Non-commutative corrections in spectral matrix gravity

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Abstract

We study a non-commutative deformation of general relativity based on spectral invariants of a partial differential operator acting on sections of a vector bundle over a smooth manifold. We compute the first non-commutative corrections to Einstein equations in the weak deformation limit and analyze the spectrum of the theory. Related topics are discussed as well.

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1. Introduction

The basic concept of general relativity is the spacetime, which is a smooth manifold with a globally hyperbolic metric $g_{\mu\nu}$. As it is well known that the dynamics of the metric is described by the Einstein–Hilbert action functional (with cosmological constant)

$$ S = \frac{1}{16\pi G} \int_M dx \, g^{1/2} (R - 2\Lambda), $$

where $dx$ denotes the standard Lebesgue measure on $M$, $g = \det g_{\mu\nu}$, $R$ is the scalar curvature of the metric, $G$ is the Newtonian gravitational constant and $\Lambda$ is the cosmological constant.

It is a rather old idea that the metric can be described by the properties of light propagating in the spacetime (that is, by studying the properties of the wave equation) and the action of general relativity can be ‘induced’ by spectral invariants of a second-order partial differential operator of the Laplace type (for more details, see [9]). In a series of recent papers [8–10] a new theory of gravity has been developed, called matrix gravity, in which the main idea is to describe the gravitational field by studying the propagation of fields with some internal structure rather than light. This idea follows from the fact that at a more fundamental level (short distances or high energies) the role of the photon could be played by a multiplet of gauge fields. In the papers [9, 10] the action was constructed by directly generalizing the language of Riemannian geometry to the non-commutative case. In [16] we studied the non-commutative limit of the theory proposed in [9, 10]. It turned out that this deformation was non-unique.
That is why, in [8], a new model called spectral matrix gravity was developed, which is based on spectral invariants of non-Laplace type partial differential operators. In the present paper we study the weak deformation limit in spectral matrix gravity.

We would like to stress, at this point, that our approach to deforming general relativity is different from the more standard approach of noncommutative extension of gravity on noncommutative spaces. In flat space one usually introduces noncommutative coordinates satisfying the commutation relations

\[ [x^\mu, x^\nu] = i \theta^{\mu\nu}, \]  

(1.2)

where \( \theta^{\mu\nu} \) is a real constant anti-symmetric matrix, and one replaces the standard algebra of functions with the non-commutative algebra with the Moyal star product

\[ f(x) \star g(x) = \exp \left( i \frac{\theta^{\mu\nu}}{2} \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial z^\nu} \right) f(x + y)g(x + z) \bigg|_{y = z = 0}. \]  

(1.3)

An extensive review of different realizations of gravity in the framework of noncommutative geometry, especially in connection with string theory, can be found in [23]. An extension of the star product and noncommutativity from flat to curved spacetime can be found in [2, 19].

We list below the most relevant differences between the two approaches, a more detailed and extensive discussion can be found in [9–11, 16].

The biggest problem with the curved manifolds is the nature of the object \( \theta^{\mu\nu} \). All these models are defined, strictly speaking, only in perturbation theory in the deformation parameter, that is, one takes \( \theta^{\mu\nu} \) as a formal parameter and considers formal power series in \( \theta^{\mu\nu} \). In our approach to matrix gravity the deformation parameters are not formal and the theory is defined for all finite values of the deformation parameter.

In the standard non-commutative approach the coordinates themselves are non-commutative. This condition raises the questions of whether the spacetime has the structure of a manifold, and how one can define analysis on such spaces. Moreover, one needs a way to relate the non-commutative coordinates to the usual (commutative) coordinates. In matrix gravity we do not have non-commutative coordinates. Our spacetime is a proper smooth manifold with the standard analysis defined on it.

In the non-commutative geometry approach the deformation parameter \( \theta^{\mu\nu} \) is non-dynamical, therefore there are no dynamical equations for it. This poses the question of what kind of physical, or mathematical, conditions can be used in order to determine it. In addition, in many models of non-commutative gravity (as in [2]) \( \theta^{\mu\nu} \) is a non-tensorial object which makes it dependent on the choice of the system of coordinates. In matrix gravity we do not have non-commutative coordinates. Our spacetime is a proper smooth manifold with the standard analysis defined on it.

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In our approach, we do not introduce any non-tensorial objects. As a result our theory is diffeomorphism-invariant. So, there are no problems related to the violation of Lorentz invariance, etc and there are no preferred systems of coordinates. Moreover, the non-commutative part of the metric in our approach is dynamical. We have non-commutative
Einstein equations for it. The goal of this paper is, in particular, to derive these dynamical equations in the perturbation theory.

In [19] the authors assume that $\theta^{\mu\nu}$ is a covariantly constant tensor. But then, there are strict algebraic constraints on the Riemann curvature tensor of the commutative metric (obtained by a commutator of second covariant derivatives). In matrix gravity such algebraic constraints are absent—the commutative metric is arbitrary.

In the usual approach of non-commutative geometry (as in, for example, [2]), when one defines the affine connection, the covariant derivative, the curvature and the torsion the ordering of factors is not unique. There is no natural reason why one should prefer one ordering over the other. That is, the connection coefficients can be placed on the left, or on the right (or one could symmetrize over these two possibilities) from the object of differentiation. Another aspect of the ordering problem is the fact that there is no unique way to raise and lower indices. One can act with the metric from the left or from the right. Spectral matrix gravity, instead, is pretty much unique. There is no need to define the affine connection, the covariant derivative, the curvature and the torsion. There is no ordering problem.

The definition of a ‘measure’, in standard non-commutative geometry, as a star determinant (as in [2]) does not guarantee its positivity. It only guarantees the positivity in the zero order of the perturbation theory. With our definition, the measure is positive even in the strongly non-commutative regime.

Moreover, the Moyal star product is non-local which makes the whole theory non-local with possible unitarity problems. In the spectral matrix gravity approach the action functional is a usual local functional of sigma-model type (like general relativity, but with additional non-commutative degrees of freedom). There may be problems with the renormalizability (which requires further study) but not with unitarity.

We consider the spacetime as a smooth manifold in which the gravitational field is described by propagation of gauge fields rather than light. Therefore one is naturally led to consider the non-Laplace type partial differential operator for which the leading symbol is now a matrix. One can evaluate the heat kernel asymptotic coefficients for such an operator and construct, from the first two, the action for matrix gravity as in (1.10). Similar calculations have been performed in noncommutative geometry regarding heat kernel asymptotics expansion. In [21, 22] the author evaluates the relevant geometric quantities from an approximate power expansion of the trace of the heat kernel for a Laplace operator on a compact fuzzy space. In [24], the author studies the quantization of noncommutative gravity in two dimensions by considering a noncommutative deformation (using the Moyal product) of the Jackiw–Teitelboim model for gravity. In this case the path integral can be evaluated exactly and the operator for the quantum fluctuations can be found. Once the operator is known one can study the first two heat kernel asymptotic coefficients and obtain information about the conformal anomaly and the Polyakov action. Our analysis is rather different because we consider a standard smooth manifold and a non-Laplace operator with matrix valued leading symbol without considering the Moyal product.

For the sake of completeness we describe, briefly, the framework of matrix gravity (a more detailed discussion can be found in [8–10]). Let us consider a smooth compact orientable $n$-dimensional manifold $M$ without boundary. Let $V(\omega)$ be a $N$-dimensional vector bundle over $M$ of densities of weight $\omega$ and $C^\infty(V(\omega))$ be the space of smooth sections of the vector bundle $V(\omega)$. Locally, a section $\phi$ of the vector bundle $V$ is described by the $N$-vector $\phi^A$. Let $\text{End}(V)$ be the bundle of endomorphisms of the vector bundle $V$; a section $Q$ of the endomorphism bundle is represented locally by the matrix $Q^{AB}$. Next, we assume that the vector bundle $V(\omega)$ is endowed with an Hermitian metric $G$ of weight $(1 - 2\omega)$ defining the fiber inner product $\langle \phi, \psi \rangle = \phi^\dagger G \psi$. This enables one to define the $L^2$-product of two
sections of $V$ by
\begin{equation}
(\phi, \psi)_{L^2} = \int_M dx \langle \phi(x), \psi(x) \rangle,
\end{equation}
(1.4)

The completion of the space $C^\infty(V[\omega])$ in the corresponding norm defines the Hilbert space $L^2(V[\omega])$.

Now, let us consider a second-order partial differential operator
\begin{equation}
L : C^\infty(V[\omega]) \to C^\infty(V[\omega]),
\end{equation}
acting on smooth sections of the vector bundle $V[\omega]$, with the endomorphism-valued coefficient functions
\begin{equation}
L = -a^{\mu\nu}(x) \partial_\mu \partial_\nu + b^\mu(x) \partial_\mu + c(x).
\end{equation}
(1.5)

The leading symbol of the operator $L$ is defined as
\begin{equation}
\sigma_L(L; x, \xi) = a^{\mu\nu}(x) \xi_\mu \xi_\nu,
\end{equation}
where $\xi_\mu$ is a cotangent vector at the point $x$.

We consider two cases: the Euclidean case when the operator $L$ is elliptic, and the pseudo-Euclidean case when the operator $L$ is hyperbolic. These cases are related by an analytical continuation and, for the sake of simplicity, we will restrict our calculation to the elliptic case. The formulae for the hyperbolic case look exactly the same (for more details, see [8–10]). The operator $L$ is elliptic if the leading symbol $\sigma_L(L; x, \xi)$ is non-degenerate for any $x$ and any $\xi \neq 0$. The operator $L$ is of Laplace type if the leading symbol is essentially scalar, that is,
\begin{equation}
\sigma(L; x, \xi) = g^{\mu\nu}(x) \xi_\mu \xi_\nu,
\end{equation}
where \( g \) is the identity matrix. The geometric invariants we need can be derived from the spectrum of the operator $L$.

The $L^2$-trace of the heat semigroup, $\exp(-tL)$, usually called the heat trace, contains all the information about the spectrum of the operator $L$. It is well known that the asymptotic expansion of the heat trace as $t \to 0^+$ has the form [17]
\begin{equation}
\text{Tr}_{L^2} \exp(-tL) \sim (4\pi)^{-\frac{n}{2}} \sum_{k=0}^{\infty} t^{k-\frac{n}{2}} A_k,
\end{equation}
(1.8)

where the coefficients $A_k$ are called the global heat kernel coefficients, which are, of course, spectral invariants of the operator $L$. They have the form
\begin{equation}
A_k = \int_M dx \text{tr}_V a_k,
\end{equation}
(1.9)

where $\text{tr}_V$ is the fiber trace on the vector bundle $V$ and the coefficients $a_k$ are some endomorphism-valued densities of weight one, called the local heat kernel coefficients. These are the coefficients $A_0$ and $A_1$ that we study in this paper.

The action of spectral matrix gravity proposed in [10] is a linear combination of the heat kernel coefficients $A_0$ and $A_1$, more precisely,
\begin{equation}
S = \frac{1}{16\pi GN} [6A_1 - 2A A_0].
\end{equation}
(1.10)

We would like to point out here that the above action can be also thought of as a particular case of the spectral action principle introduced in the framework of noncommutative geometry in [12, 13].

For the Laplace operator, $L = -\Delta$, the heat kernel coefficients are well known
\begin{equation}
a_0 = \mathbb{I}, \quad a_1 = \frac{1}{6} R \mathbb{I}.
\end{equation}
(1.11)
In this case, the action of spectral matrix gravity reduces to the standard Einstein–Hilbert action (1.1).

We would like to stress, here, that we are interested, in this paper, in a much more complicated general case of an arbitrary non-Laplace type operator (with a non-scalar leading symbol). In this case there is no preferred Riemannian metric and the whole language of Riemannian geometry is not very helpful in computing the heat kernel asymptotics. That is why, until now, there are no explicit general formulae for the coefficient $A_1$. A class of so-called natural non-Laplace type operators was studied in [6, 7] where this coefficient was computed explicitly.

The main goal of this paper is to study the action of spectral matrix gravity in the weak deformation limit and to describe the corresponding corrections to Einstein equations. For the sake of completeness we will briefly describe, in the following section, the properties of some geometric two-point quantities that are widely used in the calculation of the heat kernel asymptotics.

2. Geometric background

We introduce now some geometric two-point quantities following mainly [3, 5] (see also, [3–5, 15, 20]). Let $x'$ be a fixed point on the manifold $M$. There always exists a sufficiently small neighborhood of the point $x'$ such that every point $x$ in this neighborhood can be connected to the point $x'$ by a unique geodesic. In the following, then, we will restrict ourselves to this neighborhood. Moreover we will denote tensor indices at point $x'$ by primed letters. The non-primed and primed indices will be raised and lowered by the metric at the points $x$ and $x'$, respectively. Also, the derivative with respect to the coordinates of the point $x'$ will be denoted by primed indices as well. We will use the standard notation of square brackets to denote the coincidence limit of two-point functions, more precisely, for any function of $x$ and $x'$ we define

$$[f](x) \equiv \lim_{x \to x'} f(x, x').$$

First of all, we define the world function $\sigma(x, x')$ as one half of the square of the length of the geodesic between the points $x$ and $x'$. Next, we define the first and second derivatives of the world function

$$\sigma_\mu = \nabla_\mu \sigma, \quad \sigma_{\mu'} = \nabla_{\mu'} \sigma, \quad (2.2)$$

and the Van Vleck–Morette determinant

$$\Delta(x, x') = e^{2\xi(x, x')} = g^{-\frac{1}{2}}(x) \det[-\nabla_{\mu} \nabla_{\nu} \sigma(x, x')] g^{-\frac{1}{2}}(x'), \quad (2.4)$$

where $\xi = \log \Delta^{1/2}$. Now, let $\varphi$ be a section of a vector bundle $V$, $\nabla_{\mu}$ be a connection on the vector bundle $V$ and $R_{\mu\nu}$ be the curvature of this connection. Next, let $\mathcal{P}(x, x')$ be the parallel displacement operator of the section $\varphi$ along the geodesic from $x'$ to $x$; obviously,

$$[\nabla_{\mu}, \nabla_{\nu}]\mathcal{P} = R_{\mu\nu}\mathcal{P}. \quad (2.5)$$

Finally, we define the first derivative of the operator of parallel transport

$$E_{\nu} = \mathcal{P}^{-1} \nabla_{\nu} \mathcal{P}. \quad (2.6)$$

One can show that these functions satisfy the equations [3, 5]

$$\sigma = \frac{1}{2} \sigma_\mu \sigma^\mu = \frac{1}{2} \sigma_{\mu'} \sigma^{\mu'}. \quad (2.7)$$
\[\xi_{\mu}^v \sigma^v = \sigma^\mu, \quad \eta^{\mu'}_{v} \sigma^v = \sigma^{\mu'}, \quad \eta^{\mu'}_{v} \sigma_{v} = \sigma_{\nu}, \quad (2.8)\]

\[\sigma^\mu \nabla_\mu \xi = \frac{1}{z} (n - \xi^\mu_{\mu}). \quad (2.9)\]

\[\sigma^\mu \nabla_\mu \mathcal{P} = \sigma^\mu E_\mu = 0, \quad (2.10)\]

and the boundary conditions

\[[\sigma] = [\sigma^{\mu}] = [\sigma^{\mu'}] = 0, \quad (2.11)\]

\[[\xi_{\mu}^\nu] = \delta_{\mu}^\nu, \quad [\eta^{\mu'}_{v}] = -\delta_{\mu}^\nu. \quad (2.12)\]

\[[\Delta] = 1, \quad [\xi] = 0, \quad (2.13)\]

\[[\mathcal{P}] = \mathbb{I}. \quad (2.14)\]

The coincidence limits of higher derivatives of the two-point functions introduced above are expressed in terms of the curvature. In particular, the ones that we will need below are [3, 5, 14, 15]

\[[\xi_{,\mu}] = 0 \quad (2.15)\]

\[[\xi_{,\mu\nu}] = \frac{1}{2} R_{\mu\nu}, \quad (2.16)\]

\[[\eta^{\mu'}_{v,\alpha}] = 0, \quad (2.17)\]

\[[\eta^{\mu'}_{v,\alpha\beta}] = -\frac{1}{3} (R^{\mu}_{v\alpha\beta} + R^{\mu}_{\alpha v\beta}), \quad (2.18)\]

\[[E_{v}] = 0, \quad (2.19)\]

\[[E_{v,\mu}] = -\frac{1}{3} R_{v\mu}, \quad (2.20)\]

where \(R^{\mu}_{\nu\rho\sigma}\) is the Riemann tensor and \(R_{\mu\nu} = R^{\rho}_{\mu\rho\nu}\) is the Ricci tensor.

### 3. Heat kernel

We will use a method for the calculation of the heat kernel developed in [4, 6], which is based on a covariant Fourier transform proposed in [3, 5]. The heat kernel for the operator \(L\) is the kernel of the heat semigroup, that is,

\[U(t|x, x') = \exp(-tL)\mathcal{P}(x, x') \delta(x, x'), \quad (3.1)\]

where \(\delta(x, x')\) is the delta-function (in the density form). Now, following [3] we define the covariant Fourier transform as follows. First, we define

\[\delta(x, x') = \Delta^\frac{1}{2}(x, x') \int_{\mathbb{R}^n} \frac{d\xi}{(2\pi)^n} \exp[t\xi_{\mu}^v \sigma^v(x, x')]. \quad (3.2)\]

Then, by using this representation and the equation

\[\exp(-tL)f = f \exp(-t(f^{-1}Lf)),\]

we obtain

\[U(t|x, x') = \Delta^\frac{1}{2}(x, x')\mathcal{P}(x, x') \int_{\mathbb{R}^n} \frac{d\xi}{(2\pi)^n} \exp[t\xi_{\mu}^v \sigma^v(x, x')] \Phi(t|k, x, x'), \quad (3.3)\]

where

\[\Phi(t|k, x, x') = \exp(-tA) \cdot \mathbb{I}, \quad (3.4)\]
By using the coincidence limits of the two-point functions we obtain the heat kernel diagonal

\[
U(t|x,x) = \int_{\mathbb{R}^n} \frac{\text{d}^n k}{(2\pi)^n} \Phi(t|k,x,x). \tag{3.6}
\]

Now, we represent the operator \( L \) in the form

\[
L = -\rho^{-1} \nabla\mu \rho a^{\mu\nu} \rho \nabla_\nu \rho^{-1} + \bar{Q}, \tag{3.7}
\]

where \( a^{\mu\nu} \) is a matrix-valued symmetric tensor of type \((2, 0)\), \( \rho \) is a matrix-valued density of weight \(1/2\) and \( \bar{Q} \) is a matrix-valued function. By substituting the operator (3.7) in equation (3.5), we get

\[
A = -e^{-i\xi_{\mu}\xi_{\nu}} P^{-1} \Delta^{-\frac{1}{2}} \rho^{-1} \nabla\mu \rho a^{\mu\nu} \rho \nabla_\nu \rho^{-1} \Delta^{\frac{1}{2}} \mathcal{P} e^{i\xi_{\mu}\xi_{\nu}} + \bar{Q}, \tag{3.8}
\]

where \( \bar{Q} = P^{-1}\bar{Q}\mathcal{P} \). We rewrite this operator in a more convenient form

\[
A = -\bar{X}_\mu a^{\mu\nu} X_\nu + \bar{Q}, \tag{3.9}
\]

where

\[
X_\nu = e^{-i\xi_{\mu}\xi_{\nu}} P^{-1} \Delta^{-\frac{1}{2}} \rho^{-1} \nabla_\mu \rho^{-1} \Delta^{\frac{1}{2}} \mathcal{P} e^{i\xi_{\mu}\xi_{\nu}},
\]

\[
\bar{X}_\mu = e^{i\xi_{\mu}\xi_{\nu}} \mathcal{P} \Delta^{-\frac{1}{2}} \rho^{-1} \nabla_\mu \rho^{-1} \Delta^{\frac{1}{2}} e^{-i\xi_{\mu}\xi_{\nu}}. \tag{3.10}
\]

It is useful to introduce, now, two quantities

\[
C_\nu = -\rho_\nu \rho^{-1} \quad \text{and} \quad \bar{C}_\nu = -\rho^{-1} \rho_\nu. \tag{3.11}
\]

Then we get

\[
X_\nu = \nabla_\nu + C_\nu + \xi_\nu + E_\nu + i\xi_\mu \eta^\mu_\nu,
\]

\[
\bar{X}_\mu = \nabla_\nu - \bar{C}_\mu + \xi_\mu + E_\mu + i\xi_\nu \eta^\nu_\mu,
\]

and

\[
A = -(\nabla_\mu - \bar{C}_\mu + \xi_\mu + E_\mu + i\xi_\nu \eta^\nu_\mu) a^{\mu\nu} (\nabla_\nu + C_\nu + \xi_\nu + E_\nu + i\xi_\mu \eta^\mu_\nu) + \bar{Q}. \tag{3.14}
\]

Finally, a straightforward calculation gives

\[
A = H + K + \mathcal{L}. \tag{3.15}
\]

Here

\[
H = \xi_\mu \xi_\nu \eta^\mu_\nu \eta^\nu_\mu a^{\mu\nu},
\]

\[
K = -i\xi_\nu (B^{\nu\mu} \nabla_\mu + G^\nu),
\]

\[
\mathcal{L} = -\tilde{\mathcal{D}}_\mu a^{\mu\nu} \mathcal{D}_\nu + \bar{Q},
\]

where

\[
B^{\nu\mu} = 2\bar{\eta}^\mu_\nu a^{\mu\nu},
\]

\[
G^\nu = a^{\mu\nu} \eta^\nu_\mu + a^{\mu\nu} \eta^\nu_\nu_\nu + \bar{C}_\mu a^{\mu\nu} \eta^\nu_\nu + \eta^\nu_\nu a^{\mu\nu} C_\mu
\]

\[
\quad + E_\mu a^{\mu\nu} \eta^\nu_\nu + \eta^\nu_\nu a^{\mu\nu} E_\mu + 2\xi_\mu a^{\mu\nu} \eta^\nu_\nu,
\]

\[
\tilde{\mathcal{D}}_\nu = \nabla_\nu + \mathcal{A}_\nu = \nabla_\nu - \bar{C}_\nu + \xi_\nu + E_\nu,
\]

\[
\mathcal{D}_\nu = \nabla_\nu + \mathcal{A}_\nu = \nabla_\nu + C_\nu + \xi_\nu + E_\nu,
\]

with \( C_\nu, \bar{C}_\mu, \xi \) and \( E_\mu \) defined in (3.11), (2.4) and (2.6).
More explicitly we can also write that
\[ \mathcal{L} = -a^{\mu\nu} \nabla_\mu \nabla_\nu + \Psi^{\mu} \nabla_\nu + \mathcal{Z}, \] (3.21)
where
\[ \Psi^{\mu} = -a^{\mu\nu}_\tau \phi^{\nu} + \bar{C}_\nu \phi^{\mu} - 2a^{\mu\nu}_\tau \xi_\nu - a^{\mu\nu}_\tau E_\nu - E_\nu a^{\mu\nu}, \] (3.22)
\[ \mathcal{Z} = -a^{\mu\nu}_\tau C_\nu - a^{\mu\nu}_\tau C_\nu + \bar{C}_\mu a^{\mu\nu}_\tau C_\nu - a^{\mu\nu}_\tau \xi_\nu - a^{\mu\nu}_\tau E_\nu + \bar{C}_\mu a^{\mu\nu}_\tau E_\nu - E_\nu a^{\mu\nu}_\tau C_\nu - E_\nu a^{\mu\nu}_\tau E_\nu - \xi_\nu a^{\mu\nu}_\tau E_\nu - \xi_\nu E_\nu a^{\mu\nu}_\tau + Q. \] (3.23)

Thus, by using equation (3.15) we obtain
\[ U(t|x, x) = \int_{\mathbb{R}^n} \frac{d^d \xi}{(2\pi)^d} e^{-(H \phi + \mathcal{L})} \Bigg|_{x=x'}, \] (3.24)
which, by scaling the integration variable \( \xi \rightarrow t^{-\frac{1}{2}} \xi \), takes the form
\[ U(t|x, x) = (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^n} \frac{d^d \xi}{(2\pi)^d} e^{-(\bar{H} \phi + \mathcal{L})} \Bigg|_{x=x'}, \] (3.25)
It is convenient, to rewrite this equation as
\[ U(t|x, x) = (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^n} \frac{d^d \xi}{(2\pi)^d} e^{-|\xi|^2} \exp(-\bar{H} - \sqrt{t} \mathcal{K} - t \mathcal{L}) \Bigg|_{x=x'}, \] (3.26)
where \( |\xi|^2 = g^{\mu\nu} \xi_\mu \xi_\nu \) and
\[ \bar{H} = H - |\xi|^2. \] (3.27)

In order to evaluate the first three coefficients of the asymptotic expansion of (3.26) as \( t \rightarrow 0 \) we use the Volterra series for the exponent of a sum of two non-commuting operators
\[ e^{A+B} = e^A + \sum_{k=1}^{\infty} \prod_{l=0}^{k-1} d\tau_l \int_{0}^{\tau_l} d\tau_{k-l} \cdots \int_{0}^{\tau_1} d\tau_0 e^{(1-\tau_l)A} B e^{(1-\tau_{k-l})A} \cdots e^{(1-\tau_0)A} B e^{\tau_0 A}. \] (3.28)

We obtain
\[ \exp(-\bar{H} - \sqrt{t} \mathcal{K} - t \mathcal{L}) = e^{-\bar{H}} - \sqrt{t} \Omega + t \Psi + O(t^{\frac{3}{2}}), \] (3.29)
where
\[ \Omega = \int_{0}^{1} d\tau_1 e^{-(1-\tau_0)\bar{H}} K e^{-\tau_0 \bar{H}}, \] (3.30)
and
\[ \Psi = \int_{0}^{1} d\tau_2 \int_{0}^{\tau_2} d\tau_1 e^{-(1-\tau_0)\bar{H}} K e^{-(1-\tau_1)\bar{H}} K e^{-\tau_1 \bar{H}} - \int_{0}^{1} d\tau_3 e^{-(1-\tau_2)\bar{H}} L e^{-\tau_2 \bar{H}}. \] (3.31)

We are only interested in the terms \( a_0 \) and \( a_1 \) of the heat kernel expansion, namely the terms of zero order and linear in the parameter \( t \). These terms can be written, respectively, as
\[ a_0 = g^{\frac{1}{2}} \tilde{a}_0, \] (3.32)
\[ a_1 = g^{\frac{1}{2}} \tilde{a}_1, \] (3.33)
where
\[ \tilde{a}_0 = \int_{\mathbb{R}^n} \frac{d^d \xi}{(2\pi)^d} g^{\frac{1}{2}} e^{-|\xi|^2} \exp(-\bar{H}) \Bigg|_{x=x'}, \] (3.34)
\[ \tilde{a}_1 = \int_{\mathbb{R}^n} \frac{d\xi}{\pi^{\frac{n}{2}}} g^{-\frac{1}{2}} e^{-|\xi|^2} \psi \cdot \bigg|_{x=x'} \cdot \bigg|_{x=x'}. \quad (3.35) \]

The term \( t_1 \) of the heat kernel expansion vanishes identically. This happens because the heat kernel coefficients are defined as \( \xi \)-integrals over the whole \( \mathbb{R}^n \) and the term \( \Omega \) in (3.30) is an odd function of \( \xi \).

4. Evaluation of the heat kernel coefficients

4.1. Local coefficient \( \tilde{a}_0 \)

We will evaluate the heat kernel coefficients \( A_0 \) and \( A_1 \) using the perturbation theory. The main idea is to introduce a small deformation parameter \( \lambda \) and evaluate the non-commutative corrections to the action of spectral matrix gravity. For this purpose we write the matrix \( a_{\mu\nu} \) as

\[ a_{\mu\nu} = g_{\mu\nu} I + \lambda h_{\mu\nu}, \quad (4.1) \]

where \( h_{\mu\nu} \) is a traceless matrix-valued tensor field (a non-commutative perturbation of the Riemannian metric), satisfying

\[ \text{tr}_V h_{\mu\nu} = 0. \quad (4.2) \]

Furthermore, we parametrize the matrix-valued density \( \rho \) introduced in (3.7) as

\[ \rho = g^{-\frac{1}{4}} e^\phi e^{\lambda \sigma}. \quad (4.3) \]

Here \( \sigma \) is a traceless matrix-valued scalar field and \( \phi \) is a scalar field. (Do not confuse it with the world function introduced in the previous sections!) Finally, we also decompose the endomorphism \( Q \),

\[ Q = q \cdot I + \lambda \Theta, \quad (4.4) \]

where \( \Theta \) is a traceless matrix-valued scalar field.

Now we expand all the quantities in powers of \( \lambda \). On doing so the matrix \( \rho \) and its inverse read

\[ \rho = g^{\frac{1}{4}} e^\phi \left( 1 + \lambda \sigma + \frac{\lambda^2}{2} \sigma_2 \right) + O(\lambda^3), \]

\[ \rho^{-1} = g^{-\frac{1}{4}} e^{-\phi} \left( 1 - \lambda \sigma + \frac{\lambda^2}{2} \sigma_2 \right) + O(\lambda^3), \quad (4.5) \]

and its derivative is

\[ g^{-\frac{1}{4}} \rho_{\mu\nu} = e^\phi \left[ \lambda \sigma_{\mu\nu} + \frac{\lambda^2}{2} \left( \sigma_{\mu\nu} + \sigma \sigma_{\mu\nu} \right) \right] + e^\phi \phi_{\mu\nu} \left( 1 + \lambda \sigma + \frac{\lambda^2}{2} \sigma_2 \right) + O(\lambda^3). \quad (4.6) \]

From the last two expressions one can easily evaluate the operators \( C_\nu \) and \( \tilde{C}_\nu \) obtaining explicitly

\[ C_\nu = -\rho_{\nu\rho}^{-1} = -\phi_{\nu\rho} - \lambda \sigma_{\nu\rho} + \frac{\lambda^2}{2} \left[ \sigma_{\nu\rho}, \sigma \right] + O(\lambda^3), \quad (4.7) \]

\[ \tilde{C}_\nu = -\rho^{-1}_{\nu\rho} = -\phi_{\nu\rho} - \lambda \sigma_{\nu\rho} - \frac{\lambda^2}{2} \left[ \sigma_{\nu\rho}, \sigma \right] + O(\lambda^3). \quad (4.8) \]
The operators $\hat{H}$, $K$ and $\mathcal{L}$ introduced above in (3.16), (3.17) and (3.18) depend on the deformation parameter $\lambda$ as well. By expanding them in terms of the deformation parameter we get

\begin{align}
\hat{H} &= H_0 + \lambda H_1,
K &= K_0 + \lambda K_1 + \lambda^2 K_2 + O(\lambda^3), \quad (4.9)
\mathcal{L} &= \mathcal{L}_0 + \lambda \mathcal{L}_1 + \lambda^2 \mathcal{L}_2 + O(\lambda^3),
\end{align}

where

\begin{align}
H_0 &= \xi_\nu \xi_\mu (\eta^{\nu}_{\mu} \eta^{\rho}_{\sigma} g^{\alpha\nu} - g^{\alpha\rho}), \\
H_1 &= \xi_\nu \xi_\mu \eta^{\nu}_{\mu} \eta^{\rho}_{\sigma} h^{\alpha\rho}, \\
K_0 &= -t \xi_\nu (2\eta^{\nu}_{\mu} b^{\mu\nu} \nabla_v + 2\eta^{\nu}_{\mu} b^{\mu\nu} \xi_\nu + \eta^{\nu}_{\mu} j^{\nu}_{\mu} + E^{\mu} \eta^{\nu}_{\mu} + \eta^{\nu}_{\mu} E^{\mu}), \\
K_1 &= -t \xi_\nu (2\eta^{\nu}_{\mu} h^{\mu\nu} \nabla_v + \eta^{\nu}_{\mu} h^{\mu\nu} \xi_\nu + h^{\mu\nu} \eta^{\nu}_{\mu} \xi_\nu + \eta^{\nu}_{\mu} h^{\mu\nu} E_v + \eta^{\nu}_{\mu} E_v h^{\mu\nu}), \\
K_2 &= -t \xi_\nu [\eta^{\nu}_{\mu} (f_{\mu\nu} + [\sigma_{\mu\nu}, \sigma])] + [\sigma^{\mu\nu}, \sigma], \\
L_0 &= -\nabla_v^2 - (2\eta^{\nu}_{\mu} + 2E^{\mu}) \nabla_v + \phi_{\mu\nu} \phi^{\mu\nu} - \xi_{\mu\nu}^{\mu\nu} - \xi_{\mu\nu}^{\mu\nu} - E_{\mu\nu} - E_{\mu\nu} E_v - 2\xi_{\mu\nu} E_v + q, \\
L_1 &= -h^{\mu\nu} \nabla_v \nabla_v - \left[ h^{\mu\nu} \nabla_v + (h^{\mu\nu} E_v + E_v h^{\mu\nu}) \nabla_v + \sigma_{\mu\nu}^{\mu\nu} + h^{\mu\nu} \phi_{\mu\nu} + h^{\mu\nu} \phi_{\mu\nu} + 2\phi_{\mu\nu} \sigma_{\mu\nu} + \phi_{\mu\nu} h^{\mu\nu} \phi_{\mu\nu} - h^{\mu\nu} \xi_\mu \xi_\nu - h^{\mu\nu} \xi_\nu h^{\mu\nu} \xi_\nu - h^{\mu\nu} \phi_{\mu\nu} \xi_\mu \xi_\nu - h^{\mu\nu} \phi_{\mu\nu} \xi_\nu h^{\mu\nu} \xi_\nu - \xi_\mu h^{\mu\nu} E_v - \xi_\nu h^{\mu\nu} E_v + \Theta, \\
L_2 &= (h^{\mu\nu}, \sigma_{\mu\nu}) - [\sigma_{\mu\nu}, \sigma] \nabla_v - \frac{1}{2} (\sigma_{\mu\nu}, \sigma) + h^{\mu\nu} \sigma_{\mu\nu} + h^{\mu\nu} \sigma_{\mu\nu} + \sigma_{\mu\nu} \sigma_{\mu\nu} + \sigma_{\mu\nu} h^{\mu\nu} \phi_{\mu\nu} + \phi_{\mu\nu} h^{\mu\nu} \sigma_{\mu\nu}, \\
&\quad + \sigma_{\mu\nu} h^{\mu\nu} \sigma_{\mu\nu} - \sigma_{\mu\nu} h^{\mu\nu} \xi_\mu - \frac{1}{2} (\sigma_{\mu\nu}, \sigma) E_{\mu\nu} + E^\eta (\sigma_{\mu\nu}, \sigma) \\
&\quad + E_{\mu\nu} h^{\mu\nu} \sigma_{\mu\nu} + h^{\mu\nu} \sigma_{\mu\nu} E_{\mu\nu} + \sigma_{\mu\nu} h^{\mu\nu} E_{\mu\nu}. \\
\end{align}

In the framework of perturbation theory we write, then, the coefficients $\bar{a}_0$ and $\bar{a}_1$ of the heat kernel expansion in (3.34) and (3.35) in terms of the deformation parameter $\lambda$, namely

\begin{align}
\bar{a}_0(\lambda) &= a_0^{(0)} + \lambda a_0^{(1)} + \lambda^2 a_0^{(2)} + O(\lambda^3), \\
\bar{a}_1(\lambda) &= a_1^{(0)} + \lambda a_1^{(1)} + \lambda^2 a_1^{(2)} + O(\lambda^3). \\
\end{align}

By using the explicit formulae obtained in (4.10) through (4.17), we will be able to evaluate all the coefficients of the Taylor expansions in (4.18).

Next, we introduce a notation that will be useful in the following calculations. Let $f$ be a function of $\xi$. We define the Gaussian average of the function $f$ as

\begin{equation}
\langle f \rangle = \int_{\mathbb{R}^n} \frac{d\xi}{\pi^2} e^{-|\xi|^2} f(\xi).
\end{equation}

The Gaussian averages of the polynomials is well known

\begin{align}
\langle \xi_{\mu_1} \cdots \xi_{\mu_{2n+1}} \rangle &= 0, \\
\langle \xi_{\mu_1} \cdots \xi_{\mu_{2n}} \rangle &= (2n)! \sum_{\permutations{2n}{n}} g_{\mu_1 \mu_2} \cdots g_{\mu_{2n-1} \mu_{2n}},
\end{align}

where the parentheses () denote the symmetrization over all the included indices.
For the coefficient of order zero of the heat kernel expansion we consider the first equation in (4.18). From formula (3.34), it is easy to see that the only non-vanishing contribution to \( \tilde{a}_0 \) is

\[
\tilde{a}_0 = \left( 1 - \lambda N + \frac{\lambda^2}{2} N^2 \right) + O(\lambda^3).
\]  

(4.21)

By using equations (4.10), (4.11) and after taking the coincidence limit we obtain the expression

\[
\tilde{a}_0 = \left( 1 - \lambda h^\mu_\nu \xi_\mu \xi_\nu + \frac{\lambda^2}{2} h^\mu_\nu h^\sigma_\mu h_\sigma^\nu \right) + O(\lambda^3),
\]

(4.22)

and then, by performing the Gaussian averages, we get

\[
\tilde{a}_0 = 1 - \frac{\lambda}{2} h + \frac{\lambda^2}{8} (h^2 + 2 h^\mu_\nu h_\mu^\nu) + O(\lambda^3),
\]

(4.23)

where \( h = g_\mu^\nu h^\mu_\nu \). In order to evaluate the global coefficient \( A_0 \), given in (1.9), we need the trace of (4.23). Since \( h^\mu_\nu \) is traceless we immediately obtain

\[
\text{tr}_V \tilde{a}_0 = \text{tr}_V \left( 1 + \frac{\lambda^2}{8} h^2 + \frac{\lambda^2}{4} h^\mu_\nu h_\mu^\nu \right) + O(\lambda^3).
\]

(4.24)

4.2. Local coefficient \( \tilde{a}_1 \)

Now we evaluate the coefficient \( \tilde{a}_1 \). By using expressions (3.35), (3.31) and (3.18) we have

\[
\text{tr}_V \tilde{a}_1 = \left\{ \left[ \text{tr}_V [ - e^{-\tilde{H}} Q ] \right] + \left[ \text{tr}_V \left[ \int_0^1 \int_0^{t_2} \int_0^{t_1} e^{-(1-t_2)\tilde{H}} K e^{-(t_2-t_3)\tilde{H}} K e^{-t_3\tilde{H}} \right] \right] \right\}_{t_1 = t_2'}. 
\]

(4.25)

where the first term of expression (4.25) has been obtained by simply using the cyclic property of the trace.

In the following we will evaluate the terms in (4.25) separately. We start with the simplest of them, namely the one involving the endomorphism \( Q \). By using the Taylor expansion in \( \lambda \) of \( \tilde{H} \) in (4.9) and the coincidence limits (2.12) we obtain

\[
\text{tr}_V [ - e^{-\tilde{H}} Q ] = \left. \text{tr}_V \left( -Q + \lambda \xi_\mu \xi_\nu h^\mu_\nu Q - \frac{\lambda^2}{2} \xi_\mu \xi_\nu \xi_\sigma h^\mu_\nu h^\sigma_\nu Q \right) \right|_{t_1 = t_2'}. + O(\lambda^3).
\]

(4.26)

By expanding \( Q \) as in (4.4) and by performing the Gaussian averages we obtain

\[
\text{tr}_V [ - e^{-\tilde{H}} Q ] = -N q + \lambda^2 \text{tr}_V \left( \frac{1}{2} h \Theta - \frac{1}{2} h^2 q - \frac{1}{2} h^\mu_\nu h_\mu^\nu q \right) + O(\lambda^3),
\]

(4.27)

where we used the property (4.2).

For the second term in equation (4.25) we get, by using the definition (3.17),

\[
\left. \text{tr}_V \left[ \int_0^1 \int_0^{t_2} \int_0^{t_1} e^{-(1-t_2)\tilde{H}} K e^{-(t_2-t_3)\tilde{H}} K e^{-t_3\tilde{H}} \right] \right|_{t_1 = t_2'} = \left. \left( \int_0^{t_2} \int_0^{t_1} \int_0^{t_2} \int_0^{t_1} \int_0^{t_3} \int_0^{t_2} \int_0^{t_1} e^{-(1-t_2)\tilde{H}} B^\sigma e^{-(t_2-t_3)\tilde{H}} B^\rho e^{-t_3\tilde{H}} \right) \right|_{t_1 = t_2'}.
\]

(4.28)
It is straightforward to notice that in the last expression we need to compute first and second derivatives of the exponentials containing the operator $\hat{H}$. These derivatives are computed by using integral representations, i.e. for the first derivative we have \[10\]

$$\nabla_\mu e^{-s\hat{H}} = -\beta_\mu(\tau) e^{-s\hat{H}},$$

where

$$\beta_\mu(\tau) = \int_0^\tau ds e^{-s\hat{H}} \hat{H}_{;\mu} e^{s\hat{H}}.$$  \tag{4.30}$$

This last integral can be evaluated by referring to the following formula and by integrating over $s$ \[10\]

$$e^{-s\hat{H}} \hat{H}_{;\mu} e^{s\hat{H}} = \sum_{k=0}^\infty \frac{(-1)^k}{k!} [\hat{H}, \hat{H}_{;\mu}] \cdots [\hat{H}, \hat{H}_{;\mu}] \cdots .$$ \tag{4.31}$$

By expanding (4.30) in $s$, up to the second order in $\lambda$, we obtain

$$\nabla_\mu e^{-s\hat{H}} = - (\tau \hat{H};_{\mu\nu} + \frac{1}{2}\tau^2[\hat{H};_{\mu\nu}, \hat{H}]) e^{-s\hat{H}} + O(\lambda^3).$$ \tag{4.32}$$

This last expression can be obtained by recalling that the coincidence limit for $\hat{H}$ and its derivatives is of order $\lambda$ without the zeroth-order term (see appendix A). For the second derivative we write \[10\]

$$\nabla_\mu \nabla_\nu e^{-s\hat{H}} = - \int_0^\tau ds_1 e^{-s_1\hat{H}} \hat{H}_{;\mu\nu} e^{-s_1\hat{H}} + \int_0^\tau ds_2 \int_0^{s_2} ds_1 (e^{-s_2-s_1\hat{H}} \hat{H}_{;\nu}, e^{-s_1\hat{H}} \hat{H}_{;\mu} e^{-(s_2-s_1)\hat{H}} + e^{-s_2-s_1\hat{H}} \hat{H}_{;\mu} e^{-(s_2-s_1)\hat{H}} \hat{H}_{;\nu} e^{-s_1\hat{H}}).$$ \tag{4.33}$$

We can express this formula in the same form as (4.32), i.e.

$$\nabla_\mu \nabla_\nu e^{-s\hat{H}} = - (\tau \hat{H};_{\mu\nu\nu} + \frac{1}{2}\tau^2[\hat{H};_{\mu\nu\nu}, \hat{H}] - \frac{1}{2}\tau^2[\hat{H};_{\mu\nu}, \hat{H}_{;\mu}]) e^{-s\hat{H}} + O(\lambda^3).$$ \tag{4.34}$$

Now that we have expressions (4.29)–(4.34), we can substitute them in (4.28) and we can expand the remaining exponentials in $\tau$ up to orders $\lambda^2$. After the expansion of the exponentials and after taking the coincidence limit, we have to evaluate the double integrals of polynomials in $\tau_1$ and $\tau_2$ which will yield the numerical coefficients for the various terms in (4.28). The most general double integral that we need to evaluate is the following:

$$I_1(\alpha, \beta, \gamma, \delta) = \int_0^1 d\tau_2 \int_0^{\tau_1} d\tau_1 \tau_1^\alpha (1 - \tau_2)^\beta (\tau_2 - \tau_1)^\gamma (1 - \tau_2 + \tau_1)^\delta,$$ \tag{4.35}$$

with $(\alpha, \beta, \gamma, \delta)$ being positive integers so that the integral is well defined. The solution in closed form of (4.35) is (see appendix B)

$$I_1(\alpha, \beta, \gamma, \delta) = \frac{\Gamma(1 + \alpha)\Gamma(1 + \beta)\Gamma(1 + \gamma)\Gamma(2 + \alpha + \beta + \delta)}{\Gamma(3 + \alpha + \beta + \gamma + \delta)\Gamma(2 + \alpha + \beta)},$$ \tag{4.36}$$

where $\Gamma(x)$ is the Euler gamma function.

By using the technical details described above and the coincidence limits for the various terms in (4.28) (see appendix A), we obtain

$$\left\langle \text{tr}_Y \left[ \int_0^1 d\tau_2 \int_0^{\tau_1} d\tau_1 e^{-(1-\tau_2)\hat{H}} \hat{K} e^{-(\tau_2-\tau_1)\hat{H}} K e^{-\tau_1\hat{H}} \right] \right|_{x=x'} = \text{tr}_Y (\Omega_0) + \lambda \text{tr}_Y (\Omega_1) + \lambda^2 \text{tr}_Y (\Omega_2) + O(\lambda^3),$$ \tag{4.37}$$
where

\[ \Omega_0 = \frac{1}{6} R, \quad (4.38) \]

\[ \Omega_1 = \frac{1}{6} h^{\mu\nu;\alpha} x_1 - \frac{1}{6} h^{\mu\nu;\mu\nu} - \frac{1}{12} h R + \frac{1}{6} h^{\mu\nu} R_{\mu\nu}, \quad (4.39) \]

\[ \Omega_2 = -\frac{1}{16} h_{\mu;\nu} h^{\mu\nu} + \frac{1}{6} h^{\mu\nu;\mu\nu} h_{\nu,\nu} - \frac{1}{6} h^{\mu\nu;\mu\nu} h_{\nu,\nu} - \frac{1}{12} h h^{\mu\nu} - \frac{1}{12} h h^{\mu\nu} + \frac{1}{2} h^{\mu\nu} h_{\mu;\nu} + \frac{1}{2} h^{\mu\nu} h_{\mu;\nu} - \frac{1}{12} h^{\mu\nu} R_{\mu\nu} \]
\[ + \frac{1}{2} h^{\mu\nu} h_{\mu;\nu} + \frac{1}{2} h^{\mu\nu} h_{\mu;\nu} + \frac{1}{12} h^{\mu\nu} R_{\mu\nu} + \frac{1}{12} h^{\mu\nu} h_{\mu;\nu}, \quad (4.40) \]

We can finally evaluate the last term in equation (4.25). By using the definitions (3.18) and (3.20) we can write that

\[
\left\{ \text{tr}_V \left[ \int_0^1 d\tau_1 e^{-1/\tau_1} \bar{D}_\mu a^{\mu\nu} \bar{D}_\nu e^{-1/\tau_1} \right] \right\}_{\lambda = \tau'}
\]
\[ = \left\{ \text{tr}_V \left[ \int_0^1 d\tau_1 e^{-1/\tau_1} \left( a^{\mu\nu;\mu} \nabla_v + a^{\mu\nu;\mu} \nabla_v + a^{\mu\nu;\mu} \nabla_v \right) + a^{\mu\nu} \nabla_{\mu,\nu} + a^{\mu\nu} \nabla_{\nu,\mu} + a^{\mu\nu} \nabla_{\nu,\mu} \right] \right\}_{\lambda = \tau'}. \quad (4.41)\]

In order to evaluate this term we use the derivatives in (4.32) and (4.34) and we expand the remaining exponentials of \( \bar{H} \) in \( \tau \) up to terms in \( \lambda^2 \). During the calculation the numerical coefficients of the various terms can be evaluated by referring to the following general integral

\[ I_2(\alpha, \beta) = \int_0^1 d\tau_1 \tau_1^{\alpha} (1 - \tau_1)^\beta, \quad (4.42) \]

where \((\alpha, \beta)\) are positive integers, and for which the general solution in closed form is (see appendix B)

\[ I_2(\alpha, \beta) = \frac{\Gamma(1 + \alpha) \Gamma(1 + \beta)}{\Gamma(2 + \alpha + \beta)}. \quad (4.43) \]

The explicit form of (4.41) can be obtained with the help of (4.43) and the coincidence limits in appendix A. After a straightforward calculation one gets

\[ -\left\{ \text{tr}_V \left[ \int_0^1 d\tau_1 e^{-1/\tau_1} \bar{D}_\mu a^{\mu\nu} \bar{D}_\nu e^{-1/\tau_1} \right] \right\}_{\lambda = \tau'} \]
\[ = \text{tr}_V (\Sigma_0) + \lambda \text{tr}_V (\Sigma_1) + \lambda^2 \text{tr}_V (\Sigma_2) + O(\lambda^3), \quad (4.44) \]

where

\[ \Sigma_0 = -\phi_{\mu,\nu} - \phi_{\mu}, \phi^{\mu} , \quad (4.45) \]

\[ \Sigma_1 = -\phi_{\mu,\nu} - h^{\mu\nu;\mu} \phi_{\mu} - h^{\mu\nu;\mu} \phi_{\mu} - 2\phi^{\mu;\nu} \phi_{\nu} - \phi_{\mu,\nu} h^{\mu\nu} \phi_{\mu} \]
\[ + \frac{1}{2} h (\phi_{\mu,\nu} + \phi_{\nu,\mu}) - \frac{1}{2} h^{\mu\nu} \phi_{\mu}, \quad (4.46) \]

\[ \Sigma_2 = \frac{1}{8} h h^{\mu\nu;\mu} + \frac{1}{2} h^{\mu\nu;\mu} h_{\nu,\nu} + \frac{1}{2} h_{\mu;\nu} h^{\mu\nu} + \frac{1}{2} h_{\mu;\nu} h^{\mu\nu} - \frac{1}{2} h^{\mu\nu} h_{\nu,\nu} \]
\[ - \frac{1}{2} h^{\mu\nu;\mu} h_{\nu,\nu} - h^{\mu\nu;\mu} \phi_{\nu} = h^{\mu\nu;\nu} \phi_{\nu} - \phi_{\mu,\nu} h^{\mu\nu} - \frac{1}{2} h_{\mu;\nu} \phi^{\mu;\nu} - \frac{1}{2} h_{\mu;\nu} \phi^{\mu;\nu} - \frac{1}{2} h_{\mu;\nu} \phi^{\mu;\nu} \]
\[ + \frac{1}{2} h \phi_{\mu,\nu} + \frac{1}{2} h h^{\mu\nu} \phi_{\mu,\nu} - h^{\mu\nu} \phi_{\mu,\nu} - \frac{1}{2} h^2 \phi_{\mu,\nu} - \frac{1}{2} h^2 \phi_{\mu,\nu} \phi_{\mu,\nu} \]
\[ + h \phi_{\mu,\nu} \phi_{\mu,\nu} - \frac{1}{2} h^{\mu\nu} \phi_{\mu,\nu} - 2 h^{\mu\nu} \phi_{\mu,\nu}. \quad (4.47) \]
In the notation of equation (4.18) we can write, now, the different contributions, in increasing order of $\lambda$, to the coefficient $\tilde{a}_1$. In more detail, by using the results (4.27), (4.38), (4.45) and recalling that
\[
\tilde{a}_1(\lambda) = a_1^{(0)} + \lambda a_1^{(1)} + \lambda^2 a_1^{(2)} + O(\lambda^3),
\]
we get
\[
a_1^{(0)} = \frac{1}{6}R - \phi_{\mu\nu}\phi_{\mu\nu} - \phi_{\mu\nu}\phi_{\mu\nu} - q. \tag{4.48}
\]
Moreover, by using (4.39) and (4.46) we obtain
\[
a_1^{(1)} = \frac{1}{2}\sigma^{\mu\nu} - h^{\mu\nu}\phi_{\mu\nu} \sigma^{\nu} - 2\phi^{\mu\nu}\phi_{\mu\nu} + \frac{1}{12}h_{\mu\nu}h_{\mu\nu} + \frac{1}{48}h_{\mu\nu}h_{\mu\nu} + \frac{1}{16}h_{\mu\nu}h_{\mu\nu} + \frac{1}{8}h_{\mu\nu}h_{\mu\nu} + \frac{1}{48}h_{\mu\nu}h_{\mu\nu}, \tag{4.49}
\]
Finally, by combining the results in (4.27), (4.40) and (4.47), we have the following expression for the term of order $\lambda^2$ in $\tilde{a}_1$, i.e.
\[
a_1^{(2)} = -\frac{1}{2}\lambda^2 R + \frac{1}{12}h_{\mu\nu}h_{\mu\nu} + \frac{1}{24}h_{\mu\nu}h_{\mu\nu} + \frac{1}{8}h_{\mu\nu}h_{\mu\nu} + \frac{1}{48}h_{\mu\nu}h_{\mu\nu} + \frac{1}{8}h_{\mu\nu}h_{\mu\nu}, \tag{4.50}
\]
\section{5. Construction of the action}

In order to write the action of spectral matrix gravity, we need to evaluate the global heat kernel coefficients $A_0$ and $A_1$. As we already mentioned above, the coefficients $A_k$ are expressed in terms of integrals of the local heat kernel coefficients $a_k$ (which are densities) or the coefficients $\tilde{a}_k$ (which are scalars). Namely,
\[
A_k = \int_M dx g^{-1} trV(\tilde{a}_k). \tag{5.1}
\]
By using equation (4.24), we get
\[
A_0 = \int_M dx g^{-1} trV \left[ \frac{\lambda^2}{8}(h^2 + 2h_{\mu\nu}h^{\mu\nu}) \right] + O(\lambda^3). \tag{5.2}
\]
Now we use equations (4.18) and (4.48)–(4.50) to compute the coefficient $A_1$. By integrating by parts and by noticing that the trace of a commutator of any two matrices vanishes, up to terms of order $\lambda^2$, we obtain
\[
A_1 = \int_M dx g^{-1} trV \left[ -q - \phi_{\mu\nu}\phi_{\mu\nu} + \frac{1}{6}R + \lambda^2 \left( -\sigma^{\mu\nu}\sigma_{\mu\nu} + \frac{1}{2}h_{\theta} \right) - \frac{1}{2}h_{\mu\nu}R_{\mu\nu} + \frac{1}{6}h_{\mu\nu}h_{\mu\nu} - \frac{1}{12}h_{\mu\nu}h_{\mu\nu} + \frac{1}{24}h_{\mu\nu}h_{\mu\nu} + \frac{1}{8}h_{\mu\nu}h_{\mu\nu} + \frac{1}{48}h_{\mu\nu}h_{\mu\nu} \right] + O(\lambda^3), \tag{5.3}
\]
The invariant action functional is written as linear combination of the coefficients $A_0$ and $A_1$ as shown in (1.10)

$$S = \frac{1}{16\pi G} \int_M dx \, g^{\frac{3}{2}} \left\{ -6q - 6\phi^{;\mu\nu} \phi_{;\mu\nu} + R - 2\Lambda \right. $$

$$+ \frac{\lambda^2}{N} \text{tr} \left( -6\sigma^{;\mu} \sigma_{;\mu} - 3h^{;\mu} \sigma_{;\mu} + 3h\Theta + 3hh^{\mu\nu} \phi_{;\mu \nu} + 6h\sigma^{;\nu} \phi_{;\nu} \right)$$

$$- 3h^{\mu\nu};h_{;\mu \nu} - 12\sigma_{;\nu}h^{\mu\nu};\phi_{;\mu} - \frac{1}{2}h^{\mu\nu}R_{\mu \nu} + \frac{1}{2}h^{\mu\nu}h^{\sigma\rho}R_{\sigma \mu \rho \nu} - \frac{1}{2}h^\nu h^\rho h^\mu R_{\mu \nu \rho \nu}$$

$$- \frac{1}{2}h^{\mu\nu};h_{;\nu} + \frac{1}{2}h^{\mu\nu};h_{;\rho \mu \nu} + \frac{1}{8}h^2 R + \frac{1}{4}h^{\mu\nu}h_{\mu \nu} R - \frac{1}{8}h_{;\mu \mu}h_{;\nu \nu} - \frac{1}{4}h^{\mu\nu;\rho}h_{\mu \nu \rho}$$

$$- \frac{3}{4}h^{\mu\nu}h_{\mu \nu} - \frac{3}{4}h_2^{\mu \phi;\mu \tau} - \frac{3}{4}h_{;\mu \mu}h_{;\nu \nu} + \frac{3}{2}h^{\mu\nu}h_{\mu \nu} \phi_{;\rho\beta}$$

$$\left. - \frac{3}{2}h^{\mu\nu}h_{\mu \nu \phi} - \frac{\Lambda^2}{4} h^2 - \frac{\Lambda}{2} h^{\mu \nu} \phi_{;\rho} \right\} + O(\lambda^3). \quad (5.4)$$

It is straightforward to show that the action functional that we obtained is invariant under the diffeomorphisms and the gauge transformation $h^{\mu\nu} \rightarrow Uh^{\mu\nu}U^{-1}$.

The next task is to find the equations of motion for the fields $\sigma$, $h_{\mu \nu}$ and $\phi$ by varying the action functional. In this way we will explicitly find the noncommutative corrections to Einstein’s equations.

6. The equations of motion

By performing the variation with respect to the field $\sigma$, we obtain the equation

$$4 \Box \sigma + \Box h - 2h \Box \phi - 2h^{\mu\nu} \phi_{;\mu \nu} + 4h^{\mu\nu};\phi_{;\mu \nu} + 4h^{\mu\nu} \phi_{;\mu \nu} + O(\lambda^3) = 0. \quad (6.1)$$

Here $\Box$ is the Laplacian in the Euclidean case and the D’Alembertian in the pseudo-Euclidean case. For the matrix-valued field $h^{\mu\nu}$ we obtain the equation

$$g^{\mu\nu} \Box \sigma + g^{\mu\nu} \Theta + h^{;\mu \nu};\phi_{;\mu \nu} + 2g^{\mu\nu} \sigma^{;\mu \nu} \phi_{;\rho} + 2g^{\mu\nu} \sigma^{;\rho} \phi_{;\nu}$$

$$- h^{;\mu \nu};\phi_{;\nu} - 4\sigma^{;\mu \nu};\phi_{;\nu} + g^{\mu\nu}h^{\rho \sigma};\phi_{;\rho \sigma} + g^{\mu\nu}h^{\rho \sigma} \phi_{;\rho \sigma} - \frac{1}{6}g^{\mu\nu}h^{\rho \sigma} R_{\rho \sigma}$$

$$- \frac{1}{6}h R^{\mu \nu} + \frac{1}{6}h^{\rho \sigma} R_{\rho \sigma}^{\mu \nu} + \frac{1}{3}h^{\rho (\mu} R^{\nu \rho)} + \frac{1}{6}h^{\rho(\mu} R_{\nu \rho)}^{\nu} + \frac{1}{6}g^{\mu \nu}h^{\rho \sigma};\phi_{;\rho \sigma} - \frac{1}{3}h^{(\mu | \rho^{(\nu \rho)}|)}$$

$$+ \frac{1}{12}g^{\mu \nu}h R_{\mu \nu} + \frac{1}{12}g^{\mu \nu} \Box h + \frac{1}{12}h^{\mu \nu} \Box \theta + \frac{1}{12}g^{\mu \nu} \Box h_{\mu \nu} + \frac{1}{2}g^{\mu \nu} h q - \frac{1}{2}g^{\mu \nu} h \Box \phi$$

$$- \frac{1}{2}g^{\mu \nu} h \phi_{;\mu \nu} - \frac{\Lambda}{3} h^{\mu \nu} + O(\lambda^3) = 0. \quad (6.2)$$

The variation of the action with respect to the scalar field $\phi$ yields

$$4 \Box \phi = - \frac{\lambda^2}{N} \text{tr} \left( -2h h^{\mu \nu} \phi_{;\mu \nu} - 2h_{;\mu \nu} h^{\mu \nu};\phi_{;\mu \nu} - 2h^{\mu \nu};h_{;\mu \nu} \phi_{;\nu} \right)$$

$$- 2h_{;\mu \nu} \sigma^{;\nu} - 2h \Box \sigma + h^{\mu \nu};h_{;\mu \nu} + h^{\mu \nu} h_{;\mu \nu} + 4\sigma_{;\mu \nu} h^{\mu \nu}$$

$$+ 4\sigma_{;\mu \nu} h^{\mu \nu} + h_{;\mu \nu} h^{\mu \nu} + h_{;\mu \nu} \phi_{;\nu} + h^{\mu \nu} \Box \phi - h^{\mu \nu} \Box h_{\mu \nu}$$

$$- h_{;\mu \nu} h^{\mu \nu};\phi_{;\rho} + 2h_{\mu \nu} h^{\mu \nu};\phi_{;\rho} + 2h^{\mu \nu} h_{\mu \nu} \Box \phi \right) + O(\lambda^3). \quad (6.3)$$
The equation of motion for the field $g^{\mu\nu}$ can be written in the following form

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} = T^{\mu\nu} + \frac{\lambda^2}{N} \text{tr}_{V} \mathcal{A}^{\mu\nu}. \quad (6.4)$$

Here the tensor $T^{\mu\nu}$ is

$$T^{\mu\nu} = 6 \phi^{(\mu} \phi^{\nu)} - 3 g^{\mu\nu} \phi_{\rho} \phi^{\rho} - 3 q g^{\mu\nu},$$

which represents the stress–energy tensor for a massless scalar field. The tensor $\mathcal{A}^{\mu\nu}$ represents, instead, the stress–energy tensor for the fields $h^{\mu\nu}$ and $\sigma$.

The equation (6.4) is the main result of our paper. As we can see, the new fields of our model, $h^{\mu\nu}$ and $\sigma$, contribute to modify the standard Einstein equations. More precisely they contribute to an additional term in the stress–energy tensor.

The tensor $\mathcal{A}^{\mu\nu}$ can be written as the sum of six terms

$$\mathcal{A}^{\mu\nu} = \mathcal{A}^{(1)\mu\nu} + \mathcal{A}^{(2)\mu\nu} + \mathcal{A}^{(3)\mu\nu} + \mathcal{A}^{(4)\mu\nu} + \mathcal{A}^{(5)\mu\nu} + \mathcal{A}^{(6)\mu\nu}. \quad (6.6)$$

In the first term we have only derivatives of the field $\sigma$

$$\mathcal{A}^{(1)\mu\nu} = 6 \sigma^{(\mu} \sigma^{\nu)} - 3 g^{\mu\nu} \sigma_{\rho} \sigma^{\rho} - 6 \sigma h^{(\mu} \phi^{\nu)} + 3 \sigma h^{(\mu} h^{\nu)} + 6 h^{\mu\nu} \sigma^{(\mu} \phi^{\nu)} + 3 \sigma_{\rho} h^{\mu\nu} + 3 g^{\mu\nu} (h \sigma^{(\mu} \phi^{\nu)} - \frac{1}{2} \sigma^{(\mu} h_{\rho}^{\nu)} - 2 \sigma_{\rho} \phi^{(\mu} h^{\nu)}). \quad (6.7)$$

The second term only contains derivatives of the scalar field $\phi$, namely

$$\mathcal{A}^{(2)\mu\nu} = 3 h^{\sigma\rho} \phi_{\sigma} h_{\mu}^{\nu} + 3 \phi_{\sigma} (h^{\rho\sigma} h^{\mu\nu} - \frac{1}{2} g^{\mu\nu} h^{\rho\sigma} h_{\rho})$$

The third term only contains second derivatives of the matrix-valued tensor field $h^{\mu\nu}$,

$$\mathcal{A}^{(3)\mu\nu} = h^{(\mu}_{\rho} h^{\nu)} + \frac{1}{4} h^{\sigma\rho} \omega_{\sigma} h^{\mu\nu} - \frac{1}{2} h^{(\mu}_{\rho} h^{\nu)} + \frac{1}{4} g^{\mu\nu} h^{\rho}_{\sigma} h_{\rho_{\sigma}} - \frac{1}{4} h^{\rho}_{\sigma} h^{\mu\nu} h^{(\rho}_{\sigma} + \frac{1}{4} g^{\mu\nu} h^{\rho}_{\sigma} h_{\rho_{\sigma}}. \quad (6.9)$$

The fourth term contains only first derivatives of $h^{\mu\nu}$, namely

$$\mathcal{A}^{(4)\mu\nu} = - \frac{1}{2} h^{\sigma\rho} h^{\mu\nu} + \frac{1}{2} h^{\mu\nu} h^{\rho}_{\sigma} - \frac{1}{4} h^{(\mu}_{\rho} h^{\nu)} h^{\rho}_{\sigma} + \frac{1}{4} g^{\mu\nu} h^{\rho}_{\sigma} h^{\rho}_{\sigma} + \frac{1}{4} g^{\mu\nu} h^{\rho}_{\sigma} h^{\rho}_{\sigma} - \frac{1}{4} h^{(\mu}_{\rho} h^{\nu)} h^{\rho}_{\sigma} + \frac{1}{4} g^{\mu\nu} h^{\rho}_{\sigma} h^{\rho}_{\sigma} h^{\rho}_{\sigma}. \quad (6.10)$$

The fifth coefficient contains only first and second derivatives of $h$

$$\mathcal{A}^{(5)\mu\nu} = \frac{1}{2} h^{\tau}_{\rho} h^{\mu\nu} + \frac{1}{2} h^{(\mu}_{\rho} h^{\nu)} + \frac{1}{4} h h^{\rho}_{\sigma} h^{\rho}_{\sigma} h^{\rho}_{\sigma}$$

The last term, $\mathcal{A}^{(6)\mu\nu}$, does not contain any derivative of $h^{\mu\nu}$, namely

$$\mathcal{A}^{(6)\mu\nu} = 3 \phi (h^{(\mu}_{\rho} + \frac{1}{2} g^{\mu_{\rho}} h) - \frac{1}{2} \left[ h^{(\mu}_{\rho} + \frac{1}{2} g^{\mu_{\rho}} h \right]$$

where

$$\frac{1}{2} R_{\sigma \tau} (g^{(\mu}_{\sigma} g^{\nu)\rho} + h^{(\mu}_{\sigma} h^{\nu)} + g^{(\mu}_{\sigma} h^{\nu)} + h^{(\mu}_{\sigma} h^{\nu)} + h^{(\mu}_{\sigma} h^{\nu)} + h^{(\mu}_{\sigma} h^{\nu)}$$

The tensor $\mathcal{A}^{(6)\mu\nu}$ is

$$\mathcal{A}^{(6)\mu\nu} = \frac{1}{16} (R - 6q)(g^{\mu\nu} h_{\rho\sigma} + 4h_{\rho\sigma} h^{\mu\nu} + \frac{1}{3} \Lambda). \quad (6.12)$$
The dynamics described by equations (6.1), (6.2), (6.3) and (6.4) can be studied by using an iterative method. Let us write the solution for the background fields $\phi$ and $g^{\mu\nu}$ as Taylor expansion in the deformation parameter $\lambda$ as follows

\[ \phi = \phi_0 + \lambda \phi_1 + \lambda^2 \phi_2 + O(\lambda^3), \]
\[ g^{\mu\nu} = g_0^{\mu\nu} + \lambda g_1^{\mu\nu} + \lambda^2 g_2^{\mu\nu} + O(\lambda^3). \]  

By substituting these expressions in equations (6.3) and (6.4) we obtain, for the terms of order $\lambda^0$, the dynamical equations

\[ \Box \phi = 0, \]
\[ R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} = 0. \]  

As we can see from the last equations the term $g_0^{\mu\nu}$ is nothing but the solution of the ordinary Einstein equation in vacuum with cosmological constant. By substituting the solutions to (6.14) back into the equations of motion for the fields $\sigma$ and $h^{\mu\nu}$ we get equations of the form

\[ \Phi_1(g_0^{\mu\nu}, \phi_0) \sigma = O(\lambda^2), \]
\[ \Phi_2(g_0^{\mu\nu}, \phi_0) h^{\mu\nu} = O(\lambda^2), \]  

where $\Phi_1(g_0^{\mu\nu}, \phi_0)$ and $\Phi_2(g_0^{\mu\nu}, \phi_0)$ are linear second-order partial differential operators.

By iterating this process we can, in principle, find the solution to our dynamical equations in the form of a Taylor series in $\lambda$.

7. Spectrum of matrix gravity on de Sitter space

The action for matrix gravity obtained in the previous section is a functional of the fields $\phi, \sigma, h^{\mu\nu}$ and $g^{\mu\nu}$. The dynamics is described by a system of nonlinear partial differential equations coupled with each other. We analyze, now, the dynamics of the theory. For simplicity we will set, from now on, $Q = 0$. This particular value for the matrix-valued scalar $Q$ will not affect our analysis.

As already mentioned above, from the equation of motion (6.3) for the field $\phi$, we can see that to the zeroth order in the deformation parameter $\lambda$ the field $\phi$ satisfies the following equation

\[ \Box \phi = 0. \]  

As it is well known, the solution of the last equation represents a wave propagating in the whole space. Since we require that $\phi$ vanishes at infinity, the only solution is $\phi = O(\lambda^2)$ in the whole space.

With this solution for the field $\phi$, the matrix-valued function $\rho$ defined in (4.3) becomes

\[ \rho = e^{i\sigma}. \]  

A deeper analysis shows that the matrix-valued scalar field $\sigma$ is not an independent field. Following [9, 10] the general form of $\rho$ can be written as

\[ \rho = \omega^{-i}, \]  

where

\[ \omega = -\frac{1}{m^2} \epsilon^{\mu_1...\mu_a} \epsilon_{\nu_1...\nu_a} d^{\mu_1\nu_1} \cdots d^{\mu_a\nu_a}. \]  

By using the decomposition (4.1) of $h^{\mu\nu}$ in equation (7.4) we get the following formula, up to the term linear in the deformation parameter $\lambda$,

\[ \sigma = -\frac{1}{4} h + \frac{\lambda}{8} h^{\mu\nu} + O(\lambda^2). \]
We can write down, now, the action by imposing the constraints $\phi = O(\lambda^2)$ and (7.5). The final result is the following:

$$S = \frac{1}{16\pi G} \int_M dx \, g^{\frac{1}{2}} \left\{ -6\phi^{,\mu\nu} \phi_{,\mu\nu} + R - 2\Lambda + \frac{\lambda^2}{N} \right\} \left[ \frac{1}{4} h^{\mu\nu} h_{,\mu\nu} - \frac{1}{2} h h^{\mu\nu} R_{\mu\nu} 
+ \frac{1}{2} h^{\mu\nu} h^{\sigma\rho} R_{\sigma\mu\rho\nu} - \frac{1}{2} h^{\mu\nu} h^{\rho\mu} R_{\rho\nu} - \frac{1}{2} h^{\mu\nu} h_{,\mu\nu} 
+ \frac{1}{2} h^{\mu\nu\rho} h_{,\mu\rho\nu} + \frac{1}{8} h^2 R + \frac{1}{4} h^{\mu\nu} h_{,\mu\nu} R - \frac{1}{4} h^{\mu\nu\rho} h_{\mu\nu\rho} 
- \frac{\Lambda}{4} h^2 - \frac{\Lambda}{2} h_{\mu\nu} h^{\mu\nu} ) \right\} + O(\lambda^3).$$

The action depends, now, only on the independent tensor fields $g^{\mu\nu}, h^{\mu\nu}$ and the scalar field $\phi$. Therefore, we will have only two equations that describe the dynamics of the theory. These dynamical equations can be easily derived from those given in the previous section by imposing the conditions (7.5) and $\phi = O(\lambda^2)$. The action (7.6) and the equations of motions for the fields evaluated in the previous section, assume a simple form on maximally symmetric background geometries. As we mentioned in the previous section, the $\lambda^0$ term of the background field $g_0^{\mu\nu}$ is solution of the Einstein equations in vacuum with cosmological constant (6.14). In this section we consider the de Sitter solution to equation (6.14). In this maximally symmetric case the Ricci and Riemann tensors take the following form:

$$R^{\mu\nu} = \frac{1}{n(n-1)} \left( \delta^{\mu\nu} g_{\rho\sigma} - \delta^{\mu\sigma} g_{\rho\nu} \right) R \quad \text{and} \quad R_{\mu\nu} = \frac{1}{n} g_{\mu\nu} R.$$  \hspace{1cm} (7.7)

The de Sitter metric gives a solution of the classical equations provided

$$R = \frac{2n}{n-2} \Lambda.$$  \hspace{1cm} (7.8)

By substituting the expressions in equation (7.7) in the action (7.6), we find a form of the action functional valid in de Sitter geometry, namely

$$S = \frac{1}{16\pi G} \int_M dx \, g^{\frac{1}{2}} \left\{ -6\phi^{,\mu\nu} \phi_{,\mu\nu} + R - 2\Lambda + \frac{\lambda^2}{N} \right\} \left[ \frac{1}{4} h(-\Box + \mu_1) h 
- \frac{1}{4} h_{\mu\nu}(-\Box + \mu_2) h^{\mu\nu} + \frac{1}{2} h^{\mu\nu} h_{,\mu\nu} - \frac{1}{2} h^{\mu\nu} h_{,\mu\nu} R + \frac{1}{4} h^{\mu\nu\rho} h_{\mu\nu\rho} 
+ \frac{1}{4} h_{,\mu\nu} h^{\mu\nu} R - \frac{\Lambda}{4} h^2 - \frac{\Lambda}{2} h_{\mu\nu} h^{\mu\nu} ) \right\} + O(\lambda^3).$$

where the terms $\mu_1$ and $\mu_2$ are defined as follows

$$\mu_1 = \frac{n^2 - 5n + 8}{2n(n-1)} R - \Lambda,$$  \hspace{1cm} (7.10)

$$\mu_2 = - \frac{n - 3}{n-1} R + 2\Lambda.$$  \hspace{1cm} (7.11)

It is interesting, at this point of the discussion, to derive explicitly the spectrum of the theory. In order to achieve this result we need to decompose the field $h_{\mu\nu}$ in its irreducible modes: traceless transverse tensor mode, transverse vector mode, scalar mode and trace part. In other words we can write $h_{\mu\nu}$ as

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} + \frac{1}{n} g_{\mu\nu} \psi + 2\chi_{\mu\nu} + \psi_{,\mu\nu},$$

where the scalar field $\psi$ is defined as follows:

$$\psi = h - \Box \psi.$$
and the fields $\tilde{h}^{\perp}_{\mu\nu}$ and $\tilde{\zeta}^{\perp}_{\mu}$ satisfy the conditions

$$\nabla^\mu \tilde{h}^{\perp}_{\mu\nu} = 0, \quad g^{\mu\nu} \tilde{h}^{\perp}_{\mu\nu} = 0, \quad \nabla^\mu \tilde{\zeta}^{\perp}_{\mu} = 0. \quad (7.13)$$

We can now substitute expression (7.10) in action (7.9), and evaluate the terms separately. Explicitly we obtain

$$\int_M dx \ g^{\frac{1}{2}} \left( \frac{1}{2} h^{\mu\nu} ; h_{\mu\nu} \right) = \int_M dx \ g^{\frac{1}{2}} \left[ -\frac{1}{2n^2} \psi \Box \psi - \frac{1}{n} \psi \left( \Box + \frac{R}{n} \right) \Box \psi + \frac{1}{2} \psi \left( \Box + \frac{R}{n} \right)^2 \Box \psi \right] + \frac{1}{2} \psi \left( \Box + \frac{R}{n} \right) \Box \psi, \quad (7.14)$$

$$- \int_M dx \ g^{\frac{1}{2}} \left( \frac{1}{2} h^{\mu\nu} ; h_{\mu\nu} \right) = \int_M dx \ g^{\frac{1}{2}} \left[ \frac{1}{2n} \psi \Box \psi + \frac{1}{2} \left( \frac{n + 1}{n} \right) \psi \left( \Box + \frac{R}{n} \right) \Box \psi \right] + \frac{1}{2} \psi \left( \Box + \frac{R}{n} \right) \Box \psi, \quad (7.15)$$

$$\int_M dx \ g^{\frac{1}{2}} h_{\mu\nu} (\Box + \mu_2) h^{\mu\nu} = \int_M dx \ g^{\frac{1}{2}} \left[ \frac{1}{2} \psi \left( \Box + \mu_2 \right) \Box \psi + 2 \tilde{\zeta}^{\perp}_{\mu} \left( \Box + \frac{R}{n} \right) \Box \psi + \frac{2}{n} \psi \left( \Box + \mu_2 \right) \Box \psi + \frac{2}{n} \psi \left( \Box + \mu_2 \right) \Box \psi \right] \Box \psi. \quad (7.16)$$

and finally

$$\int_M dx \ g^{\frac{1}{2}} h (\Box + \mu_1) h = \int_M dx \ g^{\frac{1}{2}} \left[ \psi (\Box + \mu_1) \psi + 2 \phi (-\Box + \mu_1) \Box \psi + \psi (-\Box + \mu_1) \Box \psi \right]. \quad (7.17)$$

By using the decompositions (7.14)–(7.17) we rewrite the action (7.9) in terms of the irreducible modes of $h^{\mu\nu}$, namely

$$S = \frac{1}{16\pi G} \int_M dx \ g^{\frac{1}{2}} \left[ -6 \phi_{\mu}^2 \phi_{,\mu} + R - 2 \lambda + \frac{3}{2} \nabla^2 \left[ -\frac{1}{4} \tilde{h}^{\perp}_{\mu\nu} (\Box + \mu_2) \tilde{h}^{\perp}_{\mu\nu} \right] + \frac{(n - 1)(n - 2)}{4n^2} \psi \left( \Box + \frac{n}{2(n - 1)} R - \frac{n(n + 2)}{(n - 1)(n - 2)} \right) \Box \psi - \frac{1}{2} \zeta^{\perp}_{\mu} \left( \Box + \frac{R}{n} \right) \left[ n - 2 \right] \left[ n - 2 \right] \left[ R - 2 \lambda \right] \zeta^{\perp}_{\mu} + \frac{2}{4n} \psi \left( \Box + \frac{n - 2}{n} R - 2 \lambda \right) \Box \psi + \frac{3}{8} \psi \left( \Box + \frac{2R}{3n} \right) \Box \psi \right] + O(\lambda^3). \quad (7.18)$$

It is straightforward to show now that on the mass shell, (7.8), the terms containing the fields $\zeta^{\perp}_{\mu}$ and $\psi$ vanish identically. More precisely we obtain the following form for the action functional

$$S = \frac{1}{16\pi G} \int_M dx \ g^{\frac{1}{2}} \left[ \frac{4 \lambda}{n - 2} + \frac{3}{N} \nabla^2 \left[ -\frac{1}{4} \tilde{h}^{\perp}_{\mu\nu} \left( -\Box + \frac{4 \lambda}{(n - 1)(n - 2)} \right) \tilde{h}^{\perp}_{\mu\nu} \right] + \frac{(n - 1)(n - 2)}{4n^2} \psi \left( \Box - \frac{2n \lambda}{(n - 1)(n - 2)} \right) \Box \psi \right] + O(\lambda^3). \quad (7.19)$$

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Thus on the mass shell the only remaining fields are the (traceless) matrix-valued traceless transverse tensor $\bar{h}_{\mu\nu}$ and the (traceless) matrix-valued scalar field $\varphi$. This action looks exactly the same as in general relativity, the only difference being that the fields are matrix-valued and traceless. Therefore, it describes $(N - 1)$ spin-2 particles and $(N - 1)$ spin-0 particles. Note also, that exactly as in general relativity the scalar conformal mode is unstable.

8. Concluding remarks

In this paper we studied a noncommutative deformation of general relativity (called spectral matrix gravity) proposed in [10] where the noncommutative limit has been explicitly evaluated. The approach of the paper [10] to construct the action for matrix gravity differs from the one proposed in [9, 8, 16]. In the latter the action of our model was a straightforward generalization of the Einstein–Hilbert action in which the measure and the scalar curvature were matrix-valued quantities. This last approach seems to have some intrinsic arbitrariness due to the freedom of choosing the particular form of the matrix-valued measure (for a discussion see [16]). In order to avoid these issues, in [10] the action of matrix gravity was defined as a linear combination of the first two global heat kernel coefficients (1.10) of a non-Laplace type partial differential operator.

By using the covariant Fourier transform method we were able to evaluate the coefficients $A_0$ and $A_1$, and as a result, the action functional within the perturbation theory in the deformation parameter $\lambda$. The main result of this paper is the derivation of the modified Einstein equations in (6.4) in the weak deformation limit. In this case the pure noncommutative fields, namely $h^{\mu\nu}$ and $\sigma$, contribute to the right-hand side of the Einstein equation, that is, the stress–energy tensor. The explicit form of these non-commutative correction terms has been derived in (6.7)–(6.12) for the first time.

It is worth noting that the dynamical equations for the fields, found in this paper, are classical and therefore they should be studied from the classical point of view. It is an intriguing question to study the physical and mathematical effects of the non-commutative corrections. First of all, this is the problem of singularities in general relativity. Another very interesting task would be to find some basic particular solutions, in particular, static spherically symmetric solutions (non-commutative Schwarzschild) and time-dependent homogeneous solutions (non-commutative Robertson–Walker). Finally, it is the question of whether the non-commutative fields could account for the dark matter and the dark energy. All these questions require further study.

Some of the physical implications of matrix gravity have been extensively discussed in [8–10]. This theory exhibits non-geodesic motion which can be related to a violation of the equivalence principle, moreover, because of the new gauge symmetry, there are new physical conserved charges. At last, this theory represents a consistent model of interacting spin-2 particles on curved space which usually was a problem. An interesting question is the limit as $N \to \infty$ of our model, this might be related to matrix models and string theory.

As it is outlined in the introduction matrix gravity can be considered as a gravitational chromodynamics describing the gravitational interaction of a new degree of freedom that we call gravitational color. Whether or not it is related to the color of QCD is an open question. Let us suppose for simplicity that it is the same, and that the gauge group of matrix gravity is nothing but $SU(3)$. If one pushes this analogy with QCD to its logical limit then this would mean that the theory predicts that the gravitational interaction of quarks depends on their colors. Exactly as in QCD the strong interaction between quark of color $i$ and a quark of color $j$ is transmitted by gluon of type $(ij)$, the gravitational interaction between quark of color $i$ and a quark of color $j$ is transmitted by the graviton of type $(ij)$. In this case, all particles
in the electro-weak sector, including photon, do not feel the gravitational color. In that sense it is ‘dark’. Note that in the non-relativistic limit the Newtonian potential will also become ‘matrix-valued’. One can go even further. Since the usual (white) mass is determined by the sum of the color masses, one can assume that the color masses can be even negative. Then the gravitational interaction of such particles would include not only attractive forces but also repellent forces (antigravity?). This feature could then solve the mystery of singularities in general relativity.

The consequences of our model in the ambit of cosmology are easily seen by inspecting equations (6.4) and (6.5). The deformation of the energy–momentum tensor in (6.5) is written only in terms of the noncommutative part \( h^{\mu\nu} \). It would be interesting to study whether or not \( h^{\mu\nu} \) could account for a field of negative pressure. If so, our model could describe the dynamics of dark energy. Furthermore, the distortion of the gravitation expansion of the universe due to the non-commutative degrees of freedom of the gravitational field will certainly have some effect on the anisotropy of the cosmic background radiation, nucleosynthesis and structure formation. Of course, a detailed analysis of these effects requires a careful study of the fluctuations in the early universe.

Of course the validity of this statements require further investigations. The ultimate goal of this theory is to construct a consistent theory of the gravitational field which is compatible with the standard model and able to solve the current open issues which afflict general relativity, i.e. the problems of the origin of dark matter and dark energy, the recent anomalies found in the solar system (Pioneer anomaly, flyby anomaly, etc) and last, but not least, the problem of quantization of the gravitational field.

In summary, we would like to stress that our model makes it possible to make a number of very specific predictions that can serve as experimental tests of the theory.

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Appendix A. Coincidence limits

In this appendix we will list the various coincidence limits used during the calculations performed in this paper.

Through all this appendix the subscripts 0, 1 and 2 will be used to denote terms of different order in the deformation parameter \( \lambda \). More precisely for any quantity \( X \) which contains different orders of \( \lambda \) we write

\[
X = X_0 + \lambda X_1 + \lambda^2 X_2 + O(\lambda^3),
\]

where \( X_0, X_1 \) and \( X_2 \) denote, respectively, the zeroth, first and second order in \( \lambda \).

We start with the coincidence limit of the operator \( \tilde{H} \) in (3.16) and its derivatives. More precisely we have

\[
[\tilde{H}] = \lambda \xi_\mu \xi_\nu h^{\mu\nu}.
\]  
(A.1)

For the first derivative we obtain

\[
[\tilde{H};_\mu] = \lambda \xi_\rho \xi_\sigma h^{\rho\sigma;_\mu}.
\]  
(A.2)

For the second derivative we get the following formula

\[
[\tilde{H},_\mu;_\nu] = -\frac{1}{2} \lambda \xi_\alpha \xi_\beta R^{\alpha\beta}_{\mu\nu} - \frac{3}{2} \lambda \xi_\alpha \xi_\beta h^{\alpha\beta} (R^{\mu\nu}_{\rho\sigma} + R^{\rho\sigma}_{\mu\nu}) + \lambda \xi_\alpha \xi_\beta h^{\alpha\beta;_\mu;_\nu}.
\]  
(A.3)
Recall, now, definition (3.17) for the operator $K$. The coincidence limits of the terms in $K$ are
\[ [B_0^{\rho\nu}] = -2g^{\rho\nu}, \quad (A.4) \]
\[ [B_1^{\rho\nu}] = -2h^{\rho\nu}. \quad (A.5) \]

For the derivatives of these quantities we have
\[ [(\nabla_\mu B_0^{\rho\nu})_0] = 0, \quad (A.6) \]
\[ [(\nabla_\mu B_0^{\rho\nu})_1] = -2h^{\rho\nu}; \quad (A.7) \]

For the other terms we have
\[ [G_0^{\rho\nu}] = 0, \quad (A.8) \]
\[ [G_1^{\rho\nu}] = -h^{\rho\nu}; \quad (A.9) \]

for the derivatives of (A.8) we get
\[ [(\nabla_\mu G_0^{\rho\nu})_0] = \frac{1}{3} R^{\rho\nu} + \mathcal{R}^{\rho\nu}, \quad (A.10) \]
\[ [(\nabla_\mu G_0^{\rho\nu})_1] = -h^{\rho\nu}; \quad (A.11) \]
\[ [(\nabla_\mu G_0^{\rho\nu})_2] = -[\sigma_{\rho\nu}, h^{\rho\nu}] - [\sigma_{\rho\nu}, h^{\rho\nu}; \sigma] - [\sigma_{\rho\nu}, \sigma_{\rho\nu}]. \quad (A.12) \]

For the operator $\mathcal{L}$ in (3.18) we need the following coincidence limits
\[ [(A_{\nu})_0] = -[(\bar{A}_{\nu})_0] = -\phi_{\nu}, \quad (A.13) \]
\[ [(A_{\nu})_0] = -[(\bar{A}_{\nu})_0] = -\sigma_{\nu}, \quad (A.14) \]
\[ [(A_{\nu})_0] = [(\bar{A}_{\nu})_0] = \frac{1}{2} [\sigma_{\nu}, \sigma]. \quad (A.15) \]

We also used, during the calculation, the coincidence limits for the derivatives of $A_{\nu}$, namely
\[ [(\nabla_\mu A_{\nu})_0] = -[\phi_{\nu\mu}] + \frac{1}{2} R_{\nu\mu} - \frac{1}{2} \mathcal{R}_{\nu\mu}, \quad (A.16) \]
\[ [(\nabla_\mu A_{\nu})_1] = -[\sigma_{\nu\mu}], \quad (A.17) \]
\[ [(\nabla_\mu A_{\nu})_2] = \frac{1}{2} [\sigma_{\nu\mu}, \sigma] + [\sigma_{\nu\mu}, \sigma_{\nu\mu}], \quad (A.18) \]

**Appendix B. Integrals**

In this appendix we will explicitly derive the solution to the integrals in (4.35) and (4.42). The first integral that we want to evaluate is the following
\[ I_1(\alpha, \beta, \gamma, \delta) = \int_0^1 d\tau_2 \int_0^\tau_2 d\tau_1 \, \tau_1^\alpha (1 - \tau_2)^\beta (\tau_2 - \tau_1)^\gamma (1 - \tau_2 + \tau_1)^\delta. \quad (B.1) \]

This integral is well defined for $\text{Re}(\alpha) > -1$, $\text{Re}(\gamma) > -1$ and $0 < \tau_2 < 1$. By using the integral representation of the hypergeometric function we can evaluate the integral in $\tau_1$, i.e.
\[ I_1 = \frac{\Gamma(1+\alpha)\Gamma(1+\gamma)}{\Gamma(2+\alpha+\gamma)} \int_0^1 d\tau_2 \, \tau_2^{1+\alpha+\gamma} (1 - \tau_2)^{\beta+\delta} \, \frac{\tau_2}{1-\tau_2}. \quad (B.2) \]
Now, by using the linear transformation formula for the hypergeometric function we get [1]

\[
\begin{align*}
_2F_1 \left( \frac{1 + \alpha, -\delta}{1 - \tau_2} \right) \\
= (1 - \tau_2)^{1+\alpha} \, _2F_1 \left( 1 + \alpha, 2 + \alpha + \gamma; \frac{\tau_2}{1 - \tau_2} \right).
\end{align*}
\]

(B.3)

By substituting the last expression in the integral (B.2) we obtain

\[
I_1 = \frac{\Gamma(1 + \alpha) \Gamma(1 + \gamma)}{\Gamma(2 + \alpha + \gamma)} \int_0^1 d\tau_2 \tau_2^{1+\alpha} (1 - \tau_2)^{1+\alpha+\beta+\delta} \, _2F_1 \left( 1 + \alpha, 2 + \alpha + \gamma; \frac{\tau_2}{1 - \tau_2} \right).
\]

(B.4)

This integral over \( \tau_2 \) is, now, of the following general form

\[
\int_0^1 dx \, x^{c-1} (1 - x)^{d-1} \, _2F_1 \left( a, b, c; x \right),
\]

which is well defined for \( \text{Re}(c) > 0 \) and \( \text{Re}(d) > 0 \) and has a solution in a closed form [18], namely

\[
I = \frac{\Gamma(c) \Gamma(d) \Gamma(c + d - a - b)}{\Gamma(c + d - a) \Gamma(c + d - b)}.
\]

(B.6)

By using the results (B.6) in the integral (B.4), we get the solution (4.36), i.e.

\[
I_1 = \frac{\Gamma(1 + \alpha) \Gamma(1 + \beta) \Gamma(1 + \gamma) \Gamma(2 + \alpha + \beta + \delta)}{\Gamma(3 + \alpha + \beta + \gamma + \delta) \Gamma(2 + \alpha + \beta)}.
\]

(B.7)

The next integral that we used in our paper is the one in (4.42), namely

\[
I_2(\alpha, \beta) = \int_0^1 d\tau_1 \, \tau_1^\alpha (1 - \tau_1)^\beta.
\]

(B.8)

The solution to this integral is easily found by recalling the integral representation of the hypergeometric function [1]

\[
_2F_1 \left( a, b, c; z \right) = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c - b)} \int_0^1 dt \, t^{b-1}(1 - t)^{c-b-1}(1 - tz)^{-a},
\]

where \( \text{Re}(c) > 0 \) and \( \text{Re}(b) > 0 \). From this last general expression we obtain the integral (B.8) by setting \( z = 0, \alpha = b - 1 \) and \( \beta = c - b - 1 \). By recalling that \( _2F_1 \left( a, b, c; 0 \right) = 1 \), we finally get

\[
I_2(\alpha, \beta) = \frac{\Gamma(1 + \alpha) \Gamma(1 + \beta)}{\Gamma(2 + \alpha + \beta)} = B(1 + \alpha, 1 + \beta),
\]

(B.10)

where \( B(a, b) \) denotes the Euler beta function.

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