Interacting Massless Infraparticles in 1+1 Dimensions

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Received: 12 October 2021 / Accepted: 22 June 2022
Published online: 12 September 2022 – © The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract: The Buchholz’ scattering theory of waves in two dimensional massless models suggests a natural definition of a scattering amplitude. We compute such a scattering amplitude for charged infraparticles that live in the GNS representation of the 2d massless scalar free field and obtain a non-trivial result. It turns out that these excitations exchange phases, depending on their charges, when they collide.

1. Introduction

Construction of interacting quantum field theories, even in low-dimensional spacetime, is notoriously difficult, cf. e.g. [GJ,Le08,Ta14,CT15,Di18,GH21,GHW20]. Apart from the well known technical problems there is also a conceptual difficulty: what does it mean that a quantum field theory is interacting? Non-triviality of the $S$-matrix is a clear-cut criterion only for collisions of Wigner particles 1. Outside of this restrictive setting, excluding most physical particles [Bu86], there is no generally accepted criterion for interaction and one has to proceed on a case by case basis. In this note we formulate a natural criterion for interaction for massless two-dimensional theories, which is suggested by the Buchholz’ collision theory of waves [Bu75]. It allows us to exhibit a subtle scattering between charged excitations in massless free field theory in two dimensions, which consists in exchanging phases depending on their charges. This interaction, which coexists with a linear field equation, is effected by the exotic infrared structure of the vacuum in the model. We remark that a similar effect was anticipated by Streater in [St10].

Let us now introduce our criterion for interaction for Haag-Kastler theories $(\mathcal{F}, U, \Omega)$ of massless particles on $\mathbb{R}^2$. Here $\mathcal{F} \subset B(\mathcal{H})$ is a $C^*$-algebra 2 acting irreducibly on a

1 We refer to Chapters II.4 and VI of [Ha] for classical results and to [Dy05,DH15,Du17,Du18] for more recent developments.
2 It will be interpreted as the $C^*$-algebra of charge carrying fields. The sub-algebra of observables $\mathcal{A}$ will appear in a concrete model in Sect. 2.
Hilbert space $\mathcal{H}$, $U$ is a strongly continuous unitary representation of the Poincaré group satisfying the spectrum condition and the unit vector $\Omega \in \mathcal{H}$ is the vacuum vector, that is, $\Omega$ is cyclic for $\mathcal{F}$ and invariant under $U$. Furthermore, $\mathcal{F}$ is the global algebra of a net of $C^*$-algebras $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O})$, labelled by open bounded regions $\mathcal{O} \subset \mathbb{R}^2$, which is local and isotonous. Following [Bu75], for any $F \in \mathcal{F}$ we define the averaged operators at time $|T| \geq 1$

$$F_{\pm}(h_T) \doteq \int dt \ h_T(t) F(t_{\pm}), \quad t_{\pm} \doteq (t, \pm t),$$

(1.1)

where $F(t_{\pm}) \doteq U(t_{\pm})FU(t_{\pm})^*, \ h \in \mathcal{D}(\mathbb{R}), \ h \geq 0, \ h(t) = h(-t), \ \int dt h(t) = 1 \text{ and } h_T(t) \doteq \frac{1}{s(T)} h\left(\frac{T-t}{s(T)}\right), s(T) \doteq \ln |T|$. For any $F, G \in \mathcal{F}$ we define vectors

$$\Psi_T \doteq F_+(h_T)G_-(h_T)\Omega \in \mathcal{H},$$

$$\Psi_T^\circ \doteq F_+(h_T)\Omega \otimes G_-(h_T)\Omega \in \mathcal{H} \otimes \mathcal{H}. \quad (1.2)$$

In this work we are interested in scattering amplitudes of massless excitations of the net $(\mathcal{F}, U, \Omega)$ which are defined by

$$S(F, G) \doteq \lim_{T \to \infty} \frac{\langle \Psi_T, \Psi_T^{-} \rangle}{\langle \Psi_T^\circ, \Psi_T^{-} \rangle} = \lim_{T \to \infty} \frac{\langle \Omega, G^+(h_T)F^+(h_T)F_+(h_T)G_-(h_T)\Omega \rangle}{\langle \Omega, F^+(h_T)F_+(h_T)G_-(h_T)\Omega \rangle},$$

(1.4)

provided that the denominator is non-zero for sufficiently large $T$ and the limit exists and satisfies $|S(F, G)| \leq 1$. If, in addition, $0 \neq S(F, G) \neq 1$ for some $F, G \in \mathcal{F}$, then we say that the Haag-Kastler theory is interacting. The denominator in (1.4) ensures that $S(\mathbb{1}, \Gamma) = S(\mathbb{1}, G) = 1$. Hence, like in the conventional setting discussed below, there is no one-body interaction.

Let us now point out that also at the two-body level this concept of interaction generalizes the conventional one from [Bu75]. The latter work studies vectors (1.2) under the assumption that the invariant subspaces $\mathcal{H}_\pm \{ \Psi \in \mathcal{H} | U(t_{\pm})\Psi = \Psi, t \in \mathbb{R} \}$ contain some non-zero vectors orthogonal to the vacuum. These vectors describe massless particles in the sense of Wigner, which in this context are called waves. For any non-zero $\Psi_\pm \in \mathcal{H}_\pm$ one can find $F, G \in \mathcal{F}$ s.t.

$$\Psi_+ = \lim_{T \to \pm \infty} F_+(h_T)\Omega, \quad \Psi_- = \lim_{T \to \pm \infty} G_-(h_T)\Omega. \quad (1.5)$$

It is shown in [Bu75] that the limits

$$\Psi_\text{out}^\pm = \lim_{T \to \infty} \Psi_T = \lim_{T \to -\infty} F_+(h_T)G_-(h_T)\Omega, \quad \Psi_\text{in}^\pm = \lim_{T \to \infty} \Psi_T^{-} = \lim_{T \to -\infty} F_+(h_T^{-})G_-(h_T^{-})\Omega \quad (1.6)$$

exist, are different from zero, depend only on the single-particle vectors $\Psi_\pm$ and satisfy $\|\Psi_\text{out}\| = \|\Psi_+ \otimes \Psi_-\| = \|\Psi_\text{in}\|$. Thus one can define the (isometric) scattering matrix by

$$S\Psi_\text{out} = \Psi_\text{in}. \quad (1.7)$$

Then the scattering amplitude (1.4) has the form

$$S(F, G) = \frac{\langle \Psi_\text{out}^\circ, S\Psi_\text{out} \rangle}{\langle \Psi_\text{out}, \Psi_\text{out} \rangle}. \quad (1.8)$$
and satisfies $|S(F,G)| \leq 1$. In this context $S(F,G) \neq 1$ implies $S \neq 1$, that is, interaction in the conventional sense. We remark as an aside, that scattering theory of waves was generalized to wedge-local Haag-Kastler nets and interacting examples were found in this broader setting in [DT11, BT13].

For $F, G$ s.t. the vectors $F\omega, G\omega$ are orthogonal to the subspaces of waves $\mathcal{H}_{\pm}$, the resulting scattering states (1.6) vanish. This fact alone should not be interpreted as interaction but as a manifestation of the infraparticle problem. Our criterion for interaction, $0 \neq S(F,G) \neq 1$, accounts for this, and may still hold due to cancellations between the numerator and the denominator in (1.4).

Such a situation occurs for charge carrying fields in massless free field theory in two dimensions and the resulting scattering amplitude is computed for specific $F, G$ in Theorem 2.1 below. The result is

$$S(F,G) = e^{-\frac{i}{2}q_f q_g}, \tag{1.9}$$

where $q_f, q_g$ are charges of the excitations created by $F, G$ from the vacuum. Such excitations can be called infraparticles as they are not eigenvectors of the relativistic mass operator. While it is well known that massless two dimensional theories contain infraparticles [Sch63, Bu96, DT12, DT13], we give the first example in which they interact by exchanging charge-dependent phases in collisions. Similarly to the Ising or Federbush model (cf. [Le05, Ta14]), the scattering amplitude (1.9) is independent of momenta.

Our paper is organized as follows: In Sect. 2 we define the model and state our main result. Section 3 contains some preparations on the behaviour of vacuum expectation values under large spacetime translations. In Sect. 4 we give the proof of the main result. Some technical details of the discussion are postponed to the appendices.

2. The Model and the Main Result

We outline the massless scalar free field theory on two-dimensional Minkowski spacetime, referring to [AAR, AMS92, AMS93, BFR17, BLOT, SW70, Ci09, DM06, MPS90, Sch13] for more information. We denote the momentum by $p$ and the energy-momentum of a particle of mass $m \geq 0$ by $p \doteq (\omega_m(p^1), p^1)$, $\omega_m(p^1) \doteq \sqrt{(p^1)^2 + m^2}$. The two-point function $w_m$ of the free field $\phi_m$ in spacetime dimension $d = 2$ with mass $m > 0$ is

$$w_m(x) \doteq (2\pi)^{-1} \int d\mu_m(p)e^{-ip\cdot x}, \tag{2.1}$$

where $d\mu_m(p^1) \doteq dp^1/2\omega_m(p^1)$. The limit $w \doteq \lim_{m \to 0} w_m$ is only well-defined for test functions from $\mathcal{D}(\mathbb{R}^2)$ whose charge $q_f \doteq \int f(x)d^2x$ is equal to zero. On the other hand, the commutator function

$$iD_0(x) \doteq \lim_{m \to 0} \left( w_m(x) - w_m(-x) \right) \tag{2.2}$$

is well-defined for all test functions from $\mathcal{D}(\mathbb{R}^2)$. This gives a non-degenerate symplectic form on the vector space $\mathcal{L} \doteq \mathcal{D}(\mathbb{R}^2)/\square \mathcal{D}(\mathbb{R}^2)$, where $\square$ is the d’Alembertian. The non-degeneracy can be shown as in [BGP, Theorem 3.4.7] using explicit formulas for the propagators, see e.g. [CRV21, formula (3.3)]. Thus, by [BGP, Theorem 4.2.9] we obtain a unique $C^*$-algebra $\mathcal{F}$ generated by the abstract Weyl operators $W(f), f \in \mathcal{D}(\mathbb{R}^2)$, with the Weyl relations

$$W(f)W(g) = e^{-iD_0(f,g)/2}W(f + g) \quad \text{and} \quad W(f)^* = W(-f), \tag{2.3}$$
where $D_0(f, g) = \int f(x)D_0(x-y)g(y)d^2xd^2y$. For any open bounded region $\mathcal{O} \subset \mathbb{R}^2$, the subspace $\mathcal{L}(\mathcal{O}) = \mathcal{D}_\mathbb{R}(\mathcal{O})/\mathcal{D}_\mathbb{R}(\mathbb{R}^2)$ of $\mathcal{L}$ gives rise to a $C^*$-algebra $\mathcal{F}(\mathcal{O})$ which is naturally a subalgebra of $\mathcal{F}$. The resulting isotonous net $\mathcal{O} \mapsto \mathcal{F}(\mathcal{O})$ is local, since $D_0$ is supported on lightlike vectors. Denoting by $\mathcal{P}_{\uparrow}$ the proper orthochronous Poincaré group, for any $L \in \mathcal{P}_{\uparrow}$ we define an automorphism of $\mathcal{F}$ by

$$\alpha_L(W(f)) = W(f_L), \quad f_L(x) = f(L^{-1}x), \quad (2.4)$$

which acts covariantly on the net. Next, we introduce a linear functional $\langle \cdot \rangle$ on $\mathcal{F}$ by

$$\langle W(f) \rangle = \lim_{m \to 0} e^{-\frac{1}{2}w_m(f,f)} = \begin{cases} e^{-\frac{1}{2}w(f,f)} & \text{if } q_f = 0, \\ 0 & \text{else}. \end{cases} \quad (2.5)$$

This agrees with $w_m(f,f) = \int f(x)w_m(x-y)f(y)d^2xd^2y$ which diverges to $+\infty$ with $m \to 0$ if $q_f \neq 0$. By [AMS92, AMS93] this functional defines a state on $\mathcal{F}$ and the resulting GNS representation $(\pi, \mathcal{H}, \Omega)$ is irreducible. This state is invariant under the Poincaré transformations (2.4) and the corresponding group of automorphisms is unitarily implemented by a strongly continuous group of unitaries $U$ on $\mathcal{H}$ satisfying the spectrum condition (cf. Appendix A). Thus the functional (2.5) is a pure vacuum state on $\mathcal{F}$ in the sense of [Ar] and $(\pi(\mathcal{F}), U, \Omega)$ is a Haag-Kastler net as defined in the Introduction.

Let $\mathcal{A} \subset \mathcal{F}$ be the subalgebra generated by $W(f)$, $q_f = 0$. We can use it to decompose the Hilbert space into the neutral and charged subspace:

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_ch, \quad \mathcal{H}_0 = \pi(\mathcal{A})\Omega, \quad \mathcal{H}_ch = \mathcal{H}_0^\perp. \quad (2.6)$$

By cyclicity of $\Omega$, the subspace $\mathcal{H}_ch$ is spanned by all $W(f)$ s.t. $q_f \neq 0$. As shown in Appendix A

$$\mathcal{H}_\pm \subset \mathcal{H}_0 \quad (2.7)$$

thus all the single-wave vectors in the model have zero charge. In Sect. 4 of this paper we compute the scattering amplitude (1.4) both for neutral and charged excitations in this model and obtain the following result:

**Theorem 2.1.** Let $(\pi(\mathcal{F}), U, \Omega)$ be the Haag-Kastler net of the massless scalar free field, as defined above. Then, for $F = \pi(W(f))$, $G = \pi(W(g))$, $f, g \in \mathcal{D}_\mathbb{R}(\mathbb{R}^2)$, the scattering amplitude (1.4) exists and equals

$$S(F, G) = e^{-\frac{i}{2}q_fq_g}, \quad (2.8)$$

where $q_f = \int f(x)d^2x$ and $q_g = \int g(x)d^2x$.

We infer from this theorem that colliding excitations in the model exchange phases depending on their charges. Clearly, a non-trivial effect occurs only if both excitations are charged, i.e. $q_fq_g \neq 0$. In this case neither of them can be a wave, cf. (2.7), thus they should be considered infraparticles. To resolve the apparent paradox that a free field is interacting, one should recall that the $2d$ massless free field has a much richer vacuum structure than its higher dimensional counterparts: The representation $\pi$ is not regular, so the field $\phi$ does not exist. The Hilbert space $\mathcal{H}$ is not separable, so it is not the usual Fock space. In fact it is an uncountable direct sum of Fock spaces labelled by the charges, on which the operators $\pi(W(f))$ act in a way dictated by $q_f$ (cf. [AMS93, formula (4.4)]). Considering all this, it is not a surprise that ‘free field’ turns out to be a misnomer for this model.
3. Spacetime Asymptotics of the Vacuum State

A standard tool to deal with the state (2.5) is the regularized two-point function:

\[ w_v(x) = (2\pi)^{-1} \int d\mu_0(p)(e^{-ip \cdot x} - v(p)), \tag{3.1} \]

where \( v: \mathbb{R}^2 \to \mathbb{R} \) is a measurable function s.t. \( v(|p^1|, p^1) = v(|p^1|, -p^1), v(0, 0) = 1 \) and for some \( r, \epsilon, c > 0 \)

\[ |v(|p^1|, p^1) - v(0, 0)| \leq c|p^1|^\epsilon \text{ for } |p^1| < r \quad \text{and} \quad \int_{|p^1| > r} \frac{dp^1}{|p^1|}|v(|p^1|, p^1)| < \infty. \tag{3.2} \]

(Note that different functions \( v \) give rise to an additive constant, \( w_v(x) = w_v(x) + c' \)).

Due to (3.2), \( w_v(f, f) \) is finite for all \( f \in \mathcal{D}(\mathbb{R}^2) \) and we can define the regularized Wick exponentials by

\[ \langle W(f) \rangle_v = e^{-\frac{1}{2} w_v(f, f)}. \tag{3.3} \]

Needless to say, they generate the same \( C^* \)-algebra \( \mathcal{F} \). The restriction of the regularized functional \( \langle \cdot \rangle_v \) to the neutral subalgebra \( \mathcal{A} \) coincides with the physical vacuum state \( \langle \cdot \rangle \), since \( w(f, f) = w_v(f, f) \) if \( q f = 0 \), cf. (3.1). However, the regularized functional is non-positive [AAR, p.27] and should only be seen as a device for organizing computations. In particular, it is easy to see that the Weyl relations and the vacuum expectation values of products of regularized Wick exponentials have the form

\[ \langle \cdots : W(f_1) : W(f_2) : \cdots : W(f_n) : \rangle_v = \delta_{0,q} e^{-\sum_{i<j} w_v(f_i, f_j)}, \tag{3.5} \]

Equality (3.5) follows from \( \langle \cdots : W(f) : \rangle_v = \delta_{0,q} \) and we stress that there is the physical (non-regularized) vacuum state (2.5) on its left hand side.

It is well known [AAR, p.27] and very useful for our investigation that (3.1) can be rewritten in the form

\[ w_v(x) = \lim_{\epsilon \searrow 0} \frac{-1}{4\pi} \ln(-\mu_v^2 x^2 + i\epsilon x^0), \tag{3.6} \]

where the scale \( \mu_v > 0 \) depends on \( v \) and the limit is in \( \mathcal{D}'(\mathbb{R}^2) \). The precise definition of this logarithm is often glossed over in the literature. Here and below the complex logarithm will always be understood with a cut along the negative real axis. As the imaginary part of this logarithm gives rise to the non-trivial amplitude in (2.8), we provide a detailed proof of the equality between (3.1) and (3.6) in Appendix B.

Using (3.6) we will study the asymptotic behaviour of the regularized two-point function in Lemmas 3.1, 3.2 below. Then, exploiting formula (3.5), we will compute the scattering amplitude (1.4) in the next section.
Lemma 3.1. Let $f \in \mathcal{D}(\mathbb{R}^2)$ and $h_1, h_2$ be real polynomials in $x^0, x^1$ which are not identically zero. Then

$$\lim_{\varepsilon \downarrow 0} \int \ln[h_1(x) + i\varepsilon h_2(x)] f(x) d^2x = \int (\ln |h_1(x)| + \theta(-h_1(x))\text{sgn}(h_2(x))i\pi) f(x) d^2x. \quad (3.7)$$

Proof. For $0 < \eta < 1$, denote $N(\eta) = \{ x \in \text{supp}(f) | |h_1(x)| < \eta \}$ and $N(\eta)' = \mathbb{R}^2 \setminus N(\eta)$. We decompose the region of integration:

$$\int \ln[h_1(x) + i\varepsilon h_2(x)] f(x) d^2x = \int_{N(\eta)} \ln[h_1(x) + i\varepsilon h_2(x)] f(x) d^2x \quad (3.8)$$

$$+ \int_{N(\eta)'} \ln[h_1(x) + i\varepsilon h_2(x)] f(x) d^2x. \quad (3.9)$$

We intend to first take the limit $\varepsilon \downarrow 0$ and then $\eta \downarrow 0$. In the leading term (3.9) the limit $\varepsilon \downarrow 0$ can be computed by the dominated convergence, which gives

$$\lim_{\varepsilon \downarrow 0} \int_{N(\eta)'} \ln[h_1(x) + i\varepsilon h_2(x)] f(x) d^2x$$

$$= \int_{N(\eta)'} (\theta(h_1(x)) \ln |h_1(x)| + \theta(-h_1(x))(\ln |h_1(x)| + \text{sgn}(h_2(x))i\pi)) f(x) d^2x$$

$$= \int_{N(\eta)'} (\ln |h_1(x)| + \theta(-h_1(x))\text{sgn}(h_2(x))i\pi) f(x) d^2x. \quad (3.10)$$

where $\theta$ is the Heaviside function. Now in the limit $\eta \downarrow 0$ dominated convergence gives the expression on the r.h.s. of (3.7).

Now we consider the rest term (3.8). We rewrite it as follows

$$\int_{N(\eta)} \ln[h_1(x) + i\varepsilon h_2(x)] f(x) d^2x = \int_{N(\eta)} \frac{1}{2} \ln[h_1(x)^2 + \varepsilon^2 h_2(x)^2] f(x) d^2x \quad (3.11)$$

$$+ \int_{N(\eta)} \varepsilon \varphi_\varepsilon(x) f(x) d^2x, \quad (3.12)$$

where $\varphi_\varepsilon(x)$ is the phase of $h_1(x) + i\varepsilon h_2(x)$. By dominated convergence, expression (3.12) has the limit $\varepsilon \downarrow 0$ equal to $\int_{N(\eta)} (\pm i\pi) f(x) d^2x$, where $\pm$ may depend on $x$. The subsequent limit $\eta \downarrow 0$ is equal to zero due to the shrinking of the region of integration. As for (3.11), after decomposing the region of integration into $N_{\pm}(\eta) = \{ x \in N(\eta) | \pm f(x) \geq 0 \}$ and choosing $\varepsilon$ small enough, so that $h_1(x)^2 + \varepsilon^2 h_2(x)^2 \leq 1$ in the region of integration (recall that $\eta < 1$) we use the monotone convergence to compute the limit $\varepsilon \downarrow 0$. The resulting expression $\int_{N(\eta)} \frac{1}{2} \ln[h_1(x)^2] f(x) d^2x$ tends also to zero with $\eta \downarrow 0$ by dominated convergence. This concludes the proof. \[\square\]

Lemma 3.2. Let $e_0 \doteq (1, 0)$, $e_1 \doteq (0, 1)$, $e_\pm \doteq e_0 \pm e_1$, be vectors in $\mathbb{R}^2$. Furthermore, let $t \mapsto r_t \in \mathbb{R}^2$ be a family of vectors s.t. $\|r_t\| = O(|t|^{-\alpha})$ for some

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0 < α < 1 in the Euclidean norm of \( \mathbb{R}^2 \). Then, for any fixed \( f \in \mathcal{D}(\mathbb{R}^2) \) and \( t \to \pm \infty \), there holds

\[
\int w_v(x - t(e(1) + r_1)) f(x) d^2x = \frac{-1}{4\pi} 2 \ln(|t| \mu_v) q_f + O(|t|^{-\alpha}), \quad (3.13)
\]

\[
\int w_v(x - t(e(0) + r_1)) f(x) d^2x = \frac{-1}{4\pi} 2 \ln(|t| \mu_v) + i \frac{\pi}{2} q_f + O(|t|^{-\alpha}), \quad (3.14)
\]

\[
\int w_v(x - te(\lambda)) f(x) d^2x = \int w_v^{-\lambda}(x) f(x) d^2x = \frac{-1}{4\pi} \{ \ln(2 \mu_v |t|) &+ i \frac{\pi}{2} \} q_f + O(|t|^{-1}), \quad (3.15)
\]

where \( w_v^\pm(x) \equiv \lim_{\epsilon \downarrow 0} \frac{-1}{4\pi} \ln \left[ i \mu_v x^\pm + \epsilon \right] \).

**Proof.** We will apply Lemma 3.1 considering that \( w_v(x) = \lim_{\epsilon \downarrow 0} \frac{-1}{4\pi} \ln(-\mu_v^2 x^2 + i \epsilon x^0) \).

In the case of (3.13), we have \( w_v(x - t(e(1) + r_1)) = \lim_{\epsilon \downarrow 0} \frac{-1}{4\pi} \ln[h_1(x) + i \epsilon h_2(x)] \), where \( h_1(x) = -\mu_v^2(x - t(e_1 + r_1))^2 \) and \( h_2(x) = (x - t(e_1 + r_1))^0 \). Thus, for any \( f \in \mathcal{D}(\mathbb{R}^2) \),

\[
\int w_v(x - t(e(1) + r_1)) f(x) d^2x = \frac{-1}{4\pi} \int (\ln|h_1(x)| + \theta(-h_1(x)) \text{sgn}(h_2(x)) i \pi) f(x) d^2x. \quad (3.16)
\]

As the following equality holds (3.17)

\[
h_1(x) = -\mu_v^2(x - t(e_1 + r_1))^2 = t^2 \mu_v^2 \left( 1 + \frac{2(x - tr_1) \cdot e(1)}{t} - \frac{(x - tr_1)^2}{t^2} \right), \quad (3.17)
\]

for any compactly supported smearing function \( f \) we can choose \( t \) large enough so that the underbraced term has modulus smaller than \( \frac{1}{2} \) and in particular \( h_1(x) > 0 \) on the support of \( f \). Thus we can omit the term involving \( \theta(-h_1(x)) \) in (3.16). We are left with

\[
\int w_v(x - t(e(1) + r_1)) f(x) d^2x = \frac{-1}{4\pi} \ln[t^2 \mu_v^2] q_f + O(|t|^{-\alpha}), \quad (3.18)
\]

where we used \(| \ln(1 + y) | \leq (2 \ln 2) |y| \) for \(|y| \leq \frac{1}{2} \).

In the case of (3.14), we have \( w_v(x - t(e(0) + r_1)) = \lim_{\epsilon \downarrow 0} \frac{-1}{4\pi} \ln[h_1(x) + i \epsilon h_2(x)] \), where \( h_1(x) = -\mu_v^2(x - t(e_0 + r_1))^2 \), \( h_2(x) = (x - t(e_0 + r_1))^0 \). In this case we have

\[
h_1(x) = -t^2 \mu_v^2 \left( 1 - \frac{2(x - tr_1) \cdot e(0)}{t} + \frac{(x - tr_1)^2}{t^2} \right). \quad (3.19)
\]

By choosing \( t \) sufficiently large, we obtain that \( h_1(x) < 0 \) and \( \text{sgn}(h_2(x)) = \text{sgn}(-t) \) for all \( x \in \text{supp}(f) \). Thus a counterpart of formula (3.16) gives

\[
\int w_v(x - t(e(0) + r_1)) f(x) d^2x = \frac{-1}{4\pi} \{ \ln[t^2 \mu_v^2] + \text{sgn}(-t) i \pi \} + O(|t|^{-\alpha}). \quad (3.20)
\]

As for (3.15), we proceed as follows: Write \( x^\pm \equiv x^0 \pm x^1 \) and note

\[
(x - te(\lambda))^2 = x \cdot x - 2tx^{-\lambda} = -2tx^{-\lambda} \left( 1 - \frac{x^\lambda}{2t} \right), \quad (3.21)
\]
where we have used that $x \cdot t e(\lambda) = t x^{-\lambda}$ and $x \cdot x = x^\lambda x^{-\lambda}$. Then by equation (3.6), for $|t|$ sufficiently large, depending on the support of the smearing function $f$ (which we omit below in the notation, i.e., the limit is in $\mathcal{D}'(\mathbb{R}^2)$)

$$w_v(x - t e(\lambda)) = \lim_{\varepsilon \downarrow 0} \frac{-1}{4\pi} \ln \left[ -\mu_v^2 (x - t e(\lambda))^2 + i\varepsilon (x - t e(\lambda))^0 \right]$$

$$= \lim_{\varepsilon \downarrow 0} \frac{-1}{4\pi} \ln \left[ 2t \mu_v^2 x^{-\lambda} (1 - \frac{x^\lambda}{2t}) + i\varepsilon (x - t e(\lambda))^0 \right]$$

$$= \frac{-1}{4\pi} \left( \ln |h_1(x)| - \theta(-h_1(x))\text{sgn}(t)i\pi \right)$$

$$= \frac{-1}{4\pi} \left( \ln |2t \mu_v| + \ln |\mu_v x^{-\lambda}| + \ln \left| 1 - \frac{x^\lambda}{2t} \right| - \theta(-t \mu_v x^{-\lambda})\text{sgn}(t)i\pi \right)$$

(3.22)

for $h_1(x) = 2t \mu_v^2 x^{-\lambda} (1 - \frac{x^\lambda}{2t})$ and $h_2(x) = (x - t e(\lambda))^0$ with $\text{sgn}(h_2) = -\text{sgn}(t)$.

On the other hand, we observe that for $u \in \mathbb{R}$

$$\ln |u| - \theta(-tu) \text{sgn}(t)i\pi = \lim_{\varepsilon \downarrow 0} \ln (iu + \varepsilon) - \text{sgn}(t)\frac{\pi}{2}. \quad (3.23)$$

For this is equivalent to the relation

$$\lim_{\varepsilon \downarrow 0} \ln (iu + \varepsilon) = \ln |u| + i\frac{\pi}{2} \left( \text{sgn}(t)(1 - 2\theta(-tu)) \right) = \ln |u| + i\frac{\pi}{2} \text{sgn}(u). \quad (3.24)$$

(We used the fact that $\text{sgn}(t)(1 - 2\theta(-tu)) = \text{sgn}(u)$.) Thus we can write

$$\int w_v(x - t e(\lambda)) f(x) d^2 x$$

$$= \lim_{\varepsilon \downarrow 0} \frac{-1}{4\pi} \int \left( \ln |2t \mu_v| + \ln |\mu_v x^{-\lambda}| + \varepsilon - \text{sgn}(t)i\pi \right) f(x) d^2 x + O(|t|^{-1}). \quad (3.25)$$

This concludes the proof. □

4. Computation of the Scattering Amplitude

In this section we prove Theorem 2.1. We choose $F = \pi(W(f))$, $G = \pi(W(g))$ as in the statement of Theorem 2.1 and denote by $S_T(F, G)$ the approximating sequence of the scattering amplitude from (1.4). To simplify the expressions, we write

$$f^{(1)} \doteq -g, \quad f^{(2)} \doteq -f, \quad f^{(3)} \doteq f, \quad f^{(4)} \doteq g. \quad (4.1)$$

The corresponding charges are

$$q_1 \doteq -q_g, \quad q_2 \doteq -q_f, \quad q_3 \doteq q_f, \quad q_4 \doteq q_g. \quad (4.2)$$
We have by the Wick theorem (3.5):

\[
S_T(F, G) = \int d^4t h(t_1) \ldots h(t_4) \langle: W(f_{(t_1^T)_{-} - (t_2^T)_{+}}) :_v W(f_{(t_3^T)_{+} - (t_4^T)_{+}}) :_v W(f_{- (t_1^T)_{+}}) :_v W(f_{- (t_2^T)_{+}}) :_v \rangle
\]

\[
\int d^4t h(t_1) \ldots h(t_4) \langle: W(f_{(t_1^T)_{-} - (t_2^T)_{+}}) :_v W(f_{(t_3^T)_{+} - (t_4^T)_{+}}) :_v W(f_{- (t_1^T)_{+}}) :_v W(f_{- (t_2^T)_{+}}) :_v \rangle
\]

\[
\int d^4t h(t_1) \ldots h(t_4) \langle: W(f_{(t_1^T)_{-}}) :_v W(f_{(t_2^T)_{+}}) :_v W(f_{- (t_1^T)_{+}}) :_v W(f_{- (t_2^T)_{+}}) :_v \rangle,
\]

(4.3)

where \( t_i^T = T + s(T)t_i \), \( s(T) = \ln |T| \). (Actually, for negative times we get \( t_i^T = T - s(T)t_i \), but this can be readjusted using that \( h \) is symmetric). We note that the middle line in the numerator of (4.3) has the same structure as the denominator and only the time averaging prevents immediate cancellation. As we will see, the cancellation actually takes place in the limit \( T \to \infty \). Let us now list the differences of time arguments appearing in the numerator of formula (4.3):

\[
(t_1^T)_{-} - (t_2^T)_{+} = 2T \left( (0, -1) + \frac{1}{2} \frac{s(T)}{T} (t_1 - t_2, t_1 + t_2) \right) = -2T(e_{(1)} + r_{1,2}^{t_1,t_2}),
\]

(4.4)

\[
(t_1^T)_{-} + (t_3^T)_{+} = 2T \left( (1, 0) + \frac{1}{2} \frac{s(T)}{T} (t_1 + t_3, t_3 - 1) \right) = 2T(e_{(0)} + r_{1,3}^{t_1,t_3}),
\]

(4.5)

\[
(t_1^T)_{-} + (t_4^T)_{-} = 2T \left( (1, -1) + \frac{s(T)}{2T} (t_1 + t_4, -(t_1 + t_4)) \right)
\]

\[
= 2T \left( 1 + \frac{s(T)}{2T} (t_1 + t_4) \right) e_{(-)},
\]

(4.6)

\[
(t_2^T)_{+} + (t_3^T)_{+} = 2T \left( (1, 1) + \frac{s(T)}{2T} (t_2 + t_3, t_2 + t_3) \right)
\]

\[
= 2T \left( 1 + \frac{s(T)}{2T} (t_2 + t_3) \right) e_{(+)},
\]

(4.7)

\[
(t_2^T)_{+} + (t_4^T)_{-} = 2T \left( (1, 0) + \frac{s(T)}{2T} (t_4 + t_2, t_2 - t_4) \right) = 2T(e_{(0)} + r_{2,4}^{t_2,t_4}),
\]

(4.8)

\[-(t_2^T)_{+} + (t_4^T)_{-} = 2T \left( (0, -1) + \frac{s(T)}{2T} (t_4 - t_3, t_4 + t_3) \right) = -2T(e_{(1)} + r_{2,4}^{t_2,t_4}).
\]

(4.9)

Here the vectors \( r_{i,j}^{t_i,t_j} \) differ from line to line, but they satisfy \( \| r_{i,j}^{t_i,t_j} \| = O(T^{-\alpha}) \), \( 0 < \alpha < 1 \), uniformly on compact sets in \( t_i, t_j \).

Concerning the first line in the numerator of formula (4.3), we obtain from Lemma 3.2

\[
\langle: W(f_{(t_1^T)_{-}}) :_v W(f_{(t_2^T)_{+}}) :_v \rangle = \exp \left( - \int f^{(1)}(x) w_v(x - y + ((t_1^T)_{-} - (t_2^T)_{+})) f^{(2)}(y) d^2 x d^2 y \right)
\]

\[
= \exp \left( - \frac{1}{4\pi} 2 \ln(2T \mu_v) \sum_{q_1 q_2} g_1 g_2 + O(T^{-\alpha}) \right),
\]

(4.10)
Now substituting (4.10)–(4.15) to (4.3) we note the cancellation of all terms involving \( \frac{1}{4\pi} \int f(x) w(x - y + ((t^T_1)_+ + (t^T_2)_+)) f^{(y)}(y) d^2x d^2y \)

\[
= \exp \left( - \int f^{(1)}(x) w(x - y + ((t^T_1)_- + (t^T_2)_-)) f^{(y)}(y) d^2x d^2y \right)
= \exp \left( \frac{1}{4\pi} \left( \ln(2T \mu_v) + i \frac{\pi}{2} \right) q_1 q_3 + O(T^{-\alpha}) \right). \tag{4.11}
\]

In the middle line in the numerator of formula (4.3), as well as in its denominator, we have translations in lightlike directions. We set \( \beta_T^{t_i, t_j} = 1 + \frac{s(T)}{2T} (t_i + t_j) \) and compute using Lemma 3.2

\[
\langle: W(f^{(1)}_{(t^T_1)_-}) : \rangle v : W(f^{(4)}_{-(t^T_1)_-}) : \rangle v = \exp \left( - w_v(f^{(1)}_{(t^T_1)_-}, f^{(4)}_{-(t^T_1)_-}) \right)
= \exp \left( - \int f^{(1)}(x) w_v(x - y + ((t^T_1)_- + (t^T_2)_-)) f^{(4)}(y) d^2x d^2y \right)
= \exp \left( - \frac{1}{4\pi} \{ \ln(4\mu_v T \beta_T^{t_i, t_j}) + i \frac{\pi}{2} \} q_1 q_4 + O(T^{-1}) \right). \tag{4.12}
\]

\[
\langle: W(f^{(2)}_{(t^T_2)_+}) : \rangle v : W(f^{(3)}_{-(t^T_2)_+}) : \rangle v = \exp \left( - w_v(f^{(2)}_{(t^T_2)_+}, f^{(3)}_{-(t^T_2)_+}) \right)
= \exp \left( - \int f^{(2)}(x) w_v(x - y + ((t^T_2)_+ + (t^T_3)_+)) f^{(3)}(y) d^2x d^2y \right)
= \exp \left( - \frac{1}{4\pi} \{ \ln(4\mu_v T \beta_T^{t_i, t_j}) + i \frac{\pi}{2} \} q_2 q_3 + O(T^{-1}) \right). \tag{4.13}
\]

Finally, in the bottom line of the numerator of formula (4.3) we have

\[
\langle: W(f^{(2)}_{(t^T_2)_+}) : \rangle v : W(f^{(4)}_{-(t^T_2)_-}) : \rangle v = \exp \left( - w_v(f^{(2)}_{(t^T_2)_+}, f^{(4)}_{-(t^T_2)_-}) \right)
= \exp \left( - \int f^{(2)}(x) w_v(x - y + ((t^T_2)_+ + (t^T_4)_-)) f^{(4)}(y) d^2x d^2y \right)
= \exp \left( - \frac{1}{4\pi} \{ \ln(2T \mu_v) + i \frac{\pi}{2} \} q_2 q_4 + O(T^{-\alpha}) \right), \tag{4.14}
\]

\[
\langle: W(f^{(3)}_{-(t^T_2)_+}) : \rangle v : W(f^{(4)}_{-(t^T_2)_-}) : \rangle v = \exp \left( - w_v(f^{(3)}_{-(t^T_2)_+}, f^{(4)}_{-(t^T_2)_-}) \right)
= \exp \left( - \int f^{(3)}(x) w_v(x - y + ((t^T_2)_+ + (t^T_4)_-)) f^{(4)}(y) d^2x d^2y \right)
= \exp \left( - \frac{1}{4\pi} \{ \ln(2T \mu_v) q_3 q_4 + O(T^{-\alpha}) \right). \quad \tag{4.15}
\]

Now substituting (4.10)–(4.15) to (4.3) we note the cancellation of all terms involving \( \frac{1}{4\pi} \{ \ln(2T \mu_v) q_g q_f \} \) in the numerator and the contributions \( \exp \left( - \frac{1}{4\pi} \{ \ln(4\mu_v T) + i \frac{\pi}{2} \} q^2_{f/g} \right) \) between the numerator and the denominator. Thus we obtain (4.16)
\[ S_T(F, G) = e^{-\frac{i}{2}q_f q_g} \int d^4 t h(t_1) \cdots h(t_4) \exp \left( - \frac{q_f^2}{4\pi} \ln(\beta_T^{-1} t_1) - \frac{q_g^2}{4\pi} \ln(\beta_T^{2} t_4) + O(T^{-\alpha}) \right) \]

where the crucial pre-factor \( e^{-\frac{i}{2}q_f q_g} \) originates from the \( i \frac{\pi}{2} \)-terms in (4.11), (4.14).

Now since \( h \) is compactly supported and \( \lim_{T \to \infty} \beta_T^{t, t'} = \lim_{T \to \infty} \left( 1 + \frac{s(T)}{2T}(t + t') \right) = 1 \) for any \( t, t' \in \mathbb{R} \), the dominated convergence gives

\[ \lim_{T \to \infty} S_T(F, G) = e^{-\frac{i}{2}q_f q_g}. \]  

This concludes the proof. \( \square \)

Acknowledgements. Both authors were partially supported by the Emmy Noether grant DY107/2-2 of the DFG. W.D. also acknowledges support of the National Science Centre, Poland, within the Grant ‘Sonata Bis’ 2019/34/E/ST1/00053. J.M. received financial support from the Brazilian research agency CNPq, and is also grateful to CAPES and Finep.

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A. Spectral Properties of the Vacuum

Continuity of the unitary group implementing the Poincaré transformations is a consequence of the continuity of the following function in some neighbourhood of unity in the Poincaré group

\[ L \mapsto \langle \Omega, \pi(W(f))U(L)\pi(W(g))\Omega \rangle = \langle W(f)W(g_L) \rangle = \delta_{q_f + q_g, 0} \langle W(f)\nu(W(g))e^{-w_{\nu}(f,g)} \rangle. \]  

Here \( f, g \in D_{\mathbb{R}}(\mathbb{R}^2) \) and we made use of (3.5). The continuity of \( L \mapsto w_{\nu}(f, g_L) \) can be conveniently checked using formulas (3.6), (3.7) and the dominated convergence. Now restricting attention to translations in (A.1), one can determine the support of the Fourier transform of this function using its analyticity properties via [RSII, Theorem IX.16]. This gives the spectrum condition, i.e., the spectral measure of \( a \mapsto U(a) \) is supported in the future lightcone.

Proceeding to more detailed properties of the spectrum, let us show the inclusion (2.7), i.e., neutrality of the waves. By the ergodic theorem, the orthogonal projection on the single-wave subspaces \( H_\lambda, \lambda = \pm \), can be written as follows:

\[ E_\lambda = \lim_{T \to \infty} \int dt h_T(t) U(t, \lambda t), \]  

where \( h_T \) was defined below formula (1.1). Thus to prove (2.7), it suffices to show that

\[ \lim_{t \to \infty} \langle \Omega, \pi(W(g))\pi(W(f(t, \lambda t)))\Omega \rangle = 0 \]  

unless \( q_g = q_f = 0 \). Suppose that one of the charges is different from zero. Then the scalar product in (A.3) is zero unless \( q_g + g_f = 0 \). Now making use of the lightlike asymptotics in (3.15), we can write

\[ \langle \Omega, \pi(W(g))\pi(W(f(t, \lambda t)))\Omega \rangle = \langle W(g)\nu(W(f))e^{-\left( \int w_{\nu}(x)(g*f_\lambda)(x)d^2x - \frac{1}{2\pi} \ln(2\mu_v|t|) - i\frac{\pi}{2} \right) q_f q_g + O(|t|^{-1})} \rangle, \]  

where \( (g*f_\lambda)(x) \equiv \int g(x - y)f(-y)d^2y \). Since \( q_f q_g < 0 \), the above expression tends to zero with \( t \to \infty \).
B. Regularized Two-Point Function

Lemma B.1. Under the assumptions on \( v \) specified in Sect. 3, there exists \( \mu_v > 0 \) such that

\[
w_v(x) \doteq (2\pi)^{-1} \int d\mu_0(p) (e^{-ip\cdot x} - v(p)) = \lim_{\varepsilon \downarrow 0} \frac{-1}{4\pi} \ln(-\mu_v^2x^2 + i\varepsilon x^0), \tag{B.1}
\]

\[
w_v^\pm(x) \doteq (2\pi)^{-1} \int_{p^1 \leq 0} d\mu_0(p) (e^{-ip\cdot x} - v(p)) = \lim_{\varepsilon \downarrow 0} \frac{-1}{4\pi} \ln(i\mu_v x^\pm + \varepsilon). \tag{B.2}
\]

Proof. We first calculate \( w_v^- \), i.e., we consider \( p^1 > 0 \) in (B.2). In this case, \( p \cdot x = |p^1|x^0 - p^1x^1 = p^1x^- \), where \( x^- = x^0 - x^1 \). Thus, the LHS of (B.2) is \((-4\pi)^{-1}I(x^-)\), with

\[
I(u) = \int_0^\infty \frac{dp^1}{p^1} (v(p^1, p^1) - e^{-ip^1u}), \tag{B.3}
\]

understood as a distribution on \( \mathbb{R} \).

Its restriction \( I_+ \) to \( \mathcal{D}(\mathbb{R}^+) \) has the imaginary part \( \text{Im} I_+(u) = \int_0^\infty \frac{dp^1}{p^1} \sin(p^1) = \frac{\pi}{2} \)
and satisfies the ODE \((I_+(u))' = 1/u\). This implies that there is a \( \mu_+ > 0 \) such that

\[
I_+(u) = \ln(\mu_+ u) + i\frac{\pi}{2}, \quad u > 0. \tag{B.4}
\]

Similarly, one finds that the restriction \( I_- \) to \( \mathcal{D}(\mathbb{R}^-) \) is \( I_-(u) = \ln(\mu_- |u|) - i\frac{\pi}{2} \)
for some \( \mu_- \). But the relation \( I_-(u) = I_+(u) \), that follows from (B.3), implies that \( \mu_+ = \mu_- \doteq \mu_v \). Thus,

\[
I(u) = \ln(\mu_v |u|) + \text{sgn}(u)\frac{\pi}{2} = \lim_{\varepsilon \downarrow 0} \ln(i\mu_v u + \varepsilon) \tag{B.5}
\]
on \( \mathbb{R} \setminus \{0\} \). Thus, as distributions on \( \mathbb{R} \) the LHS and the RHS can only differ by a delta distribution or derivatives thereof. But, as one readily checks, both distributions have scaling degree at most 0, that is e.g. \( \lim_{\varepsilon \downarrow 0} \lambda^\varepsilon I(\lambda u) = 0 \) for any \( \varepsilon > 0 \). This would be spoiled by the (derivative of) a delta distribution. Therefore, equation (B.5) holds in the sense of distributions on \( \mathbb{R} \). This proves equation (B.2) for \( w_v^- \), i.e., for the case \( p^1 > 0 \).

In the case \( p^1 < 0, \ p \cdot x = -p^1x^0 - p^1x^1 = -p^1x^+ \), and the LHS of (B.2) is

\[
-\frac{1}{4\pi} \int_{-\infty}^0 \frac{dp^1}{|p^1|} (v(|p^1|, p^1) - e^{ip^1x^+}) = -\frac{1}{4\pi} \int_0^\infty \frac{dp^1}{p^1} (v(p^1, -p^1) - e^{-ip^1x^+}) = -\frac{1}{4\pi} I(x^+), \tag{B.6}
\]
since we assumed \( v(|p^1|, p^1) = v(|p^1|, -p^1) \).

Equation (B.1) follows from \( w_v(x) = w_v^+(x) + w_v^-(x) \). In more detail, we have

\[
w_v(x) = w_v^+(x) + w_v^-(x) = -\frac{1}{4\pi} \left( \ln(\mu_v |x^-|) + \ln(\mu_v |x^+|) + \frac{1}{2} (\text{sgn}(x^-) + \text{sgn}(x^+)) i\pi \right)
\]

\[
= -\frac{1}{4\pi} \left( \ln(\mu_v^2x^2) + \theta(x^2)\text{sgn}(x^0)i\pi \right)
\]

\[
= \lim_{\varepsilon \downarrow 0} \frac{-1}{4\pi} \ln(-\mu_v^2x^2 + i\varepsilon x^0), \tag{B.7}
\]
where in the second step one simply checks all the possibilities and in the last step we used Lemma 3.1.

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Communicated by Y. Ogata