Path algebra algorithm for finding longest increasing subsequence

Anatoly Rodionov
ayarodionov@gmail.com
September 11, 2014

Abstract

New algorithm for finding longest increasing subsequence is discussed. This algorithm is based on the ideas of idempotent mathematic and uses Max-Plus idempotent semiring. Problem of finding longest increasing subsequence is reformulated in a matrix form and solved with linear algebra.

1 Introduction

The purpose of this short article is to bring attention to unifying approach to software and hardware design suggested and developed by Grigory Litvinov, Viktor Maslov and coworkers[1]. The unifying approach is based on observation that many algorithms do not depend on particular models of a numerical domain and even on the domain itself. Algorithms of linear algebra (matrix multiplication, Gauss elimination etc.) are good examples of algorithms of this type. One can do linear algebra with field of rational numbers, complex numbers and so on. But it is not obvious that the same algorithms (slightly reformulated) may be used also for idempotent semirings of different kind. Why this is interesting? Because many problems, which never were considered as problems that have anything to do with linear algebra, can be solved with standard linear algebra methods. For example problem of finding shortest, critical, maximal capacity or most reliable path on graph can be solved using linear algebra over different idempotent semirings. Object oriented languages make it possible to reuse the same codes for solving entirely different problems without changing a single line! In this way all the knowledge and power of linear algebra can be used much wider area. In fact almost all problems of dynamic programming can be solved in this way[2].

In this article I show how a very simple and classical problem of dynamic programming – finding of the longest increasing subsequence, can be solved using max-plus idempotent semiring and linear algebra. The problem has simple formulation and requires the simplest linear algebra method for its solution. So I think that it can be used as a good illustration of the beauty of unifying
approach. Because of its simplicity the algorithm can be already found and discussed. I never found any references on this subject. If you know about such publication please let me know and I will add the corresponding reference.

I used path algebra in the title to emphasize that in this article I follow ideas of Bernard Carré who wrote pioneer work \[3\] on using linear algebra methods in graph theory. There many names for the same thing now: idempotent mathematics, tropical mathematics . . . Bernard Carre used path algebra \[4\]. This name I prefer to use in the title while in the article itself I will use more familiar max-plus idempotent semiring.

2 Formulation of the problem

The problem is: in a given sequence of elements find length of the longest increasing subsequence. (Here we assume that all elements of the sequence can be compared.)

Let $S$ be a sequence of elements $S = \{s_0, s_1, \ldots, s_{n-1}\}$. Find the largest length $l$ on increasing subsequence $\{s_{i_0}, s_{i_1}, \ldots, s_{i_l-1}\}$ such that $s_i \in S$ and $s_i < s_j$ if $i < l$.

For example for $\{5, 2, 8, 6, 3, 6, 9, 7\}$ the longest increasing subsequence $\{2, 3, 6, 9\}$ has length 4.

The usual algorithm for solving this problem looks like:

```
for i = 0, \ldots, n-1
    L[j] = 1 + \max(L[i] \text{ such that } S[i] < S[j])
return \max(L[j])
```

The algorithm requires at most $O(n^2)$ operations. (In the special case of numerical elements algorithm of $O(n \ln(n))$ complexity exists and even $O(n \ln(\ln(n)))$ \[5\].) The purpose of this note is not to improve algorithm’s speed but to show how it can be reformulated as a linear matrix problem. For this Max-Plus algebra is used.

3 Max-Plus algebra

Let me recall definition of max-plus algebra.

Denote by $R_{\max}$ set of real number with minus infinity $R_{\max} = R \cup \{-\infty\}$. For elements from $R_{\max}$ we can define operations $\oplus$ and $\odot$:

$$
a \oplus b = \max(a, b) \\
a \odot b = a + b
$$

In this algebra $-\infty$ plays role of zero, and 0 - of unit element. We will use symbol $\phi$ for zero and $e$ for unit element.

It can be easily proved that
• operation ⊕ is
  commutative: \( a \oplus b = b \oplus a \),
  associative: \((a \oplus b) \oplus c = a \oplus (b \oplus c)\),
  and idempotent: \( a \oplus a = a \).

• operation ⊙ is
  associative: \((a \odot b) \odot c = a \odot (b \odot c)\),
  distributive over ⊕:
  \[ a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c) \text{ and } (b \oplus c) \odot a = (b \odot a) \oplus (c \odot a). \]

\[ \phi \oplus a = a \]
\[ \phi \odot a = a \odot \phi = \phi \]
\[ e \odot a = a \odot e = e \]

This construction is called Max-Plus algebra\(^1\) which is an example of idempotent semiring. This algebra has many applications and was studied in many works (see for example [6], [7]).

We will use notations \( a^n \) for product of \( n \) elements: \( a^0 = e, a^n = a \odot a \ldots \odot a \); \( a^+ \) for infinite sum of nonzero powers of \( a \):
\[ a^+ = a^1 \oplus a^2 \oplus a^3 \ldots \text{and } a^* = e + a^+. \]
In what follows these sums will contains only finite number of nonzero elements.
\( a^* \) gives solution for equation
\[ y = a \odot y \oplus b \] (1)
Solution is \( y = a^* \odot b \). The result can be proved by direct substitution and using trivial facts that \( a \odot a^* = a^+ \) and \( a^* = e + a^+ \). Indeed, the right hand side after the substitution is \( a \odot a^+ \odot b \odot b = a^+ \odot b \odot b = (a^+ \odot e) \odot b = a^* \odot b \).

Over elements of Max-Plus algebra we can construct matrices \( A = \{a_{i,j}\} \).
We define \( \oplus \) and \( \odot \) operations for matrices as
\[ A \oplus B = \{a_{i,j} \oplus b_{i,j}\} \]
\[ A \odot B = \left\{ \oplus \sum_{k=0}^{n} a_{i,k} \odot b_{k,j} \right\} \]
where \( \oplus \sum_{k=0}^{n} a_k = a_0 \oplus a_1 \oplus \ldots a_n \). Zero element \( \Phi \) is matrix filled with \( \phi \); unit element \( E \) - matrix with \( e \) on diagonal and \( \phi \) in all other positions.

As in scalar case define \( A^n \) for product of \( n \) elements: \( A^0 = E \), \( A^n = A \odot A \ldots \odot A \); \( A^+ \) for infinite sum of nonzero powers of \( A \):
\[ A^+ = A^1 \oplus A^2 \oplus A^3 \oplus \ldots \text{and } A^* = E + A^+. \]

\(^1\)It is also called \( P_3 \) in Bernard Carré classification.
The constructed algebra of matrices is itself an idempotent semiring. In this algebra equation

$$Y = A \odot Y \oplus B$$  \hspace{1cm} (2)$$

has solution $$Y = A^* \odot B$$.

If $$B = E$$ then $$Y = A^*$$. This fact will be used later in the algorithm for efficient calculation of $$A^*$$. 

4 Path algebra algorithm

The algorithm contains two steps: constructing of incidence matrix and solving linear equation.

4.1 Constructing of incidence matrix

We will construct graph which vertices represents elements of the sequence $$S$$. Two vertices $$s_i$$ and $$s_j$$ are connected by directed edge $$s_i \to s_j$$ if $$s_i < s_j$$ and $$i < j$$.

We will use incidence matrix for graph representation. Element $$a_{i,j}$$ of incidence matrix is equal to 1 if $$s_i$$ and $$s_j$$ are connected, else $$\phi$$:

$$a_{i,j} = 1$$ if $$s_i < s_j$$ and $$i < j$$ else $$\phi$$

By construction this matrix is upper triangle.

For the example above \{5, 2, 8, 6, 3, 6, 9, 7\} the matrix looks like:

$$A = \begin{bmatrix}
\phi & 1 & 1 & \phi & 1 & 1 & 1 \\
\phi & 1 & 1 & 1 & 1 & 1 & 1 \\
\phi & \phi & \phi & \phi & \phi & \phi & 1 & \phi \\
\phi & \phi & 1 & 1 & 1 & 1 & 1 & 1 \\
\phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi & \phi & \phi & 1 & 1 \\
\phi & \phi & \phi & \phi & \phi & \phi & 1 & 1 \\
\phi & \phi & \phi & \phi & \phi & \phi & \phi & \phi \\
\end{bmatrix}$$

Matrix $$A$$ describes all path of length one in the graph, matrix $$A^2$$

$$A^2 = \begin{bmatrix}
\phi & \phi & \phi & \phi & 2 & 2 & 2 \\
\phi & \phi & \phi & \phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi & \phi & \phi & \phi \\
\phi & \phi & \phi & \phi & \phi & \phi & \phi \\
\end{bmatrix}$$
describes all paths of length two and so on. Because the graph is acyclic all paths are shorter than number of vertices \( N \), so matrix \( A^N \) contains only zero elements.

Indeed, the problem may be solved by subsequently calculating powers of matrix \( A \): \( A^2, A^3, \ldots \); the last power \( k \) such that \( A^k \) contains at least one element not equal to \( \phi \) is the required maximum length minus one. This solves the problem. For the example above the power is equal to 3.

Unfortunately the cost of this solution is \( N^3 \): in the worst case we have to make \( N \) matrix multiplications and each multiplication requires \( N^2 \) operations. There is a more efficient way to find the maximum length - solving linear equation (2).

### 4.2 Solving linear equation

The \( n \)-th power of matrix \( A \) contains all paths of length \( n \), so sum of powers of matrix \( A \) (which is \( A^* \)) contains all possible maximum length paths in graph. We can add zero power of \( A \) which by definition equals to unity matrix \( E \). Unity matrix has Max-Plus unit elements on its diagonal which are real zeros, so we may think about \( A^0 \) as matrix of paths of zero length connecting edges to themselves. Adding \( A^0 \) to \( A^* \) gives \( A^* \). So \( A^* \) contains all maximum length paths. If we know \( A^* \) then can find length of the longest path. But direct calculation of \( A^* \) is costly operation.

To solve the problem let’s use (2) with \( B = E \).

\[
Y = A \odot Y \oplus E
\]

Solution of this equation is \( Y = A^* \).

Let \( Y^i \) be the \( i \)-th column of matrix \( Y \) and \( E^i \) be the \( i \)-th column of matrix \( E \). Then the equation can be rewritten as system of \( N \) equations:

\[
Y^i = A \odot Y^i \oplus E^i \text{ where } i = 0, \ldots, N
\]

Elements \( y^j_i \in Y^j \) are maximum path length from \( s_i \) to \( s_j \). Elements \( x_j \) of vector \( X = Y^0 \oplus Y^1 \oplus \ldots Y^{N-1} \) (here superscript is index, not power) give maximum length to element \( s_j \). Equation to \( X \) is:

\[
X = A \odot X \oplus U \tag{3}
\]

where \( U \) is \( N \) column and one row matrix each all elements are equal to \( e \).

Because \( A \) is upper diagonal matrix the equation can be easily solved by direct substitutions:
\[ L = \bigoplus_{i=0}^{N-1} x_i \text{ where} \]

\[ x_{N-1} = e \]

\[ x_i = e \oplus (\bigoplus_{j=i+1}^{N-1} a_{i,j} \odot x_j) \text{ where } (i = N - 2, \ldots, 0) \]

(here \( \bigoplus \sum \) is sum over operation \( \oplus \)). \( L + 1 \) gives maximum length on an increasing subsequence in \( S \).

This formula gives us solution in \( \frac{N(N+1)}{2} \) time which is the same as in classical dynamic programming algorithm. In fact the last formula can be rewritten in the usual way:

\[ L = \max_{i=0,\ldots,N-1} (x_i) \text{ where} \]

\[ x_{N-1} = 0 \]

\[ x_i = \max_{j=i+1,\ldots,N-1} (a_{i,j} + x_j) \text{ where } (i = N - 2, \ldots, 0) \text{ and } a_{i,j} \neq 0 \]

5 New class of generic algorithms

As it was pointed out by David Musser and Alexander Stepanov "generic programming centers around the idea of abstracting from concrete, efficient algorithms to obtain generic algorithms that can be combined with different data representations to produce a wide variety of useful software" [8].

But what are these generic algorithms? How to find them? The great success of standard template library (STL) is based greatly on containers and algorithms specific for containers - iteration through container, finding elements, sorting. But what are other classes of generic algorithms? Paths algebra gives an answer to this question - linear algebra algorithms used on idempotent semirings. The fact that all dynamic programming algorithms can be reformulated in terms of linear algebra should not be overlooked. In fact, we already have many different parametrized representation of vectors and matrices (for example in BOOST C++ library [9]). But what kind of elements for these matrices can we picked in BOOST? Integer numbers, floating point numbers, rational numbers, complex numbers, octonions, quaternions, intervals... - all from so called numerical domain. Why don’t we add some more interesting objects - semirings? As it was shown in [1] and [2] this can be very useful for solving a wide variety of problems.
References

[1] G.L. Litvinov, V.P. Maslov, A.Ya.Rodionov. A unifying approach to software and hardware design for scientific calculations and idempotent mathematics. arxiv.org/abs/math/0101069.

[2] G.L. Litvinov, V.P. Maslov, A.N. Sobolevsky, S.N. Sergeev, A.Ya Rodionov. Universal algorithms for solving the matrix Bellman equations over semirings Soft Computing DOI 10.1007/s00500-012-1027-5 (2012).

[3] B.A. Carré. Analgebra for Network Routine Problems, J. Inst. Math. 7 (1971), pages 273–294.

[4] Bernard Carré. Graphs and Networks. Oxford University Press, 1979.

[5] Sergei Bespamyatnikh, Michael Segal. Enumerating longest increasing subsequences and patience sorting. Information Processing Letters 76 (2000), pages 7–11.

[6] Peter Bitkovič. Max-linear Systems: Theory and Algorithms. Springer-Verlag London Limited, 2010.

[7] Bernard Heidergott, Geert Jan Olsder, Jacob van der Woude. Max Plus at Work. Princeton University Press, 2006.

[8] David R. Musser and Alexander A. Stepanov. Generic Programming. ISSAC 1988, pages 13-25.

[9] Boost (C++ libraries). http://www.boost.org/