Acyclic Colorings of Directed Graphs

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Abstract

The acyclic chromatic number of a directed graph $D$, denoted $\chi_A(D)$, is the minimum positive integer $k$ such that there exists a decomposition of the vertices of $D$ into $k$ disjoint sets, each of which induces an acyclic subgraph. For any $m \geq 1$, we introduce a generalization of the acyclic chromatic number, namely $\chi_m(D)$, which is the minimum number of sets into which the vertices of a digraph can be partitioned so that each set is weakly $m$-degenerate. We show that for all digraphs $D$ without directed 2-cycles, $\chi_m(D) \leq \frac{4\Delta(D)}{4m+1} + o(\Delta(D))$. Because $\chi_1(D) = \chi_A(D)$, we obtain as a corollary that $\chi_A(D) \leq \frac{4}{5} \cdot \Delta(D) + o(\Delta(D))$, significantly improving a bound of Harutyunyan and Mohar.

1. Introduction

A proper vertex coloring of an undirected graph partitions its vertices into independent sets. To extend this notion to directed graphs (digraphs), we consider acyclic sets instead of independent sets. An acyclic set in a digraph is a set of vertices whose induced subgraph contains no directed cycle. The acyclic chromatic number of a digraph $D$, denoted $\chi_A(D)$, is then defined to be the minimum number of acyclic sets into which the vertices of $D$ can be partitioned.

Many upper bounds on the chromatic number of undirected graphs are phrased in terms of $\Delta(G)$, the maximum degree of $G$. To extend this notion to directed graphs, there are a few options which measure the maximum degree. Given a digraph $D$, $\tilde{\Delta}(D)$ is the maximum geometric mean of the in-degree and the out-degree of a vertex in $D$, and $\Delta(D)$ is the maximum arithmetic mean of the in-degree and the out-degree of a vertex in $D$. Notice that if the out-degrees and in-degrees of all vertices in $D$ are equal, then $\tilde{\Delta}(D) = \Delta(D)$.

The acyclic chromatic number $\chi_A(D)$ is one of many chromatic numbers which have been defined for digraphs. Bokal et al. [3] introduced the circular chromatic number of a digraph $D$, denoted $\chi_c(D)$, as a generalization of the acyclic chromatic number. Let $S_p$ denote the circle with perimeter $p$, and for $x, y \in S_p$, let $d(x, y)$ denote the clockwise distance from $x$ to $y$. Then the circular chromatic number $\chi_c(D)$ is the infimum of all positive real numbers $p$ for which there exists a function $c : V(D) \to S_p$ such that for each edge $uv$ of

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factor. Kim, Johansson, and Jamall all used the pseudo-random method to prove upper
bounds on the chromatic number. The central idea of this method is that given a graph
\( G \), and the goal of using \( \Delta(G)/k \) colors, where \( k \) is a positive integer, the algorithm goes
through several rounds and assigns colors randomly to a subset of the uncolored vertices at

In this paper, we use the following generalization of the acyclic chromatic number and
deduce new bounds on the acyclic chromatic number itself as a special case of our results.
For a positive integer \( m \), a digraph \( D \) is said to be weakly \( m \)-degenerate if for every induced
subgraph of \( D \), there is a vertex of out-degree or in-degree strictly less than \( m \). Therefore,
a digraph is weakly 1-degenerate if and only if it is acyclic. Given a positive integer \( k \), a
\((k,m)\)-degenerate coloring of \( D \) is a partition of \( V(D) \) into \( k \) sets, each of which is weakly
\( m \)-degenerate. Given a positive integer \( m \), we denote \( \chi_m(D) \), the \( m \)-degenerate chromatic
number of \( D \), as the smallest positive integer \( k \) such that \( D \) has a \((k,m)\)-degenerate coloring.
Notice that \( \chi_1(D) = \chi_A(D) \); hence the parameter \( \chi_m(D) \) is a generalization of \( \chi_A(D) \).
Bokal et al. \cite{3} showed some further connections between weak degeneracy and acyclic colorings.
For instance, if a digraph is weakly \( m \)-degenerate, then \( \chi_A(D) \leq m + 1 \); moreover, this
bound is tight for each positive integer \( m \).

The acyclic chromatic number has received the most attention among digraph chromatic
numbers because recent results \cite{1, 3, 15, 12, 2} suggest that the acyclic chromatic number
in digraphs behaves similarly to the chromatic number in undirected graphs. Much still
remains to be learned however. For instance, it is easily proved using the greedy algorithm
that \( \chi_A(D) \leq \Delta(D) + 1 \); this is analogous to the fact that in an undirected graph \( G \),
\( \chi(G) \leq \Delta(G) + 1 \). However, this bound is not tight for most digraphs.

In the case of undirected graphs, Brooks \cite{7} made the first improvement on the obvious
bound of \( \chi(G) \leq \Delta(G) + 1 \); he showed that \( \chi(G) \leq \Delta(G) \) unless \( G \) is a complete graph or
an odd cycle. Borodin and Kostochka \cite{6} and Catlin \cite{9} then independently strengthened
Brooks’ theorem, showing that if \( G \) is \( K_4 \)-free, then \( \chi(G) \leq \left\lfloor \frac{3\Delta(G)+1}{4} \right\rfloor \).
Mohar \cite{18} recently proved an analogue of Brooks’ theorem for digraphs such that at most one edge connects
any pair of vertices; such a digraph is called an oriented graph. It follows from Mohar’s
results that if \( D \) is an oriented graph, then \( \chi_A(D) \leq \left\lceil \Delta(D) \right\rceil \) unless \( D \) is a directed cycle.
However, it appears that this upper bound on \( \chi_A(D) \) can be significantly improved further;
Harutyunyan and Mohar \cite{12} credit McDiarmid and Mohar with the following conjecture.

**Conjecture 1.1** (McDiarmid & Mohar, 2002 \cite{12}). *Every oriented graph \( D \) satisfies \( \chi_A(D) = \Omega \left( \frac{\Delta(D)}{\log \Delta(D)} \right) \).

Conjecture 1.1 is analogous to a result for undirected graphs by Kim \cite{16}, who showed
that \( \chi(G) \leq (1 + o(1)) \frac{\Delta(G)}{\log \Delta(G)} \) if the girth (length of the shortest cycle) of \( G \) is greater than
4. Johansson \cite{14} later extended Kim’s bound to graphs \( G \) of girth greater than 3, and
Jamall \cite{13} has since used a simpler proof to strengthen Johansson’s bound by a constant
factor. Kim, Johansson, and Jamall all used the pseudo-random method to prove upper
bounds on the chromatic number. The central idea of this method is that given a graph
\( G \), and the goal of using \( \Delta(G)/k \) colors, where \( k \) is a positive integer, the algorithm goes
through several rounds and assigns colors randomly to a subset of the uncolored vertices at
each round. Eventually a proper coloring of all the vertices of $G$ will be found with positive probability.

Harutyunyan and Mohar [11] applied the pseudo-random method to digraphs to show that $\chi_A(D) \leq (1 - e^{-13})\tilde{\Delta}(D)$, which only slightly improves upon the trivial bound of $\chi_A(D) \leq \Delta(D) + 1$ and is far from the bound given in Conjecture 1.1. Therefore, Harutyunyan and Mohar [11] posed the following relaxation of Conjecture 1.1.

Conjecture 1.2 (Harutyunyan & Mohar, 2011 [11]). Let $D$ be an oriented graph. Then $\chi_A(D) \leq \left\lceil \frac{\tilde{\Delta}(D)}{2} \right\rceil + 1$.

In Theorem 1.3, we prove an upper bound on the $m$-degenerate chromatic number of any digraph $D$ in terms of $\tilde{\Delta}(D)$. It is a special case of Theorem 1.3 that we can significantly improve Harutyunyan and Mohar’s bound, thus making progress towards Conjecture 1.2.

Theorem 1.3. Let $m$ be a positive integer. Suppose we are given an oriented graph $D$ with $\tilde{\Delta}(D) \geq 2m$. Then

$$\chi_m(D) \leq \left\lceil \frac{\tilde{\Delta}(D) - \left(\frac{1}{2}\right)\left\lceil \frac{\tilde{\Delta}(D)+1/2}{2m+1/2} \right\rceil}{m} \right\rceil + 1.$$ 

By taking $m = 1$, the following corollary follows immediately from Theorem 1.3.

Corollary 1.4. For an oriented graph $D$ with $\tilde{\Delta}(D) \geq 2$, $\chi_A(D) \leq \left\lceil \frac{4}{5} \cdot \tilde{\Delta}(D) + \frac{2}{5} \right\rceil + 1$.

We prove Theorem 1.3 by using a strategy similar to one originally introduced by Borodin and Kostochka [5] and by Catlin [9] to prove an upper bound on the chromatic number in undirected graphs. Specifically, we show that the vertices of a directed graph $D$ can be partitioned into several subsets, each inducing a subgraph $D_i \subset D$, such that $\tilde{\Delta}(D_i)$ is small. We then use this to show that $\chi_m(D_i)$ is also small, meaning that we can find an upper bound on $\chi_m(D) \leq \sum_i \chi_m(D_i)$.

The organization of this paper is as follows. In Section 2, we prove Theorem 1.3, giving an upper bound on $\chi_m(D)$ for any digraph $D$ in terms of $\tilde{\Delta}(D)$. In Section 3, we improve upon this bound for $m = 1$ for a particular class of digraphs. We give some concluding remarks in the final section.

2. Acyclic colorings

Recall that Harutyunyan and Mohar [11] proved that given a digraph $D$, if $\tilde{\Delta}(D)$ is large enough, then $\chi_A(D) \leq (1 - e^{-13})\tilde{\Delta}(D)$. They used a non-constructive method to do so, and posed the problem of improving this bound, remarking that a different technique may be necessary. In this section, we use a constructive technique to prove Theorem 1.3, a specific case of which (Corollary 1.4) is the significantly stronger upper bound of $\chi_A(D) \leq \frac{4}{5} \cdot \tilde{\Delta}(D) + \frac{2}{5}$. 


would have at least \( \frac{4}{6} \cdot \Delta(D) + o(\Delta(D)) \). The outline of our proof of Theorem 1.3 is somewhat similar to that of an undirected analogue proved independently by Borodin and Kostochka [6] and Catlin [9].

It is easy to show that \( \chi_m(D) \leq \left\lceil \frac{\Delta(D)}{m} \right\rceil + 1 \); the proof is similar to that of the fact that \( \chi_A(D) \leq \left\lceil \Delta(D) \right\rceil + 1 \). In particular, we color the vertices of \( D \) greedily, in any order. At each step, the next vertex \( v \) to be colored has either out-degree or in-degree at most \( \Delta(D) \), suppose without loss of generality out-degree. Therefore, there are at most \( \left\lceil \frac{\Delta(D)}{m} \right\rceil \) colors which are represented in at least \( m \) out-neighbors of \( v \). We now color \( v \) using one of the remaining colors which is represented in less than \( m \) out-neighbors of \( v \). The resulting coloring is indeed \( m \)-degenerate, since in any subset of any color class, the vertex in that subset colored last must have less than \( m \) in-neighbors or out-neighbors in that subset.

Note that Theorem 1.3 gives a significant improvement to \( \chi_m(D) \leq \left\lceil \frac{\Delta(D)}{m} \right\rceil + 1 \) for any oriented graph \( D \). To prove Theorem 1.3, we begin by generalizing a directed graph analogue of Brooks’ theorem [7] due to Mohar [18] to the framework of \( m \)-degenerate colorings. Our proofs follow similar outlines to those of Mohar.

A few definitions are needed to state Lemma 2.1. Given a digraph \( D \) and \( u \in V(D) \), we denote the subgraph induced on \( V(D) \setminus \{u\} \) by \( D - u \). Moreover, given a positive integer \( m \), a critical vertex is a vertex \( v \in V(D) \) such that \( \chi_m(D - v) < \chi_m(D) \). Every vertex of \( D \) is critical and \( \chi_m(D) = k \), then we define \( D \) to be a \((k, m)\)-critical digraph. Lemma 2.1 shows that critical vertices in a digraph must have large in-degree and out-degree.

**Lemma 2.1.** Suppose \( v \) is a critical vertex in a digraph \( D \), \( m \geq 1 \), and \( \chi_m(D) = k \). Then \( d^+(v), d^-(v) \geq (k - 1)m \).

*Proof.* Suppose for the purpose of contradiction that \( d^+(v) < (k - 1)m \). We will show that we can find a \((k - 1, m)\)-degenerate coloring of \( D \), a contradiction to the fact that \( \chi_m(D) = k \). Since \( v \) is \((k, m)\)-critical, we can find a \((k - 1, m)\)-degenerate coloring of \( D - v \). At least one color class \( c \) must be represented in less than \( m \) out-neighbors of \( v \) because otherwise \( v \) would have at least \( (k - 1)m \) out-neighbors. Now we color \( v \) color \( c \), and claim that the subgraph \( H \) induced by all vertices of color \( c \) is \( m \)-degenerate. To see this, let \( H' \) be an induced subgraph of \( H \). If \( v \in V(H') \), then notice that \( v \) has at most \( m - 1 \) out-neighbors in \( H' \). Otherwise, note that \( H' \) is a subset of a color class in a \((k - 1, m)\)-degenerate coloring of \( D - v \), meaning that there is some vertex in \( H' \) of in-degree or out-degree less than \( m \).

A similar argument shows that \( d^-(v) \geq (k - 1)m \). \( \square \)

A digraph is weakly connected if the underlying undirected graph is connected. We next state and prove Lemma 2.2, which states that we only need to consider the weakly connected components of a digraph to find its \( m \)-degenerate chromatic number.

**Lemma 2.2.** If \( D \) is a digraph and \( D_1, \ldots, D_l \) are its weakly connected components for some positive integer \( l \), then for any \( m \geq 1 \), \( \chi_m(D) = \max_{1 \leq i \leq l} \chi_m(D_i) \).

*Proof.* Let \( k = \max_{1 \leq i \leq l} \chi_m(D_i) \). We can find a \((k, m)\)-degenerate coloring of each \( D_i \), for \( 1 \leq i \leq l \), and the resulting composite coloring is a \((k, m)\)-degenerate coloring of \( D \) since
there is no edge between any two weakly connected components of \( D \).

By Lemma 2.2, a digraph \( D \) which is \((k, m)\)-critical is also weakly connected.

Lemma 2.3 shows that in a \((k, m)\)-critical oriented graph, we can find a certain set of vertices, which, when removed, does not break the weakly connectedness of the oriented graph.

**Lemma 2.3.** Suppose \( m \geq 1 \), \( k \geq 3 \), and \( D \) is a \((k, m)\)-critical oriented graph on \( n \) vertices in which each vertex \( v \) satisfies \( d^+(v) = d^-(v) = (k-1)m \). Then there exists a set of vertices \( u_1, u_2, \ldots, u_m, u_n \in V(D) \) such that \( u_1, \ldots, u_m+1 \) are all out-neighbors or all in-neighbors of \( u_n \) and the digraph induced by \( V(D) - \{u_1, \ldots, u_m+1\} \) is weakly connected.

**Proof.** Pick any \( m + 1 \) vertices \( u_1, \ldots, u_m+1 \) which are all in-neighbors or out-neighbors of another vertex \( u_n \), and let \( u_m+2, \ldots, u_{n-1} \) be the remaining vertices of \( D \). Let \( D_0 \) be the subgraph induced by \( \{u_m+2, \ldots, u_n\} \). Assume that \( D_0 \) is not weakly connected; if it is, the proof is complete.

Let \( w \) be any vertex in \( D_0 \). Since \( D \) is weakly connected, there is some path (not necessarily directed) from \( w \) to \( u_i \), for each \( 1 \leq i \leq m + 1 \). Therefore, any weakly connected component of \( D_0 \) must contain a vertex which is adjacent to some \( u_i \), for \( 1 \leq i \leq m + 1 \). Given a weakly connected component \( C \) of \( D_0 \), let \( s(C) \) be the set of all \( u_i \) (\( 1 \leq i \leq m + 1 \)) which are adjacent to some vertex of \( C \), and let \( N(C) \) be the subgraph of \( D \) induced by \( V(C) \cup s(C) \).

We now perform a multi-step process, considering many possible choices of \( u_1, \ldots, u_m+1, u_n \) until we find one which forces \( D_0 \) to be weakly connected. If \( D_0 \) is not weakly connected, then choose a weakly connected component \( C_1 \), and suppose without loss of generality that \( s(C_1) = \{u_1, \ldots, u_h\} \) for some \( 1 \leq h \leq m + 1 \). We first mark all vertices of \( C_1 \). Let \( C_2 \) be another weakly connected component of \( D_0 \). We next claim that there is some vertex \( v \in V(C_2) \) with at least \( m + 1 \) in-neighbors or \( m + 1 \) out-neighbors in \( V(C_2) \). To show this, we must consider two cases:

**Case 1.** \( m \geq 2 \) or \( k \geq 4 \). Notice that each \( v \in V(C_2) \) has \((k-1)m\) in-neighbors and \((k-1)m\) out-neighbors, all of which must belong to \( V(N(C_2)) \) by assumption. Since \( m \geq 2 \) and \( k \geq 4 \), we have that \( (2k-4)m > m + 1 \geq |s(C_2)| \). Therefore, since \( D \) is oriented, each \( v \in V(C_2) \) must have strictly fewer than \((k-2)m\) in-neighbors or strictly fewer than \((k-2)m\) out-neighbors in \( s(C_2) \). Thus \( v \) has at least \( m + 1 \) in-neighbors or \( m + 1 \) out-neighbors in \( V(C_2) \).

**Case 2.** \( m = 1 \) and \( k = 3 \). In this case, we consider acyclic colorings, and moreover, since each \( v \in V(C_2) \) has 2 in-neighbors and 2 out-neighbors, our claim is true unless \(|s(C_2)| = 2\) and each \( v \in V(C_2) \) has 1 in-neighbor and 1 out-neighbor in \( s(C_2) \). Moreover, some vertex of \( s(C_2) \) must have an in-neighbor or out-neighbor which does not belong to \( V(N(C_2)) \); if this were not true, then \( D \) would not be weakly connected. Since this vertex has a total of 4 neighbors and must be joined to each vertex of \( C_2 \), we have \(|V(C_2)| \leq 3\).

We first assume that \(|V(C_2)| = 3\), and let \( s(C_2) = \{w_1, w_2\} \). Since each vertex of \( D \) has 2 in-neighbors and out-neighbors, and since \( w_1 \) and \( w_2 \) are interchangeable, \( N(C_2) \) is
and since

We can make this assumption because the red vertices form an induced acyclic subgraph

\[ \text{we finally mark all of} \quad N \]

\[ \text{namely} \quad u \]

\[ \text{its out-neighbor and} \quad w \]

\[ \text{find an acyclic 2-coloring of} \quad D \]

\[ \text{loss of generality that there is no monochromatic directed path from} \quad w_1 \text{ to} \quad w_2 \text{ in} \quad D - V(C_2) \]

We can make this assumption because the red vertices form an induced acyclic subgraph

\[ \text{and} \quad w_1 \text{ and} \quad w_2 \text{ are interchange} \]

\[ \text{we pick a vertex} \quad x \in V(C_2) \text{ which has} \quad w_1 \text{ as its out-neighbor and} \quad w_2 \text{ as its in-neighbor; there exists such an} \quad x \]

\[ \text{least} \quad m \]

\[ \text{we color the other 2 vertices in} \quad V(C_2) \text{ the other color, suppose blue. As shown in} \]

\[ \text{is isomorphic to the digraph shown in Figure 1(a). Moreover, since} \quad D \]

\[ \text{is (3, 1)-critical, we can find an acyclic 2-coloring of} \quad D - V(C_2) \]

If \( w_1 \) and \( w_2 \) have the same color, suppose red, in this coloring, then suppose without loss of generality that there is no monochromatic directed path from \( w_1 \) to \( w_2 \) in \( D - V(C_2) \). We can make this assumption because the red vertices form an induced acyclic subgraph and since \( w_1 \) and \( w_2 \) are interchangeable. Next, pick a vertex \( x \in V(C_2) \) which has \( w_1 \) as its out-neighbor and \( w_2 \) as its in-neighbor; there exists such an \( x \) since \( |V(C_2)| = 3 \) and at most 2 vertices in \( V(C_2) \) have \( w_1 \) as an in-neighbor and \( w_2 \) as an out-neighbor. Color vertex \( x \) red, and color the other 2 vertices in \( V(C_2) \) the other color, suppose blue. As shown in Figure 1(b), this completes an acyclic 2-coloring of \( D \), contradicting its (3, 1)-criticality.

If \( w_1 \) and \( w_2 \) have different colors in the acyclic 2-coloring of \( D - V(C_2) \), then no monochromatic directed cycle in \( D \) can contain a vertex in \( V(C_2) \) and a vertex in \( D - V(N(C_2)) \). Assuming that \( w_1 \) is colored red and \( w_2 \) is colored blue, we pick 2 vertices of \( V(C_2) \), which, when colored red, do not form a monochromatic directed cycle with \( w_1 \), and color these vertices red. We color the third vertex of \( V(C_2) \) blue. As shown in Figure 1(c), this completes the acyclic 2-coloring of \( D \) in this case also, contradicting the (3, 1)-criticality of \( D \).

In the case that \( C_2 \) has 2 or fewer vertices, the proof that we can find an acyclic 2-coloring of \( D \) is nearly identical.

Hence we can find a \( v \in V(C_2) \) with \( m + 1 \) in-neighbors or \( m + 1 \) out-neighbors in \( V(C_2) \). We let \( u'_n = v \), and let the \( m + 1 \) in-neighbors or out-neighbors of \( v \) in \( V(C_2) \) be \( u'_1, \ldots, u'_{m+1} \). We finally mark all of \( u_1, \ldots, u_k \); notice that the subgraph induced by the marked vertices, namely \( N(C_1) \), is weakly connected.

We next repeat the above process with our choice of \( u'_1, \ldots, u'_{m+1}, u'_n \). Notice that the marked vertices must all belong to the same weakly connected component of \( D'_0 := D - \{u'_1, \ldots, u'_{m+1}\} \), which we call \( C'_1 \). We then find a second weakly connected component of \( D_0, C'_2 \), and notice that by the same reasoning as above, there is some \( v' \in V(C'_2) \) with at least \( m + 1 \) in-neighbors or \( m + 1 \) out-neighbors in \( C'_2 \). We let \( u''_n = v' \), and let the \( m + 1 \) in-neighbors or out-neighbors of \( v \) in \( C'_2 \) be \( u''_1, \ldots, u''_{m+1} \). We finally mark all vertices of \( N(C'_1) \), noting that we mark at least 1 vertex, namely all vertices in \( s(C'_1) \), for the first time.
and the subgraph induced by the marked vertices, namely \( N(C'_i) \), is weakly connected.

Since at least 1 new vertex is being marked with each iteration of the process, it must eventually end at some step \( t \). That is, we can eventually find \( u_1^{(t)}, \ldots, u_{m+1}^{(t)} \) so that all vertices in \( D_0^{(t)} = D - \{ u_1^{(t)}, \ldots, u_{m+1}^{(t)} \} \) are marked, meaning that \( D_0^{(t)} \) is weakly connected. This is a contradiction to our original assumption.

\[ \square \]

Theorem 2.4 states that the in-degree and out-degree of every vertex cannot be too small in a \((k, m)\)-critical oriented graph.

**Theorem 2.4.** Suppose that \( m \geq 1 \) and \( D \) is a \((k, m)\)-critical oriented graph in which each vertex \( v \) satisfies \( d^+(v) = d^-(v) = (k - 1)m \). Then \( k \leq 2 \).

Theorem 2.4 is of particular interest since it generalizes the following theorem of Mohar, who proved the case \( m = 1 \), which is a statement about acyclic colorings.

**Theorem 2.5** (Mohar [18]). If \( D \) is a \((k, 1)\)-critical oriented graph in which each vertex \( v \) satisfies \( d^+(v) = d^-(v) = k - 1 \), then \( k \leq 2 \).

We now prove Theorem 2.4, using Lemmas 2.3 and 2.2.

Proof of Theorem 2.4. We assume that \( k \geq 3 \) for the purpose of contradiction and create a linear ordering of the vertices of \( D \), as follows. Pick a \( u_n \) and choose \( m + 1 \) of its out-neighbors (or in-neighbors) \( u_1, u_2, \ldots, u_{m+1} \), so that the digraph \( D - \{ u_1, \ldots, u_{m+1} \} \) is weakly connected. This construction is possible by Lemma 2.3. Let \( D' = D - \{ u_1, \ldots, u_{m+1} \} \). Now, since \( u_n \) has \((k - 1)m \) in-neighbors (or out-neighbors), there is some \( u_{n-1} \in D' \) apart from \( u_1, u_2, \ldots, u_{m+1} \) such that \( u_n \) is an out-neighbor or in-neighbor of \( u_{n-1} \). Thus, \( u_{n-1} \) has at most \((k - 1)m - 1 \) out-neighbors (or in-neighbors) in \( D' - u_n \). Now continue in a similar manner, using the weakly connectedness of \( D' \) to find \( \{ u_{m+2}, \ldots, u_{n-1} \} \) such that each of these vertices \( v_i \) \((m + 2 \leq i \leq n - 1) \) has out-degree or in-degree less than \((k - 1)m \) in \( D - \{ u_{i+1}, \ldots, u_n \} \).

We now color the vertices of \( D \) as follows: we give \( u_1, u_2, \ldots, u_{m+1} \) the same color; at this point, the coloring of \( \{ u_1, \ldots, u_{m+1} \} \) is \((k - 1, m)\)-degenerate since each vertex among \( u_1, \ldots, u_{m+1} \) has less than \( m \) in-neighbors or out-neighbors. Then for \( m + 2 \leq i \leq n - 1 \), we start with a \((k - 1, m)\)-degenerate coloring of \( \{ u_1, \ldots, u_{i-1} \} \) and extend this coloring to \( u_i \). This is possible since in the subgraph of \( D \) induced by \( \{ u_1, \ldots, u_i \} \), \( u_i \) has in-degree or out-degree less than \((k - 1)m \). Therefore, in the \((k - 1, m)\)-degenerate coloring of \( \{ u_1, \ldots, u_{i-1} \} \), one of the color classes contains fewer than \( m \) in-neighbors or out-neighbors of \( u_i \). We finally extend the \((k - 1, m)\)-degenerate coloring of \( \{ u_1, \ldots, u_{n-1} \} \) to \( u_n \). This is possible since \( u_n \) has \( m + 1 \) in-neighbors or out-neighbors, namely \( u_1, \ldots, u_{m+1} \), of the same color, but exactly \((k - 1)m \) in-neighbors and out-neighbors in total, so we can find a color represented among fewer than \( m \) in-neighbors or out-neighbors of \( u_n \), and color \( u_n \) this color.

At the end of the process, we claim that each color class \( c \) is \( m \)-degenerate. To show this, for any color class \( c \) and subset \( S \) of the vertices colored \( c \), pick \( u_i \in S \) so that \( i \) is as large as possible. Then since \( u_i \) is the vertex in \( S \) that was colored last, \( u_i \) has at most \( m - 1 \) in-neighbors or out-neighbors in \( S \), completing the proof.

\[ \square \]
Lemma 2.6 uses Lemma 2.1 to extend Theorem 2.4 to digraphs that are not \((k, m)\)-critical. Intuitively, this is possible because \((k, m)\)-critical digraphs are the worst case for finding an \(m\)-degenerate coloring with few colors.

**Lemma 2.6.** Suppose that \(m \geq 1\), \(\chi_m(D) = k + 1\), for some integer \(k \geq 2\), and that \(D\) is an oriented graph. Then \(\bar{\Delta}(D) > km\).

**Proof.** Fix \(m \geq 1\). Suppose for the purpose of contradiction that for some \(k \geq 2\), there is an oriented graph \(D\) with as few vertices as possible, such that \(\chi_m(D) = k + 1\) and \(\bar{\Delta}(D) \leq km\). Notice that if \(D\) were not \((k + 1, m)\)-critical, we could remove some vertex \(v\) to form \(D' = D - v\), and we would have \(\chi_m(D') = k + 1\) and \(\Delta(D') \leq \bar{\Delta}(D) \leq km\). This contradicts the fact that \(D\) has as few vertices as possible such that \(\bar{\Delta}(D) \leq km\) holds. Hence \(D\) is \((k + 1, m)\)-critical.

By Lemma 2.1, for each \(v \in V(D)\), we have that \(d^+(v) \geq km\) and \(d^-(v) \geq km\). In order to have \(\bar{\Delta}(D) \leq km\), we must have \(d^+(v) = d^-(v) = km\) for all \(v \in V(D)\). But then by Theorem 2.4, we have that \(k + 1 \leq 2\), contradicting the fact that \(k + 1 \geq 3\). \(\square\)

The following corollary follows from Lemma 2.6. It is a directed analogue of a theorem of Borodin [4].

**Corollary 2.7.** If \(D\) is an oriented graph such that \(\bar{\Delta}(D) \geq 2m\), then \(\chi_m(D) \leq \lceil \frac{\bar{\Delta}(D)}{m} \rceil\).

**Proof.** Let \(k = \lceil \frac{\bar{\Delta}(D)}{m} \rceil\). Then \(k \geq 2\). If \(\chi_m(D) \geq k + 1\), then by Lemma 2.6, \(\bar{\Delta}(D) > km\), meaning that \(\frac{\bar{\Delta}(D)}{m} > k\), so it is impossible that \(k = \lceil \frac{\bar{\Delta}(D)}{m} \rceil\). \(\square\)

To prove Theorem 1.3, we also use a theorem of Lovász [17], which states that the vertices of a graph can be decomposed into sets so that the sum of the maximal degrees of all the sets is less than the maximal degree of the graph.

**Theorem 2.8** (Lovász [17]). For an undirected graph \(G\), suppose that for some \(s \geq 1\) and positive integers \(\Delta_1, \ldots, \Delta_s\), we have \(\Delta(G) = (s - 1) + \sum_{i=1}^{s} \Delta_i\). Then there is a covering of \(V(G)\) with \(s\) subgraphs \(G_i\) \((1 \leq i \leq s)\), so that \(\Delta(G_i) \leq \bar{\Delta}_i\) for \(1 \leq i \leq s\).

We deduce as a corollary of Theorem 2.8 a version for directed graphs.

**Corollary 2.9.** Given a digraph \(D\), \(s \geq 1\), and rational numbers \(\bar{\Delta}_1, \ldots, \bar{\Delta}_s \geq 1\) so that for \(1 \leq i \leq s\), \(2\bar{\Delta}_i\) is an integer, suppose \(\bar{\Delta}(D) = \frac{s - 1}{2} + \sum_{i=1}^{s} \bar{\Delta}_i\). Then there is a covering of \(V(D)\) with \(s\) subgraphs \(D_i\) \((1 \leq i \leq s)\) such that \(\bar{\Delta}(D_i) \leq \bar{\Delta}_i\).

**Proof.** Given a digraph \(D\), consider the underlying undirected graph \(G\). Notice that \(\Delta(G) = 2\bar{\Delta}(D)\). By Theorem 2.8, we can partition \(G\) into vertex-disjoint subgraphs \(G_1, \ldots, G_s\), so that the maximum total degree in \(G_i\) is at most \(2\bar{\Delta}_i\). For \(1 \leq i \leq s\), we let \(D_i\) be the directed subgraph of \(D\) induced by \(V(G_i)\). Thus, for \(1 \leq i \leq s\), \(\bar{\Delta}(D_i) \leq \bar{\Delta}_i\). \(\square\)
We now prove Theorem 1.3, using Lemma 2.6 and Corollary 2.9 to find an upper bound on $\chi_m(D)$.

**Proof of Theorem 1.3.** Set

$$s = \left\lfloor \frac{\bar{\Delta}(D) + 1/2}{2m + 1/2} \right\rfloor, \quad r = \bar{\Delta}(D) + 1/2 - s(2m + 1/2).$$

Then $\bar{\Delta}(D) = \frac{s}{2} + (\sum_{i=1}^{s} 2m) + (r - \frac{1}{2})$, meaning that, by Corollary 2.9, the vertices of $D$ can be covered with $s + 1$ subgraphs $D_1, \ldots, D_{s+1}$, which satisfy:

$$\Delta(D_i) \leq \begin{cases} 2m & \text{if } 1 \leq i \leq s \\ r - \frac{1}{2} & \text{if } i = s + 1. \end{cases}$$

By Lemma 2.6, if $\chi_m(D_i) \geq 3$, then we would have that $\bar{\Delta}(D_i) > 2m$. Moreover, recalling the trivial bound that $\chi_m(D) \leq \left\lfloor \frac{\bar{\Delta}(D)}{m} \right\rfloor + 1$, we have

$$\chi_m(D_i) \leq \begin{cases} 2 & \text{if } 1 \leq i \leq s \\ 1 + \left\lfloor \frac{(r-1/2)}{m} \right\rfloor & \text{if } i = s + 1. \end{cases}$$

We thus have

$$\chi_m(D) \leq \sum_{i=1}^{s+1} \chi_m(D_i) \leq 2s + \left\lfloor \frac{(r-1/2)}{m} \right\rfloor + 1.$$

3. Better bounds for $m = 1$ with improved decomposition lemma

In this section, we show that the bounds in Theorem 2.8 can be improved for any digraph which does not contain any of the graphs shown in Figure 2 as an induced subgraph. This leads to improvements to the bounds in Theorem 1.3 in the case $m = 1$ for these classes of digraphs. We first introduce some notation. Given disjoint sets of vertices $V_1$ and $V_2$ which
belong to a digraph $D$, we let $E(V_1, V_2)$ be the set of all edges which point from a vertex in $V_1$ to a vertex in $V_2$. Moreover, we define $e(V_1, V_2) = |E(V_1, V_2)|$. Finally, given a vertex $u$, we define $d^+_V(u)$ as the number of out-neighbors of $u$ in $V_1$, $d^-_V(u)$ as the number of in-neighbors of $u$ in $V_1$, and $d_V(u) = d^+_V(u) + d^-_F(u)$. We let $F_1, F_2, G_1$, and $G_2$ be the 4-vertex digraphs shown in Figure 2. For simplicity, we say that $D$ avoids $F$ if $D$ contains neither $F_1$ nor $F_2$ as an induced subgraph and that $D$ avoids $G$ is $D$ contains neither $G_1$ not $G_2$ as an induced subgraph.

We let $A$ be the set of all $i$ such that $\bar{\Delta}_i = 1$. Notice that if $i \in A$, then any directed cycle in $V_i$ must be both induced and a component of the subgraph induced by $V_i$. Following notation of Catlin [8], we call any such directed cycle a Brooks cycle. Now, choose a partition of the vertices of $D$ into subsets $V_1, \ldots, V_s$ so that, (i), $f(V_1, \ldots, V_s)$ is maximized, and (ii), the number of Brooks cycles is minimized, subject to (i).

We first claim that any partition which maximizes $f$ has $\bar{\Delta}(D_i) \leq \bar{\Delta}_i$ for all $i$. Notice that for any $u \in V_1$, and for $2 \leq j \leq s$, we have

$$f(V_1, \ldots, V_s) - f(V_1 - u, V_2, \ldots, V_j + u, \ldots, V_s) \geq 0,$$
by maximality of $f$. But

$$f(V_1, \ldots, V_s) - f(V_1 - u, V_2, \ldots, V_j + u, \ldots, V_s) = e(V_1, V_j) + e(V_j, V_1) - (e(V_1 - u, V_j + u) + e(V_j + u, V_1 - u)) + \bar{\Delta}_1 - \bar{\Delta}_j$$

which is impossible. Hence there is some $j$.

Since $2\bar{\Delta}$ is an integer, we must have that $\bar{d}_V(u) \leq \bar{\Delta}_j$. By averaging (3) over all choices of $j$, we obtain

$$\bar{d}_V(u) \leq \bar{\Delta}_1 - \bar{\Delta}_j + \bar{d}_V(u).$$

Therefore, for $1 \leq j \leq s$, we have

$$\bar{d}_V(u) \leq \bar{\Delta}_1 - \bar{\Delta}_j + \bar{d}_V(u).$$

By averaging (3) over all choices of $j$, $1 \leq j \leq s$, we obtain

$$\bar{d}_V(u) \leq \frac{\sum_{j=1}^{s} \bar{\Delta}_j}{s} + \frac{\bar{d}_G(u)}{s} = \bar{\Delta}_1 - \bar{\Delta}(D) - (s-2)/2 + \bar{d}_G(u).$$

But $\bar{d}_G(u) \leq \bar{\Delta}(D)$, so

$$\bar{d}_V(u) \leq \bar{\Delta}_1 + \frac{s-2}{2s}.$$

Since $2\bar{\Delta}_1$ and $2\bar{d}_V(u)$ are integers, we must have that $\bar{d}_V(u) \leq \bar{\Delta}_1$. Since $u$ can be any vertex in $V_1$, we have $\bar{\Delta}(D_1) \leq \bar{\Delta}_1$. Moreover, we can repeat the above process with $V_1$ replaced by $V_i$, for $2 \leq i \leq s$, and we have that $\bar{\Delta}(D_i) \leq \bar{\Delta}_i$.

Therefore, if $\bar{\Delta}_i \geq 2$, that is, if $i \notin A$, then by Lemma 2.6, we have that $\chi_A(D_i) \leq \bar{\Delta}(D_i) \leq \bar{\Delta}_i$. We now claim that since the partition $V_1, \ldots, V_s$ minimizes the total number of Brooks cycles subject to the fact that $f(V_1, \ldots, V_s)$ is maximized, the total number of Brooks cycles is 0. For the purpose of contradiction, suppose there is some Brooks cycle $C_0 \subset V_i$ where $i \in A$, and let $v_0 \in V(C_0)$. If $\bar{d}_V(v_0) \geq \bar{\Delta}_j + \frac{1}{2}$ for each $j \neq i$, then since $\bar{d}_V(v_0) = \bar{\Delta}_i = 1$,

$$\bar{d}(v_0) \geq \bar{\Delta}_i + \sum_{j \neq i} (\bar{\Delta}_j + \frac{1}{2}) = \bar{\Delta}(D) + \frac{1}{2} > \bar{\Delta}(D),$$

which is impossible. Hence there is some $j \neq i$ so that $\bar{d}_V(v_0) \leq \bar{\Delta}_j$. Thus, noting that $\bar{d}_V(u) = \bar{\Delta}_i$, and by equality of (1) with (2), moving $v_0$ from $V_i$ to $V_j$ does not decrease $f(V_1, \ldots, V_s)$. Moreover, since moving $v_0$ removes a Brooks cycle from $V_i$, it must create a Brooks cycle $C_1$ in $V_j$. Therefore, by our definition of Brooks cycle, $\bar{\Delta}_j = 1$, so $j \in A$. We then pick $v_1 \neq v_0$ such that $v_1 \in V(C_1)$, and repeat the process.

Eventually, since the total number of vertices is finite, there must be some $V_i$ so that the process generates an infinite sequence of Brooks cycles in $V_i$, which we call $C'_1, C'_2, \ldots$. Since each of the steps in our process moves only a single vertex, we must be able to find Brooks cycles $C'_p$ and $C'_q$ that differ in only one vertex. Specifically, suppose that $a \geq 2$ and the vertices of $C'_p$ are $x_1, \ldots, x_a, x_p$ while the vertices of $C'_q$ are $x_1, \ldots, x_a, x_q$. 

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If $a = 2$, then the vertices $x_1, x_a, x_p, x_q$ form an induced subgraph isomorphic to either $G_1$ or $G_2$, depending on whether there is an edge between $x_p$ and $x_q$. If $a \geq 3$, then since $C'_p$ and $C'_q$ are induced, there is no edge between $x_a$ and $x_1$, meaning that the vertices $x_1, x_a, x_p, x_q$ form an induced subgraph isomorphic to either $F_1$ or $F_2$. In either case, we have a contradiction to the fact that $D$ is $F$-free and $G$-free.

Thus there are no Brooks cycles, meaning that for $i \in A$, $D_i$ is acyclic, so $\chi_A(D_i) = 1 = \bar{\Delta}_i$ for all $i \in A$.

Our main theorem of this section improves the bound of $\chi_A(D) \leq \frac{4}{5} \cdot \bar{\Delta}(D) + o(\bar{\Delta}(D))$ in Corollary 1.4 to $\chi_A(D) \leq \frac{2}{3} \cdot \bar{\Delta}(D) + o(\bar{\Delta}(D))$ for oriented graphs $D$ which avoid $F$ and $G$.

**Theorem 3.2.** Suppose $D$ is an oriented graph which avoids $F$ and $G$. Then

$$\chi_A(D) \leq \left\lfloor \frac{2}{3} \cdot \bar{\Delta}(D) + 1/2 \right\rfloor + 1.$$  

**Proof.** Set

$$s = \left\lfloor \frac{\bar{\Delta}(D) + 1}{3/2} \right\rfloor, \quad r = \bar{\Delta}(D) + 1 - 3/2s.$$  

Then $\bar{\Delta}(D) = (r - 1) + \frac{s}{2} + \sum_{i=1}^{s} 1$, meaning that, by Lemma 3.1, the vertices of $D$ can be covered with $s + 1$ subgraphs $D_1, \ldots, D_{s+1}$, which satisfy:

$$\chi_A(D_i) \leq \begin{cases} 1 & \text{if } 1 \leq i \leq s \\ \lfloor r - \frac{1}{2} \rfloor + 1 & \text{if } i = s + 1. \end{cases}$$  

Thus

$$\chi_A(D) \leq \sum_{i=1}^{s+1} \chi_A(D_i)$$

$$\leq s + \left\lfloor \bar{\Delta}(D) + 1/2 - 3/2s \right\rfloor + 1$$

$$= \left\lfloor \frac{\bar{\Delta}(D) + 1}{3/2} \right\rfloor + \left\lfloor \bar{\Delta}(D) + 1/2 - 3/2 \left\lfloor \frac{\bar{\Delta}(D) + 1}{3/2} \right\rfloor \right\rfloor + 1$$

$$= \left\lfloor \bar{\Delta}(D) + 1/2 - 1/2 \left\lfloor \frac{\bar{\Delta}(D) + 1}{3/2} \right\rfloor \right\rfloor + 1$$

$$\leq \left\lfloor 2/3 \cdot \bar{\Delta}(D) + 1/2 \right\rfloor + 1.$$  

4. **Concluding remarks**

In this paper, we have proved an upper bound on a generalization of the acyclic chromatic number, the $m$-degenerate chromatic number. Moreover, the special case of $m = 1$ gives a bound on the acyclic number which significantly improves current bounds. However, the
bound in Theorem 1.3 differs from the conjectured bound in Conjectures 1.1 by a factor of \( \log \Delta(D) \). It seems that a new technique is necessary to obtain the additional factor of \( \log \Delta(D) \), if it is indeed correct. Moreover, it seems that Conjecture 1.1 can be extended to the \( m \)-degenerate chromatic number.

**Conjecture 4.1.** If \( m \) is a positive integer and \( c \) is a constant which does not depend on \( m \), then every oriented graph \( D \) has \( \chi_m(D) \leq (c + o(1)) \frac{(\Delta(D)/m)}{\log(\Delta(D)/m)} \).

We finally note that the bounds obtained in this paper were all phrased in terms of \( \Delta(D) \). By the arithmetic mean - geometric mean inequality, for any digraph \( D \), we have \( \Delta(D) \leq \bar{\Delta}(D) \), meaning that bounds phrased in terms of \( \bar{\Delta}(D) \) would be slightly stronger. It does not seem that our methods can produce bounds which would hold with \( \Delta(D) \) replaced by \( \bar{\Delta}(D) \). However, most graphs which are interesting for the problem of acyclic chromatic number have the in-degree and out-degree of the vertices approximately equal, meaning that \( \Delta(D) \approx \bar{\Delta}(D) \).

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