On the causal properties of symmetric Lorentzian spaces

Ya. V. Bazaikin

Abstract

The causal properties of Lorentzian symmetric spaces are investigated in the paper. There is proved the global hyperbolicity of the Cahen–Wallach Lorentzian symmetric spaces, corresponding to solvable Lie algebras.

1 Introduction

Symmetric Lorentzian spaces play a significant role in the foundations of modern supergravity theories \cite{1, 2}. In \cite{3}, a classification of all Lorentzian symmetric spaces was given. Besides the universal covers of multi-dimensional de Sitter and anti-de Sitter spaces, the only simply-connected symmetric spaces are the Cahen–Wallach spaces \( CW_n(A) \), which are constructed in the following way \cite{1}. Consider the Euclidean space \( \mathbb{R}^n \) with coordinates \( (x^1, \ldots, x^n) \) and the plane \( \mathbb{R}^2 \) with coordinates \( (\xi, \eta) \). Let \( A = (a_{ij}) \) be a symmetric matrix. Then \( CW_n(A) = (\mathbb{R}^2 \times \mathbb{R}^n, ds^2_{n,A}) \), where

\[
\begin{align*}
ds^2_{n,A} = 2d\eta \left( d\xi + \sum_{i,j=1}^{n} a_{ij} x^i x^j d\eta \right) + (dx^1)^2 + \ldots + (dx^n)^2.
\end{align*}
\]

One of the main results of the paper is the following

**Theorem 1.** The space \( CW_n(A) \) is globally hyperbolic.

**Remark 1.** The spaces \( CW_n(A) \) correspond to solvable Lie algebras in the classification of \cite{3}. It can be shown in the remaining cases that de Sitter space is globally hyperbolic (for instance, \cite{2}), and anti-de Sitter space is not globally hyperbolic \cite{4}.

Theorem 1 turns out to be an immediate consequence of the following more general result.

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Theorem 2. Let \((M, g_1)\) be a Riemannian \(C^1\)-manifold and let there exist a point \(p_0\) with the property that the exponential map \(\exp_{p_0}: T_{p_0}M \to M\) is a diffeomorphism. Denote by \(\rho(p) = \rho(p_0, p)\) the Riemannian distance function to the point \(p_0\).

Consider the following Lorentzian metric \(g_2\) on \(N = \mathbb{R}^2 \times M\):

\[
g_2 = 2d\eta(d\xi - Fd\eta) + g_1,
\]

where \(\xi, \eta\) are the coordinates on \(\mathbb{R}^2\) and \(F: M \to \mathbb{R}\) is a continuous function. Assume that

\[
|F(p)| \leq c_1^2 + c_2^2 \rho(p)^2
\]

for some constants \(c_1, c_2\). Then \((N, g_2)\) is globally hyperbolic.

Remark 2. One can relax the conditions of Theorem 2: instead of the function \(\rho\) one can choose the distance function to some subset \(S \subset M\); and the exponential map condition can be substituted by the following one: the restriction of the function \(\rho\) to every piecewise \(C^1\)-smooth curve in \(M\) is a piecewise \(C^1\)-smooth real function. The last property is closely related to the cut locus structure of \(M\).

2 Proof of the main results

Let \((N, g)\) be a time-oriented Lorentzian manifold. The manifold \((N, g)\) is called strongly causal if for each point \(p \in N\) and for each neighborhood \(U\) of \(p\) there exists an open neighborhood \(V \subset U\) of \(p \in V\) such that each causal curve in \(N\), starting and ending in \(V\), is entirely contained in \(U\). As usual, a causal future (past) \(J^+(p) (J^-(p))\) of the point \(p\) is the set of all such points \(q\) that either \(p = q\) or there exists a future directed (past directed) causal curve starting at \(p\) and ending at \(q\). A strongly causal Lorentzian manifold \((N, g)\) is called globally hyperbolic if for each pair \(p, q \in N\) the set \(J^+(p) \cap J^-(q)\) is compact.

It is proved in \([6, 7]\) that global hyperbolicity of the time-oriented Lorentzian manifold \((N, g)\) is equivalent to the following condition: \((N, g)\) is isometric to the direct product \(\mathbb{R} \times S\) with the metric \(-\beta dt^2 + g_t\), where \(\beta\) is a smooth positive function, \(g_t\) is a Riemannian metric on \(S\) depending smoothly on \(t\), and each "section" \(\{t\} \times S\) is a smooth space-like Cauchy hypersurface in \(M\).

We say that a space-time \((N_1, g_1)\) is asymptotically dominated by a space-time \((N_2, g_2)\) if there exists a continuous map \(f: N_1 \to N_2\), satisfying the following two properties:

1) for each causal curve \(\gamma\) in \(N_1\) the curve \(f(\gamma)\) is causal in \(N_2\);
2) for each point \(p \in N_2\) the set \(f^{-1}(p)\) is compact.

Lemma 1. Let \((N_1, g_1)\) be asymptotically dominated by \((N_2, g_2)\). Then if \((N_2, g_2)\) is globally hyperbolic, then \((N_1, g_1)\) is also globally hyperbolic.

Proof. Since \((N_2, g_2)\) is globally hyperbolic, there exists a time function \(T': N_2 \to \mathbb{R}\). Put \(T = T' \circ f\). The property 1) of the map \(f\) implies
that $T : N_1 \to \mathbb{R}$ is a time function on $N_1$. Then $(N_1, g_1)$ is stably causal and, consequently, it is strongly causal \[3\].

Let us prove that for each compact set $K' \subset N_2$ the set $K = f^{-1}(K') \subset N_1$ is compact. Introducing Riemannian metrics on $N_1$ and $N_2$, we can assume that $N_1$ and $N_2$ are metric spaces with countable topology bases. So, it suffices for us to prove that from each sequence $p_i \in K$ we can extract a convergent subsequence. Therefore, in view of compactness of $K'$, we can assume that $f(p_i) \to q \in K'$. This means that there exists a subsequence $p_i$ converging to the set $K_0 = f^{-1}(q)$, which is compact by the property 2) of the map $f$. Consider an open $\varepsilon$-neighborhood $U$ of the compact $K_0$ with the compact closure $K_1 = \overline{U}$. Then we can assume that $p_i \in U$ and, passing to a subsequence, we obtain that $p_i \to p \in K_1$. So far as $p_i$ converges to $K_0$, we see that $p \in K_0 \subset K$.

Further, if $x \in J^+(p) \cap J^-(q)$, then $f(x) \in J^+(f(p)) \cap J^-(f(q))$. Therefore, $J^+(p) \cap J^-(q)$ is a subset of the compact set $f^{-1}(J^+(f(p)) \cap J^-(f(q)))$. So far as the set $J^+(p) \cap J^-(q)$ is closed in a strongly causal space-time \[3\], we can conclude that it is compact. This argument completes the proof of the global hyperbolicity of $(N_1, g_1)$. The lemma is proved.

Consider the following metric on $N = \mathbb{R}^3$ with coordinates $(\eta, \xi, \tau)$:

$$g = 2d\eta (d\xi - f(\tau)d\eta) + d\tau^2,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Let $(N, g)$ be oriented in time by the coordinate vector field $\frac{\partial}{\partial \tau}$.

**Theorem 3.** Let there exist constants $c_1, c_2$ such that $|f(\tau)| \leq c_1^2 \tau^2 + c_2^2$. Then the metric $g$ is globally hyperbolic.

**Remark 3.** In the paper \[8\] the global hyperbolicity of the metric $g$ was proved in the case when $f(\tau) \sim \tau$. However, the method that was used there does not work in the general case.

**Proof.** Let $f_0(\tau) = c_1^2 \tau^2 + c_2^2$. Let us consider the metric

$$\tilde{g} = 2d\eta (d\xi - f_0(\tau)d\eta) + d\tau^2,$$

which dominates $g$, and let us show its global hyperbolicity.

First of all, we construct a global time function $T = T(\eta, \xi, \tau)$. We will seek $T$ in the form:

$$T = \eta - \Phi(\xi, \tau),$$

for some smooth function $\Phi : \mathbb{R}^2 \to \mathbb{R}$. It is easy to calculate the gradient of $T$ with respect to the metric $\tilde{g}$:

$$\nabla T = (1 - 2f_0(\tau)\Phi_\xi) \partial_\xi - \Phi_\xi \partial_\eta - \Phi_\tau \partial_\tau.$$

We will try to find a time function $T$, whose gradient is time-like, that is

$$|\nabla T|^2 = 2f_0(\tau)\Phi_\xi^2 + \Phi_\tau^2 - 2\Phi_\xi < 0,$$
or, equivalently,
\[ \Phi_\xi^2 + \frac{\Phi_\tau^2}{2f_0(\tau)} < \frac{\Phi_\xi}{f_0(\tau)}. \]  \hspace{1cm} (2)

Let \( \phi, \psi : \mathbb{R} \to \mathbb{R} \) be some smooth functions. More precisely, we will try to find the function \( \Phi \) in the following form:
\[ \Phi(\xi, \tau) = \phi \left( \frac{\xi}{\psi(\tau)} \right). \]

Then the condition (2) changes into the following inequality:
\[ \phi' \left( \frac{\xi}{\psi(\tau)} \right) \left[ 1 + \frac{\psi'(\tau)^2}{2f_0(\tau)} \left( \frac{\xi}{\psi(\tau)} \right)^2 \right] < \frac{\psi(\tau)}{f_0(\tau)}. \]  \hspace{1cm} (3)

Take \( \varepsilon > 0 \) and put \( \psi(\tau) = \varepsilon f_0(\tau) = \varepsilon (c_1^2 \tau^2 + c_2^2) \), that is \( \psi(\tau)/f_0(\tau) = \varepsilon \). Then
\[ \frac{\psi'(\tau)^2}{2f_0(\tau)} = \frac{2\varepsilon^2 c_1^4 \tau^2}{c_1^2 \tau^2 + c_2^2} \leq 2\varepsilon^2 c_1^2. \]

Consequently, the inequality (3) holds everywhere if we require \( \phi \) to satisfy the following equation:
\[ \phi'(x) \left( 1 + 2\varepsilon^2 c_1^2 x^2 \right) = \frac{\varepsilon}{2}. \]

An elementary integration yields
\[ \phi_\varepsilon(x) = \frac{1}{2\sqrt{2c_1}} \arctan \left( \sqrt{2\varepsilon c_1} x \right). \]

So we have proved that the everywhere defined smooth function
\[ T_\varepsilon(\xi, \eta, \tau) = \eta - \frac{1}{2\sqrt{2c_1}} \arctan \left( \frac{\sqrt{2\varepsilon c_1} \xi}{c_1^2 \tau^2 + c_2^2} \right) \]
is a time function with the time-like gradient. In particular, this implies strong causality of the metric \( g \).

Consider two points \( p_i = (\xi_i, \eta_i, \tau_i), i = 1, 2 \) in \( N \), where \( T(p_1) < T(p_2) \) (in the opposite case the intersection of the cones \( K = J^+(p_1) \cap J^-(p_2) \) is empty). Note that for all \( \eta_0, \xi_0 \) the transformation
\[ (\xi, \eta, \tau) \mapsto (\xi + \xi_0, \eta + \eta_0, \tau) \]
is an isometry of the Lorentzian manifold \((N, g)\). Consequently, we can assume, without loss of generality, that \( \eta_1 = 0 \).

**Lemma 2.** Assuming the conditions described above, if \( \eta_2 < \frac{\pi}{4\sqrt{2c_1}} \), then \( K = J^+(p_1) \cap J^-(p_2) \) is compact.
Proof of Lemma 2. In the further calculations we put
\[ x = \frac{\xi}{c_1^2 + c_2^2}, x_i = \frac{\xi_i}{c_1^2 + c_2^2}, i = 1, 2. \]

Now choose \( \varepsilon > 0 \) small enough so that
\[ |\phi_\varepsilon(x_i)| \leq \frac{\pi}{4\sqrt{2}c_1} - \eta_2, i = 1, 2. \]

Consider some point \( p \in K \) and two causal curves: \( \gamma(T) \), starting at \( p_1 \) and ending at \( p \), and \( \delta(T) \), starting at \( p \) and ending at \( p_2 \) (it is evident that we can choose the time function \( T \) as a regular parameter on these curves). Joining these curves, we can assume that we have one piecewise causal curve \( \gamma(T) \), starting at \( p_1 \), going through \( p \) and ending at \( p_2 \). Then, using monotonicity of the functions \( T \) (strict) and \( \eta \) (non-strict) along \( \gamma \), we have
\[ \phi_\varepsilon(x(T)) \leq \phi_\varepsilon(x_1) + \frac{\pi}{4\sqrt{2}c_1} - \eta_2 < \eta(T) - \eta_1 + \frac{\pi}{4\sqrt{2}c_1} - \eta_2 \leq \frac{\pi}{4\sqrt{2}c_1}. \]

The definition of the function \( \phi_\varepsilon \) immediately implies that there exists a constant \( d \) such that \( x(T) \leq d \) for all \( T \). Since the initial point \( p \) was taken to be arbitrary, we have \( x(p) \leq d \) for all \( p \in K \). Analogously,
\[ \phi_\varepsilon(x(T)) \geq \phi_\varepsilon(x_2) - \frac{\pi}{4\sqrt{2}c_1} + \eta_2 > \eta(T) - \eta_2 - \frac{\pi}{4\sqrt{2}c_1} + \eta_2 \geq -\frac{\pi}{4\sqrt{2}c_1}. \]

Therefore, we can assume that for all \( p \in K \) the following inequality takes place:
\[ |x(p)| \leq d. \quad (4) \]

Further, from the relation (1) we have:
\[ 1 = \frac{d\eta}{dT} - \phi'_\varepsilon(x) \frac{dx}{dT}, \]
that is
\[ \frac{dx}{dT} \geq -\frac{2}{\varepsilon} \left( 1 + 2\varepsilon^2 c_1^2 x^2 \right) \geq -\frac{2}{\varepsilon} \left( 1 + 2\varepsilon^2 c_1^2 d^2 \right). \]

Consequently,
\[ \frac{dx}{dT} = \frac{1}{f_0} \frac{d\xi}{dT} - \frac{\xi}{f_0} \frac{2c_1^2 \tau}{d\tau} \frac{d\tau}{dT} = \frac{1}{f_0} \frac{d\xi}{dT} - x \frac{2c_1^2 \tau}{f_0} \frac{d\tau}{dT} \geq -\frac{2}{\varepsilon} \left( 1 + 2\varepsilon^2 c_1^2 d^2 \right). \]

This immediately implies
\[ \frac{d\xi}{dT} \geq -f_0 \left( 1 + 2\varepsilon^2 c_1^2 d^2 \right) + 2x c_1^2 \frac{d\tau}{dT} \geq -f_0 \left( 1 + 2\varepsilon^2 c_1^2 d^2 \right) - 2d_1 c_1^2 \left| \frac{d\tau}{dT} \right|. \]
Further, we use the causality condition for the curve $\gamma$:

$$0 \geq 2 \frac{d\eta}{dT} \left( \frac{d\xi}{dT} - f_0 \frac{d\eta}{dT} \right) + \left( \frac{d\tau}{dT} \right)^2 \geq$$

$$2 \frac{d\eta}{dT} \left( - f_0 \frac{2}{\varepsilon} \left( 1 + 2 \varepsilon^2 c_1^2 d^2 \right) - 2 d c_1^2 \left| \frac{d\tau}{dT} \right| - f_0 \frac{d\eta}{dT} \right) + \left( \frac{d\tau}{dT} \right)^2$$

(5)

Let us assume, first, that $\frac{d\eta}{dT} \geq 1$. Then, continuing the previous inequality, we obtain

$$-2 f_0 \left( 1 + \frac{2}{\varepsilon} \left( 1 + 2 \varepsilon^2 c_1^2 d^2 \right) \right) \left( \frac{d\eta}{dT} \right)^2 - 4 d c_1^2 \frac{d\eta}{dT} \left| \frac{d\tau}{dT} \right| - 2 \left( 1 + \frac{2}{\varepsilon} \left( 1 + 2 \varepsilon^2 c_1^2 d^2 \right) \right) \leq 0.$$

In view of our assumption about the growth of $\eta$, we can change the parameter $T$ on the curve $\gamma$, taking the coordinate $\eta$ as a new parameter. In this case (dividing by $f_0$), we have:

$$\frac{1}{f_0} \left( \frac{d\tau}{dT} \right)^2 - 4 d c_1^2 \left| \frac{\tau}{f_0 \frac{d\eta}{dT}} \right| - 2 \left( 1 + \frac{2}{\varepsilon} \left( 1 + 2 \varepsilon^2 c_1^2 d^2 \right) \right) \leq 0.$$

So far as

$$\frac{\left| \tau \right|}{\sqrt{f_0}} = \frac{1}{\sqrt{c_1^2 + \frac{\varepsilon^2}{2}}} \leq \frac{1}{c_1},$$

we have

$$\left( \frac{1}{\sqrt{f_0} \frac{d\eta}{dT}} \right)^2 - 4 d c_1 \left| \frac{\tau}{f_0 \frac{d\eta}{dT}} \right| - 2 \left( 1 + \frac{2}{\varepsilon} \left( 1 + 2 \varepsilon^2 c_1^2 d^2 \right) \right) \leq 0.$$

The last estimate implies

$$\left| \frac{1}{\sqrt{f_0} \frac{d\eta}{dT}} - 2 d c_1 \right| \leq \sqrt{4 d^2 c_1^2 + 2 \left( 1 + \frac{2}{\varepsilon} \left( 1 + 2 \varepsilon^2 c_1^2 d^2 \right) \right)}.$$

By virtue of

$$\frac{1}{\sqrt{f_0} \frac{d\eta}{dT}} = \frac{d}{d\eta} \left( \frac{1}{c_1} \text{arcsinh} \left( \frac{c_1 \tau}{c_2} \right) \right),$$

we can integrate the last inequality and, using a priori boundedness of $\eta$, we obtain that the total increment of the function $\tau$ on those subsegments of $\gamma$, where $\frac{d\eta}{dT} \geq 1$, is bounded by some constant $D$ depending only on $c_1, c_2, \eta_2$ and on the constants $\xi_1, \xi_2, d, \varepsilon$ which are determined by the choice of the points $p_1, p_2$.

If $\frac{d\eta}{dT} \leq 1$, the inequality (5) can be immediately rewritten as:

$$\left( \frac{d\tau}{dT} \right)^2 - 4 d c_1^2 \left| \frac{d\tau}{dT} \right| - 2 f_0 \left( 1 + \frac{2}{\varepsilon} \left( 1 + 2 \varepsilon^2 c_1^2 d^2 \right) \right) \leq 0.$$
As in the previous case, dividing by $f_0$ and integrating, we obtain that the total increment of $\tau$ on those subsegments, where $\frac{d\eta}{dT} \leq 1$, is also universally bounded by the constant $D$.

Thus we have proved that the coordinate $\tau$ is universally bounded along each causal curve. The estimate (4) implies that the coordinate $\xi$ is also universally bounded. Therefore, the set $K$ is contained in a bounded domain in $\mathbb{R}^3$, that is, the closure of $K$ is compact. Strong causality implies the closeness and, consequently, compactness of $K$ \[5\]. The lemma is proved.

Now let $p_1$ and $p_2$ be the same as above but without any restriction on $\eta_2$. As above, we consider the intersection of the cones $K = J^+(p_1) \cap J^-(p_2)$. Take a constant $C > 0$ and consider the following Lorentzian metric on the manifold $N' = \mathbb{R}^3(\xi', \eta', \tau')$:

$$
\tilde{g}''^2 = 2d\eta' (d\xi' - f'_0(\tau')d\eta') + d\tau'^2,
$$

where

$$
f'_0(\tau') = \frac{C_1^2}{C^2} \tau'^2 + \frac{C_2^2}{C^2} = C_1^2 \tau'^2 + C_2^2.
$$

Now choose $C$ large enough for the inequality

$$
\eta_2 < \frac{\pi}{4\sqrt{2}C_1}
$$

(6) to hold. Consider the transformation $\sigma : N \rightarrow N'$:

$$
\sigma(\xi, \eta, \tau) = \left( \frac{\xi}{C^2}, \eta, \frac{\tau}{C} \right).
$$

It is evident that the transformation of the Lorentzian spaces $\sigma : (N, ds^2) \rightarrow (N', ds'^2)$ is a conformal diffeomorphism. Consequently, it maps homeomorphically the intersection $K$ of the causal future and past of the points $p_1, p_2$ to the intersection $K'$ of the corresponding causal future and past of the points $p'_1 = (\xi_1/C^2, 0, \tau_1/C)$ and $p'_2 = (\xi_2/C^2, \eta_2, \tau_2/C)$. Inequality (6) and Lemma 2 imply that $K'$ is compact and, therefore, $K$ is also compact. Our theorem is proved.

**Remark 4.** The asymptotical growth of the function $f_0$ in Theorem 3 is optimal in the class of power functions. Indeed, if, in the conditions of Theorem 3, we consider $f = \tau^{2+\varepsilon}$ for $\varepsilon > 0$, then it is sufficient to take a class of the causal curves of the form $\gamma(s) = (\xi_0, \eta(s), \tau(s))$. Such curves are causal on the plane with coordinates $(\eta, \tau)$ with respect to the Lorentzian metric

$$
-2d\eta^2 + \frac{d\tau^2}{f(\tau)}.
$$

An elementary integration shows that this metric is not globally hyperbolic: there exist light-like curves, both future- and past-directed, which escape to $+\infty$ with respect to the variable $\tau$ for a finite increment of $\eta$. 

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Proof of Theorem 2. It suffices to note that the metric $g_2$ in the conditions of Theorem 2 is asymptotically dominated by the metric

$$ ds^2 = g_2 = 2d\eta \left(d\xi - (c_1^2 + c_2^2 \rho^2)\, d\eta\right) + d\rho^2. $$

Indeed, let $\gamma(s) = (\xi(s), \eta(s), p(s))$ be a causal curve in $N$. Let $\delta_s(t), 0 \leq t \leq \rho(p(s))$ be the shortest normal geodesic joining the points $p_0 = \delta_s(0)$ and $p(s) = \delta_s(p(s))$. Let

$$ \frac{dp}{ds} = dp \left(\frac{dp}{ds}\right) \frac{\partial}{\partial \rho} + \left(\frac{dp}{ds} - dp \left(\frac{dp}{ds}\right) \frac{\partial}{\partial \rho}\right) $$

be the decomposition of the tangent vector into radial and tangential components with respect to the level hypersurface of the function $\rho$ (which is, in fact, the geodesic sphere of radius $\rho$ centered at $p_0$). Moreover, we can interpret $\frac{\partial}{\partial \rho}$ as a tangent vector field to the normal geodesic $\delta_s$. In this case

$$ \left|\frac{dp}{ds}\right| \geq \left|dp \left(\frac{dp}{ds}\right)\right| = \frac{dp(p(s))}{ds}. \quad (7) $$

By the assumptions of the theorem, each piecewise $C^1$-smooth curve in $N$ is projected under the map $(\xi, \eta, p) \mapsto (\xi, \eta, \rho(p))$ to a piecewise $C^1$-smooth curve in $\mathbb{R}^3$. Then, (7) implies that each causal curve in $N$ is projected under the same map to a causal curve in $\mathbb{R}^3$ with the Lorentzian metric

$$ g = 2d\eta \left(d\xi - (c_1^2 + c_2^2 \rho^2)\, d\eta\right) + d\rho^2. $$

Obviously, the level sets of the function $\rho$ (that is, the geodesic spheres in $M$) are compact. It remains to apply Theorem 3 and Lemma 1.

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