Towards Skyrmion Stars: Large Baryon Configurations in the Einstein-Skyrme Model

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Abstract

We investigate the large baryon number sector of the Einstein-Skyrme model as a possible model for baryon stars. Gravitating hedgehog skyrmions have been investigated previously and the existence of stable solitonic stars excluded due to energy considerations[1]. However, in this paper we demonstrate that by generating gravitating skyrmions using rational maps, we can achieve multi-baryon bound states whilst recovering spherical symmetry in the limit where $B$ becomes large.

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1 Introduction

The Skyrme model, in its initial form, was proposed and developed by T.H.R. Skyrme in a series of papers as a non-linear field theory of pions [2], [3]. Skyrme's initial idea was to think of baryons (in particular the nucleons) as secondary structures arising from a more fundamental mesonic fluid. The key property of the model was that the baryons arose as solitons in a topological manner and thus possessed a conserved topological charge identified with the baryon number.

The lowest energy stable solutions of the model are termed Skyrmions and can be thought of as baryonic solitons. The Skyrme model has been very successful in modelling the structures of various nuclei and has been shown by Witten et al. [4] to possess the general features of a low energy effective field theory for QCD.

Some studies of the Skyrme model coupled to gravity have previously been undertaken [1], [5], [6], mainly with the motivation of a comparison of its features with those of other non-linear field theories coupled to gravity. Of particular note is the Einstein-Yang-Mills theory, in which gravitationally bound configurations of non-abelian gauge fields are produced.

Other reasons for studying the Einstein-Skyrme model are cosmological and astrophysical ones. Various authors have studied black hole formation in the model, with the conclusion that the so-called no-hair conjecture may not hold [7], [8].

The purpose of this paper is to study large baryon number Skyrmions or configurations of Skyrmions in the Einstein-Skyrme model. In particular, we wish to investigate if stable solitonic stars could exist within the model and to compare their properties to those of neutron stars.

Preliminary studies of Skyrmion stars have predicted instability to single particle decay [1]. However this was done using the hedgehog ansatz for baryon number larger than 1 which is known to lead to unstable solutions even for the usual Skyrme model. Since then, it has been shown that the Skyrme model has stable shell-like solutions [9] which can be well approximated by the so called rational map ansatz [10].

In this paper we use the rational map ansatz and its extension to multiple shells to construct configurations in the Gravitating Skyrme model that have a very large number of baryon. We show that those configurations, contrary to the hedgehog ansatz are bound even for very large baryon numbers.

To construct configurations that have a baryon number comparable to that of neutron star, we have to introduce a further approximation, which we call the ramp ansatz. We show
that this ansatz introduces further errors of only a few percent and we use it to compute very large Skyrmion configurations.

The paper is organised as follows: first we outline the Einstein-Skyrme model and discuss the main features of the results on static gravitating SU(2) hedgehogs obtained by Bizon and Chmaj [1]. We then use the rational map ansatz to construct shell-like gravitating multibaryon configurations and show that for a fixed value of the coupling constant, the configurations exist only when the baryon number is below a certain critical value. Finally we introduce a ramp profile approximation to construct solutions with extremely high baryon numbers. We show how accurate it is and use it to construct Skyrmion stars configuration.

2 The Einstein-Skyrme Model

The action for gravitating Skyrmions is formed from the standard Skyrme action for the matter field and the Einstein-Hilbert action for the gravitational field.

\[ S = \int_M \sqrt{-g} \left( \mathcal{L}_{Sk} - \frac{R}{16\pi G} \right) d^4x. \]  

(1)

Here \( \mathcal{L}_{Sk} \) is the Lagrangian density for the Skyrme model defined on the manifold \( M \):

\[ \mathcal{L}_{Sk} = \frac{F^2}{16} \text{Tr}(\nabla_\mu U \nabla^K U^{-1}) + \frac{1}{32e^2} \text{Tr}[(\nabla_\mu U)U^{-1}, (\nabla_\nu U)U^{-1}]^2, \]

(2)

where \( U \) belongs to SU(2). As we eventually wish to study baryon stars, we take a spherically symmetric metric, such as associated with the line element

\[ ds^2 = -A^2(r) \left( 1 - \frac{2m(r)}{r} \right) dt^2 + \left( 1 - \frac{2m(r)}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]

(3)

where \( A(r) \) and \( m(r) \) are two profile functions that must be determined by solving the Einstein equations for the model. Our choice of ansatz is motivated by the fact that although in some cases we will be studying non-spherical Skyrmion configurations, the regime we are primarily interested in (i.e. Skyrmions of extremely high baryon number) will be shown to admit quasi spherical solutions. Also, for realistic values of the couplings, the gravitational interaction is small compared to the Skyrme interaction and thus the use of a spherical metric even with non-spherical configurations, is not a great problem.

From (3), it can be shown that the Ricci scalar is

\[ R = \frac{-2}{Ar^2} \left( -A''r^2 - 2A'r + 2A''rm + A'm + 3A'r'm' + Am'' + 2Am' \right) \]  

(4)
which, after integrating various terms by parts and noting that asymptotic flatness requires both $A(r)$ and $m(r)$ to take a constant value at spatial infinity, reduces the gravitational part of the action to

$$S_{gr} = \int A(r) \left( \frac{-m'(r)}{G} \right) dr + \frac{m(\infty)}{G}. \tag{5}$$

For what follows, it will be convenient to scale to dimensionless variables by defining $x = e^{F_{\pi} r}$ and $\mu(x) = e^{F_{\pi} m(r)/2}$, resulting in one dimensionless coupling parameter for the model, $\alpha = \frac{\pi F_{\pi}^2}{G}$. We note that taking $F_\pi = 186\text{MeV}$ and $G = 6.72 \times 10^{-45}\text{MeV}^{-2}$, then the physical value of the coupling is $\alpha = 7.3 \times 10^{-40}$.

As the Skyrme field is an $SU(2)$ valued scalar field, at any given time one can think of it as a map from $\mathbb{R}^3$ to the $SU(2)$ manifold. Finite energy considerations impose that the field at spatial infinity should map to the same point on $SU(2)$, say the identity. Thus, one can simply think of the Skyrme field as a map between three-spheres. All such maps fall into disjoint homotopy classes characterised by their winding number. This winding number is a conserved topological charge because no continuous deformation of the field and thus no time evolution, can allow transitions between homotopy classes. It is this topological charge that is interpreted as the baryon number.

### 3 Gravitating Hedgehog Skyrmions

Gravitating Skyrmions were first studied by Bizon and Chmaj[1] who analysed the properties of static spherically symmetric gravitating $SU(2)$ skyrmions. Taking the Hedgehog Ansatz for the Skyrme field

$$U = \exp(i \sigma \cdot \hat{r} F(r)) \tag{6}$$

subject to the boundary conditions

$$F(r = 0) = B\pi \tag{7}$$
$$F(r = \infty) = 0 \tag{8}$$

where $B$ is the Baryon number associated with the Skyrmion configuration, they derived the Euler-Lagrange equation for the profiles $F(r)$, $A(r)$ and $m(r)$ and found that the model admit two branches of global solitonic solutions at each given baryon number, which annihilate at a critical value of the coupling parameter. Above $\alpha_{crit}$ no further solutions were found. In particular the value of the critical coupling decreased quite considerably with increasing baryon number as $\alpha_{crit} \approx 0.040378/B^2$. It appears that the existence of a
critical coupling does not signal the collapse of a Skyrmion to form a black hole. In fact
the metric factor \( S(x) = (1 - \frac{2\mu(x)}{x}) \) is non-zero at \( \alpha_{\text{crit}} \); there simply ceases to be any
stationary points of the action above the critical coupling.

The major problem with the ansatz (7) is that it leads to unstable solutions, i.e. for any
given value of \( \alpha \), \( M_{\text{ADM}}(B = N) > N M_{\text{ADM}}(B = 1) \). This is actually the case for the pure
Skyrme model as well where the hedgehog ansatz (7) with \( B > 1 \) does not correspond to the
lowest energy solution for the model. The solutions of the pure Skyrme model when \( B > 1 \)
are known not to be spherically symmetric[11] but are stable i.e. \( E(B = N) < N*E(B = 1) \).

It was actually shown by Houghton et al [10], [12] that the multi-baryon solutions of
the pure Skyrme model can be well approximated by the so called rational maps ansatz
which is a generalisation of the hedgehog ansatz. While not radially symmetric, the ansatz
separates its radial and angular dependence through a profile function and a rational map
respectively.

In the following sections we will generalise the construction of Houghton et al to approx-
imate the solution of the Einstein-Skyrme model.

4 The Rational Map Ansatz

The rational map ansatz introduced by Houghton et al.[10] works by decomposing the
field into angular and radial parts. Using the polar coordinates in \( \mathbb{R}^3 \) and defining the
stereographic coordinates \( z = \tan(\theta/2) \exp^{i\phi} \) the ansatz reads [10]

\[
U = \exp(i \vec{\sigma} \cdot \hat{n}_R F(r,t))
\]  

(9)

where

\[
\hat{n}_R = \frac{1}{1 + |R|^2} \left(2R(R), 2\Im(R), 1 - |R|^2\right)
\]  

(10)
is a unit vector where \( R \) is a rational function of \( z \).

It can be shown that the baryon number for Skyrmions constructed in this way, is equal
to the degree of the rational map providing we take the boundary conditions

\[
F(r = 0) = \pi
\]

\[
F(r = \infty) = 0.
\]  

(11)

Substituting the ansatz (10) into the action for the model and scaling to dimensionless
variables as earlier, we obtain the following reduced Hamiltonian
\[
H = \frac{16\pi F_e}{e} \int_0^\infty A(x) \left( \frac{1}{2} S(x) F(x)^2 x^2 + B \sin^2 F(x) (1 + S(x) F(x)^2) \right. \\
\left. + \frac{\mathcal{I} \sin^4 F(x)}{2x^2} \frac{x^2}{\alpha} + \frac{\mu'(x)}{\alpha} \right) dx + \mu(\infty) \tag{12}
\]
where
\[
S(x) = 1 - \frac{2\mu(x)}{x} \tag{13}
\]
From which one obtains the following field equations
\[
\mu'(x) = \alpha \left( \frac{1}{2} S(x) x^2 F(x)^2 + B \sin^2 F(x) + S(x) B F(x)^2 \sin^2 F(x) + \frac{\mathcal{I} \sin^4 F(x)}{2x^2} \right) \tag{14}
\]
\[
F''(x) = \frac{1}{S(x)V(x)} \left[ \sin 2F(x) \left( B + S(x) B F(x)^2 + \frac{\mathcal{I} \sin^2 F(x)}{x^2} \right) \right. \\
\left. \frac{\alpha S(x) F(x)^4 V(x)^2}{x} - S(x)' F(x)' V(x) - S(x) F(x)' V(x)' \right] \tag{15}
\]
and
\[
A'(x) = \alpha A(x) F(x)^2 \left( x + 2B \sin^2 F(x) \right) \tag{16}
\]
where, for convenience, we have defined \( V(x) \) as
\[
V(x) = x^2 + 2B \sin^2 F(x). \tag{17}
\]
\( B \) is the baryon number and
\[
\mathcal{I} = \frac{1}{4\pi} \int \left( \frac{1 + |z|^2}{1 + |R|^2} \left| \frac{dR}{dz} \right| \right)^4 \frac{2dxdz}{(1 + |z|^2)^2} \tag{18}
\]
Its value depends on the chosen rational map \( R \). To compute low energy configurations for a given baryon charge \( B \) one must find the rational map \( R \) or degree \( B \) that minimize \( \mathcal{I} \). This has been done in [10] and [11] for several values of \( B \). Moreover when \( b \) is large, one can use the approximation [11] \( \mathcal{I} \approx 1.28B^2 \). The value of \( \mathcal{I} \) so obtained is then used as a parameter and one can solve equations (14) - (16) for the radial profiles \( F(x), A(x) \) and \( \mu(x) \).

We should point out here that for the pure Skyrme model the rational map ansatz produce very good approximation to the multi skyrmion solutions [10]: the energies are only 3 or 4 percent higher and the energy densities exhibit the same symmetries and differ by very little. All the solutions computed by Battye and Sutcliffe [11], when \( B \) is not too small, have somehow the shape of a hollow shell. The baryon density is very small everywhere outside the shell, while on the shell itself, it forms a lattice of hexagons and pentagons.
5 Hollow Skyrmion Shells

Using the rational map ansatz, we will now solve the field equations (14) - (16) to compute some low action configurations. These solutions will correspond, initially, to a hollow shell of Skyrmions similar to the configuration obtained with the rational map ansatz for the pure Skyrme model. In the following sections we will show how our ansatz can be generalised to allow for more realistic configuration made out of embedded shells.

The first thing to note about our solutions is that we again obtain two branches of solutions at each baryon number (Fig. 1). Obtaining this same qualitative behaviour is not surprising when one considers that the $B = 1$ rational map Skyrmion reproduces the usual $B = 1$ hedgehog. However, the behaviour of the critical coupling itself is drastically altered for the rational map generated configurations. Namely, we observe that it decreases as approximately $0.040378/B^{1/2}$ (Fig. 2). In particular this means that for a given value of the coupling, the rational map generated skyrmions can possess a much higher topological charge than their hedgehog counterparts, before there ceases to be any solutions. Quantitatively if $B_{\text{hedgehog}}$ is the maximum baryon number for which hedgehog solutions can be found at a given value of the coupling, then the highest baryon number rational map solution found at the same value of $\alpha$ will be approximately $B_{\text{hedgehog}}^4$. Again we observe that the metric function $S(x)$ is non-zero at the critical coupling for all the solutions we have found and as such a horizon has not formed.

In Table 1 we present the radius, ADM mass per baryon and minimum value of the metric function, $S(x)$, for configurations up to the maximum baryon number allowed at $\alpha = 1 \times 10^{-6}$. These values were obtained by direct numerical solution of equations (14) - (16), where we have used the boundary data as specified in (11).

We didn’t use the physical value of $\alpha$ ($7.3 \times 10^{-40}$) because for this value, the ratio between the width of the shell and its radius is so small when we reach the maximum value of $B$ that it becomes very difficult to solve the equation reliably. The value $\alpha = 1 \times 10^{-6}$ is small enough to allow for a shell with a large baryon number to exist but large enough to make it possible to compute these solution nears the critical value of $B$ for a single shell configuration.
Figure 1: Plot of the two branches of solutions found for $B=2$ configurations generated with the rational map ansatz.

The major difference between these configuration and the solutions of Bizon and Chmadj is that the rational map ansatz configurations become more bound when the baryon number increases. This suggests the possibility that giant gravitating Skyrmions can be bound and consequently, that the Skyrme model can be used to study baryon stars.

Another interesting feature of the data is the observed change in the radius of the solutions with increasing baryon number. We note that the radius grows as approximately $B^{1/2}$. However there are two main deviations from this. Firstly, the constant of proportionality relating the radius to the square root of the baryon number decreases slightly but persistently as we increase the baryon number, indicating the gravitational interaction becoming more important as the number of baryons increases.

As we approach the maximum baryon charge that can exist at $\alpha = 1 \times 10^{-6}$, we also notice that the radius of the skyrmion actually decreases as we add more baryons. This shows that the gravitation pull plays a crucial role near the critical value of the skyrmion. This is a tantalising property when one considers that generally a neutron star’s radius must decrease for an increase in mass in order to achieve sufficient degenerate neutron pressure.
Figure 2: Plot of the decrease in $\alpha_{\text{crit}}$ with increasing baryon number, for configurations generated with the rational map ansatz. +: $\alpha_{c}\gamma$ for the minimum value of $I$; curve: $\alpha_{c}\gamma = 0.0404B^{-1/2}$.

To support the star.

To motivate the further approximation that we will introduce in the next section, we now look at the profiles of the configuration that we have computed. First of all, we observe that the profile function $F(x)$ stays approximately at its boundary value, $\pi$, for a finite radial distance before decreasing monotonically over some small region and finally attaining its second boundary value, 0. A similar behaviour is seen for both the mass field $\mu(x)$ and the metric field $A(x)$ (see Fig. 3). Furthermore, as we increase the baryon number, the structure becomes more pronounced, with the distance before the fields change (shell radius) increasing significantly, whilst the distance over which the fields change (shell width) settles to a constant size. We conclude that at large baryon numbers, those configurations correspond to hollow shells where the baryons are distributed on a tight lattice over the shell. As such the, structures are nearly spherical, validating our choice of radial metric.

Such structures immediately pose an interesting question. Can the gravitating Skyrmions exist as shells with more than one layer? To investigate this we note that it is possible to
\[ B \quad R\left(\frac{1}{F_{\pi}}\right) \quad M_{ADM}\left(\frac{F_{\pi}}{2\text{topconv}}\right) \quad S_{\text{min}} \]

| \( B \) | \( R\left(\frac{1}{F_{\pi}}\right) \) | \( M_{ADM}\left(\frac{F_{\pi}}{2\text{topconv}}\right) \) | \( S_{\text{min}} \) |
|-----|----------------|-----------------|--------|
| 1   | 0.8763         | 1.2315          | 1.0000 |
| 4   | 1.7728         | 1.1365          | 1.0000 |
| 8   | 2.5065         | 1.1180          | 1.0000 |
| 100 | 8.6829         | 1.0845          | 0.9999 |
| 500 | 19.3994        | 1.0827          | 0.9998 |
| \( 1 \times 10^3 \) | 27.4314 | 1.0825 | 0.9997 |
| \( 1 \times 10^4 \) | 86.7192 | 1.0821 | 0.9989 |
| \( 1 \times 10^5 \) | 274.0397 | 1.0814 | 0.9963 |
| \( 1 \times 10^6 \) | 864.6968 | 1.0792 | 0.9883 |
| \( 1 \times 10^7 \) | 2715.0729 | 1.0722 | 0.9628 |
| \( 1 \times 10^8 \) | 8377.4601 | 1.0500 | 0.88192 |
| \( 1 \times 10^9 \) | 23585.5315 | 0.9743 | 0.6107 |
| \( 1.5 \times 10^9 \) | 26860.2040 | 0.9463 | 0.5020 |
| \( 1.8 \times 10^9 \) | 27470.2449 | 0.9302 | 0.4256 |
| \( 1.81 \times 10^9 \) | 27456.5804 | 0.9296 | 0.4225 |
| \( 1.85 \times 10^9 \) | 27357.9201 | 0.9274 | 0.4090 |
| \( 1.9 \times 10^9 \) | 27078.6014 | 0.9246 | 0.3886 |
| \( 1.95 \times 10^9 \) | 26126.5508 | 0.9217 | 0.3517 |
| \( 1.951 \times 10^9 \) | 26050.7695 | 0.9217 | 0.3495 |
| \( 1.952 \times 10^9 \) | 25937.4210 | 0.9216 | 0.3463 |

Table 1: Properties of the one shell low energy configuration for \( \alpha = 1 \times 10^{-6} \)
modify the boundary condition (11) to read

\[ F(r = 0) = N\pi \] \hspace{1cm} (19)

\[ F(r = \infty) = 0 \] \hspace{1cm} (20)

whilst still ensuring that the Skyrme field is well defined at the origin. This idea was first used in [12] to construct two shell configurations for the pure Skyrme model.

The baryon charge is now \( N \) times the degree of the rational map. Fig. 4 shows the structure of the solutions we find in this case when \( N = 2 \). They suggest that the Skyrmion now exists as a \( N \)-layered structure. This is exhibited in the form of the profile, mass and metric functions which interpolate between the boundary values in \( N \) distinct steps of equal
size stacked next to each other.

We can therefore think of this as a naive way of constructing a gravitating Skyrmion. Instead of using the boundary conditions as in (11) and a rational map of degree $B$ we consider constructing the $B$-Skyrmion using a rational map of degree $B/N$ (with the associated value of $I$) and the boundary condition (20). This is a crude construction as we are effectively considering $N$ adjacent shells of baryons, all with the same baryon number. We might realistically expect that the baryon number per shell and distribution of shells may vary significantly for the minimum energy configuration. Nevertheless we shall study the properties of such structures. In fact, in the case where the baryon number is large and the number of shells is small, we expect this crude construction to be quite valid. That is, we do not expect the baryon number to change significantly over the few shells at large radius.

Figure 4: Numerical solutions for the profiles $F(x)$, $\mu(x)$ and $A(x)$ for 2 layers configurations ($F(0) = 2\pi$) when $B = 2 \times 10^6$ and $\alpha = 1 \times 10^{-6}$. 
\[ B \quad R \left( \frac{F_{\pi}}{F_{\text{ext}} \sigma} \right) \quad M_{\text{ADM}} \left( \frac{F_{\pi}}{F_{\text{ext}} \sigma} \right) \quad S_{\text{min}} \]

| \( B \)  | \( R \left( \frac{F_{\pi}}{F_{\text{ext}} \sigma} \right) \) | \( M_{\text{ADM}} \left( \frac{F_{\pi}}{F_{\text{ext}} \sigma} \right) \) | \( S_{\text{min}} \) |
|----|-----------------|-----------------|-------|
| 4  | 1.2898          | 1.6179          | 1.0000|
| 8  | 1.7858          | 1.4072          | 1.0000|
| 100 | 6.1754          | 1.1363          | 0.9999|
| \( 1 \times 10^3 \) | 19.4157        | 1.0913          | 0.9996|
| \( 1 \times 10^4 \) | 61.3207        | 1.0833          | 0.9985|
| \( 1 \times 10^5 \) | 193.7006       | 1.0812          | 0.9949|
| \( 1 \times 10^6 \) | 610.6271       | 1.0779          | 0.9835|
| \( 1 \times 10^7 \) | 1911.3704      | 1.0680          | 0.9475|
| \( 1 \times 10^8 \) | 5825.2626      | 1.0362          | 0.8325|
| \( 9.0 \times 10^8 \) | 13736.9982     | 0.9302          | 0.4258|
| \( 9.7 \times 10^8 \) | 13263.0853     | 0.9224          | 0.3644|
| \( 9.76 \times 10^8 \) | 12998.2817     | 0.9217          | 0.3480|
| \( 9.764 \times 10^8 \) | 12931.5189     | 0.9216          | 0.3444|
| \( 9.7647 \times 10^8 \) | 12895.6984     | 0.9216          | 0.3425|
| \( 9.76472 \times 10^8 \) | 12891.4247     | 0.9216          | 0.3423|
| \( 9.764724 \times 10^8 \) | 12889.8645     | 0.9216          | 0.3422|

Table 2: Table of properties of double layer solutions obtained numerically at \( \alpha = 1 \times 10^{-6} \)

For the remainder of this section we will restrict ourselves to the case where \( N = 2 \).

Table 2 summarises the properties of double layered gravitating skyrmions up to the maximum baryon charge allowed at \( \alpha = 1 \times 10^{-6} \). Briefly, we note the main features. Firstly, for all baryon numbers, the radius of the double layered solutions is significantly less than their single layered counterparts. This is not surprising as the baryon charge exists over a thicker region and so the mean radius can decrease with the baryon density remaining the same. Secondly, when \( B \) is large enough, i.e. when the double layer starts to make sense, the double layer solutions are energetically favourable when one compares the ADM mass with the single layer solutions. Finally we note that the maximum baryon number allowed (at the given coupling) is almost twice as much in the case of the single layer skyrmions.

Of course the results of this section are not really the main regime of interest. We clearly
need to study configurations of extremely high baryon number (of order $10^{58}$) relevant for baryon stars. We will now discuss this high baryon number regime.

6 The Ramp-profile Approximation

Unfortunately, at very high baryon numbers, eqns. (14) - (16) become difficult to handle numerically. This is largely because the radius of the solutions becomes much larger than the distance over which the fields change. That is, we need to integrate over a region which is much less than $10^{-16}$ radius, and so even double precision data types have insufficient precision.

Moreover, single shell configurations are not physically relevant and multiple shells will only yield configuration that looks like a star if the number of layers is very large, typically well over $10^{17}$. With such a large number of layers we won’t be able to solve the equation numerically as we will need at least 10 times as many sampling points for the profile functions. We must thus resort to another level of approximation: approximate the profile functions by profiles that are piecewise linear. This is inspired by the work of Kopeliovich [13] [14] except that our ansatz has to be piecewise linear to be able to generate configurations with a huge number of layers. After defining the ansatz for an arbitrary number of layers, we will show that for a single layer configuration the ansatz produces configurations that are in good agreements with the rational map ansatz configuration. Then we will use the new ansatz to construct configurations that are made out of a very large no of layers.

We have shown, in the previous section, that one can construct shell like structures with very large Baryon numbers. At large baryon numbers, the Skyrmions resemble shell like structures. That is, the fields are constant nearly everywhere except in a small region corresponding to the shell. In that region, the profile look like linear functions smoothly linked to the constant parts at the edges (cfr. Fig 3). Motivated by this we approximate the fields by the ramp-functions

$$F(x) = \frac{N\pi}{2} - (x - x_0)\frac{\pi}{W}, \quad (x_0 - NW/2) \leq x \leq (x_0 + NW/2) \quad (21)$$

$$\mu(x) = \frac{M}{2} + (x - x_0)\frac{M}{NW}, \quad (x_0 - NW/2) \leq x \leq (x_0 + NW/2) \quad (22)$$

$$A(x) = \frac{(1 + A_0)}{2} + (x - x_0)\frac{(1 - A_0)}{NW}, \quad (x_0 - NW/2) \leq x \leq (x_0 + NW/2) \quad (23)$$

In the above there are four free parameters, namely the central radius $x_0$ of the shell
over which the fields change, the width of the shell $W$, the mass field at spatial infinity $M$ and the value of the metric field at the origin $A_0$ such that $\lim_{x \to \infty} = 0$. $N$ is the number of layers we wish to study and, as such, is treated as an input parameter.

The picture is of a gravitating skyrmion with very high baryon number existing as $N$ thin layers or shell of small thickness.

The above ansatz, allow us to find an approximation to the integrated energy. To do this we use the fact that the shell width is much smaller than the radius at large baryon numbers. In particular to evaluate the action integral we can approximate expressions of the type $\int G(x) \sin^p F(x)$ for any function $G(x)$ that varies very little over the width of the shell by $\int G(x_0) \sin^p F(x)$. We then use the fact that

$$\int_{-\frac{N W}{2}}^{\frac{N W}{2}} \sin^p F(x) = \frac{N \pi}{W} \int_0^{\pi} \sin^p y \, dy.$$  

(24)

This leads to the following expression for the energy:

$$E = -\frac{16 \pi F_e}{e} \left[ \frac{M}{\alpha} \left( \frac{1 + A_0}{2} \right) - \frac{\pi^2}{W} \left( \frac{1 + A_0}{2} \right) \left( \frac{x_0^2}{2} - M x_0 + \frac{W^2}{12} - \frac{M W}{6} \right) + \left( \frac{1 - A_0}{W} \right) \left( \frac{W^2 x_0}{6} - \frac{M W^2}{12} - \frac{M W x_0}{6} \right) \right] + B \left( \frac{1 + A_0}{2} \right) \frac{1}{2} \left( \frac{M \pi^2}{W^2} - W - \frac{\pi^2}{W} \right) - \frac{32 W}{16 \pi^3} \left( \frac{1 + A_0}{2} \right) - \frac{M}{\alpha}.$$  

(25)

To find the configurations which minimize this energy we first minimised it with respect to $A_0$ and $M$ algebraically in order to find an expression for the energy as a function of the width and radius only. Then we minimised this numerically using Mathematica. We will now discuss the features of these configurations.

First of all, we must compare the results obtained with the ramp-profile when $N = 1$ and compare them to the result obtained with the full profile. Tables. 3 and 4 show the properties of solutions we obtained using the ramp-profile approximation, again at $\alpha = 1 \times 10^{-6}$. All the general features of the full numerical solutions are reproduced. In particular, the approximate $B^{\frac{1}{2}} \alpha$ scaling and then decrease of the radius, the decreasing ADM mass and the differences between the double and single layer solutions are all exhibited by the data obtained using the ramp-profile approximation.

Quantitatively though, there are some differences. The approximation allows a significant increase in the maximum allowed baryon charge. Also, the radius of configurations obtained using the approximation, tend to be smaller than those obtained numerically. If we concentrate on the baryon numbers greater than $10^5$ so as to ensure our approximation,
| $B$   | $R(\frac{M}{\pi e})$ | $W$  | $M_{ADM}(\frac{L}{\pi e})$ | $S_{min}$ |
|-------|----------------------|------|----------------------------|-----------|
| 100   | 8.3063               | 3.1286 | 1.1023                      | 0.9999    |
| 500   | 18.6031              | 3.1386 | 1.1160                      | 0.9997    |
| $1 \times 10^3$ | 26.313              | 3.1397 | 1.1195                      | 0.9996    |
| $1 \times 10^4$ | 83.206               | 3.1396 | 1.1254                      | 0.9987    |
| $1 \times 10^5$ | 262.94               | 3.1357 | 1.1266                      | 0.9960    |
| $1 \times 10^6$ | 829.60               | 3.1230 | 1.1251                      | 0.9872    |
| $1 \times 10^7$ | 2604.2               | 3.0825 | 1.1186                      | 0.9595    |
| $1 \times 10^8$ | 8032.8               | 2.9512 | 1.0972                      | 0.8713    |
| $1 \times 10^9$ | 22899                | 2.4837 | 1.0272                      | 0.5772    |
| $2 \times 10^9$ | 29121                | 2.1092 | 0.9818                      | 0.3645    |
| $2.8 \times 10^9$ | 29098                | 1.6623 | 0.9505                      | 0.1380    |
| $2.83 \times 10^9$ | 28514                | 1.6066 | 0.9495                      | 0.1119    |
| $2.839 \times 10^9$ | 28024                | 1.5671 | 0.94922                     | 0.09373   |
| $2.8397 \times 10^9$ | 27869.3              | 1.5556 | 0.94924                     | 0.08845   |
| $2.83975 \times 10^9$ | 27869.8              | 1.5524 | 0.94925                     | 0.08699   |
| $2.839752 \times 10^9$ | 27822                | 1.5521 | 0.94925                     | 0.08687   |

Table 3: Table of properties of the single layer step ansatz configurations for varying the baryon number at fixed $\alpha = 1 \times 10^{-6}$
Table 4: Table of properties of the double layer step ansatz configurations for varying the baryon number at fixed $\alpha = 1 \times 10^{-6}$

| $B$       | $R(\frac{2}{\pi r})$ | $W$     | $M_{ADM}(\frac{2}{\pi r})$ | $S_{\text{min}}$ |
|-----------|----------------------|---------|---------------------------|------------------|
| 100       | 5.7924               | 3.0428  | 1.0692                    | 0.9999           |
| $1 \times 10^3$ | 18.5788             | 3.1305  | 1.1047                    | 0.9995           |
| $1 \times 10^4$ | 58.8202             | 3.1380  | 1.1201                    | 0.9983           |
| $1 \times 10^5$ | 185.8420            | 3.1332  | 1.1246                    | 0.9944           |
| $1 \times 10^6$ | 585.7950            | 3.1153  | 1.1233                    | 0.9820           |
| $1 \times 10^7$ | 1833.0500           | 3.0578  | 1.1143                    | 0.9428           |
| $1 \times 10^8$ | 5587.3600           | 2.8688  | 1.0840                    | 0.8172           |
| $9 \times 10^8$ | 14147.1782          | 2.1859  | 0.9900                    | 0.4065           |
| $9.764724 \times 10^8$ | 14472.3851        | 2.1276  | 0.9837                    | 0.3746           |
| $1 \times 10^9$ | 14560.5000          | 2.1092  | 0.9818                    | 0.3646           |
| $1.4 \times 10^9$ | 14549.0000         | 1.6623  | 0.9505                    | 0.1381           |
| $1.41963 \times 10^9$ | 13994.0523        | 1.5644  | 0.9492                    | 0.0926           |
| $1.419635134 \times 10^9$ | 13993.2000       | 1.5643  | 0.9492                    | 0.0925           |
that the width is much smaller than the radius, is valid, then at worst we find a discrepancy in the ADM mass of 11% and in the radius of 7%.

In general then, the data seems to confirm the reliability of the ramp-profile approximation. In fact the approach will be even more reliable at the extremely high values of the baryon number that we are interested in. This is because the radius of solutions is of orders of magnitudes greater than the width in such a regime, consistent with the approximations we have made.

Moreover, whilst searching for minima of the energy does not allow us to probe both branches of solutions, it does allow us to locate the value of $\alpha_{\text{crit}}$. We again obtain the approximate trend $\alpha_{\text{crit}} \propto B^{-\frac{1}{2}}$, for large $B$.

Now in order to say anything about the possibility of baryon stars in the Skyrme model we need to be able to verify that the decrease in the ADM mass per baryon we observed at low and moderate baryon numbers, extends to baryon numbers of order $10^{58}$ for $\alpha = 7.3 \times 10^{-40}$.

Table. 5 summarizes our solutions in such a regime. Firstly we consider constructing a single layer self-gravitating Skyrmion with these values. We do indeed see that the configuration is bound. This is verified by checking that the ADM mass is lower (even at this significantly lower value of $\alpha$) than for the $B = 1$ hedgehog. So the possibility of baryon stars in the Einstein-Skyrme model cannot be ruled out on the grounds of energy.

The Skyrmion exists as a giant thin shell, and the large baryon charge is distributed as a tight lattice over this. However a hollow shell is clearly not a realistic construction for a neutron star. This fact manifests itself in the extremely high radius of the configuration. Transferring to standard units, the single layer $B = 10^{58}$ gravitating Skyrmion has a radius of $2.42 \times 10^{10}$ km!

To address this issue, we can use a large number of layered Skyrmions as discussed earlier. This has several benefits. Firstly, as we are distributing the baryon number through a larger volume, then at a given baryon density the necessary radius can decrease. Similar to what we see in the double layer results. On top of this, we expect the radius to decrease further due to extra gravitational compression, as the outer layers of the Skyrmions feel the attraction of inner layers. Finally, the many layer approach is also a more realistic construction of a solid baryon star.

The results for using more and more layers in the construction (for fixed $B$ and $\alpha$), are also presented in Fig. 5. We note that not only does the radius decrease significantly, but
the added gravitational binding further improves the energies of the configurations, reflected in the low ADM masses obtained. There appears to be a critical number of layers that can be used before there ceases to be any solutions and although the value of $S_{\text{min}}$ is close to zero at this point, the star still has not collapsed to form a black hole. Finally, we note that the radius of the Skyrmion at the critical number of layers is approximately $20.91\text{km}$. This is comparable to a real neutron star, with a typical radius of $10\text{km}$.

We reemphasise here that our approach to embedding shells of baryons is quite crude. For few shells and large baryon number, we might reasonably believe that baryon number does not change significantly from one shell to the next. However, when we embed many shells we should really consider that the baryon number of the inner most shells would likely be significantly less than that of the outer shells. Nevertheless, our naive embedding has produced some interesting properties. In a future work we hope to improve our multi-layer construction to obtain a more realistic description of a baryon star.
| $N_{\text{Shell}}$ | $R(\frac{2}{\pi})$ | $W$ | $M_{\text{ADM}}/(6\pi^2B)$ | $S_{\text{min}}$ |
|-----------------|------------------|-----|---------------------|--------|
| $1 \times 10^2$ | $8.3236 \times 10^{27}$ | 3.1416 | 1.1285 | 1.0000 |
| $1 \times 10^3$ | $2.6321 \times 10^{27}$ | 3.1416 | 1.1285 | 1.0000 |
| $1 \times 10^4$ | $8.3236 \times 10^{26}$ | 3.1416 | 1.1285 | 1.0000 |
| $1 \times 10^5$ | $2.6321 \times 10^{26}$ | 3.1416 | 1.1285 | 1.0000 |
| $1 \times 10^6$ | $8.3236 \times 10^{25}$ | 3.1416 | 1.1285 | 1.0000 |
| $1 \times 10^7$ | $2.6321 \times 10^{25}$ | 3.1416 | 1.1285 | 1.0000 |
| $1 \times 10^8$ | $8.3236 \times 10^{24}$ | 3.1416 | 1.1285 | 1.0000 |
| $1 \times 10^9$ | $2.6321 \times 10^{24}$ | 3.1415 | 1.1285 | 1.0000 |
| $1 \times 10^{10}$ | $8.3234 \times 10^{23}$ | 3.1415 | 1.1285 | 0.9999 |
| $1 \times 10^{11}$ | $2.6319 \times 10^{23}$ | 3.1412 | 1.1285 | 0.9997 |
| $1 \times 10^{12}$ | $8.3216 \times 10^{22}$ | 3.1402 | 1.1283 | 0.9991 |
| $1 \times 10^{13}$ | $2.6301 \times 10^{22}$ | 3.1373 | 1.1278 | 0.9971 |
| $1 \times 10^{14}$ | $8.3034 \times 10^{21}$ | 3.1280 | 1.1263 | 0.9907 |
| $1 \times 10^{15}$ | $2.6118 \times 10^{21}$ | 3.0986 | 1.1213 | 0.9705 |
| $1 \times 10^{16}$ | $8.1147 \times 10^{20}$ | 3.0037 | 1.1057 | 0.9063 |
| $1 \times 10^{17}$ | $2.4001 \times 10^{20}$ | 2.6810 | 1.0552 | 0.6977 |
| $5 \times 10^{17}$ | $8.2066 \times 10^{19}$ | 1.7888 | 0.9552 | 0.2036 |
| $5.3 \times 10^{17}$ | $7.4172 \times 10^{19}$ | 1.6227 | 0.9491 | 0.1247 |
| $5.33 \times 10^{17}$ | $7.1871 \times 10^{19}$ | 1.5625 | 0.94866 | 0.0971 |
| $5.3306 \times 10^{17}$ | $7.1597 \times 10^{19}$ | 1.5549 | 0.94868 | 0.0936 |
| $5.33065 \times 10^{17}$ | $7.1525 \times 10^{19}$ | 1.5528 | 0.948694 | 0.0927 |
| $5.330657 \times 10^{17}$ | $7.1506 \times 10^{19}$ | 1.5523 | 0.948692 | 0.0924 |

Table 5: Table of properties of the step ansatz configurations for varying the number of embedded shells at fixed $B = 10^{58}$ and $\alpha = 7.3 \times 10^{-40}$.  

7 Conclusions

Previous work on the Einstein-Skyrme model highlighted a considerable problem with using the Skyrmions as a model for baryon stars. Namely, multibaryon hedgehog Skyrmions were simply not energetically favourable states. We have shown that this is simply a consequence of a poor ansatz for the true Skyrmion and, having used the more appropriate rational map ansatz, we have generated energetically favourable configurations of multibaryons.

We also observe the interesting property that near the critical coupling, the Skyrmions can decrease in radius as we add more baryons. This hints towards the similar behaviour exhibited by real neutron stars.

Although the rational map ansatz does not have an exact radial symmetry, at large scale it does. The anisotropy only appears at the nucleon scale.

Finally, since we started with the motivation of studying baryon stars within the Skyrme model, it is interesting to compare the features of our configurations with those of neutron stars. For realistic values, $B = 10^{58}$ and $\alpha = 7.3 \times 10^{-40}$ we find a minimal energy single layer configuration with radius=$2.42 \times 10^{10}$km. This is clearly too large for a neutron star (which is of order $10\text{km.}$ in radius). This is to be expected however due to the shell model we have taken. Firstly, as we are distributing the baryons over the surface area rather than throughout the volume of the star we naturally must require a much larger star for a given baryon number. This effect is two-fold in that if we were distributing the baryons throughout the volume, outer layers would feel the attraction of inner layers and enhanced radial compression would occur. The loss of such an effect is pronounced when we are considering realistically small values of the coupling.

It seems therefore that the way to construct baryon stars in the Skyrme model is to consider embedding shells of baryons within shells. This gives rise to more appropriate specifications for the star and is also more realistic. We do indeed observe such improvements for a many layered configuration. In fact the radius of $B = 10^{58}$ gravitating Skyrmion (at realistic $\alpha$), can be decreased in this manner to approximately $20.91\text{km.}$

We note however that this approach to shell embedding has only been done naively thus far. We have only considered the case where the baryon number is equal for each shell. We really should allow the baryon number(and hence the rational map quantities) to vary over the shells. One approach towards this would be to assume that the baryon density is a constant over the shells. An even better approach would be to allow this to be a smoothly varying function that must be determined by minimising the energy. This will give a more realistic
description of baryon stars within the Einstein-Skyrme model, as traditional descriptions of neutron stars also involve many strata, of differing neutron density. We are currently investigating such configurations.

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