On sound ranging in some non-proper metric spaces

Sergij V. Goncharov*

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Abstract

We consider the sound ranging, or source localization, problem — find the unknown source-point from known moments when the spherical wave of linearly, with time, increasing radius reaches known sensor-points — in some non-proper metric spaces (closed ball is not always compact). Under certain conditions we approximate the solution to arbitrary precision by the iterative processes with and without a stopping criterion. We also consider this problem in normed spaces with a strictly convex norm when the sensors are dense on the unit sphere.

Appended is the implementation of the approximation algorithm in Julia language.

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Introduction

Let \((X; \rho)\) be a metric space, i.e. the set \(X\) with the metric \(\rho: X \times X \rightarrow \mathbb{R}_+\). At unknown moment \(t_0 \in \mathbb{R}\) the unknown “source” \(s\) ∈ \(X\) “emits the (sound) wave”, which is the sphere \(\{x \in X \mid \rho(x; s) = v(t - t_0)\}\) at any moment \(t \geq t_0\). We assume that “sound velocity” \(v = 1\).

Let \(\{r_i\}_{i \in I} \subseteq X\) be an indexed set of known “sensors”. For each sensor we also know the moment \(t_i = t_0 + \rho(r_i; s)\) when it was reached by the expanding wave.

The sound ranging problem (SRP), also called source localization, is to find \(s\) from \((\{r_i\}; \{t_i\})\). Another name is time-difference-of-arrival (TDOA) problem, because to obtain \(s\), when possible, it suffices to know the delays \(t_i - t_j\) rather than “absolute” \(t_i\).

SRPs, usually in Euclidean space and with noisy measurements, appear in many circumstances; see e.g. [6, 1], [12, 9.1], [17, 6], [19, 1] for further references on quite large and diverse literature. For the first time they were considered and studied at least one century ago (see [2] pp. 33–39).

This paper is a generalization of [11], where we looked into SRP in proper (also called finitely compact or Heine-Borel) metric spaces, in which any closed ball is compact. Without Euclidicity in general case, the classical approach “solve equations \((t_i - t_0)^2 = \sum (\rho^{(i)} - s_j)^2\), where \(t_0\) and coordinates \(\{s_j\}\) of \(s\) are unknowns” (which we applied in [10] investigating the noiseless SRP in \(l_2\)) does not work even if the space is provided with some coordinates, because they are not so easily “extractable” from the equations \(t_i - t_0 = \rho(r_i; s)\). Instead, we described the “approximating” approach — the iterative process that converges to the source.

Here we propose the variations of more or less the same approach, adjusted to regard non-properness of underlying space. Being more general, they work in proper spaces as well. For the sake of convenience, some content from [11] is repeated with necessary modifications.

We introduce some notions — functions, sets, constructions etc. and the constraints they must satisfy — to formulate the approximation algorithm using them. Thus, if they are instantiated in any given space, the algorithm can be implemented in that space accordingly; see Appendix.

*Faculty of Mechanics and Mathematics, Oles Honchar Dnipro National University, 72 Gagarin Avenue, 49010 Dnipro, Ukraine. E-mail: goncharov@mmf.dnu.edu.dp.ua
Indeed, we are interested in an algorithm that does not require its “executor” (computer) to be too far beyond mental and physical reach “of sentient life in this universe”, particularly in terms of the elementary actions the executor can perform. Our executor, for example, cannot run card \( X \geq 2^{30} \) calculations in parallel (then we would simply “verify each \( x \in X \), if it is a solution, simultaneously”). However, some data it needs to operate on may be considered as obtained from an “oracle” that is able e.g. to calculate the exact sum of an infinite series in a finite time.

We deal with SRP in an “empty” space without other waves, reverberation, varying propagation velocity, imprecise measurements etc. — without “physics”; this simplification makes the delays \( t_i - t_j \) known exactly. For certain non-negative function that depends on these delays, we perform a root finding of “exclude & enclose” type (see [4]), — we search for its unique zero instead of search for its extremum, the latter would be an optimization approach, of a kind widely used in solving SRP with noised data in Euclidean space, particularly based on the maximum likelihood estimation, though other methods exist (see e.g. [1], [3], [5], [7], [9], [12], [16], [19]).

“•” Well-known statement (see [8], [13], [14], [15], [18]), included for the sake of completeness.

“♣” Additional assumption or constraint.

One may feel that this paper (except for Appendix) should belong to 1920–30s.

1 Preparations

We recall some basic terms and denotations to avoid ambiguity.

• “2nd \( \Delta \) inequality”: \( \forall x, y, z \in X \, |\rho(x; z) - \rho(x; y)| \leq \rho(x; y) \).

• Let \( f : A \times B \to \mathbb{R}_+ \). Then \( \forall u, v \in B : \sup_{w \in A} f(w; u) - \sup_{w \in A} f(w; v) \leq \sup_{w \in A} |f(w; u) - f(w; d)| \).

As usual, \( x_k \xrightarrow{k \to \infty} y \) means \( \rho(x_k; y) \xrightarrow{k \to \infty} 0 \).

• Continuity of metric: \( x_k \xrightarrow{k \to \infty} y \implies \rho(x_k; z) \xrightarrow{k \to \infty} \rho(y; z) \).

\( B(c; r) = \{ x \in X \mid \rho(x; c) < r \} \), \( B(c; r) = \{ x \in X \mid \rho(x; c) \leq r \} \), and \( S(c; r) = \{ x \in X \mid \rho(x; c) = r \} \) denote open ball, closed ball, and sphere with center \( c \) and of radius \( r \).

• For any \( B(c; r) \) and any \( a \in X \) with \( \rho(a; c) = d \) we have \( \forall x \in B(c; r) : d - r \leq \rho(x; a) \leq d + r \).

The closure of the set \( A \subseteq X \) is \( \bar{A} = \{ y \in X \mid \exists \{x_k\}_{k \in \mathbb{N}} \subseteq A : x_k \xrightarrow{k \to \infty} y \} \).

\( A \subseteq X \) is said to be compact if \( \forall \{x_k\}_{k \in \mathbb{N}} \subseteq A : \exists \{x_k\}_{k \in \mathbb{N}}, \exists x \in A : x_k \xrightarrow{l \to \infty} x \).

\( A \subseteq X \) is said to be relatively compact if \( \forall \{x_k\}_{k \in \mathbb{N}} \subseteq A : \exists \{x_k\}_{k \in \mathbb{N}} : \exists x \in X : x_k \xrightarrow{l \to \infty} x \).

• \( A \subseteq X \) is relatively compact iff \( \bar{A} \) is compact.

• If \( A \) is compact, then any closed subset of \( A \) is compact too.

The family of sets \( \{ C_j \}_{j \in J} \), \( C_j \subseteq X \), is said to be a cover of \( A \subseteq X \) if \( A \subseteq \bigcup_{j \in J} C_j \).

• The closed \( A \subseteq X \) is compact iff any open cover of \( A \) has finite subcover.

\( A \subseteq X \) is called bounded if \( \text{diam } A = \sup_{x, y \in A} \rho(x; y) < \infty \).

• \( A \) is bounded iff \( \exists B(c; r) : A \supseteq B(c; r) \).

The norm \( \| \cdot \| \) of a normed space \( (X; \| \cdot \|) \) is called strictly convex if spheres do not contain segments: \( \forall x, y \in S[\theta; 1] \) (where \( \theta \) is zero of \( X \) as linear vector space) such that \( x \neq y \), and \( \forall \lambda \in (0; 1) : \| \lambda x + (1 - \lambda y) \| < 1 \).

• The norm is strictly convex iff \( \Delta \) inequality becomes equality only for positively proportional elements: \( \forall x, y \in X, \| x + y \| = \| x \| + \| y \| \) and \( x \neq \theta \), then \( y = \lambda x \) for some \( \lambda \geq 0 \).

Now we proceed to the SRP. The source \( s \in X \) and the emission moment \( t_0 \in \mathbb{R} \) are unknown.

\( \blacksquare \) 1. The set of sensors \( R = \{ r_i \}_{i \in I} \) is bounded.
These sensors and the moments
\[ t_i = t_0 + \rho(r_i; s), \ i \in I \]
define the SRP (\( \{ r_i \}; \{ t_i \} \)) . Any pair \((s'; t') \in X \times \mathbb{R}\) satisfying the set of equations
\[ t_i = t' + \rho(r_i; s'), \ i \in I \]
is a solution of this SRP. Or, we call \( s' \in X \) itself a solution when \( t_i - \rho(r_i; s') \equiv \text{const} (= t') \).

\[ \text{Lemma 2.} \] The solution \( s \) of the SRP (\( \{ r_i \}; \{ t_i \} \)) is unique in \( X \).

This is not the general case; for example, when all \( r_i \) are equal, any \( s' \in X \) is a solution, with \( t' = t_1 - \rho(s'; r_1) \). On the other hand, in some spaces we can ensure such uniqueness by placing the sensors appropriately: in \( l_2 \) we take \( r_2 = \theta, r_i = e_{i-2} \) for \( i \geq 3 \), and \( r_1 = -e_1 \) (\[\text{[10} \) Prop. 4\]).

Definition 1. For any \( x \in X \) the backward moments \( \tau_i(x) := t_i - \rho(x; r_i), \ i \in I \).

\( \tau_i(x) \) is the moment when the wave must be emitted from \( x \) to reach \( r_i \) at the moment \( t_i \).

Since \( \sum_i p_i = 1 \) implies \( R \subseteq \mathcal{B}(c; r) \), we have \( \forall i \in I : |\tau_i(x)| \leq |t_0| + \rho(s; r_i) + \rho(x; r_i) \leq \sum_i p_i |\tau_i(x) - \tau_j(x)| \leq D_\infty(x) \leq 2T(x) \).

Hereinafter the usage of \( D_1 \) requires \( R \) to be finite or countable, while there is no such restriction for \( D_\infty \). We consider \( D_1 \) because in some circumstances its calculation may be “easier”.

By \( D(x) \) we denote either \( D_\infty(x) \) or \( D_1(x) \), though we assume the same choice for all \( x \in X \).

Proposition 1. \( s' \in X \) is the solution of the SRP \iff \( D(s') = 0 \).

Proof. If \( s' \) is such solution, then \( \tau_i(s') \equiv t' \Rightarrow \tau_i(s') - \tau_j(s') \equiv 0 \Rightarrow D(s') = 0 \). Contrariwise, \( D(s') = 0 \) (along with \( p_i > 0 \) for \( D_1 \)) implies \( \tau_i(s') \equiv t' \), and \( (s'; t') \) is the solution. \( \square \)

Corollary 1. \( D(x) \) has exactly one zero in \( X \), at \( x = s \). (Follows from \[ \text{Lemma 2} \].

Proposition 2. \( \forall x, y \in X : |D(x) - D(y)| \leq 2\rho(x; y) \).

Proof. \( |\tau_i(x) - \tau_i(y)| = |t_k - \rho(x; r_k) - t_k + \rho(y; r_k)| \leq \rho(x; y) \), therefore \( |D_\infty(x) - D_\infty(y)| \leq \sup_{i,j \in I} |\tau_i(x) - \tau_j(x) - \tau_i(y) + \tau_j(y)| \leq \sup_{i \in I} \tau_i(x) - \tau_i(y) + \sup_{j \in I} \tau_j(x) - \tau_j(y) = \sup_{i \in I} |\tau_i(x) - \tau_i(y)| + \sup_{j \in I} |\tau_j(x) - \tau_j(y)| \leq 2\rho(x; y) \).

Similarly, \( |D_1(x) - D_1(y)| = \sum_{i \in I} p_i \bigg| \bigg( \sum_{j \in I} p_j \tau_j(x) \bigg) - \bigg( \sum_{j \in I} p_j \tau_j(y) \bigg) \bigg| \leq \sum_{i \in I} p_i \bigg| \bigg( \sum_{j \in I} p_j \tau_j(x) \bigg) - \bigg( \sum_{j \in I} p_j \tau_j(y) \bigg) \bigg| \leq \sum_{i \in I} p_i \bigg| \bigg( \sum_{j \in I} p_j \tau_j(x) - \tau_i(y) \bigg) \bigg| \leq \sum_{i \in I} p_i \bigg| \bigg( \sum_{j \in I} p_j \tau_j(x) - \tau_i(y) \bigg) \bigg| \leq \sum_{i \in I} p_i \bigg| \bigg( \sum_{j \in I} p_j \tau_j(x) - \tau_j(y) \bigg) \bigg| \leq 2\rho(x; y) \) \( \square \)

\( D(\cdot) \) is a Lipschitz function (\[\text{[11} \) 9.4\]).

Corollary 2. \( D(x) \) is uniformly continuous on \( X \).
**Proposition 3.** Any relatively compact set in $X$ has SDN property.

**Proof.** Let $A$ be such set, $s \in A$, then $\forall \delta > 0$ let $G = A \setminus B(s; \delta)$, $\varepsilon = \inf_{x \in G} D(x)$; we claim that $\varepsilon > 0$. Indeed, $\forall k \in \mathbb{N}$ $\exists x_k \in G$: $\varepsilon \leq D(x_k) \leq \varepsilon + \frac{1}{k}$. Due to relative compactness $\exists \{x_k\}_{k \in \mathbb{N}}$.

Now, continuity of metric implies $\rho(x; s) \geq \delta$, hence by Cor. 1 $\varepsilon > 0$.

**Corollary 3.** Any compact set in $X$ has SDN property.

**Non-SDN example.** Let $(X, \rho)$ be $l_2$. By $E = \{e_i\}_{i \in \mathbb{N}}$ we denote the usual orthonormal basis of $l_2$, that is, the coordinates $e_j^{(i)} = \delta_{ij}$. Let $I = \mathbb{Z} \setminus \{0\}$, $R = \{r_i\}_{i \in I} := E \cup (-E) = \{\ldots; -e_2; -e_1; e_1; e_2; \ldots\}$ $\forall i \in \mathbb{N}$ $r_i = e_i$, $r_{-i} = -e_i$.

We claim that $A = B[\theta; 1]$ does not have SDN property.

$\Leftarrow$ Obviously, $[\text{1}]$ is satisfied. Let $s = \theta$ and $t_0 = 0$, then $\forall r_1 \in R$ $t_1 = 1$; $[\text{2}]$ holds because, assuming that $x \in l_2$ is a solution, it must be equidistant from all $r_i$, and $\forall i \in \mathbb{N}$ $\rho(x; r_i) = \rho(x; r_{-i})$ $\therefore$ $\sum_{j \in \mathbb{N}, j \neq i} x_j^2 + (x_i - 1)^2 = \sum_{j \in \mathbb{N}, j \neq i} x_j^2 + (x_i + 1)^2 \Rightarrow |x_i - 1| = |x_i + 1| \Rightarrow x_i = 0$ i.e. $x = s$.

Consider $\{x_n\}_{n \in \mathbb{N}}$ with the coordinates $x_j^{(n)} = 1/\sqrt{n}$ for $j \leq n$ and $x_j^{(n)} = 0$ for $j > n$.

$\rho(x_n; \theta) = \sqrt{\sum_{j=1, n}^{n} \frac{1}{n}} = 1$, so $x_n \in A \setminus B(\theta; \delta)$ for any $\delta \in (0; 1]$, $\forall i \in \mathbb{N}$

$\rho(x_n; r_i) = \begin{cases} \sqrt{\sum_{j=1, n, j \neq i}^{n} \frac{1}{n} + \frac{1}{n} + (1 - \frac{1}{\sqrt{n}})^2} = \sqrt{\frac{2}{\sqrt{n}}}, & i \leq n, \\ \sqrt{\sum_{j=1, n, j \neq i}^{n} \frac{1}{n} + 1} = \sqrt{2}, & i > n. \end{cases}$

Therefore $D_\infty(x_n) = \sup_{i \in I} \tau_i(x_n) - \inf_{i \in I} \tau_i(x_n) = \sup_{i \in I} \left[1 - \rho(x_n; r_i)\right] - \inf_{i \in I} \left[1 - \rho(x_n; r_i)\right] = \sqrt{2 - \frac{2}{\sqrt{n}}} - \sqrt{\frac{2}{\sqrt{n}}} \to 0$, which implies $\forall \delta \in (0; 1]$: $\inf_{x \in A \setminus B(\theta, \delta)} D_\infty(x) = 0$. $\therefore$

Take $x_n = \frac{i}{\sqrt{n}} \sum_{i=1}^{n} e_i$, to show that, with this $R$, $\forall r > 0$ $B[\theta; r]$ in $l_2$ is non-SDN. Cf. the reasonings from the end of Section 3 though, where $A$ is of the same kind, but $R = S[\theta; 1]$.

**Proposition 4.** If $s \in B[c; r]$, then $D(c) \leq 2r$.

**Proof.** If $s \in B[c; r]$, then $\rho(r_i; c) - r \leq \rho(r_i; s) \leq \rho(r_i; c) + r, i \in I$ $\Leftarrow$

$\Leftarrow t_0 = \left[t_i - \rho(r_i; s)\right]$, $i \in I$

hence $t_0 \in \bigcap_{i \in I} \left[t_i - \rho(r_i; c) - r; t_i - \rho(r_i; c) + r\right]$, $i \in I$

it is easy to see that $C \neq \emptyset$ iff $D_\infty(c) \leq 2r$. And $D_1(c) \leq D_\infty(c)$.

**Corollary 4.** If $D(c) > 2r$, then $s \notin B[c; r]$.

The balls $B[c; r]$ that pass the test $D(c) \leq 2r$ are “suspicious”: more “sophisticated” tests may or may not prove that $s \notin B[c; r]$.

**Proposition 5.** If $D(x) > 0$, $x \in B[y; r]$, and $r < \frac{1}{2}D(x)$, then $s \notin B[y; r]$.

**Proof.** By Prop. 2 $\rho(y; x) \leq r < \frac{1}{2}D(x) \Rightarrow |D(y) - D(x)| < \frac{1}{2}D(x) \Rightarrow D(y) > \frac{1}{2}D(x) > 0$.

Since $r < \frac{1}{2}D(x) < \frac{1}{2}D(y)$, Cor. 4 implies $s \notin B[y; r]$. $\blacksquare$
Covershapes and coverands. Covershape is a certain way to define the subset of $X$ by its anchor $x \in X$ and its size $r > 0$, such subset to contain at least $x$ and to be contained in $B[x; r]$. 

Coverand, denoted by $C[x; r]$, is the covershape defined by given $x$ and $r$.

The example of a covershape is “an intersection of $S \subseteq X$ and a closed ball whose center is in $S$”, one of corresponding coverands is e.g. $B[\theta; 1] \cap S$ ($x = \theta, r = 1$).

We distinguish them because, for a given space, we can use a single covershape, based on the properties of that space, while taking many instances of this “shape”, which are the coverands.

On the other hand, for one and the same space there are usually many covershapes as well.

Put differently, covershape is a type, and coverand is an object of that type.

We consider covershapes with the following 2 properties:

\begin{itemize}
  \item [\textbullet\textsuperscript{3}] For any coverand $C[c; r]$, there is a finite cover by $C[c_i; \frac{r}{2}]$, where $c_i \in C[c; r]$.
  \item [\textbullet\textsuperscript{4}] For any coverand $C[c; r] = F_0$ let $F_1 = \bigcup_{x \in F_0} C[x; \frac{r}{2}]$, ..., $F_k = \bigcup_{x \in F_{k-1}} C[x; \frac{r}{2}]$, ... Then $F_{\infty} = \bigcup_{k \in \mathbb{Z}_+} F_k$ has SDN property.
\end{itemize}

Note that $F_{k-1} \subseteq F_k$. Also, $F_{\infty} \subseteq B[c; 2r]$, $\forall x \in F_{\infty}$: $x \in F_k$ for some $k$, hence $x \in C[c_k; \frac{r}{2}] \subseteq B[c_k; \frac{r}{2}]$. In turn, $c_k \in C[c_{k-1}; \frac{r}{2}] \subseteq B[c_{k-1}; \frac{r}{2}]$, ..., $c_1 \in B[c; r]$. Thus $\rho(x; c) \leq \rho(x; c_k) + \rho(c_k; c_{k-1}) + ... + \rho(c_1; c) \leq \frac{r}{2} + \frac{r}{2} + ... + r \leq 2r$.

One general way to covershapes and corresponding coverands that satisfy \textbullet\textsuperscript{3} \textbullet\textsuperscript{4} lies in considering any $S \subseteq X$ which is proper under the same metric $\rho$: $\forall x \in S$, $\forall r > 0$ $B_S[x; r] := B[x; r] \cap S$ is compact. Then a closed ball in $S$ is a covershape in $X$, and for any given $x \in S$, $r > 0$ $B_S[x; r]$ is the coverand. \textbullet\textsuperscript{3} holds because $\{B(y; \frac{r}{2}) \cap S\}_{y \in B_S[x; r]}$ is the open (in $S$) cover of $B_S[x; r]$, which, due to properness of $S$, has a finite subcover $\{B(y_i; \frac{r}{2}) \cap S\}^n_{i=1}$, so much the more $\{B_S[y_i; \frac{r}{2}]\}^n_{i=1}$ covers $B_S[x; r]$; \textbullet\textsuperscript{4} holds too because $F_{\infty} \subseteq B_S[x; 2r]$, which is compact.

2 RC-algorithm

(RC is Refining Cover.) We assume that its “executor” calculates $D(x)$ at given $x$, builds a finite cover of $C[x; r]$ from \textbullet\textsuperscript{3} etc., and completes these actions in a finite time.

For the sake of simplicity, we add one more assumption, probably the most “restrictive” (and thus reducing the generality of our approach) one:

\begin{itemize}
  \item [\textbullet\textsuperscript{5}] Some coverand $C[c; r] \ni s$ is known.
\end{itemize}

Let $\delta > 0$ be any precision chosen in advance; our goal is to obtain $x \in X$ such that $\rho(x; s) < \delta$.

Step 0. Let $k := 1$, $C_0 := \{C[c; r]\}$, and $r_0 := r$. Also, choose $D = D_{\infty}$ or $D = D_1$.

Step 1. Let $C_k := \emptyset$.

For each coverand $C = C[y; r_{k-1}] \in C_{k-1}$, where $r_{k-1} = \frac{r}{2^k}$, by \textbullet\textsuperscript{3} there is the finite cover $\mathcal{C}$ of $C$, which consists of the coverands $C' = C[z; r_k]$, where $z \in C$ and $r_k = \frac{r}{2^k}$.

Consider each $C'$ in turn and test it as the corresponding $B' = B[z; r_k]$; if $D(z) \leq 2r_k$, then add $C'$ to $\mathcal{C}_k$, that is, let $\mathcal{C}_k := \mathcal{C}_k \cup \{C'\}$.

Since $s \in \bigcup_{C \in C_{k-1}} C$, at least 1 coverand $C'$ from these $\mathcal{C}$ contains $s$, thus by Prop. \textbullet\textsuperscript{4} $C'$ passes the test and appears in $\mathcal{C}_k$. Therefore, at the end of this step $\mathcal{C}_k \neq \emptyset$ and $s \in \bigcup_{C' \in \mathcal{C}_k} C'$.
Step 2. Let $c_k$ be the anchor of the arbitrarily chosen coverand from $C_k$.

Step 3. Let $d_k := r_k + \max_{\rho(c_k; c') \in C | c' \in C_k} \rho(c_k; c')$. If $d_k < \delta$, then let $x := c_k$ and halt; else let $k := k + 1$ and goto Step 1.

Proposition 6. This algorithm halts after a finite number of iterations, at that $\rho(x; s) < \delta$.

Proof. By $\bigcup_{\epsilon_k \in C_k} C \subseteq F_\infty$, which has SDN property. Hence $\exists \varepsilon > 0$: $z \in F_\infty$, $D(z) < \varepsilon$ imply $\rho(z; s) < \frac{1}{2} \delta$. When $r_k = \frac{\varepsilon}{2} < \frac{1}{2} \delta \Rightarrow k > \log_2 \frac{2\varepsilon}{\delta}$, for the coverands $C[c_k; r_k]$ and any $C[c'; r_k]$ to be in $C_k$ it is necessary that $D(c_k), D(c') \leq 2r_k < \varepsilon$, thus $\rho(c_k; c') \leq \rho(c_k; s) + \rho(s; c') < \frac{1}{2} \delta$.

As soon as we reach $k$ such that $r_k < \frac{1}{2} \delta$ and $r_k < \frac{1}{2} \delta$ (the latter holds when $k > \log_2 \frac{2\varepsilon}{\delta}$), we have $d_k < \frac{1}{2} \delta + \frac{1}{2} \delta = \delta$, and the algorithm halts with $x = c_k$, where $C[c_k; r_k] \in C_k$.

Of course, $d_k < \delta$ may become true for $k$ even smaller than $\max\{\log_2 \frac{2\varepsilon}{\delta} + \log_2 \frac{2\varepsilon}{\delta}\}$. Suppose $s \in C[c'; r_k] \in C_k$, then $\rho(x; s) \leq \rho(c_k; c') + \rho(c'; s) \leq d_k < \delta$.

If we replace Step 3 by

Step 3'. Let $k := k + 1$, goto Step 1.

then we get the infinite sequence of $c_k \xrightarrow{k \to \infty} s$. Indeed, $\forall \delta > 0$ the same reasonings provide $\exists \varepsilon > 0$: $z \in F_\infty$, $D(z) < \varepsilon \Rightarrow \rho(z; s) < \delta$. Then $\forall k \geq k_0$, where $r_{k_0} < \frac{1}{2} \varepsilon$ (e.g. $k_0 = \max\{1; \lfloor \log_2 \frac{2\varepsilon}{\delta}\rfloor + 1\}$), we have $C[c_k; r_k] \in C_k$, thus $D(c_k) \leq 2r_k \leq 2r_{k_0} < \varepsilon$, so $\rho(c_k; s) < \delta$.

Remark. One can “weaken” $\boxdot_3$ to $\boxdot_4$, which requires the cover of $C[c; r]$ to consist of no more than $N \geq N_0$ coverands $C[c_j; r_j], c_j \in C[c; r]$, but then one has to “strengthen” the algorithm’s executor accordingly, so that it is able to build such cover and test $N$ coverands in a finite time (also, $d_k := r_k + \sup_{\rho(c_k; c') \in C | c' \in C_k} \rho(c_k; c')$). In case card $X = \aleph$ (or even card $A = \aleph, \overline{A} = X$) it seems easier for the executor to verify all $x \in X$ for being $s$ in a more direct way.

As the next section illustrates, in certain spaces, when $\{r_i\}$ are at specific positions and $\{t_i\}$ take specific values, there are “better”/faster methods to approximate $s$ or even obtain it exactly.

3 Dense sensors and normed spaces

Here we consider $R$ consisting of “much more” sensors, — in terms of density in $X$ rather than in terms of cardinality. On the other hand, the components of the algorithm described above, — the refining cover and the defect, — if needed, become much simpler.

We recall that the set $A \subseteq X$ is called dense in the set $B \subseteq X$ if $\overline{A} \supseteq B$. In particular, when $\overline{A} = X$, $A$ is everywhere dense.

Suppose $R \subseteq \overline{R}$. We can assume that the set of sensors is $\overline{R}$ from the start, because $\forall r \in \overline{R}, t_r = t_0 + \rho(s; r) = \lim_{i \to \infty} t_{r, i} = t_0 + \lim_{i \to \infty} \rho(s; r_{i})$ for $\forall r_{i}, r_{i} \in R$, due to continuity of metric; that is, the original sensors uniquely define the moments when the wave reaches new sensors from the closure. From now on, $\overline{R} = R$, or, equivalently, $R$ is closed.

The easiest case is when we know that $s \in R$: $t_s = t_0$, while $\forall r \in R, r \neq s: t_r = t_0 + \rho(s; r) > t_0$, so $t_s = \inf_{r \in \overline{R}} t_r$. In other words, the solution then is the sensor where $t_s$ attains its infimum. In general case, $a \in R$ such that $t_a = \inf_{r \in \overline{R}} t_r = t_0 + \sup_{r \in \overline{R}} \rho(s; r)$ is the best approximant (BA) of $s$ in $R$. 
SRP is simplified when $R$ is \textit{``conceal''} enough to\textit{ get} the \textquote{shape} of expanding wave at some moment(s), and from that shape, in turn, derive the position of the source. In this section we consider spherical sensor-sets in normed spaces.

Precisely, hereinafter in this section
\textbullet\ (X; $\rho$) is a normed space $(X; \| \cdot \|)$ with a strictly convex norm, and $\dim X \geq 2$;
\textbullet\ $R = S[\theta; 1]$.
(Also, we assume that we can determine, in a finite time, $r \in R$ where $t_r$ attains its inf or sup.)

\textbf{Case} $\|s\| < 1$. Consider $s \neq \theta$.

\[
\forall x \in X, \|x - s\| < 1 - \|s\|: \|x\| = \|(x - s) + s\| < 1 - \|s\| + \|s\| = 1, \text{ while for } b = s/\|s\|: \\
\|b\| = 1 \text{ and } \|b - s\| = 1/\|s\| - 1 \cdot \|s\| = 1 - \|s\|. \text{ Thus } b \text{ is BA of } s \text{ in } R, \text{ and } \rho(s; R) = 1 - \|s\|.
\]

Suppose $u$ is BA of $s$ in $R$. Then $\|u - s\| = 1 - \|s\|$ and $\|u\| = 1$. We have $\|(u - s) + s\| = \|u\| = 1 = 1 - \|s\| + \|s\| = \|u - s\| + \|s\|$, hence the strict convexity of $\| \cdot \|$ implies $u - s = \lambda s \Leftrightarrow u = (1 + \lambda)s, \lambda \geq 0$. Since $1 = \|u\| = (1 + \lambda)\|s\|, u = s/\|s\| = b, \text{ BA of } s \text{ in } R$ is unique.

Now, $vu \in R$: $\|u - s\| < \|u\| + \|s\| = 1 + \|s\|$, while for $w = -s/\|s\|: \|w - s\| = 1/\|s\| + 1 \cdot \|s\| = 1 + \|s\|$. Thus $w$ is the \textit{worst approximate (WA)} of $s$ in $R$: $\rho(s; w) = \rho(s; r)$.

Analogously, if $u$ is WA of $s$ in $R$, then $\|u - s\| = 1 + \|s\| = \|u\| + \|s\| = 1 - \|s\|$, so $u = \lambda(-s)$, at that $\lambda \geq 0; 1 = \|u\| = \|s\| \Rightarrow u = -s/\|s\| = w, \text{ WA is unique as well.}$

Let $t_b = \inf_{r \in R} t_r$ and $t_w = \inf_{r \in R} t_r$. We see that $t_b = t_0 + 1 - \|s\|$ is attained only at $b$ and $t_w = t_0 + 1 + \|s\|$ is attained only at $w$. Hence $t_w - t_b = 2\|s\|$, implying $s = \frac{1}{2}(t_w - t_b)b$.

This method of obtaining $s$ works for $s = \theta$ too, when $t_r = t_0 + 1 \Rightarrow t_w - t_b = 0$.

\textbf{Case} $\|s\| \geq 1$. It is an easy exercise to show that $b = s/\|s\|$ is the unique BA of $s$ in $R$ and $w = -b$ is the unique WA of $s$ in $R$ again. However, $t_b = \inf_{r \in R} t_r = t_0 + \|s\| - 1$ (attained at $b$) and $t_w = \sup_{r \in R} t_r = t_0 + \|s\| + 1$ (attained at $w$), which isn’t enough to determine $\|s\|$.

Since $\dim X \geq 2, \exists r \in R: r \neq b$ and $r \neq w$, with corresponding $t_r = t_0 + \|r - s\|$. We claim that $(b; t_b), (w; t_w), \text{ and } (r; t_r)$ determine $s = \text{db}$ uniquely on the ray $L = \{db \mid d \geq 1\}$.

Indeed, $d_1 = \|s\|$ satisfies all 3 equations. Assume that there is another solution $d_2 \geq 1, d_2 \neq d_1$. Then $t_0 = 2(t_0 + |s|) = 2(t_0 + t_w) - \|s\|$. We rewrite $t_r = \frac{1}{2}(t_0 + t_w) - d - \|r - db\|$, or $|r - db| = d - t_0 - \frac{1}{2}(t_0 + t_w)$.


a) \text{If } \|r - db\| = \|d_1 - d_2\|, \text{ then from } \|r - db\| = \|d_1 - d_2\| + \|d_2 - d_1\| \text{ and strict convexity of } \| \cdot \| \text{ it follows that } r - d_2b = \lambda(d_2 - d_1)b, \text{ hence } r = \gamma b, \text{ which is impossible, because then } \|r\| = \|b\| = 1 \text{ and we would require } |\gamma| = 1 \text{ and we would obtain } r = \pm b \text{ a contradiction.}

b) \text{If } |r - d_1b| - |r - d_2b| = - \|d_1 - d_2b\|, \text{ then } |r - d_2b| = \|r - d_1b\| + \|d_1 - d_2b\| \text{ likewise implies a contradiction.}

Thus the SRP in $L$ defined by $(b; t_b), (w; t_w), (r; t_r)$ satisfies $\Box$ and $\Box$ (note that $w, r \notin L$). We approximate $s$ to arbitrary precision using the RC-algorithm, and for that we need

1) \text{The defect } D(x) = D_\infty(x): \text{ since } \tau_p(x) = t_b - \|x - b\| = t_0 + \|s\| - \|x\| = t_w - \|x - w\| = \tau_w(x), \text{ we have } D(x) = |\tau_r(x) - \tau_p(x)| = |t_r - t_0 - \|x - r\| + \|x\| - 1|.

2) \text{Covershape is a closed segment on the ray } L, \text{ and the coverand } C[c; r] = \{c + \text{ub} \mid |u| \leq r\}, \text{ where } c \in L. \text{ Moreover, we can assume that an upper estimate of } \|s\| \text{ is known, } \|s\| \leq M + 1, \text{ and consider only } K = \cap \cap B[\theta; M + 1] \ni s, \text{ which is compact. Then } \forall k \in 2^\infty \text{ the coverands are given explicitly as } C_{k; 0} = \{x \in [M \frac{k}{M + 1} H]| H \leq r\} \subseteq K, i = 0, 2^k - 1, \text{ at that } C_{k,i} = C_{k+1,2i} \cup C_{k+1,2i+1}, \text{ in other words, } \Box, \Box \text{ and } \Box \text{ hold.}

Next, we choose small enough $\delta > 0$, run the algorithm, and obtain $x \in K, \|x - s\| < \delta$.

These cases are distinguished by $t_w - t_b$, which is $2\|s\| < 2$ when $\|s\| < 1$, and 2 when $\|s\| \geq 1$. 


SDN without relative compactness. We claim that $A = B[\theta; 1]$ has SDN property. Indeed, let $s \in A$; consider $\delta > 0$ and $x \in A$ such that $|x-s| \geq \delta$. The “straight line” $L = \{s+dv \mid d \in \mathbb{R}\}$, where $v = (x-s)/\|x-s\|$, intersects $R$ at 2 points: $u_+$ for $d = d_+ \geq \|x-s\|$, and $u_-$ for $d = d_- \leq 0$ (it follows from continuity of $f(d) = |s+dv|$, $f(0) = \|s\| \leq 1$, $f(d) \geq |d| - \|s\| > 1$ when $|d| > \|s\| + 1$, and strict convexity of $\|\cdot\|$).

$$D_\infty(x) \geq |\tau_{u_+}(x) - \tau_{u_-}(x)| = \left|\|s - u_+\| - \|s - u_-\| - \|x - u_+\| + \|x - u_-\|\right| =$$

$$= \left|d_+ - (|s - v|)\right| - \left|d_+ - \|u_- - s\|\right| - \left|d_+ - \|x - s\|\right| = 2\|x - s\| \geq 2\delta$$

thus $\inf_{x \in A \setminus B(x,\delta)} D_\infty(x) \geq 2\delta$. If dim $X = \infty$, $A$ is not relatively compact ($\mathbb{R}$ 8.30, 8.28).

4 Refining $\varepsilon$-neighborhood-covers of compact sets

In this section we keep $\mathbb{R}^2$ but discard $\mathbb{R}^3$, $\mathbb{R}^4$, and replace them by

$\bullet$ 3. There is a known $\{K_n\}_{n \in \mathbb{N}}$, $K_n \subseteq X$, such that $K_n$ are compact and $\rho(s; K_n) \xrightarrow{n \to \infty} 0$.

$\bullet$ 4. $\bigcup_{n \in \mathbb{N}} K_n \subseteq A \supseteq s$, diam $A \leq r$, where $A$ has SDN property and $r > 0$ is known.

For example, let $(X; \rho) = l_p$, $1 \leq p < \infty$, and $|s| \leq M$, at that $M$ is known. Let the finite-dimensional subspaces $L_n := \{(x_1; \ldots; x_n; 0; \ldots) \mid x_j \in \mathbb{R}, j = 1, n\}$, $n \in \mathbb{N}$. It is well known that $s_1; \ldots; s_n; 0; \ldots$ is a BA of $s$ in $L_n$, $\rho(s; L_n) = \inf_{u \in L_n} \rho(s; u) = \rho(s; s_n) = \left(\sum_{j=n+1}^{\infty} |s_j|^p\right)^{\frac{1}{p}} \xrightarrow{n \to \infty} 0$, and $|s_n| \leq |s| \leq M$. If the arrangement of the sensors $R$ provides SDN property of $B[\theta; M]$, then $K_n = B[\theta; M] \cap L_n$ satisfy $\mathbb{R}^3$ and $\mathbb{R}^4$ with $A = B[\theta; M]$, $r = 2M$.

**Definition.** Let $A \subseteq X$ and $\varepsilon > 0$. $\{C_j\}_{j \in J}$, where $C_j \subseteq X$, is called an $\varepsilon$-neighborhood-cover of $A$ if $\forall x \in X$, $\rho(x; A) \leq \varepsilon : \exists j \in J : x \in C_j$.

In other words, $(\bigcup_{x \in A} B(x; \varepsilon)) \cup \{x \in X \mid \rho(x; A) = \varepsilon\} \subseteq \bigcup_{j \in J} C_j$.

**Proposition.** For any compact $K \subseteq X$ and any $\varepsilon > 0$ there exists a finite $\varepsilon$-neighborhood-cover $\{B(c_j; 2\varepsilon)\}_{j=1}^m$ of $K$, at that $c_j \in K$.

**Proof.** $K$ is totally bounded ($\mathbb{R}$ 8.28), so there is a finite $\varepsilon$-net ($\mathbb{R}$ 8.24) $\{c_{j_1}\}_{j_1=1}^m \subseteq K$ for $K$. Then $\{B(c_{j_1}; 2\varepsilon)\}_{j_1=1}^m$ is a sought cover: indeed, $\forall x \in X$ such that $\rho(x; K) \leq \varepsilon$, by Weierstrass theorem, the continuous $f(u) = \rho(x; u)$ attains its infimum on $K$ at some $u$, $\rho(x; u) = \rho(x; K)$, and by definition of $\varepsilon$-net $\exists c_{j_1} : \rho(x; c_{j_1}) \leq \varepsilon$. Hence $\rho(x; c_{j_1}) \leq \rho(x; u) + \rho(u; c_{j_1}) \leq 2\varepsilon$. $\square$

Our goal here is to obtain (or rather “construct”) $\{x_n\}_{n \in \mathbb{N}} \subseteq X : x_n \xrightarrow{n \to \infty} s$.

Take $\forall n \in \mathbb{N}$ and consider the sequence of $\frac{\varepsilon}{2^n}$-neighborhood-covers of $K_n$ from Prop. 7. $\forall k \in \mathbb{N}$

$$\mathcal{C}_{n,k} = \left\{B(c_{j,n}; \frac{\varepsilon}{2^n})\right\}_{j=1}^{m_{n,k}}, c_{j,n} \in K_n$$

Let $r_k = \frac{\varepsilon}{2^n}$. The balls from $\mathcal{C}_{n,k}$ that pass the test are the elements of

$$\mathcal{G}_{n,k} = \left\{B(c; r_k) \in \mathcal{C}_{n,k} \mid D(c) \leq 2r_k\right\}$$

diam $A \leq r \Rightarrow \forall x \in A$ $s \in B(x; r)$, and Prop. 4 implies $\mathcal{G}_{n,1} = \mathcal{C}_{n,1}$.

As for $k > 1$, there are 2 alternatives:

1) if $s \in K_n$, then $\forall k \in \mathbb{N}$ $\exists B(c_{j,n}; r_k) \in \mathcal{C}_{n,k}$: $s \in B(c_{j,n}; r_k) \Rightarrow B(c; r_k) \in \mathcal{G}_{n,k}$.

2) if $s \notin K_n$, then $\rho(s; K_n) > 0$ due to compactness, thus by SDN $\varepsilon = \inf_{x \in K_n} D(x) > 0$. As soon as $k$ is big enough so that $r_k < \varepsilon/2$, $\forall x \in K_n D(x) \geq \varepsilon > 2r_k$, and by Cor. 4 $\mathcal{G}_{n,k} = \mathcal{G}_{n,1}$.

Let $\mu_n = \min\{k \in \mathbb{N} \mid \mathcal{G}_{n,k+1} = \mathcal{G}_{n,k}\}$ ($\mu_n = \infty$ if $s \notin K_n$), then $\forall k < \mu_n + 1 : \mathcal{G}_{n,k} \neq \mathcal{G}_{n,k+1}$. $\mu_n \geq 1$.

Let $\nu = \min\{n \in \mathbb{N} \mid \mu_n = \infty\}$ ($\nu = \infty$ if $s \notin \bigcup_{n \in \mathbb{N}} K_n$), then $\forall n < \nu : \mu_n < \infty$. 

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Now we proceed to the algorithm that defines \( \{x_n\}_{n \in \mathbb{N}} \):

Let \( n = 1 \). Build \( C_{1,k}, \mathcal{G}_{1,k} \), let \( x_k \) be the center of the arbitrarily chosen ball from \( \mathcal{G}_{1,k} \), consecutively for \( k = 1, 2, \ldots \), until \( \mathcal{G}_{1,k} = \emptyset \) (\( k = \mu_1 + 1 \) then) or, if that never happens, eternally (when \( \mu_1 = \infty \)).

\[ \ldots \]

If \( \mu_n < \infty \), let \( x_n := x_{n-1} + \mu_n \) (that is, replace it by the center of some ball from \( \mathcal{G}_{n,\mu_n} \), part of the “finest” cover of \( K_n \) preceding the cover that fails the test entirely) and discard \( x_{n+1}, x_{n+2}, \ldots, x_{n-1} + \mu_n \).

Increment \( n := n + 1 \). Build \( C_{n,k}, \mathcal{G}_{n,k} \), let \( x_{n-1 + k} \) be the center of the arbitrarily chosen ball from \( \mathcal{G}_{n,k} \), for \( k = 1, 2, \ldots \), until \( \mathcal{G}_{n,k} = \emptyset \) (\( k = \mu_n + 1 \)) or eternally (\( \mu_n = \infty \)).

\[ \ldots \]

This algorithm never halts. One the one hand, while it runs, however long the span \( x_n, \ldots, x_{n-1 + \mu_n} \) is at some \((n;k)\)-step, at the next step this span can be almost completely erased, leaving \( x_{n-1} + \mu_n \) in place of \( x_n \). On the other hand, for a given \( j \in \mathbb{N} \) \( x_j \) has only a finite number of changes (undefined or gets new value).

Eventually this algorithm comes into one of the following mutually exclusive eternal loops:

**Case** \( \nu = \infty \).

By (4) \( \forall \delta > 0 \exists \varepsilon > 0 : \text{if } x \in A \text{ and } D(x) < \varepsilon \text{, then } \rho(x; s) < \delta. \)

\( \exists m \in \mathbb{N} : \frac{1}{m} \varepsilon \leq r \text{ (e.g. } m = \max \{1 : \lfloor \log_2 \frac{1}{\varepsilon} \rfloor \}). \)

By (4) \( \exists N \in \mathbb{N} : \forall n \geq N \rho(s; K_n) < \frac{1}{m+1} \varepsilon. \)

Consider any such \( n \).

It is easy to see that \( \exists k \in \mathbb{N} : r_k = \left( \frac{1}{m+1} \varepsilon ; \frac{1}{m+1} \varepsilon \right) \). Indeed, \( \log_2 r_1 = \log_2 r > \log_2 \varepsilon - m \), therefore for some \( k \) : \( \log_2 r_k = \log_2 r - (k-1) \in [\log_2 \varepsilon - m - 1 ; \log_2 \varepsilon - m) = \left( \log_2 \left( \frac{1}{m+1} \varepsilon ; \frac{1}{m+1} \varepsilon \right) \right) \).

Construction of \( C_{n,k} \) implies that \( \exists B[c ; r_k] \in C_{n,k} : s \in B[c ; r_k] (r_k > 2 \rho(s; K_n)) \), and it follows from Prop. (4) that \( B[c ; r_k] \in C_{n,k} \neq \emptyset. \) Moreover, \( \forall k' < k : r_{k'} > r_k \Rightarrow C_{n,k'} = \emptyset. \) Thus \( \mu_n \geq k. \)

Since \( x_n \) is the center of some ball from \( \mathcal{G}_{n,\mu_n} \), we have \( D(x_n) \leq 2r_{\mu_n} \leq 2r_k < \frac{1}{m+1} \varepsilon \leq \varepsilon \), so \( \rho(x_n; s) < \delta \) for \( n \geq N. \)

**Case** \( \mu_n = \infty \). For the sake of simplicity we denote \( x_n := x_{n-1} + n \in K_n \), \( n \in \mathbb{N} \), that is, we skip \( x_1, \ldots, x_{n-1} \).

The reasonings as in Section 2 then follow: due to (4) \( \forall \delta > 0 \exists \varepsilon > 0 : \text{if } x \in K_n \text{ and } D(x) < \varepsilon \text{, then } \rho(x; s) < \delta. \)

Let \( N = \max \{1 : \lfloor \log_2 \frac{1}{\varepsilon} \rfloor + 2 \} \). \( \forall n \geq N : r_n < \frac{1}{m} \varepsilon \), and \( B[x_n; r_n] \in \mathcal{G}_{n,n} \Rightarrow D(x_n) \leq 2r_n < \varepsilon \Rightarrow \rho(x_n; s) < \delta. \)

Either way, \( x_n \xrightarrow{n \to \infty \infty} s. \)

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Appendix

The ancillary files to this paper are the implementation of the RC-algorithm from Section 2 for $\mathbb{R}^m$, $1 \leq p < \infty$ (it is proper, but can be viewed as a sub/superspace of some non-proper space), in Julia language [https://julialang.org].

| FILE               | SIZE | SHA2-256   | SHA3-256   |
|--------------------|------|------------|------------|
| space.jl           | 2397 | d991 cca8 6b22 dba8 | 26d7 0de7 72b5 6e8d |
| sr_ms_rc.jl        | 1699 | 6197 9bde 1f0a cac8 | 0946 f95d 5288 23b5 |

sr_ms_rc.jl, with the algorithm itself and the functions it relies on, almost does not use anything specific to the space. To change the space, it is enough to modify only space.jl.

Typical execution result for $m = 2$, $p = 5.6789$, 64 sensors, and $\delta = 0.1$ is shown below.

Note that the number of coverands does not decrease, and the distance error is smaller than the precision $\delta$.

```
$ julia sr_ms_rc.jl
```

| Iteration 1: | 1 coverands
| Iteration 2: | 2 coverands
| Iteration 3: | 4 coverands
| Iteration 4: | 4 coverands
| Iteration 5: | 12 coverands
| Iteration 6: | 12 coverands
| Iteration 7: | 48 coverands
| Iteration 8: | 192 coverands
| Iteration 9: | 576 coverands

Approximated source: $\text{Point([7.7343749999999999, -9.34895833333337])}$
Real source: $\text{Point([7.701565893029412, -9.36462698238313])}$
Distance error: $0.03289547227878735$
Time: 1.59940107 sec

However, even for $m = 3$ both memory and time requirements increase significantly.