ON VERY GENERIC DISCRIMINANTAL ARRANGEMENTS

AUTHOR: C.P. ANIL KUMAR*

ABSTRACT. In this article we prove two main results. Firstly, we show that any six-line arrangement, consisting of three pairs of mutually perpendicular lines, does not give rise to a “very generic or sufficiently general” discriminantal arrangement in the sense of C. A. Athanasiadis [2]. We give two proofs of the first result. The second result is as follows. The codimension-one boundary faces of (a region) a convex cone of a very generic discriminantal arrangement has not been characterised and is not known even though the intersection lattice of a very generic discriminantal arrangement is known. So secondly, we show that the number of simplex cells of the very generic hyperplane arrangement $H_n^m = \{ H_i : \sum_{j=1}^m a_{ij} x_j = c_i, 1 \leq i \leq n \}$ may not be not precisely equal to the number of codimension-one boundary hyperplanes of $\mathbb{R}^n$ of the convex cone $C$ containing $(c_1, c_2, \ldots, c_n)$ in the associated very generic discriminantal arrangement. That is, for $1 \leq i_1 < i_2 < \ldots < i_m < i_{m+1} \leq n$, if $\Delta_{H_{i_1}H_{i_2} \ldots H_{i_m}H_{i_{m+1}}}$ is a simplex cell of the hyperplane arrangement $H_n^m$ then it need not give rise to a codimension-one boundary hyperplane of the convex cone $C$ containing $(c_1, c_2, \ldots, c_n)$ in the associated very generic discriminantal arrangement. We finally mention an interesting open-ended remark before the appendix section.

In the appendix section we give a self contained exposition and describe combinatorially the intersection lattice of a (Zariski open and dense) class of “very generic or sufficiently general” discriminantal arrangements. An important ingredient involved in this exposition is the Crapo’s characterization ([5], Section 6, Page 149, Theorem 2) of the matroid of circuits of the configuration of $n$ “generic” points in $\mathbb{R}^m$ as an application of Hall’s marriage theorem and a result in linear algebra about a minor and its complementary minor of an orthogonal matrix. We mention this characterization here in detail. Another aspect of this exposition is Conjecture 4.3 stated in M. Bayer and K. Brandt [3] and its proof which is given in C. A. Athanasiadis [2]. We mention this aspect also in detail. As a consequence, we give a geometric description of the lattice elements as sets of concurrencies of the hyperplane arrangements which give the same “very generic or sufficiently general” discriminantal arrangement.

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*The work is done when the author is a Post Doctoral Fellow at HRI, Allahabad.
1. Introduction

Concurrency Geometries have been studied by H. H. Crapo in [4] in order to solve problems on configurations of hyperplane arrangements in a Euclidean space. Imagine a finite set \( H^m_n \) of \( n \)-hyperplanes in \( \mathbb{R}^m \) for some \( n, m \in \mathbb{N} \) which can move freely and whose normal directions arise from a fixed finite set \( N \subset \mathbb{R}^m \) of cardinality \( n \) which is generic, that is, any subset \( M \subset N \) of cardinality at most \( m \) is linearly independent. They give rise to different hyperplane arrangements in \( \mathbb{R}^m \). The combinatorial aspects of such configurations of hyperplane arrangements is studied by associating with them another arrangement which is known in the literature as Discriminantal arrangements or Manin-Schechtman arrangements (refer to Page 205, Section 5.6 in P. Orlik and H. Terao [14]). Some of the authors who have worked on the discriminantal arrangements are C. A. Athanasiadis [2], M. Bayer and K. Brandt [3], M. Falk [8], Yu. I. Manin and V. V. Schechtman [13] and more recently A. Libgober and S. Settepanella [11].

In this paper we prove in first main Theorem A that, a six-line arrangement, consisting of three pairs of mutually perpendicular lines, does not give rise to a “sufficiently general” discriminantal arrangement in the sense of C. A. Athanasiadis [2]. In second main Theorem B, we prove that there exists a generic six-line arrangement \( H^2_6 = \{ H_i : a_{i1}x_1 + a_{i2}x_2 = c_i, 1 \leq i \leq 6 \} \) which give rise to very generic discriminantal arrangement \( C^6_6 = \{ M_{\{i_1, i_2, i_3\}} | 1 \leq i_1 < i_2 < i_3 \leq 6 \} \) such that not every triangular cell (2-simplex region) of \( H^2_6 \) gives rise to a codimension-one boundary of the convex cone \( C \) containing \( (c_1, c_2, c_3, c_4, c_5, c_6) \) in the associated very generic discriminantal arrangement \( C^6_6 \).

2. Definitions and Statements of the main results

We begin with some definitions before stating the main results.

**Definition 2.1** (A Hyperplane Arrangement, An Essential Hyperplane Arrangement, A Generic Hyperplane Arrangement, A Central Hyperplane Arrangement).

Let \( m, n \) be positive integers. We say a set

\[ H^m_n = \{ H_1, H_2, \ldots, H_n \} \]

of \( n \) affine hyperplanes in \( \mathbb{R}^m \) forms a hyperplane arrangement. We say that the hyperplane arrangement \( H^m_n \) is essential if the normals of the hyperplanes \( H_i, 1 \leq i \leq n \) span \( \mathbb{R}^m \). We say that they form a generic hyperplane arrangement or a hyperplane arrangement in general position, if Conditions 1,2 are satisfied.
• Condition 1: For $1 \leq r \leq m$, the intersection of any $r$ hyperplanes has dimension $m - r$.

• Condition 2: For $r > m$, the intersection of any $r$ hyperplanes is empty.

We say that the hyperplane arrangement $\mathcal{H}_m^n$ is central if $\bigcap_{i=1}^n H_i \neq \emptyset$.

Remark 2.2. A very generic hyperplane arrangement is defined below in Definition 2.6 and more specifically in Definition 7.13.

Definition 2.3 (Discriminantal Arrangement-A Central Arrangement). Let $m, n$ be positive integers. Let

$$\mathcal{H}_m^n = \{H_1, H_2, \ldots, H_n\}$$

be a generic hyperplane arrangement of $n$ hyperplanes in $\mathbb{R}^m$. Let the equation for $H_i$ be given by

$$\sum_{j=1}^m a_{ij} x_j = b_i, \text{ with } a_{ij}, b_i \in \mathbb{R}, 1 \leq j \leq m, 1 \leq i \leq n.$$ 

For every $1 \leq i_1 < i_2 < \ldots < i_{m+1} \leq n$ consider the hyperplane $M_{\{i_1, i_2, \ldots, i_{m+1}\}}$ passing through the origin in $\mathbb{R}^n$ in the variables $y_1, y_2, \ldots, y_n$ whose equation is given by

$$\text{Det}\begin{pmatrix}
        a_{i_11} & a_{i_12} & \cdots & a_{i_1(m-1)} & a_{i_1m} & y_{i_1} \\
        a_{i_21} & a_{i_22} & \cdots & a_{i_2(m-1)} & a_{i_2m} & y_{i_2} \\
        \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
        a_{i_{m-1}1} & a_{i_{m-1}2} & \cdots & a_{i_{m-1}(m-1)} & a_{i_{m-1}m} & y_{i_{m-1}} \\
        a_{i_{m}1} & a_{i_{m}2} & \cdots & a_{i_{m}(m-1)} & a_{i_{m}m} & y_{i_m} \\
        a_{i_{m+1}1} & a_{i_{m+1}2} & \cdots & a_{i_{m+1}(m-1)} & a_{i_{m+1}m} & y_{i_{m+1}}
\end{pmatrix} = 0$$

Then the associated discriminantal arrangement of hyperplanes passing through the origin in $\mathbb{R}^n$ is given by

$$C_n^{\mathcal{H}_m^n}_{m+1} = \{M_{\{i_1, i_2, \ldots, i_{m+1}\}} \mid 1 \leq i_1 < i_2 < \ldots < i_{m+1} \leq n\}.$$ 

It is a central arrangement consisting of hyperspaces, that is, linear subspaces of codimension one in $\mathbb{R}^n$.

Note 2.4. Even though the definition of hyperplanes $M_{\{i_1, i_2, \ldots, i_{m+1}\}}$ of the discriminantal arrangement $C_n^{\mathcal{H}_m^n}_{m+1}$ involves the coefficients $[a_{ij}]_{1 \leq j \leq m, 1 \leq i \leq n}$ of the variables $x_i, 1 \leq i \leq m$ we can pick and fix any one set of equations for the hyperplanes $H_i, 1 \leq i \leq n$ of the hyperplane arrangement to associate the discriminantal arrangement.
Theorem 2.3 in [2] intersection lattice has been already characterised by C. A. Athanasiadis (refer to that any subset $V \subseteq U$ of cardinality at most $m$ is a linearly independent set. We fix the matrix $[a_{ij}]_{1 \leq i \leq n, 1 \leq j \leq m} \in M_{n \times m}(\mathbb{R})$. Let $H^m_n = \{H_1, H_2, \ldots, H_n\}$ be any hyperplane arrangement such that the normal vector of $H_i$ is $v_i, 1 \leq i \leq n$. When we write equations for the hyperplane $H_i$, we use the fixed matrix and write

$$H_i : \sum_{j=1}^{m} a_{ij}x_j = b_i \text{ for some } b_i \in \mathbb{R}.$$ 

Using this coefficient matrix, we define the discriminant arrangement $C^n_m$ which depends only on set $\mathcal{U}$. Any such hyperplane arrangement gives a vector $(b_1, b_2, \ldots, b_n)$ and conversely any vector $(b_1, b_2, \ldots, b_n)$ gives such a hyperplane arrangement $\mathcal{H}^m_n$. If the vector $(b_1, b_2, \ldots, b_n)$ lies in the interior of a cone of the discriminant arrangement $C^n_m$, then the hyperplane arrangement $\mathcal{H}^m_n$ is generic. The combinatorics of such arrangements $\mathcal{H}^m_n$ is studied by using the discriminant arrangement $C^n_m$.

The combinatorics of the discriminantal arrangement – more precisely the intersection lattice of the discriminantal arrangement – has been studied very well. The intersection lattice for a certain class of “sufficiently general” (Definition 2.1 in [2]) discriminant arrangements which maximises the $f$-vector of the intersection lattice has been already characterised by C. A. Athanasiadis (refer to Theorem 2.3 in [2]). An important ingredient in its proof is the Crapo’s characterisation of the matroid $M(n, m)$ of circuits of the configuration of $n$-generic points in $\mathbb{R}^m$. This matroid is introduced in H. H. Crapo [4] and characterised in H. H. Crapo [5], Chapter 6, when the coordinates of the $n$-points are generic indeterminates, as the Dilworth completion of $D_m(B_n)$ of the $m^{th}$-lower truncation of the Boolean algebra of rank $n$ (see H. H. Crapo and G. C. Rota [6], Chapter 7). The intersection lattice of “sufficiently general” discriminantal arrangements coincides with the lattice $L(n, m)$ of flats of $M(n, m)$. In C. A. Athanasiadis [2], it is proved that this lattice is isomorphic to the lattice $P(n, m)$ (refer to Theorem 3.2 in [2]). $P(n, m)$ is the collection of all sets of the form $S = \{S_1, S_2, \ldots, S_r\}$, where $S_i \subseteq \{1, 2, \ldots, n\}$, each of cardinality at least $m + 1$, such that

$$| \bigcup_{i \in I} S_i | > m + \sum_{i \in I} (| S_i | - m)$$

for all $I \subseteq \{1, 2, \ldots, r\}$ with $| I | \geq 2$. They partially order $P(n, m)$ by letting $\{S_1, S_2, \ldots, S_r\} = S \leq T = \{T_1, T_2, \ldots, T_p\}$, if, for each $1 \leq i \leq r$ there exists $1 \leq j \leq p$ such that $S_i \subseteq T_j$. This isomorphism between the lattices $L(n, m)$ and
\( P(n, m) \) was initially conjectured (refer to Definition 4.2 and Conjecture 4.3) in M. Bayer and K. Brandt [3].

**Definition 2.6 (Generic Sets, Very Generic Sets and Very Generic Hyperplane Arrangements).** Let \( m, n \) be positive integers. A finite set \( \mathcal{U} = \{ v_i = (a_{i1}, a_{i2}, \ldots, a_{im}) \mid 1 \leq i \leq n \} \subset \mathbb{R}^m \) is said to be generic if any subset \( \mathcal{V} \subseteq \mathcal{U} \) of cardinality at most \( m \) is a linearly independent set. A generic set \( \mathcal{U} \) is said to be very generic if it gives rise to a “sufficiently general” discriminantal arrangement in sense of C. A. Athanasiadis (Definition 2.1 in [2]). Also refer to Definition 7.13 for more details. This implies that the discriminantal arrangement is “very generic” in the sense of M. Bayer and K. Brandt (Definition 4.2 in [3]) by using Theorem 2.3 in [2]. A generic hyperplane arrangement \( \mathcal{H}^m_n = \{ H_i : \sum_{j=1}^{m} a_{ij} x_j = b_i \mid 1 \leq i \leq n \} \) in \( \mathbb{R}^m \) is very generic if \( \mathcal{U} = \{ v_i = (a_{i1}, a_{i2}, \ldots, a_{im}) \mid 1 \leq i \leq n \} \subset \mathbb{R}^m \) is a very generic set. The space \( \mathcal{O}(n, m) \) of very generic hyperplane arrangements is zariski open and dense in the space of all generic hyperplane arrangements.

We state the main theorems.

**Theorem A.** Let \( \mathcal{M} = \{ v_i \mid 1 \leq i \leq 6 \} \subset \mathbb{R}^2 \) be a generic set such that the set \( \mathcal{M} \) gives rise to six-line arrangements where there are three pairs of mutually perpendicular lines. Then \( \mathcal{M} \) is not very generic, that is, it gives rise to a discriminantal arrangement which is not a “sufficiently general” discriminantal arrangement in the sense of C. A. Athanasiadis (Definition 2.1 in [2]).

**Remark 2.7.** We will give two proofs of Theorem A. The first one uses elementary techniques and only uses the fact that the space of “sufficiently general” discriminantal arrangements is an open set in the space of discriminantal arrangements associated to line arrangements where the lines are in general position and a topological fact in Remark 3.5 which can be deduced. This proof also reveals much of the content of Theorem A. The second proof uses the result of C. A. Athanasiadis [2] that the intersection lattice of a “sufficiently general” discriminantal arrangement is isomorphic to the lattice \( P(n, m) \) (refer to Theorem 3.2 in [2]). This proof does not reveal much, the content of Theorem A.

**Theorem B.** There exists a generic six-line arrangement \( \mathcal{H}^2_6 = \{ H_i : a_{i1} x_1 + a_{i2} x_2 = c_i, 1 \leq i \leq 6 \} \) which give rise to a very generic or a sufficiently general discriminantal arrangement \( \mathcal{C}^6_{\binom{6}{4}} = \{ M_{\{i_1, i_2, i_3\}} \mid 1 \leq i_1 < i_2 < i_3 \leq 6 \} \), that is, the set \( \mathcal{V} = \{ (a_{i1}, a_{i2}) \mid 1 \leq i \leq 6 \} \) is a very generic set, such that not every triangular cell (2-simplex region) of \( \mathcal{H}^2_6 \) gives rise to a codimension-one boundary of the convex cone \( C \) containing \( (c_1, c_2, c_3, c_4, c_5, c_6) \) in the associated very generic discriminantal arrangement \( \mathcal{C}^6_{\binom{6}{3}} \).
3. First Proof of the First Main Theorem

We begin with a lemma.

**Lemma 3.1.** Let $0 < m_1 < m_2$ be two real numbers. Let $\mathcal{M} = \{v_1 = (0,1), v_2 = (-\frac{1}{m_1},1), v_3 = (-\frac{1}{m_2},1), v_4 = (1,0), v_5 = (m_1,1), v_6 = (m_2,1)\}$ be a generic set containing six vectors in $\mathbb{R}^2$. Then for any generic six-line arrangement $L_6^3 = \{L_1, L_2, L_3, L_4, L_5, L_6\}$ such that a normal vector of $L_i$ is $v_i$, there exists three subscripts $1 \leq i < j < k \leq 6$ such that the triangle $\Delta L_i L_j L_k$, that is, the triangle formed by the lines or sides $L_i, L_j, L_k$ is an acute-angled triangle.

**Proof.** We can assume that the line $L_1$ is the X-axis, $L_4$ is the Y-axis, $L_i$ has slope $m_{i-1}$ for $i = 2, 3, L_j$ has slope $-\frac{1}{m_{j-1}}$ for $j = 5, 6$. So $L_i \perp L_{i+3}$ for $i = 1, 2, 3$. Let $A = L_1 \cap L_2, B = L_1 \cap L_6, C = L_2 \cap L_6, D = L_4 \cap L_2, E = L_4 \cap L_6$. Then the angles $\angle CAB, \angle ABC, \angle CDE, \angle CED$ are acute angles. If $\angle ACB$ is also acute then the triangle $\Delta L_1 L_2 L_6$ is an acute-angled triangle. If $\angle ACB$ is obtuse then the angle $\angle DCE$ is acute. Hence the triangle $\Delta L_4 L_2 L_6$ is an acute-angled triangle. This proves the lemma.

Now we have the corollary given below follows using the above lemma.

**Corollary 3.2.** In any generic six-line arrangement with three pairs of mutually perpendicular lines, the lines can be individually translated to form an arrangement of six lines such that some three lines form an acute-angled triangle and the remaining three lines are the altitudes of the triangle.

Now we prove two propositions regarding such six-line arrangements.

**Proposition 3.3.** Consider Figure 1 which is obtained by translating the altitudes of the $\Delta L_1 L_3 L_5$ to $L_4, L_6, L_2$ in a manner given in the figure. Then the distance of the perpendicular dropped from $A = L_2 \cap L_6$ to the line $L_4$ is greater than the distance of the perpendicular dropped from $L_3 \cap L_5$ to the line $L_4$. The distance of the perpendicular dropped from $B = L_2 \cap L_4$ to the line $L_6$ is greater than the distance of the perpendicular dropped from $L_1 \cap L_5$ to the line $L_6$. The distance of the perpendicular dropped from $C = L_4 \cap L_6$ to the line $L_2$ is greater than the distance of the perpendicular dropped from $L_1 \cap L_3$ to the line $L_2$.

**Proof.** We prove that, the distance of the perpendicular dropped from $A = L_2 \cap L_6$ to the line $L_4$ is greater than the distance of the perpendicular dropped from $L_3 \cap L_5$ to the line $L_4$. The proof of the remaining two assertions are similar.

Let $L_1$ be the X-axis, $L_4$ be the Y-axis. Let the equation of $L_i$ be $y = m_{i-1}x + c_{i-1}$ for $i = 2, 3$ for some real numbers $m_1, m_2, c_1, c_2$. Then the equation of $L_j$ is $y = -\frac{1}{m_{j-4}}x + d_{j-4}$ for $j = 5, 6$ and some real numbers $d_1, d_2$. Now we observe
Figure 1. A Generic Six-Line Arrangement with Three Pairs of Mutually Perpendicular Lines

from the figure that \(0 < m_1 < m_2\) and using the points on the Y-axis we have \(c_2 > d_1 > c_1 > d_2 > 0\). Also we have \(-\frac{c_1}{m_1} < -\frac{c_2}{m_2} < 0 < d_2m_2 < d_1m_1\) using the points on the X-axis. The distance of the perpendicular dropped from \(A = L_2 \cap L_6\) to the line \(L_4\) is \(\frac{m_2(c_1 - d_2)}{1 + m_1m_2}\). The distance of the perpendicular dropped from \(L_3 \cap L_5\) to the line \(L_4\) is \(\frac{m_1(c_2 - d_1)}{1 + m_1m_2}\). Now we observe the following.

\[
\frac{c_1}{m_1} - \frac{c_2}{m_2} > 0 > \frac{d_2m_2 - d_1m_1}{m_1m_2} = \frac{d_2}{m_1} - \frac{d_1}{m_2} \Rightarrow \frac{c_1 - d_2}{m_1} > \frac{c_2 - d_1}{m_2} \Rightarrow
\]
\[
m_2(c_1 - d_2) > m_1(c_2 - d_1) \Rightarrow \frac{m_2(c_1 - d_2)}{1 + m_1m_2} > \frac{m_1(c_2 - d_1)}{1 + m_1m_2}.
\]

This proves the proposition.

\[\blacksquare\]

**Proposition 3.4.** Consider Figure 1 and Figure 2. Suppose the order of points

Figure 2. A Generic Six-Line Arrangement
If \( L_1 \perp L_4, L_2 \perp L_5, L_3 \perp L_6 \) then, as given in Figure 1, the order of points on lines \( L_2, L_4, L_6 \) as given in Figure 2 is not possible.

The order of points on lines \( L_2, L_4, L_6 \) as given in Figure 2 is not possible.

Proof. It is easy to see that given the order of points on \( L_1, L_3, L_5 \), the points of intersections \( L_2 \cap L_4, L_4 \cap L_6, L_6 \cap L_2 \) lie in the interior of the triangle \( \Delta L_1 L_3 L_5 \). This can be proved as follows. Consider the convex quadrilateral \( PQRS \) whose vertices are \( P = L_4 \cap L_6, Q = L_2 \cap L_5, R = L_1 \cap L_4, S = L_3 \cap L_2 \). This quadrilateral lies in the interior of the triangle \( \Delta L_1 L_3 L_5 \) and the lines \( L_2, L_4 \) form the diagonals of the quadrilateral. Hence \( L_2, L_4 \) lies in the interior of the triangle \( \Delta L_1 L_3 L_5 \). The cases of the points \( L_4 \cap L_6, L_2 \cap L_6 \) are similar.

Now there are two possibilities for orders of points on lines \( L_2, L_4, L_6 \) as shown in Figure 1 and Figure 2. We need to show that if \( L_1 \perp L_4, L_2 \perp L_5, L_3 \perp L_6 \) then the orders of points on lines \( L_2, L_4, L_6 \) given in Figure 1 occur and the orders of points on lines \( L_2, L_4, L_6 \) given in Figure 2 does not occur.

We prove this as follows. Assume that \( L_1 \perp L_4, L_2 \perp L_5, L_3 \perp L_6 \) and \( L_1 \) is the X-axis and \( L_4 \) is the Y-axis. Assume without loss of generality by using a reflection about the X-axis, that the triangle \( \Delta L_1 L_3 L_5 \) is in the closed upper half-plane. Now assume without loss of generality by using a reflection about the Y-axis, that the extreme point \( L_1 \cap L_5 \) on \( L_1 \) lies on the positive X-axis and the extreme point \( L_1 \cap L_2 \) on \( L_1 \) lies on the negative X-axis. Note that under reflections, the orders of intersection points on the lines do not change.

Now the point \( L_2 \cap L_5 \) lies in the Quadrant I, the point \( L_5 \cap L_6 \) lies in Quadrant IV, the points \( L_3 \cap L_5, L_3 \cap L_6, L_3 \cap L_2 \) lie in Quadrant II, the points \( L_2 \cap L_1, L_3 \cap L_1 \) lie on the negative X-axis, the points \( L_1 \cap L_6, L_1 \cap L_5 \) lie on the positive X-axis. The points \( L_i \cap L_4 \) for \( i = 2, 3, 5, 6 \) lie on the positive Y-axis.

Now we observe that the order of intersection points on a large circle \( C \) enclosing all the intersection points \( L_i \cap L_j, 1 \leq i < j \leq 6 \) in an anticlockwise manner is the \( L_1 \cap C, L_2 \cap C, L_3 \cap C, L_4 \cap C, L_5 \cap C, L_6 \cap C \). Also if we let the equation of \( L_i \) be given by \( y = m_i x + c_i \) for \( i = 2, 3 \) then we have \( 0 < m_1 < m_2, 0 < c_1 < c_2 \) and the equation of the line \( L_j \) is given by \( y = -\frac{1}{m_j} x + d_j \) for \( j = 5, 6 \) for some \( 0 < d_2 < d_1 \).

Using the order of intersection points on the X-axis, we get \( -\frac{c_1}{m_1} < -\frac{c_2}{m_2} < 0 < d_2 m_2 < d_1 m_1 \). Using the orders of intersection points on the lines \( L_3 \) and \( L_5 \), we
also have $c_2 > d_1 > \max(c_1, d_2) > 0$. Now we observe

$$\frac{c_1}{m_1} - \frac{c_2}{m_2} > 0 > \frac{d_2m_2 - d_1m_1}{m_1m_2} = \frac{d_2}{m_1} - \frac{d_1}{m_2} \Rightarrow \frac{c_1 - d_2}{m_1} > \frac{c_2 - d_1}{m_2} \Rightarrow$$

$$m_2(c_1 - d_2) > m_1(c_2 - d_1) \Rightarrow \frac{m_2(c_1 - d_2)}{1 + m_1m_2} > \frac{m_1(c_2 - d_1)}{1 + m_1m_2} > 0.$$

So we obtain that $c_1 > d_2$ and the point $L_2 \cap L_6$ lies in the Quadrant II since its X-coordinate is $-\frac{m_2(c_1 - d_2)}{1 + m_1m_2}$ which is negative.

Now we conclude that the orders of intersection points on the lines $L_2, L_4, L_6$ is as given in Figure 1 and not as given in Figure 2. This proves the proposition. \[\blacksquare\]

Now we prove the first main theorem after a remark.

**Remark 3.5.** The main idea behind the first proof of Theorem A is that if a line arrangement $\mathcal{L}_n^2$ arises from a discriminantal arrangement $\mathcal{C}_n^m$ which is “sufficiently general” which in turn is an open condition, a line arrangement $\tilde{\mathcal{L}}_n^2$ obtained by small perturbations of the slopes of the lines in $\mathcal{L}_n^2$ must also arise from a “sufficiently general” discriminantal arrangement $\mathcal{C}_n^{m'}$. Moreover, all the finitely many configurations of line arrangements arising from $\mathcal{C}_n^{m'}$ are also given by the line arrangements arising from $\mathcal{C}_n^{m''}$ and vice versa.

**Proof of Theorem A.** Consider the open set $\mathcal{U}(6, 2)$ in $(\mathbb{R}^3)^6$ given by

$$\mathcal{U}(6, 2) = \{((a_i, b_i, c_i))_{i=1}^6 \in (\mathbb{R}^3)^6 \mid \det\begin{pmatrix} a_i & b_i & c_i \\ a_j & b_j & c_j \\ a_k & b_k & c_k \end{pmatrix} \neq 0, 1 \leq i < j < k \leq 6, \det\begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix} \neq 0, 1 \leq i < j \leq 6\}. $$

If $((a_i, b_i, c_i))_{i=1}^6 \in \mathcal{U}(6, 2)$ then the line arrangement $\mathcal{L}_6^2 = \{L_i : a_ix + b_iy = c_i \mid 1 \leq i \leq 6\}$ is a generic line arrangement. Let $\mathcal{V}(6, 2) \subset \mathcal{U}(6, 2)$ denote the open and zariski dense subset of those points $((a_i, b_i, c_i))_{i=1}^6 \in \mathcal{U}(6, 2)$ such that the generic set $\mathcal{E} = \{(a_i, b_i) \mid 1 \leq i \leq 6\}$ give rise to sufficiently general discriminantal arrangements in the sense of C. A. Athanasiadis (Definition 2.1 in [2]). It is an open subset using Proposition 2.2 in [2]. Let $\mathcal{G}(6, 2) = \{((a_i, b_i))_{i=1}^6 \in (\mathbb{R}^2)^6 \mid \det\begin{pmatrix} a_i & b_i \\ a_j & b_j \end{pmatrix} \neq 0, 1 \leq i < j \leq 6\}$. It is an open subset in $(\mathbb{R}^2)^6$.

Consider the projection $\pi : (\mathbb{R}^3)^6 \to (\mathbb{R}^2)^6$ which is an open map given by $((a_i, b_i, c_i))_{i=1}^6 \to ((a_i, b_i))_{i=1}^6$. Then we have a surjection $\pi : \mathcal{U}(6, 2) \to \mathcal{G}(6, 2)$. Now $V = \pi(\mathcal{V}(6, 2))$ is an open set in $\mathcal{G}(6, 2)$. 


Let $\mathcal{M} = \{v_i = (a_i, b_i) \mid 1 \leq i \leq 6\}$ be a generic set which gives rise to three pairs of mutually perpendicular lines. Suppose $((a_i, b_i))_{i=1}^6 \in V$. Assume, using Corollary 3.2 that, we have after renumbering the subscripts, an acute-angled triangle $\Delta L_1 L_2 L_3$ and $L_4 \perp L_1, L_5 \perp L_2, L_6 \perp L_3$ where the direction cosines of $L_i$ is either $\frac{1}{\sqrt{a_i^2+b_i^2}}(a_i, b_i)$ or $-\frac{1}{\sqrt{a_i^2+b_i^2}}(a_i, b_i), 1 \leq i \leq 6$. If $((a_i, b_i))_{i=1}^6$ gives rise to Figure 1 which is in the set $\pi^{-1}(((a_i, b_i))_{i=1}^6) \subset \pi^{-1}(V)$, then after a small perturbation of $((a_i, b_i))_{i=1}^6$ in $V$ to $((a'_i, b'_i))_{i=1}^6$, that is, there exists an open set $W \subset V, ((a_i, b_i))_{i=1}^6 \in W, ((a'_i, b'_i))_{i=1}^6 \in W$ such that $((a'_i, b'_i))_{i=1}^6$ gives rise to Figure 2 in $\pi^{-1}(((a'_i, b'_i))_{i=1}^6) \subset \pi^{-1}(W)$ where the orders of points on all the lines in Figure 2 are represented by those of some element in $\pi^{-1}(((a_i, b_i))_{i=1}^6)$. Now this is a contradiction to Proposition 3.4. Hence main Theorem A follows.

4. Second Proof of the First Main Theorem

Now as mentioned in Remark 2.7, we use the full strength of C. A. Athanasiadis [2] result in the second proof of Theorem A.

Proof. If we have a “sufficiently general” discriminantal arrangement $\mathcal{C}^n_{m+1}$, then the hyperplane arrangements that arise from it have the following nice property. Let $\mathcal{H}^m_n$ be any hyperplane arrangement that arises from $\mathcal{C}^n_{m+1}$. Let $S = \{S_1, S_2, \ldots, S_r\}$ be the sets of concurrencies that exist in $\mathcal{H}^m_n$. Then we have that $S \in P(n, m)$, that is,

$$| \bigcup_{i \in I} S_i | > m + \sum_{i \in I}(| S_i | - m)$$

for all $I \subseteq \{1, 2, \ldots, r\}$ with $| I | \geq 2$.

Now in the case of a line arrangement of six lines in the plane which form a triangle with three altitudes, there are four sets of concurrencies where three lines concur at each of these four points. If $S_i, i = 1, 2, 3, 4$ are the sets of concurrences then we have $| \bigcup_{i=1}^4 S_i | = 6$ and $2 + \sum_{i=1}^4(| S_i | - 2) = 2 + (3 - 2) + (3 - 2) + (3 - 2) + (3 - 2) = 6$. Hence such a configuration cannot arise from a “sufficiently general” discriminantal arrangement. This proves Theorem A.

Remark 4.1. The same conclusion as in the second proof of Theorem A can be obtained for six lines forming a quadrilateral and two diagonals. In fact there is a projective transformation which takes a line arrangement of six lines which form a triangle with the three altitudes to a line arrangement of six lines which form a quadrilateral and two diagonals. Also refer to Theorem 8, page 163 in H. H. Crapo [5].
5. Computational Verification

In this section we computationally verify and cohere with the fact that indeed a generic six-line arrangement, which consists of three pairs of mutually perpendicular lines, does not give rise to a very generic discriminantal arrangement with two cases of examples. For computing the number of convex cones of a discriminantal arrangement, we compute its characteristic polynomial. The method of computing the characteristic polynomial for hyperplane arrangements is a well established method. Articles by T. Zaslavsky [16], [17], F. Ardila [1], E. Katz [9], and books by A. Dimca [7], P. Orlik & H. Terao [14], R. Stanley [15], in which this concept is explained, are relevant.

Assume without loss of generality that, \( L_1 \) is the X-axis, \( L_4 \) is the Y-axis and \( L_i \) has slope \( m_i \) for \( i = 2, 3, 5, 6 \) with \( m_5 = -\frac{1}{m_2} > m_6 = -\frac{1}{m_3} > m_1 = 0 > m_2 > m_3 \).

We consider two cases: \( m_2 = \frac{1}{m_3} \) and \( m_2 \neq \frac{1}{m_3} \).

5.1. The Case \( m_2 = \frac{1}{m_3} \). We mention an example here.

Example 5.1.
Consider the following generic six-line arrangement \( L_6^2 = \{L_1, L_2, L_3, L_4, L_5, L_6\} \) given by

\[
L_1 : x_1 = 0, L_2 : 2x_1 + 3x_2 = -2, L_3 : 3x_1 + 2x_2 = 3,  \\
L_4 : x_2 = 0, L_5 : 3x_1 - 2x_2 = 5, L_6 : 2x_1 - 3x_2 = 5.
\]

We have \( L_1 \perp L_4, L_2 \perp L_5, L_3 \perp L_6 \). The \( \binom{6}{3} = 20 \) hyperplanes of its discriminantal arrangement in \( \mathbb{R}^6 \) is given by

\[
\begin{align*}
M_{\{1,2,3\}} &= \{(y_1, y_2, y_3, y_4, y_5, y_6) \in \mathbb{R}^6 | -5y_1 - 2y_2 + 3y_3 = 0\}, \\
M_{\{1,2,4\}} &= \{(y_1, y_2, y_3, y_4, y_5, y_6) \in \mathbb{R}^6 | 2y_1 - y_2 + 3y_4 = 0\}, \\
M_{\{1,2,5\}} &= \{(y_1, y_2, y_3, y_4, y_5, y_6) \in \mathbb{R}^6 | -13y_1 + 2y_2 + 3y_5 = 0\}, \\
M_{\{1,2,6\}} &= \{(y_1, y_2, y_3, y_4, y_5, y_6) \in \mathbb{R}^6 | -12y_1 + 3y_2 + 3y_6 = 0\}, \\
M_{\{1,3,4\}} &= \{(y_1, y_2, y_3, y_4, y_5, y_6) \in \mathbb{R}^6 | 3y_1 - y_3 + 2y_4 = 0\}, \\
M_{\{1,3,5\}} &= \{(y_1, y_2, y_3, y_4, y_5, y_6) \in \mathbb{R}^6 | -12y_1 + 2y_3 + 2y_5 = 0\}, \\
M_{\{1,3,6\}} &= \{(y_1, y_2, y_3, y_4, y_5, y_6) \in \mathbb{R}^6 | -13y_1 + 3y_3 + 2y_6 = 0\}, \\
M_{\{1,4,5\}} &= \{(y_1, y_2, y_3, y_4, y_5, y_6) \in \mathbb{R}^6 | -3y_1 + 2y_4 + y_5 = 0\}, \\
M_{\{1,4,6\}} &= \{(y_1, y_2, y_3, y_4, y_5, y_6) \in \mathbb{R}^6 | -2y_1 + 3y_4 + y_6 = 0\}, \\
M_{\{1,5,6\}} &= \{(y_1, y_2, y_3, y_4, y_5, y_6) \in \mathbb{R}^6 | -5y_1 + 3y_5 - 2y_6 = 0\},
\end{align*}
\]
Now the characteristic polynomial of a very generic discriminantal arrangement

\[ \chi_M \] is the characteristic polynomial of \( A \). Formula 5.1 can be derived from Proposition 3.11.3 on Page 283 of R. Stanley [15]. We take here \( A = C_6^6 = B \) to simplify notation. Here \( B \) denotes the \((20 \times 6)\) matrix of coefficients of the variables \( y_1, \ldots, y_6 \) in the hyperplanes \( M_{i<j<k}, 1 \leq i < j < k \leq 6 \). Then we can compute the ranks of all submatrices corresponding to any finite subset \( B \) of the rows of \( B = A \), and use the above formula to compute \( r(C_6^6) = (-1)^6 \chi_{C_6^6}(-1) \).

Upon computation we obtain that there are

\[ r(C_6^6) = (-1)^6 \chi_{C_6^6}(-1) = 884 \text{ convex cones}. \]

In this example we have the slopes \( m_i \) of the lines \( L_i, 1 \leq i \leq 6 \) are given by \( m_1 = 0, m_2 = -\frac{2}{3}, m_3 = -\frac{3}{2}, m_4 = \infty, m_5 = \frac{3}{2}, m_6 = \frac{2}{3} \) with \( m_2 = \frac{1}{m_5} \).

Now the characteristic polynomial of a very generic discriminantal arrangement corresponding is given by

\[
\chi_{\text{Very Generic}}(x) = x^6 - 20x^5 + 145x^4 - 426x^3 + 300x^2 = x^2(x-1)(x^3 - 19x^2 + 126x - 300).
\]

For this we may refer to Y. Numata, A. Takemura [12] and H. Koizumi, Y. Numata, A. Takemura [10]. Hence we have

\[ r_{\text{Very Generic}} = (-1)^6 \chi_{\text{Very Generic}}(-1) = 892 \neq 884. \]
5.2. The Case $m_2 \neq \frac{1}{m_3}$. We mention another example here.

Example 5.2. If we consider six lines $L_i, 1 \leq i \leq 6$ with slopes $m_i, 1 \leq i \leq 6$ respectively such that $m_1 = 0, m_2 = -1, m_3 = -2, m_4 = \infty, m_5 = 1, m_6 = \frac{1}{2}$ then we have $m_2 \neq \frac{1}{m_3}$ and a similar computation as in the previous case yields $888 \neq 892$ convex cones of its associated discriminant arrangement.

6. Proof of the Second Main Theorem

We begin with a proposition.

 Proposition 6.1. Consider the generic line arrangement given in Figure 3. Here $L_1$ is the X-axis, $L_4$ is the Y-axis. The equations for $L_1, L_2, L_3, L_4, L_5, L_6$ are given as follows.

$$L_1 : y = 0, L_2 : x - 2y = 7, L_3 : -2x + y = 4,$$

$$L_4 : x = 0, L_5 : 5x + y = 2, L_6 : x + y = -3.$$ 

Let $A = L_1 \cap L_5, B = L_1 \cap L_2, C = L_2 \cap L_5$. By translating the lines $L_i, 1 \leq i \leq 6$ in the plane and keeping the orders of the remaining points of intersections on each of the lines unchanged, the interchange of points $A, B$ on the line $L_1, B, C$ on the line $L_2$ and $A, C$ on the line $L_5$ cannot be done, that is, the orientation of the triangle $\Delta ABC$ cannot be flipped. Equivalently the point $C$ alone cannot be pushed to the first quadrant by the translations of the lines $L_2, L_3, L_5, L_6$ and keeping the orders of intersections of points the same on all the lines $L_i, 1 \leq i \leq 6$ except for the interchange of points $A, B$ on the line $L_1, B, C$ on the line $L_2$ and $A, C$ on the line $L_5$.

Proof. First of all we need not need to translate all the lines to prove the proposition. We fix the lines $L_1, L_4$ as X and Y-axes respectively. Upon translations of the lines $L_2, L_3, L_5, L_6$, let the equations of the lines be given by

$$L_1 : y = 0, L_2 : x - 2y = a, L_3 : -2x + y = b,$$

$$L_4 : x = 0, L_5 : 5x + y = c, L_6 : x + y = d.$$ 

Now we must have that $L_3 \cap L_5$ is in quadrant II, $L_3 \cap L_6, L_2 \cap L_3$ is in quadrant III and $L_2 \cap L_6, L_5 \cap L_6$ is in quadrant IV. We need to prove that $C = L_2 \cap L_5$ cannot be in quadrant I. Also we must have by observing the X-intercept and Y-intercept, $a > 0, b > 0, c > 0, d < 0$. By observing the order of points on the Y-axis we have $-\frac{a}{2} < d < 0 < c < b$. By observing the order of points on the X-axis we have $d < -\frac{b}{2} < 0$. This implies $0 < c < b < -2d < a$. Now $C = L_2 \cap L_5 = \left(\frac{a+2c}{11}, \frac{c-5a}{11}\right)$ and here we have $\frac{a+2c}{11} > 0$ and $\frac{c-5a}{11} < \frac{c-a}{11} < 0$. Hence the point $C$ must lie in the quadrant IV and not in quadrant I. This proves the proposition.

Now we prove the second main theorem.
Proof of Theorem B. If the set $\mathcal{U} = \{v_1 = (0, 1), v_2 = (1, -2), v_3 = (-2, 1), v_4 = (1, 0), v_5 = (5, 1), v_6 = (1, 1)\}$ given by Figure 3 is a very generic set then we are done. This is because the hyperplane $M_{\{1,2,5\}}$ of the associated discriminantal arrangement $C_{6 \choose (5)}$ is not a codimension-one boundary hyperplane of the cone $C$ containing the point $(b_1, b_2, b_3, b_4, b_5, b_6) = (0, 7, 4, 0, 2, -3)$ using Proposition 6.1.

If the set $\mathcal{U}$ is not very generic then we use a density argument. The space of sufficiently general discriminantal arrangements $C_{n \choose (m+1)}$ is dense in the space of discriminantal arrangements given by generic finite subsets of $\mathbb{R}^m$ of cardinality $n$. (Here $n = 6, m = 2, {n \choose (m+1)} = \binom{6}{3} = 20$). The space of very generic (ordered) sets in $(\mathbb{R}^2)^n$ is dense in the space of generic (ordered) sets in $(\mathbb{R}^2)^n$. There exists
a small neighbourhood $V$ of the point $U = (v_1, \ldots, v_6)$ in $(\mathbb{R}^2)^6$ which contains a very generic point $V = (w_1, \ldots, w_6) \in (\mathbb{R}^2)^6$, $w_i = (a_i, a_i)$, $1 \leq i \leq 6$ which gives rise to a very generic six-line arrangement \{\(L_i : a_i x_1 + a_i x_2 = c_i \mid 1 \leq i \leq 6\)\} with the constant coefficient vector $(c_1, \ldots, c_6)$ in a small neighbourhood of $(b_1, \ldots, b_6)$. For this very generic discriminantal arrangement given by $V$, the hyperplane $M_{\{1,2,5\}}$ is not a codimension-one boundary hyperplane of the cone $C$ containing the point $(c_1, \ldots, c_6)$. This follows since it is an open condition. This proves the second main theorem. ■

We finally mention a remark before the appendix section.

Remark 6.2. I believe that most of the discriminantal arrangements $\mathcal{C}_n^{m+1}$ which are “sufficiently general” satisfy the property that there is a convex cone $C$ such that codimension-one boundary hyperplanes of $\mathbb{R}^n$ of the convex cone $C$ containing $(c_1, c_2, \ldots, c_n)$ need not bijectively give rise to simplex cells of the hyperplane arrangement $\mathcal{H}_n^m$ corresponding to $(c_1, c_2, \ldots, c_n)$. However it is still an interesting exercise to characterize “sufficiently general” discriminantal arrangements where a correspondence exists.

7. Appendix

In the appendix section we discuss the intersection lattice of a (Zariski open and dense) class of “very generic” or “sufficiently general” discriminantal arrangements and give a combinatorial description of the lattice.

7.1. The matroid of circuits of the configuration of $n$ “generic” points in $\mathbb{R}^k$. First we mention a remark concerning the “generic” condition.

Remark 7.1. A “generic” set $O \subset \mathcal{X}$ in an affine or quasi-affine irreducible algebraic set $\mathcal{X} \subseteq \mathbb{R}^k$ is a nonempty Zariski open subset given by non-vanishing of certain finite collection of polynomials in $k$ variables. Hence such an open set $O$ is dense in $\mathcal{X}$.

Definition 7.2. A matroid $M$ is a ordered pair $(E, \mathcal{I})$ consisting of a finite set $E$ and collection $\mathcal{I}$ of subsets of $E$ having the following properties:

(1) $\emptyset \in \mathcal{I}$.
(2) If $I \in \mathcal{I}$, $I' \subseteq I$ then $I' \in \mathcal{I}$.
(3) If $I_1$ and $I_2$ are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there exists an element $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

The members of $\mathcal{I}$ are called independent sets of $M$. A subset of $E$ that is not in $\mathcal{I}$ is called dependent. A minimal dependent set is called a circuit of $M$. The set of all circuits of $M$ is denoted by $\mathcal{C}(M)$. A circuit of $M$ having $n$ elements is
called an $n$-circuit. A maximal independent set is called a basis of $M$. It follows that any two bases of $M$ has the same cardinality. The set of all bases of $M$ is denoted by $B(M)$.

**Example 7.3.** Let $V = \{v_i = (a_{i1}, a_{i2}, \ldots, a_{ik}) \mid 1 \leq i \leq n\} \subset \mathbb{R}^k$ be a set of $n$-vectors. Let $\mathcal{I}$ be the collection of the linearly independent subsets of $E$. Then $M[V] = M[A] = (V, \mathcal{I})$ is a matroid where $A = [a_{ij}]_{1 \leq i \leq n, 1 \leq j \leq k}$. It is called the vector matroid.

**Example 7.4.** Let $E = \{1, 2, \cdots, n\}$ for a positive integer $n$. Let $\mathcal{I} = \{X \subseteq E$ such that $|X| \leq k\}$. Then $G = (E, \mathcal{I})$ is a matroid. The matroid $G$ is called uniform matroid. It is also denoted by $U_{k,n}$. The collection of circuits of $U_{k,n}$ is given by $\mathcal{C}(U_{k,n}) = \{X \subseteq E$ such that $|X| = k + 1\}$. The collection of bases of $U_{k,n}$ is given by $\mathcal{B}(U_{k,n}) = \{X \subseteq E$ such that $|X| = k\}$.

**Remark 7.5.** For a set $V = \{v_1, v_2, \ldots, v_n\}$ of $n$-vectors in $\mathbb{R}^k$ if, in addition, any $k \leq n$ elements in $V$ are linearly independent then $M[V]$ is a vector matroid realizing the uniform matroid $U_{k,n}$. We say that the matroid $U_{k,n}$ is representable. Sometimes when there is no ambiguity, the matroid $M[V]$ is also denoted by $U_{k,n}$. In the collection of all possible ordered $n$-tuples of vectors in $\mathbb{R}^k$, that is, in the space $(\mathbb{R}^k)^n$, the subset of all ordered $n$-tuples of vectors such that, any $k \leq n$ of each $n$-tuple is linearly independent, is a “generic” set as it is given by non-vanishing of certain determinant polynomials.

Now we form another matroid on the collection $\mathcal{C}(U_{k,n})$ of circuits of the matroid $U_{k,n}$.

**Definition 7.6** (The matroid $D(U_{k,n})$, $D$ stands for Dilworth). The Dilworth matroid is given by $D(U_{k,n}) = (\mathcal{C}(U_{k,n}), \mathcal{D})$ where $\mathcal{D}$ is defined as follows. We say a collection $\mathcal{S} = \{C_1, C_2, \cdots, C_m\}$ of circuits of $U_{k,n}$ is independent in $D(U_{k,n})$, that is, $\mathcal{S} \in \mathcal{D}$ if for any non-empty subset $J \subseteq \{1, 2, \cdots, m\}$ we have $|\bigcup_{j \in J} C_j| \geq k + |J|$. The collection $\mathcal{B}(D(U_{k,n})) \subset \mathcal{D}$ of bases of $D(U_{k,n})$ is precisely those elements of $\mathcal{D}$ which have cardinality $n - k$.

**7.1.1. Construction of a Representation of the Dilworth Matroid $D(U_{k,n})$.** Now we construct, from a representation $M[V]$ of the uniform matroid $U_{k,n}$, a representation of the Dilworth matroid $D(U_{k,n})$ when $V$ is restricted in a “further generic” subset of $(\mathbb{R}^k)^n$. This will occupy a few pages below. Let $V = \{v_i = (a_{i1}, a_{i2}, \cdots, a_{ik}) \mid 1 \leq i \leq n\} \subset \mathbb{R}^k$ be denoted by the $(n \times k)$-matrix $A = [a_{ij}]_{1 \leq i \leq n, 1 \leq j \leq k}$ with all the $k \times k$ minors non-zero. Let $S = \{i_1 < i_2 < \cdots < i_k < i_{k+1}\} \subset \{1, 2, \cdots, n\}$ be a $(k + 1)$-subset. Let $X = [x_{ij}]_{1 \leq i \leq n, 1 \leq j \leq k}$
be a \( n \times k \) matrix of indeterminates and \( P_S(X) = \text{Det}([x_{ij}]_{1 \leq i,j \leq k}) \). The “generic” subset to which \( V \) belongs right now is given by

\[
\mathcal{O} = \{ A = [a_{ij}]_{1 \leq i \leq n, 1 \leq j \leq k} \mid P_S(A) = \text{Det}([a_{ij}]_{1 \leq i,j \leq k}) \neq 0 \}
\]

for every \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \} \subset \mathbb{R}^{kn} \).

Here \( \mathcal{O} \) is non-empty Zariski open and hence a dense subset of \( \mathbb{R}^{kn} \) as \( \mathbb{R}^{kn} \) is irreducible in Zariski topology.

Let us order the collection of ascendingly sorted \((k+1)\)-subsets of \( \{1, 2, \ldots, n\} \) in dictionary order. For example if \( n = 4, k = 2 \) then

\[
\{1, 2, 3\} < \{1, 2, 4\} < \{1, 3, 4\} < \{2, 3, 4\}.
\]

Let \( y_1, y_2, \ldots, y_n \) be \( n \) distinct indeterminates. For every \((k+1)\)-subset \( S = \{ i_1 < i_2 < \cdots < i_k < i_{k+1} \} \subset \{1, 2, \ldots, n\} \) consider the following linear polynomial in the \( n \) variables \( y_1, y_2, \ldots, y_n \).

\[
L_{s,x} = [x_{ij}]_{1 \leq i,j \leq k} (y_1, y_2, \ldots, y_n) = \text{Det} \begin{pmatrix}
  x_{i_11} & x_{i_21} & x_{i_31} & \cdots & x_{i_k1} & y_{i_1} \\
  x_{i_12} & x_{i_22} & x_{i_32} & \cdots & x_{i_k2} & y_{i_2} \\
  x_{i_13} & x_{i_23} & x_{i_33} & \cdots & x_{i_k3} & y_{i_3} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  x_{i_1k} & x_{i_2k} & x_{i_3k} & \cdots & x_{i_kk} & y_{i_k} \\
  x_{i_{k+1}1} & x_{i_{k+1}2} & x_{i_{k+1}3} & \cdots & x_{i_{k+1}k} & y_{i_{k+1}}
\end{pmatrix}
\]

Now form the \( \binom{n}{k+1} \times n \) matrix \( \text{Disc}(A) \) using the ordering mentioned on the collection of \((k+1)\)-subsets of \( \{1, 2, \ldots, n\} \) where the row corresponding to \((k+1)\)-subset \( S \) is the \( n \)-dimensional real coefficient vector of the linear polynomial

\[
L_{s,a} = [a_{ij}]_{1 \leq i \leq n, 1 \leq j \leq k} (y_1, y_2, \ldots, y_n).
\]

Note that each row of \( \text{Disc}(A) \) has \( (k+1) \) nonzero entries and the remaining \( n-k-1 \) entries are zero.

We will show that there exists a zariski open and hence dense subset \( \mathcal{U} \subset \mathcal{O} \subset \mathbb{R}^{kn} \) such that for \( A \in \mathcal{U} \) the vector matroid \( M[A] \) represents the uniform matroid \( U_{k,n} \) and the vector matroid \( M[\text{Disc}(A)] \) represents the Dilworth matroid \( D(U_{k,n}) \). This result is due to H. Crapo [5], Section 6, Page 149, Theorem 2.

Let \( X = [x_{ij}]_{1 \leq i \leq n, 1 \leq j \leq k} \) be the \( \binom{n}{k+1} \times n \) matrix where the row corresponding to the \((k+1)\)-subset \( S = \{ i_1 < i_2 < \cdots < i_k < i_{k+1} \} \subset \{1, 2, \ldots, n\} \) is the \( n \)-dimensional coefficient vector of the linear polynomial \( L_{s,x} = [x_{ij}]_{1 \leq i \leq n, 1 \leq j \leq k} (y_1, y_2, \ldots, y_n) \). Then we immediately observe that \( \text{Disc}(X)X = 0 \) the zero matrix of size \( \binom{n}{k+1} \times k \). This follows because the determinant of a square matrix with a repeated row or a repeated column is zero. So if \( A \in \mathcal{O} \) then the rows of \( \text{Disc}(A) \) are orthogonal to the
columns of $A$. Since $A$ has rank $k$ the rank of $\text{Disc}(A)$ is at most $n - k$. Let $n - k < l \leq n$. So the determinant of any $l \times l$ minor in $\text{Disc}(A)$ is zero. Since this is true for every $A \in \mathcal{O}$ and $\mathcal{O}$ is Zariski dense in $\mathbb{R}^{kn}$, the determinant polynomial in the variables $X = [x_{ij}]_{1 \leq i \leq n, 1 \leq j \leq k}$ of any $l \times l$ minor of $\text{Disc}(X)$ is an identically zero polynomial. What about $l \times l$ minors of the matrix $\text{Disc}(X)$ for $1 \leq l \leq n - k$? The following theorem answers this question to some extent.

**Theorem 7.7.** Let $S_1, S_2, \cdots, S_l$ be $(k+1)$-subsets of $\{1, 2, \cdots, n\}$ and suppose $\bigcup_{i=1}^l S_i = \{j_1, j_2, \ldots, j_m\}$.

1. For $1 \leq l < m - k$, the determinant polynomial of some $l \times l$ minor in the corresponding $l \times m$ submatrix of the $\left(\begin{smallmatrix} n \\ k+1 \end{smallmatrix}\right) \times n$ matrix $\text{Disc}(X)$ in the variable entries $X = [x_{ij}]_{1 \leq i \leq n, 1 \leq j \leq k}$ is not identically zero if the collection $\{S_1, S_2, \cdots, S_l\}$ is independent in the Dilworth matroid $D(U_{k,n})$.

2. For $l = m - k$, the determinant polynomial of any $l \times l$ minor in the corresponding $l \times m$ submatrix of the $\left(\begin{smallmatrix} n \\ k+1 \end{smallmatrix}\right) \times n$ matrix $\text{Disc}(X)$ in the variable entries $X = [x_{ij}]_{1 \leq i \leq n, 1 \leq j \leq k}$ is not identically zero if the collection $\{S_1, S_2, \cdots, S_l\}$ is independent in the Dilworth matroid $D(U_{k,n})$.

3. Also the determinant polynomial of any $l \times l$ minor in the corresponding $l \times m$ submatrix of the $\left(\begin{smallmatrix} n \\ k+1 \end{smallmatrix}\right) \times n$ matrix $\text{Disc}(X)$ in the variable entries $X = [x_{ij}]_{1 \leq i \leq n, 1 \leq j \leq k}$ is identically zero if the collection $\{S_1, S_2, \cdots, S_l\}$ is not independent in the Dilworth matroid $D(U_{k,n})$.

**Proof.** Because of the nature of the entries of the matrix $\text{Disc}(X)$ and the fact that any $l \times l$ minor is zero for $l > n - k$, it is enough to prove the theorem for $l = n - k$. So we assume that $l = n - k$. Even in this case $l = n - k$, if $\{S_1, S_2, \cdots, S_{n-k}\}$ is not independent in $D(U_{k,n})$ then it is clear that the determinant polynomial of any $(n - k) \times (n - k)$ minor in the corresponding $(n - k) \times m$ submatrix of the $\left(\begin{smallmatrix} n \\ k+1 \end{smallmatrix}\right) \times n$ matrix $\text{Disc}(X)$, in the variable entries $X = [x_{ij}]_{1 \leq i \leq n, 1 \leq j \leq k}$ is identically zero. An example is given in 7.10 illustrating this case. Also an example is given in 7.11 illustrating assertion Theorem 7.7(1).

So we assume that $\{S_1, S_2, \cdots, S_{n-k}\}$ is a basis of $D(U_{k,n})$ and so $m = n$. Here we produce a real matrix $A = [a_{ij}]_{1 \leq i \leq n, 1 \leq j \leq k}$ such that every $k \times k$ minor is nonzero such that for the associated matrix $\text{Disc}(A)$, in the $(n - k) \times n$ submatrix corresponding to the rows $S_1, S_2, \cdots, S_{n-k}$, every $(n - k) \times (n - k)$ minor is nonzero.

Consider $(k + 1)\left(\begin{smallmatrix} n \\ k+1 \end{smallmatrix}\right)$ variables $X_{a,S}$ for $a \in S$ and any $(k + 1)$-subset $S$ of $\{1, 2, \cdots, n\}$. Replace the nonzero polynomial entries of the matrix $\text{Disc}(X)$ by these variables in their exact positions. Let us denote the new matrix by $Y$. 
Claim 7.8. Any \((n-k) \times (n-k)\) minor of the \((n-k) \times n\) submatrix of \(Y\) corresponding to the rows \(S_1, S_2, \ldots, S_{n-k}\) is nonzero where \(\{S_1, S_2, \ldots, S_{n-k}\}\) is a basis of \(D(U_{k,n})\).

Proof of the Claim. This is proved by an application of Hall’s marriage theorem. Let 
\[ T = \{i_1 < i_2 < \cdots < i_{n-k}\} \subset \{1, 2, \ldots, n\} \] and denote 
\[ T^c = \{1, 2, \ldots, n\} \setminus \{i_1 < i_2 < \cdots < i_{n-k}\}. \] Then we have for any subset 
\[ J \subset \{1, 2, \ldots, (n-k)\}, \] 
\[ |\bigcup_{j \in J} (S_j \cap T)| = |(\bigcup_{j \in J} S_j) \cap T| = |(\bigcup_{j \in J} S_j)^c| \geq |\bigcup_{j \in J} S_j| - |T^c| \geq k + |J| - k = |J|. \] Hence Hall’s marriage condition is satisfied for the sets \(S_i \cap T, i = 1, 2, \ldots, n - k\). So there exists \(a_i \in S_i \cap T, 1 \leq i \leq n-k, a_i \neq a_j, 1 \leq i \neq j \leq n-k\). Hence there is a diagonal of variables \(X_{a_i, S_i}\) in the \((n-k) \times (n-k)\) in the submatrix corresponding to \(S_1, S_2, \ldots, S_{n-k}\) and 
\[ T = \{i_1 < i_2 < \cdots < i_{n-k}\} \] which give rise to a monomial term in determinant expansion of its minor of \(Y\). So the \((n-k) \times (n-k)\) minor of \(Y\) is nonzero. This holds for any set \(T\) of cardinality \(\frac{n}{2}\). Hence the claim follows.

Continuing with the proof of Theorem 7.7, we can choose a set of real numbers for the entries \(X_{a, S}\) in \(Y\) such that any \((n-k) \times (n-k)\) minor of the \((n-k) \times n\) submatrix of \(Y\) corresponding to the rows \(S_1, S_2, \ldots, S_{n-k}\) is nonzero where \(\{S_1, S_2, \ldots, S_{n-k}\}\) is a basis of \(D(U_{k,n})\). Let us denote this \((n-k) \times n\) submatrix by \(C\). Let \(D\) be a \(n \times k\) real matrix such that the columns of \(D\) span the \(k\)-dimensional space in \(\mathbb{R}^n\) which is orthogonal to the \((n-k)\)-dimensional row subspace of \(C\) in \(\mathbb{R}^n\). Then every \(k \times k\) minor of \(D\) is nonzero. This follows from the claim below.

Claim 7.9. Let \(Q\) be a non-singular matrix \(n \times n\) matrix such that first \(k\) rows \(q_1, q_2, \cdots, q_k\) span a space \(W_1 \subset \mathbb{R}^n\) and the remaining \((n-k)\) rows \(q_{k+1}, q_{k+2}, \cdots, q_n\) span a space \(W_2 \subset \mathbb{R}^n\). Assume also that \(W_1 \perp W_2\), that is, \(W_1\) is orthogonal to \(W_2\). Then a \(k \times k\) minor in the first \(k\) rows is nonzero if and only if its complementary \((n-k) \times (n-k)\) minor in the remaining \(n-k\) rows is nonzero.

Proof of the Claim. We have \(\dim(W_1) = k, \dim(W_2) = n-k\). Let \(f_1, f_2, \cdots, f_k\) be an orthonormal row basis of \(W_1\) and \(f_{k+1}, f_{k+2}, \cdots, f_n\) be an orthonormal row basis of \(W_2\) such that the matrix \(n \times n\) matrix of rows \(f_1, \cdots, f_n\) is a special orthogonal matrix \(O\). Now there exists a square matrix \(F\) of size \(k\) and a square matrix \(H\) of size \(n-k\) such that
\[
F \begin{pmatrix} q_1 \\ \vdots \\ q_k \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_k \end{pmatrix} \quad \text{and} \quad H \begin{pmatrix} q_{k+1} \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} f_{k+1} \\ \vdots \\ f_n \end{pmatrix} \quad \text{where} \quad Q = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}.
\]
A $k \times k$ minor in the first $k$ rows of $Q$ is nonzero if and only if the corresponding $k \times k$ minor is nonzero in $O$. A similar assertion holds for the remaining $n-k$ rows of $Q$ and $O$. Now we have reduced the claim to a special orthogonal matrix $O$. By using appropriate permutation matrices, it is enough to prove the claim for the principal $k \times k$ minor of $O$ and its complementary $(n-k) \times (n-k)$ minor of $O$.

Now we have if $O = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$ then $O^{-1} = O^t = \begin{pmatrix} M_1^t & M_3^t \\ M_2^t & M_4^t \end{pmatrix}$ and $\det(O) = \det(O^t) = 1$. So we have

$$
\begin{pmatrix} M_1 & M_2 \\ 0_{(n-k) \times k} & I_{(n-k) \times (n-k)} \end{pmatrix} \begin{pmatrix} M_1^t & M_3^t \\ M_2^t & M_4^t \end{pmatrix} = \begin{pmatrix} I_{k \times k} & 0_{k \times (n-k)} \\ 0_{(n-k) \times k} & I_{(n-k) \times (n-k)} \end{pmatrix}.
$$

Hence $\det(M_1) = \det(M_1^t) = \det(M_4)$. This proves the claim. ■

Continuing with the proof of Theorem 7.7, we have that all $k \times k$ minors of $D$ are nonzero. Hence $M[D]$ represents the uniform matroid $U_{k,n}$. Form the matrix $\text{Disc}(D)$ and consider the $(n-k) \times n$ submatrix $C'$ of rows of $\text{Disc}(D)$ corresponding to the basis \{${S_1, S_2, \cdots, S_{n-k}}$\} of $D(U_{k,n})$. Now any two corresponding rows of the matrices $C'$ and $C$ are both orthogonal to the columns of $D$, that is, $C'D = 0 = CD$ and they both give linear dependence relations of same $k+1$ rows of $D$. Hence each row of $C'$ is proportional to corresponding row of $C$ and not all entries of these two corresponding rows are zero. Therefore every $(n-k) \times (n-k)$ minor of $C'$ is nonzero, since every $(n-k) \times (n-k)$ minor of $C$ is nonzero. The matrix $D$ is the required matrix that we were looking for.

This proves that the determinant polynomial of each $(n-k) \times (n-k)$ minor of the submatrix of $\text{Disc}(X)$ corresponding to the rows $S_1, S_2, \cdots, S_{n-k}$ is not identically zero, if $\{S_1, S_2, \cdots, S_{n-k}\}$ is a basis for $D(U_{k,n})$. Hence Theorem 7.7 follows.

So we define the Zariski dense open subset $\mathcal{U} \subset \mathcal{O} \subset \mathbb{R}^{kn}$ as follows. Let $X$ and $\text{Disc}(X)$ be defined as before. Let $\mathcal{S} = \{S_1, S_2, \cdots, S_l\}$ be a collection of $(k+1)$-subsets of $\{1, 2, \cdots, n\}$. Consider the $l \times n$ submatrix $Y_\mathcal{S}$ of $\text{Disc}(X)$ corresponding to the rows $S_1, S_2, \cdots, S_l$. Let $P_\mathcal{S}(X)$ be the sum of squares of $l \times l$ minors of the matrix $Y_\mathcal{S}$. Then we have using Theorem 7.7 that, $P_\mathcal{S}(X) \in \mathbb{R}[x_{ij}]$ is not identically zero if and only if the collection $\mathcal{S}$ is an independent set in the Dilworth matroid $D(U_{k,n})$. So let

$$
\mathcal{U} = \{A = [a_{ij}]_{1 \leq i \leq n, 1 \leq j \leq k} \in \mathcal{O} \subset \mathbb{R}^{kn} \mid P_\mathcal{S}(A) \neq 0 \text{ whenever } P_\mathcal{S}[X] \text{ is not an identically zero polynomial}\}.
$$

(7.1)
Example 7.10. Let \( n = 6, k = 2 \) and \( l = 4 \) with \( S_1 = \{1, 2, 3\}, S_2 = \{1, 2, 4\}, S_3 = \{2, 3, 4\}, S_4 = \{4, 5, 6\} \). Then the collection \( \{S_1, S_2, S_3, S_4\} \) is not independent in \( D(U_{k,n}) \) because \( |S_1 \cup S_2 \cup S_3| = 4 < 2 + 3 = 5 \). So all the \( 3 \times 3 \) minors of the submatrix corresponding to the rows \( S_1, S_2, S_3 \) are zero because \( 3 > 4 - 2 = 2 \). Hence all the \( 4 \times 4 \) minors of the submatrix corresponding to the rows \( S_1, S_2, S_3, S_4 \) are zero.

Example 7.11. Let \( n = 9, k = 2, l = 3 \) with \( S_1 = \{1, 2, 3\}, S_2 = \{4, 5, 6\}, S_3 = \{7, 8, 9\} \). Then the collection \( \{S_1, S_2, S_3\} \) is independent in \( D(U_{k,n}) \). In the \( 3 \times 9 \) submatrix corresponding to rows \( S_1, S_2, S_3 \), there are \( 3 \times 3 \) minors which are nonzero as well as there are \( 3 \times 3 \) minors which are identically zero in the variables \( [x_{ij}]_{1 \leq i \leq 9, 1 \leq j \leq 2} \).

So we have proved the following theorem.

**Theorem 7.12.** If \( A \in U \) then \( M[A] \) represents the uniform matroid \( U_{k,n} \) and \( M[\text{Disc}(A)] \) represents the Dilworth matroid \( D(U_{k,n}) \).

7.2. The Intersection Lattice of a Very Generic Discriminantal Arrangement.

Now we define a very generic hyperplane arrangement and a very generic discriminantal arrangement, though it is mentioned in Definition 2.6.

**Definition 7.13.** Let \( k, n \) be positive integers. Let

\[ \mathcal{H}_n^k = \{H_1, H_2, \ldots, H_n\} \]

be a generic hyperplane arrangement of \( n \) hyperplanes in \( \mathbb{R}^k \). Let the equation for \( H_i \) be given by

\[ \sum_{j=1}^{k} a_{ij} x_j = b_i, \text{ with } a_{ij}, b_i \in \mathbb{R}, 1 \leq j \leq k, 1 \leq i \leq n. \]

Let \( U \) be the zarski dense open set in \( \mathbb{R}^{kn} \) as mentioned in Theorem 7.12 and in Equation 7.1. We say the generic hyperplane arrangement \( \mathcal{H}_n^k \) is very generic if \( A = [a_{ij}]_{1 \leq i \leq n, 1 \leq j \leq k} \in U \). Also here the associated discriminantal arrangement of hyperplanes passing through the origin in \( \mathbb{R}^n \) is given by

\[ \mathcal{C}_{(k+1)}^n = \{M_{\{i_1, i_2, \ldots, i_{k+1}\}} \mid 1 \leq i_1 < i_2 < \ldots < i_{k+1} \leq n\} \]
where the hyperplane \(M_{\{i_1,i_2,\ldots,i_{k+1}\}}\) passing through the origin in \(\mathbb{R}^n\) in the variables \(y_1, y_2, \ldots, y_n\) has the equation given by

\[
\begin{vmatrix}
  a_{i_11} & a_{i_12} & \cdots & a_{i_1(k-1)} & a_{i_1k} & y_{i_1} \\
  a_{i_21} & a_{i_22} & \cdots & a_{i_2(k-1)} & a_{i_2k} & y_{i_2} \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  a_{i_{k-1}1} & a_{i_{k-1}2} & \cdots & a_{i_{k-1}(k-1)} & a_{i_{k-1}k} & y_{i_{k-1}} \\
  a_{i_{k1}} & a_{i_{k2}} & \cdots & a_{i_{k1}(k-1)} & a_{im} & y_{i_k} \\
  a_{i_{k+1}1} & a_{i_{k+1}2} & \cdots & a_{i_{k+1}(k-1)} & a_{i_{k+1}k} & y_{i_{k+1}} \\
\end{vmatrix} = 0
\]

is said to be a very generic discriminantal arrangement.

Now we describe a lattice which arises from the Dilworth matroid \(D(U_{k,n})\).

### 7.2.1. Dilworth Lattice \(P(n,k)\).

**Definition 7.14.** The set \(P(n,k)\) consists of collections of all sets of the form \(S = \{S_1, S_2, \ldots, S_r\}\), where \(S_i \subseteq \{1,2,\ldots,n\}\) where \(S_i \subseteq \{1,2,\ldots,n\}\), each of cardinality at least \(k+1\) such that \(|\bigcup_{i \in I} S_i| \geq k + \sum_{i \in I} (|S_i| - k)\) for all \(I \subseteq \{1,2,\ldots,r\}\) with \(|I| \geq 2\). They partially order \(P(n,k)\) by letting \(\{S_1, S_2, \ldots, S_r\} \leq T = \{T_1, T_2, \ldots, T_p\}\), if, for each \(1 \leq i \leq r\) there exists \(1 \leq j \leq p\) such that \(S_i \subseteq T_j\).

Now we show that this lattice is isomorphic to the intersection lattice of a very generic discriminantal arrangement. Let \(A \in \mathcal{U}\) which represents the uniform matroid \(U_{k,n}\) and \(M[\text{Disc}(A)]\) which represents the Dilworth matroid \(D(U_{k,n})\) be such that \(A\) gives rise to a very generic discriminantal arrangement \(C_{n\choose k+1}^n\).

Now the intersection lattice of \(C_{n\choose k+1}^n\) is isomorphic to the lattice of flats \(L(n,k)\) of that arises from the rows of \(\text{Disc}(A)\) by taking orthogonal complement. The lattice \(L(n,k)\) is the lattice of subspaces of \(\mathbb{R}^n\) which are spanned by the rows of \((n\choose k+1) \times n\) matrix \(\text{Disc}(A)\). For any collection \(\{S_1, S_2, \ldots, S_m\}\) of \((k+1)\)-subsets we have

\[
\bigcap_{i=1}^m M_{S_i} = \langle a_{S_i} : 1 \leq i \leq m \rangle^\perp
\]

where \(a_{S_i}, 1 \leq i \leq m\) are the rows in \(\text{Disc}(A)\) corresponding to rows \(S_1, S_2, \ldots, S_m\).

For an arbitrary antichain \(S = \{S_1, S_2, \ldots, S_m\}\) of subsets of \(\{1,2,\ldots,n\}\) define \(V_S\) as the row span of those rows in \(\text{Disc}(A)\) which correspond to sets \(S\) of cardinality \(k+1\) such that \(S \subseteq S_i\) for some \(i, 1 \leq i \leq m\). If \(S = \{S\}\) then denote \(V_S\) by \(V_S\). So if \(|S| \geq k + 1\) then \(\dim V_S = |S| - k\). If \(S \in P(n,k)\) and \(S_1, S_2 \in S\) are two distinct elements then \(|S_1 \cap S_2| < k\). So here \(S\) is an antichain. Now we define a map \(\phi : P(n,k) \to L(n,k)\) as \(\phi(S) = V_S\).

**Theorem 7.15.** The map \(\phi : P(n,k) \to L(n,k)\) is an isomorphism of posets.
We give a proof of this theorem after proving the following three lemmas.

**Definition 7.16.** Let $v(S) = \max(0, |S| - k)$. For $\mathcal{F} = \{S_1, S_2, \ldots, S_m\}$ a collection of subsets of $\{1, 2, \ldots, n\}$ define

$$\Delta(\mathcal{F}) = v(\bigcup_{i=1}^{m} S_i) - \sum_{i=1}^{m} v(S_i).$$

**Remark 7.17.** If $S = \{S_1, S_2, \ldots, S_m\}$ is a collection of $(k+1)$-subsets independent in $U_{k,n}$ then $\Delta(\mathcal{F}) \geq 0$ for all $\mathcal{F} \subseteq S$. If $S \in P(n, k)$ then $\Delta(\mathcal{F}) > 0$ whenever $\mathcal{F} \subseteq S$ and it has at least two elements.

**Lemma 7.18.** Let $S$ be an antichain with the properties $|S| \geq k + 1$ for all $S \in S$ and $\Delta(\mathcal{F}) \geq 0$ for all $\mathcal{F} \subseteq S$. Suppose we have that $V_S = \sum_{S \in S} V_S = \bigoplus_{S \in S} V_S$, that is, the sum is direct. If $S \notin P(n, k)$ then there exists an antichain $S'$ with the same three properties as $S$ and such that $V_S = V_{S'}$, $|S| > |S'|$ and $S < S'$.

**Proof.** By assumption, $\Delta(\mathcal{F}) = 0$ for some $\mathcal{F} \subseteq S$ with at least two elements.

Let $U = \bigcup\mathcal{F}$. Now we observe that $\bigoplus_{S \in \mathcal{F}} V_S = \sum_{S \in \mathcal{F}} V_S \subseteq V_U$ and $\sum_{S \in \mathcal{F}} \dim V_S = \sum_{S \in \mathcal{F}} v(S) = v(U) = \dim V_U$. Hence we have $\bigoplus_{S \in \mathcal{F}} V_S = \sum_{S \in \mathcal{F}} V_S = V_U$. Replace the sets $S \in \mathcal{F} \subseteq S$ with their union $U$ to get a collection $S'$. Now $S'$ is antichain.

Suppose not, that is, there exists an element $S \in S \setminus \mathcal{F}$ such that $S \subseteq U = \bigcup\mathcal{F}$. Then we have $0 \leq \Delta(\bigcup\mathcal{F} \cup S) = v(U) - \sum_{F \in \mathcal{F}} v(F) - v(S) = \Delta(\mathcal{F}) - v(S) = -v(S) < 0$ which is a contradiction. Hence $S'$ is an antichain and satisfies the property that $\mathcal{E} \subseteq S' \Rightarrow \Delta(\mathcal{E}) \geq 0, |S| > |S'|$ and $S < S'$. This proves the lemma. \hfill \blacksquare

**Lemma 7.19.** If $S$ and $\mathcal{T}$ are any antichains with $S \leq \mathcal{T}$ and $\Delta(\mathcal{F}) \geq 0$ for all $\mathcal{F} \subseteq S$, then $\sum_{S \in S} v(S) \leq \sum_{T \in T} v(T)$.

**Proof.** The proof is by induction on the cardinality of $S$. Choose a set $T \in \mathcal{T}$ so that the subfamily $\mathcal{F} = \{F \in \mathcal{S} : F \subseteq T\}$ is non-empty. Then we have

$$\sum_{S \in S} v(S) = \sum_{F \in \mathcal{F}} v(F) + \sum_{S \in S \setminus \mathcal{F}} v(S) \leq v(\bigcup\mathcal{F}) + \sum_{S \in S \setminus \mathcal{F}} v(S) \leq v(T) + \sum_{S \in S \setminus \mathcal{F}} v(S).$$

Since $S \setminus \mathcal{F} \leq \mathcal{T} \setminus \{T\}$, the lemma follows by induction. \hfill \blacksquare

**Lemma 7.20.** If $S \in P(n, k), T \subseteq \{1, 2, \ldots, n\}$ with $|T| \geq k + 1$ and suppose for all $R \subseteq T$ with $|R| = k + 1$ we have $\{R\} \leq S$, then $\{T\} \leq S$.

**Proof.** Suppose $S_1, S_2 \in S$ are two distinct sets. Then $|S_1 \cup S_2| > |S_1| + |S_2| - k$. So we get $|S_1 \cap S_2| < k$. If $R_1 \subseteq T, R_2 \subseteq T$ are two $(k + 1)$-subsets such
that \(| R_1 \cap R_2 | = k \) and \( R_1 \subseteq S_1, R_2 \subseteq S_2 \) then we have \( S_1 = S_2 \). So \( R_1 \cup R_2 \subseteq S_1 = S_2 \). By applying this procedure repeatedly we conclude that \( \{ T \} \subseteq S \).

Now we prove Theorem 7.15.

**Proof.** If \( S, T \in P(n,k), S \leq T \) the \( V_S \subseteq V_T \). So the map \( \phi \) is order preserving. Let \( V \) be a span of some rows of \( \text{Disc}(A) \), that is, \( V \in L(n,k) \). Let \( S_1, S_2, \ldots, S_m \) be an independent set in \( D(U_{k,n}) \) such that the rows corresponding to them in \( \text{Disc}(A) \) span \( V \), that is, \( S = \{ S_1, S_2, \ldots, S_m \} \) then \( V = V_S \). Now \( S \) is an antichain satisfying the hypothesis of Lemma 7.18. If \( S \in P(n,k) \) then \( \phi(S) = V_S \).

If not, then by the repeated application of Lemma 7.18, we obtain an antichain \( T \) such that \( S \leq T \) such that \( V = V_S = V_T \) and \( T \in P(n,k) \) also \( T \) satisfies the hypothesis of Lemma 7.18, that is, \( V_T = \sum_{T \in T} V_T = \bigoplus_{T \in T} V_T \), the sum is direct and \( \phi(T) = V_T \). So the map \( \phi \) is surjective.

Now we prove that \( \phi \) is injective. Suppose on the contrary, we have that \( \phi(S) = \phi(T) \) for two distinct elements \( S, T \in P(n,k) \). We assume without loss of generality that \( T \leq S \) is not valid. By Lemma 7.20 there exists a \((k+1)\)-set \( R \) which is contained in subset of \( T \) but not contained in any subset of \( S \). Since \( V_S = V_T \), we can choose a minimal collection \( \mathcal{F} \leq S \) of \((k+1)\)-subsets such that \( \mathcal{F} \) is a independent in the Dilworth lattice \( D(U_{k,n}) \) and the row corresponding to \( R \) in \( \text{Disc}(A) \) is linearly dependent on the rows corresponding to \( F \in \mathcal{F} \) in \( \text{Disc}(A) \).

Hence we have by minimality of \( \mathcal{F} \),

\[
v(\cup \mathcal{F} \cup R) < | \mathcal{F} | + 1 \quad \text{and on the other hand} \quad v(\cup \mathcal{F}) \geq | \mathcal{F} | .
\]

Hence we conclude that \( R \subset \cup \mathcal{F}, \Delta(\mathcal{F}) = 0 \), that is, \( v(\cup \mathcal{F}) = | \mathcal{F} | \).

Let \( \mathcal{E} \) be the subcollection consisting of those subsets \( E \in S \) which contain some \( F \in \mathcal{F} \). Let \( \mathcal{E}' \) be the subcollection obtained from \( \mathcal{E} \) by intersecting all sets in \( \mathcal{E} \) with \( \cup \mathcal{F} \). So we have \( \mathcal{F} \leq \mathcal{E}' \) and

\[
v(\cup \mathcal{E}') = v(\cup \mathcal{F}) = \sum_{F \in \mathcal{F}} v(F) \leq \sum_{E' \in \mathcal{E}'} v(E')
\]

where the last inequality follows form Lemma 7.19. Now by adding further elements to each \( E' \in \mathcal{E}' \) we get that

\[
v(\cup \mathcal{E}) \leq \sum_{E \in \mathcal{E}} v(E) \Rightarrow \Delta(\mathcal{E}) \leq 0.
\]

By the choice of \( R \) we have that \( \mathcal{E} \) has at least two elements. Hence we arrive at a contradiction to \( S \in P(n,k) \). So \( \phi \) is injective.

Now we prove that the inverse of \( \phi \) is order preserving. Let \( S, T \in P(n,k) \) be such that \( \phi(S) = V_S \subseteq V_T = \phi(T) \). Choose a basis of rows in \( \text{Disc}(A) \) for
V_{S_i}$ that is, an independent set $S_0$ in $D(U_{k,n})$ and extend it to a basis of rows in $\text{Disc}(A)$ for $V_T$, that is, an independent set $T_0$ in $D(U_{k,n})$. So we have $S_0 \subseteq T_0$ are a collection of $(k+1)$-subsets. Now we apply Lemma 7.18 repeatedly to obtain $S_1$ such that $S_0 \leq S_1$ and $\phi(S) = V_S = V_{S_0} = V_{S_1} = \phi(S_1)$. So $S = S_1$ by injectivity of $\phi$. Now the collection $S_1 \cup (T_0 \setminus S_0)$ is an antichain because the sum $V_{S_1} + V_{T_0 \setminus S_0}$ is direct and all elements of $T_0 \setminus S_0$ are independent $(k+1)$-subsets. Also we have $\Delta(E) \geq 0$ for all $E \subseteq S_1 \cup (T_0 \setminus S_0)$ we have $\Delta(E) \geq 0$. Now we apply Lemma 7.18 to obtain a set $T_1$ such that $S_1 \cup (T_0 \setminus S_0) \leq T_1$ and $\phi(T) = V_T = V_{T_0} = V_{T_1} = \phi(T_1)$. So we have $T = T_1$ as $\phi$ is injective and $S_1 \leq T_1 \Rightarrow S \leq T$. This proves that the inverse of $\phi$ is order preserving. Hence Theorem 7.15 follows.

7.3. New Description of the Lattice Elements in $P(n,k)$. In this section we describe the lattice elements in a more geometric manner. Here again we assume that $A \in \mathcal{U}$ which represents the uniform matroid $U_{k,n}$ and $M[\text{Disc}(A)]$ which represents the Dilworth matroid $D(U_{k,n})$ be such that $A$ gives rise to a very generic discriminantal arrangement $\mathcal{C}^n_{\binom{n}{k+1}}$.

**Definition 7.21** (Concurrency Closed Sub-collection and Concurrency Closure). Let $n > k$ be two positive integers. Let

$$E = \{\{i_1, i_2, \ldots, i_{k+1}\} \mid 1 \leq i_1 < i_2 < \cdots < i_k < i_{k+1} \leq n\}$$

be the collection of all subsets of cardinality $k+1$. Let $D \subset E$ be any arbitrary collection.

We say $D$ is concurrency closed if the following criterion for any element $S \in E$ is satisfied with respect to $D$. Suppose there exists $\{S_1, S_2, \ldots, S_r\} \subseteq D$ and $S_i \neq S, 1 \leq i \leq r$ such that for every $J \subseteq \{1,2,\ldots,r\}$ we have $\big| \bigcup_{j \in J} S_j \big| \geq k + |J|$ and $\big| \bigcup_{i=1}^{r} S_i \cup S \big| < k + r + 1$ then $S \in D$. This definition is motivated by the notion of independence in the Dilworth matroid $D(U_{k,n})$.

We observe that the collection $E$ is concurrency closed. Now let $D \subset E$ be any arbitrary collection. Construct the concurrency closure $\overline{D}$ of $D$ as follows. First set $D_0 = D$ and add those elements $S \in E$ to $D_0$ if these $S$ satisfy the criterion mentioned above, to obtain $D_1$. Now construct $D_2$ from $D_1$ similarly and so on. We have

$$D_0 = D \subsetneq D_1 \subsetneq D_2 \subsetneq \cdots \subsetneq D_n = \overline{D}.$$ 

Since $E$ is a finite set we obtain $\overline{D}$ from $D_0$ in finitely many steps. Actually it can be shown that $D_1$ itself is concurrency closed and $D_1 = \overline{D}$. This is because $D_1$ is set of all elements in $D(U_{k,n})$ which are dependent on $D_0$. Hence we get $D_1 = D_2 = D_3 = \cdots = \overline{D}$. 
Definition 7.22 (Base Collection).
Let \( n > k \) be two positive integers. Let
\[
E = \{ \{i_1, i_2, \ldots, i_{k+1}\} \mid 1 \leq i_1 < i_2 < \cdots < i_k < i_{k+1} \leq n \}
\]
be the collection of all subsets of cardinality \( k + 1 \). Let \( D \subset E \) be any arbitrary collection. We say \( \tilde{D} \) is a base collection for \( D \) if \( \overline{\tilde{D}} = \overline{D} \) and \( \tilde{D} \) is minimal, that is, if \( D' \) is any other collection such that \( \overline{D'} = \overline{D} \) and \( D' \subset D \) then we have \( \tilde{D} = D' \). We can actually show that all minimal bases have equal cardinality. This follows from the property of bases (a standard fact) in matroid theory.

Definition 7.23 (Construction of a Base for a Concurrency Closed Collection).
Let \( n > k \) be two positive integers. Let
\[
E = \{ \{i_1, i_2, \ldots, i_{k+1}\} \mid 1 \leq i_1 < i_2 < \cdots < i_k < i_{k+1} \leq n \}
\]
be the collection of all subsets of cardinality \( k + 1 \). Let \( D \subset E \) be a concurrency closed subcollection. We say there is a concurrency of order \( m \geq k + 1 \) in \( D \), if there exists a concurrency set \( D \subset \{1,2,\ldots,n\} \) of size \( m \) such that all \( \binom{m}{k+1} \) subsets of \( D \) of size \( k + 1 \) are in the collection \( D \). Moreover \( D \) should be maximal with respect to this property, that is, there does not exist a set \( E \supset D \) of size more than \( m \) such that all \( \binom{|E|}{k+1} \) subsets of size \( k + 1 \) are in the collection \( D \). Let \( m_1, m_2, \cdots, m_r \) be the orders of concurrencies that exist in \( D \) with \( m_i \geq k + 1, 1 \leq i \leq r \). Then the cardinality of a base collection \( D' \) for \( D \) is given by
\[
(m_1 - k) + (m_2 - k) + \cdots + (m_r - k) = \left( \sum_{i=1}^{r} m_i \right) - rk.
\]
Also see Corollary 3.6 in C. A. Athanasiadis [2]. Let \( D_i \subset \{1,2,\ldots,n\} \) be the concurrency set of size \( m_i \) which gives rise to the order \( m_i \) concurrency in the concurrency closed subcollection \( D \). Let \( S = \{D_1, D_2, \ldots, D_r\} \). We have that
\[
\#(D' = Base of (D)) = n - \text{dim} \left( \bigcap_{(i_1,j_2,\ldots,i_{k+1}) \in D} M_{i_1 < j_2 < \cdots < i_k < i_{k+1}} \right)
\]
\[
= \sum_{D \in S} v(D)
\]
in the notation of Corollary 3.6 in [2] where \( v(D) = \max(0, |D| - m) \) for \( D \subset \{1,2,\cdots,n\} \). We also can show in this case that \( S \in P(n,k) \) (See Theorem 7.24). Here we do something more. We actually construct a base collection \( D' \) for \( D \). Let the concurrency sets be given by
\[
D_i = \{ j^i_1 < j^i_2 < \cdots < j^i_{m_i} \}, 1 \leq i \leq r.
\]
Then a base collection \( \mathcal{D}' \) for \( \mathcal{D} \) is given by

\[
\{ \{ j_i^1, j_i^2, \ldots, j_i^k \} \mid k + 1 \leq l \leq m_i, 1 \leq i \leq r \}.
\]

This collection \( \mathcal{D}' \) has the required cardinality. We denote

\[
\text{rank}(\mathcal{D}) = \sum_{D \in \mathcal{S}} v(D) \quad \text{(is the cardinality of any base collection of } \mathcal{D}).
\]

**Theorem 7.24.** Let \( n > k \) be two positive integers. Let

\[
\mathcal{E} = \{ \{ i_1, i_2, \ldots, i_{k+1} \} \mid 1 \leq i_1 < i_2 < \cdots < i_k < i_{k+1} \leq n \}
\]

be the collection of all subsets of cardinality \( k + 1 \). Let \( \mathcal{D} \subset \mathcal{E} \) be a concurrency closed subcollection. Let \( \mathcal{S} = \{ D_1, D_2, \ldots, D_r \} \) be the collection of sets of concurrences in \( \mathcal{D} \). Then

1. \( (a) \) \( S \in P(n,k) \).
2. \( (b) \) \( V_S = \sum_{E \in D_i, \text{ for some } 1 \leq i \leq r, |E| = k+1} V_E = \sum_{E \in \mathcal{D}} V_E = V_D. \)

Moreover \( \mathcal{S} \) is uniquely determined with these two properties \((a),(b)\).

3. \( (c) \) Let the concurrency sets be given by \( D_i = \{ j_i^1 < j_i^2 < \cdots < j_i^{n-1} \}, 1 \leq i \leq r \). Then a base collection \( \mathcal{D}' \) for \( \mathcal{D} \) is given by \( \{ \{ j_i^1, j_i^2, \ldots, j_i^k \} \mid k + 1 \leq l \leq m_i, 1 \leq i \leq r \} \). The

4. \( \text{The cardinality of any base of } \mathcal{D} \text{ is } \sum_{D \in \mathcal{S}} v(D). \)

**Proof.** Let \( \mathcal{T}_0 = \{ T_1, T_2, \ldots, T_p \} \subseteq \mathcal{D} \) be a base for the antichain \( \mathcal{D} \). So \( V_{T_0} = \bigoplus_{i=1}^{p} V_{T_i} = V_D \). We have \( \Delta(F) \geq 0 \) for all \( F \subseteq \mathcal{T}_0 \) and \( \mathcal{T}_0 \) is an antichain. So we apply Lemma 7.18 repeatedly to obtain a collection \( \mathcal{T} \in P(n,k) \) such that \( \mathcal{T}_0 \leq \mathcal{T} \) and \( V_{T} = V_{T_0} = V_D \). We prove that \( \mathcal{T} = \mathcal{S} \).

Let \( T \in \mathcal{T} \) and \( T = T_1 \cup T_2 \cup \cdots \cup T_l \) after renumbering the subsets \( \mathcal{T}_0 \) and this union is obtained using Lemma 7.18. Then we have \( \Delta(\{ T_1 \cup T_2 \cup \cdots \cup T_l \}) = 0 \).

Let \( E \subseteq T \) of cardinality \( k + 1 \). Then we have \( k + l + 1 > k + l = | T_1 \cup T_2 \cup \cdots \cup T_l | = | T_1 \cup T_2 \cup \cdots \cup T_l \cup E |. \) Hence \( \{ E, T_1, \ldots, T_l \} \) is independent and \( \{ T_1, \ldots, T_l \} \) is independent, that is, \( V_{E} \subseteq \bigoplus_{i=1}^{l} V_{T_i} \). So \( E \in \mathcal{T}_0 = \mathcal{D} \). This shows that \( \mathcal{P}_{k+1}(T) \subseteq \mathcal{D} \) where \( \mathcal{P}_{k+1}(T) \) is the collection of all \((k+1)\)-subsets of \( T \). By the definition of concurrency orders and by the definition of \( \mathcal{S} \) there exists \( 1 \leq i \leq r \) such that \( T \subseteq D_i \). So we have proved that \( \mathcal{T} \leq \mathcal{S} \).

Now we prove that \( \mathcal{S} \leq \mathcal{T} \). Let \( E \subseteq D \in \mathcal{S} \) be a \((k+1)\)-subset. Then there exists \( \mathcal{T}'_0 = \{ E = T_1', T_2', \ldots, T_p' \} \) a base of \( \mathcal{D} \). So \( V_{\mathcal{T}'_0} = \bigoplus_{i=1}^{p} V_{T_i'} = V_D \). We have \( \Delta(F) \geq 0 \) for all \( F \subseteq \mathcal{T}'_0 \) and \( \mathcal{T}'_0 \) is an antichain. So we apply Lemma 7.18 repeatedly to obtain a collection \( \mathcal{T}' \in P(n,k) \) such that \( \mathcal{T}'_0 \leq \mathcal{T}' \) and \( V_{\mathcal{T}'} = V_{\mathcal{T}'_0} = V_D \). This
implies $\phi(T') = V_D = \phi(T)$ and $T, T' \in P(n, k)$. So by injectivity of $\phi$ we have that $T = T'$. Now clearly $\{E\} = \{T'_1\} \leq T'_0 \leq T' = T$ for every $(k + 1)$-subset $E \subseteq D \in S$. Using Lemma 7.20 we conclude that $\{D\} \leq T$ as $T \in P(n, k)$ for every $D \in S$. This implies $S \leq T$. So we have $S = T$ and have proved Theorem 7.24(1).

Now we prove (2). Since $\{D_1, D_2, \ldots, D_r\} = S = T \in P(n, k)$ obtained by applying Lemma 7.18 repeatedly to a base of $D$, the sum $\sum_{i=1}^{r} V_{D_i} = \bigoplus_{i=1}^{r} V_{D_i}$ is direct and if $D_i = \{j^i_1 < j^i_2 < \cdots < j^i_{m_i}\}, 1 \leq i \leq r$ then the set $D' = \{\{j^i_1, j^i_2, \ldots, j^i_{l_i}, j^i_{l_i}\} \mid k + 1 \leq l \leq m_i, 1 \leq i \leq r\}$ gives a base for $D$. This proves (2).

To prove (3) we observe that the cardinality of $D'$ is $\sum_{D \in S} v(D)$. Hence the cardinality of any base of $D$ is also given by the same value.

Hence Theorem 7.24 follows.

Now we consider a new poset which is isomorphic to $P(n, k)$.

**Definition 7.25.** Let $C(n, k)$ be the collection of all concurrency closed subcollections of $\mathcal{E} = \{\{i_1, i_2, \ldots, i_k, i_{k+1}\} \mid 1 \leq i_1 < \cdots < i_{k+1} \leq n\}$. Let $\mathcal{D}_1, \mathcal{D}_2$ be two concurrency closed subcollections of $\mathcal{E}$. We say $\mathcal{D}_1 \subseteq \mathcal{D}_2$ if $\mathcal{D}_1 \subseteq \mathcal{D}_2$.

Define a map $\psi : P(n, k) \rightarrow C(n, k)$ given by

$$\psi(S) = \{E \subset \{1, 2, \ldots, n\} \text{ such that } |E| = k + 1, E \subset S \text{ for some } S \in S\}.$$

In fact we will show that $\psi(S)$ is concurrency closed and hence is an element of $C(n, k)$. Define another map $\psi$ is given by $\sigma : C(n, k) \rightarrow P(n, k)$ defined as

$$\sigma(D) = S \text{ where } S \text{ is the collection of sets of concurrency in } D.$$

**Theorem 7.26.** The maps $\psi$ and $\sigma$ are poset isomorphisms and inverses of each other.

**Proof.** First we prove $\psi(S)$ is concurrency closed. Let $E$ be a $(k + 1)$-subset which is dependent on $\psi(S)$. So we have $V_E \subseteq V_S$. Then there exists a basis for $V_S$ whose corresponding $(k + 1)$-subsets has the form $T_0 = \{T_1 = E, T_2, \ldots, T_p\}$. Now the hypothesis of Lemma 7.18 is satisfied for $T_0$. So by applying Lemma 7.18 repeatedly there exists $T \in P(n, k)$ such that $T_0 \leq T$ and $\phi(T) = V_T = V_{T_0} = V_S = \phi(S)$. So by injectivity of $\phi$ we get that $T = S$. Since $\{E\} \leq T_0 \leq T = S$ there exists $S \in S$ such that $E \subseteq S$. So $E \in \psi(S)$. So $\psi(S)$ is concurrency closed.

Now let $\psi(S) = \mathcal{D}$ and $\sigma(D) = T$ then we get $V_S = V_D$ and $V_T = V_D$ by Theorem 7.24(1)(b). Hence $\phi(S) = \phi(T) \Rightarrow S = T$. This proves that $\sigma \circ \psi = 1_{P(n,k)}$. 


Let $\mathcal{D}$ be concurrency closed and let $\sigma(\mathcal{D}) = \mathcal{S} = \{D_1, D_2, \ldots, D_r\}$. Then every $(k + 1)$-set $E \in \mathcal{D}$ belongs to $\mathcal{D}_i$ for some $1 \leq i \leq r$ by the definition of sets of concurrences in $\mathcal{D}$. Hence $E \in \psi(\mathcal{S})$. So $\mathcal{D} \subseteq \psi(\mathcal{S})$. Similarly it clear that $\psi(\mathcal{S}) \subseteq \mathcal{D}$. Hence we get $\psi \circ \sigma = 1_{\mathcal{C}(n,k)}$.

Now we show that $\psi$ is order preserving. If $\mathcal{S} \subseteq \mathcal{T}$ then it is clear that $\psi(\mathcal{S}) \subseteq \psi(\mathcal{T})$. Similarly if $\mathcal{D}_1 \subseteq \mathcal{D}_2$ then by definition of sets of concurrences in $\mathcal{D}_1$ and $\mathcal{D}_2$ we have $\sigma(\mathcal{D}_1) \leq \sigma(\mathcal{D}_2)$. So $\sigma$ is also order preserving.

This proves Theorem 7.26. ■

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Post Doctoral Fellow in Mathematics, Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Prayagraj (Allahabad)-211019, Uttar Pradesh, INDIA

Email address: akcp1728@gmail.com