Two dimensional Sen connections and quasi-local energy-momentum

L. B. Szabados
Research Institute for Particle and Nuclear Physics
H–1525 Budapest 114, P.O.Box 49, Hungary
E-mail: lbszab@rmki.kfki.hu

Abstract
The recently constructed two dimensional Sen connection is applied in the problem of quasi-local energy-momentum in general relativity. First it is shown that, because of one of the two 2 dimensional Sen–Witten identities, Penrose’s quasi-local charge integral can be expressed as a Nester–Witten integral. Then, to find the appropriate spinor propagation laws to the Nester–Witten integral, all the possible first order linear differential operators that can be constructed only from the irreducible chiral parts of the Sen operator alone are determined and examined. It is only the holomorphy or anti-holomorphy operator that can define acceptable propagation laws. The 2 dimensional Sen connection thus naturally defines a quasi-local energy-momentum, which is precisely that of Dougan and Mason. Then provided the dominant energy condition holds and the 2-sphere is convex we show that the following statements are equivalent: i. the quasi-local mass (energy-momentum) associated with a 2-sphere $ is zero; ii. the Cauchy development $D(\Sigma)$ is a pp-wave geometry with pure radiation ($D(\Sigma)$ is flat), where $\Sigma$ is a spacelike hypersurface with $\partial \Sigma = \$$; iii. there exist a Sen–constant spinor field (two spinor fields) on $. Thus the pp-wave Cauchy developments can be characterized by the geometry of a two rather than a three dimensional submanifold.

Introduction
This paper is the second part of a four part series on the theory and applications of the two dimensional Sen connection in general relativity. In the first of this series [1] a covariant spinor formalism was developed which is the two dimensional version of the usual (three dimensional) Sen connection. As the first application of this formalism quasi-local energy-momentum expressions based on the spinorial Nester–Witten 2-form will be examined [2-6]. The different constructions correspond to different additional spinor propagation laws within the 2-surface, and the solutions of these spinor equations are interpreted as the spinor constituents of the ‘quasi-translations’ of the 2-surface. The question is therefore how to define the ‘quasi-translations’ of the 2-surface $$. If $$ is in the Minkowski spacetime then the ‘quasi-translations’ can be expected to coincide with the familiar translational Killing vectors at the points of $$. To ensure this coincidence the spinor propagation laws must contain some extrinsic geometrical properties of $$. The usual formalism in the spinorial approaches of the quasi-local energy-momentum is the GHP formalism [7-10]. In the GHP formalism the two edth operators, $\partial$ and $\partial'$, are the covariant directional derivations with respect to the induced intrinsic Levi-Cività connection; and the extrinsic curvatures of $$
are encoded into the spin coefficients $\rho$, $\sigma$ and $\rho'$, $\sigma'$. The thorn operators, $\triangleright$ and $\triangleright'$, and the remaining spin coefficients $\kappa$, $\tau$ and $\kappa'$, $\tau'$ all depend not only on the geometry of $\mathcal{S}$ but the way how the normals to $\mathcal{S}$ are extended off $\mathcal{S}$. The quasi-local energy-momentum, however, is expected to depend only on the (intrinsic and extrinsic) geometry of $\mathcal{S}$. Thus in constructing the propagation laws we can use only the operators $\partial$ and $\partial'$, the spin coefficients $\sigma$, $\rho$ and $\sigma'$, $\rho'$ and possibly those components of the curvature that are determined by $\mathcal{S}$. In spite of these restriction there remain too much freedom to construct the propagation laws. The two dimensional Sen connection, on the other hand, contains all the information on the extrinsic geometry of $\mathcal{S}$ since the Sen operator is the sum of the intrinsic covariant derivation and the (boost-gauge invariant combination of the) extrinsic curvatures [1]. Thus the 2 dimensional Sen operator alone might be, and as we will show, is enough to construct the spinor propagation laws.

In the first section of the present paper the 2-surface integral of the spinorial Nester–Witten 2-form will be examined. As a consequence of the 2 dimensional Sen–Witten identities Penrose’s construction [2] can also be considered as a Nester–Witten integral. The possible propagation laws will be considered and discussed in section 2. All the possible first order differential operators, acting on the covariant spinor fields, that can be constructed out of the the chiral irreducible parts of the 2 dimensional Sen operator will be determined. In particular, the properties of the holomorphy/anti-holomorphy operators will be examined in detail. Here we use the GHP form of the chiral irreducible parts of the Sen operator. To clarify the kernel space of the possible first order operators we need to know the structure of the kernel spaces of the edth operators. This clarification is made in the Appendix for 2-surfaces homeomorphic to $S^2$, using only Liouville’s theorem and Baston’s formula [10] for the analitic index of $\partial$ and $\partial'$. We will show that the Sen operator naturally defines a quasi-local energy-momentum, which energy-momentum turns out to be just that proposed by Dougan and Mason [5] within the GHP formalism. This justifies the rather heuristic argumentation-based choice of Dougan and Mason for the spinor propagation laws. The part of section 2 dealing with the holomorphy/anti-holomorphy operators can therefore be considered as the investigation of the conditions under which the Dougan–Mason construction can(not) be done. Although there are ‘exceptional’ 2-surfaces, e.g. the marginally trapped surfaces, for which the Dougan–Mason construction does not work (at least in its present form), for ‘strictly convex’ 2-spheres the construction seems to be well defined.

In section 3 first the Dougan–Mason energy and mass nonnegativity proof will be reviewed in the covariant spinor formalism and then some recent results on the Dougan–Mason energy-momentum will be generalized and discussed from new points of view too. We give equivalent statements for the vanishing of the quasi-local energy-momentum and for the vanishing of the quasi-local mass. In particular we will see that the vanishing of the Dougan–Mason mass (energy-momentum) is equivalent to the existence of one (two) Sen–constant spinor field(s) on the 2-surface $\mathcal{S}$. Furthermore, it will be clear that the $pp$-wave Cauchy developments with pure radiation can be characterized not only by the usual Cauchy data on a finite three dimensional Cauchy surface $\Sigma$, but by the two dimensional Sen–geometry of the boundary of $\Sigma$ too.

Finally we examine the possibility of defining the quasi-local mass of marginally trapped surfaces as the limit of the masses of a family of non-exceptional 2-spheres. We will see that this definition is ambiguous since the limit depends not only on the geometry of the trapped surface but the family as well.
The notations and conventions are the same that used in [1]. In particular, abstract index formalism [8] will be used unless otherwise stated. The ‘name’ or component indices will be underlined.

1. Quasi-local Nester–Witten integrals

First recall that for any two spinor fields $\lambda_A$, $\mu_A$ the spinorial Nester–Witten 2-form is defined [12,13] by:

$$u(\lambda, \bar{\mu})_{ab} := \frac{i}{2} \langle \bar{\mu}_A \nabla_{BB'} \lambda_A - \bar{\mu}_B \nabla_{AA'} \lambda_B \rangle.$$  (1.1)

Apart from an exact form this is ‘hermitian’ in the sense that

$$u(\lambda, \bar{\mu})_{ab} = u(\mu, \lambda)_{ab} - \frac{i}{2} \langle \nabla_a K_b - \nabla_b K_a \rangle, \quad K_a := \lambda_A \bar{\mu}_A.$$

Thus its real and imaginary parts are $F_{ab} := u(\lambda, \bar{\mu})_{ab} + u(\mu, \lambda)_{ab} = u(\lambda, \bar{\mu})_{ab} + u(\mu, \lambda)_{ab} + \frac{i}{2} \langle \nabla_a K_b - \nabla_b K_a \rangle$ and $iK_{ab} := u(\lambda, \bar{\mu})_{ab} - u(\lambda, \bar{\mu})_{ab} = u(\lambda, \bar{\mu}) - u(\mu, \lambda)_{ab} - \frac{i}{2} \langle \nabla_a K_b - \nabla_b K_a \rangle$, respectively. For $\mu_A = \lambda_A$ $F_{ab}$ is just the Reula–Tod 2-form [14] (and its dual *$F_{ab}$ is the Ludvigsen–Vickers 2-form [15]) by means of which they proved the positivity of the Bondi–Sachs mass. In general $K_{ab}$ is not exact. In the non-abstract index formalism Sparling’s form is

$$\Gamma(\lambda, \bar{\mu}) := i \nabla_{BB'} \lambda_A \nabla_{CC'} \bar{\mu}_A dx^a \wedge dx^b \wedge dx^c.$$  (1.3)

This is ‘hermitian’: $\bar{\Gamma}(\bar{\lambda}, \mu) = \Gamma(\lambda, \bar{\mu})$, and the Sparling equation is

$$du = \Gamma - \frac{1}{2} \lambda^A \bar{\mu}^A' G_{AA'} \Sigma_b,$$  (1.4)

where $G_{ab}$ is the Einstein tensor, $\Sigma_a := \frac{1}{2} \varepsilon_{abcd} dx^b \wedge dx^c \wedge dx^d$ and Einstein’s equations are written in the form $G_{ab} = -\kappa T_{ab}$.

Let us define the quasi-local Nester–Witten integral by

$$H_\Sigma[\lambda_R, \bar{\mu}_S] := \frac{2}{\kappa} \oint_\Sigma u(\lambda, \bar{\mu})_{ab} dx^a \wedge dx^b = \frac{4}{\kappa} \oint_\Sigma t^a u^b \ast u_{ab} d\Sigma,$$  (1.5)

which in the formalism developed in [1] takes the following form

$$H_\Sigma[\lambda_R, \bar{\mu}_S] = \frac{2}{\kappa} \oint_\Sigma \gamma^{R'S'} \bar{\mu}_R' \Delta_S' S' d\Sigma = \frac{2}{\kappa} \oint_\Sigma \bar{\mu}' (\Delta_R^{R'} \lambda_R - \Delta_{R'}^{R} \lambda_R) d\Sigma.$$  (1.6)

By virtue of (1.2) $H_\Sigma$ is a hermitian bilinear functional on the space $C^\infty(\Sigma, S_A)$ of smooth spinor fields on $\Sigma$. The importance of the quasi-local Nester–Witten integral is shown by Sparling’s equation (1.4): $H_\Sigma$ is connected to the energy-momentum of gravitating systems [12-15]. It might be interesting to note that the integral of the conformal invariant hermitian scalar product of local twistors on $\Sigma$ can also be expressed by (1.6): If $Z^a = (\lambda^A, i\Delta_{AA'} \lambda^B)$ and are $\tilde{W}^a = (\mu^B, i\Delta_{BB'} \lambda^K)$ are local twistors on $\Sigma$ [1] and $(Z, \tilde{W}) := -\lambda^A i \Delta_{AA'} \bar{\mu}^{B'} + \bar{\mu}^B i \Delta_{BB'} \lambda^A$ then

$$h_\Sigma[\lambda_R, \bar{\mu}_S'] := H_\Sigma[\gamma^{R'} K_K, \bar{\mu}_S'] - H_\Sigma[\lambda_R, \gamma^{R'} K_K, \bar{\mu}_S'] = \frac{2}{\kappa} \oint_\Sigma \langle Z, \tilde{W} \rangle d\Sigma.$$  (1.7)
If in (1.6) the spinor field $\mu_S$ is chosen to be $\Delta_S^S \bar{\omega}_S'$ for some spinor field $\omega_S$ then by the first Sen–Witten type identity (7.3) of [1]

$$A_S[\lambda_R, \omega_S] := H_S[\lambda_R, \Delta_S^S \omega_S] =$$
$$= \frac{2}{k} \int_8 \left( (\Delta_R^+ A^A \omega_A)(\Delta - R^B \lambda_B) + (\Delta_R^+ A^A \lambda_A)(\Delta - R^B \omega_B) \right) dS =$$
$$= \frac{2}{k} \int_8 \left( (T^+_R R S^A \omega_A)(T - R^B S B \lambda_B) + (T^+_R R S^A \lambda_A)(T - R^B S B \omega_B) \right) dS -$$
$$- \frac{i}{k} \int_8 \lambda^A \omega^B R_{AB,cd} dx^c \wedge dx^d. \tag{1.8}$$

This is a symmetric bilinear functional on $C^\infty(\mathcal{S}, S_A)$. If at least one of $\lambda_R$ and $\omega_S$ is a solution of the 2-surface twistor equation, or both $\lambda_R$, $\omega_S$ belong to the kernel space of $T^+$ or to the kernel space of $T^-$ then $A_S$ reduces to Penrose’s charge integral; i.e. to the expression of the kinematic twistor [2,12,16]. Or, in other words, for spinor fields satisfying the twistor equation Penrose’s charge integral can be expressed as a quasi-local Nester–Witten integral too (see also [13]). In a similar way one can choose $\mu_S := \gamma_S^K \Delta_K^R \bar{\omega}_K'$ for some spinor field $\omega_S$ in (1.6). Then

$$I_S[\lambda_R, \omega_S] := H_S[\lambda_R, \gamma_S^K \Delta_K^R \omega_K] =$$
$$= \frac{2}{k} \int_8 \left( (\Delta_R^+ A^A \omega_A)(\Delta - R^B \lambda_B) - (\Delta_R^+ A^A \lambda_A)(\Delta - R^B \omega_B) \right) dS \tag{1.9}$$

is an antisymmetric bilinear functional on $C^\infty(\mathcal{S}, S_A)$ and by the second Sen–Witten type identity (7.4) of [1] $I_S[\lambda_R, \omega_S]$ can be rewritten as the integral of a quadratic expression of the chiral twistor-derivatives and the charge integrals of the curvature and the torsion. The kernel space of these functionals is

$$\text{ker } H_S := \{ \lambda_A \in C^\infty(\mathcal{S}, S_A) \mid H_S[\lambda, \tilde{\mu}| = 0 \ \forall \mu_A \in C^\infty(\mathcal{S}, S_A) \} =$$
$$= \text{ker } A_S = \text{ker } I_S = \text{ker } (\Delta^- \oplus \Delta^+) \tag{1.10}.$$ 

ker $\Delta^\pm$ are infinite dimensional subspaces of $C^\infty(\mathcal{S}, S_A)$.

In the present paper we are interested in the possibility of finding quasi-local energy-momentum (and possibly angular momentum) expressions for gravitating systems in the form of Nester–Witten integrals. However these quantities are expected to be in the dual space of the four real dimensional vector space of the ‘quasi-translations’ and of the six real dimensional vector space of the ‘quasi-rotations’ of $\mathcal{S}$, respectively. Furthermore in order to define the quasi-local mass as the length of the quasi-local energy-momentum the space of ‘quasi-translations’ must have a Lorentzian metric. Thus what we need is a rule to reduce the infinite dimensional complex vector space $C^\infty(\mathcal{S}, S_A)$ to a finite dimensional subspace that can be interpreted as the space of the spinor constituents of the ‘quasi-translations’/‘quasi-rotations’ of $\mathcal{S}$. In other words, propagation law(s) for the spinor fields $\lambda_R$ and $\mu_S$ should be prescribed. It is natural to look for these propagation laws in the form $\Phi \lambda = 0$ where $\Phi$ is a differential operator acting on the space of the spinor fields. Since the ‘quasi-translations’ (the ‘quasi-rotations’) should form finite dimensional vector space(s) $\Phi$ must be a linear differential operator with finite dimensional kernel.

Since ker $H_S = \text{ker } A_S = \text{ker } I_S = \text{ker } (\Delta^- \oplus \Delta^+)$ one can see that the full 2 dimensional Weyl–Sen–Witten equation, in contrast to the 3 dimensional Sen–Witten equation, cannot be used
to define propagation law(s) within $$. Thus to find the appropriate propagation law(s) a detailed and systematic study of the linear differential operators on $$C^\infty(S, S_A)$$ is needed.

2. Propagation laws

In this paper we restrict our considerations to those first order linear differential operators $D$ on $$C^\infty(S, S_A)$$ that can be constructed only from the chiral irreducible parts of the Sen operator $\Delta_c$. (See the Introduction.) In the GHP formalism that we will use in this section are the chiral irreducible parts $\Delta^\pm$ and $T^\pm$ of $\Delta_c$. They are differential operators on $E^\infty(-1, 0) \oplus E^\infty(1, 0) \simeq C^\infty(S, S_A)$:

$$\begin{align*}
-\Delta^- \lambda &= \partial \lambda^0 - \rho \lambda^1, \\
-\Delta^+ \lambda &= \partial^\prime \lambda^1 - \rho \lambda^0,
\end{align*}$$

where the spinor components are defined by $\lambda^R = \lambda^0 \rho R + \lambda^1 l^R = : \lambda_1 o R - \lambda_0 t R$ (see (6.9-12) of [1]). These are the ‘elementary operators’ by means of which we construct all the first order operators. Higher order operators can also be constructed from $\Delta^\pm$ and $T^\pm$ taking into account that by (6.13) of [1] they cannot be composed in any way. The properties of the edth operators we need are clarified in the Appendix.

The irreducible chiral operators $\Delta^\pm$ and $T^\pm$ have infinite dimensional kernel spaces, and the direct sums $\Delta^\pm \oplus \Delta^\pm$ and $T^\pm \oplus T^\pm$ are obviously equivalent to $\Delta^\pm$ and $T^\pm$ themselves, respectively. The remaining direct sums consisting of two terms are the operators

$$\begin{align*}
-\Delta^- &:= \Delta^- \oplus \Delta^+: E^\infty(-1, 0) \oplus E^\infty(1, 0) \rightarrow E^\infty(0, -1) \oplus E^\infty(0, 1) \\
-\mathcal{H}^- &:= \Delta^- \oplus T^- : E^\infty(-1, 0) \oplus E^\infty(1, 0) \rightarrow E^\infty(0, -1) \oplus E^\infty(2, -1) \\
-\mathcal{C}^- &:= T^- \oplus \Delta^- : E^\infty(-1, 0) \oplus E^\infty(1, 0) \rightarrow E^\infty(-2, 1) \oplus E^\infty(0, -1) \\
-\mathcal{H}^+ &:= T^- \oplus \Delta^+ : E^\infty(-1, 0) \oplus E^\infty(1, 0) \rightarrow E^\infty(-2, 1) \oplus E^\infty(0, 1) \\
-\mathcal{T} &:= T^- \oplus T^- : E^\infty(-1, 0) \oplus E^\infty(1, 0) \rightarrow E^\infty(-2, 1) \oplus E^\infty(2, -1).
\end{align*}$$

$\Delta$, $\mathcal{T}$ and $\mathcal{H}^\pm$ are elliptic operators, and since $S$ is compact they have finite dimensional kernels. To determine the dimension of the kernels first calculate their analytic index following Baston’s calculations [10] for the twistor operator. Recall that the index of an elliptic operator $P$ is $\dimker P - \dimker P^\dagger$, where $P^\dagger$ is the adjoint of $P$ with respect to some hermitian scalar products on the space of the smooth sections of the vector bundles. By the Atiyah–Singer index theorem the index of the elliptic linear differential operators is a topological invariant of the bundles and the operators. Thus their index coincides with that of the operators

$$\begin{align*}
\begin{pmatrix} \partial(-1, 0) & 0 \\ 0 & \partial^\prime(-1, 0) \end{pmatrix}, \begin{pmatrix} \partial^\prime(-1, 0) & 0 \\ 0 & \partial(-1, 0) \end{pmatrix}, \begin{pmatrix} \partial(-1, 0) & 0 \\ 0 & \partial^\prime(-1, 0) \end{pmatrix}, \begin{pmatrix} \partial(-1, 0) & 0 \\ 0 & \partial(-1, 0) \end{pmatrix}, \begin{pmatrix} \partial(-1, 0) & 0 \\ 0 & \partial^\prime(-1, 0) \end{pmatrix}
\end{align*},$$

respectively. Then the index of the elliptic operators can be calculated from eq.(A.1) of the Appendix. If $\mathcal{G}$ is the genus of $S$ then
index $\Delta = 0$,
index $\mathcal{T} = 4(1 - \mathcal{G})$,
index $\mathcal{H}^\pm = 2(1 - \mathcal{G})$. 
(2.4)

Thus from the index theorem it does not follow the existence of non trivial solutions of $\Delta_{R^K} \lambda_R = 0$. Since however $\ker \Delta$ is precisely the kernel of $H_8$, $A_8$ and $I_8$, $\Delta_{R^K}$ could not be used to define propagation laws even if there were nontrivial solutions of $\Delta_{R^K} \lambda_R = 0$.

By (2.4) for topological 2-spheres $T_{R^K} \Sigma_{RS^K} \lambda_K = 0$ has at least four independent solutions [10]. One way of determining dim $\ker T$ is to consider the adjoint $T^\dagger$ of the twistor operator. For any fixed nowhere vanishing $h \in E^\infty(1, 1) \langle (\phi^0, \phi^1), (\psi^0, \psi^1) \rangle := (\phi^0, \psi^0)_{(p, q)} + (\phi^1, \psi^1)_{(p', q')}$. $T$ is a hermitian scalar product on the space of the smooth sections of the Whitney sum $E(p, q) \oplus E(p', q')$, where $(\phi, \psi)$ is defined by (A.3) in the Appendix. With respect to this scalar product the adjoint of the twistor operator is

$$
T^\dagger : E^\infty(-2, 1) \oplus E^\infty(2, -1) \to E^\infty(-1, 0) \oplus E^\infty(1, 0)
$$

(\omega^0, \omega^1) \mapsto \left( -\frac{1}{|h|} \frac{\partial}{h} (h | \omega^0) + \bar{\sigma} \omega^1 | h \right), -|h| \left( \partial' (\omega^1 | h) + \bar{\sigma}' | h | \omega^0 \right).
(2.5)

Thus with the definitions $\mu^0 := |h| \omega^0 \in E^\infty(-1, 2)$, $\mu^1 := |h|^{-1} \omega^1 \in E^\infty(1, -2)$ the adjoint of the twistor equations are read

$$
\partial (-1, 2) \mu^0 + \bar{\sigma} \mu^1 = 0
$$
$$
\bar{\sigma}' (-1, -2) \mu^1 + \bar{\sigma}' \mu^0 = 0.
(2.6)
$$

Since by (A.9) dim $\ker \partial (-1, 2) = \ker \bar{\sigma}' (1, -2) = 0$, dim $\ker T^\dagger = 0$ if $\sigma = 0$ or $\sigma' = 0$ on $\mathcal{S}$, and hence in these cases $\ker T$ is precisely four dimensional. In particular, for round spheres (i.e. for 2-spheres of spherical symmetry in spherically symmetric spacetimes [17]) the number of independent solutions of $T_{R^K} \Sigma_{RS^K} \lambda_K = 0$ is in fact precisely four [12]. However, as Jeffreys [18] has shown, in general dim $\ker T$ may be greater than index $\mathcal{T}$. Thus for the choice $\Phi = T$ we have at least 4 real and 6 complex integrals $H_8$, 10 complex integrals $A_8$ and 6 complex integrals $I_8$. Since we would like to have four real integrals for the energy-momentum and six real integrals for the angular momentum none of the expressions (1.6), (1.8) and (1.9) seems to yield the expected number of kinematical quantities unless an extra structure is used to reduce the number of them. In fact, in certain special cases (1.9) defines a real, skew and simple twistor $A_{\alpha \beta}$, the so-called infinity twistor, so that $I_{\alpha \beta}$, together with the hermitian metric $h_{\alpha \beta}$, defined by (1.7), can be used to reduce the ten complex components of the kinematic twistor $A_{\alpha \beta}$, defined by (1.9), to ten real components [2,12,16]. This is the original twistor-theory-motivated proposal of Penrose for the four quasi-local energy-momentum and six angular momentum. In general, however, no such infinity twistor exists [19].

Since $\mathcal{S}$ is an oriented closed 2 dimensional Riemannian submanifold it is a compact Riemann surface with the naturally defined complex structure [9,20-22]. This is precisely the integrable almost complex structure that the projections $\pi^{\pm a} := \pi^{\pm A} B \bar{\pi}^{\mp A'} B'$ define (see [1]). One can therefore define holomorphic functions, the multiplicity of the zeros and the order of the poles of meromorphic functions. If $(U, \xi)$ is a local holomorphic coordinate system and $m^a = P(\xi, \bar{\xi}) (\frac{\partial}{\partial \xi})^a$ for some nonzero smooth $P(\xi, \bar{\xi})$ then $f : U \to \mathbb{C}$ is holomorphic iff $\bar{m}^a \delta_a f = 0$ iff $\bar{m}^a \Delta_a f = 0$. 

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Thus the notion of holomorphic (meromorphic) functions and the multiplicity of their zeros and
the order of their poles are independent of the operators \( \delta_a, \Delta_a \). The notion of the holomorphic
tensor and spinor fields, however, does depend on the choice of the differential operator. The
spinor field \( \phi_{A...B...}' \) is said to be holomorphic with respect to the induced Levi–Civita connection
if \( m^a \delta_a \phi_{A...B...}' = 0 \). In the case of surface vector fields this notion of holomorphy coincides with
the holomorphy with respect to the intrinsic complex structure of \$ (see the Appendix). The
point \( p \in \$ \) is said to be a zero of the \( \delta_a \)-holomorphic spinor field \( \phi_{A...B...}' \) with multiplicity \( m \) if
\( \phi_{A...B...}' \) and its first \( (m - 1) \) \( m^a \delta_a \)-derivatives vanish at \( p \) but its \( m \)th \( m^a \delta_a \)-derivative is not zero
there. It is not difficult to show that \( p \) is a zero of the \( \delta_a \)-holomorphic \( \phi_{A...B...}' \) with multiplicity \( m \) if
and only if there is a \( \delta_a \)-holomorphic spinor field \( \psi_{A...B...}' \) and a holomorphic function \( f \) on an open
neighbourhood \( W \) of \( p \) such that \( \phi_{A...B...}' = f \psi_{A...B...}' \) and \( \psi_{A...B...}' \) is nonzero on \( W \) and \( p \) is a zero of \( f \) with
multiplicity \( m \). (Hint: On a sufficiently small neighbourhood \( W \) of \( p \) there are \( \delta_a \)-holomorphic
and \( \delta_a \)-anti-holomorphic spinor fields \( \lambda^A_R \) and \( \mu^A_B \), \( A, B = 0, 1 \), respectively, which form bases in
the spinor space at each point of \( W \). Then the components \( \phi_{A...BC'D'} \) of \( \phi_{A...BC'D'} \) in the
\( \delta_a \)-holomorphic basis \( \lambda^A_R...\lambda^B_B...\mu^C_C'...\mu^D_D' \) are holomorphic.) As a consequence the zeros of a not
identically vanishing \( \delta_a \)-holomorphic spinor field are isolated. The spinor field \( \phi_{A...B...}' \) is said to be
holomorphic with respect to the 2 dimensional Sen connection if \( m^a \Delta_a \phi_{A...B...}' = 0 \). Thus the spinor
field \( \lambda_R \) is holomorphic with respect to \( \Delta_a \) iff \( \Delta_a \lambda_R = 0 \) and \( T_{RS} \lambda_R = 0 \) by (6.10) and
(6.12) of [1]; and \( \lambda_R \) is anti-holomorphic with respect to \( \Delta_a \) iff \( \Delta_a \lambda_R = 0 \) and \( T_{RS} \lambda_R = 0 \) by
(6.9) and (6.11) of [1]. Thus \( \mathcal{H}^\pm \) may be called the holomorphy/anti-holomorphy operators. One
can define the multiplicity of the zeros of \( \Delta_a \)-holomorphic spinor fields too, and their zeros are
also isolated. The notion of holomorphic spinor fields defined by Dougan and Mason [5] coincides
with this notion of holomorphy, and, apart from the Appendix, in the rest of this paper
‘holomorphy’ will mean ‘holomorphy with respect to \( \Delta_a \)’ unless otherwise stated.

Since \( \mathcal{H}^- \) is elliptic and \$ is compact, \( \ker \mathcal{H}^- \) is finite dimensional. By (2.4) for topological
2-spheres index \( \mathcal{H}^- = 2 \); i.e. there are at least two linearly independent anti-holomorphic spinor
fields, say \( \lambda^0_R \) and \( \lambda^1_R \), on \$. (If a spinor field has a ‘name’ index too then the spinor index will
be written as a subscript and the ‘name’ index as a superscript. Thus for example \( \lambda^0_A \) is the
1 component of the zeroth spinor: \( \lambda^0_A = \lambda^0_{AR} \), while \( \lambda^0_1 = 0 \) component of the first spinor:
\( \lambda^1_0 = \lambda^1_{AR} \). Then \( m^a \nabla_e (\varepsilon^{RS} \lambda^A_R \lambda^B_S) = 0 \), \( A, B = 0, 1 \); and hence by Liouville’s theorem \( \varepsilon^{AB} := \varepsilon^{RS} \lambda^{0A}_R \lambda^{0B}_S \)
is constant on \$. If this constant is not zero (such two-surfaces will be called generic)
then it can be chosen to be \( \varepsilon^{AB} \), the Levi–Civita alternating symbol, and hence \( \lambda^0_R \) and \( \lambda^1_R \) form
a normalized basis in the spinor space at each point of \$. Thus for any spinor field \( \lambda_R \) there are
complex functions \( \alpha \) and \( \beta \) on \$ such that \( \lambda_R = \alpha \lambda^0_R + \beta \lambda^1_R \). If \( \lambda_R \) is anti-holomorphic then \( \alpha \) and
\( \beta \) are anti-holomorphic on \$. Then by Liouville’s theorem they must be constant; i.e. for generic
2-spheres \( \dim C \ker \mathcal{H}^- = 2 \). If \( \lambda^0_R \lambda^1_R \varepsilon^{RS} = 0 \) (such two-surfaces will be called exceptional) then
there must be a function \( f : \$ \to C \) such that \( m^a \Delta_a f = 0 \) and \( \lambda^1_R = f \lambda^0_R \). But \( \lambda^1_R \) and \( \lambda^1_R \) can be
independent solutions to \( \mathcal{H}^- \lambda = 0 \) only if \( \lambda^0_R \) has at least one zero and \( f \) is only anti-meromorphic
with pole(s) at the zero(s) of \( \lambda^0_R \). Since the sum of the order of the poles of an anti-meromorphic
function on a sphere is equal to the sum of the multiplicities of its zeros, the sum of the multiplicities
of the zeros of \( \lambda^0_R \) is equal to the sum of the multiplicity of the zeros of \( \lambda^1_R \). If \( \lambda_R \) is any anti-
holomorphic spinor field on \$ then \( \lambda_R \lambda^0_S \varepsilon^{RS} = 0 \) since \( \lambda_R \lambda^0_S \varepsilon^{RS} \) is constant on \$ and \( \lambda^0_R \) has
zeros. Thus for exceptional 2-spheres any anti-holomorphic spinor field on \$ is proportional to a
‘basis solution’, say \( \lambda^0_R \), and the factor of proportionality is an anti-meromorphic function. These
functions can be given explicitly in a coordinate system: Let $n \in \mathbb{S}$ (‘north pole’), $U := \mathbb{S} - \{n\}$ and $\xi : U \rightarrow \mathbb{C}$ a holomorphic coordinate. Let the zeros of $\lambda_R^0$ be $z_0 = \infty$ and $z_1, \ldots, z_k \in \xi(U) = \mathbb{C}$ with multiplicities $m_0, m_1, \ldots, m_k$, respectively, and define $m := m_0 + m_1 + \ldots + m_k$. Then the most general anti-meromorphic function on $\mathbb{S}$ whose possible poles are $z_0, \ldots, z_k$ with maximal order $m_0, \ldots, m_k$, respectively, is
\[
f(\xi) = \frac{a_m \xi^m + a_{m-1} \xi^{m-1} + \ldots + a_1 \xi + a_0}{(\xi - z_1)^{m_1} \cdots (\xi - z_k)^{m_k}},
\]
where $a_0, a_1, \ldots, a_k \in \mathbb{C}$. These functions form a complex vector space of dimension $m + 1$, and hence $\dim \ker H^\perp$ is not exceptional provided $\dot{f}(\xi) = 0$ for any of these conditions is satisfied then $\dim \ker H^\perp$ is still two dimensional. First consider the special case $\lambda_R^0 \neq 0$ we have $H^\perp \lambda = 0$. This reads
\[
\begin{align*}
\partial^\perp \mu^0 + \sigma \mu^1 &= 0, \\
\partial^\perp \mu^1 + \rho \mu^0 &= 0,
\end{align*}
\]
where $\mu^0 \in E^\infty(1, 0)$ and $\mu^1 \in E^\infty(1, -2)$. By (A.9) $\dim \ker \partial^\perp(1,0) = \dim \ker \partial^\perp(1,-2) = 0$, thus the adjoint equation (2.8) has only the trivial solution if $\rho = 0$ or $\sigma = 0$ on the whole $\mathbb{S}$, and $\ker H^\perp$ is still two dimensional. First consider the special case $\rho = 0$, which turns out to characterize an exceptional 2-sphere.

**Lemma 2.9:**
If $\lambda_R \in \ker H^\perp$ then the following statements are equivalent:

i. $\lambda_R = \lambda^1 R$, i.e. $\lambda^0$ is zero on $\mathbb{S}$,

ii. $\mathbb{S}$ is a future marginally trapped surface, i.e. $\rho = 0$ on $\mathbb{S}$,

iii. $\lambda_R$ is anti-holomorphic with respect to $\delta_e$ too.

If any of these conditions is satisfied then $\dim \ker H^\perp = 2$, the two independent anti-holomorphic spinor fields are proportional, $\lambda_R^1 = f \lambda_R^0$, and each has a single zero with multiplicity 1.

This lemma is a simple consequence of (A.9), (A.11), (A.18) and the GHP form of (A.17).

Now consider a smooth 1 parameter family $(u) \in (-\epsilon, \epsilon)$ of spacelike 2-spheres, $u \in (-\epsilon, \epsilon)$, such that $(0) = \mathbb{S}$ is a future marginally trapped surface. We show that for sufficiently small $|u| \neq 0$ $(u)$ is not exceptional provided $\dot{\rho} := (\frac{d}{du} \rho(u))_{u=0}$ is nowhere zero on $\mathbb{S}$. If $H_u^\perp$ is the anti-holomorphic operator for $(u)$ then in the complex coordinate system $(\xi, \bar{\xi})$ above the equation $H_u^\perp \lambda(u) = 0$ takes the following form
\[
P(u) \frac{\partial \lambda_1(u)}{\partial \xi} + \beta(u) \lambda_1(u) + \rho'(u) \lambda_0(u) = 0
\]
\[
P(u) \frac{\partial \lambda_0(u)}{\partial \bar{\xi}} - \beta(u) \lambda_0(u) + \sigma(u) \lambda_1(u) = 0.
\]

Here $\beta$ is the spin coefficient $-B_e m^e = m^e \nabla_e A^o A$ and $\lambda_R(u) = \lambda_1(u) \rho_R - \lambda_0(u) \sigma_R$. Since index $H_u^\perp = 2$, there certainly exist two linearly independent solutions $\lambda_R^A(u), A = 0, 1$, to $H_u^\perp \lambda(u) = 0$ for any $u \in (-\epsilon, \epsilon)$; and these solutions can be chosen so that $\lambda_R^A(0)$ to coincide with the independent solutions (A.18) guaranteed by Lemma 2.9. Then taking the derivative of (2.19) with respect to $u$ at $u = 0$ we have
\[ \lambda' \delta \lambda^A + \rho' \lambda^A = 0 \]  
\[ \delta \lambda^A + \sigma \lambda^A = 0, \]  
(2.11)

(2.12)

where \( \delta \lambda^A := \hat{P} \frac{\partial \lambda^A}{\partial \lambda} - \beta \lambda^A \). \( \lambda^A \) therefore depend only on \( \lambda^A \) and \( \rho' \), but are independent of \( \hat{P} \), \( \beta \) and \( \sigma \). Since \( \epsilon RS \lambda^A_R(u) \lambda^A_S(u) \) is constant on \( \mathcal{S}(u) \), its derivative is also constant on \( \mathcal{S} \), and since

\[ \frac{d}{du} \left( \epsilon RS \lambda^0_R(u) \lambda^0_S(u) \right)_{u=0} = \lambda^0_0(\lambda^0_1 - f \lambda^0_1), \]

(2.13)

it is zero if and only if \( \lambda^0_1 = f \lambda^0_1 \). Since by (A.9) \( \dim ker \delta_{(1,0)} = 0 \), \( \lambda^0_1 = f \lambda^0_1 \) iff \( f \lambda^0_1 \) is smooth; i.e. iff \( \lambda^0_1 \) has a zero precisely at the pole of \( f \). In other words the derivative (2.13) is zero if and only if the solution \( \lambda^0_1 \) of (2.11) has a zero and the zero of \( \lambda^0_1 \) and \( \lambda^0_0 \) coincide. We will show that the zero of \( \lambda^0_1 \) and \( \lambda^0_0 \) do not coincide provided \( \rho' \neq 0 \) everywhere on \( \mathcal{S} \); and hence for sufficiently small nonzero \( |u| \) the 2-surface \( \mathcal{S}(u) \) is generic. The coordinate system \( (\xi, \bar{\xi}) \) can always be chosen so that \( \hat{P}(\xi, \bar{\xi}, u) = e^{-\omega(u)}(1 + \xi \bar{\xi}) \), where \( \omega(u) \) is a smooth real function on the whole \( \mathcal{S} \); and by (A.18) \( \lambda^0_0 = i(\frac{\nu}{\bar{\nu}})^\frac{1}{2} \) where \( \nu = \exp(-\int \xi') \lambda^0_1 \). With these choices the solution of (2.11) is

\[ \lambda^0_1(\xi, \bar{\xi}) = -ie^{-\frac{1}{2}\omega + \frac{1}{2}\int A_\xi} \sqrt{1 + \xi \bar{\xi}} \int_0^{\xi} \frac{\xi'}{(1 + \xi \bar{\xi})^2} e^{2\omega - \int A_\xi} d\xi', \]

(2.14)

and hence

\[ |\lambda^0_1(\xi, \bar{\xi})| \geq \min_{\mathcal{S}} \{|e^{-\frac{1}{2}\omega + \frac{1}{2}\int A_\xi}| \} \min_{\mathcal{S}} \{|e^{2\omega - \int A_\xi}| \} \frac{|\xi|}{\sqrt{1 + \xi \bar{\xi}}}. \]

(2.15)

Thus \( \lambda^0_1(\xi, \bar{\xi}) \) may have a zero only in the south pole \( (\xi = 0) \) while \( \lambda^0_0 \) has a single zero in the north pole \( (\xi = \infty) \). This result can also be interpreted as the perturbations of a future marginally trapped surface yield generic 2-spheres provided the perturbations satisfy \( \rho' \neq 0 \) everywhere. The marginally trapped surfaces are therefore really ‘exceptional’.

Next suppose that \( \sigma = 0 \). Then for the two linearly independent solutions, \( \lambda^0_R \) and \( \lambda^1_R \), the equation \( \mathcal{H}^{-} \lambda = 0 \) reduces to

\[ \delta \lambda^A + \rho' \lambda^A = 0 \]  
\[ \delta \lambda^A = 0. \]  
(2.16a)

(2.16b)

But (2.16b) is just the GHP form of (A.17), furthermore (2.16a) has the same structure as (2.11). Thus the pair \( (\lambda^0_0, \lambda^1_0) \) is a solution of (2.16) where in the complex coordinate system above \( \lambda^0_0 = i(1 + \xi \bar{\xi}) - \frac{1}{2} e^{\frac{1}{2}\omega - \frac{1}{2}\int A_\xi} \) and \( \lambda^1_0 \) is given by (2.14) (without the dots). \( \lambda^0_0 \) has a zero in the north pole \( (\xi = \infty) \), but in general (e.g. if \( \rho' \geq 0 \) but not identically zero on \( \mathcal{S} \); i.e. \( \mathcal{S} \) is ‘convex’) the solution \( (\lambda^0_0, \lambda^1_0) \) does not have a zero. However with an appropriately chosen \( \rho' \lambda^1_0 \) will have a zero in the north pole too. For example with \( \rho' = \frac{1-\xi}{1+i\xi} \) we have
\[ | \lambda^0_\epsilon(\xi, \bar{\xi}) | \leq \max_{\mathfrak{h}} \{ | e^{\frac{i}{2} \omega + \frac{i}{2} \int A_\nu |} \} \max_{\mathfrak{h}} \{ | e^{2\omega - \int A_\nu |} \sqrt{1 + \xi \bar{\xi}}\int_0^\xi \rho'(\xi', \bar{\xi}) (1 + \xi' \bar{\xi})^2 d\xi' | = \]

\begin{equation}
= \text{const} \frac{| \xi |}{(1 + \xi \bar{\xi}) \sqrt{1 + \xi \bar{\xi}}}
\end{equation}

which has a zero in the north and the south poles of $\mathcal{S}$. Hence the anti-holomorphic spinor field $\lambda^0_R$ has a single zero, implying that although $\dim \ker \mathcal{H}^- = 2$ and $\mathcal{S}$ is not future marginally trapped, it is exceptional.

Finally we note that Jeffryes’s construction [18] can be repeated to show that there might be 2-spheres on which the adjoint equation (2.8) has at least one nontrivial solution; implying that for such surfaces $\dim \ker \mathcal{H}^- \geq 3$. For the existence of the nontrivial solution of (2.8) in Jeffryes’s construction, however, the vanishing both of $\sigma$ and $\rho'$ and some of their derivatives at least at two different points is needed. Thus the strict positivity of $\rho'$ on $\mathcal{S}$ excludes all the three forms of exceptional 2-spheres considered here. The ‘strict convexity’ of $\mathcal{S}$ seems therefore to ensure the genericity of $\mathcal{S}$. This proposition is however not yet proved.

If $\mathcal{S}$ is generic then $\ker \mathcal{H}^\pm$ are therefore two dimensional complex vector spaces and $\varepsilon_{\mathcal{A}\mathcal{B}} := \varepsilon_{\mathcal{A}\mathcal{B}}^\mathcal{S} \lambda^0_{\mathcal{B}}$ is a naturally defined constant symplectic inner product. Then $\varepsilon_{\mathcal{A}\mathcal{B}}$ is invertible and the space of the holomorphic/anti-holomorphic spinor fields is an $\text{SL}(2, \mathbb{C})$-spinor space. In Minkowski spacetime the restriction to $\mathcal{S}$ of the constant spinor fields (i.e. the spinor constituents of the restriction to $\mathcal{S}$ of the translation Killing vectors) can thus be recovered as the solutions of $\mathcal{H}^\pm \lambda = 0$. Substituting the holomorphic or anti-holomorphic spinor fields into (1.6) we obtain four real integrals, while both $A_\xi$ and $I_\xi$ are identically zero.

The operators $\mathcal{C}^\pm$ are not elliptic, and hence although $\mathcal{S}$ is compact $\dim \ker \mathcal{C}^\pm$ are not necessarily finite. In fact, if for example $\mathcal{G} = 0$ and $\rho' = 0$ on $\mathcal{S}$ then by (A.9) $\lambda^0$ must be zero and hence $\dim \ker \mathcal{C}^- = 0$ if $\sup \rho' = \mathcal{S}$, while $\dim \ker \mathcal{C}^- = \infty$ if $\sup \rho' \neq \mathcal{S}$. On the other hand if $\sigma' = 0$ then by (A.11) there are two independent solutions for $\lambda^0$ and hence $\dim \ker \mathcal{C}^- = 2$ if $\sup \rho' = \mathcal{S}$ (non-trapped round spheres, for example), while $\dim \ker \mathcal{C}^- = \infty$ if $\sup \rho' \neq \mathcal{S}$. Thus the operators $\mathcal{C}^\pm$ do not define acceptable spinor propagation laws. The spinor fields belonging to $\ker \mathcal{C}^-$ are precisely those satisfying $\Delta_a \lambda_R \pi^{+R}_K = 0$. The integrability condition of this equation is

\[ \lambda^K F_{K\mathcal{R}\mathcal{A}A} \pi^{+R}_S = -\Delta_a \lambda_K \pi^{-K}_L (\delta^c_a Q^L_{\mathcal{B}B' R} - \delta^c_L Q^A_{\mathcal{A}A' R}) \pi^{+R}_S, \]

which for any pair $\lambda_R$, $\mu_R$ of spinor fields from $\ker \mathcal{C}^-$ implies

\[ \left( \lambda^R \mu^\mathcal{S}_\mathcal{R} \right) \pi^{-K}_C \pi_{C\mathcal{B}d} \pi^{-D}_L = \Delta_a \left( \lambda^R \mu^\mathcal{S}_\mathcal{R} \right) \pi^{+K}_C \left( \delta^c_a Q^C_{\mathcal{B}B' R} - \delta^c_L Q^C_{\mathcal{A}A' D} \right) \pi^{-D}_L. \]

The independent direct sums of $\Delta^\pm$ and $\mathcal{T}^\pm$ consisting of three terms are

\[ \mathcal{J}^- := \mathcal{T}^- \oplus \Delta^+ \oplus \Delta^- \approx \mathcal{T}^- \oplus \Delta \approx \mathcal{H}^- \oplus \Delta^+ \]

\[ \mathcal{J}^+ := \mathcal{T}^+ \oplus \Delta^+ \oplus \Delta^- \approx \mathcal{T}^+ \oplus \Delta \approx \mathcal{H}^+ \oplus \Delta^- \]

\[ \mathcal{K}^+ := \mathcal{T}^+ \oplus \mathcal{T}^- \oplus \Delta^+ \approx \mathcal{T} \oplus \Delta^+ \approx \mathcal{T}^+ \oplus \mathcal{H}^+ \]

\[ \mathcal{K}^- := \mathcal{T}^+ \oplus \mathcal{T}^- \oplus \Delta^- \approx \mathcal{T} \oplus \Delta^- \approx \mathcal{T}^+ \oplus \mathcal{H}^- \]
Although they are not elliptic, they have finite dimensional kernels. Since \( \ker \mathcal{J}^\pm \subset \ker \Delta \) the operators \( \mathcal{J}^\pm \) cannot be used to define propagation laws.

Since \( \ker \mathcal{K}^\pm = \ker \mathcal{T}^\pm \cap \ker \mathcal{H}^\pm \) the elements of \( \ker \mathcal{K}^\pm \) are special holomorphic/anti-holomorphic spinor fields, and hence for generic 2-spheres \( \dim \ker \mathcal{K}^\pm \leq 2 \). Then if there were two independent spinor fields in \( \ker \mathcal{K}^- \), say \( \lambda R \) and \( \mu R \), then \( \lambda R \mu R \varepsilon_{RS} \) would be a nonzero constant on \( \mathcal{S} \), and hence by (2.19) and (2.20) \( \pi^- K C F^C D_{AB} \varepsilon^{+D} \) would have to be zero. Thus for generic topological 2-spheres \( \dim \ker \mathcal{K}^\pm \leq 1 \) and \( \mathcal{K}^\pm \) do not yield the appropriate number of quasi-local integrals.

Finally the direct sum of all the irreducible chiral operators is

\[
\mathcal{C} := \mathcal{T}^+ \oplus \mathcal{T}^- \oplus \Delta^+ \oplus \Delta^- \approx \mathcal{T} \oplus \Delta \approx \mathcal{H}^+ \oplus \mathcal{H}^-.
\]

The spinor field \( \lambda_R \) is holomorphic and antiholomorphic iff it is \( \Delta_a \)-constant, which is equivalent to \( \Delta_R(R \lambda_S) = 0 \) by eqs.(6.9-12) and (6.1) of [1]. Thus the elements of the kernel space of \( \mathcal{C} \) are precisely the \( \Delta_b \)-constant spinor fields on \( \mathcal{S} \). If \( \lambda_R \) is a \( \Delta_b \)-constant spinor field on \( \mathcal{S} \) then by (4.1) of [1] \( \lambda A F_{ABcd} = 0 \), which is obviously equivalent to

\[
\lambda^A_i F_{ABcd} t^e v^f \varepsilon_{ef} c^d = \lambda^A \left( \phi_{ABCD} \gamma^{CD} - \phi_{A'B'B} \gamma^{A'B'} + 2 \lambda \gamma_{AB} \right) = 0.
\]

Thus \( \mathcal{S} \) admits a \( \Delta_b \)-constant spinor field \( \lambda_R \) only if the two principle spinors of \( F_{ABcd} t^e v^f \varepsilon_{ef} c^d \) coincide and are proportional to \( \lambda_R \); i.e. in algebraically general spacetimes \( \dim \ker \mathcal{C} = 0 \). If \( \lambda_R^0 \) and \( \lambda_R^1 \) are \( \Delta_b \)-constant spinor fields then \( \lambda_R^0 \lambda_S \varepsilon^{RS} \) is constant on \( \mathcal{S} \); and hence either \( \lambda_R^1 = c \lambda_R^0 \) for some nonzero \( c \in \mathbb{C} \) or \( \lambda_R^0 \) and \( \lambda_R^1 \) form a basis in the spinor spaces at each point of \( \mathcal{S} \). If \( \lambda_R^0 \), \( \lambda_R^1 \), \( \Delta = 0, 1 \), are independent \( \Delta_b \)-constant spinor fields then for any spinor field \( \lambda_R \) there are functions \( \alpha \) and \( \beta \) on \( \mathcal{S} \) such that \( \lambda_R = \alpha \lambda_R^0 + \beta \lambda_R^1 \), and if \( \lambda_R \) is \( \Delta_b \)-constant then \( \alpha \) and \( \beta \) are constant. Thus there are at most two independent \( \Delta_b \)-constant spinor fields on \( \mathcal{S} \), when by (2.22) \( F_{ABcd} = 0 \) on \( \mathcal{S} \). In a \( pp \)-wave spacetime the constituent spinor field \( \lambda_A \) of the constant null vector can always be chosen to be constant; i.e. to satisfy \( \nabla_b \lambda_A = 0 \). Its restriction to \( \mathcal{S} \) is \( \Delta_b \)-constant and hence in a \( pp \)-wave spacetime \( \dim \ker \mathcal{C} \geq 1 \). In Minkowski spacetime there are two linearly independent constant spinor fields whose restriction to \( \mathcal{S} \) are the two independent \( \Delta_b \)-constant spinor fields and \( \dim \ker \mathcal{C} = 2 \). In the next section we will show that the converse of these statements is also true, namely assuming \( \mathcal{S} \) is a generic topological 2-sphere bounding a spacelike hypersurface \( \Sigma \) on which the dominant energy condition holds and \( \mathcal{S} \) is ‘convex’, the existence of one/two \( \Delta_b \)-constant spinor field(s) on \( \mathcal{S} \) implies that the Cauchy development \( D(\Sigma) \) of \( \Sigma \) is a \( pp \)-wave/flat spacetime geometry (and hence in a nonflat \( pp \)-wave spacetime \( \ker \mathcal{C} \) is precisely 1 dimensional).

To summarize our results on the kernel spaces of the first order operators we have the following theorem:

**Theorem 2.23:**

The only first order linear differential operators on \( C^\infty(\mathcal{S}, \mathbb{S}_A) \) that are constructed only from the chiral irreducible parts of the 2 dimensional Sen operator and have generically 2 dimensional kernels are the holomorphic and anti-holomorphic operators \( \mathcal{H}^\pm \).

The ‘natural’ propagation laws for \( \lambda_R \) are therefore \( \lambda \in \ker \mathcal{H}^\pm \). With this choice in the generic case we have four real quasi-local Nester–Witten integrals and there is some hope to obtain reasonable energy-momentum expressions. In fact, this is precisely the Dougan–Mason energy-momentum...
is just the pull back of the Sparling equation (1.4) along the natural imbedding condition on $$. Then by the Reula–Tod form [14] of the 3 dimensional Sen–Witten identity, which have

smooth spacelike 3 dimensional submanifold $$\Sigma$$. Let

the solution of the Sen–Witten equation $$D_{R} \lambda_{R} = 0$$ with an as yet unspecified boundary condition on $$. Then by the Reula–Tod form [14] of the 3 dimensional Sen–Witten identity, which is just the pull back of the Sparling equation (1.4) along the natural imbedding $$i : \Sigma \to M$$, we have

$$H_{\Sigma}[\lambda, \overline{\lambda}] = \frac{2}{\kappa} \int_{\Sigma} \left\{ - h^{ab} R_{R}^{C} \left( D_{a} \lambda_{R} \right) \left( D_{b} \lambda_{R} \right) - \frac{1}{2} \kappa^{A} \lambda^{A} G_{ab}^{b} \right\} d\Sigma,$$

(3.1)

where $$d\Sigma$$ is the induced volume element on $$\Sigma$$. Thus if the dominant energy condition holds on $$\Sigma$$ then $$H_{\Sigma}[\lambda, \overline{\lambda}] \geq 0$$. On the other hand, using $$\Delta_{R} \lambda_{R} = 0$$ and equations (3.4), (4.4), (4.6) and (5.3) of [1]

$$H_{\Sigma}[\lambda, \overline{\lambda}] - H_{\Sigma}[\overline{\lambda}, \lambda] = \frac{2}{\kappa} \int_{\Sigma} \left\{ - Q_{R}^{R_{B} B} \left( \lambda^{B} - \lambda^{B} \right) \left( \overline{\lambda}^{B} - \lambda^{B} \right) + \left( \overline{\lambda}^{R} - \lambda^{R} \right) \pi_{K}^{K} \lambda^{K} + \left( \lambda^{R} - \lambda^{R} \right) \pi_{K}^{K} \lambda^{K} \right\} d\Sigma.$$  

The most natural boundary condition to $$D_{R} \lambda_{R} = 0$$ would therefore be $$\lambda^{R} \mid \Sigma = \lambda^{R}$$, which would ensure the non-negativity of $$H_{\Sigma}[\lambda, \overline{\lambda}]$$ too. The Sen–Witten equation, however, does not have in general a solution on $$\Sigma$$ with this boundary condition. We should therefore relax this boundary condition, and it seems natural next to choose $$\pi_{- R} \left( \lambda^{R} \mid \Sigma - \lambda^{R} \right) = 0$$. With this choice we have

$$H_{\Sigma}[\lambda, \overline{\lambda}] - H_{\Sigma}[\overline{\lambda}, \lambda] = - \frac{2}{\kappa} \int_{\Sigma} Q_{R}^{R_{B} B} \pi^{+ B_{A}} \pi^{+ B_{A}} \left( \lambda^{A} - \lambda^{A} \right) \left( \overline{\lambda}^{A} - \lambda^{A} \right) d\Sigma.$$

(3.2)

If, following Dougan and Mason, we assume that the outgoing null geodesics orthogonal to $$\Sigma$$ are not contracting on $$\Sigma$$, i.e. $$\rho' \geq 0$$, then, because of (4.9) of [1], this integral is nonnegative. Furthermore $$\rho' \geq 0$$ and the dominant energy condition on $$\Sigma$$ ensure the existence of a solution $$\lambda^{R}$$ to the Sen–Witten equation with the boundary condition above [5]. Thus $$H_{\Sigma}$$ is a nonnegative hermitian scalar product on ker $$\Delta^{-}$$, and hence $$H_{\Sigma}$$ satisfies the Cauchy–Schwartz inequality:

$$H_{\Sigma}[\lambda, \overline{\lambda}] H_{\Sigma}[\mu, \overline{\mu}] \geq H_{\Sigma}[\lambda, \overline{\mu}] H_{\Sigma}[\mu, \overline{\lambda}]$$

(3.3)
for any $\lambda_R, \mu_S \in \ker \Delta^-$. Similarly, the dominant energy condition and $\rho \leq 0$ ensure the non-negativity of $H_\$ on $\ker \Delta^+$. Thus all the quasi-local energy-momentum expressions in which $\lambda_R \in \ker \Delta^-$ or $\lambda_R \in \ker \Delta^+$ is a part of the complete propagation law have this non-negativity property. Such are for example the Ludvigsen–Vickers [3], the Dougan–Mason [5] and the Bergqvist [6] propagation laws.

The quasi-local energy-momentum is defined to be an element of the dual space to the vector space of the ‘quasi-translations’ of $\$. Explicitly, if $\$ is generic and $\{\lambda^A_R\}$ is a basis in $\ker H^\pm$ then for any constant hermitian matrix $K^B_A$, the vector field $K^a_a := K^B_A \lambda^A_R \overline{\lambda^A_R}$ can be interpreted as a ‘quasi-translation’ of $\$. These ‘quasi-translations’ form a four real dimensional subspace of $\ker H^\pm \otimes \ker H^\mp$ and span the four dimensional tangent spaces at the points of $\$. Then the Dougan–Mason quasi-local energy-momentum is defined by $K^a_a := K^B_A H_\$[\lambda^A_R, \overline{\lambda^A_R}]$; i.e. if $\{\lambda^R\}$ is a *normalized* spinor basis in $\ker H^\pm$ then the components of the quasi-local energy-momentum, the quasi-local energy and mass are defined by

\begin{align}
P^{AB'}_\$ &:= H_\$[\lambda^A, \overline{\lambda^{B'}}], \\
E_\$ &:= \frac{1}{\sqrt{2}} (P^{00'}_\$ + P^{11'}_\$), \\
m^2_\$ &:= \varepsilon^{AB} \varepsilon^{A'B'} P^{AB'}_\$ P^{B'A'}_\$ = 2 (P^{00'}_\$ P^{11'}_\$ - P^{01'}_\$ P^{10'}_\$),
\end{align}

respectively. Thus if the dominant energy condition holds and in the anti-holomorphic case $\rho' \geq 0$ on $\$ (and $\rho \leq 0$ on $\$ in the holomorphic case) then $E_\$ \geq 0$ [5] and by the Cauchy–Schwartz inequality $m^2_\$ \geq 0$; i.e. $P^a_a$ is a future directed nonspacelike vector [11]. This energy-momentum gives the correct, expected value in the weak field approximation [5]. The quasi-local energy-momentum is calculated for round spheres and small and large spheres [17] and at the horizon of the Reissner–Nordström and Kerr spacetimes [24] and compared with other definitions. At future null infinity the quasi-local energy-momentum defined by the anti-holomorphic spinor fields tends to the Bondi–Sachs four-momentum. The expression based on the holomorphic spinor fields in general tends to infinity. That yields the Bondi–Sachs four-momentum at past null infinity [17]. At spacelike infinity both definitions give the ADM energy-momentum.

Recently it has been shown that for generic $\$ $P^{AB'}_\$ = 0$ iff the Cauchy development of $\Sigma$ is flat; but the vanishing of the mass alone does not imply flatness. The zero-mass Cauchy developments are precisely the $pp$-wave geometries with pure radiation [11]. In the rest of this section only the anti-holomorphic expression will be examined and we prove two theorems which give further equivalent statements for the zero energy-momentum and zero mass spacetime configurations.

**Theorem 3.7:**

Let $\$ be a generic 2-sphere for which $\rho' \geq 0$, let $\Sigma$ be a spacelike hypersurface such that $\partial \Sigma = \$ and let the dominant energy condition hold on $\Sigma$. Then the following statements are equivalent:

1. $P^{AB'}_\$ = 0$,
2. $E_\$ = 0$,
3. $D(\Sigma)$, the Cauchy development of $\Sigma$, is flat,
4. There exist two linearly independent $\Delta_e$–constant spinor fields on $\$.

*Proof:* 1. obviously implies 2. Since $H_\$ is non-negative, $E_\$ = 0$ implies both $H_\$[\lambda^0, \overline{\lambda^0'}] = 0$ and $H_\$[\lambda^1, \overline{\lambda^1'}] = 0$. They, as it was shown in [11], imply the flatness of $D(\Sigma)$. If $D(\Sigma)$ is flat then,
at least in an open neighbourhood of $ \mathcal{S}$ in $D(\Sigma)$, there exist two linearly independent $\nabla_e$–constant spinor fields. Their restriction to $\mathcal{S}$ are independent $\Delta_e$–constant spinor fields on $\mathcal{S}$. If $\lambda^0_R$ and $\lambda^1_R$ are $\Delta_e$–constant on $\mathcal{S}$ then by $\lambda^R \in \ker(\Delta^+ + \Delta^- + T^+ + T^-)$ $H_\mathcal{S}\left[\lambda^0, \bar{\lambda}^0'\right] = H_\mathcal{S}\left[\lambda^1, \bar{\lambda}^1'\right] = 0$, which by the Cauchy–Schwartz inequality imply $m^2_\mathcal{S} = 0$.

The equivalence of 1. and 3. has been discussed in [11], thus we discuss only the equivalence of 2. and 3. and of 3. and 4. In the (classical and quantum) theory of fields one can define the vacuum state as the minimal energy state of the system and the ground state as in which all the particle fields and field strength (of the gauge fields) are zero. These states do not necessarily coincide even if the energy functional is bounded below, as for example in the $\phi^4$–theory. The strict positivity of the ADM and Bondi–Sachs masses [14,15] implies that the ground state (i.e. the flat spacetime) is the minimal energy state among the states describing asymptotically flat spacetimes. This however does not necessarily exclude the existence of non-asymptotically flat spacetimes with negative quasi-local energy somewhere. Having accepted the Dougall–Mason energy-momentum as 'the' correct gravitational energy-momentum, we can define quasi-locally the vacuum state of Einstein's theory by $E_\mathcal{S} = 0$. Then the equivalence of the statements 2. and 3. thus means that the (quasi-locally defined) vacuum state is the uniquely determined ground state, and hence no spontaneous symmetry breaking can occur in Einstein's theory.

The fact whether there exist two independent $\Delta_e$–constant spinor fields on $\mathcal{S}$ depends only on the 2 dimensional Sen–geometry of $\mathcal{S}$. On the other hand the equivalence of 3. and 4. of Theorem 3.7 means that gravitation together with matter fields satisfying the dominant energy condition is so ‘rigid’ a system that the information that $D(\Sigma)$ is flat is completely encoded into the Sen–geometry of $\mathcal{S}$. In other words flat Cauchy developments of a finite Cauchy surface can be characterized not only by the usual Cauchy data on a three dimensional $\Sigma$ but by the Sen–geometry of a spacelike two dimensional sphere too.

**Theorem 3.8:**
Under the conditions of Theorem 3.7 the following statements are equivalent:
1. $m^2_\mathcal{S} = 0$,
2. $D(\Sigma)$ is a $\text{pp}$-wave geometry with pure radiation; i.e. there exists a constant nonzero null vector field $L^a$ on $D(\Sigma)$ such that $L^a T_{ab} = 0$,
3. There exists a $\Delta_e$–constant spinor field on $\mathcal{S}$.

**Proof:** The fact that 1. implies 2. was proved in [11]. If there is a future directed constant nonzero null vector field $L^a$ on $D(\Sigma)$ then there exists a $\nabla_e$–constant spinor field $\lambda^A$ on $D(\Sigma)$ such that $L^a = \lambda^A \bar{\lambda}^A'$. The restriction of $\lambda^A$ to $\mathcal{S}$ is a nonzero $\Delta_e$–constant spinor field on $\mathcal{S}$. If $\lambda^A$ is $\Delta_e$–constant on $\mathcal{S}$ then by $\lambda_R \in \ker\left(\Delta^+ + \Delta^- + T^+ + T^-\right)$ $H_\mathcal{S}\left[\lambda, \bar{\lambda}\right] = 0$. However $\lambda_R$ can be chosen to be $\lambda^0_R$, one of the normalized basis spinors in $\ker H^-$. But then $P^0_\mathcal{S} = 0$, which by the Cauchy–Schwartz inequality implies $m^2_\mathcal{S} = 0$. 

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The equivalence of 1. and 2. was discussed in [11], thus we consider only the equivalence of 2. and 3. Again, the existence of a $\Delta\pi$-constant spinor field depends only on the two dimensional Sen–geometry of $\$. The equivalence of the statements 2. and 3. of Theorem 3.8, on the other hand, means that the information that $D(\Sigma)$ is a $pp$-wave geometry with pure radiation is completely encoded into the Sen–geometry of $\$. There is, however, an essential difference between the zero-energy and zero-mass cases. Namely while in the zero-energy case we could determine the metric of $D(\Sigma)$, that is flat, in the zero-mass case we can determine only the class of the metric of $D(\Sigma)$: that is $pp$-wave plus pure radiation. Thus naturally arises the question whether all the information on the metric of $D(\Sigma)$ itself are encoded into the Sen–geometry of $\$. The answer obviously depends on the details of the field equations for the matter fields. For vacuum the answer is affirmative as there is a smooth function $\Phi : \$ \to \mathbb{C}$ whose second Sen–derivatives determine completely the geometry of $D(\Sigma)$. $\Phi$ is constant iff $D(\Sigma)$ is flat. The (vacuum) $pp$-wave Cauchy developments can therefore be characterized not only by the usual Cauchy data on a three dimensional hypersurface $\Sigma$ but by the two dimensional Sen–geometry of $\$. The details of this analysis will be published in a separate paper.

If $\$ is exceptional and dim ker $\mathcal{H}^- = 2$ (e.g. if $\$ is future marginally trapped) then the quasi-translations are null and are proportional with each other, but the components $P_{AB}^{s}$ with respect to a basis $\{\lambda_{\mathcal{H}}\}$ of ker $\mathcal{H}^-$ can still be defined by (3.4). One can, however, see from (1.6) and Lemma 2.9.i that for the physically important special case of future marginally trapped surfaces $P_{s}^{AB}$ are zero. But in this case $\varepsilon_{AB} := R^{RS}_{\Delta R} \lambda_{R}^{\Sigma}$ is singular, thus one might conjecture that the vanishing of $P_{s}^{AB}$ does not mean the vanishing of the quasi-local four-momentum, and the quasi-local mass of the marginally trapped surfaces can be defined in a limiting procedure. In fact, the quasi-local mass was calculated for round spheres and it was found that the quasi-local mass has a well defined and nonzero limit even if the round spheres tend to a marginally trapped surface [17]. Furthermore, the thermodynamical analysis shows that a positive mass is associated with the marginally trapped surfaces [25], and in general the irreducible mass $(\frac{\text{Area}(\$)}{2})^{\frac{1}{2}}$ is expected (see for example [26]). Here we show that although for the family $\$ (u) of 2-surfaces considered in section 2 $m_{\Sigma(u)}^{2}$ has a well defined positive limit, the limiting value does depend on the family $\$ (u).

The solution of (2.11) with $\lambda_{s}^{1} = i(\frac{n}{p})^{\frac{1}{2}} \xi$ is

$$
\bar{\lambda}_{s}^{1}(\xi, \bar{\xi}) = -ie^{-\frac{\bar{\omega} + \frac{1}{2} \bar{t}}{\bar{\xi}}} \int A_{s} \sqrt{1 + \xi \bar{\xi}} \left\{ c + \int_{0}^{\xi} \frac{\bar{\rho}(\xi', \bar{\xi})}{(1 + \xi \bar{\xi})^{2}} \bar{\xi} e^{2\omega - \int A_{s} d\xi'} \right\},
$$

where the constant $c$ is the value of

$$
c(\xi, \bar{\xi}) := -\int_{0}^{\xi} \frac{\bar{\rho}(\xi', \bar{\xi})}{(1 + \xi \bar{\xi})^{2}} \bar{\xi} e^{2\omega - \int A_{s} d\xi'}
$$

in the north pole of $\$. Then by (2.14) and (3.9) the derivative of $\Lambda(u) := \varepsilon_{AB}^{RS} \lambda_{B}^{\Sigma}(u) \lambda_{s}^{1}(u)$ with respect to $u$ at $u = 0$, given by (2.13), is just the constant $c$. Let $\$ (u) be generic for any nonzero $u$ (e.g. if $\bar{\rho} > 0$ on $\$). Then $c \neq 0$, $\varepsilon_{AB}^{s}(u)$ is invertible and $m_{\Sigma(u)}^{2}$ can be defined (see (3.6)):

$$
m_{\Sigma(u)}^{2} = 2 \left( \frac{P_{s}^{00'}(u) P_{s}^{11'}(u)}{\Lambda(u)} - \frac{P_{s}^{10'}(u) P_{s}^{01'}(u)}{\Lambda(u)} \right).
$$

But by the L’Hospital rule and (2.11)
inner product is given by \[ E \text{defines a hermitian inner product on } \phi, \psi \in \mathcal{E}(\Sigma) \text{ if } \phi \psi = \int_{\Sigma} \phi \psi \, d\mathcal{S}, \]

and hence \( \lim_{u \to 0} m^2_{\Sigma(u)} \) does depend not only on the geometry of \( \Sigma = \Sigma(0) \) but on \( \rho' \) too, i.e. on the family of the 2-surfaces \( \Sigma(u) \) as well. In particular for the 2-sphere with \( \rho' = 0, P = \frac{1}{\sqrt{2}} (1 + \xi) \) and \( A_c = 0 \) the choice \( \rho' = \text{const} \) yields the expected value \( \lim_{u \to 0} m^2_{\Sigma(u)} = \frac{4\pi}{\kappa^2} \text{Area} (\Sigma) \), while the choice \( \rho' = \frac{\text{const}}{1+\xi} \) yields \( \frac{32\pi}{9\kappa^2} \text{Area} (\Sigma) \).

**Appendix: The \( \partial \) operator**

The aim of this appendix is to clarify the structure of the kernel spaces of the edth operators if \( \Sigma \) is homeomorphic to a 2-sphere. First recall [7-10] that the spin connection on \( (B, \Sigma, \mathcal{C}^*) \) determines a connection on the associated vector bundle \( E(p,q) \) of scalars of weight \( (p,q) \), \( p - q \in \mathbb{Z} \). The corresponding covariant directional derivations in the directions \( m_a \) and \( \bar{m}_a \) are the usual edth operators \( \partial_{(p,q)} \) and \( \partial'_{(p,q)} \) sending smooth cross sections of \( E(p,q) \) to smooth cross sections of \( E(p+1,q-1) \) and \( E(p-1,q+1) \), respectively. They are elliptic operators and their index was calculated by Baston [10]:

\[
\begin{align*}
\text{index } \partial_{(p,q)} &= \left(1 + p - q\right)(1 - G), \\
\text{index } \partial'_{(p,q)} &= \left(1 - p + q\right)(1 - G);
\end{align*}
\]

where \( G \) is the genus of the closed two-surface \( \Sigma \), and \( G = 0 \) for \( \Sigma \) homeomorphic to a 2-sphere.

The smooth section \( \phi \in E^\infty(p,q) \) is called anti-holomorphic with respect to \( \delta_c \) if \( \partial_{(p,q)} \phi = 0 \). \((E^\infty(p,q) \text{ is the space of the smooth cross sections of } E(p,q)).\) The multiplicity of the zeros of anti-holomorphic sections can also be defined and the zeros are isolated. Since for any \( \phi \in E^\infty(p,q) \) and \( \psi \in E^\infty(p',q') \) \( \phi \psi \in E^\infty(p+p',q+q') \), the Leibnitz rule \( \partial_{(p+p',q+q')} (\phi \psi) = \psi \partial_{(p,q)} \phi + \phi \partial_{(p',q')} \psi \) implies the inequality

\[
\dim \ker \partial_{(p+p',q+q')} \geq \max \left\{ \dim \ker \partial_{(p,q)}, \dim \ker \partial'_{(p',q')} \right\} \quad \text{if} \quad (\dim \ker \partial_{(p,q)})(\dim \ker \partial'_{(p',q')} ) \neq 0.
\]

There is a similar inequality for the primed edth too. If \( h \) is any fixed nowhere zero scalar of weight \( (1,1) \) then for any \( \phi, \psi \in E^\infty(p,q) \)

\[
\langle \phi, \psi \rangle_{(p,q)} := \oint_{\Sigma} \left| h \right|^{-(p+q)} \phi \psi \, d\mathcal{S}
\]

defines a hermitian inner product on \( E^\infty(p,q) \). The adjoint of \( \partial_{(p,q)} \) and \( \partial'_{(p,q)} \) with respect to this inner product is given by

\[
\left( \partial_{(p,q)} \right)^\dagger = -| h |^{(p+q)} \partial_{(-q+1,-p-1)} | h |^{-(p+q)}, \\
\left( \partial'_{(p,q)} \right)^\dagger = -| h |^{(p+q)} \partial_{(-q-1,-p+1)} | h |^{-(p+q)};
\]
Thus there is precisely one scalar $h$ and hence the kernel spaces $\ker(\partial_{(p,q)})^\dagger$ and $\ker(\partial'_{(p,q)})^\dagger$ depend on $h$. Since however (A.4) is a similarity transformation between $(\partial_{(p,q)})^\dagger$ and $\partial'_{(-q+1, -p-1)}$ and $(\partial'_{(p,q)})^\dagger$ and $\partial_{(-q-1, -p+1)}$, respectively, we have

$$\dim \ker(\partial_{(p,q)})^\dagger = \dim \ker(\partial'_{-q+1, -p-1}),$$
$$\dim \ker(\partial'_{(p,q)})^\dagger = \dim \ker(\partial_{-q-1, -p+1}).$$

(A.5)

In the rest of this appendix $G = 0$ will be assumed. Then Liouville’s theorem implies

$$\dim \ker(\partial_{(0,0)}) = \dim \ker(\partial'_{(0,0)}) = 1.$$  

(A.6)

By (A.1,5) this is equivalent to $\dim \ker(\partial_{(-1,1)}) = \dim \ker(\partial'_{(-1,1)}) = 0$, which, by (A.1,2,5), implies $\dim \ker(\partial_{(1,-1)}) = \dim \ker(\partial'_{(1,-1)}) = 3$. Using (A.1,2,5,6) it is not difficult to show by induction that

$$\dim \ker(\partial_{n,-n}) = \dim \ker(\partial'_{-n,n}) = 2n + 1$$
$$\dim \ker(\partial_{-n,n}) = \dim \ker(\partial'_{n,-n}) = 0 \quad \forall n \in \mathbb{N}.$$  

(A.7)

Since by (A.1) $\dim \ker(\partial_{(p,p)}) \geq 1$ and $\dim \ker(\partial'_{(p,p)}) \geq 1$ $\forall p \in \mathbb{R}$, (A.2) and (A.6) imply

$$\dim \ker(\partial_{(p,p)}) = \dim \ker(\partial'_{(p,p)}) = 1 \quad \forall p \in \mathbb{R}.$$  

(A.8)

Thus there is precisely one scalar $h_0 \in E^\infty(1,1)$ satisfying $\partial h_0 = 0$ and precisely one scalar $h_0' \in E^\infty(-1, -1)$ satisfying $\partial' h_0' = 0$ (and hence $h_0' = \partial h_0$).

For any $p \in \mathbb{R}$ and $n \in \mathbb{N}$ (A.1) implies $\dim \ker(\partial_{-p+n,-p}) \geq 1 + n$ and $\dim \ker(\partial'_{-p,-p+n}) \geq 1 + n$. Thus by (A.2) and (A.6) we have

$$\dim \ker(\partial_{(p,-p-n)}) = \dim \ker(\partial'_{(p,-p-n)}) = 0 \quad \forall p \in \mathbb{R}, \quad n \in \mathbb{N}.$$  

(A.9)

Similarly, (A.1) implies $\dim \ker(\partial_{(p+2n,p)}) \geq 1 + 2n$ and $\dim \ker(\partial'_{(p+2n,p)}) \geq 1 + 2n$ and then by (A.2), (A.7) and (A.8)

$$\dim \ker(\partial_{(p+2n,p)}) = \dim \ker(\partial'_{(p+2n,p)}) = 1 + 2n \quad \forall p \in \mathbb{R}, \quad n \in \mathbb{N}.$$  

(A.10)

Finally by (A.1),(A.5) and (A.9)

$$\dim \ker(\partial_{(p+2n-1,p-1)}) = \dim \ker(\partial'_{(p+2n-1,p-1)}) = 2n \quad \forall p \in \mathbb{R}, \quad n \in \mathbb{N}.$$  

(A.11)

(A.8)-(A.11) is the complete list of the dimension of the kernel spaces of the edth operators.

For certain combinations of $p$ and $q$ the elements of these kernel spaces can be realized by special vector and spinor fields on $\mathbb{S}$. First consider surface vector fields $K^a$ satisfying

$$m^a \delta_a K^b = 0 \quad \left(\Pi^c_a K^a = K^b\right);$$  

(A.12)

i.e. that are anti-holomorphic with respect to the induced Levi–Civita connection, or, in other words, with respect to the intrinsic complex structure of $\mathbb{S}$. The components of $K^b$, $\lambda := m_b K^b$ and $\tilde{\lambda} := \bar{m}_b K^b$, are scalars of weight $(1,-1)$ and $(-1,1)$, respectively; and (A.12) is equivalent to $\partial_{(1,-1)} \lambda = 0$ and $\partial_{(-1,1)} \tilde{\lambda} = 0$. Since $\dim \ker(\partial_{(1,-1)}) = 0$, $K^b$ satisfying (A.12) must be proportional to $\bar{m}^b$. The explicit solutions of (A.12) can be given in a coordinate system. Let $\{(U, \xi), (V, \eta)\}$ be the usual complex analytic atlas for $\mathbb{S}$ (i.e. $n \in \mathbb{S}$ (‘north pole’), $U := \mathbb{S} - \{n\}$, $\xi : U \to \mathbb{C}$ and
homeomorphism, $s \in \mathcal{S}$ such that $\xi(s) = 0$ ('south pole'), $V := \mathcal{S} - \{s\}$ and $\eta : V \to \mathbb{C}$ such that $\eta = \frac{1}{\xi}$ on $U \cap V$ and $\eta(n) = 0$; and let the vector field $m^a$ be chosen to be proportional to $\frac{\partial}{\partial \xi}$: $m^a = P(\xi, \bar{\xi}) \left( \frac{\partial}{\partial \xi} \right)^a$ (see e.g. [8]). Then in $(U, \xi)$ the general solution of (A.12) is a countable linear combination of the vector fields

$$K_m^a := \xi^m \left( \frac{\partial}{\partial \xi} \right)^a, \quad m = 0, 1, 2, ...$$  \hspace{1cm} (A.13)

These are the independent conformal Killing vectors of $(U, q_{ab})$, their Lie algebra is the Virasoro algebra (without the central extension): $[K_m, K_n]^a = (n - m)K_m^a + n_{m,n-1}$, and $k_2^a := iK_1^a$, $k_y^a := -\frac{1}{\sqrt{2}}(K_0^a + \frac{1}{2}K_2^a)$, $k_2^a := \frac{i}{\sqrt{2}}(K_0^a - \frac{1}{2}K_2^a)$ are the usual generators of its $so(3, \mathbb{C})$ subalgebra. The behaviour of the vector fields $K_m^a$ in the 'north pole' $n$ can be clarified in $(V, \eta)$: Only $K_0^a$, $K_1^a$ and $K_2^a$ are well defined on the whole $\mathcal{S}$ and $K_0^a$ has a zero in $n$ (of multiplicity 2), $K_1^a$ has zeros in $n$ and $s$ (both of multiplicity 1) and $K_2^a$ has a zero in $s$ (of multiplicity 2). The corresponding (1,-1) weight scalars (in the $(U, \xi)$ coordinate system): $\lambda_0 = -\bar{P}^{-1}$, $\lambda_1 = -\bar{P}^{-1}\bar{\xi}$ and $\lambda_2 = -\bar{P}^{-1}\bar{\xi}^2$, respectively. They are the independent elements of $\ker \partial_{(1,-1)}$ and obviously $\lambda_0\lambda_2 = (\lambda_1)^2$.

Next consider normal vector fields $N^a$ satisfying

$$m^a \delta_a N^b = 0 \quad \left( \Pi^b_N N^c = 0 \right);$$  \hspace{1cm} (A.14)

i.e. which are anti-holomorphic with respect to the induced Levi-Civita connection. Then $\nu := l_a N^a$ and $\nu := n_a N^a$ are scalars of weight (1,1) and (-1,-1), respectively; and (A.14) is equivalent to $\partial_{(1,1)} \nu = 0$ and $\partial_{(-1,-1)} \nu = 0$. Since $\dim \ker \partial_{(1,1)} = \dim \ker \partial_{(-1,-1)} = 1$, (A.14) has precisely two independent solutions: $m^a$ and $\nu^b$; furthermore if $\partial_{(1,1)} \nu = 0$ then, provided $\nu$ is nowhere zero, $\nu = \nu^{-1}$. In the coordinate system $(U, \xi)$ above (A.14) takes the form

$$P \frac{\partial \nu}{\partial \xi} + \nu A_e m^e = 0,$$  \hspace{1cm} (A.15)

where $A_e$ is the 'boost gauge potential'. The general solution of (A.15) on $U$ is the countable linear combination of the scalars

$$\nu_n := \bar{\xi}^n \exp \left( - \int_{\xi_0}^\xi A_e \right), \quad n = 0, 1, 2, ...$$  \hspace{1cm} (A.16)

Here the integration is taken along the complex path whose tangent is $m^a$. However $\nu_n, n^b$ is well defined on the whole $\mathcal{S}$ only for $n = 0$. This serves the only independent element of $\ker \partial_{(1,1)}$, and since this is nowhere zero, the only independent element of $\ker \partial_{(-1,-1)}$ too.

Finally consider spinor fields satisfying

$$m^a \delta_a \omega^R = 0;$$  \hspace{1cm} (A.17)

i.e. that are anti-holomorphic with respect to the induced Levi-Civita connection. $\omega := \omega^Ro_R$ and $\omega := \omega^Rt_R$ are scalars of weight (1,0) and (-1,0), respectively, and (A.17) is equivalent to $\partial_{(1,0)} \omega = 0$ and $\partial_{(-1,0)} \omega = 0$. Since however $\dim \ker \partial_{(1,0)} = 0$ and $\dim \ker \partial_{(-1,0)} = 2$, there are two independent solutions of (A.17), and both must have the form $\omega^R = \omega^R t_R$. Let $(\sigma^A, t^A)$ be a local cross section of the spin frame bundle $(B, \mathcal{S}, \mathbb{C}^*)$ on $U$ so that $\sigma^A t^A = m^a = P(\xi, \bar{\xi}) \left( \frac{\partial}{\partial \xi} \right)^a$, and let $\omega_0$ and $\omega_1$ be the two independent solutions. If $\nu$ is the solution of (A.15) then $\omega_0^2 \nu^{-1}$,
ω₀ω₁ν⁻¹ and ω²ν⁻¹ are independent elements of ker ∂(1,−1), and hence they may be chosen to be λ₀, λ₁ and λ₂ above, respectively. Therefore on U

\[ \omega_0 = i(\nu')^{\frac{1}{2}}, \quad \omega_1 = i(\nu')^{\frac{1}{2}} \xi. \]  

The corresponding spinor fields have single zeros of multiplicity 1 in the ‘north pole’ n and in the ‘south pole’ s, respectively. These serve the independent elements of ker ∂(1,0), and \{ω₀ⁿ, ω₀⁻¹ω₁, ..., ω₁ⁿ\} is a basis in ker ∂(n,0). The sum of the multiplicity of the zeros of the elements of ker ∂(n+p,p) is precisely n for any n ∈ N and p ∈ R.

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References

[1] L.B. Szabados, Two dimensional Sen connections in general relativity, preprint, 1994
[2] R. Penrose, Proc.Roy.Soc.Lond. A 381 53 (1982)
[3] M. Ludvigsen, J.A.G. Vickers, J.Phys.A: Math.Gen. 16 1155 (1983)
[4] G. Bergqvist, M. Ludvigsen, Class.Quantum Grav. 6 L133 (1988); G. Bergqvist, M. Ludvigsen, Class.Quantum Grav. 8 L29 (1991)
[5] A.J. Dougan, L.J. Mason, Phys.Rev.Lett. 67 2119 (1991)
[6] G. Bergqvist, Class.Quantum Grav. 9 1917 (1992)
[7] R. Geroch, A. Held, R. Penrose, J.Math.Phys. 14 874 (1973)
[8] R. Penrose, W. Rindler, Spinors and Spacetime, Vol.1, Cambridge Univ. Press, 1984
[9] S.A. Hugget, K.P. Tod, An Introduction to Twistor Theory, (London Mathematical Society Texts 4) Cambridge University Press, Cambridge 1985
[10] R.J. Baston, Twistor Newsletter, 17 31 (1984)
[11] L.B. Szabados, Class.Quantum Grav. 10 1899 (1993)
[12] R. Penrose, W. Rindler, Spinors and Spacetime, Vol.2, Cambridge Univ. Press, 1986
[13] L.J. Mason, J. Frauendiener, The Sparling 3-form, Ashtekar Variables and Quasi-local Mass, in Twistors in Mathematics and Physics (London Math. Soc. Lecture Notes No. 156) Ed.: T.N. Bailey and R.J. Baston, Cambridge Univ. Press, Cambridge 1990
[14] O. Reula, K.P. Tod, J.Math.Phys. 25 1004 (1984)
[15] M. Ludvigsen, J.A.G. Vickers, J.Phys.A: Math.Gen. 15 L67 (1982)
[16] K.P. Tod, Penrose’s Quasi-local Mass, in Twistors in Mathematics and Physics (London Math. Soc. Lecture Notes No. 156) Ed.: T.N. Bailey and R.J. Baston, Cambridge Univ. Press, Cambridge 1990
[17] A.J. Dougan, Class.Quantum Grav. 9 2461 (1992)
[18] B.P. Jeffryes, Class.Quantum Grav. 3 L9 (1986)
[19] A.D. Helfer, Class.Quantum Grav. 9 1001 (1992)
[20] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, Vol 2, Interscience, 1968
[21] R.O. Wells, Differential Analysis on Complex Manifolds, Prentice–Hall, INC., Englewood Cliffs, New Jersey 1973
[22] R.C. Gunning, Lectures on Riemann Surfaces, (Princeton Mathematical Notes) Princeton Univ. Press, Princeton, New Jersey 1966
[23] A. Sen, J.Math.Phys. 22 1781 (1981)
[24] G. Bergqvist, Class.Quantum Grav. 9 1753 (1992)
[25] R.M. Wald, The first law of black hole mechanics, University of Chicago preprint, 1993
[26] D. Christodoulou, S.-T. Yau, Some remarks on the quasi-local mass, in Mathematics and General Relativity, (Contemporary Mathematics No 71) Ed.: J.A. Isenberg, AMS, New York 1988