Abstract. We give a formal account of Bénabou’s theorem for pseudoadjunctions in the context of Gray-categories. We prove that to give a pseudoadjunction $F \dashv U : A \to X$ with unit $\eta$ in a Gray-category $K$ is precisely to give an absolute left (Kan) pseudoextension $U$ of $1_X$ along $F$ witnessed by $\eta$.

Bénabou showed in [1] that an adjunction

$\begin{array}{c}
\mathcal{A} \\
F \\
\downarrow
\end{array} \quad \begin{array}{c}
\mathcal{X} \\
U \\
\downarrow
\end{array}
\begin{array}{c}
\Rightarrow
\end{array}$

(with unit $\eta$) between categories $\mathcal{A}$ and $\mathcal{X}$ can be characterised as an absolute left Kan extension

$\begin{array}{c}
\mathcal{X} \xrightarrow{F} \mathcal{A} \\
\downarrow \eta \\
1 \\
\downarrow U
\end{array}$

of $1$ along $F$. In this note we are interested in proving a correspondence very similar to the above: a pseudoadjunction $F \dashv U : A \to X$ (below left)

$\begin{array}{c}
A \\
F \\
\downarrow
\end{array} \quad \begin{array}{c}
X \\
U \\
\downarrow
\end{array}
\begin{array}{c}
\Rightarrow
\end{array}$

in a Gray-category $K$ is precisely an absolute left pseudoextension (above right) of $1$ along $F$ in $K$.

Thus our aim is to reproduce Bénabou’s result for the weaker notion of a pseudoadjunction. Pseudoadjunctions are interesting: they abound, e.g., in the study of pseudomonads. An important example is the theory of free cocompletions of categories [6]. The notion of a pseudoextension already appears in [7].

Instead of working with 2-categories, pseudofunctors and pseudonatural transformations and studying pseudoadjunctions in this setting, we work in the framework of Gray-categories, that is, categories enriched in the category $\mathcal{V} = \text{Gray}$ of 2-categories and 2-functors, equipped with Gray-tensor product [5].

We will structure this note as follows:

- The necessary background is covered in Section 1.
• Section 2 contains the definitions of pseudoadjunctions and pseudoextensions.
• The proof of Bénabou’s theorem appears in Section 3.

1. Background

In this note we shall work with Gray-categories, i.e., categories enriched in the category $\mathcal{V} = \text{Gray}$ (see [6] for the theory of enriched categories), where Gray is a symmetric monoidal closed category of 2-categories with the Gray tensor product (as described in [3] or [4]).

Gray-categories. Recall that a Gray-category $K$ with objects $A$, $B$, $C$, has a hom-2-category $K(A, B)$ for each pair $A$, $B$ of objects, the unit 2-functor $u_A : I \rightarrow K(A, A)$ sends the unique $i$-cell ($i = 0, 1, 2$) to the identity $(i+1)$-cell of $A$ (e.g., the unique object $\ast$ of $I$ gets sent to $1_A : A \rightarrow A$), and the composition

$$K(B, C) \otimes K(A, B) \rightarrow K(A, C)$$

is essentially the composition cubical functor

$$K(B, C) \times K(A, B) \rightarrow K(A, C)$$

yielding, for any $F$ in $K(A, B)$ and any $G$ in $K(B, C)$, two 2-functors

$$(-)F : K(B, C) \rightarrow K(A, C)$$

$$G(-) : K(A, B) \rightarrow K(A, C)$$

(“precomposition” and “postcomposition”), with $(-)F$ acting on the data

$$\begin{array}{ccc}
B & \overset{H}{\Rightarrow} & C \\
\downarrow & \alpha & \downarrow \beta \\
K & \downarrow \tau
\end{array}$$

and $G(-)$ acting on

$$\begin{array}{ccc}
B & \overset{HF}{\Rightarrow} & C \\
\downarrow & \alpha F & \downarrow \beta F \\
K & \downarrow \tau F
\end{array}$$

$$\begin{array}{ccc}
A & \overset{H}{\Rightarrow} & B \\
\downarrow & \alpha & \downarrow \beta \\
K & \downarrow \tau
\end{array}$$

and

$$\begin{array}{ccc}
A & \overset{GH}{\Rightarrow} & B \\
\downarrow & G\alpha & \downarrow G\beta \\
K & \downarrow G\tau
\end{array}$$

The 2-functors $(-)F$ and $G(-)$ are subject to the equality

$$(G)F = G(F) = GF$$
and for every pair

\[
\begin{array}{ccc}
A & \xymatrix{ F \ar[r]^{\alpha} & B } & B & \xymatrix{ G \ar[r]_{\beta} & C }
\end{array}
\]

there is an isomorphism

\[
\begin{array}{ccc}
GF & \xymatrix{ \beta F \ar[d]_{G\alpha} & G'F \ar[l]_{G'\alpha} }
\end{array}
\]

subject to the cubical functor axioms:

1. Composition axioms

\[
\begin{array}{ccc}
GF & \xymatrix{ \beta F \ar[r] & G'F \ar[d]_{G\alpha} }
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} }
\end{array}
\]

\[
\begin{array}{ccc}
GF'' & \xymatrix{ \beta F'' \ar[r] & G''F'' }
\end{array}
\]

\[
\begin{array}{ccc}
GF'' & \xymatrix{ \beta F'' \ar[r] & G''F'' }
\end{array}
\]

and

\[
\begin{array}{ccc}
GF & \xymatrix{ \beta F \ar[r] & G'F \ar[d]_{G\alpha} } & \delta F \ar[r] & G''F
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F
\end{array}
\]

\[
\begin{array}{ccc}
GF'' & \xymatrix{ \beta F'' \ar[r] & G''F'' }
\end{array}
\]

\[
\begin{array}{ccc}
GF'' & \xymatrix{ \beta F'' \ar[r] & G''F'' }
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F'
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F'
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F'
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F'
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F'
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F'
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F'
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F'
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F'
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F'
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F'
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F'
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F'
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F'
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F'
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F'
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F'
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F'
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F'
\end{array}
\]

\[
\begin{array}{ccc}
GF' & \xymatrix{ \beta F' \ar[r] & G'F' \ar[d]_{G'\alpha} } & \delta F' \ar[r] & G'''F'
\end{array}
\]
(2) “Modification” axioms

\[
\begin{align*}
GF &\xrightarrow{\beta F} GF' > G'F \\
G &\xrightarrow{\beta \alpha} G' \xleftarrow{G' \alpha} G' \\
GF' &\xrightarrow{\beta F'} GF' \\
&= \\
GF'' &\xrightarrow{\beta F''} GF'' > G'F'' \\
G &\xrightarrow{G' \alpha} G' \xleftarrow{G' \alpha} G' \\
GF'' &\xrightarrow{\beta F''} GF'' \\
&= \\
\end{align*}
\]

for any 3-cell \(s : \alpha' \Rightarrow \alpha\) and

\[
\begin{align*}
GF &\xrightarrow{\beta F} G'F \\
G &\xrightarrow{\beta \alpha} G' \xleftarrow{G' \alpha} G' \\
GF' &\xrightarrow{\beta F'} G'F' \\
&= \\
GF'' &\xrightarrow{\beta F''} G'F'' \\
G &\xrightarrow{\beta \alpha} G' \xleftarrow{G' \alpha} G' \\
GF'' &\xrightarrow{\beta F''} G'F'' \\
&= \\
\end{align*}
\]

for any 3-cell \(t : \beta' \Rightarrow \beta\).

Given any triple

\[
\begin{align*}
A &\xrightarrow{\alpha} B \\
\xrightarrow{F} F' & B \\
\xrightarrow{G} G' & B \xrightarrow{\beta} C \\
\xrightarrow{H} H' & C \xrightarrow{\gamma} D \\
\end{align*}
\]

the associativity equalities

\[
\gamma_{(\beta F)} = (\gamma_{\beta}) F, \quad \gamma_{(G \alpha)} = (\gamma G) \alpha, \quad H(\beta \alpha) = (H \beta) \alpha
\]

hold, allowing us to relax the notation when working with the invertible 3-cells \(\beta \alpha\). Finally, the unit equalities

\[
1F = F, \quad G1 = G
\]

hold, where 1 can stand for the identity 1-cell, 2-cell or 3-cell on \(B\).

**Duality.** Bénabou’s theorem admits variations based on duality, both in the case of ordinary categories and in the setting of Gray-categories. We introduce two dual constructions on a Gray-category \(K\).

- The **horizontal dual** \(K^{op}\) of \(K\) is defined by reversing the 1-cells of \(K\). That is,

\[
K^{op}(A, B) = K(B, A).
\]

Composition in \(K^{op}\) is defined by the symmetry of the Gray-tensor product, as is usual in the context of enriched categories.
The vertical dual \( K^{co} \) of \( K \) is defined by reversing the 2-cells of \( K \). That is, we put
\[
K^{co}(A, B) = (K(A, B))^\text{op};
\]
obs...
Duality operations with the Gray-category $K$ transform pseudoadjunctions into pseudoadjunctions. The roles of the defining data have to be swapped accordingly.

**Remark 2.2.** Suppose we are given the category $K$ as in Definition 2.1 and the data for the pseudoadjunction $F \dashv U : A \to X$.

1. In $K^{\text{op}}$, the same data transform into a pseudoadjunction $U \dashv F : X \to A$ due to the reversal of 1-cells. The unit $\eta$ and counit $\varepsilon$ stay the same, as well as the coherence 3-cells $s$ and $t$.
2. In $K^{\text{co}}$, the same data transform into a pseudoadjunction $U \dashv F : X \to A$, but with unit $\varepsilon$ and counit $\eta$; the coherence 3-cells $s$ and $t$ stay the same, although their role as witnesses for the triangle isomorphisms is swapped.

The notion of a (left) pseudoextension is the appropriate weakening of the usual notion of a (left) Kan extension.

**Definition 2.3** (Left pseudoextension [2, 7]). In a Gray-category $K$, we say that

$$
\begin{array}{c}
X \xrightarrow{J} A \\
\downarrow \eta \Downarrow \Rightarrow \\
H \xrightarrow{\Downarrow} L \\
\downarrow \Rightarrow B
\end{array}
$$

exhibits $L$ as a left pseudoextension of $H$ along $J$ if for each

$$
\begin{array}{c}
X \xrightarrow{J} A \\
\downarrow f \Downarrow \Rightarrow \\
H \xrightarrow{\Downarrow} K \\
\downarrow \Rightarrow B
\end{array}
$$

(i.e., $f : H \Rightarrow KJ$) there is a 2-cell $f^\sharp : L \Rightarrow K$ and an isomorphism 3-cell

$$
\begin{array}{c}
H \xrightarrow{\eta} LJ \\
\downarrow \mu(f) \Downarrow \Rightarrow \\
f \xrightarrow{\Downarrow} f^\sharp J \\
\downarrow \Rightarrow KJ
\end{array}
$$

such that for each $k : L \Rightarrow K$ and a 3-cell

$$
\begin{array}{c}
H \xrightarrow{\eta} LJ \\
\downarrow \omega \Downarrow \Rightarrow \\
f \xrightarrow{\Downarrow} kJ \\
\downarrow \Rightarrow KJ
\end{array}
$$
there is a unique 3-cell $\tilde{\omega} : k \Rightarrow f^2$ such that

$$\begin{array}{c}
\begin{array}{c}
H \xrightarrow{\eta} LJ \\
\downarrow f \quad \downarrow \mu(f) \\
KJ \xleftarrow{\tilde{\omega}J}
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
H \xrightarrow{\eta} LJ \\
\downarrow f \quad \downarrow \omega \\
KJ \xleftarrow{kJ}
\end{array}
\end{array}$$

We say that the pseudoextension $\eta : H \to LJ$ is preserved by $G : B \to C$ if the 2-cell

$$X \xrightarrow{J} A$$

exhibits $GL$ as a left pseudoextension of $GH$ along $J$.

The pseudoextension $\eta : H \to LJ$ is said to be absolute if it is preserved by any 1-cell $G : B \to C$.

The various notions of duality for Gray-categories allow us to express compactly the definition of (left/right) pseudoextensions and pseudoliftings via the definition of a left pseudoextension.

**Definition 2.4.** Given a left pseudoextension

$$X \xrightarrow{J} A$$

in a Gray-category $K$, we call it

1. a right pseudoextension of $H$ along $J$ in $K^{op}$.
2. a left pseudolifting of $H$ through $J$ in $K^{op}$.
3. a right pseudolifting of $H$ through $J$ in $K^{coop}$.

With these definitions at hand, we can move to the statement and proof of Bénabou’s theorem.

### 3. Bénabou’s theorem for pseudoextensions

Let us first recall ordinary Bénabou’s theorem [1].

**Theorem 3.1.** For functors $U : \mathcal{A} \to \mathcal{X}$ and $F : \mathcal{X} \to \mathcal{A}$ the following are equivalent:

1. $F \dashv U$ holds with unit $\eta$.
2. $\eta$ exhibits $U$ as an absolute left extension of $1_\mathcal{X}$ along $F$.
3. $\eta$ exhibits $U$ as a left extension of $1_\mathcal{X}$ along $F$, and this extension is preserved by $F$.
4. $\eta$ exhibits $F$ as an absolute left lifting of $1_\mathcal{X}$ through $U$.
5. $\eta$ exhibits $F$ as a left lifting of $1_\mathcal{X}$ through $U$, and this lifting is preserved by $U$.

The pseudo-version of Bénabou’s theorem can be stated in the same way, changing the notions of adjunction and extension/lifting to the notions of pseudoadjunction and pseudoextension/pseudolifting.
Theorem 3.2. Given two 1-cells \( U : A \to X \) and \( F : X \to A \) in a \( \text{Gray} \)-category \( K \), the following are equivalent:

1. \( F \dashv U \) is a pseudoadjunction with unit \( \eta \).
2. \( \eta \) exhibits \( U \) as an absolute left pseudoextension of \( 1_X \) along \( F \).
3. \( \eta \) exhibits \( U \) as a left pseudoextension of \( 1_X \) along \( F \), and this extension is preserved by \( F \).
4. \( \eta \) exhibits \( F \) as an absolute left pseudolifting of \( 1_X \) through \( U \).
5. \( \eta \) exhibits \( F \) as a left pseudolifting of \( 1_X \) through \( U \), and this lifting is preserved by \( U \).

The proof strategy in the ordinary case and in the pseudo-case is the same: it is enough to prove the implications (1) \( \Rightarrow \) (2) and (3) \( \Rightarrow \) (1). This is because (2) \( \Rightarrow \) (3) is trivial, and because the equivalence of (1), (4) and (5) follows by duality. Moreover, the ordinary proofs can serve as a guidance for the proofs of the pseudo-case.

Lemma 3.3 (The implication (3) \( \Rightarrow \) (1)). Suppose that

\[
\begin{array}{ccc}
X & \xrightarrow{F} & A \\
\downarrow \eta & & \downarrow U \\
X & & \\
\end{array}
\]

is a left (Kan) pseudoextension preserved by \( F \). Then \( \eta \) can be made a unit of a pseudoadjunction \( F \dashv U \).

Proof. Recall that

\[
\begin{array}{ccc}
X & \xrightarrow{F} & A \\
\downarrow \eta & & \downarrow U \\
X & & \\
\end{array}
\]

is a left pseudoextension if for each

\[
\begin{array}{ccc}
X & \xrightarrow{F} & A \\
\downarrow f & & \downarrow K \\
X & & \\
\end{array}
\]

(i.e., \( f : 1_X \Rightarrow KF \)) there is a 2-cell \( f^\sharp : U \Rightarrow K \) and an isomorphism 3-cell

\[
\begin{array}{ccc}
1_X & \xrightarrow{\eta} & UF \\
\downarrow \mu(f) & & \downarrow f^\sharp F \\
\downarrow f & & \downarrow KF \\
\end{array}
\]
such that for each $k : U \Rightarrow K$ and a 3-cell

\[
\begin{array}{ccc}
1_X & \xrightarrow{\eta} & UF \\
\downarrow & \Downarrow \omega & \downarrow f \\
& \hat{k}F & \\
& \hat{\omega}F & \uparrow \eta \\
& KF & \end{array}
\]

there is a unique 3-cell $\hat{\omega} : k \Rightarrow f^2$ such that

\[
\begin{array}{ccc}
1_X & \xrightarrow{\eta} & UF \\
\downarrow & \Downarrow \mu(f) & \downarrow f \\
& \hat{\omega}F & \uparrow \eta \\
& \hat{k}F & \uparrow \omega \\
& KF & \end{array}
\]

For the purpose of establishing notation, we describe the data concerning the left pseudoextension

\[
\begin{array}{ccc}
X & \xrightarrow{F} & A \\
\downarrow F\eta & \Downarrow & \downarrow FU \\
F & \Rightarrow & F \\
\downarrow & \downarrow \mu & \downarrow \nu \\
& 1_A & \Rightarrow \Rightarrow \\
& A & \end{array}
\]

Given, e.g., the identity

\[
\begin{array}{ccc}
X & \xrightarrow{F} & A \\
\downarrow 1_F & \Downarrow & \downarrow 1_A \\
F & \Rightarrow & 1A \\
\downarrow & \downarrow \epsilon & \downarrow \epsilon \\
& A & \end{array}
\]

(the 2-cell $1_F : F \Rightarrow F$), we have a 2-cell $(1_A)^2 : FU \Rightarrow 1_A$ that we will denote by $\varepsilon$ and which will be the counit of the pseudoadjunction we construct. With this counit comes an isomorphism

\[
\begin{array}{ccc}
F & \xrightarrow{F\eta} & FUF \\
\downarrow \Downarrow s^{-1} & \downarrow \varepsilon & \downarrow F \\
1_F & \Rightarrow & hF \\
\downarrow & \downarrow & \downarrow \\
F & \Rightarrow & F
\end{array}
\]

such that for each $h : FU \Rightarrow 1_A$ and a 3-cell

\[
\begin{array}{ccc}
F & \xrightarrow{F\eta} & FUF \\
\downarrow \Downarrow \nu & \downarrow hF & \downarrow \eta \\
1_F & \Rightarrow & \Rightarrow \\
\downarrow & \downarrow & \downarrow \\
F & \Rightarrow & F
\end{array}
\]
there is a unique $\hat{\nu} : h \Rightarrow \varepsilon$ satisfying

$$
\begin{array}{ccc}
F & \xrightarrow{F\eta} & FUF \\
\downarrow s^{-1} & \Leftrightarrow & \downarrow \hat{\nu} \\
1_F & \Leftrightarrow & hF \\
F & \leftarrow & F
\end{array}
\quad
\begin{array}{ccc}
F & \xrightarrow{F\eta} & UF \\
\downarrow kF = & \Leftrightarrow & \downarrow hF \\
1_F & \Leftrightarrow & F
\end{array}
$$

Let us first observe that $s^{-1}$ (or, equivalently, its inverse $s$) witnesses the first triangle axiom of a pseudomonad. To obtain the second triangle isomorphism, consider that $\eta : 1_X \Rightarrow UF$ lifts to the identity $1_U = \eta^t : U \Rightarrow U$ with the identity 3-cell $(\mu(\eta) = 1_\eta)$

$$
\begin{array}{ccc}
1_X & \xrightarrow{\eta} & UF \\
\downarrow \eta & \Leftrightarrow & \downarrow 1_{UF} \\
UF & \leftarrow & UF
\end{array}
$$

By the universal property of the left pseudoextension given by $\eta$ we get that for the 2-cell

$$
U \xRightarrow{\eta_U} UFU \xRightarrow{U\varepsilon} U
$$

and the 3-cell

$$
\begin{array}{ccc}
1_X & \xrightarrow{\eta} & UF \\
\downarrow \eta & \Leftrightarrow & \downarrow \eta UF \\
UF & \leftarrow & UF
\end{array}
$$

there is a unique 3-cell

$$
\begin{array}{ccc}
U & \xrightarrow{\eta_U} UFU \\
\downarrow t & \Leftrightarrow & \downarrow U\varepsilon \\
1_U & \Leftrightarrow & U
\end{array}
$$

such that the 3-cell

$$
\begin{array}{ccc}
1_X & \xrightarrow{\eta} & UF \\
\downarrow tF & \Leftrightarrow & \downarrow U\varepsilon F \\
1_{UF} & \Leftrightarrow & UF
\end{array}
$$
equals

\[
\begin{array}{c}
1_x & \xrightarrow{\eta} & UF \\
\downarrow & & \downarrow \\
UF & \xrightarrow{UF\eta} & UFUF \\
\downarrow & & \downarrow \\
UF & \xrightarrow{U^{-1}} & UF \\
\end{array}
\]

or, written differently, that the 3-cell

\[
\begin{array}{c}
UF & \xrightarrow{1_{UF}} & UF \\
\downarrow & & \downarrow \\
UF & \xrightarrow{UF\eta \ U} & UFUF \\
\downarrow & & \downarrow \\
UF & \xrightarrow{tF} & UF \\
\end{array}
\]

is equal to identity. This is precisely the first coherence axiom for pseudoadjunctions.

For the other coherence axiom, we need the 3-cell

\[
\begin{array}{c}
FU & \xrightarrow{1_{FU}} & FU \\
\downarrow & & \downarrow \\
FU & \xrightarrow{F\eta} & FUFU \\
\downarrow & & \downarrow \\
FU & \xrightarrow{FU\epsilon} & FU \\
\end{array}
\]

to be equal to identity as well.
We shall use that for each \( h : FU \Rightarrow 1_A \) and a 3-cell

\[
\begin{array}{c}
F \\
\downarrow \scriptstyle F\eta \\
FUF \\
\downarrow \scriptstyle \nu \\
hF \\
\end{array}
\]

there is a unique \( \hat{\nu} : h \Rightarrow \varepsilon \) satisfying

\[
\begin{array}{c}
F \\
\downarrow \scriptstyle s^{-1} \\
\varepsilon F \\
\downarrow \scriptstyle 1_F \\
F \\
\end{array}
\rightleftharpoons
\begin{array}{c}
F \\
\downarrow \scriptstyle F\eta \\
FUF \\
\downarrow \scriptstyle 1_F \\
F \\
\end{array}
\rightleftharpoons
\begin{array}{c}
F \\
\downarrow \scriptstyle kF \\
F \\
\end{array}
\]

Thus if we find two 3-cells \( \alpha, \beta : h \Rightarrow \varepsilon \) with

\[
\begin{array}{c}
F \\
\downarrow \scriptstyle s^{-1} \\
\varepsilon F \\
\downarrow \scriptstyle 1_F \\
F \\
\end{array}
\rightleftharpoons
\begin{array}{c}
F \\
\downarrow \scriptstyle F\eta \\
FUF \\
\downarrow \scriptstyle 1_F \\
F \\
\end{array}
\rightleftharpoons
\begin{array}{c}
F \\
\downarrow \scriptstyle \alpha F \\
F \\
\end{array}
\]

it means that \( \alpha = \beta \). Take now the 3-cell

\[
\begin{array}{c}
FU \\
\downarrow \scriptstyle F\eta U \\
FUFU \\
\downarrow \scriptstyle F\iota \\
FU \varepsilon \\
\downarrow \scriptstyle 1_{FU} \\
FU \Rightarrow \varepsilon \\
\end{array}
\rightleftharpoons
\begin{array}{c}
FU \varepsilon \\
\downarrow \scriptstyle 1_A \\
\end{array}
\]

for \( \alpha \) and the 3-cell

\[
\begin{array}{c}
FU \\
\downarrow \scriptstyle s^{-1}U \\
\varepsilon FU \\
\downarrow \scriptstyle 1_{FU} \\
FU \\
\end{array}
\rightleftharpoons
\begin{array}{c}
FU \varepsilon \\
\downarrow \scriptstyle \varepsilon^{-1} \\
\varepsilon \Rightarrow \varepsilon \\
\end{array}
\rightleftharpoons
\begin{array}{c}
FU \Rightarrow \varepsilon \\
\downarrow \scriptstyle \varepsilon \Rightarrow 1_A \\
\end{array}
\]
for $\beta$. We ask whether the 3-cells

$$
\begin{align*}
F & \xrightarrow{F\eta} FUF & \xrightarrow{F\etaUF} & FUUF & \xrightarrow{FU\varepsilon F} & FUF
\end{align*}
$$

and

$$
\begin{align*}
FUF & \xrightarrow{F\etaUF} FUUF & \xrightarrow{FU\varepsilon F} FUF
\end{align*}
$$

are equal. Pasting $s$ and $F\eta$, we can equivalently ask whether the 3-cells

$$
\begin{align*}
F & \xrightarrow{F\eta} FUF & \xrightarrow{FUUF} & FUUF & \xrightarrow{FU\varepsilon F} & FUUF
\end{align*}
$$

are equal.
and

\[
\begin{array}{c}
\xymatrix{
\text{F} & \text{F}\eta \ar[r] & \text{FU}\times\text{FU} \\
\ar[d]_{\text{F}\eta} & \ar[r]_{\text{F}\eta\times\text{F}\eta} & \\
\text{FU}\ar[r]_{\text{F}\times\text{FU}} & \text{FU}\times\text{FU} \\
\ar[r]_{\varepsilon\times\varepsilon} & \text{FU} \\
\ar[d]_{\varepsilon} & \\
\text{F} \\
\end{array}
\]

are equal. Using the first coherence axiom, the diagram (3.1) is equal to

\[
\begin{array}{c}
\xymatrix{
\text{F} & \text{F}\eta \ar[r] & \text{FU}\times\text{FU} \\
\ar[d]_{\text{F}\eta} & \ar[r]_{\text{F}\eta\times\text{F}\eta} & \\
\text{FU}\ar[r]_{\text{F}\times\text{FU}} & \text{FU}\times\text{FU} \\
\ar[r]_{\varepsilon\times\varepsilon} & \text{FU} \\
\ar[d]_{\varepsilon} & \\
\text{F} \\
\end{array}
\]

Let us take diagrams (3.2) and (3.3) and paste $\varepsilon_{\text{F}\eta}$ and $\varepsilon_{\text{F}\eta}$. The resulting diagrams

\[
\begin{array}{c}
\xymatrix{
\text{F} & \text{F}\eta \ar[r] & \text{FU}\times\text{FU} \\
\ar[d]_{\text{F}\eta} & \ar[r]_{\text{F}\eta\times\text{F}\eta} & \\
\text{FU}\ar[r]_{\text{F}\times\text{FU}} & \text{FU}\times\text{FU} \\
\ar[r]_{\varepsilon\times\varepsilon} & \text{FU} \\
\ar[d]_{\varepsilon} & \\
\text{F} \\
\end{array}
\]

\[
\begin{array}{c}
\xymatrix{
\text{F} & \text{F}\eta \ar[r] & \text{FU}\times\text{FU} \\
\ar[d]_{\text{F}\eta} & \ar[r]_{\text{F}\eta\times\text{F}\eta} & \\
\text{FU}\ar[r]_{\text{F}\times\text{FU}} & \text{FU}\times\text{FU} \\
\ar[r]_{\varepsilon\times\varepsilon} & \text{FU} \\
\ar[d]_{\varepsilon} & \\
\text{F} \\
\end{array}
\]
and

\[(3.5)\]

are equal by using the identities

and

The proof is therefore finished. \qed

**Lemma 3.4** (The implication \((1) \implies (2)\)). Suppose that \(F \dashv U : A \to X\) is a pseudoadjunction in a Gray-category \(K \) with

\[
\begin{align*}
X & \xrightarrow{F} A \\
\eta & \xrightarrow{1} U \\
X & \xrightarrow{\varepsilon} A
\end{align*}
\]
Then

\[
\begin{array}{ccc}
X & \xrightarrow{F} & A \\
\downarrow{\eta} & & \downarrow{U} \\
1 & \downarrow & X
\end{array}
\]

is an absolute left pseudolifting.

**Proof.** We need to show that for each \( G : X \to Y \) and \( g : G \Rightarrow KF \) there is a 2-cell \( g^\sharp : GU \Rightarrow K \) and an isomorphism

\[
\begin{array}{ccc}
G & \xrightarrow{G\eta} & GU \\
\downarrow{g} & \downarrow{\mu(g)} & \downarrow{g^\sharp F} \\
G & \xrightarrow{\nu(g)} & KF
\end{array}
\]

satisfying that for each \( k : GU \Rightarrow K \) and

\[
\begin{array}{ccc}
G & \xrightarrow{G\eta} & GU \\
\downarrow{\omega} & \downarrow{g} & \downarrow{\kappa F} \\
G & \xrightarrow{\nu(g)} & KF
\end{array}
\]

there is a unique 3-cell \( \hat{\omega} : k \Rightarrow g^\sharp \) such that

\[
\begin{array}{ccc}
G & \xrightarrow{G\eta} & GU \\
\downarrow{\mu(g)} & \downarrow{g^\sharp F} & \downarrow{\omega F} \\
K & \xrightarrow{\kappa F} & KF
\end{array}
\]

\[
\begin{array}{ccc}
G & \xrightarrow{G\eta} & GU \\
\downarrow{\kappa F} & \downarrow{g} & \downarrow{\kappa F} \\
K & \xrightarrow{K\epsilon} & K
\end{array}
\]

We shall define \( g^\sharp : GU \Rightarrow K \) as the 2-cell

\[
GU \xrightarrow{gU} KFU \xrightarrow{K\epsilon} K
\]
Then, for the 2-cell \( g : G \Rightarrow KF \) we define \( \mu(g) \) to be the 3-cell

\[
\begin{array}{ccc}
G & \xrightarrow{G\eta} & GU \\ \\
\downarrow{g} & \cong & \downarrow{gUF} \\
KF & \xrightarrow{KF\eta} & KFU \\ \\
\downarrow{1_{KF}} & \cong & \downarrow{K\varepsilon F} \\
KF & \xrightarrow{Ks^{-1}} & KF
\end{array}
\]

Now given a 3-cell

\[
\begin{array}{ccc}
G & \xrightarrow{G\eta} & GU \\ \\
\downarrow{\omega} & \cong & \downarrow{kF} \\
KF & \xrightarrow{\varepsilon} & KF
\end{array}
\]

we will show that the “lifted” 3-cell \( \hat{\omega} : k \Rightarrow g^2 \) is the 3-cell

\[
\begin{array}{ccc}
GU & \xleftarrow{1_{GU}} & GU \\ \\
gU & \cong & \downarrow{G\eta U} \\
KFU & \xleftarrow{kFU} & GUFU \\ \\
\downarrow{k\varepsilon^{-1}} & \cong & \downarrow{GU\varepsilon} \\
K & \xleftarrow{k} & GU
\end{array}
\]

Indeed, observe that the 3-cell

\[
\begin{array}{ccc}
G & \xleftarrow{G\eta} & GU \\ \\
\downarrow{gU} & \cong & \downarrow{G\eta U} \\
K & \xleftarrow{k} & GU \\ \\
\downarrow{kF} & \cong & \downarrow{kF}
\end{array}
\]
is the composite

which is equal (transforming the green subdiagram) to the 3-cell
and, transforming $Ks^{-1}$ to $GU s^{-1}$, the 3-cell

\[
\begin{array}{c}
\xymatrix{
K F & G U F \\
\omega \ar[r] & G U F}
\end{array}
\]

is equal to

\[
\begin{array}{c}
\xymatrix{
G & G U F \\
\omega \ar[r] & G U F}
\end{array}
\]

And since the whole coloured subdiagram equals identity by the pseudomonad coherence axiom, the diagram simplifies to $\omega$, showing that our choice of $\hat{\omega}$ was correct. Indeed, the choice of $\hat{\omega}$ is even the only possible one: each diagram in the following series is equal to $\hat{\omega}$.

(3.6)

\[
\begin{array}{c}
\xymatrix{
G U & K \\
\hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r] & \hat{\omega} \ar[r}
\[
\begin{array}{c}
\begin{array}{c}
1_{GU} \\
\downarrow \\
Gt^{-1} \\
\downarrow \\
GU FU \\
\downarrow \\
GU \\
gU \\
\downarrow \\
KFU \\
\end{array} \\
\begin{array}{c}
G\eta U \\
\downarrow \\
Gt \\
\downarrow \\
GU \varepsilon \\
\downarrow \\
k \\
gU \\
\downarrow \\
K \varepsilon \\
\end{array} \\
\begin{array}{c}
k \\
\downarrow \\
K \varepsilon \\
\end{array}
\end{array}
\]

\[\text{(3.7)}\]

\[
\begin{array}{c}
\begin{array}{c}
1_{GU} \\
\downarrow \\
Gt^{-1} \\
\downarrow \\
GU FU \\
\downarrow \\
GU \\
gU \\
\downarrow \\
KFU \\
\end{array} \\
\begin{array}{c}
G\eta U \\
\downarrow \\
Gt \\
\downarrow \\
GU \varepsilon \\
\downarrow \\
k \\
gU \\
\downarrow \\
K \varepsilon \\
\end{array} \\
\begin{array}{c}
k \\
\downarrow \\
K \varepsilon \\
\end{array}
\end{array}
\]

\[\text{(3.8)}\]
The proof is therefore complete. □

**Remark 3.5.** Having proved the implications \( (1) \implies (2) \) and \( (3) \implies (1) \), the proof of Theorem 3.2 is complete.

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