Monte Carlo construction of cubature on Wiener space

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Abstract

In this paper, we investigate application of mathematical optimization to construction of a cubature formula on Wiener space, which is a weak approximation method of stochastic differential equations introduced by Lyons and Victoir (Cubature on Wiener Space, Proc. R. Soc. Lond. A 460, 169–198). After giving a brief review of the cubature theory on Wiener space, we show that a cubature formula of general dimension and degree can be obtained through a Monte Carlo sampling and linear programming. This paper also includes an extension of stochastic Tchakaloff’s theorem, which technically yields the proof of our main result.

1 Introduction

The cubature formulas on Wiener space (Lyons & Victoir 2004) is a certain family of numerical formulas for approximating the expectation of functionals of diffusion processes, which are significant in mathematical finance and other related fields. The existence of cubature formulas of general dimension and degree has been known, but the constructions given in the literature were based on algebraic structure among continuous paths and Brownian motion, and limited to ones of low degree and dimension. Our aim in this paper is to obtain a method of constructing general cubature formulas on Wiener space through mathematical optimization.

In this introduction, we shall briefly explain the background of this study. Section 1.1 gives a brief explanation of the cubature theory on Wiener space (Lyons & Victoir 2004) and identifies the problem to be solved. Section 1.2 introduces the recent Monte Carlo approach by Hayakawa (2020) to general cubature construction, on which our method in this paper is based. We explain the relation between cubature on Wiener space and a generalized one, as well as the particular difficulty that arises on the Wiener space. The contribution of this paper is summarized in Section 1.3, and Section 1.4 shows the organization of the rest of the paper.

1.1 Cubature on Wiener space

Let $B = (B^1, \ldots, B^d)$ be a $d$-dimensional standard Brownian motion. Let $C^\infty_b(\mathbb{R}^N; \mathbb{R}^N)$ be the space of infinitely differentiable $\mathbb{R}^N$-valued functions defined on $\mathbb{R}^N$ whose every order of derivative is bounded. Consider the following $N$-dimensional Storatonovich stochastic differential equation (SDE):

$$
\begin{align*}
    dX_t &= \sum_{i=1}^{d} V_i(X_t) \circ dB^i_t + V_0(X_t) \, dt, \\
    X_0 &= x,
\end{align*}
$$

where $x \in \mathbb{R}^N$, $V_i \in C^\infty_b(\mathbb{R}^N; \mathbb{R}^N)$ for $i = 0, \ldots, d$. As the process $X_t$ is dependent on the initial value $x$, we denote it by $X_t(x)$ if necessary. We may assume the solution $X_t(x)$ is continuous with respect to $t$ and $x$. Our aim is to efficiently compute or approximate the expectation $E[f(X_t)]$ with $t > 0$ and some smooth or Lipschitz $f$. This sort of approximation is called a weak approximation of
SDE and well-studied in the literature (Kloeden & Platen 1992, Kusuoka 2001, Kusuoka 2004, Lyons & Victoir 2004, Ninomiya & Victoir 2008).

We here focus on the approach introduced in Lyons & Victoir (2004) called cubature on Wiener space. Broadly speaking, a cubature formula on Wiener space (of the time interval $[0, T]$) is the approximation

$$P \simeq \sum_{i=1}^{n} \lambda_{j} \delta_{w_{j}},$$

where $P$ is the Wiener measure on the Wiener space $C_{0}^{d}([0, T]; \mathbb{R}^{d})$ (the space of $\mathbb{R}^{d}$-valued continuous function in $[0, T]$ starting at the origin),

$$w_{j} = (w_{j}^{0}, w_{j}^{1}, \ldots, w_{j}^{d}) \in C_{0}^{d}([0, T]; \mathbb{R} \oplus \mathbb{R}^{d})$$

for $j = 1, \ldots, n$, and $\lambda_{1}, \ldots, \lambda_{n}$ are positive real weights whose sum equals one. Instead of polynomials in conventional cubature formulas, we adopt iterated integrals as the test functionals, that is, we want to find paths $w_{i}$ satisfying

$$E \left[ \int_{0<t_{1}<\cdots<t_{k}<T} \circ dB_{t_{1}}^{i} \cdots \circ dB_{t_{k}}^{i} \right] = \sum_{j=1}^{n} \lambda_{j} \int_{0<t_{1}<\cdots<t_{k}<T} dw_{j}^{i}(t_{1}) \cdots dw_{j}^{i}(t_{k})$$

(1.2)

over some set of multi-indices $(i_{1}, \ldots, i_{k})$, where the iterated Stratonovich integral appears in the left-hand side. Precisely speaking, we formally set $B_{t}^{i} = t$ for $t \geq 0$ and assume $w_{j}$ is of bounded variation for each $j = 1, \ldots, n$. Although $w_{j}^{i}(t) = t$ is also assumed in Lyons & Victoir (2004), we here may generalize and remove this condition.

The iterated integrals appearing in (1.2) have a rich algebraic structure (see Section 3.1), so algebraic approaches have been adopted in the literature (Lyons & Victoir 2004, Ninomiya & Victoir 2008, Gyurko & Lyons 2011). However, solving complicated equations of Lie algebra is required in those approaches, and constructions of the formula is limited to a small range (see Section 2.3). Our objective is to give a construction method for a general setting where there are no limitations on the number of iterations of the integral ($k$ in (1.2)). To do so, we adopt an optimization-based viewpoint instead of algebraic ones and extend the result in Hayakawa (2020), which gives a randomized construction of general cubature formulas, to our situation.

We should mention the Kusuoka approximation (Kusuoka 2001, Kusuoka 2004, Shinozaki 2017), which is closely related to cubature on Wiener space. However, our objective in this paper is limited to the construction of the cubature on Wiener space, and the optimization-based approach to the general Kusuoka approximation is deferred for future work.

1.2 Monte Carlo approach to generalized cubature

A cubature formula is originally a numerical integration formula on some Euclidean space that exactly integrates polynomials of up to a certain degree (Steinitz 1916), the theory of which underlies cubature on Wiener space introduced in the previous section. We shall explain it in a generalized setting in the following and briefly explain the idea of Hayakawa (2020) for constructing general cubature formulas.

Let $\Omega$ be some measurable space and $X$ be a random variable on it. A generalized cubature formula with respect to $X$ and integrable functions $\varphi_{1}, \ldots, \varphi_{D}: \Omega \to \mathbb{R}$ is a set of points $x_{1}, \ldots, x_{n} \in \Omega$ and positive weights $\lambda_{1}, \ldots, \lambda_{n}$ such that

$$E\left[\varphi_{i}(X)\right] = \sum_{j=1}^{n} \lambda_{j} \varphi_{i}(x_{j}), \quad i = 1, \ldots, D.$$

For simplicity, we require $\varphi_{1} \equiv 1$ in the following. In this setting, $\lambda_{1} + \cdots + \lambda_{n} = 1$ must hold. We can also regard the above condition as one vector valued equality $E[\varphi(X)] = \sum_{j=1}^{n} \lambda_{j} \varphi(x_{j})$, where $\varphi: \Omega \to \mathbb{R}^{D}$ is defined as $\varphi = (\varphi_{1}, \ldots, \varphi_{D})^{T}$. The existence of such a formula is assured by the following theorem (Tchakaloff 1957, Bayer & Teichmann 2006, Sawa et al. 2019):
Theorem 1.1 (Generalized Tchakaloff’s theorem). Under the above setting, there exists a cubature formula whose number of points satisfies $n \leq D$. Moreover, we can take points $x_1, \ldots, x_n$ so as to satisfy $\varphi(x_j) \in \text{supp } P(\varphi(X))$ for each $j = 1, \ldots, n$.

In the above statement, $\text{supp } P(\varphi(X))$ is the support of the distribution of the vector-valued random variable $\varphi(X)$. Equivalently, this is the smallest closed set $A \subset \mathbb{R}^D$ satisfying $P(\varphi(X) \in A)$. Generalized Tchakaloff’s theorem can be understood as an immediate consequence of a discrete-geometric argument. Indeed, $E[\varphi(X)]$ is contained in the convex hull of $\text{supp } P(\varphi(X))$ (the convex hull of an $A \subset \mathbb{R}^D$ is defined as $\text{conv } A := \{ \sum_{m=1}^{m} \lambda_i x_i \mid m \geq 1, \lambda_i \geq 0, \sum_{m=1}^{m} \lambda_i = 1, x_i \in A \}$). Therefore, the generalized Tchakaloff’s theorem follows from the following Carathéodory’s theorem (note that we assume $\varphi_1 \equiv 1$):

Theorem 1.2 (Carathéodory). For an arbitrary $A \subset \mathbb{R}^D$ and $x \in \text{conv } A$, there exists $D+1$ points $x_1, \ldots, x_{D+1} \in A$ such that $x \in \text{conv } \{x_1, \ldots, x_{D+1}\}$.

Although the above argument cannot directly be used in the construction of cubature, we can make use of its nature by introducing the concept of relative interior. For a set $A \subset \mathbb{R}^D$, define its affine hull by

$$\text{aff } A := \left\{ \sum_{i=1}^{m} \lambda_i x_i \mid m \geq 1, \lambda_i \in \mathbb{R}, \sum_{i=1}^{m} \lambda_i = 1, x_i \in A \right\}$$

Then, the relative interior of $A$ is the interior of $A$ with respect to the subspace topology on $\text{aff } A$ and denoted by $\text{ri } A$. In terms of this relative interior, the following generalization of Carathéodory’s theorem holds:

Theorem 1.3. (Steinitz 1916, Bonnice & Klee 1963) Suppose an $A \subset \mathbb{R}^D$ satisfy $\text{aff } A$ is a $k$-dimensional affine subspace of $\mathbb{R}^D$. Then, for each $x \in \text{ri conv } S$, there exists some subset $B \subset A$ composed of at most $2k$ points such that $\text{aff } B = \text{aff } A$ and $x \in \text{ri conv } B$.

From this generalization and the fact that

$$E[\varphi(X)] \in \text{ri conv } \text{supp } P(\varphi(X))$$

holds (see, e.g., Hayakawa (2020) for proof), we obtain the following randomized construction of cubature formulas from i.i.d. copies of $X$:

Theorem 1.4 (Hayakawa 2020). Let $X_1, X_2, \ldots$ be i.i.d. copies of $X$. Then there exists almost surely a positive integer $n$ satisfying $E[\varphi(X)] \in \text{conv } \{\varphi(X_1), \ldots, \varphi(X_n)\}$.

Though the determination of weights remains, for a sufficiently large $n$, it suffices to take a basic feasible solution of the linear programming problem

$$\text{minimize } 0 \quad \text{subject to } \sum_{j=1}^{n} \lambda_j \varphi(X_j), \lambda_j \geq 0.$$ 

Indeed, its basic feasible solution satisfies the bound of points used in a cubature given in Tchakaloff’s theorem (Theorem 1.1). This sort of technique reducing the number of points in a discrete measure is called Carathéodory-Tchakaloff subsampling (Piazzi et al. 2017).

Here, if we formally write the iterated integral $\int_{0 \leq t_1 < \cdots < t_k \leq T} dw^{i_1}(t_1) \cdots dw^{i_k}(t_k)$ appearing in (1.2) as $\varphi_{(i_1, \ldots, i_k)}(w)$ for a valid $w$ (and the Brownian motion $B$), then the cubature on Wiener space is a set of paths $w_j$ and weights $\lambda_j$ formally satisfying

$$E[\varphi_W(B)] = \sum_{j=1}^{n} \lambda_j \varphi_W(w_j),$$

where $\varphi_W$ denotes a vector of some functions of the form $\varphi_{(i_1, \ldots, i_k)}$. Therefore, if we could directly generate sample paths of the Brownian motion, then Theorem 1.4 should be applicable. In reality, it is impossible to generate a Brownian motion on a computer, and it is not even of bounded variation.
However, the assumption $X_1, X_2, \ldots$ having the same distribution as $X$ in Theorem 1.4 can be weakened; the same conclusion follows from the following condition for the i.i.d. sequence:

$$\text{aff sup} P\varphi(X_1) = \text{aff sup} P\varphi(X), \quad \text{sup} P\varphi(X_1) \supset \text{sup} P\varphi(X). \quad (1.5)$$

From this fact, it suffices to investigate the distribution of iterated integrals, and we indeed show the Wiener space counterpart of the condition (1.5) in Proposition 3.5 and Corollary 4.2.

1.3 Contribution of this study

Broadly speaking, the contribution of this study is the following two items:

- (main contribution) We show that one can construct a cubature formula on Wiener space of general dimension and degree with a randomized algorithm.

- (technical contribution) To apply the technique of Hayakawa (2020) to the problem of cubature on Wiener space, we characterize the affine hull of the distribution of iterated Stratonovich integrals and prove stochastic Tchakaloff’s theorem in a stronger way.

The main result with a simple Monte Carlo construction is given in Proposition 4.5. It asserts that a certain random generation of piecewise linear paths combined with a linear programming yields a cubature formula on Wiener space.

As a technical contribution, we extend stochastic Tchakaloff’s theorem (Lyons & Victoir 2004), which assures the existence of cubature formulas on Wiener space. Although the original statement was just that there exists a cubature formula, we show that the expectation of iterated Stratonovich integrals of a Brownian motion ($E[\varphi(W)]$ in (1.4)) is contained in the relative interior of $\text{conv}\{\varphi_W(w)\}$ with a valid range of paths $w$ of bounded variation (Theorem 3.8). This stronger statement with “relative interior” follows from our characterization (Proposition 3.5) of $\text{aff sup} P\varphi_W(u)$ in terms of (1.4) and is essential in exploiting the existing construction of general cubature formulas (Hayakawa 2020).

1.4 Outline

We give a brief overview of the following sections.

Section 2 is devoted to a concise review of cubature on Wiener space by Lyons & Victoir (2004). We introduce the facts around vector fields and relevant notations used throughout the paper in Section 2.1. The precise definition and error estimate of cubature of Wiener space is given in Section 2.2, and Section 2.3 provides information about known constructions based on algebraic arguments.

In Section 3, we give the extended statement of stochastic Tchakaloff’s theorem and its proof. Section 3.1 and 3.2 provide algebraic background of cubature construction. Section 3.3 is devoted to the proof of our version of stochastic Tchakaloff’s theorem, and it also includes the characterization of the distribution of iterated Stratonovich integrals from the viewpoint of the affine hull of the support.

We discuss a way to obtain piecewise linear cubature formulas on Wiener space in Section 4. After giving some general properties of continuous BV-paths in our context in Section 4.1, we prove our main result in Section 4.2 that we can generally construct cubature formulas on Wiener space with simple randomized algorithms. Section 4.3 gives the numerical verification of our result in a small range of parameters.

We finally give our conclusion in Section 5.

2 Theoretical background of cubature on Wiener space

In this section, we provide a review of the cubature theory on Wiener space, which was first introduced by Lyons & Victoir (2004). We quickly introduce basic notions concerning multi-dimensional stochastic flows, and give the error estimate of cubature formula. Additionally, we shall see a few examples of concrete construction of cubature formulas on Wiener space in Section 2.3.
2.1 Vector fields

In this section, we define vector fields on \( \mathbb{R}^N \) and see the correspondence between vector fields and vector-valued functions. Let \( C^\infty(\mathbb{R}^N) \) be the set of real-valued smooth functions over \( \mathbb{R}^N \).

**Definition 2.1.** A vector field on \( \mathbb{R}^N \) is a \( (\mathbb{R},\mathbb{R}) \)-linear mapping \( V : C^\infty(\mathbb{R}^N) \to C^\infty(\mathbb{R}^N) \) such that \( V(fg) = (Vf)g + fVg \) holds for arbitrary \( f, g \in C^\infty(\mathbb{R}^N) \).

Due to the condition, a vector field on \( \mathbb{R}^N \) has to be a differential operator \( \sum_{i=1}^N V^i \partial_i \) where \( V^i \in C^\infty(\mathbb{R}^N) \) and \( \partial_i \) denotes the \( i \)-th partial derivative for \( i = 1, \ldots, d \). Therefore, a vector field corresponds to the vector-valued smooth function \( (V^1, \ldots, V^N)^\top \in \mathbb{R}^N \). By abuse of notation, we also denote this vector-valued function by \( V \).

If \( A \) and \( B \) are vector fields on \( \mathbb{R}^N \), we define the Lie bracket \( [A, B] := AB - BA \). This \( [A, B] \) is also a vector field because the second derivatives vanish. Note that \( [A, B] \) corresponds to the vector \( (\partial B)A - (\partial A)B \), where \( A, B \) are regarded as function and \( \partial C \) denotes the Jacobian matrix of \( C \) (see, e.g., Hairer 2011).

Reciprocally, the coefficients of the SDE (1.1) can be regarded as vector fields. These vector fields are closely related to the behavior of \( X_t \).

Let \( V_0, \ldots, V_d \) be the vector fields (operators) induced by the coefficients of (1.1), and define the operator \( L := V_0 + \frac{1}{2}(V_1^2 + \cdots + V_d^2) \). Consider the parabolic partial differential equation (PDE),

\[
\begin{align*}
\frac{\partial}{\partial t} u(t, x) &= Lu(t, x), \\
\phantom{u(t, x)} &= u(0, x) = f(x),
\end{align*}
\]

with a Lipschitz function \( f : \mathbb{R}^N \to \mathbb{R}^N \). Because \( u(T, x) = E[f(X_T(x))]) \) holds (Ikeda & Watanabe 1989), we can exploit the numerical schemes in PDE theory to get \( E[f(X_T(x))] \) and vice versa.

We also introduce several conditions on the vector fields. They are assumed in order to obtain the estimate given in the Proposition 2.2. Before we state those, we introduce some notations based on Kusuoka (2001).

Let \( \mathcal{A} := \{\emptyset\} \cup \bigcup_{k=1}^\infty \{0, 1, \ldots, d\}^k \). For \( \alpha \in \mathcal{A} \), define \( |\alpha| := 0 \) if \( \alpha = \emptyset \) and \( |\alpha| := k \) if \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \{0, \ldots, d\}^k \). We also define \( \|\alpha\| := |\alpha| + \{|1 \leq j \leq |\alpha| \mid \alpha_j = 0|\} \). For \( \alpha, \beta \in \mathcal{A} \), define \( \alpha \ast \beta := (\alpha_1, \ldots, \alpha_{|\alpha|}, \beta_1, \ldots, \beta_{|\beta|}) \). Let \( \mathcal{A}_0 := \mathcal{A} \setminus \{\emptyset\} \) and \( \mathcal{A}_1 := \mathcal{A} \setminus \{\emptyset \cup (0)\} \). We also define, for each integer \( m \geq 1 \),

\[
\mathcal{A}(m) := \{\alpha \in \mathcal{A} \mid \|\alpha\| \leq m\}, \quad \mathcal{A}_0(m) := \mathcal{A}(m) \cap \mathcal{A}_0, \quad \mathcal{A}_1(m) := \mathcal{A}(m) \cap \mathcal{A}_1.
\]

Define a vector field \( V_{[\alpha]} \) for each \( \alpha \in \mathcal{A} \) inductively by \( V_{[\emptyset]} = 0 \) and

\[
V_{[i]} := V_{[\{i\}]}, \quad V_{[\alpha_1, \ldots, \alpha_j]} := [V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_j}], \quad (|\alpha_j| \geq 1, i = 0, \ldots, d).
\]

We can now state the uniformly finitely generated (UFG) condition (Kusuoka & Stroock 1987, Kusuoka 2001):

(UFG) There exists a positive integer \( L \geq 1 \) such that, for an arbitrary \( \alpha \in \mathcal{A}_1 \), there exists \( \varphi_{\alpha, \beta} \in C^\infty_b(\mathbb{R}^N) \) for each \( \beta \in \mathcal{A}_1(L) \) satisfying

\[
V_{[\alpha]} = \sum_{\beta \in \mathcal{A}_1(L)} \varphi_{\alpha, \beta} V_{[\beta]}.
\]

This is equivalent to the statement that the \( C^\infty_b(\mathbb{R}^N) \)-module generated by \( \{V_{[\alpha]} \mid \alpha \in \mathcal{A}_1\} \) is finitely generated. Note that (UFG) is known to be strictly weaker than the uniform Hörmander condition (see, e.g., Example 2.2 in Crisan & Ottobre 2016), which is one of the typical assumptions on vector fields.

Although only the condition (UFG) was assumed in Lyons & Victoir (2004), it was pointed out in Crisan & Ghazali (2007) that the following condition is also essential:
(V0) There exists \( \varphi_\beta \in C_0^\infty(\mathbb{R}^N) \) for each \( \beta \in A_1(2) \) such that

\[
V_0 = \sum_{\beta \in A_1(2)} \varphi_\beta V_\beta.
\]

For a function \( f \in C_b^\infty(\mathbb{R}^N) \), define \( (P_t f)(x) := \mathbb{E}[f(X_t(x))] \). The following estimate is essential.

**Proposition 2.2** (Kusuoka & Stroock 1987, Crisan & Ghazali 2007). Assume both (UFG) and (V0) hold. Then, for any positive integer \( r \) and \( \alpha_1, \ldots, \alpha_r \in A \), there exists a constant \( C > 0 \) such that

\[
\|V_{[\alpha_1]} \cdots V_{[\alpha_r]} P_t f\|_\infty \leq \frac{Ct^{1/2}}{\ell(\|\alpha_1\| + \cdots + \|\alpha_r\|)} \|\nabla f\|_\infty
\]

(2.2)

Although we can obtain a weaker bound without assuming (V0), we later exploit this bound assuming both (UFG) and (V0) for simplicity.

We finally state the stochastic Taylor formula in terms of the vector-field notation introduced above. By Itô’s formula, we obtain for any positive integer \( m \)

\[
\begin{align*}
&\int_0^t V_i(X_s) dX_s = f(x) + \sum_{i=0}^d \int_0^t V_i(X_s) \partial_j f(X_s) \circ dB_s^i \\
&= f(x) + \sum_{i=0}^d \int_0^t (V_i f)(X_s) \circ dB_s^i,
\end{align*}
\]

where we denote \( ds \) by \( \circ dB_s^0 \). Therefore, the repetition of Itô’s formula yields

\[
f(X_t) = f(x) + \sum_{i=0}^d (V_i f)(x) \int_0^t \circ dB_s^i + \sum_{i,j=0}^d (V_i V_j f)(x) \int_0^t \circ dB_{t_1}^i \circ dB_{t_2}^j \cdots .
\]

This is the stochastic Taylor formula, which is rigorously stated as follows. For a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_k) \in A \), we denote by \( V_\alpha \) the operator \( V_{\alpha_1} \cdots V_{\alpha_k} \).

**Proposition 2.3** (Lyons & Victoir 2004, Proposition 2.1; Kloeden & Platen 1992). Let \( f \in C_b^\infty(\mathbb{R}^N) \) and \( m \) be a positive integer. Then, we have

\[
f(X_t(x)) = \sum_{\alpha \in A(m)} (V_\alpha f)(x) + R_m(t, x, f),
\]

where the remainder term satisfies, for some constant \( C > 0 \),

\[
\sup_{x \in \mathbb{R}^N} \sqrt{\mathbb{E} [R_m(t, x, f)^2]} \leq C t^{\frac{m+1}{2}} \sup_{\beta \in A(m+2), A(m)} \|V_\beta f\|_\infty.
\]

### 2.2 Formulation and evaluation of cubature on Wiener space

We can now state the precise definition of the cubature formula (Lyons & Victoir 2004).

**Definition 2.4.** Let \( T > 0 \), and let \( m \) be a positive integer. The paths \( w_1, \ldots, w_n \in C_0^\infty([0, T]; \mathbb{R} \oplus \mathbb{R}^d) \) with bounded variation and the weights \( \lambda_1, \ldots, \lambda_n > 0 \) with \( \sum_{i=1}^n \lambda_i = 1 \) are said to define a cubature formula on Wiener space of degree \( m \) at time \( T \), if and only if,

\[
\mathbb{E} \left[ \int_{0 < t_1 < \cdots < t_k < T} \circ dB_{t_1}^{i_1} \cdots \circ dB_{t_k}^{i_k} \right] = \sum_{j=1}^n \lambda_j \int_{0 < t_1 < \cdots < t_k < T} dw_{j_1}^{i_1}(t_1) \cdots dw_{j_k}^{i_k}(t_k).
\]

holds for all \( (t_1, \ldots, t_k) \in A(m) \). Here, \( \circ dB_s^i \) represents \( ds \).
To construct a cubature formula, it suffices to find it over \([0,1]\). Indeed, when \(w_1, \ldots, w_n \in C_b^0([0,1]; \mathbb{R} \oplus \mathbb{R}^d)\) form the cubature at time one,
\[
 w_{1,T,J}(t) := \begin{cases} T w_1^j(t/T) & (i=0) \\ \sqrt{T} w_i^j(t/T) & (i=1, \ldots, d) \\ \end{cases}
\]
with the same weights define the cubature at time \(T\). This is an immediate consequence of the scaling property of the Brownian motion.

Once such paths are given, we can easily compute each evolution driven by \(w_i\) as it is just an ODE. For a path \(w \in C_b^0([0,T]; \mathbb{R} \oplus \mathbb{R}^d)\) of bounded variation, define \(\tilde{X}_t(x,w)\) as the solution of the ODE
\[
d\tilde{X}_t(x,w) = \sum_{i=0}^d V_i(\tilde{X}_t(x,w))dw^i(t), \quad \tilde{X}_0(x,w) = x.
\]
Then, \(\sum_{j=1}^n \lambda_j f(\tilde{X}_T(x,w_{r,j}))\) should well approximate \(E[f(X_T(x))]\). Indeed, the estimate in Proposition [2.3] holds for \(t = T\) if we replace the Wiener measure by the discrete measure \(\sum_{j=1}^n \lambda_j \delta_{w_j}\).

Therefore, (by applying Cauchy-Schwarz) we obtain the evaluation:
\[
\sup_{x \in \mathbb{R}^N} \left| E[f(X_T(x))] - \sum_{j=1}^n \lambda_j f(\tilde{X}_T(x,w_{r,j})) \right| \leq C T^{m+1} \sup_{\beta \in A(m+2)\backslash A(m)} \|V_\beta f\|_\infty \tag{2.3}
\]
with the constant \(C > 0\) depending only on \(w_1, \ldots, w_n\).

The above formula does not work as a good approximation unless \(T\) is small. Therefore, we divide \([0,T]\) into smaller time intervals as \(0 = t_0 < t_1 < \cdots < t_k = T\). If we consider the repeated application of the cubature formula over each subinterval \([t_{\ell-1}, t_\ell]\), we can use
\[
\sum_{j_1, \ldots, j_k=1}^n \lambda_{j_1} \cdots \lambda_{j_k} f(\tilde{X}_{t_\ell}(x,w_{s_{\ell-1}}, \cdots, w_{s_\ell}), \cdots, w_{s_k}), \tag{2.4}
\]
where \(w \ast v\) denotes the concatenation of two paths and \(s_\ell := t_\ell - t_{\ell-1}\) for each \(\ell = 1, \ldots, k\), as an approximation of the expectation \(E[f(X_T(x))]\). If we define discrete Markov random variables \(Y_0, \ldots, Y_k\) independent of the Brownian motion as
\[
Y_0 = x, \quad P(Y_\ell = \tilde{X}_{s_{\ell-1}}w(s_{\ell-1}, y) | Y_{\ell-1} = y) = \lambda_j \quad (\ell = 1, \ldots, k, j = 1, \ldots, n),
\]
\(E[Y_k]\) coincides with the approximation [2.4]. Then, combining an estimate for
\[
\sup_{x \in \mathbb{R}^N} |E[f(Y_k)] - E[f(X_T(x))]|
\]
with Proposition 2.2, we can prove the following assertion.

**Proposition 2.5** (Lyons & Victoir 2004, Proposition 3.6). Let \(f\) be a bounded Lipschitz function in \(\mathbb{R}^N\). Then, under (UFG) and (V0), we have
\[
\sup_{x \in \mathbb{R}^N} |E[f(Y_k)] - E[f(X_T(x))]| \leq C\|\nabla f\|_\infty \left( s_k^{1/2} + \sum_{\ell=1}^{k-1} \frac{s_{\ell}^{(m+1)/2}}{(T-t_{\ell})^{m/2}} \right)
\]
for some constant \(C > 0\), which is dependent only on \(m\) and \(w_1, \ldots, w_n\).

The equally spaced partition \(t_\ell = \ell T/k\) for \(\ell = 0, \ldots, k\) is not the optimal one in terms of asymptotic error bound with \(k \to \infty\). Consider taking \(t_\ell = T (1 - (1 - \delta_{\ell})\gamma)\) with a constant \(\gamma > 0\) independent of \(k\) (\(\gamma = 1\) corresponds to the equally spaced partition). By taking \(\gamma > m - 1\), we have the following estimate (Kusuoka 2001):
\[
\sup_{x \in \mathbb{R}^N} |E[f(Y_k)] - E[f(X_T(x))]| \leq C k^{-(m-1)/2} \|\nabla f\|_\infty.
\]
Therefore, a cubature formula of degree \(m\) with an appropriate time partition achieves the error rate \(O(k^{-(m-1)/2})\) where \(k\) is the number of partitions.
Remark 2.6. If we are to compute all the $k$-times concatenation of a cubature formula composed of $n$ sample paths, we have to solve $\frac{k+1}{n-1}$ ODEs in total (Lyons & Victoir 2004). When this is too large, we reduce the computational complexity through some Monte Carlo simulation or subsampling method (Litterer & Lyons 2012, Tchernychova 2015, Piazzon et al. 2017).

2.3 Known construction of cubature on Wiener space

We should note that some concrete examples of cubature formulas on Wiener space are already known. The simplest case treated in Lyons & Victoir (2004) is $m = 3$, where we have a cubature formula composed of linear paths (i.e., with only one linear segment).

Let $n := 2^d$ and $z_1, \ldots, z_n \in \mathbb{R}^d$ be all the elements of $\{-1,1\}^d$. Then, paths

$$w_i(t) := t(1, z_i) = (t, tz_1^i, \ldots, tz_n^i), \quad 0 \leq t \leq 1, \quad i = 1, \ldots, n$$

with weights $\lambda_1 = \cdots = \lambda_n = 2^{-d}$ construct a cubature formula with $m = 3$. Although $2^d$ is much larger than $|A(3)| = O(d^3)$, we can reduce the number of paths, e.g., by using Carathéodory-Tchakaloff subsampling.

Constructions for the case $d = 2$ and $m = 5$ are also given in Lyons & Victoir (2004), where the authors give cubature formulas using only $O(d^3)$ paths. Moreover, Gyurkó & Lyons (2011) constructed higher order cubature formulas up to $m = 11$, but the construction is limited to one dimensional space-time ($d = 1$).

All the examples itemized above are derived by solving equations in terms of Lie algebra (see Section 3.1), which can be directly written by making use of the Campbell-Baker-Hausdorff formula. However, as a different approach, we address an optimization-based construction in the next section.

3 Stochastic Tchakaloff’s theorem

We have seen in the previous section the theoretical support of cubature on Wiener space. However, it is important to know if such formulas can actually be constructed. In this section, we shall state stochastic Tchakaloff’s theorem, which assures the existence of cubature formulas on Wiener space. Though the stochastic Tchakaloff’s theorem is originally given in Lyons & Victoir (2004), we state it in a stronger way by using the concept of relative interior.

Before doing so, we shall introduce the rich algebraic structures behind the theory of cubature on Wiener space in the following first two sections, which are also essential in our proof of stochastic Tchakaloff’s theorem.

3.1 Tensor and Lie algebra

We introduce a tensor algebra which is suitable to our case (Lyons 1998, Lyons & Victoir 2004, Kusuoka 2004). Denote $\mathbb{R} \oplus \mathbb{R}^d$ by $E$. Define $U_0(E) := \mathbb{R} (=: E^{\otimes 0})$. Let $A_0 := \mathbb{R}$ and $A_1 := \mathbb{R}^d$, and define

$$U_n(E) := \bigoplus_{(i_1, \ldots, i_k) \in \{0,1\}^k, \quad 2k - (i_1 + \cdots + i_k) = n} A_{i_1} \otimes \cdots \otimes A_{i_k}$$

for each positive integer $n$. Here, the condition for $(i_1, \ldots, i_k)$ means that $A_{i_1} \otimes \cdots \otimes A_{i_k}$ takes all the arrangement of $\mathbb{R}$ and $\mathbb{R}^d$ such that $2(\# \text{ of } \mathbb{R}) + (\# \text{ of } \mathbb{R}^d) = n$. Then we consider the tensor algebra of formal series

$$T((E)) := \bigoplus_{n=0}^{\infty} U_n \left( \simeq \bigoplus_{n=0}^{\infty} E^{\otimes n} \right),$$

where the direct sum is regarded as a series, i.e., $T$ is the set of all the infinite sequences $(a_n)_{n=0}^{\infty}$ where $a_n \in U_n$ for each $n \geq 0$. Let $T^{(n)}(E) := \bigoplus_{k=0}^{n} U_k$, and let $\pi_n : T((E)) \to T^{(n)}(E)$ be the canonical projection for each $n \geq 0$. 

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where the latter two operations are limited for a \( \pi \) commute with each projection homomorphism \( R \) and \( L \). We shall introduce the signature (or Chen series, Chen 1957) of a path, which summarizes the algebraic structure of iterated integrals. Let

\[
\text{3.2 Signature of a path}
\]

Additionally, we define the exponential, inverse and logarithm:

\[
\exp(a) := \sum_{k=0}^{\infty} \frac{a^\otimes k}{k!}, \quad a^{-1} := \frac{1}{a_0} \sum_{k=0}^{\infty} \left( 1 - \frac{a}{a_0} \right)^\otimes k, \quad \log a := \log a_0 - \sum_{k=1}^{\infty} \frac{1}{k} \left( 1 - \frac{a}{a_0} \right)^\otimes k,
\]

where the latter two operations are limited for \( a \in T((E)) \) with \( a_0 \neq 0 \). Note that these operations commute with each projection homomorphism \( \pi_n \).

Let us introduce the space of Lie series. Define

\[
L((E)) := 0 \oplus E \oplus [E, E] \oplus [E, [E, E]] \oplus \cdots \subset T((E)) \simeq \bigoplus_{n=0}^{\infty} E^\otimes n,
\]

where, for linear subspaces \( A, B \in T((E)) \), \([A, B]\) is the linear subspace of \( T((E)) \) spanned by Lie brackets \([a, b] := a \otimes b - b \otimes a \ (a \in A, b \in B)\). \( L((E)) \) is the so-called free Lie algebra generated by \( E \) (Reutenauer 1993). The elements of \( L((E)) \) are called Lie series. We also define \( L^{(n)}(E) := \pi_n(L((E))) \), the elements of which are called Lie polynomials.

### 3.2 Signature of a path

We shall introduce the signature (or Chen series, Chen 1957) of a path, which summarizes the algebraic structure of iterated integrals. Let \( w = (w^0, \ldots, w^d) \in C_0^0([0, T]; \mathbb{R} \oplus \mathbb{R}^d) \) be a path of bounded variation. We define the signature of a path.

**Definition 3.1.** For \( 0 \leq s \leq t \leq T \), define \( S(w)_{s,t} \in T((E)) \) by

\[
S(w)_{s,t} := \sum_{n=0}^{\infty} \int_{s<t_1<\cdots<t_n<t} dw(t_1) \otimes \cdots \otimes dw(t_n).
\]

We call it the signature of \( w \) over \([s, t]\).

Here, the above definition can also be understood in terms of \( \mathbb{R}[[Z_0, Z_1, \ldots, Z_d]] \) as follows:

\[
S(w)_{s,t} = \sum_{n=0}^{\infty} \sum_{(i_1, \ldots, i_k) \in A(n) \setminus A(n-1)} \int_{s<t_1<\cdots<t_n<t} dw^{i_1}(t_1) \cdots dw^{i_k}(t_n) Z_{i_1} \cdots Z_{i_k}.
\]

In both cases, we think of the 0-th (or constant) term of \( S(w)_{s,t} \) as 1. The following is Chen’s theorem.

**Theorem 3.2** (Chen 1957, Lyons 1998). The process \( S(w) \) satisfies \( S(w)_{s,t} \otimes S(w)_{t,u} = S(w)_{s,u} \) for arbitrary \( 0 \leq s \leq t \leq u \leq T \). It also holds that \( \log S(w)_{s,t} \in L((E)) \) and therefore \( \pi_n(\log S(w)_{s,t}) \in L^{(n)}(E) \).

Moreover, the inverse of this correspondence holds, i.e., for an arbitrary Lie polynomial \( \mathcal{L} \in L^{(n)}(E) \subset T(E) \) and arbitrary \( 0 \leq s < t \leq T \), there exists a bounded-variation path \( w \in C_0^0([0, T]; \mathbb{R} \oplus \mathbb{R}^d) \) such that \( \pi_n(\log S(w)_{s,t}) = \mathcal{L} \).
The following theorem from Lyons & Victoir 2004 asserts a stronger statement.

**Theorem 3.3** (Lyons & Victoir 2004, Theorem 4.4). Let $\mathcal{L} \in L^{(m)}(E)$ be an arbitrary Lie polynomial such that the coefficient of the term $Z_0$ of $\exp(\mathcal{L}) \in \mathbb{R}[[Z_0, Z_1, \ldots, Z_d]]$ is equal to 1. Then, there exists a bounded-variation path $w \in C_0^0([0, 1]; \mathbb{R} \oplus \mathbb{R}^d)$ that satisfies

$$w^0(t) = t \quad (0 \leq t \leq 1) \quad \text{and} \quad \pi_n(\log S(w)_{0,1}) = \mathcal{L}.$$  

By virtue of these assertions, we see that the problem to find paths constructing a cubature formula is equivalent to the problem to find the corresponding Lie polynomials. The following Brownian-motion version of this result is also important.

**Proposition 3.4** (Kusuoka 2004, Lyons & Victoir 2004). Define the (Storatonovich) signature of the Brownian motion as an element of $T((E)) = \mathbb{R}[[Z_0, Z_1, \ldots, Z_d]]$ by

$$S(B)_{s,t} = \sum_{n=0}^{\infty} \sum_{(i_1, \ldots, i_k) \in \mathcal{A}(n) \setminus \mathcal{A}(n-1)} \left( \int_{s < t_1 < \cdots < t_n < t} \circ dB_{t_1}^{i_1} \cdots \circ dB_{t_n}^{i_k} \right) Z_{i_1} \cdots Z_{i_k}$$

for each $0 \leq s \leq t$. Then, $\log S(B)_{s,t}$ is almost surely a Lie series.

As we mainly deal with the signature over $[0,1]$, in the following let $S(w)$ and $S(B)$ represent $S(w)_{0,1}$ and $S(B)_{0,1}$, respectively. We also define, for each $\alpha = (i_1, \ldots, i_k) \in \mathcal{A}$,

$$I^\alpha(w) := \int_{0 < t_1 < \cdots < t_k < 1} dw^{i_1}(t_1) \cdots dw^{i_k}(t_k), \quad I^\alpha(B) := \int_{0 < t_1 < \cdots < t_k < 1} \circ dB_{t_1}^{i_1} \cdots \circ dB_{t_k}^{i_k}.$$  

Note that we set $I^\alpha(w) = I^\alpha(B) = 1$ if $\alpha = \emptyset$.

If we define $\mathcal{L} := \pi_n (\log S(B))$, we of course have the expression

$$E[\pi_n(\log S(B))] = E[\pi_n(\exp \mathcal{L})].$$

As $\mathcal{L}$ is a random Lie polynomial from the previous assertion, roughly speaking, the generalized Tchakaloff’s theorem (Theorem 1.1) and the inverse statement in Theorem 3.2 yield the existence of a cubature formula on Wiener space. However, we show the result in a stronger way in the following section for the sake of application in the next section.

### 3.3 Proof of stochastic Tchakaloff’s theorem

Throughout the section, we fix a positive integer $m$ and consider elements in $T^{(m)}(E)$. Note that $T^{(m)}(E)$ can naturally be regarded in the same light as $F := \mathbb{R}^d$. Define a set $G$ (as a subset of $F$) by Define a set $G$ (as a subset of $F$) by

$$G := \{ S(w) \mid w \in C_0^0([0, 1]; \mathbb{R} \oplus \mathbb{R}^d) \text{ is of bounded variation, } w^0(1) = 1 \}$$

From Theorem 3.2 this coincides with the set of $\exp(\mathcal{L})$, where $\mathcal{L}$ is a Lie polynomial with its coefficient of $Z_0$ is 1.

We denote the distribution of $S(B)$ over $F$ by $P_{S(B)}$. We shall argue the relation of $G$ and $\text{supp} P_{S(B)}$ in the following.

**Proposition 3.5.** It holds that $\text{aff supp} P_{S(B)} = \text{aff} G$.

**Proof.** From Proposition 3.2 $\text{supp} P_{S(B)} \subseteq \text{aff} G$ holds (as $\text{aff} G$ includes the closure of $G$). Therefore, its suffices to show $G \subseteq \text{aff supp} P_{S(B)}$.

As $\text{aff supp} P_{S(B)}$ is the intersection of all the hyperplane which includes $\text{supp} P_{S(B)}$, it can be represented as

$$\text{aff supp} P_{S(B)} = \bigcap_{(c,d) \in H} \{ v \in F \mid c^\top v = d \},$$
The result for the Itô integrals. Define

\[ c^\top S(B) = d \quad \text{a.s.} \quad \implies \quad c^\top S(w) = d \]

for every \( w \) appearing in the definition of \( G \). This follows from the following lemma as \( \pi_0(S(B)) = \pi_0(S(w)) = 1 \) always holds.

Lemma 3.6. Let \( (c_\alpha)_{\alpha \in \mathcal{A}} \in \mathbb{R}^d \) be a vector whose all but finite entries are zero. Then, if \( \sum_{\alpha \in \mathcal{A}(m)} c_\alpha I^\alpha(B) = 0 \) holds almost surely,

\[ \sum_{\alpha \in \mathcal{A}(m)} c_\alpha I^\alpha(w) = 0 \]

holds for every bounded-variation path \( w \in C^0_0([0,1];\mathbb{R} \oplus \mathbb{R}^d) \) with \( w^0(1) = 1 \).

Proof. This proof exploits several arguments in Kloeden & Platen (1992, Chapter 5). First, we prove the result for the Itô integrals. Define

\[ \mathcal{A}' := \{ \alpha \in \mathcal{A} \mid \alpha \text{ contains at least one non-zero index} \}. \]

We define for \( t \geq 0 \) that \( \mathcal{I}_t^\alpha := 1 \) and

\[ \mathcal{I}_t^\alpha := \int_{0<t_1<\ldots<t_k<t} dB_{t_1}^i \cdots dB_{t_k}^i \]

for each \( \alpha = (i_1, \ldots, i_k) \in \mathcal{A} \). We also define \( \alpha^- := (i_1, \ldots, i_{k-1}) \) and \( s(\alpha) = i_k \) for \( \alpha \neq \emptyset \). Note that it holds that

\[ \mathcal{I}_t^\alpha = \int_0^t \mathcal{I}_r^{\alpha^-} dB_r^{(s(\alpha))}. \]

For a multi-index \( \alpha = (i_1, \ldots, i_k) \in \mathcal{A} \), let \( \alpha^+ \) be the sequence defined by nonzero indices of \( \alpha \). For example, if \( \alpha = (0,2,0,1,0) \), then \( \alpha^+ = (2,1,1) \).

Let \( u = (u_t)_{t \geq 0} \) be a progressively measurable and second mean integrable stochastic process. Then, for each index \( i \neq 0 \),

\[ \int_0^t \left( \int_0^s u_r dB_r^i \right) ds = \int_0^t \left( \int_0^t u_r 1_{\{s>r\}}(r,s) ds \right) dB_r^i = \int_0^t (t-r) u_r dB_r^i \]

holds. By using Itô isometry and this relation repeatedly, we can show that

\[ \mathbb{E} \left[ \mathcal{I}_t^\alpha \mathcal{I}_t^\beta \right] = 0, \quad \text{if } \alpha, \beta \in \mathcal{A}' \text{ and } s(\alpha^+) \neq s(\beta^+). \quad (3.1) \]

Indeed, in the case \( k \)-times zeros appear in the suffix of \( \alpha \), we can show inductively

\[ \int_{0<t_1<\ldots<t_k<t} \left( \int_0^{t_1} u_r dB_r^i \right) dt_1 \cdots dt_k \]

\[ = \cdots = \int_{0<t_1<\ldots<t_k<t} \left( \int_0^{t_j} \frac{(t_j-r)^{j-1}}{(j-1)!} u_r dB_r^i \right) dt_j \cdots dt_k \]

\[ = \cdots = \int_0^t \frac{(t-r)^k}{k!} u_r dB_r^i, \quad (3.2) \]

so we can finally apply the following Itô isometry:

It holds, for progressively measurable and second mean integrable stochastic process \( u \) and \( v \), that

\[ \mathbb{E} \left[ \left( \int_0^t u_s dB_s^i \right) \left( \int_0^t v_s dB_s^j \right) \right] = \delta^{ij} \mathbb{E} \left[ \int_0^t u_s v_s ds \right], \]

where \( i, j \in \{1, \ldots, d\} \) and \( \delta^{ij} \) is Kronecker’s delta.
We then prove a stronger assertion than (3.1), which states
\[ E \left[ I_0^\alpha I_0^\beta \right] = 0, \quad \text{if } \alpha \lor \beta \in \mathcal{A}', \text{ and } \alpha^+ \neq \beta^+. \] (3.3)

Let \( \alpha = \tilde{\alpha} \ast (i, 0, \ldots, 0) \) and \( \beta = \tilde{\beta} \ast (j, 0, \ldots, 0) \). It suffices to consider the case \( i = j \neq 0 \), and we have
\[
E \left[ I_t^\alpha I_t^\beta \right] = E \left[ \left( \int_0^t \frac{(t-s)^k}{k!} I_s^\alpha \, dB_s^i \right) \left( \int_0^t \frac{(t-s)^\ell}{\ell!} I_s^\beta \, dB_s^j \right) \right]
= E \left[ \int_0^t \frac{(t-s)^{k+\ell}}{k! \ell!} I_s^\alpha I_s^\beta \, ds \right]
= \int_0^t \frac{(t-s)^{k+\ell}}{k! \ell!} E \left[ I_s^\alpha I_s^\beta \right] \, ds.
\] (3.4)

Therefore, by an inductive argument, it only remains to prove \( E \left[ I_t^\alpha I_t^\beta \right] = 0 \) in the case \( \alpha \lor \beta \) only contain zeros and \( \alpha^+ \neq \beta^+ \), but this case is trivial since the one is a constant and the other’s expectation becomes zero from (3.2). We now have completed the proof of (3.3).

From these results, we have the decomposition
\[
E \left[ \left( \sum_{\alpha \in \mathcal{A}} c_{\alpha} I_t^\alpha \right)^2 \right] = \sum_{\beta \in \mathcal{A}'} E \left[ \left( \sum_{\alpha \in \mathcal{A}, \alpha^+ = \beta} c_{\alpha} I_t^\alpha \right)^2 \right],
\]
where all but finite \( c_{\alpha} \) equal to zero and \( \mathcal{A}' \) denotes the set of all multi-indices which contain no zeros. Therefore, it suffices to consider sums of the form \( \sum_{i=1}^n c_i I_t^{\alpha_i} \), with \( c_1, \ldots, c_n \in \mathbb{R} \) and \( \alpha^+_1 = \cdots = \alpha^+_n \neq 0 \). Note that \( \alpha_1, \ldots, \alpha_n \) can be taken pairwise different. Assume this sum almost surely equals to zero. We next prove \( \sum_{i=1}^n c_i I_t^{\alpha_i} (w)_{0,1} \) is also zero for the \( w \) with \( w(s) = s \) (\( I_t^{\alpha}(w)_{s,t} \) is \( S(w)_{s,t} 's \) entry corresponding to the multi-index \( \alpha \)). Let each \( c_{\alpha_i} \) have the form \( \alpha_i = \beta_i \ast (j, 0, \ldots, 0) \),

where \( j \neq 0 \) is an index independent of \( i \).

From (3.2), we have
\[
E \left[ \left( \sum_{i=1}^n c_i I_t^{\alpha_i} \right)^2 \right] = E \left[ \left( \int_0^t \frac{(t-r)^{k_i}}{k_i!} c_i I_r^{\alpha_i} \, dB_r \right)^2 \right]
= E \left[ \int_0^t \left( \frac{n}{k_i!} \sum_{i=1}^n \frac{(t-r)^{k_i}}{k_i!} c_i I_r^{\alpha_i} \right)^2 \, dr \right]
= \int_0^t E \left[ \left( \sum_{i=1}^n \frac{(t-r)^{k_i}}{k_i!} c_i I_r^{\alpha_i} \right)^2 \right] \, dr.
\] (3.5)

If the left-hand side is zero, the integrand in the right-hand side is also zero for all \( 0 \leq r \leq t \) because it is continuous in \( r \).

We here note that the deterministic counterpart of (3.2) holds. Indeed, if \( u = (u_t)_{t \geq 0} \) is a continuous \( \mathbb{R} \)-valued path, then for a \( w \in C_0^1([0, 1]; \mathbb{R} \oplus \mathbb{R}^d) \) with \( w^0(t) = t \), we have
\[
\int_{0 < t_1 < \cdots < t_k < t} \left( \int_0^{t_1} u_r \, dw_r^i \right) \, dt_1 \cdots dt_k
= \cdots = \int_{0 < t_j < \cdots < t_k < t} \left( \int_0^{t_j} \frac{(t_j - r)^{j-1}}{(j-1)!} u_r \, dw_r^i \right) \, dt_j \cdots dt_k
= \cdots = \int_0^t \frac{(t-r)^k}{k!} u_r \, dw_r^i.
\] (3.6)
Therefore, we obtain
\[ \sum_{i=1}^{n} c_i I^\alpha_i(w)_{0,t} = \int_{0}^{t} \left( \frac{(t-r)^{k_i}}{k_i!} c_i I^{\beta_i}(w)_{0,r} \right) dr. \]  
(3.7)

As we have already obtained \( \sum_{i=1}^{n} \frac{(t-r)^{k_i}}{k_i!} c_i I^{\beta_i} = 0 \) for \( 0 \leq r \leq t \), the problem reduces to that of \( \beta_1, \ldots, \beta_n \). Therefore, by inductive arguments with respect to \( |\alpha_i^+| \), we can prove
\[ \sum_{i=1}^{n} c_i I^{\alpha_i} = 0 \implies \sum_{i=1}^{n} c_i I^{\alpha_i}(w)_{0,t} = 0, \]
where the base step is trivial since it only contains the integral with respect to time.

Though we have to extend this result when \( t = 1 \) to all the \( w \) satisfying only \( w^0(1) = 1 \), it can be proved by using perturbation arguments. More precisely, we can prove the result for piecewise linear paths with \( w^0(1) = 1 \) since \( \sum_i c_i I^{\alpha_i} \) is a polynomial of increments in the path and equals to zero at infinitely many points (if \( w^0 \) is monotone increasing then we can re-parametrize it so that \( w^0(t) = t \)). Indeed, proving for piecewise linear paths is sufficient (Friz & Victoir 2010, Theorem 7.28).

It remains to modify the result for Stratonovich integrals. The relation between multiple Stratonovich integrals and Ito integrals is known as follows (Kloeden & Platen 1992):
\[
\begin{align*}
I^0(B)_{0,t} &= T^0_t = 1, \\
I^{(i)}(B)_{0,t} &= T^{(i)}_t \quad (i = 0, 1, \ldots, d), \\
I^{\alpha}(B)_{0,t} &= \int_{0}^{t} I^{\alpha-}(B)_{0,r} dB_r^{(\alpha)} + \frac{1}{2} \cdot 1_{\{s(\alpha) = s(\alpha-) \neq 0\}} \int_{0}^{t} I^{(\alpha-)}(B)_{0,r} dr \quad (|\alpha| \geq 2).
\end{align*}
\]

Therefore, each \( I^{\alpha}(B)_{0,t} \) is represented as a positive combination of \( T^\beta_t \) such that \( \beta \) can be acquired by replacing two consecutive same nonzero indices by a zero some times. Assume \( \sum_{i=1}^{n} c_i I^{\alpha_i}(B) = 0 \) holds almost surely (\( c_i \neq 0 \) for all \( i \)). We shall again prove \( \sum_{i=1}^{n} c_i I^{\alpha_i}(w) \) by inductive arguments.

Let \( \alpha_0 \in \arg \max_{\alpha_+} |\alpha_+| \) and \( \beta := \alpha_0^+ \). Then, by the above expansion, \( \sum_{i=1}^{n} c_i I^{\alpha_i}(B) \) can be rewritten as a sum of Ito integrals. In particular,
\[
\sum_{i=1}^{n} c_i I^{\alpha_i}(B) - \sum_{i: \alpha_i^+ = \beta} c_i I^{\alpha_i}
\]
is represented as a weighted sum of \( T^\beta_t \) with \( \alpha^+ \neq \beta \). Therefore, \( \sum_{i: \alpha_i^+ = \beta} c_i I^{\alpha_i} = 0 \) holds almost surely, and so \( \sum_{i: \alpha_i^+ = \beta} c_i I^{\alpha_i}(w) = 0 \) holds for all the valid \( w \).

From Theorem 3.2 and Proposition 3.4 with probability one there exists some (random) \( w \) such that \( \pi_n(S(B)) = \pi_n(S(w)) \), where \( n \) is taken sufficiently large. By using such a \( w \), we obtain
\[
\sum_{i: \alpha_i^+ = \beta} c_i I^{\alpha_i}(B) = \sum_{i: \alpha_i^+ = \beta} c_i I^{\alpha_i}(w) = 0.
\]

Therefore, if \( \sum_{i=1}^{n} c_i I^{\alpha_i}(B) = 0 \) holds almost surely, then we can prove inductively (with respect to some order over \( \mathcal{A}^+ \)), for each \( \beta \in \mathcal{A}^+ \) and \( w \),
\[
\sum_{i: \alpha_i^+ = \beta} c_i I^{\alpha_i}(w) = 0, \quad \sum_{i: \alpha_i^+ = \beta} c_i I^{\alpha_i}(B) = 0.
\]
By considering the sum, we obtain \( \sum_{i=1}^{n} c_i I^{\alpha_i}(w) = 0 \) and the proof is completed. \( \square \)
Remark 3.7. In the last part of the above proof, we have essentially used the assertion (actually the inverse also follows from the above proof)

\[ \sum_{\alpha} c_{\alpha} I_{\alpha}^t = 0 \implies \sum_{\alpha} c_{\alpha} I(B)_{0,t} = 0. \]

Although we have proved it via Lie-algebraic arguments which exploit Theorem 3.2 and Proposition 3.3, it can be directly proved by repeatedly using the relations (3.2) and (3.5).

The following is a stochastic version of Tchakaloff’s theorem, which assures the existence of cubature formulas on Wiener space. It is stated in a little stronger way than Lyons & Victoir (2004, Theorem 2.4), in which the “relative interior” did not appear. Note that the time interval considered in the following is \([0, 1]\).

Theorem 3.8. Let \(m\) be a positive integer. There exist \(n\) paths \(w_1, \ldots, w_n \in C^0([0, 1]; \mathbb{R} \oplus \mathbb{R}^d)\) of bounded variation and \(n\) positive weights \(\lambda_1, \ldots, \lambda_n\) whose sum is 1 that satisfy \(n \leq |A(m)|\) and

\[ E[\pi_m(S(B))] = \sum_{i=1}^{n} \lambda_i \pi_m(S(w_i)). \]

Moreover, if we loosen the condition to be \(n \leq 2|A(m)|\), \(w_1, \ldots, w_n\) can be taken such that \(\pi_m(S(w_i))\) is contained in \(G\) for each \(i\), \(\text{aff}\{\pi_m(S(w_1)), \ldots, \pi_m(S(w_n))\} = \text{aff} G\), and

\[ E[\pi_m(S(B))] \in \text{ri conv}\{\pi_m(S(w_1)), \ldots, \pi_m(S(w_n))\}. \]

Proof. By virtue of Carathéodory’s theorem, the former part follows from the latter part. We here show the latter part.

From Theorem 1.3 and Proposition 3.5 we can find \(n\) Lie polynomials \(L_1, \ldots, L_n\) such that each \(\pi_m(\exp L_i)\) is contained in \(G\) and \(E[\pi_m(S(B))]\) is contained in the relative interior of their convex hull. Here, \(n\) can actually be taken such that \(n \leq 2 \dim G \leq 2|A(m)|\) because of Theorem 1.3. From the correspondence stated in Chen’s theorem (Theorem 3.2), we can find a desired set of paths in \(C^0([0, 1]; \mathbb{R} \oplus \mathbb{R}^d)\). Note that the condition \(\pi_m(\exp L_i) \subset G\) implies that the corresponding path satisfies \(w_0^i(1) = 1\).

Remark 3.9. By exploiting Theorem 3.3, we can also prove the same result even if we require \(w_0^i(t) = t\) for each \(i = 1, \ldots, n\) and \(0 \leq t \leq 1\). The modification is clear as \(\exp(L_i)\) in the above proof already satisfies that its coefficient of \(Z_0\) in the sense \(\exp(L_i) \in \mathbb{R}[Z_0, Z_1, \ldots, Z_d]\) is indeed 1.

4 Monte Carlo approach to cubature on Wiener space

In this section, we investigate a way to construct cubature formulas, which is based on mathematical optimization instead of Lie algebra. We limit the arguments to cubature formulas composed of continuous piecewise linear paths, and propose a construction based on Monte Carlo sampling, which is the application of Hayakawa (2020) to our case. We also carry out numerical experiments in concrete cases.

Although existing constructions of cubature formulas on Wiener space are based on Lie-algebraic equations, we can simply regard the cubature construction as an optimization problem. One such way is to consider an LP problem, which is analogous to ordinary cubature problems treated in Section 1.2. From this viewpoint, we can naively generate many sample paths and then reduce their number by using Carathéodory-Tchakaloff subsampling. We later see that this approach is applicable at least theoretically (Section 4.2).
4.1 Signature of continuous BV-paths

In this section, we see the properties of paths of bounded variation and their signature. We also see that the truncated signature of continuous BV paths can be approximated with any accuracy by that of piecewise linear paths.

Let \( w = (w^0, w^1, \ldots, w^d) \in C^0_0([0,1]; \mathbb{R} \oplus \mathbb{R}^d) \) be of bounded variation. We define the total variation of \( w \) as

\[
\|w\|_1 := \sup_{\Delta} \sum_{i=1}^{k} \max_{0 \leq j \leq d} \left| w^j(t_i) - w^j(t_{i-1}) \right| = \sup_{\Delta} \sum_{i=1}^{k} \|w(t_i) - w(t_{i-1})\|_{\infty},
\]

where \( \Delta \) is the partition of \([0,1]\) by \( 0 = t_0 < t_1 < \ldots < t_k = 1 \) and \( k \) also varies in \( \sup_{\Delta} \). We call \( w \) of bounded variation if \( \|w\|_1 < \infty \) holds. Note that other norms are also equivalent as the space \( \mathbb{R} \oplus \mathbb{R}^d \) is finite-dimensional though we are using the sup-norm \( \|\cdot\|_{\infty} \) of \( \mathbb{R} \oplus \mathbb{R}^d \).

We can re-parameterize \( w \) so that it becomes Lipschitz continuous if necessary. Indeed, if we let \( \|w\|_{[s,t]}_1 \) be the total variation of \( w \) over \([s,t] \) and

\[
\tau(t) := \frac{\|w\|_{[0,t]}_1}{\|w\|_1}
\]

for a non-constant \( w \), then \( w \circ \tau \) is a well-defined Lipschitz path (\( \tau \) becomes a non-decreasing function onto \([0,1]\)). It is also important that the signature is invariant under this re-parameterization (see, e.g., Friz & Victoir 2010, Proposition 1.42 and 7.10).

Hereafter, we may assume that there exist \( d+1 \) derivative functions \( f^0, f^1, \ldots, f^d \in L^\infty([0,1]; \mathbb{R}) \) such that

\[
w^j(t) = \int_0^t f^j(s) \, ds, \quad t \in [0,1], \ j = 0, 1, \ldots, d.
\]

In this case, the total variation of \( w \) can be written as

\[
\|w\|_1 = \int_0^1 \max_{0 \leq j \leq d} |f^j(s)| \, ds
\]

Signature can also be represented by using the derivatives as

\[
I^\alpha(w) = \int_{0 < t_1 < \ldots < t_k < 1} d\mu^{i_1}(t_1) \cdots d\mu^{i_k}(t_k)
= \int_{0 < t_1 < \ldots < t_k < 1} f^{i_1}(t_1) \cdots f^{i_k}(t_k) \, dt_1 \cdots dt_k.
\]

for each multi-index \( \alpha = (i_1, \ldots, i_k) \in \mathcal{A} \).

As a special case of BV paths, we are interested in (continuous) piecewise linear paths, which are easy to implement on computers. Let \( 0 = s_0 < s_1 < \ldots < s_n = 1 \) be a partition of \([0,1]\). Then, we can define a path \( w \in C^0_0([0,1]; \mathbb{R} \oplus \mathbb{R}^d) \) which is linear on each interval \([s_{j-1}, s_j] \) \((j = 1, \ldots, n)\) by determining the slope vector in \( \mathbb{R} \oplus \mathbb{R}^d \) at each interval. For the sake of computation, we here give the calculation of the signature explicitly. Let \( g_j = (g^{0}_j, g^{1}_j, \ldots, g^{d}_j) \) be the slope of \( w \) over \([s_{j-1}, s_j] \). Then, for \( \alpha = (i_1, \ldots, i_k) \in \mathcal{A} \),

\[
I^\alpha(w) = \sum_{1 \leq \ell_1 < \cdots < \ell_k \leq n+1} \Pi_{j=1}^{n} \left( s_j - s_{j-1} \right) \left( \ell_j - \ell_{j-1} \right)! \prod_{\ell=\ell_{j-1}}^{\ell_j-1} g^{i_{\ell}}_j
\]

holds. We can derive this by dividing the domain of integral into disjoint segments which are compatible with the partition \( 0 = s_0 < s_1 < \cdots < s_n = 1 \). If we adopt the notation \( g^\alpha_j := g^{i_1}_j \cdots g^{i_k}_j \) for each \( \alpha \in \mathcal{A} \), \( I^\alpha(w) \) can also be written as

\[
I^\alpha(w) = \sum_{\alpha \in \mathcal{A}} \prod_{j=1}^{n} \frac{\left( s_j - s_{j-1} \right)^{\left| \alpha_j \right|}}{\left| \alpha_j \right|!} g^\alpha_j.
\]
The latter expression can also be easily derived from Chen’s theorem (Theorem 3.2).

The following proposition evaluates the approximation error of “discretized” signature.

**Proposition 4.1.** Let \( w \in C^0_0([0, 1]; \mathbb{R} \oplus \mathbb{R}^d) \) be a Lipschitz path and consider a time partition \( 0 = s_0 < s_1 < \ldots < s_n = 1 \). Let \( \tilde{w} \) be a path which linearly connects the points \( w(s_0), w(s_1), \ldots, w(s_n) \). Then, we have

\[
|I^\alpha(w) - I^\alpha(\tilde{w})| \leq \frac{2|\alpha|!}{(|\alpha| - 1)!} \max_{1 \leq i \leq n} \|w|_{[s_{i-1}, s_i]}\|_1
\]

for each \( \alpha \in \mathcal{A} \setminus \{\emptyset\} \). In particular, this error is of the order \( O(1/n) \) when \( n \to \infty \) if we appropriately choose the sequence of partitions of \([0, 1]\) into \( n \) intervals.

**Proof.** Let \( f^0, f^1, \ldots, f^d \) be the derivatives of \( w \). Also, let \( \tilde{f}^0, \tilde{f}^1, \ldots, \tilde{f}^d \) be the derivatives of \( \tilde{w} \). Let \( \alpha = (i_1, \ldots, i_k) \in \mathcal{A} \). We can rewrite the difference \( I^\alpha(w) - I^\alpha(\tilde{w}) \) by using these derivatives as follows:

\[
I^\alpha(w) - I^\alpha(\tilde{w}) = \int_{0< t_1< \ldots< t_k< 1} f^{i_1}(t_1) \cdots f^{i_k}(t_k) \, dt_1 \cdots dt_k - \int_{0< t_1< \ldots< t_k< 1} \tilde{f}^{i_1}(t_1) \cdots \tilde{f}^{i_k}(t_k) \, dt_1 \cdots dt_k
\]

\[
= \sum_{j=1}^k \int_{0< t_1< \ldots< t_k< 1} \left( \prod_{\ell=1}^{j-1} f^{i_\ell}(t_\ell) \right) (f^{i_j}(t_j) - \tilde{f}^{i_j}(t_j)) \left( \prod_{\ell=j+1}^k \tilde{f}^{i_\ell}(t_\ell) \right) \, dt_1 \cdots dt_k.
\]

Let \( t_0 := 0 \) and \( t_{k+1} := 1 \). Here, the integral \( \int_{t_{j-1}}^{t_j} (f^{i_j}(s) - \tilde{f}^{i_j}(s)) \, ds \) appearing in the above expression can be estimated as in the following. Note that \( f^{i_j}(s) - \tilde{f}^{i_j}(s) \) becomes zero when integrated over each \([s_{i-1}, s_i]\). Let \( t_{j-1} \in [s_{i-1}, s_i] \) and \( t_{j+1} \in [s_{i'}, s_{i'}] \) hold. Then, we have

\[
\left| \int_{t_{j-1}}^{t_{j+1}} (f^{i_j}(s) - \tilde{f}^{i_j}(s)) \, ds \right| \\
\leq \left| \int_{t_{j-1}}^{s_i} (f^{i_j}(s) - \tilde{f}^{i_j}(s)) \, ds \right| + \left| \int_{s_{i'}-1}^{t_{j+1}} (f^{i_j}(s) - \tilde{f}^{i_j}(s)) \, ds \right|
\]

\[
= \frac{1}{2} \left( \left| \int_{s_{i-1}}^{t_{j-1}} (f^{i_j}(s) - \tilde{f}^{i_j}(s)) \, ds \right| + \left| \int_{t_{j-1}}^{s_{i'}} (f^{i_j}(s) - \tilde{f}^{i_j}(s)) \, ds \right| \right)
\]

\[
+ \frac{1}{2} \left( \left| \int_{s_{i'-1}}^{t_{j+1}} (f^{i_j}(s) - \tilde{f}^{i_j}(s)) \, ds \right| + \left| \int_{t_{j+1}}^{s_{i'}} (f^{i_j}(s) - \tilde{f}^{i_j}(s)) \, ds \right| \right)
\]

\[
\leq \max_{1 \leq i \leq n} \int_{s_{i-1}}^{s_i} |f^{i_j}(s) - \tilde{f}^{i_j}(s)| \, ds \leq 2 \max_{1 \leq i \leq n} \int_{s_{i-1}}^{s_i} |f^{i}(s)| \, ds
\]

Let \( \varepsilon_{ij} \) be the value of this right-hand side and define \( \varepsilon := \max_{0 \leq i \leq d} \varepsilon_i \). We also define a function on \([0, 1]\) as \( g := \max_{0 \leq i \leq d} \{|f^i|, |\tilde{f}^i|\} \). Remark that

\[
\int_0^1 g(s) \, ds \leq 2 \int_0^1 \max_{0 \leq i \leq d} |f^i(s)| \, ds
\]

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holds. From above arguments, we obtain
\[ |I^\alpha(w) - I^\alpha(\tilde{w})| \]
\[ \leq \varepsilon k \sum_{j=1}^{k} \int_{0<t_1<\cdots<t_{j-1}<t_{j+1}<\cdots<t_k} \prod_{\ell=1}^{j-1} |f^{i_\ell}(t_\ell)| \prod_{\ell=j+1}^{k} |\tilde{f}^{i_\ell}(t_\ell)| dt_1 \cdots dt_{j-1} dt_j \cdots dt_k \]
\[ \leq \varepsilon k \int_{0<t_1<\cdots<t_k} g(t_1) \cdots g(t_{k-1}) dt_1 \cdots dt_{k-1} \]
\[ = \frac{\varepsilon k}{(k-1)!} \left( \int_0^1 g(s) \, ds \right)^{k-1}. \]

We finally obtain the desired estimate by replacing these estimates in terms of \( \| \cdot \|_1 \).

The following assertion is an immediate consequence of the above proposition, combined with Theorem 3.2 and Theorem 3.3.

**Corollary 4.2.** The set of continuous piecewise linear paths is dense in the set of paths in \( C^0_0([0,1]; \mathbb{R} \oplus \mathbb{R}^d) \) of bounded variation, in terms of the pseudo-metric induced by the Euclidean distance of their signatures in \( T^{(m)}(E) \) for each \( m \geq 1 \).

The statement still holds even if the piecewise linear paths are restricted to ones such that \( w^0(t) = t \) \((0 \leq t \leq 1)\).

**Remark 4.3.** With regard to the former part of the above corollary, a stronger result is known (Friz & Victoir 2010, Theorem 7.28). Every Lie polynomial can be exactly (not approximately) represented as a (truncated) logarithm of some continuous piecewise linear path with a finite number of linear intervals.

### 4.2 Piecewise linear cubature

The following theorem assures the existence of a cubature formula on Wiener space composed of continuous piecewise linear paths.

**Theorem 4.4.** For each positive integer \( m \), there exist \( n \) paths \( w_1, \ldots, w_n \in C^0_0(\mathbb{R} \oplus \mathbb{R}^d) \) which are piecewise linear and \( n \) positive weights \( \lambda_1, \ldots, \lambda_n \) whose sum is 1 that satisfy \( n \leq |A(m)| \) and
\[ E[\pi_m(S(B))] = \sum_{i=1}^{n} \lambda_i \pi_m(S(w_i)). \]

The statement still holds even if we require \( w_i^0(t) = t \) for \( i = 1, \ldots, n \) and \( 0 \leq t \leq 1 \).

**Proof.** From Theorem 3.8 (see also Remark 3.9 and Corollary 4.2), we can easily deduce that there exist a set of at most \( 2|A(m)| \) continuous piecewise linear paths whose truncated (by \( \pi_m \)) signatures’ convex hull contains \( E[\pi_m(S(B))] \) (in its relative interior). Rigorously, we can apply the same argument as the proof of Theorem 1.4. Finally, by Carathéodory’s theorem, \( n \leq |A(m)| \) can actually be achieved.

Based on this theorem, it suffices for us to look for cubature formulas within piecewise linear paths. Our approach to construction of a piecewise linear cubature is an application of “Monte Carlo cubature construction” (Hayakawa 2020). Of course, we are not able to generate a Brownian motion and use it as a candidate for sample points of cubature formulas, because it is not of bounded variation and it cannot be implemented on computers anyway. However, the methods in Hayakawa (2020) is still applicable here as we see in the following proposition.

**Proposition 4.5.** Let \( m \) be a positive integer. Then, for a sufficiently large \( M \) (the lower bound of \( M \) depends on \( m \)), the following statement holds:

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Let a sequence of continuous piecewise linear paths \( w_1, w_2, \ldots \) be generated identically and independently. Assume also each \( w_i \) satisfies that

- \( w_i \) is a path which linearly connects points \( w_i(k/M) \) \((0 \leq k \leq M)\);
- one of (a) and (b) holds:
  - (a) \( w_i^d(t) = t \) \((0 \leq t \leq 1)\) holds;
  - (b) \( w_i^d(0) = 0 \) and \( w_i^d(1) = 1 \) hold, and \( w_i^d\left(\frac{k}{M}\right) - w_i^d\left(\frac{k-1}{M}\right) \) \((1 \leq k \leq M - 1)\) are independent random variables and have a density on \( \mathbb{R} \) which is positive almost everywhere;
- random variables \( w_i^j\left(\frac{k}{M}\right) - w_i^j\left(\frac{k-1}{M}\right) \) \((1 \leq j \leq d, 1 \leq k \leq M)\) are independent (also from the ones of zero-th coordinate) and have a density on \( \mathbb{R} \) which is positive almost everywhere.

Then, with probability one, there exists an \( N \) such that a subset of \( \{w_1, w_2, \ldots, w_N\} \) can construct a cubature on Wiener space of degree \( m \).

**Proof.** Take \( M \) as large as we can find (at most) \( 2|A(m)| \) piecewise linear paths (denoted by \( \hat{w}_i \)) with at most \( M \) linear segments such that \( \text{conv}\{\pi_m(S(\hat{w}_i))\} \subseteq \text{int} \). The existence of such an \( M \) is assured by the proof of Theorem 4.4 (or see Theorem 3.8, Remark 3.9 and Corollary 4.2).

From (4.2), the truncated signature of each \( w_i \) is a polynomial of random variables \( w_i^j\left(\frac{k}{M}\right) - w_i^j\left(\frac{k-1}{M}\right) \) \((0 \leq j \leq d, 1 \leq k \leq M, (j, k) \neq (0, M))\). Our assumption assures that these variables take values in every neighborhood of some point with a positive probability. In particular, it implies that

\[
P\left(\|\pi_m(S(\hat{w}_i)) - \pi_m(S(w_i))\| < \varepsilon\right) > 0,
\]

where \( \| \cdot \| \) is the Euclidean norm on \( T^{(m)}(E) \simeq F \), holds for each \( i, \ell \) and \( \varepsilon > 0 \). Note that the left-hand side probability does not depend on \( i \) by i.i.d. assumption. Therefore, the argument in the proof of Theorem 4.4 holds here again and we obtain the desired assertion.

**Remark 4.6.** For the scheme (b), the condition \( w_i^d(1) = 1 \) is necessary to assure that each \( \pi_m(S(w_i)) \) is contained in \( G \) defined in Section 4.3.

Note that the generation rule of sample paths in this proposition is just one of infinitely many possible examples. We may alternatively, for instance, directly generate \( w_i^j(k/M) \) independently.

Proposition 4.5 assures only the existence of “some large” \( M \) and \( N \), so it might be numerically hard to find cubature formulas from this approach. However, it is beneficial to know that the construction can be reduced to at least the stage of machine power.

### 4.3 Numerical experiments

We explain our simple numerical method based on a Monte Carlo approach. We first describe the algorithm for computing the signature of piecewise linear paths, and then we give our Monte Carlo approach and its result in some pairs of \( (d, m) \).

**Calculation of signature** Note that in the following \( d \) and \( m \) are regarded as parameters already given. For positive integers \( M \) and \( N \), we generate \( N \) piecewise linear paths (denoted by \( w_1, \ldots, w_N \)) with \( M \) intervals of time (see Proposition 4.5). The time complexity of this paths generation is \( O(NMd) \). In the following, we address each component of generated paths by

\[
\text{PATH}[i, j, k] := w_i^k\left(\frac{j}{M}\right) - w_i^k\left(\frac{j-1}{M}\right) \quad (1 \leq i \leq N, 1 \leq j \leq M, 0 \leq k \leq d).
\]

In all the experiments, we generated \( \text{PATH}[i, j, k] \) (with \( k \neq 0 \)) so that it follows the centered normal distribution of variance \( 1/M \). We set \( \text{PATH}[i, j, 0] = 1/M \) for each \( i, j \). We also denote \( w_i \) by just writing \( \text{PATH}[i] \) for each \( i \).
As the $M$ we consider is not so large in this study, we calculate the signature of generated paths by a simple dynamic programming (Algorithm 1). In the algorithm, we calculate the signature of $w_i$ over $[0, k/M]$ for $k = 1, \ldots, M$, by using the expression (4.2). The time complexity of this algorithm is $O(M|A(m)|^2)$, though a pruning of possible multi-indices $(\alpha, \beta)$ works a little.

Algorithm 1 Calculation of (i-th) signature

Input: $M, \text{PATH}[i]$

Initialize:
- $\text{SIGNATURE}[i, \emptyset] = 1$
- $\text{SIGNATURE}[i, \alpha] = 0$ ($\alpha \in A_0(m)$)
- $\text{TEMPORARY}[\alpha]$ ($\alpha \in A(m)$)
- $\text{NEXT}[\alpha]$ ($\alpha \in A(m)$)

for $j = 1, \ldots, M$ do
  for $\alpha \in A(m)$ do
    $\text{TEMPORARY}[\alpha] = \text{SIGNATURE}[i, \alpha]$
    $\text{NEXT}[\alpha] = 1 / |\alpha| \prod_{k=1}^{|\alpha|} \text{PATH}[i, j, \alpha_k]$
    $\text{SIGNATURE}[i, \alpha] = 0$
  end for
  for $(\alpha, \beta) \in A(m) \times A(m)$ do
    if $\alpha * \beta \in A(m)$ then
      $\text{SIGNATURE}[i, \alpha * \beta] += \text{TEMPORARY}[\alpha] \cdot \text{NEXT}[\beta]$
    end if
  end for
end for

Output: $\text{SIGNATURE}[i]$

Monte Carlo approach In the approach based on Monte Carlo sampling, we simply generate many paths and determine, by solving an LP problem, whether or not we can construct a cubature formula of desired degree from generated paths.

The part of solving an LP problem was done by using IBM ILOG CPLEX Optimization Studio version 12.10 (we shall call it simply CPLEX in the following). From Proposition 4.5, for a sufficiently large $N$ we can construct a cubature formula using a subset of paths $\{w_1, \ldots, w_N\}$.

We conducted experiments for six cases $(d, m) = (2, 3), (3, 3), (4, 3), (2, 5), (3, 5), (2, 7)$ and, for each $(d, m)$, set $N = 2|A(m)|, 4|A(m)|, 8|A(m)|, M = 2, 4, 8, 16, 32$ and examined if we could construct a cubature formula by using CPLEX (note that $|A(m)|$ depends on $d$). The following tables show how many times out of 10 trials we successfully obtained cubature formulas. Blanks in the tables mean that the corresponding experiments were not done because we already got 10 successes out of 10 with a smaller $N$.

Table 1: $(d, m) = (2, 3), |A(m)| = 20$

| N \backslash M | 2  | 4  | 8  | 16 | 32 |
|--------------|----|----|----|----|----|
| 2|A(m)| | 3  | 4  | 3  | 2  | 2  |
| 4|A(m)| | 10 | 10 | 10 | 10 | 10 |
| 8|A(m)| | 10 | 10 | 10 | 10 | 10 |

Table 2: $(d, m) = (3, 3), |A(m)| = 47$

| N \backslash M | 2  | 4  | 8  | 16 | 32 |
|--------------|----|----|----|----|----|
| 2|A(m)| | 1  | 2  | 1  | 1  |
| 4|A(m)| | 10 | 10 | 10 | 10 | 10 |
| 8|A(m)| | 10 | 10 | 10 | 10 | 10 |

From these results, we may expect that the change in $m$ is more essential than $d$ in that we need larger number of partitions and ratio $N/|A(m)|$ as $m$ gets larger, though more experiments are necessary.

1https://www.ibm.com/analytics/cplex-optimizer
Table 3: \((d, m) = (4, 3), |\mathcal{A}(m)| = 94\)

| N \(\backslash\) M | 2  | 4  | 8  | 16 | 32 |
|-------------------|----|----|----|----|----|
| 2|\(\mathcal{A}(m)\) | 2  | 0  | 0  | 2  | 1  |
| 4|\(\mathcal{A}(m)\) | 10 | 10 | 10 | 10 | 10 |
| 8|\(\mathcal{A}(m)\) | 10 |

Table 4: \((d, m) = (2, 5), |\mathcal{A}(m)| = 119\)

| N \(\backslash\) M | 2  | 4  | 8  | 16 | 32 |
|-------------------|----|----|----|----|----|
| 2|\(\mathcal{A}(m)\) | 0  | 0  | 0  | 0  | 0  |
| 4|\(\mathcal{A}(m)\) | 7  | 10 | 10 | 10 | 10 |
| 8|\(\mathcal{A}(m)\) | 10 |

Table 5: \((d, m) = (3, 5), |\mathcal{A}(m)| = 516\)

| N \(\backslash\) M | 2  | 4  | 8  | 16 | 32 |
|-------------------|----|----|----|----|----|
| 2|\(\mathcal{A}(m)\) | 0  | 0  | 0  | 0  | 0  |
| 4|\(\mathcal{A}(m)\) | 2  | 10 | 10 | 10 | 10 |
| 8|\(\mathcal{A}(m)\) | 10 |

Table 6: \((d, m) = (2, 7), |\mathcal{A}(m)| = 696\)

| N \(\backslash\) M | 2  | 4  | 8  | 16 | 32 |
|-------------------|----|----|----|----|----|
| 2|\(\mathcal{A}(m)\) | 0  | 0  | 0  | 0  | 0  |
| 4|\(\mathcal{A}(m)\) | 0  | 0  | 0  | 1  | 8  |
| 8|\(\mathcal{A}(m)\) | 10 |

5 Concluding remarks

In this paper, we have seen that we can construct piecewise linear cubature formulas on Wiener space through a Monte Carlo sampling and an LP problem. Our construction is supported by the technical contribution which extends stochastic Tchakaloff’s theorem by using our characterization of the distribution of Storatonovich iterated integrals. We confirmed that for small pairs of \((d, m)\) our algorithm actually works in numerical experiments.

Although we have shown that one can theoretically construct cubature formulas of any dimension and degree, the number of paths used in our construction only attains the Tchakaloff bound and so it requires too much computational cost for large \((d, m)\) in practice. Therefore, we may consider reducing the number of paths by using additional optimization techniques.

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