Exact Calculation of the Ground State
Single-Particle Green’s Function for the $1/r^2$
Quantum Many Body System at Integer Coupling

P.J. Forrester*
Department of Mathematics
University of Melbourne
Parkville, Victoria 3052
Australia

Abstract

The ground state single particle Green’s function describing hole propagation is calculated exactly for the $1/r^2$ quantum many body system at integer coupling. The result is in agreement with a recent conjecture of Haldane.

1 Introduction

The $1/r^2$ quantum many body problem, and the closely related Haldane-Shastry $1/r^2$ Heisenberg chain, are the subject of intense theoretical study at present, due in part to their relationship to an ideal gas obeying fractional statistics (see [1] for a review). One direction of study has been the exact calculation of some ground state correlation functions [1-5]. Let us briefly summarize the main results of these works.

The static ground state correlations (one-body density matrix, two-point distribution function) for the $1/r^2$ Hamiltonian in periodic boundary conditions

$$H := -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2q(q-1) \left( \frac{\pi}{L} \right)^2 \sum_{1 \leq j < k \leq N} \frac{1}{\sin^2 \pi (x_k - x_j)/L}, q \in \mathbb{Z}^+$$

(1.1)

(here we have set $\hbar^2/2m := 1$), for which the exact ground state wave function is

$$|0> := \psi_0(x_1, \ldots, x_N) := \prod_{1 \leq j < k \leq N} \left( \sin \pi (x_k - x_j)/L \right)^q,$$

(1.2)

have been calculated exactly in terms of a class of generalized hypergeometric functions in several variables based on the Jack symmetric polynomials [2]. These functions are defined in terms of power series, and in the cases of interest an explicit expression for each coefficient is known. However, the coefficient of the $N$th term involves a sum over all partitions of $N$ into $q$ parts and is thus impractical to compute except for small $N$ and $q$. To overcome this difficulty, recurrence formulas have been presented [3] which allows for the rapid numerical evaluation of the ground state correlations in the finite system. Furthermore, integral representations of the generalized hypergeometric functions are known [4,5], from which the density matrix and

*e-mail: matpjf@maths.mu.oz.au
two-point distribution function have been written in terms of \( q \) and \( 2q \) dimensional integrals respectively.

The thermodynamic limit can be taken in the integral formulas, giving similar integral formulas for the density matrix and two-point distribution function in the thermodynamic limit \([5]\). This integral formula for the density matrix has recently been generalized by Haldane \([1]\) to an integral formula for the retarded single-particle Green’s function

\[
-iG\left((x,t),(0,0)\right) := TL < 0 \mid \psi^+(x,t)\psi(0,0) \mid 0 >
\]

where \( TL \) denotes the thermodynamic limit \( N, L \to \infty, N/L = \rho \). Haldane conjectures that for general \( q \in \mathbb{Z}^+ \)

\[
-iG\left((x,t),(0,0)\right) = \frac{\rho}{2} B_q \int_{[-1,1]^q} dv_1 \ldots dv_q \prod_{j=1}^q (1 - v_j^2)^{-1+1/q} 
\times \prod_{1 \leq j < k \leq q} |v_k - v_j|^{2/q} \prod_{j=1}^q e^{i\pi \rho v_j} e^{-iq(\pi \rho)^2(1-v_j^2)t/\hbar}
\]

(1.4)

where

\[
B_q := \prod_{j=1}^q \frac{\Gamma(1+1/q)}{\left(\Gamma(j/q)\right)^2} = \frac{q}{(2\pi)^{q-1}} \left(\frac{\Gamma(1+1/q)}{\Gamma(1+1/2)}\right)^q
\]

(1.5)

As \( t \to 0^+ \) the results of \([5]\) for the density matrix are reclaimed. For general \( t \), this result was proved in \([6]\) for \( q = 2 \) using a mapping to a dynamical matrix model; for \( q = 1 \) - free fermions - it is easy to verify from the differential equation satisfied by \(-iG((x,t),(0,0))\) (see e.g. \([7]\)).

### 2 Derivation of Haldane’s Conjecture

#### 2.1 Action of the Hamiltonian on a hole state

Our starting point is to write the retarded single-particle Green’s function for the finite system in a more explicit form (see \([6]\), eq. (12):

\[
-iG_{N+1}\left((x,t);(0,0)\right) = (N + 1) < 0 \mid \prod_{l=1}^N (\sin \pi(x_l - x)/L)^q e^{-i(H-E_0)t/\hbar} 
\times \prod_{l=1}^N (\sin \pi x_l/L)^q \mid 0 >
\]

(2.1)

Next we introduce the auxiliary variables \( y_1, \ldots, y_q \) and take up the task of calculating

\[
(N + 1) < 0 \mid \prod_{l=1}^N \prod_{j=1}^q \sin \pi(x_l - y_j)/L e^{-i(H-E_0)t/\hbar} \prod_{l=1}^N (\sin \pi x_l/L)^q \mid 0 >.
\]

(2.2)

We do this by first expanding the exponential

\[
e^{-i(H-E_0)t/\hbar} = \sum_{j=0}^{\infty} \frac{(-it/\hbar)^j}{j!} (H - E_0)^j
\]

(2.3)
and considering the action of the operator $H - E_0$ on the hole state

$$|\phi> := \prod_{j=1}^{q} \sin \pi (x_l - y_j)/L |0>.$$  \hfill (2.4)

We have the following result.

**Lemma 1**

With $H$ given by (1.1), $E_0$ the corresponding ground state energy and $|0>$ given by (2.4) we have

$$(H - E_0) |\phi> = T_{\{y\}} |\phi>$$ \hfill (2.5a)

where

$$T_{\{y\}} := q \left[ \sum_{j=1}^{q} \frac{\partial^2}{\partial y_j^2} + \frac{\pi^2}{qL} \sum_{j_1 \neq j_2} \cot \pi (y_{j_1} - y_{j_2})/L \right.$$

$$\left. \times \left( \frac{\partial}{\partial y_{j_1}} - \frac{\partial}{\partial y_{j_2}} \right) + \left( \frac{\pi}{L} \right) \left( qN + N(N - 1) \right) \right]$$ \hfill (2.5b)

**Proof**

Direct differentiation gives

$$(H - E_0) |\phi> =$$

$$- \left( \frac{\pi}{L} \right)^2 \left[ -qN + \sum_{l=1}^{N} \sum_{j_1 \neq j_2} \cot \pi (x_l - y_{j_1})/L \cot \pi (x_l - y_{j_2})/L$$

$$+ 2q \sum_{l=1}^{N} \sum_{l' \neq l} \cot \pi (x_l - x_{l'})/L \sum_{j=1}^{q} \cot \pi (x_l - y_j)/L \right] |\phi> \hfill (2.6)$$

We rewrite the second summations by grouping together the summand with the summand with $l$ and $l'$ interchanged. Simple manipulation gives

$$\cot \pi (x_l - x_{l'})/L [\cot \pi (x_l - y_j)/L - \cot \pi (x_{l'} - y_j)/L]$$

$$= - (1 + \cot \pi (x_l - y_j)/L \cot \pi (x_{l'} - y_j)/L).$$ \hfill (2.7)

In the first summations we use (2.7) to rewrite the summand:

$$\cot \pi (x_l - y_{j_1})/L \cot \pi (x_l - y_{j_2})/L$$

$$= -1 - \cot \pi (y_{j_1} - y_{j_2})/L [\cot \pi (y_{j_1} - x_l)/L - \cot \pi (y_{j_2} - x_l)/L].$$ \hfill (2.8)

Substituting (2.8) and (2.7) in the first and second summations respectively of (2.6) we obtain

$$(H - E_0) |\phi> = - \left( \frac{\pi}{L} \right)^2 \left[ -q^2 N - \sum_{j_1 \neq j_2} \cot \pi (y_{j_1} - y_{j_2})/L$$

$$\times \sum_{l=1}^{N} \cot \pi (y_{j_1} - x_l)/L - \cot \pi (y_{j_2} - x_l)/L - qN(N - 1)$$

$$- q \sum_{j=1}^{q} \sum_{l_1 \neq l_2} \cot \pi (x_l - y_j)/L \cot \pi (x_{l'} - y_j)/L \right]$$ \hfill (2.9)
But from (2.4) and (1.2)

$$\frac{\partial}{\partial y_j} | \phi >= - \left( \frac{\pi}{L} \right) \sum_{l=1}^{N} \cot \pi (x_l - y_j)/L | \phi >$$  \hspace{1cm} (2.10)

and

$$\frac{\partial^2}{\partial y_j^2} | \phi >= \left( \frac{\pi}{L} \right)^2 \sum_{l',l=1}^{N} \cot \pi (x_l - y_j)/L \cot \pi (x_{l'} - y_j)/L | \phi > .$$  \hspace{1cm} (2.11)

Substituting (2.10) in (2.9) we obtain (2.5), as required.

Remark: The operator $T \{ y \}$ has occurred previously in the study of the $1/r^2$ quantum many body system [8]. Thus if $\psi$ denotes an eigenfunction of $H$ with $q$ replaced by $1/q$ (recall (1)), and $\psi = \Phi \mid 0 >$, then $\Phi$ is an eigenfunction of $T_y$. Furthermore, it is known [4] that the eigenfunctions of $T_y$ are the Jack polynomials $C^q(\kappa) e^{2\pi iy_1/L}, \ldots, e^{2\pi iy_q/L}$ where $\kappa$ is a partition which labels the eigenfunction. This latter fact will play a crucial role in our subsequent analysis.

Using Lemma 1 and (2.3), we see that

$$-iG_{N+1} ( (x, t), (0, 0) )$$

$$= (N + 1) \sum_{j=0}^{\infty} \left( \frac{-it/\hbar}{j!} \right)^j (T_{\{ y \}})^j \phi \mid \prod_{l=1}^{N} \sin \pi x_l/L | \phi > | y_1 = \ldots = y_q = x .$$  \hspace{1cm} (2.12)

We have previously evaluated the inner product in (2.11), which at the specified point is precisely the static one-body density matrix. Thus [4, eq.(3.4)]

$$\left( \frac{N + 1}{L} \right) C_{q, N} \prod_{l=1}^{q} e^{-\pi i q l} 2F_1(q)(-N, 1; -N + 1 - 1/q; e^{2\pi iy_1/L}, \ldots, e^{2\pi iy_q/L}),$$  \hspace{1cm} (2.13)

where

$$C_{q, N} = \frac{A_q(1/q, -1, 2/q)}{A_q(1/q, -N - 1, 2/q)}$$  \hspace{1cm} (2.14)

with

$$A_n(\lambda_1, \lambda_2, \lambda) = \prod_{j=1}^{n} \frac{\Gamma(1 + \lambda/2)\Gamma(\lambda_1 + \lambda_2 + \lambda(n + j - 2)/2)}{\Gamma(1 + \lambda j/2)\Gamma(\lambda_1 + \lambda(j - 1)/2)\Gamma(\lambda_2 + \lambda(j - 1)/2)}.$$

The generalized hypergeometric function of several variables $2F_1(\lambda)(a, b; c; x_1, \ldots, x_m)$ is the unique symmetric power series solution of the partial differential equations

$$x_j(1 - x_j) \frac{\partial^2 F}{\partial x_j^2} + \left[ \frac{c - 2}{\lambda} (m - 1) - (a + b + 1 - \frac{2}{\lambda} (m - 1))x_j \right] \frac{\partial F}{\partial x_j}$$

$$+ 2 \frac{2}{\lambda} \sum_{k=1}^{m} \frac{1}{x_k - x_j} \left( x_j(1 - x_j) \frac{\partial}{\partial x_j} - x_k(1 - x_k) \frac{\partial}{\partial x_k} \right) F - abF = 0$$  \hspace{1cm} (2.15)

\footnote{This equation is erroneously missing the factor $C_{q, N}$}
with the initial condition \(2F_1^{(\lambda)} = 1\) when \(x_1 = \ldots = x_m = 0\) [9,10] (when \(m = 1\) - single variable case - the differential equation satisfied by the Gauss hypergeometric equation results).

We need to be able to compute the action of the operator \((T_{(q)})^2\) on (2.12). For this purpose we first write \(2F_1^{(q)}\) in integral form.

### 2.2 An integral formula for \(2F_1^{(q)}\)

The following integral representation is due to Zan [10]:

\[
2F_1^{(2/\lambda)}(a; \lambda_1, \lambda_2, \lambda_1 + \lambda(n - 1)/2; \lambda_1 + \lambda_2 + \lambda(n - 1); z_1, \ldots, z_n) = A_n(\lambda_1, \lambda_2, \lambda) \int_{I_n} 1F_0^{(2/\lambda)}(a; z_1, \ldots, z_n; s_1, \ldots, s_n) \times D_{\lambda_1, \lambda_2, \lambda}(s_1, \ldots, s_n) ds_1 \ldots ds_n
\tag{2.16}
\]

where

\[
D_{\lambda_1, \lambda_2, \lambda}(s_1, \ldots, s_n) := \prod_{l=1}^q s_l^{\lambda_1 - 1} (1 - s_l)^{\lambda_2 - 1} \prod_{1 \leq j < k \leq q} |s_k - s_j|^\lambda
\tag{2.17}
\]

and

\[
1F_0^{(2/\lambda)}(a; z_1, \ldots, z_n; s_1, \ldots, s_n) := \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{|\kappa|=k} [a]_{\kappa}^{(2/\lambda)} C_{\kappa}^{(2/\lambda)}(z_1, \ldots, z_n) C_{\kappa}^{2/\lambda}(s_1, \ldots, s_n)
\tag{2.18}
\]

In (2.18) the second sum is over all partitions \((\kappa_1, \ldots, \kappa_n)\) of \(k\) into \(n\) parts, \(C_{\kappa}^{(2/\lambda)}\) denotes a suitably normalized Jack polynomial [9], and

\[
[a]_{\kappa}^{(\alpha)} := \prod_{j=1}^n \left( a - \frac{1}{\alpha} (j - 1) \right)_{\kappa_j}
\tag{2.19a}
\]

where

\[
(a)_k := a(a + 1) \ldots (a + k - 1).
\tag{2.19b}
\]

In general no formulas expressing \(1F_0^{(2/\lambda)}\) in terms of simpler functions are known. However, an exception is the equal variable case \(z_1 = \ldots = z_n = z\) when [4]

\[
1F_0^{(2/\lambda)}(a; z, \ldots, z; s_1, \ldots, s_n) = \prod_{l=1}^n (1 - zs_l)^{-a}
\tag{2.20}
\]

The interval of integration for each variable \(s_1, \ldots, s_n\) given in [10] is \(I = [0, 1]\).

The l.h.s. of (2.16) corresponds to the \(2F_1^{(q)}\) function in (2.12) if we take

\[
\frac{2}{\lambda} = q, \quad a = -N, \quad n = q, \quad \lambda_1 = 1/q, \quad \lambda_2 = -N - 1, \quad \lambda_j = e^{2\pi iy_j/L}
\tag{2.21}
\]

in the former. However, we see from the definition (2.17) that in this case the integrand in (2.16) is not integrable at \(s_j\) due to the factor \((1 - v_j)^{-(N+2)}\), for each \(j = 1, \ldots, q\). In this circumstance, in the one variable case (Gauss hypergeometric function) we know that the contour of integration \(I\) can be shifted to

\[
I = (-\infty, 0]
\tag{2.22}
\]

and the integration formula (2.16) is both well defined and satisfies (2.15). We expect this property to carry over to the \(m\)-variable case, and indeed it does with the change of normalization

\[
A_n(\lambda_1, \lambda_2, \lambda) \mapsto (-1)^{\lambda_1 n} A_n(\lambda_1, -\lambda(n - 1) - \lambda_1 - \lambda_2 + 1, \lambda)
\tag{2.23}
\]

as is shown in Appendix A. Thus we can use the integral representation (2.16) with parameters given by (2.21), interval of integration (2.22) and change of normalization (2.23).
2.3 An equivalent operator

Using the integral representation in (2.12) we see from (2.16) and (2.11) that we must calculate

\[ \left( T\{y\} \right)^j \prod_{l=1}^q e^{-\pi i qy_l} \mathcal{F}_0^{(q)} (-N; e^{2\pi i y_1/L}, \ldots, e^{2\pi i y_q/L}; s_1, \ldots, s_q) \bigg|_{y_1=\ldots=y_q=x} \]  

(2.24)

To accomplish this task we recall [4] that \( T\{y\} \) has

\[ \prod_{l=1}^q e^{-\pi i qy_l} C_\kappa^{(q)}(e^{2\pi i y_1/L}, \ldots, e^{2\pi i y_q/L}) \]

as an eigenfunction, with eigenvalue \( t_\kappa \) say, for each partition \( \kappa \). Hence, using (2.18), we see that (2.24) is equal to

\[ e^{-\pi i p q x} \sum_{j=0}^\infty \frac{1}{j!} \sum_{|\kappa|=j} [a]^{(2/\lambda)}(t_\kappa)^j \frac{C_\kappa^{(q)}(e^{2\pi i x/L}, \ldots, e^{2\pi i x/L})C_\kappa^{(q)}(s_1, \ldots, s_q)}{C_\kappa^{(q)}(1, \ldots, 1)} \]

(2.25)

If we know introduce an operator \( T\{s\} \) acting on the coordinates \( s_1, \ldots, s_q \) such that

\[ T\{s\} C_\kappa^{(q)}(s_1, \ldots, s_q) = t_\kappa C_\kappa^{(q)}(s_1, \ldots, s_q) \]

(2.26)

we see from (2.18) that

\[ \left( T\{s\} \right)^j e^{-\pi i p q x} \mathcal{F}_0^{(q)} (-N; e^{2\pi i x/L}, \ldots, e^{2\pi i x/L}; s_1, \ldots, s_q) \]

(2.27)

is identical to (2.25), so we can replace (2.24) by the equivalent operation (2.27). The construction of \( T\{s\} \) is simple, due to the symmetry between the eigenfunctions \( C_\kappa^{(q)}(z_1, \ldots, z_q) \) and \( C_\kappa^{(q)}(s_1, \ldots, s_q) \) exhibited in (2.18). In Appendix B we show that

\[ T\{s\} = -2 \left( \frac{2\pi}{L} \right)^2 \left[ \frac{q}{2} \sum_{j=1}^q s_j^2 \frac{\partial^2}{\partial s_j^2} + \sum_{j,k=1}^q \frac{s_j^2}{s_j - s_k} \frac{\partial}{\partial s_j} \right] \]

\[ + \left( q(4\pi^2 \rho/L) - \left( 2\pi/L \right)^2 \right) \sum_{j=1}^q s_j \frac{\partial}{\partial s_j} + q(1-q) \left( \frac{\pi}{L} \right)^2 N \]

(2.28)

The calculation of (2.27) is straightforward since the special case of the \( \mathcal{F}_0^{(q)} \) function therein can be evaluated according to (2.20).

Substituting the evaluation (2.20) in (2.27) and replacing (2.24) by the resulting formula we thus have from (2.11), (2.12) and (2.16) (with the further specifications made after (2.23)) the formula

\[ -i G_{N+1}(x, t, (0, 0)) \]

\[ = -C_{N, q} A_q(1/q, N + 1/q, 2/q) e^{-\pi i p x} \sum_{j=0}^\infty (-i t/h)^j j! \]

\[ \times \int_{(-\infty, 0]^q} ds_1 \ldots ds_q \left[ \left( T\{s\} \right)^j \prod_{l=1}^q \left( 1 - e^{2\pi i x/L s_l} \right)^N \right] \]

\[ \times D_{1/q, -N-1, 2/q}(s_1, \ldots, s_q) \]

(2.29)
2.4 The thermodynamic limit

From the explicit form (2.28), the action of the operator \( (T'_{(s)})^j \) in (2.29) can readily be computed using a computer algebra package for given \( N \) and \( j \). However, many terms result and it doesn’t seem possible to sum the series in \( j \). However, an analytic calculation is possible in the large \( N,L \) limit. From (2.28) we see that to leading order in \( N \) with respect to its action on the function

\[
\prod_{l=1}^{q}(1 - e^{2\pi i x/L s_l})^N
\]  

we can replace \( (T'_{(s)})^j \) by

\[
\left(-q \left(\frac{2\pi}{L}\right)^2 \sum_{j=1}^{q} s_j^2 \frac{\partial^2}{\partial s_j^2} + \pi q \rho \left(\frac{4\pi}{L}\right) \sum_{j=1}^{q} s_j \frac{\partial}{\partial s_j}\right)^j
\]  

The operator (2.31) gives terms which are \( O(1) \) and \( O(1/N) \) whereas the remaining part of \( (T'_{(s)})^j \) gives terms \( O(1/N) \) only, which can therefore be ignored. The \( O(1) \) action of (2.31) on (2.30) is readily computed, and we conclude

\[
(T'_{(s)})^j \prod_{l=1}^{q}(1 - e^{2\pi i x/L s_l})^N
\]

\[
= \left(-q(2\pi \rho)^2 \sum_{j=1}^{q} \frac{s_j^2}{(1 - s_j)^2} + \frac{s_j}{1 - s_j}\right)^j \prod_{l=1}^{q}(1 - e^{2\pi i x/L s_l})^N + O(1/N)
\]  

Substituting (2.32) in (2.29) we recognise that the sum over \( j \) is simply the power series for the exponential. Hence

\[
-iG_{N+1}((x, t), (0, 0))
\]

\[
\sim -C_{N,q} A_q(1/q, N + 1/q, 2/q)e^{-\pi i q x} \times \int_{(-\infty,0)^q} ds_1 \ldots ds_q \prod_{l=1}^{q} \exp \left[ it/h \left( \frac{s_j^2}{(1 - s_j)^2} + \frac{s_j}{1 - s_j} \right) \right] \prod_{l=1}^{q}(1 - e^{2\pi i x/L s_l})^N
\]

\[
\times D_{1/q,-N-2/q}(s_1, \ldots, s_q)
\]  

We are now close to deriving the result (1.4).

To finish off the calculation, we first note from (2.13) and (2.14) that

\[
\lim_{N \to \infty} C_{N,q} A_q(1/q, N + 1/q, 2/q) = B_q
\]  

where \( B_q \) is given by (1.4b). Next we change variables

\[
\frac{s_j}{1 - s_j} = -t_j
\]  

in the integral (2.33) so that it reads

\[
-\int_{[0,1]^q} dt_1 \ldots dt_q \prod_{l=1}^{q} \exp \left[ it/h(t_j^2 - t_j) \right] (1 - t_j(1 - e^{2\pi i x/L}))^N
\]
\[ \times D_{1/q,1/q,2/q}(t_1, \ldots, t_q) \]  (2.36)

But
\[ \lim_{N,L \to \infty} \frac{N}{L=\rho} (1 - t_j (1 - e^{2\pi ix/L}))^N = e^{2\pi i t_j \rho x}. \]  (2.37)

Substituting (2.37) in (2.36), changing variables
\[ t_j = \frac{(v_j + 1)}{2} \]  (2.38)

and substituting the resulting expression in (2.33), we obtain the formula (1.4) for the single-particle Green’s function (1.3).

Note added: Since completing this work, the generalization of the formula (1.3) to all rational values of the coupling \( q \), together with a derivation using Jack polynomials, has been announced by Ha [12].

ACKNOWLEDGEMENTS
I thank Professor F.D.M. Haldane for sending me [1]. This work was supported by the Australian Research Council.
Appendix A

Here we derive the integral representation (2.16) with interval of integration (2.22) and change of normalization (2.23). We start with the transformation formula [4, eq.(3.6)]

\[
{_{2}F_{1}}^{(2/\lambda)}(a, b; c; t_{1}, \ldots, t_{m}) = C {_{2}F_{1}}^{(2/\lambda)}(a, b; a+b+1+\lambda(n-1)/2-c; 1-t_{1}, \ldots, 1-t_{m})
\]

where

\[
C := C_{m}(a, b, c; \lambda) := \frac{A_{n}(b-\lambda(n-1)/2, c-b-\lambda(n-1)/2, \lambda)}{A_{n}(b-\lambda(n-1)/2, c-b-a-\lambda(n-1)/2, \lambda)}
\]

which is valid for \(a \in \mathbb{Z}_{\leq 0}\). Use of (A1) in (2.16) gives

\[
{_{2}F_{1}}^{(2/\lambda)}(a, \lambda_{1} + \lambda(n-1)/2; a+1-\lambda_{2}; z_{1}, \ldots, z_{n}) = \frac{A_{n}(\lambda_{1}, \lambda_{2}, \lambda)}{C_{n}(a, \lambda_{1} + \lambda(n-1)/2, \lambda_{1} + \lambda_{2} + \lambda(n-1)/2, \lambda)} \int_{[0,1]^{n}} {_{1}F_{0}}^{(2/\lambda)}(a; 1-z_{1}, \ldots, 1-z_{n}; s_{1}, \ldots, s_{n})
\]

\[
\times D_{\lambda_{1}, \lambda_{2}, \lambda}(s_{1}, \ldots, s_{n})ds_{1} \ldots ds_{n}
\]

where again it is assumed \(a \in \mathbb{Z}_{\leq 0}\).

Next we change variables

\[
s_{i} = -\frac{u_{i}}{1-u_{i}} \quad (i = 1, \ldots, n).
\]

To do this we note

\[
D_{\lambda_{1}, \lambda_{2}, \lambda}(s_{1}, \ldots, s_{n})ds_{1} \ldots ds_{n} = (-1)^{\lambda n} D_{\lambda_{1}+\lambda_{2}+\lambda(n-1)-1, \lambda}(u_{1}, \ldots, u_{n})du_{1} \ldots du_{n}
\]

Furthermore we can make use of the transformation formula given by the following result

\textbf{Lemma}

We have

\[
{_{1}F_{0}}^{(2/\lambda)}(a; 1-z_{1}, \ldots, 1-z_{n}; -\frac{u_{1}}{1-u_{1}}, \ldots, -\frac{u_{n}}{1-u_{n}}) = \prod_{j=1}^{n} (1-u_{j})^{a}{_{1}F_{0}}^{(2/\lambda)}(a; z_{1}, \ldots, z_{n}; u_{1}, \ldots, u_{n})
\]

\textbf{Proof}

We know that [10]

\[
{_{2}F_{1}}^{(2/\lambda)}(a, b; c; y_{1}, \ldots, y_{n}) = \prod_{j=1}^{n} (1-y_{j})^{-a}{_{2}F_{1}}^{(2/\lambda)}(a, c-b; c; -\frac{y_{1}}{1-y_{1}}, \ldots, -\frac{y_{n}}{1-y_{n}})
\]

Expressing both sides of (A6) in terms of the integral representation (2.16) and changing variables \(s_{j} \mapsto 1-s_{j}\) on the r.h.s. shows

\[
\int_{[0,1]^{n}} {_{1}F_{0}}^{(2/\lambda)}(a; y_{1}, \ldots, y_{n} ; s_{1}, \ldots, s_{n})
\]

\footnote{this formula is erroneously missing \(C\)}
Since the terms in the square brackets are independent of \( \lambda \) vanishes identically. The identity (A6) then follows by noting from (2.18) that in general
\[
\lambda \text{ zero for all } \Re(\lambda_1, \lambda_2) > 0, \text{ we can conclude the combination of terms in the square brackets vanishes identically.}
\]

The identity (A6) then follows by noting from (2.18) that in general
\[
1 - \frac{y_1}{1 - y_1}, \ldots, \frac{y_n}{1 - y_n}; 1 - s_1, \ldots, 1 - s_n
\]
is unchanged by interchanging all the variables \( y_1, \ldots, y_n \) with the variables \( s_1, \ldots, s_n \).

Substituting (A4) and (A5) in (A2), noting that in general
\[
\prod_{j=1}^{n} (1 - u_j)^a D_{\lambda_1, \lambda_2, \lambda}(u_1, \ldots, u_n) = D_{\lambda_1, \lambda_2 + a, \lambda}(u_1, \ldots, u_n),
\]
and replacing \( \lambda_2 \) by \( a + 1 - (\lambda_1 + \lambda_2 + \lambda(n - 1)) \) shows that (A2) reduces to (2.16) with the substitutions (2.22) and (2.23), as required.

**Appendix B**

Here we construct the operator \( T'_{(s)} \) defined in subsection 2.3 by the eigenvalue equation
\[
T'_{(s)} C^{(q)}(s_1, \ldots, s_q) = t_\kappa C^{(q)}(s_1, \ldots, s_q),
\]
where the eigenvalue \( t_\kappa \) is first to be calculated from the eigenvalue equation
\[
T_{(y)} \prod_{l=1}^{q} e^{-\pi i y l} C^{(q)}(e^{2\pi i y_1/L}, \ldots, e^{2\pi i y_q/L})
\]
\[
= t_\kappa \prod_{l=1}^{q} e^{-\pi i y l} C^{(q)}(e^{2\pi i y_1/L}, \ldots, e^{2\pi i y_q/L})
\]
with \( T_{(y)} \) given by (2.5b).

To calculate \( t_\kappa \) we recall that the Jack symmetric polynomial
\[
C^{(q)}(e^{2\pi i y_1/L}, \ldots, e^{2\pi i y_q/L}),
\]
is labelled by a partition \( \kappa = (\kappa_1, \ldots, \kappa_m) \) of non-negative integers such that
\[
\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_m \quad \text{and} \quad \sum_{j=1}^{m} \kappa_j = k,
\]
is the unique (up to normalization) solution of the eigenvalue equation [4, eq. (2.11) with \( \alpha = m = q \)]
\[
T_{(y)} C^{(q)}(e^{2\pi i y_1/L}, \ldots, e^{2\pi i y_q/L})
\]
\[
= \left( -q e'_{\kappa}(q) + \left( \frac{\pi}{L} \right)^2 \left[ qN + N(N - 1) \right] \right) C^{(q)}(e^{2\pi i y_1/L}, \ldots, e^{2\pi i y_q/L})
\]
where
\[
e'_{\kappa}(q) = \frac{2}{q} \left( \frac{2\pi}{L} \right)^2 (e_{\kappa}(q) + k/2)
\]
\[
e_{\kappa}(q) = q \sum_{j=1}^{q} \kappa_j(\kappa_j - 1)/2 - \sum_{q=1}^{q} (j - 1)\kappa_j + (q - 1)k
\]
Using (B4) and the fact that \( C_{\kappa}^{(q)} \) is homogeneous of order \( k \), which implies

\[
\sum_{j=1}^{q} \frac{\partial}{\partial \theta_j} C_{\kappa}^{(q)}(e^{2\pi iy_1/L}, \ldots, e^{2\pi iy_q/L}) = \frac{2\pi i}{L} k C_{\kappa}^{(q)}(e^{2\pi iy_1/L}, \ldots, e^{2\pi iy_q/L}),
\]

the l.h.s. of (B2) is readily computed, and we find

\[
t_{\kappa} = 2(2\pi/L)^2 e_{\kappa}(q) + k[q(4\pi^2 \rho/L) - (2\pi/L)^2] + q(q - 1)(\pi/L)^2 N \quad \text{(B7)}
\]

To construct \( T'_{\{s\}} \) with this eigenvalue according to (B1), we recall [9] that

\[
\Delta_{\{s\}} C_{\kappa}^{(q)}(s_1, \ldots, s_q) = e_{\kappa}(q) C_{\kappa}^{(q)}(s_1, \ldots, s_q),
\]

where

\[
\Delta_{\{s\}} := \frac{q}{2} \sum_{j=1}^{q} s_j^2 \frac{\partial^2}{\partial s_j^2} + \sum_{j,k=1}^{q} \frac{s_j^2}{s_j - s_k} \quad \text{(B9)}
\]

Furthermore, we can rewrite (B6) as

\[
\sum_{j=1}^{q} s_j \frac{\partial}{\partial s_j} C_{\kappa}^{(q)}(s_1, \ldots, s_q) = k C_{\kappa}^{(q)}(s_1, \ldots, s_q) \quad \text{(B10)}
\]

Hence

\[
\left( -2 \left(2\pi/L\right)^2 \Delta_{\{s\}} + [q(4\pi^2 \rho/L) - (2\pi/L)^2] \sum_{j=1}^{q} s_j \frac{\partial}{\partial s_j} + q(q - 1)(\pi/L)^2 N \right) C_{\kappa}^{(q)}(s_1, \ldots, s_q)
\]

\[
= t_{\kappa} C_{\kappa}^{(q)}(s_1, \ldots, s_q) \quad \text{(B11)}
\]

from which we read off that the explicit form of \( T'_{\{s\}} \) is given by (2.28).
References

1 F.D.M. Haldane, in *Proceedings of the 16th Taniguchi Symposium*, Kashikojima, Japan, 1993, eds. A. Okiji and N. Kawakami, Springer-Verlag, 1994.

2 P.J. Forrester, Nucl. Phys. B **388** (1992) 671.

3 P.J. Forrester, J. Stat. Phys. **72** (1993) 39.

4 P.J. Forrester, Nucl. Phys. B **416** (1994) 377.

5 P.J. Forrester, Phys. Lett. A **179** (1993) 127.

6 F.D.M. Haldane and M.R. Zirnbauer, Phys. Rev. Lett. **71** (1993) 4055.

7 E.K.U. Gross, E. Runge and O. Heinonen, *Many Particle Theory*, (Adam Hilger, Bristol, 1991).

8 B. Sutherland, Phys. Rev. A **4** (1971) 2019.

9 J. Kaneko, SIAM J. of Math. Analysis **44** (1993) 1086.

10 Y. Zan, Canad. J. of Math. **44** (1992) 1317.

11 I.G. Macdonald, *Lecture Notes in Math.* vol. 1271 (Springer, Berlin, 1987) 189.

12 Z.N.C. Ha, "Exact Dynamical Correlation Functions of Calogero-Sutherland Model and One-dimensional Fractional Statistics", preprint, submitted Phys. Rev. Lett.