IMAGINARY VERMA MODULES FOR $U_q(\widehat{\mathfrak{sl}(2)})$ AND CRYSTAL-LIKE BASES

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Abstract. We consider imaginary Verma modules for quantum affine algebra $U_q(\widehat{\mathfrak{sl}(2)})$ and define a crystal-like base which we call an imaginary crystal basis using the Kashiwara algebra $K_q$ constructed in earlier work of the authors. In particular, we prove the existence of imaginary crystal-like bases for a suitable category of reduced imaginary Verma modules for $U_q(\widehat{\mathfrak{sl}(2)})$.

1. Introduction

We consider imaginary Verma modules for quantum affine algebra $U_q(\widehat{\mathfrak{sl}(2)})$ and define a crystal-like base which we call an imaginary crystal basis using the Kashiwara algebra $K_q$ constructed in earlier work of the authors. In particular, we prove the existence of imaginary crystal-like bases for a suitable category of reduced imaginary Verma modules for $U_q(\widehat{\mathfrak{sl}(2)})$.

Consider the affine Lie algebra $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}(2)}$ with Cartan subalgebra $\widehat{\mathfrak{h}}$. Let $\{\alpha_0, \alpha_1\}$ be the simple roots, $\delta = \alpha_0 + \alpha_1$ the null root and $\Delta$ the set of roots for $\widehat{\mathfrak{g}}$ with respect to $\widehat{\mathfrak{h}}$. Then we have a natural (standard) partition of $\Delta = \Delta_+ \cup \Delta_-$ into set of positive and negative roots. Corresponding to this standard partition we have a standard Borel subalgebra from which we induce the standard Verma module. Let $S = \{\alpha_1 + k\delta \mid k \in \mathbb{Z}\} \cup \{l\delta \mid l \in \mathbb{Z}_{>0}\}$. Then $\Delta = S \cup -S$ is another closed partition of the root system $\Delta$ which is not Weyl group conjugate to the standard partition. The classification of closed partitions of the root system for affine Lie algebras was obtained by Jakobsen and Kac [JK85, JK89], and independently by Futorny [Fut90, Fut92]. In fact for affine Lie algebras there exists a finite number $(\geq 2)$ of inequivalent Weyl group orbits of closed partitions. For the affine Lie algebra $\widehat{\mathfrak{g}}$ the partition $\Delta = S \cup -S$ is the only nonstandard closed partition which gives rise to a nonstandard Borel subalgebra. The Verma module $M(\lambda)$ with highest weight $\lambda$ induced by this nonstandard Borel subalgebra is called the imaginary Verma module for $\widehat{\mathfrak{g}}$. Unlike the standard Verma module, the imaginary Verma module $M(\lambda)$ contain both finite and infinite dimensional weight spaces.

For generic $q$, consider the associated quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ ([Dri85], [Jim85]). Lusztig [Lus88] proved that the integrable highest weight modules of $\widehat{\mathfrak{g}}$ can be deformed to those over $U_q(\widehat{\mathfrak{g}})$ in such a way that the dimensions of the weight spaces are invariant under the deformation. Following the framework of [Lus88] and [Kan95], it was shown in ([CFKM97], [FGM98]) that the imaginary Verma modules $M(\lambda)$ can also be $q$-deformed to the quantum imaginary Verma modules $M_q(\lambda)$ in

1991 Mathematics Subject Classification. Primary 17B37, 17B15; Secondary 17B67, 17B69.

Key words and phrases. Quantum affine algebras, Imaginary Verma modules, Kashiwara algebras, crystal bases.

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such a way that the weight multiplicities, both finite and infinite-dimensional, are preserved.

Lusztig [Lus90] from a geometric view point and Kashiwara [Kas91] from an algebraic view point introduced the notion of canonical bases (equivalently, global crystal bases) for standard Verma modules $V_q(\lambda)$ and integrable highest weight modules $L_q(\lambda)$. The crystal base ([Kas90, Kas91]) can be thought of as the $q = 0$ limit of the global crystal base or canonical base. An important ingredient in the construction of crystal base by Kashiwara in [Kas91], is a subalgebra $B_q$ of the quantum group which acts on the negative part of the quantum group by left multiplication. This subalgebra $B_q$, which we call the Kashiwara algebra, played an important role in the definition of the Kashiwara operators which defines the crystal base. In [CFM10] we constructed an analog of Kashiwara algebra, denoted by $K_q$ for the imaginary Verma module $M_q(\lambda)$ for the quantum affine algebra $U_q(\hat{g})$ by introducing certain Kashiwara-type operators. Then we proved that a certain quotient $N_q^-$ of $U_q(\hat{g})$ is a simple $K_q$-module and gave a necessary and sufficient condition for a particular quotient $\tilde{M}_q(\lambda)$ (called reduced imaginary Verma module) of $M_q(\lambda)$ to be simple. These results were generalized to any affine Lie algebra of $ADE$ type in [CFM14].

In this paper we consider a category $O_{red,im}^q$ of $U_q(\hat{g})$-modules and define a crystal-like basis which we call imaginary crystal basis for modules in this category. We show that the reduced imaginary Verma modules $\tilde{M}_q(\lambda)$ are in $O_{red,im}^q$. Then we show that any module in $O_{red,im}^q$ is a direct sum of reduced imaginary Verma modules for $U_q(\hat{g})$. Finally we prove the existence of imaginary crystal basis for the reduced imaginary Verma module $\tilde{M}_q(\lambda)$.

The paper is organized as follows. In Sections 2 we recall necessary definitions and properties about the algebra $U_q(\hat{g})$ that we need. In Section 3 we recall the definitions and relations of $\Omega$-operators defined in [CFM10]. In Section 4, we recall the definition of the Kashiwara algebra $K_q$ and the symmetric bilinear form $\langle , \rangle$ on the simple $K_q$-module $N^-_q$ from [CFM10] and show that this form satisfies certain orthonormality condition modulo $q^2$ and is non-degenerate. In Section 5 we recall the definitions and properties of imaginary Verma modules $M(\lambda)$ for the affine Lie algebra $\hat{g}$ and the reduced imaginary Verma modules $\tilde{M}(\lambda)$. In Section 6 we define the category $O_{red,im}^q$ of $\hat{g}$-modules and show that this category is a Serre category and any module in this category is a direct sum of some simple reduced imaginary Verma modules. In Section 7 we recall some basic results about quantized imaginary Verma modules and reduced quantized imaginary Verma modules for $U_q(\hat{g})$. In Section 8 we define the category $O_{red,im}^q$ of $U_q(\hat{g})$-modules containing the reduced quantized imaginary modules $\tilde{M}_q(\lambda)$ and define Kashiwara type operators $\Omega_q(m)$ and $\tilde{x}_m$ on $\tilde{M}_q(\lambda)$. In Section 9 we define the imaginary crystal basis for any module $M \in O_{red,im}^q$ and prove the existence of an imaginary crystal basis for any reduced quantized Verma module $\tilde{M}_q(\lambda)$.

2. Notation

2.1. Let $F$ denote a field of characteristic zero. The quantum group $U_q(A_1^{(1)})$ is the $\mathbb{F}(q^{1/2})$-algebra with 1 generated by

$$e_0, e_1, f_0, f_1, K_0^{\pm 1}, K_1^{\pm 1}, D^{\pm 1}$$
with defining relations:

\[
DD^{-1} = D^{-1}D = K_i K_i^{-1} = K_i^{-1}K_i = 1,
\]
\[
e_i f_j - f_j e_i = \frac{\delta_{ij}}{q - q^{-1}} (K_i - K_i^{-1}),
\]
\[
K_i e_i K_i^{-1} = q^{2}e_i, \quad K_i f_j K_i^{-1} = q^{-2}f_j, \quad \text{if } i \neq j,
\]
\[
K_i K_j - K_j K_i = 0, \quad K_i D - DK_i = 0,
\]
\[
D e_i D^{-1} = q^{\delta_{i,0}} e_i, \quad D f_i D^{-1} = q^{-\delta_{i,0}} f_i,
\]
\[
e_i^3 e_j - [3] e_i^2 e_j e_i + [3] e_i e_j e_i^2 - e_j e_i^3 = 0, \quad i \neq j,
\]
\[
f_i^3 f_j - [3] f_i^2 f_j f_i + [3] f_i f_j f_i^2 - f_j f_i^3 = 0, \quad i \neq j,
\]

where, \([n] = \frac{q^n - q^{-n}}{q - q^{-1}}\).

The quantum group \(U_q(A_1^{(1)})\) can be given a Hopf algebra structure with a comultiplication given by

\[
\Delta(K_i) = K_i \otimes K_i,
\]
\[
\Delta(D) = D \otimes D,
\]
\[
\Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i,
\]
\[
\Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i,
\]

and an antipode given by

\[
s(e_i) = -e_i K_i^{-1},
\]
\[
s(f_i) = -K_i f_i,
\]
\[
s(K_i) = K_i^{-1},
\]
\[
s(D) = D^{-1}.
\]

There is an alternative realization for \(U_q(A_1^{(1)})\), due to Drinfeld [Dri85], which we shall also need. Let \(U_q\) be the associative algebra with 1 over \(\mathbb{F}(q^{1/2})\) generated by the elements \(x^\pm_k (k \in \mathbb{Z}), h_l (l \in \mathbb{Z} \setminus \{0\}), K^{\pm 1}, D^{\pm 1}\), and \(\gamma^{\pm \frac{1}{2}}\) with the following
Drinfeld’s relations (3), (8)-(10) can be written as

\[(2.13)\]

\[DD^{-1} = D^{-1}D = KK^{-1} = K^{-1}K = 1,\]

\[(2.2)\]

\[\gamma \pm \frac{1}{2}, u] = 0 \ \forall u \in U,\]

\[(2.3)\]

\[h_k, h_l = \delta_{k+l,0} \frac{2k}{k} \gamma^k - \gamma^{-k},\]

\[(2.4)\]

\[h_k, K] = 0, \ [D, K] = 0,\]

\[(2.5)\]

\[Dh_kD^{-1} = q^k h_k,\]

\[(2.6)\]

\[Dx_k^\pm D^{-1} = q^k x_k^\pm,\]

\[(2.7)\]

\[Kx_k^\pm K^{-1} = q^{\pm 2} x_k^\pm,\]

\[(2.8)\]

\[[h_k, x_l^\pm] = \pm \frac{[2k]}{k} \gamma^\pm \frac{\mu_k}{v} x_k^\pm,\]

\[(2.9)\]

\[x_{k+1}^\pm x_l^\mp - q^{\pm 2} x_l^\mp x_{k+1}^\pm = q^{\pm 2} x_k^\mp x_{k+1}^\pm - x_{k+1}^\pm x_k^\pm,\]

\[(2.10)\]

\[x_k^\pm, x_l^\mp = \frac{1}{q - q^{-1}} \left( \gamma^{\pm 1} \psi(k + l) - \gamma^{\pm 1} \phi(k + l) \right),\]

\[(2.11)\]

\[\sum_{k=0}^{\infty} \psi(k) z^{-k} = K \exp \left( (q - q^{-1}) \sum_{k=1}^{\infty} h_k z^{-k} \right),\]

\[(2.12)\]

\[\sum_{k=0}^{\infty} \phi(-k) z^k = K^{-1} \exp \left( -(q - q^{-1}) \sum_{k=1}^{\infty} h_k z^k \right).\]

The algebras $U_q(A_1^{(1)})$ and $U_q$ are isomorphic [Dri85]. The action of the isomorphism, which we shall call the Drinfeld Isomorphism, on the generators of $U_q(A_1^{(1)})$ is given by:

\[e_0 \mapsto x_1^1 K^{-1}, \ f_0 \mapsto K x_1^{-1},\]

\[e_1 \mapsto x_0^1, \ f_1 \mapsto x_0^{-1},\]

\[K_0 \mapsto \gamma K^{-1}, \ K_1 \mapsto K, \ D \mapsto D.\]

If one uses the formal sums

\[(2.13)\]

\[\phi(u) = \sum_{p \in \mathbb{Z}} \phi(p) u^{-p}, \ \psi(u) = \sum_{p \in \mathbb{Z}} \psi(p) u^{-p}, \ x^\pm(u) = \sum_{p \in \mathbb{Z}} x^\pm(p) u^{-p}\]

Drinfeld’s relations (3), (8)-(10) can be written as

\[(2.14)\]

\[[\phi(u), \phi(v)] = 0 = [\psi(u), \psi(v)]\]

\[(2.15)\]

\[\phi(u)x^\pm v)\phi(u)^{-1} = g(uv^{-1} \gamma^{\mp 1/2}; x^\pm(v)\]

\[(2.16)\]

\[\psi(u)x^\pm v)\phi(u)^{-1} = g(vu^{-1} \gamma^{\mp 1/2}; x^\pm(v)\]

\[(2.17)\]

\[(u - q^{\pm 2} v)x^\pm(u)x^\pm(v) = (q^{\pm 2} u - v)x^\pm(v)x^\pm(u)\]

\[(2.18)\]

\[x^+(u), x^-(v) = (q - q^{-1})^{-1} \delta(u/v) \psi(v') - \delta(u/v) \phi(u')\]

where $g(t) = g_q(t) = \sum_{k \geq 0} g_k t^k$ is the Taylor series at $t = 0$ of the function $(q^2 t - 1)/(t - q^2)$ and $\delta(z) = \sum_{k \in \mathbb{Z}} z^k$ is the formal Dirac delta function.
Remark 2.1.1. Writing $g(t) = g_q(t) = \sum_{r \geq 0} g(r)t^r$ we have

$$
(2.19) \quad g(r) = g_q(r) = g_{q-1}(r) = \begin{cases} 
q^2 & \text{if } r = 0 \\
(1 - q^{-4})q^{2(r+1)} = (q^4 - 1)q^{2(r-1)} & \text{if } r > 0.
\end{cases}
$$

Considering Serre’s relation with $k = l$, we get

$$
(2.20) \quad x_k^- x_{k+1}^- = q^2 x_{k+1}^- x_k^-
$$

The product on the right side is in the correct order for a basis element. If $k + 1 > l$ and $k \neq l$ in (2.9), then $k + 1 > l + 1$ so that $k \geq l + 1$, and thus we can write

$$
(2.21) \quad x_k^- x_{k+1}^- = q^2 x_{k+1}^- x_k^- - x_k^- x_{k+1}^- + q^2 x_{k+1}^- x_k^-
$$

and then after repeating the above identity, we will eventually arrive at sums of terms that are in the correct order. This is the opposite ordering of monomials as we had previously.

3. $\Omega$-operators and their relations

Let $\mathbb{N}^{\mathbb{N}^r}$ denote the set of all functions from $\{k \delta \mid k \in \mathbb{N}^r\}$ to $\mathbb{N}$ with finite support. Then we can write

$$
\hat{h}^+ = (s_k) := h_{r_1}^{s_1} \cdots h_{r_l}^{s_l}, \quad \hat{h}^- := (s_k) := h_{r_1}^{-s_1} \cdots h_{r_l}^{-s_l},
$$

for $f = (s_k) \in \mathbb{N}^{\mathbb{N}^r}$ whereby $f(r_k) = s_k$ and $f(t) = 0$ for $t \neq r_i, 1 \leq i \leq l$.

Consider now the subalgebra $\mathcal{N}_q^-$, generated by $\gamma^{\pm 1/2}$, and $x_l^-, l \in \mathbb{Z}$. Note that the corresponding relations (9) hold in $\mathcal{N}_q^-$. Consider $x^-(v) = \sum_m x_m^- v^m$ as a formal power series of left multiplication operators $x_m^- : \mathcal{N}_q^- \to \mathcal{N}_q^-$. As in our previous paper we set

$$
P = x^-(v_1) \cdots x^-(v_k)
$$

and

$$
P_l = x^-(v_1) \cdots x^-(v_{l-1})x^-(v_{l+1}) \cdots x^-(v_k),
$$

and

$$
G_l = G_{l/q}^{1/q} := \prod_{j=1}^{l-1} g_{q-1}(v_j/v_l), \quad G_l^q = \prod_{j=1}^{l-1} g(v_j/v_l)
$$

where $G_1 := 1$. As in our previous work we define a collection of operators $\Omega_\psi(k), \Omega_\phi(k) : \mathcal{N}_q^- \to \mathcal{N}_q^-$, $k \in \mathbb{Z}$, in terms of the generating functions

$$
\Omega_\psi(u) = \sum_{l \in \mathbb{Z}} \Omega_\psi(l)u^{-l}, \quad \Omega_\phi(u) = \sum_{l \in \mathbb{Z}} \Omega_\phi(l)u^{-l}
$$

by setting

$$
(3.1) \quad \Omega_\psi(u)(P) := \gamma^m \sum_{l=1}^{k} G_l \hat{P}_l \delta(u/v_l \gamma)
$$

$$
(3.2) \quad \Omega_\phi(u)(P) := \gamma^m \sum_{l=1}^{k} G_l^q \hat{P}_l \delta(u \gamma/v_l).
$$

Note that $\Omega_\psi(u)(1) = \Omega_\phi(u)(1) = 0$. More generally let us write

$$
P = x^-(v_1) \cdots x^-(v_k) = \sum_{n \in \mathbb{Z}} \sum_{n_1, n_2, \ldots, n_k} x_{n_1}^- \cdots x_{n_k}^- v_1^{-n_1} \cdots v_k^{-n_k}
$$
Then
\[
\psi(u^{-1}/2)\Omega_p(u)(\bar{P}) = \sum_{k \geq 0} \sum_{p \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \gamma^{k/2} \psi(k)\Omega_p(p)(x_{-1} \cdots x_{-k})v_1^{n_1} \cdots v_k^{n_k} u^{-k-p}
\]
\[
= \sum_{n_1 \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \sum_{k \geq 0} \gamma^{k/2} \psi(k)\Omega_p(m-k)(x_{-1} \cdots x_{-k})v_1^{n_1} \cdots v_k^{n_k} u^{-m}
\]
while
\[
[x^+(u), \bar{P}] = \sum_{m \in \mathbb{Z}} \sum_{n_1, n_2, \ldots, n_k \in \mathbb{Z}} [x^+_m, x^-_{n_1} \cdots x^-_{n_k}]v_1^{n_1} \cdots v_k^{n_k} u^{-m}.
\]
Thus for a fixed \(m\) and \(k\)-tuple \((n_1, \ldots, n_k)\) the sum
\[
\sum_{k \geq 0} \gamma^{k/2} \psi(k)\Omega_p(m-k)(x_{-1} \cdots x_{-k})
\]
must be finite. Hence
\[
(3.3) \quad \Omega_p(m-k)(x_{-1} \cdots x_{-k}) = 0,
\]
for \(k\) sufficiently large.

**Proposition 3.0.2.** Then
\[
(3.4) \quad \Omega_p(u)x^-(v) = \delta(v\gamma/u) + g_{\gamma-1}(v\gamma/u)x^-(v)\Omega_p(u),
\]
\[
(3.5) \quad \Omega_p(u)x^-(v) = \delta(u\gamma/v) + g(u\gamma/v)x^-(v)\Omega_p(u)
\]
\[
(3.6) \quad (q^2u_1 - u_2)\Omega_p(u_1)\Omega_p(u_2) = (u_1 - q^2u_2)\Omega_p(u_2)\Omega_p(u_1)
\]
\[
(3.7) \quad (q^2u_1 - u_2)\Omega_p(u_1)\Omega_p(u_2) = (u_1 - q^2u_2)\Omega_p(u_2)\Omega_p(u_1)
\]
\[
(3.8) \quad (q^2 \gamma u - u_2)\Omega_p(u_1)\Omega_p(u_2) = (\gamma^2u_1 - q^2u_2)\Omega_p(u_2)\Omega_p(u_1)
\]
The identities in Proposition 3.0.2 can be rewritten as
\[
(3.9) \quad (q^2v\gamma - u)\Omega_p(u)v^-(v) = (q^2v\gamma - u)\delta(v\gamma/u) + (q^2v\gamma - u)v^-(v)\Omega_p(u),
\]
\[
(3.10) \quad (q^2v - u\gamma)\Omega_p(u)v^-(v) = (q^2v - u\gamma)\delta(v/u\gamma) + (v - q^2u\gamma)v^-(v)\Omega_p(u)
\]
which may be written out in terms of components as
\[
(3.11) \quad q^2\gamma\Omega_p(m)x^-(n+1) - \Omega_p(m+1)x^- = (q^2 - \gamma)\delta_{m,n} + \gamma x_{n+1}^m\Omega_p(m) - q^2x_n^m\Omega_p(m+1),
\]
\[
(3.12) \quad q^2\Omega_p(m)x^-(n+1) - \gamma\Omega_p(m+1)x^-(n) = (q^2 - \gamma)\delta_{m,n} + x^-(n+1)\Omega_p(m) - q^2x^-n\Omega_p(m+1).
\]
We also have by (3.8)
\[
(3.13) \quad \Omega_p(k)\Omega_p(m) = \sum_{r \geq 0} g(r)\gamma^{2r}\Omega_p(r + m)\Omega_p(k - r),
\]
as operators on \(N^q_{\gamma}\).

We can also write (3.4) in terms of components and as operators on \(N^q_{\gamma}\)
\[
(3.14) \quad \Omega_p(k)x^-(m) = \delta_{k,m}\gamma^k + \sum_{r \geq 0} g_{\gamma-1}(r)x^-(m + r)\Omega_p(k - r)\gamma^r.
\]
The sum on the right hand side turns into a finite sum when applied to an element in \(N^q_{\gamma}\), due to (3.3).
4. The Kashiwara algebra $\mathcal{K}_q$

The Kashiwara algebra $\mathcal{K}_q$ is defined to be the $\mathbb{F}(q^{1/2})$-subalgebra of $\text{End}(N_q)$ generated by $\Omega_\psi(m), x_n^-, \gamma_{\pm 1/2}, m, n \in \mathbb{Z}, \gamma_{\pm 1/2}$. Then the $\gamma_{\pm 1/2}$ are central and the following relations (which are implied by (3.14)) are satisfied

\begin{align}
(4.1) \quad & q^2\gamma \Omega_\psi(m)x_{n+1} - \Omega_\psi(m+1)x_n^- = (q^2\gamma - 1)\delta_{m,-n-1} + \gamma x_{n+1}^- \Omega_\psi(m) - q^2x_n^- \Omega_\psi(m+1) \\
(4.2) \quad & q^2\Omega_\psi(k+1)\Omega_\psi(l) - \Omega_\psi(k)\Omega_\psi(l+1) = \Omega_\psi(k)\Omega_\psi(l) - q^2\Omega_\psi(l+1)\Omega_\psi(k) \\
(4.3) \quad & x_i x_{k+1}^- - q^2x_{k+1}^- x_i = q^2x_{i+1}^- x_k - x_k^- x_{i+1}
\end{align}

together with

\[ \gamma^{1/2}\gamma^{-1/2} = 1 = \gamma^{-1/2}\gamma^{1/2}. \]

**Proposition 4.0.3.** [CFM10] There is a unique symmetric bilinear form $(\ , \ )$ defined on $N_q^-$ satisfying

\[(x_m^-, b) = (a, \Omega_\psi(-m)b), \quad (1, 1) = 1.\]

For $m = (m_1, \ldots, m_n)$ set

\[ x_m = x_{m_1}^- \cdots x_{m_n}^- \]

and define the length of such a Poincare-Birkhoff-Witt basis element to be $|m| = n$.

**Proposition 4.0.4.** For $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, and $k = (k_1, \ldots, k_l) \in \mathbb{Z}^l$, if $n > l$, then

\[(4.4) \quad (x_m, x_k) = 0.\]

On the other hand if $n = l$ with

\[ m_1 \geq m_2 \geq \cdots \geq m_n, \quad k_1 \geq k_2 \geq \cdots \geq k_n, \quad \sum_{i=1}^n m_i = \sum_{i=1}^n k_i \]

we have

\[(4.5) \quad (x_m, x_k) \equiv \delta_{m,k} \mod q^2\mathbb{Z}[q].\]

and the form is symmetric.

**Proof.** The fact that the form is symmetric comes from Proposition 4.0.3 above. Suppose $n > l$. Then

\[
(x_{m_1} \cdots x_{m_n}, x_{k_1} \cdots x_{k_l}) = (x_{m_2} \cdots x_{m_n}, \Omega_\psi(-m_1)x_{k_1} \cdots x_{k_l})
= \delta_{m_1,k_1}(x_{m_2} \cdots x_{m_n}, x_{k_2} \cdots x_{k_n})
+ \sum_{r \geq 0} g_q^{-r}(r)(x_{m_2} \cdots x_{m_n}, x_{k_1+r}\Omega_\psi(-m_1-r)x_{k_2} \cdots x_{k_l}).
\]

By the Serre relations (2.20) and (2.21)

\[ x_{k_1+r}\Omega_\psi(-m_1-r)x_{k_2} \cdots x_{k_l} \]

is a sum of monomials of length $l - 1$ we can use induction to see that

\[(x_{m_2} \cdots x_{m_n}, x_{k_1+r}\Omega_\psi(-m_1-r)x_{k_2} \cdots x_{k_l}) = 0.\]
Hence \((x_{m_1} \cdots x_{m_n}, x_{k_1} \cdots x_{k_l}) = 0\).

Now suppose \(n = l\). For \(n = 1\) we have

\[
(x_{m}, x_{k}) = (1, \Omega_{\psi}(-m)x_{m}) = \delta_{m,k}
\]

by (3.14).

For \(n = 2\) we have by (3.14) for \(m_1 \geq m_2\), \(k_1 \geq k_2\) and \(m_1 + m_2 = k_1 + k_2\)

\[
(x_{m_1}, x_{m_2}, x_{k_1}, x_{k_2}) = (x_{m_2}, \Omega_{\psi}(-m_1)x_{k_1}, x_{k_2})
\]

\[
= \delta_{m_1,k_1}(x_{m_2}, x_{k_2}) + \sum_{r \geq 0} g_{q^{-1}}(r)(x_{m_2}, x_{k_1+r}, \Omega_{\psi}(-m_1 - r)x_{k_2})
\]

\[
= \delta_{m,k} + \sum_{r \geq 0} g_{q^{-1}}(r)(x_{m_2}, x_{k_1+r})\delta_{m_1+r,k_2}
\]

\[
= \delta_{m,k} + \sum_{r \geq 0} g_{q^{-1}}(r)\delta_{m_2,k_1+r}\delta_{m_1+r,k_2}
\]

\[
= \delta_{m,k} + H(k_2 - m_1)g_{q^{-1}}(k_2 - m_1)\delta_{m_2-k_1,k_2-m_1}
\]

where \(H\) is the Heaviside function given by \(H(n) = 1\) if \(n \geq 0\) and \(H(n) = 0\) otherwise. Interchanging \((m_1, m_2) \leftrightarrow (k_1, k_2)\) in the above calculation we see that the \((x_{m_1}, x_{m_2}, x_{k_1}, x_{k_2}) = (x_{k_1}, x_{k_2}, x_{m_1}, x_{m_2})\). Now if \(k_2 - m_1 \neq 1\), then it is clear from (2.19), that \((x_{m_1}, x_{m_2}, x_{k_1}, x_{k_2}) \in \delta_{m,k} + q^2Z[q]\). If \(k_2 - m_1 = 1\), then the second summand above is nonzero if and only if \(m_2 - k_1 = 1\). But then

\[
m_1 \geq m_2 = k_1 + 1 > k_1 \geq k_2 = m_1 + 1
\]

which is impossible. Hence for \(n = 2\), we have (4.5).

Assume that (4.5) holds up to Poincare-Birkhoff-Witt monomials of length \(n - 1\). Let us first prove by induction that for all \(1 \leq i \leq n - 1\) and any \(p \in \mathbb{N}\),

\[
(x_{m_1}^{-1} \cdots x_{m_i}^{-1}, x_{s_1}^{-1} x_{s_2}^{-1} \cdots x_{s_i}^{-1}, \Omega_{\psi}(-m_1 - p)x_{s_{i+1}}^{-1} \cdots x_{s_n}^{-1}) \in Z[q],
\]

(4.6)

\[
(x_{m_2}^{-1} \cdots x_{m_n}^{-1}, x_{s_2}^{-1} \cdots x_{s_n}^{-1}) \in Z[q]
\]

(4.7)

for any \(s = (s_2, \ldots, s_n) \in \mathbb{Z}\) (so that \(x_{s_2}^{-1} \cdots x_{s_n}^{-1}\) is not necessarily a PBW monomial).

We say that \((s_2, \ldots, s_n)\) has \(k\) ascending inversions if the number of pairs of indices \((i, l)\) with \(i < j\) and \(s_i < s_j\) is \(k\). Recall the Serre relations (2.20) and (2.21). Suppose there is an ascending inversion at the pair of indices \((i, i+1)\) with \(s_i = k\) and \(s_{i+1} = k + 1\), then

\[
(x_{m_2}^{-1} \cdots x_{m_n}^{-1}, x_{s_2}^{-1} \cdots x_{s_{i+1}}^{-1} x_{s_i}^{-1} x_{s_{i+1}}^{-1} \cdots x_{s_n}^{-1}) = q^2(x_{m_2}^{-1} \cdots x_{m_n}^{-1}, x_{s_2}^{-1} \cdots x_{s_{i+1}}^{-1} x_{s_i}^{-1} x_{s_{i+1}}^{-1} \cdots x_{s_n}^{-1}).
\]

(4.8)

Then we have decreased the number of ascending inversions and by induction on the number of inversion on products of length \(n - 1\) we conclude

\[
(x_{m_2}^{-1} \cdots x_{m_n}^{-1}, x_{s_2}^{-1} \cdots x_{s_i}^{-1} x_{s_{i+1}}^{-1} \cdots x_{s_n}^{-1}) \in Z[q].
\]
Suppose there is an ascending inversion at the pair of indices \((i, i+1)\) with \(s_i = l\) and \(s_{i+1} = k + 1\) with \(l < k\), then
\[
(x_{m_2} \cdots x_{m_n}, x_{s_2}^{-}, \cdots, x_{s_i}^{-}, x_{s_{i+1}}^{-}, \cdots, x_{s_n}^{-}) = (x_{m_2}^{-} \cdots x_{m_n}^{-}, x_{s_2}^{-} \cdots x_{l}^{-} x_{k+1}^{-} \cdots x_{s_n}^{-})
\]
\[
= q^2 (x_{m_2}^{-} \cdots x_{m_n}^{-}, x_{s_2}^{-} \cdots x_{k}^{-} x_{l+1}^{-} \cdots x_{s_n}^{-})
\]
\[
= (x_{m_2}^{-} \cdots x_{m_n}^{-}, x_{s_2}^{-} \cdots x_{k}^{-} x_{l+1}^{-} \cdots x_{s_n}^{-})
\]
\[
+ q^2 (x_{m_2}^{-} \cdots x_{m_n}^{-}, x_{s_2}^{-} \cdots x_{l}^{-} x_{k+1}^{-} \cdots x_{s_n}^{-})
\]
Observe that the number of ascending inversions in the first two summands has decreased by one and the last summand can also be rewritten as a sum of terms that have a decrease in the number of ascending inversions. By induction on the number of inversion on products of length \(n - 1\) we again conclude
\[
(x_{m_2}^{-} \cdots x_{m_n}^{-}, x_{s_2}^{-} \cdots x_{s_i}^{-}, x_{s_{i+1}}^{-} \cdots x_{s_n}^{-}) \in \mathbb{Z}[q],
\]
For the first statement \((4.6)\) we begin at \(i = n - 1\). By \((3.14)\) and \((4.9)\) this is
\[
(x_{m_2}^{-} \cdots x_{m_n}^{-}, x_{s_2}^{-} \cdots x_{s_{n-1}}^{-}, \Omega_{\psi}(-m_1 - p)x_{s_n}^{-}) \in \mathbb{Z}[q].
\]
Suppose \((4.6)\) is true for \(i + 1 \leq n - 1\). Then
\[
(x_{m_2}^{-} \cdots x_{m_n}^{-}, x_{s_2}^{-} x_{s_2}^{-} \cdots x_{s_{n-1}}^{-}, x_{s_{n+1}}^{-} \cdots x_{s_n}^{-}) \in \mathbb{Z}[q]
\]
\[
= \delta_{m_1 + p, s_n} (x_{m_2}^{-} \cdots x_{m_n}^{-}, x_{s_2}^{-} \cdots x_{s_{n-1}}^{-}) \in \mathbb{Z}[q].
\]
Hence \((4.6)\) is proved.
Now we want to prove a refined special case of \((4.6)\): For any \(1 \leq i \leq n - 1\) and \(t \in \mathbb{Z}_{\geq 0}\) one has
\[
(x_{m_2}^{-} \cdots x_{m_n}^{-}, x_{s_2}^{-} x_{s_2}^{-} \cdots x_{s_{i+1}}^{-} x_{s_{i+1}}^{-} x_{s_{i+2}}^{-} \cdots x_{s_n}^{-}) \equiv \delta_{m_1 + t, s_{i+1}} (x_{m_2}^{-} \cdots x_{m_n}^{-}, x_{s_2}^{-} \cdots x_{s_i}^{-}, x_{s_{i+1}}^{-} x_{s_{i+2}}^{-} \cdots x_{s_n}^{-}) \mod \mathbb{Z}[q]
\]
\[
\equiv 0 \mod \mathbb{Z}[q].
\]
Here we assume \(k_1 \geq k_2 \geq \cdots \geq k_n\). For \(i = n - 1\) this is just \((4.10)\). Now for any \(t \geq 0\) we assume that \((4.11)\) is true for \(i + 1 \leq n - 1\). Then by \((4.6)\) and induction
we have
\[ (x_{m_2} \cdots x_{m_n}, x_{k_1+1} \cdots x_{k_{1+1}} \cdots x_{k_{n-1}} \cdots x_{k_{n-1}} \cdots x_{k_1} ) \]
\[ = \delta_{m_1+t,k_1} (x_{m_2} \cdots x_{m_n}, x_{k_1+1} \cdots x_{k_{1+1}} \cdots x_{k_{n-1}} \cdots x_{k_1} ) \]
\[ + \sum_{r \geq 0} g_{q-1}(r) (x_{m_2} \cdots x_{m_n}, x_{k_1+1} x_{k_2+1} \cdots x_{k_{1+1}} \cdots x_{k_{n-1}} \cdots x_{k_1} ) \Omega_{q}(-m_1 - r) x_{k_{1+1}} \cdots x_{k_n} \]
\[ = \delta_{m_1+t,k_1} (x_{m_2} \cdots x_{m_n}, x_{k_1+1} \cdots x_{k_{1+1}} \cdots x_{k_{n-1}} \cdots x_{k_1} ) \]
\[ + g_{q-1}(1) (x_{m_2} \cdots x_{m_n}, x_{k_1+1} \cdots x_{k_{1+1}} \cdots x_{k_{n-1}} \cdots x_{k_1} ) \Omega_{q}(-m_1 - 1) x_{k_{1+1}} \cdots x_{k_n} \]
\[ \equiv \delta_{m_1+t,k_1} (x_{m_2} \cdots x_{m_n}, x_{k_1+1} \cdots x_{k_{1+1}} \cdots x_{k_{n-1}} \cdots x_{k_1} ) \]
\[ + g_{q-1}(1) \delta_{m_1+t+1,k_{i+1}} \delta_{m_2,k_1+1} \cdots \delta_{m_{i+1},k_{i+1}} \cdots \delta_{m_n,k_n} \mod q^2 \mathbb{Z}[q]. \]

where we used the fact that all monomials appearing are of PBW type with weakly decreasing indices. But the second summand in the last congruence above is nonzero only if
\[ m_1 \geq m_{i+1} = k_1 + 1 > k_i \geq k_{i+1} = m_1 + t + 1 \]
which is impossible for \( t \geq 0 \). Hence the second summand is zero modulo \( q^2 \mathbb{Z}[q] \). This completes the proof of (4.11).

Now we show the induction step to complete the proof of the proposition:
\[ (x_{m_1} \cdots x_{m_n}, x_{k_1} \cdots x_{k_n}) = (x_{m_2} \cdots x_{m_n}, \Omega_{q}(-m_1) x_{k_1} \cdots x_{k_n}) \]
\[ = \delta_{m_1,k_1} (x_{m_2} \cdots x_{m_n}, x_{k_2} \cdots x_{k_n}) + \sum_{r \geq 0} g_{q-1}(r) (x_{m_2} \cdots x_{m_n}, x_{k_1+r} \cdots x_{k_2} \cdots x_{k_{n-1}} \cdots x_{k_1} ) \]
\[ = \delta_{m,k} + g_{q-1}(1) (x_{m_2} \cdots x_{m_n}, x_{k_1} \cdots x_{k_{n-1}} \cdots x_{k_1} ) \mod q^2 \mathbb{Z}[q] \]
\[ = \delta_{m,k} + g_{q-1}(1) \delta_{m_1+1,k_1} \cdots \delta_{m_n,k_n} \mod q^2 \mathbb{Z}[q] \]
where we used (4.6) in the third line and (4.11) in the fourth line. The second summand in the last congruence is nonzero if and only if \( m_1 + 1 = k_2, m_2 = k_1 + 1, m_3 = k_3, \ldots, m_n = k_n \). But this means that
\[ m_1 \geq m_2 = k_1 + 1 > k_1 \geq k_2 = m_1 + 1 \]
which is a contradiction. This completes the proof of the proposition.

\[ \square \]

**Corollary 4.0.5.** The form \( \langle \chi, \chi \rangle \) is non-degenerate.

**Proof.** Suppose \( u \in \mathcal{N}_q^- \), with \( (u,v)=0 \) for all \( v \in \mathcal{N}_q^- \) and say \( u = \sum_{m} a_m x_{m_1} \cdots x_{m_n} \), then in particular this holds for any \( v = x_{k_1} \cdots x_{k_n} \). Hence
\[ 0 = (u,x_{k_1} \cdots x_{k_n}) = \sum_{m} a_m (x_{m_1} \cdots x_{m_n}, x_{k_1} \cdots x_{k_n}) = a_k. \]

Thus \( a_k = 0 \) for all \( k \).

\[ \square \]
5. Imaginary Verma Modules for $A_1^{(1)}$

We begin by recalling some basic facts and constructions for the affine Kac-Moody algebra $A_1^{(1)}$ and its imaginary Verma modules. See [Kac90] for Kac-Moody algebra terminology and standard notations.

5.1. The algebra

The algebra $A_1^{(1)}$ is the affine Kac-Moody algebra over field $\mathbb{F}$ with generalized Cartan matrix $A = (a_{ij})_{0 \leq i, j \leq 1} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. The algebra $A_1^{(1)}$ has a Chevalley-Serre presentation with generators $e_0, e_1, f_0, f_1, h_0, h_1, d$ and relations

$$[h_i, h_j] = 0, \quad [h_i, d] = 0,$$
$$[e_i, f_j] = 0,$$
$$[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j,$$
$$[d, e_j] = \delta_{0,j} e_j, \quad [d, f_j] = -\delta_{0,j} f_j,$$
$$(\text{ad } e_i)^3 e_j = (\text{ad } f_i)^3 f_j = 0, \quad i \neq j.$$

Alternatively, we may realize $A_1^{(1)}$ through the loop algebra construction

$$A_1^{(1)} \cong \mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}] \oplus \mathbb{F}c \oplus \mathbb{F}d$$

with Lie bracket relations

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m} + n\delta_{n+m,0}(x, y)c,$$
$$[x \otimes t^n, c] = 0 = [d, c], \quad [d, x \otimes t^n] = nx \otimes t^n,$$

for $x, y \in \mathfrak{sl}_2$, $n, m \in \mathbb{Z}$, where $(\ , \ )$ denotes the Killing form on $\mathfrak{sl}_2$. For $x \in \mathfrak{sl}_2$ and $n \in \mathbb{Z}$, we write $x(n)$ for $x \otimes t^n$.

Let $\Delta$ denote the root system of $A_1^{(1)}$, and let $\{\alpha_0, \alpha_1\}$ be a basis for $\Delta$. Let $\delta = \alpha_0 + \alpha_1$, the minimal imaginary root. Then

$$\Delta = \{ \pm \alpha_1 + n\delta \mid n \in \mathbb{Z} \} \cup \{ k\delta \mid k \in \mathbb{Z} \setminus \{0\} \}.$$

5.2. The universal enveloping algebra $U(A_1^{(1)})$ of $A_1^{(1)}$ is the associative algebra over $\mathbb{F}$ with 1 generated by the elements $h_0, h_1, d, e_0, e_1, f_0, f_1$ with defining relations

$$[h_0, h_1] = [h_0, d] = [h_1, d] = 0,$$
$$h_i e_j - e_j h_i = a_{ij} e_j, \quad h_i f_j - f_j h_i = -a_{ij} f_j,$$
$$d e_j - e_j d = \delta_{0,j} e_j, \quad d f_j - f_j d = -\delta_{0,j} f_j,$$
$$e_i f_j - f_j e_i = \delta_{ij} h_i,$$
$$e_i e_i^3 - 3e_i e_j e_i + 3e_i^2 e_j e_i - e_i^3 e_j = 0 \text{ for } i \neq j,$$
$$f_i f_j - f_j f_i = 3f_i^2 f_j f_i - f_i^3 f_j = 0 \text{ for } i \neq j.$$

Corresponding to the loop algebra formulation of $A_1^{(1)}$ is an alternative description of $U(A_1^{(1)})$ as the associative algebra over $\mathbb{F}$ with 1 generated by the elements...
e(k), f(k) \ (k \in \mathbb{Z}), h(l) \ (l \in \mathbb{Z} \setminus \{0\}), c, d, h, with relations
\[
[c, u] = 0 \quad \text{for all } u \in U(A_1^{(1)}),
\]
\[
[h(k), h(l)] = 2k\delta_{k+l,0}c,
\]
\[
[h, d] = 0, \quad [h, h(k)] = 0,
\]
\[
[d, h(l)] = lh(l), \quad [d, e(k)] = ke(k), \quad [d, f(k)] = kf(k),
\]
\[
[h, e(k)] = 2e(k), \quad [h, f(k)] = -2f(k),
\]
\[
[h(k), e(l)] = 2e(k+l), \quad [h(k), f(l)] = -2f(k+l),
\]
\[
[e(k), f(l)] = h(k+l) + k\delta_{k+l,0}c.
\]

5.3. A subset \(S\) of the root system \(\Delta\) is called \textit{closed} if \(\alpha, \beta \in S\) and \(\alpha + \beta \in \Delta\) implies \(\alpha + \beta \in S\). The subset \(S\) is called a \textit{closed partition} of the roots if \(S\) is closed, \(S \cap (-S) = \emptyset\), and \(S \cup -S = \Delta\) \([JK85],[JK89],[Fut90],[Fut92]\). The set
\[
S = \{\alpha_1 + k\delta \mid k \in \mathbb{Z}\} \cup \{l\delta \mid l \in \mathbb{Z}_{>0}\}
\]
is a closed partition of \(\Delta\) and is \(W \times \{\pm 1\}\)-inequivalent to the standard partition of the root system into positive and negative roots \([Fut94]\).

For \(\hat{\mathfrak{g}} = A_1^{(1)}\), let \(\mathfrak{g}^{(S)}_{\pm} = \sum_{\alpha \in S} \hat{\mathfrak{g}}_{\pm \alpha}\). In the loop algebra formulation of \(\hat{\mathfrak{g}}\), we have that \(\mathfrak{g}^{(S)}_{-}\) is the subalgebra generated by \(e(k) \ (k \in \mathbb{Z})\) and \(h(l) \ (l \in \mathbb{Z}_{>0})\) and \(\mathfrak{g}^{(S)}_{+}\) is the subalgebra generated by \(f(k) \ (k \in \mathbb{Z})\) and \(h(-l) \ (l \in \mathbb{Z}_{>0})\). Since \(S\) is a partition of the root system, the algebra has a direct sum decomposition
\[
\hat{\mathfrak{g}} = \mathfrak{g}^{(S)}_{-} \oplus \mathfrak{h} \oplus \mathfrak{g}^{(S)}_{+}.
\]
Let \(U(\mathfrak{g}^{(S)}_{\pm})\) be the universal enveloping algebra of \(\mathfrak{g}^{(S)}_{\pm}\). Then, by the PBW theorem, we have
\[
U(\hat{\mathfrak{g}}) \cong U(\mathfrak{g}^{(S)}_{-}) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{g}^{(S)}_{+}),
\]
where \(U(\mathfrak{g}^{(S)}_{-})\) is generated by \(e(k) \ (k \in \mathbb{Z})\), \(h(l) \ (l \in \mathbb{Z}_{>0})\), \(U(\mathfrak{g}^{(S)}_{+})\) is generated by \(f(k) \ (k \in \mathbb{Z})\), \(h(-l) \ (l \in \mathbb{Z}_{>0})\) and \(U(\mathfrak{h})\), the universal enveloping algebra of \(\mathfrak{h}\), is generated by \(h, c\) and \(d\).

Let \(\lambda \in P\), the weight lattice of \(\hat{\mathfrak{g}} = A_1^{(1)}\). A \(U(\hat{\mathfrak{g}})\)-module \(V\) is called a \textit{weight} module if \(V = \oplus_{\mu \in \mathfrak{p}} V_{\mu}\), where
\[
V_{\mu} = \{v \in V \mid h \cdot v = \mu(h)v, c \cdot v = \mu(c)v, d \cdot v = \mu(d)v\}.
\]
Any submodule of a weight module is a weight module. A \(U(\hat{\mathfrak{g}})\)-module \(V\) is called an \(S\)-\textit{highest weight module} with highest weight \(\lambda\) if there is a non-zero \(v_{\lambda} \in V\) such that
\begin{enumerate}[(i)]
\item \(u^+ \cdot v_{\lambda} = 0\) for all \(u^+ \in U(\mathfrak{g}^{(S)}_{+}) \setminus \mathbb{F}^*\),
\item \(h \cdot v_{\lambda} = \lambda(h)v_{\lambda}, c \cdot v_{\lambda} = \lambda(c)v_{\lambda}, d \cdot v_{\lambda} = \lambda(d)v_{\lambda}\),
\end{enumerate}
any \(S\)-highest weight module is a weight module.

For \(\lambda \in P\), let \(I_S(\lambda)\) denote the ideal of \(U(A_1^{(1)})\) generated by \(e(k) \ (k \in \mathbb{Z})\), \(h(l) \ (l > 0)\), \(h - \lambda(h)1, c - \lambda(c)1, d - \lambda(d)1\). Then we define \(M(\lambda) = U(A_1^{(1)})/I_S(\lambda)\) to be the \textit{imaginary Verma module} of \(A_1^{(1)}\) with highest weight \(\lambda\). Imaginary Verma modules have many structural features similar to those of standard Verma modules, with the exception of the infinite-dimensional weight spaces. Their properties were investigated in \([Fut94]\), from which we recall the following proposition \([Fut94,\text{ Proposition 1, Theorem 1}]\).
Proposition 5.3.1. (i) $M(\lambda)$ is a $U(g^\ast)$-free module of rank 1 generated by the $S$-highest weight vector $1 \otimes 1$ of weight $\lambda$.

(ii) $\dim M(\lambda)_\lambda = 1$; $0 < \dim M(\lambda)_{\lambda - k\delta} < \infty$ for any integer $k > 0$; if $\mu \neq \lambda - k\delta$ for any integer $k \geq 0$ and $M(\lambda)_\mu \neq 0$, then $\dim M(\lambda)_\mu = \infty$.

(iii) Let $V$ be a $U(A_1^{(1)})$-module generated by some $S$-highest weight vector $v$ of weight $\lambda$. Then there exists a unique surjective homomorphism $\varphi : M(\lambda) \to V$ such that $\varphi(1 \otimes 1) = v$.

(iv) $M(\lambda)$ has a unique maximal submodule.

(v) Let $\lambda, \mu \in P$. Any non-zero element of $\Hom_{U(A_1^{(1)})}(M(\lambda), M(\mu))$ is injective.

(vi) $M(\lambda)$ is irreducible if and only if $\lambda(c) \neq 0$. □

Suppose now that $\lambda(c) = 0$. Consider an ideal $J(\lambda)$ of $U(A_1^{(1)})$ generated by $I_S(\lambda)$ and $h(l)$ for all $l$. Set

$$\tilde{M}(\lambda) = U(A_1^{(1)})/J(\lambda).$$

Then $\tilde{M}(\lambda)$ is a homomorphic image of $M(\lambda)$ which we call the reduced imaginary Verma module. The module $\tilde{M}(\lambda)$ has a $\Lambda$-gradation:

$$\tilde{M}(\lambda) = \sum_{\xi \in \Lambda} \tilde{M}(\lambda)_\xi.$$

6. The category $\mathcal{O}_{\text{red,im}}$

Let $G$ be the Heisenberg subalgebra, $G = \sum_{k \in \mathbb{Z}\{0\}} \mathfrak{g}_{k\delta} \oplus \mathbb{F}c$. We say that a non-zero $\tilde{\mathfrak{g}}$-module $V$ is $G$-compatible if

i. $V$ has a decomposition $V = TF(V) \oplus T(V)$ into a sum of nonzero $G$-submodules such that

ii. $G$ is bijective on $TF(V)$ (that is, any non-zero element $g \in G$ is a bijection on $TF(V)$) and $TF(V)$ has no nonzero $\mathfrak{g}$-submodule,

iii. $G \cdot T(V) = 0$.

Consider the set

$$h^*_\text{red} := \{ \lambda \in h^* \mid \lambda(c) = 0, \lambda(h) \notin \mathbb{Z}_{\geq 0} \}.$$

As usual let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The category $\mathcal{O}_{\text{red,im}}$ has as objects $\tilde{\mathfrak{g}}$-modules $M$ such that

$$M = \bigoplus_{\nu \in h^*_\text{red}} M_\nu, \quad \text{where} \quad M_\nu = \{ m \in M \mid hm = \nu(h)m \}.$$

Note $\dim M_\nu$ may be infinite dimensional.

(2) $e_n = e \otimes t^n$ acts locally nilpotently for any $n \in \mathbb{Z}$.

(3) $M$ is $G$-compatible.

The morphisms in the category are $\tilde{\mathfrak{g}}$-module homomorphisms. For example direct sums of reduced imaginary Verma modules $M(\lambda)$ are in the category $\mathcal{O}_{\text{red,im}}$. In this case $TF(\tilde{M}(\lambda)) = \oplus_{k \in \mathbb{Z}, n \in \mathbb{Z}_{+}} M(\lambda)_{\lambda - n \delta + k\delta}$ and $T(\tilde{M}(\lambda)) = \tilde{M}(\lambda)_\lambda \cong \mathbb{F}$.

A loop module for $\mathfrak{g}$ is any representation of the form $\tilde{M} := M \otimes \mathbb{F}[t, t^{-1}]$ where $M$ is a highest weight module for $\mathfrak{sl}(2, \mathbb{F})$ and

$$(x \otimes t^k)(m \otimes t^l) := x \cdot m \otimes t^{k+l}, \quad c(m \otimes t^l) = 0.$$
Here \( x \cdot m \) is the action of \( x \in \mathfrak{sl}(2, \mathbb{F}) \) on \( m \in M \).

**Proposition 6.0.2.**

(1) The loop modules \( \hat{M} \) with \( M \) in the category \( \mathcal{O} \) for \( \mathfrak{sl}(2, \mathbb{F}) \) are not in \( \mathcal{O}_{\text{red,im}} \).

(2) For \( \lambda, \mu \in \mathfrak{h}^*_{\text{red}} \) one has \( \text{Ext}^1_{\mathcal{O}}(\hat{M}(\lambda), \hat{M}(\mu)) = 0 \).

**Proof.** Suppose \( \hat{M} \) is a loop module with \( M \in \mathcal{O} \) for \( \mathfrak{sl}(2, \mathbb{F}) \). Then \( \hat{M} \) satisfies condition (1) and (2) from above. Assume \( \hat{M} = TF(\hat{M}) \oplus T(\hat{M}) \) satisfies (i)–(iii) above. Now take any \( \sum_{i=-k}^k m_i \otimes t^i \in T(\hat{M}) \) with \( m_i \in M_\mu \) for some weight \( \mu \).

Then by (iii) we have
\[
0 = h \otimes t^r \cdot \left( \sum_{i=-k}^k m_i \otimes t^i \right) = \lambda(h) \left( \sum_{i=-k}^k m_i \otimes t^{i+k} \right)
\]
so that \( \lambda(h) = 0 \) which contradicts \( \lambda \in \mathfrak{h}^*_{\text{red}} \). Then \( T(\hat{M}) = 0 \) and \( \hat{M} = TF(\hat{M}) \) which is a \( \mathfrak{g} \)-module contradicting (i) and (ii) and thus (3).

For (2) we need to show that there are no nontrivial extensions between reduced imaginary Verma modules \( \hat{M}(\lambda) \) and \( \hat{M}(\mu) \). If \( \mu = \lambda + k\delta \) for some integer \( k \) then any extension of \( \hat{M}(\lambda) \) by itself has a two dimensional highest weight space of weight \( \lambda \). Any highest weight vector in this space generates an irreducible submodule and thus the extension splits as a direct sum of two submodules each isomorphic to \( \hat{M}(\lambda) \).

Indeed suppose now \( \mu = \lambda + k\delta = sa \) for some integers \( k \) and \( s > 0 \). Consider a short exact sequence
\[
0 \longrightarrow \hat{M}(\lambda) \longrightarrow M \longrightarrow \hat{M}(\mu) \longrightarrow 0,
\]
where we view \( \iota \) as just the inclusion map. For any preimage weight vector \( \bar{v}_\mu \) of a highest weight vector \( v_\mu \) in \( \tilde{M}(\mu) \) one has \( G\bar{v}_\mu \in \tilde{M}(\lambda) \). On the other hand \( G\bar{v}_\mu = 0 \). Suppose \( 0 \neq v = h_m\bar{v}_\mu \). Then \( h_m\bar{v}_\mu = h_m v' \) for some \( v' \in M(\lambda) \) (one cannot have \( h_m\bar{v}_\mu = \alpha v_\lambda \), \( \alpha \in \mathbb{F} \) as otherwise \( \mu + m\delta = \lambda \) and \( s = 0 \)). Then \( h_n(v' - \bar{v}_\mu) = 0 \) and so \( v' - \bar{v}_\mu \in T(M) = \mathbb{F} v_\lambda \) which is a contradiction to the fact \( \bar{v}_\mu \notin M(\lambda) \).

Recall that \( e_0 \) acts locally nilpotently on \( \bar{v}_\mu \). Moreover, \( e_0^t \bar{v}_\mu \neq 0 \) if \( t < s \), otherwise \( e_0^{s-1} \bar{v}_\mu \) would generate a submodule in \( \tilde{M}(\lambda) \) which is a contradiction.

So, \( e_0^k \bar{v}_\mu = 0 \) if \( k = 0 \) and \( e_0^{s-k} \bar{v}_\mu = 0 \) if \( k \neq 0 \). Without loss of generality we assume the latter. Suppose \( s > 1 \). Consider an \( \mathfrak{sl}(2) \)-subalgebra \( \mathfrak{a} \) generated by \( e_0 \) and \( f_0 \) and an \( \mathfrak{a} \)-module generated by \( \bar{v}_\mu \). This module is a non trivial extension of two Verma modules over \( \mathfrak{a} \) with highest weights \( \lambda + k\delta - \alpha \) and \( \lambda + k\delta - sa \). But this is impossible (e.g. these modules have different central characters).

Suppose now \( s = 1 \). Then apply the same argument to an \( \mathfrak{sl}(2) \)-subalgebra generated by \( e_k \) and \( f_k \). Assuming \( e_k \bar{v}_\mu \neq 0 \) we obtain a contradiction as above. Therefore, \( M = \tilde{M}(\lambda) \oplus \tilde{M}(\mu) \) completing the proof.

**□**

**Proposition 6.0.3.** If \( M \in \mathcal{O}_{\text{red,im}} \) is a simple object, then \( M \cong \hat{M}(\lambda) \) for some \( \lambda \in \mathfrak{h}^*_{\text{red}} \).

**Proof.** Consider any simple \( M \in \mathcal{O}_{\text{red,im}} \). Let \( v \in T(M) \) be a nonzero element of some weight \( \lambda \in \mathfrak{h}^*_{\text{red}} \). Then \( Gv = 0 \) and \( e_0^N v = 0 \) for some positive integer \( N \).

Choose \( N \) to be the least possible with such property. If \( N = 1 \) then \( e_n v = 0 \) for all integers \( n \) and hence \( M \) is a quotient of the reduced imaginary Verma module
$\hat{M}(\lambda)$ with highest weight $\lambda$. Since $\lambda \in \hat{h}^*_{\text{red}}$ then $\hat{M}(\lambda)$ is simple and thus $M \simeq \hat{M}(\lambda)$. Assume now that $N > 1$ and set $w = e_0^{N-1}v$. Then $e_0w = 0$. We have $0 = h_{\beta}e_0^{N}v = 2N\epsilon_k e_0^{N-1}v = 2N\epsilon_k w$ for all integers $k$. Therefore $M$ is a quotient of the loop module induced from $U(G)w$ (with $e_k U(G)w = 0$ for all integers $k$). If $w \in T(M)$ then we are done. Suppose $w \notin T(M)$ so $0 \neq w \in TF(M)$ may be assumed to be a weight vector of weight $\mu$. Then $W = U(G)w$ is a $G$-submodule of $TF(M)$. Consider the induced module $I(W) = \text{Ind}_{G+N+H}^G \hat{M}(\lambda)_{\bar{w}}$ where $N = \oplus_{n \in \mathbb{Z}} F e_n$ acts by zero on $W$, $H = \mathbb{F} h + \mathbb{F} d$ acts by $hw = \mu(h)w$ and $dw = \mu(d)w$. Since $U(G)w \subset TF(M)$ then it is easy to see that $TF(I(W)) = I(W)$. Hence the same holds for any of its quotients by the Short Five Lemma, i.e. $TF(M) = M$ which is a contradiction. Therefore $w \in T(M)$ which completes the proof. \qed

**Theorem 6.0.4.** If $M \in \mathcal{O}_{\text{red,im}}$ is any object then $M = \oplus_{\lambda_i \in \hat{h}^*_{\text{red}}} \hat{M}(\lambda_i)$, $i \in I$ for some weights $\lambda_i$’s.

**Proof.** Consider the subspace $T(M)$. Since the weights of $M$ are in $\hat{h}^*_{\text{red}}$, $T(M)$ is not a $\hat{g}$-submodule. Let $w \in T(M)$ be a nonzero element, $W = U(G)w \subset T(M)$. Arguing as in the proof of Proposition 6.0.3 we find a nonzero element $w' \in M$ such that $e_k w' = 0$ for all integers $k$. If $U(G)w' \neq CW'$ then $w' \in TF(M)$ which is a contradiction. Hence $w'$ generates a submodule isomorphic to a reduced imaginary Verma module containing $W$. Thus each nonzero element of $T(M)$ generates $\hat{M}(\lambda)$ for some $\lambda$.

**Corollary 6.0.5.** The category $\mathcal{O}_{\text{red,im}}$ is closed under taking subquotients and direct sums so it is a Serre category.

### 7. Quantized Imaginary Verma modules

Let $\Lambda$ denotes the weight lattice of $\hat{g} = A^{(1)}_1$, $\lambda \in \Lambda$. Denote by $I^q(\lambda)$ the ideal of $U_q = U_q(\hat{g})$ generated by $x^+(k), k \in \mathbb{Z}, a(l), l > 0$, $K^{\pm 1} - q^{\lambda(h)1}$, $\gamma^{\pm 1} - q^{\pm 1}$ and $D^{\pm 1} - q^{\pm 1}1$. The imaginary Verma module with highest weight $\lambda$ is defined to be ([CFK97])

$$M_q(\lambda) = U/I^q(\lambda).$$

**Theorem 7.0.6 ([CFK97], Theorem 3.6).** The imaginary Verma module $M_q(\lambda)$ is simple if and only if $\lambda(c) \neq 0$.

Suppose now that $\lambda(c) = 0$. Then $\gamma^{\pm 1}$ acts on $M_q(\lambda)$ by 1. Consider the ideal $J^q(\lambda)$ of $U_q$ generated by $I^q(\lambda)$ and $a(l)$ for all $l$. Denote

$$\tilde{M}_q(\lambda) = U_q/J^q(\lambda).$$

Then $\tilde{M}_q(\lambda)$ is a homomorphic image of $M_q(\lambda)$ which we call the reduced quantized imaginary Verma module. The module $\tilde{M}_q(\lambda)$ has a $\Lambda$-gradation:

$$\tilde{M}_q(\lambda) = \bigoplus_{\xi \in \Lambda} \tilde{M}_q(\lambda)_{\xi}.$$

**Theorem 7.0.7 ([CFM10]).** Let $\lambda \in \Lambda$ be such that $\lambda(c) = 0$. Then module $\tilde{M}_q(\lambda)$ is simple if and only if $\lambda(h) \neq 0$. 
8. The category $\mathcal{O}_e^{\eta}$

Consider the set
$$\mathfrak{h}_{\text{red}}^* := \{ \lambda \in \mathfrak{h}^* \mid \lambda(c) = 0, \lambda(h) \neq 0 \}.$$

The category $\mathcal{O}_e^{\eta}$ has as objects $U_q(\hat{g})$-modules $M$ such that there exists $\lambda_i \in \mathfrak{h}_{\text{red}}^*$, $i \in I$, with
$$M \cong \bigoplus_{i \in I} \tilde{M}_q(\lambda_i).$$

The morphisms in the category are just $U_q(\hat{g})$-module homomorphisms. Since $\tilde{M}_q(\lambda)$ is a quantization of $M(\lambda)$ in the sense of Lusztig, modules in $\mathcal{O}_e^{\eta}$ are quantizations of modules in $\mathcal{O}_{\text{red,im}}$. So equivalently the category $\mathcal{O}_e^{\eta}$ can be defined as follows:

Let $G_q$ be the quantized Heisenberg subalgebra generated by $h_k, k \in \mathbb{Z} \setminus \{0\}$ and $\gamma$. We say that a nonzero $U_q(\hat{g})$-module $V$ is $G_q$-compatible if

i). $V$ has a decomposition $V = TF(V) \oplus T(V)$ into a sum of nonzero $G_q$-submodules such that

ii). $G_q$ is bijective on $TF(V)$ (that any nonzero element $g \in G_q$ is a bijection on $TF(V)$) and $TF(V)$ has no nonzero $U_q(\hat{g})$-submodule,

iii). $G_q \cdot T(V) = 0$.

The category $\mathcal{O}_e^{\eta}$ has as objects $U_q(\hat{g})$-modules $M$ such that

(1)
$$M = \bigoplus_{\nu \in \mathfrak{h}_{\text{red}}^*} M_{\nu}, \quad \text{where} \quad M_{\nu} = \{ m \in M \mid K_m = K^{\nu(h)} m, \quad Dm = q^{\nu(d)} m \},$$

(2) $x^+_n$, $n \in \mathbb{Z}$ act locally nilpotently,

(3) $M$ is $G_q$-compatible.

If $M \in \mathcal{O}_e^{\eta}$, we can write $M = \bigoplus \tilde{M}_q(\lambda_i)$ with $\tilde{M}_q(\lambda_i) = \bigoplus \mathbb{F}[q^{1/2}] x_{n_1}^+ \cdots x_{n_k}^- v_{\lambda_i}$.

We define $\tilde{\Omega}_\psi(m)$ and $\tilde{x}_m^-$ on each $\tilde{M}_q(\lambda_i)$ as in (3.1):

\begin{align*}
\tilde{\Omega}_\psi(m)(x_{n_1}^- \cdots x_{n_k}^- v_{\lambda_i}) & := \Omega_\psi(m)(x_{n_1}^- \cdots x_{n_k}^-) v_{\lambda_i}, \\
\tilde{x}_m^- (x_{n_1}^- \cdots x_{n_k}^- v_{\lambda_i}) & := x_m^- x_{n_1}^- \cdots x_{n_k}^- v_{\lambda_i}.
\end{align*}

Hence the following result follows.

**Theorem 8.0.8.** The operators $\tilde{\Omega}_\psi(m)$ and $\tilde{x}_m^-$ are well defined on objects in the category $\mathcal{O}_e^{\eta}$. Moreover on each summand $\tilde{M}_q(\lambda_i) \cong N_q^-$ they agree with the $\Omega_\psi(m)$ respectively left multiplication by $x_m^-$ defined as in (3.1).

9. Imaginary $\mathfrak{a}$-lattices and imaginary crystal basis

Let $\mathfrak{a}_0$ (resp. $\mathfrak{a}_\infty$) to be the ring of rational functions in $q^{1/2}$ with coefficients in a field $\mathbb{F}$ of characteristic zero, regular at $0$ (resp. at $\infty$). Let $\mathfrak{a} = \mathbb{F}[q^{1/2}, q^{-1/2}, \frac{1}{n} | n > 1]$, and $P = \{ -k\alpha + m\delta \mid k > 0, m \in \mathbb{Z} \} \cup \{0\}$. Let $M$ be a $U_q(\hat{g})$-module in the category. We call a free $\mathfrak{a}_0$-submodule $\mathcal{L}$ of $M$ an imaginary crystal $\mathfrak{a}_0$-lattice of $M$ if the following hold

(i). $\mathbb{F}[q^{1/2}] \otimes_{\mathfrak{a}_0} \mathcal{L} \cong M$,

(ii). $\mathcal{L} = \bigoplus_{\lambda} \mathcal{L}_\lambda$ and $\mathcal{L}_\lambda = \mathcal{L} \cap M_\lambda$,

(iii). $\tilde{\Omega}_\psi(m) \mathcal{L} \subseteq \mathcal{L}$ and $\tilde{x}_m^- \mathcal{L} \subseteq \mathcal{L}$ for all $m \in \mathbb{Z}$. 

We now show that the above definition is not vacuous. Let \( \lambda \in \mathfrak{h}^* \) and define

\[
\mathcal{L}(\lambda) := \bigoplus_{i_1 \geq \cdots \geq i_k \geq 0} \mathbb{A} x_{i_1}^{-1} \cdots x_{i_k}^{-1} v_\lambda \subset \mathcal{N}_q v_\lambda = \tilde{M}_q(\lambda)
\]

and also operators \( \tilde{\Omega}_\psi(m) : \tilde{M}_q(\lambda) \to \tilde{M}_q(\lambda) \) and \( \tilde{x}_m : \tilde{M}_q(\lambda) \to \tilde{M}_q(\lambda) \) where \( \tilde{x}_m \) is the left multiplication operator by \( x_m \) and \( \tilde{\Omega}_\psi(m)(x_{i_1}^{-1} \cdots x_{i_k}^{-1} v_\lambda) := \Omega_\psi(m)(x_{i_1}^{-1} \cdots x_{i_k}^{-1}) v_\lambda \) for \( i_1 \geq \cdots \geq i_k \).

For \( \mu = \lambda - k \alpha + m \delta \),

\[
\tilde{M}_q(\lambda)_\mu = \left\{ \bigoplus_{i_j = m, i_1 \geq \cdots \geq i_k} \mathbb{Q}(q^{1/2}) x_{i_1}^{-1} \cdots x_{i_k}^{-1} v_\lambda \right\}
\]

Now observe (i) is satisfied for \( \mathcal{L} = \mathcal{L}(\lambda) \) as well as

1. For \( \mathcal{L}(\lambda)_\mu := \mathcal{L}(\lambda) \cap \tilde{M}_q(\lambda)_\mu \) one has \( \mathcal{L}(\lambda) = \bigoplus_{\lambda \in \mu} \mathcal{L}(\lambda)_\mu \), and

2. If \( \mathbb{F} = \mathbb{Q} \), then

\[
\mathcal{L}(\lambda) = \left\{ u \in \tilde{M}_q(\lambda) \mid (u, \tilde{M}_q(\lambda)) \subset \mathbb{A}_0 \right\}
\]

If \( \mathbb{F} = \mathbb{Q} \), then

\[
\mathcal{L}(\lambda) = \left\{ u \in \tilde{M}_q(\lambda) \mid (u, u) \in \mathbb{A}_0 \right\}
\]

**Proof.** Let \( R \) denote the right hand side of the above equality. We have the inclusion \( \mathcal{L}(\lambda) \subseteq R \) by Proposition 4.0.4. For the other inclusion let \( u \in R \) and by clearing denominators we can find a smallest \( n \geq 0 \) such that \( q^{n/2} u \in \mathcal{L}(\lambda) \). If \( n > 1 \) then

\[
(q^{n/2} u, \tilde{M}_q(\lambda)) \equiv 0 \mod q^{n/2} \mathbb{A}_0.
\]

By Proposition 4.0.4 (iv) \( \mathcal{L}(\lambda) \) is non-degenerate modulo \( q^2 \mathcal{L}(\lambda) \), we must have \( q^{n/2} u \equiv 0 \mod q^2 \mathcal{L}(\lambda) \). Hence \( q^{(n/2) - 2} u \in \mathcal{L}(\lambda) \) which contradicts the minimality of \( n \). Thus \( u \in \mathcal{L}(\lambda) \). \( \square \)

For \( \lambda \in \mathfrak{h}^* \) define

\[
\mathcal{B}(\lambda) := \{ \tilde{x}_{i_1}^{-1} \cdots \tilde{x}_{i_k}^{-1} v_\lambda + q \mathcal{L}(\lambda) \in \mathcal{L}(\lambda)/q \mathcal{L}(\lambda) \mid i_1 \geq \cdots \geq i_k \}.
\]

An imaginary crystal basis of a \( U_q(\hat{\mathfrak{g}}) \)-module \( M \) in the category \( \mathcal{O}_q^{\text{red,im}} \) is a pair \((\mathcal{L}, \mathcal{B})\) satisfying

1. \( \mathcal{L} \) is an imaginary crystal lattice of \( M \),
2. \( \mathcal{B} \) is an \( \mathbb{F} \)-basis of \( \mathcal{L}/q \mathcal{L} \cong \mathbb{F} \otimes_{\mathbb{A}_0} \mathcal{L} \),
3. \( \mathcal{B} = \bigcup_{\mu \in \mathcal{P}} \mathcal{B}_\mu \), where \( \mathcal{B}_\mu = \mathcal{B} \cap (\mathcal{L}_\mu/q \mathcal{L}_\mu) \),
4. \( \tilde{x}_m \mathcal{B} \subset \pm \mathcal{B} \cup \{0\} \) and \( \tilde{\Omega}_\psi \mathcal{B} \subset \pm \mathcal{B} \cup \{0\} \),
5. For \( m \in \mathbb{Z} \), if \( \tilde{\Omega}_\psi(-m)b \neq 0 \) and \( \tilde{x}_m b \neq 0 \) for \( b \in \mathcal{B} \), then \( \tilde{x}_m \tilde{\Omega}_\psi(-m)b = \tilde{\Omega}_\psi(-m) \tilde{x}_m b \).
Theorem 9.0.10. For $\lambda \in \hat{\mathfrak{h}}_{\text{red}, \text{sm}}^*$, the pair $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ is an imaginary crystal basis of the reduced imaginary Verma module $\tilde{\mathcal{M}}_\theta(\lambda)$.

Proof. Conditions (i)-(ii) are clear. For (iii) consider $b = \tilde{x}_{i_1}^- \cdots \tilde{x}_{i_k}^- v_\lambda + q\mathcal{L}(\lambda)$ with $i_1 \geq i_2 \geq \cdots \geq i_k$. If $m \geq i_1$, then

$$\tilde{x}_m^- b = \tilde{x}_m^- \tilde{x}_{i_1}^- \cdots \tilde{x}_{i_k}^- v_\lambda + q\mathcal{L}(\lambda) \in \mathcal{B}.$$ 

If $m = i_1 - 1$, then by (2.20) we have

$$\tilde{x}_m^- b = q^2 \tilde{x}_m^- \tilde{x}_{i_1}^- \cdots \tilde{x}_{i_k}^- v_\lambda + q\mathcal{L}(\lambda) = 0 \mod q\mathcal{L}(\lambda).$$

If $m < i_1 - 1$, ($l = m$ and $k + 1 = i_1$ so $k = i_1 - 1$) then by (2.21) we have

$$(\ref{eq:9.1})$$

$$\tilde{x}_m^- b = -\tilde{x}_{i_1-1}^- \tilde{x}_{i_2}^- \cdots \tilde{x}_{i_k}^- v_\lambda + q\mathcal{L}(\lambda)$$

and $i_1 - 1 \geq m + 1$. By induction this is either $0 \mod q\mathcal{L}(\lambda)$ if $i_j = m + j$ for some $j$ or in $\pm \mathcal{B}$. To sum it up we have

$$(\ref{eq:9.2})$$

$$\tilde{x}_m^- b \equiv \begin{cases} 
\tilde{x}_m^- \tilde{x}_{i_1}^- \cdots \tilde{x}_{i_k}^- v_\lambda + q\mathcal{L}(\lambda) & \text{if } m \geq i_1 \\
0 & \text{if } m + j = i_j \text{ for some } 1 \leq j \leq l, \\
(\ref{eq:9.1}) & \text{if } m + j > i_j \text{ but } m + j - 1 < i_{j-1}, \\
0 & \text{for some } 1 \leq j \leq l, \\
(\ref{eq:9.1}) & \text{if } m + j \neq i_j. 
\end{cases}$$

Next we have

$$\tilde{\Omega}_\psi(k) \tilde{x}_{i_1}^- \cdots \tilde{x}_{i_k}^- v_\lambda = \delta_{k,-1} \tilde{x}_{i_1}^- \cdots \tilde{x}_{i_k}^- v_\lambda + \sum_{r \geq 0} g_{q-1}(r) \tilde{x}^-_{i_{1+r}} \tilde{\Omega}_\psi(k - r) \tilde{x}^-_{i_2} \cdots \tilde{x}^-_{i_k} v_\lambda$$

$$\equiv \delta_{k,-1} \tilde{x}_{i_1}^- \cdots \tilde{x}_{i_k}^- v_\lambda - \tilde{x}^-_{i_1+1} \tilde{\Omega}_\psi(k - 1) \tilde{x}^-_{i_2} \cdots \tilde{x}^-_{i_k} v_\lambda \mod q\mathcal{L}(\lambda)$$

$$\equiv \delta_{k,-1} \tilde{x}_{i_2}^- \cdots \tilde{x}_{i_k}^- v_\lambda - \sum_{j=2}^l (-1)^{j-2} \delta_{k,-j+1,-i_j} \tilde{x}^-_{i_{1+j}} \tilde{x}^-_{i_{2+j}} \cdots \tilde{x}^-_{i_{j-1+1+j}} v_\lambda \mod q\mathcal{L}(\lambda)$$

$$\equiv \sum_{j=1}^l (-1)^{j-1} \delta_{k,-j+1,-i_j} \tilde{x}^-_{i_{1+j}} \cdots \tilde{x}^-_{i_{j-1+1+j}} \tilde{x}^-_{i_k} v_\lambda \mod q\mathcal{L}(\lambda)$$

Observe that each summand on the right is ordered so that it is in $\pm \mathcal{B} \cup \{0\}$. The only way in which the whole summation is not in $\pm \mathcal{B} \cup \{0\}$ is if there are at least two indices $r < s$ such that $i_r = -k + r - 1$ and $i_s = -k + s - 1$. But $i_r \geq i_s$, which is a contradiction.

Condition (v) is satisfied by (3.14). Indeed we begin by induction. Now if $b = \tilde{x}^-_{i_1} v_\lambda + q\mathcal{L}(\lambda)$, then we have

$$\tilde{\Omega}_\psi(k) \tilde{x}^-_m b = \delta_{k,-m} b + \sum_{r \geq 0} g_{q-1}(r) \tilde{x}^-_{m+r} \tilde{\Omega}_\psi(k - r) \tilde{x}^-_{i_1} v_\lambda$$

$$= \delta_{k,-m} b - \tilde{x}^-_{m+1} \tilde{\Omega}_\psi(k - 1) \tilde{x}^-_{i_1} v_\lambda$$

$$= \delta_{k,-m} b - \delta_{k,-1} \tilde{x}^-_{m+1} v_\lambda.$$
Thus if $k = -m$ and $k - 1 = -i_1$, then $i_1 = m + 1$ which is a contradiction to the assumption $\tilde{x}_m b \neq 0$. Hence $\Omega_{\psi}(-m)\tilde{x}_m b = b$.

Now assuming $\tilde{\Omega}_{\psi}(-m)\tilde{x}_{i_1}^{-1} v_\lambda = \Omega_{\psi}(-m)b = 0$ by (9.3) we have

$$\tilde{\Omega}_{\psi}(-m)\tilde{x}_{i_1}^{-1} v_\lambda \equiv \delta_{m,i_1} v_\lambda \mod q\mathcal{L}(\lambda)$$

and we must have $m = i_1$. Thus $\tilde{x}_m \tilde{\Omega}_{\psi}(-m)\tilde{x}_{i_1}^{-1} v_\lambda \equiv \tilde{x}_m v_\lambda = b$.

Next take $b = \tilde{x}_{i_1}^{-1} \cdots \tilde{x}_{i_k}^{-1} v_\lambda$ with $i_1 \geq i_2 \geq \cdots \geq i_k$ and if $\tilde{x}_m b \neq 0$, then $i_j \neq i_1$ for all $1 \leq j \leq l$ by (9.2) and we first consider the case $m \geq i_1$:

$$\tilde{\Omega}_{\psi}(-m)\tilde{x}_m b \equiv \tilde{\Omega}_{\psi}(-m)\tilde{x}_m \tilde{x}_{i_1}^{-1} \cdots \tilde{x}_{i_k}^{-1} v_\lambda$$

$$\equiv \tilde{x}_{i_1}^{-1} \cdots \tilde{x}_{i_k}^{-1} v_\lambda - \tilde{x}_{m+1} \tilde{\Omega}_{\psi}(-m-1)\tilde{x}_{i_1}^{-1} \cdots \tilde{x}_{i_k}^{-1} v_\lambda$$

If $\tilde{\Omega}_{\psi}(-m-1)b \neq 0$, then by (9.3) $m + j = i_j$ for some $1 \leq j \leq l$ but this contradicts $\tilde{x}_m b \neq 0$. Hence

$$\tilde{\Omega}_{\psi}(-m)\tilde{x}_m b \equiv b$$

for $m \geq i_1$.

On the other hand assuming $\tilde{\Omega}_{\psi}(-m)b \neq 0$, by (9.3) $m + j - 1 = i_j$ for some $1 \leq j \leq l$, so that

$$\tilde{\Omega}_{\psi}(-m)b \equiv (-1)^{j-1} \delta_{m-j+1,-i_j} \tilde{x}_{i_1+1}^{-1} \cdots \tilde{x}_{i_{j-1}+1}^{-1} \tilde{x}_{i_j}^{-1} \cdots \tilde{x}_{i_k}^{-1} v_\lambda \mod q\mathcal{L}(\lambda)$$

we have by (9.2)

$$\tilde{x}_m \tilde{\Omega}_{\psi}(-m)b \equiv (-1)^{j-1} \delta_{m-j+1,-i_j} \tilde{x}_m \tilde{x}_{i_1+1}^{-1} \cdots \tilde{x}_{i_{j-1}+1}^{-1} \tilde{x}_{i_j}^{-1} \cdots \tilde{x}_{i_k}^{-1} v_\lambda \mod q\mathcal{L}(\lambda)$$

$$\equiv \delta_{m-j+1,-i_j} \tilde{x}_{i_1}^{-1} \cdots \tilde{x}_{i_{j-1}}^{-1} \tilde{x}_{i_j+1}^{-1} \cdots \tilde{x}_{i_k}^{-1} v_\lambda \mod q\mathcal{L}(\lambda)$$

$$\equiv b.$$

Finally if $\tilde{x}_m b \neq 0$, by (9.2) $m + j \neq i_j$ and

$$\tilde{x}_m b \equiv (-1)^{j-1} \tilde{x}_{i_1}^{-1} \cdots \tilde{x}_{i_{j-1}}^{-1} \tilde{x}_{i_j+1}^{-1} \cdots \tilde{x}_{i_k}^{-1} v_\lambda + q\mathcal{L}(\lambda)$$

so that by (9.3)

$$\Omega_{\psi}(-m)\tilde{x}_m b \equiv (-1)^{j-1} \Omega_{\psi}(-m)\tilde{x}_{i_1}^{-1} \cdots \tilde{x}_{i_{j-1}}^{-1} \tilde{x}_{i_j+1}^{-1} \cdots \tilde{x}_{i_k}^{-1} v_\lambda + q\mathcal{L}(\lambda)$$

$$= \tilde{x}_{i_1}^{-1} \tilde{x}_{i_2}^{-1} \cdots \tilde{x}_{i_j}^{-1} \cdots \tilde{x}_{i_k}^{-1} v_\lambda + q\mathcal{L}(\lambda)$$

$$= b.$$

\[\Box\]

Acknowledgement

The first two authors would like to thank the Mittag-Leffler Institute for its hospitality during their stay where part of this work was done. The first author was partially support by a Simons Collaboration Grant (#319261). The second author was supported in part by the CNPq grant (301320/2013-6) and by the FAPESP grant (2014/09310-5). The third author was partially support by the Simons Foundation Grant #307555. The authors would like to acknowledge the valuable comments of Masaki Kashiwara on earlier versions of this paper.
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