Stationary States of Dissipative Quantum Systems

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Abstract

In this Letter we consider stationary states of dissipative quantum systems. We discuss stationary states of dissipative quantum systems, which coincide with stationary states of Hamiltonian quantum systems. Dissipative quantum systems with pure stationary states of linear harmonic oscillator are suggested. We discuss bifurcations of stationary states for dissipative quantum systems which are quantum analogs of classical dynamical bifurcations.

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1 Introduction

The dissipative quantum systems are of strong theoretical interest. As a rule, any microscopic system is always embedded in some (macroscopic) environment and therefore it is never really isolated. Frequently, the relevant environment is in principle unobservable or it is unknown. This would render theory of dissipative quantum systems a fundamental generalization of quantum mechanics.

Spohn derives sufficient condition for existence of an unique stationary state for dissipative quantum system described by Lindblad equation. The irreducibility condition given by defines stationary state of dissipative quantum systems. An example, where the stationary state is unique and approached by all states for long times is considered by Lindblad for Brownian motion of quantum harmonic oscillator. The stationary solution of Wigner function evolution equation for dissipative quantum system was discussed in. Quantum effects in the steady states of the dissipative map are considered in.
2 Definition of stationary states

In the general case, the time evolution of quantum state $\rho_t$ is described by Liouville-von Neumann equation

$$\frac{d}{dt}\rho_t = \hat{\Lambda}\rho_t, \quad (1)$$

where $\hat{\Lambda}$ is a quantum Liouville operator. For Hamiltonian systems quantum Liouville operator has the form

$$\hat{\Lambda}\rho_t = -\frac{i}{\hbar}[H, \rho_t], \quad (2)$$

where $H = H(q, p)$ is a Hamilton operator. If quantum Liouville operator $\hat{\Lambda}$ cannot be represented in the form (2), then quantum system is called non-Hamiltonian or dissipative quantum system. Stationary state is defined by the condition

$$\hat{\Lambda}\rho_t = 0.$$

For Hamiltonian systems this condition has the form

$$[H, \rho_t] = 0. \quad (3)$$

3 Pure stationary states of Hamiltonian systems

A pure state $\rho_\Psi = |\Psi><\Psi|$ is a stationary state of Hamiltonian quantum system, if $|\Psi>$ is an eigenvector of Hamilton operator $H = H(q, p)$. Using $<\Psi|\Psi> = 1$, we get the equality (3) in the form

$$H|\Psi> = |\Psi> E, \quad (4)$$

where $E = <\Psi|H|\Psi>$. Equation (4) defines pure stationary states $|\Psi>$ of Hamiltonian systems. Eigenvalues of Hamilton operator are identified with the energy of the system. It is known, that Hamilton operator for linear harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{m\omega^2q^2}{2}. \quad (5)$$

Equation (4) has the solution if

$$E_n = \frac{1}{2}\hbar\omega(2n + 1). \quad (6)$$

In coordinate representation stationary states of linear harmonic oscillator are

$$\Psi_n(q) = \frac{1}{q_0}exp\left(-\frac{q^2}{2q_0^2}\right)\mathcal{H}_n\left(\frac{q}{q_0}\right), \quad q_0 = \sqrt{\frac{\hbar}{m\omega}}, \quad (7)$$

where $\mathcal{H}_n(q/q_0)$ is Hermitian polynomial of order $n$. 

2
4 Pure stationary states of dissipative systems

Let us consider Liouville-von Neumann equation (11) of the form

$$\frac{d}{dt} \rho_t = -\frac{i}{\hbar} [H, \rho_t] + \sum_{k=1}^{s} \hat{F}_k N_k(\hat{L}_H, \hat{R}_H) \rho_t. \quad (8)$$

Here $\hat{F}_k$ are operators act on operator space, $\hat{L}_A$ and $\hat{R}_A$ are operators of left and right multiplication \cite{13} defined by

$$\hat{L}_A B = AB, \quad \hat{R}_A B = BA,$$

for all operators $B$.

Let $\rho_\Psi = |\Psi><\Psi|$ is a pure state with eigenvector $|\Psi>$ of the Hamilton operator $H$. If equation (14) is satisfied, then the state $\rho_\Psi = |\Psi><\Psi|$ is a stationary state of Hamilton system

$$\frac{d}{dt} \rho_t = -\frac{i}{\hbar} [H, \rho_t], \quad (9)$$

associated with dissipative system (8).

If the vector $|\Psi>$ is eigenvector of $H$, then Liouville-von Neumann equation (8) for pure state $\rho_\Psi = |\Psi><\Psi|$ has the form

$$\frac{d}{dt} \rho_\Psi = \sum_{k=1}^{s} N_k(E, E) \hat{F}_k \rho_\Psi,$$

where the functions $N_k(E, E)$ are defined by

$$N_k(E, E) = \langle \Psi | (N_k^\dagger(\hat{L}_H, \hat{R}_H) I) | \Psi \rangle.$$

Operator $N_k^\dagger(\hat{L}_H, \hat{R}_H)$ is adjoint operator on operator space defined by

$$(N_k^\dagger(\hat{L}_H, \hat{R}_H) A | B) = (A | N_k(\hat{L}_H, \hat{R}_H) B),$$

where $(A | B) = Tr(A^\dagger B)$. If all functions $N_k(E, E)$ are equal to zero

$$N_k(E, E) = 0, \quad (10)$$

then the stationary state of Hamiltonian quantum system (9) is stationary state of dissipative quantum system (8).

Note, that functions $N_k(E, E)$ are eigenvalues and $|\Psi>$ is eigenvector of operators $N_k(H, H) = N_k^\dagger(\hat{L}_H, \hat{R}_H) I$, since

$$N_k(H, H) | \Psi >= | \Psi > N_k(E, E).$$

Therefore stationary states of dissipative quantum system (8) are defined by zero eigenvalues of operators $N_k(H, H) = N_k^\dagger(\hat{L}_H, \hat{R}_H) I$. 

3
5 Dissipative systems with oscillator stationary states

In this section we consider simple examples of dissipative quantum systems (5).

1) Let us consider nonlinear oscillator with friction defined by the equation

\[ \frac{d}{dt}\rho_t = -\frac{i}{\hbar}[H_{nl},\rho_t] + \frac{i}{\hbar}\beta[q^2, p^2 \circ \rho_t], \]  

where Hamilton operator $H_{nl}$ is

\[ H_{nl} = \frac{p^2}{2m} + \frac{m\Omega^2 q^2}{2} + \frac{\gamma q^4}{2}, \]

and

\[ A \circ B = \frac{1}{2}(AB + BA). \]

Equation (11) can be rewritten in the form

\[ \frac{d}{dt}\rho_t = -\frac{i}{\hbar}[H,\rho_t] + \frac{2im\beta}{\hbar}[q^2, (\frac{p^2}{2m} + \frac{\gamma q^2}{2m\beta} - \frac{\Delta}{4\beta} I) \circ \rho_t], \]  

where $\Delta = \Omega^2 - \omega^2$, and $H$ is Hamilton operator of linear harmonic oscillator (5). Equation (12) has the form (8), where

\[ \hat{F} = \frac{2im\beta}{\hbar}(\hat{L}q^2 - \hat{R} q^2), \]

\[ N(\hat{L}_H, \hat{R}_H) = \frac{1}{2}(\hat{L}_H + \hat{R}_H) - \frac{\Delta}{2\beta} \hat{L}_I, \]

\[ N(E, E) = <\Psi|H - \frac{\Delta}{2\beta} I|\Psi> = E - \frac{\Delta}{2\beta}. \]

Let $\gamma = \beta m^2 \omega^2$. The dissipative system (11) has one stationary state (7) of harmonic oscillator with energy $E_n = (\hbar \omega/2)(2n + 1)$, if

\[ \Delta = 2\beta \hbar \omega(2n + 1), \]

where $n$ is an integer nonnegative number. This stationary state is one of stationary states of linear harmonic oscillator with the mass $m$ and frequency $\omega$. In this case, we can have the quantum analog [14] of dynamical Hopf bifurcation [15, 16].

2) Let us consider dissipative system described by evolution equation

\[ \frac{d}{dt}\rho_t = -\frac{i}{\hbar}[H,\rho_t] + \frac{i}{\hbar}[q, N(\hat{L}_H, \hat{R}_H)\rho_t], \]  

where the Hamilton operator is defined by (5) and

\[ N(\hat{L}_H, \hat{R}_H) = \cos\left(\frac{\pi}{2\varepsilon_0}(\hat{L}_H + \hat{R}_H)\right) = \sum_{m=0}^{\infty} \frac{1}{(2m)!}\left(\frac{i\pi}{2\varepsilon_0}\right)^{2m}(\hat{L}_H + \hat{R}_H)^{2m}. \]
The operator $\hat{F}$ on operator space is

$$\hat{F} = \frac{i}{\hbar}(\hat{L}_q - \hat{R}_q).$$

The function $N(E, E)$ has the form

$$N(E, E) = \cos\left(\frac{\pi E}{\varepsilon_0}\right) = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \left(\frac{i\pi E}{\varepsilon_0}\right)^{2m}.$$

The stationary state condition (10) has the solution

$$E = \frac{\varepsilon_0}{2}(2n + 1),$$

where $n$ is an integer number. If parameter $\varepsilon_0$ is equal to $\hbar \omega$, then quantum system (13), (14) has stationary states (7) with the energy (6). As the result stationary states of dissipative quantum system (13) coincide with stationary states (7) of the linear harmonic oscillator.

If the parameter $\varepsilon_0$ is equal to $\hbar \omega(2l + 1)$, then quantum system (13), (14) has stationary states (7) with $n(k, m) = 2kl + k + l$ and

$$E_{n(k,l)} = \frac{\hbar \omega}{2}(2k + 1)(2l + 1).$$

3) Let us consider the operators $N_k(\hat{L}_H, \hat{R}_H)$ in the form

$$N_k(\hat{L}_H, \hat{R}_H) = \frac{1}{2\hbar} \sum_{n,m} v_{kn}v_{km}^* (2\hat{L}_H^n \hat{R}_H^m - \hat{L}_H^{n+m} - \hat{R}_H^{n+m}),$$

and $\hat{F}_k = \hat{L}_I$. In this case, Liouville-von Neumann equation (8) has the form of Lindblad equation [17, 18, 19]:

$$\frac{d}{dt}\rho_t = -\frac{i}{\hbar}[H, \rho_t] + \frac{1}{2\hbar} \sum_{k=1}^{m} ([V_k \rho_t, V_k^\dagger] + [V_k, \rho_t V_k^\dagger]). \quad (15)$$

with operators

$$V_k = \sum_n v_{kn}H^n, \quad V_k^\dagger = \sum_m v_{km}^* H^m.$$

If $\rho_\Psi = |\Psi><\Psi|$ is a pure stationary state, then $N_k(E, E) = 0$ and this state is a stationary state of the dissipative quantum system (15).
6 Dynamical bifurcations and catastrophes

Let us consider a special case of dissipative quantum systems such that the function $N_k(E, E)$ be a potential function, i.e., we have a potential $V(E)$ such that

$$\frac{\partial V(E)}{\partial E_k} = N_k(E, E),$$

where $E_k = <\Psi|H_k|\Psi>$, and

$$N_k(E, E) = <\Psi|(N_k^1(\hat{L}_H, \hat{R}_H)I)|\Psi>,
\quad H = \sum_{k=1}^s H_k, \quad H_k|\Psi> = |\Psi > E_k.$$

In this case, the stationary condition for dissipative system is defined by critical points of the potential $V(E)$. If the system has one variable $E$, then the function $N(E, E)$ is always potential function. In the general case, the functions $N_k(E, E)$ are potential, if

$$\frac{\partial N_k(E, E)}{\partial E_l} = \frac{\partial N_l(E, E)}{\partial E_k}.$$

Stationary states of dissipative quantum system with potential functions $N_k(E, E)$ are depend on critical points of potential $V(E)$. If the system has one variable $E$, then the function $N(E, E)$ is always potential function. In the general case, the functions $N_k(E, E)$ are potential, if

$$\frac{\partial N_k(E, E)}{\partial E_l} = \frac{\partial N_l(E, E)}{\partial E_k}.$$

For polynomial operators $N_k(\hat{L}_H, \hat{R}_H)$ we have

$$N_k(\hat{L}_H, \hat{R}_H)\rho = \sum_{n=0}^N \sum_{m=0}^n a^{(k)}_{n,m} H^m \rho H^{n-m}.$$

In the general case, $m$ and $n$ are multi-indeces. The function $N_k(E, E)$ is a polynomial

$$N_k(E, E) = \sum_{n=0}^N \alpha_n^{(k)} E^n,$$

where

$$\alpha_n^{(k)} = \sum_{m=0}^n a^{(k)}_{n,m}.$$

We can define the variable $x = E - a$, such that function $N_k(E, E) = N_k(x + a, x + a)$ has no the term $x^{n-1}$.

$$N_k(x + a, x + a) = \sum_{n=0}^N \alpha_n^{(k)} (x + a)^n =$$
\[
= \sum_{n=0}^{N} \sum_{m=0}^{n} \alpha_n^{(k)} \frac{n!}{m!(n-m)!} x^m (a^{(k)})^{n-m}.
\]

If the coefficient of the term \( x^{n-1} \) is equal to zero
\[
\alpha_n^{(k)} \frac{n!}{(n-1)!} a^{(k)} + \alpha_{n-1}^{(k)} = \alpha_n^{(k)} n a^{(k)} + \alpha_{n-1}^{(k)} = 0,
\]

then we have
\[
a^{(k)} = -\frac{\alpha_{n-1}^{(k)}}{n \alpha_n^{(k)}}.
\]

If we change parameters \( \alpha_n^{(k)} \), then can arise stationary states of dissipative quantum systems. For example, the bifurcation with birth of linear oscillator stationary state is a quantum analog of dynamical Hopf bifurcation [15, 16].

If the function \( N(E, E) \) is equal to \( N(E, E) = \pm \alpha_n E^n + \sum_{j=1}^{n-1} \alpha_j E^j \ n \geq 2, \)
then potential \( V(x) \) is
\[
V(x) = \pm x^{n+1} + \sum_{j=1}^{n-1} a_j x^j \ n \geq 2,
\]
and we have catastrophe of type \( A_{\pm n} \).

If we have \( s \) variables \( E_k \), where \( k = 1, 2, ..., s \), then quantum analogous of elementary catastrophes \( A_{\pm n}, \ D_{\pm n}, \ E_{\pm 6}, \ E_7 \) and \( E_8 \) can be realized. Let us write the full list of typical set of potentials \( V(x) \), which leads to elementary catastrophes (zero-modal) defined by \( V(x) = V_0(x) + Q(x) \), where
\[
A_{\pm n} : \ V_0(x) = \pm x_1^{n+1} + \sum_{j=1}^{n-1} a_j x_1^j \ n \geq 2,
\]
\[
D_{\pm n} : \ V_0(x) = x_1^2 x_2 + x_2^{n-1} + \sum_{j=1}^{n-3} a_j x_2^j + \sum_{j=n-2}^{n-1} x_1^{j-(n-3)},
\]
\[
E_{\pm 6} : \ V_0(x) = (x_1^3 \pm x_2^4) + \sum_{j=1}^{2} a_j x_2^j + \sum_{j=3}^{5} a_j x_1 x_2^{j-3},
\]
\[
E_7 : \ V_0(x) = x_1^3 + x_1 x_2^3 + \sum_{j=1}^{4} a_j x_2^j + \sum_{j=5}^{6} a_j x_1 x_2^{j-5},
\]
\[ E_8: V_0(x) = x_1^3 + x_2^5 + \sum_{j=1}^{3} a_j x_j^3 + \sum_{j=4}^{7} a_j x_1 x_j^{j-4}. \]

Here \( Q(x) \) is nondegenerate quadratic form with variables \( x_2, x_3, \ldots, x_s \) for \( A_{\pm n} \) and parameters \( x_3, \ldots, x_s \) for other cases.

## 7 Fold catastrophe

In this section, we suggest an example of catastrophe \( A_2 \) called fold.

Let us consider Liouville-von Neumann equation for nonlinear quantum oscillator with friction

\[ \frac{d}{dt} \rho_t = -\frac{i}{\hbar} [H, \rho_t] + \alpha_0 \frac{i}{\hbar} [q, p \circ \rho_t] + \alpha_1 [q, p \circ (H \circ \rho_t)] + \alpha_2 [q, p \circ (H \circ (H \circ \rho_t))], \tag{16} \]

where \( H \) is Hamilton operator defined by (5).

In this case, we have

\[ \hat{F} = \frac{i}{\hbar} (\hat{L}_q - \hat{R}_q)(\hat{L}_p + \hat{R}_p), \]

\[ N(\hat{L}_H, \hat{R}_H) = \alpha_0 \hat{L}_I + \frac{\alpha_1}{2} (\hat{L}_H + \hat{R}_H) + \frac{\alpha_2}{4} (\hat{L}_H + \hat{R}_H)^2, \]

\[ N(E, E) = \langle \Psi | N(H, H) | \Psi \rangle = \alpha_0 + \alpha_1 E + \alpha_2 E^2. \]

Stationary state \( \rho_\Psi = |\Psi \rangle \langle \Psi | \) of harmonic oscillator is stationary state of dissipative quantum system (16), if

\[ \alpha_0 + \alpha_1 E + \alpha_2 E^2 = 0. \]

If we define the variable \( x \) and parameter \( \lambda \) by

\[ x = E - a, \quad a = -\frac{\alpha_1}{2\alpha_2}, \quad \lambda = \frac{4\alpha_0\alpha_2 - \alpha_1^2}{4\alpha_2^2}, \]

then we have stationary condition \( N(E, E) = 0 \) in the form

\[ x^2 - \lambda = 0. \]

If \( \lambda \leq 0 \), then we have no stationary states. If \( \lambda > 0 \), then we have stationary states for discrete set of parameter values \( \lambda \). If the parameters \( a \) and \( \lambda \) are equal to

\[ a = \frac{\hbar \omega}{2}(n_1 + n_2 + 1), \quad \lambda = \hbar^2 \omega^2 \frac{(n_1 - n_2)^2}{4}, \]

where \( n_1 \) and \( n_2 \) are nonnegative integer numbers, then dissipative quantum system has two stationary state (17) of linear harmonic oscillator. The energy of these states is equal to

\[ E_{n_1} = \hbar \omega(n_1 + \frac{1}{2}), \quad E_{n_2} = \hbar \omega(n_2 + \frac{1}{2}). \]
8 Conclusion

Dissipative quantum systems can have stationary states. Stationary states of non-Hamiltonian and dissipative quantum systems can coincide with stationary states of Hamiltonian systems. As an example we suggest quantum dissipative systems with pure stationary states of linear harmonic oscillator. Using (8), it is easy to get dissipative quantum systems with stationary states of hydrogen atom. For a special case of dissipative systems we can use usual bifurcation and catastrophe theory. It is easy to derive quantum analogous of classical dynamical bifurcations.

Dissipative quantum systems with two stationary states can be considered as qubit. It allows to consider quantum computer with dissipation as non-dissipative quantum computer. In the general case, we can consider dissipative n-qubit quantum systems as quantum computer with mixed states and quantum operations, not necessarily unitary, as gates [20, 21]. A mixed state (operator of density matrix) of n two-level quantum systems is an element of $4^n$-dimensional operator Hilbert space. It allows to use quantum computer model with 4-valued logic [21]. The gates of this model are general quantum operations which act on the mixed state.

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References

[1] U. Weiss, Quantum dissipative systems (World Scientific, Singapore, 1993).
[2] V.S. Maskevich, Laser kinetics (Elsevier, Amsterdam, 1967).
[3] V.E. Tarasov, ”Bosonic string in affine-metric curved space” Phys. Lett. B 323 (1994) 296-304. (arXiv:hep-th/0401223)
[4] I. Prigogine, From being to becoming (Freeman and Co., San Francisco, 1980).
[5] H. Spohn, ”Approach to equilibrium for completely positive dynamical semigroups of N-level systems” Rep. Math. Phys. 10 (1976) 189.
[6] H. Spohn, ”An algebraic condition for the aproach to equilibrium of an open N-level system” Lett. Math. Phys. 2 (1977) 33-38.
[7] H. Spohn, J.L. Lebowitz, ”Stationary non-equilibrium states of infinite harmonic system” Commun. Math. Phys. 54 (1977) 97-120.
[8] E.P. Davies, ”Quantum stochastic processes II” Commun. Math. Phys. 19 (1970) 83-105.
[9] G. Lindblad, ”Brownian motion of a quantum harmonic oscillator” Rep. Math. Phys. 10 (1976) 393-406.
[10] C. Anastopoulous, J.J. Halliwell, ”Generalized uncertainty relations and long-time limits for quantum Brownian motion models” Phys. Rev. D 51 (1995) 68706885.
[11] A. Isar, A. Sandulescu, W. Scheid, "Phase space representation for open quantum systems with the Lindblad theory" Int. J. Mod. Phys. B 10 (1996) 2767-2779. (arXiv:quant-ph/9605041)

[12] T. Dittrich, R. Craham, "Quantum effects in the steady state of the dissipative standard map" Europhys. Lett. 4 (1987) 263.

[13] V.E. Tarasov, "Weyl quantization of dynamical systems with flat phase space" Moscow Univ. Phys. Bull. 56 (6) (2001) 5-10.

[14] V.E. Tarasov, "Quantization of non-Hamiltonian and dissipative systems" Phys. Lett. A 288 (2001) 173-182. (arXiv:quant-ph/0311159)

[15] J.M.T. Thompson, T.S. Lunn, "Resonance-sensitivity in dynamic Hopf bifurcations under fluid loading" Appl. Math. Model ng, 5 (1981) 143-150.

[16] J. Marsden, M. McCracken, The Hopf bifurcation and its applications (Springer, Berlin, 1976).

[17] V.E. Tarasov, Mathematical introduction to quantum mechanics (MAI, Moscow, 2000).

[18] G. Lindblad, "On the generators of quantum dynamical semigroups" Commun. Math. Phys. 48 (1976) 119-130.

[19] R. Alicki and K. Lendi, Quantum dynamical semigroups and applications (Springer-Verlag, Berlin, 1987).

[20] D. Aharonov, A. Kitaev, N. Nisan, "Quantum circuits with mixed states" in Proceedings of the Thirtieth Annual ACM Symposium on Theory of Computation (STOC, 1997). pp. 20-30; (arXiv:quant-ph/9806029)

[21] V.E. Tarasov, "Open n-qubit systems as a quantum computer with four-valued logic" Preprint SINP MSU. 2001-31/671. (arXiv:quant-ph/0112023)

V.E. Tarasov, "Quantum computation by quantum operations on mixed states" (arXiv:quant-ph/0201033)

V.E. Tarasov, "Quantum computer with mixed states and four-valued logic" J. Phys. A 35 (2002) 5207-5235. (arXiv:quant-ph/0312131)