FREE BOUNDARY REGULARITY
FOR A CLASS OF ONE-PHASE PROBLEMS WITH
NON-HOMOGENEOUS DEGENERACY

BY

JOÃO VITOR DA SILVA

Department of Mathematics, Universidade Estadual de Campinas - UNICAMP
Campinas - SP, Brazil
e-mail: jdasilva@unicamp.br, girampasso@ime.unicamp.br

AND

GIANE CASARI RAMPASSO

Instituto de Matemática e Computação, Universidade Federal de Itajubá - UNIFEI
Campus Prof. José Rodrigues Seabra, CEP: 37500-903, Itajubá - MG, Brazil
e-mail: gianecr@unifei.edu.br

AND

GLEYDSON CHAVES RICARTE

Department of Mathematics, Universidade Federal do Ceará - UFC
Fortaleza - CE 60455-760, Brazil
e-mail: ricarte@mat.ufc.br

AND

HERNÁN AGUSTÍN VIVAS

Centro Marplatense de Investigaciones Matemáticas
Universidad Nacional de Mar del Plata
Mar del Plata, Argentina
e-mail: havivas@mdp.edu.ar

Received April 16, 2021 and in revised form September 30, 2021
ABSTRACT
We consider a one-phase free boundary problem governed by doubly degenerate fully nonlinear elliptic PDEs with non-zero right hand side, which should be understood as an analog (non-variational) of certain double phase functionals in the theory of non-autonomous integrals. By way of brief elucidating example, such nonlinear problems in force appear in the mathematical theory of combustion, as well as in the study of some flame propagation problems. In such an environment we prove that solutions are Lipschitz continuous and they fulfill a non-degeneracy property. Furthermore, we address the Caffarelli’s classification scheme: Flat and Lipschitz free boundaries are locally $C^{1,\beta}$ for some $0 < \beta(\text{universal}) < 1$. Particularly, our findings are new even in the toy model
\[ G_{p,q}[u] := [|
abla u|^p + a(x)|\nabla u|^q] \Delta u, \quad \text{for } 0 < p < q < \infty \text{ and } 0 \leq a \in C^0(\Omega). \]
We also bring to light other interesting doubly degenerate settings where our results still work. Finally, we present some key tools in the theory of degenerate fully nonlinear PDEs, which may have their own mathematical importance and applicability.

1. Introduction
In this work, we establish regularity estimates to solutions and their interfaces in a so-named one-phase free boundary problems (for short FBPs) for some very degenerate elliptic operators. Precisely, given a bounded domain $\Omega \subset \mathbb{R}^n$, we consider the doubly degenerate fully nonlinear elliptic problem
\[
\begin{aligned}
\mathcal{H}(x, \nabla u)F(x, D^2 u) &= f(x) \quad \text{in } \Omega_+(u), \\
|\nabla u| &= Q(x) \quad \text{on } \mathcal{F}(u),
\end{aligned}
\]
where $\mathcal{H} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ and $F : \Omega \times \text{Sym}(n) \to \mathbb{R}$ satisfy appropriate structural assumptions (to be clarified soon), $Q \geq 0$ is a continuous function, $f \in L^\infty(\Omega) \cap C(\Omega)$ and
\[ u \geq 0 \text{ in } \Omega, \quad \Omega_+(u) := \{ x \in \Omega : u(x) > 0 \} \quad \text{and} \quad \mathcal{F}(u) := \partial\Omega_+(u) \cap \Omega. \]
One of the main signatures of the model case (1.1) is its interplay between two different kinds of degeneracy laws, in accordance with the values of the modulating function $a$.

$$\Omega \times \mathbb{R}^n \ni (x, \xi) \mapsto H(x, \xi) \propto |\xi|^p + a(x)|\xi|^q,$$

$0 < p < q < \infty$ and $0 \leq a \in C^0(\Omega)$.

Therefore, the diffusion process exhibits a non-uniformly elliptic and doubly degenerate feature, which mixes up two power-degenerate type operators (cf. [ART15], [ART17], [BD14], [BD15], [BDL19], [daSLR21], [daSV20], [daSV21], [IS12] and [IS16] for related regularity estimates and free boundary problems driven by second order operators with a single degeneracy law).

Mathematically, (1.1) consists of a model equation for a fully nonlinear prototype enjoying a non-homogeneous degeneracy law, which constitutes an analogous non-divergence form of certain variational integrals from the Calculus of Variations with double phase structure:

\[ (DPF) \quad (W^{1,p}_0(\Omega)+u_0, L^m(\Omega)) \ni (w, f) \mapsto \min \int_{\Omega} \left( \frac{1}{p} |\nabla w|^p + \frac{a(x)}{q} |\nabla w|^q - fw \right) dx, \]

where $a \in C^{0,\alpha}(\Omega, [0, \infty])$, for some $0 < \alpha \leq 1$, $1 < p \leq q < \infty$ and $m \in (n, \infty]$, see [BCM15], [CM15], [DeFM19] and [DeFO19] and the references therein.

Let us remember that the mathematical studies for the model case given by (DPF) date back to Zhikov’s fundamental works in the context of Homogenization problems and Elasticity theory, and they also represent new examples of the occurrence of the Lavrentiev phenomenon; see, e.g., [Zhi93] (see also [BCM15] and [CM15]). Moreover, minimizers to (DPF), i.e., weak solutions of

$$-\text{div}(A(x, \nabla u)\nabla u) = f(x) \quad \text{with} \quad A(x, \xi) := |\xi|^{p-2} + a(x)|\xi|^{q-2}$$

play an important role in some contexts of Materials Science and engineering, where they describe the behavior of certain strongly anisotropic materials, whose hardening estates, connected to the gradient’s growth exponents, change point-wisely. In a precise way, a mixture of two heterogeneous materials, with hardening ($p$&$q$)-exponents, can be performed according to the intrinsic geometry of the null set of the modulation function $x \mapsto a(x)$.

Let us come back to our main purpose. Historically the mathematical investigation of the regularity of the free boundary $\mathcal{F}(u)$ in problems like (1.1) has a large literature and it has presented wide advances in the last three decades or so. Let us summarize the state-of-the-art of such progress:
Uniform elliptic case—Variational approach. The case $f = 0$ and $\mathcal{H}(x, \xi) = 1$ (for a second order linear operator), was widely studied in the seminal works of Caffarelli et al. [AC81], [Caf87], [Caf89] by minimizing

$$ J(u) := \int_{\Omega \cap \{u > 0\}} f(x, u(x), \nabla u(x)) dx \longrightarrow \min, $$

where they proved the existence of a minimum and regularity of the free boundary via blow-up techniques, or via singular perturbation methods for the problem $\Delta u_\varepsilon = \beta_\varepsilon(u_\varepsilon)$; see also Caffarelli-Salsa’s nowadays classic monograph [CS05].

Degenerate cases—Variational approach. In this setting, we may firstly quote the work due to Danielli and Petrosyan in [DP05], in which they established the regularity near “flat points” of the free boundary of non-negative solutions to the minimization problem

$$ \min J_p(u) \quad \text{with} \quad J_p(u) := \int_{\Omega} (|\nabla u|^p + \lambda_0^p \chi_{\{u > 0\}}) dx, $$

which is governed by the $p$-Laplacian operator, for $f = 0$, $1 < p < \infty$ and $\lambda > 0$. Further, Martínez and Wolanski in [MW08] completely address the optimization problem of minimizing

$$ \min J_G(u) \quad \text{with} \quad J_G(u) := \int_{\Omega} (G(|\nabla u|) + \lambda_0 \chi_{\{u > 0\}}) dx, $$

in an Orlicz–Sobolev scenario (cf. [Chl18]), thereby extending the Alt–Caffarelli’s theory in [AC81] for such a context. In the aftermath, Fernández Bonder et al. in [FBMW10], and Lederman and Wolanski in [LW17], [LW19] and [LW21] completed the study of existing, Lipschitz regularity and regularity of the free boundary for homogeneous/inhomogeneous free boundary problems driven by $p(x)$-Laplacian type operators as follows:

$$ \begin{cases} 
\text{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x) & \text{in } \Omega_+(u), \\
|\nabla u| = \lambda^*(x) & \text{on } \Gamma(u). 
\end{cases} $$

Uniform elliptic case—Non-variational approach. In the context of fully nonlinear elliptic equations, the homogeneous problem, i.e., $f = 0$ (with $\mathcal{H}(x, \xi) \equiv 1$), we refer the reader to [Fel01] and the references therein. On the other hand, the non-homogeneous case, $f \neq 0$ (with $\mathcal{H}(x, \xi) \equiv 1$),
was studied in the series of De Silva et al.’s fundamental works [DeS11] and [DFS15] in the one- and two-phase scenarios respectively.

In spite of these prolific references, one-phase problems are still far from being completely understood from a regularity perspective. In particular, according to our scientific knowledge, there are no results in this direction for problems like (1.1) with \(0 < p < q < \infty\) (i.e., a doubly degenerate, non-variational structure). We also point out that the regularity of the free boundary for the inhomogeneous problem \(f \neq 0\) has not been obtained even in the case of

\[
G(u) := \left| \nabla u \right|^p + a(x) \left| \nabla u \right|^q \text{tr}(A(x) D^2 u) \quad \text{with (A2) and (1.3) in force,}
\]

where \(A : \Omega \to \text{Sym}(n)\) is a uniform elliptic and continuous matrix.

In this manuscript, precisely, we will study a very general form of such a problem with free boundaries, in which the diffusion process degenerates in a non-homogeneous fashion. In such a scenario, we establish optimal regularity results for solutions and fine properties of the free boundary. Furthermore, such findings are relevant from both the pure and the applied mathematics viewpoints, and also they may open the possibility of dealing with other interesting scenarios in elliptic regularity theory.

In turn, the FBP considered in (1.1) also appears as the limit of certain inhomogeneous singularly perturbed problems in the non-variational context of high energy activation models in combustion and flame propagation theories (cf. [ART17], [RS15] and [RT11] for related topics), whose simplest mathematical model (in this case) is given by: for each \(\varepsilon > 0\) fixed, we seek a non-negative profile \(u^\varepsilon\) satisfying

\[
\begin{cases}
[|\nabla u^\varepsilon|^p + a(x)|\nabla u^\varepsilon|^q] \Delta u^\varepsilon = \frac{1}{\varepsilon} \beta(u^\varepsilon) + f^\varepsilon(x) & \text{in } \Omega, \\
u^\varepsilon(x) = g(x) & \text{on } \partial \Omega,
\end{cases}
\]

in the viscosity sense for suitable data \(p, q \in (0, \infty)\), \(a, g\), where \(\beta_\varepsilon\) behaves singularly of order \(o(\varepsilon^{-1})\) near \(\varepsilon\)-level surfaces. In such a scenario, existing solutions are locally (uniformly) Lipschitz continuous (see [BJdaSR21, Theorem 1.4]). Thus, up to a subsequence, there exists a function \(u_0\), obtained as the uniform limit of \(u^{\varepsilon_k}\), as \(\varepsilon_k \to 0\). Furthermore, such a limiting profile satisfies

\[
[|\nabla u_0|^p + a(x)|\nabla u_0|^q] \Delta u_0 = f_0(x)
\]

for an appropriate RHS \(f_0 \geq 0\) in the viscosity sense (see also [RST17] for a non-variational one-phase problem driven by a strongly degenerate operator).
Therefore, we are able to apply our results to limit functions of such inhomogeneous singular perturbation problems for the doubly degenerate fully nonlinear operator that we have addressed in [BJdaSR21].

It is worthwhile to highlight that we can consider/recover fully nonlinear models of degenerate/singular type as follows:

\[ G_p(x, \xi, X) := |\xi|^p F(x, X) \quad \text{(with (A0)–(A2) in force)} \]

in any Euclidean dimension and such that \(-1 < p < \infty\) (cf. [LR18]). Particularly, we may apply our findings to limit profiles of inhomogeneous singular perturbation problems for fully nonlinear operators of degenerate type, which were addressed in [ART17] by the third author et al.

1.1. Assumptions and statement of the main results. Next, we will present the structural assumptions to be made throughout this work.

(A0) (Continuity and normalization condition)

Fixed \( \Omega \ni x \mapsto F(x, \cdot) \in C^0(\operatorname{Sym}(n)) \) and \( F(\cdot, O_n) = 0 \)

where \( O_n \) is the zero matrix; this normalizing assumption can be imposed without loss of generality.

(A1) (Uniform ellipticity) For any pair of matrices \( X, Y \in \operatorname{Sym}(n) \)

\[ \mathcal{P}_{-\lambda, \Lambda}(X - Y) \leq F(x, X) - F(x, Y) \leq \mathcal{P}_{+\lambda, \Lambda}(X - Y) \]

where \( \mathcal{P}_{\pm \lambda, \Lambda} \) stand for Pucci’s extremal operators given by

\[ \mathcal{P}_{-\lambda, \Lambda}(X) := \lambda \sum_{e_i > 0} e_i(X) + \Lambda \sum_{e_i < 0} e_i(X) \]

and

\[ \mathcal{P}_{+\lambda, \Lambda}(X) := \Lambda \sum_{e_i > 0} e_i(X) + \lambda \sum_{e_i < 0} e_i(X) \]

for ellipticity constants \( 0 < \lambda \leq \Lambda < \infty \), where \( \{e_i(X)\}_i \) are the eigenvalues of \( X \).

Moreover, for our Lipschitz estimates, we must require some sort of uniform continuity assumption on the coefficients:

(A2) (\( \omega \)-continuity of coefficients) There exist a uniform modulus of continuity \( \omega : [0, \infty) \rightarrow [0, \infty) \) and a constant \( C_F > 0 \) such that

\[ \Omega \ni x, x_0 \mapsto \Theta_F(x, x_0) := \sup_{X \in \operatorname{Sym}(n)} \frac{|F(x, X) - F(x_0, X)|}{\|X\|} \leq C_F \omega(|x - x_0|), \]
which measures the oscillation of coefficients of $F$ around $x_0$. We will denote $\Theta_F(x, 0)$ simply by $\Theta_F(x)$. Finally, we define
\[
\|F\|_{C^\omega(\Omega)} := \inf \left\{ C_F > 0 : \frac{\Theta_F(x, x_0)}{\omega(|x - x_0|)} \leq C_F, \forall x, x_0 \in \Omega, x \neq x_0 \right\}.
\]

In our studies, the diffusion properties of the model (1.1) degenerate along an a priori unknown set of singular points of existing solutions:
\[
S(u, \Omega) := \{ x \in \Omega : |\nabla u(x)| = 0 \}.
\]
For this reason, we will enforce that $H : \Omega \times \mathbb{R}^n \to [0, \infty)$ behaves as
\[
(1.2) \quad L_1 \cdot K_{p, q, a}(x, |\xi|) \leq H(x, \xi) \leq L_2 \cdot K_{p, q, a}(x, |\xi|)
\]
for constants $0 < L_1 \leq L_2 < \infty$, where
\[
(N{-}HDeg) \quad K_{p, q, a}(x, |\xi|) := |\xi|^p + a(x)|\xi|^q, \quad \text{for } (x, \xi) \in \Omega \times \mathbb{R}^n.
\]
In addition, for the non-homogeneous degeneracy (N{-}HDeg), we suppose that the exponents $p, q$ and the modulating function $a(\cdot)$ fulfil
\[
(1.3) \quad 0 < p \leq q < \infty \quad \text{and} \quad a \in C^0(\Omega, [0, \infty)).
\]
Finally, we will assume the following condition: there exist a universal constant $C_a > 0$ and a modulus of continuity $\omega_a : [0, \infty) \to [0, \infty)$ such that
\[
(1.4) \quad |H(x, \xi) - H(y, \xi)| \leq C_a \omega_a(|x - y|)|\xi|^q \quad \forall (x, y, \xi) \in \Omega \times \Omega \times \mathbb{R}^n.
\]
Now, we can start stating our main results. In a first point, we will establish optimal Lipschitz regularity to solutions of (1.1) (see Section 2 for the definition of viscosity solutions).

**Theorem 1.1 (Optimal Lipschitz regularity):** Let
\[
Q \in C^0(B_1; [0, \infty)) \cap L^\infty(B_1; [0, \infty))
\]
and $u$ be a bounded viscosity solution to (1.1) in $B_1$. Then, there exists a universal constant $C_1 = C_1(n, \lambda, \Lambda, a, L_1, p, q) > 0$ such that
\[
(1.5) \quad u(x_0) \leq C_1 \cdot (\|u\|_{L^\infty(B_1)} + \|Q\|_{L^\infty(B_1)}) + \max\{\|f\|_{L^\infty(B_1)}^{\frac{1}{p+1}}, \|f\|_{L^\infty(B_1)}^{\frac{1}{q+1}}\}) \text{dist}(x_0, \mathfrak{F}(u)),
\]
for all $x_0 \in B_{1/2}$; i.e., solutions have at most linear behavior close to free boundary points. Particularly, there exists $C_2 = C_2(n, \lambda, \Lambda, L_1, p, q, \|F\|_{C^\omega}) > 0$ such that

$$
\|\nabla u\|_{L^\infty(B_{1/2})} \leq C_2 \cdot (\|u\|_{L^\infty(B_1)} + \|Q\|_{L^\infty(B_1)}) + \max\{\|f\|_{L^\infty(B_1)}, \|f\|_{C^\omega(B_1)}\} + 1.
$$

Once the (optimal) growth control away from the free boundary is obtained, the next point of interest is the non-degeneracy of solutions, that controls the behavior from below:

**Theorem 1.2 (Non-degeneracy of solutions):** Let

$$Q \in C^0(B_1; [0, \infty)) \cap L^\infty(B_1; [0, \infty))$$

and $u$ be a bounded viscosity solution to (1.1) in $B_1$. Further, suppose that $\mathcal{F}(u)$ is a Lipschitz graph in $B_1$ with $\mathcal{F}(u) \cap B_{1/2}^+(u) \neq \emptyset$. There exist a universal $\eta_0 \in (0, 1)$ and a universal constant $C_* = C(n, \lambda, \Lambda, p, q, \|F\|_{C^\omega(B_1)}) > 0$ such that if

$$\|Q - 1\|_{L^\infty(B_1)} < \eta_0,$$

then

$$u(x_0) \geq C_* \cdot \text{dist}(x_0, \mathcal{F}(u)),$$

for all $x_0 \in B_{1/2}^+(u)$; i.e., solutions grow at least in a linear fashion close to free boundary points.

We will also develop the regularity theory of $\mathcal{F}(u)$. Precisely, we will adapt the technique presented in [DeS11] to prove that flat free boundaries are $C^{1,\beta}$:

**Theorem 1.3 (Flatness implies $C^{1,\beta}$):** Let $u$ be a viscosity solution to (1.1) in $B_1$. Suppose that $0 \in \mathcal{F}(u)$, $Q(0) = 1$ and $F(0, X)$ is uniformly elliptic. Then, there exists a constant $\bar{\varepsilon}(\text{universal}) > 0$ such that, if the graph of $u$ is $\bar{\varepsilon}$-flat in $B_1$, i.e.,

$$(x_n - \bar{\varepsilon})^+ \leq u(x) \leq (x_n + \bar{\varepsilon})^+ \quad \text{for } x \in B_1$$

and

$$\max\{\|f\|_{L^\infty(B_1)}, \[Q\]_{C_0^\infty(B_1)}, \|F\|_{C^\omega(B_1)}\} \leq \bar{\varepsilon},$$

then $\mathcal{F}(u)$ is $C^{1,\beta}$ in $B_{1/2}$ for some (universal) $\beta \in (0, 1)$.

Finally, through a blow-up from Theorem 1.3 and the approach used in [Caf87], we obtain our last main result:
Theorem 1.4 (Lipschitz implies $C^{1,\beta}$): Let $u$ be a viscosity solution for the free boundary problem (1.1). Assume further that $0 \in \mathcal{F}(u)$, $f \in L^\infty(B_1)$ is continuous in $B_1^+(u)$ and $Q(0) > 0$. If $\mathcal{F}(u)$ is a Lipschitz graph in a neighborhood of $0$, then $\mathcal{F}(u)$ is $C^{1,\beta}$ in a (smaller) neighborhood of $0$.

In Theorem 1.4, the size of the neighborhood where $\mathcal{F}(u)$ is $C^{1,\beta}$ depends on the radius $r$ of the ball $B_r$ where $\mathcal{F}(u)$ is Lipschitz, the Lipschitz norm of $\mathcal{F}(u)$, $n$, and $\|f\|_\infty$. We also emphasize that to obtain Theorems 1.3 and 1.4 via the improvement of flatness property for the graph of $u$, we will need a version of a Hopf type estimate, Harnack inequality, Lipschitz regularity, and non-degeneracy for $u$. We stress that all of these key tools will be developed either in the first part or the Appendix of our manuscript and they have their own independent mathematical relevance.

1.2. Major obstacles and strategy for the free boundary regularity. Let us comment on the main obstacles we came across in order to obtain an improvement of flatness property for the graph of a solution of (1.1) and how to overcome them.

As stated in [DeS11], the strategy of proving Theorem 1.3 is to obtain an improvement of flatness property for the graph of a solution $u$: if the graph of $u$ oscillates away $\varepsilon$ from a hyperplane in $B_1$, then in $B_{\delta_0}$ it oscillates $\frac{\delta_0 \varepsilon}{2}$ away from possibly a different hyperplane. We stress that fundamental tools to achieve this property are a Harnack type inequality and characterizing limiting solutions. By way of information, the structure of the operator $G_{p,q}[u] := H(x, \nabla u) F(x, D^2 u)$ requires some non-trivial adaptations.

(1) Harnack type inequality. When we consider the problem (1.1) for $0 < p \le q < \infty$, the first difficulty we find lies in the following fact: in general, if $\ell$ is an affine function and $u$ is a solution to the problem

$$(1.7) \quad H(x, \nabla u) F(x, D^2 u) = f(x) \quad \text{in } B_r(x_0), \text{ where } x_0 = \frac{e_n}{10},$$

we cannot conclude that $u + \ell$ is a solution to the equation (1.7) yet. In contrast, for $p = q = 0$ we know $u + \ell$ is still a solution for (1.7). In effect, in [DeS11], De Silva has used this fact, thereby allowing us to apply the Harnack inequality for $v(x) = u(x) - x_n$, which plays a crucial role in reaching an improvement of flatness for the graph of $u$. We will overcome this difficulty as follows:
Step 1. We notice that the function $v(x) = u(x) - x_n$ is a solution to the problem

$$
H(x, \nabla v + e_n)F(x, D^2 v) = f(x) \quad \text{in } B_r(x_0).
$$

Then, we know (see Appendix) that $v$ satisfies the following Harnack Inequality:

$$
\sup_{B_{r/2}(x_0)} v \leq C \cdot \left\{ \inf_{B_{r/2}} v + (q + 1) \frac{1}{r^{1+}} \max\{r^{\frac{q+2}{p+1}}, r^{\frac{q+2}{p+1}}\} \Pi_{p, q}^{f, a} \right\},
$$

where $C(\text{universal}) > 0$.

Step 2. Since we will make use of a blow-up procedure to prove our results (Theorem 1.3 and Theorem 1.4), we may assume without loss of generality that $\|f\|_{\infty}$ is small. Hence, we can consider the scaled function $v_r(x) = \frac{v(rx + x_0)}{r}$ and apply (1.9) to get

$$
\sup_{B_{r/2}(x_0)} v \leq C \left\{ \inf_{B_{r/2}(x_0)} v + \max\left\{r^{\frac{q+2}{p+1}}, r^{\frac{q+2}{p+1}}\right\} \right\},
$$

for a constant $C = C(n, p, q, a, \lambda, \Lambda) > 0$.

Step 3. Notice that the Harnack Inequality (1.10) is slightly different from the one addressed in [DeS11]. In effect, for $\varepsilon \in (0, 1)$, De Silva used the inequality

$$
\sup_{B_{r/2}(x_0)} v \leq C \left\{ \inf_{B_{r/2}(x_0)} v + \|f\|_{L^{\infty}(B_{r/2}(x_0))} \right\}
$$

(1.11)

to show that if $\|f\|_{\infty}$ satisfies the smallness assumption

$$
\|f\|_{L^{\infty}(B_{r/2}(x_0))} \leq \varepsilon^2,
$$

then we are able to build radial barriers $w_{r, x_0}$ and apply comparison techniques to achieve an appropriate Harnack type inequality to establish the desired improvement flatness (see [DeS11, Theorem 3.1 and Lemma 3.3] for more details). A careful analysis of the behavior of $v = u - x_n$ (or $v = x_n - u$) in a ball $B_{r_1}(x_0)$ with

$$
|\nabla u| < \frac{1}{2} \quad \text{in } B_{r_1}(x_0),
$$

and $r_1 = r_1(\mu) > 0$, reveals that if we consider radial barriers $w_{r, x_0 + r_2 e_n}$, the condition $\|f\|_{\infty} \leq \varepsilon^2$ used in (1.11) can be replaced by an adequate smallness condition of the radius $r = r_2$ in (1.10) to obtain a Harnack type inequality, where $r_2 = r_2(r_1)$. 

1.3. SOME EXTENSIONS AND FURTHER COMMENTS. In conclusion, we stress that our approach is particularly refined and quite far-reaching in order to be employed in other classes of operators. As a matter of fact, we can also extend our results for nonlinear elliptic equations with non-homogeneous terms as follows:

(1) **Multi-degenerate operators in non-divergence form.** We stress that an extension of our results also holds for general multi-degenerate fully nonlinear models given by

\[ G(x, Du, D^2 u) := \left( |Du|^p + \sum_{i=1}^N a_i(x)|Du|^{q_i} \right) F(x, D^2 u) \]

(with (A0)–(A2) in force),

where \( 0 \leq a_i \in C^0(\Omega), i \in \{1, \ldots, N\} \), and \( 0 < p \leq q_1 \leq \cdots \leq q_N < \infty \), which are a natural non-variational counterpart of certain multi-phase variational problems treated in [DeFO19] (see [daSR20, p. 8] for related discussion).

(2) **Doubly degenerate \((p, q)\)-Laplacian in non-divergence form.** We would also like to highlight that another interesting class of degenerate operators where our results work out is the double degenerate \(p\)-Laplacian type operator, in non-divergence form (cf. [APR17] and [daSR20, Section 5.1] for related regularity aspects), for \( 2 < p_0 \leq q_0 < \infty \) and \( 1 < p < \infty \):

\[ G_{p_0, q_0}(x, \xi, X) = H_{p_0, q_0}(x, \xi) F_p(\xi, X), \]

where

\[ H_{p_0, q_0}(x, \xi) := |\xi|^{p_0-2} + a(x)|\xi|^{q_0-2} \quad \text{(with (1.3) in force)} \]

and

\[ F_p(\xi, X) := \text{tr} \left[ (\text{Id}_n + (p - 2) \frac{\xi \otimes \xi}{|\xi|^2}) X \right] \]

is the well-known Normalized \(p\)-Laplacian operator. Notice that \( F_p \) satisfies assumptions (A0)–(A2) with

\[ \lambda_p = \min\{p - 1, 1\} \quad \text{and} \quad \Lambda_p = \max\{p - 1, 1\} \quad \text{and} \quad \omega \equiv 0. \]
Fully nonlinear models with non-standard growth. We would like to stress the class of variable-exponent, degenerate elliptic equations in non-divergence form, which is, to some extent, the non-variational counterpart of certain non-homogeneous functionals satisfying nonstandard growth conditions (see [BPRT20] for an enlightening essay). Particularly, such models encompass problems ruled by the $p(x)$-Laplacian operator (cf. [LW17], [LW19] and [LW21]).

An archetypical example we have in mind concerns models of the form

$$G_{p(x), q(x)}(x, \xi, X) := (|\xi|^{p(x)} + a(x)|\xi|^{q(x)})F(x, X)$$

(with (A0)–(A2) and (1.3) in force),

for rather general exponents $p, q \in C^0(\Omega; (0, \infty)$ (see [BPRT20] for details).

The rest of the paper is organized as follows. In Section 2 we define the notion of viscosity solution to the free boundary problem (1.1) and gather a few tools that we shall use in the proofs of Theorem 1.1 and Theorem 1.2, which are the contents of Section 3. In Section 4 we present the proof of Harnack type inequality which in turn is used in Section 5 to prove the improvement of flatness result. In Section 6 we establish the regularity of the free boundary $\mathcal{F}(u)$, i.e., Theorems 1.3 and 1.4. Finally, in the Appendix we present a discussion of the Harnack inequality, which was of use throughout the paper.

2. Preliminaries and some auxiliary results

We start by giving the definition of a viscosity solution for problems of the form (1.1). First, recall that given two continuous functions $u$ and $\phi$ defined in an open set $\Omega$ and a point $x_0 \in \Omega$, we say that $\phi$ touches $u$ from below (resp. above) at $x_0$ whenever $u(x_0) = \phi(x_0)$:

$$u(x) \geq \phi(x) \quad (\text{resp. } u(x) \leq \phi(x)) \quad \text{in a neighborhood } \mathcal{O} \text{ of } x_0.$$

If this inequality is strict in $\mathcal{O} \setminus \{x_0\}$, we say that $\phi$ touches $u$ strictly from below (resp. above).
Definition 2.1: Let $u \in C(\Omega)$ be nonnegative. We say that $u$ is a viscosity supersolution (resp. subsolution) to
\[
\begin{cases}
H(x, \nabla u)F(x, D^2u) = f(x) & \text{in } \Omega_+(u), \\
|\nabla u| = Q(x) & \text{on } \mathcal{F}(u),
\end{cases}
\]
if and only if the following conditions are satisfied:

(F1) If $\phi \in C^2(\Omega^+(u))$ touches $u$ from below (resp. above) at $x_0 \in \Omega^+(u)$, then
\[
H(x_0, \nabla \phi(x_0))F(x_0, D^2\phi(x_0)) \leq f(x_0) 
\]
(resp. $H(x_0, \nabla \phi(x_0))F(x_0, D^2\phi(x_0)) \geq f(x_0)$).

(F2) If $\phi \in C^2(\Omega)$ and $\phi$ touches $u$ below (resp. above) at $x_0 \in \mathcal{F}(u)$ and $|\nabla \phi|(x_0) \neq 0$, then
\[
|\nabla \phi|(x_0) \leq Q(x_0) \quad (\text{resp. } |\nabla \phi|(x_0) \geq Q(x_0)).
\]

We say that $u$ is a viscosity solution if it is a viscosity supersolution and a viscosity subsolution.

We will further need the notion of a comparison subsolution/supersolution:

Definition 2.2: We say $u \in C(\Omega)$ is a strict comparison subsolution (resp. supersolution) to
\[
\begin{cases}
H(x, \nabla u)F(x, D^2u) = f(x) & \text{in } \Omega_+(u), \\
|\nabla u| = Q(x) & \text{on } \mathcal{F}(u),
\end{cases}
\]
if and only if $u \in C^2(\Omega^+(u))$ and the following conditions are satisfied:

(G1) $H(x, \nabla u)F(x, D^2u) > f(x)$ in $\Omega^+(u)$ (resp. $H(x, \nabla u)F(x, D^2u) < f(x)$).

(G2) If $x_0 \in \mathcal{F}(u)$, then
\[
|\nabla u|(x_0) > Q(x_0) \quad (\text{resp. } 0 < |\nabla u|(x_0) < Q(x_0)).
\]

In the sequel, let us remember the following notion of convergence of sets:

Definition 2.3: A sequence of sets $\{\mathcal{A}_k\}$ is said to converge (locally) to a set $\mathcal{A}$ in the Hausdorff distance if, given a compact set $K$ and a $\delta > 0$, there exists a $k = k(\delta, K) \in \mathbb{N}$, and the following inclusions hold:
\[
K \cap \mathcal{A}_k \subset \mathcal{N}_\delta(\mathcal{A}) \cap K \quad \text{and} \quad K \cap \mathcal{A} \subset \mathcal{N}_\delta(\mathcal{A}_k) \cap K,
\]
where $\mathcal{N}_\delta(E) := \{x \in \mathbb{R}^n : \text{dist}(x, E) < \delta\}$. 


The next Lemma provides a crucial comparison device for solutions to FBP (1.1).

**Lemma 2.4:** Let $u, v$ be respectively a solution and a strict subsolution to (1.1) in $\Omega$. If $u \geq v^+$ in $\Omega$, then

$$u > v^+ \text{ in } \Omega^+(v) \cup \overline{\mathcal{F}(v)}.$$

As stated in [DeS11], another crucial piece of information in proving Theorem 1.3 is the regularity of solutions to the homogeneous problem with a Neumann boundary condition:

$$(2.1) \begin{cases} F(D^2 u_\infty) = 0 & \text{in } B^+_{\rho_0}, \\ \frac{\partial u_\infty}{\partial x_n} = g(x) & \text{on } \Upsilon_{\rho_0}, \end{cases}$$

which arises from a specific blowing-up procedure, where we denote

$$B^+_{\rho_0} := \{(x, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x| < \rho_0 \text{ and } x_n > 0\}$$

and

$$\Upsilon_{\rho_0} := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < \rho_0 \text{ and } x_n = 0\}.$$

In the sequel, we present the notion of a viscosity solution employed in (2.1).

**Definition 2.5:** We say that $u_\infty \in C^0(B_\rho \cap \{x_n \geq 0\})$ is a viscosity solution to (2.1) if, given $P$ a quadratic polynomial touching $u_\infty$ from below (resp. above) at $x_0 \in B_\rho \cap \{x_n \geq 0\}$, then:

(i) If $x_0 \in B^+_{\rho}$, then $F(D^2 P(x_0)) \leq 0$ (resp. $F(D^2 P(x_0)) \geq 0$).

(ii) If $x_0 \in \Upsilon_\rho$, then $\frac{\partial P(x_0)}{\partial x_n} \leq g(x_0)$ (resp. $\frac{\partial P(x_0)}{\partial x_n} \geq g(x_0)$).

It is worth highlighting that, in the above definition, we may choose polynomials $P$ touching $u_\infty$ strictly from below/above. Furthermore, it suffices to check that (ii) does hold for polynomials $\tilde{P}$ such that $F(D^2 \tilde{P}) > 0$ (see [DeS11] for such details).

The following regularity estimate for solutions of (2.1) will play a key role in the improvement of the flatness process. It holds as a consequence of the boundary regularity results addressed by Milakis–Silvestre in [MS06, Theorem 6.1].

**Lemma 2.6:** Let $u$ be a viscosity solution to

$$\begin{cases} F(D^2 u) = 0 & \text{in } B^+_{\frac{\rho}{2}} \\ \frac{\partial u}{\partial \nu} = g(x) & \text{on } \Upsilon_{\frac{\rho}{2}} \end{cases}$$
with \( \| u \|_{L^\infty(B_{\frac{1}{2}}^+)} \leq 1 \). Then, there exist universal constants \( \alpha \in (0, 1) \) and \( C_0 > 0 \) such that
\[
\sup_{B_{\rho}^+} \frac{|u(x) - u(0) - \nabla u(0) \cdot x|}{\rho^{1+\alpha}} \leq C_0 \quad \text{for} \ \rho \in \left(0, \frac{1}{2}\right).
\]

In this last part we will collect some fundamental auxiliary results for our purposes. The first one is a well known result, the so-called ABP estimate. We refer the reader to [BJdaSR21, Theorem 8.6] for an exact proof.

**Theorem 2.7** (Alexandroff–Bakelman–Pucci estimate): Assume that assumptions (A0)–(A2) hold. Then, there exists \( C = C(n, \lambda, p, q, \text{diam}(\Omega)) > 0 \) such that for any \( u \in C^0(\overline{\Omega}) \) the viscosity solution
\[
\mathcal{H}(x, \nabla u)F(x, D^2 u) = f(x) \quad \text{in} \ \Omega
\]
satisfies
\[
\| u \|_{L^\infty(\Omega)} \leq \| u \|_{L^\infty(\partial \Omega)} + C \cdot \text{diam}(\Omega) \max \left\{ \left\| \frac{f}{1+a} \right\|_{L^\infty(\Omega)}, \left\| \frac{f}{1+a} \right\|_{L^\infty(\Omega)} \right\}
\]

We close this section with the following \( C^{1,\beta}_{1,\text{loc}} \) regularity result for doubly degenerate, fully nonlinear elliptic problems.

**Theorem 2.8** (Gradient estimates at interior points): Assume that assumptions (A0)–(A2), (1.2) and (1.3) hold. Let \( u \) be a bounded viscosity solution to
\[
\mathcal{H}(x, \nabla u)F(x, D^2 u) = f(x) \quad \text{in} \ \Omega
\]
with \( f \in L^\infty(\Omega) \). Then, \( u \) is \( C^{1,\beta} \), at interior points, for \( \beta \in (0, \alpha_{\text{Hom}}) \cap (0, \frac{1}{p+1}] \). More precisely, for any point \( x_0 \in \Omega' \subseteq \Omega \)
\[
[u]_{C^{1,\beta}(B_r(x_0))} \leq C \cdot \left( \| u \|_{L^\infty(\Omega)} + 1 + \| f \|_{L^\infty(\Omega)}^{\frac{1}{p+1}} \right)
\]
holds for \( 0 < r < \frac{1}{2} \) where \( C > 0 \) is a universal constant.\(^1\)

For a proof of Theorem 2.8, we refer the reader to [daSR20, Theorem 1.1] and [DeF20].

---

\(^1\) A constant is said to be universal if it depends only on dimension, degeneracy and ellipticity constants, \( \alpha_{\text{Hom}}, \beta, L_1, L_2 \) and \( \| F \|_{C^\omega(\Omega)} \).
3. Lipschitz regularity and non-degeneracy of solutions

At this point, we are in a position to prove the optimal Lipschitz regularity in Theorem 1.1. Nevertheless, in contrast with [DeS11, Lemma 6.1], the proof can be obtained by employing some ideas as those in [ART17], [BJdaSR21], [RS15], [RST17] and [RT11] for the scenario of singularly perturbed FBP.

Proof of Theorem 1.1. Let $x_0 \in B_{1/2}$ such that $x_0 \in B_{1/2}^+(u)$. Then, we define

$$d_0 := \text{dist}(x_0, \overline{\mathcal{F}(u)}).$$

We will suppose that $d_0 \leq \frac{1}{2}$. Let us consider the scaled function $v_{x_0,d_0} : B_1 \to \mathbb{R}$ defined by

$$v_{x_0,d_0}(x) := \frac{u(x_0 + rd_0x)}{d_0},$$

for $r \in (0,1)$ to be chosen later. At this point, it will be enough to prove that $v_{x_0,d_0}(0) \leq C_0$ for some constant $C_0(\text{universal}) > 0$.

Indeed, note that $v_{x_0,d_0}$ is a non-negative viscosity solution of

$$\mathcal{H}_{x_0,d_0}(x, \nabla v_{x_0,d_0})F_{x_0,d_0}(x, D^2 v_{x_0,d_0}) = f_{x_0,d_0}(x) \quad \text{in } B_1$$

where

$$\begin{cases}
F_{x_0,d_0}(x, X) := r^2d_0F(x_0 + rd_0x, \frac{1}{r^2d_0}X), \\
\mathcal{H}_{x_0,d_0}(x, \xi) := rp\mathcal{H}(x_0 + rd_0x, \frac{1}{r}\xi), \\
a_{x_0,d_0}(x) := r^{p-q}a(x_0 + rd_0x), \\
f_{x_0,d_0}(x) := r^{p+2}d_0f(x_0 + rd_0x), \\
Q_{x_0,d_0}(x) := rQ(x_0 + rd_0x).
\end{cases}$$

Furthermore, $F_{x_0,d_0}, \mathcal{H}_{x_0,d_0}$ and $a_{x_0,d_0}$ satisfy the structural assumptions (A0)–(A2), (1.2) and (1.3).

Now, let us consider the annulus

$$\mathcal{A}_{\frac{1}{2}, 1} := B_1 \setminus B_{\frac{1}{2}}$$

and the barrier function $\Phi : \overline{\mathcal{A}_{\frac{1}{2}, 1}} \to \mathbb{R}_+$ given by

$$(3.1) \quad \Phi(x) = \mu_0 \cdot (e^{-\delta|x|^2} - e^{-\delta})$$

where $\mu_0, \delta > 0$ will be chosen a posteriori. Next, we observe that the gradient and the Hessian of $\Phi$ in $\mathcal{A}_{\frac{1}{2}, 1}$ are

$$\nabla \Phi(x) = -2\mu_0\delta xe^{-\delta|x|^2} \quad \text{and} \quad D^2 \Phi(x) = 2\mu_0\delta e^{-\delta|x|^2}(2\delta x \otimes x - \text{Id}_n).$$
In the sequel, we will show that Φ is a strict viscosity subsolution to
\begin{equation}
(3.2) \quad \mathcal{H}_{x_0,d_0}(x, \nabla \Phi) F_{x_0,d_0}(x, D^2 \Phi) = f_{x_0,d_0}(x) \quad \text{in} \ A_{\frac{1}{2},1}
\end{equation}
provided we may adjust appropriately the values of μ₀, δ > 0 and r > 0.

First, notice that if we have \( \delta > \frac{\Lambda(n-1)}{2\Lambda} \), then Φ is a convex and decreasing function in the annular region \( A_{\frac{1}{2},1} \). This and the ellipticity of \( F_{x_0,d_0} \) (see (A1)) give
\[
F_{x_0,d_0}(x, D^2 \Phi) \geq 2\mu_0 \delta e^{-\delta |x|^2} [2\delta \lambda - \Lambda(n-1)] \quad \text{in} \ A_{\frac{1}{2},1}.
\]

Now (1.2) further gives
\[
\mathcal{H}_{x_0,d_0}(x, \nabla \Phi) = r^p \mathcal{H}(x_0 + rd_0x, \frac{1}{r} \nabla \Phi)
\geq r^p \left( \frac{1}{r^p} |\nabla \Phi|^p + a(x_0 + rd_0x) \frac{1}{r^q} |\nabla \Phi|^q \right)
\geq (2\delta \mu_0 e^{-\delta})^p \quad \text{in} \ A_{\frac{1}{2},1}
\]
(recall \( q \geq p \)). These two expressions together give
\[
\mathcal{H}_{x_0,d_0}(x, \nabla \Phi) F_{x_0,d_0}(x, D^2 \Phi) \geq (2\delta \mu_0 e^{-\delta})^{p+1} [2\delta \lambda - \Lambda(n-1)]
\geq r^{p+2} d_0 \|f\|_{L^\infty(A_{\frac{1}{2},1})},
\]
which holds true provided we choose \( r \ll 1 \) small (depending on \( \mu_0 \) and \( \delta \)). Therefore Φ is a strict subsolution.

Furthermore, we choose \( \mu_0 := (e^{-\delta/4} - e^{-\delta})^{-1} \cdot \inf_{\partial B_{\frac{1}{2}}} v_{x_0,d_0}(x) > 0 \). It follows that
\[
\Phi(x) \leq v_{x_0,d_0}(x) \quad \text{on} \ \partial A_{\frac{1}{2},1}.
\]
Hence, from the Comparison Principle (see Theorem A.5), we can conclude that
\begin{equation}
(3.3) \quad \Phi(x) \leq v_{x_0,d_0}(x) \quad \text{in} \ A_{\frac{1}{2},1}.
\end{equation}

Now, let \( z_0 \in \overline{\mathcal{F}}(v_{x_0,d_0}) \) be a point that achieves the distance, i.e., \( rd_0 = |x_0 - z_0| \), and consider \( y_0 := \frac{z_0 - x_0}{rd_0} \in \partial B_1 \). Therefore, taking into account the free boundary condition, we obtain the following concerning the normal derivatives in the direction \( \nu \) at \( x_0 \):
\begin{equation}
(3.4) \quad \mu_0 \delta e^{-\delta} \leq \frac{\partial \Phi(y_0)}{\partial \nu} \leq r Q(y_0) \leq \|Q\|_{L^\infty(B_1)}.
\end{equation}
Therefore,
\[
\inf_{\partial B_{\frac{1}{2}}} v_{x_0,d_0}(x) \leq \|Q\|_{L^\infty(B_1)} \delta^{-1} \cdot (e^{\frac{2\delta}{q}} - 1) = \|Q\|_{L^\infty(B_1)} C(\delta).
\]

Now, by invoking the Harnack inequality (see Theorem A.3) we conclude that
\[
\sup_{B_{\frac{1}{2}}} v_{x_0,d_0}(x)
\leq C \cdot \left\{ \inf_{\partial B_{\frac{1}{2}}} v_{x_0,d_0} + (q + 1) \frac{1}{q+1} \max\{ (r^{p+2}d_0)^{\frac{1}{p+1}}, (r^{p+2}d_0)^{\frac{1}{q+1}} \} \Pi_{p,q}^{f,a_{x_0,d_0}} \right\}
\leq C \cdot \left\{ \inf_{\partial B_{\frac{1}{2}}} v_{x_0,d_0} + (q + 1) \frac{1}{q+1} \max\{ (r^{p+2}d_0)^{\frac{1}{p+1}}, (r^{p+2}d_0)^{\frac{1}{q+1}} \} \Pi_{p,q}^{f,a_{x_0,d_0}} \right\}
\leq C \cdot \|Q\|_{L^\infty(B_1)} C(\delta) + (q + 1) \frac{1}{q+1} \max\{ (r^{p+2}d_0)^{\frac{1}{p+1}}, (r^{p+2}d_0)^{\frac{1}{q+1}} \} \Pi_{p,q}^{f,a_{x_0,d_0}} \},
\]
and from the definition of \(v_{x_0,d_0}\) it follows that
\[
\sup_{B_{\frac{1}{2}}} u(x) \leq C_0(\text{universal}) d_0
\cdot \left\{ \|Q\|_{L^\infty(B_1)} + \max\{ (r^{p+2}d_0)^{\frac{1}{p+1}}, (r^{p+2}d_0)^{\frac{1}{q+1}} \} \Pi_{p,q}^{f,a_{x_0,d_0}} \right\}.
\]

Finally, by \(C^{1,\beta}_{\text{loc}}\)-estimates (see Theorem 2.8) we have
\[
(3.5) \quad |\nabla u(x_0)| = |\nabla v(0)| \leq C \cdot (\|v_{x_0,d_0}\|_{L^\infty(B_{\frac{1}{2}})} + 1 + f_{x_0,d_0}) \frac{1}{\|L^\infty(B_{\frac{1}{2}})\}). \tag{3.5}
\]

We now prove our non-degeneracy result. After constructing the appropriate barrier, the argument follows as that in [DeS11], but we sketch it for completeness.

**Proof of Theorem 1.2.** Let \(x_0 \in B_{1/2}^+(u)\) and define, as in the previous proof,
\[
d_0 := \text{dist}(x_0, \mathcal{F}(u)).
\]

We will suppose that \(d_0 \leq \frac{1}{2}\) and consider again
\[
v_{x_0,d_0}(x) := \frac{u(x_0 + rd_0 x)}{d_0}.
\]
The aim is to show that
\[
v_{x_0,d_0}(0) \geq C_\ast.
\]

Let \(\Phi\) be defined as in (3.1) and
\[
\hat{\Phi}(x) := c \cdot (1 - \Phi(x))
\]
with c to be determined a posteriori. On one hand, we can repeat the argument of the previous proof to get that $-\Phi$ is a strict viscosity supersolution to

$$H_{x_0,d_0}(x, \nabla \Phi)F_{x_0,d_0}(x, D^2\Phi) = f_{x_0,d_0}(x) \quad \text{in } A_{1/2}$$

(and hence so is $\hat{\Phi}$). Further, we choose $c$ so that

$$|\nabla \hat{\Phi}| = c|\nabla \Phi| < 1 - \eta_0$$

on $\partial B_{1/2}$ so that $\hat{\Phi}$ is a strict supersolution for the free boundary problem.

Assume, without loss of generality, that $\mathcal{F}(u)$ is a graph in the $x_n$ direction with Lipschitz constant $L$ and, as a final piece of notation, let us denote

$$\hat{\Phi}_t(x) := c \cdot (1 - \Phi(x + te_n)).$$

Note that for $t$ sufficiently large depending on $L$ we have that $\hat{\Phi}_t$ lies above $v_{x_0,d_0}$ (which will be identically 0 eventually); we set $\hat{t}$ the smallest of such $t$. Next, we note that the touching point between $\hat{\Phi}_t$ and $v_{x_0,d_0}$ has to occur at some point $\hat{x}$ on the $c$ level set,

$$v_{x_0,d_0}(\hat{x}) = c$$

(since $v_{x_0,d_0}$ is a solution and $\hat{\Phi}_t$ is a supersolution), and $\text{dist}(\hat{x}, \mathcal{F}(u)) \leq L$. Now,

$$0 < v_{x_0,d_0}(\hat{x}) = c \leq C_1 \text{dist}(\hat{x}, \mathcal{F}(u))$$

where $C_1$ is the constant from Theorem 1.1, hence

$$\frac{c}{C_1} \leq \text{dist}(\hat{x}, \mathcal{F}(u)) \leq L.$$ 

This control above and below $\text{dist}(\hat{x}, \mathcal{F}(u))$ and the fact that $\mathcal{F}(u)$ is Lipschitz allow us to connect $0$ and $\hat{x}$ with a (universal, depending on $\eta_0$ small) number of intersecting balls in which the Harnack inequality applies and we get

$$\frac{u(x_0)}{d_0} = v_{x_0,d_0}(0) \geq Cv_{x_0,d_0}(\hat{x}) = Cc := C_*,$$

for a $C > 0$ coming from the Harnack inequality (Theorem A.3). This completes the proof. \qed
4. A Harnack type inequality

In this section we will establish a Harnack type inequality (namely Theorem 4.2) for solutions to the free boundary problem (1.1) under the following smallness regime on the right-hand side, the normal derivative and the oscillation of the coefficients:

\[(4.1) \quad \|f\|_{L^\infty(\Omega)} \leq \varepsilon^2,\]
\[(4.2) \quad \|Q - 1\|_{L^\infty(\Omega)} \leq \varepsilon^2,\]
\[(4.3) \quad \Theta_F(x) \leq \varepsilon^2,\]

for \(0 < \varepsilon < 1.\)

The proof relies on the following auxiliary Lemma:

**Lemma 4.1:** There exists a constant \(\tilde{\varepsilon}(\text{universal}) > 0\) such that if \(0 < \varepsilon \leq \tilde{\varepsilon},\) \(u\) is a viscosity solution to (1.1) in \(\Omega\) and (4.1)-(4.3) hold, then: if

\[p^+(x) \leq u(x) \leq (p(x) + \varepsilon)^+, \quad |\sigma| < \frac{1}{20} \text{ in } B_1,\]

with

\[p(x) = x_n + \sigma\]

and at \(x_0 = \frac{1}{10} e_n\)

\[(4.4) \quad u(x_0) \geq \left( p(x_0) + \frac{\varepsilon}{2} \right)^+,\]

then

\[(4.5) \quad u(x) \geq (p(x) + c\varepsilon)^+ \text{ in } \overline{B}_{\frac{1}{2}},\]

for some \(0 < c < 1.\) Analogously, if

\[(4.6) \quad u(x_0) \leq \left( p(x_0) + \frac{\varepsilon}{2} \right)^+,\]

then

\[(4.7) \quad u(x) \leq (p(x) + (1 - c)\varepsilon)^+ \text{ in } \overline{B}_{\frac{1}{2}}.\]

**Proof.** We will verify (4.5); the proof of (4.7) is similar. Furthermore, the degeneracy character of the operator naturally leads to splitting the proof into two steps, according to whether the gradient is “large or small”. Before that, we recall that the interior estimates from [daSR20] and [DeF20] and the fact that

\[B_{\frac{1}{2\eta}}(x_0) \subset B_1^+(u)\]
ensure, together with the ABP estimate from Theorem 2.7, that $u \in C^{1,\alpha}(B_{\frac{1}{40}}(x_0))$ and
\[ [u]_{1+\alpha,B_{\frac{1}{40}}(x_0)} \leq C(\|u\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}^{\frac{1}{p+1}} + 1) \leq C \]
for a constant $\alpha$ (universal) $\in (0,1)$ and $C > 1$ depending also on $\|f\|_{L^\infty(\Omega)}$ and diam$(\Omega)$.

Now we consider two cases:

**Case 1:** If $|\nabla u(x_0)| < \frac{1}{4}$.

We can choose $r_1 = r_1(p,q) > 0$ such that
\[ |\nabla u| \leq \frac{1}{2} \text{ in } B_{r_1}(x_0). \]

Further, if we take $r_2$ small enough depending on $r_1$ we can have
\[ (x - r_2 e_n) \in B_{r_1}(x_0), \quad \text{for all } x \in B_{\frac{1}{40}}(x_0). \]

Finally, let
\[ r_3 = \min \left\{ \frac{r_1}{4}, \frac{r_2}{8} \right\}. \]

In $B_{2r_3}(x_0)$ we can apply the Harnack inequality for nonhomogeneous operators (see Remark A.4 in the Appendix) to get
\[ u(x) - p(x) \geq c_0(u(x_0) - p(x_0)) \geq \frac{c_0 \varepsilon}{2} \]
for all $x \in B_{r_3}(x_0)$ or, denoting $\overline{x}_0 = x_0 - r_2 e_n$,
\[ u(x) - p(x) \geq \frac{c_0 \varepsilon}{2} \]
for all $x \in B_{r_3}(\overline{x}_0)$.

Let now $w := \overline{D} \to \mathbb{R}$ be defined by
\[ w(x) = ce^{-\delta|x-\overline{x}_0|} - e^{-\frac{4\delta}{9}}, \]
where $D := B_{\frac{1}{6}}(\overline{x}_0) \setminus \overline{B}_{r_3}(\overline{x}_0)$ and $\delta$ is a positive constant to be suitably chosen later, and
\[ c := (e^{-\delta r_3} - e^{-\frac{4\delta}{9}})^{-1} \]
so that
\[ w = \begin{cases} 0 & \text{on } \partial B_{\frac{1}{6}}(\overline{x}_0), \\ 1 & \text{on } \partial B_{r_3}(\overline{x}_0), \end{cases} \]
and we further extend $w \equiv 1$ in $B_{r_3}(\overline{x}_0)$.
Next define

\[ v(x) = p(x) + \frac{c_0 \varepsilon}{2} (w(x) - 1), \quad x \in \overline{B}_{\frac{1}{2}}(\overline{x}_0), \]

and for \( t \geq 0, \)

\[ v_t(x) = v(x) + t, \quad x \in \overline{B}_{\frac{1}{2}}(\overline{x}_0) \]

and note that

\[ v_0(x) = v(x) \leq p(x) \leq u(x), \quad x \in \overline{B}_{\frac{1}{2}}(\overline{x}_0). \]

Consider

\[ t_0 = \sup \{ t \geq 0 : v_t \leq u \text{ in } \overline{B}_{\frac{1}{2}}(\overline{x}_0) \}. \]

We claim that if we can show that \( t_0 \geq \frac{c_0 \varepsilon}{2} \) we conclude the proof; indeed, in this scenario the definition of \( v \) implies

\[ u(x) \geq v(x) + t_0 \geq p(x) + \frac{c_0 \varepsilon}{2} w(x), \quad x \in B_{\frac{1}{2}}(\overline{x}_0). \]

Also, note that \( \overline{B}_{\frac{1}{2}} \subset B_{\frac{1}{2}}(\overline{x}_0) \) and \( w \geq \tilde{c} > 0 \) in \( B_{\frac{1}{2}}(\overline{x}_0) \). Hence, we conclude (\( \varepsilon \) small) that

\[ u(x) - p(x) \geq c_1 \varepsilon \quad \text{in } B_{1/2}, \]

as desired.

The remainder of the proof is therefore dedicated to showing that indeed \( t_0 \geq \frac{c_0 \varepsilon}{2} \). We suppose for the sake of contradiction that \( t_0 < \frac{c_0 \varepsilon}{2} \). Then, there would exist \( y_0 \in \overline{B}_{\frac{1}{2}}(\overline{x}_0) \) such that

\[ v_{t_0}(y_0) = u(y_0). \]

In the sequel, we show that \( y_0 \in B_{r_0}(\overline{x}_0) \). In fact, from the definition of \( v_t \) and \( w \) (that vanishes on \( \partial B_{\frac{1}{2}} (\overline{x}_0) \)) we have

\[ v_{t_0}(x) = p(x) - \frac{c_0 \varepsilon}{2} + t_0 \leq p(x) \leq u(x) \quad \text{on } \partial B_{\frac{1}{2}} (\overline{x}_0) \]

so \( y_0 \notin \partial B_{\frac{1}{2}} (\overline{x}_0) \).

Let us show that \( y_0 \) cannot belong to \( D \) either. We compute directly

\[ \partial_i w = -2c\delta e^{-\delta |x-x_0|} (x_i - x_{0i}) \]

where \( x_{0i} \) is the \( i \)-th component of \( \overline{x}_0 \) and

\[ \partial_{ij} w = \begin{cases} 
4c\delta^2 e^{-\delta |x-x_0|} (x_i - x_{0i})(x_j - x_{0j}) & \text{if } i \neq j, \\
-2c\delta e^{-\delta |x-x_0|} (1 - 2\delta (x_i - x_{0i})^2) & \text{if } i = j.
\end{cases} \]
Recall that (4.3) implies that \( F(x, D^2 w) \) is uniformly elliptic and note that if \( \delta > \frac{\Lambda(n-1)}{2\lambda} \) we have
\[
F(x, D^2 w) \geq \mathcal{P}_{\varphi, \Lambda}(D^2 w(x)) = 2c\delta e^{-\delta|x-x_0|}(2\delta\lambda - \Lambda(n-1)) \\
\geq 2c\delta e^{-\frac{4\delta}{\Lambda}}(2\delta\lambda - \Lambda(n-1)).
\]

Furthermore,
\[
\nabla v_{t_0} = e_n + \frac{c_0\varepsilon}{2} \nabla w \quad \text{and} \quad D^2 v_{t_0} = \frac{c_0\varepsilon}{2} D^2 w.
\]
Now
\[
|\nabla w(x)| \leq 2c\delta e^{-\delta r_3} \quad \text{in} \ D,
\]
so for \( \varepsilon > 0 \) small enough we have in \( D \)
\[
(4.17) \quad |\nabla v_{t_0}| \geq \frac{1}{2}.
\]
This and (1.2) imply
\[
\mathcal{H}(x, \nabla v_{t_0}) \geq \eta
\]
for some \( \eta > 0 \). This, the ellipticity of the operator and the previous computations give
\[
\mathcal{H}(x, \nabla v_{t_0}) F(x, D^2 v_{t_0}) \geq \eta F\left(x, \frac{c_0\varepsilon}{2} D^2 w\right) \geq \eta c_0\varepsilon c\delta e^{-\frac{4\delta}{\Lambda}}(2\delta\lambda - \Lambda(n-1)).
\]
Notice that
\[
h(\delta) := e^{-\frac{4\delta}{\Lambda}}\eta c_0\delta(2\delta\lambda - \Lambda(n-1))
\]
satisfies
\[
h(\delta) > 0 \quad \text{for} \ \delta \in \left(\frac{\Lambda(n-1)}{2\lambda}, \infty\right),
\]
and
\[
h\left(\frac{\Lambda(n-1)}{2\lambda}\right) = 0 \quad \text{and} \quad \lim_{\delta \to \infty} h(\delta) = 0.
\]
Then, choosing \( \delta \) maximizing \( h \) and for \( \varepsilon \) small enough,
\[
\mathcal{H}(x, \nabla v_{t_0}) F(x, D^2 v_{t_0}) \geq \varepsilon^2 \geq f(x).
\]
On the other hand, recall that
\[
(4.18) \quad |\nabla v_{t_0}| \geq |\partial_n v| = |1 + \frac{c_0\varepsilon}{2} \partial_n w| \quad \text{in} \ D.
\]
By radial symmetry of \( w \), we have
\[
(4.19) \quad \partial_n w(x) = |\nabla w(x)||\langle \nu_x, e_n \rangle|, \quad x \in D,
\]
where $\nu_x$ is the unit vector in the direction of $x - \mathbf{x}_0$. From (4.15) we have

\begin{equation}
|\nabla w(x)| = 2c\delta e^{-\delta|x-x_0|}|x_i - x_{0i}| \geq 2c\delta e^{-\delta r_3 r_3} > 0.
\end{equation}

Also we have $\langle \nu_x, e_n \rangle \geq c$ in $\{v_{t_0} \leq 0\} \cap D$ (for $\varepsilon$ small enough). In fact, if $\varepsilon$ is small enough

$$
\{v_{t_0} \leq 0\} \cap D \subset \left\{ p \leq \frac{c_0 \varepsilon}{2} \right\} = \left\{ x_n \leq \frac{c_0 \varepsilon}{2} - \sigma \right\} \subset \left\{ x_n < \frac{1}{20} \right\}.
$$

We therefore conclude that

$$
\langle \nu_x, e_n \rangle = \frac{1}{|x - \mathbf{x}_0|} \langle x - \mathbf{x}_0, e_n \rangle
\geq \frac{5}{4} \langle x - \mathbf{x}_0, e_n \rangle
\geq \frac{5}{4} \left( \frac{1}{10} - r_2 - x_n + \frac{1}{20} - \frac{1}{20} \right)
> c_7
$$

in $\{v_{t_0} \leq 0\} \cap D$.

From this, (4.18), (4.19) and (4.20) we obtain

$$
|\nabla v_{t_0}|^2 \geq \left( 1 + \frac{c_0 \varepsilon}{2} |\nabla w(x)| \langle \nu_x, e_n \rangle \right)^2
= 1 + 2c_6 \varepsilon + c_6 \varepsilon^2 |\nabla w|^2
\geq 1 + 2c_9 \varepsilon + c_{10} \varepsilon^2
\geq 1 + \varepsilon^2
$$

and hence

$$
|\nabla v_{t_0}|^2 \geq 1 + \varepsilon^2 > Q^2 \quad \text{in} \quad \{v_{t_0} \leq 0\} \cap D.
$$

In particular, we have

$$
|\nabla v_{t_0}| > Q \quad \text{in} \quad D \cap \mathcal{F}(v_{t_0}).
$$

Thus, $v_{t_0}$ is a strict subsolution in $D$ and by Lemma 2.4 we conclude that $y_0$ cannot belong to $D$, as desired.

In conclusion, $y_0 \in B_{r_3}(\mathbf{x}_0)$; but then

$$
u(y_0) = v_{t_0}(y_0) = v(y_0) + t_0 = p(y_0) + t_0 < p(y_0) + \frac{c_0 \varepsilon}{2},$$

which drives us to a contradiction to (4.9), so this part of the proof is finished.
Case 2: If $|\nabla u(x_0)| \geq \frac{1}{4}$.

Since $u \in C^{1,\alpha}(B_{\frac{1}{4}}(x_0))$, there exists a constant $r_0 > 0$ such that

$$|\nabla u| \geq \frac{1}{8} \text{ in } B_{r_0}(x_0),$$

which in particular implies (as in the previous step)

$$\mathcal{H}(x, \nabla u) \geq c > 0.$$

Then, it is straightforward to see that $u$ satisfies

$$F(x, D^2 u) = \frac{f(x)}{\mathcal{H}(x, |\nabla u|)} \text{ in } B_{r_0}(x_0),$$

in the viscosity sense with

$$\left| \frac{f(x)}{\mathcal{H}(x, |\nabla u|)} \right| \leq \frac{\varepsilon^2}{c}.$$

Thus, by the classical Harnack Inequality (found, for instance, in [CC95]) we obtain

$$u(x) - p(x) \geq c_0(u(x_0) - p(x_0)) - C\|f\|\infty \geq \frac{c_0\varepsilon}{2} - C_1\varepsilon^2 \geq c_1\varepsilon,$$

for all $x \in B_{1/40}(x_0)$, if $\varepsilon > 0$ is sufficiently small. The rest of the proof follows as in the previous case considering the same barrier $w$ defined in $B_{\frac{1}{6}}(x_0) \setminus B_{\frac{1}{40}}(x_0)$ instead of $B_{\frac{1}{6}}(x_0) \setminus B_{r_0}(x_0)$.

Now, we establish the main result of this section, which is a straightforward consequence of the previous Lemma:

**Lemma 4.2:** Let $u$ be a viscosity solution to (1.1) in $\Omega$ under assumptions (4.1)–(4.2). There exists a constant $\tilde{\varepsilon}(\text{universal}) > 0$ such that, if $u$ satisfies at some $x_0 \in \Omega^+(u) \cup \mathfrak{F}(u)$,

$$ (x_n + a_0)^+ \leq u(x) \leq (x_n + b_0)^+ \text{ in } B_r(x_0) \subset \Omega, $$

and

$$ b_0 - a_0 \leq \varepsilon r, \quad \varepsilon \leq \tilde{\varepsilon}, $$

then

$$ (x_n + a_1)^+ \leq u(x) \leq (x_n + b_1)^+ \text{ in } B_{\frac{r}{40}}(x_0) $$

with

$$ a_0 \leq a_1 \leq b_1 \leq b_0, \quad b_1 - a_1 \leq (1 - c)\varepsilon r, $$

and $0 < c(\text{universal}) < 1$. 

Proof. With no loss of generality, we can assume $x_0 = 0$ and $r = 1$. Let us call $p(x) = x_n + a_0$ and notice that by (4.23)
\[ p^+(x) \leq u(x) \leq (p(x) + \varepsilon)^+ \quad \text{with } d_0 = a_0 + \varepsilon. \]

Now, if $|a_0| < \frac{1}{20}$ we can apply the previous Lemma 4.2 to get the desired result (either (4.4) or (4.6) must happen). The possibility $a_0 < -\frac{1}{20}$ is a contradiction, since it implies that $0 \in \{u = 0\}$. Finally, if $a_0 > \frac{1}{20}$ we are strictly contained in the positivity set of $u$ and the result follows by the Harnack inequality. \[ \]

An immediate consequence of Lemma 4.2 is the following Hölder estimate:

**Corollary 4.3:** Let $u$ be a viscosity solution to (1.1) in $\Omega$ under assumptions (4.1)–(4.2). If $u$ satisfies (4.23) then, in $B_1(x_0)$ for $r = 1$, the function
\[ u_\varepsilon(x) := \frac{u(x) - x_n}{\varepsilon} \]
has a Hölder modulus of continuity at $x_0$ outside of a ball of radius $\varepsilon/\tilde{\varepsilon}$; i.e., for all $x \in (\Omega^+(u) \cup \overline{\delta}(u)) \cap B_1(x_0)$ with $|x - x_0| \geq \varepsilon/\tilde{\varepsilon}$,
\[ |\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq C|x - x_0|^\gamma. \]

**Proof.** The proof holds by a standard iteration scheme. If $u$ satisfies (4.23) for $r = 1$, then
\[ (x_n + a_1)^+ \leq u(x) \leq (x_n + b_1)^+ \quad \text{in } B_{\frac{\varepsilon}{40}}(x_0) \]
with
\[ b_1 - a_1 \leq (1 - c)\varepsilon. \]

Nevertheless, if
\[ (1 - c)\varepsilon \leq 40^{-1}\tilde{\varepsilon} \]
we can apply Lemma 4.2 once more and get
\[ (x_n + a_2)^+ \leq u(x) \leq (x_n + b_2)^+ \quad \text{in } B_{\frac{\varepsilon}{40^2}}(x_0) \]
with
\[ b_2 - a_2 \leq (1 - c)^2\varepsilon, \]
and we can repeat this argument as long as
\[ (1 - c)^m\varepsilon \leq 40^{-m}\tilde{\varepsilon}. \]
This means that the oscillation of $u_\varepsilon$ in $B_r(x_0)$ is smaller than $(1-c)^m = 40^{-\gamma m}$ as long as $r \geq \frac{\varepsilon}{\tilde{\varepsilon}}$, which yields the desired estimate. \[ \]
5. An improvement of flatness scheme

In this section we prove an improvement of flatness lemma, from which the proof of Theorem 1.3 will follow via an iterative scheme.

**Lemma 5.1** (Improvement of flatness): Let \( u \) be a viscosity solution to (1.1) in \( \Omega \) under assumptions (4.1)–(4.2) with \( 0 \in \mathfrak{F}(u) \), and assume it satisfies

\[
(x_n - \varepsilon)^+ \leq u(x) \leq (x_n + \varepsilon)^+ \quad \text{for } x \in B_1.
\]

Then there exists a constant \( r_0(\text{universal}) > 0 \) such that if \( 0 < r \leq r_0 \) and \( 0 < \varepsilon \leq \varepsilon_0 \) (with \( \varepsilon_0 \) depending on \( r \)), then

\[
\left( \langle x, \nu \rangle - r\varepsilon \frac{\varepsilon}{2} \right)^+ \leq u(x) \leq \left( \langle x, \nu \rangle + r\varepsilon \frac{\varepsilon}{2} \right)^+, \quad x \in B_r,
\]

for some \( \nu \in \mathbb{S}^{n-1} \) (unity sphere) and

\[
|\nu - e_n| \leq C\varepsilon^2
\]

for a constant \( C(\text{universal}) > 0 \).

**Proof.** We will split the proof into three steps. From now on, we will use the following notation:

\[
\Omega_\rho(u) := (B_1^+(u) \cup \mathfrak{F}(u)) \cap B_\rho.
\]

**Step 1—Compactness Lemma:** Fix \( r \leq r_0 \) with \( r_0(\text{universal}) \) (\( r_0 \) will be chosen in Step 3). Assume, for the sake of contradiction, that we can find sequences \( \varepsilon_k \to 0 \) and \( \{u_k\}_{k \geq 1} \subset C(\Omega) \) viscosity solutions to

\[
\begin{cases}
\mathcal{H}(x, \nabla u_k)F_k(x, D^2 u_k) = f_k(x) & \text{in } \Omega_+(u_k), \\
|\nabla u_k| = Q_k(x) & \text{on } \mathfrak{F}(u_k),
\end{cases}
\]

with

\[
\max\{\|f_k\|_{L^\infty(\Omega)}, \|Q_k - 1\|_{L^\infty(\Omega)}, \Theta_{F_k}(x)\} \leq \varepsilon_k^2,
\]

and

\[
(x_n - \varepsilon_k)^+ \leq u_k(x) \leq (x_n + \varepsilon_k)^+ \quad \text{for } x \in B_1,
\]

but it does not satisfy the conclusion (5.2) of Lemma 5.1.

Let \( v_k : \Omega_1(u_k) \to \mathbb{R} \) be defined by

\[
v_k(x) := \frac{u_k(x) - x_n}{\varepsilon_k}.
\]
Then (5.4) gives
\[ -1 \leq v_k(x) \leq 1 \quad \text{for } x \in \Omega_1(u_k). \]
From Corollary 4.3, it follows that the function \( v_k \) satisfies
\[ |v_k(x) - v_k(y)| \leq C|x - y|^\gamma \]
for \( C(\text{universal}) \) and
\[ |x - y| \geq \varepsilon_k/\bar{\varepsilon}, \quad x, y \in \Omega_{1/2}(u_k). \]
From (5.4) it follows that \( \mathcal{F}(u_k) \to B_1 \cap \{x_n = 0\} \) in the Hausdorff distance (see Definition 2.3). This fact and (5.6) together with Arzelà-Ascoli give that as \( \varepsilon_k \to 0 \), the graphs of the \( v_k \) over \( \Omega_{1/2}(u_k) \) converge (up to a subsequence) in the Hausdorff distance to the graph of a Hölder continuous function \( u_\infty \) over \( B_{1/2} \cap \{x_n \geq 0\} \).

**Step 2—Limiting PDE:** We claim that \( u_\infty \) is a solution of the problem
\[
\begin{cases}
F_\infty(D^2u_\infty) = 0 & \text{in } B_{1/2} \cap \{x_n > 0\} \\
\frac{\partial u_\infty}{\partial x_n} = 1 & \text{on } B_{1/2} \cap \{x_n = 0\}
\end{cases}
\]
in the viscosity sense, for some \( F_\infty \) uniformly elliptic with constant coefficients.

Notice that a stability argument and the uniform modulus of continuity of the operators \( F_k \) give the existence of such \( F_\infty \) and uniform convergence of \( F_k \) (see, for instance, [daSV21, Lemma 2.2]), and the smallness assumption on \( \Theta_{F_k} \) gives zero oscillation (for the coefficients) in the limit profile.

Now let \( P(x) \) be a quadratic polynomial touching \( u_\infty \) at \( x_0 \in B_{1/2} \cap \{x_n \geq 0\} \) strictly from below. As shown in [DeS11], there exist points \( x_k \in \Omega_{1/2}(u_k), x_k \to x_0, \) and constants \( c_k \to 0 \) such that
\[ u_k(x_k) = \tilde{P}(x_k) \]
and
\[ u_k(x) \geq \tilde{P}(x) \quad \text{in a neighborhood of } x_k, \]
where
\[ \tilde{P}(x) = \varepsilon_j(P(x) + c_k) + x_n. \]

We need to prove the following:

(i) If \( x_0 \in B_{1/2} \cap \{x_n > 0\} \) then \( F_\infty(D^2P) \leq 0. \)
(ii) If \( x_0 \in B_{1/2} \cap \{x_n = 0\} \) then \( \frac{\partial P}{\partial x_n}(x_0) \leq 0. \)
(i) If $x_0 \in B_{\frac{1}{2}} \cap \{x_n > 0\}$ then, since $P$ touches $u_k$ from below at $x_k$, we estimate

$$\varepsilon^2_k \geq f_k(x_k) \geq H(x_k, \nabla \tilde{P}) F_k(x_k, D^2 \tilde{P}).$$

Now note that

$$\nabla \tilde{P} = \varepsilon_k \nabla P + e_n$$

such that

$$H(x_k, \nabla \tilde{P}) \geq c$$

for $k$ large enough ($c \in (0, 1)$), so we may take the limit in

$$\frac{\varepsilon^2_k}{H(x, \nabla P)} \geq F_k(x_k, D^2 \tilde{P})$$

to get

$$0 \geq F_\infty(D^2 \tilde{P}).$$

(ii) If $x_0 \in B_{\frac{1}{2}} \cap \{x_n = 0\}$, we can assume that

$$F_\infty(D^2 P) > 0.$$  \hspace{1cm} (5.8)

Notice that for $k$ sufficiently large we have $x_k \in \mathcal{G}(u_k)$. In fact, suppose by contradiction that there exists a subsequence $x_{k_j} \in B_1^+(u_{k_j})$ such that $x_{k_j} \to x_0$. Then arguing as in (i) we obtain

$$F_k(x_{k_j}, D^2 P) \leq C \varepsilon_k$$

whose limit implies

$$F_\infty(D^2 P) \leq 0,$$

which contradicts (5.8). Therefore, there exists $k_0 \in \mathbb{N}$ such that $x_k \in \mathcal{G}(u_k)$ for $k \geq k_0$.

Moreover, as in the previous step,

$$|\nabla \tilde{P}| \geq \frac{1}{2},$$

for $k$ sufficiently large. Since that $\tilde{P}^+$ touches $u_k$ from below we have

$$|\nabla \tilde{P}|^2 \leq Q_k(x_k) \leq (1 + \varepsilon_k^2).$$

Then, we obtain

$$|\nabla \tilde{P}|^2 \leq (1 + \varepsilon_k^2).$$

Moreover,

$$|\nabla \tilde{P}|^2 = \varepsilon_k^2 |\nabla P(x_k)|^2 + 1 + 2 \varepsilon_k \frac{\partial P}{\partial x_n}(x_k).$$
Putting the last two equations together and dividing by $\varepsilon_k$ we get

$$
\varepsilon_k |\nabla P(x_k)|^2 + 2 \frac{\partial P}{\partial x_n}(x_k) \leq \varepsilon_k,
$$

and taking $j \to \infty$ we conclude that $\frac{\partial P}{\partial x_n}(x_0) \leq 0$.

**Step 3—Improvement of flatness:** So far we have that $u_\infty$ solves (5.7)
and, from (5.5), it satisfies

$$
-1 \leq u_\infty \leq 1 \quad \text{in } B_{1/2} \cap \{x_n \geq 0\}.
$$

From Lemma 2.6 and the bound above we obtain that, for the given $r$,

$$
|u_\infty(x) - u_\infty(0) - \langle \nabla u_\infty(0), x \rangle| \leq C_0 r^{1+\alpha} \quad \text{in } B_r \cap \{x_n \geq 0\},
$$

for a constant $C_0(\text{universal})$. In particular, since $0 \in \mathcal{K}(u_\infty)$ and $\partial_n u_\infty(0) = 0$,

$$
\langle \tilde{x}, \tilde{\nu} \rangle - C_1 r^{1+\alpha} \leq u_\infty(x) \leq \langle \tilde{x}, \tilde{\nu} \rangle + C_0 r^{1+\alpha} \quad \text{in } B_r \cap \{x_n \geq 0\},
$$

where $\tilde{\nu}_i = \langle \nabla u_\infty(0), e_i \rangle$ ($i = 1, \ldots, n-1$), $|\tilde{\nu}| \leq \tilde{C}$ and $\tilde{C}(\text{universal})$. Therefore, for $k$ large enough we get

$$
\langle \tilde{x}, \tilde{\nu} \rangle - C_1 r^{1+\alpha} \leq v_k(x) \leq \langle \tilde{x}, \tilde{\nu} \rangle + C_1 r^{1+\alpha} \quad \text{in } \Omega_r(u_k).
$$

From the definition of $v_k$ the inequality above reads

$$
\varepsilon_k \langle \tilde{x}, \tilde{\nu} \rangle + x_n - C_1 \varepsilon_k r^{1+\alpha} \leq v_k(x) \leq \langle \tilde{x}, \tilde{\nu} \rangle + x_n + \varepsilon_k C_1 r^{1+\alpha} \quad \text{in } \Omega_r(u_k).
$$

Now, let us define

$$
\nu := \frac{1}{\sqrt{1 + \varepsilon_k^2}}(\varepsilon_k \tilde{\nu}, 1).
$$

Since, for $k$ large,

$$
1 \leq \sqrt{1 + \varepsilon_k^2} \leq 1 + \frac{\varepsilon_k^2}{2},
$$

we conclude from (5.10) that

$$
\langle x, \nu \rangle - \frac{\varepsilon_k^2}{2} r - C_1 r^{1+\alpha} \varepsilon_k \leq u_k \leq \langle x, \nu \rangle + \frac{\varepsilon_k^2}{2} r + C_1 r^{1+\alpha} \varepsilon_k \quad \text{in } \Omega_r(u_k).
$$

In particular, if $r_0$ is such that $C_1 r_0^\alpha \leq \frac{1}{4}$ and also $k$ is large enough so that $\varepsilon_k \leq \frac{1}{2}$, we find that

$$
\langle x, \nu \rangle - \frac{\varepsilon_k}{2} r \leq u_k \leq \langle x, \nu \rangle + \frac{\varepsilon_k}{2} r \quad \text{in } \Omega_r(u_k),
$$


which together with (5.4) implies that

\[
\left( \langle x, \nu \rangle - \frac{\varepsilon_k}{2} r \right)^+ \leq u_k \leq \left( \langle x, \nu \rangle + \frac{\varepsilon_k}{2} r \right)^+ \text{ in } B_r.
\]

Finally, such a \( u_k \) satisfies the conclusion of Lemma 5.1, thereby yielding a contradiction. 

6. Regularity of the free boundary

In this section we will present the proof of Theorem 1.3. Subsequently, via a blow-up argument performed in such a result, we will deliver the proof of Theorem 1.4. The proof of Theorem 1.3 is based on an improvement of flatness coming from Harnack type estimates and it follows closely the ideas of [DeS11].

Proof of Theorem 1.3. The idea of the proof is to iterate Lemma 5.1 in an appropriate geometric scaling. To that end, let \( u \) be a viscosity solution to the free boundary problem

\[
\begin{cases}
\mathcal{H}(x, \nabla u) F(x, D^2 u) = f(x) & \text{in } B_1^+(u), \\
|\nabla u| = Q(x) & \text{on } \mathfrak{F}(u),
\end{cases}
\]

with \( 0 \in \mathfrak{F}(u) \) and \( Q(0) = 1 \). Now, assume further that

\[
(x_n - \varepsilon)^+ \leq u(x) \leq (x_n + \varepsilon)^+ \text{ for } x \in B_1,
\]

and

\[
\max\{\|f\|_{L^\infty(B_1)}, [Q]_{C^{0,\alpha}(B_1)}, \|F\|_{C^\infty(B_1)}\} \leq \bar{\varepsilon},
\]

with \( \bar{\varepsilon} > 0 \) to be fixed soon.

Let us start by fixing \( r(\text{universal}) > 0 \) to be a constant such that

\[
(6.1) \quad \overline{r} \leq \min \left\{ r_0, \left( \frac{1}{4} \right)^\frac{1}{\alpha} \right\},
\]

with \( r_0(\text{universal}) \) the constant in Lemma 5.1. After choosing \( \overline{r} \), let \( \varepsilon_0 := \varepsilon_0(\overline{r}) \) be the constant given by Lemma 5.1.

Now, let

\[
(6.2) \quad \tilde{\varepsilon} := \varepsilon_0^2 \quad \text{and} \quad \varepsilon_k := 2^{-k} \varepsilon_0.
\]

Notice that our choice of \( \tilde{\varepsilon} \) ensures that

\[
(x_n - \varepsilon_0)^+ \leq u(x) \leq (x_n + \varepsilon_0)^+ \text{ in } B_1.
\]
Thus by Lemma 5.1 there exists $\nu$ with $|\nu| = 1$ and $|\nu - e_n| \leq C \varepsilon_0^2$ such that
\[
(x, \nu) - r \frac{\varepsilon_0}{2} \leq u(x) \leq (x, \nu) + r \frac{\varepsilon_0}{2} \quad \text{in } B_{\bar{r}}.
\]

**Smallness regime.** Consider the sequence of re-scaling profiles $u_k : B_1 \to \mathbb{R}$ given by
\[
u_k := \frac{u(\lambda_k x)}{\lambda_k}
\]
with $\lambda_k = \bar{r}^k$, $k = 0, 1, 2, \ldots$, for a fixed $\bar{r}$ as in (6.1). Then, we observe that $u_k$ fulfils in the viscosity sense the following free boundary problem:

\[
\begin{align*}
\mathcal{H}(\lambda_k x, \nabla u_k) F_k(x, D^2 u_k) &= f_k(x) \quad \text{in } B_1^+(u_k), \\
|\nabla u_k| &= Q_k \quad \text{on } \mathcal{F}(u_k),
\end{align*}
\]

where
\[
\begin{align*}
F_k(x, X) &:= \lambda_k F(\lambda_k x, \lambda_k^{-1} X), \\
\mathcal{H}_k(x, \xi) &:= \mathcal{H}(\lambda_k x, \xi), \\
a_k(x) &:= a(\lambda_k x), \\
f_k(x) &:= \lambda_k f(\lambda_k x), \\
Q_k(x) &:= Q(\lambda_k x).
\end{align*}
\]

Furthermore, $F_k, \mathcal{H}_k$ and $a_k$ fulfil the structural assumptions (A0)–(A2), (1.2) and (1.3).

Now, we also claim that for the choices made in (6.2) the assumptions (4.1)–(4.2) hold true. Indeed, in $B_1$ we have
\[
|f_k(x)| \leq \lambda_k \|f\|_{L^\infty(B_1)} \leq \bar{\varepsilon} \bar{r}^k \leq \varepsilon^2_k, \\
|Q_k(x) - 1| = |Q(\lambda_k x) - Q_k(0)| \leq [Q]_{C^\alpha(B_1)} \lambda_k^\alpha \leq \bar{\varepsilon} \bar{r}^{k\alpha} \leq (\varepsilon_0 2^{-k})^2 = \varepsilon^2_k, \\
\Theta_{F_k}(x) \leq \varepsilon^2_k.
\]

Therefore, we can iterate the above argument and obtain that
\[
(x, \nu_k) - \varepsilon_k \leq u_k(x) \leq (x, \nu_k) + \varepsilon_k \quad \text{in } B_1,
\]
with $|\nu_k| = 1$, $|\nu_k - \nu_{k+1}| \leq C \varepsilon_k$ (with $\nu_0 = e_n$, with $C(\text{universal})$. Thus, we have
\[
\begin{align*}
(x, \nu_k) - \frac{\varepsilon_0}{2k} \leq u(x) \leq (x, \nu_k) + \frac{\varepsilon_0}{2k} \bar{r}^k \quad \text{in } B_{\bar{r}^k},
\end{align*}
\]
with
\[
|\nu_{k+1} - \nu_k| \leq C \frac{\varepsilon_0}{2k}.
\]
Furthermore, (6.4) implies that

\[ \partial \{ u > 0 \} \cap B_{\bar{r}^k} \subset \left\{ |\langle x, \nu_k \rangle| \leq \frac{\varepsilon_0}{2^k \bar{r}^k} \right\}, \]

which implies that \( B_{3/4} \cap \mathcal{F}(u) \) is a \( C^{1,\beta} \) graph. In fact, by (6.5) we have that \( \{ \nu_k \}_{k \geq 1} \) is a Cauchy sequence. Hence, there exists the limit

\[ \nu_\infty := \lim_{k \to \infty} \nu_k. \]

In addition, from (6.5) we conclude that

\[ |\nu_k - \nu_\infty| \leq C \frac{\varepsilon_0}{2^k} \]

for \( k \) large enough.

Now, from (6.6) we have

\[ |\langle x, \nu_k \rangle| \leq \frac{\varepsilon_0}{2^k \bar{r}^k}. \]

Next, fix \( x \in B_{3/4} \cap \partial \{ u > 0 \} \) and choose \( k \) an integer such that

\[ \bar{r}^{k+1} \leq |x| \leq \bar{r}^k. \]

Then,

\[ |\langle x, \nu_\infty \rangle| \leq |\langle x, \nu_\infty - \nu_k \rangle| + |\langle x, \nu_k \rangle| \]

\[ \leq |\nu(0) - \nu_k||x| + \frac{\varepsilon_0}{2^k \bar{r}^k} \leq C \frac{\varepsilon_0}{2^k} |x| + \frac{\varepsilon_0}{2^k \bar{r}^k} \]

\[ \leq C \frac{\varepsilon_0}{2^k} (|x| + \bar{r}^k) \]

\[ = C \frac{\varepsilon_0}{2^k} \left( |x| + \frac{\bar{r}^{k+1}}{\bar{r}} \right) \]

\[ \leq C \frac{\varepsilon_0}{2^k} \left( 1 + \frac{1}{\bar{r}} \right) |x|. \]

From the choice of \( k \) we have \( |x| \geq \bar{r}^{k+1} \). Hence, if we define \( 0 < \beta < 1 \) such that

\[ \frac{1}{2} = \bar{r}^\beta \iff \beta := \frac{\ln(2)}{\ln(\bar{r}^{-1})}, \]

we have

\[ |\langle x, \nu_\infty \rangle| \leq C \left( \frac{1}{2} \right)^k \left( 1 + \bar{r}^{-1} \right) |x| = C \left( \frac{1}{2} \right)^{k+1} \left( 1 + \bar{r}^{-1} \right) 2 |x| \]

\[ \leq C (1 + \bar{r}^{-1}) \varepsilon_0 |x|^{1+\beta} \]

\[ \leq C \varepsilon_0 |x|^{1+\beta}. \]

Finally, we obtain

\[ \partial \{ u > 0 \} \cap B_{\bar{r}^k} \subset \{ \langle x, \nu_\infty \rangle \leq C \varepsilon_0 \bar{r}^{k(1+\beta)} \}, \]
which implies that $\partial\{u > 0\}$ is a differentiable surface at 0 with normal $\nu_\infty$. By applying this argument at all points in $\partial\{u > 0\} \cap B_{3/4}$ we see that $\partial\{u > 0\} \cap B_{3/4}$ is a $C^{1,\beta}$ surface.

Remark 6.1: By appropriately modifying the above proof, the same result could be proved if the free boundary condition is assumed to satisfy the more general continuity assumption

$$|Q(x) - Q(y)| \leq \omega(|x - y|)$$

for some modulus of continuity which is not necessarily a power.

The next point to address is the proof of Theorem 1.4. We present first two preliminary Lemmas. The first lemma is standard and follows essentially a similar result as in [DeS11].

**Lemma 6.2 (Compactness):** Let $u_k$ be a sequence of (Lipschitz) viscosity solutions to

$$\begin{cases}
H(x, \nabla u_k) F_k(x, D^2 u_k) = f_k(x) & \text{in } \Omega^+(u_k), \\
|\nabla u_k| = Q_k(x) & \text{on } \mathcal{F}(u_k).
\end{cases}$$

Assume further that

(i) $u_k \rightarrow u_\infty$ uniformly on compacts

(ii) $F_k \rightarrow F_\infty$ locally uniformly on $\text{Sym}(n) \times \mathbb{R}^n$;

(iii) $\partial\{u_k > 0\} \rightarrow \partial\{u_\infty > 0\}$ locally in the Hausdorff distance

(iv) $\max\{\|f_k\|_{L^\infty}, \|Q_k - 1\|_{L^\infty}, \Theta_F\} = o(1)$, as $k \rightarrow \infty$.

Then, $u_\infty$ is a viscosity solution of

$$\begin{cases}
F_\infty(D^2 u_\infty) = 0 & \text{in } \Omega^+(u_\infty), \\
\frac{\partial u_\infty}{\partial x_n} = 1 & \text{on } \mathcal{F}(u_\infty).
\end{cases}$$

**Proof.** The proof is rather similar to the one of Step 2 in Lemma 5.1 so we omit it.

Finally, we will use the following Liouville type result for global viscosity solutions to a one-phase homogeneous free boundary problem. The result is more general and is proved in [DFS15], but we restate it here in a way suitable for our context:
Lemma 6.3: Let \( v : \mathbb{R}^n \to \mathbb{R} \) be a non-negative viscosity solution to
\[
\begin{cases}
F(D^2 v) = 0 & \text{in } \{ v > 0 \} \\
\frac{\partial v}{\partial \nu} = 1 & \text{on } \mathcal{F}(v)
\end{cases}
\]
Assume further that \( \mathcal{F}(v) = \{ x_n = g(x') \text{ for } x' \in \mathbb{R}^{n-1} \} \) with \( \text{Lip}(g) \leq M \). Then, \( g \) is a linear function and
\[
v(x) = (x \cdot e)^+, \]
for some unit vector \( e \), i.e., \( v \) is a one-dimensional linear profile.

Finally, we are in a position to present the proof of Theorem 1.4.

Proof of Theorem 1.4. Let \( \bar{\varepsilon}(\text{universal}) > 0 \) be the constant from Theorem 1.3. Without loss of generality, we may assume \( Q(0) = 1 \). Now, consider the rescaled functions
\[
u_k(x) := u_{\delta_k}(x) = \frac{u(\delta_k x)}{\delta_k},
\]
with \( \delta_k \to 0 \) as \( k \to \infty \). Notice that \( u_k \) solves in the viscosity sense
\[
\begin{cases}
\mathcal{H}_k(x, \nabla u_k) F_k(x, D^2 u_k) = f_k(x) & \text{in } B_1^+(u_k), \\
|\nabla u_k| = Q_k(x) & \text{on } \mathcal{F}(u_k),
\end{cases}
\]
where
\[
\begin{aligned}
F_k(x, X) &:= \delta_k F(\delta_k x, \delta_k^{-1} X), \\
\mathcal{H}_k(x, \xi) &:= \mathcal{H}(\delta_k x, \xi), \\
a_k(x) &:= a(\delta_k x), \\
f_k(x) &:= \delta_k f(\delta_k x), \\
Q_k(x) &:= Q(\delta_k x),
\end{aligned}
\]
with \( F_k, \mathcal{H}_k \) and \( a_k \) fulfilling the structural assumptions (A0)–(A2), (1.2) and (1.3). Furthermore, for \( k \) large, the smallness conditions are satisfied for a constant \( \bar{\varepsilon}(\text{universal}) \). In fact, in \( B_1 \) we have
\[
\begin{align*}
|f_k(x)| &= \delta_k |f(\delta_k x)| \leq \delta_k \| f \|_{L^\infty} \leq \bar{\varepsilon}^2, \\
|Q_k(x) - 1| &= |Q_k(x) - Q_k(0)| \leq \tau(1) \delta_k^\beta \leq \bar{\varepsilon}^2, \\
\Theta_{F_k}(x) &\leq \bar{\varepsilon}^2,
\end{align*}
\]
for $k$ large enough. Therefore, using non-degeneracy and uniform Lipschitz continuity of the $u_k$’s provided by Theorems 1.1 and 1.2, standard arguments imply that, up to a subsequence:

(i) There exists $u_\infty \in C(\Omega)$ such that $u_k \to u_\infty$ uniformly on compact sets;
(ii) there exists $F_\infty : \text{Sym}(n) \to \mathbb{R}$ such that $F_k \to F_\infty$ locally uniformly;
(iii) $\partial \{u_k > 0\} \to \partial \{u_\infty > 0\}$ locally in the Hausdorff distance;
(iv) $\max\{\|f_k\|_{L^\infty(B_1)}, \|Q_k - 1\|_{L^\infty(B_1)}, \Theta F_k(x)\} = o(1)$ as $k \to \infty$.

Now, as in Lemma 6.2 and using a Cutting Lemma as that in [IS12, Lemma 6], the blow-up limit $u_\infty$ solves the global homogeneous one-phase free boundary problem

$$
\begin{cases}
F_\infty(D^2 u_\infty) = 0 & \text{in } \{u_\infty > 0\}, \\
|\nabla u_\infty| = 1 & \text{on } \mathcal{F}(u_\infty).
\end{cases}
$$

Furthermore, since $\mathcal{F}(u_k)$ is a Lipschitz graph in a neighborhood of 0, we also have from items (i)–(iii) that $\mathcal{F}(u_\infty)$ is Lipschitz continuous. Thus, from Lemma 6.3 we conclude that $u_\infty$ is a one-phase solution, i.e., up to rotations,

$$u_\infty(x) = x_n^+.$$

Thus, for $k$ large enough we have

$$\|u_k - u_\infty\|_{L^\infty(B_1)} \leq \tilde{\varepsilon}.$$

By combining the above facts, one concludes that for all $k$ large enough, $u_k$ is $\tilde{\varepsilon}$-flat in $B_1$ (see Theorem 1.3). Thus,

$$(x_n - \tilde{\varepsilon})^+ \leq u_k(x) \leq (x_n + \tilde{\varepsilon})^+, \quad x \in B_1.$$

Therefore, $\mathcal{F}(u_k)$ is a graph $C^{1,\gamma}$ and consequently $\mathcal{F}(u)$ are $C^{1,\gamma}$, for some $\gamma \in (0, 1)$. This completes the proof.

\textbf{Appendix A.}

\textbf{A.1. A Harnack inequality for doubly degenerate elliptic PDEs.} For the reader’s convenience, in what follows we gather the statements of two fundamental results in elliptic regularity, namely the weak Harnack inequality and the local maximum principle. Such pivotal tools will provide a Harnack inequality (resp. local Hölder regularity) to viscosity solutions.
Theorem A.1 (Weak Harnack inequality, [I11, Theorem 2]): Let $u$ be a non-negative continuous function such that
\[ F_0(x, \nabla u, D^2u) \leq 0 \quad \text{in } B_1 \]
in the viscosity sense. Assume that $F_0$ is uniformly elliptic in the $X$ variable (see condition (A1)) and $F_0 \in C^0(B_1 \times (\mathbb{R}^n \setminus B_{M_F}) \times \text{Sym}(n))$ for some $M_F \geq 0$. Further assume that
\[ |\xi| \geq M_F \quad \text{and} \quad F_0(x, \xi, X) \leq 0 \implies \mathcal{P}_{\lambda, \Lambda}^{-}(X) - \sigma(x)|\xi| - f_0(x) \leq 0 \]
for continuous functions $f_0$ and $\sigma$ in $B_1$. Then, for any $q_1 > n$,
\[ \|u\|_{L^{p_0}(B_1^+)} \leq C\{\inf_{B_1^+} u + \max\{M_F, \|f_0\|_{L^n(B_1)}\}\} \]
for some $p_0(\text{universal}) > 0$ and a constant $C > 0$ depending on $n, q_1, \lambda, \Lambda$ and $\|\sigma\|_{L^{q_1}(B_1)}$.

Theorem A.2 (Local maximum principle, [I11, Theorem 3]): Let $u$ be a continuous function satisfying
\[ F_0(x, \nabla u, D^2u) \geq 0 \quad \text{in } B_1 \]
in the viscosity sense. Assume that $F_0$ is uniformly elliptic in the $X$ variable (see condition (A1)) and $F_0 \in C^0(B_1 \times (\mathbb{R}^n \setminus B_{M_F}) \times \text{Sym}(n))$ for some $M_F \geq 0$. Further assume that
\[ |\xi| \geq M_F \quad \text{and} \quad F_0(x, \xi, X) \geq 0 \implies \mathcal{P}_{\lambda, \Lambda}^{+}(X) + \sigma(x)|\xi| + f_0(x) \geq 0 \]
for continuous functions $f_0$ and $\sigma$ in $B_1$. Then, for any $p_1 > 0$ and $q_1 > n$,
\[ \sup_{B_1^+} u \leq C\{\|u^+\|_{L^{p_1}(B_1^+)} + \max\{M_F, \|f_0\|_{L^n(B_1)}\}\} \]
where $C > 0$ is a constant depending on $n, q_1, \lambda, \Lambda, \|\sigma\|_{L^{q_1}(B_1)}$ and $p_1$.

Let us recall that such results were proved in Imbert’s manuscript [I11] by following the strategy of the uniformly elliptic case; see [CC95, Section 4.2]. Such a strategy is based on the so-called $L^\varepsilon$-Lemma, which establishes a polynomial decay for the measure of the super-level sets of a non-negative supersolution for the Pucci extremal operator $\mathcal{P}_{\lambda, \Lambda}^{+}$:
\[ \mathcal{L}^n(\{x \in B_1 : u(x) > t\} \cap B_1) \leq \frac{C}{t^\varepsilon} . \]
Unfortunately, Imbert’s manuscript has a gap in the proof of (A.3). Such an error was recently made up in a joint work with Silvestre, see [IS16], where an appropriate $L^\varepsilon$-estimate was addressed. In fact, their proof holds for “Pucci extremal operators for large gradients” defined, for a fixed $\tau$, by

$$\tilde{P}^+_{\lambda,\Lambda}(D^2 u, \nabla u) := \begin{cases} P^+_{\lambda,\Lambda}(D^2 u) + \Lambda |\nabla u| & \text{if } |\nabla u| \geq \tau, \\ +\infty & \text{otherwise}; \end{cases}$$

$$\tilde{P}^-_{\lambda,\Lambda}(D^2 u, \nabla u) := \begin{cases} P^-_{\lambda,\Lambda}(D^2 u) - \Lambda |\nabla u| & \text{if } |\nabla u| \geq \tau, \\ -\infty & \text{otherwise}. \end{cases}$$

The $L^\varepsilon$-estimate is proved to hold whenever $\tau \leq \varepsilon_0$ is universal (see [IS16, Theorem 5.1]). Moreover, note that the ellipticity condition $\tilde{P}^-_{\lambda,\Lambda}$ is consistent with (A.1) if we take $\sigma(x) \equiv \Lambda$. Precisely, if (A.1) and $u$ is a super-solution for $F_0$, then it is also a supersolution for $\tilde{P}^-_{\lambda,\Lambda}$ with right hand side $f_0$. An analogous reasoning is valid for $\tilde{P}^+_{\lambda,\Lambda}$ and (A.2).

At this point, once the $L^\varepsilon$ is derived, the proof of Theorem A.1 is exactly as the one in [I11] which is, in turn, a modification of the uniformly elliptic case in [CC95, Theorem 4.8, a]. As for Theorem A.2, it also follows from (A.3) by assuming (in a first moment) that the $L^\varepsilon$ norm of $u^+$ is small and then obtaining the general result by interpolation. Indeed, the smallness of the $L^\varepsilon$ norm readily implies (A.3), which in turn gives that $u$ is bounded (see [CC95, Lemma 4.4], which is adapted in [I11, Section 7.2]).

Notice that our class of operators fits in this scenario by setting

$$F_0(x, \nabla v, D^2 v) := \mathcal{H}(x, \nabla v) F(x, D^2 v) - f(x)$$

and

$$f_0(x) := \frac{L_1^{-1} f^+(x)}{\varepsilon_0^p + a(x) \varepsilon_0^q}$$

for suitable $\varepsilon_0 > 0$.

In effect, we have that whenever

$$\mathcal{H}(x, \nabla v + \xi) F(x, D^2 v) \leq f(x) \quad \text{in } B_1$$

in the viscosity sense, then the ellipticity condition of $F$, namely (A1), ensures us that

$$\mathcal{P}^-_{\lambda,\Lambda}(D^2 v) \leq F(x, D^2 v) \leq \frac{f(x)}{\mathcal{H}(x, \nabla v + \xi)} \leq \frac{f^+(x)}{\mathcal{H}(x, \nabla v + \xi)} \quad \text{in } B_1 \cap \{|\nabla v + \xi| > \varepsilon_0\},$$
whenever $|\nabla v| > M_F = \frac{3}{2} \varepsilon_0$ and $|\xi| \leq \frac{1}{2} \varepsilon_0$, so that
\[
\mathcal{P}_{\lambda, \Lambda}^{-}(D^2v) - \Lambda |\nabla v| - f_0(x) \leq \left( \frac{1}{\mathcal{H}(x, \nabla v + \xi)} - \frac{L_1^{-1}}{\varepsilon_0^p + a(x)\varepsilon_0^q} \right) f^+(x) \leq 0
\]
in $B_1 \cap \{ |\nabla v| > \frac{3}{2} \varepsilon_0 \}$.

Remember that the constants obtained in [IS16] are monotone with respect to $\tau$ and bounded away from zero and infinity, so we get a uniform estimate as (A.3) for supersolutions of
\[
\mathcal{G}_\xi[v] := \mathcal{H}(x, \nabla v + \xi)F(x, D^2v).
\]

Therefore, in such a situation we have (recall $\sigma(x) \equiv \Lambda$) from Theorem A.1
\[
(A.4) \quad \|v\|_{L^p(\Omega)} \leq C \cdot \left\{ \inf_{B_{\frac{1}{2}}} v + \frac{3}{2} \varepsilon_0 + \|f_0\|_{L^\infty(B_1)} \right\} \leq \Xi_0,
\]
where
\[
\Xi_0 := \begin{cases} 
C \cdot \left\{ \inf_{B_{\frac{1}{2}}} v + \min \left\{ \frac{3}{2}, \left( (q + 1) \sqrt{B_1} |L_1^{-1}| \frac{f^+}{1 + a} \right)^{\frac{1}{1+q}} \right\} \right\} & \text{if } \varepsilon_0 \in (0, 1] \\
C \cdot \left\{ \inf_{B_{\frac{1}{2}}} v + \min \left\{ \frac{3}{2}, \left( (p + 1) \sqrt{B_1} |L_1^{-1}| \frac{f^+}{1 + a} \right)^{\frac{1}{1+p}} \right\} \right\} & \text{if } \varepsilon_0 \in (1, \infty). 
\end{cases}
\]
Notice that we have used in the above inequalities that the function
\[
(0, \infty) \ni t \mapsto h(t) = \frac{3}{2} t + \frac{1}{t^s} \left( \sqrt{B_1} |L_1^{-1}| \frac{f^+}{1 + a} \right)_{L^\infty(B_1)}
\]
is optimized (lowest upper bound) when
\[
t^* = \left( \frac{2s}{3} \sqrt{B_1} |L_1^{-1}| \frac{f^+}{1 + a} \right)^{\frac{1}{s+1}}_{L^\infty(B_1)}
\]
for $s \in (0, \infty)$.

In conclusion, in any case, we obtain (since $0 < p \leq q < \infty$)
\[
\|v\|_{L^p(\Omega)} \leq C \cdot \left\{ \inf_{B_{\frac{1}{2}}} v + (q + 1) \right\}^{\frac{1}{q+1}} \Pi_{p, q}^{f^+, a}
\]
where
\[
\Pi_{p, q}^{f^+, a} := \max \left\{ \left[ \frac{n}{\sqrt{B_1} |L_1^{-1}|} \frac{f^+}{1 + a} \right]^{\frac{1}{p+1}}, \left[ \frac{n}{\sqrt{B_1} |L_1^{-1}|} \frac{f^+}{1 + a} \right]^{\frac{1}{q+1}} \right\}.
\]
In a similar way, from Theorem A.2, if
\[ \mathcal{H}(x, \nabla v + \xi) F(x, D^2 v) \geq f(x) \quad \text{in } B_1 \]
in the viscosity sense, we again have for suitable \( \varepsilon_0 > 0 \)
\[ \mathcal{D}_{\lambda, \Lambda}^+(D^2 v) \geq F(x, D^2 v) \geq \frac{f(x)}{\mathcal{H}(x, \nabla v + \xi)} \geq -\frac{f^{-}(x)}{\mathcal{H}(x, \nabla v + \xi)} \]
in \( B_1 \cap \{ |\nabla v + \xi| > \varepsilon_0 \} \).

Whenever \( |\nabla v| > M_F = \frac{3}{2} \varepsilon_0 \) and \( |\xi| \leq \frac{1}{2} \varepsilon_0 \), we can set
\[ f_0(x) := \frac{L^{-1} f^{-}(x)}{\varepsilon_0^p + a(x) \varepsilon_0^p} \]
to get
\[ \mathcal{D}_{\lambda, \Lambda}^+(D^2 v) + \Lambda |\nabla v| + f_0(x) \geq \left( \frac{L^{-1}}{\varepsilon_0^p + a(x) \varepsilon_0^p} - \frac{1}{\mathcal{H}(x, \nabla v + \xi)} \right) f^{-}(x) \geq 0 \]
in \( B_1 \cap \{ |\nabla v| > \frac{3}{2} \varepsilon_0 \} \).

Therefore, we have from Theorem A.2
\[ \sup_{B_\frac{1}{2}} v \leq \| u^+ \|_{L^p(B_1)} + \frac{3}{2} \varepsilon_0 + \| f_0 \|_{L^\infty(B_1)} \]
\[ \leq \Xi_1, \]
where, as before, we can estimate
\[ \Xi_1 := \begin{cases} 
C \cdot \left\{ \| v^+ \|_{L^p(B_1)} + \min \left\{ \frac{3}{2}, \left( q+1 \right)^{\frac{1}{q}} \sqrt{|B_1| L_1^{-1}} \left\| \frac{f^{-}}{1+a} \right\|_{L^\infty(B_1)} \right\} \right\} & \text{if } \varepsilon_0 \in (0, 1], \\
C \cdot \left\{ \| v^+ \|_{L^p(B_1)} + \min \left\{ \frac{3}{2}, \left( p+1 \right)^{\frac{1}{p}} \sqrt{|B_1| L_1^{-1}} \left\| \frac{f^{-}}{1+a} \right\|_{L^\infty(B_1)} \right\} \right\} & \text{if } \varepsilon_0 \in (1, \infty). 
\end{cases} \]

Therefore, in any setting (since \( 0 < p \leq q < \infty \))
\[ \sup_{B_\frac{1}{2}} v \leq C \cdot \{ \| v^+ \|_{L^p(B_1)} + (q+1)^{\frac{1}{q}} \Pi_{p,q}^{f^{-},a} \} \]
thereby concluding this analysis.

Finally, combining (A.4) and (A.5), we obtain the following Harnack inequality:
Theorem A.3 (Harnack inequality): Let \( u \) be a non-negative viscosity solution to
\[
F_0(x, \nabla v + \xi, D^2v) = 0 \quad \text{in } B_1.
\]
Then
\[
\sup_{B_{\frac{1}{2}}} u(x) \leq C \cdot \{ \inf_{B_{\frac{1}{2}}} u(x) + (q + 1) \frac{1}{q+1} \Pi f, a \},
\]
where \( C = C(n, \lambda, \Lambda) > 0 \).

Remark A.4 (Harnack inequality—scaled version): For our purposes, it will be useful to obtain a scaled version of the Harnack inequality: Let \( v \) be a non-negative viscosity solution to
\[
F_0(x, \nabla v + \xi, D^2v) = 0 \quad \text{in } B_r \quad \text{for a fixed } r \in (0, \infty) \text{ and any } \xi \in \mathbb{R}^n,
\]
where (A0)–(A2), (1.2) and (1.3) are in force. Then
\[
\sup_{B_{\frac{r}{2}}} v(x) \leq C \cdot \{ \inf_{B_{\frac{r}{2}}} v(x) + (q + 1) \frac{1}{q+1} \max\{r^{\frac{p+2}{p+1}}, r^{\frac{p+2}{q+1}}\} \Pi f, a \},
\]
where \( C = C(n, \lambda, \Lambda) > 0 \).

A.2. The Comparison Principle: Doubly degenerate scenario. In this subsection, we prove the classical result known as the comparison principle. In order to do so, we consider the following approximation problem:
\[
(A.6) \quad \mathcal{G}_\varepsilon [u] := \mathcal{H}_\varepsilon (x, \nabla u)[\varepsilon u + F(x, D^2u)] = f(x) \quad \text{in } \Omega,
\]
where \( \mathcal{H}_\varepsilon : \Omega \times \mathbb{R}^n \rightarrow [0, \infty) \) behaves as
\[
(A.7) \quad L_1 \cdot \mathcal{K}^{\varepsilon}_{p,q,a}(x, |\xi|) \leq \mathcal{H}_\varepsilon(x, \xi) \leq L_2 \cdot \mathcal{K}^{\varepsilon}_{p,q,a}(x, |\xi|)
\]
for constants \( 0 < L_1 \leq L_2 < \infty \), with \( (A.8) \quad \mathcal{K}^{\varepsilon}_{p,q,a}(x, |\xi|) := (\varepsilon + |\xi|)^p + a(x)(\varepsilon + |\xi|)^q, \) for \( (x, \xi) \in \Omega \times \mathbb{R}^n \) for \( 0 < \varepsilon < 1 \) and \( u \in C^0(\overline{\Omega}) \). The desired comparison result will hold true by letting \( \varepsilon \rightarrow 0^+ \). The main idea of the proof follows essentially from [CIL92] (see also [HPRS] for more details). As a separate interesting point, such a result is a non-homogeneous degenerate counterpart for the degenerate one in [BD04, Theorem 1.1].

Theorem A.5 (Comparison principle): Assume that assumptions (A0)–(A1), (1.2) and (1.3) hold. Let \( f \in C^0(\overline{\Omega}) \). Suppose \( u_1 \) and \( u_2 \) are respectively a viscosity supersolution and subsolution of (A.6). If \( u_1 \leq u_2 \) on \( \partial \Omega \), then \( u_1 \leq u_2 \) in \( \Omega \).
Proof. We shall prove this result by contradiction. To this end, suppose that such a statement is false. Then, we can assume that

$$M_0 := \max_{x \in \Omega} (u_1(x) - u_2(x)) > 0.$$ 

For $\delta > 0$, let us define

$$M_\delta := \max_{x,y \in \Omega} \left[ u_1(x) - u_2(y) - \frac{|x-y|^2}{2\delta} \right].$$

Assume that the maximum $M_\delta$ is attained at a point $(x_\delta, y_\delta) \in \overline{\Omega} \times \overline{\Omega}$ and notice that $M_\delta \geq M_0$.

From [CIL92, Lemma 3.1], we have that

$$\lim_{\delta \to 0} \frac{|x_\delta - y_\delta|}{\delta} = 0.$$ 

This implies that $x_\delta, y_\delta \in \Omega$ for $\delta$ sufficiently small. Moreover, [CIL92, Theorem 3.2 and Proposition 3] assures the existence of a limiting super-jet $(\frac{x_\delta - y_\delta}{\delta}, X)$ of $u_1$ at $x_\delta$ and a limiting sub-jet $(\frac{x_\delta - y_\delta}{\delta}, Y)$ of $u_2$ at $y_\delta$, where the matrices $X$ and $Y$ satisfy the inequality

$$-\frac{3}{\delta} \begin{pmatrix} \text{Id}_n & 0 \\ 0 & \text{Id}_n \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{3}{\delta} \begin{pmatrix} \text{Id}_n & -\text{Id}_n \\ -\text{Id}_n & \text{Id}_n \end{pmatrix},$$

$\text{Id}_n$ being the identity matrix.

Observe that the assumption (A1) implies that the operator $F$ is, in particular, degenerate elliptic. It implies that

$$F(x, X) \geq F(x, Y)$$

for every $x \in \Omega$ fixed, since $X \leq Y$ from (A.10).

Therefore, as a consequence of the two viscosity inequalities (and relation (A.11))

$$\mathcal{H}_\varepsilon \left(x_\delta, \frac{x_\delta - y_\delta}{\delta}\right)[\varepsilon u_1(x_\delta) + F(x_\delta, X)] \leq f(x_\delta)$$

and

$$\mathcal{H}_\varepsilon \left(y_\delta, \frac{x_\delta - y_\delta}{\delta}\right)[\varepsilon u_2(y_\delta) + F(y_\delta, Y)] \geq f(y_\delta),$$

we get

$$\frac{\varepsilon M_0}{2} \leq \varepsilon (u_1(x_\delta) - u_2(y_\delta))$$

and

$$\frac{f(x_\delta)}{\mathcal{H}_\varepsilon \left(x_\delta, \frac{x_\delta - y_\delta}{\delta}\right)} + [F(x_\delta, Y) - F(y_\delta, Y)] - \frac{f(y_\delta)}{\mathcal{H}_\varepsilon \left(y_\delta, \frac{x_\delta - y_\delta}{\delta}\right)}.$$
Now, observe that, from assumption (A2), we can estimate
\[
|F(x_\delta, Y) - F(y_\delta, Y)| \leq C_F \omega(|x_\delta - y_\delta|)\|Y\|.
\]
Moreover, if \(\omega_f\) is a modulus of continuity of \(f\) on \(\Omega\), from assumptions (1.2) and (1.4) we obtain
\[
\frac{f(x_\delta)}{H_\epsilon(x_\delta, \frac{x_\delta - y_\delta}{\delta})} - \frac{f(y_\delta)}{H_\epsilon(y_\delta, \frac{x_\delta - y_\delta}{\delta})} \\
\leq \frac{f(x_\delta) - f(y_\delta)}{H_\epsilon(x_\delta, \frac{x_\delta - y_\delta}{\delta})} + f(y_\delta) \left[ \frac{1}{H_\epsilon(x_\delta, \frac{x_\delta - y_\delta}{\delta})} - \frac{1}{H_\epsilon(y_\delta, \frac{x_\delta - y_\delta}{\delta})} \right]
\leq \frac{\omega_f(|x_\delta - y_\delta|)}{L_1 \cdot K_{p,q,a}(x_\delta, \frac{x_\delta - y_\delta}{\delta})} \\
+ \|f\|_{L^\infty(\Omega)} \cdot \left| \frac{H_\epsilon(y_\delta, \frac{x_\delta - y_\delta}{\delta}) - H_\epsilon(x_\delta, \frac{x_\delta - y_\delta}{\delta})}{L_1^2 \cdot K_{p,q,a}(x_\delta, \frac{x_\delta - y_\delta}{\delta}) K_{p,q,a}(y_\delta, \frac{x_\delta - y_\delta}{\delta})} \right|
\leq \frac{\omega_f(|x_\delta - y_\delta|)}{L_1(\epsilon + \frac{|x_\delta - y_\delta|}{\delta})^p} + C_a \|f\|_{L^\infty(\Omega)} \frac{\omega_a(|x_\delta - y_\delta|)(\epsilon + \frac{|x_\delta - y_\delta|}{\delta})^q}{L_1^2(\epsilon + \frac{|x_\delta - y_\delta|}{\delta})^{2p}}.
\]
In conclusion, by combining (A.12), (A.13) and (A.14) we obtain
\[
\frac{\varepsilon M_0}{2} \leq C_F \omega(|x_\delta - y_\delta|)\|Y\| + \frac{\omega_f(|x_\delta - y_\delta|)}{L_1(\epsilon + \frac{|x_\delta - y_\delta|}{\delta})^p} \\
+ C_a \|f\|_{L^\infty(\Omega)} \frac{\omega_a(|x_\delta - y_\delta|)(\epsilon + \frac{|x_\delta - y_\delta|}{\delta})^q}{L_1^2(\epsilon + \frac{|x_\delta - y_\delta|}{\delta})^{2p}}
= o(1) \quad \text{as } \delta \to 0^+,
\]
which yields a contradiction, thereby proving the Comparison Principle by letting \(\varepsilon \to 0^+\).  

ACKNOWLEDGMENTS. J. V. da Silva and G. C. Ricarte have been partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq-Brazil) under Grants No. 310303/2019-2 and No. 304239/2021-6. G. C. Rampasso was partially supported by CAPES-Brazil. This research was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - (CAPES - Brazil) - Finance Code 001 and FAPDF Demanda Espontânea 2021. H. A. Vivas was partially supported by Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET-Argentina).
References

[AC81] H. W. Alt and L. A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, Journal für die Reine und Angewandte Mathematik 325 (1981), 105–144.

[ART15] D. J. Araújo, G. C. Ricarte and E. V. Teixeira, *Geometric gradient estimates for solutions to degenerate elliptic equations*, Calculus of Variations and Partial Differential Equations 53 (2015), 605–625.

[ART17] D. J. Araújo, G. C. Ricarte and E. V. Teixeira, *Singularly perturbed equations of degenerate type*, Annales de l’Institut Henri Poincaré C. Analyse Non Linéaire 34 (2017), 655–678.

[APR17] A. Attouchi, M. Parviainen and E. Ruosteenoja, *$C^{1,\alpha}$ regularity for the normalized $p$-Poisson problem*, Journal de Mathématiques Pures et Appliquées 108 (2017), 553–591.

[BCM15] P. Baroni, M. Colombo and G. Mingione, *Regularity for general functionals with double phase*, Calculus of Variations and Partial Differential Equations 57 (2018), Article no. 62.

[BD04] I. Birindelli and F. Demengel, *Comparison principle and Liouville type results for singular fully nonlinear operators*, Annales de la Faculté des Sciences de Toulouse. Mathématiques 13 (2004), 261–287.

[BD14] I. Birindelli and F. Demengel, *$C^{1,\beta}$ regularity for Dirichlet problems associated to fully nonlinear degenerate elliptic equations*, ESAIM. Control, Optimisation and Calculus of Variations 20 (2014), 1009–1024.

[BD15] I. Birindelli and F. Demengel, *Hölder regularity of the gradient for solutions of fully nonlinear equations with sub linear first order term*, in *Geometric Methods in PDE’s*, Springer INdAM Series, Vol. 13, Springer, Cham, 2015, pp. 257–268.

[BDL19] I. Birindelli, F. Demengel and F. Leoni, *$C^{1,\gamma}$ regularity for singular or degenerate fully nonlinear equations and applications*, NoDEA Nonlinear Differential Equations and Applications 26 (2019), Article no. 40.

[BJdaSR21] E. C. Bezerra Júnior, J. V. da Silva, and G. C. Ricarte, *Fully nonlinear singularly perturbed models with non-homogeneous degeneracy*, Revista Matemática Iberoamericana, https://doi.org/10.4171/rmi/1319.

[BPRRT20] A. Bronzi, E. Pimentel, G. Rampasso and E. V. Teixeira, *Regularity of solutions to a class of variable-exponent fully nonlinear elliptic equations*, Journal of Functional Analysis 279 (2020), Article no. 108781.

[Caf87] L. A. Caffarelli, *A Harnack inequality approach to the regularity of free boundaries. Parte I: Lipschitz free boundaries are $C^{1,\alpha}$*, Revista Matemática Iberoamericana 3 (1987), 139–162.

[Caf89] L. A. Caffarelli, *A Harnack inequality approach to the regularity of free boundaries. Parte II: Flat free boundaries are Lipschitz*, Communications in Pure and Applied Mathematics 42 (1989), 55–78.

[CC95] L. A. Caffarelli and X. Cabré, *Fully Nonlinear Elliptic Equations*, American Mathematical Society Colloquium Publications, Vol. 43, American Mathematical Society, Providence, RI, 1995.
[CS05] L. Caffarelli and S. Salsa, A Geometric Approach to Free Boundary Problems, Graduate Studies in Mathematics, Vol. 68, American Mathematical Society, Providence, RI, 2005.

[Chl18] I. Chlebicka, A pocket guide to nonlinear differential equations in Musielak–Orlicz spaces, Nonlinear Analysis 175 (2018), 1–27.

[CM15] M. Colombo and G. Mingione, Regularity for double phase variational problems, Archive for Rational Mechanics and Analysis 215 (2015), 443–496.

[CIL92] M. Crandall, H. Ishii and P.-L. Lions, User’s guide to viscosity solutions of second order partial differential equations, Bulletin of the American Mathematical Society 27 (1992), 1–67.

[DP05] D. Danielli and A. Petrosyan, A minimum problem with free boundary for a degenerate quasilinear operator, Calculus of Variations and Partial Differential Equations 23 (2005), 97–124.

[daSLR21] J. V. da Silva, R. A. Leitão and G. C. Ricarte, Geometric regularity estimates for fully nonlinear elliptic equations with free boundaries, Mathematische Nachrichten 294 (2021), 38–55.

[daSR20] J. V. da Silva and G. C. Ricarte, Geometric gradient estimates for fully nonlinear models with non-homogeneous degeneracy and applications, Calculus of Variations and Partial Differential Equations 59 (2020), Article no. 161.

[daSV20] J. V. da Silva and H. Vivas, The obstacle problem for a class of degenerate fully nonlinear operators, Revista Matemática Iberoamericana 37 (2021), 1991–2020.

[DaSV21] J. V. da Silva and H. Vivas, Sharp regularity for degenerate obstacle type problems: a geometric approach, Discrete and Continuous Dynamical Systems 41 (2021), 1359–1385.

[DeF20] C. De Filippis, Regularity for solutions of fully nonlinear elliptic equations with nonhomogeneous degeneracy, Proceedings of the Royal Society of Edinburgh. Section A. Mathematics 151 (2021), 110–132.

[DeFM19] C. De Filippis and G. Mingione, On the Regularity of Minima of Non-autonomous functionals, Journal of Geometric Analysis 30 (2020), 1584–1626.

[DeFO19] C. De Filippis and J. Oh, Regularity for multi-phase variational problems, Journal of Differential Equations 267 (2019), 1631–1670.

[DeS11] D. De Silva, Free boundary regularity for a problem with right hand side, Interfaces and Free Boundaries 13 (2011), 223–238.

[DFS15] D. De Silva, F. Ferrari and S. Salsa, Free boundary regularity for fully nonlinear non-homogeneous two-phase problems, Journal de Mathématiques Pures et Appliquées 103 (2015), 658–694.

[Fel01] M. Feldman, Regularity of Lipschitz free boundaries in two-phase problems for fully nonlinear elliptic equations, Indiana University Mathematics Journal 50 (2001), 1171–1200.

[FBMW10] J. Fernández Bonder, S. Martínez and N. Wolanski, A free boundary problem for the p(x)-Laplacian, Nonlinear Analysis 72 (2010), 1078–1103.

[HPRS] G. Huaroto, E. A. Pimentel, G. C. Rampasso and A. Święch, A fully nonlinear degenerate free transmission problem, https://arxiv.org/abs/2008.06917.
[I11] C. Imbert, *Alexandroff-Bakelman-Pucci estimate and Harnack inequality for degenerate/singular fully nonlinear elliptic equations*, Journal of Differential Equations 250 (2011), 1553–1574.

[IS12] C. Imbert and L. Silvestre, *$C^{1,\alpha}$ regularity of solutions of some degenerate fully nonlinear elliptic equations*, Advances in Mathematics 233 (2012), 196–216.

[IS16] C. Imbert and L. Silvestre, *Estimates on elliptic equations that hold only where the gradient is large*, Journal of the European Mathematical Society 18 (2016), 1321–1338.

[LW17] C. Lederman and N. Wolanski, *Weak solutions and regularity of the interface in an inhomogeneous free boundary problem for the $p(x)$-Laplacian*, Interfaces and Free Boundaries 19 (2017), 201–241.

[LW19] C. Lederman and N. Wolanski, *Inhomogeneous minimization problems for the $p(x)$-Laplacian*, Journal of Mathematical Analysis and Applications 475 (2019), 423–463.

[LW21] C. Lederman and N. Wolanski, *Lipschitz continuity of minimizers in a problem with nonstandard growth*, Mathematics in Engineering 3 (2021), Article no. 009.

[LR18I] R. Leitão and G. Ricarte, *Free boundary regularity for a degenerate fully nonlinear elliptic problem with right hand side*, https://arxiv.org/abs/1810.07840.

[MW08] S. Martínez and N. Wolanski, *A minimum problem with free boundary in Orlicz spaces*, Advances in Mathematics 218 (2008), 1914–1971.

[MS06] E. Milakis and L. Silvestre, *Regularity for fully nonlinear elliptic equations with Neumann boundary data*, Communications in Partial Differential Equations 31 (2006), 1227–1252.

[RS15] G. C. Ricarte and J. V. Silva, *Regularity up to the boundary for singularly perturbed fully nonlinear elliptic equations*, Interfaces and Free Boundaries 17 (2015), 317–332.

[RST17] G. C. Ricarte, J. V. Silva and R. Teymurazyan, *Cavity type problems ruled by infinity Laplacian operator*, Journal of Differential Equations 262 (2017), 2135–2157.

[RT11] G. Ricarte and E. Teixeira, *Fully nonlinear singularly perturbed equations and asymptotic free boundary*, Journal of Functional Analysis 261 (2011), 1624–1673.

[Zhi93] V. V. Zhikov, *Lavrentiev phenomenon and homogenization for some variational problems*, Comptes Rendus de l’Académie des Sciences. Série I. Mathématique 316 (1993), 435–439.