Deterministic Leader Election in $O(D + \log n)$ Time with Messages of Size $O(1)$

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Abstract. This paper presents a distributed algorithm, called $S\!ST\!T\!T$, for electing deterministically a leader in an arbitrary network, assuming processors have unique identifiers of size $O(\log n)$, where $n$ is the number of processors. It elects a leader in $O(D + \log n)$ rounds, where $D$ is the diameter of the network, with messages of size $O(1)$. Thus it has a bit round complexity of $O(D + \log n)$. This substantially improves upon the best known algorithm whose bit round complexity is $O(D \log n)$. In fact, using the lower bound by Kutten et al. [26] and a result of Dinitz and Solomon [17], we show that the bit round complexity of $S\!ST\!T\!T$ is optimal (up to a constant factor), which is a step forward in understanding the interplay between time and message optimality for the election problem. Our algorithm requires no knowledge on the graph such as $n$ or $D$.

1 Introduction

The election problem in a network consists of distinguishing a unique node, the leader, which can subsequently act as coordinator, initiator, and more generally performs some special role in the network (see [40] p. 262). Once a leader is established, many problems become simpler. For this reason, election algorithms are often considered as building blocks for other distributed algorithms and the election problem, together with consensus, has probably been the most studied task in distributed computing literature [15], starting with the works of Le Lann [29] and Gallager [19] in the late 70’s.

A distributed algorithm solves the election problem if it always terminates and in the final configuration exactly one process (or node) is in the elected state and all others are in the non-elected state. It is also required that once a process becomes elected or non-elected, it remains so for the rest of the execution. The vast body of literature on election (see [4,30,36,41] and references therein) actually covers a number of different topics. They include the feasibility of deterministic election in anonymous networks, starting with the seminal paper of Angluin [2] and the key role of coverings; the complexity of deterministic election in networks with identifiers; and the complexity of probabilistic election in anonymous (or sometimes identified) networks.

The present work is in the second category. We assume that each node has a unique identifier which is a positive integer of size $O(\log n)$, and the nodes exchange messages with their neighbours in synchronous rounds. The exact complexity of deterministic leader election in this setting has proven elusive for decades and even some simple

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questions remain open [26]. Assuming the size of messages is logarithmic (i.e. messages of size $O(\log n)$), we know since Peleg [33] that $O(D)$ rounds are sufficient to elect a leader in arbitrary networks. This was recently proven optimal by Kutt et al. [26] using a very general $\Omega(D)$ lower bound (that applies even in the probabilistic setting). Independently, Fusco and Pelc [18] showed that the time complexity of leader election is $\Omega(D + \lambda)$ where $\lambda$ is the smallest depth at which some node has a unique view, called the level of symmetry of the network. (The view at depth $t$ from a node is the tree of all paths of length $t$ originating at this node.) If nodes have unique identifiers, then $\lambda = 0$, which indeed implies the same $\Omega(D)$ bound as in [26].

Regarding message complexity, Gallager [19] presents the first election algorithm for general graphs with $O(m + n \log n)$ messages, where $m$ is the number of edges, and whose running time is $O(n \log n)$. Santoro [35] proves a matching $\Omega(m + n \log n)$ lower bound for the number of messages. A few years later, Awerbuch [5] presents an algorithm whose message complexity is again $O(m + n \log n)$, but time complexity is taken down to $O(n)$.

A number of questions remain open for election. Peleg asks in [33] whether an algorithm could be both optimal in time and in number of messages. The answer depends on the setting, but remains essentially open [26]. In the conclusion of their paper, Fusco and Pelc [18] also observe that it would be interesting to investigate other complexity measures for the leader election problem, such as bit complexity. This measure can be viewed as a natural extension of communication complexity (introduced by Yao [44]) to the analysis of tasks in a distributed setting.

Following [24], the bit round complexity of an algorithm $\mathcal{A}$ is the total number of bit rounds it takes for $\mathcal{A}$ to terminate, where a bit round is a round with single bit messages. This measure has become popular recently, as it captures into a single quantity aspects that relate both to time and to the amount of information exchanged. In this framework, the time-optimal algorithm of Peleg [33] results in a bit round complexity of $O(D \log n)$ (i.e. $O(D)$ rounds with $O(\log n)$ message size), and the message-optimal algorithm of [5] results in a $O(n \log n)$ bit round complexity (i.e. $O(n)$ time with $O(\log n)$ message size).

In this paper, we present a bit round complexity optimal leader election algorithm for synchronous arbitrary networks. Precisely, our algorithm requires $O(D + \log n)$ bit rounds, and we show that this is optimal by combining the lower bound from [26] and a recent communication complexity result by Dinitz and Solomon [17]. This algorithm is thus a step forward in the knowledge of the election problem, and a partial answer to the question of whether an algorithm can be both optimal in time and in the amount of information exchanged. (This question is perhaps more general than that of measuring time on the one hand, and the number of messages of a given size on the other hand.) In this respect, our result illustrates the benefits of studying optimality under the unified lenses of bit complexity.
1.1 Contributions

We present an election algorithm, denoted $STT$, using messages of size $O(1)$ and having a time complexity of $O(D + \log n)$, where $D$ is the diameter of the network. Algorithm $STT$ solves the explicit (i.e. strong) variant of the problem defined in [26], namely, the identifier of the elected node is eventually known to all the nodes. It also fulfills requirements from [17], such as ensuring that every non-leader node knows which local link is in direction of the leader, and these nodes learn the maximal id network-wide ($MaxF$), as a by-product of electing this specific node in the explicit variant.

The general architecture of our algorithm follows the same abstract principle as many election algorithms, such as those of Gallager [19] or Peleg [33]. That is, it relies on a competition of spanning tree constructions that works by extinction of those trees originating at nodes with lower identifiers (see Algorithm 4 in [4] and discussion therein). Eventually a single spanning tree survives, whose root is the node with highest identifier. This node becomes elected when it detects termination (recursively from the leaves up the root). Difficulty arises from designing such algorithms with the extra constraint that only constant size messages must be used. Of course, one might simulate $O(\log n)$-size messages in the obvious way paying $O(\log n)$ bit rounds for each message. But then, the bit round complexity would remain $O(D \log n)$. Our algorithm takes it down to $O(D + \log n)$.

For ease of exposition, we split the $STT$ algorithm into three components described below, whose execution is however joint in a specific way.

1. A spreading algorithm $S$ which pipelines the maximal identifier bitwise to each node, in a mix of battles (comparisons), conquests (progress of locally higher prefixes), and correction waves whose amplitude is fortunately bounded;
2. A spanning tree algorithm that executes in parallel of $S$ and whose union with $S$ is denoted $ST$. It essentially consists in updating the tree relations depending on what neighbour brought the highest prefix so far;
3. A termination detection algorithm that executes in parallel of $ST$ and whose union with $ST$ is denoted $STT$. This component enables the node with highest identifier (and only this one) to detect termination of the spanning tree construction rooted whose root it is.

An extra component can be added to broadcast a (constant size) termination signal from the root down the tree, once election is complete. This component is trivial and therefore not described here.

**Lower Bound:** Dinitz and Solomon [17] prove a lower bound (Theorem 1 below) on the leader election problem among two nodes.

**Theorem 1 ([17]).** Let $M$ be an integer such that $M \geq 2$. Let $G$ be the graph with two nodes linked by an edge each node has a unique identifier taken from the set $\mathbb{Z}_M = \{0, 1, \ldots, M-1\}$.
The bit round complexity of the Leader task and of the MaxF version is exactly $2\lceil \log_2((M + 2)/3.5) \rceil$.

This theorem implies that the time complexity of an election algorithm with messages of size $O(1)$ is $\Omega(\log n)$, and thus the bit round complexity of Algorithm $\text{STT}$ is $\Omega(\log n)$.

On the other hand, the lower bound by Kutten et al. in [25], establishing that $\Omega(D)$ time is required with logarithmic size messages, obviously extends to constant size messages. Put together, these results imply that the bit complexity of leader election with messages of size $O(1)$ and identifiers of size $O(\log n)$ is $\Omega(D + \log n)$, which makes our algorithm bit-optimal (up to a constant factor).

In fact, the lower bound holds for arbitrary sizes $|id|$ of the identifiers (necessarily larger than $\log n$, though, since they are unique). Likewise, the complexity of our algorithm is formulated relative to identifiers of arbitrary sizes (see Theorem 25). Hence, the bit round complexity of the election problem is in fact $\Theta(D + |id|)$.

Table 1 summarises the main elements.

|                | Time   | Number of messages | Size of messages | Bit round complexity |
|----------------|--------|--------------------|-----------------|----------------------|
| Awerbuch [5]   | $O(n)$ | $\Theta(m + n \log n)$ | $O(\log n)$ | $O(n \log n)$ |
| Peleg [33]     | $\Theta(D)$ | $O(Dm)$             | $O(\log n)$ | $O(D \log n)$ |
| This paper     | $O(D + \log n)$ | $O((D + \log n)m)$ | $O(1)$ | $\Theta(D + \log n)$ |

Table 1. Best known solutions in terms of time and number of messages, compared to our algorithm.

Outline: The paper is organised as follows. Section 2 starts by providing general definitions. Then Section 3, 4 and 5 present the three components of the algorithm, which are the spreading algorithm $S$, its joint use with the spanning tree algorithm $(ST)$, and the adjunction of termination detection $(STT)$. Extra literature on the election problem is provided in Section 6. We conclude in Section 7 with some remarks.

2 Model and definitions

2.1 The Network

We consider a failure-free message passing model for distributed computing. The communication model consists of a point-to-point communication network described by a connected graph $G = (V, E)$ where the nodes $V$ represent network processes (or nodes) and the edges $E$ represent bidirectional communication channels. Processes communicate by message passing: a process sends a message to another by depositing the message in the corresponding channel.

Let $n$ be the size of $V$. We assume that each node $u$ is identified by a unique positive integer of $O(\log n)$ bits, called identifier and denoted $Id_u$ (in fact, $Id_u$ denotes both the
identifier and its *binary representation*. We do not assume any global knowledge on the network, not even the size or an upper bound on the size, neither do the nodes require position or distance information. Every node is equipped with a port numbering function (i.e. a bijection between the set of incident edges $I_u$ and the integers in $[1, |I_u|]$), which allows it to identify which channel a message was received from, or must be sent to. Two nodes $u$ and $v$ are said to be neighbours if they can communicate through a port.

Finally, we assume the system is fully synchronous, namely, all processes start at the same time and time proceeds in synchronised rounds composed of the following three steps:

1. Send messages to (some of) the neighbours,
2. Receive messages from (some of) the neighbours,
3. Perform local computation.

The time complexity of an algorithm is the number of such rounds needed to complete the execution in the worst case.

### 2.2 Further definitions

The paper uses a number of definitions from graph theory and formal language theory. Although most readers may be familiar with them, we remind the most important ones. Next we define the bit round complexity.

**Definitions on graphs:** These definitions are selected from [34] (Chapter 8). A tree is a connected acyclic graph. A rooted tree is a tree with one distinguished node, called the root, in which all edges are implicitly directed away from the root. A spanning tree of a connected graph $G = (V, E)$ is a tree $T = (V, E')$ such that $E' \subseteq E$. A forest is an acyclic graph. A spanning forest of a graph $G = (V, E)$ is a forest whose node set is $V$ and edge set is a subset of $E$. A rooted forest is a forest such that each tree of the forest is rooted. A child of a node $u$ in a rooted tree is a node that is the immediate successor of $u$ on a path from the root. A descendant of a node $u$ in a rooted tree is $u$ itself or any node that is a successor of $u$ on a path from the root. The parent of a node $u$ in a rooted tree is a node that is the immediate predecessor of $u$ on a path to $u$ from the root.

**Definitions on languages:** These definitions are selected from [34] (Chapter 16). Let $A$ be an alphabet, $A^*$ is the set of all words over $A$, the empty word is denoted by $\epsilon$. If $x$ is a non empty word over the alphabet $A$ of length $p$ then $x$ can be written as the concatenation of $p$ letters, i.e., $x = x[1]x[2]\cdots x[p]$ with each $x[i]$ in $A$. If $a \in A$ and $i$ is a positive integer then $a^i$ is the word equal to the concatenation $i$ times of the letter $a$. Let $x$ and $y$ be two words over alphabet $A$, $x$ is said to be a prefix (resp. proper prefix) of $y$ if there exists a word (resp. non-empty word) $z$ such that $y = xz$. 

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**Bit round complexity:** The bit complexity in general may be viewed as a natural extension of communication complexity (introduced by Yao [44]) to the analysis of tasks in a distributed setting. An introduction to the area can be found in Kushilevitz and Nisan [25]. In this paper, we follow the definition from [24], that is, the bit round complexity of an algorithm $A$ is the total number of *bit rounds* it takes for $A$ to terminate, where a bit round is a synchronous round with single bit messages. This measure captures into a single quantity aspects that relate both to time and to the amount of information exchanged. Various definitions are considered in the literature, e.g. in [6,7,8,16] the bit complexity is the total number of bits sent until every node has completed its computation. In [38], bit complexity is the maximum number of bits sent through a same channel. In both variants, silences may convey much information, while not being measured as a cost. In contrast, the definition from [24] in terms of *round* complexity accounts for silences and therefore seems to be a more comprehensive measure.

3 A spreading algorithm

This section presents a distributed spreading algorithm using only messages of size $O(1)$ which allows each node to know the highest identifier among the set of all identifiers with a time complexity of $O(D + \log n)$, where $D$ is the diameter of $G$.

3.1 Preamble

Given a node $u$ and the binary representation $Id_u$ of its identifier. We define $\alpha(Id_u)$ as the word

$$\alpha(Id_u) = 1^{\|Id_u\|}0Id_u.$$ 

For instance, if $u$ has identifier 23, then $Id_u = 10111$ and $\alpha(Id_u) = 11111010111$. This encoding has the nice property that it extends the natural order $<$ of integers into a lexicographic order $\prec$ on their $\alpha$-encoding (Remark 2).

**Remark 2.** Let $u$ and $v$ be two nodes of $G$ whose identifiers are $Id_u$ and $Id_v$. Then:

$$Id_u < Id_v \Leftrightarrow \alpha(Id_u) < \alpha(Id_v).$$

As a result, the order between two identifiers $Id_u$ and $Id_v$ is the order induced by the first letter which differs in $\alpha(Id_u)$ and $\alpha(Id_v)$. This property is key to our algorithm, in which the spreading of identifiers progresses bitwise and comparisons occur consistently.

3.2 The algorithm $S$

**Variables:** Each node can be *active* or *follower*, depending on whether this node is still a candidate for becoming the leader (i.e. no higher identifier was detected so far). Each node $u$ also has variables $Y_u$, $Z_u$ and $Z^v_u$ (one for each neighbour $v$ of $u$) which are
words over the alphabet \{0, 1\}. \(Y_u\) is a shorthand for \(\alpha(Id_u)\), it is set initially and never changes afterwards. \(Z_u\) is a prefix of \(Y_u\), for some node \(w\) (possibly \(u\) itself). It indicates the highest prefix known so far by \(u\). On each node, this variable will eventually converge to the \(\alpha\)-encoding of the highest identifier. Finally, for each neighbour \(v\) of \(u\), \(Z_u^v\) is the lastest value of \(Z_v\) known to \(u\).

**Initialisation:** Initially every node \(u\) is active, all the \(Z_u\)'s are set to the empty word \(\epsilon\), and the \(Z_u^v\)'s are accordingly set to the empty word (wlog, we assume that a preliminary round made it possible for all nodes to know what neighbours they have).

**Main loop:** In each round, the algorithm consists of the following actions.

1. update \(Z_u\),
2. send to all neighbours a signal indicating how \(Z_u\) was updated,
3. receive such signals from neighbours,
4. update all the \(Z_u^v\) accordingly.

The main action is the update of \(Z_u\) (step 1). It depends on the values of \(Z_u^v\) for all neighbours \(v\) and \(Z_u\) itself at the end of the previous round. This update is done according to a number of rules. For instance, as long as \(u\) remains active and \(Z_u\) is a proper prefix of \(Y_u\), the update will simply consist in appending the next bit of \(Y_u\) to \(Z_u\). Most updates are however more complex and we described them in a dedicated paragraph below. The three other actions (step 2, 3, and 4 above) only serve the purpose of informing the neighbours as to how \(Z_u\) was updated, so that all \(Z_u^v\) are correctly updated. In fact, \(Z_u\) can only be updated in seven possible ways, each causing the sending of a particular signal among \{append0, append1, delete1, delete2, delete3, change, null\}, with following meaning:

- **append0** or **append1**: \(Z_u\) was updated by appending a single 0 or a single 1;
- **delete1**, **delete2**, or **delete3**: \(Z_u\) was updated by deleting one, two or three letters from the end;
- **change**: \(Z_u\) was updated by changing the last letter from 0 to 1;
- **null**: \(Z_u\) was not modified.

Based on these signals, each node can update its variables \(Z_u^v\) consistently (step 4).

**Remark 3.** By the end of each round, it holds that \(Z_u^v = Z_v\) for any neighbour \(v\) of \(u\). Thus from now on, \(Z_u^v\) is simply written \(Z_v\).

We now describe the way \(Z_u\) is updated by each node \(u\). One property that the update guarantees is that by the end of each round, if \(u\) and \(v\) are two neighbours, then \(Z_u\) and \(Z_v\) must have a common prefix followed, in each case, by at most six letters. This fact is later used for analysis.
The computation of $\mathcal{Z}$ of round $t$ results from $u$ being active or follower, and the values of $\mathcal{Z}^{t-1}_u$ and $\mathcal{Z}^{t-1}_v$ for all neighbours $v$ of $u$. It is done according to the following rules given in order of priority, i.e., $R_{1.1}$ has a higher priority than $R_{1.2}$, having itself a higher priority than $R_{2}$, etc. Whenever a rule is applied, the subsequent rules are ignored.

- $R_1$ (delete). The relationship between $\mathcal{Z}^{t-1}_u$ and $\mathcal{Z}^{t-1}_v$ for any neighbour $v$ of $u$ may mean that a delete operation is possible. If any delete is possible, one will be carried out; if more than one is possible, the greatest will be carried out.
  - $R_{1.1}$ If some $\mathcal{Z}^{t-1}_v$ is a proper prefix of $\mathcal{Z}^{t-1}_u$ and $v$’s last action was a delete, delete $\min\{|\mathcal{Z}^{t-1}_u|-|\mathcal{Z}^{t-1}_v|,3\}$ letters from the end of $\mathcal{Z}^{t-1}_u$.
  - $R_{1.2}$ If $\mathcal{Z}^{t-1}_u = z0x$ with $x \neq \epsilon$ and some $\mathcal{Z}^{t-1}_v = z1y$, delete $|x|$ letters from the end of $\mathcal{Z}^{t-1}_u$.

- $R_2$ (change). If $\mathcal{Z}^{t-1}_u = z0$ and some $\mathcal{Z}^{t-1}_v = z1y$ then change $\mathcal{Z}^{t-1}_u$ to $z1$ and change $u$’s state to follower if it is active;

- $R_3$ (append). if for some $v$, $\mathcal{Z}^{t-1}_v = \mathcal{Z}^{t-1}_u1x$, then $\mathcal{Z}^{t}_u$ is obtained by appending 1 to $\mathcal{Z}^{t-1}_u$;

- $R_4$ (append). if for some $v$, $\mathcal{Z}^{t-1}_v = \mathcal{Z}^{t-1}_u0x$, then $\mathcal{Z}^{t}_u$ is obtained by appending 0 to $\mathcal{Z}^{t-1}_u$;

- $R_5$ (append). if $u$’s state is active and $t < |\mathcal{Y}_u|$, append $\mathcal{Y}_u[t]$ to $\mathcal{Z}^{t}_u$.

If none of these actions apply, then $\mathcal{Z}_u$ remains unchanged and a null signal is sent. Otherwise, a signal corresponding to the resulting action is sent. We now prove some properties on Algorithm $\mathcal{S}$.

**Lemma 4.** Whenever a node $u$ carries out a delete operation at round $t$, $u$’s operation at round $t+1$ must be another delete operation or a change operation.

**Proof.** By induction on $t$. The delete operation actually carried out at round $t$ on $u$ was made possible by one or more neighbours of $u$ according to rule 1. Let $v$ be one such neighbour.

- If $R_{1.1}$ was applied at round $t$ on $u$:
  - $\mathcal{Z}^{t}_v$ is a proper prefix of $\mathcal{Z}^{t}_u$,
  - $v$ did a delete at round $t-1$.

Thus $\mathcal{Z}^{t}_u = \mathcal{Z}^{t}_v d$ (for some non empty $d$), and $\mathcal{Z}^{(t+1)}_u$ is obtained from $\mathcal{Z}^{t}_u$ by erasing at most $|d|$ letters at the end, i.e., $\mathcal{Z}^{(t+1)}_u = \mathcal{Z}^{t}_v d'$ for some $d'$.

By induction, $v$’s action at round $t$ is another delete operation or a change.

- If it is a delete operation, then $\mathcal{Z}^{(t+1)}_u$ is again obtained by erasing some letters at the end of $\mathcal{Z}^{t}_v$ thus it is a proper prefix of $\mathcal{Z}^{t}_u$ and a proper prefix of $\mathcal{Z}^{t}_v d' = \mathcal{Z}^{(t+1)}_u$ and $R_{1.1}$ applies again at $(t+1)$ on $\mathcal{Z}^{(t+1)}_u$.
• If it is a change operation then \( Z_v^t = w0 \) (for some \( w \)), \( Z_v^{t+1} = w1 \). Finally, \( Z_u^{t+1} = Z_u^t d' = w0d' \) and either \( d' \) is a non empty word and \( R_{1.2} \) applies on \( Z_u^{(t+1)} = Z_u^t d' = w0d' \), or \( d' \) is the empty word and \( R_2 \) applies with \( y = \epsilon \) on \( Z_u^{(t+1)} = Z_u^t = w0 \): it is a change.

– Otherwise \( R_{1.2} \) was applied at round \( t \) on \( u \):
  • \( Z_u^t = w0x = w0 \) with \( x \neq \epsilon \),
  • \( Z_v^t = y1y \) for some \( v \), and
  • \( Z_u^{(t+1)} = w0 \) by the delete operation at round \( t \).

Then:
• If the operation at round \( t \) on \( Z_v^t \) was a delete operation, according to at least \( 1y \) is deleted or not, the operation on \( Z_u^{(t+1)} \) at round \( (t + 1) \) is a delete or a change.
• If the operation at round \( t \) on \( Z_v^t \) is a change or an append then the operation on \( Z_u^{(t+1)} \) at round \( t + 1 \) is also a change (unless a delete applies the erase of another \( v \)).

\( \square \)

Lemma 4 induces immediately:

**Corollary 5.** A sequence of delete operations on a node \( u \) ends with a change operation on \( u \).

**Remark 6.** If a node \( u \) applies \( R_{1.1}, R_{1.2}, R_2, R_3, \) or \( R_4 \) then there exists a node \( v \) such that \( Y_u \prec Y_v \).

**Remark 7.** Let \( u \) be a node. If there exists a neighbour \( v \) of \( u \) and a round \( t \) such that \( |Z_u^t| < |Z_v^t| \) then \( u \) becomes follower.

**Lemma 8.** Let \( u \) and \( v \) be two neighbours. Let \( t \) be a round number. The words \( Z_u^t \) and \( Z_v^t \) will always take one of the following forms (up to renaming of \( u \) and \( v \)) where \( p \) and \( w \) are words and \( a \) is \( 1 \) or \( 0 \):

1. \( Z_u^t = p \) and \( Z_v^t = p \),
2. \( Z_u^t = p \) and \( Z_v^t = pw \) with \( 1 \leq |w| \leq 2 \),
3. \( Z_u^t = p0 \) and \( Z_v^t = p1a \),
4. \( Z_u^t = p1 \) and \( Z_v^t = p0w \) and \( |w| \leq 3 \),
5. \( Z_u^t = p \) and \( Z_v^t = pw \) and \( 3 \leq |w| \leq 6 \) and \( u \) has just performed a delete.

**Proof.** By induction on \( t \).

At round \( t = 0 \), \( Z_u^t = \epsilon \) and \( Z_v^t = \epsilon \).

Without loss of generality, we will always consider the form given in the lemma and not the reverse.

We note that the induction implies that if \( R_{1.2} \) is applied then necessarily \( y = \epsilon \) and \( |x| \leq 3 \) (item 4).

We consider the five possible relations between \( Z_u^{t-1} \) and \( Z_v^{t-1} \) and show that in each case \( Z_u^t \) and \( Z_v^t \) still have one of the five forms.
Corollary 9. If $R_{1.2}$ is applied then $0 < |x| \leq 3$ and $y = \epsilon$.

Lemma 8 implies:
Theorem 10. Let $G$ be a graph of size $n$ and diameter $D$ such that each node $u$ is endowed with a unique identifier $Id_u$ which is a non negative integer. Let $X$ be the highest identifier. After at most $|\alpha(X)| + 6D$ rounds, algorithm $S$ terminates and for each node $u$, $Z_u = \alpha(X)$.

Proof. Let $u_0$ be the node endowed with the highest identifier. Let $k$ be a non negative integer. By induction on $k$ we prove that after at most $|\alpha(X)| + 6k$ rounds each node at distance at most $k$ from $u_0$ knows the highest identifier.

It is true for $k = 0$: as long as $|\alpha(Id_{u_0})| > t$, $u_0$ applies $R_5$ at round $t$.

We assume that each node at distance at most $k$ from $u_0$ knows the highest identifier after at most $|\alpha(X)| + 6k$ rounds. Let $v$ be a node at distance $k + 1$ from $u_0$. Let $u$ be a node at distance $k$ from $u_0$ and neighbour of $v$. The node $u$ knows the highest identifier after at most $|\alpha(X)| + 6k$ rounds, i.e., $Z_u = \alpha(X)$. From Lemma 8 and knowing that $Z_u = \alpha(X)$ where $X$ is the highest identifier, we deduce that words $Z_u$ and $Z_v$ will always take one of the following forms at round $|\alpha(X)| + 6k$ where $p$ and $w$ are words and $a$ is the bit 1 or the bit 0:

1. $Z_u = p$ and $Z_v = p$,
2. $Z_v = p$ and $Z_u = pw$ with $1 \leq |w| \leq 2$,
3. $Z_v = p0$ and $Z_u = p1a$,
4. $Z_u = p1$ and $Z_v = p0w$ and $|w| \leq 3$,
5. $Z_v = p$ and $Z_u = pw$ and $3 \leq |w| \leq 6$ and $u$ has just performed a delete.

As $Z_u = \alpha(X)$ where $X$ is the highest identifier, the fourth form is impossible. The other cases imply that $Z_v$ will be equal to $Z_u$ after at most 6 rounds and the result follows. □

4 A Spanning Tree Algorithm

This section explains how the computation of a spanning tree may be associated to the spreading algorithm $S$ by selecting for each node $u$ the edge through which $Z_u$ was modified.

Let $u$ be a node, we add for each neighbour $v$, a variable $status_u^v$ whose possible values are in $\{child, parent, other\}$: it indicates the status of $v$ for $u$; initially $status_u^v = other$. The computation of the spanning tree occurs concurrently with the spreading algorithm $S$ as follows. If $R_2$, $R_3$, or $R_4$ is applied at round $t$ relative to neighbour $v$, then $u$ choses $v$ as parent if $v$ is not already $u$’s parent. Then, in addition to the signals of the spreading algorithm (indicating how $Z_u$ was updated), $u$ sends a signal $parent$ to $v$ and a signal $other$ to its previous parent (if different from $v$).

After receiving signals from neighbours, in addition to the computation of the new value of $Z_v$ for each neighbour $v$ by Algorithm $S$, $u$ updates $status_u^v$.

Algorithm $ST$ denotes the algorithm obtained with Rules of the spreading algorithm $S$ and actions described just above.
Remark 11. A node has no parent if and only if it is active.

Remark 12. A node has at most one parent.

The next definition introduces for each node $u$ a word $T_u$ that we will use to prove that the relation induced by all the parent relations has no cycle.

Definition 13. Let $u$ be a node, let $t$ be a round number of the spreading algorithm $S$; $T_u^t$ is equal to:

- $Z_u^t$ if $t = 0$ or if $Z_u^t$ has been obtained from $Z_u^{t-1}$ thanks to $R_2$ or $R_3$ or $R_4$ or $R_5$;
- $Z_u^{t'}$ if $Z_u^t$ has been obtained from $Z_u^{t-1}$ thanks to $R_{1,1}$ or $R_{1,2}$ and $t' < t$ is the last round where $Z_u^{t'}$ has not been obtained by a delete operation.

The following lemma is a direct consequence of the definition of $T_u^t$, and of $R_2$, $R_3$ and $R_4$:

Lemma 14. Let $t$ be a round number of the spreading algorithm $S$. If $v$ is parent of $u$ then $T_u^t \leq T_v^t$; furthermore if $v$ becomes parent of $u$ at round $t$ then $T_u^t \times T_v^t$ or $T_u^t = T_v^t$ and $T_u^{t-1} \leq T_v^{t-1}$.

Corollary 15. Let $t$ be a round number. Let $u_1$ be a node. Let $(u_i)_{1 \leq i \leq p}$ be nodes of $G$ such that, at round $t$, for $2 \leq i \leq p$ $u_i$ is parent of $u_{i-1}$ Then $u_1 \neq u_p$.

Proof. Let $t$ be a round, and let $u_1$ be a node. Let $(u_i)_{1 \leq i \leq p}$ be nodes of $G$ such that, at round $t$, for $2 \leq i \leq p$ $u_i$ is parent of $u_{i-1}$. The previous lemma implies that $(T_{u_i}^t)_{1 \leq i \leq p}$ is increasing. Considering a couple $(u_i, u_{i+1})$ where $R_2$, or $R_3$, or $R_4$ has been applied for the last time before $t$, we obtain the result. □

Corollary 16. Let $t$ be a round number. Let $u_1$ be a node. Then either $u_1$ is active or there exist $(u_i)_{1 \leq i \leq p}$ nodes of $G$ such that, for $2 \leq i \leq p$ $u_i$ is parent of $u_{i-1}$ and $u_p$ is active.

Definition 17. We denote by $ST(G)$ the subgraph of $G = (V, E)$ having $V$ as node set and there is an edge between the node $u$ and the node $v$ if $u$ is the parent of $v$ or $v$ is the parent of $u$ when algorithm $ST$ terminates.

When Algorithm $ST$ terminates there is exactly one active node: the node with highest identifier. Now, from Remark 12 and Corollary 16

Proposition 18. Let $G$ be a connected graph such that each node is endowed with a unique identifier. Let $u$ be the node with the highest identifier. When algorithm $ST$ terminates, the graph $ST(G)$ is a spanning tree of $G$. 

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5 Termination Detection of Algorithm \( ST \)

This section presents some actions which, added to algorithm \( ST \), enable the node with the highest identifier to detect termination of algorithm \( ST \); furthermore, as it is the only one, when it detects the termination it becomes elected. Our solution is a bitwise adaptation of the propagation process with feedback introduced in [39] and further formalised and studied in Chapter 6 and 7 of [41].

**Definition 19.** Let \( v \) be a node. Let \( t \) be a round number of the spreading algorithm. The variable \( Z^t_v \) is said to be well-formed if there exists an identifier \( Id \) such that \( Z^t_v = \alpha(Id) \).

Each node \( v \) is equipped with a boolean variable \( \text{Term}_v \) which is true iff \( v \) and all of its subtree have terminated. Whenever a rule of the spreading algorithm is applied to node \( v \), the variable \( \text{Term}_v \) is set to false, and a signal is sent to its neighbours to indicate that \( \text{Term}_v = \text{false} \). Indeed, this variable can be updated several times for a same node before stabilizing to true.

We describe an extra rule to be added to the \( ST \) algorithm in order to allow the node with highest identifier to learn that it is so by detecting termination of the spanning tree algorithm. This rule is considered after those of algorithm \( ST \) in each round.

Let us denote by \( N_v \) the set of neighbours of \( v \), and by \( Ch_v \subseteq N_v \) those which are \( v \)'s children. Also recall that we omit the round number in the expression on variables when it is non ambiguous.

**The rule:** Given a node \( v \), if (\( v \) is follower) and (\( \text{Term}_v = \text{false} \)) and (\( Z_v \) is well-formed) and (\( \forall w \in N_v \ Z_w = Z_v \)) and (\( \forall w \in Ch_v \text{Term}_w = \text{true} \)) then \( \text{Term}_v := \text{true} \). Furthermore \( v \) sends to his parent a signal indicating that \( \text{Term}_v = \text{true} \).

We denote by \( \mathcal{STT} \) the algorithm obtained by putting together the rules of Algorithm \( ST \) and this extra rule for termination detection.

**Remark 20.** Let \( v \) be a node, if \( \text{Term}_v = \text{true} \) then \( Z_v \) has the same value it had when \( \text{Term}_v \) became \text{true} the last time.

**Remark 21.** If \( Ch_v = \emptyset \), i.e., \( v \) is a leaf, and \( Z_v \) is well-formed and for each neighbour \( w \) of \( v \) \( Z_w = Z_v \) then \( v \) sets \( \text{Term}_v \) to \text{true} right away (and \( v \) sends to his parent a signal indicating that \( \text{Term}_v = \text{true} \)).

**Remark 22.** Let \( u \) be the node with highest identifier. Let \( v \) be a node. If \( Z_v = \alpha(Id_u) \) then \( Z_v \) will never change.

Theorem[10] and Proposition[18] imply:

**Proposition 23.** Let \( G \) be a graph such that each node has a unique identifier which is an integer. Algorithm \( \mathcal{STT} \) terminates. Furthermore, if the node \( u \) has the highest identifier then, after a run of algorithm \( \mathcal{STT} \), for each neighbour \( v \) of \( u \) \( Z_v = \alpha(Id_u) \) and \( \text{Term}_v = \text{true} \) and the node \( u \) receives from each node \( v \) in \( Ch_u \) the signal indicating that \( \text{Term}_u = \text{true} \).
The next proposition established that only the node with highest identifier can receive a termination signal from all neighbors.

**Proposition 24.** Let \( G \) be a graph such that each node has a unique identifier. Let \( v \) be a node which has not the highest identifier and such that \( Z_v = \alpha(Id_v) \) and for each neighbour \( w \) of \( v \), \( Z_w = Z_v \). Then there exists a neighbour \( v' \) of \( v \) such that \( \text{Term}_{v'} = \text{false} \).

**Proof.** If a node \( v \) has not the highest identifier, then there exists a node \( w' \) such that \( \alpha(v) < \alpha(w') \). The graph \( G \) is connected thus there exists a path \( (v_i)_{0 \leq i \leq k} \) such that: \( v_0 = v, v_k = w' \) and, for each \( 0 \leq i < k \), \( \{v_i, v_{i+1}\} \) is an edge of \( G \).

We have: \( Z_v = \alpha(Id_v) \), and \( Z_{v_1} = Z_{v_0} = Z_v \) thus there is a path \( (s_i)_{0 \leq i \leq j} \) between \( v_1 \) and \( v_0 \) with \( s_0 = v_1, s_j = v_0 \) and for each \( 0 \leq i < j \), \( s_{i+1} \) is parent of \( s_i \). If \( \text{Term}_{s_{j-1}} = \text{true} \) then necessarily \( \text{Term}_{v_1} = \text{true} \) and \( Z_{v_2} = Z_v \).

We can make the same construction with \( v_2 \): there exists a path between \( v_2 \) and \( v \) and if the last node on this path, which is a neighbour of \( v \), has its variable \( \text{Term} \) at \( \text{true} \) we deduce that \( Z_{v_3} = Z_v \). Finally, by iterating this argument and if at each step the variable \( \text{Term} \) of the neighbour of \( v \) is \( \text{true} \), we deduce that \( Z_{v_{k-1}} = Z_v \): that is impossible since \( \alpha(Id_v) < \alpha(Id_{w'}) \) and \( v_{k-1} \) is a neighbour of \( w' \). \( \square \)

If the node with the highest identifier, denoted \( u \), becomes elected as soon as, for each neighbour \( v \) of \( u \), \( Z_v = \alpha(Id_u) \) and \( \text{Term}_v = \text{true} \) and it receives from each child \( v \) the signal indicating that \( \text{Term}_v = \text{true} \) we deduce:

**Theorem 25.** Let \( G \) be a graph such that each node has a unique identifier which is an integer. Let \( u \) be the node with the highest identifier. There exists an election algorithm for \( G \) with messages of size \( O(1) \) which terminates after at most \( |\alpha(Id_u)| + 6D \) rounds.

6 Further Related Work

The election problem is fundamental in distributed computing and there exists a large number of papers: see for example [41,4,30,36]. It is very close to the problem of the computation of a spanning tree. It was first posed by LeLann [29]. As indicated in [26], some simple problems are still open. We will not discuss of all results. The election problem is investigated in at least three directions:

- characterisation of graphs for which there exists a deterministic election algorithm;
- lower and upper bounds of the time complexity and the message complexity of deterministic election algorithms according the knowledge on the graph, it is assumed that each node has a unique identifier;
- randomised election algorithms for anonymous graphs depending on the knowledge on the graph such as the size, the diameter or the topology (trees, complete graphs...).
For the first item the starting point is the seminal work of Angluin [2] which highlights, in particular, the key role of coverings: a graph $G$ is a covering of a graph $H$ if there is a surjective homomorphism $\varphi$ from $G$ to $H$ which is locally bijective (the restriction of $\varphi$ to incident edges of any node $v$ is a bijection between incident edges of $v$ and incident edges of $\varphi(v)$). More general definitions may be found in [10]. Characterisations of graphs for which there exists an election algorithm depend on the model. The first characterisations have been obtained in [9,43,31]. The fundamental tool in [9,43] is the notion of view: the view from a node $v$ of a labelled graph $G$ is an infinite labelled tree rooted in $v$ obtained by considering all labelled walks in $G$ starting from $v$. The characterisation in [31] used non-ambiguous graphs: a graph labelling is said to be locally bijective if vertices with the same label are not in the same ball and have isomorphic labelled neighbourhoods. A graph $G$ is ambiguous if there exists a non-bijective labelling of $G$ which is locally bijective. In [21], authors prove that the non-ambiguous graphs, as introduced by Mazurkiewicz, are exactly the covering-minimal graphs. The main ideas of the election algorithm developed in [31] have been applied to some other models in [13,14,11] by adapting the notion of covering. A characterisation of families of graphs which admit an election algorithm (i.e., the same algorithm works on each graph of the family) has been obtained in [12].

Concerning lower bounds or upper bounds for deterministic algorithms when nodes have a unique identifier which is a non negative integer of size $O(\log n)$:

- for the time complexity: Peleg presents in [33] a simple time optimal election algorithm for general graphs: its time complexity is $O(D)$; the size of messages is $O(\log n)$ thus its bit round complexity is $O(D \log n)$ and the message complexity is $O(D|E|)$ where $|E|$ is the size of the edge set. More recently, Kutten et al. [26] prove the lower bound $\Omega(D)$ for the time complexity in a very general context which contains the deterministic case studied in this paper. Fusco and Pelc [18] show that the time complexity of the election problem is $\Omega(D + \lambda)$ where $\lambda$ is the level of symmetry of the graph $G$ (Let $G$ be graph. The view at depth $t$ from a node is the tree of all paths of length $t$ originating at this node. The symmetry of $G$ is the smallest depth at which some node has a unique view of $G$). In our case, each node has a unique identifier thus $\lambda = 0$, and we obtain the same bound as [26].

- for the message complexity: Gallager [19] presents the first election algorithm for general graphs with $O(m + n \log n)$ messages, where $m$ is the number of edges, and Santoro [35] proves the $\Omega(m + n \log n)$ lower bound. The election problem is equivalent to constructing a spanning tree; from this consideration we may deduce that the message complexity of the election problem is $O(m + n \log n)$ as explained by Peleg in [33] from observations in [20] (see also the discussions in [41] Chapter 7). The work presented in [20] had a great influence on many papers, the time complexity of the algorithm is $O(n \log n)$ and the message complexity is optimal in the worst case. Optimal message complexity in $O(m + n \log n)$ has been obtained also in [5], in this case the time complexity is $O(n)$, the size of message is $O(\log n)$ and the
bit round complexity is $O(n \log n)$. We can note that very efficient algorithms for both election and spanning tree computation are presented in [23]. The table below summarises some elements on different complexities of the election problem.

A Las Vegas algorithm is a probabilistic algorithm which terminates with a positive probability (in general 1) and always produces a correct result. A Monte Carlo algorithm is a probabilistic algorithm which always terminates; nevertheless the result may be incorrect with a certain probability. Some results on graphs having $n$ vertices are expressed with high probability, meaning with probability $1 - o(n^{-1})$ (w.h.p. for short). Chapter 9 of [41] and [28] give a survey of what can be done and of impossibility results in anonymous networks concerning the election problem. In particular, no deterministic algorithm can elect (see Angluin [2], Attiya et al. [3] and Yamashita and Kameda [42]): furthermore, with no knowledge on the network, there exists no Las Vegas election algorithm [22]. Some results on the randomised complexity of leader election are obtained in [27]. Monte Carlo election algorithms for anonymous graphs without knowledge are presented in [22, 1, 37]. They are correct with probability $1 - \epsilon$, where $\epsilon$ is fixed and known to all vertices. [32] presents Monte Carlo algorithms which solve the problems discussed above w.h.p. and which ensure for each node $v$ an error probability bounded by $\epsilon_v$, where $\epsilon_v$ is determined by $v$ in a fully decentralised way. To be more precise, these algorithms ensure an error probability bounded by $\epsilon$ where $\epsilon$ is the smallest value among the set of error probabilities determined independently by each node. If the network size is known then Las Vegas election algorithms exist see for example [22].

7 Conclusion

Concerning deterministic election algorithms for graphs such that nodes have unique identifiers, we may consider three complexity measures: time complexity, message complexity, and bit (round) complexity. The work of Santoro [35] proved that $\Omega(|E| + n \log n)$ is a lower bound for the number of messages required and Awerbuch [5] presented an algorithm that matches this bound. Kutten et al. [26] shows that concerning the time complexity $\Omega(D)$ is a lower bound and [33] implies that $O(D)$ is a tight upper bound. For bit (round) complexity, we deduced from [26] and [17] that $\Omega(D + \log n)$ is a lower bound and we presented an algorithm that matches this bound with a running time of $O(D + \log n)$ bit rounds. Furthermore, our algorithm needs no knowledge on the graph such as the size or the diameter.

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