Soma Maity · Gautam Neelakantan Memana

Uniform Poincaré inequalities on measured metric spaces

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Abstract. Consider a proper geodesic metric space $(X, d)$ equipped with a Borel measure $\mu$. We establish a family of uniform Poincaré inequalities on $(X, d, \mu)$ if it satisfies a local Poincaré inequality $P_{loc}$, and a condition on the growth of volume. Consequently, if $\mu$ is doubling and supports $P_{loc}$ then it satisfies a uniform $(\sigma, \beta, \sigma)$-Poincaré inequality. If $(X, d, \mu)$ is a Gromov-hyperbolic space, then using the volume comparison theorem in Besson et al. (Curvature-free Margulis lemma for Gromov-hyperbolic spaces, 2020), we obtain a uniform Poincaré inequality with the exponential growth of the Poincaré constant. Next, we relate the growth of Poincaré constants to the growth of discrete subgroups of isometries of $X$, which act on it properly. We show that if $X$ is the universal cover of a compact $CD(K, \infty)$ space with $K \leq 0$, it supports a uniform Poincaré inequality, and the Poincaré constant depends on the growth of the fundamental group of the quotient space.

1. Introduction

Cheeger, Hajłasz, and Koskela showed the importance of local Poincaré inequalities in geometry and analysis on metric spaces with doubling measures in [9, 15]. In this paper, we establish a family of global Poincaré inequalities on geodesic spaces equipped with Borel measures, which satisfy a local Poincaré inequality along with certain other geometric conditions.

Let $(X, d)$ be a proper geodesic metric space, i.e., all closed balls in $(X, d)$ are compact and, any two points can be joined by a geodesic. Consider a Borel measure $\mu$ on $X$ such that every closed ball has a finite positive measure. We call the triple $(X, d, \mu)$ a measured metric space. A complete Riemannian manifold with distance and volume measure induced from the Riemannian metric is an example of a measured metric space. Riemannian manifolds with Ricci curvature bounded below may have polynomial or exponential growth of volume depending on bounds on curvature. Suppose there exists a non-decreasing function $f : (0, \infty) \to \mathbb{R}$ such that

$$\frac{\mu(B(x, R))}{\mu(B(x, \frac{1}{2}))} \leq f(R), \quad \forall x \in X, \forall R \geq \frac{1}{2}. \quad (1.1)$$

This is a condition on the growth of volume on large scales.
Definition 1. Let \( C(X) \) denote the space of continuous functions on \( X \). An upper gradient of \( u \in C(X) \) is a Borel function \( g_u : X \to [0, \infty] \) such that for each curve \( \gamma : [0, 1] \to X \) with finite length \( l(\gamma) \) and constant speed,
\[
|u(\gamma(1)) - u(\gamma(0))| \leq l(\gamma) \int_0^1 g(\gamma(t))\,dt.
\]
Let \( u_R \) denote the mean of \( u \) on balls of radius \( R \), i.e.,
\[
u R(x) = \frac{1}{\mu(B(x, R))} \int_{B(x, R)} u \,d\mu.
\]
Definition 2. Let \( 1 \leq \sigma < \infty \). \((X, d, \mu)\) is said to satisfy a local Poincaré inequality \( P_{\text{loc}} \) if there exist positive constants \( C(\sigma), r_0 \) such that for every \( u \in C(X) \) and its upper gradient \( g_u : X \to [0, \infty] \),
\[
\int_{B(x, R)} |u - u_R|^\sigma \,d\mu \leq C \int_{B(x, R)} g_u^\sigma \,d\mu, \quad \forall x \in X \text{ and } 0 < R \leq r_0.
\]
When the radius of the ball on the right-hand side is bigger than \( R \), the inequality is called a weak Poincaré inequality. If \((X, d, \mu)\) supports a weak local Poincaré inequality then it satisfies a Poincaré inequality as above possibly with a different constant \([14,15]\).

Theorem 1.1. Let \((X, d, \mu)\) be a measured metric space which satisfies the growth condition \((1.1)\) and \( P_{\text{loc}} \) for \( r_0 \geq 1 \) and \( \sigma \geq 1 \). Then for any \( u \in C(X) \) and its upper gradient \( g_u \),
\[
\int_{B(x, R)} |u - u_R|^\sigma \,d\mu \leq C_0 R^{\sigma - 1} f(4\lambda R) \int_{B(x, 2\lambda R)} g_u^\sigma \,d\mu, \quad \forall R \geq 4\lambda
\]
where \( \lambda = f(8.5) + 1 \) and \( C_0 = 2^{4\sigma - 2}\lambda^3 f(12.5)C \).

The constant \( C_0 R^{\sigma - 1} f(4\lambda R) \) is called the Poincaré constant throughout this paper. It is an upper bound on the best constant for which the above inequality holds. In light of the Bishop-Gromov volume comparison theorem, it is interesting to study the growth of the Poincaré constant when \((X, d, \mu)\) satisfies a notion of a lower bound on Ricci curvature.

Non-negative Ricci curvature on Riemannian manifolds plays a crucial role in establishing the existence of positive Green’s functions and bounded harmonic functions, estimates on heat kernel and green functions, parabolic Harnack inequalities [18]. These results are also proved using variants of doubling measures and local Poincaré inequalities to replace non-negative Ricci curvature condition [15,24]. We refer to Sect. 5 for the definition of a doubling measure. If \( \mu \) is a doubling measure on a measured metric space \((X, d, \mu)\), then the growth of volume is polynomial. If it also supports \( P_{\text{loc}} \) then as a consequence of Theorem 1.1 \((X, d, \mu)\) supports a uniform \((\sigma, \beta, \sigma)\)-Poincaré inequality (see Corollary 4.2), i.e., there exist positive constants \( C_0, r, \beta \) and \( \lambda \geq 1 \) such that for any \( u \in C(X) \) and its upper gradient \( g_u \),
\[
\int_{B(x, R)} |u - u_R|^\sigma \,d\mu \leq C_0 R^{\beta} \int_{B(x, \lambda R)} g_u^\sigma \,d\mu(z), \quad \forall x \in X, \forall R \geq r.
\]
Some interesting examples of such metric spaces are complete Riemannian manifolds with non-negative Ricci curvature, finitely generated groups with polynomial growth, Lie groups with Carnot-Carathéodory metrics, topological manifolds with Ahlfors regular measures [15]. Besson, Courtois, and Hersonsky established a family of \((\sigma, \beta, \sigma)\)-Poincaré inequality on complete Riemannian manifolds with Ricci curvature bounded below when the growth of volume is polynomial, and volumes of unit balls are bounded below by a positive constant in [3]. However, by Theorem 1.1, for a Riemannian manifold with Ricci curvature bounded below it suffices to satisfy (1.1) for some polynomial \(f(R) = vR^\alpha \) to support a \((\sigma, \beta, \sigma)\)-Poincaré inequality.

Next, we consider \(\delta\)-hyperbolic metric spaces in the sense of Gromov. The volume of a ball on a Riemannian manifold with negative sectional curvature grows exponentially as a function of its radius. Since \(\delta\)-hyperbolic spaces are defined generalizing certain metric properties of negatively curved Riemannian manifolds, it is natural to consider a Borel measure with exponential growth on them. The entropy of a measured metric space \((X, d, \mu)\) is defined by

\[
\text{Ent}(X, d, \mu) = \lim \inf_{R \to \infty} \frac{1}{R} \ln(\mu(B(x, R))).
\]

It is independent of the choice of \(x\). Besson, Courtois, Gallot, and Sambusetti established a Bishop-Gromov volume comparison theorem on \(\delta\)-hyperbolic measured metric spaces with volume entropy bounded above [4]. They showed that if the measure is invariant under certain group action, then the growth of volume is exponential and, depends on the entropy of the space. They also pointed out that an upper bound on entropy may be considered as a lower bound on Ricci curvature in a weak sense. As a consequence of Theorem 1.1 and the volume comparison theorem on \(\delta\)-hyperbolic spaces, we obtain the following theorem relating the growth of the Poincaré constant to the entropy of the space.

**Theorem 1.2.** Let \((X, d, \mu)\) be a measured \(\delta\)-hyperbolic space which supports \(P_{loc}\) for \(r_0 \geq 1\) and \(\sigma \geq 1\). Let \(\Gamma\) be a group acting on \(X\) isometrically and properly such that the diameter of the quotient space \(\Gamma \backslash X\) is bounded by \(D\). Suppose the action of \(\Gamma\) is also measure preserving and the entropy of \((X, d, \mu)\) is bounded by \(H\). Then there exist \(C_0(\delta, D, H, \mu, \sigma) > 0\) and \(\lambda \geq 1\) such that for any \(u \in C(X)\) and its upper gradient \(g_u\),

\[
\int_{B(x, R)} |u - u_R|^\sigma d\mu \leq C_0 R^{\sigma + 6HD + \frac{21}{2}} e^{12\lambda HR} \int_{B(x, 2\lambda R)} g_u^\sigma d\mu, \quad \forall R \geq \frac{5}{2}(7D + 4\delta), \forall x \in X.
\]

For an explicit description of the constants, we refer to Theorem 5.1 in Sect. 5. The above theorem generalizes the Poincaré inequality on Riemannian manifolds with Ricci curvature bounded below stated in section 10.1 in [15] on Gromov hyperbolic spaces. Cheeger showed that if a measured metric space admits \(P_{loc}\), it has a nice local structure [9]. Since on a \(\delta\)-hyperbolic space, local geometry and topology may be anything within a radius of \(\delta\) of a point, \(P_{loc}\) is a crucial assumption in the above theorem. An upper bound of volume entropy only gives a
bound on the growth of volume asymptotically. Hence a stronger notion of lower bound on Ricci curvature is required to replace $P_{loc}$ in the theorem above. If a group $\Gamma$ acts on a Riemannian manifold cocompactly, then it satisfies the $P_{loc}$ condition. K. Akutagawa, G. Carron, and R. Mazzeo showed that a large class of singular Riemannian manifolds, namely stratified spaces, also support $P_{loc}$ [2]. Hence a $\delta$-hyperbolic stratified space admitting a group action as described in Theorem 1.2 satisfies a global Poincaré inequality with the exponential growth of Poincaré constant. We refer to Section 7 in [4] for examples of such $\delta$-hyperbolic spaces.

More generally, when a discrete group acts on a measured metric space, we study the growth of the Poincaré constant in terms of the growth of the group. Consider a discrete subgroup $\Gamma\backslash\Gamma_1$ of isometries of $(X,d,\mu)$ acting on it properly such that the quotient space $\Gamma\backslash\Gamma_1\backslash X$ is compact. Define,

$$F_{\Gamma}(R) = |\Gamma x \cap B(x,R)|. \quad (1.4)$$

Here $|.|$ denotes the cardinality of the set. $F_{\Gamma}(R)$ is independent of the choice of $x$, and it determines the growth of $\Gamma$ with respect to $R$. Suppose the action of $\Gamma$ on $X$ is free, measure-preserving, and the quotient space $\Gamma\backslash\Gamma_1\backslash X$ is compact. If the volume and the diameter of $\Gamma\backslash X$ are bounded above by $V$ and $D$, respectively, then

$$\mu(B(x,R)) \leq V F_{\Gamma}(R + D), \quad \forall R \geq D.$$ 

When $\Gamma\backslash X$ supports a Poincaré inequality, we show that $(X,d,\mu)$ supports a $P_{loc}$ in Sect. 6. Then Theorem 1.1 implies that $(X,d,\mu)$ admits a uniform Poincaré inequality (see Theorem 6.1).

A lower bound on Ricci curvature plays an important role in establishing a local Poincaré inequality on Riemannian manifolds [3,7,15]. Sturm, Lott, and Villani defined notions of lower bound on Ricci curvature on length spaces with probability measures in terms of optimal transports on Wasserstein spaces in the seminal papers [19,26,27]. These are called $CD(K,N)$ spaces, where $K$ is a lower bound on Ricci curvature, and $N$ is an upper bound on the dimension. Lott and Villani proved Poincaré inequality on $CD(K,N)$ spaces, which are non-branching in [19,20]. Later Rajala proved a local Poincaré inequality on $CD(K,\infty)$ spaces in the sense of Lott-Villani and Sturm without imposing the non-branching condition in [22,23]. We refer to [22] for the definition of $CD(K,\infty)$ spaces.

**Theorem 1.3.** [22,23] Suppose that $(X,d,\mu)$ is a $CD(K,\infty)$ space with $K \leq 0$. Then for any continuous function $u$ on $X$ and for any upper gradient $g_u$ of $u$

$$\int_{B(x,R)} |u - u_{B(x,R)}| \, d\mu \leq c(K,R) \int_{B(x,2R)} g_u \, d\mu, \quad \forall R > 0, \quad \forall x \in X. \quad (1.5)$$

From the proof of the above theorem, we observe the following result after applying Jensen’s inequality.
Proposition 1.1. Suppose that \((X, d, \mu)\) is a \(CD(K, \infty)\) space with \(K \leq 0\). Then there exists a positive constant \(c(K, \sigma, R)\) such that for any continuous function \(u\) on \(X\) and for any upper gradient \(g_u\) of \(u\)

\[
\int_{B(x, R)} |u - u_{B(x, R)}|^\sigma d\mu \leq c(K, \sigma, R) \int_{B(x, 2R)} g_u^{\sigma} d\mu, \quad \forall R > 0,
\]

\[
\forall x \in X, \quad \sigma \geq 1.
\]

(1.6)

\(c(K, \sigma, R)\) is continuous in \(R\).

As a consequence of Theorem 1.1, we have the following theorem.

Theorem 1.4. Let \(\Gamma\) be a discrete subgroup of isometries of a measured metric space \((X, d, \mu)\) acting on it freely and properly such that the diameter of the quotient space \(/\Gamma \backslash X\) is bounded by \(D\). Suppose the action of \(\Gamma\) is measure preserving and \(\bar{d}, \bar{\mu}\) denote the quotient metric and the quotient measure, respectively. If \((/\Gamma \backslash X, \bar{d}, \bar{\mu})\) is a \(CD(K, \infty)\) space with \(K \leq 0\) then there exist positive constants \(C(K, \sigma)\) such that for any \(\sigma \geq 1\), \(u \in C(X)\) and its upper gradient \(g_u\),

\[
\int_{B(x, R)} |u - u_{B(x, R)}|^\sigma d\mu \leq 24 \sigma^{-2} CV_0 \lambda^4 R^{\sigma - 1} F_{/\Gamma}(2\lambda R) \int_{B(x, 2\lambda R)} g_u^{\sigma} d\mu,
\]

\[
\forall R \geq r, \quad x \in X.
\]

(1.7)

where \(V_0\) is the volume of \(/\Gamma \backslash X\) and \(\lambda = V_0 + D\).

Therefore, any covering space of a compact \(CD(K, \infty)\) space with \(K \leq 0\) satisfies a global Poincaré inequality as above. Moreover, if \(X\) is simply connected, then the growth of the Poincaré constant depends on the growth of the fundamental group of the quotient space as described by Theorem 1.4. The space of all compact \(CD(K, \infty)\) metric measured spaces is quite large. An interesting class of examples arises from differential geometry as Gromov-Hausdorff limits of compact Riemannian manifolds with Ricci curvature bounded below and limits of geometric flows if they exist. In particular if the quotient space \(/\Gamma \backslash X\) in Theorem 1.2 is a \(CD(K, \infty)\) space, then \(X\) satisfies \(P_{loc}\).

Idea of proof and structure of the paper: In this paper, the scheme of the proof of the existence of such a uniform Poincaré inequality is similar to the one used in [3] and in [11]. The authors showed that a complete Riemannian manifold \((M, g)\) with polynomial growth of volume supports a uniform Poincaré inequality if and only if it satisfies a local Poincaré inequality and a graph approximation of \((M, g)\) supports a discrete version of Poincaré inequality [3,11]. By considering a more general growth condition (1.1) we are able to express the Poincaré constant in terms of the growth of volume, which allows us to apply the volume comparison theorem in various contexts. Techniques from the proof of classical Bishop-Gromov volume comparison theorem are used crucially to prove existing Poincaré inequalities on Riemannian manifolds. It was known that the growth of the Poincaré constant depends on the growth of volume. Theorem 1.1 of this paper shows this dependency explicitly. The assumption on the lower bound on \(r_0\) in Theorem 1.1 is required to choose a graph discretization of \(X\) with canonical combinatorial distance one. If \(r_0\)
is bounded below by a positive constant then the required bound may be achieved
by scaling the metric \( d \) suitably.

In Sect. 2, we establish a Poincaré inequality for a measured metric graph when it
satisfies (1.1), which is an improvement of the Poincaré inequality established in [3].
In [3], a weak Poincaré inequality for a measured metric graph is established under
the assumption of polynomial growth of the measure of balls and a uniform lower
bound on the measure of vertices, whereas, with an improvement in the proof, we
obtain a strong Poincaré inequality without the extra assumption of uniform lower
bound on the measure of vertices. Moreover, we show that the Poincaré inequality
can be further improved if we assume a uniform lower bound on the measure of
vertices.

In Sect. 3, we first prove that an \( \epsilon \)-discretization of \((X, d, \mu)\), which is a mea-
sured metric graph, is roughly isometric to \((X, d, \mu)\) under the assumption of growth
condition (1.1). Moreover, we obtain a growth function for the \( \epsilon \)-discretization sat-
sifying (1.1) in terms of the growth function for \((X, d, \mu)\) satisfying (1.1). In this
method of approximation by a graph, we try to emulate the foundational work of
Kanai in [16,17] and its later improvements made by Coulhon and Saloff-Coste in
[11]. Later in this section, with the assumption of the existence of a local Poincaré
inequality on \((X, d, \mu)\) and the growth condition (1.1), we get a uniform Poincaré
inequality from the Poincaré inequality established on its \( \epsilon \)-discretization which
completes the proof of Theorem 1.1.

In Sect. 4 we discuss examples of measured metric spaces with polynomial
growth of volume and establish a \((\sigma, \beta, \sigma)\)-type uniform Poincaré inequality. We
study a family of uniform Poincaré inequalities on a Gromov hyperbolic space
satisfying \( P_{loc} \) with a bound on volume entropy and prove Theorem 1.2 in Sect. 5.
The inspiration to consider Gromov \( \delta \)- hyperbolic spaces under such conditions is
obtained from the volume comparison theorems proved in [4], which gives us an
exponential growth function satisfying (1.1). Section 6 is devoted to developing a
relationship between the growth of Poincaré constants and the growth of groups.
The existence of a Poincaré inequality on a covering space is also discussed when its
quotient space admits a Poincaré inequality. We prove Theorem 1.4 in this section.

2. Poincaré inequality on metric measured graphs

Let \( Y = (V, E) \) be a connected graph with a measure \( \nu \), where \( V, E \) denote the set
of vertices and edges, respectively. We denote \( x \sim y \) if \( x \) is adjacent to \( y \). Define
the distance \( \rho \) on \( Y \) as the canonical combinatorial distance as follows:
\[
\rho(x, y) = 1 \quad \text{if and only if} \quad x \sim y.
\]
The length of a path \( \gamma_{x, y} \) joining two points \( x \) and \( y \) is the number of edges in \( \gamma_{x, y} \). Define \( \rho(x, y) \) as the infimum of lengths
of paths joining \( x \) and \( y \). A graph \( Y \) with a canonical combinatorial distance \( \rho \) and
a measure \( \nu \), is called a metric measured graph.

Let \( u : V \to \mathbb{R} \) be a function. The integration with respect to a measure \( \nu \) is
defined as
\[
\int_F u(x) d\nu(x) = \sum_{x \in F} u(x) \nu(x) \quad \text{for any} \quad F \subset Y.
\]  
(2.1)
The point-wise $l^\sigma$-norm of the gradient of $u$ at a vertex $x$ is defined as
\[
|\delta u|_\sigma(x) = \left( \sum_{x \sim y} |u(x) - u(y)|^\sigma \right)^{\frac{1}{\sigma}}. \quad (2.2)
\]

$L^\sigma$-norm of the gradient of $u$ with respect to $\nu$ over a $F \subset Y$ is
\[
\|\delta u\|_{\sigma,F} = \int_F |\delta u|_\sigma^\sigma d\nu. \quad (2.3)
\]

Next, we establish a Poincaré inequality on a metric measured graph $(Y, \rho, \mu)$ that satisfies a growth condition defined in (1.1).

**Theorem 2.1.** Let $(Y, \rho, \nu)$ be a metric measured graph and $f : (0, \infty) \to \mathbb{R}$ be a function such that
\[
\nu(B(x, R)) \leq f(R), \quad \forall x \in Y \text{ and } R \geq r_0 > 0.
\]

Then for any $u : Y \to \mathbb{R}$, $\sigma \geq 1$ and $R \geq r_0 > 0$,
\[
\int_{B(p, R)} |u - u_R|^\sigma d\nu \leq 2^\sigma R^{\sigma-1} f(2R) \int_{B(p, R)} |\delta u|^\sigma d\nu, \quad \forall p \in Y.
\]

**Proof.** Consider $u : Y \to \mathbb{R}$ and $R \geq r_0$. By applying Jensen’s inequality, we have,
\[
u(B(x, R)) \leq f(R), \quad \forall x \in Y \text{ and } R \geq r_0 > 0.
\]

Minkowski inequality implies that
\[
\left( \int_{B(p, R) \times B(p, R)} |u(x) - u(y)|^\sigma d(v \otimes v) \right)^{\frac{1}{\sigma}}
\]
\[
\leq \left( \int_{B(p, R) \times B(p, R)} |u(x) - u(p)|^\sigma d(v \otimes v) \right)^{\frac{1}{\sigma}}
\]
\[
+ \left( \int_{B(p, R) \times B(p, R)} |u(y) - u(p)|^\sigma d(v \otimes v) \right)^{\frac{1}{\sigma}}
\]
\[
= 2 \left( \int_{B(p, R) \times B(p, R)} |u(x) - u(p)|^\sigma d(v \otimes v) \right)^{\frac{1}{\sigma}}
\]
\[
= \left( 2^\sigma \nu(B(p, R)) \int_{B(p, R)} |u(x) - u(p)|^\sigma d\nu(x) \right)^{\frac{1}{\sigma}}.
\]

Therefore,
\[
\int_{B(p, R)} |u(x) - u_R|^\sigma d\nu(x) \leq 2^\sigma \int_{B(p, R)} |u(x) - u(p)|^\sigma d\nu(x). \quad (2.4)
\]
Let $\omega$ be the counting measure on $Y$. Let $\gamma_{p,x} = \{p = v_0, v_1, \ldots, v_k = x\}$ be a minimal geodesic joining $p$ and $x$ for $x \in B(p, R)$. Then, using Jensen’s inequality,

$$|u(x) - u(p)|^\sigma \leq k^{\sigma - 1} \sum_{i=0}^{k-1} |u(v_{i+1}) - u(v_i)|^\sigma \leq k^{\sigma - 1} \sum_{i=0}^{k-1} |\delta u|^\sigma(v_i)$$

$$\leq l_{p,x}^{\sigma - 1} \int_{\gamma_{p,x}} |\delta u|^\sigma d\omega$$

where $l_{p,x} = \text{length}(\gamma_{p,x})$. Since $l_{p,x} \leq R$ we have,

$$|u(x) - u(p)|^\sigma \leq R^{\sigma - 1} \int_{\gamma_{p,x}} |\delta u|^\sigma d\omega \leq R^{\sigma - 1} \int_{B(p, R)} |\delta u|^\sigma(y) d\omega(y). \quad (2.5)$$

Now, \[
\int_{B(p, R)} |\delta u|^\sigma(y) d\omega(y) = \sum_{y \in B(p, R)} |\delta u|^\sigma(y) = \frac{1}{v(B(p, R))} \sum_{y \in B(p, R)} |\delta u|^\sigma(y) v(B(p, R)) v(y). \quad (2.6)
\]

For any $y \in B(p, R), B(p, R) \subset B(y, 2R)$. Therefore,

$$\frac{v(B(p, R))}{v(y)} \leq \frac{v(B(y, 2R))}{v(y)} \leq f(2R).$$

Hence from (2.6), we have,

$$\int_{B(p, R)} |\delta u|^\sigma(y) d\omega(y) \leq \frac{f(2R)}{v(B(p, R))} \int_{B(p, R)} |\delta u|^\sigma(y) d\sigma(y).$$

Combining (2.4), (2.5) and the above inequality, we have,

$$\int_{B(p, R)} |u(x) - u_R|^\sigma d\sigma(y) \leq \int_{B(p, R)} \left( \frac{2^\sigma R^{\sigma - 1} f(2R)}{v(B(p, R))} \int_{B(p, R)} |\delta u|^\sigma(y) d\sigma(y) \right) d\sigma(x)$$

$$\leq 2^\sigma R^{\sigma - 1} f(2R) \int_{B(p, R)} |\delta u|^\sigma(y) d\sigma(y).$$

If there exists $c > 0$ such that $\mu(x) \geq \frac{1}{c}$ for all $x \in X$ then we have the following theorem.

**Theorem 2.2.** Let $(Y, \rho, \nu)$ be a metric measured space. Suppose there exists a constant $c > 0$ and a function $f : (0, \infty) \to \mathbb{R}$ such that

$$v(B(x, R)) \leq f(R) \quad \text{and} \quad \nu(x) \geq \frac{1}{c}, \quad \forall x \in Y \quad \text{and} \quad R \geq r_0 > 0.$$\n
Then for any $u : Y \to \mathbb{R}, \sigma \geq 1$ and $R \geq r_0 > 0$,

$$\int_{B(p, R)} |u - u_R|^\sigma d\nu \leq 2^\sigma c R^{\sigma - 1} f(R) \int_{B(p, R)} |\delta u|^\sigma d\nu, \quad \forall p \in Y.$$
Proof. From (2.6) we have,
\[
\int_{B(p,R)} |\delta u|^\sigma \, d\omega = \sum_{y \in B(p,R)} |\delta u|^\sigma(y) \leq c \sum_{y \in B(p,R)} |\delta u|^\sigma(y) v(y) \\
\leq c \int_{B(p,R)} |\delta u|^\sigma(y) d\nu(y).
\]

From (2.4) and (2.5) we have,
\[
\int_{B(p,R)} |u(x) - u_R|^\sigma \, d\nu(y) \leq \int_{B(p,R)} \left( 2^\sigma c R^{\sigma - 1} \int_{B(p,R)} |\delta u|^\sigma(y) \, d\nu(y) \right) \, d\nu(x) \\
\leq 2^\sigma c R^{\sigma - 1} v(B(p,R)) \int_{B(p,R)} |\delta u|^\sigma(y) \, d\nu(y) \\
\leq 2^\sigma c R^{\sigma - 1} f(R) \int_{B(p,R)} |\delta u|^\sigma(y) \, d\nu(y).
\]

Since \( f \) is an increasing function, \( f(2R) \geq f(R) \). Hence the Poincaré constant here is slightly better than that of the previous theorem for sufficiently large \( R \). In Theorem 4.2 in [3], the authors established a weak Poincaré inequality on a metric measured graph that satisfies a polynomial growth of measure and a uniform lower bound on \( v(x) \). Theorem 2.2 and Theorem 2.1 improve the Poincaré constant in Theorem 4.2 in [3] in the case of polynomial growth.

3. Uniform Poincaré inequalities on metric spaces

In this section, we prove the main theorem. Consider a geodesic measured metric space \( (X, d, \mu) \).

**Definition 3.** A graph \( Y \) with a metric \( \rho \) and a measure \( \mu \) on it, is said to be an \( \epsilon \)-discretization of \( (X, d, \mu) \) for any \( \epsilon > 0 \) if the following conditions hold:

(i) \( Y \) is a maximal \( \epsilon \)-separated set in \( X \) i.e. \( d(y_i, y_j) \geq \epsilon, \forall y_i, y_j \in Y \) with \( y_i \neq y_j \).

(ii) \( \rho(y_i, y_j) = \epsilon \) for any \( y_i, y_j \in Y \) if \( d(y_i, y_j) < 2\epsilon \) and \( y_i \neq y_j \); \( \rho(y_i, y_j) = 0 \) if and only if \( y_i = y_j \).

(iii) \( \nu(y) = \mu(B_X(y, \epsilon)) \), \( \forall y \in Y \).

(iv) Define a graph with \( Y \) as the set of vertices. Any \( y_i \sim y_j \) if \( \rho(y_i, y_j) = \epsilon \) for all \( y_i, y_j \in Y \).

Observe that, given \( \epsilon > 0 \), \( (X, d, \mu) \) admits such an \( \epsilon \)-discretization by Zorn’s lemma and \( \{B_X(y, \epsilon) : y \in Y\} \) covers \( X \). For any \( L \geq 1 \), the multiplicity of the covering \( \{(B_X(y, L\epsilon))\}_{y \in Y} \) is defined as
\[
\mathcal{M}(Y, L\epsilon) = \sup_{y \in Y} |\{z \in Y : B_X(y, L\epsilon) \cap B_X(z, L\epsilon) \neq \phi\}|.
\]

We obtain an estimate of the multiplicity of an \( \epsilon \)-discretization of \( X \) in terms of the growth function \( f \) following similar steps as in [3].
Lemma 3.1. Let \((Y, \rho, \nu)\) be an \(\epsilon\)-discretization of \((X, d, \mu)\) with \(\epsilon \geq 1\). If \(X\) satisfies (1.1) and \(\mathcal{M}(Y, L\epsilon)\) denotes the multiplicity of the covering \(\{B(x, L\epsilon)\}_{x \in Y}\) then

\[
\mathcal{M}(Y, L\epsilon) \leq f\left(4L\epsilon + \frac{1}{2}\right), \quad \forall L \geq 1.
\]

Proof. Every ball considered in this lemma are with respect to the distance \(d\) in \(X\). Since \(\epsilon \geq 1\) observe that \(\{B(x, \frac{1}{2}))\}_{x \in Y}\) is a disjoint family of balls in \(X\). Now, consider the set \(Z \subset Y\) such that for all \(z \in Z\), \((B(x, L\epsilon) \cap B(z, L\epsilon)) \neq \emptyset\) for some fixed \(x \in Y\). Hence \(Z \subset B(x, 2L\epsilon)\) and \(\{B(z, \frac{1}{2}))\}_{z \in Z}\) is a disjoint family of balls contained in \(B(x, 2L\epsilon + \frac{1}{2})\). Hence,

\[
\sum_{z \in Z} \mu\left(B\left(z, \frac{1}{2}\right)\right) \leq \mu\left(B\left(x, 2L\epsilon + \frac{1}{2}\right)\right). \quad (3.1)
\]

Then, by using the condition on the growth of volume of balls

\[
\mu\left(B\left(z, 4L\epsilon + \frac{1}{2}\right)\right) \leq f(4L\epsilon + \frac{1}{2}) \mu\left(B\left(z, \frac{1}{2}\right)\right), \quad \forall z \in Y.
\]

Now by using the above equation and the fact that for \(z \in Z\), \(B(x, 2L\epsilon + \frac{1}{2}) \subset B(z, 4L\epsilon + \frac{1}{2})\) we get

\[
\sum_{z \in Z} \mu\left(B\left(z, \frac{1}{2}\right)\right) \geq \frac{|Z|}{f(4L\epsilon + \frac{1}{2})} \mu\left(B\left(x, 2L\epsilon + \frac{1}{2}\right)\right). \quad (3.2)
\]

Now, combining equations (3.1) and (3.2) we get \(|Z| \leq f(4L\epsilon + \frac{1}{2})\). Hence the lemma follows.

In next two lemmas we establish relations between distances and growth of measures of \((X, d, \mu)\) and \((Y, \rho, \nu)\) in terms of the growth function \(f\). We used ideas from similar lemmas proved in [17], and [11] on Riemannian manifolds.

Lemma 3.2. Let \((Y, \rho, \nu)\) be an \(\epsilon\)-discretization of \((X, d, \mu)\) with \(\epsilon \geq 1\). If \(X\) satisfies (1.1) then

\[
d(x, y) \leq 2\rho(x, y) \leq 2f\left(8\epsilon + \frac{1}{2}\right) (d(x, y) + 2\epsilon) \quad \forall x, y \in Y.
\]

Proof. The first inequality follows easily from triangle inequality and Definition 3. To prove the second inequality, consider a geodesic \(\gamma\) in \(X\) joining \(x\) and \(y\). Let \(Y_\gamma = \{z \in Y : B(z, \epsilon) \cap \gamma \neq \emptyset\}\). Clearly, \(\{B(z, \epsilon) : z \in Y_\gamma\}\) covers \(\gamma\) and \(\rho(x, y) \leq \epsilon |Y_\gamma|\). Consider the positive integer \(k\) such that \(k - 1 < d(x, y)/\epsilon \leq k\). Let \((x = x_0, x_1, ..., x_{k-1}, x_k = y)\) be points on \(\gamma\) such that \(d(x_{j-1}, x_j) = d(x, y)/k\) for \(j = 1, ..., k\). Since \(Y_\gamma\) is contained in an \(\epsilon\) neighbourhood of \(\gamma\), \(Y_\gamma \subset \bigcup_{j=0}^{k} \{z \in Y : x_j \in B(z, 2\epsilon)\}\). By Lemma 3.1,

\[
\rho(x, y) \leq \epsilon |Y_\gamma| \leq \epsilon \sum_{j=0}^{k} |\{z \in Y : x_j \in B(z, 2\epsilon)\}| \leq \epsilon(k + 1)f\left(8\epsilon + \frac{1}{2}\right)
\]

\[
< f\left(8\epsilon + \frac{1}{2}\right) (d(x, y) + 2\epsilon).
\]
Hence we have the required inequality.

**Lemma 3.3.** Let \( (Y, \rho, \nu) \) be an \( \epsilon \)-discretization of \( (X, d, \mu) \) with \( \epsilon \geq 1 \). If \( X \) satisfies (1.1), then for all \( x \in Y \)

\[
\nu(B_Y(x, R)) \leq f(\epsilon) \mu \left( B_X \left( x, 2R + \frac{1}{2} \right) \right), \quad \text{and} \\
\frac{\nu(B_Y(x, R))}{\nu(B_Y(x, \frac{1}{2}))} \leq f(\epsilon) f \left( 2R + \frac{1}{2} \right). \tag{3.3}
\]

If \( R' = f(8\epsilon + \frac{1}{2})(R + 3\epsilon) \) then

\[
\mu(B_X(x, R)) \leq v(B_Y(x, R')), \quad \text{and} \\
\frac{\mu(B_X(x, R))}{\mu(B_X(x, \frac{1}{2}))} \leq f(\epsilon) \frac{v(B_Y(x, R'))}{v(B_Y(x, \frac{1}{2}))}.
\]

**Proof.**

\[
\nu(B_Y(x, R)) = \sum_{y \in B_Y(x, R)} \nu(y) = \sum_{y \in B_Y(x, R)} \mu(B_X(y, \epsilon)) \\
= f(\epsilon) \sum_{y \in B_Y(x, R)} \mu \left( B_X \left( y, \frac{1}{2} \right) \right).
\]

From Lemma 3.2 we obtain that \( y \in B_X(x, 2R) \) for all \( y \in B_Y(x, R) \). Observe that \( \{B_X(y, \frac{1}{2})\}_{y \in Y} \) are mutually disjoint and contained in \( B_X(x, 2R + \frac{1}{2}) \). Hence,

\[
\nu(B_Y(x, R)) \leq f(\epsilon) \mu \left( B_X \left( x, 2R + \frac{1}{2} \right) \right).
\]

Using the above inequality and (1.1), we get that

\[
\frac{\nu(B_Y(x, R))}{\nu(x)} \leq f(\epsilon) \frac{\mu \left( B_X \left( x, 2R + \frac{1}{2} \right) \right)}{\mu(B_X(x, \epsilon))} \leq f(\epsilon) f \left( 2R + \frac{1}{2} \right).
\]

Let \( R' = f(8\epsilon + \frac{1}{2})(R + 3\epsilon) \) and \( y \in B_X(x, R + \epsilon) \cap Y \). By Lemma 3.2, \( y \in B_Y(x, R') \). Hence,

\[
B_X(x, R) \subset \bigcup_{y \in B_Y(x, R + \epsilon)} B_X(y, \epsilon) \subset \bigcup_{y \in B_Y(x, R')} B_X(y, \epsilon).
\]

This implies

\[
\mu(B_X(x, R)) \leq \sum_{y \in B_Y(x, R')} \mu(B_X(y, \epsilon)) = v(B_Y(x, R')).
\]

Therefore,

\[
\frac{\mu(B_X(x, R))}{\mu(B_X(x, \frac{1}{2}))} \leq f(\epsilon) \frac{\mu(B_X(x, R))}{\mu(B_X(x, \epsilon))} \leq f(\epsilon) \frac{v(B_Y(x, R'))}{v(B_Y(x, \frac{1}{2}))}.
\]
Let $||u||_{\sigma, E}$ denote the $L^\sigma$-norm of a Borel function $u$ on a Borel set $E$ for any $\sigma \geq 1$ with respect to $\mu$. Lemma 3.5 in [3] relates the $L^\sigma$-norm of $\delta \tilde{u}$ and the $L^\sigma$-norm of the gradient of $u$ in the case of Riemannian manifolds. Following the same idea, we prove the following lemma.

Lemma 3.4. Let $(Y, \rho, v)$ be an $\epsilon$-discretization of $(X, d, \mu)$ with $\epsilon \geq 1$ and let $g_u$ be an upper gradient of $u$ in $Y$. If $X$ satisfies (1.1) and the $P_{loc}$ condition in (1.2) for $r_0 \geq \epsilon$ then $\forall x \in Y$ and $R \geq r_0$,

$$ ||\delta \tilde{u}||^\sigma_{\sigma, B(x, R) \cap Y} \leq 2^{\sigma - 1} C \left( 1 + f \left( 4\epsilon + \frac{1}{2} \right) \right)^2 f \left( 12\epsilon + \frac{1}{2} \right) ||g_u||^\sigma_{\sigma, B(x, R+3\epsilon)}.$$ 

Proof. Let $\epsilon > 0$ and $z, z' \in X$ such that $d(z, z') < 2\epsilon$. Then

$$ \tilde{u}(z) - \tilde{u}(z') = \frac{1}{\mu(B(z, \epsilon))} \frac{1}{\mu(B(z', \epsilon))} \int_{B(z, \epsilon)} \int_{B(z', \epsilon)} (u(s) - u(t)) d\mu(s) d\mu(t). $$

Using Jensen’s inequality, we have,

$$ |\tilde{u}(z) - \tilde{u}(z')|^{\sigma} \leq \frac{1}{\mu(B(z, \epsilon))} \frac{1}{\mu(B(z', \epsilon))} \int_{B(z, \epsilon)} \int_{B(z', \epsilon)} |u(s) - \tilde{u}(z)|^{\sigma} + |u(t) - \tilde{u}(z')|^{\sigma} d\mu(s) d\mu(t). $$

Now using Minkowski’s inequality and then $P_{loc}$, we get,

$$ |\tilde{u}(z) - \tilde{u}(z')|^{\sigma} \leq 2^{\sigma - 1} \frac{1}{\mu(B(z, \epsilon))} \frac{1}{\mu(B(z', \epsilon))} \int_{B(z, \epsilon)} \int_{B(z', \epsilon)} (|u(s) - \tilde{u}(z)|^{\sigma} + |u(t) - \tilde{u}(z')|^{\sigma}) d\mu(s) d\mu(t) $$

$$ \leq \frac{2^{\sigma - 1} C}{\mu(B(z, \epsilon))} \int_{B(z, \epsilon)} g_u^{\sigma} + \frac{2^{\sigma - 1} C}{\mu(B(z', \epsilon))} \int_{B(z', \epsilon)} g_u^{\sigma}. $$

Since $d(z, z') < 2\epsilon$, $B(z', \epsilon) \subset B(z, 3\epsilon)$. Therefore,

$$ |\tilde{u}(z) - \tilde{u}(z')|^{\sigma} \leq \frac{2^{\sigma - 1} C}{\mu(B(z, \epsilon))} \left( 1 + \frac{\mu(B(z, \epsilon))}{\mu(B(z', \epsilon))} \right) \int_{B(z, 3\epsilon)} g_u^{\sigma}. $$

As $B(z, \epsilon) \subset B(z', 3\epsilon)$ we have

$$ |\tilde{u}(z) - \tilde{u}(z')|^{\sigma} \mu(B(z, \epsilon)) \leq 2^{\sigma - 1} C \left( 1 + \frac{\mu(B(z', 3\epsilon))}{\mu(B(z', \epsilon))} \right) \int_{B(z, 3\epsilon)} g_u^{\sigma}. $$
The growth of volume condition implies,
\[ |\tilde{u}(z) - \tilde{u}(z')|^\sigma \mu(B(z, \epsilon)) \leq 2^{\sigma-1} C (1 + f(3\epsilon)) \int_{B(z, \epsilon)} g_u^\sigma d\mu \]
\[ \leq 2^{\sigma-1} C (1 + f(3\epsilon)) \int_{B(x, R+3\epsilon)} g_u^\sigma \chi_{B(z, \epsilon)} d\mu \tag{3.4} \]
where \( \chi_S \) denotes the characteristic function of a set \( S \). Let
\[ B(x, R) \cap Y = \{ z_\alpha : \alpha \in \Lambda \} \] and \( g_{u\alpha} = g_u \cdot \chi_{B(z_\alpha, 3\epsilon)} \).

Therefore,
\[ \|\delta \tilde{u}\|_{\sigma, B(x, R) \cap Y}^\sigma = \sum_{\alpha \in \Lambda} |\delta \tilde{u}|_{\sigma, B(z_\alpha, \epsilon)}^\sigma \mu(B(z_\alpha, \epsilon)) \]
\[ \leq \sum_{\alpha \in \Lambda} \sum_{z_\beta \in B(z_\alpha, 2\epsilon)} |\tilde{u}(z_\beta) - \tilde{u}(z_\alpha)|^\sigma \mu(B(z_\alpha, \epsilon)) \]
\[ \leq 2^{\sigma-1} C (1 + f(3\epsilon)) \mathcal{M}(Y, \epsilon) \sum_{\alpha \in \Lambda} \int_{B(x, R+3\epsilon)} |g_{u\alpha}|^\sigma d\mu \]
\[ \leq 2^{\sigma-1} C (1 + f(3\epsilon)) \mathcal{M}(Y, \epsilon) \int_{B(x, R+3\epsilon)} \sum_{\alpha \in \Lambda} |g_{u\alpha}|^\sigma d\mu. \]

The third line follows from (3.4). Now for any fixed \( z \in B(x, R + 3\epsilon) \), \( g_{u\alpha}(z) \) is non-zero only if \( z \in B(z_\alpha, 3\epsilon) \) for some \( \alpha \in \Lambda \). Hence,
\[ \sum_{\alpha \in \Lambda} |g_{u\alpha}|^\sigma (z) \leq \mathcal{M}(Y, 3\epsilon) g_u^\sigma (z) \quad \forall z \in B(x, R + 3\epsilon). \]

Therefore,
\[ \|\delta \tilde{u}\|_{\sigma, B(x, R) \cap Y}^\sigma \leq 2^{\sigma-1} C (1 + f(3\epsilon)) \mathcal{M}(Y, \epsilon) \mathcal{M}(Y, 3\epsilon) \int_{B(x, R+3\epsilon)} g_u^\sigma d\mu. \]

The required result follows from Lemma 3.1.

Next, we prove the main theorem.

**Proof of Theorem 1.1**

**Proof.** Let \( (X, d, \mu) \) be a measured metric space which satisfies \( P_{\text{loc}} \) and the growth condition as defined in (1.2) and (1.1) respectively. Fix \( R > 0 \). Let \( (Y, \rho, v) \) be a fixed \( \epsilon \)-discretization of \( (X, d, \mu) \) with \( \epsilon = 1 \). Since \( B(x, R) \subset \bigcup_{y \in Y \cap B(x, R+1)} B(y, 1) \), for any \( \eta \in \mathbb{R} \) we have,
\[ \int_{B(x, R)} |u - \eta|^\sigma d\mu \leq \sum_{y \in Y \cap B(x, R+1)} \int_{B(y, 1)} |u - \eta|^\sigma d\mu. \]
By applying Jensen’s inequality, we have,
\[
\int_{B(x, R)} |u - \eta|^\sigma \, d\mu \leq 2^{\sigma - 1} \sum_{y \in Y \cap B(x, R + 1)} \int_{B(y, 1)} |u - \tilde{u}(y)|^\sigma \, d\mu \\
+ 2^{\sigma - 1} \sum_{y \in Y \cap B(x, R + 1)} n(y)|\tilde{u}(y) - \eta|^\sigma.
\] (3.5)

Let us denote by (I) and (II), the first and the second term of the right-hand side of the last inequality, respectively. One can bound (I) by local Poincaré inequality (1.2) for radius 1 since \( r_0 \geq 1 \).

\[
(I) \leq 2^{\sigma - 1} C \sum_{y \in Y \cap B(x, R + 1)} \int_{B(y, 1)} g_u^\sigma \, d\mu.
\]

Using the same argument as in Lemma 3.4 we have,

\[
(I) \leq 2^{\sigma - 1} C M(Y, 1) \int_{B(x, R + 2)} g_u^\sigma \, d\mu \\
\leq 2^{\sigma - 1} C f(4.5) \int_{B(x, R + 2)} g_u^\sigma \, d\mu.
\] (3.6)

We obtain a bound on (II) for a certain value of \( \eta \) using the Poincaré inequality on \( (Y, \rho, \nu) \). We choose \( x_0 \in Y \) such that \( d(x, x_0) < 1 \). Hence,

\[
Y \cap B(x, R + 1) \subset B(x_0, R + 2).
\]

To apply the Poincaré inequality on \( Y \), consider \( r = f(8.5)(R + 4), h(r) = f(1)f(2r + \frac{1}{2}) \). Then using Lemma 3.2 and Lemma 3.3 we have, \( B_X(x_0, R + 2) \cap Y \subset B_Y(x_0, r) \) and \( \frac{\nu(r)}{\nu(\frac{1}{2})} \leq h(r) \). Let

\[
\tilde{u}_r = \frac{1}{\nu(B(x_0, r))} \sum_{y \in B_Y(x_0, r)} \tilde{u}(y)\nu(y).
\]

Next we choose \( \eta = \tilde{u}_r \) and estimate (II) for this particular value of \( \eta \). From Theorem 2.1 we obtain,

\[
(II) \leq 2^{\sigma - 1} r^{\sigma - 1} h(2r) \sum_{y \in B_Y(x_0, r)} |\delta \tilde{u}(y)|^\sigma \nu(y).
\]

As \( B_Y(x_0, r) \subset B_X(x_0, 2r) \) from Lemma 3.2, we have,

\[
(II) \leq 2^{\sigma - 1} r^{\sigma - 1} h(2r) \sum_{y \in B_X(x_0, 2r) \cap Y} |\delta \tilde{u}(y)|^\sigma \nu(y).
\]

Now, by Lemma 3.4 we have,

\[
(II) \leq 2^{3\sigma - 2} r^{\sigma - 1} C(1 + f(4.5))^2 f(12.5) f(4r + 0.5) \int_{B(x_0, 2r + 3)} g_u^\sigma \, d\mu \\
\leq 2^{3\sigma - 2} r^{\sigma - 1} C(1 + f(4.5))^2 f(4.5) f(12.5) f(4r + 0.5) \int_{B(x, 2r + 4)} g_u^\sigma \, d\mu.
\]
Let $\lambda = f(8.5) + 1$. Then for all $R \geq 1$ and $\eta = \tilde{u}_r$, we have from (3.6),

$$(I) + (II) \leq 2^{3\sigma - 2} r^{\sigma - 1} C \lambda^3 f(12.5) f(4r + 0.5) \int_{B(x,2r+4)} g_\sigma^\sigma d\mu.$$ 

Assuming $R \geq 4\lambda$ we have,

$$\int_{B(x,R)} |u - \tilde{u}_r|^\sigma d\mu \leq 2^{3\sigma - 2} \lambda^3 f(12.5) C R^{\sigma - 1} f(4\lambda R) \int_{B(x,2\lambda R)} g_\sigma^\sigma d\mu. \tag{3.7}$$

We have the required uniform Poincaré inequality from the following.

$$\int_{B(x,R)} |u - u_R|^\sigma d\mu \leq 2^\sigma \inf_{\tau \in \mathbb{R}} \int_{B(x,R)} |u - \tau|^\sigma d\mu \leq \int_{B(x,R)} |u - \tilde{u}_r|^\sigma d\mu. \tag{3.8}$$

We refer to [3] for proof of the above inequality.

Moreover, if the measure of every ball of radius $\frac{1}{2}$ is bounded away from zero, then, with some additional assumption on the growth of the measure, we can improve the constants in Theorem 1.1.

**Theorem 3.1.** Let $(X, d, \mu)$ be a measured metric space which satisfies $P_{loc}$ for some $r_0 \geq 1$, $\sigma \geq 1$ as in (1.2). Suppose there exist a non-decreasing function $V : (0, \infty) \to \mathbb{R}$ such that

$$\mu(B(x, R)) \leq V(R) \quad \text{and} \quad \mu(B \left( x, \frac{1}{2} \right)) \geq 1, \ \forall x \in X, \ \forall R > 0. \tag{3.9}$$

Then for any $u \in C(X)$ and its upper gradient $g_u$,

$$\int_{B(x,R)} |u - u_R|^{\sigma} \leq 2^{4\sigma - 2} C \lambda^4 R^{\sigma - 1} V(2\lambda R) \int_{B(x,2\lambda R)} g_\sigma^\sigma d\mu \tag{3.10}$$

for all $R \geq 4\lambda$ where $\lambda = \max\{V(6.5), V(4.5) + 1\}$.

**Proof.** Let $(Y, \rho, \nu)$ be an $\varepsilon$-discretization of $(X, d, \mu)$ with $\varepsilon = 1$. From the inequality (3.1) we obtain the multiplicity of the covering $\{B(y, L)\}_{y \in Y}$ for any $L \geq 1$ as follows.

$$\mathcal{M}(Y, L) \leq V \left( 2L + \frac{1}{2} \right). \tag{3.11}$$

Using the growth of volume (3.9) we obtain the following inequalities from the proof of Lemma 3.2 and Lemma 3.3 respectively.

$$\rho(x, y) \leq V(4.5)(d(x, y) + 2) \tag{3.12}$$

and

$$\nu(B_Y(x, R)) \leq V(1)V(2R). \tag{3.13}$$

Using (3.11) in Lemma 3.4, we obtain

$$||\delta \hat{t}||_{\sigma, B_x(x, R) \cap Y}^\sigma \leq 2^{\sigma - 1} C (1 + V(3))^2 V(6.5)||g_u||_{\sigma, B_x(R, R+3)}^\sigma. \tag{3.14}$$
For $u \in C(X)$, let $g_u$ be an upper gradient of $u$. From (3.5) we have for any $\eta > 0$,
\[
\int_{B(x, R)} |u(z) - \eta|^\sigma \leq (I) + (II)
\]
where $(I)$ and $(II)$ are the first term and the second term in (3.5), respectively. From (3.6) we have,
\[
(I) \leq 2^{\sigma-1} CV(2.5) \int_{B(x, R+3)} g_u^\sigma d\mu.
\]
Next to obtain an estimate on $(II)$ we choose $x_0 \in Y$ such that $d(x, x_0) < 1$. Hence,
\[
Y \cap B(x, R + 1) \subset B(x_0, R + 2).
\]
To apply discrete Poincaré inequality on $Y$ we choose $r = V(4.5)(R + 4)$ and $h(r) = V(1)V(2r)$. Therefore, using (3.12) and (3.13) we obtain,
\[
Y \cap B_Y(x_0, R + 2) \subset B_Y(x_0, r) \quad \text{and} \quad v(B_Y(x_0, r)) \leq h(r).
\]
Next we choose $\eta = \tilde{u}_r$ and estimate $(II)$ for this particular value of $\eta$. From Theorem 2.2 we have,
\[
(II) \leq 2^{2\sigma-1} r^{\sigma-1} h(r) \| \delta u \|_{\sigma, B_Y(x_0, r)}^\sigma.
\]
As $B_Y(x_0, r) \subset B_X(x_0, 2r),
\[
(II) \leq 2^{2\sigma-1} r^{\sigma-1} h(r) \| \delta u \|_{\sigma, B_X(x_0, 2r) \cap Y}^\sigma.
\]
(3.14) implies that
\[
(II) \leq 2^{3\sigma-2} C(1 + V(3))^2 V(6.5) V(1) V(2r) \int_{B(x_0, 2r+3)} g_u^\sigma d\mu.
\]
\[
(II) \leq 2^{3\sigma-2} C(1 + V(3))^2 V(1) V(6.5) V(2r) \int_{B(x, 2r+4)} g_u^\sigma d\mu.
\]
Since $R \geq 1$ from (3.15) we have,
\[
(II) \leq 2^{3\sigma-2} C(1 + V(3))^3 V(6.5) V(2r) \int_{B(x, 2r+4)} g_u^\sigma d\mu.
\]
Let $\lambda = V(4.5) + 1$ and $R \geq 4\lambda$. Then putting the value of $r$ in (3.16) we have,
\[
\int_{B(x, R)} |u - \tilde{u}_r|^\sigma d\mu \leq 2^{3\sigma-2} CV(6.5) \lambda R^{\sigma-1} V(2\lambda R) \int_{B(x, 2\lambda R)} g_u^\sigma d\mu.
\]
Now the required result follows from the following inequality.
\[
\int_{B(x, R)} |u - u_R|^\sigma d\mu \leq 2^{\sigma} \inf_{\tau \in \mathbb{R}} \int_{B(x, R)} |u - \tau|^\sigma d\mu \leq \int_{B(x, R)} |u - \tilde{u}_r|^\sigma d\mu.
\]
It would be interesting to improve the Poincaré constants we obtained in Theorem 1.1 and Theorem 3.1. Moreover, it is not clear to us how the Poincaré constant changes if the ball of radius $2\lambda R$ on the right-hand side of the inequality is replaced by a ball of radius $R$. 
4. Poincaré inequalities on measured metric spaces with polynomial growth

If the growth function in (1.1) is a polynomial, then as a consequence of Theorem 1.1 or Theorem 3.1, the growth of the Poincaré constant is also polynomial. We discuss this case in this section.

Corollary 4.1. Let \((M, g)\) be a complete Riemannian manifold with dimension \(n\) and \(\text{Ric} \geq -kg\) for some \(k > 0\). If \(\frac{\text{Vol}(B(x, R))}{\text{Vol}(B(x, \frac{1}{2}))} \leq V_0 R^\sigma\) for all \(R \geq r > 0\) then for any \(\sigma \geq 1\) there exist constants \(C_0(n, k, r, \sigma, V_0, \alpha) > 0\) and \(\lambda(V_0, \alpha) \geq 1\) such that

\[
\int_{B(x, R)} |u - u_R|\sigma \, dv_g \leq C_0 R^{\alpha + \sigma - 1} \int_{B(x, \lambda R)} |\nabla u|\sigma \, dv_g, \quad \forall u \in C^1(M), \forall R \geq r
\]

where \(\nabla u, dv_g\) denote the gradient of \(u\) and the volume form of \((M, g)\) respectively.

Proof. From Theorem 1.14 in [3], if \(\text{Ric} \geq -kg\) then \((M, g)\) satisfies a local Poincaré inequality (1.2) and the Poincaré constant \(C(n, k, R)\) depends on \(k, n\) and \(R\). Let \(C = \sup_{R \leq 1} C(n, k, R)\). Then \((M, g)\) satisfies a Poincaré inequality as in (1.2). Now the result follows from Theorem 1.1.

The above result is proved in [3] under an additional assumption of a positive lower bound on unit balls. In [10], Croke and Karcher gave examples of complete Riemannian manifolds with positive Ricci curvature such that the infimum of unit balls is zero. The main theorem in [3] does not hold in this case. More generally, let us consider a measured metric space \((X, d, \mu)\).

Definition 4. A measure \(\mu\) is called doubling for \(r \geq r_1\) if there exists \(C_0 \geq 1\) such that

\[
\frac{\mu(B(x, 2r))}{\mu(B(x, r))} \leq C_0, \quad \forall x \in X, \quad \forall r \geq r_1.
\]

\(\mu\) is doubling if the growth of volume is polynomial.

Corollary 4.2. Let \((X, d, \mu)\) be a measured metric space. Suppose \((X, d, \mu)\) satisfies \(P_{loc}\) for \(r_0, \sigma \geq 1\) as in (1.2) and \(\mu\) is doubling for \(R > 0\) with the doubling constant \(C_0\). Then there exist positive constants \(s(C_0), C_1(\sigma, C_0, C)\) and \(\lambda(C_0) \geq 1\) such that for any \(u \in C(X)\) and its upper gradient \(g_u\),

\[
\int_{B(x, R)} |u - u_R|\sigma \, d\mu \leq C_1 R^{\sigma + s - 1} \int_{B(x, \lambda R)} g_u^\sigma \, d\mu, \quad \forall R \geq 2\lambda,
\]

where \(C\) is the constant term in \(P_{loc}\).

Proof. If \(\mu\) is doubling, then from Lemma 5.2.4 in [24],

\[
\frac{\mu(B(x, R))}{\mu(B(x, \frac{1}{2}))} \leq C_0^2 R^s \quad \text{with} \quad s = \frac{\log C_0}{\log 2}.
\]
for all \( x \in X \) and \( R \geq \frac{1}{2} \). Define,

\[
f(R) = C_0^2 R^s, \quad \forall R \geq \frac{1}{2};
\]

\[
e = 1 \text{ otherwise.}
\]

Then \( \frac{\mu(B(x,R))}{\mu(B(x,\frac{1}{2}))} \leq f(R), \) for all \( R > 0 \). Hence the result follows from Theorem 1.1.

G. Carron showed that a complete Riemannian manifold satisfying the doubling condition has finitely many ends in [8]. Riemannian manifolds satisfying \( P_{loc} \) and the doubling condition also satisfy the parabolic Harnack inequality and Li-Yau type heat kernel estimates [12]. The above corollary gives a curvature-free criterion on a Riemannian manifold with doubling measure to support a \( (\sigma, \beta, \sigma) \)-type uniform Poincaré inequality. Some interesting examples of measured metric spaces which are not Riemannian manifolds but satisfy the criterion of Corollary 4.2 are as follows. Topological manifolds with Ahlfors regular measures, which also satisfy a local contractibility condition, support \( P_{loc} \) [25]. Carnot groups with Carnot-Carathéodory metrics satisfy \( P_{loc} \) and a polynomial growth condition. We refer to Sections 10 and 11 in [15] for more details on these two types of examples.

**Corollary 4.3.** Let \((X, d, \mu)\) be a measured metric space which satisfies \( P_{loc} \) for \( r_0, \sigma \geq 1 \) as in (1.2) and \( \mu(B(x, \frac{1}{2})) \geq 1 \). Suppose \( \mu \) is doubling for \( R \geq r_1 \) with the doubling constant \( C_0 \) and \( \mu(B(x, r_1)) \leq V_0, \forall x \in X \). Then there exist positive constants \( s(C_0), C_1(\sigma, C_0, C, V_0) \) and \( \lambda(C_0, V_0) \geq 1 \) such that for any \( u \in C(X) \) and its upper gradient \( g_u \),

\[
\int_{B(x,R)} |u - u_R|^{\sigma} \, d\mu \leq C_1 R^{\sigma s-1} \int_{B(x,\lambda R)} g_u^\sigma \, d\mu, \quad \forall R \geq 2\lambda,
\]

where \( C \) is the constant term in \( P_{loc} \).

**Proof.** If \( \mu \) is doubling then for all \( R \geq r \) and \( x \in X \) from Lemma 5.2.4 in [24],

\[
\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq C_0 \left( \frac{R}{r} \right)^s \text{ with } s = \frac{\log C_0}{\log 2},
\]

Define,

\[
f(R) = V_0, \quad \forall \frac{1}{2} \leq R \leq r_1;
\]

\[
e = \frac{V_0}{r^s} R^s, \quad \forall R > r_1.
\]

Now the required result is an immediate consequence of Theorem 3.1.

Hajlasz and Koskela established a different type of global Poincaré inequality on metric spaces with doubling measures, which satisfies a chain condition \( C(\lambda, M) \) for \( \lambda, M \geq 1 \) in [14]. They also showed that every ball in a geodesic metric space satisfies this chain condition for every \( \lambda \geq 1 \). Now, consider a subset \( \Omega \) on a geodesic metric space with a doubling measure, which satisfies the chain condition \( C(\lambda, M) \) for some \( \lambda, M \geq 1 \). From Theorem 1 in [14], \( \Omega \) supports a Poincaré inequality in the sense of [14] if a Poincaré inequality holds on every ball \( B \) with
We observe that a \( \delta \)-hyperbolic space for some \( \delta \geq 0 \) satisfies \( DV_{loc} \) if for any \( r > 0 \) there exists a constant \( C(r) > 0 \) such that

\[
\frac{\mu(B(x, 2r))}{\mu(B(x, r))} \leq C(r), \quad \forall x \in X.
\]

We observe that \((X, d, \mu)\) satisfies \( DV_{loc} \) for \( r \geq \frac{1}{2} \) if and only if there exists \( f : (0, \infty) \to (0, \infty) \) such that \((1.1)\) holds. Next we study Poincaré inequalities on \((X, d, \mu)\) when \( f(r) \) grows exponentially in Sect. 5.

### 5. Poincaré inequality on Gromov hyperbolic spaces

Complete simply connected Riemannian manifolds with negative sectional curvature have exponential growth of volume. Gromov-hyperbolic spaces are generalizations of them. A Gromov hyperbolic space is a \( \delta \)-hyperbolic space for some \( \delta \geq 0 \). \( \delta \)-hyperbolic spaces are defined in various ways, but the value of \( \delta \) changes if we change the definition. As theorems in this section depend on the precise value of \( \delta \), we consider the following definition of \( \delta \)-hyperbolic spaces in this paper.

**Definition 5.** Let \((X, d)\) be a proper geodesic metric space. A geodesic triangle \( T \) is called \( \delta \)-thin if any point on one of its sides is contained in the \( \delta \)-neighbourhood of the union of the other two sides. \((X, d)\) is called a \( \delta \)-hyperbolic space if all of its geodesic triangles are \( \delta \)-thin.

Let \( \Gamma \) be a finitely generated group with a finite set of generators \( S \). \((\Gamma, S)\) is called a hyperbolic group if the Cayley graph \( \Gamma / S \) is \( \delta \)-hyperbolic as a metric space. If a discrete group \( \Gamma \) acts on a \( \delta \)-hyperbolic space properly and co-compactly, then \( \Gamma \) is a hyperbolic group.

**Theorem 5.1.** Let \((X, d, \mu)\) be a measured \( \delta \)-hyperbolic space which supports \( P_{1oc} \) for some \( r_0, \sigma \geq 1 \) as in \((1.2)\). Let \( \Gamma \) be a discrete group acting on \( X \) isometrically and properly such that the diameter of the quotient space \( \Gamma \setminus X \) is bounded by \( D \geq 1 \). Suppose the action of \( \Gamma \) is measure preserving, the entropy of \((X, d, \mu)\) is bounded by \( H \) and \( \mu(B(x, \frac{1}{2})) \geq 1 \) for all \( x \in X \). Then there exist \( C_0(\delta, D, H, \mu) > 0 \) and \( \lambda \geq 1 \) such that for any \( u \in C(X) \) and its upper gradient \( g_u \),

\[
\int_{B(x, R)} |u - u_R|^{\sigma} d\mu \leq \frac{2^{4\sigma} + 6HD + \frac{25}{4} + \frac{41}{4} + 6HD C V_0}{5^6HD + \frac{25}{4} + 6HD + \frac{25}{4} e^{H(12\lambda - D)}} \times R^{\sigma \delta + 6HD + \frac{21}{4} e^{12\lambda H R}} \int_{B(x, 2\lambda R)} g_u^{\sigma} d\mu,
\]

\( \forall R \geq \frac{5\lambda}{4} \) and \( \forall z \in X \) where \( r = 7D + 4\delta, V_0 = \sup_{x \in X} \{\mu(B(x, \frac{5\lambda}{2}))\} \) and \( \lambda = V_0 + 1 \).
Proof. Let $V_0 = \sup \{ \mu(B(x, \frac{5r}{2})) | x \in X \}$. Since the action of $\Gamma$ is co-compact, $V_0$ is finite. From Theorem 1.9 part (i) in [4] we have,

$$\mu(B(x, R)) \leq \left(\frac{2}{5}\right)^{\frac{25}{4} + 6HD} \frac{3V_0R^{\frac{25}{4} + 6HD}e^{6HR}}{r^{\frac{25}{4} + 6HD}e^{H(12r-D)}}.$$

Define,

$$V(R) = V_0, \quad \forall \ R < \frac{5r}{2};$$

$$= \left(\frac{2}{5}\right)^{\frac{25}{4} + 6HD} \frac{3V_0R^{\frac{25}{4} + 6HD}e^{6HR}}{r^{\frac{25}{4} + 6HD}e^{H(12r-D)}}, \quad \forall \ R \geq \frac{5r}{2}. $$

Since $D \geq 1, r \geq 7, \frac{5r}{2} \geq 6.5$. Hence $V(4.5) = V(6.5) = V_0$. Now from Theorem 3.1 for all $R \geq \frac{5r}{2},$

$$\int_{B(x,R)} |u(z) - u_R|^\sigma d\mu(z) \leq 2^{4\sigma-2} C\lambda^{2\sigma+1} R^{\sigma-1} V(2\lambda R) \int_{B(x,2\lambda R)} g_u(z) \sigma d\mu(z) \leq \left(\frac{2}{5}\right)^{\frac{25}{4} + 6HD} \frac{3V_0R^{\frac{25}{4} + 6HD}e^{6HR}}{r^{\frac{25}{4} + 6HD}e^{H(12r-D)}} \int_{B(x,2\lambda R)} g_u(z) \sigma d\mu(z).$$

Since $\Gamma \setminus X$ is compact $\inf_{x \in X} \mu(B(x, \frac{1}{2})) > 0$. If this quantity is less than 1, one can scale the metric suitably to obtain the required lower bound. Clearly, if $H$ is zero then the Poincaré constant in the above theorem is a polynomial in $R$ and $(X, d, \mu)$ satisfies a $(\sigma, \beta, \sigma)$-type uniform Poincaré inequality as stated in Sect. 4. $X$ is called elementary if the boundary of $X$ contains at most two elements. If $X$ is elementary, then the entropy of the space is zero (see Proposition 8.43 in [4]). Therefore the Poincaré constant of an elementary $\delta$-hyperbolic space grows polynomially. We recall Proposition 1.3 from [4] on non-elementary $\delta$-hyperbolic spaces.

**Proposition 5.1.** [4] For every non-elementary $\delta$-hyperbolic metric space $(X, d)$ and for every group $\Gamma$ acting on it properly by isometries, if the diameter of $\Gamma \setminus X \leq D < \infty$ then,

$$\text{Ent}(X, d, \mu) \geq \frac{\ln 2}{27\delta + 10D}$$

where $\mu$ is any measure on $(X, d)$ preserved by the action of $\Gamma$.

Therefore, when $(X, d)$ is non-elementary the growth of the Poincaré constant is exponential. Theorem 5.1 only gives an upper bound of the best constant for which a global Poincaré inequality on $(X, d, \mu)$ holds. It would be interesting to know if the best Poincaré constant also grows exponentially in this case.

If a discrete group acts on a Riemannian manifold $(M, g)$ as mentioned in Theorem 5.1, then the Ricci curvature of $M$ is bounded. Hence $(M, g)$ supports $P_{loc}.$
Consequently, \((M, g)\) satisfies a global Poincaré inequality as stated in Theorem 5.1. A large class of compact singular Riemannian spaces with almost smooth metrics also support a local Poincaré inequality. K. Akutagawa, G. Carron, and R. Mazzeo showed that compact stratified spaces with iterated edge metrics satisfy local Poincaré and Sobolev inequality \cite{2}. We refer to \cite{1,5} for a detailed geometric description of stratified spaces, examples, and more analytic properties on them. If the quotient space \(\Gamma \backslash X\) in Theorem 5.1 is a stratified space then \(X\) support a \(P_{loc}\) by Theorem 6.1. Hence \(X\) satisfies a global Poincaré inequality. Consequently, \(\delta\)-hyperbolic polyhedral spaces with piece-wise smooth Riemannian metric admitting a co-compact group action as stated in Theorem 5.1 support a global Poincaré inequality.

When \((X, d)\) is a \(\delta\)-hyperbolic graph an assumption on local Poincaré inequality is not required.

**Corollary 5.1.** Let \(\Gamma\) be a \(\delta\) hyperbolic Cayley graph of a hyperbolic group \(G\) equipped with a measure \(\mu\). Let \(|g_u|_\sigma\) denote the point-wise \(l^\sigma\) -norm of the gradient of a function \(u\) on \(\Gamma\). Suppose \(c \leq \mu(x) \leq C\) for all \(x \in \Gamma\) and the entropy of \((\Gamma, \mu)\) is bounded by \(H\). Then for \(R \geq r = 10(1 + \delta)\), \(\sigma \geq 1\) and \(u : \Gamma \to \mathbb{R}\)

\[
\int_{B(p, R)} |u - u_R|^\sigma d\mu \leq \frac{2^{\sigma+2}Cv(r)}{cr^{\frac{2\sigma+2}{4}}} R^{\sigma+\frac{21}{4}} e^{6HR} \int_{B(p, R)} |g_u|^\sigma d\mu, \forall p \in X
\]

where \(v(r)\) is the number of elements of \(\Gamma\) in a ball of radius \(r\).

**Proof.** Let \(\nu\) be the counting measure on \(\Gamma\). Then \(\nu\) is a \(G\)-invariant measure on \(\Gamma\). Hence, for any \(R > 0\), \(\nu(B(x, R))\) is same for all \(x \in \Gamma\). We define \(\nu(R) = \nu(B(x, R))\). \(G\) acts on the set of vertices of \(\Gamma\) transitively. Hence the diameter of \(G \backslash \Gamma\) bounded above by 1. Therefore, from Theorem 1.9 part (ii) in \cite{4} we have,

\[
\nu(R) < 3v(r) \left(\frac{R}{r}\right)^{\frac{25}{4}} e^{6H(R - \frac{4r}{5})}, \forall R \geq r = 10(1 + \delta).
\]

Since \(\mu(B(x, R)) \leq Cv(R)\), applying Theorem 2.2 for all \(R \geq r\) we have,

\[
\int_{B(p, R)} |u(x) - u_R|^\sigma d\mu(x) \leq \frac{2^{\sigma+2}Cv(r)}{cr^{\frac{2\sigma+2}{4}}} R^{\sigma+\frac{21}{4}} e^{6H(R - \frac{4r}{5})} \int_{B(p, R)} |g_u|^\sigma d\mu(x)
\]

\[
\leq \frac{2^{\sigma+2}Cv(r)}{cr^{\frac{2\sigma+2}{4}}} R^{\sigma+\frac{21}{4}} e^{6HR} \int_{B(p, R)} |g_u|^\sigma d\mu(x).
\]

Hence the proof follows.

Examples of hyperbolic groups are fundamental groups of negatively curved manifolds, finitely generated free groups, or any discrete group acting on a \(\delta\)-hyperbolic space properly and isometrically such that the quotient space is compact. Hyperbolic groups play an essential role in geometric group theory \cite{6}. A general \(\delta\)-hyperbolic graph satisfies the following Poincaré inequality.
Corollary 5.2. Let \((X, d, \mu)\) be a \(\delta\)-hyperbolic graph. Let \(\Gamma\) be a discrete group acting on \(X\) isometrically and properly such that the diameter of the quotient space \(\Gamma \backslash X\) is bounded by \(D \geq 1\). Let \(|g_u|_\sigma\) denote the point-wise \(L^\sigma\)-norm of the gradient of a function \(u\) on \(\Gamma\). Suppose the action of \(\Gamma\) is measure preserving, and the entropy of \((X, d, \mu)\) is bounded by \(H\). Then for any \(u \in C(X), \forall R \geq \frac{5r}{2}\) and \(\forall z \in X\),
\[
\int_{B(x, R)} |u - u_R|^\sigma d\mu \leq \frac{2^\sigma + 8 + 6HD_c V_0}{56(1 + HD)r^6 + 6HD e^{6H} R^6 + 4 + 6HD e^{6H} R^6 + 4 + 6HD e^{6H} R^6} R^{\frac{25}{4} + 6HD e^{6H} R} \int_{B(x, R)} |g_u|^\sigma d\mu
\]
where \(r = 7D + 4\delta\), \(V_0 = \sup_{x \in X} \{\mu(B(x, \frac{5r}{2}))\}\), and \(\frac{1}{c} = \inf_{x \in X} \mu(x)\).

Proof. Let \(V_0 = \sup\{\mu(B(x, \frac{5r}{2}))\}|x \in X\). Since the action of \(\Gamma\) is co-compact \(V_0\) is finite. From Theorem 1.9 part (i) in [4] we have, for all \(R \geq \frac{5r}{2}\),
\[
\mu(B(x, R)) \leq \left(\frac{2}{\frac{5r}{2}}\right)^{25 + 6HD} \frac{3V_0 R^{25 + 6HD} e^{6H} R}{r^{25 + 6HD} e^{H(12r - D)}},
\]
Define,
\[
f(R) = V_0, \quad \forall R \leq \frac{5r}{2};
\]
\[
= \left(\frac{2}{\frac{5r}{2}}\right)^{25 + 6HD} \frac{3V_0 R^{25 + 6HD} e^{6H} R}{r^{25 + 6HD} e^{H(12r - D)}}, \quad \forall R > \frac{5r}{2}.
\]
Since \(\Gamma \backslash X\) is compact and the action of \(\Gamma\) is measure preserving, \(\inf_{x \in X} \mu(x) > 0\). Now, as a consequence of Theorem 2.2 we have,
\[
\int_{B(p, R)} |u(x) - u_R|^\sigma d\mu(x) \leq 2^{\sigma} c R^{\sigma - 1} f(R) \int_{B(p, R)} |g_u|^\sigma(y) d\mu(y)
\]
\[
\leq \frac{2^{\sigma} + 8 + 6HD_c V_0}{56(1 + HD)r^6 + 6HD e^{6H} R^6 + 4 + 6HD e^{6H} R^6} R^{\frac{25}{4} + 6HD e^{6H} R} \int_{B(p, R)} |g_u|^\sigma(y) d\mu(y).
\]

6. Uniform Poincaré inequalities and growth of groups

Relations between the growth of groups, volume, and curvature motivate us to understand the dependence of the growth of Poincaré constants on the growth of groups. Let \(\Gamma\) be a discrete subgroup of isometries of a measured metric space \((X, d, \mu)\) acting on it freely and properly such that \(\Gamma \backslash X\) is compact. The point-wise systole is defined as \(\text{sys} \Gamma(x) = \inf_{y \neq x} d(x, y)\). The systole \(\text{sys} \Gamma\) of \(X\) is the infimum of \(\text{sys} \Gamma(x)\) over \(x \in X\). Since \(\Gamma \backslash X\) is compact \(\text{sys} \Gamma\) is non-zero. Define the quotient metric \(\bar{d}\) on \(\Gamma \backslash X\) as follows. For \(x, y \in \Gamma \backslash X\) choose \(\bar{x} \in p^{-1}(x)\) and \(\bar{y} \in p^{-1}(y)\). Then
\[
\bar{d}(x, y) = \inf_{\gamma \in \Gamma} d(\bar{x}, \gamma \bar{y}) = \inf_{\gamma_1, \gamma_2 \in \Gamma} d(\gamma_1 \bar{x}, \gamma_2 \bar{y}). \tag{6.1}
\]

The quotient map \(p\) is a covering map and \(p : B(\bar{x}, \frac{\text{sys} \Gamma}{4}) \to B(x, \frac{\text{sys} \Gamma}{4})\) is an isometry for all \(x \in \Gamma \backslash X\) and \(\bar{x} \in p^{-1}(x)\). Suppose the action of \(\Gamma\) is measure-preserving. Define, the quotient Borel measure \(\bar{\mu}\) on \(X \backslash \Gamma\) such that \(p\) restricted to each \(B(\bar{x}, \frac{\text{sys} \Gamma}{4})\) is measure-preserving for all \(\bar{x} \in X\).
Lemma 6.1. Suppose \((X \setminus \Gamma, \bar{d}, \bar{\mu})\) satisfies a Poincaré inequality, i.e. for a fixed \(\sigma \geq 1\) there exist a constant \(C(R) > 0\) such that for any \(u \in C(X)\) and its upper gradient \(g_u\),
\[
\int_{B(x, R)} |u - u_R|_\sigma^\sigma \, d\bar{\mu} \leq C(R) \int_{B(x, R)} g_u^\sigma \, d\bar{\mu}, \quad \forall R \in \left(0, \frac{\text{sys}_\Gamma}{4}\right]
\tag{6.2}
\]
then \((X, \bar{d}, \bar{\mu})\) satisfies \(P_{loc}\) for any \(R \leq \frac{\text{sys}_\Gamma}{4}\).

Proof. Consider a continuous function \(\tilde{u} : X \to \mathbb{R}\) and an upper gradient \(g_{\tilde{u}}\) of \(\tilde{u}\). Given \(R < \frac{\text{sys}_\Gamma}{4}\) choose \(R'\) such that \(R < R' < \frac{\text{sys}_\Gamma}{4}\) and a bump function \(\phi : X \to \mathbb{R}\) such that \(\phi = 1\) on \(B(\tilde{x}, R)\) and \(\phi = 0\) outside \(B(\tilde{x}, R')\). Define, \(u : \Gamma \setminus X \to \mathbb{R}\) as
\[
u(y) = \sum_{\tilde{y} \in p^{-1}(y)} \tilde{u}(\tilde{y})\phi(\tilde{y}).
\]
Corresponding to the upper gradient \(g_{\tilde{u}}\) we can define an upper gradient for \(u\) on \(B(x, R)\) as
\[
g_u(y) = \sum_{\tilde{y} \in p^{-1}(y)} g_{\tilde{u}}(\tilde{y})\phi(\tilde{y}).
\]
To see that \(g_u\) is indeed an upper gradient for \(u\) on \(B(x, R)\) choose a unit speed curve \(\alpha : [0, 1] \to B(x, R)\) and let \(\tilde{\alpha}\) be its lift via the map \(p\) passing through \(B(\tilde{x}, R)\).

\[
|u(\alpha(1)) - u(\alpha(0))| = |u(\tilde{\alpha}(1)) - u(\tilde{\alpha}(0))| \leq \int_0^1 g_{\tilde{u}}(\tilde{\alpha}(t)) dt = \int_0^1 g_u(\alpha(t)) dt.
\]
Let \(C = \sup\{C(R) : R \leq \frac{\text{sys}_\Gamma}{4}\}\). Now we have the required local Poincaré inequality as
\[
\int_{B(\tilde{x}, R)} |\tilde{u} - \tilde{u}_R|_\sigma^\sigma \, d\bar{\mu} = \int_{B(x, R)} |u - u_R|_\sigma^\sigma \, d\bar{\mu} \leq C \int_{B(x, R)} g_u^\sigma \, d\bar{\mu} = C \int_{B(\tilde{x}, R)} g_{\tilde{u}}^\sigma \, d\mu.
\]

Suppose \(\Gamma\) is a discrete group acting on \((X, \bar{d})\) freely, properly, and isometrically. Then recall that
\[
F_{\Gamma}(R) = |\Gamma x \cap \overline{B(x, R)}|.
\]

Theorem 6.1. Consider a measured metric space \((X, d, \mu)\). Let \(\Gamma\) be a discrete subgroup of isometries of \((X, d, \mu)\) acting on it freely and properly such that \(\Gamma \setminus X\) is compact and \(\text{sys}_\Gamma \geq 4\). Suppose \(\mu(B(x, \frac{1}{2})) \geq 1\) for all \(x \in X\). Let the diameter and the volume of \(\Gamma \setminus X\) be bounded above by \(D\) and \(V_0\), respectively. If \((\Gamma \setminus X, \bar{d}, \bar{\mu})\) satisfies a Poincaré inequality (6.2) then for any \(u \in C(X)\) and its upper gradient \(g_u\),
\[
\int_{B(x, R)} |u - u_R|_\sigma^\sigma \, d\mu \leq 2^{2\sigma - 2} CV_0\lambda^4 R^{\sigma - 1} F_{\Gamma}(2\lambda R) \int_{B(x, 2\lambda R)} g_u^\sigma \, d\mu \tag{6.3}
\]
for all \(R \geq 4\lambda\) and \(\sigma \geq 1\), where \(\lambda = V_0 + D\).
Let $\lambda(\cdot)$ and above by $\lambda(\cdot)$ and $\kappa$ respectively. Then there exist positive constants $C$ and $K$ such that for any $u \in \mathcal{C}^1(M)$ and $R \geq 4\lambda$,
\[
\int_{B(x, R)} |u - u_R|^\sigma \, dV_g \leq C R^{-\sigma} F(2\lambda R) \int_{B(x, 2\lambda R)} |\nabla u|^\sigma \, dV_g, \quad \forall \ x \in M
\]
where $dV_g$ is the volume form induced from $g$.

**Proof.** Consider a Riemannian manifold $(M, g)$ and a group $\Gamma$ which satisfy the assumptions of Corollary 6.1. Then for any $x \in M$, $\{B(y, x, D)\}_{y \in \Gamma}$ covers $M$. By continuity of Riemannian curvature, sectional curvatures of $(M, g)$ are bounded on each $\overline{B(y, x, D)}$. Since $\Gamma$ acts isometrically, sectional curvatures of $(M, g)$ are bounded. Let $\kappa \leq \sec \leq K$. From Theorem 1.14 in [3] there exists a constant $C(n, \kappa, R) > 0$ such that for any $u \in \mathcal{C}^1(M)$,

\[
\int_{B(x, R)} |u - u_R|^\sigma \, dV_g \leq C(n, \kappa, R) \int_{B(x, R)} |\nabla u|^\sigma \, dV_g, \quad \forall \ R > 0, \ \forall \ x \in M.
\]

Let $C = \sup_{0 < R \leq 1} C(n, \kappa, R)$. Hence $(M, g)$ satisfies $P_{loc}$ as in (1.2). Let $Vol(B(x, R))$ denote the volume of $B(x, R)$. Since $\Gamma$ acts isometrically,

\[
\sup_{x \in M} Vol(B(x, R)) = \sup_{y \in \overline{B(x, D)}} \{Vol(B(y, R)) : y \in \overline{B(x, D)}\}.
\]

Hence it is bounded above. Similarly $\inf_{x \in M} Vol(B(x, R))$ is also bounded below by a positive constant for any $R > 0$. From Lemma 3.5 in [4] for any $R > 0$,

\[
Vol(B(x, R)) \leq Vol(B(x, D)) F_{\Gamma}(R + D).
\]
Let $V^K_n(R)$ denote the volume of a ball of radius $R$ in the space of constant curvature $K$. From the Bishop-Gromov volume comparison theorem $\text{Vol}(B(x, D)) \leq V^K_n(D)$ and $\text{Vol}(B(x, \frac{1}{2})) \geq V^K_n(\frac{1}{2})$ for all $x \in M$. Hence,

$$\text{Vol}(B(x, R)) \leq V^K_n(D)F_{\Gamma}(R + D).$$

Now the required result follows from Theorem 3.1. \hfill \Box

When $M$ is simply connected, Corollary 6.1 shows the dependence of the growth of the Poincaré constant on the growth of the fundamental group of the quotient space. If $(M, g)$ is the universal cover of a compact Riemannian manifold with non-negative Ricci curvature, then the growth of volume and the growth of the fundamental group $\Gamma$, is polynomial [21]. Hence, $(M, g)$ satisfies a $(\sigma, \beta, \sigma)$-uniform Poincaré inequality as in Corollary 3.2. If $(M, g)$ is the universal cover of a compact negatively curved Riemannian manifold, then the growth of the fundamental group $\Gamma$ is exponential [21].

More generally, consider a simply connected measured metric space $(X, d, \mu)$. As a consequence of Theorem 6.1, the growth of the Poincaré constant with respect to $R$ is polynomial (or exponential) if the growth of $\Gamma$ is polynomial (or exponential). Given a fixed measured metric space $(X, d, \mu)$, multiple groups may act isometrically, properly on $X$, preserving the measure $\mu$. In this case, the group with minimal growth gives a better Poincaré constant. It would be interesting to find out how the best Poincaré constant of $(X, d, \mu)$ is related to the growth of a group with minimal growth.

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**Declarations**

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

**Data availability** Data sharing not applicable to this article as nodatasets were generated or analysed during the current study.

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