

Landscape Theory for Schrödinger Operators with General Hopping Terms on a Finite Lattice

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Abstract

Findings by M. L. Lyra, S. Mayboroda and M. Filoche relate invertibility and positivity of a class of discrete Schrödinger matrices with the existence of the “Landscape Function”, which provides an upper bound on all eigenvectors simultaneously. Their argument is based on the variational principles. We consider an alternative method of proving these results, based on the power series expansion, and demonstrate that it naturally extends the original findings to the case of long range operators. The method of proof by power series expansion can also be employed in other scenarios, such as higher dimensional lattices.

1 Introduction

The findings of Lyra, Mayboroda, and Filoche [6] that we are concerned with are summarized below. Their paper examined the following Schrödinger matrix \( H \), a matrix form of the equation \( [-\Delta + V(\vec{x})] \Psi(\vec{x}) = E \Psi(\vec{x}) \), on a finite, discrete lattice with a hopping distance of one:

\[
H = \begin{pmatrix}
v_1 & -1 & 0 & \cdots & 0 \\
-1 & v_2 & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & -1 & v_n
\end{pmatrix}
\]

\( v_j \geq 2, j = 1, \ldots, n. \) (1.1)

It was shown in the original paper [6] that there exists a unique solution \( \vec{u} \) to the “Landscape Function” equation

\[
H \vec{u} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.
\] (1.2)

The solution vector \( \vec{u} \) is also called the “Landscape Function,” and was shown to have the property that for any eigenvector \( \vec{x} \) and corresponding eigenvalue \( \lambda \) of the matrix, so that \( H \vec{x} = \lambda \vec{x} \), and the above conditions on \( v_j \), we have, for all \( j = 1, \ldots, n \)

\[
\frac{|x_j|}{\max_{1 \leq k \leq n} |x_k|} \leq \lambda u_j
\] (1.3)

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The above result is what we will call throughout the “Landscape Theory.” We were motivated by these initial results to extend them to more complicated versions of the Schrödinger matrix. Our results allow more interesting applications and models to be explored and developed. We present our results in two sections. The first covers an extension of a slightly weakened version of the initial Landscape Theory to a “Long Range” Schrödinger matrix. We then provide the findings necessary to strengthen our theory so that it is a complete extension of the original results. It must also be highlighted that our extensions are built off a new method of proof of the Landscape Theory — which also holds for the original model with hopping distance one. We incorporate the power series expansion of matrices and all our results will derive from there. This method is widely applicable not only in potentially extending Landscape Theory even further, but as a tool in proving similar problems from other areas — for example, higher dimensional lattice models. We take a moment in the following paragraph to provide some explanation of these applications, which provides some physical motivation for our results. The rest of the paper will then be devoted to extending and proving our results.

As is well-established in physics, the eigenvalues of the Schrödinger matrix represent the allowed energies of a single particle on an $N$-dimensional lattice. Then, as the number of dimensions in the model increases, this corresponds to bigger matrices as directions of hopping are possible. For example, a 1D chain model of length $n$ has $n^2$ entries, and a 2D lattice model of size $n$ has $n^4$ possible hopping directions and entries. In this case, the matrix used to describe the system becomes much larger. The presence of interatomic matrix elements between different sites leads to a probability of hopping between the sites, the general case of which is covered by the general landscape theory given here in our study of the general Schrödinger matrix.

Before we move to our results, we want to make a few more comments about the original Landscape Theory. Anderson localization [1] in a disordered medium is one of the most important and popular topics in condensed matter physics. A new concept landscape function for an elliptic differential operator $L$, was first introduced in 2012 by Filoche and Mayboroda [4], and was shown to be extremely adept at predicting the location of regions of low energy eigenstates of $L$. The concept of landscape function was generalized from the continuous case to the discrete case in [6], for which our current paper is based on. The Landscape Theory was further developed e.g. in mathematics [8, 2], as well as in theoretical and experimental physics [5, 7]. We refer readers to the above papers and the references therein for more background and details of the Landscape Theory.

## 2 Strict Hamiltonian Potential Inequality

Let the Long Range Schrödinger matrix be given by $H = -H_0 + V$, where, for $n \in \mathbb{N}$ and $n \geq 2$,

$$V = \begin{pmatrix} v_1 & 0 & 0 & \cdots & 0 \\ 0 & v_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & v_{n-1} & 0 \\ 0 & \cdots & 0 & 0 & v_n \end{pmatrix} \quad (2.1)$$
and
\[
H_0 = \begin{pmatrix}
0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\
-1 & 0 & a_1 & \cdots & a_{n-2} & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
& \cdots & \cdots & 0 & a_1 & a_2 \\
a_{n-2} & \cdots & \cdots & a_1 & 0 & a_1 \\
a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 & 0
\end{pmatrix}, \quad a_i \geq 0, \ i = 1, \cdots, n-1. \tag{2.2}
\]

**Theorem 1** (Landscape Theory for general hopping matrix on a finite lattice). Let \( H \) be given as above. We consider the eigenvector \( \vec{x} \) of \( H \) with corresponding eigenvalue \( \lambda \), so that \( H\vec{x} = \lambda \vec{x} \).

Assume that
\[
v_j > 2 \sum_{i=1}^{n-1} a_i, \quad j = 1, \cdots, n \tag{2.3}
\]

Then, the solution to the Landscape Function equation, \( \vec{u} \in \mathbb{R}^n \), exists, satisfying \( H\vec{u} = \vec{1}^T \). Further, for all \( j = 1, \cdots, n \)
\[
\frac{|x_j|}{\max_{1 \leq k \leq n} |x_k|} \leq \lambda u_j, \tag{2.4}
\]

**Remark 2.** If \( a_1 = 1 \) and \( a_2 = \cdots = a_{n-1} = 0 \), then Theorem 1 under condition (2.3) gives a weakened version of the original landscape theory. It is also easy to check that if \( a_1 = \cdots = a_{n-1} = 0 \), Theorem 1 holds trivially for the diagonal matrix \( H = V \). The stronger version of Theorem 1 with a soft inequality of (2.3), also holds true. We will discuss that in Section 3.

The outline of the proof will be similar to the original Landscape Theory: (i) the existence of the inverse and the Landscape Function solution; (ii) the positivity of the inverse and Landscape Function. However, we use an alternative proof, built off the power series expansion of the pertinent matrices, to show these results.

**Lemma 3.** If \( v_j \) satisfies (2.3), then \( H \) is invertible. As a consequence, there is always a vector \( \vec{u} \in \mathbb{R}^n \) satisfying \( H\vec{u} = \vec{1}^T \), with the explicit expression
\[
u_j = \sum_{k=1}^{n} G_{jk}, \tag{2.5}
\]

where \( G_{ij} = H^{-1}(i,j) \) is the \((i,j)\)th entry of the inverse of \( H \).

**Remark 4.** Moreover, all eigenvalues of \( H \) are strictly positive. This is easily deduced from the fact that the matrix \( H \) is both self-adjoint and strictly diagonally dominant by construction—due to our conditions. Together, this means, due in part to positive semi-definiteness, that all eigenvalues of \( H \) are real and greater than or equal to zero. Further, as \( H \) is invertible, zero cannot be an eigenvalue and the result follows.

**Lemma 5.** For \( j = 1, \cdots, n \),
\[
u_j > 0. \tag{2.6}
\]

\(^1\text{We use the notation } \vec{1} = (1, \cdots, 1)^T.\)
We will prove Lemma 3 and Lemma 5 later. We will first complete the proof of Theorem 1.

Proof of Theorem 1. Let \( \vec{x} \) be an eigenvector of \( H \) with eigenvalue \( \lambda \):

\[
H\vec{x} = \lambda \vec{x}
\]
(2.7)

\[
\vec{x} = \lambda H^{-1}\vec{x}
\]
(2.8)

\[
x_j = \lambda \sum_{k=1}^{n} G_{jk} x_k,
\]
(2.9)

where (2.8) follows from Lemma 3 and (2.9) follows from Lemma 3 and matrix multiplication. We now examine the vector \( \vec{x} \) scaled by its maximum value, entry-wise. Without loss of generality, we can assume \( \lambda > 0 \):

\[
\frac{|x_j|}{\max_{1 \leq k \leq n} |x_k|} = \lambda \left| \sum_{k=1}^{n} G_{jk} \frac{x_k}{\max_{1 \leq k \leq n} |x_k|} \right|
\]
(2.10)

\[
\leq \lambda \sum_{k=1}^{n} G_{jk} \left| \frac{x_k}{\max_{1 \leq k \leq n} |x_k|} \right|
\]
(2.11)

\[
\leq \lambda \sum_{k=1}^{n} G_{jk}
\]
(2.12)

\[
= \lambda u_j,
\]
(2.13)

where (2.11) follows by the triangle inequality, (2.12) by Lemma 5, (2.13) by our upper bound being one, and (2.14) by Lemma 3 and Remark 4.

Our proofs for Lemma 3 and Lemma 5 rely on the following result which pertains to power series expansion of matrices.

Lemma 6. For two \( n \times n \) matrices \( A \) and \( B \), if \( A \) is invertible and \( \| A^{-1} B \| < 1 \)\(^2\) then \( A - B \) is invertible and

\[
(A - B)^{-1} = \sum_{k=0}^{\infty} (A^{-1} B)^k A^{-1}.
\]
(2.15)

Lemma 6 is quite standard. For the sake of completeness, we include the proof here for the readers’ convenience.

Proof of Lemma 6. It is a known theorem in Analysis that, provided \( \| A \| < 1 \) and letting \( I \) denote the identity matrix

\[
(I - A)^{-1} = \sum_{k=0}^{\infty} A^k.
\]
(2.16)

Thus, we can see from the following manipulations that

\[
(A - B)^{-1} = (A(I - A^{-1} B))^{-1} = (I - A^{-1} B)^{-1} A^{-1}.
\]
(2.17)

\(^2\) \( \| \cdot \| \) denotes the operator norm.
Hence, provided \( \|A^{-1}B\| < 1 \),
\[
(I - A^{-1}B)^{-1}A^{-1} = \sum_{k=0}^{\infty} (A^{-1}B)^k A^{-1}.
\tag{2.18}
\]

In order to apply Lemma 6 we also need estimates of the matrix norms of \( V \) and \( H_0 \). We can prove that

**Lemma 7.**
\[
\|V^{-1}\| \leq \max_{1 \leq j \leq n} v_j^{-1}
\tag{2.19}
\]
and
\[
\|H_0\| \leq 2 \sum_{i=1}^{n-1} a_i
\tag{2.20}
\]

**Proof.** We will start with (2.19). It is observed that \( V \) is a diagonal matrix with all diagonal entries strictly greater than zero (in both Theorems 1 and 3 which we will discuss later). Thus we can safely say that \( V^{-1} \) is well defined and has diagonal entries \((v_j)^{-1} = \frac{1}{v_j}\). Further, as \( V^{-1} \) is a diagonal, self-adjoint matrix, we know that \( \|V^{-1}\| \) is equal to its maximal eigenvalue, which in this case is simply \( \max_{1 \leq j \leq n} v_j^{-1} \).

Now we move on to (2.20). We start with a definition of the notation \( R_k \) and \( L_k \) as the matrices with ones on the \( k^{th} \) right and left off-diagonals, respectively, and zero elsewhere. Now we can see that \( H_0 \), as given in initial equation (2.2), has the following decomposition
\[
H_0 = a_1 R_1 + a_1 L_1 + \cdots + a_{n-1} R_{n-1} + a_{n-1} L_{n-1}.
\tag{2.21}
\]
Thus the norm of \( H_0 \) obeys the following statement, based of the triangle inequality and properties of norms
\[
\|H_0\| = \|a_1 R_1 + a_1 L_1 + \cdots + a_{n-1} R_{n-1} + a_{n-1} L_{n-1}\|
\tag{2.22}
\]
\[
\|H_0\| \leq |a_1| \|R_1\| + |a_1| \|L_1\| + \cdots + |a_{n-1}| \|R_{n-1}\| + |a_{n-1}| \|L_{n-1}\|
\tag{2.23}
\]

Further, we can show that all norms \( \|R_i\| \) and \( \|L_i\|, \ i = 1, \cdots, n-1 \) are less than or equal to one. This is demonstrated in several steps. First, it can be shown by direct computation that \( R_i = (R_1)^i \) and \( L_i = (L_1)^i \). Then, one can see
\[
\|R_i\| = \|(R_1)^i\| \leq \|R_1\|^i
\tag{2.24}
\]

The same results apply to all \( L_i \). The final step to reach our assertion that all the pertinent norms are less than or equal to one is to show that \( \|R_1\| \) and \( \|L_1\| \) are less than or equal to one, which is done below.
\[
L_1 := L = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 0 & 0 \\
0 & \cdots & 0 & 1 & 0
\end{pmatrix}, \quad R_1 := R = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 0 & 1 \\
0 & \cdots & 0 & 0 & 0
\end{pmatrix}.
\tag{2.25}
\]

This relationship between a self-adjoint matrix’s operator norm and maximum eigenvalue is a known—albeit more advanced—result in Linear Algebra.
For any $\vec{x} = (x_1, x_2, \cdots, x_n)^T$, direct computation shows that

$$R\vec{x} = R \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-2} \\ x_n - 1 \end{pmatrix}$$

(2.26)

Therefore,

$$\|R\vec{x}\|^2 = x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \leq x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + x_n^2 = \|\vec{x}\|^2$$

(2.27)

which implies $\|R\vec{x}\| \leq \|\vec{x}\|$. According to the definition of the matrix operator norm and (2.27)

$$\|R\| = \max_{\vec{x} \neq 0} \frac{\|R\vec{x}\|}{\|\vec{x}\|} \leq 1.$$  

(2.28)

Exactly the same argument shows that $\|L\| \leq 1$, which completes the assertion. Putting all of the above findings together, we can substitute all $\|R_i\|$s and $\|L_i\|$s in our $H_0$ equation with ones via an inequality.

$$\|H_0\| \leq \sum_{i=1}^{n-1} |a_i| + \sum_{i=1}^{n-1} |a_i| = 2 \sum_{i=1}^{n-1} a_i$$

(2.29)

This is what we desire. The absolute value falls away because all $a_i$s are positive by construction. $\square$

Now we can proceed to prove Lemma 3 and Lemma 5 using eq. (2.15) and Lemma 7.

**Proof of Lemma 3.** Given the structure of $V$ and $H_0$ given in Theorem 1, in particular (2.3), we can see that by Lemma 7

$$\|V^{-1}\| \leq \max_{1 \leq j \leq n} v_j^{-1} < \frac{1}{2 \sum_{i=1}^{n-1} a_i}$$

(2.30)

and, as proven

$$\|H_0\| \leq 2 \sum_{i=1}^{n-1} a_i.$$  

(2.31)

Thus, we see that

$$\|V^{-1}H_0\| \leq \|V^{-1}\|\|H_0\| < \frac{2 \sum_{i=1}^{n-1} a_i}{2 \sum_{i=1}^{n-1} a_i} = 1.$$  

(2.32)

Hence, we meet the conditions to satisfy Lemma 6 and have invertibility of our matrix $H$. $\square$

**Proof of Lemma 5.** With the same conditions as for the proof of Lemma 3 we see that every element of $V^{-1}$—whose diagonal elements are solely elements $v_j^{-1}$, $j = 1, \cdots, n$ which are the reciprocals of the diagonal elements of $V$—and every element of $H_0$ are non-negative, based on the structure of these two matrices as given in Theorem 1. We thus have non-negativity of $(V - H_0)^{-1}$ by the structure of the power series expansion given in Lemma 6. Next, since $V - H_0$ is invertible, we have that none of the rows of $(V - H_0)^{-1}$ are identically zero. Together with (2.5), this implies $u_j > 0$ for $j = 1, \cdots, n$.

$\square$
3 Soft Hamiltonian Potential Inequality

In this section, we would like to study the optimality of condition (2.3). We now prove that

**Theorem 8.** Let $H$ be given as in the Theorem 1. All conclusions of Theorem 1 hold true under the condition

$$v_j \geq 2 \sum_{i=1}^{n-1} a_i, \quad j = 1, \cdots, n.$$  \hspace{1cm} (3.1)

As we see from the method of proof employed in Section 2, it is enough to show

**Lemma 9.**

$$\|H_0\| < 2 \sum_{i=1}^{n-1} a_i.$$  \hspace{1cm} (3.2)

It follows from

**Lemma 10.** If $A_j(n)$ is an $n \times n$ matrix given by

$$A_j(n) = \begin{pmatrix}
0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
1 & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 1 & \cdots & 0 & 0
\end{pmatrix},$$  \hspace{1cm} (3.3)

where the distance of the 1 off-diagonal—which is symmetric—to the diagonal is $j$, $1 \leq j \leq n - 1$, then all the eigenvalues of $A_j(n)$ are strictly between $-2$ and 2. As a consequence, due to the fact that $A_j(n)$ is self-adjoint, $\|A_j\| < 2$.

**Remark 11.** If $n = 1$, we allow $j = 1$, and denote the trivial case $A_1(1) = (0)$.

**Lemma 12.** Let $\mathcal{O} = \{\hat{e}_1, \hat{e}_2, \cdots, \hat{e}_n\}$ be an ordered basis of $\mathbb{R}^n$, where $\hat{e}_j, j = 1 \cdots, n$ are the standard basis vectors of $\mathbb{R}^n$. For any $j$ such that $2 \leq j \leq n - 1$, there is a rearrangement of $\mathcal{O}$ that is an ordered basis $\tilde{\mathcal{O}} = \{\hat{e}_{i_1}, \hat{e}_{i_2}, \cdots, \hat{e}_{i_n}\}$ such that, under $\tilde{\mathcal{O}}$, $A_j(n)$ has the following block matrix representation, with blocks of either the form (3.3) above, denoted $A_1(k_i)$, or completely 0, denoted $O$.

$$\tilde{A}_j(n) = \begin{pmatrix}
A_1(k_1) & O & \cdots & O \\
O & A_1(k_2) & \ddots & \vdots \\
\vdots & \ddots & \ddots & O \\
O & \cdots & O & A_1(k_j)
\end{pmatrix}$$  \hspace{1cm} (3.4)

where $k_1 + k_2 + \cdots k_j = n$.

\footnote{Clearly, as discussed in Remark 12 if $a_1 = \cdots = a_{n-1} = 0$, we still need the strict inequality $v_j > 0$ for the trivial diagonal case to be true.}
Proof of Lemma 12. We use an explicit construction as our method of proof. Consider the matrix $A_j(n)$ as seen in Lemma 10. To transform this matrix into one of the form $A_j(n)$ (as seen in 3.4), we need to perform a change of coordinates that involves permuting the order of the standard basis vectors. To pick a specific ordering, we perform the following algorithm:

1. Start with the standard basis vector $e_1$. Add $j$ to the subscript. If $j + 1 \leq n$, continue to add $j$ until $kj + 1 > n$ for some $k \in \mathbb{Z}$. Group all of the basis vectors $e_1, e_{j+1}, \ldots, e_{(k-1)j+1}$. Let the set $\{e_1, e_{j+1}, \ldots, e_{(k-1)j+1}\}$ be called $[e_1]$. The order of the elements of $[e_1]$ and all future $[e_i]$ will be important.

2. Repeat this process with $e_2$. Add $j$ to the subscript. If $j + 2 \leq n$, continue to add $j$ until $kj + 2 > n$ for some $k \in \mathbb{Z}$. Group all of the basis vectors $e_2, e_{j+2}, \ldots, e_{(k-1)j+2}$. Let the set $\{e_2, e_{j+2}, \ldots, e_{(k-1)j+2}\}$ be called $[e_2]$.

3. Continue this process until $e_j$ is reached. Again, add $j$ to the subscript. If $2j \leq n$, continue to add $j$ until $kj > n$ for some $k \in \mathbb{Z}$. Group all of the basis vectors $e_j, e_{2j}, \ldots, e_{(k-1)j}$. Let the set $\{e_j, e_{2j}, \ldots, e_{(k-1)j}\}$ be called $[e_j]$. At this point, all of the standard basis vectors are an element of some $[e_i]$ for $i = 1, \ldots, j$. Now, the process can be stopped.

4. Rewrite the matrix $A_j(n)$ in the coordinate system $[[e_1], \ldots, [e_j]]$. The order of the elements does matter. Call this matrix $\tilde{A}_j(n)$.

Note that this algorithm can be applied to any $A_j(n)$ and that $\tilde{A}_j(n)$ will be a matrix of the form seen in 3.4. This is illustrated by considering $A_{1j}(n)$, a matrix under the basis system $[e_1]$ along with the remaining basis elements, unaltered—excepting any perturbations that may have occurred by grouping $[e_1]$. In other words, the elements of $[e_1]$ come first and then the remaining elements follow in the same order as they were originally. Thus, $A_{1j}(n)$ is matrix $A_j(n)$ after performing the above algorithm for only one of the potentially many basis groupings, and then changing $A_j(n)$ to representation under the new basis $[[e_1], \text{the rest}]$. We know $[e_1]$ is self-contained with respect to matrix multiplication by ordered $[e_1]$ elements against $A_{1j}(n)$—they will not leave the set or interfere with any other outside basis multiplications. Further and more generally, one notes that the only non-zero elements of matrices of the form $A_j(n)$ are at positions $(k, l)$, where $|k - l| = j$. By construction, both $e_k$ and $e_l$, where $k$ and $l$ satisfy the previous equality, will be in the same block $A_1(k_l)$ after the above algorithm and change of coordinates are employed. Looking at the alternate situation, positions $(k, l)$ of $A_j(n)$ where $|k - l| \neq j$, we have zero entries. This means that these basis elements do not interact via $A_j(n)$. Therefore, all elements resulting from interactions between basis vectors in different blocks of $\tilde{A}_j(n)$ will be zero. We also know that within the grouping given by the algorithm, basis elements are shifted strictly to “adjacent” elements under matrix multiplication. This suggests a matrix of the following form

$$A_{1j}(n) = \begin{pmatrix} A_1(k_1) & O \\ O & B \end{pmatrix} \quad (3.5)$$

Where $A_1(k_1)$ is of the form 3.3, given above and $B$ represents the part of matrix $A_{1j}(n)$ not yet reordered—which may not exist at all in certain cases. The subsequent basis groups follow in the same manner, culminating in $\tilde{A}_j(n)$. This completes the proof.  \[\square\]

We can now proceed to prove Lemmas 9 and 10.

Proof of Lemma 10. We see by Lemma 12 that all matrices of the form $A_j(n)$ are similar—via a change of coordinates—to a matrix $\tilde{A}_j(n)$ that can colloquially be described as a “first off-diagonal matrix—with one entries—with potentially several one entries missing.” We know via our work on the proof of Lemma 7 that the norm of a matrix consisting of the two first off-diagonals is less than
or equal to two. We can make this inequality strict by considering this “almost” first off-diagonal matrix as a finite portion of an infinite dimensional lattice.

Consider the \( n \times n \) matrix \( A_1(n) \) consisting of the complete first off-diagonals.

\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 1 \\
0 & \cdots & 0 & 1 & 0
\end{pmatrix}
\]

As the matrix is self-adjoint, we know that all eigenvalues are less than or equal to two—due to the bound on the norm given as part of our proof of Lemma 7. To make this inequality strict\(^5\), we will contradict the following difference equation.

\[H_0 \vec{x} = 2 \vec{x}\]

Which is the same as

\[x_{k-1} + x_{k+1} = 2x_k, \quad k = 1, \ldots, n\]

Where \( \vec{x} = (x_1, \ldots, x_n) \) is part of an infinite system

\[x_{k-1} + x_{k+1} = 2x_k, \quad k \in \mathbb{Z}, \quad (3.6)\]

with a zero boundary condition so that \( x_0 = x_{n+1} = 0 \). We know that the only solutions to the difference equation above come from the fundamental set of solutions formed by the following two basis elements (i.e., these two solutions are the only ones we must check).

\[\vec{\alpha} = \vec{1}, \quad \vec{\gamma} = (1, \ldots, n)\]

Note that each of these solutions technically solves the infinite system identified in (3.6), but we have used our zero boundary condition to examine a finite subsystem, specifically, one that arises from the \( n \times n \) version of the above matrix. First, let us try solution \( \vec{\alpha} \).

\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 1 \\
0 & \cdots & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
1 \\
\vdots \\
\vdots \\
\vdots \\
1 \\
1
\end{pmatrix} = 2
\begin{pmatrix}
1 \\
\vdots \\
\vdots \\
\vdots \\
1 \\
1
\end{pmatrix}
\]

Examine the first place

\[1 = 2\]

Thus, we have a contradiction as only the trivial solution where we multiply \( \vec{\alpha} \) by 0 solves the above equation at the first place. Now, let us examine solution \( \vec{\gamma} \).

\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 1 \\
0 & \cdots & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
1 \\
\vdots \\
\vdots \\
\vdots \\
1 \\
\vdots
\end{pmatrix} = 2
\begin{pmatrix}
1 \\
\vdots \\
\vdots \\
\vdots \\
1 \\
\vdots
\end{pmatrix}
\]

\[\text{Actually, it is well known that all the } n \text{ eigenvalues of } A_1(n) \text{ can be computed explicitly using the } n\text{-th Chebyshev polynomial of the second kind, see e.g. } [3, 9]. \text{ The explicit expression shows that all the eigenvalues of } A_1(n) \text{ are strictly in between } -2 \text{ and } 2, \text{ for any } n. \text{ Here, instead of using the explicit expression, we present a self-consistent proof for this fact.}\]

9
Examine the nth place

\[ n - 1 = 2n. \]

Once again, we have a contradiction, as only \( n = -1 \) solves the equation at the nth place. This is impossible as the vector \( \vec{\gamma} \) is not structured so as to allow this. Thus, our setup and assumptions in constructing the initial difference equation must have been incorrect due to contradiction, and 2 cannot be an eigenvalue of \( H_0 \). The fact that \(-2\) cannot be an eigenvalue follows from the same argument.

Now, the “almost” first off-diagonal matrix that our reordered basis matrix \( \tilde{A}_j(n) \)—provided by Lemma 12—can be shown by direct computation to have a norm less than or equal to the norm of a full first off-diagonal matrix, and thus this matrix, too, has a norm less than two. Further, this matrix is similar to the matrix \( A_j(n) \) and the operator norm—what we are employing—of two similar matrices is the same. We are thus given Lemma 10.

**Proof of Lemma 9.** We can show the desired result simply by applying Lemma 10 to each off-diagonal set. Letting \( A_j(n), j = 1, \cdots , n - 1 \) be given as in (3.3), we have

\[
\| H_0 \| = \| a_1 A_1(n) + \cdots + a_{n-1} A_{n-1}(n) \|
\]

\[
\| H_0 \| \leq a_1 \| A_1(n) \| + \cdots + a_{n-1} \| A_{n-1}(n) \| \quad \text{by the triangle inequality}
\]

\[
\| H_0 \| < 2 \sum_{j=1}^{n-1} a_j
\]

The last statement is reached by applying Lemma 10 to all \( A_j(n) \) and rewriting the result as a summation.

Lastly, we will complete our paper with the proof of Theorem 8.

**Proof of Theorem 8.** We simply need to show that our matrix satisfies Lemma 6. Invertibility of \( H \) and positivity of the inverse and Landscape Function will follow from there via Lemmas 3 and 5 which are—as in Theorem 1—completely proven by Lemma 6. To show the desired result, we note that, given the conditions in Theorem 8

\[
\| V^{-1} \| \leq \max_{1 \leq j \leq n} v_j^{-1} \leq \frac{1}{2 \sum_{i=1}^{n-1} a_i}
\]

and, as shown by Lemma 9

\[
\| H_0 \| < 2 \sum_{i=1}^{n-1} a_i
\]

Thus, we see that

\[
\| V^{-1} H_0 \| \leq \| V^{-1} \| \| H_0 \| < \frac{2 \sum_{i=1}^{n-1} a_i}{2 \sum_{i=1}^{n-1} a_i} = 1
\]

Hence, we meet the conditions to satisfy Lemma 6 and have invertibility and positivity of our matrix \( H \) under our extended Theorem.
4 Conclusion

In summary, we have extended the findings of, specifically, Lyra, Mayboroda, and Filoche [6] by proving the invertibility and positivity of a more general “Long Range” Schrödinger Matrix. Our method of proof includes employment of the power series expansion of matrices, with said method now available to solve similar problems currently under investigation. In addition to the mathematical findings of this paper, our results allow for an investigation of more complex applications in the realm of Landscape Theory. Curious readers are directed to the references section for a further treatment of these. It is likely that our methods can be utilized in further extensions of the theory, including higher dimensional lattice models.

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