SOME REMARKS ON CHARACTERIZATION OF T-NORMED INTEGRALS ON COMPACTA

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Abstract. A characterization of t-normed integrals was obtained in [1] for finite compacta and in [15] for the general case. Such characterization establishes a correspondence between the space of capacities and homogeneous respect t-norm monotone normed functionals preserving the maximum operation of comonotone functions. In fact these theorems we can consider as non-additive and non-linear analogues of well-known Riesz Theorem about a correspondence between the set of σ-additive regular Borel measures and the set of linear positively defined functionals. We discuss optimality of such characterization.

1. Introduction

Capacities (non-additive measures, fuzzy measures) were introduced by Choquet in [2] as a natural generalization of additive measures. They found numerous applications (see for example [4], [5], [16]). Capacities on compacta were considered in [10] where the important role plays the upper-semicontinuity property which connects the capacity theory with the topological structure. Categorical and topological properties of spaces of upper-semicontinuous normed capacities on compact Hausdorff spaces were investigated in [13].

In fact, the most of applications of non-additive measures to game theory, decision making theory, economics etc deal not with measures as set functions but with integrals which allow to obtain expected utility or expected pay-off. Several types of integrals with respect to non-additive measures were developed for different purposes (see for example [6], [7], [9], and [3]). Such integrals are called fuzzy integrals. The most known are the Choquet integral based on the addition and the minimum operations [2] and the Sugeno integral based on the maximum and the minimum operations [18]. If we change the minimum operation by any t-norm, we obtain the generalization of the Sugeno integral called t-normed integrals [17].

One of the important problems of the fuzzy integrals theory is characterization of integrals as functionals on some function space (see for example subchapter 4.8 in [6] devoted to characterizations of the Choquet integral and the Sugeno integral). A characterization of t-normed integrals was obtained in [1] for finite compacta and in [15] for the general case.

It was remarked in [6] that in particular case, when we consider the Sugeno integral on finite sets, some of conditions used in the characterization theorem in [1] are superfluous and a simpler characterization of the Sugeno integral on finite sets was given. Another simpler characterization of the Sugeno integral was obtained in [11] for finite sets and [12] or [14] for any compactum.
The main aim of this paper is to analyze if we can generalize one of mentioned simplification to the general case of t-normed integral changing the minimum operation by any t-norm. We will answer this question in negative. We obtain in Section 2 a characterization of the Sugeno integral which is simpler then one in \[11\]. But we show in Section 3 that the characterization from \[11\] can not be generalized for any t-normed integral even for the finite case. However we generalize the characterization of Sugeno integral given in \[6\] for any t-normed integral and any compactum in Section 4. Finally, we discuss an open problem in Section 5.

2. Capacities and t-normed integrals

In what follows, all spaces are assumed to be compacta (compact Hausdorff space) except for \(\mathbb{R}\) and the spaces of continuous functions on a compactum. All maps are assumed to be continuous. By \(\mathcal{F}(X)\) we denote the family of all closed subsets of a compactum \(X\).

We shall denote the Banach space of continuous functions on a compactum \(X\) endowed with the sup-norm by \(C(X)\). For any \(c \in \mathbb{R}\) we shall denote the constant function on \(X\) taking the value \(c\) by \(c_X\). We also consider the natural lattice operations \(\lor\) and \(\land\) ( on \(C(X)\) and its sublattices \(C(X,[0, +\infty))\) and \(C(X,[0,1])\).

We need the definition of capacity on a compactum \(X\). We follow a terminology of \[13\].

**Definition 1.** A function \(\nu : \mathcal{F}(X) \to [0,1]\) is called an upper-semicontinuous capacity on \(X\) if the three following properties hold for each closed subsets \(F\) and \(G\) of \(X\):

1. \(\nu(X) = 1,\) \(\nu(\emptyset) = 0,\)
2. if \(F \subseteq G,\) then \(\nu(F) \leq \nu(G),\)
3. if \(\nu(F) < a\) for a \(\in [0,1],\) then there exists an open set \(O \supseteq F\) such that \(\nu(B) < a\) for each compactum \(B \subseteq O.\)

By \(M X\) we denote the space \(M X\) of all upper-semicontinuous capacities on a compactum \(X\).

Remind that a triangular norm \(\ast\) is a binary operation on the closed unit interval \([0,1]\) which is associative, commutative, monotone and \(s \ast 1 = s\) for each \(s \in [0,1]\) \[8\]. Let us remark that the monotonicity of \(\ast\) implies distributivity, i.e. \((t \lor s) \ast l = (t \ast l) \lor (s \ast l)\) for each \(t, s, l \in [0,1]\). We consider only continuous t-norms in this paper.

Integrals generated by t-norms are called t-normed integrals and were studied in \[19\], \[20\] and \[17\]. Denote \(\varphi_t = \varphi^{-1}([t,1])\) for each \(\varphi \in C(X,[0,1])\) and \(t \in [0,1]\). So, for a continuous t-norm \(\ast,\) a capacity \(\mu\) and a function \(f \in C(X,[0,1])\) the corresponding t-normed integral is defined by the formula

\[
\int_X f d\mu = \max\{\mu(f_t) * t \mid t \in [0,1]\}.
\]

Let \(X\) be a compactum. We call two functions \(\varphi, \psi \in C(X,[0,1])\) comonotone (or equiordered) if \((\varphi(x_1) - \varphi(x_2)) \cdot (\psi(x_1) - \psi(x_2)) \geq 0\) for each \(x_1, x_2 \in X\). Let us remark that a constant function is comonotone to any function \(\psi \in C(X,[0,1])\).

Let \(\ast\) be a continuous t-norm. We denote for a compactum \(X\) by \(\mathcal{T}^*(X)\) the set of functionals \(\mu : C(X,[0,1]) \to [0,1]\) which satisfy the conditions:

1. \(\mu(1_X) = 1\) (\(\mu\) is normed);
2. \(\mu(\varphi) \leq \mu(\psi)\) for each functions \(\varphi, \psi \in C(X,[0,1])\) such that \(\varphi \leq \psi\) (\(\mu\) is monotone);
3. \(\mu(\psi \lor \varphi) = \mu(\psi) \lor \mu(\varphi)\) for each comonotone functions \(\varphi, \psi \in C(X,[0,1])\) \((\mu\) preserves \(\lor\) for comonotone functions);
(4) \( \mu(cX \ast \varphi) = c \ast \mu(\varphi) \) for each \( c \in [0, 1] \) and \( \varphi \in C(X, [0, 1]) \) (\( \mu \) is \( * \)-homogeneous).

It was proved in [1] for finite compacta and in [15] for the general case that a functional \( \mu \) on \( C(X, [0, 1]) \) belongs to \( T^*(X) \) if and only if there exists a unique capacity \( \nu \) such that \( \mu \) is the t-normed integral with respect to \( \nu \).

Let us remark that the above characterization can be simplified in the particular case for the Sugeno integral (when \( * = \wedge \)). We can replace Property 3 by a weaker condition: \( \mu(cX \vee \varphi) = c \vee \mu(\varphi) \) for each \( c \in [0, 1] \) and \( \varphi \in C(X, [0, 1]) \) (see [1] for finite sets and [12] for any compactum. See also [14] where some modification of the Sugeno integral was considered).

The following lemma implies another simplification of this characterization of the Sugeno integral.

**Lemma 1.** Let \( X \) be a compactum and \( \mu : C(X) \to [0, 1] \) be a \( \vee \)-homogeneous functional. Then \( \mu \) is normed. If \( \mu \) is additionally \( \wedge \)-homogeneous, then it is monotone.

**Proof.** Let \( \mu : C(X) \to [0, 1] \) be a \( \vee \)-homogeneous functional. We have \( \mu(1_X) = \mu(1_X \vee 1_X) = 1 \vee \mu(1_X) = 1 \). Hence \( \mu \) is normed.

Now, let \( \mu : C(X) \to [0, 1] \) be a \( \vee \)-homogeneous and \( \wedge \)-homogeneous functional. Take any \( \varphi, \psi \in C(X, [0, 1]) \) such that \( \psi \leq \varphi \). Suppose the contrary \( \mu(\varphi) = b < a = \mu(\psi) \). Choose \( c, d \in [0, 1] \) such that \( b < c < d < a \).

Since \( \psi \leq \varphi \), we have \( \varphi^{-1}([0, c]) \cap \psi^{-1}([d, 1]) = \emptyset \).

Choose a function \( \xi \in C(X, [0, 1]) \) such that
\[
\xi|_{\varphi^{-1}([0, c])} = \varphi|_{\varphi^{-1}([0, c])}, \quad \xi|_{\psi^{-1}([d, 1])} = \psi|_{\psi^{-1}([d, 1])}
\]
and
\[
\xi(X \setminus (\varphi^{-1}([0, c]) \cup \psi^{-1}([d, 1]))) \subset [c, d].
\]
Then we have
\[
\xi \wedge cX = \varphi \wedge cX \quad \text{and} \quad \xi \vee dX = \psi \vee dX.
\]

Since \( \mu \) is \( \wedge \)-homogeneous, we have
\[
\mu(\xi \wedge c) = \mu(\xi) \wedge c = \mu(\varphi \wedge cX) = \mu(\varphi) \wedge c = b \wedge c = b,
\]
what implies \( \mu(\xi) = b \).

On the other hand, using \( \vee \)-homogeneity of \( \mu \) we obtain
\[
\mu(\xi) \vee d = \mu(\xi \vee dX) = \mu(\psi \vee dX) = \mu(\psi) \vee d = a \vee d = a,
\]
what implies \( \mu(\xi) = a \). We have a contradiction. \( \square \)

So, we obtain a further simplification of characterization of the Sugeno integral.

**Theorem 1.** A functional \( \mu : C(X, [0, 1]) \to [0, 1] \) is \( \vee \)-homogeneous and \( \wedge \)-homogeneous if and only if there exists a unique capacity \( \nu \) such that \( \mu \) is the Sugeno integral with respect to \( \nu \).

We will show that we can not generalize the above theorem for any t-norm, moreover we can not even generalize above mentioned results from [11] and [12] for any t-norm. Let us formulate it more precisely. Let \( * \) be a continuous t-norm. Denote for a compactum \( X \) by \( T^*_c(X) \) the set of monotone \( \vee \)-homogeneous and \( * \)-homogeneous functionals \( \mu : C(X, [0, 1]) \to [0, 1] \). The following question arises naturally: if \( \mu \in T^*_c(X) \) implies that there exists a unique capacity \( \nu \) such that \( \mu \) is the t-normed integral with respect to \( \nu \)? We answer this question in negative in the next section building a functional \( \mu \in T^*_c(\{1, 2, 3\}) \setminus T(\{1, 2, 3\}) \), where \( \cdot \) is usual multiplication on \([0, 1]\).
3. An example of monotone \(\sim\)-homogeneous and \(\lor\)-homogeneous functional which does not preserve \(\lor\) for comonotone functions.

The main aim of this section is to build a functional \(\mu \in \mathcal{T}_1((\{1, 2, 3\})/\mathcal{T} ((\{1, 2, 3\}))\), where \(\cdot\) is usual multiplication on \([0, 1]\).

We will develop some technique of extending of monotone \(\sim\)-homogeneous and \(\lor\)-homogeneous functionals defined on some subsets of \(C(X, [0, 1])\) for a compactum \(X\). By \(C_0\) we denote the subset of \(C(X, [0, 1])\) consisting of all constant functions.

**Definition 2.** A subset \(A \subset C(X, [0, 1])\) is called a \((\lor, \cdot)\)-subspace of \(C(X, [0, 1])\) if \(C_0 \subset A\) and \(e_X \lor \psi \in A\) and \(e_X \cdot \psi \in A\) for each \(e \in [0, 1]\) and \(\psi \in A\).

We will use simpler denotations \(e \lor \psi, e \cdot \psi\) instead \(e_X \lor \psi, e_X \cdot \psi\) and \(e \leq (\geq)\psi\) instead \(e_X \leq (\geq)\psi\) in the following.

The following theorem can be considered as partial idempotent version of Hahn-Banach Theorem and the proof uses the main idea of the proof of Hahn-Banach Theorem.

**Theorem 2.** Let \(\mu : A \to [0, 1]\) be a monotone \(\sim\)-homogeneous and \(\lor\)-homogeneous functional where \(A\) is a \((\lor, \cdot)\)-subspace of \(C(X, [0, 1])\). Then there exists a monotone \(\sim\)-homogeneous and \(\lor\)-homogeneous functional \(\mu_1 : C(X, [0, 1]) \to [0, 1]\) such that \(\mu_1|A = \mu\).

**Proof.** Consider any \((\lor, \cdot)\)-subspace \(A\) of \(C(X, [0, 1])\) and a monotone \(\sim\)-homogeneous and \(\lor\)-homogeneous functional \(\mu : A \to [0, 1]\). Let \(\varphi \in C(X, [0, 1])\). Put \(A' = \{d \lor (e \cdot \varphi) \mid d, e \in [0, 1]\} \cup A\). It is easy to check that \(A'\) is a \((\lor, \cdot)\)-subspace of \(C(X, [0, 1])\). We will build a monotone \(\sim\)-homogeneous and \(\lor\)-homogeneous functional \(\mu' : A' \to [0, 1]\) which is an extension of \(\mu\).

Define a subset of \(\mathbb{R}\) as follows

\[ H = \{c^{-1} \cdot \mu(\psi) \mid c \in [0, 1]\} \text{ and } \psi \in A \text{ such that } \psi \geq c \cdot \varphi\}. \]

Put \(a = \inf H\). It is easy to check that \(a \in [0, 1]\).

Define a functional \(\mu' : A' \to [0, 1]\) as follows \(\mu'(\psi) = \mu(\psi)\) for each \(\psi \in A\) and \(\mu'(d \lor (e \cdot \varphi)) = d \lor (e \cdot a)\) for \(d, e \in [0, 1]\). It is easy to see that \(\mu'\) is a \(\sim\)-homogeneous and \(\lor\)-homogeneous functional.

Let us show that \(\mu'\) is monotone. It is obvious for a pair of functions from \(A\). It is easy to check for a pair of functions from \(\{d \lor (e \cdot \varphi) \mid d, e \in [0, 1]\}\). Now, consider any \(\psi \in A\) and \(d, e \in [0, 1]\) such that \(d \lor (e \cdot \varphi) \leq \psi\). We can assume that \(e > 0\). We have \(d \leq \psi\) and \(e \cdot \varphi \leq \psi\). We also have \(c^{-1} \cdot \mu(\psi) \geq a\) by the choice of \(a\). Since \(\mu\) is monotone on \(A\) and \(C_0 \subset A\), we have \(\mu(\psi) \geq d\). Hence \(\mu'(\psi) = \mu(\psi) \geq d \lor (e \cdot a) = \mu'(d \lor (e \cdot \varphi))\).

Conversely, consider any \(\psi \in A\) and \(d, e \in [0, 1]\) such that \(d \lor (e \cdot \varphi) \geq \psi\). We can assume that \(e > 0\). Suppose the contrary \(d \lor (e \cdot a) < \mu(\psi)\). Since \(a = \inf \{s^{-1} \cdot \mu(\xi) \mid s \in [0, 1]\}\) and \(\xi \in A\) such that \(\xi \geq s \cdot \varphi\), we can choose \(\xi \in A\) such that \(\xi \geq s \cdot \varphi\) and \(s^{-1} \cdot \mu(\xi) < c^{-1} \cdot \mu(\psi)\).

If \(s \leq c\) we put \(\psi' = s^{-1} \cdot \psi \in A\). Then we have \(\mu(\psi') = s^{-1} \cdot \mu(\psi)\). Hence \(\mu(\xi) < \mu(\psi')\). Since \(\xi \in A\), we have \(\mu(s^{-1} \cdot d \lor \xi) = s^{-1} \cdot d \lor \xi \leq s^{-1} \cdot \mu(\xi) \leq \mu(\psi')\). On the other hand \(\psi' \leq s^{-1} \cdot \psi\) and \(\psi' \leq s \cdot \varphi \leq \xi\). Hence \(\psi' \leq s^{-1} \cdot d \lor \xi\) and we obtain a contradiction with the monotonicity of \(\mu\).

Now, let us consider \(s > c\). Put \(\xi' = c \cdot s^{-1} \cdot \xi \in A\). We have \(c^{-1} \cdot \mu(\xi') = s^{-1} \cdot \mu(\xi) \leq c^{-1} \cdot \mu(\psi')\). Hence \(\mu(\xi') < \mu(\psi')\) and \(\mu(d \lor \xi') = d \lor \mu(\xi') < \mu(\psi')\). On the other hand \(\psi' \leq d \lor c \cdot \varphi = d \lor c \cdot s^{-1} \cdot (s \cdot \varphi) \leq d \lor c \cdot s^{-1} \cdot \xi \leq d \lor \xi'\) and we obtain a contradiction with monotonicity of \(\mu\).

Zorn Lemma implies existence of a monotone \(\sim\)-homogeneous and \(\lor\)-homogeneous functional \(\mu_1 : C(X, [0, 1]) \to [0, 1]\) such that \(\mu_1|A = \mu\). □
Now we are able construct the announced example. By $X = \{1, 2, 3\}$ we denote 3-point space with the discrete topology. Define two functions $\varphi_1, \varphi_2 \in C(X, [0, 1]) = [0, 1]^3$ as follows $\varphi_1(1) = 0, \varphi_1(2) = 1/3, \varphi_1(3) = 2/3$ and $\varphi_2(1) = 1/3, \varphi_2(2) = 1/3, \varphi_2(3) = 1$. Put $\varphi_3 = \varphi_1 \lor \varphi_2$. Then we have $\varphi_3(1) = 1/3, \varphi_3(2) = 1/2$ and $\varphi_3(3) = 1$. Let us remark that $\varphi_1$ and $\varphi_2$ are comonotone.

For $i \in \{1, 2, 3\}$ define sets $H_i \subset C(X, [0, 1])$ as follows $H_i = \{d \lor (c \cdot \varphi_i) \mid d, c \in [0, 1]\}$ and put $H = \bigcup_{i=1}^3 H_i$. It is easy to check that $H$ is a $(\lor, \cdot)$-subspace of $C(X, [0, 1])$.

Put $m_1 = m_2 = 1/3$ and $m_3 = 1/2$. Define a functional $\mu : H \to [0, 1]$ by the formula $\mu(d \lor (c \cdot \varphi_i)) = d \lor (c \cdot m_i)$ for $i \in \{1, 2, 3\}$ and $d, c \in [0, 1]$. It is easy to see that $\mu$ is $\cdot$-homogeneous and $\lor$-homogeneous. Let us show that $\mu$ is monotone. It is a routine checking for a pair of functions belonging to the same $H_i$ and we omit it. Let us check it for functions from distinct $H_i$.

Let $\psi_1 \in H_1$ and $\psi_2 \in H_2$. Then we have

$$\psi_1 = d_1 \lor (c_1 \cdot \varphi_1), \quad \psi_2 = d_2 \lor (c_2 \cdot \varphi_2)$$

for some $d_1, d_2, c_1, c_2 \in [0, 1]$ and

$$\mu(\psi_1) = d_1 \lor (c_1/3), \quad \mu(\psi_2) = d_2 \lor (c_2/3).$$

Consider the case when $\psi_1 \leq \psi_2$. In particular, we have

$$d_1 \lor (c_1/3) \leq d_1 \lor (c_1/2) = \psi_1(2) \leq \psi_2(2) = d_2 \lor (c_2/3).$$

We obtain $\mu(\psi_1) \leq \mu(\psi_2)$. Conversely, let $\psi_1 \geq \psi_2$. In particular, we have

$$d_1 \lor (c_1/3) \geq d_1 = \psi_1(1) \geq \psi_2(1) = d_2 \lor (c_2/3).$$

Hence $\mu(\psi_1) \geq \mu(\psi_2)$.

Let $\psi_1 \in H_1$ and $\psi_3 \in H_3$. Then we have

$$\psi_1 = d_1 \lor (c_1 \cdot \varphi_1), \quad \psi_3 = d_3 \lor (c_3 \cdot \varphi_3)$$

for some $d_1, d_3, c_1, c_3 \in [0, 1]$ and

$$\mu(\psi_1) = d_1 \lor (c_1/3), \quad \mu(\psi_3) = d_3 \lor (c_3/2).$$

Consider the case when $\psi_1 \leq \psi_3$. In particular, we have

$$d_1 \lor (c_1/3) \leq d_1 \lor (c_1/2) = \psi_1(2) \leq \psi_3(2) = d_3 \lor (c_3/2).$$

We obtain $\mu(\psi_1) \leq \mu(\psi_3)$. Conversely, let $\psi_1 \geq \psi_3$. In particular, we have

$$d_1 = \psi_1(1) \geq \psi_3(1) = d_3 \lor (c_3/3).$$

Suppose the contrary

$$d_1 \lor (c_1/3) < d_3 \lor (c_3/2).$$

The above inequalities imply

$$d_1 < c_3/2 = d_3 \lor (c_3/2).$$

On the other hand we have

$$d_1 \lor (2c_2/3) = \psi_3(3) \geq \psi_3(3) = d_3 \lor c_3 = c_3.$$

Thus, we obtain $2c_2/3 \geq c_3$ or $c_2/3 \geq c_3/2$ and we have a contradiction. Hence $\mu(\psi_1) \geq \mu(\psi_3)$.

Let $\psi_2 \in H_2$ and $\psi_3 \in H_3$. Then we have

$$\psi_2 = d_2 \lor (c_2 \cdot \varphi_2), \quad \psi_3 = d_3 \lor (c_3 \cdot \varphi_3)$$

for some $d_2, d_3, c_2, c_3 \in [0, 1]$ and

$$\mu(\psi_2) = d_2 \lor (c_2/3), \quad \mu(\psi_3) = d_3 \lor (c_3/2).$$
Then we have
\[ \mu(\psi_2) = \psi_2(2), \text{ and } \mu(\psi_3) = \psi_3(2). \]
Hence \( \psi_2 \leq (\geq) \psi_3 \) implies \( \mu(\psi_2) \leq (\geq) \mu(\psi_3) \) and we obtain that \( \mu \) is monotone.

Let us remark that our arguments also imply that \( \mu \) is correctly defined.

Thus the functional \( \mu \) is \(-\)-homogeneous, \(\vee\)-homogeneous and monotone. But we have
\[ \mu(\psi_1) \vee \mu(\psi_2) = 1/3 \neq 1/2 = \mu(\psi_3) = \mu(\psi_1 \vee \psi_2), \]
hence \( \mu \) does not preserve \( \vee \) for comonotone functions. By Theorem 2 we can extend \( \mu : H \rightarrow [0, 1] \) to a \(-\)-homogeneous, \(\vee\)-homogeneous and monotone functional \( \mu_1 : C^{+}(X, [0, 1]) \rightarrow [0, 1] \) which does not preserve \( \vee \) for comonotone functions.

4. Functionals which preserve \( \vee \) for comonotone functions.

Another characterization of the Sugeno integral on finite sets was given by Theorem 4.58 in \[6\]. Let \( X \) be a finite set and \( |X| = n \in \mathbb{N} \). The characterization theorem is formulated for the Sugeno integral defined on functions from \( C^{+}(X, [0, +\infty)) = [0, +\infty]^n \). Reformulating this theorem for functions from \( C(X, [0, 1]) = [0, 1]^n \) we obtain that a functional \( \mu : [0, 1]^n \rightarrow [0, 1] \) has the properties

1. \( \mu(1_X) = 1; \)
2. \( \mu \) preserves \( \vee \) for comonotone functions;
3. \( \mu(c \chi \land \chi_A) = c \land \mu(\chi_A) \) for each \( c \in [0, 1] \) and \( A \subset X \) (by \( \chi_A \) we denote the characteristic function of the set \( A \)).

if and only if there exists a unique capacity \( \nu \) on \( X \) such that \( \mu \) is the Sugeno integral with respect to \( \nu \).

We can rewrite the proof of Theorem 4.58 from \[5\] to obtain its generalization.

**Theorem 3.** Let \( X \) be a finite set and \( |X| = n \in \mathbb{N} \) and \( * \) is a continuous \( t \)-norm. A functional \( \mu : [0, 1]^n \rightarrow [0, 1] \) has the properties

1. \( \mu(1_X) = 1; \)
2. \( \mu \) preserves \( \land \) for comonotone functions;
3. \( \mu(c \chi \land \chi_A) = c \land \mu(\chi_A) \) for each \( c \in [0, 1] \) and \( A \subset X \) (by \( \chi_A \) we denote the characteristic function of the set \( A \)).

if and only if there exists a unique capacity \( \nu \) on \( X \) such that \( \mu \) is the \( t \)-normed integral with respect to \( \nu \).

Let us consider the general case of any compactum. Since the characteristic function \( \chi_A \) generally speaking is not continuous, we will use the \(*\)-homogeneity of functionals instead the second property from the above theorem.

For \( A \in \mathcal{F}(X) \) put \( \Upsilon_A = \{ \varphi \in C(X, [0, 1]) \mid \varphi(a) = 1 \text{ for each } a \in A \} \). If \( A = \emptyset \) we put \( \Upsilon_A = C(X, [0, 1]) \).

**Lemma 2.** Let \( \varphi \in C(X, [0, 1]), \delta < \xi < 1 \). Then there exists \( \psi \in \Upsilon_{\varphi_\xi} \) such that \( \psi|_{\varphi^{-1}[0, \delta]} = \varphi|_{\varphi^{-1}[0, \delta]} \) and \( \psi \) is comonotone with \( \varphi \).

**Proof.** Consider the linear function \( \kappa : [\delta, \xi] \rightarrow [1, \xi^{-1}] \) such that \( \kappa(\delta) = 1 \) and \( \kappa(\xi) = \xi^{-1} \).

Put
\[
\psi(x) = \begin{cases} 
\varphi(x), & \varphi(x) \leq \delta, \\
\kappa(\xi)\varphi(x), & \delta \leq \varphi(x) \leq \xi, \\
1, & \xi \leq \varphi(x). 
\end{cases}
\]

It is easy to check that \( \psi' \) is a function we are looking for. \( \square \)
The following characterization theorem is a modification of the above mentioned result from \[15\]. In fact, we will see that the property of monotonicity is obsolete. The proof is also a modification of the proof given in \[15\]. We simply reduce using monotonicity to the pairs of comonotone functions.

We denote by $\mathcal{B}$ the set of functionals $I : C(X, [0, 1]) \rightarrow [0, 1]$ which satisfy the properties

1. $\mu(1_X) = 1$;
2. $I$ preserves $\vee$ for comonotone functions;
3. $I(c x \ast \varphi) = c \ast I(\varphi)$ for each $c \in [0, 1]$ and $\varphi \in C(X, [0, 1])$.

**Theorem 4.** Let $X$ be a compactum and $\ast$ is a continuous t-norm. A functional $I : C(X, [0, 1]) \rightarrow [0, 1]$ belongs to the set $\mathcal{B}$ if and only if there exists a unique capacity $\nu$ on $X$ such that $I$ is the t-normed integral with respect to $\nu$.

**Proof.** Sufficiency is proved in Lemma 4 from \[15\].

Necessity. Take any $I \in \mathcal{B}$. Define $\nu : \mathcal{F}(X) \rightarrow [0, 1]$ as follows $\nu(A) = \inf\{I(\varphi) \mid \varphi \in \Upsilon_A\}$ if $A \neq \emptyset$ and $\nu(\emptyset) = 0$. It is easy to see that $\nu$ satisfies Conditions 1 and 2 from the definition of capacity.

Let $\nu(A) < \eta$ for some $\eta \in [0, 1]$ and $A \in \mathcal{F}(X)$. Then there exists $\varphi \in \Upsilon_A$ such that $I(\varphi) < \eta$. Choose $\beta \in [0, 1]$ such that $I(\varphi) < \beta < \eta$. Since the operation $\ast$ is continuous, $\eta \ast 1 = \eta$ and $\eta \ast 0 = 0$, there is $\delta \in [0, 1]$ such that $\eta \ast \delta = \beta$. Evidently, $\delta < 1$. Choose $\zeta \in [0, 1]$ such that $\delta < \zeta < 1$. We can choose a function $\psi \in \Upsilon_{\varphi}$ comonotone to $\varphi$ such that $\psi|_{\varphi^{-1}([0, \delta])} = \varphi|_{\varphi^{-1}([0, \delta])}$ by Lemma 2. Then we have $\delta \ast \psi \leq \varphi$ and $\delta \ast I(\psi) \leq I(\varphi) < \beta = \delta \ast \eta$. Hence $I(\psi) < \eta$. Put $U = \psi^{-1}(\{\zeta, 1\})$. Evidently $U$ is open and $U \supset A$. We have $\nu(K) \leq I(\psi) < \eta$ for each compactum $K \subset U$. Hence $\nu \in M_X$.

Let us show that $\int_X^* \varphi d\nu = I(\varphi)$ for each $\varphi \in C(X, [0, 1])$. We have $\int_X^* \varphi d\nu = \max\{\inf\{I(\chi) \mid \chi \in \Upsilon_{\varphi}\} \ast t \mid t \in [0, 1]\} = \max\{\inf\{I(\ast \chi) \mid \chi \in \Upsilon_{\varphi}\} \ast t \mid t \in [0, 1]\}$.

The inequality $\inf\{I(\chi) \mid \chi \in \Upsilon_{\varphi}\} \ast t \leq I(\varphi)$ is obvious for each $t \in [0, I(\varphi)]$. Consider any $t > I(\varphi)$. By Lemma 2 for each $\delta < t$ we can choose a function $\chi^\delta \in \Upsilon_{\varphi}$, such that $\chi^\delta|_{\varphi^{-1}([0, \delta])} = \varphi|_{\varphi^{-1}([0, \delta])}$. Then we have $\delta \ast \chi^\delta \leq \varphi$ and $\delta \ast I(\chi^\delta) \leq I(\varphi)$. Since the operation $\ast$ is continuous, $\inf\{I(\chi) \mid \chi \in \Upsilon_{\varphi}\} \ast t \leq I(\varphi)$.

Hence $\int_X^* \varphi d\nu \leq I(\varphi)$.

Suppose $b = \int_X^* \varphi d\nu < I(\varphi) = a$. Put $m = \max_{x \in X} \varphi(x)$. Then for each $t \in [0, m]$ there exists $\chi^t \in \Upsilon_{\varphi}$ such that $I(t \ast \chi^t) < a$. We can assume that $\chi^t$ is comonotone with $\varphi$ by Lemma 4 from \[15\]. For each $t \in [0, m]$ choose $t' > t$ such that $t' \ast I(\chi^t) < a$. The set $V_t = \{y \mid t' \ast \chi^t(y) > \varphi(y)\}$ is an open neighborhood for each $x \in X$ with $\varphi(x) = t$.

Now we will choose an open neighborhood for the set $\varphi_m$. Put $\xi = \min\{\eta \in [0, 1] \mid \eta \ast m \leq m\}$. We have $\xi \ast m = m$. Since $a \leq m$, using arguments as before we can find $\gamma \in [0, 1]$ such that $\gamma \ast m = a$. Then we have $\xi \ast a = \xi \ast \gamma \ast m = \gamma \ast m = a$.

Choose $\lambda < \xi$ such that $b < \lambda \ast a$. Then there exists $\omega \in \Upsilon_{\varphi_{m-a}}$ such that $m \ast \lambda \ast I(\omega) < \lambda \ast a$, hence $m \ast I(\omega) < a$. We can assume that $\omega$ is comonotone with $\varphi$ by Lemma 4 from \[15\]. Since $m \ast \lambda < m$, the open set $W = \varphi^{-1}(m \ast \lambda, m)$ contains the set $\varphi_m$. Let us remark that $m \ast \omega(x) = m$ for each $x \in W$.

We can choose a finite subcover $\{W, V_{i_1}, \ldots, V_{i_k}\}$ of the open cover $\{W \cup \{V_t \mid t \leq m\}$. We can assume that $\chi^0 = m \ast \lambda$ and $t_i \in (0, m) \setminus \{t_0\}$ for $i > 0$.

Then we have that the functions $t_i \ast \chi^t$ and $\omega$ are pairwise comonotone, hence $I(\omega \vee V_{i_1}^c \{t_i \ast \chi^t\}) < a$. On the other hand $\varphi \leq \omega \vee V_{i_1}^c \{t_i \ast \chi^t\}$ and we obtain a contradiction. \[\square\]
5. Monotonicity of functionals: open problem.

Following [6] we say that a functional $f : C(X, [0, 1]) \rightarrow [0, 1]$ is comonotonically maxitive if it preserves preserves $\vee$ for comonotone functions. We compare the properties of monotonicity and comonotonical maxitivity in this section. Results of Section 3 in particular demonstrate that monotonicity does not imply comonotonical maxitivity. It is easy to see that a comonotonically maxitive functional is monotone for pairs of comonotone functions. Theorem 4 with Lemma 4 from [15] implies that each normed, $\ast$-homogeneous respect a continuous t-norm and comonotonically maxitive functional is monotone. The following theorem shows that the first and the second conditions are superfluous for a finite compactum.

**Theorem 5.** Let $X$ be a finite set and a functional $\mu : C(X, [0, 1]) = [0, 1]^n \rightarrow [0, 1]$ is comonotonically maxitive. Then $\mu$ is monotone.

**Proof.** Let $X = \{x_1, \ldots, x_n\}$ and $\varphi, \psi : X \rightarrow [0, 1]$ are two functions such that $\psi \leq \varphi$. We can assume that that the function $\varphi$ is nondecreasing (if not, apply some permutation on $X$).

We build recursively a finite sequence $\psi_1, \ldots, \psi_{n-1}$ of functions as follows. Put $\psi_1(x) = \max\{\psi(x), \psi(x_1)\}$. For $i \in \{1, \ldots, n - 2\}$ we define the function $\psi_{i+1}$ by the formula

$$\psi(x_j) = \begin{cases} \psi_i(x_j), & j \leq i, \\ \max\{\psi_i(x_{j+1}), \psi_i(x_j)\}, & j \geq i + 1. \end{cases}$$

It is easy to check that $\psi \leq \psi_1 \leq \ldots \leq \psi_{n-1} \leq \varphi$, $\psi_i$ is comonotone to $\psi_1$, $\psi_i$ is comonotone to $\psi_{i+1}$ and $\psi_{n-1}$ is nondecreasing, thus comonotone to $\varphi$. Hence we have $\mu(\psi) \leq \mu(\psi_1) \leq \cdots \leq \mu(\psi_{n-1}) \leq \mu(\varphi)$.

But the problem is still open in the general case.

**Problem 1.** Let $X$ be a compactum and a functional $\mu : C(X, [0, 1]) \rightarrow [0, 1]$ is comonotonically maxitive. Is $\mu$ monotone?

**References**

[1] Luis M. de Campos, Maria T. Lamata and Serafín Moral *A unified approach to define fuzzy integrals*, Fuzzy Sets and Systems **39** (1991), 75–90.

[2] G. Choquet *Theory of Capacity*, An.l’Institue Fourie **5** (1953-1954), 13–295.

[3] D. Denneberg, *Non-Additive Measure and Integral*. Kluwer, Dordrecht, 1994.

[4] J. Eichberger, D. Kelsey, *Non-additive beliefs and strategic equilibria*.

[5] I. Gilboa, *Expected utility with purely subjective non-additive probabilities*, J. of Mathematical Economics **16** (1987) 65–88.

[6] Michel Grabisch, *Set Functions, Games and Capacities in Decision Making*. Springer, 2016.

[7] A. Kolesarova, R. Mesiar, *Discrete Universal Fuzzy Integrals*. In Cornejo, M.E., Harmati, I.A., Koczy, L.T., Medina-Moreno, J. (eds) *Computational Intelligence and Mathematics for Tackling Complex Problems 4. Studies in Computational Intelligence*, vol 1040. p. Springer. 2023.

[8] E.P. Klement, R. Mesiar and E.Pap. *Triangular Norms*. Dordrecht: Kluwer. 2000.

[9] J. Li, R. Mesiar, Y. Ougang, A. Seliga, *A new class of decomposition integrals on finite spaces*, International Journal of Approximate Reasoning **149** (2022) 192–205.

[10] Lin Zhou, *Integral representation of continuous comonotonically additive functionals*, Transactions of the American Mathematical Society **350** (1998) 1811–1822.

[11] J.-L. Marichal, *On Sugeno integral as an aggregation function*, Fuzzy Sets and Systems **114** (2000) 347–365.

[12] O.R. Nykyforchyn, *The Sugeno integral and functional representation of the monad of lattice-valued capacities*, Topology **48** (2009) 137–148.

[13] O.R. Nykyforchyn, M.M. Zarichnyi, *Capacity functor in the category of compacta*, Mat.Sh. **199** (2008) 3–26.

[14] T. Radul, *A functional representation of capacity monad*, Topology **48** (2009) 100–104.

[15] T. Radul, *Games in possibility capacities with payoff expressed by fuzzy integral*, Fuzzy Sets and systems **434** (2022) 185-197.
[16] D. Schmeidler, *Subjective probability and expected utility without additivity*, Econometrica **57** (1989) 571–587.

[17] F. Suarez, *Familias de integrales difusas y medidas de entropia relacionadas*, Thesis, Universidad de Oviedo, Oviedo (1983).

[18] M. Sugeno, *Fuzzy measures and fuzzy integrals*, A survey. In Fuzzy Automata and Decision Processes. North-Holland, Amsterdam: M. M. Gupta, G. N. Saridis et B. R. Gaines editeurs. 89–102. 1977

[19] S. Weber *Decomposable measures and integrals for archimedean t-conorms*, J. Math. Anal. Appl. **101** (1984), 114–138.

[20] S. Weber *Two integrals and some modified versions - Critical remarks*, Fuzzy Sets and Systems **20** (1986), 97–105.