Microrotation Waves Propagating in a Cylindrical Waveguide

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Abstract. The present study is devoted to analysis of coupled harmonic waves of translational displacements and microrotations propagating along the axis of a long cylindrical waveguide with circular cross-section. Microrotation waves modelling is realized within the frameworks of linear micropolar elasticity by introducing microrotations as independent degrees of freedom of the elastic continuum. The coupled system of vector differential equations of the micropolar elasticity is given. The translational displacements and microrotations vectors in the coupled wave are decomposed into potential and vortex parts. The coupled differential equations are uncoupled for some distinguished cases. The Helmholtz equations solutions for the translational and microrotation waves are obtained for a high-frequency waves in a cylindrical domain.

1. Introduction

By now the micropolar continuum models are intensively used in applied mechanics in order to resolve various problems arising while employing the classical theory of elasticity. Asymmetric stress and strain tensors are intrinsic to the micropolar theory of elasticity (see discussion in [1] for the subject). Mechanics of metamaterials are often requires the micropolar elastic models. The aim of the present study is to give a method which provides determination of the translational and microrotation waves propagating along an infinite waveguide with circular cross-section. Microrotation waves in an elastic solid are modelled within the frameworks of linear micropolar elasticity by introducing in the basic equations of the continuum mechanics microrotations as independent degrees of freedom. Such a modelling can be regularly carried out in terms of the classical field theory [2], starting from the action integral and the least action variational principle.

2. Boundary Value Problem

The vector partial differential equations determining the coupled translational and microrotation waves in a micropolar elastic continuum are formulated as (see [1]):

\[
\begin{aligned}
(\lambda + 2\mu)\nabla\cdot u - (\mu + \eta)\nabla \times (\nabla \times u) + 2\eta\nabla \times \phi &= \rho \ddot{u}, \\
(\beta + 2\gamma)\nabla\cdot \phi - (\gamma + \varepsilon)\nabla \times (\nabla \times \phi) + 2\eta\nabla \times u - 4\eta \phi &= \zeta \ddot{\phi},
\end{aligned}
\]  

where \(\lambda, \mu, \gamma, \beta, \varepsilon, \eta\) are the constitutive constants of micropolar elastic continuum; \(\nabla\) is the three-dimensional Hamilton operator (Hamilton nabla); \(u\) is the translational displacement vector; \(\phi\) is the microrotation vector; \(\rho\) and \(\zeta\) are the density and elastic compliance of the medium, respectively.

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is the microrotation vector; $\rho$ is the mass density; $\mathcal{Z}$ is the microelement inertia constant; superimposed dot denotes partial differentiation with respect to time.

The constitutive equations for the force stress tensor $\sigma$ and the moment stress tensor $\mu$ can be represented as follows

$$
\sigma = (\mu + \eta)\gamma + (\mu - \eta)\gamma^T + \lambda I \text{tr}\gamma,
$$

$$
\mu = (\gamma + \epsilon)\kappa + (\gamma - \epsilon)\kappa^T + \beta I \text{tr}\kappa.
$$

(2)

In the given equations $I$ denotes the three-dimensional unit tensor, $\gamma$ is the strain tensor, $\kappa$ is the bending-torsion tensor. The latter tensors are expressed via translational displacement vector and the microrotation vector according to

$$
\gamma = \nabla \otimes u - \phi \cdot \varepsilon,
$$

$$
\kappa = \nabla \otimes \phi.
$$

(3)

We employ the notation $\varepsilon$ for the three-dimensional permutation tensor.

In view of equations (2) and (3) after chain of transformations the following equations for the traction vector $t$ and the moment vector $m$ can be obtained:

$$
t = \lambda (\nabla \cdot u) n + 2\mu (n \cdot \nabla) u + (\mu - \eta)(n \times \nabla) \times u + 2\eta n \times \phi,
$$

$$
m = \beta (\nabla \cdot \phi) n + 2\gamma (n \cdot \nabla) \phi + (\gamma - \epsilon)(n \times \nabla) \times \phi,
$$

(4)

where $n$ is the unit normal vector determining the spatial orientation of the plane element.

These equations are used in order to formulate boundary conditions on the free from tractions and moments sidewall of the waveguide:

$$
t = 0, \quad m = 0.
$$

(5)

The system of vector partial differential equations (1) is coupled and can be uncoupled following a scheme described below. First, introduce dynamic potentials of the translational displacements and microrotations. They are decomposed into potential and vortex parts according to the Helmholtz representations:

$$
u = \nabla \Phi + \nabla \times \Psi,
$$

$$
\phi = \nabla \Sigma + \nabla \times \mathcal{H},
$$

(6)

where $\Phi$, $\Sigma$ are the scalar potentials, and $\Psi$, $\mathcal{H}$ are the vector potentials of translational displacements and microrotations respectively.

Second, in the present study we shall impose the usual calibration conditions for the dynamic potentials

$$
\nabla \cdot \Psi = 0,
$$

$$
\nabla \cdot \mathcal{H} = 0.
$$

(7)

Then, after substituting equations (6) in (1) and taking into consideration the calibrating equations (7), it is seen that the system (1) is satisfied if the scalar and the vector potentials are to be solutions of the uncoupled scalar equations

$$
\Phi - \frac{1}{c_s^2} \ddot{\Phi} = 0,
$$

$$
\Sigma - \frac{1}{\sqrt{c_s^2}} \ddot{\Sigma} - \frac{\Omega^2}{\sqrt{c_s^2}} \Sigma = 0,
$$

(8)

and the following coupled system of vector partial differential equations:
\[ \begin{align*}
\Psi - \frac{1}{\xi_2^2} \ddot{\Psi} + 2d_2^2 \nabla \times H &= 0, \\
H - \frac{1}{\rho_2^2} \ddot{H} - \frac{\Omega^2}{\rho_2^2} H + \frac{\Omega^2}{2 \rho_2^2} \nabla \times \Psi &= 0,
\end{align*} \tag{9} \]

wherein we have introduced the notations for a set of constitutive constants
\[ \Omega^2 = \frac{4\eta}{3}, \quad \xi_{\parallel}^2 = \frac{\lambda + 2\mu}{\rho}, \quad \xi_{\perp}^2 = \frac{\beta + 2\gamma}{3}, \quad \xi_{\perp}^2 = \frac{\mu + \eta}{3}, \]
\[ \gamma_{\perp}^2 = \frac{\mu + \eta}{\rho}, \quad \gamma_{\parallel}^2 = \frac{\eta}{\rho}, \quad d_2^2 = \frac{\gamma_{\parallel}^2}{\gamma_{\perp}^2}. \tag{10} \]

In the present study only coupled high-frequency waves of translational displacements and microrotations are considered (i.e. \( \omega > \Omega \)). In such a case the following differential equations for the dynamic potentials can be obtained
\[ \begin{align*}
( + \alpha_{\parallel}^2) \Phi &= 0, \\
( + \beta_{\parallel}^2) \Sigma &= 0; \\
( + \alpha_{\parallel}^2) \Psi + 2d_2^2 \nabla \times H &= 0, \\
( + \beta_{\parallel}^2) H + \frac{\Omega^2}{2 \gamma_{\perp}^2} \nabla \times \Psi &= 0.
\end{align*} \tag{11} \]

In the above equations we have employed the constants defined as
\[ \alpha_{\parallel}^2 = \frac{\omega^2}{\xi_{\parallel}^2}, \quad \beta_{\parallel}^2 = \frac{\omega^2 - \Omega^2}{\xi_{\parallel}^2}; \]
\[ \alpha_{\perp}^2 = \frac{\omega^2}{\gamma_{\perp}^2}, \quad \beta_{\perp}^2 = \frac{\omega^2 - \Omega^2}{\gamma_{\perp}^2}. \tag{12} \]

The equations (11) for the vortex potentials \( \Psi \) and \( H \) are still coupled. Uncoupled equations can be derived, however this requires differential operators of the fourth order. At last, for the vortex potentials \( \Psi \) and \( H \) the separate equations are furnished by
\[ \begin{align*}
( + K_{\parallel}^2)( + K_{\parallel}^2) \Psi &= 0, \\
( + K_{\parallel}^2)( + K_{\parallel}^2) H &= 0,
\end{align*} \tag{13} \]

where
\[ K_{\parallel}^2 = -\Delta_{12}, \]
\[ \Delta_{12} = \frac{2}{d_2^2 \Omega^2} \left( \alpha_{\parallel}^2 + \beta_{\parallel}^2 + \sigma_{\perp}^2 \right) \pm \sqrt{(\alpha_{\parallel}^2 - \beta_{\parallel}^2 + \sigma_{\perp}^2)^2 + 4\beta_{\parallel}^2 \sigma_{\perp}^2}, \tag{14} \]
\[ \sigma_{\perp}^2 = \frac{d_2^2 \Omega^2}{\beta_{\parallel}^2}. \]

The coupled translational and microrotation wave fields in a cylindrical domain are determined in the cylindrical coordinates \( r, \varphi, z \) by the separation of variables technique (see details in [2]). This technique permits investigation of waves of an arbitrary azimuthal number \( n \) propagating along the waveguide.
For the scalar potentials \( \Phi, \Sigma \) the following representations are obtained:
\[ \Phi = C_1 I_n(p_Ir) \left[ \begin{array}{c} \cos n\phi \\ -\sin n\phi \end{array} \right] e^{\pm ikz}, \]

\[ \Sigma = -C_2 I_n(p_Ir) \left[ \begin{array}{c} \sin n\phi \\ \cos n\phi \end{array} \right] e^{\pm ikz}, \]

wherein the harmonic exponents have been omitted, \( k \) denotes the wavenumber; \( C_1 \) and \( C_2 \) are arbitrary constants; \( I_n(\cdot) \) is the Bessel function of the first kind of an imaginary argument.

The vortex potentials of translational displacements and microrotations are given by the following formulae for their physical components in the cylindrical coordinate net:

\[ \Psi_r = \left[ C'_3 I_{n-1}(q_1r) + C'_{I_{n+1}}(q_1r) + C''_{I_{n-1}}(q_2r) + C''_{I_{n+1}}(q_2r) \right] \left[ \begin{array}{c} \sin n\phi \\ -\sin n\phi \end{array} \right] e^{\pm ikz}, \]

\[ \Psi_\phi = \left[ C'_3 I_{n-1}(q_1r) - C'_{I_{n+1}}(q_1r) + C''_{I_{n-1}}(q_2r) - C''_{I_{n+1}}(q_2r) \right] \left[ \begin{array}{c} \cos n\phi \\ \cos n\phi \end{array} \right] e^{\pm ikz}, \]

\[ \Psi_z = \left[ C'_3 I_n(q_1r) + C''_n(q_2r) \right] \left[ \begin{array}{c} \sin n\phi \\ \cos n\phi \end{array} \right] e^{\pm ikz}; \]

\[ H_r = \left[ L'_3 I_{n-1}(q_1r) + L'_{I_{n+1}}(q_1r) + L''_{I_{n-1}}(q_2r) + L''_{I_{n+1}}(q_2r) \right] \left[ \begin{array}{c} -\cos n\phi \\ \sin n\phi \end{array} \right] e^{\pm ikz}, \]

\[ H_\phi = \left[ L'_3 I_{n-1}(q_1r) - L'_{I_{n+1}}(q_1r) + L''_{I_{n-1}}(q_2r) - L''_{I_{n+1}}(q_2r) \right] \left[ \begin{array}{c} \cos n\phi \\ \sin n\phi \end{array} \right] e^{\pm ikz}, \]

\[ H_z = \left[ L'_3 I_n(q_1r) + L''_n(q_2r) \right] \left[ \begin{array}{c} \cos n\phi \\ -\sin n\phi \end{array} \right] e^{\pm ikz}, \]

wherein the harmonic exponents have been omitted as previously, \( C'_3 - C'_5 \), \( C''_3 - C''_5 \), \( L'_3 - L'_5 \) and \( L''_3 - L''_5 \) are arbitrary constants and

\[ q_1^2 = k^2 - \alpha_1^2, \quad q_2^2 = k^2 - \beta_1^2. \]

The coordinate representations (16) and (17) of the vortex potentials are based on a solution

\[ \Gamma = \begin{bmatrix} \Gamma_r \\ \Gamma_\phi \\ \Gamma_z \end{bmatrix} = \begin{bmatrix} C_3 I_n(q_1r) + C'_{I_{n+1}}(q_1r) \\ C'_3 I_{n-1}(q_1r) - C'_{I_{n+1}}(q_1r) \\ C_3 I_n(q_1r) \end{bmatrix} \]

of the vector Helmholtz equation

\[ \Delta \Gamma + k^2 \Gamma = 0. \]
which can be obtained by separating of variables in the cylindrical coordinates \( r, \varphi, z \); we should note that

\[ q_s^2 = k^2 - k_e^2. \]

In view of equations (15), (16) and (17) the physical components of displacement and microrotation vectors in the propagating coupled wave can be obtained. We give these components for any azimuthal number and omitting as usually the harmonic exponents:

\[
\begin{align*}
\mathbf{u}_r &= \left\{ p_1 I_{n+1}(p_1 r) + \frac{n}{r} I_n(p_1 r) + \frac{n}{r} \left( \frac{C'}{3} I_{n-1}(q_1 r) + \frac{C''}{3} I_{n-1}(q_2 r) \right) + \right. \\
&\quad \left. + (\mp ik) \left( \frac{C'}{3} I_{n-1}(q_1 r) - \frac{C'}{4} I_{n+1}(q_1 r) + \frac{C''}{3} I_{n+1}(q_2 r) - \frac{C''}{4} I_{n+1}(q_2 r) \right) \right\} \times \\
&\quad \left\{ \cos n\varphi \right. \left. - \sin n\varphi \right\} e^{\pm ikz},
\end{align*}
\]

\[
\mathbf{u}_\varphi = \left\{ \frac{n}{r} I_n(p_1 r) - C'_3 \right\} \left\{ q_1 I_{n+1}(q_1 r) + \frac{n}{r} I_n(q_1 r) \right\} + \\
\left\{ \mp ik \right\} \left\{ C'_3 I_{n-1}(q_1 r) + \frac{n}{r} I_n(q_1 r) \right\} + \\
\left\{ \frac{n}{r} I_n(q_1 r) \right\} \times \\
\left\{ \cos n\varphi \right. \left. - \sin n\varphi \right\} e^{\pm ikz},
\]

\[
\begin{align*}
\mathbf{u}_z &= \left\{ \frac{n}{r} I_n(p_1 r) + \left( C'_3 - C'_4 \right) q_1 I_n(q_1 r) + \left( C''_3 - C''_4 \right) q_2 I_n(q_2 r) \right\} \times \\
&\quad \left\{ \cos n\varphi \right. \left. - \sin n\varphi \right\} e^{\pm ikz};
\end{align*}
\]

\[
\begin{align*}
\mathbf{\phi}_r &= \left\{ \frac{n}{r} I_n(p_2 r) + \frac{n}{r} I_n(p_2 r) + \frac{n}{r} \left( \frac{L'_1}{3} I_{n+1}(q_1 r) + \frac{L''_1}{3} I_{n+1}(q_2 r) \right) + \right. \\
&\quad \left. + (\mp ik) \left( \frac{L'_1}{3} I_{n-1}(q_1 r) - \frac{L'_1}{4} I_{n-1}(q_1 r) + \frac{L''_1}{3} I_{n+1}(q_2 r) - \frac{L''_1}{4} I_{n+1}(q_2 r) \right) \right\} \times \\
&\quad \left\{ \cos n\varphi \right. \left. - \sin n\varphi \right\} e^{\pm ikz},
\end{align*}
\]

\[
\begin{align*}
\mathbf{\phi}_\varphi &= \left\{ \frac{n}{r} I_n(p_2 r) - \frac{n}{r} I_n(p_2 r) + \frac{n}{r} \left( \frac{L'_1}{3} I_{n+1}(q_1 r) + \frac{L''_1}{3} I_{n+1}(q_2 r) \right) + \right. \\
&\quad \left. + (\mp ik) \left( \frac{L'_1}{3} I_{n-1}(q_1 r) - \frac{L'_1}{4} I_{n-1}(q_1 r) + \frac{L''_1}{3} I_{n+1}(q_2 r) - \frac{L''_1}{4} I_{n+1}(q_2 r) \right) \right\} \times \\
&\quad \left\{ \cos n\varphi \right. \left. - \sin n\varphi \right\} e^{\pm ikz},
\end{align*}
\]
\[
\phi_z = \left( - \frac{1}{2} (\pm ik) I_n (p_2 r) + (L_3' - L_4') q_1 I_n (q_1 r) + (L_3'' - L_4'') q_2 I_n (q_2 r) \right) \times \\
\times \begin{cases} 
\sin n \varphi \\
\cos n \varphi
\end{cases} e^{\pm ikz}.
\]

Arbitrary constants in the formulae (18)-(23) are related by the linear equations

\[
(C'_3 + C'_4) q_1 + (\pm ik) C'_5 = 0, \\
(C''_3 + C''_4) q_2 + (\pm ik) C''_5 = 0; \\
(L'_3 + L'_4) q_1 + (\pm ik) L'_5 = 0, \\
(L''_3 + L''_4) q_2 + (\pm ik) L''_5 = 0.
\]

which follows from the calibration conditions (7). Additional linear algebraic equations for them are derived from the boundary conditions (5) on sidewall of the waveguide. These conditions are to be expanded in virtue of formulae (4), representing the traction and moment vectors via the displacement and microrotation vectors, and the Helmholtz decompositions (6).

References

[1] Nowacki W 1986 Theory of Asymmetric Elasticity. (Pergamon Press, Oxford)
[2] Kovalev V A and Radayev Y N 2010 Wave Problems of the Field Theory and Thermomechanics. (Saratov University Press)

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