Extremal Radii, Diameter, and Minimum Width in Generalized Minkowski Spaces

Thomas Jahn

We discuss the notions of circumradius, inradius, diameter, and minimum width in generalized Minkowski spaces (that is, with respect to gauges), i.e., we measure the “size” of a given convex set in a finite-dimensional real vector space with respect to another convex set. This is done via formulating some kind of containment problem incorporating homothetic bodies of the latter set or strips bounded by parallel supporting hyperplanes thereof. The paper can be seen as a theoretical starting point for studying metrical problems of sets in generalized Minkowski spaces.

Keywords: circumradius, containment problem, convex distance function, diameter, gauge, generalized Minkowski space, inradius, Minkowski functional, minimum width

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1 Introduction

The celebrated Sylvester problem, which was originally posed in [25], asks for a point that minimizes the maximum distance to points from a given finite set of points in the Euclidean plane. There are at least two ways to generalize this problem. From a first point of view, we might keep the participating geometric configuration – given a set, we are searching a point – but change the distance measurement. Classically, distance measurement is provided by the Euclidean norm or, equivalently, by its unit ball, which is a centered, compact, convex set having the origin as interior point. Then the Sylvester problem asks for the least scaling factor (called circumradius) such that there is a correspondingly scaled version of the unit ball that contains the given set. In the literature [7], this setting has already been relaxed by using norms [1, 15, 19] and even by dropping the centeredness and the boundedness of the unit ball as well as the finite cardinality of the given set [6–8]. Vector spaces equipped with such a unit ball shall be called generalized Minkowski spaces. The corresponding analogue of the norm is the Minkowski functional of the unit ball, which is also called gauge or convex distance function in the literature. A second possibility to change the setting of the Sylvester problem is as follows. We keep the Euclidean distance measurement, but instead of asking for a point which approximates the given set in a minimax sense, we ask for an affine flat of certain dimension doing this.
In this paper, we focus on generalizing the distance measurement and, after obtaining an appropriate notion of circumradius, discuss how to define the notions of inradius, diameter, and minimum width within the general setting of generalized Minkowski spaces.

The paper is organized as follows. In Section 2, we introduce our notation and recall some basic facts regarding support functions and width functions. Elementary properties of the four classical quantities circumradius, inradius, diameter, and minimum width are investigated in Section 3. The paper is finished by a collection of open questions in Section 4.

2 Preliminaries

Four classical quantities for measuring the size of a given set are: the maximum distance between two of its points (its diameter), the minimal distance between two parallel supporting hyperplanes (its minimum width or thickness), the radius of the smallest ball containing the set (its circumradius), and the radius of the largest ball that is contained in the set (its inradius). In the framework of convex geometry, the definitions of these quantities refer to Euclidean distance measurement, that is, we compare the size of the given set with the size of the Euclidean unit ball. In the following, we will describe how diameter, minimum width, circumradius, and inradius can be defined precisely, in which way the definition can be translated when comparing sizes with a centered convex body (not necessarily the Euclidean unit ball), and what can be done when we even drop the centeredness of the measurement body. At first, we have a look at support and width functions, which, for convex sets, are related to some kind of signed Euclidean distances between supporting hyperplanes and the origin and between parallel supporting hyperplanes, respectively.

Throughout this paper, we shall be concerned with the vector space \( \mathbb{R}^d \), with the topology generated by the usual inner product \( \langle \cdot | \cdot \rangle \) and the norm \( |\cdot| = \sqrt{\langle \cdot | \cdot \rangle} \) or, equivalently, its unit ball \( B \). For the extended real line, we write \( \mathbb{R} := \mathbb{R} \cup \{+\infty, -\infty\} \) with the conventions \( 0(+\infty) := +\infty, 0(-\infty) := 0 \) and \((+\infty) + (-\infty) := +\infty\). We use the notation \( 'C^d \) for the family of non-empty closed convex sets in \( \mathbb{R}^d \). We denote the class of bounded sets that belong to \( 'C^d \) by \( 'K^d \). We also write \( 'C_0^d \) and \( 'K_0^d \) for the classes of sets having non-empty interior and belonging to \( 'C^d \) and \( 'K^d \), respectively. The line segment between \( x \) and \( y \) shall be denoted by \([x, y]\). The abbreviations cl, int and co stand for closure, interior and convex hull, respectively. A set \( K \) is centrally symmetric iff there is a point \( z \in \mathbb{R}^n \) such that \( K = 2z - K \), and \( K \) is said to be centered iff \( K = -K \).

Definition 2.1. The support function of a set \( K \subseteq \mathbb{R}^d \) is defined as \( h_K : \mathbb{R}^d \to \mathbb{R} \), \( h_K(x) := \sup \{ \langle x | y \rangle | y \in K \} \). Its sublevel set 

\[ K^\circ := \{ x \in \mathbb{R}^d \mid h_K(x) \leq 1 \} \]

is called the polar set of \( K \).

Lemma 2.2 ( [4, Proposition 7.11], [5, § 15]). Let \( K, K' \subseteq \mathbb{R}^d \), \( x, y \in \mathbb{R}^d \), and \( \alpha > 0 \). We have
(a) $h_K = h_{\text{cl}(K)} = h_{\text{co}(K)}$
(b) $h_{K+K'} = h_K + h_{K'}$,
(c) sublinearity: $h_K(x + y) \leq h_K(x) + h_K(y)$, $h_K(ax) = ah_K(x)$,
(d) $h_{\alpha K}(x) = \alpha h_K(x)$, $h_{-K}(x) = h_K(-x)$.

The previous lemma tells us that it suffices to consider closed and convex sets for the study of support functions. As mentioned before, there is a link between support functions and Euclidean distances between the origin and supporting hyperplanes. These are given by

$$H_K(u) = \left\{ y \in \mathbb{R}^d \mid \langle u, y \rangle = h_K(u) \right\}$$

(provided $h_K(u) < +\infty$), and $u$ is then called the outer normal vector of the supporting hyperplane. The Euclidean distance function of a set $K$ evaluated at $x \in \mathbb{R}^d$ is given via minimal distances, namely $\text{dist}(x, K) := \inf \{ |y - x| \mid y \in K \}$.

**Proposition 2.3** ([23, Theorem 1.1]). Let $K \in \mathcal{C}^d$ and $u \in \mathbb{R}^d$ such that $|u| = 1$. We have

$$\frac{|h_K(u)|}{|u|} = \text{dist}(0, H_K(u)).$$

The distances between parallel supporting hyperplanes of a set $K$ are encoded by its width function.

**Definition 2.4.** The width function of a set $K \subseteq \mathbb{R}^d$ is defined as $w_K : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}$, $w_K(x) := h_K(x) + h_K(-x)$.

**Lemma 2.5.** Let $K, K' \subseteq \mathbb{R}^d$, $u \in \mathbb{R}^d$, and $\alpha > 0$. We have

(a) $w_K = h_{K-K}$,
(b) $w_K = w_{\text{cl}(K)} = w_{\text{co}(K)}$,
(c) $w_K$ is sublinear and non-negative,
(d) $w_{K+K'} = w_K + w_{K'}$,
(e) $w_{\alpha K} = \alpha w_K$, $w_{-K} = w_K$,
(f) if $w_K(u) < +\infty$, then $w_K(u) = \text{dist}(0, H_K(u)) + \text{dist}(0, H_K(-u)) = \text{dist}(y, H_K(-u))$ for all $y \in H_K(u)$. 

3
The four classical quantities

3.1 Circumradius: measuring from outside

The definition of the circumradius can be found, e.g., in [9, 10, 13] for the case $C = B$ (Euclidean space), and in [11, 19] for the case $C = -C \in \mathcal{K}_d^0$ (normed spaces).

Definition 3.1 ([6–8]). The circumradius of $K \subseteq \mathbb{R}^d$ with respect to $C \in \mathcal{C}^d$ is defined as

$$R(K, C) = \inf_{x \in \mathbb{R}^d} \inf\{\lambda > 0 | K \subseteq x + \lambda C\}.$$  

If $K \subseteq x + R(K, C)C$, then $x$ is a circumcenter of $K$ with respect to $C$.

![Figure 1](image)

**Figure 1.** Circumradius: The set $C$ is a Reuleaux triangle (bold line, left), the set $K$ is a triangle (bold line, right). The circumradius $R(K, C)$ is determined by the smallest homothet of $C$ that contains $K$ (thin line).

**Proposition 3.2.** Let $K, K' \subseteq \mathbb{R}^d$, $C, C' \in \mathcal{C}^d$, and $\alpha, \beta > 0$. Then

(a) $R(K', C') \leq R(K, C)$ if $K' \subseteq K$ and $C \subseteq C'$,

(b) $R(K, C) = R(\text{cl}(K), C) = R(\text{co}(K), C)$,

(c) $R(K + K', C) \leq R(K, C) + R(K', C)$,

(d) $R(x + K, y + C) = R(K, C)$ for all $x, y \in \mathbb{R}^d$,

(e) $R(\alpha K, \beta C) = \alpha \beta R(K, C)$,

(f) $R(K, C') \leq R(K, C)R(C, C')$.

**Proof.** For $x \in \mathbb{R}^d$ and $\lambda \geq 0$, we have $\text{cl}(K) \subseteq x + \lambda C \iff K \subseteq x + \lambda C \iff \text{co}(K) \subseteq x + \lambda C$. This proves (b). Statement (c) is also rather simple: If $K \subseteq z + \lambda C$ and $K' \subseteq z' + \lambda' C$ for some $z, z' \in \mathbb{R}^d$, $\lambda, \lambda' > 0$, then $K + K' \subseteq (z + z') + (\lambda + \lambda')C$. In order to show (f), note that there exist numbers $\lambda, \lambda' > 0$ and points $z, z' \in \mathbb{R}^d$ such that $K \subseteq z + \lambda C$ and $C \subseteq z' + \lambda' C'$. Substituting the latter inclusion into the former one, we obtain $K \subseteq z + \lambda z' + \lambda \lambda' C'$. \qed
Remark 3.3. (a) The following implication is wrong: If $K, K' \in \mathcal{C}^d$ and $C \in \mathcal{K}^d_0$, then $R(K + K', C + K') = R(K, C)$. For example, take

$$C = \left\{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid \|x\|_\infty \leq 1 \right\},$$

$$K = \left\{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d \mid \|x\|_1 \leq 1, x_1, \ldots, x_d \geq 0 \right\},$$

$K' = [0, e_d]$, where $e_d \in \mathbb{R}^d$ denotes the vector whose entries are 0 except for the $d$th one, which is 1. Then $R(K, C) = \frac{1}{2}$ and $R(K + K', C + K') = \frac{2}{3}$. But the following implication is true: If $K, C' \in \mathcal{C}^d$, $K \in \mathcal{K}^d$, and $R(K, C) = 1$, then $R(K + K', C + K') = R(K, C)$. This is because of the cancellation rule, which reads as

"Let $K_1, K_2 \in \mathcal{C}^d$ and $K \in \mathcal{K}^d$. If $K_1 + K \subseteq K_2 + K$, then $K_1 \subseteq K_2$."

and can easily be proved via support functions. For all $\lambda > 1$, there exists $z \in \mathbb{R}^d$ such that $K \subseteq z + \lambda C$. It follows that $K + K' \subseteq z + \lambda C + K' \subseteq z + \lambda(C + K')$. In other words, $R(K + K', C + K') \leq 1$. Conversely, assume that $R(K + K', C + K') < 1$. Then there are $\lambda < 1$ and $z \in \mathbb{R}^d$ such that $K + K' \subseteq z + \lambda(C + K') \subseteq z + \lambda C + K'$. By virtue of the cancellation rule, we have $K \subseteq z + \lambda(C + K') \subseteq z + \lambda C$, which is a contradiction to $R(K, C) = 1$.

(b) Proposition 3.2(c) holds with equality if $K' = aC$ for all $a > 0$.

The previous lemma tells us that the circumradius of $K$ with respect to $C$ is invariant under translations of both $K$ and $C$. So, without loss of generality, we may assume that $0 \in \text{relint}(C)$. Then the circumradius can be equivalently written as

$$R(K, C) = \inf_{x \in \mathbb{R}^d} \sup_{y \in K} \gamma_C(y - x),$$

(1)

where $\gamma_C : \mathbb{R}^d \to \overline{\mathbb{R}}$ is the Minkowski functional defined by $\gamma_C(x) := \inf \{ \lambda > 0 \mid x \in \lambda C \}$.

Lemma 3.4. Given $x, y, z \in \mathbb{R}^d$, $a > 0$, and $C \in \mathcal{K}^d_0$ with $0 \in \text{int}(C)$, we define $g(x, y) := 2R((x, y), C)$. The following statements are true:

(a) $g(x, y) \geq 0$ with equality if and only if $x = y$,

(b) $g(x, y) = g(y, x)$,

(c) $g(x + z, y + z) = g(x, y)$,

(d) $g(ax, ay) = ag(x, y)$,

(e) $g(x, y) \leq g(x, w) + g(w, y)$.

Proof. The non-negativity follows from the definition. The characterization of the equality case is a consequence of the more general result that $R(K, C) = 0$ if and only if $K$ is contained in a translate of the cone $\{ y \in \mathbb{R}^d \mid y + C \subseteq C \}$, see [7, Lemma 2.2]. (Note that the authors of [7] omit the important assumption that the set of extreme points of $C$ is...
bounded.) The symmetry is clear. The invariance under translations and the compatibility with scaling follow from Proposition 3.2(d), (e). Finally, since \( g \) is translation-invariant and symmetric, we only have to check \( g(0,x+y) \leq g(0,x)+g(0,y) \) for the triangle inequality. But we have

\[
g(0,x+y) = R((0,x+y),C) \leq R((0,x,y,x+y),C) \\
\leq R((0,x),C)+R((0,y),C) = g(0,x)+g(0,y)
\]

by Proposition 3.2(c).

Since the triangle inequality for \( g \) turns out to be true, the mapping \( x \mapsto 2R((0,x),C) \) defines a norm on \( \mathbb{R}^d \). The unit ball of this norm is \( \frac{1}{2}(C-C) \). This fact can be proved as follows. First, we show that if \( x \in \frac{1}{2}(C-C) \), then \( R((0,x),\frac{1}{2}C) \leq 1 \). Namely, there exist \( y_1, y_2 \in \frac{1}{2}C \) such that \( x = y_1 - y_2 \). Thus \( R((0,x),\frac{1}{2}C) = R((y_1,y_2),\frac{1}{2}C) \leq 1 \). The reverse implication is as easy as the first one. If \( R((0,x),\frac{1}{2}C) > 1 \), then there is no \( z \in \mathbb{R}^d \) such that \( \{0,x\} \in z + \frac{1}{2}C \) or, equivalently, such that \( \{z,x-z\} \in \frac{1}{2}C \). Thus there is no representation \( x = (x-z)-(z) \in \frac{1}{2}(C-C) \).

**Definition 3.5.** The maximal chord-length function of \( K \subseteq \mathbb{R}^d \) is defined as \( l_K : \mathbb{R}^d \rightarrow \mathbb{R} \),

\[
l_K(x) = \sup \{ \alpha > 0 \mid \alpha x \in K - K \}.
\]

The radius function \( r_K : \mathbb{R}^d \rightarrow \mathbb{R} \), defined as

\[
r_K(u) = \sup \{ \alpha > 0 \mid \alpha u \in K \},
\]

is the pointwise inverse to the Minkowski functional \( \gamma_K \).

**Lemma 3.6.** Let \( K \subseteq \mathbb{R}^d \), \( x,y \in \mathbb{R}^d \), and \( \beta_1, \beta_2 > 0 \). We have

\[
(a) \; l_{\beta_2 K}(\beta_1 x) = \frac{\beta_2}{\beta_1} l_K(x), \\
(b) \; l_{\gamma K}(x) = l_K(-x) = l_{-K}(x) = l_K(x).
\]

**Proof.** We only prove the first part, because the second one is an easy consequence of the centeredness of \( K - K \). We have

\[
l_{\beta_2 K}(\beta_1 x) = \sup \{ \alpha > 0 \mid a \beta_1 x \in \beta_2 K \} = \sup \left\{ \alpha > 0 \mid a \frac{\beta_1}{\beta_2} x \in K \right\} \\
= \sup \left\{ \gamma \frac{\beta_2}{\beta_1} \mid \gamma > 0, \gamma x \in K \right\} = \frac{\beta_2}{\beta_1} l_K(x). \quad \square
\]

For centered sets \( K \), the maximal circumradius of two-element subsets is attained at antipodal points of \( K \).

**Lemma 3.7.** Let \( K \subseteq \mathbb{R}^d \) be a bounded set, and let \( C \in \mathcal{K}_0^d \). If \( K = -K \), then

\[
\sup \{ R((-x,x),C) \mid x \in K \} = \sup \{ R((x,y),C) \mid x,y \in K \}.
\]
Proof. Using Proposition 3.2, we have
\[
\sup \{ R((x,y),C) | x,y \in K \} \\
\leq \sup \{ R((0,x,y,x+y),C) | x \in K \} \\
\leq \sup \{ R((0,x),C) | x \in K \} + \sup \{ R((0,y),C) | y \in K \} \\
= 2 \sup \{ R((0,x),C) | x \in K \} \\
= \sup \{ R((0,2x),C) | x \in K \} \\
= \sup \{ R((−x,x),C) | x \in K \} \\
\leq \sup \{ R((x,y),C) | x,y \in K \}.
\]

Similarly, the maximum chord length of centered convex bounded sets is attained at antipodal points of \( K \).

Lemma 3.8. Let \( K \in \mathcal{K}^d \), \( K = −K \), and \( u \in \mathbb{R}^d \) be such that \( |u| = 1 \). Then there is \( z \in K \) such that \( l_K(u) = |z − (−z)| = 2|z| \).

Proof. We have
\[
l_K(u) = \sup \{ α > 0 | αu \in K − K \} = \sup \{ α > 0 | αu \in 2K \} \\
= \sup \left\{ α > 0 \mid \frac{1}{2} αu \in K \right\} = 2 \sup \{ α > 0 | αu \in K \}. \tag{2}
\]

Since \( K \) is compact, we have \( \sup \{ α > 0 | αu \in K \} < +\infty \), and therefore
\[
z := \sup \{ α > 0 | αu \in K \} u \in K.
\]

If both \( K \) and \( C \) are centered, there is another nice representation of the circumradius.

Lemma 3.9 (\( \{11, (1.1)\} \)). Let \( K \subseteq \mathbb{R}^d \), \( C \in \mathcal{C}^d_0 \), \( 0 \in \text{int}(C) \), \( C = −C \), \( K = −K \). Then
\[
R(K,C) = \sup \{ γ_C(x) | x \in K \}.
\]

Proof. If \( K \subseteq z + λC \) for suitable \( z \in \mathbb{R}^d \) and \( λ > 0 \), then \( K \subseteq −z + λC \) due to the centeredness of \( K \) and \( C \). It follows that
\[
K \subseteq \frac{1}{2} K + \frac{1}{2} K \subseteq \frac{1}{2} (z + λC) + \frac{1}{2} (−z + λC) = λC.
\]

In other words, the circumradius is already determined by the sets \( λC \) with \( λ > 0 \):
\[
R(K,C) = \inf \{ λ > 0 | K \subseteq λC \} = \sup \{ γ_C(x) | x \in K \}.
\]

\[\square\]
Inradius: The set $K$ is a Reuleaux triangle (bold line, right), the set $C$ is a triangle (bold line, left). The circumradius $R(K,C)$ is determined by the largest homothet of $C$ that is contained in $K$ (thin line).

### 3.2 Inradius: measuring from inside

The definition of the inradius can be found, e.g., in [9, 10, 13] for the case $C = B$ (Euclidean space) and in [11] for the case $C = -C \in \mathcal{K}_0^d$ (normed spaces).

**Definition 3.10.** The inradius of $K \subseteq \mathbb{R}^d$ with respect to $C \in \mathcal{C}^d$ is defined as

$$r(K,C) = \sup_{x \in \mathbb{R}^d} \sup_{\lambda \geq 0} \{\lambda \geq 0 \mid x + \lambda C \subseteq K\}.$$

This definition is similar to the definition of the circumradius, and so are the corresponding basic properties.

**Proposition 3.11.** Let $K,K' \subseteq \mathbb{R}^d$, $C,C' \in \mathcal{C}^d$, and $\alpha, \beta > 0$. Then

(a) $r(K',C') \geq r(K,C)$ if $K' \subseteq K$ and $C \subseteq C'$,

(b) $r(K,C) = r(\cl(K),C)$ if $K$ is convex,

(c) $r(K + K',C) \geq r(K,C) + r(K',C)$,

(d) $r(x + K, y + C) = r(K,C)$ for all $x, y \in \mathbb{R}^d$,

(e) $r(\alpha K, \beta C) = \frac{\alpha}{\beta} r(K,C)$,

(f) $r(K,C') \geq r(K,C) r(C,C')$.

**Proof.** For $x \in \mathbb{R}^d$ and $\lambda \geq 0$, we have $\cl(K) \subseteq x + \lambda C \iff K \subseteq x + \lambda C$. This proves (b). Statement (c) is rather simple: If $K \supseteq z + \lambda C$ and $K' \supseteq z' + \lambda' C$ for some $z,z' \in \mathbb{R}^d$, $\lambda, \lambda' > 0$, then $K + K' \supseteq (z + z') + (\lambda + \lambda')C$. In order to show (f), note that there exist numbers $\lambda, \lambda' > 0$ and points $z,z' \in \mathbb{R}^d$ such that $K \supseteq z + \lambda C$ and $C \supseteq z' + \lambda' C'$. Substituting the latter inclusion into the former one, we obtain $z + \lambda z' + \lambda \lambda' C' \subseteq K$. 

### 3.3 Diameter

In Euclidean geometry, the diameter of a given set is usually defined as the maximum distance of two points of this set. But there are several other representations of this quantity which do not coincide when replacing the Euclidean unit ball by a convex body $C$ in general (but at least if $C = -C$). This offers various possibilities to think about an
appropriate extension of the notion of diameter. At first, let us consider the interpretation of the diameter as maximum distance between points of the set. Here, the distance notion is provided by the Minkowski functional of $C$. Then we can rewrite the expression for the diameter as the supremum of the Euclidean width function over the polar set of $C$.

**Theorem 3.12.** For $K \subseteq \mathbb{R}^d$ and $C \in \mathcal{K}_0^d$ with $0 \in \text{int}(C)$, the following numbers are equal:

(a) $\sup \{ \gamma_C(x-y) \mid x, y \in K \}$,

(b) $\sup \{ w_K(u) \mid u \in C^o \}$,

(c) $\sup \{ \langle u \mid x \rangle \mid u \in C^o, x \in K - K \}$.

If $K \in \mathcal{K}^d$, then the following number also belongs to this set of equal quantities:

(d) $\sup \{ \frac{l_K(u)}{r_C(u)} \mid u \in \mathbb{R}^d \setminus \{0\} \}$.

**Proof.** Using [16, Lemma 2.1], we have

$$
sup_{x, y \in K} \gamma_C(x-y)
= sup_{x, y \in K} sup_{u \in C^o} \langle u \mid x-y \rangle
= sup_{u \in C^o} sup_{x, y \in K} \langle u \mid x-y \rangle
= sup_{u \in C^o} (\langle u \mid x \rangle + \langle -u \mid y \rangle)
= sup_{u \in C^o} (h_K(u) + h_K(-u))
= sup_{u \in C^o} w_K(u)
= sup_{u \in C^o} h_K(u)
= sup_{u \in C^o} \{ \langle u \mid x \rangle \mid u \in C^o, x \in K - K \}.
$$

If $K \in \mathcal{K}^d$, then

$$
sup_{x, y \in K} \gamma_C(x-y)
= sup_{u \in \mathbb{R}^d \setminus \{0\}} sup_{a > 0: au \in K - K} \gamma_C(au)
= sup_{u \in \mathbb{R}^d \setminus \{0\}} sup_{a > 0: au \in K - K} ay_C(u)
= sup_{u \in \mathbb{R}^d \setminus \{0\}} l_K(u)\gamma_C(u)
= sup_{u \in \mathbb{R}^d \setminus \{0\}} \frac{l_K(u)}{r_C(u)}
$$

$\square$
Other representations of the diameter in the Euclidean case are written in terms of circumradii, see [2, Theorem 5.2]. Together with the representation from Theorem 3.12, we obtain a chain of inequalities.

**Theorem 3.13.** If \( K \subseteq \mathbb{R}^d \) and \( C \in \mathcal{K}_0^d \) with \( 0 \in \text{int}(C) \), then

\[
2 \sup \left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \middle| u \in \mathbb{R}^d \setminus \{0\} \right\}
= 2 \sup \{R((x,y),C) \middle| x, y \in K\}
\leq R(K-K, \frac{1}{2}(C-C))
\leq \sup \{\gamma_C(x-y) \middle| x, y \in K\},
\]

with equality if \( C = -C \). If \( K \in \mathcal{E}^d \), then we have also

\[
\sup \{R((x,y),C) \middle| x, y \in K\} = \sup \left\{ \frac{l_{K(u)}}{l_{C(u)}} \middle| u \in \mathbb{R}^d \setminus \{0\} \right\}.
\]

**Proof.** If \( C = -C \), then we have

\[
2 \sup \left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \middle| u \in \mathbb{R}^d \setminus \{0\} \right\}
= \sup \left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \middle| u \in \mathbb{R}^d \setminus \{0\} \right\}
= \sup \left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \middle| u \in \mathbb{B} \setminus \{0\} \right\}
= \sup \left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \middle| u \in \mathbb{B} \setminus \{0\} \right\}
= \sup \{h_{K-K}(x) \middle| x \in C^c \setminus \{0\}\}
= \sup \{h_{K-K}(x) \middle| x \in C^c\}
= \sup \{\gamma_C(x-y) \middle| x, y \in K\}
= R(K-K, C)
\]

by using Theorem 3.12, Lemma 3.9, Lemma 3.4, and Proposition 3.2. Note that Lemma 3.4 is independent of centeredness of \( C \) and, therefore, can be equally used in the general case, which comes next. From now on, we do not assume \( C = -C \). We apply the calculations for
the symmetric case and obtain
\[
2 \sup \left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \mid u \in \mathbb{R}^d \setminus \{0\} \right\} = 2 \sup \left\{ \frac{h_{(K-K)-(K-K)}(u)}{h_{(C-C)-(C-C)}(u)} \mid u \in \mathbb{R}^d \setminus \{0\} \right\} = R \left( (K - K) - (K - K), \frac{1}{2}((C - C) - (C - C)) \right)
\]
\[
= R \left( K - K, \frac{1}{2}(C - C) \right)
\]
\[
= 2 \sup \{R((x, y), C) \mid x, y \in K\}
\]
\[
= 2 \sup \{R((0, x), C) \mid x \in K - K\}
\]
\[
= \sup \{R((-x, x), C) \mid x \in K - K\}
\]
\[
\leq R(K - K, C)
\]
\[
\leq \inf(\lambda > 0 | K - K \subseteq \lambda C)
\]
\[
= \sup_{x \in K - K} \gamma_C(x)
\]
\[
= \sup_{x, y \in K} \gamma_C(x - y).
\]

In order to prove the addendum (4), let \( K \in \mathbb{E}^d \). Then
\[
2 \sup \{R((x, y), C) \mid x, y \in K\}
\]
\[
= 2 \sup \left\{ \frac{|x - y|}{l_C \left( \frac{x - y}{|x - y|} \right)} \mid x, y \in K, x \neq y \right\}
\]
\[
= 2 \sup_{u \in \mathbb{R}^d \setminus \{0\}} \left\{ \frac{\alpha}{l_C(u)} \mid \alpha > 0, au \in K - K \right\}
\]
\[
= 2 \sup_{u \in \mathbb{R}^d \setminus \{0\}} \left\{ \frac{\sup\{\alpha \mid \alpha > 0, au \in K - K\}}{l_C(u)} \right\}
\]
\[
= 2 \sup \left\{ \frac{l_K(u)}{l_C(u)} \mid u \in \mathbb{R}^d \setminus \{0\} \right\}.
\]

The following examples show that the inequalities in (3) need not be strict if \( K \) and \( C \) are not centrally symmetric but, on the other hand, can be strict even if \( K \) is centrally symmetric. An illustration of these examples is provided by Figure 3.

**Example 3.14.**  (a) Let \( d = 2 \) and
\[
C = -K = ((2, 0) + 2\sqrt{3}B) \cap ((-1, \sqrt{3}) + 2\sqrt{3}B) \cap ((-1, -\sqrt{3}) + 2\sqrt{3}B).
\]
Then \( K - K = C - C = 2\sqrt{3}B \), i.e.,
\[
2 \sup \left\{ \frac{1}{2} \left( \frac{h_{K-K}(u)}{h_{C-C}(u)} \mid u \in \mathbb{R}^d \setminus \{0\} \right) \right\} = 2,
\]
but
\[
R(K - K, C) = \sup \{\gamma_C(x - y) \mid x, y \in K\} = \frac{1}{3}(3 + \sqrt{3}) \approx 2.366025.
\]
(b) Let \( d = 2, C = \co \{(2, 0), (-1, \sqrt{3}), (-1, -\sqrt{3})\} \), and 
\[
K = \left\{ x \in \mathbb{R}^2 \mid \|x\|_\infty \leq \sqrt{3} \right\}.
\]
Then 
\[
\sup \{ \gamma_C(x-y) \mid x, y \in K \} = 3 + \sqrt{3} \approx 4.732, 
R(K-K,C) = 2 + \frac{4}{\sqrt{3}} = 4.3094, 
2 \sup \{ R((x,y),C) \mid x, y \in K \} = \frac{2}{3} (3 + \sqrt{3}) \approx 3.1547, 
2 \sup \left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \right\} = \frac{2}{3} (3 + \sqrt{3}) \approx 3.1547.
\]

![Figure 3. Illustration of Example 3.14.](image)

Note that usually the diameter is defined on the lines of Theorem 3.12(a), see [9, 10, 13] for the Euclidean case (i.e., \( C = B \)) and [11] for the normed case (i.e., \( C = -C \in \mathcal{K}_0^d \)).

In the general setting, each of the representations may have its own benefits. However, following [7, Definition 5.2], we can define the notion of diameter via circumradii of two-element subsets which is, by Lemma 3.4, the usual diameter with respect to the norm generated by \( \frac{1}{2}(C-C) \).

**Definition 3.15.** The **diameter** \( K \) with respect to \( C \) is 
\[
D(K,C) = 2 \sup \{ R((x,y),C) \mid x, y \in K \}.
\]

The diameter also behaves nicely under hull operations and Minkowski sums in the first arguments, as well as under independent translations and scalings of both arguments.

**Proposition 3.16.** Let \( K, K' \subseteq \mathbb{R}^d, C, C' \in \mathcal{C}^d \), and \( \alpha, \beta > 0 \). Then we have
(a) $D(K', C') \leq D(K, C)$ if $K' \subseteq K$ and $C \subseteq C'$,

(b) $D(K, C) = D(\text{cl}(K), C) = D(\text{co}(K), C)$,

(c) $D(K + K', C) \leq D(K, C) + D(K', C)$,

(d) $D(x + K, y + C) = r(K, C)$ for all $x, y \in \mathbb{R}^d$,

(e) $D(\alpha K, \beta C) = \frac{\alpha}{\beta} D(K, C)$.

(f) $D(K, C') \leq D(K, C)D(C, C')$.

Proof. Statement (a) is a consequence of Proposition 3.2(a). Clearly, we have $D(K, C) \leq D(\text{cl}(K), C)$ and $D(K, C) \leq D(\text{co}(K), C)$. Furthermore, we obtain

$$D(\text{co}(K), C) = \sup \{ R([x, y], C) | x, y \in \text{co}(K) \}$$

$$= \sup \{ R([0, z], C) | z \in \text{co}(K - K) \}$$

and

$$D(\text{cl}(K), C) = \sup \{ R([x, y]), C) | x, y \in \text{cl}(K) \}$$

$$= \sup \{ R([x_k, y_k], C) | x_i, y_i \in K, i \in \in \mathbb{N}, x_k \rightarrow x, y_k \rightarrow y \}$$

$$\leq \sup \{ \sup \{ R([w, z], C) | w, z \in K \} | x_i, y_i \in K, i \in \in \mathbb{N}, x_k \rightarrow x, y_k \rightarrow y \}$$

$$= \sup \{ R([w, z], C) | w, z \in K \}.$$

This yields claim (b). In order to prove part (c), we observe that

$$D(K + K', C) = \sup \{ R([x, y]), C) | x, y \in K + K' \}$$

$$= \sup \{ R([w + w', z + z'], C) | w, z \in K, w', z' \in K' \}$$

$$\leq \sup \{ R([w + w', z + z', w + z, w' + z'], C) | w, z \in K, w', z' \in K' \}$$

$$\leq \sup \{ R([w, z], C) + R([w', z'], C) | w, z \in K, w', z' \in K' \}$$

$$= \sup \{ R([w, z], C) | w, z \in K \} + \sup \{ R([w', z'], C) | w', z' \in K' \}$$

and

$$= D(K, C) + D(K', C).$$
In order to prove (f), we use the representation $D(K, C) = \sup \{ \gamma_{C - C}(x) \mid x \in K - K \}$. Without loss of generality, we may assume that $K$ is bounded since otherwise $D(K, C) = D(K, C') = +\infty$ due to the positive homogeneity of Minkowski functionals. Furthermore, we assume that $0 \in \text{int}(C) \cap \text{int}(C')$ due to (d). We have

$$D(K, C') = \sup \{ \gamma_{C - C}(x) \mid x \in K - K \}$$

$$= \sup \{ \gamma_{C - C}(au) \mid u \in \text{bd}(B), \alpha \in [0, r_{K-K}(u)] \}$$

$$= \sup \left\{ \frac{r_{K-K}(u)\gamma_{C-C}(u)}{\gamma_{C-C}(u)} \mid u \in \text{bd}(B) \right\}$$

$$\leq \sup \left\{ \frac{r_{K-K}(u)\gamma_{C-C}(u)}{\gamma_{C-C}(u)} \mid u \in \text{bd}(B) \right\}$$

$$\leq \sup \{ \gamma_{C-C}(x) \mid x \in K - K \}$$

$$\leq \sup \{ \gamma_{C-C}(x) \mid x \in C - C \}$$

$$= D(K, C)D(C, C').$$

\[ \square \]

**Lemma 3.17.** Let $L \subseteq \mathbb{R}^d$ and $C \in \mathcal{K}_0^d$ with $0 \in \text{int}(C)$. Then

$$D(K, C) \leq 2R(K, C),$$

with equality if, e.g., $C = -C$ and $K = -K$.

**Proof.** Apply Proposition 3.2(a) and Theorem 3.13. \[ \square \]

### 3.4 Minimum Width

In Euclidean space, the notion of minimum width is intimately related to the notion of diameter. The latter is the maximum of the width function (see Theorem 3.12), the former is classically defined as the corresponding infimum. Here the reference to the (possibly non-centered) "unit ball" is done by considering the ratio of the width functions. At first, we collect relations between several representations of minimum width in normed spaces [3, Theorem 5.3] and within the general setting.

**Lemma 3.18.** Let $K \subseteq \mathbb{R}^d$, $C \in \mathcal{K}_0^d$. If $C = -C$, we have

$$2\inf \left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \mid u \in \mathbb{R}^d \setminus \{0\} \right\} = \inf \left\{ \frac{h_{K-K}(u)}{\gamma_C(u)} \mid u \in \mathbb{R}^d \setminus \{0\} \right\}. \tag{5}$$

In other words, the minimal ratio of the (Euclidean) width functions is equal to the minimal distance of parallel supporting hyperplanes of $K$, measured by the norm $\gamma_C$. If $h_{K-K}(u) > 0$ for all $u \in \mathbb{R}^d \setminus \{0\}$, also the reverse implication is true.

**Proof.** The first statement is clear because of the relations $\gamma_C^* = h_C$, $C - C = 2C$, and Lemma 2.2(d). For the reverse statement, assume that $h_{K-K}(u) > 0$ for all $u \in \mathbb{R}^d \setminus \{0\}$ and

$$2\inf \left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \mid u \in \mathbb{R}^d \setminus \{0\} \right\} < \inf \left\{ \frac{h_{K-K}(u)}{h_C(u)} \mid u \in \mathbb{R}^d \setminus \{0\} \right\}. \tag{6}$$
Then
\[ \frac{1}{2} \frac{h_{K-K}(u)}{h_{C-C}(u)} < \frac{h_{K-K}(u)}{h_C(u)} \]
for all \( u \in \mathbb{R}^d \setminus \{0\} \), which is equivalent to
\[ \frac{h_{C-C}}{2} < h_C \]
or \( h_C > h_{-C} \). Since \( C \) is convex, this means \(-C \not\subset C\) which is impossible. We obtain the same result if we assume the reverse inequality in (6).

**Lemma 3.19.** If \( K, C \in \mathcal{K}_0^d \), then
\[ r(K - K, C) = R(C, K - K)^{-1} = \{ \sup \{ \langle u \mid x \rangle \mid u \in (K - K)^o, x \in C \} \}^{-1}. \]

**Proof.** Note that \( K - K \) is centered and apply Theorem 3.13.

**Remark 3.20.** The claim of the previous lemma fails if \( K \) is not convex. For example, if \( K \) is a finite set, then \( r(K - K, C) = 0 \). But
\[ (\text{co}(K) - \text{co}(K))^o = (\text{co}(K - K))^o = \{ y \in \mathbb{R}^d \mid h_{\text{co}(K-K)}(y) \leq 1 \} \]
\[ = \{ y \in \mathbb{R}^d \mid h_{K-K}(y) \leq 1 \} = (K - K)^o, \]
where we only used Lemma 2.2(a). It follows that
\[ \sup \{ \langle u \mid x \rangle \mid u \in (K - K)^o, x \in C \} = \sup \{ \langle u \mid x \rangle \mid u \in \text{co}(K) - \text{co}(K))^o, x \in C \}, \]
which is apparently not equal to zero in general.

**Lemma 3.21.** Let \( K \in \mathcal{K}^d, C \in \mathcal{K}_0^d \). Then
\[ r(K - K, C) \leq 2 \inf \left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \middle| u \in \mathbb{R}^d \setminus \{0\} \right\} \]
with equality if \( C = -C \).

**Proof.** For all \( \alpha < r(K - K, C) \), there exists \( z \in \mathbb{R}^d \) such that \( z + \alpha C \subset K - K \). Using Lemma 2.5, we obtain
\[ w_{z + \alpha C}(u) \leq w_{K-K}(u) \text{ for all } u \in \mathbb{R}^d \setminus \{0\} \]
\[ \iff w_{\alpha C}(u) \leq w_{K-K}(u) \text{ for all } u \in \mathbb{R}^d \setminus \{0\} \]
\[ \iff a w_{C}(u) \leq 2 h_{K-K}(u) \text{ for all } u \in \mathbb{R}^d \setminus \{0\} \]
\[ \iff \alpha \leq 2 \frac{h_{K-K}(u)}{h_{C-C}(u)} \text{ for all } u \in \mathbb{R}^d \setminus \{0\}. \]
Passing \( \alpha \) to \( r(K - K, C) \), we obtain (5). Now let \( C = -C \) and assume that (5) is a strict inequality, i.e., there exists \( \alpha \) such that
\[ r(K - K, C) < \alpha < 2 \inf \left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \mid u \in \mathbb{R}^d \setminus \{0\} \right\}. \]
Like above, we obtain \( w_{\alpha C} < w_{K-K} \). Dividing by 2, we have \( h_{\alpha C} < h_{K-K} \). It follows that \( r(K - K, C) C \subset \alpha C \subset K - K \). This is a contradiction to the definition of \( r(K - K, C) \).

15
Remark 3.22. If $C \neq -C$, we may have a strict inequality in (5). In the situation of Example 3.14(a), we obtain

$$2 \inf \left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \mid u \in \mathbb{R}^d \setminus \{0\} \right\} = 2,$$

but obviously $r(K-K,C) = \sqrt{3} \neq 2$, see Figure 4.

![Figure 4. Illustration of Remark 3.22: K and C are Reuleaux triangles (bold lines).](image)

Summarizing, we obtain the following theorem on the notion of minimum width in normed spaces.

**Theorem 3.23** ([3, Theorem 5.3]). For $K \subseteq \mathcal{K}^d$ and $C \in \mathcal{K}_0^d$ with $C = -C$, the following numbers are equal:

(a) $r(K-K,C)$,

(b) $2 \inf \left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \mid u \in \mathbb{R}^d \setminus \{0\} \right\}$,

(c) $\inf \left\{ \frac{h_{K-K}(u)}{\gamma_C(u)} \mid u \in \mathbb{R}^d \setminus \{0\} \right\}$,

(d) $(\sup \| \langle u \mid x \rangle \| u \in (K-K)^*, x \in C \})^{-1}$.

**Proof.** This is a combination of the previous lemmas. If $h_{K-K} = +\infty$, then $K = \mathbb{R}^d$ and all the numbers equal $+\infty$. Similarly, if there is $u \in \mathbb{R}^d \setminus \{0\}$ such that $h_{K-K}(u) = 0$, then all the numbers equal 0. (For the last item use the convention $1/0 = +\infty$, $1/(+\infty) = 0$.)

Since we focus on containment problems (that is, finding in some sense extremal scaling factors), it is useful to take the minimal ratio of support functions as definition of minimum width.

**Definition 3.24** ([7, Definition 2.6]). The **minimum width** of $K$ with respect to $C$ is

$$\omega(K,C) := 2 \inf \left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \mid u \in \mathbb{R}^d \setminus \{0\} \right\}$$

$$= 2 \inf \left\{ R(K,C+L) \mid L \in \mathcal{L}_{d-1}^d \right\},$$

where $\mathcal{L}_{d-1}^d$ denotes the family of $(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$. 

16
Proposition 3.25. Let $K, K' \subseteq \mathbb{R}^d$, $C, C' \in \mathcal{C}^d$, and $\alpha, \beta > 0$. Then we have

(a) $\omega(K', C') \geq \omega(K, C)$ if $K' \subseteq K$ and $C \subseteq C'$,

(b) $\omega(K, C) = \omega(\text{cl}(K), C)$ if $K$ is convex,

(c) $\omega(K + K', C) \geq \omega(K, C) + \omega(K', C)$,

(d) $\omega(x + K, y + C) = \omega(K, C)$ for all $x, y \in \mathbb{R}^d$,

(e) $\omega(\alpha K, \beta C) = \frac{\alpha}{\beta} \omega(K, C)$,

(f) $\omega(K, C') \geq \omega(K, C) \omega(C, C')$.

Proof. For $x \in \mathbb{R}^d$, $\lambda \geq 0$, and $L \in \mathcal{L}_{d-1}^d$, we have $\text{cl}(K) \subseteq x + L + \lambda C \iff K \subseteq x + L + \lambda C$. This proves (b). In order to prove part (c), we observe

$$
\omega(K + K', C) = \inf \left\{ \frac{h_{K+K'-K-K}(u)}{h_{C-C}(u)} \middle| u \in \mathbb{R}^d \setminus \{0\} \right\}
$$

$$= \inf \left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} + \frac{h_{K'-K}(u)}{h_{C-C}(u)} \middle| u \in \mathbb{R}^d \setminus \{0\} \right\}
$$

$$\geq \inf \left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \middle| u \in \mathbb{R}^d \setminus \{0\} \right\} + \inf \left\{ \frac{h_{K'-K}(u)}{h_{C-C}(u)} \middle| u \in \mathbb{R}^d \setminus \{0\} \right\}
$$

$$= \omega(K, C) + \omega(K', C).
$$

Finally, we have

$$
\omega(K, C') = \inf \left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \frac{h_{C-C}(u)}{h_{C'-C}(u)} \middle| u \in \mathbb{R}^d \setminus \{0\} \right\}
$$

$$\geq \inf \left\{ \frac{h_{K-K}(u)}{h_{C-C}(u)} \middle| u \in \mathbb{R}^d \setminus \{0\} \right\} \inf \left\{ \frac{h_{C-C}(u)}{h_{C'-C}(u)} \middle| u \in \mathbb{R}^d \setminus \{0\} \right\}
$$

$$= \omega(K, C) \omega(C, C').\qedfill
$$

4 Open questions

The increasing interest in vector spaces equipped with Minkowski functionals can be observed in various directions. For example, gauges or convex distance functions occur in computational geometry, operations research, and location science (see, e.g., [12, 14, 16, 18, 23, 24]). In the present paper, a gentle start is provided for applying this setting to basic metrical notions of convex geometry. Various further natural questions occur immediately. For example, are there reverse inequalities for Proposition 3.25(c) and Proposition 3.11(c) like in [9, Theorems 1.1, 1.2]? How can we use these quantities to obtain generalizations of diametrical maximality [21, 22], constant width [20, § 2], and reducedness [17]?
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