A STUDY OF THE APPROXIMATE SINGULAR LAGRANGIAN - CONDITIONAL NOETHER SYMMETRIES AND FIRST INTEGRALS

SAMEERAH JAMAL

School of Mathematics, University of the Witwatersrand, Johannesburg, South Africa.
Sameerah.Jamal@wits.ac.za

The investigation of approximate symmetries of reparametrization invariant Lagrangians of \( n + 1 \) degrees of freedom and quadratic velocities is presented. We show that extra conditions emerge which give rise to approximate and conditional Noether symmetries of such constrained actions. The Noether symmetries are the simultaneous conformal Killing vectors of both the kinetic metric and the potential. In order to recover these conditional symmetry generators which would otherwise be lost in gauge fixing the lapse function entering the perturbative Lagrangian, one must consider the lapse among the degrees of freedom. We establish a geometric framework in full generality to determine the admitted Noether symmetries. Additionally, we obtain the corresponding first integrals (modulo a constraint equation). For completeness, we present a pedagogical application of our method.

Keywords: Approximate symmetries; Noether symmetries; Conservation laws.

PACS Nos.: 04.20.Fy; 02.20.Sv; 02.40.Ky.

1. Introduction

The Action Principle

\[
S = \int L(t, x, \dot{x}, N) \, dt = \int \left( \frac{1}{2N} g_{ij}(x) \dot{x}^i \dot{x}^j - NV(x) \right) dt, \quad \det g_{ij} \neq 0, \quad i, j = 1, \ldots, n, \quad (1)
\]

represents a singular system consisting of \( n + 1 \) degrees of freedom and is quadratic in the velocities. Due to the reparametrization invariance of this action, the system of Euler-Lagrange equations

\[
\mathcal{H} \doteq \frac{1}{2N^2} g_{ij} \dot{x}^i \dot{x}^j + V = 0, \quad (2)
\]

\[
\mathcal{E}^k \doteq \ddot{x}^k + \Gamma^k_{ij} \dot{x}^i \dot{x}^j - \frac{\dot{N}}{N} \dot{x}^k + N^2 g^{ik} V_i = 0, \quad (3)
\]
which describe its dynamics is singular. The lapse function $N = N(t)$ is a singular degree of freedom with Euler-Lagrange equation (4) also written as

$\frac{\partial L}{\partial N} = 0.$

These Lagrangians are encountered in various cosmological models - particularly minisuperspace models where the $x^i$ represents the scale factor components and/or possible matter fields. Subsequently new dynamical systems of such Lagrangians which admit symmetries were found in other related areas such as general relativity and classical mechanics.

Per contra, in much of the past and present literature, the singular Lagrangian of the type (1) was relegated to that of a regular Lagrangian by setting the lapse function to $N = 1$, or fixing it to some other convenient value. Under this assumption of pseudo-regularity, Christodoulakis et al. [1] proved how this leads to the loss of a class of symmetries called conditional symmetries [2]. Furthermore, they found that the presence of the lapse function affects the corresponding reparametrisation generator, i.e. the quadratic constraint which finds importance in solving the classical equations of motion.

To encompass the notion of no regularity, instead of gauge fixing the lapse function one considers the lapse as an independent degree of freedom, thereby stipulating that (4) is a constraint equation. Intrinsically, this influences the space of dependent variables while simultaneously placing additional restrictions on the symmetry determining conditions which will now involve $N$ and its derivatives.

In a previous paper [3], we investigated approximate regular Lagrangians and proved that where the perturbation terms do not affect the kinetic energy, approximate symmetries exist if and only if the kinetic metric admits a nontrivial Homothetic algebra. This planted the seed that there exists a strong and deep connection between geometry and approximations. This idea was extended to Lagrangians of partial differential equations [4-6] and moreover a self-contained approximate Lie symmetry determining method was successfully devised in [7]. In the variational studies, we unveiled new higher-order approximate versions of Noether’s theorem. A slightly different geometric approach can be found in [8]. Important approximate and regular Lagrangians were considered in [9]. In some related work, [10] used Noether symmetries to select viable theories of gravity, whilst in [11] and [12] these symmetries form a method to single out classical universes in quantum cosmology. Invariant solutions emerging from Noether symmetries are discussed in [13]. In these references, the minisuperspaces considered are similar to the ones considered here.

In the present work we start from the point of view of perturbed singular Lagrangians in order to define a conditional approximate symmetry. The method essentially consists of applying a geometric approach to determine the conditional variational symmetries. Specifically, we consider an approximate singular Lagrangian of the form:

$L (t, x^k, \dot{x}^k, N, \varepsilon) = L_0 (t, x^k, \dot{x}^k, N) + \varepsilon L_1 (t, x^k, \dot{x}^k, N) + O (\varepsilon^2),
(5)$
where we stipulate that the exact and approximate terms are defined by the Lagrangians

\[ L_0(t, x^k, \dot{x}^k, N) = \frac{1}{2N} g_{ij} \dot{x}^i \dot{x}^j - NV_0(x^k), \]

\[ L_1(t, x^k, \dot{x}^k, N) = \frac{1}{2N} h_{ij} \dot{x}^i \dot{x}^j - NV_1(x^k), \]

respectively, where the metric tensors \( g_{ij} = g_{ij}(x^k), h_{ij} = h_{ij}(x^k) \), \( \varepsilon \) is the perturbation parameter and dot denotes total derivative with respect to the independent parameter \( t \), i.e. \( \dot{x}^i = \frac{dx^i}{dt} \).

**Structure of the paper:** In §2 we introduce the main ideas of our approach: the determining system of approximate conditional symmetries to first-order. A natural extension to \( n \)-th order Noether conditional symmetries is given in §3. In §4, we frame Noether’s theorem in the context of our theory. Finally, we give explicit examples to showcase the applicability of our results and conclude in §5 and §6, respectively.

2. First-Order Approximations (\( \varepsilon^1 \))

The basic principles of our pedagogy is presented here. We point out that the nature of our geometric approach contributes to the success of our method. We begin with the standard first-order approximate generator

\[ X = X_0 + \varepsilon X_1 + O(\varepsilon^2), \]

where we define

\[ X_A = \xi_A(t, x^k, N) \partial_t + \eta_A^i(t, x^k, N) \partial_i + \omega_A(t, x^k, N) \partial N, \quad A = 0, 1. \]

We reiterate that since \( N \) is considered as a degree of freedom, it appears in the same context as the \( x^k \) terms. The parameter value \( A = 0 \) represents the exact symmetry vector field while \( A = 1 \) corresponds to the approximate part. The generator \( X_0 \) is a Noether point symmetry satisfying the condition

\[ X_0^{[1]} L + L \frac{d \xi}{dt} = \frac{df}{dt}, \]

or in its approximate expanded form

\[ \left( X_0^{[1]} + \varepsilon X_1^{[1]} \right) (L_0 + \varepsilon L_1) + (L_0 + \varepsilon L_1) \frac{d}{dt} (\xi_0 + \varepsilon \xi_1) - \frac{d}{dt} (f_0 + \varepsilon f_1) = O(\varepsilon^2), \]

where \( f_A = f_A(t, x^i, N) \) is the Noether boundary function. The first prolongation is

\[ X_A^{[1]} = \xi_A \partial_t + \eta_A^i \partial_i + \left( \dot{\eta}_A^i - \dot{\xi}_A \xi_A \right) \partial \dot{x}^i, \]

with

\[ \dot{\eta}_A^i - \dot{\xi}_A \dot{x}^i = \eta_{,A}^i - \xi_{,A} \dot{x}^i + \eta_{,A}^j \dot{x}^j - \xi_{,A} \dot{x}^i \dot{x}^j + \dot{N} \eta_{A,N}^i - \xi_{A,N} \dot{N} \dot{x}^i, \]
minus $\partial_N$ in \([12]\) since the $\dot{N}$ terms are absent in the Lagrangian.

A substitution of the Lagrange functions \([10]\) and \([7]\) into the left-hand-side of the Noether condition \([11]\), we find

$$X_0^{[1]}L_0 = \left( \xi_0 \partial_x + \eta_0^k \partial_{x^k} + \left( \eta_0^k - \dot{x}^k \dot{x}_0 \right) \partial_{\dot{x}_k} \right) \left( \frac{1}{2N} g_{ij} \dot{x}^i \dot{x}^j - NV_0 \left( x^k \right) \right)$$

$$= \frac{1}{2N} \eta_{ij} k \eta_0^k \dot{x}^j - N \eta_0^k V_0, k - \omega_1 \frac{1}{2N^2} g_{ij} \dot{x}^i \dot{x}^j - \omega_0 V_0 + \frac{1}{N} \left( \eta_{0, i \dot{g}} k \dot{x}^i - \xi_0, i g \dot{x}^i \dot{x}^k - \xi_0, j g \dot{x}^i \dot{x}^j \dot{x}^k + \dot{N} \eta_0^k N g \dot{x}^k \right) - \frac{1}{N} \left( \dot{N} \xi_0, N g \dot{x}^i \dot{x}^j \dot{x}^k - \frac{1}{2} \eta_{0, j} g \dot{x}^i \dot{x}^j \dot{x}^k - \frac{1}{2} \eta_{0, k} g \dot{x}^i \dot{x}^k \right).$$

Similarly,

$$\varepsilon X_0^{[1]}L_1 = \varepsilon \left( \frac{1}{2N} h_{ij, k} \eta_0^k \dot{x}^i \dot{x}^j - N \eta_0^k V_1, k - \omega_1 \frac{1}{2N^2} g_{ij} \dot{x}^i \dot{x}^j - \omega_1 V_1 + \frac{1}{N} \left( \eta_{0, i \dot{g}} k \dot{x}^i - \xi_0, i g \dot{x}^i \dot{x}^k - \xi_1, j g \dot{x}^j \dot{x}^i \dot{x}^k \dot{x}^k - \xi_0, N g \dot{x}^k \right) \right) - \frac{1}{N} \left( \dot{N} \xi_1, N g \dot{x}^i \dot{x}^j \dot{x}^k - \frac{1}{2} \eta_{0, j} g \dot{x}^i \dot{x}^j \dot{x}^k - \frac{1}{2} \eta_{0, k} g \dot{x}^i \dot{x}^k \right).$$

$$\varepsilon^2 X_0^{[1]}L_1 = \varepsilon^2 \left( \frac{1}{2N} h_{ij, k} \eta_0^k \dot{x}^i \dot{x}^j - N \eta_0^k V_1, k - \omega_1 \frac{1}{2N^2} g_{ij} \dot{x}^i \dot{x}^j - \omega_1 V_1 + \frac{1}{N} \left( \eta_{0, i \dot{g}} k \dot{x}^i - \xi_0, i g \dot{x}^i \dot{x}^k - \xi_1, j g \dot{x}^j \dot{x}^i \dot{x}^k \dot{x}^k - \xi_0, N g \dot{x}^k \right) \right) - \frac{1}{N} \left( \dot{N} \xi_1, N g \dot{x}^i \dot{x}^j \dot{x}^k - \frac{1}{2} \eta_{0, j} g \dot{x}^i \dot{x}^j \dot{x}^k - \frac{1}{2} \eta_{0, k} g \dot{x}^i \dot{x}^k \right).$$

For the middle terms,

$$(L_0 + \varepsilon L_1) \frac{d}{dt} \left( \xi_0 + \varepsilon \xi_1 \right) = \dot{\xi}_0 L_0 + \varepsilon \dot{\xi}_1 L_0 + \varepsilon^2 \dot{\xi}_1 L_1,$$

and therefore

$$\dot{\xi}_0 L_0 = \left( \xi_0, t + \xi_0, k \dot{x}^k + \dot{N} \xi_0, N \right) \left( \frac{1}{2N} g_{ij} \dot{x}^i \dot{x}^j - NV_0 \left( x^k \right) \right)$$

$$= \frac{1}{2N} \left( \xi_0, t g_{ij} \dot{x}^i \dot{x}^j + \xi_0, k g \dot{x}^i \dot{x}^j \dot{x}^k + \dot{N} \xi_0, N g \dot{x}^k \right) - \frac{1}{\xi_0, t N V_0} - \xi_0, k N V_0, \dot{x}^k + \dot{N} \xi_0, N V_0.$$
Similarly, the rest of the expression is
\[\varepsilon L_1 \dot{\xi}_0 = \varepsilon \left( \frac{1}{2N} \left( \xi_{0,t} h_{ij} \dot{x}^i \dot{x}^j + \xi_{0,k} h_{ij} \dot{x}^i \dot{x}^j \dot{x}^k + \dot{N} \xi_{0,N} h_{ij} \dot{x}^i \dot{x}^j \right) - \xi_{0,t} N V_1 - \xi_{0,k} N V_1 \dot{x}^k + \dot{N} N \xi_{0,N} V_1 \right),\]
\[\varepsilon \dot{\xi}_1 L_0 = \varepsilon \left( \frac{1}{2N} \left( \xi_{1,t} g_{ij} \dot{x}^i \dot{x}^j + \xi_{1,k} g_{ij} \dot{x}^i \dot{x}^j \dot{x}^k + \dot{N} \xi_{1,N} g_{ij} \dot{x}^i \dot{x}^j \right) - \xi_{1,t} N V_0 - \xi_{1,k} N V_0 \dot{x}^k + \dot{N} N \xi_{1,N} V_0 \right),\]
\[\varepsilon^2 \dot{\xi}_1 L_1 = \varepsilon^2 \left( \frac{1}{2N} \left( \xi_{1,t} h_{ij} \dot{x}^i \dot{x}^j + \xi_{1,k} h_{ij} \dot{x}^i \dot{x}^j \dot{x}^k + \dot{N} \xi_{1,N} h_{ij} \dot{x}^i \dot{x}^j \right) - \xi_{1,t} N V_1 - \xi_{1,k} N V_1 \dot{x}^k + \dot{N} N \xi_{1,N} V_1 \right).
\]

The boundary terms produce the expression
\[\dot{f}_0 + \varepsilon \dot{f}_1 = \left( f_{0,t} + f_{0,k} \dot{x}^k + \dot{N} f_{0,N} \right) + \varepsilon \left( f_{1,t} + f_{1,k} \dot{x}^k + \dot{N} f_{1,N} \right).
\]

Next, we distinguish terms based on the order of \(\varepsilon\), viz. we obtain the equation for \(\varepsilon^0\):
\[
\begin{align*}
\frac{1}{2N} g_{ij,k} \eta_0^k \dot{x}^i \dot{x}^j - N \eta_0^k V_{0,k} - \omega_0 \frac{1}{2N} g_{ij,k} \dot{x}^i \dot{x}^j - \omega_0 V_0 &+ \\
\frac{1}{N} \left( \eta_{0,t}^{k} g_{ik} \dot{x}^i - \xi_{0,t} g_{ik} \dot{x}^i \dot{x}^k - \xi_{0,j} g_{ik} \dot{x}^i \dot{x}^j \dot{x}^k + \dot{N} \eta_{0,N}^k g_{ik} \dot{x}^i \right) &- \\
\frac{1}{N} \left( \dot{N} \xi_{0,N} g_{ik} \dot{x}^i \dot{x}^k - \frac{1}{2} \eta_{0,j} g_{ik} \dot{x}^i \dot{x}^j \dot{x}^k - \frac{1}{2} \eta_{0,j} g_{jk} \dot{x}^i \dot{x}^j \right) &+ \\
\frac{1}{2N} \left( \xi_{0,t} g_{ij} \dot{x}^i \dot{x}^j + \xi_{0,k} g_{ij} \dot{x}^i \dot{x}^j \dot{x}^k + \dot{N} \xi_{0,N} g_{ij} \dot{x}^i \dot{x}^j \right) &- \\
\xi_{0,t} N V_0 - \xi_{0,k} N V_0 \dot{x}^k + \dot{N} N \xi_{0,N} V_0 &\quad = f_{0,t} + f_{0,k} \dot{x}^k + \dot{N} f_{0,N}.
\end{align*}
\]

It is now necessary to separate monomials to obtain the determining system of
equations:

\[
\begin{align*}
\left( \dot{N} \dot{x}^i \dot{x}^j \right) : & \quad - \frac{1}{2N} \xi_{0,N} g_{ij} = 0, \\
\left( \dot{x}^i \dot{x}^j \dot{x}^k \right) : & \quad - \frac{1}{2N} \xi_{0,k} g_{ik} = 0, \\
\left( \dot{N} \dot{x}^i \right) : & \quad \frac{1}{N} \eta_{0,N} g_{ik} = 0, \\
\left( \dot{x}^i \dot{x}^j \right) : & \quad \mathcal{L}_{\eta_0} g_{ik} = \left( \eta_0^k (\ln V_0) \right)_k - \frac{f_{0,t}}{NV_0} g_{ik}, \\
\left( \dot{x}^i \right) : & \quad - f_{0,k} - \xi_{0,k} NV_0 + \frac{1}{N} \eta_{0,t} g_{ik} = 0, \\
\left( \dot{N} \right) : & \quad - f_{0,N} + N \xi_{0,N} V_0 = 0, \\
(1) : & \quad \omega_0 = - N \eta_{0}^k (\ln V_0)_k - N \xi_{0,t} \frac{f_{0,t}}{V_0}, \quad V_0 \neq 0.
\end{align*}
\]

The $L_{\eta_j}$ refers to the geometric derivative or Lie derivative operator along $\eta_j$. From Eq. (13)-(19), we have that $f_0 = f_0 \left( t, x^i \right)$, $\eta_0^k = \eta_0^k \left( t, x^k \right)$ and $\xi_0 = \xi_0 \left( t \right)$. 
For the determining equation involving $\varepsilon^1$, we find:

$$
\begin{align*}
&\frac{1}{2N} h_{ij,k} \xi^k_0 \dot{x}^i \dot{x}^j - N \eta^k_0 V_{1,k} - \omega_0 \frac{1}{2N^2} h_{ij} \dot{x}^i \dot{x}^j - \omega_0 V_1 + \\
&\frac{1}{N} \left( \frac{\eta^k_0 h_{ik} \dot{x}^i}{2} - \xi_0, t h_{ik} \dot{x}^i \dot{x}^k - \xi_0, j h_{ik} \dot{x}^i \dot{x}^j \dot{x}^k + \hat{N} \eta^k_{0,N} h_{ik} \dot{x}^i \right) - \\
&\frac{1}{N} \left( \hat{N} \xi_0, N h_{ik} \dot{x}^i \dot{x}^k - \frac{1}{2} \eta^k_{0,j} h_{ik} \dot{x}^i \dot{x}^j - \frac{1}{2} \eta^k_{0,i} h_{jk} \dot{x}^i \dot{x}^j \right) + \\
&\frac{1}{N} \left( \frac{\eta^k_1 g_{ik} \dot{x}^i}{2} - \xi_1, t g_{ik} \dot{x}^i \dot{x}^k - \xi_1, j g_{ik} \dot{x}^i \dot{x}^j \dot{x}^k + \hat{N} \eta^k_{1,N} g_{ik} \dot{x}^i \right) - \\
&\frac{1}{N} \left( \hat{N} \xi_1, N g_{ik} \dot{x}^i \dot{x}^k - \frac{1}{2} \eta^k_{1,j} g_{ik} \dot{x}^i \dot{x}^j - \frac{1}{2} \eta^k_{1,i} g_{kj} \dot{x}^i \dot{x}^j \right) + \\
&\frac{1}{2N} \left( \xi_0, t h_{ij} \dot{x}^i \dot{x}^j + \xi_0, k h_{ij} \dot{x}^i \dot{x}^j \dot{x}^k + \hat{N} \xi_0, N h_{ij} \dot{x}^i \dot{x}^j \right) - \\
&\xi_0, t N V_1 - \xi_0, k N V_{1,k} + \hat{N} N \xi_0, N V_1 + \\
&\frac{1}{2N} \left( \xi_1, t g_{ij} \dot{x}^i \dot{x}^j + \xi_1, k g_{ij} \dot{x}^i \dot{x}^j \dot{x}^k + \hat{N} \xi_1, N g_{ij} \dot{x}^i \dot{x}^j \right) - \\
&\xi_1, t N V_0 - \xi_1, k N V_{0,k} + \hat{N} N \xi_1, N V_0 = \\
f_{1,t} + f_{1,k} \dot{x}^k + \hat{N} f_{1,N}.
\end{align*}
$$

Note that we impose the restriction that terms involving $\varepsilon^2$ vanish. Separating
coefficients here leads to the determining system:

\[
\begin{align*}
(\dot{N}\dot{x}^j) : & \quad -\frac{1}{2N}\xi_{1,N}g_{ij} = 0, \\
(\dot{x}^i\dot{x}^j\dot{x}^k) : & \quad -\frac{1}{2N}\xi_{1,k}g_{ik} = 0, \\
(\dot{N}\dot{x}^i) : & \quad \frac{1}{N}\eta_{k,N}g_{ik} = 0, \\
(\dot{x}^i\dot{x}^j) : & \quad \mathcal{L}_{\eta_0}h_{ik} + \mathcal{L}_{\eta_1}g_{ik} = \left(\xi_{0,t} + \frac{u_0}{N}\right)h_{ik} + \left(\xi_{1,t} + \frac{u_1}{N}\right)g_{ik}, \\
(\dot{x}^k) : & \quad -f_{1,k} - \xi_{0,k}NV_1 - \xi_{1,k}NV_0 + \frac{1}{N}\eta_{0,t}h_{ik} + \frac{1}{N}\eta_{1,t}g_{ik} = 0, \\
(\dot{N}) : & \quad -f_{1,N} + N\xi_{0,N}V_1 + N\xi_{1,N}V_0 = 0, \\
(1) : & \quad \frac{\omega_0}{V_0} + \frac{\omega_1}{V_1} = -N\eta_0\left(\frac{\ln V_1}{V_0}\right)_k - N\xi_{0,t}\frac{\ln V_0}{V_1} - \frac{N\xi_{1,t}}{V_1} - f_{1,t}V_0V_1, \quad V_0V_1 \neq 0.
\end{align*}
\]

Also $f_1 = f_1\left(t, x^i\right)$, $\eta_1^k = \eta_1^k\left(t, x^k\right)$ and $\xi_1 = \xi_1\left(t\right)$. Finally, the system of equations (13)-(19) and (20)-(26) provide the approximate Noether symmetry conditions for the perturbed Lagrangian (5) defined by equations (6), (7) and in correspondence with the Noether symmetry vector (8). In order to enhance the applicability of these geometric conditions, we now derive the higher-order version of this approach.

3. Generalizations to ($\varepsilon^n$).

Let us extend our analysis to approximations of any order, that is $\varepsilon^n$. To initiate this generalization, we consider the approximate symmetry generator

\[
X = X_0 + \sum_{\gamma=1}^{n} \varepsilon^{\gamma}X_{\gamma} + O\left(\varepsilon^{\gamma+1}\right).
\] (27)

Analogous to the Noether condition (11), we impose the generalized condi-
In lieu of these generalizations, we state the following determining system for higher-order approximate Noether symmetries. The derivation of this system follows the same procedure outlined in Section 2, but for the economy of space we simply state the relevant formulae. To this end, the Noether symmetry conditions for \( (\varepsilon)^0 \) are the same as before, namely Eqs. (13)-(19). On the other hand for \( (\varepsilon)^\gamma, \gamma = 1 \ldots n \) : we obtain the determining system

\[
\begin{align*}
\dot{N} \dot{x}^i & = 0, \\
\frac{1}{2N} \xi_{\gamma,N} g_{ij} & = 0, \\
\dot{x}^i \dot{x}^j \dot{x}^k & = 0, \\
\frac{1}{N} \eta_{\gamma,N} g_{ik} & = 0, \\
\frac{1}{N} h^k_{\gamma, N} g_{ik} & = 0, \\
\mathcal{L}_{\gamma_{\gamma-1}} h_{ik} + \mathcal{L}_{\gamma} g_{ik} & = \left( \xi_{\gamma-1,t} + \frac{w_{\gamma-1}}{N} \right) h_{ik} + \left( \xi_{\gamma,t} + \frac{w_{\gamma}}{N} \right) g_{ik}, \\
\dot{x}^k & = -f_{\gamma,k} - \xi_{\gamma-1,k} N V_1 - \xi_{\gamma,k} N V_0 + \frac{1}{N} h^k_{\gamma-1,t} h_{ik} + \frac{1}{N} h^k_{\gamma,t} g_{ik} = 0, \\
\dot{N} & = -f_{\gamma,N} + N \xi_{\gamma-1,N} V_1 + N \xi_{\gamma,N} V_0 = 0, \\
\omega_{\gamma} V_0 + \omega_{\gamma-1} V_1 & = -N h^k_{\gamma-1} \frac{(\ln V_1)_k}{V_0} - N h^k_{\gamma} \frac{(\ln V_0)_k}{V_1} - N \xi_{\gamma-1,t} \frac{V_0}{V_1} - f_{\gamma,t} V_0 V_1, \quad V_{0-1} \neq 0.
\end{align*}
\]

The importance of this system is that its solution provides approximate and conditional symmetries at higher-order perturbations.
4. Noether Integrals

By Noether’s theorem [14] the symmetry vector field \( \xi \) with \( A = 0 \) for the Lagrangian (9) with boundary term \( f \), admits the conservation law:

\[
I_0 = \xi_0 \left( \dot{x}_i \frac{\partial L_0}{\partial \dot{x}_i} + \dot{N} \frac{\partial L_0}{\partial \dot{N}} - L_0 \right) - \frac{\partial L_0}{\partial \dot{N}} \eta_0^i - \frac{\partial L_0}{\partial \dot{N}} w_0 + f_0,
\]

\[
= \xi_0 \left( \frac{1}{2N} g_{ij} \dot{x}_i \dot{x}_j + NV_0 \right) - \frac{1}{2N} g_{ik} \dot{x}_i \eta_0^k + f_0
\]

\[
= \xi_0 H_0 - \frac{1}{2N} g_{ij} \dot{x}_i \eta_0^j + f_0,
\]

since there are no \( \dot{N} \) terms in the Lagrangian and where \( H_0 \) is the exact Hamiltonian.

In a similar way, for any approximate Lagrangian (5) and an approximate Noether generator (8), we derive the first-order approximate part, \( I_1 \) as follows:

\[
I_1 = H_0 \xi_1 - \frac{1}{2N} g_{ij} \dot{x}_i \eta_1^j + f_1 + \xi_0 H_1 - \frac{1}{2N} h_{ij} \dot{x}_i \eta_0^j.
\]

A generalization of this idea to the higher-order case of \( \varepsilon \), with the symmetry generator (27) leads us to deduce the formulae for the associated Noether integrals, viz.

\[
I_\gamma = H_0 \xi_\gamma - \frac{1}{2N} g_{ij} \dot{x}_i \eta_\gamma^j + f_\gamma + \xi_0 H_1 - \frac{1}{2N} h_{ij} \dot{x}_i \eta_\gamma^j.
\]

The function \( I_0 \) or \( I_\gamma \) gives rise to a so called ‘weak’ conservation law in the sense that one needs to impose the constraint condition \( \frac{\partial L}{\partial \dot{N}} = 0 \) in order for \( D_t I = 0 \); we illustrate this in the next section.

5. Applications

Now that we have developed an explicit method of deriving the conditional and approximate Noether symmetry conditions, we may tackle some examples. The progression from the determining system to the symmetry generator is as follows. With the help of Eqs. (13)-(19) and Eqs. (29)-(35) we will be led to the conditions that the Noether symmetries must satisfy. The solution of the set of these conditions must be done sequentially in order to acquire the conditional symmetry generators. This process is straightforward, albeit lengthy and so we merely list the pertinent results. That is, we present the approximate Noether symmetries and first integrals for each example.

5.1. Case A:

Consider the Lagrange functions

\[
L_0 = \frac{1}{2N} \dot{x}^2 - \frac{N}{2} x^2, \quad L_1 = -NV \frac{x^3}{3},
\]
with corresponding Hamiltonian functions

\[ H_0 = \frac{1}{2N} \dot{x}^2 + \frac{N}{2} x^2, \quad H_1 = NV_1 \frac{x^3}{3}. \]

Since we would like to compare the symmetries obtained under fixing the lapse function with allowing the lapse to be a degree of freedom, for this first case we present both results. As the reader shall see, the results differ substantially.

- For the exact and approximate Noether symmetries under constant lapse \( N = 1 \), the approximate symmetries are [15],

\[
Y^1 = \partial_t, \quad f = 0,
\]

\[
Y^2 = \sin(2t) \partial_\phi + \cos(2t) x \partial_x, \quad f = -x^2 \sin(2t),
\]

\[
Y^3 = \cos(2t) \partial_\phi - \sin(2t) x \partial_x, \quad f = -x^2 \cos(2t),
\]

\[
Y^4 = \sin(t) \partial_x, \quad f = x \cos(t),
\]

\[
Y^5 = \cos(t) \partial_x, \quad f = -x \sin(t),
\]

\[
Y^6 = \sin(t) \partial_x + \varepsilon \left( \frac{4}{3} V_1 \cos(t) \partial_t - \frac{2}{3} V_1 \sin(t) \partial_x \right), \quad f = x \cos(t) - \varepsilon \left( \frac{V_1 x^2}{3} \cos(t) \right),
\]

\[
Y^7 = \cos(t) \partial_x + \varepsilon \left( -\frac{4}{3} V_1 \sin(t) \partial_t - \frac{2}{3} V_1 \cos(t) \partial_x \right), \quad f = -x \sin(t) + \varepsilon \left( \frac{V_1 x^2}{3} \sin(t) \right).
\]

- For the conditional Noether symmetries with \( N = N(t) \) we obtain

\[
X^{Ai} = T(t) \partial_t + \frac{1}{x} \partial_x - N \left( \frac{d}{dt} T(t) + \frac{2}{x^2} \right) \partial_N, \quad f = 0,
\]

\[
X^{Ai} = T(t) \partial_t + \frac{1}{x} \partial_x - N \left( \frac{d}{dt} T(t) + \frac{2}{x^2} \right) \partial_N + \varepsilon \left( T(t) \partial_t - \frac{V_1}{3} \partial_x - N \frac{d}{dt} T(t) \partial_N \right), \quad f = 0.
\]

- For the approximate Noether integrals we have

\[
I(X^{Ai}) = T(t)H_0 - \frac{1}{x} \frac{\dot{x}}{N},
\]

\[
I(X^{Ai}) = T(t)H_0 - \frac{1}{x} \frac{\dot{x}}{N} + \varepsilon \left( T(t)H_0 + T(t)H_1 + \frac{V_1}{3} \frac{\dot{x}}{N} \right).
\]

With regard to the conservation laws corresponding to conditional symmetry vectors, as an example:

\[
D_t I(X^{Ai}) = \frac{2}{x^2} \frac{\partial L}{\partial N},
\]

is a multiple of the constraint equation, rather than strictly zero.

5.2. Case B:

In this case we take the Lagrangians

\[ L_0 = \frac{1}{2N} 6x\dot{x}^2 + 2\Lambda N x^3, \quad L_1 = N \frac{V_1}{x^2}, \]

with Hamiltonians

\[ H_0 = \frac{1}{2N} 6x\dot{x}^2 - 2\Lambda N x^3, \quad H_1 = -N \frac{V_1}{x^2}. \]
For the exact and approximate Noether symmetries we find
\[ X^{Bi} = T(t) \frac{\partial}{\partial t} + \frac{1}{x^2} \frac{\partial}{\partial x} - N \left( \frac{d}{dt} T(t) + \frac{3}{x^3} \right) \frac{\partial}{\partial N}, \quad f = 0, \]
\[ X^{Bii} = T(t) \frac{\partial}{\partial t} + \frac{1}{x^2} \frac{\partial}{\partial x} - N \left( \frac{d}{dt} T(t) + \frac{3}{x^3} \right) \frac{\partial}{\partial N} + \varepsilon \left( \frac{5 V_1}{32 x^8 \Lambda} \frac{\partial}{\partial x} + N \frac{65 V_1}{32 \Lambda x^8} \frac{\partial}{\partial N} \right), \quad f = 0, \]
\[ X^{Biii} = \varepsilon \left( t \frac{\partial}{\partial t} + x \ln x \frac{\partial}{\partial x} - 3N \left( \ln x + \frac{1}{3} \right) \frac{\partial}{\partial N} \right), \quad f = 0, \]
\[ X^{Biv} = \varepsilon \frac{\partial}{\partial t}, \quad f = 0, \]
\[ X^{Bv} = \varepsilon \left( x \frac{\partial}{\partial x} - 3N \frac{\partial}{\partial N} \right), \quad f = 0. \]

The approximate Noether integrals are
\[ I(X^{Bi}) = T(t) H_0 - \frac{1}{x^2} \frac{x}{N}, \]
\[ I(X^{Bii}) = T(t) H_0 - \frac{1}{x^2} \frac{x}{N} - \varepsilon \left( \frac{30 V_1}{32 \Lambda x^6} \right), \]
\[ I(X^{Biii}) = \varepsilon \left( t H_0 - 6 \frac{x^2 \ln x}{N} \right), \]
\[ I(X^{Biv}) = \varepsilon H_0, \]
\[ I(X^{Bv}) = -\varepsilon \frac{6 \frac{x^2}{N}}{N}. \]

5.3. Case C:

Suppose we have the Lagrangians
\[ L_0 = \frac{1}{2N} x^2 - NV_0^2 x^2, \quad L_1 = -NV_1 \frac{\exp(x)}{2}, \]
and Hamiltonian functions
\[ H_0 = \frac{1}{2N} \frac{x^2}{N} + NV_0^2 x^2, \quad H_1 = NV_1 \frac{\exp(x)}{2}. \]

The Noether symmetries are
\[ X^{Ci} = T(t) \frac{\partial}{\partial t} + \frac{1}{x} \frac{\partial}{\partial x} - N \left( \frac{d}{dt} T(t) + \frac{2}{x^2} \right) \frac{\partial}{\partial N}, \quad f = 0, \]
\[ X^{Cii} = T(t) \frac{\partial}{\partial t} + \frac{1}{x} \frac{\partial}{\partial x} - N \left( \frac{d}{dt} T(t) + \frac{2}{x^2} \right) \frac{\partial}{\partial N} + \varepsilon \left( T(t) \frac{\partial}{\partial t} - \frac{1}{4} \frac{V_1 \exp^2}{V_0 x^3} \frac{\partial}{\partial x} - N \left( \frac{d}{dt} T(t) + \frac{1}{2} \frac{V_1 (x - 3) \exp^2}{V_0 x^4} \right) \right), \quad f = 0. \]

For the approximate Noether integrals we have
\[ I(X^{Ci}) = T(t) H_0 - \frac{1}{x} \frac{\dot{x}}{N}, \]
\[ I(X^{Cii}) = T(t) H_0 - \frac{1}{x} \frac{\dot{x}}{N} + \varepsilon \left( T(t) H_0 + T(t) H_1 - \frac{1}{4} \frac{V_1 \exp^2 \dot{x}}{V_0 x^3 N} \right). \]
5.4. Case D:

In the last case, let us consider

\[ L_0 = \frac{1}{2N} \dot{x}^2 + N \frac{x^3}{3}, \quad L_1 = N \frac{V_1 x^n}{n}, \quad n \neq 0, 3 \]

and

\[ H_0 = \frac{1}{2N} \dot{x}^2 - N \frac{x^3}{3}, \quad H_1 = -N \frac{V_1 x^n}{n}. \]

Now, the Noether symmetries are found to be

\[ X^{Di} = T(t) \partial_t + 1 \frac{1}{x^{3/2}} \partial_x - N \left( \frac{d}{dt} T(t) + 3 \frac{3}{x^{5/2}} \right) \partial_N, \quad f = 0, \]

\[ X^{Dii} = T(t) \partial_t + 1 \frac{1}{x^{3/2}} \partial_x - N \left( \frac{d}{dt} T(t) + 3 \frac{3}{x^{5/2}} \right) \partial_N \]

\[ + \varepsilon \left( T(t) \partial_t + 3 \frac{x^3 V_1 (n + 3)}{2 \frac{x^{3/2}}{n - 3} n} \partial_x \right) \]

\[ - N \left( \frac{\frac{1}{x^3} x^3 V_1 (n + 3) x^{n-3} + (\frac{d}{dt} T(t) x^{11/2}) n - 3 V_1 (n + 3) x^n}{x^{11/2} (n - 3) n} \right) \partial_N, \]

\[ f = 0. \]

The corresponding Noether integrals are

\[ I(X^{Di}) = T(t) H_0 - \frac{1}{x^{3/2} N} \dot{x}, \]

\[ I(X^{Dii}) = T(t) H_0 - \frac{1}{x^{3/2} N} \dot{x} + \varepsilon \left( T(t) H_0 + T(t) H_1 - \frac{3}{2} \frac{x^3 V_1 (n + 3) x^{n-3}}{x^{3/2} (n - 3) n} \dot{x} \right). \]

6. Conclusion and Outlook

In this work, we bypassed the usual procedure of gauge fixing the lapse function to obtain a constrained and approximate action quadratic in velocities. This combination created a challenging problem from a symmetry perspective, especially in the presence of a broader space of variables and/or increasingly higher-order perturbations. The addition of a geometric approach allowed us to examine the fate of the resultant Noether symmetries. We encountered the coefficient of \( \partial_t \) as an unrestricted function of time, both exactly and approximately, a special feature of singular Lagrangians owing to the time reparametrisation invariance. This coincides with the results found in [16] for the exact case. Lastly, this study showed that constraint dependent variational symmetries are obtainable in an approximate sense.

Acknowledgments: We acknowledge the financial support from the National Research Foundation of South Africa (99279).
References

[1] T. Christodoulakis, N. Dimakis, P. A. Terzis, G. Doulis, Th. Grammenos, E. Melask, A. Spanou, Conditional Symmetries and the canonical quantization of constrained minisuperspace actions: the Schwarzschild case, *J. Geom. Phys.* **71** 127 (2013).

[2] K.V. Kuchar, Conditional symmetries in parametrized field theories. *J. Math. Phys.* **23** 1647 (1982).

[3] A. Paliathanasis, S. Jamal, Approximate Noether symmetries and collineations for regular perturbative Lagrangians, *J. Geom. Phys.* **124** 300 (2018).

[4] S. Jamal, Perturbative manifolds and the Noether generators of nth-order Poisson equations, *J. Diff. Eqs.* (2018) DOI: 10.1016/j.jde.2018.09.025.

[5] S. Jamal, N. Mnguni, Approximate conditions admitted by classes of the Lagrangian $L = \frac{1}{2} \left( u'^2 + u^2 \right) + \epsilon^i G_i (u, u', u'')$, *App. Math. Comp.* **335** 65 (2018).

[6] S. Jamal, nth -Order approximate Lagrangians induced by perturbative geometries, *Math. Phys. Anal. Geom.* **21**(25) 1 (2018).

[7] S. Jamal, Geometrization of heat conduction in perturbative spacetimes, *Can. J. Phys.* (2018) DOI: 10.1139/cjp-2018-0017.

[8] U. Camci, The geometric nature of approximate Noether gauge symmetries, *Gen. Relativ. Gravit.* **46** (2014) 1824.

[9] P.G.L. Leach, S. Moyo, S. Cotsakis, R.L. Lemmer, Symmetry, singularities and integrability in complex dynamics III: Approximate symmetries and invariants, *J. Nonl. Math. Phys.* **8** (2001) 139.

[10] K. F. Dialektopoulos, S. Capozziello, Noether symmetries as a geometric criterion to select theories of gravity, *Int. J. Geom. Meth. Mod. Phys.* **15** (2018) 1840007.

[11] S. Capozziello, M. De Laurentis, S.D. Odintsov, Hamiltonian dynamics and Noether symmetries in Extended Gravity Cosmology, *Eur. Phys. J. C* **72** (2012) 2068.

[12] S. Capozziello and M. De Laurentis, Noether symmetries in extended gravity quantum cosmology, *Int. J. Geom. Meth. Mod. Phys.* **11** (2014) 1460004.

[13] A. Borowiec, S. Capozziello, M. De Laurentis, F. S. N. Lobo, A. Paliathanasis, M. Paolella, A. Wojnar, Invariant solutions and Noether symmetries in hybrid gravity, *Phys. Rev. D* **91** (2015) 023517.

[14] E. Noether, *Invariante Variationsprobleme*, Nachr. d. König. Gesellsh. d. Wiss. zu Göttingen, Math-Phys. Klasse, (1918) 235.

[15] K.S. Govinder, T.G. Heil and T. Uzer, Approximate Noether symmetries, *Phys. Lett. A* **240** (1998) 127

[16] T. Christodoulakis, N. Dimakis, P. A. Terzis, Lie - point and variational symmetries in minisuperspace Einsteins gravity, *J. Phys. A: Math. Theo.* **47**(9) (2014) 095202.