Spacetime singularities in string and its low dimensional effective theory

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Spacetime singularities are studied in both the $D + d$-dimensional string theory and its $D$-dimensional effective theory, obtained by the Kaluza-Klein compactification. It is found that spacetime singularities in the low dimensional effective theory may or may not remain after lifted to the $D + d$-dimensional string theory, depending on particular solutions. It is also found that in some cases solutions of the low dimensional effective theory are not singular, but after they are lifted to high dimensions of string theory, the higher dimensional spacetimes become singular. A closer analysis shows that it is due to the collapse of the extra dimensions.

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I. INTRODUCTION

String and M-theory all suggest that we may live in a world that has more than three spatial dimensions 1. Because only three of these are presently observable, one has to explain why the others are hidden from detection. One such explanation is the so-called Kaluza-Klein (KK) compactification, according to which the size of the extra dimensions is very small (often taken to be on the order of the Planck length) 2. As a consequence, modes that have momentum in the directions of the extra dimensions are excited at currently inaccessible energies.

In such a frame of compactifications, one often finds that the low-dimensional effective theories are usually singular, and such singularities might disappear when we consider the problem in the original high dimensional spacetimes. In particular, it was shown that the 4-dimensional extreme black hole with a special dilaton coupling constant can be interpreted as a completely non-singular, non-dilaton, black p-brane in (4+p) dimensions, provided that $p$ is odd 3.

Recently, three of the current authors 4 studied the problem in the frame of colliding two timelike branes in string theory. After developing the general formulas to describe such events, we studied a particular class of exact solutions first in the 5-dimensional effective theory, and then lifted it to the 10-dimensional spacetime. In general, the 5-dimensional spacetime is singular, due to the mutual focus of the two colliding 3-branes. Non-singular cases also exist, but with the price that both of the colliding branes violate all the three energy conditions, weak, dominant, and strong 5. After lifted to 10 dimensions, we found that the spacetime remains singular, whenever it is singular in the 5-dimensional effective theory. In the cases where no singularities are formed after the collision, we found that the two 8-branes necessarily violate all the energy conditions.

In this paper we shall address the same problem, but for the sake of simplicity, we consider only the case where no branes are present. Specifically, the paper is organized as follows. In the next section, we will set up the model to be studied in the framework of both $D + d$-dimensional string theory and its $D$-dimensional effective theory. In Section III, we study two classes of exact solutions, by paying particular attention on their local and global singular behavior. In Section IV, we first lift these solutions to the $D + d$-dimensional spacetimes of string theory, and then study their singular behavior. Section V contains our conclusions and discussing remarks.

II. TOROIDAL COMPACTIFICATION OF THE EFFECTIVE ACTION

Let us consider the toroidal compactification of the NS-NS sector of the action in $(D+d)$ dimensions, $M_{D+d} = M_D \times M_d$, where for the string theory we have $D + d = 10$. Then, the action takes the form 6,

$$S_{D+d} = - \frac{1}{2\kappa_{D+d}^2} \int d^{D+d}x \sqrt{|g_{D+d}|} e^{-\Phi} \left\{ \hat{R}_{D+d} + \hat{g}^{AB} (\hat{\nabla}_A \hat{\Gamma}) (\hat{\nabla}_B \hat{\Gamma}) - \frac{1}{12} \hat{H}^2 \right\},$$

(2.1)

where in this paper we consider the $(D + d)$-dimensional spacetimes described by the metric,

$$d\hat{s}^2_{D+d} = \hat{g}_{AB} dx^A dx^B = \gamma_{\mu\nu} (x^\lambda) dx^\mu dx^\nu + \phi^2 (x^\lambda) \gamma_{ab} (z^c) dz^a dz^b,$$

(2.2)

with $\gamma_{\mu\nu} (x^\lambda)$ and $\phi^2 (x^\lambda)$ depending only on the coordinates $x^\lambda$ of the spacetime $M_D$, and $\gamma_{ab} (z^c)$ only on the internal coordinates $x^c$, where $\mu, \nu, \lambda = 0, 1, 2, \ldots, D - 1$; $a, b, c = D, D + 1, \ldots, D + d - 1$; and $A, B, C = 0, 1, 2, \ldots, D + d - 1$. Assuming that matter fields are all independent of $z^c$, one finds that the internal space $M_d$ must be Ricci flat $\hat{R} [\gamma] = 0$ 4. For the purpose of the current work, it is sufficient to assume that $M_d$ is a
\[ -\frac{2}{\phi d} \gamma^\mu\nu \left( \frac{\nabla_{\mu} \phi}{\sqrt{\eta_{\nu}}} \right). \] (2.3)

Ignoring the dilaton \( \hat{\Gamma} \) and the form field \( \hat{H} \), the integration of the action (2.1) over the internal space yields,

\[ S_{D}^{eff.} = -\frac{1}{2\kappa^{2}_{D}} \int d^{D}x \sqrt{|g|} \{ R_{D} [g] + \frac{\kappa^{2}_{D}}{2} \left( \nabla \phi \right)^{2} \} \equiv V_{s}, \] (2.4)

where

\[ \kappa^{2}_{D} \equiv \frac{\kappa^{2}_{D+2}}{V_{s}}, \] (2.5)

and \( V_{s} \) is defined as

\[ V_{s} = \int \sqrt{\hat{g}} d^{d}z. \] (2.6)

For a string scale compactification, we have \( V_{s} = \left( \frac{2\pi}{\alpha'} \right)^{d} \), where \( (2\pi\alpha') \) is the inverse string tension.

After the conformal transformation,

\[ g_{\mu\nu} = \hat{g}_{\mu\nu} \gamma_{\mu\nu}, \] (2.7)

the D-dimensional effective action of Eq. (2.4) can be cast in the minimally coupled form,

\[ S_{D}^{eff.} = -\frac{1}{2\kappa^{2}_{D}} \int d^{D}x \sqrt{|g|} \{ R_{D} [g] - \kappa^{2}_{D} \left( \nabla \phi \right)^{2} \}, \] (2.8)

where

\[ \phi \equiv \pm \left( \frac{(D + d - 2)/2}{\kappa^{2}_{D} (D - 2)} \right)^{\frac{1}{2}} \ln \left( \frac{\hat{g}}{\phi} \right). \] (2.9)

In this paper, we refer the actions of Eqs. (2.1) and (2.8) to as the string frames, and the one of Eq. (2.8) the Einstein frames. It should be noted that solutions related by this conformal transformation can have completely different physical and geometrical properties in the two frames. In particular, in one frame a solution can be singular, while in the other it can be totally free from any kind of singularities. A simple example is the conformally-flat spacetimes \( g_{AB} = \Omega^{2}(x) \eta_{AB} \), where the spacetime described by \( g_{AB} \) can have a completely different spacetime structure from that of the Minkowski, \( \eta_{AB} \).

Although we are mainly interested in the string theory with the split \( D = 5 \) and \( d = 5 \), in most cases considered in this paper we shall not impose these restrictions here, so that our results obtained in this paper can be applied to other situations.

### III. SOLUTIONS IN D-DIMENSIONAL SPACETIMES IN THE EINSTEIN FRAME

The variation of the action (2.8) with respect to \( g_{\mu\nu} \) and \( \phi \) yields the D-dimensional Einstein-scalar field equations,

\[ R_{\mu\nu} = \kappa^{2}_{D} \phi_{,\mu} \phi_{,\nu}, \] (3.1)

\[ \nabla_{\lambda} \phi = 0, \] (3.2)

where \( (\ )_{,\mu} \equiv \partial(\ )/\partial x^{\mu} \).

In this paper, we consider the D-dimensional spacetimes described by the metric

\[ ds^{2}_{D,E} = 2e^{2\sigma(u,v)} du dv - e^{2\varphi(u,v)} d\Sigma^{2}, \] (3.3)

where

\[ d\Sigma^{2}_{D-2} = \sum_{i=2}^{D-1} (dx^{i})^{2}. \] (3.4)

Clearly, the \((D - 2)\)-dimensional space \( \Sigma \), spanned by \( d\Sigma^{2}_{D-2} \), is Ricci flat, and its topology remains unspecified. In this paper, we assume that this space is compact. One example is that it is a \((D - 2)\)-dimensional torus.

It should be noted that metric (3.3) is invariant under the coordinate transformation,

\[ u = f(\bar{u}), \quad v = g(\bar{v}), \] (3.5)

where \( f(\bar{u}) \) and \( g(\bar{v}) \) are arbitrary functions of their indicated arguments.

Introducing the following two null vectors [7],

\[ l_{\mu} \equiv \frac{\partial u}{\partial x^{\mu}} = \delta_{\mu}^{u}, \]

\[ n_{\mu} \equiv \frac{\partial v}{\partial x^{\mu}} = \delta_{\mu}^{v}, \] (3.6)

we can see that both of them are future-directed and orthogonal to \( \Sigma \). In addition, each of these two null vectors defines an affinely parameterized null geodesic congruence,

\[ t^{\lambda} \nabla_{\lambda} l_{\mu} = 0 = n^{\lambda} \nabla_{\lambda} n_{\mu}. \] (3.7)

Then, the expansions of the null ray \( u = \text{Const.} \) and the one \( v = \text{Const.} \) are defined, respectively, by

\[ \theta_{t} \equiv \nabla^{\lambda} l_{\lambda} = (D - 2) e^{-2\alpha} h_{\alpha}, \]

\[ \theta_{n} \equiv \nabla^{\lambda} n_{\lambda} = (D - 2) e^{-2\alpha} h_{\alpha}. \] (3.8)

It should be noted that the two null vectors are uniquely defined only up to a factor [6, 7]. In fact,

\[ \tilde{l}_{\mu} = f(u)\delta_{\mu}^{u}, \quad \tilde{n}_{\mu} = g(v)\delta_{\mu}^{v}, \] (3.9)

represent another set of null vectors that also define affinely parameterized null geodesics, and the corresponding expansions are given by

\[ \tilde{\theta}_{+} = f(u)\theta_{+}, \quad \tilde{\theta}_{-} = g(v)\theta_{-}. \] (3.10)
However, since along each curve \( u = \text{Const.} \) \((v = \text{Const.})\), the function \( f(u) \) \((g(v))\) is constant, this does not affect the definition of trapped surfaces in terms of the expansions (See [3] in details). Thus, without loss of generality, in the following definitions of trapped surfaces and black holes we consider only the expressions given by Eq. (3.8).

**Definitions** [3, 4]: The spatial \((D - 2)\)-dimensional surface \( S \) of constant \( u \) and \( v \) is said trapped, marginally trapped, or untrapped, according to whether \( \theta_{n|S} > 0 \), \( \theta_{n|S} = 0 \), or \( \theta_{n|S} < 0 \). Assuming that on the marginally trapped surfaces \( S \) we have \( \theta_{n|S} = 0 \), then an apparent horizon is the closure \( \Sigma \) of a three-surface \( \Sigma \) foliated by the trapped surfaces \( S \) on which \( \theta_{n|S} \neq 0 \). It is said outer, degenerate, or inner, according to whether \( \mathcal{L}_{u} \theta_{v} = 0 \), \( \mathcal{L}_{v} \theta_{u} = 0 \), or \( \mathcal{L}_{n} \theta_{v} > 0 \), where \( \mathcal{L}_{a} \) denotes the Lie derivative along the normal direction \( n_{\mu} \), given by,

\[
\mathcal{L}_{n} \theta_{l} = (D - 2) e^{-4\sigma} (h_{uv} - 2\sigma_{u} h_{v}), \\
\mathcal{L}_{l} \theta_{n} = (D - 2) e^{-4\sigma} (h_{uv} - 2\sigma_{u} h_{v}).
\] (3.11)

In addition, if \( \theta_{n|S} < 0 \) then the apparent horizon is said future, and if \( \theta_{n|S} > 0 \) it is said past.

**Black holes** are usually defined by the existence of future outer apparent horizons [3, 4, 7, 10]. However, in a definition given by Tipler [11] the degenerate case was also included.

For the metric (3.23), we find that Eqs. (3.11) and (3.2) yield

\[
h_{uu} + h_{v}^{2} - 2h_{u} \sigma_{u} = -\frac{k_{D}^{2}}{D - 2} \phi_{u}^{2}, \]

\[
h_{uv} + h_{v}^{2} - 2h_{v} \sigma_{v} = -\frac{k_{D}^{2}}{D - 2} \phi_{v}^{2}.
\] (3.12)

\[
2\sigma_{uv} + (D - 2) (h_{uv} + h_{u} h_{v}) = -k_{D}^{2} \phi_{u} \phi_{v}.
\] (3.13)

\[
\sigma_{uv} + (D - 2) h_{u} h_{v} = 0, \]

\[
2\phi_{uv} + (D - 2) (h_{u} \phi_{v} + h_{v} \phi_{u}) = 0.
\] (3.14)

\[
h(u, v) = \frac{1}{D - 2} \ln (F(u) + G(v)),
\] (3.17)

where \( F(u) \) and \( G(v) \) are arbitrary functions. Then, one may integrate Eq. (3.17) to find \( h \). Once \( h \) and \( \phi \) are found, \( \sigma \) can be obtained by integrating Eqs. (3.12) and (3.13). The general solutions for \( \sigma \) and \( \phi \) are unknown. In the following, we shall consider some specific solutions. In particular, we shall consider the three cases separately:

\( a) \) \( F'(u) \neq 0, G'(v) = 0; \)

\( b) \) \( F'(u) = 0, G'(v) \neq 0; \)

\( c) \) \( F'(u) G'(v) \neq 0, \)

where a prime denotes the ordinary differentiation. The second case can be obtained from the first one by exchanging the \( u \) and \( v \) coordinates. Thus, without loss of generality, we need consider only Cases (a) and (c).

Before proceeding further, we note that for the solution of Eq. (3.17), Eqs. (3.8) and (3.11) reduce to

\[
\theta_{l} = e^{-2\sigma} \frac{G'(v)}{F(u) + G(v)},
\]

\[
\theta_{n} = e^{-2\sigma} \frac{F'(u)}{F(u) + G(v)},
\] (3.18)

and

\[
\mathcal{L}_{n} \theta_{l} = -\theta_{l} (\theta_{n} + 2e^{-2\sigma} \sigma_{u}),
\]

\[
\mathcal{L}_{l} \theta_{n} = -\theta_{n} (\theta_{l} + 2e^{-2\sigma} \sigma_{v}).
\] (3.19)

\( A. \) \( F'(u) \neq 0, G'(v) = 0 \)

In this case, from Eq. (3.13) we find that \( \phi \neq \phi(u) \). Hence, Eq. (3.14) yields

\[
\sigma(u, v) = a(u) + b(v). \] (3.20)

where \( a(u) \) and \( b(v) \) are other arbitrary functions. Using the gauge freedom of Eq. (3.23), without loss of generality we can always set \( a(u) = 0 = b(v) \), so that this class of solutions are given by

\[
\sigma(u, v) = 0,
\]

\[
h(u, v) = \ln a(u),
\]

\[
\phi(u, v) = \pm \int^{u} \left( -\frac{\alpha''(u')}{\alpha(u')} \right)^{1/2} du' + \phi_{0},
\] (3.21)

where \( \alpha(u) \equiv F(u)^{1/(D-2)} \), and \( \phi_{0} \) is an integration constant. Inserting Eq. (3.21) into Eq. (3.8), we find that

\[
\theta_{l} = 0, \quad \theta_{n} = (D - 2) \frac{\alpha'(u)}{\alpha(u)},
\] (3.22)

for which we have \( \theta_{l} \theta_{n} = 0 \) identically. Then, according the above definition, the \((D - 2)\)-surface \( S \) is always marginally trapped. Since

\[
\mathcal{L}_{l} \theta_{l} = 0 = \mathcal{L}_{l} \theta_{n},
\] (3.23)

\( S \) is also degenerate.

To study these solutions further, we notice that for these solutions all the scalars built from the Riemann curvature tensor are zero, therefore, in the present case scalar curvature singularities are always absent [12]. However, non-scalar curvature singularities might also exist. In particular, tidal forces experienced by an observer may become infinitely large under certain conditions [12]. To see how this can happen, let us consider
the timelike geodesics in the \((u, v)\)-plane, which in the present case are simply given by
\[
\dot{u} = \gamma_0, \quad \dot{v} = \frac{1}{2\gamma_0}, \quad \dot{x}^i = 0, \quad (3.24)
\]
where \(i = 2, \ldots, D - 1\), \(\gamma_0\) is an integration constant, and an overdot denotes the ordinary derivative with respect to the proper time, \(\lambda\), of the timelike geodesics. Defining \(e^\mu_0 = dx^\mu/d\lambda\), we find that the unit vectors, given by
\[
e^\mu_{(0)} = \gamma_0 \delta^\mu_u + \frac{1}{2\gamma_0} \delta^\mu_v, \\
e^\mu_{(1)} = \gamma_0 \delta^\mu_u - \frac{1}{2\gamma_0} \delta^\mu_v, \\
e^\mu_{(i)} = \frac{1}{\alpha(u)} \delta^\mu_i, \quad (3.25)
\]
form a freely falling frame,
\[
e^\mu_{(\alpha)} e_{(\beta)} g_{\mu\nu} = \eta_{\alpha\beta}, \quad e^\mu_{(\alpha)} e^\nu_{(\beta)} = 0, \quad (3.26)
\]
where \(\eta_{\alpha\beta} = \text{diag.} \{-1, 1, \ldots, 1\}\). Projecting the Ricci tensor onto the above frame, we find that
\[
R_{(\alpha)(\beta)} = R_{\mu\nu} e^\mu_{(\alpha)} e^\nu_{(\beta)} \\
= -\gamma_0^2 (D - 2) \left(\frac{\alpha''(u)}{\alpha(u)}\right) \left(\delta^\nu_{\alpha} \delta^\nu_{\beta}\right) \\
- \left(\delta^\nu_{\alpha} \delta^\nu_{\beta} + \delta^\nu_{\alpha} \delta^\nu_{\beta} + \delta^\nu_{\beta} \delta^\nu_{\beta}\right). \quad (3.27)
\]
Clearly, the tidal forces remain finite over the whole spacetime, as long as \(\alpha''/\alpha\) is finite. To see this clearly, let us consider the following solutions,
\[
\frac{\alpha''(u)}{\alpha(u)} = -\frac{\omega^2}{(u - u_0)^\gamma}, \quad (3.28)
\]
for which we have
\[
\phi(u, v) = \phi_0 + \left(\frac{\omega^2(D - 2)}{\kappa_2^2}\right)^{1/2} \times \left\{ \begin{array}{l l}
\frac{2}{D} (u - u_0)^{1-\gamma/2}, & \gamma \neq 2, \\
\ln (u - u_0), & \gamma = 2,
\end{array}\right. \quad (3.29)
\]
where \(u_0\) is an arbitrary constant, and without loss of generality, we can always set \(u_0 = 0\), an assumption we shall adopt in the following discussions. The constants \(\omega\) and \(\gamma\) have to satisfy the conditions \(\alpha''(u) > 0\) and \(\alpha''(u)/\alpha(u) < 0\), so that the metric has the correct signs and the scalar field is real.

From Eq. (3.24), we find that \(u \sim \gamma_0 \lambda\), where the proper time \(\lambda\) was chosen such that \(u = 0\) corresponds to \(\lambda = 0\). Then, the distortion, which is proportional to the twice integrals of \(R_{(\alpha)(\beta)}\) with respect to the proper time \(\lambda\), is given by
\[
D_{(\alpha)(\beta)} \equiv \int d\lambda \int R_{(\alpha)(\beta)} d\lambda \\
\sim \left\{ \begin{array}{l l}
\frac{\lambda (\ln \lambda - 1)}{\ln \lambda}, & \gamma = 1, \\
\frac{\lambda^2}{\ln \lambda}, & \gamma = 2, \\
\lambda^{2-\gamma}, & \gamma \neq 1, 2,
\end{array}\right. \quad (3.30)
\]

\[\text{FIG. 1: The Penrose diagram for the case } \gamma < 0 \text{ of the solution (3.28) in the Einstein frame. The double solid lines } AB \text{ and } CD \text{ represent null infinities } u = \pm \infty \text{ and denote strong spacetime singularities, where both the tidal forces and distortions exerting on a freely falling observer become unbound. The spacetime is regular at } u = 0, \text{ and its extension to Region } II \text{ where } u < 0 \text{ is unique only when } -\gamma \text{ is an integer.}\]

As \(\lambda \to 0\). To study these solutions further, let us consider the following cases separately.

\[\text{1. } \gamma < 0\]

In this case, Eqs. (3.27), (3.28) and (3.30) show that both the tidal forces and distortions are finite at \(u = 0\). Therefore, to have a geodesically maximal spacetime, we need to extend the solutions across this surface to the region \(u < 0\). When \(\gamma = -n\), where \(n(= 1, 2, \ldots)\) is an integer, the extension is simple, and can be obtained by simply taking \(u\) for \(u \in (-\infty, \infty)\). However, When \(-\gamma\) is not an integer, the solution is not analytically at \(u = 0\), and the extension is not unique. One possible extension is to replace \(u\) by \(|u|\). Once such an extension is done, it can be seen that both the tidal forces and distortions diverge as \(|u| \to \infty\). Therefore, the spacetimes are singular at the null infinities, and the nature of the singularities is strong, as both of them diverge. The corresponding Penrose diagram is given by Fig. 1.

\[\text{2. } \gamma = 0\]

In this case, \(\alpha(u)\) and \(\phi(u)\) can be given explicitly,
\[
\alpha(u) = \alpha_0 \sin (\omega u + \Delta),
\]
\[ \phi(u) = \pm \sqrt{\frac{D-2}{\kappa_D^2}} \omega u + \phi_0, \]  
(3.31)

where \( \Delta \) and \( \phi_0 \) are the integration constants. From the above we can see that the \((D-2)\)-dimensional space \( S \) collapses to a point at \( \omega u + \Delta = n\pi \). This can also be seen from the expansion given by Eq. (3.22), which now reads,

\[ \theta_n = \omega (D-2) \cos \left( \frac{\omega u + \Delta}{\sin(\omega u + \Delta)} \right). \]

(3.32)

Clearly, \( \theta_n \) diverges at \( \omega u + \Delta = n\pi \). However, a closer investigation of this solution shows that the spacetime is not singular at these points. For example, one may consider the tidal forces measured by observers that move along timelike geodesics not perpendicular to \( S \). Without loss of generality, let us consider the timelike geodesics, described by \( u = u(\lambda) \), \( v = v(\lambda) \), \( x^2 = x^2(\lambda) \) and \( x^i = x^i_0 = \text{Const.} \). Then, it can be shown that the timelike geodesical equation allows the first integration, and yields,

\[ \dot{u} = \gamma_0, \quad \dot{v} = \frac{1}{2\gamma_0} \left( \frac{\beta^2_0}{\alpha^2(u)} + 1 \right), \]

\[ \dot{x}^2 = \frac{\beta^2_0}{\alpha^2(u)}, \quad \dot{x}^i = 0, \quad (i = 3, 4, \ldots, D - 1), \]

(3.33)

where \( \gamma_0 \) and \( \beta_0 \) are two integration constants. From the above we find the following unit vectors,

\[ u^{(0)} \equiv \frac{dx^\mu}{d\lambda} = \gamma_0 \delta^\mu_0 + \frac{1}{2\gamma_0} \left( \frac{\beta^2_0}{\alpha^2(u)} + 1 \right) \delta^\mu_0 + \frac{\beta_0}{\alpha^2(u)} \delta^\mu_2, \]

\[ u^{(1)} = \gamma_0 \delta^\mu_0 + \frac{1}{2\gamma_0} \left( \frac{\beta^2_0}{\alpha^2(u)} - 1 \right) \delta^\mu_0 + \frac{\beta_0}{\alpha^2(u)} \delta^\mu_2, \]

\[ u^{(2)} = \frac{\beta_0}{\gamma_0 \alpha(u)} \delta^\mu_0 + \frac{1}{\alpha(u)} \delta^\mu_2, \]

\[ u^{(i)} = \frac{1}{\alpha(u)} \delta^\mu_i, \quad (i = 3, 4, \ldots, D - 1). \]

(3.34)

It can be shown that they satisfy Eq. (3.20), that is, they also form a freely-falling frame. Then, we find that

\[ R^{(\alpha)(\beta)} = R^{\mu
u} e^\mu_{(\alpha)} e^\nu_{(\beta)} = \omega^2 (D-2) e^\mu_{(\alpha)} e^\nu_{(\beta)}, \]

(3.35)

which is always finite, as can be seen from Eq. (3.34). Therefore, although the \((D-2)\)-dimensional space \( S \) focuses at \( \alpha(u) = 0 \), no spacetime singularities appear at these points, because no trapped compact surface exists in the present case. The Hawking-Penrose singularity theorems require the existence of both a compact trapped surface and the focusing of geodesics [5]. The extension across these focusing points can be done by simply taking \( u \) as any real value, i.e., \( u \in (\infty, \infty) \). Once such an extension is made, Eq. (3.31) shows that we may have spacetime singularities at \( u = \pm \infty \). As Eq. (3.30) shows that indeed the distortion at these null infinities diverge, although the tidal forces still remain finite. Then, the corresponding Penrose diagram is given by Fig. 2.

\[ 3. \ 0 < \gamma < 2 \]

In this case, Eqs. (3.27), (3.28) and (3.30) show that at \( u = 0 \) the tidal forces become unbounded, while the distortions remain finite. This type of singularities is usually said weak, and the spacetime beyond this surface may be extendible [1], although it is still unclear how to carry out specifically such extensions. As \( u \to \infty \), from Eqs. (3.27), (3.28) and (3.30) we find that the tidal forces are bounded, but now the distortions become unbounded. The corresponding Penrose diagram is given by Fig. 3.

\[ 4. \ \gamma = 2 \]

When \( \gamma = 2 \), Eq. (3.28) has the solution

\[ \alpha(u) = \alpha_0 u^\delta, \]

(3.36)

where \( \omega^2 = \delta(1-\delta) \) with \( 0 < \delta < 1 \). For such a solution, Eq. (3.22) reads,

\[ \theta_n = \frac{\delta(D-2)}{u}. \]

(3.37)

Then, from Eqs. (3.27), (3.28) and (3.30) we find that in the present case both the tidal forces and distortions become unbound at \( u = 0 \), so a strong spacetime singularity appears at \( u = 0 \). In this case, we also have \( \theta_n(u = 0) = \infty \). However, at \( u = \infty \), the tidal forces remain finite, while the distortions become unbounded. The
FIG. 3: The Penrose diagram for the case $0 < \gamma < 2$ of the solution (3.28). The spacetime is singular at $u = 0$ in the sense that the tidal forces become unbounded, while the distortions remain bounded. It is also singular at the null infinity $u = \infty$, denoted by the line CD, where the tidal forces remain finite, but the distortions become unbounded.

The corresponding Penrose diagram in this case is given by Fig. 4.

5. $\gamma > 2$

When $\gamma \geq 2$, both the tidal forces and the distortion become unbound at $u = 0$, so the spacetime has a strong singularity along this surface. However, at $u = \infty$ all of them remain finite. Therefore, in the present case the spacetime is free of spacetime singularity at the null infinity $u = \infty$. The corresponding Penrose diagram in this case is given by Fig. 5.

B. $F'(u)G'(\nu) \neq 0$

In this case to solve Eqs. (3.12)-(3.16), it is found convenient first to introduce two new coordinates $\bar{u}$ and $\bar{v}$ via the relations $\bar{u} \equiv F(u)$ and $\bar{v} \equiv G(v)$, using the gauge freedom (3.5). In terms of these new coordinates, the metric (3.3) takes the form,

$$ds_{D,E}^2 = \bar{g}_{\mu\nu} d\bar{x}^{\mu} d\bar{x}^{\nu}$$

$$= 2e^{2\Sigma(\bar{u},\bar{v})} d\bar{u}d\bar{v} - e^{2H(\bar{u},\bar{v})} d\Sigma^{2}_{D-2},$$  \hspace{1cm} (3.38)

where

$$H(\bar{u},\bar{v}) \equiv h(u,v) = \frac{1}{D-2} \ln (\bar{u} + \bar{v}),$$

FIG. 4: The Penrose diagram for the case $\gamma = 2$ of the solution (3.28). The spacetime is singular at $u = 0$ in the sense that the tidal forces become unbounded, while the distortions remain bounded. It is also singular at the null infinity $u = \infty$, denoted by the line CD, where the tidal forces remain finite, but the distortions become unbounded.

FIG. 5: The Penrose diagram for the case $\gamma > 2$ of the solution (3.28). The spacetime is singular at $u = 0$ in the sense that both the tidal forces and distortions become unbounded. At the null infinity $u = \infty$, denoted by the line CD, it is non-singular, because now both the tidal forces and distortions remain finite.
\[ \Sigma(\bar{u}, \bar{v}) \equiv \sigma(u, v) - \frac{1}{2} \ln \left[ F'(u) G'(v) \right]. \]  

Then, it can be shown that Eqs. (3.12)–(3.16) reduce to
\[ M_{,t} = \frac{1}{2} \left( \phi_{,t}^2 + \phi_{,y}^2 \right), \]  
\[ M_{,y} = t \phi_{,t} \phi_{,y}, \]  
\[ M_{,tt} - M_{,yy} = -\frac{1}{2} \left( \phi_{,t}^2 - \phi_{,y}^2 \right), \]  
\[ \phi_{,tt} + \frac{1}{t} \phi_{,t} - \phi_{,yy} = 0, \]  
where
\[ \Sigma \equiv -\frac{D - 3}{2(D - 2)} \ln(t) + \kappa_D^2 M, \]
\[ t \equiv \bar{u} + \bar{v}, \quad y \equiv \bar{u} - \bar{v}. \]  

Eq. (3.42) is the integrability condition of Eqs. (3.40) and (3.41). Thus, once a solution for \( \phi \) is found from Eq. (3.43), the remaining is to find \( M \) from Eqs. (3.40) and (3.41) by quadratures.

Using the freedom on the choice of the two null vectors defined by Eq. (3.33), for the metric (3.38) we define \( \bar{l}_\mu = \delta^\mu_\alpha \) and \( \bar{n}_\mu = \delta^\mu_\beta \). Then, we find that,
\[ \bar{\theta}_t \equiv \bar{g}^{\alpha\beta} \bar{l}_\alpha \bar{\theta}_\beta = \frac{e^{-2\Sigma}}{t}, \]
\[ \bar{\theta}_n \equiv \bar{g}^{\alpha\beta} \bar{n}_\alpha \bar{\theta}_\beta = \frac{e^{-2\Sigma}}{t}, \]  
from which we find that \( \bar{\theta}_t \bar{\theta}_n \geq 0 \), where equality holds only when \( t = \infty \) or/and \( \Sigma = \infty \). Therefore, in the present case, the \( (D - 2) \)-dimensional surface \( S \) is always trapped or marginally trapped. However, it must be noted that the coordinates \((t, y)\) or \((\bar{u}, \bar{v})\) do not always cover the whole spacetime. There exist cases where the hypersurface \( \Sigma = \infty \) does not represent the boundary of the spacetime. To have a geodesically maximal spacetime, extension beyond this surface is needed. In the extended region(s), one may have \( \bar{\theta}_t \bar{\theta}_n < 0 \), that is, the spacetime is not trapped. Typical examples of this kind can be found in [3].

In the following, we consider three classes of solutions, to be referred, respectively, to as, Class IIa, IIb, and IIc, and show explicitly that such cases happen here, too.

1. **Class IIa Solutions**

This class of solutions is given by
\[ M = \frac{1}{2} c^2 \ln(t) + M_0, \]
\[ \phi = c \ln(t) + \phi_0, \]  
where \( c \), \( \phi_0 \) and \( M_0 \) are integration constants. Then, from Eq. (3.33) we find that
\[ \bar{\theta}_t = \bar{\theta}_n = \frac{e^{-\kappa_D^2 M_0}}{t^{\kappa_D^2} + \frac{\phi_0}{\phi_0}}. \]

2. **Class IIb Solutions**

This class of solutions is given by
\[ M = \frac{1}{2} c^2 \ln(t) + M_0, \]
\[ \phi = c \ln(t) + \phi_0, \]  
for which from Eq. (3.47) we find that
\[ \bar{\theta}_t = \bar{\theta}_n = \frac{\left( y^2 - t^2 \right) \left[ y + \sqrt{y^2 - t^2} \right]}{e^{2\kappa_D^2 M_0} t^{4\kappa_D^2} + \frac{\phi_0}{\phi_0}}. \]  

Clearly, in the present case the hypersurface \( t = 0 \) still represents a focusing point. In addition, on the hypersurfaces \( y^2 = t^2 \), we have \( \Sigma = \infty \) and \( \bar{\theta}_t \bar{\theta}_n = 0 \). Thus, the surfaces \( S \) now become marginally trapped along \( y^2 = t^2 \). If the spacetime is not singular on these null surfaces, extension beyond them is needed. To study the singular behavior along these surfaces, let us consider the quantity,
\[ R_D[g] = \kappa_D^2 \cdots \]
\[\Sigma = \left(2\chi^2 - \frac{D - 3}{2(D - 2)}\right) \ln(\tilde{u}^n - (-\tilde{v})^n)
+ \frac{D - 3}{2(D - 2)} \ln(\tilde{u}^n + (-\tilde{v})^n) + \Sigma_0,\]
\[\phi = \frac{2\chi}{\kappa_D} \ln(\tilde{u}^n - (-\tilde{v})^n) + \phi_0,\] (3.55)

where \(\Sigma_0 \equiv \kappa_D^2 M_0 + \ln \left(\frac{g^{2/2 - \chi^2}}{u}\right)\). Clearly, the coordinate singularity at \(y = t\) or \(\tilde{v} = 0\) disappears, and the solutions can be considered as valid for \(\tilde{v} > 0\). To study the properties of the spacetime in the extended region, let us consider the quantities,

\[\theta_t = (D - 2)e^{-2\Sigma} \tilde{H} \tilde{v} = 2ne^{-2\Sigma} \frac{(-\tilde{v})^{2n-1}}{\tilde{u}^2n - (-\tilde{v})^{2n}},\]
\[\theta_n = (D - 2)e^{-2\Sigma} \tilde{H} \tilde{u} = 2ne^{-2\Sigma} \frac{(-\tilde{v})^{2n-1}}{\tilde{u}^2n - (-\tilde{v})^{2n}},\]
\[\mathcal{L}_n \tilde{g}_t = -4n^2 e^{-2\Sigma} \frac{\tilde{u}^{n-1} (-\tilde{v})^{2n-1}}{(\tilde{u}^2n - (-\tilde{v})^{2n})^2} \times \left\{\left(2\chi^2 + \frac{1}{D - 2}\right) \tilde{u}^n + 2\chi^2 (-\tilde{v})^n\right\},\]
\[R_D[\tilde{g}] = 8n^2 \chi^2 e^{-2\Sigma} \frac{(-\tilde{v})^{2n-1}}{(\tilde{u}^2n - (-\tilde{v})^{2n})^2}.\] (3.56)

Then, we find that

\[\tilde{g}_t \tilde{g}_n = 4n^2 e^{-4\Sigma} \frac{(-\tilde{v})^{2n-1}}{(\tilde{u}^2n - (-\tilde{v})^{2n})^2} \times \left\{\begin{array}{l}
< 0, \tilde{u} \tilde{v} > 0, \\
= 0, \tilde{u} = \tilde{v} = 0, \\
> 0, \tilde{u} \tilde{v} < 0.
\end{array}\right.\] (3.57)

That is, now in the extended region where \(\tilde{u} \tilde{v} > 0\) the spacetime becomes untrapped. Across the hypersurface \(\tilde{v} = 0\), \(\tilde{u} \geq 0\), we have

\[\tilde{g}_t (\tilde{u} > 0, \tilde{v} = 0) = 0, \quad \tilde{g}_n (\tilde{u} > 0, \tilde{v} = 0) > 0,\]
\[\mathcal{L}_n \tilde{g}_t (\tilde{u} > 0, \tilde{v} = 0) = 0.\] (3.58)

Therefore, the half infinite line \(\tilde{v} = 0\), \(\tilde{u} \geq 0\) represents a past degenerate apparent horizon.

To study the singular behavior of the spacetime in the extended region, we need to distinguish the case where \(n\) is an even integer from the one where \(n\) is an odd integer. When \(n\) is an even integer, Eq. (3.56) shows that the spacetime is singular along the line \(\tilde{u} = \tilde{v}\), denoted by the vertical line \(0\tilde{C}\) in Fig. 8.

Because of the symmetry of the spacetime, one can make a similar extension across the half line \(\tilde{u} = 0\) and \(\tilde{v} \geq 0\), but now with

\[\tilde{u} = -(-\tilde{u})^n, \quad \tilde{v} = \tilde{v}^2.\] (3.59)
sents a big bang singularity. The \((D-2)\)-dimensional surfaces of constant \(\bar{u}\) and \(\bar{v}\) are always trapped in Regions \(I\) and \(I'\), but not in Regions \(II\) and \(II'\). The spacetime along the vertical line \(0C\) is singular. The two regions \(II\) and \(II'\) are physically disconnected. The lines \(0E\) and \(0D\) represent past degenerate apparent horizons.

Then, one finds that this half line also represents a past degenerate apparent horizon, and the extended region \(II'\) is not trapped, although it is disconnected with Region \(II\), because of the timelike singularity along the vertical line \(0C\).

When \(n\) is an odd integer, Eq. (3.56) shows that the spacetime is not singular along the line \(u = \bar{v}\), as shown in Fig. 9 where the lines \(0E\) and \(0D\) represent past degenerate apparent horizons. Then, Regions \(I\) and \(I'\) act as white holes. Since the solutions are symmetric with respect to \(t\), one may consider the case where \(t \leq 0\). Then, the spacetime will be given by the lower half part of Fig. 9 in which the \((D-2)\)-dimensional surfaces \(S\) of constant \(\bar{u}\) and \(\bar{v}\) are trapped in Regions \(III\) and \(III'\), and is not trapped in Region \(IV\). The lines \(0D'\) and \(0E'\) act as future degenerate apparent horizons, so Regions \(III\) and \(III'\) now represent black holes. The spacetimes of the lower half part is disconnected with that of the upper half by the spacelike singularity located along \(t = 0\).

**Case B.2.3:** \(\chi^2 \geq 1\). When \(\chi^2 \geq 1\), the hypersurfaces \(y = \pm t\) represent spacetime null infinities, and the solutions are already geodesically maximal. Indeed, it is found that the null geodesics \(\bar{u} = \text{Constant}\) have the integral,

\[
\eta = \left\{ \begin{array}{ll}
\eta_0 (\bar{v})^{1-\chi^2}, & \chi^2 > 1, \\
\eta_0 \ln(\bar{v}), & \chi^2 = 1, 
\end{array} \right. \quad (3.60)
\]

near the hypersurface \(y = t\) \((\bar{v} = 0)\), where \(\eta_0\) is an integration constant, and \(\eta\) denotes the affine parameter along the null geodesics. Thus, as \(\bar{v} \to 0^-\), we always have \(|\eta| \to \infty\).

Similar, it can be shown that the hypersurface \(y = -t\) also represents a null infinity of the spacetime. Therefore, the corresponding Penrose diagram is given by Fig. 10.

![Fig. 8: The Penrose diagram in the D-dimensional spacetime for \(\chi^2 < 1\) for the solutions given by Eq. (3.54) with \(n\) being an even integer, as well as for the ones given by Eq. (3.61) with \(n\) being an odd integer. The horizontal line \(t = 0\) represents a big bang singularity. The \((D-2)\)-dimensional surfaces of constant \(\bar{u}\) and \(\bar{v}\) are always trapped in Regions \(I\) and \(I'\), but not in Regions \(II\) and \(II'\). The spacetime along the vertical line \(0C\) is singular. The two regions \(II\) and \(II'\) are physically disconnected. The lines \(0E\) and \(0D\) represent past degenerate apparent horizons.](image1)

![Fig. 9: The Penrose diagram in the D-dimensional spacetime for \(\chi^2 < 1\) for the solutions given by Eq. (3.54) with \(n\) being an odd integer, as well as for the ones given by Eq. (3.61) with \(n\) being an even integer. The horizontal line \(t = 0\) represents a spacetime singularity. The \((D-2)\)-dimensional surfaces of constant \(\bar{u}\) and \(\bar{v}\) are always trapped in Regions \(I, I', III\) and \(III'\), but not in Regions \(II\) and \(IV\). The upper half part is disconnected with the lower half part by the spacetime singularity along the line \(AB\) where \(t = 0\). The lines \(0E\) and \(0D\) represent past degenerate apparent horizons, while the ones \(0E'\) and \(0D'\) represent future degenerate apparent horizons. Regions \(I\) and \(I'\) represent white holes, while Regions \(III\) and \(III'\) represent black holes.](image2)

3. **Class IIc Solutions**

This class of solutions is given by

\[
M = \frac{1}{2} c^2 \ln \left( \frac{\left( y + \sqrt{y^2 - t^2} \right)^2}{y^2 - t^2} \right) + M_0, \quad \phi = c \ln \left( y + \sqrt{y^2 - t^2} \right) + \phi_0, \quad (3.61)
\]

for which we find

\[
\tilde{\theta}_t = \tilde{\theta}_n = \frac{e^{-2n\tilde{v}M_0} (y^2 - t^2)^{\chi^2}}{t^{1/\chi}} \left( y + \sqrt{y^2 - t^2} \right)^{2\chi},
\]

\[
\propto \left\{ \begin{array}{ll}
t^{\frac{n\tilde{v}}{\chi}} & y > 0, \\
t^{-(4\chi^2 + n\tilde{v})} & y < 0,
\end{array} \right. \quad (3.62)
\]

as \(t \to 0\). Therefore, in this case the hypersurface \(t = 0\) represents a focusing point, while the ones \(y = \pm t\)
when $\chi^2 \geq 1$.

Thus, the nature of the spacetime singularity at $t = 0$ is different on the half line $y > 0$ from that on the other half line $y < 0$. In particular, we find

$$R_D[g] \propto \begin{cases} -\frac{t}{y^2} \rightarrow 0, & y > 0, \\ -\frac{t}{-(4\chi^2 + t^2)} \rightarrow -\infty, & y < 0, \end{cases}$$

as $t \rightarrow 0$. Therefore, $R_D[g]$ is singular only on the half line $t = 0, y \leq 0$. However, the studies of other scalars show that the other half line, $t = 0$ and $y > 0$, is also singular. For example, the corresponding Kretschmann scalar is given by

$$I_D \equiv R^{\mu \nu \lambda \sigma} R_{\mu \nu \lambda \sigma} = \frac{(y^2 - t^2)^{2\chi^2 - 5}}{8t^3 \left( y + \sqrt{y^2 - t^2} \right)^{4\chi^2}} I_D^{(0)}(t, y) \bigg|_{D=4},$$

where, as $t \rightarrow 0$, we have

$$I_D^{(0)} \approx \begin{cases} -\frac{9y^{10}}{t}, & y > 0, \\ -(1024\chi^6 - 320\chi^4 + 96\chi^2 + 9)y^{10}, & y < 0. \end{cases}$$

On the other hand, depending on the value of $\chi$, the spacetime may or may not be singular on the two null surfaces $y = \pm t$, similar to the last case. In particular, when $\chi^2 < 1/2$, the spacetime is singular there, and the corresponding Penrose diagram is also given by Fig. 4.

When $\chi^2 < 1$, the spacetime is not singular at $y = \pm t$, although the metric coefficients are. Then, extending the solutions to the region $v > 0$ is needed. As in the last case, the extension across this surface is unique only when $n$ defined by Eq. (3.53) is an integer. Thus, in the following we shall consider only this case. Then, the extension can be done by introducing the new coordinates $\tilde{u}$ and $\tilde{v}$ defined by Eq. (3.52), for which we find the extended metric takes the same form as that given by Eq. (3.54), but now with

$$\tilde{H} = \frac{1}{D-2} \ln \left( \tilde{u}^{2n} - \tilde{v}^{2n} \right),$$

$$\tilde{\Sigma} = \left( 2\chi^2 - \frac{D-3}{2(D-2)} \right) \ln \left( \tilde{u}^n + (-\tilde{v})^n \right) - \frac{D-3}{2(D-2)} \ln \left( \tilde{u}^n - (-\tilde{v})^n \right) + \tilde{\Sigma}_0,$$

$$\tilde{\Sigma}_0 \equiv \kappa_0^2 M_0 + \frac{1}{2} \ln \left( 4\chi^2 n^2 \right),$$

$$\phi = 2e \ln \left( \tilde{u}^n + (-\tilde{v})^n \right) + \phi_0.$$  

Thus, the coordinate singularity at $\tilde{v} = 0$ disappears, and the solutions can be considered as valid for $v > 0$, too. In terms of $\tilde{u}$ and $\tilde{v}$, we find

$$\tilde{\theta}_{\tilde{t}} = (D-2) e^{-2\Sigma} \tilde{H}_{\tilde{t}} = 2ne^{-2\Sigma} \left( \tilde{u}^{2n-1} \right. \tilde{u}^n - \left. \tilde{v}^{2n-1} \right),$$

$$\tilde{\theta}_{\tilde{n}} = (D-2) e^{-2\Sigma} \tilde{H}_{\tilde{n}} = 2ne^{-2\Sigma} \tilde{u}^n - \tilde{v}^{2n-1},$$

$$L_n \tilde{\theta}_{\tilde{t}} = -4n^2 e^{-2\Sigma} \tilde{u}^{n-1} \left( \tilde{u}^{2n-1} \right. \tilde{u}^n - \left. \tilde{v}^{2n-1} \right),$$

$$R_D[\tilde{g}] = -4\chi^2 + \frac{5}{2} \chi^2 e^{-2\Sigma} M_0,$$

$$\times \left( \tilde{u}^n - (-\tilde{v})^n \right) \left( \tilde{u}^n + (-\tilde{v})^n \right) \chi^2 + \frac{5}{2} \chi^2.$$  

From the above expressions, we can see that in the extended region where $\tilde{u} \tilde{v} > 0$ the spacetime becomes untrapped. Across the hypersurface $\tilde{v} = 0, \tilde{u} \geq 0$, we have $\tilde{\theta}_{\tilde{t}} = 0, \tilde{\theta}_{\tilde{n}} > 0$ and $L_n \tilde{\theta}_{\tilde{t}} = 0$. That is, the half infinite line $\tilde{v} = 0, \tilde{u} \geq 0$ acts as a past degenerate apparent horizon.

In addition, in contrast to the last case, the expression of $R_D[\tilde{g}]$ given above tells us that the spacetime is singular along the vertical line $\tilde{u} = \tilde{v}$ for $n$ being an odd integer, and not singular for $n$ being an even integer. The corresponding Penrose diagram is given, respectively, by Figs. 7 and 8.

When $\chi^2 \geq 1$, the hypersurfaces $y = \pm t$ already represent the null infinities of the spacetime, and the corresponding Penrose diagram is that of Fig. 10, where the hypersurfaces $\tilde{u} = 0$ and $\tilde{v} = 0$ are not singular, and the two regions, $II$ and $II'$, are still connected.
IV. SOLUTIONS IN \((D + d)\)-DIMENSIONAL SPACETIMES IN THE STRING FRAME

In \((D + d)\)-dimensions, the metric in the string frame takes the form of Eq. (2.22). Considering Eq. (3.3), we find that it can be written in the form,

\[
d^2 \hat{S}_{D+d} = \hat{g}_{AB}dx^Adx^B = \gamma_{\mu\nu}dx^\mu dx^\nu + \hat{\Phi}\gamma_{ab}dz^a dz^b
\]

\[
= 2e^{2\hat{g}(\hat{u},\hat{v})}d\hat{u}d\hat{v} - e^{2\hat{h}(\hat{u},\hat{v})}d\Sigma^2_{D-2} - e^{2\hat{g}(\hat{u},\hat{v})}\hat{\gamma}_{ab}dz^a dz^b,
\]

where

\[
\gamma_{\mu\nu} = \exp \left\{ \epsilon_\alpha \left( \frac{4\kappa^2_d d}{(D-2)(D-d)} \right)^{1/2} \right\} g_{\mu\nu},
\]

\[
\hat{\Phi} = e^\hat{g} = \exp \left\{ -\epsilon \left( \frac{(D-2)\kappa^2_d d}{(D-d)^2d} \right)^{1/2} \right\},
\]

as can be seen from Eqs. (2.7) and (2.9), where \(\epsilon_\alpha = \pm 1\).

Introducing the two null vectors, \(\hat{i}_A\) and \(\hat{n}_A\) via

\[
\hat{i}_A = \delta^A_\hat{i}, \quad \hat{n}_A = \delta^A_\hat{n},
\]

once can show that they also define two null affinely defined geodesic congruences, \(\hat{i}_A;B\hat{B} = 0 = \hat{n}_A;B\hat{B}\). The corresponding expansions are given by

\[
\hat{\theta}_l = \hat{i}_A;B\hat{g}^{AB} = 2\epsilon \left[ (D-2)\hat{h}_{\hat{u}\hat{v}} + d\hat{g}_{\hat{u}\hat{v}} \right],
\]

\[
\hat{\theta}_n = \hat{n}_A;B\hat{g}^{AB} = 2\epsilon \left[ (D-2)\hat{h}_{\hat{u}\hat{v}} + d\hat{g}_{\hat{u}\hat{v}} \right],
\]

from which we find

\[
\mathcal{L}_{\hat{n}}\hat{\theta}_l = e^{-\hat{g}} \left\{ (D-2)\hat{h}_{\hat{u}\hat{v}} + d\hat{g}_{\hat{u}\hat{v}} \right\}
\]

\[
- 2\hat{g}_{\hat{v}} \left\{ (D-2)\hat{h}_{\hat{u}\hat{v}} + d\hat{g}_{\hat{u}\hat{v}} \right\},
\]

\[
\mathcal{L}_{\hat{l}}\hat{\theta}_n = e^{-\hat{g}} \left\{ (D-2)\hat{h}_{\hat{u}\hat{v}} + d\hat{g}_{\hat{u}\hat{v}} \right\}
\]

\[
- 2\hat{g}_{\hat{v}} \left\{ (D-2)\hat{h}_{\hat{u}\hat{v}} + d\hat{g}_{\hat{u}\hat{v}} \right\}. \tag{4.5}
\]

A. \(F'(u) \neq 0, G'(u) = 0\)

In this case, the modulus \(\phi(u, v)\) is given by Eq. (3.29).

In the \((D + d)\)-dimensional spacetime, the metric can be written in the form

\[
d^2 \hat{S}_{D+d} = 2d\hat{u}d\hat{v} - e^{2\hat{h}(\hat{u},\hat{v})}d\Sigma^2_{D-2} - \hat{\Phi}^2(\hat{u})\hat{\gamma}_{ab}(z)dz^a dz^b,
\]

where

\[
\hat{d}\hat{u} = e^{2\hat{g}(\hat{u},\hat{v})} du,
\]

\[
\hat{h} = \frac{1}{2}au^{1-\gamma/2} + \alpha(u),
\]

\[
\hat{\Phi} = e^{b^{1-\gamma/2}}, \tag{4.12}
\]

where

\[
\alpha = \frac{2\epsilon_a}{2 - \gamma} \left( \frac{4\omega^2 d}{D + d - 2} \right)^{1/2},
\]

\[
b = -\frac{2\epsilon_a}{2 - \gamma} \left( \frac{\omega^2 (D - 2)^2}{(D + d - 2)d} \right)^{1/2}. \tag{4.13}
\]
Then, the non-vanishing frame components of the Riemann tensor are given by

\[
\begin{align*}
\hat{R}^{(i)}_{(0)(j)(0)} &= \hat{s}_0^2 e^{-2a u^{1-\gamma}/2} \\
&\times \left\{ a\gamma(2-\gamma) + a^2(2-\gamma)^2 + 16\omega^2 \right\} \delta_j^i,
\end{align*}
\]

\[
\begin{align*}
\hat{R}^{(a)}_{(0)(b)(0)} &= \hat{s}_0^2 \frac{b(2-\gamma)}{4} e^{-2a u^{1-\gamma}/2} \\
&\times \left\{ \frac{\gamma}{u^{1/2}} - \frac{(b-a)(2-\gamma)}{u^2} \right\} \delta_b^a. \tag{4.14}
\end{align*}
\]

Therefore, for the choice \( \epsilon_a = +1 \) we have

\[
\begin{align*}
\hat{R}_{(0)(B)(0)} &= \left\{ \begin{array}{ll} 
\infty, & \gamma > 0, \\
\text{constant}, & \gamma = 0, \\
\infty, & -2 < \gamma < 0, \tag{4.15} \\
\text{finite}, & \gamma \leq -2,
\end{array} \right.
\end{align*}
\]

\[
\hat{n}_n = \left\{ \begin{array}{ll} 
\infty, & \gamma > 2, \\
(D-2)\alpha'(0), & \gamma \leq 2, \tag{4.16}
\end{array} \right.
\]
as \( u \to 0 \), but now with \( A, B = i, a \). Thus, the tidal forces experiencing by a free-falling observer remain finite in the string frame at \( u = 0 \) for all the cases, except for the ones where \( 0 < \gamma \) or \(-2 < \gamma < 0\). As a result, the spacetime is singular at \( u = 0 \) for these latter solutions. However, the singularity is weak for \(|\gamma| < 2\), because the distortion exerting on the observer is still finite,

\[
\int \! d\lambda \int \! \hat{R}_{(0)(B)(0)} \! d\lambda \sim A_1 \lambda^{2-\gamma} + A_2 \lambda^{1-\gamma}/2
\]

\[
\sim \text{finite}, \tag{4.17}
\]
as \( \lambda \to 0 \) (or \( u \to 0 \)) when \(|\gamma| < 2\), where \( A_1 \) and \( A_2 \) are finite constants.

Therefore, for the choice \( \epsilon_a = +1 \) the strong singularities of the solutions with \( \gamma > 2 \) at \( u = 0 \) in the Einstein frame now remain in the string frame. The singularities of the solutions with \( 0 < \gamma < 2 \) are weak in both of the two frames. The solutions with \(-2 < \gamma < 0\) is free from singularity in the Einstein frame, while they become singular in the string frame, although the nature of the singularities is still weak. The solutions with \( \gamma = 0 \) and \( \gamma \leq -2 \) are free from singularity at \( u = 0 \) in both of the two frames.

Note that for \( \gamma = 0 \) Eq.\,(4.14) shows that

\[
\hat{R}_{(0)(B)(0)} \to 0, \tag{4.18}
\]
as \( u \to \infty \). Thus, in this case the spacetime singularity at \( u = \infty \) appearing in the Einstein frame now disappears in the \((D+d)\)-dimensional string frame, although the null infinity \( u = \infty \) still remains singular [cf. Fig. \( \ref{fig:fig} \)].

When \( \epsilon_a = -1 \), from Eq.\,(4.10) we find that

\[
\hat{R}_{(0)(B)(0)} = \left\{ \begin{array}{ll} 
0, & \gamma > 2, \\
\infty, & 2 > \gamma > 0, \\
\text{constant}, & \gamma = 0, \\
\infty, & -2 < \gamma < 0, \\
\text{finite}, & \gamma \leq -2, \tag{4.19}
\end{array} \right.
\]
as \( u \to 0 \). It can be shown that in this case the nature of the singularities of the solutions remains the same in both of the two frames for \( 2 > \gamma \geq 0 \) and \( \gamma \leq -2 \), that is, in both frames it is weak for \( 0 < \gamma < 2 \), and free of singularities for \( \gamma = 0 \) and \( \gamma \leq -2 \). For \(-2 < \gamma < 0\), the solutions are free of singularities in the Einstein frame, but singular in the string frame with the nature of the singularities being weak. For \( 2 < \gamma \), on the other hand, the solutions are free of singularities in the string frame, but singular in the Einstein frame with the nature of the singularities being strong.

Similarly, one can show that for \( \gamma = 0 \) the spacetime singularity at \( u = -\infty \) appearing in the Einstein frame now disappears in the \((D+d)\)-dimensional string frame, although the spacetime is still singular at the null infinity \( u = +\infty \) [cf. Fig. \( \ref{fig:fig} \)].

2. \( \gamma = 2 \)

When \( \gamma = 2 \), the corresponding solutions in the Einstein frame are given by Eqs.\,(3.29) and \,(3.36). The solutions have a strong singularity at \( u = 0 \). In the string frame, the corresponding solutions are given by Eq.\,(4.6) but with

\[
\hat{h}(\hat{u}) = \frac{a + 2\delta}{2(1 + a)} \ln |\hat{u}|, \tag{4.20}
\]

\[
\hat{\Phi}(\hat{u}) = \left((1 + a)\hat{u}\right)^{-2\delta},
\]

where

\[
\hat{u} = \frac{1}{1 + a} u^{1+\alpha} = \left\{ \begin{array}{ll} 
0, & a > -1, \\
-\infty, & a < -1, \tag{4.21}
\end{array} \right.
\]
as \( u \to 0 \). Thus, when \( a > -1 \) the half plane \( u \geq 0 \) is mapped to the half plane \( \hat{u} \geq 0 \), and the hypersurface \( u = 0 \) (\( u = \infty \)) is mapped to the one \( \hat{u} = 0 \) (\( \hat{u} = \infty \)). When \( a < -1 \) the half plane \( u \geq 0 \) is mapped to the one \( \hat{u} \leq 0 \), and the hypersurface \( u = 0 \) (\( u = \infty \)) corresponds to the one \( \hat{u} = -\infty \) (\( \hat{u} = 0 \)).

It can be shown that now we have

\[
\hat{R}^{(i)}_{(0)(j)(0)} = -\hat{s}_0^2 \frac{(a + 2\delta)(2\delta - a - 2)}{4(1 + a)^2 \delta^2} \delta_j^i,
\]

\[
\hat{R}^{(a)}_{(0)(b)(0)} = -\hat{s}_0^2 \frac{b(a - 1)}{(1 + a)^2 \delta^2} \delta_b^a. \tag{4.22}
\]

Clearly, the spacetime is singular at \( \hat{u} = 0 \), and the nature of the singularity is strong, because

\[
\int \! d\lambda \int \! \hat{R}_{(0)(B)(0)} \! d\lambda \sim \ln \lambda \rightarrow -\infty, \tag{4.23}
\]
as \( \lambda \to 0 \) (or \( \hat{u} \to 0 \)). Note that the distortion also becomes unbound as \( |\hat{u}| \to \infty \) (\( |\lambda| \to \infty \)), although the tidal forces vanish there.

It should be noted that the above analysis is valid only for \( a \neq -1 \). When \( a = -1 \), we find that

\[
\omega^2 = \frac{D + d - 2}{4d}, \quad b = \frac{D - 2}{2d}. \tag{4.24}
\]
we arrive at the form:

\[ \hat{h}(\hat{u}) = \left( \delta - \frac{1}{2} \right) \hat{u}, \]

\[ \hat{\Phi}(\hat{u}) = e^{\frac{D-2}{2} \hat{u}}, \quad (4.25) \]

where \( u = e^\hat{u}. \) Then, we find that

\[ \hat{R}^{(i)}_{\ (0)(j)(0)} = -\frac{2}{d} \left( \delta - \frac{1}{2} \right)^2 \delta_i^j, \]

\[ \hat{R}^{(a)}_{\ (0)(b)(0)} = -\frac{2}{d} \left( \frac{D-2}{2d} \right)^2 \delta^a_b, \quad (4.26) \]

which are finite (constants). However, at the null infinities \( \hat{u} = \pm \infty, \) which correspond, respectively, to \( u = 0 \) and \( u = \infty, \) the distortions are still unbound. As a result, the \((D + d)\)-dimensional spacetimes remain singular on these surfaces.

Therefore, when \( \gamma = 2 \) the corresponding Penrose diagram of the \((D + d)\)-dimensional spacetimes is that of Fig. 1 where the two hypersurfaces \( u = 0 \) and \( u = \infty \) remain singular.

B. \( F'(u)G'(v) \neq 0 \)

In this case, three classes of solutions were studied in the last section. To have manageable, in this subsection we shall restrict ourselves only to \( D = d = 5. \) We generalize the metric (4.10) and using \( t = u + v, \) \( y = u - v \) we arrive at the form:

\[ ds_{10}^2 = \frac{1}{2} e^{2A(t, u)} (dt^2 - dy^2) - e^{2B(t, y)} d\Sigma_3^2 \]

\[ -e^{2C(t, y)} d\Sigma_{5, z}^2, \quad (4.27) \]

where \( d\Sigma_{5, z}^2 \equiv \gamma_{ab} (z^c) dz^a dz^b (a, b = 1, 2, \ldots, 5), \)

\[ A = \sigma - \frac{5}{3} \beta \phi = \frac{1}{3} \ln(t) + \kappa_5^2 M - \frac{5}{3} \beta \phi, \]

\[ B = h - \frac{5}{3} \beta \phi = \frac{1}{3} \ln(t) - \frac{5}{3} \beta \phi, \]

\[ C = \beta \phi, \quad \beta = e^{\sqrt{\frac{3\kappa_5^2}{40}}} \epsilon_a = \pm 1. \quad (4.28) \]

1. Class IIa Solutions

In this case, substituting the solution (4.35) into Eq.(4.28) and setting \( M_0 = \phi_0 = 0 \) without loss of generality, we obtain that

\[ A = \frac{1}{2} \left[ \left( \kappa_5 - \epsilon \sqrt{\frac{5}{24}} \right)^2 - \frac{7}{8} \right] \ln(t), \]

\[ B = \left( \frac{1}{3} - \epsilon \sqrt{\frac{5}{24} \kappa_5} \right) \ln(t), \]

\[ C = \epsilon_\alpha \sqrt{\frac{3}{40} \kappa_5} \ln(t). \quad (4.29) \]

The corresponding Kretschmann scalar is given by,

\[ I_{10} \equiv R_{abcd} R^{abcd} = \frac{\tilde{I}_{10}}{t^{\alpha_0}} \quad (4.30) \]

where

\[ \alpha_0 = 2 \left( \kappa_5 - \epsilon \sqrt{\frac{5}{24}} \right)^2 + \frac{9}{4} > 0, \]

\[ \tilde{I}_{10} = \frac{1}{45} \left[ 9 \chi^2 (40 \chi + 143) - \epsilon \alpha_5 \sqrt{30 \chi^{3/2}} (3 \chi + 4) + 80 (5 \chi + 2) \right], \quad (4.31) \]

now with \( \chi \equiv c^2 \kappa_5^2. \) Clearly, the spacetime is always singular at \( t = 0 \) for any given \( c, \) similar to that in the 5-dimensional case. Therefore, in the present case, the spacetime singularity remains even after lifted from the effective 5-dimensional spacetime to the 10 dimensional bulk.

2. Class IIb Solutions

In this case, the combination of Eqs.(4.35) and (4.28) yields

\[ A = \left[ 2 \left( \kappa_5 - \epsilon \sqrt{\frac{5}{24}} \right)^2 - \frac{7}{16} \right] \ln(t), \]

\[ - \left[ \left( \kappa_5 - \epsilon \sqrt{\frac{5}{24}} \right)^2 - \frac{5}{16} \right] \ln \left( y + \sqrt{y^2 - t^2} \right) \]

\[ - \frac{1}{2} \chi^2 \ln \left( y^2 - t^2 \right), \]

\[ B = \left( \frac{1}{3} - \epsilon \sqrt{\frac{5}{24} \kappa_5} \right) \ln(t), \]

\[ - \epsilon_\alpha \sqrt{\frac{3}{40} \kappa_5} \ln \left( y + \sqrt{y^2 - t^2} \right), \]

\[ C = \epsilon_\alpha \sqrt{\frac{3}{40} \kappa_5} \ln(t), \]

\[ - \epsilon_\alpha \sqrt{\frac{3}{40} \kappa_5} \ln \left( y + \sqrt{y^2 - t^2} \right), \quad (4.32) \]

for which we find that

\[ I_{10} = \frac{- \tilde{I}_{10}}{t^{\alpha_0} (y^2 - t^2)^{\alpha_1} (y + \sqrt{y^2 - t^2})^{\alpha_2}} \quad (4.33) \]

where \( \alpha_0 \) is given by Eq.(4.31),

\[ \alpha_1 \equiv 2 \left( \frac{3}{4} - \chi \right), \]

\[ \alpha_2 \equiv \frac{101}{24} - \left( 2 \kappa_5 - \epsilon_\alpha \sqrt{\frac{5}{24}} \right)^2, \quad (4.34) \]
and \( \tilde{I}_{10} = \tilde{I}_{10}(t, y) \), which is non-zero for \( t = 0 \) and \( y^2 = t^2 \), but its expression is too complicated to give it here explicitly.

From the above expression, it can be seen that the spacetime is always singular at \( t = 0 \), but the strength of the singularity for \( y > 0 \) and \( y < 0 \) is different, because when \( t = 0 \) we have \( y + \sqrt{y^2 - t^2} = 0 \) for \( y \leq 0 \) and \( y + \sqrt{y^2 - t^2} \neq 0 \) for \( y > 0 \). In particular, when \( t = 0 \) and \( y > 0 \), we find that

\[
I_{10} \bigg|_{t=0, y>0} \approx \frac{\tilde{I}_{10}}{t^{3/2}},
\]

where \( \tilde{I}_{10} \) is still given by Eq. (4.36).

Eqs. (4.33) and (4.34) also show that the spacetime is singular when \( y^2 - t^2 = 0 \) for \( \chi < 3/4 \),

\[
I_{10(b)} \bigg|_{t^2 = y^2} \propto \frac{\tilde{I}_{10}}{(y^2 - t^2)^{\alpha_1}},
\]

but now with

\[
\tilde{I}_{10} \bigg|_{t^2 = y^2} = \pm \frac{8}{15} \chi \left[ 20(6\chi^2 - \chi - 1) - 13\epsilon \sqrt{30\chi} (2\chi^2 - 1) \right].
\]

The corresponding Penrose diagram for \( \chi < 3/4 \) is given by Fig. 7. It is remarkable to note that the solutions with \( 1/2 \leq \chi < 3/4 \) is not singular in the 5-dimensional effective theory, as shown explicitly in Sec. III.B.2.

Since the solution in the 10-dimensional bulk is not singular across the hypersurfaces \( y^2 = t^2 \) for \( \chi \geq 3/4 \), one must extend the solutions beyond these surfaces. The extension is quite similar to the 5-dimensional case for the ones with \( \chi \geq 1/2 \). In particular, for \( 3/4 \leq \chi < 1 \), setting

\[
\bar{u} = (y + t)^{2n}, \quad \bar{v} = (y - t)^{2n},
\]

where \( n \) is given by Eq. (4.38), one can show that the coordinate singularity at \( y^2 = t^2 \) disappears in terms of \( \bar{u} \) and \( \bar{v} \). Then, the Penrose diagram for the extended solutions is given exactly by Fig. 6.

When \( \chi \geq 1 \), the hypersurfaces \( y^2 = t^2 \) already represent the null infinities, and the corresponding Penrose diagram is given by Fig. 6, but now the hypersurfaces \( y^2 = t^2 \), represent, respectively, by the lines \( 0D \) and \( 0E \), are non-singular. The two regions \( I \) and \( I' \) are physically disconnected.

3. Class IIc Solutions

In this case, from Eq. (5.31), we find that

\[
A = -\frac{1}{3} \ln(t) - \frac{1}{2} \frac{\chi}{t^2} \ln(y^2 - t^2)
\]

\[
+ \left[ \left( c\kappa_5 - \epsilon \sqrt{\frac{5}{96}} \right)^2 - \frac{5}{96} \right] \ln \left( y + \sqrt{y^2 - t^2} \right),
\]

\[
B = \frac{1}{3} \ln(t) - \epsilon \sqrt{\frac{5}{24}} c\kappa_5 \ln \left( y + \sqrt{y^2 - t^2} \right),
\]

\[
C = \epsilon \sqrt{\frac{5}{40}} c\kappa_5 \ln \left( y + \sqrt{y^2 - t^2} \right),
\]

and that

\[
I_{10} = \frac{\tilde{I}_{10}}{t^{8/3}(y^2 - t^2)^{\alpha_3}(y + \sqrt{y^2 - t^2})^{\alpha_3}},
\]

where

\[
\alpha_3 = \frac{91}{24} + \left( 2c\kappa_5 - \epsilon \sqrt{\frac{5}{24}} \right)^2 > 0,
\]

and \( \tilde{I}_{10} = \tilde{I}_{10}(t, y) \) is non-zero and finite at \( t = 0 \), but too complicated to be written out here.

It is very surprising to note that the spacetime is now always singular on the hypersurface \( t = 0 \), in contrast to the 5-dimensional case in which the spacetime is free of spacetime singularities, as shown in Sec. III.B.3. The strength of the singularities once again depends on \( y < 0 \) and \( y > 0 \). In particular, when \( t = 0 \) and \( y > 0 \), we find that

\[
I_{10} \bigg|_{t=0, y<0} \approx \frac{\tilde{I}_{10}}{t^{8/3}},
\]

with \( \tilde{I}_{10} \approx 512y^7/9 \). But for \( t = 0 \) and \( y < 0 \), we find that

\[
I_{10} \bigg|_{t<0, y<0} = \frac{\tilde{I}_{10}(y - \sqrt{y^2 - t^2})^{\alpha_3}}{t^{8/3} + 2^{3/2}(y^2 - t^2)^{\alpha_3}},
\]

where \( \alpha_3 > 0 \), as shown by Eq. (4.43). Therefore, we again have a spacetime singularity at \( t = 0 \) for \( y < 0 \), but with more singular strength.

The singular behavior of the spacetime along the hypersurfaces \( y = \pm t \) are similar to the last case. In particular, it is singular for \( \chi < 3/4 \), and corresponding Penrose diagram is given by Fig. 7. It is interesting to note again that the solutions with \( 1/2 \leq \chi < 3/4 \) is not singular in the 5-dimensional effective theory.

When \( 1 > \chi \geq 3/4 \) the corresponding solutions in the 10-dimensional bulk is not singular across the hypersurfaces \( y^2 = t^2 \), one must extend the solutions beyond these surfaces. The extension is quite similar to the last case by simply introducing two null coordinates, defined by
Eq. (4.40). Then, the Penrose diagram for the extended solutions is given exactly by Fig. 6.

When $\chi \geq 1$, the hypersurfaces $y^2 = t^2$ already represent the null infinities, and the corresponding Penrose diagram is given by Fig. 7.

V. CONCLUSIONS AND DISCUSSING

Remarks

According to the singularity theorems [5], the 4-dimensional spacetimes are generically singular, if matter filled the spacetimes satisfies certain energy condition(s). It is a common belief that new physics involved with quantum mechanics will play an important role when the spacetime curvature is very high. The new physics gives us hope that the classical singularities might be finally removed.

However, it was shown that certain singularities can be also removed by simply passing to a higher-dimensional theory of gravity, for which spacetime is only effectively four-dimensional below some compactification scale [3, 15]. This is because when lifted to higher-dimensions, the conditions required by the singularity theorems of Penrose and Hawking do not hold any more.

In this paper, we have investigated this problem, by first studying the local and global properties of the spacetimes in the low dimensional effective theory, and then lifting them to the corresponding high dimensional spacetimes. We have shown explicitly that spacetime singularities may or may not remain after lifted to higher dimensions, depending on the particular solutions considered. We have also found that there exist cases in which the spacetimes of low dimensional effective theory do not possess any singularities, but when lifted to high dimensions of the string theory, new spacetime singularities rise. This seems a little bit odd, and a closer analysis shows that it is due to the gravitational collapse of the extra dimensions [described by $\Phi^2 \delta_{ab} dz^n dz^b$, cf. Eq. (2.2)]. To prevent such singularities to occur, one needs to stabilize the extra dimensions, for example, using the KKLT mechanism [10].

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