ON THE CLASSIFICATION OF ALMOST CONTACT METRIC MANIFOLDS

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Abstract. On connected manifolds of dimension higher than three, the non-existence of 132 Chinea and González-Dávila types of almost contact metric structures is proved. This is a consequence of some interrelations among components of the intrinsic torsion of an almost contact metric structure. Such interrelations allow to describe the exterior derivatives of some relevant forms in the context of almost contact metric geometry.

1. Introduction

In [2] Chinea and González-Dávila displayed a Gray-Hervella type classification for almost contact metric structures. Such a classification is based on the decomposition of the space possible intrinsic torsions into irreducible $U(n)$-modules ($U(n) \times 1$ is the structural group in case of almost contact metric structure). Since in general dimensions they obtained a decomposition of the intrinsic torsion $\xi$ of the structure into twelve $U(n)$-components $\xi_i$ respectively belonging to irreducible $U(n)$-modules $\mathcal{C}_i$, from algebraic point view, there are potentially $2^{12}$ classes. However, because of geometry, some of these classes could not exist on connected manifolds. For instance, in [10] Marrero has proved the non-existence of almost contact metric structure of strict type $\mathcal{C}_5 \oplus \mathcal{C}_6$ defined on a connected manifold of dimension higher than 3. A similar fact of non-existence has been proved for $G_2$-structures in [11] and for $SU(3)$-structures on six dimensional manifolds in [12]. Each one of these results shows the non-existence of only one type of the considered $G$-structure. In the present paper we prove the non-existence of 132 types of almost contact metric structures defined on a connected manifold of dimension higher than 3. This is a consequence of Theorem 4.1 below. Concretely we prove

For an almost contact metric connected manifold of dimension $2n + 1$, $n > 1$:

(i) If the structure is of type $\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_8 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{11}$ with $(\xi_5, \xi_6) \neq (0, 0)$, then it is of type $\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 \oplus \mathcal{C}_5 \oplus \mathcal{C}_8 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{11}$, or of type $\mathcal{C}_2 \oplus \mathcal{C}_6 \oplus \mathcal{C}_9$.

(ii) If the structure is of type $\mathcal{C}_2 \oplus \mathcal{C}_5 \oplus \mathcal{C}_7 \oplus \mathcal{C}_9$ with $(\xi_5, \xi_7) \neq (0, 0)$, then it is of type $\mathcal{C}_2 \oplus \mathcal{C}_7 \oplus \mathcal{C}_9$ or of type $\mathcal{C}_2 \oplus \mathcal{C}_5 \oplus \mathcal{C}_9$.

In the proof of these results we make use of some interrelations among components of the intrinsic torsion which are consequences of the identities $d^2\eta = 0$ and $d^2F = 0$, where $\eta$ is the one-form metrically equivalent to the Reeb vector field $\zeta$ and $F$ is the fundamental two-form of the structure. Such interrelations are interesting by their own and give rise to expressions for the exterior derivatives of the functions $d^*\eta$, $d^*F(\zeta)$ and the one-forms $\xi_4, \xi_5$ and $\sum_{i=1}^{2n+1} (\xi_i)_{e_i}e_i \parallel = \frac{n-1}{2}\theta$, where $\theta$ denotes the Lee form considered in [8], $\{e_1, \ldots, e_{2n+1}\}$.
is an orthonormal basis for vectors and $X^\flat$ is the one-form metrically equivalent to the vector $X$. These functions and one-forms determine the components of the intrinsic torsion in $\mathcal{C}_5$, $\mathcal{C}_6$, $\mathcal{C}_{12}$ and $\mathcal{C}_4$, respectively.

Finally, we describe how to use the exterior derivatives $d\eta$, $dF$ and the Nijenhuis tensor $N_\varphi$ to determine the type of almost contact metric structure. This is used in some examples.

2. Preliminaries

An almost contact structure $(\varphi, \zeta, \eta)$ on a manifold $M$ consists of a $(1,1)$-tensor $\varphi$, a vector field $\zeta$, called the Reeb vector field, and a one-form $\eta$ such that

$$\varphi^2 = -I + \eta \otimes \zeta, \quad \eta(\zeta) = 1.$$ 

The dimension of $M$ must be $2n + 1$. The presence of an almost contact structure is equivalent to say that there is a $GL(n, \mathbb{C}) \times 1$-structure defined on $M$. A manifold $M$ is said to be equipped with an almost contact metric structure, if there is an almost contact structure and a Riemannian metric $\langle \cdot, \cdot \rangle$ on $M$ such that the following compatibility condition is satisfied

$$\langle \varphi X, \varphi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y).$$

This is equivalent to say that there is a $U(n) \times 1$-structure defined on $M$. We will make reiterated use of the musical isomorphisms $b : TM \to T^*M$ and $\sharp : T^*M \to TM$, induced by $\langle \cdot, \cdot \rangle$, defined respectively by $X^\flat = \langle X, \cdot \rangle$ and $\langle \theta^\sharp, \cdot \rangle = \theta$. Thus one has $\eta = \zeta^\flat$ and $\zeta = \eta^\sharp$.

Associated with an almost contact metric structure, the tensor $F = \langle \cdot, \varphi \cdot \rangle$, called the fundamental two-form, is usually considered. Using $F$ and $\eta$, $M$ can be oriented by fixing a constant multiple of $F^n \wedge \eta = F^{(n)} \wedge F \wedge \eta$ as volume form.

For almost contact metric structures, the cotangent space on each point $T^*_m M$ is not irreducible under the action of the group $U(n) \times 1$. In fact, $T^*_m M = \eta^\perp \oplus \mathbb{R}\eta$, where $\eta^\perp$ is the image under by $b$ of the distribution $\zeta^\perp$ orthogonal to $\zeta$. Taking this into account, it follows

$$\text{so}(2n + 1) \cong \Lambda^2 T^*_m M = \Lambda^2 \eta^\perp \oplus \eta^\perp \wedge \mathbb{R}\eta.$$ 

From now on we will denote $X_{\zeta^\perp} = X - \eta(X)\zeta$, for all vector field $X$. Since $\Lambda^2 \eta^\perp = u(n) \oplus u(n)_{\zeta^\perp}^\perp$, where $u(n)$ ($u(n)_{\zeta^\perp}^\perp$) consists of those two-forms $b$ such that $b(\varphi X, \varphi Y) = b(X_{\zeta^\perp}, Y_{\zeta^\perp})$ ($b(\varphi X, \varphi Y) = -b(X_{\zeta^\perp}, Y_{\zeta^\perp})$), we have

$$\text{so}(2n + 1) = u(n) \oplus u(n)^\perp, \quad \text{with } u(n)^\perp = u(n)_{\zeta^\perp}^\perp \oplus \eta^\perp \wedge \mathbb{R}\eta.$$ 

Denoting by $\nabla$ the Levi Civita connection, the minimal connection $\nabla^{U(n)}$ is the unique $U(n)$-connection such that $\xi_X = \nabla^{U(n)}_X - \nabla_X$ satisfies the condition $\xi_X \in u(n)^\perp$. The tensor $\xi$ is referred to as the intrinsic torsion of the almost contact metric structure [4]. The space $T^*_m M \otimes u(n)^\perp$ of intrinsic torsions has the following first decomposition into $U(n)$-modules:

$$T^*_m M \otimes u(n)^\perp = (\eta^\perp \otimes u(n)_{\zeta^\perp}^\perp) \oplus (\eta \otimes u(n)_{\zeta^\perp}^\perp) \oplus (\eta^\perp \otimes \eta^\perp \wedge \eta) \oplus (\eta \otimes \eta^\perp \wedge \eta).$$ 

Chinea and González-Dávila [2] showed that $T^*_m M \otimes u(n)^\perp$ is decomposed into twelve irreducible $U(n)$-modules $\mathcal{C}_1, \ldots, \mathcal{C}_{12}$, where

$$\eta^\perp \otimes u(n)_{\zeta^\perp}^\perp = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 \oplus \mathcal{C}_4, \quad \eta^\perp \otimes \eta^\perp \wedge \eta = \mathcal{C}_5 \oplus \mathcal{C}_8 \oplus \mathcal{C}_9 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_{10},$$

$$\eta \otimes u(n)_{\zeta^\perp}^\perp = \mathcal{C}_{11}, \quad \eta \otimes \eta^\perp \wedge \eta = \mathcal{C}_{12}.$$
The $U(n)$-modules $\mathcal{C}_1, \ldots, \mathcal{C}_4$ are isomorphic to the Gray and Hervella's ones given in [3]. Furthermore, note that $\varphi$ restricted to $\xi^\perp$ works as an almost complex structure and, if one considers the $U(n)$-action on the bilinear forms $\otimes^2 \eta^\perp$, then one has the decomposition

$$\otimes^2 \eta^\perp = \mathbb{R} \langle \cdot, \cdot \rangle_{\xi^\perp} \oplus \mathfrak{su}(n)_s \oplus [\sigma^{2,0}] \oplus \mathbb{R} F \oplus \mathfrak{su}(n)_a \oplus \mathfrak{u}(n)^\perp_{\xi^\perp}.$$  

The modules $\mathfrak{su}(n)_s (\mathfrak{su}(n)_a)$ consists of Hermitian symmetric (skew-symmetric) bilinear forms orthogonal to $\langle \cdot, \cdot \rangle_{\xi^\perp} (F)$, and $[\sigma^{2,0}] (\mathfrak{u}(n)^\perp_{\xi^\perp})$ is the space of anti-Hermitian symmetric (skew-symmetric) bilinear forms. With respect to the modules $\mathcal{C}_i$, one has $\eta^\perp \otimes \eta^\perp \cap \mathbb{R} \eta \cong \otimes^2 \eta^\perp$ and, using the $U(n)$-map $\xi U(n) \rightarrow -\xi U(n) \eta = \nabla \eta$, it is obtained

$$\mathcal{C}_5 \cong \mathbb{R} \langle \cdot, \cdot \rangle_{\xi^\perp}, \quad \mathcal{C}_8 \cong \mathfrak{su}(n)_s, \quad \mathcal{C}_9 \cong [\sigma^{2,0}], \quad \mathcal{C}_6 \cong \mathbb{R} F; \quad \mathcal{C}_7 \cong \mathfrak{su}(n)_a, \quad \mathcal{C}_{10} \cong \mathfrak{u}(n)^\perp_{\xi^\perp}.$$  

In summary, the space of intrinsic torsions $T^* M \otimes \mathfrak{u}(n)^\perp$ consists of those tensors $\xi$ such that

$$\varphi \xi_X Y + \xi_X \varphi Y = \eta(Y) \varphi \xi_X \zeta + \eta(\xi_X \varphi Y) \zeta$$

(2.1)

and, under the action of $U(n) \times 1$, is decomposed into:

1. if $n = 1$, $\xi \in T^* M \otimes \mathfrak{u}(n)^\perp = \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{12}$;
2. if $n = 2$, $\xi \in T^* M \otimes \mathfrak{u}(n)^2 = \mathcal{C}_2 \oplus \mathcal{C}_4 \oplus \cdots \oplus \mathcal{C}_{12}$;
3. if $n \geq 3$, $\xi \in T^* M \otimes \mathfrak{u}(n)^{\perp} = \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_{12}$.

Some of these classes are referred to, by diverse authors [1][2], as:

$\{ \xi = 0 \}$ = cosymplectic manifolds or integrable almost contact metric structure, $\mathcal{C}_1$ = nearly-K-cosymplectic manifolds, $\mathcal{C}_5$ = $\alpha$-Kenmotsu manifolds, $\mathcal{C}_6$ = $\alpha$-Sasakian manifolds, $\mathcal{C}_9$ = trans-Sasakian manifolds, $\mathcal{C}_2 \oplus \mathcal{C}_9$ = almost cosymplectic manifolds, $\mathcal{C}_6 \oplus \mathcal{C}_7$ = quasi-Sasakian manifolds, $\mathcal{C}_1 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6$ = nearly-trans-Sasakian manifolds, $\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10}$ = quasi-K-cosymplectic manifolds, $\mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8$ = normal manifolds, $\mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_8$ = integrable almost contact structure, etc.

The intrinsic torsion is given by

$$\xi_X = -\frac{1}{2} \varphi \circ \nabla_X \varphi + \nabla_X \eta \otimes \zeta - \frac{1}{2} \eta \otimes \nabla_X \zeta = \frac{1}{2} (\nabla_X \varphi) \circ \varphi + \frac{1}{2} \nabla_X \eta \otimes \zeta - \eta \otimes \nabla_X \zeta$$

(see [7]). If the almost contact metric structure is of type $\mathcal{C}_5 \oplus \cdots \oplus \mathcal{C}_{10} \oplus \mathcal{C}_{12}$, then the expression for the intrinsic torsion is reduced to $\xi_X = \nabla_X \eta \otimes \zeta - \eta \otimes \nabla_X \zeta$.

The tensor $\xi(i)$ will denote the component of $\xi$ obtained by the $U(n)$-isomorphism $(\nabla F)_{(i)} = (-\xi F)_{(i)} \in \mathcal{C}_i \rightarrow \xi_{(i)}$. In this way we are using the same terminology used in [2] by Chinea and González-Dávila when we are referring to classes.

Some vector fields are involved in the characterization of certain types of almost contact metric manifolds. For instance, if $d^*$ denotes the coderivative and $\{ e_1, \ldots, e_{2n+1} \}$ is a local orthonormal frame field, the vector field

$$\sum_{i=1}^{2n+1} \xi e_i e_i = -\frac{1}{2} \varphi (d^* F)^2 - d^* \eta \zeta - \frac{1}{2} \nabla \zeta.$$  

Because

$$\sum_{i=1}^{2n+1} \xi(4)e_i e_i = -\frac{1}{2} \varphi (d^* F)^2 + \frac{1}{2} \nabla \zeta,$$

$$\sum_{i=1}^{2n+1} \xi(5)e_i e_i = -d^* \eta \zeta,$$

this vector field is contributed by the components of $\xi$ in $\mathcal{C}_4$, $\mathcal{C}_5$ and $\mathcal{C}_{12}$.

Moreover, one has the vector field

$$\sum_{i=1}^{2n+1} \xi e_i \varphi e_i = -\frac{1}{2} (d^* F)^2 - \frac{1}{2} d^* F(\zeta) \zeta - \varphi \nabla \zeta.$$
This second vector field is contributed by the components of $\xi$ in $\mathcal{C}_4$ and $\mathcal{C}_6$. In fact,
\[
\sum_{i=1}^{2n+1} \xi_{(4)i} \varphi e_i = -\frac{1}{2} (d^* F)^{\sharp} - \varphi \nabla \zeta \zeta + \frac{1}{2} d^* F(\zeta), \quad \sum_{i=1}^{2n+1} \xi_{(6)i} \varphi e_i = -d^* F(\zeta) \zeta.
\]

Remark 2.1. For using simpler and standard notation, we recall that $\lambda_0^{p,q}$ is a complex irreducible $U(n)$-module coming from the $(p, q)$-part of the complex exterior algebra, and that its corresponding dominant weight in standard coordinates is given by $\{1, \ldots, 1, 0, \ldots, 0, -1, \ldots, -1\}$, where $1$ and $-1$ are repeated $p$ and $q$ times, respectively. By analogy with the exterior algebra, there are also complex irreducible $U(n)$-modules $\sigma_0^{p,q}$, with dominant weights $(p, 0, \ldots, 0, -q)$ coming from the complex symmetric algebra. The notation $[V]$ stands for the real vector space underlying a complex vector space $V$, and $[W]$ denotes a real vector space that admits $W$ as its complexification.

The space of two forms $\Lambda^2 T^* M$ is decomposed into irreducible $U(n)$-components as follows:
\[
\Lambda^2 T^* M = \mathbb{R} F + \mathbb{R}^{\lambda_0^{1,1}} + \mathbb{R}^{\lambda_0^{1,0}} + \mathbb{R}^{\lambda_0^{0,2}} + \mathbb{R} \wedge \mathbb{R}^{\lambda_0^{1,0}}.
\]

The components of a two-form $\alpha$ are given by
\[
2\alpha_{[\lambda_1,1]}(X, Y) = \alpha(\varphi X, \varphi Y), \quad 2\alpha_{[\lambda_2,0]}(X, Y) = \alpha(\varphi^2 X, \varphi^2 Y) - \alpha(\varphi X, \varphi Y), \quad \alpha_{\eta \wedge [\lambda_1,0]} = \eta \wedge (\zeta \alpha),
\]
where $\cdot$ denotes the interior product. We will use the natural extension to forms of the metric $\langle \cdot, \cdot \rangle$. Thus, for all $p$-forms $\alpha$, $\beta$,
\[
\langle \alpha, \beta \rangle = \frac{1}{p!} \sum_{i_1, \ldots, i_p=1}^{2n+1} \alpha(e_{i_1}, \ldots, e_{i_p}) \beta(e_{i_1}, \ldots, e_{i_p}).
\]
Using this product we have $\alpha_{\mathbb{R} F} = \frac{1}{n} \langle \alpha, F \rangle F$.

In the sequel, we will consider the orthonormal basis for vectors $\{e_1, \ldots, e_{2n}, e_{2n+1} = \zeta\}$. Likewise, we will use the summation convention. The repeated indexes will mean that the sum is extended from $i = 1$ to $i = 2n$. Otherwise, the sum will be explicitly written.

3. Exterior derivatives of relevant forms of the structure

In this section we will display several identities relating components of the intrinsic torsion which are consequences of the equalities $d^2 F = 0$ and $d^2 \eta = 0$. They are interesting by their own and we will use later some of them. Some of those identities were already obtained in [7] for the particular case of almost contact metric structures of type $\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_{10}$. Here the parts $\xi_{(11)}$ and $\xi_{(12)}$ of the intrinsic torsion are also considered.

As applications of the identities we will obtain the $U(n)$-components of the exterior derivatives of the one-forms $\theta$ and $\xi_{\zeta} \eta$, and the functions $d^* \eta$, $d^* F(\zeta)$. These one-forms and functions determine $\xi_{(4)}$, $\xi_{(12)}$, $\xi_{(5)}$ and $\xi_{(6)}$, respectively.
Lemma 3.1. For almost contact metric manifolds of dimension $2n+1$, $n > 1$, the following identity is satisfied
\[
0 = \frac{n-2}{n} \langle \nabla U^{(n)}(\xi(4)e_i, e_i, Y) - \frac{n-2}{n} \langle \nabla U^{(n)}(\xi(4)e_i, e_i, X) - \frac{n-2}{n} \langle \nabla U^{(n)}(\xi(4)e_i, e_i) \rangle \rangle \rangle (\varphi e_j) F(X, Y) \\
- 2 \langle (\nabla U^{(n)}(\xi(3))_X Y, e_i) + 2 \langle (\nabla U^{(n)}(\xi(3))_Y X, e_i) - \frac{n-2}{n} \langle \nabla e_i \xi(4)e_i, \varphi Y \rangle \\
+ \frac{n-4}{4} \langle \nabla U^{(n)}(\xi(4)e_i, \varphi X) - 3 \langle \xi(1)_X e_i, \xi(2)_Y e_i \rangle + 3 \langle \xi(1)_Y e_i, \xi(2)_X e_i \rangle \\
- \frac{1}{n} d^* \eta^* F(\zeta) F(X, Y) + \frac{1}{n} \langle d^* \eta(\xi(7)_Y \eta)(Y) - \frac{1}{n} \langle d^* F(\zeta)(\xi(8)_X \eta)(\varphi Y) \\
+ 4 \langle \xi(7)_X \zeta, \xi(8)_Y \zeta \rangle - 4 \langle \xi(7)_Y \zeta, \xi(8)_X \zeta \rangle + 4 \langle \xi(11)_X e_i, \xi(10)_Y \zeta \rangle - 4 \langle \xi(11)_Y e_i, \xi(10)_X \zeta \rangle.
\]

Proof. The proof follows in a similar way as in [7] Lemma 4.5, page 163 for other identities below. Firstly we note that $d^2 F \in \Lambda^4 T^* M$ and $\Lambda^4 T^* M$ has the following $U(n)$-decomposition
\[
\Lambda^4 T^* M = \langle \Lambda^{4,0} \rangle \oplus \langle \Lambda^{3,1} \rangle \oplus \langle \Lambda^{2,2} \rangle \wedge F \oplus \langle \Lambda^{1,1} \rangle \wedge F \oplus \Lambda^2 F \wedge F \\
\oplus \langle \Lambda^{3,0} \rangle \otimes \eta \oplus \langle \Lambda^{2,1} \rangle \wedge \eta \oplus \langle \Lambda^{1,0} \rangle \wedge F \wedge \eta.
\]

Then $d^2 F$ is written in terms of $\nabla U^{(n)}$ and $\xi$, i.e.
\[
d^2 F(X_1, X_2, X_3, X_4) = \sum_{1 \leq a < b \leq 4} (-1)^{a+b} \left( \left( \langle \nabla U^{(n)}(\xi)_X \rangle \right)_b \right) (X_c, X_d) \\
+ \sum_{1 \leq a < b < c \leq 4} (-1)^{a+b} \langle \xi X_a \rangle (X_b, X_c) F(X_d), \\
- \sum_{1 \leq a < b < c < d \leq 4} (-1)^{a+b} \langle \xi X_a \rangle (X_b, X_c) F(X_d),
\]
where $c < d, \{a, b\} = \{1, \ldots, 4\} - \{a, b\}$ in each case and $[\xi X_a, \xi X_b] = \xi X_a \xi X_b - \xi X_b \xi X_a$. Now contracting with $F$ on the first two arguments and then taking the corresponding projection to $[\Lambda^{1,1}]$, it is computed the $[\Lambda^{1,1}]$-component of $d^2 F$. Using the symmetries of the components of $\xi$ and the fact $d^2 F = 0$, one has the required identity.

In previous Lemma, if we use the equality
\[
\langle \nabla U^{(n)}(\xi(4)e_i, e_i) \rangle (Y) - \langle \nabla U^{(n)}(\xi(4)e_i, e_i) \rangle (X) = d(\langle \xi(4)e_i, e_i \rangle (X, Y) - \langle \xi X Y - \xi Y X, \xi(4)e_i, e_i \rangle,
\]
we will obtain the $[\Lambda^{1,1}]$-component of the exterior derivative of the Lee form $\theta$.

Proposition 3.2. For almost contact metric manifolds of dimension $2n+1$, $n > 1$, we have
\[
\frac{n-2}{2} d\theta_{[\Lambda^{1,1}]}(X, Y) = \frac{1}{2} \langle d\theta, F \rangle F(X, Y) - \langle \langle \nabla e_i \xi(3) \rangle X Y, e_i \rangle + \langle \langle \nabla e_i \xi(3) \rangle Y X, e_i \rangle \\
- \frac{1}{n} d^2 \eta^* F(\zeta) F(X, Y) + \frac{1}{n} \langle d^* \eta(\xi(7)_Y \eta)(Y) - \frac{1}{n} \langle d^* F(\zeta)(\xi(8)_X \eta)(\varphi Y) \\
+ 2 \langle \xi(7)_X \zeta, \xi(8)_Y \zeta \rangle - 2 \langle \xi(7)_Y \zeta, \xi(8)_X \zeta \rangle \\
+ 2 \langle \xi(11)_X e_i, \xi(10)_Y \zeta \rangle - 2 \langle \xi(11)_Y e_i, \xi(10)_X \zeta \rangle.
\]
and $(d\theta)_{\mathbb{R}}(X, Y) = \frac{1}{n} \langle d\theta, F \rangle F(X, Y)$, where
\[
\langle d\theta, F \rangle = \frac{1}{n} d^* \eta^* F(\zeta) - 2 \langle \xi(7)_\varphi e_i, \xi(8)_e \zeta \rangle - 2 \langle \xi(11)_\varphi e_i, \xi(10)_e \zeta \rangle.
\]

The identity in next Lemma is also a consequence of $d^2 F = 0$. 
Lemma 3.3. For almost contact metric manifolds of dimension $2n + 1$, $n > 1$, the following identity is satisfied

$$0 = 3(\langle \nabla^{U(n)} e_i \xi(1) \rangle_{e_i} X, Y) - \langle \nabla^{U(n)} e_i \xi(3) X, Y \rangle + (n - 2)\langle \nabla^{U(n)} e_i \xi(4) X, Y \rangle$$

$$+ \langle \xi(3) X e_i, \xi(2) X e_i \rangle - \frac{1}{n+2} \langle \xi(1) X e_i, X, Y \rangle + \frac{1}{n+2} \langle \xi(2) X e_i, X, Y \rangle + \langle \xi(3) e_i, e_i, X, Y \rangle$$

$$+ \langle \xi(6) \eta \rangle \langle \xi(11) X, \varphi Y \rangle + (n - 2)\langle \xi(5) \eta \rangle \langle \xi(10) \eta \rangle (X, Y)$$

$$- 2\langle \xi(7) \eta \rangle \langle \xi(10) \eta \rangle (X, Y) - 2\langle \xi(8) \eta \rangle \langle \xi(10) \eta \rangle (X, Y)$$

$$- 2\langle \xi(7) \eta \rangle(\xi(10) \eta \rangle (X, Y) - 2\langle \xi(8) \eta \rangle \langle \xi(10) \eta \rangle (X, Y)$$

$$+ 2\langle \xi(7) \eta \rangle(\xi(10) \eta \rangle (X, Y) - 2\langle \xi(7) \eta \rangle \langle \xi(10) \eta \rangle (X, Y).$$

Proof. The proof follows in a similar way as in [7, Lemma 4.5]. There the proof is given with more details. We begin by considering Lemma 3.3. Next we give a third consequence of Lemma 3.3. Finally we conclude the proof.

Proposition 3.4. For almost contact metric manifolds of dimension $2n + 1$, $n > 1$, the following identity is satisfied

$$\frac{n-2}{2} d\theta_{[\lambda^2, \eta]}(X, Y) = -3(\langle \nabla^{U(n)} e_i \xi(1) \rangle_{e_i} X, Y) + \langle \nabla^{U(n)} e_i \xi(3) X, Y \rangle + \langle \xi(3) X e_i, \xi(1) X e_i \rangle$$

$$- \langle \xi(3) \eta \rangle e_i, \xi(1) X e_i \rangle - \frac{1}{2} \langle \xi(3) X e_i, \xi(2) X e_i \rangle + \frac{1}{2} \langle \xi(3) \eta \rangle e_i, \xi(2) X e_i \rangle$$

$$+ \frac{3(n-3)}{2} \langle \xi(1) \eta \rangle X, Y \rangle - \frac{n-1}{2} \langle \xi(3) \eta \rangle X, Y \rangle - d^* F(\xi(11) X, \varphi Y)$$

$$- \frac{n-2}{2} d^* F(\xi(10) \varphi X) \eta \rangle (Y) + \frac{n-2}{2} d^* F(\xi(10) \varphi X) \eta \rangle (Y)$$

$$+ 2\langle \xi(7) \eta \rangle \langle \xi(9) \eta \rangle (X, Y) + 2\langle \xi(7) \eta \rangle \langle \xi(10) \eta \rangle (X, Y)$$

$$+ 2\langle \xi(8) \eta \rangle \langle \xi(9) \eta \rangle (X, Y) + 2\langle \xi(8) \eta \rangle \langle \xi(10) \eta \rangle (X, Y)$$

Next we have a third consequence of $d^2 F = 0$.

Lemma 3.5. For almost contact metric manifolds of dimension $2n + 1$, the following identity is satisfied

$$0 = -3(\langle \nabla^{U(n)} e_i \xi(1) \rangle_{e_i} X, Y) - (n - 1) \langle \nabla^{U(n)} e_i \xi(3) \eta \rangle (X) + \langle \nabla^{U(n)} e_i \xi(3) \eta \rangle (X)$$

$$- \langle \xi(3) \eta \rangle X e_i, \xi(3) \eta \rangle (X) + \langle \xi(3) e_i, \xi(2) e_i \rangle - \frac{1}{2} \langle \xi(3) e_i, \xi(2) e_i \rangle$$

$$- \langle \xi(3) e_i, \xi(3) e_i \rangle + \langle \xi(10) e_i \eta \rangle \langle \xi(10) e_i \eta \rangle \langle \xi(3) e_i, X \rangle$$

$$+ \langle \xi(10) e_i \eta \rangle \langle \xi(10) e_i \eta \rangle \langle \xi(10) e_i \eta \rangle \langle \xi(3) e_i \rangle$$

$$- \frac{1}{n+2} \langle \xi(8) \xi(4) e_i \eta \rangle (X) + \langle \xi(6) \xi(4) e_i \eta \rangle (X) - \langle \xi(6) \xi(4) e_i \eta \rangle (X)$$

$$- \langle \xi(7) \xi(4) e_i \eta \rangle (X) - (n - 1) \langle \xi(5) \xi(4) e_i \eta \rangle (X) - \langle \xi(10) \xi(4) e_i \eta \rangle (X) - \langle \xi(11) X, \xi(4) \xi \rangle.$$
Next by noting that \((\nabla^U_X(\xi(4))_e, e_i) = \nabla^U_X(\xi(4))_e, e_i\) and using the identities

\[
(\xi(5)\eta)(Y) = \frac{d^\eta}{2n}(Y, X - \eta(X)\eta(Y)), \quad (\xi(6)\eta)(Y) = -\frac{d^\eta}{2n}(X, Y),
\]

\[
(\nabla^U_X(\xi(5))_Y\eta)(Z) = \frac{d^\eta}{2n}(Y, Z) - \eta(Y)\eta(Z),
\]

\[
(\nabla^U_X(\xi(4))_e, e_i, X) = d(\xi(4))_e, e_i)(\xi(X) - (\xi(11)_X, \xi(4))_e, e_i) - (\xi(\varphi X)\eta)(\xi(4))_e, e_i),
\]

another version of the identity in the previous Lemma is given in next Proposition. Such a version relates the exterior derivatives of the Lee form \(\varphi\) and the coderivative \(d^\eta\).

**Proposition 3.6.** For almost contact metric manifolds of dimension \(2n + 1\), we have

\[
\frac{n-1}{2} d\theta(\xi, X) = \frac{n-1}{2n} d(\varphi^2)(X) + (\nabla^U_X(\xi(8))_e, e_i)(X) - (\nabla^U_X(\xi(10))_e, e_i)(X)
\]

In particular, if the almost contact metric structure is of type \(\mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3 \oplus \mathbb{E}_5 \oplus \mathbb{E}_6 \oplus \mathbb{E}_9 \oplus \mathbb{E}_{12}\) or \(\mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3 \oplus \mathbb{E}_6 \oplus \mathbb{E}_7 \oplus \mathbb{E}_9 \oplus \mathbb{E}_{12}\) and \(n > 1\), then \(d(\varphi^2)(X)\) is given by \(\frac{n-1}{2} d\theta(\xi, X)\). Likewise, for the type \(\mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3 \oplus \mathbb{E}_5 \oplus \mathbb{E}_9 \oplus \mathbb{E}_{12}\) and \(n > 1\), the one-form \(\text{div}(\xi)\) is closed.

If we consider the identity \(d^2\eta = 0\), we will obtain an expression for \(d(\varphi^2)(X)\).

**Proposition 3.7.** For almost contact metric manifolds of dimension \(2n + 1\), the exterior derivative \(d(\varphi^2)(X)\) is given by

\[
\frac{n-1}{2n} d(\varphi^2)(X) = ((\nabla^U_X(\xi(7))_e, e_i))(\varphi X) - (\nabla^U_X(\xi(10))_e, e_i)(\varphi X) - (\xi(7)_X, \xi) - 2(\xi(7)\eta)(\xi(10)_X, \xi) - \frac{n-1}{2n} d^\theta(\xi, X)
\]

In particular, if the almost contact metric structure is of type \(\mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3 \oplus \mathbb{E}_4 \oplus \mathbb{E}_5 \oplus \mathbb{E}_6 \oplus \mathbb{E}_8 \oplus \mathbb{E}_9 \oplus \mathbb{E}_{11} \oplus \mathbb{E}_{12}\) and \(n > 1\), then the exterior derivative \(d(\varphi^2)(X)\) is given by

\[
d(\varphi^2)(X) = \frac{1}{n} d\eta d\varphi^2 + (\xi(7)_e, e_i)(\varphi X, \xi) + (\xi(8)_e, e_i)(\varphi X, \xi).
\]

In particular, if the almost contact metric structure is of type \(\mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3 \oplus \mathbb{E}_4 \oplus \mathbb{E}_5 \oplus \mathbb{E}_6 \oplus \mathbb{E}_8 \oplus \mathbb{E}_9 \oplus \mathbb{E}_{11} \oplus \mathbb{E}_{12}\) and \(n > 1\), then the exterior derivative \(d(\varphi^2)(X)\) is given by

\[
d(\varphi^2)(X) = -d^\theta(\xi, X) + d^\theta(\xi, X) \xi - d^\theta(\xi, X) \xi + (\xi(11)_X, \xi) \xi).
\]
Proof. Using $\nabla = \nabla^{U(n)} - \xi$ and $\nabla^{U(n)}\eta = 0$, the form $d^2\eta$ is written in terms of $\nabla^{U(n)}$ and $\xi$, i.e.

$$d^2\eta(X_1, X_2, X_3) = \sum_{1 \leq a < b \leq 3} (-1)^{a+b} \left( \left( (\nabla^{U(n)}_{X_a})_b \right)(X_e) \right) \eta(X_e)$$

$$+ \sum_{1 \leq a < b \leq 3} (-1)^{a+b} (\xi_{X_a}X_b - \xi_{X_b}X_a) \eta(X_e)$$

$$- \sum_{1 \leq a < b \leq 3} (-1)^{a+b} (\xi_{X_a}, \xi_{X_b}) \eta(X_e, X_d),$$

where $\{e\} = \{1, 2, 3\} - \{a, b\}$ in each case. Note that, $\Lambda^3 T^* M$ is decomposed into

$$\Lambda^3 T^* M = [\lambda^3, 0] + [\lambda^0, 2, 1] + [\lambda^1, 0] \land F + [\lambda^2, 0] \land \eta + \mathbb{R} F \land \eta + [\lambda^0, 1] \land \eta, \quad (3.3)$$

under the action of $U(n) \times 1$. Now, we compute the parts of $d^2\eta$ in $[\lambda^1, 0] \land F$ and $\mathbb{R} F \land \eta$, by contracting $d^2\eta$ with $F$ on the first to arguments. Then we have

$$0 = \frac{1}{2} d^2\eta(e_i, \varphi e_i, X) = -((\nabla^{U(n)}_{e_i})_{\varphi e_i} \eta)(X) + ((\nabla^{U(n)}_{\varphi e_i})_{e_i} \eta)(\varphi e_i) - ((\nabla^{U(n)}_{\varphi e_i})_{\varphi e_i} \eta)(\varphi e_i)$$

$$+ (\xi_{e_i} \eta)(\varphi e_i) + (\xi_{\varphi e_i} \eta)(\varphi e_i) - (\xi_{\varphi e_i} \eta)(\varphi e_i)$$

$$- (\xi_{\varphi e_i} \eta)(\varphi e_i) - (\xi_{\varphi\varphi e_i} \eta)(X) + (\xi_{\varphi e_i} \eta)(\varphi e_i). \quad (3.4)$$

Now, by considering $X = X_\xi$ and each component $\xi^i$ of the intrinsic torsion beside with its properties, it follows the first required identity. Here it is used

$$(\nabla^{U(n)}_{Z \xi})_{\eta}(X) = \frac{1}{2n} (d^2\eta)(\varsigma)(Z)(F(X, Y) \varsigma + \eta(Y) \varphi(X)).$$

From this, it is obtained $((\nabla^{U(n)}_{\xi})_{e_i} \eta)(Z) = -\frac{1}{2n} (d^2\eta)(\varsigma)(\varphi Z).$

For the second identity required in Lemma, by taking $X = \xi$ in the identity (3.4) and considering each component $\xi^i$ of the intrinsic torsion beside with its properties, it follows the second required identity. In the computation it is used

$$((\nabla^{U(n)}_{\xi})_{\eta})\varphi e_i = - (d\xi \eta, F) - (\xi_{(4)} e_i, \varphi \xi \varsigma) = \text{div} \varphi \xi \eta + (\xi_{(4)} e_i, \varphi \xi \varsigma). \quad \square$$

Next, using the identity $d^2\eta = 0$, we will study the contributions of the components $\xi^i$ in the exterior derivative $d\xi \eta$. Firstly we give an expression for $d\xi \eta$ in terms of $\nabla^{U(n)}$ and $\xi$.

**Proposition 3.8.** For almost contact metric manifolds, the exterior derivative of the one-form $\xi \eta$ which determines the $\xi_{12}$-component of the intrinsic torsion is given by

$$d\xi \eta(X, Y) = ((\nabla^{U(n)}_{\xi}) X \eta)(Y) - ((\nabla^{U(n)}_{\xi}) Y \eta)(X) + (\xi X \eta)(\xi Y) - (\xi Y \eta)(\xi X)$$

$$- (\xi_{XX} \xi \eta)(Y) + (\xi_{XY} \eta)(X) + (\xi_{XX} \eta)(Y) - (\xi_{XX} \xi \eta)(X).$$

**Proof.** It follows from the identity $d^2\eta(X, Y, \varsigma) = 0$. In fact, we have

$$(d^2\eta)(X, Y, \varsigma) = ((\nabla^{U(n)}_{X} \xi) \eta)(Y) - ((\nabla^{U(n)}_{Y} \xi) \eta)(X) - ((\nabla^{U(n)}_{\varsigma} \xi) X \eta)(Y) + ((\nabla^{U(n)}_{\varsigma} \xi) Y \eta)(X)$$

$$+ (\xi \eta)(X Y - (\xi \eta)(X Y) - (\xi \eta)(X Y) + (\xi \eta)(X Y) + (\xi \eta)(X Y) + (\xi \eta)(X Y) + (\xi \eta)(X Y).$$

Since we have $((\nabla^{U(n)}_{X} \xi) \eta)(Y) = ((\nabla^{U(n)}_{X} \xi) \eta)(Y)$ and

$$d\xi \eta(X, Y) = ((\nabla^{U(n)}_{X} \xi) \eta)(Y) - (\nabla^{U(n)}_{Y} \xi \eta)(X) + (\xi \eta)(X Y) - (\xi \eta)(X Y),$$

it is obtained the required identity. \quad \square
In next lemma, \( (d\xi, \eta)_V \) will denote the projection of \( d\xi, \eta \) on the \( U(n) \)-space \( V \).

**Lemma 3.9.** The \( U(n) \)-components of \( d\xi, \eta \) are given by:

\[
(d\xi, \eta)_{U(n)} = \frac{1}{n}(d\xi, \eta, F)^n F,
\]

where

\[
(d\xi, \eta, F) = -d(d^* F(\zeta))(\zeta) + \frac{1}{n}d^* \eta d^* F(\zeta) + 2(\xi(7) e_i, F(\varphi \xi(8) e_i, \zeta)) + 2(\xi(10) e_i, \eta) (\varphi(\xi(11) e_i, \zeta)),
\]

\[
(d\xi, \eta, [\lambda, 1, 1])_{(X, Y)} = -\frac{1}{n}d(d^* F(\zeta))(\zeta)(F(X, Y))^n + \frac{1}{n}d^* \eta d^* F(\zeta)(F(X, Y)^n) + 2(((\nabla^{U(n)}_\zeta)_{\xi(7) X})(\eta(Y)) - \frac{2}{n}d^* \eta(\xi(7) X)(\eta(Y) + \frac{2}{n}d^* F(\zeta)(\xi(8) Y)(\eta(Y) - 2(\xi(7) X, Y)) (\varphi Y) - 2(\xi(7) X, Y)(\xi(8) Y)(\zeta) + 2(\xi(7) X, Y)(\xi(8) Y)(\zeta) + 2(\xi(9) X, Y)(\xi(10) Y)(\zeta) - 2(\xi(9) Y, \zeta)(\xi(10) X, Y) + 2(\xi(10) X, Y)(\xi(11) Y, \zeta) - 2(\xi(10) Y, \zeta)(\xi(11) X, Y),
\]

\[
(d\xi, \eta, [\lambda, 2, 0])_{(X, Y)} = 2(((\nabla^{U(n)}_\zeta)_{\xi(10) X})(\eta(Y)) - \frac{2}{n}d^* F(\zeta)(\xi(11) X, Y)(\eta(Y) + 2(\xi(7) Y, \zeta)(\xi(9) X, Y) + 2(\xi(10) Y, \zeta)(\xi(11) X, Y) - 2(\xi(7) Y, \zeta)(\xi(11) X, Y) + 2(\xi(10) Y, \zeta)(\xi(11) X, Y) - 2(\xi(11) Y, \zeta)(\xi(10) X, Y),
\]

\[
(d\xi, \eta, [\lambda, 1, 0])_{\eta} = \eta \wedge \zeta d\xi, \eta, \text{ where}
\]

\[
\zeta d\xi, \eta(X) = ((\nabla^{U(n)}_\zeta)_{\xi(12) X})(\eta(X) + (\xi(12) \eta)(\xi(11) X) - \frac{1}{2n}d^* \eta(\xi(12) \eta)(X) - (\xi(9) \xi(12) \eta)(X) - (\xi(9) \xi(12) \eta)(X) - \frac{1}{2n}d^* F(\zeta)(\xi(12) \eta)(\varphi X) + (\xi(7) \xi(12) \eta)(X) + (\xi(10) \xi(12) \eta)(X).
\]

**Proof.** It follows from the expression for \( d\xi, \eta \) given in Proposition 3.8.

Now we give some sufficient conditions for the vanishing of components of \( d\xi, \eta \).

**Proposition 3.10.**

(i) If the structure is of type \( C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5 \oplus C_9 \oplus C_{12} \oplus C_x \oplus C_y, \) where \( (x, y) \in \{(7, 10), (7, 11), (8, 10), (8, 11)\}, \) then \( \langle d\xi, \eta, F \rangle = 0. \) In such a case, we have \( \text{div } \varphi_{\zeta, \xi} = \left< n-1 \right> \eta \left( \varphi_{\zeta, \xi} \right) \).

(ii) If the structure is of type \( C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5 \oplus C_9 \oplus C_{11} \oplus C_{12} \) or \( C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5 \oplus C_x \oplus C_10 \oplus C_{12}, \) where \( x \in \{6, 8\}, \) then \( \langle d\xi, \eta, [\lambda, 1, 1] \rangle = 0. \)

(iii) If the structure is of type \( C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5 \oplus C_9 \oplus C_{11} \oplus C_{12} \) or \( C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5 \oplus C_{10} \oplus C_{12}, \) then \( \langle d\xi, \eta, [\lambda, 1, 0] \rangle = 0. \)

(iv) If the structure is of type \( C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5 \oplus C_8 \oplus C_{12} \oplus C_x \oplus C_y, \) then \( \langle d\xi, \eta, [\lambda, 2, 0] \rangle = 0. \)

(v) If the structure is of type \( C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5 \oplus C_8 \oplus C_9 \oplus C_{11} \oplus C_{12}, \) then \( d(\xi, \eta, [\lambda, 1, 1] = 0 \) and \( \langle d\xi, \eta, [\lambda, 2, 0] \rangle = 0. \) That is, \( d\xi, \eta \in \eta \wedge [\lambda, 1, 0]. \)

**Proof.** All parts are direct consequences of Lemma 3.9.

4. **Non-existence of certain types of almost contact metric structures**

The main purpose of this section is to prove that there are certain types of almost contact metric structure, which were initially possible from algebraic point of view, that do not exist because of geometry. A first result in this sense was proved by Marrero in [10]. He showed
that if a connected manifold of dimension higher than 3 is equipped with an almost contact metric structure of type \( C_5 \oplus C_6 \), then it must be of one of the singles types \( C_5 \) or \( C_6 \). This is a particular case of the result proved here.

**Theorem 4.1.** For a connected almost contact metric manifold of dimension \( 2n + 1 \), \( n > 1 \), we have:

(i) If the structure is of type \( C_1 \oplus C_2 \oplus C_3 \oplus C_5 \oplus C_6 \oplus C_8 \oplus C_9 \oplus C_{11} \oplus C_{12} \) with \( \langle d\xi \eta, F \rangle = 0 \) and \( (\xi(5), \xi(6)) \neq (0,0) \), then it is of type \( C_1 \oplus C_2 \oplus C_3 \oplus C_5 \oplus C_6 \oplus C_8 \oplus C_9 \oplus C_{11} \oplus C_{12} \), i.e. \( \xi(6) = 0 \), or of type \( C_2 \oplus C_6 \oplus C_9 \oplus C_{12} \) with \( \xi(6) \neq 0 \). Likewise, for this last type (with the previously fixed condition \( \langle d\xi \eta, F \rangle = 0 \)), \( d^* F(\xi) = d^* F(\xi) \xi \eta \eta \) and \( \xi \eta \eta \) is closed.

(ii) If the structure is of type \( C_2 \oplus C_5 \oplus C_7 \oplus C_9 \) being \( (\xi(5), \xi(7)) \neq (0,0) \), then it is of type \( C_2 \oplus C_7 \oplus C_9 \), or of type \( C_2 \oplus C_5 \oplus C_9 \).

**Proof.** For (i), from [3,2], one has \( d(d^* F(\xi)) = d^* F(\xi) \xi \eta \eta + \frac{1}{n} d^* \eta d^* F(\xi) \eta \). This implies

\[
0 = \frac{1}{n} d^* \eta d^* F(\xi) \eta \wedge \xi \eta \eta + d^* F(\xi) d\xi \eta + \frac{1}{n} d^* F(\xi) d(d^* \eta) \wedge \eta + \frac{1}{n} d^* \eta d^* F(\xi) \xi \eta \wedge \eta \\
+ \frac{1}{n} d^* \eta d^* F(\xi) \xi \eta \eta.
\]

Therefore,

\[
0 = d^* F(\xi) d\xi \eta + \frac{1}{n} d^* F(\xi) d(d^* \eta) \wedge \eta + \frac{1}{n} d^* \eta d^* F(\xi) \xi \eta \eta.
\]

Since the third summand must be zero, one has \( d^* \eta d^* F(\xi) = 0 \). From this it follows

\[
d(d^* F(\xi)) = d^* F(\xi) \xi \eta \eta, \quad d^* F(\xi) d\xi \eta = 0, \quad d^* F(\xi) d(d^* \eta) = 0.
\]

If \( d^* F(\xi) = 0 \) on a point, then \( d(d^* F(\xi)) = 0 \) on that point and \( d^* F(\xi) = 0 \) on the whole connected manifold. Hence \( \xi(6) = 0 \).

If \( d^* F(\xi) \) is non-zero, then \( d(d^* F(\xi)) = 0 \) on that point and \( d(d^* \eta) = 0 \) in the whole manifold. Hence \( \xi(5) = 0 \), \( \xi(6) \neq 0 \), and, using Lemma [3.9] we have

\[
0 = (d\xi \eta)_{[X,Y]} = 2 d^* F(\xi) (\xi(8) \xi \eta)(\varphi Y) = \frac{1}{n} d^* F(\xi) (\xi(11) \xi X, \varphi Y).
\]

Therefore, \( \xi(5) = 0 \) and \( \xi(11) = 0 \).

On other hand, since \( d\eta = \frac{d^* F(\xi)}{n} F + \xi \eta \wedge \eta \), we have \( 0 = \frac{1}{n} d^* F(\xi) d F \). This implies \( \xi(11) = 0 \) and \( \xi(3) = 0 \) (see comments about \( d F \) in Section [5]).

For (ii), from Lemma [3.1], we have \( d^* \eta (\xi(11) \eta) = 0 \) which, in this case, is equivalent to \( d^* \eta d\eta = 0 \). Then, doing the exterior derivative in both sides and taking Proposition [3.1] into account, we obtain \( d(d^* \eta) (\xi \eta \eta \eta) = 0 \), where \( d(d^* \eta) = d(d^* \eta) (\xi \eta) \eta \). If there is a point where \( \xi(7) \neq 0 \), then \( d^* \eta = 0 \) and \( d(d^* \eta) = 0 \) on the whole manifold. Thus \( \xi(7) \neq 0 \) everywhere with \( \xi(5) = 0 \) or \( \xi(7) = 0 \) everywhere with \( \xi(5) \neq 0 \).

The following result is an immediate consequence of previous Theorem.

**Corollary 4.2.** For a connected almost contact metric manifold of dimension \( 2n + 1 \), \( n > 1 \), if the structure is of type \( C_1 \oplus C_2 \oplus C_3 \oplus C_5 \oplus C_6 \oplus C_8 \oplus C_9 \oplus C_{11} \) with \( (\xi(5), \xi(6)) \neq (0,0) \), then it is of type \( C_1 \oplus C_2 \oplus C_3 \oplus C_5 \oplus C_8 \oplus C_9 \oplus C_{11} \) or of type \( C_2 \oplus C_6 \oplus C_9 \).
Remark 4.3. This implies that, for higher dimensions, the possible number of classes to really consider is $2^{12} - (2^6 + 2^4 + 2^2 + 2^2) = 3964$. That is, 132 classes do not properly exist because of geometry. At the beginning, the number of algebraically possible classes is $2^{12} = 4096$.

The type $C_2 \oplus C_0 \oplus C_0$ could be referred to as almost a-Sasakian structure, where $a = \frac{d^* F(\xi)}{n}$ which is constant for this type in case of $n > 1$. Almost Sasakian structure, $a = 1$, have been considered in references (see [9][13]).

5. Examples

In this section we will display some examples. But previously, we will describe some preliminary material which will help us to understand them. We will begin computing some components of the intrinsic torsion $\xi$ by using exterior algebra. Thus we will compute the $U(n)$-components of the exterior derivatives $d\eta$ and $dF$. To complete the information about $\xi$, we will also need to compute $U(n)$-components of the Nijenhuis tensor $N_\varphi$ of the tensor $\varphi$.

The form $d\eta$ is in $\bigwedge T^*M = RF \oplus [\lambda_0^{1,1}] \oplus [\lambda_0^{2,0}] \oplus \eta^* \Lambda^1 \Lambda^0$. Using the $U(n)$-map $\xi \to \text{alt}(\xi\eta)$, where $\text{alt}(\xi\eta)(X,Y) = (\xi_X\eta)(Y) - (\xi_Y\eta)(X) = d\eta(X,Y)$, the information about some components of $\xi$ is translated to the components of $d\eta$: $d\eta_{R,F} = -\text{alt}(\xi(6)\eta) = 2\xi(6)\eta$, $d\eta_{\lambda_0^{1,1}} = -\text{alt}(\xi(7)\eta) = 2\xi(7)\eta$, $d\eta_{\lambda_0^{2,0}} = -\text{alt}(\xi(10)\eta) = 2\xi(10)\eta$ and $d\eta'_{\lambda_0^{1} \wedge \lambda_0^{1}} = -\text{alt}(\xi(12)\eta) = 2\xi(12)\eta$. The form $dF$ is in $\bigwedge T^*M$ which is decomposed as in [33]. In a similar way as above, by using $U(n)$-map $\xi \to \text{alt}(\xi F)$, where $\text{alt}(\xi F)(X,Y,Z) = (\xi_X F)(Y,Z) + (\xi_Y F)(X,Z) = dF(X,Y,Z)$, the information about some components of $\xi$ is translated to the components of $dF$: $dF_{\lambda_0^{3,0}} = -\text{alt}(\xi(1)\eta)$, $dF_{\lambda_0^{2,1}} = -\text{alt}(\xi(3)\eta)$, $dF_{\lambda_0^{1} \wedge \lambda_0^{1}} = -\text{alt}(\xi(4)\eta)$, $dF_{\lambda_0^{1} \wedge \eta} = -\text{alt}(\xi(5)\eta)$, $dF_{\lambda_0^{2,0} \wedge \eta} = -\text{alt}(\xi(8)\eta)$ and $dF_{\lambda_0^{2,0} \wedge \eta} = -\text{alt}(\xi(10) \wedge \xi(11))\eta$. Note that the component $dF_{\lambda_0^{2,0} \wedge \eta}$ contains partial information of $\xi(10)$ and partial information of $\xi(11)$. This is because some diagonal of the space $C_{10} \oplus C_{11}$ is included in $\ker(\text{alt}(\xi F))$ and $\text{alt}(\xi F)(C_{10} \oplus C_{11})$ is the isomorphic image of the orthogonal complementary in $C_{10} \oplus C_{11}$ of the mentioned diagonal, $dF_{\lambda_0^{2,0} \wedge \eta} = \eta \wedge (2\xi(10)\eta \circ \varphi - \xi(11)\eta)$. However, if one considers together $d\eta_{\lambda_0^{2,0}}$ and $dF_{\lambda_0^{2,0} \wedge \eta}$, the whole information about both, $\xi(10)$ and $\xi(11)$, is available.

From all of this, $\xi(2)$ and $\xi(9)$ are the only components of the intrinsic torsion that we have not information yet. Thus we need also to consider the Nijenhuis tensor $N_\varphi$ of $\varphi$. It is defined by $N_\varphi(X,Y) = -\varphi^2 X, Y - [\varphi X, \varphi Y] + [\varphi X, Y] + [\varphi X, Y]$ and contains information about $\xi(2)$ and $\xi(9)$. Hence by analyzing $d\eta$, $dF$ and $N_\varphi$, we will completely determine $\xi$ and locate the type of almost contact metric structure.

Next we describe some properties of the tensor $N_\varphi$ (see [2]). The tensor $N_\varphi$ is in $\bigwedge T^*M \otimes TM$ and satisfies the properties:

\[
N_\varphi(\varphi X, \varphi Y) = -N_\varphi(X, Y) + \eta(X)N_\varphi(\zeta, Y) - \eta(Y)N_\varphi(\zeta, X) + d\eta(\varphi X, \varphi Y)\zeta + d\eta(\varphi^2 X, \varphi^2 Y)\zeta,
\]

\[
\varphi N_\varphi(X, Y) = -N_\varphi(X, \varphi Y) + \eta(Y)N_\varphi(\zeta, \varphi X) + d\eta(\varphi X, \varphi^2 Y)\zeta,
\]

\[
\eta(N_\varphi(X, Y))\zeta = d\eta(\varphi X, \varphi Y)\zeta.
\]
Note that, in particular, \( N_\varphi(\zeta, \varphi X) = -\varphi N_\varphi(\zeta, X) \) and \( \eta(N_\varphi(\zeta, X)) = 0 \). If we consider the almost contact structure without metric, just as a \( GL(n, \mathbb{C}) \)-structure, the tensor \( N_\varphi \) is in \( W_1 \oplus W_2 \oplus W_3 \oplus W_4 \)

\[
W_1 = \{ t \in \Lambda^2 T^*M \otimes TM \mid t(\varphi X, \varphi Y) = -t(X, Y), \varphi t(X, Y) = -t(X, \varphi Y) \}, \\
W_2 = \{ t \in \Lambda^2 T^*M \otimes TM \mid t(X, Y) = \eta(X)t(\zeta, Y) - \eta(Y)t(\zeta, X), \eta(t(X, Y)) = 0 \}, \\
W_3 = \{ t \in \Lambda^2 T^*M \otimes TM \mid t(X, Y) = \eta(t(X, Y))\zeta, \varphi t(X, \varphi Y) = t(X, Y) \}, \\
W_4 = \{ t \in \Lambda^2 T^*M \otimes TM \mid t(X, Y) = \eta(t(X, Y))\zeta, \varphi t(X, \varphi Y) = -t(X, Y) \}.
\]

The \( GL(n, \mathbb{C}) \)-components of \( N_\varphi \) are given by

\[
N_{\varphi W_1}(X, Y) = N_{\varphi}(X, Y) - \eta(X)N_{\varphi}(\zeta, X) - \eta(Y)N_{\varphi}(\zeta, X) - d\eta(\varphi X, \varphi Y)\zeta, \\
N_{\varphi W_2}(X, Y) = \theta(X)N_{\varphi}(\zeta, Y) - \eta(Y)N_{\varphi}(\zeta, X), \\
N_{\varphi W_3}(X, Y) = \frac{1}{2}(d\eta(\varphi X, \varphi Y) + d\eta(\varphi^2 X, \varphi^2 Y))\zeta, \\
N_{\varphi W_4}(X, Y) = \frac{1}{2}(d\eta(\varphi X, \varphi Y) - d\eta(\varphi^2 X, \varphi^2 Y))\zeta.
\]

**Remark 5.1.** For an almost contact structure, the structure tensor (a notion defined in [5, 6]) is determined by the part \( T_\xi \) of the torsion \( T \) of a \( GL(n, \mathbb{C}) \)-connection in the \( GL(n, \mathbb{C}) \)-complementary part \( \xi \) of the image by the Spencer operator of \( T^*M \otimes \mathfrak{gl}(n, \mathbb{C}) \) in \( \Lambda^2 T^*M \otimes TM \).

It turns out \( \xi = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus W_5 \), where

\[
W_5 = \{ t \in \Lambda^2 T^*M \otimes TM \mid t(X, Y) = \eta(X)\eta(t(\zeta, X))\zeta - \eta(Y)\eta(t(\zeta, X))\zeta \}.
\]

The \( GL(n, \mathbb{C}) \)-components of \( T_\xi \) are given by

\[
8T_{W_1}(X, Y) = -2N_{\varphi}(X, Y) + \eta(X)N_{\varphi}(\zeta, X) - \eta(Y)N_{\varphi}(\zeta, X) + 2d\eta(\varphi X, \varphi Y)), \\
2T_{W_2}(X, Y) = -\eta(X)N_{\varphi}(\zeta, X), \\
2T_{W_3}(X, Y) = (d\eta(\varphi^2 X, \varphi^2 Y) + d\eta(\varphi X, \varphi Y))\zeta, \\
2T_{W_4}(X, Y) = (d\eta(\varphi^2 X, \varphi^2 Y) - d\eta(\varphi X, \varphi Y))\zeta, \\
T_{W_5}(X, Y) = (\eta \wedge (\zeta \land d\eta))(X, Y)\zeta.
\]

The existence of a torsion free \( GL(n, \mathbb{C}) \)-connection is equivalent to the vanishing of the structure tensor. Therefore, in the case of almost contact structure is equivalent to \( N_\varphi = 0 \) and \( d\eta = 0 \). These last conditions are also equivalent to the integrability of the almost contact structure. We recall that, in general, the integrability of a \( G \)-structure implies that the corresponding structure tensor is zero. The converse is not true. Only for certain particular cases, as almost complex structure, the vanishing of the structure tensor implies integrability (Newlander-Nirenberg’s Theorem). For almost contact structures, this is also the case.

Now, in the presence of a compatible metric, one can relate \( N_\varphi \) with \( \xi \) by the \( U(n) \)-map \( \xi \to N(\xi) \), where

\[
N(\xi)(X, Y) = -\varphi(\xi X \varphi X) Y + \varphi(\xi Y \varphi Y) X + (\xi \varphi X \varphi Y) Y - (\xi \varphi Y \varphi X) X = N_\varphi(X, Y).
\]

It turns out that \( \ker(N) = \mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_8 \oplus \mathcal{C}_{10,11} \oplus \mathcal{C}_{12} \)

\[
N_{\varphi W_1} = N(\xi_1 + \xi_2), \quad N_{\varphi W_2} = N(\xi_9 + \xi_10 + \xi_{11}) - 2(\xi_{10} \eta) \otimes \zeta, \\
N_{\varphi W_3} = N(\xi_6 + \xi_7) = d\eta_{\Lambda^{1,1}} \otimes \zeta, \quad N_{\varphi W_4} = -d\eta_{\Lambda^{2,0}} \otimes \zeta = 2(\xi_{10} \eta) \otimes \zeta,
\]
where \( \mathcal{C}_{10,11} \cong [\lambda^{2,0}] \) denotes certain diagonal in the space \( \mathcal{C}_{10} \oplus \mathcal{C}_{11} \). Therefore, the remaining information about \( \xi \) above mentioned, included in \( \xi_{(2)} \) and \( \xi_{(9)} \), is located in \( N_{\varphi}W_1 \) and \( N_{\varphi}W_2 \).

**Example 5.2 (The hyperbolic space).** The following example has been already considered in [2]. Let \( \mathcal{H} = \{(x_1, \ldots, x_{2n+1}) \in \mathbb{R}^{2n+1} : x_1 > 0 \} \) be the \((2n+1)\)-dimensional hyperbolic space with the Riemannian metric

\[
\langle \cdot, \cdot \rangle = \frac{1}{x_1^2} (dx_1 \otimes dx_1 + \cdots + dx_{2n+1} \otimes dx_{2n+1}).
\]

With respect to this metric, \( \{E_1, \ldots, E_{2n+1}\} \) is an orthonormal frame field, where \( E_i = cx_1 \frac{\partial}{\partial x_i}, \)

\( i = 1, \ldots, 2n+1 \). For the Lie brackets, one has \( [E_1, E_j] = cE_j, \) \( j = 2, \ldots, 2n+1 \). The remaining Lie brackets relative to this frame are zero.

The corresponding metricaly equivalent coframe is \( \{e_1, \ldots, e_{2n+1}\} \), where \( e_i = \frac{1}{cx_1} dx_i \). Note that \( de_i = -ce_1 \wedge e_i, \) \( i = 1, \ldots, 2n+1 \).

The almost contact metric structure \( (\varphi = \sum_{j=1}^{2n+1} \varphi_j e_j \otimes E_i, \xi, \eta, \langle \cdot, \cdot \rangle) \) is considered in [2]. The functions \( \varphi_j \) are constant, \( n \geq 2 \) and \( \xi = \sum_{i=1}^{2n+1} x_1 k_i \frac{\partial}{\partial x_i} = \sum_{i=1}^{2n+1} k_i E_i \), being \( k_i = \) constant and \( k_1^2 + \cdots + k_{2n+1}^2 = c^2 \). The one-form \( \eta \) and the fundamental form \( F \) are given by

\[
\eta = \sum_{i=1}^{2n+1} k_i e_i, \quad F = \sum_{i,j=1}^{2n+1} \varphi_j e_i \wedge e_j.
\]

Then their exterior derivatives are expressed as \( d\eta = -ce_1 \wedge \eta, \) \( dF = -2ce_1 \wedge F \). Hence \( d\eta = \xi \eta \wedge \eta \in [\lambda^{1,0}] \cong \mathcal{C}_{12} \), where \( \xi \eta = k_1 \eta - ce_1 \), and \( dF = 2(k_1 \eta - ce_1) \wedge F - 2k_1 \eta \wedge F \in [\lambda^{1,0}] \wedge F + \mathbb{R} \eta \wedge F \cong \mathcal{C}_4 \oplus \mathcal{C}_5 \). Since \( dF_{[\lambda^{1,0}] \wedge F} = \theta \wedge F \) and \( dF_{\mathbb{R} \eta \wedge F} = -\varphi \eta \wedge F \), we have

\[
\theta = 2\xi \eta, \quad d^* \eta = 4nk_1.
\]

Now, by using the Lie brackets described above, one can check that \( N_{\varphi}(E_i, E_j) = 0 \). From all of this, the structure is of type \( \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_{12} \) as it is shown in [2].

Note that \( d\theta = k_1 \theta \wedge \eta, \) \( d\xi \eta = k_1 \xi \eta \wedge \eta \) and \( d(d^* \eta) = 0 \). From \( d\xi \eta = k_1 \xi \eta \wedge \eta \), using Lemma 3.3 we deduce \( \nabla^{U(n)} \xi \eta = 0 \). This can be checked by using \( \nabla^{U(n)} = \nabla + \xi \) and taking into account that for the Levi Civita connection one has

\[
\nabla_{E_i} E_i = cE_i, \quad \nabla_{E_i} E_1 = -cE_i, \quad i = 2, \ldots, 2n+1,
\]

being \( \nabla_{E_i} E_j = 0 \) for the remaining cases. In fact,

\[
\nabla^{U(n)} \xi \eta = -c\nabla \xi E_1 - c\xi_{(12)} E_1 = c\sum_{i=2}^{2n+1} k_i E_i - c(c \xi - k_1 E_1) = 0.
\]

From these comments, it is also immediate to check the identities for the components of \( d\theta \) given in Propositions 3.2, 3.4 and 3.6.

Particular cases are:

(i) \( k_1 = 0 \) and \( n > 1 \). The structure is of strict type \( \mathcal{C}_4 \oplus \mathcal{C}_{12} \). The one-forms \( \theta \) and \( \xi \eta \) are closed. In fact, \( \xi \eta = -d(\ln x_1) \). If we do the conformal change of metric \( x_1^2 \langle \cdot, \cdot \rangle \), we obtain the flat cosymplectic structure on \( \mathcal{H} \) as an open set of \( \mathbb{R}^{2n+1} \) with the Euclidean metric.

(ii) \( k_1 = 0 \) and \( n = 1 \). The structure is of strict type \( \mathcal{C}_{12} \) and \( \xi \eta \) is closed.

(iii) \( k_1 = 1 \). The structure is of strict type \( \mathcal{C}_5 \) with \( d^* \eta = 2n \).
Some part of $\mathcal{H}$ with another metric. Now we take the subset \( \{ p \in \mathcal{H} \mid x_2(p) > 0 \} \) and, on this set, consider \( c = 1 \), the one-form \( e_0 = \frac{x_2}{x_1} dx_1 \) and the metric
\[
\langle \cdot, \cdot \rangle_0 = e_0 \otimes e_0 + \frac{1}{x_1^2} \sum_{i=2}^{2n+1} dx_i \otimes dx_i.
\]

An orthonormal frame field is given by \( \{ E_0 = \frac{x_1}{x_2} \frac{\partial}{\partial x_1}, E_2, \ldots, E_{2n+1} \} \).

A first almost contact metric structure: Now we consider the almost contact metric structure
\[
(\varphi = \sum_{i,j=2}^{2n+1} \varphi^j e_j \otimes E_i, \zeta = E_0, \eta = e_0, \langle \cdot, \cdot \rangle_0),
\]
such that \( \varphi^j \) are constant and \( n > 1 \).

The exterior derivative \( d\eta \) is given by
\[
d\eta = dx_2 \wedge \frac{1}{x_2^2} dx_1 = \frac{x_2}{x_1^2} e_2 \wedge \eta \in \left[ \lambda^{1,0} \right] \wedge \eta \cong \mathcal{C}_{12}.
\]

Hence it is obtained \( \xi_{\zeta} \eta = \frac{x_2}{x_1^2} e_2 = d(\ln x_2) \) which is closed.

Now taking \( de_i = -\frac{1}{x_2} \eta \wedge e_i, i = 2, \ldots, 2n + 1 \), into account, we have
\[
dF = -\frac{2}{x_2} \eta \wedge F \in \mathbb{R} \eta \wedge F \cong \mathcal{C}_5.
\]

Therefore, in this case \( d^* \eta = \frac{2n}{x_2} \) which is not constant. However, \( d(d^* \eta) = -d^* \eta \xi_{\zeta} \eta \) as it is expected by Proposition 3.6. In this case, \( d(d^* \eta)(\zeta) = 0 \).

For the Lie brackets, one has \( \left[ \zeta, E_2 \right] = \frac{x_2}{x_1^2} \zeta + \frac{1}{x_1} E_2, \left[ \zeta, E_i \right] = \frac{1}{x_2} E_i, i = 3, \ldots, 2n + 1 \). The remaining Lie brackets are zero. Taking this into account, it is obtained \( N_{\varphi} = 0 \).

From all of this, we conclude that the almost contact metric structure is of type \( \mathcal{C}_5 \oplus \mathcal{C}_{12} \).

Finally, if we do a conformal change of the metric \( e^a \langle \cdot, \cdot \rangle_0 \), where \( e^a = x_2^{-1} \), we will obtain an almost contact metric structure of type \( \mathcal{C}_4 \oplus \mathcal{C}_5 \). Denoting the new intrinsic torsion by \( \xi_a \) and the one-form \( \eta_a \) is the one metrically equivalent to the new Reeb vector field, we have \( \theta = -2d(\ln x_2) \), and \( d^* \eta_a = 2n \).

A second almost contact metric structure: Now we consider the almost contact metric structure
\[
(\varphi = \sum_{i,j=2}^{2n+1} \varphi^j e_j \otimes E_i, \zeta = E_0, \eta = e_0, \langle \cdot, \cdot \rangle_0),
\]
where \( \varphi^j \) are constant and \( n > 1 \).

The exterior derivatives of \( \eta \) and \( F \) are given by
\[
d\eta = -\frac{1}{x_2} e_0 \wedge \eta \in \left[ \lambda^{1,0} \right] \wedge \eta \cong \mathcal{C}_{12}.
\]

\[
dF = -\frac{2}{x_2} e_0 \wedge F + \frac{x_2}{x_1^2} \eta \wedge \varphi e_0 \wedge e_0 \in \left[ \lambda^{1,0} \right] \wedge F \wedge \eta \wedge \left[ \lambda^{1,1} \right] \cong \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_8.
\]

For the Nijenhuis tensor we obtain
\[
N_{\varphi} = \frac{1}{x_2} e_0 \wedge \varphi e_0 \otimes \varphi E_0 + \frac{1}{x_2} \sum_{i=3}^{2n+1} e_0 \wedge e_i \otimes E_i - \frac{x_2}{x_1^2} \eta \wedge e_0 \otimes E_0 + \frac{x_2}{x_1} \eta \wedge \varphi e_0 \otimes \varphi E_0.
\]

Thus in this case \( N_{\varphi} \in \mathcal{N}(\mathcal{C}_2) \oplus \mathcal{N}(\mathcal{C}_9) \). Therefore, the almost contact structure is of type \( \mathcal{C}_2 \oplus \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_8 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{12} \).

Note that in this case \( \theta = 2x_2 \xi_{\zeta} \eta - x_2 e_0 \), \( d^* \eta = -x_2 \). Hence \( d\theta = d\xi_{\zeta} \eta = 0 \), \( d(d^* \eta) = d^* \eta \xi_{\zeta} + (d^* \eta)^2 \eta \) and \( d(d^* \eta) = 0 \). Note that \( d(d^* \eta)(\zeta) = (d^* \eta)^2 \) which is not constant.

Finally, for the conformal change of the metric \( e^{2a} \langle \cdot, \cdot \rangle_0 \), where \( e^a = x_1 x_2^{-\frac{1}{n}} \), we will obtain an almost contact metric structure of type \( \mathcal{C}_2 \oplus \mathcal{C}_8 \oplus \mathcal{C}_9 \). In fact, \( \eta_a \) is closed and
\[
dF_a = x_2^{\frac{1}{n}} \eta_a \wedge \varphi a e_a \wedge e_a - \frac{1}{n} x_2^{\frac{1}{n}} \eta_a \wedge F_a \in \eta \wedge \left[ \lambda^{1,1} \right] \cong \mathcal{C}_8.
\]
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