Evidence for a floating phase of the transverse ANNNI model at high frustration

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Abstract

We study the transverse quantum ANNNI model in the region of high frustration (κ > 0.5) using the DMRG algorithm. We obtain a precise determination of the phase diagram, showing clear evidence for the existence of a floating phase, separated from the paramagnetic modulated phase by a high-order critical line ending at the multicritical point. We obtain simple and accurate formulae for the two critical lines.

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I. THE MODEL

The ANNNI model is an axial Ising model with competing next-nearest-neighbor antiferromagnetic coupling in one direction. It is a paradigm for the study of competition between magnetic ordering, frustration and thermal disordering effects.

In the Hamiltonian limit, we consider a one-dimensional quantum spin $S = \frac{1}{2}$ chain interacting with an external magnetic field, called the TAM model (transverse ANNNI).

The TAM Hamiltonian for $L$ spins with open boundary conditions reads$^{1,2}$

$$H = -J_1 \sum_{i=1}^{L-1} \sigma_i^z \sigma_{i+1}^z - J_2 \sum_{i=1}^{L-2} \sigma_i^z \sigma_{i+2}^z - B \sum_{i=1}^{L} \sigma_i^x.$$  \hspace{1cm} (1)

We use the “traditional” notation $\kappa = -J_2/J_1$. The notations $\lambda = J_1/B$ and $\Gamma = B$ are sometimes used in the literature.

The sign of $J_1$ is immaterial, since the Hamiltonian is invariant under the transformation

$$J_1 \rightarrow -J_1, \quad \sigma_i^y \rightarrow (-1)^i \sigma_i^y, \quad \sigma_i^x \rightarrow (-1)^i \sigma_i^x.$$ \hspace{1cm} (2)

Likewise, the sign of $B$ is immaterial. Without loss of generality, we set $J_1 = 1$. We restrict ourselves to positive $\kappa$ and even $L$.

We also consider fixed boundary conditions, where we add to the extremities of the chain two fixed spins $\sigma_0$ and $\sigma_{L+1}$, with the possibilities of parallel ($\sigma_0, \sigma_{L+1} = \uparrow\uparrow$) or antiparallel ($\sigma_0, \sigma_{L+1} = \uparrow\downarrow$) boundary conditions.

In the region of high frustration ($\kappa > 0.5$), despite extensive studies$^{3-7}$, the phase diagram of the transverse ANNNI model is not well known. For low $B$, the model is known to be in the gapless “antiphase” $\uparrow\uparrow\downarrow\downarrow$. It undergoes a second-order phase transition at a magnetic field $B_1(\kappa)$. The existence of a “floating” phase, massless and with slowly decaying spin correlation functions, up to a Kosterliz-Thouless phase transition at a magnetic field $B_2(\kappa)$, is an open question. For high $B$, the TAM is known to be in a paramagnetic modulated phase.

II. OBSERVABLES

We measure the two lowest energies $E_0$ and $E_1$, the mass gap $\Delta = E_1 - E_0$, the entanglement entropy $S_A$ (see below), and two spin-spin correlation functions: the “slow” correlation
function
\[ c_s(d) = \langle \sigma_{L/2+1}^z \sigma_{L/2+1+d}^z \rangle, \quad 1 \leq d \leq L/2 \] (3)

and the “fast” correlation function
\[ c_f(d) = \langle \sigma_{L/2-d}^z \sigma_{L/2+1}^z \rangle, \quad 0 \leq d \leq L/2. \] (4)

Interesting quantities related to the correlation functions are:
- the overlap \( o \) of \( c_s(d) \) with the antiphase correlation function
  \[ c_a(d) = (-1)^{\lfloor (d-L/2)/2 \rfloor}, \quad o = \frac{2}{L} \sum_{d=1}^{L/2} c_s(d) c_a(d); \] (5)
- the average fast correlation function (times an oscillating sign)
  \[ \bar{c}_f = (-1)^{L/2} \frac{2}{L+2} \sum_{d=0}^{L/2} c_f(d); \] (6)
- the range of the fast correlation function
  \[ R = \frac{\sum_{d=0}^{L/2} d c_f^2(d)}{\sum_{d=0}^{L/2} c_f^2(d)}. \] (7)

A. Entanglement entropy

It is possible to study an order-disorder phase transition using the entanglement entropy\(^8\). We divide the system of size \( L \) into a left subsystem of size \( \ell \) and a right subsystem of size \( L - \ell \), and define
\[ S_A(\ell; L) = - \text{Tr}(\rho_A \ln \rho_A), \] (8)
where \( A \) denotes the degrees of freedom of the left subsystem, \( B \) the degrees of freedom of the right subsystem, and \( \rho_A = \text{Tr}_B |\Psi_0\rangle\langle \Psi_0| \); note that \( S_A(\ell; L) = S_A(L-\ell; L) \). For a critical system we expect (neglecting lattice artifacts)
\[ S_A(\ell; L) \sim \left( \frac{c}{6} \right) \log(L \sin(\pi \ell/L)), \] (9)

where \( c \) is the conformal anomaly number (central charge) of the corresponding conformal field theory, and \( \sim \) means “up to a (non-universal) additive constant”; for the case of interest for the infinite-volume DMRG, \( \ell = \frac{1}{2} L \), and \( \sin(\pi \ell/L) \) only shifts the additive constant. For a noncritical system, we expect
\[ S_A(\frac{1}{2} L; L) \sim \left( \frac{c}{6} \right) [\log L + s(L/\xi)], \] (10)
where \( s(x) \) is a *universal* finite-size scaling function satisfying the constraints \( s(0) = 0 \) and 
\[ s(x) \sim - \log x \] 
for large \( x \).

### B. Domain-wall energy

So far, we only considered open boundary conditions. Following Ref. 9, we define the domain-wall energy (note that our definition of \( L \) differs by 2 from the definition of Ref. 9)

\[
E_{\text{DW}}(\kappa, B, L) = (-1)^{L/2+1} \left[ E_{0}^{\uparrow\uparrow}(\kappa, B, L) - E_{0}^{\uparrow\downarrow}(\kappa, B, L) \right],
\]

(11)

where \( E_{0}^{\uparrow\uparrow} \) and \( E_{0}^{\uparrow\downarrow} \) are the ground state energies with parallel and antiparallel boundary conditions respectively.

### III. THE ALGORITHM

We implement the density matrix renormalization group (DMRG) algorithm described in Ref. 10. We sample the \( n_s \) lowest energy levels with equal weights, i.e., we use the reduced density matrix

\[
\hat{\rho}_S = \frac{1}{n_s} \text{Tr}_E \sum_{i=0}^{n_s-1} |\psi_i\rangle \langle \psi_i |
\]

(12)

(see Eq. (26) of Ref. 10). Usually, since we are interested in the mass gap \( \Delta \), we set \( n_s = 2 \). We identify system and environment (for antiparallel boundary conditions, up to a spin flip \( \sigma_i^y \rightarrow -\sigma_i^y \), \( \sigma_i^z \rightarrow -\sigma_i^z \)). The typical dimensions of the truncated system and environment \( M \) range from 80 to 160; in the following, \( M = 80 \) will be understood, unless \( M \) is explicitly quoted.

The crucial part of the numerical computation is finding the lowest eigenvalues and eigenvectors of the superblock Hamiltonian; we employ the Implicitly Restarted Arnoldi Algorithm implemented in Arpack\textsuperscript{11}, in the routine \texttt{dsaupd} used in mode 1. We use (typically) 100 Lanczos vectors and require convergence to machine precision, obtaining residual norms 
\[ |Hx - \lambda x|/|\lambda| \sim 10^{-14}. \]

We observe a truncated weight (the sum of the eigenvalues of the density matrix whose eigenvectors are dropped in the truncation) \( \varepsilon \sim 10^{-8} \) for “normal” configurations, and 
\[ \varepsilon \sim 10^{-7} \] 
for peaks of \( \Delta \) (see below).
We managed to diagonalize the system exactly up to \( L = 22 \); both the finite- and the infinite-volume DMRG algorithm reproduce the results of exact diagonalization. For moderate \( L \), finite- and the infinite-volume DMRG give consistent results. For higher \( L \), discrepancies between finite- and infinite-volume DMRG and \( M \)-dependence of \( \Delta \) becomes noticeable; they are strongly observable-dependent, and they will be discussed below, where results on observables are presented.

During a run of the finite-volume DMRG algorithm on a system with \( L_n \) sites, information about the system/environment for all smaller system is available. It is therefore possible, with a moderate extra numerical effort, to estimate the observables for all the systems with \( L < L_n \) sites. These estimates almost coincide with the results obtained running independently at each \( L \). The additional errors introduced by this procedure will also be discussed below.

The finite-volume DMRG algorithm at large \( L \) requires a very large amount of memory; however, since the observables at each lattice size are accessed only twice per cycle, they can be conveniently kept on disk, requiring only a very large amount of disk space; for \( L = 600 \) and \( M = 80 \), e.g., ca. 6 Gbytes are required.

IV. PHASES AT \( \kappa = 0.75 \)

We will first focus our attention on the model at \( \kappa = 0.75 \), and later on extend the study to other values of \( \kappa \).

Running the infinite-volume DMRG algorithm at \( \kappa = 0.75 \) and \( B \leq 0.257 \), with open boundary conditions, we observe that the mass gap \( \Delta \) vanishes exponentially in \( L \), apart from numerical errors due to the fact that \( \Delta \) is computed as \( E_1 - E_0 \): see, e.g., Fig. 1.

The slow correlation function \( c_s(d) \) almost coincides with \( c_a(d) \); the overlap \( o \) approaches a value very close to 1 with corrections proportional to \( 1/L \). The fast correlation function \( c_f(d) \) is constant and close to \( \pm 1 \), apart from \( d = 0 \) and \( ds \) close to \( L/2 \); the range \( R \) is almost exactly \( L/4 \) (the value for a constant \( c_f(d) \)) and the average \( \bar{c}_f \) approaches a value very close to \( -1 \) with corrections proportional to \( 1/L \).

There is a very sharp phase transition at \( 0.257 < B_1 < 0.258 \). We will postpone its detailed study, since it is best done using \( E_{DW} \).

Running at \( \kappa = 0.75 \) and \( B \geq 0.258 \), the mass gap \( \Delta \) as a function of \( L \) at fixed \( B \) shows
FIG. 1: Mass gap vs. $L$ for $\kappa = 0.75$ and $B = 0.257$. To appreciate the effect of numerical errors, note that $E_0(B=0.257) \approx -0.8L$.

sharp peaks with a frequency increasing with $B$: see Fig. 2. Each peak match exactly a change of sign of $\overline{\sigma}_f$. At each $B$, for $L$ smaller than the first peak we observe signals very similar to the case $B \leq 0.257$; for higher $L$, we observe that the minima of $\Delta$ seem to go to zero for $B \leq 0.4$ and to a nonzero limit for $B \geq 0.5$; however, the determination of $\Delta$ from the infinite-volume DMRG is not accurate for $L \gtrsim 200$; finite-volume DMRG data with $M = 80$ become unreliable for $L > 300$; we show in Fig. 3 the case $B = 0.3$.

We performed a finite-size analysis of $\Delta$, similar to the analysis of Ref. 2, but with a complication arising from the peak structure. We run the finite-volume DMRG algorithm for $B = 0.4, 0.41, 0.42, 0.43, 0.44, 0.45, 0.46$ and $L_n \geq 292$ corresponding to a minimum of $\Delta$. For each $B$, we select the minima of $\Delta$ and define $\Delta_{L}(B, \kappa)$ outside the minima by interpolation in $L$. We now take two values $L_1$ and $L_2$ and look for the intersection $B_i(L_1, L_2)$ of the two curves $L_1\Delta_{L_1}(B, \kappa)$ and $L_2\Delta_{L_2}(B, \kappa)$ vs. $B$ (interpolating in $B$ at fixed $L$ and $\kappa$ as needed). The results are shown in Fig. 4: we note that $M = 120$ and $M = 160$ data almost coincide, and even $M = 80$ data are adequate in the range of $L$s considered; we quote as a final result $B_2 = 0.424(3)$. The data presented here were obtained from a run at a single $L_n$ for each $B$ (see Sect. III); in order to check that the error introduced is under control, we
FIG. 2: Mass gap peaks in the $L$–$B$ plane for $\kappa = 0.75$.

FIG. 3: Minima of mass gap $\Delta$ vs. $L$ for $\kappa = 0.75$ and $B = 0.3$.

also performed separate runs for all the values of $L$ required, for $N = 80$ and for $N = 120$ at $B = 0.42$, and repeated the analysis: the values of $B_i(L_1, L_2)$ never change by more than 0.0005.
For $0.258 \leq B \lesssim 0.45$, the slow correlation function $c_s(d)$ at fixed $L$ shows oscillations with power-law damping, in rough agreement with

$$c_s(d) \approx ad^{-\eta} \cos(qd + \phi),$$

(13)

with $\eta \sim 0$ for $B = 0.258$, increasing with $B$ but remaining smaller than $\frac{1}{2}$. For $B \gtrsim 0.5$, $c_s(d)$ at fixed $L$ shows oscillations with exponential damping. We tried to extract $\eta$ by fitting $c_s^2(d)$, smoothed by taking a running average over $\lfloor 2\pi/q + \frac{1}{2} \rfloor$ points, to the form $ad^{-2\eta}$. In Fig. 5 we show the typical case $B = 0.425$; $c_s^2(d)$ for different values of $L$ converge not to a single curve but to two separate curves, with different values of $\eta$, preventing a precise determination of $\eta$.

The range of the fast correlation function $R$ should distinguish clearly the floating phase, where $R \to \infty$ as $L \to \infty$ (since $\eta < \frac{1}{2}$), from the paramagnetic phase, where $R$ has a finite limit as $L \to \infty$. A first problem is the presence of oscillations, with dips corresponding to the peaks of $\Delta$ (see Fig. 6); it is solved by selecting the values of $R$ at the $L$s corresponding to the peaks of $\Delta$. After this operation, $R$ vs. $L$ at fixed $B$ and $M$ is well fitted to the form

$$R(L) = \frac{s_R L^2 + pL}{L + q};$$

(14)

if we plot the asymptotic slope $s_R$ vs. $B$ we should be able to see a drop towards 0 in cor-
FIG. 5: The smoothed squared slow correlation function $c_s^2(d)$ for $\kappa = 0.75$ and $B = 0.425$, from the finite-volume DMRG at $L_n = 710$ and $M = 120$, for values of $L$ corresponding to minima of $\Delta$. (Data at $M = 80$ give a very similar plot.)

FIG. 6: $R$, at the $L$s corresponding to minima of $\Delta$, with fits to Eq. (14), for $\kappa = 0.75$, $B = 0.45$ and different $M$s. For $M = 160$, all the values of $R$ are also plotted.
A. Entanglement entropy

We show in Fig. 7 the entanglement entropy $S_A$ for the typical case $B = 0.425$. Finite-volume DMRG essentially reproduces the results of infinite-volume DMRG at the same $M$. We can estimate that the DMRG determination of $S_A$ is reliable up to $L = 200$ for $M = 80$, up to $L = 300$ for $M = 120$, and up to $L = 400$ for $M = 160$. For each value of $M$, within the given range of $L$, the difference between finite- and infinite-volume DMRG results is less than 0.001. We can therefore compute $S_A$ using the faster infinite-volume DMRG, and this allows us to work at larger values of $M$.

In the antiphase, $S_A$ is essentially constant, indicating a very small correlation length.
FIG. 8: The reciprocal correlation length $1/\xi$ (determined from $S_A$) vs. $B$, for $\kappa = 0.75$. Fitting the $M = 120$ and $M = 160$ data for $20 \leq L \leq 200$ we obtain results almost identical to the $M = 80$ results plotted; likewise, fitting the $M = 160$ data for $20 \leq L \leq 300$ we obtain results almost identical to the $M = 120$ results plotted; we omit these results from the plot for readability. All data for $B \leq 0.43$ are consistent with zero, with a very small error ($\sim 10^{-6}$) which is not visible at the scale of the plot.

The simple Ansatz for the finite-size scaling function entering Eq. (10),

$$s(x) = -\ln(x + e^{-\alpha x})$$  \hspace{1cm} (15)

with $\alpha \equiv 1$, is found to fit the entanglement entropy data very well (excluding just the very smallest lattices with $L \lesssim 10$) in all cases for the floating and paramagnetic phases; $c$ is always compatible with 1. The best determination of $\xi$, obtained by fitting $S_A$ with $c \equiv 1$ fixed, is shown in Fig. 8; our final estimate is $B_2 = 0.44(1)$.

B. Domain-wall energy

So far, we only considered open boundary conditions. We now switch to fixed boundary conditions, in order to compute the domain-wall energy $E_{DW}$; with fixed boundary conditions, there are no problems with quasi-degenerate energy levels (typically, $\Delta > 0.01$) or
peaks in $\Delta$ associated with level crossings, and truncated weights are $\varepsilon \sim 10^{-9}$ or smaller for $M = 80$. Even if we are not interested in the mass gap, we run with $n_s = 2$, which gives results more stable than $n_s = 1$.

We may fit $E_{DW}$ to the form

$$E_{DW} = a \exp(-dL)L^{-\nu} + E_{\infty}$$

(16)
in the antiphase and

$$E_{DW} = \frac{a \exp(-dL)}{L} \left[ |\cos(kL + \phi)| - |\sin(kL + \phi)| \right]$$

(17)
in the floating phase (with $d = 0$) and in the paramagnetic phase$^9$. The fits, excluding (typically) lattices with $L < 16$, are of very good quality and stable.

Eq. (16), fits perfectly $E_{DW}$ for $B \leq 0.257$, giving $E_{\infty} \to 0$, constant $\nu \simeq 1.6$ and $d \simeq 0.008$ for $B \nearrow B_1$. Eq. (17), with $d = 0$, fits perfectly $E_{DW}$ for $B > 0.257$ and gives $k \to 0$ for $B \searrow B_1$. The best estimators of $B_1$ are $E_{\infty}$ in the antiphase and $k^2$ in the floating phase, both vanishing linearly at $B_1$, see Figs. 9 and 10; the final estimate of the critical field is $B_1 = 0.2574(2)$.

So far, we obtained results very similar to those of Ref. 9. We turn now to the problem of identifying the floating phase, i.e., a region with $d = 0$. The data generated with the
infinite-volume DMRG at 0.3 < B < 0.4 seem to indicate d < 0; this appears to be an artifact of the infinite-volume DMRG, as we can see from the comparison of $E_{DW}$ evaluated with the finite- and infinite-volume DMRG at $B = 0.3$, shown in Fig. 11.

We must therefore resort to the resource-consuming finite-volume DMRG. We checked in several instances that $M = 80$ is sufficient to obtain accurate results and that obtaining the data for all $L$s from a run at a single $L_n$ is acceptable: in all cases, the determinations of $d$ are well within the error quoted.

We show the results in Fig. 12. The value of $d$ obtained from the finite-volume DMRG is consistent with zero up to $B = 0.425$, where we estimate $\xi \equiv 1/d > 10^4$. The smoothness of $d$ vs. $B$ suggest a higher-order, possibly Kosterliz-Thouless, phase transition. Note that $d$ is quite compatible with $1/\xi$ of Fig. 8 (apart from a normalization stemming from the different definition of correlation length), and so is the resulting $B_2 = 0.435(10)$.

The precise determination of the transition point of a Kosterliz-Thouless phase transition is a notoriously difficult problem: it is always possible that a system with a huge correlation length is mistaken for a critical system\textsuperscript{12}. Indeed, the correlation length is expected to diverge very rapidly when the Kosterliz-Thouless critical coupling is approached; following Ref. 13, we could conjecture a behavior $\xi \propto \exp(b/(B - B_2))$ for $B \searrow B_2$. A fit to the above

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig10}
\caption{The squared modulation parameter $k^2$ vs. $B$, for $\kappa = 0.75$.}
\end{figure}
FIG. 11: Determinations of the domain-wall energy $E_{DW}$, multiplied by $L$, vs. $L$, for $\kappa = 0.75$ and $B = 0.3$: finite-volume DMRG at $M = 80, 120, 160$; infinite-volume DMRG at $L_n = 368, M = 80$; fit to Eq. (17) of the $L_n = 368$ data.

FIG. 12: The decay parameter $d$ of the domain-wall energy $E_{DW}$, computed with the finite-volume DMRG, vs. $B$, for $\kappa = 0.75$. 
form of the data of Fig. 12 gives unstable results for $B_2$, indicating that the determination of $B_2$ from $E_{DW}$ should be treated with some caution. On the other hand, a fit to the data of Fig. 8 gives results which are stable and fully consistent with the estimate of $B_2$ from $S_A$ given above.

The two determinations of $B_2$ from $\Delta_L$ and $S_A$, obtained by quite different methods, are accurate and in agreement with each other (and with the less reliable determination from $E_{DW}$); moreover, for each method, we see no trend of $B_2$ decreasing with increasing $L$ or $M$ (see especially Figs. 4 and 8). We can conclude that, while it is possible that the errors on $B_2$ are underestimated, it is difficult to believe that the floating phase might disappear completely on larger systems.

V. PHASE DIAGRAM

The study of the phase transitions at other values of $\kappa$s is very similar to the one at $\kappa = 0.75$ presented in Sect. IV and we can avoid repeating the details. We selected for our analysis the values $\kappa = 0.5, 0.52, 0.55, 0.6, 0.75, 1.0, 1.25, 1.5, 2.0, 5.0$.

At $\kappa = 0.5$, the DMRG algorithm becomes inefficient at low $B$, and we are unable to run at $B < 0.01$. We see no sign of a floating phase: the curves $L\Delta_L(B, \kappa)$ vs. $B$ almost coincide for $B < 0.06$, and the intersections are very unstable. Determining $\xi$ by fitting $S_A$ with $c \equiv 1$ fixed, we see no sign of $\xi = \infty$, see Fig. 13; given the poor convergence in $M$ for small $B$, we estimate $B_2 < 0.04$. The analysis of $E_{DW}$ does not give precise results for $B_2$. The behavior of the modulation parameter $k$ (see Fig. 14), which goes to a nonzero value as $B \to 0$, hints at the very peculiar nature of the multicritical point at $\kappa = 0.5, B = 0^{1,14}$.

For $\kappa = 0.52$ and 0.55, the quality of the determinations of $B_1$ and of the determination of $B_2$ from $S_A$ is similar to those at $\kappa = 0.75$; on the other hand, the analysis of $\Delta_L$ and $E_{DW}$ do not give precise results for $B_2$. We present the plot of $1/\xi$ for $\kappa = 0.52$ in Fig. 15: the difference from Fig. 13 is remarkable.

For $0.6 \leq \kappa \leq 1.5$, there are no relevant differences from the case $\kappa = 0.75$ described in Sect. IV; we only present the determinations of $B_1$ and $B_2$ in Table I. For $\kappa = 2$, the only difference is that $E_{DW}$ is not fitted well by Eq. (17) in the floating and paramagnetic phases, and therefore the determination of $B_2$ from $E_{DW}$ is unreliable.

In the case $\kappa = 5$, the determination of $B_1$ and $B_2$ is rather imprecise: $E_{DW}$ is not fitted
well by Eq. (16) in the antiphase, and it is fitted poorly by Eq. (17) in the floating and paramagnetic phases; it is very hard to get precise results from $\Delta L$, since the modulation parameter is very small ($k \lesssim 0.01$ in the floating phase). It is still possible to estimate $B_1$
FIG. 15: The reciprocal correlation length $1/\xi$ (determined from $S_A$) vs. $B$, for $\kappa = 0.52$.

| $\kappa$ | $B_1 (E_{DW})$ | $B_1 (o)$ | $B_2 (\Delta L)$ | $B_2 (S_A)$ | $B_2 (E_{DW})$ |
|-----------|----------------|-----------|-----------------|-------------|----------------|
| 0.5       | 0              | 0         | < 0.06          | < 0.04      | < 0.08         |
| 0.52      | 0.0201(1)      | 0.021(1)  | 0.095(15)       | 0.115(5)    | 0.12(3)        |
| 0.55      | 0.0501(2)      | 0.052(2)  | 0.160(15)       | 0.175(5)    | 0.18(2)        |
| 0.6       | 0.1015(2)      | 0.103(2)  | 0.235(6)        | 0.25(1)     | 0.25(1)        |
| 0.75      | 0.2574(2)      | 0.2575(5) | 0.424(3)        | 0.44(1)     | 0.425(10)      |
| 1.0       | 0.5213(2)      | 0.522(2)  | 0.700(5)        | 0.72(1)     | 0.71(1)        |
| 1.25      | 0.7867(2)      | 0.785(2)  | 0.972(4)        | 1.00(1)     | 0.98(1)        |
| 1.5       | 1.0514(2)      | 1.045(5)  | 1.235(3)        | 1.26(1)     | 1.26(1)        |
| 2.0       | 1.5775(2)      | 1.576(2)  | 1.756(6)        | 1.79(1)     | 1.79(3)        |
| 5.0       | —              | 4.667(3)  | —               | 4.88(1)     | —              |

TABLE I: Determinations of the transition fields $B_1$ and $B_2$ by different techniques.

from $o$ and $B_2$ from $S_A$.

For all the values of $\kappa$ considered, the different determinations of $B_1$ and $B_2$ are in substantial agreement with each other: this is a strong argument supporting the reliability of our results. It should be noticed, however, that the determination of $B_2$ from $\Delta L$ is
FIG. 16: Phase diagram in the $\kappa$–$B$ plane; the solid lines for $\kappa \geq 0.5$ correspond to Eq. (18).

systematically lower than the determination from $S_A$, possibly indicating that the error on the determination from $\Delta_L$ reported in Table I is underestimated.

We can beautifully summarize all the above results by noticing that all the determinations of $B_1$ and $B_2$ are consistent with

$$B_1(\kappa) \approx 1.05(\kappa - \frac{1}{2}), \quad B_2(\kappa) \approx 1.05\sqrt{(\kappa - \frac{1}{2})(\kappa - 0.1)}.$$  \hspace{1cm} (18)

Finally, we draw the phase diagram in the $\kappa$–$B$ plane in Fig. 16. The region $\kappa < 0.5$ was studied in Ref. 2; the critical line separating the paramagnetic modulated and paramagnetic unmodulated phases is known analytically. The data in the region $\kappa > 0.5$ are taken from the present work; note that earlier results provided only a qualitative picture of the phase diagram in this region.

A very interesting question is whether the floating phase extends up to $\kappa = \infty$ or it terminates at finite $\kappa$; we found that the floating phase extends at least up to $\kappa = 5$.

VI. SUMMARY AND CONCLUSIONS

We applied the DMRG algorithm to the study of the quantum transverse ANNNI model in the region of high frustration ($\kappa > 0.5$).
We obtained clear evidence for the existence of a floating phase for $\kappa > 0.5$, extending at least up to $\kappa = 5$. The floating phase is separated from the paramagnetic modulated phase by a high-order (possibly Kosterliz-Thouless) critical line, ending at the multicritical point ($\kappa = 0.5, B = 0$); the corresponding central charge is $c = 1$. In Ref. 7, the floating phase was shown to have a finite extent at $\kappa = 0.5$; our study cannot exclude a floating phase of very small extent, i.e., $0 < B_2(\kappa=0.5) \lesssim 0.04$.

We obtained precise estimates for the critical points, verifying that different methods give consistent results. Simple and accurate formulae for the two critical lines are reported in Eq. (18).

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The labels “slow” and “fast” only refer to the different distances involved and have no physical meaning.