An elementary proof that vector bundles on $\mathbb{P}^1$ split

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Abstract

This paper gives a new elementary proof of the theorem that all vector bundles on $\mathbb{P}^1$ split into the direct sum of line bundles. The proof is based on the study of divisors associated to germs of sections at the generic point.

Dedekind and Weber first proved that all vector bundles on $\mathbb{P}^1$ split into the direct sum of line bundles, in terms of the factorization of matrices, in [DWS2]. Grothendieck later proved it using a cohomology argument in [Gro57]. This paper presents a new elementary proof, based on studying divisors of meromorphic sections. We will use elementary standard facts from algebraic geometry which can be found in [Hat77, Chapter II].

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Let $X$ be a nonsingular irreducible algebraic curve, and $\mathcal{E}$ a vector bundle on $X$ (i.e. a locally free sheaf of finite rank). Let $K$ be the field of rational functions on $X$. If we localize $\mathcal{E}$ at the generic point $\xi$, we get a $K$-vector space, $\mathcal{E}_\xi$. We begin by attaching some additional structure to that vector space.

Definition 1. Let $P$ be a point on $X$ and $\alpha$ an element of $\mathcal{E}_\xi$. Let $t$ be a local parameter at $P$. Since there is an injection from $\mathcal{E}_P$ to $\mathcal{E}_\xi$, we can view the former as an $O_P$-submodule of the latter. We define the order of $\alpha$ at $P$ to be

$$\delta_P(\alpha) = \max\{n | t^{-n}\alpha \in \mathcal{E}_P\}.$$ 

This number does not depend on our choice of $t$.

Definition 2. We define the divisor $\delta(\alpha)$ of $\alpha$ to be the divisor whose value at each $P$ is $\delta_P(\alpha)$. The degree $\deg(\alpha)$ of $\alpha$ is the degree of $\delta(\alpha)$.

It is easy to check that $\delta_P(\alpha)$ is always finite, and that $\delta(\alpha)$ has finite support. The proofs are mostly analogous to the proofs for the divisor function on $K$, which we will also refer to as $\delta$.

Lemma 3. The divisor function $\delta$ has the following properties.

1. For $f \in K$, $\alpha \in \mathcal{E}_\xi$, we have

$$\delta(f\alpha) = \delta(f) + \delta(\alpha).$$
2. For \( \alpha_1, \ldots, \alpha_n \in \mathcal{E}_\xi \), we have

\[
\delta \left( \sum_{i=1}^{n} \alpha_i \right) \geq \min_i \delta(\alpha_i).
\]

**Proof.** Analogous to the corresponding properties of the usual divisor function on \( K \).

**Remark 4.** Each point on a curve corresponds to a valuation, which can be converted into an absolute value by the formula \( |f| = q^{-\delta_P(f)} \) for \( q > 1 \), the same process used to create p-adic absolute values. The corresponding function \( q^{-\delta_P(\alpha)} \) on \( \mathcal{E}_\xi \) gives it the structure of a Banach space. This sort of space is discussed in [GI63].

**Definition 5.** Recall that when we identify a vector bundle with its sheaf of sections, then line bundles correspond to locally free sheaves of rank 1. Given an injective map \( F \hookrightarrow E \) of locally free sheaves, we call \( F \) a subbundle of \( E \) if for every point \( f : P \rightarrow X \) the map on fibers \( f^*F \rightarrow f^*E \) is injective.

**Proposition 6.** The following hold.

1. Let \( F \) and \( E \) be vector bundles. The isomorphisms \( E \cong F \) are in one-to-one correspondence with those isomorphisms \( \mathcal{E}_\xi \cong \mathcal{F}_\xi \) which commute with \( \delta \).

2. For a vector bundle \( E \), every 1-dimensional subspace \( L \subset \mathcal{E}_\xi \) corresponds to a unique subline bundle \( L \hookrightarrow E \) such that \( L = \mathcal{L}_\xi \).

3. Let \( \{e_i\} \) be a basis of \( \mathcal{E}_\xi \). \( E \) splits into the line bundles corresponding to the spaces generated by the \( e_i \) if and only if, for all \( f_1, \ldots, f_n \in K \)

\[
\delta \left( \sum_{i=1}^{n} f_i e_i \right) = \min_i \delta(f_i e_i). \tag{1}
\]

**Proof.** To prove part 1, we will reconstruct the modules and maps that make up the sheaf \( E \) from \( \delta \). First note that \( \delta_P(\alpha) \geq 0 \) if and only if \( \alpha \in \mathcal{E}_P \). This means that \( \alpha \in \mathcal{E}(U) \) if and only if for all \( P \in U, \delta_P(\alpha) \geq 0 \). Therefore we can reconstruct \( \mathcal{E}(U) \) by taking the set of all elements in \( \mathcal{E}_\xi \) with that property. Since each section is identified with its germ at \( \xi \), the restriction maps are given by the natural inclusion maps. Since this construction is unique, the result follows.

To prove part 2, we use the argument from part 1 to see how the subspace \( L \) corresponds to a subsheaf \( \mathcal{L} \hookrightarrow \mathcal{E} \). Fix a nonzero \( \alpha \in L \). It is clear from part 1 of Lemma 3 that \( \mathcal{L} \) is the sheaf of sections of the line bundle corresponding to the divisor \( \delta(\alpha) \). It remains to verify that \( \mathcal{L} \hookrightarrow \mathcal{E} \) is a subbundle. This follows since \( \mathcal{L} \) is locally generated by the nonvanishing section \( t^{-\delta_P(\alpha)} \alpha \).

To prove part 3, first assume that \( E \) splits into a direct sum of line bundles. Then the divisor function on \( \mathcal{E}_\xi \) satisfies (1). To obtain the other direction, note that, by part 1, property (1), combined with the divisors of the line bundles, characterizes the sheaf \( \mathcal{E} \) up to isomorphism.

**Lemma 7.** Let \( \alpha_1, \ldots, \alpha_n \in \mathcal{E}_\xi \) and suppose

\[
\delta_P \left( \sum_{i=1}^{n} \alpha_i \right) > \min_i (\delta_P(\alpha_i)).
\]
1. Let $J$ be the set of $i$ such that $\delta_P(\alpha_i) = \min_{i \in J}(\delta_P(\alpha_i))$. Then
\[
\delta_P \left( \sum_{i \in J} \alpha_i \right) > \min_{i \in J}(\delta_P(\alpha_i)).
\]

2. Assume $X = \mathbb{P}^1$. Further, assume that for all $i$, we have $\delta_P(\alpha_i) = \min_{i \in J}(\delta_P(\alpha_i))$. Let $\alpha_j$ be an element of the smallest degree among $\{\alpha_i\}$. Then we can choose $f_1, \ldots, f_n \in K$ such that
\[
\delta \left( \sum_{i=1}^n f_i \alpha_i \right) > \delta(\alpha_j).
\]

**Proof.** To prove part 1
\[
\delta_P \left( \sum_{i \in J} \alpha_i \right) = \delta_P \left( \sum_{i=1}^n \alpha_i - \sum_{i \notin J} \alpha_i \right) \geq \min \left( \delta_P \left( \sum_{i=1}^n \alpha_i \right), \min_{i \notin J} \delta_P(\alpha_i) \right)
\]
and since both $\delta_P \left( \sum_{i=1}^n \alpha_i \right)$ and each $\delta_P(\alpha_i)$ with $i \notin J$ are greater than $\min_i \delta_P(\alpha_i)$, their sum is as well.

To prove part 2, for each $i$ we choose $D \geq \delta(\alpha_j)$ with $\deg(D) = \delta(\alpha_i)$ and $\delta_P(D) = \delta_P(\alpha_j)$. Since by assumption $X = \mathbb{P}^1$, there exists $f_i \in K$ such that $\delta(f_i \alpha_i) = D$ and we may further assume that $f_i(P) = 1$.

Since $\delta(f_i \alpha_i) \geq \delta(\alpha_j)$, we know that
\[
\delta \left( \sum_{i=1}^n f_i \alpha_i \right) \geq \delta(\alpha_j).
\]

To prove strict inequality, we will show that the order of the divisor of the sum is strictly larger at $P$. Intuitively this is because multiplying by $f_i$ did not change the value of the sections at $P$, so they still cancel. Algebraically, we have
\[
\delta_P \left( \sum_{i=1}^n f_i \alpha_i \right) = \delta_P \left( \sum_{i=1}^n (f_i - 1) \alpha_i + \sum_{i=1}^n \alpha_i \right) > \delta_P(\alpha_j)
\]

The two parts of Lemma 7 naturally combine, as the sum produced by the first satisfies the conditions required by the second. We can now prove our theorem.

**Theorem 8.** All vector bundles on $\mathbb{P}^1$ split into a direct sum of line bundles.

**Proof.** Consider a vector bundle $\mathcal{E}$. First we show that there must be a maximal degree among the divisors of the sections. If $\mathcal{E}_\xi$ contains a section of degree $d$, then $\mathcal{E}$ contains the line bundle generated by that section, which has degree $d$ and so has a $d+1$-dimensional vector space of global sections. All these global sections will also be global sections of the vector bundle. Since it has only a finite-dimensional vector space of global sections, there must be a maximum degree.
Choose a section $e_1 \in \mathcal{E}_\xi$ of highest degree. For each $i$ choose an $e_i$ of highest degree among sections not in the span of $e_1$ through $e_{i-1}$. Continue until the whole space is spanned. The set \{\(e_1, ..., e_n\)\} clearly form a basis of $\mathcal{E}_\xi$.

We will now argue that, for all $f_1, ..., f_n \in K$, we have
\[
\delta \left( \sum_{i=1}^{n} f_i e_i \right) = \min_i \delta(f_i e_i)
\]

By Proposition 6 this proves our claim. Assume that this equation fails for some \{\(f_i\)\}. We know that the left hand side cannot be less at any point, so it must be greater somewhere.

Choose a $P$ where equality does not hold. We use the first part of Lemma 7 then the second, to choose a new weighted sum $\sum g_i e_i$ of the $e_i$. Because some of the elements are removed by the first part, some of the $g_i$ will be 0. Let $e_j$ be the last, and therefore lowest-degree, basis element with nonzero coefficient $g_j$. The sum will satisfy
\[
\delta \left( \sum_{i=1}^{n} g_i e_i \right) > \delta(f_j e_j)
\]
As $g_j \neq 0$, $\sum_{i=1}^{n} g_i e_i$ is not in the span of $e_1$ through $e_{j-1}$, but its degree is greater than $e_j$’s. This is a contradiction.

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