 Integrable extensions of the rational and trigonometric $A_N$ Calogero Moser potentials

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Abstract

We describe the $R$-matrix structure associated with integrable extensions, containing both one-body and two-body potentials, of the $A_N$ Calogero-Moser $N$-body systems. We construct non-linear, finite dimensional Poisson algebras of observables. Their $N \to \infty$ limit realize the infinite Lie algebras $\text{Sdiff}(\mathbb{R} \times S^1)$ in the trigonometric case and $\text{Sdiff}(\mathbb{R}^2)$ in the rational case. It is then isomorphic to the algebra of observables constructed in the two-dimensional collective string field theory.
1 Introduction

Recent studies have lead to a number of results on the Calogero-Moser $N$-body classical models \[1, 2\]. In particular the $R$-matrix structure for the Lax operator of the $A_N$ Calogero models was derived in the case of rational and trigonometric two-body potentials \[3\]. This structure was interpreted and generalized to other algebras in \[3\]. It follows from the realization of these models by hamiltonian reduction \[3\] of free or harmonic motions on a suitable symmetric space, introduced in \[3\]. It provides us with the complete algebraic framework for integrability of the two-body potential Calogero-Moser models.

We here extend \[4\] to integrable spinless Calogero models containing an extra one-body term. We restrict ourselves to the $A_N$ case. Such generalizations for the rational and trigonometric case were introduced and studied in \[7, 8, 9, 10, 11\].

The systems under consideration have the following Hamiltonians:

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i\neq j}^{N} v_2(q_i - q_j) + \sum_{i=1}^{N} v_2(q_i)$$

with $v_2 = \frac{1}{(\sinh or \ sine)^2(q_i - q_j)}$ ; $v_1 = (c_1 e^{2q_i} + c_2 e^{-2q_i} + c_0)^2$

and $v_2 = \frac{1}{(q_i - q_j)^2}$ ; $v_1 = (aq_i^2 + bq_i + c_0)^2$ (1)

Note that they are not the most general Calogero systems of this type since integrability can be proved directly when $v_1$ is any linear combination of the monomials involved in (1) \[8, 9\]. However their associated algebraic structures can be derived from basic principles without much computational difficulties, and this will be our major interest here.

We first describe the particular scheme of Hamiltonian reduction from which these models naturally arise. This enables us to derive their associated $R$-matrix structure. It is in fact a modified structure involving an extra term in the Poisson brackets of the Lax operators. It is however shown to imply the existence of a canonical $R$-matrix structure of the usual form \[12, 13\].

Using this extended $R$-matrix structure we then construct closed Poisson algebras of functions on the phase space, naturally including the commuting Hamiltonians, providing us with a purely algebraic framework for the eventual explicit resolution of the models.

In order to proceed, we recall two fundamental results of the theory of integrable systems \[12, 13, 14, 15\]:

**Proposition 1**:

Given a Lax operator $L$, valued in a Lie algebra $G$ and depending on canonically conjugate phase space variables $p_i, q_j$; its adjoint invariant functions $\{\text{Tr}L^n\}$
Poisson-commute if and only if there exists an $R$-matrix in $\mathcal{G} \otimes \mathcal{G}$, a priori depending on the phase space variables, such that:

$$\{L \otimes L\} = [R, L \otimes 1] - [R^\pi, 1 \otimes L]$$

where $L \equiv \sum \alpha l^\alpha t_\alpha$; $\{L \otimes L\} \equiv \sum \sum \{l^\alpha, l^\beta\} t_\alpha \otimes t_\beta$; $R^\pi$ denotes the action on $R$ of the permutation operator $\Pi$ of the two algebras in $\mathcal{G} \otimes \mathcal{G}$; $\{t_\alpha\}$ is a basis of $\mathcal{G}$.

Proposition 2:

If a Lax operator $L$ has an $R$-matrix (2), any conjugated Lax operator $L^u = u(p, q)Lu^{-1}(p, q)$ has a conjugated $R$-matrix:

$$R^u = u \otimes u R u^{-1} \otimes u^{-1} + 1 \otimes u \{u \otimes L\} u^{-1} \otimes u^{-1} + 1/2 \left\{u \otimes u\right\} u^{-1} \otimes u^{-1}, 1 \otimes L \right\}$$

2 Calogero models from Hamiltonian reduction

The pattern to obtain integrable trigonometric/hyperbolic Calogero-Moser models runs as follows [4, 6, 10]. Starting from a Lie group $G$ with a subgroup $K$ such that $G/K$ be a symmetric space, one takes as phase space the cotangent bundle of $G/K$ suitably parametrized by $N \times N$ matrices $(X, Y)$ with the symplectic form $\Omega = \text{Tr} dX \wedge dY$. The free motion described by the Hamiltonian: $H = \text{Tr} XY XY$, is Liouville-integrable due to the existence of Poisson-commuting quantities $H^{(n)} = \text{Tr}(XY)^n$ including the Hamiltonian itself [16].

There is a natural left action of $G$, and particularly its subgroup $K$, on $G/K$, which is symplectic and has a momentum map $\mu = [X, Y]$ in our coordinates. Explicit Hamiltonian reduction is achieved by first introducing a Cartan decomposition of $X = uQu^{-1}$, $u \in H$, $Q \in \text{exp}\Lambda$ where $\Lambda$ is the Cartan algebra of the symmetric space $G/K$, and similarly conjugating $Y$ as $\tilde{Y} = u^{-1}Yu$ [6].

We choose from now on $G = SL(N, \mathbb{C})$ and $K = SU(N)$. Fixing the value of the momentum $\mu$ to $\mu_0 = v^T v - 1$, $v = (1, ...1)$ and dividing out by the invariance group $SU(N-1) \times U(1)$ of this fixed momentum achieves the Hamiltonian reduction to the Calogero-Moser trigonometric two-body model [5], leaving as canonical variables set defined by $Q = \text{diag} \exp q_i$ and $\tilde{Y}_{ii} \equiv p_i \exp -q_i$. The off-diagonal part of the Lax operator $L = QY$ is completely determined once the momentum $\mu = [X, Y] \equiv [Q, \tilde{Y}]$ has been fixed.

This specific choice of $\mu_0$ made in [4, 6] is crucial to preserve the integrability properties of the model, as was first shown in [6]: it guarantees that the variables $u$ and the quantities $[Q, \tilde{Y}]$ are canonically conjugate in the neighborhood of the surface $u = 1, \mu = \mu_0$. Precisely $\Omega$ reads:

$$\Omega = \text{Tr}(\partial_X \text{ln} \exp u \text{exp} -\mu) \text{Tr} dX \wedge dY$$
\[ \Omega = \text{Tr}\{dQ \wedge d\tilde{Y} + du.u^{-1} \wedge d[Q, \tilde{Y}] + du.u^{-1} \wedge [Q, [du.u^{-1}, \tilde{Y}]] \} \] (4)

and when \( \mu = \mu_0 \) as defined above the last term vanishes exactly on the surface \( u = 1, \mu = \mu_0 \). Hence we have:

**Proposition 3**

Given a set of operators \( L(X, Y) \) on the initial phase space, such that \( L(uXu^{-1}, uYu^{-1}) = uL(X, Y)u^{-1} \), the Poisson structure of the reduced operators \( L(Q, \tilde{Y}) \) is obtained by conjugation of the original Poisson structure and subsequent elimination of the extra variables \( u \) and \([Q, \tilde{Y}]\).

It follows that if the original \( L \) operators have \( R \)-matrix structures, the reduced Lax operators will have \( R \)-matrix structures obtained by conjugation of the initial structure according to Proposition 2 and subsequent elimination of \( u \) and \([Q, \tilde{Y}]\). Moreover one has:

**Proposition 4**

If a set of adjoint-invariant functions on the large phase space \( F_n(X, Y) \) realizes a closed Poisson algebra, the set of functions \( F_n(Q, \tilde{Y}) \) on the reduced phase space realizes an isomorphic algebra.

This comes from the fact that adjoint-invariance automatically eliminates \( u \) as a relevant variable in the set of \( F_n \).

The construction of integrable two-body plus one-body potentials now follows from Propositions 3 and 4. We introduce a pair of Lax operators \( X\tilde{Y} \pm V(X) \equiv L^\pm \) where \( V \) is some (finite) Laurent series. The candidate Hamiltonian is chosen to be \( H = \text{Tr} L^+L^- \). As follows from Proposition 4, if the original Hamiltonian \( H = \text{Tr}(XYXY - V^2(X)) \) is integrable, and the commuting action variables are invariant functions of \( X \) and \( Y \), the reduction procedure immediately guarantees that the reduced Calogero-type system will be integrable. We consider the set \( \{\text{Tr}(L^+L^-)^n\} \) as natural candidate action-variables, and investigate under which condition they will commute by deriving the Poisson structure of the operator \( L^+L^- \).

**Remark:** When one considers non simply-laced Lie algebras instead of \( A_N \), Calogero models generically contain one-body plus two-body terms, related to the structure of the roots of such algebras. These models have a Lax representation with a single Lax operator; the Hamiltonians have additional terms \( 1/\sinh^2(q_i + q_j) \) \[17] \[18\]. It is possible that the more general models studied in \[14\] \[15\] \[16\] are related to such algebras.

\[^3\]These references were brought to our attention by E. K. Sklyanin
3 Extended trigonometric potentials

We first compute the Poisson structure for $L^+, L^-$ and $L^+ L^-$. For $V(X) = X^n, n > 0$ one has:

\[
\{L^\pm \otimes L^\pm\} = [c^{sl(N)}, L^\pm \otimes 1 - 1 \otimes L^\pm] \\
\{L^+ \otimes L^-\} = [c^{sl(N)}, L^+ \otimes 1 - 1 \otimes L^-] \\
\quad + \sum_{m=1}^n c^{sl(N)} (X^{n-m} \otimes X^m + X^m \otimes X^{n-m})
\] (5)

Here $c^{sl(N)}$ is the quadratic Casimir operator of $sl(N)$ represented as:

\[
c^{sl(N)} = \sum_{i \neq j} e_{ij} \otimes e_{ji} + \sum_i e_{ii} \otimes e_{ii}
\]

From (5) and denoting $R = 1 \otimes L^+ c^{sl(N)} + c^{sl(N)} L^- \otimes 1$ one has:

\[
\{L^+ L^- \otimes L^+ L^-\} = [R, L^+ L^- \otimes 1] - [R^\pi, 1 \otimes L^+ L^-] \\
\quad + c^{sl(N)} \left\{ \sum_{m=1}^n L^+ (X^{n-m} L^- \otimes X^m + X^m L^- \otimes X^{n-m}) \\
\quad - (X^{n-m} \otimes L^+ X^m L^- + X^m \otimes L^+ X^{n-m} L^-) \right\}
\] (6)

For $n < 0$, one gets similar formulas where the summation goes from $m = 1$ to $-n$ and the extra terms in (5, 6) become $-c^{sl(N)} (X^{m+1} \otimes X^{-m+1} - X^{-m+1} \otimes X^{m+1})$. Finally if $V$ is a sum of monomials, every monomial contributes as one such sum multiplied by $c^{sl(N)}$.

At this point it is important to recall the following property of $c^{sl(N)}$

\[
\forall a, b \in \mathcal{G} \equiv sl(N, \mathbb{C}), \ Tr_{\mathcal{G} \otimes \mathcal{G}} c^{sl(N)} a \otimes b = Tr_{\mathcal{G}} ab
\] (7)

The following result is a consequence of (7):

**Proposition 5:**

If a Lax operator $L$ has a Poisson bracket structure of the extended form:

\[
\{L \otimes L\} = [R, L \otimes 1] - [R^\pi, 1 \otimes L] + c^{sl(N)} \sum_{i=1}^k (a_i \otimes b_i - b_i \otimes a_i)
\] (8)

and if for all $i = 1 \cdots k, a_i = L^p$ or $b_i = L^p$ for some value of $p$ in $\mathbb{Z}$ then $\forall q, m \in \mathbb{Z}, \{\text{Tr} L^q, \text{Tr} L^m\} = 0$ and $L$ has therefore a genuine $R$-matrix structure according to Proposition 1.

Explicit derivation of this canonical $R$-matrix structure is a question which we do not wish to address here. In fact the Poisson structure (8) is perfectly
suitable for the purpose of proving integrability (as shown in Proposition 5) and deriving the algebraic structure of observables as will soon be clear.

If now terms of the form \( C_{\text{sl}}(N)(a_i \otimes b_i - b_i \otimes a_i) \) occur with neither \( a_i \) nor \( b_i \) equal to a power of \( L \), they will contribute to the Poisson bracket \( \{ \text{Tr} L^p, \text{Tr} L^q \} \) as \( \text{Tr} (L^{p-1}aL^{q-1}b - L^{q-1}aL^{p-1}b) \) which does not vanish a priori. Hence such a Lax operator does not in general lead to an integrable hierarchy of Hamiltonians, and may not have an \( R \)-matrix structure.

From this result and examining (6) we deduce:

a) For \( n = 1, 0, -1 \) and linear combinations thereof \( (L^\pm = XY \pm c_1 X \pm c_2 X^{-1} \pm c_0 1) \), the Lax operator \( L = L^+L^- \) has an \( R \)-matrix structure. The Hamiltonian \( H = \text{Tr} (L^+L^-) \) is therefore integrable; the action variables are the higher traces \( H^{(m)} = \text{Tr} (L^+L^-)^m \). After Hamiltonian reduction, this gives rise to a set of two-body plus one-body hyperbolic or trigonometric Hamiltonians of the form (redefining \( q_i \leftarrow 2q_i \)):

\[
H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i \neq j=1}^{N} \frac{1}{(\text{sine or sinh})^2(q_i - q_j)} + \sum_{j=1}^{N} c_1^2 e^{4(i \eta_j)} + c_2^2 e^{-4(i \eta_j)} + 2c_1c_0 e^{2(i \eta_j)} + 2c_2c_0 e^{-2(i \eta_j)} \tag{9}
\]

This case was described in [10], where use was made of the Poisson structure of the Lax matrix \( L \pm c_1 X \pm c_2 X^{-1} \) providing a first example of our general structure (6).

b) For all other exponents \( n \), \( L = L^+L^- \) does not have a priori an \( R \)-matrix structure. This does not eliminate definitely corresponding extensions of the Calogero potentials; however the occurrence of terms \( C_{\text{sl}}(N)(a_i \otimes b_i - b_i \otimes a_i) \) will plague every attempt at defining higher conserved Hamiltonians of the form \( \text{Tr Polynomials} (L^+L^-) \).

Finally the conjugation formula in Proposition 3 can be extended to a Poisson bracket structure of the form (6), where the supplementary term \( \sum C_{\text{sl}}(N) \ldots \) is simply conjugated by \( u \otimes u \). This leads to a Poisson bracket structure for the Lax operators of extended Calogero models of the form

\[
\{L^\pm \otimes L^\pm\} = [R, L^\pm \otimes 1] - [R^\mp, 1 \otimes L^\pm] \\
\{L^+ \otimes L^-\} = [R, L^+ \otimes 1] - [R^-, 1 \otimes L^-] + \sum_{m=1}^{n} C_{\text{sl}}(N)(Q^{n-m} \otimes Q^m + Q^m \otimes Q^{n-m}) \tag{10}
\]

for any \( V(X) = X^n, n \in \mathbb{N} \). The signs and indices are changed accordingly
for $V(X) = X^{-n}$. The $R$-matrix in [10] is the same as for the pure two-body case.

\[ R = \sum_{i \neq j} \coth(q_i - q_j) e_{ij} \otimes e_{ji} \]
\[ + \frac{1}{2} \sum_{i \neq j} (\coth(q_i - q_j) + 1)(e_{ii} + e_{jj}) \otimes e_{ij} + \sum_{i} e_{ii} \otimes e_{ii} \quad (11) \]

4 Algebra of observables

We shall first examine the case where $V$ is a monomial $V(X) = X$ or $V(X) = X^{-1}$. Since $[\tilde{Y}, Q] = \mu_0$, one has

\[ [L^+, L^-] = \mu_0(L^+ - L^-); \quad V = Q \]
\[ [L^+, L^-] = -(L^+ - L^-)\mu_0; \quad V = Q^{-1} \quad (12) \]

This allows us to introduce a notion of normal-ordered observables.

**Proposition 6:**

For any monomial trace $\text{Tr}(L_1 \cdots L_n)$ where $L_i = L^\pm$ and $V(Q) = Q$ or $Q^{-1}$, there exists a polynomial expression of this trace in terms of normal ordered traces $\text{Tr}(L^+)^a (L^-)^m$.

The proof runs as follows. Denoting the length of the monomials by $l(A)$ one first checks the statement for $l(A) = 1, 2$ in which case it is trivial. One then assume that the normal ordering is proved for $(L_1 \cdots L_n)$ until $n = n_0 + 1$. Consider now $\text{Tr}(L_1 \cdots L_{n_0+2})$. Normal-ordering is achieved by commuting every $L^+$ with every $L^-$ on its left, leading to

\[ \text{Tr}(L_1 \cdots L_{n_0+2}) = \text{Tr}(L^+)^a (L^-)^b + \sum \text{Tr}([L^+, L^-] L'_1 \cdots L'_{n_0}) \quad (13) \]

One can rewrite every supplementary term in (13) as:

\[ \text{Tr}([L^+, L^-] L'_1 \cdots L'_{n_0}) = \text{Tr}[L^+, L^- L'_1 \cdots L'_{n_0}] + \sum \text{Tr}([L^+, L^-] L''_1 \cdots L''_{n_0}) \quad (14) \]

\footnote{The discrepancy with formulas in [4, 3] is due to an extra conjugation of $L$ by $\text{Diag \ exp } q_i/2$}
where $L''_1 \cdots L''_{n_0}$ denotes a reordering of $L'_1 \cdots L'_{n_0}$ each time $L^+$ commutes with one $L^-$ in the expansion of $\text{Tr} \ [L^+, L^- L'_1 \cdots L'_{n_0}]$. Hence $L''_1 \cdots L''_{n_0}$ can be reexpressed as $L'_1 \cdots L'_{n_0}$ up to extra commutators, leading to the equation:

$$0 = \text{Tr}[L^+, L^- L'_1 \cdots L'_{n_0}] = (1 + k) \text{Tr}([L^+, L^-] L'_1 \cdots L'_{n_0}) + \sum \text{Tr}([L^+, L^-] A[L^+, L^-] B)$$

(15)

In (15) $k$ denotes the number of $L^-$ terms in $L'_1 \cdots L'_{n_0}$; $A$ and $B$ are monomials in $L^\pm$ of total length $l(A) + l(B) = n_0 - 2$. Using (12) all remaining terms in (15) have the structure (respectively for the potential $X^{-1}$ or $X$):

$\text{Tr}(\mu_0(L^+ - L^-) A \mu_0(L^+ - L^-) B)$ or $((L^+ - L^-) \mu_0 A(L^+ - L^-) \mu_0 B)$

Since $\mu_0 = \sum_{ij} e_{ij}$ one has the remarkable property:

$$\text{Tr}((\mu_0 + 1) A (\mu_0 + 1) B) = \text{Tr}((\mu_0 + 1) A) \text{Tr}((\mu_0 + 1) B)$$

(16)

hence one gets:

$$\text{Tr} \ (\mu_0(L^+ - L^-) A \mu_0(L^+ - L^-) B) = \text{Tr} \ ((\mu_0 + 1)(L^+ - L^-) A) \text{Tr}((\mu_0 + 1)(L^+ - L^-) B) - \text{Tr} \ (\mu_0(L^+ - L^-) B(L^+ - L^-) A) - \text{Tr} \ (\mu_0(L^+ - L^-) A(L^+ - L^-) B) - \text{Tr} \ (L^+ - L^-) B(L^+ - L^-) A)$$

(17)

Since $l(A) + l(B) = n_0 - 2$, every monomial trace on the right hand side of (17) falls under the recursion hypothesis. In particular one replaces the expression $\mu_0(L^+ - L^-)$ by the commutator $[L^+, L^-]$ thereby generating in (17) monomials of length at most $n_0 + 1$ to which the recursion hypothesis applies.

This demonstration is here specifically given for $V(X) = X^{-1}$. In the other case one considers expressions of the form $\text{Tr} \ (L^+ - L^-) \mu_0 A(L^+ - L^-) \mu_0 B)$; one simply needs to apply cyclicity of the trace operation in order to get terms of the form (17). The conclusion is preserved.

This ends the proof of the recursion at the level $n_0 + 2$. Hence the normal-ordering Proposition is proved.

We now derive the algebra of observables $\{W^n_n, W^n_p\} = \text{Tr}(L^+)^n (L^-)^m$. From the Poisson structure (10) it follows that (for $V(X) = X$):

$$\{W^n_n, W^n_p\} = - \sum_{a,b} \text{Tr}(L^+ - L^-) ((L^-)^{a-1}(L^+)^a - (L^-)^{m-a}(L^+)^b(\text{Tr}(L^-)^{a-1}(L^+)^a - (L^-)^{m-a}) + \sum_{c,d} \text{Tr}(L^+ - L^-) ((L^+)^{c-1}(L^-)^m - (L^+)^{m-c}(L^-)^{d-1}(L^+)^p - (L^-)^{d-1}(L^+)^p (L^-)^{q-d} + (L^-)^{d-1}(L^+)^p (L^-)^{q-d}(L^+)^{c-1}(L^-)^m - (L^-)^m (L^+)^{m-c})$$

(18)
For $V(X) = X^{-1}$ the global sign changes. The normal-ordering mechanism now leads to:

\[
\{W^m_n, W^q_p\} = (mp - nq)\{W^{m+q-1}_{n+p} + W^{m+q}_{n+p-1}\} + \mathcal{P}(W^a_b, a \leq n + p - 2; b \leq m + q - 2)
\]

where $\mathcal{P}$ can be explicitly computed using the previous recursion procedure.

The Poisson algebra of observables therefore realizes a structure evoking a non-linear deformation of a $W_\infty$ algebra on a cylinder. Indeed the linear terms in (19) are reproduced as the Poisson bracket structure of the set: $W^m_n \equiv (p + e^{iq})^n(p - e^{iq})^m$ with $\{p, q\} = 1$.

One must emphasize however that the normal-ordering procedure is not univocally defined: another ordering of the steps in the recursion may lead to a different but equivalent expression in term of the explicit matrices given their non-generic features. Moreover, there exists necessarily an ideal of strictly zero polynomials of the adjoint-invariant quantities $\text{Tr}(L^+_n)L^+_m$, since when the matrix $L$ is of size $N$ the quantities $\text{Tr}(L^p), p > N$ are polynomial functions of the independent quantities $\text{Tr}(L^p), p \leq N$. Hence (19) is in fact an ambiguously defined expression as long as the quotient by this trivial ideal is not explicitly achieved. In particular this algebra can only have a finite number of generators as long as $N$ is finite.

The infinite-dimensional linearized limit can however be well-defined. First of all, the recursive proof shows that the next-to-longest term in the normal-ordering procedure of $\text{Tr}(L_1 \cdots L_m)$ has a total length $m - 1$. Hence if one renormalizes: $\tilde{W}^p_q \equiv N^{-p-q+1}W^p_q$ one gets:

\[
\{\tilde{W}^m_n, \tilde{W}^q_p\} = (mp - nq)\{\tilde{W}^{m+q-1}_{n+p} + \tilde{W}^{m+q}_{n+p-1}\} + N^{-1}\mathcal{P}(\tilde{W}^a_b)
\]

(20)

Moreover we expect that when $N$ becomes large, at least all normal-ordered monomials $\tilde{W}^m_n$ up to a value of $n + m$ of order $N$ will be algebraically independent. The $N \to \infty$ limit of (20) is then well-defined and coincides with $\text{Sdiff}(S_1 \times \mathbb{R})$.

In particular, recalling that $H = \text{Tr}L^+L^- = W^1_1$ one has:

\[
\{H, W^m_n\} = (n - m)(W^m_{n-1} + W^m_{n+1}) + O(N^{-2})...
\]

(21)

This does not allow to solve the eigenfunction equation $\{H, \mathcal{O}\} = \epsilon\mathcal{O}$ by setting $\mathcal{O}$ to be a finite linear combination of normal-ordered observables. A related discussion appears in [18] for collective string field theory.

Let us consider now the case when both monomials $X$ and $X^{-1}$ are retained in the potential (keeping the term $c_01$ does not modify our generating set of observables). The Poisson brackets (3) cannot be rewritten purely in terms of $L^+ - L^-$ due to the change of relative sign between $X$ and $X^{-1}$ contributions.
To get a closed Poisson algebra, one needs a priori to introduce a three-index set of observables generated by terms of the form $\text{Tr} \left( Q^n (L^+)^m (L^-)^p \right)$.

First of all the problem of normal ordering must be revisited. The relevant commutators are:

$$[L^+, L^-] = c_2 \mu_0 Q^{-1} - c_1 Q \mu_0; \quad [Q, L^\pm] = Q \mu_0; \quad [Q^{-1}, L^\pm] = -\mu_0 Q^{-1}$$  \hspace{1cm} (22)

The normal-ordering Proposition states:

**Proposition 7**

Any monomial trace $\text{Tr}(A_1 \cdots A_n)$ with $A_i \in \{L^+, L^-, Q, Q^{-1}\}$ can be re-expressed as a polynomial of normal-ordered traces $\text{Tr} Q^m (L^+)^n (L^-)^p$.

The proof involves a double recursion procedure. One denotes by $n(L^\pm)$ the number of such generators in a given monomial.

**Step 1:** Proof of Proposition 7 for $\text{Tr}(A_1 \cdots A_{N_0+1})$, $n(L^-) = 0$.

For $N = 1$ normal-ordering is obvious. Assume it is proved up to order $N_0$ (recursion hypothesis R1). Take $\text{Tr}(A_1 \cdots A_{N_0+1})$. Normal-ordering this expression involves commutators of the form (22), hence one has:

$$\text{Tr}(A_1 \cdots A_{N_0+1}) = \text{Tr} Q^p (L^+)^{n(L^+)} + \sum \text{Tr} Q \mu_0 A_1 \cdots A_{N_0-1} + \sum \text{Tr} \mu_0 Q^{-1} A_1 \cdots A_{N_0-1}$$  \hspace{1cm} (23)

Any term of the form $\text{Tr} Q \mu_0 A_1 \cdots A_{N_0-1}$ can be rewritten as in (14):

$$0 = \text{Tr}[Q, L^+ A_1 \cdots A_{N_0-1}] = (1 + n(L^+)) \text{Tr} Q \mu_0 A_1 \cdots A_{N_0-1} + \sum \text{(reordering corrections)}$$  \hspace{1cm} (24)

The corrections take the form $\text{Tr} Q \mu_0 AQ \mu_0 B$ and $\text{Tr} Q \mu_0 A Q^{-1} B$ with a length $l(A) + l(B)$ at most $N_0 - 3$. Using the decoupling property of $\mu_0$ (13) generates:

1) from $\text{Tr} Q \mu_0 AQ \mu_0 B : \text{Tr} QAQB ; \text{Tr} Q \mu_0 AQ \mu_0 B ; \text{Tr} Q \mu_0 B QA ; \text{Tr} Q \mu_0 A$; $\text{Tr} Q \mu_0 B$. Reexpressing $Q \mu_0$ as $[Q, L^+]$ turns all such terms into monomials of length $N_0$ or less to which R1 applies.

2) from $\text{Tr} Q \mu_0 A Q^{-1} B : \text{Tr} QA Q^{-1} B ; \text{Tr} Q \mu_0 A Q^{-1} B ; \text{Tr} QA Q^{-1} B ; \text{Tr} \mu_0 A$; $\text{Tr} \mu_0 Q^{-1} B Q$. Reexpressing $Q \mu_0$ and $\mu_0 Q^{-1}$ as commutators, and $\mu_0$ as $Q^{-1}[Q, L^+]$ turns all these terms into monomials of length $N_0$ or less to which R1 applies.

Hence (24) leads to an explicit normal-ordered expression for any term $\text{Tr} Q \mu_0 A_1 \cdots A_{N_0-1}$. A similar argument holds for $\text{Tr} \mu_0 Q^{-1} A_1 \cdots A_{N_0-1}$. Therefore (23) gives a normal-ordered expression for any monomial of length $N_0 + 1$. This ends the proof of Proposition 7 for $n(L^-) = 0$.  


Step 2 : Assume that Proposition 7 holds for any monomial with $n(L^-)$ up to $n_0$ (recursion hypothesis R2).

Consider now $n(L^-) = n_0 + 1$, and a general monomial in this subset $TrA_1 \cdots A_N, N \geq n_0 + 1$. For $N = n_0 + 1$ normal-ordering is already achieved! Assume now that it has been proved for $n(L^-) = n_0 + 1$ and up to a total length $N_0$ (recursion hypothesis R1). Take a monomial $TrA_1 \cdots A_{N_0+1}$. The general normal-ordering equation (24) still holds since all commutators are of the form $[Q, L]$. The residual terms $A_1 \cdots A_{N_0-1}$ contain at most $n_0 + 1$ terms $L^-$. Hence all terms in the second line of (25) can therefore be rewritten as normal-ordered polynomials due to hypothesis R1 and R2.

Now extra contributions $TrQ\mu_0 A_1' \cdots A_{N_0-1}'$ have appeared in (25) where $A_1' \cdots A_{N_0-1}'$ contains $1 + n(L^+)$ terms $L^+$ and at most $n_0$ terms $L^-$. Replacing in these terms the original $[Q, L^-] \equiv Q\mu_0$ by $[Q, L^+]$ which is identical, turns them into monomials with at most $n(L^-) = n_0$. The recursion hypothesis R2 thus holds for these monomials. Hence all terms in the second line of (25) can be normal-ordered, and so can $TrQ\mu_0 A_1 \cdots A_{N_0-1}$. Consequently the normal-ordering proposition is proved for $n(L^-) = n_0 + 1, N = N_0 + 1$. Hence it is proved by recursion for $n(L^-) = n_0 + 1$, and any total length $N$. Together with Step 1, this finally proves it by recursion for any $n(L^-)$.

Two degeneracy relations must now be implemented in order to eliminate redundant generators:

$$L^+ - L^- = 2c_1Q - 2c_2Q^{-1} \Rightarrow Q^{-1} = 1/2c_2(L^+ - L^- - 2c_1Q)$$

$$QQ^{-1} = 1 = \frac{Q}{2c_2}(L^+ - L^- - 2c_1Q) \Rightarrow Q^2 = -\frac{c_1}{c_2}(1 - \frac{1}{2c_2}(QL^+ - QL^-) \tag{26}$$

It follows that one shall consider as algebra of normal-ordered generators the set \{Tr$Q^n(L^+)^m(L^-)^p, n \in \{0, 1\}, m, p \in \mathbb{Z}$.\}

In order to get the Poisson structure of this algebra, it is here easier to use Proposition 4: one computes the Poisson brackets of observables on the full phase space \{Tr$X^n(L^+)^m(L^-)^p, n \in \{0, 1\}, m, p \in \mathbb{Z}$.\}. From Proposition 4 the Poisson algebra of reduced observables \{Tr$Q^n(L^+)^m(L^-)^p, n \in \{0, 1\}, m, p \in \mathbb{Z}$.\} is isomorphic to it. One then applies the normal-ordering procedure.
Although the finite $N$ algebra is quite complicated and again ambiguously defined, we can easily derive its leading linear order -i.e. the large $N$ limit. It is obtained by reordering all terms in the full phase space algebra as $\text{Tr}(X^n(L^+)^m(L^-)^p)$ and dropping all induced commutators. Hence, using (3) and 

\[
\{ X \otimes L^\pm \} = 1 \otimes X C^{sl(n)}
\]

one ends up with a general formula for $n_1, n_2 \in \{0, 1\}$:

\[
\{ \text{Tr}(Q^{n_1}(L^+)^{m_1}(L^-)^{p_1}), \text{Tr}(Q^{n_2}(L^+)^{m_2}(L^-)^{p_2}) \} = \left( (n_1 - p_1)m_2 - (n_2 - p_2)m_1 \right) \text{Tr}(Q^{n_1+n_2}(L^+)^{m_1+m_2-1}(L^-)^{p_1+p_2}) + \left( (n_1 - m_1)p_2 - (n_2 - m_2)p_1 \right) \text{Tr}(Q^{n_1+n_2}(L^+)^{m_1+m_2}(L^-)^{p_1+p_2-1}) + 2c_1(m_1p_2 - m_2p_1)\text{Tr}(Q^{n_1+n_2+1}(L^+)^{m_1+m_2-1}(L^-)^{p_1+p_2-1}) + \cdots \text{lower orders}
\]

(27)

One must also implement the second degeneracy condition from (25) and eliminate $Q^2$ wherever it appears in (27) in order to get the algebraic structure in terms of the independent generators.

The large $N$ limit is then achieved by redefining generators $\text{Tr}(Q^n(L^+)^m(L^-)^p) \to N^{n+m+p-1} \text{Tr}(Q^n(L^+)^m(L^-)^p)$. (27) then leads to a lengthy linear algebraic structure mixing both types of generators $\text{Tr}(L^+)^n(L^-)^m$ and $\text{Tr}Q(L^+)^n(L^-)^m$. We shall not write it explicitly, since anyhow we lack for the moment an interpretation of this algebra in term of diffeomorphisms of a surface. It is relevant for the study of the collective field theory corresponding to particular unitary matrix models [19, 20]. Finally note that adding the term $\pm c_0 I$ only modifies the degeneracy relations (26) by adding lower-order terms which will disappear from (27) after taking the large $N$ limit.

5 Extended rational potentials

Rational pure two-body potentials are obtained by normalizing the $q$ coordinates in $Q$ as $Q = \exp aq_I$ and sending $a$ to zero. Their original Lax operator is then $Y$. Extended potentials arise from considering shifted Lax operators $L^\pm \equiv Y \pm V(X)$. The Poisson brackets for $L^\pm$ are essentially identical to (3) up to one power of $X$ in the extra $\sum$. It follows from similar considerations that integrable potentials -for which the Poisson brackets take the form (8)- are $V(X) = X^2$, $X$ and $1$. We shall restrict ourselves to the monomial cases.

The case $V(X) = X^2$ leads to an integrable quartic Hamiltonian:

\[
H_0 = \sum_{i=1}^n \frac{p_i^2}{2m} + \sum_{i \neq j = 1}^n \frac{1}{(q_i - q_j)^2} + \sum_{i=1}^n gq_i^4.
\]

(28)

A similar problem arises in this case to define a normal-ordered set of observables. Poisson brackets of $L^+$ with $L^-$ contain terms $C^{sl(N)} (Q \otimes 1 + 1 \otimes Q)$ which cannot be rewritten as linear functions of $L^+$ and $L^-$. This here requires
adding extra observables of the form $\text{Tr } QL_1 \cdots L_m$ in order to have a closed Poisson algebra. Contrary to the trigonometric case this algebra of observables will exhibit a structure of symmetric algebra as $\{\text{Tr}QL_1 \cdots L_m\} \oplus \{\text{Tr}L_1 \cdots L_m\}$.

The relevant commutators for the normal ordering procedure are:

$$[L^+, L^-] = \mu_0 Q + Q\mu_0; [Q, L^\pm] = \mu_0$$

(29)

A normal-ordering procedure expresses monomials of the form $\text{Tr}A_1 \cdots A_N$ with $A_i \in \{L^\pm, Q\}$ as polynomials of $\text{Tr}Q^m(L^+)^n(L^-)^p$ The commutator structure defined in (29) again triggers a double recursion proof:

**Step 1.** Normal-ordering is proved for $n(L^-) = 0$ on the same lines as in the previous cases, since any commutator decreases the length of the monomial by two units:

$$\text{Tr}A_1 \cdots A_N = \text{Tr}Q^m(L^+)^n + \sum \text{Tr}\mu_0 A_1 \cdots A_{N-2}$$

(30)

Then since $[Q, L^\pm] = \mu_0$, one rewrites

$$0 = \text{Tr}[Q, L^+ A_1 \cdots A_{N-2}] = (1 + n(L^+))\text{Tr}\mu_0 A_1 \cdots A_{N-2} + \text{reordering terms}$$

(31)

Reordering corrections have the form $\text{Tr}\mu_0 A_1 \cdots A_{N-1} B$ with $l(A) + l(B) = N_0 - 4$. Hence the decoupling mechanism [4] generates, after replacing $\mu_0$ by $[Q, L^+]$, monomials of maximal length $N_0 - 2$ allowing the recursion to hold.

**Step 2.** Recursion proof on $n(L^-)$.

The normal-ordering hypothesis is assumed to be proved up to $n(L^-) = n_0$ (hypothesis R2).

Here one needs to be more careful when considering the reordering terms and the replacement of commutators in them since, contrary to all previous cases, the commutators of fundamental generators here assume qualitatively different forms $\mu_0$ or $\mu_0 Q + Q\mu_0$. This is dealt with by introducing a different recursion hypothesis:

For $n(L^-) = n_0 + 1$, we assume that the normal-ordering proposition is proved BOTH for terms of the form $\text{Tr}A_1 \cdots A_N$ and $\text{Tr}\mu_0 A_1 \cdots A_N$ for $N$ up to $N_0$ (hypothesis R1). For $N = n(L^-)$ the proposition is trivial since $\text{Tr}\mu_0(L^\pm)^m = 0$.

Take now $\text{Tr}A_1 \cdots A_{N_0+1}$. Normal-ordering of this expression generates corrective terms of the form $\text{Tr}\mu_0 A_1 \cdots A_{N_0}$ or even $\text{Tr}\mu_0 A_1 \cdots A_{N_0-1}$ to which the recursion hypothesis applies.

Now take $\text{Tr}\mu_0 A_1 \cdots A_{N_0+1}$. It is rewritten as:

$$0 = \text{Tr}[Q, L^+ A_1 \cdots A_{N_0+1}] = (1 + n(L^+))\text{Tr}\mu_0 A_1 \cdots A_{N_0+1}$$

$$+ \sum \text{Tr}\mu_0 A_1' \cdots A_{N_0+1} + \sum \text{Tr}\mu_0 A_1 B \text{ (reordering corrections)}$$

(32)
The term $Tr\mu_0 A'_1 \cdots A'_{N_0 + 1}$, contributed to by commutators $[Q, L^{-}]$, contains at most $n(L^{-}) = n_0$; hence the recursion hypothesis R2 can be applied after reexpressing $\mu_0$ as $[Q, L^+]$.

The corrective terms $Tr\mu_0 A\mu_0 B$ have a total length of monomials of at most $N_0$: They generate monomials $Tr AB, Tr A\mu_0 B, Tr\mu_0 A, Tr\mu_0 B$ to which R1 applies globally. Hence normal-ordering is proved for all values of $n$ for which $R_1$ applies. Hence it is proved by recursion for any $N$ and $n(L^{-}) = n_0 + 1$. Finally Step 1 and Step 2 guarantee that the normal ordering procedure holds for all values of $n(L^{-})$.

One degeneracy relation exists here, namely $Q^2 = L^+ - L^-$. Hence one shall define the relevant set of observables as \( \{ V^m_n = Tr Q (L^+)^n (L^-)^m \, ; \, W^m_n = Tr (L^+)^n (L^-)^m \} \).

We compute again the leading order of the Poisson algebra by simply considering the reordered Poisson algebra without the extra commutators on the full phase space of $X, Y$. Recalling the basic Poisson bracket $\{ X \otimes Y \} = C^{s(N)}$ we get a symmetric Lie algebra structure:

\[
\begin{align*}
\{ V^m_{n_1}, V^n_{m_2} \} &= (2(n_1 m_2 - n_2 m_1) + m_2 - m_1)V^{n_1+n_2}_{m_1+m_2-1} \\
&- (2(n_1 m_2 - n_2 m_1) + n_2 - n_1)V^{n_1+n_2-1}_{m_1+m_2} + \text{lower orders} \\
\{ V^m_{n_1}, W^n_{m_2} \} &= (2(n_1 m_2 - n_2 m_1) + m_2)W^{n_1+n_2}_{m_1+m_2-1} \\
&- (2(n_1 m_2 - n_2 m_1) + n_2)W^{n_1+n_2-1}_{m_1+m_2} + \text{lower orders} \\
\{ W^m_{n_1}, W^n_{m_2} \} &= 4(n_1 m_2 - n_2 m_1)V^{n_1+n_2-1}_{m_1+m_2-1} + \text{lower orders}
\end{align*}
\]

The large $N$ limit is achieved by redefining $V^m_n \rightarrow N^{n+m-1}V^m_n$ and $W^m_n \rightarrow N^{n+m-3/2}W^m_n$.

The algebra of $V$ generators recalls quite closely the algebra $S_{diff}(\mathbb{R} \times S^1)$ for trigonometric potentials (20), up to a shift of the $n$ indices by $1/2$.

The case $V(X) = X$ leads to the original so-called Type-V potential [8]. In this case $[L^+, L^-] = \mu_0$. Here the normal-ordering procedure can be derived in an exactly similar way to the first case of trigonometric potentials.

The observables are defined as $W^m_n \equiv \{ Tr (L^+)^n (L^-)^m \}$. The algebra then reads:

\[
\{ W^m_n, W^p_q \} = (mp - nq)W^{m+n+q-1}_{n+p-1} + \text{lower orders}
\]

The large $N$ limit is there achieved by defining $W^p_q \equiv N^{-(p-q+2)}W^p_q$.

A major difference with the trigonometric case is that the leading terms of the Poisson algebra, which survive in the large $N$ limit to give the linearized algebra, realize the $W_\infty$ algebra $S_{diff}(\mathbb{R}^2)$. The $N \rightarrow \infty$ limit of the algebra is therefore isomorphic to the algebra of observables for the collective string
field theory considered in [18]. This statement is the correct formulation of the identification indicated in [21] between observables of the matrix model and observables of the collective theory.

Finally the linearized Poisson algebra (34) is in fact exact when \(n = m = 0, 1\). This implies:

**Proposition 8:**

\(W^m_n\) are eigenfunctions of the Calogero Hamiltonian flow with energy \(n - m\).

The proof runs as follows. The Poisson bracket \(\text{Tr}\{L^+ L^- \otimes (L^+)^n (L^-)^m\}\) only gets contributions from the extra terms in (34). Hence:

\[
\begin{align*}
\text{Tr}\{L^+ L^- \otimes (L^+)^n (L^-)^m\} &= \sum_{a,b} \text{Tr}\{-L^+ \otimes (L^+)^{n-a} C^{sl(n)} 1 \otimes (L^+)^{a-1} (L^-)^m + L^- \otimes (L^-)^{m-b} C^{sl(n)} (L^-)^{b-1}\} \\
&= \sum_{a,b} \text{Tr}\{-L^+ L^- \otimes (L^+)^{n-a} (L^-)^{m-a} + -L^- L^+ (L^+)^{n} (L^-)^{m-b}\} \\
&= (m-n) \text{Tr}(L^+)^n (L^-)^m
\end{align*}
\]

This construction was first used in [22] although the proof is rather involved. The Type V potential realizes a consistent discretization of the hermitian one-matrix model [22]. The corresponding derivation of eigenoperators for the associated collective field Hamiltonian was given in [18].

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**References**

[1] F. Calogero, Lett.Nuov. Cim. 13, 411 (1975); ibid 16, 77 (1976).

[2] J. Moser, Adv. Math. 16, 1 (1976); S. Wojciechowski, Lett. Nuov. Cim 18, 13 (1977).

[3] J. Avan, M. Talon, Phys. Lett. B 303, 33 (1993).
[4] J. Avan, O. Babelon, M. Talon, to appear in Journ. Geom. Phys. (1994).

[5] R. Abraham, J.E. Marsden, "Foundations of Mechanics", ed. Benjamin-Cumings (1978).

[6] M.A. Olshanetskii, A.M. Perelomov, Inv. Math. 37, 76 (1976); id. Lett. Nuov. Cim. 16, 333 (1976).

[7] M. Adler, Comm. Math. Phys. 55, 195 (1977).

[8] V.I. Inozemtsev, Phys. Lett. A 98, 416 (1983); Physica Scripta 29, 518 (1984).

[9] S. Wojciechowski, Phys. Lett. A 102, 85 (1984); ibid. 105, 188 (1984).

[10] A. P. Polychronakos, Phys. Lett. B 276, 341 (1992); B 277, 102 (1992).

[11] A. Perelomov. Integrable systems of classical mechanics and Lie algebras (Birkhauser, 1992).

[12] M. A. Semenov-Tjan-Shanskii, Funct. Anal. Appl. 17, 17 (1983).

[13] E. K. Sklyanin, Preprint LOMI E-3-79 (1979).

[14] O. Babelon, C. M. Viallet, Phys. Lett B 237, 411 (1989).

[15] J. M. Maillet, Phys. Lett B 162, 137 (1985).

[16] V. I. Arnol’d, "Mathematical Methods in Classical Mechanics”, Graduate Texts in Mathematics, 60, Springer Verlag (1978).

[17] V.I. Inozemtsev, Physica Scripta 39, 289 (1989); Lett. Math. Phys. 17, 11 (1989)

[18] J. Avan. A. Jevicki, Comm. Math. Phys. 150, 149 (1992).

[19] A. Jevicki, B. Sakita, Phys. Rev D 22, 467 (1980).

[20] A. Jevicki, Nucl. Phys. B 376, 75 (1992); M. Douglas, Preprint Rutgers 93-13 (1993).

[21] A. Jevicki, J. Rodrigues, A. van Tonder, Nucl. Phys. B 404, 91 (1993).

[22] J. Avan, A. Jevicki, Phys. Lett. B 266, 35 (1991).

[23] M. A. Perelomov, Sov. Journ. Part. Nucl. 10, 336 (1979).