AUSLANDER’S DEFECT FORMULA AND A COMMUTATIVE TRIANGLE IN AN EXACT CATEGORY

PENGJIE JIAO

Abstract. We prove the Auslander’s defect formula in an exact category, and obtain a commutative triangle involving the Auslander bijections and the generalized Auslander-Reiten duality.

1. Introduction

Throughout $k$ denotes a commutative artinian ring. We consider a $k$-linear Hom-finite Krull-Schmidt exact category $C$, which is skeletally small.

Recall that Auslander’s defect formula appeared as [1, Theorem II.1.4.1] for the first time. Krause [7] gave a short proof for the formula on modules categories. Ringel [9] introduced the notion of Auslander bijection on module categories. Chen [3] established a commutative triangle involving Auslander bijections, universal extensions and the Auslander-Reiten duality.

More recently, the notion of generalized Auslander-Reiten duality was introduced in [5, Section 3]. Using this notion, we prove the Auslander’s defect formula in an exact category $C$ based on some results in [3]; see Theorem 3.3. Compared with the proof of [3, Theorem 3.7], it seems that the treatment here can be applied to the study of higher Auslander-Reiten theory.

We generalize the commutative triangle established by Chen [3, Theorem 4.6]; see Theorem 4.2. The Auslander bijection in the commutative triangle suggests the usage of universal extensions in the study of morphisms determined by objects in an exact category.

In Section 2, we recall the convariant defect and the contravariant defect. Sections 3 and 4 are dedicated to the proofs of Theorems 3.3 and 4.2 respectively.

2. Convariant defect and contravariant defect

Let $k$ be a commutative artinian ring and $\bar{k}$ be the minimal injective cogenerator. We denote by $\text{mod} \ k$ the category of finitely generated $k$-modules and by $D = \text{Hom}_k(-, \bar{k})$ the Matlis duality. We recall the convariant defect and the contravariant defect on an exact category.

Let $C$ be a $k$-linear Hom-finite Krull-Schmidt exact category, which is skeletally small. Recall that an exact category is an additive category $C$ together with a collection $E$ of exact pairs $(i, d)$, which satisfies the axioms in [6, Appendix A].

Here, an exact pair $(i, d)$ means a sequence of morphisms $X \xrightarrow{i} Y \xrightarrow{d} Z$ such that
Let $\xi: X \to E \to Y$ be a conflation. For each morphism $f: Z \to Y$ we let $\xi.f$ be the conflation obtained by pullback of $\xi$ along $f$; for each morphism $g: X \to Z$ we let $g.\xi$ be the conflation obtained by pushout of $\xi$ along $g$.

Recall from \[\text{Section IV.4}\] that a morphism $f: X \to Y$ is called *projectively trivial* if for each object $Z$, the induced map $\text{Ext}_d^1(f, Z): \text{Ext}_C^1(Y, Z) \to \text{Ext}_C^1(X, Z)$ is zero. We observe that $f$ is projectively trivial if and only if $f$ factors through any deflation ending at $Y$. Dually, we call $f$ *injectively trivial* if for each object $Z$, the induced map $\text{Ext}_{C}^{1}(Z, f): \text{Ext}_{C}^{1}(Z, X) \to \text{Ext}_{C}^{1}(Z, Y)$ is zero.

Given a pair of objects $X$ and $Y$, we denote by $\mathcal{P}(X, Y)$ the set of projectively trivial morphisms $X \to Y$. Then $\mathcal{P}$ forms an ideal of $\mathcal{C}$. We set $\mathcal{C} = \mathcal{C}/\mathcal{P}$. Given a morphism $f: X \to Y$, we denote by $f$ its image in $\mathcal{C}$. We denote by $\text{Hom}_C(X, Y) = \text{Hom}_C(X, Y)/\mathcal{P}(X, Y)$ the set of morphisms $X \to Y$ in $\mathcal{C}$.

Dually, we denote by $\mathcal{I}(X, Y)$ the set of injectively trivial morphisms $X \to Y$. Set $\mathcal{I} = \mathcal{C}/\mathcal{I}$. Given a morphism $f: X \to Y$, we denote by $f$ its image in $\mathcal{I}$. We denote by $\text{Hom}_C(X, Y) = \text{Hom}_C(X, Y)/\mathcal{I}(X, Y)$ the set of morphisms $X \to Y$ in $\mathcal{I}$.

Let $\xi: X \to E \to Y$ be a conflation and $K$ be an object in $\mathcal{C}$. We have the *connecting map*

$$c(\xi, K): \text{Hom}_C(X, K) \to \text{Ext}_C^1(Y, K), \quad f \mapsto f.\xi.$$ 

We mention that $c(\xi, K)$ is natural in both $\xi$ and $K$. Moreover, we have the exact sequence in functor category

$$\text{Hom}_C(E, -) \xrightarrow{\text{Hom}_C(i,-)} \text{Hom}_C(X, -) \xrightarrow{c(-,-)} \text{Ext}_C^1(Y, -).$$

Recall from \[\text{Section IV.4}\] that the *covariant defect* of the conflation $\xi$ is a covariant functor $\xi_*: \mathcal{C} \to \text{mod} k$ satisfying the following exact sequence

$$0 \to \text{Hom}_C(Y, -) \xrightarrow{\text{Hom}_C(d,-)} \text{Hom}_C(E, -) \xrightarrow{\text{Hom}_C(i,-)} \text{Hom}_C(X, -) \xrightarrow{u} \xi_* \to 0$$
in functor category. Then we have the following natural isomorphism

$$(2.1) \quad \xi_* \simeq \text{Im} c(\xi, -).$$

For any injectively trivial morphism $f: K \to K'$, we have the following commutative diagram

$$\begin{array}{ccc}
\text{Hom}_C(E, K) & \xrightarrow{\text{Hom}_C(i, K)} & \text{Hom}_C(X, K) \\
\downarrow{\text{Hom}_C(E, f)} & & \downarrow{\text{Hom}_C(X, f)} \\
\text{Hom}_C(E, K') & \xrightarrow{\text{Hom}_C(i, K')} & \text{Hom}_C(X, K')
\end{array}$$

We observe that for any morphism $g: X \to K$, the morphism $f \circ g$ is injectively trivial, and hence factors through $i$. Therefore $f \circ g$ lies in $\text{Im} \text{Hom}_C(i, K')$. We obtain $u_{K'}(f \circ g) = 0$. It follows that

$$(\xi_*(f) \circ u_{K'})(g) = (u_{K'} \circ \text{Hom}_C(X, f))(g) = u_{K'}(f \circ g) = 0.$$
We obtain $\xi_*(f) \circ u_K = 0$. Therefore $\xi_*(f) = 0$ since $u_K$ is surjective. We obtain the induced functor $\mathcal{C} \rightarrow \text{mod } k$, which we still denote by $\xi_*$.

We observe that $c(\xi, K)(f) = 0$ for any object $K$ and any injectively trivial morphism $f: X \rightarrow K$. We obtain the natural transformation

$$c(\xi, -): \text{Hom}\mathcal{C}(X, -) \rightarrow \text{Ext}^1_\mathcal{C}(Y, -).$$

Dually, we have the connecting map

$$c(K, \xi): \text{Hom}\mathcal{C}(K, Y) \rightarrow \text{Ext}^1_\mathcal{C}(K, X), \ f \mapsto \xi.f.$$ 

We mention that $c(K, \xi)$ is natural in both $\xi$ and $K$.

The contravariant defect of the conflation $\xi$ is a contravariant functor $\xi^*: \mathcal{C} \rightarrow \text{mod } k$ satisfying the following exact sequence

$$0 \rightarrow \text{Hom}\mathcal{C}(-, X) \rightarrow \text{Hom}\mathcal{C}(-, E) \rightarrow \text{Hom}\mathcal{C}(-, Y) \rightarrow \xi^* \rightarrow 0$$

in functor category. We have the following natural isomorphism

$$\xi^* \simeq \text{Im } c(-, \xi).$$

We mention that $\xi^*$ vanishes on projectively trivial morphisms and $c(K, \xi)(f) = 0$ for any object $K$ and any projectively trivial morphism $f: K \rightarrow Y$. We then have the induced functor $\xi^*: \mathcal{C} \rightarrow \text{mod } k$ and the natural transformation

$$c(-, \xi): \text{Hom}\mathcal{C}(-, Y) \rightarrow \text{Ext}^1_\mathcal{C}(-, X).$$

3. Auslander’s defect formula

Recall from [5, Section 2] two full subcategories

$$\mathcal{C}_r = \{ X \in \mathcal{C} | \text{the functor } D \text{Ext}^1_\mathcal{C}(X, -): \mathcal{C} \rightarrow \text{mod } k \text{ is representable} \}$$

and

$$\mathcal{C}_l = \{ X \in \mathcal{C} | \text{the functor } D \text{Ext}^1_\mathcal{C}(-, X): \mathcal{C} \rightarrow \text{mod } k \text{ is representable} \}.$$ 

We have a pair of mutually quasi-inverse equivalences

$$\tau: \mathcal{C}_l \sim \mathcal{C}_r$$

and

$$\tau^{-1}: \mathcal{C}_r \sim \mathcal{C}_l.$$ 

For each object $Y \in \mathcal{C}_r$, we have a natural isomorphism

$$\phi_Y: \text{Hom}\mathcal{C}(-, \tau Y) \sim D \text{Ext}^1_\mathcal{C}(Y, -).$$

We let

$$\alpha_Y = \phi_{Y, \tau Y}(\text{Id}_{\tau Y}) \text{ and } \mu_Y = \psi_{\tau Y, Y}^{-1}(\alpha_Y).$$

For each object $X \in \mathcal{C}_l$, we have a natural isomorphism

$$\psi_X: \text{Hom}\mathcal{C}(\tau^{-1}X, -) \sim D \text{Ext}^1_\mathcal{C}(-, X).$$

We let

$$\beta_X = \psi_{X, \tau^{-1}X}(\text{Id}_{\tau^{-1}X}) \text{ and } \nu_X = \phi_{\tau^{-1}X, X}^{-1}(\beta_X).$$

The following lemma is contained in the proof of [5, Proposition 3.4].

Lemma 3.1. For each object $Y \in \mathcal{C}_r$, we have

$$\alpha_Y = \beta_{\tau Y} \circ \text{Ext}^1_\mathcal{C}(\mu_Y, \tau Y);$$

for each object $X \in \mathcal{C}_l$, we have

$$\beta_X = \alpha_{\tau^{-1}X} \circ \text{Ext}^1_\mathcal{C}(\tau^{-1}X, \nu_X).$$
Proof. We only prove the first equality. We observe that \( \alpha_Y = \psi_{Y,Y}(\mu_Y) \). Consider the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_C(\tau^{-\tau}Y, \tau^{-\tau}Y) & \xrightarrow{\psi_{Y,Y}} & D\text{Ext}_C^1(\tau^{-\tau}Y, \tau Y) \\
\text{Hom}_C(\tau^{-\tau}Y, \mu_Y) & \downarrow & D\text{Ext}_C^1(\mu_Y, \tau Y) \\
\text{Hom}_C(\tau^{-\tau}Y, Y) & \xrightarrow{\psi_{Y,Y}} & D\text{Ext}_C^1(Y, \tau Y).
\end{array}
\]

By a diagram chasing, we obtain

\[
\alpha_Y = \psi_{Y,Y}(\mu_Y) = (\psi_{Y,Y} \circ \text{Hom}_C(\tau^{-\tau}Y, \mu_Y))(\text{Id}_{\tau^{-\tau}Y}) = (D\text{Ext}_C^1(\mu_Y, \tau Y) \circ \psi_{Y,Y}(\tau^{-\tau}Y))(\text{Id}_{\tau^{-\tau}Y}) = D\text{Ext}_C^1(\mu_Y, \tau Y)(\beta_{\tau Y}) = \beta_{\tau Y} \circ \text{Ext}_C^1(\mu_Y, \tau Y).
\]

Here, the third equality holds by the commutative diagram, and the fourth equality holds by the definition of \( \beta_{\tau Y} \). \( \square \)

We mention the following fact. Here, we give a proof.

**Lemma 3.2** ([3, Lemma 4.3]). Let \( \xi: X \to E \to Y \) be a conflation in \( C \).

1. For each object \( K \in C_r \), there exists a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_C(X, \tau K) & \xrightarrow{\psi_{\tau K, X}} & \text{Ext}_C^1(Y, \tau K) \\
\phi_{K,X} & \downarrow & D(\psi_{\tau K, X} \circ \text{Hom}_C(\mu_K, Y)) \\
D\text{Ext}_C^1(K, X) & \xrightarrow{D(\xi, K)} & D\text{Hom}_C(K, Y),
\end{array}
\]

which is natural in both \( \xi \) and \( K \).

2. For each object \( K \in C_l \), there exists a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_C(\tau^{-\tau}K, Y) & \xrightarrow{\psi_{\tau^{-\tau}K, Y}} & \text{Ext}_C^1(\tau^{-\tau}K, X) \\
\phi_{K,Y} & \downarrow & D(\phi_{\tau^{-\tau}K, X} \circ \text{Hom}_C(X, \mu_K)) \\
D\text{Ext}_C^1(Y, K) & \xrightarrow{D(\xi, K)} & D\text{Hom}_C(X, K),
\end{array}
\]

which is natural in both \( \xi \) and \( K \).

**Proof.** We only prove (1). We set

\[
\psi' = \psi_{\tau K, Y} \circ \text{Hom}_C(\mu_K, Y): \text{Hom}_C(K, Y) \to D\text{Ext}_C^1(Y, \tau K).
\]

We observe that each conflation \( \zeta \) in \( \text{Ext}_C^1(Y, \tau K) \) induces a \( k \)-linear map

\[
D\text{Ext}_C^1(Y, \tau K) \to \hat{k}, \quad f \mapsto f(\zeta).
\]
For any morphisms $g: X \to \tau K$ and $h: K \to Y$, we have
\[
(D\psi' \circ c(\xi, \tau K))(\overline{h})(\overline{h}) = (c(\xi, \tau K)(\overline{h}) \circ \psi')(\overline{h}) = (g(\xi)(\psi'(\overline{h}))) = (\psi'(\overline{h}))(g(\xi)) = \psi_{\tau K,Y}(h \circ \mu_K)(g(\xi)).
\]

Here, the second equality holds by the definition of $c(\xi, \tau K)$, and the third equality holds by the canonical isomorphism $\Ext^1_C(Y, \tau K) \cong DD\Ext^1_C(Y, \tau K)$. It follows by the naturality of $\psi_{\tau Y}$ that
\[
\psi_{\tau K,Y}(h \circ \mu_K)(g(\xi)) = \beta_{\tau Y}(g(\xi), h, \mu_K).
\]

On the other hand, we have
\[
(Dc(K, \xi) \circ \phi_{K,X})(\overline{h})(\overline{h}) = (\phi_{K,X}(\overline{h}) \circ c(K, \xi))(\overline{h}) = \phi_{K,X}(\overline{h})(\xi, h).
\]

It follows by the naturality of $\phi_K$ that
\[
\phi_{K,X}(\overline{h})(\xi, h) = \alpha_K(g(\xi), h).
\]

By Lemma 3.1 we have $\alpha_K = \beta_{\tau K} \circ \Ext^1_C(\mu_K, \tau K)$ and then
\[
\alpha_K(g(\xi), h) = (\beta_{\tau K} \circ \Ext^1_C(\mu_K, \tau K))(g(\xi), h) = \beta_{\tau K}(g(\xi), h, \mu_K).
\]

It follows that
\[
(D\psi' \circ c(\xi, \tau K))(\overline{h})(\overline{h}) = (Dc(K, \xi) \circ \phi_{K,X})(\overline{h})(\overline{h}).
\]

By the arbitrariness of $h$ and $g$, we obtain
\[
D\psi' \circ c(\xi, \tau K) = Dc(K, \xi) \circ \phi_{K,X}.
\]

The naturality is a direct verification. \qed

Now, we prove Auslander’s defect formula; compare \cite[Theorem III.4.1]{11} and \cite[Theorem]{7}.

**Theorem 3.3.** Let $\xi: X \overset{1}{\to} E \overset{d}{\to} Y$ be a conflation in $\mathcal{C}$.

1. For each object $K \in \mathcal{C}_r$, there exists an isomorphism $\xi_*(\tau K) \cong D\xi^*(K)$, which is natural in both $\xi$ and $K$.
2. For each object $K \in \mathcal{C}_l$, there exists an isomorphism $\xi^*(\tau^- K) \cong D\xi_*(K)$, which is natural in both $\xi$ and $K$.

**Proof.** We only prove (1). By Lemma 3.2(1), we obtain the following commutative diagram
\[
\begin{array}{ccc}
\Hom_C(X, \tau K) & \overset{c(\xi, \tau K)}{\longrightarrow} & \Ext^1_C(Y, \tau K) \\
\downarrow{\cong} & & \downarrow{\cong} \\
D\Ext^1_C(K, X) & \overset{Dc(K, \xi)}{\longrightarrow} & D\Hom_C(K, Y).
\end{array}
\]

Then the result follows by the above commutative diagram, since we have the natural isomorphisms \cite[2.1]{2} and \cite[2.2]{2}. \qed
4. A Commutative Triangle

Recall that two morphism \( f: X \to Y \) and \( f': X' \to Y \) are called right equivalent if \( f \) factors through \( f' \) and \( f' \) factors through \( f \). We denote by \( \{ f \} \) the right equivalence class containing \( f \), and by \( \{ \cdot \} \) the class of right equivalence classes of morphisms ending to \( Y \). We mention that \( \{ - \} \) is a set and has a natural lattice structure; see [3] Section I.2. We denote by \( \{ - \} \text{epi} \) the subclass of \( \{ - \} \) formed by deflations.

For an object \( K \) in \( \mathcal{C} \), we set \( \Gamma(K) = \text{End}_\mathcal{C}(K) \), and denote by \( \text{add} K \) the category of direct summands of finite direct sums of \( K \). For a \( \Gamma(K) \)-module \( M \), we denote by \( \text{sub}_{\Gamma(K)} M \) the lattice of \( \Gamma(K) \)-submodules of \( M \). We obtain a morphism of posets

\[
\eta_{C,Y}: \{ Y \} \text{epi} \to \text{sub}_{\Gamma(C)} \text{Hom}_\mathcal{C}(C,Y), \quad [d] \mapsto \text{Im} \text{Hom}_\mathcal{C}(C,d).
\]

Recall that a morphism \( f: X \to Y \) is called right \( K \)-determined if the following condition is satisfied: each morphism \( g: T \to Y \) factors through \( f \), provided that for each morphism \( h: C \to T \) the morphism \( g \circ h \) factors through \( f \).

For an object \( K \) in \( \mathcal{C} \), we denote by \( K \{ - \} \text{epi} \) the subclass of \( \{ - \} \text{epi} \) formed by right \( K \)-determined deflations, and by \( \text{add} K \{ - \} \text{epi} \) the subclass of \( \{ - \} \text{epi} \) formed by deflations whose kernel lies in \( \text{add} K \). We mention that \( \{ f \} \in K \{ - \} \text{epi} \) means that there exists some deflation \( f': X' \to Y \) with \( f' \in \{ f \} \) and \( \text{Ker} f' \in \text{add} K \).

For a deflation \( d: X \to Y \), we denote by \( \xi_d \) the conflation \( \text{Ker} d \to X \xrightarrow{\delta_d} Y \). We set \( \delta_{K,Y}(d) = \text{Im} c(\xi_d,K) \), which is a \( \Gamma(K) \)-submodule of \( \text{Ext}^1_{\mathcal{C}}(Y,K) \). We mention the following fact; see [3] Propositions 2.4 and 4.5.

**Lemma 4.1.** Let \( K \) be an object in \( \mathcal{C} \).

1. There exists an anti-isomorphism of posets

\[
\delta_{K,Y}: K \{ Y \} \text{epi} \to \text{sub}_{\Gamma(K)} \text{Ext}^1_{\mathcal{C}}(Y,K).
\]

2. If \( K \in \mathcal{C}_l \), then

\[
K \{ Y \} \text{epi} = \tau^{-1} K \{ Y \} \text{epi}.
\]

We mention that the natural isomorphism

\[
\psi_{K,Y}: \text{Hom}_\mathcal{C}(\tau^{-1} K,Y) \to D \text{Ext}^1_{\mathcal{C}}(Y,K)
\]

induces an anti-isomorphism of posets

\[
\gamma_{K,Y}: \text{sub}_{\Gamma(K)} \text{Ext}^1_{\mathcal{C}}(Y,K) \to \text{sub}_{\Gamma(\tau^{-1} K)} \text{Hom}_\mathcal{C}(\tau^{-1} K,Y)
\]

\[
L \mapsto \psi_{K,Y}^{-1}(D(\text{Ext}^1_{\mathcal{C}}(Y,K)/L)).
\]

The following result establishes Auslander bijection. This is a slight generalization of [3] Theorem 4.6]. The proof is similar.

**Theorem 4.2.** Let \( Y \in \mathcal{C} \) and \( K \in \mathcal{C}_l \) be objects. Then we have the following commutative bijection triangle

\[
\begin{array}{ccc}
\text{sub}_{\Gamma(K)} \text{Ext}^1_{\mathcal{C}}(Y,K) & \xrightarrow{\delta_{K,Y}} & \text{sub}_{\Gamma(\tau^{-1} K)} \text{Hom}_\mathcal{C}(\tau^{-1} K,Y) \\
\tau^{-1} K \{ Y \} \text{epi} & \xrightarrow{\eta_{K,Y}} & \text{sub}_{\Gamma(\tau^{-1} K)} \text{Hom}_\mathcal{C}(\tau^{-1} K,Y).
\end{array}
\]
Proof. It is sufficient to show that the triangle is commutative. For each deflation $d: X \rightarrow Y$, we have an exact sequence

$$
\text{Hom}_C(\tau^{-K},X) \xrightarrow{\text{Hom}_C(\tau^{-K},d)} \text{Hom}_C(\tau^{-K},Y) \xrightarrow{c(\tau^{-K},\xi_d)} \text{Ext}^1_C(\tau^{-K},\text{Ker} d)
$$

in mod $k$. We obtain

$$
\eta_{\tau^{-K},Y}([d]) = \text{Im} \text{Hom}_C(\tau^{-K},d) = \text{Ker} c(\tau^{-K},\xi_d).
$$

Applying $D$ to the exact sequence

$$
\text{Hom}_C(X,K) \xrightarrow{c(\xi_d,K)} \text{Ext}^1_C(Y,K) \rightarrow \text{Ext}^1_C(Y,K)/\text{Im} c(\xi_d,K) \rightarrow 0
$$

in mod $k$, we obtain

$$
D(\text{Ext}^1_C(Y,K)/\text{Im} c(\xi_d,K)) = \text{Ker} D c(\xi_d,K).
$$

By the commutative diagram in Lemma 3.2, we obtain

$$
\psi_{K,Y}(\text{Ker} c(\tau^{-K},\xi_d)) = \text{Ker} D c(\xi_d,K) = D(\text{Ext}^1_C(Y,K)/\text{Im} c(\xi_d,K)).
$$

It follows that

$$
(\gamma_{K,Y} \circ \delta_{K,Y})([d]) = \psi^{-1}_{K,Y}(D(\text{Ext}^1_C(Y,K)/\text{Im} c(\xi_d,K)))
$$

$$
= \text{Ker} c(\tau^{-K},\xi_d)
$$

$$
= \eta_{\tau^{-K},Y}([d]).
$$

Here, the first equality holds by the definitions of $\delta_{K,Y}$ and $\gamma_{K,Y}$. The second and third equalities are just (4.2) and (4.1), respectively. □

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School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, PR China
E-mail address: jiaopjie@mail.ustc.edu.cn