \langle Rz, w - Tw \rangle \geq 0 \}.
$$

Letting $C_w$ be the half-space as in Proposition V.2, so that $x^+_{GKM} = \Pi_{C_w} x$ (for simplicity we set $\lambda = 1$), for the half-spaces (23) it holds that

$$
\text{zer } R \subseteq C_w \subseteq H_w
$$

the last inclusion holding as equality iff $Rw = 0$. This means that in the GKM scheme, the same $w$ yields an iterate $x^+_{GKM}$ which is closer to any $z \in \text{zer } R$ with respect to $x^+$ (cf. Fig. 4). Notice further that the hyperplanes delimiting the two half-spaces are parallel, with $\text{bdry } C_w$ passing by $Tw$ (or $T_{1/2\alpha}w$ for generic $\alpha$’s) and $\text{bdry } H_w$ by $w$.

The requirement of positive definiteness of matrix $G$ in defining the update direction $d$ is due to the fact that [25] addresses a broader class of monotone operators; we instead exploited at full the nonexpansiveness of id $- R$ and as a result have complete freedom in selecting $d$ [see Fig. 4(a)] and better projections [see Fig. 4(c)].

Moreover, it can be easily verified that the proposed extra-gradient step does not extend the classical KM iteration unless $T$ has a very peculiar property, namely that $Rx = \frac{(RTx,Rx)}{\|Rx\|^2}RTx$ for every $x$. (In particular, for such a $T$ necessarily $\|Rx\| = \|RTx\|$ for all $x$, and consequently there cannot exist $\alpha \in (0, 1)$ for which $T$ is $\alpha$-averaged).

3) Line-Search for KM: Giselsson et al. [13] propose an acceleration of the classical KM scheme for finding a fixed point of an $\alpha$-averaged operator $T$ based on a linesearch on the relaxation parameter. Namely, instead of the nominal update $\bar{x} = T_\lambda x$ with $\lambda \in [0, \gamma)$ as in (9), values $\lambda' > \gamma$, are first tested and the update $x^{+} = T_{\lambda'}x$ is accepted as long as $\|Rx^{+}\| \leq c_1 \|\bar{x}\|$ holds for some constant $c_1 \in (0, 1)$.

In the setting of the SuperMann scheme, this corresponds to selecting $d_k = -Rx_k$, discarding blind updates (i.e., setting $c_0 = 0$, foretracking educated updates, and using plain KM iterations as safeguard steps. Convergence can be enhanced and the method is attractive when $T = S_1 \circ S_2$ is the composition of an affine mapping $S_1$ and a cheap operator $S_2$, in which case the linesearch is inexpensive. However, though preserving the same theoretical convergence guarantees of KM (hence of the SuperMann scheme), it does not improve its best case local linear rate.

Although other choices $d_k$ may also be considered, however, fast directions such as Newton-type ones would be discarded and replaced by nominal KM updates every time the candidate point $x_k + d_k$ does not meet some requirements. Avoiding this take-it-or-leave-it behavior is exactly the primary goal of GKM iterations, so that candidate good directions are never discarded.

4) Smooth Optimization With Envelope Functions: For solving nonsmooth minimization problems in composite form, Patrinos et al. [26], [27] introduced forward–backward envelope and Douglas–Rachford envelope functions. The original nonsmooth problem is recast into the minimization of continuous (possibly continuously differentiable) real-valued exact penalty functions closely related to FBS and DRS, named envelopes due...
to their kinship with the Moreau envelope and the proximal point algorithm. This paved the way for the employment of fast methods for smooth unconstrained minimization problems [26]–[28], or for globalizing convergence of fast methods for solving non-linear equations [29], [30]. Although they have the advantage of being suited for nonconvex problems, however, their employment is limited to composite operators as described earlier and they cannot handle, for instance, saddle-point convex-concave optimization problems typically arising from primal-dual splittings such as Vĩ–Condat (VC) [31]. The SuperMann scheme instead offers a unifying framework that is based uniquely on evaluations of the nonexpansive mapping \( T \), regardless of their structure.

VII. SIMULATIONS—LINEAR OPTIMAL CONTROL

For matrices \( A_t \) and \( B_t \) of suitable size, \( t = 0,\ldots,N-1 \), consider a state-input dynamical system

\[
x_{t+1} = A_t x_t + B_t u_t, \quad t = 0,\ldots,N-1
\]

where the \( x_0 \in \mathbb{R}^{n_x} \) is given, and the next states \( x_t \in \mathbb{R}^{n_x} \) are determined by the user-defined inputs \( u_t \in \mathbb{R}^{n_u}, \tau = 0,\ldots,t-1 \). States \( x = (x_1,\ldots,x_N) \) can be expressed in terms of the inputs \( u = (u_0,\ldots,u_{N-1}) \) through a linear operator \( L \in \mathbb{R}^{N x_n \times N_u} \) as \( x = Lu + b \) for some constant \( b \in \mathbb{R}^{n_x} \).

The goal is to choose inputs that minimize a cost

\[
\ell(u, x) = \sum_{t=0}^{N-1} \ell_t(u_t, x_t) + \ell_N(x_N)
\]

subject to some constraints

\[
x_{t+1} \in X_{t+1}, \quad u_t \in U_t, \quad t = 0,\ldots,N-1
\]

A. VC Splitting

The constraint sets in (24c) are typically simple and easy to project onto (boxes, Euclidean balls, etc.). However, while simple input constraints can be easily handled due to the coupling enforced by the dynamics (24a) expressing \( X_{t+1} \) in terms of the optimization variable \( u \) results in much more complicated sets (polyhedra, ellipsoids, etc.). To avoid this complication, we make use of the extremely versatile algorithm that VC three-term splitting offers [31, Algorithm 3.1]. In its general form, the algorithm addresses problems of the form

\[
\min_{x \in \mathbb{R}^n} f(x) + g(x) + h(Lx)
\]

where \( f : \mathbb{R}^n \to \mathbb{R} \) is convex with \( L_f \)-Lipschitz continuous gradient, \( g : \mathbb{R}^n \to \mathbb{R} \) and \( h : \mathbb{R}^m \to \mathbb{R} \) are convex, and \( L \in \mathbb{R}^{n \times m} \), by iterating the following steps:

\[
\begin{align*}
\{ x^+ \} &= \text{prox}_{\sigma g}(x - \tau (\nabla f(x) + L y)) \\
y^+ &= \text{prox}_{\sigma h}(y + \sigma L (2x^+ - x)).
\end{align*}
\]

Here, \( 0 < \tau < \frac{\sigma}{\|L\|} \) and \( 0 < \sigma < \frac{\|L\|}{1 - \|L\|^2} \) are stepizes, and \( y \in \mathbb{R}^m \) is a Lagrange multiplier. VC splitting is a primal-dual method that generalizes FBS by allowing an extra non-smooth term \( h \) and a linear operator \( L \) (by neglecting \( h \) and \( L \) one recovers the proximal gradient iterations of FBS).

The optimal control problem (24) can be cast into VC splitting form (25) by letting \( f(u) = \ell(u, Lu), g = \delta u, \) and \( h = \delta C \cdot + b \) where \( \mathcal{U} = \mathcal{U}_0 \times \cdots \times \mathcal{U}_{N-1} \) and \( X = X_1 \times \cdots \times X_N \) (in particular, \( n = Nn_u \) and \( m = Nn_x \)). Then, \( \text{prox}_{\sigma g} = \Pi_\mathcal{U} \) and \( \text{prox}_{\sigma h} = \Pi_\mathcal{X} \) (by neglecting \( \Pi_\mathcal{X} \) of suitable size, \( u = (u_0,\ldots,u_{N-1}) \) through a linear operator \( L \in \mathbb{R}^{N x_n \times N_u} \) as \( x = Lu + b \) for some constant \( b \in \mathbb{R}^{n_x} \). The goal is to choose inputs that minimize a cost

\[
\ell(u, x) = \sum_{t=0}^{N-1} \ell_t(u_t, x_t) + \ell_N(x_N)
\]

subject to some constraints

\[
x_{t+1} \in X_{t+1}, \quad u_t \in U_t, \quad t = 0,\ldots,N-1
\]

\[\text{B. Oscillating Masses Experiment}\]

We tried this approach on the benchmark problem of controlling a chain of oscillating masses connected by springs and with both ends attached to walls. The chain is composed of \( 2K \) bodies of unit mass subject to a viscous friction of 0.1, the springs have elastic constant 100, and no damping, and the system is controlled through \( K \) actuators, each being a force acting on a pair of masses, as depicted in Fig. 5. Therefore, \( n_u = 4K \) (the states are the displacement from the rest position and velocity of each mass) and \( n_x = K \). The inputs are constrained in \([−2, 2]\), while the position and velocity of each mass is constrained in \([−5, 5]\).

The continuous-time system was discretized with a sampling time \( T_s = 0.1s \). We considered quadratic stage costs \( \frac{1}{2} x^T Q x \) for the states and \( \frac{1}{2} u^T u \) for the inputs, where \( Q \) is diagonal positive definite with random diagonal entries, and generated a random (feasible) initial state \( x_0 \). Notice that a QP reformulation would require the computation of the full cost matrix, differently from the splitting approach where only the small dynamics matrices \( A \) and \( B \) are needed, as \( L \) and \( L^T \) can be abstract operators.

We simulated different scenarios for all combinations of \( K \in \{ 8, 16 \} \) and \( N \in \{ 10, 20, 30, 40, 50 \} \). We compared VC splitting with its “super” enhancement (SuperVC); parameters were set as detailed in Section VI-D. Fig. 6 shows a comparison of the convergence rates for one problem instance, whereas Table I offers an overview of the whole experiment: SuperVC is roughly 13 times faster on average and 21 times better in worst

\[\delta_C \text{ denotes the indicator function of the nonempty closed convex set } C, \text{ namely } \delta_C(x) = 0 \text{ if } x \in C \text{ and } \delta_C(x) = \infty \text{ otherwise.}\]
case performance than VC algorithm in reaching the termination criterion \(||Rx^k|| < 10^{-4}||Rx^0||\).

**TABLE I**

COMPARISON BETWEEN VC ALGORITHM AND ITS “SUPER” ENHANCEMENT (SUPERVC) IN SOLVING THE OSCILLATING MASSES PROBLEM WITH \(||Rx^k|| < 10^{-4}||Rx^0||\) AS TERMINATION CRITERION

| K = 8   | \(N = 10\) | \(N = 20\) | \(N = 30\) | \(N = 40\) | \(N = 50\) |
|---------|------------|------------|------------|------------|------------|
| VC      | avg max   | avg max   | avg max   | avg max   | avg max   |
| 19.0 337.1 | 10.0 174.4 | 10.0 193.6 | 21.0 136.5 | 50.0 109.8 | 2.0 6.6   |
| SVC     | avg max   | avg max   | avg max   | avg max   | avg max   |
| 1.0 5.5 | 1.0 4.3   | 2.0 19.3  | 2.0 10.9  | 2.0 6.6   |

| K = 16  | \(N = 10\) | \(N = 20\) | \(N = 30\) | \(N = 40\) | \(N = 50\) |
|---------|------------|------------|------------|------------|------------|
| VC      | avg max   | avg max   | avg max   | avg max   | avg max   |
| 62.0 400+ | 30.0 344.9 | 30.0 400+ | 65.0 400+ | 29.0 318.6 |
| SVC     | avg max   | avg max   | avg max   | avg max   | avg max   |
| 3.0 39.5 | 3.0 11.6  | 3.0 46.6  | 3.0 30.4  | 2.6 26.1  |

Average and worst performances among 25 simulations with randomly generated starting point \(x_0\) for each combination of \(K \in \{8, 16\}\) and \(N \in \{10, 20, 30, 40, 50\}\). The tables compare the number of calls to the operators \(L\) and \(L^*\), which are the expensive operations (the rest are projections on boxes). In four problems VC exceeded \(4 \times 10^5\) many calls (corresponding to \(10^5\) iterations) and was stopped prematurely.

**VIII. CONCLUSION**

We proposed the SuperMann scheme (see Algorithm 2), a novel algorithm for finding fixed points of an NE operator \(T\) that generalizes and greatly improves the classical KM scheme, enjoying the same favorable properties: global convergence with worst case sublinear rate, cheap iterations based solely on evaluations of \(T\), and easy codability. The SuperMann scheme is an extremely versatile algorithm, its flexibility being twofold: on one hand it works for any NE operator \(T\) by requiring only the oracle \(x \mapsto TXx\); on the other hand it allows for the integration of any fast local method for solving nonlinear equations, leaving much freedom for trading off cheap iterations or faster convergence. The remarkable performance of the method is supported both in practice with promising simulations and in theory where the employment of quasi-Newton directions is shown to yield asymptotic superlinear convergence rates provided a condition analogous to the famous result by Dennis and Moré is satisfied. Most importantly, superlinear convergence does not require nonsingularity of the Jacobian of the residual at the solution but merely metric subregularity, and as such can be achieved even when the solution is not isolated.

We encourage the employment of the SuperMann scheme to improve and robustify convex splitting algorithms; in particular, we strongly believe that its integration in generic solvers which are based on fixed-point iterations of NE operators such as splitting conic solver (SCS) [32] would be extremely beneficial.

**APPENDIX A PROOFS OF SECTION IV**

**Proof of Theorem IV.1**

1) **Theorem IV.1(i):** We start observing that because of (6) and the triangular inequality, for all \(k \in K_0 \cup K_1\) we have

\[ ||x_{k+1} - z|| \leq ||x_k - z|| + D||Rx_k|| \quad \forall z \in \text{fix} T \]  

(27)

and since \(R\) is \(2\alpha\)-Lipschitz continuous we also have that

\[ ||Rx_{k+1}|| \leq (1 + 2\alpha D)||Rx_k||. \]  

(28)

By combining [15, Proposition 3.2(i)] with (4) and (27), it follows that in order to prove quasi-Fejér monotonicity it suffices to show that the sequence \(\{||Rx_k||\}_{k \in K_0 \cup K_1}\) is summable. Let \(K_0\) and \(K_1\) be indexed as in (5). Since \(\eta_k\) is kept constant whenever \(k \not\in K_0\)

\[ \eta_k = ||Rx_{k-1}|| = c_0 \eta_{k-1} \cdots \leq c_{k-1} \eta_1 = c_{k-1} \eta_0. \]  

(29)

In particular, \(\{||Rx_k||\}_{k \in K_1}\) is summable (regardless of whether \(K_0\) is finite or not).

As for \(K_1\), the safeguard parameter \(r_{safe}\) ensures that

\[ ||Rx_{k+1}|| \leq ||Rx_{k+1}|| + q^{k+1} \leq c_1 ||Rx_{k+1}|| + q^{k+1} \leq c_1 ||Rx_{k+1}|| + q^{k+1}. \]

By iterating the inequality, for any \(\rho \in (0, 1)\) such that \(\rho > \max\{c_1, q\}\), we have

\[ ||Rx_{k+1}|| \leq \rho^{k+1}||Rx_k|| + \sum_{i=1}^{k} c_1^{i-1} \rho^{k-i} \leq C \rho^k. \]  

(30)

where \(C := \frac{1}{\rho}||Rx_k|| + \sum_{i \in \mathbb{N}} (c_1 \rho)^i < \infty\). In particular, also \(\{||Rx_k||\}_{k \in K_1}\) is summable.

2) **Theorem IV.1(ii):** Due to quasi-Fejér monotonicity, for all \(z \in \text{fix} T\) there exists \((e_k(z))_{k \in \mathbb{N}} \in \ell^1_+\) such that

\[ ||x_{k+1} - z||^2 \leq ||x_k - z||^2 + e_k(z). \]

By combining this with (4) and telescoping the inequalities, we obtain that for all \(z \in \text{fix} T\)

\[ ||x_0 - z||^2 \geq \sum_{k \in K_2} ||Rx_k||^2 - \sum_{k \in K_0 \cup K_1} e_k(z). \]  

(31)

Since the sequence \((e_k(z))_{k \in K_0 \cup K_1}\) is summable, then so is \((||Rx_k||^2)_{k \in K_2}\). In turn, since \((||Rx_k||)_{k \in K_0 \cup K_1}\) is also
Theorem IV.1(iii): Follows by combining IV.1(ii) with Theorem III.3.

4) Theorem IV.1(iv): Trivially follows from the already proven point Theorem IV.1(ii), together with the observation that since \( \eta_k \) is kept constant whenever \( k \notin K_0 \), the condition \( \|Rx_k\| \leq c_0\eta_k \) will be satisfied infinitely often if \( c_0 > 0 \). ■

We now state two lemmas which will be needed in the proof of Theorem IV.3.

Lemma VIII.1 (Asymptotic properties of \( K_0 \) and \( K_1 \)): Suppose the hypotheses of Theorem IV.1 hold and let \( (x_k)_{k \in \mathbb{N}} \) be the sequence generated by Algorithm 1. Then,

i) \( \|Rx_k\|_{k \in K_0} \) is Q-linearly convergent;

ii) \( \|Rx_k\|_{k \in K_1} \) is R-linearly convergent;

iii) if \( c_0 > 0 \), then for some \( \rho \in (0, 1) \) and \( \beta \in \mathbb{R} \)

\[ \ell_0(k) \geq \rho \ell_1(k) - \beta \quad \forall k \in \mathbb{N} \]

where \( \ell_j(k) := \# \{k' \in K_j \mid k' \leq k \} \), \( j = 0, 1, 2 \), is the number of times \( K_j \) was visited up to iteration \( k \).

Proof:

1) Lemma VIII.1(i) and (ii): Already shown in (29) and (30).

2) Lemma VIII.1(iii): If \( c_1 = 0 \), then \( K_1 = \emptyset \) and the claim trivially holds with \( \rho = 1 \) and \( \beta = 0 \). Otherwise, from (30) and due to the definition of \( \ell_1(k) \) there exist \( C > 0 \) and \( \rho \in (0, 1) \) such that

\[ \|Rx_k\| \leq C \rho^{\ell_1(k)} \quad \forall k \in K_1. \]

If \( k \in K_1 \), then \( \|Rx_k\| \) did not pass the test at step 2, therefore

\[ C \rho^{\ell_1(k)} \geq \|Rx_k\| \geq \eta_k^{(29)} = \|Rx_0\|c_0^{\ell_0(k)}. \]

The proof now follows by simply taking the logarithm on the outer inequality. ■

Lemma VIII.2: Let \( (u_k)_{k \in \mathbb{N}} \subset [0, +\infty) \) be a sequence, and let \( K_1, K_2 \subseteq \mathbb{N} \) be such that \( \mathbb{N} = K_1 \cup K_2 \). Let \( K_1 \) be indexed as \( K_1 = \{k'_0, k'_1, \ldots\} \), and suppose that there exist \( a, b > 0 \) and \( \rho \in (0, 1) \) such that

\[
\begin{align*}
    u_{k+1} &\leq au_k \quad \text{for all } k \in \mathbb{N} \\
    u_{k+1} &\leq bp^k \quad \text{for all } k' \in K_1 \\
    u_{k+1} &\leq \rho u_k \quad \text{for all } k \in K_2
\end{align*}
\]

Then, there exists \( \sigma \in (0, 1) \) such that \( u_k \leq ab^k \).

Proof: To exclude trivialities we assume that \( K_1 \) and \( K_2 \) are both infinite. To arrive to a contradiction, for all \( \sigma \in (0, 1) \), let \( k = k(\sigma) \) be the minimum such that \( u_k > ab^k \). Let \( \sigma \geq \rho \) be fixed. If \( k - 1 \in K_2 \), then

\[ \rho u_{k-1} \geq u_k > ab^k \geq ab\rho^{k-1} \]

and, therefore, \( u_{k-1} > ab^{k-1} \) which contradicts minimality of \( k \). It follows that necessarily \( k - 1 \in K_1 \), hence, \( k - 1 = k' \in K_1 \) for some \( \ell \in \mathbb{N} \). For all \( n \in \mathbb{N} \), let \( k'_{n+1} = k(\rho^{\ell_n}) - 1 \), i.e., the minimum such that \( u_{k'_{n+1}} > abp^\ell \). By combining with the property of \( K_1 \), we obtain

\[ abp^{\ell_n+1} < u_{k'_{n+1}} \leq au_{k'_{n}} \leq abp^\ell \]

and in particular \( \ell_n \leq \frac{k'_n}{n} \). This means that up to \( k = k'_n \) there are at most \( \frac{1}{n} \) elements in \( K_1 \), and consequently at least \( k - \frac{1}{n} \) in \( K_2 \). Therefore,

\[ b\rho^{\ell_n} < u_k \leq a^n\rho ^{b-k/n}u_0. \]

After taking the \( k \)th square root on the outer inequality we are left with

\[ (\frac{1}{n})^{1-\frac{1}{n}} < \frac{a^n}{b^n} \]

By letting \( n \to +\infty \), so that also \( k \to +\infty \), we arrive to the contradiction \( \rho \geq 1 \).

Proof of Theorem IV.3: Letting \( e_k := \text{dist}(x_k, \text{fix T}) \), because of (28) and (8) there exists \( B > 1 \) such that

\[ \|Rx_{k+1}\| \leq B\|Rx_k\| \quad \text{and} \quad e_{k+1} \leq Be_k \quad \forall k \in \mathbb{N}. \]

Suppose that \( R \) is metrically subregular at \( x_0 \) with radius \( \varepsilon > 0 \) and modulus \( \gamma > 0 \); since \( x_k \to x_* \), up to an index shifting without loss of generality we may assume that \( (x_k)_{k \in \mathbb{N}} \subset B(x_0; \varepsilon) \). Let \( z_k = \Pi_{B(x_0; \varepsilon)} x_k \), so that \( e_k = \|x_k - z_k\| \); by combining (4) and (8), we obtain that for all \( k \in K_2 \)

\[ e_{k+1}^2 \leq \|x_{k+1} - z_k\|^2 \leq \|x_k - z_k\|^2 - \sigma \|Rx_k\|^2 \leq \rho^2 e_k^2 \]

where \( \rho := \sqrt{1 - \eta/z} \in (0, 1) \). By possibly enlarging \( \rho \) we may assume \( \rho > \max\{c_0, c_1\} \).

If \( c_0 = 0 \), then \( K_0 = \emptyset \) and using Lemma VIII.1(ii) and (33) we may invoke Lemma VIII.2 to infer \( R \)-linear convergence of the sequence \( (e_k)_{k \in \mathbb{N}} \) and conclude the proof.

Therefore, let us suppose that \( c_0 > 0 \), so that by Theorem IV.1(iv) the set \( K_0 \) contains infinite many indices. We now show that there exists \( n \in \mathbb{N} \) such that every \( n \) consecutive indices at least one is in \( K_0 \). Let \( k \in K_0 \) be fixed and suppose that \( k + 1, \ldots, k + n + 1 \notin K_0 \).

1) If \( c_1 = 0 \), then \( K_1 = \emptyset \) and all such indices belong to \( K_2 \).

Then,

\[ \|Rx_{k+n+1}\| \leq 2\alpha e_{k+n+1} \leq 2\alpha \rho^n e_{k+1} \leq 2\alpha B\rho^n e_k \]

since \( n + 1 \notin K_0 \), then \( \|Rx_{k+n+1}\| \) failed the test at step 3 and therefore

\[ c_0 \|Rx_k\| = c_0\eta_{k+n+1} < \|Rx_{k+n+1}\| \leq 2\alpha B\rho^n \|Rx_k\| \]

which proves that \( n \) cannot be arbitrarily large.

2) If instead \( c_1 > 0 \), let \( n_1 \) be the number of indices among \( k + 1, \ldots, k + n \) that belong to \( K_1 \), and \( n_2 = n - n_1 \) those belonging to \( K_2 \). Then, from iteration \( k + 1 \) to \( k + n + 1 \) the distance from the fixed set has reduced \( n_2 \) times (at least) by a factor \( \rho \) and, due to (33), increased...
at most by a factor $B$ the remaining $n_1$ times

$$\|Rx_{k+n+1}\| \leq 2\alpha e_{k+n+1} \leq 2\alpha\rho^{n_2}B^{n_1}e_{k+1} \tag{8}$$

$$\leq 2\alpha\rho^{n_2}B^{n_1+1}e_{k} \leq 2\alpha\rho^n B^{n_1+1}\|Rx_k\|. \tag{33}$$

Again, since $k+n+1 \notin K_0$ we have $c_0\|Rx_k\| < 2\alpha\rho^n B^{n_1+1}\|Rx_k\|$, and therefore

$$n_1 > \sum_{i=0}^{\infty} \frac{e_{k+i}}{\ln \rho_B} + 1 + \frac{\ln \rho}{\ln \rho_B} n_2.$$ 

In particular, for large $n$ the number $n_1$ of indices in $K_1$ grows proportionally with respect to $n$, and from Lemma VIII.1(iii) we conclude once again that $n$ cannot be arbitrarily large (since the number of visits to $K_0$ does not change from $k+1$ to $k+n$).

So far we proved that there exists $n \in \mathbb{N}$ such that every $n$ indices at least one belongs to $K_0$. In particular, indexing $K_0 = \{k_0, k_1, \ldots\}$ we have that $k_1 \leq n\epsilon$, hence, for all $k_t \in K_0$

$$\|Rx_{k_t}\| \leq c_{\ell 0}\|Rx_0\| \leq (c_{\ell 0}^{\ell 1})^{k_t}\|Rx_0\|. \tag{35}$$

Moreover, any $k \in \mathbb{N}$ is at most $n-1$ indices away from the nearest previous index $k_t \in K_0$; combined with (35) and invoking (33) we obtain

$$\|Rx_{k}\| \leq B^{n-1}\|Rx_0\| (c_{\ell 0}^{\ell 1})^{k} \leq B^{n-1}\|Rx_0\| (c_{\ell 0}^{\ell 1})^{k},$$

proving the sought $R$-linear convergence of $(\|Rx_k\|)_{k \in \mathbb{N}}$. It follows that for some $b > 0$ and $r \in (0, 1)$, we have $\|Rx_k\| \leq br^k$ for all $k \in \mathbb{N}$; then

$$\|x_k - x\| \leq \sum_{j=0}^{\infty} \|x_{j+1} - x_j\| \leq D \sum_{j=k}^{\infty} \|Rx_j\| \leq \frac{bD}{1 - r}\|x_k - x\|,$$

where in the second inequality we used the bound (6), which also holds for $k \in K_2$ (up to possibly enlarging $D$) due to the fact that for $k \in K_2$ under metric subregularity we have

$$\|x_{k+1} - x_k\| \leq \|x_{k+1} - z_k\| + \|x_k - z_k\| \leq 2\epsilon_{k} \leq \gamma\|Rx_k\|. \tag{8}$$

This shows that $(x_k)_{k \in \mathbb{N}}$ is $R$-linearly convergent too. 

**Appendix B**

**Proofs of Section VI**

**Proof of Theorem VI.1:** Because of Theorem V.4 we know that for any direction $d_k$, a feasible stepsize $\tau_k$ complying with the requirements of step 5(b) will eventually be found, lower bounded as in Theorem VI.1(ii) due to Theorem V.4 and Assumption 2. In particular, the scheme is well defined. Moreover, from Proposition V.2(ii) we have that there exists a constant $\sigma > 0$ such that

$$\|x_{k+1} - z\|^2 \leq \|x_k - z\|^2 + \sigma\|Rx_k\|^2$$

for all $k \in K_2$ and $z \in \text{fix}T$. It follows that the SuperMann scheme is a special case of Algorithm 1 and the proof entirely follows from Theorems IV.1 and IV.3.

**Proof of Theorem VI.4:**

1) **Theorem VI.4(i):** Let $w_k := x_k + d_k$. Superlinear convergence of $(d_k)_{k \in \mathbb{N}}$ then reads $\|Rw_k\| \rightarrow 0$. In particular, if $c_0 > 0$ then there exists $k \in \mathbb{N}$ such that $\|Rw_k\| \leq c_1\|Rx_k\|$ for all $k \geq k$, i.e., the point $w_k = x_k + d_k$ will always pass condition at step 5(a) resulting in

$$x_{k+1} = w_k = x_k + d_k$$

for all $k \geq k$. Similarly, if $c_0 > 0$ then $K_0$ is infinite as shown in Theorem VI.1(iii); moreover, for $\ell \in \mathbb{N}$

$$\|Rx_{k+1}\| = \|Rx_k\| = \|Rx_{k_1} + d_k\| \rightarrow 0$$

as $\ell \rightarrow \infty$ and, therefore, the ratio eventually is always smaller than $c_0$, resulting in $k_{l+1} \in K_0$ for $\ell$ large enough. Consequently, the sequence will eventually reduce to $x_{k+1} = x_k + d_k$.

2) **Theorem VI.4(ii) and (iii):** $Q$-superlinear convergence of the sequence $(Rx_k)_{k \in \mathbb{N}}$ follows from the fact that $x_{k+1} = x_k + d_k$ for $k \geq k$. In particular, $(\|Rx_k\|)_{k \in \mathbb{N}}$ is summable and there exists a sequence $(\delta_k)_{k \in \mathbb{N}} \rightarrow 0$ such that $\|Rx_{k+1}\| \leq \delta_k\|Rx_k\|$ for all $k$. If $\|d_k\| \leq D\|Rx_k\|$ for some $D > 0$, then

$$\sum_{k \geq k} \|x_{k+1} - x_k\| \leq D \sum_{k \geq k} \|Rx_k\| < \infty$$

which implies that $(x_k)_{k \in \mathbb{N}}$ is a Cauchy sequence and, hence, converges to a point, be it $x^*$. Moreover, by possibly enlarging $D$ so as to account for the iterates $k < k$, we have

$$\|x_k - x^*\| \leq \sum_{j \geq k} \|x_{j+1} - x_j\| \leq D \sum_{j \geq k} \|Rx_j\| \leq \sum_{j \geq k} \|Rx_j\| =: \Delta_k.$$ 

This shows that $(x_k)_{k \in \mathbb{N}}$ is $R$-superlinearly convergent, since $\Delta_{k-1} \rightarrow 0$.

3) **Theorem VI.4(iv):** Already shown in the proof of VI.4(i).

**Proof of Theorem VI.8:** Let $G_* := JRx_* \in \mathbb{R}^{n \times n}$ and let $\|\cdot\|$ denote the Euclidean norm. From [22, Lemma 2.2], we have that there exist a constant $L$ and a neighborhood $U_{x_*}$ of $x_*$ such that

$$\frac{\|y_k - G_* s_k\|}{\|s_k\|} = \frac{\|Rw_k - Rx_k - G_* (w_k - x_k)\|}{\|w_k - x_k\|} \leq L \max\{\|x_k - x_*\|, \|w_k - x_*\|\}.$$ 

Because of (17), the fact that $\tau_k \leq 1$, and the triangular inequality we have $\|w_k - x_*\| \leq \|x_k - x_*\| + D\|Rx_k\|$ and consequently

$$\sum_{k \in \mathbb{N}} \frac{\|y_k - G_* s_k\|}{\|s_k\|} \leq L \sum_{k \in \mathbb{N}} \left(\|x_k - x_*\| + D\|Rx_k\|\right) < \infty$$

where the last inequality follows from Theorem VI.1(i).

Let $E_k := B_{x_*} - G_*$ and let $\|\cdot\|_F$ denote the Frobenius norm. With a simple modification of the proofs of [22, Th. 4.1] and [24, Lemma 4.4] that takes into account the scalar $\delta_k \in [\delta, 2 - \delta]$
we obtain
\[ \|E_{k+1}\|_F \leq \|E_k\|_F - \frac{\tilde{\vartheta}^2}{2\|E_k\|_F^2} \|y_k - G_{s\vartheta} s_k\|_F^2 \]
\[ \leq \|E_k\|_F - \vartheta \left(2 - \tilde{\vartheta}\right) \|y_k - G_{s\vartheta} s_k\|_F^2 \]
\[ + \left(2 - \tilde{\vartheta}\right) \|y_k - G_{s\vartheta} s_k\|_F \].

The last term on the right-hand side, be it \( \sigma_k \), is summable and, therefore, the sequence \((E_k)_{k\in\mathbb{N}}\) is bounded. Let \( E := \sup\{\|E_k\|_F \}_{k\in\mathbb{N}} \), then
\[ \|E_{k+1}\|_F - \|E_k\|_F \leq \sigma_k - \frac{\vartheta^2}{2\|E_k\|_F} \left(\|y_k - G_{s\vartheta} s_k\|_F^2 \right)^2 \].

Telescoping the above-mentioned inequality, summability of \( \sigma_k \) ensures that \( \|y_k - G_{s\vartheta} s_k\|_F \leq \sigma_k \), proving in particular the claimed Dennis–Moré condition (20).

### References

[1] N. Parikh and S. Boyd, “Proximal algorithms,” Found. Trends Optim., vol. 1, no. 3, pp. 127–239, Jan. 2014.
[2] G. Stathopoulos, H. Shukla, A. Szucs, Y. Yu, and C. N. Jones, “Operator splitting methods in control,” Found. Trends Syst. Control, vol. 3, no. 3, pp. 249–362, 2016.
[3] L. Combettes and J.-C. Pesquet, Proximal Splitting Methods in Signal Processing, New York, NY, USA: Springer, 2011, pp. 118–212.
[4] W. Ben-Ameur, P. Bianchi, and J. Jakowicz, “Robust distributed consensus using total variation,” IEEE Trans. Autom. Control, vol. 61, no. 6, pp. 1550–1564, Jun. 2016.
[5] F. Iutzeler, P. Bianchi, P. Ciblat, and W. Hachem, “Explicit convergence rate of a distributed alternating direction method of multipliers,” IEEE Trans. Autom. Control, vol. 61, no. 4, pp. 892–904, Apr. 2016.
[6] P. L. Combettes and B. C. Vu, “Variable metric forward-backward splitting with applications to monotone inclusions in duality,” Optimization, vol. 63, no. 9, pp. 1289–1318, 2014.
[7] Y. Zheng, G. Fantuzzi, A. Papachristodoulou, P. Goulart, and A. Wynn, “Fast ADMM for semidefinite programs with chordal sparsity,” in Proc. Amer. Control Conf., May 2017, pp. 3335–3340.
[8] F. Facchinei and J.-S. Pang, Finite-Dimensional Variational Inequalities and Complementarity Problems, vol. 2, New York, NY, USA: Springer, 2003.
[9] A. F. Izmailov and M. V. Solodov, Newton-Type Methods for Optimization and Variational Problems, New York, NY, USA: Springer, 2014.
[10] P. L. Combettes, “A hybrid method for nonlinear equations,” in Numerical Methods for Nonlinear Algebraic Equations, New York, NY, USA: Gordon & Breach, 1970, pp. 87–144.
[11] M. A. Krasnosel’ski, “Two remarks on the method of successive approximations,” Uspekhi Matematicheskikh Nauk, vol. 10, no. 1, pp. 123–127, 1955.
[12] W. R. Mann, “Mean value methods in iteration,” Proc. Amer. Math. Soc., vol. 4, no. 3, pp. 506–510, 1953.
[13] P. Giselsson, M. Fält, and S. Boyd, “Line search for averaged operator iteration,” in Proc. IEEE 55th Conf. Decis. Control, Dec. 2016, pp. 1015–1022.
[14] H. H. Bauschke and P. L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, New York, NY, USA: Springer, 2011.
[15] P. L. Combettes, “Quasi-fejérian analysis of some optimization algorithms,” Stud. Comput. Math., vol. 8, pp. 115–152, 2001.
[16] J. Ermol’ev and A. D. Tuniev, “Random Fejér and quasi-Fejér sequences,” in Selected Translations in Mathematical Statistics and Probability, vol. 13, Providence, RI, USA: Amer. Math. Soc., 1973, 143–148.
[17] X. Chen and M. Fukushima, “Proximal quasi-Newton methods for nondifferentiable convex optimization,” Math. Program., vol. 85, no. 2, pp. 313–334, 1999.
[18] H. H. Bauschke, D. Noll, and H. M. Phan, “Linear and strong convergence of algorithms involving averaged nonexpansive operators,” J. Math. Anal. Appl., vol. 421, no. 1, pp. 1–20, 2015.
[19] A. L. Dontchev and R. T. Rockafellar, “Regularity and conditioning of solution mappings in variational analysis,” Set-Valued Anal., vol. 12, no. 1/2, pp. 79–109, 2004.
[20] R. T. Rockafellar and R. J.-B. Wets, Variational Analysis, vol. 317, New York, NY, USA: Springer, 2009.
[21] S. M. Robinson, Some Continuity Properties of Polyhedral Multifunctions, Berlin, Germany: Springer, 1981, pp. 206–214.
[22] C.-M. Ip and J. Kyparisis, “Local convergence of quasi-Newton methods for B-differentiable equations,” Math. Program., vol. 56, no. 1–3, pp. 71–89, 1992.
[23] J.-S. Pang, “Newton’s method for B-differentiable equations,” Math. Oper. Res., vol. 15, no. 2, pp. 311–341, 1990.
[24] F. J. Aragón Artacho, A. Belyakov, A. L. Dontchev, and M. Lópe, “Local convergence of quasi-Newton methods under metric regularity,” Comput. Optim. Appl., vol. 58, no. 1, pp. 225–247, 2014.
[25] M. V. Solodov and B. F. Svaiter, “A globally convergent inexact Newton method for systems of monotone equations,” in Reformulation: Nonsmooth, Piecewise Smooth, Semismooth and Smoothing Methods, New York, NY, USA: Springer, 1999, pp. 355–369.
[26] P. Patrinos and A. Bemporad, “Proximal Newton methods for convex composite optimization,” in Proc. IEEE Conf. Decis. Control, 2013, pp. 2358–2363.
[27] P. Patrinos, L. Stella, and A. Bemporad, “Douglas-Rachford splitting: Complexity estimates and accelerated variants,” in Proc. 53rd IEEE Conf. Decis. Control, Dec. 2014, pp. 4234–4239.
[28] L. Stella, A. Themelis, and P. Patrinos, “Forward-backward quasi-Newton methods for nonsmooth optimization problems,” Comput. Optim. Appl., vol. 67, no. 3, pp. 443–487, Jul. 2017.
[29] A. Themelis, L. Stella, and P. Patrinos, “Forward-backward envelope for the sum of two nonconvex functions: Further properties and nonmonotone linesearch algorithms,” SIAM J. Optim., vol. 28, no. 3, pp. 2274–2303, 2018.
[30] L. Stella, A. Themelis, P. Sopasakis, and P. Patrinos, “A simple and efficient algorithm for nonlinear model predictive control,” in Proc. IEEE 56th Annu. Conf. Decis. Control, Dec. 2017, pp. 1939–1944.
[31] L. Condat, “A primal-dual splitting method for convex optimization involving Lipschitzian, proximable and linear composite terms,” J. Optim. Theory Appl., vol. 158, no. 2, pp. 460–479, 2013.
[32] B. O’Donoghue, E. Chu, N. Parikh, and S. Boyd, “Conic optimization via operator splitting and homogeneous self-dual embedding,” J. Optim. Theory Appl., vol. 169, pp. 1042–1068, 2016.