The Adiabatic Limit of Fu–Yau Equations

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Abstract  In this paper, we consider the adiabatic limit of Fu–Yau equations on a product of two Calabi–Yau manifolds. We prove that the adiabatic limit of Fu–Yau equations are quasilinear equations.

Keywords  The Fu–Yau equation, adiabatic limit, Strominger system, 2nd Hessian equation

MR(2010) Subject Classification  58J05, 53C55, 35J60

1 Introduction

To study supergravity in theoretical physics, Strominger [16] proposed a new system of equations, now referred to as the Strominger system, on 3-dimensional complex manifolds. It can be viewed as a generalization of the Calabi equation for Ricci flat Kähler metric to non-Kähler spaces [19]. In [8], Fu–Yau gave non-perturbative, non-Kähler solutions of the Strominger system on a toric fibration over a K3 surface constructed by Goldstein–Prokushki [9] and proposed a modified Strominger system in higher dimensions. They [7, 8] proved that it suffices to solve the Fu–Yau equation for the modified Strominger system (Strominger system) in higher dimensions (dimension 3) on the toric fibration over a Calabi–Yau manifold and derived the existence of the Fu–Yau equation in dimension 2. The Fu–Yau equation has been studied extensively, for example [3, 4, 11–13] and the references therein.

In this paper, we consider the adiabatic limit of the Fu–Yau equation. The adiabatic limit of partial differential equations (system) have been studied extensively. For example, Donaldson-Tomas [5] considered the adiabatic limit of holomorphic instanton equations. The adiabatic limit of anti-self-dual connections was studied by Chen [1, 2] on unitary bundles over a product of two compact Riemann surfaces and Calabi–Yau surfaces, respectively. Tosatti [17] researched adiabatic limits of Ricci-flat Kähler metrics on a Calabi–Yau manifold. Fine [6] proved the existence of constant scalar curvature Kähler metrics on certain complex surfaces, via an adiabatic limit. For more interesting papers about the adiabatic limit, we refer to [14, 15] and the references therein.

Now we recall the construction of the toric fibration over a Calabi–Yau manifold in [9]. Let $\beta_1$ and $\beta_2$ be primitive harmonic $(1, 1)$ forms on an $n$-dimension Calabi–Yau manifold $(M, \omega)$...
(ω is a Kähler-Ricci flat metric on M) with a nowhere vanishing holomorphic \((n,0)\)-form \(Ω\) satisfying: \[\left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right] = 0\). The toric fibration \(π: X → M\) is determined by \(β_1\) and \(β_2\).

In this paper, we consider \((M = M_1 × M_2, ω_ε = ε^2 ω_1 ⊕ ω_2)\), where \((M_1, ω_1), (M_2, ω_2)\) are \(n_1\)-dimensional, \(n_2\)-dimensional Ricci flat Kähler manifolds, respectively. Let \(E_i\) be degree zero stable holomorphic vector bundles with Hermitian–Einstein metric \(H_i\) on \(M_i\) with respect to \(ω_i, i = 1, 2\). Denote by \(π_i: M → M_i, i = 1, 2\) the projection map. Consider the bundle \((E = π_1^* E_1 ⊕ π_2^* E_2, H = π_1^* H_1 ⊕ π_2^* H_2)\), then \(H\) is a Hermitian–Einstein metric with respect to \(ω_ε\). Let \(F\) be the Chern curvature of \(H\). Suppose there exist \(β_1, β_2 ∈ H_1^1(M_1, Z), β_3, β_4 ∈ H_1^1(M_2, Z)\) that are primitive harmonic \((1, 1)\) forms with respect to \(ω_1, ω_2\), respectively. Set \(β = π_1^* β_1 ⊕ π_2^* β_3, β_2 = π_1^* β_2 ⊕ π_2^* β_4\), then \(β_1, β_2 ∈ H_1^1(M, Z)\) are primitive harmonic \((1, 1)\) forms with respect to \(ω_ε\) for any \(ε\). Then Fu–Yau equations have the following form [8]

\[
\sqrt{-1} ∂\bar{∂}(e^{φ_ε} ω_ε - α e^{-φ_ε} ρ_ε) ∧ ω_ε^{n-2} + nα_ε \sqrt{-1} ∂\bar{∂}φ_ε^2 ∧ ω_ε^{n-2} + μ_ε ω_ε^n = 0, \tag{1.1}
\]

where

\[
ρ_ε = \frac{1}{2} \sqrt{-1} tr(B ∧ \bar{B}^* · g_ε^{-1})
\]

are smooth \((1, 1)\) forms and

\[
μ = -\frac{α_ε (tr R_ε ∧ R_ε - tr F ∧ F) ∧ ω_ε^{n-2}}{ω_ε^n} + \frac{β_1^2 ∧ ω_ε^{n-2}}{ω_ε^n} + \frac{β_2^2 ∧ ω_ε^{n-2}}{ω_ε^n}. \tag{1.2}
\]

are smooth functions satisfying \(∫_M μ_ε ω_ε^n = 0\). Denote by \(R_ε\) the curvature tensor of \(ω_ε\). Here \(B\) is given in [8, p. 380] which only depends on \(β_1, β_2\) and \(B ∧ \bar{B}^* · g_ε^{-1}\) is a \((1, 1)\) form with value in the endomorphism bundle of \(TM\). \(B^*\) is the conjugate transpose of \(B\). It follows

\[
α_ε = \left(∫_M (tr R_ε ∧ R_ε - tr F ∧ F) ∧ ω_ε^{n-2}\right)^{-1} ∫_M (β_1^2 ∧ ω_ε^{n-2} + β_2^2 ∧ ω_ε^{n-2}).
\]

By \(tr\), we denote the trace of the endomorphism bundle of \(TM\) or \(E\). Since \(tr R_ε ∧ R_ε\) and \(F\) are independent of \(ε\),

\[
||ρ||_{C^0(ω)} ≤ C ε^{-2}, \quad ||μ||_{L^∞} ≤ C ε^{-t}, \tag{1.3}
\]

where \(||ρ||_{C^0(ω)}\) means the \(C^0\) norm of \(ρ\) with respect to \(ω, ω = ω_1 ⊕ ω_2\) and \(t ≥ 4\) is a constant.

We consider general Fu–Yau equations,

\[
\sqrt{-1} ∂\bar{∂}(e^{φ_ε} ω_ε - α e^{-φ_ε} ρ_ε) ∧ ω_ε^{n-2} + nα \sqrt{-1} ∂\bar{∂}φ_ε^2 ∧ ω_ε^{n-2} + μ_ε ω_ε^n = 0, \tag{1.4}
\]

where \(ρ_ε\) is a \((1, 1)\) form and \(μ_ε\) is a smooth function satisfying \(∫_M μ_ε ω_ε^n = 0\). Suppose there exist constants \(t ≥ 4\) and uniform constant \(K\) such that

\[
||e^t μ_ε||_{C^2(ω_ε)} ≤ K, \quad ||e^t ρ_ε||_{C^2(ω_ε)} ≤ K, \quad |e^{-4} α_ε| ≤ K. \tag{1.5}
\]

Without confusion, we also denote \(α_ε\) by \(α\).

Using the argument of [3], we have the following existences and a priori estimates of solutions:

**Theorem 1.1** We can find a uniform small constant \(A_0 > 0\) and a uniform constant \(M_0\) depending only on \(K\) and \((M, ω)\) such that for any positive \(A ≤ A_0\), there exist solutions \(φ_ε\) of (1.4) satisfying the elliptic condition

\[
\bar{ω}_ε = e^{φ_ε} ω_ε + α e^{-φ_ε} ρ_ε + 2nα \sqrt{-1} ∂\bar{∂}φ_ε ∈ Γ_2(M), \tag{1.6}
\]
and the normalization condition
\[
\|e^{-\varphi_\epsilon}\|_{L^1(\omega_\epsilon)} = Ae^{2\eta_1 + t},
\]
where $\Gamma_2(M)$ is the space of the second convex $(1,1)$-forms. In addition, we have
\[
\frac{1}{M_0 Ae^t} \leq e^{\varphi_\epsilon} \leq \frac{M_0}{Ae^t}
\]
and
\[
|\partial \varphi_\epsilon|_{\omega_\epsilon} + \sqrt{-1} \partial \bar{\partial} \varphi_\epsilon|_{\omega_\epsilon} \leq M_0.
\]

Note there exists a subsequence of $\epsilon^t \mu_\epsilon$, still denoted by $\epsilon^t \mu_\epsilon$, weakly converging to a function $\mu_0$ in $L^2(M, \omega)$. We further prove

**Theorem 1.2** Let $\varphi_\epsilon$ be the solution of (1.4) defined in Theorem 1.1. Assume that $t \geq 4$ is a fixed constant and $\mu_\epsilon$ is a function satisfying (1.5) and $\int_M \mu_\epsilon \omega_\epsilon^n = 0$. There exists a subsequence, still denoted by $\varphi_\epsilon$, such that $\varphi_\epsilon + t \ln \epsilon$ converges to a function $\hat{\varphi}$ in $C^{1,\beta}$ on $M_2$ which is a weak solution of
\[
\sqrt{-1} \partial \bar{\partial} (e^{\hat{\varphi}}) \wedge \omega_2^{n_2-1} + \tilde{\mu} \omega_2^{n_2} = 0,
\]
(1.8)
where $\tilde{\mu} = \frac{\int_M \mu_0 \omega_1^n}{\int_M \omega_1^n}$.

For (1.1), choose $t \geq 4$ and uniform constant $K$ such that
\[
\|\epsilon^t \mu\|_{C^2(\omega_\epsilon)} \leq K, \quad \|\epsilon^t \rho\|_{C^2(\omega_\epsilon)} \leq K, \quad |\epsilon^{t-4} \alpha| \leq K.
\]

We have
\[
\epsilon^t \mu \to \mu_0 = \lim_{\epsilon \to 0} \epsilon^t \mu_\epsilon = \lim_{\epsilon \to 0} \epsilon^t \mu_\epsilon - \epsilon^{t-4} \alpha_0 (\text{tr} R_{1\epsilon} \wedge R_{1\epsilon} - \text{tr} F_{1} \wedge F_{1}) \wedge \omega_1^{n-2} \omega_1^4 + \epsilon^{t-4} \left( \frac{\tilde{\beta}_2 \wedge \omega_1^{n_1-2}}{\omega_1^4} + \frac{\tilde{\beta}_2 \wedge \omega_1^{n_1-2}}{\omega_1^4} \right),
\]
where $\alpha_0$ is a constant such that $\int_M \mu_0 \omega^n = 0$. It is a function on $M_1$, therefore $\hat{\mu}$ can be viewed as a constant on $M_2$. Note $\int_M \mu_0 \omega^n = 0$. We have $\hat{\mu} = 0$. It follows from Theorem (1.2):

**Corollary 1.3** Let $\varphi_\epsilon$ be the solution of (1.1) defined in Theorem 1.1. Assume $\mu_\epsilon$ is the function defined in (1.2). There exists a subsequence, still denoted by $\varphi_\epsilon$, such that $\varphi_\epsilon + t \ln \epsilon$ converges to a function $\hat{\varphi}$ in $C^{1,\beta}$ on $M_2$ which is a weak solution of
\[
\sqrt{-1} \partial \bar{\partial} (e^{\hat{\varphi}}) \wedge \omega_2^{n_2-1} = 0.
\]
It follows $\hat{\varphi}$ is a constant.

In Section 2, we give the uniform estimates, using the Chu–Huang–Zhu’s argument [3]. Compared to [3], there are something that are different. First, the Sobolev inequality with respect to $\omega_\epsilon$ depends on $\epsilon$. In addition, there are terms in equations (1.4) involving $\frac{1}{\epsilon}$. We need some delicate calculations in the $C^0$ estimates. In fact, we consider the function $\varphi_\epsilon + \frac{1}{\epsilon}$ which satisfy the Fu–Yau equation that the terms in equation are uniformly bounded for $\epsilon$, but $\alpha$ will tend to 0. It is also important to choose an appropriate $L^1$ norm of $e^{-\varphi_\epsilon}$. When we obtain the $C^0$ estimates, we can use the Chu–Huang–Zhu’s result to prove the $C^1$ and $C^2$
estimates. However, there are $\frac{1}{\alpha}$ in the proof of [3]. We show that Chu–Huang–Zhu’s $C^1$ and $C^2$ estimates are uniform when $\alpha \to 0$ (see Subsection 2.2).

In Section 3, to prove the weak convergence, we use an argument of Tosatti [17]. By the $L^\infty$ estimates for Monge–Ampère equation [10], we can conclude that $\varphi_\epsilon - \frac{1}{\text{Vol}(M)} \int_M \varphi_\epsilon \omega_1^{n_1}$ tends to zero. Then, we can prove that a subsequence of $\varphi_\epsilon$ converges to a function on $M_2$, by Theorem 1.1.

## 2 Proof of Theorem 1.1

In this section, we prove Theorem 1.1.

### 2.1 $C^0$ Estimates

First, we give the zero order estimates.

**Proposition 2.1** Let $\varphi_\epsilon$ be a smooth solution of (1.4). There exist constants $A_0$ and $M_0$ depending only on $K$ and $(M, \omega)$ such that if

$$e^{-\varphi_\epsilon} \leq \epsilon^t \delta_0 := \epsilon^t \frac{1}{2\alpha \|\rho_1\|_{C^0(\omega)} + 1} \leq \epsilon^t \sqrt{\frac{1}{2e^{2t}\|\rho_1\|_{C^0(\omega)} + 1}} \quad (2.1)$$

and

$$\|e^{-\varphi_\epsilon}\|_{L^1(\omega_\epsilon)} = A \epsilon^{2n_1+t} \leq A_0 \epsilon^{2n_1+t}, \quad (2.2)$$

we have

$$\frac{1}{M_0 A \epsilon^t} \leq e^{\varphi_\epsilon} \leq \frac{M_0}{A \epsilon^t}.$$

We need the following the Sobolev inequality and the Poincaré inequality:

**Lemma 2.2** For any $u \in C^\infty(M)$, there exists a uniform constant $C$ such that

$$\|u\|_{L^{2np}(\omega_\epsilon)} \leq C \epsilon^{-\frac{n_1}{p}} (\|u\|_{L^p(\omega)} + \|\partial u\|_{L^p(\omega_\epsilon)}), \quad (2.3)$$

and

$$\int_M \left| u - \frac{1}{\text{Vol}(M)} \int_M u \omega_\epsilon^n \right|^2 \omega_\epsilon^n \leq C \int_M |\partial u|^2 \omega_\epsilon^n \quad (2.4)$$

where $\|u\|_{L^{2np}(\omega_\epsilon)} = (\int_M |u|^{\frac{2np}{n_1(n_1-1)p}} \omega_\epsilon^n)^{\frac{n_1(n_1-1)p}{2np}}$ and $\text{Vol}(M) = \int_M \omega_\epsilon^n$. Here $|\partial u|_\epsilon$ is the norm of gradient of $u$ with respect to $\omega_\epsilon$ and $\|\partial u\|_{L^p(\omega_\epsilon)} = (\int_M |\partial u|_\epsilon^p \omega_\epsilon^n)^{\frac{1}{p}}$.

**Proof** By the Sobolev inequality with respect to $\omega$, we have

$$\|u\|_{L^{\frac{2np}{n_1(n_1-1)p}}(\omega_\epsilon)} \leq \epsilon^{-\frac{n_1(n_1-1)p}{2np}} \|u\|_{L^{2np}(\omega)}.$$ 

Denote $V = \int_M \omega^n$. We have, by the Poincaré inequality with respect to $\omega$,

$$\int_M \left| u - \frac{1}{\text{Vol}(M)} \int_M u \omega_\epsilon^n \right|^2 \omega_\epsilon^n = \epsilon^{2n_1} \left( \int_M u^2 \omega^n - \frac{1}{V} \left( \int_M \omega^n \right)^2 \right)$$
Proof of Proposition 2.1. Multiplying the equation (1.4) by $e^\varphi$, we have
\begin{equation}
\sqrt{-1}\partial\bar{\partial}(e^{\tilde{\varphi}}\omega - \tilde{\alpha} e^{-\tilde{\varphi}} \tilde{\rho}_e) \wedge \omega_{e,-2}^n + nk\sqrt{-1}\partial\bar{\partial}\tilde{\phi}_e^2 \wedge \omega_{e,-2}^n + \tilde{\mu}_e \omega_e^n = 0,
\end{equation}
where $\tilde{\alpha} = e^\varphi \tilde{\alpha}, \tilde{\varphi}_e = \varphi_e + t \ln e, \tilde{\rho}_e = e^\varphi \rho_e$ and $\tilde{\mu}_e = e^\varphi \mu_e$. By (1.5),
\begin{equation}
\|\tilde{\rho}_e\|_{C^2(\omega_e)} + \|\tilde{\mu}_e\|_{C^2(\omega_e)} \leq 2K, \quad \tilde{\alpha} \to 0.
\end{equation}
Then we have, by (2.2),
\begin{equation}
\|e^{-\tilde{\varphi}_e}\|_{L^1(\omega_e)} = A e^{2n_1}.
\end{equation}
First, we estimate the infimum. By Stokes' formula, we derive
\begin{align*}
-2k \int_M e^{-k\tilde{\varphi}_e} (e^{\tilde{\varphi}_e} \omega_e + \tilde{\alpha} e^{-\tilde{\varphi}_e} \tilde{\rho}_e) & \wedge \sqrt{-1}\partial\bar{\partial}\tilde{\phi}_e \wedge \omega_{e,-2}^n \\
= -2k \int_M e^{-k\tilde{\varphi}_e} \sqrt{-1}\partial\bar{\partial}\tilde{\phi}_e \wedge \bar{\partial}(e^{\tilde{\varphi}_e} \omega_e - \tilde{\alpha} e^{-\tilde{\varphi}_e} \tilde{\rho}_e) \wedge \omega_{e,-2}^n \\
& \quad - 2\tilde{\alpha}k \int_M e^{-(k+1)\tilde{\varphi}_e} \sqrt{-1}\partial\bar{\partial}\tilde{\phi}_e \wedge \bar{\partial}\tilde{\rho}_e \wedge \omega_{e,-2}^n \\
= -2 \int_M e^{-k\tilde{\varphi}_e} \sqrt{-1}\partial\bar{\partial}(e^{\tilde{\varphi}_e} \omega_e - \tilde{\alpha} e^{-\tilde{\varphi}_e} \tilde{\rho}_e) \wedge \omega_{e,-2}^n \\
& \quad - 2\tilde{\alpha}k \int_M e^{-(k+1)\tilde{\varphi}_e} \sqrt{-1}\partial\bar{\partial}\tilde{\phi}_e \wedge \bar{\partial}\tilde{\rho}_e \wedge \omega_{e,-2}^n.
\end{align*}
Now we deal with the first term on the right hand side in (2.8).

By (1.6), it follows for $k \geq 2$,
\begin{equation}
k \int_M e^{-k\tilde{\varphi}_e} \sqrt{-1}\partial\bar{\partial}\tilde{\phi}_e \wedge \bar{\partial}\tilde{\phi}_e \wedge \tilde{\omega}_e \wedge \omega_{e,-2}^n \geq 0.
\end{equation}
Using (2.5), (1.6) and integration by parts, we have
\begin{align*}
-2 \int_M e^{-k\tilde{\varphi}_e} \sqrt{-1}\partial\bar{\partial}(e^{\tilde{\varphi}_e} \omega_e - \tilde{\alpha} e^{-\tilde{\varphi}_e} \tilde{\rho}_e) \wedge \omega_{e,-2}^n & = 2n\tilde{\alpha} \int_M e^{-k\tilde{\varphi}_e} \sqrt{-1}\partial\bar{\partial}\tilde{\phi}_e \wedge \sqrt{-1}\partial\bar{\partial}\tilde{\phi}_e \wedge \omega_{e,-2}^n + 2 \int_M e^{-k\tilde{\varphi}_e} \tilde{\mu}_e \frac{\omega_e^n}{n!} \\
& = -2n\tilde{\alpha} \int_M e^{-k\tilde{\varphi}_e} \sqrt{-1}\partial\bar{\partial}\tilde{\phi}_e \wedge \sqrt{-1}\partial\bar{\partial}\tilde{\phi}_e \wedge \omega_{e,-2}^n + 2 \int_M e^{-k\tilde{\varphi}_e} \tilde{\mu}_e \frac{\omega_e^n}{n!} \\
& = 2n\tilde{\alpha} \int_M e^{-k\tilde{\varphi}_e} \sqrt{-1}\partial\bar{\partial}\tilde{\phi}_e \wedge \sqrt{-1}\partial\bar{\partial}\tilde{\phi}_e \wedge \omega_{e,-2}^n + 2 \int_M e^{-k\tilde{\varphi}_e} \tilde{\mu}_e \frac{\omega_e^n}{n!} \\
& \geq -k \int_M e^{-k\tilde{\varphi}_e} (e^{\tilde{\varphi}_e} \omega_e + \tilde{\alpha} e^{-\tilde{\varphi}_e} \tilde{\rho}_e) \wedge \sqrt{-1}\partial\bar{\partial}\tilde{\phi}_e \wedge \bar{\partial}\tilde{\phi}_e \wedge \omega_{e,-2}^n + 2 \int_M e^{-k\tilde{\varphi}_e} \tilde{\mu}_e \frac{\omega_e^n}{n!}.
\end{align*}
Substituting (2.9) into (2.8), we obtain
\[
\begin{align*}
&k \int_M e^{-k \tilde{\varphi}_*} (e^{\tilde{\varphi}_*} \omega_\epsilon + \tilde{\alpha} e^{-\tilde{\varphi}_*} \tilde{\rho}_\epsilon) \wedge \sqrt{-1} \partial \bar{\partial} \tilde{\varphi}_* \wedge \omega_\epsilon^{n-2} \\
&\leq 2\tilde{\alpha} k \int_M e^{-(k+1) \tilde{\varphi}_*} \sqrt{-1} \partial \bar{\partial} \tilde{\varphi}_* \wedge \omega_\epsilon^{n-2} - 2 \int_M e^{-k \tilde{\varphi}_*} \tilde{\mu}_\epsilon \omega_\epsilon^n.
\end{align*}
\]

(2.10)

From the definition of \(\delta_0\) and the condition \(e^{-\tilde{\varphi}_*} \leq \delta_0\), we get
\[
\omega_\epsilon + \tilde{\alpha} e^{-2 \tilde{\varphi}_*} \tilde{\rho}_\epsilon \geq \frac{1}{2} \omega_\epsilon.
\]

(2.11)

By (2.6), (2.10), (2.11) and the Cauchy–Schwarz inequality, we conclude
\[
\begin{align*}
k \int_M e^{-(k-1) \tilde{\varphi}_*} |\partial \tilde{\varphi}_*|_\epsilon^2 \omega_\epsilon^n &\leq Ck \int_M (e^{-(k+1) \tilde{\varphi}_*} |\partial \tilde{\varphi}_*|_\epsilon + e^{-k \tilde{\varphi}_*}) \omega_\epsilon^n \\
&\leq \frac{k}{2} \int_M e^{-(k-1) \tilde{\varphi}_*} |\partial \tilde{\varphi}_*|_\epsilon^2 \omega_\epsilon^n \\
&+ Ck \int_M (e^{-(k+3) \tilde{\varphi}_*} + e^{-k \tilde{\varphi}_*}) \omega_\epsilon^n.
\end{align*}
\]

Note \(e^{-\tilde{\varphi}_*} \leq \delta_0\). It follows
\[
\frac{k}{2} \int_M e^{-(k-1) \tilde{\varphi}_*} |\partial \tilde{\varphi}_*|_\epsilon^2 \omega_\epsilon^n \leq Ck(\delta_0^4 + \delta_0) \int_M e^{-(k-1) \tilde{\varphi}_*} \omega_\epsilon^n,
\]

which implies
\[
\int_M |\partial e^{-(k-1) \tilde{\varphi}_*}|_\epsilon^2 \omega_\epsilon^n \leq Ck^2 \int_M e^{-(k-1) \tilde{\varphi}_*} \omega_\epsilon^n.
\]

Replacing \(k-1\) by \(k\), for \(k \geq 1\),
\[
\|e^{-\tilde{\varphi}_*}\|_{L^{2n}((\omega_\epsilon))} \leq C \frac{1}{\epsilon} e^{-2n_1} \frac{1}{k^{2n_1}} \|e^{-\tilde{\varphi}_*}\|_{L^1((\omega_\epsilon))},
\]

Using the Morse iteration and (2.7), we obtain
\[
\|e^{-\tilde{\varphi}_*}\|_{L^\infty} \leq C \epsilon^{-2n_1} \|e^{-\tilde{\varphi}_*}\|_{L^1((\omega_\epsilon))} \leq CA.
\]

(2.12)

Now we estimate the supremum. By the similar calculation of (2.8)–(2.10), for \(k \geq 1\), we have
\[
\begin{align*}
k \int_M e^{k \tilde{\varphi}_*} (e^{\tilde{\varphi}_*} \omega_\epsilon + \tilde{\alpha} e^{-\tilde{\varphi}_*} \tilde{\rho}_\epsilon) \wedge \sqrt{-1} \partial \bar{\partial} \tilde{\varphi}_* \wedge \omega_\epsilon^{n-2} \\
&\leq 2\tilde{\alpha} k \int_M e^{(k-1) \tilde{\varphi}_*} \sqrt{-1} \partial \bar{\partial} \tilde{\varphi}_* \wedge \omega_\epsilon^{n-2} + 2 \int_M e^{k \tilde{\varphi}_*} \tilde{\mu}_\epsilon \omega_\epsilon^n.
\end{align*}
\]

Combining this with (2.11), we obtain
\[
\int_M (e^{(k+1) \tilde{\varphi}_*} |\partial \tilde{\varphi}_*|_{g_\epsilon}^2 \omega_\epsilon^n \leq C \int_M (e^{(k-1) \tilde{\varphi}_*} |\partial \tilde{\varphi}_*|_{g_\epsilon} + e^{k \tilde{\varphi}_*}) \omega_\epsilon^n.
\]

Then, by \(e^{-\tilde{\varphi}_*} \leq \delta_0\) and the Cauchy–Schwarz inequality,
\[
\int_M e^{(k+1) \tilde{\varphi}_*} |\partial \tilde{\varphi}_*|_{g_\epsilon}^2 \omega_\epsilon^n \leq C \int_M e^{k \tilde{\varphi}_*} \omega_\epsilon^n.
\]

(2.13)

We use (2.12) to get
\[
\int_M e^{k \tilde{\varphi}_*} |\partial \tilde{\varphi}_*|_{g_\epsilon}^2 \omega_\epsilon^n \leq C \int_M e^{k \tilde{\varphi}_*} \omega_\epsilon^n.
\]

(2.13)
By (2.3), it follows \( \|e^{\tilde{\varphi}}\|_{L^{\infty}(\omega_{\epsilon})} \leq C\epsilon^{-\frac{2n_1}{n}}\|e^{\tilde{\varphi}}\|_{L^{1}(\omega_{\epsilon})} \). By the Morse inequality,
\[
\|e^{\tilde{\varphi}}\|_{L^{\infty}} \leq C\epsilon^{-2n_1}\|e^{\tilde{\varphi}}\|_{L^{1}(\omega_{\epsilon})}.
\] (2.14)

Now it suffices to prove the \( L^{1} \) estimate.

Without loss of generality, we assume that Vol\((M, \omega_{\epsilon}) = \epsilon^{2n_1} \). We define a set by
\[
U = \left\{ x \in M \left| e^{-\tilde{\varphi}}(x) \geq \frac{A}{2} \right. \right\}.
\]
Then by (2.12) and (2.7), we have
\[
\epsilon^{2n_1}A = \int_{U} e^{-\tilde{\varphi}} \omega_{\epsilon}^{n} + \int_{M \setminus U} e^{-\tilde{\varphi}} \omega_{\epsilon}^{n} \leq e^{-\inf_{M} \tilde{\varphi}} \text{Vol}_{\epsilon}(U) + \frac{A}{2} (\epsilon^{2n_1} - \text{Vol}_{\epsilon}(U)) \leq \left( C - \frac{1}{2} \right) A \text{Vol}_{\epsilon}(U) + \epsilon^{2n_1} \frac{A}{2},
\]
where Vol\(_{\epsilon}(U)\) is the volume of \( U \) with respect to \( \omega_{\epsilon} \). It implies
\[
\text{Vol}_{\epsilon}(U) \geq \frac{\epsilon^{2n_1}}{C_{0}}.
\] (2.15)

By the Poincaré inequality and (2.13) (taking \( k = 1 \)), we have
\[
\int_{M} e^{2\varphi} \omega_{\epsilon}^{n} - \frac{1}{\epsilon^{2n_1}} \left( \int_{M} e^{\varphi} \omega_{\epsilon}^{n} \right)^{2} \leq C \int_{M} |\partial e^{\varphi}| \omega_{\epsilon}^{n} \leq C \int_{M} e^{\tilde{\varphi}} \omega_{\epsilon}^{n}.
\]
By (2.15) and the Cauchy–Schwarz inequality, we obtain
\[
\left( \int_{M} e^{\tilde{\varphi}} \omega_{\epsilon}^{n} \right)^{2} \leq (1 + C_{0}) \left( \int_{M} e^{\varphi} \omega_{\epsilon}^{n} \right)^{2} + \left( 1 + \frac{1}{C_{0}} \right) \left( \int_{M \setminus U} e^{\tilde{\varphi}} \omega_{\epsilon}^{n} \right)^{2} \leq \frac{4(1 + C_{0})}{A^{2}} \left( \text{Vol}_{\epsilon}(U) \right)^{2} + \left( 1 + \frac{1}{C_{0}} \right) \left( \epsilon^{2n_1} - \text{Vol}_{\epsilon}(U) \right) \int_{M} e^{2\varphi} \omega_{\epsilon}^{n} \leq \frac{4(1 + C_{0})}{A^{2}} \epsilon^{4n_1} + \left( 1 - \frac{1}{C_{0}} \right) \left( \int_{M} e^{\varphi} \omega_{\epsilon}^{n} \right)^{2} + C\epsilon^{2n_1} \int_{M} e^{\tilde{\varphi}} \omega_{\epsilon}^{n}.
\]
It follows
\[
\|e^{\tilde{\varphi}}\|_{L^{1}} \leq \frac{C\epsilon^{2n_1}}{A}.
\] (2.16)

By (2.16) and (2.14), we see that \( \|e^{\tilde{\varphi}}\|_{L^{\infty}} \leq C\epsilon^{-2n_1}\|e^{\tilde{\varphi}}\|_{L^{1}} \leq \frac{C}{A} \). Note \( e^{\varphi} = e^{-t^{2}}e^{\tilde{\varphi}} \). We complete the proof. \( \Box \)

### 2.2 \( C^{1} \) and \( C^{2} \) Estimates

In this subsection, we recall the \( C^{1} \) and \( C^{2} \) estimates in [3]. In fact, their estimates are uniform for \( \alpha \) when \( |\alpha| \) is bounded. However there are some \( \frac{1}{\alpha} \) in the proof. In fact, the \( \frac{1}{\alpha} \) can be cancelled. In the following, we verify this in general manifolds. Let \((M, \omega)\) be an \( n \)-dimension Kähler manifold. In this subsection, we consider the following equation
\[
\sqrt{-1}\partial\bar{\partial}(e^{\varphi}\omega - \alpha e^{-\varphi} \rho) \wedge \omega^{n-2} + n\alpha \sqrt{-1} \partial\bar{\partial} \varphi \wedge \sqrt{-1} \partial\bar{\partial} \varphi \wedge \omega^{n-2} + \mu \frac{\omega^{n}}{n!} = 0,
\] (2.17)
where \( \rho \) is a real-valued smooth \((1, 1)\)-form and \( \mu \) is a smooth function.
Proposition 2.3  Let $\varphi$ be a solution of (2.17) satisfying
\[ \tilde{\omega} = e^{\varphi} \omega + \alpha e^{-\varphi} \rho + 2n\alpha \sqrt{-1} \partial \bar{\partial} \varphi \in \Gamma_2(M). \]  
(2.18)
Assume that
\[ \frac{A}{M_0} \leq e^{-\varphi} \leq M_0 A \quad \text{and} \quad |\partial \bar{\partial} \varphi|_g \leq D, \]
where $M_0$ is a uniform constant. Then there exists a uniform constant $C_0$ such that if
\[ A \leq A_D := \frac{1}{C_0 M_0 D}, \]
then
\[ |\partial \varphi|^2_g \leq C_1, \]
where $C_1$ is a uniform constant for $\alpha$.

Proof  The proof is similar to [3, Proposition 3.1]. It suffices to show that the “$C$”, from [3, (3.9)] to [3, (3.11)], does not depend on $\frac{1}{\alpha}$. For reader’s convenience, we include the proof. We apply the maximum principle to the quantity
\[ Q = \log |\partial \varphi|^2_g + \frac{\varphi}{B}, \]
where $B > 1$ is a large uniform constant to be determined later.

Assume that $Q$ achieves its maximum at $x_0$. Let \( \{e_i\}_{i=1}^n \) be a local unitary frame in a neighborhood of $x_0$ such that, at $x_0$, with respect to $g$
\[ \tilde{g}_{ij} = \delta_{ij} \tilde{g}_{ii} = \delta_{ij} (e^\varphi + \alpha e^{-\varphi} \rho_{ii} + 2n\alpha \varphi_{ii}). \]  
(2.19)
For convenience, we use the following notation:
\[ \tilde{\omega} = e^{-\varphi} \omega, \quad \tilde{g}_{ij} = e^{-\varphi} \tilde{g}_{ij} \quad \text{and} \quad F_{ij} = \frac{\partial \sigma_2(\tilde{\omega})}{\partial \tilde{g}_{ij}}. \]

By [3, (3.9)],
\[ \sum_k F_{ij} (e_k(\varphi_{ij}) \varphi_k^i + \bar{e}_k(\varphi_k^i) \varphi_k) \]
\[ \geq -Ce^{-\varphi}|\partial \varphi|^2_g - Ce^{-\varphi}|\partial \varphi|^2_g + 2|\partial \varphi|^2_g F_{ij} \varphi_{ij} - 2(n-1)\text{Re} \left( \sum_k (|\partial \varphi|^2_g)_{k} \varphi_k^i \right) \]
\[ + 2(n-1)|\partial \varphi|^4_g - \frac{n-1}{\alpha} e^{-\varphi} |\partial \varphi|^2_g f + \frac{n-1}{2\alpha} e^{-\varphi} \text{Re} \left( \sum_k f_k \varphi_k^i \right). \]  
(2.20)

There are two terms containing $\frac{1}{\alpha}$, i.e.
\[ -\frac{n-1}{\alpha} e^{-\varphi} |\partial \varphi|^2_g f + \frac{n-1}{2\alpha} e^{-\varphi} \text{Re} \left( \sum_k f_k \varphi_k^i \right). \]
Recall the definition of $f$ [3, (3.2)]. It can be written as
\[ f = \alpha \tilde{f}, \]  
(2.21)
where
\[ \tilde{f} \omega^n = 2\rho \wedge \omega^{n-1} + \alpha e^{-2\varphi} \rho^2 \wedge \omega^{n-2} - 4n\mu \frac{\omega^n}{n!}. \]
$$+4n\alpha e^{-\psi}\sqrt{-1}(\partial\varphi \wedge \overline{\partial} \varphi \wedge \rho - \partial\varphi \wedge \overline{\partial} \rho - \partial\rho \wedge \overline{\partial} \varphi + \partial\overline{\partial} \rho) \wedge \omega^{n-2}. \quad (2.22)$$

Hence,

$$-\frac{n-1}{\alpha} e^{-\psi}|\partial\varphi|^2 f + \frac{n-1}{2\alpha} e^{-\psi} \text{Re} \left( \sum_k f_k \varphi_k \right)$$

$$= -(n-1) e^{-\psi}|\partial\varphi|^2 \tilde{f} + \frac{n-1}{2} e^{-\psi} \text{Re} \left( \sum_k \tilde{f}_k \varphi_k \right). \quad (2.23)$$

By (2.22), a direct calculation shows that

$$-\frac{n-1}{\alpha} e^{-\psi}|\partial\varphi|^2 f + \frac{n-1}{2\alpha} e^{-\psi} \text{Re} \left( \sum_k f_k \varphi_k \right)$$

$$= -(n-1) e^{-\psi}|\partial\varphi|^2 \tilde{f} + \frac{n-1}{2} e^{-\psi} \text{Re} \left( \sum_k \tilde{f}_k \varphi_k \right)$$

$$\geq -C(e^{-2\psi}|\partial\varphi|_g^2 + e^{-2\psi}|\partial\varphi|_g) \sum_{i,j} (|e_i \varphi_j| + |e_j \varphi_i|) - Ce^{-\psi}|\partial\varphi|^4$$

$$- Ce^{-\psi}|\partial\varphi|^3 - Ce^{-\psi}|\partial\varphi|^2 - Ce^{-\psi}|\partial\varphi|_g$$

$$\geq -\frac{1}{10} \sum_{i,j} (|e_i \varphi_j|)^2 + |e_i e_j| - Ce^{-\psi}|\partial\varphi|^4 - Ce^{-\psi}. \quad (2.24)$$

Here $C$ does not depend on $\frac{1}{\alpha}$. Substitute (2.24) into (2.20), we have

$$\sum_k F^{\alpha}(e_k(\varphi_\alpha)) \varphi_k + \overline{e}_k(\varphi_\alpha) \varphi_k$$

$$\geq 2|\partial\varphi|_g^2 F^{\alpha} \varphi_\alpha + 2(n-1)|\partial\varphi|^4 - 2(n-1) \text{Re} \left( \sum_k (|\partial\varphi|_g^2 \varphi_k) \right)$$

$$- \frac{1}{10} \sum_{i,j} (|e_i \varphi_j|)^2 + |e_i e_j| - Ce^{-\psi}|\partial\varphi|^4 - Ce^{-\psi}. \quad (2.25)$$

The rest of the proof is similar to the proof in [3].

**Proposition 2.4** Let $\varphi$ be a solution of (2.17) satisfying (2.18) and $\frac{A}{M_0} \leq e^{-\psi} \leq M_0 A$ for some uniform constant $M_0$. There exist uniform constants $D_0$ and $C_0$ such that if

$$|\partial\overline{\partial} \varphi|_g \leq D, \quad D_0 \leq D \quad \text{and} \quad A \leq A_D := \frac{1}{C_0 M_0 D},$$

then

$$|\partial\overline{\partial} \varphi|_g \leq \frac{D}{2}.$$  

**Proof** The proof is similar to [3, Proposition 4.1]. It suffices to show that the “$C$”, from [3, (4.6)] to [3, (4.7)], do not depend on $\frac{1}{\alpha}$.

First, we recall the notation in [3]. Consider the following quantity

$$Q = |\partial\overline{\partial} \varphi|_g^2 + B |\partial\varphi|^2,$$

where $B > 1$ is a uniform constant to be determined later. Assume that $Q(x_0) = \max_M Q$. Choose a local $g$-unitary frame $\{e_i\}_{i=1}^n$ for $T^{(1,0)}_c M$ in a neighborhood $x_0$ such that $\bar{g}_c(x_0)$ is
diagonal. Denote
\[ \hat{\omega} = e^{-\varphi}\hat{\omega}, \quad \hat{g}_{ij} = e^{-\varphi}\tilde{g}_{ij}, \quad F^\iota = \frac{\partial\sigma_2(\hat{\omega})}{\partial \hat{g}_{ij}} \quad \text{and} \quad F^\iota,kl = \frac{\partial^2\sigma_2(\hat{\omega})}{\partial \hat{g}_{ij}\partial \hat{g}_{kl}}. \]

It follows
\[ F^\iota = \delta_{ij}F^\iota = \delta_{ij}e^{-\varphi}\sum_{k\neq i}\tilde{g}_{ki}. \]

and
\[ F^\iota,kl = \begin{cases} 1, & \text{if } i = j, k = l, i \neq k; \\ -1, & \text{if } i = l, k = j, i \neq k; \\ 0, & \text{otherwise}. \end{cases} \]

By the assumption of Proposition 2.4, assume that
\[ e^{-\varphi}|\partial\varphi|_g \leq \frac{1}{1000n^3(|\alpha| + 1)B}. \tag{2.26} \]

Then
\[ |F^\iota - (n - 1)| \leq \frac{1}{100} \quad \text{and} \quad |F^\iota,kl| \leq 1. \tag{2.27} \]

By [3, (4.4)],
\[ F^\iota e_k\tilde{\tau}_l(e^{-\varphi}\tilde{g}_{ij}) = I_1 + I_2 + I_3, \tag{2.28} \]

where
\[ I_1 = -F^\iota,kl e_k(e^{-\varphi}\tilde{g}_{ij})\tilde{\tau}_l(e^{-\varphi}\tilde{g}_{ij}), \]
\[ I_2 = -2n(n - 1)\alpha e_k\tilde{\tau}_l(e^{-\varphi}|\partial\varphi|^2)_g, \]
\[ I_3 = \frac{n(n - 1)}{2} e_k\tilde{\tau}_l(e^{-2\varphi}f). \]

In the following, we deal with each term in (2.28) below. Recall
\[ \varphi_{ij} = \sqrt{-1}\partial\varphi(e_i, \tilde{\tau}_j) = e_i\tilde{\tau}_j(\varphi) - [e_i, \tilde{\tau}_j]^{0,1}(\varphi). \tag{2.29} \]

For \( I_1 \), by (2.27), Proposition 2.3 and the Cauchy–Schwarz inequality, we derive
\[ |I_1| \leq \sum_{i,j,k} |e_k(\alpha e^{-2\varphi}\rho_{ij} + 2\alpha e^{-\varphi}\varphi_{ij})|^2 \]
\[ \leq 2\sum_{i,j,k} |e_k(2\alpha e^{-\varphi}\varphi_{ij})|^2 + 2\sum_{i,j,k} |e_k(\alpha e^{-2\varphi}\rho_{ij})|^2 \]
\[ \leq 8n^2\alpha^2e^{-2\varphi}\sum_{i,j,k} |e_ke_i\tilde{\tau}_j(\varphi) - e_k[e_i, \tilde{\tau}_j]^{0,1}(\varphi) - \varphi_k\varphi_{ij}|^2 + C^2e^{-4\varphi} \]
\[ \leq \alpha^2\left(16n^2e^{-2\varphi}\sum_{i,j,k} |e_ke_i\tilde{\tau}_j(\varphi)|^2 + Ce^{-2\varphi}\sum_{i,j} |e_i\tilde{\tau}_j(\varphi)|^2 + |e_ie_j(\varphi)|^2 + Ce^{-2\varphi}\right). \]

Similarly, for \( I_2 \) and \( I_3 \), by (2.21), we get
\[ |I_2| \leq \alpha \left( Ce^{-\varphi}\sum_{i,j,p} |e_pe_i\tilde{\tau}_j(\varphi)| + Ce^{-\varphi}\sum_{i,j} |e_i\tilde{\tau}_j(\varphi)|^2 + |e_ie_j(\varphi)|^2 + Ce^{-\varphi}\right) \]
and
\[ |I_3| = \frac{n(n-1)}{2} e^{-2\varphi} |4\varphi_k \varphi_l f - 2e_k \varphi_l f| - 2\varphi_k f_l - 2\varphi f_k + e_k \varphi_l f| \]
\[ = \alpha \left( \frac{n(n-1)}{2} e^{-2\varphi} |4\varphi_k \varphi_l f - 2e_k \varphi_l f| - 2\varphi_k f_l - 2\varphi f_k + e_k \varphi_l f| \right) \]
\[ \leq \alpha \left( C e^{-2\varphi} \sum_{i,j,p} |e_p e_i \varphi_j(\varphi)| + C e^{-2\varphi} \sum_{i,j} (|e_i \varphi_j(\varphi)|^2 + |e_i e_j(\varphi)|^2) + C e^{-2\varphi} \right), \]
where we used Proposition 2.3 and (2.22). Thus substituting these estimates into (2.28), we conclude
\[ |F^\bar{\eta} e_k \varphi_l(e^{-\varphi} \tilde{g}_{\eta})| \]
\[ \leq 16n^2 \alpha^2 e^{-2\varphi} \sum_{i,j,p} |e_p e_i \varphi_j(\varphi)|^2 + C a e^{-\varphi} \sum_{i,j,p} |e_p e_i \varphi_j(\varphi)| \]
\[ + C a e^{-\varphi} \sum_{i,j} (|e_i \varphi_j(\varphi)|^2 + |e_i e_j(\varphi)|^2) + C a e^{-\varphi}. \]  
(2.30)

In addition, by the definition of \( \tilde{g}_{\eta} \) and (2.29), we have
\[ F^\bar{\eta} e_k \varphi_l(e^{-\varphi} \tilde{g}_{\eta}) \]
\[ = \alpha F^\bar{\eta} e_k \varphi_l(e^{-2\varphi} \rho_{\eta}) + 2\alpha F^\bar{\eta} e_k \varphi_l(e^{-\varphi} \varphi_l \varphi_l) \]
\[ = \alpha F^\bar{\eta} e_k \varphi_l(e^{-2\varphi} \rho_{\eta}) + 2\alpha F^\bar{\eta} e_k \varphi_l(e^{-\varphi} e_i \varphi_l(\varphi)) \]
\[ - 2\alpha F^\bar{\eta} e_k \varphi_l(e^{-\varphi} |e_i, \varphi_l|^{(0,1)}(\varphi)). \]

Then by (2.27) and Proposition 2.3, it follows that
\[ |2\alpha e^{-\varphi} F^\bar{\eta} e_k \varphi_l e_i \varphi_l(\varphi)| \leq |F^\bar{\eta} e_k \varphi_l(e^{-\varphi} \tilde{g}_{\eta})| + C a e^{-\varphi} \sum_{i,j,p} |e_p e_i \varphi_j(\varphi)| \]
\[ + C a e^{-\varphi} \sum_{i,j} (|e_i \varphi_j(\varphi)|^2 + |e_i e_j(\varphi)|^2) + C a e^{-\varphi}. \]

Thus substituting (2.30) into the above inequality, we derive
\[ |F^\bar{\eta} e_k \varphi_l e_i \varphi_l(\varphi)| \leq 8n |\alpha| e^{-\varphi} \sum_{i,j,p} |e_p e_i \varphi_j(\varphi)|^2 + C \sum_{i,j,p} |e_p e_i \varphi_j(\varphi)| \]
\[ + C \sum_{i,j} (|e_i \varphi_j(\varphi)|^2 + |e_i e_j(\varphi)|^2) + C. \]  
(2.31)

Here \( C \) does not depend on \( \frac{1}{\alpha} \). The rest of the proof is similar to the proof of [3, Proposition 4.1]. \( \square \)

Then we have

**Proposition 2.5** Let \( \varphi_\epsilon \) be the solution of (1.4). Assume that
\[ \frac{A \epsilon^4}{M_0} \leq e^{-\varphi_\epsilon} \leq M_0 A \epsilon^4 \quad \text{and} \quad |\partial \varphi_\epsilon|_g \leq D. \]

Then there exists a uniform constant \( C_0 \) such that if
\[ A \leq A_D := \frac{1}{C_0 M_0 D}. \]
then
\[ |\partial \varphi_\epsilon|_{\omega_\epsilon}^2 \leq M_1, \quad |\sqrt{-1} \partial \varphi_\epsilon|_{\omega_\epsilon} \leq \frac{D}{2}, \]
where \( M_1 \) is a uniform constant only depending on \( K \) and \( (M, \omega) \).

Proof. Replacing \( \omega, \varphi, \rho, \alpha \) and \( \mu \) by \( \omega_\varepsilon, \tilde{\varphi}_\varepsilon, \tilde{\rho}_\varepsilon, \tilde{\alpha} \) and \( \tilde{\mu}_\varepsilon \) in the proof of Proposition 2.3 and Proposition 2.4, we obtain it. \( \square \)

### 2.3 Proof of Theorem 1.1

Now, we complete the proof of Theorem 1.1.

Proof. We consider the family equation:

\[
\sqrt{-1}\partial \overline{\partial}(e^{\tilde{\varphi}_\varepsilon C\omega_{\varepsilon}} - s\tilde{\omega}_{\varepsilon} e^{-\tilde{\varphi}_\varepsilon}) = n\tilde{\alpha}(-1)\partial \overline{\partial} \tilde{\varphi}_\varepsilon)^2 \land \omega_{\varepsilon}^{-2} + s\tilde{\mu}_\varepsilon \omega_{\varepsilon}^n = 0. \tag{2.32}
\]

We define the following sets of functions on \( M \),

\[
B = \left\{ u \in C^{2, \beta}(M) \left| \int_M e^{-u} \omega^n = A e^{2n1} \right. \right\},
\]

\[
B_1 = \{(u, s) \in B \times [0, 1] \left| u \text{ satisfies } e^u \omega_{\varepsilon} + \alpha e^{-u} \rho_{\varepsilon} + 2n\alpha \sqrt{-1}\partial \overline{\partial} \omega \in \Gamma_2(M) \right. \}. \tag{2.33}
\]

Note that at \( s = 0, \tilde{\varphi}_\varepsilon = \ln A \). Let \( I \) be the set

\[
\{ s \in [0, 1] \left| \text{ there exists } (\tilde{\varphi}, s) \in B_1 \text{ satisfying } (2.32) \right. \}. \tag{2.34}
\]

By the [3, Subsection 5.1], \( I \) is open. Now we prove \( I \) is closed. The proof is similar to [3, Proposition 5.1]. For reader’s convenience, we include the proof. Let \( \tilde{\varphi}_{s\varepsilon} \) be the solution of \( (2.32) \) at time \( s \). Without confusion, we use \( \tilde{\varphi}_{s} \) to denote \( \tilde{\varphi}_{s\varepsilon} \). We have

**Proposition 2.6** If \( \varphi_s \) satisfies (1.6) and (1.7), there exists a constant \( C_A \) depending only on \( A, K, \beta \) and \( (M, \omega) \) such that

\[
\sup_M e^{-\varphi_s} \leq 2M_0 A, \quad \sup_M |\partial \overline{\partial} \varphi_s|_\varepsilon \leq D_0, \quad \sup_M |\partial \overline{\partial} \varphi_s|^2 \leq C, \quad \|\tilde{\varphi}_s\|_{C^{2, \beta}} \leq C_A, \quad s \in [0, s_0).
\]

Here \( C, M_0, D_0 \) are uniform constant only depending on \( (M, \omega) \) and \( K \).

Proof. First, we give the zero order estimate. In fact, we have

**Claim 1**

\[
\sup_M e^{-\varphi_s} \leq 2M_0 A, \quad s \in [0, s_0), \tag{2.35}
\]

where \( M_0 \) is the constant in Proposition 2.1.

Note that \( \tilde{\varphi}_0 = -\ln A \). Then \( \sup_M e^{-\varphi_0} \leq M_0 A \), which satisfies (2.33). If (2.33) is false, we can find \( \tilde{s} \in (0, s_0) \) such that

\[
\sup_M e^{-\varphi_{\tilde{s}}} = 2M_0 A. \tag{2.36}
\]

Recall \( \delta_0 = \sqrt{\frac{1}{2|\alpha|\|\rho_1\|_{C^{0}(\omega)}}} \) is chosen as in Proposition 2.1. Assume \( 2M_0 A \leq \delta_0 \), then \( e^{-\varphi_{\tilde{s}}} \leq \delta_0 \). Applying Proposition 2.1 to \( \varphi_{s\varepsilon} \), we derive \( e^{-\varphi_{\tilde{s}}} \leq M_0 A \), which contradicts to (2.36). Then we obtain (2.33). Combining (2.33) and Proposition 2.1, the zero order estimate follows.

Using the similar argument, we obtain the second order estimate

\[
\sup_M |\partial \overline{\partial} \varphi_s|_\varepsilon \leq D_0, \tag{2.37}
\]
for any $s \in (0, s_0)$, where $D_0$ is the constant as in Proposition 2.4.

By (2.35) and Proposition 2.5, we have the first order estimate
\[
\sup_M |\partial^2 \tilde{\psi}|^2 \leq C. \tag{2.36}
\]
On the other hand, (2.5) can be written as a 2nd Hessian equation of the form (see [3, 1.8])
\[
\sigma_2(\tilde{\phi}) = F(z, \tilde{\phi}, \partial \tilde{\phi}), \tag{2.37}
\]
where
\[
F(z, \tilde{\phi}, \partial \tilde{\phi}) = \frac{n(n-1)}{2} (e^{2\tilde{\phi}} - 4ae^{\tilde{\phi}}|\partial \tilde{\phi}|^2) + \frac{n(n-1)}{2} f(z, \tilde{\phi}, \partial \tilde{\phi}).
\]
In fact, $f(z, \tilde{\phi}, \partial \tilde{\phi})$ satisfies
\[
|f(z, \tilde{\phi}, \partial \tilde{\phi})| \leq C(e^{-2\tilde{\phi}} + e^{-\tilde{\phi}}|\partial \tilde{\phi}|^2 + 1).
\]

By (2.35), (2.36) and (2.32), we get
\[
\left| \sigma_2(\tilde{\phi}) - \frac{n(n-1)}{2} e^{2\tilde{\phi}} \right| \leq Ce^{\tilde{\phi}}. \tag{2.38}
\]
Using the zero order estimate, we deduce $\frac{1}{e^{\tilde{\phi}} A} \leq \sigma_2(\tilde{\phi}) \leq \frac{C}{e^{\tilde{\phi}}}$. Hence, (2.32) is uniformly elliptic and non-degenerate.

By the $C^{2,\alpha}$-estimate (cf. [18, Theorem 1.1]), we obtain $||\tilde{\phi}_s||_{C^{2,\alpha}} \leq CA$.

Then $I = [0,1]$. By Proposition 2.6, Theorem 1.1 follows. \hfill \Box

3 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. First, we have the following proposition. It plays an important role in the proof of Theorem 1.2.

**Proposition 3.1** Let $\tilde{\phi}_\epsilon = \frac{1}{\text{Vol}(M)} \int_M \tilde{\phi},(\omega_1^n)$. Then
\[
|\tilde{\phi}_\epsilon - \tilde{\phi}_{\tilde{\epsilon}}| \leq Ce^2, \tag{3.1}
\]
where $C$ is a uniform constant only depending on $(M,\omega)$ and $K$.

**Proof** Let $\tilde{\psi} = \frac{1}{\epsilon^2} (\tilde{\phi}_\epsilon - \tilde{\phi}_{\tilde{\epsilon}})$. By Theorem 1.1, we can choose $A$ small enough such that $e^{\tilde{\phi}}(\omega_1^n + \sqrt{-1} \partial \bar{\partial} \tilde{\phi}_\epsilon) > 0$. Then, by Theorem 1.1, we have
\[
(e^{\tilde{\phi}}(\omega_1^n + \sqrt{-1} \partial \bar{\partial} \tilde{\psi})) \leq C.
\]
Using the $L^\infty$ estimate for complex Monge–Ampère equation [10], we have $\|\tilde{\psi}\|_{L^\infty} \leq C$, which implies $|\tilde{\phi}_\epsilon - \tilde{\phi}_{\tilde{\epsilon}}| \leq Ce^2$. \hfill \Box

Now we prove the Theorem 1.2.

**Proof of Theorem 1.2** By Theorem 1.1, there exists a subsequence, still denoted by $\sqrt{-1} \partial \bar{\partial} \tilde{\phi}_\epsilon$, and a function $\tilde{\phi}$ such that $\tilde{\phi}_\epsilon$ converges to $\tilde{\phi}$ in $C^{1,\beta}(M,\omega)$ and $\sqrt{-1} \partial \bar{\partial} \tilde{\phi}_\epsilon$ weakly converges to $\sqrt{-1} \partial \bar{\partial} \tilde{\phi}$. By Proposition 3.1, $\tilde{\phi}$ is a function on $M_2$. Notate that $\tilde{\mu}_\epsilon$ are uniformly bounded. There exists a subsequence, still denoted by $\tilde{\mu}_\epsilon$, weakly converging to a function $\mu_0$ in $L^2(M,\omega)$. Let $\eta$ be a smooth function on $M_2$. Then, by (2.5), we have
\[
\frac{1}{\epsilon^{2n_1}} \int_M \eta \sqrt{-1} \partial \bar{\partial} \tilde{\phi}_\epsilon \wedge \omega_\epsilon^{n-1} - \frac{1}{\epsilon^{2n_1-2} \alpha} \int_M \eta \sqrt{-1} \partial \bar{\partial} (e^{-2\tilde{\phi}_\epsilon} \rho_\epsilon) \wedge \omega_\epsilon^{n-2}
\]
\[ + \frac{1}{\epsilon^{2n_1-t}} \int_M n\alpha(n\sqrt{-1}\partial\bar{\partial}\tilde{\varphi}_\epsilon)^2 \wedge \omega_\epsilon^{n-2} + \frac{1}{\epsilon^{2n_1-t}} \int_M \mu_\epsilon \eta \omega_\epsilon^{2n} = 0. \tag{3.2} \]

First, we estimate the first term. Note
\[ \frac{1}{\epsilon^{2n_1}} \int_M \eta \sqrt{-1}\partial\bar{\partial}e^{\tilde{\varphi}_\epsilon} \wedge \omega_\epsilon^{n-1} \]
\[ = \frac{1}{\epsilon^{2n_1}} \int_M \eta \sqrt{-1}\partial\bar{\partial}(e^{\tilde{\varphi}_\epsilon} - e^{\tilde{\varphi}}) \wedge \omega_\epsilon^{n-1} + \frac{1}{\epsilon^{2n_1}} \int_M \eta \sqrt{-1}\partial\bar{\partial}e^{\tilde{\varphi}} \wedge \omega_\epsilon^{n-1} \]
\[ = \frac{1}{\epsilon^{2n_1}} \int_M (e^{\tilde{\varphi}_\epsilon} - e^{\tilde{\varphi}}) \sqrt{-1}\partial\bar{\partial}\eta \wedge \omega_\epsilon^{n-1} + \frac{1}{\epsilon^{2n_1}} \int_M e^{\tilde{\varphi}} \sqrt{-1}\partial\bar{\partial}\eta \wedge \omega_\epsilon^{n-1} \]
\[ = \int_M (e^{\tilde{\varphi}_\epsilon} - e^{\tilde{\varphi}}) \sqrt{-1}\partial\bar{\partial}\eta \wedge \omega_1^{n_1} \wedge \omega_2^{n_2-1} + \int_M e^{\tilde{\varphi}} \sqrt{-1}\partial\bar{\partial}\eta \wedge \omega_1^{n_1} \wedge \omega_2^{n_2-1} \]
\[ := A_1 + A_2. \tag{3.3} \]

For the first term on the right hand side, using Theorem 1.1 and (3.1), we have
\[ A_1 \to 0. \]

Note \( \tilde{\varphi}_\epsilon \) converges to \( \tilde{\varphi} \). Combining with (3.1), we have
\[ \tilde{\varphi}_\epsilon \to \tilde{\varphi}. \]

Hence, we have
\[ A_2 \to \int_M e^{\tilde{\varphi}} \sqrt{-1}\partial\bar{\partial}\eta \wedge \omega_1^{n_1} \wedge \omega_2^{n_2-1}. \]

Therefore,
\[ \frac{1}{\epsilon^{2n_1-t_2}} \int_M \eta \sqrt{-1}\partial\bar{\partial}(e^{\tilde{\varphi}_\epsilon} \rho_\epsilon) \wedge \omega_\epsilon^{n-2} \]
\[ = \frac{1}{\epsilon^{2n_1-t_2}} \int_M e^{-\tilde{\varphi}_\epsilon} \rho_\epsilon \sqrt{-1}\partial\bar{\partial}\eta \wedge \omega_\epsilon^{n-2} \]
\[ = e^{t_2-2} \int_M e^{-\tilde{\varphi}_\epsilon} \rho_{1\epsilon} \sqrt{-1}\partial\bar{\partial}\eta \wedge \omega_1^{n_1-1} \wedge \omega_2^{n_2-1} \]
\[ + e^{t_2} \int_M e^{-\tilde{\varphi}_\epsilon} \rho_{2\epsilon} \sqrt{-1}\partial\bar{\partial}\eta \wedge \omega_1^{n_1} \wedge \omega_2^{n_2-2} \to 0. \tag{3.5} \]

For the third term in (3.2), by Theorem 1.1, we have
\[ \left| \frac{1}{\omega^n \epsilon^{2n_1-2t}} (\sqrt{-1}\partial\bar{\partial}\tilde{\varphi}_\epsilon)^2 \wedge \omega_\epsilon^{n-2} \right| \leq C \epsilon^{2t}, \]

which implies
\[ \frac{1}{\epsilon^{2n_1-t}} \int_M n\alpha(n\sqrt{-1}\partial\bar{\partial}\tilde{\varphi}_\epsilon)^2 \wedge \omega_\epsilon^{n-2} \to 0. \tag{3.6} \]

For the last term, we have
\[ \frac{1}{\epsilon^{2n_1}} \int_M \tilde{\mu}_\epsilon \eta \omega_\epsilon^{2n} \to \int_{M_2} \eta \left( \int_{M_1} \mu_\eta \omega_1^{n_1} \right) \omega_2^{n_2}. \]
In conclusion, using (3.2), (3.4), (3.5) and (3.6), when $\epsilon \to 0$, we obtain
\[
\int_M e^\hat{\varphi} \sqrt{-\text{I}\partial\bar{\partial}\eta} \, \omega_1^n \wedge \omega_2^{n-1} + \int_{M_2} \eta \left( \int_{M_1} \mu_0 \omega_1^n \right) \omega_2^n = 0.
\]
Recall $\hat{\varphi}$ is independent of the points on $M_1$. It implies
\[
\int_{M_2} e^\hat{\varphi} \sqrt{-\text{I}\partial\bar{\partial}\eta} \, \omega_2^{n-1} + \int_{M_2} \eta \hat{\mu} \omega_2^n = 0,
\]
where $\hat{\mu} = \frac{1}{\int_{M_1} \omega_1^n} \int_{M_1} \mu_0 \omega_1^n$. Therefore, $\hat{\varphi}$ is a weak solution of
\[
\sqrt{-\text{I}\partial\bar{\partial}\varphi} \, \omega_2^{n-1} + \hat{\mu} \omega_2^n = 0.
\]

Acknowledgements

I would like to thank Professor J. Y. Chen for suggesting this problem and for very useful discussions. This work was carried out while I was visiting the Department of Mathematics at the University of British Columbia, supported by the China Scholarship Council (File No. 201906340217). I would like to thank UBC for the hospitality and support.

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