On unitarity of a Yang-Mills type formulation for massless and massive gravity with propagating torsion

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Abstract

A perturbative regime based on contorsion as a dynamical variable and metric as a (classical) fixed background, is performed in the context of a pure Yang-Mills formulation based on \( GL(3,R) \) gauge group. In the massless case we show that the theory propagates three degrees of freedom and only one is a non-unitary mode. Next, we introduce quadratical terms dependent on torsion, which preserve parity and general covariance. The linearized version reproduces an analogue Hilbert-Einstein-Fierz-Pauli unitary massive theory plus three massless modes, two of them non-unitary ones. Finally we confirm the existence of a family of unitary Yang-Mills-extended theories which are classically consistent with Einstein’s solutions coming from non massive and topologically massive gravity.

1 Introduction

There were some contributions on the exploration of classical consistency of a pure Yang-Mills (YM) type formulation for gravity, including the cosmological extension [1, 2] (and the references therein), among others. In those references, Einstein’s theory is recovered after the imposition of torsion constraints.

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Unfortunately, the path to a quantum version (if it is finally possible) is not straightforward. For example, it is well known that the Lagrangian of a pure YM theory based on the Lorentz group $SO(3, 1) \simeq SL(2, C)$ \[3\] leads to a non-positive Hamiltonian (due to non-compactness of the aforementioned gauge group) and, then the canonical quantization procedure fails. However, there is a possible way out if it is considered an extension of the YM model thinking about a theory like Gauss-Bonnet with Torsion\[3\] and this is confirmed because the existence of a possible family of quadratical curvature theories from which can be recovered unitarity\[10\].

A first aim of this work is to expose, with some detail, a similar (and obvious) situation about non-unitarity in a YM formulation with $GL(3, R)$ as a gauge group in both massless and massive theories. There is an interest focussed in the study of massive gravity and propagating torsion\[11\], among others. Particularly, the massive versions that we shall explore here arise, on one hand from some quadratical terms set ($T^2$-terms) preserving parity which depends on torsion (the old idea about considering $T^2$-terms in a dynamical theory of torsion has been considered in the past\[4\]) and, at a perturbative regime they give rise to a Fierz-Pauli’s massive term. On the other hand, we review the topologically massice version of the YM gravity\[2\] which do not preserves parity and how is the way to reach unitarity.

Whatever the model considered, throughout this work we follow the spirit of Kibble’s idea\[5\] treating the metric as a fixed background, meanwhile the torsion (contorsion) shall be considered as a dynamical field and it would be thought as a quantum fluctuation around a classical fixed background.

This paper is organized as follows. The next section is devoted to a brief re- view on notation of the cosmologically extended YM formulation\[1\] in $N$-dimensions and its topologically massive version in $2 + 1$ dimension\[2\]. In section 3, we consider the scheme of linearization of the massless theory around a fixed Minkowskian background, allowing fluctuations on torsion. Next, the Lagrangian analysis of constraints and construction of the reduced action is performed, showing that this theory does propagate degrees of freedom, including a ghost. In section 4, we introduce an
appropriate $T^2$-terms, which preserve parity, general covariance, and its linearization gives rise to a Fierz-Pauli mass term. There, the non-positive definite Hamiltonian problem gets worse: the Lagrangian analysis shows that the theory has more non-unitary degrees of freedom and we can’t expect other thing. Gauge transformations are explored in section 5. Although $T^2$-terms provide mass only to some spin component of contorsion, the linearized theory loses the gauge invariance and there is no residual invariance. This is clearly established through a standard procedure for the study of possible chains of gauge generators[6]. In section 6 we confirm the well known fact that there exists a family of theories which can cure the ghost problem[10] and they are classically consistent when it is shown that the set of solutions contains the Einsteinians ones. We end up with some concluding remarks.

2 A pure Yang-Mills formulation for gravity: massless and topological massive cases

Let $M$ be an $N$-dimensional manifold with a metric, $g_{\mu\nu}$ provided. A (principal) fiber bundle is constructed with $M$ and a 1-form connection is given, $(A_\lambda)^\mu$ which will be non metric dependent. The affine connection transforms as $A_\lambda' = U A_\lambda U^{-1} + U \partial_\lambda U^{-1}$ under $U \in GL(N, R)$. Torsion and curvature tensors are $T^\mu{}_{\lambda\nu} = (A_\lambda)^\mu{}_{\nu} - (A_\nu)^\mu{}_{\lambda}$ and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, respectively. Components of the Riemann tensor are $R_\sigma{}_{\alpha\mu\nu} \equiv (F_{\nu\mu})^\sigma{}_{\alpha}$. The gauge invariant action with cosmological contribution is

$$S^{(N)}_0 = \kappa^2 (4-N) \langle -\frac{1}{4} \text{tr} F^{\alpha\beta} F_{\alpha\beta} + q(N) \lambda^2 \rangle ,$$

where $\kappa^2$ is in length units, $\langle \ldots \rangle \equiv \int d^N x \sqrt{-g(\ldots)}$, $\lambda$ is the cosmologic constant and the parameter $q(N) = 2(4-N)/(N-2)^2(N-1)$ depends on dimension. The shape of $q(N)$ allows the recovering of (free) Einstein’s equations as a particular solution when the torsionless Lagrangian constraints are imposed and $q(N)$ changes it sign when $N > 5$. The field equations are $T^\alpha{}_{\alpha\beta} = -\kappa^2 g^{\alpha\beta} \lambda^2$ where $T^\alpha{}_{\alpha\beta} \equiv \kappa^2 \text{tr} F^{\alpha\sigma} F_{\beta}^{\sigma}$
$\frac{g^{\alpha\beta}}{4} F_{\mu\nu} F_{\mu\nu}$ is the energy-momentum tensor of gravity, and equation coming from variation of connection is
\[ \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} F^{\alpha\lambda}) + [A_\alpha, F^{\alpha\lambda}] = 0, \]
which can be rewritten as follows
\[ \nabla_\mu R_{\sigma\lambda} - \nabla_\lambda R_{\sigma\mu} = 0, \tag{2} \]
and the trace $\sigma - \lambda$ gives the expected condition $R = constant$.

It is well known that the introduction of a Chern-Simons lagrangian term (CS) in the Hilbert-Einstein formulation of gravity provides a theory which describes a massive excitation of a graviton in 2+1 dimensions[7]. If a cosmological term is included, the cosmologically extended topological massive gravity (TMG$\lambda$) arises[8]. The aforementioned action is
\[ S = \frac{1}{\kappa^2} \int d^3 x \sqrt{-g} (R + \lambda) + \frac{1}{\kappa^2 \mu} S_{CS}, \tag{3} \]
where $\mu$ is in (length)$^{-1}$ units and $S_{CS}$ is the CS action. In a Riemannian space-time, the action (3) gives the field equation, \( R^{\mu\nu} - \frac{g^{\mu\nu}}{2} R - \lambda g^{\mu\nu} + \frac{1}{\mu} C^{\mu\nu} = 0 \) where $C^{\mu\nu}$ is the (traceless) Cotton tensor. The trace of the field equation gives a consistency condition on the trace of the Ricci tensor (this means, $R = -6\lambda$). Starting with the field equation, it is possible to write down an hyperbolic-causal equation which describes a massive propagation for the Ricci tensor as follows
\[ (\nabla_\mu \nabla^\mu - \mu^2) R_{\mu\nu} - R^{\alpha\beta} R_{\alpha\beta} g_{\mu\nu} + 3 R^\alpha_\mu R_{\alpha\nu} + \frac{\mu^2}{3} R g_{\mu\nu} \]
\[ - \frac{3}{2} R R_{\mu\nu} + \frac{1}{2} R^2 g_{\mu\nu} = 0, \tag{4} \]
where $\nabla_\mu$ is taken with Christoffel’s symbols.

Next, we can explore consistence of a Yang-Mills type formulation for topological massive gravity with cosmological constant (GTMG$\lambda$), verifying the existence of causal propagation and the fact that standard TMG$\lambda$ can be recovered from GTMG$\lambda$ at the torsionless limit. The GTMG$\lambda$ model is[2]
\[ S_{GTMG\lambda} = S^{(3)}_0 + \frac{m\kappa^2}{2} \langle \varepsilon^{\mu\nu\lambda} tr \left( A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \right) \rangle, \tag{5} \]
which does not preserve parity and $S^{(3)}_0$ is given by (11) for $N = 3$. Moreover this model is gauge variant because the Chern-Simons transforms like

$$\delta U S_{CS} = - \frac{m\kappa^2}{2} \int d^3x \, e^{\mu\nu\lambda} \, \text{tr} \, [A_\mu \partial_\nu U U^{-1} - 4\pi^2 \kappa^2 m W(U)] ,$$

(6)

where $W(U) \equiv \frac{1}{2\kappa^2} \int d^3x \, e^{\mu\nu\lambda} \, \text{tr} \, (U^{-1} \partial_\mu U U^{-1} \partial_\nu U U^{-1} \partial_\lambda U)$ is the "winding number" of the gauge transformation $U$.

The torsionless limit of (5) can be explored by introducing nine torsion’s constraints through the new action $S' = S_{GTMGX} + \kappa^2 \int d^3x \sqrt{-g} \, b_{\alpha\beta} \varepsilon^{\beta\gamma\delta}(A_\lambda)^\alpha\gamma\sigma$, where $b_{\alpha\beta}$ are Lagrange multipliers. Variation on connection and metric gives rise the following field equations

$$\nabla_\mu R_\sigma\lambda - \nabla_\lambda R_{\sigma\mu} - m \varepsilon^{\mu\nu\sigma}(g_{\lambda\nu} R_{\rho\mu} - g_{\mu\nu} R_{\lambda\rho} - \frac{2}{3} R g_{\lambda\nu} g_{\mu\rho}) = 0 ,$$

(7)

$$R_{\sigma\mu} R^\sigma_{\nu} - R R_{\mu\nu} + \frac{g_{\mu\nu}}{4} R^2 - g_{\mu\nu} \lambda^2 = 0 ,$$

(8)

and Lagrange multipliers are

$$b_{\mu\nu} = \frac{mR}{6} g_{\mu\nu} .$$

(9)

The trace $\sigma - \lambda$ of (7) leads to the following consistency condition

$$R = \text{constant} ,$$

(10)

and due to this condition on the Ricci scalar, we can test particular solutions of the type $R_{\mu\nu} = \frac{R}{3} g_{\mu\nu}$, by plugging them in (8), and this gives

$$R = \pm 6 | \lambda | ,$$

(11)

verifying the existence of (Anti) de Sitter solutions.

A quick look on causal propagation of the theory can be performed writing a second order equation from (7), this means

$$(\nabla_\alpha \nabla^\alpha - m^2) R_{\mu\nu} - R^{\alpha\beta} R_{\alpha\beta} g_{\mu\nu} + 3 R^\alpha_{\mu} R_{\alpha\nu} + \frac{m^2 R}{3} g_{\mu\nu} + \frac{3R}{2} R_{\mu\nu} + \frac{R^2}{2} g_{\mu\nu} = 0 ,$$

(12)
which describes a massive hyperbolic-causal propagation of graviton. So, GTMG contains as a particular case the TMG classical formulation (at the torsionless limit) if we take the mass value \( m \) as the CS (\( m = \mu \)) and the consistency condition (10) is fixed as (11).

We underline that GTMG is gauge variant under \( GL(3, R) \), due to the presence of the CS term. However, it is well known by taking boundary conditions on the elements \( U \), the term \( \int d^3 x \epsilon^{\mu \nu \lambda} \text{tr} \partial_{\nu} [A_{\mu} \partial_{\lambda} U U^{-1}] \) in (10), goes to zero and the transformation rule now is \( \delta_U S = -4 \pi^2 \kappa^2 m W(U) \). If we demand that the expectation value of a gauge invariant operator (i.e., \( \langle \mathcal{O} \rangle \equiv Z^{-1} \int \mathcal{D}A \mathcal{O}(A) e^{iS} \) with the gauge invariant measure \( \mathcal{D}A \) and the normalization constant \( Z \)) must be gauge invariant too, it is required that \( -4 \pi^2 \kappa^2 m W(U) \) be an integral multiple of \( 2\pi \) and a quantization condition on the parameter \( \kappa^2 m \) must arises. This fact occurs, at least, by performing a restriction on the covariance of the theory, this means, taking a compact subgroup of \( GL(3, R) \) (i.e., \( SO(3) \)).

3 Linearization of the massless theory

With a view on the performing of a perturbative study of the massive model, we wish to note some aspects of the variational analysis of free action (1) in 2 + 1 dimensions. As we had said above, the connection shall be considered as a dynamical field whereas the space-time metric would be a fixed background, in order to explore (in some sense) the isolated behavior of torsion (contorsion) and avoid higher order terms in the field equations. For simplicity we shall assume \( \lambda = 0 \).

Then, let us consider a Minkowskian space-time with a metric \( \text{diag}(-1, 1, 1) \) provided and, obviously with no curvature nor torsion. The notation is

\[
\bar{g}_{\alpha \beta} = \eta_{\alpha \beta} ,
\]

\[
F^{\alpha \beta} = 0 ,
\]

\[
\bar{T}^\lambda_{\mu \nu} = 0 .
\]
It can be observed that curvature $F^{\alpha\beta} = 0$ and torsion $T^{\lambda}_{\mu\nu} = 0$, in a space-time with metric $g_{\alpha\beta} = \eta_{\alpha\beta}$ satisfy the background equations, 
\[
\frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} F^{\alpha\lambda}) + [A_\alpha, F^{\alpha\lambda}] = 0
\]
and $T_g^{\alpha\beta} = 0$, identically.

Thinking in variations

\[ A_\mu = \overline{A}_\mu + a_\mu , \quad |a_\mu| \ll 1 , \quad (16) \]

for this case $\overline{A}_\mu = 0$. Then, action (11) takes the form

\[ S^{(3)}L_0 = \kappa^2 \left\langle -\frac{1}{4} tr f^{\alpha\beta}(a)f_{\alpha\beta}(a) \right\rangle , \quad (17) \]

where $f_{\alpha\beta}(a) = \partial_\alpha a_\beta - \partial_\beta a_\alpha$ and (17) is gauge invariant under

\[ \delta a_\mu = \partial_\mu \omega , \quad (18) \]

with $\omega \in G = U(1) \times \ldots 3^2 \ldots \times U(1)$.

In order to describe in detail the action (17), let us consider the following decomposition for perturbed connection

\[ (a_\mu)^\alpha_\beta = \epsilon^\sigma_\alpha_\beta k^\mu_\sigma + \delta^\alpha_\mu v^\beta - \eta^\mu_\beta v^\alpha , \quad (19) \]

where $k_{\mu\nu} = k_{\nu\mu}$ and $v_\mu$ are the symmetric and antisymmetric parts of the rank two perturbed contorsion (i.e., the rank two contorsion is $K_{\mu\nu} \equiv -\frac{1}{2} \epsilon^{\sigma\rho_\nu} K_{\sigma\rho\mu}$), respectively. It can be noted that decomposition (19) has not been performed in irreducible spin components and explicit writing down of the traceless part of $k_{\mu\nu}$ would be needed. This component will be considered when the study of reduced action shall be performed. Using (19) in (17), we get

\[ S^{(3)}L_0 = \kappa^2 \left\langle k_{\mu\nu} \Box k_{\mu\nu} + \partial_\mu k^{\mu\sigma} \partial_\nu k^{\nu}_\sigma - 2 \epsilon^{\sigma\alpha_\beta} \partial_\alpha v_\beta \partial_\nu k^{\nu}_\sigma - v_\mu \Box v^\mu + (\partial_\mu v^\mu)^2 \right\rangle , \quad (20) \]

which is gauge invariant under the following transformation rules (induced by (18))

\[ \delta k_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu , \quad (21) \]
\[ \delta v_\mu = -\epsilon^{\sigma\rho}_\mu \partial_\sigma \xi_\rho, \] (22)

with \( \xi_\mu \equiv \frac{1}{4} \epsilon^{\alpha\beta}_\mu w^{\alpha\beta} \). These transformation rules clearly show that only the antisymmetric part of \( w \) is needed (i.e.: only three gauge fixation would be chosen).

In expression (20) we can observe that the term \( v_\mu \Box v^\mu \) has a wrong sign, telling us about the non-unitarity property of the theory. However, field equations are

\[
2 \Box k_{\mu \nu} - \partial_\mu \partial_\sigma k^{\sigma \nu} - \partial_\nu \partial_\sigma k^{\sigma \mu} + \epsilon^{\rho \mu \nu} \partial_\rho \partial_\sigma v_\rho + \epsilon^{\rho \nu \mu} \partial_\rho \partial_\sigma v_\rho = 0,
\] (23)

\[
\epsilon^{\sigma \rho \beta} \partial_\sigma k_{\mu \rho} + \Box v^{\beta} + \partial^\beta \partial_\mu v^\mu = 0,
\] (24)

and note that (23) satisfies the consistency condition

\[ \Box k - \partial_\mu \partial_\nu k^{\mu \nu} = 0. \] (25)

Divergence of (24) says that \( \partial_\mu v^\mu \) is a massless 0-form then, if we define \( \hat{\partial}_\sigma \equiv \Box - \frac{1}{2} \partial_\sigma \), the following relation can be written

\[ v^{\beta} = -\epsilon^{\sigma \rho \beta} \hat{\partial}_\sigma \hat{\partial}_\mu k^{\mu \rho}, \] (26)

up to a massless-transverse 1-form. Using (26) in (23), gives rise to

\[ \Box k_{\mu \nu} - \partial_\mu \partial_\sigma k^{\sigma \nu} - \partial_\nu \partial_\sigma k^{\sigma \mu} + \partial_\mu \partial_\nu k = 0, \] (27)

up to a massless 0-form. This last equation with condition (25) would suggests a possible equivalence with the model for gravitons of the linearized Hilbert-Einstein theory (i.e.: free gravity in 2 + 1 does not propagate degrees of freedom). However, this suggestion is wrong because we were dropped out some light modes and, then it is necessary to take into account both massive and massless complete sets of modes (light modes are relevant at the lower energy regime).

Now, let us study the system of Lagrangian constraints in order to explore the number of degrees of freedom. A possible approach consists in a 2 + 1 decomposition
of the action (20) in the way

\[
S^{(3)L}_0 = \kappa^2 \langle \left[ -\hat{k}_{0i} + 2\partial_i k_{00} - 2\partial_n k_{ni} - 2\epsilon_{in} \dot{v}_n + 2\epsilon_{in} \partial_n v_0 \right] \dot{k}_{0i} \\
+ \dot{k}_{ij} \dot{k}_{ij} + \left[ 2\epsilon_{nj} \partial_n k_{00} + 2\epsilon_{nj} \partial_m k_{nm} - \dot{v}_j - 2\partial_j v_0 \right] \dot{v}_j \\
+2(\dot{v}_0)^2 + k_{00} \Delta k_{00} - 2k_{0i} \Delta k_{0i} + k_{ij} \Delta k_{ij} - (\partial_i k_{i0})^2 \\
+\partial_n k_{ni} \partial_m k_{mi} - 2\epsilon_{ij} \partial_i \dot{v}_j \partial_n k_{n0} - 2\epsilon_{lm} \partial_m v_0 \partial_n k_{nl} + \\
v_0 \Delta v_0 - v_i \Delta v_i + (\partial_n v_n)^2 \rangle
\]  

(28)

and using a Transverse-Longitudinal (TL) decomposition\cite{9} with notation

\[ k_{00} \equiv n, \]

(29)

\[ h_{i0} = h_{0i} \equiv \partial_k k^L + \epsilon_{il} \partial_l k^T, \]

(30)

\[ k_{ij} = k_{ji} \equiv (\eta_{ij} \Delta - \partial_i \partial_j) k^{TT} + \partial_i \partial_j k^{LL} + \]

\[ + (\epsilon_{ik} \partial_k \partial_j + \epsilon_{jk} \partial_k \partial_i) k^{TL} \]

(31)

\[ v_0 \equiv q, \]

(32)

\[ v_i \equiv \partial_i v^L + \epsilon_{il} \partial_l v^T, \]

(33)
where $\Delta \equiv \partial_i \partial_i$, eq. (28) can be rewritten as follows

$$S^{(3)L}_0 = \kappa^2 \langle \dot{k}^L \Delta \dot{k}^L + \dot{k}^T \Delta \dot{k}^T + \dot{v}^L \Delta \dot{v}^L + \dot{v}^T \Delta \dot{v}^T + 2 \dot{v}^L \Delta \dot{k}^T - 2 \dot{v}^T \Delta \dot{k}^L + (\Delta \dot{k}^{TT})^2 + (\Delta \dot{k}^{LL})^2 + 2(\Delta \dot{k}^{TT})^2 + 2(\dot{q})^2 - 2n\Delta \dot{k}^L + 2n\Delta \dot{v}^T + 2q\Delta \dot{v}^L - 2q\Delta \dot{k}^T + 2\Delta k^{LL} \Delta \dot{k}^L + 2\Delta k^{TL} \Delta \dot{k}^T + 2\Delta k^{LL} \Delta \dot{v}^T - 2\Delta k^{TL} \Delta \dot{v}^L + q\Delta q + n\Delta n + (\Delta k^L)^2 + 2(\Delta k^T)^2 + 2(\Delta v^L)^2 + (\Delta v^T)^2 + 2\Delta v^T \Delta k^L + 2q\Delta^2 k^{TL} + \Delta k^{TT} \Delta^2 k^{TT} + \Delta k^{TL} \Delta^2 k^{TL} \rangle \quad (34)$$

Primary Lagrangian constraints, joined to some links among accelerations, can be obtained through an inspection on field equations, which arise from (34). A "Coulomb" gauge is defined by the constraints $\partial_i k_{i\mu} = 0$, which can be rewritten in terms of the TL-decomposition as follows (up to harmonics)

$$k^L = k^{LL} = k^{TL} = 0 \quad , \quad (35)$$

and preservation provides the next conditions for longitudinal velocities and accelerations

$$\dot{k}^L = \dot{k}^{LL} = \dot{k}^{TL} = 0 \quad , \quad (36)$$

$$\ddot{k}^L = \ddot{k}^{LL} = \ddot{k}^{TL} = 0 \quad . \quad (37)$$

Equations (35) and (36) are six Lagrangian constraints.

Field equations with the help of gauge constraints, give the following five (primary) constraints

$$n = 0 \quad , \quad (38)$$

$$v^T = 0 \quad , \quad (39)$$
\[ \dot{v}^T = 0, \quad (40) \]
\[ \dot{k}^T - \dot{v}^L + q = 0, \quad (41) \]
\[ \Delta k^T - \Delta v^L + \dot{q} = 0, \quad (42) \]
\[ n = \dot{n} = v^T = \dot{v}^T = 0 \dot{k}^T - \dot{v}^L + q = \Delta k^T - \Delta v^L + \dot{q} = 0 \text{ and accelerations are related through} \]
\[ \ddot{v}^T = -\dot{n}, \quad (43) \]
\[ \ddot{k}^T + \ddot{v}^L = \ddot{q} + 2\Delta k^T, \quad (44) \]
\[ \ddot{k}^{TT} = \Delta k^{TT}, \quad (45) \]
\[ \ddot{q} = \Delta q. \quad (46) \]

Systematic preservation of constraints (38) and (42) provide a new constraint
\[ \dot{n} = 0, \quad (47) \]
and accelerations
\[ \ddot{n} = 0, \quad (48) \]
\[ \ddot{v}^T = 0, \quad (49) \]
\[ \ddot{k}^T = \Delta k^T, \quad (50) \]
\[ \ddot{v}^L = \Delta v^L. \quad (51) \]

In short, there is a set of twelve constraints
\[ n = \dot{n} = v^T = \dot{v}^T = k^L = \dot{k}^L = k^{LL} = \dot{k}^{LL} = k^{TL} = \dot{k}^{TL} = 0, \quad (52) \]
\[ \dot{k}^T - \dot{v}^L + q = 0, \quad (53) \]
\[ \Delta k^T - \Delta v^L + \dot{q} = 0, \quad (54) \]
then, there are three degrees of freedom, and the constraint system give rise to reduced action

\[ S^{(3)L^*}_0 = \kappa^2 \langle 4\dot{k}^T \Delta k^T + 4(\Delta k^T)^2 + 4(\dot{q})^2 
+ 4q\Delta q + (\Delta k^{TT})^2 + \Delta k^{TT} \Delta^2 k^{TT} \rangle \]  

(55)

Introducing notation

\[ Q \equiv 2q \]  

(56)

\[ Q^T \equiv 2(-\Delta)^{\frac{1}{2}} k^T \]  

(57)

\[ Q^{TT} \equiv \Delta k^{TT} \]  

(58)

the reduced action is rewritten as follows

\[ S^{(3)L^*}_0 = \kappa^2 \langle Q \Box Q - Q^T \Box Q^T + Q^{TT} \Box Q^{TT} \rangle \]  

(59)

showing two unitary and one non-unitary modes, then the Hamiltonian is not positive definite. This study could also have considered from the point of view of the exchange amplitude procedure, in which is considered the coupling to a (conserved) energy-momentum tensor of some source, trough Lagrangian terms \( \kappa k_{\mu\nu} T^\mu\nu \) and \( \chi v_\mu J^\mu \).

4 YM gravity with parity preserving massive term

It can be possible to write down a massive version which respect parity, for example

\[ S^{(3)}_m = S^{(3)}_0 - \frac{m^2 \kappa^2}{2} \langle T^\sigma_{\sigma\nu} T^\rho_{\rho\nu} - T^\lambda_{\lambda\mu\nu} T_{\mu\lambda\nu} - \frac{1}{2} T^{\lambda\mu\nu} T_{\lambda\mu\nu} \rangle . \]  

(60)

In a general case, if we allow independent variations on metric and connection two types of field equations can be obtained. On one hand, variations on metric give rise to the expression of the gravitacional energy-momentum tensor, \( T^\alpha_{\alpha\beta} \equiv \kappa^2 tr[F^{\alpha\sigma} F^\beta_{\sigma} - \frac{2\delta^\beta}{4} F^{\mu\nu} F_{\mu\nu}] \), in other words

\[ T^\alpha_{\alpha\beta} = -T^\alpha_{\alpha\beta} - \kappa^2 g^{\alpha\beta} \lambda^2 , \]  

(61)
where \( T^{\alpha\beta} \equiv -m^2 \kappa^2 t^{\alpha} \sigma t^\beta \sigma - t^{\alpha} \sigma t^\beta \sigma - (t^{\alpha\beta} + t^{\beta\alpha}) t_\sigma \sigma - \frac{5 \sigma^\alpha\beta}{2} t^{\mu\nu} t_{\mu\nu} + \frac{3 \sigma^\alpha\beta}{2} t^{\mu\nu} t_{\nu\mu} + \frac{\sigma^\alpha\beta}{2} (t_\sigma \sigma)^2 \) is the torsion contribution to the energy-momentum distribution and \( t^{\alpha\beta} \equiv \frac{\epsilon^{\mu\nu\alpha\beta}}{2} T_\nu^\lambda \). This says, for example, that the quest of possible black hole solutions must reveal a dependence on parameters \( m^2 \) and \( \lambda^2 \).

On the other hand, variations on connection provide the following equations

\[
\frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} F^{\alpha\lambda}) + [A_\alpha, F^{\alpha\lambda}] = J^\lambda ,
\]

where the current is \( (J^\lambda)^\nu_\sigma = m^2 (\delta^\lambda_\nu K^\rho_\sigma - \delta^\nu_\sigma K^\rho_\lambda + 2 K^\nu_\lambda) \) and the contorsion \( K^{\lambda\mu\nu} \equiv \frac{1}{2} (T^{\lambda\mu\nu} + T^{\mu\lambda\nu} + T^{\nu\lambda\mu}) \). We can observe in (62) that contorsion and metric appear as sources of gravity, where the cosmological contribution is obviously hide in space-time metric. In a weak torsion regime, equation (62) takes a familiar shape, this means \( \nabla_\alpha F^{\alpha\lambda} = J^\lambda \).

Now we explore the perturbation of the massive case given at (60) and with the help of (19), the linearized action is

\[
S^{(3)L}_m = \kappa^2 \langle k_{\mu\nu} \Box k^{\mu\nu} \rangle + \partial_\mu k^{\mu\sigma} \partial_\nu k^{\nu\sigma} - 2 \epsilon^{\sigma\alpha\beta} \partial_\alpha v_\beta \partial_\nu k^{\nu\sigma} - v_\mu \Box v^{\mu} - (\partial_\mu v^{\mu})^2 - m^2 (k_{\mu\nu} k^{\mu\nu} - k^2) \rangle .
\]

Using a TL-decomposition defined by (29)–(33), we can write (63) in the way

\[
S^{(3)L}_m = \kappa^2 \langle k^L \Delta k^L + \dot{k}^T \Delta \dot{k}^T + \dot{v}^L \Delta \dot{v}^L + \dot{v}^T \Delta \dot{v}^T + 2 \dot{v}^L \Delta \dot{k}^L - 2 \dot{v}^T \Delta \dot{k}^L \\
+ (\Delta \dot{k}^{TT})^2 + (\Delta \dot{k}^{LL})^2 + 2 (\Delta \dot{k}^{TL})^2 + 2 (\dot{q})^2 - 2 n \Delta \dot{k}^L + 2 n \Delta \dot{v}^T \\
+ 2q \Delta \dot{v}^L - 2q \Delta \dot{k}^L + 2k^{LL} \Delta \dot{k}^L + 2k^{TL} \Delta \dot{k}^T + 2k^{LL} \Delta \dot{v}^T \\
- 2k^{TL} \Delta \dot{v}^L + q \Delta q + n \Delta n + (\Delta k^L)^2 + 2 (\Delta k^T)^2 + 2 (\Delta v^L)^2 \\
+ (\Delta v^T)^2 + 2 \Delta v^T \Delta k^L + 2q \Delta^2 k^{TL} + \Delta k^{TT} \Delta^2 k^{TT} + \Delta k^{TL} \Delta^2 k^{TL} \\
+ m^2 [ -2k^L \Delta k^L - 2k^T \Delta k^T - 2 (\Delta k^{TT})^2 - 2 n (\Delta k^{TT} + \Delta k^{LL}) \\
+ 2 \Delta k^{TT} \Delta k^{LL} ] \rangle .
\]
Here, there is no gauge freedom (as it shall be confirmed in next section) and field equations provide primary constraints and some accelerations. The preservation procedure gives rise to expressions for all accelerations

\[ \ddot{n} = \Delta \dot{k}^L, \]  

\[ \ddot{k}^L = -\Delta \dot{k}^{TT} + \dot{n}, \]  

\[ \ddot{k}^T = \Delta \dot{k}^{TL}, \]  

\[ \ddot{k}^{LL} = \dot{k}^L + \dot{v}^T + m^2 k^{TT} - m^2 \Delta^{-1} n, \]  

\[ \ddot{k}^{TL} = \frac{1}{2}(\dot{k}^T - \dot{v}^L + \Delta \dot{k}^{TL} + q - 2m^2 k^{TL}), \]  

\[ \ddot{v}^L = -\dot{q} + 2\Delta v^L, \]  

\[ \ddot{v}^T = -\Delta \dot{k}^{TT} - \Delta \dot{k}^{LL} + \Delta k^L + \Delta v^T, \]  

and a set of eight constraints

\[ \dot{v}^T - \dot{k}^L + n - m^2 (k^{TT} + k^{LL}) = 0, \]  

\[ \Delta \dot{k}^{LL} - \Delta k^L - \Delta v^T + m^2 k^L = 0, \]  

\[ \Delta \dot{k}^{TL} - \dot{q} - \Delta k^T + \Delta v^L + m^2 k^T = 0, \]  

\[ \Delta \dot{k}^{LL} - \Delta k^L - \Delta v^T + m^2 (\dot{k}^{TT} + \dot{k}^{LL}) = 0, \]  

\[ \dot{k}^L + \Delta k^{TT} - n = 0, \]  

\[ \dot{k}^T - \Delta k^{TL} = 0, \]  

\[ \dot{v}^T + \Delta k^{TT} + m^2 (k^{TT} + k^{LL}) - 2m^2 \Delta^{-1} n = 0, \]  

\[ \dot{n} - \Delta k^L = 0, \]
which says that this massive theory propagates five degrees of freedom. In order to explore the physical content, we can take a short path to this purpose and it means to start with a typical transverse-traceless (Tt) decomposition instead the TL-decomposition one. Notation for the Tt-decomposition of fields is

\[ k_{\mu
u} = k^{Tt}_{\mu\nu} + \hat{\theta}_\mu \theta^T_{\nu} + \hat{\theta}_\nu \theta^T_{\mu} + \hat{\mu} \hat{\nu} \psi + \eta_{\mu\nu} \phi , \]  

(82)

\[ v_\mu = v^T_\mu + \hat{\theta}_\mu v , \]  

(83)

with the subsidiary conditions

\[ k^{Tt}_{\mu\nu} = 0 , \quad \partial^\mu k^{Tt}_{\mu\nu} = 0 , \quad \partial^\mu \theta^T_{\mu} = 0 , \quad \partial^\mu v^T_{\mu} = 0 . \]  

(84)

Action (63) is

\[ S^{(3)}_{Lm} = \kappa^2 \langle k^{Tt}_{\mu\nu}(\Box - m^2)k^{Tt\mu\nu} - \theta^T_{\mu}(\Box - 2m^2)\theta^T_{\nu} - 2\epsilon^{\sigma\alpha\beta} \partial_\alpha v^T_{\beta} \Box \theta^T_{\sigma} - v^T_{\mu} \Box v^T_{\mu} + 2v \Box v + 2\phi \Box \phi + 4m^2 \psi \phi + 6m^2 \phi^2 \rangle . \]  

(85)

A new transverse variable, \( a^T_{\mu} \) is introduced through

\[ \theta^T_{\mu} \equiv \epsilon^{\mu}_{\alpha\beta} \hat{\alpha} a^T_{\beta} , \]  

(86)

and the action (85) is rewritten as

\[ S^{(3)}_{Lm} = \kappa^2 \langle k^{Tt}_{\mu\nu}(\Box - m^2)k^{Tt\mu\nu} - a^T_{\mu}(\Box - 2m^2)a^T_{\mu} - 2a^T_{\mu} \Box v^T_{\mu} - v^T_{\mu} \Box v^T_{\mu} + 2v \Box v + 2\phi \Box \phi + 4m^2 \psi \phi + 6m^2 \phi^2 \rangle . \]  

(87)

The field equations are

\[ (\Box - m^2)k^{Tt}_{\mu\nu} = 0 , \]  

(88)

\[ \Box v^T_{\mu} = 0 , \]  

(89)

\[ \Box v = 0 , \]  

(90)

\[ a^T_{\mu} = 0 , \]  

(91)
\[ \psi = \phi = 0 , \tag{92} \]

and reduced action is

\[ S^{(3)L_m} = \kappa^2 \langle k^T_{\mu\nu}(\Box - m^2)k^{T\mu\nu} + 2\nu \Box v - v^T \mu \Box v^T \rangle , \tag{93} \]

saying that the contorsion propagates two massive helicities \( \pm 2 \), one massless spin-0 and two massless ghost vectors. Then, there is not positive definite Hamiltonian. This observation can be confirmed in the next section when we shall write down the Hamiltonian density and a wrong sign appears in the kinetic part corresponding to the canonical momentum of \( v_i \) (see eq. (105)).

5 Gauge transformations

The quadratical Lagrangian density dependent in torsion and presented in (60), has been constructed without free parameters, with the exception of \( m^2 \), of course. It has a particular shape which only gives mass to the spin 2 component of the contorsion, as we see in the perturbative regime. Let us comment about de non existence of any possible ”residual” gauge invariance of the model. The answer is that the model lost its gauge invariance and it can be shown performing the study of symmetries through computation of the gauge generator chains. For this purpose, a \( 2 + 1 \) decomposition of (63) is performed, this means

\[
S^{(3)L_m} = \kappa^2 \langle [-\dot{k}_{0i} + 2\partial_i k_{00} - 2\partial_n k_{ni} - 2\epsilon_{in} \dot{v}_n + 2\epsilon_{in} \partial_n v_0] \dot{k}_{0i} + \dot{k}_{ij} \dot{k}_{ij} \\
+ [2\epsilon_{nj} \partial_n k_{00} + 2\epsilon_{nj} \partial_m k_{nm} - \dot{v}_j - 2\partial_j v_0] \dot{v}_j + 2(\dot{v}_0)^2 + k_{00}\Delta k_{00} \\
-2k_{0i}\Delta k_{0i} + k_{ij}\Delta k_{ij} - (\partial_i k_{0i})^2 + \partial_n k_{ni}\partial_m k_{mi} - 2\epsilon_{ij} \partial_i \dot{v}_j \partial_n k_{n0} \\
-2\epsilon_{im} \partial_m v_0 \partial_n k_{ni} + v_0 \Delta v_0 - v_i \Delta \dot{v}_i + (\partial_n v_n)^2 \\
+ m^2 \langle 2k_{0i} k_{0i} - k_{ij} k_{ij} - 2k_{00} k_{ii} + (k_{ii})^2 \rangle \rangle , \tag{94} \]

where \( \epsilon_{ij} \equiv \epsilon^0_{ij} \) and \( \Delta \equiv \partial_i \partial_i \).
Next, the momenta are
\[ \Pi \equiv \frac{\partial L}{\partial \dot{k}_{00}} = 0 , \tag{95} \]
\[ \Pi^i \equiv \frac{\partial L}{\partial \dot{k}_{0i}} = -2\dot{k}_{0i} - 2\epsilon_m \dot{v}_n + 2\partial_i \dot{k}_i0 - 2\partial_n \dot{k}_{ni} + 2\epsilon_m \partial_n v_0 , \tag{96} \]
\[ \Pi^{ij} \equiv \frac{\partial L}{\partial \dot{k}_{ij}} = 2\dot{k}_{ij} , \tag{97} \]
\[ P \equiv \frac{\partial L}{\partial \dot{v}_0} = 4\dot{v}_0 , \tag{98} \]
\[ P^j \equiv \frac{\partial L}{\partial \dot{v}_j} = -2\epsilon_n \dot{k}_{0n} - 2\dot{v}_j + 2\epsilon_n \partial_n k_{00} + 2\epsilon_n \partial_n k_{mn} - 2\partial_j v_0 , \tag{99} \]
and we establish the following commutation rules
\[ \{k_{00}(x), \Pi(y)\} = \{v_0(x), P(y)\} = \delta^2(x - y) , \tag{100} \]
\[ \{k_{0i}(x), \Pi^j(y)\} = \{v_i(x), P^j(y)\} = \delta^j_i \delta^2(x - y) , \tag{101} \]
\[ \{k_{ij}(x), \Pi^{mn}(y)\} = \frac{1}{2}(\delta^n_i \delta^m_j + \delta^m_i \delta^n_j) \delta^2(x - y) . \tag{102} \]

It can be noted that (95) is a primary constraint that we name
\[ G^{(K)} \equiv \Pi , \tag{103} \]
where \( K \) means the initial index corresponding to a possible gauge generator chain, provided by the algorithm developed in reference[6]. Moreover, manipulating (97) and (99), other primary constraints appear
\[ G^{(K)}_i \equiv \partial_n k_{ni} - \epsilon_m \partial_n v_0 - \frac{\epsilon_m}{4} P^n + \frac{1}{4} \Pi^i , \tag{104} \]
and we observe that \( G^{(K)} \) and \( G^{(K)}_i \) are first class.

The preservation of constraints requires to obtain the Hamiltonian of the model. First of all, the Hamiltonian density can be written as \( \mathcal{H}_0 = \Pi^i \dot{h}_{0i} + \Pi^{ij} \dot{h}_{ij} + P \dot{v}_0 + \)
\( P^i v_i - \mathcal{L} \), in other words

\[
\mathcal{H}_0 = \frac{P^i \Pi^{ij} P_j}{4} + \frac{P^2}{8} - \frac{P^i P_i}{4} + \epsilon_{nj} \partial_m k_{nm} P^j + v_0 \left[ \partial_i P^i + 4 \epsilon_{ni} \partial_m \partial_n k_{nm} \right] 
+ k_{00} \left[ 2 \partial_m \partial_n k_{nm} - \epsilon_{nm} \partial_n P^m + 2 m^2 k_{ii} \right] + 2 k_{0i} \Delta k_{0i} - k_{ij} \Delta k_{ij} 
+ (\partial_i k_{i0})^2 - 2 \partial_n k_{ni} \partial_m k_{mi} + 2 \epsilon_{ij} \partial_i v_j \partial_n k_{n0} + v_i \Delta v_i - (\partial_n v_n)^2 
- m^2 \left[ 2 k_{0i} k_{0i} - k_{ij} k_{ij} + (k_{ii})^2 \right].
\] 

(105)

Then, the Hamiltonian is \( H_0 = \int dy^2 \mathcal{H}_0(y) \equiv \langle \mathcal{H}_0 \rangle_y \) and the preservation of \( G^{(i)} \), defined in (103) is

\[
\{ G^{(i)}(x), H_0 \} = -2 \partial_m \partial_n k_{mn}(x) + \epsilon_{nm} \partial_n P^m(x) - 2 m^2 k_{ii}(x). \]

(106)

The possible generators chain is given by the rule: 

\[
G^{(i-1)}(x) = 2 \partial_m \partial_n k_{mn}(x) - \epsilon_{nm} \partial_n P^m(x) + 2 m^2 k_{ii}(x)
+ \langle a(x,y)G^{(i)}(y) + b(x,y)G^{(i)}(y) \rangle_y.
\]

(107)

The preservation of \( G^{(i)}_i \), defined in (104), is

\[
\{ G^{(i)}_i(x), H_0 \} = \frac{\partial_i \Pi^{ni}(x)}{2} - \frac{\epsilon_{in} \partial_n P(x)}{4} + \frac{\epsilon_{in} \Delta v_n(x)}{2} + \frac{\epsilon_{in} \partial_n \partial_m v_m(x)}{2} + \epsilon_{nm} \partial_i \partial_n v_m(x) - (\Delta - m^2) k_{0i}(x),
\]

(108)

then

\[
G^{(i-1)}_i(x) = - \frac{\partial_i \Pi^{ni}(x)}{2} + \frac{\epsilon_{in} \partial_n P(x)}{4} - \frac{\epsilon_{in} \Delta v_n(x)}{2} - \frac{\epsilon_{in} \partial_n \partial_m v_m(x)}{2}
- \epsilon_{nm} \partial_i \partial_n v_m(x) + (\Delta - m^2) k_{0i}(x)
+ \langle a^i(x,y)G^{(i)}(y) + b^i(x,y)G^{(i)}(y) \rangle_y.
\]

(109)

The undefined objects \( a(x,y), b^i(x,y), a^i(x,y) \) and \( b^i_j(x,y) \) in expressions (107) and (109), are functions or distributions. If it is possible, they can be fixed in a way
that the preservation of $G^{(K-1)}(x)$ and $G'_{i}^{(K-1)}(x)$ would be combinations of primary constraints. With this, the generator chains could be interrupted and we simply take $K = 1$. Of course, the order $K - 1 = 0$ generators must be first class, as every one. Next, we can see that all these statements depend on the massive or non-massive character of the theory.

Taking a chain with $K = 1$, the candidates to generators of gauge transformation are [103], (104), (107) and (109). But, the only non null commutators are

$$\{G^{(1)}_i(x), G^{(0)}_j(y)\} = \frac{m^2}{4} \eta_{ij} \delta^2(x - y) ,$$

$$\{G^{(0)}(x), G^{(0)}_i(y)\} = m^2 (\partial_i \delta^2(x - y) + \frac{b^i(x, y)}{4}) ,$$

saying that the system of "generators" is not first class. Moreover, the unsuccessful conditions (in the $m^2 \neq 0$ case) to interrupt the chains, are

$$\{G^{(0)}(x), H_0\} = m^2 (\Pi^{nn}(x) - 2 \partial_n k_{0n}(x)) ,$$

$$\{G^{(0)}_i(x), H_0\} = m^2 (\partial_i k_{in}(x) + \partial_i k_{00}(x) - \partial_i k_{nn}(x)) ,$$

where we have fixed

$$a(x, y) = 0 ,$$

$$b^i(x, y) = -2 \partial^i \delta^2(x - y) ,$$

$$a^i(x, y) = 0 ,$$

$$b^i_j(x, y) = 0 .$$

All this indicates that in the case where $m^2 \neq 0$ there is not a first class consistent chain of generators and, then there is no gauge symmetry.
However, if we revisit the case $m^2 = 0$, conditions (112) and (113) are zero and the chains are interrupted. Now, the generators $G^{(1)}$, $G_i^{(1)}$, $G^{(0)}$ and $G_i^{(0)}$ are first class. Using (114)-(115), the generators are rewritten again

$$G^{(1)} \equiv \Pi,$$  \hspace{2cm} (118)

$$G_i^{(1)} \equiv \partial_n k_{ni} - \epsilon_{in} \partial_n v_0 - \frac{\epsilon_{in}}{4} P^m + \frac{1}{4} \Pi^i,$$  \hspace{2cm} (119)

$$G^{(0)} = -\frac{\epsilon_{nm}}{2} \partial_n P^m - \frac{\partial_n \Pi^m}{2},$$  \hspace{2cm} (120)

$$G_i^{(0)} = -\frac{\partial_n \Pi^m}{2} + \frac{\epsilon_{in}}{4} \partial_n P - \frac{\epsilon_{in}}{2} \Delta v_0 - \frac{\epsilon_{in}}{2} \partial_n \partial_m v_m - \frac{\epsilon_{nm}}{2} \partial_i \partial_n v_m + \Delta k_{0i}.$$  \hspace{2cm} (121)

Introducing the parameters $\varepsilon(x)$ and $\varepsilon^i(x)$, a combination of (118)-(121) is taken into account in the way that the gauge generator is

$$G(\ddot{\varepsilon}, \ddot{\varepsilon}^i, \varepsilon, \varepsilon^i) = \left( \dot{\varepsilon}(x) G^{(1)}(x) + \dot{\varepsilon}^i(x) G_i^{(1)}(x) + \varepsilon(x) G^{(0)}(x) + \varepsilon^i(x) G_i^{(0)}(x) \right),$$  \hspace{2cm} (122)

and with this, for example the field transformation rules (this means, $\delta(...)=\{(...),G\}$) are written as

$$\delta k_{00} = \dot{\varepsilon},$$  \hspace{2cm} (123)

$$\delta k_{0i} = \frac{\dot{\varepsilon}^i}{4} + \frac{\partial_i \varepsilon}{2},$$  \hspace{2cm} (124)

$$\delta k_{ij} = \frac{1}{4} (\partial_i \varepsilon_j + \partial_j \varepsilon_i),$$  \hspace{2cm} (125)

$$\delta v_0 = \frac{\epsilon_{nm}}{4} \partial_n \varepsilon_m,$$  \hspace{2cm} (126)

$$\delta v_i = \frac{\epsilon_{in}}{4} \dot{\varepsilon}_n - \frac{\epsilon_{in}}{2} \partial_n \varepsilon,$$  \hspace{2cm} (127)

and, redefining parameters as follows: $\varepsilon \equiv 2\xi_0$ and $\varepsilon^i = 4\xi^i$, it is very easy to see that these rules match with (21) and (22), as we expected.

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6 YM-extended formulation

Here we review a possible quadratical term family which allows to eliminate non-unitary propagations in the contorsion (torsion) perturbative regime in 2 + 1 dimension. The most general shape of a Lagrangian counter terms set is

\[
S^{(3)}_0 = \kappa^2 \left\langle -\frac{1}{4} (F^\mu{}^\nu)^\sigma{}^\rho (F_{\mu\nu})^\rho{}^\sigma + a_1 (F_{\mu\nu})^\sigma{}^\rho (F^\mu{}^\rho)^{\nu\rho} + a_2 (F_{\mu\nu})^\sigma{}^\rho (F^\mu{}^\rho)^{\nu\rho} 
+ a_3 (F_{\mu\sigma})^\sigma{}^\rho (F^\mu{}^\rho)^{\sigma\nu} + a_4 (F_{\mu\sigma})^\sigma{}^\rho (F^\nu{}^\rho)^{\rho\mu} + a_5 ((F_{\mu\nu})^{\mu\nu})^2 \right\rangle ,
\]

(128)

where \(a_1, a_2, a_3, a_4\) and \(a_5\) are real parameters.

A naive try to reach unitarity consists to perform a direct matching between the perturbative action coming from (128) and the linearized Hilbert-Einstein one, given by

\[
S_{HE}^L = -2\kappa^2 \left\langle h_{\mu\nu} G_{L}^{\mu\nu} \right\rangle = \kappa^2 \left\langle h_{\mu\nu} \Box h^{\mu\nu} + 2\partial_\mu h^{\mu\sigma} \partial_\nu h^{\nu\sigma} + 2h \partial_\mu \partial_\nu h^{\mu\nu} - h \Box h \right\rangle ,
\]

(129)

where \(h_{\mu\nu}\) is the metric perturbation and \(G_{L}^{\mu\nu}\) is the linearized Einstein’s tensor. Then, under perturbations of the contorsion (torsion), one can use again eq. (19), this time in (128). Next, making comparison with (129), a two free parameter system can be obtained (i.e., \(a_3 \equiv \alpha\) and \(a_5 \equiv \beta\)) and possible unitary theories which propagates massless spin 2 in 2+1 dimension, are

\[
S^{(3)}_{(\alpha,\beta)} = \kappa^2 \left\langle -\frac{1}{4} (F^\mu{}^\nu)^\sigma{}^\rho (F_{\mu\nu})^\rho{}^\sigma - (1 + \alpha) (F_{\mu\nu})^\sigma{}^\rho (F^\mu{}^\rho)^{\nu\rho} 
+ \frac{5}{8} + \alpha + 4\beta) (F_{\mu\sigma})^\sigma{}^\rho (F^\mu{}^\rho)^{\sigma\nu} + \alpha (F_{\mu\sigma})^\sigma{}^\rho (F^\nu{}^\rho)^{\rho\mu} 
- \frac{1}{2} + \alpha + 4\beta) (F_{\mu\sigma})^\sigma{}^\rho (F^\nu{}^\rho)^{\rho\mu} + \beta ((F_{\mu\nu})^{\mu\nu})^2 \right\rangle ,
\]

(130)

and they are labeled with parameters \(\alpha\) and \(\beta\). There are two possible massive cases. On one hand, can be considered the topological massive model \([5]\), which is sensitive under parity. On the other hand, there is a “Fierz-Pauli” model \([60]\). Our main purpose in this section is to study the classical consistence of field equations, focusing the attention at the massless and topological massive cases.
In the massless theory with cosmological constant, $\lambda$ in $2 + 1$ dimension, we introduce a cosmological term as follows

$$S^{(3)}_{(\alpha, \beta, \lambda)} = S^{(3)}_{(\alpha, \beta)} + \kappa^2 \langle q(\alpha, \beta) \lambda^2 \rangle,$$  \hspace{1cm} (131)

where $q(\alpha, \beta)$ is a (unknown) real function of family’s parameters. Next, in order to consider classical consistence at the torsionless regime, we take into account some auxiliary fields (Lagrange multipliers), $b_{\mu\nu}$ and the action with torsion constraints is given by

$$S'_{(3)}^{(3)}(\alpha, \beta, \lambda) = S^{(3)}_{(\alpha, \beta)} + \kappa^2 \langle q(\alpha, \beta) \lambda^2 \rangle + \kappa^2 \langle b_{\alpha\beta} \varepsilon^{\beta\lambda\sigma} (A_{\lambda})^\alpha_{\sigma} \rangle,$$  \hspace{1cm} (132)

where arbitrary variations on fields $b_{\mu\nu}$, obviously provide the condition $T^{\alpha}_{\mu\nu} = 0$. Then, the field equation coming from variations of connection is

$$\nabla_{\mu}(F_{\mu\nu})^{\sigma}_{\rho} + b_{\rho\mu} \varepsilon^{\mu\sigma} = 0,$$  \hspace{1cm} (133)

where $F_{\mu\nu}$ is defined in terms of the Yang-Mills curvature, $F_{\mu\nu}$ in the way

\[
(F_{\mu\nu})^{\sigma}_{\rho} \equiv (F_{\mu\nu})^{\sigma}_{\rho} + 2(1 + \alpha)[(F_{\mu})^{\sigma}_{\rho} - (F_{\nu})^{\mu}_{\rho}] + \left(\frac{5}{4} + \alpha + 2\beta\right)[(F_{\rho})^{\sigma}_{\mu} - (F_{\rho})^{\mu\nu}] \\
+ 2\alpha[(F_{\nu})^{\lambda\sigma\delta\rho}_{\lambda} - (F_{\mu})^{\lambda\sigma\delta\rho}_{\mu}] + (1 + 2\alpha + 8\beta)[(F_{\lambda})^{\lambda\mu\delta\nu}_{\lambda} - (F_{\sigma})^{\lambda\mu\delta\nu}_{\rho}] \\
+ 2\beta(F_{\lambda\kappa})^{\lambda\kappa}(g^{\mu\sigma}\delta^{\nu}_{\rho} - g^{\nu\sigma}\delta^{\mu}_{\rho}),
\]

\hspace{1cm} (134)

and now, we can match the YM curvature with the Riemann-Christoffel one (i.e., $(F_{\mu\nu})_{\alpha\beta} = R_{\alpha\beta\nu\mu}$), which satisfies the well known algebraic properties and Bianchi identities recalling as follows

**Symmetry:** $R_{\alpha\beta\nu\mu} = R_{\nu\mu\alpha\beta}$, \hspace{1cm} (135)

**Antisymmetry:** $R_{\alpha\beta\nu\mu} = -R_{\beta\alpha\nu\mu} = R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\mu\nu}$, \hspace{1cm} (136)
Cyclicity: \( R_{\alpha\beta\nu\mu} + R_{\alpha\mu\beta\nu} + R_{\alpha\nu\mu\beta} = 0 \), \hspace{1cm} (137)

Bianchi identities: \( \nabla_\sigma R_{\alpha\beta\nu\mu} + \nabla_\mu R_{\alpha\beta\sigma\nu} + \nabla_\nu R_{\alpha\beta\mu\sigma} = 0 \). \hspace{1cm} (138)

In 2 + 1 dimension, the curvature tensor can be written in terms of Ricci’s tensor \( R_{\mu\sigma} \equiv R^\lambda_{\mu\lambda\sigma} \) and its trace \( R \equiv R^\lambda_{\lambda\lambda} \) in the way \( R_{\lambda\mu\nu\sigma} = g_{\lambda\nu} R_{\mu\sigma} - g_{\lambda\sigma} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\sigma} + g_{\mu\sigma} R_{\lambda\nu} - \frac{R}{2} (g_{\lambda\nu} g_{\mu\sigma} - g_{\lambda\sigma} g_{\mu\nu}) \). So, the object defined in (134) takes the shape

\[
(F_{\alpha\nu})_{\lambda\mu} = \left(\frac{3}{2} + 4\beta\right) R_{\lambda\mu\nu\sigma} + (1 + 8\beta)(g_{\mu\nu} R_{\lambda\sigma} - g_{\mu\sigma} R_{\lambda\nu}) + 2\beta R(g_{\lambda\nu} g_{\mu\sigma} - g_{\lambda\sigma} g_{\mu\nu}) ,
\]

which do not depend on parameter \( \alpha \). Moreover, if \( \beta \) is fixed as \( \beta = -\frac{1}{8} \), \hspace{1cm} (140)

then, relation (139) leads to

\[
(F_{\alpha\nu})_{\lambda\mu} \big|_{\beta=-\frac{1}{8}} = R_{\lambda\mu\nu\sigma} - \frac{R}{4} (g_{\lambda\nu} g_{\mu\sigma} - g_{\lambda\sigma} g_{\mu\nu}) ,
\]

and this one satisfies all symmetry properties of a curvature, showing in relations (135)-(137) with the exception of the Bianchi identities, (138). It can be noted that the trace of (131), this means \( (F_{\alpha\lambda})^\lambda_{\mu} \) is the Einstein’s tensor.

Next, some discussion on the critical value (140) shall be performed when the connection’s field equation is taking into account. With the help of symmetry properties, Bianchi’s identities, and relationship between Riemann-Christoffel and Ricci tensor, the field equation (133) can be rewritten as follows

\[
(\frac{1}{2} - 4\beta) \nabla_\rho R_{\nu\sigma} - (\frac{3}{2} + 4\beta) \nabla_\sigma R_{\nu\rho} + (\frac{1}{2} + 2\beta) g_{\nu\rho} \partial_\sigma R + 2\beta g_{\nu\sigma} \partial_\rho R + b_{\rho\mu} \varepsilon^\mu_{\nu\sigma} = 0 ,
\]

and with some algebraic computation, it can be shown (for all \( \beta \)) the next symmetry property

\[
b_{\nu\mu} = b_{\mu\nu} ,
\]

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and

\[(\beta - \frac{5}{8}) b_{\mu \nu} = 0 \, , \quad \text{(144)}\]

\[(\beta + \frac{1}{8}) \partial_{\mu} R = 0 . \quad \text{(145)}\]

Consistence condition (144) establishes that the work out of Lagrange multipliers depends on the following restriction

\[\beta \neq \frac{5}{8} . \quad \text{(146)}\]

then, \(b_{\mu \nu} = 0\). Condition (146) induces a wide set of possible vacuum’s solutions, including non-Einsteinian ones beside (A)dS, because eq. (145) becomes an identity when it is evaluated on the critical \(\beta\) given by (140). This fact is confirmed when \(\beta = -\frac{1}{8}\) is introduced in eq. (142), in other words

\[\nabla_{\mu} R_{\nu \rho} - \nabla_{\sigma} R_{\nu \rho} = 0 \, , \quad \text{(147)}\]

where notation means

\[R_{\mu \nu} \equiv R_{\mu \nu} - \frac{g_{\mu \nu}}{4} R . \quad \text{(148)}\]

It can be observed that equation (147) looks like eq. (2), but here, as one can expect the trace \(\sigma - \lambda\) of (147) is an identity.

In order to conclude the comments on the massless theory, next we consider the field equation which comes from variations on metric of the action (132) and it can be written in terms of Ricci’s tensor and Ricci’s scalar as follows

\[\left(\frac{3}{2} - \alpha + 12 \beta\right) R_{\alpha \mu} R_{\nu \sigma} - \left(\frac{1}{2} - \alpha + 6 \beta\right) R R_{\mu \nu} - \left(1 - \alpha + 4 \beta\right) R^{\rho \sigma} R_{\sigma \rho} g_{\mu \nu} \]

\[+ \left(\frac{5}{16} - \frac{\alpha}{2} + 2 \beta\right) R^2 g_{\mu \nu} + \frac{q}{2} \lambda^2 g_{\mu \nu} = 0 . \quad \text{(149)}\]

Immediately, the consistence with (A)dS solutions is evaluated by replacing the contractions of \(R_{\mu \nu \sigma} = \lambda (g_{\rho \sigma} g_{\mu \nu} - g_{\mu \nu} g_{\rho \sigma})\) in (149). This gives

\[q(\alpha) = \frac{3}{2} - 4 \alpha , \quad \text{(150)}\]
and this indicates that if $\alpha = \frac{3}{8}$ is introduced in action (132) get implicit (A)dS solutions from its field equations.

Now we take a look on the $GTMG\lambda$ formulation, considering the YM-extended action at the torsionless limit, this means

$$S' = S^{(3)}_{(\alpha, \beta)} + \frac{m\kappa^2}{2} \langle \epsilon_{\mu\nu\lambda} tr \left( A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \right) \rangle + \kappa^2 \langle q(\alpha) \lambda^2 \rangle + \kappa^2 \langle b_{\alpha\beta} \epsilon^{\beta\lambda\sigma} (A_\lambda)^{\alpha}_{\sigma} \rangle,$$

(151)

where $q(\alpha)$ is defined by (150) then, this action is consistent with (A)dS solutions when $m = 0$. Variations on the metric conduce to the known equations (149). So, the connection field equation is

$$\nabla_\mu (F^{\mu\nu})^{\sigma}_{\rho} + \frac{m}{2} \epsilon^{\alpha\beta\nu} (F_{\alpha\beta})^{\sigma}_{\rho} + b_{\rho\mu} \epsilon^{\mu\nu\sigma} = 0,$$

(152)

and $(F^{\mu\nu})^{\sigma}_{\rho}$ is defined in (134). Recalling that $(F_{\mu\nu})_{\alpha\beta} = R_{\alpha\beta\nu\mu}$ in a torsionless space-time, equation (152) can be rewritten in terms of Ricci’s tensor as follows

$$\left(\frac{1}{2} - 4\beta\right) \nabla_\rho R_{\nu\sigma} - \left(\frac{3}{2} + 4\beta\right) \nabla_\sigma R_{\nu\rho} + \left(\frac{1}{2} + 2\beta\right) g_{\nu\rho} \partial_\sigma R + 2\beta g_{\nu\sigma} \partial_\rho R$$

$$- m \epsilon^{\alpha\beta\nu} (g_{\alpha\sigma} R_{\beta\rho} - g_{\alpha\rho} R_{\beta\sigma} - \frac{R}{2} g_{\alpha\sigma} g_{\beta\rho}) + b_{\rho\mu} \epsilon^{\mu\nu\sigma} = 0.$$

(153)

Performing some algebraic manipulation on this last equation, conditions (143) and (145), which establish the symmetry property of Lagrange multipliers and the indetermination of scalar curvature when $\beta = -\frac{1}{8}$, rise again in a similar way that they do in the massless theory.

Then, using condition (146), the Lagrange multipliers are given by

$$b_{\mu\nu} = 2 \left(\frac{\beta + \frac{1}{8}}{\beta - \frac{3}{8}}\right) m R_{\mu\nu} - \left(\frac{\beta + \frac{3}{8}}{\beta - \frac{3}{8}}\right) \frac{m R}{2} g_{\mu\nu},$$

(154)

and if (140) is fixed, the result (9) is recovered. So, evaluating the theory on $\beta = -\frac{1}{8}$, the action (153) becomes in a similar form as in (7), this means

$$\nabla_\mu R_{\sigma\lambda} - \nabla_\lambda R_{\sigma\mu} - m \epsilon^{\nu\rho}_{\sigma} (g_{\lambda\nu} R_{\mu\rho} - g_{\mu\nu} R_{\lambda\rho} - \frac{2}{3} R_{\nu} g_{\lambda\rho} g_{\mu\sigma}) = 0,$$

(155)
where again $\mathcal{R}_{\mu\nu}$ is defined as in (148) and the trace $\sigma - \lambda$ is an identity, as one can expect.

7 Concluding remarks

A perturbative regime based on arbitrary variations of the contorsion and metric as a (classical) fixed background, is performed in the context of a pure Yang-Mills formulation of the $GL(3, R)$ gauge group. There, we analyze in detail the physical content and the well known fact that a variational principle based on the propagation of torsion (contorsion), as dynamical and possible candidate for a quantum canonical description of gravity in a pure YM formulation gets serious difficulties.

In the $2+1$ dimensional massless case we show that the theory propagates three massless degrees of freedom, one of them a non-unitary mode. Then, introducing appropriate quadratical terms dependent on torsion, which preserve parity and general covariance, we can see that the linearized limit do not reproduc es an equivalent pure Hilbert-Einstein-Fierz-Pauli massive theory for a spin-2 mode and, moreover there is other non-unitary modes. Roughly speaking, at first sight one can blame it on the kinetic part of YM formulation because the existence of non-positive Hamiltonian connected with non-unitarity problem. Nevertheless there are other possible models of Gauss-Bonnet type which could solve the unitarity problem.

Exploring the massless and the topological massive gravity models in $2+1$ dimension, the well known existence of a YM-extended theories family is noted. This family is labeled with two free parameters, $\alpha$ and $\beta$ and can cure non-unitary propagations.

Nevertheless, when the classical consistence between these type of theories and the Einstein’s one is tackled, what we have mentioned as torsionless limit, it is shown that the parameter $\alpha$ is related with the coupling of the cosmological constant in the action.

Meanwhile, the parameter $\beta$ get two types of critical values. On one side, the number $\beta = \frac{5}{8}$ is connected to the classical consistence requirement which demands the
introduction of torsion’s Lagrangian constraints with solvable Lagrange multipliers. On the other side, the value $\beta = -\frac{1}{8}$ establishes a wide set of theories, including the Einstein’s solutions after the imposition of a auxiliary condition $R = constant$ and non-Einsteinian ones when the Ricci scalar became an arbitrary function. But, even though the Lagrangian extension of the YM formulation for gravity conduces to the well known fact that there exists unphysical classical solutions, the same occurs (in a much less severe way) without these corrections and one can recall the YM pure formulation gives rise a set of solutions for the massless and topological massive gravity with the property $R = constant$ and only Einsteinian results can be obtained if the auxiliary condition $R = -6\lambda$ is fixed.

A picture with a little bit of generalization including a dynamic metric and non-Minkowskian background in the perturbative analysis would be considered elsewhere.

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