FAST-SLOW PARTIALLY HYPERBOLIC SYSTEMS:
BEYOND AVERAGING.
PART I (LIMIT THEOREMS)

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Abstract. We prove several limit theorems for a simple class of partially hyperbolic fast-slow systems. We start with some well known results on averaging, then we give a substantial refinement of known large (and moderate) deviation results and conclude with a completely new result (a local limit theorem) on the distribution of the process determined by the fluctuations around the average. The method of proof is based on a mixture of standard pairs and Transfer Operators that we expect to be applicable in a much wider generality.

1. Introduction

In this paper we analyze various limit theorems for a class of partially hyperbolic systems of the fast-slow type. Such systems are very similar to the ones studied by Dolgopyat in [18]: in such paper the fast variables are driven by an hyperbolic diffeomorphism or flow (see also [3, 33, 38, 23] for related results), here we consider the case in which they are driven by an expanding map. Notwithstanding the fact that we are not aware of an explicit treatment of the latter case, the difference is not so relevant to justify, by itself, a paper devoted to it. In fact, we chose to deal with one dimensional expanding maps only to simplify the exposition. The point here is that, on the one hand, we propose a different approach and, more importantly, on the other hand, we show that by such an approach it is possible to obtain a much sharper results: a sharp Moderate and Large Deviation Theorem and a Local Limit Theorem. To the best of our knowledge, this is the first time a precise rate function is computed for moderate deviations and a local limit type theorem is obtained for a deterministic evolution converging to a diffusion process. Admittedly, the present is not the most general case one would like to deal with, it is just a primer. However, it shows that local limit results are attainable with an appropriate combination/refinement of present days techniques (see the discussion below on how general our approach really is).

The importance of local limit theorems hardly needs to be emphasized but, for the skeptical reader, it can be nicely illustrated by the companion paper [11]. Indeed, in such a paper the present moderate deviations and local limit results are used in a fundamental manner to obtain sharps results on the statistical properties.

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(e.g. existence and properties of the SRB measure, decay of correlations, metastability etc... for the same class of systems but for a fixed rate between the speeds of the slow and fast motions, not just in the limit of the rate going to zero. This provides a new class of partially hyperbolic systems for which precise quantitative statistical properties can be established (the only other cases, to our knowledge, are flows, group extensions and some total automorphisms). In addition, contrary to the other cases, our results apply to an open set of systems (in the $C^4$ topology).

For partially hyperbolic fast-slow systems several results concerning limit laws have already been obtained. In [3, 32] it is proven that the motion converges in probability to the motion determined by the averaged equation (morally a law of large numbers). In [18] there are important results on the fluctuations around the average. In particular, both large deviations and converges in law to a diffusion for the fluctuation field (morally a central limit theorem) are obtained. In [33] some more general and precise large deviation results are obtained, yet they do not suffice for the present purposes as we need explicit uniform estimates allowing to treat moderate deviations as well. To obtain such a result it is necessary to compute the rate function with a precision considerably higher than the $o(1)$ achieved in [33]. Here we present independent proofs of the above facts (or, better, of the aforementioned substantial refinements of the above facts) and, most importantly, we make a further step forward by addressing the issue of the local central limit theorem, a result out of the reach of all previous approaches.

The lesson learned from [18] is that the standard pair technique is the best suited to investigate partially hyperbolic systems. Nevertheless, in the uniformly hyperbolic case, techniques based on the study of the spectrum of the Transfer Operator are usually much more efficient. It is then tempting to try to mix the two points of view as much as possible. This is one of the goals of our work and, to simplify matters, we attempt it in the simplest possible setting (one dimensional expanding maps). Nevertheless, we like to remark that extending the present results to hyperbolic maps or flows is just a technical, not a conceptual problem. Indeed, till the recent past the use of Transfer Operators was limited to the expanding case (or could be applied only after coding the system via Markov partitions, greatly reducing the effectiveness of the method). Yet, recently, starting with [8] and reaching maturity with [25, 5, 26, 37, 43, 21, 14, 15, 19, 20, 22], it has been clarified how to fully exploit the power of Transfer Operators in the hyperbolic, partially hyperbolic and piecewise smooth setting. Accordingly, it is now totally reasonable to expect that any proof developed in the expanding case can be extended to the hyperbolic one, whereby making the following arguments of a much more general interest.

The structure of the paper is as follows: we first describe the class of systems we are interest in, and state precisely the main results. Then we discuss in detail the standard pair technology. This must be done with care as we will need higher smoothness as well as complex standard pairs, which have not been previously considered. In Section 4.1 we show that, for relatively short times, our dynamics can be shadowed by much simpler ones and we provide precise error estimates. In the following section we use the tools so far introduced to establish an averaging theorem.

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1 In particular, as far as we know, it represents the most efficient way to “condition” with respect to the past in a field (deterministic systems) where conditioning poses obvious conceptual problems.
As already explained this result is not new, but it serves the purpose of illustrating the generals strategy to the reader and the proof contains some fact needed in the following arguments. Section 6 establishes the large and moderate deviations of our dynamics from the average. We compute with unprecedented precision the rate function of the large deviation principle. We stop short of providing a full large and moderate deviations theory only to keep the exposition simple and since it is not needed for our later purposes. Nevertheless, we improve considerably on known results. Finally in Section 7 we build on the previous work and prove a local limit result for our dynamics. The proof is a bit lengthy but it follows the usual approach: compute the Fourier transform of the distribution. To do so we divide the time interval in shorter blocks (this is done in Section 8), then estimate carefully the contribution of each block (this is done in Sections 9 and 10) and we conclude by combining together the contributions of the single blocks (done in section 11). Some fundamental technical tools needed to perform such computations are detailed in the appendices. Appendix A contains a manifold of results on Transfer Operators and their perturbation theory. In fact, not only it collects, for the reader convenience, many results scattered in the literature, but also provides some new results. In addition, it contains a discussion of the genericity of various conditions used in the paper including the, to us, unexpected results that for smooth maps aperiodicity and not being cohomologous to a constant are equivalent. Appendix B contains the computation of the moment generating function needed to establish our large deviation results. Finally, Appendix C provides a detailed discussion of transfer operators associated to semiflows that, although essentially present in the literature, was not in the form needed for our needs (in particular we need uniform results for a one parameter family of systems).
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Notation. Through the paper we will use $C_\#$ and $c_\#$ to designate a generic constant, depending only on our dynamical system, whose value can change form an occurrence to the next even in the same line. We will use $C_{a,b,...}$ to designate generic constants that depend on the quantities $a,b,...$.

Also we use the notation $o^*(\cdot)$ to mean a function of $\varepsilon$ that is $o(\varepsilon^{\alpha}\cdot)$ for some (arbitrarily small) $\alpha > 0$. 
2. The system and the results

For $\varepsilon > 0$ let us consider the map $F_\varepsilon \in C^4(T^2, T^2)$ defined by

$$F_\varepsilon(x, \theta) = (f(x, \theta), \theta + \varepsilon \omega(x, \theta)).$$

(2.1)

We assume that the $f_\theta = f(\cdot, \theta)$ are uniformly expanding, i.e. there exists a real number $\lambda > 1$ so that:

$$\inf_{(x, \theta) \in T^2} \partial_x f(x, \theta) \geq \lambda.$$  

(2.2)

This fact is well known to imply that each $f(\cdot, \theta)$ has a unique invariant measure absolutely continuous with respect to Lebesgue (often called SRB measure), which we denote by $\mu_\theta$. Also, we assume that, for all $\theta \in \mathbb{T}$, $\omega$ is not cohomologous to a constant function, i.e. it cannot be written as

$$\omega(x, \theta) = g_\theta(f(x, \theta)) - g_\theta(x) + a_\theta$$

(2.3)

for functions $g_\theta$ and constants $a_\theta$. Note that, (2.3) can hold only if for any invariant probability measure $\nu$ of $f_\theta$, $\nu(\omega(\cdot, \theta)) = \nu(a_\theta) = a_\theta$. In particular, if $\omega$ has different averages along two different periodic orbits, then (2.3) cannot hold. It is then fairly easy to check such a condition. Also it should be remarked that the smoothness of the dynamics provides some kind of rigidity which automatically implies aperiodicity (see Lemma A.18). In particular note that our assumption holds generically (see Appendix A.5 for a more complete discussion of these issues).

Given $(x, \theta) \in T^2$, let us define the trajectory $(x_k, \theta_k) = F_k \varepsilon(x, \theta)$ for any $k \in \mathbb{N}$.

Here we describe a sequence of increasingly sharper results on the behavior of the dynamics for times of order $\varepsilon^{-1/2}$. We start with well known facts, but we provide complete proofs both for the reader convenience and because they are a necessary preliminary to tackle our main results.

If we take the formal average with respect to the SRB measure of (2.1) we obtain the following first order differential equation

$$\frac{d\bar{\theta}}{dt} = \bar{\omega}(\bar{\theta}) \quad \bar{\theta}(0) = \theta_0,$$  

(2.4)

where $\bar{\omega}(\theta) = \mu_\theta(\omega(\cdot, \theta))$. For future use, let us also define the function $\hat{\omega}(x, \theta) = \omega(x, \theta) - \bar{\omega}(\theta)$. Note that, by the differentiability of $\mu_\theta$ with respect to $\theta$ (see [25, Section 8] and Section A), the above equation has a unique solution, which we denote by $\theta(t, \theta_0)$. We expect $\theta_k$ to be close to $\theta(\varepsilon k, \theta_0)$ at least for times of order $\varepsilon^{-1}$. In fact, it is natural to expect more: let $d \in \mathbb{N}$, $B \in C^2(T^2, \mathbb{R}^{d-1})$, and fix $\zeta_0 = 0$; for any $n \in \mathbb{N}$ let us define

$$\zeta_{n+1} = \varepsilon B(x_n, \theta_n) + \zeta_n.$$  

(2.5)

This equation describes the evolution of a passive quantity and we will need to consider such a situation in the companion paper [11]. Then the motion of $\zeta_k$ should be close to $\zeta(\varepsilon k, \theta_0)$, which denotes the unique solution of the differential equation

$$\dot{\zeta}(t, \theta_0) = B(\theta(t, \theta_0)) \quad \zeta(0, \theta_0) = 0,$$  

(2.6)

In some cases it is also possible to obtain information for times of the order $\varepsilon^{-2}$ (see [18]). Yet, as far as we currently see, not of the quantitative type we are interested in.
where we introduced the averaged function $\bar{B}(\theta) = \mu_\theta(B(\cdot, \theta))$. It is then convenient to consider the variables $x, z = (\theta, \zeta) \in \mathbb{R}^d$ (for convenience we have lifted $\theta$ to its universal cover) whose evolution is given by the map
\begin{equation}
F_\epsilon(x, z) = (f(x, \theta), \theta + \epsilon \omega(x, \theta), \zeta + \epsilon B(x, \theta)); \tag{2.7}
\end{equation}
again we set $(x_k, z_k) = F_\epsilon^k(x, z)$, for $k \in \mathbb{N}$. A first relevant fact is that the above averaging approximation can be justified rigorously. These type of results are well known and go back, at least, to Anosov \[.\] Let
\begin{equation}
z_\epsilon(t) = (\theta_\epsilon(t), \zeta_\epsilon(t)) = z_{\lfloor t \epsilon^{-1} \rfloor + \lfloor t \epsilon^{-1} \rfloor} + (t \epsilon^{-1} - \lfloor t \epsilon^{-1} \rfloor)[z_{\lfloor t \epsilon^{-1} \rfloor + 1} - z_{\lfloor t \epsilon^{-1} \rfloor}]. \tag{2.8}
\end{equation}
Then $z_\epsilon \in C^0([0, T], \mathbb{R}^d)$, and we can consider it as random variables with values in the path space $C^0([0, T], \mathbb{R}^d)$, the randomness being determined by the distribution of the initial condition.

**Theorem 2.1 (Averaging).** Let $\theta_0 \in T^1$ and $x_0$ be distributed according to a smooth distribution $\mu_0$; then for all $T > 0$:
\[
\lim_{\epsilon \to 0} z_\epsilon(t) = \bar{z}(t, \theta_0) = (\bar{\theta}(t, \theta_0), \bar{\zeta}(t, \theta_0))
\]
where the limit is in probability with respect to the measure $\mu_0$ and the uniform topology in $C^0([0, T], \mathbb{R}^d)$.

The proof is more or less standard. We provide it in Section 5 for reader’s convenience. Indeed, we provide a proof that contains, in an elementary form, some of the ideas that will be instrumental in the following. The reader not very familiar with the transfer operator or standard pairs technology is advised to read Sections 3 and 5 first.

For the following let $d \in \mathbb{N}$ and $A = (A_1, \ldots, A_d) \in C^2(T^2, \mathbb{R}^d)$, with $A_1(x, \theta) = \omega(x, \theta)$ and $A_{i+1} = B_i$ for $i \in \{1, \ldots, d - 1\}$.

The next natural question concerns the behavior of deviations from the average. To this end it is more convenient to view $C^0([0, T], \mathbb{R}^d)$ as the fundamental probability space. Then the $\sigma$-algebra is just the Borel $\sigma$-algebra and the probability measure is $\mathbb{P}_{A, \epsilon} = (z_\epsilon)_*, \mu_0$, that is the law of $z_\epsilon$ under $\mu_0$. Note that the paths $z_\epsilon$ are all Lipschitz with Lipschitz constant bounded by $\|A\|_{C^0}$. Thus each $\mathbb{P}_{A, \epsilon}$ is supported on a compact set.

To obtain complete results we need some extra hypothesis. Also assume that, for each $\sigma \in \mathbb{R}^d$, $\langle \sigma, A \rangle$ is not cohomologous to a constant (this implies, in particular, (2.3)). Note that such a condition is implied by the existence of $d + 1$ periodic orbits for which the differences of the averages of $A$ spans $\mathbb{R}^d$. Hence our condition is generic.

We prove an upper bound for the probability of large and moderate deviations. The result is much sharper than the one contained in [33]. It is of a more quantitative nature (in the spirit of [18] where the rate function is only estimated near zero and in a much rougher manner). In particular, it provides bounds on the rate function that allow to treat both large and moderate deviations for all $\epsilon$ small enough (not just asymptotically). We refrain from developing a more complete theory because on the one hand it would not change substantially the result, on the other
hand it would increase the length of an already long paper and, finally, since the
results presented here already more than suffices for our purposes (i.e. both for our
later use and to pedagogically illustrates few ideas used in the following). In fact,
the next Theorem does not contain even the full force of what we prove in Section 6,
nevertheless its statement requires already quite bit of preliminary notations. We
advise the reader that wants a quick, but sub-optimal, idea of the type of results
that can be obtained to jump directly to the Corollaries 2.4, 2.5 and 2.6.

For each interval \( J \subset C^0([0, T], \mathbb{R}^d) \) and path \( \gamma \in C^0([0, T], \mathbb{R}^d) \) we will systematically use the notation

\[
B_J(\gamma, r) = \left\{ \gamma \in C^0([0, T], \mathbb{R}^d) : \sup_{t \in J} \| \gamma(t) - \bar{\gamma}(t) \| < r \right\},
\]

to designate a ball of radius \( r \) subordinated to the interval \( J \).

The first object we need, as in any respectable large deviation theory, are lower
semicontinuous rate functions \( \mathcal{F}_\theta : C^0([0, T], \mathbb{R}^d) \to \mathbb{R}_+ \). The precise
properties are specified in Section 6.1, here we just summarize the main facts.

The effective domain \( \mathbb{D}_\theta = \{ \gamma \in C^0([0, T], \mathbb{R}^d) \mid \mathcal{F}_\theta (\gamma) < \infty \} \) consists of Lipschitz paths such that \( \gamma(0) = 0 \) and, for almost all \( t \in [0, T] \), there exists an invariant measure \( \nu \) of the map \( f(\cdot, \gamma_1(t) + \theta) \) such that \( \gamma'(t) = \nu(A(\cdot, \gamma_1(t) + \theta)) \). In
particular, this implies \( ||\gamma'||_\infty \leq ||A||_\infty \). Note that \( \mathbb{D}_\theta \) can be determined with
arbitrary precision by studying the periodic orbits of the dynamics (see Remark 6.9
for details). Also let

\[
\Sigma^2(\theta) = \mu_\theta \left( \hat{A}(\cdot, \theta) \otimes \hat{A}(\cdot, \theta) \right) + 2 \sum_{m=1}^{\infty} \mu_\theta \left( \hat{A}(f^m_\theta(\cdot, \theta) \otimes \hat{A}(\cdot, \theta) \right),
\]

where \( \hat{A}(\theta) = \mu_\theta(A) \) and \( \hat{A} = A - \bar{A} \). It turns out that \( \Sigma \in C^1(\mathbb{T}, M_d) \), where \( M_d \)
is the space of \( d \times d \) symmetric non negative matrices. If \( \Sigma \) is invertible, then, for
each \( \theta_0 \in \mathbb{T} \) and \( \gamma \in \mathbb{D}_{\theta_0} \), setting \( \theta^\ast(t) = \gamma(t) + \theta_0 \),

\[
\mathcal{F}_{\theta_0}(\gamma) = \frac{1}{2} \int_0^T \left( \gamma'(s) - \hat{A}(\theta^\ast(s)) \Sigma^2(\theta^\ast(s))^{-1} \left[ \gamma'(s) - \hat{A}(\theta^\ast(s)) \right] \right) ds
+ O(||\gamma' - \bar{A} \cdot (\theta^\ast)||_{L^2}),
\]

The above would suffice to state a standard large deviation result, but we are
interested in stating estimates valid for all, sufficiently small, \( \varepsilon \) not just in the
limit \( \varepsilon \to 0 \). To do so the most convenient way is introduce slightly modified rate
functions and to state the result by saying that the probability of a set is controlled
from above by the inf of the rate function on a slightly larger set and form below
from the inf on a slightly smaller set. Unfortunately, to make precise these notions
is a bit tricky, so we ask for the reader patience.

First, for each \( \Delta_\ast > 0 \), we introduce \( \mathcal{F}_{\theta_0, \Delta_\ast} \), which agree with \( \mathcal{F} \) outside a \( \Delta_\ast \)
neighborhood of \( \partial \mathbb{D} \) and so that \( \mathcal{F}_{\theta_0, \Delta_\ast} \leq \mathcal{F}_{\theta_0} \leq \mathcal{F}_{\theta_0}^+ \). By lower semi-continuity
it follows that

\[
\lim_{\Delta_\ast \to 0} \mathcal{F}_{\theta_0, \Delta_\ast} = \mathcal{F}_{\theta_0} \leq \mathcal{F}_{\theta_0}^+ = \lim_{\Delta_\ast \to 0} \mathcal{F}_{\theta_0, \Delta_\ast},
\]

where \( \mathcal{F}_{\theta_0}^+ \) agrees with \( \mathcal{F}_{\theta_0} \) everywhere away from \( \partial \mathbb{D} \) where it has value +\( \infty \).

\footnote{Here \( \gamma(s) \) is the first component of the vector \( \gamma(s) \), the one that corresponds to the \( \theta \) motion.}

\footnote{Essentially \( \mathcal{F}_{\theta_0}^+ = \infty \) in a \( \Delta_\ast \) neighborhood of \( \partial \mathbb{D} \) while \( \mathcal{F}_{\theta_0}^- < \infty \), see Section 6.1
for a precise definition.}
Second, let \( \theta_0 \in T \); let \( R_\pm : C^0([0,T], \mathbb{R}^d) \to \mathbb{R}_{\geq 0} \) be defined by \( R_- = C_\# \varepsilon \frac{\cdot}{\gamma} \) and \( R_+ (\gamma) = \left[ \sqrt{\varepsilon} |\gamma - \bar{\varepsilon}(\cdot, \theta_0)|_{\infty}^{-1} \right]^+ |\gamma - \bar{\varepsilon}(\cdot, \theta_0)|_{\infty} \).

For each \( Q \subset C^0([0,T], \mathbb{R}^d) \) define \( Q^+ = \cup_{\gamma \in Q} B_{0,T}(\gamma, R_+ (\gamma)) \), \( Q^- = \{ \gamma \in Q : B_{0,T}(\gamma, R_-(\gamma)) \subset Q \} \) and \( \varrho (\theta_0, Q) = \inf_{\gamma \in Q} |\gamma - \bar{\varepsilon}(\cdot, \theta_0)|_{\infty} \).

We are now ready to state our first main result.

**Theorem 2.2** (Large and Moderate deviations). Consider the initial condition specified by \( (\theta_0, 0) \in \mathbb{R}^d \), and \( x_0 \) distributed according to a smooth distribution. Let \( C_0 > 1 \) and \( T > 0 \). If \( A(\theta) \) is never cohomologous to a constant (with respect to the dynamics \( f(\cdot, \theta) \)), then \( \Sigma \) is invertible and for all \( \Delta_* > 0 \) and any \( \varepsilon \) small enough (depending on \( \Delta_* \)), for any sequence of events \( Q_\varepsilon \), holds true

\[
\mathbb{P}_{A, \varepsilon}(\{ \varepsilon \in Q_\varepsilon \}) \leq e^{-\varepsilon^{-1} \left[ 1 + C_{\Delta_*} T \varepsilon^{\frac{1}{12}} \right] \inf_{\gamma \in Q_\varepsilon^+} \mathcal{I}_{\theta_0, \Delta_*} (\gamma) - C_T \varepsilon}
\]

and

\[
\mathbb{P}_{A, \varepsilon}(\{ \varepsilon \in Q_\varepsilon \}) \geq e^{-\varepsilon^{-1} \left[ 1 - C_{\Delta_*} T \varepsilon^{\frac{1}{12}} \right] \inf_{\gamma \in Q_\varepsilon^-} \mathcal{I}_{\theta_0, \Delta_*} (\gamma) + C_T \varepsilon}
\]

where the value of the inf is taken to be \( +\infty \) if the set is empty.

The proof can be found in Section 6.6, in fact we will prove the result for more general initial condition, see Remark 6.2.

**Remark 2.3.** Note that, using the results of Section 6 (in particular Lemmata 6.13 and 6.16) one could have stated the theorem for the case of \( T \) depending on \( \varepsilon \), provided \( T > \varepsilon^{\frac{1}{12}} \). This is in fact not necessary, indeed if one wants to study an event in \( C^0([0,C_\# \varepsilon^\alpha], \mathbb{R}^d) \), \( \alpha \in [0,1) \), then it can always be seen as an event in \( C^0([0,T], \mathbb{R}^d) \). One can then check, using (2.10), that the times larger than \( C_\# \varepsilon^\alpha \) do no contribute to the inf as the event contains trajectories for which \( \gamma' = A \) for all \( t \geq C_\# \varepsilon^\alpha \).

The statement of Theorem 2.2, due to its precise quantitative nature, may still feel a bit cumbersome. To help the reader understand its force let us spell out few easy consequences.

**Corollary 2.4** (Large deviations). Given any event \( Q \subset C^0([0,T], \mathbb{R}^d) \), then

\[
- \inf_{\gamma \in Q} \mathcal{I}_{\theta_0} (\gamma) \leq \lim_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_{A, \varepsilon}(Q) \leq \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_{A, \varepsilon}(Q) \leq - \inf_{\gamma \in \overline{Q}} \mathcal{I}_{\theta_0} (\gamma).
\]

**Proof.** If \( \varrho (\theta_0, \overline{Q}) = 0 \), then \( \inf_{\gamma \in \overline{Q}} \mathcal{I}_{\theta_0}^{+ \Delta_*} (\gamma) = 0 \), and the Lemma flows trivially. Otherwise, let \( 2\Delta_* < \varrho (\theta_0, Q) \) and let \( Q^{+2\Delta_*} = \cup_{\gamma \in Q} B(\gamma, 2\Delta_*) \) and \( Q^{-2\Delta_*} = \{ \gamma \in Q \mid B(\gamma, 2\Delta_*) \subset Q \} \). Hence, for \( \varepsilon \) small enough, \( Q^+ \subset Q^{+2\Delta_*} \) and \( Q^{-2\Delta_*} \subset Q^- \). Thus Theorem 2.2 implies

\[
- \inf_{\gamma \in \overline{Q}^-} \mathcal{I}_{\theta_0}^{\pm \Delta_*} (\gamma) \leq \liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_{A, \varepsilon}(Q) \leq \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_{A, \varepsilon}(Q) \leq - \inf_{\gamma \in \overline{Q}^+} \mathcal{I}_{\theta_0}^{\pm \Delta_*} (\gamma)
\]

which, taking the limit for \( \Delta_* \to 0 \), remembering the lower semicontinuity of \( \mathcal{I}_{\theta_0} \) and by the exact definition of \( \mathcal{I}_{\theta_0}^{\pm \Delta_*} \) (see (6.34), where \( \mathcal{I}_{\theta_0}^{\pm \Delta_*} \) is called, more precisely, \( \mathcal{I}_{\theta_0, \Delta_*}^{\pm} \), [0,T]), and (6.11)) implies the statement.

The above is the usual asymptotic large deviation principle, similar to what can be found in [33], yet the finite size version provided by Theorem 2.2 implies much more. Also note that, although the statement of Corollary 2.4 looks very clean, it is not very easy to use it since the inf involved is often very hard to compute, even
for a simple event like $Q = \{ \gamma \in C^0([0, T], \mathbb{R}) \mid \| \gamma(s) - \tilde{z}(s, \theta_0) \| \geq Cs, s \in [0, T] \}$. For not too large deviations, one can get some estimates from the expansion of $J_{\theta_0}$ stated in (2.10). One can be more explicit for moderate deviations.

**Corollary 2.5** (Moderate deviations). Given $T > 0$ and any event $Q \subset C^0([0, T], \mathbb{R}^d)$, setting $Q_{\varepsilon} = \{ \varepsilon^2 \gamma(\cdot) - (1 - \varepsilon^2) \tilde{z}(\cdot, \theta_0) \}_{\gamma \in Q}$, for some $\beta \in (0, \frac{1}{2})$, we have

$$\limsup_{\varepsilon \to 0} \varepsilon^{1-2\beta} \log P_{A,C}(Q_{\varepsilon}) \leq - \inf_{\gamma \in Q} J_{\text{Lin}}(\gamma),$$

where

$$J_{\text{Lin}}(\gamma) = \frac{1}{2} \int_0^T \langle \gamma'(s) - \tilde{A}(\tilde{\theta}(s, \theta_0)), \Sigma(\tilde{\theta}(s, \theta_0))^{-1} [\gamma'(s) - \tilde{A}(\tilde{\theta}(s, \theta_0))] \rangle ds.$$

Moreover, if $\beta > \frac{3}{8}$, then

$$\liminf_{\varepsilon \to 0} \varepsilon^{1-2\beta} \log P_{A,C}(Q_{\varepsilon}) \geq - \inf_{\gamma \in Q} J_{\text{Lin}}(\gamma).$$

**Proof.** Let us start with the upper bound, if $\tilde{z} \in Q$, then the bound on the probability gives one, which is trivially true. Otherwise let $\gamma = \varepsilon^2 \gamma - (1 - \varepsilon^2) \tilde{z}$. For $\varepsilon$ small enough, $R_\varepsilon(\gamma) = C\# \varepsilon^{\frac{3}{8}} + \varepsilon^{\beta}$. Also by (2.10) and the smoothness of $\Sigma$ we have

$$J_{\theta_0,\varepsilon}(\gamma) = J_{\theta_0}(\gamma) = \frac{\varepsilon^{2\beta}}{2} \int_0^T \langle \gamma'(s) - \tilde{A}(\tilde{\theta}(s)), \Sigma(\tilde{\theta}(s))^{-1} [\gamma'(s) - \tilde{A}(\tilde{\theta}(s))] \rangle ds + o(\varepsilon^{2\beta}).$$

Substituting the above estimate in Theorem 2.2 yields the wanted result. The lower bound follows by similar arguments. \hfill $\square$

For the reader convenience we given an explicit simple example.

**Example.** Let $Q = \{ \gamma \in C^0([0, T], \mathbb{R}) \mid \| \gamma(s) - \tilde{\theta}(s, \theta_0) \| \geq Cs \}$, then we have $Q_{\varepsilon} = \{ \gamma \in C^0([0, T], \mathbb{R}) \mid \| \gamma(s) - \theta(s, \theta_0) \| \geq \varepsilon^2 Cs \}$. To apply Corollary 2.5 we have to compute the inf on $Q$. This might be a bit tricky, to simplify matter let us assume that we are interested in an initial condition $\theta_0$ such that $\tilde{\omega}(\theta_0) = 0$, then $\theta(s, \theta_0) = \theta_0$. Then, let $\gamma \in Q$, suppose $\gamma - \tilde{\theta} \geq 0$ (the other possibility being similar), and setting $\gamma_0(s) = \gamma(s) + \lambda(Cs + \tilde{\theta}(s, \theta_0) - \gamma(s))$ we have

$$\partial_\lambda J_{\text{Lin}}(\gamma_0) \big|_{\lambda=0} = - \int_0^T (\gamma' - \tilde{A})^2 (\Sigma_{1,1})^{-1} + C \int_0^T (\gamma' - \tilde{A}) (\Sigma_{1,1})^{-1}.$$

Next, let $\eta(s) = \gamma(s) - \tilde{\theta}(s, \theta_0) - Cs$ and note that, by definition, $\eta \geq 0$, then

$$\partial_\lambda J_{\text{Lin}}(\gamma_0) \big|_{\lambda=0} = - \int_0^T (\eta')^2 + 2C\eta' + C^2 (\Sigma_{1,1})^{-1} + C \int_0^T \eta' + C \Sigma_{1,1}^{-1} \leq - \int_0^T (\eta')^2 + C\eta' (\Sigma_{1,1})^{-1} = -[\Sigma_{1,1}(\theta_0)]^{-1} \left[ \int_0^T (\eta')^2 + C\eta(T) \right] \leq 0.$$

Hence the inf is realized at the boundary of $Q$ and Corollary 2.5 yields the estimate

$$\limsup_{\varepsilon \to 0} \varepsilon^{1-2\beta} \log P_{A,C}(Q_{\varepsilon}) \leq -C^2[\Sigma_{1,1}(\theta_0)]^{-1}T.$$
Moreover, for $\beta > \frac{3}{5}$, Corollary 2.5 yields also the lower bound
\[
\liminf_{\varepsilon \to 0} \varepsilon^{1-2\beta} \log \mathbb{P}_{A,\varepsilon}(Q_\varepsilon) \geq -C^2[\Sigma_{1,1}^2(\theta_0)]^{-1}T.
\]

In fact, Theorem 2.2 allows to estimate the probability of even smaller deviations, up to the scale of the Central Limit Theorem.

**Corollary 2.6 (Small deviations).** For each $\vartheta \in (0,1)$ and $T > 0$ there exists $\varepsilon_0, C_\varepsilon > 0$ such that, for each $\varepsilon \in (0,\varepsilon_0)$, given any event $Q \subset C^0([0,T],\mathbb{R}^d)$ such that $\eta(\theta_0, Q) \geq C_\varepsilon \sqrt{\varepsilon}$ and setting $Q_\varepsilon = \{ \varepsilon^{\vartheta} \gamma(:) - (1 - \varepsilon^{\vartheta}) \tilde{\varepsilon}(\cdot, \theta_0) \}_{\varepsilon \in Q}$, we have
\[
\mathbb{P}_{A,\varepsilon}(Q_\varepsilon) \leq e^{-\vartheta \inf_{\varepsilon \in Q^\vartheta} \mathcal{A}_{\varepsilon}(\gamma)+C_\vartheta},
\]
where $Q^\vartheta = \bigcup_{\varepsilon \in Q} B(\gamma, \vartheta \| \gamma - \tilde{\varepsilon} \| \infty)$.

**Proof.** Let $C_\vartheta$ large enough and set $\gamma_\varepsilon = \varepsilon^{\vartheta} \gamma - (1 - \varepsilon^{\vartheta}) \tilde{\varepsilon}$. For each $\gamma \in Q$ we have $R_+(\gamma_\varepsilon) \leq \vartheta \| \gamma - \tilde{\varepsilon} \| \infty$. Thus, in the notation of Theorem 2.2, $(Q^\vartheta)_\varepsilon \supset Q^\vartheta _\varepsilon$. Since (2.10) implies
\[
\mathcal{A}_{\varepsilon}(\gamma_\varepsilon) = \varepsilon \mathcal{A}_{\varepsilon}(\gamma) + O(\varepsilon^{\vartheta}),
\]
the result follows directly by Theorem 2.2. \qed

Note that, taking $Q = \{ \gamma \mid |\gamma_1(s) - \tilde{\theta}(s, \theta_0)| \geq C \sqrt{\varepsilon} s \}$ and $\theta_0$ as in the above example, the Corollary gives, for $\varepsilon$ small enough,
\[
\mathbb{P}_{A,\varepsilon}(Q_\varepsilon) \leq e^{-\frac{\varepsilon^2}{2}[\gamma(\theta_0)]^2 T}.
\]

Given that in many cases we have seen that the upper bound is sharp, one expects that typical deviations are of order $\sqrt{\varepsilon} T$. It is then natural to wonder about their distribution. It is possible to prove (see e.g. [18, Theorem 5], where a slightly different class of systems is investigated, or [12] for a pedagogical exposition of the present case) that the deviation of $z_\varepsilon$ from the average, when rescaled by $\varepsilon^{-\vartheta}$, converges towards a diffusion process. To simplify matters we will discuss only the case $d = 1$, but similar results hold for any $d$.

Let us describe the statement more precisely. Once again fix $\theta_0$ and let $x_0$ be distributed according to a smooth probability; define $\Delta_\varepsilon(t) = \varepsilon^{-1/2} \left[ \theta_0(t) - \tilde{\theta}(t, \theta_0) \right]$. Then, as $\varepsilon \to 0$, the deviation $\Delta_\varepsilon(t)$ converges weakly to $\Delta(t)$, the solution of:
\[
d\Delta(t) = \hat{\omega}(\hat{\theta}(t, \theta_0)) \Delta(t) dt + \hat{\sigma}^2(\hat{\theta}(t, \theta_0)) dB(t)
\]
where $B(t)$ is a standard Brownian motion and\(^6\)
\[
\hat{\sigma}^2(\theta) = \mu_\theta \left( \hat{\omega}^2(\cdot, \theta) + 2 \sum_{m=1}^{\infty} \hat{\omega}(f^m_\theta(\cdot), \theta) \hat{\omega}(\cdot, \theta) \right).
\]

We avoid to prove such a result to get right away to the other main result of this paper.

**Theorem 2.7.** For any $T > 0$, there exists $\varepsilon_0 > 0$ so that the following holds. For any compact interval $I \subset \mathbb{R}$, real numbers $\kappa > 0$, $\varepsilon \in (0,\varepsilon_0)$ and $t \in [\varepsilon^{1/2000}, T]$, any fixed $\theta_0 \in T^1$, if $x_0$ is distributed according to a smooth distribution, we have:
\[
\varepsilon^{-1/2} \mathbb{P}_{\varepsilon}(\Delta_\varepsilon(t) \in \varepsilon^{1/2} I + \kappa) = \text{Leb} I \cdot \frac{e^{-\kappa^2/2\sigma_\varepsilon^2(\theta_0)}}{\sigma_\varepsilon(\theta_0) \sqrt{2\pi}} + o^+(1).
\]

\(^6\) Note this is noting else that $\Sigma_{1,1}^2$, which appeared in the moderate deviations.
where the variance $\sigma^2_\theta(\theta_0)$ is given by

$$\sigma^2_\theta(\theta_0) = \int_0^t e^{2\int_0^s \omega'(\tilde{\theta}(r, \theta_0))dr} \sigma^2(\tilde{\theta}(s, \theta_0))ds,$$

(2.14)

Remark 2.8. Observe that the above result is compatible with (2.11): in fact, (2.11) implies the following ODE for the variance of $\Delta(t)$:

$$\frac{dE(\Delta(t)^2)}{dt} = 2\omega'(\tilde{\theta}(t))E(\Delta(t)^2) + \sigma^2(\tilde{\theta}(t, \theta_0));$$

it is then trivial to check that (2.14) is the unique solution of the above equation with initial condition $E(\Delta(0)^2) = 0$. Moreover, one can check that the main term of (2.13) provides a solution of (2.11).

3. Standard pairs and families

In this section we introduce standard pairs and families for our system. As mentioned in the introductory section, this tool proved quite powerful in obtaining quantitative statistical results in systems with some degree of hyperbolicity. The first step is thus to establish some hyperbolicity result.

3.1. Dominated splitting.
Let us start with a preliminary inspection of the geometry of our system: for $c > 0$, consider the cones $K_c = \{(\xi, \eta) \in \mathbb{R}^2 : |\eta| \leq c|\xi|\}$. Note that

$$dF_\varepsilon = \begin{pmatrix} \partial_x f & \partial_\theta f \\ \varepsilon \partial_x \omega & 1 + \varepsilon \partial_\theta \omega \end{pmatrix}.$$ 

Thus if $(1, \varepsilon u) \in K_c$,

$$d_p F_\varepsilon(1, \varepsilon u) = (\partial_x f(\varepsilon u, \varepsilon u) + \varepsilon \partial_\theta f(\varepsilon u, \varepsilon u) + \varepsilon^2 \partial_\theta \omega(\varepsilon u)u)\partial_x f(\varepsilon u, \varepsilon u)$$

$$= \partial_x f(p) + \varepsilon \partial_\theta f(p) \left(1 + \varepsilon \partial_\theta \omega(p)\right)(1, \varepsilon \Xi_p(u))$$

where

$$\Xi_p(u) = \frac{\partial_x \omega(p) + (1 + \varepsilon \partial_\theta \omega(p))u}{\partial_x f(p) + \varepsilon \partial_\theta f(p)u}.$$

We conclude that, choosing $\varepsilon$ small enough and

$$(3.2) \quad \|\partial_x \omega\|_{\lambda - 1} < c < K \varepsilon^{-1} \quad \|\partial_\theta f\|_{\lambda - 1} < \lambda - 1,$$

the cone $K_c$ is invariant under $dF_\varepsilon$ and $K_{cK^{-1}}$ is invariant under $dF^{-1}_\varepsilon$. Hence, for any $p \in \mathbb{R}^2$ and $n \in \mathbb{N}$, we can define the quantities $\nu_n, u_n, s_n, \mu_n$ as follows:

$$d_p F_\varepsilon^n(1, 0) = \nu_n(1, \varepsilon u_n) \quad d_p F_\varepsilon^n(s_n, 1) = \nu_n(0, 1)$$

with $|u_n| \leq c$ and $|s_n| \leq K^{-1}$. Notice that $d_p F_\varepsilon^n(s_n(p), 1) = \mu_n / \mu_{n-1}(s_{n-1}(F_\varepsilon(p), 1));$ therefore, there exists a constant $b$ such that:

$$(3.4) \quad \exp(-bc) \leq \frac{\mu_n}{\mu_{n-1}} \leq \exp(bc).$$

Furthermore, define $\Gamma_n = \prod_{k=0}^{n-1} \partial_x f \circ F_\varepsilon^k$, and let

$$a = c \|\partial_\theta f\|_{\lambda - 1}.$$
Clearly

\[(3.5) \quad \Gamma_n \exp(-a\varepsilon n) \leq \nu_n \leq \Gamma_n \exp(a\varepsilon n).\]

3.2. Standard pairs: definition and properties.
We now proceed to define standard pairs for our system: we begin by introducing
real standard pairs, and then proceed to extend our definitions to complex standard pairs.

Let us fix a small \(\delta > 0\), and \(D_1, D'_1 > 0\) large to be specified later; for any \(c_1 > 0\) let us define the set of functions

\[\Sigma_{c_1} = \{G \in C^3([a, b], \mathbb{T}^1) \mid a, b \in \mathbb{T}^1, \ b - a \in [\delta/2, \delta],\]

\[\|G\| \leq \varepsilon c_1, \ |G'| \leq \varepsilon D_1 c_1, \ |G''| \leq \varepsilon D'_1 c_1\}.\]

Let us associate to each \(G \in \Sigma_{c_1}\) the map \(\mathbb{G}(x) = (x, G(x))\) whose image is a curve
– the graph of \(G\) – which will be denoted by \(\gamma_G\); such curves are called standard curves. For any \(c_2, c_3 > 0\) define the set of \((c_2, c_3)\)-standard probability densities on the standard curve \(\gamma_G\) as

\[D_{c_2, c_3}^\mathbb{R}(G) = \left\{ \rho \in C^2([a, b], \mathbb{R}_{>0}) \mid \int_a^b \rho(x)dx = 1, \ \left\| \frac{\rho'}{\rho} \right\| \leq c_2, \ \left\| \frac{\rho''}{\rho} \right\| \leq c_3 \right\}.\]

A real \((c_1, c_2, c_3)\)-standard pair \(\ell\) is given by \(\ell = (\mathbb{G}, \rho)\) where \(G \in \Sigma_{c_1}\) and \(\rho \in D_{c_2}^\mathbb{R}(G)\). A real standard pair \(\ell = (\mathbb{G}, \rho)\) induces a probability measure \(\mu_\ell\) on \(\mathbb{T}^2\) defined as follows: for any Borel-measurable function \(g\) on \(\mathbb{T}^2\) let

\[\mu_\ell(g) := \int_a^b g(x, G(x))\rho(x)dx.\]

We define\(^{7}\) a standard family \(\mathcal{L} = (\{\ell_j\}, \nu)\) as a (finite or) countable collection of standard pairs \(\{\ell_j\}\) endowed with a finite factor measure \(\nu\), i.e. we associate to each standard pair \(\ell_j\) a positive weight \(\nu_j\) so that \(\sum_{\ell \in \mathcal{L}} \nu_\ell < \infty\). A standard family \(\mathcal{L}\) naturally induces a finite measure \(\mu_\mathcal{L}\) on \(\mathbb{T}^2\) defined as follows: for any Borel-measurable function \(g\) on \(\mathbb{T}^2\) we let

\[\mu_\mathcal{L}(g) := \sum_{\ell \in \mathcal{L}} \nu_\ell \mu_\ell(g).\]

A standard family is a standard probability family if the induced measure is a probability measure (i.e. if \(\nu\) is itself a probability measure). Let us denote by \(\sim\) the equivalence relation induced by the above correspondence i.e. we let \(\mathcal{L} \sim \mathcal{L'}\) if and only if \(\mu_\mathcal{L} = \mu_\mathcal{L'}\).

We now proceed to generalize the above definitions in order to introduce complex standard pairs. Let us first define the set of complex standard densities:

\[D_{c_2, c_3}^\mathbb{C}(G) = \left\{ \rho \in C^2([a, b], \mathbb{C}^*) \mid \int_a^b \rho(x)dx = 1, \ \left\| \frac{\rho'}{\rho} \right\| \leq c_2, \ \left\| \frac{\rho''}{\rho} \right\| \leq c_3 \right\},\]

where we denote \(\mathbb{C}^* = \mathbb{C} \setminus \{0\}\). Yet, this time, for technical reasons, we require\(^{8}\) \(c_2\delta \leq \pi/10\). A complex standard pair is then given by \(\ell = (\mathbb{G}, \rho)\) where \(G \in \Sigma_{c_1}\) and \(\rho \in D_{c_2, c_3}^\mathbb{C}(G)\); a complex standard pair induces a natural complex measure on \(\mathbb{T}^2\). A complex standard family \(\mathcal{L}\) is defined as its real counterpart, but now we

\(^{7}\) We remark that this is not the most general definition of standard family, yet it suffices for our purposes and it allows to greatly simplify our notations.

\(^{8}\) This requirement is indeed quite arbitrary, but see Lemma 3.1.
allow \( \ell \)'s to be complex standard pairs and \( \nu \) to be a complex measure so that \( \sum_{\ell \in \mathcal{E}} |\nu_\ell| < \infty \). Clearly, a complex standard family naturally induces a complex measure on \( T^2 \).

**Lemma 3.1** (Variation). Let \( G \) be a standard curve and \( \rho \in D^C_{c_2,c_3}(G) \); if \( \delta \) is sufficiently small we have:

\[
\text{Range } \rho \subset \{ z = re^{i\theta} \in \mathbb{C} \mid e^{-2c_2\delta} < r(b-a) < e^{2c_2\delta}, |\theta| < c_3\delta \}.
\]

**Proof.** Observe that, by definition of standard density we have \( \| (\log \rho) \| \leq c_2 \); since we are assuming \( c_2\delta \leq \pi/10 \), we can unambiguously define the function \( \log \rho \), which is contained in a square \( S \subset \mathbb{C} \) of side \( c_2\delta \). Thus, \( \text{Range } \rho \subset \exp S \), which is an annular sector. The normalization condition \( \int \rho = 1 \) and the Mean Value Theorem imply that \( \exp S \) must non-trivially intersect the sets \( \{ \text{Re } z = (b-a)^{-1} \} \) and \( \{ \text{Im } z = 0 \} \); these two conditions immediately imply that \( \exp S \subset \{ re^{i\theta} \in \mathbb{C} \mid |\theta| < c_2\delta \} \). It is then immediate to show that

\[
\exp S \subset \left \{ re^{i\theta} \mid e^{-c_2\delta} < r(b-a) < \frac{1}{\cos(c_2\delta)} e^{c_2\delta} \right \} ,
\]

which concludes the proof. \( \square \)

**Remark 3.2.** The above lemma also implies a uniform \( C^2 \) bound on standard densities given by \( \| \rho \| c_2 \leq e^{2c_2\delta}c_3 \). Moreover, we have \( |\mu_\ell| < e^{2c_2\delta} \).

The key property of the class of real standard pairs is its invariance under push-forward by the dynamics; we are now going to prove a more general result. Let \( \Psi \subset C^2(T^2,\mathbb{C}) \) be a family of smooth functions with uniformly bounded \( C^2 \)-norm; we denote by \( \| \Psi \|_{C^r} = \sup_{\Omega \in \Psi} \| \Omega \|_{C^r} \). For any \( \Omega \in \Psi \) define the operator \( F_{\varepsilon,\Omega} \) which acts on a complex measure \( \mu \) as follows: for any measurable function \( g \) of \( T^2 \):

\[
[F_{\varepsilon,\Omega}] (g) := \mu(e^{\Omega} \circ g \circ F_\varepsilon).
\]

We call \( F_{\varepsilon,\Omega} \) the weighted push-forward operator with potential \( \Omega \); observe that \( F_{\varepsilon,0} = F_\varepsilon \) the usual push-forward.

**Proposition 3.3** (Invariance). Given a family of complex potentials \( \Psi \), there exist \( c_1, c_2, c_3 \) and \( \delta \) such that the following holds. For any \( \Omega \in \Psi \) and complex \((c_1, c_2, c_3)\)-standard family \( \mathcal{J} \), the complex measure \( F_{\varepsilon,\Omega} \mu_{\mathcal{J}} \) can be decomposed in complex \((c_1, c_2, c_3)\)-standard pairs, i.e. there exists a complex \((c_1, c_2, c_3)\)-standard family \( \mathcal{J}_\Omega' \) such that \( F_{\varepsilon,\Omega} \mu_{\mathcal{J}} = \mu_{\mathcal{J}_\Omega'} \). We say that \( \mathcal{J}_\Omega' \) is a \((c_1, c_2, c_3)\)-standard decomposition of \( F_{\varepsilon,\Omega} \mu_{\mathcal{J}} \) and we write –with a little abuse of notation– \( \mathcal{J}_\Omega' \sim F_{\varepsilon,\Omega} \mathcal{J} \). Moreover, the constant \( c_1 \) does not depend on \( \Psi \), whereas the constants \( c_2 \) and \( c_3 \) (and consequently \( \delta \)) can be chosen as follows (see discussion before (B.3)):

\[
c_2 \geq C\#(1 + \| \Psi \|_{C^1}) \quad \quad c_3 \geq C\#(1 + \| \Psi \|_{C^2} + \| \Psi \|_{C^2}^2).
\]

**Proof.** For simplicity, let us assume that \( \mathcal{J} \) is given by a single complex standard pair \( \ell \); the general case does not require any additional ideas and it is left to the reader.

Let then \( \ell = (G, \rho) \) be a complex \((c_1, c_2, c_3)\)-standard pair. For any sufficiently smooth function \( A \) on \( T^2 \), by the definition of standard curve, it is trivial to check
that:

\begin{align}
(3.6a) & \quad \|(A \circ G)\| \leq \|dA\|(1 + \varepsilon c_1) \\
(3.6b) & \quad \|(A \circ G)''\| \leq \varepsilon \|dA\|D_1 c_1 + \|dA\|_{C^1}(1 + \varepsilon c_1)^2 \\
(3.6c) & \quad \|(A \circ G)'''\| \leq \varepsilon \|dA\|D_1' c_1 + \|dA\|_{C^2}(1 + \varepsilon(1 + D_1) c_1)^3.
\end{align}

Let us then introduce the maps \( f_G = f \circ G, \omega_G = \omega \circ G \) and \( \Omega_G = \Omega \circ G \). We will assume \( \varepsilon \) to be small enough (depending on our choice of \( c_1 \)) so that \( f_G \geq \lambda - \varepsilon c_1 \|\partial_x f\| > 3/2 \); in particular, \( f_G \) is an expanding map. Provided \( \delta \) has been chosen small enough, \( f_G \) is invertible. Let \( \varphi(x) = f_G^{-1}(x) \). Differentiating we obtain

\begin{equation}
(3.7) \quad \varphi' = \frac{1}{f_G} \circ \varphi \quad \varphi'' = \frac{f_G''}{f_G} \circ \varphi \quad \varphi''' = \frac{3f_G'' - f_G'''}{f_G} \circ \varphi.
\end{equation}

Then, by definition, for any measurable function \( g \):

\[
F_{x,\Omega,\mu}(g) = \mu_\varepsilon(e^{\lambda \cdot g \circ F_\varepsilon}) = \int_a^b g(f_G(x), \tilde{G}(x)) \cdot e^{\Omega_G(x)} \rho(x) dx,
\]

where \( \tilde{G}(x) = G(x) + \varepsilon \omega_G(x) \). Then, fix a partition (mod 0) \([f_G(a), f_G(b)] = \cup_{j \in J} [a_j, b_j] \), with \( b_j - a_j \in [\delta/2, \delta] \) and \( b_j = a_{j+1} \). We can thus write:

\begin{equation}
(3.8) \quad F_{x,\Omega,\mu}(g) = \sum_j \int_{a_j}^{b_j} g(x, G_j(x)) \tilde{\rho}_j(x) dx,
\end{equation}

provided that \( G_j = \tilde{G} \circ \varphi_j \) and \( \tilde{\rho}_j(x) = e^{\Omega_G(x)} \rho(x) \circ \varphi_j \), where \( \varphi_j = \varphi|_{[a_j, b_j]} \).

In order to conclude our proof it suffices to show that (i) there exists \( c_1 \) large enough so that if \( G \in \Sigma_{c_1} \), then \( G_j \in \Sigma_{c_1} \), and (ii) there exist \( c_2, c_3 \) large enough and \( \delta \) small enough so that if \( \rho \in D_{c_2, c_3}^C(G) \), \( \tilde{\rho}_j \) can be normalized to a complex standard density belonging to \( D_{c_2, c_3}^C(G_j) \).

Item (i) follows from routine computations: differentiating the above definitions and using (3.7) we obtain

\begin{align}
(3.9a) & \quad G_j' = \frac{\tilde{G}'}{f_G} \circ \varphi_j \\
(3.9b) & \quad G_j'' = \frac{\tilde{G}''}{f_G} \circ \varphi_j - G_j' \cdot \frac{f_G''}{f_G^2} \circ \varphi_j \\
(3.9c) & \quad G_j''' = \frac{\tilde{G}'''}{f_G} \circ \varphi_j - 3G_j'' \cdot \frac{f_G''}{f_G^2} \circ \varphi_j - G_j' \cdot \frac{f_G'''}{f_G^2} \circ \varphi_j
\end{align}

Using (3.9a), the definition of \( \tilde{G} \) and (3.6a) we obtain, for small enough \( \varepsilon \):

\[
\|G_j'\| \leq \left\| \left( G_j' + \varepsilon \omega_G \right) \right\|_{f_G} \leq \frac{2}{3}(1 + \varepsilon \|\partial_x G\|)\varepsilon c_1 + \frac{2}{3}\varepsilon \|\partial_x G\| \leq \frac{3}{4}\varepsilon c_1 + \varepsilon D_1
\]

where \( D_1 = \frac{3}{4}\|\partial_x G\| \). We can then fix \( c_1 \) large enough so that the right hand side of the above inequality is less than \( c_1 \). Next we will use \( C_* \) for a generic constant...
depending on $c_1, D_1, D'_1$ and $C_\#$ for a generic constant depending only on $F_*$. Then, we find
\[
\|G''_j\| \leq \frac{3}{4} \varepsilon [c_1 D_1 + C_\#] + \varepsilon^2 C_*;
\]
\[
\|G''''_j\| \leq \frac{3}{4} \varepsilon [c_1 (D'_1 + D_1 C_\# + C_\#) + C_\#] + \varepsilon^2 C_*.
\]
We can then fix $c_1, D'_1$ sufficiently large and then $\varepsilon$ sufficiently small to ensure that the $G_j$’s are $c_1$-standard pairs. We now proceed with item (ii); by differentiating the definition of $\tilde{\rho}_j$ we obtain
\[
\begin{aligned}
\frac{\tilde{\rho}'_j}{\tilde{\rho}_j} &= \frac{\rho'}{\rho} \circ \varphi_j - \frac{f''_G}{f'_G} \circ \varphi_j + \frac{\Omega''_G}{f''_G} \circ \varphi_j \\
\frac{\tilde{\rho}''_j}{\tilde{\rho}_j} &= \frac{\rho''}{\rho} \circ \varphi_j - 3 \frac{\rho'}{\rho} \cdot \frac{f''_G}{f'_G} \circ \varphi_j - \frac{f''_G}{f''_G} \circ \varphi_j + \frac{2 \rho' \Omega'_G}{\rho f'_G} \circ \varphi_j + \frac{\Omega''_G}{f''_G} \circ \varphi_j.
\end{aligned}
\]
From the first of the above expressions and (3.6a) we gather:
\[
\left\| \frac{\tilde{\rho}'_j}{\tilde{\rho}_j} \right\|_{c_0} \leq \frac{2}{3} \left\| \frac{\rho'}{\rho} \right\|_{c_0} \ + \ D + C_\# \| \Omega \|_{c_1},
\]
where $D$ is a uniform constant related to the distortion of the maps $f(\cdot, \theta)$, which can be obtained using our uniform bounds on $\|G'_j\|$ and $\|G''_j\|$. The above expression implies that we can choose $c_2 = O(1 + \|\Omega\|_{c_1})$ so that if $\|\rho'/\rho\|_{c_0} \leq c_2$, then $\|\tilde{\rho}'_j/\tilde{\rho}_j\|_{c_0} \leq c_2$. A similar computation, using (3.10b), yields:
\[
\left\| \frac{\tilde{\rho}''_j}{\tilde{\rho}_j} \right\| \leq \frac{4}{9} \left\| \frac{\rho''}{\rho} \right\| \ + \ C_\# (\|\Omega\|_{c_2} + \|\Omega\|_{c_1}^2 + c_2 (\|\Omega\|_{c_1} + D) + D'),
\]
where, once again, $D'$ is uniformly bounded thanks to our bounds on $\|G'_j\|$, $\|G''_j\|$ and $\|G''''_j\|$. As before, this implies the existence of $c_3 = O(1 + \|\Omega\|_{c_2} + \|\Omega\|_{c_1}^2)$ so that if $\|\rho'/\rho\|_{c_0} \leq c_3$, then $\|\tilde{\rho}'_j/\tilde{\rho}_j\|_{c_0} \leq c_3$.

We are now left to show that, using our requirement on $\delta$:
\[
\nu_j := \int_{a_j}^{b_j} \tilde{\rho}_j \, dx \neq 0;
\]
this implies that $\rho_j := \nu_j^{-1} \tilde{\rho}_j \in D^\infty_{c_2, c_3} (G_j)$, which concludes our proof: in fact, define the standard family $\mathcal{S}'_{\Omega}$ given by $\{\ell_j, \nu\}$, where $\nu_{\ell_j} = \nu_j$; then we can rewrite (3.8) as follows:
\[
F_{c_*, \Omega} \mu_{\nu_j}(g) = \sum_{\ell \in \mathcal{S}'_{\Omega}} \nu_{\ell} \mu_{\ell}(g) = \mu_{\mathcal{S}'_{\Omega}}(g).
\]
The proof of (3.11) follows from arguments similar to the ones used in the proof of Lemma 3.1: in fact $\delta$ is sufficiently small so that the function $\log \tilde{\rho}_j$ can be defined and it is contained in a square of side $c_2 \delta$. Therefore, Range $\tilde{\rho}_j$ is contained in an annular sector of small aperture, whose convex hull is bounded away from 0; this implies that $\nu_j \neq 0$.  

\[9\] The reader can easily fill in the details of the computations.
Remark 3.4. Assume $\ell$ is a real standard pair and $\Omega \in C^2(\mathbb{T}^2, \mathbb{R})$: then $\mathcal{L}_{\ell, \Omega}$ is a real standard family. Moreover, $\mathcal{L}_{\ell, 0}$ is a standard probability family.

We say that $\ell$ is a $\mathcal{P}$-standard pair if $c_1, c_2, c_3$ and $\delta$ are so that Proposition 3.3 holds with respect to the family $\mathcal{P}$. Given a $\mathcal{P}$-standard pair $\ell$ and a sequence of potentials $(\Omega_k)_{k \in \mathbb{N}} = \Omega \in \mathcal{P}^\mathbb{N}$, we denote (again with an abuse of notation) by $\mathcal{L}_{\ell, \Omega}^{(n)}$ a standard decomposition of $F_{\epsilon, \Omega}^{(n)} \mu_\ell = F_{\epsilon, \Omega_{n-1}} \cdots F_{\epsilon, \Omega_0} \mu_\ell$, which we obtain by iterating the above proposition. By definition, therefore, we have, for any sufficiently smooth function $g$ of $\mathbb{T}^2$:

$$F_{\epsilon, \Omega}^{(n)} \mu_\ell(g) = \sum_{\tilde{\ell} \in \mathcal{L}_{\ell, \Omega}^{(n)}} \nu_{\tilde{\ell}} \mu_{\tilde{\ell}}(g) = \mu_\ell \left( e^{S_{\ell, \Omega}} g \circ F_{\epsilon}^n \right),$$

where we have defined the “Birkhoff sum” $S_{\ell, \Omega} = \sum_{k=0}^{n-1} \Omega_k \circ F_{\epsilon}^k$. In particular, (3.12) implies that $\mu_\ell \left( e^{S_{\ell, \Omega}} \right) = \sum_{\tilde{\ell} \in \mathcal{L}_{\ell, \Omega}^{(n)}} \nu_{\tilde{\ell}}$.

Remark 3.5. The proof of Proposition 3.3 allows to define, for any $\tilde{\ell} \in \mathcal{L}_{\ell, \Omega}^{(n)}$ the corresponding characteristic function $1_{\tilde{\ell}}$, that is a random variable on $\ell$ which equals 1 on points which are mapped to $\tilde{\ell}$ by $F_{\epsilon}^n$ and 0 elsewhere. This allows to write:

$$\nu_{\tilde{\ell}} = \mu_\ell \left( e^{S_{\ell, \Omega}} 1_{\tilde{\ell}} \right)$$

(3.13a)

$$\mu_{\tilde{\ell}}(g) = \nu_{\tilde{\ell}}^{-1} \mu_\ell \left( e^{S_{\ell, \Omega}} 1_{\tilde{\ell}} \cdot g \circ F_{\epsilon}^n \right),$$

(3.13b)

Observe that (3.13a) immediately implies that, for any $n \in \mathbb{N}$:

$$\sum_{\tilde{\ell} \in \mathcal{L}_{\ell, \Omega}^{(n)}} |\nu_{\tilde{\ell}}| \leq e^{\sum_{k=0}^{n-1} \max \Re \Omega_k} e^{2c_2 \delta}.$$  

(3.14)

Moreover,

$$\sum_{\ell_1 \in \mathcal{L}_{\ell_1, \Omega}^{(s)}} \sum_{\ell_2 \in \mathcal{L}_{\ell_2, \Omega_2}^{(s)}} \cdots \sum_{\ell_m \in \mathcal{L}_{\ell_m, \Omega_m}^{(s)}} \prod_{j=1}^{m} |\nu_{\ell_j}| \leq e^{\sum_{n=0}^{m-1} \max \Re \Omega_k} e^{2c_2 \delta},$$

(3.15)

where $s$ is the one-sided shift acting naturally on $\mathcal{P}^\mathbb{N}$. In fact, the above is just a special choice of standard decomposition for $F_{\epsilon, \Omega}^{(nm)} \mu_\ell$, indexed by a $m$-tuple of standard pairs $(\ell_1, \ldots, \ell_{m-1})$ selected at intermediate steps of length $n$.

Remark 3.6. Given a standard pair $\ell = (\mathbb{G}, \rho)$, we will interpret $(x_k, \theta_k)$ as random variables defined as $(x_k, \theta_k) = F_{\epsilon}^k(x, G(x))$, where $x$ is distributed according to $\rho$.

4. Approximations of the trajectory

In this section we describe some preliminary results that allow to compare the true dynamics with simpler ones.

First, in the next subsection, we prove some quantitative shadowing results. This allows to compare the true dynamics with much simpler ones for times of order $\epsilon^{-\frac{1}{2}}$. Then, in the following subsection, we construct a random variable that is close to the true trajectory for times of order $\epsilon^{-1}$. 


4.1. Shadowing.

First, given $\theta^* \in \mathbb{T}^1$, let us introduce $F_\varepsilon(x, \theta) = (f_\varepsilon(x), \theta)$, where $f_\varepsilon(x) = f(x, \theta^*)$; our first result allows to compare orbits of $F_\varepsilon$ with orbits of $F_\ast$.

**Lemma 4.1.** Consider a standard pair $\ell = (\mathbb{G}, \rho)$ and fix $\theta^* \in \mathbb{T}^1$ at a distance at most $\varepsilon$ from the range of $G$. For any $n \in \mathbb{N}$ so that $cn + n^2\varepsilon \leq C_\#$, there exists a diffeomorphism $\gamma_n : [a, b] \to [a^*, b^*]$ such that $(x_n, \theta_n) = F^n_\varepsilon(x, G(x)) = (f^n_\ast(Y_n(x)), \theta_n)$, where $C_\#$ is introduced in (4.1).

In addition, for all $k \in \{0, \ldots, n\}$ and setting $x_k^* = f^k_\ast(Y_n(x))$,

$$\left\| \theta_k - \theta^* - \varepsilon \sum_{j=0}^{k-1} \omega(x_j^*, \theta^*) \right\|_{c^0} \leq C_\# |\varepsilon + \varepsilon^2 k^2|,$$

$$\left\| x_k^* - x_k + \varepsilon \sum_{j=k}^{n-1} \Lambda_{k,j}^* \partial_\theta f(x_j^*, \theta^*) \sum_{\ell=0}^{j-1} \omega(x_\ell^*, \theta^*) \right\|_{c^0} \leq C_\# |\varepsilon + \varepsilon^2 k^2|,$$

$$\left\| 1 - Y_n \prod_{k=0}^{n-1} f'_k(x_k^*) \right\|_{c^0} \leq C_\# n \varepsilon,$$

where we defined $\Lambda_{k,j}^* = \prod_{i=k}^{j} f'_i(x_i^*)^{-1} = \frac{(f_i^*)^{-k-1}}{f_i^*} \leq \lambda^{k-j-1}$.

**Proof.** Let us denote with $\pi_x : \mathbb{T}^2 \to T$ the canonical projection on the $x$ coordinate; then, for $\varrho \in [0, 1]$, define

$$H_n(x, z; \varrho) = \pi_x F^n_\varepsilon(x, \theta^* + \varrho(G(x) - \theta^*)) - f^n_\ast(z).$$

Note that, $H_n(x, x; 0) = 0$, in addition, for any $x, \varrho$:

$$\partial_\varrho H_n(x, z; \varrho) = -(f^n_\ast)'(z) \neq 0.$$

Accordingly, by the Implicit Function Theorem, for any $n \in \mathbb{N}$ and $\varrho \in [0, 1]$, there exists $Y_n(x; \varrho)$ such that $H_n(x, Y_n(x; \varrho); \varrho) = 0$; from now on $Y_n(x)$ stands for $Y_n(x; 1)$. Observe moreover that

$$(4.1) \quad Y_n' = \frac{(\pi_x F^n_\varepsilon \circ G)' \circ Y_n}{(f^n_\ast)' \circ Y_n} = \frac{(1 - G(s_n))\nu_n}{(f^n_\ast)' \circ Y_n}.$$

where we have used the notations introduced in (3.3).

Next, we want to estimate to which degree $\{(x_k^*, \theta^*)\}_{k=0}^n$ shadows the true trajectory. Observe that

$$\theta_k = \varepsilon \sum_{j=0}^{k-1} \omega(x_j, \theta_j) + \theta_0$$

thus $|\theta_k - \theta^*| \leq C_\# \varepsilon k + \varepsilon$. Accordingly, let us set $\xi_k = x_k^* - x_k$; then by the Intermediate Value Theorem we obtain, for some $x, \theta \in \mathbb{T}$:

$$\xi_{k+1} = f'_k(x) \cdot \xi_k + \partial_\theta f(x_k, \theta) \cdot (\theta_k - \theta^*) \geq \lambda \xi_k - C_\# (\theta_k - \theta^*)$$

$^{10}$The reader should not confuse the notation $F_\ast$, which is a map of $\mathbb{T}^2$, with the push-forward $F_\varepsilon$ introduced in the previous section.
which, by backward induction, using the fact that \( \xi_n = 0 \) and our previous estimates on \( |\theta_k - \theta^*| \), yields \( |\xi_k| \leq C_\# (\epsilon + \epsilon k) \). We thus obtain:

\[
\theta_k - \theta^* = \theta_0 - \theta^* + \epsilon \sum_{j=0}^{k-1} \omega(x_j^*, \theta^*) + O(\epsilon k + \epsilon k^2)
\]

\[
\xi_k = -\sum_{j=k}^{n-1} \Lambda_{j,k} \theta_0 f(x_j^*, \theta^*) \left( \epsilon \sum_{l=0}^{j-1} \omega(x_l^*, \theta^*) + O(\epsilon + \epsilon^2 j^2) \right).
\]

Finally, recalling (4.1), (3.5) and using invariance of the center cone, we have

\[
e^{-\epsilon \# n} \prod_{k=0}^{n-1} \frac{\partial f(x_k, \theta_k)}{f_0(x_k)} \leq \left| \frac{(1 - G's_n)_\nu}{f_0(x)} \right| \leq e^{-\epsilon \# n} \prod_{k=0}^{n-1} \frac{\partial f(x_k, \theta_k)}{f_0(x_k)}.
\]

Accordingly, \( W_n \) is invertible with uniformly bounded derivative, since we assume \( n\epsilon + n^2 \epsilon \leq C_\# \). \( \square \)

Next, given a standard pair \( \ell = (\mathbb{S}, \rho) \), let \( \theta^*_\ell = \mu_\ell(\theta) \) and define \( \bar{\theta}_k = \overline{\theta}(\epsilon k, \theta^*_\ell) \), where \( \theta \) is the unique solution of (2.4). Recall that we consider \( x_k \) and \( \theta_k \) as random variables under \( \mu_\ell \), (see Remark 3.6). Let us define the shorthand notations \( f_k = f(\cdot, \theta_k) \) and \( f^{(n)} = f_{n-1} \circ \cdots \circ f_0 \); finally, let \( F_k(x, \theta) = (f(x, \theta), \theta(\epsilon, \theta)) \) so that \( F_k(x, \theta) = (f(x, \theta), \theta(\epsilon k, \theta)). \)

**Lemma 4.2.** For any \( n \in \mathbb{N} \) so that \( n^2 \epsilon \leq C_\# \), there exists a diffeomorphism \( Y_n : [a, b] \rightarrow [\bar{a}, \bar{b}] \) such that \( (x_n, \theta_n) = F^n(x, G(x)) = (f^{(n)}(\bar{Y}_n(x)), \theta_n) \). Additionally, for all \( k \in \{0, \cdots, n\} \), let us introduce the notation \( \bar{x}_k = \bar{f}^{(k)}(\bar{Y}_n(x)) \) and define the random variables \( \xi_k := x_k - \bar{x}_k \) and \( \Delta_k := \theta_k - \bar{\theta}_k \),\(^{11}\) then

(4.2a) \[ \Delta_k = \epsilon \bar{W}_k + \mathcal{E}_{\text{quad},k} \]

(4.2b) \[ \xi_k = \epsilon \bar{\mathcal{W}}_k + \mathcal{E}_{\text{quad},k} \]

where we defined the random variables

\[
\bar{W}_k = \sum_{j=0}^{k-1} \bar{\omega}(\bar{x}_j, \bar{\theta}_j) + \Phi_k^0(\bar{x}_0), \quad \Phi_k = \epsilon^{-1}(G \circ \bar{Y}_n - \bar{\theta}_0)
\]

\[
\bar{\mathcal{W}}_k = \sum_{l=k}^{n-1} \Lambda_{l,\ell} f_0(\bar{x}_l, \bar{\theta}_l), \quad \Lambda_{k,\ell} = \prod_{r=k}^{l} f(x_r, \bar{\theta}_r)^{-1},
\]

and \( \mathcal{E}_{\text{quad},k} \) so that \( |\mathcal{E}_{\text{quad},k}| \leq C_\# \epsilon^2 k^2 \) and \( |\mathcal{E}_{\text{quad},k}'| \leq C_\# \epsilon k \max_{j \leq k} (x_j, \bar{x}_j) \), while \( \mathcal{E}_{\text{quad},k} \) is a random variable so that \( |\mathcal{E}_{\text{quad},k}| \leq C_\# \epsilon^2 k^2 \) and \( |\mathcal{E}_{\text{quad},k}'| \leq C_\# \epsilon k (n - k) \max_{j \leq k} (x_j, \bar{x}_j) \), where the ' denotes differentiation with respect to \( \bar{x}_0 \).

The proof of the above lemma is a variation on the one given for Lemma 4.1.

**Proof.** Let us once again introduce the projection on the \( x \)-coordinate \( \pi_x : \mathbb{T}^2 \rightarrow \mathbb{T} \), and define, for \( s \in [0, 1] \):

\[
H_n(x, z; s) = \pi_x F^n_x(x, \bar{\theta}_0 + s(G(x) - \bar{\theta}_0)) - \pi_x F^n_x(z, \bar{\theta}_0).
\]

\(^{11}\)Note that the functions \( \xi_k, \Delta_k \) depend explicitly on the standard pair via the initial condition \( \theta^*_\ell \), so to make the notation precise they should have an index \( \ell \). We omit it to ease notation and since it should not create any confusion.
Note, as before, that \( \dot{H}_n(x, x; 0) = 0 \), and moreover \( \partial_x \dot{H}_n = -\partial_x (\pi_x F_{\pi_x}(\cdot, \bar{\theta}_0)) < -\lambda^n \); using once again the Implicit Function Theorem, for any \( n \in \mathbb{N} \) and \( s \in [0, 1] \) there exists \( \Upsilon_n(n; s) \) so that \( \dot{H}_n(x, \Upsilon_n(x; s); s) = 0 \). Let \( \Upsilon_n(x) = \Upsilon_n(x; 1) \). As before:

\[
\frac{d\Upsilon_n}{dx_0} = \frac{(1 - G's_n)\nu_n}{(f^{(n)})' \circ \Upsilon_n},
\]

where once again we used the notation introduced in (3.3). Note that

\[
\bar{\theta}_k+1 = \bar{\theta}_k + \varepsilon \bar{\omega}(\bar{\theta}_k) + O(\varepsilon^2).
\]

by the above equation we obtain that \( \Delta_k = \varepsilon \Phi(\bar{x}_0) + \varepsilon \sum_{j=0}^{k-1} \bar{\omega}(x_k, \theta_k) + O(\varepsilon^2 k) \), which in particular implies that \( |\Delta_k| < C\varepsilon(k + 1) \). Notice that, by definition, \( \xi_n = 0 \) and

\[
\xi_{k+1} = f(x_k, \theta_k) - f(\bar{x}_k, \bar{\theta}_k).
\]

As before, the Intermediate Value Theorem allows to conclude that in fact \( |\xi_n| \leq C\varepsilon \sum_{j=0}^{n-1} \lambda^{k-j} |\Delta_j| \leq C\varepsilon(k + 1) \); in particular we obtain that \( |\xi_0| = |\Upsilon(x) - x| = O(\varepsilon) \). Notice that by invariance of standard pairs (see Section 3) we obtain that \( |\Delta_k| \leq C\varepsilon \sum_{j=0}^{n-1} \lambda^{k-j} \), which immediately implies (4.2a). Using first order Taylor expansion\(^{12}\) for (4.4) and substituting \( \Delta_k \) using (4.2a) we immediately obtain (4.2b) \( \square \)

Note that we cannot hope for the second derivative \( Y''_n \) and \( \Upsilon''_n \) to be under control;\(^{13}\) yet, we can nonetheless obtain some useful estimates.

**Lemma 4.3.** We have

\[
\left| \frac{d}{dx} (1 - G's_n) \right| \leq \varepsilon n C\varepsilon C\varepsilon \varepsilon^n
\]

**Proof.** From (3.3) it follows that \( s_0(x_n) = 0 \) and

\[
s_{n-k}(x_k) = \frac{s_{n-k-1}(x_{k+1})(1 + \varepsilon \partial_\theta \bar{\omega}(x_k, \theta_k)) - \partial_\theta f(x_k, \theta_k)}{\partial_\theta f(x_k, \theta_k) - \varepsilon \partial_\theta \bar{\omega}(x_k, \theta_k) s_{n-k-1}(x_{k+1})}.
\]

We proceed by induction: suppose \( |\partial_x s_{n-k-1}| \leq C(n - k - 1) e^{C\varepsilon(n-k-1)} |\partial_{x_{k+1}}| \), then

\[
|\partial_x s_{n-k}| \leq C\varepsilon |\partial_{x_{k+1}}| + |\partial_x s_{n-k-1}| e^{C\varepsilon} \varepsilon |\partial_{x_{k+1}}| \leq C(n - k) e^{C\varepsilon(n-k)} |\partial_{x_{k+1}}|,
\]

provided \( C \) is large enough. Since \( \|G''\|_{C^1} = O(\varepsilon) \), we conclude the proof. \( \square \)

**4.2. Random approximation.**

In order to obtain Theorem 2.7 we will need to control deviations from the average up to order \( \varepsilon \), which requires much finer bounds than the ones already obtained in Subsection 4.1. Let \( \ell \) be a standard pair and \( \bar{\theta}_k = \bar{\theta}(\varepsilon_k, \theta_k') \) as in Section 4.1.

Then, by definition

\[
\theta_{k+1} - \theta_k = \varepsilon \bar{\omega}(\theta_k) + \varepsilon \bar{\omega}(x_k, \theta_k)
\]

\[
\bar{\theta}_{k+1} - \bar{\theta}_k = \varepsilon \bar{\omega}(\bar{\theta}_k) + \frac{1}{2} \varepsilon^2 \bar{\omega}'(\bar{\theta}_k) \bar{\omega}(\bar{\theta}_k) + O(\varepsilon^3).
\]

\(^{12}\) The attentive reader will notice that we need to use the integral form of the Taylor remainder term in order to obtain a bound for the derivative of the error term.

\(^{13}\) The reader can easily check that \( \|\Upsilon''_n\|_{\infty} \sim \lambda^n \).
Since $|\theta_k - \theta_0| \leq C_{\#} \varepsilon k$ and $|\tilde{\theta}_k - \tilde{\theta}_0| \leq C_{\#} \varepsilon k$, we trivially find $|\Delta_k| < C_{\#} \varepsilon k$; moreover, the above equations imply that $\Delta_k$ satisfies the following difference equation:

$$
\Delta_{k+1} - \Delta_k = \varepsilon \hat{\omega}(x_k, \theta_k) + \varepsilon \hat{\omega}'(\tilde{\theta}_k) \Delta_k + \frac{1}{2} \varepsilon \hat{\omega}''(\tilde{\theta}_k) \Delta_k^2 + \frac{1}{2} \varepsilon^2 \hat{\omega}'(\tilde{\theta}_k) \hat{\omega}''(\tilde{\theta}_k) + \mathcal{O}(\varepsilon \Delta_k^3 + \varepsilon^3).
$$

Define now the auxiliary random variables

$$
H_{t,k} = \varepsilon \sum_{j=0}^{k-1} \Xi_{t,j,k} \hat{\omega}(x_j, \theta_j), \quad \text{where} \quad \Xi_{t,j,k} = \exp \left( \frac{\varepsilon}{t} \sum_{l=j+1}^{k-1} \hat{\omega}'(\tilde{\theta}_l) \right);
$$

Remark 4.4. As here $\ell$ is fixed and no ambiguity can arise, to ease notation in the rest of the section we will drop the subscript $\ell$ in the above quantities.

Lemma 4.5. For any $k \leq T \varepsilon^{-1}$ we have

$$
|H_k - \Delta_k| \leq C_T (k \varepsilon^2 + \varepsilon \sum_{j=0}^{k-1} [H_j^2 + \varepsilon H_j]).
$$

In addition, given $N \in \mathbb{R}_{\geq 0}$ such that $|H_j - \Delta_j| \leq \frac{1}{2 C_T N}$, for all $j \leq N \varepsilon^{-1}$, we have

$$
\sup_{k \leq N \varepsilon^{-1}} |H_k - \Delta_k| \leq C_T (N \varepsilon + \varepsilon \sum_{j=0}^{N \varepsilon^{-1}-1} [\Delta_j^2 + \varepsilon \Delta_j]).
$$

Proof. It is immediate to check that, by definition, $H_k$ solves the following difference equation:

$$
H_{k+1} - H_k = \varepsilon \hat{\omega}(x_k, \theta_k) + \varepsilon \hat{\omega}'(\tilde{\theta}_k) H_k + \mathcal{O}(\varepsilon^2) H_k.
$$

Observe that the above implies

$$
|H_k| \leq \varepsilon C_{\#} + (1 + C_{\#} \varepsilon) |H_{k-1}| \leq \varepsilon C_{\#} (1 + C_{\#} \varepsilon)^k k,
$$

thus, since $k \leq C_{\#} T \varepsilon^{-1}$ we also obtain $|H_k| \leq C_{\#} \varepsilon k \leq C_{\#}$. We can subtract the two difference equations for $H_k$ and $\Delta_k$ obtaining:

$$
|H_{k+1} - \Delta_{k+1}| \leq (1 + C_{\#} \varepsilon) |H_k - \Delta_k| + \mathcal{O}(\varepsilon^2 + \varepsilon^2 H_k + \varepsilon H_k^2)
$$

$$
\leq C_{\#} (1 + C_{\#} \varepsilon)^k \left(k \varepsilon^2 + \varepsilon \sum_{j=0}^{k} [\varepsilon H_j + H_j^2]\right),
$$

where we repeatedly used the fact that both $\Delta_k$ and $H_k$ are bounded, e.g. to ensure that $|H_k^2 - \Delta_k^2| \leq C_{\#} |H_k - \Delta_k|$.

Next, since

$$
|H_k - \Delta_k| \leq C_T (k \varepsilon^2 + \varepsilon \sum_{j=0}^{k-1} [\Delta_j^2 + |H_j - \Delta_j|^2 + \varepsilon \Delta_j + \varepsilon |H_j - \Delta_j|]),
$$

setting $|H - \Delta| = \sup_{k \leq \varepsilon^{-1} N} |H_k - \Delta_k|$, if $|H - \Delta| \leq \frac{1}{2 C_T N}$, we have

$$
|H - \Delta| \leq C_T (N \varepsilon + \varepsilon \sum_{j=0}^{N \varepsilon^{-1}-1} [\Delta_j^2 + \varepsilon \Delta_j]). \quad \square
$$
5. Averaging

This section is devoted to the proof of Theorem 2.1. Note that it may be possible to obtain this result almost surely rather than in probability. We do not push this venue since it is irrelevant for our purposes. Since the aim of this section is mostly notational and didactic we keep things as simple as possible and provide the proof only for the variable $\theta$, the argument for $\zeta$ being exactly the same.

**Proof of Theorem 2.1.** We prove the theorem in a slightly more general form which is better adapted to our needs: let the initial condition be given by a probability measure supported on a sequence of real standard pairs $\ell_\varepsilon$ that weakly converge, for $\varepsilon \to 0$, to a standard pair $(G_0, \rho)$ where $G_{\theta_0}(x) = (x, \theta_0)$ is the type of initial condition mentioned in the statement of Theorem 2.1. In the following we will drop the subscript $\varepsilon$ in quantities related to the standard pair when this does not create confusion.

Let $h > 0$ and note that

$$|\theta_{(t+h)\varepsilon^{-1}} - \theta_{t\varepsilon^{-1}}| \leq C|h|.$$  \hfill (5.1)

Then the functions $\theta_\varepsilon \in C^0([0, T], \mathbb{R})$, defined in (2.8), are uniformly Lipschitz, hence they form a compact set (by Ascoli-Arzela). Consider then a converging subsequence $\theta_{\varepsilon_j}$. Recall that we defined $\hat{\omega}(x, \theta) = \omega(x, \theta) - \tilde{\omega}(\theta)$; let us start by computing

$$\mu_\ell \left( \varepsilon \sum_{k=1}^{\lfloor h \varepsilon^{-1} \rfloor} \hat{\omega}(x_k, \theta_k) \right)^2 = \sum_{\ell_1 \in \mathcal{C}_1^{(t\varepsilon^{-1})}} \sum_{k=0}^{\lfloor h \varepsilon^{-1} \rfloor} \varepsilon^2 \nu_{\ell_1} \mu_{\ell_1} (\hat{\omega}^2 \circ F^k_{\varepsilon}) +$$

$$\sum_{\ell_1 \in \mathcal{C}_1^{(t\varepsilon^{-1})}} \sum_{j=0}^{\lfloor h \varepsilon^{-1} \rfloor - 1} \sum_{k=j+1}^{\lfloor h \varepsilon^{-1} \rfloor - 1} \varepsilon^2 \nu_{\ell_1} \nu_{\ell_2} \mu_{\ell_1} (\hat{\omega} \circ F^{k-j} \cdot \hat{\omega}),$$

where we repeatedly used Proposition 3.3 and the notation introduced before (3.12) without $\Omega$, since in this case $\Omega = 0$.

Next, we use Lemma 4.1 to introduce, for any standard pair $\tilde{\ell} = (\tilde{G}, \tilde{\rho})$, the diffeomorphisms $Y = Y_{h \varepsilon^{-1}}$ and let $[a^*, b^*] = Y([a, b])$. Let us call $\rho^* = \frac{\rho^* \circ Y^{-1}}{\int_{\mathbb{R}} \rho^* \circ Y^{-1}}$ the push-forward of $\rho$ by $Y$, also let $\theta^*_k = \mu_{\ell_1}(\theta)$. For any functions $\varphi, g \in C^1(\mathbb{T}^2)$ and $k \in \mathbb{N}$, Lemma 4.1 implies

$$\mu_{\ell_1}(g \circ F_{\varepsilon}^k \cdot \varphi) = \int_a^{b^*} \rho^*(x) \varphi(Y^{-1}(x), \theta^*_k) \cdot g(f_{\theta_1}^k(x), \theta_1^k) dx + \mathcal{O}(k \varepsilon \|g\|_{C^1} ||\varphi||_{C^1})$$

$$= \int_a^b \tilde{\rho}(x) f_{\theta_1}^k(x) \cdot g(f_{\theta_1}^k(x), \theta_1^k) dx + \mathcal{O}(k \varepsilon \|g\|_{C^1} ||\varphi||_{C^1}).$$

To continue we introduce one of the main tools in the study of uniformly hyperbolic systems: the Transfer Operator (for now, without potential). Let

$$\mathcal{L}_\theta g(x) = \sum_{y \in f_{\theta_1}^{-1}(x)} \frac{g(y)}{f_{\theta_1}(y)},$$

\hfill 14 But see [33] for a discussion of possible counterexamples.
The basic properties of these operators are well known (see e.g. [4]) but in the following we need several very sophisticated facts that are either not easily found or absent altogether in the literature. To help the reader we have collected all the needed properties in Appendix A.\textsuperscript{15} We can thus estimate the quantity in the second line of (5.2) as

$$\mu_{\ell}(\dot{w} \circ F_{\varepsilon} \cdot \dot{w}) = \int [\mathcal{L}_{\theta_{\ell}^*} (\mathbb{1}_{[a,b]} \rho \dot{w})](x, \theta_{\ell}^*) \cdot \dot{w}(x, \theta_{\ell}^*) \, dx + \mathcal{O}(t^2 \varepsilon) =$$

$$= \int_{T_1} h_{\theta_{\ell}^*}(x) \dot{w}(x, \theta_{\ell}^*) \, dx \int_a^b \rho(x) \dot{w}(x, \theta_{\ell}^*) \, dx + \mathcal{O}(t^2 \varepsilon + \tau^2) =$$

$$= \mathcal{O}(t^2 \varepsilon + \tau^2),$$

since \(\mu_{\theta}(\dot{w}(\cdot, \theta)) = 0\) by construction.

Collecting all the above considerations we obtain

(5.3)

$$\mu_{\ell} \left( \varepsilon \sum_{k=\lfloor \ell \varepsilon^{-1} \rfloor}^{\lfloor \ell \varepsilon^{-1} \rfloor - 1} \dot{w}(x_k, \theta_k) \right)^2 \leq C_{\#}\varepsilon^2 \sum_{k=0}^{\lfloor \ell \varepsilon^{-1} \rfloor - 1} \left[ 1 + \sum_{j=1}^{\lfloor \ell \varepsilon^{-1} \rfloor - k - 1} \{\tau^j + j^2 \varepsilon\} \right]$$

$$\leq C_{\#}[\varepsilon \ell + \varepsilon^{-1} \ell^4].$$

The above implies

$$\mu_{\ell} \left( \left| \theta_{\varepsilon}(t) - \theta_{\varepsilon}(0) - \int_0^t \dot{w}(\theta_{\varepsilon}(s)) \, ds \right|^2 \right)$$

$$\leq t \ell^{-1} \sum_{r=0}^{t \ell^{-1} - 1} \mu_{\ell} \left( \left| \theta_{\varepsilon}(r h + 1) - \theta_{\varepsilon}(r h) - \int_{r h}^{(r+1) h} \dot{w}(\theta_{\varepsilon}(s)) \, ds \right|^2 \right)$$

$$\leq t \ell^{-1} \sum_{r=0}^{t \ell^{-1} - 1} \mu_{\ell} \left( \left| \varepsilon \sum_{k=\lfloor r h \varepsilon^{-1} \rfloor}^{\lfloor (r+1) h \varepsilon^{-1} \rfloor - 1} \dot{w}(x_k, \theta_k) + \mathcal{O}(\varepsilon h) \right|^2 \right)$$

$$\leq C_{\#} t^2 [\varepsilon \ell^{-1} + \varepsilon^{-1} \ell^2 + \varepsilon] \leq C_{\#} t^2 \ell^{2/3},$$

where we have chosen \(h = \varepsilon^{2/3}\). Using Chebyshev inequality we thus have, for any \(t \leq T\),

(5.4) \(\mu_{\ell} \left( \left| \theta_{\varepsilon}(t) - \theta_{\varepsilon}(0) - \int_0^t \dot{w}(\theta_{\varepsilon}(s)) \, ds \right| \geq C_{\#} \varepsilon^{1/8} \right) \leq C_{\#} T^2 \varepsilon^{1/3} \varepsilon^{-1/4} \).

Recalling (5.1), we finally have\textsuperscript{16}

(5.5) \(\mu_{\ell} \left( \left| \theta_{\varepsilon}(t) - \theta_{\varepsilon}(0) - \int_0^t \dot{w}(\theta_{\varepsilon}(s)) \, ds \right| \geq C_{\#} \varepsilon^{1/24} \right) \leq C_{\#} T \varepsilon^{1/24} \).

\textsuperscript{15} For the time being we need only that \(\int g \mathcal{L}_{\phi} \phi = \int g \mathcal{L}_{\theta} \phi\) and that, seen as an operator acting on \(\text{BV}\), \(\mathcal{L}_{\phi}\) has \(1\) as a maximal eigenvalue, a spectral gap, and \(h_{\phi}\) (the eigenfunction associated to the eigenvalue \(1\)) is the \(C^{-1}\) density of the unique absolutely continuous invariant measure of \(\mathcal{L}_{\phi}\).

\textsuperscript{16} Since it suffices to use (5.4) only once in each time interval of length \(\varepsilon^{2/3}\), hence \(\varepsilon^{-2/3}\) times.
Choosing $\varepsilon = \varepsilon_j$ and taking the limit $j \to \infty$ it follows that all accumulation points are measures supported on paths satisfying the integral version of (2.4). Since such differential equation admits a unique solution, the limit exists and is given by the solution of (2.4).

If we consider now the initial conditions of the Lemma, which allow to consider all random variables on the same probability space, we immediately have the result for $\theta$. The results for $\zeta$ is more of the same.

Remark 5.1. The bound (5.5) was obtained by estimating the second moment. This gives a simple argument but the result is not sufficient for our later needs. To get sharper bounds we will need to estimate the exponential moment, which is tantamount to studying large deviations.

6. Deviations from the average

In this section we study the deviations from the average behavior described in the previous section.

Remark 6.1. The results presented here are in the spirit of [18] although more precise, insofar in [18], only a rough upper bound on the rate function is provided. For classical large deviations the exact rate function was derived in [33], but with an estimate of the error largely insufficient to handle moderate deviations, as the function is computed with a mistake of order $o(1)$. Here we estimate the error much more precisely and we are therefore able to study accurately also deviations of order $\varepsilon^n$, with $\alpha < 1/2$. In addition, contrary to [18], we derive not only the upper bound but a lower bound as well, at least for deviations larger than $\varepsilon^{5/12}$. We refrain from obtaining completely optimal results (which can be obtained using the techniques developed in this paper) only to keep the length of the paper (somewhat) under control.

Remark 6.2. Note that we will treat initial conditions that are more general than the ones specified in Theorem 2.2. We will let the random variables $(x, \theta)$, at time 0, to be distributed according to the measures $\mu_\varepsilon$, where $\ell_\varepsilon$ is a sequence of standard pairs that converge to some standard pair $\ell_0$ with $G'_{\ell_0} = 0$, which correspond to the initial condition in Theorem 2.2. In fact, to ease notation, we will often use just $\ell_0$ to mean the sequence $\{\ell_\varepsilon\}$. Also, in the following, we will often not make explicit the dependence of various object on $\ell_0$, which is anyhow assumed fixed (although arbitrary) throughout.

Consider $d \in \mathbb{N}$ and $A = (A_1, \cdots, A_d) \in C^2(T^2, \mathbb{R}^d)$, with $A_1(x, \theta) = \omega(x, \theta)$. As explained in Section 2 we assume that, for each $\sigma \in \mathbb{R}^d$, $(\sigma, A)$ is not cohomologous to a constant. Note that, for convenience, we will often, implicitly, lift $\theta$ to its universal cover.

Let us define\(^{17}\) the polygonalization

\begin{equation}
\gamma_{A, \varepsilon}(t) = \varepsilon \sum_{j=0}^{\lfloor te^{-1} \rfloor - 1} A \circ F_j(x, \theta) + (t - \varepsilon \lfloor te^{-1} \rfloor) A \circ F_{\lfloor te^{-1} \rfloor}(x, \theta).
\end{equation}

observe that $\gamma_{A, \varepsilon} \in C^0([0, T], \mathbb{R}^d) = \{ \gamma \in C^0([0, T], \mathbb{R}^d) \mid \gamma(0) = 0 \}, \gamma_{A, \varepsilon}(t) = (\theta_\varepsilon(t) - \theta, \zeta_\varepsilon(t))$ and, by the same argument as in (5.1) it is uniformly Lipschitz, in

\(^{17}\) Using the notation of (2.8), we have $\gamma_{A, \varepsilon}(t) = z_\varepsilon(t) - z_\varepsilon(0)$.
fact differentiable at all points not multiple of $\varepsilon$. Given a (sequence of) standard pair $\ell_0$ we can consider $\gamma_{A,\varepsilon}$ as a random element of $\gamma_{A,\varepsilon} \in C^0([0,T],\mathbb{R}^d)$ by assuming that $(x, \theta)$ are distributed according to $\ell_0$. In fact, as already mentioned in Section 2, it is more convenient to work directly in the probability space $C^0([0,T],\mathbb{R}^d)$ endowed with the probability measure $\mathbb{P}_{\ell_0,A,\varepsilon}$ determined by the law of $\gamma_{A,\varepsilon}$ under $\ell_0$, that is $\mathbb{P}_{\ell_0,A,\varepsilon} = (\gamma_{A,\varepsilon})_* \mu_{\ell_0}$. In particular, for all functions $g \in C^0(\mathbb{R}^d,\mathbb{R})$ and standard pairs $\ell$, we have defined $\bar{\ell}$ and define the function $Z$ associated to a Transfer Operators associated to a path $\gamma$, the path $\gamma$ is unique solution of equations (2.4) and (2.6) with initial conditions $(\theta_{\ell_0}^0, 0)$. Here we study the deviations from such a limit.

The fundamental object in the theory of large deviations is the rate function. As its definition is a bit involved we start by discussing it in some detail. The reader that is not familiar with the meaning and the use of such a function may have a preliminary look at Section 6.4 where it is made clear the role of the rate function in the statements of the various large and moderate deviations results.

### 6.1. The rate function: definition and properties.

The first object we need to define is a class of Transfer Operators associated to a function $A \in C^2(T^2,\mathbb{R}^d)$ and a parameter $\sigma \in \mathbb{R}^d$: for any function $g \in C^1(T)$, let

$$
(\mathcal{L}_{\theta,\sigma,\mu} g)(x) = \sum_{j_{\mu}(y) = x} \frac{e^{\ell(\gamma_{A}(y,\theta))}}{f_{\mu}(y)} g(y) = e^{\ell(\mu)}(\mathcal{L}_{\theta,\sigma,\gamma}(g))(x),
$$

where, letting $\mu_\theta$ be the unique absolutely continuous invariant measure of $\theta$, we have defined $\mu_\theta(A(\cdot, \theta))$ and $A = A - \bar{A}$. The above operators (acting on $C^1$) are of Perron–Frobenius type (see Lemma A.1), that is, they have a simple maximal eigenvalue and a spectral gap. Let $e^{\chi_\sigma(\mu, \theta)}$ and $e^{\overline{\chi}_\sigma(\mu, \theta)}$ be their maximal eigenvalue, respectively. Obviously $\chi_\sigma(\mu, \theta) = (\sigma, \bar{A}(\theta)) + \overline{\chi}_A(\sigma, \theta)$. For each $\sigma, b \in \mathbb{R}^d$ and $\theta \in T^1$, let

$$
\kappa(\sigma, b, \theta) = (\sigma, b - \bar{A}(\theta)) - \overline{\chi}_A(\sigma, \theta),
$$

and define the function $Z : \mathbb{R}^d \times T \to \mathbb{R} \cup \{+\infty\}$ as

$$
Z(b, \theta) = \sup_{\sigma \in \mathbb{R}^d} \kappa(\sigma, b, \theta).
$$

We can finally give our first definition of the rate function $I_{\text{pre}} : C^0([0,T],\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$

$$
I_{\text{pre}}(\gamma) = \begin{cases} +\infty & \text{if } \gamma \text{ is not Lipschitz, or } \gamma(0) \neq 0 \\ \int_0^T Z(\gamma'(s), \bar{\theta}(s, \theta_{\ell_0}^0)) \, ds & \text{otherwise.} \end{cases}
$$

18 According to the usual probabilistic notation $\gamma(t)$ stands both for the numerical value of the path $\gamma$ at time $t$ and for the function $C^0([0,T],\mathbb{R}^d) \to \mathbb{R}^d$ defined by $\gamma(t)(\tilde{\gamma}) = \tilde{\gamma}(t)$, for all $\tilde{\gamma} \in C^0([0,T],\mathbb{R}^d)$.
Our next task is to investigate the properties of $I$, hence of $Z$. Let $D(\theta)$ be the effective domain of $Z(\cdot, \theta)$, i.e. $D(\theta) = \{b \in \mathbb{R}^d : Z(b, \theta) < +\infty\}$.

**Lemma 6.3.** Assume that, for all $\sigma \in \mathbb{R}^d$ and $\theta \in T^1$, $\langle \sigma, \hat{A}(\cdot, \theta) \rangle$ is not cohomologous to a constant function. Then the following properties hold:

1. $Z(\cdot, \theta)$ is a convex lower semi-continuous function;
2. $D(\theta) = \partial_\sigma X_A(\mathbb{R}^d, \theta) := D_\ast(\theta)$ and is a convex set;
3. setting $U = \{(b, \theta) : \theta \in T, b \in D(\theta)\}, Z \in C^2(U, \mathbb{R}_{\geq 0})$ and analytic in $b$;
4. $D(\theta)$ contains a neighborhood of $\hat{A}(\theta)$;
5. $Z(\hat{A}(\theta), \theta) = 0, \partial_b Z(\hat{A}(\theta), \theta) = 0, \partial_b Z(\hat{A}(\theta), \theta) = 0$, and $Z \geq 0$;
6. $\partial_\theta^2 Z(b, \theta) > 0$, and setting $[\partial_\theta^2 Z(\hat{A}(\theta), \theta)]^{-1} = \Sigma^2(\theta)$ we have

$$
\Sigma^2(\theta) = \mu_\theta \left( \hat{A}(\cdot, \theta) \otimes \hat{A}(\cdot, \theta) \right) + 2 \sum_{m=1}^\infty \mu_\theta \left( \hat{A}(f^m(\cdot), \theta) \otimes \hat{A}(\cdot, \theta) \right)
$$

**Proof.** The zero point follows since, for each $\theta$, $Z(\cdot, \theta)$ is the (convex) conjugate function of a proper function, hence a convex lower semi-continuous function, [41].

By equations (A.10a), (A.10b) and Lemma A.14 it follows that $\chi_A(\cdot, \theta)$ is a strictly convex function. Accordingly $\partial_\sigma X_A$ in an injective map and hence, by the Invariance of Domain Theorem $D_\ast(\theta)$ is open. The equality $D(\theta) = D_\ast(\theta)$ follows then from [41, Corollary 26.4.1]. We have thus proved point one.

Since, for $b \in D_\ast(\theta)$, $Z(b, \theta) = \kappa(a_\ast, \sigma, \theta)$ where $a_\ast = a_*(b, \theta)$ is the unique solution of

$$
(6.6) \quad b = \partial_\sigma X_A(a_\ast, \theta),
$$

point two follows by the Implicit Function Theorem and the perturbation results collected in Appendix A.2, A.3.

By (A.9a) $\partial_\sigma X_A(\sigma, \theta) = \nu_{\theta, (\sigma, A)}(A(\cdot, \theta))$, where $\nu_{\theta, (\sigma, A)}$ is the invariant probability measure associated to the operator (6.2), in particular $\nu_{0, \theta} = \mu_\theta$. Then $\partial_\sigma X_A(0, \theta) = 0$ which implies $\hat{A}(\theta) \in D_\ast(\theta)$, hence proving point three.

Moreover $Z(\hat{A}(\theta), \theta) = 0$ and $Z(b, \theta) \geq -\chi_A(0, \theta) = 0$. Next, a direct computation show that

$$
\partial_b Z(b, \theta) = a_*(b, \theta).
$$

In particular $\partial_b Z(\hat{A}(\theta), \theta) = 0$. Next, $(\partial_b Z)(\hat{A}(\theta), \theta) = -\partial_\theta X_A(0, \theta) = 0$ by (A.19a) and (A.5). This proves point five.

Finally, by [41, Theorem 26.5], $\partial_\theta^2 Z(b, \theta) = [\partial_\sigma^2 X_A(a(b, \theta), \theta)]^{-1}$. This and (A.10a) imply point five. 

**Remark 6.4.** By (A.9a), $\|\partial_\sigma X_A\|_\infty \leq \|A\|_\infty$, thus $D_\ast(\theta)$ is uniformly bounded. It follows that, if $\gamma'(s) \in D_\ast(\theta(s, \theta^*_\alpha))$ and $\sigma_\ast(s)$ is the solution of

$$
(6.7) \quad \gamma'(s) = \partial_\sigma X_A(\sigma, \theta(s, \theta^*_\alpha))
$$

then

$$
(6.8) \quad Z(\gamma'(s), \theta(s, \theta^*_\alpha)) = \kappa(\sigma_\ast(s), \gamma'(s), \theta(s, \theta^*_\alpha))
$$

and $\partial_b Z(\gamma'(s), \theta(s, \theta^*_\alpha)) = \sigma_\ast(s)$.

**Remark 6.5.** Using the above facts, it would be possible to show that $I_{pre}$ is lower semi-continuous with respect to the uniform topology. We refrain from proving it here because the proof will be given later in Lemma 6.11.
6.2. The rate function: entropy.
As already mentioned the rate function can be expressed in terms of entropy (e.g. see [33]).

Lemma 6.6. For each $b \in \mathbb{R}^d, \theta \in \mathbb{T}^1$ we have\footnote{We adopt the convention that the sup over an empty set yields $-\infty$.}
\[
Z(b, \theta) = - \sup_{\nu \in \mathcal{M}_\theta(b)} \{ h_\nu - \nu(\log f'_\theta) \},
\]
where $\mathcal{M}_\theta(b) = \{ \nu \in \mathcal{M}_\theta : \nu(A(\cdot, \theta)) = b \}$, $\mathcal{M}_\theta$ being the set of $f_\theta$-invariant probability measures, and $h_\nu$ is the Kolmogorov-Sinai metric entropy. In particular, $\mathcal{D}(\theta) = \{ b \in \mathbb{R}^d \mid \mathcal{M}_\theta(b) \neq \emptyset \}$.

Proof. It is well known [4, Remark 2.5], that
\[
\chi_A(\sigma, \theta) = \sup_{\nu \in \mathcal{M}_\theta} \{ h_\nu + \nu(\langle \sigma, A \rangle - \log f'_\theta) \} = h_{\nu_{\theta, \langle \sigma, A \rangle}} + \nu_{\theta, \langle \sigma, A \rangle}(\langle \sigma, A \rangle - \log f'_\theta)
\]
where $\nu_{\theta, \langle \sigma, A \rangle}(g) = m_{\theta, \langle \sigma, A \rangle}(gh_{\theta, \langle \sigma, A \rangle})$, $m_{\theta, \langle \sigma, A \rangle}$ and $h_{\theta, \langle \sigma, A \rangle}$ are respectively the left and right eigenvectors of $L_{\theta, \langle \sigma, A \rangle}$ corresponding to the eigenvalue $e^{\chi_A}$, normalized so that $\nu_{\theta, \langle \sigma, A \rangle}$ is a probability measure. We record, for future use, some properties of the entropy: since each $f_\theta$ is expansive, $h_\nu$ is an upper-semicontinuous function of $\nu$, with respect to the weak topology [30, Theorem 4.5.6]. Also $h_\nu$ is a convex affine function of $\nu$, [30, Theorem 3.3.2].\footnote{That is, as a function of $\nu$ is both convex and concave.} Incidentally, this implies that the sup in (6.9) is the same if taken only on the ergodic measures, see [30, Theorem 4.3.7]. Then
\[
Z(b, \theta) = \sup_\sigma \left\{ (\sigma, b) - \sup_{\nu \in \mathcal{M}_\theta} \{ h_\nu + \nu(\langle \sigma, A \rangle - \log f'_\theta) \} \right\} \leq - \sup_{\nu \in \mathcal{M}_\theta(b)} \{ h_\nu - \nu(\log f'_\theta) \}.
\]
On the other hand
\[
Z(b, \theta) = \sup_\sigma \left\{ -h_{\nu_{\theta, \langle \sigma, A \rangle}} + \nu_{\theta, \langle \sigma, A \rangle}(\langle \sigma, b - A \rangle) - \nu_{\theta, \langle \sigma, A \rangle}(\log f'_\theta) \right\}.
\]
Note that the first term on the right hand side is bounded by the topological entropy [30, Theorem 4.2.3], while the last is bounded because $f' > 1$. Thus, if $\sup_\nu |\nu_{\theta, \langle \sigma, A \rangle}(\langle \sigma, b - A \rangle)| = \infty$, it must be $Z(b, \theta) = +\infty$.\footnote{Since $Z(b, \theta) \geq 0$.} Suppose next that $\sup_\nu |\nu_{\theta, \langle \sigma, A \rangle}(\langle \sigma, b - A \rangle)| < \infty$. For each $\lambda \in \mathbb{R}$ consider the function $K_\lambda \in C^0(\mathbb{R}^d, \mathbb{R}^d)$ defined by $K_\lambda(\sigma) = \nu_{\theta, \lambda, \langle \sigma, A \rangle}(b - A)$. Clearly $K_\lambda(\mathbb{R}^d) \subset B = \{ x \in \mathbb{R}^d \mid \|x\| \leq \|b - A\| \}$, by Brouwer fixed-point theorem it follows that there exists $\sigma_\lambda \in B$ such that $K_\lambda(\sigma_\lambda) = \sigma_\lambda$. Accordingly, for each sequence $\lambda_j$, such that $\lambda_j \to +\infty$, we have
\[
\langle \lambda_j \sigma_{\lambda_j}, \nu_{\theta, \langle \lambda_j \sigma_{\lambda_j}, A \rangle}(b - A) \rangle = \lambda_j \|\nu_{\theta, \langle \lambda_j \sigma_{\lambda_j}, A \rangle}(b - A)\|^2 \geq 0.
\]
Since the left hand side is bounded by hypothesis, it must be $\lim_{j \to \infty} \nu_{\theta, \langle \lambda_j \sigma_{\lambda_j}, A \rangle}(b - A) = 0$. We can then consider a subsequence $\{j_k\}$ such that, setting $\sigma_k = \lambda_k \sigma_{\lambda_k}$, $\nu_{\theta, \langle \sigma_k, A \rangle}$ converges weakly to a measure $\nu_*$. Then, for each $k \in \mathbb{N}$
\[
Z(b, \theta) \geq -h_{\nu_{\theta, \langle \sigma_k, A \rangle}} - \nu_{\theta, \langle \sigma_k, A \rangle}(\log f'_\theta).
\]
Given the upper-semicontinuity of the entropy, we have
\[ Z(b, \theta) \geq -\limsup_{k \to \infty} \left[ h_{\nu_b, \sigma_k, A} + \nu_{\theta, i} \sigma_k, A \right] \log f_{\theta}^k \geq -h_{\nu_*} \nu_*(\log f_\theta^k). \]

Finally notice that \( \nu_* \in \mathcal{M}_\theta \) and \( \nu_*(b - A) = 0 \), hence \( \nu_* \in \mathcal{M}_\theta(b) \). Thus we have
\[ Z(b, \theta) \geq -\sup_{\nu \in \mathcal{M}_\theta(b)} \{ h_{\nu_*} + \nu_*(\log f_{\theta}^k) \}, \]

whereby concluding the proof of the Lemma.

\[ \square \]

**Remark 6.7.** Note that Lemmata 6.3 and 6.6 show that \( \mathcal{D}(\theta) \) is a compact set and characterize \( \partial \mathcal{D}(\theta) \) as those values that can be attained as averages of \( A \) with respect to an invariant measure which is not associated to a transfer operator of the type (6.2).

The above Lemma allows to specify exactly the effective domain \( \mathcal{D}(I) \) of \( I: I(\gamma) < \infty \) if and only if \( \gamma(0) = 0 \), \( \gamma \) is Lipschitz and \( \mathcal{M}_{\theta_{i(t, \theta_0)}}(\gamma'(t)) \neq \emptyset \) for almost all \( t \in [0, T] \). We will call \( s \)-admissible the paths such that \( \gamma \in \mathcal{D}(I) \).

To use effectively such a fact, it would be convenient if one could characterize \( s \)-admissibility in terms of periodic orbits. To this end, given a periodic orbit \( p \), let \( \nu_p \) the measure determined by the average along the orbit of \( p \).

**Lemma 6.8.** Given \( \theta \in \mathbb{T} \) and \( b \in \mathbb{R}^d \), \( b \) belongs to the interior of \( \mathcal{D}(\theta) \) if and only if there exist \( d + 1 \) periodic orbits \( \{ p_i \} \) of \( f(\cdot, \theta) \) such that the convex hull of \( \nu_{p_i} (b - A(\cdot, \theta)) \) contains a neighborhood of zero. Also if there exists \( n \in \mathbb{N} \) such that
\[ \inf_{x \in \mathbb{T}} \left| \sum_{k=0}^{n-1} A(f_\theta^k(x), \theta) \right| > 0, \text{ then } b \notin \mathcal{D}(\theta). \]

**Proof.** If there exists \( \delta > 0 \) such that, for all \( b' \in \mathbb{R}^d \), \( \| b - b' \| < \delta \), then there exists \( \{ \alpha_i \}_{i=1}^{d+1} \subset \mathbb{R}_{\geq 0}, \sum_{i=1}^{d+1} \alpha_i = 1 \) such that \( \sum_{i=1}^{d+1} \alpha_i \nu_{p_i}(A(\cdot, \theta)) = b', \) hence \( b' \in \mathcal{D}(\theta) \) and \( b \in \mathcal{D}_*(\theta) \). On the other hand if \( b \in \mathcal{D}_*(\theta) \) then there are \( \{ b_i \}_{i=1}^{d+1} \subset \mathcal{D}_*(\theta) \) such that \( b \) belongs to the interior of their convex hull. Hence there exists \( \nu_i \in \mathcal{M}_\theta(b_i) \) such that their convex combination gives an element of \( \mathcal{M}_\theta(b) \). Since the measures supported on periodic orbits are weakly dense in the set of the invariant measures [39], it is possible to find periodic orbits \( \{ p_i \} \) such that the convex hull of \( \nu_{p_i}(A(\cdot, \theta)) \) contains a neighborhood of \( b \), hence the necessity of the condition.

To prove the other necessary condition, note that, by (A.9a), equation (6.7) reads
\[ b = \nu_{\theta, i} \sigma(A(\cdot, \theta)). \]

Thus \( b \in \mathcal{D}_*(\theta) \) if and only if (6.10) has a solution. If the second condition in the lemma is satisfied, then for each invariant measure \( \nu \) we have \( |\langle b, b - \nu (A(\cdot, \theta)) \rangle| \geq 0 \), hence equation (6.10) cannot be satisfied. Moreover, the same conclusion holds for any \( \bar{b} \) in a small neighborhood of \( b \), hence the claim.

\[ \square \]

**Remark 6.9.** From the above Lemma we see that it is not obvious how to identify exactly the domain where the rate function is infinite, i.e. how to distinguish between impossible and almost impossible paths. Yet, this has limited importance in applications. In particular for the applications we are interested in, it will suffice the characterization stated in Lemma 6.8. To simplify the next discussion it

\[ \text{22 In fact the proof there is for the invertible case but it applies almost verbatim to the present one.} \]
is convenient to build such a limitation into the rate function: for each \( \epsilon > 0 \) let \( \partial_\epsilon \mathcal{D}(\theta) = \{ b \in \mathbb{R}^d : \text{dist}(b, \partial \mathcal{D}(\theta)) < \epsilon \} \) and define

\[
\begin{align*}
Z^+_{\epsilon}(b, \theta) &= \begin{cases} 
\mathcal{Z}(b, \theta) & \text{if } b \notin \partial_\epsilon \mathcal{D}(\theta) \\
+\infty & \text{otherwise,}
\end{cases} \\
Z^-_{\epsilon}(b, \theta) &= \begin{cases} 
\mathcal{Z}(b, \theta) & \text{if } b \notin \partial_\epsilon \mathcal{D}(\theta) \\
\mathcal{Z}(b + \lambda_{b, \theta}(A(\theta) - b), \theta) & \text{otherwise,}
\end{cases}
\end{align*}
\]

(6.11)

where \( \lambda_{b, \theta} = \inf \{ \lambda' \mid b + \lambda'(A(\theta) - b) \in \mathcal{D}(\theta) \setminus \partial_\epsilon \mathcal{D}(\theta) \} \). Note that, by Lemma 6.8, the set \( \partial_\epsilon \mathcal{D}(\theta) \) can be explicitly determined for arbitrarily small \( \epsilon \) by computing longer and longer periodic orbits and ergodic averages (see [10] for discussions on the speed of such approximation).

All the above discussion will suffice to describe the deviation from the average behavior for relatively short times. If we want to study longer times, then we must consider slightly different rate functions. That is, for each \( \epsilon > 0 \), setting \( \theta^\gamma(s) = (\gamma(s))_1 + \theta^\epsilon_{s_0} \),

\[
\mathcal{F}_{\epsilon, \theta^\epsilon_{s_0}}(\gamma) = \begin{cases} 
+\infty & \text{if } \gamma \text{ is not Lipschitz, or } \gamma(0) \neq 0 \\
\int_0^T \mathcal{Z}^\pm(\gamma'(s), \theta^\gamma(s)) \, ds & \text{otherwise.}
\end{cases}
\]

Finally, we define \( \mathcal{F}_{\theta^\epsilon_{s_0}}(\gamma) = \lim_{\epsilon \to 0} \mathcal{F}_{\epsilon, \theta^\epsilon_{s_0}}(\gamma) \). The effective domain \( \mathcal{D}(\mathcal{F}_{\theta^\epsilon_{s_0}}) = \{ \gamma \in C^0 \mid \mathcal{F}_{\theta^\epsilon_{s_0}}(\gamma) < \infty \} \) is given by those paths \( \gamma \) that are Lipschitz, \( \gamma(0) = 0 \) and such that \( \mathcal{M}_{\gamma, t}(\gamma'(t)) \neq 0 \) for almost all \( t \in [0, T] \). On the other hand, the effective domain of \( \mathcal{F}_{\theta^\epsilon_{s_0}} \) is given by the interior of \( \mathcal{D}(\mathcal{F}_{\theta^\epsilon_{s_0}}) \). We call the paths in \( \mathcal{D}(\mathcal{F}_{\theta^\epsilon_{s_0}}) \) admissible. Note that the two functionals only differ on \( \partial \mathcal{D}(\mathcal{F}_{\theta^\epsilon_{s_0}}) \), moreover \( \mathcal{F}_{\theta^\epsilon_{s_0}} \) is lower-semicontinuous. Given the customary use in large deviation theory, it is then natural to consider \( \mathcal{F}_{\theta^\epsilon_{s_0}} = \mathcal{F}_{\theta^\epsilon_{s_0}} \) as the large deviation function.

### 6.3. The rate function: an equivalent definition.

Unfortunately, in our subsequent discussion the rate function will appear first in a much less transparent form, possibly different from the definition (6.5). Namely, recall the notation (6.3) and let, for \( \sigma \in \mathbb{R}^d \),

\[
\tilde{\kappa}_{\gamma}(\sigma, s) = \kappa(\sigma, \gamma'(s), \bar{\theta}(s, \theta^\epsilon_{s_0})).
\]

Also let

\[
\text{Lip}_{C, s} = \{ \gamma \in C^0([0, T], \mathbb{R}^d) : \gamma(0) = 0, ||\gamma(t) - \gamma(s)|| \leq C|t - s| \forall t, s \in [0, T] \},
\]

for some fixed \( C \geq 2\|A\|_{C^0} \). Then, \( I : C^0([0, T], \mathbb{R}^d) \to \mathbb{R} \cup \{ +\infty \} \) will appear naturally as

\[
I(\gamma) = \begin{cases} 
+\infty & \text{if } \gamma \notin \text{Lip}_{C, s} \\
\sup_{\sigma \in \text{BV}} \int_0^T \tilde{\kappa}_{\gamma}(\sigma(s), s) \, ds & \text{otherwise.}
\end{cases}
\]

(6.13)

It is the task of this subsection to show that the two definitions (6.5) and (6.13) coincide. At a superficial level, it amounts to prove that we can bring the sup inside the integral. This will be proved essentially via a compactness argument.

First, since \( I \) is the conjugate function of a proper function it is convex. Moreover, \( I \geq 0 \) (just consider \( \sigma = 0 \) in the sup) and \( I(\hat{A}) = 0 \) (since \( \hat{\chi}_A \geq 0 \)). Our first task is to show that we can replace the sup on \( \sigma \in \text{BV} \) with a sup on \( \sigma \in L^1 \).
Lemma 6.10. For each $\gamma \in \text{Lip}_C$, we have

$$I(\gamma) = \sup_{\sigma \in L^1} \int_0^T \bar{\kappa}_\gamma(\sigma(s), s)ds.$$ 

Proof. Let us start noticing that, by (A.9a), we have $\|\bar{\kappa}_\gamma(\sigma(s))\| \leq C_\# \|\sigma\|$ and $\|\partial_\sigma \bar{\kappa}_\gamma(\sigma(s))\| \leq C_\# \|\sigma\|$ for all $s \in [0, T]$. It follows that, for all $\gamma \in \text{Lip}_C$, the functional $J_\gamma(\sigma) = \int_0^T \bar{\kappa}_\gamma(\sigma(s), s)ds$ is continuous in the $L^1$ topology.

Let $\sigma \in L^1$. Since BV is dense in $L^1$, for each $\epsilon > 0$ there exists $\sigma_\epsilon \in \text{BV}$ such that

$$\int_0^T \bar{\kappa}_\gamma(\sigma(s), s)ds \leq C_\# \epsilon + \int_0^T \bar{\kappa}_\gamma(\sigma_\epsilon(s), s)ds \leq C_\# \epsilon + \sup_{\sigma \in \text{BV}} \int_0^T \bar{\kappa}_\gamma(\bar{\sigma}(s), s)ds.$$ 

Taking the limit $\epsilon \to 0$ first and then sup on $\sigma \in L^1$ we have that the sup on BV equals the sup on $L^1$, proving the lemma. \qed

Lemma 6.11. The functional $I$ is lower semi-continuous on $C^0([0, T], \mathbb{R}^d)$.

Proof. Let $\{\gamma_n\} \subset C^0([0, T], \mathbb{R}^d)$ that converges uniformly to $\gamma$. If $\liminf_{n \to \infty} I(\gamma_n) = +\infty$, then obviously $\liminf_{n \to \infty} I(\gamma_n) \geq I(\gamma)$. Otherwise, there exists a subsequence $\{\gamma_{n_j}\}$, $M > 0$ and $j_0 \in \mathbb{N}$ such that

$$\liminf_{n \to \infty} I(\gamma_n) = \lim_{j \to \infty} I(\gamma_{n_j}),$$ 

and $I(\gamma_{n_j}) \leq M$ for all $j \geq j_0$. This implies that the $\gamma_{n_j}$ are Lipschitz and that $\gamma_{n_j}(0) = 0$, $\|\gamma_{n_j}\|_{\infty} \leq C$. It follows that the same is true for $\gamma$. This implies that, for all $\sigma \in L^1$,

$$\lim_{j \to \infty} \int_0^T \langle \sigma, \gamma_{n_j} \rangle = \int_0^T \langle \sigma, \gamma \rangle.$$ 

Indeed, for all $\epsilon > 0$ let $\sigma_\epsilon \in C^1$, such that $\|\sigma - \sigma_\epsilon\|_{L^1} \leq \epsilon$, then

$$\left| \int_0^T \langle \sigma, \gamma_{n_j} \rangle - \int_0^T \langle \sigma, \gamma \rangle \right| \leq 2C\epsilon + \int_0^T |\langle \sigma_\epsilon(T), \gamma_{n_j}(T) - \gamma(T) \rangle|.$$ 

Thus, for each $\sigma \in L^1$,

$$\liminf_{n \to \infty} I(\gamma_n) \geq \lim_{j \to \infty} \int_0^T \left( \langle \sigma(s), \gamma_{n_j}(s) \rangle - \chi_\Lambda(\sigma(s), \bar{\theta}(s, \theta_{\epsilon_0}^*)) \right) ds$$

$$= \int_0^T \left( \langle \sigma(s), \gamma'(s) \rangle - \chi_\Lambda(\sigma(s), \bar{\theta}(s, \theta_{\epsilon_0}^*)) \right) ds.$$ 

The proof follows by taking the sup on $\sigma$. \qed

At last we can show that the definition of $I$ given in the current section coincides with (6.5).

Lemma 6.12. If $\gamma$ is Lipschitz and $\gamma(0) = 0$, then

$$I(\gamma) = \int_0^T \mathcal{Z}(\gamma(s), \bar{\theta}(s, \theta_{\epsilon_0}^*)) ds = I_{\text{pre}}(\gamma).$$
Proof: If $\gamma \notin \text{Lip}_{C, \ast}$, then, provided $C$ has been chosen large enough, there is a positive measure set in which $\gamma'(t) \notin \mathbb{D}(\theta)$ an hence $\int_0^T Z(\gamma'(s), \bar{\theta}(s, \theta_0^s))ds = \infty$ as required. We can then assume $\gamma \in \text{Lip}_{C, \ast}$. 

Suppose that $z_\ast(t) = Z(\gamma'(t), \bar{\theta}(t, \theta_0^t))$ is not in $L^1$. Let $M > 0$. By Lusin theorem and Lebesgue monotone convergence Theorem there exists $\lambda > 0$ and a compact set $E$ such that $\gamma'$ and $\min\{\lambda, z_\ast(t)\}$ are continuous on $E$ and $\int_E \min\{\lambda, z_\ast(t)\}dt \geq M$. Then for each $t \in E$ let $\sigma_\lambda(t)$ be such that $\kappa(\sigma_\lambda(t), t) \geq \frac{1}{4}\min\{\lambda, z_\ast(t)\}$. Since $\kappa(\sigma(t), s)$ is continuous in $s \in E$, it follows that, for each $t \in E$, there exists and open set $U(t) \ni t$ such that $\kappa(\sigma_\lambda(t), s) \geq \frac{1}{4}\min\{\lambda, z_\ast(t)\}$ for all $s \in U(t) \cap E$. We can then extract a finite sub cover $\{U(t_i)\}$ of $E$ and define $k(s) = \inf\{i : s \in U(t_i)\}$. 

Consequently, setting $z_\lambda(s) = 1_{U(t)}(t) \cdot \min\{\lambda, z_\ast(t)\}$, by Lemma 6.10 we have

$$I(\gamma) \geq \int_0^T \kappa(\sigma_\lambda(s), s)ds \geq \frac{1}{4} \int_0^T z_\lambda(s)ds \geq \frac{M}{4}.$$ 

By the arbitrariness of $M$ it follows $I(\gamma) = +\infty$.

On the other hand, if $z_\ast \in L^1$, then by Lemma 6.10

$$I(\gamma) = \sup_{\sigma \in L^1} \int_0^T \kappa(\sigma(s), \gamma'(s), \bar{\theta}(s, \theta_0^s))ds \leq \int_0^T z_\ast(s)ds < +\infty.$$ 

In addition, $\gamma'(s) \in D(\bar{\theta}(s, \theta_0^s))$ for almost every $s \in [0, T]$. Let us define $\gamma_\varrho(s) = \gamma(s) + \varrho(\bar{A}(\bar{\theta}(s, \theta_0^s)) - \gamma(s))$ for $\varrho \in (0, 1)$ and $s \in [0, T]$. Since $\bar{A}(\bar{\theta}(s, \theta_0^s))$ belongs to the interior of $\mathbb{D}(\theta)$, it follows that, for each $\varrho \in (0, 1)$, there exists a compact set $K_\varrho \subset D_\ast(\theta)$, such that $\gamma_\varrho(s) \in K_\varrho$ for almost all $s \in [0, T]$. Since the inverse of $\partial_s \chi_A(\cdot, \theta)$ is a continuous function, it follows that the preimage of $K_\varrho$ is a compact set. Hence, there exists $\sigma_\varrho \in L^\infty$ such that $\gamma_\varrho(s) = \partial_s \chi_A(\sigma_\varrho(s), \bar{\theta}(s, \theta_0^s))$ for almost all $s \in [0, T]$.

$$I(\gamma) = \lim_{\varrho \to 0} (1 - \varrho) I(\gamma) \geq \liminf_{\varrho \to 0} I(\gamma_\varrho) \geq \liminf_{\varrho \to 0} \int_0^T \kappa(\sigma_\varrho(s), \gamma_\varrho'(s), \bar{\theta}(s, \theta_0^s))ds$$

$$= \liminf_{\varrho \to 0} \int_0^T Z(\gamma_\varrho'(s), \bar{\theta}(s, \theta_0^s))ds \geq \int_0^T \liminf_{\varrho \to 0} Z(\gamma_\varrho'(s), \bar{\theta}(s, \theta_0^s))ds$$

where we have used the convexity of $I$ first, then Lemma 6.10, then equation (6.8), then Fatou Lemma and finally point (0) of Lemma 6.3. The above concludes the proof.

6.4. Large deviations: upper bound (short times).

We are now at last ready to state and prove our results. We start by establishing large deviations results that are optimal only for relatively short times. Then we will use such preliminary results to prove Theorem 2.2.

Note that, by the above mentioned uniform Lipschitz property, there exists $C > 0$ such that, $\exists_{L, A, \varepsilon}(\text{Lip}_C) = 1$ where

$$\text{Lip}_C = \{\gamma \in C^0([0, T], \mathbb{R}^d) : \|\gamma(t) - \gamma(s)\| \leq C|t - s| \forall t, s \in [0, T]\}.$$
Recalling Definition (2.9), for each \( \tilde{\gamma} \in C^0([0, T], \mathbb{R}^d) \) and \( r > 0 \), we set \( B(\tilde{\gamma}, r) = B_{[0, T]}(\tilde{\gamma}, r) \)
and
\[
B_{C, r}(\tilde{\gamma}, r) = \{ \gamma \in B(\tilde{\gamma}, r) \mid \gamma(0) = 0, \gamma \in \text{Lip}_C \}.
\]
By (5.1), we have
\[
P_{\ell, A, \varepsilon}(B(\tilde{\gamma}, r) \setminus B_{C, r}(\tilde{\gamma}, r)) = 0.
\]

**Lemma 6.13** (Upper bound). There exist \( \varepsilon_0, C_0, C_0^* > 0 \) such that, for all \( \varepsilon \leq \varepsilon_0 \), \( T \in [c_1^0 \sqrt{\bar{T}}, c_0^1] \), \( r > 0 \), \( \nu \in (c_1^0 \sqrt{\bar{T}}, \frac{1}{2}) \) and \( \tilde{\gamma} \in C^0([0, T], \mathbb{R}^d) \),
\[
P_{\ell, A, \varepsilon}(B(\tilde{\gamma}, r)) \leq e^{-\varepsilon^{-1} \inf_{\gamma \in B(\tilde{\gamma}, r)} I(\gamma)},
\]
where we defined
\[
R_{\tilde{\gamma}} = \begin{cases} 
C\sqrt{T} ||\Gamma||_{\infty} & \text{if } ||\Gamma||_{\infty} \geq C_0(\varepsilon T) \frac{1}{4} \\
2 \max\{\bar{\gamma} ||\Gamma||_{\infty}, C_0^* \varepsilon T \sqrt{\bar{T}}\} & \text{if } ||\Gamma||_{\infty} \leq C_0(\varepsilon T) \frac{1}{4},
\end{cases}
\]
with \( \bar{\Gamma}(s) = \tilde{\gamma}(s) + (\theta_{t_o}^* - 0, -\tilde{z}(s, \theta_{t_o}^*)) \).

**Proof.** For any \( \nu \in M^d([0, T]) = C^0([0, T], \mathbb{R}^d)' = [C^0([0, T], \mathbb{R})']^d \), recalling (6.15), we have
\[
P_{\ell, A, \varepsilon}(B(\tilde{\gamma}, r)) \leq e^{-\inf_{\nu \in B(\tilde{\gamma}, r)} \nu(\gamma)} \leq e^{-\inf_{\nu \in B_{C, r}(\tilde{\gamma}, r)} \nu(\gamma)}.
\]
It is now time to introduce another of the basic objects in the theory of Large Deviations: the logarithmic moment generating function and its conjugate function
\[
(6.17) \quad \Lambda_{\varepsilon}(\nu) = \varepsilon \log E_{\ell, A, \varepsilon}(\nu)'; \quad \Lambda^*_{\varepsilon}(\gamma) = \sup_{\nu \in M([0, T])} (\nu(\gamma) - \Lambda_{\varepsilon}(\nu)).
\]
Note that \( |\Lambda_{\varepsilon}(\nu)| \leq \varepsilon C_0^* ||\nu|| < \infty \), hence \( \Lambda_{\varepsilon} \) is a convex function. Since \( \Lambda_{\varepsilon}(0) = 0 \), we have \( \Lambda^*_{\varepsilon} \geq 0 \). Moreover \( \Lambda^*_{\varepsilon} : C^0([0, T], \mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\} \) is convex as well and lower semi-continuous (with respect to the uniform topology), being a conjugate function of a proper function. We can then follow the strategy of [13, Exercise 4.5.5]. Note that \( g(\nu, \gamma) = \nu(\gamma) - \Lambda_{\varepsilon}(\nu) \) is a function concave in \( \nu \), continuous in \( \gamma \) w.r.t. the \( C^0 \) topology, also it is convex in \( \gamma \) for any \( \nu \in M([0, T]) \). Finally, \( B_{C, r}(\tilde{\gamma}, r) \) is compact in \( C^0 \). Thus, the Minimax Theorem ([42], but see [34] for an elementary proof)
\[
\sup_{\nu \in M^d} \inf_{\gamma \in B_{C, r}(\tilde{\gamma}, r)} g = \inf_{\gamma \in B_{C, r}(\tilde{\gamma}, r)} \sup_{\nu \in M} g,
\]
The above implies, taking the inf on \( \nu \) in (6.16),
\[
P_{\ell, A, \varepsilon}(B(\tilde{\gamma}, r)) \leq e^{-\varepsilon^{-1} \inf_{\gamma \in B_{C, r}(\tilde{\gamma}, r)} \sup_{\nu \in M} \nu(\gamma) - \Lambda_{\varepsilon}(\nu)} \leq e^{-\varepsilon^{-1} \inf_{\gamma \in B_{C, r}(\tilde{\gamma}, r)} \Lambda^*_{\varepsilon}(\gamma)}.
\]

---

\[\text{23 It follows from the Hölder inequality since, for all } t \in [0, 1], \text{ and } \nu, \nu' \in M([0, T]), \]
\[
\Lambda_{\varepsilon}(t\nu + (1 - t)\nu') = \varepsilon \log E_{\ell, A, \varepsilon} \left( (\nu')^t (\nu')^{1-t} \right)
\leq \varepsilon \log \left( E_{\ell, A, \varepsilon} (\nu')^t E_{\ell, A, \varepsilon} (\nu')^{1-t} \right) = t\Lambda_{\varepsilon}(\nu) + (1 - t)\Lambda_{\varepsilon}(\nu').
\]
The latter estimate looks quite promising; unfortunately it is completely useless
without sharp information on \( \Lambda^*_r \).

We are thus left with the task of computing \( \Lambda^*_r \). It turns out to be convenient
to introduce the function \( \sigma(s) = \varepsilon \nu((s, T]) \). Note that \( \sigma = (\sigma_1, \ldots, \sigma_d) \) with \( \sigma_i \in \BV \),
thus \( \sigma \in \BV^d \), that we will simply call \( \BV \) to ease notation. By definition \( \| \nu \| = \varepsilon^{-1} \| \sigma \|_{\BV} \) and, for any \( \gamma \in B\mathcal{C}_+\),
\begin{equation}
\nu(\gamma) = \varepsilon^{-1} \int_0^T \langle \gamma'(s), \sigma(s) \rangle ds.
\end{equation}

Moreover (recall definition (6.1)):
\begin{align}
\nu(\gamma_{A, \varepsilon}) &= \varepsilon^{-1} \int_0^T \langle \frac{d}{ds} \gamma_{A, \varepsilon}(s), \sigma(s) \rangle ds \\
&= \sum_{k=0}^{[T \varepsilon^{-1}] - 1} \varepsilon^{-1} \langle A \circ F_{\varepsilon}^k, \int_{k \varepsilon}^{(k+1) \varepsilon} \sigma(s) ds \rangle + O(\| \sigma \|_{\BV}(\varepsilon, [T \varepsilon^{-1}], T)) \\
&= \sum_{k=0}^{[T \varepsilon^{-1}] - 1} \langle A \circ F_{\varepsilon}^k, \sigma(\varepsilon k) \rangle + O(\| \sigma \|_{\BV}).
\end{align}

Hence, for all \( \sigma \in \BV \),
\begin{equation}
\mathbb{E}_{A, \varepsilon}(\nu') = \mu_{\varepsilon} \left( \sum_{k=0}^{[T \varepsilon^{-1}] - 1} \langle \sigma(\varepsilon k), A \circ F_{\varepsilon}^k \rangle + O(\| \sigma \|_{\BV}) \right) \\
\leq \varepsilon^{-1} \int_0^T \langle \sigma(s), \tilde{\chi}_A(\sigma(s), \tilde{\theta}(s, \theta_{t_0}^*) \rangle \rangle ds + O(\mathcal{R}(\sigma, T))
\end{equation}

where, in the second line, we have used Proposition B.1 which, using Cauchy–
Schwartz inequality, implies that, for all \( L \in [C_0 \varepsilon_0^{-1/2}] \):
\begin{equation}
\mathcal{R}(\sigma, T) = \left\{ \begin{array}{ll}
L \| \sigma \|_{\BV} + \varepsilon^{-1} \left[ L^{-1} + \min\{ \| \sigma \|_{L_1}, T \} \right] \| \sigma \|_{L_1} + LT & \text{for all } \sigma \in \BV \\
\| \sigma \|_{\BV} + LT + (\log \varepsilon^{-1})^2 | \sigma |_{L_1} + \varepsilon^{-1} | | \sigma |^2_{L_1} + \varepsilon^{-1} L^{-1} | | \sigma |^2_{L_2} & \text{if } \| \sigma \|_{L_\infty} < \sigma_*,
\end{array} \right.
\end{equation}

for some \( \sigma_* > 0 \) small enough. The above implies
\begin{align}
\Lambda^*_r(\gamma) &\geq \sup_{\sigma \in \BV} \int_0^T \{ \langle \sigma(s), \Gamma'(s) \rangle - \tilde{\chi}_A(\sigma(s), \tilde{\theta}(s, \theta_{t_0}^*) \rangle \} ds - \varepsilon C_{r, \mathcal{R}}(\sigma, T) \\
\Gamma(s) &= \gamma(s) - \int_0^s \tilde{\chi}(\tilde{\theta}(s_1, \theta_{t_0}^*) ds_1.
\end{align}

Formula (6.22) closely resembles the rate function discussed in the previous sections.
Unfortunately there is an obvious obstacle: the last term in the first line can in
principle dominate the first one. This possibility can be ruled out, but to see how
it requires a detailed discussion.

Recall (see (6.15)) that we only have to worry about \( \gamma \in \text{Lip}_C \). In order to
control \( \mathcal{R} \) it is convenient to define a polygonalization \( \Gamma_h \) of \( \Gamma \) made of segments of
length \( h \), where \( h = T/N_h \), for \( N_h \in \mathbb{N} \) suitably large to be chosen later. Note that
\( \Gamma'_h = \Pi_h(\Gamma') \) where we have introduced the projector
\begin{equation}
\Pi_h(g)(s) = h^{-1} \int_{h[s h^{-1}]}^{h(s h^{-1} + 1)} g(r) dr.
\end{equation}
Noted that $\|\Pi_h\sigma\|_{\text{BV}} \leq C_h \dot{h}^{-1} \|\Pi_h\sigma\|_{L^1}$. Also, for further use, remark that, if $\|\sigma\|_{L^\infty} > \sigma_*$, then $T^{-1}\|\sigma\|_{\text{BV}} \geq \|\sigma\|_{\text{BV}} + 1$ and, if $\|\sigma\|_{L^\infty} \leq \sigma_*$, then $\|\sigma\|_{L^2}^2 \leq \sigma_* \|\sigma\|_{L^1}$. Hence, since $\Pi_h$ is a contraction in the BV and $L^p$, $p \in [1, \infty)$, norms, we have

$$\mathcal{R}(\Pi_h\sigma, T) \leq \mathcal{R}_1(\sigma, T, h).$$

where $\mathcal{R}_1(\sigma, T, h)$ is equal to $\mathcal{R}(\sigma, T)$ but with all the occurrences of $\|\sigma\|_{\text{BV}}$ replaced by $h^{-1}\|\sigma\|_{L^1}$. The basic advantage of this regularization procedure lies in the following: for any $\gamma \in \text{Lip}_C$ with $\gamma' \in \text{BV}$ define

$$\Omega(\gamma) := \sup_{\sigma \in \text{BV}} \int_0^T \left[ \langle (\Pi_h\sigma(s), \Gamma'(s)) \rangle - \hat{\chi}_A(\Pi_h\sigma(s), \bar{\vartheta}(s, \theta^*_n)) \right] \, ds - \varepsilon C_h \mathcal{R}(\sigma, T)$$

(6.24)

$$\Omega_h(\gamma) := \sup_{\sigma \in \text{BV}} \int_0^T \left[ \langle (\sigma(s), \gamma'(s)) \rangle - \hat{\chi}_A(\sigma(s), \bar{\vartheta}(s, \theta^*_n)) \right] \, ds - \varepsilon C_h \mathcal{R}_1(\sigma, T, h).$$

Then we have:

$$\Omega(\Gamma) \geq \sup_{\sigma \in \text{BV}} \int_0^T \left[ \langle (\Pi_h\sigma(s), \Gamma'(s)) \rangle - \hat{\chi}_A(\Pi_h\sigma(s), \bar{\vartheta}(s, \theta^*_n)) \right] \, ds - \varepsilon C_h \mathcal{R}(\Pi_h\sigma, T)$$

(6.25)

$$\geq \sup_{\sigma \in \text{BV}} \int_0^T \left[ \langle (\sigma(s), \Pi_h\Gamma'(s)) \rangle - \Pi_h \hat{\chi}_A(\sigma(s), \bar{\vartheta}(s, \theta^*_n)) \right] \, ds - \varepsilon C_h \mathcal{R}_1(\sigma, T, h)$$

$$\geq \Omega_h(\hat{\Gamma}_h)$$

where, in the first inequality above we used that $\text{BV} \supset \Pi_h\text{BV}$, the second one follows by $\int_0^T f \Pi_h g = \int_0^T \Pi_h f g$, the convexity of $\chi$ together with Jensen inequality, and (6.23). Finally, in last line we have used $\Pi_h 1 = 1$ and the definition of $\Omega_h$.

Note that, for $s \in [nh, (n + 1)h]$, we have

$$|\hat{\Gamma}_h(s) - \Gamma(s)| \leq \int_{nh}^{(n+1)h} |\hat{\Gamma}'_h(r) - \Gamma'(r)| \, dr \leq 2 \int_{nh}^{(n+1)h} |\Gamma'(r)| \, dr.$$ 

Thus $|\hat{\Gamma}_h - \Gamma|_{\infty} \leq 2Ch$ and $\hat{\Gamma}_h \in \text{Lip}_{2C}$. By (6.22) and (6.25), follows

$$\inf_{\gamma \in \text{BV}, (\gamma, r)} \Lambda^*_\varepsilon(\gamma) \geq \inf_{\gamma \in \text{BV}, (\gamma, r)} \Omega(\Gamma) \geq \inf_{\gamma \in \text{BV}, (\gamma, r + 2Ch)} \Omega_h(\Gamma).$$

The next Sub-Lemma describes the connection between the functional $\Omega$ and the rate function $I$. Observe that, while not being as sharp as it could be, yet it suffices for our purposes while retaining a simple formulation.

**Sub-lemma 6.14.** Let $\gamma \in \text{Lip}_{3C,0}$. If $T \geq \varepsilon^\delta$, and $h = C_h \sqrt[4]{\varepsilon T}$, then, for any $\bar{\vartheta} \in (0, 1)$, we have

$$\Omega_h(\Gamma) \geq \begin{cases} \inf_{\tilde{\gamma} \in B(\gamma, C_{\#} \sqrt[4]{T} \|\Gamma\|_{\infty})} \tilde{I}(\tilde{\gamma}) & \text{if } \|\Gamma\|_{\infty} \geq C_{\#}(\varepsilon T)^{\bar{\vartheta}}, \\ \inf_{\tilde{\gamma} \in B(\gamma, C_{\#} \sqrt[4]{T} \|\Gamma\|_{\infty} + C_{\#} \varepsilon^{\bar{\vartheta}-2} \sqrt[4]{T})} \tilde{I}(\tilde{\gamma}) & \text{if } \|\Gamma\|_{\infty} \in [0, C_{\#}(\varepsilon T)^{\bar{\vartheta}}]. \end{cases}$$

Before proving the above sub-lemma, let us notice that, together with (6.18) and (6.26), it immediately allows to conclude our proof. □

**Proof of Sub-Lemma 6.14.** Before getting to the heart of the argument we need a few preliminary results. To start with, we claim that, for any $\theta$ fixed, if $\Gamma', \sigma \in \mathbb{R}^n$ are such that $\Gamma' = \partial_\sigma \hat{\chi}_A(\sigma, \theta)$, then

$$C_{\#}^{-1} \min\{\|\sigma\|, \|\sigma\|^2\} \leq \langle \sigma, \Gamma' \rangle \leq C_{\#} \min\{\|\sigma\|, \|\sigma\|^2\}.$$
In fact, if \(|\sigma| \geq \sigma_*\), then let \(\hat{\sigma} = |\sigma|^{-1}\sigma\), then, since \(\partial_\sigma^2 \hat{\chi}_A(\cdot, \sigma) \geq C_# \mathbb{1}\) (as quadratic forms) in a neighborhood of zero,

\[
(\sigma, \Gamma') = \int_0^{||\sigma||} \langle \sigma, \partial_\sigma^2 \hat{\chi}_A(\lambda \hat{\sigma}, \theta \hat{\sigma}) \rangle d\lambda \geq C_# ||\sigma||. 
\]

(6.28)

On the other hand, If \(|\sigma| < \sigma_*\), then \(\partial_\sigma^2 \hat{\chi}_A(\lambda \sigma, \theta) \geq C_# \mathbb{1}\) for all \(\lambda \in [0, 1]\) and hence \(\langle \sigma, \Gamma' \rangle \geq C_# ||\sigma||^2\). We have thus obtained the lower bound in (6.27). On the other hand, since \(\Gamma'\) is bounded (recall we assume \(\gamma\) to be Lipschitz), we immediately obtain \(\langle \sigma, \Gamma' \rangle \leq C_# ||\sigma||\); moreover, by the first equality in (6.28) and since \(\partial_\sigma^2 \hat{\chi}_A\) is uniformly bounded from above for all \(\sigma\) (see Remark A.7), we conclude that \(\langle \sigma, \Gamma' \rangle \leq C_# ||\sigma||^2\), which concludes the proof of (6.27).

Next, we need an upper bound on \(I(\gamma)\) when it is finite. Let us introduce the shorthand notation \(\tilde{\chi}_A(\sigma) = \hat{\chi}_A(\sigma(\cdot), \tilde{\theta}(\cdot, \theta_{t_0}^s))\). Let \(\tilde{\sigma}_\lambda\) be the solution of \(\lambda \Gamma' = \partial_\sigma \tilde{\chi}_A(\tilde{\sigma}_\lambda)\) for each \(\lambda \in [0, 1]\). Then let

\[
\varphi(s, \lambda) = \langle \lambda \Gamma'(s), \tilde{\sigma}_\lambda(s) \rangle - \tilde{\chi}_A(\tilde{\sigma}_\lambda(s)).
\]

Note that \(\partial_\lambda \varphi(s, \lambda) = \langle \Gamma'(s), \partial_\lambda \tilde{\sigma}_\lambda(s) \rangle = \langle \partial_\lambda \tilde{\sigma}_\lambda(s), \partial_\sigma^2 \tilde{\chi}_A(\tilde{\sigma}_\lambda) \partial_\lambda \tilde{\sigma}_\lambda(s) \rangle \geq 0\).

Using the upper bound in (6.27), we have

\[
\varphi(s, 1) \leq \partial_\lambda \varphi(s, 1) \leq C_# \min \{||\tilde{\sigma}_1(s)||, ||\tilde{\sigma}_1(s)||^2\}.
\]

Thus, we obtained the bounds

\[
I(\gamma) = \int_0^T \varphi(s, 1) ds \leq C_# \int_0^T \min \{||\tilde{\sigma}_1(s)||, ||\tilde{\sigma}_1(s)||^2\} ds,
\]

(6.29)

\[
I(\gamma) \geq \int_0^T \int_0^{\frac{T}{2}} d\lambda \int_0^{\lambda} \nu(\Gamma'(s), \partial_\sigma^2 \tilde{\chi}_A(\tilde{\sigma}_\nu(s))^{-1} \Gamma'(s)) \geq C_# ||\Gamma'||^2_{L^2}.
\]

We are now ready to prove the Sub-Lemma. Given \(\varrho \in (0, 1)\), recall the notation \(\gamma_\varrho(s) = \gamma(s) + \varrho \left(\int_0^s \tilde{A}(\tilde{\theta}(\tau, \theta_{t_0}^s)) d\tau - \gamma(s)\right)\) and let

\[
\Gamma_\varrho(s) = \gamma_\varrho(s) - \int_0^s \tilde{A}(\tilde{\theta}(\tau, \theta_{t_0}^s)) = (1 - \varrho) \left[\gamma(s) - \int_0^s \tilde{A}(\tilde{\theta}(\tau, \theta_{t_0}^s))\right] = (1 - \varrho) \Gamma(s).
\]

Let \(\varrho = C_# \sqrt{T}\) and assume that \(I(\gamma_\varrho) = \infty\). Then, we want to prove that \(\Omega_h(\Gamma) = \infty\). For each \(M > 0\), there exists \(\sigma \in BV\) such that \(\int_0^T k(\sigma, \gamma_\varrho' - \tilde{\theta}) \geq M\). Define the set \(K = \{s \in [0, T] : ||\sigma(s)|| \geq \sigma_*\}\). Then

\[
M \leq \int_0^T \langle \sigma, \Gamma_\varrho' - \tilde{\chi}_A(\sigma, \tilde{\theta}) \rangle - \int_0^T \langle \sigma, \Gamma'_\varrho \rangle - C_# \int_K ||\sigma|| \leq \int_0^T \langle \sigma, \Gamma'_\varrho \rangle - C_# ||\sigma|| L^1 - T \sigma_*.\]

Accordingly, using the first possibility of (6.21) with \(h = C_# \sqrt{\varepsilon T}\), \(L = T^\frac{1}{4} \varepsilon^{-\frac{1}{4}}\), yields

\[
\Omega_h(\Gamma) \geq \int_0^T \langle \sigma, \Gamma'_\varrho \rangle - \tilde{\chi}_A(\sigma, \tilde{\theta}) + \varrho \int_0^T \langle \sigma, \Gamma' \rangle - C_# \varepsilon T^\frac{1}{4} T^\frac{3}{4} - \left\{\varepsilon^{\frac{1}{4}} T^{-\frac{1}{4}} + T\right\} ||\sigma|| L^1
\]

\[
\geq \frac{M}{2},
\]

provided \(\varepsilon\) is small enough. By the arbitrariness of \(M\) it follows \(\Omega_h(\Gamma) = \infty\).

Next, suppose \(I(\gamma_\varrho) < \infty\) (recall \(\varrho = C_# \sqrt{T}\)). Since there exists a compact set \(K_T\) such that \(\gamma_\varrho(s) \in K_T \subset D_*(s)\), for almost all \(s \in [0, T]\), then \(\Gamma'_\varrho(s) = \partial_\sigma \tilde{\chi}_A(\sigma)\)
has a (unique) solution for almost all \( s \in [0, T] \), let \( \tilde{\sigma}_\theta \in L^\infty \) be such a solution. Again define the set \( K = \{ s \in [0, T] : \|\tilde{\sigma}_\theta(s)\| \geq \sigma_\ast \} \) and the complement \( K^c = [0, T] \setminus K \). Then, from (6.24), using the first possibility of (6.21) with \( h = C_\# \sqrt{T} \), \( L = T^{\frac{1}{4}} \varepsilon^{-\frac{3}{4}} \) and recalling (6.27), yields

\[
\Omega_h(\Gamma) \geq \int_0^T (\tilde{\sigma}_\theta, \Gamma') - \chi_A(\tilde{\sigma}_\theta) + C_\# \int_K \|\tilde{\sigma}_\theta\| + C_\# \int_{K^c} \|\tilde{\sigma}_\theta\|^2 - C_\# T^{\frac{1}{2}} \varepsilon^{\frac{3}{2}}
- \left\{ \varepsilon^{\frac{3}{2}} T^{-\frac{1}{2}} + \min\{T, \|\tilde{\sigma}_\theta\|_{L^1}\} \right\} \|\tilde{\sigma}_\theta\|_{L^1}.
\]

Next, we need divide the discussion in further cases. If \( \|\tilde{\sigma}_\theta\|_{L^1} \geq C_\# T \), then, since \( \int_{K^c} \|\tilde{\sigma}_\theta\| \leq \sigma_\ast T \), \( \int_K \|\tilde{\sigma}_\theta\| \geq \int_{K^c} \|\tilde{\sigma}_\theta\| \). It follows that, given the constraints on \( T \),

\[
\Omega_h(\Gamma) \geq I(\gamma_\theta) \geq \inf_{\tilde{\gamma} \in B(\gamma, C_\# \sqrt{T \|\Gamma\|_\infty})} I(\tilde{\gamma}).
\]

On the other hand, if \( \|\tilde{\sigma}_\theta\|_{L^1} \leq C_\# T \), then

\[
\min\{T, \|\sigma\|_{L^1}\} \|\sigma\|_{L^1} \leq C_\# T \int_K \|\sigma\| + \left( \int_{K^c} \|\sigma\| \right)^2 \leq C_\# T \int_K \|\sigma\| + T \int_{K^c} \|\sigma\|^2.
\]

Thus,

\[
\Omega_h(\Gamma) \geq I(\gamma_\theta) + C_\# \sqrt{T} \left[ \int_K \|\tilde{\sigma}_\theta\| + \int_{K^c} \|\tilde{\sigma}_\theta\|^2 \right] - C_\# T^{\frac{1}{2}} \varepsilon^{\frac{3}{2}} - C_\# \left( \varepsilon T \right)^{\frac{1}{2}} \sqrt{\int_{K^c} \|\tilde{\sigma}_\theta\|^2}
\geq I(\gamma_\theta),
\]

provided \( \int_{K^c} \|\tilde{\sigma}_\theta\|^2 \geq C_\# \varepsilon^{\frac{3}{2}} T^{-\frac{1}{2}} \) or \( \int_K \|\tilde{\sigma}_\theta\| \geq C_\# \max\{\varepsilon^{\frac{3}{2}} T^{-\frac{1}{2}} \sqrt{\int_{K^c} \|\tilde{\sigma}_\theta\|^2}, (T \varepsilon)^{\frac{3}{4}}\} \).

We are then left with the case in which \( \int_{K^c} \|\tilde{\sigma}_\theta\|^2 \leq C_\# \varepsilon^{\frac{3}{2}} T^{-\frac{1}{2}} \) and \( \int_K \|\tilde{\sigma}_\theta\| \leq C_\# \max\{\varepsilon^{\frac{3}{2}} T^{-\frac{1}{2}} \sqrt{\int_{K^c} \|\tilde{\sigma}_\theta\|^2}, (T \varepsilon)^{\frac{3}{4}}\} \), hence \( \int_K \|\tilde{\sigma}_\theta\| \leq C_\# \varepsilon^{\frac{3}{2}} T^{-\frac{1}{2}} \). Note that this, by (6.29), implies

\[
\|\Gamma\|_{L^1} \leq \sqrt{T} \|\Gamma\|_{L^2} \leq C_\# \left( \varepsilon T \right)^{\frac{1}{4}}.
\]

In such a case we repeat verbatim the above discussion but, this time, choosing \( \theta = \frac{1}{2} \theta \in \left[ T^{\frac{1}{2}}, \frac{1}{2} \right] \). It follows \( \Omega_h(\Gamma) \geq I(\gamma_\theta) \) unless \( \int_{K^c} \|\tilde{\sigma}_\theta\|^2 \leq C_\# \varepsilon^{-2} \varepsilon^{\frac{3}{2}} T^{\frac{1}{2}} \) and \( \int_K \|\tilde{\sigma}_\theta\| \leq C_\# \varepsilon^{-1} \max\{\varepsilon^{\frac{3}{2}} T^{\frac{1}{2}} \sqrt{\int_{K^c} \|\tilde{\sigma}_\theta\|^2}, T^{\frac{1}{2}} \varepsilon^{\frac{3}{4}}\} \). This implies that

\[
\int_K \|\Gamma\| = \int_K \|\tilde{\sigma}_\theta\| \leq C_\# \int_K \|\tilde{\sigma}_\theta\| \leq C_\# \varepsilon^{-2} \varepsilon^{\frac{1}{2}} T.
\]

This means that if we define \( \tilde{\gamma} \) such that \( \gamma'(s) = \tilde{\gamma}'(s) \) for \( s \notin K \) and \( \tilde{\gamma}'(s) = 0 \) for \( s \in K \), then \( \tilde{\gamma} \in B(\gamma, C_\# \varepsilon^{-2} \varepsilon^{\frac{1}{2}} T) \) and the corresponding \( \tilde{\sigma}_\theta \) agrees with \( \tilde{\sigma}_\theta \) outside \( K \) and is zero on \( K \), thus \( \|\tilde{\sigma}_\theta\|_{L^\infty} \leq \sigma_\ast \). We can then use the second possibility of (6.21) with \( L = C_\# \varepsilon \varepsilon^{-2} \varepsilon^{\frac{1}{2}} T \) to write

\[
\Omega_h(\tilde{\Gamma}) \geq I(\tilde{\gamma}_\theta) + C_\# \int_{K^c} \|\tilde{\sigma}_\theta\|^2 - C_\# \varepsilon^{\frac{3}{2}} - \mathcal{R}(\tilde{\sigma}_\theta, T, h)
\geq I(\tilde{\gamma}_\theta) + C_\# \varepsilon \|\tilde{\sigma}_\theta\|^2 - C_\# \varepsilon \varepsilon^{\frac{1}{2}} T \|\tilde{\sigma}_\theta\|_{L^2} - C_\# T \|\tilde{\sigma}_\theta\|^2_{L^2} + C_\# \varepsilon
\geq I(\tilde{\gamma}_\theta),
\]

provided \( \|\tilde{\sigma}_\theta\|_{L^1} \geq C_\# \varepsilon^{-1} \varepsilon^{\frac{1}{2}} T \). If the latter inequality is violated, then (6.29) implies \( \|\Gamma\|_{L^1} \leq C_\# \varepsilon^{-1} T \varepsilon^{\frac{1}{2}} \), which concludes the Lemma since, in such a case, \( \gamma_A \in B(\gamma, C_\# \varepsilon^{-2} \varepsilon^{\frac{1}{2}} T) \), thus the inf on \( I \) on the latter set is zero. \( \square \)
Remark 6.15. The above Lemma is based on a trade-off: for small deviations (up to the ones predicted by the CLT) it gives rough estimates, but up to times of order one; while for larger deviations it provides much sharper results but only for short times. To obtain sharper results for short times would entail more work, in particular a sharper version of Lemma B.1 (which can be achieved by using the techniques that we will employ in the next section to prove a local CLT), while to extend the above sharp results to longer times one can simply divide the time interval in shorter ones and use Lemma 6.13 repeatedly (see Theorem 2.2 for a simple implementation).

6.5. Large deviation: lower bound (short times).
Next we study the lower bound. It turns out that in [11] we will need such type of estimates only for large deviations, hence we do not insist in obtaining optimal bounds for all moderate deviations, as this would require considerable additional work. Yet, we can, and will, obtain results for not too small deviations at essentially no extra cost. Also, to simplify the exposition, we will consider trajectories that are not arbitrarily close to being impossible, that is that belong to the domains \( \mathcal{D}_\delta(\theta) = \mathcal{D}(\theta) \setminus \partial_{\delta}\mathcal{D}(\theta) \) for some arbitrarily small, but fixed, \( \delta \).

Lemma 6.16 (Lower bound). For each \( \delta \in (0, \frac{1}{2}) \) there exists \( \varepsilon_\delta, c_0, c_1 > 0 \) such that, for all \( \varepsilon \leq \varepsilon_\delta \), \( T \in [c_1 \varepsilon^{\frac{1}{4}}, c_0] \) and \( \bar{\gamma} \in \text{Lip}_{C,\nu}([0, T]) \), \( \bar{\gamma}'(s) \in \mathcal{D}_\delta(\bar{\theta}(s, \sigma_0^*)) \), for almost all \( s \in [0, T] \),

\[
\mathbb{P}_{\ell, A, \varepsilon}(B(\bar{\gamma}, 5T^{\frac{1}{2}})) \geq e^{-\varepsilon^{-1}I(\bar{\gamma}) - c_\delta \varepsilon^{-1}T^{\frac{1}{2}}}
\]

Remark 6.17. Note that the above estimate converges to the one for the upper bound in Lemma 6.13 only for small \( T \) and deviations large enough such that \( I(\bar{\gamma}) \geq C \# T^{\frac{1}{2}} \).

Proof of Lemma 6.16. First, it is convenient to have a reference path of some controlled smoothness. To this end we introduce again the polygonalization defined just before (6.23): let us call \( \bar{\gamma}_* \) the polygonalization of \( \bar{\gamma} \), so that \( \bar{\gamma}_* = \Pi_h \bar{\gamma}' \). Note that, setting \( \bar{\sigma} \) so that \( \Pi_h \bar{\gamma} = \partial_{\bar{\sigma}} X_A(\bar{\sigma}) \), we have that \( \bar{\sigma} \) is constant on each interval \( [kh, (k + 1)h] \) and, for \( s \in [kh, (k + 1)h] \),

\[
\|\bar{\sigma}(s)\| \leq C_\delta \|\Pi_h \bar{\gamma}((s))\| \leq C_\delta h^{-1} \int_{kh}^{(k+1)h} ||\bar{\gamma}'(s)||ds \leq C_\delta C.
\]

It follows that, choosing \( h = C_\delta T^{\frac{1}{2}} \), \( ||\bar{\sigma}||_{BV} \leq C_\delta Th^{-1} = C_\delta T^{-\frac{1}{2}} \). Also note that we have \( ||\gamma - \gamma_*||_{\infty} \leq T^{\frac{1}{2}} \), provided the constant in the definition of \( h \) has been chosen small enough. It follows that \( B(\bar{\gamma}_*, 5T^{\frac{1}{2}}) \supset B(\bar{\gamma}_*, 4T^{\frac{1}{2}}) \).

Let us now define \( \tilde{\nu} \in \mathcal{M}(\ell, [0, T]) \) such that \( \tilde{\sigma}(s) = \varepsilon \tilde{\nu}((s, T]) \). We define the measure \( \mathbb{F}_{\ell, A, \varepsilon, \tilde{\nu}} \) on \( C^0([0, T], \mathbb{R}^d) \) defined by

\[
\mathbb{F}_{\ell, A, \varepsilon, \tilde{\nu}}(\mathcal{A}) = \frac{\mathbb{E}_{\ell, A, \varepsilon}(\tilde{\nu}^p \mathcal{A})}{\mathbb{E}_{\ell, A, \varepsilon}(\tilde{\nu}^p)},
\]

for each continuous functional \( \mathcal{A} \) on \( C([0, T], \mathbb{R}^d) \).

Sublemma 6.18. There exists \( \varepsilon_\delta > 0 \) such that, setting \( K = \{ \gamma \in C^0([0, T], \mathbb{R}^d) : ||\gamma - \gamma_*||_{\infty} \geq 4T^{\frac{1}{2}} \} \), for each \( \varepsilon \leq \varepsilon_\delta \), we have

\[
\mathbb{F}_{\ell, A, \varepsilon, \tilde{\nu}}(K) \leq \frac{1}{2}.
\]
Proof. The idea is to cover the support of $\mathbb{P}_{\ell,A,\varepsilon,\varphi}$ with finitely many sufficiently small balls whose measure we can estimate using the previously obtained upper bound Lemma 6.13. In order to do so we divide the interval $[0, T]$ in subintervals of length $c_\# r_{\varepsilon} r_{\varepsilon} = \varepsilon T^{-\frac{3}{2}}$, so that a path in the support of the measure can move in any given subinterval by at most $2r_{\varepsilon}$. This means that there exist $e^{c_\# r_{\varepsilon}^{-1} T}$ paths $\{\gamma_i\}$ such that we can cover the support of the measure with $\bigcup_i B(\gamma_i, r_{\varepsilon})$. Thus, define $J = \{i : \|\gamma_i - \gamma_i\|_\infty \geq 3T^\frac{3}{2}\}$; then

$$\mathbb{P}_{\ell,A,\varepsilon,\varphi}(K) \leq \sum_{i \in J} \mathbb{P}_{\ell,A,\varepsilon,\varphi}(B(\gamma_i, r_{\varepsilon})).$$

Using (6.17), (6.18) and (6.26) yields

$$\mathbb{P}_{\ell,A,\varepsilon,\varphi}(B(\gamma_i, r_{\varepsilon})) = e^{-c_\# \lambda_i(\varphi)} \mathbb{P}_{\ell,A,\varepsilon}(e^{\varphi} \mathbb{1}_{B(\gamma_i, r_{\varepsilon})})$$

$$= e^{-c_\# \lambda_i(\varphi)} + O(e^{c_\# r_{\varepsilon}\|\sigma\|_\infty}) \mathbb{P}_{\ell,A,\varepsilon}(B(\gamma_i, r_{\varepsilon})).$$

Next, we set $\Xi_i = \inf_{\gamma \in B(\gamma_i, r_{\varepsilon})} J(\gamma)$, with $R = 2T^\frac{3}{2}$, and we apply Lemma 6.13. To do so, we start by noticing that if $\|\Gamma_i\|_\infty \geq 2CT$, then $\mathbb{P}_{\ell,A,\varepsilon}(B(\gamma_i, r_{\varepsilon})) = 0$. Next, if $2CT > \|\Gamma_i\|_\infty \geq C_0\varepsilon T^\frac{3}{2}$, then $R_{\gamma_i} = C\#\sqrt{T}\|\Gamma_i\|_\infty \leq C\# T^\frac{3}{2} \leq R$. If $C\#\varepsilon T^\frac{3}{2} \geq \|\Gamma_i\|_\infty \geq C\#\sqrt{T}$, then choose $\vartheta = C\# T^\frac{3}{2}$ it follows $R_{\gamma_i} = C\# T^\frac{3}{2} \|\Gamma_i\|_\infty \leq R$. If $C\# T^{-\frac{3}{2}} \|\Gamma_i\|_\infty \geq C\#\sqrt{T}$, then we choose $\vartheta = C\# T^\frac{3}{2}$, and $R_{\gamma_i} \leq C\# T^{\frac{3}{2}} \|\Gamma_i\|_\infty \leq R$, thanks to the hypothesis on $T$. Finally, if $C\#\varepsilon T \geq \|\Gamma_i\|_\infty$, we again choose $\vartheta = 0$ and $R_{\gamma_i} \leq C\#\varepsilon T \leq R$. In conclusion Lemma 6.13 yields

$$\mathbb{P}_{\ell,A,\varepsilon}(B(\gamma_i, r_{\varepsilon})) \leq e^{-c_\# -\Xi_i}.$$ 

Substituting the above estimate in (6.30), recalling (6.3), (6.4), (6.5), and using (6.20) we have

$$\mathbb{P}_{\ell,A,\varepsilon,\varphi}(B(\gamma_i, r_{\varepsilon})) \leq e^{-c_\# |\mathcal{R}(\sigma, T)| + C\# r_{\varepsilon}\|\sigma\|_\infty} \mathbb{P}_{\ell,A,\varepsilon}(B(\gamma_i, r_{\varepsilon})).$$

Note that we need to worry about only the $\gamma$ for which $\mathcal{Z}(\gamma', \vartheta)$ is integrable. To do so, define the function $H(\lambda) := (\lambda(\gamma' - \gamma'_*) \vartheta) - \mathcal{Z}(\gamma'_* + \lambda(\gamma' - \gamma'_*), \vartheta) + \mathcal{Z}(\gamma'_*, \vartheta)$. Note that, by the discussion around (6.8), we have that the function is concave and $H(0) = H'(0) = 0$,

$$H''(\lambda) = -\langle \gamma' - \gamma'_*, \partial_\sigma \hat{\lambda} (\sigma) \rangle^{-1} (\gamma' - \gamma'_*) \leq -C\# \|\gamma' - \gamma'_*\|^2,$$

where $\gamma'_* = \partial_\sigma \hat{\lambda} (\sigma)$. Thus

$$\langle \gamma' - \gamma'_*, \vartheta \rangle - \mathcal{Z}(\gamma'_*, \vartheta) - \mathcal{Z}(\gamma'_*, \vartheta) = H(1) \leq -C\# \|\gamma' - \gamma'_*\|^2.$$

Remember that, choosing $L = T^{1/3}$ in the first possibility of the statement of Proposition B.1,

$$\mathcal{R}(\sigma, T) \leq C\# \left\{ T^{-\frac{3}{2}} + \varepsilon^{-1} T^{\frac{3}{2}} \right\} \leq C\# \varepsilon^{-1} T^\frac{3}{2},$$

provided $\varepsilon$ is small enough and where we have used that $T \geq \varepsilon^{1/3}$. Thus,

$$\mathbb{P}_{\ell,A,\varepsilon,\varphi}(B(\gamma_i, r_{\varepsilon})) \leq e^{-C\# \varepsilon^{-1} \inf_{\gamma \in B(\gamma_i, r_{\varepsilon})} \|\gamma' - \gamma'_*\|^2} + C\# T^\frac{3}{2}.$$
To conclude, note that, for \( i \in J, \| \gamma' - \gamma'_i \|_{L^2}^2 \geq C_\# T^\frac{5}{2} \).\(^{24}\) Accordingly,

\[
\mathbb{P}_{\ell,A,\varepsilon} \left( K \right) \leq e^{-C_\# \varepsilon^{-1} T^\frac{5}{2} + C_\# \varepsilon^{-1} T^\frac{5}{2}} \leq \frac{1}{2},
\]

provided \( \varepsilon \) is small enough. \( \square \)

To conclude one must compute the relation between the measure \( \mathbb{P}_{\ell,A,\varepsilon} \) and the measure \( \mathbb{P}_{\ell,A,\varepsilon} \). By Sub-Lemma 6.18 and using again (6.32) we have

\[
\frac{1}{2} \leq \mathbb{P}_{\ell,A,\varepsilon}(B(\tilde{\gamma}_s, 4T^{\frac{5}{2}})) = \frac{\mathbb{E}_{\ell,A,\varepsilon}(e^{\bar{p} B(\tilde{\gamma}_s, 4T^{\frac{5}{2}})})}{\mathbb{E}_{\ell,A,\varepsilon}(e^{\bar{p}})}
\leq C_\# e^{-T (|\tilde{\gamma}_s| - c_\# T^{\frac{5}{2}})} \mathbb{E}_{\ell,A,\varepsilon}(e^{\bar{p} B(\bar{\gamma}_s, 4T^{\frac{5}{2}})})
\leq C_\# e^{-T (|\tilde{\gamma}_s| - c_\# T^{\frac{5}{2}})} \mathbb{E}_{\ell,A,\varepsilon}(B(\bar{\gamma}_s, 4T^{\frac{5}{2}})).
\]

By the definition of \( \bar{\sigma} \) and recalling (6.13) we have

\[
\mathbb{P}_{\ell,A,\varepsilon}(B(\bar{\gamma}_s, 5T^{\frac{5}{2}})) \geq \mathbb{P}_{\ell,A,\varepsilon}(B(\bar{\gamma}_s, 4T^{\frac{5}{2}})) \geq e^{-T (|\tilde{\gamma}_s| - c_\# T^{\frac{5}{2}})}.
\]

The Lemma follows since, arguing like in (6.25), \( I(\bar{\gamma}) \geq I(\bar{\gamma}_s) \). \( \square \)

### 6.6. Large and moderate deviations: long times.

Lemmata 6.13, 6.16 are the basic ingredients to prove all wanted results, in particular Theorem 2.2. Their only drawback is that they are really effective only for small times. It is then natural to subdivide a trajectory in shorter time subintervals and apply the mentioned Lemmata to each such subinterval. To this end, some type of Markov-like property is needed. Before introducing such property we need a bit of notation.

Let us consider a path \( \bar{\gamma} \in C^0([0, T], \mathbb{R}^d) \), a standard pair \( \ell_0 \) and a number \( r > 0 \). For each standard pair \( \ell \) and times \( t \in [0, T] \) let \( \mathbb{1}_{\ell,r} (\ell, t) = 1 \) if \( \| \theta^*_\ell - \gamma_1(t) - \theta^*_{\ell_0} \| < r \) and zero otherwise. Analogously we define \( \mathbb{1}_{\ell,r} (\ell, t) = 1 \) if \( \sup_{s \in [0, T]} \| (\mathbb{G}(x) - \gamma_1(t)) \| < r \) and zero otherwise. Finally let, for each interval \( E = [a, b] \),

\[
P_-(E, \bar{\gamma}, r) = \sup_{\{ \ell : \mathbb{1}_{\ell,r} (\ell, b) = 1 \}} \mathbb{P}_{\ell,A,\varepsilon}(B_{[0,b-a]}(\bar{\gamma}(a + \cdot) - \gamma(a), r)),
\]

\[
P_+(E, \bar{\gamma}, r) = \inf_{\{ \ell : \mathbb{1}_{\ell,r} (\ell, b) = 1 \}} \mathbb{P}_{\ell,A,\varepsilon}(B_{[0,b-a]}(\bar{\gamma}(a + \cdot) - \gamma(a), r)).
\]

**Lemma 6.19.** For each \( \tau \in \mathbb{R}_{\geq 0} \) and \( k \in \{ 0, 1, \ldots, T \tau^{-1} \} \),\(^{25}\) let \( t_k = k \tau \in [0, T] \), \( K = T \tau^{-1} \) and \( E_k = [t_k, t_{k+1}] \), then

\[
\prod_{k=0}^{K-1} P_+(E_k, \bar{\gamma}, r/2) \leq \mathbb{P}_{\ell,A,\varepsilon}(B_{[0,T]}(\bar{\gamma}, r)) \leq \prod_{k=0}^{K-1} P_-(E_k, \bar{\gamma}, r + C_\# r),
\]

where the first inequality holds only if \( \tau \leq C_\# r \).

---

\(^{24}\) Note that there exists \( a \in [0, T] \) such that \( T^{\frac{5}{2}} \leq \| \gamma' - \gamma'_i \|_{L^2} \leq \| \gamma' - \gamma'_i \|_{L^2} \sqrt{T}. \)

\(^{25}\) For simplicity we assume that \( T \tau^{-1}, \varepsilon^{-1} \tau \in \mathbb{N} \), the general case being identical a part for the need of a heavier notation.
Proof. We discuss only the upper bound, the lower bound being completely similar.

Applying (3.12), with $\Omega = 0$ and $n = \varepsilon^{-1} \tau$, and recalling the definition of $P_{\ell_0, A, \varepsilon}$ at the beginning of Section 6, we have

$$P_{\ell_0, A, \varepsilon} \left( B_{[0, T]}(\bar{\gamma}, r) \right) \leq P_{\ell_0, A, \varepsilon} \left( \bigcap_{k=1}^{K} B_{\ell_k}(\bar{\gamma}, r) \right)$$

$$\leq \bar{\mu}_{\ell_0} \left( \bigcap_{k=1}^{K} \left\{ (x, z) \in \mathbb{T}^1 \times \mathbb{R}^d \mid \| \bar{\gamma}(t_k) + (\theta_{t_0}^r, 0) - P^{k\tau \varepsilon^{-1}}(x, z) \| < r + C\# \varepsilon \right\} \right)$$

$$\leq \sum_{\ell_1 \in \mathcal{L}_{0_{\ell_0}}} \nu_{\ell_1} \mathbb{1}_{B_{E_{0_{\ell_1}}}((\bar{\gamma}, r))}((\ell_1, \tau)) \bar{\mu}_{\ell_1} \left( \bigcap_{k=1}^{K-1} \left\{ (\| \bar{\gamma}(t_{k+1}) + (\theta_{t_1}^r, 0) - P^{k\tau \varepsilon^{-1}}(x, z) \| < r + C\# \varepsilon \right\} \right)$$

$$\leq \sum_{\ell_1 \in \mathcal{L}_{0_{\ell_0}}} \cdots \sum_{\ell_{K_{1}} \in \mathcal{L}_{0_{\ell_{K_{1}-1}}}} \prod_{k=1}^{K} \nu_{\ell_k} \mathbb{1}_{B_{E_{0_{\ell_k}}}((\bar{\gamma}, r))}((\ell_k, k\tau)) \leq \prod_{k=1}^{K} P_{E_k, (\bar{\gamma}, r + C\# \varepsilon)}.$$ 

We are now ready to give the promised

**Proof of Theorem 2.2.** The following is just a refinement of the argument in Sub-Lemma 6.18. As discussed in Remark 6.2, we consider a slightly more general case than in the statement of the Theorem: the initial condition is distributed according to a (sequence of) standard pair $\ell_0$.

Let us fix some $\varepsilon > 0$. The following discussion will not depend from the choice of this parameter, apart from the numerical values of certain constants that we will generically designate by $C_{\varepsilon}$. For further use we introduce the notations,

$$\mathcal{S}_{\theta, \varepsilon, [a, b]}(\gamma) = \begin{cases} +\infty, & \text{if } \gamma(0) \neq 0 \text{ or } \gamma \not\in \text{Lip} \\ \int_a^b \mathcal{Z}_{\varepsilon}(\gamma'(s), \gamma(s) + \theta) \, ds, & \text{otherwise.} \end{cases}$$

(6.34)

$$I_{\theta, [a, b]}(\gamma) = \begin{cases} +\infty, & \text{if } \gamma(0) \neq 0 \text{ or } \gamma \not\in \text{Lip} \\ \int_a^b \mathcal{Z}(\gamma'(s), \theta(s - a, \theta)) \, ds, & \text{otherwise.} \end{cases}$$

We will also use the shorthand notation $I_{\theta}$ and $\mathcal{S}_{\theta, \varepsilon}$ for $I_{\theta, [0, T]}$ and $\mathcal{S}_{\theta, \varepsilon, [0, T]}$ respectively.

To start with we divide $[0, T]$ in subintervals of length $\zeta_{\varepsilon} = C_{\varepsilon} \int_{\varepsilon}^{1}$, for some fixed $C_{0}$ small enough, and construct a special class of paths. Let $t_j = j \zeta_{\varepsilon}$ and consider the paths $\gamma$ that, in each subinterval $[t_j, t_{j+1}]$, satisfy $\gamma'(s) = \bar{A}(\gamma(s)) + G_{t_0} + a_j$ with $a_j \in \mathbb{R}^d$ and where $\gamma(s)_1$ is the first component of the vector $\gamma(s)$. Let $\xi(a_j) = \gamma(t_{j+1} - a_j) - \gamma(t_j)$ and notice that $\partial_{a_j} \xi(a_j) = \zeta_{\varepsilon} \mathbb{1} + \mathcal{O}(\zeta_{\varepsilon}^{2})$. Thus we can choose the $a_j$ such that $\bar{A}(\gamma(s)) \in \{ \zeta_{\varepsilon} \hat{a} \}_{\hat{a} \in \mathbb{Z}^d}$. Note that, for such a set, there exist some fixed $c > 0$ such that $c^{-1} \geq ||a_j|| \geq c$. Hence, given the value at the left hand side of a sub-interval on the right hand side the polygonal can take, at most, $C_{\#}$ different relevant values while the other possibilities yield paths with a large Lipschitz constants that, consequently, do not belong to the support of the measure. Let $\{ \gamma_i \}$ be a collection of paths as above and such that $\bigcup_i B_{[0, T]}(\gamma_i, \zeta_{\varepsilon})$ covers the $C$-Lipschitz functions.

Let $\mathcal{A}_{i,j} = ||\gamma_i'(s) - \bar{A}(\gamma_i(s))||$ for $s$ belonging to the sub-interval $j$. Let $K_i = \# \{ j : a_{i,j} \neq 0 \}$, by (6.29),

$$\mathcal{S}_{\theta_{1,0}}(\gamma) \geq C_{\#} \sum_j a_{i,j}^2 \zeta_{\varepsilon} \geq C_{\#} \zeta_{\varepsilon} K_i.$$
Thus, given $M > 0$, the sets $I^\pm_M = \{ i : A^\pm_{\theta_{i\ell}}(\gamma_i) \in [M, 2M] \}$, for $M > \varsigma$, and $I^\pm_M = \{ i : A^\pm_{\theta_{i\ell}}(\gamma_i) \in [0, M] \}$, otherwise, consists of paths with $K_i \leq C_M \varsigma^{-1}$.

Hence, for $M \leq C_M (\log \varsigma^{-1})^2$, by Stirling formula we have
\begin{equation}
\# I^\pm_M \leq \left( \frac{C_M}{\varsigma} \right)^{M \varsigma^{-1}} \leq C_M M \varsigma^{-1} \log \varsigma^{-1}.
\end{equation}

While for larger $M$ we have the trivial bound $e^{C_M \varsigma^{-1}}$. Let $Q \subset C([0, T], \mathbb{R}^d)$ and $I^+_{Q,M} = \{ i \in I^+_M : B_{[0,T]}(\gamma_i, \varsigma) \cap Q \neq \emptyset \}$, $I^-_{Q,M} = \{ i \in I^-_M : B_{[0,T]}(\gamma_i, \varsigma) \subset Q \}$.

It then convenient to write
\begin{equation}
P_{\ell, A, \epsilon}(Q) \leq \sum_{n=0}^{\infty} \sum_{i \in I^-_{Q,2^n \varsigma}} \mathbb{E}_{\ell, A, \epsilon} \left( \mathbf{1}_{B_{[0,T]}(\gamma_i, \varsigma)} \right).
\end{equation}

Conversely,
\begin{equation}
P_{\ell, A, \epsilon}(Q) \geq \sup_{n \in \mathbb{N}} \sup_{i \in I^+_{Q,2^n \varsigma}} \mathbb{E}_{\ell, A, \epsilon} \left( \mathbf{1}_{B_{[0,T]}(\gamma_i, \varsigma)} \right).
\end{equation}

We start by computing the expectations on the right hand side of (6.36) and (6.37). The basic idea is to apply Lemma 6.19 with appropriate time steps $\tau$. More precisely, the choice of $\tau$ will depend on $\gamma_i$ and will be done at the end of the argument.

For each standard pair $\ell$ such that $\mathbf{1}_{\gamma_i, \varsigma}(\ell, t_k) = 1$ we have the average behavior given by $z^*_{k,\ell}(s) = \tilde{A}(z_{k,\ell}(s)), s \in E_k = [t_k, t_{k+1}]$ and $z_{k,\ell}(t_k) = (\theta^*_{\epsilon} t_k, 0)$.

For convenience we will at time $\tilde{A}_{k,\ell}(s)$ to designate $z_{k,\ell}(s + t_k)$ and set $\tilde{A}_{k,\ell}(t) = z_{k,\ell}(t) - (\theta^*_{\epsilon} t_k, 0)$. Note that $|\gamma_i(t_k) + \theta^*_{\epsilon} t_k - \theta^*_{\epsilon} t_k| \leq C_M \varsigma$, moreover a direct computation shows that $\Delta_i(k) = \sup_{s \in E_k} \|\gamma_i(s) - \gamma_i(t_k)\| + C_M \varsigma$. Since it will be often convenient to work in the interval $E_0$, we define the translation of the path $\gamma_i$ from $E_k$ to $E_0$, that is
\[ \gamma^k_i(s) = \gamma_i(t_k + s) - \gamma_i(t_k). \]

In the case $\Delta_i(k) = 0$ we use the trivial bound
\[ P_{\ell, A, \epsilon}(B_{E_0}(\gamma_i^k, \varsigma)) \leq 1. \]

Otherwise $\Delta_i(k) \in [C_M \varsigma, C_M T]$ and some computation is in order. We apply Lemma 6.13 in each interval $E_k$ with the choice\footnote{Note that we can choose the constant so that $\vartheta_i, k \in (c_i \sqrt{T}, \frac{1}{2})$ as required in Lemma 6.13.}
\[ \vartheta_i, k = C_M (\epsilon T)^{1/2} \Delta_i(k), \]
\[ R_i(k) = \left\{ \begin{array}{ll}
\epsilon T^{1/2} \Delta_i(k), & \text{if } \Delta_i(k) \geq C_0(\epsilon T)^{1/2} \\
(\epsilon T)^{1/2} \Delta_i(k), & \text{otherwise}.
\end{array} \right. \]

Note that, by choosing the constants properly, one can ensure that $R_i(k) < C_M \Delta_i(k)$, for any fixed constant $C_M > 0$. Thus
\begin{equation}
P_{\ell, A, \epsilon}(B_{E_0}(\gamma_i^k, \varsigma)) \leq e^{-\epsilon^{-1} \inf_{\gamma \in E_0} \tau_{\epsilon, \gamma_i, r_i, k}} f_{\epsilon, \gamma_i, r_i, k}^{\tau_{\epsilon, \gamma_i, r_i, k}}(\gamma, r_i, k, \gamma_i). 
\end{equation}

where $\gamma_i^k(s) = \gamma_i(t_k + s) - \gamma_i(t_k)$. 

In each interval $E_k$ with the choice
\[ \vartheta_i, k = C_M (\epsilon T)^{1/2} \Delta_i(k), \]
\[ R_i(k) = \left\{ \begin{array}{ll}
\epsilon T^{1/2} \Delta_i(k), & \text{if } \Delta_i(k) \geq C_0(\epsilon T)^{1/2} \\
(\epsilon T)^{1/2} \Delta_i(k), & \text{otherwise}.
\end{array} \right. \]

Note that, by choosing the constants properly, one can ensure that $R_i(k) < C_M \Delta_i(k)$, for any fixed constant $C_M > 0$. Thus
\begin{equation}
P_{\ell, A, \epsilon}(B_{E_0}(\gamma_i^k, \varsigma)) \leq e^{-\epsilon^{-1} \inf_{\gamma \in E_0} \tau_{\epsilon, \gamma_i, r_i, k}} f_{\epsilon, \gamma_i, r_i, k}^{\tau_{\epsilon, \gamma_i, r_i, k}}(\gamma, r_i, k, \gamma_i). 
\end{equation}

where $\gamma_i^k(s) = \gamma_i(t_k + s) - \gamma_i(t_k)$. 

\[ \gamma_i^k(s) = \gamma_i(t_k + s) - \gamma_i(t_k). \]
Given $\gamma \in B_{E_0}(\gamma^k, R_{i,k})$ if $I_{0^\epsilon, E_0}(\gamma) = \infty$, then obviously $I_{0^\epsilon, E_0}(\gamma) \geq I_{0^\epsilon, E_0}(\gamma)$, we can then assume that $Z(\gamma'(s), \tilde{\theta}_{k,l}(s)) < \infty$, i.e. $\gamma'(s) \in \mathbb{D}(\tilde{\theta}_{k,l}(s))$, almost surely. By definition (6.11), setting $\varphi_{i,k} = \gamma_i(t_k) + \theta_{i,0}^*$, we have that
\[ Z^-(\gamma'(s), \gamma(s)_1 + \varphi_{i,k}) = Z(\gamma'_{\lambda}(s), \gamma(s)_1 + \varphi_{i,k}), \]
where $\lambda$ is such that $\gamma'_{\lambda}(s) = \gamma'(s) + \lambda(\tilde{A}(\gamma(s)_1 + \varphi_{i,k}) - \gamma'(s)) \notin \partial_s \mathbb{D}(\gamma(s)_1 + \varphi_{i,k})$.

Recalling (6.29), we have
\[ \|\tilde{\theta}_{k,l} - (\gamma)_1 - \varphi_{i,k}\|_{L^\infty(E_0)} \leq \int_{E_0} \|\tilde{A}(\tilde{\theta}_{k,l}(s)) - \gamma'(s)\| ds + C_\# \zeta \epsilon \]
(6.40)
\[ \leq C_\# \left[ \tau \int_{E_0} \|\tilde{A}(\tilde{\theta}_{k,l}(s)) - \gamma'(s)\| ds \right]^\frac{1}{2} + C_\# \zeta \epsilon \]
\[ \leq C_\# \sqrt{\tau I_{0^\epsilon, E_0}(\gamma)} + C_\# \zeta \epsilon. \]

By Lemma 6.3-(2) we can expand $Z$ to second order around the point $(\gamma'_{\lambda}(s), \tilde{\theta}_{k,l}(s))$
\[ Z^-(\gamma'(s), \gamma(s)_1 + \varphi_{i,k}) = Z(\gamma'_{\lambda}(s), \tilde{\theta}_{k,l}(s)) + \partial_s Z(\gamma'_{\lambda}(s), \tilde{\theta}_{k,l}(s))(\gamma(s)_1 + \varphi_{i,k} - \tilde{\theta}_{k,l}(s)) + C_\mathcal{O}((\gamma(s)_1 + \varphi_{i,k} - \tilde{\theta}_{k,l}(s))^2). \]

Next, we can expand $\partial_s Z$ and $\partial_t Z$ around the point $(\tilde{A}(\tilde{\theta}_{k,l}(s)), \tilde{\theta}_{k,l}(s))$. Recalling Lemma 6.3-(2,4) and (6.40), we have
\[ Z^-(\gamma'(s), \gamma(s)_1 + \varphi_{i,k}) = Z(\gamma'_{\lambda}(s), \tilde{\theta}_{k,l}(s)) + C_\mathcal{O}(\zeta^2 + \tau I_{0^\epsilon, E_0}(\gamma)) \]
(6.41)
\[ + C_\mathcal{O} \left( \|\tilde{A}(\tilde{\theta}_{k,l}(s)) - \gamma'(s)\| \left[ \sqrt{\tau I_{0^\epsilon, E_0}(\gamma)} + \zeta \right] \right). \]

Since Lemma 6.3-(0,2,4) implies $Z(\gamma'_{\lambda}(s), \tilde{\theta}_{k,l}(s)) \leq Z(\gamma'(s), \tilde{\theta}_{k,l}(s))$, we have
(6.42)
\[ \mathcal{J}^-_{\varphi_{i,k}, E_0}(\gamma) \leq (1 + C_\epsilon \tau) I_{0^\epsilon, E_0}(\gamma) + C_\mathcal{O} \sqrt{\tau I_{0^\epsilon, E_0}(\gamma)} + C_\# \zeta. \]

It remains the problem of joining the estimates in the different time intervals. This can be done in various ways. We choose to control the trajectories at the endpoint of the intervals so that the paths corresponding to different time intervals will join naturally into a continuous path. To this end let us define the sets
\[ B^*_{i,\tau, k} = \{ \gamma \in B_{(0,\tau]}(\gamma^k, 2R_{i,k}) \mid \gamma(0) = 0 ; \gamma(\tau) = \gamma_i(t_{k+1}) - \gamma_i(t_k) \} \]
Then, for each $\gamma \in B_{E_0}(\gamma^k, R_{i,k})$ we define, for each $s \in E_0$,
\[ \gamma^*(s) = \gamma(s) + \frac{\gamma_i(t_{k+1}) - \gamma_i(t_k) - \gamma(\tau)}{\tau} s. \]

One can check that $\gamma^* \in B^*_{i,\tau, k}$. Again, it suffices to consider the case $\mathcal{J}^-_{\varphi_{i,k}, E_0}(\gamma) < \infty$. Assume that $\gamma'(s) \notin \partial_s \mathbb{D}(\gamma(s)_1 + \theta_{i,0}^*)$ and $(\gamma^*)'(s) \notin \partial_s \mathbb{D}(\gamma^*(s)_1 + \theta_{i,0}^*)$ (the other cases being similar) and set $\gamma_{\lambda} = \gamma + \lambda(\gamma^* - \gamma)$. Using Lemma 6.3-(2,4) we can write
\[ Z(\gamma', (\gamma)_1 + \theta_{i,0}^*) = Z((\gamma^*)', (\gamma^*)_1 + \theta_{i,0}^*) + \int_0^1 \partial_s Z((\gamma^*_{\lambda})_1 + \theta_{i,0}^*), (\gamma^* - \gamma)_1) d\lambda \]
\[ + \int_0^1 \partial_t Z((\gamma^*_1 + \theta_{i,0}^*), (\gamma^*)_1 + \theta_{i,0}^*), (\gamma^* - \gamma)_1) d\lambda \]
\[ = Z((\gamma^*)', (\gamma^*)_1 + \theta_{i,0}^*) + C_\mathcal{O} \left( \|(\gamma^* - \gamma)'\| \|(\gamma^*)_1 + \theta_{i,0}^*\| \right) \]
\[ + C_\mathcal{O} \left( \|\gamma^* - \gamma\| \|(\gamma^*)_1 + \theta_{i,0}^*\| \right) + \|(\gamma^* - \gamma)'\| R_{i,k} \tau^{-1}. \]
It follows that
\[
\mathcal{J}_{\varphi_{i,k},E_0}^-(\gamma) = \mathcal{J}_{\varphi_{i,k},E_0}^-(\gamma^*) + C_\varepsilon \mathcal{O}\left(\frac{R_{i,k}}{\tau} \int_{E_0} \|((\tau^*)') - \tilde{A}((\tau^*)_1 + \theta_0^*\rangle\| + \frac{R_{i,k}^2}{\tau}\right)
\]
\[
= \mathcal{J}_{\varphi_{i,k},E_0}^-(\gamma^*) + C_\varepsilon \mathcal{O}\left(\frac{R_{i,k}}{\sqrt{\tau}} \sqrt{\mathcal{J}_{\varphi_{i,k},E_0}^-(\gamma^*)} + \frac{R_{i,k}^2}{\tau}\right).
\]

By (6.42) we have
\[
\Delta_{i,k} \leq C_\# R_{i,k} + \|\tilde{\gamma}_{k,t} - \gamma^* - \gamma_i(t_k)\| \leq C_\# R_{i,k} + C_\# \sqrt{\tau} \mathcal{J}_{\theta_{i,k},E_0}^-(\gamma^*).
\]

We can then assume that \( \tau \) is small enough and (6.38) implies \( C_\# R_{i,k} \leq \frac{1}{2} \Delta_{i,k} \), hence
\[
\Delta_{i,k} \leq C_\# \sqrt{\tau} \mathcal{J}_{\theta_{i,k},E_0}^-(\gamma^*).
\]

Accordingly, if \( \Delta_{i,k} \geq C_0(\varepsilon \tau)^{\frac{1}{4}} \), then \( R_{i,k} \leq \tau \sqrt{\mathcal{J}_{\varphi_{i,k},E_0}^-(\gamma^*)} \) and
\[
\mathcal{J}_{\varphi_{i,k},E_0}^-(\gamma) = \left(1 + C_\varepsilon \mathcal{O}(\sqrt{\tau})\right) \mathcal{J}_{\varphi_{i,k},E_0}^-(\gamma^*).
\]

While, if \( \Delta_{i,k} \leq C_0(\varepsilon \tau)^{\frac{1}{4}} \), then \( R_{i,k} \leq C_\# \varepsilon \tau^{-\frac{1}{2}} \mathcal{J}_{\varphi_{i,k},E_0}^-(\gamma^*) \). If \( \mathcal{J}_{\varphi_{i,k},E_0}^-(\gamma^*) \geq \varepsilon \tau^{-\frac{1}{2}} \), then
\[
\mathcal{J}_{\varphi_{i,k},E_0}^-(\gamma) = \left(1 + C_\varepsilon \mathcal{O}\left(\frac{\varepsilon}{\mathcal{J}_{\varphi_{i,k},E_0}^-(\gamma^*)}\right)^{\frac{1}{2}}\right) \mathcal{J}_{\varphi_{i,k},E_0}^-(\gamma^*).
\]

Collecting all the above facts we finally have an estimate with all the wanted properties
\[
(6.43) \quad \mathcal{J}_{\varphi_{i,k},E_0}^-(\gamma) = \left(1 + C_\varepsilon \mathcal{O}\left(\tau^{\frac{1}{4}}\right)\right) \mathcal{J}_{\varphi_{i,k},E_0}^-(\gamma^*) + \mathcal{O}(\varepsilon \tau^{-\frac{1}{4}}).
\]

Using the above estimate together with (6.42) in (6.39) and recalling Lemma 6.19 we finally have
\[
P_{\ell_0,A,\varepsilon}(B_{[0,T]}(\gamma_i,\varsigma_i)) \leq e^{-\varepsilon^{-1}(1+\mathcal{O}(\varepsilon^{\frac{1}{3}}))}\left[\sum_{k} \inf_{\gamma \in B_{[0,T]}(\gamma_i,\varsigma_i)} \mathcal{J}_{\varphi_{i,k},E_0}^-(\gamma) + C_\# \varepsilon \tau^{-\frac{1}{4}}\right].
\]

Next, note that, given \( \tilde{\gamma}_k \in B_{R_k}^\ast \), if we define \( \tilde{\gamma}(s) = \tilde{\gamma}_k(s-t_k) + \gamma_i(t_k) \) for \( s \in E_k \), then \( \tilde{\gamma} \in B_{[0,T]}(\gamma_i, R_i) \), where \( R_i = \max\{R_{i,k}\} \). In addition, \( \mathcal{J}_{\varphi_{i,k},E_0}^-(\tilde{\gamma}_k) = \mathcal{J}_{\theta_{i,k}}^-(\tilde{\gamma}) \). We can then choose \( \tau = \varepsilon \tau^{-\frac{1}{4}} \), whereby obtaining
\[
P_{\ell_0,A,\varepsilon}(B_{[0,T]}(\gamma_i,\varsigma_i)) \leq e^{-\varepsilon^{-1}\left[\inf_{\gamma \in B_{[0,T]}(\gamma_i, R_i)} (1 - C_\# \varepsilon \varepsilon^{\frac{1}{3}} \|\Gamma_i\| - \mathcal{J}_{\theta_{i,k}}^-(\gamma)) + C_\# \varepsilon \varepsilon^{\frac{1}{3}} \|\Gamma_i\|^{-\frac{1}{2}}\right]}.
\]

Since for each \( \gamma \in B_{[0,T]}(\gamma_i, R_i) \) we have \( \|\Gamma_i\| \leq C_\# \|\Gamma\| \leq C_\# \sqrt{\mathcal{J}_{\theta_{i,k}}^-(\gamma)} \), we have (provided \( \mathcal{J}_{\theta_{i,k}}^-(\gamma) \geq C_\# \varepsilon\))
\[
\varepsilon \varepsilon^{\frac{1}{3}} \|\Gamma_i\|^{-\frac{1}{2}} \mathcal{J}_{\theta_{i,k}}^-(\gamma) \geq C_\# \varepsilon \varepsilon^{\frac{1}{3}} \|\Gamma_i\|^{-\frac{1}{2}}.
\]

The two facts above imply the estimate
\[
(6.44) \quad P_{\ell_0,A,\varepsilon}(B_{[0,T]}(\gamma_i,\varsigma_i)) \leq e^{-\varepsilon^{-1}\inf_{\gamma \in B_{[0,T]}(\gamma_i, R_i)} (1 - C_\# \varepsilon \varepsilon^{\frac{1}{3}} \|\Gamma_i\| - \mathcal{J}_{\theta_{i,k}}^-(\gamma)) + C_\#}.
\]
Last we would like to estimate $R_i$. In order to obtain a simple formula we will provide a sub-optimal bound, sharper bounds are easily obtainable. Let $R_{i,k} = R_i$, then there are two possibilities: either

$$R_{i,k} = \left[ \varepsilon \frac{1}{2} \| \Gamma_i \|^{-\frac{3}{2}} \right] \Delta_{i,k} \leq \left[ \frac{\sqrt{\varepsilon}}{\| \Gamma_i \|} \right]^\frac{1}{3} ||\Gamma_i||.$$

or

$$R_{i,k} = \varepsilon \frac{1}{2} \left[ \frac{1}{2} \| \Gamma_i \|^{-\frac{3}{2}} \right] \Delta_{i,k} \leq \varepsilon \frac{1}{2} \| \Gamma_i \|^{-\frac{3}{2}} = \left[ \frac{\sqrt{\varepsilon}}{\| \Gamma_i \|} \right]^\frac{1}{3} ||\Gamma_i||.$$

Since for $\Gamma_i \leq \sqrt{\varepsilon}$ the radius is larger than the distance from the averaged trajectory, both formulae above will yield the same trivial estimate. We can then choose

$$R_i = \left[ \frac{\sqrt{\varepsilon}}{\| \Gamma_i \|} \right]^\frac{1}{3} ||\Gamma_i||.$$  

(6.45)

To conclude the estimate we must perform the sum in (6.36). To this end define $\varrho(\ell_0, Q) = \inf_{\gamma \in Q} \| \gamma - \bar{\xi}(\cdot, \theta_{\ell_0}^\gamma) \|_\infty$, $T^*- = \{ i \in B_{[0,T]}(\gamma_i, R_i) \cap Q \neq \emptyset \}$ and $Q^+ = \cup_{i \in T^*} B_{[0,T]}(\gamma_i, 2R_i)$. Also let $\eta = \inf_{\gamma \in Q^+} J_{\theta_{\ell_0}^\gamma}(\gamma)$. Note that $R_Q = \max R_i = \varepsilon \frac{1}{2} \varrho(\ell_0, Q)^{\frac{3}{2}}$, also let $\Delta_Q = \varepsilon \frac{1}{2} \log \varrho(\ell_0, Q)^{\frac{3}{2}}$. We can use (6.35), (6.36) and (6.44) to write

$$\mathbb{P}_{\ell_0,A,\varepsilon}(Q) \leq \sum_{n=0}^{\infty} \sum_{\gamma \in T^*} \mathbb{P}_{\ell_0,A,\varepsilon} \left( B_{[0,T]}(\gamma_i, \varepsilon) \right)$$

$$\leq \sum_{\{ n \leq 2^n \eta \leq (\log \varepsilon^{-1})^{-1} \}} e^{-\varepsilon^{-1}|(1-C_\sigma \Delta_Q)2^n \eta-C_\# \varepsilon|} + C_\# \varepsilon^{-\frac{3}{2}} \log \varepsilon^{-1}$$

$$+ \sum_{\{ n \leq 2^n \eta \geq (\log \varepsilon^{-1})^{-1} \}} e^{-\varepsilon^{-1}|(1-C_\sigma \Delta_Q)2^n \eta-C_\# \varepsilon|} + C_\# \varepsilon^{-\frac{3}{2}}$$

$$\leq e^{-\varepsilon^{-1}|(1-C_\sigma \Delta_Q)\eta-C_\# \varepsilon|}.$$

The above concludes the discussion of the upper bound.

To treat the lower bound we argue similarly but now we consider only the case $\varrho(\ell_0, Q) \geq C_\# (T \varepsilon)^{\frac{1}{3}}$ and divide the time, as before, in time steps of length $C_\# \varepsilon^{\frac{1}{3}}$ and we set $R = C_\# \varepsilon^{\frac{1}{3}}$. Then Lemmata 6.16 and 6.19, and arguing similarly to (6.40), (6.41), imply

$$\mathbb{P}_{\ell_0,A,\varepsilon} \left( B_{[0,T]}(\gamma_i, R) \right) \geq e^{-\varepsilon^{-1}\left[ (1+C_\sigma \varepsilon^{\frac{1}{3}}) \bar{\xi}_{\theta_{\ell_0}^\gamma}(\gamma_i) + C_\# \varepsilon^{\frac{1}{3}} \bar{\xi}_{\theta_{\ell_0}^\gamma}(\gamma_i) \right]}.$$

Define $Q^- = \{ \gamma \in Q : B_{[0,T]}(\gamma, R) \subset Q \}$ and set $\bar{\eta} = \inf_{\gamma \in Q^-} J_{\theta_{\ell_0}^\gamma}(\gamma)$ where the inf is taken to be $\infty$ if $Q^- = \emptyset$. Finally, by the same type of argument used in the upper bound, (6.37) implies

$$\mathbb{P}_{\ell_0,A,\varepsilon}(Q) \geq e^{-\varepsilon^{-1}\left[ (1+C_\sigma \varepsilon^{\frac{1}{3}}) \bar{\eta} + C_\# \varepsilon^{\frac{1}{3}} \right]},$$

whereby concluding the argument. \qed
7. Local Limit Theorem

The results of the last section allow to study deviations $\Delta_n = \theta_n - \bar{\theta}(\varepsilon n, \theta^\varepsilon_n)$ from the average larger than $\sqrt{\varepsilon}$ but give no information on smaller fluctuations, apart from the fact that with very high probability the fluctuations are of order $\sqrt{\varepsilon}$ or smaller. In fact, with techniques inspired by [18], it is possible to prove that the fluctuation from the average, once renormalized multiplying it by $\varepsilon^{-1/2}$, converges in law to a diffusion process. We refrain from proving this fact as it is subsumed by our, much stronger, Theorem 2.7 and it would not have much pedagogical value either since we have already introduced all needed tools in the previous sections.

For simplicity, we will study only the fluctuations of $\theta$, although the same type of arguments would yield similar results for $\zeta$.

Our first goal, and the main technical task, is to obtain a Local Central Limit Theorem for the random variable $H_{t,k}$ introduced in Subsection 4.2. Let us state the precise result although we will postpone its proof after showing how it is used to prove Theorem 2.7.

Proposition 7.1. For any $T > 0$ there exists $\varepsilon_0$ so that the following holds. For any real numbers $\varepsilon \in (0, \varepsilon_0)$, $\kappa > 0$ and compactly supported $\psi \in C_0^\infty(\mathbb{R}, \mathbb{R}_{\geq 0})$ with $r$ large enough (e.g. $r \geq 8$), let us define the renormalized translations $\psi_{\varepsilon,\kappa}(\cdot) = \psi(\varepsilon^{-1}(\cdot - \kappa \varepsilon^{1/2})$). Then for any $t \in [\varepsilon^{1/1000}, T]$ and any standard pair $\ell_0$ we have\(^{27}\)

\[
\varepsilon^{-1/2} \mu_{\ell_0}(\psi_{\varepsilon,\kappa}(H_{\ell_0,1\varepsilon^{-1}})) = \text{Leb} \psi \cdot \frac{e^{-\kappa^2/(\sigma^2_\varepsilon(\theta^\varepsilon_\ell))}}{\sigma(\theta^\varepsilon_\ell)^{\sqrt{\pi}}} + \|\psi(r)\|_{L^2} \varepsilon^{1/100} + C\# \|\psi\|_{L^1} \left[1 + \|\psi(r)\|_{L^2}^{1/3} \right] \varepsilon^{-1/100}.
\]

Remark 7.2. Observe that the power $1/100$ has been arbitrarily chosen and is by all means sub-optimal. On the other hand, our later computations will also give errors which are bounded by some small power of $\varepsilon$, and, even if it would be possible to optimize our choice we do not pursue this task for sake of readability.

Remark 7.3. Observe that, by the Large Deviations estimate (Theorem 2.2 and Corollary 2.5), we can ensure that, for any $\beta \in (1/2, 1)$, $|\Delta_k| \leq \varepsilon^{1-\beta}$ on the complement of a set of exponentially small probability.\(^{28}\) Thus, by Lemma 4.5 we have that, for $k \leq C\# \varepsilon^{-1}, |H_{t,k} - \Delta_k| \leq k \varepsilon^{-1-2\beta} < \varepsilon^{2-2\beta}$, again in the complement of a set of exponentially small probability. Unfortunately, this error is still too large to achieve the resolution $O(\varepsilon)$ we require in Theorem 2.7.

From the above remark it seems that Proposition 7.1 does not suffice to prove Theorem 2.7, even when taking into account the large deviations results detailed in Theorem 2.2. Nevertheless, we are going to show that it indeed implies Theorem 2.7, provided that one argues in a less naive manner. The basic idea is to apply Proposition 7.1 twice, first up to a time $\varepsilon^{\frac{1}{10}}$ before the final time, but estimating the probability of belonging to intervals of length $\varepsilon^{\frac{1}{2}}$. Then applying it again, for the remaining time, to each standard pair in such intervals, but this time estimating the probability that the random variable belongs to intervals of order $\varepsilon$, which is the wanted resolution.

\(^{27}\)The formula for $\sigma(\theta)$ is given by (2.14).

\(^{28}\)That is smaller than $e^{-c\# \varepsilon^{1-2\beta}}$. 

Proof of Theorem 2.7. First let us extend the result stated in Proposition 7.1 to indicator functions of compact intervals rather than for smooth compactly supported positive functions.

For any compact interval $J = [a, b] \subset \mathbb{R}$, $\varepsilon^{1/1000} \leq |b - a| \leq C_\varepsilon \varepsilon^{-1/3}$, let $\varepsilon^* = \varepsilon^{1/800}$, define $J^* = [a - \varepsilon^*, b + \varepsilon^*]$ and $J_* = [a + \varepsilon^*, b - \varepsilon^*]$. Define $\psi^*, \psi_* \in C^3_0(\mathbb{R}, \mathbb{R}_{\geq 0})$ so that $\psi^*(x) = 1$ if $x \in J$ (resp. $\psi_*(x) = 1$ if $x \in J_*$), $\psi^*(x) = 0$ if $x \notin J^*$ (resp. $\psi_*(x) = 0$ if $x \notin J_*$) and $\|\psi^*\|_{L^1}, \|\psi_*\|_{L^1}, \|\psi^*(\varepsilon^{1/4})\|_{L^2}, \|\psi^*(\varepsilon^{1/4})\|_{L^2} = O(\varepsilon^{-15/1600})$. Then clearly $\psi_* \leq \varepsilon^* \leq \psi^*$, and Leb $\psi^*$, Leb $\psi_*$ = Leb $J + O(\varepsilon^{1/800})$. Also note that since $t \geq \varepsilon^{1/1000}$ we have $\sigma_t \geq \varepsilon^{1/1000}$. From (7.1) we thus conclude that

\begin{equation}
(7.2) \quad e^{-\frac{3}{2} \mu_{t_0}}(H_{t_0, [t \varepsilon^{-1}]} \in \varepsilon J + \kappa \varepsilon^{1/2}) = \frac{e^{-\kappa^2/(\sigma^2_\varepsilon(t^*)^2)}}{\sigma_t(\theta^*_\varepsilon)} \sqrt{\pi} \text{Leb } J + O(\varepsilon^{1/800}).
\end{equation}

We now use the above improved version of Proposition 7.1. Fix $t_* = t - \varepsilon^{1/1000}$ and let $m_* = [t_* \varepsilon^{-1}]$. Observe that, as noted in Remark 7.3, by Theorem 2.2, Remark 6.2 and Lemma 4.5 we obtain

$$|\Delta_{m_*} - H_{t_0, m_*}| \leq \varepsilon^{3/4}$$

except on a set of exponentially small probability. Let us then choose $J_\varepsilon = [-\varepsilon^{-1/3}, \varepsilon^{-1/3}]$ and for $j \in \mathbb{Z}$ let $\kappa_j = 2j^{1/6}$; applying (7.2), we obtain

$$\varepsilon^{-1/2} \mu_{t_0}(\Delta_{m_*} \in \varepsilon J_\varepsilon + \kappa_j \varepsilon^{1/2}) = \frac{e^{-\kappa_j^2/(\sigma^2_\varepsilon(t^*_\varepsilon)^2)}}{\sigma_t(\theta^*_\varepsilon)} \sqrt{\pi} \text{Leb } J_\varepsilon + O(\varepsilon^{1/2000}).$$

Let us now consider the standard family $\xi_* = \xi_*^{t_0}$; for any standard pair $\ell_* \in \xi_*$ we can once again apply (7.2) for time $s_* = \varepsilon^{1/1000}$; as before, notice that, except on a set of exponentially small probability, $|\Delta_{s_* \varepsilon^{-1}} - H_{s_* \varepsilon^{-1}}| = o^*(\varepsilon)$.

Our next step is to observe that the random variable $\Delta_{t, s, \varepsilon^{-1} t}$ under $\ell_0$, conditioned to $(x_{t, s, \varepsilon^{-1} t}, \theta_{t, s, \varepsilon^{-1} t}) \in \ell_*$, has the same distribution of the random variable $\Delta_{t, s, \varepsilon^{-1} t} + \tilde{\theta}(s, \theta^*_s) - \tilde{\theta}(t, \theta^*_t)$ under $\ell_*$. On the other hand, setting $\xi(t, \theta_0) = \partial_{\theta_0}(\tilde{\theta}(t, \theta_0))$, we have

$$\tilde{\xi} = \tilde{\omega}(\tilde{\theta}) \xi \quad \xi(0) = 1,$$

thus $\xi(\tau) = e^{\int_0^\tau \omega(\tilde{\theta}(\tau)) d\tau}$. By the smoothness of the solutions of the ODE with respect to the initial conditions it follows

$$\tilde{\theta}(t, \theta^*_t) - \tilde{\theta}(s, \theta^*_s) = e^{\int_0^s \omega(\tilde{\theta}(\tau + t, \theta^*_t)) d\tau} [\tilde{\theta}(t_* + t, \theta^*_s) - \theta^*_t] + O \left( \left[ \tilde{\theta}(t_* + t, \theta^*_s) - \theta^*_t \right]^2 \right) ;$$

The basic idea is then to partition the standard family $\xi_*$ in standard families $\xi_{s,j}$ in such a way that $\ell_* \in \xi_{s,j}$ if $\theta^*_s - \tilde{\theta}(t, x^*_t) \in \varepsilon J + \kappa_j \varepsilon^{1/2}$. In doing so we actually need to neglect standard pairs which lie too close (e.g. $\varepsilon^{3/4}$) to the boundary of the intervals.\footnote{That is at a distance of order $\varepsilon^{-1/4}$ from the boundary of $J_\varepsilon$.}

Let us call $\xi_{s,j}$ such restricted families. Also, we will neglect all the $\xi_{s,j}$ for $|j| \geq \varepsilon^{1/6 - 1/10000} = N_\varepsilon$. This poses no real trouble since, once again by (7.2), the total mass of all such neglected standard pairs is $o^*(1)$.

To conclude, fix $I$ and $\kappa$, and set, for $\ell_* \in \xi_{s,j}$,

$$\kappa_{j, \ell_*} = \kappa - \varepsilon^{-\frac{1}{4}} \left[ \theta(t, \theta^*_t) - \tilde{\theta}(s, \theta^*_s) \right] = (\kappa - \kappa_j) + O(\varepsilon^{1/6}).$$
In addition,

\[ \Delta_{[\varepsilon^{-1} t]} = \Delta_{[\varepsilon^{-1} s, \varepsilon]} + \varepsilon^2 \langle \kappa - \kappa_j, t, \ell \rangle. \]

Also, for the following, it is convenient to set \( \sigma_* = \sigma_{s, s, \ell} (\tilde{\theta}(t, \theta_t^?) ) \) and note that \( \sigma_*^2 + \sigma_* (\theta_t^?)^2 = \sigma_t (\theta_t^?)^2 + o^* (1) \).

Therefore, calling \( I_{\pm} \) the enlargement and restriction of \( I \) by \( \varepsilon^\pm \), we have

\[
e^{-1/2} \mu_{ou}(\Delta_{[\varepsilon^{-1} t]} \in \varepsilon I + \kappa \varepsilon^{-1/2})
\leq e^{-1/2} \sum_{|j| \leq N_\varepsilon} \sum_{\ell, s \in \Sigma, j} \nu_{\ell, s} \mu_{\ell, s} (\Delta_{[\varepsilon^{-1} s, \varepsilon]} + \varepsilon^2 \langle \kappa_j, \ell, \varepsilon \rangle) + o^* (1)
= \text{Leb} \sum_{|j| \leq N_\varepsilon} \sum_{\ell, s \in \Sigma, j} \nu_{\ell, s} \frac{e^{-\kappa_j^2/\langle \sigma_* \rangle^2}}{\sigma_* (\theta_{t, \ell}^?) \sqrt{\pi}} + o^* (1)
= \text{Leb} \sum_{|j| \leq N_\varepsilon} \sum_{\ell, s \in \Sigma, j} \frac{e^{-\kappa_j^2/\langle \sigma_* \rangle^2}}{\sigma_* (\theta_{t, \ell}^?) \sqrt{\pi}} + e^{1/2} \text{Leb} \sum_{|j| \leq N_\varepsilon} \frac{\nu_{\ell, s}}{\sigma_* (\theta_{t, \ell}^?) \sqrt{\pi}} + o^* (1)
= \text{Leb} \sum_{|j| \leq N_\varepsilon} \frac{e^{-\kappa_j^2/\langle \sigma_* \rangle^2}}{\sigma_* (\theta_{t, \ell}^?) \sqrt{\pi}} + o^* (1).
\]

The Lemma follows since we can obtain a similar lower bound by repeating the above computation with \( I_{-} \).

We thus proceed to give the

**Proof of Proposition 7.1.** We start with some preliminaries; denote by \( \mathcal{F} \) the Fourier transform, then

\[
\mathcal{F}[\psi_{\varepsilon, \kappa}](\xi) = e^{-i \varepsilon^{1/2} \kappa \xi} \mathcal{F}[\psi](\varepsilon \xi)
\]

therefore,

\[
e^{-1/2} \mu_{ou}(\psi(H_{t, \varepsilon^{1/100}})) = \frac{1}{2 \pi} \int_{\mathbb{R}} e^{-i \eta} \mathcal{F}[\psi](\eta) e^{1/2} \mu_{ou}(e^{i \varepsilon^{1/2} \eta \varepsilon^{1/100}}) d\eta.
\]

We are thus reduced to compute the latter expectation: we will find convenient to split the integral in four regimes: let us fix \( \kappa_0 > 0 \), \( \kappa > 1/2 \) to be determined later, and partition \( \mathbb{R} = J_0 \cup J_1 \cup J_2 \cup J_3 \), where

\[
J_0 = \{ \eta | \leq \varepsilon^{1/100} \}, \\
J_1 = \{ \varepsilon^{1/100} < |\eta| \leq \kappa_0 \varepsilon^{-1/2} \}, \\
J_2 = \{ \kappa_0 \varepsilon^{-1/2} < |\eta| \leq \kappa \}, \\
J_3 = \{ \kappa < |\eta| \}.
\]

Correspondingly, we can rewrite (7.3) as

\[
e^{-1/2} \mu_{ou}(\psi(H_{t, \varepsilon^{1/100}})) = I_0 + I_1 + I_2 + I_3,
\]

where each \( I_j \) denotes the contribution of \( J_j \) to the integral on the right hand side of (7.3). Clearly, \( |I_0| \leq C_{\#} \| \psi \|_{L^1} \varepsilon^{1/100} \). The contribution of \( I_3 \) can also be
neglected, since
\[
|I_3| \leq \frac{1}{2\pi} \varepsilon^{-1/2} \int_{s^{1/2} J_3} |F[\psi](\sigma)| \, d\sigma
\]
\[
\leq C_\# \varepsilon^{-1/2} \|\psi^{(r)}\|_{L^2} \left[ \int_{s^{1/2} J_3} \sigma^{-2r} \, d\sigma \right]^{1/2}
\]
\[
\leq C_\# \|\psi^{(r)}\|_{L^2} \varepsilon^{(\varepsilon-1/2)(r-1/2)-1/2}.
\]

If we assume
\[
r > \frac{1}{2} + \frac{1/2 + 1/100}{\varepsilon - 1/2},
\]
we can thus conclude that \(|I_3| \leq C_\# \|\psi^{(r)}\|_{L^2} \varepsilon^{1/100}\). We are then left to estimate the contribution of \(I_1\) and \(I_2\) to the integral \((7.3)\); we will (impressionistically) call \(J_1\) the small \(\eta\) regime and \(J_2\) the large \(\eta\) regime. Our basic, not very original, idea is to perform the computation in blocks of length \(L \in \{1, \ldots, \varepsilon^{-1}\}\). Observe that, by definition of \(H_{\ell_0, k}\) we have,
\[
\mu_{\ell_0} \left( e^{i\sigma \varepsilon^{-1/2} H_{\ell_0, \varepsilon^{-1}}(1)} \right) = \mu_{\ell_0} \left( e^{i\sigma \sum_{j=0}^{\ell_0} \omega_j(x_j, \theta_j)} \right)
\]
where \(\sigma \equiv \varepsilon^{1/2}\) and \(\hat{\omega}_j = \hat{\Xi}_{\ell_0, j, \varepsilon^{-1}} \hat{\omega}\). Let us now consider the (finite) sequence of potentials \(\Omega = (i\sigma \hat{\omega}_k)_{k \in \{0, \ldots, \varepsilon^{-1}\}}\) and consider complex \(\Omega\)-standard pairs. Remark that the \(C^2\)-norm of the potentials \(\Omega\) is bounded by \(C_\# |\sigma|\), where \(C_\#\) depends only on \(T\) and on the \(C^2\)-norm of \(\omega\). It follows by the definition of complex standard pairs, that we can consider only curves of length \(\min\{c_\# c_2^{-1}, C_T |\sigma|^{-1}\}\).

This can always be achieved by cutting the real standard pair \(\ell_0\) in sufficiently short pieces. Since we will see that the contribution of each piece yields essentially the same quantity (multiplied by the mass of the piece), they can be added to obtain the final result. Thus we can, from now on, assume that \(\ell_0\) stands for one of such pieces rather that for the original standard pair. Note however that such a cutting is necessary only if \(\eta \in J_2\), provided \(\sigma_0\) has been fixed small enough. Then we can use \((3.12)\) and write
\[
\mu_{\ell_0} \left( e^{i\sigma \sum_{k=0}^{\ell_0} \omega_k(x_k, \theta_k)} \right) = F_{\hat{\ell}_0, \Omega}^{(\ell_0-1)}(1) \mu_{\ell_0}(1) =
\]
\[
= \underbrace{F_{\hat{\ell}_0, \Omega}^{(L)}(1)}_{R \text{ times}} \cdots \underbrace{F_{\hat{\ell}_0, \Omega}^{(L)}(1)}_{R \text{ times}} \underbrace{\mu_{\ell_0}(1)}_{R \text{ times}} = \sum_{\ell_1 \in \Omega_{\ell_0, \Omega}} \cdots \sum_{\ell_R \in \Omega_{\ell_{R-1}, \ell_{R-1}}(L) \Omega} \prod_{r=1}^{R} \nu_{\ell_r},
\]
where \(R \approx \varepsilon^{-1} L^{-1}\). From now on, since it is fixed once and for all, for ease of notation, we will often omit the potential sequence in the notation of standard families. Observe that \((3.15)\) gives the following trivial bound on \((7.4)\):
\[
\sum_{\ell_1 \in \Omega_{\ell_0}^{(L)}} \cdots \sum_{\ell_R \in \Omega_{\ell_{R-1}}^{(L)}} \prod_{r=1}^{R} |\nu_{\ell_r}| \leq C_\#.
\]

The above estimates completely ignore possible cancellations among complex phases; our next step are the following—much stronger—results.
Proposition 7.4 (Large $\eta$ regime). Assume $1/2 < \kappa < 4/7$; then for any $\eta \in J_2$ and $L \sim C_{#} \varepsilon^{-1/3}$, any complex standard pair $\ell \in L_{R-2}$

$$\sum_{\ell \in \mathcal{L}_R^L} \nu_{\ell} = O(\varepsilon^{\kappa+1/100}).$$

Proposition 7.5 (Small $\eta$ regime). For $\sigma_0$ small enough, for any complex standard pair $\ell$, for any $\eta \in J_1$ and $L \in (\varepsilon^{-1/3}, \varepsilon^{-3/8})$:

$$\mu_{\ell_0}(e^{i\eta \varepsilon^{-1/2} H_{(\ell_0-1)}}) = \sum_{\ell_1 \in \mathcal{L}_{\ell_0}} \cdots \sum_{\ell_{R-1} \in \mathcal{L}_{\ell_{R-1}}} \prod_{r=1}^{R} \nu_{\ell_r} =
$$

$$= e^{-\frac{1}{2} \eta} \int_{\mathcal{J}_1} e^{\int_{\mathcal{J}_1} R(F(\theta(s, \theta_0^{\eta})) ds + R(\eta, \varepsilon)),
$$

where $\sigma \in C^1(T, \mathbb{R}_{>0})$ is the function defined in (2.12) and $R$ is a small remainder term in the sense that $\int_{\mathcal{J}_1} |R(\eta, \varepsilon)| d\eta = O(\varepsilon^{1/100})$.

The above two propositions will be proved in Section 9 and Section 11, respectively. We can now conclude the proof of Proposition 7.1: in fact, Proposition 7.4 together with (7.5) implies that $|\mathcal{I}_2| \leq C_{#} \|\psi\|_{L^1} \varepsilon^{1/100}$. On the other hand, by Proposition 7.5 we have

$$\mathcal{I}_1 = \frac{1}{2\pi} \int_{|\eta| \leq \varepsilon^{-1/100}} e^{-i \eta \varepsilon^{1/2} \mathcal{F}[\psi](\eta)} e^{-\frac{\varepsilon}{2} \sigma_0^2(\theta_0^\eta)} d\eta + \|\psi\|_{L^1} O(\varepsilon^{1/100})
$$

$$= \frac{1}{2\pi} \mathcal{F}[\psi](0) \int_{\mathbb{R}} e^{-i \eta \varepsilon^{1/2} \mathcal{F}[\psi](\eta)} d\eta + (1 + |\text{supp}(\psi)| \varepsilon^{1/3}) \|\psi\|_{L^1} O(\varepsilon^{1/100})
$$

$$= \mathcal{F}[\psi](0) e^{-\frac{\varepsilon^2}{2(2\sigma_0^2(\theta_0^\eta))}} (1 + \varepsilon^{1/3} |\text{supp}(\psi)|) \|\psi\|_{L^1} O(\varepsilon^{1/100}),$$

from which we obtain (7.1) and conclude the proof. \hfill \Box

8. ONE BLOCK ESTIMATE: TECHNICAL PRELIMINARIES

The rest of the paper is dedicated to proving Propositions 7.4 and 7.5. We start by analyzing the contribution of a block of length $L$; later on we will use (7.4) to combine together the estimates of the different blocks in order to obtain the result we need. The basic idea is to reduce the computation of the contribution of one block to estimating the action of suitable Transfer Operators with a complex weight. This subsection is devoted to the proof of a proposition which contains the main technical ingredients for our estimates.

Let us fix the length of a block $L = (\ell \varepsilon^{-\beta}$, where $\beta \in (1/3, 3/8)$. We will apply the following Proposition to a generic block, that is at the time interval $[\varepsilon^{-1} L, \varepsilon^{-1} L + L - 1]$. Since the potential depends on the block, it is convenient to state our result for a generic potential.

Proposition 8.1. Let $\Omega = (i \sigma \omega_k)_{k \in \{0, \cdots, L\}}$, with $\|\omega_k\|_{C^2} \leq C_{#}$, and $\varepsilon^{-\hat{\theta}} \sigma = \eta \in J_1 \cup J_2$; then for $k = 0, \cdots, L - 1$ and complex standard pair $\ell = (\mathcal{G}, \rho)$, we have

$$\sum_{\ell \in \mathcal{G}_L^{\mathcal{L}_R^L}} \nu_{\ell} \rho_{\ell} \mathcal{I}_{[\alpha, \beta]} = e^{\Delta_0} \mathcal{L}_L \cdots \mathcal{L}_0 (1 + \Delta_2 + \Delta_{\text{err}}) \hat{\mu} + O((\varepsilon L)^{1+1/100}),$$

where:
Proof. (8.5) ∆

(8.2) \[ L_k \] = \sum_{y \in f_k^{-1}(x)} e^{\Omega_k(y, \bar{\theta}_k)} f_k(y) g(y),

with \( \bar{\theta}_k = \theta(\varepsilon_k, \theta^*_k) \) and potential \( \Omega_k \):

(8.3) \[ \Omega_k = i\sigma w_k + \varepsilon \Psi_k \]

and \( \Psi_k : \mathbb{T}^2 \to \mathbb{C} \) are smooth functions such that \( \mu_{\bar{\theta}_k}(\Psi_k) = 0, \| \Psi_k \|_{C^3} = O((1 + |\sigma|)(L - k) + \sqrt{L}) \)

In particular:

(8.4) \[ \| \Omega_k \|_{C^3} = O(\sigma + L\varepsilon) \]

Moreover, we can decompose \( \Phi_j = \Phi_j^* + \Phi_j^{**} \), with

\[ \| \Phi_j^* \|_{C^1} = O((1 + |\sigma|)\varepsilon^{-1}) \]

\[ \| \Phi_j^{**} \|_{C^0} = O(\varepsilon^{100}) \]

\[ \| \Phi_j^{**} \|_{C^1} = O((1 + |\sigma|)(L - j)) \]

and \( \hat{\Phi}_k = \Phi_k - \mu_{\bar{\theta}_k}(\Phi_k) \). On the other hand, \( \Phi_\ell \) is a smooth function

(8.5) \[ \Delta_2 \circ Y_L = \varepsilon \sum_{0 \leq k < j < L} \hat{\omega}(\bar{x}_k, \hat{\theta}_k) \hat{\Phi}_j(\bar{x}_j, \hat{\theta}_j) + \varepsilon \sum_{0 \leq j < L} \Phi_j(\bar{x}_j, \hat{\theta}_j) \Phi_\ell(\bar{x}_0), \]

where \( \Phi_j : \mathbb{T}^2 \to \mathbb{C} \) are smooth functions so that \( \| \Phi_j \|_{C^0} = O(1 + |\sigma|) \) and \( \| \Phi_j \|_{C^1} = O((1 + |\sigma|)(L - j)) \). Moreover, we can decompose \( \Phi_j = \Phi_j^* + \Phi_j^{**} \), with

\[ \| \Phi_j^* \|_{C^1} = O((1 + |\sigma|)\varepsilon^{-1}) \]

\[ \| \Phi_j^{**} \|_{C^0} = O(\varepsilon^{100}) \]

\[ \| \Phi_j^{**} \|_{C^1} = O((1 + |\sigma|)(L - j)) \]

and \( \hat{\Phi}_k = \Phi_k - \mu_{\bar{\theta}_k}(\Phi_k) \). On the other hand, \( \Phi_\ell \) is a smooth function

(8.6a) \[ \| \Delta_{err} \|_{C^0} = |\sigma|O(\varepsilon L + o^*(\varepsilon L)) \]

(8.6b) \[ \| \Delta'_{err} \|_{C^0} = O((1 + |\sigma|)\varepsilon L) \sum_{k=0}^{L-1} \max(x_k', \bar{x}_k')(L - k)^2 \]

where \( ' \) denotes differentiation with respect to \( \bar{x}_0 \);

(e) \( \hat{\rho}_\ell \in C^2(\mathbb{T}^1, \mathbb{R}) \) is close to \( \rho \) in the sense that

(8.7) \[ \| \hat{\rho}_\ell - \rho \|_{L^1} = O((\varepsilon L)^{1 + 1/100}). \]

Moreover, \( \| \hat{\rho}_\ell \|_{W^{3,1}} = C# \max \{ (\varepsilon L)^{-(1 + 1/100)}, |\sigma| \}^{-1}. \)

Proof. Observe that, by definition, for any smooth function \( g : \mathbb{T} \to \mathbb{R} \), we have

\[ \int_{\mathbb{T}} g \cdot \sum_{\ell \in \omega_k} \nu_\ell \rho_\ell \mathbb{1}_{[a_\ell, b_\ell]} dx = \int_{\mathbb{T}} g(x_L) \cdot e^{i\sigma} \sum_{k=0}^{L-1} \omega_k \cdot \mathbb{1}_{[a_\ell, b_\ell]} dx \]

First of all, let us express the sum of \( \omega_k \) along the real trajectory appearing at the exponent as a sum along the averaged trajectory plus an error, for which we
compute precise bounds. Observe that, by Lemma 4.2 (of which we use the notation $\mathcal{E}_{\text{quad},k}, \tilde{\mathcal{E}}_{\text{quad},k}$ with $n = L$), we have

\begin{equation}
\varpi_k(x_k, \theta_k) = \varpi_k(x_k, \theta_k) + \varpi_{k,x}(\tilde{x}_k, \tilde{\theta}_k)\xi_k + \varpi_{k,\theta}(\tilde{x}_k, \tilde{\theta}_k)\Delta_k + \mathcal{E}_{\text{quad},k}.
\end{equation}

We can then write

\begin{equation}
\varpi_k(x_k, \theta_k) = \sigma \sum_{k=0}^{L-1} \varpi_k(x_k, \theta_k) = \sigma \sum_{k=0}^{L-1} \varpi_k(x_k, \theta_k) + \Delta^L + \Delta^L_R,
\end{equation}

where, by equations (4.2) and (8.8):

\begin{equation}
\Delta^L = \sigma \sum_{k=0}^{L-1} \{ \varpi_{k,x}(\tilde{x}_k, \tilde{\theta}_k)\mathcal{W}_k + \varpi_{k,\theta}(\tilde{x}_k, \tilde{\theta}_k)W_k \}
\end{equation}

and we used the random variables $W_k$ and $\mathcal{W}_k$ introduced in Lemma 4.2. Thus, we can write:

\[
\int_{\mathbb{T}} g \cdot \sum_{i \in A(L)} \nu_{i\beta} \rho \mathbb{1}_{[a_i, b_i]} \, dx = \int_{\mathbb{T}}^\prime g(x_L) \cdot e^{i\sigma} \sum_{k=0}^{L-1} \varpi_{k,\theta}(\tilde{x}_k, \tilde{\theta}_k)\xi_k + \varpi_{k,x}(\tilde{x}_k, \tilde{\theta}_k)\mathcal{W}_k + \varpi_{k,\theta}(\tilde{x}_k, \tilde{\theta}_k)W_k \, dx
\]

we have used the notation, introduced before Lemma 4.2, $\tilde{F}_k(x, \tilde{\theta}_k) = (\tilde{f}(k)(x), \tilde{\theta}_k)$ together with $\mathcal{W}_k \rho = (\mathcal{W}_k^L)^{\prime} \cdot [\rho \mathbb{1}_{[a, b]}] \circ \mathcal{W}_k^{L-1}$. We now turn to analyze the push-forward $\mathcal{W}_k \rho$: our main obstacle is that $(\mathcal{W}_k^L)^{\prime}$ has huge derivatives, and thus it is not a good density. In fact, using (4.3) we obtain

\[
(\mathcal{W}_k^L)^{\prime} = \frac{(\tilde{f}(k))^\prime}{\left(1 - G^L S_L \nu_L \right) \circ \mathcal{W}_k^{L-1}},
\]

where

\begin{equation}
\begin{aligned}
\nu_{k+1} &= (f_x(x_k, \theta_k) + \varepsilon u_k f_\theta(x_k, \theta_k))\nu_k \quad \text{and, by (3.1) and (3.3)},
\end{aligned}
\end{equation}

and $r_u$ is a remainder term of bounded $C^1$-norm.

To overcome such problems let us decompose $\mathcal{W}_k \rho = \rho \ast e^{\tilde{\Delta}^L \ast}$ where:

\[
\rho \ast = \left[ \frac{\rho \mathbb{1}_{[a, b]} \circ \mathcal{W}_k^{L-1} \circ \mathcal{W}_k^{L-1}}{1 - G^L S_L \circ \mathcal{W}_k^{L-1} \circ \mathcal{W}_k^{L-1}} \right] \rho \ast
\]

Then, by Lemma 4.3, $\|\rho \ast\|_{BV} \leq C_\#$; moreover, recalling also Lemma 4.2 and since $\rho$ is a $c_2 \max \{1, \|\sigma\|\}$-standard density, we have $\|\rho \ast - \rho\|_{L^1} \leq C_\# \varepsilon \max \{1, \|\sigma\|\}$. Yet, in the sequel we will need to deal with smooth density functions. We can then consider densities $\tilde{\rho}$ such that $\|\tilde{\rho} - \rho \ast\|_{L^1} \leq C_\# (\varepsilon L)^{1+\beta}$ and $\|\tilde{\rho}\|_{W^{1,1}} \leq
$C_\# \max\{\varepsilon L\}^{-\left(1+\beta'\right)} \|\sigma\|^{r-1}$, where $\beta'$ is a small positive real number. Notice moreover that by (8.11) and Lemma 4.2 we have

$$u_{k+1} = \frac{u_k + \omega_x(x_k, \bar{\theta}_k)}{f_x(x_k, \theta_k)} + \mathcal{E}_{\text{lin},k},$$

where $|\mathcal{E}_{\text{lin},k}| \leq C_\# \varepsilon \kappa$ and $|(\mathcal{E}_{\text{lin},k})'| \leq C_\# \max_{j \leq k} \{|x'_j|, |\bar{x}'_j|\}$. Define $\bar{u}_k$ as the sequence given by the following inductive relation:

$$\bar{u}_{k+1} = \frac{\bar{u}_k + \omega_x(x_k, \bar{\theta}_k)}{f_x(x_k, \theta_k)} \quad \text{for} \quad \bar{u}_0 = 0;$$

so that we can conclude that

$$u_k = \bar{u}_k + \mathcal{E}_{\text{lin},k}.$$

Thus, we can write $\Delta \Re = \Delta \Re_{\text{lin}} + \Delta \Re_{\text{R}}$, where

$$\Delta \Re \circ \mathcal{Y}_L = \varepsilon \sum_{k=0}^{L-1} f_x(x_k, \bar{\theta}_k)^{-1} \left\{ f_{xx}(x_k, \bar{\theta}_k) \mathfrak{M}_k + f_{x\theta}(x_k, \bar{\theta}_k)W_k \right\} + \varepsilon \sum_{k=0}^{L-1} \bar{u}_k f_{\theta}(x_k, \bar{\theta}_k)$$

$$\Delta \Re_{\text{R}} = \sum_{k=0}^{L-1} \mathcal{E}_{\text{quad},k}.$$

Let us define $\Delta = \Delta \Re + i \Delta \Im$ and $\Delta \mathcal{R} = \Delta \Re_{\text{lin}} + i \Delta \Re_{\text{lin}}$; we thus obtain

$$\int_{\mathbb{T}^1} g \cdot \sum_{l \in \mathbb{Z}_L} \nu_{l} f_{\mathcal{Y}}(l^j, l^\ell) d\xi = \int_{\mathbb{T}^1} \left( g \circ \int_{\mathbb{T}^1} e^{i\sigma} \sum_{k=0}^{L-1} \omega_k \circ \mathbb{F}_{\mathcal{Y}}^{-1} + \Delta + \Delta \varepsilon \right) d\xi + O(\varepsilon L^{1+\beta'} \|g\|_{L^1}).$$

The proposition follows from a careful inspection of the term $\Delta$; using (8.10) and (8.12), we rewrite $\Delta$ as

$$\Delta \circ \mathcal{Y}_L = \varepsilon \sum_{k=0}^{L-1} \left\{ a_k(x_k, \bar{\theta}_k)W_k + b_k(x_k, \bar{\theta}_k) \mathfrak{M}_k + c_k(x_k, \bar{\theta}_k) \right\}$$

with

$$a_k = f_{x}^{-1} f_{x\theta} + i \sigma \omega_{k,\theta} \quad b_k = f_{x}^{-1} f_{xx} + i \sigma \omega_{k,xx} \quad c_k = \bar{\omega}_x \sum_{j=k+1}^{L-1} \Lambda^{(j-k)} f_{\theta} \circ \mathbb{F}_{\mathcal{Y}}^{j-k},$$

where we introduced the notation $\Lambda^{(\ell)} = \left[ \prod_{j=0}^{\ell-1} f_x \circ \mathbb{F}_{\mathcal{Y}}^j \right]^{-1}$, so that $\Lambda^{(j-k)}(x_k, \bar{\theta}_k) = \Lambda_{x, j-k}$. Let us first collect those terms that multiply the function $\Phi_\ell$ (see the definition of $W_\ell$ in Lemma 4.2); an immediate inspection shows that such terms contribute with $\varepsilon \sum_{k=0}^{L-1} \Phi_k(x_k, \bar{\theta}_k) \Phi_\ell(\bar{x}_0)$, where we defined the function

$$\Phi_k = a_k - b_k \sum_{l=k}^{L-1} \Lambda^{(l-k)} f_{\theta} \circ \mathbb{F}_{\mathcal{Y}}^{l-k}. $$
We then proceed to rearrange the remaining terms as follows:

\[
\begin{align*}
\sum_{k=0}^{L-1} a_k(x_k, \bar{\theta}_k) & \sum_{j=0}^{k-1} \hat{\omega}(\bar{x}_j, \bar{\theta}_j) + \\
+ \sum_{k=0}^{L-1} b_k(x_k, \bar{\theta}_k) & \sum_{l=k}^{L-1} \Lambda^{(l-k)}(x_k, \bar{\theta}_k) f_\theta(x_l, \bar{\theta}_l) \sum_{j=0}^{l-1} \hat{\omega}(\bar{x}_j, \bar{\theta}_j) + c_k(x_k, \bar{\theta}_k) \\
= & \sum_{k=0}^{L-2} \sum_{j=k+1}^{L-1} \hat{\omega}(\bar{x}_k, \bar{\theta}_k) \Phi_j(x_k, \bar{\theta}_j) + \sum_{k=0}^{L-1} A_k(x_k, \bar{\theta}_k),
\end{align*}
\]

where we defined

\[
A_k = b_k \sum_{j=k}^{L-2} \Lambda^{(j-k)} \hat{\omega} \circ \bar{F}_\varepsilon^{j-k} \sum_{l=j+1}^{L-1} \Lambda^{(l-j)} \circ \bar{F}_\varepsilon^{l-k} \cdot f_\theta \circ \bar{F}_\varepsilon^{l-k} + c_k.
\]

In summary, we have

\[
\Delta = \varepsilon \sum_{k=0}^{L-2} \sum_{j=k+1}^{L-1} \hat{\omega}(\bar{x}_k, \bar{\theta}_k) \Phi_j(x_k, \bar{\theta}_j) + \varepsilon \sum_{k=0}^{L-1} A_k(x_k, \bar{\theta}_k) + \varepsilon \sum_{k=0}^{L-1} \Phi_k(x_k, \bar{\theta}_k) \Phi_\ell(x_0).
\]

Observe that, even if \(\|A_k\|_{C^0} = \mathcal{O}(1 + |\sigma|)\), we do not have good control on its higher order norms; it is thus necessary to introduce a cutoff regularization: let us write \(A_k = A_k^* + A_k^{**}\) where

\[
A_k^* = b_k \sum_{j=k}^{L-2} \Lambda^{(j-k)} \hat{\omega} \circ \bar{F}_\varepsilon^{j-k} \sum_{l=j+1}^{L-1} \Lambda^{(l-j)} \circ \bar{F}_\varepsilon^{l-k} \cdot f_\theta \circ \bar{F}_\varepsilon^{l-k} + \hat{\omega}_2 \sum_{j=k+1}^{L-2} \Lambda^{(j-k)} f_\theta \circ \bar{F}_\varepsilon^{j-k},
\]

with \(M_k = \min \{L - 1, k + C_\# \log \varepsilon^{-1}\}\). Observe that \(A_k^*\) is \(C^0\)-close but clearly not \(C^1\)-close to \(A_k\); in fact \(\|A_k^*\|_{C^0} = o^*(1 + |\sigma|)\) but we necessarily have \(\|A_k^{**}\|_{C^1} = \mathcal{O}((L - k)^2 (1 + |\sigma|))\). On the other hand we have \(\|A_k^*\|_{C^1} = \mathcal{O}(\log \varepsilon^{-1} (1 + |\sigma|))\) and we can choose \(C_\#\) so to ensure that \(\|A_k^*\|_{C^3} = o((1 + |\sigma|)\sqrt{L})\).

We can at last conclude our analysis, and decompose \(\Delta = \Delta_0 + \Delta_1 + \Delta_2 + \Delta_{\text{err}}\), where

\[
\begin{align*}
\Delta_0 & = \varepsilon \sum_{k=0}^{L-2} \Re \mu_{\bar{\theta}_k}(A_k^*) \\
\Delta_1 & = \varepsilon \sum_{k=0}^{L-2} \Psi_{k}(x_k, \bar{\theta}_k) \\
\Delta_2 & = \varepsilon \sum_{k=0}^{L-2} \sum_{j=k+1}^{L-1} \hat{\omega}(\bar{x}_k, \bar{\theta}_k) \Phi_j(x_k, \bar{\theta}_k) + \varepsilon \sum_{k=0}^{L-1} \Phi_k(x_k, \bar{\theta}_k) \Phi_\ell(x_0) \\
\Delta_{\text{err}} & = \varepsilon \sum_{k=0}^{L-2} \{\Im \mu_{\bar{\theta}_k}(A_k^*) + A_k^{**}(x_k, \bar{\theta}_k)\} + \Delta_R
\end{align*}
\]
where we defined
\[ \Psi_k = \hat{\omega} \sum_{j=k+1}^{L-1} \mu_{\tilde{b}_j}(\Phi_j) + \hat{A}^c_k \]
and we introduced the notations \( \hat{A}^c_k = A^c_k - \mu_{\tilde{b}_k}(A^c_k) \) and \( \hat{\Phi}_k = \Phi_k - \mu_{\tilde{b}_k}(\Phi_k) \).

From the above estimates it follows that \( \Delta = o^*(1) \), provided that \( \sigma \in L^2 = o^*(1) \), which we can assume provided that \( r \) is sufficiently large. We thus write:
\[ e^{\sum_{k=0}^{L-1} \Omega_t(\bar{x}_k, \bar{\theta}_k) + \Delta_2 + \Delta_{err}} = e^{\sum_{k=0}^{L-1} \Omega_t(\bar{x}_k, \bar{\theta}_k)(1 + \Delta_2 + \hat{\Delta}_{err})}, \]
where \( \hat{\Delta}_{err} = \Delta_2 + \Delta_{err} - 1 - \Delta_2 \). Thus
\[ \int_{\mathbb{T}^1} g \cdot \sum_{\ell \in \mathbb{Z}_L^d} \nu_t F_\ell [\theta_k, b_k] \, d\lambda = e^{\Delta_0} \int_{\mathbb{T}^1} g \circ f^{L} \cdot e^{\sigma \sum_{k=0}^{L-1} \Omega_t(\bar{x}_k, \bar{\theta}_k)(1 + \Delta_2 + \hat{\Delta}_{err}) \hat{\rho}_\ell} \, d\lambda \]
\[ = e^{\Delta_0} \int_{\mathbb{T}^1} g \, L_{L-1} \cdots L_0 \left( (1 + \Delta_2 + \hat{\Delta}_{err}) \hat{\rho}_\ell \right) \, d\lambda \]
\[ + O((\varepsilon L)^{1+\beta'} \|g\|_{L^1}). \]
Formula (8.1) follows then by the arbitrariness of \( g \). Observe that all bounds stated in items (b), (d) and (e) immediately follow from the above definitions. In order to conclude, observe that one can obtain the decomposition described in item (c) by applying to \( \Psi_j \) (that is, to \( A^c_k \)) a regularization completely analogous to the one described above.

\[ \square \]

9. One block estimate: the large \( \eta \) regime

In the large \( \eta \) regime it suffices to estimate the contribution of the first block. This is the content of Proposition 7.4 and this short section is devoted to its proof.

**Proof of Proposition 7.4.** The basic idea is to apply Proposition 8.1 with \( L = C_\# \varepsilon^{-1/3} \) and \( \varpi_k = \Xi_{\ell_0, (R-1)L + k, \varepsilon^{-1}j} \hat{\omega} \). Let us first estimate the contribution coming from \( \hat{\Delta}_{err} \). By item (d) of said proposition, we have that
\[ \|\hat{\Delta}_{err}\|^{c_0} \leq C_\# \|\Delta_{err}\|^{c_0} + C_\# \|\Delta_2\|^{c_0} \]
\[ = o(\varepsilon L) + O(\sigma \varepsilon L) + O(\varepsilon^2 L^4) \leq C_\# \varepsilon^{7/6 - \kappa}, \]
where we have used (8.5) and (8.6). Also, a direct computation shows that
\[ \|\varepsilon^{\Delta_0} L_{L-1} \cdots L_0\|^{c_0} \leq C_\#. \]
Thus the integral over \( J_2 \) of the above contributions is bounded by \( O(\varepsilon^{\kappa+1/100}) \) provided \( \kappa < 7/12 - 1/200 \).

We are thus left with the task of estimating \( \|L_{L-1} \cdots L_0(1 + \Delta_2) \hat{\rho}_\ell\|^{c_0} \). To this end recall the definition of Transfer Operator (8.2) and potential (8.3). Recall moreover that \( \hat{\omega} \) is not cohomologous to a piecewise constant function, thus the same applies to \( \varpi_k \). Hence the \( \varpi_k \) satisfy UUNI (see Corollary C.4), thus the imaginary part of the potentials satisfies (C.4) (see Remark C.5). We can thus apply Theorem C.6, and hence Lemma A.20. This, together with Lemma A.19,
and trivial perturbation theory arguments, implies that there exists \( \tau \in (0, 1) \) such that, for each \( \eta \in \mathcal{J}_2, i \in \{0, \ldots, L\} \) and \( n \geq n_\varepsilon = [C\# \log \varepsilon^{-1}] \)
\[
\|L_i^n\|_{c^1} \leq \tau^n.
\]

On the other hand, by (A.7) and (8.3), we have
\[
\|L_i - L_k\|_{c^1 \rightarrow c^0} \leq C\# \varepsilon (1 + |\sigma|) (|k - i| + (\log \varepsilon^{-1})^2),
\]
thus, for \( n \geq n_\varepsilon \),
\[
\|L_{i+n} \cdots L_i g\|_{c^0} \leq \sum_{k=0}^{n} \|L_{i+n} \cdots L_{i+k+1} (L_{i+k} - L_i) L_i^{k-1} g\|_{c^0}
\leq C\# \left[ \sum_{k=0}^{n_\varepsilon} \varepsilon^{3/2 - \kappa} \max \{k, (\log \varepsilon^{-1})^2\} + \sum_{k=n_\varepsilon}^{n-1} \varepsilon^{3/2 - \kappa} \max \{k, (\log \varepsilon^{-1})^2\} \tau^k + \tau^n \right] \|g\|_{c^1}
\leq C\# \varepsilon^{3/2 - \kappa} (\log \varepsilon^{-1})^3 \|g\|_{c^1}.
\]

Finally, note that, by the Lasota–Yorke inequality (A.2) and item (e) of Proposition 8.1,
\[
\|L_n \cdots L_0 \tilde{\rho}_\varepsilon\|_{c^1} \leq C\# (\lambda^{-n} \varepsilon^{-2/3} (1 + 1/100) + |\sigma|).
\]
Recalling the form of \( \Delta_2 \) (see (8.5)) we can use the above estimates to obtain
\[
\|L_{L-1} \cdots L_0 (1 + \Delta_2) \tilde{\rho}_\varepsilon\|_{c^0} \leq C\# (\tau^{-\varepsilon^{-1/3}} \varepsilon^{-1} + \varepsilon^{50}) + \varepsilon \sum_{0 \leq k < j < L} \|L_{L-1} \cdots L_j \tilde{\Phi}_j L_{j-1} \cdots L_k \tilde{\rho}_\varepsilon L_{k-1} \cdots L_0 \tilde{\rho}_\varepsilon\|_{c^0}
+ \varepsilon \sum_{k=0}^{L-1} \|L_{L-1} \cdots L_k \tilde{\Phi}_k L_{k-1} \cdots L_0 \tilde{\rho}_\varepsilon\|_{c^0}
\leq C\# \varepsilon^{5/2 - \kappa} (\log \varepsilon^{-1})^3 (\log \varepsilon^{-1} + |\sigma|) \tilde{L}^2 \leq C\# \varepsilon^{7/3 - 2\kappa} (\log \varepsilon^{-1})^4,
\]
where we have used that there must necessarily be a long block of Transfer Operators. Thus, the proposition follows by the integrated version of (8.1), provided \( \kappa < 7/9 - 1/100 \).

10. One block estimate: the small \( \eta \) regime

From the above discussion it is clear that we could treat also some smaller \( \eta \) by the same one block estimate; on the other hand, to treat all \( \eta \) we will have to combine the contraction of all blocks. In any case an estimate of the one block contraction for \( \eta \in \mathcal{J}_1 \) is inevitably our next step; the precise statement of such bound will be given in Proposition 10.4. To this end a more precise knowledge of the Transfer operator for such \( \eta \)’s is needed; we begin with an obvious \( L^1 \) estimate that follows directly from the second estimate in (8.4):
\[
\|L_{\theta, \Omega_k}\|_{L^1} \leq e \cdot L.
\]

The above equation and the iterated version of (A.2) allows us to conclude that, for any \( k > s \) such that \( (k - s) \leq C\# (\varepsilon L)^{-1} \):
\[
\|(L_k L_{k-1} \cdots L_s g)'\|_{L^1} \leq C\# (\|g\|_{L^1} + \|\Lambda_{s,k} g\|_{L^1}).
\]
Next, we obtain some more sophisticated piece of information. Assume we are considering the block indexed by \( \bar{r} \) in our decomposition (i.e. the block starting at \( k = \bar{r}L \) and ending at \( k = (\bar{r} + 1)L - 1 \) where \( L \) is as defined at the beginning of Section 8); then let \( \ell \in \Omega^L_{\bar{r} - 1} \) define \( K = L(R - \bar{r}) \), where \( R = \lfloor t\varepsilon^{-1}L \rfloor \),
\[
\varpi_k = \hat{\varpi}_k \hat{\omega}
\]
for some \( \hat{\varpi}_k \in \mathbb{R}_{\geq 0} \) with \( \sup_k \hat{\varpi}_k \leq C_\# \), and the operators \( L_k \) as in Proposition 8.1.

**Lemma 10.1.** For small enough \( \sigma_0 \) in the definition of the sets \( J_k \) we have the following. For any \( \sigma \in \varepsilon^{1/2}J_1 \) and \( k \in \{0, \ldots, L - 1\} \):

(a) \( L_k \) admits a spectral decomposition, i.e. we can write \( L_k = e^{\chi_k}P_k + Q_k \), where \( e^{\chi_k} \) is the maximal eigenvalue of \( L_k \) (as an operator acting on \( C^1, W^{1,1} \) or \( BV \)), \( P_k, Q_k \) are such that \( P_k^2 = P_k, P_kQ_k = Q_kP_k = 0 \) and there exists \( \tau \in (0, 1) \) so that
\[
\|Q_k^n\|_{C^1, W^{1,1}, BV} \leq C\tau^n |e^{\chi_k}|.
\]

(b) we have
\[
\chi_k = \frac{\sigma^2}{2} \hat{\varpi}_k^2 \hat{\sigma}^2(\tilde{\theta}_k) + O(|\sigma|^3 + |\sigma|\varepsilon L) \leq -\frac{1}{4} \sigma^2 \hat{\varpi}_k^2 \hat{\sigma}^2(\tilde{\theta}_k)
\]
where \( \hat{\sigma}^2 \in C^1(\mathbb{R}_{\geq 0}) \) is given by the Green-Kubo formula (2.12).

**Proof.** Recall that \( L_k = L_{\tilde{\theta}_k, c_0, \Omega_k} \) is the Transfer operator associated to \( f_k = f(\cdot, \tilde{\theta}_k) \) with potential \( \Omega_k(\cdot, \tilde{\theta}_k) \) and, by (8.3), \( \Omega_k = i\sigma\varpi_k + \varepsilon \Psi_k \) with \( \| \Psi_k \|_{C^3} = O(L - k) + O(\sqrt{L}) \). In order to apply the results of Appendix A, let us consider the Transfer Operator given by \( L_{\tilde{\theta}_k, c_0, \Omega_k} \), for \( \varsigma \in [0, 1] \). Since the operator, for \( \varsigma = 0 \), has 1 as a simple maximal eigenvalue and a spectral gap (in any of the above mentioned space), it follows that there exists \( \sigma_0 \) such that, for any \( \sigma \in [0, \sigma_0] \), the operator for \( \varsigma \in [0, 1] \) has still a simple maximal eigenvalue and a spectral gap (assuming \( \varsigma \) to be sufficiently small); observe that, since the resolvent is continuous in \( \theta \), \( \sigma_0 \) can be chosen uniformly in \( \theta \) and, consequently, since we have good control on all terms appearing in \( \Omega_k \), \( \sigma_0 \) can be chosen to be uniform in \( k \) as well. This proves item (a).

We now prove item (b): for fixed \( k \), we use (A.10a) with \( \varsigma = 0 \) and (A.11) to obtain:
\[
e^{\chi_k} = 1 + \frac{1}{2} \mu_{\bar{\theta}_k}(\Omega_k(\cdot, \bar{\theta}_k)^2) + 2 \sum_{m=1}^{\infty} \mu_{\bar{\theta}_k}(\Omega_k(f_k^{(m)}(\cdot, \bar{\theta}_k)\Omega_k(\cdot, \bar{\theta}_k)) + O(|\sigma^3|) =
\]
\[
e^{-\frac{\sigma}{2} \hat{\varpi}_k \hat{\sigma}^2(\bar{\theta}_k)} - 2\sigma^2 \sum_{m=1}^{\infty} \mu_{\bar{\theta}_k}(\varpi_k f_k^{(m)}(\varpi_k) + O(|\sigma^3| + |\sigma|\varepsilon L + \varepsilon^2 L^2)
\]
where we have used the fact that \( \sigma \in \varepsilon^{1/2}J_1 \) and the decay of correlations implied by item (a). The proof then follows by the definition of \( \varpi_k \).

We remark the fact that, since \( P_k \) is a one dimensional projector, it can be written as \( P_k = h_k \otimes m_k \), where \( h_k \) and \( m_k \) are normalized according to Lemma A.5. The following lemma is an impressionistic version of a spectral decomposition (with error) for a block of operators along the averaged dynamics.

**Lemma 10.2.** For any \( 0 \leq s < k < L \) we can write:
\[
L_{k-1}L_{k-2} \cdots L_s = B_{k,s}^P + B_{k,s}^Q + B_{k,s}^R
\]
where $B^P_{k,s} = \varepsilon \sum_{i=0}^{k-1} \chi_i h_{k-1} \otimes m_s$ and we have

\begin{align}
(10.6) \quad \|B^Q_{k,s}\|_{W^{1,1}} & \leq C_\#(r^{k-s} + \varepsilon); \\
(10.7) \quad \|B^R_{k,s}\|_{W^{1,1}} & \leq C_\# \varepsilon \sum_{i=0}^{k-1} \chi_i \sigma^2 + C_\# \varepsilon^2 L^3; \\
\end{align}

additionally, the following estimate holds true:

\begin{equation}
(10.8) \quad \|B^Q_{k,s} \otimes Q_{\delta_{s-1,0}} \cdots Q_{\delta_{0,0}}\|_{W^{1,1}} \leq C_\#. 
\end{equation}

**Proof.** Let us write $L_{k-1} L_{k-2} \cdots L_s = \sum_{j=s}^k E_j$, where, for $s < j < k$,

\[ E_j = \varepsilon \sum_{i=0}^{j-1} \chi_i L_{k-1} \cdots L_{j+1} Q_j P_{j-1} \cdots P_s, \]

and $E_k = \varepsilon \sum_{i=0}^{k-1} \chi_i P_{k-1} P_{k-2} \cdots P_s$, $E_s = L_{k-1} L_{k-2} \cdots Q_s$. In order to study the operators $E_j$ we first need to obtain a few useful bounds.

**Sub-lemma 10.3.** The following estimates hold true for any $0 \leq j \leq l \leq L$:

\begin{align}
(10.9a) \quad \|Q_{l} h_j\|_{C^1} & \leq C_\# \varepsilon(l - j) + (\log \varepsilon^{-1})^2; \\
(10.9b) \quad P_l P_{l-1} \cdots P_j = (1 + O(\sigma^2 + \varepsilon^2 L^3)) h_l \otimes m_j. 
\end{align}

**Proof.** Let us recall that, by definition, $h_j = h_{\tilde{\theta}, \Omega_j}$; thus:

\begin{equation}
(10.10) \quad h_j = (h_{\tilde{\theta}, \Omega_j} - h_{\tilde{\theta}, \Omega_l}) + (h_{\tilde{\theta}, \Omega_l} - h_{\tilde{\theta}, \Omega_l}) + h_l.
\end{equation}

Using (A.17) and (A.20a) we obtain:

\begin{align}
(10.11a) \quad \|h_{\tilde{\theta}, \Omega_j} - h_{\tilde{\theta}, \Omega_l}\|_{C^k} & \leq C_\# |\tilde{\theta}_j - \tilde{\theta}_l|(1 + \|\Omega_j\|_{C^{k+1}}) \\
(10.11b) \quad \|h_{\tilde{\theta}, \Omega_l} - h_{\tilde{\theta}, \Omega_l}\|_{C^1} & \leq C_\# \|\Omega_j - \Omega_l\|_{C^1}; 
\end{align}

Hence, applying $Q_l$ to both sides of (10.10), we get:

\[ \|Q_l h_j\|_{C^1} \leq C_\# \varepsilon(l - j)(1 + \|\Omega_j\|_{C^2}) + C_\# \|\Omega_j - \Omega_l\|_{C^1}, \]

which immediately implies (10.9a) by the estimates on $\Omega_j$ reported in Proposition 8.1.

We now proceed with the proof of (10.9b): let us fix $k \leq j < r < l$; then:

\[ P_{r+1}(P_r - P_j)P_j = P_{r+1}((P_{\tilde{\theta}, \Omega_r} - P_{\tilde{\theta}, \Omega_j}) + (P_{\tilde{\theta}, \Omega_j} - P_{\tilde{\theta}, \Omega_j}))P_j \]

\[ = \left[ \int_{\tilde{\theta}_j}^{\tilde{\theta}_l} \frac{\partial}{\partial \theta} m_{r+1} \partial \theta P_{\tilde{\theta}, \Omega_r} h_j + \int_0^1 \frac{\partial}{\partial \theta} m_{r+1} \partial \theta P_{\tilde{\theta}, \Omega_r} h_j \right] h_{r+1} \otimes m_j \]

where we defined the convex interpolation $\Omega_\theta = \Omega_j + \theta(\Omega_l - \Omega_j)$. Observe that, by definition of $m$ and $h$, we have, for any $\theta$:

\[ m_{\theta, \Omega_j} \partial \theta P_{\tilde{\theta}, \Omega_r} h_{\theta, \Omega_j} = 0 \quad \text{and} \quad m_{\theta, \Omega_j} \partial \theta P_{\tilde{\theta}, \Omega_r} h_{\theta, \Omega_r} = 0, \]

moreover, using bounds (A.17) and (A.20), we immediately obtain:

\[ \|\partial \theta P_{\tilde{\theta}, \Omega_r} g\|_{C^k} \leq C_\# (1 + \|\Omega_j\|_{C^{k+1}}) \|g\|_{C^2} \]

\[ \|\partial \theta P_{\tilde{\theta}, \Omega_l} g\|_{C^k} \leq C_\# \|\Omega_j - \Omega_r\|_{C^1} \|g\|_{C^k}. \]
Thus, we can estimate:

\[
m_{r+1} \partial_\theta P_{\tilde{\theta}, \Omega} h_j = [(m_{r+1} - m_{\tilde{\theta}, \Omega}) + (m_{\tilde{\theta}+1, \Omega} - m_{\tilde{\theta}, \Omega})] \partial_\theta P_{\tilde{\theta}, \Omega} h_{\tilde{\theta}, \Omega} + m_{r+1} \partial_\theta P_{\tilde{\theta}, \Omega} (h_j - h_{\tilde{\theta}, \Omega})
\]

\[
\leq C_\#(1 + \|\Omega_j\| c^2) \|\Omega_j - \Omega_l\| c^2 + C_\#(1 + \|\Omega_j\| c^2)(1 + \|\Omega_j\| c^2) \varepsilon L + C_\#(1 + \|\Omega_j\| c^2) \varepsilon L \leq C_\# L
\]

where we used repeatedly estimates (A.17) and (A.20), and Proposition 8.1 in the last step. Likewise, we can estimate:

\[
m_{r+1} \partial_\epsilon P_{\tilde{\theta}, \Omega} h_j = [(m_{r+1} - m_{\tilde{\theta}, \Omega}) + (m_{\tilde{\theta}+1, \Omega} - m_{\tilde{\theta}, \Omega})] \partial_\epsilon P_{\tilde{\theta}, \Omega} h_{\tilde{\theta}, \Omega} + m_{r+1} \partial_\epsilon P_{\tilde{\theta}, \Omega} (h_j - h_{\tilde{\theta}, \Omega})
\]

\[
= C_\# \varepsilon \|\Omega_j - \Omega_l\| c^2 + C_\# \|\Omega_j - \Omega_l\| c^2 + C_\# \|\Omega_j - \Omega_l\| c^2 \leq \varepsilon^2 L^2,
\]

since \(\|\Omega_j - \Omega_l\| c^2 \leq C_\# |\sigma| \varepsilon |j - l| + \varepsilon (|\psi_j| c^2 + |\psi_l| c^2) \leq C_\# \varepsilon L\). Thus, we conclude that \(P_{r+1}(P_r - P_j)P_j = O(\varepsilon^2 L^2) h_{r+1} \otimes m_j\), which implies by iteration that

\[P_j P_{j-1} \cdots P_1 = P_j \approx O(\varepsilon^2 L^3) h_l \otimes m_j.\]

We therefore reduced ourselves to obtain an estimate for

\[P_j P_j - h_l \otimes m_j = [m_j h_j - 1] h_l \otimes m_j;\]

we claim that \(|m_j h_j - 1| = O(\sigma^2)|\); in fact let us now write

\[m_l = \text{Leb} + \int_0^1 \partial_\lambda m_{\tilde{\theta}, \lambda, \Omega} d\kappa';\]

\[h_j = h_{\tilde{\theta}, 0} + \int_{\tilde{\theta}_j}^{\delta_j} \partial_\theta h_{\tilde{\theta}, \kappa} d\kappa' + \int_0^1 \partial_\kappa h_{\tilde{\theta}, \kappa, \Omega} d\kappa'.\]

By (A.15) we know that \(|m_l(h_{\tilde{\theta}, 0}) - 1| = O(\sigma^2)|\); in order to conclude we thus need to estimate the remaining pieces, which we write as follows:

\[
\int_{\tilde{\theta}_j}^{\delta_j} \text{Leb} (\partial_\theta h_{\tilde{\theta}, \theta_0}) d\kappa' + \int_0^1 \text{Leb} (\partial_\kappa h_{\tilde{\theta}, \kappa, \Omega}) d\kappa' + \int_0^1 \partial_\lambda m_{\tilde{\theta}, \lambda, \Omega} (\partial_\theta h_{\tilde{\theta}, \theta_0}) d\kappa' d\kappa'' + \int_0^1 \partial_\kappa m_{\tilde{\theta}, \kappa, \Omega} (\partial_\theta h_{\tilde{\theta}, \theta_0}) d\kappa' d\kappa'';
\]

the first term is 0 by Lemma A.5; the second one is \(\text{Leb}(h_{\tilde{\theta}, \lambda, \Omega}) - 1\) which equals \(O(\sigma^2)\) by (A.15); the third term, by (A.14b), (A.20) and recalling the ranges of \(\sigma, L\), is \(O(|\sigma| \varepsilon L) = o(\sigma^2)\); finally, for the fourth term, we use the explicit expressions (A.12a) and (A.12b), which yield once again a bound \(O(\sigma^2)\). This proves (10.9b).

Let us resume the proof of Lemma 10.2: we first examine terms \(E_j\) for \(s < j < k\): by (10.9b) we can write

\[
E_j = O(1) \varepsilon \sum_{j-1}^{j-1} E_{k-1} \cdots E_j h_{j-1} \otimes m_s.
\]
Let us now examine the operators $L_{k-1} \cdots L_{j+1} Q_j$: let us write
\[ L_{k-1} \cdots L_{j+1} Q_j = \sum_{l=j+1}^{k-1} L_{k-1} \cdots L_{l+1}(L_l - L_j)Q_j^{l-j} + Q_j^{k-j} ; \]
observe that \((A.6)\) and Proposition 8.1-(b) imply $\|L_l - L_j\|_{W^{1,1} \to L^1} \leq C\# \varepsilon (l - j)^2$.
Therefore, using \((10.1)\), and \((10.3)\) when summing over $l$, we can conclude that
\[ \|L_{k-1} \cdots L_{j+1} Q_j - Q_j^{k-j}\|_{W^{1,1} \to L^1} \leq C\# \varepsilon , \]
which implies that
\[ \|L_{k-1} \cdots L_{j+1} Q_j\|_{W^{1,1} \to L^1} \leq C\# (\tau^{k-j} + \varepsilon) . \]
Observe now that, by iterating \((A.2)\), and using the above estimate, since $\tau > \lambda^{-1}$, we obtain that $\|(L_{k-1} \cdots L_{j+1} Q_j)g\|_{L^1} \leq C\# (\tau^{k-j} + \varepsilon)$ which finally implies
\[(10.13) \quad \|L_{k-1} \cdots L_{j+1} Q_j\|_{W^{1,1}} \leq C\# (\tau^{k-j} + \varepsilon) . \]
Applying the above estimate to \((10.12)\), for $s < j < k$, and using \((10.9a)\) we thus obtain that
\[(10.14) \quad \sum_{j=s+1}^{k-1} \|\xi_j\|_{W^{1,1}} = O(\varepsilon (\log \varepsilon^{-1})^2) . \]
Let us set $B_{k,s}^Q = \xi_s$ and $B_{k,s}^P = \sum_{j=s+1}^{k} \xi_j - B_{k,s}^Q$. Then, by \((10.9b)\) we obtain that $\|\xi_k - B_{k,s}^P\|_{W^{1,1}} = O(\sigma^2 + \varepsilon^2 L^3 + \varepsilon (\log \varepsilon^{-1})^2)$, which, together with \((10.14)\), implies \((10.7)\); on the other hand, \((10.13)\) immediately implies \((10.6)\).

In order to conclude the proof it thus suffices to prove \((10.8)\); this amounts to write:
\[ L_{k-1} \cdots L_{s+1} Q_s = \sum_{l=s+1}^{k-1} L_{\theta_{k-1},0} \cdots L_{\theta_{l+1},0}(L_l - L_{\theta_{l},0})L_{l-1} \cdots L_{s+1} Q_s + L_{\theta_{l-1},0} \cdots L_{\theta_{s+1},0}(Q_s - Q_{\theta_{s},0}) + Q_{\theta_{k-1},0} \cdots Q_{\theta_{s},0} . \]
Then, observe that \((10.8)\) immediately follows, using \((10.13)\) and the fact that $\|\partial L_{\theta,\sigma,\Omega}\|_{W^{1,1}}$, $\|\partial Q_{\theta,\sigma,\Omega}\|_{W^{1,1}} = O(\sigma)$.

Now that we obtained all necessary properties of the Transfer Operators in the small $\eta$ regime, we can state and prove the main result of this section.

**Proposition 10.4.** Let $\hat{\Xi} \in C^1$ and numbers $\{\hat{\Xi}_j\}_{j=0}^L \subset \mathbb{R}_{\geq 0}$ such that $\|\hat{\Xi}\|\in L^1 \leq C\#$ and $\|\hat{\Xi}(\varepsilon j) - \hat{\Xi}_j\| \leq C\# \varepsilon$. Then, for any complex standard pair $\ell$, function $\varphi \in C^0$ and $\eta \in J_1$:
\begin{align*}
(10.15a) \quad \mu_{\ell} \left( e^{i\sigma} \sum_{j=0}^{L-1} \hat{\Xi}_j \hat{\omega}(x_j, \theta_j) \varphi \right) &= \sum_{\tilde{\ell} \in E_\ell^L} \nu_{\tilde{\ell}} \int_{\bar{\alpha}_{\tilde{\ell}}}^{b_{\tilde{\ell}}} \rho_{\tilde{\ell}} \varphi \; ; \\
(10.15b) \quad \sum_{\tilde{\ell} \in E_\ell^L} \nu_{\tilde{\ell}} \rho_{\tilde{\ell}} &= e^{-\frac{\pi}{2} L} \int_{0}^{L} \hat{\Xi}(s) \hat{\omega}(s, \theta_s) ds + O(\omega^3 L) (h_{L-1} m_0 \hat{\rho}_{\ell} + P) + Q
\end{align*}

\[31\text{Recall that, for all } \theta, \theta' \in \mathbb{T}, \rho_{\theta,0} Q_{\theta',0} = 0.\]
where \( \hat{\rho}_t \) is defined in Proposition 8.1–(e). \( P \) and \( Q \) are error terms such that 
\[
\|P\|_{w^{1,1}} \leq \sum_{k=0}^{2} p_k |\eta|^k, \quad \|Q\|_{L^1} \leq q_0 + q_1 |\eta| \quad \text{and} \quad \|Q\|_{L^1} = o(\sqrt{\varepsilon}L), \quad \text{where}
\]
\[
(10.16a) \quad p_0 = o^*(\varepsilon L) \quad p_1 = o^*(\varepsilon L) \quad p_2 = o^*((\varepsilon L)^{3/2})
\]
\[
(10.16b) \quad q_0 = o^*(\varepsilon L) \quad q_1 = o^*(\varepsilon).
\]

**Proof.** Observe that (10.15a) is just a reminder of (3.12). Let us thus prove (10.15b). Proposition 8.1, with \( \tau_j = \hat{\xi}_j \hat{\omega}_j \), implies that 
\[
\sum_{\ell \in \Omega^k_t} \nu_{\ell} \hat{q}_{[\nu_{\ell}, \nu_{\ell+1}]} = e^{\Delta_0} L_{L-1} L_{L-2} \cdots L_0 (1 + \Delta_2 + \hat{\Delta}_{\text{err}}) \hat{\rho}_t + O((\varepsilon L)^{1+1/100});
\]
the following sub-lemma shows that the contribution of the term \( \hat{\Delta}_{\text{err}} \) is compatible with the error term \( Q \).

**Sub-lemma 10.5.** The following estimates hold:
\[
(10.17a) \quad \|L_{L-1} L_{L-2} \cdots L_0 \hat{\Delta}_{\text{err}} \hat{\rho}_t\|_{L^1} = o^*(\varepsilon L) + |\eta| o(\varepsilon)
\]
\[
(10.17b) \quad \left\| \frac{d}{d\bar{x}_L} L_{L-1} L_{L-2} \cdots L_0 \hat{\Delta}_{\text{err}} \hat{\rho}_t \right\|_{L^1} = O(\varepsilon L).
\]

**Proof.** Let us recall (see Proposition 8.1–d) that \( \hat{\Delta}_{\text{err}} = e^{\Delta_2 + \hat{\Delta}_{\text{err}}} - 1 - \Delta_2 \), which implies:
\[
|\hat{\Delta}_{\text{err}}| \leq C_\# |\Delta_{\text{err}}| + C_\# |\Delta_2^2|
\]
\[
|\hat{\Delta}'_{\text{err}}| \leq C_\# |\Delta'_{\text{err}}| + C_\# ((|\Delta_{\text{err}}| + |\Delta_2|)|\Delta_2'|),
\]
where we denote with \('\) the derivative with respect to \( x_0 \). In turn, the estimates given in Proposition 8.1–(c,d) immediately imply that
\[
|\Delta_{\text{err}}| = o^*(\varepsilon L) + |\eta| o(\varepsilon); \quad \quad |\Delta_2| = O(\varepsilon L^2).
\]

The above \( C^0 \)-estimate for \( \Delta_2 \) is not sharp enough for our needs; in order to obtain a more useful bound we need to study its \( L^1 \) norm, on which we have better control by means of our knowledge of large deviation bounds. By Lemma 4.2 we have
\[
\left| \sum_{l=0}^{k} \hat{\omega}(\bar{x}_l \circ Y_{\bar{x}_l}^{-1}, \hat{\theta}_l) - \sum_{l=0}^{k} \hat{\omega}(x_l, \theta_l) \right| \leq C_\# \varepsilon k.
\]

Thus
\[
\Delta_2 = \varepsilon \sum_{j=1}^{L-1} \hat{\phi}_j \sum_{k=0}^{j-1} \hat{\omega}(x_k, \theta_k) + O(\varepsilon L + \varepsilon^2 L^3) = \varepsilon \sum_{j=1}^{L-1} \hat{\phi}_j \cdot (\theta_j - \bar{\theta}_j) + O(\varepsilon L + \varepsilon^2 L^3).
\]

Fix \( \alpha \in (1/2, 2/3) \), set \( A = \{ \max_{0 \leq k < L} |\theta_j - \bar{\theta}_j| \leq C L^{-\alpha} \} \) and decompose the density \( \hat{\rho}_t \) as \( \hat{\rho}_t = (1 - \delta_A) \hat{\rho}_t + \delta_A \hat{\rho}_t \). Theorem 2.2, Remark 6.2 and Remark 2.3 implies that \( \| (1 - \delta_A) \hat{\rho}_t \|_{L^1} = O(\varepsilon^{-C L^{2-\alpha}}) \), which can be taken to be arbitarily small; on the other hand \( \| \delta_A \Delta_2 \|_{C^0} = O(\varepsilon L^{1+\alpha}) \). Summarizing, we conclude that \( \| \Delta_2 \|_{L^1} = O(\varepsilon L^{1+\alpha}) \), that yields
\[
\| \Delta_2^2 \|_{L^1} \leq \| \Delta_2 \|_{C^0} \| \Delta_2 \|_{L^1} = O(\varepsilon^2 L^{3+\alpha}) = o^*(\varepsilon L),
\]

\[32 \] Use the expansion (2.10) and the fact that for a trajectory to deviate from the average by at least \( \varepsilon L^\alpha \) in a time \( \varepsilon L \) and minimise the rate function, it must have a derivative of order \( L^{\alpha-1} \) on the all interval (see the discussion of the example after Corollary (2.5) for details).
which, using (10.1), implies (10.17a).

We now proceed to prove (10.17b): Proposition 8.1–(c) gives:

$$|\Delta'_2| = O(\varepsilon L) \sum_{k=0}^{L-1} (L-k)\bar{x}_k',$$

which, together with (8.6b) immediately implies:

$$|\bar{\Delta}_{\text{err}}'| = O(\varepsilon L) \sum_{k=0}^{L-1} \max(x'_k, \bar{x}'_k)(L-k)^2.$$

In order to obtain (10.17b) it thus suffices to notice that $\Lambda_{0, L} \max(\bar{x}'_k, x'_k) \leq \lambda^{k-L}$ and apply (10.2).

We are thus left to analyze the term $e^{\Delta_0} L_{L-1} L_{L-2} \cdots L_0 (1+\Delta_2) \tilde{\rho}_\ell$; Lemma 10.2 implies that

$$L_{L-1} \cdots L_0 \tilde{\rho}_\ell = e^{\sum_{l=0}^{L-1} \chi_l} \left[h_{L-1} m_0 \tilde{\rho}_\ell + O_{W^{1,1}}(\sigma^2)\right] + O_{W^{1,1}}(\varepsilon^2 L^3 + \tau L).$$

Notice that Lemma 10.1 allows us to write:

$$\sum_{l=0}^{L-1} \chi_l = -\frac{\sigma^2}{2} \sum_{l=0}^{L-1} \tilde{\Xi}_l^2 \hat{\sigma}_l^2(\bar{\theta}_l) + O(|\sigma^3| L + |\sigma| \varepsilon L^2)
= -\frac{\sigma^2}{2} \int_0^\varepsilon L \tilde{\Xi}(s)^2 \hat{\sigma}_l^2(\bar{\theta}(s, \bar{\theta}_0))ds + O(|\sigma^3| L + \sigma^2 \varepsilon L + |\sigma| \varepsilon L^2),$$

which allows us to conclude that

$$L_{L-1} \cdots L_0 \tilde{\rho}_\ell = e^{-\frac{\sigma^2}{2} \int_0^\varepsilon L \tilde{\Xi}(s)^2 \hat{\sigma}_l^2(\bar{\theta}(s, \bar{\theta}_0))ds} e^{O(|\sigma^3| L)(h_{L-1} m_0 \tilde{\rho}_\ell + P)} + Q.$$  

Likewise, observe that, by (8.5) we can write:

$$(10.18a) \quad L_{L-1} \cdots L_0 \Delta_2 \tilde{\rho}_\ell = \varepsilon \sum_{0 \leq k<j<L} L_{L-1} \cdots L_j \tilde{\Psi}_j L_{j-1} \cdots L_k \hat{\psi}_k L_{k-1} \cdots L_0 \tilde{\rho}_\ell +$$

$$(10.18b) \quad + \varepsilon \sum_{0 \leq j<k<L} L_{L-1} \cdots L_j \hat{\psi}_j L_{j-1} \cdots L_k \tilde{\Psi}_k L_{k-1} \cdots L_0 \tilde{\rho}_\ell$$

In order to analyze the above terms, we first observe that, for any $\Phi$ smooth:

$$m_{k+1}(\Phi h_k) = \mu_{\tilde{\theta}_{k+1}}(\Phi) + O(\sigma) ||\Phi||_{W^{1,1}};$$

in fact, let us write:

$$m_{k+1}(\Phi h_k) = \text{Leb}(\Phi h_{\tilde{\theta}_{k+1}, 0}) + \text{Leb} \left( \Phi \int_{\tilde{\theta}_{k+1}}^{\theta_k} \partial_h h_{\tilde{\theta}_{k+1}, 0} d\tilde{\theta} \right) +$$

$$+ \text{Leb} \left( \Phi \int_0^1 \partial_{l_h} h_{\tilde{\theta}_{k+1}, c_{l_h}} d\xi \right) + \int_0^1 \partial_{l_h} m_{\tilde{\theta}_{k+1}, c_{l_h}} (\Phi h_k) d\xi.$$

The first term equals $\mu_{\tilde{\theta}_{k+1}}(\Phi)$, the second one is 0, and the remaining terms contribute for $O(\sigma) ||\Phi||_{C^1}$ by estimates (A.14), from which we obtain (10.19a).

By similar (and actually simpler) arguments we can also show that

$$m_0(\Phi \tilde{\rho}_\ell) = \text{Leb}(\Phi \tilde{\rho}_\ell) + O(\sigma) ||\Phi||_{W^{1,1}}. \tag{10.19b}$$

---

33 By $O_{W^{1,1}}(h)$ we mean a generic function whose $W^{1,1}$ norm is bounded by $h$.

34 Here and in the following $P, Q$ stand for generic functions that satisfy the inequalities specified in the statement of Proposition 10.4.
Our next step is to show that the contribution of \( \Phi^*_j \) to (10.18) is negligible: observe that, by (10.1) and (10.2), using the estimates of Proposition 8.1–(c), for any smooth function \( g \):

\[
\| \mathcal{L}_{L-1} \cdots \mathcal{L}_j \Phi^*_j g \|_{W^{1,1}} \leq C_g (\varepsilon^{100} + (L-j)\tau^{L-j}) \| g \|_{W^{1,1}}.
\]

Using Lemma 10.2, bounds (10.19) and Proposition 8.1–(c), we obtain:

\[
\sum_{k=0}^{j-1} \| \mathcal{L}_{L-1} \cdots \mathcal{L}_k \hat{\mathcal{L}}_k \cdots \mathcal{L}_0 \hat{\varphi}_k \|_{W^{1,1}} = O(1 + |\sigma|L)
\]

Plugging in (10.20) in (10.18) and using the above estimates, we obtain that the contribution of \( \Phi^*_j \) to (10.18) is \( O(\sigma L + \varepsilon) \), that is compatible with our requirements for \( Q \).

Thus, we are left to analyze the contribution of \( \Phi^* \); let us start with (10.18a). Let us first consider the case \( k = 0 \): by applying Lemma 10.2 we obtain\(^{35}\)

\[
\sum_{j=1}^{L-1} \| \mathcal{L}_{L-1} \cdots \mathcal{L}_j \hat{\mathcal{L}}_j \cdots \mathcal{L}_0 \hat{\varphi}_k \|_{W^{1,1}} = O(1 + |\sigma|L),
\]

which implies that the contribution of the term \( k = 0 \) is \( O(\sigma \varepsilon L + \varepsilon) \), which can be added to \( Q \). Let us now assume \( k > 0 \); we need to apply Lemma 10.2 to each of the three strings of Transfer Operators appearing in (10.18a) and study the sum:

\[
\sum_{0<k<j<L} \sum_{\alpha_0, \alpha_1, \alpha_2 \in \{P, Q\}} B^{0}_{0j} \hat{\Phi}_{j,k} B_{j,k}^{01} \hat{\varphi}_{k,0}.
\]

First, let us show that we can neglect terms such that at least one of the \( \alpha \)'s is equal to \( R \); assume for the moment that both remaining \( \alpha \)'s equal \( P \); then, by definition of \( B^P \) and by (10.7), the total contribution of such terms is

\[
e \sum_{k=0}^{L-1} \chi_k \mathcal{O}_{W^{1,1}} (\sigma^2 \varepsilon L^2 \log \varepsilon^{-1}) + \mathcal{O}_{W^{1,1}} (\varepsilon^3 L^5 \log \varepsilon^{-1});
\]

the first term can thus be added to \( P \) and the second one to \( Q \). On the other hand, the total contribution of terms so that at least one of the remaining \( \alpha \)'s is a \( Q \) is bounded by

\[
\mathcal{O}_{W^{1,1}} (\sigma \varepsilon L \log \varepsilon^{-1} + \varepsilon^2 L^2 \log \varepsilon^{-1})
\]

which can also be added to \( Q \). At last, if all three \( \alpha \)'s are \( R \), then we have a bound

\[
e \sum_{k=0}^{L-1} \chi_k \mathcal{O}_{W^{1,1}} (\sigma^2 \log \varepsilon^{-1}) + \mathcal{O}_{W^{1,1}} (\varepsilon^2 L^3 \log \varepsilon^{-1});
\]

which can be again added to \( P \) and \( Q \). Hence, we are now reduced to study the case in which \( \alpha_1, \alpha_2, \alpha_3 \in \{P, Q\} \). We first consider the case in which at least two of the \( \alpha \)'s are equal to \( Q \); the total contribution of such terms is then bounded by \( \mathcal{O}(\varepsilon \log \varepsilon^{-1}) \) and can thus be added to \( Q \). On the other hand, if all \( \alpha \)'s are equal to \( P \), the contribution is bounded, using (10.19a) and Proposition 8.1–(c), by

\[
e \sum_{k=0}^{L-1} \chi_k \mathcal{O}_{W^{1,1}} (\sigma^2 L^2 \varepsilon \log \varepsilon^{-1}) = e \sum_{k=0}^{L-1} \chi_k \mathcal{O}_{W^{1,1}} (\eta^2 L^2 \varepsilon^2 \log \varepsilon^{-1})
\]

which can be added to \( P \).

We thus conclude by studying the case in which exactly one of the \( \alpha \)'s is equal to \( Q \) and the remaining two are equal to \( P \). If either \( \alpha_0 \) or \( \alpha_2 \) are equal to \( Q \), then

\(^{35}\) Consistently with the notation in Proposition 8.1, \( \hat{\Phi}_j = \Phi^*_j - \mu_{\hat{\Phi}_j} (\Phi^*_j) \).
using once again \((10.19a)\) and \((10.6)\), we obtain a bound of \(\mathcal{O}(\sigma \varepsilon L \log \varepsilon^{-1})\), which can be added to \(Q\). The only remaining term is the correlation term:

\[
\sum_{0 < \kappa < \gamma < L} \mathcal{B}_{\gamma, \kappa}^P \hat{\mathcal{B}}_{\delta, \kappa}^Q \omega \mathcal{B}_{\mu, 0}^P
\]

\[
= e \sum_{j=1}^{L-1} \chi_{j} \sum_{\kappa=0}^{k-1} \chi_{\kappa} h_{L-1} m_{0}(\tilde{\mu}_{\kappa}) \cdot \sum_{0 < \kappa < \gamma < L} m_{\gamma} \hat{\mathcal{B}}_{\delta, \kappa}^Q \omega h_{k-1};
\]

we now apply \((A.14b)\), \((A.20a)\), and use \((10.8)\) (when \(k-j \leq C \log \varepsilon^{-1}\)) and \((10.6)\) (otherwise) to obtain:

\[
\varepsilon \sum_{0 < \kappa < \gamma < L} m_{\gamma} \hat{\mathcal{B}}_{\delta, \kappa}^Q \omega h_{k-1}
\]

\[
= \varepsilon \sum_{0 < \kappa < \gamma < L} \text{Leb} \hat{\mathcal{B}}_{\delta, \kappa}^Q \omega h_{k-1} + \mathcal{O}(\sigma \varepsilon L (\log \varepsilon^{-1})^{2});
\]

note that the error term can be added once again to \(Q\). Observe that \((10.4)\) implies that

\[
e^{-\sum_{j=k}^{j} \chi_{j}} = \mathcal{O}(\sigma^{2}(j-k)).
\]

Thus, we conclude that

\[
\text{Leb} \hat{\mathcal{B}}_{\delta, \kappa}^Q \omega h_{k-1} - \varepsilon \sum_{j=k}^{j} \chi_{j} C_{k,j} = \mathcal{O}(\sigma(j-k) \tau^{j-k})
\]

where we defined the correlation

\[
C_{k,j} = \mu_{\hat{\theta}}([\text{Re} \hat{\mathcal{B}}_{\delta, \kappa}^Q \circ \text{Im} \hat{\mathcal{B}}_{\delta, \kappa}^Q \circ \cdots \circ \text{Im} \hat{\mathcal{B}}_{\delta, \kappa}^Q]),
\]

and used that \(\|\text{Im} \hat{\mathcal{B}}_{\delta, \kappa}^Q\|_{L^1} = \mathcal{O}(\sigma)\). We thus conclude that

\[
\varepsilon \sum_{0 < \kappa < \gamma < L} \mathcal{B}_{\gamma, \kappa}^P \hat{\mathcal{B}}_{\delta, \kappa}^Q \omega \mathcal{B}_{\mu, 0}^P
\]

\[
e^{-\sum_{j=0}^{j} \chi_{j} h_{L-1} m_{0}(\tilde{\mu}_{\kappa})} \sum_{0 < \kappa < \gamma < L} \varepsilon \sum_{0 < \kappa < \gamma < L} C_{k,j} + \mathcal{O}(\sigma \varepsilon L (\log \varepsilon^{-1})^{2}).
\]

The latter term can again be absorbed in \(Q\). The first term instead seems too big to be absorbed in \(P\). Nevertheless, we will see at the end of the proof that such term provides a much needed cancellation.

We are left to analyze \((10.18b)\): the computation is similar to the one we did above for the case \(k = 0\): by applying Lemma 10.2 to \((10.18b)\) we need to study

\[
\varepsilon \sum_{j=0}^{L-1} \alpha_{\kappa,j} \mathcal{B}_{\gamma, \kappa}^P \hat{\mathcal{B}}_{\delta, \kappa}^Q \omega \mathcal{B}_{\mu, 0}^P
\]

Arguments similar to the ones we used before imply that the contribution of terms where at least one of the \(\alpha\)’s equal either \(Q\) or \(R\) is negligible (i.e. can be added to either \(P\) or \(Q\)); on the other hand, if both \(\alpha\)’s equal \(P\), then \((10.19b)\) and Proposition 8.1–(c) imply that this contribution is \(\mathcal{O}(\sigma \varepsilon L)\), which is also negligible.

Summarizing, we obtain that

\[
\sum_{i \in \Gamma_{L}^{\mu}} \nu_{i} \rho_{i} = e^{\Delta_{0} \varepsilon} \int_{0}^{L \varepsilon} E_{0} \hat{\sigma}^{2}(\tilde{\sigma}(\tilde{\sigma}_{0}, \tilde{\sigma}_{i})) ds e^{\mathcal{O}(\varepsilon L) h_{L-1}(m_{0}(\tilde{\mu}_{\kappa}) + P) + Q}
\]
where \( \hat{\Delta}_0 = \Delta_0 + \varepsilon \sum_{k=0}^{L-1} \sum_{j=k+1}^{L-1} C_{kj} \) is a constant term \( \mathcal{O}(\varepsilon L) \) which does not depend on \( \sigma \). We claim that in fact \( \hat{\Delta}_0 = o^*(\varepsilon L) \): indeed, by definition, we have that \( 1 = \mu_{\ell}(1) = \text{Leb} \sum_{j \in \mathcal{L}_{\ell}} \nu_{j} \hat{\rho}_{j}|_{\sigma=0} \). Hence:

\[
1 = \text{Leb} \sum_{j \in \mathcal{L}_{\ell}} \nu_{j} \hat{\rho}_{j}|_{\sigma=0} = e^{\hat{\Delta}_0} + o^*(\varepsilon L),
\]

which implies that \( \hat{\Delta}_0 = o^*(\varepsilon L) \). Hence it can indeed be added to \( Q \), concluding our proof.

\[
\square
\]

11. Combining many blocks

This section contains the proof of Proposition 7.5, which follows from combining together the one block estimates obtained in the previous section (i.e. Proposition 10.4); we also rely on the results we obtain in Appendix A. The proof essentially follows from the next technical lemma.

For convenience, given \( \theta \in \mathbb{T} \), let us define

\[
\tilde{H}_{\ell, \theta} = \sum_{j=0}^{L-1} \hat{\omega}(x_j, \theta) e^{\varepsilon \sum_{i=j+1}^{(R-\ell)L-1} \hat{\omega}(\hat{\theta}(i \varepsilon, \theta))}
\]

(11.1)

\[
\tilde{H}_{\ell, \theta} = \sum_{j=0}^{(R-\ell)L-1} \hat{\omega}(x_j, \theta) e^{\varepsilon \sum_{i=j+1}^{(R-\ell)L-1} \hat{\omega}(\hat{\theta}(i \varepsilon, \theta))}
\]

\[
\sigma_{\ell}^2(x, \theta) = \int_{0}^{1} e^{2 \int_{0}^{s} \hat{\omega}(\hat{\theta}(s, \theta))ds} d\theta
\]

where \((x_j, \theta_j)\) are random variables under some \( \ell \). It is also useful to define the potentials \( \Omega_{\ell, \theta} \) by

\[
\Omega_{\ell, \theta} = \int_{0}^{1} e^{2 \int_{0}^{s} \hat{\omega}(\hat{\theta}(s, \theta))ds} d\theta
\]

Lemma 11.1. For all \( \ell \in \{0, \ldots, R-1\} \) and \( \theta \in \mathbb{T} \)

\[
\mu_{\ell}(e^{i\sigma \tilde{H}_{\ell, \theta}}) = e^{-\frac{\sigma^2}{2} \sigma_{\ell}^2(\theta) + \mathcal{O}(R^{-\ell})} + \nu_{\theta, \sigma_{\ell}}(\hat{\rho}_{\ell} + R_{\ell}(\eta)),
\]

where \( \Omega_{0, \ell, \theta} = \Omega_{0, \ell, \theta} + e\Psi_{0} \) as defined in (8.3), \( \hat{\rho}_{\ell} \) is the density obtained applying Proposition 8.1 to the standard pair \( \ell \) and \( R_{\ell}(\eta) \) is a remainder term satisfying the following bound:

\[
|R_{\ell}(\eta)| \leq C_{\#} \sum_{l=0}^{R-1} e^{-\eta^2(\ell-R) \varepsilon L} \left( \hat{\sigma}_{\theta}^2 - C_{\#} \sigma_{\ell} \right) \left( (R-l) \varepsilon \hat{\sigma}_{\theta}^2 L^{1+\alpha} + \hat{P}_{\ell}(\eta) \right) + \hat{Q}(\eta) + C_{\#} (R-\ell) e^{-C_{\#} \ell L^{2n-1}},
\]

with \( \hat{\sigma}_{\theta}^2 = \min_{\theta} \sigma_{\theta}^2(\theta) \) and where \( \hat{P}_{\ell} \) and \( \hat{Q} \) are remainder terms which satisfy estimates similar to (10.16): \( \hat{P}_{\ell}(\eta) = \sum_{s=0}^{\ell} \hat{p}_{s} |\eta|^s \) and \( \hat{Q}(\eta) = q_0 + q_1 |\eta| \) where \( \hat{p}_{s} \)

\[\text{Note that the definition of } \sigma_{\ell}^2(\theta_0, \theta_1) \text{ is a slight generalization of the definition of } \sigma_{\ell}^2(\theta_0) \text{ in (2.14). In fact, } \sigma_{\ell}^2(\theta, \theta) = \sigma_{\ell}^2(\theta).\]
and $\bar{q}_n$ are non-negative numbers satisfying
\begin{align}
\bar{p}_0 &= o^*(\varepsilon L) \\
\bar{p}_1 &= o^*(\varepsilon L) \\
\bar{p}_2 &= o^*((\varepsilon L)^{3/2}) \\
\bar{q}_0 &= o^*(\varepsilon L) \\
\bar{q}_1 &= o^*(\varepsilon). 
\end{align}

**Proof.** We proceed by backward induction. The base step $\bar{r} = R - 1$ immediately follows by (10.15b). In fact, by definition,
\begin{align}
\mu_{\bar{r}-1}(e^{i\bar{\sigma}_{\bar{r}-1,n}}) &= \sum_{\bar{r} \in G_{\bar{r}-1,n}} \nu_{\bar{r},R} \mathbb{I}_{[\bar{a}_{\bar{r}},b_{\bar{r}}]} \\
&= e^{-\eta^2_{\bar{r}} \sigma_{\bar{r}-1,n} \bar{\theta}_{\bar{r},\bar{r}-1,n}} e^{O(\sigma^3 L)} \\
&\cdot ((1 + O(\sigma^2)) m_{\ell_{\bar{r}-1}} \Omega_{\ell_{\bar{r}-1}} \left( \bar{\mu}_{\bar{r}-1} + \text{Leb} P \right) + \text{Leb} Q).
\end{align}

The required bounds (11.2) immediately follow from (10.16).

Next, assuming that our statement holds for $0 < \bar{r} \leq R - 1$, we proceed to prove it for $\bar{r} - 1$. Then, by (3.12) and the induction hypothesis, we have
\begin{align}
\mu_{\bar{r}-1}(e^{i\bar{\sigma}_{\bar{r}-1,n}}) &= \mu_{\bar{r}-1}(e^{i\bar{\sigma}_{\bar{r}-1,n} + i\alpha_{\bar{r},\bar{r}-1,n}}) \\
&= \sum_{\bar{r} \in G_{\bar{r}-1,n}} \nu_{\bar{r},R} (e^{i\bar{\sigma}_{\bar{r}-1,n} + i\alpha_{\bar{r},\bar{r}-1,n}}) \\
&= \sum_{\bar{r} \in G_{\bar{r}-1,n}} \nu_{\bar{r}} \left[ e^{-\frac{\eta^2_0}{2} \sigma_{\bar{r}-1,n} \bar{\theta}_{\bar{r},\bar{r}-1,n} + O((\bar{r} - \bar{r})^3 \sigma^3 L)} m_{\ell_{\bar{r}},\Omega_{\ell_{\bar{r}},\bar{r},\bar{r} - 1}} \left( \bar{\mu}_{\bar{r}-1} + \text{R}_\bar{r}(\eta) \right) \right].
\end{align}

Next, observe that Theorem 2.2 (see also Remark 6.2) implies that, for any $\alpha \in (0,1)$:
\begin{align}
\sum_{\ell_{\bar{r}} \in G_{\bar{r}-1,n}} \mathbb{I}_{[\bar{\theta}_{\bar{r}-1,n} - \theta_{\ell_{\bar{r}}},\bar{\theta}_{\bar{r}-1,n} + \varepsilon \sigma L]} |\nu_{\ell_{\bar{r}}}| \leq C\# e^{-C_\# L^{2\alpha - 1}}.
\end{align}

Since $\|\sigma\|_{C^1} \leq C\#$, using (7.5) we have
\begin{align}
\mu_{\bar{r}-1}(e^{i\bar{\sigma}_{\bar{r}-1,n}}) &= e^{-\frac{\eta^2_0}{2} \sigma_{\bar{r}-1,n} \bar{\theta}_{\bar{r},\bar{r}-1,n} + O((\bar{r} - \bar{r})^3 \sigma^3 L)} m_{\ell_{\bar{r}},\Omega_{\ell_{\bar{r}},\bar{r},\bar{r} - 1}} \left( \bar{\mu}_{\bar{r}-1} + \text{R}_\bar{r}(\eta) \right).
\end{align}

with $C = \frac{\eta^2}{2} \int_0^\varepsilon L \int_{s}^{\varepsilon L} \int_{s,1}^{\varepsilon L} ds \mathcal{W}_0(\bar{\theta}(s,\theta_{\ell_{\bar{r}-1,n}})) \mathcal{W}_0(\bar{\theta}(s,\theta_{\ell_{\bar{r}-1,n}}))$. For simplicity, let us set $\Omega_0 = \Omega_{\bar{r},\bar{r}-1,n}$. We now proceed to take the distributions $m$ out of the sum on standard pairs; in order to do so we compare $m_{\ell_{\bar{r}},\Omega_0}$ with $m_{\bar{\sigma}_{\bar{r}-1,n} \bar{\theta}_{\bar{r},\bar{r}-1,n}} \bar{\theta}_{\bar{r},\bar{r}-1,n}$. Therefore, using once again (11.3) together with (A.20b) and Proposition 8.1-(c), we obtain
\begin{align}
\sum_{\ell_{\bar{r}} \in G_{\bar{r}-1,n}} \nu_{\ell_{\bar{r}}} m_{\ell_{\bar{r}},\Omega_0} \bar{\theta}_{\ell_{\bar{r}}} = m_{\bar{\sigma}_{\bar{r}-1,n} \bar{\theta}_{\bar{r},\bar{r}-1,n}} \bar{\theta}_{\bar{r},\bar{r}-1,n} + O(\sigma L^{2\alpha}) \|\bar{\mu}_{\bar{r}}\|_{W^{2,1}} + O(e^{-C\# L^{2\alpha - 1}})
\end{align}
by Proposition 8.1–(e), we have \( \| \hat{\theta}_{t_r} \|_{W^{2,1}} = \mathcal{O}(L^{-1/2}) \), which implies that we can bound the first error term above by \( \eta \mathcal{O}(L^{-2/3} + \log L^{-1/2}) \), and can thus be added to \( \tilde{P}_r \). We now use (8.7) and (A.25) and obtain:

\[
m_{\tilde{\theta}_r, \Omega_0} \sum_{t_r \in \Sigma^{(L)}_{t-1}} \nu_{t_r} \hat{\theta}_{t_r} = m_{\tilde{\theta}_r, \Omega_0} \sum_{t_r \in \Sigma^{(L)}_{t-1}} \nu_{t_r} \theta_{t_r} + o(\varepsilon L) + \mathcal{O}(\sigma^2).
\]

Once again, the error term can be added to \( \tilde{P}_r \), since \( \mathcal{O}(\sigma^2) = \eta^2 \mathcal{O}(\varepsilon L^{3/2}) \). At last, we use (10.15b) to obtain

\[
m_{\tilde{\theta}_r, \Omega_0} \sum_{t_r \in \Sigma^{(L)}_{t-1}} \nu_{t_r} \theta_{t_r} = e^{-\frac{\alpha}{2} \int_0^{\varepsilon L} e^{\sum_{s=0}^{t_r} \sigma^2(s, \varepsilon L)} ds + \mathcal{O}(\sigma^3 L)} \cdot \left[m_{\tilde{\theta}_r, \Omega_0} \hat{\theta}_{t_r} \cdot m_{\tilde{\theta}_r, \Omega_0} \hat{\theta}_{t_r} + F_{\Omega(\varepsilon L)} + \mathcal{O}(\sigma^3 L), \right] + \mathcal{O}(\sigma^2) \mathcal{O}(\varepsilon L^{1/2}) + \mathcal{O}(\sigma^3 L),
\]

Then, by (A.18) we have \( m_{\tilde{\theta}_r, \Omega_0} \hat{\theta}_{t_r} \cdot m_{\tilde{\theta}_r, \Omega_0} \hat{\theta}_{t_r} - 1 = \mathcal{O}(\sigma^2) \), which can thus be added to \( \tilde{P}_r \), on the other hand

\[|m_{\tilde{\theta}_r, \Omega_0} F_{\Omega(\varepsilon L)}| \leq C_\# \| F_{\Omega(\varepsilon L)} \|_{W^{1,1}}\]

and thus can be added to \( \tilde{P}_r \) since \( F \) satisfies (10.16a). Finally we use (A.25) of to estimate the last term:

\[|m_{\tilde{\theta}_r, \Omega_0} \mathcal{O}(\sigma^2) \mathcal{O}(\varepsilon L^{1/2}) + \mathcal{O}(\sigma^3 L)|_{W^{1,1}} \]

using the estimates for \( F \) given in Proposition 10.4 we can thus add the error terms to \( \tilde{P}_r \), concluding our inductive step. 

We are now, at the very last, ready to conclude.

**Proof of Proposition 7.5.** First of all note that \( \tilde{H}_{0, \theta_{t_0}} = \varepsilon^{-1} H^{\theta_{t_0}}(e^{-1}) \) (see (11.1) and (4.5)). Thus, by Lemma 11.1 with \( r = 0 \) and \( \theta = \theta_{t_0} \), we can write

\[
\mu_{\theta_{t_0}}(e^{-\frac{\alpha}{2} H^{\theta_{t_0}}(e^{-1})}) = e^{-\frac{\alpha}{2} \sigma^2(\theta_{t_0})} + \mathcal{R},
\]

where we defined

\[
\mathcal{R} = \mathcal{R}_0(\varepsilon) + e^{-\frac{\alpha}{2} \sigma^2(\theta_{t_0})}(e^{\mathcal{O}(\sigma^2 \varepsilon^{-1})} m_{\theta_{t_0}, \Omega_0} \hat{\theta}_{t_0} - 1).
\]

Recall that, by (10.19b) and Proposition 8.1–(e), we have

\[
m_{\theta_{t_0}, \Omega_0} \hat{\theta}_{t_0} = \text{Leb}(\hat{\theta}_{t_0}) + \mathcal{O}_{W^{1,1}}(\varepsilon \eta \hat{\theta}_{t_0}) = 1 + \mathcal{O}(\varepsilon L) + \mathcal{O}(\eta \varepsilon^{1/2}).
\]
In order to conclude, we thus need to bound the integral of $R$:

$$\int_{J_1} |R| \, d\eta \leq C_\# \int_{J_1} \sum_{l=0}^{R-1} e^{-c_\# \eta^2 (R-l) \epsilon L} \left[ (R-l) |\eta|^2 \epsilon^2 L^{1+\alpha} + \tilde{P}_l(\eta) \right] \, d\eta$$

$$+ \int_{J_1} \tilde{Q}(\eta) \, d\eta + C_\# R \epsilon^{-\frac{1}{2}} \sigma_0 e^{-C_\# L^{2\alpha-1}}$$

$$+ C_\# \int_{J_1} e^{-c_\# \eta^2} \left[ o^*(\epsilon L) + \mathcal{O}(|\eta|^3 \epsilon^{1/2}) + \mathcal{O}(|\eta| \epsilon^{1/2}) \right] \, d\eta$$

$$\leq C_\# \int_{J_1} \sum_{l=1}^{(\epsilon L)^{-1}} (\epsilon L)^{-\frac{1}{2}} e^{-z^2 l c_\#} \left[ |z|^2 l \epsilon L^\alpha + \tilde{P}_{R-l}(z(\epsilon L)^{-1/2}) \right] \, dz$$

$$+ \int_{J_1} \tilde{Q}(\eta) \, d\eta + C_\# R \epsilon^{-\frac{1}{2}} \sigma_0 e^{-C_\# L^{2\alpha-1}} + C_\# \epsilon^{1/2}.$$ 

Let us analyze each contribution separately. The first term can be bounded by:

$$\epsilon^{1/2} L^{\alpha - 1/2} \sum_{l=1}^{(\epsilon L)^{-1}} l \int_{\mathbb{R}} e^{-z^2 l c_\#} |z|^2 \, dz = \epsilon^{1/2} L^{\alpha - 1/2} \sum_{l=1}^{(\epsilon L)^{-1}} l^{-1/2} = L^{\alpha - 1} = o^*(1).$$

The second one is bounded, for any fixed $s = 0, 1, 2$, by:

$$\sum_{l=1}^{(\epsilon L)^{-1}} \left( \frac{\epsilon L}{l} \right)^{\frac{s+1}{2}} \tilde{p}_s \int_{\mathbb{R}} e^{-c_\# z^2} z^s \, dz \leq \left( \frac{\epsilon L}{l} \right)^{\frac{s+1}{2}} \tilde{p}_s \int_{\mathbb{R}} e^{-\epsilon L^2 y^2} y^s \, dy \sum_{l=1}^{(\epsilon L)^{-1}} l^{-\frac{s+1}{2}}$$

$$\leq C_\# \left( \frac{\epsilon L}{l} \right)^{\frac{s+1}{2}} \tilde{p}_s \cdot \begin{cases} \left( \frac{\epsilon L}{l} \right)^{1/2} & \text{if } s = 0 \\ \log \epsilon^{-1} & \text{if } s = 1 \\ 1 & \text{if } s \geq 2. \end{cases}$$

The third term gives:

$$\tilde{q}_s \int_{J_1} \eta^s \, d\eta \leq C_\# \tilde{q}_s \epsilon^{-\frac{s+1}{2}},$$

whereas the fourth term can be taken to be as small as needed. By the estimates (11.2) we thus have $\int_{J_1} |R| \, d\eta = o^*(1)$, which concludes the proof. \hfill \Box

**Appendix A. Transfer Operators**

In this appendix we collect some known and less known (or possibly unknown) results on Transfer Operators that are used in the main part of this paper. Let $f_\theta \in C^r(T, \mathbb{T})$, $r \geq 3$, $\theta \in \mathbb{T}$, be a one parameter family of orientation preserving expanding maps (i.e., there exists $\lambda > 1$ such that $\inf_{x, \theta} f_\theta'(x) \geq \lambda$). Let $\Omega_\theta \in C^{r-1}(T, \mathbb{C})$ be a family of potentials. Assume that $f, \Omega$ are twice differentiable with respect to $\theta$. For any $\zeta \in \mathbb{R}$ we can then consider the operators

$$\mathcal{L}_{\theta, \zeta} \Omega g(x) = \sum_{y \in f_\theta^{-1}(x)} \frac{e^{\zeta \Omega(y)}}{f_\theta'(y)} g(y).$$

It is well known that the spectrum of such operators depend drastically on the space on which they act. We will be interested in $BV, W^{s,1}$ and $C^s$, $s \leq r - 1$. 
A.1. General facts.

Let us start with a useful result for the case of real potentials.

**Lemma A.1.** If $\Omega_\theta$ is real, then for any $\zeta \in \mathbb{R}$, $\theta \in T$, the operator $L_{\theta, \zeta, \Omega} : C^1 \to C^1$ is of Perron–Frobenius type. That is, it has a simple maximal eigenvalue $e^{\lambda \theta, \zeta, \Omega}$ with left and right eigenvectors $m_{\theta, \zeta, \Omega}, h_{\theta, \zeta, \Omega}$. In addition, $m_{\theta, \zeta, \Omega}$ is a measure. Also, the spectral gap is continuous in $\zeta, \theta$ and the leading eigenvalue and eigenprojector are analytic in $\zeta$ and differentiable in $\theta$.

**Proof.** This can be proven by reducing the system to symbolic dynamics and then using results on the induced transfer operator. Yet, a much more efficient and direct proof can be obtained by reducing the system to symbolic dynamics and then using results on the induced transfer operator. It follows that $\theta, \zeta \in C$.

Accordingly, there exists $\alpha_n \in [e^{-a}, e^a]$ such that

$$\| L^n_{\theta, \zeta, \Omega} f - \alpha_n L^n_{\theta, \zeta, \Omega} g \|_{L^\infty} \leq C_\# \Theta( L^n_{\theta, \zeta, \Omega} f, L^n_{\theta, \zeta, \Omega} g ) \| L^n_{\theta, \zeta, \Omega} 1 \|_{L^\infty}$$

Next, let $f, g \in K_\alpha$, $m(f) = m(g) = 1$. Then $e^{-a} \leq f, g \leq e^a$ hence $e^{-a} L^n_{\theta, \zeta, \Omega} 1 \leq L^n_{\theta, \zeta, \Omega} f, L^n_{\theta, \zeta, \Omega} g \leq e^a L^n_{\theta, \zeta, \Omega} 1$, this means that, for each $n \in \mathbb{N}$, there exists $\alpha_n \in [e^{-a}, e^a]$ such that

$$\frac{\lambda \zeta}{4}$$

Let $e^{\lambda \theta, \zeta, \Omega}$ be the maximal eigenvalue of $L_{\theta, \zeta, \Omega}$ when acting on $C^1$. The above displayed equations, together with (A.2), imply that $L_{\theta, \zeta, \Omega}$, when acting on $C^1$, has a simple maximal eigenvalue and a spectral gap of size at least $e^{\lambda \theta, \zeta, \Omega} (1 - \tanh \frac{\lambda \zeta}{4})$.

Accordingly, there exists $m_{\theta, \zeta, \Omega} \in C^1$ and a distribution $m_{\theta, \zeta, \Omega} \in (C^1)'$ such that

$$L_{\theta, \zeta, \Omega} (g) = e^{\lambda \theta, \zeta, \Omega} m_{\theta, \zeta, \Omega} g + Q_{\theta, \zeta, \Omega} (g) =: e^{\lambda \theta, \zeta, \Omega} P_{\theta, \zeta, \Omega} (g) + Q_{\theta, \zeta, \Omega} (g),$$

where, for each $n \in \mathbb{N}$, $\| Q^n_{\theta, \zeta, \Omega} \|_{C^1} \leq C_{\theta, \zeta, \Omega} e^{\lambda \theta, \zeta, \Omega} \tau_{\theta, \zeta, \Omega}$, with $\tau_{\theta, \zeta, \Omega} \in (0, 1 - \tanh \frac{\lambda \zeta}{4})$, and $m_{\theta, \zeta, \Omega} (h_{\theta, \zeta, \Omega}) = 1$. Moreover, by standard perturbation theory all the above quantities are analytic in $\zeta$. In addition, it is important to realize that $m_{\theta, \zeta, \Omega}$

---

37 It suffices to compute the distance of a function from the constant function and remember that, for $f, g \in K_\alpha$, $\Theta(f, g)$ is defined as $\log \frac{1}{\mu}$ where $\mu$ is the inf of the $\alpha$ such that $\alpha f - g \in K_\alpha$ and $\lambda$ is the sup of the $\beta$ such that $f - \beta g \in K_\alpha$. 
is not simply an element of \((C^1)^\prime\), as follows automatically from the general theory, but a measure. Indeed, by the iterated version of (A.2),

\[
|m_{\theta,\Sigma}(g)| = e^{-\lambda_0 x,\Sigma} |m_{\theta,\Sigma}(L_{\theta,\Sigma}^n g)| \\
\leq e^{-\lambda_0 x,\Sigma} C \left( \|L_{\theta,\Sigma}^n g\|_{C^1} + \|L_{\theta,\Sigma}^n g\|_{C^0} \right) \\
\leq e^{-\lambda_0 x,\Sigma} \left( C \lambda^{-n} \|g\|_{C^1} + (1 + s) \|\partial_\Sigma g\|_{C^0} \right) \|L_{\theta,\Sigma}^n 1\|_{C^0}
\]

(A.4)

which, taking the limit \(n \to \infty\), implies that \(m_{\theta,\Sigma}\) is a measure. Finally, the perturbation theory in [25, Section 8] implies that \(\chi_{\theta,\Sigma}\) and \(h_{\theta,\Sigma} \otimes m_{\theta,\Sigma}\) are differentiable in \(\theta\) (the latter with respect to the \(L(C^2, C^0)\) topology) and \(C_{\theta,\Sigma}, \tau_{\theta,\Sigma}\) can be taken continuous in \(\theta\).

Indeed, a direct computation shows that, setting

\[
(A.5) \quad \Psi_{\theta,\Sigma}(g) = -\left[ \frac{\partial_\theta f_\theta}{f_\theta} g \right] + \zeta \left[ \partial_\theta \Omega_\theta - \left( \frac{\partial_\theta f_\theta}{f_\theta} \Omega_\theta \right) \right] g,
\]

we have

\[
(A.6) \quad L_{\theta+s,\Sigma} = L_{\theta,\Sigma} + \int_0^{\theta+s} L_{\varphi,\Sigma} \Psi_{\varphi,\Sigma} d\varphi = L_{\theta,\Sigma}(1 + s\Psi_{\theta,\Sigma}) + \frac{s^2}{2} R_{\theta,s,\Sigma},
\]

and

\[
[\text{A.7}] \quad \|L_{\theta+s,\Sigma}^k - L_{\theta,\Sigma}^k\|_{C^n} \leq C_{\theta,\Sigma} e^{k x,\Sigma} \\
\|L_{\theta+s,\Sigma} - L_{\theta,\Sigma}\|_{C^{n+1}} \leq C_{\theta,\Sigma} \|s\| \sup_{\varphi \in [\theta, \theta+s]} \|\Psi_{\varphi,\Sigma}\|_{C^{n+1}} \\
\|R_{\theta,s,\Sigma}\|_{C^{n+1}} \leq C_{\theta,\Sigma} (\lambda_{\text{ess}} + |\partial_\theta \Omega_\theta|)
\]

where \(R_{\theta,s,\Sigma} = \frac{s}{2} [L_{\theta+s,\Sigma} - L_{\theta,\Sigma}(1 - s\Psi_{\theta,\Sigma})]\). Hence the hypotheses of [25, Theorem 8.1] are satisfied and the resolvent \(1 - L_{\theta,\Sigma}\), viewed as an operator from \(C^2\) to \(C^0\), is differentiable in \(\theta\). This implies the same for all the spectral data since they can be recovered by integrating the resolvent over the complex plane.

It is possible to obtain some information on the spectrum also for complex potentials.

**Lemma A.2.** For each \(\varsigma \in \mathbb{R}, \theta \in T\), let \(e^{x,\Sigma}\) be the spectral radius of \(L_{\theta,\Sigma}\) as an element of \(L(C^0, C^0)\). Then the spectral radius \(e^{x,\Sigma}\) of \(L_{\theta,\Sigma}\) as an element of \(L(C^1, C^1)\) is bounded by \(e^{x,\varsigma}\). In addition, the essential spectral radius is bounded by \(\lambda^{-1} e^{x,\varsigma}\). Finally, the spectrum outside the disk of radius \(\lambda^{-1} e^{x,\varsigma}\) is the same when \(L_{\theta,\Sigma}\) acts on all \(C^1\), \(s \in \{1, \ldots, r - 1\}\).

**Proof.** Note that the computation yielding (A.2) holds also for powers \(f_\theta\) yielding

\[
(A.8) \quad \frac{d}{dx} L_{\theta,\Sigma}^n g = L_{\theta,\Sigma}^n \left( g' \left( \frac{f_\theta}{f_\theta'} \right) + \zeta \frac{g'_\theta}{f_\theta} - \frac{g(f_\theta')''}{(f_\theta')^2} \right)
\]

where \(\Omega_{\theta,n} = \sum_{k=0}^{n} \Omega_\theta \circ f_\theta^k\). Then a direct computation yields

\[
\|L_{\theta,\Sigma}^n g\|_{C^1} \leq \|L_{\theta,\Sigma}^n\|_{C^0} \|\lambda^{-n} g\|_{C^1} + C\|g\|_{C^0}.
\]

---

38 The Banach spaces \(B^i\) in [25, Section 8] here are taken to be \(C^i\).

39 To get the first inequality, use the spectral decomposition together with (A.2) and its obvious analogous for higher derivatives.
Hence the spectral radius of \( L_{\theta,c,\Omega} \) as an element of \( L(C^1, C^1) \) is bounded by \( e^{\tau_0,\Omega} \).

In addition, it follows from the usual Hennion's argument [28] that the essential spectral radius is bounded by \( \lambda^{-1} e^{\tau_0,\Omega} \). To conclude note that, by differentiating (A.8) one see that the essential spectral radius on \( C^* \) is bounded by \( \lambda^{-s} e^{\tau_0,\Omega} \).

On the other hand, an eigenvalue in \( C^* \) is also an eigenvalue in \( C^1 \). To prove the contrary let \( Q_\tau g(x) = \int e^{-1} q(\epsilon^{-1}(x-y))g(y)dy \), where \( q \in C_0^\infty(\mathbb{R}, \mathbb{R}_{\geq 0}) \) with \( \int q = 1 \), and define \( L_\epsilon = Q_\tau L_{\theta,c,\Omega} \). By the perturbation theory in [31] the spectrum of \( L_\epsilon \) and \( L_{\theta,c,\Omega} \) are close on each \( C^* \). On the other hand \( L_\epsilon \) is a compact operator and its spectrum is the same on each \( C^* \) since each eigenvalue belongs to \( C^\infty \). \( \square \)

Note that, in general, it could happen that the spectral radius of \( L_{\theta,c,\Omega} \) on \( C^1 \) is smaller than \( \lambda^{-1} e^{\tau_0,\Omega} \). In such a case the second part of the above Lemma is of limited interest. Luckily, for special potentials, we can obtain more explicit results.

**Lemma A.3.** If \( i\Omega \) is real, then for any \( \varsigma \in \mathbb{R}, \theta \in \mathbb{T} \), the operator \( L_{\theta,c,\Omega} : C^1 \rightarrow C^1 \) has essential spectral radius bounded by \( \lambda^{-1} \) and the spectral radius by one. Instead, for real potentials the spectral radius of \( L_{\theta,c,\Omega} \) on \( C^1 \) and \( C^0 \) are the same.

**Proof.** First of all notice that, for purely imaginary potentials, \( \| L_{\theta,c,\Omega} \|_{L^1} \leq 1 \). And, from (A.2), it follows that the essential spectral radius on \( W^{1,1} \) is bounded by \( \lambda^{-1} \) and the spectral radius by one. Accordingly \( L_{\theta,c,\Omega} \) is power bounded on \( C^0 \). Our claim follows then by Lemma A.2.

Assume instead that the potential is real. Then, for each \( \tau < \tau_{\theta,c,\Omega} \) and \( \chi > \chi_{\theta,c,\Omega} \), there exists \( n \in \mathbb{N} \) and \( g \in C^1 \) such that, for all \( n \geq \bar{n} \),

\[
e^{-\tau n} \| g \|_{C^0} \leq \| L_{\theta,c,\Omega}^n g \|_{C^0} \leq \| L_{\theta,c,\Omega}^n \| \cdot \| g \|_{C^0} \leq e^{\tau n} \| g \|_{C^0}.
\]

The proof follows then by Lemma A.2. \( \square \)

Finally, it is worth to remark that the functional \( m_{\theta,c,\Omega} \) is guaranteed to be a measure only assuming the potential \( \Omega \) to be real: this is essentially due to the fact that, because of cancellations of complex phases, the spectral radius on \( C^1 \) might be smaller than the spectral radius on \( C^0 \) if the potential has a non-zero imaginary part.

**A.2. Perturbation Theory with respect to \( \varsigma \).**

In this and the following sections we will consider only the case in which there is a unique maximal eigenvalue \( e^{\chi_{\varsigma,c,\Omega}} \) which is simple. Hence \( m_{\theta,c,\Omega} \) and \( h_{\theta,c,\Omega} \) are well defined, a part from a normalization which is not determined by the spectral projector \( P_{\theta,c,\Omega} = h_{\theta,c,\Omega} \otimes m_{\theta,c,\Omega} \) associated to \( e^{\chi_{\varsigma,c,\Omega}} \). Note that \( \chi_{\theta,0} = 0 \); moreover, for \( \varsigma = 0 \) there exists a natural normalization for \( m_{\theta,0} \) and \( h_{\theta,0} \) so that \( m_{\theta,0} \) is the Lebesgue measure and \( h_{\theta,0} \) is the density of the invariant SRB probability measure \( \mu_\theta \); there is, however, no natural normalization for \( \varsigma \neq 0 \).

**Remark A.4.** Recall that the spectral data is analytic in \( \varsigma \) in a neighborhood of zero. Standard perturbation theory implies that such neighborhood contains the \( \varsigma \) such that \( \| \Omega \|_{C^1} |\varsigma| \leq \sigma_1 \) for some fixed \( \sigma_1 > 0 \) small enough. From now on we will assume \( \varsigma \) in the latter set unless otherwise specified.

Let us differentiate the relation \( m_{\theta,c,\Omega} L_{\theta,c,\Omega} h_{\theta,c,\Omega} = e^{\chi_{\varsigma,c,\Omega}} \) and obtain

(A.9a) \[
\partial_\varsigma \chi_{\theta,c,\Omega} = \nu_{\theta,c,\Omega}(\Omega_\theta)
\]
where \( \nu_{\theta,\Omega}(g) = m_{\theta,\Omega}(g h_{\theta,\Omega}) \); let us introduce the renormalized operators \( \tilde{\mathcal{L}}_{\theta,\Omega} = e^{-\chi_{\theta,\Omega}} \mathcal{L}_{\theta,\Omega} \). Note that, for all \( n \in \mathbb{N} \),

\[
(A.9b) \quad \nu_{\theta,\Omega}(\phi g \circ f_n^{\theta}) = m_{\theta,\Omega}(\tilde{\mathcal{L}}_{\theta,\Omega}(\phi g \circ f_n^{\theta} h_{\theta,\Omega})) = m_{\theta,\Omega}(g \tilde{\mathcal{L}}_{\theta,\Omega}(\phi h_{\theta,\Omega})).
\]

The above and the iteration of (A.2) imply, setting \( g = 1 \) and taking the limit for \( n \to \infty \), that \( \nu_{\theta,\Omega} \) is measure if \( \mathcal{L}_{\theta,\Omega} \) is power bounded as an operator on \( C^0 \). In addition, taking \( \phi = 1 \) we see that, in general, it is an invariant distribution for \( f_{\theta} \). Also, notice that \( \tilde{\mathcal{L}}_{\theta,\Omega} = \mathcal{P}_{\theta,\Omega} + \tilde{Q}_{\theta,\Omega} \), where \( \tilde{Q}_{\theta,\Omega} = e^{-\chi_{\theta,\Omega}} Q_{\theta,\Omega} \). Then by (A.9a) and the definition of \( \mathcal{L}_{\theta,\Omega} \) we obtain

\[
\partial_\nu \tilde{\mathcal{L}}_{\theta,\Omega}(g) = \tilde{\mathcal{L}}_{\theta,\Omega}(\Omega_{\theta,\Omega} g)
\]

with \( \Omega_{\theta,\Omega} = \Omega - \nu_{\theta,\Omega}(\Omega) \). Thus, differentiating the relations \( \tilde{\mathcal{L}}_{\theta,\Omega} h_{\theta,\Omega} = h_{\theta,\Omega} \) and \( m_{\theta,\Omega}(\tilde{\mathcal{L}}_{\theta,\Omega} g) = m_{\theta,\Omega}(g) \) yields

\[
(A.9c) \quad \partial_\nu h_{\theta,\Omega} = [1 - \tilde{Q}_{\theta,\Omega}]^{-1} \tilde{\mathcal{L}}_{\theta,\Omega} (\Omega_{\theta,\Omega} h_{\theta,\Omega}) - C(\theta,\varsigma) h_{\theta,\Omega}
\]

\[
(A.9d) \quad \partial_\nu m_{\theta,\Omega}(g) = m_{\theta,\Omega}(\Omega_{\theta,\Omega} [1 - \tilde{Q}_{\theta,\Omega}]^{-1} \tilde{g} + C(\theta,\varsigma) m_{\theta,\Omega}(g)
\]

where \( \tilde{g} = (1 - \mathcal{P}_{\theta,\Omega}) g = g - h_{\theta,\Omega} m_{\theta,\Omega}(g) \) and \( C(\theta,\varsigma) \) depends on the normalization of \( h_{\theta,\Omega} \) and \( m_{\theta,\Omega} \). Using the above expressions, and differentiating (A.9a), it is immediate to obtain

\[
(A.9e) \quad \partial_\nu^2 \chi_{\theta,\Omega} = m_{\theta,\Omega}(\Omega_{\theta,\Omega} [1 - \tilde{Q}_{\theta,\Omega}]^{-1} (1 + \tilde{\mathcal{L}}_{\theta,\Omega})\Omega_{\theta,\Omega} h_{\theta,\Omega}),
\]

which yields

\[
(A.10a) \quad \partial_\nu^2 \chi_{\theta,\Omega} = \nu_{\theta,\Omega}(\Omega_{\theta,\Omega}^2) + 2 \sum_{k=1}^{\infty} \nu_{\theta,\Omega}(\Omega_{\theta,\Omega} \circ f_k^{\theta}) =
\]

\[
(A.10b) \quad = \lim_{n \to \infty} \frac{1}{n} \nu_{\theta,\Omega} \left( \left[ \sum_{k=0}^{n-1} \Omega_{\theta,\Omega} \circ f_k^{\theta} \right]^2 \right)
\]

where we used the identity \( m_{\theta,\Omega}(g_1 \tilde{\mathcal{L}}_{\theta,\Omega}(g_2)) = m_{\theta,\Omega}(g_1 \circ f_k^{\theta} g_2) \), which is obtained directly by definition of the Transfer operator \( \mathcal{L}_{\theta,\Omega} \). By further differentiation of (A.9e) it is simple to show that

\[
\partial_\nu^3 \chi_{\theta,\Omega} = \nu_{\theta,\Omega}(\Omega_{\theta,\Omega}^3) + 3 \sum_{k=1}^{\infty} \nu_{\theta,\Omega}(\Omega_{\theta,\Omega} \circ f_k^{\theta} \Omega_{\theta,\Omega}^2) + \nu_{\theta,\Omega}(\Omega_{\theta,\Omega} \circ f_k^{\theta} \Omega_{\theta,\Omega}) +
\]

\[
+ 6 \sum_{k=1}^{\infty} \sum_{j=k+1}^{\infty} \nu_{\theta,\Omega}(\Omega_{\theta,\Omega} \circ f_k^{\theta} \Omega_{\theta,\Omega} \circ f_j^{\theta} \Omega_{\theta,\Omega})
\]

which implies the useful estimate

\[
(A.11) \quad |\partial_\nu^3 \chi_{\theta,\Omega}| \leq C_{\theta,\varsigma} \|\Omega\|_{C^1}^3.
\]

**Lemma A.5.** There exists a normalization for \( h_{\theta,\Omega} \) and \( m_{\theta,\Omega} \) so that \( m_{\theta,0} = \text{Leb} \) and the corresponding \( C(\theta,\varsigma) \) is identically 0, that is:

\[
(A.12a) \quad \partial_\nu h_{\theta,\Omega} = [1 - \tilde{Q}_{\theta,\Omega}]^{-1} \tilde{\mathcal{L}}_{\theta,\Omega} (\Omega_{\theta,\Omega} h_{\theta,\Omega})
\]

\[
(A.12b) \quad \partial_\nu m_{\theta,\Omega}(g) = m_{\theta,\Omega}(\Omega_{\theta,\Omega} [1 - \tilde{Q}_{\theta,\Omega}]^{-1} \tilde{g}),
\]

provided \( \Omega \) is real or, for general potentials, if \( \|\Omega\|_{C^1} \leq \sigma_1 \).
Proof. Let us fix temporarily a normalization which defines \( \bar{h}_{\theta,\varsigma} \) and \( \bar{m}_{\theta,\varsigma} \) so that \( \text{Leb}(\bar{h}_{\theta,\varsigma}) = 1 \). Note that for real potentials this can always be done since \( h_{\theta,\varsigma} > 0 \) due to Lemma A.1. For general potentials it is possible only if \( \text{Leb}(\mathcal{P}_{\theta,\varsigma}(\phi)) \neq 0 \) for some \( \phi \in \zeta^0 \). This is the case for small \( \varsigma \) due to \( \text{Leb}(\mathcal{P}_{\theta,\varsigma}(\phi)) = \text{Leb}(\phi) \) and the continuity of \( \mathcal{P}_{\theta,\varsigma} \). Using (A.9c) and differentiating this normalization condition with respect to \( \varsigma \) we obtain
\[
(A.13) \quad C(\theta, \varsigma) = \text{Leb}(1 - \widehat{Q}_{\theta,\varsigma})^{-1} \widehat{L}_{\theta,\varsigma}(\Omega_{\theta,\varsigma} \bar{h}_{\theta,\varsigma})).
\]
Define \( \alpha(\theta, \varsigma) = \int_{0}^{c} C(\theta, \varsigma^{'}) d\varsigma^{'} \) and choose a new normalization so that \( h_{\theta,\varsigma} = e^{\alpha(\theta, \varsigma)} \bar{h}_{\theta,\varsigma} \) (and consequently \( m_{\theta,\varsigma} = e^{-\alpha(\theta, \varsigma)} \bar{m}_{\theta,\varsigma} \)). Then, an immediate computation shows that \( h_{\theta,\varsigma} \) and \( m_{\theta,\varsigma} \) satisfy equations (A.12).

Lemma A.6. Choosing the normalization of \( h_{\theta,\varsigma} \) and \( m_{\theta,\varsigma} \) obtained in Lemma A.5, we have
\[
(A.14a) \quad \| \partial_{\varsigma} h_{\theta,\varsigma} \|_{C^1} \leq C_{\#} \| \Theta_{\theta} \|_{C^1}
\]
\[
(A.14b) \quad | \partial_{\varsigma} m_{\theta,\varsigma} g | \leq C_{\#} \| \Theta_{\theta} \|_{C^1} \| g \|_{C^1};
\]
and, moreover
\[
(A.15) \quad | m_{\theta,0} h_{\theta,\varsigma} - 1 | \leq C_{\#} \varsigma^2 \| \Theta_{\theta} \|_{C^1}^2 \quad | m_{\theta,\varsigma}(h_{\theta,0}) - 1 | \leq C_{\#} \varsigma^2 \| \Theta_{\theta} \|_{C^1}^2.
\]

Proof. Since all the quantities are analytic in \( \varsigma \) and \( \varsigma \) belongs to a fixed compact set, we have uniform bounds on \( \| h_{\theta,\varsigma} \|_{C^1} \) and \( \| m_{\theta,\varsigma} \|_{C^1} \). Thus, by (A.12a), taking the \( C^1 \)-norm, we obtain:
\[
\| \partial_{\varsigma} h_{\theta,\varsigma} \|_{C^1} \leq \| 1 - \widehat{Q}_{\theta,\varsigma} \|^{-1} \widehat{L}_{\theta,\varsigma}(\Omega_{\theta,\varsigma} h_{\theta,\varsigma}) \|_{C^1} \leq C_{\#} \| \Theta_{\theta} \|_{C^1},
\]
which implies \( \| h_{\theta,\varsigma} \|_{C^1} \leq C_{\#} (1 + |\varsigma| \| \Theta_{\theta} \|_{C^1}) \). Similar computations yield an analogous result for \( m_{\theta,\varsigma} \).

Finally, in order to obtain equations (A.15), observe that \( m_{\theta,0} h_{\theta,\varsigma} - 1 = m_{\theta,0} \left( \int_{0}^{\varsigma} d\varsigma^{'} \partial_{\varsigma} \bar{h}_{\theta,\varsigma} \right) \); then, since \( m_{\theta,0} \partial_{\varsigma} \bar{h}_{\theta,0} \) we can write
\[
| m_{\theta,0} h_{\theta,\varsigma} - 1 | \leq \int_{0}^{\varsigma} d\varsigma^{'} \int_{0}^{\varsigma} d\varsigma^{''} \| \partial_{\varsigma} \left( \| 1 - \widehat{Q}_{\theta,\varsigma} \|^{-1} \widehat{L}_{\theta,\varsigma}(\Omega_{\theta,\varsigma} \bar{h}_{\theta,\varsigma}) \right) \|_{C^0}
\]
from which follows the first of (A.15); a similar computation yields the remaining estimate, which concludes the proof.

Remark A.7. Note that (A.9a) implies that \( \sup_{\varsigma} \| \partial_{\varsigma} \chi_{\theta,\varsigma} \| \leq C_{\#} \| \Theta_{\theta} \|_{C^1} \). On the other hand (A.10b) implies that the second derivative is non negative if \( \Theta_{\theta} \) is real. Accordingly, \( \lim_{\varsigma \to \pm \infty} \partial_{\varsigma} \chi_{\theta,\varsigma} = 0 \), and hence \( \partial_{\varsigma} \chi_{\theta,\varsigma} \) is uniformly bounded.

We will also need to deal with Transfer Operators weighted with two different families of potentials, \( \Omega^{(0)} \) and \( \Omega^{(1)} \); if \( \| \Omega^{(0)} - \Omega^{(1)} \|_{C^1} \) is small enough, we can once again use perturbation theory to compare spectral data. For \( \varrho \in [0, 1] \), let us define the convex interpolation \( \Omega^{(\varrho)} = \Omega^{(0)} + \varrho (\Omega^{(1)} - \Omega^{(0)}) \), and let \( \delta_{\varrho} = \partial_{\varrho} \Omega^{(\varrho)} \). Then, by arguments analogous to the ones leading to (A.9), we obtain
\[
(A.16a) \quad \partial_{\varrho} h_{\varrho} = \left( \| 1 - \widehat{Q}_{\varrho} \|^{-1} \widehat{L}_{\varrho}(\delta_{\varrho} - m_{\varrho}(\delta_{\varrho} h_{\varrho})) + C_{\#} \right) h_{\varrho}
\]
\[
(A.16b) \quad \partial_{\varrho} m_{\varrho} g = m_{\varrho} ((\delta_{\varrho} - m_{\varrho}(\delta_{\varrho} h_{\varrho})) (1 - \widehat{Q}_{\varrho})^{-1} (g - h_{\varrho} m_{\varrho} g)) + C_{\#} m_{\varrho} g
\]
and by arguments similar to the ones given in the proof of Lemma A.5 we have that $|C(\rho)| \leq C_\#||\delta\Omega||_{C^1}$. Hence we obtain:

\begin{equation}
\tag{A.17}
\|\partial_\theta h_{\theta,\Omega}\|_{C^k} \leq C_\#||\delta\Omega||_{C^k} \quad |\partial_\theta m_{\theta,\Omega}| \leq C_\#||\delta\Omega||_{C^1} \|g\|_{C^1}.
\end{equation}

We conclude the section with a simple, but useful, inequality.

**Lemma A.8.** For any \( \theta \) the following estimate holds:

\begin{equation}
\tag{A.18}
|m_{\theta,\Omega}(0) h_{\theta,\Omega(1)} - 1| \leq C_\#(||\Omega_0^{(0)}||_{C^1} + ||\Omega_0^{(1)}||_{C^1})^2
\end{equation}

**Proof.** Let us write

\[ m_{\theta,\Omega}(0) h_{\theta,\Omega(1)} = \left[ m_{\theta,0} + \int_0^1 dk \partial_\theta m_{\theta,k,\Omega(0)} \right] \left[ h_{\theta,0} + \int_0^1 dk \partial_\theta m_{\theta,k,\Omega(0)} \right] \]

then, Lemma A.6 immediately implies (A.18). \( \square \)

**A.3. Perturbation Theory with respect to \( \theta \).**

Recalling the notation and computations at the end of the proof of Lemma A.1, differentiating with respect to \( \theta \) we have:

\begin{align}
\tag{A.19a}
\partial_\theta \chi_{\theta,\Omega} &= m_{\theta,\Omega}(\Psi_{\theta,\Omega} h_{\theta,\Omega}) \\
\tag{A.19b}
\partial_\theta h_{\theta,\Omega} &= [1 - \tilde{Q}_{\theta,\Omega}]^{-1} \tilde{L}_{\theta,\Omega} \Psi_{\theta,\Omega} h_{\theta,\Omega} - D(\theta, \varsigma) h_{\theta,\Omega} \\
\tag{A.19c}
\partial_\theta m_{\theta,\Omega} &= m_{\theta,\Omega}(\Psi_{\theta,\Omega}^2 [1 - \tilde{Q}_{\theta,\Omega}]^{-1} \tilde{g}) + D(\theta, \varsigma) m_{\theta,\Omega} g
\end{align}

where \( \Psi_{\theta,\Omega} = \Psi_{\theta,\Omega} h_{\theta,\Omega} m_{\theta,\Omega}(\Psi_{\theta,\Omega} g) \), once more \( D(\theta, \varsigma) \) is a function which depends on the normalization for \( h_{\theta,\Omega} \) and \( m_{\theta,\Omega} \). Note that we cannot, in general, assume that \( D = 0 \); since \( m_{\theta,0} = \text{Leb} \), it is however true that \( D(\theta, 0) = 0 \) for any \( \theta \). Also, since \( \chi_{\theta,0} = 0 \), we have \( \partial_\theta \chi_{\theta,0} = 0 \); moreover, \( |\partial_\theta \chi_{\theta,\Omega}| \leq C_\#|\varsigma| \|\Omega_0||_{C^1} \).

**Lemma A.9.** With the normalization chosen in Lemma A.5, we have, for any \( k \leq 1 \):

\begin{align}
\tag{A.20a}
\|\partial_\theta h_{\theta,\Omega}\|_{C^k} &\leq C_\#(1 + \|\Omega\|_{C^{k+1}}) \\
\tag{A.20b}
|\partial_\theta m_{\theta,\Omega}| &\leq C_\#\|\Omega\|_{C^2} \|g\|_{C^2} \\
\tag{A.20c}
|\partial_\theta \chi_{\theta,\Omega}| &\leq C_\#\|\Omega\|_{C^1}. \\
\tag{A.20d}
|\partial_\theta^2 \chi_{\theta,\Omega}| &\leq C_\#\|\Omega\|_{C^2}.
\end{align}

Additionally, \( \partial_\chi h_{\theta,\Omega}, \partial_\chi m_{\theta,\Omega} \) and \( \partial_\chi^2 h_{\theta,\Omega} \) are differentiable in \( \theta \).

**Proof.** Recall that in Lemma A.5 we have set \( h_{\theta,\Omega} = e^{\alpha(\theta, \varsigma)} \tilde{h}_{\theta,\Omega} \), where \( \tilde{h}_{\theta,\Omega} \) is such that \( \text{Leb}(\tilde{h}_{\theta,\Omega}) = 1 \) and \( \alpha(\theta, \varsigma) = \int_0^1 \tilde{C}(\theta, \varsigma) \tilde{c}' \), \( \tilde{C} \) being given by (A.13). Observe that, differentiating with respect to \( \theta \) the normalization condition for \( \tilde{h}_{\theta,\Omega} \), we obtain, using equations (A.19):

\[ \tilde{D}(\theta, \varsigma) = \text{Leb}(1 - \tilde{Q}_{\theta,\Omega})^{-1} \tilde{L}_{\theta,\Omega} \tilde{\Psi}_{\theta,\Omega} \tilde{h}_{\theta,\Omega}. \]

Then, by definition of \( h_{\theta,\Omega} \) we get:

\[ D(\theta, \varsigma) = \tilde{D}(\theta, \varsigma) + \partial_\theta \alpha(\theta, \varsigma). \]

Thus, using the explicit formula (A.19b) and the definition of \( \alpha(\theta, \varsigma) \), the proof of (A.20a) amounts to a simple computation which is left to the reader. The
Next, we estimate the derivative with respect to different potentials. Considerations and the formulae \( (A.19b) \) is the same but easier. Finally, the last statement follows from the above considerations and the formulæ \( (A.12) \) and \( (A.9e) \).

We conclude this discussion with a useful comparison of the left eigenvalue for different potentials.

**Lemma A.10.** For each \( \theta_0, \theta_1 \in \mathbb{T}^1 \), real potentials \( \Omega^{(0)}, \Omega^{(1)} \), \( \|\Omega^{(i)}\|_{c^2} \leq C_\# \), interval \( I \), \( |I| \geq \delta \), and function \( A \in C^1(I, \mathbb{R}_{\geq 0}) \), with \( |A'(x)| + |A''(x)| \leq cA(x) \), for all \( x \in I \), we have

\[
|m_{\theta_1, \Omega^{(1)}}(\mathbf{1}_I A) - m_{\theta_0, \Omega^{(0)}}(\mathbf{1}_I A)| \leq C_\# m_{\theta_0, \Omega^{(0)}}(\mathbf{1}_I A) \left\{ \|\Omega_1 - \Omega_0\|_{L^\infty} + |\theta_1 - \theta_0||\Omega_1|_{c^2} \left[ \log \varepsilon^{-1} \right]^2 + \varepsilon \left[ 1 + \sum_{i=0}^{1} \|\Omega_i\|_{L^\infty} \right] \right\}.
\]

**Proof.** Define the convex interpolation \( \Omega^{(\varepsilon)} = \Omega^{(0)} + \varepsilon(\Omega^{(1)} - \Omega^{(0)}) \), \( \varepsilon \in [0, 1] \). Also, it is convenient to introduce \( A_\varepsilon \in C^2(\mathbb{T}, \mathbb{R}_{\geq 0}) \) such that \( \|A_\varepsilon\|_{L^\infty} \leq C_\# m_{\theta_0, \Omega^{(\varepsilon)}}(A_\varepsilon)|I| \), \( \|A_\varepsilon - \mathbf{1}_I A\|_{L^1} \leq \varepsilon \text{Leb}(A) \) and \( \|A_\varepsilon\|_{W^{1, 1}} \leq C_\# \text{Leb}(A) \) and \( \|A_\varepsilon\|_{W^{1, 1}} \leq C_\# \varepsilon^{-1} \text{Leb}(A) \). By Lemma A.5, we have

\[
|m_{\theta_0, \Omega^{(\varepsilon)}}(A_\varepsilon)| = \left| m_{\theta_0, \Omega^{(\varepsilon)}}\left( (\Omega^{(1)} - \Omega^{(0)})[\mathbf{1} - \tilde{Q}_{\theta_0, \Omega^{(\varepsilon)}}]^{-1} \tilde{A}_\varepsilon \right) \right| \leq C_\# \|\Omega^{(1)} - \Omega^{(0)}\|_{L^\infty} m_{\theta_0, \Omega^{(\varepsilon)}}(A_\varepsilon).
\]

Thus,

\[
m_{\theta_0, \Omega^{(1)}}(A_\varepsilon) = e^{\Omega(\|\Omega^{(1)} - \Omega^{(0)}\|_{L^\infty})} m_{\theta_0, \Omega^{(0)}}(A_\varepsilon).
\]

Next, we estimate the derivative with respect to \( \theta \). By equation \( (A.19c) \) we have

\[
|\partial_\theta m_{\theta, \Omega^{(1)}}(A_\varepsilon)| \leq \left| m_{\theta, \Omega^{(1)}} \left( \left[ \partial_\theta f_\theta \right] \tilde{Q}_{\theta, \Omega^{(1)}}^{-1} \tilde{A}_\varepsilon \right) \right|
\]

\[
+ \left| m_{\theta, \Omega^{(1)}} \left( \left[ \partial_\theta f_\theta \right] \Omega^{(1)} \tilde{Q}_{\theta, \Omega^{(1)}}^{-1} \tilde{A}_\varepsilon \right) \right|
\]

\[
+ \|\Omega^{(1)}\|_{c^2} \left| m_{\theta, \Omega^{(1)}}(A_\varepsilon) \right|
\]

where, in the last line, we have used the fact that \( D(\theta, \phi) \) is analytic in \( \phi \) and \( D(\theta, 0) = 0 \). To continue note that, by \( (A.12b) \), for all \( \phi \in C^1(\mathbb{T}^1, \mathbb{R}_{\geq 0}) \), we have

\[
\partial_\theta m_{\theta, \Omega^{(1)}}(\phi) = m_{\theta, \Omega^{(1)}}(\Omega^{(1)}[\mathbf{1} - \tilde{Q}_{\theta, \Omega^{(1)}}]^{-1} \phi) = \sum_{n=0}^{\infty} m_{\theta, \Omega^{(1)}}(\Omega^{(1)} \tilde{L}_{\theta, \Omega^{(1)}}^n \phi).
\]
Thus, there exists $\tau > 0$ such that, for each $R \in \mathbb{N}$,

$$m_{\theta, t\Omega(1)}(\phi) = \text{Leb}(\phi) + \sum_{n=0}^{R} \int_{0}^{A} ds \ m_{\theta, s\Omega(1)}(\Omega(1)^{1} \tilde{L}_{\theta, s\Omega(1)}^{n}(\phi)) + C_{\#} e^{-\tau R} \|\Omega(1)^{1}\|_{L^{\infty}} \|\phi\|_{W^{1,1}}$$

(A.22)

$$\leq \text{Leb}(\phi) + C_{\#} R \|\Omega(1)^{1}\|_{L^{\infty}} \int_{0}^{A} ds \ m_{\theta, s\Omega(1)}(\phi) + C_{\#} e^{-\tau R} \|\Omega(1)^{1}\|_{L^{\infty}} \|\phi\|_{W^{1,1}}.$$  

By Gronwall inequality and provided $R \leq \|\Omega(1)^{1}\|_{L^{1}}^{-1}$, it follows

$$m_{\theta, t\Omega(1)}(\phi) \leq C_{\#} \left[\text{Leb}(\phi) + C_{\#} e^{-\tau R} \|\Omega(1)^{1}\|_{L^{\infty}} \|\phi\|_{W^{1,1}}\right].$$

Plugging the above estimate into the integral in (A.22), we have

$$|m_{\theta, t\Omega(1)}(\phi) - \text{Leb}(\phi)| \leq C_{\#} R \|\Omega(1)^{1}\|_{L^{\infty}} \|\phi\|_{L^{1}}$$

$$+ C_{\#} e^{-\tau R} \|\Omega(1)^{1}\|_{L^{\infty}} (1 + R \|\Omega(1)^{1}\|_{L^{\infty}}) \|\phi\|_{W^{1,1}}.$$

Since $\text{Leb}(\phi') = 0$, using the above expression in (A.21) and setting $R = 101 \tau^{-1} \log \epsilon^{-1}$, we have

$$|\partial_{\theta} m_{\theta, t\Omega(1)}(A_{c})| \leq C_{\#} (\log \epsilon^{-1})^{2} \|\Omega(1)^{1}\|_{C^{2}} \|m_{\theta, t\Omega(1)}(A_{c})|.$$  

Collecting the above facts the Lemma follows. \hfill \Box

A.4. Results for functions of Bounded Variation.

In certain parts of the paper we will need to consider Transfer Operators acting on the Sobolev Space $W^{1.1}$ or on the space of function of bounded variations $BV$. As $W^{1.1} \subset BV$, with the same norm, we will limit our discussion to the latter, more general, case.

For functions of bounded variation the Lasota–Yorke inequality reads as follows: setting $\Omega_{\theta, n} = \sum_{k=0}^{n-1} \Omega_{\theta} \circ f_{\theta}^{k}$, for each $\phi \in C^{1}$

$$\left| \int \phi' \mathcal{L}_{\theta, \Omega_{n}}^{n} g \right| \leq \int ge^{\Omega_{\theta, n}(\phi')} \circ f_{\theta}^{n}$$

(A.23)

$$\leq \left| \int g \left[ e^{\Omega_{\theta, n}(\phi')} \circ f_{\theta}^{n} \right] \right| + C_{\#} \int |\partial_{\theta} \mathcal{L}_{\theta, 0}^{n} \left[ e^{\text{Re}(\Omega_{\theta, n})} |g| \right]|.$$  

Thus, if $\tilde{\chi}_{n} = \log \|\mathcal{L}_{\theta, \text{Re}(\Omega_{\theta, n})}^{n}\|_{L^{1}} = \log \|e^{\text{Re}(\Omega_{\theta, n})}\|_{L^{1}}$ and $\tau_{n} = \log \|e^{\text{Re}(\Omega_{\theta, n})}\|_{L^{\infty}}$ we have

$$\|\mathcal{L}_{\theta, \text{Re}(\Omega_{\theta, n})}^{n} g\|_{BV} \leq e^{\tilde{\chi}_{n}} \|g\|_{BV} + C_{\#} e^{\tilde{\chi}_{n}} \|g\|_{L^{1}}.$$  

By the usual Hennion argument [28], the spectral radius of $\mathcal{L}_{\theta, \Omega}$ is bounded by $e^{\tilde{\chi}_{n}/n}$ and the essential spectral radius by $e^{\tau_{n}/n}$. Note that (A.23) implies as well

$$\|\mathcal{L}_{\theta, \text{Re}(\Omega_{\theta, n})}^{n} g\|_{BV} \leq \left[ e^{\tilde{\chi}_{n} + C_{\#} \mathcal{L}_{\theta, 0}^{n} e^{\text{Re}(\Omega_{\theta, n})}} \right] \|g\|_{BV}.$$  

In addition, calling $\mathcal{H}_{n}$ the set of inverse branches of $f_{\theta}^{n}$, we have

$$\int_{T^{1}} e^{\text{Re}(\Omega_{\theta, n})} = \int_{T^{1}} \mathcal{L}_{\theta, 0}^{n} e^{\text{Re}(\Omega_{\theta, n})} = \sum_{h \in \mathcal{H}_{n}} \int_{T^{1}} e^{\text{Re}(\Omega_{\theta, n})} h(x) h'(x) dx \geq C_{\#} e^{\tilde{\chi}_{n}}.$$  

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40 Remember that, in one dimension, $\|g\|_{L^{\infty}} \leq \|g\|_{BV}$. 

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Thus our bound on the spectral radius is larger or equal than our bound on the essential spectral radius. Nevertheless, this are just estimates: the real values could be much smaller.

**Remark A.11.** For real potentials and \( \varsigma \geq 0 \), we can say a bit more. Indeed, if \( \chi \) is the spectral radius of \( \mathcal{L}_{\varsigma,\Omega} \) when acting on \( C^1 \) and \( e^\chi \) its maximal eigenvalue, then

\[
C_\# e^{n \chi} \leq \int_{\mathbb{T}^1} L_{\theta,\varsigma,\Omega}^n 1 = \int_{\mathbb{T}^1} L_{\theta,0}^n e^{\varsigma \Omega g} \leq C_\# e^{n \chi}
\]

This implies that the spectral radius on BV coincides with the spectral radius on \( C^1 \) and, more, \( e^{-\chi} \mathcal{L}_{\varsigma,\Omega} \) is power bounded. Note however that this does not imply that \( \mathcal{L}_{\varsigma,\Omega} \) is always of Perron–Frobenius type also when acting on BV, since, for large \( \varsigma \) the essential spectral radius could coincide with the spectral radius. We can nevertheless find a simple condition that prevents such a pathology. Let \( \gamma_\theta = \sup \Omega - \inf \Omega \) we have

\[
e^{\gamma_n} \leq \log \left[ \lambda^{-n} e^{\varepsilon \gamma_\theta} \right] \leq \log \left[ \lambda^{-n} e^{\varepsilon \gamma_\theta} \right] \| e^{\varsigma \Omega g} \|_{L^1}.
\]

The above implies that if \( |\varsigma| \gamma_\theta < \log \lambda \), then the spectral radius is strictly larger than the essential spectral radius.

**Remark A.12.** For complex potentials, no such general bounds are available, we must then rely on perturbation theory. If the potential is complex, then (A.24) implies that the spectral radius is smaller than \( \lambda^{-1} \). Since the point spectrum is independent on the space on which the operators act,\(^{41}\) it follows that the spectrum outside the disk \( \{|z| \leq \lambda^{-1}\} \) on BV coincides with the spectrum on \( C^1 \).

We conclude this brief discussion with an estimate on the left eigenvalue.

**Lemma A.13.** There exists \( \sigma_2 > 0 \) such that, provided \( |\varsigma| \|\Omega\|_{C^1} < \sigma_2 \), for each \( g \in C^1 \) we have

\[
(A.25) |m_{\theta,\varsigma,\Omega} g - \text{Leb} g| \leq C_\# |\varsigma| \|\Omega\|_{C^0} \left[ |\log |\varsigma||\Omega\|_{C^0}|^2 \|g\|_{L^1} + |\varsigma| \|\Omega\|_{C^1} \|g\|_{BV}\right].
\]

**Proof.** First of all we choose \( \sigma_2 \leq \sigma_1 \) so that \( \mathcal{L}_{\varsigma,\varsigma,\Omega} \) is of Perron–Frobenius type and \( h_{\theta,\varsigma,\Omega} \otimes m_{\theta,\varsigma,\Omega} \) is the eigenprojector associated to its maximal eigenvalue \( e^{\lambda_{\theta,\varsigma,\Omega}} \).

Integrating (A.12b) from zero to \( \varsigma \) we have, for all \( g \in C^1 \),

\[
(A.26) |m_{\theta,\varsigma,\Omega} g| \leq |\text{Leb} g| + \int_0^\varsigma \text{d}\varsigma \left[m_{\theta,\varsigma,\Omega}(\Omega_{\theta,\varsigma,\Omega}[1 - \widehat{\Theta}_{\theta,\varsigma,\Omega}]^{-1}\tilde{g})\right]
\]

\[
\leq |\text{Leb} g| + \int_0^\varsigma \text{d}\varsigma \left[|\text{Leb}(\Omega_{\theta,\varsigma,\Omega}[1 - \widehat{\Theta}_{\theta,\varsigma,\Omega}]^{-1}\tilde{g})| + C_\# |\varsigma|^2 \|\Omega\|_{C^1} \|\tilde{g}\|_{C^1}\right]
\]

where, in the second line, we have iterated the first inequality. On the other hand,

\[
|\text{Leb}(\Omega_{\theta,\varsigma,\Omega}[1 - \widehat{\Theta}_{\theta,\varsigma,\Omega}]^{-1}\tilde{g})| \leq \sum_{k=0}^\infty |\text{Leb}(\Omega_{\theta,\varsigma,\Omega}\Delta_{\theta,\varsigma,\Omega}^k \tilde{g})| \leq C_\# \|\Omega_{\theta,\varsigma,\Omega}\|_{C^0} \|\tilde{g}\|_{BV}.
\]

We can now choose \( \sigma_2 \) so that \( C_\# |\varsigma| \|\Omega_{\theta,\varsigma,\Omega}\|_{C^0} \leq \frac{1}{16} \), then we have

\[
|m_{\theta,\varsigma,\Omega} g| \leq |\text{Leb} g| + \frac{2}{3} |m_{\theta,\varsigma,\Omega} g| + \frac{1}{16} \|g\|_{BV} + C_\# |\varsigma|^2 \|\Omega\|_{C^1} \|g\|_{C^1}.
\]

\(^{41}\)This can be proven as in Lemma A.2.
It thus looks possible to bootstrap away the dependence on the $C^1$ norm. To do so let $b \geq 0$ be the smallest number such that
\[
|m_{\theta, c, \Omega} g| \leq |\text{Leb } g| + \frac{1}{2}|m_{\theta, c, \Omega} g| + \frac{1}{8} \|g\|_{BV} + b\|g\|_{C^1},
\]
for all $g \in C^1$. Then, substituting the above in the integral of the first line of (A.26) we have
\[
|m_{\theta, c, \Omega} g| \leq |\text{Leb } g| + 2 \int_0^\infty \text{d}k' |\text{Leb}(\Omega_{\theta, c', \Omega}[I - \hat{Q}_{\theta', c']^{-1} \hat{g}])| + \frac{1}{4} C_{\#} \|\Omega\|_{C^0} \|\hat{g}\|_{BV} + C_{\#} \|s\|_{C^0} \|g\|_{C^1},
\]
and
\[
\leq |\text{Leb } g| + \frac{1}{2} |m_{\theta, c, \Omega} g| + \frac{1}{8} \|g\|_{BV} + \frac{b}{2} \|g\|_{C^1}.
\]
It follows $b = 0$, that is
\[
(A.27) \quad |m_{\theta, c, \Omega} g| \leq 2|\text{Leb } g| + \frac{1}{8} \|g\|_{BV}.
\]
We are now ready to conclude
\[
|\text{Leb}(\Omega_{\theta, c', \Omega}[I - \hat{Q}_{\theta', c']^{-1} \hat{g}])| \leq \sum_{k=0}^{\infty} |\text{Leb}(\Omega_{\theta, c, \Omega}\hat{L}_{\theta, c, \Omega}^k \hat{g})|
\]
\[
\leq C_{\#} \|\Omega\|_{C^0} \|g\|_{L^1} + n |m_{\theta, c, \Omega} g| + e^{-a n} \|g\|_{BV}
\]
where $a$ is determined by the spectral gap in $BV$. Note that in the considered range of $c$ we can take a constant. We can now choose $n = A \log(|\|\Omega_{\theta, c, \Omega}\|_{C^0}|)^{-1}$, for some $A > 0$ to be specified shortly. Integrating (A.12b) again and using (A.27), we have
\[
|m_{\theta, c, \Omega} g - \text{Leb } g| \leq 2 \int_0^\infty \text{d}k' |\text{Leb}(\Omega_{\theta, c', \Omega}[I - \hat{Q}_{\theta', c']^{-1} \hat{g}])| + C_{\#} \|s\|_{C^0} \|\hat{g}\|_{BV}
\]
\[
\leq C_{\#} \|\Omega\|_{C^0} \|g\|_{L^1} + \frac{1}{2} |m_{\theta, c, \Omega} g - \text{Leb } g| + C_{\#} \|s\|_{C^0} \|\hat{g}\|_{BV},
\]
where we have chosen $A$ large enough and $\sigma_2$ small enough. Iterating the estimate one more time we finally have the claim of the Lemma. \qed

A.5. Generic conditions.
Here we discuss some conditions that prevent non generic behavior of the Transfer Operator. They are arranged by (apparent) increasing strength. Yet, we will see at the end of the section that, although in general they are all different, in the specially simple case we are interested in they are in fact all equivalent to the first: not being a coboundary. As the latter is a generic condition, all the condition below are generic.

**Lemma A.14.** Let $\theta, c$ be values for which $\hat{L}_{\theta, c, \Omega}$ has a spectral gap and $m_{\theta, c, \Omega}$ is a measure. If $\partial_2^2 \chi_{\theta, c, \Omega}$ is zero, then there exists $\beta \in \mathbb{R}$ and $\phi \in C^1$ such that
\[
\Omega_{\theta} = \beta + \phi - \phi \circ f_{\theta}.
\]
That is, $\Omega_{\theta, c, \Omega}$ is cohomologous to a constant.
Proof. Note that if the second derivative is zero for some $\theta$ and $\varsigma$ then, by the computation implicit in (A.10b), it follows that the sequence $\sum_{k=0}^{n} \Omega_{\theta,\varsigma,\Omega} \circ f_{\theta}^{k}$ is uniformly bounded in $L^2(\mathbb{T}, m_{\theta,\varsigma,\Omega})$ and hence weakly compact.\footnote{Indeed, recalling (A.10a),}

Let $\sum_{k=0}^{n} \Omega_{\theta,\varsigma,\Omega} \circ f_{\theta}^{k}$ be a weakly convergent subsequence and let $\phi \in L^2$ be its limit. Hence, for any $\varphi \in L^2$ holds

$$\lim_{j \to \infty} \nu_{\theta,\varsigma,\Omega}(\varphi \sum_{k=0}^{n} \Omega_{\theta,\varsigma,\Omega} \circ f_{\theta}^{k}) = \nu_{\theta,\varsigma,\Omega}(\varphi \phi).$$

It follows that, for any $\varphi \in C^1$,\n
$$\nu_{\theta,\varsigma,\Omega}(\varphi[\Omega_{\theta,\varsigma,\Omega} - \phi + \phi \circ f_{\theta}]) =$$

$$= \nu_{\theta,\varsigma,\Omega}(\varphi[\Omega_{\theta,\varsigma,\Omega}]) + \lim_{j \to \infty} \sum_{k=0}^{n} \nu_{\theta,\varsigma,\Omega}(\varphi[\Omega_{\theta,\varsigma,\Omega} \circ f_{\theta} - \Omega_{\theta,\varsigma,\Omega} \circ f_{\theta}^{k}])$$

$$= \lim_{j \to \infty} \nu_{\theta,\varsigma,\Omega}(\varphi \Omega_{\theta,\varsigma,\Omega} \circ f_{\theta}^{n}) = \lim_{j \to \infty} m_{\theta,\varsigma,\Omega}(\Omega_{\theta,\varsigma,\Omega} \tilde{L}_{\theta,\varsigma,\Omega}(\varphi h_{\theta,\varsigma,\Omega}))$$

$$= \nu_{\theta,\varsigma,\Omega}(\varphi) \nu_{\theta,\varsigma,\Omega}(\Omega_{\theta,\varsigma,\Omega}) = 0.$$\n
Since $C^1$ is dense in $L^2$, it follows\n
(A.28) $\Omega_{\theta,\varsigma,\Omega} = \phi - \phi \circ f_{\theta}$, \quad $m_{\theta,\varsigma,\Omega} - a.s.$\n
A function with the above property is called a coboundary, in this case an $L^2$ coboundary. In fact, more is true: $\phi \in C^1$. Indeed, first of all notice that $\nu_{\theta,\varsigma,\Omega}(\phi) = 0$. Then

$$\tilde{L}_{\theta,\varsigma,\Omega}(\Omega_{\theta,\varsigma,\Omega} h_{\theta,\varsigma,\Omega}) = \tilde{L}_{\theta,\varsigma,\Omega} (\phi h_{\theta,\varsigma,\Omega}) - \phi h_{\theta,\varsigma,\Omega} = (1 - \tilde{L}_{\theta,\varsigma,\Omega})(\phi h_{\theta,\varsigma,\Omega}).$$

Note that the above equation has a unique $L^2(\mathbb{T}, m_{\theta,\varsigma,\Omega})$ solution.\footnote{Indeed, otherwise the equation $\tilde{L}_{\theta,\varsigma,\Omega} \psi = \psi$ would have more than one solution in $L^2$. But for any such solution let $\{\psi_n\} \subset C^1$ be a sequence that converges to $\psi$ in $L^2$, then it converges in $L^1$, moreover $m_{\theta,\varsigma,\Omega}((\tilde{L}_{\theta,\varsigma,\Omega}^{n}(\psi - \psi_n)) \leq m_{\theta,\varsigma,\Omega} |\psi - \psi_n|$. Thus, $\psi = \tilde{L}_{\theta,\varsigma,\Omega}^{n} \psi + o(1) = h_{\theta,\varsigma,\Omega} m_{\theta,\varsigma,\Omega} \psi + o(1)$. Hence $\psi = h_{\theta,\varsigma,\Omega} m_{\theta,\varsigma,\Omega} \psi$.}\n
Remark A.15. Note that the above lemma applies in particular to the case of real potentials (when $m_{\theta,\varsigma,\Omega}$ is a measure by Lemma A.1) and for $\varsigma = 0$ (since $m_{\theta,0}$ is Lebesgue).

Following [27] we introduce
Definition A.16. A real function $A \in C^1$ is called aperiodic, with respect to the dynamics $f$, if there is no BV function $\beta$ and $\nu_0, \nu_1 \in \mathbb{R}$ such that $A + \beta \circ f - \beta$ is constant on each domain of invertibility of $f$, and has range in $2\pi\nu_1\mathbb{Z} + \nu_0$.

Also in the following we will need the, seemingly stronger, condition.

Definition A.17. A real function $A \in C^1$ is called $c$-constant, with respect to the dynamics $f$, if there is a BV function $\beta$ such that $A + \beta \circ f - \beta$ is constant on each domain of invertibility of $f$.

We conclude with the announced proof that all the above properties are equivalent in our special case.

Lemma A.18. If $f \in C^2(T, \mathbb{T})$ and expanding, then each $c$-constant zero average function $A \in C^1(T, \mathbb{R})$ is necessarily a coboundary.

Proof. By definition there exists $\beta \in \text{BV}$ such that $\alpha = A + \beta \circ f - \beta$ where $\alpha$ is constant on the invertibility domains of $f$. If we apply the normalised transfer operator $\hat{\mathcal{L}}$ to the previous relation we have $\beta = (1 - \hat{\mathcal{L}})^{-1}\hat{\mathcal{L}}(\alpha - A)$. Note that, by hypothesis, $\hat{\mathcal{L}}(\alpha - A)$ is $C^1$ apart from at most one point (the image of the boundary points of invertibility domains), for each $k > 0$. Thus $\beta$ has at most one (jump) discontinuity, which we assume without loss of generality to be at $x = 0$. Since $A$ is smooth on $\mathbb{T}$, calling $\mathcal{P}$ the partition of invertibility domains, we have:

$$0 = \int_0^1 A' \, dx = \sum_{p \in \mathcal{P}} \int_p (\beta' - (\beta \circ f)') \, dx = (1 - d)(\beta(1^-) - \beta(0^+))$$

where $d$ is the number if invertibility domains of $f$, i.e. its topological degree. We thus conclude that $\beta$ is in fact continuous on $\mathbb{T}$ and therefore $\alpha$ has to be constant on $\mathbb{T}$ (hence identically zero). Then $\beta$ must be smooth and $A$ is then a $C^1$ coboundary.

A.6. Non perturbative results.

In this section we collect some results that hold for large $\varsigma$, that is well outside the perturbative regime. They hold under the generic conditions discussed in the previous section. Even though we have proven that all the conditions are equivalent we state the Lemma under the conditions that are most natural in the proof (and for which the Lemma might naturally hold in greater generality).

Lemma A.19. If $\Delta \omega_0$ is real, of zero average with respect to $\nu_{0,0}$ and is aperiodic, then the spectral radius of $\mathcal{L}_{\theta,\varsigma}$, when acting both on $C^1$ and BV, for all $\varsigma \neq 0$ is strictly less than one and varies continuously unless it is less than $\lambda^{-1}$.

Proof. We start by noticing that, for $\varsigma = 0$, the maximal eigenvalue is one and all the other eigenvalues have modulus strictly smaller. As pointed out in Remark A.12 the relevant spectrum on $C^1$ and BV is the same. Hence, for small $\varsigma$ we can apply perturbation theory and the first and last of (A.9) imply that $\chi_{\theta,\varsigma} = -\frac{\Sigma(\theta)}{2} \varsigma^2 + O(\varsigma^3)$ for some $\Sigma(\theta) > 0$. Note that $\Sigma(\theta)$ is continuous in $\theta$; hence, by Lemma A.14, $\inf_\theta \Sigma(\theta) > 0$. On the other hand, suppose that, for some $\varsigma \in \mathbb{R}$, $\mathcal{L}_{\theta,\varsigma} h_* = e^{i\beta} h_*$ for some $h_* \in C^1(T, \mathbb{C})$. Then $|h_*| \leq \mathcal{L}_{\theta,0} |h_*|$, but $\int |\mathcal{L}_{\theta,0}| h_* \, | - |h_*| | = 0$ implies $|h_*| = \mathcal{L}_{\theta,0} |h_*|$, so $|h_*| = h_0$, the maximal eigenvalue of $\mathcal{L}_{\theta,0}$. Accordingly, $h_* = e^{i\beta} h_0$. Note that we can choose $\beta$ so that it is smooth a

\footnote{This follows from (A.10b) and the perturbation theory in [31].}
part, at most, a jump of $2\pi n$, for some $n \in \mathbb{N}$, at a fixed point of $f_\theta$. Next, notice that
\[ h_0 = e^{-it^2 - i\vartheta}L_{\theta,x,t}^*h_+ = L_{\theta,0}\left(e^{it^2 - i\vartheta}\bar{\alpha} + \beta f_\theta - \vartheta\right)h_0.\]
If we set $\alpha = -i\vartheta + \beta - \beta \circ f_\theta - \vartheta$ and we take the real part of the above we get
\[ 0 = L_{\theta,0}\left(1 - \cos \alpha\right)h_0.\]
Since the function to which the operator is applied is non negative the range of $\alpha$ must be a subset of $2\pi \mathbb{Z}$ and can have discontinuities only at the preimages of the discontinuity of $\beta$.

Finally, the continuity of the maximal eigenvalue follows from standard perturbation theory [29] unless the essential spectral radius coincides with the spectral radius.

The above theorem implies that the spectral radius is smaller than one but does not provide any uniform bound. Since we will need a uniform bound more information is necessary. This, as already noticed in [6, 24], can be gained by using Dolgopyat’s technique [16].

**Lemma A.20.** If $i\Omega_\theta$ is real, of zero average with respect to $\nu_{\theta,0}$ and is a non c-constant function constant with respect to $f_\theta$, then for each $\zeta > 0$ there exists $\tau \in [0,1)$, such that, for any $\varsigma \in [-\zeta, \zeta]$ and $\theta \in \mathbb{T}^1$, the spectral radius of $L_{\theta,\varsigma}^\theta$, when acting on $C^1$, is less than $\tau$.

**Proof.** By Lemma C.2 and Theorem C.6 of Appendix C, there exists $\varsigma_1, A > 0$ and $\tau \in (0,1)$ such that for all $\varsigma \in [-\varsigma_1, \varsigma_1]$, $n \geq A \log |\varsigma|$ and $\theta \in \mathbb{T}^1$,

(A.29) \[ \|L_{\theta,\varsigma}^\theta\|_{1,\varsigma} \leq \tau^n \]

where $\|f\|_{1,\varsigma} = |f|_{\infty} + |\varsigma|^{-1}|f'|_{\infty}$. Next, by Lemma A.19, the spectral radius of $L_{\theta,\varsigma}^\theta$, for $|\varsigma| \leq |\varsigma_0, \varsigma_1|$ is uniformly smaller than one, hence (A.29) is valid also in such a range provided $A$ and $\tau$ are properly chosen. Also note that (A.2) implies, for all $n \in \mathbb{N}$, $\|L_{\theta,\varsigma}^\theta\|_{1,\varsigma} \leq C_{\#}\|\Omega\|_{C^1}^\theta$. In particular, by expanding via the Newman series, for any $z \in \mathbb{C}$, $\tau < |z| \leq 1$:

(A.30) \[ \|(1 - L_{\theta,\varsigma}^\theta)^{-1}\|_{1,\varsigma} \leq C_{\#}\left(\|\Omega\|_{C^1}A \log |\varsigma| + \frac{(\tau|z|^{-1})A \log |\varsigma|}{1 - \tau|z|^{-1}}\right). \]

Hence the spectral radius is bounded by $\tau$ while (A.2) implies that the essential spectral radius is bounded by $\lambda^{-1}$.

**APPENDIX B. MOMENT GENERATING FUNCTION**

In this section, given a $(c_1, c_2, c_3)$-standard pair $\ell = (G, \rho)$, we will call it simply a $c_2$-standard pair, since $c_1$ will be fixed always as in the rest of the paper and $c_2$ is irrelevant for the following estimates. Given $A \in C^2(\mathbb{T}^2, \mathbb{R}^d)$, $A_1 = \omega$, define $\tilde{A}(\theta) = \mu_{\theta}(A(\cdot, \theta))$ and $\tilde{A} = A - \tilde{A}$. Let $\theta^*_\ell = \mu_{\ell}(G) = \int_{\sigma} \rho(x) G(x) dx$ (hence it belongs to the range of a standard pair $\ell$). Moreover, as in Section 2 we assume that, for all $\sigma \in \mathbb{R}^d$, $(\sigma, A)$ is not cohomologous to a constant. Finally, let $e^{\chi_{\ell}(\sigma, \theta)}$ be the maximal eigenvalue of the operator $L_{\theta^*_{\ell}(\sigma, \tilde{A})}$, defined in (A.1), when acting on $C^1$. Note that the results of Appendices A.2 and A.3 imply $\tilde{X}_A \in C^2(\mathbb{R}^d \times \mathbb{T}, \mathbb{R})$. 
Proposition B.1. There exists $\varepsilon_0, \sigma_*, > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, $T > \varepsilon^{\frac{1}{2}}$, $\sigma \in BV([0, T])$, and any standard pair $\ell_0 = (G_0, \rho_0)$ we have,

$$\mu_{\ell_0} \left( e^{\sum_{k=0}^{T \varepsilon^{-1} - 1} (\sigma(\varepsilon k), A \circ F_{\varepsilon}^k) \right) = e^{-1} \int_0^T \left[ (\sigma(s), \hat{A}(\delta(s,\theta_0^s))) + \hat{\chi}_A(\sigma(s), \delta(s,\theta_0^s)) \right] ds,$$

where

$$\mathcal{R}(\sigma, T) = L \|\sigma\|_{BV} + LT + \varepsilon^{-1} \left[ L^{-1} - \min\{T, \|\sigma\|_{L^1}\}\right] \|\sigma\|_{L^1},$$

and $L$ is some natural number with $L \in \left[ C_\#, \varepsilon_0 \sqrt{\varepsilon^2} \right]$. If, additionally, $\|\sigma\|_{L^\infty} < \sigma_*$, we obtain the better estimate:

$$\mathcal{R}(\sigma, T) = \|\sigma\|_{BV} + LT + (\log \varepsilon^{-1})^2 \|\sigma\|_{L^1} + \varepsilon^{-1} \|\sigma\|_{L^2}^2 + \varepsilon^{-1} - L^{-1} \|\sigma\|_{L^2}^2.$$

Proof. Define $\sigma_k = \sigma(\varepsilon k)$. By (3.12) and proceeding with a number $K$ of time steps of length $L, K = \lceil T\varepsilon^{-1}\rceil L^{-1}$, we have that, for any $g \in L^\infty(T^2, \mathbb{R})$,

$$\mu_{\ell} \left( \prod_{k=0}^{\lfloor T\varepsilon^{-1} \rfloor - 1} e^{(\sigma_k, A \circ F_{\varepsilon}^k) \cdot g \circ F_{\varepsilon}^{\lfloor T\varepsilon^{-1} \rfloor}} \right) = \sum_{\ell_1 \in \mathcal{G}_{\varepsilon_{\ell}}} \cdots \sum_{\ell_K \in \mathcal{G}_{\varepsilon_{\ell - K}}} \prod_{j=1}^{K} \nu_j \mu_{\ell_{K}}(g) =$$

$$= \sum_{\ell_1 \in \mathcal{G}_{\varepsilon_{\ell}}} \cdots \sum_{\ell_K \in \mathcal{G}_{\varepsilon_{\ell - K}}} \prod_{j=1}^{K} \nu_j \int_{\mathcal{A}_{\varepsilon_{\ell}}} \rho_{\ell_{K}} g,$$

where, to ease the notation, we dropped the subscript potentials from the symbols for standard families. To further shorten notation, given a standard pair $\ell$, let $I_{\ell} = [a_{\ell}, b_{\ell}]$ and $\hat{\rho}_{\ell} = \mathbb{1}_{I_{\ell}} \rho_{\ell}$. The next result, whose proof we briefly postpone, contains our basic computational tool.

Lemma B.2. There exists $c > 0$, such that, for each $c_2$-standard pair $\ell, \sigma \in BV, L \leq c_2^{-\frac{2}{3}}$, and $g \in C^2(T, \mathbb{R})$ we have,

$$\sum_{\ell_1 \in \mathcal{G}_{\ell}} \nu_{\ell_1} \hat{\rho}_{\ell_1}(x) e^{-1} \gamma(\theta_1^x) = e^{O(L \|\sigma\|_{BV} + L \|\sigma\|_{L^1} + L^2 \varepsilon_{\ell}(\varepsilon_{\ell}^2)) \mathcal{D}_2 \min\{e^{-1} \|\sigma\|_{L^1}, L\}},$$

where

$$\mathcal{D}_2 = \|g\|_{C^1} + \|g''\|_{L^\infty}.$$

In addition, there exists $c_* \in (0, 1)$ so that if $\|\sigma\|_{L^\infty} \leq \sigma_*$, $\mathcal{D}_2 \leq c_* T$ and $c_2 \leq c_*$, we have the much sharper estimate

$$\sum_{\ell_1 \in \mathcal{G}_{\ell}} \nu_{\ell_1} \hat{\rho}_{\ell_1}(x) e^{-1} \gamma(\theta_1^x) = \hat{h}_{\ell} \mathcal{M}_{\ell} \left( e^{O(L \|\sigma\|_{BV} + L^2 \varepsilon_{\ell}(\varepsilon_{\ell}^2)) \mathcal{D}_2 \sum_{j=0}^{L-1} |\sigma_j| + L^{-1} \sum_{j=0}^{L-1} \sigma_j^2} \right).$$

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45 For simplicity of notation we ignore that $K$ may not be an integer, as such a problem can be fixed trivially.

46 In this section we will use the formula only with $g = 1$, yet in Section 6.6 this more general formulation will be used.

47 The $\{\sigma_0, \ldots, \sigma_{L-1}\}$ in Lemma B.2 will correspond to a generic block $\{\sigma_L, \ldots, \sigma_{(j+1)L-1}\}$ in equation (B.2).
where $m_\ell$ is the left eigenvector of the transfer operator $\mathcal{L}^{\psi_2}_{\theta_2}(\sigma, \tilde{A}(\cdot, \psi_2^1))$, $h_\ell$ is the right eigenvector of $\mathcal{L}^{\psi_2}_{\theta_2}(\sigma, \tilde{A}(\cdot, \psi_2^1))$, and $\tilde{e} \chi_A(\cdot, \theta)$ is the maximal eigenvalue of $\mathcal{L}^{\psi_2}_{\theta_2}(\sigma, \tilde{A}(\cdot, \psi_2^1))$.

We are now ready to begin our computation: for each $n \in \{0, \ldots, K-1\}$ and $\varphi \in T$, let

$$g_n(\varphi) = \int_{\varepsilon n L}^{\varepsilon K L} [\sigma(s), \tilde{A}(\varphi(s - \varepsilon n L, \varphi)) + \tilde{e} \chi_A(s, \varphi(s - \varepsilon n L, \varphi))] ds.$$

Note that equations (A.20c) and (A.20d) imply that $g_n \in C^2(T, \mathbb{C})$ and, for each $n$, $\mathcal{D}_2 \leq C_{\#} \|\sigma\|_{L^1([0, T])}$. Also, the proof of Proposition 3.3 implies that, if $\rho_{\ell_{j-1}}$ is a $c_2$-standard density, then the $\rho_{\ell_j}$'s are $c_2'$-standard densities with

$$c_2' = \max\{c_2 e^{-c_2 L}, C_\# \|\sigma\|_{L^\infty(J)}\},$$

where $J_j = [(j-1)\varepsilon L, j\varepsilon L]$. Iterating the above formula it follows that $\rho_{\ell_j}$ is a $c_{2j}$-standard density with

$$c_{2j} \leq \|\sigma\|_{L^\infty([0, T])} e^{-c_2 L} + C_\# (L^{-1} e^{-1} \|\sigma\|_{L^1(J_j)} + \|\sigma\|_{BV(J_j)}).$$

Let us now apply the first part of Lemma B.2 to the standard pairs $\ell_K$ in (B.2), choosing $g = 1$, $L \geq C_\# \log \varepsilon^{-1}$. We obtain:

$$\mu_0 \left( \prod_{k=0}^{K-1} e^{\langle \sigma_k, A_0 F^{k}_{\psi_2^1} \rangle} \right) = \sum_{\ell_1 \in \mathcal{E}_{\ell_0}} \cdots \sum_{\ell_{K-1} \in \mathcal{E}_{\ell_{K-2}}} \prod_{j=1}^{K-1} \nu_{\ell_j} e^{O(L\|\sigma\|_{BV(J_{K-L})})},$$

$$\cdot e^{O \left( e^{-L} \|\sigma\|_{BV(J_{K-L})} + \mathcal{D}_2 \min \{ \|\sigma\|_{L^1(J_{K-L})}, L \} + c_{2K} + (1 + \mathcal{D}_2) L^2 \right)},$$

$$\cdot e^{-\varepsilon \nu_{\ell_{K-1}} \sigma_{\ell_{K-1}}} \text{Leb}(\hat{\rho}_{\ell_{K-1}}).$$

Iterating the argument yields

(B.3) \quad \mu_0 \left( \prod_{k=0}^{K-1} e^{\langle \sigma_k, A_0 F^{k}_{\psi_2^1} \rangle} \right) = e^{-\varepsilon \nu_0 \theta_0} + O \left( L\|\sigma\|_{BV} + L T + \frac{\min \{ \|\sigma\|_{BV(J_{K-L})}, L \} + L^{-1}\|\sigma\|_{L^1}}{\varepsilon} \right).$$

The above estimate is effective for $\|\sigma\|_{L^\infty} \geq \sigma_\ast$. For $\|\sigma\|_{L^\infty} < \sigma_\ast$ it is mostly useless and one needs to use the second estimate in Lemma B.2 and keep carefully track of the mistakes. That is, we claim that, for all $n \in \{0, \ldots, K\}$, we have

(B.4) \quad \mu_0 \left( \prod_{k=0}^{K-1} e^{\langle \sigma_k, A_0 F^{k}_{\psi_2^1} \rangle} \right) = \sum_{\ell_1 \in \mathcal{E}_{\ell_0}} \cdots \sum_{\ell_n, \ell_n \in \mathcal{E}_{\ell_{n-1}}} \prod_{j=1}^{n} \nu_{\ell_j} m_{\ell_j} \left( \hat{\rho}_{\ell_n} e^{-\varepsilon \nu_0 (G_{\ell_n}(\cdot))} \right),$$

$$\cdot e^{O \left( \|\sigma\|_{BV(J_{[n L, \varepsilon K L])})} (1+\varepsilon (\log L^{-1}+L^2) \right)},$$

$$\cdot e^{O \left( e^{-L} \|\sigma\|_{BV(J_{[n L, \varepsilon K L])})} + L^{-1}\|\sigma\|_{L^2([n L, \varepsilon K L])} \right)},$$

$$\cdot e^{O \left( \|\sigma\|_{BV(J_{[n L, \varepsilon K L])})} (\log L^{-1}+1) \right)}.$$
For $n < K$, we proceed by backward induction: suppose it true for $n + 1 \leq K$, then we must compute
\[
\sum_{\ell_{n+1} \in \mathcal{C}_{\ell_{n}}^L} \nu_{\ell_{n+1}} m_{\ell_{n+1}} \left( \hat{\rho}_{\ell_{n+1}} e^{\varepsilon^{-1} g_{\ell_{n+1}}(G_{\ell_{n+1}}(\cdot))} \right).
\]

The basic idea is to apply Lemma B.2. Unfortunately, to do so we have to eliminate the dependence from $m_{\ell_{n+1}}$ in the above formula. This must be done with some care in order not to introduce a too large error. The most efficient strategy seems to substitute $m_{\ell_{n+1}}$ with $m_{\ell_{n}}$. In order to do so we use Lemma A.10 to write (B.5) as
\[
\sum_{\ell_{n+1} \in \mathcal{C}_{\ell_{n}}^L} m_{\ell_{n+1}} \left( \nu_{\ell_{n+1}} \hat{\rho}_{\ell_{n+1}} e^{\varepsilon^{-1} g_{\ell_{n+1}}(G_{\ell_{n+1}}(\cdot))} \right) e^{O(\|\sigma\|_{\text{BV}(J_n)})},
\]
\[
\cdot e^{O(\varepsilon |\log \varepsilon|) \sum_{j=n+1}^{K-1} |\sigma_j|},
\]
where $J_n = [n \varepsilon L, (n+1) \varepsilon L]$ and we have used
\[
L^{-1} \sum_{j=0}^{L-1} \|\sigma_j\| \geq L^{-1} \sum_{j=0}^{L-1} \|\sigma_0\| - \|\sigma_j - \sigma_0\| \geq \|\sigma_0\| - \|\sigma\|_{\text{BV}([0, \varepsilon L])}.
\]

We can then apply Lemma B.2 to the above expression. Note that, in this case, by equation (A.20d) we have $D_2 \leq C \# J_{\varepsilon(n+1)L} |\sigma(s)|$. Plugging the results into equation (B.4) at step $n + 1$ we complete the induction using once again A.10 to estimate $m_{\ell_{n+1}}$ and prove (B.4) at step $n$. The proof then follows from (B.4) with $n = 0$ and by using the bound on $L$ specified in the statement of the Proposition. □

Proof of Lemma B.2. Let $\phi \in C^0(\mathbb{T}^1, \mathbb{R}_{\geq 0})$, and $\beta_L = \prod_{k=0}^{L-1} e^{(\sigma_k, A \circ F^k_{\varepsilon})}$, and write
\[
\mu_{\ell} \left( \beta_L \phi(x_L) e^{\theta_L \varepsilon^{-1}} \right) = \sum_{\ell_1 \in \mathcal{C}_{\ell}^L} \nu_{\ell_1} \mu_{\ell_1} \left( \phi e^{\theta_0 \varepsilon} \right),
\]
\[
\mu_{\ell} \left( \beta_L \phi(x_L) e^{\theta_L \varepsilon} \right) = \int_{\mathbb{R}^d} \rho_L(x) \phi(x_L(x)) e^{\sum_{k=0}^{L-1} (\sigma_k, A \circ F^k_{\varepsilon}(x, \theta_0(x))) + \varepsilon^{-1} g(\theta_L(x))}
\]
where (B.7) follows by (3.12) and $\theta_0(x) = G_{\ell}(x)$. First of all, observe that, if $\theta_0$ is distributed according to $\ell$, then
\[
|g(\theta_0) - g(\theta^*_{\varepsilon})| \leq D \varepsilon.
\]
Next, note that, by Lemma 4.5, and setting $\bar{\theta}_j = \bar{\theta}(\varepsilon j, \theta_0)$, we have that, for all $k \in \{0, \ldots, L\}$,
\[
|\theta_k - \bar{\theta}_k - H_k| \leq C_\# k \varepsilon^2.
\]
Hence, we have
\[
g(\theta_L(x)) = g(\bar{\theta}_L(x)) + g'(\bar{\theta}_L(x)) \cdot (\theta_L - \bar{\theta}_L) + O(D_2 \varepsilon^2 L^2)
\]
\[
= g(\bar{\theta}_L(x)) + \varepsilon g'(\bar{\theta}(\varepsilon L, \theta^*_{\varepsilon})) \sum_{j=0}^{L-1} \mathcal{E}_{j,L} \hat{w}(x_j, \theta_j) + O(D_2 \varepsilon^2 L^2).
\]
Let $\Gamma_{a,k} = (\mathbf{g}'(\theta L, \theta_t^*)_{a,k}(x)) \Xi_{a,k, L, 0}$. We can then rewrite (B.8) as
\[
\mu_k (\beta L \phi(x_L) e^\theta (x)\zeta^{-1}) = e^{\mathcal{O}(\mathcal{D}_2 \varepsilon L^2)} \int_{a_k}^{b_k} \tilde{\rho}(x) \phi(x_L) e^{\sum_{k=0}^{L-1} ([\sigma_k, A(x_k, \theta_k)] + [\Gamma_{a,k}, \hat{A}(x_k, \theta_k)])} \, dx,
\]
where $\tilde{\rho}(x) := \rho(x) e^{\epsilon^{-1} \theta (\delta L(x))}$. Note that, $\tilde{\rho}$ is a $(c_2 + C_\mathcal{D}_2)$-standard density.

To estimate the integral in the above equation we use Lemma 4.1 and write, using the notations already defined there, 48
\[
\int_{a}^{b} \tilde{\rho}(x) \phi(x_L) e^{\sum_{k=0}^{L-1} ([\sigma_k, A(x_k, \theta_k)] + [\Gamma_{a,k}, \hat{A}(x_k, \theta_k)])} \, dx = 
\int_{a}^{b} \tilde{\rho}(x) \phi(x_L) e^{\sum_{k=0}^{L-1} \Omega_k(f^L_{x}(x) + \mathcal{O}(\varepsilon L\mathcal{D}_2 + \varepsilon L \sum_{k=0}^{L-1} \|\sigma_k\|) \, dx = 
\int_{a^*}^{b^*} \tilde{\rho} \circ Y^{-1}(x) \phi(f^L_{x}(x)) e^{\sum_{k=0}^{L-1} \Omega_k(f^L_{x}(x)) + \mathcal{O}(\varepsilon L\mathcal{D}_2 + \varepsilon L \sum_{k=0}^{L-1} \|\sigma_k\|) \, dx,}
\]
where $\Omega_k(x) = (\sigma_k, A(x, \theta^*_t)) + (\Gamma_{a,k}, \hat{A}(x, \theta^*_t))$. The problem with the above expression is that $Y'$ has a very large derivative (see footnote 13) and hence it cannot be effectively treated as a BV function. In Section 10 we deal with this problem in a more sophisticated way. However, for the current purposes, it suffices the following rougher estimate based, again, on Lemma 4.1:
\[
\frac{1}{Y^\gamma \circ Y^{-1}} = e^{\mathcal{O}(\varepsilon L)} \prod_{k=0}^{L-1} \frac{f^L_{x}(x_k)}{\partial_x f(x_k, \theta_k)} = e^{\mathcal{O}(\varepsilon L^2)}.
\]
Also note that, setting
\[
\hat{\rho} = \tilde{\rho} \circ Y^{-1} \mathbb{1}_{[a^*, b^*]} = (\tilde{\rho} \mathbb{1}_{[a,b]}) \circ Y^{-1},
\]
we have that $\hat{\rho}$ is a $C_\mathcal{D}_2(c_2 + \mathcal{D}_2)$-standard density and $\|\hat{\rho}\|_{BV} \leq C_\mathcal{D}_2\|\rho\|_{BV}$. 49 All the above imply that the last line of (B.11) is given by
\[
e^{\mathcal{O}(\varepsilon L^2(1+\mathcal{D}_2) + \varepsilon L \sum_{k=0}^{L-1} \|\sigma_k\|)} \int_T \hat{\rho}(x) \phi(f^L_{x}(x)) e^{\sum_{k=0}^{L-1} \Omega_k(f^L_{x}(x))} \, dx.
\]
It is well known that such integrals can be computed by introducing suitable weighted Transfer Operators of the form:
\[
(\mathcal{L}_\theta, \Omega g)(x) = \sum_{f_0(y) = x} \frac{e^{\Omega(y, \theta)}}{f^L_\theta(y)} g(y).
\]
Indeed, setting $\mathcal{L}_k = \mathcal{L}_{\theta^*_k} \Omega_k$, we can rewrite the integral in (B.13) as
\[
\int_T \hat{\rho}(x) \phi(f^L_{x}(x)) e^{\sum_{k=0}^{L-1} \Omega_k(f^L_{x}(x))} \, dx = \text{Leb}(\mathcal{L}_L \cdots \mathcal{L}_0[\rho]).
\]

48 Choosing $n = L, \theta^* = \theta_t^*$ and setting $Y = Y_L$.

49 Indeed, for any $\varphi \in C^1, \left| \int \varphi' (\rho \mathbb{1}_{[a,b]}) \circ Y^{-1} \right| = \left| \int_{a}^{b} (\varphi \circ Y)' \rho \right| \leq 3\|\rho\|_{BV} \|\varphi\|_{C^0}$. 

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We will now adopt the following strategy: first we obtain a rough bound for the above quantity valid for any $\sigma$. Then, we will obtain a sharper bound, that is however valid only for $\sigma$ with a relatively small $L^\infty$ norm.

Note that, for $g \geq 0$,

$$L_k g = e^{O(\|\sigma_k - \sigma_0\| + D_k \varepsilon L)} L_0 g = e^{O(\|\sigma\|_{BV([0, \varepsilon L])} + D_k \varepsilon L)} L_0 g.$$  

By Remark A.11 it follows, calling $e^{\chi_k}$ the maximal eigenvalue of $L_k$,

$$L_L \cdots L_0 [\hat{\rho}] = e^{O(L\|\sigma\|_{BV([0, \varepsilon L])} + D_k \varepsilon L^2)} L_0^L [\hat{\rho}] =$$

$$= e^{O(L\|\sigma\|_{BV([0, \varepsilon L])} + C_{\varepsilon} (c_2 + D_k \varepsilon L^2)) + L \chi_0 + \varepsilon^{-1} g(\theta(\varepsilon L, \theta^*_L)).}$$

Finally, using (A.9a), we have

$$L_L \cdots L_0 [\hat{\rho}] = e^{O(L\|\sigma\|_{BV([0, \varepsilon L])} + C_{\varepsilon} (c_2 + D_k \varepsilon L^2)) + \sum_{k=0}^{L-1} \chi_k + \varepsilon^{-1} g(\theta(\varepsilon L, \theta^*_L)).}$$

Moreover, by equations (A.9a), (A.9e), (A.20c) and since $m_{\theta^*_L} (\tilde{A}(\theta^*_L)) = 0$, we have

$$\chi_k = \tilde{\chi}_A (\sigma_j + \Gamma_{\varepsilon, j}; \theta^*_L) = \tilde{\chi}_A (\sigma_j; \theta^*_L) + O(\min \{\|\sigma_j\|_1, 1\} \|\Gamma_{\varepsilon, j}\|) =$$

$$= \tilde{\chi}_A (\sigma_j, \theta(\varepsilon, \theta^*_L)) + O(\min \{\|\sigma_j\|_1, 1\} \|\Gamma_{\varepsilon, j}\| + \varepsilon \|\sigma_j\|).$$

Substituting the above estimate in (B.14), then backtracking from (B.13), (B.11), (B.8) and (B.7), using (B.9) and since $\phi$ is arbitrary yields

$$\sum_{\ell \in \mathbb{Z}_L} u_{\ell_1, \ell_2} (x) \rho_{\ell_1} (x) e^{-1} g(\theta^*_L) = e^{O(L\|\sigma\|_{BV([0, \varepsilon L])} + L\|\sigma\|_{L^1([0, \varepsilon L])} + c_2 (1 + D_k \varepsilon L^2))} \cdot$$

$$\cdot e^{O(D_k \min \{\varepsilon^{-1} \|\sigma\|_{L^1([0, \varepsilon L])}, L\})} e^{\sum_{j=0}^{L-1} \|\sigma_j, \tilde{A}(\theta(\varepsilon, \theta^*_L)) + \tilde{\chi}_A (\sigma_j, \theta(\varepsilon, \theta^*_L)) + g(\theta(\varepsilon L, \theta^*_L))\varepsilon^{-1}}.$$

We have thus obtained the announced general, but rough, bound. Next, we will get a sharper bound for $\|\sigma\|_{L^\infty} \leq \sigma_*$ for some fixed $\sigma_* \leq \sigma_2$ (from Lemma A.13) to be chosen shortly. The basic idea is to use the perturbation theory of Section A to better estimate (B.14).

For each $k, q \in \mathbb{N}$ consider the operators $A_{k,q} = L_{(k+1)q-1} \cdots L_{kq}$. Since, by hypothesis, $\|\tilde{\Omega}_k\|_{\infty} \leq C_{\#} \sigma_*$, the $L_k$ are small perturbations of the Perron–Frobenius operator $L_{\theta^*_L, 0}$. By Lemma A.1 it follows that there exists $q \in \mathbb{N}$ such that $L_{\theta^*_L, 0}^q = P + Q$ such that $P$ is a projector and $\|Q\|_{c^1} \leq \frac{1}{4}$ and $PQ = QP = 0$. The theory in Section A.2 implies that the $A_{k,q}$ have a simple maximal eigenvalue $e^{\chi(\sigma, k, q)}$. Moreover, we can write $A_{k,q} = e^{\chi(\sigma, k, q)} P_{k,q} + Q_{k,q}$, where $P_{k,q}^2 = P_{k,q}$ and $P_{k,q} Q_{k,q} = Q_{k,q} P_{k,q} = 0$, $\|Q_{k,q}\|_{c^1} \leq \frac{1}{4} e^{\chi(\sigma, k, q)}$.

By standard perturbation theory all the above objects are analytic in $\sigma$. Thus, similarly to (A.17),

$$\|Q_{k,q} - Q_{k+1,q}\|_{c^1} + \|P_{k,q} - P_{k+1,q}\|_{c^1} \leq C_{\#} \Delta_{k+1},$$

where $\Delta_{k+1} = \min \{q\|\sigma\|_{BV([kq, (k+1)q])}, \sum_{j=qk}^{(k+1)q} |\sigma_j|\} + q \varepsilon$ and we assume conventionally $\Delta_0 = 1$.\footnote{The estimate of $\Delta_{k+1}$ seems a bit cumbersome. The reason is that the second possibility is good locally to verify the condition $\Delta_k \leq 3q\sigma_*$ in Sub-Lemma B.3 but is otherwise a bad choice since it gives a too large cumulative mistake.} For further use note that, calling $e^{\tilde{\chi}(\sigma, \theta)}$ the maximal eigenvalue
where we have used (A.9a). Note that, due to Remark A.11, the above facts hold also if we consider the operators acting on BV, provided \( \sigma_0 \) is chosen small enough.

Let \( \rho_k := \mathcal{L}_{k,q-1} \cdots \mathcal{L}_0[\hat{\rho}] \), thus \( \rho_{k+1} = A_{k,q} \rho_k \), and \( P_{k,q} \alpha = h_k m_k(g) \), normalized as in Lemma A.5. The next step is to control the growth of \( \rho_k \). First of all, recall that, by the hypothesis of the present lemma, \( \rho \) is a \( C^\# \)-standard density. Note that \( |\rho_{k+1}| = |A_{k,q} \rho_k| \leq |A_{k,q}^1||\rho_k|_\infty \), and \( |A_{k,q} 1| \leq |L^*_{k,q}|e^{|\theta^1_{\sigma}(\mathcal{L}_{k,q-1},\alpha)}\|A\|_\infty \). Also, by (A.2) it follows that \( h_k(x) \leq e^C \# \), Note that by the same arguments we can get similar lower bounds on the previous quantities.

**Sublemma B.3.** There exists \( \sigma_* > 0 \) such that, if, for all \( k \in \{1, \ldots, [T^{\epsilon^{-1}}q^{-1}] \} \), \( |\sigma_k| \leq \sigma_* \), then we have

\[
|\rho_k - e^{\sum_{j=1}^k \chi_j m_0(\hat{\rho})} h_k| \leq C_\# e^{\sum_{j=1}^k \chi_j m_0(\hat{\rho})} \sum_{j=0}^k \Delta_j h_k.
\]

**Proof.** Note that, \( \Delta_k \leq 3q \sigma_* \). Let \( \rho_{k+1} = \gamma_{k+1} h_k + \varphi_{k+1}, m_k(\varphi_{k+1}) = 0 \). Then

\[
\begin{align*}
\gamma_{k+1} & = m_k(\rho_{k+1}) = e^{\chi_k} m_k(\rho_k) = e^{\chi_k} \gamma_k + e^{\chi_k} (m_k - m_{k-1})(\rho_k) \\
\varphi_{k+1} & = \rho_{k+1} - \gamma_{k+1} h_k = A_{k,q}(\rho_k - m_k(\rho_k) h_k) \\
& = Q_{\sigma,q} \rho_k - m_k(\rho_k) h_k.
\end{align*}
\]

By (B.17) we have \( |(m_k - m_{k-1})(\rho_k)| \leq C_\# \Delta_k \|\rho_k\|_{BV} \). Accordingly, setting \( \alpha = \log 2 \),

\[
\begin{align*}
\|\varphi_{k+1}\|_{BV} & \leq e^{\chi_k - \alpha} \|\varphi_{k}\|_{BV} + C_\# \Delta_{k+1} \|\rho_k\|_{BV} \\
& \leq e^{\chi_k - \alpha} + C_\# \Delta_{k+1} \|\varphi_{k}\|_{BV} + C_\# e^{-\alpha} \Delta_{k+1} \gamma_{k+1} \\
\gamma_{k+1} & = e^{\chi_k} \gamma_k + O(e^{\chi_k} \Delta_{k+1} \|\rho_k\|_{BV}) \\
& = e^{\chi_k + O(\Delta_{k+1})} \gamma_k + O(e^{\chi_k} \Delta_{k+1} \|\varphi_k\|_{BV}).
\end{align*}
\]

Next, we prove, by induction, that there exists \( C_* > 1 \) such that,

\[
\begin{align*}
\|\varphi_k\|_{BV} & \leq C_* \gamma_k \sum_{j=0}^{k} e^{-(k-j)\alpha/2} \Delta_j.
\end{align*}
\]

Note that \( \gamma_0 = m_0(\hat{\rho}) \geq C_\# \|\rho\|_{BV} \geq C_\# \|\varphi_0\|_{BV} \), thus the relation is satisfied for \( k = 0 \) provided \( C_* \) is chosen large enough. Next, equation (B.22) implies

\[
\begin{align*}
\gamma_{k+1} & \geq e^{\chi_k - C_\# \Delta_{k+1} \gamma_k} - C_\# C_* \gamma_k \Delta_{k+1} e^{\chi_k} \sum_{j=0}^{k} e^{-(k-j)\alpha/2} q \sigma_* \geq e^{\chi_k - C_\# q \sigma_*} - C_\# C_* q^2 \sigma_*^2 \gamma_k.
\end{align*}
\]

Substituting the above into (B.22) and then in (B.20) yields,

\[
\begin{align*}
\|\varphi_{k+1}|_{BV} & \leq e^{-\alpha + C_\# q \sigma_* + C_\# q^2 C_* \sigma_*^2 \gamma_{k+1} C_* \sum_{j=0}^{k} e^{-(k-j)\alpha/2} \Delta_j} + C_\# \Delta_{k+1} \gamma_{k+1} \\
& \leq C_* \gamma_k \sum_{j=0}^{k+1} e^{-(k+1-j)\alpha/2} \Delta_j
\end{align*}
\]
provided \( C_\# \) is chosen large enough and \( C_\# \sigma_* + C_\# C_* q \sigma_*^2 \leq \frac{q}{2} \), which can always be satisfied by choosing \( \sigma_* \) small enough. Finally, substituting again (B.22) into (B.21) implies

\[
\gamma_{k+1} = e^{\gamma_k + O(\Delta_{k+1})} \gamma_k
\]

and hence the Lemma (recalling, from Lemma A.1 that \( h_k \geq C_\# \)).

We can thus use the above Sub-Lemma to estimate (B.14) whereby obtaining, for each \( \| \sigma \|_{L^\infty} \leq \sigma_* \),

\[
\int \phi \mathcal{L} \cdots \mathcal{L} \phi \| \bar{\rho} \|_{L^q} = e^{O(\| \sigma \|_{BV([0, \varepsilon L])} + L \varepsilon)} e^{L^* \sum_{j=0}^{L-1} [\langle \sigma_j, \mathcal{A}(\theta_j^*) \rangle + \mathcal{A}(\sigma_j + \Gamma_{g_{\theta_j} \theta_j})]},
\]

(B.23)

\[m_0(\bar{\rho} \mathbb{I}_{[a,b]} \circ Y^{-1}) = m_0(\bar{\rho} \mathbb{I}_{[a,b]} + O(\| \sigma_0 \|^2 \| \bar{\rho} \|_{BV}) + O(1 + \| \sigma_0 \|_{L^1}(\| \mathbb{I}_{[a,b]} \|_{L^1} \circ Y^{-1} - \bar{\rho} \mathbb{I}_{[a,b]} \|_{L^1})).\]

Since, for each \( \varphi \in L^\infty \) we have

\[
\left| \int \varphi \left[ (\bar{\rho} \mathbb{I}_{[a,b]} \circ Y^{-1} - \bar{\rho} \mathbb{I}_{[a,b]} \right) \right| = \int_a^b \left| (\varphi \circ Y \circ Y^{-1} - \varphi) \bar{\rho} \right|
\]

\[= \int \bar{\rho} \partial_x \int_0^x \mathbb{I}_{[a,b]} (\varphi \circ Y \circ Y^{-1} - \varphi)
\]

\[\leq C_\# \| \bar{\rho} \|_{BV} \| \varphi \|_{L^\infty} \| \bar{\rho} \|_{L^\infty} \| \varphi \| \leq C_\# m_0(\bar{\rho} \mathbb{I}_{[a,b]}) \| \varphi \|_{L^\infty} \leq L^2
\]

where, in the last line, we have used that \( \| \mathbb{1} - Y \|_{L^\infty} \leq C_\# L^2 \) (which follows from (B.12)) and, in the last inequality, the fact that, since \( g \geq 0, \| (g \circ \theta_k) \|_{L^\infty} \leq C_\# \varepsilon \) and \( \rho \) is a standard density, we have \( \| \bar{\rho} \|_{BV} \leq C_\# m_0(\bar{\rho} \mathbb{I}_{[a,b]}). \)

Accordingly,

\[m_0((\bar{\rho} \mathbb{I}_{[a,b]} \circ Y^{-1}) = m_0(\bar{\rho} \mathbb{I}_{[a,b]}) e^{O(\varepsilon L^2 + \| \sigma \|_{BV([0, \varepsilon L])}).
\]

Finally,

\[L^{-1} \sum_{j=0}^{L-1} \sigma_j^2 \geq L^{-1} \sum_{j=0}^{L-1} \sigma_j^2 - 2\sigma_* | \sigma_j - \sigma_0 | \geq \sigma_0^2 - 2\sigma_* \| \sigma \|_{BV([0, \varepsilon L])}
\]

Thus, we have

\[m_0((\bar{\rho} \mathbb{I}_{[a,b]} \circ Y^{-1}) = m_0(\bar{\rho} \mathbb{I}_{[a,b]}) e^{O(\varepsilon L^2 + \| \sigma \|_{BV([0, \varepsilon L])} + L^{-1} \sum_{j=0}^{L-1} \sigma_j^2}
\]

At last, substituting (B.26) into (B.24), the second part of Lemma B.2 follows. □

\[\text{The latter follows again from (A.25) and the fact that we are free to choose, once and for all, } \sigma_* \text{ small enough.}\]
Appendix C. Dolgopyat’s theory

In this appendix we prove a bound for the Transfer Operator for large $\varsigma$. The proof is after the work of Dolgopyat on the decay of correlation in Anosov flows [16]. Unfortunately, we need uniform results in $\theta$, so we cannot use directly the results in [40, 7, 2]. Although the results below can be obtained by carefully tracing the dependence on the parameters in published proofs, e.g., in [2, 7], this is a non trivial endeavor. Therefore we believe the reader will appreciate the following presentation that collects a variety of results and benefits from several simplifications allowed by the fact that we treat smooth maps (even though the arguments can be easily upgraded to cover all the results in the above mentioned papers).

C.1. Setting.
Let $f \in C^r(T^2, T^1)$ and $\omega \in C^{r-1}(T^2, \mathbb{R})$, $r \geq 2$. We will consider the one parameter family of dynamics $f_\theta(x) = f(x, \theta)$, of potentials $\Omega_\theta(x) = \omega(x, \theta)$ and the associated Transfer Operators

$$L_{\theta, i, \Omega_\theta} g(x) = \sum_{y \in f_\theta^{-1}(x)} \frac{e^{i / \Omega_\theta(y)} g(y)}{f_\theta'(y)}.$$  

Also, we assume that there exists $\lambda > 1$ such that $\inf_{x, \theta} f_\theta'(x) \geq \lambda$ (uniform expansivity). It is convenient to fix a partition $P_\theta = \{I_i\}$ of $T^1$, such that each $I_i$ is a maximal invertibility domain for $f_\theta$. We adopt the convention that the leftmost point of the interval $I_i$ is always zero (which, without loss of generality, can be assumed to be a fixed point for every $f_\theta$). Note that all the following is independent of such a choice of the partition.

Remark C.1. Since the maps $f_\theta$ are all topologically conjugate (by structural stability of smooth expanding maps), there is a natural isomorphism between $P_0$ and $P_\theta$, $\theta \in T^1$. From now on we will implicitly identify elements of the partitions (and their corresponding inverse branches) for different $\theta$ via this isomorphism and will therefore drop the subscript $\theta$ when this does not any create confusion.

At last we require that the $\Omega_\theta$ satisfies a condition (in general, although not in the present context, see Appendix A.5) stronger than aperiodicity; namely we assume it is not c-constant (see Definition A.17).

Let $H_n$ be the collection of the inverse branches of $f_\theta^n$ as defined by the partition $P$. Note that an element of $H_n$ can be written as $h_1 \circ \cdots \circ h_n$ where $h_i \in H_1$, thus $H_n$ is isomorphic to $H_1^n$. It is then natural to define $H_\infty = H_1^\mathbb{N}$.

C.2. Uniform uniform non integrability (UUNI).

The first goal of this section is to prove the following fact.

Lemma C.2. In the hypotheses specified in Subsection C.1 there exist $C_0 > 0$ and $n_0 \in \mathbb{N}$ such that, for each $n \geq n_0$ and $\theta \in T^1$,

$$(C.1) \quad \sup_{h_\theta, \kappa_\theta \in H_n} \inf_{x \in T^1} \left| \frac{d}{dx}(\Omega_{n, \theta} \circ h_\theta)(x) - \frac{d}{dx}(\Omega_{n, \theta} \circ \kappa_\theta)(x) \right| \geq C_0,$$

where, $\Omega_{n, \theta} := \sum_{k=0}^{n-1} \Omega_\theta \circ f_\theta^k$.

Remark C.3. Condition (C.1), when referred to a single map, is commonly called uniform non integrability (UNI for short) and has been originally introduced by Chernov in [9], a remarkable paper which constituted the first breakthrough in the
quantitative study of decay of correlations for flows. The difference here is due to the fact that we have a family of dynamics, rather that only one, and we require a further level of uniformity. The relation between UNI and not being cohomologous to a piecewise constant function was first showed in [2, Proposition 7.4]. The above Lemma constitutes a not very surprising extension of the aforementioned proposition.

Proof of Lemma C.2. Suppose the lemma to be false, then given $a \in \mathbb{N}$ large enough to be chosen later, there exist sequences $\{n_j, \theta_j\}$, $\lambda^{-n_j} < \frac{1}{2j^a}$, such that for each $h, \kappa \in \mathcal{H}_{\theta_j, n_j}$, there exists $x_{j,h,\kappa} \in \mathbb{T}^1$ such that

$$\left| \frac{d}{dx} (\Omega_{n_j, \theta_j \circ h}) (x_{j,h,\kappa}) - \frac{d}{dx} (\Omega_{n_j, \theta_j \circ \kappa}) (x_{j,h,\kappa}) \right| \leq \frac{1}{j^a}.$$ 

Start by noting that if $h \in \mathcal{H}_n$, then $h = h_1 \circ \cdots \circ h_n$ with $h_i \in \mathcal{H}_1$, and

$$\Omega_{n, \theta \circ h} = \sum_{k=0}^n \Omega_\theta \circ h_{k+1} \circ \cdots \circ h_n(x_0),$$

for some fixed $x_0 \in \mathbb{T}^1$. We remark that, by usual distortion arguments, for each $h \in \mathcal{H}_\infty$, we have $\|\Omega_{n, \theta \circ h}\|_{C^1} \leq C_\#$. For each $h \in \mathcal{H}_n$ and $k \leq n$ let $\bar{h}_k := f_{\theta}^k \circ h$. Next, for each $j \in \mathbb{N}$ and $p_j \leq n_j$, let $\ell \in \mathcal{H}_{\theta_j, p_j}$. Then, for each $h, \kappa \in \mathcal{H}_{\theta, n_j, -p_j}$, letting $x_{j, ho, k \circ \ell} = z$,

$$\frac{1}{j^a} \geq \left| \frac{d}{dx} (\Omega_{n_j, \theta_j \circ h \circ \ell}(z)) - \frac{d}{dx} (\Omega_{n_j, \theta_j \circ \kappa \circ \ell}(z)) \right| = \left| \sum_{k=0}^{n_j-p_j} \Omega_{n_j, \theta_j \circ \bar{h}_k \circ \ell \circ (\bar{h}_k \circ \ell)'}(z) - \Omega_{n_j, \theta_j \circ \bar{h}_k \circ \ell \circ (\bar{h}_k \circ \ell)'}(z) \right| \geq \left| \frac{d}{dx} (\Omega_{n_j, -p_j, \theta_j \circ h}(\ell(z))) - \frac{d}{dx} (\Omega_{n_j, -p_j, \theta_j \circ \kappa}(\ell(z))) \right| \Lambda^{-p_j},$$

where $\Lambda = \sup_{\theta, x} |f_{\theta}'(x)|$. Accordingly, setting $\mathcal{P}_{\theta, n} = \{h(\mathbb{T}^1)\}_{h \in \mathcal{H}_n}$, for each $I \in \mathcal{P}_{\theta, n}$ we have

$$\sup_{x \in I} \left| \frac{d}{dx} (\Omega_{n_j, -p_j, \theta_j \circ h}(x)) - \frac{d}{dx} (\Omega_{n_j, -p_j, \theta_j \circ \kappa}(x)) \right| \leq \frac{\Lambda^{p_j}}{j^a} + C_\# \lambda^{-p_j}.$$ 

Thus, setting $\bar{n}_j = n_j - p_j$, choosing $p_j = C_\# \log j$ and provided that $a$ has been chosen large enough, we have that for each $h, \kappa \in \mathcal{H}_{\theta_j, \bar{n}_j}$,

$$\left| \frac{d}{dx} (\Omega_{\bar{n}_j, \theta_j \circ h}) - \frac{d}{dx} (\Omega_{\bar{n}_j, \theta_j \circ \kappa}) \right|_\infty \leq \frac{C_\#}{j}.$$ 

Next, for $h_{\theta, 1}, \ldots, h_{\theta, m} \in \mathcal{H}_1$, let $C_m = \sup_{\theta} \sup_{\{h_{\theta, i}\}} \|\partial_{\theta} [h_{\theta, 1} \circ \cdots \circ h_{\theta, m}]\|_\infty$. Then

$$\partial_{\theta} [h_{\theta, 1} \circ \cdots \circ h_{\theta, m}] = [\partial_{\theta} h_{\theta, 1}] h_{\theta, 2} \cdots \circ h_{\theta, m} + h_{\theta, 1} \circ h_{\theta, 2} \circ \cdots \circ h_{\theta, m} \cdot \partial_{\theta} [h_{\theta, 2} \cdots \circ h_{\theta, m}].$$

implies $C_m \leq C_\# + \lambda^{-1} C_m - 1$, that is $C_m \leq C_\#$. This implies that

$$\|\partial_{\theta} [\Omega_{n, \theta \circ h}]\|_{\infty} \leq C_\#.$$
We can then consider a subsequence \( \{j_k\} \) such that \( \{\theta_{j_k}\} \) converges, let \( \bar{\theta} \) be its limit. Also, without loss of generality, we can assume that \( j_k \geq 2C_\#\lambda_k \) and \( |\theta_{j_k} - \bar{\theta}| \leq k^{-1} \). Thus, for \( k \) large enough and for each \( h, \kappa \in H_{\bar{\theta}, \bar{\theta}} \), we have
\[
\left\| \frac{d}{dx}(\Omega_{n, j_k} \circ h)(x) - \frac{d}{dx}(\Omega_{n, \bar{\theta}} \circ \kappa)(x) \right\|_{\infty} \leq \frac{1}{k}.
\]
We are now done with the preliminary considerations and we can conclude the argument. Let \( \Xi_{k, h} = \Xi_{\bar{\theta}, n, j_k} \). Since
\[
|\Omega_\theta \circ h_{k+1} \circ \cdots \circ h_n(x) - \Omega_\theta \circ h_{k+1} \circ \cdots \circ h_n(x_0)| \leq \| \frac{d}{dx}\Omega_\theta \circ h_{k+1} \circ \cdots \circ h_n \|_{\infty}
\]
\[
\leq C_\#(x - x_0)^k,
\]
it follows that the limit \( \Xi_h = \lim_{k \to \infty} \Xi_{k, h} \) exists in the uniform topology. Note that, since the derivative of \( \Xi_{k, h} \) are uniformly bounded, \( \Xi_h \) is Lipschitz in \( T^1 \setminus \{0\} \). In addition, since \( \Xi_h(x_0) = 0 \), and equation (C.3) implies, for each \( h, \kappa \in H_{\bar{\theta}, \kappa} \),
\[
\left\| \frac{d}{dx}[\Xi_{k, h} - \Xi_{k, \kappa}] \right\|_{\infty} \leq \frac{2}{k}.
\]
It follows that \( \Xi_h = \Phi \) is independent of \( h \). Finally, choose \( h \in H_1 \) and \( \bar{h} \in H_{\infty} \) such that \( \bar{h} = h \circ h \circ \ldots \circ h \), then, if \( x \in h^{-1}(T^1) \),
\[
\Xi_{\bar{\theta}, n, \bar{h}} \circ f_\theta(x) - \Xi_{\bar{\theta}, n, \bar{h}}(x) = \Omega_\theta - \Omega_\theta(h^n(x)).
\]
Since \( f_\theta \) has exactly one fixed point \( x_I \) in each \( I \in P_\theta \), \( \lim_{n \to \infty} h^n(x) = x_I \), where \( I = h(T^1) \). From the above considerations it follows
\[
\Phi \circ f_\theta - \Phi = \Omega_\theta + \Psi
\]
where \( \Psi \) is constant on the elements of \( P_\theta \) and \( \Phi \in BV \). That is, \( \Omega_\theta \) is c-constant, contrary to the hypothesis.

It is now easy to obtain the result we are really interested in.

**Corollary C.4.** In the hypotheses of Lemma C.2 there exists \( n_1 \in \mathbb{N} \) and \( h, \kappa \in H_{n_1} \) such that, for each \( n \geq n_1 \), \( \theta \in T^1 \) and \( \ell \in H_{n-n_1} \),
\[
(C.4) \quad \inf_{x \in T^1} \left| \frac{d}{dx}(\Omega_n, \theta \circ \ell \circ h)(x) - \frac{d}{dx}(\Omega_n, \theta \circ \ell \circ \kappa)(x) \right| \geq \frac{C_0}{2},
\]
\[
\text{Proof.} \quad \text{Let } n_1 \geq n_h. \text{ Then, for } h \in H_{n_1} \text{ and } \ell \in H_{n-n_1} \text{,}
\]
\[
\Omega_n, \theta \circ \ell \circ h = \Omega_{n_1, \theta} \circ h + \Omega_{n-n_1, \theta} \circ \ell \circ h.
\]
Thus, by Lemma C.2, we can choose \( h, \kappa \) so that
\[
\inf_{x \in T^1} \left| \frac{d}{dx}(\Omega_n, \theta \circ \ell \circ h - \Omega_n, \theta \circ \ell \circ \kappa)(x) \right| \geq \frac{3}{4}C_0 - C_\#|h'|_{\infty} + |k'|_{\infty}
\]
\[
\geq \frac{3}{4}C_0 - C_\#\lambda^{-n_1}.
\]
The result follows by choosing \( n_1 \) large enough.

**Remark C.5.** Note that the proof of Corollary C.4 implies that any family of potentials close enough to \( \Omega_\theta \) will satisfy (C.4) with the same \( n_1 \) and \( C_0/2 \) replaced by \( C_0/4 \).
C.3. Dolgopyat inequality.
In order to investigate the operator $\mathcal{L}_{\theta,i\Omega_0}$ for large $\zeta$ it is convenient to use slightly different norms and operators. The reason is that on the one hand, the main estimate is better done in a $\zeta$ dependent norm and, on the other hand, it is convenient to have operators that are contractions. Let $\rho := h_{\theta,0}[\int h_{\theta,0}]^{-1}$ be the invariant density of the operator $\mathcal{L}_{\theta,0}$ and define

$$\tilde{\mathcal{L}}_{\theta,i\Omega_0}(g) = \rho^{-1} \mathcal{L}_{\theta,i\Omega_0}(\rho g), \quad \|g\|_{1,\zeta} = \|g\|_{c_0} + \frac{\|g\|_{c_0}}{|\zeta|},$$

Then we have\(^{52}\)

(C.5) \[\|\tilde{\mathcal{L}}_{\theta,i\Omega_0}(g)\|_{c_0} \leq \|g\|_{c_0}, \quad \|\tilde{\mathcal{L}}_{\theta,i\Omega_0}(g)\|_{L^1_{\zeta}} \leq \|g\|_{L^1_{\zeta}},\]
as announced. Moreover, by (A.2), it follows that, for $\zeta \geq \zeta_0$, with $\zeta_0 > 0$ large enough, and $n \in \mathbb{N}$,

(C.6) \[\|\tilde{\mathcal{L}}_{\theta,i\Omega_0}^n(g)\|_{1,\zeta} \leq C_\# \lambda^{-n\zeta} \|g\|_{1,\zeta} + B_0 \|g\|_{c_0},\]
for a fixed constant $B_0$. Fix $\lambda \in (\lambda, 1)$ and choose $n_2 \in \mathbb{N}$ such that $C_\# \lambda^{-n_2} \leq \lambda^{-n_2}$. Also, for future use, we chose $n_2$ so that $\lambda^{-n_2} \leq \frac{1}{2}$. Iterating the above inequalities by steps of length $n_3 \in \mathbb{N}$, $n_3 \geq n_2$, we have

(C.7) \[\|\tilde{\mathcal{L}}_{\theta,i\Omega_0}^{kn_3}(g)\|_{1,\zeta} \leq \lambda^{-kn_3} \|g\|_{1,\zeta} + B_0 \sum_{j=0}^{k-1} \lambda^{-(k-j-1)n_3} \|\tilde{\mathcal{L}}_{\theta,i\Omega_0}^{jn_3}(g)\|_{c_0}.\]

Theorem C.6. If condition (C.4) is satisfied, then there exists $A, B > 0$, and $\gamma < 1$ such that for all $|\zeta| \geq B$ and $n \geq A \log |\zeta|$, we have

$$\sup_{\theta \in \mathbb{T}} \|\tilde{\mathcal{L}}_{\theta,i\Omega_0}^n\|_{1,\zeta} \leq \gamma^n.$$

Remark C.7. In fact, Theorem C.6, for fixed $\theta$, is a special case of [7, Theorem 1.1]. To be precise, [7, Theorem 1.1] is stated for a single map and with strictly positive roof functions (a role here played by $\Omega_0$). The latter can easily be arranged by multiplying the transfer operator by $e^{2\|\Omega_0\|}$, which does not change the norm. In addition, a careful look at the proof should show that $A, B, \gamma$ depend on the map and potential only via $n_1, C_0$ of (C.4) and $\|f_0\|_{\infty}, \|f_0^{-1}\|_{\infty}, \|f_0^{-1}(f_0')^{-1}\|_{\infty}, \|\Omega_0\|_{c^2}$ which, in the present case, are all uniformly bounded. Nevertheless, we think the reader may appreciate the following simpler, self-contained, proof rather than being referred to the guts of [7].

Proof of Theorem C.6. For each $g \in \mathcal{C}^1$ set $g_k = \tilde{\mathcal{L}}_{\theta,i\Omega_0}^k g$, with $n_3 \geq n_1$ from Corollary C.4 and $n_3 \geq n_2$ as in equation (C.7). The basic idea, going back to Dolgopyat [17], is to construct iteratively functions $u_k \in \mathcal{C}^1(\mathbb{T}, \mathbb{R}_{\geq 0})$ such that $|g_k(x)| \leq u_k(x)$ for all $k \in \mathbb{N}$ and $x \in \mathbb{T}$ and on which one has good bounds. More precisely:

Lemma C.8. There exists constants $K, \beta, B_1, \zeta_0 > 1, \tau > 0, n_3 \geq \max\{n_1, n_2\}$ and, for all $g \in \mathcal{C}^1$ with $\|g\|_{c_0} \leq \beta |\zeta|^{-\tau} \|g\|_{c_0}$, functions $\Gamma_{g,k} \in \mathcal{C}^1(\mathbb{T}, [4/5, 1])$ such that, for all $|\zeta| \geq \zeta_0$ and $k \in \mathbb{N}$,

(C.8) \[\|\Gamma_{g,k}\|_{L^\infty} \leq B_1 |\zeta|,\]

\[^{52}\] The first follows trivially from $|\tilde{\mathcal{L}}_{\theta,i\Omega_0} g| \leq \|g\|_{L^\infty} \tilde{\mathcal{L}}_{\theta,0} 1$ and $\mathcal{L}_{\theta,0} \rho = \rho$. The second from the standard $\|\tilde{\mathcal{L}}_{\theta,i\Omega_0}(g)\|_{L^1} \leq \|g\|_{L^1}$. 


and, setting $u_0 = \|g\|_{\infty} + \beta^{-1}|\varsigma|^{-1}\|g'\|_{\infty}$ and $u_{k+1} = \tilde{L}_{\theta,0}^{n_3}(\Gamma_{g,k} u_k)$, we have, for any $x \in \mathbb{T}^1$,

$$\max\{|u_k^j(x)|, |g_k^j(x)|\} \leq \beta|\varsigma|u_k(x), \quad |g_k(x)| \leq u_k(x)$$

and, for any $I = [a_1, a_2] \subset \mathbb{T}^1$ so that $|a_2 - a_1| \geq 4K|\varsigma|^{-1}$:

$$\int_I \tilde{L}_{\theta,0}^{n_3} \Gamma_{g,k} \leq e^{-3\tau}\|I\|.$$

Let us postpone the proof of Lemma C.8 and see how it implies the wanted result. First of all, note that if $\|g_k^j\|_{c_0} \geq \beta|\varsigma|^{-1}\|g_k\|_{c_0}$, then equation (C.7) implies $\|g_{k+1}\|_{1,\varsigma} \leq \gamma\|g_k\|_{1,\varsigma}$ provided $\beta$ has been chosen large enough. We can thus assume $\|g'\|_{c_0} \leq \beta|\varsigma|^{-1}\|g\|_{c_0}$ without loss of generality.

Next, note that, for any $j_0 \in \mathbb{N}$, by equation (A.2) and choosing $\bar{\lambda}$ as in equation (C.7), we can write

$$\left|\frac{d}{dx} \tilde{L}_{j_0,0}^{n_3}(u_k)\right| \leq \bar{\lambda}^{j_0n_3} \tilde{L}_{j_0,0}^{n_3}(|u_k|) + B\tilde{L}_{j_0,0}^{n_3}(u_k) \leq (\bar{\lambda}^{j_0n_3}\beta|\varsigma| + B)\tilde{L}_{j_0,0}^{n_3}(u_k).$$

By eventually increasing $\varphi_0$, we can choose $j_0$ so that, for all $\varsigma \geq \varphi_0$,

$$\left(\frac{d}{dx} \rho \tilde{L}_{j_0,0}^{n_3}(u_k)^2\right) \leq \frac{\tau|\varsigma|}{4K}\rho \tilde{L}_{j_0,0}^{n_3}(u_k)^2. \tag{C.9}$$

Thus, given any partition $\{p_m\}$ of $\mathbb{T}^1$ in intervals of size between $3K|\varsigma|^{-1}$ and $4K|\varsigma|^{-1}$ we have

$$\int_{\mathbb{T}^1} u_{k+j_0}^2 \rho \leq \int_{\mathbb{T}^1} \left(\tilde{L}_{\theta,0}^{n_3}(\Gamma_{k+j_0-1}(\tilde{L}_{\theta,0}^{j_0n_3-1} u_k))^2\right) \rho \leq \sum_m \int_{\mathbb{T}^1} \tilde{L}_{\theta,0}^{n_3} \Gamma_{k+j_0-1}^{n_3} \tilde{L}_{\theta,0}^{n_3} u_k^2 \int_{\mathbb{T}^1} \rho \tilde{L}_{j_0,0}^{n_3} u_k^2 \leq \int_{\mathbb{T}^1} \tilde{L}_{\theta,0}^{n_3} u_k^2 \int_{\mathbb{T}^1} \rho \tilde{L}_{j_0,0}^{n_3} u_k^2 \rho,$$

where in the second inequality of the first line we have used Schwartz with respect to the sum implicit in $\tilde{L}_{\theta,0}^{n_3}$ and $\tilde{L}_{\theta,0}^{n_3-1}$; the second line follows from (C.9); the first inequality of the third line follows from the last assertion of Lemma C.8, while the last inequality follows from the well known contraction of $\tilde{L}_{\theta,0}$ in $L_1^1$.

Finally, iterating the above equation, we obtain

$$\|u_{kn_0}\|_{L_2^2} \leq e^{-k\tau}\|u_0\|_{L_2^2}.$$

Accordingly, there exists $A > 0$ such that, for all $n \geq \frac{A}{2}\log|\varsigma|$ we have $\bar{\lambda} \leq |\varsigma|^{-2}$ and

$$\|g_n\|_{L_2^2} \leq \|u_n\|_{L_2^2} \leq |\varsigma|^{-4}\|u_0\|_{L_2^2}.$$

The above equation together with (C.7) and the fact that, for all $g \in \mathcal{C}_1$, \footnote{Indeed, $|\tilde{g}(x)|^2 \leq \|\tilde{g}\|_{L_2^2}^2 + 2\int_{\mathbb{T}^1} |\tilde{g}| |\tilde{g}'|$.}

$$\|	ilde{g}\|_{L_2^2} \leq \|	ilde{g}\|_{L_2^2}^2 + 2\|	ilde{g}\|_{L_2^2} + 2\|	ilde{g}'\|_{L_2^2} = \left[\|	ilde{g}\|_{L_2^2} + 2\|	ilde{g}'\|_{L_2^2}\right]^2$$

yields $\|\tilde{L}_{\theta,0}(g)\|_{1,\varsigma} \leq |\varsigma|^{-1}\|g\|_{1,\varsigma}$ for all $n \in \mathbb{Z}$. The latter readily implies Theorem C.6.
Proof of Lemma C.8. Since $u_0 = \|g\|_\infty + \beta^{-1} |\varsigma|^{-1} \|g'\|_\infty$, trivially, max\{\|u_0\|, |g_0|\} \leq \beta |u_0| and \|g(x)\| \leq u_0(x) for all $x \in \mathbb{T}^1$. Suppose, by induction, that max\{|x|, |g_k|\} \leq \beta |u_k| and \|g_k(x)\| \leq u_k(x) for all $x \in \mathbb{T}^1$, then (A.2) implies
\begin{equation}
|g_{k+1}(x)| \leq \lambda^{-n_3} \hat{\lambda}^{n_3} \|g_k\|(x) + B |\varsigma| \hat{\lambda}^{n_3} \|g_k\|(x)
\end{equation}

(C.10)
\begin{equation}
\leq \beta |\varsigma| \left( \lambda^{-n_3} + B \beta^{-1} \right) \hat{\lambda}^{n_3} \|g_k\|(x) \leq \beta |\varsigma| \frac{5}{4} \left( \lambda^{-n_3} + B \beta^{-1} \right) u_{k+1}
\end{equation}

where we have assumed the existence of the wanted $\Gamma_{g,k}$ that remains to be constructed. By choosing $\beta$ large enough it follows
\begin{equation}
|g_{k+1}(x)| \leq \beta |u_{k+1}|
\end{equation}

The proof of the analogous inequality for $u_k$ being similar, but it uses C.8.

Next, let $h, \kappa \in H_{\mathcal{N}}$ be two branches satisfying (C.4), whose existence follows by Corollary C.4, and let us define the set $\mathcal{H} = H_{\mathcal{N}} \setminus \{h, \kappa\}$. Then,
\begin{equation}
|g_{k+1}(x)| \leq \sum_{h \in \mathcal{H}} \frac{(u_{k0}) h(x)}{\rho(x)} + \left| \sum_{h \in \mathcal{H}} \frac{e^{i\mathcal{N}_\mathcal{H} \phi(x)} (g_{k0}) h(x)}{\rho(x)} \right|.
\end{equation}

To conclude we need a sharp estimate for the second term in (C.11), where a cancellation may take place. To this end it is helpful to introduce a partition of unity. This can be obtained by a function $\phi \in C^2(\mathbb{R}_{\geq 0}, [0, 1])$ such that $\phi(x) = 1$ for $|x| \leq \frac{1}{2}$ and $\phi(x) = 0$ for $|x| \geq 1$ and $\phi(x) = \frac{1}{2} \sum_{n=1}^{\infty} \phi(x-n)$, for all $x \in \mathbb{R}$. We then define $L_\varsigma = [K^{-1} |\varsigma|]$, $\psi_m(x) = \phi(L_\varsigma x - m)$. Note that, by construction, $\sum_{m=-\infty}^{\infty} \psi_m = 1$ (here we are interpreting the $\psi_m$s as functions on $\mathbb{T}^1$). Let $I_m = \text{supp} \psi_m$ and let $x_m$ be its middle point. Note that $K |\varsigma| \leq |I_m| \leq \frac{2K}{|\varsigma|}$.

To continue, for each $m \in \{1, \ldots, L_\varsigma - 1\}$, we must consider two different cases. First suppose that there exists $h_m \in \{h, \kappa\}$ such that $g_k(h_m(x_m)) \leq \frac{1}{2} u_k(h_m(x_m))$. Note that, for $z \in h(I_m)$,
\begin{equation}
e^{-2K \lambda^{-n_3} \beta} u_k(z) \leq u_k(h_m(x_m)) \leq e^{2K \lambda^{-n_3} \beta} u_k(z)
\end{equation}

hence
\begin{equation}
\left|g_k(z)\right| \leq \beta |u_k(z)| \leq \frac{3}{2} \beta |u_k(h_m(x_m))|
\end{equation}

provided $K \lambda^{-n_3} \beta \leq \frac{1}{8}$. Which implies, for all $x \in I_m$,
\begin{equation}
\left|g_k \circ h_m(x)\right| \leq \frac{3}{4} \left|u_k \circ h_m(x)\right|
\end{equation}

provided $K \lambda^{-n_3} \beta \leq \frac{1}{8}$.

Second, suppose that, for each $h \in \{h, \kappa\}$, $\left|g_k(h_m(x_m))\right| \geq \frac{1}{2} u_k(h_m(x_m))$. Then
\begin{equation}
\left|g_k(z)\right| \leq \beta |u_k(z)| \leq 2 \beta |u_k(h_m(x_m))| \leq 4 \beta \left|g_k(h_m(x_m))\right|
\end{equation}

The above implies $\frac{1}{4}\left|g_k(h_m(x_m))\right| \leq |g_k(z)| \leq 2 \left|g_k(h_m(x_m))\right|$ provided $K \lambda^{-n_3} \beta \leq \frac{1}{8}$.

Thus, setting $A_h = \frac{\left(e^{i\mathcal{N}_\mathcal{H} \phi(x)}\right)_{h(x)}}{\rho(x) \left(f^{n_3}_h\right)_{h(x)}}$, we have
\begin{equation}
\left|A_h(x)\right| \leq C_\#(h'(x))^2 \beta |\varsigma| \left|h'(x)\right| \left|u_k(h(x))\right| \leq C_\#(h'(x))^2 \beta |\varsigma| + h'(x)) \left|g_k(h(x))\right| \leq C_\#(h'(x))^2 \beta |\varsigma| + 1) \left|A_h(x)\right|.
\end{equation}

Defining $A_h = e^{i\theta_h} B_h$, with $\theta_h, B_h$ real and $B_h \geq 0$, we have
\begin{equation}
\left|A_h'\right| \geq \frac{1}{\sqrt{2}} \left(|\theta_h' B_h| + |B_h'|\right).
\end{equation}
The above implies that, given $\beta$, we can chose $n_3$ and $\varsigma_0$ large enough so that

$$|\theta'(x)| \leq C_\#(h'(x)|\varsigma| + 1) \leq \frac{C_0|\varsigma|}{32\pi},$$

$$|B'_h(x)| \leq \frac{C_0|\varsigma|}{32\pi} B_h(x).$$

Hence, setting $\Theta := \Omega_{\nu_2, \theta \circ h} - \Omega_{\nu_2, \theta \circ \kappa_\ast} + |\varsigma|^{-1}(\theta_{h_\ast} - \theta_{\kappa_\ast}),$

$$C_\# \geq \left| \frac{d}{dx} \Theta \right| \geq \frac{C_0}{4}.$$

In turns, this implies that the phase $\Theta$ has at least one full oscillation in $I_m$ provided $K \geq \frac{2\pi}{C_\#}$. Also, $\inf_{I_m} B_h \geq \frac{1}{2} \sup_{I_m} |B_h|$, provided $K \leq \frac{10\pi}{C_0}$. Next, suppose that $||B_{h_\ast}||_{\infty} \geq ||B_{\kappa_\ast}||_{\infty}$, (hence $4|B_{h_\ast}(x)| \geq |B_{\kappa_\ast}(x)|$), and set $h_m = \kappa_\ast$, the other case being treated exactly in the same way (interchanging the role of $h_\ast$ and $\kappa_\ast$, hence setting $h_m = \kappa_\ast$). Given the above notation, the last term of (C.11) reads

$$|B_{h_\ast} - e^{i\kappa_\ast} B_{h_\ast}| = \left[ B_{h_\ast}^2 + B_{\kappa_\ast}^2 - 2B_{h_\ast} B_{\kappa_\ast} \cos \varsigma \Theta \right].$$

It follows that there exists a constant $C_\ast > 0$ and intervals $J \subset I_m$, $\frac{4\pi}{32\pi} \geq |J| \geq \frac{C_\#}{C_0}$ on which $\cos \Theta \geq 0$. Then, on each such interval $J$,

$$|B_{h_\ast} - e^{i\kappa_\ast} B_{h_\ast}| \leq \left[ B_{h_\ast}^2 + B_{\kappa_\ast}^2 \right]^{1/2} \leq B_{h_\ast} + \frac{4}{5} B_{\kappa_\ast}.$$

We can then define $\Xi_m \in C^\infty(I_m, [\frac{4\pi}{32\pi}, 1])$ such that $\Xi_m(x) = 1$ outside the intervals $J, \Xi_m(x) = \frac{1}{5}$ on the mid third of each $J$ and $||\Xi_m||_{C^1} \leq C_\#|\varsigma|$. It follows

$$|B_{h_\ast} - e^{i\kappa_\ast} B_{h_\ast}| \leq \left| \frac{(g_k \rho) \circ h_\ast}{(\rho f^{m_3}_g)' \circ h_\ast} \right| + \Xi_m \left| \frac{(g_k \rho) \circ \kappa_\ast}{(\rho f^{m_3}_g)' \circ \kappa_\ast} \right|$$

$$\leq \left| \frac{(u_k \rho) \circ h_\ast}{(\rho f^{m_3}_g)' \circ h_\ast} \right| + \Xi_m \left| \frac{u_k \rho \circ h_m}{(\rho f^{m_3}_g)' \circ h_m} \right|.$$  (C.13)

We can finally define the function $\Gamma_{g, k} \in C^1(\mathbb{T}, [0, 1])$ as

$$\Gamma_{g, k}(x) = \sum_{m=0}^{L, -1} \psi_m \circ f^{m_3}(x) \Gamma_{k, m}(x).$$

where

$$\Gamma_{k, m}(x) = \begin{cases} 1 & \text{if } x \in h(I_m), h \neq h_m, \\ \Xi_m \circ f^{m_3}(x) & \text{if } x \in h_m(I_m). \end{cases}$$

Note that with the above definition, condition (C.8) is satisfied. Also, by equations (C.12) and (C.13), it follows $|g_{k+1}| \leq u_{k+1}$. Finally, we must check the last claim of the Lemma. Note that it suffices to consider intervals $J$ of size between $4K|\varsigma|^{-1}$ and $8K|\varsigma|^{-1}$.

$$\int_J \rho^{m_3}_\kappa \Gamma_{g, k}^2 \leq \sum_m \int_J \rho \circ h \cdot \rho \circ h' \cdot \rho \circ \tilde{h}_m \cdot \tilde{h}'_m.$$  (C.12)

Note that there exists at least one $m_4$ such that $I_{m_4} \subset I$. Moreover, at least $\frac{C_\#}{4\pi}$ of $I_{m_4}$ (hence at least $\frac{C_\#}{8\pi}$ of $I$) is covered by intervals $J$ on which $\Xi_{m_4} = 4/5$ and
\[ \psi_m = 1. \] Let \( J_\ast \) be the union of such intervals. Since \( \frac{\rho(x)}{\rho(y)} \leq e^{C_{\ast} |x-y|} \), for each \( \eta \in (0, 1) \),

\[
\int_I \psi_m \Xi_m \frac{\rho \circ \tilde{h}_m \cdot \tilde{h}'_m}{\rho} \leq \int_{I \setminus J_\ast} \psi_m \frac{\rho \circ \tilde{h}_m \cdot \tilde{h}'_m}{\rho} + \frac{16}{25} \int_{J_\ast} \frac{\rho \circ \tilde{h}_m \cdot \tilde{h}'_m}{\rho} \\
\leq (1-\eta) \int_{I \setminus J_\ast} \psi_m \frac{\rho \circ \tilde{h}_m \cdot \tilde{h}'_m}{\rho} + \left( \eta \frac{|I|}{|J_\ast|} + \frac{16}{25} \right) \int_{J_\ast} \psi_m \frac{\rho \circ \tilde{h}_m \cdot \tilde{h}'_m}{\rho}.
\]

Thus, choosing \( \eta = \frac{9C_{\ast}}{25C_{\ast} + 24K} \) we have

\[
\int_I \psi_m \Xi_m \frac{\rho \circ \tilde{h}_m \cdot \tilde{h}'_m}{\rho} \leq (1-\eta) \int_I \psi_m \frac{\rho \circ \tilde{h}_m \cdot \tilde{h}'_m}{\rho}.
\]

Also note that there exists \( M > 0 \) such that, for all \( h \in \mathcal{H}_{\ast} \) and \( m \in \{1, \ldots, L_{\ast} - 1\} \),

\[
\int_I \psi_m \frac{\rho \circ hh'}{\rho} \leq M \int_I \psi_m \frac{\rho \circ \tilde{h}_m \cdot \tilde{h}'_m}{\rho}.
\]

Moreover, note that \( I \) can intersect at most 9 intervals \( I_m \). By an argument similar to the above it then follows that there exists \( \tau > 0 \) such that

\[
\int_I \tilde{L}_{\theta, 0}^3 |y|^2 \leq e^{-3\tau} \sum_m \sum_{h \in \mathcal{H}_{\ast}} \int_I \psi_m \frac{\rho \circ h \cdot h'}{\rho} = e^{-3\tau} \int_I \tilde{L}_{\theta, 0}^3 = e^{-3\tau} |I|.
\]

\[ \Box \]

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