Effective dielectric constant for a random medium

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In this paper, we present an approximate expression for determining the effective permittivity describing the coherent propagation of an electromagnetic wave in random media. Under the Quasicrystalline Coherent Potential Approximation (QC-CPA), it is known that multiple scattering theory provided an expression for this effective permittivity. The numerical evaluation of this one is, however, a challenging problem. To find a tractable expression, we add some new approximations to the (QC-CPA) approach. As a result, we obtained an expression for the effective permittivity which contained at the same time the Maxwell-Garnett formula in the low frequency limit, and the Keller formula, which has been recently proved to be in good agreement for particles exceeding the wavelength.

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I. INTRODUCTION

The description of electromagnetic waves propagation in random media in term of the properties of the constituents has been studied extensively in the past decades [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. In most of works, the basic idea is to calculate several statistical moments of the electromagnetic field to understand how the wave interact with the the random medium [8, 11, 12, 13, 16]. In this paper, we are concerned by the first moment which is the average electric field. Under some assumption, it can be shown that the average electric field propagates as if the medium were homogeneous but with a renormalized permittivity, termed effective permittivity. The calculation of this parameter as a long history which dates back from the work of Clausius-Mossotti and Maxwell Garnett [24]. Since then, most of the study are concerned with the quasi-static limit where retardation effect are neglected [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34]. In order to take into account scattering effects, quantum multiple scattering theory has been transposed in the electromagnetic case [8, 11, 12, 13, 16], but as a rigorous analytical answer is unreachable, several approximation schemes have been developed [6, 8, 11, 12, 16, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48]. One of the most advanced is the Quasicrystalline Coherent Potential Approximation (QC-CPA) which takes into account the correlation function (QC-CPA) which is the Quasicrystalline Coherent Potential Approximation (QC-CPA), it is known that multiple scattering theory provided an expression for this effective permittivity. The numerical evaluation of this one is, however, a challenging problem. To find a tractable expression, we add some new approximations to the (QC-CPA) approach. As a result, we obtained an expression for the effective permittivity which contained at the same time the Maxwell-Garnett formula in the low frequency limit, and the Keller formula, which has been recently proved to be in good agreement for particles exceeding the wavelength.

II. DYSON EQUATION AND EFFECTIVE PERMITTIVITY

In the following, we consider harmonic waves with $e^{-i\omega t}$ pulsation. We consider an ensemble of $N \gg 1$ identical spheres of radius $r_s$ with dielectric function $\varepsilon_s(\omega)$ within an infinite medium with dielectric function $\varepsilon_1(\omega)$. The field produced at $r$ by a discrete source located at $r_0$ is given by the dyadic Green function $G(r, r_0, \omega)$, which verifies the following propagation equation:

$$\nabla \times \nabla \times G(r, r_0, \omega) - \varepsilon_1(r, \omega) K_{\text{vac}}^2 G(r, r_0, \omega) = \delta(r - r_0) \mathbf{T} \quad (1)$$
where $K_{\text{vac}} = \omega/c$ with $c$ the speed of light in vacuum and

$$
\epsilon_{V}(r, \omega) = \epsilon_{1}(\omega) + \sum_{j=1}^{N}(\epsilon_{s}(\omega) - \epsilon_{1}(\omega)) \Theta_{s}(r - r_{j}),
$$

where $r_{1}, \ldots, r_{N}$ are the center of the particles and $\Theta_{s}$ describes the spherical particle shape:

$$
\Theta_{s}(r) = \begin{cases} 
1 & \text{if } \|r\| < r_{s} \\
0 & \text{if } \|r\| > r_{s} .
\end{cases}
$$

The solution of equation (1) is uniquely defined if we impose the radiation condition at infinity.

The multiple scattering process by the particles is mathematically decomposed in introducing the Green function $\mathbf{G}_{1}$, describing the propagation within an homogeneous medium with permittivity $\epsilon_{1}(\omega)$, which verifies the following equation:

$$
\nabla \times \nabla \times \mathbf{G}_{1}(r, r_{0}, \omega) - \epsilon_{1}(\omega)K_{\text{vac}}^{2} \mathbf{G}_{1}(r, r_{0}, \omega) = \delta(r - r_{0})I ,
$$

with the appropriate boundary conditions. In an infinite random medium, we have [11, 12, 13, 16]:

$$
\mathbf{G} = \mathbf{G}_{1}^{\infty} + \mathbf{G}_{1}^{\infty} \cdot \nabla \cdot \mathbf{G} ,
$$

where $K_{1}^{2} = \epsilon_{1}(\omega)K_{\text{vac}}^{2}$.

In using this Green function, we decompose the Green function $\mathbf{G}(r, r_{0}, \omega)$ under the following form [6, 11, 13, 16]:

$$
\mathbf{G} = \mathbf{G}_{1}^{\infty} + \mathbf{G}_{1}^{\infty} \cdot \nabla \cdot \mathbf{G} ,
$$

where the following operator notation is used:

$$
[\mathbf{A} \cdot \mathbf{B}](r, r_{0}) = \int d^{3}r_{1} \mathbf{A}(r, r_{1}) \cdot \mathbf{B}(r_{1}, r_{0}) .
$$

The potential $\mathbf{V}$, which describes the interaction between the wave and the particles, is given by:

$$
\mathbf{V} = \sum_{i=1}^{N} \mathbf{v}_{r_{i}} ,
$$

$$
\mathbf{v}_{r_{i}}(r, r_{i}, \omega) = (2\pi)^{2} \delta(r - r_{0}) \mathbf{v}_{r_{i}}(r, \omega) ,
$$

$$
\mathbf{v}_{r_{i}}(r, \omega) = [K_{s}^{2} - K_{1}^{2}] \Theta_{d}(r - r_{i})I .
$$

with $K_{s}^{2} = \epsilon_{s}(\omega)K_{\text{vac}}^{2}$. It is useful to introduce the T matrix defined by [6, 11, 13, 14]:

$$
\mathbf{G} = \mathbf{G}_{1}^{\infty} + \mathbf{G}_{1}^{\infty} \cdot \mathbf{T} \cdot \mathbf{G}_{1}^{\infty} .
$$

In iterating equation (16) and comparing it with the definition (10), we show that the T matrix verifies the following equation:

$$
\mathbf{T} = \mathbf{V} + \nabla \cdot \mathbf{G}_{1}^{\infty} \cdot \mathbf{T} .
$$

If we introduce the T matrix of each scatterer by:

$$
\mathbf{I}_{r_{i}} = \mathbf{V}_{r_{i}} + \mathbf{G}_{r_{i}}^{\infty} \cdot \mathbf{T}_{r_{i}} ,
$$

we can decompose the T matrix for the whole system, in a series of multiple scattering processes by the particles [6, 11, 12, 16]:

$$
\mathbf{T} = \sum_{i=1}^{N} \mathbf{I}_{r_{i}} + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \mathbf{I}_{r_{i}} \cdot \mathbf{G}_{1}^{\infty} \cdot \mathbf{I}_{r_{j}} \cdots .
$$

This T matrix is useful to calculate the average field $< \mathbf{G} >$ since we have:

$$
< \mathbf{G} > = \mathbf{G}_{1}^{\infty} + \mathbf{G}_{1}^{\infty} \cdot < \mathbf{T} > \mathbf{G}_{1}^{\infty} .
$$

The equivalent of the potential operator $\mathbf{V}$ for the average Green function $< \mathbf{G} >$ is the mass operator $\Sigma$ defined by:

$$
< \mathbf{G} > = \mathbf{G}_{1}^{\infty} + \mathbf{G}_{1}^{\infty} \cdot \Sigma \cdot < \mathbf{T} > .
$$

Similarly to equation (16), we have the following relationship between the average T matrix and the mass operator:

$$
< \mathbf{T} > = \Sigma + \Sigma \cdot \mathbf{G}_{1}^{\infty} \cdot < \mathbf{T} >
$$

or equivalently,

$$
\Sigma = < \mathbf{T} > \cdot \left[ \mathbf{T} + \mathbf{G}_{1}^{\infty} \cdot < \mathbf{T} > \right]^{-1} .
$$

The operator correspond to all irreducible diagrams in the Feynman representation [4, 8, 11, 12]. The equation (16) written in differential form is:

$$
\nabla \times \nabla \times < \mathbf{G}(r, r_{0}, \omega) > - \epsilon_{1}(\omega)K_{\text{vac}}^{2} < \mathbf{G}(r, r_{0}, \omega) >

- \int d^{3}r_{1} \Sigma(r, r_{1}, \omega) \cdot < \mathbf{G}(r_{1}, r_{0}, \omega) > = \delta(r - r_{0})I .
$$

For a statistical homogeneous medium we have:

$$
\Sigma(r, r_{1}, \omega) = \Sigma(r - r_{1}, \omega) ,
$$

$$
< \mathbf{G}(r, r_{0}, \omega) > = < \mathbf{G}(r - r_{0}, \omega) > .
$$

Thus, we can use a Fourier transform:

$$
\mathbf{\Sigma}(k, \omega) = \int d^{3}r \exp(-ik \cdot r) \mathbf{\Sigma}(r, \omega) ,
$$

$$
\mathbf{\overline{G}}(k, \omega) = \int d^{3}r \exp(-ik \cdot r) \mathbf{\overline{G}}(r, \omega) ,
$$

and equation (19) becomes:

$$
\left[ ||k||^{2} (\mathbf{T} - ik) - \epsilon_{1}(\omega)K_{\text{vac}}^{2} \mathbf{T} - \mathbf{\Sigma}(k, \omega) \right] \cdot < \mathbf{G}(k, \omega) > = \mathbf{T} .
$$
For a statistical isotropic medium, we have:
\[ \Sigma(k, \omega) = \Sigma_{ \perp} (|k|, \omega) (\mathbf{T} - \hat{k} \hat{k}) + \Sigma_{ \parallel} (|k|, \omega) \hat{k} \hat{k}. \] (25)
with \( \hat{k} = k / |k| \) and then:
\[ < \mathbf{G}(k, \omega) > = \left[ |k|^2 (\mathbf{T} - \hat{k} \hat{k}) - \epsilon_1(\omega) K_{\text{vac}}^2 \mathbf{T} - \Sigma(k, \omega) \right]^{-1} \]
\[ = \frac{\mathbf{T} - \hat{k} \hat{k}}{|k|^2 - (\epsilon_1(\omega) K_{\text{vac}}^2 + \Sigma_{ \perp} (|k|, \omega))} \]
\[ - \frac{\epsilon_1(\omega) K_{\text{vac}}^2 + \Sigma_{ \parallel} (|k|, \omega)}{\hat{k} \hat{k}} \] (26)

In the following, we introduce two effective permittivity function \( \epsilon_1^\perp \) and \( \epsilon_1^\parallel \) defined by:
\[ \epsilon_1^\perp(|k|, \omega) K_{\text{vac}}^2 = \epsilon_1(\omega) K_{\text{vac}}^2 + \Sigma_{ \perp} (|k|, \omega), \]
\[ \epsilon_1^\parallel(|k|, \omega) K_{\text{vac}}^2 = \epsilon_1(\omega) K_{\text{vac}}^2 + \Sigma_{ \parallel} (|k|, \omega), \] (27)
and (28) is written:
\[ < \mathbf{G}(k, \omega) > = \frac{\mathbf{T} - \hat{k} \hat{k}}{\epsilon_1^\perp(|k|, \omega) K_{\text{vac}}^2 - \epsilon_1^\parallel(|k|, \omega) K_{\text{vac}}^2} \frac{1}{|k|^2 - \epsilon_1^\perp(|k|, \omega) K_{\text{vac}}^2} \]
\[ + \frac{\epsilon_1^\perp(|k|, \omega) K_{\text{vac}}^2 - \epsilon_1^\parallel(|k|, \omega) K_{\text{vac}}^2}{\hat{k} \hat{k}} \] (29)
The Green function in the space domain is:
\[ < \mathbf{G}(r, \omega) > = \int \frac{d^3k}{(2\pi)^3} \mathbf{T} + \frac{\nabla \nabla}{\epsilon_1^\perp(|k|, \omega) K_{\text{vac}}^2} \frac{e^{i k \cdot r}}{|k|^2 - \epsilon_1^\perp(|k|, \omega) K_{\text{vac}}^2} \]
\[ + \int \frac{d^3k}{(2\pi)^3} \left[ \epsilon_1^\perp(|k|, \omega) K_{\text{vac}}^2 - \epsilon_1^\parallel(|k|, \omega) K_{\text{vac}}^2 \right] \frac{\hat{k} \hat{k}}{|k|^2}. \] (30)

After an integration on the solid angle in equation (29) given by the expression (A1) in the appendix, we obtain:
\[ < \mathbf{G}(r, \omega) > = \int \frac{dK}{|r|} \int_{-\infty}^{+\infty} \frac{dK}{(2\pi)^2} \left[ < \mathbf{G}_{\text{vac}} > + \frac{\nabla \nabla}{\epsilon_1^\perp(K, \omega) K_{\text{vac}}^2} \right] \frac{K e^{i K \cdot |r|}}{K^2 - \epsilon_1^\perp(K, \omega) K_{\text{vac}}^2} \]
\[ + \frac{1}{|r|} \nabla \nabla \int_{-\infty}^{+\infty} \frac{dK}{(2\pi)^2} \left[ \epsilon_1^\perp(K, \omega) K_{\text{vac}}^2 - \epsilon_1^\parallel(K, \omega) K_{\text{vac}}^2 \right] \frac{1}{K} \] (31)

Furthermore, we see that the contribution due to pole \( K = 0 \) in the second term of equation (32) is null; in fact, the dyadic \( \nabla \nabla \) operate on a constant since we have \( \epsilon_1^\perp(K, |r|) = 1 \) for this pole. Hence, we obtain the following expression for the Green function:
\[ < \mathbf{G}(r, \omega) > = \sum_{i=1}^{n} \left[ \mathbf{T} + \frac{\nabla \nabla}{K_{\text{ei}}^2} \right] e^{i K_{\text{ei}} |r|} \frac{4\pi |r|}{K_{\text{ei}}^2}, \] (33)
where \( K_{\text{ei}} \) are the roots of \( K_{\text{ei}}^2 = \epsilon_1^\perp(K_{\text{ei}}, \omega) K_{\text{vac}}^2 \) such as \( \text{Im}(K_{\text{ei}}) > 0 \) to insure that the radiation condition at infinity is verified. Sheng has called the roots \( K_{\text{ei}} \) the quasi-modes of the random media. If we only consider the root \( K_e = K_{ej} \) which has the smallest imaginary part \( \text{Im}(K_{ej}) = \min_i [\text{Im}(K_{ei})] \) and then the smallest exponential factor in equation (33), we define the effective permittivity by \( \epsilon_e(\omega) = \epsilon_1^\perp(K_e, \omega) \). The average Green function is then equal to the Green function for an infinite homogenous medium with permittivity \( \epsilon_e(\omega) \):
\[ < \mathbf{G}(r, \omega) > = \mathbf{G}_{\text{e}}^\infty (r, \omega), \] (34)
where
\[ \mathbf{G}_{\text{e}}^\infty (r, \omega) = \left[ \mathbf{T} + \frac{\nabla \nabla}{K_{\text{e}}^2} \right] e^{i K_{\text{e}} |r|} \frac{4\pi |r|}{K_{\text{e}}^2}. \] (35)

Thus, the effective medium approach is valid if we neglect the longitudinal excitation in the medium and if the propagative mode with the smallest imaginary part is the primary contribution in the development (33).

III. THE COHERENT-POTENTIAL AND QUASI-CRYSTALLINE APPROXIMATIONS

Previously, we have shown how the mass operator is related to the effective permittivity. To calculate the mass operator, we can use equations (14) and (18). However, we can improve this system of equations in rewriting the Green function development (5) in replacing the Green function \( \mathbf{G}^\infty_1 \) by \( \mathbf{G}_e^\infty \):
\[ \mathbf{G} = \mathbf{G}_{\text{e}}^\infty + \mathbf{G}_e^\infty \cdot \nabla \cdot \mathbf{G}, \] (36)
where we have to introduce a new potential \( \mathbf{V}_e \):
\[ \mathbf{V}_e = \sum_{i=1}^{N} \mathbf{v}_{e, r_i}, \]
\[ \mathbf{v}_{e, r_i}(r, \omega) = (2\pi)^2 \delta(r - r_0) \mathbf{v}_{e, r_i}(r, \omega), \]
\[ \mathbf{v}_{e, r_i}(r, \omega) = [K_e^2 - K_{\text{ei}}^2] \left( \epsilon_{\text{iei}}(r - r_i) \mathbf{T} \right. \]
\[ + \left. [K_1^2 - K_{\text{ei}}^2] \mathbf{T} \right]. \] (37)

Similarly to the previous section, we introduce a T matrix such that:
\[ \mathbf{G} = \mathbf{G}_{\text{e}}^\infty + \mathbf{G}_e^\infty \cdot \mathbf{T} \cdot \mathbf{G}_e^\infty. \] (40)
which admits the following decomposition:

\[
\mathbf{T}_e = \sum_{i=1}^{N} \mathbf{t}_{e,r_i} + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \mathbf{t}_{e,r_i} \cdot \mathbf{G}_e^\infty \cdot \mathbf{t}_{e,r_j} + \ldots
\]  

(41)

where we have defined a renormalized T matrix for the particles:

\[
\tilde{\mathbf{t}}_{e,r_i} = \tilde{\mathbf{v}}_{e,r_i} + \mathbf{G}_e^\infty \cdot \mathbf{t}_{e,r_i}.
\]  

(42)

In supposing that the effective medium approach is correct, we impose the following condition on the average field:

\[
\langle \mathbf{G}(r, \omega) \rangle = \mathbf{G}_e(r, \omega),
\]  

(43)

or equivalently,

\[
\langle \mathbf{T}_e \rangle = 0,
\]  

(44)

due to equation (40). The condition (43) is the Coherent-Potential Approximation (CPA) [12, 13, 16, 25, 51]. The expression (44) and (45) form a closed system of equations on the unknown permittivity \( \varepsilon_e(\omega) \). To the first order in density of particles, this system of equations gives equation:

\[
\sum_{i=1}^{N} \langle \mathbf{t}_{e,r_i} \rangle = 0.
\]  

(45)

In Fourier-space, the T matrix for one scatterer verifies the following form:

\[
\sum_{i=1}^{N} v(e,o)(r_i) = \mathbf{T}(r,0) + \mathbf{O},
\]  

(46)

where \( \tilde{\mathbf{v}}_{e,o} \) is the T matrix for a particle located at the origin of coordinate. The average of the exponential term, as it is not the T matrix describing the scattering by a particle of permittivity \( \varepsilon_e(\omega) \) surrounded by a medium of permittivity \( \varepsilon_e(\omega) \), is quiet different from the equation (49), and in particular we have \( \mathbf{T}(r,0) = [K^2_1 - K^2_0] \mathbf{T} \) for \( ||r - r_i|| > r_s \) contrary to the definition (50), where \( \tilde{\mathbf{v}}_{e,r}(r, \omega) = 0 \) when \( r \) is outside the particle. Thus, the operator \( \tilde{\mathbf{t}}_{e,r} \) is non-local and cannot be obtained from the classical Mie theory [2, 4, 33]. To overcome this difficulty, the operator \( \tilde{\mathbf{t}}_{e,r} \) is replaced by the scattering operator of a “structural unit” in the works [11, 21, 57]. Nevertheless, this approach doesn’t seem to have any theoretical justification.

Hence, we prefer to use the more rigorous approach introduced in the scattering theory by disorder liquid metal [55, 59] and adapted in the electromagnetic case by Tsang et al. [13, 16]. In this approach, the non-local term \( [K^2_1 - K^2_0] \mathbf{T} \) is correctly taking into account by averaging equations (41) where the correct potential \( \tilde{\mathbf{v}}_{e,r} \), defined by (49), is used. A system of hierarchic equations is obtained where correlation functions between two or more particles are successively introduced. The chain of equations is closed in using the Quasi-Crystalline Approximation (QCA), which neglect the fluctuation of the effective field, acting on a particle located at \( r_j \), due to a deviation of a particle located at \( r_i \) from its average position (50). This approximation describes the correlation between the particles, only with a two-point correlation function \( g(r_i, r_j) = g(||r_i - r_j||) \). Under the QCA scheme, we obtain the following expression for the mass operator [13, 16, 55, 58, 59]:

\[
\tilde{\mathbf{V}}(k_0, \omega) = n \mathbf{G}_e(o,k_0,k_0),
\]  

(51)

\[
\mathbf{G}_e(o,k_0,k_0) = \mathbf{T}(e,o)(k|k_0)
\]  

(52)

\[
+ n \int \frac{d^3k_1}{(2\pi)^3} h(k - k_1) \tilde{\mathbf{v}}(e,o)(k_1) \cdot \mathbf{G}_e^\infty(k_1) \cdot \mathbf{G}_e(o,k_1,k_0),
\]  

(53)

where

\[
\tilde{\mathbf{v}}(e,o) = \mathbf{v}(e,o) + \mathbf{G}_e^\infty \cdot \mathbf{v}(e,o),
\]  

(54)

\[
\mathbf{G}_e^\infty(k) = \int d^3r \exp(-ik \cdot r) \mathbf{G}_e^\infty(r).
\]  

(55)

and

\[
h(r) = g(r) - 1,
\]  

(56)

\[
h(k - k_1) = \int d^3r \exp(-ik \cdot r) h(r),
\]  

(57)

\[
\mathbf{G}_e^\infty(r) = \int d^3r \exp(-ik \cdot r) \mathbf{G}_e^\infty(r).
\]  

(58)

If we rewrite the potential (55) under the following form:

\[
\mathbf{v}(e,o)(r) = [K^2_s - K^2_0] \Theta_s(r) \mathbf{T},
\]  

(59)

where we have defined a new wave number \( K^2_s = K^2_0 - K^2_1 + K^2_2 \), we see that the operator \( \tilde{\mathbf{v}}(e,o) \) is the T matrix for a scatterer of permittivity \( \varepsilon_s = \varepsilon_0 - \varepsilon_1 + \varepsilon_e \) in a medium of permittivity \( \varepsilon_e \).
As it is described in the previous section, the effective propagation constant is the root, which has the smallest imaginary part, of the equation:

$$K_e^2 = K_0^2 + \Sigma(K_e, \omega), \quad (60)$$

where the mass operator is decomposed under the form \(K_0^2\). Once the effective wave number \(K_e\) obtained, the effective permittivity is given by:

$$\varepsilon_e(\omega) = K_e^2 / K_{vac}^2. \quad (61)$$

**IV. SOME FURTHER APPROXIMATIONS**

As it can be guessed, solving numerically the previous system of equations \(25\) is full of complexities. However, the low frequency limit of this system of equations has been obtained analytically and has shown to be in good agreement with the experimental results \(13\). We have also to mention that the numerical solution of the quasicrystalline approximation (but without the coherent potential approximation) has been developed \(17\).

To reduce the numerical difficulties in the system of equations \(25\), we add two new approximations to the QC-CPA scheme:

- **A far-field approximation**: For an incident plane wave:

  $$E^i(r) = E^i(k_0) e^{i k_0 \cdot r}, \quad (62)$$

  transverse to the propagation direction \(k_0\):

  $$E^i(k_0) \cdot k_0 = 0, \quad (63)$$

  where \(k_0 = K_e k_0\) and \(k_0 \cdot k_0 = 1\), the scattered far-field, by a particle within a medium of permittivity \(\varepsilon_e(\omega)\), is described by an operator \(\hat{f}(k|k_0)\) such that:

  $$E^s(r) = \frac{e^{i K_e ||r||} \hat{f}(k|k_0) \cdot E^i(k_0)}{||r||}. \quad (64)$$

  which verifies transversality conditions:

  $$\hat{f}(k|k_0) \cdot k_0 = 0, \quad (65)$$

  $$k \cdot \hat{f}(k|k_0) = 0. \quad (66)$$

  Moreover, the scattered field in the general case is expressed with the operator \(T_{e,o}\) by:

  $$E^s(r) = \int d^3 r_1 d^3 r_2 \mathcal{G}^\infty_e (r, r_1) \cdot T_{e,o}(r_1|r_2) \cdot E^i(r_2). \quad (67)$$

  In using the phase perturbation method in equation \(67\), the scattered far-field is obtained in function of the operator \(T_{e,o}(k|k_0)\), and in comparing the result with equation \(64\), we obtain the following relationship:

  $$4\pi \hat{f}(k|k_0) = (T - k k) \cdot T_{e,o}(K_e k|K_e k_0) \cdot (T - k_0 k_0). \quad (68)$$

  Our far-field approximation consist in neglecting the longitudinal component and the off-shell contribution in the operator \(T_{e,o}\), and we write:

  $$\hat{T}(K_e k|K_e k_0) \simeq 4\pi \hat{f}(k|k_0), \quad (69)$$

  $$= 4\pi (T - k k) \cdot \hat{f}(k|k_0) \cdot (T - k_0 k_0). \quad (70)$$

  where the last equality comes from the properties \(15\).

- **A forward scattering approximation**: For scatterers large compared to a wavelength, the scattered field is predominantly in the forward direction (i.e. \(|f(k_0|k_0)\| \gg |f(-k_0|k_0)|\)). Our forward approximation consist in keeping only the contribution of the amplitude of diffusion \(f(k|k_0)\) in the direction of the incident wave \(k_0\). We write in using the hypothesis \(76\):

  $$T_{e,o}(K_e k|K_e k_0) = 4\pi \hat{f}(k|k_0), \quad (71)$$

  $$\simeq 4\pi \hat{f}(k_0|k_0), \quad (72)$$

  $$= 4\pi (T - k_0 k_0) f(K_e, \omega), \quad (73)$$

  where

  $$f(K_e, \omega) = \frac{i}{K_e} S_1(0) = \frac{i}{K_e} S_2(0), \quad (74)$$

  with \(S_1(0) = S_2(0)\) given by the Mie theory \(1\). It is worth mentioning that the approximation \(72\) is also valid for small scatterers (Rayleigh scatterers). In this case, the scattering amplitude \(f(k|k_0)\) doesn’t depend on the direction of the incident and scattered wave vector \(k\) and \(k_0\), since we have

  $$T_{e,o}(k|k_0) = \hat{t}_{e,o}(\omega) T. \quad (75)$$

  From equation \(68\), we show that:

  $$\hat{f}(k|k_0) = \hat{f}(k_0|k_0). \quad (76)$$

  and we also obtain the coefficient \(f(K_e, \omega)\):

  $$4\pi f(K_e, \omega) = \hat{t}_{e,o}(\omega). \quad (77)$$

  Furthermore, we see from equation \(52\), that to zero order in density:

  $$\hat{C}_{e,o}(k|k_0) = \hat{t}_{e,o}(k|k_0), \quad (78)$$

  and that the forward approximation \(72\) applied to the operator \(\hat{C}_{e,o}(k|k_0)\) in the low density limit.
We will suppose that the forward approximation is valid whatever the order in density for the operator $\mathcal{C}_{e,o}(k|k_0)$ and we write:

$$\mathcal{C}_{e,o}(k|k_0) \approx \mathcal{C}_{e,o}(k_0|k_0),$$

$$\approx \left( I - k_0\hat{k}_0 \right) C_{e,o}^{\perp}(|k_0|, \omega).$$

(79)

(80)

With this hypothesis, only the path of kind 1 are taken into account in our forward scattering approximation (80). Figure 1 shows the two different paths which contribute to the mass operator $S(|k_0|, \omega) = n \mathcal{C}_{e,o}(k_0|k_0)$ in the (QC-CPA) approach. Only the path of kind 1 are taken into account in our forward scattering approximation (80).

From the previous hypothesis and the QC-CPA equations (82), we obtain an equation on $\mathcal{C}_{e,o}(k_0|k_0)$:

$$\mathcal{C}_{e,o}(k_0|k_0) = 4\pi f(K_e, \omega) T_\perp(k_0)$$

$$+ 4\pi n f(K_e, \omega) \overline{m}(k_0) \cdot \mathcal{C}_{e,o}(k_0|k_0)$$

(81)

where we have introduced the notation:

$$T_\perp(k_0) = (I - k_0\hat{k}_0),$$

$$\overline{m}(k_0) = \int \frac{d^3k_1}{(2\pi)^3} \ h(k_0 - k_1) \cdot T_\perp(k_0) \cdot \mathcal{G}_e^\infty(k_1) \cdot T_\perp(k_0).$$

(82)

Then, we have:

$$\mathcal{C}_{e,o}(k_0|k_0) = \left[ T_\perp(k_0) - 4\pi n f(K_e, \omega) \overline{m}(k_0) \right]^{-1}$$

$$\cdot T_\perp(k_0) 4\pi f(K_e, \omega).$$

(83)

In using the classical properties of the Fourier transform, we write:

$$\overline{m}(k_0) = T_\perp(k_0) \cdot \int d^3r e^{-ik_0\cdot r} h(r) \mathcal{G}_e^\infty(r) \cdot T_\perp(k_0).$$

(84)

where we have used the translation invariance of the green function: $\mathcal{G}_e^\infty(r - r_0) = \mathcal{G}_e^\infty(r, r_0)$. We know that the Dyadic Green function has a singularity which can be separated in introducing the principal value of the Green function (11, 12, 53, 54):

$$\mathcal{G}_e^\infty(r) = P.V.\mathcal{G}_e(r) - \frac{1}{3K_e^2} \delta(r) \hat{T}.$$  

(86)

where the principal value is defined by:

$$P.V. \int d^3r_0 \mathcal{G}_e^\infty(r - r_0) \cdot \overline{\phi}(r_0)$$

$$= \lim_{a \to 0} \int_{S_a(r)} d^3r_0 \mathcal{G}_e^\infty(r - r_0) \cdot \overline{\phi}(r_0),$$

(87)

with $\overline{\phi}(r_0)$ a test function and $S_a(r)$ a spherical volume of radius $a$ centered at $r$. This principal value can be easily calculated, and we obtain (12, 16, 60):

$$P.V.G_e(r) = e^{iK_e||r||} \left[ \left( 1 - \frac{1}{ik_e||r||^2} \right) \mathcal{G}_e(r) \right],$$

$$- \left( 1 - \frac{3}{ik_e||r||^2} \right) \mathcal{G}_e(r).$$

(88)

In using polar coordinate in the integral (88):

$$\int d^3r = \int_0^{+\infty} r^2 dr \int_0^{2\pi} d^2\hat{r},$$

(89)

and the integral on solid angles given in Appendix A, we obtain the following result:

$$\overline{m}(k_0) = \left[ -\frac{h(0)}{3K_e^2} + m(K_e) \right] T_\perp(k_0).$$

(90)

with

$$m(K_e) = \int_0^{+\infty} r dr p(K_e, r) [g(r) - 1] e^{iK_e r},$$

(91)

$$p(x) = \frac{\sin x}{x} - \left( \frac{\sin x}{x^2} \cos x \right),$$

$$- \left( \frac{1}{ix} + \frac{1}{x^2} \right) \left( \frac{\sin x}{x^2} - 3 \sin x \cos x \right).$$

(92)

where we have assumed that $||k_0|| = K_e$ since the mass operator (51) is evaluated with this value in the equation (81) to obtain the effective permittivity. As the particle cannot penetrate, we have $g(0) = 0$ and then $h(0) = -1$. With equations (41, 42, 51, 53, 54, 100), we derive an expression for the effective wave number $K_e$:

$$K_e^2 = K_1^2 + \frac{4\pi n f(K_e, \omega)}{1 - 4\pi n f(K_e, \omega) \left( \frac{1}{ix} + m(K_e) \right)},$$

(93)

where the scalar $m(K_e)$ is defined by equations (11, 12) and the scalar $f(K_e, \omega)$ is the forward scattering amplitude: $\mathcal{F}(k_0|k_0) = (I - k_0\hat{k}_0) f(K_e, \omega)$ for a particle of
permittivity $\varepsilon_s = \varepsilon_e - \varepsilon_c$ within a medium of permittivity $\varepsilon_e$. The relationship between the effective wave number $K_e$ and the effective permittivity $\varepsilon_e$ is given by:

$$\varepsilon_e(\omega) = K_e^2 / K_{vac}^2.$$  

(94)

The formula (91-94) are the main results of this paper.

V. RAYLEIGH SCATTERERS

We now show how to recover the low-frequency limit of the QC-CPA approach. First, we have to find an expression for the $T$ matrix for a single scatterer when its size is small compared to a wavelength ($K_{vac} r_s \ll 1$). The $T$ matrix $\mathbf{T}_{e,o}$ verifies the following equation:

$$\mathbf{T}_{e,o}(r, r_0) = \mathbf{v}_{e,o}(r) \delta(r - r_0) + \int d^3 r_1 \mathbf{v}_{e,o}(r) - \mathbf{T}_{e,o}(r_1, r_0),$$  

(95)

where the potential is defined by:

$$\mathbf{v}_{e,o}(r) \equiv K_{vid}^2 (\varepsilon_e - \varepsilon_s) \Theta_s(r) \mathbf{T}.$$  

(96)

If we extract the singularity of the Green dyadic function $\mathbf{G}_e^{\infty}(r, r_1)$ in using equation (50), we obtain:

$$\mathbf{T}_{e,o}(r, r_0) = \mathbf{v}_{dip}(r) \delta(r - r_0) + \int d^3 r_1 \mathbf{v}_{dip}(r) \cdot \left[ P.V. \mathbf{G}_e^{\infty}(r, r_1) \right] \cdot \mathbf{T}_{e,o}(r_1, r_0),$$  

(97)

where

$$\mathbf{v}_{dip}(r) \equiv \left[ 1 + \frac{\mathbf{v}_{e,o}(r)}{3 K_e^2} \right]^{-1} \mathbf{v}_{e,o}(r),$$  

(98)

$$= K_{vid}^2 \alpha_{dip} \frac{\Theta_s(r)}{v_s} \mathbf{T}.$$  

(99)

with

$$v_s = \frac{4\pi}{3} r_s^3$$  

(100)

$$\alpha_{dip} = 3 \frac{\varepsilon_e - \varepsilon_s}{\varepsilon_s + 2 \varepsilon_c} v_s.$$  

(101)

It’s easy to recognize that the coefficient $\alpha_{dip}$ is the polarization factor of a dipole. Hence, the singularity in the Green dyadic function describes the depolarization factor due to the induced field in the particles. The relationship between the singularity of the Green function and the depolarization field acting on a particle has been described in numerous works. From the meaning of the coefficient $\alpha_{dip}$, we inferred that equation (97) describes the multiple scattering process by the dipoles inside the particle (where $\Theta_d(r) \neq 0$).

As the particles are small compared to a wavelength, we use a point scatterer approximation:

$$\Theta_s(r) \approx \delta(r)$$  

(102)

and the potential (99) becomes:

$$\mathbf{v}_{dip}(r) \approx K_{vid}^2 \alpha_{dip} \delta(r) \mathbf{T}.$$  

(103)

In introducing the approximation (103) in equation (97), the Dirac distribution provides an analytical answer for the $T$ matrix of a single particle:

$$\mathbf{T}_{e,o}(r, r_0) = \delta(r - r_0) \delta(r_0) \mathbf{T}_{e,o}(\omega),$$  

(104)

$$\mathbf{T}_{e,o}(\omega) = K_{vid}^2 \alpha_{dip} \left[ \mathbf{T} - K_{vid}^2 \alpha_{dip} P.V. \mathbf{G}_e^{\infty}(r = 0) \right]^{-1}. $$  

(105)

The principal value of the Green function at the origin can be evaluated in using a regularization procedure (60):

$$P.V. \mathbf{G}_e^{\infty}(r = 0) = \left[ \frac{\Lambda_T}{6\pi} + \frac{i K_e}{6\pi} \right] \mathbf{T},$$  

(106)

where the term $\Lambda_T$ is proportional to the inverse of the real size of the scatterer [7, 60]. Finally, the $T$ matrix for a single particle is:

$$\mathbf{T}_{e,o}(r_1, r_2) = \delta(r_1 - r_2) \delta(r_1) \mathbf{t}_{e,o}(K_e, \omega),$$  

(107)

$$\mathbf{t}_{e,o}(K_e, \omega) = \frac{K_{vid}^2 \alpha_{dip}}{1 - K_{vid}^2 \alpha_{dip} \left( \frac{\Lambda_T}{6\pi} + \frac{i K_e}{6\pi} \right)}.$$  

(108)

It has been shown that the $T$ matrix (107, 108) verifies the optical theorem and can present a resonant behavior due to the $\Lambda_T$ term [1, 60]. The validity of the optical theorem is an important point, to insure that the attenuation of the coherent wave due to scattering is correctly taken into account in the (QC-CPA) approach. Hence, the expressions (108) must be used rather than the usual $T$ matrix for a Rayleigh scatterer $\mathbf{t}_{e,o}(K_e, \omega) = K_{vid}^2 \alpha_{dip}$ which doesn’t verify the optical theorem [7, 10]. Furthermore, from equations (68), we notice that:

$$4\pi f(K_e, \omega) = \mathbf{t}_{e,o}(K_e, \omega).$$  

(109)

The small size of the scatterers allow us also to approximate the term $m(K_e)$ in equation (93). In fact, as there is no long range correlation in a random medium ($g(r) - 1 \approx 0$ for $r \gg r_s$) and as $K_e r_s \ll 1$, we can evaluate the function $p(K_e r)$ in the integral (91) for $K_e r$ close to zero. In using the limit:

$$p(x) = \frac{2}{3} + o(x) \quad x \to 0,$$  

(110)

we obtain the following leading term of the real and imaginary part of $m(K_e)$:

$$m(K_e) = \frac{2}{3} \int_0^{+\infty} r \, dr \, [g(r) - 1] + \frac{2i K_e}{3} \int_0^{+\infty} r^2 \, dr \, [g(r) - 1] + \ldots.$$  

(111)
If we keep only these two terms, equation 93 becomes in using the results \[109,108,1036\]:

\[
\epsilon_e = \epsilon_1 + \frac{3(\epsilon_s - \epsilon_1) \epsilon_e f_v}{(\epsilon_s - \epsilon_1)(1 - f_v^\text{vol}) + 3 \epsilon_1} + \frac{2}{3}(\epsilon_e^{1/2} K_{\text{vac}} r_s)^3 \frac{w_1}{K_e} + i \epsilon_2, 
\]

(112)

with \(f_v = n v_s\) the fractional volume occupied by the particles and \(w_1, w_2\) defined by:

\[
4\pi n \int_0^{+\infty} r \, dr \, [g(r) - 1] = w_1 - \Lambda T, 
\]

(113)

\[
4\pi n \int_0^{+\infty} r^2 \, dr \, [g(r) - 1] = w_2 - 1, 
\]

(114)

For non resonant Rayleigh scatterers, we can neglect the term \(\Lambda T \approx 0\), and also neglect the term \(\frac{2}{3}(\epsilon_e^{1/2} K_{\text{vac}} r_s)^3 \frac{w_1}{K_e}\) compare to 1 - \(f_v\). Moreover, we have usually \(Re(\epsilon_e) >> Im(\epsilon_e)\) and equation 112 can be simplified into:

\[
\epsilon_e = \epsilon_1 + \frac{3(\epsilon_s - \epsilon_1) \epsilon_e f_v}{(\epsilon_s - \epsilon_1)(1 - f_v^\text{vol}) + 3 \epsilon_1} + \frac{i}{2} \frac{(K_{\text{side}} r_d)^3}{(\epsilon_s - \epsilon_1)^2 + 5/2} \frac{f_v^\text{vol}}{f_v} w_2, 
\]

(115)

where a Percus-Yevick correlation function for \(g(r)\) gives 13, 15:

\[
w_2 = \frac{(1 - f_v)^4}{(1 + 2 f_v)^2}. 
\]

(116)

The equation 115 is the usual low-frequency limit of the QC-CPA approach obtained by Tsang et al. 13, 16. In particular, we see that in the static case (\(\omega = 0\)) the imaginary term in the right hand side of equation 115 is null, and if we replace the effective permittivity \(\epsilon_e\) by \(\epsilon_1\) in the right hand side of equation 110, we recover the classical Maxwell Garnett formula:

\[
\epsilon_e = \epsilon_1 + \frac{3(\epsilon_s - \epsilon_1) \epsilon_1 f_v}{(\epsilon_s - \epsilon_1)(1 - f_v^\text{vol}) + 3 \epsilon_1}, 
\]

(117)

which is usually written in the following form:

\[
\frac{\epsilon_e - \epsilon_1}{\epsilon_e + 2 \epsilon_1} = f_v \frac{\epsilon_s - \epsilon_1}{\epsilon_s + 2 \epsilon_1}. 
\]

(118)

In comparing equation 117 and 118, we see that the scattering process modified the Maxwell Garnett formula by adding a new term:

\[
-\frac{2}{3}(\epsilon_e^{1/2} K_{\text{vac}} r_s)^3 \frac{w_1}{K_e} + i w_2, 
\]

(119)

whose imaginary part describes the attenuation of the coherent wave, and then, the transfer to the incoherent part due to the scattering of the wave.

VI. KELLER FORMULA

We are now going to show that the relation 99 that we have obtained contain also the Keller formula 52. This formula has recently been shown to be in good agreement with experimental results for particles larger than a wavelength \[53,54\]. The Keller formula can be obtained considering the QC-CPA approach in the scalar case 13. 16. The equations are formerly identical to equation 51 where the dyadic Green function \(G_e^\infty\) gives by:

\[
G_e^\infty(r, r_0, \omega) = \left[ I + \frac{\nabla \nabla}{K_e^2} \right] \frac{e^{i K_e ||r - r_0||}}{4\pi ||r||}, 
\]

(120)

have to be replaced by the scalar Green function \(G_e\) given by:

\[
G_e^\infty(r, r_0, \omega) = \frac{e^{i K_e ||r - r_0||}}{4\pi ||r||}. 
\]

(121)

The first iteration of the scalar version of equation 52 gives in using equation A1:

\[
K_e^2 = K_1^2 + (4\pi)^2 n f(K_e, \omega) 
\]

\[
+ (4\pi)^2 n^2 f^2(K_e, \omega) \int_0^{+\infty} dr \, \frac{\sin K_e r}{K_e} [g(r) - 1] e^{i K_e r} + \ldots. 
\]

(122)

As was shown by Waterman et al, this development is valid if the following condition is verified:

\[
(4\pi)^2 n |f(K_e, \omega)|^2 / K_e \ll 1. 
\]

(123)

In the geometric limit, the scattering cross section \(\sigma_s\) for a single particle is in good approximation given by \(\sigma_s \approx 2\pi r_s^2\). As the cross section is connected to scattering amplitude by the relation \(\sigma_s = \frac{4\pi}{K_e^2}|f(K_e, \omega)|^2\) and as the maximum density is \(n = 1/v_s\), we see that for particles larger than a wavelength the condition 123 is satisfied:

\[
(4\pi)^2 n |f(K_e, \omega)|^2 / K_e \ll 1/k_s r_s \ll 1. 
\]

(124)

The equation 122, which has been derived by Keller 52, has proven to be in good agreement with experiments for particles larger than a wavelength. If we now use a Taylor development in equation 53, we obtain

\[
K_e^2 = K_1^2 + (4\pi)^2 n f(K_e, \omega) 
\]

\[
+ (4\pi)^2 n^2 f^2(K_e, \omega) \left[ \frac{1}{3 K_e^2} + m(K_e) \right] + \ldots. 
\]

(125)

This development is valid if the condition 125 is satisfied. In the geometric limits, we can approximate the function \(p(K_e r)\) in the definition 111 of \(m(K_e)\), since for \(K_e r \gg 1\) we have from equation 122:

\[
p(x) \simeq \frac{\sin x}{x}, \quad x \gg 1, 
\]

(126)
The relation \( K^2_e = K^2_i + 4\pi n f(K_e, \omega) \) becomes:
\[
K^2_e = K^2_i + 4\pi n f(K_e, \omega) + (4\pi)^2 n^2 f^2(K_e, \omega) \left[ \frac{1}{3} K^2_e + \int_0^{+\infty} dr \frac{K_e}{K} g(r) - 1 \right] e^{iK_e r} \tag{127}
\]

We see that equation (127) differ from the equation (122), only by the factor \( 1/3K^2_e \), which is due to singularity of the vectorial Green function and consequently cannot be derived from the scalar theory developed by Keller. We also remarks that that to solve numerically our new equation (128), we can use the same procedure that is used to solve the original Keller formula (122) with the Muller theory [54]. Consequently, we have derived a numerical tractable approximation to the (QC-CPA) scheme.

\[ \int 4\pi d^2\vec{r} e^{-iK_0 \cdot \vec{r}} = 4\pi \frac{\sin ||\vec{k}_0|| ||\vec{r}||}{||\vec{k}_0|| ||\vec{r}||} \tag{A1} \]
\[ \int 4\pi d^2\vec{r} e^{-iK_0 \cdot \vec{r}} \cdot \vec{r} \cdot \vec{r} = 4\pi \frac{\cos ||\vec{k}_0|| ||\vec{r}|| \cdot (T - \hat{k}_0 \hat{k}_0)}{||\vec{k}_0||^2 ||\vec{r}||^2} + 4\pi \frac{\sin ||\vec{k}_0|| ||\vec{r}||}{||\vec{k}_0|| ||\vec{r}||} + 2 \frac{\cos ||\vec{k}_0|| ||\vec{r}||}{||\vec{k}_0||^2 ||\vec{r}||^2} - 2 \frac{\sin ||\vec{k}_0|| ||\vec{r}||}{||\vec{k}_0||^3 ||\vec{r}||^3} \hat{k}_0 \hat{k}_0 \tag{A2} \]

VII. CONCLUSION

The intent of this paper has been to establish a new formula for the effective dielectric constant which characterize the coherent part of an electromagnetic wave propagating in a random medium. The starting point of our theory has been the quasicrystalline coherent potential approximation which takes into account the correlation between the particles. As the numerical calculation of the effective permittivity is still a difficult task under the (QC-CPA) approach, we have added a far-field and a forward scattering approximations to (QC-CPA) scheme. In the low frequency limit, equation is identical with the usual result obtained under the (QC-CPA) scheme, and in the high frequency limit the expression include the generalization, in the vectorial case, of the result obtained by Keller. Further study is necessary to assess the limitation of this approach on the intermediate frequency regime.

APPENDIX A: APPENDIXES

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