Receding horizon decoding of convolutional codes

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Abstract

Decoding of convolutional codes poses a significant challenge for coding theory. Classical methods, based on e.g. Viterbi decoding, suffer from being computationally expensive and are restricted therefore to codes of small complexity. Based on analogies with model predictive optimal control, we propose a new iterative method for convolutional decoding that is cheaper to implement than established algorithms, while still offering significant error correction capabilities. The algorithm is particularly well-suited for decoding special types of convolutional codes, such as e.g. doubly cyclic convolutional codes.

Key words: Convolutional codes, Hamming distance, decoding, receding horizon

1 Introduction

A central aim of coding theory is to protect transmitted or stored information against errors. A common technique is to split the sequence of information symbols into blocks of constant length and map each block injectively to a codeword of larger length. This map is called the encoding map and its image space is referred to as a block code. To protect the process from transmission errors requires a procedure that enables one to recover the sent message from the received one by projecting it back to the code. This is called decoding and forms the basis of error correction algorithms of current codes. Decoding is an inherently difficult task, but for block codes effective decoding algorithms

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are available that depend on the special algebraic structure of the codes; e.g. BCH codes, Reed-Solomon codes, or list decoding techniques, see e.g. [7,6].

In the sequel, we will consider block codes only as an intermediate step for decoding convolutional codes. Such convolutional codes are a natural generalization of block codes and have found widespread applications; see [10,12]. Algebraically, they are defined as submodules of $\mathbb{F}[z]^n$, spanned by the columns of a full rank rectangular polynomial matrix $G(z) \in \mathbb{F}^{n \times k}[z]$. In contrast to block codes, where a rather rich theory is available, there is no really efficient decoding algorithm known for convolutional codes. Classical decoding algorithms for convolutional codes such as e.g. Viterbi decoding [6] work well only for codes of moderate dimensions. There is thus considerable interest in developing efficient decoding algorithms for convolutional codes that improve known algorithms, such as Viterbi decoding.

Since convolutional codes can be interpreted as linear control systems defined over a finite field $\mathbb{F}$, see the survey paper [14] and the references therein, it is possible to apply the rich tools from linear systems theory and optimal control to study such codes. This approach offers a better understanding of convolutional codes and led already to new construction methods and algorithms; see [7,2,4,14,15]. From a systems theory point of view, decoding of convolutional codes can be interpreted in at least two different ways [13]. One interpretation treats decoding as a tracking problem, where the decoder attempts to track the received message by the most probable codeword sent. Another perspective lies in treating it as a filtering problem, where the decoder is requested to filter the noise introduced by the channel.

In this work we will follow the first approach, by treating the decoding problem of convolutional codes as a tracking exercise for linear systems to replace the received message by its closest polynomial codeword [13]. Motivated by analogies with model predictive control, we propose a simple receding horizon algorithm for convolutional decoding, that combines an arbitrary block decoding algorithm with a few receding horizon steps. The proposed method has the advantage of being computationally cheaper than well-known Viterbi decoding, while still achieving substantial error correction. In a companion paper [3] we show that our algorithm can lead to competitive decoding results for the class of doubly-cyclic convolutional codes.

The paper is structured as follows. Section 2 introduces basic terminology from coding theory and outlines a decoding procedure for convolutional codes via the Bellman optimality principle. In Section 3, the new algorithm is proposed and error correction capabilities of the method are established. Examples are discussed in section 4 to illustrate the decoding properties of the algorithm.
2 Dynamic programming approach to convolutional decoding

2.1 Coding theory terminology

We begin with a brief summary of basic notions from coding theory. For further information we refer to standard textbooks as, e.g., [7,9]. Throughout this paper, $\mathbb{F}$ denotes a finite field with $q$ elements. Given any full column rank matrix $G \in \mathbb{F}^{n \times k}$, a linear block code of length $n$ and dimension $k$ is a $k$-dimensional vector space

$$C := \{Gv | v \in \mathbb{F}^k\} \subset \mathbb{F}^n$$

and $G$ is called a generator matrix of $C$. If, possibly after a permutation of the rows, $G \in \mathbb{F}^{n \times k}$ is of the form

$$G = \begin{pmatrix} A \\ I_k \end{pmatrix},$$

then $G$ is called a systematic generator matrix. The check matrix of the code then is defined as

$$S = \begin{pmatrix} I_{n-k} - A \end{pmatrix}$$

and, since $C = \ker S$, it provides a kernel description of the code. A natural metric on a code is defined by the Hamming distance. The Hamming distance of vectors $x, y \in \mathbb{F}^n$ is defined by $d(x, y) := w(x - y)$, where

$$w(c) := \# \{i | c_i \neq 0\}.$$ 

denotes the Hamming weight. The minimum distance of a code $C$ then is the minimum among the distances between any pair of codewords, i.e.

$$d(C) := \min_{c \in C \backslash \{0\}} w(c).$$

It measures the error correcting capabilities for the code. In case a received vector $x \in \mathbb{F}^n$ does not belong to $C$, i.e. if some error has occurred in the encoding process, a natural way to recover the sent message is by taking the maximum likelihood estimate of $x$, i.e. to replace $x$ by the closest vector from $C$. This process is called decoding. A decoder is a map $\pi : \mathbb{F}^n \rightarrow C$ that is the identity on $C$. A special case in point are maximum likelihood decoders that are defined by the set valued map

$$\pi_C(x) := \arg \min_{c \in C} d(x, c) \subset C.$$
In order to achieve the best error correction properties of decoding, the vectors from \( C \) should be as far apart as possible. Thus one is searching for codes whose minimal distance is as large as possible. The maximum number of errors that a code can correct is known as the *error correction capacity* of the code, which is half the minimum distance of the code minus one. If a code of length \( n \) has minimum distance \( d = d(C) \), then the spheres centered at the codewords with radius \( \lfloor \frac{d-1}{2} \rfloor \) are disjoint. Any vector from \( F^n \) contained in one of these spheres can thus be decoded to its unique nearest codeword, the center of the sphere. But the union of these spheres may not cover all vectors of the ambient space. The *covering radius* \( \rho_C \) of a code \( C \) is the maximum distance from any vector of \( F^n \) to its nearest codeword. The covering radius of a code is the smallest radius needed for spheres centered at the codewords to cover the whole ambient space.

2.2 Convolutional codes and Bellman principle

In this paper we follow the well-established approach, see e.g. [13], that regards a convolutional code \( C \) of length \( n \), dimension \( k \) and complexity \( \delta \) as a submodule

\[
C := \{ G(z)v(z) | v(z) \in F^k[z] \} \subset F^n[z],
\]

defined by a full column rank polynomial matrix \( G(z) \in F^{n \times k}[z], \ k \leq n. \) Here the complexity \( \delta \) of \( C \), also referred to as the “constraint length” of the code [11], is defined as the maximal degree of all \( k \times k \) minors of \( G(z) \).

After a suitable permutation of the rows, we can assume that the generator matrix is of the form

\[
G(z) = \begin{pmatrix} P(z) \\ Q(z) \end{pmatrix}
\]

with right coprime polynomial factors \( P(z) \in F^{(n-k) \times k} \) and \( Q(z) \in F^{k \times k} \), respectively. Here \( \delta = \deg \det Q(z) \) is assumed to be the maximal degree of all \( k \times k \) minors of \( G \). Therefore the transfer function \( P(z)Q(z)^{-1} \in F^{(n-k) \times k}(z) \) is proper rational of McMillan degree \( \delta \) and thus has a minimal (i.e. controllable and observable) state space realization

\[
x_{t+1} = Ax_t + Bu_t, \quad x_0 = 0,
\]

\[
y_t = Cx_t + Du_t,
\]

\( A \in F^{\delta \times \delta}, \ B \in F^{\delta \times k}, \ C \in F^{(n-k) \times \delta}, \ D \in F^{(n-k) \times k}, \ x \in F^{\delta}, \ u \in F^k, y \in F^{n-k}. \)

Conversely, given any such linear linear systems representation over the field \( F \), then a right coprime factorization of the transfer function \( C(zI_\delta - A)^{-1}B + D = P(z)Q(z)^{-1} \) defines a convolutional code with generator matrix [11]. Thus the
complexity of the code corresponds to the McMillan degree of the associated rational transfer function, i.e. to the dimension of the state space of the associated controllable and observable linear system.

In the above framework, codewords of a convolutional code correspond to polynomials \( c(z) = \sum c_i z^i \in C \) with a finite number of coefficients \( c_0, \ldots, c_\gamma \in \mathbb{F}^n \). We extend the Hamming distance on \( \mathbb{F}^n \) to a metric on vector polynomials \( \mathbb{F}^n[z] \) via \( \text{dist}(\tilde{c}(z), c(z)) := \sum_{i=0}^{\infty} d(\tilde{c}_i, c_i) \). Given any polynomial \( \tilde{c}(z) = \sum_{i=0}^{T} \tilde{c}_i z^i \in \mathbb{F}^n[z] \) the task of minimal distance decoding then asks to find a code word \( c_{\text{opt}} \in C \) that minimizes the distance to \( \tilde{c} \), i.e.

\[
c_{\text{opt}} := \arg \min_{c \in C} \text{dist}(\tilde{c}, c)
\]

Such an optimal codeword always exists, since \( \mathbb{F}^n \) is finite, but need not be unique. The map \( \tilde{c} \mapsto c_{\text{opt}} \) is called a minimal distance decoder for \( C \). The method of dynamic programming yields a way to calculate this optimal code word \( c_{\text{opt}} \) for each specific choice of \( \tilde{c} \).

To see how this works we first give a linear systems interpretation of the code words of \( C \). This requires that \( \text{(2)} \) is observable. Rosenthal [13] has shown that a polynomial \( c(z) = \sum_{i=0}^{\gamma} c_i z^i \) is a code word of \( C \) if and only if and only if

\[
\begin{pmatrix}
0 & A \gamma B & A^{\gamma-1} B & \ldots & B \\
D \\
C B & D \\
-I & CAB & CB & D \\
\vdots & \ddots & \ddots \\
CA^{\gamma-1} B & \ldots & CB & D
\end{pmatrix}
\begin{pmatrix}
y_0 \\
\vdots \\
y_\gamma \\
u_0 \\
\vdots \\
u_\gamma
\end{pmatrix} = 0,
\]

(3)

where \( I \) denotes the \((\gamma + 1)(n - k) \times (\gamma + 1)(n - k)\)-identity matrix and

\[
c_t = \begin{pmatrix}
y_t \\
u_t
\end{pmatrix} \in \mathbb{F}^{(n-k)} \times \mathbb{F}^k, \quad t = 0, \ldots, \gamma.
\]

The input sequences \( u_0, \ldots, u_\gamma \) in \( \text{(2)} \) that define an admissible code word \( c \) are therefore just those controls that steer the initial condition \( x_0 = 0 \) in finite time back to \( x_0 = 0 \). It is easily seen that there exist always a nontrivial choice of such input sequences. Moreover, the minimal time to steer back to \( x_0 = 0 \) via a non-zero input is at least \( \gamma \geq \kappa_{\text{min}} \); \( \kappa_{\text{min}} \leq \left\lfloor \frac{\gamma}{k} \right\rfloor \) being the smallest
controllability index of \(2\). Let \(U = \text{proj}(C)\) denote the submodule of \(\mathbb{F}^k[z]\), obtained by projection of \(C\) onto \(\mathbb{F}^k[z]\). By coprimeness of \(P, Q\) it follows that
\[
U := Q(z)\mathbb{F}^k[z] = \{ Q(z)u(z) \in \mathbb{F}^k[z] \mid u(z) \in \mathbb{F}^k[z] \}.
\]

Moreover, expressed in terms of controllable and observable realizations \((A, B, C, D)\) of \(P(z)Q(z)^{-1}\), we obtain the state space description
\[
U = \{ u(z) = \sum_{i=0}^{\gamma} u_i z^i \in \mathbb{F}^k[z] \mid \sum_{i=0}^{\gamma} A^i Bu_i = 0, \gamma \in \mathbb{N}_0 \}.
\]
In linear systems theory, \(U\) is therefore referred to as the module of zero return.

Now assume, that we want to decode a polynomial \(\tilde{c} \in \mathbb{F}^n[z]\) via \((2)\) with coefficients
\[
\tilde{c}_t = \begin{pmatrix} \tilde{y}_t \\ \tilde{u}_t \end{pmatrix} \in \mathbb{F}^{(n-k)} \times \mathbb{F}^k, \quad t = 0, \ldots, T
\]

By the above, this means to solve the optimal control problem of finding an admissible input function \(u \in U\) that minimizes the tracking error cost functional
\[
J(x_0, u) := \text{dist}(\tilde{c}, c) = \sum_{t=0}^{\infty} k_t(x_t, u_t)
\]
where
\[
k_t(x, u) := w(u - \tilde{u}_t) + w(Cx - \tilde{y}_t + Du), \quad t \in \mathbb{N}_0.
\]

Note that the above series is always finite, as the inputs \(u(z)\) are constrained to be admissible polynomials. Thus this is an optimal control problem where the classical \(l^2\)-distance from linear quadratic controller design is replaced by the Hamming distance. Let
\[
V_\infty(x_0) = \inf_{u \in U} J(x_0, u)
\]
denote the optimal value function. The optimal control can then be computed via the Bellman principle, although this is a bit complicated here due to the varying length of the inputs. Thus we do not pursue this approach here. A simplified analysis can be given under the assumption that all data are available over a fixed horizon \(T\). Thus assume, we consider the task of minimizing
\[
J(x_0, u(\cdot), T) = \sum_{t=0}^{T} k_t(x_t, u_t)
\]
with \(k_t(x, u)\) as in \((4)\) and we optimize over all input sequences \(u_0, \ldots, u_T\). For \(N = 0, \ldots, T\) let
\[
V_N(x) = \min_{t=0}^{T} k_t(x_t, u_t)
\]
denote the $N$-th value function, $x_N := x$, where minimization occurs over all sequences $u_N, \ldots, u_T$. By the Bellman principle, these functions satisfy the functional equation

$$V_N(x) = \min_u \{ k_N(x, u) + V_{N+1}(Ax + Bu) \}; \quad N = 0, \ldots, T - 1.$$

For

$$u_N(x) = \arg \min_u \{ k_N(x, u) + V_{N+1}(Ax + Bu) \}$$

the optimal control strategy for $x_0 = 0$ then becomes

$$x_{t+1} = Ax_t + Bu_t(x_t), \quad t = 0, \ldots, T - 1.$$

Applied this approach to the negative log-likelihood function of the channel output, this then becomes exactly the Viterbi decoding algorithm \[11\]. However, the computation of the value functions is still computationally expensive and success of this method is therefore restricted to codes of small complexity. In the next section we will give a somewhat easier model predictive control approach.

### 3 Convolutional decoding via receding horizon

We now show how a modification of the *receding horizon method* from optimal control leads to an effective decoding algorithm for convolutional codes. The solution is achieved by means of a slight modification of the classical receding horizon method were $L$, instead of 1, vectors are taken to update the solution. Moreover, we carry out the main minimization step by a suitable decoding algorithm for a certain block code associated to the system. Throughout this section we assume that \[2\] is controllable and observable. This is actually a natural and frequently used assumption \[13\].

We assume the following data are given:

1. A finite sequence of received, to be decoded, words

$$\tilde{c}_t := \begin{pmatrix} \tilde{y}_t \\ \tilde{u}_t \end{pmatrix} \in \mathbb{F}^{n-k} \times \mathbb{F}^k, \quad t = 0, \ldots, T$$

for arbitrary $T \in \mathbb{N}_0$.

2. Positive integers $L \leq N \leq T$.

We then attempt to iteratively minimize the finite cost function

$$J(x_0, u, T) = \sum_{i=0}^{T-1} (w(y_i - \tilde{y}_i) + w(u_i - \tilde{u}_i)) \quad \text{(6)}$$
over the set of admissible inputs \( u \in \mathcal{U} \). For this we note, by controllability of \([2]\), that any unconstrained minimum \( u^* \) of the cost function \([6]\) can be extended to an admissible input \( u^*_{\text{adm}} \in \mathcal{U} \) by steering the terminal state \( x_T \) to zero. Of course, this admissible solution \( u^*_{\text{adm}} \) will not necessarily be an optimal solution of \([6]\).

The iteration steps to be followed at every time instant \( t \in \mathbb{N}_0 \) are

1. Consider the initial (known) state as \( x_t \).
2. Solve an \( N \)-step finite horizon tracking problem, i.e., find the unconstrained input sequence \( \{u_{t+i}\}_{i=0}^{N-1} \) which minimizes
   \[
   J(x_t, u, N) = \sum_{i=0}^{N-1} [w(y_{t+i} - \tilde{y}_{t+i}) + w(u_{t+i} - \tilde{u}_{t+i})]
   \]
3. Update the solution input with \( \{u_t, \ldots, u_{t+L-1}\} \) and use it to update the solution output with \( \{y_t, \ldots, y_{t+L-1}\} \) and to calculate \( x_{t+L} \).
4. Update the time instant \( t \) with \( t + L \) until \( t = T \).

The last step of the algorithm then results in an input sequence \( u_0, \ldots, u_{T-1} \). By controllability of \((A, B)\), we can extend this sequence to an admissible input sequence \( u_0, \ldots, u_T, u_{T+1}, \ldots, u_{T+\tau} \), by steering the final state \( x_T \) into \( x_{T+\tau} = 0 \). Here, \( \tau \leq \kappa_{\text{max}} \), with \( \kappa_{\text{max}} \) the largest controllability index of \((A, B)\).

The obtained input sequence \( u_0, \ldots, u_{T+\tau} \) with associated outputs \( y_0, \ldots, y_{T+\tau} \) and
\[
c_t := \begin{pmatrix} y_t \\ u_t \end{pmatrix} \in \mathbb{F}^{(n-k)} \times \mathbb{F}^k, \quad t = 0, \ldots, T + \tau,
\]
then defines a codeword
\[
c(z) = \sum_{i=0}^{T+\tau} c_t z^i \in \mathcal{C}
\]
that serves as a decoding for \( \tilde{c} \).
We emphasize again, that due to the unconstrained minimization, this is not an optimal solution to the above tracking problem.

Step 2 represents the main problem to be solved, which we now replace by a general block decoding step. By inspection, \( J(x_t, c, N) = w(\xi_N) \), for a vector \( \xi_N = z_{t,N} + B_N u_{t,N} \in \mathbb{R}^{Nn} \) defined by
\[
z_{t,N} := \begin{pmatrix} \mathcal{C}A^{N-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathcal{C} \end{pmatrix} x_t - \begin{pmatrix} \tilde{y}_{t+N-1} \\ \vdots \\ \tilde{y}_t \\ \tilde{u}_{t+N-1} \\ \vdots \\ \tilde{u}_t \end{pmatrix}, \quad B_N = \begin{pmatrix} D & \mathcal{C}B & \cdots & \mathcal{C}A^{N-2}B \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & D & \mathcal{C} \\ 0 & \cdots & 0 & I_{nk} \end{pmatrix}, \quad u_{t,N} = \begin{pmatrix} u_{t+N-1} \\ \vdots \\ u_{t+1} \\ u_t \end{pmatrix}
\]
Thus, each minimization step of length $N$ can be solved by decoding the vector $z_{t,N}$ with respect to the block code $C_N$ generated by $B_N$. At this point we make contact with coding theory. Consider the block code

$$C_N = \{B_N u \mid u \in \mathbb{F}^{Nk}\} \subset \mathbb{F}^{Nn}$$

with generator matrix $B_N$. The maximum likelihood decoder of $C_N$ then is

$$\pi_{C_N}(z_{t,N}) := \arg \min_{v \in C_N} d(z_{t,N}, v).$$

In particular, if we decompose any vector $v \in \pi_{C_N}(z_{t,N})$ as $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{F}^{n-k} \times \mathbb{F}^k$, then the optimal vector $u$ is equal to $-v_2$.

By replacing step (2) by a suitable decoding algorithm for $C_N$, we obtain the following algorithm where decoding of $C_N$ is represented by the function $\text{Decoding}$. Note that, for ease of notation, we have written the column vectors $w, z-e, u$ in row vector form.

**Receding Horizon Decoding Algorithm**

| Input:  | $x_0, \{\tilde{y}_t\}_0, A, B, C, D$ (*they describe the system*),  |
|---------|*length of finite horizon*) |
| U = Empty List (*the solution sequence*) |
| $t = 0$ |
| While Exit=NO |
| $w = (CA^{N-1}x_t - \tilde{y}_{t+N-1}, \ldots, Cx_t - \tilde{y}_t, 0, \ldots, 0)$ |
| $e = \text{Decoding}(B_N, w)$ |
| $(c'_N, \ldots, c'_1, c''_N, \ldots, c''_1) = z - e$ |
| $u = -(c''_1, \ldots, c''_L)$ |
| U = Append $u$ to U |
| $x_{t+L} = A^L x_t - \sum_{i=1}^{L} A^{L-i} B c''_i$ |
| $t = t + L$ |
| If $t > T$, Exit=YES |
| End-While |

Extend U to an admissible input sequence

Output: U

The whole algorithm takes at most $\lceil \frac{T}{L} \rceil$ many steps. Parameter $L$ is directly related with the number of $u_j$ that can be correctly decoded at each $N$-step problem. The precise relationship depends on the code $C_N$ and it is given in Theorem 3.2. Moreover, in each step the decoding problem may have more than one solution. We address this issue in the final section of the paper. Although convergence to the optimal solution of the tracking problem may not occur, we can at least derive an upper bound for the achieved cost.
Proposition 3.1 Let $\rho_N$ be the covering radius of the code $C_N$ and let $\tilde{u}$ denote any input sequence that is produced by the algorithm. Then

$$J(x_0, \tilde{u}, T) \leq \left\lceil \frac{T}{L} \right\rceil \rho_N$$

Proof. For any vector $z \in \mathbb{F}_{N}^n$, there exists a codeword at distance less or equal to $\rho_N$. Thus the cost added to the functional at every iteration step $t = 0, L, 2L, \cdots, \left\lceil \frac{T}{L} \right\rceil$ is upper bounded by

$$\min_{\tilde{u}} \{w(z_t,N + B_N u)\} = w(e) \leq \rho_N.$$ 

Therefore, the total cost obtained after $\left\lceil \frac{T}{L} \right\rceil$ steps is upper bounded by $\left\lceil \frac{T}{L} \right\rceil \rho_N$. □

The number $L$ of steps to update can be chosen in dependence on the correction properties of the code $C_N$. Recall the notion of decoding error, which occurs when the decoding algorithm outputs a codeword different to the original one. A case in point here is where in the iterative method the solution is not updated with the entire codeword from $C_N$ but with the components corresponding to $u_t, \ldots, u_t+L-1$. Those decoding errors which do not affect these components will be called admissible decoding errors. The precise connection between $L$ and $N$ is given by the following theorem.

Theorem 3.2 Let $d_N$ be the minimum distance of the code $C_N$. The decoding scheme can correct $\left\lfloor \frac{d'}{2} \right\rfloor$ errors, $d' \geq d_N - 1$, up to an admissible decoding error, if and only if in every codeword from $c \in C_N$ of weight $w(c) \leq d'$ all the components $c((N-L)(n-k)+1, \ldots, c_N(n-k))$ and $c_N-(L+1), \ldots, c_N$ (i.e., those that don't admit a decoding error) are zero.

Proof. Note that the generator matrix $B_N$ is systematic, and the check matrix of the code is well known to be

$$H_N = \left( \begin{array}{cccc} D & CB & CAB & \ldots & CA^{N-2}B \\
0 & D & CB & \ldots & CA^{N-3}B \\
-I d_{N(n-k)} & \vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & CB & \\
0 & \ldots & \ldots & 0 & D \end{array} \right)$$

Note that $H_N$ corresponds to the kernel representation matrix (3), up to a change of ordering of time indices to decreasing order and removing the first block row (3), that corresponds to the zero return condition.

The minimum distance $d_N$ of the code $C_N$ is precisely the minimum number of linearly dependent columns of $H_N$, [7], as the coefficients of one such linear
dependency would be the components of a codeword from \( C_N \). This bounds the number of errors that can be corrected in an \( N \)-step. Note, however, that after every \( N \)-step decoding, the method updates the partial solution just with \( u_t, \ldots, u_{t+L-1} \), i.e., decoding errors that occur in the components corresponding to \( u_{t+L}, \ldots, u_{t+N} \) (and hence also in those corresponding to \( y_{t+L}, \ldots, y_{t+N} \)) are admissible. The set of components corresponding to these vectors that allow errors is \( \alpha = \{1, \ldots, (N-L)(n-k), N(n-k)+1, \ldots, Nn-Lk\} \). Let us denote its complementary by \( \bar{\alpha} \).

An admissible error, a vector with support in \( \alpha \), is also a codeword: an admissible error is the difference between the codeword sent and the codeword wrongly decoded, and by linearity the difference of two codewords is also a codeword.

Let us assume that the code doesn’t allow to correct error vectors of weight \( t' = \lfloor \frac{d'}{2} \rfloor \) up to an admissible decoding error. Then, there exists a vector \( v \) such that for two different codewords \( c, c' \in C_N \) it can be written as \( v = c + e \) and \( v = c' + e' \) with \( w(e), w(e') \leq t' \). Since decoding up to an admissible error is not possible, we have that \( c_{\bar{\alpha}} \neq c'_{\bar{\alpha}} \), i.e., \( e_{\bar{\alpha}} \neq e'_{\bar{\alpha}} \). Then, \( c + e = c' + e' \) and by linearity \( c - c' = e' - e = e'' \) is a codeword from \( C_N \) with weight \( w(e'') \leq w(e) + w(e') \leq 2t' \leq d' \) and such that \( c''_{\bar{\alpha}} \neq 0 \), which contradicts the assumption of the theorem. The inverse implication is immediate.

As a consequence, the decoding property of our algorithm is as follows.

**Corollary 3.3** Let \( L, N, C_N \) and \( d' \) be as in Theorem 3.2 Then:

(1) The output of the algorithm is a codeword from the convolutional code.

(2) If in every subsequence

\[
(c_{jL}, c_{jL+1}, \ldots, c_{j(L-1)}), \quad j \geq 0,
\]

of the received sequence the Hamming weight of the error is at most \( d' \), then the algorithm recovers the original convolutional codeword.

**Proof.** (1) Every subsequence \( (c_{jL}, c_{jL+1}, \ldots, c_{(j+1)L-1}) \) that is generated by the algorithm at each step is in the kernel of \( H_L \). Moreover, considering the way in which \( x_{t+L} \) is updated in the algorithm, it follows that the sequence \( \{c_t\}_{t=0}^T \) is in the right kernel of the submatrix obtained by removing the first block row in (3). The last step of the algorithm consists in extending the sequence so that the last state becomes zero. Therefore the output of the algorithm is also in the kernel of the first block row of (3) and is therefore a codeword.

(2) is a direct consequence of Theorem 3.2.

**Example 3.4** Let us consider the convolutional code over \( \mathbb{F}_5 \) generated by the
which as we have seen before can be regarded as the linear system described by the equations

\[ x_{t+1} = (1, 2) u_t \]
\[ y_t = 4x_t + (1, 3) u_t , \]
\[ x_0 = 0 \]

i. e., it has a minimal realization \((A, B, C, D) = ((0), (1, 2), (4), (1, 3))\).

Let us fix \(N = 2\). Then our algorithm will work with the received vectors \(v_t, v_{t-1}\) at each time instant \(t\), when a vector must be decoded with respect to the code that has as check matrix

\[
H_N = \begin{pmatrix}
1 & 0 & 1 & 3 & 4 & 3 \\
0 & 1 & 0 & 0 & 1 & 3
\end{pmatrix}
\]

where the columns 1, 3, 4 correspond to the coordinates of \(v_t\). We observe that although the minimum distance of the the code is 2, there is no codeword of weight \(\leq 2\) with support in the positions 2, 5, 6. Then if we fix \(L = 1\) we can allow errors in coordinates 1, 3, 4 (\(v_{t-1}\) will be correctly decoded) and in exchange be able to correct one error. In this way our scheme will produce the correct \(v_{t-1}\) in each decoding step.

The convolutional code has parameters \([n, k, \delta, d_{\text{free}}] = [3, 2, 1, 3]\), and in particular it allows the correction of one error. Hence our algorithm takes full advantage of the error correcting capacities of the code.

4 Appendix: Uniqueness of decoded sequences

An important aspect of decoding is whether it has one or more solutions. Since our algorithm works sequentially we are interested to know whether there is only one list of vectors \(u_t, \ldots, u_{t+L-1}\) in each block decoding step. Note that since the matrix \(B_N\) has maximum rank, then \(u_t, \ldots, u_{t+L-1}\) are uniquely determined by a given block codeword at each step of the algorithm. Hence the question reduces to the one of how many convolutional code words are closest to a received message.
Example 4.1 Let us consider over $\mathbb{F}_2$ the finite time tracking problem problem with $N = L = 1$ defined by the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Assume that the problem is solved up to a time instant $t$ and let $x_t = (1, 0)$, $\tilde{u}_{t+1} = (0, 0)$ and $\tilde{y}_{t+1} = (0, 0)$. We need to decode the vector $z_{t,1} = (0, 1, 0, 0)\top$ with respect to the code generated by the matrix

$$B_1 = \begin{pmatrix} D \\ I_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

However both the vectors $(0, 0, 0, 0)\top$ and $(0, 1, 1, 1)\top$ belong to the code and are at a Hamming distance of 1 from the vector that we want to decode. Hence both the corresponding values for $u_t$, $(0, 0)$ and $(1, 0)$, would be equally valid.

We consider then a problem with window length $N = 2$. Let $\tilde{u}_{t+2} = (0, 0)$, $\tilde{y}_{t+2} = (1, 0)$. Then we have to decode the vector $(1, 1, 0, 1, 0, 0, 0, 0)\top$ with respect to the code generated by the matrix

$$B_2 = \begin{pmatrix} D \cdot C \cdot B \\ 0 \\ 0 & D \\ I \\ 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The (unique) closest codeword to that vector is $(1, 1, 0, 1, 0, 0, 1, 0)\top$ and hence there is a unique solution which yields $u_t = (1, 0)$.

To study the probabilities of multiple solutions we count the number of vectors that can be uniquely decoded, which are those inside the largest disjoint balls, centered at the codewords.

Consider a code $C$ of length $n$, dimension $k$ and minimum distance $d$ defined over the field $\mathbb{F}$ with $q$ elements. Its error correction capacity is $t = \lfloor \frac{d-1}{2} \rfloor$. Each ball of radius $t$ contains exactly $\sum_{i=0}^{t} \binom{n-i}{i} (q-1)^i$ vectors, and there are $q^k$ of such balls (as many as codewords). Then, the ratio of the number of uniquely decodable vectors with respect to the cardinality of the whole ambient space $\mathbb{F}^n$, 

13
known as the *density of* $\mathcal{C}$, is
\[
\delta_{\mathcal{C}} = \frac{q^k \sum_{i=0}^{t} \binom{n}{i} (q-1)^i}{q^n} = \frac{\sum_{i=0}^{t} \binom{n}{i} (q-1)^i}{q^{n-k}}.
\]

The probability that a randomly chosen vector is out of all these balls is $P_{\mathcal{C}}^o = 1 - \delta_{\mathcal{C}}$.

Before addressing the next result, and for the sake of simplicity, we fix the following notation
\[
E_{k,t} = \sum_{i=0}^{t} \binom{k}{i} (q-1)^i.
\]

**Theorem 4.2** Given a finite horizon tracking problem of window length $N$, the probability that there are $M$ different solutions which differ in $\Delta$ consecutive vectors can be upper-bounded by
\[
\frac{\delta_{\mathcal{C}_1}^{M-1}}{E_{k,t}} \prod_{i=N}^{\Delta} P_{\mathcal{C}_i}^o.
\]

**Proof.** To have $M$ different solutions that differ in $\Delta$ consecutive vectors means that at some step of the algorithm all finite horizon tracking problems with window length $\leq \Delta$ have more than one solution, and that they cannot be discriminated with a larger window, i.e., all solutions agree in the last input vector $u_{t+\Delta}$.

On each tracking problem with window lengths $N \leq l \leq \Delta$ the vector to be decoded is further than the error correction capacity of the code $\mathcal{C}_l$, and as seen before, the probability of this to happen is $P_{\mathcal{C}_l}^o$. Since this is the case for all $l = N, \ldots, \Delta$, the probability that all finite horizon tracking problems with window length $\leq \Delta$ have more than one solution can be upper-bounded by
\[
P_{\mathcal{C}_N}^o \cdots P_{\mathcal{C}_\Delta}^o = \prod_{i=N}^{\Delta} P_{\mathcal{C}_i}^o.
\]

Let us study now the probability that $M$ different optimal solutions of a finite horizon tracking problem of length $\Delta$ have the same solution vector $u_{t+\Delta}$. Let $w_{t,N} = (w_{1,\Delta}, w_{1,\Delta-1}, \ldots, w_{1,1}, w_{2,\Delta}, w_{2,\Delta-1}, \ldots, w_{2,1})$ be the vector to be decoded. For each of the $M$ solutions, $\{u_{t+i}^j\}_{i=0}^{\Delta}$ with $j \leq M$ and $u_{t+\Delta}^1 = \ldots = u_{t+\Delta}^M$, let
\[
w_{1,\Delta}^j = w_{1,\Delta} - \sum_{i=1}^{\Delta} C A^{i-1} B u_{t+\Delta-i}^j.
\]
Note that the fact that the $M$ solutions have the same $u_{t+\Delta}$ is equivalent to the $M$ vectors $v_j = (w_{1,\Delta}^j, w_{2,\Delta}^j)$ are decoded to the same codeword of the block code $C_1$ and all of them have error of the same weight (since all $M$ solutions up to instant $\Delta - 1$ are also optimal, otherwise some would have been discarded for smaller window lengths, and hence contribute the same to the cost functional).

Then, let $c \in C_{B_1}$ be the codeword to which $v_1$ is decoded, $c = (c_1, c_2)$ according to the splitting of the $v_j$. We have $d(c, v_1) = e \leq t = \lceil \frac{d-1}{2} \rceil$. The probability that for all $j = 2, \ldots, M$, $v_j$ is also decoded to $c$ and $d(c, v_j) = e$ depends on:

- the probability that there are $\alpha$ error components in the part $w_{2,\Delta}$ (which is a common part of length $k$ for all $v_j$)

$$P(d(c_2, w_{2,\Delta}) = \alpha) = \frac{\#\{v | d(c_2, v) = \alpha\}}{\#\{v | d(c_2, v) \leq t\}} = \frac{\binom{k}{\alpha}(q-1)^\alpha}{\sum_{r=0}^{t} \binom{k}{r}(q-1)^r} := P_{c,\alpha}$$

- the probability that $d(c, v_1) = e = \alpha + \beta$ ($e \leq t$) provided that $d(c_2, w_{2,\Delta}) = \alpha$

$$P(d(c, v_1) = \alpha + \beta | d(c_2, w_{2,\Delta}) = \alpha) = \frac{\binom{n-k}{\beta}(q-1)^\beta}{\sum_{s=0}^{t-\alpha} \binom{n-k}{s}(q-1)^s} := P_{e|c,\alpha}$$

- the probability that for each $j = 2, \ldots, M$ $d(c, v_j) = e$ provided that $d(c_2, w_{2,\Delta}) = \alpha$ and $d(c, v_1) = e$

$$P(d(c, v_j) = e | d(c_2, w_{2,\Delta}) = \alpha, d(c, v_1) = e) = \frac{\binom{n-k}{\beta}(q-1)^\beta}{q^{n-k}} := P_{v_j|c,\alpha,e}$$

Considering all the possibilities for values of $\alpha$ and $\beta$ (and hence of $e$) we have that the probability that all vectors $v_j$ are decoded to the same codeword of $C_1$ and that their errors have the same weight is

$$\sum_{\alpha=0}^{t} \sum_{\beta=0}^{t-\alpha} P_{c,\alpha} \cdot P_{e|c,\alpha} \cdot \prod_{j=2}^{M} P_{v_j|c,\alpha,e} = \sum_{\alpha=0}^{t} \binom{k}{\alpha}(q-1)^\alpha \sum_{\beta=0}^{t-\alpha} \left( \binom{n-k}{\beta}(q-1)^\beta \right)^{M} \left( \sum_{s=0}^{t-\alpha} \binom{n-k}{s}(q-1)^s \right)^{-1} \cdot q^{(M-1)(n-k)} \sum_{r=0}^{t} \binom{k}{r}(q-1)^r$$

(8)
Taking into account that
\[ \sum_{\beta=0}^{t-\alpha} \left( \begin{array}{c} n-k \\ \beta \end{array} \right)(q-1)^\beta )^M \leq \left( \sum_{\beta=0}^{t-\alpha} \left( \begin{array}{c} n-k \\ \beta \end{array} \right)(q-1)^\beta )^M \right) \]

\[ \left( \begin{array}{c} k \\ \alpha \end{array} \right)(q-1)^\alpha \leq \left( \begin{array}{c} k \\ \alpha \end{array} \right)(q-1)^\alpha )^{M-1} \]

\[ \sum_{\alpha=0}^{t} \left( \begin{array}{c} k \\ \alpha \end{array} \right)(q-1)^\alpha \leq \sum_{\alpha=0}^{t} \left( \begin{array}{c} n-k \\ \alpha \end{array} \right)(q-1)^\alpha \leq \sum_{i=0}^{t} \left( \begin{array}{c} n \\ i \end{array} \right)(q-1)^i \quad \text{(equality } \Leftrightarrow k, n-k \geq t) \]

then (8) is upper-bounded by
\[ \left( \sum_{n=0}^{t} \left( \begin{array}{c} k \\ \alpha \end{array} \right)(q-1)^\alpha \sum_{\beta=0}^{t-\alpha} \left( \begin{array}{c} n-k \\ \beta \end{array} \right)(q-1)^\beta \right)^{M-1} \]

\[ \frac{E_{k,t} q^{(M-1)(n-k)}}{\delta_{E_{k,t}}^{M-1} \prod_{i=N}^{\Delta} \mathcal{P}_{c_i}} \leq \frac{E_{k,t} q^{(M-1)(n-k)}}{\delta_{E_{k,t}}^{M-1} \prod_{i=N}^{\Delta} \mathcal{P}_{c_i}} \]

Thus, the probability that there are \( M \) different solutions which differ in \( \Delta \) consecutive vectors is upper-bounded by the product of (7) and (9):
\[ \frac{\delta_{E_{k,t}}^{M-1} \prod_{i=N}^{\Delta} \mathcal{P}_{c_i}}{E_{k,t} q^{(M-1)(n-k)}} \]

\[ \boxed{10} \]

5 Conclusion

We develop a system theoretic approach towards convolutional decoding, following the well-known interpretation of convolutional codes as linear systems. The Bellman optimality principle, applied to optimizing the Hamming distance function for linear systems over finite fields, then yields an optimal control decoding algorithm that is closely related to the Viterbi algorithm.

To obtain an algorithm with lower computational cost, we propose a model predictive control algorithm, using a receding horizon iteration. This new algorithm has good decoding properties, as it yields desired codeword as long
as there are not too many errors on $N$ consecutive received vectors. We also estimate the probability that the algorithm computes a unique solution.

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