STRING TOPOLOGY FOR SPHERES.

LUC MENICHI*

WITH AN APPENDIX BY GERALD GAUDENS AND LUC MENICHI

Abstract. Let $M$ be a compact oriented $d$-dimensional smooth manifold. Chas and Sullivan have defined a structure of Batalin-Vilkovisky algebra on $\mathbb{H}_\ast(LM)$. Extending work of Cohen, Jones and Yan, we compute this Batalin-Vilkovisky algebra structure when $M$ is a sphere $S^d$, $d \geq 1$. In particular, we show that $\mathbb{H}_\ast(LS^2; \mathbb{F}_2)$ and the Hochschild cohomology $HH^\ast(H^\ast(S^2); H^\ast(S^2))$ are surprisingly not isomorphic as Batalin-Vilkovisky algebras, although we prove that, as expected, the underlying Gerstenhaber algebras are isomorphic. The proof requires the knowledge of the Batalin-Vilkovisky algebra $H_\ast(\Omega^2 S^3; \mathbb{F}_2)$ that we compute in the Appendix.

Dedicated to Jean-Claude Thomas, on the occasion of his 60th birthday

1. Introduction

Let $M$ be a compact oriented $d$-dimensional smooth manifold. Denote by $LM := \text{map}(S^1, M)$ the free loop space on $M$. In 1999, Chas and Sullivan [2] have shown that the shifted free loop homology $\mathbb{H}_\ast(LM) := H_{\ast+d}(LM)$ has a structure of Batalin-Vilkovisky algebra (Definition 5). In particular, they showed that $\mathbb{H}_\ast(LM)$ is a Gerstenhaber algebra (Definition 8). This Batalin-Vilkovisky algebra has been computed when $M$ is a complex Stiefel manifold [25] and very recently over $\mathbb{Q}$ when $M$ is a $K(\pi, 1)$ [28]. In this paper, we compute the Batalin-Vilkovisky algebra $\mathbb{H}_\ast(LM; k)$ when $M$ is a sphere $S^n$, $n \geq 1$ over any commutative ring $k$ (Theorems 10, 16, 17, 24 and 25).

In fact, few calculations of this Batalin-Vilkovisky algebra structure or even of the underlying Gerstenhaber algebra structure have been done because the following conjecture has not yet been proved.

Conjecture 1. (due to [2, “dictionary” p. 5] or [7 ?])

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If $M$ is simply connected then there is an isomorphism of Gerstenhaber algebras $\mathbb{H}_*(LM) \cong HH^*(S^*(M); S^*(M))$ between the free loop space homology and the Hochschild cohomology of the algebra of singular cochains on $M$.

In [7, 5], Cohen and Jones proved that there is an isomorphism of graded algebras over any field

$$\mathbb{H}_*(LM) \cong HH^*(S^*(M); S^*(M)).$$

Over the reals or over the rationals, two proofs of this isomorphism of graded algebras have been given by Merkulov [23] and Félix, Thomas, Vigué-Poirrier [11]. Motivated by this conjecture, Westerland [30] has computed the Gerstenhaber algebra $HH^*(S^*(M; \mathbb{F}_2); S^*(M; \mathbb{F}_2))$ when $M$ is a sphere or a projective space.

What about the Batalin-Vilkovisky algebra structure?

Suppose that $M$ is formal over a field, then since the Gerstenhaber algebra structure on Hochschild cohomology is preserved by quasi-isomorphism of algebras [10, Theorem 3], we obtain an isomorphism of Gerstenhaber algebras

$$HH^*(S^*(M); S^*(M)) \cong HH^*(H^*(M); H^*(M)).$$

Poincaré duality induces an isomorphism of $H^*(M)$-modules

$$\Theta : H^*(M) \to H^*(M)^\vee.$$

Therefore, we obtain the isomorphism

$$HH^*(H^*(M); H^*(M)) \cong HH^*(H^*(M); H^*(M)^\vee)$$

and the Gerstenhaber algebra structure on $HH^*(H^*(M); H^*(M))$ extends to a Batalin-Vilkovisky algebra [26, 22, 19] (See above Proposition [20] for details). This Batalin-Vilkovisky algebra structure is further extended in [27, 9, 20, 21] to a richer algebraic structure. It is natural to conjecture that this Batalin-Vilkovisky algebra on $HH^*(H^*(M); H^*(M))$ is isomorphic to the Batalin-Vilkovisky algebra $\mathbb{H}_*(LM)$. We show (Corollary [30]) that this is not the case over $\mathbb{F}_2$ when $M$ is the sphere $S^2$. See [6, Comments 2. Chap. 1] or the papers of Tradler and Zeinalian [26, 27] for related conjecture when $M$ is not assumed to be necessarily formal. On the contrary, we prove (Corollary [23]) that Conjecture [1] is satisfied for $M = S^2$ over $\mathbb{F}_2$.

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2. The Batalin-Vilkovisky algebra structure on $H_*(LM)$.

In this section, we recall the definition of the Batalin-Vilkovisky algebra on $H_*(LM; k)$ given by Chas and Sullivan [2] over any commutative ring $k$ and deduce that this Batalin-Vilkovisky algebra $H_*(LM; k)$ behaves well with respect to change of rings.

We first recall the definition of the loop product following Cohen and Jones [7, 6]. Let $M$ be a closed oriented smooth manifold of dimension $d$. The inclusion $e : map(S^1 \vee S^1, M) \hookrightarrow LM \times LM$ can be viewed as a codimension $d$ embedding between infinite dimension manifolds [24, Proposition 5.3]. Denote by $\nu$ its normal bundle. Let $\tau_e : LM \times LM \mapsto map(S^1 \vee S^1, M)$ its Thom-Pontryagin collapse map. Recall the umkehr (Gysin) map $e!$ is the composite of $\tau_e$ and the Thom isomorphism:

$$H_*(LM \times LM; k) \xrightarrow{H_*(\tau_e; k)} H_*(map(S^1 \vee S^1, M); k) \xrightarrow{(u_k)} H_{*-d}(map(S^1 \vee S^1, M); k)$$

The Thom isomorphism is given by taking a relative cap product $\cap$ with a Thom class for $\nu$, $u_k \in H^d(map(S^1 \vee S^1, M); k)$. A Thom class with coefficients in $\mathbb{Z}$, $u_\mathbb{Z}$, gives rise a Thom class $u_k$ with coefficients in $k$, under the morphism

$$H^d(map(S^1 \vee S^1, M); \mathbb{Z}) \rightarrow H^d(map(S^1 \vee S^1, M); k)$$

induced by the ring homomorphism $\mathbb{Z} \rightarrow k$ [16, p. 441-2]. So we have the commutative diagram

$$\begin{array}{ccc}
H_*(LM \times LM; \mathbb{Z}) & \xrightarrow{\tau_e} & H_*(map(S^1 \vee S^1, M); \mathbb{Z}) \\
\downarrow & & \downarrow \\
H_*(LM \times LM; k) & \xrightarrow{\tau_e} & H_{*-d}(map(S^1 \vee S^1, M); k)
\end{array}$$

Let $\gamma : map(S^1 \vee S^1, M) \rightarrow LM$ be the map obtained by composing loops. The loop product is the composite

$$H_*(LM; k) \otimes H_*(LM; k) \rightarrow H_*(LM \times LM; k) \xrightarrow{\gamma} H_{*-d}(LM; k)$$

So clearly, we have proved

**Lemma 3.** The morphism of abelian groups $H_*(LM; \mathbb{Z}) \rightarrow H_*(LM; k)$ induced by $\mathbb{Z} \rightarrow k$ is a morphism of graded rings.
Suppose that the circle $S^1$ acts on a topological space $X$. Then we have an action of the algebra $H_*(S^1)$ on $H_*(X)$,

$$H_*(S^1) \otimes H_*(X) \to H_*(X).$$

Denote by $[S^1]$ the fundamental class of the circle. Then we define an operator of degree 1, $\Delta : H_*(X; k) \to H_{*+1}(X; k)$ which sends $x$ to the image of $[S^1] \otimes x$ under the action. Since $[S^1]^2 = 0$, $\Delta \circ \Delta = 0$. The following lemma is obvious.

**Lemma 4.** Let $X$ be a $S^1$-space. We have the commutative diagram

$$
\begin{array}{ccc}
H_*(X; \mathbb{Z}) & \xrightarrow{\Delta} & H_{*+1}(X; \mathbb{Z}) \\
\downarrow & & \downarrow \\
H_*(X; \mathbb{k}) & \xrightarrow{\Delta} & H_{*+1}(X; \mathbb{k})
\end{array}
$$

where the vertical maps are induced by the ring homomorphism $\mathbb{Z} \to \mathbb{k}$.

The circle $S^1$ acts on the free loop space on $M$ by rotating the loops. Therefore we have a operator $\Delta$ on $\mathbb{H}_*(LM)$. Chas and Sullivan \[2\] have showed that $\mathbb{H}_*(LM)$ equipped with the loop product and the $\Delta$ operator, is a Batalin-Vilkovisky algebra.

**Definition 5.** A **Batalin-Vilkovisky algebra** is a commutative graded algebra $A$ equipped with an operator $\Delta : A \to A$ of degree 1 such that $\Delta \circ \Delta = 0$ and

\begin{enumerate}
  \item $\Delta(abc) = \Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{|a|-1}|b|b\Delta(ac) - (\Delta a)bc - (-1)^{|a|}a(\Delta b)c - (-1)^{|a|+|b|}ab(\Delta c)$.
\end{enumerate}

Consider the bracket $\{ , \}$ of degree $+1$ defined by

$$\{a, b\} = (-1)^{|a|} (\Delta(ab) - (\Delta a)b - (-1)^{|a|}a(\Delta b))$$

for any $a, b \in A$. \[6\] is equivalent to the following relation called the **Poisson relation**:

\begin{enumerate}
  \item $\{a, bc\} = \{a, b\}c + (-1)^{|a|+1}|b|b\{a, c\}$.
\end{enumerate}

Getzler \[14, Proposition 1.2\] has shown that the $\{ , \}$ is a Lie bracket and therefore that a Batalin-Vilkovisky algebra is a Gerstenhaber algebra.

**Definition 8.** A **Gerstenhaber algebra** is a commutative graded algebra $A$ equipped with a linear map $\{-, -\} : A \otimes A \to A$ of degree 1 such that:

a) the bracket $\{-, -\}$ gives $A$ a structure of graded Lie algebra of degree 1. This means that for each $a, b$ and $c \in A$
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\[ \{a, b\} = -(-1)^{|a|+1}|b|+1 \{b, a\} \quad \text{and} \]
\[ \{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{|a|+1}|b|+1 \{b, \{a, c\}\}. \]

b) the product and the Lie bracket satisfy the Poisson relation (\( \mathbb{Z} \)).

Using Lemma 3 and Lemma 4 we deduce

**Proposition 9.** The \( k \)-linear map

\[ H^*_a(LM; \mathbb{Z}) \otimes \mathbb{Z} k \hookrightarrow H^*_a(LM; k) \]

is an inclusion of Batalin-Vilkovisky algebras.

In particular, by the universal coefficient theorem,

\[ H^*_a(LM; \mathbb{Z}) \otimes \mathbb{Z} \mathbb{Q} \cong H^*_a(LM; \mathbb{Q}). \]

More generally, this Proposition tell us that if \( \text{Tor}^\mathbb{Z}_n(H^*_a(LM; \mathbb{Z}), k) = 0 \) then the Batalin-Vilkovisky algebra \( H^*_a(LM; \mathbb{Z}) \) determines the Batalin-Vilkovisky algebra \( H^*_a(LM; k) \).

3. THE CIRCLE AND AN USEFUL LEMMA.

In this section, we compute the structure of the Batalin-Vilkovisky algebra on the homology of the free loop space on the circle \( S^1 \) using a Lemma which gives information on the image of \( \Delta \) on elements of lower degree in \( H_*(LM) \).

**Theorem 10.** As Batalin-Vilkovisky algebras, the homology of the free loop space on the circle is given by

\[ H^*_a(LS^1; \mathbb{Z}) \cong \mathbb{Z} [Z] \otimes \Lambda a_{-1}. \]

Denote by \( x \) a generator of \( \mathbb{Z} \). The operator \( \Delta \) is

\[ \Delta(x^i \otimes a_{-1}) = i(x^i \otimes 1), \quad \Delta(x^i \otimes 1) = 0 \]

for all \( i \in \mathbb{Z} \).

Let \( X \) be a pointed topological space. Consider the free loop fibration \( \Omega X \hookrightarrow LX \xrightarrow{ev} X \). Denote by \( hur_X : \pi_n(X) \to H_n(X) \) the Hurewicz map.

**Lemma 11.** Let \( n \in \mathbb{N} \). Let \( f \in \pi_{n+1}(X) \). Denote by \( \tilde{f} \in \pi_n(\Omega X) \) the adjoint of \( f \). Then

\[ (H_a(ev) \circ \Delta \circ H_a(j) \circ hur_{\Omega X}) (\tilde{f}) = hur_X(f). \]
Proof. Take in homology the image of $[S^1] \otimes [S^n]$ in the following commutative diagram

$$
\begin{array}{ccc}
S^1 \times \Omega X & \xrightarrow{S^1 \times j} & S^1 \times LX \\
& \downarrow \quad \text{act}_{LX} \quad \downarrow ev & \quad \downarrow \quad \text{ev} \\
S^1 \times S^n & \xrightarrow{f} & S^1 \wedge S^n \\
\end{array}
$$

where $\text{act}_{LX} : S^1 \times LX \to LX$ is the action of the circle on $LX$. □

Proof of Theorem \[10\]. More generally, let $G$ be a compact Lie group. Consider the homeomorphism $\Theta_G : \Omega G \times G \xrightarrow{\sim} LG$ which sends the couple $(w, g)$ to the free loop $t \mapsto w(t)g$. In fact, $\Theta_G$ is an isomorphism of fiberwise monoids. Therefore by \[15, \text{part 2) of Theorem 8.2}\],

$$
\mathbb{H}^*_\ast(\Theta_G) : H^*_\ast(\Omega G) \otimes \mathbb{H}^*_\ast(G) \to \mathbb{H}^*_\ast(LG)
$$

is a morphism of graded algebras. Since $H^*_\ast(S^1)$ has no torsion,

$$
\mathbb{H}^*_\ast(\Theta_{S^1}) : H^*_\ast(\Omega S^1) \otimes \mathbb{H}^*_\ast(S^1) \cong \mathbb{H}^*_\ast(LS^1)
$$

is an isomorphism of algebras. Since $\Delta$ preserve path-connected components,

$$
\Delta(x^i \otimes a_{-1}) = \alpha(x^i \otimes 1)
$$

where $\alpha \in k$. Denote by $\varepsilon_{k[Z]}$ is the canonical augmentation of the group ring $k[Z]$. Since $H^*_\ast(\Theta_{S^1}) = \varepsilon_{k[Z]} \otimes H^*_\ast(S^1),$

$$
(H^*_\ast(ev) \circ \Delta)(x^i \otimes a_{-1}) = \alpha 1.
$$

On the other hand, applying Lemma \[11\] to the degree $i$ map $S^1 \to S^1$, we obtain that $(H^*_\ast(ev) \circ \Delta \circ H^*_\ast(j))(x^i) = i 1$. Therefore $\alpha = i$. □

4. Computations using Hochschild homology.

In this section, we compute the Batalin-Vilkovisky algebra $\mathbb{H}^*_\ast(LS^n)$, $n \geq 2$, using the following elementary technique:

The algebra structure has been computed by Cohen, Jones and Yan using the Serre spectral sequence \[8\]. On the other hand, the action of $H^*_\ast(S^1)$ on $H^*_\ast(LS^n)$ can be computed using Hochschild homology. Using the compatibility between the product and $\Delta$, we determine the Batalin-Vilkovisky algebra $\mathbb{H}^*_\ast(LS^n)$ up to isomorphisms. This elementary technique will fail for $\mathbb{H}^*_\ast(LS^2)$.

Let $A$ be an augmented differential graded algebra. Denote by $sA$ the suspension of the augmentation ideal $A$, $(sA)_i = A_{i-1}$. Let $d_1$ be the differential on the tensor product of complexes $A \otimes T(sA)$. The
Here of the Hochschild chain complex, denoted $C_*(A; A)$, is the complex $(A \otimes T(sA), d_1 + d_2)$ where
\[
d_2a[sa_1] \cdots [sa_k] = (-1)^{|a|} aa_1[sa_2] \cdots [sa_k]
\]
\[
+ \sum_{i=1}^{k-1} (-1)^{\varepsilon_i} a[sa_1] \cdots [sa_ia_{i+1}] \cdots [sa_k]
\]
\[- (-1)^{|sak|\varepsilon_{k-1}} a_k [sa_1] \cdots [sa_{k-1}];
\]
Here $\varepsilon_i = |a| + |sa_1| + \cdots + |sa_i|.$
Connes boundary map $B$ is the map of degree +1
\[
B : A \otimes (sA)^{\otimes p} \to A \otimes (sA)^{\otimes p+1}
\]
defined by
\[
B(a_0[sa_1] \cdots [sa_p]) = \sum_{i=0}^{p} (-1)^{|s_{a_0} \cdots s_{a_{i-1}}| + |s_{a_i} \cdots s_{a_p}|} [sa_i] \cdots [sa_p] [sa_0] \cdots [sa_{p-1}].
\]
Up to the isomorphism $s^p(A^{\otimes (p+1)}) \to A^{\otimes (sA)^{\otimes p}}$, $s^p(a_0[sa_1] \cdots [ap]) \mapsto (-1)^{|a_0|+|p-1||a_1|+\cdots+|a_{p-1}|} a_0[sa_1] \cdots [sa_p]$, our signs coincides with those of [29].

The Hochschild homology of $A$ (with coefficient in $A$) is the homology of the Hochschild chain complex:
\[
HH_*(A; A) := H_*(C_*(A; A)).
\]
The Hochschild cohomology of $A$ (with coefficient in $A^\vee$) is the homology of the dual of the Hochschild chain complex:
\[
HH^*(A; A^\vee) := H_*(C_*(A; A)^\vee).
\]
Consider the dual of Connes boundary map, $B^\vee(\varphi) = (-1)^{|\varphi|} \varphi \circ B$. On $HH^*(A; A^\vee)$, $B^\vee$ defines an action of $H_*(S^1)$.

Example 12. Let $n \geq 2$. Let $k$ be any commutative ring. Let $A := H^*(S^n) = \Lambda x_{-n}$ be the exterior algebra on a generator of lower degree $-n$. Denote by $[sx]^k := 1[sx] \cdots [sx]$ and $x[sx]^k := x[sx] \cdots [sx]$ the elements of $C_*(A; A)$ where the term $sx$ appears $k$ times. These elements form a basis of $C_*(A; A)$. Denote by $[sx]^{k^\vee}$, $x[sx]^{k^\vee}$, $k \geq 0$, the dual basis. The differential $d^\vee$ on $C_*(A; A)^\vee$ is given by $d^\vee([sx]^{k^\vee}) = 0$ and $d^\vee(x[sx]^{k^\vee}) = \pm (1 - (-1)^{k(n+1)}) [sx]^{(k+1)^\vee}$. The dual of Connes boundary map $B^\vee$ is given by
\[
B^\vee([sx]^{k^\vee}) = \begin{cases} 
(-1)^{n+1} k x[sx]^{(k-1)^\vee} & \text{if } (k+1)(n+1) \text{ is even,} \\
0 & \text{if } (k+1)(n+1) \text{ is odd}
\end{cases}
\]
Theorem 17. \( B' x [sx]^{k' v} = 0 \). We remark that \([sx]^{k' v}\) is of (lower) degree \( k(n - 1) \) and \( x [sx]^{k' v}\) of degree \( n + k(n - 1) \).

Theorem 13. \( \text{Let } X \text{ be a simply connected space such that } H_*(X; k) \) is of finite type in each degree. Then there is a natural isomorphism of \( H_*(S^1)\)-modules between the homology of the free loop space on \( X \) and the Hochschild cohomology of the algebra of singular cochain \( S^*(X; k) \): \( (14) \quad H_*(LX) \cong HH^*(S^*(X; k); S^*(X; k)^{\vee}) \).

In this paper, when we will apply this theorem, \( H_*(X; k) \) is assumed to be \( k\)-free of finite type in each degree and \( X \) will be always \( k\)-formal: the algebra \( S^*(X; k) \) will be linked by quasi-isomorphisms of cochain algebras to \( H_*(X; k) \). Therefore \( (15) \quad HH^*(S^*(X; k); S^*(X; k)^{\vee}) \cong HH^*(H^*(X; k); H^*(X; k)^{\vee}) \).

Theorem 16. For \( n > 1 \) odd, as Batalin-Vilkovisky algebras,
\[
\mathbb{H}_*(LS^n; \mathbb{Z}) = \mathbb{k}[u_{n-1}] \otimes \Lambda a_{-n},
\]
\[
\Delta(u_{n-1} \otimes a_{-n}) = i(u_{n-1}^{i-1} \otimes 1),
\]
\[
\Delta(u_{n-1}^{i-1} \otimes 1) = 0.
\]

Proof. As algebras, Cohen, Jones and Yan [8] proved that \( \mathbb{H}_*(LS^n; \mathbb{Z}) = \mathbb{k}[u_{n-1}] \otimes \Lambda a_{-n} \) when \( k = \mathbb{Z} \). Their proof works over any \( k \) (alternatively, using Proposition 9 we could assume that \( k = \mathbb{Z} \)). Computing Connes boundary map on \( HH^*(H^*(S^n); H_*(S^n)) \) (Example 12), we see that \( \Delta \) on \( \mathbb{H}_*(LS^n; \mathbb{k}) \) is null in even degree and in degree \(-n\), and is an isomorphism in degree \(-1\). Therefore \( \Delta(u_{n-1}^{i-1} \otimes 1) = 0, \Delta(1 \otimes a_{-n}) = 0 \) and \( \Delta(u_{n-1} \otimes a_{-n}) = a_1 \) where \( a \) is invertible in \( \mathbb{k} \). Replacing \( a_{-n} \) by \( \frac{1}{a} a_{-n} \) or \( u_{n-1} \) by \( \frac{1}{a} u_{n-1} \), we can assume up to isomorphisms that \( \Delta(u_{n-1} \otimes a_{-n}) = 1 \). Therefore \( \{u_{n-1}, a_{-n}\} = 1 \). Using the Poisson relation (7), \( \{u_{n-1}^{i-1}, a_{-n}\} = i u_{n-1}^{i-1} \). Therefore \( \Delta(u_{n-1}^{i-1} \otimes a_{-n}) = i (u_{n-1}^{i-1} \otimes 1) \).

Theorem 17. For \( n \geq 2 \) even, there exists a constant \( \varepsilon_0 \in \mathbb{F}_2 \) such that as Batalin-Vilkovisky algebra,
\[
\mathbb{H}_*(LS^n; \mathbb{Z}) = \Lambda b \otimes \frac{\mathbb{Z}[a, v]}{(a^2, ab, 2av)}
\]
\[
= \bigoplus_{k=0}^{+\infty} \mathbb{Z} v_{2(n-1)}^{k} \oplus \bigoplus_{k=0}^{+\infty} \mathbb{Z} b_{-1} v^{k} \oplus \mathbb{Z} a_{-n} \oplus \bigoplus_{k=1}^{+\infty} \mathbb{Z} 2^{n} a^{k}
\]
with \( \forall k \geq 0, \Delta(v^k) = 0, \Delta(ab^k) = 0 \) and
\[
\Delta(bv^k) = \begin{cases} 
(2k + 1)v^k + \varepsilon_0 av^{k+1} & \text{if } n = 2 \\
(2k + 1)v^k & \text{if } n \geq 4.
\end{cases}
\]
Proof. As algebras, Cohen, Jones and Yan [8] proved the equality. Computing Connes boundary map on $HH^*(H^*(S^n); H^*(S^n))$ (Example 12), we see that $\Delta$ on $H^*(LS^n; k)$ is null in even degree and is injective in odd degree.

Case $n \neq 2$: this case is simple, since all the generators of $H^*(LS^n)$, $v^k$, $bv^k$ and $av^k$, $k \geq 0$, have different degrees. Using Example 12 we also see that for all $k \geq 0$,

$$\Delta : H_{−1+2k(n−1)} = \mathbb{Z}b_{−1}v^k \hookrightarrow H_{2k(n−1)} = \mathbb{Z}v^k$$

has cokernel isomorphic to $\mathbb{Z}/(2k+1)\mathbb{Z}$. Therefore $\Delta(bv^k) = \pm(2k+1)v^k$. By replacing $b_{−1}$ by $−b_{−1}$, we can assume up to isomorphims that $\Delta(b) = 1$. Let $k \geq 1$. Let $\alpha_k \in \{-2k−1, 2k+1\}$ such that $\Delta(bv^k) = \alpha_kv^k$. Using formula (6), we obtain that $\Delta(bv^k,v^l) = (2\alpha_k−1)v^k$. We know that $\Delta(bv^{2k}) = \pm(4k+1)v^{2k}$. Therefore $\alpha_k$ must be equal to $2k+1$.

Case $n = 2$: this case is complicated, since for $k \geq 0$, $v^k$ and $av^{k+1}$ have the same degree. Using Example 12, we also see that

$$\Delta : H_{−1+2k} = \mathbb{Z}b_{−1}v^k \hookrightarrow H_{2k} = \mathbb{Z}v^k \oplus \mathbb{Z}/2\mathbb{Z}av^{k+1}$$

has cokernel, denoted Coker$\Delta$, isomorphic to $\mathbb{Z}/(2k+1)\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. There exists unique $\alpha_k \in \mathbb{Z}^*$ and $\varepsilon_k \in \mathbb{Z}/2\mathbb{Z}$ such that $\Delta(bv^k) = \alpha_kv^k + \varepsilon_kav^{k+1}$. The injective map $\Delta$ fits into the commutative diagram of short exact sequences (Noether’s Lemma)

$$
\begin{array}{cccccc}
0 & \rightarrow & H_{−1+2k} & \overset{id}{\rightarrow} & H_{−1+2k} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H_{−1+2k} & \overset{\times 2}{\rightarrow} & H_{−1+2k} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & H_{−1+2k} & \overset{\Delta}{\rightarrow} & H_{2k} & \rightarrow & \text{Coker}\Delta \\
\downarrow & & \downarrow & & \downarrow & \cong & \downarrow \\
0 & \rightarrow & \mathbb{Z}/2\mathbb{Z} & \overset{\Delta}{\rightarrow} & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \rightarrow & \text{Coker}\Delta \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}
$$

The cokernel of $\Delta$, denoted Coker$\Delta$ is of cardinal $2|\alpha_k|$. So $|\alpha_k| = 2k+1$. Therefore $\Delta(bv^k) = \pm(2k+1)v^k + \varepsilon_kav^{k+1}$.

By replacing $b_{−1}$ by $−b_{−1}$, we can assume up to isomorphims that $\Delta(b) = 1 + \varepsilon_0av$. Using formula (6), we obtain that

$$\Delta(bv^k,v^l) = (\alpha_k + \alpha_l − 1)v^{k+l} + (\varepsilon_k + \varepsilon_l − \varepsilon_0)av^{k+l+1}.$$
Therefore
\[ \Delta(bv^kv^l) = (2\alpha_k - 1)v^{2k} + \varepsilon_0av^{2k+1} = \pm(4k + 1)v^{2k} + \varepsilon_{2k}av^{2k+1}. \]
So \( \alpha_k = 2k + 1, \varepsilon_{2k} = \varepsilon_0 \) and \( \varepsilon_{2k+1} = \varepsilon_{2k} + \varepsilon_1 - \varepsilon_0 = \varepsilon_1. \)

The map \( \Theta : \mathbb{H}_*(LS^2) \to \mathbb{H}_*(LS^2) \) given by \( \Theta(b_{-1}v^k) = b_{-1}v^k, \Theta(v^k) = v^k + kav^{k+1}, \Theta(av^k) = av^k, k \geq 0 \) is an involutive isomorphism of algebras. Therefore, by replacing \( v \) by \( v + av^2 \), we can assume that \( \varepsilon_1 = \varepsilon_0 \).

So we have proved
\[ \Delta(bv^k) = (2k + 1)v^k + \varepsilon_0av^{k+1}, \quad k \geq 0. \]

These two cases \( \varepsilon_0 = 0 \) and \( \varepsilon_0 = 1 \) correspond to two non-isomorphic Batalin-Vilkovisky algebras whose underlying Gerstenhaber algebras are the same. Therefore even if we have not yet computed the Batalin-Vilkovisky algebra \( \mathbb{H}_*(LS^2; \mathbb{Z}) \), we have computed its underlying Gerstenhaber algebra. Using the definition of the bracket, straightforward computations give the following corollary.

**Corollary 18.** For \( n \geq 2 \) even, as Gerstenhaber algebra
\[ \mathbb{H}_*(LS^n; \mathbb{Z}) = \Lambda b_{-1} \otimes \mathbb{Z}[a_{-n}, v_{2(n-1)}] / \langle a^2, ab, 2av \rangle \]
with \( \{ v^k, v^l \} = 0, \{ bv^k, v^l \} = -2lv^{k+l}, \{ bv^k, bv^l \} = 2(k - l)bv^{k+l}, \{ a, v^l \} = 0, \{ av^k, bv^l \} = -(2l + 1)av^{k+l} \) and \( \{ av^k, av^l \} = 0 \) for all \( k, l \geq 0 \).

5. When Hochschild cohomology is a Batalin-Vilkovisky algebra

In this section, we recall the structure of Gerstenhaber algebra on the Hochschild cohomology of an algebra whose degrees are bounded. We recall from [26, 22, 27, 19] the Batalin-Vilkovisky algebra on the Hochschild cohomology of the cohomology \( H^*(M) \) of a closed oriented manifold \( M \). We compute this Batalin-Vilkovisky algebra \( HH^*(H^*(M); H^*(M)) \) when \( M \) is a sphere.

Through this section, we will work over the prime field \( \mathbb{F}_2 \). Let \( A \) be an augmented graded algebra such that the augmentation ideal \( \overline{A} \) is concentrated in degree \( \leq -2 \) and bounded below (or concentrated in degree \( \geq 0 \) and bounded above). Then the (normalized) Hochschild cochain complex, denoted \( C^*(A, A) \), is the complex
\[ \text{Hom}(Ts\overline{A}, A) \cong \oplus_{p \geq 0} \text{Hom}((s\overline{A})^p, A) \]
with a differential $d_2$. For $f \in \text{Hom}((sA)^{\otimes p}, A)$, the differential $d_2 f \in \text{Hom}((sA)^{\otimes p+1}, A)$ is given by

$$(d_2 f)([sa_1|\cdots|sa_{p+1}]) := a_1 f([sa_2|\cdots|sa_{p+1}])$$

$$+ \sum_{i=1}^{p} f([sa_1|\cdots|s(a_i a_{i+1})|\cdots|sa_{p+1}]) + f([sa_1|\cdots|sa_p])a_p$$

The Hochschild cohomology of $A$ with coefficient in $A$ is the homology of the Hochschild cochain complex:

$$HH^\ast(A; A) := H_\ast(C^\ast(A; A))$$

We remark that $HH^\ast(A; A)$ is bigraded. Our degree is sometimes called the total degree: sum of the external degree and the internal degree. The Hochschild cochain complex $C^\ast(A, A)$ is a differential graded algebra. For $f \in \text{Hom}((sA)^{\otimes p}, A)$ and $g \in \text{Hom}((sA)^{\otimes q}, A)$, the (cup) product of $f$ and $g$, $f \cup g \in \text{Hom}((sA)^{\otimes p+q}, A)$ is defined by

$$(f \cup g)([sa_1|\cdots|sa_{p+q}]) := f([sa_1|\cdots|sa_p])g([sa_{p+1}|\cdots|sa_{p+q}]).$$

The Hochschild cochain complex $C^\ast(A, A)$ has also a Lie bracket of (lower) degree $+1$.

$$(f \circ g)([sa_1|\cdots|sa_{p+q-1}]) :=$$

$$\sum_{i=1}^{p} f([sa_1|\cdots|sa_{i-1}|sg([sa_{i}|\cdots|sa_{i+q-1}])sa_{i+q}|\cdots|sa_{p+q-1}]).$$

{f, g} = f \circ g - g \circ f. Our formulas are the same as in the non graded case [13]. We remark that if $A$ is not assumed to be bounded, the formulas are more complicated. Gerstenhaber has showed that $HH^\ast(A; A)$ equipped with the cup product and the Lie bracket is a Gerstenhaber algebra.

Let $M$ be a closed $d$-dimensional smooth manifold. Poincaré duality induces an isomorphism of $H^\ast(M; \mathbb{F}_2)$-modules of (lower) degree $d$.

$$(19) \quad \Theta : H^\ast(M; \mathbb{F}_2) \cong [M]^\vee \rightarrow H_\ast(M; \mathbb{F}_2) \cong H^\ast(M; \mathbb{F}_2)^\vee.$$ 

More generally, let $A$ be a graded algebra equipped with an isomorphism of $A$-bimodules of degree $d$, $\Theta : A \cong [A] \rightarrow A^\vee$. Then we have the isomorphism

$$HH^\ast(A, \Theta) : HH^\ast(A, A) \cong [A] \rightarrow HH^\ast(A, A^\vee).$$

Therefore on $HH^\ast(A, A)$, we have both a Gerstenhaber algebra structure and an operator $\Delta$ given by the dual of Connes boundary map $B$. 
Motivated by the Batalin-Vilkovisky algebra structure of Chas-Sullivan on $\mathbb{H}_*(LM)$, Thomas Tradler [26] proved that $HH^*(A,A)$ is a Batalin-Vilkovisky algebra. See [22] Theorem 1.6 for an explicit proof. In [19] or [27] Corollary 3.4] or [9] Section 1.4] or [20] Theorem B] or [21] Section 11.6], this Batalin-Vilkovisky algebra structure on $HH^*(A,A)$ extends to a structure of algebra on the Hochschild cochain complex $C^*(A,A)$ over various operads or PROPs: the so-called cyclic Deligne conjecture. Let us compute this Batalin-Vilkovisky algebra structure when $M$ is a sphere.

**Proposition 20.** ([30] and [31] Corollary 4.2) Let $d \geq 2$. As Batalin-Vilkovisky algebra, the Hochschild cohomology of $H^*(S^d;F_2) = \Lambda x_{-d}$, 

$$HH^*(H^*(S^d;F_2); H^*(S^d;F_2)) \cong \Lambda g_{-d} \otimes F_2[f_{d-1}]$$

with $\Delta(g_{-d} \otimes f_{d-1}^k) = k(1 \otimes f_{d-1}^{k-1})$ and $\Delta(1 \otimes f_{d-1}^k) = 0, k \geq 0$. In particular, the underlying Gerstenhaber algebra is given by $\{f^k, f^l\} = 0$, $\{gf^k, g^l\} = (k-l)gf^{k+l-1}$ for $k, l \geq 0$.

**Proof.** Denote by $A := H^*(S^d;F_2)$. The differential on $C^*(A;A)$ is null. Let $f \in \text{Hom}(sA, A) \subset C^*(A;A)$ such that $f([sx]) = 1$. Let $g \in \text{Hom}(F_2, A) = \text{Hom}((sA)^{\otimes 0}, A) \subset C^*(A;A)$ such that $g([]) = x$. The $k$-th power of $f$ is the map $f^k \in \text{Hom}((sA)^{\otimes k}, A)$ such that $f^k([sx] \cdots [sx]) = 1$. The cup product $g \cup f^k \in \text{Hom}((sA)^{\otimes k}, A)$ sends $[sx] \cdots [sx]$ to $x$. So we have proved that $C^*(A;A)$ is isomorphic to the tensor product of graded algebras $\Lambda g_{-d} \otimes F_2[f_{d-1}]$.

The unit 1 and $x_{-d}$ form a linear basis of $H^*(S^d)$. Denote by $1^\vee$ and $x^\vee$ the dual basis of $A^\vee = H^*(S^d)^\vee$. Poincaré duality induces the isomorphism $\Theta : H^*(S^d) \xrightarrow{\cong} H^*(S^d)^\vee$, $1 \mapsto x^\vee$ and $x \mapsto 1^\vee$. The two families of elements of the form $1[sx] \cdots [sx]$ and of the form $x[sx] \cdots [sx]$ form a basis of $C_*(A;A)$. Denote by $1[sx] \cdots [sx]^\vee$ and $x[sx] \cdots [sx]^\vee$ the dual basis in $C_*(A;A)^\vee$. The isomorphism $\Theta$ induces an isomorphism of complexes of degree $d$, $\widehat{\Theta} : C^*(A;A) \xrightarrow{\cong} C^*(A;A)^\vee$. Explicitly [22] Section 4] this isomorphism sends $f \in \text{Hom}((sA)^{\otimes p}, A)$ to the linear map $\widehat{\Theta}(f) \in (A \otimes (sA)^{\otimes p})^\vee \subset C_*(A;A)^\vee$ defined by 

$$\widehat{\Theta}(f)(a_0 [sa_1] \cdots [sa_p]) = ((\Theta \circ f)[sa_1] \cdots [sa_p])(a_0).$$

Here with $A = \Lambda x, \widehat{\Theta}(f^k) = x[x] \cdots [sx]^\vee$ and $\widehat{\Theta}(g \cup f^k) = 1[sx] \cdots [sx]^\vee$. Computing Connes boundary map $B^\vee$ on $C_*(A;A)^\vee$ (Example [12]) and using that by definition of $\Delta$, $\widehat{\Theta} \circ \Delta = B^\vee \circ \widehat{\Theta}$, we obtain the desired formula for $\Delta$. \qed
6. The Gerstenhaber Algebra $\mathbb{H}_*(LS^2; \mathbb{F}_2)$

Using the same Hochschild homology technique as in section 4, we compute up to an indeterminacy, the Batalin-Vilkovisky algebra $\mathbb{H}_*(LS^2; \mathbb{F}_2)$. Nevertheless, this will give the complete description of the underlying Gerstenhaber algebra on $\mathbb{H}_*(LS^2; \mathbb{F}_2)$.

Lemma 21. There exist a constant $\varepsilon \in \{0, 1\}$ such that as Batalin-Vilkovisky algebra, the homology of the free loop on the sphere $S^2$ is

$$\mathbb{H}_*(LS^2; \mathbb{F}_2) = \Lambda a_{-2} \otimes \mathbb{F}_2[u_1],$$

$$\Delta(a_{-2} \otimes u_1^k) = k(1 \otimes u_1^{k-1} + \varepsilon a_{-2} \otimes u_1^{k+1}) \quad \text{and} \quad \Delta(1 \otimes u_1^k) = 0, \quad k \geq 0.$$

Proof. In [8], Cohen, Jones and Yan proved that the Serre spectral sequence for the free loop fibration $\Omega M \xrightarrow{\iota} LM \xrightarrow{ev} M$ is a spectral sequence of algebras converging toward the algebra $\mathbb{H}_*(LM)$. Using Hochschild homology, we see that there is an isomorphism of vector spaces $\mathbb{H}_*(LS^2; \mathbb{F}_2) \cong \mathbb{H}_*(S^2; \mathbb{F}_2) \otimes H_*(\Omega S^2; \mathbb{F}_2)$. Therefore the Serre spectral sequence collapses. Since there is no extension problem, we have the isomorphism of algebras

$$\mathbb{H}_*(LS^2; \mathbb{F}_2) \cong \mathbb{H}_*(S^2; \mathbb{F}_2) \otimes H_*(\Omega S^2; \mathbb{F}_2) = \Lambda(a_{-2}) \otimes \mathbb{F}_2[u_1].$$

Computing Connes boundary map on $HH^*(H^*(S^2; \mathbb{F}_2); H_*(S^2; \mathbb{F}_2))$ (Example 12), we see that $\Delta$ on $\mathbb{H}_*(LS^2; \mathbb{F}_2)$ is null in even degree and that $\Delta : \mathbb{H}_{2k-1} \to \mathbb{H}_{2k}$ is a linear map of rank 1, $k \geq 0$. In particular $\Delta$ is injective in degree $-1$.

Applying Lemma 11 to the identity map $id : S^2 \to S^2$, we see that the composite

$$H_1(\Omega S^2; \mathbb{F}_2) \xrightarrow{H_1(id; \mathbb{F}_2)} H_1(LS^2; \mathbb{F}_2) \xrightarrow{\Delta} H_2(LS^2; \mathbb{F}_2) \xrightarrow{H_2(ev; \mathbb{F}_2)} H_2(S^2; \mathbb{F}_2)$$

is non zero. Since $\mathbb{H}_*(ev)$ is a morphism of algebras, $\mathbb{H}_0(ev)(a_{-2} u_1^2) = 0$. And so $\Delta(a_{-2} u_1) = 1 + \varepsilon a_{-2} u_1^2$ with $\varepsilon \in \mathbb{F}_2$.

We remark that when $b = c$, formula (6) takes the simple form

$$(22) \quad \Delta(ab^2) = \Delta(a)b^2 + a\Delta(b^2).$$

Using this formula, we obtain that

$$\Delta(a_{-2} u_1^{2k+1}) = \Delta((a_{-2} u_1) u_1^{k+2}) = u_1^{2k} + \varepsilon a_{-2} u_1^{2k+2} \quad k \geq 0.$$

Since $\Delta : \mathbb{F}_2 a_{-2} u_1^2 \otimes \mathbb{F}_2 u_1 \to \mathbb{H}_2$ is of rank 1 and $\Delta(a_{-2} u_1^2) \neq 0$, $\Delta(u_1) = \lambda \Delta(a_{-2} u_1^2)$ with $\lambda = 0$ or $\lambda = 1$. Using again formula (22), we have that

$$\Delta(u_1^{2k+1}) = \Delta(u_1 u_1^{k+2}) = \lambda \Delta(a_{-2} u_1^3) u_1^{2k} = \Delta(a_{-2} u_1^{2k+3}), \quad k \geq 0.$$
So finally
\[\Delta(a_{-2}u_1^k) = ku_1^{k-1} + \varepsilon k a_{-2}u_1^{k+1} \quad \text{and} \quad \Delta(u_1^k) = \lambda \Delta(a_{-2}u_1^{k+2}), \quad k \geq 0.\]
The cases \(\lambda = 0\) and \(\lambda = 1\) correspond to isomorphic Batalin-Vilkovisky algebras: Let \(\Theta : \mathbb{H}_*(LS^2; \mathbb{F}_2) \to \mathbb{H}_*(LS^2; \mathbb{F}_2)\) be an automorphism of algebras which is not the identity. Since \(\Theta(a_{-2}) \neq 0\) and \(\Theta(a_{-2}) = a_{-2}\). Since \(\Theta(a_{-2}) \) and \(\Theta(u_1)\) must generate the algebra \(\Lambda a_{-2} \otimes \mathbb{F}_2[u_1]\), \(\Theta(u_1) \neq a_{-2}u_1^3\). Therefore there is an unique automorphism of algebras \(\Theta : \mathbb{H}_*(LS^2; \mathbb{F}_2) \to \mathbb{H}_*(LS^2; \mathbb{F}_2)\) which is not the identity. Explicitly, \(\Theta\) is given by \(\Theta(u_1^k) = u_1^k + k a_{-2}u_1^{k+2}\), \(\Theta(a_{-2}u_1^k) = a_{-2}u_1^k\), \(k \geq 0\). One can check that \(\Theta\) is an involutive isomorphism of Batalin-Vilkovisky algebras who transforms the cases \(\lambda = 0\) into the cases \(\lambda = 1\) without changing \(\varepsilon\). Therefore, by replacing \(u_1\) by \(u_1 + a_{-2}u_1^3\), we can assume that \(\lambda = 0\).

Consider the four Batalin-Vilkovisky algebras \(\Lambda a_{-2} \otimes \mathbb{F}_2[u_1]\) with \(\Delta(a_{-2} \otimes u_1^k) = k(1 \otimes u_1^{k-1} + \varepsilon a_{-2} \otimes u_1^{k+1})\), \(\Delta(1 \otimes u_1^k) = \lambda \Delta(a_{-2}u_1^{k+2}),\) \(k \geq 0\), given the different values of \(\varepsilon, \lambda \in \{0, 1\}\). These four Batalin-Vilkovisky algebras have only two underlying Gerstenhaber algebras given by \(\{u_1^k \} = 0\), \(\{a_{-2}u_1^k, u_1^l\} = l u_1^{k+l-1} + \lambda(\varepsilon - \lambda) a_{-2}u_1^{k+l+1}\) and \(\{a_{-2}u_1^k, a_{-2}u_1^l\} = (k - l)a_{-2}u_1^{k+l-1}\) for \(k, l \geq 0\). Via the above isomorphism \(\Theta\), these two Gerstenhaber algebras are isomorphic.

**Corollary 23.** The free loop space modulo 2 homology \(\mathbb{H}_*(LS^2; \mathbb{F}_2)\) is isomorphic as Gerstenhaber algebra to the Hochschild cohomology of \(H^*(S^2; \mathbb{F}_2), HH^*(H^*(S^2; \mathbb{F}_2); H^*(S^2; \mathbb{F}_2))\).

7. **The Batalin-Vilkovisky algebra \(\mathbb{H}_*(LS^2)\)**

In this section, we complete the calculations of the Batalin-Vilkovisky algebras \(\mathbb{H}_*(LS^2; \mathbb{F}_2)\) and \(\mathbb{H}_*(LS^2; \mathbb{Z})\) started respectively in sections 6 and 7 using a purely homotopic method.

**Theorem 24.** As Batalin-Vilkovisky algebra, the homology of the free loop space on the sphere \(S^2\) with mod 2 coefficients is
\[
\mathbb{H}_*(LS^2; \mathbb{F}_2) = \Lambda a_{-2} \otimes \mathbb{F}_2[u_1],
\]
\(\Delta(a_{-2} \otimes u_1^k) = k(1 \otimes u_1^{k-1} + a_{-2} \otimes u_1^{k+1})\) and \(\Delta(1 \otimes u_1^k) = 0, k \geq 0.\)

**Theorem 25.** With integer coefficients, as Batalin-Vilkovisky algebra,
\[
\mathbb{H}_*(LS^2; \mathbb{Z}) = \Lambda b \otimes \frac{\mathbb{Z}[a, v]}{(a^2, ab, 2av)}
\]
\[= \bigoplus_{k=0}^{+\infty} \mathbb{Z}v_2^k \oplus \bigoplus_{k=0}^{+\infty} \mathbb{Z}b_{-1}v^k \oplus \mathbb{Z}a_{-2} \oplus \bigoplus_{k=1}^{+\infty} \mathbb{Z}2av^k\]
with \( \forall k \geq 0, \Delta(v^k) = 0, \Delta(av^k) = 0 \) and \( \Delta(bv^k) = (2k + 1)v^k + av^{k+1} \).

Denote by \( s : X \hookrightarrow LX \) the trivial section of the evaluation map \( ev : LX \twoheadrightarrow X \).

**Lemma 26.** The image of \( \Delta : H_1(LS^2; \mathbb{F}_2) \to H_2(LS^2; \mathbb{F}_2) \) is not contained in the image of \( H_2(s; \mathbb{F}_2) : H_2(S^2; \mathbb{F}_2) \hookrightarrow H_2(LS^2; \mathbb{F}_2) \).

**Lemma 27.** The image of \( \Delta : H_1(LS^2; \mathbb{Z}) \to H_2(LS^2; \mathbb{Z}) \) is not contained in the image of \( H_2(s; \mathbb{Z}) : H_2(S^2; \mathbb{Z}) \hookrightarrow H_2(LS^2; \mathbb{Z}) \).

**Proof of Lemma 27 assuming Lemma 26.** Consider the commutative diagram

\[
\begin{array}{ccc}
H_1(LS^2; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2 & \cong & H_1(LS^2; \mathbb{F}_2) \\
\Delta \otimes_{\mathbb{Z}} \mathbb{F}_2 & \cong & \Delta \\
H_2(LS^2; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2 & \cong & H_2(LS^2; \mathbb{F}_2) \\
H_2(s; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2 & \cong & H_2(s; \mathbb{F}_2) \\
H_2(S^2; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2 & \cong & H_2(S^2; \mathbb{F}_2)
\end{array}
\]

Since \( H_1(LS^2; \mathbb{Z}) \cong H_0(LS^2; \mathbb{Z}) \cong \mathbb{Z} \), the horizontal arrows are isomorphisms by the universal coefficient theorem. The top rectangle commutes according Lemma 4.

Suppose that the image of \( \Delta : H_1(LS^2; \mathbb{Z}) \to H_2(LS^2; \mathbb{Z}) \) is included in the image of \( H_2(s; \mathbb{Z}) \). Then the image of \( \Delta \otimes_{\mathbb{Z}} \mathbb{F}_2 \) is included in the image of \( H_2(s; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2 \). Using the above diagram, the image of \( \Delta : H_1(LS^2; \mathbb{F}_2) \to H_2(LS^2; \mathbb{F}_2) \) is included in the image of \( H_2(s; \mathbb{F}_2) \). This contradicts Lemma 26. \( \square \)

**Proof of Theorem 25 assuming Lemma 27.** It suffices to show that the constant \( \varepsilon \) in Lemma 21 is not zero. Suppose that \( \varepsilon = 0 \). Then by Lemma 21 \( \Delta(a_{-2} \otimes u_1) = 1 \).

It is well known that \( H_*(s) : H_*(M) \to H_*(LM) \) is a morphism of algebras. In particular, let \( [S^2] \) be the fundamental class of \( S^2 \), \( H_2(s)([S^2]) \) is the unit of \( H_*(LS^2) \). So \( \Delta(a_{-2} \otimes u_1) = H_2(s)([S^2]) \). This contradicts Lemma 26. \( \square \)

The proof of Theorem 25 assuming Lemma 27 is the same. To complete the computation of this Batalin-Vilkovisky algebra on the homology of the free loop space of a manifold, we will relate it to another structure of Batalin-Vilkovisky algebra that arises in algebraic topology: the homology of the double loop space.

Let \( X \) be a pointed topological space. The circle \( S^1 \) acts on the sphere \( S^2 \) by “rotating the earth”. Therefore the circle also acts on...
We view $\Omega^2 X = \text{map}((S^2, \text{North pole}), (X, *))$. So we have an induced operator $\Delta : H_*(\Omega^2 X) \to H_{*-1}(\Omega^2 X)$. With Theorem 32 and the following Proposition, we will able to prove Lemma 26.

**Proposition 28.** Let $X$ be a pointed topological space. There is a natural morphism $r : L\Omega X \to \text{map}_*(S^2, X)$ of $S^1$-spaces between the free loop on the pointed loop of $X$ and the double pointed loop space of $X$ such that:

- If we identify $S^2$ and $S^1 \wedge S^1$, $r$ is a retract up to homotopy of the inclusion $j : \Omega(\Omega X) \hookrightarrow L(\Omega X)$,
- The composite $r \circ s : \Omega X \leftrightarrow L(\Omega X) \to \text{map}_*(S^2, X)$ is homotopically trivial.

**Proof.** Let $\sigma : S^2 \to \frac{S^1 \times S^1}{S^1 \times \ast} = S^1 \wedge S^1$ be the quotient map that identifies the North pole and the South pole on the earth $S^2$. The circle $S^1$ acts without moving the based point on $S^1 \wedge S^1$ by multiplication on the first factor. On the torus $S^1 \times S^1$, the circle can act by multiplication on both factors. But when you pinch a circle to a point in the torus, the circle can act only on one factor. If we make a picture, we easily see that $\sigma : S^2 \to S^1 \wedge S^1$ is compatible with the actions of $S^1$. Therefore $r := \text{map}_*(\sigma, X) : L\Omega X \to \text{map}_*(S^2, X)$ is a morphism of $S^1$-spaces.

- Let $\pi : S^1_+ \wedge S^1 \to S^1 \wedge S^1 = \frac{S^1 \wedge S^1}{S^1 \times \ast}$ be the quotient map. The inclusion map $j : \Omega(\Omega X) \to L(\Omega X)$ is $\text{map}_*(\pi, X)$. The composite $\pi \circ \sigma : S^2 \to S^1 \wedge S^1$ is the quotient map obtained by identifying a meridian to a point in the sphere $S^2$. The composite $\pi \circ \sigma$ can also be viewed as the quotient map from the non reduced suspension of $S^1$ to the reduced suspension of $S^1$. So the composite $\pi \circ \sigma : S^2 \to S^1 \wedge S^1$ is a homotopy equivalence. Let $\Theta : S^1 \wedge S^1 \xrightarrow{\sim} S^2$ be any given homeomorphism. The composite $\Theta \circ \pi \circ \sigma : S^2 \to S^2$ is of degree $\pm 1$. The reflection through the equatorial plane is a morphism of $S^1$-spaces. By replacing eventually $\sigma$ by its composite with the previous reflection, we can suppose that $\Theta \circ \pi \circ \sigma : S^2 \to S^2$ is homotopic to the identity map of $S^2$, i.e. $\sigma \circ \Theta$ is a section of $\pi$ up to homotopy. Therefore $\text{map}_*(\sigma \circ \Theta, X) = \text{map}_*(\Theta, X) \circ r$ is a retract of $j$ up to homotopy.

- Let $\rho : S^1_+ \wedge S^1 \xrightarrow{S^1 \times \ast} S^1$ be the map induced by the projection on the second factor. Since $\pi_2(S^1) = \ast$, the composite $\rho \circ \sigma$ is homotopically trivial. Therefore $r \circ s$, the composite of $r = \text{map}_*(\sigma, X)$ and $s = \text{map}_*(\rho, X) : \Omega X \to L(\Omega X)$ is also homotopically trivial. $\square$

**Proof of Lemma 26.** Denote by $ad_{S^n} : S^n \to \Omega S^{n+1}$ the adjoint of the identity map $id : S^{n+1} \to S^{n+1}$. The map $L(ad_{S^2}) : LS^2 \to L\Omega S^3$ is obviously a morphism of $S^1$-spaces. Therefore using Proposition 28...
the composite \( r \circ L(ad_{S^2}) : LS^2 \rightarrow L\Omega S^3 \rightarrow \Omega^2 S^3 \) is also a morphism of \( S^1 \)-spaces. Therefore \( H_*(r \circ L(ad_{S^2})) \) commutes with the corresponding operators \( \Delta \) in \( H_*(LS^2) \) and \( H_*(\Omega^2 S^3) \).

Consider the commutative diagram up to homotopy

\[
\begin{array}{ccc}
\Omega S^2 & \xrightarrow{j} & LS^2 \\
\Omega(ad_{S^2}) & \xrightarrow{L(ad_{S^2})} & S^2 \\
\Omega^2 S^3 & \xrightarrow{j} & L\Omega S^3 \\
\Omega^2 S^3 & \xrightarrow{id} & \Omega^2 S^3 \\
\end{array}
\]

Using the left part of this diagram, we see that \( \pi_1(r \circ L(ad)) \) maps the generator of \( \pi_1(LS^2) = \mathbb{Z}(j \circ ad_{S^1}) \) to the composite \( \Omega(ad_{S^2}) \circ ad_{S^1} : S^1 \rightarrow \Omega S^2 \rightarrow \Omega^2 S^3 \) which is the generator of \( \pi_1(\Omega^2 S^3) \cong \mathbb{Z} \). Therefore \( \pi_1(r \circ L(ad)) \) is an isomorphism.

So we have the commutative diagram

\[
\begin{array}{ccc}
\pi_1(LS^2) \otimes \mathbb{F}_2 & \xrightarrow{hur} & H_1(LS^2; \mathbb{F}_2) \\
\pi_1(r \circ L(ad_{S^2}) \otimes \mathbb{F}_2) & \cong & H_1(r \circ L(ad_{S^2}); \mathbb{F}_2) \\
\pi_1(\Omega^2 S^3) \otimes \mathbb{F}_2 & \xrightarrow{hur} & H_1(\Omega^2 S^3; \mathbb{F}_2) \\
\end{array}
\]

By Theorem 32 \( \Delta : H_1(\Omega^2 S^3; \mathbb{F}_2) \rightarrow H_2(\Omega^2 S^3; \mathbb{F}_2) \) is non zero. Therefore using the above diagram, the composite \( H_2(r \circ L(ad_{S^2}) \circ \Delta \) is also non zero. On the other hand, using the right part of diagram (29), we have that the composite \( H_2(r \circ L(ad_{S^2})) \circ H_2(s) \) is null. \( \square \)

**Corollary 30.** The free loop space modulo 2 homology \( \mathbb{H}_*(LS^2; \mathbb{F}_2) \) is not isomorphic as Batalin-Vilkovisky algebras to the Hochschild cohomology of \( H^*(S^2; \mathbb{F}_2) \), \( HH^*(H^*(S^2; \mathbb{F}_2); H^*(S^2; \mathbb{F}_2)) \).

This means exactly that there exists no isomorphism between \( \mathbb{H}_*(LS^2; \mathbb{F}_2) \) and \( HH^*(H^*(S^2; \mathbb{F}_2); H^*(S^2; \mathbb{F}_2)) \) which at the same time,

- is an isomorphism of algebras and
- commutes with the \( \Delta \) operators,

although separately

- there exists an isomorphism of algebras between \( \mathbb{H}_*(LS^2; \mathbb{F}_2) \) and \( HH^*(H^*(S^2; \mathbb{F}_2); H^*(S^2; \mathbb{F}_2)) \) (Corollary 23) and
- there exists also an isomorphism commuting with the \( \Delta \) operators between them.

**Proof.** By Proposition 20 \( HH^*(H^*(S^2); H^*(S^2)) \) is the Batalin-Vilkovisky algebra given by \( \varepsilon = 0 \) in Lemma 21. On the contrary, by Theorem 24...

\[
\begin{array}{ccc}
\Omega S^2 & \xrightarrow{j} & LS^2 \\
\Omega(ad_{S^2}) & \xrightarrow{L(ad_{S^2})} & S^2 \\
\Omega^2 S^3 & \xrightarrow{j} & L\Omega S^3 \\
\Omega^2 S^3 & \xrightarrow{id} & \Omega^2 S^3 \\
\end{array}
\]
$H_*(LS^2; \mathbb{F}_2)$ is the Batalin-Vilkovisky algebra given by $\varepsilon = 1$. At the end of the proof of Lemma 21, we saw that the two cases $\varepsilon = 0$ and $\varepsilon = 1$ correspond to two non isomorphic Batalin-Vilkovisky algebras. □

More generally, we believe that for any prime $p$, the free loop space modulo $p$ of the complex projective space $H_*(\mathbb{C}P^{n-1}; \mathbb{F}_p)$ is not isomorphic as Batalin-Vilkovisky algebras to the Hochschild cohomology $HH^*(H^*(\mathbb{C}P^{n-1}; \mathbb{F}_p); H^*(\mathbb{C}P^{n-1}; \mathbb{F}_p))$. Such phenomena for formal Manifolds should not appear over a field of characteristic 0.

Recall that by Poincaré duality, we have the isomorphism
\[ \Theta : H^*(S^2) \xrightarrow{\cong} H^*(S^2)^\vee. \]
Therefore we have the isomorphism
\[ HH^*(H^*(S^2); \Theta) : HH^*(H^*(S^2); H^*(S^2)) \xrightarrow{\cong} HH^*(H^*(S^2); H^*(S^2)^\vee). \]
Consider any isomorphism of graded algebras
\[ H_*(LS^2) \cong HH^*(S^*(S^2); S^*(S^2)). \]
By Corollary 23 such isomorphism exists. Cohen and Jones ([7, Theorem 3] and [5]) proved that such isomorphism exists for any manifold $M$. Since $S^2$ is formal, we have the isomorphism of algebras
\[ HH^*(S^*(S^2); S^*(S^2)) \xrightarrow{\cong} HH^*(H^*(S^2); H^*(S^2)). \]
By [17], we have the isomorphisms of $H_*(S^1)$-modules
\[ H_*(LS^2) \xrightarrow{\cong} HH^*(S^*(S^2); S^*(S^2)^\vee) \xrightarrow{\cong} HH^*(H^*(S^2); H^*(S^2)^\vee). \]
Corollary 30 implies that the following diagram does not commute over $\mathbb{F}_2$:
\[ HH^*(S^*(S^2); S^*(S^2)^\vee) \xrightarrow{\cong} HH^*(H^*(S^2); H^*(S^2)^\vee) \]
\[ H_*(LS^2) \xrightarrow{\cong} HH^*(S^*(S^2); S^*(S^2)) \xrightarrow{2} HH^*(H^*(S^2); H^*(S^2)). \]
This is surprising because as explained by Cohen and Jones [7, p. 792], the composite of the isomorphism \(14\) given by Jones in [17] and an isomorphism induced by Poincaré duality should give an isomorphism of algebras between $H_*(LS^2)$ and $HH^*(S^*(S^2); S^*(S^2))$.

\[ ^1\text{Bökstedt and Ottosen [1] have recently announced the computation of Batalin-Vilkovisky algebra } H_*(\mathbb{C}P^n; \mathbb{F}_p). \]
8. Appendix by Gerald Gaudens and Luc Menichi.

Let \( X \) be a pointed topological space. Recall that the circle \( S^1 \) acts on the double loop space \( \Omega^2 X \). Consider the induced operator \( \Delta : H_*(\Omega^2 X) \to H_{*+1}(\Omega^2 X) \). Getzler [13] has showed that \( H_*(\Omega^2 X) \) equipped with the Pontryagin product and this operator \( \Delta \) forms a Batalin-Vilkovisky algebra. In [12], Gerald Gaudens and the author have determined this Batalin-Vilkovisky algebra \( H_*(\Omega^2 S^3; \mathbb{F}_2) \). The key was the following Theorem. In [18, Proposition 7.46], answering to a question of Gerald Gaudens, Sadok Kallel and Paolo Salvatore give another proof of this Theorem.

**Theorem 32.** [12] The operator \( \Delta : H_1(\Omega^2 S^3; \mathbb{F}_2) \to H_2(\Omega^2 S^3; \mathbb{F}_2) \) is non trivial.

Both proofs [12] and [18] Proposition 7.46] are unpublished and publicly unavailable yet. So the goal of this section is to give a proof of this theorem which is as simple as possible.

Denote by \( * \) the Pontryagin product in \( H_*(\Omega^2 X) \) and by \( \circ \) the map induced in homology by the composition map \( \Omega^2 X \times \Omega^2 S^2 \to \Omega^2 X \). Denote by \( \Omega^2_n S^2 \), the path-connected component of the degree \( n \) maps. Denote by \( v_1 \) the generator of \( H_1(\Omega^2_0 S^2; \mathbb{F}_2) \) and by [1] the generator of \( H_0(\Omega^2_0 S^2; \mathbb{F}_2) \).

**Lemma 33.** For \( x \in H_*(\Omega^2 X; \mathbb{F}_2) \), \( \Delta x = x \circ (v_1 \ast [1]) \).

**Proof.** The circle \( S^1 \) acts on the sphere \( S^2 \). Therefore we have a morphism of topological monoids \( \Theta : (S^1, 1) \to (\Omega^2 S^2, id_{\Omega^2 S^2}) \). The action of \( S^1 \) on \( \Omega^2 X \) is the composite \( S^1 \times \Omega^2 X \xrightarrow{\Theta \times \Omega^2 X} \Omega^2_1 S^2 \times \Omega^2 X \xrightarrow{\circ} \Omega^2 X \).

Therefore for \( x \in H_*(\Omega^2 X; \mathbb{F}_2) \), \( \Delta x = x \circ (H_1(\Theta)[S^1]) \).

Suppose that \( H_1(\Theta)[S^1] = 0 \). Then for any topological space \( X \), the operator \( \Delta \) on \( H_*(\Omega^2 X; \mathbb{F}_2) \) is null. Therefore, for any \( x \) and \( y \) in \( H_*(\Omega^2 X; \mathbb{F}_2) \), \( \{ x, y \} = \Delta(xy) - (\Delta x)y - x(\Delta y) = 0 \). That is the modulo 2 Browder brackets on any double loop space are null. This is obviously false. For example, Cohen in [3] explains that the Gerstenhaber algebra \( H_*(\Omega^2 \Sigma^2 Y) \) has in general many non trivial Browder brackets. So the assumption \( H_1(\Theta)[S^1] = 0 \) is false.

Since the loop multiplication by \( id_{\Omega^2 S^2} \) in the \( H \)-group \( \Omega^2 S^2 \), is a homotopy equivalence, the Pontryagin product by [1], \( \ast[1] : H_*(\Omega^2_0 S^2) \xrightarrow{\cong} H_*(\Omega^2_1 S^2) \) is an isomorphism. Therefore \( v_1 \ast [1] \) is a generator of \( H_1(\Omega^2_1 S^2) \). So \( H_1(\Theta)[S^1] = v_1 \ast [1] \). So finally

\[
\Delta x = x \circ (H_1(\Theta)[S^1]) = x \circ (v_1 \ast [1]).
\]

\[\square\]
Recall that $v_1$ denote the generator of $H_1(\Omega^2_0 S^2; \mathbb{F}_2)$.

**Lemma 34.** In the Batalin-Vilkovisky algebra $H_*(\Omega^2 S^2; \mathbb{F}_2)$, $\Delta(v_1) = v_1 \ast v_1$.

**Proof.** Recall that $[1]$ is the generator of $H_0(\Omega^2_1 S^2)$. By Lemma 33,

$$\Delta[1] = [1] \circ (v_1 \ast [1]) = (v_1 \ast [1]).$$

Denote by $Q : H_q(\Omega^2_0 S^2) \to H_{2q+1}(\Omega^2_2, S^2)$ the Dyer-Lashof operation. It is well known that $Q[1] = v_1 \ast [2]$. So by [4, Theorem 1.3 (4) p. 218]

$$\{v_1 \ast [2], [1]\} = \{Q[1], [1]\} = \{(1), \{1, [1]\}\}.$$

By [4] Theorem 1.2 (3) p. 215, $\{[1], [1]\} = 0$. Therefore on one hand, $\{v_1 \ast [2], [1]\}$ is null. And on the other hand, using the Poisson relation (7), since $\{2, [1]\} = \{[1] \ast [1, [1]\} = 2\{[1], [1]\} \ast [1] = 0,$

$$\{v_1 \ast [2], [1]\} = \{v_1, [1]\} \ast [2] + v_1 \ast \{[2], [1]\} = \{v_1, [1]\} \ast [2].$$

Since $*\{1\} : H_*(\Omega^2 S^2) \xrightarrow{\cong} H_*(\Omega^2 S^2)$ is an isomorphism, we obtain that Browder bracket $\{v_1, [1]\}$ is null. Therefore,

$$\Delta(v_1 \ast [1]) = (\Delta v_1) \ast [1] + v_1 \ast (\Delta [1]) = ((\Delta v_1) - v_1 \ast v_1) \ast [1].$$

But $\Delta(v_1 \ast [1]) = (\Delta \circ \Delta)([1]) = 0$. Therefore $\Delta(v_1)$ must be equal to $v_1 \ast v_1$. \hfill $\square$

**Proof of Theorem 32.** We remark that since $\Delta$ preserves path-connected components and since the loop multiplication of two homotopically trivial loops is a homotopically trivial loop, $H_*(\Omega^2 S^2)$ is a sub Batalin-Vilkovisky algebra of $H_*(\Omega^2 S^2)$.

Let $S^1 \hookrightarrow S^3 \xrightarrow{\eta} S^2$ be the Hopf fibration. After double loop ing, the Hopf fibration gives the fibration $\Omega^2 S^1 \hookrightarrow \Omega^2 S^3 \xrightarrow{\Omega^2 \eta} \Omega^2_0 S^2$ with contractile fiber $\Omega^2 S^1$ and path-connected base $\Omega^2_0 S^2$. Therefore $\Omega^2 \eta : \Omega^2 S^3 \xrightarrow{\cong} \Omega^2_0 S^2$ is a homotopy equivalence. And so $H_*(\Omega^2 \eta) : H_*(\Omega^2 S^3) \xrightarrow{\cong} H_*(\Omega^2_0 S^2)$ is an isomorphism of Batalin-Vilkovisky algebras.

Let $u_1$ be the generator of $H_1(\Omega^2 S^3)$. Lemma 34 implies that $\Delta(u_1) = u_1 \ast u_1$. Since $u_1 \ast u_1$ is non zero in $H_*(\Omega^2 S^3; \mathbb{F}_2)$, $\Delta(u_1)$ is non trivial. \hfill $\square$

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UMR 6093 associée au CNRS, Université d’Angers, Faculté des Sciences, 2 Boulevard Lavoisier, 49045 Angers, FRANCE

E-mail address: firstname.lastname at univ-angers.fr