On the Gradient of Harmonic Functions

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Abstract. For a harmonic function $u$ on a domain in $\mathbb{R}^n$, this note shows that $\|\nabla u\|$ is essentially determined by the geometry of level hypersurfaces of $u$. Specifically, the factor by which $\|\nabla u\|$ changes along a gradient flow is completely determined by the mean curvature of the level hypersurfaces intersecting the flow.

1 Introduction

Let $u$ be a harmonic function on an open connected subset $\Omega$ of $\mathbb{R}^n$. Suppose, for simplicity, that $u$ has no critical points in $\Omega$. Starting at a point $p_0 \in \Omega$, follow the gradient flow to reach another point $p \in \Omega$. By how much has $\|\nabla u\|$ changed along the flow? This note seeks to generalize the answer indicated in [1] for the case $n = 2$; see Remark 2 in §3.

For $a \in u(\Omega)$, let $S_a$ denote the level-$a$ hypersurface $\{p | u(p) = a\}$. Orient $S_a$ by prescribing the normal field $N = \nabla u / \|\nabla u\|$. Define $H : \Omega \to \mathbb{R}$ by letting $H(p)$ be the mean curvature of $S_{u(p)}$ at $p$.

Theorem: Let $p$ be a point on the gradient flow originating from $p_0$; let $\tilde{p}_0 p$ be the arc on the flow between $p_0$ and $p$. Then,

$$\|\nabla u(p)\| = \|\nabla u(p_0)\| \exp \left((n-1) \int_{p_0 \tilde{p}_0} H \, ds\right),$$

where $s$ denotes arc length along $\tilde{p}_0 p$.

It turns out that the case $n = 3$ completely embodies the general case. For simplicity, we treat this case.

2 Preliminaries

We begin by considering the mean curvature $H$ of level surfaces of a $C^2$ function $f$ on a domain in $\mathbb{R}^3$.

Suppose that 0 is a regular value of $f$ and let $S$ denote the level-0 set $\{p | f(p) = 0\}$. Let $N = \nabla f / \|\nabla f\|$. For $p \in S$, let $T_p S$ denote the tangent plane of $S$ at $p$. For each unit vector $v \in T_p S$, let $\gamma_v$ be the unit-speed parametrization of the normal section of $S$ with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$, then the (signed) curvature $\kappa_v(p)$ of $\gamma_v$ at $p$ is defined by the equation $\gamma_v''(0) = \kappa_v(p) N(p)$. The mean curvature $H(p)$ of $S$ at $p$ is the mean of $\kappa_v(p)$ with $v$ ranging over the unit circle in $T_p S$, i.e.,

$$H(p) := \frac{1}{2\pi} \int_{v \in T_p S : \|v\| = 1} \kappa_v(p) \, d\sigma,$$

where $d\sigma$ is the arc length element of the circle $\|v\| = 1$.

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As $\gamma_v(t) \in S$, the two vectors $\nabla f(\gamma_v(t))$ and $\gamma_v'(t)$ are orthogonal; hence

$$0 = \frac{d}{dt} \langle \nabla f(\gamma_v(t)), \gamma_v'(t) \rangle$$

$$= \left\langle \frac{d}{dt} \nabla f(\gamma_v(t)), \gamma_v'(t) \right\rangle + \langle \nabla f(\gamma_v(t)), \gamma_v''(t) \rangle$$

Note that

$$\langle \nabla f(\gamma_v(0)), \gamma_v''(0) \rangle = \kappa_p(v) \|\nabla f(p)\|,$$

whereas

$$\left\langle \frac{d}{dt} \bigg|_{t=0} \nabla f(\gamma_v(t)), \gamma_v'(0) \right\rangle = Q(p)(v, v),$$

where $Q(p) : T_p\mathbb{R}^3 \times T_p\mathbb{R}^3 \to \mathbb{R}$ is the Hessian quadratic form, implying that

$$\kappa_p(v) = -\frac{1}{\|\nabla f(p)\|} Q(p)(v, v).$$

To average $\kappa_p(v)$ over the unit circle, it suffices to average $Q(p)(v, v)$. To that end, let $(\xi, \eta)$ be an orthonormal basis for $T_pS$; then $(\xi, \eta, \mathbf{N})$ is an orthonormal basis for $T_p\mathbb{R}^3$. It is simple to verify that

$$\frac{1}{2\pi} \int_{\nu \in T_pS : \|\nu\|=1} Q(p)(v, v) d\sigma = \frac{1}{2} (Q(p)(\xi, \xi) + Q(p)(\eta, \eta)).$$

Note that

$$Q(p)(\xi, \xi) + Q(p)(\eta, \eta) + Q(p)(\mathbf{N}, \mathbf{N}) = \text{Tr} Q(p) = \Delta f(p).$$

Thus, we have

$$H(p) = \frac{Q(p)(\mathbf{N}, \mathbf{N}) - \Delta f(p)}{2 \|\nabla f(p)\|}.$$  \hfill (1)

### 3 Proof of Theorem

Let $u$ be a harmonic function on an open connected subset $\Omega$ of $\mathbb{R}^3$ without critical points. Let $\mathbf{N} = \nabla u / \|\nabla u\|$. Let $s \mapsto \varphi(s)$ be the unit-speed gradient flow originating from $p_0$; $\varphi$ is such that

$$\varphi'(s) = \mathbf{N}(\varphi(s)) \quad \text{and} \quad \varphi(0) = p_0.$$

Consider $g(t) := u(\varphi(t))$. Then,

$$g'(s) = \langle \nabla u(\varphi(s)), \varphi'(s) \rangle = \|\nabla u(\varphi(s))\|,$$

$$g''(s) = \left\langle \frac{d}{ds} \nabla u(\varphi(s)), \varphi'(s) \right\rangle + \langle \nabla u(\varphi(s)), \varphi''(s) \rangle.$$

As $\|\varphi\| \equiv 1$, the vectors $\varphi'$ and $\varphi''$, and hence $\nabla u$ and $\varphi''$, are always orthogonal. Thus,

$$g''(s) = Q(\varphi(s))(\mathbf{N}, \mathbf{N}).$$
By (1) in §2,

\[ H(\varphi(s)) = \frac{Q(\varphi(s)) (N,N)}{2 \|\nabla u(\varphi(s))\|^2} = \frac{1}{2} \frac{g''(s)}{g'(s)} = \frac{1}{2} \frac{d}{ds} \log g'(s) \]

\[ = \frac{1}{2} \frac{d}{ds} \log \|\nabla u(\varphi(s))\|. \tag{2} \]

Integrating both sides yields the result.

Remarks:

1. If \( u \) is a harmonic function of two variables, the above analysis can be easily adapted to show that

\[ \|\nabla u(p)\| = \|\nabla u(p_0)\| \exp \left( \int_{p_0}^p \kappa \, ds \right), \]

where \( \kappa \) is the curvature (signed, according to choice made of \( N \)) of the level curves of \( u \). This formula was suggested in [1] by a purely complex-analytic argument, as \( u \) is locally the real part of a holomorphic function. However, this argument does not apply when \( n > 2 \).

2. For \( n > 3 \), it suffices to notes that a level hypersurface of an \( n \)-variable \( C^2 \) function \( f \) has at \( p \) mean curvature

\[ H(p) = \frac{Q(p)(N,N) - \Delta f(p)}{(n-1)\|\nabla f(p)\|} , \]

where the factor \( 1/(n-1) \), the counterpart of the factor \( 1/2 \) in (1), stems from averaging a quadratic form on \( \mathbb{R}^n \) over a unit \((n-2)\)-sphere. Omitting the obvious details, we conclude, as a generalization of (2), that, for a harmonic function \( u \) of \( n \) variables,

\[ H = \frac{1}{(n-1)} D_N \log \|\nabla u\| , \]

from which the general case of the Theorem follows.

References

[1] R. P. Jerrard and L. A. Rubel, On the Curvature of the Level Lines of a Harmonic Function, Proceedings of the American Mathematical Society, Vol. 14, No. 1 (1963), pp.29-32.

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