TWISTED ACTIONS AND THE OBSTRUCTION TO EXTENDING UNITARY REPRESENTATIONS OF SUBGROUPS

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Abstract. Suppose that $G$ is a locally compact group and $\pi$ is a (not necessarily irreducible) unitary representation of a closed normal subgroup $N$ of $G$ on a Hilbert space $\mathcal{H}$. We extend results of Clifford and Mackey to determine when $\pi$ extends to a unitary representation of $G$ on the same space $\mathcal{H}$ in terms of a cohomological obstruction.

Let $G$ be a group and $\pi : N \to U(\mathcal{H})$ a unitary representation of a normal subgroup $N$ of $G$. When is $\pi$ the restriction of a unitary representation of $G$?

If $\pi$ does extend to a representation $\rho$ of $G$, then $\rho(s)$ implements a unitary equivalence between $\pi$ and $\pi^s : n \mapsto \pi(\pi(n)s^{-1})$. So an obvious necessary condition is that $\pi$ should be equivalent to $\pi^s$ for each $s \in G$ (we say that $\pi$ is $G$-invariant), and the problem is to decide when a $G$-invariant representation extends.

Clifford answered this extension problem in [2] when $G$ is discrete, $\pi$ is irreducible and $\mathcal{H}$ is finite-dimensional. In modern language, Clifford showed that if $\pi$ is $G$-invariant, then there is an obstruction to extending the representation in the cohomology group $H^2(G/N, \mathbb{T})$, where $\mathbb{T}$ is the unit circle. Mackey extended Clifford’s result to irreducible unitary representations of normal closed subgroups of locally compact groups [7, Theorem 8.2]. Mackey’s solution involves Borel cocycles, so his obstruction lies in a cohomology theory where all the cochains are Borel. The resulting cohomology groups were subsequently analysed by Moore in [9, 10, 11, 12].

This extension problem has recently resurfaced in the context of compact Lie groups in [1], where it was tackled using the structure theory of Lie groups, and in [3], where it was studied in the context of nonabelian duality for locally compact groups and crossed products of $C^*$-algebras. Here we investigate a cohomological obstruction to the extension of an arbitrary $G$-invariant unitary representation $\pi$ of $N$, and its relationship to the results in [11] and [3]. Our obstruction is a twisted action of $G/N$ on the von Neumann algebra $\pi(N)'$ of operators which commute with every $\pi(n)$; the representation extends if and only if this twisted action is equivalent, in a natural sense, to an ordinary action. We then use a stabilisation trick to show that if $\pi$ is $G$-invariant then infinite multiples $\pi \otimes 1$ of $\pi$ always extend.

Preliminaries. Let $G$ be a second-countable locally compact group with a closed normal subgroup $N$. We endow the group $U(\mathcal{H})$ of all unitary operators on a separable Hilbert space $\mathcal{H}$ with the strong operator topology, and note that $U(\mathcal{H})$ is a Polish group (in the sense that the topology is given by a complete metric). A unitary representation $\rho$ of $G$
is a continuous homomorphism $\rho : G \to U(\mathcal{H})$. A function $f : G \to \mathcal{H}$ is Borel if $f^{-1}(O)$
is a Borel set for each open set $O$ of $\mathcal{H}$; equivalently, if $s \mapsto (f(s)|h) : G \to \mathbb{C}$ is a Borel
function for each $h \in \mathcal{H}$. We use a left-invariant Haar measure on $G$.

Let $\mathcal{A}$ be a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$. The group
$U(\mathcal{A})$ of unitary elements in $\mathcal{A}$ is a Polish group in the ultra-weak topology, and it is
then a closed subgroup of $U(\mathcal{H})$. The group $\text{Aut}(\mathcal{A})$ of automorphisms of $\mathcal{A}$ is Polish
in the topology of pointwise ultra-weak convergence; this is called the $u$-topology in [1]. For $u \in U(\mathcal{A})$, we denote by $\text{Ad} \, u$ the automorphism $a \mapsto uau^*$ of $\mathcal{A}$, and
note that $\text{Ad} : U(\mathcal{A}) \to \text{Aut}(\mathcal{A})$ is a continuous homomorphism.

**Definition 1.** A twisted action of $G$ on $\mathcal{A}$ is a pair $(\alpha, \sigma)$ of maps $\alpha : G \to \text{Aut}(\mathcal{A})$ and
$\sigma : G \times G \to U(\mathcal{A})$ such that

1. $\alpha$ and $\sigma$ are Borel,
2. $\alpha_e = \text{id}, \sigma(e, s) = \sigma(s, e) = 1$ for $s \in G$,
3. $\alpha_s \circ \alpha_t = \text{Ad} \, \sigma(s, t) \circ \alpha_{st}$ for $s, t \in G$, and
4. $\sigma(r, s) \sigma(r, st) = \sigma(r, s) \sigma(rs, t)$ for $r, s, t \in G$.

Two twisted actions $(\alpha, \sigma)$ and $(\beta, \omega)$ of $G$ on $\mathcal{A}$ are exterior equivalent if there is a Borel
map $\nu : G \to U(\mathcal{A})$ such that

1. $\beta_s = \text{Ad} \, \nu_s \circ \alpha_s$, and
2. $\omega(s, t) = \nu_s \alpha_s(\nu_t) \sigma(s, t) \nu_t^*$.

These definitions are the von-Neumann algebraic analogues of [13] Definitions 2.1 and 3.1. Our definition of twisted action is slightly different from the one used in [16] Definition 2.1, where the map $s \mapsto \alpha_s$ is required to be continuous.

**Main Results.** In Theorem 2 we prove that the obstruction to extending a $G$-invariant
unitary representation $\pi$ of $N$ is a twisted action of $G/N$ on the von Neumann algebra
$\pi(N)'$, and in Theorem 4 we discuss the extension problem in the context of non-abelian
duality for amenable groups $G$. We reconcile the two approaches in Remark 5 to do so
one needs to understand not only the statement of Theorem 2 but also its proof.

**Theorem 2.** Let $N$ be a closed normal subgroup of a second-countable locally compact
group $G$. Suppose $\pi : N \to U(\mathcal{H})$ is a unitary representation of $N$ which is $G$-invariant.
Then there is a twisted action $(\alpha, \sigma)$ of $G/N$ on the commutant $\pi(N)'$ of $\pi(N)$ such that $\pi$
extends to a unitary representation $\rho$ of $G$ on $\mathcal{H}$ if and only if $(\alpha, \sigma)$ is exterior equivalent
to an action.

**Proof.** We start by constructing the twisted action $(\alpha, \sigma)$. Since $\pi^*$ is unitarily equivalent
to $\pi$ for all $s \in G$, there exist unitary operators $W_s \in U(\mathcal{H})$ such that $W_s \pi(n) W_s^* = \pi(sns^{-1})$. We claim that we can choose $W_s$ such that the map $s \mapsto W_s$ is Borel. To see
this, let

$$
H = \{(W, s) : W \in U(\mathcal{H}), s \in G \text{ and } W \pi(n) W^* = \pi(sns^{-1}) \text{ for } n \in N\}.
$$

Then $H$ is a subgroup of $U(\mathcal{H}) \times G$; we claim that $H$ is closed. So suppose that the net
$(W_\beta, s_\beta) \in H$ converges to $(W, s)$. Then $W_\beta \pi(n)$ converges strongly to $W \pi(n)$ for each
$n \in N$, and since multiplication in $U(\mathcal{H})$ is jointly continuous, $\pi(s_\beta sns_\beta^{-1}) W_\beta$ converges
strongly to $\pi(sns^{-1}) W$. Thus $W \pi(n) = \pi(sns^{-1}) W$, and $H$ is closed. Now $H$ is Polish
since it is a closed subgroup of a Polish group, and the quotient map \( H \to G : (W, s) \mapsto s \) has a Borel section \( s \mapsto (W, s) \) by [11, Proposition 4].

Another application of [11, Proposition 4] shows that the quotient map \( G \to G/N \) admits a Borel section \( c : G/N \to G \). We set

\[
V_s = W_{c(sN)}\pi(c(sN)^{-1} s).
\] (1)

Then \( s \mapsto V_s \) is Borel and \( V_{sn} = V_s \pi(n) \) for \( s \in G \) and \( n \in N \). We define \( \alpha_s = \text{Ad} V_s \). For \( T \in \pi(N)' \) and \( s \in G \), we have

\[
\alpha_s(T)\pi(n) = V_s T V_s^* \pi(n) = V_s T (V_s^* \pi(n) V_s) V_s^* = V_s T \pi(s^{-1} ns) V_s^*
\]

\[
= V_s \pi(s^{-1} ns) T V_s^* = V_s \pi(s^{-1} ns) V_s^* V_s T V_s^* = \pi(n) \alpha_s(T);
\]

thus \( \alpha_s : \pi(N)' \to \pi(N)' \), and \( \alpha_s \) is an automorphism of \( \pi(N)' \) because \( V_s \) is unitary.

To see that \( s \mapsto \alpha_s : G \to \text{Aut}(\pi(N)') \) is Borel, we will show that if \( V_\beta \) converges to \( V \) in the strong operator topology and \( \text{Ad} V_\beta \) and \( \text{Ad} V \) leave \( \pi(N)' \) invariant, then \( \text{Ad} V_\beta \) converges to \( \text{Ad} V \) in \( \text{Aut}(\pi(N)') \). It then follows that \( s \mapsto \alpha_s : G \to \text{Aut}(\pi(N)') \) is Borel because \( s \mapsto V_s \) is Borel. The topology on \( \text{Aut}(\pi(N)') \) is the topology generated by the seminorms \( \alpha \mapsto \| f \circ \alpha \| \), where \( f \in \pi(N)' \) and the pre-dual \( \pi(N)' \) has been identified with the ultra-weakly continuous functionals on \( \pi(N)' \). The ultra-weakly continuous functionals on \( \pi(N)' \) have the form \( \pi(n)^{\prime} \) have the form \( \sum_{n=1}^{\infty} \| T h_n | k_n \| \) where \( h_n, k_n \in H \) satisfy \( \sum_{n=1}^{\infty} \| h_n \|^2, \sum_{n=1}^{\infty} \| k_n \|^2 < \infty \) (see, for example, [12] pages 482–483). Let \( \epsilon > 0 \). If \( K \) is the maximum of \( \left( \sum_{n=1}^{\infty} \| h_n \|^2 \right)^{1/2} \) and \( \left( \sum_{n=1}^{\infty} \| k_n \|^2 \right)^{1/2} \), then

\[
\| f \circ \text{Ad} V_\beta - f \circ \text{Ad} V \| = \sup \left\{ \| f(V_\beta T V_\beta^* - V T V^*) \| : \| T \| = 1, T \in \pi(N)' \right\}
\]

\[
= \sup \left\{ \left( \sum_{n=1}^{\infty} \| (V_\beta^* h_n | T^* V_\beta^* k_n) - (TV^* h_n | V^* k_n) \| : \| T \| = 1, T \in \pi(N)' \right\} \right.
\]

\[
\leq \sup \left\{ \sum_{n=1}^{\infty} \| (V_\beta^* - V^*) h_n \| \| T^* V_\beta^* k_n \| + \| TV^* h_n \| \| (V_\beta^* - V^*) k_n \| \right\}
\]

\[
\leq \sum_{n=1}^{\infty} \| (V_\beta^* - V^*) h_n \| \| k_n \| + \| h_n \| \| (V_\beta^* - V^*) k_n \|
\]

\[
\leq K \left( \sum_{n=1}^{\infty} \| (V_\beta^* - V^*) h_n \|^2 \right)^{1/2} + K \left( \sum_{n=1}^{\infty} \| (V_\beta^* - V^*) k_n \|^2 \right)^{1/2}
\] (2)

by Hölder’s inequality. Since each \( V_\beta \) is a normal operator, we have \( V_\beta^* \to V^* \). Now choose \( N > 0 \) such that \( \sum_{n=N}^{\infty} \| h_n \|^2 < \epsilon^2 (16 K^2)^{-1} \) and \( \sum_{n=N}^{\infty} \| k_n \|^2 < \epsilon^2 (16 K^2)^{-1} \). Then, for each \( n < N \), choose an open neighbourhood \( O_n \) of \( V^* \) such that

\[
\| (V_\beta^* - V^*) h_n \|^2 < \frac{\epsilon^2}{8(N - 1) K^2} \quad \text{and} \quad \| (V_\beta^* - V^*) k_n \|^2 < \frac{\epsilon^2}{8(N - 1) K^2}
\]

whenever \( V_\beta^* \in O_n \), and check that if \( V_\beta^* \in \bigcap_{n=1}^{N-1} O_n \) then (2) \( \epsilon \). This proves that \( \text{Ad} V_\beta \) converges to \( \text{Ad} V \), and it follows that \( \alpha : G \to \text{Aut}(\pi(N)') : s \mapsto \text{Ad} V_s \) is Borel.
Next we define \( \sigma(s, t) = V_s V_t V_{st}^* \). Then
\[
\sigma(s, t)\pi(n) = V_s V_t V_{st}^* \pi(n) = V_s V_t \pi((st)^{-1} nst) V_{st}^* = V_s V_t \pi((st)^{-1} nst) V_t^* V_s^* = V_s \pi(s^{-1} ns) V_t^* V_s^* = \pi(n) V_s V_t V_{st}^*
\]
for all \( n \in N \), so \( \sigma : G \times G \to U(\pi(N)) \). Note that \( \sigma \) is Borel because \( s \mapsto V_s \) is Borel and both \( V_s \mapsto V_s^* \) and \( (s, t) \mapsto st \) are continuous. The equation \( V_{sn} = V_s \pi(n) \) implies that \( \sigma(s, n) = 1 = \sigma(n, s) \) for \( s \in G \) and \( n \in N \). We have
\[
\sigma(r, s)\sigma(r, t) = V_r V_s V_{rs} V_{rs} V_{rs}^* = V_r V_t V_{rst}^* = V_r V_t V_{rst}^* = \alpha_s(\sigma(r, t)) \sigma(r, st),
\]
and, for \( T \in \pi(N)' \),
\[
\alpha_s(\alpha_t(T)) = V_s V_t T V_s^* V_t^* = V_r V_s V_{rs} V_{rs} V_{rs}^* = \sigma(s, t) \alpha_{st}(T) \sigma(s, t)^*.
\]
Thus \((\alpha, \sigma)\) is a twisted action of \( G \) on \( \pi(N)' \). But \( \alpha_s \) depends only on \( sN \) since
\[
\alpha_{sn}(T) = V_s \pi(n) T \pi(n)^* V_s^* = V_s \pi(n) \pi(n)^* T V_s^* = \alpha_s(T)
\]
for all \( n \in N \). We also have
\[
\sigma(s, tn) = V_s V_t V_{stn}^* = V_s V_t \pi(n) V_{stn}^* = \sigma(s, t),
\]
and \( \sigma(sn, t) = \sigma(sn^{-1} s, t) = \sigma(s, t) \) because \( \sigma(ns, t) = \sigma(s, t) \). So we can view \((\alpha, \sigma)\) as a twisted action of \( G/N \) on \( \pi(N)' \).

Now suppose that \( \pi \) extends to a continuous representation \( \rho \) of \( G \) on \( \mathcal{H} \). Let \( V_s \) be as in [I] and define \( \nu : G/N \to U(\mathcal{H}) \) by \( \nu_{sn} = \rho(s) V_s^* \). Then \( \nu \) is Borel because \( s \mapsto V_s^* \) is Borel and \( \rho \) is continuous, and
\[
\omega(sN, tn) := \nu_{sn} \alpha_s(\nu_{tn}) \sigma(sN, tn) V_{stn}^* = 1.
\]
If \( \beta_{sn} := \text{Ad} \nu_{sn} \circ \alpha_{sn} \), then we can deduce from [I] and [II] that
\[
\beta_{sn} \circ \alpha_{tn} = (\text{Ad} \nu_{sn} \circ \alpha_{sn}) \circ (\text{Ad} \nu_{tn} \circ \alpha_{tn}) = \text{Ad} \nu_{sn} \circ \alpha_{sn} \circ \text{Ad} \nu_{tn} \circ \alpha_{tn}
\]
\[
= \text{Ad}(\nu_{sn} \alpha_s(\nu_{tn})) \circ \alpha_{sn} \circ \alpha_{tn} = \text{Ad}(\nu_{sn} \circ \alpha_{sn} \circ \alpha_{tn})
\]
\[
= \text{Ad} \nu_{sn} \circ \alpha_{tn} = \beta_{stn}.
\]
Since \( \beta : G/N \to \text{Aut}(\pi(N)) \) is a Borel homomorphism between Polish groups, it is continuous by [II], Proposition 5]. Thus \((\beta, 1)\) is an ordinary action, and \((\alpha, \sigma)\) is exterior equivalent to an action.

Conversely, if \((\alpha, \sigma)\) is exterior equivalent to an action, then there exists a Borel map \( \nu : G/N \to U(\mathcal{H}) \) such that \( \nu_{sn} \alpha_s(\nu_{tn}) \sigma(sN, tn) V_{stn}^* = 1 \). Set \( \rho(s) = \nu_{sn} V_s \). Then
\[
\rho(s) \rho(t) = \nu_{sn} V_s \nu_{tn} V_t = \nu_{sn} \nu_{stn} \nu_{tn} V_{st} V_t = \nu_{sn} \nu_{stn} \nu_{tn} \sigma(sN, tn) V_{st} = \nu_{stn} V_{st} = \rho(st).
\]
Thus \( \rho : G \to U(\mathcal{H}) \) is a Borel homomorphism between Polish groups, and hence is continuous by [II], Proposition 5]; \( \rho \) is the required extension of \( \pi \).

Corollary 3. If \( \pi : N \to U(\mathcal{H}) \) is a \( G \)-invariant unitary representation of \( N \), then there is a unitary representation \( \rho \) of \( G \) on \( \mathcal{H} \otimes L^2(G/N) \) such that \( \rho|_N = \pi \otimes 1 \).
Proof. Let \((\alpha, \sigma)\) be the twisted action of \(G/N\) on \(\pi(N)'\) constructed above. Then the twisted action \((\beta, \omega) := (\alpha \otimes \text{id}, \sigma \otimes 1)\) of \(G/N\) on \(\pi(N)' \otimes B(L^2(G/N)) = (\pi \otimes 1)(N)''\) is the obstruction to extending \(\pi \otimes 1\). We will show that \((\beta, \omega)\) is exterior equivalent to an action. Similar “stabilisation tricks” have been used in [17, Proposition 2.1.3] and [13, Theorem 3.4], for example.

We begin by identifying \(\mathcal{H} \otimes L^2(G/N)\) with the space \(L^2(G/N, \mathcal{H})\) of Bochner square-integrable functions. Since \(\mathcal{H}\) is separable, \(\xi \in L^2(G/N, \mathcal{H})\) if and only if \(\xi\) is a Borel function from \(G/N\) to \(\mathcal{H}\) and \(\int_{G/N} ||\xi(sN)||^2 d(sN) < \infty\). Define \(\nu : G/N \to U(L^2(G/N, \mathcal{H}))\) by

\[
(\nu_{tN}\xi)(rN) = \sigma(tN, t^{-1}r^{-1}N)^\ast \xi(rtN)\Delta(tN)^{1/2},
\]

where \(\Delta\) is the modular function of \(G/N\) and \(\xi \in L^2(G/N, \mathcal{H})\). (The modular function is necessary to ensure that \(\nu_{tN}\) is unitary.) Then

\[
(\nu_{tN}\xi)(rN) = \sigma(tN, r^{-1}N)\xi(rt^{-1}N)\Delta(tN)^{-1/2},
\]

and hence

\[
(\beta_{sN}(\nu_{tN}^\ast)\nu_{sN}^\ast\nu_{stN}\xi)(rN) = \alpha_{sN}(\sigma(tN, r^{-1}N))(\nu_{stN}\xi)(rN)\Delta(tN)^{-1/2}
\]

\[
= \alpha_{sN}(\sigma(tN, r^{-1}N)\sigma(sN, tr^{-1}N))(\nu_{stN}\xi)(rN)\Delta(stN)^{-1/2}
\]

\[
= \alpha_{sN}(\sigma(tN, r^{-1}N))\sigma(sN, tr^{-1}N)(\nu_{stN}\xi)(rN)
\]

\[
= \sigma(sN, tN)\xi(rN)
\]

\[
= (\sigma(sN, tN) \otimes 1)\xi(rN)
\]

\[
= \omega(sN, tN)\xi(rN).
\]

It follows that

\[
\nu_{sN}\beta_{sN}(\nu_{tN})\omega(sN, tN)\nu_{stN}^\ast = 1.
\]

(5)

If we now define \(\gamma : G/N \to \text{Aut}(\pi(N)')\) by \(\gamma_{sN} = \text{Ad} \nu_{sN} \circ \beta_{sN}\), then [3] implies that \(\gamma\) is a homomorphism. It remains to show that \(\nu\) is Borel, and it then follows from [11, Proposition 5] that \(\gamma = \text{Ad} \nu \circ \beta : G/N \to \text{Aut}(\pi(N)')\) is continuous.

Since \(U(L^2(G/N, \mathcal{H}))\) has the strong operator topology, \(\nu\) is Borel if and only if \(sN \mapsto \nu_{sN}\xi\) is Borel for every \(\xi \in L^2(G/N, \mathcal{H})\), and hence if and only if \(sN \mapsto (\nu_{sN}\xi \mid \eta)\) is Borel for every \(\xi, \eta \in L^2(G/N, \mathcal{H})\). Since \((U, h) \mapsto Uh\) is continuous, the map \((sN, tN, rN) \mapsto (\sigma(sN, tN)\xi(rN)) \mapsto \sigma(sN, tN)\xi(rN)\) is Borel, and hence so is

\[
(tN, rN) \mapsto \left| \left( \sigma(tN, t^{-1}r^{-1}N)^\ast \xi(rtN) \mid \eta(rN) \right) \right|
\]

(6)

Since (6) is dominated by \(||\xi(rtN)|| \cdot ||\eta(rN)||\), and an application of Tonelli’s Theorem shows that this is integrable over \(G/N \times G/N\), it follows from Fubini’s Theorem that

\[
tN \mapsto \int_{G/N} \left( \sigma(tN, t^{-1}r^{-1}N)^\ast \xi(rtN) \mid \eta(rN) \right) d(rN)
\]

defines, almost everywhere, an integrable (and therefore Borel) function. Multiplying by \(\Delta(tN)^{1/2}\) shows that \(tN \mapsto (\nu_{tN}\xi \mid \eta)\) is Borel. Thus \(\nu\) is Borel and \(\gamma\) is continuous.

Thus \(\nu\) implements an exterior equivalence between \((\beta, \omega)\) and the ordinary action \((\gamma, 1)\). It now follows from Theorem 2 that there is a representation \(\rho\) of \(G\) with \(\rho|_N = \pi \otimes 1\).

\( \square \)
The irreducible case. When the representation \( \pi \) of \( N \) is irreducible, the commutant \( \pi(N)' \) is \( \mathbb{C}I \), the action \( \alpha \) is trivial, and the obstruction \( \sigma \) to extending \( \pi \) is a Borel cocycle in the Moore cohomology group \( H^2(G/N, \mathbb{T}) \). Thus we recover Mackey’s \cite{7} Theorem 8.2 as it applies to ordinary (that is, non-projective) irreducible representations.

When the obstruction \( \sigma \) is non-trivial, we can recover Corollary \cite{3} from another important part of the Mackey machine \cite{7} Theorem 8.3: \( \pi \) extends to a projective representation \( U \) of \( G \) with cocycle \( \sigma \circ (q \times q) \), and tensoring with an irreducible \( \sigma \)-representation \( W \) of \( G/N \) gives an irreducible representation \( U \otimes (W \circ q) \) of \( G \) whose restriction to \( N \) is a multiple \( \pi \otimes 1 \) of \( \pi \).

Applications to compact Lie groups. When \( \Gamma \) is a compact connected Lie group, Moore computed \( H^2(\Gamma, \mathbb{T}) \) as follows. Let \( \Tilde{\Gamma} \) be the simply connected covering group of \( \Gamma \); then the fundamental group \( \pi_1(\Gamma) \) is isomorphic to a central subgroup of \( \Tilde{\Gamma} \) and \( \Gamma \cong \Tilde{\Gamma}/\pi_1(\Gamma) \). An inflation and restriction sequence identifies \( H^2(\Gamma, \mathbb{T}) \) with the quotient of the dual group \( \pi_1(\Gamma)^\wedge = \text{Hom}(\pi_1(\Gamma), \mathbb{T}) \) by the image of the restriction map \( \text{Res} : (\Tilde{\Gamma})^\wedge \to \pi_1(\Gamma)^\wedge \ [9] \text{ page 55} \).

When \( \Gamma = \mathbb{T}^n \), we have \( \pi_1(\Gamma) = \mathbb{Z}^n \) and \( \Tilde{\Gamma} = \mathbb{R}^n \), and the restriction map \( \mathbb{R}^n = (\mathbb{R}^n)^\wedge \to \mathbb{T}^n = (\mathbb{Z}^n)^\wedge \) is onto by duality. Thus \( H^2(\mathbb{T}^n, \mathbb{T}) = 0 \). Theorem \cite{2} thus implies that if \( G/N \cong \mathbb{T}^n \), then every \( G \)-invariant irreducible unitary representation of \( N \) extends to \( G \). Because representations of compact groups are direct sums of irreducible representations, this observation includes \cite{11} Corollary 3.5, and hence also \cite{11} Theorem 1.1.

For non-compact groups \( G \), it is not clear how one might prove Corollary \cite{3} by reducing to the irreducible case: the analogue of the direct-sum decomposition would be a direct-integral decomposition, but not every unitary representation is a direct integral of irreducible representations. (See \cite{6} §10 for a discussion of decomposing representations as direct integrals.)

The nonabelian duality approach. If \( \alpha : G \to \text{Aut}(A) \) is a strongly continuous action of a locally compact group \( G \) on a \( C^* \)-algebra \( A \), a covariant representation of \( (A, G, \alpha) \) consists of a representation \( \mu \) of \( A \) and a unitary representation \( U \) of \( G \) such that
\[
\mu(\alpha_t(a)) = U_t\mu(a)U^*_t \quad \text{for } a \in A \text{ and } t \in G;
\]
covariant representations can take values either in abstract \( C^* \)-algebras or in the concrete \( C^* \)-algebra \( B(\mathcal{H}) \). The crossed product \( A \times_\alpha G \) is the \( C^* \)-algebra generated by a universal covariant representation in the multiplier algebra \( M(A \times_\alpha G) \) (see \cite{12} for details of what this means). The covariant representations \( (\mu, U) \) of \( (A, G, \alpha) \) therefore give representations \( \mu \times U \) of \( A \times_\alpha G \), and all representations of \( A \times_\alpha G \) have this form. We shall be particularly interested in the actions \( \text{lt} : G \to \text{Aut}(C_0(G/N)) \) and \( \text{rt} : G/N \to \text{Aut}(C_0(G/N)) \) defined by
\[
\text{lt}_s(f)(uN) = f(s^{-1}uN) \quad \text{and} \quad \text{rt}_t(f)(uN) = f(utN).
\]
The automorphisms \( \text{rt}_{tN} \) commute with the automorphisms \( \text{lt}_s \), and hence induce an action \( \beta \) of \( G/N \) on the crossed product \( C_0(G/N) \times_{\beta} G \).

If \( \pi \) is a unitary representation of \( N \), then the induced representation \( \text{Ind} \pi \) of \( G \) acts in the completion \( \mathcal{H}(\text{Ind} \pi) \) of
\[
\{ \xi \in C_b(G, \mathcal{H}) : \xi(tn) = \pi(n)^{-1}(\xi(t)) \text{ and } (tN \mapsto ||\xi(t)||) \in C_c(G/N) \}
\]
with respect to the inner product $(\xi | \eta) = \int_{G/N} (\xi(t) | \eta(t)) \, d(tN)$, according to the formula $(\text{Ind} \pi)_r(\xi)(r) = \xi(t^{-1}r)$. (See, for example, [15] page 296; because $N$ is normal there is a $G$-invariant measure on $G/N$, and we can take the rho-function in the usual formula to be 1.)

Let $M$ be the representation of $C_0(G/N)$ by multiplication operators on $\mathcal{H}(\text{Ind} \pi)$, and note that $(M, \text{Ind} \pi)$ is a covariant representation of $(C_0(G/N), G, \text{lt})$. The nonabelian duality approach to the extension problem yields the following theorem.

**Theorem 4.** Suppose that $N$ is a closed normal subgroup of an amenable and second-countable locally compact group $G$, and suppose that $\pi$ is a unitary representation $\pi : N \to U(\mathcal{H})$. Then $\pi$ extends to a unitary representation of $G$ if and only if there exists a unitary representation $Q : G/N \to U(\mathcal{H}(\text{Ind} \pi))$ such that $(M \times \text{Ind} \pi, Q)$ is a covariant representation of $(C_0(G/N) \times_{\text{lt}} G, G/N, \beta)$.

**Proof.** The induction-restriction theory of [3] says that $\pi$ is the restriction of a representation of $G$ if and only if $M \times \text{Ind} \pi$ is induced, in a dual sense, from a representation of the group $C^*\text{-algebra } C^*(G) = \mathbb{C} \times G$. To deduce this from [3] Theorem 5.16], we need to recall some ideas of nonabelian duality. The group $C^*\text{-algebra } C^*(G)$ is generated by a universal unitary representation $\iota : G \to UM(C^*(G))$. The comultiplication $\delta : C^*(G) \to M(C^*(G) \otimes C^*(G))$ is the representation corresponding to the unitary representation $\iota \otimes \iota$; it has a restriction $\delta|_N$ which is a coaction of $G/N$ on $C^*(G)$. Since $G$ is amenable, $C^*(G)$ coincides with the reduced group $C^*\text{-algebra } C^r_*(G)$, and hence we can apply results from [3] and [8] concerning reduced crossed products. In particular, we can induce representations from $C^*(G)$ to the coaction crossed product $C^*(G) \times_\delta G/N$ by tensoring with a $(C^*(G) \times_\delta G/N)\text{-C}^*(N)$ bimodule $Y$ constructed by Mansfield [8]; the resulting map on representations is denoted by $Y\text{-Ind}.$

We recall from [15] Theorem C.23 that there is a Morita equivalence between $C_0(G/N) \times_{\text{lt}} G$ and $C^*(N)$ which is implemented by an imprimitivity bimodule $X$; we denote by $X\text{-Ind}$ the corresponding map on representations. The algebras $C_0(G/N) \times_{\text{lt}} G$ and $C^*(G) \times_\delta G/N$ have exactly the same covariant representations, and hence are isomorphic (see, for example, [3] Theorem A.64]). Thus we can view $X$ as a $(C^*(G) \times_\delta G/N)\text{-C}^*(N)$ bimodule. Theorem 5.16 of [3] (with $A = \mathbb{C}$ and $M = G$) says that, provided $G$ is amenable, we have a commutative diagram

$$\begin{array}{ccc}
\text{Rep} C^*(G) & \xrightarrow{\text{Res}} & \text{Rep} C^*(N) \\
\downarrow \text{Y-Ind} & & \downarrow \text{X-Ind} \\
\text{Rep}(C^*(G) \times_\delta G/N) & & \\
\end{array}$$

Since $X\text{-Ind}$ is a bijection, it follows that a representation $\pi$ of $C^*(N)$ extends to a representation of $C^*(G)$ if and only if $X\text{-Ind} \pi$ is in the range of $Y\text{-Ind}.$

To derive Theorem 4 from this, we have to make two observations. First, the representation $X\text{-Ind} \pi$ of $C_0(G/N) \times_{\text{lt}} G$ is equivalent to $M \times \text{Ind} \pi$. To see this, note that the intertwining unitary isomorphism $W$ of $(X \otimes_{C^*(N)} \mathcal{H}, X\text{-Ind} \pi)$ onto $(\mathcal{H}(\text{Ind} \pi), \text{Ind} \pi)$ constructed in the proof of [15] Theorem C.33] carries the left action of $C_0(G/N)$ into
operators defines a unitary operator $U$

Remark 5. Comparing Theorem 4 with Theorem 2, it is natural to ask what happened to the hypothesis “π is $G$-invariant”. Suppose π is $G$-invariant, so that there exist unitary operators $W_s$ on $\mathcal{H}$ such that $W_s \pi(n) W_s^* = \pi(sns^{-1})$. Then

$$U_s(\xi)(t) = W_s(\xi(ts))\Delta(sN)^{1/2}$$

defines a unitary operator $U_\lambda$ on $\mathcal{H}(\text{Ind} \pi)$ which intertwines the covariant representations $(M, \text{Ind} \pi)$ and $(M \rtimes \text{rt}_s N, \text{Ind} \pi)$. So $R_{sN} := U_{c(sN)}$ defines a map $R : G/N \to U(\mathcal{H}(\text{Ind} \pi))$ which formally satisfies the covariance relations but is not necessarily a representation.

Our original extension problem for a $G$-invariant representation $\pi : N \to U(\mathcal{H})$ therefore reduces to:

Given a representation $\phi$ of $C^*(G) \rtimes G/N$ such that $\phi \circ (\delta)_s N$ is equivalent to $\phi$ for every $sN \in G/N$, is there a representation $Q$ of $G/N$ such that $(\phi, Q)$ is covariant for $(C^*(G) \rtimes G/N, \delta_\lambda)$?

Since there are by hypothesis unitary operators $R_{sN}$ such that $\phi \circ (\delta)_s N = \text{Ad} R_{sN} \circ \phi$, we can repeat the analysis of Theorem 2 to see that there is a twisted action $(\beta, \omega)$ of $G/N$ on the commutant of the range of $\phi$, such that $(\beta, \omega)$ is exterior equivalent to an ordinary action if and only if we can adjust the $R_{sN}$ to obtain the required representation $Q$. Thus $\beta_{sN} = \text{Ad} R_{sN}$ and, for $\xi \in \mathcal{H}(\text{Ind} \pi)$,

$$\omega(rN, sN)(\xi)(t) = R_{rN} R_{sN} R_{rsN}^*(\xi)(t) = U_{c(rN)} U_{c(sN)} U_{c(rsN)}^*(\xi)(t) = W_{c(rN)} W_{c(sN)} W_{c(rsN)}^*(\xi(tc(rN)c(sN)c(rsN)^{-1}))(\xi(t)) = W_{c(rN)} W_{c(sN)} W_{c(rsN)}^* \pi(c(rN)c(sN)c(rsN)^{-1})(\xi(t)) = W_{c(rN)} W_{c(sN)} W_{c(rsN)}^* \pi(c(rN)c(sN)^{-1}c(rN)^{-1})(\xi(t)).$$

We claim that the obstruction $(\beta, \omega)$ is essentially the same as the obstruction $(\alpha, \sigma)$ to extending $\pi$ from Theorem 2. To see this, we first identify $\pi(N)'$ with $\phi(C^*(G) \rtimes G/N)'$ when $\phi = M \times \text{Ind} \pi$. If $T \in \pi(N)'$, then the formula $1 \otimes T(\xi)(t) = T(\xi(t))$ defines an operator in $\phi(C^*(G) \rtimes G/N)'$. When we view $\mathcal{H}(\text{Ind} \pi)$ as $X \otimes C^*(N) \mathcal{H}$, then we recover $\mathcal{H}$ as $\bar{X} \otimes C^*(G) \rtimes G/N (X \otimes C^*(N) \mathcal{H})$, where $\bar{X}$ is the dual imprimitivity bimodule, and the natural isomorphism of $\mathcal{H}$ onto $\bar{X} \otimes C^*(G) \rtimes G/N (X \otimes C^*(N) \mathcal{H})$ takes $T$ to $1 \otimes 1 \otimes T$. Thus $T \mapsto 1 \otimes T$ is an isomorphism of $\phi(C^*(G) \rtimes G/N)'$.

With $V$ as in Equation (11), the cocycle $\sigma$ in the twisted action $(\alpha, \sigma)$ satisfies

$$\sigma(rN, sN) = V_r V_s V_{rs}^* = W_{c(rN)} \pi(c(rN)^{-1}r) W_{c(sN)} \pi(c(sN)^{-1}s) \pi(c(rsN)^{-1}rs)^{-1} W_{c(rsN)}^* = W_{c(rN)} W_{c(sN)} \pi(c(sN)^{-1}c(rN)^{-1}r c(sN)) \pi(c(sN)^{-1}s) \pi(s^{-1}r^{-1}c(rsN)) W_{c(rsN)}^* = W_{c(rN)} W_{c(sN)} \pi(c(sN)^{-1}c(rN)^{-1}c(rsN)) W_{c(rsN)}^* = W_{c(rN)} W_{c(sN)} W_{c(rsN)}^* \pi(c(rsN)c(sN)^{-1}c(rN)^{-1}).$$
Thus with this choice of $R_{sN}$, we have $\omega(rN, sN) = 1 \otimes \sigma(rN, sN)$, and for $T \in \pi(N)'$,
\[
\beta_{sN}(1 \otimes T)(\xi)(t) = R_{sN}(1 \otimes T)R_{sN}^*(\xi)(t)
\]
\[
= W_{c(sN)}TW_{c(sN)}^*(\xi)(t))
\]
\[
= (1 \otimes V_sTV_s^*)(\xi)(t)
\]
\[
= (1 \otimes \alpha_s(T))(\xi)(t).
\]
So the isomorphism of $\pi(N)'$ onto $\phi(C^*(G) \times_{\delta|} G/N)'$ carries $(\alpha, \sigma)$ into the twisted action $(\beta, \omega)$ which obstructs the existence of $Q$. Thus, reassuringly, the cohomological obstruction to finding $Q$ is identical to the obstruction to extending $\pi$.

References

[1] J.W. Cho, M.K. Kim and D.Y Suh, On extensions of representations for compact Lie groups, J. Pure Appl. Algebra 178 (2003), 245–254.
[2] A.H. Clifford, Representations induced in an invariant subgroup, Ann. of Math. 38 (1937), 533–550.
[3] S. Echterhoff, S. Kaliszewski, J. Quigg and I. Raeburn, A categorical approach to imprimitivity theorems for $C^*$-dynamical systems, preprint, arXiv:math.OA/0205322.
[4] U. Haagerup, The standard form of von Neumann algebras, Math Scand. 37 (1975), 271–283.
[5] R.V. Kadison and J.R. Ringrose, Fundamentals of the theory of operator algebras II, Graduate Studies in Mathematics, vol. 16, Amer. Math. Soc., Providence, 1997.
[6] G.W. Mackey, Borel structure in groups and their duals, Trans. Amer. Math. Soc. 85 (1957), 134–165.
[7] G.W. Mackey, Unitary representations of group extensions. I, Acta Math 99 (1958), 265–311.
[8] K. Mansfield, Induced representations of crossed products by coactions, J. Funct. Anal. 97 (1991), 112–161.
[9] C.C. Moore, Extensions and low dimensional cohomology theory of locally compact groups. I, Trans. Amer. Math. Soc. 113 (1964), 40–63.
[10] C.C. Moore, Extensions and low dimensional cohomology theory of locally compact groups. II, Trans. Amer. Math. Soc. 113 (1964), 64–86.
[11] C.C. Moore, Group extensions and cohomology for locally compact groups. III, Trans. Amer. Math. Soc. 221 (1976), 1–33.
[12] C.C. Moore, Group extensions and cohomology for locally compact groups. IV, Trans. Amer. Math. Soc. 221 (1976), 35–58.
[13] J.A. Packer and I. Raeburn, Twisted crossed products of $C^*$-algebras, Math. Proc. Camb. Phil. Soc. 106 (1989), 293–311.
[14] I. Raeburn, On crossed products and Takai duality, Proc. Edinburgh Math. Soc. 31 (1988), 321–330.
[15] I. Raeburn and D.P. Williams, Morita Equivalence and Continuous-Trace $C^*$-Algebras, Math. Surveys and Monographs, vol. 60, Amer. Math. Soc., Providence, 1998.
[16] C.E. Sutherland, Cohomology and extensions of von Neumann algebras. I, Publ. Res. Inst. Math. Sci. 16 (1980), 105–133.
[17] C.E. Sutherland, Cohomology and extensions of von Neumann algebras. II, Publ. Res. Inst. Math. Sci. 16 (1980), 135–174.

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