ON COORDINATIZATION OF MATHEMATICS

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Abstract. The problem of advancing coordinatization of mathematics is considered. The need to develop a theory for measuring value and complexity of mathematical implications and proofs is discussed including motivations, benefits and implementation problems. Examples of mathematical considerations for such a theory are given. Arguments supporting such an advance related to applications in mathematical research guidance, publication standards and education are given.

1. Introduction.

1.1. History and the current state. Humans have always been involved in mental activities now called mathematics - mental operations on abstract notions having numerical or geometrical origin and logical implications involving these objects. The usefulness and the exceptional role of mathematics have been best expressed by the Pythagorean saying "all is number" which asserts that all physical objects, systems and processes may be precisely modelled using numerical constructions. Using a philosophical point of view and language mathematics can be thought as a universal epistemological framework created by the human intellect in order to justify knowledge or to perform regress steps for knowledge from various areas in a uniform way.

Mathematical activities have been mostly related to applied mathematics - mathematical modelling for other sciences and nontrivial processing of numerical and geometrical data. Some of these activities philosophically can be interpreted as generalized regress steps, see [15]. All sciences and disciplines involving numerical or geometrical data have been shaped and supervised by applied mathematicians.

Relatively smaller but important part of mathematical activities is pure mathematics - development of mathematical areas and mathematical language which are not directly related to applications but are derived from applied mathematics. Pure mathematics research is considered to be an important collective activity which guides and affects both applied mathematics and other disciplines - exact sciences, engineering and social sciences.

Mathematical activities have produced both pure and applied results: mathematical theories - collections of related mathematical facts, algorithms and proof techniques, nontrivial properties of mathematical objects, nonobvious logical implications
The progress of both pure and applied mathematics has been greatly influenced by advances in formalization and coordinatization of mathematical objects and the mathematical language. Encoding techniques of mathematical objects, statements and computational steps have influenced the progress of all sciences and most human activities. As encoding breakthroughs one can mention the introduction of the positional notation, algebraic operations, Cartesian coordinates, calculus, matrix algebra, mathematical logic, category theory etc. See Shafarevich [12] for a treatment and examples of coordinatizations in algebra. An important work advancing statement and implication encoding was the influential book "Principia Mathematica" by Russell and Whitehead [11]. It is accepted that mathematical statements can be encoded as well defined mathematical objects - sequences of symbols in a fixed alphabet which form words defined by certain languages. Areas of mathematical logic such as the first-order logic were founded, developed and applied. For some areas of mathematics minimal systems of axioms and inference rules have been proposed, for example, the Peano arithmetic. Another important breakthrough for the mathematical language and thinking was the category theory introduced by Eilenberg and Mac Lane, see Mac Lane [9]. The category theory is a successful attempt to advance unified encoding of the mathematical language and establish maps between different mathematical theories.

One can notice two major longterm trends in mathematics and its applications. First of all, the areas having precise mathematical models and being served by applied mathematics are constantly enlarging. Even the most "unmathematical" activities and processes may be subject to regress steps in the philosophical sense - mathematical modelling. Secondly, mathematical notations and the mathematical language are steadily getting more rigorous and well defined over time. We call these two trends the Pythagorean process.

Currently mathematical texts such as original research papers, monographs and textbooks are written in a mixed language which contains both mathematical symbols and human language. New mathematical results are being produced by mathematicians and many various mathematical theories are being developed. New mathematical results are usually related to new properties or classifications of mathematical objects, functors between different categories and other similar issues. The mathematical community has identified a number of important unsolved problems which can be roughly divided in two groups. First of all, there are important unsolved problems originating from conceptual issues of the applied or pure mathematics, such as the $P \neq NP$ problem, solutions of such problems will be quite important for the areas which have originated them and for other areas as well. Problems of this class can be further divided into subclasses such as classification-type or functor-type prob-
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problems. Secondly, there are challenging and potentially deep unsolved problems which seem to originate from computational observations, such as the Collatz conjecture, see Lagarias [8].

Computers are constantly being used by mathematicians to obtain both numerical and symbolical computational results for research purposes such as examples and counterexamples, to make computations termed "automated theorem proving" and "mechanical theorem proving" which are equivalent to statement proving in certain areas of mathematics, see Robinson [10] and Chou [1].

1.2. Possible nexts steps. One can ask whether the Pythagorean process will continue and what may be its next steps. In order to predict and stimulate nontrivial and important advances we must look at those features of mathematical activities which have not been yet coordinatized and measured. The goal of this paper is to show a possible direction for the Pythagorean process.

One feature that is still missing in the mathematical culture is a precise expression of all mathematical implications, proofs and algorithms as well defined mathematical objects. Another missing feature is mathematical structure of mathematical theories - collections of related mathematical results and proof techniques. In philosophical terms these missing features correspond to a regress step with respect to the mathematics itself - a justification of mathematics in terms of relatively simple mathematical concepts.

We conjecture that there will be new developments of mathematics which will continue the Pythagorean process - there will be advances of the mathematical thinking and encoding which will allow us to comprehensively coordinatize and precisely interpret proofs and mathematical theories as numerical or geometrical mathematical objects, known or new ones. Given a mathematical theory \( A \) (a structure containing objects of study, first-order or higher-order logic statements, proofs etc.) we may look for a mathematical object \( \alpha \) which would be a good model of \( A \): elements of \( A \) such as logical implications, proofs and subsets of mathematical statements in \( A \) would be defined as substructures or quotient structures of \( \alpha \). The transfer from \( A \) to \( \alpha \) should be thought philosophically as a regress step. For example, we can model a mathematical theory as an oriented graph with vertices corresponding to statements and oriented edges corresponding to implications and transformations of objects, in this model statements are not interpreted as implications. We call such a graph a proof graph. The proof graph coresponds to the binary relation in the set of statements with two statement \( P \) and \( Q \) being related provided \( P \) implies \( Q \), we will call it the implication relation. In terms of the proof graph model we can try to assign a mathematical structure to edges and an extra structure to vertices. We can also try to investigate the graph-theoretical structure of the proof graph. Our current mathemat-
ical experience may limit us to weighted graphs which we will often consider in this paper but other models for mathematical implications and theories may be needed and developed. Any reasonable mapping from a mathematical theory to a formally defined and simpler mathematical object should be investigated. Implications and proofs may be considered as being embedded in some suitable ambient space.

Such models would allow to increase the speed and improve the quality of progress of a given mathematical theory, improve understanding of various theories, compare different theories and improve understanding of their relations, measure mathematical results, classify mathematical theories up to isomorphism in a right sense, consider and interpret maps between mathematical theories. These models may also provide new research questions about the object $\alpha$ or stimulate development of new mathematical objects for the described modelling purposes. It may allow to substitute logical implication making by computations. The coordinatization should be essentially unique. It must be accessible and convincing to a majority of active researchers and advanced educators. It would be considered as a technique for reducing complexity and creativity needed for work in an area, i.e. the mathematical research, to a minimal admissible level. It can also lead to generalizations of the implication concept.

Such a development would allow to describe the current state and development, in narrow or wide areas, and maybe even the whole history of mathematics as a mathematical object, to measure mathematical proofs and results in a mathematically well defined way, to classify and measure mathematical creativity, to derive canonical solution paths for unsolved problems. It would also be used to guide researchers, show them the most important research directions, problems and milestones in a rigorous and quantitative way. The mathematical creativity, the progress of mathematics itself and the goal of mathematics would be defined as mathematical objects. Such an advance of the Pythagorean process may generate new encodings and metalanguages for mathematical statements and proofs. Its successes in pure mathematics may be transferred to other sciences through applied mathematics and thus mathematics one more time may play a decisive role in human history.

Computers or their future descendants will be eventually used to perform mathematical research therefore we need to create theories which would interpret and model human mathematical thinking using mathematical objects which can be processed by computers, reduce mathematical goal setting and creative theorem proving to computation, define the goal of mathematics as a computational result. It may be impossible to change human thinking but it may be realistic to organize and emulate a mathematical research process which would be performed by computers.

There may already exist scattered examples which are known to experts and the Pythagorean process may proceed in the proposed direction spontaneously. Ne-
vertheless relevant results and examples should be integrated into a single project. Even if the proposed coordinatization direction and related research projects are not considered as successful the related work may have partial successes - it may generate nontrivial mathematical results, new encodings and new thinking features and philosophy.

1.3. Applications. A mathematically sound method for measuring value or complexity of mathematical results would also allow to set rigorous standards for research publications in indexed and internationally accepted journals and other information depositories. Currently mathematical results and authors are evaluated by the number of references to these results in other mathematical texts - the number of other publications where authors have mentioned these results. There are no strict widely accepted guidelines determining referencing rules and in many cases it is not clear whether the reference or its absence is adequate. Such an evaluation method of results and merits can not be considered justified in mathematics which is the very source and center of the culture of unbiased logical reasoning and numerical analysis. In the author’s opinion, it can not be considered justified in other sciences either but this is not within the scope of this paper. A rigorous evaluation method based on mathematical analysis of results and techniques must be found. Mathematics is getting broader and deeper - the number of mathematical research results is steadily increasing therefore rigorous content measuring and research guidance techniques are needed. A mathematically justified content evaluation method would allow to establish really valuable mathematical results, proof methods and research directions, to measure and classify research results by their creativity. Such a method could be thought as an advance of the Pythagorean process into the domain of research results. If such an evaluation method is accepted by the mathematical community then mathematicians would be judged in a justified and uniform way.

Rigorous measuring methods of mathematical concepts, implications and proofs may also have an impact on education. A theory for measuring difficulty of mathematical concepts, learning activities such as computations, logical implications and proofs may be useful in planning education and designing study courses and programs. It could provide educators with a guidance for establishing goals and contents of study programs, standards for mathematical texts. Such a theory could be combined with other methods for measuring difficulty of mathematical concepts and problem solving activities, see Daugulis [2], Jentzen [7].

2. Main research and application directions.

2.1. Coordinatization of implications and proofs. Logical implications and computations are the basic components of any mathematical activity.
In recent decades there has been a considerable progress in measuring computations - computability theory and computational complexity theory are the branches of mathematics that study and classify computations. The computability theory is being developed since 1930s. Computation process has been understood as a sequence of elementary steps which are performed according to a program. This approach is the basis of both theoretical computer science and practical software development. A number of simply formulated undecidable problems have been discovered and studied.

The computational complexity theory became an active research area in the 1960s with the beginning of the widespread computer use. The well known $P \neq NP$ problem has been recognized as one of the main unsolved mathematical problems.

Logical foundations and consistency of mathematics have been intensively studied since the design of the Hilbert’s program, see Simpson [13]. Decidability of mathematical statements and inherent limitations of axiomatic mathematical systems have been investigated since Gödel, see Gödel [6] and Davis [4]. The first Gödel incompleteness theorem is obtained using arithmetization - encoding of Peano axioms using natural numbers. It is essentially linked to the fact that there are sets having cardinality higher than the cardinality of the set of natural numbers. Thus Gödel results are examples of successful instances of coordinatization and modelling of a mathematical theory and language: encoding of mathematical statements, modelling the set of mathematical statements using set theory and interpreting set-theoretic results in terms of statements and proofs.

Mathematical structure, complexity of mathematical proofs and precise value of mathematical results have not been paid adequate amount of attention of wide mathematical community. Automated theorem proving of first-order and second-order logic statements still does not substitute the human intellect. Apparently nontrivial mathematical implication making, thinking and creativity are considered to be areas which can not be mathematically modelled and computerized. Another reason for this may be that the natural limit of human intelligence has been reached - human brain can not operate at such an abstraction level. This limit will be reached in any case at some point in history, we must prepare for it.

Proofs of mathematical statements are sequences and, more generally, networks of logical implications. Therefore one approach to the study of proofs would be to study relatively simple logical implications and their networks. Research may also be needed to determine right definitions of irreducible implications, various types of implications and their linkings, embeddings of the objects corresponding to implications in suitable ambient spaces, definitions of creativity. The concept of proof may be generalized for undecidable statements. For such statements an analogue of proof may be an infinite process converging in a right sense and infinitary logic may need to be used.
Logical implications can be defined as operations on logical predicates in first-order or higher-order logic using logical connectives, especially the material condition connective $\Rightarrow$. Given two predicates $P(x)$ and $Q(x)$ defined for all $x \in X$ we say that $P$ implies $Q$ ($P \rightarrow Q$) provided

$$\bigwedge_{x \in X} \left( P(x) \Rightarrow Q(x) \right) = \text{true}.$$  

The support $\text{supp}(A)$ of a predicate $A$ may be defined as the set of $A$ argument values $x$ for which $A(x) = \text{true}$, thus $\text{supp}(A) \subseteq X$. Validity of a predicate implication $P \rightarrow Q$ is equivalent to the set-theoretic inclusion of the support of $P(x)$ into the support of $Q(x)$: $P \rightarrow Q$ is a true statement if and only if $\text{supp}(P) \subseteq \text{supp}(Q)$. We could try to coordinatise the implication $P \rightarrow Q$ by set-theoretical, combinatorial, geometrical, topological and computational properties of the sets $\text{supp}(P)$ and $\text{supp}(Q)$ such as

1) absolute and relative sizes and shapes of $\text{supp}(P)$, $\text{supp}(Q)$ and $\text{supp}(Q) \setminus \text{supp}(P)$,
2) properties of the boundaries of $\text{supp}(P)$ and $\text{supp}(Q)$. We conjecture that

1) the implication $P \rightarrow Q$ can be considered easy if $\text{supp}(P)$ is a relatively small, e.g. low-dimensional, subset of $\text{supp}(Q)$;
2) implications $P \rightarrow Q_1$ and $P \rightarrow Q_2$ can be considered distinct in a proper sense if $(\text{supp}(Q_1) \cap \text{supp}(Q_2)) \setminus \text{supp}(P)$ is relatively small.

Proofs as sequences of implications $P_1 \rightarrow P_2 \rightarrow \ldots \rightarrow P_n$ may be considered as sequences of set-theoretic inclusions $\text{supp}(P_1) \subseteq \text{supp}(P_2) \subseteq \ldots \subseteq \text{supp}(P_n)$.

Coordinatization and measurement of logical implications may also be related or even reduced to computational complexity if computations are involved determining the inclusion $\text{supp}(P) \subseteq \text{supp}(Q)$.

Viewing proofs as directed paths in a proof graph or other ambient structure with edges corresponding to simple implications we can try to use the graph-theoretical or topological intuition describing properties of proofs.

Additional idea is to generalize implications, to define other binary relations in statement sets or consider weighted graphs. In terms of generalized implications standard implications would be their special case. Given two predicates $P(x)$ un $Q(x)$ we can consider another properties of sets $\text{supp}(P)$ and $\text{supp}(Q)$ (instead of inclusion) for this purpose. For example, we can define that $P$ almost implies $Q$ if $\text{supp}(P) \setminus \text{supp}(Q)$ is relatively small or simple in a suitable sense.

**Example 2.1.** We give a candidate definition for irreducible implications in the case of propositional logic. Suppose $p(X_1, \ldots, X_n)$ and $q(X_1, \ldots, X_n)$ are formulae in propositional Boolean variables $X_1, \ldots, X_n$ and the implication $p(X_1, \ldots, X_n) \rightarrow q(X_1, \ldots, X_n)$ is true. We call the implication $p(X_1, \ldots, X_n) \rightarrow q(X_1, \ldots, X_n)$ irreducible if the full disjunctive normal form (DNF) of $q$ has exactly one more disjunct-
tive term than the full DNF of $p$. The implication $p \rightarrow q$ is not a composition of two noninvertible implications.

2.2. Modelling approaches.

2.2.1. Graph-theoretical modelling of mathematical theories. As we noted in a previous section a mathematical theory can be interpreted as a directed graph corresponding to the implication relation which we call proof graph $\Pi = (\Sigma, \Lambda)$ with vertices in the set $\Sigma$ being statements (which are not interpreted as implications) and directed edges in the set $\Lambda$ being relatively simple logical implications. As we noted above the implication concept may need to be generalized thus $\Lambda$ may be the edge set of a generalized proof graph corresponding to a suitable binary relation, in this paper we consider mainly proof graphs with edges corresponding to classical implications. Graph theory, see [5], can be considered as a candidate theory whose concepts and methods may be used to coordinatise, measure, compare and visualize proofs and mathematical theories. Graph theory - the visualisation of binary relations has shown itself to be useful in many areas of mathematics such as linear algebra or group theory as well as in modelling for a large number of application areas. Further in this subsection we give a few examples of graph-theoretical considerations which may be useful in the Pythagorean process.

Strongly connected components of the proof graph. Existence of a directed cycle of implications with the vertex set $P_1, P_2, ..., P_m$ means that all the corresponding statements $P_i$ are logically equivalent, this should remind the reader a well known theorem proving technique. Therefore a binary logical equivalence relation may be defined in the vertex set of the proof graph. Strongly connected components of the proof graph can be identified with logical equivalence classes of statements. Edges between vertices in different components should correspond to implications which are logically noninvertible. The corresponding acyclic quotient graph of the proof graph could be studied. Unidirectional connected components of proof graphs could also be studied.

Metric properties of the proof graph. Assume that any edge of a proof graph $\Pi$ is given a weight which measures the complexity or some other correctly defined property of the corresponding implication. In the simplest naive cases weights could be positive numbers but other weight sets can not be excluded from consideration. Assume that we are given a directed path between two vertices $P$ and $Q$ having edges $e_1, e_2, ..., e_n$ with weights $w_1, w_2, ..., w_n$ which corresponds to a proof $P \rightarrow Q$. Complexity or other measure of the proof could be defined as an appropriate function of weights $w_1, w_2, ..., w_n$, for example, the sum $w_1 + w_2 + ... + w_n$. Most likely, more complex weight functions dictated by mathematical logic will be used to describe
and classify implications and proofs. For example, a weight function may assign each edge the corresponding implication type. Having a proof graph invariant which would correspond to proof weight or metric we could investigate problems such as, for example, the problem of finding all statements within a fixed distance from a given statement or axiom.

**Vertices and edges with special connectivity properties.** It may happen that a vertex \( v \in \Sigma \) in a proof graph \( \Pi = (\Sigma, \Lambda) \) has special properties with respect to connectivity, e.g. the removal of \( v \) increases the number of strongly connected components, makes the modified graph disconnected etc. Vertices of the proof graph may be studied with respect to this property, e.g. all cutvertices and their generalizations for higher order connectivity may be found etc. Similarly, an edge \( e \) corresponding to a certain logical implication such as an induction argument or its weight \( w(e) \) may have special properties, e.g. all paths running between two vertices in certain areas of the graph may contain \( e \) or have an edge having weight \( w(e) \). Edges with these special properties also may be studied.

**Path systems.** Different paths between vertices \( P \) and \( Q \) represent different proofs between the corresponding statements. Having fixed vertices \( P \) and \( Q \) we can study all \((P, Q)\)-paths, e.g. we can pose the problem of finding all \((P, Q)\)-proofs up to a certain equivalence relation. We can also try to find vertices with special properties, e.g. vertices which are in more than one \((P, Q)\)-path. In topological models for proof spaces topological ideas such as homotopy classes of path systems and homology-type invariants can not be excluded from consideration.

**Shortest paths.** Given two statements \( P \) and \( Q \) in a proof graph we could look for \((P, Q)\)-paths with some special or extremal properties such as the paths having minimal weight. That would correspond to finding \((P, Q)\)-proofs with some special properties, for example, the proof of minimal complexity.

**Spanning subgraphs of the proof graph.** A right interpretation of a spanning subgraph of the proof graph having extremal properties would correspond to a special system of proofs, for example, a minimal system of proofs of all statements of the mathematical theory. Analogues of spanning distance trees and spanning normal trees may be useful, for definitions see Diestel [2].

**Sources and sinks.** Strong equivalence classes of vertices of the proof graph having no incoming or outgoing edges can be interpreted as axioms and terminal statements, respectively.
Graph homomorphisms. Given two proof graphs we can consider maps between them which preserve desired graph properties similarly to graph homomorphisms or isomorphisms. Such maps, perhaps linked with category-theoretic constructions, would allow to define maps between corresponding theories, to compare and classify theories, construct mathematics as a single object.

Line graphs. Various derived graphs of the proof graph such as the line graph can be considered and studied. Given a proof graph $\Pi = (\Sigma, \Lambda)$ we can construct its line graph $\Pi' = (\Lambda, \Lambda')$ with elements of $\Lambda$ being directed incident in $\Pi'$ provided they are directed adjacent in $\Pi$.

Proof hypergraphs. We may need to consider sequences of adjacent implications which can lead to a proof hypergraph concept. For example, all pairs $f, g$ of directed adjacent implications $P \xrightarrow{f} Q \xrightarrow{g} R$ define a ternary relation containing all ordered triples of form $(P, Q, R)$, this relation corresponds to a 3-uniform oriented hypergraph.

2.2.2. An algebraic approach to modelling of implications and proofs. A mathematical theory can also be interpreted as an algebraic structure as follows. Given two directed adjacent implications $f : P \rightarrow Q$ and $g : Q \rightarrow R$ their composition $g \circ f : P \rightarrow R$ is an implication. The composition of implications can be interpreted as a binary associative operation on the set of implications. Additionally the operation has to be defined for nonadjacent implications. The implication set $\Lambda$ thus has a natural monoid structure $(\Lambda, \circ)$, algebraic questions may be asked and algebraic methods may be used to study $\Lambda$. For example, submonoids, ideals, congruence relations and quotient structures of $\Lambda$ could be studied and interpreted. In this approach we also may consider generalizations of the implication notion.

2.2.3. A topological approach to modelling of implications and proofs. A mathematical theory $(\Sigma, \Lambda)$ can also be endowed a topological space structure as follows. We start with noting that the implication binary relation $\rightarrow$ is a preorder relation since it is reflexive and transitive. We can view the implication relation as a specialization preorder for the Alexandrov topology $\tau$ on $\Sigma$ corresponding to $\leftarrow$: the open sets for $\tau$ are the upper sets with respect to the relation $\leftarrow$. We remind the reader that a set $U$ is an upper set with respect to $\leftarrow$ provided $Q \in U$ and $P \rightarrow Q$ implies $P \in U$. Thus we can investigate the given mathematical theory via the topology $\tau$ with respect to standard problems of general and algebraic topology such as interpretation of continuity or homology invariants.

2.3. Research guidance and requirements for mathematical texts. Mathematical results in form of correct proofs are described in research papers
and other documents such as monographs. Research problems and new mathematical objects are often insufficiently motivated. Some mathematicians seem to value a result just because it describes an object, a property or a case which has not been described before, this resembles publishing a computation result just because it has not been published before and should not be accepted in mathematics. Many mathematicians seem to prefer research-like activities on insignificant problems or cases which would guarantee a publishable result instead of working on important hard problems which may be hopeless. Since the research merit of a mathematician is effectively determined as a weighted sum of numbers representing her/his publications in indexed journals there are mathematical publications in these journals which seem to have been published only to increase the number of publications of their authors. There is no justified method to determine the value and the nontriviality/creativity of a mathematical result as a mathematical object. An advance of the Pythagorean process is needed.

The following features of mathematical texts and research organization could be improved by a proof coordinatization and complexity theory. The research organization and the quality of mathematical texts should be considered jointly.

2.3.1. Research guidance. Mathematical research processes, problems and conjectures should be motivated by rigorous analysis based on a proof and statement coordinatization theory. Such a theory would show problems which need to be studied in order to advance the understanding of a given domain, concepts that need to be introduced, proofs that need to be modified, mathematical regress steps that need to be done etc. It would direct and enforce development of mathematics, link various areas of mathematics more closely than the category theory. Advanced mathematical results obtained in the last decades are steadily getting more difficult to produce and understand, the current language and encodings of mathematics may need a change. Therefore the conjectured progress of the Pythagorean process may bring an impetus for mathematics. The progress of mathematics must be defined mathematically and research must be performed in an optimal way. The global mathematical research process should be more clearly than now subdivided into a number of subprocesses corresponding to well defined longterm problems. As examples of such longterm problems one can mention longstanding conjectures in number theory and various classification problems in algebra and topology.

A rigorous research guidance is necessary and its experts should be valued higher than mainstream mathematicians no matter of how skilled these mathematicians are similarly as mathematically educated people are valued higher than mental calculators.
2.3.2. Standardization of mathematical proofs and texts. A standard proof format could be designed for mainstream mathematical publications. For example, the description of a proof would be organized so that the steps are uniform and clearly shown both in terms of implications and intermediate statements, it may be helpful to clearly separate computational steps and proving steps. It can not be excluded that the evolution of mathematical culture will lead to a new formal language for mathematical research results. Standart layout formats for monographs and textbooks showing the structure of mathematical theories and complexities of each proof step would improve the quality of mathematical texts. Mathematical contents of a paper or a monograph should be given both in standard form and in terms of new coordinates, encodings and metalanguages.

2.3.3. Historical research. Existing mathematical texts could be exaustively studied with respect to proof complexity, development of concepts and proof structure. Proof structure of mathematical results which are considered important should be analyzed. More generally, events which have formed mathematics should be identified rigorously even though they are already known. A longterm collective goal in this direction for researchers could be a mapping of the body of most known and reasonably important mathematical results produced by mankind to a suitable mathematical object which would record objects of study, statements and proofs.

2.4. Control of the publishing process. Apart from guiding mathematical research and improving mathematical texts new advances in proof coordinatization and complexity theory could control the flow of published mathematical texts and formation of research merits of mathematicians.

2.4.1. Reviewing and research merit formation. Currently there is a large number of scientific journals, internet based archives and other publishing opportunities. In most journals the value and the originality of a correct new mathematical result submitted to a journal is vaguely determined as an emotional (not being rigorously determined) opinion of one or more reviewers or even just the responsible editor of the journal. Since the amount of published mathematical results is steadily increasing it may be difficult for a reviewer to determine the value of a new result. The appearance of documents like *Code of Practice* created inside the European Mathematical Society, see [3], indicates a necessity of changes in the research result value determination process. Mathematical texts submitted for publication in journals or internet archives should be analyzed much more carefully than it is done now. The research merit of a mathematician is estimated by employers and colleagues as a weighted sum of her/his publications in journals included in citation databases controlled by private companies. It does not make much sense to use mathematics and compute various indices such as the h-index and university ratings using well defined
formulas if scientific results are published based upon emotional opinions of reviewers estimating their value. In the author’s opinion, most "indices" which are supposed to measure scientific productivity and research merits of individuals are examples of wrong and potentially professionally harmful coordinatizations and models serving career, institutional and national needs more than needs of research and scientific progress.

Currently most mathematicians seem to believe that research process can not be mathematically modelled, computerized and, therefore, rigorously measured. In the author’s opinion, this view is wrong, unreasonably conservative, exceedingly focusing on the previous experience, this view of the research process resembles performing arithmetical operations before introduction of a numeral system.

A rigorous proof complexity and value theory would allow to define and determine values of correct submitted or published results and research merits more rigorously and set standards for them. Values of research results may need to be defined locally (considering the state of mathematics in a relatively short period of time) and globally (considering a relatively long period of time). Research result evaluation would be reduced to computation, reviewing would become more time efficient, open, some of its current features such as, for example, anonymity, would become redundant. It may allow to rigorously compare and uniformize different areas, projects and activities of mathematics. Correctness and originality checking may also benefit from a proof coordinatization since proofs will be written in a more formal metalanguage. Information about all known mathematical results could be stored as a single database or similar structure. Results having low value or low proof complexity should be marked as such. Research driven results and career driven results should be separated, the publication system should support only research interests and research oriented people. Although it may be an overdoing mathematicians may need to create an additional theory to determine research merit values of authors of mathematical results. Mathematical culture should change and the mathematical community may need to accept the fact that the value of results and the complexity of proofs can be measured.

The history of the publishing process also should be modelled and analyzed using such a theory.

2.5. Education.

2.5.1. School education. Currently mathematics study programs and courses are designed following internationally accepted standards. National traditions are followed by major nations. Globalization processes such as the olympiad movement are uniformizing mathematical education. Possible advances such as a proof complexity measuring and new encodings used in mathematics could modernise mathematical
studies and make them more effective. Networks of mathematical concepts and skills taught at school should be reviewed and updated to correspond to current advances of computing technologies which are available to children.

Apart from measuring complexity and weights of proofs research could also be conducted to measure how easy or difficult (absolutely or relatively) a definition, an implication, a proof or other mathematical activity is to understand psychologically or perform for learners of various abilities and background. Separate or related learning, teaching and problem solving coordinatization theories may be necessary to reform education and make it optimal, truly differentiated and individualized.

**Ideas for dependence based learning and teaching coordinatization theories.** For any study course or program we can define a directed learning graph $\Gamma$: the vertices are knowledge units such as definitions, facts or skills; a weighted directed edge $a \xrightarrow{w} b$ means that $b$ must be taught after $a$, the learning difficulty is encoded in the edge weight $w$. Graph-theoretical considerations given in 2.2.1 can be modified for learning graphs. Given a learning graph $\Gamma$ we can define for any student $x$ her/his $\Gamma$-profile (**knowledge profile**) $\pi_{x,\Gamma}$: a function from the vertex set $V(\Gamma)$ to a suitable set $K$ which assigns to every knowledge unit $v \in V(\Gamma)$ the level of knowledge $\pi_{x,\Gamma}(v)$ (an element in $K$) the student has with respect to $v$, thus $\pi_{x,\Gamma} \in \text{Fun}(V(\Gamma), K)$, the knowledge profile can also be interpreted as a set of weights for the $\Gamma$-vertex set - the weight $\pi_{x,\Gamma}(v)$ of $v$ describes the knowledge or competence of $v$ which $x$ has. Course goals may be defined in terms of knowledge profiles, these goals may be defined both individually and collectively. For any teaching activity $\tau$ we can consider its impact $f_{\tau}$ on a knowledge profile - $f_{\tau}$ is an endofunction on $\text{Fun}(V(\Gamma), K)$. Apart from serving educational needs theories of this kind may be generalized to describe sociological and time dependent aspects of research and other related activities.

**Ideas for competence based problem solving theories.** Given a problem $P$ of mathematics or some other discipline we can define a $P$-solving graph $\Gamma$ as follows: the vertices are states of the solution process, the edges denote transitions between these states weighted by the skills which are necessary to perform these transitions. Such graph models would allow to investigate typical problems used in a given course or other learning unit, analyze the necessary facts and skills that need to be taught. They also may be helpful teaching nontrivial (olympiad) problem solving skills.

Online computerized tests could be developed and used for these purposes, and experimental results should be compared with theoretical expectations. Computerized tests could be used to measure individual parameters of learners such as mathematical reaction time, short-time and long-time memory, memory access time etc. Tests could also be used to measure how difficult a proof step or a nontrivial mathematical activity
is difficult for an individual. Additionally computerized tests and research could be carried out to study links between encodings of mathematical concepts (including mental encodings - language skills and acquired mentality features) and quality of mathematical thinking. Mental chronometry, see Daugulis [2] and Jensen [7], could be used to estimate learning difficulty and teaching effectiveness.

2.5.2. Higher education. University mathematics study programs can also benefit from a comprehensive proof coordinatization and complexity theory since such a theory may bring a significant revision, innovative structuring and modernization of the body of mathematical knowledge accessible to university students and graduate students. This revision may affect lists of mathematical areas, list of concepts and main facts, course layout designs, theorems and proof techniques. Proofs which are easy to understand or proofs containing typical implications could be collected as a spanning tree for the given theory and the spanning tree proofs could be organized into a study course or program. Algorithms for designing standard (canonical) solutions for problems should be developed and taught.

2.6. Possible future development and some problems.

2.6.1. Next steps. Initiation and development of the proposed proof coordinatization theory, framework and culture may consist of the following steps.

*Phase 1 - development of a mathematical theory.* The main development step clearly will be the development of a rigorous mathematical theory by dedicated researchers. Since mathematics is quite diverse mathematicians of different backgrounds should work in this direction. Logicians may dominate such projects and the researchers should have sufficient background in mathematical logic. We predict that an intensive research process will start after the first few convincing proof coordinatization examples. Research may need to be supervised by mathematical logicians and start with propositional logic and popular first-order theories, in the first stages it may develop independently in various areas such as combinatorics or topology. The theory may not be finished in the foreseeable future therefore this step may not be completed. It makes sense to talk only about the phase of building a sufficient body of knowledge - theory, algorithms, examples which would be sufficient for working mathematicians.

*Phase 2 - spreading of the theory to the mathematical community and establishment of a new culture.* After a sufficient amount of the new theory is developed for the main areas of mathematics the theory will start to spread to the whole mathematical community. This spreading will be driven by the professional coordinatization urge which is characteristic for mathematicians. This step will be completed when most
mathematicians will use the new theory as an indispensable research tool.

Phase 3 - adoption of the new culture by scientific and engineering communities. If the proof coordinatization framework is successful in mathematics it will continue to spread further to other sciences and engineering domains which use applied mathematics and mathematical modelling. First of all it may spread to sciences such as computer science, physics and linguistics since these are the sciences most closely related to mathematical modelling and computations. A possible change of the mathematical culture related to rigorous determination of values of research results and changes in education may lead to similar changes in other sciences and disciplines.

2.6.2. Epistemological problems. A successful progress of the project discussed in this paper may allow and need philosophical interpretations.

Methodological Cartesian scepticism. A desire to model and coordinatize mathematical implications, proofs and creativity represents an instance of methodological scepticism in mathematics which may be close in spirit to Cartesian doubt. Everyday mental experiences make most people believe that implication making is basic mental activity and creative mathematical thinking can not be mechanised, the proposed research project would challenge these beliefs. Researchers involved in this project will have to keep this in mind.

A mathematical regress step. Coordinatization of mathematical implications and creativity involves mappings of mathematical proofs to simpler mathematical objects. As it was noted above this may also be viewed as a generalization of regress steps in the standard philosophical sense. In this program logical implications which are creative mathematical acts are supposed to be explained (justified, in philosophical language) in terms of simpler mathematical objects. Reduction of an applied or pure mathematical problem or a model to a simple mathematical object often happens in mathematics. Moreover, mathematical modelling, the whole applied mathematics and a part of the Pythagorean process itself philosophically should be thought as a collection of regress steps. Mathematics should be thought as the common regress framework for all sciences. It may be too early to speculate about philosophical problems related to the regress step discussed here such as the (infinite) diallelus problem or the corresponding case of the Münchhausen trilemma, see [14]. The Münchhausen trilemma case (i.e. whether the proposed intra-mathematical regress is cyclic, infinite non-cyclic or finite) seems to be an important problem.

2.6.3. Methodological problems. The main methodological problem of the proposed advance of the Pythagorean process is the creation of adequate mappings of logical implications and mathematical theories to simpler mathematical structures.
A need to create a new language and encodings is another methodological problem. The transfer of the new mathematical theories to other sciences may need additional research in modelling and related applied mathematics.

2.6.4. Practical problems. In the theory development step - Phase 1 the main practical problem may be finding enough employment opportunities for mathematicians working full time in this direction. Employers and colleagues may be sceptical to support activities in a direction with unclear perspectives and few results. Initially it may be realistic to organize a small number of research groups in mathematical logic only in leading universities of major countries. Organization, management and support of an innovative distributed research project may be another practical problem.

In Phase 2 the spreading of the new culture to wider mathematical community may encounter resistance from conservative mathematicians and educators. The strongest resistance will be directed against possible practical introduction of research result value measuring practices and related publishing flow control. Values of research results which form research merits for many people may be lower than expected according to new theories, this will generate negative opinions and resistance. Innovations in education may also meet resistance. There must be even more resistance during Phase 3. At these stages the process will be most enthusiastically supported by relatively young researchers.

3. Conclusion. We have given a number of arguments which justify, encourage and describe possible research on comprehensive coordinatization of mathematical implications and proofs, or more generally, mathematical results. The main argument is a possibility to formalize, map into simpler mathematical objects and measure mathematical creativity, to make nontrivial and creative mathematical theorem proving a computation. Another argument is a possibility to rigorously measure mathematical results and to guide the mathematical research in a rigorous and optimal way. The mathematical culture would greatly benefit from rigorous standards for mathematical research publications and other texts. New encoding paradigms for the mathematical language and modernized mathematical education may also be generated by introducing coordinates in the space of proofs.

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