Orbit equivalence of linear systems on manifolds and semigroup actions on homogeneous spaces

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Resumo

In this paper we introduce the notion of orbit equivalence for semigroup actions and the concept of generalized linear control system on smooth manifold. The main goal is to prove that, under certain conditions, the semigroup system of a generalized linear control system on a smooth manifold is orbit equivalent to the semigroup system of a linear control system on a homogeneous space.

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1 Introduction

Although the control theory originated about a century ago, there is no global theory yet with general hypothesis. However, in special cases, the study of control theory have made rapid progress in the last decades. For example, the control theory on Lie groups has achieved significant advances due especially its relationship with the actions of semigroups on Lie groups, implying in good results in the study of control sets and controllability (see e.g. Elliott [5], Jurdjevic [7], Rocio, San Martin and Santana [8], Rocio, Santana and Verdi [9] and Sachkov [10]).

Until the 1990s the theory of control systems on Lie groups was restricted, basically, to the control system of invariant vector fields. In Ayala and Tirao
3, this study was expanded with the introduction of linear control systems on Lie groups and developed rapidly in recent years, the first papers on this subject concern about controllability (see e.g. [2], [3] and [13]). In Jouan [12], considering a control system on a manifold given by complete linear vector fields that generate a finite dimensional Lie algebra, it was showed a equivalence between this system and a linear control system on homogeneous space.

In our paper, initially we formalize the notion of orbit equivalence for semigroups actions on manifolds. Then we establishes conditions for an action of a semigroup of a control system on a manifold is orbit equivalent to the action of a semigroup in a homogeneous space. The main result of this paper establishes conditions under which the action of the semigroup associated to the generalized linear control system is orbit equivalent to the action of a semigroup on a homogeneous space.

We now touch some control theoretic aspects related with our work. Consider $G$ a (finite dimensional) connected and simply connected Lie group. Suppose that $G$ acts transitively on a manifold $M$ and take $H$ a closed subgroup. Let $\pi : G \to G/H$ be the canonical projection. A linear control system on $G/H$ is a special case of control systems where the drift is $\pi$-related with a linear vector field on $G$ and the controlled vector fields are projections of right invariant vector fields on $G$. Take $\mathfrak{g}$ the Lie algebra given by the right invariant vector fields on $G$. Using the same notations of Ayala and San Martin [2] and [3] and denoting by $e$ the identity of $G$, a vector field $\mathcal{X}$ on $G$ is called linear if for all $Y \in \mathfrak{g}$ we have $[\mathcal{X}, Y] \in \mathfrak{g}$ and $\mathcal{X}(e) = 0$. Hence a linear control system on $G$ is defined as

$$\dot{x} = \mathcal{X}(x) + \sum_{j=1}^{m} u_j Y_j(x),$$

where the drift $\mathcal{X}$ is a linear vector field on $G$, $Y_1, Y_2, \ldots, Y_m \in \mathfrak{g}$ and $u = \{u_1, u_2, \ldots, u_m\} \in \mathbb{R}^n$.

We recall the definition of linear control system on $G/H$ given in [12]. A vector field on $G/H$ is called invariant if it is the $\pi^*$-image of some right invariant vector field on $G$ and is called linear if is $\pi$-related with a linear vector field on $G$. Hence if the drift of a control system on $G/H$ is linear and the controlled vector fields are invariant then the system is called linear control system on $G/H$.

In this direction, the main concept of our paper is the generalized linear control system on manifolds. Take $\mathcal{L}(TM)$ the Lie algebra of the differentiable vector fields on $M$. A generalized linear control system on $M$ is a control system

$$\dot{x} = \mathcal{F}(x) + \sum_{j=1}^{m} u_j Y_j(x),$$

where $\mathcal{F}, Y_j \in \mathcal{L}(TM)$ for every $j = 1, \ldots, m$, $\Gamma = \{Y_1, \ldots, Y_m\}$ generates a finite dimensional Lie subalgebra $\mathcal{L}(\Gamma)$ of $\mathcal{L}(TM)$, every vector field $Y_i \in \Gamma$ is complete, $[\mathcal{F}, X] \in \mathcal{L}(\Gamma)$ for all $X \in \mathcal{L}(\Gamma)$ and there exists $x_0 \in M$ such that $\mathcal{F}_{x_0} = 0$. The motivation to study these systems come from the need to
formalize concepts involved in control theory on manifolds that transfer several issues, such as the controllability, to be treated in more pleasant state space such as Lie groups.

About the structure of this paper, in the second section we introduce the notion of orbit equivalence and topological conjugacy for semigroup actions and give some properties related with control sets. In the third section we fix the control theoretic notations and relates state equivalent control systems with diffeomorphic control systems. In the fourth section we prove that given a control system on \( M \), the semigroup system on \( M \) is orbit equivalent to a semigroup action on a homogeneous space. In the last section we prove our main result which states that supposing \( \Gamma \) is transitive on \( M \) and taking \( G \) the connected and simply connected Lie group with Lie algebra \( \mathcal{L}(\Gamma) \), then the semigroup system of the above system is orbit equivalent to a semigroup system of a linear control system on a \( G \)-homogeneous space.

2 Orbit equivalence

In this section, we define the notions of orbit equivalence for semigroups actions and topological conjugacy for skew product, this concepts will be necessary in the next sections. We establish some relations between orbit equivalence and control sets for semigroups actions. We begin recalling some concepts of the theory of control sets (for more details see e.g. San Martin [15] and San Martin and Tonelli [16]). Take a non empty semigroup \( S \) acting on a topological space \( M \). The semigroup \( S \) is said to be accessible if \( \text{int}Sx \neq \emptyset \) for every \( x \in M \). A control set for the \( S \)-action on \( M \) is a subset \( C \subset M \) such that \( \text{int}C \neq \emptyset \), \( C \subset \text{cl}(Sx) \) for all \( x \in C \) and \( C \) is maximal with the first two properties. If \( \text{cl}C = \text{cl}(Sx) \) for all \( x \in C \), the control set \( C \) will be named invariant control set. We also recall the partial ordering between control sets given by \( C_1 < C_2 \) if there exists \( x \in C_1 \) such that \( \text{cl}(Sx) \cap C_2 \neq \emptyset \).

Now about equivalence of semigroups we have the following definitions.

**Definition 1** Let \( M_1 \) and \( M_2 \) be topological spaces. Consider \( S \) and \( T \) semigroups. The actions \( (M_1, S) \) and \( (M_2, T) \) are called orbit equivalent, if there exists an homeomorphism \( f : M_1 \to M_2 \) such that \( f(Sx) = Tf(x) \) for all \( x \in M_1 \). The map \( f \) is called orbit equivalence map.

Some authors call the pair \( (M, S) \) as transformation semigroup (see e.g. Ellis in [6] and Sousa in [11]).

Locally, we have that the actions \( (M_1, S) \) and \( (M_2, T) \) are called orbit equivalent restricted to a subset \( C \subset M_1 \) if there exists an homeomorphism \( f : M_1 \to M_2 \) such that \( f(Sx) = Tf(x) \) for all \( x \in C \).

Now supposing the existence of control sets we give some properties of orbit equivalent actions. Recall that taking the topological space as flag manifolds, there exist always control sets (see e.g. [15] and [16]).
Proposition 2 Suppose that \((M_1, S)\) and \((M_2, T)\) are orbit equivalent. Hence if \(C_S\) is a control set for \(S\) then \(f(C_S)\) is a control set for \(T\) in \(M_2\). On the other hand, if \(C_T\) is a control set for \(T\) in \(M_2\) then \(f^{-1}(C_T)\) is a control set for \(S\) in \(M_1\).

Proof. Note that \(\text{int}(f(C_S)) \neq \emptyset\) and \(f(C_S) \subset \text{fe}(Ty)\) for every \(y \in f(C_S)\). By hypotheses it follows \(\text{fe}(f(Sx)) \subset \text{fe}(Tf(x))\), for all \(x \in C_S\). The proof of the converse is analogous.

Moreover, the orbit equivalence preserves the order of control sets.

Proposition 3 The topological conjugacy preserves the order of control sets.

Proof. Take the \(S\) control sets \(C_1\) and \(C_2\). Suppose that \(C_1 < C_2\), then there exists \(x \in C_1\) such that \(\text{fe}(Sx) \cap C_2 \neq \emptyset\). Take \(f : M_1 \to M_2\) a topological conjugate of the actions \((M_1, S)\) and \((M_2, T)\) and consider the control sets \(f(C_1)\) and \(f(C_2)\) for the \(T\) action. Take \(f(x) \in f(C_1)\) then \(\text{fe}(Tf(x)) \cap f(C_2) = \text{fe}(f(Sx)) \cap f(C_2)\). But \(\emptyset \neq f(\text{fe}(Sx) \cap C_2) \subset f(\text{fe}(Sx)) \cap f(C_2) \subset f(\text{fe}(Sx)) \cap f(C_2)\). Then \(\text{fe}(Tf(x)) \cap f(C_2) \neq \emptyset\). ■

For the next proposition we recall the definition of the set of transitivity \(C^0\) of a control set \(C\): \(C^0 = \{x \in C : x \in (\text{int}S)x\}\). It holds \(C^0 = (\text{int}S)x\) for all \(x \in C^0\) (see [13]).

Proposition 4 With the same notations, suppose that there exists a homeomorphism \(f : M_1 \to M_2\) that send set of transitivity in set of transitivity, that is, if \(C \subset M_1\) is the \(S\) invariant control set and \(C^0\) its set of transitivity then \(f(C)\) is the invariant control set for \(T\) with \(f(C^0)\) its set of transitivity. Suppose also that \(S\) and \(T\) are accessible. With this hypotheses we have \((M_1, \text{int}S)\) and \((M_2, \text{int}T)\) are orbit equivalent restricted to \(C\).

Proof. Take \(x \in C_0\) and \(a \in \text{int}S\), then \(f(ax) \in f(C_0) = (\text{int}T)y\) for all \(y \in f(C_0)\), in particular to \(y = f(x)\). Then \(f((\text{int}S)x) \subset (\text{int}T)f(x)\), for all \(x \in C_0\). It easy to prove that \((\text{int}T)f(x) \subset f((\text{int}S)x)\). Now take \(x \in C = \text{fe}C_0\), then exists a sequence \(x_n \in C_0\) such that \(x_n\) converge to \(x\). Moreover, we have \(f((\text{int}S)x_n) = (\text{int}T)f(x_n)\) for all \(n\). As \(f\) is homeomorphism it follows that \(f(x_n)\) converge to \(f(x)\).

It follows that for all \(g \in \text{int}S\), exists \(h \in \text{int}T\) such that \(f(x_n) = hf(x_n)\). Hence, we have that for all \(g \in \text{int}S\) exists \(h \in \text{int}T\) such that \(f(gx) = hf(x)\).

Analogously, taking \(x \in C\) then for all \(h \in \text{int}T\) exists \(g \in \text{int}S\) such that \(hf(x) = f(gx)\). Hence \(f((\text{int}S)x) = (\text{int}T)f(x)\), for all \(x \in C\). ■

It is not difficult to prove this kind of converse:

Proposition 5 Consider the notations and assumptions as above. Suppose that \((M_1, \text{int}S)\) and \((M_2, \text{int}T)\) are orbit equivalent. Then \(f(C^0) = (f(C))^0\).

To finish this section, we establish a relation between the concepts of conjugation and orbit equivalence.
Now suppose that $S$ and $T$ have the identities $e_S$ and $e_T$. Let $\varphi$ be a cocycle on $X$ to $T$, that is, $\varphi : S \times X \to T$ continuous with

$$
\varphi(st,x) = \varphi(s,tx) \varphi(t,x) \quad \text{for all } s,t \in S, x \in X, \text{ and}
$$

$$
\varphi(e_{S},x) = e_T \quad \text{for all } x \in X.
$$

The cocycle property is appropriate to define the skew-product transformation semigroup on the product space $X \times Y$ given by the mapping

$$
\Phi : S \times X \times Y \to X \times Y, \quad \Phi(s,x,y) = (sx, \varphi(s,x)y).
$$

(1)

We might write $s(x,y)$ instead of $\Phi(s,x,y)$.

We define the following subsemigroup of $T$, called system semigroup,

$$
S = \{ \varphi(s_n,x_n) \varphi(s_{n-1},x_{n-1}) \cdots \varphi(s_1,x_1) : s_j \in S, x_j \in X, n \in \mathbb{N} \}.
$$

(2)

By considering the action $\sigma$ restricted to the product $S \times Y$, we have the transformation semigroup $(S,Y,\sigma)$ associated to the skew-product transformation semigroup $(S,X \times Y,\Phi)$.

To introduce the concepts of topological conjugacy and state equivalence we consider, for $i = 1, 2$, the following two skew-product transformation semigroups

$$
\Phi^i : S \times X^i \times Y^i \to X^i \times Y^i, \quad \Phi^i(s,x,y) = (sx, \varphi^i(s,x)y)
$$

Definition 6 Let $\xi : Y^1 \to Y^2$ and $\iota : X^1 \to X^2$ be maps such that $\xi$ is continuous and satisfy:

$$
\xi(\varphi^1(s,x)y) = \varphi^2(s,\iota(x))\xi(y), \text{ for all } (s,x,y) \in S \times X^1 \times Y^1.
$$

In this case, we say that the skew product $\Phi^1$ is topologically semi conjugate to $\Phi^2$. If $\xi$ is a homeomorphism and $\iota$ is invertible, then the skew products are called topologically conjugate.

In the particular case where $\Phi^1$ and $\Phi^2$ are topologically conjugate, $\iota = id_X$ and $\xi$ is a diffeomorphism, we say that $\Phi^1$ and $\Phi^2$ are state equivalent. This terminology is inspired by the concept of state equivalence of control systems (for more details see Agrachev and Sachkov in [1]).

Now we prove a result that relates the concepts of conjugation and orbit equivalence.

Proposition 7 If $\Phi^1$ and $\Phi^2$ are topologically conjugate then the actions $(Y^1, S^1)$ and $(Y^2, S^2)$ are orbit equivalent, where $S^1$ and $S^2$ are the semigroup system of $\Phi^1$ and $\Phi^2$ respectively.

Proof. By hypothesis, there exists a homeomorphism $\xi : Y^1 \to Y^2$ and an invertible map $\iota : X^1 \to X^2$ such that $\xi(\varphi^1(s,x)y) = \varphi^2(s,\iota(x))\xi(y)$, for all $(s,x,y) \in S \times X^1 \times Y^1$.

Consider the following semigroups associated to $\Phi^i$ for $i = 1, 2$

$$
S^i = \{ \varphi^i(s_n,x_n) \cdots \varphi^i(s_1,x_1) : s_j \in S, x_j \in X^i, n \in \mathbb{N} \}.
$$
Define the homeomorphism $h$ as $\xi$. Then given $a \in h(S^1 y)$, we have $a = h(b)$, where $b \in S^2 h(y)$, i.e., $b = \varphi^1(s_n, x_n) \cdots \varphi^1(s_1, x_1)y = \varphi^1(s_n \cdots s_1, x)y$. Hence $a \in S^2 h(y)$, in fact, $a = \xi(\varphi^1(s_n \cdots s_1, x)y) = \varphi^2(s_n \cdots s_1, \iota(x))h(y)$.

For the opposite inclusion, consider $a \in S^2 h(y)$, then $a = bh(y)$, with $b \in S^2$, hence $b = \varphi^2(s_m, v_m) \cdots \varphi^2(s_1, v_1) = \varphi^2(s_m \cdots s_1, v)$. Then, using a similar idea as above we prove that $a \in h(S^1 y)$. ■

3 Conjugacy and state equivalence of control systems

In this section we prove that if two systems are diffeomorphic then they are state equivalent.

Take $M$ a differentiable and connected $d$-dimensional manifold. Consider in $M$ the following control system

$$(\Sigma) \quad \dot{x}(t) = X_0(x(t)) + \sum_{j=1}^{m} u_j X_j(x(t)),$$

where $u : \mathbb{R} \to U$ is a piecewise constant map with $U \subset \mathbb{R}^n$ compact and convex, and $X_i$ are differentiable vector fields on $M$. Denote by $\mathcal{U}$ the set of the maps $u$. It is well known that $\mathcal{U}$ is a metric space (see e.g. Colonius and Kliemann in [4]). We assume that for each $u$ and $x \in M$ this system has a unique solution $\phi(t, u, x), t \in \mathbb{R}$, with $\phi(0, u, x) = x$.

As defined in [4], take

$$\Phi : \mathbb{R} \times \mathcal{U} \times M \to \mathcal{U} \times M, \Phi(t, u, x) = (\Theta_t(u), \phi(t, u, x)),$$

the control flow of this system, we know that it is a special case of skew-product transformation semigroup (see [4]).

Then as a consequence of the previous theorem we consider two control systems $\Sigma_1$ and $\Sigma_2$ as above, take their control flows $\Phi_1$ and $\Phi_2$ and their correspondent system semigroups $S_{\Sigma_1}$ and $S_{\Sigma_2}$. Now we recall the construction of these semigroups, take the map $\varphi_{1i}^{u_i} : M_1 \to M_2$ given by $\varphi_{1i}^{u_i}(x) = \varphi(t_1, u_1, x)$ then we have that $\Sigma_1$ is a semigroup of diffeomorphisms of $M_1$ given by

$$S_{\Sigma_1} = \{ \varphi_{1r}^{u_r} \circ \cdots \circ \varphi_{11}^{u_1} : u_i \in \mathcal{U}, t_i \geq 0, r \in \mathbb{N} \}.$$

The natural action of $S_{\Sigma_1}$ on $M_1$ is defined as $\varphi \cdot x = \varphi(x)$. In the same way we have the semigroup $S_{\Sigma_2}$. Recall that $(\Sigma_1)$ and $(\Sigma_2)$ are called topologically conjugate if there exist a homeomorphism $\xi : M_1 \to M_2$ and an invertible map $\iota : \mathcal{U} \to \mathcal{V}$ such that $\xi(\varphi(t, u, x)) = \psi(t, \iota(u), \xi(x))$, for all $(t, u, x) \in \mathbb{R} \times \mathcal{U} \times M_1$. Then as a consequence of Proposition 7 we have the following proposition:

Proposition 8 Suppose that $\Sigma_1$ and $\Sigma_2$ are topologically conjugate then the actions $(M_1, S_{\Sigma_1})$ and $(M_2, S_{\Sigma_2})$ are orbit equivalent.
Another important concept used to classify control systems is the notion of state equivalence, as defined in the previous section, in this case $\xi$ is a diffeomorphism and $U = V$. This concept is used to classify control systems preserving differentiable properties. A sufficient condition to guarantee that $(\Sigma_1)$ and $(\Sigma_2)$ be state equivalent is the existence of a diffeomorphism from $M_1$ to $M_2$ that preserves the control systems. Precisely, suppose that $U = V$ and that $\xi : M_1 \to M_2$ be a diffeomorphism. For each $u \in U$ consider the vector fields $Z_u$ in $M_1$ and $W_u$ in $M_2$ given by

$$Z_u(x) = X_0(x) + \sum_{j=1}^{m} u_j X_j(x)$$

and

$$W_u(\xi(x)) = Y_0(\xi(x)) + \sum_{j=1}^{m} u_j Y_j(\xi(x)),$$

where $x \in M_1$. In this conditions we have:

**Proposition 9** If $\xi : M_1 \to M_2$ is a diffeomorphism such that $\xi_*(Z_u(x))_x = W_u(\xi(x))$, for all $u \in U$ and $x \in M_1$ then the control systems $(\Sigma_1)$ and $(\Sigma_2)$ are state equivalent.

**Proof.** Given $u \in U$ and $x \in M_1$ denote by $\varphi(t,u,x)$ the unique solution of the system $(\Sigma_1)$ such that $\varphi(0,u,x) = x$ and by $\psi(t,u,\xi(x))$ the unique solution of $(\Sigma_2)$ such that $\psi(0,u,\xi(x)) = \xi(x)$. Then $\frac{d}{dt}\varphi(t,u,x) = Z_u(\varphi(t,u,x))$, for all $t \in \mathbb{R}$ and hence $\frac{d}{dt}\xi(\varphi(t,u,x)) = \xi_*(\varphi(t,u,x)) = \xi_*(Z_u(\varphi(t,u,x))) = W_u(\xi(\varphi(t,u,x)))$, showing that $\xi(\varphi(t,u,x))$ is also the solution of the differential equation $\dot{y}(t) = Y_0(y(t)) + \sum_{j=1}^{m} u_j Y_j(y(t))$ in $M_2$, with initial value $\xi(\varphi(0,u,x)) = \xi(x)$. Therefore $\xi(\varphi(t,u,x)) = \psi(t,u,\xi(x))$, for all $(t,u,x) \in \mathbb{R} \times U \times M_1$. ■

Another well known concept (see e.g. [12]) is the diffeomorphic control systems.

**Definition 10** Using the above notations, the control systems $(\Sigma_1)$ and $(\Sigma_2)$ are diffeomorphic if there exists a diffeomorphism $\xi : M_1 \to M_2$ such that $\xi_*(X_i) = Y_i$ for $0 \leq i \leq m$.

Then we have the following result that relates state equivalent control systems with diffeomorphic control systems

**Proposition 11** If the control systems $(\Sigma_1)$ and $(\Sigma_2)$ are diffeomorphic then they are state equivalent.

**Proof.** Let $\xi : M_1 \to M_2$ be a diffeomorphism such that $\xi_*(X_i) = Y_i$ for $0 \leq i \leq m$. It is easy to see that $\xi_*(Z_u(x))_x = W_u(\xi(x))$. Then, by Proposition 9 the control systems $(\Sigma_1)$ and $(\Sigma_2)$ are state equivalences. ■
4 Orbit equivalence of semigroup system on homogeneous space

Take a control system $\Sigma$ on a manifold $M$. The purpose of this section is to prove that $(M,\Sigma)$ is orbit equivalent to a semigroup action on a homogeneous space. The Lie-Palais theorem is fundamental to obtain this result.

We begin supposing that there exists a Lie group $G$ acting transitively on $M$. In this case, $M$ is diffeomorphic to a homogeneous space of $G$. From this we prove that there exist a control system $\tilde{\Sigma}$ on a homogeneous space of $G$ such that $\Sigma$ be orbit equivalent to $\tilde{\Sigma}$. Then take the control system

$$(\Sigma) \quad \dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i X_i(x(t))$$

on $M$ with the same hypothesis of the previous section. Consider the Lie algebra $\mathcal{L}(TM)$ of all vector fields on $M$ and take its Lie algebra $\mathcal{L}(\Gamma)$, generated by the set of vector field $\Gamma = \{X_0, X_1, \ldots, X_m\}$. Supposing that $\mathcal{L}(\Gamma)$ has finite dimension we take the connected and simply connected Lie group $G$ with Lie algebra $\mathcal{L}(\Gamma)$. A natural way to define the action of $G$ on $M$ is given in the following way. Denote by $\Psi^X_t$ the flow of $X \in \mathcal{L}(\Gamma)$. As every $g \in G$ can be written as $g = e^{t_1 X_1} \cdots e^{t_s X_s}$, for some $t_1, \ldots, t_s \in \mathbb{R}$ and $X_1, \ldots, X_s \in \mathcal{L}(\Gamma)$, we can try to define an action $\phi : G \times M \to M$ by $\phi(g, x) = \Psi^{X_1 t_1}_t \circ \cdots \circ \Psi^{X_s t_s}_t(x)$. The problem is that there is not just one way to write $g \in G$ as product of exponentials. But using Lie-Palais theorem, we can guarantee that this definition does not depend on this fact. Before, we define the concept of infinitesimal action.

**Definition 12** Let $\mathfrak{g}$ be a Lie algebra and take $M$ a differentiable manifold. An infinitesimal action of $\mathfrak{g}$ on $M$ is a homeomorphism $\theta : \mathfrak{g} \to \mathcal{L}(TM)$.

It is easy see that a differentiable action $\phi : G \times M \to M$ induces an infinitesimal action $\theta : \mathfrak{g} \to \mathcal{L}(TM)$, in fact, define $\theta(X)(x) = d\phi_x|_1(X)$, where $x \in M$ and $1$ denote the identity element of $G$. One kind of converse is the Lie-Palais Theorem.

**Theorem 13** [Lie-Palais] Let $\mathfrak{g}$ be a real and finite dimensional Lie algebra. Take $G$ the connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Consider $\theta : \mathfrak{g} \to \mathcal{L}(TM)$ an infinitesimal action of $\mathfrak{g}$ and suppose that the vector fields $\theta(X), X \in \mathfrak{g}$ be complete. Then exists a differentiable action $\phi : G \times M \to M$ such that $\theta$ is the correspondent infinitesimal action.

The proof of Lie-Palais theorem can be found in San Martin [14].

**Proposition 14** Let $\Gamma = \{X_0, X_1, \ldots, X_m\}$ be a family of complete and differentiable vector fields on the manifold $M$ such that the Lie algebra $\mathcal{L}(\Gamma)$ has finite dimension. Denote by $G$ the connected and simply connected Lie group whose Lie algebra is $\mathcal{L}(\Gamma)$. Then we can define the following action $\phi : G \times M \to M$. Take $g \in G$ hence $g = e^{t_1 X_1} \cdots e^{t_s X_s}$, for some $t_1, \ldots, t_s \in \mathbb{R}$ and $X_1, \ldots, X_s \in \Gamma$. Therefore $\phi(g, x) = \Psi^{X_1 t_1}_t \circ \cdots \circ \Psi^{X_s t_s}_t(x)$.
Proof. Note that the inclusion map $\theta : \mathcal{L}(\Gamma) \to \mathcal{L}(TM)$ is an infinitesimal action of the Lie algebra $\mathcal{L}(\Gamma)$ on $M$. By Lie-Palais theorem, there exists a differential action $\phi : G \times M \to M$ such that $X_x = d\phi_x|_1(X)$, $\forall x \in M$, $\forall X \in \mathcal{L}(\Gamma)$. From the description of this action, take $X \in \mathcal{L}(\Gamma)$ and consider the field $(X, \theta(X))$ on $G \times M$. The trajectory of this field beginning in $(1, x)$ is $(e^{tX}, \Psi_t^X(x))$, that is,
\[
\frac{d}{dt} \big|_{t=0} (e^{tX}, \Psi_t^X(x)) = (X, \theta(X)_x).
\]
On the other hand, taking $\phi_x : G \to M$, applying in $e^{tX}(1) \in G$ and using Lie-Palais Theorem we have also
\[
\frac{d}{dt} \big|_{t=0} (e^{tX}, \phi_x(e^{tX})) = (X, \theta(X)_x).
\]
Then $(e^{tX}, \Psi_t^X(x)) = (e^{tX}, \phi_x(e^{tX}))$, i.e.,
\[
\Psi_t^X(x) = \phi_x(e^{tX}) = \phi(e^{tX}, x), \forall x \in M, \forall X \in \mathcal{L}(\Gamma), \forall t \in \mathbb{R}. \quad (3)
\]
Hence, if $X, Y \in \mathcal{L}(\Gamma), x \in M \in t, \tau \in \mathbb{R}$, then
\[
\Psi^Y_{-\tau}(\Psi_t^X(x)) = \phi_{\phi_x(e^{\tau Y})}(e^{\tau X}) = \phi_{e^{\tau Y}e^{\tau X}}(x).
\]
By induction we have for all $x \in M, X_1, \ldots, X_n \in \mathcal{L}(\Gamma)$ and $t_1, \ldots, t_n \in \mathbb{R}$
\[
\Psi_{t_1}^{X_1} \circ \cdots \circ \Psi_{t_n}^{X_n}(x) = \phi_{e^{t_1}X_1 \cdots e^{t_n}X_n}(x). \quad \blacksquare
\]
If we suppose that the family $\Gamma$ is transitive we have:

Theorem 15 Let $\Gamma = \{X_0, X_1, \ldots, X_m\}$ be a family of transitive, complete and differentiable vector fields on the connected manifold $M$. Suppose that the Lie algebra $\mathcal{L}(\Gamma)$ has finite dimension and take $G$ its associated connected and simply connected Lie group. Then, $M$ is diffeomorphic to a $G$-homogeneous space.

Proof. Consider the action $\phi : G \times M \to M$ the action given in the previous proposition. Then, $\phi(g, x) = \Psi_{t_1}^{X_1} \circ \cdots \circ \Psi_{t_n}^{X_n}(x)$ and as $\Gamma$ is transitive we have that this action is transitive. Hence, fixing $x_0 \in M$ and considering the isotropy subgroup $H_{x_0} = \{g \in G : \phi(g, x_0) = x_0\}$ we that $M$ is diffeomorphic to the homogeneous space $G/H_{x_0}$. $\blacksquare$

Now we describe this above diffeomorphism. If $x \in M$, as $\Gamma$ is transitive, there exist $X_{i_1}, \ldots, X_{i_s} \in \Gamma$ and $t_{i_1}, \ldots, t_{i_s} \in \mathbb{R}$ such that
\[
x = \Psi_{t_{i_1}}^{X_{i_1}} \circ \cdots \circ \Psi_{t_{i_s}}^{X_{i_s}}(x_0) = \phi_{e^{t_{i_1}X_{i_1}} \cdots e^{t_{i_s}X_{i_s}}}(x_0).
\]
In this case, the above diffeomorphism, denoted by $\xi : M \longrightarrow G/H_{x_0}$, is defined by $\xi(x) = (e^{t_{i_1}X_{i_1}} \cdots e^{t_{i_s}X_{i_s}})H_{x_0}$ and its inverse is given in the following way. Given $g \in G$, there exist $X_{i_1}, \ldots, X_{i_s} \in \Gamma$ and $t_{i_1}, \ldots, t_{i_s} \in \mathbb{R}$ such that $g =\ldots
\( e^{t_1 X_i} \cdots e^{t_s X_i} \). Remember that this choices are not unique. In this case, define

\[
\xi^{-1}(g H_{x_0}) = \Psi_{t_{i_1}}^X \circ \cdots \circ \Psi_{t_{i_s}}^X(x_0),
\]

where by Proposition 14 this definition does not depend on the exponential form of \( g \).

To finish this section we prove a result that relates a control system on \( M \) with his induced system on \( G/H_{x_0} \). But first we show an important lemma to the sequence of this paper. Consider the map \( f \) defined as \( \xi^{-1} \circ \pi : G \rightarrow M \), where \( \pi : G \rightarrow G/H_{x_0} \) is the canonical projection. With this, \( \pi(g) = \xi(f(g)), \forall g \in G \), and as \( \xi^{-1} \) and \( \pi \) are surjective maps it follows that \( f \) is surjective.

**Lemma 16** If \( X \in \mathcal{L}(\Gamma) \) then \( \pi_*(X) = \xi_*(X) \).

**Proof.** Take \( X \in \mathcal{L}(\Gamma) \), \( g \in G \) and \( x \in M \) such that \( f(g) = x \). Consider \( e^{tX} g \) the trajectory of \( X \) in \( G \) with initial point \( g \in G \). Consider \( \Psi_t^X(x) \) the trajectory of \( X \) in \( M \) with initial point \( x \in M \). Then,

\[
\frac{d}{dt}|_{t=0}(\xi(\Psi_t^X(x))) = d\xi|_{x}(X_x) \quad \text{and} \quad \frac{d}{dt}|_{t=0}(\pi(e^{tX} g)) = d\pi|_{g}(X_g).
\]

Note that there exist \( X_i, \ldots, X_k \in \Gamma \) and \( t_i, \ldots, t_k \in \mathbb{R} \) such that \( g = e^{t_1 X_{i_1}} \cdots e^{t_s X_{i_s}} \). Also there is \( g_1 \in G \) such that \( x = \phi(g_1, x_0) \). Analogously, there are \( X_{j_1}, \ldots, X_{j_k} \in \Gamma \) and \( t_j, \ldots, t_k \in \mathbb{R} \) such that \( g_1 = e^{t_{j_1} X_{j_1}} \cdots e^{t_{j_k} X_{j_k}} \). Hence

\[
\xi(\phi(g, x)) = \xi(\phi(g, g_1)) = g_1 H_{x_0}.
\]

As \( \pi(gg_1) = gg_1 H_{x_0} \), then \( \pi(gg_1) = \xi(\phi(g, x)) \). In particular, given \( X \in \mathcal{L}(\Gamma) \) and \( t \in \mathbb{R} \), \( \pi(e^{tX} g) = \xi(\phi(e^{tX}, x)) \). By [13], we have \( \pi(e^{tX} g) = \xi(\Psi_t^X(x)) \).

Hence, from (5) we have \( \pi_*(X) = \xi_*(X) \).

Returning to the control system \( (\Sigma) \) on \( M \) and taking the vector fields \( \tilde{X}_i = \pi_*(X_i) \), \( 0 \leq i \leq m \) on \( G/H_{x_0} \), we define the following control system on \( G/H_{x_0} \):

\[
(\tilde{\Sigma}) \quad \tilde{x}(t) = \tilde{X}_0(\tilde{x}(t)) + \sum_{i=1}^m u_t \tilde{X}_i(\tilde{x}(t)).
\]

Note that by Lemma 15 \( \xi_*(X_i) = \tilde{X}_i \) for \( 0 \leq i \leq m \), and knowing that \( \xi : M \rightarrow G/H_{x_0} \) is a diffeomorphism, we have that the control systems \( (\Sigma) \) and \( (\tilde{\Sigma}) \) are diffeomorphic. Consequently, by Proposition 11 it follows that \( (\Sigma) \) and \( (\tilde{\Sigma}) \) are state equivalent. Denoting by \( S_\Sigma \) and \( S_{\tilde{\Sigma}} \) the associated semigroups, using the Proposition 8 and recalling that state equivalent systems are topologically conjugate we conclude the following theorem

**Theorem 17** Let \( M \) be a connected and differentiable manifold and consider \( (\Sigma) \) the above control system. Suppose that \( \Gamma = \{X_0, X_1, \ldots, X_m\} \) is transitive and complete on \( M \). Suppose also that the Lie subalgebra of \( \mathcal{L}(TM) \), generated by \( \Gamma \), has finite dimension. Then, the action \( (M, S_\Sigma) \) is orbit equivalent to a semigroup action on a homogeneous space.

**Proof.** As we see above, the action \( (M, S_\Sigma) \) is orbit equivalent to the action \( (G/H_{x_0}, S_{\tilde{\Sigma}}) \).
5 Generalized linear system on manifolds

Our goal in this section is to introduce the concept of linear control systems on general manifolds and using the results of the previous sections show that, under certain conditions, a linear control system on a manifold is orbit equivalent to a linear control system on a homogeneous space.

Recall that the concept of linear control system depends on the structure of the Lie group. Then to define this concept on general manifolds we must work around the lack of the Lie group. Now we define the generalized linear control system. Let \( M \) be a connected manifold with finite dimension and denote by \( \mathcal{L}(TM) \) the Lie algebra of the differentiable vector fields on \( M \).

**Definition 18** A generalized linear control system on \( M \) is a control system

\[
(\Lambda) \quad \dot{x} = F(x) + \sum_{j=1}^{m} u_j Y_j(x)
\]

where

1. the set of vector fields \( \Gamma = \{ Y_1, \ldots, Y_m \} \) generates the finite dimensional Lie subalgebra \( \mathcal{L}(\Gamma) \) of \( \mathcal{L}(TM) \) and every vector field \( Y_i \in \Gamma \) is complete,
2. \( F \in \mathcal{L}(TM), [F, X] \in \mathcal{L}(\Gamma), \forall X \in \mathcal{L}(\Gamma) \) and there exists \( x_0 \in M \) such that \( F_{x_0} = 0 \),
3. and \( u = (u_1, \ldots, u_m) \in \mathbb{R}^m \).

It is clear that a linear control system on a Lie group is a generalized linear control system, but not all generalized linear control system is a linear control system. In fact, in case of generalized linear control system, the vector fields \( Y_i \) are not necessarily invariants.

Now we have our main result

**Theorem 19** Consider \( M \) a connected and simply connected differentiable manifold. Let \( (\Lambda) \) be the above generalized linear control system on \( M \). If \( \Gamma = \{ Y_1, \ldots, Y_m \} \) is transitive on \( M \), then the action \((M, S_\Lambda)\) is orbit equivalent to a semigroup action associated to a linear control system on a homogeneous space.

**Proof.** By Theorem 17 we need define a diffeomorphism \( \xi \) that carries \( \Lambda \) in a linear control system \( \tilde{\Lambda} \) on a homogeneous space. Now we define this homogeneous space, by Theorem 16 we take \( G \) the connected and simply connected Lie group with Lie algebra \( \mathcal{L}(\Gamma) \). Note that \( G \) acts transitively on \( M \). From this action, take \( H \subset G \), the isotropy subgroup in \( x_0 \in M \), then we have the diffeomorphism \( \xi : M \rightarrow G/H \) given by \( \xi(g \cdot x_0) = gH \), where \( \cdot \) denotes the action of \( G \) on \( M \). Hence, we need to show that when we apply \( \xi \) in \( (\Lambda) \) we get a linear control system on \( G/H \), i.e., \( \xi_* (F) \) is a linear vector field and \( \xi_* (Y_j) \) is right invariant vector field for \( i = \{1, \ldots, m\} \).

As \( \xi \) is a diffeomorphism, then \( \xi_* (Y_j) \) and \( Y_j \) are \( \xi \)-related. Then, as \( Y_j \) is invariant we have that \( \pi_* (Y_j) \) is invariant on \( G/H \). Moreover, by Lemma 16 we have that \( \pi_* (Y_j) = \xi_* (Y_j) \), for all \( X \in \mathfrak{g} \), therefore \( \xi_* (Y_j) \) is invariant on \( G/H \).
We need to show that $\xi_*(\mathcal{F})$ is a linear vector field, i.e., $\xi_*(\mathcal{F})$ is $\pi$-related with a linear vector field on $G$. First, we find this linear vector field on $G$. By Lemma 10 if $X \in \mathfrak{g}$ then

$$[\xi_*(\mathcal{F}), \pi_*(X)] = \pi_*[\mathcal{F}, X].$$

(6)

Let $D : \mathfrak{g} \rightarrow \mathfrak{g}$ be a derivation defined by $D(X) = [\mathcal{F}, X]$. As $G$ is connected and simply connected, there exists a linear vector field $\mathcal{X}$ on $G$ such that $D(X) = \mathcal{X}$, $\forall X \in \mathfrak{g}$.

Then we prove that $\xi_*(\mathcal{F})$ is $\pi$-related with $\mathcal{X}$. To do this, we prove that $\pi_*(\mathcal{X})$ is $\pi$-related with $\mathcal{X}$ and then we show that $\pi_*(\mathcal{X}) = \xi_*(\mathcal{F})$.

Hence we first show that $H$ is invariant by the flow $\phi_t$ of $\mathcal{X}$. Note that the vector field $\xi_*(\mathcal{F})$ in the point $H \in G/H$, $\xi_*(\mathcal{F})_H$, is equal to

$$d\xi \mid_{x_0} (\mathcal{F}_x) = d\xi \mid_{x_0} (0) = 0,$$

(7)

since $\xi(x_0) = 1 \cdot x_0 = 1H = H$.

Note also that, $\pi_*(Y)_H = 0$ for all $Y$ in the Lie algebra $\mathfrak{h}$ of $H$. In fact, as $Y \in \mathfrak{h}$ then $\exp(tY) \in H$, for all $t \in \mathbb{R}$. So, $\exp(tY) \cdot x_0 = x_0$, for all $t \in \mathbb{R}$. Hence, $\pi_*(Y)_H = 0$. Therefore, as $D(Y) = 0$ and $\xi_*(\mathcal{F})_H = 0$, we have that

$$[\xi_*(\mathcal{F}), \pi_*(Y)]_H = 0, \forall Y \in \mathfrak{h}.$$

Note that $\pi_*(\mathcal{X}, Y)]_H = 0$. Hence, its flow given by $gH \mapsto (\exp(t\mathcal{X}Y))_HgH$ satisfies $(\exp(t\mathcal{X}Y)) \cdot H = H$, for all $t \in \mathbb{R}$. Therefore, $\exp(t\mathcal{X}Y) \in H$, then $D(Y) = \mathcal{X}$, $\forall Y \in \mathfrak{h}$.

This implies that

$$\phi_t(\exp Y) = \exp(e^{tD}Y) = \exp(I + tD + \frac{t^2D^2}{2!} + \cdots)Y \in H.$$

Then, $\phi_t(\exp Y) \in H$, $\forall t \in \mathbb{R}$ $\forall Y \in \mathfrak{h}$. As $M$ is connected, simply connected and diffeomorphic to $G/H$, it follows that $G/H$ is simply connected. Then $H$ is connected. Hence, every element of $H$ is product of exponentials of elements of $\mathfrak{h}$ and as $\phi_t$ is an isomorphism then $H$ is invariant by the flow $\phi_t$. Consequently, $\pi_*(\mathcal{X})$ is a vector field on $G/H$ $\pi$-related with $\mathcal{X}$.

To conclude the proof, we show that $\pi_*(\mathcal{X}) = \xi_*(\mathcal{F})$. In fact, if $X \in \mathfrak{g}$, then $[\xi_*(\mathcal{F}), \pi_*(X)] = \pi_*[\mathcal{F}, X]$. Note that $[\pi_*(\mathcal{X}), \pi_*(X)]$ and $[\mathcal{X}, X]$ are $\pi$-related, hence $\pi_*[\mathcal{X}, X] = [\pi_*(\mathcal{X}), \pi_*(X)]$, therefore $[\pi_*(\mathcal{X}) - \xi_*(\mathcal{F}), \pi_*(X)] = 0, \forall X \in \mathfrak{g}$.

Then the flow of $\pi_*(\mathcal{X}) - \xi_*(\mathcal{F})$ on $G/H$, denoted by $\alpha_t$, commute with the flow of $\pi_*(X)$, given by $gH \mapsto (\exp(tX))gH$.

As $\mathcal{X}$ is linear, then $\pi_*(\mathcal{X})_H = 0$. Moreover, from (7) we have $\xi_*(\mathcal{F})_H = 0$, then $\pi_*(\mathcal{X})_H = \xi_*(\mathcal{F})_H = 0$. Hence, $(\pi_*(\mathcal{X}) - \xi_*(\mathcal{F}))_H = 0$, so $\alpha_t(H) = H, \forall t \in \mathbb{R}$.

Consider, $g \in G$, as $G$ is connected, there exist $Y_1, \ldots, Y_r \in \mathfrak{g}$ $\epsilon t_{i_1}, \ldots, t_{i_r} \in \mathbb{R}$ such that $g = \exp(t_{i_1}Y_{i_1}) \cdots \exp(t_{i_r}Y_{i_r})$. Then

$$\alpha_t(gH) = gH, \forall t \in \mathbb{R}.$$
Therefore, \((\pi_*(\mathcal{X}) - \xi_*(\mathcal{F}))_{\gamma H} = 0\), i.e., \(\pi_*(\mathcal{X}) = \xi_*(\mathcal{F})\). \(\blacksquare\)

In the previous theorem the hypothesis \(M\) simply connected is fundamental. In fact, this implies that \(H\) is connected. Then, consequently every element of \(H\) is a product of exponentials of elements of \(h\). Now we show that a generalization of this last theorem, where it is not necessary has \(M\) simply connected. To get this, the concept of universal covering is essential. Then, consequently every element of \(H\) is connected. Then, we show that a generalization of this last theorem, where it is not necessary has \(M\) simply connected. To get this, the concept of universal covering is essential. Then, recall that given a universal covering \(f : \tilde{M} \to M\), where \(\tilde{M}\) is a differential manifold such that \(f\) is differentiable. We can lift the vector fields on \(M\) to \(\tilde{M}\), that is, given \(Z \in TM\), the vector field \(\tilde{Z} \in T\tilde{M}\) is defined in the following way. Given \(\tilde{x} \in \tilde{M}\), as \(f\) is differentiable covering, there exist open neighborhoods \(\tilde{U}\) of \(\tilde{x}\) in \(\tilde{M}\) and \(U\) of \(x\) in \(M\) such that \(f|_\tilde{U} : \tilde{U} \to U\) is diffeomorphism. Then we define

\[
\tilde{Z}_{\tilde{x}} = d(f|_\tilde{U})^{-1}|_x(Z_x).
\]

Consider \(M\) a differentiable connected manifold and take in \(M\) the generalized linear control system

\[
(\Lambda) \quad \dot{x} = \mathcal{F}(x) + \sum_{j=1}^{m} u_j Y_j(x).
\]

Then we have the following theorem

**Theorem 20** Suppose that the family of vector fields \(\Gamma = \{Y_1, \ldots, Y_m\}\) is transitive on \(M\). Then the action \((M, S_\Lambda)\) is orbit equivalent to a semigroup action associated to a linear control system on a homogeneous space.

**Proof.**
Let \(f : \tilde{M} \to M\) be the above differentiable covering. Then from \(\Lambda\) we define the following system in \(\tilde{M}\):

\[
(\tilde{\Lambda}) \quad \dot{x} = \tilde{\mathcal{F}}(x) + \sum_{j=1}^{m} u_j \tilde{Y}_j(x),
\]

where \(\tilde{\mathcal{F}}\) and \(\tilde{Y}_j\) are as defined above.

Consider \(\tilde{\Gamma} = \{\tilde{Y}_1, \ldots, \tilde{Y}_m\}\). By definition of \(\tilde{Y}_j\) we have that the family \(\tilde{\Gamma}\) is complete and that \(L(\Gamma)\) is isomorphic to \(L(\tilde{\Gamma})\).

Note that \(\tilde{\Gamma}\) is transitive. In fact, every \(f\)-image of orbit is an orbit in \(M\), moreover, the rank of \(f\) is constant in every orbit. As \(\Gamma\) is transitive in \(M\), the \(\Gamma\)-orbit has the same dimension as \(M\), and therefore, as \(\tilde{\Gamma}\). Then, the \(\tilde{\Gamma}\)-orbits in \(\tilde{M}\) are submanifolds of the same dimension of \(M\). As \(\tilde{M}\) is connected and is the union of the \(\tilde{\Gamma}\)-orbits, it follows that exists just one \(\tilde{\Gamma}\)-orbit. Therefore, \(\tilde{\Gamma}\) is transitive.

Moreover, we have that

\[
[\tilde{\mathcal{F}}, \tilde{Y}_i] \in L(\tilde{\Gamma}),
\]

and as \(\mathcal{F}_{x_0} = 0\) it follows that \(\tilde{\mathcal{F}}_{\tilde{x}_0} = 0\) for all \(\tilde{x}_0 \in f^{-1}(x_0)\).
Consider the connected and simply connected Lie group \( G \) with Lie algebra \( \mathcal{L}(\Gamma) \) (and \( \mathcal{L}(\tilde{\Gamma}) \)).

Then by Proposition 14 we have the actions

\[
G \times M \to M \quad \text{and} \quad G \times \tilde{M} \to \tilde{M}.
\]

Take \( \tilde{x}_0 \in f^{-1}(x_0) \) then we have the isotropy subgroups

\[
H = \{ g \in G; gx_0 = x_0 \} \quad \text{and} \quad \tilde{H} = \{ g \in G; g\tilde{x}_0 = \tilde{x}_0 \}.
\]

Hence we have the diffeomorphisms \( \xi : M \to G/H \) given by \( \xi(g \cdot x_0) = gH \) and \( \tilde{\xi} : \tilde{M} \to G/\tilde{H} \) with \( \xi(g \cdot \tilde{x}_0) = gH \), here \( \cdot \) denote the action of \( G \) on \( M \) or \( \tilde{M} \).

As \( M \) is simply connected then \( \tilde{H} \) is connected. As we see in the demonstration of the previous result, we have that \( \Lambda \) is diffeomorphic to the linear control system in \( G/\tilde{H} \). Now we describe this system on \( G/\tilde{H} \).

Consider \( D : \mathcal{L}(\tilde{\Gamma}) \to \mathcal{L}(\Gamma) \) given by \( D(Y) = [\tilde{\mathcal{F}}, Y] \), note that \( D \) is derivation. Then there exists a linear vector field \( \mathcal{X} \) on \( \tilde{G} = G \) such that \( D(Y) = [\mathcal{X}, Y] \) for every \( Y \in \mathcal{L}(\Gamma) \). Let \( \tilde{\pi} : G \to G/\tilde{H} \) be the canonical projection. By previous result, we have that \( \Lambda \) is diffeomorphic to the following linear control system in \( G/\tilde{H} \):

\[
(\Lambda_{\tilde{x}}) \quad \dot{x} = \tilde{\pi}_*(\mathcal{X}) + \sum_{j=1}^{m} u_j \tilde{\pi}_*(\tilde{Y}_j),
\]

where \( \tilde{\pi}_*(\mathcal{X}) = \tilde{\xi}_*(\tilde{\mathcal{F}}) \) and \( \tilde{\pi}_*(\tilde{Y}_j) = \tilde{\xi}_*(\tilde{Y}_j) \).

Note that \( \tilde{\pi}_*(\mathcal{X}) \) exists, i.e., \( \dot{\tilde{x}} \) is invariant by the flow of \( \mathcal{X} \).

It is not difficult to see that \( \tilde{l} : G/\tilde{H} \to G/H \) defined by \( \tilde{l}(g\tilde{H}) = gH \) is a differentiable covering.

Recall that we need to show that \( \xi_*(\mathcal{F}) \) is linear vector field on \( G/H \) and that \( \xi_*(Y_j) \) are right invariant vector field for \( i = 1, \ldots, m \). As \( Y_j \) is invariant we have that \( \pi_*(Y_j) \) is invariant on \( G/H \). By Lemma 16 we have that \( \pi_*(Y_j) = \xi_*(\tilde{Y}_j) \), then \( \xi_*(\tilde{Y}_j) \) are invariants.

The vector field \( \xi_*(\mathcal{F}) \) is linear if \( \xi_*(\mathcal{F}) \) is \( \pi \)-related with a linear vector field on \( G \). Then, we first show that \( \pi_*(\mathcal{X}) \) is linear on \( G/H \), i.e., \( \mathcal{X} \) is \( \pi \)-related with \( \pi_*(\mathcal{X}) \) in \( G/H \). After this, we prove that \( \xi_*(\mathcal{F}) = \pi_*(\mathcal{X}) \).

First we note that \( \xi_*(\mathcal{F}) \) is null in \( H/\tilde{H} \). As \( \xi_*(\mathcal{F}) = \pi_*(\mathcal{X}) \) then \( \pi_*(\mathcal{X}) \) is null in \( H/\tilde{H} \). Then we can prove that \( H \) is invariant by the flow of \( \mathcal{X} \). So \( \mathcal{X} \) is \( \pi \)-related with the vector field \( \pi_*(\mathcal{X}) \) on \( G/H \).

Now we must prove that \( \xi_*(\mathcal{F}) = \pi_*(\mathcal{X}) \). As \( \tilde{\mathcal{F}_x} = d(f|_{\tilde{\mathcal{F}}})^{-1} \pi_*(\mathcal{F}_x) \) and \( \tilde{\xi} \) and \( \xi \) are diffeomorphisms it follows that

\[
\xi_*(\mathcal{F})|_{gH} = d(l|_{\pi})^{-1} (\xi_*(\mathcal{F})|_{gH})
\]

and

\[
\pi_*(\mathcal{X})|_{gH} = d(l|_{\tilde{\mathcal{F}}})^{-1} (\pi_*(\mathcal{X})|_{gH})
\]

then \( \xi_*(\mathcal{F}) = \pi_*(\mathcal{X}) \).
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