NONDENTABLE SETS IN BANACH SPACES

S. J. DIL WORTH, CHRIS GARTLAND, DENKA KUTZAROVA, AND N. LOVASOA RANDRIANARIVONY

Abstract. In his study of the Radon-Nikodým property of Banach spaces, Bourgain showed (among other things) that in any closed, bounded, convex set \( A \) that is nondentable, one can find a separated, weakly closed bush. In this note, we prove a generalization of Bourgain’s result: in any bounded, nondentable set \( A \) (not necessarily closed or convex) one can find a separated, weakly closed approximate bush. Similarly, we obtain as corollaries the existence of \( A \)-valued quasimartingales with sharply divergent behavior.

1. Introduction

We were motivated by the question of whether using the Kuratowski measure of noncompactness in place of diameter leads to a different notion of dentability of (not necessarily closed or convex) subsets of \( X \). Proposition 3.1 shows that they do not. This generalizes results from [Bou79, Chapitre 4] where \( A \) is assumed to be closed, bounded, and convex. In Section 2, we obtain as corollaries \( A \)-valued quasimartingales and \( \overline{co}(A) \)-valued martingales with sharply divergent behavior (Corollaries 3.3 and 3.4) whenever \( A \) is non-\( \varepsilon \)-dentable. In Section 4 we improve the results of Section 3 by showing that the range of the quasimartingale can be made weakly closed. As a further corollary, we show that one can find a countable set \( F \) with \( \lim_{F \varepsilon \to \infty} d(f, A) \to 0 \) such that \( \overline{co}(F) \cap \text{Ext}(\overline{co}_{w^*}(F)) = \emptyset \) (Corollary 4.9).

2. Preliminaries

For any topological vector space \( V \) over \( \mathbb{R} \) and \( E \subseteq V \), let \( \text{co}(E) \) denote the convex hull of \( E \), and \( \overline{co}(E) \) the closure of \( \text{co}(E) \) in \( V \). Henceforth, let \( (X, \| \cdot \|) \) be a Banach space over \( \mathbb{R} \). For \( r > 0 \) and \( x \in X \), \( B_r(x) \) denotes the open ball of radius \( r \) centered at \( x \). \( B_X \) denotes the closed unit ball of \( X \).

Definition 2.1. For any \( A \subseteq X \), let \( \alpha(A) \) be the infimum over all \( \varepsilon > 0 \) so that \( A \) can be covered by finitely many sets of diameter at most \( \varepsilon \). \( \alpha(A) \) is called the Kuratowski measure of noncompactness of \( A \).

Definition 2.2. For any bounded, nonempty \( A \subseteq X, f \in B_{X^*} \) (unit ball of \( X^* \)), and \( \delta > 0 \), we define the slice \( S(f, A, \delta) \), to be the set \( \{ a \in A : f(a) > \sup f(A) - \delta \} \). A slice of \( A \) is a set \( S(f, A, \delta) \) for some \( f \in B_{X^*} \) and \( \delta > 0 \).

Remark 2.3. Geometrically, a slice of \( A \) is a nonempty intersection of \( A \) with an open half-plane. Note that if \( S(f, \overline{co}(A), \delta) \) is a slice of \( \overline{co}(A) \), then \( S(f, \overline{co}(A), \delta) \cap A = \ldots \)

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$S(f, A, \delta)$ is a slice of $A$. This is due to the fact that
$$\sup(f(\overline{\co}(A))) = \sup\left(\overline{f(\co}(A))\right) = \sup(f(\co(A))) = \sup(\co(a)) = \sup(f(A))$$

**Definition 2.4.** A bounded set $A \subseteq X$ is called $\varepsilon$-dentable if there exists a slice of $A$ with $\text{diam}(A) \leq \varepsilon$, and non-$\varepsilon$-dentable otherwise. $A$ is dentable if it is $\varepsilon$-dentable for every $\varepsilon > 0$, and nondentable otherwise.

**Remark 2.5.** By Remark 2.3, if $\overline{\co}(A)$ is $\varepsilon$-dentable, $A$ is $\varepsilon$-dentable.

**Definition 2.6.** If $V$ is a topological vector space, $E \subseteq V$ and $e \in E$, $e$ is called a denting point of $E$ if $e \notin \overline{\co}(E \setminus U)$ for every neighborhood $U$ of $e$. Special cases are when $V$ is a Banach space equipped with the weak topology, or a dual Banach space equipped with the weak* topology, in which case we call $e$ a weak denting point or a weak* denting point, respectively.

**Definition 2.7.** Let $\mathbb{N}^{\times \omega}$ denote the set of finite length sequences of natural numbers. A tree is a nonempty set $T$ such that if $b \in T$ and $b = (b', i)$ for some $b' \in \mathbb{N}^{\times \omega}$ and $i \in \mathbb{N}$, then $b' \in T$. In this case, $b$ is called a child of $b'$. We say that $T$ is finitely branching if each $b \in T$ has only finitely many children. If $b \in T$ has $k$ children, we assume that they are $(b, 1), \ldots, (b, k)$. Given a sequence $b \in \mathbb{N}^{\times \omega}$, we let $|b|$ denote its length. For $n \in \mathbb{N}$, we let $T_{\leq n} = \{b \in T : |b| \leq n\}$, $T_n = \{b \in T : |b| = n\}$, and $T_{\geq n} = \{b \in T : |b| \geq n\}$. Given a positive sequence $(\delta_n)_{n \geq 0}$, finitely branching tree $T$, and subset $(x_b)_{b \in T}$ indexed by $T$, we say that $(x_b)_{b \in T}$ is a $(\delta_n)_{n \geq 0}$-approximate bush if for each $n \in \mathbb{N}$ and $b \in T_n$ with children $(b, 1), \ldots, (b, k)$, $b \in \overline{\co(x_{(b, 1)}, \ldots, x_{(b, k)})} + B_{\delta_n}(0)$. If it always holds that $b \in \overline{\co(x_{(b, 1)}, \ldots, x_{(b, k)})}$, then $(x_b)_{b \in T}$ is a bush. An approximate bush $(x_b)_{b \in T}$ is $\delta$-separated if for each $n \in \mathbb{N}$ and $b \in T_n$ and child $(b, i)$, $\|x_b - x_{(b, i)}\| > \delta$.

**Definition 2.8.** Given a filtration $(\mathcal{A}_n)_{n \geq 0}$ and a positive sequence $(\delta_n)_{n \geq 0}$, we say that a sequence of $X$-valued, $(\mathcal{A}_n)_{n \geq 0}$-adapted random variables $(M_n)_{n \geq 0}$ is a $(\delta_n)_{n \geq 0}$-quasimartingale if

$$\|E(M_{n+1} | \mathcal{A}_n) - M_n\|_\infty \leq \delta_n$$

for all $n \geq 0$. If $\|E(M_{n+1} | \mathcal{A}_n) - M_n\|_\infty = 0$ always holds, $(M_n)_{n \geq 0}$ is a martingale.

The following proposition can be found in [Bou79 Lemme 4.2]. For the sake of self-containment, we include our own proof here.

**Proposition 2.9.** Let $\varepsilon > 0$ and $\delta > 0$. Suppose that $C$ and $C_1$ are closed, bounded, convex sets with $C_1$ properly contained in $C$. If $C = \overline{\co}(C_1 \cup C_2)$, where $C_2$ is a convex subset of $C$ and $\text{diam}(C_2) < \varepsilon$, then there exists a slice $S$ of $C$ with $S \subseteq C_2 + B_\delta(0)$. In particular, $C$ is $\varepsilon$-dentable.

**Proof.** We may assume that $\text{diam}(C) \leq 1$. Since $C_1$ is a proper convex subset of $C$, by Hahn-Banach separation there exists $f \in B_X^*$ such that

$$\sup f(C_1) < M := \sup f(C)$$

Hence $C_1 \subseteq C \setminus S(f, C, \alpha)$ for some $\alpha > 0$. So

$$C = \overline{\co}(C \setminus S(f, C, \alpha)) \cup C_2$$
For \( \gamma > 0 \), let \( S_\gamma = S(f, C, \gamma) \). Consider \( y \in S_\gamma \). There exist \( \lambda \in [0, 1] \), \( z_1 \in \text{co}(C \setminus S(f, C, \alpha)) \), and \( z_2 \in C_2 \) such that \( \|y - \lambda z_1 - (1 - \lambda)z_2\| < \gamma \). Hence
\[
M - \gamma < f(y) \\
\leq f(\lambda z_1 + (1 - \lambda)z_2) + \|y - \lambda z_1 - (1 - \lambda)z_2\| \\
\leq \lambda f(z_1) + (1 - \lambda)f(z_2) + \gamma \\
\leq \lambda(M - \alpha) + (1 - \lambda)M + \gamma \\
= M - \lambda \alpha + \gamma.
\]
Hence \( \lambda < 2\gamma/\alpha \). So
\[
\|y - z_2\| < \lambda \|z_1 - z_2\| + \gamma \leq (2\gamma/\alpha) \text{diam}(C) + \gamma \leq 2(\gamma/\alpha + 1)
\]
So, setting \( \gamma := \frac{\delta}{2(\alpha + 1)} \), we get \( S := S_\gamma \subseteq C_2 + B_\delta(0) \). Note that \( \text{diam}(S) \leq \text{diam}(C_2) + 2\delta < \varepsilon \) for \( \delta \) sufficiently small. So \( C \) is \( \varepsilon \)-dentable. \( \Box \)

We now derive a corollary of this proposition that will play a crucial role in the proof of Lemma 4.3.

**Corollary 2.10.** For any closed, bounded, convex, non-\( \varepsilon \)-dentable \( C \subseteq X \), any closed, convex \( C' \subseteq C \), and any \( D \subseteq C \) with \( \alpha(D) < \varepsilon \), if \( C = \overline{\text{co}}(C' \cup D) \), then \( C = C' \).

**Proof.** Let \( C, C' \), and \( D \) be as above. Assume \( C = \overline{\text{co}}(C' \cup D) \). Since \( \alpha(D) < \varepsilon \), \( D = B_1 \cup B_2 \cup \ldots B_n \) for some \( B_i \subseteq D \) with \( \text{diam}(B_i) < \varepsilon \). Let \( C_i = \overline{\text{co}}(B_i) \). Then \( \text{diam}(C_i) = \text{diam}(B_i) < \varepsilon \), and \( C = \overline{\text{co}}(C' \cup C_1 \cup C_2 \cup \ldots C_n) \).

Since \( C \) is closed, bounded, convex, and not \( \varepsilon \)-dentable, and since \( C_n \subseteq C \) is closed, convex with \( \text{diam}(C_n) < \varepsilon \), Proposition 2.9 (with \( C_2 = C_n \) and \( C_1 = \overline{\text{co}}(C' \cup C_1 \cup C_2 \cup \ldots C_{n-1}) \)) implies that \( C = \overline{\text{co}}(C' \cup C_1 \cup C_2 \cup \ldots C_{n-1}) \). Since \( \text{diam}(C_{n-1}) < \varepsilon \), we may apply Proposition 2.9 again to obtain \( C = \overline{\text{co}}(C' \cup C_1 \cup C_2 \cup \ldots C_{n-2}) \). Iterating, we get \( C = C' \). \( \Box \)

3. \( \delta \)-Separated Martingales and Bushes

**Proposition 3.1.** Let \( A \subseteq X \) be bounded, and let \( \varepsilon > 0 \). The following are equivalent:

1. \( \alpha(S) \geq \varepsilon \) for every slice \( S \subseteq A \).
2. \( \text{diam}(S) \geq \varepsilon \) for every slice \( S \) of \( A \) (\( A \) is non-\( \varepsilon \)-dentable).
3. \( \text{diam}(S) \geq \varepsilon \) for every slice \( S \) of \( \overline{\text{co}}(A) \) (\( \overline{\text{co}}(A) \) is non-\( \varepsilon \)-dentable).

**Proof.** Let \( A, \varepsilon \) be as above. (1) \( \rightarrow \) (2) is clear from definition of \( \alpha \). (2) \( \rightarrow \) (3) follows from the fact that every slice of \( \overline{\text{co}}(A) \) contains a slice of \( A \). We now show (3) \( \rightarrow \) (1) by contradiction. Let \( C = \overline{\text{co}}(A) \), assume that \( C \) is non-\( \varepsilon \)-dentable and that there exists a slice \( S = S(f, A, \delta) \) of \( A \) with \( \alpha(S) < \varepsilon \). Set \( S_C = S(f, C, \delta) \).

Then since \( C \setminus S_C \) is a closed convex subset of \( C \) and \( C = \overline{\text{co}}((C \setminus S_C) \cup S) \), then Corollary 2.10 implies \( C = C \setminus S_C \), a contradiction since \( S_C \subseteq C \) and \( S_C \) is nonempty. \( \Box \)

As in [Bon79, Chapitre 4], we obtain several corollaries.

**Corollary 3.2.** For any \( A \subseteq X \) bounded and \( \varepsilon > 0 \), if \( A \) is non-\( \varepsilon \)-dentable, then for all \( \delta < \frac{\varepsilon}{2} \) and all \( a_1, a_2, \ldots a_n \in A \), \( \overline{\text{co}}(A) = \overline{\text{co}}(A \setminus (B_\delta(a_1) \cup B_\delta(a_2) \cup \ldots B_\delta(a_n))) \).
Proof. Let $A$, $\varepsilon$, $\delta$, and $a_1, a_2, \ldots, a_n$ be as above. Suppose there exists $x \in \overline{\sigma}(A) \setminus \overline{\sigma}(A \setminus (B_3(a_1) \cup B_3(a_2) \cup \ldots B_3(a_n)))$. By Hahn-Banach separation, we can pick a slice $S$ of $\overline{\sigma}(A)$ containing $x$ and disjoint from $\overline{\sigma}(A \setminus (B_3(a_1) \cup B_3(a_2) \cup \ldots B_3(a_n)))$. Then $S \cap A$ is a slice of $A$ disjoint from $A \setminus (B_3(a_1) \cup B_3(a_2) \cup \ldots B_3(a_n))$, and thus $S \cap A \subseteq B_3(a_1) \cup B_3(a_2) \cup \ldots B_3(a_n)$, which implies $\alpha(S \cap A) \leq 2\delta < \varepsilon$, contradicting Proposition 3.1. \qed

We can use Corollary 3.2 to construct $A$-valued quasimartingales and $\overline{\sigma}(A)$-valued martingales that diverge in a sharp manner.

Corollary 3.3. For any nonempty, bounded, non-$\varepsilon$-dentable $A \subseteq X$, any $\delta < \frac{\varepsilon}{2}$, and any positive, summable sequence $(\delta_n)_{n \geq 0}$, there exists a filtration of finite $\sigma$-algebras $(A_n)_{n \geq 0}$, with each of whose atoms is intervals, and an $(A_n)_n$-adapted sequence of random variables $(M_n)_{n \geq 0}$ such that, for all $s, t \in [0, 1]$ and $m \neq n \geq 0$,

1. $M_n$ takes values in $A$.
2. $\|M_n(t) - M_n(s)\| > \delta$.
3. $(M_n)_{n \geq 0}$ is a $(\delta_n)_{n \geq 0}$-quasimartingale: $\|\mathbb{E}(M_{n+1}|A_n) - M_n\|_{\infty} < \delta_n$.

Proof. Let $A \subseteq X$ and $\delta > 0$ be as above. We construct the martingale inductively. Let $x_0$ be any point of $A$, $A_0$ the trivial $\sigma$-algebra, and $M_0 \equiv x_0$. Suppose that, for some $N \in \mathbb{N}$, $A_n$ and $M_n$ have been constructed for all $n \leq N$ and satisfy the conclusion of the Corollary 3.3. Let $J$ be an atom of $A_N$, and let $x_J$ be the value of $M_N$ on $J$. Let $\{a_1, a_2, \ldots, a_k\} \subseteq A$ be the set of all elements in the image of any one of the $M_n$, $n \leq N$. By Corollary 3.2, $x_J \in \overline{\sigma}(A \setminus (B_3(a_1) \cup B_3(a_2) \cup \ldots B_3(a_k)))$. Thus, there exists $z_j \in \overline{\sigma}(A \setminus (B_3(a_1) \cup B_3(a_2) \cup \ldots B_3(a_k)))$ such that $|x_J - z_j| < \delta_N$. Since $z_j \in \overline{\sigma}(A \setminus (B_3(a_1) \cup B_3(a_2) \cup \ldots B_3(a_k)))$, $z_j = \lambda_1 z_j^1 + \lambda_2 z_j^2 + \ldots \lambda_m z_j^m$ for some $z_j^1, z_j^2, \ldots, z_j^m \in A$ and $\lambda_1, \lambda_2, \ldots, \lambda_m \in (0, 1)$ with $\lambda_1 + \lambda_2 + \ldots \lambda_m = 1$ and $|z_j^i - a_j| > \delta$ for all $i \leq m$ and $j \leq k$. Now we subdivide the interval $J$ into $m$ pairwise disjoint subintervals, $J_1, J_2, \ldots, J_m$, with $|J_i| = \lambda_i|J|$ for each $i$. Repeating this process for each atom $J \in A_N$ gives us a collection of pairwise disjoint intervals, and we define $A_{N+1}$ to be the $\sigma$-algebra that they generate. On each $J_i$, we define $M_{N+1}$ to be $z_j^i$. Then conclusions (1) and (2) hold, and (3) holds since $\|\mathbb{E}(M_{N+1}|A_{N}) - M_N\|_{\infty} = \sup_{J_i} |z_j^i - z_j^j| < \delta_N$. \qed

Corollary 3.4. For any nonempty, bounded, non-$\varepsilon$-dentable $A \subseteq X$, any $\delta < \frac{\varepsilon}{2}$, and any positive, summable sequence $(\delta_n)_{n \geq 0}$ on $[0, 1]$, each of whose atoms is intervals, an $(A_n)_n$-adapted quasimartingale $(M_n)_{n \geq 0}$, and an $(A_n)_n$-adapted martingale $(\overline{M}_n)_{n \geq 0}$ such that, for all $s, t \in [0, 1]$ and $m \neq n \geq 0$,

1. $M_n$ takes values in $A$.
2. $\overline{M}_n$ takes values in $\overline{\sigma}(A)$.
3. $\|M_n - \overline{M}_n\|_{\infty} < \delta_n$.
4. $\|M_n(t) - M_m(t)\|, \|\overline{M}_n(t) - \overline{M}_m(t)\| > \delta$.

Proof. Let $A \subseteq X$, and $\delta > 0$ be as above. Choose $\delta' \in (\delta, \frac{\varepsilon}{2})$ and assume $\sum_{n=0}^{\infty} \delta_n < \delta' - \delta$. Choose a positive sequence $(\gamma_k)_{k \geq 0}$ such that $\sum_{k=n}^{\infty} \gamma_k < \delta_n$, and note that this implies $\sum_{n=0}^{\infty} \gamma_n < \sum_{n=0}^{\infty} \delta_n < \delta' - \delta$. By Corollary 3.3, there is a filtration $(A_n)_n$ and an $A$-valued $(A_n)_n$-adapted quasimartingale $(M_n)_{n \geq 0}$ such that $\|M_n(s) - M_n(t)\| > \delta'$ for all $s, t \in [0, 1]$, $m \neq n$, and $\|\mathbb{E}(M_{n+1}|A_n) - M_n\|_{\infty} < \gamma_n$. This inequality, together with the fact that $(\delta_n)_{n \geq 0}$


is summable (and thus convergent to 0), implies, for each $n \geq 0$, the sequence $(E(M_k|A_n))_{k \geq n}$ is Cauchy in $L^\infty(I;X)$. Indeed, for $k > j \geq n$,
\[
\|\mathbb{E}(M_k - M_j|A_n)\|_{L^\infty(I;X)} \leq \sum_{r=j}^{k-1} \|\mathbb{E}(M_{r+1} - M_r|A_n)\|_{L^\infty(I;X)}
\]
\[
\leq \sum_{r=j}^{k-1} \|\mathbb{E}(M_{r+1} - M_r|A_r)\|_{L^\infty(I;X)} = \sum_{r=j}^{k-1} \|\mathbb{E}(M_{r+1}|A_r) - M_r\|_{L^\infty(I;X)}
\]
\[
\leq \sum_{r=j}^{k-1} \gamma_r \leq \delta_j
\]
Thus we may set $\overline{M}_n := \lim_{k \to \infty} E(M_k|A_n)$. Clearly, $(\overline{M}_n)_{n \geq 0}$ is adapted to $(A_n)_{n \geq 0}$ and takes values in $\overline{\mathfrak{a}}(A)$, showing (2). Let us check the martingale property:
\[
\mathbb{E}(\overline{M}_{n+1}|A_n) = \mathbb{E}(\lim_{k \to \infty} E(M_k|A_{n+1})|A_n) = \lim_{k \to \infty} \mathbb{E}(E(M_k|A_{n+1})|A_n)
\]
\[
= \lim_{k \to \infty} E(M_k|A_n) = \overline{M}_{n+1}
\]
showing (1). Next,
\[
\|\overline{M}_n - M_n\|_{\infty} \leq \sum_{k=n}^\infty \|E(M_{k+1} - M_k|A_n)\|_{\infty} \leq \sum_{k=n}^\infty \|E(M_{k+1} - M_k|A_k)\|_{\infty}
\]
\[
= \sum_{k=n}^\infty \|E(M_{k+1}|A_k) - M_k\|_{\infty} \leq \sum_{k=n}^\infty \gamma_k < \delta_n
\]
showing (3). We then use (3) to show (4):
\[
\|\overline{M}_n(s) - \overline{M}_m(t)\| \geq \|M_n(s) - M_m(t)\| - \delta_n - \delta_m > \delta' - (\delta' - \delta) = \delta
\]
\[\square\]

**Remark 3.5.** The union over $n$ of the image of $M_n$ forms a $\delta$-separated bush in $\overline{\mathfrak{a}}(A)$. It is norm closed and lacks extreme points.

4. **Weakly Closed $\delta$-separated Martingales and Bushes**

In this section, we sharpen our results from the previous section by constructing an $A$-valued $\delta$-separated approximate bush that is weakly closed. The argument is more involved than those of the previous section. This again extends results from Bourgain in [Bou79]. $A$ is not assumed to be closed or convex in our case.

**Definition 4.1.** Let $A \subseteq B_X$ and let $C = \overline{\mathfrak{a}}(A)$. For any $\gamma \in (0,1)$ and slice $S = S(f,C,\delta)$ of $C$, we define $S^\gamma = S(f,C,\frac{\gamma\delta}{2})$. $S^\gamma$ is called a $\gamma$-shallow parallel of $S$.

**Lemma 4.2.** For any $C \subseteq B_X$ closed and convex, any $\gamma \in (0,1)$, and any slice $S$ of $C$, $S^\gamma \subseteq S$. For any $E \subseteq C$ for which $C = \overline{\mathfrak{a}}((C \setminus S) \cup E)$, $S^\gamma \subseteq \overline{\mathfrak{a}}(E) + \overline{B}_\gamma(0) \subseteq co(E) + B_{2\gamma}(0)$.
Proof. Let $\gamma \in (0, 1)$ and $S = S(f, C, \delta)$ a slice of $C$. Since $\gamma \in (0, 1)$, $\frac{2\delta}{\gamma} < \delta$ implying $S^\gamma = S \left( f, C, \frac{2\delta}{\gamma} \right) \subseteq S(f, C, \delta) = S$. For the second part, let $E \subseteq C$ such that $C = \overline{co}(C \setminus S \cup E)$. Let $y \in S^\gamma, \epsilon > 0$, and $M := \sup(f(C))$. Since $y \in C = \overline{co}(C \setminus S \cup E)$, there exist $\lambda \in (0, 1], z_1 \in (C \setminus S), z_2 \in \overline{co}(E)$, and $u \in X$ with $\|u\| < \epsilon$ such that $y = \lambda z_1 + (1 - \lambda) z_2 + u$. Then we have

\[
M - \frac{2\delta}{\gamma} < f(y) = \lambda f(z_1) + (1 - \lambda) f(z_2) + f(u) < \lambda (M - \delta) + (1 - \lambda) M + \epsilon
\]

implying $\lambda < \frac{2}{\gamma} + \frac{\epsilon}{2\delta}$. Hence,

\[
\|y - z_2\| \leq \|y - (1 - \lambda) z_2\| + \|(1 - \lambda) z_2 - z_2\| = \|\lambda z_1 + u\| + \|\lambda z_2\| \leq 2\lambda + \epsilon < \gamma + \frac{\epsilon}{2\delta} + \epsilon
\]

Since $\epsilon > 0$ was arbitrary, this shows $y \in B_\gamma(z_2) \subseteq \overline{co}(E) + B_\gamma(0)$. The final containment $\overline{co}(E) + B_\gamma(0) \subseteq E + B_\gamma(0)$ obviously holds. \hfill \Box

Lemma 4.3. Let $A \subseteq X$ be bounded, nonempty, and non-$\epsilon$-dencable, and let $C = \overline{co}(A)$ (by Remark 2.3, $C$ is non-$\epsilon$-dencable). For any slice $S_0$ of $C$, $D \subseteq C$ with $\alpha(D) < \epsilon$, and $\gamma \in (0, 1)$, let $S(S_0, D)$ be the collection of all slices $S$ of $C$ with $S \subseteq S_0 \setminus D$ and $S^\gamma(S_0, D) = \{S^\gamma\}_{S \in S(S_0, D)}$. Let $\Lambda = \Lambda(S_0, D, \gamma) \subseteq C$ denote the union of all sets in $S^\gamma(S_0, D)$. Then $C = \overline{co}(C \setminus S_0 \cup (\Lambda \cap A))$.

Proof. Let $S_0, D, \gamma$, and $\Lambda$ be as above. By Corollary 2.10 (with $C' = \overline{co}(C \setminus S_0 \cup (\Lambda \cap A)$) and $D = D$), it suffices to prove $C = \overline{co}(C \setminus S_0 \cup D \cup (\Lambda \cap A))$. Assume $C \neq \overline{co}(C \setminus S_0 \cup D \cup (\Lambda \cap A))$. Then by Hahn-Banach separation, there exists a slice $S$ of $C$ such that $S \subseteq C \setminus \overline{co}(C \setminus S_0 \cup D \cup (\Lambda \cap A))$. This implies $S \subseteq S_0, S \cap D = \emptyset$, and $S \cap (\Lambda \cap A) = \emptyset$. Then $S \subseteq S_0 \setminus D$. Thus, $S \in S(S_0, D)$, so $S^\gamma \in S^\gamma(S_0, D)$, and finally $S^\gamma \subseteq A$. But since we also have $S^\gamma \subseteq S$ and $S \cap (\Lambda \cap A) = \emptyset$, $(S^\gamma \cap A) = S^\gamma \cap (\Lambda \cap A) = \emptyset$, a contradiction since $S^\gamma \cap A$ is a slice of $A$ (since $S^\gamma$ is a slice of $C = \overline{co}(A)$) and slices of nonempty sets are nonempty. \hfill \Box

4.1. The Construction.

Theorem 4.4. Let $A \subseteq B_X$ be nonempty and non-$\epsilon$-dencable (not necessarily closed or convex), and $C = \overline{co}(A)$ so that $C$ is also non-$\epsilon$-dencable. Fix $\delta < \frac{\epsilon}{2}$, and assume that $A$ is separable. Then $C$ is separable as well, so $C = \bigcup_{i=0}^\infty B_i$ for some open $B_i$ (relative to $C$) with $\text{diam}(B_i) < \epsilon$. Let $(\delta_n)_{n \geq 0}$ be a sequence of numbers in $(0, 1)$. There exist a finitely branching tree $T \subseteq \mathbb{N}^{\omega_1}$, an $(2\delta_n)_{n \geq 0}$-approximate bush $(x_b)_{b \in T} \subseteq A$, and slices $(S_b)_{b \in T}$ of $C$ such that, for all $n \in \mathbb{N}$,

1. For all $b \in T_n$, $x_b \in S_b^n \cap A \subseteq S_b$.
2. If $n \geq 1$, then for all $b \in T_n, S_b \cap B_{n-1} = \emptyset$ and $S_b \cap \left( \bigcup_{\|x\| \leq n-1} B_{\delta}(x_p) \right) = \emptyset$.
3. If $n \geq 1$, then for all $b \in T_{n-1}$, if $(b, 1), \ldots, (b, q)$ are the children of $b$, then $S_{(b, i)} \subseteq S_b$ and the approximate bush property is satisfied: $x_b \in co(x_{(b, 1)}, \ldots, x_{(b, q)}) + B_{2\delta_{n-1}}(0)$.

Proof. The proof is by induction on $n$. For the base case, let $S_0 = C$ and let $x_0$ be any element of $S_0^n$. For the inductive step, let $n \geq 0$ and assume $T_{\leq n}$, $(x_b)_{b \in T_{\leq n}} \subseteq A$, and $(S_b)_{b \in T_{\leq n}} \subseteq C$ have been constructed, and satisfy (1)-(3). Let $b \in T_n$. Let
\( D := B_\epsilon \cup \bigcup_{i \leq n} B_\delta(x_i) \), so that \( \alpha(D) < \epsilon \). As in Lemma 1.3, let \( S(S_0, D) \) be the collection of all slices \( S \) of \( C \) such that \( S \subseteq S_0 \setminus D \), \( S^{d_n+1}(S_0, D) = \{ S^{d_n+1} \} \), and \( \Lambda = \bigcup S^{d_n+1}(S_0, D) \). By Lemma 1.3, \( C = \overline{co}(C \setminus S_0) \cup (\Lambda \cap A) \). Then by Lemma 1.2, \( S^{d_n}_b \subseteq \overline{co}(A \cap S_b) \cup (\Lambda \cap A) \). Then since \( x_b \in S^{d_n}_b \), there exists \( z \in \overline{co}(A \cap S_b) \) such that \( \| x_b - z \| < 2\delta_n \). Let \( z_1, \ldots, z_q \in \Lambda \cap A \) and \( \lambda_1^b, \ldots, \lambda_q^b \in [0,1] \) such that \( z = \lambda_1^bz_1 + \ldots + \lambda_q^bz_q \). For each \( i = 1, \ldots, q \), since \( z_i \in \Lambda \), there are slices \( S_{z_i} \subseteq S(S_0, D) \) of \( C \) with \( z_i \in S^{d_n+1}_{z_i} \), by definition of \( \Lambda \). We now define the children of \( b \) to be \( (b, 1), \ldots, (b, q) \), \( x(b, i) \) to be \( z_i \), and \( S_{b(i)} \) to be \( S_{z_i} \). Repeating this process for each \( b \in T_n \) gives us \( T_{n+1}, (x_b)_{b \in T_{n+1}} \subseteq A \), and \( (S_b)_{b \in T_{n+1}} \subseteq C \).

(1) and (3) hold immediately by construction. It is also clear that (2) holds by recalling that \( S_{b(i)} \subseteq S(S_0, D) \), and then examining the definition of \( D \) and \( S(S_0, D) \).

**Remark 4.5.** The assumption that \( A \) is separable can be removed (at the penalty of replacing \( \epsilon \) by \( \epsilon/2 \)) because of the following result: under the hypothesis of Theorem 4.4, \( A \) contains a countable subset that is non-\( \epsilon/2 \)-dentable. This is essentially proved in [May73, Lemma 2.2], but we'll include the argument here. Since \( \text{diam}(S) > \epsilon \) for every slice \( S \) of \( A \), it follows that no slice is contained in a closed ball \( B_{\epsilon/2}(x) \). Hence, if \( a \in A \), then \( a \in \overline{co}(A \setminus B_{\epsilon/2}(a)) \). So there exists a countable set \( T(a) \subseteq A \setminus B_{\epsilon/2}(a) \) such that \( a \in \overline{co}(T(a)) \). By applying this fact iteratively as in [May73, Lemma 2.2], we can construct a countable \( A \subseteq A \) such that for every \( a \in A \), we have \( a \in \overline{co}(A \setminus B_{\epsilon/2}(a)) \). Hence every slice \( S \) of \( A \) satisfies \( \text{diam}(S) > \epsilon/2 \). Hence \( A \) is non-\( \epsilon/2 \)-dentable.

**Corollary 4.6.** For any separable \( A \subseteq B_X \) nonempty and non-\( \epsilon \)-dentable, any \( \delta < \frac{\epsilon}{2} \), and any positive \( \delta_n \)，there exists a \( \delta \)-separated, \( \delta_n \)-approximate bush \( (x_b)_{b \in T} \) in \( A \) such that any other set \( (y_b)_{b \in T} \subseteq C = \overline{co}(A) \), with \( \sup_{b \in T} \| y_b - x_b \| < \gamma_n \) for some \( \gamma_n \rightarrow 0 \), is weakly closed and discrete. In particular, \( (x_b)_{b \in T} \) is weakly closed and discrete.

**Proof.** Let \( A, \delta, \delta_n \) be as above. Applying the construction of Theorem 4.4 with \( \delta_n \) in place of \( \delta_n \), yields a bush \( (x_b)_{b \in T} \). By Theorem 4.4, \( x_b \in A \) for all \( b \in T \). Suppose \( b_1, b_2 \in T \) with \( |b_2| > |b_1| \). Then by Theorem 4.4, \( S_{b_2} \cap B_{\delta}(x_{b_1}) = \emptyset \), so \( \| x_{b_2} - x_{b_1} \| > \delta \). This means the bush is \( \delta \)-separated. By Theorem 4.4(3), if \( b \in T \) and \( (b, 1, \ldots, b, q) \) are the children of \( b \), then \( x_b \in \text{co}(x(b, 1), \ldots, x(b, q)) + B_{\delta_n}(0) \). This means the bush is \( \delta_n \)-approximate.

Finally, let \( (y_b)_{b \in T} \subseteq C \), with \( \sup_{b \in T} \| y_b - x_b \| < \gamma_n \) for some \( \gamma_n \rightarrow 0 \), and let \( z \) belong to the weak closure of \( (y_b)_{b \in T} \). Since \( C \) is norm closed and convex, it is weakly closed, and thus \( z \in C \). Then \( z \in B_i \) for some \( i \). Consider \( S_b \) for \( |b| = i + 1 \). Then \( S_b = S(\delta_n, C, \alpha_b) \) for some \( \delta_n \) and \( \alpha_b > 0 \). Hence \( z \in B_i \subseteq C \setminus S_b = \{ x \in C : f_b(x) \leq \sup f(C) - \alpha_b \} \).

Since \( B_i \) is open in the norm topology relative to \( C \) and \( C \) is convex, it follows that \( B_i \subseteq \{ x \in C : f_b(x) < \sup f(C) - \alpha_b \} \). Since \( \gamma_n \rightarrow 0 \), we can find \( \gamma > 0 \) and \( N \) large enough so that \( B_i \subseteq \{ x \in C : f_b(x) < \sup f(C) - \alpha_b \} \), \( N \geq i + 1 \), and \( \gamma_n < \gamma \) for all \( n \geq N \). Then we set \( U_b := \{ x \in C : f_b(x) < \sup f_b(C) - \alpha_b \} \) and observe that it is a weak neighborhood of \( z \) in \( C \). Hence \( U := \cap_{|b|=i+1} U_b \) is a weak neighborhood of \( z \) in \( C \). Now we wish to show the set \( U \cap (y_b)_{b \in T} \) is finite, which will imply our desired conclusion that \( (y_b)_{b \in T} \) is weakly closed and
Consider $b \in T$ with $|b| \geq N$. Then $\|y_b - x_b\| < \gamma |b| < \gamma$. Let $b_{i+1} \in T$ denote the unique predecessor of $b$ with $|b_{i+1}| = i + 1$. Then $x_{b_{i+1}} \in S_b \subseteq S_{b_{i+1}}$, and hence $f_{b_{i+1}}(x_b) > \sup f_{b_{i+1}}(C) - \alpha_{b_{i+1}}$. Since $f_{b_{i+1}} \in B_X$ and $\|y_b - x_b\| < \gamma$, this implies $f_{b_{i+1}}(y_b) > \sup f_{b_{i+1}}(C) - \alpha_{b_{i+1}} - \gamma$. Thus, by definition of $U_{b_{i+1}}$, $y_b \notin U_{b_{i+1}}$. By definition of $U$ this proves $U \cap \{y_b \in T \geq N \} = \emptyset$.

**Corollary 4.7.** For any $A \subseteq B_X$ nonempty and non-$\varepsilon$-dentable, any $\delta < \frac{\varepsilon}{2}$, and any positive, summable sequence $(\delta_n)_{n \geq 0}$, there exists a filtration of finite $\sigma$-algebras $(A_n)_{n \geq 0}$, an $A$-valued, $(A_n)_{n \geq 0}$-adapted $(\delta_n)_{n \geq 0}$-quasimartingale $(M_n)_{n \geq 0}$ with $\|M_n(s) - M_n(t)\| > \delta$ for all $n \geq m \geq 0$ and $s, t \in [0, 1]$, and the range of this quasimartingale is weakly closed and discrete.

**Proof.** Let $A, \delta, (\delta_n)_{n \geq 0}$ be as above, and apply Corollary 4.6 to obtain a $(\delta_n)_{n \geq 0}$-approximate bush $(x_b)_{b \in T}$ which is weakly closed and discrete. We define the filtration $(A_n)_{n \geq 0}$ on $[0, 1]$ recursively: Let $A_0$ be the trivial $\sigma$-algebra. Suppose $A_n$ has been defined as a finite whose atoms are intervals, the atoms are in bijection with $T_n$ via $b \mapsto I_b$, and for any $b \in T_{n-1}$ and child $(b, i) \in T_n$, $\mathcal{L}(I_b, i) = \mathcal{L}(I_b)\lambda^1_i$. Then for any $b' \in T_n$ with children $(b', 1), \ldots, (b', q)$, we pick any subdivision of $I_{b'}$ into intervals $I_{b'(1)}, \ldots, I_{b'(q)}$ so that $\mathcal{L}(I_{b'(i)}) = \mathcal{L}(I_{b'})\lambda^q_i$. Take $A_n$ to be the $\sigma$-algebra generated by these intervals. Then we define $M_n$ to be $\sum_{|b|=n} x_b$. We then have $\|E(M_{n+1}|A_n) - M_n\|_\infty = \sup_{b \in T_n} \|x_b - \lambda_1x_{(b, 1)} - \ldots - \lambda_qx_{(b, q)}\| < \delta_n$. The range of this quasimartingale is exactly the bush, and thus weakly closed and discrete. $$\square$$

**Corollary 4.8.** For any $A \subseteq B_X$ nonempty and non-$\varepsilon$-dentable, $\delta < \frac{\varepsilon}{2}$, and positive, summable sequence $(\delta_n)_{n \geq 0}$, there exist a filtration of finite $\sigma$-algebras $(A_n)_{n \geq 0}$, an $A$-valued, $(A_n)_{n \geq 0}$-adapted $(\delta_n)_{n \geq 0}$-quasimartingale $(M_n)_{n \geq 0}$ and $\mathfrak{C}(A)$-valued, $(A_n)_{n \geq 0}$-adapted martingale $(\overline{M}_n)_{n \geq 0}$ with, for all $n \neq m \geq 0$ and $s, t \in [0, 1]$,

(1) $\|\overline{M}_n(s) - \overline{M}_m(t)\| > \delta$.

(2) $\|M_n - \overline{M}_n\|_\infty < \delta_n$.

(3) The range of $(\overline{M}_n)_{n \geq 0}$ is weakly closed and discrete.

**Proof.** Let $A, \delta, (\delta_n)_{n \geq 0}$ be as above, and apply Corollary 4.7 to obtain the $\sigma$-algebra $(A_n)_{n \geq 0}$ and $A$-valued, $(\delta_n)_{n \geq 0}$-quasimartingale $(M_n)_{n \geq 0}$ with weakly closed and discrete range. Construct $(\overline{M}_n)_{n \geq 0}$ from $(M_n)_{n \geq 0}$ just as in the proof of Corollary 3.4 so that $(\overline{M}_n)_{n \geq 0}$ is $\mathfrak{C}(A)$-valued and (1) and (2) hold. To see (3), again note that the range of $(M_n)_{n \geq 0}$ is exactly $(x_b)_{b \in T_n}$ from Corollary 4.6. Since $(\overline{M}_n)_{n \geq 0}$ is adapted to the same finite filtration as $(M_n)_{n \geq 0}$, (2) implies that the range of $\overline{M}_n$ equals $(y_b)_{b \in T_n}$ for some $y_b \in \mathfrak{C}(A)$ and $\sup_{b \in T_n} \|y_b - x_b\| < \delta_n$, and they have $\overline{M}_n = M_n$ and $\sup_{b \in T_n} \|y_b - \lambda_x(x_{(b, 1)} - \ldots - \lambda_qx_{(b, q)})\| < \delta_n$. Then Corollary 4.6 implies (3). $$\square$$

**Corollary 4.9.** For any $A \subseteq B_X$ nonempty and nondentable, there exists a countable set $F \subseteq \mathfrak{C}(A)$ such that

(1) $\lim_{F \geq f \to \infty} d(f, A) = 0$

(2) $F$ is weakly closed and discrete and $\text{Ext}(F) = \emptyset$.

(3) $\mathfrak{C}(F)$ has no weak denting point.

(4) $\mathfrak{C}(F) \cap \text{Ext}(\mathfrak{C}(\mathfrak{C}(F))) = \emptyset$.

**Proof.** Let $A$ be as above. Let $\varepsilon > 0$ such that $A$ is non-$\varepsilon$-dentable and let $\delta < \frac{\varepsilon}{2}$. Let $\delta_n$ be any positive, summable sequence, and let $(A_n)_{n \geq 0}$, $(M_n)_{n \geq 0}$, and...
\((M_n)_{n \geq 0}\) be the filtration, \((\delta_n)_{n \geq 0}\)-quasimartingale, and martingale afforded to us by Corollary 4.8. Let \(F \subseteq \overline{\sigma}(A)\) be the range of the martingale. Since \((M_n)_{n \geq 0}\) is \(A\)-valued and \(\|M_n - M_{n-1}\|_\infty < \delta_n\), \(\lim_{F \ni f \rightarrow \infty} d(f, A) = 0\), showing (1).

By Corollary 4.8, \(F\) is weakly closed and discrete and clearly has no extreme point since it is a \(\delta\)-separated bush, showing (2).

Since weak denting points of \(\overline{\sigma}(F)\) are extreme points, and since \(F\) has no extreme points, the set of weak denting points of \(\overline{\sigma}(F)\) is contained in \(\overline{\sigma}(F) \setminus F\). But since \(\overline{\sigma}(F) \setminus F\) is weakly open in \(\overline{\sigma}(F)\), it follows that \(\overline{\sigma}(F) \setminus F\) contains no weak denting point. This shows (3).

For (4), we first observe that the converse of the Krein-Milman theorem ([DS58, Lemma 8.5]) implies that every extreme point of \(\overline{\sigma}w^*(F)\) is a weak* denting point of \(\overline{\sigma}w^*(F)\). To see this, let \(x\) be an extreme point of \(\overline{\sigma}w^*(F)\) and assume \(x\) is not a weak* denting point. Then there is an open neighborhood \(U \subseteq X^*\) of \(x\) such that \(x \in \overline{\sigma}w^*( \overline{\sigma}w^*(F) \setminus U)\). Then since \(\overline{\sigma}w^*(F) \setminus U\) is weak* compact, the converse to Krein-Milman implies every extreme point of \(\overline{\sigma}w^*(\overline{\sigma}w^*(F) \setminus U)\), in particular \(x\), is contained in \(\overline{\sigma}w^*(F) \setminus U\), a contradiction. Then (4) follows from (3) since weak* denting points of \(\overline{\sigma}(F) \cap \overline{\sigma}w^*(F) \subseteq X^*\) are the same as weak denting points of \(\overline{\sigma}(F) \cap \overline{\sigma}w^*(F) \subseteq X\).

□

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Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA
E-mail address: dilworth@math.sc.edu

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA
E-mail address: cgarlta2@illinois.edu

Department of Mathematics University of Illinois at Urbana-Champaign Urbana, IL 61801, USA and Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria
E-mail address: denka@math.uiuc.edu

Department of Mathematics and Statistics, Saint Louis University, St. Louis, MO 63103, USA
E-mail address: nrandria@slu.edu