A note on pseudo-Anosov maps with small growth rate

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Abstract

We present an explicit sequence of pseudo-Anosov maps $\phi_k : S_{2k} \to S_{2k}$ of surfaces of genus $2k$ whose growth rates converge to one.

Introduction

In this note, we present an explicit sequence $\phi_k$ of pseudo-Anosov of surfaces of genus $2k$ whose growth rates converge to one. This answers a question of Joan Birman, who had previously asked whether such growth rates are bounded away from one. Norbert A’Campo, Mladen Bestvina, and Klaus Johannson independently communicated this question to me. McMullen previously obtained a similar result using quite different techniques [McM00].

The growth of the genus is not an artifact of our construction. For a surface $S$ of fixed genus $g$, the growth rates of pseudo-Anosov maps of $S$ are clearly bounded away from one, for they are Perron-Frobenius eigenvalues of irreducible integral $m \times m$ matrices, with $m \leq 6g - 3$ [BH95]. Finding the smallest possible growth rate for each genus is an interesting problem that remains open.

One curious observation, due to Norbert A’Campo, is that our sequence of pseudo-Anosov maps is a sequence of monodromies of w-slalom knots, as defined in [AC98].

In Section 1, we review the part of the theory of train tracks [BH92] [BH95] that we use in this paper. Section 2 explains the intuition that led to the result, and Section 3 contains the statement and proof of the main results (Theorem 3.2 and Corollary 3.3).

The result of this paper grew out of massive computer experiments with my software package XTrain [Bri00] [BS01] in the context of the REU program.
at the University of Illinois at Urbana-Champaign. I would like to thank Vamshidhar Kommineni for collecting much of the experimental data that started this project. I am indebted to the Department of Mathematics at UIUC for funding the computer experiments. Finally, this paper would not have existed if Saul Schleimer had not encouraged me to write it up.

1 Train tracks

We present a brief review of train tracks as defined in [BH92]. Let $G$ be a finite graph without vertices of valence one or two, and let $f : G \to G$ be a homotopy equivalence of $G$ that maps vertices to vertices. The map $f$ is said to be a train track map if for every integer $n \geq 1$ and every edge $e$ of $G$, the restriction of $f$ to the interior of $e$ is an immersion.

If $E_1, \cdots, E_m$ is the collection of edges of $G$, the transition matrix of $f$ is the nonnegative $m \times m$-matrix $M$ whose $ij$-th entry is the number of times the $f$-image of $E_j$ crosses $E_i$, regardless of orientation. $M$ is said to be irreducible if for every tuple $1 \leq i, j \leq m$, there exists some exponent $n > 0$ such that the $ij$-th entry of $M^n$ is nonzero. If $M$ is irreducible, then it has a maximal real eigenvalue $\lambda \geq 1$ (see [Sen73]). We call $\lambda$ the growth rate of $f$.

The following theorem from [BH92] will be our main tool. Recall that an outer automorphism $\omega$ of a free group $F$ is called reducible if there are proper free factors $F_1, \ldots, F_r$ of $F$ such that $\omega$ permutes the conjugacy classes of the $F_i$s and $F_1 \ast \cdots \ast F_r$ is a free factor of $F$; $\omega$ is irreducible if it is not reducible. Also, note that $\pi_1 G$ is a finitely generated free group, and that a homotopy equivalence $f : G \to G$ induces an outer automorphism of $\pi_1 G$.

Theorem 1.1 ([BH92, Theorem 4.1]). Let $\omega$ be an outer automorphism of a finitely generated free group $F$. Suppose that each positive power of $\omega$ is irreducible and that there is a nontrivial word $s \in F$ such that $\omega$ preserves the conjugacy class of $s$ (up to inversion). Then $\omega$ is geometrically realized by a pseudo-Anosov homeomorphism $\phi : S \to S$ of a surface with one puncture.

Remark 1.2. If $f : G \to G$ is a train track map that induces an outer automorphism $\omega$ as in Theorem 1.1, then the transition matrix of $f$ is irreducible, and the growth rate of $f$ is the same as the pseudo-Anosov growth rate of $\phi$.

Moreover, if $f : G \to G$ is a train track map such that all positive powers of its transition matrix $M$ are irreducible, then all positive powers of the
induced outer automorphism $\omega$ are irreducible [BH92].

Remark 1.3. The proof of Corollary 3.3 uses an explicit construction of invariant foliations for pseudo-Anosov maps. This construction is straightforward but too long to be reviewed in this note, so that we just point the reader to [BH95] for details.

2 Motivation

Warning 2.1. The discussion in this section is not supposed to present any rigorous mathematical reasoning. Rather, the purpose of this section is to explain the origin of the technical definitions and computations of Section 3.

One crucial tool in the development of the intuition behind Theorem 3.2 was XTrain [Bri00] [BS01], a software package that implements algorithms from [BH92] [BH95], among others. In particular, the software allows users to define homeomorphisms of surfaces with one puncture as a composition of Dehn twists with respect to the curves shown in Figure 1. When computing Dehn twists, we adopt the following convention: We equip the surface with an outward pointing normal vector field. When twisting with respect to a curve $c$, we turn right whenever we hit $c$. We denote by $D_c$ the twist with respect to $c$.

The software represents a surface homeomorphism $\phi$ of a punctured surface $S$ as a homotopy equivalence $f$ of a graph $G$ that is embedded in as well as homotopy equivalent to $S$. There exists a loop $\sigma$ in $G$ that corresponds to a short loop around the puncture of $S$. In particular, $f$ preserves the free homotopy class of $\sigma$ (up to orientation).

The first ingredient is the observation that a homeomorphism of a surface
of genus $g$ given by
\[ \phi_g = D_{c_0} \cdots D_{c_{g-1}} D_{d_0} \cdots D_{d_{g-1}} \] (1)
can be represented by a train track map of a graph $H_g$ as in Figure 2 such that $x_0 \mapsto x_1, x_1 \mapsto x_2, \ldots, x_{2g} \mapsto x_0^{-1}$ with $\sigma_g = x_0 x_1 \cdots x_{2g} x_0^{-1} x_1^{-1} \cdots x_{2g}^{-1}$.

Note, in particular, that this map cyclically permutes the edges of $H_g$ (up to orientation).

The second ingredient comes from certain PV-automorphisms $\psi_n$ [Sta82] of a free group $F = \langle y_0, \ldots, y_n \rangle$ given by $y_0 \mapsto y_1, y_1 \mapsto y_2, \ldots, y_n \mapsto y_0 y_1$.

Mathematically, these automorphisms are very different from the maps constructed in the previous paragraph (after all, PV automorphisms are non-geometric and of exponential growth, whereas the maps of the previous paragraph are geometric and periodic).

Superficially, though, these two classes of maps look strikingly similar. Moreover, the growth rates of the maps $\psi_n$ converge to one. These two observations prompted me to investigate maps that are built from blocks as in Equation 1.

Maps of surfaces of genus $2k$ of the form
\[ \phi_k = D_{c_0} \cdots D_{c_{k-1}} D_{d_0} \cdots D_{d_{k-1}} (D_{c_k} \cdots D_{c_{2k-1}} D_{d_k} \cdots D_{d_{2k-1}})^{-1} \]
turned out to be pseudo-Anosov with rather small growth rate. Computer experiments suggested that the growth rates of these maps converges to one, and the same experiments suggested that train tracks representing these maps conform to a describable pattern, which gave rise to Definition 3.1 and Theorem 3.2. Notice how Definition 3.1 seems reminiscent of both PV automorphisms as well as homeomorphisms as in Equation 1.
3 The sequence

Motivated by the discussion of Section 2, we now define a sequence of surface homeomorphisms.

Definition 3.1. Let \( k \geq 1 \) be an integer, and choose the graph \( G_k \) be as in Figure 3. We define a map \( f_k: G_k \to G_k \) by letting

\[
\begin{align*}
    a &\mapsto ax_0 y_0 \\
    b &\mapsto by_0^{-1}x_0^{-1} \\
    c &\mapsto d \\
    d &\mapsto dy_1 x_0 \\
    x_0 &\mapsto x_1 \\
    x_1 &\mapsto x_2 \\
    &\vdots \\
    x_{2k-1} &\mapsto a^{-1} by_0^{-1} \\
    y_0 &\mapsto y_1 \\
    y_1 &\mapsto y_2 \\
    &\vdots \\
    y_{2k-1} &\mapsto c^{-1} b.
\end{align*}
\]

Finally, let

\[
\sigma_k = x_0 y_0 x_1 y_1 \cdots x_{2k-1} y_{2k-1} a^{-1} b y_0^{-1} c^{-1} d \\
\qquad b^{-1} c x_{2k-2} y_{2k-1}^{-1} x_{2k-3} y_{2k-2}^{-1} \cdots x_0^{-1} y_1^{-1} d^{-1} a.
\]

We are now ready to state and prove the main result of this note.

Theorem 3.2. The sequence of maps \( f_k: G_k \to G_k \) is a sequence of homotopy equivalences induced by pseudo-Anosov maps \( \phi_k: S_{2k} \to S_{2k} \) of surfaces of genus \( 2k \) with one puncture. If \( \lambda_k \) is the pseudo-Anosov growth rate of \( \phi_k \), then

\[
\lim_{k \to \infty} \lambda_k = 1.
\]

Proof. A number of tedious but straightforward checks yields the following facts:
1. The maps $f_k$ are train track maps.

2. All positive powers of the transition matrix $M_k$ of $f_k$ are irreducible.

3. The map $f_k$ preserves the free homotopy class of the loop $\sigma_k$.

Hence, by Theorem 1.1 and Remark 1.2, the outer automorphism induced by $f_k$ is induced by a pseudo-Anosov map $\phi_k : S_k \to S_k$, and a quick computation of Euler characteristics shows that the genus of $S_k$ is $2k$. Finally, a simple induction shows that the characteristic polynomial of the transition matrix $M_k$ is of the form

$$\chi(\lambda) = (\lambda - 1)^2(\lambda^{4k+2} - \lambda^{4k+1} - 4\lambda^{2k+1} - \lambda + 1).$$

Solving for the growth rate $\lambda_k$, we obtain

$$\lambda_k = 1 + \lambda_k^{4k+2} - \lambda_k^{4k+1} - 4\lambda_k^{2k+1}. \quad (2)$$

Note that the polynomial $\chi$ is palindromic (this is no surprise as $f_k$ is induced by a surface homeomorphism), i.e., $\chi(\lambda) = \lambda^{4k+4}\chi(\frac{1}{\lambda})$. Hence, Equa-
tion 2 also holds for \( \lambda_k^{-1} \):
\[
\lambda_k^{-1} = 1 + \lambda_k^{-(4k+2)} - \lambda_k^{-(4k+1)} - 4\lambda_k^{-(2k+1)} \\
\geq 1 - \lambda_k^{-(4k+1)} - 4\lambda_k^{-(2k+1)}. \tag{3}
\]

Recall that \( \lambda_k^{-1} < 1 \). Let \( 0 < u < 1 \) be some real number. We have
\[
\lim_{k \to \infty} 1 - u^{4k+1} - 4u^{2k+1} = 1,
\]
which implies that \( u \) only satisfies Inequality 3 for finitely many values of \( k \). Hence, for any such \( u \), the set \( \{\lambda_k | \lambda_k^{-1} < u\} \) is finite. This immediately implies that \( \lim_{k \to \infty} \lambda_k^{-1} = 1 \), hence
\[
\lim_{k \to \infty} \lambda_k = 1.
\]

\[\square\]

**Corollary 3.3.** The maps \( \phi_k : S_{2k} \to S_{2k} \) from Theorem 3.2 can be extended to pseudo-Anosov maps of closed surfaces. The growth rates of the extended maps are the same as those of the original maps.

**Proof.** A lengthy but straightforward computation of invariant foliations (see Remark 1.3) yields that the four outer vertices of the graph in Figure 3 give rise to singularities of index \( \frac{1}{2} - k \), while the central vertex does not give rise to any singularity. Hence, the sum of the indices of all singularities coming from vertices of the graph is \( 2 - 4k \), which is the Euler characteristic of a closed surface of genus \( 2k \).

Hence, the foliations have no singularity at the puncture, which implies that the extension of \( \phi_k \) to the closed surface obtained by filling in the puncture is pseudo-Anosov, with the same growth rate as \( \phi_k \). \[\square\]

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