Gauge invariant composite operators of QED in the exact renormalization group formalism

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Abstract

Using the exact renormalization group (ERG) formalism, we study the gauge invariant composite operators in QED. Gauge invariant composite operators are introduced as infinitesimal changes of the gauge invariant Wilson action. We examine the dependence on the gauge fixing parameter of both the Wilson action and gauge invariant composite operators. After defining ‘gauge fixing parameter independence,’ we show that any gauge independent composite operators can be made ‘gauge fixing parameter independent’ by appropriate normalization. As an application, we give a concise but careful proof of the Adler–Bardeen non-renormalization theorem for the axial anomaly in an arbitrary covariant gauge by extending the original proof by A Zee.

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1. Introduction

In QED, as in any gauge theories, all physical quantities are gauge invariant. They are often given as gauge invariant composite operators such as the electric current or energy–momentum tensor. The purpose of this paper is to study gauge invariant composite operators in QED formulated with the exact renormalization group (ERG)\(^1\). We are especially interested in the dependence of the gauge invariant composite operators on the covariant gauge fixing parameter. We will see that particular normalization convention must be adopted for the independence of the composite operators on the gauge fixing parameter.

There are many formulations for QED (and YM theories), and accordingly gauge invariant composite operators have already been studied in various formulations. Most notably, gauge invariant composite operators in YM theories have been studied with the dimensional

\(^{1}\) There are many reviews of ERG, [1], which gives references to some of the earlier reviews, has an emphasis on perturbative applications of ERG. A most recent review is [2].
regularization. (See chapters 12 and 13 of [3] and references therein.) Within the ERG formalism, however, gauge invariant composite operators have not been fully studied, and we wish to fill the gap in this paper.

The advantage of formulating QED with ERG is three-fold: first, the ease of renormalization. The Wilson action $S_\Lambda$ [4] with a finite UV cutoff $\Lambda$ is obtained as a perturbative solution to the ERG differential equation [5]. There is no need for regularization or taking a limit for renormalization. Renormalization is done with a selection of solutions with appropriate behaviors for large $\Lambda$. Second, the gauge invariance is incorporated nicely as the Ward–Takahashi (WT) identity among composite operators, even though the gauge invariance is not manifest in the Wilson action. Third, we work with a fixed number of dimensions for space(time), and the chirality of fermion fields can be introduced with ease. Hence, this formalism is suited for the study of axial or chiral anomalies.

In any perturbative formulation of QED, including ERG, gauge invariance is incorporated as the WT identities for the renormalized correlation functions. Within the ERG formalism alone, there are several ways [7–10]. The discussion in this paper is based on a formulation developed in [11] (which is based closely on the earlier work of Becchi [7]). The main tool is composite operators that are defined as infinitesimal variations of the Wilson action. Since the Wilson action and composite operators are well-defined functionals of field variables, formal manipulations acting on these functionals, such as functional differentiation, are also well defined. We have no regularization to take to a limit; we need not worry if the operator equations remain valid in the limit.

The main subject of this paper is the dependence of the Wilson action and composite operators on the covariant gauge fixing parameter, denoted as $\xi$. Introduction of $\xi$ is only for the convenience of perturbative expansions, and physics should not depend on the arbitrary choice of $\xi$. Our first task is to translate this simple requirement into an equation satisfied by gauge invariant composite operators. After defining '$\xi$-independence' of composite operators, we will show that any gauge invariant composite operators can be made '$\xi$-independent' by appropriate normalization. We will then study the anomalous dimensions of gauge invariant and $\xi$-independent composite operators, and show them to be independent of $\xi$. Given an anomalous dimension as a function of the squared gauge coupling $e^2$, a composite operator is determined uniquely except for a constant factor.

We apply our results to the Adler–Bardeen non-renormalization theorem for the axial anomaly [12]. We follow the proof given by A Zee who applied the renormalization group for the first time to the non-renormalization of the axial anomaly [13]. Our treatment resembles chapter 13 of [3], where the dimensional regularization is used instead of ERG. The apparent resemblance is inevitable; the main difference is in the technique of constructing gauge invariant composite operators. We pay a careful attention to the gauge fixing parameter, necessary for the perturbative formulation of QED. Since the anomaly equation is formulated as a linear relation among gauge invariant and $\xi$-independent composite operators, our proof is valid in any gauge.

The organization of the paper is as follows. In section 2 we review the ERG formalism for QED that gives the WT identities in terms of an operator identity. For the reader unfamiliar with perturbative ERG, we have prepared a short summary of the formalism in appendix A, using the example of the $\phi^4$ theory. The formalism is simple both logically and technically, and we hope that section 2 and appendix A are more than enough to prepare the reader to follow the remaining sections. In section 3 we introduce a WT identity for composite operators to define their gauge invariance. In sections 4 and 5, we discuss the main subject of this paper,

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2 A notable exception is the manifestly gauge invariant ERG formalism of Arnone, Morris, and Rosten. (See [6] and references therein. See also IX B of [2].)
the $\xi$-dependence. In section 4, we discuss the $\xi$-dependence of the Wilson action, and derive an explicit formula for the $\xi$-dependence of renormalized correlation functions of elementary fields. This extends the formula originally given by Landau and Khalatnikov for the bare correlation functions [14–17]. In section 5, we extend the discussion to composite operators. We first define what we mean by $\xi$-independent composite operators, and then show that any gauge invariant composite operator can be made $\xi$-independent by normalization. In section 6 we relate ERG to the standard RG by discussing the $\mu$-dependence of the Wilson action [18].

A renormalization scale $\mu$ is introduced to specify a Wilson action as a unique solution to the ERG differential equation. We introduce the beta functions of the parameters and anomalous dimensions of the elementary fields. Then, in section 7, we define the anomalous dimensions of composite operators. We show that the anomalous dimension of a gauge invariant and $\xi$-independent composite operator is independent of $\xi$. In section 8, we apply the results to the non-renormalization of the axial anomaly.

Most of the technicalities have been relegated to the appendices which are almost as long as the main text. We wish to avoid interrupting the simple logical flow of the paper by technical details, even though the details of how to use the techniques of ERG constitute an essential part of this work. The appendices give all the necessary details except the actual 1-loop calculations of which we only enumerate the results.

Throughout the paper we work with the four dimensional Euclidean space. We find the following shorthand notations convenient and use them frequently:

$$\int_p \equiv \int \frac{d^4p}{(2\pi)^4}, \quad \delta(p) \equiv (2\pi)^4\delta^{(4)}(p).$$

2. The Wilson action for QED

In the ERG formalism, the main object of our study is a Wilson action [4]. In the case of QED, a Wilson action $S_\Lambda[A_\mu, \psi, \bar{\psi}]$ is a functional of the photon field $A_\mu$ and Dirac spinor fields $\psi, \bar{\psi}$ with a finite UV cutoff $\Lambda$. We define the correlation functions of elementary fields by

$$\langle \cdots \rangle_{S_\Lambda} = \int \left[ dA_\mu \, d\psi \, d\bar{\psi} \right] e^{S_\Lambda} \cdots \int \left[ dA_\mu \, d\psi \, d\bar{\psi} \right] e^{S_\Lambda}$$

where the dots denote a product of elementary fields. Given the free action

$$S_{F,\Lambda} \equiv -\int_k K\left(\frac{k}{\Lambda}\right) \left( k^2 \delta_{\mu\nu} - \left( 1 - \frac{1}{\xi} \right) k_\mu k_\nu \right) \frac{1}{2} A_\mu(k)A_\nu(-k)$$

$$-\int_p K\left(\frac{p}{\Lambda}\right) \bar{\psi}(-p) \left( p^2 + im \right) \psi(p)$$

where $\xi$ is introduced as a gauge fixing parameter, the propagators are obtained as

$$\langle A_\mu(k)A_\nu(k') \rangle_{S_{F,\Lambda}} = \frac{K\left(\frac{k}{\Lambda}\right)}{k^2} \left( \delta_{\mu\nu} - (1 - \frac{1}{\xi}) \frac{k_\mu k_\nu}{k^2} \right) \delta(k + k')$$

$$\langle \psi(p)\bar{\psi}(-q) \rangle_{S_{F,\Lambda}} = \frac{K\left(\frac{p}{\Lambda}\right)}{p^2 + im} \delta(p - q).$$

The cutoff function $K\left(\frac{k}{\Lambda}\right)$ is a decreasing but positive function of $k^2/\Lambda^2$; it is 1 for $k^2 < \Lambda^2$, and is nowhere zero except at infinity so that the division by $K\left(\frac{k}{\Lambda}\right)$ makes sense for finite $k^2$.

The cutoff function $K\left(\frac{k}{\Lambda}\right)$ is a decreasing but positive function of $k^2/\Lambda^2$; it is 1 for $k^2 < \Lambda^2$, and is nowhere zero except at infinity so that the division by $K\left(\frac{k}{\Lambda}\right)$ makes sense for finite $k^2$. We also assume that it decays fast enough as $k^2 \to \infty$ so that the functional integrals are well

\[3\] For the reader unfamiliar with the (perturbative) ERG formalism, we have prepared a short summary in appendix A.
functions are obtained as where the dots indicate replacement of more pairs. Especially, the cutoff independent two-point defined, free of UV divergences. The cutoff dependence of the Wilson action is given by the ERG differential equation [5]:

\[
- \Lambda \partial_{\Lambda} e^{S_\Lambda} = \int_k \Delta \left( \frac{k}{\Lambda} \right) \left[ \frac{1}{K \left( \frac{k}{\Lambda} \right)} A_{\mu}(k) \frac{\delta}{\delta A_{\mu}(k)} \right] e^{S_\Lambda} + \frac{1}{k^2} \left[ \delta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right] \frac{1}{2} \delta^2 \delta A_{\mu}(k) \delta A_{\nu}(-k) \] 

\[
+ \int_p \Delta \left( \frac{p}{\Lambda} \right) \left[ \frac{1}{K \left( \frac{p}{\Lambda} \right)} \left( e^{S_\Lambda} \tilde{\delta}_\mu(p) \psi(p) + \tilde{\psi}(-p) \tilde{\delta} \tilde{\psi}(-p) e^{S_\Lambda} \right) \right] + \text{Tr} \left( \delta \tilde{\psi}(-p) e^{S_\Lambda} \frac{\delta}{\delta \psi(p)} \frac{1}{\hat{p} + im} \right) \] 

(7)

where we denote

\[
\Delta \left( \frac{k}{\Lambda} \right) = \Lambda \frac{\partial}{\partial \Lambda} K \left( \frac{k}{\Lambda} \right) \] 

(8)

(7) gives the cutoff dependence of the Wilson action, but at the same time it assures that the following linear combination of correlation functions remains independent of \( \Lambda \) [11] (see also ‘dual actions’ in [2]):

\[
\langle A_{\mu_1}(k_1) \cdots A_{\mu_L}(k_L) \psi(p_1) \cdots \psi(p_N) \tilde{\psi}(-q_1) \cdots \tilde{\psi}(-q_N) \rangle_{\Lambda} = \prod_{i=1}^{L} K \left( \frac{k_\mu_i}{\Lambda} \right) \prod_{j=1}^{N} \frac{1}{K \left( \frac{p_\nu_j}{\Lambda} \right) K \left( \frac{q_\mu_j}{\Lambda} \right)} \left[ \langle A_{\mu_1}(k_1) \cdots \rangle_{S_\Lambda} + \frac{K \left( \frac{k_\mu_i}{\Lambda} \right) (K \left( \frac{k_\mu_i}{\Lambda} \right) - 1)}{k^2} \delta_{\mu_i,1} - (1 - \xi) \frac{k_{\mu_i} k_{\mu_j}}{k^2} \right] 

\times \delta(k_i + q_j) \left[ \cdots A_{\mu_i}(k_i) \cdots A_{\mu_i}(k_i) \cdots \right]_{S_\Lambda} + \sum_{i,j} \frac{K \left( \frac{k_\mu_i}{\Lambda} \right) (K \left( \frac{k_\mu_i}{\Lambda} \right) - 1)}{\hat{p} + im} \delta(p_i - q_j) \left[ \cdots \psi(p_i) \cdots \tilde{\psi}(-q_j) \cdots \right]_{S_\Lambda} + \cdots \] 

(9)

where the dots indicate replacement of more pairs. Especially, the cutoff independent two-point functions are obtained as

\[
\langle A_{\mu}(k) A_{\nu}(k') \rangle_{\Lambda} = \frac{1}{K \left( \frac{k}{\Lambda} \right)^2} \langle A_{\mu}(k) A_{\nu}(k') \rangle_{S_\Lambda} + \frac{1 - 1/K \left( \frac{k}{\Lambda} \right)}{k^2} \left( \delta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right) \delta(k + k') \] 

(10)

\[
\langle \psi(p) \tilde{\psi}(-q) \rangle_{\Lambda} = \frac{1}{K \left( \frac{p}{\Lambda} \right)^2} \langle \psi(p) \tilde{\psi}(-q) \rangle_{S_\Lambda} + \frac{1 - 1/K \left( \frac{p}{\Lambda} \right)}{\hat{p} + im} \delta(p - q). \] 

(11)

To specify a solution of the ERG differential equation uniquely, we can impose an asymptotic condition on the Wilson action for large \( \Lambda \). Perturbative renormalizability of QED amounts to the existence of Wilson actions for which the coefficients of higher dimensional terms vanish.

\( K(k) = o(k^{-4}) \) for \( k^2 \gg 1 \) is sufficient.
as $\Lambda \to \infty$ \cite{19}. More specifically, for the field momenta small compared with $\Lambda$, we can expand the action as

$$S_\Lambda \xrightarrow{\Lambda \to \infty} \int d^4x \left[ \frac{1}{2} \partial_\mu A_\nu \partial_\nu A_\mu - \left(1 - \frac{1}{\xi} \right) \frac{1}{2} (\partial_\mu A_\mu)^2 + \bar{\psi} \left( \frac{1}{i} \gamma + im \right) \psi \right]$$

$$+ \int d^4x \left[ a_2(\Lambda) \frac{1}{2} A^2_\mu + z(\Lambda) \frac{1}{2} (\partial_\mu A_\mu)^2 + \bar{\psi}(\Lambda) \left( \frac{1}{i} \gamma + zm(\Lambda) im \right) \psi + a_3(\Lambda) \bar{\psi} A \psi + a_4(\Lambda) \frac{1}{8} (A^2_\mu)^2 \right]$$

(12)

where the first integral comes from $S_{F,\Lambda}$, and the rest from the interaction part

$$S_{I,\Lambda} \equiv S_\Lambda - S_{F,\Lambda}.$$  

(13)

The cutoff dependence of the coefficients are determined by (7), but their values at a finite momentum scale $\mu$ can be chosen arbitrarily. Hence, the Wilson action is parametrized by seven arbitrary parameters:

$$a_2(\mu), \; z(\mu), \; \bar{z}(\mu), \; z_2(\mu), \; zm(\mu), \; a_3(\mu), \; a_4(\mu).$$

All are dimensionless, except for $a_2(\mu)$ which has mass dimension 2. The WT identity, reviewed in the next section, reduces the number of arbitrary parameters from 7 to 3 \cite{11}.

3. Gauge invariant composite operators

A composite operator $O_\Lambda[A_{\mu}, \psi, \bar{\psi}]$ is a functional which can be regarded as an infinitesimal change of the Wilson action \cite{1, 7}. Hence, $O_\Lambda e^{S_\Lambda}$ satisfies the same ERG differential equation as $e^{S_\Lambda}$:

$$-\Lambda \partial_\Lambda \left( O_\Lambda e^{S_\Lambda} \right) = \int_k \Lambda \left( \frac{k}{\Lambda} \right) \left[ \frac{1}{K \left( \frac{k}{\Lambda} \right)} A_{\mu}(k) \frac{\delta}{\delta A_{\mu}(k)} \right.$$ 

$$+ \frac{1}{k^2} \left( \delta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right) \frac{1}{2} \delta A_{\mu}(k) \delta A_{\nu}(-k) \right] (O_\Lambda e^{S_\Lambda})$$

$$+ \int_p \Lambda \left( \frac{p}{\Lambda} \right) \left[ \frac{1}{K \left( \frac{p}{\Lambda} \right)} \left( O_\Lambda e^{S_\Lambda} \right) \frac{\delta}{\delta \bar{\psi}(p)} \psi(p) 

+ \bar{\psi}(-p) \frac{\delta}{\delta \bar{\psi}(-p)} (O_\Lambda e^{S_\Lambda}) \right] \right.$$ 

$$+ \text{Tr} \frac{\delta}{\delta \bar{\psi}(-p)} (O_\Lambda e^{S_\Lambda}) \frac{\delta}{\delta \bar{\psi}(p)} \left( \frac{1}{i} \gamma + im \right).$$

(14)

This is obtained from (7) by replacing $e^{S_\Lambda}$ by $O_\Lambda e^{S_\Lambda}$. The above equation implies that the correlation functions

$$\langle O_{A_{\mu_1}}(k_1) \cdots A_{\mu_N}(k_L) \bar{\psi}(p_1) \cdots \bar{\psi}(p_N) \bar{\psi}(-q_1) \cdots \bar{\psi}(-q_N) \rangle^\infty$$

$$\equiv \prod_{i=1}^L \frac{1}{K \left( \frac{k_i}{\Lambda} \right)} \prod_{j=1}^N \left[ \frac{1}{K \left( \frac{p_j}{\Lambda} \right)} K \left( \frac{q_j}{\Lambda} \right) \right] \langle O_{A_{\mu_1}}(k_1) \cdots \rangle_{S_\Lambda}$$

$$+ \sum_{i<j} K \left( \frac{k_i}{\Lambda} \right) K \left( \frac{k_j}{\Lambda} \right) \left( \delta_{\mu_i\mu_j} - (1 - \xi) \frac{k_{i\mu} k_{j\mu}}{k_i^2} \right)$$

$$\times \delta(k_i + k_j) \langle O_{A_{\mu_1}}(k_1) \cdots \bar{A}_{\mu_i}(k_i) \cdots A_{\mu_j}(k_j) \cdots \rangle_{S_\Lambda}$$




are independent of the cutoff $\Lambda$.

The simplest examples of composite operators are given by

\[
[A,\mu(k)]_\Lambda \equiv \frac{1}{K} A,\mu(k) + \frac{1 - K (\xi)}{k^2} \left( \delta_{\mu\nu} - (1 - \xi) \frac{k_{\mu} k_{\nu}}{k^2} \right) \frac{\delta S_{\Lambda}}{\delta A_{\nu}(-k)} \tag{16}
\]

\[
[\psi(p)]_\Lambda \equiv \frac{1}{K} \psi(p) + \frac{1 - K (\xi)}{p^2} \frac{\delta}{\delta \psi(-p)} S_{\Lambda} \tag{17}
\]

\[
[\tilde{\psi}(-p)]_\Lambda \equiv \frac{1}{K} \tilde{\psi}(-p) + S_{\Lambda} \frac{\delta}{\delta \tilde{\psi}(p)} \frac{1 - K (\xi)}{p^2 + im}. \tag{18}
\]

These correspond to the respective elementary fields in the sense that

\[
\begin{aligned}
\langle [A,\mu(k)] \ldots \rangle_\infty &= \langle A,\mu(k) \ldots \rangle_\infty \\
\langle [\psi(p)] \ldots \rangle_\infty &= \langle \psi(p) \ldots \rangle_\infty \\
\langle [\tilde{\psi}(-p)] \ldots \rangle_\infty &= \langle \tilde{\psi}(-p) \ldots \rangle_\infty
\end{aligned}
\tag{19}
\]

where the dots represent the same product of elementary fields on both sides; the left-hand sides are defined by (15) while the right-hand sides are defined by (9).

In constructing QED, the gauge invariance plays the most important role. The gauge invariance of QED is realized as the WT identities among the correlation functions. In [11] the WT identity of the Wilson action has been given concisely as the current conservation equation:

\[
k_{\mu} J_{\mu}(k) = e \Phi(k) \tag{20}
\]

where the two composite operators are defined by\(^5\)

\[
J_{\mu}(k) \equiv \frac{\delta S_{\Lambda}}{\delta A_{\mu}(-k)} \tag{21}
\]

\[
\Phi(k) \equiv \int_p K \left( \frac{p}{\Lambda} \right) e^{-S_{\Lambda}} \left[ - \text{Tr}(e^{S_{\Lambda}}[\psi(p + k)]_\Lambda) \frac{\delta}{\delta \psi(p)} + \text{Tr} \frac{\delta}{\delta \tilde{\psi}(-p)} (e^{S_{\Lambda}}[-\tilde{\psi}(p + k)]_\Lambda) \right] \tag{22}
\]

where $J_{\mu}(k)$ defines the electric current. $\Phi(k)$ is an 'equation-of-motion' composite operator whose correlation functions are given exactly by\(^6\)

\[
\langle \Phi(k) A_{\mu_1}(k_1) \ldots \psi(p_1) \ldots \tilde{\psi}(-q_1) \ldots \rangle_\infty
\]

\[
= \sum_i \left( - \langle A_{\mu_i}(k_1) \ldots \psi(p_i + k) \ldots \rangle_\infty + \langle A_{\mu_i}(k_1) \ldots \tilde{\psi}(-q_i + k) \ldots \rangle_\infty \right) \tag{23}
\]

where each $\psi(p_i)$ is replaced by $-\psi(p_i + k)$, and each $\tilde{\psi}(-q_i)$ by $+\tilde{\psi}(-q_i + k)$.

We can make the implication of (20) more transparent by rewriting it as

\[
k_{\mu}^3 k_{\mu_i} [A,\mu_i(k)] = D(k + e \Phi(k) \tag{24}
\]

\(^5\) In [11], $-e \Phi(k)$ is denoted as $\Phi(k)$.

\(^6\) An 'equation-of-motion' composite operator $O_{\Lambda}$ has the property that $O_{\Lambda} e^{S_{\Lambda}}$ is a total derivative with respect to fields.
To obtain this, we consider operators to define their gauge invariance. We introduce the WT identity of a composite section.

The product is defined so that

\[ \langle D(k)A_{\mu_1}(k_1) \cdots A_{\mu_i}(k_l)\psi(p_1) \cdots \rangle^\infty = 0 \]

where \( D(k) \equiv -\frac{\delta S_\Lambda}{\delta A_{\mu}(-k)} \)

is also an equation-of-motion composite operator just like \( \Phi(k) \); it eliminates photon fields one by one:

\[ \langle D(k)A_{\mu_1}(k_1) \cdots A_{\mu_i}(k_l)\psi(p_1) \cdots \rangle^\infty = \sum_{i=1}^L \delta(k + k_i)k_{\mu_i}\langle A_{\mu_1}(k_1) \cdots A_{\mu_{i-1}}(k_{i-1}) A_{\mu_i}(k_i) \cdots A_{\mu_L}(k_L)\psi(p_1) \cdots \rangle^\infty. \]

For the correlation functions of elementary fields, (24) gives immediately

\[ \frac{1}{\xi}k_{\mu}\langle A_{\mu}(k)\rangle^\infty = \frac{1}{k^2} \left[ \sum_i k_{\mu_i} \delta(k + k_i) \left( \cdots A_{\mu_i}(k_i) \right)^\infty \right. \]

\[ \left. + e \sum_i \left\{ - \langle \cdots \psi(p_i) \cdots \rangle^\infty + \langle \cdots \bar{\psi}(-q_i + k) \cdots \rangle^\infty \right\} \right] \]

which is the usual form of the WT identities in QED.

As a consequence of (24) (or equivalently (20)), the number of arbitrary parameters is reduced from 7 to 3, as has been discussed in [11]. We can take the three parameters as

\( z(\mu), z_\pi(\mu), \quad z_m(\mu) \)

corresponding to the freedom of normalizing \( A_\mu, \psi \) and \( \bar{\psi} \), and the mass parameter \( m \). The parameter \( e \) is introduced via the WT identity (24). In general we can choose the above three parameters as arbitrary functions of \( e^2 \) and \( \xi \) (the gauge fixing parameter). In [11] we have chosen these three parameters as zero. Though this choice is practical, it is not a good choice if we wish to control the dependence of \( S_\Lambda \) on \( \xi \). This will be explained in the next section.

Before proceeding to the next section, let us generalize the WT identity for the composite operators to define their gauge invariance. We introduce the WT identity of a composite operator \( O_\Lambda \) by

\[ \frac{1}{\xi}k_{\mu}\langle A_{\mu}(k)O_\Lambda \rangle = \frac{1}{k^2} (D(k) + e\Phi(k)) * O_\Lambda. \]

The left-hand side is a composite operator corresponding to the product of \( A_{\mu}(k) \) with \( O_\Lambda \):

\[ [A_{\mu}(k)O_\Lambda] = [A_{\mu}(k)]_\Lambda O_\Lambda + \frac{1 - K \xi}{k^2} \left( \delta_{\mu\nu} - (1 - \xi) k_{\mu}k_{\nu} \right) \frac{\delta O_\Lambda}{\delta A_{\nu}(-k)} \]

\[ = e^{-S_\Lambda} \left[ \frac{1}{K} A_{\mu}(k) + \frac{1 - K \xi}{k^2} \left( \delta_{\mu\nu} - (1 - \xi) k_{\mu}k_{\nu} \right) \frac{\delta}{\delta A_{\nu}(-k)} \right] \times (e^{S_\Lambda} O). \]

To obtain this, we consider \( [A_{\mu}(k)]_\Lambda e^{S_\Lambda} \), vary \( S_\Lambda \) infinitesimally by \( O_\Lambda \), and divide the result by \( e^{S_\Lambda} \). (See appendix A.3 for the product of a composite operator with an elementary field.) The product is defined so that

\[ \langle [A_{\mu}(k)O] \cdots \rangle^\infty = \langle A_{\mu}(k)O \cdots \rangle^\infty. \]

The star products on the right-hand side of (28) are defined as the following equations-of-motion operators:

\[ D(k) * O_\Lambda \equiv e^{-S_\Lambda} \left( \frac{\delta}{\delta A_{\mu}(-k)} (k_{\mu}O_\Lambda e^{S_\Lambda}) \right) \]

\[ \text{Note } D(k) \text{ of (25) is the same as } D(k) * 1. \text{ Similarly, } \Phi(k) = \Phi(k) * 1. \]
\[ \Phi(k) * O_\lambda \equiv e^{-S_\lambda} \int \frac{d^4p}{(2\pi)^4} K \left( \frac{p}{\lambda} \right) \left[ -\text{Tr}[O\psi(p+k)]_\lambda e^{S_\lambda} \right] \delta \left( \frac{\delta}{\delta \psi(p)} \right) + \text{Tr} \left[ \delta \psi(-p) (O\psi(-p+k))_\lambda e^{S_\lambda} \right]. \]  

As we obtain \([A_\mu(k)O]_\lambda e^{S_\lambda}\) from \([A_\mu(k)]_\lambda e^{S_\lambda}\), we obtain \(e^{S_\lambda}D(k) * O_\lambda\) and \(e^{S_\lambda}\Phi(k) * O_\lambda\) from \(e^{S_\lambda}D(k)\) and \(e^{S_\lambda}\Phi(k)\) by changing \(S_\lambda\) infinitesimally by \(O_\lambda\). The correlation functions of these composite operators are given by

\[ \langle D(k) * O \cdots \rangle \approx = \sum_i \delta(k+k_i)k_i \langle O \cdots \hat{A}_\mu(k) \cdots \rangle \]  

\[ \langle \Phi(k) * O \cdots \rangle \approx = \sum_i \left\{ -\langle O \cdots \psi(p_i + k) \cdots \rangle + \langle O \cdots \hat{\psi}(-q_i + k) \cdots \rangle \right\}. \]

4. Dependence of the action on the gauge fixing parameter \(\xi\)

We are now ready to discuss the main subject of this paper: how the Wilson action and gauge invariant composite operators depend on the gauge fixing parameter \(\xi\). Of course we do not expect any physical quantities to depend on \(\xi\), but we cannot remove \(\xi\)-dependence entirely from either the Wilson action or the gauge invariant composite operators. Fortunately, for QED, we can derive the \(\xi\)-dependence of the correlation functions explicitly.

In this section we consider only the Wilson action and derive the renormalized Landau–Khalatnikov relation that gives the \(\xi\) dependence of the correlation functions [14–17]. In order to present the logical flow clearly, we defer technical details to appendix C.

We first introduce a composite operator \(O_\xi\) that generates an infinitesimal variation of \(\xi\):

\[ O_\xi \equiv -e^{-S_\lambda} \left[ \partial_\xi + \int \frac{d^4k}{(2\pi)^4} \frac{K(\frac{k}{\lambda})}{k^4} \left( k_\mu k_\nu \frac{1}{2} \delta^2 A_\mu(k) \delta A_\nu(-k) \right) e^{S_\lambda}. \]  

We may call \(O_\xi\) a composite operator ‘conjugate to’ \(\xi\). It has the following correlation functions:

\[ \langle O_\xi \cdots \rangle \approx = -\frac{\partial}{\partial \xi} \langle \cdots \rangle \]

where the dots represent a product of elementary fields. The second term of (36) arises due to the \(\xi\) dependence of the photon propagator.

Let us recall that our Wilson action depends not only on \(m, e, \xi\), but also on \(z_m(\mu), \xi(\mu), \xi \xi(\mu)\). In perturbation theory we can expand the latter three in powers of \(e^2\), but the coefficients of the expansions can be given arbitrary \(\xi\)-dependence. We would like to choose

\[ \partial_\xi z_m(\mu), \quad \partial_\xi \xi(\mu), \quad \partial_\xi \xi \xi(\mu) \]
in such a way that $O_\xi$ becomes an equation-of-motion composite operator. Without this choice, $O_\xi$ would mix with the composite operators conjugate to $\epsilon$ and $m$ ($O_\epsilon$ and $O_m$, to be defined in section 6) so that $O_\xi$ would not be an equation-of-motion composite operator. This convenient choice will give us immediately a renormalized Landau–Khalatnikov relation for the correlation functions of the elementary fields.

To motivate the desired rewriting of $O_\xi$, let us first compute it classically. Since the classical action is

$$S_{cl} = -\int d^4x \left[ \frac{1}{2} \left( \partial_\mu A_\nu \right)^2 - \left( 1 - \frac{1}{\xi} \right) \frac{1}{2} \left( \partial_\mu A_\mu \right)^2 + \tilde{\psi} \left( \frac{i}{\xi} \partial + im - eA \right) \psi \right]$$

(38)

we obtain

$$(O_\xi)_{cl} = -\partial_\mu S_{cl} = -\frac{1}{\xi^2} \int d^4x \frac{1}{2} \left( \partial_\mu A_\mu \right)^2.$$ (39)

We would like to find a composite operator that corresponds to this classical expression.

We first try

$$-\frac{1}{\xi^2} \int \frac{1}{2} k_\mu k_\nu [A_\mu(k)A_\nu(-k)]_\Lambda.$$ (40)

This does not work for two reasons. First, it contains the contribution to the vacuum functional integral. Since

$$k_\mu k_\nu [A_\mu(k)A_\nu(k')]^\infty = -\xi \delta(k + k')$$ (41)

we should try instead

$$\int \frac{1}{2 \xi^2} k_\mu k_\nu [A_\mu(k)A_\nu(-k)]_\Lambda + \frac{1}{2 \xi^2} \delta(0)$$ (42)

by subtracting the contribution to the vacuum functional. (This amounts to normal ordering.) Second, the integral over $k$ is UV divergent. Subtracting the divergence, we obtain a proper generalization of (39) as

$$O'_\xi = \int \left[ -\frac{1}{2 \xi^2} k_\mu k_\nu [A_\mu(k)A_\nu(-k)]_\Lambda + \frac{1}{2 \xi^2} \delta(0) - \frac{e^2 f(k/\mu)}{2 k^4 - N_F} \right].$$ (43)

where $f(k)$ is an arbitrary function that approaches 1 as $k^2 \to \infty$, and vanishes for small $k^2$. $N_F$ is an equation-of-motion composite operator defined by

$$N_F = \int_p \left( \frac{p}{\Lambda} \right) e^{-S_A} \left\{ \Tr \left( [\psi(p)]_\Lambda \psi^{\Lambda} \right) \frac{\delta}{\delta \psi(p)} + \Tr \frac{\delta}{\delta \psi(-p)} \left( [\bar{\psi}(-p)]_\Lambda \bar{\psi}^{\Lambda} \right) \right\}.$$ (44)

This counts the number of fermionic fields:

$$[N_F A_{\mu_1}(k_1) \cdots A_{\mu_n}(k_n) \psi(p_1) \cdots \psi(p_N) \bar{\psi}(-q_1) \cdots \bar{\psi}(-q_N)]^\infty$$

$$= 2N [A_{\mu_1}(k_1) \cdots \bar{\psi}(-q_N)]^\infty.$$ (45)

To see the UV finiteness of (43), we need the following equality:

$$-\frac{1}{\xi^2} k_\mu k_\nu [A_\mu(k)A_\nu(-k)]_\Lambda + \frac{1}{\xi^2} \delta(0) = \frac{1}{k^2} (D(-k) + e\Phi(-k)) \ast (D(k) + e\Phi(k)).$$ (46)

A proof is given in the first part of appendix C.

Now, using (46), we can rewrite $O'_\xi$ as

$$O'_\xi = \frac{1}{2} \int \frac{1}{k^2} [(D(k) + e\Phi(k)) \ast (D(-k) + e\Phi(-k)) - e^2 f(k/\mu)N_F].$$ (47)
This is manifestly an equation-of-motion composite operator. To see the UV finiteness of the integral, we evaluate its correlation function with elementary fields:

\[
\langle O'_\xi A_{\mu_1}(k_1) \cdots A_{\mu_L}(k_L) \psi(p_1) \cdots \psi(p_N) \bar{\psi}(-q_1) \cdots \bar{\psi}(-q_N) \rangle^\infty
\]

\[
= - \sum_{i<j} \frac{(k_i)_{\mu_1}(k_j)_{\mu_2}}{k^2_i} \delta(k_i + k_j) \{ \cdots \hat{A}_{\mu_i}(k_i) \cdots A_{\mu_j}(k_j) \cdots \}^\infty
\]

\[
+ e \sum_{i=1}^L \frac{(k_i)_{\mu_1}}{k^2_i} \sum_{j=1}^N \{ \{ \cdots \hat{A}_{\mu_i}(k_i) \cdots \psi(p_j + k_i) \cdots \}^\infty
\]

\[
- \{ \cdots \hat{A}_{\mu_i} \cdots \bar{\psi}(-q_j + k_i) \cdots \}^\infty
\]

\[
+ e^2 \int \frac{1}{k^4} \sum_{i<j} \{ \{ \cdots \psi(p_i + k) \cdots \psi(p_j - k) \cdots \}^\infty
\]

\[
+ \{ \cdots \bar{\psi}(-q_i + k) \cdots \bar{\psi}(-q_j - k) \cdots \}^\infty
\]

\[
- \sum_{i,j} \{ \cdots \psi(p_i + k) \cdots \bar{\psi}(-q_j - k) \cdots \}^\infty + N (1 - f(k/\mu)) \langle \cdots \rangle^\infty \}
\]

(48)

Look at the integrand of the \( k \)-integral without the factor \( \frac{1}{k^2} \). It vanishes as \( k^2 \rightarrow \infty \) except for the last term with the \( k \)-independent correlation function. If we choose \( f(\infty) = 1 \), its coefficient vanishes, and the \( k \)-integral becomes UV finite. We also note that the integrand (without \( \frac{1}{k^2} \)) vanishes at \( k = 0 \) if we choose \( f(0) = 0 \). We have thus justified the counterterm proportional to \( \Lambda_f \) in (43) and (47).

We are left with showing

\[
O_\xi = O'_\xi.
\]

(49)

This is not valid in general. The equality requires tuning the \( \xi \)-dependence of \( z_m(\mu), z(\mu), \) and \( z_F(\mu) \). We give the details in the second part of appendix C, where we first note that the difference \( O_\xi - O'_\xi \) is gauge invariant so that it has three parameters just as the Wilson action. The parameters can be taken as \( \delta z_m(\mu), \delta z(\mu), \delta z_F(\mu) \). Hence, we can satisfy (49) by tuning these parameters. For the reader’s convenience, we give the 1-loop results for \( z_m(\mu), z(\mu), z_F(\mu) \) in appendix F.

Given (37) and (49), the right-hand side of (48) gives \( -\partial_\xi \) of the correlation function. This is the renormalized version of the Landau–Khalatnikov relation, usually given for unrenormalized correlation functions [14].

5. Dependence of the gauge invariant composite operators on \( \xi \)

In the previous section we have shown that the Wilson action can be made to satisfy (49). We obtain the definition of \( \xi \)-independent composite operators by taking an infinitesimal variation of this equality.

We first recall the definition of \( O_\xi \) given by (36). Consider \( e^{S_\xi}(-O_\xi) \), and replace \( e^{S_\xi} \) by \( O_\Lambda e^{S_\xi} \). Dividing the result by \( e^{S_\xi} \), we obtain

\[
d_\xi O_\Lambda \equiv e^{-S_\xi} \left[ \partial_\xi + \int_k \frac{K (\frac{k^2}{4}) (K (\frac{k^2}{4}) - 1)}{k^4} k_\mu k_\nu \frac{1}{2} \delta A_{\mu}(k) \bar{\delta} A_{\nu}(-k) \right] (O_\Lambda e^{S_\xi}).
\]

(50)

This is a composite operator whose correlation functions with elementary fields are given by

\[
\langle d_\xi O \cdots \rangle^\infty = \partial_\xi \langle O \cdots \rangle^\infty.
\]

(51)
Note that we can write
\[ O_\xi = -d_\xi 1. \]  

We next recall the alternative definition (47) of \( O'_\xi \). Consider \( e^{5_\xi}(-O'_\xi) \) and replace \( e^{5_\xi} \) by \( O_\Lambda e^{5_\Lambda} \). Dividing the result by \( e^{5_\Lambda} \), we obtain
\[ d'_\xi O_\Lambda \equiv -\frac{1}{2} \int_k \frac{1}{k^4} \left[ (D(k) + e\Phi(k)) * (D(-k) + e\Phi(-k)) - e^2 f(k/\mu)N_F \right] * O_\Lambda \]  
where we define
\[ N_F * O_\Lambda \equiv \int_p K \left( \frac{p}{\Lambda} \right) e^{-S_F} \left\{ \text{Tr}(\bar{[O\psi(p)]}_\Lambda e^{5_\Lambda}) \frac{\delta}{\delta \bar{\psi}(p)} + \text{Tr} \frac{\delta}{\delta \psi(-p)}([\bar{\psi}(-p)O_\Lambda e^{5_\Lambda}] \right\}. \]  
The latter has the correlation function
\[ \langle N_F * O \cdots \rangle_\infty = 2N\langle O \cdots \rangle_\infty \]  
where \( N \) is the number of \( \psi \)'s (equivalently \( \bar{\psi} \)'s) contained in the dots. Note that we can write
\[ O'_\xi = -d'_\xi 1 \]  
\( d'_\xi O_\Lambda \) is an equation-of-motion composite operator with the correlation functions:
\[ -\langle d'_\xi O \cdots \rangle_\infty = -\sum_{i,j} \frac{\langle k_i \rangle_{\mu_i} \langle k_j \rangle_{\mu_j}}{k_i^2} \delta(k_i + k_j) \langle O \cdots A_{\mu_i}(k_i) \cdots \hat{A}_{\mu_j}(k_j) \cdots \rangle_\infty \]
\[ + e \sum_{i=1}^N \frac{\langle k_i \rangle_{\mu_i}}{k_i^2} \sum_{j=1}^N \left\{ \langle O \cdots \hat{A}_{\mu_i}(k_i) \cdots \psi(p_j + k_i) \cdots \rangle_\infty \right\} \]
\[ - \langle O \cdots \hat{\psi}(-q_j + k_i) \cdots \rangle_\infty \]
\[ + e^2 \int_k \frac{1}{k^4} \left[ \sum_{i,j} \left\{ \langle O \cdots \psi(p_i + k) \cdots \psi(p_j - k) \cdots \rangle_\infty \right\} \right. \]
\[ - \sum_{i,j} \left\{ \langle O \cdots \psi(p_i + k) \cdots \bar{\psi}(-q_j - k) \cdots \rangle_\infty \right\} \]
\[ + N(1 - f(k/\mu)) \langle O \cdots \rangle_\infty \right\}. \]  

We wish to define ‘\( \xi \)-independent’ composite operators such that \( d'_\xi = d_\xi \) acting on them. With the notation
\[ D_\xi \equiv d_\xi - d'_\xi \]  
we then define the ‘\( \xi \)-independence’ of a composite operator \( O_\Lambda \) by
\[ D_\xi O_\Lambda = 0. \]  
In this notation, (49) is given by
\[ D_\xi 1 = 0. \]  

In the remainder of this section, we wish to show that we can make any gauge invariant composite operator \( O_\Lambda \) ‘\( \xi \)-independent’ by taking an appropriate linear combination with other gauge invariant composite operators.

Let \( O_{\Lambda i} (i = 1, \ldots, n) \) be gauge invariant composite operators satisfying the WT identity (28). We assume that this set is closed in the sense that any gauge invariant composite operator
with the same dimension and conserved quantum numbers can be given as a linear combination of these \( n \) composite operators. Now, in appendix D, we show that \( D_\xi O_\Lambda \) satisfies the WT identity, if \( O_\Lambda \) does. Hence, \( D_\xi O_\Lambda \) must be a linear combination of these \( n \) gauge invariant composite operators:

\[
D_\xi O_\Lambda = \sum_{j=1}^{n} C_{ij}(e^2, \xi) O_{\Lambda j} \tag{61}
\]

where \( C_{ij} \) are functions of \( e^2 \) and \( \xi \), independent of the cutoff \( \Lambda \). Let

\[
O'_{\Lambda i} \equiv \sum_{j=1}^{n} Z_{ij}(e^2, \xi) O_{\Lambda j} \quad (i = 1, \ldots, n)
\]

be a new basis of gauge invariant composite operators. We find

\[
D_\xi O'_{\Lambda i} = \sum_{j=1}^{n} \left( \partial_\xi Z_{ij}(e^2, \xi) + \sum_{k} Z_{ik}(e^2, \xi) C_{kj}(e^2, \xi) \right) O_{\Lambda j}. \tag{63}
\]

For an arbitrary initial condition \( Z_{ij}(e^2, 0) \), we can solve the homogeneous equations

\[
\partial_\xi Z_{ij}(e^2, 0) + \sum_{k} Z_{ik}(e^2, \xi) C_{kj}(e^2, \xi) = 0. \tag{64}
\]

Thus, \( O'_{\Lambda i} \) with these coefficients are the desired composite operators, that are both gauge invariant and \( \xi \)-independent.

6. RG equations for QED

In this and the next sections we wish to show that the anomalous dimension of a gauge invariant and \( \xi \)-independent composite operator is independent of \( \xi \). Since no discussion of the \( \mu \)-dependence of the Wilson action of QED seems available in the literature, we devote this section for its discussion, rather than giving it in an appendix. Then, in the next section, we introduce anomalous dimensions of composite operators and derive the desired result.

The Wilson action depends not only on the UV cutoff \( \Lambda \), but also on the renormalization scale \( \mu \). Since \( -\mu \partial_\mu S_\Lambda \) can be regarded as an infinitesimal change of \( S_\Lambda \), it satisfies the WT identity:

\[
1/\xi [A_\mu(k)(-\mu \partial_\mu S_\Lambda)]_\Lambda = 1/k^2 (D(k) + e\Phi(k)) * (-\mu \partial_\mu S_\Lambda). \tag{65}
\]

Thus, \( -\mu \partial_\mu S_\Lambda \) is a gauge invariant composite operator.

Since the WT identity leaves the Wilson action with three degrees of freedom, we expect to find three gauge invariant composite operators.

(i) The composite operator conjugate to the mass parameter \( m \) is given by

\[
O_m \equiv e^{-S_\Lambda} \left[ \partial_m e^{S_\Lambda} + \int p K \left( \frac{p}{\Lambda} \right) K \left( \frac{p}{\Lambda} \right) - 1 \right] \text{Tr} \left[ \frac{\delta}{\delta \bar{\psi}(-p)} e^{S_\Lambda} \frac{\delta}{\delta \psi(p)} \left( \hat{p} + im \right)^2 \right]. \tag{66}
\]

This has the following correlation functions:

\[
\langle O_m \cdots \rangle^\infty = -\partial_m \langle \cdots \rangle^\infty \tag{67}
\]

where the dots denote a product of elementary fields.
(ii) The fermion counting operator $N_F$ is given by (44).

(iii) The third gauge invariant composite operator is given by

$$N_A - eO_e + 2\xi O_\xi$$

where $O_\xi$, the composite operator conjugate to $\xi$, is defined by (36). Let us define the other two: $N_A$ is the photon counting operator

$$N_A \equiv -\int_k K \left( \frac{k}{A} \right) e^{-S_{\Lambda}} \frac{\delta}{\delta A_\mu(k)} \left( [A_\mu(k)]_\Lambda e^{S_{\Lambda}} \right)$$

and $O_e$ is the composite operator conjugate to $e$

$$O_e \equiv -\partial_e S_{\Lambda}$$

$N_A$ and $O_e$ have the correlation functions

$$\langle N_A \cdots \rangle_\infty = L \langle \cdots \rangle_\infty$$

$$\langle O_e \cdots \rangle_\infty = -\partial_e \langle \cdots \rangle_\infty$$

where $L$ is the number of elementary photon fields in the dots.

The WT identities for $O_m$ and $N_F$ can be derived most easily from the WT identity (27) given for the correlation functions. Differentiating (27) with respect to $-m$, we obtain

$$\frac{1}{\xi} k_\mu \langle A_\mu(k) O_m \cdots \rangle_\infty = \frac{1}{\xi^2} \left[ \sum_{i=1}^L k_\mu \delta(k + k_i) \langle O_m \cdots \tilde{A}_{\mu}(k_i) \cdots \rangle_\infty \right. \\
+ e \sum_i \left\{ -\langle O_m \cdots \tilde{\psi}(p_i + k) \cdots \rangle_\infty + \langle O_m \cdots \bar{\psi}(-q_i + k) \cdots \rangle_\infty \right\} \right].$$

This implies the WT identity for $O_m$. Similarly, we can show the gauge invariance of $N_F$. See appendix E for the WT identity satisfied by the linear combination $N_A - eO_e + 2\xi O_\xi$.

Since there are only three composite operators that are gauge invariant, we must find a linear relation

$$-\mu \partial_\mu S_\Lambda = m\beta_m O_m + \gamma_F N_F + \gamma_A (N_A - eO_e + 2\xi O_\xi)$$

where $\beta_m$, $\gamma_F$, $\gamma_A$ are functions of $e^2$ and $\xi$. This implies nothing but the trace anomaly, giving rise to the usual RG equation for the correlation functions:

$$(-\mu \partial_\mu + \beta \partial_\mu + \beta_\xi \partial_\xi + m\beta_m \partial_m - L\gamma_A - 2N\gamma_F) \\
\times \langle A_{\mu_1}(k_1) \cdots A_{\mu_L}(k_L) \tilde{\psi}(p_1) \cdots \tilde{\psi}(p_N) \bar{\psi}(-q_1) \cdots \bar{\psi}(-q_N) \rangle_\infty = 0$$

where $\beta_m$ is the anomalous dimension of $m$, and $\gamma_A$, $\gamma_F$ are the anomalous dimensions of the photon and fermion fields. Note that the beta functions of $e$ and $\xi$ are given in terms of the anomalous dimension $\gamma_A$ by

$$\beta = -e\gamma_A$$

$$\beta_\xi = 2\xi \gamma_A.$$
7. ξ-independence of the anomalous dimensions

In this section, we wish to show that the anomalous dimension of a composite operator that is both gauge invariant and ξ-independent does not depend on ξ.

Let us first introduce the anomalous dimension for an arbitrary composite operator \( O_\Lambda \).

We define a new composite operator by

\[
\varrho \equiv (-\mu d_\mu + m \beta_m d_m + \beta d_e + \beta_\xi d_\xi - \gamma_F N_F \ast - \gamma_A N_A \ast) O_\Lambda
\]

(78)

where we define

\[
e^{S_\Lambda} \varrho \equiv \partial_\mu (e^{S_\Lambda} O_\Lambda)
\]

(79)

\[
e^{S_\Lambda} d_\mu O_\Lambda \equiv k \left( \frac{p}{\Lambda} \right) \left( k \left( \frac{p}{\Lambda} \right) - 1 \right) \operatorname{Tr} \frac{\delta}{\delta \psi (-p)} \left( e^{S_\Lambda} O_\Lambda \right) \frac{\delta}{\delta \bar{\psi} (p)} \left( \bar{p} + i m \right)^2
\]

(80)

\[
e^{S_\Lambda} d_m O_\Lambda \equiv \partial_m (e^{S_\Lambda} O_\Lambda)
\]

(81)

so that

\[
\langle d_\mu O \cdots \rangle_\infty = \partial_\mu \langle O \cdots \rangle_\infty
\]

(82)

\[
\langle d_m O \cdots \rangle_\infty = \partial_m \langle O \cdots \rangle_\infty
\]

(83)

\[
\langle d_e O \cdots \rangle_\infty = \partial_e \langle O \cdots \rangle_\infty
\]

(84)

Hence, we can write

\[
d_\mu 1 = \partial_\mu S_\Lambda, \quad d_m 1 = -\varrho_m, \quad d_e 1 = -\varrho_e.
\]

(85)

Using this notation, we can give the trace anomaly (74) as

\[
d_1 1 = 0.
\]

(86)

To see that \( d_1 \) defines the anomalous dimension of a composite operator, let us suppose \( \{ O_{\Lambda i} \}_{i=1,\ldots,N} \) are a basis of composite operators that are both gauge invariant and ξ-independent. The composite operators mix under the change of renormalization scale \( \mu \):

\[
d_1 O_{\Lambda i} = \sum_{j=1}^{N} \gamma_{ij} O_{\Lambda j}
\]

(87)

where \( \gamma_{ij} \) are functions of \( e^2, \xi \). This implies the RG equations

\[
(-\mu \partial_\mu + \beta_\mu d_\mu + \beta_\xi d_\xi + \beta \varrho_m d_m - L_{\gamma_A} - 2N \gamma_F) \langle O_\cdots \rangle_\infty = \sum_j \gamma_{ij} \langle O_j \cdots \rangle_\infty
\]

(88)

where the dots stand for products of elementary fields.

In appendix H we show that \( d_1 O_{\Lambda i} \) are both gauge invariant and ξ-independent. Assuming this result, we obtain

\[
\sum_j D_\xi (\gamma_{ij} O_{\Lambda j}) = \sum_j (\partial_\xi \gamma_{ij} \cdot O_{\Lambda j} + \gamma_{ij} D_\xi O_{\Lambda j}) = 0.
\]

(89)

Since \( D_\xi O_{\Lambda j} = 0 \), this gives

\[
\partial_\xi \gamma_{ij} = 0,
\]

(90)

Thus, the anomalous dimensions are ξ-independent.
8. The axial anomaly

The axial anomaly is a linear relation among the four gauge invariant pseudoscalar composite operators:

\[ k_\mu J_\mu (k), \quad J_5 (k), \quad \left[ 1 + \frac{1}{4} F \tilde{F} \right] (k), \quad \Phi_5 (k). \]

Except for the last one, which is completely determined by the Wilson action, the operators must be carefully defined. Since \( e^2 \xi \) is an RG invariant, specifying the anomalous dimensions of these composite operators leaves their normalization with arbitrary dependence on \( e^2 \xi \). Demanding \( \xi \)-independence, however, specifies them uniquely only up to constant normalization. Thus, we demand these composite operators be both gauge invariant and \( \xi \)-independent. (This is automatic for \( \Phi_5 \).)

(i) Axial vector current \( J_\mu (k) \)—The asymptotic behavior for large cutoff \( \Lambda \) is given by

\[
J_\mu (k) \xrightarrow{\Lambda \to \infty} a_3 (\Lambda) \int_p \bar{\psi} (p - k) \gamma_5 \gamma^\mu \psi (p) + a_5 (\Lambda) \varepsilon_{\mu \alpha \beta \gamma} \int_l A_\alpha (k - l) l_\beta A_\gamma (l)
\]

where

\[
a_3 (\Lambda) = 1 + O (e^2), \quad a_5 (\Lambda) = O (e^2).
\]

The tree level value \( a_3^{(0)} = 1 \) is a normalization condition. The vanishing of the anomalous dimension specifies the axial vector current unambiguously.

(ii) Pseudoscalar \( J_5 (k) \)—This is determined uniquely by the vanishing of the anomalous dimension of \( m J_5 (k) \). The asymptotic behavior is given by

\[
J_5 (k) \xrightarrow{\Lambda \to \infty} j (\Lambda) \int_p \bar{\psi} (p - k) \gamma_5 \psi (p)
\]

where \( j (\Lambda) \) is normalized as

\[
j (\Lambda) = 1 + O (e^2).
\]

(iii) FF dual \( \left[ \frac{1}{2} F \tilde{F} \right] (k) \)—This is determined uniquely by the vanishing of the anomalous dimension of \( e^2 \left[ \frac{1}{2} F \tilde{F} \right] (k) \). The asymptotic behavior is given by

\[
\left[ \frac{1}{4} F \tilde{F} \right] \xrightarrow{\Lambda \to \infty} f_3 (\Lambda) \int_p \bar{\psi} (p - k) \gamma_5 k \psi (p) + f_5 (\Lambda) k_\mu \varepsilon_{\mu \alpha \beta \gamma} \int_l A_\alpha (k - l) l_\beta A_\gamma (l)
\]

where the normalization condition is

\[ f_3 (\Lambda) = 1 + O (e^2). \]

Somewhat unexpectedly, \( f_3 \) is also of order 1. If \( f_3 \) were of order \( e^2 \), the operator would mix with \( k_\mu J_\mu (k) \) under \( d_e \).

(iv) Equation-of-motion \( \Phi_5 (k) \) is defined by

\[
\Phi_5 (k) \equiv - \int_p K \left( \frac{p}{\Lambda} \right) e^{-5_5} \left[ \text{Tr} (e^{5_5} \gamma_5 [\psi (p - k) \Lambda] \frac{\delta}{\delta \bar{\psi} (p)} \right] + \text{Tr} \frac{\delta}{\delta \bar{\psi} (-p)} (e^{5_5} [\psi (p - k) \Lambda] \gamma_5 )
\]

We choose \( \epsilon_{1234} = 1 \).
\begin{align}
\frac{\Lambda}{\Lambda_1} \to \infty & \quad \phi'_j(\Lambda) \int_p \psi(-p-k) \gamma_5 \bar{k} \psi(p) + \phi_5(\Lambda) \epsilon_{\mu \alpha \beta \gamma} \int_I A_\mu(-k-l) \epsilon_\beta A_\gamma(l) \\
& + \phi(\Lambda) \int_p \psi(-p-k) \bar{2} i m \gamma_5 \psi(p). \tag{98}
\end{align}

By definition, this is both gauge invariant and \( \xi \)-independent. It has the following correlation functions:

\begin{equation}
(\Phi_5(-k) \cdots)_{\infty} = - \sum_i \{ \langle \cdots \gamma_5 \psi(p_i-k) \cdots \rangle_{\infty} + \langle \cdots \bar{\psi}(-q_i-k) \gamma_5 \cdots \rangle_{\infty} \}. \tag{99}\end{equation}

We define the axial anomaly by

\begin{equation}
A(-k) \equiv k_\mu J_5^\mu(-k) - 2 i m J_5^\mu(-k) - \Phi_5(-k). \tag{100}\end{equation}

This is both gauge invariant and \( \xi \)-independent, and its anomalous dimension is zero. Since it vanishes at tree level, it must be proportional to \( e^2 \). The only possibility is a constant multiple of \( e^2 \left[ \frac{1}{4} \tilde{F} \tilde{F} \right]_{\Lambda_1}(-k) \):

\begin{equation}
A(-k) = \text{const.} \frac{e^2}{(4\pi)^2} \left[ \frac{1}{4} \tilde{F} \tilde{F} \right]_{\Lambda_1}(-k). \tag{101}\end{equation}

This is equivalent to

\begin{align}
\phi'_j(\Lambda) - \phi'_j(\Lambda) &= \text{const.} \frac{e^2}{(4\pi)^2} f_5(\Lambda) \tag{102} \\
\phi_5(\Lambda) - \phi_5(\Lambda) &= \text{const.} \frac{e^2}{(4\pi)^2} f_5(\Lambda) \tag{103} \\
j(\Lambda) + \phi(\Lambda) &= 0. \tag{104}\end{align}

The constant is determined by the 1-loop calculation, given in appendix I, as

\begin{equation}
\text{const.} = -4. \tag{105}\end{equation}

Hence, we obtain

\begin{equation}
A(-k) = -4 \frac{e^2}{(4\pi)^2} \left[ \frac{1}{4} \tilde{F} \tilde{F} \right]_{\Lambda_1}(-k). \tag{106}\end{equation}

### 9. Concluding remarks

We hope we have convinced the reader of the nice compatibility of QED with ERG. The main tool provided by ERG is composite operators which are introduced as infinitesimal variations of the Wilson action. We have shown in details how to formulate gauge invariance and gauge fixing parameter \( \xi \)-dependence of composite operators using the ERG formalism.

Originally we tried to formulate the \( \xi \)-dependence of QED by introducing a free scalar field (Stückelberg field). Though it is straightforward to do this for the unrenormalized QED [20], we have not been able to do so for the renormalized QED. Once the renormalized Landau–Khalatnikov relation (49) is obtained for the Wilson action, it is easy to guess the \( \xi \)-independence (57) for the correlation functions of physical composite operators. Expressing the \( \xi \)-independence as an operator equation \( D_\xi \mathcal{O}_{\Lambda} = 0 \) has taken us some efforts.

We expect that most results obtained for QED can be generalized to YM theories as formulated via ERG. Equation-of-motion composite operators will be replaced by BRST exact operators for YM theories.
Acknowledgments

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Appendix A. ERG in a nutshell

To make the paper self-contained, we would like to give a short summary of the ERG formulation applied perturbatively to the $\phi^4$ theory. See [1] for further details.

A.1. Wilson action

The Wilson action is a functional $S_\Lambda[\phi]$ of the Fourier transform $\phi(p)$ of a real scalar field in the four dimensional Euclidean space [4]. We define the correlation functions by the functional integrals:

$$\langle \cdots \rangle_{S_\Lambda} \equiv \int [d\phi] \cdots e^{S_\Lambda} / \int [d\phi] e^{S_\Lambda}. \quad (A.1)$$

The Wilson action splits into the free and interaction parts:

$$S_\Lambda[\phi] = S_{F,\Lambda}[\phi] + S_{I,\Lambda}[\phi]. \quad (A.2)$$

The free part defined by

$$S_{F,\Lambda}[\phi] = \int_p \frac{p^2 + m^2}{K \left( \frac{p}{\Lambda} \right)} \frac{1}{2} \phi(-p)\phi(p) \quad (A.3)$$
gives the propagator

$$\langle \phi(p)\phi(p') \rangle_{S_{F,\Lambda}} = \frac{K \left( \frac{p}{\Lambda} \right)}{p^2 + m^2} \cdot \delta(p+p'). \quad (A.4)$$

(We write $\delta(p)$ for $(2\pi)^4\delta(4)(p)$.) We choose the cutoff function $K \left( \frac{p}{\Lambda} \right)$ such that it is 1 for $p^2 < \Lambda^2$, and it decays fast enough for large momenta so that the correlation functions are all UV finite.

The dependence of the Wilson action on the UV momentum cutoff $\Lambda$ is given by the ERG differential equation: [5]

$$-\Lambda \frac{d}{d\Lambda} e^{S_\Lambda} = \int \Delta \left( \frac{p}{\Lambda} \right) \left[ \frac{p^2 + m^2}{K \left( \frac{p}{\Lambda} \right)} \phi(p) \frac{\delta}{\delta\phi(p)} + \frac{1}{2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right] e^{S_\Lambda} \quad (A.5)$$

where

$$\Delta \left( \frac{p}{\Lambda} \right) \equiv \Lambda \frac{\partial}{\partial\Lambda} K \left( \frac{p}{\Lambda} \right). \quad (A.6)$$

This equation guarantees the $\Lambda$ independence of

$$\langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_\Lambda} \equiv \prod_{i=1}^n \frac{1}{K \left( \frac{p_i}{\Lambda} \right) \left( K \left( \frac{p_i}{\Lambda} \right) - 1 \right) \frac{p_i^2 + m^2}{p_i^2 + m^2}} \delta(p_i + p_j) \phi(p_i) \phi(p_j) \cdots \langle \cdots \rangle_{S_\Lambda} + \cdots \quad (A.7)$$

where the dots correspond to terms for which two or more pairs of $\phi$'s are replaced by

$$\frac{K \left( \frac{p_i}{\Lambda} \right) \left( K \left( \frac{p_i}{\Lambda} \right) - 1 \right) \frac{p_i^2 + m^2}{p_i^2 + m^2}}{p_i^2 + m^2} \delta(p_i + p_j). \quad (A.8)$$
infinitesimal variations of the Wilson action. Hence, its

The coefficients are expanded in powers of

Composite operators

Alternatively, we can specify the asymptotic behavior of $S_\lambda$ for large $\Lambda$:

The existence of a perturbative solution of this type, characterized by the vanishing of higher dimensional terms, amounts to perturbative renormalizability of the theory [19].

The $\Lambda$ dependence of $a_2, z, a_4$ is determined by (A.5). Their values at an arbitrary scale $\mu$ parametrize the Wilson action:

If we choose a convention

we obtain

up to 1-loop.

Composite operators

Composite operators $\mathcal{O}_\lambda[\phi]$ are $\Lambda$-dependent functionals that can be interpreted as infinitesimal variations of the Wilson action. Hence, its $\Lambda$ dependence is given by

which is obtained from (A.5) by changing $S_\lambda$ infinitesimally by $\Lambda$. Corresponding to (A.7), the cutoff independent correlation functions are given by

The simplest example of a composite operator is

This gives the same correlation functions as the elementary field $\phi(p)$:

9 The coefficients are expanded in powers of $m^2/\Lambda^2$ and momenta divided by $\Lambda$. 
Given a composite operator $O_{\Lambda}$, its product with an elementary field $\phi(p)$ is given by the composite operator

$$[O\phi(p)]_{\Lambda} \equiv O_{\Lambda}[\phi(p)]_{\Lambda} + \frac{1 - K \left( \frac{p}{\Lambda} \right)}{p^2 + m^2} \frac{\delta}{\delta \phi(-p)} O_{\Lambda}$$

$$= e^{-S_{\Lambda}} \left( \frac{1}{K \left( \frac{p}{\Lambda} \right)} \phi(p) + \frac{1 - K \left( \frac{p}{\Lambda} \right)}{p^2 + m^2} \frac{\delta}{\delta \phi(-p)} \right) (O_{\Lambda} e^{S_{\Lambda}}). \tag{A.16}$$

As the second expression shows, $e^{S_{\Lambda}}[O\phi(p)]_{\Lambda}$ is obtained from $e^{S_{\Lambda}}[\phi(p)]_{\Lambda}$ by changing $S_{\Lambda}$ infinitesimally by $O_{\Lambda}$. We find

$$\langle [O\phi(p)]_{\Lambda} \phi(p_1) \cdots \phi(p_n) \rangle_\infty = \langle O \phi(p) \phi(p_1) \cdots \phi(p_n) \rangle_\infty. \tag{A.17}$$

Given an arbitrary composite operator $O_{\Lambda}(p)$ dependent on a momentum $p$, we can define a corresponding equation-of-motion composite operator by

$$O_{\Lambda}' \equiv - \int_p K \left( \frac{p}{\Lambda} \right) e^{-S_{\Lambda}} \frac{\delta}{\delta \phi(p)} (e^{S_{\Lambda}} O_{\Lambda}(p)). \tag{A.18}$$

In the correlation function with elementary fields, $O_{\Lambda}'$ replaces each $\phi(p_i)$ by $O_{\Lambda}(p_i)$:

$$\langle O' \phi(p_1) \cdots \phi(p_n) \rangle_\infty = \sum_{i=1}^n \langle \phi(p_1) \cdots O(p_i) \cdots \phi(p_n) \rangle_\infty. \tag{A.19}$$

We can rewrite the definition of the composite operator $[\phi(p)]_{\Lambda}$ as the operator equation

$$(p^2 + m^2)[\phi(p)]_{\Lambda} - \frac{\delta S_{\Lambda}}{\delta \phi(-p)} = -K \left( \frac{p}{\Lambda} \right) \frac{\delta S_{\Lambda}}{\delta \phi(-p)} \tag{A.20}$$

where the right-hand side is an equation-of-motion composite operator (A.18) with $O_{\Lambda}(q) = \delta(p - q)$.

### A.4. Trace anomaly—beta functions and anomalous dimension

The logarithmic derivative $-\mu \partial_\mu S_{\Lambda}$ is a composite operator corresponding to the trace anomaly [18]. Since it is a dimension 4 operator, we can expand it as

$$-\mu \partial_\mu S_{\Lambda} = \beta(\lambda) O_{\Lambda} + \beta_m(\lambda, m^2) O_m + \gamma(\lambda) N$$

where

$$O_{\Lambda} \equiv -e^{-S_{\Lambda}} \partial_\mu e^{S_{\Lambda}}$$

$$O_m \equiv -e^{-S_{\Lambda}} \left[ \partial_{m^2} - \frac{1}{p^2 + m^2} \int_p \frac{K(K-1)}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right] e^{S_{\Lambda}} \tag{A.22}$$

$$N \equiv - \int_p K \left( \frac{p}{\Lambda} \right) e^{-S_{\Lambda}} \frac{\delta}{\delta \phi(p)} (e^{S_{\Lambda}}[\phi(p)]) \tag{A.23}$$

are the three independent composite operators with dimension up to 4. The second term of (A.23) is necessary because of the $m^2$ dependence of (A.8). The $\Lambda$ independent correlation functions of these operators are given by

$$\langle O_{\Lambda} \phi(p_1) \cdots \phi(p_n) \rangle_\infty = -\partial_\lambda \langle \phi(p_1) \cdots \phi(p_n) \rangle_\infty \tag{A.24}$$

$$\langle O_m \phi(p_1) \cdots \phi(p_n) \rangle_\infty = -\partial_m \langle \phi(p_1) \cdots \phi(p_n) \rangle_\infty \tag{A.25}$$

$$\langle N \phi(p_1) \cdots \phi(p_n) \rangle_\infty = n \langle \phi(p_1) \cdots \phi(p_n) \rangle_\infty. \tag{A.26}$$

(A.27)
Thus, (A.21) gives
\[
-\mu \partial_\mu + \beta(\lambda) \partial_\lambda + \beta_m(\lambda, m^2) \partial_{m^2} - n \gamma(\lambda) (\phi(p_1) \cdots \phi(p_n))^{\infty} = 0 \tag{A.28}
\]
which is a standard RG equation, where \(\beta\) and \(\beta_m\) are the beta functions of \(\lambda\) and \(m^2\), and \(\gamma\) is the anomalous dimension of \(\phi\).

Up to 1-loop, we obtain\(^\text{10}\)
\[
\begin{cases}
\beta_m = -m^2 \frac{\lambda}{(4\pi)^2} + \frac{1}{2} \mu^2 \int \frac{\Delta_p}{p^2} \\
\beta = -3 \frac{\lambda^2}{(4\pi)^2} \\
\gamma = 0.
\end{cases}
\tag{A.29}
\]

### Appendix B. Summary of notations for composite operators

Given a composite operator \(O_{\Lambda}\), there are three ways of generating more composite operators.

#### B.1. Products with elementary fields

\[
[A_\mu(k)O]_\Lambda \equiv e^{-S_\Lambda} \left( \frac{A_\mu(k)}{K \left( \frac{k}{\Lambda} \right)} + \frac{1 - K \left( \frac{k}{\Lambda} \right)}{k^2} \left( \delta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right) \delta \Delta_\lambda(-k) \right) \tag{B.1}
\]

\[
[\psi(p)O]_\Lambda \equiv e^{-S_\Lambda} \left( \frac{\psi(p)}{K \left( \frac{p}{\Lambda} \right)} + \frac{1 - K \left( \frac{p}{\Lambda} \right)}{p + im} \delta \bar{\psi}(-p) \right) \tag{B.2}
\]

\[
[\bar{\psi}(-p)O]_\Lambda \equiv (O_{\Lambda} e^{S_\Lambda}) \left( \frac{\bar{\psi}(-p)}{K \left( \frac{p}{\Lambda} \right)} + \frac{\delta}{\delta \bar{\psi}(p)} \frac{1 - K \left( \frac{p}{\Lambda} \right)}{p + im} \right) e^{-S_\Lambda}. \tag{B.3}
\]

#### B.2. Derivatives

\[
d_\mu O_\Lambda \equiv e^{-S_\Lambda} \partial_\mu (e^{S_\Lambda} O_{\Lambda}) \tag{B.4}
\]

\[
d_\xi O_\Lambda \equiv e^{-S_\Lambda} \partial_\xi (e^{S_\Lambda} O_{\Lambda}) \tag{B.5}
\]

\[
d_m O_\Lambda \equiv e^{-S_\Lambda} \left[ \partial_m (e^{S_\Lambda} O_{\Lambda}) + \int \frac{1}{p} K \left( \frac{p}{\Lambda} \right) \left( K \left( \frac{p}{\Lambda} \right) - 1 \right) \delta \Delta_\lambda(-k) \right]
\]

\[
d_x O_{\Lambda} \equiv e^{-S_\Lambda} \left[ \partial_x + \int \frac{1}{k} K \left( \frac{k}{\Lambda} \right) \left( K \left( \frac{k}{\Lambda} \right) - 1 \right) k_\mu k_\nu \frac{\delta^2}{2 \delta A_\mu(k) \delta A_\nu(-k)} \right] (e^{S_\Lambda} O_{\Lambda}). \tag{B.7}
\]

\(^{10}\) Note that \(\beta_m\) is not proportional to \(m^2\). For mass independence, we must require \(a_2(\Lambda)\) not to have terms proportional to \(\mu^2\).
Appendix C. Proof of \( O_{\xi} = O^\prime_{\xi} \)

C.1. Proof of (46).

Using the WT identity (24), we obtain
\[
- \frac{1}{\xi^2} k_\mu k_\nu [A_{\mu}(k)A_{\nu}(-k)]_\Lambda = \frac{1}{\xi^2} k_\nu [A_{\mu}(k)(D(-k) + e\Phi(-k))]_\Lambda.
\]  

\[\text{(C.1)}\]

Using the definitions (22, 25) and (31, 32), we obtain
\[
[A_{\mu}(k)D(-k)]_\Lambda = D(-k) * [A_{\mu}(k)]_\Lambda - k_\mu \delta(0)
\]
\[\text{(C.2)}\]

\[
[A_{\mu}(k)\Phi(-k)]_\Lambda = \Phi(-k) * [A_{\mu}(k)]_\Lambda.
\]
\[\text{(C.3)}\]

Hence, we obtain
\[
- \frac{1}{\xi^2} k_\mu k_\nu [A_{\mu}(k)A_{\nu}(-k)]_\Lambda = \frac{1}{k^4} (D(-k) + e\Phi(-k)) * \frac{1}{\xi} k_\mu [A_{\mu}(k)]_\Lambda - \frac{1}{\xi} \delta(0)
\]
\[\text{\rule{1cm}{.4pt}}\]
\[
= \frac{1}{k^4} (D(-k) + e\Phi(-k)) * (D(k) + e\Phi(k)) - \frac{1}{\xi} \delta(0)\]
\[\text{(C.4)}\]

where we have used (24) once more. This is the desired equality. 

B.4. Commutation relations

Derivatives and equation-of-motion \( N_\Lambda(k) *, N_{\Phi}(k') *, N_{\S}(k'') * \) commute among themselves. The only exception is \( D(k) * \) which commutes with everything except for \( N_\Lambda(k') * \). We find in particular
\[
D(k) * N_\Lambda(0) - N_\Lambda(0) * D(k) = - D(k).
\]
\[\text{(B.15)}\]
C.2. Perturbative determination of $\partial_\xi z_m$, $\partial_\xi z$, $\partial_\xi z_F$

We wish to show that we can make $O_\xi = \partial_\xi (O_\xi' \equiv O_\xi')$ by choosing the $\xi$-dependence of the parameters $z_m(\mu)$, $z(\mu)$, $z_F(\mu)$ appropriately. Though this appendix augments the discussion in section 4, we use the notations $d_\xi$ (50) and $d_\xi'$ (53) introduced in section 5. Moreover, we use the result of appendix D. We ask the reader to glance over the beginning of section 5 for $d_\xi$, $d_\xi'$, and bare with us for our assuming the result of appendix D.

We first show that the difference $O_\xi - O_\xi'$ satisfies the WT identity:

$$
\frac{1}{\xi} k_\mu [A_\mu(k)(O_\xi - O_\xi')]|_\Lambda = \frac{1}{k^2} \left( O(k) + e\Phi(k) \right) \ast (O_\xi - O_\xi').
$$

(C.5)

Since

$$
O_\xi = -d_\xi 1, \quad O_\xi' = -d_\xi' 1
$$

we find

$$
O_\xi - O_\xi' = -D_\xi 1. \tag{C.7}
$$

In appendix D we show the gauge invariance of $D_\xi O_\Lambda$ for any gauge invariant composite operator $O_\Lambda$. Since $O_\Lambda = 1$ is gauge invariant, $O_\xi - O_\xi'$ is gauge invariant, satisfying (C.5).

Now, the gauge invariant $O_\xi - O_\xi'$ has three degrees of freedom, just as the Wilson action itself. Hence, we can make it vanish by tuning three parameters. To see this more explicitly, we examine the asymptotic behavior of $O_\xi - O_\xi'$ for $\Lambda$ much bigger than the momenta of the fields. Let $O_\Lambda^{(n)}$ be the $n$-loop part of $O_\Lambda$. We then obtain, for $n \geq 1$,

$$
O_\xi^{(n)} \xrightarrow{\Lambda \rightarrow \infty} -\partial_\xi S_\Lambda^{(n)} + \int_k \left( \frac{1}{k^2} \right) \left( \delta^2 S_\Lambda^{(n-1)} \right) \left( \frac{2}{2} k_\mu k_\nu A_\mu(k)A_\nu(-k) \right)
$$

(C.8)

and

$$
O_\xi'^{(n)} \xrightarrow{\Lambda \rightarrow \infty} -\int_k \left( \frac{1}{k^2} \right) \left( \frac{1}{2} k_\mu k_\nu \delta^2 S_\Lambda^{(n-1)} \right) \left( \frac{2}{2} k_\mu k_\nu A_\mu(k)A_\nu(-k) \right) + \frac{e^2}{k^4} f(k/\mu) N_F^{(n-1)}
$$

(C.9)

where the integration over $k$ provides one extra loop. Note $\partial_\xi S_\Lambda^{(n)}$ is parametrized by the three parameters $\partial_\xi z_m(\mu)$, $\partial_\xi z(\mu)$, $\partial_\xi z_F(\mu)$. Thus, by tuning these, we can make $O_\xi - O_\xi'$ vanish at $n$-loop. Since $O_\xi = O_\xi'$ at tree level, we have proven $O_\xi = O_\xi'$ by mathematical induction on the number of loops.

We give the 1-loop results of tuning in appendix F.

Appendix D. Gauge invariance of $D_\xi O_\Lambda$

The result of this appendix is necessary for the previous appendix and section 5. We wish to show that $D_\xi O_\Lambda = (d_\xi - d_\xi') O_\Lambda$, where $O_\Lambda$ satisfies the WT identity (28), also satisfies the WT identity:

$$
\frac{1}{\xi} k_\mu [A_\mu(k)D_\xi O_\Lambda] = \frac{1}{k^2} \left( D(k) + e\Phi(k) \right) \ast D_\xi O_\Lambda.
$$

(D.1)

In the following we will adopt a formal algebraic approach. Alternatively, we could derive (D.1) by examining the correlation functions of $D_\xi O_\Lambda$ with elementary fields. Using the definition of $D_\xi$ and the WT identity satisfied by $O_\Lambda$, we can compute

$$
\frac{1}{\xi} k_\mu [A_\mu(k)D_\xi O_\Lambda] \sim \cdot \cdot \cdot
$$

and show that this has the expected form as the WT identity for $D_\xi O_\Lambda$. 

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Let us instead give a formal algebraic proof. First, let $O_\Lambda$ be an arbitrary composite operator, not necessarily gauge invariant. By definition we have
\[ d_\xi (k_\mu [A_\mu (k) O_\Lambda]) = k_\mu [A_\mu (k) d_\xi O_\Lambda]. \quad \text{(D.2)} \]

Hence, we obtain
\[ \frac{1}{\xi} k_\mu [A_\mu (k) d_\xi O_\Lambda] - d_\xi \left( \frac{1}{\xi} k_\mu [A_\mu (k) O_\Lambda] \right) = \frac{1}{\xi^2} k_\mu [A_\mu (k) O_\Lambda]. \quad \text{(D.3)} \]

From (22, 25) and (31, 32), we obtain
\[ k_\mu [A_\mu (k) D(l) \ast O_\Lambda] = D(l) \ast k_\mu [A_\mu (k) O_\Lambda] - k^2 \delta (k + l) O_\Lambda. \quad \text{(D.4)} \]

Hence, we obtain
\[ k_\mu [A_\mu (k) \Phi (l) \ast O_\Lambda] = \Phi (l) \ast k_\mu [A_\mu (k) O_\Lambda]. \quad \text{(D.5)} \]

From (54), we also obtain
\[ k_\mu [A_\mu (k) N_F \ast O_\Lambda] = N_F \ast k_\mu [A_\mu (k) O_\Lambda]. \quad \text{(D.7)} \]

Recalling the definition of $d_\xi O_\Lambda$ (53):
\[ d_\xi O_\Lambda \equiv \frac{1}{2} \int \frac{1}{k^2} [(D(k) + e\Phi(k)) \ast (D(-k) + e\Phi(-k)) - e^2 f(k/\mu) N_F] \ast O_\Lambda \quad \text{(D.8)} \]
we obtain, from (D.6) and (D.7),
\[ k_\mu [A_\mu (k) d_\xi O_\Lambda] - d_\xi k_\mu [A_\mu (k) O_\Lambda] = \frac{1}{k^2} (D(k) + e\Phi(k)) \ast O_\Lambda. \quad \text{(D.9)} \]

Thus, from (D.3) and (D.9), we obtain
\[ \frac{1}{\xi} k_\mu [A_\mu (k) D_\xi O_\Lambda] - D_\xi \left( \frac{1}{\xi} k_\mu [A_\mu (k) O_\Lambda] \right) = \frac{1}{\xi} \left\{ \frac{1}{\xi} k_\mu [A_\mu (k) O_\Lambda] \right\} - \frac{1}{k^2} (D(k) + e\Phi(k)) \ast O_\Lambda. \quad \text{(D.10)} \]

This is valid for any composite operator $O_\Lambda$. If $O_\Lambda$ satisfies the WT identity, the right-hand side vanishes, and we obtain
\[ \frac{1}{\xi} k_\mu [A_\mu (k) D_\xi O_\Lambda] = D_\xi \left( \frac{1}{k^2} (D(k) + e\Phi(k)) \ast O_\Lambda \right). \quad \text{(D.11)} \]

Since $D_\xi$ commutes with $D(k) + e\Phi(k)$\footnote{\(d_\xi\) commutes with $D(k)\ast\Phi(k)\ast$, \(d_\xi\), consisting of $D(k')\ast\Phi(k')\ast$, and $N_F\ast$, also commutes with $D(k)\ast\Phi(k)\ast$.}, we obtain
\[ \frac{1}{\xi} k_\mu [A_\mu (k) D_\xi O_\Lambda] = \frac{1}{k^2} (D(k) + e\Phi(k)) \ast D_\xi O_\Lambda. \quad \text{(D.12)} \]

Thus, $D_\xi O_\Lambda$ satisfies the WT identity.
Appendix E. Gauge invariance of \( N_A - eO_e + 2\xi O_\xi \)

In this appendix we prove the gauge invariance of the linear combination \( N_A - eO_e + 2\xi O_\xi \), where \( N_A \) is defined by (69), \( O_e \) by (70), and \( O_\xi \) by (36). It is the simplest if we derive the WT identity for the correlation functions for \( O = N_A - eO_e + 2\xi O_\xi \).

From the original WT identity (27), we first obtain
\[
\frac{1}{\xi} k_\mu (A_\mu (k) N_A \cdots ^\infty) = \frac{1}{k^2} \left[ \sum_i k_\mu, \delta(k + k_i) \langle (N_A + 2) \cdots A_\mu(k_i) \cdots \rangle^\infty \right.
+ e \sum_i \left\{ -\langle (N_A + 1) \cdots \psi(p_i + k) \cdots \rangle^\infty \right.
\left. + \langle (N_A + 1) \cdots \hat{\psi}(-q_i + k) \cdots \rangle^\infty \right]\right].
\] (E.1)

Differentiating (27) by \( e \) and multiplying by \( e \), we obtain
\[
\frac{1}{\xi} k_\mu (A_\mu (k)(-eO_e) \cdots ^\infty) = \frac{1}{k^2} \left[ \sum_i k_\mu, \delta(k + k_i) \langle (-eO_e) \cdots A_\mu(k_i) \cdots \rangle^\infty \right.
+ e \sum_i \left\{ -\langle (-eO_e + 1) \cdots \psi(p_i + k) \cdots \rangle^\infty \right.
\left. + \langle (-eO_e + 1) \cdots \hat{\psi}(-q_i + k) \cdots \rangle^\infty \right]\right].
\] (E.2)

Finally, differentiating (27) by \(-\xi \) and multiplying by \( 2\xi \), we obtain
\[
\frac{1}{\xi} k_\mu (A_\mu (k)(2\xi O_\xi + 2) \cdots ^\infty) = \frac{1}{k^2} \left[ \sum_i k_\mu, \delta(k + k_i) \langle (2\xi O_\xi) \cdots A_\mu(k_i) \cdots \rangle^\infty \right.
+ e \sum_i \left\{ -(2\xi O_\xi + \psi(p_i + k) \cdots \rangle^\infty + \langle 2\xi O_\xi \cdots \hat{\psi}(-q_i + k) \cdots \rangle^\infty \right]\right].
\] (E.3)

Summing the three equations together, we obtain the WT identity for the linear combination:
\[
\frac{1}{\xi} k_\mu (A_\mu (k)(N_A - eO_e + 2\xi O_\xi) \cdots ^\infty) = \frac{1}{k^2} \left[ \sum_i k_\mu, \delta(k + k_i) \right.
\times \langle (N_A - eO_e + 2\xi O_\xi) \cdots A_\mu(k_i) \cdots \rangle^\infty
+ e \sum_i \left\{ -\langle (N_A - eO_e + 2\xi O_\xi) \cdots \psi(p_i + k) \cdots \rangle^\infty \right.
\left. + \langle (N_A - eO_e + 2\xi O_\xi) \cdots \hat{\psi}(-q_i + k) \cdots \rangle^\infty \right]\right].
\] (E.4)

Appendix F. Asymptotic coefficients at 1-loop

At 1-loop, the WT identity (24) alone gives the following results:
\[
\alpha^{(1)}(\Lambda) = \frac{e^2}{(4\pi)^2} \left[ -2(4\pi)^2 \Lambda^2 \int_p \frac{\Delta(p)(1 - K(p))}{p^2} + 2m^2 \right]
\] (F.1)
\[
\tilde{\alpha}^{(1)}(\Lambda) = -\tilde{\alpha}^{(1)}(\Lambda) + \frac{2}{3} \frac{e^2}{(4\pi)^2}
\] (F.2)
\[ \frac{1}{e^2} a_s^{(1)}(\Lambda) = -z_{e}^{(1)}(\Lambda) - \frac{e^2}{(4\pi)^2} \left( \frac{1}{2} \frac{K(p)(1-K(p))^2}{p^4} + 3 - \frac{3}{4} \right) \]  
(F.3)

\[ \frac{1}{e^2} a_s^{(1)}(\Lambda) = \frac{4}{3} \frac{e^2}{(4\pi)^2} \]  
(F.4)

where

\[ z_{e}^{(1)}(\Lambda) = \frac{8}{3} \frac{e^2}{(4\pi)^2} \ln \frac{\Lambda}{\mu} + z_{e}^{(1)} \]  
(F.5)

\[ z_{F}^{(1)}(\Lambda) = 2\xi \frac{e^2}{(4\pi)^2} \ln \frac{\Lambda}{\mu} + z_{F}^{(1)} \]  
(F.6)

\[ z_{m}^{(1)}(\Lambda) = 2(3 + \xi) \frac{e^2}{(4\pi)^2} \ln \frac{\Lambda}{\mu} + z_{m}^{(1)} \]  
(F.7)

Imposing \( O_{\xi} = O_\xi^{(49)} \) further, we obtain

\[ z_{e}^{(1)} = \frac{e^2}{(4\pi)^2} Z \]  
(F.8)

\[ z_{F}^{(1)} = \frac{e^2}{(4\pi)^2} \left[ Z_F + \xi \left\{ \frac{1}{4} - (4\pi)^2 \int \frac{1}{k^2} ((1-K(k))^2 - f(k)) \right\} \right] \]  
(F.9)

\[ z_{m}^{(1)} = \frac{e^2}{(4\pi)^2} \left[ Z_m - \xi (4\pi)^2 \int \frac{1}{k^2} ((1-K(k))^2 - f(k)) \right] \]  
(F.10)

where the numerical constants \( Z, Z_F, Z_m \) are still left arbitrary.

### Appendix G. Beta functions and anomalous dimensions

The \( \mu \)-dependence of the Wilson action is given by (74)

\[ -\mu \partial_\mu S_\Lambda = m_0 O_m + \gamma_F N_F + \gamma_A (N_A - e O_e + 2\xi O_\xi). \]  
(G.1)

#### G.1. Anomalous dimensions at 1-loop

To extract \( \beta_m, \gamma_F, \gamma_A \), we compare the asymptotic behaviors of both hand sides. At tree level, we find

\[ m_0 O_m^{(0)} \xrightarrow{\Lambda \to \infty} \text{im} \int d^4 x \bar{\psi} \psi \]  
(G.2)

\[ N_A^{(0)} - e O_e^{(0)} + 2\xi O_\xi^{(0)} \xrightarrow{\Lambda \to \infty} \frac{1}{2} \int d^4 x F_{\mu\nu} F_{\mu\nu} \]  
(G.3)

\[ N_F^{(0)} \xrightarrow{\Lambda \to \infty} 2 \int d^4 x \bar{\psi} \left( \frac{1}{i} \slashed{\partial} + im - e \slashed{A} \right) \psi \]  
(G.4)

On the other hand, at 1-loop level, the asymptotic coefficients of the previous appendix give

\[ -\mu \partial_\mu S_\Lambda^{(1)} \xrightarrow{\Lambda \to \infty} \frac{e^2}{(4\pi)^2} \int d^4 x \left[ \frac{4}{3} \frac{1}{2} F_{\mu\nu} F_{\mu\nu} + 2\xi \bar{\psi} \left( \frac{1}{i} \slashed{\partial} + im - e \slashed{A} \right) \psi + 6im\bar{\psi} \psi \right] \]  
(G.5)

Hence, we obtain the following 1-loop results:

\[ \beta_m^{(1)} = 6 \frac{e^2}{(4\pi)^2} \]  
(G.6)

\[ \gamma_A^{(1)} = 4 \frac{e^2}{(4\pi)^2} \]  
(G.7)

\[ \gamma_F^{(1)} = \xi \frac{e^2}{(4\pi)^2}. \]  
(G.8)
The trace anomaly (G.1) implies
\[
\left( -\mu \partial_\mu + \beta \partial_\beta + \beta_\xi \partial_\xi + m \beta_m \partial_m - L_{\gamma_A} - 2N_{\gamma_F} \right)
\times \langle A_{\mu_1}(k_1) \cdots A_{\mu_L}(k_L) \psi(p_1) \cdots \psi(p_N) \bar{\psi}(-q_1) \cdots \bar{\psi}(-q_N) \rangle_{\infty} = 0
\] (G.9)
for the correlation functions. Differentiating these with respect to the parameters \(m, e, \xi\), we can extract the anomalous dimensions for the conjugate operators:
\[
dt O_m = -\beta_m O_m
\] (G.10)
\[
dt O_e = - (\partial_e \beta_m) m O_m + \partial_e (e \gamma_A) O_e
\] (G.11)
\[
dt O_\xi = - (\partial_\xi \beta_m) m O_m + \partial_\xi (e \gamma_A) O_e
\] (G.12)
Since the number of elementary fields does not change under an infinitesimal change of \(\mu\), we obtain
\[
dt N_A = dt N_F = 0
\] (G.13)

### G.3. \(\xi\) dependence of \(\beta_m, \gamma_A, \gamma_F\)

We recall that the gauge invariance of \(-\mu \partial_\mu 1 = -\mu \partial_\mu S_\Lambda\) implies (76) and (77). We can obtain the \(\xi\)-dependence of the anomalous dimensions \(\beta_m, \gamma_A, \gamma_F\) from the \(\xi\)-independence of the action \(O_\xi = O'_\xi\). \(dt O'_\xi\) is given by (G.12). To compute \(dt O'_\xi\), we recall
\[
O'_\xi = \int_k \left[ -\frac{1}{2\xi^2} k_\mu k_\nu [A_{\mu}(k)A_{\nu}(-k)] + \frac{1}{2\xi} \delta(0) - \frac{e^2}{2} \frac{f(k/\mu)}{k^4} N_F \right].
\] (G.14)
Since
\[
dt [A_{\mu}(k)A_{\nu}(-k)] = 2\gamma_A [A_{\mu}(k)A_{\nu}(-k)]_A
\] (G.15)
\[
dt \frac{1}{\xi} = -2\gamma_A \frac{1}{\xi}
\] (G.16)
\[
dt e^2 = -2\gamma_A e^2
\] (G.17)
\[
dt N_F = 0
\] (G.18)
we obtain
\[
dt O'_\xi = -2\gamma_A O'_\xi + \frac{e^2}{2} \int_k \frac{1}{k^4} \partial_\mu f(k/\mu) N_F.
\] (G.19)
To calculate the integral, we only need the asymptotic values \(f(0) = 0, f(\infty) = 1\). We find
\[
\int_k \frac{1}{k^4} \partial_\mu f(k/\mu) N_F = -\frac{2}{(4\pi)^2}.
\] (G.20)
Hence, we obtain
\[
dt O'_\xi = -2\gamma_A O'_\xi - \frac{e^2}{(4\pi)^2} N_F.
\] (G.21)
Comparing this with (G.12), we obtain
\[
\partial_\xi \beta_m = \partial_\xi \gamma_A = 0, \quad \partial_\xi \gamma_F = \frac{e^2}{(4\pi)^2}.
\] (G.22)
Appendix H. Gauge invariance and $\xi$-independence of $d_\xi \mathcal{O}_\Lambda$

Let us assume that $\mathcal{O}_\Lambda$ is both gauge invariant and $\xi$-independent, satisfying (28) and (59)

\[
\frac{1}{\xi} [k_\mu A_\mu (k) \mathcal{O}] = \frac{1}{k^2} (D(k) + e \Phi(k)) \ast \mathcal{O}_\Lambda = \mathcal{O}_\Lambda
\] (H.1)

\[
D_\xi \mathcal{O}_\Lambda = 0.
\] (H.2)

We wish to show first that $d_\xi \mathcal{O}$ satisfies the WT identity, and then that $d_\xi \mathcal{O}$ is $\xi$-independent.

H.1. WT identity for $d_\xi \mathcal{O}$

$d_\xi \mathcal{O}$ is defined by (78):

\[
d_\xi \mathcal{O} \equiv (-\mu d_\mu + m_\mu \gamma_m + \beta d_e + \beta e d_\xi - \gamma_{F\Lambda} N_F \ast -\gamma_{N\Lambda} \ast) \mathcal{O}.
\] (H.3)

We find

\[
\frac{1}{\xi} k_\mu [A_\mu (k) d_\xi \mathcal{O}] = \frac{1}{\xi} k_\mu [A_\mu (k) (-\mu d_\mu + m_\mu \gamma_m + \beta d_e + \beta e d_\xi - \gamma_{F\Lambda} N_F \ast -\gamma_{N\Lambda} \ast) \mathcal{O}] = \frac{1}{\xi} - \mu d_\mu + \beta d_e + \beta e d_\xi + \frac{1}{\xi} k_\mu [A_\mu (k) \mathcal{O}]
\] (H.4)

Since

\[
\frac{1}{\xi} k_\mu [A_\mu (k) N_{F\Lambda} \ast \mathcal{O}] = N_{F\Lambda} \ast \frac{1}{\xi} k_\mu [A_\mu (k) \mathcal{O}] + \frac{1}{\xi} k_\mu [A_\mu (k) \mathcal{O}]
\] (H.5)

we obtain

\[
\frac{1}{\xi} k_\mu [A_\mu (k) d_\xi \mathcal{O}] = \left\{ -\mu d_\mu + \beta d_e + \beta e d_\xi + \frac{1}{\xi} k_\mu [A_\mu (k) \mathcal{O}]ight\} \times \frac{1}{\xi} k_\mu [A_\mu (k) \mathcal{O}].
\] (H.7)

Using the WT identity (H.1) and the commutator (B.15), we obtain

\[
\frac{1}{\xi} k_\mu [A_\mu (k) d_\xi \mathcal{O}] = \frac{1}{k^2} (D(k) + e \Phi(k)) \ast \left\{ -\mu d_\mu + \beta d_e + \beta e d_\xi + \frac{1}{\xi} k_\mu [A_\mu (k) \mathcal{O}]ight\} \times \frac{1}{\xi} k_\mu [A_\mu (k) \mathcal{O}]
\] (H.8)

Finally, using (76, 77), we obtain

\[
\frac{1}{\xi} k_\mu [A_\mu (k) d_\xi \mathcal{O}] = \frac{1}{k^2} (D(k) + e \Phi(k)) \ast (d_\xi \mathcal{O} + \gamma_\Lambda \mathcal{O}) - \frac{1}{k^2} \gamma_\Lambda (D(k) + e \Phi(k)) \ast \mathcal{O}
\]

\[
= \frac{1}{k^2} (D(k) + e \Phi(k)) \ast d_\xi \mathcal{O}.
\] (H.9)

Thus, $d_\xi \mathcal{O}_\Lambda$ satisfies the WT identity.
H.2. $\xi$-independence of $d_\xi O_\Lambda$

We will prove

$$D_\xi d_\xi O_\Lambda = 0$$

by deriving the commutator

$$d_\xi D_\xi - D_\xi d_\xi = -2\gamma_\lambda D_\xi.$$  \hspace{1cm} (H.10)

Assuming $D_\xi O_\Lambda = 0$, this commutator gives immediately (H.10).

We recall the definition

$$d_\xi \equiv -\mu d_\mu - e\gamma_\lambda d_\xi + 2\xi \gamma_\lambda d_\xi + \beta_\mu m d_m - \gamma_\lambda N_\Lambda - \gamma_\mu N_\mu.$$  \hspace{1cm} (H.12)

where the $\xi$-dependence of $\gamma_\lambda$, $\beta_\mu$, $\gamma_\mu$ is given by (G.22). Since $d_\xi$ commutes with derivatives and $N_\Lambda$, $N_\mu$, we obtain

$$d_\xi d_\xi - d_\xi d_\xi = 2\gamma_\lambda d_\xi - (\partial_\xi \gamma_\mu) N_\mu * = 2\gamma_\lambda d_\xi - \frac{e^2}{(4\pi)^2} N_\mu *.$$  \hspace{1cm} (H.13)

To compute the commutator of $d_\xi$ and $d_\xi'$, we recall

$$d_\xi' \equiv -\frac{1}{2} \int \frac{1}{k^4} [D(k) + e\Phi(k)] \ast (D(-k) + e\Phi(-k)) \ast - f(k/\mu) e^2 N_\mu *.$$  \hspace{1cm} (H.14)

As a preparation, we compute the commutator

$$(D(-k) + e\Phi(-k)) \ast d_\xi - d_\xi(D(-k) + e\Phi(-k)) *$$

$$= e\gamma_\lambda \Phi(-k) - \gamma_\lambda (D(-k) \ast N_\Lambda - N_\Lambda \ast D(-k) *).$$  \hspace{1cm} (H.15)

Using the commutator (B.15), we get

$$(D(-k) + e\Phi(-k)) \ast d_\xi - d_\xi(D(-k) + e\Phi(-k)) *$$

$$= \gamma_\lambda (D(-k) + e\Phi(-k)) *.$$  \hspace{1cm} (H.16)

Thus, we obtain

$$(D(k) + e\Phi(k)) \ast \{d_\xi(D(-k) + e\Phi(-k)) * + \gamma_\lambda (D(-k) + e\Phi(-k) *)\}$$

$$= d_\xi(D(k) + e\Phi(k)) \ast (D(-k) + e\Phi(-k)) *$$

$$+ 2\gamma_\lambda (D(k) + e\Phi(k)) * (D(-k) + e\Phi(-k)) *.$$  \hspace{1cm} (H.17)

Since $N_\mu$ commutes with $d_\xi$, we obtain

$$f(k/\mu) e^2 N_\mu * = d_\xi \{f(k/\mu) e^2 N_\mu *\} + (\mu \partial_\mu f(k/\mu) + 2\gamma_\lambda e^2 N_\mu *.$$  \hspace{1cm} (H.18)

Hence, using (H.17, H.18), we obtain

$$d_\xi d_\xi = d_\xi d_\xi' - \frac{1}{2} \int \frac{1}{k^4} \left[ 2\gamma_\lambda (D(k) + e\Phi(k)) \ast (D(-k) + e\Phi(-k)) * - 2\gamma_\lambda f(k/\mu) e^2 N_\mu *$$

$$- \mu(\partial_\mu f(k/\mu)) e^2 N_\mu * \right]$$

$$= d_\xi d_\xi' + 2\gamma_\lambda d_\xi' + \frac{1}{2} e^2 \int \frac{1}{k^4} \mu(\partial_\mu f(k/\mu)) N_\mu *$$

$$= d_\xi d_\xi' + 2\gamma_\lambda d_\xi' - \frac{e^2}{(4\pi)^2} N_\mu *.$$  \hspace{1cm} (H.19)

where we have used (G.20).

Combining (H.13) and (H.19), we obtain

$$d_\xi (d_\xi - d_\xi') - (d_\xi - d_\xi')d_\xi = -2\gamma_\lambda (d_\xi - d_\xi').$$  \hspace{1cm} (H.20)

This is the desired commutator (H.11).
Appendix I. Axial anomaly at 1-loop

Up to 1-loop, we obtain the following results:

(i) $J_{5\mu}$

$$a_3^5(\Lambda) = 1 + \frac{e^2}{(4\pi)^2}\left[-2\xi \ln \frac{\lambda}{\mu} + A + \xi (4\pi)^2 \int_k \frac{1}{k^4} [(1 - K(k))^3 - f(k)]\right]$$  \hspace{1cm} (I.1)

$$a_5(\Lambda) = \frac{e^2}{(4\pi)^2} - \frac{8}{3}$$  \hspace{1cm} (I.2)

$a_5$ is determined by $a_3$ by gauge invariance. The numerical constant $A$ cannot be fixed by imposing the vanishing of the 1-loop anomalous dimension.

(ii) $J_5$

$$j(\lambda) = 1 + \frac{e^2}{(4\pi)^2}\left[-2(3 + \xi) \ln \frac{\lambda}{\mu} + B + \xi (4\pi)^2 \int_k \frac{1}{k^4} [(1 - K(k))^3 - f(k)]\right]$$  \hspace{1cm} (I.3)

where the numerical constant $B$ cannot be fixed by demanding the vanishing of the anomalous dimension of $mJ_5$ at 1-loop.

(iii) $\frac{1}{2} F F_{\lambda}$

$$f_3(\Lambda) = -\frac{9}{8} + \frac{e^2}{(4\pi)^2}\left[-3 + \frac{9}{4} \xi \ln \frac{\lambda}{\mu} - \frac{9}{8} \xi (4\pi)^2 \int_k \frac{1}{k^4} [(1 - K(k))^3 - f(k)] + C\right]$$  \hspace{1cm} (I.4)

$$f_5(\Lambda) = 1 + \frac{e^2}{(4\pi)^2} D$$  \hspace{1cm} (I.5)

where $C, D$ cannot be fixed by demanding the vanishing of the anomalous dimension of $e^2[F F]^\lambda_{\lambda}$ at 1-loop.

(iv) $\Phi_5$—this is unambiguously determined by the Wilson action.

$$\phi_3^5(\Lambda) = 1 + \frac{e^2}{(4\pi)^2}\left[-2\xi \ln \frac{\lambda}{\mu} - \frac{3}{4} - Z_F + \xi (4\pi)^2 \int_k \frac{1}{k^4} [(1 - K)^3 - f]\right]$$  \hspace{1cm} (I.6)

$$\phi_5(\Lambda) = \frac{e^2}{(4\pi)^2} \left[-3 + \frac{9}{4} \xi \ln \frac{\lambda}{\mu} - \frac{9}{8} \xi (4\pi)^2 \int_k \frac{1}{k^4} [(1 - K)^3 - f]\right]$$  \hspace{1cm} (I.7)

$$\phi(\Lambda) = -1 + \frac{e^2}{(4\pi)^2}\left[2(3 + \xi) \ln \frac{\lambda}{\mu} + Z_m - \xi (4\pi)^2 \int_k \frac{1}{k^4} [(1 - K)^3 - f]\right]$$

$$+ 3(4\pi)^2 \int_k \frac{K(1 - K)^2}{k^4}$$  \hspace{1cm} (I.8)

where the undetermined numerical constants $Z_F, Z_m$ are carried over from appendix F.

We thus obtain, up to 1-loop,

$$a_3^5(\Lambda) - \phi_3^5(\Lambda) = \frac{e^2}{(4\pi)^2}\left(A + \frac{3}{4} + Z_F\right)$$  \hspace{1cm} (I.9)

$$a_5(\Lambda) - \phi_5(\Lambda) = \frac{e^2}{(4\pi)^2} (-4)$$  \hspace{1cm} (I.10)

$$j(\Lambda) + \phi(\Lambda) = \frac{e^2}{(4\pi)^2}\left[B + Z_m + 3(4\pi)^2 \int_k \frac{K(1 - K)^2}{k^4}\right].$$  \hspace{1cm} (I.11)

The last equation must vanish, and we obtain

$$B = Z_m - 3(4\pi)^2 \int_k \frac{K(1 - K)^2}{k^4}.$$  \hspace{1cm} (I.12)
Comparing the first two equations with
\[
\text{const.} \frac{e^2}{(4\pi)^2} f_3^{(0)}(A) = \frac{e^2}{(4\pi)^2} \text{const.} \frac{-9}{8}
\]  
(I.13)

\[
\text{const.} \frac{e^2}{(4\pi)^2} f_5^{(0)}(A) = \frac{e^2}{(4\pi)^2} \text{const.}
\]  
(I.14)

we obtain
\[
\text{const.} = -4
\]  
(I.15)

and
\[
A = \frac{15}{4} - Z_F.
\]  
(I.16)

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