Minimal entropy for uniform lattices in $\text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R})$

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June 18, 2014

Abstract

We prove that, among metrics on a compact quotients of $\mathbb{H}^2 \times \mathbb{H}^2$ of prescribed total volume, the product of hyperbolic metrics has minimal volume entropy.

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1 Introduction

Let $X$ be a compact Riemannian $n$-manifold and $\tilde{X}$ be the universal cover of $X$. The volume entropy is defined as

$$h(g) = \lim_{R \to \infty} \frac{1}{R} \log(\text{Vol}(B(x, R)))$$

where $B(x, R)$ is the ball of radius $R$ in $\tilde{X}$. The limit limit exists and is independent of the choice of $x$ (see [Man79] p.568).

In this paper we are interested in the following problem.
Question 1. Let $M$ be a compact locally symmetric space of noncompact type with locally symmetric metric $g_0$. Let $g$ be any other metric on $M$ such that Vol($M, g_0$) = Vol($M, g$). Do we have 

$$h(g) \geq h(g_0)?$$

M. Gromov was the first to conjecture such a result in [Gro83]. He was only interested in the real hyperbolic case. But the question still makes sense for a general symmetric space.

In the case where $\tilde{M}$ is reducible, there exists a unique locally symmetric metric of minimal entropy among locally symmetric metrics of prescribed volume, that is for scaling the metric in the factors ([CF03b]). This metric is called "the" locally symmetric metric and is denoted by $g_0$.

In this work we give a positive answer to the previous question in the case of compact quotients of products of two hyperbolic planes, that is $M = \Gamma \backslash (H^2 \times H^2)$ where $\Gamma$ is a uniform lattice in $\text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R})$. More precisely our main result is the following:

**Theorem 1 (Main theorem).** Assume that $(M, g_0)$ is a compact quotient of the product of 2 hyperbolic planes. Then, for any other metric $g$ on $M$,

$$h^4(g) \text{Vol}(M, g) \geq h^4(g_0) \text{Vol}(M, g_0).$$

Let us remark that the above inequality is sharp and no assumption is made on the metric $g$.

According to [Ebe96] p.260 there exists in $\text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R})$ two types of uniform lattices: virtually products and irreducible lattices. Then our main theorem applies for the compact products of hyperbolic surfaces or surfaces finitely covered by a product but also for quotients of $H^2 \times H^2$ which are far from being a product. Both compact examples do exist (see [Shi63] chapter 6 or [Bor63] for arithmetic examples).

In the fundamental paper [BCG95], G. Besson, G. Courtois and S. Gallot dealt with the case where $M$ is a locally symmetric space of rank one ($g_0$ is negatively curved) and obtained a similar statement than the main theorem for such spaces. The same result was obtained before (see [BCG91]) in any rank but for a metric $g$ in the conformal class of the locally symmetric metric $g_0$ (supposed to be irreducible). In the case where dim $M \geq 3$, the method is based on the barycenter map and the inequality in theorem 1 appears as an inequality of calibration (to be described below). In the case where dim $M = 2$, already proved by Katok (see [Kat88]), one can still ask if the inequality can be seen as an inequality of calibration. Besson, Courtois and Gallot showed that it is indeed true and gave another proof of the conjecture for the hyperbolic surfaces.

The barycenter method was improved by Connell and Farb in [CF03b] and, working factor by factor, they also gave a positive answer to the question in the case where $M$ is locally a product of rank one symmetric spaces with no factor $H^2$. In fact in that case,
there exists a unique locally symmetric metric (for scaling the metric in each factor) of minimal entropy.

In both of those papers, the authors pointed out that the case of products of hyperbolic surfaces still remained unknown.

Note also that there is an answer to question 1 in the same setting as [CF03b] (products of rank 1 symmetric spaces) in [BCG07] using an interesting different point of view. The inequality between volumes and entropies appears as a corollary of a general work on representations of fundamental groups of compact manifolds into Lie groups of non-compact types.

We now describe how is designed this paper. To prove our main result, we use the general outline invented in [BCG95]. It consists on an application of a method of calibration. To make this method efficient, we embed the universal cover $\mathbb{H}^2 \times \mathbb{H}^2$ in the unit sphere of $L^2$-functions of the Furstenberg boundary $\mathbb{S}^1 \times \mathbb{S}^1$ by the products of Poisson kernels. The aim is to show that this embedding has minimal volume. In order to detect this minimal property, one may use a 4-differential form taking its extremal values over orthonormal frames in tangent frames of the embedding. The method is briefly recalled in sections 2.1 and 2.2.

The hardest part is to find the calibrating form. Apart from the barycenter map, which is not efficient in the 2-dimensional case, Besson, Courtois and Gallot developed an alternative idea. In chapter 3 of [BCG91], following Gromov [Gro82], there is a general process to build suitable differential forms using bounded cocycles. The choice of the appropriate cocycle in $\mathbb{H}^2$ is then discussed in [BCG95] chapter 6. We made the slight necessary modifications of the general construction in paragraph 2.3 in our case. Indeed we had to consider the Furstenberg boundary instead of the visual boundary. Then we generalize this approach for the compact quotients of $\mathbb{H}^2 \times \mathbb{H}^2$ using the bounded 4-cocycle on $\mathbb{T}^2$ that M.Bucher exploit in [BK08b]. We describe this bounded cocycle in paragraph 3.1 and we check that the derived differential form has the required properties afterwards. The calibrating inequality is finally obtained in paragraph 3.4.

The last section is devoted to applications. We obtain a (non optimal) estimate for the minimal volume of a compact quotient $\Gamma \left( \mathbb{H}^2 \times \mathbb{H}^2 \right)$. The most spectacular application is that we are able to give an optimal bound for degrees of maps $f : Y^4 \to \Gamma \backslash (\mathbb{H}^2 \times \mathbb{H}^2)$ from any Riemannian 4-manifold. Precisely,

**Corollary 2.** Let $Y$ be a differentiable manifold of dimension 4 endowed with a Riemannian metric $g$ and let $f$ be a continuous map

$$f : (Y, g) \to (M, g_0)$$

Then

$$h(g)^4 \text{Vol}(Y, g) \geq |\text{deg } f| h(g_0)^4 \text{Vol}(M, g_0)$$

I would like thank my doctorate’s advisor C.Bavard for many useful discussions and comments. I am also grateful to G.Besson for his encouragements and the remarks he made on a preliminary version of this paper.
2 Method of calibration

2.1 The Spherical Volume

In this section, \( M \) is a quotient of \( \mathbb{H}^2 \times \mathbb{H}^2 \) by a uniform lattice \( \Gamma = \pi_1(M) \) and then \( \tilde{M} \) is the Riemannian product \( \mathbb{H}^2 \times \mathbb{H}^2 \) with usual metric of curvature \(-1\) in both factors (called \( g_0 \) in both of the manifolds \( M \) and \( \tilde{M} \)). Remark that \( g_0 \) is the best locally symmetric metric in the sense of [CF03b] chapter 2. As above, \( g \) is any other metric on \( M \).

Let us start with a few notations. We choose once and for all a basepoint \( o \in \mathbb{H}^2 \), for instance \( o = (0,0) \) in the model of the Poincaré disk. We will denote by the same letter \( o \) the basepoint in \( \mathbb{H}^2 \times \mathbb{H}^2 \). There will be no ambiguity resulting of that convention. The basepoint is used to compute the Buseman functions and to identify the boundary \( \partial \mathbb{H}^2 \) with the circle \( S^1 \) (see below).

In the sequel, \( \partial_F(\tilde{X}) \) will denote the Furstenberg boundary of \( \tilde{M} \): the space of Weyl chambers at infinity in \( M \) emanating from the same point (see [Ebe96] or [GJT98] for further discussions). There will be no conceptual difficulties coming from a general theory of Furstenberg boundary: we just use the fact that \( \partial_F(\mathbb{H}^2 \times \mathbb{H}^2) \) is identified to the 2-dimensional torus \( T^2 = S^1 \times S^1 \), after we choose a basepoint. The Furstenberg boundary is more adapted to the case of higher rank symmetric spaces and it is one of the key point in [CF03b]. The Furstenberg boundary and the visual boundary are the same in the rank one case and that’s why the distinction doesn’t appear in [BCG95].

The Furstenberg boundary turns out to be a probability space in the following way. The circle \( S^1 = \mathbb{R}/2\pi \mathbb{Z} \) is endowed with the Lebesgue probability \( d\theta \) (normalized in such a way that \( d\theta(S^1) = 1 \)). The 2-torus is the product (in the sense of probability spaces) of two such circles. The spaces of \( L^2 \) functions on \( T^2 \) is defined with respect to this probability.

The Poisson kernel of the disk \( p_o \) is defined by:

\[
p_o(x, \theta) = e^{-B_o(x, \theta)},
\]

where \( B_o(x, \theta) \) is the Buseman function. A classical computation gives the explicit expression in the Poincaré disk

\[
p_o(x, \theta) = \frac{1 - |x|^2}{|x - \theta|^2}
\]

for any \( x \in B(0,1) \) and \( \theta \in S^1 \).

The definition of Spherical Volume in [BCG95] extends to \( \mathbb{H}^2 \times \mathbb{H}^2 \) in the following way. We consider two representations of \( \pi_1(M) \). The first in \( \text{Isom}(\tilde{M}, g_0) \) is as usual. The second is the representation in the unit sphere of \( L^2(\partial_F \tilde{M}) \), the set of \( L^2 \)-functions on the Furstenberg boundary with real values. We denote this unit sphere by \( S^\infty \) or
$S^\infty(\partial_F \tilde{M})$ if the universal cover needs to be specified. More precisely it is a unitary representation restricted to $S^\infty$. It is defined by
\[
\gamma \cdot f(\theta) = f(\gamma^{-1}(\theta)) \sqrt{p_0(\gamma(o)_1, x)} \sqrt{p_0(\gamma(o)_2, y)}
\]
where $o$ is the basepoint of $\mathbb{H}^2 \times \mathbb{H}^2$, $\gamma(o) = (\gamma(o)_1, \gamma(o)_2) \in \mathbb{H}^2 \times \mathbb{H}^2$ and $\theta = (x, y) \in \mathbb{T}^2$. This is the change of variables formula (for $L^2$ functions). Indeed the Jacobian of an isometry acting on the Furstenberg boundary is given by the product of Poisson kernels.

Then, as in [BCG95], we introduce the family $N$ of Lipschitz continuous immersions $\Phi : \tilde{M} \to L^2(\partial_F \tilde{M})$ which are equivariant for both representations of $\pi_1(M)$, that is satisfying the following equation
\[
\Phi(\gamma \cdot x) = \gamma \cdot \Phi(x)
\]
for all $\gamma \in \pi_1(M)$ and all $x \in \tilde{M}$. We also require that
\[
\forall x \in \mathbb{H}^2 \times \mathbb{H}^2 \| \Phi(x) \|_{L^2} = 1,
\]
that is $\Phi(x) \in S^\infty$ and that, for every $x \in \tilde{M}$, $\Phi(x)$ is positive almost everywhere. We can also consider those immersions as functions of two variables $\Phi : (x, \theta) \mapsto \Phi(x) (\theta)$. The product of square roots of Poisson kernels $\Phi_0 = \sqrt{p_0} \times \sqrt{p_0}$: $\mathbb{H}^2 \times \mathbb{H}^2 \to S^\infty$ is an example of such an immersion (see lemma 4 below). Moreover it is an embedding. We will think of $\mathbb{H}^2 \times \mathbb{H}^2$ embedded in $S^\infty$ by the product of Poisson kernels. The spherical volume is then defined by
\[
\text{SphereVol}(M) = \inf_{\Phi \in N} \{ \text{Vol}(\Phi(U) \subset L^2(\partial_F \tilde{M}) \}
\]
where $U$ is a fundamental domain in $\tilde{M}$ for $\pi_1(M)$. More explicitly
\[
\text{SphereVol}(M) = \inf_{\Phi \in N} \{ \int_M \sqrt{\det g(\Phi(x))} \cdot dv_g(x) \}
\]
where $g_\Phi$ denotes the pull-back of the usual Hilbertian metric on $L^2$ by the immersion $\Phi$ and $\det g(\Phi)$ is computed in any $g$-orthonormal basis.

The Spherical volume is a transitional object. We use it to make a link between entropies and volumes. We want to prove the following inequalities.

First we recall the second inequality.

**Proposition 3** ([BCG95] chapter 3). $\text{SphereVol}(M) \leq \text{Vol}(M, g) \left( \frac{h(g)^2}{16} \right)^2$.

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First we recall the second inequality.

**Proposition 3** ([BCG95] chapter 3). $\text{SphereVol}(M) \leq \left( \frac{h(g)^2}{16} \right)^2 \text{Vol}(M, g)$.
**Proof:** We refer to the chapter 3 of [BCG95] for the same proof in the rank one case. There is only very few modifications we have to make in order to drive the same proof in our setting. Let’s first reintroduce the family of immersions which satisfy the conditions above. For a real parameter \( c > h(g) \), we consider:

\[
\Psi_c(x, \theta) = \left( \int_M e^{-cd(x,y)} p_0(x,y, \theta) dv_g(y) \right)^{1/2}.
\]

The condition on \( c \) ensures that the integral converges. Indeed for uniform lattices volume entropy and critical exponent are the same. Then we define an element of \( \mathcal{N} \) by

\[
\Phi_c(x, \theta) = \frac{\Psi_c(x, \theta)}{\left( \int_{\partial M} \Psi^2_c(x, \theta) d\theta \right)^{1/2}}.
\]

We just have replaced the boundary sphere by the Furstenberg boundary. We can now perform the very same computation as [BCG95]. We get the required estimate for the Spherical volume.

\[ \square \]

It remains to check now, if \( M \) is a compact quotient of \( \mathbb{H}^2 \times \mathbb{H}^2 \), that

\[
\text{SphereVol}(M) = \left( \frac{h(g_0)^2}{16} \right)^2 \text{Vol}(M, g_0).
\]

In fact, we can find an immersion \( \Phi \) of \( \mathcal{N} \) which has precisely the volume needed. Indeed

**Lemma 4.** Let \( M \) be a compact quotient of \( \mathbb{H}^2 \times \mathbb{H}^2 \). We embed \( \mathbb{H}^2 \times \mathbb{H}^2 \) in \( L^2(T^2) \) by the square root of the product of Poisson kernels,

\[
\Phi_0(z, t, \theta_1, \theta_2) = \sqrt{\frac{1 - |z|^2}{|z - \theta_1|^2} \frac{1 - |t|^2}{|t - \theta_2|^2}}.
\]

Then \( \Phi_0 \) belongs to \( \mathcal{N} \) and

\[
\text{Vol}(\Phi_0) = \left( \frac{h(g_0)^2}{16} \right)^2 \text{Vol}(g_0) = \left( \frac{1}{8} \right)^2 \text{Vol}(g_0).
\]

**Proof:** It is enough to handle the same situation with only one factor \( \mathbb{H}^2 \), the situation appearing completely as a product. The family of measures \( (\nu_z)_{z \in \mathbb{H}^2} \) which have a Lebesgue class given by

\[
\frac{d\nu_z}{d\theta} = \frac{1 - |z|^2}{|z - \theta|^2}
\]

are in fact the so-called Patterson-Sullivan measures of the hyperbolic plane which were constructed in [Pat76]. We refer to this original paper for the equivariance relation (remark that this family of measures is \( \text{SL}_2(\mathbb{R}) \)-equivariant and not only \( \Gamma \)-equivariant).
Then it is enough to show that \( \Phi_0 \) is an immersion at the basepoint \( o \), \( \text{SL}_2(\mathbb{R}) \) acting (transitively on \( \mathbb{H}^2 \)) by diffeomorphisms. We will compute the differential later at page 8.

Finally the volume of \( \Phi_0 \) has been computed in \[BCG95\] at page 744.

\[ \square \]

### 2.2 Calibration theory

In order to show that the Spherical Volume is achieved by the map \( \Phi_0 \), we use a classical method of calibration.

Now let’s make a brief review on the method (see \[Mor90\] or \[BCG95\] chapter 4). Remember we deal with positive equivariant immersions

\[
\Phi : \mathbb{H}^2 \times \mathbb{H}^2 \to S^\infty(T^2).
\]

An alternative definition for the volume of such an immersion \( \Phi \) is given by the following formula:

\[
\text{Vol}(\Phi) = \sup \frac{1}{\text{comass } \alpha} \int_M \Phi^* \alpha,
\]

where the supremum is taken over all 4-differential forms \( \alpha \) on \( S^\infty \) and

\[
\text{comass } \alpha = \sup |\alpha_g(Y_1, Y_2, Y_3, Y_4)|,
\]

where we compute the supremum over all orthonormal basis \((Y_1, Y_2, Y_3, Y_4)\) of \( T_yS^\infty \).

One says that a closed equivariant 4-differential form \( \omega \) calibrates the immersion \( \Phi_0 \) if for every \( x \in \mathbb{H}^2 \times \mathbb{H}^2 \) and every basis \((U_1, U_2, U_3, U_4)\) of \( T_x(\mathbb{H}^2 \times \mathbb{H}^2) \) we have

\[
\text{comass}(\omega) = \frac{|\omega_{\Phi_0(x)}(d\Phi(U_1), d\Phi(U_2), d\Phi(U_3), d\Phi(U_4))|}{\|d\Phi_0(U_1) \wedge d\Phi_0(U_2) \wedge d\Phi_0(U_3) \wedge d\Phi_0(U_4)\|},
\]

that is \( \omega \) restricted to orthonormal basis is maximal on \( T\Phi_0(\mathbb{H}^2 \times \mathbb{H}^2) \).

The following result is a consequence of the definitions above (see \[BCG95\] for a proof). Just remark that here we use the hypothesis of Lipschitz continuity for the immersions \( \Phi \in \mathcal{N} \).

**Lemma 5** (\[BCG95\] p.748). If there exists a closed \( \pi_1 \)-equivariant 4-differential form \( \omega \) which calibrates the immersion \( \Phi_0 \), then

\[
\text{SphereVol}(M) = \text{Vol}(\Phi_0).
\]

From now on, it just remains to find the 4-differential form \( \omega \) which calibrates the immersion \( \Phi_0 \).
2.3 A general process to build differential forms

In order to apply the method of calibration, we need to use a closed invariant 4-form on $S^\infty(L^2(T^2))$. Let us now explain how we found this form.

We start with a $k$-bounded combinatorial $\Gamma = \pi_1(M)$-invariant cocycle $c$ on $T^2 = \partial F_M$. Then we associate the $k$-form on $S^\infty$ by

$$\Omega(c)(f_1, \ldots, f_k) = \int_{T^2} c(\theta_0, \ldots, \theta_k) \varphi^2(\theta_0) \varphi f_1(\theta_1) \cdots \varphi f_k(\theta_k) d\theta_0 \cdots d\theta_k.$$

Here $\varphi$ belongs to $S^\infty$ and the functions $f_i$ are tangent to $S^\infty$ at the point $\varphi$, that is

$$\int_{T^2} \varphi f_i(\theta) d\theta = 0.$$

**Remark** The properties of differential form we construct in this way are not representative of the cohomology class of the cocycle we use. Indeed the image of a coboundary is not necessarily a differential of some lower degree differential form. The map $c \mapsto \Omega(c)$ is then only defined at the level of cochains and does not induce a map between cohomology spaces.

The calibrating form we are looking for must take extremal values on orthonormed frames tangent to the embedding by the product of Poisson kernels. The following lemma describe this tangent space (at the basepoint).

**Lemma 6.** We consider the map $\Phi_0$,

$$\Phi_0 : \mathbb{H}^2 \times \mathbb{H}^2 \rightarrow S^\infty \quad (z,t) \mapsto (\theta_1, \theta_2) \mapsto \sqrt{\frac{1-|z|^2}{|z_1-z|^2} \cdot \frac{1-|t|^2}{|t_2-t|^2}}.$$

Let $\theta \in S^1 \times S^1$ be a boundary point and let $\theta = (x, y)$ be its coordinates in the circle. Then the four functions

$$f(\theta) = \sqrt{2} \cos(x), \quad g(\theta) = \sqrt{2} \sin(x), \quad h(\theta) = \sqrt{2} \cos(y) \quad \text{et} \quad k(\theta) = \sqrt{2} \sin(y).$$

generate $T_{\Phi_0(0,0)}S^\infty$, that is

$$T_{\Phi_0(0,0)}S^\infty = \mathbb{R}f \oplus \mathbb{R}g \oplus \mathbb{R}h \oplus \mathbb{R}k$$

and this basis is $L^2$-orthonormed.

2.4 Back to the 2-dimensional case

Let us briefly recall how the general method to build differential forms was applied to get a calibrating form for compact quotients of $\mathbb{H}^2$. It is the occasion to introduce various
tools for our proof.

The calibrating form is expressed by the following formula
\[
\omega_\varphi(f,g) = \int_{(S^1)^3} c(x_0, x_1, x_2) \varphi^2(x_0) \varphi f(x_1) \varphi g(x_2) dx_0 dx_1 dx_2
\]
where \( c \) is the Euler cocycle of the circle, that is
\[
\begin{align*}
  c(x_0, x_1, x_2) &= 1 \text{ if the points are cyclically ordered on } S^1 \\
  c(x_0, x_1, x_2) &= -1 \text{ if not }
\end{align*}
\]
In fact, the cocycle \( c \) represents combinatorially the volume of ideal triangles in \( \mathbb{H}^2 \).

**Proposition 7** ([BCG95] p.756-758). The 2-form \( \omega \) is closed and invariant with respect to the whole group \( \text{Diff}^+(S^1) \) of orientation-preserving diffeomorphisms. Its comass equals \( \frac{1}{\pi} \) and is attained on orthonormal frames tangent to the embedding by the square root of the Poisson kernel.

We will show later in our situation that the closure of \( c \) as a combinatorial cocycle implies the closure of \( \omega \). Moreover \( c \) is itself \( \text{Diff}^+(S^1) \)-invariant and this proves that \( \omega \) is \( \text{Diff}^+(S^1) \)-invariant (see again the general case below). We refer to [BCG95] p.758 for the comass’s computation. We just mention the alternative expression for \( \omega \),
\[
\omega_\varphi(f,g) = 2 \int_{S^1} F(x) dG(x)
\]
where \( F \) and \( G \) are primitive of \( f \) and \( g \) with integrals vanishing. We shall use this expression later.

### 3 Implementation of the method and proof of the main theorem

#### 3.1 A combinatorial description of the volume form in \( \mathbb{H}^2 \times \mathbb{H}^2 \)

We saw previously that the Euler class of the circle plays a fundamental role. We then look for an analogous of this class in \( \mathbb{H}^2 \times \mathbb{H}^2 \).

An important remark concerning the Euler class is that this cocycle represents the volume class of \( \mathbb{H}^2 \) (through the Van Est isomorphism, see [Gui72] chapter III paragraph 7). M.Bucher in [BK08b] described a cocycle which also represents the volume class of \( \mathbb{H}^2 \times \mathbb{H}^2 \) and having some extremal properties (see remark below). This cocycle is defined as
\[
C = \text{Atl}(c \cup c).
\]
It is a 4-cocycle on $T^2 = \partial F\tilde{M}$. More explicitly,

$$C(\theta_0, \theta_1, \theta_2, \theta_3, \theta_4) = \sum_{\sigma \in S^5} \varepsilon(\sigma) c(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}) c(y_{\sigma(2)}, y_{\sigma(3)}, y_{\sigma(4)})$$

where $\theta_i = (x_i, y_i) \in T^2$.

The Euler class $c$ is closed and so is the cup product $c \cup c$. The alternation $C$ of $c \cup c$ is then closed.

With the process we described above, we build a 4-differential form on the unit sphere $S^\infty(L^2(T^2))$ by

$$\Omega_{\varphi}(f, g, h, k) = \int_{(T^2)^5} C(\Theta) \varphi^2(\theta_0) \varphi f(\theta_1) \varphi g(\theta_2) \varphi h(\theta_3) \varphi k(\theta_4) d\Theta$$

We denote by $\Theta$ a point on $(T^2)^5 : \Theta = (\theta_0, \theta_1, \theta_2, \theta_3, \theta_4)$.

Remark It is shown in [BK08b] that the cocycle $C$ represents the Gromov norm of the volume class, that is the proportionality constant between volume and simplicial volume (see [BK08a]). This extremal property of $C$ made us want to study the corresponding differential form although we do not know how to construct a general theory making the link between extremal properties of the cocycle and the calibrating property of the form.

3.2 An invariance relation satisfied by $\Omega$

Let us show in this paragraph that the form $\Omega$ is invariant under the action of the group

$$G = \text{Diff}^+(S^1) \times \text{Diff}^+(S^1).$$

This group is embedded in $\text{Diff}^+(T^2)$ by the diagonal map The action of $G$ on $L^2(T^2)$ extends the action of $\text{Isom}^+(H^2) \times \text{Isom}^+(H^2)$ by the $(L^2)$ change of variables formula

$$\Gamma \cdot \varphi = \sqrt{\text{Jac}^{-1}(x, y)} \varphi \circ \Gamma^{-1}(x, y).$$

The inverse of $\Gamma$ appears to make a left action. The group $G$ contains $\pi_1(M)$.

Let us take a diffeomorphism of $T^2$ of the form $\Gamma : (x, y) \mapsto (\gamma_1(x), \gamma_2(y))$. One has

$$\Gamma^* \Omega_{\varphi}(f, g, h, k) = \Omega_{\Gamma \varphi}(\Gamma \cdot f, \Gamma \cdot g, \Gamma \cdot h, \Gamma \cdot k) = \Omega_{\varphi}(f, g, h, k) = \int_{(T^2)^5} C(\Theta)(\Gamma \varphi)^2(\theta_0) \Gamma \varphi f(\theta_1) \Gamma \varphi g(\theta_2) \Gamma \varphi h(\theta_3) \Gamma \varphi k(\theta_4) d\Theta$$
Then we make the change of variables \((\theta_i)_{i=0,\ldots,4} = (\Gamma(\theta_i))_{i=0,\ldots,4}\). We obtain
\[
\Gamma^* \Omega_\varphi(f, g, h, k) = \int_{(\mathbb{T}^2)^5} C(\Gamma(\theta_0), \Gamma(\theta_1), \Gamma(\theta_2), \Gamma(\theta_3), \Gamma(\theta_4)) \varphi^2(\theta_0) \varphi f(\theta_1) \varphi g(\theta_2) \varphi h(\theta_3) \varphi k(\theta_4) d\Theta
\]
We are led to show the invariance of \(C\) under the action of \(G\). Indeed,
\[
C(\Gamma(\Theta)) = \sum_{\sigma \in S_5} \varepsilon(\sigma) c(\gamma_1(x_{\sigma(0)}), \gamma_1(x_{\sigma(1)}), \gamma_1(x_{\sigma(2)})) c(\gamma_2(y_{\sigma(2)}), \gamma_2(y_{\sigma(3)}), \gamma_2(y_{\sigma(4)}))
\]
\[
= \sum_{\sigma \in S_5} \varepsilon(\sigma) c(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}) c(y_{\sigma(2)}, y_{\sigma(3)}, y_{\sigma(4)})
\]
\[
= C(\Theta)
\]
because the Euler cocycle is invariant with respect to the action of \(\text{Diff}^+(\mathbb{S}^1)\).

### 3.3 Closure of \(\Omega\)

In order to use the method of calibration, we have to deal with a closed form. This is the aim of this paragraph.

**Proposition 8.** The 4-differential form \(\Omega\) is closed.

**Proof:** To differentiate \(\Omega\), it is easier to have an expression on a space of measures instead of the unit \(L^2\)-sphere. Let us begin by a quick review on the structure of the space of measures we will deal with.

Let \(M\) be the Banach vector space dual to \(C^0(\mathbb{T}^2)\), the Banach space of continuous functions on \(\mathbb{T}^2\). The space \(M\) is also the space of Radon measures on \(\mathbb{T}^2\). We consider the affine space
\[
M_1 = \{ \mu \in M| |\mu|(\mathbb{T}^2) = 1 \}.
\]
We think as this affine space as an infinite dimensional manifold shaped on a Banach space. The tangent space in each point is the vector space of zero-mass measures.

The differential form \(\Omega\) comes from a form on \(M_1\) pulled-back by the smooth map
\[
A: S^\infty \rightarrow M_1, \quad \varphi \mapsto B \mapsto \int_B \varphi^2(\theta) d\theta.
\]
The image of a function \(\varphi\) is the measure with density \(\varphi^2\) with respect to the Lebesgue measure. Let us define a form \(\Omega'\) on \(M_1\),
\[
\Omega'_\mu(\alpha, \beta, \gamma, \delta) = \int_{(\mathbb{T}^2)^5} C(\Theta) d\mu(\theta_0) d\alpha(\theta_1) d\beta(\theta_2) d\gamma(\theta_3) d\delta(\theta_4).
\]
Clearly
\[
\Omega = A^* \Omega'.
\]
Then it is enough to show that $\Omega'$ is closed. But now, $\Omega'$ is a linear map with respect to $\mu$. Then, we have (see [Lan62] p.84 for the differential formula of a form on a Banach manifold)

$$d\Omega'_\mu(\alpha, \beta, \gamma, \delta, \varepsilon) = \partial_\mu \Omega_\mu(\beta, \gamma, \delta, \varepsilon) \cdot \alpha - \partial_\mu \Omega_\mu(\alpha, \gamma, \delta, \varepsilon) \cdot \beta + \partial_\mu \Omega_\mu(\alpha, \beta, \gamma, \delta, \varepsilon) \cdot \gamma - \partial_\mu \Omega_\mu(\alpha, \beta, \gamma, \varepsilon) \cdot \delta + \partial_\mu \Omega_\mu(\alpha, \beta, \gamma, \delta) \cdot \varepsilon.$$ 

With the skew-symmetry relations, we obtain

$$d\Omega'_\mu(\alpha, \beta, \gamma, \delta, \varepsilon) = 5 \int_{(T^2)^5} C(\Theta) d\varepsilon(\theta_0) d\alpha(\theta_1) d\beta(\theta_2) d\gamma(\theta_3) d\delta(\theta_4).$$

Each measure has a vanishing total mass. Let us show that the above expression is in fact

$$d\Omega'_\mu(\alpha, \beta, \gamma, \delta, \varepsilon) = 5 \int_{(T^2)^6} \partial C(\Theta) d\varepsilon(\theta_0) d\alpha(\theta_1) d\beta(\theta_2) d\gamma(\theta_3) d\delta(\theta_4) d\mu(\theta_5).$$

Let us recall the combinatorial boundary of a cocycle

$$\partial C(\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5) = \sum_{i=0}^{5} (-1)^i C(\theta_0, \ldots, \hat{\theta_i}, \ldots, \theta_5).$$

First we have,

$$\int_{(T^2)^5} C(\theta_0, \ldots, \theta_4) d\varepsilon(\theta_0) d\alpha(\theta_1) d\beta(\theta_2) d\gamma(\theta_3) d\delta(\theta_4)$$

$$= \int_{(T^2)^6} C(\theta_0, \ldots, \hat{\theta_i}, \ldots, \theta_5) d\varepsilon(\theta_0) d\alpha(\theta_1) d\beta(\theta_2) d\gamma(\theta_3) d\delta(\theta_4) d\mu(\theta_5)$$

because $\mu(T^2) = 1$. We recognize the last term in the expression of $\partial C$ (corresponding to the index $i = 5$). We now prove that the other terms in the expression of $\partial C$ vanish. Each of this remaining terms are in the form of

$$\int_{(T^2)^6} C(\theta_0, \ldots, \hat{\theta_i}, \ldots, \theta_5) d\varepsilon(\theta_0) d\alpha(\theta_1) d\beta(\theta_2) d\gamma(\theta_3) d\delta(\theta_4) d\mu(\theta_5)$$

with $i \neq 5$. Integrating with respect to the variable $\theta_i$ which is not involved in the cocycle, make appear a multiplicative term, the total mass of one of a tangent measure, supposed to be zero.

So finally the closure of $C$ as a combinatorial cocycle (see page 9) proves the proposition.

\[\Box\]
3.4 The calibrating inequality

M. Bucher proves in [BK08b] that the Gromov’s norm of $C$ equals $2/3$. We deduce a rough inequality for the comass of $\Omega$. Indeed

\[
|\Omega_\mu(\alpha, \beta, \gamma, \delta)| \leq \int_{(T^2)^5} |C(\Theta)|^2 \varphi^2(\theta_0) \varphi f(\theta_1) \varphi g(\theta_2) \varphi h(\theta_3) \varphi k(\theta_4) | \, d\Theta \leq \frac{2}{3} \int (T^2)^5 \varphi^2(\theta_0) |\varphi f(\theta_1) \varphi g(\theta_2) \varphi h(\theta_3) \varphi k(\theta_4)| \, d\Theta \\
\leq \frac{2}{3} \sqrt{\int f^2(\theta_1) d\theta_1} \sqrt{\int g^2(\theta_2) d\theta_2} \sqrt{\int h^2(\theta_3) d\theta_3} \sqrt{\int k^2(\theta_4) d\theta_4} \\
= \frac{2}{3} \| f \| \| g \| \| h \| \| k \| .
\]

In fact $C$ takes extremal values on a zero-measure set of $(T^2)^5$. This is why one may suspect that the comass is strictly smaller than $2/3$. More precisely, let us show the following statement.

**Proposition 9.** $\Omega$’s comass equals $\frac{1}{6\pi^2}$ and is achieved on orthonormal frames tangent to the embedding $\Phi_0$.

**Proof:** Let us introduce the 4 functions on the torus

\[
f(\theta) = \sqrt{2} \cos(x),
g(\theta) = \sqrt{2} \sin(x),
h(\theta) = \sqrt{2} \cos(y),
\]

and

\[
k(\theta) = \sqrt{2} \sin(y)
\]

where $\theta = (x, y) \in S^1 \times S^1$. This is an orthonormal family. All this 4 functions are tangent to $S^\infty$ at the point $\varphi = 1$. Indeed

\[
\int_{T^2} f(\theta) d\theta = \int_{T^2} g(\theta) d\theta = \int_{T^2} h(\theta) d\theta = \int_{T^2} k(\theta) d\theta = 0.
\]

We compute the value of $\Omega$ on this functions. Then we show that the values of $\Omega$ on other orthonormal frames are smaller. We will use the explicit definition for the cocycle

\[
C(\theta_0, \theta_1, \theta_2, \theta_3, \theta_4) = \sum_{\sigma \in S_5} \varepsilon(\sigma) c(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}) c(y_{\sigma(2)}, y_{\sigma(3)}, y_{\sigma(4)})
\]

First we look for the permutations $\sigma$ such that the integrals

\[
\int_{(T^2)^5} c(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}) c(y_{\sigma(2)}, y_{\sigma(3)}, y_{\sigma(4)}) f(\theta_1) g(\theta_2) h(\theta_3) k(\theta_4) d\Theta
\]

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are nonzero. We will show later that, for each such permutation, the (nonzero) values are all
\[ \varepsilon(\sigma) \frac{1}{\pi^2}. \]

To make an integral of the previous form nonzero, it is necessary that the set
\[ \{\sigma(0), \sigma(1), \sigma(2)\} \]
contains the set \( \{1, 2\} \) and the set \( \{\sigma(2), \sigma(3), \sigma(4)\} \) contains the set \( \{3, 4\} \). Indeed, otherwise, for example let \( \sigma \) be a permutation such that \( \sigma(i) \neq 1 \) for \( i = 1, 2, 3 \). We get immediately
\[
\int_{(\mathbb{T}^2)^5} c(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}) c(y_{\sigma(2)}, y_{\sigma(3)}, y_{\sigma(4)}) f(x_1) g(x_2) h(y_3) k(y_4) d\Theta
\]

\[
= \int_{(\mathbb{S}^1)^9} c(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}) c(y_{\sigma(2)}, y_{\sigma(3)}, y_{\sigma(4)}) \left( \int_{\mathbb{S}^1} f(x_1) dx_1 \right) g(x_2) h(y_3) k(y_4) \frac{d\Theta}{dx_1}.
\]

In order to simplify notations, we denote \( \frac{d\Theta}{dx_1} \) the measure \( dy_1 dy_2 dy_3 dy_4 \). Such a term vanishes because
\[
\int_{\mathbb{S}^1} f(x_1) dx_1 = 0.
\]

There is in fact 20 permutations satisfying the condition above. They are listed in the following table together with their signatures.

| Permutation | Signature | Permutation | Signature |
|-------------|-----------|-------------|-----------|
| 0 1 2 3 4   | 1         | 0 1 2 4 3   | -1        |
| 0 2 1 3 4   | -1        | 0 2 1 4 3   | 1         |
| 1 0 2 3 4   | -1        | 1 0 2 4 3   | 1         |
| 1 2 0 3 4   | 1         | 1 2 0 4 3   | -1        |
| 1 2 3 0 4   | -1        | 1 2 3 4 0   | 1         |
| 1 2 4 0 3   | 1         | 1 2 4 3 0   | -1        |
| 2 0 1 3 4   | 1         | 2 0 1 4 3   | -1        |
| 2 1 0 3 4   | -1        | 2 1 0 4 3   | 1         |
| 2 1 3 0 4   | 1         | 2 1 3 4 0   | -1        |
| 2 1 4 0 3   | -1        | 2 1 4 3 0   | 1         |

Then, we check, using skew-symmetry of the cocycle \( C \) that each of those 20 terms corresponding to the above permutations
\[
\int_{(\mathbb{T}^2)^5} c(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}) c(y_{\sigma(2)}, y_{\sigma(3)}, y_{\sigma(4)}) f(x_1) g(x_2) h(y_3) k(y_4) d\Theta
\]
equals \[
\varepsilon(\sigma) \frac{1}{\pi^2}.
\]

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For example, let us take the identity permutation.

\[
\int_{(T^2)^5} c(x_0, x_1, x_2) c(y_2, y_3, y_4) \cos(x_1) \sin(x_2) \cos(y_3) \sin(y_4) d\Theta
\]

\[
= \int_{(S^1)^6} c(x_0, x_1, x_2) c(y_2, y_3, y_4) \cos(x_1) \sin(x_2) dx_0 dx_1 dx_2 dy_2 dy_3 dy_4
\]

\[
= \left( \int_{(S^1)^3} c(x_0, x_1, x_2) \cos(x_1) \sin(x_2) dx_1 dx_2 \right)
\]

\[
\times \left( \int_{(S^1)^3} c(y_2, y_3, y_4) \sin(y_3) dy_2 dy_3 dy_4 \right)
\]

Each of the 2 factors equals the extremal value of the form \( \omega \) in the \( \mathbb{H}^2 \)-case, that is \( \frac{1}{\pi} \).

Let us take again the transposition \((0, 1, 2, 4, 3)\) of signature \(-1\). In this case, we have

\[
\int_{(T^2)^5} c(x_0, x_1, x_2) c(y_2, y_3, y_4) \cos(x_1) \sin(x_2) \cos(y_3) \sin(y_4) d\Theta
\]

\[
= -\int_{(T^2)^5} c(x_0, x_1, x_2) c(y_2, y_3, y_4) \cos(x_1) \sin(x_2) \cos(y_3) \sin(y_4) d\Theta.
\]

Using skew-symmetry of \( c \). As above, we get

\[
\int_{(T^2)^5} c(x_0, x_1, x_2) c(y_2, y_4, y_3) \cos(x_1) \sin(x_2) \cos(y_3) \sin(y_4) d\Theta = -\frac{1}{\pi^2}.
\]

Let us consider a last example of a permutation with larger support,

\[\sigma = (1, 2, 3, 0, 4)\]

of signature \(-1\). The cocycle is changed in the following way:

\[c(x_1, x_2, x_3) c(y_3, y_0, y_4) = -c(x_3, x_1, x_2) c(y_0, y_3, y_4).\]

The \( x \)'s transformation leave the sign unchanged, unless the \( y \)'s transformation change the sign. We obtain again

\[
\int_{(T^2)^5} c(x_1, x_2, x_3) c(y_3, y_0, y_1) \cos(x_1) \sin(x_2) \cos(y_3) \sin(y_4) d\Theta = \epsilon(\sigma) \frac{1}{\pi^2}.
\]

In this way, we check that

\[
\Omega_1(f, g, h, k) = \frac{1}{120} \frac{20}{\pi^2} = \frac{1}{6\pi^2}.
\]

Remark that, due to invariance for \( \Omega \) under \( \text{Diff}^+(S^1) \times \text{Diff}^+(S^1) \), the 4-form is constant along the embedding.
We are now going to prove that the values of the form are smaller on tangent frames which are not tangent to the embedding $\Phi_0$. We use the Fourier’s Hilbertian basis of $L^2(T^2)$. First recall we remarked at the page 12 that the expression

$$\int_{(T^2)^5} C(\Theta) f(\theta_0) g(\theta_1) h(\theta_2) k(\theta_3) t(\theta_4) d\Theta$$

vanishes if every function $f$, $g$, $h$, $k$ and $t$ have integral zero. Then,

$$\int_{(T^2)^5} C(\Theta) (\varphi^2(\theta_0) - 1) \varphi f(\theta_1) \varphi g(\theta_2) \varphi h(\theta_3) \varphi k(\theta_4) d\Theta = 0.$$

and

$$\Omega_\varphi(f, g, h, k) = \int_{(T^2)^5} C(\Theta) \varphi^2(\theta_0) \varphi f(\theta_1) \varphi g(\theta_2) \varphi h(\theta_3) \varphi k(\theta_4) d\Theta = \Omega_1(\varphi f, \varphi g, \varphi h, \varphi k)$$

So we don’t lose generality if we assume that each function is tangent at the basepoint $1$. We also recall that, if $f$ is a tangent function at $S^\infty$ at $1$, that is with integral zero, it could be written as a Fourier’s sum, as

$$f(\theta) = \sum_{(n_1, n_2) \in \mathbb{N} \times \mathbb{Z} \setminus \{(0,0)\}} \sqrt{2} a_{n_1, n_2} \cos(n_1 x + n_2 y) + b_{n_1, n_2} \sqrt{2} \sin(n_1 x + n_2 y).$$

The convergence has to be understood in the $L^2$ setting. The functions

$$\alpha_{n_1, n_2}(x, y) = \sqrt{2} \cos(n_1 x + n_2 y) \text{ et } \beta_{n_1, n_2}(x, y) = \sqrt{2} \sin(n_1 x + n_2 y)$$

are a Hilbertian basis of $T_1S^\infty$. The indices are taken in the set

$$\mathbb{N} \setminus \{0\} \times \mathbb{Z} \cup \{0\} \times \mathbb{N}.$$ 

We build a basis of $(T_1S^\infty)^4$ as usual. Let us set the following notations. The basis elements are denoted

$$\begin{pmatrix} \gamma_{n_1, n_2}, \gamma_{m_1, n_2}, \gamma_{p_1, p_2}, \gamma_{q_1, q_2} \end{pmatrix}$$

where $\gamma_{n_1, n_2}$ is either $\alpha_{n_1, n_2}$ or $\beta_{n_1, n_2}$. We denote also $\gamma$ a function which is either cos, either sin. The first step of the study of the values of $\Omega$ on this basis is given by the following statement.

**Step 1.** With the above notations, $\Omega$ vanishes on every elements

$$\begin{pmatrix} \gamma_{n_1, n_2}, \gamma_{m_1, n_2}, \gamma_{p_1, p_2}, \gamma_{q_1, q_2} \end{pmatrix}$$

as soon as one of the couple of integers, for example $(n_1, n_2)$ satisfies $n_1 n_2 \neq 0$. 

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Proof: Let us show that every term of $\Omega$ vanishes.

The easiest case of the above statement is the case where at least 3 of the 4 couples of integers have two nonzero components. We use the classical formulas

$$\alpha_{n_1,n_2}(x,y) = \cos(n_1x)\cos(n_2y) - \sin(n_1x)\sin(n_2y)$$

and

$$\beta_{n_1,n_2}(x,y) = \sin(n_1x)\cos(n_2y) + \cos(n_1x)\sin(n_2y).$$

A function

$$(x,y) \mapsto \gamma_{n_1,n_2}(x,y)$$

in the $T_1S^\infty$'s basis satisfying $n_1n_2 \neq 0$ is a function which actually depends of two variables (it is not constant in $x$ nor $y$). In this case, one has

$$\int_{S^1} \gamma_{n_1,n_2}(x,y)dx = \int_{S^1} \gamma_{n_1,n_2}(x,y)dy = 0.$$ 

Let us take one of the terms of $\Omega$,

$$\int_{\{\gamma\}} c(x_\sigma(0),x_\sigma(1),x_\sigma(2))c(y_\sigma(0),y_\sigma(3),y_\sigma(4))\gamma_{n_1,n_2}(x_1,y_1)\gamma_{m_1,m_2}(x_2,y_2)\gamma_{p_1,p_2}(x_3,y_3)\gamma_{q_1,q_2}(x_4,y_4)d\Theta.$$ 

In the case we are looking at, there at least 8 variables which actually appear in the expression

$$\gamma_{n_1,n_2}(x_1,y_1)\gamma_{m_1,m_2}(x_2,y_2)\gamma_{p_1,p_2}(x_3,y_3)\gamma_{q_1,q_2}(x_4,y_4).$$

Among those 8 variables, at least two are not in the set

$$\{x_\sigma(0),x_\sigma(1),x_\sigma(2),y_\sigma(3),y_\sigma(4)\}.$$ 

We suppose for example that $\gamma_{n_1,n_2}$ is not constant in $x_1$ and that

$$x_1 \notin \{x_\sigma(0),x_\sigma(1),x_\sigma(2),y_\sigma(3),y_\sigma(4)\}$$

(other cases are similar). We integrate with respect to $x_1$ and we use the fact that

$$\int_{S^1} \gamma_{n_1,n_2}(x_1,y_1)dx_1 = 0.$$ 

Now, we assume that 2 of the 4 functions $\gamma_{n_1,n_2}$, $\gamma_{m_1,m_2}$, $\gamma_{p_1,p_2}$ are $\gamma_{q_1,q_2}$ are constant in one of their 2 variables. Without loss of generality, we assume that the 2 functions are $\gamma_{p_1,p_2}$ and $\gamma_{q_1,q_2}$. If it is the same variable appearing in those 2 functions, that is if

$$\gamma_{p_1,p_2}(x,y) = \gamma(p_1x) \quad \text{et} \quad \gamma_{q_1,q_2}(x,y) = \gamma(q_1x)$$

for example. Then after we use the addition’s formulas for $\gamma_{n_1,n_2}$ and $\gamma_{m_1,m_2}$, we notice that the 4 variables $x_1$, $x_2$, $x_3$ and $x_4$ all appear in the expression

$$\gamma_{n_1,n_2}(x_1,y_1)\gamma_{m_1,m_2}(x_2,y_2)\gamma_{p_1,p_2}(x_3,y_3)\gamma_{q_1,q_2}(x_4,y_4).$$
Then each term of $\Omega$ vanishes as we see after we integrate with respect to the variable $x_i$ which is not represented in $c(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)})$.

So let us assume that

$$
\gamma_{p_1,p_2}(x,y) = \gamma(p_1 x) \quad \text{et} \quad \gamma_{q_1,q_2}(x,y) = \gamma(q_2 y).
$$

After transforming the first two functions with addition’s formulas, we can write each term of $\Omega$ as a sum of 4 expressions of the form

$$
\int_{(\mathbb{T}^2)^5} c(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}) c(y_{\sigma(2)}, y_{\sigma(3)}, y_{\sigma(4)}) \gamma(n_1 x_1) \gamma(n_2 y_1) \gamma(m_1 x_2) \gamma(m_2 y_2) \gamma(p_1 x_3) \gamma(q_2 y_4) d\Theta.
$$

We remark that the variable $x_1$ and $y_1$ and also the variables $x_2$ and $y_2$ actually appear. This is not possible in the cocycle: there is only one repetition between the indices corresponding to variables $x_i$ and $y_i$. So, among the 4 variables $x_1$, $x_2$, $y_1$, $y_2$ is not involved in the cocycle. We conclude by integrating with respect to this variable.

The last case we have to study is the case where only one of the 4 functions $\gamma_{n_1,n_2}$, $\gamma_{m_1,m_2}$, $\gamma_{p_1,p_2}$ and $\gamma_{q_1,q_2}$ is not constant in one of its variables. We can assume that the 3 functions $\gamma_{m_1,m_2}$, $\gamma_{p_1,p_2}$ et $\gamma_{q_1,q_2}$ are constant in one of its variables. As in the previous discussion we deal separately with the case where the 3 same variables are involved, that is if

$$
\gamma_{m_1,m_2}(x,y) = \gamma(m_1 x) \quad \text{et} \quad \gamma_{p_1,p_2}(x,y) = \gamma(p_1 x) \quad \text{et} \quad \gamma_{q_1,q_2}(x,y) = \gamma(q_1 x)
$$

for example (the other case where $y$ is involved is similar). We use the addition’s formula for the first function $\gamma_{m_1,n_2}$. We have to look at expressions of the form

$$
\int_{(\mathbb{T}^2)^5} c(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}) c(y_{\sigma(2)}, y_{\sigma(3)}, y_{\sigma(4)}) \gamma(n_1 x_1) \gamma(n_2 y_1) \gamma(m_1 x_2) \gamma(p_1 x_3) \gamma(q_1 x_4) d\Theta.
$$

The 4 variables $x_i$ appear unless only 3 are used by the cocycle. We conclude as above by integrating with respect to the remaining variable $x_i$. The very last case is then

$$
\gamma_{m_1,m_2}(x,y) = \gamma(m_1 x) \quad \text{et} \quad \gamma_{p_1,p_2}(x,y) = \gamma(p_1 x) \quad \text{et} \quad \gamma_{q_1,q_2}(x,y) = \gamma(q_2 y)
$$

and we are interested in terms of the form

$$
\int_{(\mathbb{T}^2)^5} c(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}) c(y_{\sigma(2)}, y_{\sigma(3)}, y_{\sigma(4)}) \gamma(n_1 x_1) \gamma(n_2 y_1) \gamma(m_1 x_2) \gamma(p_1 x_3) \gamma(q_2 y_4) d\Theta.
$$

There is 6 variables actually involved; so the only case where we can’t conclude as above is when the set $\{x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}\}$ equals the set $\{x_1, x_2, x_3\}$. We integrate with respect to $x_1$, $x_2$ and $x_3$. This integral equals, modulo the sign,

$$
\int_{(\mathbb{T}^1)^3} c(x_1, x_2, x_3) \gamma(n_1 x_1) \gamma(m_1 x_2) \gamma(p_1 x_3) dx_1 dx_2 dx_3.
$$

But we already remarked that, because the 3 functions

$$
\begin{align*}
&x \mapsto \gamma(n_1 x) \quad \text{et} \quad x \mapsto \gamma(m_1 x) \quad \text{et} \quad x \mapsto \gamma(p_1 x)
\end{align*}
$$

we have
have integral zero, we have
\[
\int_{(S^1)^3} c(x_1, x_2, x_3) \gamma(n_1 x_1) \gamma(m_1 x_2) \gamma(p_1 x_3) \gamma(q_1 x_4) dx_1 dx_2 dx_3 = 0.
\]
This is the end of the discussion and the proof of step 1.

So far we showed that, a necessary condition for \(\Omega\) being nonzero on the element
\[
(\gamma_{n_1, n_2}, \gamma_{m_1, m_2}, \gamma_{p_1, p_2}, \gamma_{q_1, q_2}),
\]
is that the 4 functions are constant in one of their variables. In fact, we can go a little further.

**Step 2.** Let \(A\) be a 4-uple of functions denoted as above
\[
A = (\gamma_{n_1, n_2}, \gamma_{m_1, m_2}, \gamma_{p_1, p_2}, \gamma_{q_1, q_2}).
\]
We assume
\[
n_1 n_2 = m_1 m_2 = p_1 p_2 = q_1 q_2 = 0.
\]
For \(\Omega\) being nonzero on \(A\), it is necessary that, among the 4 couples of integers \((n_1, n_2), (m_1, m_2), (p_1, p_2)\) and \((q_1, q_2)\), 2 of the first coordinates vanishes and 2 of the second coordinates vanishes.

**Proof:** By contradiction, we show that, otherwise, every term of \(\Omega\) vanishes. There is only two cases to discuss (the other are similar with the role of the variables changed).

First, assume that
\[
\gamma_{n_1, n_2}(x, y) = \gamma(n_1 x),
\]
\[
\gamma_{m_1, m_2}(x, y) = \gamma(m_1 x),
\]
\[
\gamma_{p_1, p_2}(x, y) = \gamma(p_1 x)
\]
and
\[
\gamma_{q_1, q_2}(x, y) = \gamma(q_1 x).
\]
As above
\[
\int_{(T^2)^5} c(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}) c(y_{\sigma(2)}, y_{\sigma(3)}, y_{\sigma(4)}) \gamma(n_1 x_1) \gamma(m_1 x_2) \gamma(p_1 x_3) \gamma(q_1 x_4) d\Theta
\]
must vanishes because we can integrate with respect to one of the variables which don’t appear in the cocycle (there is at least one).

So let us assume that
\[
\gamma_{n_1, n_2}(x, y) = \gamma(n_1 x),
\]
\[
\gamma_{m_1, m_2}(x, y) = \gamma(m_1 x),
\]
\[ \gamma_{p_1,p_2}(x,y) = \gamma(p_1 x) \]

and

\[ \gamma_{q_1,q_2}(x,y) = \gamma(q_2 y) . \]

This time we have to separate the permutations between two types

1. Those with \( \{x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}\} \neq \{x_1, x_2, x_3\} \). In this case, we conclude as usual.

2. Those with \( \{x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}\} = \{x_1, x_2, x_3\} \). We then integrate with respect to \( x_1, x_2 \) and \( x_3 \) in

\[
\int_{(\mathbb{T}^2)^5} c(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}) c(y_{\sigma(2)}, y_{\sigma(3)}, y_{\sigma(4)}) \gamma(n_1 x_1) \gamma(m_1 x_2) \gamma(p_1 x_3) \gamma(q_2 y_4) d\Theta,
\]

and we get a multiplicative factor

\[
\int_{(S^1)^3} c(x_1, x_2, x_3) \gamma(n_1 x_1) \gamma(m_1 x_2) \gamma(p_1 x_3) dx_1 dx_2 dx_3
\]

which is zero (3 functions with integrals zero).

\( \square \)

So far we proved that, for \( \Omega \) being nonzero on

\[
(\gamma_{n_1,n_2}, \gamma_{m_1,m_2}, \gamma_{p_1,p_2}, \gamma_{q_1,q_2})
\]

we must have (after we possibly exchange the functions),

\[
\gamma_{n_1,n_2}(x,y) = \gamma(n_1 x),
\]

\[
\gamma_{m_1,m_2}(x,y) = \gamma(m_1 x),
\]

\[
\gamma_{p_1,p_2}(x,y) = \gamma(p_2 y)
\]

and

\[
\gamma_{q_1,q_2}(x,y) = \gamma(q_2 y).
\]

We will make this assumption from now on. We also remark that the permutations for which the corresponding term of \( \Omega \) is nonzero are exactly those listed at the page \([14]\).

Every term has the same value, for example,

\[
\left( \int_{(S^1)^3} c(x_0, x_1, x_2) \gamma(n_1 x_1) \gamma(m_1 x_2) dx_0 dx_1 dx_2 \right) \left( \int_{(S^1)^3} c(y_2, y_3, y_4) \gamma(p_2 y_3) \gamma(q_2 y_4) dy_2 dy_3 dy_4 \right).
\]

The conclusion will now be provided by the study of the calibrating form on \( \mathbb{H}^2 \). Recall that we now from [BCG95], that if \( f \) and \( g \) are functions with integrals zero on the circle, then

\[
\int_{(S^1)^3} c(x_0, x_1, x_2) f(x_1) g(x_2) dx_0 dx_1 dx_2 = 2 \int_{S^1} F(x) g(x) dx
\]

where \( F \) is the primitive of \( f \) with integral zero. This explains why we get the following statement
Step 3. Among the 4-uples considered

\[ (\gamma_{n,0}, \gamma_{1,0}, \gamma_{0,p,2}, \gamma_{0,q,2}) \]

For \( \Omega \) being nonzero, we need again, after we change the order of the functions, that

\[ n_1 = m_1 \text{ et } p_2 = q_2 \]

and also

\[ \gamma_{n,0} = \alpha_{n,0}, \gamma_{1,0} = \beta_{1,0}, \gamma_{0,p,2} = \alpha_{0,p,2} \text{ et } \gamma_{0,q,2} = \beta_{0,q,2}. \]

Proof: This is indeed the only possibilities for which the primitive of \( \gamma_{n,0} \) and \( \gamma_{1,0} \) as well as the primitive of \( \gamma_{0,p,2} \) and \( \gamma_{0,q,2} \) are not \( L^2(S^1) \)-orthogonal.

Let us take 4 functions as in the last step. We denote

\[ n = n_1 = m_1 \text{ et } p = p_2 = q_2. \]

In this case, \( \Omega \) is nonzero and

\[
|\Omega(\alpha_{n,0} \land \beta_{n,0} \land \alpha_{0,p} \land \beta_{0,p})| = \frac{1}{6} 4 \left( \int_{S^1} \frac{\beta^2_{n,0}(x)}{n} dx \right) \left( \int_{S^1} \frac{\beta^2_{0,p}(y)}{p} dy \right) = \frac{1}{6np\pi^2}.
\]

The coefficient \( \frac{1}{6} \) corresponds to the proportion of nonzero terms in the sum describing \( \Omega \) (there is indeed 20 permutations among 120). The very same argument as in the computation of the value of \( \Omega \) on the orthonormed tangent to the embedding by \( \Phi_0 \) shows that each of those 20 terms have the same value.

We finally remark that the value is maximum if (and only if) \( n = p = 1 \), that is when the functions generate the tangent to \( \Phi_0(o,o) \).

We now have all the ingredients we need to establish the calibrating inequality. Let us take 4 tangent functions \( f, g, h \) and \( k \) that form an orthonormal frame. Let us then write the corresponding Fourier’s decompositions,

\[
f(\theta) = \sum_{n,p} a_{n,p} \cos(nx + py) + b_{n,p} \sin(nx + py),
\]

\[
g(\theta) = \sum_{n,p} c_{n,p} \cos(nx + py) + d_{n,p} \sin(nx + py),
\]

\[
h(\theta) = \sum_{n,p} u_{n,p} \cos(nx + py) + v_{n,p} \sin(nx + py)
\]

and

\[
k(\theta) = \sum_{n,p} r_{n,p} \cos(nx + py) + s_{n,p} \sin(nx + py).
\]
The sums are convergent because the coefficients are orthonormed because the calibrating inequality only depends on the space they span. We identified earlier the nonzero terms of $\Omega$. Then, let us write,
\[
|\Omega(f, g, h, k)| \leq \frac{1}{6\pi^2} \sum_{n>0, p>0} (|a_{n,0}d_{n,0}| |u_{0,p}s_{0,p}| + |b_{n,0}c_{n,0}| |u_{0,p}r_{0,p}|) \\
+ \frac{1}{6\pi^2} \sum_{n>0, p>0} (|a_{n,0}d_{n,0}| |v_{0,p}r_{0,p}| + |b_{n,0}c_{n,0}| |v_{0,p}r_{0,p}|) \\
+ \frac{1}{6\pi^2} \sum_{n>0, p>0} (|a_{0,p}d_{0,p}| |u_{n,0}s_{n,0}| + |b_{0,p}c_{0,p}| |u_{n,0}s_{n,0}|) \\
+ \frac{1}{6\pi^2} \sum_{n>0, p>0} (|a_{0,p}d_{0,p}| |v_{n,0}r_{n,0}| + |b_{0,p}c_{0,p}| |v_{n,0}r_{n,0}|).
\]

The sums are convergent because the coefficients are $l^2$. Then we factorize the upper bound of $\Omega$.
\[
|\Omega(f, g, h, k)| \leq \frac{1}{6\pi^2} \sum_{n>0, p>0} (|a_{n,0}d_{n,0}| + |b_{n,0}c_{n,0}|) (|u_{0,p}s_{0,p}| + |v_{0,p}r_{0,p}|) \\
+ \frac{1}{6\pi^2} \sum_{n>0, p>0} (|a_{0,p}d_{0,p}| + |b_{0,p}c_{0,p}|) (|u_{n,0}s_{n,0}| + |v_{n,0}r_{n,0}|) \\
= \frac{1}{6\pi^2} \left( \sum_{n>0} |a_{n,0}d_{n,0}| + |b_{n,0}c_{n,0}| \right) \left( \sum_{p>0} |u_{0,p}s_{0,p}| + |v_{0,p}r_{0,p}| \right) \\
+ \frac{1}{6\pi^2} \left( \sum_{p>0} |a_{0,p}d_{0,p}| + |b_{0,p}c_{0,p}| \right) \left( \sum_{n>0} |u_{n,0}s_{n,0}| + |v_{n,0}r_{n,0}| \right).
\]

We next apply the Cauchy-Schwarz inequality. The first factor for example,
\[
\sum_{n>0} |a_{n,0}d_{n,0}| + |b_{n,0}c_{n,0}|
\]

is seen as the scalar product of the vector
\[
(a_{n,0}, b_{n,0})_{n>0}
\]

by the vector
\[
(d_{n,0}, c_{n,0})_{n>0}.
\]

So
\[
|\Omega(f, g, h, k)| \leq \frac{1}{6\pi^2} \left[ \left( \sum_{n>0} a_{n,0}^2 + b_{n,0}^2 \right) \left( \sum_{n>0} a_{n,0}^2 + c_{n,0}^2 \right) \left( \sum_{p>0} u_{0,p}^2 + v_{0,p}^2 \right) \left( \sum_{n>0} s_{n,0}^2 + r_{n,0}^2 \right) \\
+ \frac{1}{6\pi^2} \left( \sum_{p>0} a_{0,p}^2 + b_{0,p}^2 \right) \left( \sum_{p>0} d_{0,p}^2 + c_{0,p}^2 \right) \left( \sum_{n>0} u_{n,0}^2 + v_{n,0}^2 \right) \left( \sum_{n>0} s_{n,0}^2 + r_{n,0}^2 \right) \right].
\]
Let us slightly simplify the notations. We denote by $\varphi$ the 8 variables function given by

$$\varphi(x, y, z, t, x', y', z', t') = xyzt' + x'y'zt.$$ 

We must show that $\varphi$ is bounded above by 1 when $x^2 + x'^2 \leq 1$, $y^2 + y'^2 \leq 1$, $z^2 + z'^2 \leq 1$, et $t^2 + t'^2 \leq 1$.

We use successively Cauchy-Schwartz inequalities

$$|\varphi(x, y, z, t, x', y', z', t')| \leq \sqrt{(xy)^2 + (x'y')^2} \sqrt{(zt)^2 + (z't')^2} \leq \sqrt{x^4 + x'^4} \sqrt{y^4 + y'^4} \sqrt{z^4 + z'^4} \sqrt{t^4 + t'^4}.$$ 

We conclude with the arithmetic/geometric mean

$$\sqrt{x^2 + x'^2} \leq x + x'.$$

We get

$$|\varphi(x, y, z, t, x', y', z', t')| \leq \sqrt{x^2 + x'^2} \sqrt{y^2 + y'^2} \sqrt{z^2 + z'^2} \sqrt{t^2 + t'^2} \leq 1,$$

which is the end of the calibrating inequality.

We succeeded in applying the method of calibration proving that

$$\text{SphereVol}(M) = \left(\frac{1}{8}\right)^2 \text{Vol}(g_0),$$

which is the end of the proof of our main theorem. \qed

4 Applications

We finally look for some consequences, suggested by the last chapter of [BCG95].

It’s possible to extend the main result to the case where $g$ lives on another differentiable manifold related to $X$ by a map of non-zero degree. The result we obtain is an optimal "degree theorem" as in the article [CF03a], who misses the case we investigate. There is also a similar result in [LS09a] but with a nonoptimal constant and with an additional hypothesis on $\text{Ric}(g)$.

**Corollary 10.** Let $Y$ be a differentiable manifold of dimension 4 endowed with a Riemannian metric $g$ and let $f$ be a continuous map

$$f : (Y, g) \rightarrow (M, g_0)$$

Then

$$h(g)^4 \text{Vol}(Y, g) \geq |\text{deg } f| h(g_0)^4 \text{Vol}(M, g_0)$$

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Proof: Observe that the inequality is trivial if deg(f) = 0. So let us assume that deg(f) is non-zero. First, one can regularize the map f in a homotopic map, still denoted f, which is $C^1$. We call $\tilde{f}$ the map induced by f from $\tilde{Y}$ to $\tilde{M}$. Let us introduce the invariant appropriated to this new situation

$$\text{SphereVol}(f) = \inf \text{Vol}((U, \Phi^*(\text{can}))$$

where $\Phi$ are Lipschitz continuous equivariant immersions from $\tilde{Y}$ to $L^2(\partial_F M)$. As before, one example is given by the product of Poisson kernels:

$$\Phi_0(y, \theta) = \sqrt{p_0(\tilde{f}(y)_1, \theta_1)p_0(\tilde{f}(y)_2, \theta_2)}$$

and a useful family of examples is given by:

$$\Phi_c(y, \theta) = \frac{\int_{\tilde{Y}} e^{-cd(y,z)}\Phi_0^2(z, \theta)dv_g(z)}{\int_{\tilde{Y}} e^{-cd(y,z)}dv_g(z)}$$

The two arguments above (page 5 for the first and page 13 for the second) give in this context

1. SphereVol $\leq \left(\frac{h(g)^2}{16}\right)^2 \text{Vol}(Y, g)$ using the computation of $\text{Vol}(\Phi_c)$
2. The image of $\Phi_0$ is still calibrated because f is surjective and then SphereVol $= \text{Vol}(\Phi_0)$. Moreover

$$\text{Vol}(\Phi_0) = |\text{deg} f| \left(\frac{h(g_0)^2}{16}\right)^2 \text{Vol}(X, g_0)$$

We also obtain an estimate for the minimal volume. Let $X$ be a compact manifold. The minimal volume is defined as

$$\text{MinVol} = \inf \{\text{Vol}(g), |K(g)| \leq 1\}$$

(see [Gro82]).

Corollary 11. Let $X = \Gamma \backslash (\mathbb{H}^2 \times \mathbb{H}^2)$ be a compact quotient of $\mathbb{H}^2 \times \mathbb{H}^2$. Then

$$\text{MinVol}(X) \geq \frac{4}{81} \text{Vol}(g_0).$$

In particular, we reprove a general theorem of [LS06] stating that MinVol is nonzero.

Proof: We still follow chapter 9 of [BCG95]. Take a metric $g$ on $X$ with $|K(g)| \leq 1$. We deduce an equality on the Ricci curvature

$$\text{Ric}(g) \geq -3g$$

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We then apply the Bishop’s inequality ([GHL04] p.144). Taking log and making the radius go to infinity, we have
\[ h(g) \leq 3. \]

We conclude introducing this inequality in
\[ \text{Vol}(g) \geq \left( \frac{\sqrt{2}}{h(g)} \right)^4 \text{Vol}(g_0). \]

\[ \square \]

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