Systemic Risk Components in a Network Model of Contagion∗

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Abstract

We show how to do systemic risk attribution in a network model of contagion with interlocking balance sheets, using the Shapley and Aumann-Shapley values. As a preliminary, we extend portfolio risk attribution to risk attribution for a single firm with a balance sheet. Along the way, we establish new results on sensitivity analysis of the Eisenberg-Noe network model of contagion, featuring a Markov-chain interpretation. We illustrate the design process for systemic risk attribution methods by developing several of them.

1 Introduction

Risk attribution is important in systemic risk management, as it is in portfolio risk management. Systemic risk involves risk that arises because of the structure of the financial system and interactions between financial institutions. Systemic risk attribution is decomposing the risk of a system into risk components that are attributed to components of the system. For an introduction and literature review, see Staum (2011a,b). We build on a basic theory of systemic risk attribution in which the key tools are the Shapley and Aumann-Shapley values, introduced briefly in Section 2.1; for further background, see Staum (2011b). Our primary contribution is to develop further the theory of systemic risk attribution by showing how to do systemic risk attribution in a network model of interlocking balance sheets. Our preliminary results in Section 2, extending portfolio risk attribution to risk attribution for a single firm with a balance sheet, are a secondary contribution.

In the network model presented in Section 3, a graph represents a financial network, with nodes representing firms, and directed edges representing loans from one firm to another. Contagion spreads through the network when the default of one firm imposes losses on its creditor, causing

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the creditor to default. The balance sheets of borrower and lender are interlocking, because the loan appears as an asset on the lender’s balance sheet and a liability on the borrower’s balance sheet. A problem of “holes in banks’ balance sheets” (Gauthier et al., 2010) arises when one attempts to use the Shapley or Aumann-Shapley values to attribute risk. As Drehmann and Tarashev (2011) do in simultaneous research, we propose a solution to this problem. Whereas they use only the Shapley value, we show how to use linear programming (LP) sensitivity analysis to compute Aumann-Shapley values for systemic risk components in a network model of contagion. The LP sensitivity analysis in Section 3.2 is one of our secondary contributions. It extends the results of Liu and Staum (2010) on sensitivity analysis of the Eisenberg and Noe (2001) network model of contagion by including sensitivity to liabilities as well as to income, and it adds a Markov-chain interpretation. We use simple, artificial examples (Section 3.3) to illustrate the workings of our methods, whereas Drehmann and Tarashev (2011) apply their method to real data on major global banks.

Our analysis goes beyond that of Drehmann and Tarashev (2011) in that we explore more than one solution to the problem of holes in banks’ balance sheets in Section 4. We contribute to the methodology of design of risk attribution methods through a discussion of the characteristics of different systemic risk attribution methods. Section 5 offers some guidance about choosing and designing systemic risk attribution methods.

2 Risk Attribution with a Single Balance Sheet

As a preliminary to risk attribution in a model of interlocking balance sheets, we extend risk attribution among assets in a portfolio to risk attribution among components of both sides of the balance sheet of a single firm. In this setting, risk attribution can be performed by applying the Shapley or Aumann-Shapley values once four steps are performed.

The first step is to select the components of the time-0 balance sheet to which risk is to be attributed, and to create a “balance-sheet scheme,” which specifies how the whole time-0 balance sheet depends on the sizes of these components. We explore some balance-sheet schemes that lead to different risk attribution methods. The remaining three steps are part of the risk management framework also for tasks aside from risk attribution. For concreteness and simplicity, throughout our study we will work with the following choices for these steps.

The second step is to model the time-1 balance sheet as a function of the time-0 balance sheet and a random vector of risk factors that vary across possible future scenarios. The time-0 balance sheet includes liabilities, $\gamma$ of equity and $\delta$ of debt, and assets, $\eta$ of cash and a row vector $\Theta$ of risky asset holdings. These quantities must satisfy the accounting equation, which says that the size of the balance sheet $\varsigma = \gamma + \delta = \eta + \Theta 1$, i.e., assets equal liabilities. At time 1, the asset portfolio value $e = \eta + \Theta a$, where $a$ is a column vector of random growth factors for each of the asset values. We do not need to discuss time-1 liabilities because of the choice we make in the third step.

The third step is to associate a cost with a time-1 balance sheet. We take the cost to be the loss $\ell = (\delta - e)^+ + e$ to creditors on their principal. With this choice, we can ignore randomness in time-1 liabilities and even any interest that the firm may owe to its creditors. In portfolio risk allocation it has been standard to ignore the liability side of the balance sheet altogether in taking cost to be the opposite of portfolio value $e$ or P&L $e - \varsigma$.

The fourth step is to choose a risk measure that maps a random vector whose value in each scenario is cost to a scalar whose interpretation is risk. We take the risk measure to be expected
cost. This choice allows us to focus henceforth on cost allocation within a single scenario. Under mild conditions, the risk attributed to a component of the balance sheet is the expectation of the cost attributed to that component by the Shapley or Aumann-Shapley value (Staum, 2011b). Staum (2011a, §6) describes how to implement our approach for other risk measures.

2.1 Cost Allocation

Here we briefly summarize a standard framework for cost allocation. Suppose that cost is to be allocated to $n$ components of a system, for example, to risky asset holdings. Let $\lambda \in [0,1]^n$ be a participation vector, where $\lambda_i$ is interpreted as the participation level of component $i$ in the system. A participation vector $\lambda$ specifies a counterfactual system: for example, one in which the risky asset holdings are $\Theta \text{diag}(\lambda)$ instead of $\Theta$ as in the real system. The key object is the cost function $c$, where $c(\lambda)$ is the cost of the counterfactual system specified by participation vector $\lambda$: for example, the creditors’ loss when risky asset holdings are $\Theta \text{diag}(\lambda)$. The cost function must make $c(1)$ be the cost of the real system and $c(0)$ be zero. There are two widely-used methods for cost allocation: the Shapley value and the Aumann-Shapley value. For any subset $S \subseteq \{1, \ldots, n\}$, let $\lambda(S)$ be such that $\lambda_i(S)$ equals 1 if $i \in S$ and 0 if $i \notin S$. The Shapley value allocates to component $i$

$$\frac{1}{n!} \sum_{S \not\ni i} (n - |S| - 1)! |S|! \left(c(\lambda(S) \cup \{i\}) - c(\lambda(S))\right).$$

(1)

This is based on the incremental costs $c(\lambda(S) \cup \{i\}) - c(\lambda(S))$ when the participation of component $i$ is added to the participation of a set $S$ of other components. The Aumann-Shapley value is

$$\int_0^1 \nabla c(\lambda 1) \, d\lambda.$$  

(2)

This vector equals the gradient $\nabla c(1)$ if $c$ is (positively) homogeneous. Its allocation to component $i$ is based on the sensitivity of cost to perturbations of the participation of component $i$. Throughout Section 2, we focus on the Aumann-Shapley value. It has attracted more attention in the portfolio risk management literature, and it requires more study because it involves gradient computation.

Next we apply this general framework to a firm’s balance sheet and the loss to its creditors. Let $(\lambda)$ indicate any quantity in the counterfactual system specified by participation vector $\lambda$. The time-0 balance sheet includes equity $\gamma(\lambda)$, debt $\delta(\lambda)$, cash $\eta(\lambda)$, and risky assets $\Theta(\lambda)$. These quantities must satisfy the accounting equation. The time-1 asset value is $e(\lambda) = \eta(\lambda) + \Theta(\lambda)a$, so the creditors’ loss is $c(\lambda) = (\delta(\lambda) - e(\lambda))^+$. The gradient in the Aumann-Shapley value is

$$\nabla c(\lambda) = 1\{e(\lambda) < \delta(\lambda)\} (\nabla \delta(\lambda) - \nabla e(\lambda))$$

$$= 1\{\eta(\lambda) + \Theta(\lambda)a < \delta(\lambda)\} (\nabla \delta(\lambda) - \nabla \eta(\lambda) - \nabla (\Theta(\lambda)a)).$$

2.2 Cost Allocation to Assets

We consider two balance-sheet schemes that allocate cost to assets. In the first, cost is allocated to all assets, and the size of the balance sheet depends on assets’ participation. In the second, cost is allocated only to risky assets, and the size of the balance sheet is fixed.

In the first scheme, the participation vector $\lambda = [\lambda^\gamma \lambda^\Theta]$, where $\lambda^\gamma$ is the participation level of cash and $\lambda^\Theta$ contains the participation levels of risky assets. The asset holdings are $\eta(\lambda) = \eta \lambda^\gamma$
of cash and \( \Theta(\lambda) = \Theta \text{diag}(\lambda^\theta) \) of risky assets, so the balance sheet size \( \varsigma(\lambda) = \eta \lambda^\eta + \Theta \lambda^\theta \) is variable. The proportions of liabilities are fixed: \( \gamma(\lambda) = \gamma \varsigma(\lambda)/\varsigma \) and \( \delta(\lambda) = \delta \varsigma(\lambda)/\varsigma \). The time-1 portfolio value \( e(\lambda) = \eta \lambda^\eta + \Theta \text{diag}(\lambda^\theta)a \), so the cost function \( c \) given by

\[
c(\lambda) = (\delta(\lambda) - e(\lambda))^+ = \left( \eta \lambda^\eta \left( \frac{\delta}{\varsigma} - 1 \right) + \Theta \text{diag}(\lambda^\theta) \left( \frac{\delta}{\varsigma}1 - a \right) \right)^+
\]

is homogeneous. Because \( \lambda = 0 \) makes the balance sheet size zero, \( c(0) = 0 \). The cost allocations to cash and to risky assets are zero if the cost is zero and otherwise are

\[
\frac{\partial c}{\partial \lambda^\eta}(1) = \left( \frac{\delta}{\varsigma} - 1 \right) \eta \quad \text{and} \quad \nabla_{\lambda^\theta} c(1) = \text{diag} \left( \frac{\delta}{\varsigma}1 - a \right) \Theta.
\]

The return \( \delta/\varsigma - 1 \) serves as a benchmark because this return for the asset portfolio puts the firm on the borderline of default. The variable balance-sheet size means that assets’ effects are best understood in terms of the magnitude of their profit or loss; the benchmark is zero profit or loss. The loss to external creditors is exacerbated by moving away from cash towards any asset with a loss, and is mitigated by moving away from cash towards any asset with a profit.

In the second scheme, only risky assets have participation levels, i.e., \( \lambda = \lambda^\theta \). The holdings of risky assets are \( \Theta(\lambda) = \Theta \text{diag}(\lambda) \). The balance sheet size is fixed and liabilities do not change: \( \varsigma(\lambda) = \varsigma, \gamma(\lambda) = \gamma, \) and \( \delta(\lambda) = \delta \). To make the accounting equation hold, cash is the substitute for risky assets: \( \eta(\lambda) = \varsigma - \Theta \text{diag}(\lambda) \). The time-1 portfolio value \( e(\lambda) = \varsigma + \Theta \text{diag}(\lambda)(a - 1) \). The cost function \( c \) given by \( c(\lambda) = (\delta - e(\lambda))^+ = (\Theta \text{diag}(\lambda)(1 - a) - \gamma)^+ \) is not homogeneous. Because \( \lambda = 0 \) makes the portfolio return zero, \( c(0) = 0 \). The cost allocations to risky assets are zero if the cost is zero. If not, they are

\[
\int_0^1 \nabla c(\lambda 1) \, d\lambda = \int_0^1 \text{diag}(\Theta)(1 - a) \, d\lambda = \frac{\ell}{\Theta(1 - a)} \text{diag}(\Theta)(1 - a)
\]

because cost is zero for \( \lambda < \gamma/\Theta(1 - a) \), and \( 1 - \gamma/\Theta(1 - a) = \ell/\Theta(1 - a) \). Each asset is allocated a fraction of the cost \( \ell \), and this fraction is the fraction of the net loss \( \Theta(1 - a) \) that is due to this asset. (This fraction is negative for assets that generate profits.) The fixed balance-sheet size means that assets’ effects are best understood in terms of the magnitude of their profit or loss; the benchmark is zero profit or loss. The loss to external creditors is exacerbated by moving away from cash towards any asset with a loss, and is mitigated by moving away from cash towards any asset with a profit.

The two schemes yield the same risk attribution method if equity \( \gamma = 0 \), in which case the cost \( \ell \) is the portfolio loss \( (\varsigma - e)^+ = (\Theta(1 - a))^+ \). Then both schemes result in an allocation of 0 to cash and an allocation to the risky assets of \( (\text{diag}(\Theta)(1 - a))^+ \). This yields the usual Euler allocation or gradient allocation in the theory of portfolio risk attribution (see, e.g., Tasche, 2008).

### 2.3 Cost Allocation to Liabilities

This scheme allocates cost to equity and debt. The participation vector \( \lambda = [\lambda^\gamma \lambda^\delta] \). The liabilities are \( \gamma \lambda^\gamma \) of equity and \( \delta \lambda^\delta \) of debt. The balance sheet has size \( \varsigma(\lambda) = \gamma \lambda^\gamma + \delta \lambda^\delta \), which
is variable. The asset proportions are fixed: \( \eta(\lambda) = \eta \kappa(\lambda)/\varsigma \) and \( \Theta(\lambda) = \Theta \kappa(\lambda)/\varsigma \). The time-1 portfolio value \( e(\lambda) = e \kappa(\lambda)/\varsigma \), so the cost function \( c \) given by \( c(\lambda) = (\lambda \delta(1 - e/\varsigma) - \lambda \gamma e/\varsigma)^+ \) is homogeneous. Because \( \lambda = 0 \) makes the balance sheet size zero, \( c(0) = 0 \). The cost allocations to debt and equity are zero if the cost is zero and otherwise are \( \delta(1 - e/\varsigma) \) and \(-\gamma e/\varsigma\).

3 A Network Model of Contagion

Consider a system containing \( N \) firms whose balance sheets are interlocking due to loans between firms. Part of the model of the system is a network or graph in which firms are nodes and directed edges represent the flow of money at time 0 when one firm lends to another; the flow of money at time 1 when loans mature is in the opposite direction. It is important to distinguish between internal assets, which are claims on other nodes, and external assets, which are claims on entities outside the system, and between internal liabilities, which are obligations to other nodes, and external liabilities, which are obligations to entities outside the system. We assume that internal assets and liabilities include only debt, not equity. Elsinger (2007) shows how to extend the analysis of Section 3.1 when equity is held inside the system. Liu and Staum (2011) handle time-1 liabilities that result from internal financial relationships other than loans, such as swaps. The vector of equity issued by each node is \( \gamma \). The vector of debt at each node and held externally is \( \delta \). Internal loans are described by an \( N \times N \) matrix \( D \), where \( D_{ij} \) is the principal lent by node \( i \) to node \( j \). The nodes’ external assets include the vector \( \eta \) of cash and the matrix \( \Theta \) of risky asset holdings, where \( \Theta_{ij} \) is the amount of asset \( j \) held by node \( i \). The accounting equation is

\[
\varsigma = \gamma + \delta + D^\top 1 = \eta + \Theta 1 + D 1. \tag{3}
\]

At time 1, the external assets of each node are worth \( e = \eta + \Theta a \). We assume that all loans have equal seniority. This assumption too can be lifted using the results of Elsinger (2007); Drehmann and Tarashev (2011) and Liu and Staum (2011) study systemic risk components in models in which external liabilities are senior to internal liabilities. For simplicity, we assume that the interest rate on a loan depends only on the borrower, not on the lender. Let \( \bar{r} - 1 \) be the vector of interest rates at which the nodes borrow. The time-1 internal liabilities of the nodes form the matrix \( L = \text{diag}(\bar{r}) D^\top \), where \( L_{ij} = \bar{r}_i D_{ji} \) is the amount owed by node \( i \) to node \( j \). Their time-1 external liabilities form the vector \( L^0 = \text{diag}(\bar{r}) \delta \), where \( L^0_i = \bar{r}_i \delta_i \) is the amount owed by node \( i \) to external creditors. We assume that all of the quantities in the model, including interest rates \( \bar{r} - 1 \), are non-negative. We assume that every external asset value in \( e \), and therefore every balance-sheet size in \( \varsigma \), is positive.¹

The nodes’ capacities to pay their debts depend on the values of their assets, internal as well as external. Because each internal loan appears as an asset on one balance sheet and as a liability on an interlocking balance sheet, contagion is a factor and it is not simple to determine the returns \( r^* - 1 \) on nodes’ debt. The next subsection describes how to compute them.

We take the systemic risk measure to be expected cost and cost to be the external creditors’ aggregate loss \( \ell = \delta^\top (1 - r^*)^+ \), not net of profits. This choice enables us to focus our systemic risk attribution schemes on contagion while ignoring effects due to the loan portfolio held by external creditors.

¹For some of our methods, existence of the gradient in Equation (2) may require that other quantities, such as cash holdings, be positive. However, we pass over this technical issue. When a derivative fails to exist for this reason, a one-sided derivative may be used instead.
creditors. With other choices, the systemic risk attributed to a node could have to do with the correlation between the losses of its external creditors and the losses of other nodes’ external creditors. Suppose that cost were the external creditors’ net loss \((\delta^\top (1 - r^*))^+\), treating the profits of one node’s external creditors as though they offset the losses of another node’s external creditors. In a scenario in which external creditors have a positive net loss, a node that generates a profit for external creditors could get a negative cost allocation due to mitigating the net loss. Such effects due to the external creditors’ portfolio, which can be understood in the systemic risk framework of Chen et al. (2011), are important in some systemic risk management applications. We choose a systemic risk measure that excludes them because they obscure the way in which systemic risk attribution can reveal contributions to losses via contagion, which it is our aim to analyze.

3.1 Clearing Payments

We use the Eisenberg and Noe (2001) analysis of the flows of money in the network at time 1. The inflows from external assets make up the vector \(e\), and \(L\) is a matrix of maximum flows from one node to another. Let the external creditors be represented by a sink node. Then \(L^0\) contains maximum flows to the sink node. Define \(\bar{p} = L^0 + Li = \text{diag}(\bar{r})(\delta + D^\top 1)\) as the vector of total time-1 liabilities of each node, i.e., maximum outflows from each node. Let \(p\) be the vector of outflows from each node. Define the payment fractions \(f = (\text{diag}(\bar{p}))^{-1}p\). The returns on debt are \(r = (\text{diag}(\delta + D^\top 1))^{-1}p = \text{diag}(f)\bar{r}\). Define the matrix \(\Pi = (\text{diag}(\bar{p}))^{-1}L = (\text{diag}(\delta + D^\top 1))^{-1}D^\top\) whose element \(\Pi_{ij} = L_{ij}/\bar{p}_i\) is the fraction of the outflow from node \(i\) that goes to node \(j\). Similarly, \(\Pi^0 = (\text{diag}(\bar{p}))^{-1}L^0 = (\text{diag}(\delta + D^\top 1))^{-1}\delta\) contains the fractions of the nodes’ outflows that go to external creditors. The preceding definitions may contain an ambiguity due to division by zero. For any \(i\) such that \(\bar{p}_i = 0\), let \(\Pi^0_{ii} = 0\), the row \(\Pi^0_i = 0\), and \(f_i = 1\).

The matrix of flows from one node to another is \(\text{diag}(p)\Pi\) and the vector of flows to external creditors is \(\text{diag}(p)\Pi^0\). The vector of internal inflows to nodes is \(\Pi^\top p\). The flows must satisfy the capacity constraints

\[
0 \leq p \leq \bar{p} \quad \text{and} \quad (I - \Pi^\top)p \leq e. \tag{4}
\]

The first capacity constraint says that the outflow from each node can neither be negative nor exceed its liabilities. The second says that the outflow of each node can not exceed its inflow from internal and external sources: \(p \leq \Pi^\top p + e\). The time-1 equity value of the nodes is \(v = e + \Pi^\top p - p\).

The flows must also satisfy the priority constraint

\[
(\bar{p} - p) \land v = 0, \tag{5}
\]

which says that any node whose outflow is less than its liabilities is left with zero equity value. Eisenberg and Noe (2001) define a clearing payment vector as a value of \(p\) that satisfies the capacity and priority constraints. Under our assumption that the external inflows are all positive, a unique clearing payment vector exists. Call it \(p^*\). Let \(f^*\), \(r^*\), and \(v^*\) be the corresponding vectors of payment fractions, returns on debt, and equity, respectively.

3.2 Sensitivity Analysis

Computing the gradient in Equation (2) for the Aumann-Shapley value requires a sensitivity analysis. The following classification of nodes is useful in sensitivity analysis.
Definition 1. Node $i$ is green if $v_i^* > 0$, yellow if $1 < r_i^* < \bar{r}_i$, and red if $r_i^* < 1$. Any other node is borderline. Node $i$ defaults if $r_i^* < \bar{r}_i$.

Let $G$, $Y$, $R$, $D$, and $N$ be the sets of green, yellow, red, defaulting, and non-defaulting nodes, respectively. Because of the priority constraint (5), a green node $i$ satisfies $r_i^* = \bar{r}_i$, i.e., does not default. The set $N$ of non-defaulting nodes contains all nodes that are green or on the borderline of default, i.e., the borderline between green and yellow. The set $D$ of defaulting nodes contains all nodes that are yellow, red, or on the borderline between yellow and red. Yellow nodes are those that default but nonetheless yield profits to their creditors. Red nodes impose losses on their creditors. Define the indicators $R = 1\{r^* < 1\}$ of red nodes, $G = 1\{v^* > 0\}$ of green nodes, $D = 1\{p < \bar{p}\}$ of defaulting nodes, and $N = 1 - D$ of non-defaulting nodes.

The sensitivity analysis involves the matrix $\nabla_e p^*$ whose $(i, j)$th element is the partial derivative of the clearing payment made by node $i$ to the income of node $j$. The formula for $\nabla_e p^*$ in Corollary 1 agrees with that of Liu and Staum (2010, Prop. 2). We derive it here via Proposition 1 because the Markov-chain computation yields insight and a good way to implement the “fictitious default algorithm” of Eisenberg and Noe (2001), which they gave in an inexplicit form. In the algorithm, $D$ represents a set of defaulting nodes, and $N$ represents a set of nodes that have not yet been identified as defaulting.

Proposition 1. The clearing payment vector $p^*$ is the ultimate value of $p$ in the following algorithm.

1. Set $D \leftarrow \emptyset$.
2. Set $N \leftarrow \{1, \ldots, N\} \setminus D$.
3. Set $p_N \leftarrow \bar{p}_N$ and $p_D \leftarrow (I - (\Pi_{DD})^\top)^{-1}(e_D + (\Pi_{ND})^\top \bar{p}_N)$.
4. Set $D' \leftarrow \{i \in N : \bar{p}_i > e_i + (\Pi_i)^\top p\}$.
5. If $D' \neq \emptyset$, set $D \leftarrow D \cup D'$ and return to Step 2.

Proof. Eisenberg and Noe (2001) proved that the fictitious default algorithm yields the clearing payment vector. Our algorithm is theirs, except that in Step 3 we have an explicit formula for the fixed point of their map $p \rightarrow \text{diag}(D)(\Pi^\top(\text{diag}(D)p + \text{diag}(N)\bar{p}) + e) + \text{diag}(N)\bar{p}$. To paraphrase, this map returns a vector containing the total liabilities for nodes in $N$ and, for defaulting nodes in $D$, the node’s value assuming that nodes in $N$ pay in full and nodes in $\bar{D}$ make payments given by $p$. That is, letting $p'$ represent the image of $p$, $p_N' = \bar{p}_N$ and $p_D' = (\Pi_D)^\top(\text{diag}(D)p + \text{diag}(N)\bar{p}) + e_D = (\Pi_{DD})^\top p_D + (\Pi_{ND})^\top p_N + e_D$. The fixed point of the map, satisfying $p' = p$, therefore satisfies $p_N = \bar{p}_N$ and $p_D = (\Pi_{DD})^\top p_D + (\Pi_{ND})^\top \bar{p}_N + e_D$. The solution is

$$p_D = (I - (\Pi_{DD})^\top)^{-1}(e_D + (\Pi_{ND})^\top \bar{p}_N).$$  \hspace{1cm} (6)

Equation (6) has an interpretation in terms of a Markov chain. Its states correspond to nodes, including the sink node, which represents external creditors. The transitions of the Markov chain represent the movements of a dollar from one node to another at time 1. The sink node and other non-defaulting nodes correspond to absorbing states: because they are already making payments
equal to their liabilities, increasing the inflow to such a node does not increase its outflow. The defaulting nodes correspond to transient states: an extra dollar received by a defaulting node will be paid out to its creditors, and eventually be absorbed elsewhere. Let the matrix of transition probabilities from transient states be \([\Pi_D^0 \quad \Pi_D]\), assigning index 0 to the sink node. If nodes \(i\) and \(j\) default, the \((i, j)\)th element of the matrix \((I - (\Pi_{DD})^\top)^{-1}\) is the expected number of visits by the Markov chain to state \(i\) given that the chain's initial state is \(j\). This can also be interpreted as the expected number of visits to node \(i\) of a dollar that starts at node \(j\) before absorption by the sink node or another non-defaulting node. The vector \(e_D + (\Pi_{N/D})^\top \bar{p}_N\) contains the inflow to each defaulting node from outside the system and from non-defaulting nodes. Therefore the right side of Equation (6) contains the outflows from each defaulting node: if node \(i\) defaults, its outflow equals its inflow, and the \(i\)th element of the right side of Equation (6) is the inflow, the number of times that a dollar enters node \(i\) from any source.

**Corollary 1.** The right derivatives \(\nabla_e^* p^*\) are given by \((\nabla_e^* p^*)_{DD} = (I - \Pi_{DD}^\top)^{-1}\), whereas the rest of the elements of \(\nabla_e^* p^*\) are zero. The left derivatives \(\nabla_e^* p^*\) are given by the same formula, but with \(D\) replaced by \(G^\delta\). If there are no nodes on the borderline of default, i.e., \(N = G\), then \(\nabla_e^* p^*\) equals the left and right derivatives.

**Proof.** A sufficiently small positive perturbation in external asset value \(e\) does not change the set \(D\) of defaulting nodes. Therefore the statement about right derivatives follows from Proposition 1, because Equation (6) holds for \(p^*\). If node \(i\) is on the borderline of default, then the time-1 equity value \(v_i = 0\) and \(\bar{p}_i = p_i^* = e_i + (\Pi.i)^\top p^*\). It follows that Equation (6) also holds for \(p^*\) but with \(D\) replaced by \(G^\delta\), the set of nodes that are not green. A sufficiently small negative perturbation in external asset value does not change the set \(G\) of green nodes. This establishes the statement about left derivatives.

\[
\ell(\delta, e, L^0, L) = \delta^\top (1 - r^*)^+ = \delta^\top \text{diag}(R)(1 - r^*) \\
= R^\top (\delta - \text{diag}(\Pi^0)p^*) = R^\top (\delta - \text{diag}(f^*L^0)).
\]

Its sensitivities are given by Proposition 2. A fundamental quantity is \(\zeta_i\), the marginal price of wealth at node \(i\): it is the rate of decrease of cost as the inflow to node \(i\) increases. Its Markov-chain interpretation is the probability of reaching the sink node from a red node when starting at node \(i\). If node \(i\) defaults, then \(\Pi_i\zeta\) is a continuation value in the Markov chain. Therefore the formula for \(\zeta\) given by Equation (10) is the solution to the system of equations

\[
\zeta_G = 0, \quad \zeta_Y = \Pi_Y \zeta, \quad \text{and} \quad \zeta_R = \Pi_R^0 + \Pi_R \zeta.
\]

The \((i, j)\)th element of the matrix \(\nabla_L \ell\) is \(\partial \ell / \partial L_{ij} = f^*_i(\partial \ell / \partial e_j - \partial \ell / \partial e_i) = f^*_i(\zeta_i - \zeta_j)\), which can be interpreted in terms of the marginal cost of moving wealth from node \(i\) to node \(j\).

**Proposition 2.** If there are no borderline nodes, then

\[
\nabla_{\delta} \ell(\delta, e, L^0, L) = R, \\
-\zeta = \nabla_e \ell(\delta, e, L^0, L) = -\nabla_e p^* \top \text{diag}(\Pi^0)R, \\
\nabla_{L^0} \ell(\delta, e, L^0, L) = \text{diag}(f^*)(\zeta - R), \quad \text{and} \\
\nabla_L \ell(\delta, e, L^0, L) = \text{diag}(f^*)(\zeta^\top - 1\zeta^\top).
\]

**Proof.** See Appendix A. 

\[\square\]
3.3 Numerical Examples

Our examples of financial systems are artificial, not realistic. They are designed to illustrate clearly the different behaviors of the systemic risk attribution methods we introduce.

Example 1. There are two nodes which are identical except that the “downstream” node 1 has lent 500000 to the “upstream” node 2. Both nodes have 50000 in equity and 1000000 in total debt, on which they owe 4% interest. All of the downstream node’s debt is external debt, whereas the upstream node’s debt is half internal and half external. Both nodes have 50000 in cash. The upstream node has 500000 each in external assets 1 and 2. The downstream node’s assets are 500000 each in external asset 3 and in the internal loan to the upstream node. Table 1 shows the external assets’ returns in each of eight scenarios, with their probabilities. The nodes’ behavior is stochastically identical, as can be seen from comparing the returns on their debt in Table 1. Both nodes have default probability 3% and their loss given default is 10%.

| Table 1: Scenarios in Example 1. |
|-----------------------------------|
| Probability | External Asset Returns | Nodes’ Debt Returns |
|            | $a_1 - 1$ | $a_2 - 1$ | $a_3 - 1$ | $r^*_1 - 1$ | $r^*_2 - 1$ |
| 93.04%     | 4% | 10% | 10% | 4% | 4% |
| 1.96%      | 4% | 10% | -34% | -10% | 4% |
| 1.96%      | -10% | 10% | 10% | 4% | 4% |
| 0.04%      | -10% | 10% | -34% | -10% | 4% |
| 1.33%      | 4% | -34% | 10% | 4% | -10% |
| 0.67%      | 4% | -34% | -20% | -10% | -10% |
| 0.67%      | -10% | -20% | 10% | 4% | -10% |
| 0.33%      | -10% | -20% | -20% | -10% | -10% |

Table 2 presents the systemic risk components for Example 1. The external creditors’ loss given default is 100000 for the downstream node and only 50000 for the upstream node, because only half of its debt is external. Therefore the expected loss to external creditors of the downstream node is twice that to those of the upstream node: 3000 vs. 1500. The systemic risk components provided by other methods are discussed below, as those methods are introduced. For the present, we merely remark on the diversity of the attributions: just the opposite of attributing to each node its own external creditors’ loss, another method attributes 1500 to the downstream node and 3000 to the upstream node.

Example 2. This example is of a single scenario involving six nodes. All nodes have 50000 in equity and borrow at an interest rate of 6%. The size of nodes 1–4 is 1000000 and the size of nodes 5 and 6 is 500000. Node 1 has lent 1500000 each to nodes 2, 3, and 4. Node 3 has lent 400000 to node 5 and 200000 to node 6. Node 4 has lent 200000 to node 6. The losses incurred by the nodes on external assets are $e - (\Theta \mathbf{1} + \eta) = [106 \ 24.5 \ -38 \ -78 \ 98 \ 98]$, in thousands. Therefore, the returns on the nodes’ debt are $r^* - 1 = [-4.12\% \ 2.68\% \ 2.53\% \ 6\% \ -10.67\% \ -10.67\%]$, so node 4 is green, nodes 2 and 3 are yellow, and nodes 1, 5, and 6 are red. Figure 1 represents the network.
Table 2: Systemic Risk Components in Example 1.

| Method               | Downstream Node 1 | Upstream Node 2 |
|----------------------|-------------------|-----------------|
| External Creditors’ Loss | both             | 3000            | 1500            |
| External Assets      | Shapley           | 2650            | 1850            |
|                      | Aumann-Shapley    | 2571            | 1929            |
| Transmission         | Shapley           | 2950            | 1550            |
| Leverage             | Aumann-Shapley    | 2707            | 1793            |
| Absorption           | Shapley           | 2200            | 2300            |
|                      | Aumann-Shapley    | 1500            | 3000            |
| Solvency             | Aumann-Shapley    | 2900            | 1600            |
| Funding              | Aumann-Shapley    | 2000            | 2500            |

Figure 1: The network of Example 2.

Table 3 presents the cost allocations for Example 2. Again, for the present, we merely comment on the diversity of the allocations. The system’s cost is 49851, but some methods attribute more than this to a single node, whereas other methods give a single node a negative allocation whose magnitude exceeds the system’s cost. The yellow nodes have positive allocations under some methods and negative allocations under others. The positive allocations to a node under different methods can vary by more than an order of magnitude. The methods disagree about whether node 2 contributes to, mitigates, or does not affect cost. Its allocation is positive under methods that focus on external assets because these assets make a loss, and positive for the leverage method because its leverage is too high. Its allocation is zero under the absorption Aumann-Shapley method because it neither imposes losses on external creditors nor absorbs losses generated elsewhere in the system. Its allocation is negative for other methods because red node 1 makes a profit in lending to it.
Table 3: Cost Allocations in Example 2.

| Method                  | Node 1 | Node 2 | Node 3 | Node 4 | Node 5 | Node 6 |
|-------------------------|--------|--------|--------|--------|--------|--------|
| External Creditors' Loss| both   | 39184  | 5333   | 5333   |        |        |
| External Assets         |        |        |        |        |        |        |
| Shapley                 |        | 35575  | 1934   | -5982  | -1491  | 9974   | 9842   |
| A-S                     |        | 33362  | 1218   | -1888  |         | 9662   | 7498   |
| Transmission            |        |        |        |        |        |        |
| Shapley                 |        | 46443  | -2013  | -3044  | -4500  | 6789   | 6175   |
| Leverage A-S            |        | 29848  | 4691   | 2982   | -283   | 6630   | 5982   |
| Absorption              |        |        |        |        |        |        |
| Shapley                 |        | 46443  | -2013  | -35044 | -15167 | 28123  | 27509  |
| A-S                     |        | 3914   | -64000 | -21333 | 48000  | 48000  |        |
| Solvency A-S            |        | 56000  | -4026  | -13895 | -9000  | 12070  | 8702   |
| Funding A-S             |        | 40418  | -636   | -15195 | -3867  | 16062  | 13068  |

4 Balance-Sheet Schemes and Systemic Risk Components

As in Section 2, the crucial step in designing a systemic risk attribution method is creating a balance-sheet scheme. For the model of interlocking balance sheets, a balance-sheet scheme specifies how every entry on the balance sheet of every node in the network depends on the vector $\lambda$ of participation levels of some components of the system: equity $\gamma(\lambda)$, external debt $\delta(\lambda)$, internal debt $D(\lambda)$, cash $\eta(\lambda)$, and risky external assets $\Theta(\lambda)$. In general, let $(\lambda)$ denote a quantity in the counterfactual system specified by participation vector $\lambda$. As in Section 2, such balance-sheet schemes must satisfy the accounting equation (3). Satisfying the accounting equation when balance sheets are interlocking is more difficult because internal loans affect two balance sheets, those of the lender and of the borrower. In this section, we explore a few balance-sheet schemes for the model of interlocking balance sheets, and derive systemic risk attribution methods from them. In designing a balance-sheet scheme, the following questions arise.

To which components of the system is risk attributed? For brevity, we consider only attribution to nodes and attribution to external assets. Nonetheless, we will address in each scheme how counterfactual changes to internal loans would affect systemic risk, because this is helpful for understanding how the corresponding methods attribute systemic risk to nodes.

Are balance-sheet sizes variable or fixed?

How is the accounting equation enforced? One way is that when the size of one entry in the balance sheet changes, another entry on the same side of the balance sheet compensates so that the other side of the balance sheet need not change. For example, in the fixed-size scheme of Section 2.2, cash serves as a substitute for all other assets. Another way is that when the size of one entry in the balance sheet changes, the size of the balance sheet changes, leading to an increase in one, some, or all entries on the other side. For example, in Section 2.3, an increase in liabilities causes the asset portfolio to grow while its proportions remain fixed.

When risk is attributed to nodes, how does the size of an internal loan depend on the participation levels of the borrower and lender? It can depend on neither, on the borrower, on the lender, or on both. Consider a loan from node $i$ to node $j$. The participation level of the lender is $\lambda_i$ and that of the borrower is $\lambda_j$. To formalize the attribution of responsibility for this internal loan, we let the size of the internal loan be $D_{ij}(\lambda) = \lambda_{ij}^D(\lambda)D_{ij}$ and make the scaling factor $\lambda_{ij}^D(\lambda)$ equal to $\lambda_j$ for borrower responsibility, $\lambda_i$ for lender responsibility, and $\sqrt{\lambda_i\lambda_j}$ for shared responsibility. The reasons to
use the function \( \lambda_{ij}^D(\lambda) = \sqrt{\lambda_i \lambda_j} \) follow. It is homogeneous. It is 1 when both lender and borrower participate fully. It is 0 if either one of them does not participate. It leads to equal sharing of responsibility for an internal loan between the lender and borrower when using the Aumann-Shapley value, as is seen from the following sensitivity analysis. The non-zero sensitivities of the internal loan scaling factor \( \lambda_{ij}^D \) to participation levels are \( \partial \lambda_{ij}^D / \partial \lambda_j = 1 \) for borrower responsibility, \( \partial \lambda_{ij}^D / \partial \lambda_i = 1 \) for lender responsibility, and \( (\partial \lambda_{ij}^D / \partial \lambda_j)(\lambda) = \frac{1}{2} \sqrt{\lambda_j / \lambda_i} \) and \( (\partial \lambda_{ij}^D / \partial \lambda_j)(\lambda) = \frac{1}{2} \sqrt{\lambda_i / \lambda_j} \) for shared responsibility. In Equation (2) for the Aumann-Shapley value, \( \lambda = \lambda_1 \), so the sensitivities associated with shared responsibility are \( (\partial \lambda_{ij}^D / \partial \lambda_j)(\lambda 1) = (\partial \lambda_{ij}^D / \partial \lambda_j)(\lambda 1) = 1/2 \). Therefore, shared responsibility generates an allocation that is the average of the those generated by borrower responsibility and lender responsibility.

Given a balance-sheet scheme, the Shapley value can be computed from Equation (1). However, further work is required to derive the Aumann-Shapley value in Equation (2) because the gradient of the cost function must be taken. In the network model of contagion, the cost function whose further work is required to derive the Aumann-Shapley value in Equation (2) because the gradient appears in Equation (2) is given by \( c(\lambda) = \ell(\delta(\lambda), e(\lambda), \text{diag}(\bar{r})\delta(\lambda), \text{diag}(\bar{r})D^\top(\lambda)) \), where external asset value \( e(\lambda) = \eta(\lambda) + \Theta(\lambda)a \). The sensitivity analysis in Section 3.2 enables computation of its gradient. Proposition 3 gives this gradient for schemes in which internal loan sizes are fixed or depend only on the participation levels of the borrower or lender. In the latter case, the number \( n \) of components in the participation vector \( \lambda \) equals the number \( N \) of nodes. Let \( \nabla \delta, \nabla e, \nabla D, \) and \( \nabla D^\top \) denote the \( n \times N \) matrices whose \( (i,j) \)th elements are \( \partial \delta_j / \partial \lambda_i, \partial e_j / \partial \lambda_i, \partial D_{ij} / \partial \lambda_i, \) and \( \partial D_{ji} / \partial \lambda_i \), respectively.

**Proposition 3.** Suppose that there are no borderline nodes in the counterfactual system specified by participation vector \( \lambda \), and that the external debt \( \delta \), external asset value \( e \), and internal debt \( D \) are differentiable at \( \lambda \). If for all \( i, j \in \{1, \ldots, N\} \) and \( \lambda' \in [0, 1]^n \), the counterfactual principal \( D_{ij}(\lambda') \) of the loan from node \( i \) to node \( j \) depends only on the participation levels \( \lambda'_i \) and \( \lambda'_j \), then the gradient of the cost function is given by

\[
\nabla c(\lambda) = (\nabla \delta(\lambda)) \left( (1 - r^*(\lambda))^+ + \text{diag}(r^*(\lambda))\zeta(\lambda) \right) \\
- (\nabla e(\lambda))\zeta(\lambda) \\
+ (\nabla D(\lambda))\text{diag}(r^*(\lambda))\zeta(\lambda) - \text{diag}(\zeta(\lambda))((\nabla D(\lambda))\text{diag}(\lambda(\lambda)))r^*(\lambda) \\
+ \text{diag}(\zeta(\lambda))\text{diag}(r^*(\lambda))((\nabla D^\top(\lambda))1 - \text{diag}(r^*(\lambda))((\nabla D^\top(\lambda))\zeta(\lambda). \tag{13}
\]

**Proof.** This is an application of the chain rule and Proposition 2. The second line, involving sensitivity to external asset value, comes directly from Equation (10). It was established in Section 3.1 that the returns on debt \( r^* = \text{diag}(f^*)\bar{r} \), which is used in deriving the remaining terms. The first line, involving sensitivity to external debt, comes from \( \text{diag}(\bar{R})(1 - r^*) = (1 - r^*)^+ \) and Equations (9) and (11). The fourth line, involving sensitivity to internal borrowing, comes from \( \partial \ell / \partial L_{ij} = f_i^*(\zeta_j - \zeta_i) \), i.e., Equation (12). The third line, involving sensitivity to internal lending, comes from Equation (12) with indices \( i \) and \( j \) reversed: \( \partial \ell / \partial L_{ji} = f_j^*(\zeta_i - \zeta_j) \). No other terms are present because all elements of \( L \) that are sensitive to \( \lambda_i \) have \( i \) as an index.

### 4.1 External Assets

The purpose of these balance-sheet schemes is to show how systemic risk arises from the risks that nodes take in investing in external assets. One may attribute systemic risk to each node, i.e., to its exposure to external risks, or to each asset.
4.1.1 Attribution to Nodes

The participation level $\lambda_i$ scales the size of the investment by node $i$ in risky assets: risky asset holdings are $\Theta(\lambda) = \text{diag}(\lambda)\Theta$. Balance sheets have fixed sizes and liabilities do not change: $\varsigma(\lambda) = \varsigma$, $\gamma(\lambda) = \gamma$, $\delta(\lambda) = \delta$, and $D(\lambda) = D$. Cash is the substitute for risky external assets: $\eta(\lambda) = \varsigma - D1 - \Theta(\lambda)1$. The external asset value $e(\lambda) = \varsigma - D1 + \Theta(\lambda)(a - 1) = \varsigma - D1 + \text{diag}(\lambda)\Theta(a - 1)$. Because $e(0) \geq \delta$, zero participation implies no defaults, so the cost function satisfies $c(0) = 0$. This scheme can be applied with the Shapley or Aumann-Shapley value.

The cost function $c$ is not homogeneous, and it is somewhat involved to compute the Aumann-Shapley value in Equation (2) by integrating $\nabla c$ along the “diagonal” $\{\lambda 1: 0 \leq \lambda 1\}$ because $c$ need not be differentiable everywhere on the diagonal, due borderline nodes in counterfactual systems. Suppose that there are borderline nodes in the counterfactual system generated by participation vector $\lambda = \lambda 1$ only for finitely many values of $\lambda \in [0, 1]$. Let these values be arranged in the decreasing sequence $\lambda_1, \ldots, \lambda_{m+1}$, in which $\lambda_1 = 1$ and $\lambda_{m+1} = 0$. For any other value of $\lambda$, in the counterfactual system generated by participation vector $\lambda 1$, there are sensitivities $\zeta(\lambda 1) = -\nabla e(\delta(\lambda 1), e(\lambda 1), L^0(\lambda 1), L(\lambda 1))$, cf. Equation (10). The sensitivities are piecewise constant in $\lambda$, with points of discontinuity contained in the set $\{\lambda_2, \ldots, \lambda_m\}$, and can be found with an application of parametric linear programming as explained in Appendix B.1. In applying Proposition 3, we observe that participation does not affect debt, and the sensitivities of external asset value to participation are $\nabla e(\lambda) = \text{diag}(\Theta(a - 1))$. Therefore $\nabla c(\lambda 1) = -\text{diag}(\Theta(a - 1))\zeta(\lambda 1)$. Define $\omega_i = \lambda_i - \lambda_{i+1}$ and $\mu_i = (\lambda_i + \lambda_{i+1})/2$, the width and midpoint of the $i$th interval. The Aumann-Shapley allocation to nodes is

$$\text{diag}(\Theta(1 - a)) \int_0^1 \zeta(\lambda 1) d\lambda = \text{diag}\left(\sum_{i=1}^m \omega_i \zeta(\mu_i 1)\right) \Theta(1 - a).$$

The allocation weights each node’s profit or loss on external assets by an average sensitivity. The sensitivity is averaged over counterfactual systems in which profit or loss is scaled down to zero.

In Example 1, the external assets of the downstream node pose a greater systemic risk than do the external assets of the upstream node. The reason is that a loss earned by the upstream node that exceeds its equity will be partially absorbed by the downstream node’s equity, whereas any loss earned by the downstream node in excess of its equity is felt entirely by external creditors. If the downstream node’s external assets were replaced by cash, the systemic risk would be $1500 = 3\% \times 50000$, coming entirely from losses of external creditors in lending to the upstream node. This is the “stand-alone systemic risk” of the upstream node in this scheme. If instead the upstream node’s external assets were replaced by cash, the systemic risk would come entirely from losses in lending to the downstream node and would equal $2300 = 2\% \times 100000 + 1\% \times 30000$, the stand-alone systemic risk of the downstream node. In this expression, 100000 and 30000 are the values of the downstream node’s loss of 500000(1 − $a_3$) on external assets minus 50000 of equity and minus the profit of 20000 on the internal loan, when the downstream node’s external asset return $a_3$ − 1 equals -34% or -20%, respectively. The systemic risk of the real system is 4500. The systemic risk accounted for by the interaction of the participation of both nodes is $700 = 4500 - (1500 + 2300)$. This interaction stems from the event of probability 1% that both nodes default: $700 = 1\% \times 70000$, where 70000 can be viewed as the part of the loss to external creditors of the downstream node associated with the downstream node’s loss on its loan to the upstream node. The Shapley value splits the interaction equally between the two nodes: adding half of the interaction to stand-alone
systemic risk yields systemic risk components of 1850 for the upstream node and 2650 for the downstream node. Because it is proportional to a node’s loss on external assets, the Aumann-Shapley value attributes more systemic risk to the upstream node, which has twice the external assets of the downstream node.

In Example 2, this scheme differentiates sharply between the yellow nodes 2 and 3. Node 2 experiences a loss on its external assets and consequently has a positive cost allocation, whereas node 3 gets a negative allocation because it makes a profit on its external assets. For a similar reason, the Shapley value yields a negative allocation to node 4, which is green: if its external assets had generated zero profits, it would have been able to pay less interest to node 1, entailing a greater loss for node 1’s external creditors. However, the Aumann-Shapley value allocates zero cost to node 4, because cost is marginally insensitive to node 4’s external assets: if their return were perturbed, node 4 would still be green and would make its full interest payment to node 1.

4.1.2 Attribution to Assets

This scheme is very similar to that of Section 4.1.1, except that participation is by risky external assets and not nodes. The participation level \( \lambda_j \) scales the size of all nodes’ investments in risky asset \( j \): risky asset holdings are \( \Theta(\lambda) = \Theta \text{diag}(\lambda) \). Therefore the external asset value \( e(\lambda) = \varsigma - D1 + \Theta(\lambda)(a - 1) = \varsigma - D1 + \Theta(\text{diag}(\lambda)a - \lambda) \). Its sensitivities are \( \nabla e(\lambda) = \Theta \text{diag}(a - 1) \).

The Aumann-Shapley allocation to risky external assets is

\[
\sum_{i=1}^{m} \omega_i \zeta(\mu_i1) \top \Theta \text{diag}(1 - a).
\]

The points of potential discontinuity \( \hat{\lambda}_2, \ldots, \hat{\lambda}_m \) and the sensitivities \( \zeta(\lambda1) \) are the same as in Section 4.1.1. The reasons is that the counterfactual system generated by the \( n \)-vector \( \lambda1 \) of risky external assets’ participation is the same as that generated by the \( N \)-vector \( \lambda1 \) of risky external assets’ participation.

In Example 1, the Aumann-Shapley systemic risk attributed to the risky external assets is 43 to asset 1, 1886 to asset 2, and 2571 to asset 3. Although assets 2 and 3 have the same return distribution, asset 3 is held at a more systemically important location in the network, namely the downstream node. The systemic risk attributed to asset 1 is far less than that to asset 2, even though both are held at the upstream node, because asset 1 has less downside and a lower correlation with the event of default of the upstream node.

4.2 Transmission and Leverage

This balance-sheet scheme leads to attribution of systemic risk to the substitution of each node’s debt for equity. There are two associated effects: the general risk-increasing effect of a node’s leverage on its creditors and the potential to transmit profits or losses to internal creditors. Balance-sheet sizes are fixed, \( \varsigma(\lambda) = \varsigma \), and each node is responsible for its debt: \( \delta(\lambda) = \text{diag}(\lambda)\delta \) and \( D(\lambda) = D\text{diag}(\lambda) \). Therefore equity \( \gamma(\lambda) = \varsigma - \text{diag}(\lambda)(\delta + D\top1) \). Risky external assets are fixed, \( \Theta(\lambda) = \Theta \), and cash is the substitute for internal assets: \( \eta(\lambda) = \varsigma - \Theta1 - D\lambda \). It follows that the external asset value \( e(\lambda) = e + D(1 - \lambda) \). This scheme can be used with the Shapley or Aumann-Shapley values. The Shapley value focuses on the transmission of profit or loss through loans. The Aumann-Shapley value focuses on leverage.
The Shapley value in this scheme is the same as that of Drehmann and Tarashev (2011). The external creditors of a non-participating node suffer no loss. A non-participating node transmits no losses internally: its internal creditors have cash in place of the internal loan asset. Therefore the losses transmitted by a borrower to its non-participating internal creditor count for nothing. Thus, non-participation of a node is equivalent to elimination of the losses it transmits or receives, externally or internally.

According to this scheme, internal loans have mixed effects on systemic risk, but tend to increase it. Considering one scenario at a time, profitable internal loans make non-positive contributions to cost, and loss-making internal loans make non-negative contributions to cost. Turning to systemic risk measured across scenarios, an internal loan increases the borrower’s leverage and the risk of the lender’s asset portfolio. It also increases the expected return of the lender’s asset portfolio. However, assuming the internal loan is fairly priced, this effect tends to be outweighed by the effects on leverage and asset risk.

In Example 1, this scheme views the effect of the internal loan as transmitting a loss of 50000 if the upstream node defaults or a profit of 20000 otherwise. It has no impact on the loss of the downstream node’s external creditors if the downstream node’s external assets gain 50000. With probability 1%, the upstream node defaults and the downstream node’s external assets lose, so that the internal loan adds 50000 to the loss of the downstream node’s external creditors. With probability 2%, the upstream node does not default and the downstream node’s external assets lose, so that the internal loan reduces the loss of the downstream node’s external creditors by 20000. According to this view, the internal loan contributes 100 to systemic risk. The Shapley value splits this 100 equally between lender and borrower. As in Section 4.1.1, the stand-alone systemic risk of the upstream node is 1500, so the Shapley value attributes a systemic risk component of 1550 to the upstream node. According to this scheme, or any other scheme in which cash is a substitute for an internal loan asset, the stand-alone systemic risk of the downstream node is 2900, the expected loss to its external creditors if cash replaced its internal loan. Therefore the Shapley value attributes a systemic risk component of 2950 to the downstream node.

The cost function \( c \) is not homogeneous, so as in Section 4.1, the Aumann-Shapley value takes the form 
\[
\int_{0}^{1} \nabla c(\lambda) \, d\lambda = \sum_{i=1}^{m} \omega_{i} \nabla c(\mu_{i})
\]
Appendix B.2 shows how to compute the interval widths \( \omega_{1}, \ldots, \omega_{m} \) and midpoints \( \mu_{1}, \ldots, \mu_{m} \), and it derives the gradient formula
\[
\nabla c(\lambda) = \text{diag}(R(\lambda)) \delta + \text{diag}(1 - \text{diag}(G(\lambda)) \bar{r}) D^{\top} \zeta(\lambda).
\]

Example 2 best illustrates the difference between the Shapley and Aumann-Shapley values. The Shapley value allocates negative cost to yellow nodes 2 and 3 because of the reduction in cost generated by the profits that node 1 makes on its loans to them. It envisions a large change that replaces node 1’s loans to nodes 2 and 3 with cash, which would be worse for the external creditors of node 1. The Aumann-Shapley value allocates positive cost to nodes 2 and 3 because a marginal decrease in their debt would help the external creditors of node 1: replacing some of the debt of nodes 2 and 3 with equity would make node 1 earn a larger profit on a smaller loan. The Shapley value describes the impact of eliminating each node from the network; the Aumann-Shapley value describes the impact of perturbing each node’s leverage.

### 4.3 Absorption and Solvency

This balance-sheet scheme can be used in two ways. When applied with borrower responsibility for internal loans, it yields a systemic risk attribution method based on nodes’ contributions to
the solvency of the system. To the extent that lenders are held responsible for internal loans, the systemic risk attribution method emphasizes the effect of internal lending in protecting external creditors by absorbing losses transmitted inside the system. Therefore we give the name “absorption” to the systemic risk attribution methods developed here based on the Shapley value with shared responsibility and the Aumann-Shapley value with lender responsibility. This scheme plugs the holes left in balance sheets by substituting cash for internal loan assets and substituting external debt for internal debt. This substitution of liabilities keeps leverage constant. In this scheme, internal loans decrease systemic risk. An internal loan reduces external creditors’ exposure to the borrower. Loss-making internal loans divert losses away from external creditors to other nodes, where they may be, at least in part, absorbed by equity instead of eventually reaching external creditors. Profitable internal loans reduce external creditors’ losses if they transmit profits through the network to red nodes.

Participation determines the balance sheets as follows. In this scheme, each node’s participation controls its size: \( \zeta(\lambda) = \text{diag}(\lambda)\zeta \). Each node’s proportions of external risky assets and of equity are fixed: \( \Theta(\lambda) = \text{diag}(\lambda)\Theta \) and \( \gamma(\lambda) = \text{diag}(\lambda)\gamma \). Cash and external debt are determined by the accounting equation. With borrower responsibility, internal loans \( D(\lambda) = D\text{diag}(\lambda) \), external debt \( \delta(\lambda) = \text{diag}(\lambda)\delta \), and cash \( \eta(\lambda) = \text{diag}(\lambda)(\zeta - \Theta(\alpha - 1)) - D\lambda \), so external asset value \( e(\lambda) = \text{diag}(\lambda)(\zeta + \Theta(\alpha - 1)) - D\lambda \). With lender responsibility, internal loans \( D(\lambda) = \text{diag}(\lambda)D \), cash \( \eta(\lambda) = \text{diag}(\lambda)\eta \) so external asset value \( e(\lambda) = \text{diag}(\lambda)e \), and external debt is \( \delta(\lambda) = \text{diag}(\lambda)(\delta + D^\top 1) - D^\top \lambda \).

The Shapley value can be applied only with shared responsibility. Under shared responsibility, in a counterfactual system in which a node does not participate, it is absent, and so are any loans in which it is involved. Attributing responsibility to the borrower only or lender only results in an infeasible counterfactual system. For example, consider borrower responsibility and a system in which it is involved. Attributing responsibility to the borrower only or lender only results in a system in which node \( i \) has lent \( D_{ij} > 0 \) to node \( j \). In a counterfactual system in which node \( i \) does not participate (\( \lambda_i = 0 \)) but node \( j \) does (\( \lambda_j = 1 \)), the balance sheet of node \( i \) has size \( \zeta_i(\lambda) = 0 \), yet it contains a loan to node \( j \) of size \( D_{ij}(\lambda) = D_{ij} \), which is impossible.

Under lender responsibility, the cost function \( c \) is homogeneous, so the Aumann-Shapley value equals \( \nabla c(1) \). By Proposition 3, if there are no borderline nodes then the Aumann-Shapley value is

\[
\nabla c(1) = \left( \text{diag}(\delta + D^\top 1) - D \right) \left( (1 - r^*)^+ + \text{diag}(r^*)\zeta \right) - \text{diag}(e)\zeta + D\text{diag}(r^*)\zeta - \text{diag}(\zeta)Dr^*
\]

\[
= \left( \text{diag}(\delta + D^\top 1) - D \right) (1 - r^*)^+ + \text{diag}(\zeta) \left( \text{diag}(r^*)(\delta + D^\top 1) - e - Dr^* \right).
\]

The capacity constraint (4) implies that the total outflow from a node can not exceed the total inflow: \( \text{diag}(r^*)(\delta^* + D^\top 1) \leq e + Dr^* \). The priority constraint (5) implies that equality holds for rows corresponding to defaulting nodes, which are obligated to pay out as much as they can. As stated in Equation (8), green nodes have zero marginal price of wealth: \( \zeta_\varnothing = 0 \). Therefore

\[
\text{diag}(\zeta)\text{diag}(r^*)(\delta + D^\top 1) = \text{diag}(\zeta)(e + Dr^*).
\]

From this it follows that the Aumann-Shapley value is

\[
\text{diag}(\delta + D^\top 1)(1 - r^*)^+ - D(1 - r^*)^+.
\]

The first term is losses transmitted to external and internal creditors of each node. The second term is losses absorbed by each node in internal lending. The allocations to green nodes consist only of the second term, and are non-positive.
Under borrower responsibility too, the cost function $c$ is homogeneous. By Proposition 3, if there are no borderline nodes, the Aumann-Shapley value is

$$
\nabla c(1) = \text{diag}(\delta)\text{diag}(R)(1 - r^*) + \text{diag}(\delta)\text{diag}(r^*)\zeta - \text{diag}\left(\delta + D^\top 1 + \gamma + \Theta(a - 1)\right)\zeta \\
+ D^\top \zeta + \text{diag}(\zeta)\text{diag}(r^*)D^\top 1 - \text{diag}(r^*)D^\top \zeta. 
$$

(16)

Because green nodes have zero marginal price of wealth, $\zeta_G = 0$, and the return on their debt equals the promised interest rate, $r^*_G = \bar{r}_G$, the allocation to green nodes is

$$
(\nabla c)_G(1) = -\left(\text{diag}(D^\top \zeta)(\bar{r} - 1)\right)_G, 
$$

which is non-positive and is based on the profits that each green node generates for internal creditors. From Equation (8) for non-green nodes and the definitions $\Pi = (\text{diag}(\bar{p}))^{-1}L = (\text{diag}(\delta + D^\top 1))^{-1}D^\top$ and $\Pi^0 = (\text{diag}(\bar{p}))^{-1}L^0 = (\text{diag}(\delta + D^\top 1))^{-1}\delta$, it follows that

$$
\left(\text{diag}(\delta)R + D^\top \zeta\right)_{(Y \cup R)} = \left(\text{diag}(\delta + D_j^\top 1)\zeta\right)_{(Y \cup R)}.
$$

(17)

Using this equation in Equation (16) and simplifying yields the allocation to non-green nodes

$$
(\nabla c)_{(Y \cup R)}(1) = (\text{diag}(\Theta(1 - a) - \gamma)\zeta)_{(Y \cup R)}, 
$$

(18)

which is based on the net contribution of each defaulting node to the solvency of the system, namely $\gamma + \Theta(a - 1)$, equity plus profit or loss on external assets. Therefore we name this the “solvency” method of systemic risk attribution.

In Example 1, this scheme sees two effects of the internal loan. One effect is transmitting profit or loss from the upstream node to the downstream node, which increases systemic risk by 100, as discussed in Section 4.2. In this scheme, the stand-alone systemic risk of the upstream node is 3000: if there were no internal loan from the downstream node, the upstream node would replace this internal debt with external debt, thus doubling the exposure and expected loss of its external creditors. That is, the second effect of the internal loan is halving the upstream node’s external debt, thus reducing the expected loss to the upstream node’s external creditors from 3000 to 1500. The net effect of the internal loan is to decrease cost by 1400 = 1500 – 100. Therefore the Shapley value attributes a systemic risk component of 2300 = 3000 – 1400/2 to the upstream node. As in Section 4.2, the stand-alone systemic risk of the downstream node is 2900, so the Shapley value attributes a systemic risk component of 2200 = 2900 – 1400/2 to the downstream node. The absorption Aumann-Shapeley method (with lender responsibility) attributes to the upstream node, which is the borrower, its stand-alone systemic risk of 3000, and attributes 1500 = 2900 – 1400 to the downstream node, which is the lender. The solvency method (Aumann-Shapeley value with borrower responsibility) attributes to the downstream node its stand-alone systemic risk of 2900, and attributes 1600 = 3000 – 1400 to the upstream node.

Example 2 illustrates a further point about the behavior of the systemic risk attribution method with borrower responsibility. It gives very different allocations of 12070 and 8702 to nodes 5 and 6, even though the balance sheets of these nodes are identical except for the identities of their creditors. The other methods discussed so far gave very similar allocations to nodes 5 and 6. Indeed, the absorption Aumann-Shapeley method gives nodes 5 and 6 identical allocations of 48000,
which is the total loss experienced by the creditors of either one of them. The difference between
nodes 5 and 6 is simply their position in the network: node 5 has only node 3 as an internal creditor,
whereas node 6 splits its internal debt between nodes 3 and 4. The default of node 3 affects the
external creditors of node 1, so losses transmitted to node 3 by its internal creditors are harmful;
node 4 does not default and is able to absorb some of the losses transmitted by node 6 as an internal
borrower. The effect of network structure that makes losses transmitted internally by node 5 more
damaging is quantified by \( \zeta_5 = 0.25 \) and \( \zeta_6 = 0.18 \), which appear in Equation (18) multiplying
48000, the amount by which both node 5 and node 6 diminish the solvency of the system.

4.4 Funding

Consider the system’s net balance sheet, consisting only of external assets and liabilities. The
previously developed schemes consider counterfactual systems with different net balance sheets. In
particular, in Section 4.3, an internal loan makes the system’s portfolio of external assets riskier by
substituting for cash as an asset, and it reduces the system’s leverage by substituting for external
debt as a liability. In the scheme developed in this section, all counterfactual systems have the same
net balance sheet as the real system. Cash is the substitute for an internal loan as an asset. There is
no substitute for an internal loan as a liability. Instead, receiving an internal loan increases the size
of the borrower’s balance sheet. Therefore we name this the “funding” scheme. In a counterfactual
system in which an internal loan is smaller, the borrower is smaller. The corresponding adjustment
on the asset side of the borrower’s balance sheet is made in cash. That is, one may imagine that the
recipient of an internal loan invests the additional funds in cash. Thus, the effect of an internal loan
is to move cash from the lender to the borrower, which has no effect on the system’s net balance
sheet.

Participation determines the balance sheets as follows. Each node’s participation controls the
amount of its equity, external liabilities, and external risky assets: \( \gamma(\lambda) = \text{diag}(\lambda)\gamma, \delta(\lambda) = \text{diag}(\lambda)\delta, \) and \( \Theta(\lambda) = \text{diag}(\lambda)\Theta. \) Cash is determined by the accounting equation. Applying this
scheme with borrower responsibility yields the same results as in Section 4.3. Therefore we focus
on lender responsibility, \( D(\lambda) = \text{diag}(\lambda)D. \) This makes size \( \varsigma(\lambda) = \text{diag}(\lambda)(\gamma + \delta) + D^\top\lambda, \) so cash and external asset value are

\[
\begin{align*}
\eta(\lambda) &= \text{diag}(\lambda)(\gamma + \delta - \Theta 1 - D1) + D^\top\lambda = \text{diag}(\lambda)\eta - \text{diag}(\lambda)D^\top1 + D^\top\lambda \quad \text{and} \\
e(\lambda) &= \text{diag}(\lambda)(\gamma + \delta + \Theta(a - 1) - D1) + D^\top\lambda = \text{diag}(\lambda)e - \text{diag}(\lambda)D^\top1 + D^\top\lambda.
\end{align*}
\]

It is not generally feasible to apply the Shapley value with this scheme. Suppose that in the actual
system, node 1 lends 10 to node 2, which holds 5 in cash. Then eliminating node 1 would leave
node 2 with negative cash, which is infeasible. However, the Aumann-Shapley value can be applied
if each element of \( \eta \) is positive, which guarantees feasibility of the counterfactual systems in a
neighborhood of the diagonal \( \{\lambda 1 : 0 \leq \lambda 1\} \) (see Staum, 2011b).

The cost function is homogeneous under lender responsibility, so the Aumann-Shapley value
equals \( \nabla c(1) \). By Proposition 3, if there are no borderline nodes, the Aumann-Shapley value is

\[
\nabla c(1) = \text{diag}(\delta)((1 - r^*)^+\text{diag}(r^*)\varsigma) - \text{diag}(e - D^\top1)\varsigma - D\varsigma + D\text{diag}(r^*)\varsigma - \text{diag}(\varsigma)Dr^* \\
= \text{diag}(\delta)((1 - r^*)^+ + \text{diag}(\delta)\text{diag}(r^*)\varsigma + \text{diag}(D^\top1)\varsigma - D\text{diag}(1 - r^*)\varsigma \\
- \text{diag}(\varsigma) (e + Dr^*) \\
= \text{diag}(\delta)((1 - r^*)^+ + \text{diag}(D^\top1)\text{diag}(\varsigma)(1 - r^*) - D\text{diag}(\varsigma)(1 - r^*)).
\]

(19)
The last equation follows from Equation (15). The first term in Equation (19) is the external creditors’ loss. The second term relates to the losses or profits transmitted to internal creditors. The third term is for the losses or profits earned on internal lending. The allocation to a green node consists only of the third term, which can be negative or positive. It is positive for a node that does more harm by absorbing interest payments from defaulting nodes than it does good by absorbing losses from them.

In Example 1, the first term, the external creditors’ loss, contributes 3000 to the systemic risk component due to the downstream node and 1500 to the component due to the upstream node. The second term is zero for the downstream node, which is not an internal borrower; the third term is zero for the upstream node, which is not an internal lender. The second term for the upstream node equals the third term for the downstream node, so that the systemic risk components sum to the systemic risk of 4500. The second term for the upstream node accounts for that part of the losses to the downstream node’s external creditors that is caused by the absorption by the downstream node of losses transmitted by the upstream node. The profits transmitted by the upstream node in some scenarios count for nothing because in those scenarios the upstream node 2 is green and \( \zeta_2 = 0 \). The valuation of the losses transmitted comes from the scenarios in which the upstream node defaults and transmits a loss of \( D_{12}(1 - r^*_2) = 50,000 \). In the event of 2% probability in which only the upstream node defaults, \( \zeta_2 = 0.5 \) because an extra dollar at the upstream node reduces the loss of its external creditors by 50 cents while the other 50 cents is absorbed by the downstream node. In the event of 1% probability in which both nodes default, \( \zeta_2 = 1 \) because an extra dollar at the upstream node goes in its entirety to reduce the losses of the external creditors of both nodes. The valuation of the losses transmitted is \( 2\% \times 50,000 \times 0.5 + 1\% \times 50,000 \times 1 = 1,000 \), so the systemic risk components are 2000 = 3000 – 1000 for the downstream node and 2500 = 1500 + 1000 for the upstream node.

5 Choosing a Method

We have explored several different methods for systemic risk attribution. There are several different methods because there are multiple causes of default, namely leverage and risky investment, and because there are multiple perspectives on responsibility for contagion, including borrower responsibility, lender responsibility, and shared responsibility. Which method should a systemic risk manager use? Section 4 showed how different methods serve different purposes. It is possible to design a systemic risk attribution method that targets external assets, leverage, inter-bank borrowing, or other phenomena as the source of systemic risk, depending on what the risk manager fears, intends to change, or blames for causing systemic risk. For example, the external-assets scheme (Section 4.1) is designed to attribute systemic risk to assets outside the financial system that are held by financial firms. Methods that use it to attribute systemic risk to nodes in the financial network reveal who is making the investments that endanger the system, but do not reveal other aspects of contagion: such methods would attribute little systemic risk to a node that takes on little risk in its external assets but plays a large role in channeling losses through the financial network to external creditors. One may also desire a general-purpose method to use when there is no specific purpose in mind. One good general-purpose method is the transmission method (Section 4.2), introduced by Drehmann and Tarashev (2011). It focuses on the effects of transmission of profits and losses, with borrower and lender sharing responsibility for internal loans. Another good one is the solvency method (Section 4.3), which focuses on the sensitivity of systemic risk to contributions
to the solvency of the system.

One consideration in choosing a method is whether to use the Shapley or Aumann-Shapley value. Each has advantages. Because the Shapley value considers counterfactual systems that are very different from the real system, it is difficult to be confident that those counterfactual systems are well-specified. For example, if a node’s external assets were entirely cash, it might borrow at a lower interest rate than in the real system, because it would be more creditworthy. The Aumann-Shapley value considers counterfactual systems that are more similar to the real system; if the cost function is homogeneous, it considers only perturbations of the real system.

One possible regulatory application of systemic risk attribution is to provide firms with incentives to lower systemic risk. Difficult questions surround the attempt to do this for contagion (Staum, 2011b, §2). However, if this is the goal, then the Shapley and Aumann-Shapley values each have some but not all of the desirable properties with respect to incentives (Staum, 2011b, §6). One form of incentive is systemic risk charges based on systemic risk components. In such a regulatory framework, it might be desirable to have a method that ensures that the systemic risk components are non-negative, avoiding political problems that could arise from payments (negative systemic risk charges) made by the regulator to financial firms. Liu and Staum (2011) developed methods that yield non-negative systemic risk components based on Staum (2010), and the same approach could be applied to the schemes in Section 4.

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A Proof of Proposition 2

Because there are no borderline nodes, sufficiently small perturbations of the data \((\delta, e, L^0, L)\) do not change the nodes’ colors. Because the clearing payment vector \(p^*\) depends only on \((e, L^0, L)\), Equation (9) follows directly from Equation (7). Likewise, Equation (10) follows directly from Equation (9). Consider the LP

\[
\min c^T x \text{ such that } Ax = b, \ x \geq 0
\]

where

\[
x = \begin{bmatrix} f \\
g \\v \\k \\\ell \end{bmatrix}, \ c = \begin{bmatrix} 0 \\
0 \\
0 \\
0 \\
1 \end{bmatrix}, \ A = \begin{bmatrix} \text{diag}(L1) - L^T & 0 & I & I & -I \\
-\text{diag}(L^0) & 0 & 0 & I & -I \\
I & I & 0 & 0 & 0 \end{bmatrix}, \text{ and } b = \begin{bmatrix} e - \delta \\
-\delta \\
1 \end{bmatrix}.
\]

There is a one-to-one correspondence between feasible solutions \(x\) of LP (20) and payment vectors \(p\) that satisfy the capacity constraints (4). The correspondence \(p = \text{diag}(\vec{p}) f\) is one-to-one for any value of \(\vec{p}\) because we stipulated \(f_i = 1\) if \(\vec{p}_i = 0\). Given \(f\), the only values of other coordinates of \(x\) that satisfy \(Ax = b\), \(k \geq 0\), and \(\ell \geq 0\) are \(g = 1 - f\), \(k = (\text{diag}(f)L^0 - \delta)^+\), \(\ell = (\text{diag}(f)L^0 - \delta)^-\), and \(v = e + L^T f - \text{diag}(f)\text{diag}(\delta + L1)\). The non-negativity of \(f\) and \(g\) is equivalent to the capacity constraint \(0 \leq p \leq \vec{p}\), and the non-negativity of \(v\) is equivalent to the capacity constraint \((I - \Pi^T)p \leq e\).

First we establish that the clearing payment vector \(p^*\) corresponds to an optimal solution \(x^*\) of LP (20). It follows from Proposition 1 of Liu and Staum (2010) that for all \(p\) satisfying the capacity constraints, \(p \leq p^*\). On the feasible set, the objective \(c^T x = 1^T \ell = 1^T (\text{diag}(f)L^0 - \delta)^- = 1^T (\delta - \text{diag}(\Pi^0)p)^+\). Therefore \(c^T x \geq c^T x^*\) for any feasible \(x\).

Next we perform a sensitivity analysis of the LP (20) at its optimal solution \(x^*\). A basis \(B\) for the solution contains the \(3N\) variables \(f, g_{\mathcal{R}_i \cup \mathcal{Y}}, v_G, k_{\mathcal{G}_i \cup \mathcal{G}},\) and \(\ell_R\). Because there are no borderline nodes, this basis is also a basis for the optimal solution when the LP data \(A\) and \(b\) are perturbed. The non-basic variables are zero. The basic variables are given by \(x_B = A_{B}^{-1} b\) and the sensitivities of the optimal objective value \(c^T x^*\) to \(b\) are given by the dual-optimal solution \(y^* = c_B^T A_{B}^{-1}\). The non-zero sensitivities of \(b\) to the underlying data are \(\nabla_b b = \text{diag}([1 \ 0 \ 0])\) and \(\nabla_\delta b = \text{diag}([-1 \ -1 \ 0])\). Then, using Equation (10), \(-\zeta = y^* \nabla_e b\), i.e., the first \(N\) components of \(y^*\) are \(-\zeta\). From this and Equation (9), it follows that \(R = y^* \nabla_\delta b\), and therefore components \(N + 1, \ldots, 2N\) of \(y^*\) are \(\zeta - R\). The sensitivity of the optimal objective value to \(A_{ij}\) is \(-y^*_i x^*_i\). Therefore its sensitivity to \(L_i^0\) is \((\zeta_i - R_i) f^*_i\), and its sensitivity to \(L_{ij}\) is \(\zeta_j f^*_i - \zeta_i f^*_i\).
B Derivation of Aumann-Shapley Values

B.1 External Assets

In Section 4.1, the points of potential discontinuity \( \tilde{\lambda}_2, \ldots, \tilde{\lambda}_m \) can be found by starting from \( \tilde{\lambda}_1 = 1 \) and descending to the next value of \( \tilde{\lambda} \) for which a node becomes borderline. The distance by which \( \lambda \) can be decreased from \( \tilde{\lambda}_i \) to the next value \( \tilde{\lambda}_{i+1} \) is which a node becomes borderline involves a comparison of the left derivatives

\[
p'(\tilde{\lambda}_1) = (\nabla_x p(\tilde{\lambda}_1))1 = (\nabla_e p(\tilde{\lambda}_1))/(\nabla_x e)1 = (\nabla_e p(\tilde{\lambda}_1))\Theta(a - 1).
\]

with the amounts by which \( p(\lambda1) \) can change until one of its components hits \( p = D^T1 + \delta \), the boundary between red and yellow, or \( \tilde{p} = \tilde{\rho} \circ p \), the boundary between yellow and green.

A green node \( j \) satisfies \( p_j(\tilde{\lambda}_1) = \tilde{p}_j \) and has positive time-1 equity \( v_j(\tilde{\lambda}_1) \). It reaches the borderline of yellow when its time-1 equity drops to 0. The left derivatives of time-1 equity are

\[
v'(\tilde{\lambda}_1) = (\nabla_e v(\tilde{\lambda}_1))\Theta(a - 1) \text{ where } \nabla_e v(\tilde{\lambda}_1) = I + (\Pi^T - I)\nabla_e p(\tilde{\lambda}_1) \text{ (Liu and Staum, 2010).}
\]

Because we are decreasing \( \lambda \), let \( \partial p_i/\partial e_j \) be the left derivative if \( \Theta_j(a - 1) > 0 \), in which case \( e_j \) decreases as \( \lambda \) decreases, and otherwise be the right derivative (see Liu and Staum, 2010). The distance \( \omega_{ij} \) associated with node \( j \) in the counterfactual system specified by \( \tilde{\lambda}1 \) is as follows.

- If node \( j \) is red and \( p'_j(\tilde{\lambda}_1) < 0 \), then \( \omega_{ij} = -(\tilde{p}_j - p_j(\tilde{\lambda}_1))/p'_j(\tilde{\lambda}_1) \) to hit red-yellow.
- If node \( j \) is yellow or red-yellow, and \( p'_j(\tilde{\lambda}_1) < 0 \), then \( \omega_{ij} = -(\tilde{p}_j - p_j(\tilde{\lambda}_1))/p'_j(\tilde{\lambda}_1) \) to hit yellow-green.
- If node \( j \) is yellow or yellow-green, and \( p'_j(\tilde{\lambda}_1) > 0 \), then \( \omega_{ij} = (p_j(\tilde{\lambda}_1) - \tilde{p}_j)/p'_j(\tilde{\lambda}_1) \) to hit red-yellow.
- If node \( j \) is green and \( v'_j(\tilde{\lambda}_1) > 0 \), then \( \omega_{ij} = v_j/v'_j(\tilde{\lambda}_1) \) to hit yellow-green.
- Otherwise, \( \omega_{ij} = \infty \) because there is no local movement of node \( j \) towards a borderline.

For \( i < m \), \( \lambda \) can decrease from \( \tilde{\lambda}_i \) by \( \omega_i = \min_{j=1,\ldots,N} \omega_{ij} \) until some node hits a borderline. The process ends at \( i = m \) because \( \min_{j=1,\ldots,N} \omega_{ij} = \infty \) indicating that no more borderlines are encountered for \( \lambda \in [0, \tilde{\lambda}_1] \). Then \( \omega_m = \lambda_m \). The sensitivities \( \zeta(\mu,1) = \zeta(\tilde{\lambda}_1) \) can be computed at participation \( \tilde{\lambda}_1 \) by evaluating Equation (10) with \( R_j = 1 \) for all red nodes and for red-yellow nodes such that \( p'_j(\tilde{\lambda}_1) > 0 \), and with the appropriate one-sided derivatives for \( \partial p_i/\partial e_j \).

B.2 Leverage

In Section 4.2, the total time-1 liabilities \( \bar{p}(\lambda) = \text{diag}(\lambda)\bar{p} \) and the outflow fractions \( \Pi(\lambda) = \Pi \). The cost is

\[
c(\lambda) = \delta(\lambda)^TR(\lambda) - p(\lambda)^T\text{diag}(R(\lambda))\Pi_0.
\]

The clearing payment vector is

\[
p(\lambda) = \text{diag}(G(\lambda))\text{diag}(\lambda)\bar{p} + (\nabla_e p(\lambda)) \left( e + D(1 - \lambda) + \Pi^T\text{diag}(G(\lambda))\text{diag}(\lambda)\bar{p} \right)
\]

where the factor multiplying \( \nabla_e p(\lambda) \) in Equation (22) contains the inflow to each node, from external and internal sources. The first term in Equation (22) contains the payments made by green
nodes as a function of their scale: the less their debt, the less they pay. The rows and columns of \( \nabla_e p(\lambda) \) corresponding to green nodes are zero. Therefore the second term in Equation (22) shows how, as scale decreases, the payments made by non-green nodes change because of their increased external asset value (due to more cash because of smaller internal loan assets) and decreased internal income, derived from decreased payments made by green nodes to non-green nodes. Because no node is both green and red, the first term (involving green nodes) contributes zero to the expression \( p(\lambda)^\top \text{diag}(R(\lambda)) \) in Equation (21). Using \( \delta(\lambda) = \text{diag}(\lambda)\delta \) and Equations (10), (21), and (22), the cost is

\[
c(\lambda) = \delta^\top \text{diag}(\lambda) R(\lambda) - \left( e^\top + (1 - \lambda)^\top D^\top + \bar{p}^\top \text{diag}(\lambda)\text{diag}(G(\lambda)) \right) \Pi \zeta(\lambda).
\]

Its gradient is \( \nabla c(\lambda) = \text{diag}(R(\lambda))\delta + (D^\top - \text{diag}(G(\lambda))\text{diag}(\bar{p})\Pi)\zeta(\lambda) \). Equation (14) for \( \nabla c(\lambda) \) follows by simplifying using \( \text{diag}(\bar{p})\Pi = \text{diag}(\bar{r})D^\top \).

The points of potential discontinuity \( \lambda_2, \ldots, \lambda_m \) are found using a procedure similar to that in Appendix B.1. The left derivatives

\[
p'(\bar{\lambda}_i 1) = \text{diag}(G(\mu_1))\bar{p} + (\nabla^+ e p(\bar{\lambda}_i 1))(\Pi^\top \text{diag}(G(\mu_1))\bar{p} - D1).
\]

In this expression, right derivatives \( \nabla^+_e p(\bar{\lambda}_i 1) = \nabla_e p(\mu_1 1) \) are used because external asset value increases as scale decreases. Because equity \( v(\lambda) = \text{diag}(G(\lambda))(\Pi^\top p(\lambda) - \bar{p}(\lambda) + e(\lambda)) \), the left derivatives

\[
v'(\bar{\lambda}_i 1) = \text{diag}(G(\mu_1)) \left( \Pi^\top p'(\bar{\lambda}_i 1) - \bar{p} - D1 \right).
\]

In the expressions for left derivatives, one does not need to analyze a perturbed scale or the counterfactual system specified by the participation vector \( \mu_1 1 \), because Lemma 1 states that the indicator \( G(\mu_1 1) = N(\bar{\lambda}_i 1) \). The distance \( \omega_{ij} \) associated with node \( j \) the counterfactual system specified by \( \bar{\lambda}_i 1 \) is as follows.

- If node \( j \) is red and \( p'_j(\bar{\lambda}_i 1) < p_j \), then \( \omega_{ij} = (\bar{\lambda}_i p_j - p_j(\bar{\lambda}_i 1))/(p_j - p'_j(\bar{\lambda}_i 1)) \) to hit red-yellow.
- If node \( j \) is neither red nor green, then \( \omega_{ij} \) is the minimum of \( \omega_{ij}^{\text{rg}} \) and \( \omega_{ij}^{\text{yr}} \) defined as follows.
  - If \( p'_j(\bar{\lambda}_i 1) < \bar{p}_j \) and node \( j \) is not yellow-green, then \( \omega_{ij}^{\text{rg}} = (\bar{\lambda}_i \bar{p}_j - p_j(\bar{\lambda}_i 1))/(\bar{p}_j - p'_j(\bar{\lambda}_i 1)) \) to hit yellow-green. Otherwise, \( \omega_{ij}^{\text{rg}} = \infty \).
  - If \( p'_j(\bar{\lambda}_i 1) > p_j \) and node \( j \) is not red-yellow, then \( \omega_{ij}^{\text{yr}} = (p_j(\bar{\lambda}_i 1) - \bar{\lambda}_i p_j)/(p'_j(\bar{\lambda}_i 1) - p_j) \) to hit red-yellow. Otherwise, \( \omega_{ij}^{\text{yr}} = \infty \).
- If node \( j \) is green and \( v'_j(\bar{\lambda}_i 1) > 0 \), then \( \omega_{ij} = v_j/v'_j(\bar{\lambda}_i 1) \) to hit yellow-green.
- Otherwise, \( \omega_{ij} = \infty \) because there is no local movement of node \( j \) towards a borderline.

**Lemma 1.** For any critical scale \( \bar{\lambda}_i \), the set \( N(\bar{\lambda}_i 1) \) of nodes that do not default equals the set \( \bar{G}(\mu_1 1) \) of nodes that are green when the scale is perturbed downwards.

**Proof.** After the perturbation, green nodes remain green and defaulting nodes are still defaulting; the only question is about the borderline yellow-green nodes. Consider a node \( i \) that is yellow-green
in the counterfactual system specified by the participation vector \( \tilde{\lambda}_i 1 \). Its inflow in Equation (22) equals its total time-1 liabilities. Expressed using left derivatives,

\[
\left( \nabla^+_e p_i(\tilde{\lambda}_i 1) \right) \left( e + D1 + \tilde{\lambda}_i \left( \Pi^\top \text{diag}(G(\mu_1))\bar{p} - D1 \right) \right) = \tilde{\lambda}_i \bar{p}_i. \tag{23}
\]

The left derivative of this inflow at scale \( \lambda = \tilde{\lambda}_i \) is

\[
\iota = \left( \nabla^+_e p_i(\tilde{\lambda}_i 1) \right) \left( \Pi^\top \text{diag}(G(\mu_1))\bar{p} - D1 \right).
\]

Therefore the left derivative \( \iota \leq \bar{p}_i \), as follows by dividing Equation (23) by \( \tilde{\lambda}_i \). This shows that, as scale is perturbed downwards from \( \tilde{\lambda}_i \), the inflow to node \( i \) decreases more slowly than its total time-1 liabilities.

\( \Box \)