Canonical Differential Calculi For Finitely Generated Abelian
Group and Their Fermionic Representations

Jian Dai, Xing-Chang Song
Theory Group, Department of Physics, Peking University
Beijing, P. R. China, 100871
jdai@mail.phy.pku.edu.cn, songxc@ibm320h.phy.pku.edu.cn

May 8th, 2001

Abstract

Canonical differential calculus is defined for finitely generated abelian group with an
involution existing consistently. Two such canonical calculi are found out. Fermionic
representation for canonical calculus is defined based on quantized calculus. Fermionic
representations for above-mentioned two canonical calculi are searched.

PACS: 02.40.Gh, 02.20.Bb

Keywords: finitely generated abelian group, canonical differential calculus, invo-
lation, quantized calculus, fermionic representation

I Introduction

Noncommutative differential calculus can be formulated on finite groups and some simple
types of discrete infinite groups within the quantum-algebraic approach towards noncom-
mutable geometry (NCG) [1][2]. There are two problems which are far from being fully
understood however: 1) The choice of differential calculus is considerably arbitrary, even
though translation invariance being included; and the arbitrariness corresponds to the
power set of the group. 2) Whether the concept of spinor can be generalized to this differential construction is unknown.

These two problems for the category of finitely generated abelian group (FGAG) whose structures have been totally clear in group theory will be addressed here. Canonical differential calculi and their fermionic representations will be considered for these groups, where the term “canonical” refers to that the choice of the calculus reflects the groups structure properly and is compatible with group involution. Note that the first question for FGAGs has been considered in [3] in a less rigid sense. This article is organized as following. In Sec.II, necessary mathematics is prepared, including differential calculus over finitely generated groups, classification of FGAGs. In Sec.III, canonical calculus is defined and two canonical calculi are found for FGAGs. In Sec.IV, fermionic representation for canonical calculi is discussed and fermionic representations for the above two canonical calculi are searched. In Sec.V, some open discussions are given.

II Preliminaries and Notations

Contents in this section are well-known, and they are rephrased here just for integrality of this article. Group $G$ is required to be finitely generated and the term “discrete” is avoided due to the subtlety in topology. Dual of $\mathbb{C}G$ is written as $\mathcal{A}[G]$, which becomes a commutative $\mathbb{C}$-algebra under pointwise product. Induced right(left) actions of $G$ on $\mathcal{A}[G]$ are written as $R_g(L_g)$ for all $g$ in $G$. $e$ is the unit of $G$ as usual. A first order differential over $\mathcal{A}[G]$ is a pair $(M, d)$, where $M$ being an $\mathcal{A}[G]$-bimodule generated by a left invariant basis $\{\xi^g : g \in G' \subset G \{e\}\}$, whose structure is characterized by the equation

$$\xi^g f = R_g(f) \xi^g, \forall f \in \mathcal{A}[G], g \in G'$$

and $d \in Hom_{\mathbb{C}}(\mathcal{A}[G], M)$ fulfills Leibnitz rule. Eq.(1) is crucially important for first order differential on $G$ in the sense that $\mathcal{A}[G]$ is a set-theoretical object on which no group structure is encoded, therefore this noncommutativity endows group structure onto
\( \mathcal{A}[G] \), providing \( G' \) generates \( G \). Generically for all \( f \) in \( \mathcal{A}[G] \), \( d(f) = \partial_g(f)\xi^g \) with \( \partial_g(f) \in \mathcal{A}[G] \); accordingly, \( \partial_g(ff') = \partial_g(f)R_g(f') + f\partial(f') \). \( \partial_g \) is fixed to be \( R_g - Id \) up to a constant depending on \( g \), which can be absorbed by rescaling \( \xi^g \). A differential calculus \( (\Omega(\mathcal{A}[G]), d) \) is \( \mathcal{A}[G] \)-tensor product \( \bigotimes_{\mathcal{A}[G]} M \) modulo equivalent relations generated by the requirement that the extension of \( d \) over this tensor product satisfying nilpotent rule and graded Leibnitz rule. In [4], Feng et al proved that if \( G \) is a direct product of subgroups \( H_1 \) by \( H_2 \), and \( G' \subset (G' \cap H_1) \cup (G' \cap H_2) \), then

\[
\xi^{h_1} \otimes \xi^{h_2} + \xi^{h_2} \otimes \xi^{h_1} \cong 0, \forall h_1 \in G' \cap H_1, h_2 \in G' \cap H_2.
\] (2)

An involution \( \dagger \) on \( G \) is an order-2 bijection; it is pulled back to be an involution on \( \mathcal{A}[G] \) as \( (\dagger \ast f)(g) = f(\dagger(g)) \). Only Those \( \dagger \in Aut(G) \) will be considered below. An involution on \( \Omega(\mathcal{A}[G]) \) is an antilinear antihomomorphism which commutes skewly with \( d \). Main involution \( \ast \) on \( \mathcal{A}([G]) \) is defined by \( \dagger = Id \); the necessary condition of extensibility of \( \ast \) to an involution on \( \Omega(\mathcal{A}[G]) \) is that \((\xi^g)^\ast = -\xi^{g^{-1}} \) and that \( g \in G' \Leftrightarrow g^{-1} \in G' \). If \( G \) is abelian, the structure of \( G \) is classified by the following theorem [5]

**Theorem 1** An abelian group is finitely generated if and only if it is a direct product of finitely many cyclic groups of infinite or prime-power orders. [7]

\( G \) will be assumed to be abelian henceforth. In this circumstance, parity \( r \) on \( G \) which is defined by \( r(g) = -g \) can be extended to be an involution over \( \Omega(\mathcal{A}[G]) \) with any \( G' \). One can check that \( r^\ast(\xi^g) = -\xi^g \).

### III Canonical Differential Calculi

As being remarked under Eq.(1), \( \mathcal{A}[G] \) is a set-theoretical algebraic object, washing all information of \( G \) out, and the structure of \( G \) is partly recovered by Eq.(1) with \( R_g \in End_G(\mathcal{A}[G]), g \in G' \). Therefore, the following definition makes good sense.

\( ^{a)}\) In mathematical literature, direct product of abelian groups is also referred as direct sum.
Definition 1. A differential calculus is canonical, if 1) $G'$ generates $G$, 2) there exists an involution which is an extension of one $\jmath^*$, 3) $G'$ is the minimum within all subsets of $G$ which satisfy 1) and 2).

Follow Theorem 1, $G$ can be uniquely decomposed as a direct product of $\mathbb{Z}^{n_I}$ with a torsion subgroup $T = \bigotimes_{p,n_{pi},k_{pi}} (\mathbb{Z}_{p_{pi}})^{k_{pi}}$ where $p$ belongs to a finite set of prime numbers, $n_I, n_{pi}, k_{pi} \in \mathbb{N}$. From group theory, there exists a basis for $G$ which is the minimum generator set of $G$ and any element in $G$ is $\mathbb{Z}$-linear combination of this basis; more specifically, the basis is formed just by generators of each factor group in the decomposition of $G$, written as $\sigma_\alpha$ for $\mathbb{Z}_2$ factors, $T_s$ for $\mathbb{Z}$ factors, and $t_a$ for others. It is easy to check the following statement.

Proposition 1. If $G'$ is taken to be the basis of $G$ and involution is taken to be parity, then the correspondent calculus is canonical.

Corollary 1. If $G'$ is taken to be the basis of $G$ and involution is taken to be parity, then $1) \xi^g$ from different factor groups anticommute.

2) 
\[
d(\xi^{\sigma_\alpha}) - 2\xi^{\sigma_\alpha} \otimes \xi^{\sigma_\alpha} \simeq 0, d(\xi^{T_s}) \simeq 0, d(\xi^{t_a}) \simeq 0, \xi^{T_s} \otimes \xi^{T_s} \simeq 0, \xi^{t_a} \otimes \xi^{t_a} \simeq 0.
\] (3)

Proof:

1) is inferred by Eq.(2). To show 2) only $\mathbb{Z}_2$, $\mathbb{Z}$ and $\mathbb{Z}_n$, $n \geq 3$ need to be considered, and Eq.(3) is inferred by $d(d(f)) = 0$. ■

The following statement is also obvious.

Proposition 2. If $G' = \{\sigma_\alpha, T_s, -T_s, t_a, -t_a : \text{for all } \alpha, s, a\}$ and the involution is taken to be the main involution, then the correspondent calculus is canonical.

The corollary below can be verified in the same way as that for Corollary 1.

Corollary 2. 1) $\xi^g$ from different factor groups anticommute.

2) 
\[
d(\xi^{\sigma_\alpha}) - 2\xi^{\sigma_\alpha} \otimes \xi^{\sigma_\alpha} \simeq 0; \xi^\tau \otimes \xi^\tau \simeq 0 \equiv \xi^{-\tau} \otimes \xi^{-\tau}, d(\xi^\tau) - \{\xi^\tau, \xi^{-\tau}\} \simeq 0.
\]
IV Fermionic Representations of Calculi

Following Connes [6], a quantized calculus over $A[G]$ is a quadruple $(H, \pi, F, \mathcal{J})$ where $H$ is separable Hilbert space, $\pi$ is a faithful representation of $A[G]$ by bounded operators on $H$, $\mathcal{J}$ is representation of $\mathcal{J}^*$ by $\pi(f)\mathcal{J}^{\dag}\mathcal{J}^{\dag}$, $\forall f \in A[G]$, hence $\mathcal{J}^2 = \text{Id}$, moreover $\mathcal{J}$ is required to be unitary, and $F$ is a $\mathcal{J}$-selfadjoint operator which satisfies $[F, \pi(f)]$ is compact for all $f \in A[G]$ and $F^2 = \text{Id}$ up to a scalar. A quantized differential is defined to be $\hat{d} : \pi(A[G]) \to \mathcal{K}(H), \pi(f) \mapsto i[F, \pi(f)]$ where $\mathcal{K}(H)$ is the algebra of compact operators on $H$. A $p$-form in quantized calculus is of the form of linear combination of $i^p \pi(f_0)[F, \pi(f_1)]...[F, \pi(f_p)]$. $\hat{d}$ is extended to be a coboundary operator through graded adjoint action by $iF$.

**Definition 2** A fermionic representation of a canonical calculus over $A[G]$ is a quantized calculus over $A[G]$ whose $\pi$ is extended to be an involutive differential representation of $\Omega(A[G])$.

The extension of $\pi$ is written as $\tilde{\pi}$ with $\tilde{\pi}(f) = \pi(f), \forall f \in A[G]$.

**Theorem 2** A fermionic representation of a canonical calculus over $A[G]$ exists, iff for an assignment of $\{\tilde{\pi}(\xi^g) : g \in G'\}$ the following conditions are satisfied

\begin{align*}
\tilde{\pi}(f \xi^g) &= \pi(f)\tilde{\pi}(\xi^g), \tilde{\pi}(\xi^g f) = \tilde{\pi}(\xi^g)\pi(f) \\
V := iF - \sum_{g \in G'} \tilde{\pi}(\xi^g) &\in \pi(A[G])' \\
\sum_{h,g-h \in G'} \tilde{\pi}(\xi^h)\tilde{\pi}(\xi^{g-h}) &= 0, g \in G \setminus \{e\} \cup G' \\
\sum_{h,g-h \in G'} \tilde{\pi}(\xi^h)\tilde{\pi}(\xi^{g-h}) + \{V, \tilde{\pi}(\xi^g)\} &= 0, g \in G' \\
\tilde{\pi}(\mathcal{J}^{\dag}(\xi^g)) &= \mathcal{J}\tilde{\pi}(\xi^g)\mathcal{J}^{\dag} \\
\end{align*}

\textsuperscript{b)} That $F$ is $\mathcal{J}$-selfadjoint is weaker than that $F$ is selfadjoint and commutes with $\mathcal{J}$, which is adopted in other literature.
where $\pi(A[G])'$ is commutant of $\pi(A[G])$ in $\mathcal{K}(\mathcal{H})$.

**Proof:**

*Necessity:* If $\hat{\pi}$ exists, then it must be a module homomorphism from $M$ to $\mathcal{K}(\mathcal{H})$, which implies Eq.(3); Eq.(3) is inferred by the commutative diagram $\hat{d} \circ \pi = \pi \circ d$; Eqs.(3)(6) are implied by $\hat{d} \circ \pi \circ d = 0$ which is the nilpotent rule of differential; Eq.(8) follows that $\hat{\pi}$ is involutive representation.

*Sufficiency:* Assume Eq.(4) holds, then $\pi$ can be extended to be module homomorphism from $\bigotimes_{A[G]} M$ to $\mathcal{K}(\mathcal{H})$; and Eq.(4) implies that Eq.(6) is realized by $\hat{\pi}(\xi^g)\pi(f) = \pi(R_g(f))\hat{\pi}(\xi^g)$, with that $\hat{\pi}(d(f)) = \left[\sum_{g \in G'} \hat{\pi}(\xi^g), \pi(f)\right]$ following. Consequently, first order differential $(M, d)$ can be implemented by $\hat{d} \circ \pi = \pi \circ d$, thanks to Eq.(3). High order representation of $d$ is realized by Eqs.(3)(6) and the identity $ABC - (-)^{p+q}BCA = (AB - (-)^p BA)C + (-)^p B(AC - (-)^q CA)$, which guarantee the nilpotent rule and graded Leinitz rule of differential, together with that quotient conditions making tensor product of $M$ be $\Omega(A[G])$ are contained in $ker(\hat{\pi})$. In the end, remember that $J^*$ is realized by $J$, involutive homomorphism is extended from Eq.(8) by antilinearity and antihomomorphic rule. That $F$ is $J$-selfadjoint implies that skew-commutativity of involution and differential is realized. ■

Fermionic representation exists for canonical calculus on $G$ with main involution. In fact, introduce a set of fermionic creation and annihilation operators for each factor group of $G$ as $b^{\sigma_o}, b^{T_o}, b^{T_o\dagger}, b^{\alpha}, b^{\alpha\dagger}$ fulfilling anticommutative relations

$$\{b^{T_o}, b^{T_o}\} = \{b^{T_o}, b^{T_o\dagger}\} = \delta^{\sigma_o}, \{b^{T_o}, b^{\beta}\} = 0, \{b^{T_o}, b^{\sigma_\beta}\} = 0, \{b^{T_o}, b^{\sigma_\beta}\} = 0, \{b^{T_o}, b^{\sigma_\beta}\} = 0, \{b^{T_o}, b^{\sigma_\beta}\} = 0, \{b^{T_o}, b^{\sigma_\beta}\} = 0, \{b^{T_o}, b^{\sigma_\beta}\} = 0,$$  

$$\{b^{T_o\dagger}, b^{T_o}\} = \delta^{\sigma_o}, \{b^{T_o\dagger}, b^{T_o\dagger}\} = 0, \{b^{T_o\dagger}, b^{\beta}\} = 0, \{b^{T_o\dagger}, b^{\sigma_\beta}\} = 0, \{b^{T_o\dagger}, b^{\sigma_\beta}\} = 0, \{b^{T_o\dagger}, b^{\sigma_\beta}\} = 0, \{b^{T_o\dagger}, b^{\sigma_\beta}\} = 0, \{b^{T_o\dagger}, b^{\sigma_\beta}\} = 0,$$  

$$\{b^{\alpha}, b^{T_o}\} = \{b^{\alpha}, b^{T_o\dagger}\} = 0, \{b^{\alpha}, b^{\beta}\} = 0, \{b^{\alpha}, b^{\sigma_\beta}\} = 0, \{b^{\alpha}, b^{\sigma_\beta}\} = 0, \{b^{\alpha}, b^{\sigma_\beta}\} = 0, \{b^{\alpha}, b^{\sigma_\beta}\} = 0, \{b^{\alpha}, b^{\sigma_\beta}\} = 0, \{b^{\alpha}, b^{\sigma_\beta}\} = 0,$$  

$$\{b^{\alpha\dagger}, b^{T_o}\} = \{b^{\alpha\dagger}, b^{T_o\dagger}\} = 0, \{b^{\alpha\dagger}, b^{\beta}\} = 0, \{b^{\alpha\dagger}, b^{\sigma_\beta}\} = 0, \{b^{\alpha\dagger}, b^{\sigma_\beta}\} = 0, \{b^{\alpha\dagger}, b^{\sigma_\beta}\} = 0, \{b^{\alpha\dagger}, b^{\sigma_\beta}\} = 2\delta^{\alpha_\beta}.$$  

in which $b^{\sigma_\alpha}$ for $\mathbb{Z}_2$ factors are selfadjoint. Let $S$ be a representation of the above fermionic operators and $\mathcal{H} = l^2(G) \otimes S$ with $\pi(f) = f \otimes Id$. Define $\hat{\pi}(\xi^g) = ib^{\sigma_o}R_{\sigma_o}, \hat{\pi}(\xi^{-T_o}) = \ldots$.
\[ib^{T_s}R_{-T_s}, \tilde{\pi}(\xi^{T_s}) = ib^{T_s\dagger}R_{T_s}, \tilde{\pi}(\xi^{-t_a}) = ib^{t_a}R_{-t_a}, \tilde{\pi}(\xi^{t_a}) = ib^{t_a\dagger}R_{t_a}\] and
\[F = b^{\sigma\alpha}R_{\sigma\alpha} + b^{T_s}R_{-T_s} + b^{T_s\dagger}R_{T_s} + b^{t_a}R_{-t_a} + b^{t_a\dagger}R_{t_a},\] (14)

\(\mathcal{J} = Id\). Then one can check that \(F^\dagger = F\), \(F^2 \sim Id\). Take Eq.(4) as definition, Eq.(5) is obvious with \(V = 0\), and Eqs.(6)(7) are implied by Eqs.(8)(13). Eq.(8) can be checked easily. Hence due to Theorem 2 above \((\mathcal{H}, \pi, \mathcal{J}, F)\) provides a fermionic representation of canonical calculus with main involution.

It is not so easy to construct fermionic representation for canonical calculus whose involution is taken to be parity. In fact, let \(\mathcal{J} = r \otimes Id\), then \(\mathcal{J}\pi(f)^\dagger \mathcal{J}^\dagger = \pi(r^*(f))\). Introduce a set of Clifford generators \(\gamma^{\sigma\alpha}, \gamma^{T_s}, \gamma^{t_a}\) for each factor group of \(G\), fulfilling that
\[\{\gamma^{\sigma\alpha}, \gamma^{\sigma\beta}\} = 2\delta^{\alpha\beta}, \{\gamma^{\sigma\alpha}, \gamma^{T_r}\} = 0, \{\gamma^{\sigma\alpha}, \gamma^{t_b}\} = 0,\]
\[\{\gamma^{T_s}, \gamma^{\sigma\beta}\} = 0, \{\gamma^{T_s}, \gamma^{T_r}\} = 2\delta^{sr}, \{\gamma^{T_s}, \gamma^{t_b}\} = 0,\]
\[\{\gamma^{t_a}, \gamma^{\sigma\beta}\} = 0, \{\gamma^{t_a}, \gamma^{T_r}\} = 0, \{\gamma^{t_a}, \gamma^{t_b}\} = 2\delta^{ab}.\]

And define \(F = \gamma^{\sigma\alpha}R_{\sigma\alpha} + \gamma^{T_s}R_{T_s} + \gamma^{t_a}R_{t_a}\). Then Eqs.(4)(5)(7)(8) are valid. However, Eq.(8) can not be realized unless requiring the product of conjunct \(\gamma^{T_s}, \gamma^{t_a}\) to be wedge product. This will also guarantee that \(F^2 \sim Id\) in the above sense. In this circumstance, a weak form of fermionic representation is found out.

V Discussions

When \(G\) is taken to be \(\mathbb{Z}^d\), a nontrivial correspondence between \(F\) in Eq.(14) and staggered Dirac operator in lattice field theory was shown in [7]. On the contrary, Eq.(14) can be understood as a generalized staggered Dirac operator on FGAGs.

Structures for non-FGAGs are far less clear in group theory, so the term canonical calculus for these groups are very blurred. Some specific choices have been explored in [2][8].
Acknowledgements
This work was supported by Climb-Up (Pan Deng) Project of Department of Science and Technology in China, Chinese National Science Foundation and Doctoral Programme Foundation of Institution of Higher Education in China.

References

[1] A. Sitarz, J. Geom. Phys. 15(1995)123, hep-th/9210098.

[2] K. Bresser, F. Müller-Hoissen, A. Dimakis, A. Sitarz, J. Phys. A: Math. Gen. 29(1996)2705, q-alg/9509004; L. Castellani, “DIFFERENTIAL CALCULI ON FINITE GROUPS”, JHEP, Proceedings of the Corfu Summer Inst. on Elementary Particle Phys. (1999), DFTT-19/2000, math.QA/0005226.

[3] H-Y. Guo, K. Wu, W. Zhang, Commun. Theor. Phys. 34(2000)245.

[4] B. Feng, J-M. Li, X-C. Song, Commun. Theor. Phys. 30(1998)257.

[5] D. J. S. Robinson, A Course in the Theory of Groups, 1982 Springer-Verlag New York Inc., GTM 80.

[6] A. Connes, Noncommutative Geometry, 1994 Academic Press; ibid, J. Math. Phys. 36(1995)6194; ibid, Commun. Math. Phy. 182(1996)155, hep-th/9603053.

[7] J. Dai, X-C. Song, “NONCOMMUTATIVE GEOMETRY OF LATTICE AND STAGGERED FERMIONS”, accepted to be published in Phys. Lett. B, hep-th/0101130.

[8] S. Majid, E. Raineri, “ELECTROMAGNETISM AND GAUGE THEORY ON THE PERMUTATION GROUP S3”, hep-th/0012123.