ON POLYNOMIALS AND SURFACES OF VARIOUSLY POSITIVE LINKS

This is a preprint. I would be grateful for any comments and corrections!

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Abstract. It is known that the minimal degree of the Jones polynomial of a positive knot is equal to its genus, and the minimal coefficient is 1, with a similar relation for links. We extend this result to almost positive links and partly identify the 3 following coefficients for special types of positive links. We also give counterexamples to the Jones polynomial-ribbon genus conjectures for a quasipositive knot. Then we show that the Alexander polynomial completely detects the minimal genus and fiber property of canonical Seifert surfaces associated to almost positive (and almost alternating) link diagrams.

1. Introduction

A link is called quasipositive if it is the closure of a braid which is the product of conjugates of the Artin generators \( \sigma_i \) [Ru2]. (We call such conjugates and their inverses positive resp. negative bands.) It is called strongly quasipositive if these conjugates are positive embedded bands in the band representation of [Ru2]. It is called positive if it has a diagram with all crossings positive (in the skein sense), and braid positive (or a positive braid link) if it has a braid representation which is a positive word in the Artin generators. It is called fibered, if its complement in \( S^3 \) is a surface bundle over the circle.

Then

\[
\{ \text{quasipositive links} \} \supset \{ \text{strongly quasipositive links} \} \supset \{ \text{positive links} \} \supset \{ \text{fibered positive links} \} \supset \{ \text{braid positive links} \}. \tag{1}
\]

The only non-obvious inclusions are the second and fourth one. The fourth inclusion is a well-known fact (it follows e.g. from [Ga]), and the second inclusion follows, as observed by Rudolph [Ru] and Nakamura [N], by applying the algorithm of Yamada [Y] or Vogel [Vo] to a positive diagram. Links in some of the above classes have been studied, beside by their intrinsic knot-theoretical interest, with different motivations and in a variety of contexts, including singularity theory [A, BoW, Mi], algebraic curves [Ru2, Ru3], dynamical systems [BW] and (in some vague and yet-to-be understood way) in 4-dimensional QFTs [Kr].

A different related class to positive links are the almost positive links, those with almost positive diagrams, which are, however, not positive. (A diagram is almost positive if it has exactly one negative crossing.)

Let \( g \) be the genus of a knot, \( g_s \) the slice genus, and \( g_r \) the ribbon genus. (Definitions are recalled below.) For links similarly write \( \chi, \chi_r \) and \( \chi_s \) for the (Seifert), ribbon and slice Euler characteristic resp. As any Seifert surface is a ribbon surface, and any ribbon surface is (deformable into) a slice surface, one has the inequalities \( g \geq g_r \geq g_s \) and \( \chi \leq \chi_r \leq \chi_s \).

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For knots we also have $u \geq g_r$, with $u$ being the unknotting number [Li]. By the work of Kronheimer–Mrowka [KM, KM2] and Rudolph [Ru2], it is now known that the slice genus is estimated below by the slice Bennequin inequality (a version of [Be, theorem 3] with $g$ replaced by $g_s$), implying that for a strongly quasipositive knot $g = g_s$, so that $u \geq g = g_s$ for positive braid knots $u \leq g$ was known by [BoW]. Thus $u = g$ in this case.

Let $V$ be the Jones polynomial [J]. Fiedler [Fi] proved that $\mindeg V = g$ for a positive braid knot, and that $\mincf V = 1$. For positive braid links $L$ of $n = n(L)$ components, $\mincf V = (-1)^{n-1}$ and $2\mindeg V = 1 - \chi$. This follows more generally for positive links $L$ by virtue of the fact that positive diagrams are semiadequate (see [LT]). Fiedler further conjectured (his Conjecture 1) that for arbitrary knots and links $L$, which have a band representation on $s$ strands with $b$ bands,

$$\mindeg V \leq \frac{b - s + 1}{2}.$$ 

He made a second conjecture (Conjecture 2), whose truth would imply that equality in the above inequality is achieved only for quasipositive links $L$.

In the paper of Kawamura [K], the theorems of Fiedler and Kronheimer–Mrowka–Rudolph have been found to imply that for a positive braid knot, $\mindeg V = u$, with a similar(ly obvious) relation for links. Then Kawamura quoted the first case of Fiedler’s first conjecture, asking whether it is true (at least) for quasipositive links, and observing that the slice Bennequin inequality would then imply the relation $\mindeg V \leq u$ for a quasipositive knot. (That $\mindeg V = u$ does not extend to quasipositive knots is easy to observe.)

In this paper we will give counterexamples to both Fiedler conjectures of several special types, in particular the case of the first conjecture addressed by Kawamura. Then we will partly identify up to 3 of the coefficients of the Jones polynomial of a positive link following the minimal one, including a handy criterion to single out positive braid links, even among fibered positive links. We will also extend Fiedler’s result to almost positive links. Some consequences are derived for the skein polynomial [H]. Our results allow also to identify up to two more coefficients of the skein polynomial of positive links. For almost positive links, we obtain a proof of the inequality conjectured by Morton [Mo2] (for which in the general case counterexamples are now known [St3]).

We will use in some of the proofs the even valence graph version of the Alexander polynomial studied in [MS] with K. Murasugi. Applying this method, we can also slightly improve and simplify the proof of results of Hirasawa [Hi] and Goda–Hirasawa–Yamamoto [GHY]. These results are amplified by showing that the Alexander polynomial completely determines the minimal genus and fiber property of canonical Seifert surfaces associated to almost positive (and almost alternating) link diagrams. At the end we will give a few examples showing that many of the possible extensions of these theorems are not true, and mention some problems.

## 2. Preliminaries

### Link polynomials

The *skein polynomial* $P$ is a Laurent polynomial in two variables $l$ and $m$ of oriented knots and links and can be defined by being 1 on the unknot and the (skein) relation

$$l^{-1}P(\chi\chi) + lP(\chi) = -mP(\chi\chi).$$ \hspace{1cm} (2)

For a diagram $D$ of a link $L$, we will use all of the notations $P(D) = P_D = P_D(l, m) = P(L)$ etc. for its skein polynomial, with the self-suggestive meaning of indices and arguments.

The *Jones polynomial* $V$, and (one variable) *Alexander polynomial* $\Delta$ are obtained from $P$ by the substitutions

$$V(t) = P(-it, i(t^{-1/2} - t^{1/2})), \hspace{1cm} \text{(3)}$$

$$\Delta(t) = P(i, i(1/2 - t^{-1/2})). \hspace{1cm} \text{(4)}$$

Hence these polynomials also satisfy corresponding skein relations. The sign ‘÷’ means that the Alexander polynomial is defined only up to units in $\mathbb{Z}[t, t^{-1}]$; we will choose the normalization depending on the context.

We denote in the following the coefficient of $t^m$ in $V(t)$ by $[V(t)]_m$. In the case of a 2-variable polynomial, we index the bracket by the whole monomial, and not just the power of the variables. The *minimal or maximal degree*
mindeg \( V \) or maxdeg \( V \) is the minimal resp. maximal exponent of \( t \) with non-zero coefficient in \( V \). An explicit (1-variable) polynomial may be denoted by the convention of [LM] by its coefficient list, when bracketing its absolute term to indicate its minimal degree, e.g. \( -3 \, [1] \, 2 = -3/t + 1 + 2t \). The minimal or leading coefficient \( \text{mincf} \, V \) of \( V \) is \( [V]_{\text{mindeg} \, V} \).

For an account on these link polynomials we refer to the papers [LM, J]. (Note: our convention here for \( P \) differs from [LM] by interchange of \( l \) and \( l - 1 \), that is, our \( P(l, m) \) is Lickorish and Millett’s \( P(l - 1, m) \).)

**Link diagrams**

A crossing \( p \) in a link diagram \( D \) is called reducible (or nugatory) if \( D \) can be represented in the form

\[
P \quad p \quad Q.
\]

\( D \) is called reducible if it has a reducible crossing, else it is called reduced.

A link diagram \( D \) is composite, if there is a closed curve \( \gamma \) intersecting (transversely) the curve of \( D \) in two points, such that both in- and exterior of \( \gamma \) contain crossings of \( D \), that is, \( D \) has the form

\[
A \quad B.
\]

Otherwise \( D \) is prime. A link is prime if any in composite diagram replacing one of \( A \) and \( B \) by a trivial (0-crossing) arc gives an unknot diagram.

The diagram is split, if there is a closed curve not intersecting it, but which contains parts of the diagram in both its in- and exterior:

\[
A \quad B.
\]

Otherwise \( D \) is connected or non-split. A link is split if it has a split diagram, and otherwise non-split.

We call a diagram \( D \) \( k \)-almost positive, if \( D \) has exactly \( k \) negative crossings. A link \( L \) is \( k \)-almost positive, if it has a \( k \)-almost positive diagram, but no \( l \)-almost positive one for any \( l < k \). We call a diagram or link positive, if it is 0-almost positive (see [Cr, O, Yo, Zu]), and almost positive if it is 1-almost positive [St2]. Similarly one defines \( k \)-almost negative, and in particular almost negative and negative links and diagrams to be the mirror images of their \( k \)-almost positive (or almost positive or positive) counterparts, and \( (k-) \)almost alternating diagrams and links [Ad, Ad2].

The valency of a Seifert circle \( s \) is the number of crossings attached to \( s \). We call such crossings also adjacent to \( s \).

**Link surfaces**

A Seifert resp. slice surface of \( L \subset S^3 = \partial B^4 \subset B^4 \) is a smoothly embedded compact orientable surface \( S \subset S^3 \) resp. \( S \subset B^4 \) with \( \partial S = L \). A ribbon surface is a smoothly immersed compact orientable surface \( S \subset S^3 \) with \( \partial S = L \), embedded except at a finite number of double transverse (ribbon) singularities, whose preimages are two arcs, one lying entirely in the interior \( \text{int}\, S \) of \( S \) and the other one, too, except for its two endpoints, which lie on \( \partial S \). A canonical (Seifert) surface is a Seifert surface obtained by Seifert’s algorithm (see[Ro]). We may allow (for links) all these surfaces to be disconnected, but they should have no closed (\( \partial = \varnothing \)) components.

The (Seifert) genus \( g \), slice genus \( g_s \), canonical genus \( \tilde{g} \) and ribbon genus \( g_r \), are defined to be the minimal genera of Seifert, slice, canonical resp. ribbon surfaces of \( L \). Similarly one can define the (Seifert), slice, canonical resp. ribbon Euler characteristic \( \chi \), \( \chi_s \), \( \tilde{\chi} \), and \( \chi_r \) to be the maximal Euler characteristic of such surfaces of \( L \).
In [Be, theorem 3], Bennequin shows that for a braid \( \beta \) on \( s(\beta) \) strands, with writhe (exponent sum) \( w(\beta) \) and with closure \( \beta = K \), we have an estimate for the Euler characteristic \( \chi(K) \) of \( K \):

\[
1 - \chi(K) \geq w(\beta) - s(\beta) + 1.
\]

This is easily observed to extend by means of the algorithm of Yamada [Y] or Vogel [Vo] to an inequality for arbitrary link diagrams \( D \) of \( K \):

\[
1 - \chi(K) \geq w(D) - s(D) + 1 =: b(D),
\]

with \( w(D) \) being the writhe of \( D \), and \( s(D) \) the number of its Seifert circles. We call the r.h.s. of (5) the Bennequin number of \( D \). (It clearly depends a lot on the diagram for a given link.)

Rudolph [Ru3] later improved this inequality, by replacing \( \chi(K) \) by \( \chi_s(K) \):

\[
1 - \chi_s(K) \geq b(D).
\]

Recently, he obtained a further improvement, this time by raising the r.h.s. [Ru2]:

\[
1 - \chi_s(K) \geq w(D) - s(D) + 1 + 2s_(D) =: rb(D),
\]

with \( s_(D) \) being the number of (\( \geq 2 \)-valent) Seifert circles of \( D \), to which only negative non-nugatory crossings are adjacent. The new quantity on the right we call Rudolph-Bennequin number of \( D \). Again \( rb(D) \) heavily depends on the diagram, even more than \( b(D) \). (For example, unlike \( b(D) \), \( rb(D) \) is no longer invariant under flypes and mutations.) Thus again one is interested in choosing for a given link \( K \) the diagram \( D \) so that \( rb(D) \) is as large as possible.

3. Counterexamples to the Jones polynomial-ribbon genus conjectures

3.1. Preparations

While the improvement (7), as compared to (6), may not seem significant at first sight, it has the advantage of eliminating the minimal \( l \)-degree in the skein polynomial \( P \) as an obstruction to increasing the estimate by proper choice of the diagram \( D \), since by [Mo] we always have \( b(D) \leq \mindeg_l P(K) \).

A practical example where this turned out helpful was given in [St4], and is recalled below, as it will be used. (The notation for knots we apply is the one of Rolfsen’s tables [Ro, appendix] for \( \leq 10 \) crossings, and of the knot table program KnotScape [HT] for 11 to 16 crossings. By \( \overline{K} \) we will denote the obverse, or mirror image, of \( K \).)

Example 1 The knot 13_{6374} has \( \mindeg_1 P = 0 \) and Alexander polynomial \( \Delta = 1 \). It has many diagrams \( D \) with \( b(D) = 0 \), but it cannot have any such diagram with \( b(D) > 0 \), because of Morton’s inequality. However, it does have diagrams \( D \) with \( rb(D) > 0 \), thus showing it not to be slice.

In order to construct our counterexamples, we need a few more simple lemmas.

Lemma 1 If \( K \) is quasipositive, then \( \mindeg_1 P(K) \geq 1 - \chi_s(K) \).

Proof. If \( D \) is a diagram of a quasipositive braid representation of \( K \), then \( 1 - \chi_s(K) = b(D) \), and \( b(D) \leq \mindeg_1 P(K) \) by Morton’s inequality. \( \square \)

In the following \( K_1 # K_2 \) denotes the connected sum of \( K_1 \) and \( K_2 \), and \( # n K \) denotes the connected sum of \( n \) copies of \( K \).

Lemma 2 If \( K_{1,2} \) have diagrams \( D_{1,2} \) which are not negative, then \( K_1 # K_2 \) has a diagram \( D \) with \( rb(D) = rb(D_1) + rb(D_2) \).

Proof. We must apply the connected sum of \( D_{1,2} \) so that the Seifert circles of \( D_{1,2} \) affected by the operation have at least one positive crossing adjacent to them. \( \square \)
3.2 Degree inequality conjecture

**Lemma 3** If $K$ is strongly quasipositive, then $\chi(K) = \chi_s(K)$.

**Proof.** For the Seifert surface $S$ associated to a strongly quasipositive braid representation diagram $D$ of $K$, we have

$$1 - \chi(K) \leq 1 - \chi(S) = b(D) \leq rb(D) \leq 1 - \chi_s(K) \leq 1 - \chi(K),$$

implying equality everywhere. \hfill \Box

3.2. Degree inequality conjecture

Fiedler’s first conjecture was whether

$$\text{mindeg}_L \leq \frac{b - s + 1}{2},$$

if $L$ has a $b$-band representation on $s$ strands, and Kawamura’s (weaker) question was whether it is true at least if this band representation is positive.

**Example 2** Consider the knot $115_{162508}$ (see figure 2). Using the method described in [St5, appendix], it was found that it is ribbon (and hence slice), and one calculates $\text{mindeg} V = 1$. It turns out to have the quasipositive 5-braid representation

$$(\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_4\sigma_3^{-1}\sigma_2\sigma_1)(\sigma_2^{-1}\sigma_1\sigma_2)(\sigma_2\sigma_3^{-1}\sigma_4\sigma_3\sigma_2^{-1})\sigma_3.$$

(The knot can be identified from this representation by the tool knotfind included in [HT]. Note that this representation also directly shows sliceness.) Thus it is a slice example answering negatively Kawamura’s question, and hence also a counterexample Fiedler’s first conjecture.

Another special type of example is

**Example 3** Consider the knot $K$ in figure 3, which is the closure of the 4-braid

$$\sigma_1^2(\sigma_1\sigma_2\sigma_1^{-1})\sigma_2\sigma_1\sigma_3(\sigma_1\sigma_2\sigma_1^{-1})\sigma_2(\sigma_2\sigma_3\sigma_2^{-1})(\sigma_1\sigma_2\sigma_1^{-1})(\sigma_2\sigma_3\sigma_2^{-1}).$$

(8)

This braid is quasipositive, in fact, strongly quasipositive. The diagram of $K$ in figure 3 was obtained from that representation.

One easily sees that $g = g_s = 4$. But $\text{mindeg} V = 5$. Thus $g_s < \text{mindeg} V$. In fact, this knot has unknotting number 4. (Switch the encircled crossings in the diagram of figure 3.) Thus even the weaker inequality, in which Kawamura was interested, $\text{mindeg} V \leq u$ is not always true.
Remark 1 \!
15_{162508} is surely not strongly quasipositive, as $g > 0 = g_s$. Thus the above example $K$ is the most special in the hierarchy (1).

The only case of some interest, remaining not covered by the above examples, is that of a slice knot with $u < \text{mindeg} V$. Very likely such examples exist, too, although I didn’t find any.

Remark 2 If one is interested in a general knot $K$ with $\text{mindeg} V > u$, then there is a much simpler and well-known example, \!
10_{132}. It has $u = 1$, but $\text{mindeg} V = 2$. However, \!
10_{132} is not quasipositive. As it is not ribbon, or slice (its determinant 5 is not a square), it has 4-genus 1, and a quasipositive representation of $n$ strands would have $n + 1$ bands. Then the untwisted 2-cable link \!
(\!
10_{132})_2 would have a representation on $2n$ strands of writhe $2n + 2$. Thus by [Mo], \!
\text{mindeg}_t P((\!
10_{132})_2) \geq 3$, but from the calculation of [MoS] we know $\text{mindeg}_t P((\!
10_{132})_2) = 1$.

Remark 3 T. Tanaka, in a preprint [Ta] sent to T. Fiedler, has claimed independently counterexamples to Fiedler’s first conjecture. On the opposite end, M. Ishikawa [I] proved Fiedler’s inequality for some links obtained by A’Campo’s method [A].
3.3 Extremal property conjecture

Fiedler also conjectured (his Conjecture 2) that if a link \( L \) has a \( b \)-band \( s \)-strand band representation with

\[
\min \deg V_L = \frac{b - s + 1}{2},
\]

then it is quasi-positive. (Fiedler’s formulation is slightly different, but easily implies the one given here.)

We will now construct a counterexample also to this conjecture, albeit some more effort is necessary, and we must use the example found previously in a related context in [St4]. Our counterexample has likely crossing number 58.

**Proposition 1** The knot \( K' = 13_{6374} \# \#^3 (!15_{162508}) \), is not quasipositive, yet it has a band representation with equality in (9).

**Proof.** We first discuss the prime factors separately.

1. Consider \( 13_{6374} \). By switching one of the crossings in the clasp in the lower right part of the diagram in figure 1, one obtains \( 4_1 \). Thus \( #^2 13_{6374} \) turns by 2 crossing changes into the slice knot \( 4_1 \# 4_1 \). Hence \( 1 - \chi_s (#^2 13_{6374}) \leq 4 \). On the other hand, as \( 13_{6374} \) has a diagram \( D \) with \( rb(D) = 2 \), we have by lemma 2 that \( 1 - \chi_s (#^2 13_{6374}) = 4 \).

As said,

\[
\min \deg L (#^2 13_{6374}) = 0 = \frac{1 - \chi_s (#^2 13_{6374})}{2} - 2.
\]

Also \( \min \deg V = -1 \) by calculation. As \( \max \deg_m P(13_{6374}) = 4 \) and it has crossing number \( < 15 \), by [St] \( 13_{6374} \) has a diagram of canonical genus 2, and thus by applying Yamada’s algorithm [Y] on it, we obtain a (n embedded) band representation with

\[
\frac{b - s + 1}{2} = 2 = \min \deg V + 3.
\]

2. For \( !15_{162508} \), we have a quasipositive band representation as 5-braid with 4 bands, and it is slice. Thus

\[
1 - \chi_s = 0 = \frac{\min \deg P}{2} \text{ and } \frac{b - s + 1}{2} = 0 = \min \deg V - 1.
\]

In summary we have the following situation for proper diagrams \( D \) and \( b \)-band \( s \)-braid representations:

| Diagram       | \( 13_{6374} \) | \( !15_{162508} \) |
|---------------|----------------|------------------|
| \( \frac{\min \deg P}{2} - \frac{rb(D)}{2} \) | -1              | 0                |
| \( \min \deg V - \frac{b - s + 1}{2} \)       | -3              | 1                |

Since both quantities are additive under connected sum for proper diagrams and band representations (by Lemma 1 resp. in the obvious way), we obtain for \( K' \) a band representation with

\[
\frac{b - s + 1}{2} = \min \deg V(K'),
\]

but also a diagram \( D \) of \( K' \# K' \) with

\[
\min \deg L (K' \# K') < rb(D) \leq 1 - \chi_s (K' \# K'),
\]

(in fact \( rb(D) = 1 - \chi_s (K' \# K') \)), so that \( K' \# K' \) is not quasipositive by lemma 1. Then \( K' \# K' \) cannot be quasipositive either. \( \square \)

One can also obtain a counterexample to an “embedded band” version of Fiedler’s conjecture, namely whether a knot \( K \) with an embedded band representation achieving equality in (9) is strongly quasipositive.
Counterexamples to the Jones polynomial-ribbon genus conjectures

Proposition 2 The knot $K' = \#^3 K \# 6_1$, with $K$ being the knot in example 3, is not strongly quasipositive, yet it has an embedded band representation satisfying equality in (9).

Proof. As $6_1$ has canonical genus 1, it has an embedded band representation with

$$\frac{b - s + 1}{2} = 1 = \mindeg V + 3.$$ 

Now consider $K$. It has a strongly quasipositive band representation with $b = 11$ bands on $s = 4$ strands, so that

$$1 - \chi_s = \frac{b - s + 1}{2} = 4.$$ 

However,

$$\mindeg V = 5.$$ 

Thus $K'$ has an embedded band representation satisfying equality in (9).

As genus is additive under connected sum, we have $g(K') = 13$. However, as $g_s$ is subadditive under connected sum, and $6_1$ is slice, we have $g_s(K') \leq 12$, so that $g > g_s$, and so $K'$ is not strongly quasipositive by lemma 3.

There is an exponentiated version of Fiedler’s conjecture, namely to ask about (non-strong) quasipositivity assuming equality in (9) for an embedded band representation. We conclude this section by showing how to construct counterexamples also for this most sharp case.

The problem to give such counterexamples reduces in one possible way to replacing $!15_{162508}$ by a strongly quasipositive knot with $\mindeg V > g$. Then the same argument as in the proof of proposition 1 goes through with embedded band representations.

Example 4 Consider the (apparently) 17 crossing knot on figure 4. It has a band representation with 7 bands on 4 strands,

$$(\sigma_2 \sigma_3 \sigma_2^{-1}) \sigma_1 \sigma_2 \sigma_1^{-1})^3 \sigma_1.$$ 

(The diagram in figure 4 was obtained again using KnotScape.) Thus its genus is $g = 2$. Also $\mindeg P = 4$, but $\mindeg V = 3$.

Figure 4

Remark 4 It is clear from example 4 that in fact we could have used it also as counterexample to Fiedler’s first conjecture. However, unlike for the knot $K$ in example 3, I cannot show $u = g(= 2)$ here. On the other hand, $K$ cannot be used in example 4, because it has $\mindeg P = 10$. ($K$ was found as a counterexample to Morton’s conjecture, as reported in [St3].) This way, any of the previous knots comes to its own right.
4. The coefficients of the Jones polynomial

Convention. It is convenient to assume in the sequel that all diagrams we consider are non-split. In particular, since non-split positive diagrams represent non-split links, we assume all positive links to be so.

Definition 1 A separating Seifert circle is a Seifert circle with non-empty interior and exterior. (That is, both interior and exterior contain crossings, or equivalently, other Seifert circles.) A diagram with no separating Seifert circles is called special.

Any diagram decomposes as Murasugi sum along its separating Seifert circles into special diagrams (see [Cr, §1]). For any diagram, any two of the properties positive, alternating and special imply the third. We call these diagrams special alternating to conform to the classical terminology of Murasugi [Mu3].

4.1. Positive braids

It was known (e.g. from [Fi]) that the minimal coefficient of the Jones polynomial of a positive braid link is $\pm 1$. We will show here a statement on the next 3 coefficients.

Theorem 1 Let $L$ be a non-split braid positive link of $n(L)$ components. Then

$$(-1)^n(L)^{-1/2}V(L)(t) = 1 + pt^2 + kt^3 + ( \text{higher order terms} ),$$

with $p = p(L)$ being the number of prime factors of $L$ and

$$-p \leq k \leq \frac{3}{2}(1 - \chi(L) - p),$$

where $\chi(L)$ is the Euler characteristic of $L$.

Note that it is a rather unusual situation to be able to read the number of prime factors off the polynomial. This is, for example, not possible for alternating links as shows the well-known pair $8_9$ and $4_{1}#4_{1}$ – the one knot is prime and the other one composite, yet they have the same Jones polynomial. Two more interesting examples of this type are as follows:

Example 5 With some effort one also finds such pairs of positive (or special) alternating knots: $12_{420}$ (figure 5) and $!3_1#9_{13}$ or $14_{4132}$ and $!5_2#9_9$.

Example 6 Even more complicated, but still existent, are such examples of fibered positive knots. The simplest group I found is a triple consisting of $14_{39977}$, $!3_1#14_{33805}$ and $!3_1#14_{37899}$ (see figure 5).

Proof of theorem 1. If $\beta$ is a positive braid diagram of $L$, then by the result of [Cr2] the number of prime factors of the link $\hat{\beta}$ is equal to the number of prime factors of the diagram $\hat{\beta}$. By [BoW] one can always choose $\hat{\beta}$ so that it contains a $\sigma^2$. Apply the skein relation at one of the crossings. Then

$$V_+ = t^2V_- + (t^{3/2} - t^{1/2})V_0,$$

with $L_-$ and $L_0$ both braid positive. Let $p_+ = p(L_+)$. By induction on the crossing number of the braid we have

$$t^2V_- = (-1)^{n(L)^{-1/2}(1-\chi(L))/2}, \quad (\begin{array}{cccc} 0 & 1 & 0 & p_- \\ 1 & -1 & p_0 & k_0 - p_0 \\ 0 & 1 & p_0 & k_0 - p_0 + p_- \\ \end{array}),$$

$$V_+ = (-1)^{n(L)^{-1/2}(1-\chi(L))/2}, \quad (\begin{array}{cccc} 0 & 1 & 0 & p_- \\ 1 & -1 & p_0 & k_0 - p_0 \\ 0 & 1 & p_0 & k_0 - p_0 + p_- \\ \end{array}).$$
The coefficients of the Jones polynomial

As \( p_+ = p_0 \) and \( 0 \leq p_- - p_+ = p_- - p_0 \leq 2 \), the claim follows by induction, once it is checked directly for connected sums of trefoils and Hopf links, except for the right inequality in (10), which follows only with the constant \( \frac{3}{2} \) replaced by 2. (Note, that \( k \) and \( p \) are both additive under connected sum.) To prove \( k \leq \frac{3}{2} (1 - \chi - p) \), we need to show that after a smoothing with \( p_- = p_+ + 2 \) we can choose another one with \( p_- \leq p_+ + 1 \).

Write

\[
\beta = \prod_{k=1}^{l} \sigma_i^{p_k} w_k
\]

with all \( w_k \) containing no \( \sigma_i \) but some of \( \sigma_i \pm 1 \). Then one of the \( k_i \), say \( k_1 \), is equal to 2, \( k_2 \geq 2 \) and \( l = 2 \). Then after smoothing out one of the crossings in the clasp, we have \( k_1 = 1 \), and then applying the skein relation at the other clasp, we have \( p_- - p_0 \leq 1 \), as desired. \( \square \)

From the proof it is clear that the second inequality in (10) is not sharp, and with some work it may be improvable. Candidates for the highest ratio \( k/(1 - \chi - p) \) are braids of the form \( (\sigma_1^2 \sigma_2^2 \ldots \sigma_l^2)^2 \), for which with \( l \to \infty \) this ratio converges upward to 1.

Contrarily, the first inequality is clearly sharp, namely for connected sums of \((2,.)\)-torus links.

**Question 1** Are the only links with \( p + k = 0 \) connected sums of \((2,.)\)-torus links?

### 4.2. Fibered positive links

We shall now prove a result on almost positive diagrams, which shows a weaker version of theorem 1 for fibered positive links. We need one definition.

**Definition 2** The Seifert graph \( \tilde{S}_D \) of a diagram \( D \) is a graph obtained by putting a vertex for each Seifert circle of \( D \) and connecting by an edge two vertices if a crossing is joining the two corresponding Seifert circles. (If two Seifert circles are connected by several crossings, \( \tilde{S}_D \) has multiple edges.) The reduced Seifert graph \( S_D \) of \( D \) is obtained by removing edges of \( \tilde{S}_D \) such that (a) \( S_D \) has no multiple edge and (b) two vertices are connected by an edge in \( \tilde{S}_D \) iff they are so in \( S_D \).

**Definition 3** For a link diagram \( D \), let \( \chi(D) = s(D) - c(D) \), where \( s(D) \) is the number of Seifert circles, and \( c(D) \) the number of crossings of \( D \). \( \chi(D) \) is the canonical Euler characteristic of \( D \).

**Theorem 2** Let \( D \) be an almost positive diagram of a link \( L \) with \( n(L) \) components, with negative crossing \( p \). If there is another crossing in \( D \) joining the same two Seifert circles as \( p \), then \( \min \deg V_L \geq (1 - \chi(D))/2 \). Otherwise, \( \min \deg V_L = (1 - \chi(D))/2 - 1 \) and \( \min \deg V_L = (-1)^{n(L)-1} \).
4.2 Fibered positive links

Recall, that the Kauffman bracket $[D] \ [\text{Ka}]$ of a link diagram $D$ is a Laurent polynomial in a variable $A$, obtained by summing over all states $S$ the terms

$$A^{\#A(S)-\#B(S)} (-A^2 - A^{-2})^{|S|-1}, \quad (12)$$

where a state is a choice of splittings of type $A$ or $B$ for any single crossing (see figure 6), $\#A(S)$ and $\#B(S)$ denote the number of type A (resp. type B) splittings and $|S|$ the number of (disjoint) circles obtained after all splittings in a state.

![Figure 6: The A- and B-corners of a crossing, and its both splittings. The corner A (resp. B) is the one passed by the overcrossing strand when rotated counterclockwise (resp. clockwise) towards the undercrossing strand. A type A (resp. B) splitting is obtained by connecting the A (resp. B) corners of the crossing.](image)

The Jones polynomial of a link $L$ is related to the Kauffman bracket of some diagram of it $D$ by

$$V_L(t) = (-t^{-3/4})^{-w(D)} [D] \bigg|_{A=t^{-1/4}}, \quad (13)$$

with $w(D)$ being the writhe of $D$.

**Proof.** The maximal possible degree of $A$ in

$$[D] = \sum_{S\text{ state}} A^{\#A(S)-\#B(S)} (-A^2 - A^{-2})^{|S|-1}, \quad (14)$$

is that of the $A$-state (the state with all crossings $A$-splitted), as under any splitting switch $A \to B$, the power of $A$ in first factor in (12) goes down by 2, and the maximal power of $A$ in second factor in (12) increases at most by 2. If $D$ is almost positive with negative crossing $p$, then the maximal possible power of $A$ in (14) is $A^{c(D)+2(s(D)-2)}$, as the $A$-state $S_A$ has $s(D)-1$ loops. They are the Seifert circles not adjacent to $p$, and a loop consisting of the 2 Seifert circles, call them $a$ and $b$, adjacent to $p$.

Now we must consider what states contribute terms of $A^{c(D)+2(s(D)-2)}$ in (14). These are exactly the states, for which, when obtained from the $A$-state by successively switching $A \to B$ splittings, $|\cdot|$ increases under any such switch.

Let $\langle S : k \rangle \in \{A, B\}$ be the split of $k$ in $S$, and let $s_k(S)$ be the state obtained by switching splitting $A \to B$ at crossing $k$ in $S$, assuming $\langle S : k \rangle = A$. Then if $|s_k(S_A)| < |S_A|$, any state $S$ with $\langle S : k \rangle = B$ is irrelevant for the highest term in (14). Clearly, this happens whenever $k$ is a crossing connecting one or two Seifert circles not adjacent to $p$.

Thus the only terms contributing to $A^{c(D)+2(s(D)-2)}$ in (14) are those for which $\langle S : k \rangle = B$ implies that $k$ has the same two adjacent Seifert circles $a$ and $b$ as $p$.

Let $p_1, \ldots, p_k = p$ be these crossings. Since any splitting switch $A \to B$ in $s_p(S_A)$ reduces $|\cdot|$, the only state $S$ with $\langle S : p \rangle = B$ relevant for the highest term in (14) is $s_p(S_A)$, whose contribution to the coefficient of this highest term is $(-1)^{|S_p(S_A)|-1} = (-1)^{s(D)-1}$.

It is also easy to see that if $\langle S : p \rangle = A$, any of the $2^{k-1}$ remaining states $S$ to consider contribute to $A^{c(D)+2(s(D)-2)}$, the coefficient being $(-1)^{s(D)+\#B(S)}$, as $|S| = s(D) - 1 + \#B(S)$. The sum over all such $S$ of these coefficients is $(-1)^{s(D)}$ times the alternating sum of binomial coefficients. Thus this sum vanishes for $k-1 > 0$, and cancels for $k-1 = 0$ the coefficient $(-1)^{s(D)-1}$ of $s_p(S_A)$.

The rest follows from (13) with $w(D) = c(D) - 2$, and the remark that $1 - \chi(D)$ and $n(L) - 1$ have the same parity. □
Corollary 1 Let $L$ be a fibered positive link of $n(L)$ components. Then $[V_L(t)]_{(3-\chi(L))/2} = 0$, that is,

$$(-1)^{n(L)-1}(3-\chi(L))/2 V_L(t) = 1 + k t^2 + \text{(higher order terms)},$$

with $k$ being some integer.

Proof. This is proved as theorem 1 by induction on the crossing number of a positive diagram $D$. Apply the skein relation at any (non-nugatory) crossing $p$ of $D$. Since the reduced Seifert graph of $D$ is a tree, there is another crossing between the same two Seifert circles. Let $D_0$ be $D$ with $p$ smoothed out, and $L_0$ be the link $D_0$ represents. $L_0$ is still fibered, because $D_0$ is positive and connected, and its reduced Seifert graph is still a tree. Similarly let $D_-$ be $D$ with $p$ switched and $L_-$ representing a link $L_-$. Then to $L_-$ we can apply the above theorem. So mindeg $V_- = (1-\chi)/2 - 1$, and the coefficients of $t^{(1-\chi)/2 + 1}$ in $t^2 V_-$ and $(t^{3/2} - t^{1/2})V_0$ cancel as in (11).

4.3. Positive and almost positive links

Corollary 1 is a special case of the following result, describing the second coefficient of the Jones polynomial for an arbitrary positive link.

Theorem 3 Let $L$ be a positive link with positive diagram $D$. Then

$$(-1)^{n(L)-1}[V_L]_{(3-\chi(L))/2} = s(D) - 1 - \# \{ (a,b) \text{ Seifert circles : there is a crossing joining } a \text{ and } b \}.$$ 

In other words, if $S_D$ is the reduced Seifert graph, then $(-1)^{n(L)}[V_L]_{(3-\chi(L))/2} = b_1(S_D)$, $b_1$ being the first Betti number.

Corollary 2 For a positive diagram $D$, $b_1(S_D)$ is an invariant of the link represented by $D$.

Note, that for the non-reduced Seifert graph $\bar{S}_D$ of $D$, $b_1(\bar{S}_D) = 1 - \chi(D) = 1 - \chi(L)$ is also a link invariant.

Corollary 3 For a positive link $L$ of $n(L)$ components, $(-1)^{n(L)}[V_L]_{(3-\chi(L))/2} \geq 0$, and this coefficient is 0 iff $L$ is fibered.

Of course, this fiberedness condition is not very useful, when a positive diagram of $L$ is given, since to decide then about fiberedness is trivial. However, applied in the opposite direction, it can prove that $L$ is not positive. This happens sometimes in a quite non-trivial way, as shows the following example.
4.3 Positive and almost positive links

Example 7 The knot $16_{1059787}$ in figure 7 satisfies all conditions on positivity known about its $V$, $V$, $P$ and $F$ polynomials. It seems useful to list all properties that hold, even if they involve invariants we did not consider here. See the given references for an accurate account. (However, keep in mind that the conventions there differ from the ones we use; for $F$ we conjugate in the $a$ variable.)

- $2 \min \deg V = \min \deg P = \max \deg mP = \max \deg V = \min \deg aF = 4$ [Cr, Fi, Yo, Zu],
- $[P]_{m^2(t)}(\sqrt{-t})$ and $V(z)$ are positive (that is, all coefficients are non-negative) [Cr],
- $P_i(l) := \sqrt{-1} [P]_{m^2(\sqrt{-t})}$ take only positive values on $l \in (0, 1)$ for $i = 0, 2, 4$ [CM].
- $[V]_{a^2} = 1$ [Fi, Zu].
- $[F]_{a^2} = \Delta(t)$ [Yo].
- $[F]_{A^2} \geq 0 \forall k \geq 0$ (that is, “critical line” polynomials are positive) [Th2].
- $16_{1059787}$ does have diagrams (with canonical Seifert surface) of genus 2, so that $g = \max \deg \Delta = 2$ [Cr]. Here $\Delta$ is normalized so that $\Delta(t) = \Delta(1/t)$ and $\Delta(1) = 1$.
- The signature $\sigma = 4$ [CG, St2], so that by Murasugi’s inequality [Mu2], $g_s = g = 2$ [Ru, St6].

However, now $[V]_{a^3} = 0$, so that if $16_{1059787}$ is positive, it must be fibered. But $[\Delta(t)]_{a^2} = 2$ contradicts this property.

Proof of theorem 3. This is proved as theorem 2 using the bracket. The term $s(D) - 1$ comes from the $A$-state, while for every pair of Seifert circles joined by (at least) one crossing, a $-1$ comes from an alternating sum of binomial coefficients coming from states in which a $B$-splitting is applied at some (non-empty) set of crossings linking $a$ and $b$.

Corollary 4 Let $L$ be an almost positive link with an almost positive diagram $D$ such that there is no positive crossing $q$ joining the same two Seifert circles as the negative crossing $p$. Then

$$\min \deg V_L = \frac{1 - \chi(D)}{2} \quad \text{and} \quad \min \text{cf } V_L = (-1)^{n(L) - 1}.$$  

Proof. Apply the skein relation at the negative crossing $p$ and use theorem 3 on $D_+$ and $D_0$ (they have the same reduced Seifert graph).

The following theorem is the key step needed to extend Fiedler’s result to almost positive links.

Theorem 4 Let $p$ be a crossing in a reduced special alternating diagram $D$ such that there is no crossing $q$ joining the same two Seifert circles as $p$. Let $D_p$ be $D$ with $p$ smoothed out. Then $\Delta_{D_p}(0) < \Delta_D(0)$, where $\Delta$ is the Alexander polynomial normalized so that $\min \deg \Delta = 0$ and $\min \text{cf } \Delta = \Delta(0) > 0$.

The proof will use the machinery of even valence graphs [MS]. We recall the basic notions from that paper.

Definition 4 The join (or block sum) `$\ast$' of two graphs is defined by

$\ast$ This operation depends on the choice of a vertex in each one of the graphs. We call this vertex the join vertex.

A cut vertex is a vertex, which disconnects the graph, when removed together with all its incident edges. (A join vertex is always a cut vertex.) Analogously a 2-cut of $G$ is a pair of edges of $G$ whose deletion disconnects $G$. 
**Definition 5** A cell $C$ is the boundary of a connected component of the complement of a graph $G$ in the plane. It consists of a set of edges. If $p$ is among these edges, then we say that $C$ contains $p$ or $p$ bounds $C$. By $G \setminus C$ we mean the graph obtained from $G$ by deleting all edges in $C$.

A cycle $C$ is a graph $G$ is a set of edges $\{p_1, \ldots, p_n\}$, such that the pairs $(p_1, p_n)$ and $(p_i, p_{i+1})$ for $1 \leq i < n$ share a common vertex, and all these vertices are different. The plane complement of a cycle in a planar graph has two components. The bounded one we will call interior $\text{int}(C)$ of $C$, and the unbounded one exterior $\text{ext}(C)$. (A cell is a cycle with one of interior or exterior being empty, that is, containing no edges.)

Before we make the next definition, first note, that the Seifert graph $\tilde{S}_D$ of any diagram $D$ is always planarly embeddable. Namely $S_D$ is the join of the Seifert graphs corresponding to the special diagrams in the Murasugi sum decomposition of $D$ along its separating Seifert circles, the join vertex corresponding to the separating Seifert circle.

The join of planar graphs is planar, and if $D$ is a special diagram, then $\tilde{S}_D$ has a natural planar embedding (shrink the Seifert circles into vertices and turn crossings into edges).

**Definition 6** Assume for a special diagram $D$ that $\tilde{S}_D$ is planarly embedded in the natural way. Its dual is called the even valence graph $G_D$ of $D$ (as the name says, all its vertices have even valence). Alternatively, $G_D$ is the checkerboard graph with vertices corresponding to the non-Seifert circle regions of $D$.

A canonical orientation is an orientation of the edges of $G_D$ so that all edges bounding a cell are oriented the same way between clockwise and counterclockwise as seen from inside this cell. (The canonical orientation is unique up to reversal of orientation of all edges in a connected component of the graph.)

**Proof of theorem 4.** Consider the plane even valence graph $G_{D_p}$ associated to $D$. Then $G_{D_p} = (G_D)_p$, where $G_p$ is $G$ with edge $p$ contracted. Both $G$ and $G_p$ are connected by assumption. We shall assume from now on that a canonical orientation is chosen in $G = G_{D_p}$, and hence also on $G_p$.

By the matrix-tree theorem (see theorem 2 of [MS]), we have that $\Delta_D(0) = \min \Delta$ is the number of index-0 spanning rooted trees of $G$, i.e. trees in which each edge, as oriented in $G$, points towards the root of the tree. We will call such trees also arborescences. Importantly, the number of arborescences does not depend on the choice of root vertex. We will exploit this property several times in the following.

Let $v_0$ be the source, and $v_1$ the target of $p$ in $G$. In $G_p$, $v_0$ and $v_1$ are identified to a vertex we call $v$.

By the proof of proposition 1, part 3), of [MS], we have that

$$\# \{ \text{index-0 sp. rooted trees with root } v \text{ in } G_p \} = \# \{ \text{index-0 sp. rooted trees with root } v_1 \text{ in } G \text{ containing } p \}.$$

Thus the statement of the theorem is equivalent to saying that $G$ has an index-0 spanning rooted tree with root $v_1$ not containing $p$. The assumption of the theorem in terms of even valence graphs means that each edge of $G$ bounds a cell not containing $p$, or equivalently, $p$ is in no 2-cut of $G$. In particular, both $v_0$ and $v_1$ have valence at least 4 in $G$.

It is easy to see that any plane even valence graph $G$ can be built up from the empty one by adding directed cycles. Moreover, if $G$ is connected, then we can achieve that all intermediate graphs are connected (or more exactly spoken, all their connected components except one are trivial, i.e. an isolated vertex of valence 0). Also, one can start the building-up with any particular cycle in $G$.

Let $E$ be a cell (cycle with empty interior) in $G$ containing $p$. We claim that then $\hat{G} = G \setminus E$ is still connected. There is a little argument needed for this. We will explain that if a disconnected graph $\hat{G}$ is connected by adding a cell $E$, then each edge in $E$ forms a 2-cut with another edge in $E$. To see this, first reduce the problem to $\hat{G}$ having two components $\hat{G}_1$ and $\hat{G}_2$. If $\hat{G}$ has further components $\hat{G}_3, \ldots, \hat{G}_n$, one can connect them to $\hat{G}_2$ by adding cells, and a 2-cut of edges in $E$ would still remain one if we undo this connecting. It also is easy to see that one can assume there are no valence-2-vertices of $E$ in $G$ (that is, each vertex of $E$ is attached to one of $\hat{G}_1$ or $\hat{G}_2$ in $G$). Then we show that there are at most two edges of $E$ connecting $\hat{G}_1$ and $\hat{G}_2$. Since $E$ is oriented, one can easily distinguish between interior or exterior of $E$ depending on the (left or right) side in orientation direction. If $\geq 4$ edges connect $\hat{G}_1$ and $\hat{G}_2$, one must attach vertices of $\hat{G}_1$ and $\hat{G}_2$ from different sides to $E$, and $E$ will not be a cell in $G$.

Let $E'$ be some other cycle passing through $v_1$, such that $p \notin E'$. (Such a cycle exists because $\text{val}_G(v_1) > 2$.)
4.3 Positive and almost positive links

Then build up $G$ by adding cycles $E_n$, such that we start with $E_1 = E'$ and finish with $E_z = E$, and all intermediate graphs $G_n$ are connected. We construct successively in each $G_n$ an index-0 spanning rooted tree $T_n$ with root $v_1$, such that in the final stage in $G_z = G$ the tree $T_z = T$ does not contain $p$.

In $G_1 = E'$, fix the root to be $v_1$ and let $T_1$ consist of all edges in $E'$ except the one outgoing from $v_1$.

Now, given an index-0 spanning rooted tree $T_n$ of $G_n$, we construct an index-0 spanning rooted tree $T_{n+1}$ of $G_{n+1} = G_n \cup E_{n+1}$ as follows.

Let $w_1, \ldots, w_k$ be the vertices of the cycle $E_{n+1}$ in cyclic order, so that $w_i$ and $w_{i+1}$ are connected by a (directed) edge $p_i$. Then there is a non-empty set $S \subset \{1, \ldots, k\}$ such that for all $s \in S$, $w_s \in G_n$, and $w_s$ is a trivial connected component (isolated vertex) in $G_n$ otherwise. Then add the following vertices to $T_n$ to obtain $T_{n+1}$: for each $i, j \in S$ such that $(i, j) \cap S = \emptyset$ add $\{p_m : m \in (i+1, j-1)\}$. Here $(i, j)$ is the interval of numbers between (but not including) $i$ and $j$, meant w.r.t. the cyclic order in $\mathbb{Z}_k$.

Here is an example of a cycle $E_{n+1}$, in which the vertices in $G_n$ are encircled, and the edges in $T_{n+1} \setminus T_n$ thickened.

Then $T_{n+1}$ is an index-0 spanning rooted tree with root $v_1$ in $G_{n+1}$.

It remains to see why $p \not\in T_z = T$. For this note that $E = E_z \ni p$ is added last, and $\text{val}_{G}(v_0), \text{val}_{G}(v_1) \geq 4$, so that $v_0, v_1 \in G_{z-1}$.

**Corollary 5** If $D$ is an almost positive diagram with negative crossing $p$ such that there is no (positive) crossing $q$ joining the same two Seifert circles as $p$, then $\text{mindeg} \Delta_D(t) = 1 - \chi(D)$, where $\Delta$ is normalized so that $\Delta(t) = \Delta(1/t)$ and $\Delta(1) = 1$. In particular, the canonical Seifert surface associated to $D$ is of minimal genus.

**Proof.** Apply the skein relation for $\Delta$ at the negative crossing to obtain the result for special diagrams. Then use the multiplicativity of $[\Delta(D)](1-\chi(D))/2$ under Murasugi sum of diagrams [Mu2] to obtain the general case.

This corollary improves the result of Hirasawa [Hi] (theorem 2.1) stating that this Seifert surface is incompressible.

Now we have all the preparations together to obtain the extension of Fiedler’s result.

**Theorem 5** If $L$ is an almost positive link, then $\text{mindeg} V_L = \frac{1}{2}(1 - \chi(L))$, and $\text{mincf} V_L = (-1)^{n(L)-1}$.
Proof. Let $D$ be an almost positive diagram of $D$ with negative crossing $p$ and canonical Seifert surface $S$. One can easily reduce the proof to the situation that $D$ is connected. Distinguish then two cases.

a) There is a (positive) crossing $q$ joining the same two Seifert circles as $p$. By theorem 2 we must show that

$$\frac{1 - \chi(L)}{2} = \frac{1 - \chi(D)}{2} - 1.$$ 

Clearly, $\frac{1 - \chi(L)}{2} \leq \frac{1 - \chi(D)}{2}$, and by Bennequin’s inequality $\frac{1 - \chi(L)}{2} \geq \frac{1 - \chi(D)}{2} - 1$.

Thus assume that $\frac{1 - \chi(L)}{2} = \frac{1 - \chi(D)}{2}$, i.e. $S$ is a minimal genus surface. By [Ga2], then this is true for the Murasugi summand of $S$, which is the canonical Seifert surface associated to an almost positive (or almost alternating) special diagram. However, by assumption this surface is clearly not of minimal genus, a contradiction.

b) There is no such crossing $q$. Then we must show by corollary 4 that $\frac{1 - \chi(L)}{2} = \frac{1 - \chi(D)}{2}$, i.e. $S$ is a minimal genus surface. This follows again from [Ga], using corollary 5. □

4.4. Skein polynomial and Morton’s inequalities

The results on the Jones polynomial and their proofs allow also some applications to the skein polynomial [H].

First, we can identify two more coefficients of the polynomial of some positive links.

Corollary 6 If $L$ is a fibered positive link of $n(L)$ components, then

$$[P_L]_{1 - \chi(L) m^{-1 - \chi(L)}} = (-1)^{n(L)} (1 - \chi(L)),$$

and if $L$ is prime and braid positive, then

$$[P_L]_{1 - \chi(L) m^{-3 - \chi(L)}} = (-1)^{n(L)-1} \frac{\chi(L)}{2} + 1.$$ 

Proof. Murasugi and Przytycki showed in [MP] that $[P_D]_{m^{1 - \chi(D)}}$ is multiplicative under Murasugi sum. (That was shown by Morton [Mo].) Since any positive diagram of a fibered positive link decomposes as Murasugi sum of connected sums of (2, .)-torus links, we have for any fibered positive link $L$,

$$[P_L]_{m^{1 - \chi(L)}} = l^{1 - \chi(L)} \cdot (-1)^{n(L)-1}.$$ 

Now apply corollary 1 and the conversion (3). □

Remark 5 A formula for the first of the coefficients in the corollary can be written for an arbitrary positive link using theorem 3 instead of corollary 1.

The proof of theorem 5 can also be applied for $P$ and $\Delta$, normalized so that $\Delta(t) = \Delta(1/t)$ and $\Delta(1) = 1$.

Theorem 6 If $L$ is an almost positive link, then

$$2 \max \deg \Delta(L) = \max \deg_m P(L) = 1 - \chi(L).$$

(16)
4.4 Skein polynomial and Morton’s inequalities

Proof. That \(2\max \deg \Delta_L = \max \deg_m P(L)\) follows from a well-known argument. From (4) clearly \(2\max \deg \Delta_L \leq \max \deg_m P(L)\). Assume \(2\max \deg \Delta_L < \max \deg_m P(L)\). This means that in the calculation of \(\Delta_L\) by some (arbitrary) skein resolution tree (see [Cr, §1]) contributions of terminal nodes of the tree occur, which affect \(\max \deg_m P(L)\), but cancel when substituting \(l = i\) (as in (4)). However, in the proof of [Cr, corollary 2.2], Cromwell argued that for an almost positive link diagram, one can choose the skein resolution tree so that no contributions cancel. Thus we are left to show the second equality in (16).

Consider the two cases in the proof of theorem 5.

If \(q\) shares its Seifert circles with another (positive) crossing in \(D\), then one of the special Murasugi sum components \(D'\) of \(D\), can be reduced to a diagram \(D''\) with \(\chi(D'') > \chi(D)\). Thus by (15),

\[
\max \deg_m P(D') = \max \deg_m P(D'') \leq 1 - \chi(D'') < 1 - \chi(D),
\]

so that \([P_{D'}]_{1-\chi(D')} = 0\). Then by [MP] the same holds for \(D\). Since we know that

\[
1 - \chi(L) = -1 - \chi(D)
\]

from the proof of theorem 5, the inequality

\[
\max \deg_m P(L) \leq 1 - \chi(L)
\]

(18)

follows. Now use that, as a consequence of [LM, proposition 21], for an arbitrary link \(L\), \(\min \deg P \leq \max \deg_m P\). From Morton’s inequalities [Mo] we have then for an almost positive diagram \(D\) of \(L\)

\[
-1 - \chi(D) = w(D) - s(D) + 1 \leq \min \deg P(D) \leq \max \deg_m P(L).
\]

(19)

Now (19) and (17) show that the inequality (18) is exact.

If \(q\) does not share its Seifert circles with another crossing in \(D\), then combining [Mo] and the argument in the proof of theorem 5, we have

\[
1 - \chi(D) = 2\max \deg \Delta(D) \leq \max \deg_m P(D) \leq 1 - \chi(D) = 1 - \chi(L),
\]

so we have the equality in (16).

\(\square\)

Remark 6 Note that the equality \(2\max \deg \Delta = 1 - \chi\) is a property of purely algebraic-topological nature, and has been examined for many links long before the skein polynomial was around. Thus the usage of the skein polynomial in its proof here is somewhat surprising. It seems essential, though, and also explains why this equality cannot be proved so for other classes. (Indeed, for example it is not always true for almost alternating links.)

As a consequence using theorem 5, we have

Corollary 7 If \(L\) is an almost positive link, then \(\max \deg \Delta_L = \min \deg V_L\).

\(\square\)

This solves conjecture 5.2 of [St2], which stated this property for almost positive knots. Using the work therein, we have

Corollary 8 There are only finitely many almost positive knots with the same Alexander polynomial.

\(\square\)

Further we have

Corollary 9 If \(L\) is an almost positive link, then \(\min \deg P(L) \leq 1 - \chi(L)\).

\(\square\)

Proof. Use again the mentioned consequence of [LM, proposition 21].

This is another special case of Morton’s conjectured inequality [Mo2] (disproved now in [St3] for arbitrary links). There is, though, much experimental evidence that we have in fact equality in Morton’s inequality.
Question 2 Is for any almost positive link \( L \), \( \mindeg_l P(L) = 1 - \chi(L) \)?

Note that in the one case in the proof of (16), we obtained this desired equality, namely when the almost positive diagram \( D \) is not of minimal genus. Latter property is to be understood so that its canonical Seifert surface does not realize the (Seifert) genus of \( L \), i.e. \( \chi(D) > \tilde{\chi}(L) \). Question 2 is thus related to the question: Does any almost positive link \( L \) have an almost positive diagram \( D \), which is not of minimal genus?

As we later found, the answer to this question is negative, and a counterexample is the knot \( !12_{1930} \) (which nevertheless satisfies \( \mindeg_l P = 1 - \chi \)). It is displayed in figure 8 of [St6] and occurs also later in this paper as \( L_4 \) in the proof of corollary 10). Beside its obvious two almost positive diagrams (considered also in the proof below), there are no other (reduced) ones. The proof of this fact will be presented elsewhere, as it requires, apart from some computation, several tools (developed in [St6, St2, St]), that go beyond the scope of the present paper.

The opposite situation to the last question is not less interesting, in particular because positive diagrams are always genus-minimizing.

Question 3 Does any almost positive link \( L \) have an almost positive diagram \( D \) of minimal genus?

A positive answer to this question will show that Morton’s inequality for \( \tilde{\chi}(L) \),

\[
\max \deg_m P(L) \leq 1 - \tilde{\chi}(L)
\]

(20)

(which is a direct consequence of (15)) is sharp. It would not be a surprise, as knots with strict inequality are hard to find. So far two methods apply: unity root values of \( V \) [St] and Gabai’s foliation algorithm [Ga3] to show that in fact \( \max \deg_m P(L) < 1 - \chi(L) \) [St3]. Latter option seems unlikely to work for almost positive links, and former option requires considerable extension of the calculations. Out of the \( \approx 4500 \) non-alternating prime knots \( K \) of \( \leq 16 \) crossings with \( \max \deg_m P(K) \leq 4 \), in [St] we obtained 28 such knots with \( 4 = \max \deg_m P(K) < 2 \tilde{g}(K) \) using values of the Jones and \( Q \) polynomial at roots of unity (and one further undecided case). An easy check shows that none of these 28 knots is almost positive.

5. Almost positive diagrams with canonical fiber Seifert surfaces

The even valence graphs can be used to give a description of almost positive diagrams whose canonical Seifert surfaces are fiber surfaces. The restriction to such surfaces is suggestive, since in general the decision of fiber property of a link or a surface may be difficult, even although both algebraic and geometric methods are known. Our result is closely related to the result for almost alternating diagrams due to Goda–Hirasawa–Yamamoto [GHY]. Our main motivation here was in fact to use the present (and quite different) tools to extend and simplify the proof of their criterion. We succeed almost completely, with the exception that we cannot recover combinatorially the fact (see their proposition 5.1) that instead of general Murasugi sum decomposability of the fiber into Hopf bands in part 1) of the theorem we will state below we have in fact stronger plumbing decomposability. On the other hand, we show in part 3) that the fiberedness condition for the Alexander polynomial is exact. Due to the copious ways to calculate the Alexander polynomial, this makes the fiberedness property even easier to detect than by the classification result 4) for such diagrams by itself. (Our version of this result is also more explicit than in the form given in [GHY].) In the next section we will give examples showing (together with the examples in [GHY]) that one cannot extend the result much further.

In the following \( \Delta \) will be normalized so that \( \Delta(t) = \Delta(t^{-1}) \) and \( \min \text{cf} \Delta > 0 \).

Theorem 7 Let \( D \) be a connected almost positive link diagram with canonical Seifert surface \( S \). Then the following conditions are equivalent:

1) \( S \) decomposes under iterated Murasugi sum (not necessarily plumbing) completely into Hopf bands (of 1 full twist).

2) \( S \) is a fiber surface.
3) $2 \text{max} \deg \Delta(D) = 1 - \chi(D)$ and $\text{min} \deg \Delta(D) = 1$.

4) Decompose $D$ along its separating Seifert circles (Seifert circles with non-empty interior and exterior) as Murasugi sum of special diagrams, and those special diagrams into prime factors. Then all these prime factors are special alternating diagrams of $(2,n)$-torus links (parallelly oriented), except for one, which after reductions of the type

$$
\begin{array}{c}
\includegraphics[width=2cm]{figure8a}\rightarrow \includegraphics[width=2cm]{figure8b}
\end{array}
$$

becomes an almost positive special diagram of the following forms:

a) a special diagram whose (even valence) checkerboard graph $G$ can be obtained as follows:  
Take a chain of circles of positive edges

$$
\begin{array}{c}
\includegraphics[width=2cm]{figure8c}
\end{array}
$$

and attach to it from outside a cell (cycle with empty exterior) with one negative edge, which joins interior points of the two 2 outermost loops in (22). (The negative edge corresponds to the crossing to be switched.) E.g. see figure 8.

or (non-exclusively)

b) a diagram of a $(2,2,\ldots,2)$-pretzel link (at least two ‘2’s), oriented to be special, with one crossing changed.

**Proof.** In the following graph pictures, we assume graphs to be canonically oriented, but do not draw edge orientation if it is not necessary. The edge of the (only) crossing $p$ of negative checkerboard sign will be distinguished by being drawn as a thickened or dashed line.

4) $\Rightarrow$ 1). The reverse of the move on (21) preserves the property the canonical Seifert surface to be a fiber, as it corresponds to plumbing of a Hopf band. That the canonical Seifert surfaces of the diagrams in 4.b) are fibers is easy to see. (The diagram $D'$ obtained from $D$ by removing the trivial clasp is a connected sum of Hopf links, its surface is clearly a fiber, which is unique, and $\chi(D') = \chi(D)$.) For 4.a) we remark that each of the graphs described turn into
under repeating the operation

\[ \xrightarrow{\text{contracting a double edge}} \]

(contracting a double edge), with the dashed line having two properties. First, it is an arc passing through edges whose total sign sum is 0 (in our case the negative edge \( p \) and one other, positive, edge), and second, it has a single vertex and no complete edge in (at least) one of its interior or exterior. This corresponds to the move on diagrams

\[ \xrightarrow{\gamma} \]

on diagrams. The dashed line \( \gamma \) must pass through interior of Seifert circles and crossings only, such that the total writhe of these crossings is 0, and must have a single non-Seifert circle region, and no complete Seifert circle in (at least) one of its interior or exterior.

Then this is a Hopf plumbing, the Hopf band being obtained by thickening \( \gamma \) into a strip, and taking the union with the two half-twisted strips and one Seifert circle on the left of (23). (The first condition on \( \gamma \) is needed to ensure the correct twisting, while the second one is needed to have the Hopf band being separated by a sphere from the rest of the surface after deplumbing.)

1) \( \Rightarrow \) 2) \( \Rightarrow \) 3) are well-known, so it remains to show the real result 3) \( \Rightarrow \) 4).

As the minimal coefficient of the Alexander polynomial, when its degree is equal to \( (1 - \chi)/2 \), is multiplicative under Murasugi sum [Mu2], we need to consider only the almost alternating special Murasugi summand.

For this we consider the canonically oriented even valence graph \( G = G_D \) of \( D \) and turn to the said in the proof of theorem 4. The condition \( 2 \min \deg \Delta = 1 - \chi \) implies that each edge of \( G \) bounds a cell not containing \( p \), \( p \) being the edge in \( G = G_D \) of negative checkerboard sign. In particular, both \( v_0 \) and \( v_1 \) have valence \( \geq 4 \).

Let \( E_1 \) and \( E_2 \) be the cells containing \( p \). Then \( E_1 \cap E_2 = \{ p \} \), since \( p \) is in no 2-cut of \( G \), and \( G \setminus E_1 \) and \( G \setminus E_2 \) are connected, by the argument in the proof of theorem 4.

Now consider the condition \( \min cf \Delta(D) = 1 \). It means that there is only one index-0 spanning rooted tree with root \( v_1 \) not containing \( p \). If \( G \setminus E_1 \) has two different index-0 spanning rooted trees with root \( v_1 \), then by the construction in the proof of theorem 4, we could extend them to index-0 spanning rooted trees of \( G \) with root \( v_1 \) not containing \( p \), which would clearly still be different. Thus \( G \setminus E_1 \) has only one index-0 spanning rooted tree (with root \( v_1 \) or any other fixed vertex). Then, by part 5) of theorem 3 of [MS], \( G \setminus E_1 \) is a join of chains (topologically, a bouquet of circles).

\[ \xrightarrow{\text{topologically, a bouquet of circles}} \]

Assume w.l.o.g. that \( G \) is embedded so that the exterior \( \text{ext}(E_1) \) of \( E_1 \) is the unbounded component. Since adding \( E_1 \) must remove all cut vertices (our diagram is prime by assumption), \( E_1 \) must touch interior points of all circles \( L_i \).
with only one cut vertex. (Here ‘interior’ is meant with the circle considered 1-dimensional with boundary being the cut vertex/vertices.) We call these \( L_i \) leaf circles; in (24) they are drawn dashed. Also, since the exterior of \( E_1 \) is the unbounded component, cut vertices coming from attaching circles within others cannot be removed by adding \( E_1 \), so assume there are no such inner circles. Thus we have a picture like

From now on, let us remove valence-2-vertices (we call this operation \textit{unbisection}) and consider only the topological type of the tree

\[ \xrightarrow{\text{unbisection}}. \] (25)

This move on graphs corresponds to the reversed move (21). (Note that \( \text{val}(v_{0,1}) \geq 4 \) by assumption, so that both edges on the left of (25) are positively signed.)

The \textit{way} between two leaf circles \( L_1 \) and \( L_2 \) is made up of those circles bounding disks whose interior is passed by a path from an interior point of the disk bounded by \( L_1 \) to an interior point of the disk bounded by \( L_2 \). Hereby we require that this path is passing only through interior points of disks bounded by loops and cut vertices, each such vertex being passed at most once.

Now use that \( G \setminus E_2 \) must also be a join of loops (or bouquet of circles). We claim that either

a) \( p \) must touch interior points of two different leaf loops \( L_1, L_2 \) and all other vertices of \( E_1 \) touch only interior points or cut vertices belonging to circles \textit{on the way} between \( L_1 \) and \( L_2 \) (as on the left of figure 8)
b) \( p \) touches two points on the same loop \( L \) (interior points or the cut vertex), and \( E_1 \) touches points only of the same loop.

Assume neither a) nor b) holds. We derive a contradiction showing that \( G \setminus E_2 \) has at least 2 arborescences. Observe that by unbisections (25) and separations

\[
\begin{array}{c}
\text{(26)} \\
\end{array}
\]

one can simplify \( G \) to obtain

\[
\begin{array}{c}
\text{(27)} \\
\end{array}
\]

(still with \( E_1 \) being the boundary of the infinite region), in the way that \( v_{0,1} \) are not involved in any of these moves. Then removing \( E_2 \) and applying unbisections, one obtains a graph \( G_0 \) with two vertices and an edge of multiplicity 4, which has 2 arborescences. It suffices now to show that bisections (reverses of unbisections) and deseparations (reverses of separations (26)) do not reduce the number of arborescences. For this we specify how to map injectively arborescences of the original graph with root \( v_1 \) to arborescences of the resulting graph.

For a bisection creating a vertex \( v \neq v_{0,1} \), add the outgoing edge of \( v \) to the arborescence, and so do with the incoming one, if the original (bisected) edge was in the (original) arborescence. The same argument, but without the restriction \( v \neq v_{0,1} \), finds 2 arborescences (with root \( v_1 \) or any other vertex) of (27), starting from those of \( G_0 \).

For a deseparation at least one of the two vertices \( v, v' \) on the right of (26) has valence 2. Let \( v \) be such a vertex. The outgoing edge \( e \) of \( v \) is in any arborescence, since \( v \neq v_{0,1} \). Remove \( e \) from the arborescence, and keep the status of the other edges, while joining \( v \) and \( v' \).

Thus \( G \) is one of the types a) and b).

In case a) the assumption there are no cut vertices in \( G \) implies that \( G \setminus E_1 \) is as in (22), and \( p \) joins interior points on the two outermost circles. Note, that the union of \( L_{1,2} \) and all loops on the way between them form a bouquet of type (22). Thus we arrive at case 4.a) in theorem 7.

In case b), \( G \setminus E_1 \) must be a single loop, and we have a picture like this

\[
\begin{array}{c}
\text{(27)} \\
\end{array}
\]

This is case 4.b) in theorem 7. \( \square \)

What we have done allows to solve an open problem in our previous work [St2] on almost positive knots and to prove
Corollary 10 There exist almost positive knots of any genus ≥ 2.

It was shown in [St], that there are no almost positive knots of genus 1.

Proof. Consider the (3,3,…,3,−1)-pretzel knots and links \( L_n \). (These are 2-component links if the number of ‘3’s is odd; in this case we orient them so that the twists counted by the ‘3’s are reverse.) As these diagrams of \( L_n \) come from the construction in part 4) of theorem 7, \( L_n \) are fibered (or see alternatively theorem 6.7 of [Ga4]). Their diagrams reduce by one crossing to \( D_n = (−2,−1,3,…,3) \) (one ‘3’ less), which are almost positive and of crossing number \( 3(1−\chi(D_n)) = 3(1−\chi(L_n)) = 3n \). If \( L_n \) were positive, by [Cr, corollary 5.1] they would have crossing number \( c(L_n) ≤ 2(1−\chi) \).

To show that this is not the case, consider the crossing number inequality of \([Ka, Mu, Th]\), \( c(L_n) ≥ \text{span} V(L_n) \).

We know that \( \text{mindeg} V(L_n) = \frac{1−\chi}{2} \). On the other hand, for \( n = 1−\chi > 2 \), \( \text{maxdeg} V(L_n) \) is easy to determine, as the diagram \( D_n \) is \( B \)-semiadequate, and thus only the contribution of the \( B \)-state \( S_B \) (\( S_B : k = k \) for any crossing \( k \)) is relevant in (14). By simple count of the loops one arrives at \( \text{maxdeg} V(L_n) = \frac{7}{2}(1−\chi)−2 \), and thus

\[
c(L_n) ≥ \text{span} V(L_n) = \text{maxdeg} V(L_n) − \text{mindeg} V(L_n) = 3(1−\chi)−2 > 2(1−\chi).
\]

Thus \( L_n \) is almost positive for \( n ≥ 3 \). (For \( n = 1 \) and \( n = 2 \) one obtains the Hopf link and trefoil, resp.) □

6. Some examples and problems

6.1. Showing almost positivity

The problem to show that a certain link is almost positive, but not positive, turned out very hard. All previously known positivity criteria either are easily provable to extend to almost positive links, or at least no examples are known where they do not. Theorem 5 is an addition to that picture.

In [St2] it was shown, in the case of knots, that any almost positive knot has only finitely many reduced almost positive diagrams. As the proof is constructive, one can, in theory, decide (for knots) about almost positivity, in the sense that for any knot one can write down a finite set of almost positive diagrams, among which one would have to check whether the knot occurs. However, this method is not generally very efficient, except for a few knots of small genus.

Cromwell’s estimate \( c ≤ 2(1−\chi) \) for fibered homogeneous links remains the only way known so far to circumvent these problems, at least in certain cases. Using theorem 7, we can construct now a plenty of examples of almost positive fibered link diagrams, which we can show to represent almost positive links using that Cromwell’s inequality is violated.

However, this inequality will still not be violated in many cases, and thus one may ask whether it can be improved.

Cromwell’s estimate is trivially sharp for alternating (prime) links (consider the rational links 222⋯ 2) and composite positive links (consider the connected sums of Hopf links). However, even for prime positive links the inequality can not be improved much.

Example 8 The (2,2,…,2,−2,−2)-pretzel link, oriented the clasps counted by ‘2’ to be reverse, and those counted by ‘−2’ to be parallel, has \( c = 2(1−\chi)−2 \). The diagram is of minimal crossing number as follows by considering linking numbers, and Murasugi desums into connected sums of Hopf bands, thus the link is fibered. (The link is also prime by [KL].)

Example 9 Even just considering knots, there exist examples of genus 4 and crossing number 16, 16_{177344} and 16_{1243226}. (For genus 3 the maximal crossing number example is the knot 11_{350} of [St3] without a minimal positive diagram.) Apparently these examples can be generalized to higher genera (although the proof of minimality of the crossing number is not straightforward).

Thus Cromwell’s estimate seems rather sharp even in our case.

A different problem in this context is the position of the class of almost positive links w.r.t. the hierarchy (1).
Some examples and problems

Figure 9: The two fibered positive knots of genus 4 and crossing number 16. (The diagrams here are chosen to be positive and reveal a plumbing structure of the fiber surfaces. 16_1243226 has also almost positive 16 crossing diagrams, and 16_1177344 even 2-almost positive ones.)

Question 4 Is any almost positive link strongly quasipositive, or at least quasipositive?

Some 2-almost positive links, like the figure-8-knot, are not quasipositive. On the other hand, all almost positive examples examined so far are strongly quasipositive.

6.2. Detecting genus and fiberedness with the Alexander polynomial

From our results in the previous two sections, we have the following

Corollary 11 If a link $L$ has a connected almost positive (or almost alternating) diagram (with canonical Seifert surface) of minimal genus, then

a) $2 \max \deg \Delta_L = 1 - \chi(L)$, and

b) $L$ is fibered if and only if $\min \deg \Delta_L = \pm 1$.

Unfortunately, we cannot decide about fiberedness if the almost positive diagram is not of minimal genus. Many almost positive knots seem to have almost positive minimal genus diagrams, but whether all have is unclear. Coming back to the inequality $g(K) \leq \tilde{g}(K)$ in question 3, it is known that almost alternating knots may fail to realize it sharply. One of the two $\Delta = 1$ knots of 11 crossings has genus two [Ga3], and is almost alternating by the verification in [Ad, Ad2], while the calculation in [LM, example 11.1] gives $\max \deg_{\text{alt}} P = 6$, so that by (20), $\tilde{g} = 3$. (A genus three canonical surface is not too hard to find.) This knot thus does not have any diagram whatsoever of minimal genus. Let us mention in contrast that among the 28 knots we found with unsharp Morton inequality $4 = \max_{\text{alt}} P(K) < 2\tilde{g}(K)$, none could be identified as almost alternating (although there are not enough tools to exclude it). However, there are several $\leq 2$-almost alternating knots, for example 15_130745 (see figure 9 of [St]).

For almost positivity the problem to find knots with $\tilde{g} > g$ seems much harder than for almost alternating.

Question 5 What is the minimal $k$ with a $k$-almost positive knot having no diagram of minimal genus?

So far it seems likely that such knots with $k = 4$ occur, but even whether $k \leq 3$ is unclear. Contrarily, there is a 2-almost positive knot, 16_1337674, with unsharp inequality (20).

Note that both statements in corollary 11 are true for many (other) links, in particular all knots in Rolfsen’s tables [Ro, appendix]. However, the following examples show that the corollary does not extend much further.
6.2 Detecting genus and fiberedness with the Alexander polynomial

**Example 10** Consider the diagram on the middle of figure 10. It is another diagram of the previously encountered knot $13_{6374}$ with unit Alexander polynomial. It is 2-almost alternating, and its canonical Seifert surface is of minimal genus (two), as can be shown by [Ga3]. Thus neither of both criteria hold for 2-almost alternating diagrams.

**Example 11** The diagram on the right of figure 10 depicts the knot $12_{1581}$ with Alexander polynomial $\Delta = (2\, [-3]\, 2)$. It is a (special) 2-almost positive diagram whose canonical Seifert surface is of minimal genus (again two). Thus criterion a) in corollary 11 is not true for 2-almost positive diagrams.

So far I have no example of a 2-almost positive knot diagram for criterion b), but one can easily obtain a link diagram.

**Example 12** Consider the diagram of $13_{6374}$ in example 10. It has a single separating Seifert circle, whose interior contains two crossings. By removing this interior (deplumbing a Hopf band), one arrives at the link diagram on the left of figure 10. Its canonical Seifert surface is still of minimal genus by [Ga2], so that $1 - \chi = 3$, but one calculates that $\Delta = t^{1/2} - t^{-1/2}$.

In all the above examples we showed a surface not to be a fiber using that the Alexander polynomial has too small degree. There are also examples, where the degree is maximal, and thus all conditions in corollary 11 taken together still do not suffice to determine a fiber.

**Example 13** The $(-2, 4, 6)$-pretzel link diagram, oriented to be special (all clasps reverse), has max deg $\Delta = 1 - \chi = 2$ and min cf $\Delta = 1$. That its canonical surface is not a fiber follows from [Ga4, theorem 6.7] (Case 1). Using properly signed Hopf plumbings, one can generate from this one many more examples of 2-almost alternating and/or 2-almost positive diagrams, in particular (diagrams of) several genus two knots. Two such knots (for 2-almost positive diagrams) are the mirror images of $15_{120617}$ and $15_{159580}$, displayed in figure 11. (These two knots have been in fact found first, by a check in the tables, and the pretzel link was obtained from them by deplumbings.)

Remark 7 For genus one canonical surfaces of knots one needs 3-almost positive (and 3-almost alternating) diagrams to have such a situation, the simplest example being the $(-3, 5, 5)$-pretzel knot [CT]. (It has the Alexander polynomial of the figure-8-knot.) For 4-almost positive (or 4-almost alternating) diagrams, even worse, one can use [Ga4, theorem 6.7] to construct diagrams differing by mutation, with the canonical surface of the one being a fiber, and of the other one not.

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