This is an exposition of recent progress in the categorical approach to D-brane physics. I discuss the physical underpinnings of the appearance of homotopy categories and triangulated categories of D-branes from a string field theoretic perspective, and with a focus on applications to homological mirror symmetry.

1. Introduction

In his 1994 ICM lecture \[57\], M. Kontsevich suggested a vast categorical extension of mirror symmetry, known as ‘homological mirror symmetry’. This proposal put forward a very ambitious program aimed at deriving mirror symmetry from a deeper correspondence between certain categories associated with a mirror pair. In physical language, Kontsevich’s proposal amounts to formulating mirror symmetry for open topological strings in the presence of the most general D-branes, and then extract the usual closed string formulation as a consequence. While far from complete, the basic ideas of this program have gradually penetrated a few areas of inquiry, leading to valuable insights into D-brane dynamics. The purpose of the present paper is to explain some results which have emerged from such investigations.

Let us start with the broad expectations (which are in many ways rather conjectural). In a nutshell, Kontsevich’s original proposal is as follows. If \( X, Y \) is a mirror pair of Calabi-Yau manifolds, then one considers two homological objects associated with these spaces. The object associated with \( X \) is \( D^b(X) \), the bounded derived category of the Abelian category of coherent sheaves on \( X \). This is a triangulated category in the sense of Verdier. The object associated with \( Y \) is a triangulated category \( DFuk(Y) \), the derived category of the Fukaya category of \( Y \), whose objects are roughly a certain ‘quantum version’ of Lagrangian cycles in \( Y \), and whose morphism spaces are given by Floer homology. The rigorous construction of this category is now almost complete \[59, 33\]. Given these two objects, Kontsevich’s conjecture states:

**Conjecture 0** For \( X \) and \( Y \) a mirror Calabi-Yau pair, \( D^b(X) \) and \( DFuk(Y) \) are equivalent as triangulated categories.

The functor implementing this equivalence should be viewed as a ‘derived’
version of the mirror map. There exists a ‘homotopical’ extension of this conjecture. For this, one notices that the two categories $D^b(X)$ and $DFuk(Y)$ can be enhanced to certain minimal $A_\infty$ categories, i.e. their associative products $r_2 : \text{Hom}(A_2, A_3) \times \text{Hom}(A_1, A_2) \to \text{Hom}(A_1, A_3)$ can be extended by higher products $r_n : \text{Hom}(A_n, A_{n+1}) \times \ldots \times \text{Hom}(A_1, A_2) \to \text{Hom}(A_1, A_{n+1})$ in order to obtain minimal $A_\infty$ categories $D^b_\infty(X)$ and $D^\infty \text{Fuk}(Y)$. The enhancement $D^b_\infty(X)$ results by considering a certain version of Massey products which is always well-defined and is induced by the deformation theory of complexes of locally free sheaves. It can also be realized more directly by constructing $D^b(X)$ as the derived category of a dG category and considering the minimal model of the category of twisted complexes appearing in that construction. The $A_\infty$ enhancement $D^\infty \text{Fuk}(Y)$ results directly from its construction as the derived category of an $A_\infty$ category, by taking the minimal model of an $A_\infty$ category of twisted complexes which appears as an intermediate step in the construction. Then the ‘enhanced’ version of homological mirror symmetry states:

**Conjecture 1** The two $A_\infty$ categories $D^b_\infty(X)$ and $D^\infty \text{Fuk}(Y)$ are homotopy equivalent.

Note that Conjecture 1, if true, immediately implies Conjecture 0.

When $X$ and $Y$ are Calabi-Yau threefolds, one way to understand the origin of this proposal is as follows. Physically, one is interested in formulating mirror symmetry for open strings. Following the approach pioneered by E. Witten, it suffices to formulate such a correspondence at topological open string tree level, i.e. in terms of open topological string amplitudes on a disk. The relevant topological string theories are the open A model on $X$ and the open B model on $Y$. One wants to consider open strings with ‘arbitrary’ boundary conditions, i.e. in the presence of the most general topological branes. This can be achieved by formulating the problem in the language of string field theory, starting with a distinguished set of boundary data (namely standard D-branes of the B-model associated with holomorphic vector bundles on $X$, respectively D-branes of the A-model described by Lagrangians in $Y$ carrying Chan-Paton bundles). Then the ‘most general boundary data’ can be recovered indirectly by requiring closure of the resulting string field theory under processes of D-brane composite formation. Modulo certain (weak) assumptions, this recovers $D^b(X)$ and $DFuk(Y)$ as the required collections of ‘generalized topological D-branes’, and leads to extended string field theories defined in terms of these categories. Thus the weak form of the conjecture (Conjecture zero) can be reformulated physically as follows:

**Conjecture 0phys** The collection of topological D-branes of the B model with target space $X$ is equivalent to that of the A-model with target space $Y$. Moreover, this correspondence preserves the triangulated structure.

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*a*Note that in this article we follow the convention that morphisms compose ‘backwards’.

*b*The case of Calabi-Yau manifolds of arbitrary dimension is less clear, since the formulation of topological string field theory used in this article is appropriate only for threefolds.
The $A_\infty$ enhancements $D^b_{\infty}(X)$ and $D_{\infty}Fuk(X)$ encode the (gauge-fixed) open string amplitudes on the disk, according to the prescription:

$$\langle \langle u_n, \ldots, u_0 \rangle \rangle_{\text{tree}} = \langle u_n, r_n(u_{n-1}, \ldots, u_0) \rangle,$$

where the left hand side is the disk amplitude for scattering of the states $u_0 \in \text{Hom}(A_0, A_1) \ldots u_{n-1} \in \text{Hom}(A_{n-1}, A_n)$ and $u_n \in \text{Hom}(A_n, A_0)$ and $\langle \ldots \rangle$ in the right hand side is a bilinear pairing which plays the role of BPZ form for the topological open string field theory under consideration.

The statement of Conjecture 1 can then be reformulated as follows:

**Conjecture 1**

For a mirror pair of Calabi-Yau threefolds $(X, Y)$, the ‘complete’ open string field theories which incorporate the most general D-branes of the topological $A$ model on $X$ and the topological $B$-model on $Y$ should be equivalent as BV systems.

By a categorical extension of the results of [35], this conjecture implies homotopy equivalence of the associated $A_\infty$ categories, as well as compatibility of this equivalence with the bilinear forms.

It is crucial for physical applications to consider the supplementary information contained in the superconformal models of which the $A$ and $B$ models are twisted versions. This line of investigation has been pioneered recently by M. Douglas and P. Aspinwall. In [5], they have proposed a way of selecting a subclass of B-type branes (an Abelian subcategory of $D^b(X)$) which should be BPS saturated (and thus stable) when embedded in the full superstring compactification on $X^c$. This fascinating proposal has many interesting aspects, for example the relevant set of stable objects must change with the stringy Kahler moduli of $X$, which gives an extremely general realization of the phenomenon of marginal stability known from supersymmetric field theories. It is currently not entirely certain (from a physical point of view) what is the proper ‘mirror’ of this stability condition, which is expected to select a triangulated subcategory of the derived category of the Fukaya category of $Y$ (though a natural conjecture follows from the work of [14] and was considered recently in [97]).

Having recalled the broad conjectures, we now turn to discussing what is known. This paper is written from a physical perspective, but must assume familiarity with certain mathematical notions which have been rarely used in the physics literature. The string field theoretic point of view used below is not the only possible approach to this subject. However, it does provide a unifying perspective on the origins of both of the categories $D^b_{\infty}(X)$ and $D_{\infty}Fuk(Y)$, and gives a natural physical context for the somewhat formidable theory of $A_\infty$ categories. For different (though related) perspectives for the case of the $B$ model, I refer the reader to the work of M. Douglas, P. Aspinwall and collaborators [14] [2] [3] [6] [22] [20].

* A mathematical formulation of this stability condition was recently given in [14].
2. Generalized D-branes and open string field theory

One of the deepest recent insights into D-brane physics is the reconsideration of the very concept of D-brane. In the traditional approach, one constructs D-branes by imposing boundary conditions on a string’s endpoints, and interprets the resulting string dynamics as a quantization of the underlying space-time object carrying the boundary data. A crucial observation is that such a description does not generally suffice, since it cannot always account for all products of open string dynamics. For example, the endpoint of tachyon condensation in a system of D-branes evolving in a nontrivial background cannot always be described directly in this language. Thus a complete description of open string dynamics in the presence of D-branes requires a more general formulation. As originally suggested by A. Sen, this can be sought in the framework of open string field theory. In this approach, one can give a conceptually clear description of D-brane composite formation by using the language of category theory. More precisely, one finds that open (topological or bosonic\(^d\)) string field theory in the presence of D-branes (and formulated with Witten’s choice of vertex) can be described in terms of a differential graded (dG) category endowed with certain supplementary data, and that ‘closure’ of this description under processes of D-brane composite formation in the topological case requires that the underlying dG category obey a certain ‘quasunitarity’ constraint. When applied to the string field theory of open topological strings (the open A and B model), this quasunitarity condition requires that one extend any initial collection of D-branes by adding the totality of their composites. In this section, I give a brief exposition of this analysis, which was carried out in \([63, 64]\). As we shall see below, the string field theory approach recovers a construction originally proposed by Kontsevich in his 1994 ICM lecture (and previously investigated in the mathematical work of \([9]\)). By generalizing to an arbitrary choice of string field vertices along the lines of \([65]\), this argument has an \(A_\infty\) version which leads to a more general construction also sketched in Kontsevich’s original talk and recently developed in \([71, 72]\).

Let us note that the open string field theory approach to the classification of D-branes can be viewed as an ‘off-shell’ extension (in the sense of removing BRST closure constraints) of the on-shell discussion given in \([72, 73]\) and \([62]\) (though currently this extension is only understood at open string tree level). As usual in physics, off-shell formulations allow one to extract more information.

2.1. Open string field theory with D-branes

Categorical constructs enter D-brane physics through the very basic structure involved in the description of open string dynamics. One way to understand their appearance is to extend the string-field theoretic framework of \([109]\) in order to in-

\(^d\)In the bosonic case, there are functional-analytic aspects which are not fully understood. In this article, we shall only need the topological case.
corporate D-branes. For this, one treats D-branes as abstract objects, whose worldsheet description is encoded in the algebraic data of string products. As explained in [63], one can abstract the following information, provided that one uses Witten’s choice of vertex in order to construct the string field action:

1. A collection of objects $a$ (the ‘D-branes’) together with complex vector spaces $\text{Hom}(a, b)$ for any two objects $a$ and $b$. These describe the state space of oriented strings stretching from $a$ to $b$, and carry an integer-valued grading given by the ghost number of such states. The degree of an element $u \in \text{Hom}(a, b)$ will be denoted by $|u|$. 

2. A collection of non-degenerate bilinear pairings $\langle \cdot, \cdot \rangle : \text{Hom}(a, b) \times \text{Hom}(b, a) \rightarrow \mathbb{C}$ which describe the BPZ forms.

3. A collection of trilinear maps $\langle \langle \cdot, \cdot, \cdot \rangle : \text{Hom}(c, a) \times \text{Hom}(b, c) \times \text{Hom}(a, b) \rightarrow \mathbb{C}$, which describe the three-point functions on a disk with ‘boundary conditions’ $a, b, c$. Upon dualizing with respect to the BPZ forms, the same information be encoded in bilinear compositions $\text{Hom}(b, c) \times \text{Hom}(a, b) \rightarrow \text{Hom}(a, c)$ defined through:

\[
\langle \langle u, v, w \rangle \rangle = ca \langle u, vw \rangle ac.
\]

4. A collection of linear operators $d : \text{Hom}(a, b) \rightarrow \text{Hom}(a, b)$ of degree $+1$, which square to zero. These are the worldsheet BRST operators acting on the spaces of open string states.

This data is subject to certain compatibility constraints, which can be expressed succinctly as follows:

(A) The objects $a$ form an (associative) category with morphism spaces $\text{Hom}(a, b)$ and the composition defined as in point (3) above. When the morphism spaces are endowed with the grading and with the BRST operators $d$, this becomes a differential graded (dG) category. This means that $d$ squares to zero, acts as a derivation of the composition operation, and that the latter has degree zero.

(B) The bilinear forms $\langle \cdot, \cdot \rangle$ are compatible with the derivations $d$ and with the morphism compositions in the following sense:

\[
\langle du, v \rangle + (-1)^{|u|} \langle u, dv \rangle = 0
\]

\[
\langle u, vw \rangle = \langle uv, w \rangle .
\]

In the situation of interest below (namely a topological string field theory defined on a Calabi-Yau threefold), the bilinear form must also satisfy the selection rule:

\[
\langle u, v \rangle = 0 \quad \text{unless} \quad |u| + |v| = 3.
\]

In the case of bosonic string field theory, one also has certain conjugation operations obeying natural compatibility conditions. We shall limit ourselves to topological string field theories for what follows. Succinctly:

An open topological string field theory with D-branes is described by a dG category $A$ together with a collection of invariant nondegenerate bilinear pairings (of total degree $−3$) on its morphism spaces.
(For what follows, we shall also assume that $A$ is endowed with direct sums. This can be easily achieved by adding such sums if necessary).

Given this data, one can write a string field action as follows. The string field is a degree one element $\phi$ of the total boundary space $\mathcal{H} = \oplus_{a,b} \text{Hom}(a,b)$. The string field action has the form:

$$S(\phi) = \frac{1}{2} \langle \phi, d\phi \rangle + \frac{1}{3} \langle \phi, \phi \phi \rangle$$

where the total BPZ form $\langle \cdot, \cdot \rangle$ is obtained from $ab\langle \cdot, \cdot \rangle ba$ by summing over the components $\text{Hom}(a,b)$. Here $d$ is the differential induced on $\mathcal{H}$ and we let $\bullet$ (usually denoted by juxtaposition) be the composition induced on this space.

2.2. Vacua and D-brane composites

String field vacua are obtained by extremizing $S(\phi)$, which gives the equation:

$$d\phi + \frac{1}{2} [\phi, \phi] = 0 \ ,$$

where $[\cdot, \cdot]$ is the graded commutator built from the associative composition $\bullet$ on $\mathcal{H}$. This can be recognized as the Maurer-Cartan equation governing deformations of the differential graded (dG) Lie algebra $(\mathcal{H}, d, [\cdot, \cdot])$ (=the commutator algebra of the differential graded associative algebra $(\mathcal{H}, d, \bullet)$). In terms of the total boundary data $d$, $\mathcal{H}$ and $\bullet$, the vacua are described in the manner familiar from open string field theory without D-branes. The new element is the category structure provided by the D-brane labels $a$, which is compatible with the total boundary data in sense that $d$ preserves the spaces $\text{Hom}(a,b)$, the boundary product $\bullet$ maps $\text{Hom}(b,c) \times \text{Hom}(a,b)$ into $\text{Hom}(a,c)$ and the bilinear form $\langle \cdot, \cdot \rangle$ vanishes on $\text{Hom}(a,b) \times \text{Hom}(c,d)$ unless $b = c$ and $a = d$.

Equations (6) have the trivial solution $\phi = 0$ (this reflects the fact that we consider a background-dependent formulation of open string field theory, expanded around a given vacuum). A nontrivial solution $\phi = \alpha$ of (5) defines a new vacuum, which can be viewed as a translation of this distinguished background. The total BRST operator for the expansion around the new background has the form:

$$d_{\alpha} = d + [\alpha, \cdot]$$

Equation (6) can be viewed as the tadpole cancellation condition for the translated background and is equivalent with the requirement $d_{\alpha}^2 = 0$. The crucial observation is that $d_{\alpha}$ need not be compatible with the original category structure. This means that shifting the string vacuum changes the D-brane content of the theory, an observation which allows one to give an abstract description of D-brane dynamics. Namely, the D-brane content described by a vacuum $\alpha$ is encoded in a category structure $A_{\alpha}$ which is compatible with the new BRST operator $d_{\alpha}$. This category describes the D-brane content obtained by expanding the theory around the vacuum $\alpha$, and was constructed explicitly in [63], where it was called the collapsed category. In that reference, it was also showed that $A_{\alpha}$ obeys all axioms of open
D-brane categories

String field theory with D-branes, with respect to bilinear forms and compositions constructed naturally by shifting the string vacuum.

Shifting the vacuum by a solution $\phi = \alpha = \bigoplus_{a,b} \alpha_{ab}$ of (6) amounts to condensing spacetime fields associated with the components $\alpha_{ab} \in \text{Hom}(a,b)$. This leads to the formation of D-brane composites out of the branes described by the objects $a$ for which there exists an object $b$ with $\alpha_{ab} \neq 0$ or $\alpha_{ba} \neq 0$ (the branes lying in the support $S$ of $\alpha$). In the simplest case when $S$ is ‘connected’ (in graph-theoretic sense explained in [63]), the collapsed category $\mathcal{A}_\alpha$ is obtained upon replacing these D-branes (the objects lying in the support of $\alpha$) with a new object $\ast$, identified with the resulting D-brane composite. The new morphism spaces, compositions and bilinear forms are constructed accordingly. The result $\mathcal{A}_\alpha$ of this construction is a dG category carrying bilinear forms subject to the axioms Section 2.1. Hence (topological) tree-level open string dynamics in the presence of D-branes can be described by passage to the collapsed string field theory associated to whatever open string background one cares to condense.

2.3. Closure under composite formation and the quasiunitary cover

A complete description of D-brane dynamics must be closed under formation of D-brane composites. This means that the theory $\mathcal{A}_\alpha$ obtained after shifting the string vacuum should be a ‘sub-theory’ of the original string field theory, in the sense that all possible D-brane composites should already be considered as objects in the original theory. This requirement was formalized in [63], as the condition that the collapsed category $\mathcal{A}_\alpha$ is dG-equivalent with a subcategory of the original dG category $\mathcal{A}$ (the bilinear forms can also be matched). In [63], an open string field theory with D-branes which satisfies this completeness requirement was called ‘quasi-unitary’. As explained in [63], any open string field theory with D-branes $\mathcal{A}$ can be extended to a quasiunitary theory which ‘contains’ it in an appropriate sense. In fact, there exists a minimal extension of this type, the so-called ‘quasi-unitary cover’ $c(\mathcal{A})$ of $\mathcal{A}$, which is constructed as follows:

(1) The objects of $c(\mathcal{A})$ are generalized complexes of degree one over $\mathcal{A}$. A generalized complex of degree one is a finite sequence $(a_j)$ of objects of $\mathcal{A}$, together with degree one morphisms $q_{ij} \in \text{Hom}^1_{\mathcal{A}}(a_i,b_j)$ such that $q := \bigoplus_{i,j} q_{ij}$ satisfies the Maurer-Cartan equation (6):

$$dq_{ij} + \sum_k q_{kj}q_{ik} = 0 \quad .$$

(8)

(2) Given two generalized complexes $A := (a_k,q_{ij})$ and $B := (b_k,q'_{ij})$, the space $\text{Hom}_{c(\mathcal{A})}(A,B)$ is given by the direct sum $\bigoplus_{i,j} \text{Hom}_{\mathcal{A}}(a_i,b_j)$, with the induced grading. The differential on this space is defined by:

$$du = \bigoplus_{i,j} \left[ du_{ij} + \sum_k q'_{kj}u_{ik} - \sum_l (-1)^{|u_{ij}|}u_{ij}q_{il} \right] \quad .$$

(9)
for $u = \oplus_{i,j} u_{ij} \in \text{Hom}(A, B)$, with $u_{ij} \in \text{Hom}(a_i, b_j)$.

(3) Given a third degree one generalized complex $C = (c_k, q''_{ij})$, the composition of morphisms $\text{Hom}(B, C) \times \text{Hom}(A, B) \to \text{Hom}(A, C)$ is given by:

$$uw = \oplus_{i,k} \left[ \sum_j u_{jk}v_{ij} \right],$$

(10)

where $u = \oplus_{j,k} u_{jk} \in \text{Hom}(B, C), v = \oplus_{i,j} v_{ij} \in \text{Hom}(A, B)$, with $u_{jk} \in \text{Hom}(b_j, c_k)$ and $v_{ij} \in \text{Hom}(a_i, b_j)$.

(4) Finally, one has natural bilinear forms on the morphism spaces of $c(A)$ and it is easy to check that the axioms of Section 2.1 are satisfied (see [63]).

This physically-motivated construction is entirely general and relies only on the most basic data which specify an open (topological) string field theory with D-branes. As we shall see below, it is realized explicitly for the open B-model, as well as in a sector of the open A-model. Moreover, a homotopy version of it is realized through Fukaya’s approach to topological A-type branes, leading to Kontsevich’s proposal [57] for the construction of the derived category of Fukaya’s category.

### 2.4. Topological A/B strings and graded topological D-branes

If one considers the open A or B model on a Calabi-Yau space $M$, then one must take into account some supplementary data which enter a complete description of boundary sectors. This is easiest to see in the context of the A-model, for which a basic D-brane is described by a Lagrangian cycle of $M$ (which carries Chan-Paton data and has vanishing Maslow index etc). As already pointed out by Kontsevich in his 1994 lecture [57] and elaborated by Seidel [88], a well-defined $\mathbb{Z}$-grading on the state spaces of open strings stretching between two Lagrangians requires that one specifies their ‘relative’ grading (this is necessary even in the case of special Lagrangians, which describe D-branes of the untwisted model). A similar grading can be introduced for the basic topological D-branes of the B-model (which are described by holomorphic vector bundles). This grading is specified by a discrete choice and arises naturally in at the topological level [57, 88] [20, 61, 23, 65]. Its effect is to shift the worldsheet $U(1)$ charge of a string stretching from a brane $a$ to a brane $b$ according to the formula:

$$|\phi_{ba}| \to |\phi_{ba}| + \text{grade}(b) - \text{grade}(a).$$

(11)

More recently, M. Douglas introduced a different type of grade (which is real-valued) [20]. This is relevant for D-branes of the full superstring compactification, to which the topological models are related by twisting. The grade introduced by M. Douglas is related to the space-time central charge of the D-brane, when the latter is embedded into the full superstring model. This real grading should be treated as supplementary data which allows one to make contact with superstring physics, by picking out those topological D-branes which are conjectured to correspond to BPS
saturated branes of the associated superstring compactification (as we discuss later). For the moment, we focus on the topological aspects of Kontsevich’s conjecture.

The effect of introducing the integer grading at the topological string field level was first discussed in [64]. As explained in [64], the combination of the string field theory analysis of the previous section with the supplementary data described by the integer grading makes immediate contact with the older work of Bondal and Kapranov [9] while providing a clear physical justification for a construction already proposed in Kontsevich’s original lecture [57]. The argument is based on the string field theoretic analysis of D-brane composite formation which was sketched above. This argument works for any graded string field theory, and in particular can be applied for both the $A$ and $B$ models. For the $A$-model, a formulation in terms of $A_\infty$ categories is more practical in view of the work of [29], as we will discuss later.

2.5. Graded open string field theory

We say that an open string field theory with D-branes is (integer) graded if the underlying dG category $\mathcal{A}$ is endowed with shift functors $[n]$ ($n \in \mathbb{Z}$) compatible with the bilinear forms. Given a theory as in Subsection 2.1, one can consider the minimal graded theory containing it, the so-called shift-completion of the original theory. This is obtained by adding objects $a[n]$ for each integer $n$ and each object $a$ of the original theory, and adding morphism spaces, differentials, compositions and bilinear forms in the obvious manner dictated by shift-invariance. In particular, the original objects $a$ can be identified with the new objects $a[0]$, and the original dG category $\mathcal{A}$ becomes a full subcategory of its shift-completion $\tilde{\mathcal{A}}$. It is easy to check that $\tilde{\mathcal{A}}$ satisfies the axioms of Section 2.1 when endowed with the bilinear forms constructed in the obvious manner.

In the shift-completed theory $\tilde{\mathcal{A}}$, one has the basic relation:

$$\text{Hom}(a[m], b[n]) = \text{Hom}(a, b)[m-n] ,$$

where $\text{Hom}(a, b)[p]^h := \text{Hom}^{h+p}(a, b)$. Combining with the construction discussed above, one finds that the quasiunitary cover $c(\tilde{\mathcal{A}})$ of the shift-completed theory is described through data familiar from the work of [9]. Namely, applying the description of generalized complexes given in the previous subsection shows that a degree one generalized complex over $\tilde{\mathcal{A}}$ is given by a finite sequence of objects $a_i[n_i]$ of $\tilde{\mathcal{A}}$ (with $a_i$ objects of $\mathcal{A}$), together with morphisms $q_{ij} \in \text{Hom}_\tilde{\mathcal{A}}^1(a_i[n_i], a_j[n_j]) = \text{Hom}^{1+n_i-n_j}(a_i, a_j)$, subject to the Maurer-Cartan equation (8) with respect to the composition on $\tilde{\mathcal{A}}$.

Such objects were originally considered in the mathematics literature [9], where they were called twisted complexes over $\mathcal{A}$. In the language of [9], the twisted complexes predicted by string field theory are two-sided, i.e. the morphisms $q_{ij}$ are not

\footnotetext{A discussion based on that of [63] and [64] was later given in [23] for the $B$ model. See also [65] for the effect of this on a certain sector of the $A$-model.}
required to vanish for $i \geq j$ (this was pointed out in [64]). If one does impose this condition (which is useful for technical reasons), then one obtains the concept of one-sided twisted complexes. The dG category $c(\mathcal{A})$ of two-sided twisted complexes over $\mathcal{A}$ will be denoted by $tw(\mathcal{A})$. As originally showed in [9], one-sided twisted complexes also form a dG category, which we shall denote by $tw^+(\mathcal{A})$. In conclusion, the quasunitary cover of the shift-completed theory $c(\mathcal{A})$ coincides with the category $tw(\mathcal{A})$ of two-sided twisted complexes:

$$c(\mathcal{A}) = tw(\mathcal{A})$$

(13)

This relation was pointed out in [64] upon building on the work of [63] (see also [23] and [65]).

As discussed in [9], the associative category $H^0(tw^+(\mathcal{A}))$ obtained by taking the cohomology of $tw^+(\mathcal{A})$ in degree zero is triangulated. This category will be called the derived category of the dG category $\mathcal{A}$ and denoted by $D(\mathcal{A})$. Passage to this category allows one to identify topological D-branes up to quasi-isomorphisms, a process which is permitted by the Batalin-Vilkovisky formalism. Since $tw^+(\mathcal{A})$ is a subcategory of $tw(\mathcal{A})$, it immediately follows that $D(\mathcal{A}) = H^0(tw^+(\mathcal{A}))$ is a subcategory of the associative category $H^0(tw(\mathcal{A}))$ predicted by string field theory. In general, there is no clear reason to believe that $D(\mathcal{A})$ and $H^0(tw(\mathcal{A}))$ are equivalent, though one can come close to finding an equivalence between them.

For various results about dG categories, their enhanced triangulated categories and the associated theory of functors I refer the reader to [9, 53, 24]. The case of two-sided twisted complexes is largely understudied.

3. The $A_\infty$ structure induced by gauge-fixing and the deformation potential

Returning to the general framework of Section 2.1, let us consider the string field action (5). This has the infinitesimal gauge-invariance:

$$\phi \rightarrow \phi + d\beta + [\phi, \beta],$$

(14)

for generators $\beta \in \mathcal{H}^0$ (the finite gauge transformations are easily obtained by exponentiating (14)). We consider the problem of building a tree-level effective potential for fluctuations of $\phi$ in the vicinity of a solution of the equations of motion (6). This was discussed in [66], with the following result. Fluctuations which are not pure gauge can be separated explicitly provided that the theory under consideration is endowed with supplementary data (technically, what is needed is a so-called cohomological splitting of the underlying dG category). The most direct physical interpretation arises in a theory satisfying (4), provided that the cohomological splitting is induced by an antilinear involution $c : \mathcal{H} \rightarrow \mathcal{H}$, which is compatible

\[\text{[Provided that one uses sequences in the construction of twisted complexes, rather than sets of objects, as discussed in [53, 24].}]


with the category structure and satisfies $|cu| = 3 - |u|$. Under this assumption, one can define a nondegenerate Hermitian metric on $\mathcal{H}$ through $h(u, v) := \langle cu, v \rangle$ and partially fix the gauge invariance \(^{(14)}\) by imposing the condition $d^\dagger \phi = 0$, where $d^\dagger$ is the Hermitian conjugate of $d$ with respect to $h$. The degree one component of the space $K = \ker d \cap \ker d^\dagger$ describes massless fluctuations around the vacuum and can be identified with the degree one cohomology $H^1_0(\mathcal{H})$. One has a natural propagator $U = \frac{1}{d} \pi_d$, where $\pi_d$ is the orthoprojector on $\text{im}d$ (with respect to the metric $h$). When expanding around the solution $\phi = 0$ of \(^{(6)}\), the massive modes can be integrated out to produce an effective potential $W$ for the massless modes $u \in K^1$, whose tree level piece $W_{\text{tree}}$ can be easily described by diagrams involving the propagator $U$. The result is:

$$W_{\text{tree}}(u) = \sum_{n \geq 3} \frac{1}{n + 1} (-1)^{n(n-1)/2} \langle \langle u_0, \ldots, u_n \rangle \rangle_{\text{tree}}^{(n+1)} ,$$

where:

$$\langle \langle u_0 \ldots u_n \rangle \rangle_{\text{tree}}^{(n+1)} = \langle u_0, r_n(u_1 \ldots u_n) \rangle$$

for $u_j \in K$. Here $r_n : K^{\otimes n} \to K$ are multilinear maps which can be described in terms of the propagator and the associative composition on $\mathcal{H}$. As discussed in \cite{66}, the tree level amplitudes satisfy certain cyclicity constraints, while the maps $(r_n)_{n \geq 2}$ form a minimal $A_\infty$ algebra. In a theory with multiple D-branes, the operator $c$ must be compatible with the category structure, in which case the products $r_n$ define a minimal $A_\infty$ category with the same objects and morphism spaces as the homology category $H^*(\mathcal{A})$. Moreover, this minimal $A_\infty$ category is quasi-isomorphic with the original dG category $\mathcal{A}$ (by results of \cite{33}), such a quasi-isomorphism is automatically a homotopy equivalence. As discussed in \cite{66}, extremizing $W_{\text{tree}}$ over $K^1$ and modding out residual complex symmetries is equivalent with solving the Maurer-Cartan equations \cite{66} and dividing through the full gauge symmetry of the original theory. This gives an alternative description of the local moduli space, and shows that $W_{\text{tree}}$ encodes the obstructions to infinitesimal deformations of the vacuum sitting at $\phi = 0^\circ$. The $A_\infty$ algebra described by $(K, \{r_n\})$ is the so-called minimal model \cite{18} of the dG algebra $(\mathcal{H}, d, \bullet)$ (a similar terminology applies at the category level). The physical construction of the minimal model which follows as in \cite{66} by integrating out the massive modes in perturbation theory coincides with a mathematical construction discussed for example in \cite{59}. This gives a physics-inspired proof of the fact that a dG algebra (and, more generally, an $A_\infty$ algebra \cite{59}) always admits a minimal model, i.e. a homotopically-equivalent $A_\infty$ algebra whose first product

\footnote{The fact that an effective potential of this type should govern vacuum deformations of pure Chern-Simons theory in three dimensions was suggested a while ago by E. Witten \cite{107}. The potential \cite{18} implements this idea in the more general case of open string field theory, whose action is formally of Chern-Simons type. More recent work on such deformation potentials can be found in \cite{49}.}
$r_1$ vanishes. A similar result holds for $A_\infty$ categories. Hence the minimal model theorem for $A_\infty$ algebras and categories is a mathematical reflection of the existence of an effective potential for massless modes. As explained in [66], this effective potential is holomorphic (when convergent) with respect to coordinates defined by a linear basis of $K^1$, and, in the general case of multiple boundary sectors, it can be viewed as a potential for the low energy dynamics of arbitrary systems of topological D-branes. In the context of the string field theory obtained by twisting a superstring compactification on a Calabi-Yau threefold, it can be viewed as a generalization of the ‘D-brane superpotential’ of [47]. The potential $W_{\text{tree}}$ is defined on the ‘virtual tangent space’ at a point $\phi$ of the moduli space (in the discussion above, we chose $\phi = 0$). The former is the space of linearized deformations described by $H^1_{\text{loc}}(\mathcal{H}) \approx K^1$, which is typically of higher dimension than the dimension of the (highest local component of the) moduli space. We note that generalized D-branes are typically heavily obstructed, hence the deformation potential rarely vanishes for a nontrivial object of a topological D-brane category.

The construction performed in [66] depends on the existence of a conjugation operator $c$ with certain properties (a formulation which was chosen there due to its physical character). However, it immediately follows from the results of [66] that the main property of this superpotential (namely that it encodes obstructions to the deformations of a given vacuum) remains valid under more general assumptions. In fact, one can replace the string scattering products $r_n$ by the products of any homotopy-equivalent model of the original dG category $\mathcal{A}$ in order to describe such deformations. This always leads to the categorical homotopy version of the Maurer-Cartan equations which is discussed, for example, in [51]. In particular, the minimal model theorem can be applied for any cohomological splitting, and thus such a ‘low energy’ description of deformations can be extracted via more abstract means.

It is clear from the discussion above that the $A_\infty$ structure is crucial both physically and mathematically, even if one formulates the underlying open string field theory by using Witten’s vertex. Physically, this structure encodes the data of the D-brane superpotential of [47] at an extremely general level. Thus the enhanced version of homological mirror symmetry (Conjecture 1 in the introduction) expresses matching of tree-level open string scattering amplitudes between mirror systems of D-branes or, equivalently, matching of the associated D-brane superpotentials.

4. Topological B-type branes and the derived category

The string field theoretic framework of the previous section can be realized very explicitly for the open sector of the topological B model originally introduced by E. Witten [102, 103, 107]. Given a Calabi-Yau threefold $X$, the simplest topological D-branes of the associated B-model are described by holomorphic vector bundles over $X$. For any two such bundles $E_1, E_2$, localization shows that the (off-shell) state space of topological open strings stretching from $E_1$ to $E_2$ can be identified with $\Omega^0,*(X, E_1^* \otimes E_2)$, while the boundary BRST operator $d$ becomes the Dolbeault
D-brane categories

The differential $\delta$ coupled to the bundle $E_1^* \otimes E_2$. Thus our original class of objects is $\text{Vect}(X)$, the collection of all holomorphic vector bundles on $X$, while the morphism spaces are given by $\text{Hom}(E_1, E_2) := \Omega^0(\text{ker}(\nabla))$ with the obvious grading. An immediate generalization of [107] shows that the string product $\text{Hom}(E_2, E_3) \times \text{Hom}(E_1, E_2) \rightarrow \text{Hom}(E_1, E_3)$ is given by the wedge product of bundle-valued forms (which includes composition in the bundle directions). Finally, the BPZ form on $\text{Hom}(E_2, E_1) \times \text{Hom}(E_1, E_2)$ is given by:

$$\langle u, v \rangle = \int_X \Omega \wedge \text{tr}(u \wedge v).$$

(17)

This data gives a dG category $\mathcal{A} = \text{Vect}_{dg}$, endowed with the nondegenerate bilinear forms (17) which are easily seen to obey all axioms of Section 2.

As explained in [20], this picture must be completed by introducing an integer-valued grade $h$ for each brane $E \in \text{Vect}(X)$, which can be described abstractly by adding all of the formal translates $E[n]$. Following the general discussion of the previous section, this leads to the shift-completed string field theory $\tilde{\mathcal{A}}$, which was originally considered in [64]. As discussed in [23], this theory admits an interesting sector which is obtained upon restricting to an object $E$ together with its degree translates $E[n]$. In this case, the string field action can be written in terms of a graded superconnection $B$ of total degree $(0, 1)$ on the graded bundle $E = \oplus_n E[n]$:  

$$S := \int_X \Omega \wedge \text{str} \left[ \frac{1}{2} B DB + \frac{1}{3} B^3 \right].$$

(18)

This can be viewed as a background independent formulation of our string field theory, when restricted to such a sector. The original formulation used above arises upon choosing a background superconnection $B_0$ and expanding $B = B_0 + \phi$, where $\phi$ plays the role of the string field of Section 2. Expanding $\phi$ into its components along $\text{Hom}(E[n], E[n])$ recovers the category-theoretic description, as it applies to this particular sector. In particular, the equations of motion of (18) become the Maurer-Cartan equations for $\phi$, which recover twisted complexes upon expanding $\phi$ in its components.

Returning to the general formulation of the previous section, one can now pass to the category $c(\tilde{\mathcal{A}}) = H^0(\text{tw}(\mathcal{A}))$. If one restricts to one-sided twisted complexes for which $q_{ij}$ vanishes unless $n_i < n_j$ (a technical step whose physical justification in topological string field theory is unclear), then one is left with usual complexes of vector bundles since these are the only one-sided twisted complexes which satisfy that condition. Because of this, one finds that $D^0(X)$ can be identified with a subcategory of $H^0(\text{tw}(\mathcal{A}))$, a relation which follows easily from the results of [9] (see [64, 23, 1]). Via this identification, the $A_\infty$ structure which governs the deformation potential of Section 8 extends to give the $A_\infty$ enhancement $D^0_{\infty}(X)$ of $D^0(X)$ entering Conjecture 1 of the introduction [83]. An immediate consequence of the

bRecall that there also exists a different type of grade, which is real-valued, and which is relevant for the untwisted model. This will be discussed later.
description of topological B-branes as objects of the derived category is a general realization of open string state spaces as Ext groups. Some aspects of how this description arises from the nonlinear sigma model were recently studied in [95].

5. Topological A branes and the Fukaya category

This section gives a brief discussion of Fukaya’s category, including an explanation of its physical origins. I also discuss work on a certain sector of this category, which leads to a graded version of Chern-Simons field theory, and describe some of the results obtained in that direction.

Since a systematic presentation of the mathematical theory of $A_\infty$ categories is outside the scope of this article, I refer the reader to the paper [33] and the thesis [71] (see [54] for an introduction), where the basic theory of such categories is developed in some detail. The role played by $A_\infty$ algebras in open string field theory (with one boundary sector) was discussed in [35]. It parallels the well-known fact [101, 98] that $L_\infty$ algebras appear naturally in closed (bosonic or topological) string field theory, as the algebraic constraints satisfied by tree-level products as a consequence of Ward identities. For open strings, $A_\infty$ categories (as opposed to $A_\infty$ algebras) arise in the framework of [35] simply by introducing a collection of D-branes. As well-known from the work of B. Zwiebach (see, for example, [108]), the construction of a (bosonic or topological) string field theory requires a choice of vertices. For (bosonic or topological) open string field theory, the most widely-used choice is that employed by E. Witten in [106], which requires only one (triple) vertex and thus leads to a single string product $r_2$. Algebraically, this choice leads to a differential graded category, as discussed in Section 2 (where $r_2$ was identified – up to sign factors – with the composition $\bullet$). On the other hand, a general choice of vertices leads to an infinity of string products which are subject to $A_\infty$ constraints, thus requiring the algebraic framework of $A_\infty$ algebras [55], which is appropriate for a single boundary sector. In the presence of a collection of D-branes, this framework must be further generalized to that of $A_\infty$ categories. It turns out that Fukaya’s construction uses a general choice of vertices, which is why $A_\infty$ structures are relevant in that situation. We also note that by results of [33], an $A_\infty$ category admits a so-called ‘anti-minimal’ model, i.e. it is homotopy-equivalent with a dG category. Therefore, one can in principle used the dG framework of [105, 63] throughout. However, this is not always technically advantageous, and the homotopy equivalence required to recover the anti-minimal description can be extremely complicated. In the case of Fukaya’s category, for example, such a homotopy equivalence will involve summation over all disk instanton contributions.

Let us mention that one of the main results of the theory of $A_\infty$ algebras is that a quasi-isomorphism between such objects is the same as a homotopy equivalence (see [29]), A similar result is proved in [33] for $A_\infty$ categories. This shows that the homological and homotopical classification of such objects coincide.
5.1. A physical overview

As compared to the B-model, the case of topological A-branes is considerably more difficult. The main reason is the presence of worldsheet instanton corrections for the open A-model, which leads to a series of phenomena studied in detail in [25, 26, 76, 32, 29, 33]. Since one attempts to build an open string field theory, it suffices to study tree level instantons, which in this case amount to (pseudo)-holomorphic maps from a disk to the target space, subject to so-called Lagrangian boundary conditions [107].

Let us first explain the main physical ideas. The open A-model with target space a Calabi-Yau threefold $Y$ was introduced in [107]. This is a cohomological field theory governing maps $\phi : \Sigma \to Y$ where $\Sigma$ is a Riemann surface with boundary. As showed in [107], the model admits Lagrangian boundary conditions, a class of boundary conditions which respect the BRST symmetry. The associated topological branes are described by pairs $(L, E)$ where $L$ is a Lagrangian submanifold of $Y$ and $E$ is a vector bundle on $L$ endowed with a connection whose curvature equals $2\pi i B|_L$ ($B$ is the B-field on $Y$). Remember that a submanifold $L$ of $Y$ is called Lagrangian if its real dimension equals the complex dimension of $Y$ and if $\omega|_L = 0$, where $\omega$ is the Kahler form of $Y$. Physically, $E$ is the bundle of Chan-Paton factors.

As we shall see in the next subsection, one should add supplementary data, namely a grading [88] and a ‘relative spin’ structure [29] for $L$.

As in the closed case, the open A-model suffers instanton corrections, with worldsheet instantons described by the condition that $\phi$ is holomorphic. One can consider strings whose endpoints end on a single brane $(L, E)$, or strings stretching between two D-branes $(L_1, E_1)$ and $(L_2, E_2)$. In the first case, open string states localize on elements of $\Omega^*(L, \text{End}(E))$, while in the second case they are described by elements of $\oplus_{p \in L_1 \cap L_2} \text{Hom}(E_{1|p}, E_{2|p})$, where we assume for simplicity that $L_1$ and $L_2$ have transverse intersection. These spaces describe off-shell open strings states in these boundary sectors. To describe physical string states in the first case is considerably more complicated (due to the effect of instanton corrections), as we shall see in a moment. The final result will be that such on-shell (physical) state spaces are described by a very general version of Floer homology developed in [29, 33].

The most general tree-level situation is to specify a finite collection of topological A-branes $(L_j, E_j)$ ($j = 1 \ldots n$) and study scattering amplitudes on the disk. Namely, one fixes some points $z_j$ on the boundary of the disk $D$ such that $\phi(z_j) = p_j \in L_j \cap L_{j+1}$ (where $L_{n+1} := L_1$) as well as mutually distinct points $w^{(j)}_i$ on $\partial D$ (with $i = 1 \ldots l_j$) such that $\phi(w^{(j)}_i) = q^{(j)}_i \in L_j$, and scatters elements of $\text{Hom}(E_{j|p_j}, E_{j|p_{j+1}})$ and $\Omega^*(L_j, \text{End}(E_j))$. Here we assumed that the ordering

1It is also possible to consider the more general case of Lagrangian immersions, but we shall not discuss that here.

2This is a slight extension of the work of [107], where only bundles carrying a flat connection (B=0) were considered.
of $z_j$ on the boundary agrees with the orientation of the disk. The scattering amplitude will involve integration over the moduli space of the boundary-punctured disk (the configuration space of boundary points $z_j$ and $w^{(i)}_j$ divided by the obvious $PSL(2,\mathbb{R})$ action). Through localization, the associated path integral gives an integral over the moduli space of holomorphic maps $\phi$ from $D$ to $Y$, subject to the conditions $\phi(D_j) \subset L_j$ and $\phi(z_j) = p_j$ (one must also integrate over $q^{(i)}_j$ in $L_j$ so these points do not give extra-constraints). Here $D_j$ is the segment along the disk’s boundary lying between the points $p_j$ and $p_{j+1}$. This moduli space is of course non-compact, and a proper definition of the integral requires finding an appropriate compactification and defining a fundamental cycle. The latter requires perturbation of the complex structure of $Y$ to an almost complex structure, which is physically allowed since the open $A$-model continues to be well-defined and cohomological in this more general situation. Such technical difficulties are familiar from the study of Gromov-Witten invariants, but they are more severe in our case since the moduli spaces involved are not complex and thus cannot be approached with the methods of algebraic geometry. Instead, an analytic approach is required. This was developed in [25, 26, 28, 32, 76, 29, 33] and its results will be briefly recalled below. As we shall see, one must use a certain modification of the description of topological $A$-branes through pairs $(L, E)$ (namely one must use ‘graded Lagrangian submanifolds’ $L$ [88] which carry a so-called ‘relative spin structure’).

The main difficulty arising in the study of open $A$-type strings is the fact that worldsheet instanton corrections induce tadpole contributions which may displace the original string background. This phenomenon was originally discovered by K. Fukaya [32] and later also noticed in [47]. It arises from strings ending on a brane $(L, E)$ (namely when $\phi(\partial D) \subset L$, without boundary insertions) and signals the fact that the description of the brane through the pair $(L, E)$ is generally only valid semiclassically (by which we mean in the absence of worldsheet instanton corrections). In [29, 30], this phenomenon is described as an ‘obstruction’ to the existence of Floer homology. To make sense of the underlying theory (in the sector where strings stretch from $L$ to itself) one must find a shift of the string vacuum which cancels the tadpole (of course, it is possible that such a deformation of $(L, E)$ does not exist, in which case the associated topological brane will be destabilized by worldsheet instanton corrections). The process of shifting the string background can be described most elegantly in the framework of string field theory. Since the approach of Fukaya uses string amplitudes directly, we must employ the general formalism of open string field theory, which results from that of [106] by generalizing the choice of vertex. Recall from [35] that:

A (topological) open string field theory with a single boundary sector, defined with a general choice of vertices and expanded around a background which satisfies the equations of motion, is described by a set of string products which satisfy the constraints of an $A_{\infty}$ algebra, together with a compatible and nondegenerate bilinear form (the BPZ form).

In the conventions used in this paper, the products $r_n$ have degree $2 - n$ and
the bilinear form has degree $-3^k$. The string field action has the form:

$$S(\phi) = \sum_{n \geq 0} \frac{1}{n+1} (-1)^{n(n-1)/2} \langle \phi, r_n(\phi^\otimes n) \rangle,$$

(19)

where $\phi$ is a degree one element of the boundary space. Compatibility of the bilinear form means that the string products satisfy certain cyclicity conditions, which are given in [35].

The statement above applies only if the string field theory is expanded around a background which satisfies the equations of motion. It can be shown [60] that expanding the string field theory around a general background (which need not satisfy the equations of motion) leads to a so-called weak $A_\infty$ algebra, namely the obvious generalizing of an $A_\infty$ algebra obtained by allowing for a supplementary product $r_0 : C \rightarrow \mathcal{H}$ (this mathematical concept was introduced in [32]). The products $r_n$ obey a slight generalization of the $A_\infty$ constraints familiar from the case of standard (‘strong’) $A_\infty$ algebras, for example:

$$r_1(r_0(1)) = 0$$

(20)

$$r_2(r_0(1), u) \pm r_2(u, r_0(1)) \pm r_1(r_1(u)) = 0$$

and higher relations. In the context of string field theory, $r_0(1)$ describes the contribution of the tadpole, which induces a linear term in the string field action. As in [35], the product $r_1$ plays the role of worldsheet BRST charge; it fails to square to zero due to the presence of tadpoles. Shifting the string background is achieved by a translation $\phi \rightarrow \phi + \alpha$ of the string field, which leads to a new set of products $r'_n$ obtained by substituting this shift in the action:

$$r'_n(u_1 \ldots u_n) := \sum_{j_0 \ldots j_n \geq 0} (-1)^N r_{n+j_0+\ldots+j_n}(\alpha^\otimes j_0, u_1, \alpha^\otimes j_1, u_2 \ldots u_n, \alpha^\otimes j_n).$$

(21)

where $(-1)^N$ is a sign factor obtained by suspension. The condition that the shifted background satisfies the equations of motion is equivalent with vanishing of the tadpole contribution $r'_0$ around the new background; this automatically assures that $(r'_n)_{n \geq 1}$ satisfy the (usual) $A_\infty$ constraints, and in particular the shifted worldsheet BRST charge $r'_1$ squares to zero. The condition $r'_0 = 0$ gives the ‘weak homotopy Maurer-Cartan equation’:

$$\sum_{n \geq 0} (-1)^{n(n-1)/2} r_n(\alpha^\otimes n) = 0$$

(22)

This equation admits a natural gauge equivalence, which allows\(^1\) for building a local moduli space as explained in [30]. It is of course possible that (22) does not have

\(^1\)This is related to the conventions of [33] by suspension, namely Fukaya’s products $m_n$ are given by $r_n = sm_n(\alpha^\otimes n)$ where $s$ is the suspension map. This introduces certain sign prefactors in our formulae. We prefer to work with $r_n$ since they are more directly related to the case of dG algebras.

\(^1\)Properly speaking, it allows one to define a (local) deformation functor. To build a moduli space one must perform a Kuranishi analysis.
any solutions, in which case the original background does not admit a continuous deformation to a string vacuum. If (22) does admit solutions, then the shifted backgrounds can be described abstractly by the pair (original background, solution \( \alpha \) of (22)). This is the physical interpretation of the procedure used in [29].

As in the case of associative string field theory (which is described by Witten’s choice of vertex [106]), the more general description of [35] admits a categorical extension. Such an extension is appropriate (and required) when one has more than one boundary sector, namely when writing the string field action in the presence of more than one D-brane. This extension [60] is formulated as follows:

A (topological) open string field theory with D-branes, constructed with a general choice of vertices and expanded around a background which satisfies the equations of motion, is described by an \( A_\infty \) category \( \mathcal{A} \) together with compatible and nondegenerate bilinear forms (of degree \(-3\)) on its morphism spaces.

Compatibility of the bilinear form means that the \( A_\infty \) products of the category satisfy cyclicity relations with respect to these forms, which are the obvious categorical generalization of the cyclicity conditions given in [35] (written there with conventions different from ours). I will not write the relevant formulae here, because they are quite complicated. Instead, the reader is referred to [33] for a clear mathematical treatment of such data (except for the bilinear forms and cyclicity, which are easily recovered by adapting the work of [35]). When expanding around a background which fails to satisfy the equations of motion, one must again include zeroth order products \( r_0 \) which describe the tadpoles correcting the D-branes. The result is that the \( A_\infty \) category is replaced by a weak \( A_\infty \) category, the obvious generalization of a weak \( A_\infty \) algebra. Finally, the procedure of shifting the string vacuum has an obvious analogue in the presence of D-branes, and the relevant formulae are obtained from those given above by expanding in boundary sectors. Once again, we shall assume that \( \mathcal{A} \) is endowed with direct sums.

Granted that one expands around a string vacuum, one can again consider the problem of D-brane composite formation. Arguments very similar to those of [63] show that closure under such processes generally requires an enlargement of the underlying \( A_\infty \) category, which is obtained by a ‘homotopy’ analogue of the discussion of Section 2.2. This shows that the original \( A_\infty \) category \( \mathcal{A} \) must be extended to its quasi-unitary cover \( c(\mathcal{A}) \), which is an \( A_\infty \) category whose objects are two-sided generalized complexes over \( \mathcal{A} \) (defined by the obvious \( A_\infty \) version of the generalized complexes discussed in Section 2).

For a graded string field theory, generalized complexes become two-sided twisted complexes (the two-sided version of the twisted complexes used in [74, 57, 33]). One can again restrict to one-sided complexes, a technically advantageous step whose physical justification is unclear at the topological string level. With this restriction, one obtains an \( A_\infty \) category \( \text{tw}^+(\mathcal{A}) \), whose construction was originally suggested in [57] and carried out in detail in [33, 71]. The (zeroth) homology category \( D\mathcal{A} \) of \( \text{tw}^+(\mathcal{A}) \) (where the homology is taken with respect to the products \( r_1 \) of \( \text{tw}^+(\mathcal{A}) \)) is a triangulated category as showed, for example, in [33, 71]. This is the derived
category of the $A_\infty$ category $A$.

Summing up, the physical description of the work of [29, 33] (at least for the case of Calabi-Yau threefolds) is as follows:

1. One builds string amplitudes on the disk associated to scattering of open string states between finite systems of ‘graded’ topological A-branes described by pairs $(L, E)$ with $L$ a graded Lagrangian submanifold carrying a relative spin structure.

2. One uses these to build an open string field theory in the general sense of [35] (and including multiple boundary sectors). Due to use of graded Lagrangians, this string field theory will be graded.

3. Because of the presence of tadpoles correcting the D-branes, one must shift the original D-brane background to a true string vacuum. This requires solving the equations of motion of the theory, which are encoded in the category-theoretic version of (22). When solutions exist, then each choice of solution specifies a deformation of the original set of D-branes, which is induced by worldsheet instanton corrections. Choose such a deformation $\alpha$ of the D-brane background and shift to the associated string vacuum to obtain an $A_\infty$ category $A_\alpha$ endowed with shift functors. This is the Fukaya category $Fuk(Y)^\alpha$.

4. To implement closure of the physical description under formation of D-brane composites, one must pass to the quasi-unitary cover of $Fuk(Y)$, which is achieved by introducing twisted complexes. Restricting to one-sided twisted complexes, one obtains an $A_\infty$ category $tw^+(Fuk(Y))$. Its zeroth cohomology $H^0(tw^+(Fuk(Y)))$ is the derived category $DFuk(Y)$ of the Fukaya category.

5.2. Graded A-type branes

As originally pointed out in [58], a well-defined $\mathbb{Z}$-grading (as opposed to $\mathbb{Z}_2$-grading) on Floer homology requires that one add certain discrete data to the description of topological A-branes originally proposed in [107]. Since Floer homology describes the on-shell state space of open A-strings (which is $\mathbb{Z}_2$-graded by worldsheet $U(1)$ charge), this fact clearly has physical relevance, which was explained in [20]. As in the case of the B-model, this becomes important when one considers at least two D-branes, and is necessary in order to fix an ambiguity in the description of the worldsheet $U(1)$ charge. The ambiguity is specified by an integer (which, following [20], we shall call the grade of the brane$^5$), which in the worldsheet language of the untwisted theory describes the winding number of the boson which appears by bosonizing the worldsheet $U(1)$ current [20]. Since this is a conformal field theory argument, it will only be valid if the branes under consideration

$^5$It is possible that more than one solution $\alpha$ of the tadpole cancellation conditions exists. In that case, one can construct a different version of Fukaya’s category by including all possible quantum deformations of the original collection of topological D-branes. This is related to the theory of deformations of $A_\infty$ categories, currently being developed by Kontsevich and collaborators.

$^6$This is the integer grade relevant at topological sigma-model level, which in particular specifies a choice of branch for the real-valued grade of [20].
are described by *special* Lagrangian cycles. It is known \[30\] that the Maslow index \( \mu : \pi_2(Y, L) \to \mathbb{Z} \) of a special Lagrangian cycle is identically zero, a condition which is believed \[96\] to be the symplectic-topology equivalent of the special Lagrangian constraint. As we recall below, a Lagrangian cycle can be graded only if its Maslow index vanishes.

For any point \( p \) in \( Y \), consider the oriented Lagrangian Grassmannian \( L(T_p Y) \), i.e. the Grassmannian of all oriented Lagrangian linear subspaces of \( T_p Y \). This is a space whose fundamental group equals \( \mathbb{Z} \). Varying \( p \) inside \( Y \), we obtain a bundle \( L(Y) \). Since \( Y \) is a Calabi-Yau manifold, we have \( c_1(Y) = 0 \) which implies \[88\] \[33\] that there exists a cover \( \tilde{L}(Y) \) of this bundle whose restriction to each point \( p \) of \( Y \) can be identified with the universal cover of \( L(T_p Y) \). Choosing such a cover \( \tilde{L}(Y) \) makes the underlying symplectic space of \( Y \) into a graded symplectic manifold (and we shall fix such a cover below). Given a Lagrangian submanifold \( L \) in \( Y \), we have the so-called Gauss map, which is the section of \( L(Y) \) over \( L \) given by:

\[
\sigma(p) := T_p Y \quad \text{for} \quad p \in L.
\]

A grading of \( L \) is a lift of this map to the cover \( \tilde{L}(Y) \). As explained in \[33\], such a lift exists only if \( L \) has vanishing Maslow index, in which case there is a countable number of such lifts. A graded Lagrangian submanifold of \( Y \) is simply a Lagrangian submanifold \( L \) endowed with such a lift. We shall denote a graded Lagrangian by the same letter \( L \), the grading being understood. We then use \( L[n] \) to denote the same cycle \( L \), but with the grading shifted by \( n \) (though the action of \( \mathbb{Z} \) on \( \tilde{L}(Y) \)). When \( L \) is special Lagrangian, there is a canonical choice of grading which defines an ‘origin’ in the set of all gradings of \( L \) \[88\]; this is why the grading of a special Lagrangian can be simply viewed as an integer. As for the B-model, the effect of the grading is to shift the worldsheet \( U(1) \) charge of string states according to relation \[11\]. The complete description of topological A-brane configurations requires the specification of this data, which is essential in the construction of Fukaya’s category.

### 5.3. Relative spin structure

As explained in \[29\] \[33\], the construction of string field products also requires that our Lagrangian submanifolds \( L \) admit a so-called ‘relative spin structure’ – this is necessary in order to give an orientation to the moduli spaces of open string instantons. To define this data, one must fix a class \( t \in H^2(Y, \mathbb{Z}_2) \). Then there is a unique rank 2 real vector bundle \( V \) on \( Y \) of Stiefel-Whitney classes \( w_1(V) = 0, w_2(V) = t \) (remember that we assume \( \dim \mathbb{C}Y = 3 \)). We shall also assume that all our Lagrangian submanifolds are oriented. Then a relative spin structure \[29\] on \( L \) with respect to \( t \) is a spin structure on the restriction of the bundle \( V|_L \oplus TL \) to the two-skeleton of \( L \). Such a structure exists if \( L \) is oriented and \( w_2(TL) = t|_L \), in which case one says that \( L \) is relatively spin. We shall use the letter \( L \) to denote a (gradable and relatively spin) Lagrangian cycle together with its grading and choice of relative spin structure.
5.4. A (very) brief review of Fukaya’s results

As explained in [29, 33], the objects of Fukaya’s category are oriented Lagrangian submanifolds of $Y$ which are both graded and endowed with a relative spin structure (for technical reasons, the construction proceeds through a choice of a countable family of such cycles). To construct the string products $r_n$, one must consider families of objects (described by pairs formed by such a cycle and a Chan-Paton bundle) $a_j$ ($j = 1 \ldots n+1$) and scatter strings as explained at the beginning of this section. However, because objects are graded, it is now possible for example that $a_{j+1} = a_j[1]$ in such a sequence. Thus one effect of the grading is that one must consider disk amplitudes in which one of the strings which are scattered stretches between an object $a$ and the object $b = a[n]$, which geometrically has the same underlying cycle $L$ but with grading shifted by $n$. In this case, the grading on the state space of strings stretching from $a$ to $b$ must be shifted according to relation (11):

$$[\text{Hom}(E_a, E_b) \otimes \Lambda^*(L)]^k = \text{Hom}(E_a, E_b) \otimes \Lambda^{k-n}(L).$$

(24)

An example of this situation is showed in figure 1.

![Figure 1. Boundary sectors for a pair of graded D-branes wrapping the same special Lagrangian cycle. The two D-branes $a$ and $b$ are thickened out for clarity, though their (classical) thickness is zero.](image)

As an extreme case of this, one can consider the sector described by a brane $(L, E)$ and all branes obtained from it by shifting the grade. This is the case considered in the work of [65, 67, 68, 69], which we discuss in the next subsection. As explained above, such D-brane sectors are required in the construction of Fukaya’s category (see Section 4 of Fukaya’s paper [33] for the discussion of these sectors). Similarly, they are required in the open B-model, as explained in the previous section. Without including such sectors, one cannot obtain a triangulated category in the end (since one needs the existence of shift functors for that purpose).

The construction of string products for an entire (countable) collection of cycles is given in [29, 33], and I will refer the reader to those references for the (very
technical) details. As described above, the result is essentially a weak $A_\infty$ category. In fact, this concept is not entirely appropriate \cite{33, 29}, since one must deal with convergence issues for the disk instanton expansion, which requires one to work over the ‘positive’ Novikov ring $\Lambda_{0,\text{nov}}$ and introduce an energy filtration. Because of this, the technically correct version of weak $A_\infty$ category employed in \cite{33, 29} is a so-called filtered $A_\infty$ category. I refer the reader to Ref. \cite{33} for a detailed description of this notion. Then the main result of \cite{29, 33} is as follows:

**Theorem 1.** Fix $t \in H^2(Y, \mathbb{Z}_2)$ and a cover $\tilde{L}(Y)$ of $L(Y)$. Also fix an appropriate countable collection of graded Lagrangian submanifolds $L$ of $X$, each endowed with a relative spin structure (with respect to $t$) and with a complex vector bundle $E$ carrying a unitary connection whose curvature equals $2\pi i B|_L$. Call these objects and denote them by $a_j$. Assume that for any two such objects, the underlying Lagrangian submanifolds either coincide or intersect transversely. Also assume that this collection of objects is closed under the action of shifts of grading. Then there exists a filtered $A_\infty$ category $A$ with objects $a_j$, which encodes the disk amplitudes of the underlying A-model in the presence of the topological D-branes described by this countable set of objects.

The morphism spaces $\text{Hom}(a, b)$ of this category are defined as follows. Let $a = (L_a, E_a, \tilde{\sigma}_a, rst(a))$ and $b = (L_b, E_b, \tilde{\sigma}_b, rst(b))$ be two objects, where $\tilde{\sigma}$ are the associated lifts of the Gauss map (which specify the grading) and $rst$ are the relative spin structures. Then:

1. If $L_a$ and $L_b$ intersect transversely, then one has $\text{Hom}(a, b) = \oplus_{p \in L_a \cap L_b} \text{Hom}(E_a|_p, E_b|_p) \otimes \Lambda_{0,\text{nov}}$, with the grading induced by the so-called absolute Maslov index (see \cite{33, 33}).

2. If $L_b = L_a[n]$, then $\text{Hom}(L_a, L_b) = C(a, b) \otimes \Lambda_{0,\text{nov}}$, where $C(a, b)$ is a certain countably-generated subcomplex of the complex $W^{-\infty}(\text{Hom}(E_a, E_b) \otimes \Lambda^s(L))$ of distribution-valued forms, with the grading induced by the following twisted grading on $\text{Hom}(E_a, E_b) \otimes \Lambda^s(L)$:

$$[\text{Hom}(E_a, E_b) \otimes \Lambda^s(L)]^g := \text{Hom}(E_a, E_b) \otimes \Lambda^{g-n}(L).$$

In the definition of $\text{Hom}(a, b)$, the symbol $\hat{\otimes}$ denotes completion of the tensor product with respect to the metric induced by the energy filtration. Consideration of distribution-valued forms and the restriction to the subcomplex $C(a, b)$ is necessary for technical reasons which are explained in \cite{29}.

As outlined above, the next step is to shift the string vacuum in order to eliminate the zero-th products $r_0$, thus obtaining a (true) $A_\infty$ category $Fuk(Y) = \mathcal{A}_\alpha$ defined over the Novikov ring. Then one constructs the category $tw^+(Fuk(Y))$ by considering (one-sided) twisted complexes over $Fuk(Y)$ (again one can consider all possible deformations $\alpha$ at the same time, which is probably more appropriate

\textit{\footnote{Strictly speaking, the result quoted below has been proved (as of the time of writing) only for the case of Chan-Paton data described by line bundles.}}
both physically and mathematically). Finally, one can pass to zeroth cohomology to obtain a triangulated category $DFuk(Y)$. As explained above, each of these steps has a direct justification in topological open string field theory (except for the restriction to one-sided twisted complexes, whose physical meaning is unclear at the topological string field theory level).

5.5. Graded Chern-Simons field theory

As mentioned above, Fukaya’s category admits a sector obtained by considering a topological brane $a$ together with all branes $a[n]$ obtained from it by shifting the grade. It is interesting to ask what happens to Fukaya’s category in such a sector. By specializing the results of [29] and [33], one finds a ($\mathbb{Z}$-graded) $A_\infty$ algebra which describes string field amplitudes in this sector\(^9\). In fact, one can consider a slightly more general case, by taking branes whose underlying Lagrangian cycles coincide, whose gradings span the set of all integers $n$ and allow the Chan-Paton bundle $E_n$ of each brane to have a different rank\(^q\). (With this generalization \(^r\), the relevant sector of Fukaya’s category need not be closed under shifts.) The relevant construction of string amplitudes is discussed in [33]. Since disk instanton effects are rather hard to compute, it is natural in first approximation to ask what can be learned by neglecting them. This is possible close to the large radius limit of $Y$ or when $Y = T^*L$ (since in the latter case no disk instanton corrections are present\(^{107}\)). For simplicity, we shall also take the B-field to vanish.

Following the approach of [63, 64], the string field theory relevant for this situation was written down in [65], and turns out to be a $\mathbb{Z}$-graded version of Chern-Simons field theory. This should be viewed as an extension of Witten’s original work\(^{107}\), obtained by adding graded topological A-branes. It differs markedly from the $\mathbb{Z}_2$-graded version (known as super-Chern-Simons theory). Given a 3-manifold $L$, a graded Chern-Simons field theory involves the choice of a $\mathbb{Z}$-graded complex superbundle $E$, whose degree $n$ component we denote by $E_n$. The dynamical field is a graded superconnection $B$ of total degree one on $E$ in the sense of [13], and the action has the standard Chern-Simons form:

$$S := \int_L str \left[ \frac{1}{2} BDB + \frac{1}{3} B^3 \right],$$

(26)

where one uses graded multiplication in the space of sections of $End(E) \otimes \Lambda^*(L)$ (endowed with the total grading) and $str$ is the associated supertrace. We refer

\(^9\)The fact that such sectors should be included in Fukaya’s category was discussed in [33] (though it was probably known to some mathematicians). The recent paper [33] does include such sectors, which are necessary if one is to obtain a triangulated category at the end of the construction.

\(^q\)As mentioned above, this corresponds to a slight extension of the set-up of [33], where only line bundles are considered.

\(^r\)In fact, the papers [65, 67, 68, 69] also consider a further extension, namely they allow the connection to be complex (rather than unitary). This can be viewed as a way to incorporate simultaneous deformations of $L$ and $E_n$. 


the reader to [65] for details. The equations of motion of (26) require the graded superconnection $B$ to be flat in the sense of [13].

When $L$ is a Lagrangian in $X$, this theory naturally captures topological open string dynamics in the absence of worldsheet instanton corrections (and the effect of open string instantons at tree level is described by a sector of Fukaya’s category as mentioned above — as we discuss below, this can be included as a correction to the superpotential of [69]). As explained in [65], fluctuations $\phi$ around a background flat graded superconnection $B_0$ describe open string states stretching between $L$ and its grade-translates $L[n]$. To see this, let us limit from now on to the case when $E_n = E$ for all $n$ (i.e. all Chan-Paton bundles coincide) and notice that $\phi$ is a degree one section of $\text{End}(E) \otimes \Lambda^*(L)$, which can be expanded as:

$$\phi = \sum_{m,n \in \mathbb{Z}} \phi_{mn},$$

(27)

where $\phi_{mn}$ are sections of $\text{Hom}(E_m, E_n) \otimes \Lambda^{1+m-n}L$. Then $\phi_{mn}$ describes a state of the open string stretching from $a[m]$ to $a[n]$ (this state has worldsheet $U(1)$ charge equal to one). The equations of motion are:

$$d\phi + \frac{1}{2} [\phi, \phi] = 0,$$

(28)

where $d$ is the de Rham differential on $L$ coupled to $B_0$ and $[\cdot, \cdot]$ is the graded commutator in the graded associative algebra $\mathcal{H}$ of sections of $\text{End}(E) \otimes \Lambda^*(L)$. Expanding $\phi$ in (28) shows that $\phi_{mn}$ satisfy equations (8), and thus the collection $(\phi_{mn})$ defines a twisted complex of degree one. Note that the twisted complexes resulting from (26) are not one-sided, since there is no physical reason to require $\phi_{mn} = 0$ if $n \leq m$. The original motivation for studying the theory (26) goes back to the announcement [64] and consists in understanding the physical role of twisted complexes which are not one-sided. As we have just seen, such complexes appear naturally in the theory (26). It is also easy to show [65, 67] that condensation of twisted complexes gives an explicit description of extended deformations of the pair $(L, E)$. By this we mean that one can integrate out finite extended deformations, namely they are explicitly represented as twisted complexes of bundle-valued forms and maps. An explicit representation of finite deformations is difficult to give at the level of the associated triangulated category, since it requires one to represent the homotopy Maurer-Cartan functor discussed in Section 4.

The innocently-looking action (26) describes a rather complicated system, a fact which becomes apparent when one expands $B$ in its components. To study its dynamics, one must at least understand how to perform appropriate gauge-fixing. Since the theory involves higher rank forms, this requires the full force of the BV formalism, which was applied to the action (26) in the papers [67] and [69]. The appropriate BV action for (26) was written down in [65] and was analyzed in detail in [67], where it was showed that it satisfies the classical master equation.

The general proof of the classical master equation given in [67] rests on a certain graded version of the geometric formalism of BV systems due to [105, 96] (this
is necessary because the ghost number is an integer rather than \( \mathbb{Z}_2 \) valued) and makes use of the graded supermanifolds of \([99]\). While this requires one to develop some formalism, it allows for a one-line proof of the final result. An interesting by-product of the analysis is a certain periodicity mod 6 of the BV description under grade shifts, which is related to the periodicity observed in \([20]\). The resulting BV action can be formulated quite succinctly in supergeometric terms (see \([65, 67, 69]\)). Gauge fixing of this action was performed in \([69]\), and requires an infinite triangle of antifields and auxiliary fields. The final result can be written in a form which resembles that of usual Chern-Simons theory, though it contains considerably more complicated dynamics.

Another result extracted in \([69]\) is an expression for the associated tree-level potential, which (as predicted by the general discussion of Section \(8\)) is encoded by a series of \(A_\infty\) products describing tree-level scattering amplitudes. The perturbative expansion of this potential can be written down explicitly by applying the method of \([66]\) to this graded set-up. In open string language, the fact that one obtains nontrivial higher products in this manner is the effect of the so-called ‘open instantons at infinity of the moduli space’ pointed out in \([107]\) and discussed (in that case) in \([31]\). The results of \([66]\) imply that the \(A_\infty\) algebra defined by the tree-level scattering products is cyclic with respect to a certain bilinear form and homotopy equivalent with the differential graded algebra \((\mathcal{H}, d, \bullet)\), where \(\mathcal{H}\) is the space of sections of \(\text{End}(E) \otimes \Lambda^*(L)\), \(d\) is the de Rham differential coupled to the background flat graded superconnection \(\mathcal{B}_0\) and \(\bullet\) is the associative product on \(\mathcal{H}\). One can further pass to a countably-generated subcomplex \(C\) of \(W^{-\infty}(\text{End}(E) \otimes \Lambda^*(L))\) whose components \(C(m, n)\) are constructed as in \([33]\) and mentioned above. It then follows from the work of \([29, 33]\) that the effect of disk instanton corrections is (up to an irrelevant homotopy equivalence which amounts to a change of coordinates on the local moduli space) to deform the \(A_\infty\) algebra obtained through gauge-fixing in \([69]\) to the \(A_\infty\) algebra of \([29, 33]\) which encodes the instanton corrections; this of course deforms the associated tree-level potential \(W_{tree}\). With the extension to complex graded superconnections \(\mathcal{B}\) used in \([65, 67, 68, 69, 66]\), the deformed potential \(W_{def}\) obtained in this manner should be viewed as a (graded) A-model version of the D-brane superpotential of \([37]\).

Since the theory is topological, the partition function of \([26]\) should provide a topological invariant of triples \((L, E, \mathcal{B})\) where \(\mathcal{B}\) is a flat graded superconnection on \(E\). This is a sort of analogue of Witten’s invariant, whose ‘quantization’ by including instanton corrections could teach us something interesting about the diagonal sector of Fukaya’s category. Unfortunately, it is currently unknown how to determine this invariant, for example whether the surgery arguments of \([104]\) admit an extension to this graded case. However, it is possible to build a simpler invariant, namely a graded version of the analytic torsion of Ray and Singer (a form of Ray-Singer torsion coupled to a graded flat superconnection). Physically, this differential invariant arises upon expanding the partition function of \([26]\) around
the given flat graded superconnection, and regularizing the relevant volume factors. The necessary analysis was performed in [70], which extracts an explicit expression for the result in terms of a graded version of the Ray-Singer norm and proves independence of this norm of the choice of auxiliary metric data used to perform the gauge fixing. An different derivation is given in [69] in the framework of the BV formalism. Note that the graded Ray-Singer torsion encodes one-loop effects in the graded Chern-Simons theory, which correspond to open string loops of the underlying topological string (in the absence of instanton corrections). As such, the information contained in this invariant goes beyond string tree level. It is currently unknown how to effectively include the effect of worldsheet instanton corrections to the graded Ray-Singer torsion. Finally, it is quite obvious that a very similar analysis can be carried out for the holomorphic graded Chern-Simons theory discussed in Section 4. Most of the relevant results can be obtained simply by performing appropriate substitutions in the final results of [65, 67, 68, 69]. In particular, one can write down a graded version of holomorphic Ray-Singer torsion.

6. Π-stability

In the paper [5] (see also [20, 21 and the earlier work 19]), M. Douglas and P Aspinwall proposed a stability condition for objects of \(D^b(X)\) when \(X\) is a Calabi-Yau threefold. This proposal was formulated rigorously in [14] for arbitrary triangulated categories. Given a triangulated category \(\mathcal{T}\), a stability condition [14] is a pair \((Z, \mathcal{P})\) where:

1. \(Z : K(\mathcal{T}) \to \mathbb{C}\) is a complex-valued linear map on the \(K\)-group \(K(\mathcal{T})\) of \(\mathcal{T}\)
2. \(\mathcal{P}\) is a family of full subcategories \(\mathcal{P}(\phi)\), indexed by a real parameter \(\phi\) and satisfying the following requirements:
   2a. \(\phi = \frac{1}{\pi} \arg Z([E]) \mod 2\) for all \(E \in \mathcal{P}(\phi)\) (where \([E]\) is the K-theory class of \(E\)).
   2b. \(\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]\) for all real \(\phi\)
   2c. \(\text{Hom}_\mathcal{T}(E_1, E_2)\) vanishes if \(E_1 \in \mathcal{P}(\phi_1)\) and \(E_2 \in \mathcal{P}(\phi_2)\) with \(\phi_1 > \phi_2\)
   2d. For every nonzero object \(E\) of \(\mathcal{T}\) there exists a finite descending sequence \(\phi_1 > \phi_2 > \cdots > \phi_n\) and a collection of triangles \(E_j \to E_{j+1} \to A_j \to E_j[1]\) (with \(j = 0, \ldots, n-1\) and \(E_n = E\)) such that \(A_j \in \mathcal{P}(\phi_j)\) for all \(j\).

With this definition, one can show [14] that each subcategory \(\mathcal{P}(\phi)\) is Abelian. The objects of \(\mathcal{P}(\phi)\) are called semistable objects of phase \(\phi\), while the simple semistable objects are called stable. One also shows [14] that the ‘decomposition’ at point (2d) is unique for every nonzero object \(E\). The objects \(A_j\) are called the semistable factors of \(E\).

In the context of an open superstring compactification on a Calabi-Yau threefold \(X\), this data arises as follows. The triangulated category \(\mathcal{T}\) is the bounded derived category \(D^b(X)\). The stringy Kahler moduli space of \(X\) (which arises from closed string mirror symmetry) coincides with the complex moduli space \(\mathcal{M}\) of the mirror \(Y\) of \(X\). Given a choice of large complex structure point of \(Y\) (identified with a
large radius point of $X$) in $\mathcal{M}$, there is an associated ‘topological mirror map’ $m : K(D^b(X)) \to H_3(Y, \mathbb{Z})$ which implements open mirror symmetry at the level of D-brane charges. Fixing a point $q$ in $\mathcal{M}$, one considers the linear map $Z_q : K(T) \to \mathbb{C}$ given by:

$$Z_q([E]) = \int_{m([E])} \Omega_q ,$$  

where $\Omega$ is the (normalized) holomorphic 3-form of $Y$ associated to the complex structure defined by $q$. The quantity $Z_q([E])$ is the central charge of any B-type brane in the K-theory class $[E]$. This naturally depends on the stringy Kahler moduli of $X$, described by the point $q$. The quantity $\phi$ defined at (2a) is the real-valued grade of $P$. In particular, the worldsheet $U(1)$ charge of a superstring state stretching between branes $A, B$ of phases $\phi_A, \phi_B$ is shifted by $\phi_B - \phi_A$. Condition (2b) reflects the relation between the real-valued grade $\phi$ and the integral grade used in the associated B-twisted version of topological strings, which was discussed above (in particular, the integral grade fixes a branch of the argument appearing in the definition of $\phi$). Condition (2c) also follows from conformal field theory considerations in the untwisted theory. Condition (2d) is related to the existence of a finite mass gap in the spectrum of BPS states. The objects of $P(\phi)$ describe the stable B-type branes whose central charge has phase $\phi$. The phase specifies the particular $\mathcal{N} = 1$ supersymmetry preserved by the associated BPS state (i.e. the particular combination of the original $\mathcal{N} = 2$ generators which survives when including this state). The ‘decomposition’ at (2d) above describes the decay products of a brane $E$; $E$ is semistable when $n = 1$. The mass of such a brane is given by $m(E) = \sum_j |Z(A_j)|$, with $A_j$ its semistable factors. By the triangle inequality, one has $m(E) \geq |Z(E)|$, with equality if $E$ is semistable; this implements the Bogomolnyi bound.

We now present a result of [14], which clarifies the relation of these notions with more standard constructions in the theory of triangulated categories. Let us return to a triangulated category $\mathcal{T}$. Remember that a $t$-structure on $\mathcal{T}$ is a full subcategory $\mathcal{F}$ of $\mathcal{T}$ such that:

1. $\mathcal{F}[1] \subset \mathcal{F}$
2. For every objects $E$ in $\mathcal{T}$, there exists a triangle $F \to E \to G \to F[1]$ of $\mathcal{T}$ with $F \in \mathcal{F}$ and $G \in \mathcal{F}^\perp$.

Here $\mathcal{F}^\perp$ is the set of objects $F$ of $\mathcal{T}$ such that $\text{Hom}_\mathcal{T}(E, F)$ vanishes for all objects $E$ of $\mathcal{F}$.

Given a $t$-structure $\mathcal{F}$, its heart is the full subcategory of $\mathcal{T}$ defined through:

$$\mathcal{A} := \mathcal{F} \cap \mathcal{F}^\perp[1] ;$$

This map depends on the choice of large radius point, which specifies a semiclassical limit of the model. There exist models (such as a model studied in [61]) which have more than one large radius limit, and where this dependence can be seen very explicitly and has pronounced physical consequences for the D-brane spectrum.
this is an Abelian category (whose short exact sequences are those triangles of $T$ all of whose vertices belong to $A$). The main use of t-structures is to identify Abelian subcategories of a triangulated category in the manner indicated by this result. A t-structure is called \textit{bounded} if $T$ equals the union of $F[m] \cap F[n]$ over $m, n \in \mathbb{Z}$; such a t-structure can be recovered from its heart.

Also recall the notion of \textit{slope function} on an Abelian category $A$ and the associated notion of stability. A slope function is a linear function $Z : K(A) \rightarrow \mathbb{C}$ such that $\phi(E) := \frac{1}{2\pi} \arg Z(E) \in (0, 1]$ for all nonzero objects $E$ of $A$ (in [14], such slope functions are called ‘centered’). Given a slope function, a nonzero object $E$ of $A$ is called \textit{semistable} if $\phi(A) \leq \phi(E)$ for any nontrivial exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $A$. A slope function satisfies the \textit{Harder-Narasimhan} property if every object of $A$ admits an ascending filtration $0 = E_0 \hookrightarrow \ldots \hookrightarrow E_n = E$ whose quotients $F_j = E_j/E_{j-1}$ are semistable and of decreasing slope (i.e. $\phi(F_1) > \ldots > \phi(F_n)$) (such a filtration is called a \textit{Harder-Narasimhan filtration}).

The following result is proved in [14]:

\textbf{Proposition 2.} \textit{Giving a stability condition on $T$ amounts to giving a bounded t-structure, together with a slope function on its heart satisfying the Harder-Narasimhan property.}

Given the stability condition $(P, Z)$, the t-structure is recovered as follows. For each interval $I$ lying along the real axis, one defines $P(I)$ to be the extension-closure in $T$ of the collection $P(\phi)$ with $\phi \in I$ (the extension closure is taken with respect to the triangles of $T$). Then $F = P(0, \infty)$ is a bounded t-structure on $T$, whose heart equals $A = P(0, 1]$. The slope function on $A$ is obtained by restricting $Z$. The semistable objects in $A$ defined by this slope function are the objects of $P(\phi)$ with $\phi \in (0, 1]$, and the decompositions at (2d) above give the Harder-Narasimhan filtrations in $A$. The converse also follows in a pretty obvious manner (see [14] for details).

This result shows that the stability condition of [20, 5] is a natural extension of the standard stability condition on an Abelian category, as formulated by Rudakov [33]. It also allows one to recover the stable objects of $T$, provided that one can detect the later on an appropriate Abelian subcategory. Among these is the beautiful fact that (under certain technical assumptions), the set of so-called \textit{numerical} stability conditions on $T$ carries a natural topology with respect to which every connected component is a manifold.

Let us now return to the physically interesting case (namely $T = D^b(X)$ with $X$ a smooth Calabi-Yau threefold) and briefly recall some other aspects discussed in [20, 5]. Since the central charge $Z$ depends on the complexified stringy Kahler moduli of $X$, the subcategory $P(\phi)$ will vary with such moduli. This phenomenon (which the authors of [20, 5] call ‘flow of gradings’) is analogous to the dependence of $\mu$-stability of coherent sheaves on the choice of Kahler class.

A related aspect is the existence of monodromies of the periods of $Y$ around components of the discriminant locus in $M$, which imply that $\phi$ is multivalued when
defined on \( \mathcal{M} \). Physical consistency requires that its monodromy transformations should correspond to autoequivalences of \( D^b(X) \), relating the subcategories \( P(\phi) \) associated to different branches of \( \phi \). Some evidence for this conjecture was given in [5], upon using previous work of [89]. Further aspects of this fascinating proposal are discussed in [2], which considers its physical consequences for certain classes of \( D^0 \)-branes and what their physics may teach us in relation to the reconstruction results of [10].

The proposal of [20, 5] amounts to a physically-motivated definition, whose applications are currently under-explored. In algebraic geometry, stability conditions traditionally play a role in the global construction of moduli spaces as algebro-geometric objects (stacks, schemes or even varieties, in a few lucky cases). From this perspective, the deeper mathematical role of the stability condition of [20, 5, 14] could be found by connecting it with moduli problems. In particular, one would like to understand the relation of such a global study to the local description afforded by the \( A_\infty \) structure on \( D^b_{\infty}(X) \).

7. Some applications

7.1. Stability, monodromies and derived equivalences

Homological mirror symmetry predicts the existence of certain autoequivalences of \( D^b(X) \) associated with monodromy transformations around the discriminant locus in \( \mathcal{M} \). This was pointed out by Kontsevich [58] and analyzed in detail in [89, 90, 91, 45]. Here is the basic idea. It is known [87, 88] that any Lagrangian sphere \( S \) in \( Y \) defines a symplectic automorphism of \( Y \) called \textit{generalized Dehn twist} along \( S \) (a symplectic version of Picard-Lefschetz transformation). Such automorphisms involve a choice of local data, but the induced functor on \( DFuk(Y) \) is expected to be independent of such choices and lead to an autoequivalence of \( DFuk(Y) \) (see [93, 94] for a study in a local context). By homological mirror symmetry, the mirror of \( S \) should be a \textit{spherical object} \( E \) of \( D^b(X) \), i.e. an object for which \( \text{Ext}^*(E) \) is concentrated in degrees zero and \( \text{dim}X \), in which degrees it equals \( \mathbb{C} \). The Dehn twist action on \( DFuk(Y) \) should correspond to an autoequivalence \( T_E \) of \( D^b(X) \) induced by \( E \).

Using the known action of Dehn twists on \( H_*(Y) \) and the topological mirror map, one finds a prediction for the action induced on \( K(D^b(X)) \), which suggests an ansatz for the autoequivalence. Namely, \( T_E \) should be given by:

\[
T_E(F) = \text{Cone}(\text{Hom}^*(E, F) \otimes_{\mathbb{C}} E \rightarrow F),
\]

where the arrow is the evaluation map. Of course, this only determines \( T_E(F) \) up to a non-canonical isomorphism, an ambiguity which is eliminated by the following precise definition:

**Definition 3.** The \textit{twist functor} \( T_E \) defined by an object \( E \) of \( D^b(X) \) is the Fourier-
Mukai transform with kernel¹:

\[ \mathcal{P} = \text{Cone}(\eta : E^\vee \boxtimes E \to \mathcal{O}_\Delta) \in D^b(X \times X) , \]

where \( \Delta \) is the diagonal of \( X \times X \), \( \eta \) is the natural pairing and \( \boxtimes \) is the exterior tensor product.

Thus \( T_E(F) = R\pi_{2*}(\pi_1^* F \boxtimes \mathcal{P}) \), where \( \pi_j \) are the two projections on the factors of \( X \times X \). Then [89] proves the following result, which agrees with homological mirror symmetry expectations:

**Theorem 4.** If \( E \) is a spherical object, then \( T_E \) is an autoequivalence of \( D^b(X) \).

In fact, the paper [89] also analyzes a more general situation, which arises for example when studying resolutions of Calabi-Yau quotient singularities. Namely, they study so-called \( A_n \)-configurations of objects in \( D^b(X) \), which are expected to be mirror to similar configurations of Lagrangians in \( DFuk(Y) \). In this case, they prove that the associated twist functors define a braid group action on \( D^b(X) \), which is faithful for \( \dim X > 1 \).

The beautiful results of [89] where discussed from a physical perspective in [5, 3] (for the case \( \dim X = 3 \)). The basic observation is that the description (31) of the twist functor can be justified by considering the effect of monodromies on the \( \Pi \)-stability condition on \( D^b(X) \). This interpretation arises when one has a spherical object \( E \) which becomes massless along a certain component of the discriminant locus. In the toric case, the work of [36, 37, 61, 46] suggests that the spherical object \( \mathcal{O}_X \) (which describes a D6-brane wrapping \( X^u \)) is always massless along the principal component of the discriminant locus. This allows one to argue [5, 3] that the associated twist functor \( T_{\mathcal{O}_X} \) is compatible with the \( \Pi \)-stability condition of [5]. Whether the \( \Pi \)-stability condition in fact implies the form (31) of this monodromy action (as conjectured in [5, 3]) can presumably be proved by using the rigorous definition given in [14].

A far-reaching extension of the results of [89] was given in [45], which constructs a large class of autoequivalences of \( D^b(X) \) induced by so-called "EZ-spherical objects". These give a generalization of the spherical objects of [89]. Such autoequivalences are associated with a flat morphism from a smooth complete subvariety \( E \) of \( X \) to another smooth subvariety \( Z \) of lower dimension. In the toric case, this situation can be realized by considering contractions associated with various components of the discriminant locus; the principal discriminant then corresponds to \( Z = \{ \text{a point} \} \) and recovers the analysis of [89]. The physical interpretation of this larger class of autoequivalences was discussed in [3], which again relates them to the effect of monodromy transformations on the \( \Pi \)-stability condition.

¹The definition works for objects \( E \) which are complexes of locally free sheaves and extends correctly to \( D^b(X) \).
²And whose expected mirror is a sphere in \( Y \) [86, 40, 41, 42].
7.2. **Autoequivalences induced by flops, worldvolume theories and ”toric duality”**

Another mathematical result which admits a physical interpretation is due to [15] (see [11] for earlier results):

**Theorem 5.** Let \( S \) be a projective threefold with terminal singularities and let \( X, X' \) be two crepant resolutions of \( S \). Then there exists an equivalence of triangulated categories between \( D^b(X) \) and \( D^b(X') \).

By results of [55], any two such resolutions are related by a finite chain of flops. For a flop, the result follows [15] by starting with \( X \) and building \( X' \to S \) as a (fine) moduli space of so-called ”perverse point sheaves” constructed from the data of \( X \to S \). Then the equivalence of the theorem is a Fourier-Mukai transform whose kernel is constructed by using perverse sheaves.

This result implies that birational Calabi-Yau threefolds have equivalent derived categories, a property which is physically quite natural since the topological B-model is independent of their Kahler class. Since the Hodge numbers of \( X \) can be extracted from the derived category, this also recovers the invariance of Hodge numbers under flops, which was proved for general dimension in [12]. For extensions of Bridgeland’s result, the reader is referred to [17, 75]. In particular, the paper [17] generalizes this to flops between threefolds \( X, X' \) which are allowed to have terminal Gorenstein singularities.

An interesting interpretation of Bridgeland’s result and its generalizations arises upon considering (possibly partial) resolutions of Calabi-Yau quotient singularities, which can be described in physical terms by introducing D-brane probes transverse to the singularity [15, 53]. In this framework, an interesting situation arises for singularities which admit multiple partial resolutions [39, 17]. Then it is possible [7, 8] that two such partial resolutions lead to distinct but equivalent descriptions of the associated worldvolume field theory. As discussed in [8], such distinct field theory descriptions can sometimes be related by Seiberg dualities. A conjectural extension of this point of view was recently proposed in [22], by making use of the derived category picture of B-type branes. In [22], it is suggested that there should be an equivalence between the derived category of (certain?) partial resolutions and the derived category of the quiver which describes the associated worldvolume theory\(^v\). Based on this identification, the appropriate generalization of Bridgeland’s equivalences would translate into autoequivalences of the Abelian category of quiver representations. It is known that such autoequivalences can be decomposed into tilts (this follows from [84] and the description of such representations as modules over the path algebra of the quiver). On the other hand, it was argued in [22] that tilting equivalences of the Abelian category of quiver representations can be sometimes related to Seiberg dualities of the associated field theory. This is an interesting

\(^v\)This should presumably follow by an extension of the categorical McKay correspondence of [15].
proposal which deserves further study. Other results on the relation between derived categories of D-branes and supersymmetric field theory can be found in [6].

8. Open questions

Perhaps the most important open question in homological mirror symmetry is to devise effective algorithms for determining the categorical mirror map. This can be done for the case of elliptic curves [70, 80, 81, 82, 83] and higher-dimensional Abelian varieties [77, 84] (see also [91, 92] for studies of complex tori). Unfortunately, very little is known in other cases. For some recent progress the reader is referred to [59, 27]. A related problem is to find effective methods for computing Fukaya’s category. Some ideas in this direction were recently proposed in [93]. Though an impressive mathematical ‘tour de force’, the construction of Fukaya’s category is somewhat unsatisfactory both from a physical and mathematical perspective. Physically, the topological string field theory description of obstructions used in Fukaya’s work seems to be exceedingly abstract, and it is hard to see how one could ever compute them through such methods. One may hope that the alternative point of view provided by graded Chern-Simons theories can be more fruitful from a physical perspective. Mathematically, the construction of [29, 33] is so complicated that one is perhaps better off viewing it as an existence proof. As such, it may be useful to attempt to characterize Fukaya’s category (up to homotopy equivalence) through some abstract conditions, and use those in order to prove results about its behavior under various geometric operations. This may allow one to compute Fukaya’s category by reduction to simpler cases.

A question of great physical importance is to understand the extension of homological mirror symmetry to the case of mixed and non-geometric phases, and to extend the results of [31, 100] to the open string level. Some recent work along these lines was carried out in [59, 78], and this entire subject deserves much more investigation. As in the closed string case, one can shed some light on this problem [43, 44] by using an open version of Witten’s linear sigma models [100], though the categorical aspects of the linear sigma model construction have not been thoroughly investigated. Finally, it would be of great physical interest to combine such an analysis with the stability proposal of [5, 14] in order to extract a more complete description of D-brane physics on Calabi-Yau manifolds.

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