CONCISE PRESENTATIONS OF DIRECT PRODUCTS

MARTIN R. BRIDSON

Abstract. Direct powers of perfect groups admit more concise presentations
than one might naively suppose. If $H_1(G, \mathbb{Z}) = H_2(G, \mathbb{Z}) = 0$, then $G^n$ has
a presentation with $O(\log n)$ generators and $O((\log n)^3)$ relators. If, in addition,
there is an element $g \in G$ that has infinite order in every non-trivial quotient of
$G$, then $G^n$ has a presentation with $d(G) + 1$ generators and $O(\log n)$ relators.
The bounds that we obtain on the deficiency of $G^n$ are not monotone in $n$; this
points to potential counterexamples for the Relation Gap Problem.

1. Introduction

If two groups are presented as $A = \langle X \mid R \rangle$ and $B = \langle Y \mid S \rangle$, then their direct
product is given by the presentation with generators $X \sqcup Y$ and relators $R, S$ and
$\{[x, y] : x \in X, y \in Y\}$. Similarly, if $A_i = \langle X_i \mid R_i \rangle$ with $|X_i| = k_i$ and $|R_i| = l_i$, then
the obvious presentation of $A_1 \times \cdots \times A_n$ has $\sum k_i$ generators and $\sum l_i + \sum_{i<j} k_i k_j$
relators. In particular, the direct product $A^n$ of $n$ copies of $A = \langle X \mid R \rangle$ with
$|X| = k$ and $|R| = l$ has a presentation with $kn$ generators and $nl + k^2 n(n - 1)/2$
relators. In the absence of further hypotheses, one cannot do better than these naive
bounds. For example, one cannot generate $\mathbb{Z}^n$ with fewer than $n$ generators, and the
number of relators needed to present $\mathbb{Z}^n$ is at least the rank of $H_2(\mathbb{Z}^n, \mathbb{Z}) \cong \mathbb{Z}^n \wedge \mathbb{Z}^n$,
which is $n(n - 1)/2$. But when $H_1(G, \mathbb{Z})$ and $H_2(G, \mathbb{Z})$ vanish, one can construct
much more concise presentations of $G^n$ – that is the main theme of this note.

We shall see that, in addition to the vanishing of homology, the existence of finite
quotients of $G$ plays a key role in determining how concise a presentation of $G^n$ can
be. The various possibilities are summarised in the following theorem, in which we
use the standard notation $d(\Gamma)$ for the minimal number of generators of $\Gamma$ and we
define $\rho(\Gamma)$ to be the minimum number of relators in any finite presentation of $\Gamma$.
All of the results concerning the growth of $d(G^n)$ are taken from [17]; they draw on
earlier results of Hall [8], Wiegold [13, 14, 15] and others. The estimates on $\rho(G^n)$
are new (or trivial).

1991 Mathematics Subject Classification. 20F05, 20J06.
Key words and phrases. Group presentations, direct products, homology of groups, relation gap.
The author is supported by a Wolfson Research Merit Award from the Royal Society.
We use the standard notation $f(n) = \Theta(g(n))$ for functions that are bounded above and below by positive multiples of $g(n)$, and for brevity we write $H_i G$ in place of $H_i(G, \mathbb{Z})$. Throughout, $G^n$ denotes the direct product of $n$ copies of $G$.

Theorem 1.1. Let $G$ be a finitely presented group.

1. If $H_1 G \neq 0$, then $d(G^n) = \Theta(n)$ and $\rho(G^n) = \Theta(n^2)$.
2. If $H_1 G = 0$ and $H_2 G \neq 0$, then $d(G^n) = O(\log n)$ and $\rho(G^n) = \Theta(n)$.
3. If $H_1 G = H_2 G = 0$, then $d(G^n) = O(\log n)$ and $\rho(G^n) = O(\log n)^3$.
4. If $H_1 G = H_2 G = 0$ and $G$ has a non-trivial finite quotient, then $d(G^n) = \Theta(\log n)$ and there are constants $c_0, c_1$ such that $c_0 \log n \leq \rho(G^n) \leq c_1 (\log n)^3$.
5. If $H_1 G = H_2 G = 0$ and there is an element $g \in G$ that has infinite order in every non-trivial quotient of $G$, then $d(G^n) \leq d(G) + 1$ for all $n$, and $\rho(G^n) = O(\log n)$.

In all cases, the upper bounds on $d(G^n)$ and $\rho(G^n)$ can be satisfied simultaneously.

I see no reason to expect that $\rho(G^n)$ is a monotone function of $n$ for all finitely presented perfect groups $G$, and this is intriguing in the context of the celebrated Relation Gap Problem [9]. Recall that the deficiency of a finite group presentation $\langle A \mid R \rangle$ is $|R| - |A|$, and the deficiency $\text{def}(G)$ of a group $G$ is defined\footnote{There are two conventions in the literature: many authors take this definition to be $-\text{def}(G)$.} to be the least deficiency among all finite presentations of $G$. Our constructions suggest that the following problem might have a positive answer. If it does, then $\Gamma^m$ would be a counterexample to the Relation Gap Problem: see remark 2.8 for an explanation and variations.

Problem 1.2. Does there exist a finitely presented perfect group $\Gamma$ and a positive integer $m$ such that $\text{def}(\Gamma^m) > \text{def}(\Gamma^{m+1})$ or $\rho(\Gamma^m) > \rho(\Gamma^{m+1})$?

2. Proofs

We shall need some basic facts about universal central extensions of groups.

A central extension of a group $G$ is a group $\widetilde{G}$ equipped with an epimorphism $\pi : \widetilde{G} \to G$ whose kernel is central in $\widetilde{G}$. Such an extension is universal if given any other central extension $\pi' : E \to G$ of $G$, there is a unique homomorphism $f : \widetilde{G} \to E$ such that $\pi' \circ f = \pi$. The standard reference for this material is [10] pp. 43–47. The properties that we need here are these: $G$ has a universal central extension $\widetilde{G}$ if (and only if) $H_1(G, \mathbb{Z}) = 0$; there is a short exact sequence

$$1 \to H_2(G, \mathbb{Z}) \to \widetilde{G} \to G \to 1;$$

and if $G$ has no non-trivial finite quotients then neither does $\widetilde{G}$.

The following result is Proposition 3.5 of [2].
Lemma 2.1. Let $G = \langle X \mid R \rangle$ be a perfect group, let $F$ be the free group on $X$ and for each $x \in X$ let $c_x \in [F, F]$ be a word such that $x = c_x$ in $G$. Then the following is a presentation of the universal central extension of $G$:

$$(2.1) \quad \tilde{G} = \langle X \mid x^{-1}c_x, [r, x] \ (\forall r \in R, x \in X) \rangle,$$

and the identity map $X \to X$ extends uniquely to an epimorphism $\tilde{G} \to G$ with kernel isomorphic to $H_2(G, \mathbb{Z})$.

Corollary 2.2. If $H_1(G, \mathbb{Z}) = H_2(G, \mathbb{Z}) = 0$, then (2.1) is a presentation of $G$.

It will be convenient to use functional notation for words. Thus, given a word $u$ in the symbols $x_1^{\pm 1}, \ldots, x_k^{\pm 1}$, we write $u(x)$ to emphasize the underlying alphabet and we write $u(y)$ for the word obtained by replacing each occurrence of each $x_i$ with $y_i$, where $y_i^{\pm 1}, \ldots, y_k^{\pm 1}$ is a second (ordered) alphabet.

Proposition 2.3. Let $G = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_l \rangle$, let $F$ be the free group on the $x_i$, suppose that $H_1(G, \mathbb{Z}) = H_2(G, \mathbb{Z}) = 0$, and for each $x_i$ fix $c_i(x) \in [F, F]$ such that $x_i = c_i(x)$ in $G$. Then the following is a presentation of $G \times G$:

$$(2.2) \quad \langle x_1, \ldots, x_k, y_1, \ldots, y_k \mid r_1, \ldots, r_l, y_i^{-1}c_i(y), [x_i, y_i^{-1}, y_j], 1 \leq i, j \leq k \rangle.$$

Proof. First observe that the last family of relations can be written as $x_i^{-1}y_i x_i = y_i^{-1}y_j y_i$, from which it follows that $x_i^{-1}ux_i = y_i^{-1}uy_i$ for all words $u$ in the free group on $\{y_1, \ldots, y_k\}$ and each $i = 1, \ldots, k$. Therefore, in the group presented, the transcription $r_j(y)$ of each relation $r_j(x)$ is central in the subgroup $G_1 := \langle y_1, \ldots, y_k \rangle$, because $y_i = y_i^{r_j(x)} = y_i^{r_j(y)}$. Thus $G_1$ (which is clearly normal) satisfies the relations that were used in Lemma 2.1 to define the universal central extension $\tilde{G}$. And $\tilde{G} = G$, because $H_2(G, \mathbb{Z}) = 0$.

At this stage we know that the group given by presentation (2.2) has the form $G_1 \rtimes G_2$, with $G_1 \cong G_2 \cong G$, where $G_1$ is the subgroup generated by the $y_i$ and $G_2$ is the subgroup generated by the $x_i$. The action $\phi : G_2 \to \text{Aut}(G_1)$ defining the semidirect product is by inner automorphisms, $x_i \mapsto \text{ad}_{y_i}$. Because this action factors $G_2 \to G_1 \to \text{Inn}(G_1)$, we have $G_1 \rtimes G_2 \cong G_1 \times G_2$; indeed an isomorphism $\phi : G_1 \times G_2 \to G_1 \rtimes G_2$ is given by $\phi(y_i) = y_i$ and $\phi(x_i) = y_i^{-1}x_i$. \hfill \Box

At first blush, this proposition seems to gain us little or nothing compared to the naive presentation of $G \times G$: we have traded the $l$ obvious relations of $G_1$ for the $k$ relations $y_i^{-1}c_i(y)$. The real benefit comes when we iterate the construction and use the fact that the number of generators that $G^n$ requires grows strikingly slowly (an old observation of Philip Hall [8]). To exploit this we need:

Lemma 2.4. Let $G = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_l \rangle$ and suppose $H_1G = H_2G = 0$. If $G^n$ requires at most $k$ generators and $2^n \leq N$, then $G^{2^n}$ has a presentation with $k$ generators and $n(k^2 + 2k) + l$ relations.
Proof. As in the previous proof, we construct a presentation of \( G^2 \) with \( 2k \) generators \( b_1, \ldots, b_{2k} \) and \( k^2 + k + l \) relations. We then make Tietze moves to add a new generating set \( a_1, \ldots, a_k \), together with \( k \) relations expressing the \( a_i \) as words in the generators \( b_i \). There are words \( u_i \) in the generators \( a_j \) such that \( b_i = u_i \) in \( G \times G \). We make further Tietze moves, removing the generators \( b_i \) and replacing each occurrence of \( b_i \) in the relators by \( u_i \). Thus we obtain a presentation of \( G \times G \) with \( k \) generators and \( k^2 + 2k + l \) relators.

Repeating the argument with \( G \times G \) in place of \( G \), we obtain a presentation for \( G^4 \) with \( k \) generators and \( 2(k^2 + 2k) + l \) relators. And continuing in this manner (provided that we stay in the range where \( G^{2^n} \) needs only \( k \) generators), we obtain a presentation for \( G^{2^n} \) with \( k \) generators and \( n(k^2 + 2k) + l \) relators. \( \square \)

**Corollary 2.5.** If \( G \) and \( N \) are as in the lemma and \( m \leq N/2 \), then \( G^m \) has a presentation with \( k \) generators and \( (k^2 + 2k)(\log_2 m + 1) + l + k \) relators.

*Proof.* Let \( n \) be the least integer such that \( m \leq 2^n \) and write \( G^{2^n} = G^m \times G^{2^n-m} \). The lemma tells us that \( G^{2^n} \) has a presentation with \( k \) generators and \( n(k^2 + 2k) + l \) relators. Moreover, as \( 2^n - m < N \), the second factor in the given decomposition is a \( k \)-generator group, and can therefore be killed by the addition of at most \( k \) relations. To complete the proof, note that \( n - 1 < \log_2 m \). \( \square \)

It is an open question as to whether every finitely generated perfect group is the normal closure of one element. If it is, then the \( k \) relations added to kill \( G^{2^n-m} \) in the above proof could be replaced by a single relation.

We shall need the following result of Wiegold and Wilson [17]; the proof presented here is new but has much in common with the original.

**Proposition 2.6.** Let \( G \) be a perfect group. If \( d(G) = r \), then \( d(G^m) \leq r(1 + \lceil \log_2 (m+1) \rceil) \).

*Proof.* Let \( M = \lceil \log_2 (m+1) \rceil \), the least integer with \( m < 2^M \). The proof uses binary expansions \( j = \sum_{i=0}^{M-1} \varepsilon_i(j)2^i \) of integers \( j = 1, \ldots, m \). Given a generating set \( \{a_1, \ldots, a_k\} \) for \( G \), for \( i = 0, \ldots, M - 1 \) we define

\[
a_{r,i} = (a_r^{\varepsilon_i(1)}, a_r^{\varepsilon_i(2)}, \ldots, a_r^{\varepsilon_i(m)}).
\]

For each pair of integers \( 1 \leq j < j' \leq m \), there is some \( i \) such that \( \varepsilon_i(j) \neq \varepsilon_i(j') \), and for that \( i \) we have \( p_{j,j'}(a_{r,i}) \in \{(a_r,1), (1,a_r)\} \), where \( p_{j,j'} : G^m \rightarrow G \times G \) is the coordinate projection to the \( j \) and \( j' \) factors. The image under \( p_{j,j'} \) of the diagonal element \( a_r := (a_r, \ldots, a_r) \) is \( (a_r, a_r) \). Thus the restriction of \( p_{j,j'} \) to the subgroup \( S \leq G^m \) generated by the set \( \{a_r, a_{r,i} \mid r = 1, \ldots, k; i = 0, \ldots, M - 1\} \) is surjective. It follows that \( S \) contains the \( (m-1) \)-st term of the lower central series of \( G^m \) (see [4] p.643). But \( G^m \) is perfect, so each term of the lower central series is the whole group, and therefore \( S = G^m \). \( \square \)
Theorem 2.7. If $G$ is a finitely presented group with $H_1G = H_2G = 0$, then $G^m$ has a finite presentation with at most $O(\log m)$ generators and $O(\log m)^3$ relators.

Proof. The preceding proposition shows that $d(G^m) = O(\log m)$. We fix a constant $k$ so that $G$ can be generated by $k$ elements and $G^m$ can be generated by $k[\log_2 m]$ elements, for all positive integers $m$. Suppose $G = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_l \rangle$. Since $G^2$ only needs $k$ generators, as in Lemma 2.4 we obtain a presentation of $G^2$ with $k$ generators and $k^2 + 2k + l$ relators. From Proposition 2.3 we then get a presentation of $G^4 = G^2 \times G^2$ with $2k$ generators and $k^2 + k + (k^2 + 2k + l) = 2k^2 + 3k + l$ relators. Applying Proposition 2.3 again we get a presentation of $G^8$ with $4k$ generators and $(2k)^2 + 2k + (k^2 + 3k + l) = 6k^2 + 5k + l$ relators. Since $G^8$ only requires $3k$ generators, as in the proof of Lemma 2.4 we can convert this to a presentation with $3k$ generators and $6k^2 + 8k + l$ relators.

Repeating this argument, we obtain a presentation of $G^{16}$ with $6k$ generators and $(3k)^2 + 3k + (6k^2 + 8k + l) = 15k^2 + 11k + l$ relators, which we convert to one with $4k$ generators and $15k^2 + 15k + l$ relators. And, proceeding by induction, we get a presentation of $G^{2^n}$ with $nk$ generators and $\sigma_n k^2 + \tau_n k + l$ relators, where $\sigma_n - 1 = n(n - 1)(2n - 1)/6$ is the sum of squares up to $(n - 1)^2$ and $\tau_n = n^2 - 1$.

Given $m$, we let $n = [\log_2 m]$, write $G^{2^n} = G^m \times G^{2^n - m}$, take the presentation of $G^{2^n}$ constructed above and kill the factor $G^{2^n - m}$ by adding relations to kill a generating set of cardinality $k[\log_2 (2^n - m)]$, which is at most $k(n - 1)$. Thus we obtain a presentation of $G^m$ with $kn = O(\log m)$ generators and $\sigma_n k^2 + \tau_n k + l + k(n - 1) = O(\log m)^3$ relators.

Proof of Theorem 1.1 All of the results that we need concerning the growth of $d(G^n)$ can be found in [17]; they draw on earlier results of Hall [8], Wiegold [13, 14, 15] and others. Thus we focus on the estimates for $\rho(G^n)$.

A simple induction using the Künneth formula shows that if $H_1G \neq 0$ then the number of generators needed for $H_2G^n$ is at least $n(n - 1)/2$, so one needs at least this number of relations to present $G^n$. The complementary upper bound is provided by the naive construction in the first paragraph of the Introduction. This proves (1).

If $G$ is perfect, then by the Künneth formula $H_2G^n$ is a direct sum of $n$ copies of $H_2G$, and therefore $d(H_2G^n)$ grows linearly if $H_2G \neq 0$. This provides the lower bound for (2). To establish a complementary upper bound, we consider the universal central extension $\tilde{G}$. Theorem 2.7 tells us that $\tilde{G}^n$ has a presentation with at most $O(\log_2 n)$ generators and $O(\log n)^3$ relations. The kernel of $\tilde{G} \to G$ is isomorphic to $H_2G$, so we need only add a further $n d(H_2G)$ relations to pass from $\tilde{G}^n$ to the quotient $G^n$. 


(3) is Theorem 2.7. The bounds on the number of relations in (4) follow from (3) and the simple observation that since \( H_1G^n = 0 \), the number of relators in any presentation is at least as great as the number of generators.

If \( G \) is perfect and \( g \in G \) has infinite order in every non-trivial quotient of \( G \), then \( G^n \) is generated by the diagonal copy of \( G \) together with \((g, g^2, \ldots, g^n)\), by Theorem 4.4 of [17]; hence \( d(G^n) \leq d(G) + 1 \), as asserted in (5). The required bound on \( \rho(G^n) \) is a special case of Corollary 2.5. \( \square \)

**Relation Gap Problem.**

**Remark 2.8.** If one expresses a finitely presented group \( G \) as a quotient of a free group \( G \cong F/R \), then the action of \( F \) by conjugation on \( M = R/[R, R] \) makes \( M \) a \( \mathbb{Z}F \)-module (and a \( \mathbb{Z}G \)-module). It is obvious that this module requires at most \( d_F(R) \) generators, where \( d_F(R) \) is the least number of elements (relations of \( G \)) that one needs to generate \( R \) as a normal subgroup of \( F \). Despite much effort, there is no example known where \( M \) is proved to require fewer than \( d_F(R) \) generators – the putative difference is the relation gap.

An elementary calculation shows that if \( N < G \) is normal and perfect, then the relation module for \( F \to G/N \) requires no more generators than \( M \) does, but one suspects that in some cases \( G/N \) is finitely presented and requires more relations than \( G \). For example, if one could prove that there is a finitely presented perfect group \( \Gamma \) such that \( \text{def}(\Gamma^n) > \text{def}(\Gamma^m) \) for some \( n < m \), then one could take a finite presentation realising the deficiency of \( \Gamma^m \) and add relations to kill a direct factor \( \Gamma^{m-n} \); the resulting presentation of \( \Gamma^n \) would have a relation gap of at least \( \text{def}(\Gamma^n) - \text{def}(\Gamma^m) \).

Similarly, if \( \rho(\Gamma^n) > \rho(\Gamma^m) \) for some \( n < m \), then by taking a presentation of \( \Gamma^m \) with \( \rho(\Gamma_m) \) relators and passing to \( \Gamma^n \) by killing a direct factor \( \Gamma^{m-n} \), we would obtain a presentation with a relation gap. More generally, it would suffice to prove that a specific map \( F \to \Gamma^n \) from a finitely generated free group factored as \( F \to \Gamma^m \to \Gamma^n \), where the second map is the quotient by a direct factor and the kernel of the first map requires fewer normal generators than the composite. The special role that powers of the form \( G^{2r} \) play in the proofs of this section is intriguing in this regard.

### 3. Examples

**3.1. Profinitely trivial examples.** In [3] Fritz Grunewald and I constructed a family of infinite super-perfect groups \( B_p \) that have no non-trivial finite quotients. The presentation given there is

\[
B_p = \langle a, b, \alpha, \beta \mid ba^p b^{-1} = a^{p+1}, \beta \alpha^p \beta^{-1} = a^{p+1}, [bab^{-1}, a] \beta^{-1}, [\beta \alpha \beta^{-1}, \alpha] b^{-1} \rangle.
\]
A 3-generator, 3-relator presentation of \( B_p \) can be obtained from this by a simple Tietze move removing the generator \( \beta \) and the third relation, replacing the occurrences of \( \beta \) in the second and fourth relations by the word \([bab^{-1}, a] \).

**Lemma 3.1.** Let \( Q \) be a quotient of \( H = \langle a, b \mid ba^pb^{-1} = a^{p+1} \rangle \). If the image of \( a \) in \( Q \) has finite order, then the image of \([bab^{-1}, a] \) is trivial.

**Proof.** If the image \( \bar{a} \) of \( a \) has finite order, then the images of \( a^p \) and \( a^{p+1} \) in \( Q \) must have the same order, since they are conjugate. But the order of \( \bar{a} \) is \( m/c \), where \( m \) is the order of \( a \) and \( c = (m, r) \) is the highest common factor. Since \( p \) and \( p+1 \) are coprime, it follows that \( \bar{a}^p \) generates \( A = \langle \bar{a} \rangle \) and the image of \( b \) conjugates \( \bar{a} \) to a power of \( \bar{a} \). In particular, the image of \([bab^{-1}, a] \) in \( Q \) is trivial. \( \square \)

We need the following strengthening of the fact that \( B_p \) has no non-trivial finite quotients.

**Proposition 3.2.** \( a \in B_p \) has infinite order in every non-trivial quotient of \( B_p \).

**Proof.** If the image of \( a \) has finite order in a quotient \( Q \), then the image of \([bab^{-1}, a] \) is trivial, by the lemma. The relations \( \beta = [bab^{-1}, a] \) and \( \beta a^p \beta^{-1} = a^{p+1} \) then force \( \beta \) and \( \alpha \) to have trivial image in \( Q \), whence \( b = [\beta a \beta^{-1}, \alpha] \) does too. So \( Q = 1 \). \( \square \)

**Theorem 3.3.** For all integers \( p, m \), the direct product of \( m \) copies of \( B_p \) has a presentation with at most 4 generators and \( 24 \lceil \log_2 m \rceil - 1 \) relations.

**Proof.** By Theorem 1.1(5) (which is from [17]) or Remark 3.4(1), we know that \( B_p^n \) requires at most 4 generators. To estimate the number of relations needed, first, as in Proposition 2.3, we present \( B_p \times B_p \) with 6 generators and 15 relations. Then, as in the proof of Lemma 2.4, we reduce this to a presentation with 4 generators and 19 relations. Continuing the argument of Lemma 2.4, we get a 4-generator presentation of \( B_p^4 \) with \( 16 + 4 + 19 + 4 = 43 \) relations, then a 4-generator presentation of \( B_p^8 \) with \( 16 + 4 + 43 + 4 = 67 \) relations, a 4-generator presentation of \( B_p^{16} \) with \( 16 + 4 + 67 + 4 = 91 \) relations, and a 4-generator presentation of \( B_p^{2^n} \) with \( 24n - 5 \) relations. As in Lemma 2.4, we conclude that \( B_p^n \) has a 4-generator presentation with at most \( 24 \lceil \log_2 m \rceil - 1 \) relations. \( \square \)

**Remarks 3.4.** (1) I do not know if the number of relations needed to present \( B_p^n \) is \( \Omega(\log m) \).

(2) In [1], Baumslag and Miller constructed a 4-generator finitely presented group \( G_p \) that admits a surjection \( G_p \to G_p \times G_p \). The group \( B_p \) is a quotient of \( G_p \), and therefore \( B_p^n \) is a quotient of \( G_p \) for all positive integers \( n \).

### 3.2. Infinite simple groups.

The Burger-Mozes groups are infinite simple groups that arise as the fundamental groups of compact non-positively squared 2-complexes [5]. Such a complex \( X \) is a classifying space for its fundamental group \( \Gamma = \pi_1 X \),
so $H_2\Gamma = H_2 X$. These complexes have many more 2-cells than 1-cells, so $H_2 X$ is a free-abelian group of non-zero rank. By combining parts (2) and (5) of Theorem 1.1, we see that $\Gamma^n$ has a finite presentation with at most $d(\Gamma) + 1$ generators but the number of relations needed to present $\Gamma^n$ grows linearly.

Rattaggi [12] refined the original construction of Burger and Mozes to produce examples with relatively small presentations. In particular he constructed an example with 3 generators and 62 relations.

Richard Thompson’s group $T$ provides a further example of a 3-generator infinite simple group [11] (see [6], for example). Ghys and Sergiescu [7] proved that $H_2(T, \mathbb{Z}) \neq 0$, so again $T^n$ needs at most 4 generators but the number of relations required to present $T^n$ grows linearly.

3.3. Finite groups. Super-perfect finite groups are covered by Theorem 1.1(4). It would be particularly interesting to improve the estimate $\rho(G^n) = O(\log n)^3$ in this case, where one has so much more structure.

To close, we follow our construction in the case of the binary icosahedral group $\bar{A}_5 \cong \text{SL}(2, 5)$. Since it is the universal central extension of $A_5$, we have $d(\bar{A}_5^n) = d(A_5^n)$. Famously, Philip Hall [8] calculated the range of $n$ in which $d(A_5^n) = 2, 3$. By following our argument in this case we get, for example, that $\bar{A}_5^{16}$ has a 2-generator presentation with 36 relators, while $\bar{A}_5^{1024}$ has a 3-generator presentation with 118 relators.

Acknowledgment. The exposition in this article benefited from the perceptive comments of a diligent referee.

References

[1] G. Baumslag and C.F. Miller III, Some odd finitely presented groups, Bull. London Math. Soc. 20 (1988), 239–244.
[2] M.R. Bridson, Decision problems and profinite completions of groups, J. Algebra 326 (2011), 59–73.
[3] M.R. Bridson and F. Grunewald, Grothendieck’s problems concerning profinite completions and representations of groups, Annals of Math. 160 (2004), 359–373.
[4] M.R. Bridson and C.F. Miller III, Structure and finiteness properties of subdirect products of groups, Proc. London Math. Soc. (3) 98 (2009), 631–651.
[5] M. Burger and S. Mozes, Lattices in product of trees, Inst. Hautes Etudes Sci. Publ. Math. No. 92 (2000), 151–194.
[6] J.W. Cannon, W.J. Floyd and W.R. Parry, Introductory notes on Richard Thompson’s groups, Enseign. Math. 42 (1996), 215–256.
[7] E. Ghys and V. Sergiescu, Sur un groupe remarquable de difféomorphismes du cercle, Comment. Math. Helv. 62 (1987), 185–239.
[8] P. Hall, The Eulerian functions of a group, Quart. J. Math 7 (1936), 134–151.
[9] J. Harlander, On the relation gap and relation lifting problem, in “Groups St Andrews 2013,” London Math. Soc. Lecture Note Ser. 422, pp. 278–285, Cambridge Univ. Press, 2015.
[10] J. Milnor, “Introduction to Algebraic K-Theory”, Ann. of Math. Stud., vol. 72, Princeton University Press, Princeton 1971.

[11] R. McKenzie and R.J. Thompson, An elementary construction of unsolvable word problems in group theory, in “Word Problems” (Conf., Univ. California, Irvine, 1969, Boone, W.W., Cannonito, F.B., Lyndon, R.C. (eds.), Studies in Logic and the Foundations of Mathematics, vol. 71, pp. 457–478. North-Holland, Amsterdam 1973.

[12] D. Rattaggi, A finitely presented torsion-free simple group, J. Group Theory 10 (2007), 363–371.

[13] J. Wiegold, Growth sequences of finite groups, Collection of articles dedicated to the memory of Hanna Neumann, VI. J. Austral. Math. Soc. 17 (1974), 133–141.

[14] J. Wiegold, Growth sequences of finite groups II, J. Austral. Math. Soc. 20 (1975), 225–229.

[15] J. Wiegold, Growth sequences of finite groups III, J. Austral. Math. Soc. 25 (1978), 142–144.

[16] J. Wiegold, Growth sequences of finite groups IV, J. Austral. Math. Soc. Ser. 29 (1980), 14–16.

[17] J. Wiegold and J.S. Wilson, Growth sequences of finitely generated groups, Arch. Math. (Basel) 30 (1978), 337–343.

Martin R. Bridson, Mathematical Institute, Andrew Wiles Building, Oxford OX2 6GG, European Union

E-mail address: bridson@maths.ox.ac.uk