Deformation Quantization of $\text{sdiff}(\Sigma_2)$ SDYM Equation

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March 27, 2022

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ABSTRACT Deformation quantization (the Moyal deformation) of SDYM equation for the algebra of the area preserving diffeomorphisms of a 2-surface $\Sigma_2$, $\text{sdiff}(\Sigma_2)$, is studied. Deformed equation we call the master equation (ME) as it can be reduced to many integrable nonlinear equations in mathematical physics. Two sets of conserved charges for ME are found. Then the linear systems for ME (the Lax pairs) associated with the conserved charges are given. We obtain the dressing operators and the infinite algebra of hidden symmetries of ME. Twistor construction is also done.

1 Introduction

In 1994 V.Husain$^1$ was able to reduce the Ashtekar-Jacobson-Smolin equations describing the metric of self-dual complex vacuum spacetimes (the heavenly spacetimes) to one equation for one holomorphic function $\Theta_0 = \Theta_0(x,y,p,q)$

$$
\partial_x^2 \Theta_0 + \partial_y^2 \Theta_0 + \{ \partial_x \Theta_0, \partial_y \Theta_0 \} P = 0 \tag{1.1}
$$

where $\{ , \}$ denotes the Poisson bracket.

Eq. (1.1) is called the Husain-Park heavenly equation (H-P equation) as it has also been found by Q.H.Park$^2$ from another point of view. Namely, in Park’s

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approach Eq. (1.3) is the principal chiral model equation for the algebra of the area preserving diffeomorphisms of a 2-surface $\Sigma^2$, $\text{sdiff}(\Sigma^2)$, and it is obtained by a symmetry reduction of $\text{sdiff}(\Sigma^2)$ SDYM equation

$$\partial_x \partial_{\tilde{x}} \Theta_0 + \partial_y \partial_{\tilde{y}} \Theta_0 + \{ \partial_x \Theta_0, \partial_y \Theta_0 \} \rho = 0 \quad (1.3)$$

where now $\Theta_0 = \Theta_0(x, y, \tilde{x}, \tilde{y}, p, q)$. A natural generalization of Eq. (1.3) can be done when the Poisson bracket is changed by the Moyal one. Thus one arrives at the following equation [3, 4]

$$\partial_x \partial_{\tilde{x}} \Theta + \partial_y \partial_{\tilde{y}} \Theta + \{ \partial_x \Theta, \partial_y \Theta \}_M = 0 \quad (1.4)$$

$\Theta = \Theta(h; x, y, \tilde{x}, \tilde{y}, p, q)$

where $\{\cdot, \cdot\}_M$ denotes the Moyal bracket

$$\{f, g\}_M := \frac{1}{i\hbar}(f * g - g * f) = \frac{\hbar^2}{2} \sin \left( \frac{\hbar^2}{2} \mathcal{P} \right) g; \quad \hbar \in \mathbb{R}$$

$$f * g := \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\hbar^n}{2^n} \omega^{i_1 j_1} \ldots \omega^{i_n j_n} \frac{\partial^n f}{\partial X^{i_1} \ldots \partial X^{i_n}} \frac{\partial^n g}{\partial X^{j_1} \ldots \partial X^{j_n}}$$

$$= f \exp(i \frac{\hbar^2}{2} \mathcal{P}) g, \quad i_1, \ldots, j_1, \ldots = 1, 2; \quad (X^1, X^2) = (q, p), \quad (\omega^{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (1.5)$$

The real parameter $\hbar$ is a deformation parameter.

Eq. (1.4) we call the master equation (ME). By a symmetry reduction, using also some representations of the Moyal bracket Lie algebra one can reduce ME to the known heavenly equations, to $su(N)$ SDYM equations or $su(N)$ principal chiral model equations and to many integrable nonlinear equations of mathematical physics [3]. We should point out that $\Theta(h; x, y, \tilde{x}, \tilde{y}, p, q)$ being a solution of ME is considered to be the formal series with respect to $\hbar$,

$$\Theta = \sum_{n=-N}^{\infty} \Theta_n \hbar^n, \quad N < \infty, \quad \Theta_n = \Theta_n(x, y, \tilde{x}, \tilde{y}, p, q) \quad (1.6)$$

In our recent paper [6] some evidence for the integrability of ME has been provided. It has been shown that ME admits infinite number of nonlocal conservation laws and linear systems (Lax pairs) for ME have been found. Moreover, a twistor construction has been also done.

The aim of the present work is to consider in some details and develop the results of [6]. First, in Section 2, we find infinite number of new nonlocal conservation laws such that the conserved "charges" define hidden symmetries of ME. In Section 3 new linear systems for ME are obtained and the dressing operators leading to solutions of these systems are found. The dressing operators appear to be the solutions of the linear systems presented in [6]. In Section 4 the infinite Lie algebra of the hidden symmetries of ME is given. Finally, Section 5 is devoted to a twistor construction for ME.
Conservation laws and hidden symmetries.

Let \( \eta^{(0)} = \eta^{(0)}(\hbar; x, y, \tilde{x}, \tilde{y}, p, q) \) be some function. Define

\[
j^{(1)}_x := D_{\tilde{x}}\eta^{(0)}, \quad j^{(1)}_y := D_{\tilde{y}}\eta^{(0)}
\]

(2.1)

where

\[
D_{\tilde{x}} := \partial_{\tilde{x}} - \frac{1}{i\hbar}\partial_y\Theta, \quad D_{\tilde{y}} := \partial_{\tilde{y}} + \frac{1}{i\hbar}\partial_x\Theta
\]

(2.2)

and \( \Theta = \Theta(\hbar; x, y, \tilde{x}, \tilde{y}, p, q) \) is a solution of ME (1.4). We have

\[
\partial_x j^{(1)}_x + \partial_y j^{(1)}_y = (\partial_x D_{\tilde{x}} + \partial_y D_{\tilde{y}})\eta^{(0)} = (D_{\tilde{x}}\partial_x + D_{\tilde{y}}\partial_y)\eta^{(0)}
\]

\[
= \partial_x \partial_x \eta^{(0)} + \partial_y \partial_y \eta^{(0)} + \frac{1}{i\hbar}(\partial_x \Theta \ast \partial_y \eta^{(0)} - \partial_y \Theta \ast \partial_x \eta^{(0)})
\]

(2.3)

Consequently \( \partial_x j^{(1)}_x + \partial_y j^{(1)}_y = 0 \) iff \( \eta^{(0)} \) satisfies the following linear equation

\[
\partial_x \partial_x \eta^{(0)} + \partial_y \partial_y \eta^{(0)} + \frac{1}{i\hbar}(\partial_x \Theta \ast \partial_y \eta^{(0)} - \partial_y \Theta \ast \partial_x \eta^{(0)}) = 0
\]

(2.4)

Given a function \( \eta^{(0)} \) satisfying Eq. (2.4), there exists a function \( \eta^{(1)} = \eta^{(1)}(\hbar; x, y, \tilde{x}, \tilde{y}, p, q) \) such that

\[
\partial_x \eta^{(1)} = D_{\tilde{y}}\eta^{(0)}, \quad \partial_y \eta^{(1)} = D_{\tilde{x}}\eta^{(0)}
\]

(2.5)

From (2.5) with (2.2) one gets

\[
(D_{\tilde{x}}\partial_x + D_{\tilde{y}}\partial_y)\eta^{(1)} = (D_{\tilde{x}}D_{\tilde{y}} - D_{\tilde{y}}D_{\tilde{x}})\eta^{(0)}
\]

\[
= \frac{1}{i\hbar}(\partial_x \partial_x \Theta + \partial_y \partial_y \Theta + \{\partial_x \Theta, \partial_y \Theta\}_{\mathcal{A}}) \ast \eta^{(0)} \text{ by ME } = 0
\]

(2.6)

Therefore, as \( \Theta \) is a solution of ME \( \eta^{(1)} \) satisfies the same equation (2.4) as \( \eta^{(0)} \) does. This enables us to define the current \( j^{(2)} \)

\[
j^{(2)}_x := D_{\tilde{x}}\eta^{(1)}, \quad j^{(2)}_y := D_{\tilde{y}}\eta^{(1)}
\]

(2.7)

which satisfies the equation \( \partial_x j^{(1)}_x + \partial_y j^{(1)}_y = 0 \). Hence there exists a function \( \eta^{(2)} \) such that

\[
\partial_x \eta^{(2)} = D_{\tilde{y}}\eta^{(1)}, \quad \partial_y \eta^{(2)} = D_{\tilde{x}}\eta^{(1)}
\]

(2.8)

Continuing this procedure we arrive at the series of functions (conserved charges) \( \eta^{(1)}, \eta^{(2)}, \ldots \) and currents \( j^{(1)}, j^{(2)}, \ldots \), defined by the recursion equations

\[
\begin{align*}
j^{(n+1)}_x &= -\partial_y \eta^{(n+1)} = D_{\tilde{x}}\eta^{(n)} \\
j^{(n+1)}_y &= \partial_x \eta^{(n+1)} = D_{\tilde{y}}\eta^{(n)}, \quad n = 0, 1, \ldots
\end{align*}
\]

(2.9)
It is evident that as \( \Theta \) satisfies ME and \( \eta^{(0)} \) satisfies (2.4) all \( \eta^{(n)} \) satisfy the linear equation

\[
\partial_x \partial_{\tilde{x}} \eta^{(n)} + \partial_y \partial_{\tilde{y}} \eta^{(n)} + \frac{1}{i\hbar}(\partial_x \Theta * \partial_y \eta^{(n)} - \partial_y \Theta * \partial_x \eta^{(n)}) = 0
\]  

(2.10)

Thus we obtain infinite number of conservation laws

\[
\partial_x j_x^{(n)} + \partial_y j_y^{(n)} = 0, \quad n = 1, 2, ...
\]  

(2.11)

and the conserved charges

\[
\eta^{(n)} = \int dx^{(n)} D_{\tilde{y}} \int dx^{(n-1)} D_{\tilde{y}} ... \int dx^{(1)} D_{\tilde{y}} \eta^{(0)}, \quad n = 1, 2, ...
\]  

(2.12)

In particular, an interesting case is when one puts

\[
\eta^{(0)} = 1.
\]  

(2.13)

Then (taking appropriate boundary conditions) we get from (2.5) with (2.13)

\[
\eta^{(1)} = \frac{1}{i\hbar} \Theta
\]  

(2.14)

Observe that Eq. (2.10) for \( \eta^{(1)} \) given by (2.13) is exactly ME. [The case of \( \eta^{(0)} = 1 \) has been analyzed in our previous work \[6\] and in fact the considerations of \[6\] closely follow E.Brezin et al \[7\], M.K.Prasad et al \[8\], L.L.Chau et al \[9\] and L.L.Chau \[10\] where nonlocal conservation laws for some 2-dimensional nonlinear field theories and for SDYM equations have been found.]

Now we are going to look for another collection of conserved charges. Here we follow V.Husain \[1\] and M.Dunajski and L.J.Mason \[11\] who obtained infinite number of conservation laws for some heavenly equations in four dimensions. \[1\]

Assume that \( \sigma^{(0)} = \sigma^{(0)}(\bar{h}; x, y, \tilde{x}, \tilde{y}, p, q) \) is any solution of the **linearized master equation** (LME)

\[
\partial_x \partial_{\tilde{x}} \sigma^{(0)} + \partial_y \partial_{\tilde{y}} \sigma^{(0)} + \{\partial_x \Theta, \partial_y \sigma^{(0)}\}_M - \{\partial_y \Theta, \partial_x \sigma^{(0)}\}_M = 0
\]  

(2.15)

Define

\[
J^{(1)}_x := L_{\tilde{x}} \sigma^{(0)}, \quad J^{(1)}_y := L_{\tilde{y}} \sigma^{(0)}
\]  

(2.16)

where

\[
L_{\tilde{x}} := \partial_{\tilde{x}} - \{\partial_y \Theta, \cdot\}_M, \quad L_{\tilde{y}} := \partial_{\tilde{y}} + \{\partial_x \Theta, \cdot\}_M.
\]  

(2.17)

Then

\[
\partial_x J^{(1)}_x + \partial_y J^{(1)}_y = (L_{\tilde{x}} L_x + L_{\tilde{y}} L_y) \sigma^{(0)} = (L_{\tilde{x}} \partial_x + L_{\tilde{y}} \partial_y) \sigma^{(0)} = \partial_x \partial_x \sigma^{(0)} + \partial_y \partial_y \sigma^{(0)} + \{\partial_x \Theta, \partial_x \sigma^{(0)}\}_M - \{\partial_y \Theta, \partial_x \sigma^{(0)}\}_M \quad \text{by (2.15)} = 0.
\]  

(2.18)

* We are indebted to Maciej Dunajski for pointing out to us the method how to obtain new conservation laws.
Hence, there exists a function $\sigma^{(1)}$ such that

$$\partial_x \sigma^{(1)} = \mathcal{L}_{\tilde{y}} \sigma^{(0)}, \quad -\partial_y \sigma^{(1)} = \mathcal{L}_{\tilde{x}} \sigma^{(0)}.$$  

(2.19)

From (2.19) we get

$$(\mathcal{L}_{\tilde{x}} \partial_x + \mathcal{L}_{\tilde{y}} \partial_y) \sigma^{(1)} = (\mathcal{L}_{\tilde{x}} \mathcal{L}_{\tilde{y}} - \mathcal{L}_{\tilde{y}} \mathcal{L}_{\tilde{x}}) \sigma^{(0)}$$

$$= \{ \partial_x \partial_{\tilde{x}} \Theta + \partial_y \partial_{\tilde{y}} \Theta + \{ \partial_x \Theta, \partial_y \Theta \}_\mathcal{M}, \sigma^{(0)} \}_\mathcal{M} \quad \text{by ME} \quad = 0.$$  

(2.20)

It means that $\sigma^{(1)}$ satisfies LME. Analogously as before we arrive at the series of conserved charges $\sigma^{(1)}, \sigma^{(2)}, \ldots$ and currents $J^{(1)}, J^{(2)}, \ldots$ which are defined by the following recursion equations

$$J^{(n+1)}_x = -\partial_y \sigma^{(n+1)} = \mathcal{L}_{\tilde{y}} \sigma^{(n)}$$

$$J^{(n+1)}_y = \partial_x \sigma^{(n+1)} = \mathcal{L}_{\tilde{x}} \sigma^{(n)}, \quad n = 0, 1, \ldots$$  

(2.21)

From the assumption that $\Theta$ is a solution of ME and $\sigma^{(0)}$ satisfies LME (2.15) it follows that all $\sigma^{(n)}$s satisfy LME

$$\partial_x \partial_{\tilde{x}} \sigma^{(n)} + \partial_y \partial_{\tilde{y}} \sigma^{(n)} + \{ \partial_x \Theta, \partial_y \sigma^{(n)} \}_\mathcal{M} - \{ \partial_y \Theta, \partial_x \sigma^{(n)} \}_\mathcal{M} = 0, \quad n = 0, 1, \ldots$$  

(2.22)

This means that $\sigma^{(n)}, \ n = 0, 1, \ldots$ are the hidden symmetries of ME. Equation (2.21) defines the recursion operator $R$ by

$$\sigma^{(n+1)} = R \sigma^{(n)}$$  

(2.23)

(Compare with [11]).

For example taking $\sigma^{(0)} = \tilde{x}$

we find

$$\sigma^{(1)} = -y + f^{(1)}(\tilde{h}; \tilde{x}, \tilde{y}, p, q),$$

$$\sigma^{(2)} = x \partial_{\tilde{y}} f^{(1)} - y \partial_{\tilde{x}} f^{(1)} + \{ \Theta, f^{(1)} \}_\mathcal{M} + f^{(2)}(\tilde{h}; \tilde{x}, \tilde{y}, p, q),$$

...etc

(2.25)

If one puts $\sigma^{(0)} = \tilde{y}$

then

$$\sigma^{(1)} = x + f^{(1)}(\tilde{h}; \tilde{x}, \tilde{y}, p, q),$$

$$\sigma^{(2)} = x \partial_{\tilde{x}} f^{(1)} - y \partial_{\tilde{y}} f^{(1)} + \{ \Theta, f^{(1)} \}_\mathcal{M} + f^{(2)}(\tilde{h}; \tilde{x}, \tilde{y}, p, q),$$

...etc

(2.27)
Linear systems for ME and dressing operators.

We deal here with conserved charges $\eta^{(0)}, \eta^{(1)}, \ldots$, etc., defined by (2.13), (2.14) and (2.12). For this especial choice we put instead of $\eta^{(0)}, \eta^{(1)}, \ldots, \eta^{(n)}, \ldots$ the symbols $\psi^{(0)}, \psi^{(1)}, \ldots, \psi^{(n)}, \ldots$ so we have

$$
\psi^{(0)} = 1, \quad \psi^{(1)} = \frac{1}{i\hbar} \Theta,
$$

$$
\psi^{(n)} = \int x^{(n)} D_y \int x^{(n-1)} D_y \ldots \int x^{(2)} D_y 1 \quad (3.1)
$$

Define

$$
\Psi(\lambda) := 1 + \sum_{n=1}^{\infty} \lambda^n \psi^{(n)}, \quad \lambda \in \mathbb{C} - \{\infty\} \quad (3.2)
$$

One can easily check that $\Psi(\lambda)$ satisfies the following linear system of differential equations

$$
\frac{\partial}{\partial x} \Psi(\lambda) = \lambda D_y \Psi(\lambda), \quad \lambda \in \mathbb{C} - \{\infty\} \quad (3.3)
$$

and, in fact, this system is a Lax pair for ME. Employing the results of [12, 13, 14] one can show that $\Psi(\lambda)$ has the form of

$$
\Psi(\lambda) = \exp\{\frac{1}{i\hbar} \sum_{n=1}^{\infty} \lambda^n \Lambda^{(n)}\}, \quad \Lambda^{(1)} = \Theta \quad (3.4)
$$

$$
\partial_x \Lambda^{(n)} = \partial_y \Lambda^{(n-1)} + \sum_{l=1}^{n-1} \frac{B_l}{l!} \sum_{k_1 + \ldots + k_l = n-1} \Lambda^{(k_1)}, \ldots, \Lambda^{(k_l)}, \partial_x \Theta, \ldots, \Lambda^{(n)} \quad (3.5)
$$

$$
\partial_y \Lambda^{(n)} = -\partial_x \Lambda^{(n-1)} + \sum_{l=1}^{n-1} \frac{B_l}{l!} \sum_{k_1 + \ldots + k_l = n-1} \Lambda^{(k_1)}, \ldots, \Lambda^{(k_l)}, \partial_y \Theta, \ldots, \Lambda^{(n)} \quad (3.6)
$$

where $B_l$ are the Bernoulli numbers, $\frac{1}{e^{\lambda} - 1} = \sum_{l=0}^{\infty} B_l \lambda^l$. The formula (3.4) proves that if $\Theta$ is analytic in $\hbar$ then all $\Lambda^{(n)}$s are also analytic in $\hbar$. Then $\Psi^{-1}(\lambda)$ defined by

$$
\Psi^{-1}(\lambda) * \Psi(\lambda) = \Psi(\lambda) * \Psi^{-1}(\lambda) = 1 \quad (3.7)
$$

has the following form

$$
\Psi^{-1}(\lambda) = \exp\{\frac{1}{i\hbar} \sum_{n=1}^{\infty} \lambda^n \Lambda^{(n)}\} \quad (3.8)
$$
and it fulfills the following system
\[
\begin{align*}
\partial_x \Psi^{-1}_x(\lambda) &= \lambda (\partial_y \Psi^{-1}_y(\lambda) - \frac{1}{i\hbar} \Psi^{-1}_y(\lambda) \ast \partial_x \Theta) \\
-\partial_y \Psi^{-1}_x(\lambda) &= \lambda (\partial_x \Psi^{-1}_x(\lambda) + \frac{1}{i\hbar} \Psi^{-1}_x(\lambda) \ast \partial_y \Theta), \quad \lambda \in \mathbb{C} - \{\infty\}
\end{align*}
\] (3.7)

Analogously, defining
\[
\sigma(\lambda) := \sum_{n=1}^{\infty} \lambda^n \sigma^{(n)}, \quad \lambda \in \mathbb{C} - \{\infty\}
\] (3.8)

where \(\sigma^{(n)}\), \(n = 0, 1, \ldots\) are the conserved charges introduced in the previous section (see (2.21)) one arrives at the system
\[
\begin{align*}
\partial_x \sigma(\lambda) &= \lambda L_y \sigma(\lambda) \\
-\partial_y \sigma(\lambda) &= \lambda L_x \sigma(\lambda), \quad \lambda \in \mathbb{C} - \{\infty\}
\end{align*}
\] (3.9)

which is also a Lax pair for ME.

Let \(F\) be a function such that
\[
\sigma(\lambda) = \Psi(\lambda) \ast F \ast \Psi^{-1}_x(\lambda)
\] (3.10)

It is evident that such a function \(F\) always exists and is uniquely defined by \(\sigma(\lambda)\). Moreover, from (3.3), (3.7), and (3.9) one quickly finds that \(F\) must be of the form
\[
F = F(h; \tilde{y} + \lambda x, \tilde{x} - \lambda y, \lambda, p, q)
\] (3.11)

It means that \(F\) is a twistor function, as the equations
\[
\tilde{y} + \lambda x =: w^1 = \text{const}, \quad \tilde{x} - \lambda y =: w^2 = \text{const}, \quad \lambda =: w^3 = \text{const}
\] (3.12)

define a totally null anti-self-dual 2-surface in \(\mathbb{C}^4\) (the twistor surface). This twistor surface is the integral manifold for the following anti-self-dual 2-form \(\omega\)
\[
\omega = (d\tilde{y} + \lambda dx) \wedge (d\tilde{x} - \lambda dy)
\]
\[
= -d\tilde{x} \wedge d\tilde{y} + \lambda (dx \wedge d\tilde{x} + dy \wedge d\tilde{y}) - \lambda^2 dx \wedge dy
\] (3.13)

The formula (3.10) says that \(\Psi(\lambda)\) is the dressing operator for the linear system (3.9). For example, taking
\[
F := \tilde{x} - \lambda(y + f^{(1)}(h; p, q))
\] (3.14)

one recovers the solution given by (2.24), (2.25) with \(f^{(1)} = f^{(1)}(h; p, q)\); taking
\[
F := \tilde{y} + \lambda(x + f^{(1)}(h; p, q))
\] (3.15)
we arrive at (2.26), (2.27). (Compare with [13].) Analogously as in our previous work [6] consider the linear system
\[
\frac{1}{\lambda} \partial_x \Phi(\frac{1}{\lambda}) = D \tilde{y} \Phi(\frac{1}{\lambda}) \\
- \frac{1}{\lambda} \partial_y \Phi(\frac{1}{\lambda}) = D \tilde{x} \Phi(\frac{1}{\lambda}) , \quad \lambda \in \mathbb{C} - \{0\}
\]
(3.16)
\[
\Phi(\frac{1}{\lambda}) = \Phi(0) + \sum_{n=1}^{\infty} \left( \frac{1}{\lambda} \right)^n \Phi^{(n)}.
\]
This is also a linear system for ME. One quickly finds that
\[
\partial_x \Theta = i \hbar \Phi(0)^* \partial_y [\Phi(0)^{-1}] \\
\partial_y \Theta = - i \hbar \Phi(0)^* \partial_x [\Phi(0)^{-1}]
\]
(3.17)
The solution \( \Phi(\frac{1}{\lambda}) \) can be written in the form
\[
\Phi(\frac{1}{\lambda}) = \exp \left\{ \frac{1}{i \hbar} \sum_{n=0}^{\infty} \left( \frac{1}{\lambda} \right)^n \Omega^{(n)} \right\} \\
\exp \left\{ \frac{1}{i \hbar} \Omega^{(0)} \right\} = \Phi^{(0)}.
\]
(3.18)
Then we consider \( \tilde{\sigma}(\frac{1}{\lambda}) = \sum_{n=0}^{\infty} \left( \frac{1}{\lambda} \right)^n \sigma^{(n)} \)
\[
\tilde{\sigma}(\frac{1}{\lambda}) = \Phi(\frac{1}{\lambda}) \ast \tilde{F} \ast \Phi^{-1}(\frac{1}{\lambda}).
\]
(3.19)
Straightforward calculations show that \( \tilde{\sigma}(\frac{1}{\lambda}) \) satisfies the following linear system
\[
\frac{1}{\lambda} \partial_x \tilde{\sigma}(\frac{1}{\lambda}) = \mathcal{L}_y \tilde{\sigma}(\frac{1}{\lambda}) \\
- \frac{1}{\lambda} \partial_y \tilde{\sigma}(\frac{1}{\lambda}) = \mathcal{L}_x \tilde{\sigma}(\frac{1}{\lambda}) , \quad \lambda \in \mathbb{C} - \{0\}
\]
(3.20)
iff the function \( \tilde{F} \) is of the form
\[
\tilde{F} = \tilde{F}(\hbar; x + \frac{1}{\lambda} \tilde{y}, - y + \frac{1}{\lambda} \tilde{x}, \frac{1}{\lambda} p, q)
\]
(3.21)
i.e., \( \tilde{F} \) is a twistor function.
Of course the system (3.20) is also a linear system for ME (a Lax pair for ME) and Eq. (3.19) expresses the fact that \( \Phi(\frac{1}{\lambda}) \) is the dressing operator for this system.
4 Infinite algebra of hidden symmetries.

From the previous sections one can conclude that the general solution of LME (2.13), i.e., the general symmetry of ME is given by

\[
\delta_{(F \tilde{F})} \Theta = \frac{1}{2\pi i} \oint_{\gamma} d\lambda \frac{d\bar{\lambda}}{\lambda^2} (-\Psi(\lambda) * \Phi(\lambda) + F_{1} * \Phi_{s}^{-1}(\lambda))
\]

\[
F = F(h; \tilde{y} + \lambda x, \tilde{x} - \lambda y, \lambda, p, q)
\]

\[
\tilde{F} = \tilde{F}(h; x + \frac{1}{\lambda} \tilde{y}, y + \frac{1}{\lambda} \tilde{x}, \lambda, p, q)
\] (4.1)

where a curve \( \gamma \) does not contain singularities of functions which are integrated. One can compare (4.1) with respective formulas given by Q.H.Park [2, 16]. In (1.1) the first sign \((-\) and the factor \(\lambda^2\) are chosen for further convenience.

Now we are looking for the algebra of hidden symmetries of ME. To this end consider the commutator

\[
[\delta_{(F_{1} \tilde{F}_{1})}, \delta_{(F_{2} \tilde{F}_{2})}] \Theta = \delta_{(F_{1} \tilde{F}_{1})}(\Theta + \delta_{(F_{2} \tilde{F}_{2})}(\Theta) - \delta_{(F_{2} \tilde{F}_{2})}(\Theta + \delta_{(F_{1} \tilde{F}_{1})}(\Theta)) + \delta_{(F_{2} \tilde{F}_{2})}(\Theta)
\] (4.2)

Simple calculations with the use of (4.1) give

\[
[\delta_{(F_{1} \tilde{F}_{1})}, \delta_{(F_{2} \tilde{F}_{2})}] \Theta = \frac{(i\hbar)}{2\pi i} \oint_{\gamma} d\lambda \frac{d\bar{\lambda}}{\lambda^2} (-\{\delta_{(F_{2} \tilde{F}_{2})} \Psi * \Phi_{s}^{-1}, \Psi * F_{1} * \Phi_{s}^{-1}\})\lambda
\]

\[
+\{\delta_{(F_{1} \tilde{F}_{1})} \Psi * \Phi_{s}^{-1}, \Psi * F_{2} * \Phi_{s}^{-1}\}\lambda
\]

\[
+\{\delta_{(F_{2} \tilde{F}_{2})} \Phi * \Phi_{s}^{-1}, \Phi * \tilde{F}_{1} * \Phi_{s}^{-1}\}\lambda
\]

\[
-\{\delta_{(F_{1} \tilde{F}_{1})} \Phi * \Phi_{s}^{-1}, \Phi * \tilde{F}_{2} * \Phi_{s}^{-1}\}\lambda
\] (4.3)

Performing variation of the system (3.3) and employing (3.7) one obtains

\[
\partial_{x}(\delta_{(F_{1} \tilde{F}_{1})} \Psi * \Psi_{s}^{-1}) = \lambda [L_{\tilde{y}}(\delta_{(F_{1} \tilde{F}_{1})} \Psi * \Psi_{s}^{-1}) + \frac{1}{i\hbar} \partial_{x}(\delta_{(F_{1} \tilde{F}_{1})} \Theta)]
\]

\[
-\partial_{y}(\delta_{(F_{1} \tilde{F}_{1})} \Psi * \Psi_{s}^{-1}) = \lambda [L_{\tilde{x}}(\delta_{(F_{1} \tilde{F}_{1})} \Psi * \Psi_{s}^{-1}) - \frac{1}{i\hbar} \partial_{y}(\delta_{(F_{1} \tilde{F}_{1})} \Theta)]
\] (4.4)

The solution of (4.4) can be found and it reads

\[
i\hbar \delta_{(F_{1} \tilde{F}_{1})} \Psi(\lambda) * \Psi_{s}^{-1}(\lambda) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \lambda^{n} \oint_{\gamma} d\lambda \frac{d\bar{\lambda}}{(\lambda')^{n+1}} (-\Psi' * F_{1}' * \Psi_{s}^{-1}' + \Phi' * \tilde{F}_{1}' * \Phi_{s}^{-1}')
\]

\[
= \frac{1}{2\pi i} \oint_{\gamma} d\lambda \lambda \frac{\lambda}{\lambda'\lambda - \lambda''} (-\Psi' * F_{1}' * \Psi_{s}^{-1}' + \Phi' * \tilde{F}_{1}' * \Phi_{s}^{-1}')
\] (4.5)
where $\gamma'$ is any curve closing a domain containing a circle $K(0; r) : \gamma \subset K(0; r)$. Then $\Psi := \Psi(\lambda')$, $F'^{i}_{\lambda} := F^{i}_{\lambda}(h; y + \lambda'x, \bar{x} - \lambda'y, \lambda', p, q)$, etc.

Analogously for $\delta_{(F_{i}, F_{j})} \Phi \ast \Phi^{-1}$ one gets the system of equations

$$
\frac{1}{\lambda} \partial_{x}(\delta_{(F_{i}, F_{j})} \Phi \ast \Phi^{-1}) = \mathcal{L}_{y}(\delta_{(F_{i}, F_{j})} \Phi \ast \Phi^{-1}) + \frac{1}{i\hbar} \partial_{\bar{z}}\delta_{(F_{i}, F_{j})} \Theta
$$

$$
-\frac{1}{\lambda} \partial_{y}(\delta_{(F_{i}, F_{j})} \Phi \ast \Phi^{-1}) = \mathcal{L}_{\bar{z}}(\delta_{(F_{i}, F_{j})} \Phi \ast \Phi^{-1}) - \frac{1}{i\hbar} \partial_{\bar{y}}\delta_{(F_{i}, F_{j})} \Theta
$$

(4.6)

The solution of (4.6) reads

$$
i\hbar\delta_{(F_{i}, F_{j})} \Phi(\frac{1}{\lambda}) \ast \Phi^{-1}(\frac{1}{\lambda}) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{1}{\lambda^{n}} \oint_{\gamma} d\lambda' (\lambda')^{n-1} (-\Psi' \Phi' \ast \Phi^{-1} + \Phi' \Phi' \ast \Phi^{-1})
$$

(4.7)

Since for calculating (4.3) we need $\lambda \in \gamma$ one can write down (4.7) in the form of

$$
i\hbar\delta_{(F_{i}, F_{j})} \Phi(\frac{1}{\lambda}) \ast \Phi^{-1}(\frac{1}{\lambda})|_{\lambda \in \gamma} = \frac{1}{2\pi i} \oint_{\gamma_{\leq}} d\lambda' \frac{\lambda}{\lambda'(\lambda' - \lambda)} (-\Psi' \Phi' \ast \Phi^{-1} + \Phi' \Phi' \ast \Phi^{-1})
$$

(4.8)

where now $\gamma_{\leq} \subset K(0; r)$ and $K(0; r)$ is a circle belonging to the domain closed by $\gamma$. Note that $\gamma$, $\gamma'$, and $\gamma_{\leq}$ must be chosen so that they close the same singularities of the integrated functions. Substituting (4.5) and (4.8) and also analogous formulas for $\delta_{(F_{i}, F_{j})} \Psi(\lambda) \ast \Psi^{-1}(\lambda)$ and $\delta_{(F_{i}, F_{j})} \Phi(\frac{1}{\lambda}) \ast \Phi^{-1}(\frac{1}{\lambda})$ into (4.3), performing then integrations and applying the residue theorem one gets

$$
[\delta_{(F_{i}, F_{j})}, \delta_{(F_{j}, F_{k})}]\Theta = 2\delta_{(F_{i}, F_{j}, M(F_{k}, F_{k}), M)} \Theta
$$

(4.9)

Hence, the hidden symmetries for ME form a closed algebra associated with the Moyal bracket Lie algebra.

This coincides with the results of [2, 11, 12] where the algebra of hidden symmetries is given for Plebański’s heavenly equations and for SDYM equations.

As ME can be reduced to all known heavenly equations, to $su(N)$ SDYM equations, to $su(N)$ chiral equations etc., the algebra (4.4) contains the hidden symmetry algebras for the reduced equations.

5 Twistor construction.

For completeness we describe here briefly a twistor construction for ME given in our previous work [3]. This construction is similar to the one for SDYM equations [17, 18].

What must be noted, and it has been pointed out by K.Takasaki [3], is that the twistor construction we propose is valid for the case of $\Theta$ being analytic in $h$, i.e., $\Theta = \sum_{n=0}^{\infty} h^{n} \Theta_{n}$, $\Theta_{n} = \Theta_{n}(x, y, \bar{x}, \bar{y}, p, q)$. The case with negative powers of $h$ should be considered separately and as we know from [3] it can be solved
by the Fairlie-Leznov method \[19\] but we still have not succeeded in a twistor
image for this case.

We start with a twistor function

\[ H = H(\lambda) = H(h; \hat{y} + \lambda x, \hat{x} - \lambda y, \lambda, p, q) = \exp\{ \frac{1}{\hbar} \sum_{n=-\infty}^{\infty} \lambda^n \Delta^{(n)} \} \]

\[ \lambda \epsilon (\overline{C} - \{0\}) \cap (\overline{C} - \{\infty\}) \]

\[ \Delta^{(n)} = \sum_{m=0}^{\infty} h^m \Delta^{(m)}_m(x, y, \hat{x}, \hat{y}, p, q) \]

(5.1)

Let \( \Psi = \Psi(\lambda) \) be a function of the form

\[ \Psi(\lambda) = \exp\{ \frac{1}{\hbar} \sum_{n=1}^{\infty} \lambda^n \Lambda^{(n)} \} , \quad \lambda \epsilon (\overline{C} - \{\infty\}) \]

\[ \Lambda^{(n)} = \Lambda^{(n)}(h; x, y, \hat{x}, \hat{y}, p, q) = \sum_{m=0}^{\infty} h^m \Lambda^{(m)}_m(x, y, \hat{x}, \hat{y}, p, q) , \]

\[ \Lambda^{(1)} = \Theta \]

(5.2)

and let \( \Phi = \Phi(\frac{1}{\lambda}) \) be a function of the form

\[ \Phi(\frac{1}{\lambda}) = \exp\{ \frac{1}{\hbar} \sum_{n=1}^{\infty} (\frac{1}{\lambda})^n \Omega^{(n)} \} , \quad \lambda \epsilon (\overline{C} - \{0\}) \]

\[ \Omega^{(n)} = \Omega^{(n)}(h; x, y, \hat{x}, \hat{y}, p, q) = \sum_{m=0}^{\infty} h^m \Omega^{(m)}_m(x, y, \hat{x}, \hat{y}, p, q) , \]

(5.3)

These functions are chosen so that the following factorization holds

\[ H(\lambda) = \Phi^{-1}(\frac{1}{\lambda}) * \Psi(\lambda) , \quad \lambda \epsilon (\overline{C} - \{0\}) \cap (\overline{C} - \{\infty\}). \]

(5.4)

(This is the Riemann–Hilbert problem or the Birkhoff factorization \[13\]).

One easily finds that from the conditions: \( (\lambda \partial_{\hat{y}} - \partial_x)H(\lambda) = 0 \) and \( (\lambda \partial_{\hat{x}} + \partial_y)H(\lambda) = 0 \) it follows that

\[ [(\lambda \partial_{\hat{y}} - \partial_x)\Psi(\lambda)] * \Psi^{-1}(\lambda) = \lambda[(\partial_{\hat{y}} - \frac{1}{\lambda} \partial_x)\Phi(\frac{1}{\lambda})] * \Phi^{-1}(\frac{1}{\lambda}) \]

\[ [(\lambda \partial_{\hat{x}} + \partial_y)\Psi(\lambda)] * \Psi^{-1}(\lambda) = \lambda[(\partial_{\hat{x}} + \frac{1}{\lambda} \partial_y)\Phi(\frac{1}{\lambda})] * \Phi^{-1}(\frac{1}{\lambda}) \]

\[ \lambda \epsilon (\overline{C} - \{0\}) \cap (\overline{C} - \{\infty\}) \]

(5.5)

The left-hand side of Eq. (5.5) can be analytically extended on \( \overline{C} \) and in the gauge \( \ref{5.2} \) we get

\[ [(\lambda \partial_{\hat{y}} - \partial_x)\Psi(\lambda)] * \Psi^{-1}(\lambda) = -\lambda \frac{1}{\hbar \lambda} \partial_x \Theta \]

\[ [(\lambda \partial_{\hat{x}} + \partial_y)\Psi(\lambda)] * \Psi^{-1}(\lambda) = \lambda \frac{1}{\hbar \lambda} \partial_y \Theta , \quad \lambda \epsilon \overline{C} \]

(5.6)

Thus we recover the linear system \( \ref{3.3} \) of ME. Substituting \( \ref{5.6} \) into \( \ref{5.5} \) one recovers the linear system \( \ref{3.16} \) of ME as well.

It means that our procedure gives a twistor construction for ME.
Acknowledgments

We are grateful to Maciej Dunajski for many discussions on the problems considered in the paper. This work was partially supported by the CONACyT (México) grant 32427-E and by KBN (Poland) grant Z/370/S.

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