3D reduction of the three-fermion Bethe-Salpeter equation

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Abstract

We present a 3D approximation of the three-fermion Bethe-Salpeter equation. Our 3D equation is covariantly cluster separable and the two-fermion cluster separated limits are exact equivalents of the corresponding two-fermion Bethe-Salpeter equations. The potentials include positive free energy projectors in order to avoid continuum dissolution.

1 Introduction

The elimination of the relative times in the three-fermion Bethe-Salpeter equation can be performed in many ways. This equation can for example be approximated by 3D Schrödinger-Pauli or Faddeev equations. In principle a lot of higher-order correction terms of various origins, often neglected, should restore the equivalence with the initial Bethe-Salpeter equation. We are searching for a 3D equation which would be an element in a chain of approximations transforming the original Bethe-Salpeter equation into a manageable equation and would also satisfy at best the following list of requirements: Lorentz invariance, cluster separability, hermiticity and slow energy dependence of the potentials, correct heavy mass limits, absence of continuum dissolution. The solutions of the corresponding two-fermion problem will provide the building blocks of our three-fermion equation.

2 The two-fermion problem.

The Bethe-Salpeter equation for the bound states of two fermions is

$$\Phi = G_0 K \Phi$$

(1)

where $\Phi$ is the Bethe-Salpeter amplitude, $K$ the Bethe-Salpeter kernel (sum of the irreducible Feynman graphs) and

$$G_0 = G_{01} G_{02}, \quad G_{0i} = \frac{1}{p_{i0} - h_i + i\epsilon h_i} \beta_i$$

(2)

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the free propagator. The \( h_i \) are the free Dirac hamiltonians

\[
h_i = \vec{\alpha}_i \cdot \vec{p}_i + \beta_i m_i \quad (i = 1, 2).
\] (3)

We shall denote the total and relative momenta, and the corresponding combinations of the free hamiltonians by

\[
P = p_1 + p_2, \quad p = \frac{1}{2}(p_1 - p_2), \quad S = h_1 + h_2, \quad s = \frac{1}{2}(h_1 - h_2).
\] (4)

We shall also need the positive and negative free energy projectors:

\[
\Lambda^{\pm \pm} = \Lambda_1^\pm \Lambda_2^\pm, \quad \Lambda_i^\pm = \frac{E_i \pm h_i}{2E_i}, \quad E_i = \sqrt{h_i^2} = (\vec{p}_i^2 + m_i^2)^{\frac{1}{2}}.
\] (5)

The free propagator \( G_0 \) will be written as the sum of an approached propagator \( G_\delta \) (combining a constraint fixing the relative energy, and a global 3D propagator) and a rest \( G_R \). Salpeter’s 3D propagator, which appears automatically in case of an “instantaneous kernel" is

\[
\int dp_0 G_0(p_0) = -2i\pi \frac{\Lambda^{++} - \Lambda^{--}}{P_0 - S} \beta_1 \beta_2.
\] (6)

We shall skip the \( \Lambda^{--} \) projector and write the free propagator as

\[
G_0 = G_\delta + G_R, \quad G_\delta(p_0) = -2i\pi \delta(p_0 - s) \frac{\Lambda^{++}}{P_0 - S} \beta_1 \beta_2.
\] (7)

The Bethe-Salpeter equation becomes then the inhomogeneous equation

\[
\Phi = \Psi + G_R K \Phi, \quad \Psi = G_\delta K \Phi.
\] (8)

Eliminating \( \Phi \):

\[
\Psi = G_\delta K(1 - G_R K)^{-1} \Psi = G_\delta K_T \Psi, \quad K_T = K + KG_R K + ...
\] (9)

The reduction series \( K_T \) re-introduces in fact the reducible graphs into the Bethe-Salpeter kernel, but with \( G_0 \) replaced by \( G_R \). The equation becomes

\[
\Psi = G_\delta K_T \Psi = -2i\pi \delta(p_0 - s) \frac{\Lambda^{++}}{P_0 - S} \beta_1 \beta_2 K_T \Psi.
\] (10)

Eliminating the relative energy dependence gives a single 3D equation:

\[
\Psi = \delta(p_0 - s) \psi, \quad \psi = \frac{\Lambda^{++}}{P_0 - E_1 - E_2} V(P_0) \psi
\] (11)

\[
V(P_0) = -2i\pi \int dp'_0 dp_0 \delta(p'_0 - s) \beta_1 \beta_2 K_T(p'_0, p_0, P_0) \delta(p_0 - s).
\] (12)

We explicitated the dependence of the operator \( K_T \) in the conserved total momentum \( P_0 \) and in the relative momentum \( p_0 \) (this last dependence being non-local).

Similar results are obtained with other constraints and 3D propagators.
3 The three-fermion problem.

The Bethe-Salpeter equation is now:

$$\Phi = [G_{01}G_{02}K_{12} + G_{02}G_{03}K_{23} + G_{03}G_{01}K_{31} + G_{01}G_{02}G_{03}K_{123}] \Phi$$  \hspace{1cm} (13)

where $K_{123}$ is the sum of the purely three-body irreducible contributions. We shall neglect it and replace the two-body kernels by instantaneous ones, equivalent at the cluster-separated limits. Among an infinity of choices, we shall use directly the two-body potentials of the previous section, putting the spectator fermion on the mass shell:

$$K_{12}(p'_{120},p_{120},P_{120})$$

$$\approx \beta_1 \beta_2 \Lambda_{12}^{++} \int dp'_{120} dp_{120} \delta(p'_{120} - s_{12}) \beta_1 \beta_2 K_{T12}(p'_{120},p_{120},P_{120} - h_3) \delta(p_{120} - s_{12}) \Lambda_{12}^{++}$$

$$= -\frac{1}{2i \pi} \beta_1 \beta_2 \Lambda_{12}^{++} V_{12}(P_0 - h_3) \Lambda_{12}^{++},...$$  \hspace{1cm} (14)

where $P$ is now the total energy-momentum of the three-fermion system. The Bethe-Salpeter equation becomes

$$\Phi = -\frac{1}{2i \pi} G_{01}G_{02}G_{03} \beta_1 \beta_2 \beta_3 \left[ \Lambda_{12}^{++} V_{12} \Lambda_{12}^{++} \psi_{12} + \cdots + \cdots \right]$$  \hspace{1cm} (15)

$$\psi_{ij}(p_{kj}) = \beta_k G^{-1}_{0k} \int dp_{ij0} \Phi.$$  \hspace{1cm} (16)

This leads to a set of three coupled integral equations in the $\psi_{ij}$. We shall search for solutions analytical in the $\text{Im}(p_{kj}) < 0$ half planes and close the integration paths clockwise in these planes. The only singularities will then be the poles of the free propagators. Performing the integrations with respect to the $p_{ij0}$ gives then

$$\psi_{12}(p_{30}) = \frac{\Lambda_{12}^{++}}{(P_0 - S) - (p_{30} - h_3) + i \epsilon} \left[ \Lambda_{12}^{++} V_{12} \Lambda_{12}^{++} \psi_{12}(p_{30}) + \Lambda_{23}^{++} V_{23} \Lambda_{23}^{++} \psi_{23}(h_1) + \Lambda_{31}^{++} V_{31} \Lambda_{31}^{++} \psi_{31}(h_2) \right]$$  \hspace{1cm} (17)

and similarly for $\psi_{23}$ and $\psi_{31}$. Solving (17) with respect to $\psi_{12}(p_{120})$ confirms its analyticity in the $\text{Im}(p_{k0}) < 0$ half plane. Furthermore, equation (13) shows that the three projections $\Lambda^+_k \psi_{ij}(h_k)$ are equal (let us call them $\psi$ ) and satisfy the 3D equation

$$\psi = -\frac{1}{P_0 - E_1 - E_2 - E_3} \left[ V_{12}(P_0 - E_3) + V_{23}(P_0 - E_1) + V_{31}(P_0 - E_2) \right] \psi.$$  \hspace{1cm} (18)

Moreover, it can be shown that $\psi$ is the integral of the Bethe-Salpeter amplitude with respect to the relative times:

$$\psi = -\frac{1}{2i \pi} \int dp_{10} dp_{20} dp_{30} \delta(p_{10} + p_{20} + p_{30} - P_0) \Phi.$$  \hspace{1cm} (19)
4 Conclusions: pro’s and con’s of our three-cluster equation.

— The positive-energy projectors included in the equation forbid the mixing of the physical bound states with a continuum combining positive and negative energy free states (equations suffering of this "continuum dissolution" disease have no normalisable solutions).

— When the two potentials acting on one of the fermions are "switched off", one gets a free Dirac equation for this fermion and a correct two-fermion equation for the two other fermions (cluster separability). Furthermore, this last equation is an exact 3D equivalent of the two-fermion Bethe-Salpeter equation.

— We did not specify our reference frame until now. Our equations are not explicitly covariant, but if we assume that they are written in the three-fermion rest frame, we can always render them covariant by using the conserved total energy-momentum vector $P$. At the cluster-separated limits, however, the cluster separability requirement forbids the use of this vector. The equations for the two-fermion clusters must then be covariant. The fact that these equations are exact equivalents of the covariant two-fermion Bethe-Salpeter equations insures an implicit covariance without introducing Lorentz boosts by hand.

— The 3D reduction of the two-fermion Bethe-Salpeter equation of section 2 is only an example. The requirement of preserving the equivalence with the original equation leaves a large freedom which could be used to suit the needs of the three-fermion phenomenology.

— The potentials are hermitian and their dependence in the total energy is an higher-order effect.

— When the mass of one of the fermions becomes infinite, its presence should be translated in the equations by a potential (Coulombian in QED) acting on the other fermions. This requirement is only approximately satisfied. Satisfying it exactly would demand the reintroduction of some of the neglected three-body terms.

— Our two-body potentials are the sum of an infinity of contributions symbolized by Feynman graphs. Keeping only the first one (Born approximation) or a finite number of them renders the Lorentz covariance of the two-fermion clusters only approximate. One can use another 3D reduction based on a covariant second-order two-body propagator of Sazdjian, combined with a covariant substitute of $A^{++}$. This leads to a 3D three-cluster equation which is covariantly Born approximable, but more complicated.

— Our equation can also be written as a set of three Faddeev equations. These Faddeev equations can also be obtained as an approximation of Gross’ spectator model equations. This approximation being of the same order than these already made in Gross’ model, further investigations would be needed to decide which model is closer to the exact Bethe-Salpeter equation.

— The higher-order three-body contributions we neglected in our approximation are explicitly given at the Bethe-Salpeter level. We are presently trying to transform them into correcting terms to our 3D equation.