On graph Laplacians eigenvectors with components in \( \{1, -1, 0\} \)

J. G. Caputo\(^1\), I. Khames\(^2\), A. Knippel\(^3\)

Laboratoire de Mathématiques, INSA Rouen Normandie. 76800 Saint-Etienne du Rouvray, FRANCE.

Abstract

We define a bivalent graph as having an eigenvector of the graph Laplacian matrix with components in \( \{-1, +1\} \) and a trivalent graph as having a Laplacian eigenvector with components in \( \{-1, 0, +1\} \). These graphs are important because they yield periodic orbits for nonlinear wave equations on networks. We characterize them by applying some transformations on graphs. Bivalent graphs are shown to be the regular bipartite graphs and their extensions obtained by adding edges between vertices with the same value for the considered eigenvector. We define a soft regular graph as having a Laplacian eigenvector whose all non zero component vertices have same degree. Trivalent graphs are shown to be extensions of these soft regular graphs via the transformation defined previously.

1. Introduction

The graph wave equation \(^1\) where the Laplacian is replaced by the graph Laplacian \(^2\), is a natural model for describing miscible flows on a network since it arises from conservation laws. The graph wave equation is well understood in terms of normal modes i.e. periodic solutions associated to the eigenvectors of the graph Laplacian. In a previous work \(^3\), we considered a nonlinear graph wave equation with a cubic on-site nonlinearity which is the discrete \( \phi^4 \) model \(^4\) and we studied the extension of the normal modes into nonlinear periodic orbits.

\(^1\)caputo@insa-rouen.fr
\(^2\)imene.khames@insa-rouen.fr
\(^3\)arnaud.knippel@insa-rouen.fr
We generalized the criterion of Aoki\textsuperscript{[5]} for paths and cycles to the case of general graphs and showed that the linear normal modes associated to eigenvectors composed of \{-1, 0, 1\} extend into nonlinear periodic orbits. We defined monovalent, bivalent and trivalent eigenvectors depending whether their components are in \{+1\} or \{-1, +1\} or \{-1, 0, +1\}. The first case is trivial as the vector of one’s is always an eigenvector of the graph Laplacian, associated to the eigenvalue 0.

The trivalent eigenvectors contain components of value 0, corresponding to vertices that we call soft nodes to insist on their special role in the dynamical systems, as analyzed in\textsuperscript{[1]}. A classification of graphs whose Laplacian matrices have eigenvectors with soft nodes, is presented in\textsuperscript{[6]}.

In\textsuperscript{[3]}, we classified the bivalent and trivalent eigenvectors in paths and cycles for which the spectrum is well-known\textsuperscript{[7]}. It is then natural to try and characterize the graphs having bivalent and trivalent eigenvectors.

Wilf\textsuperscript{[8]} asked what kind of a graph admits an adjacency matrix eigenvector consisting solely of \pm 1 entries. More recently, Stevanović\textsuperscript{[9]} proved that Wilf’s problem is NP-complete, but also that the set of graphs having a \pm 1 eigenvector of adjacency matrix is quite rich.

We ask here the same question in the case of the Laplacian matrix of a graph, and give a characterization of graphs having Laplacian eigenvectors in \{-1, 1\} or \{-1, 0, 1\}. We call these graphs respectively bivalent and trivalent. This is done using transformations of graphs, from the literature\textsuperscript{[10]} and from our own. In case of regular graphs, all results about the Laplacian spectrum of graphs carry over to results about the adjacency spectrum.

The article is organized as follows. In section 2, we introduce some preliminaries of the graph Laplacian and transformations of graphs. Section 3 presents a characterization of bivalent graphs: we show that the bivalent graphs are the regular bipartite graphs and their extensions by adding edges between two equal-valued vertices. Section 4 presents a similar characterization for trivalent graphs: we show that the trivalent graphs are obtained from what we call soft regular graphs by applying some transformations.
2. Graph Laplacian

Let \( G(\mathcal{V}, \mathcal{E}) \) be a graph with vertex set \( \mathcal{V} \) of cardinality \( N \) and edge set \( \mathcal{E} \). All graphs in this article are finite and undirected with no loops or multiple edges. Denote the degree of vertex \( j \) by \( d_j \) and let \( D \) be the \( N \times N \) diagonal matrix of vertex degrees \( D_{jj} = d_j \). We will indicate adjacency of vertices by \( i \sim j \) for \( e_{ij} \in \mathcal{E}(G) \). Let \( A \) be the \( N \times N \) \{0, 1\} adjacency matrix such that \( A_{ij} = 1 \) if and only if \( e_{ij} \in \mathcal{E}(G) \) \((i \neq j)\). The Laplacian matrix \( \Delta \) associated to the graph \( G \) is the difference of the diagonal matrix of its vertex degrees and its adjacency matrix \( \Delta = D - A \). For an extensive survey on the Laplacian matrix see Merris [11].

Since the graph Laplacian \( \Delta \) is a real symmetric positive semi-definite matrix, it is diagonalizable

\[
\Delta v^k = \lambda_k v^k. \tag{1}
\]

The eigenvectors \( v^k, k \in \{1, \ldots, N\} \) of \( \Delta \) can be chosen to be orthogonal with respect to the scalar product in \( \mathbb{R}^N \), i.e. \( \langle v^k, v^l \rangle = 0 \) for \( k \neq l \). We arrange the eigenvalues \( \lambda_k \) of \( \Delta \) as \( \lambda_1 = 0 \leq \lambda_2 \leq \cdots \leq \lambda_N \). The first eigenvalue \( \lambda_1 = 0 \) corresponds to the monovalent eigenvector \( v^1 = (1, 1, \ldots, 1)^T \).

We will abuse the language by referring to the Laplacian \( \Delta \) of the graph \( G \) as \( \Delta(G) \). Thus, \( v \) is an eigenvector of \( \Delta(G) \) affording \( \lambda \) if and only if

\[
(d_i - \lambda)v_i = \sum_{j \sim i} v_j, \quad \forall i \in \{1, \ldots, N\}, \tag{2}
\]

where \( \sum_{j \sim i} v_j \) is the sum, over the vertices \( j \) adjacent in \( G \) to \( i \), of \( v_j \).

2.1. Definitions

**Definition 1** (Soft node [1]). A node \( j \) of a graph is a soft node for an eigenvalue \( \lambda \) of the graph Laplacian if there exists an eigenvector \( v \) for this eigenvalue such that \( v_j = 0 \).

**Definition 2** (Regular graph). A graph is \( d \)-regular if every vertex has the same degree \( d \).

**Definition 3** (Soft regular graph). A graph is \( d \)-soft regular for an eigenvector \( v \) of the Laplacian if every non-soft node for \( v \) has the same degree \( d \).
The graph on the left of Fig. 1 is 3-soft regular for the eigenvector $(0, 1, 1, 0, -1, -1)^T$ since all the non-zero vertices have the same degree 3. The graph on the right of Fig. 1 is non-soft regular for the eigenvector $(0, 1, 1, 0, -1, -1, 0, 0)^T$ since the non-zero vertices have different degrees.

Figure 1: 3-soft regular graph for the Laplacian eigenvector $(0, 1, 1, 0, -1, -1)^T$ (left). Non-soft regular graph for the Laplacian eigenvector $(0, 1, 1, 0, -1, -1, 0, 0)^T$ (right).

**Definition 4** (Bivalent graph). A graph is bivalent if there exists an eigenvector of the graph Laplacian composed uniquely of $-1, +1$. Such a vector is called bivalent.

The bivalent eigenvector $v$ must have as many $-1$ and $+1$ coefficients, and thus the bivalent graph must have an even number of nodes. This is a consequence of the orthogonality of $v$ to the monovalent eigenvector $v^1$.

**Definition 5** (Trivalent graph). A graph is trivalent if there exists an eigenvector of the graph Laplacian composed uniquely of $-1, 0, +1$. Such a vector is called trivalent.

**Definition 6** ($k$-partite graph). A $k$-partite graph is a graph whose vertices can be partitioned into $k$ different independent sets so that no two vertices within the same set are adjacent.

When $k = 2$ these are the bipartite graphs, and when $k = 3$ they are called the tripartite graphs.

**Definition 7** (Perfect matching). A perfect matching of a graph $G$ is a matching (i.e., an independent edge set) in which every vertex of the graph is incident to exactly one edge of the matching.
**Definition 8** (Alternate perfect matching). An alternate perfect matching for a vector \( v \) on the nodes of a graph \( G \) is a perfect matching for the nonzero nodes such that edges \( e_{ij} \) of the matching satisfy \( v_i = -v_j \) (\( \neq 0 \)).

The left of Fig. 1 shows the alternate perfect matching (represented by red lines) for the eigenvector \( (0, -1, -1, 0, 1, 1)^T \) on the nodes of cycle 6.

### 2.2. Transformations of graphs

Merris [10] considers several transformations of graphs based on Laplacian eigenvectors. In the following we review three of them and we present another transformation.

#### 2.2.1. Transformations preserving eigenvalues

**Theorem 9** (Link between two equal nodes [10]). Let \( v \) be an eigenvector of \( \Delta(G) \) affording an eigenvalue \( \lambda \). If \( v_i = v_j \), then \( v \) is an eigenvector of \( \Delta(G') \) affording the eigenvalue \( \lambda \), where \( G' \) is the graph obtained from \( G \) by deleting or adding the edge \( e_{ij} \) depending whether \( e_{ij} \) is an edge of \( G \) or not.

Fig. 2 shows how Th. 9 can be used to extend an eigenvector and its eigenvalue to the transformed graphs by adding edges (represented by red lines) between nodes having the same value. Notice that this transformation does not preserve the regularity of the graph.

![Figure 2: Three graphs obtained by adding or deleting edges between equal nodes, affording (the same eigenvalue) \( \lambda = 2 \).](image)

**Theorem 10** (Extension of soft nodes [10]). For a graph \( G(V, E) \) fix a nonempty subset \( W \) of \( V \). Delete all the vertices in \( V \setminus W \) that are adjacent in \( G \) to no vertex of \( W \). Remove any remaining edges that are incident with no vertex of \( W \). Suppose \( v \) is an eigenvector of the Laplacian of the reduced graph \( G\{W\} \) that affords \( \lambda \) and is supported by \( W \) in the sense that if \( v_i \neq 0 \), then \( i \in W \). Then the extension \( v' \) with \( v'_j = v_j \) for \( j \in W \) and \( v'_j = 0 \) otherwise is an eigenvector of \( \Delta(G) \) affording \( \lambda \).
Figure 3: Extension at soft node of the eigenvector \((-1, 0, 1)^T\) by adding soft nodes. The eigenvectors afford (the same eigenvalue) \(\lambda = 1\).

We introduce the following transformation which preserves the eigenvalues and does not preserve the soft regularity of the graph.

**Theorem 11 (Replace an edge by a soft square).** Let \(v\) be an eigenvector of \(\Delta(G)\) affording an eigenvalue \(\lambda\). Let \(G'\) be the graph obtained from \(G\) by deleting an edge \(e_{ij} \in \mathcal{E}(G)\) such that \(v_i = -v_j\) and adding two soft nodes \(k, l \in V(G')\) for the extension \(v'\) of \(v\) (such that \(v'_m = v_m\) for \(m \in V(G)\) and \(v'_k = v'_l = 0\)) and adding four edges \(e_{ik}, e_{kj}, e_{il}, e_{lj} \in \mathcal{E}(G')\). Then, \(v'\) is an eigenvector of \(\Delta(G')\) for the eigenvalue \(\lambda\).

**Proof.** Suppose the edge \(e_{ij} \in \mathcal{E}(G)\) joining two nodes having opposite values \(v_i = -v_j\), is replaced by a square \(e_{ik}, e_{kj}, e_{il}, e_{lj} \in \mathcal{E}(G')\) of soft nodes \(k, l \in V(G')\). The eigenvector condition

\[
((d + 1) - \lambda) v_i = v_i + \sum_{m \sim i} v_m = \sum_{m \sim i, m \neq j} v_m = 2 \times 0 + \sum_{m \sim i, m \neq j} v_m,
\]

is the condition that must be met at vertex \(i\) for the extension \(v'\) of \(v\), by defining \(v'_m = 0\) for \(m \in V(G')\setminus V(G)\), to be an eigenvector of \(\Delta(G')\) affording \(\lambda\). The eigenvector condition at vertex \(j\) is confirmed similarly, and the conditions at the other vertices are the same for \(G'\) as they are for \(G\). □

Fig. 4 shows how Th. 11 can be used to transform a soft regular graph to a non-soft regular graph without changing the eigenvalue. Note that a square of soft nodes can be replaced by an edge between opposite nodes.
Figure 4: Replacing an edge between opposite nodes by a square of soft nodes. The eigenvectors afford (the same eigenvalue) $\lambda = 3$.

### 2.2.2. Transformations changing eigenvalues

The following transformation allows to extend graphs by changing the eigenvalues and preserving the soft regularity of the graph.

**Theorem 12 (Add/Delete an alternate perfect matching [10]).** Let $v$ be an eigenvector of $\Delta(\mathcal{G})$ affording an eigenvalue $\lambda$. Let $\mathcal{G}'$ be the graph obtained from $\mathcal{G}$ by adding (resp. deleting) an alternate perfect matching for $v$. Then, $v$ is an eigenvector of $\Delta(\mathcal{G}')$ affording the eigenvalue $\lambda + 2$ (resp. $\lambda - 2$).

Adding an alternate perfect matching is illustrated in Fig. 5. This transformation preserves the soft regularity of the graph and increases the eigenvalue by 2.

Figure 5: Graphs obtained by adding alternate perfect matching for the eigenvector $(0, 1, 1, 0, -1, -1)^T$. The eigenvalues are $\lambda = 1$ (left), $\lambda = 3$ (middle) and $\lambda = 5$ (right).

### 3. Bivalent graphs

For bivalent graphs, we give the following characterization.
Theorem 13 (Bivalent graphs). The bivalent graphs are the regular bipartite graphs and their extensions obtained by adding edges between nodes having the same value for a bivalent eigenvector.

Proof. Let $G$ be a graph having a bivalent eigenvector $v$ affording $\lambda$. We reduce $G$ by deleting all the edges between equal nodes Th.9, thus obtaining a graph where edges only connect $+1$ to $-1$. This is a bipartite graph.

We write the eigenvector condition for nodes $j$ (with degree $d_j$) such that $v_j = 1$

$$(d_j)(1) + \sum_{i \sim j} (-1)(-1) = 2d_j = \lambda. \quad (3)$$

Similarly for nodes $j$ such that $v_j = -1$.

The satisfaction of the eigenvector condition for all vertices of $G$ requires that $\lambda = 2d_j$, $\forall j \in \{1, \ldots, N\}$ so that $d_j = d$, $\forall j \in \{1, \ldots, N\}$. Thus, $G$ is $d$-regular graph. Hence, the bivalent graphs are the $d$-regular bipartite graphs and their extensions obtained by adding edges between equal nodes Th.9.

Reciprocally, a bipartite $d$-regular graph $G$ has an even number of nodes and satisfies the eigenvalue condition (3) so that $G$ is bivalent.

As an example, Fig.6 shows the smallest bivalent graph, with eigenvalue $\lambda = 2$. It is a 1-regular graph.

![Figure 6: A 1-regular bivalent graph $(d = 1, \, \lambda = 2)$.](image)

The extension of two copies of chain of length 1 seen in Fig.6 by adding an alternate perfect matching Th.12 produces the 2-regular bivalent graph shown in the right of Fig.7.

![Figure 7: Construction of 2-regular bivalent graph $d = 2, \, \lambda = 4$ (right) from the 1-regular bivalent graph $d = 1, \, \lambda = 2$ (left) by adding an alternate perfect matching.](image)
The extension of three copies of chain of length 1 seen in Fig. 6 by adding an alternate perfect matching Th.12 (two times) gives the 3-regular bivalent graph shown in the right of Fig. 8.

![Figure 8](image.png)

Figure 8: Construction of 3-regular bivalent graph $d = 3$, $\lambda = 6$ (right) from the 1-regular bivalent graph $d = 1$, $\lambda = 2$ (left) by adding two alternate perfect matchings.

Adding edges between equal nodes Th.9 to three copies of chain of length 1 seen in Fig. 6 produces the bivalent eigenvector of the non-regular graphs shown in Fig. 9 affording the same eigenvalue $\lambda = 2$.

![Figure 9](image.png)

Figure 9: Two bivalent graphs obtained from the 1-regular graph by adding edges between equal nodes, that afford (the same eigenvalue) $\lambda = 2$.

More generally, note that a bivalent eigenvector affords an eigenvalue $\lambda \in \{2, 4, \ldots, 2d_{\text{min}}\}$ where $d_{\text{min}}$ is the smallest degree of nodes in the graph.

**Theorem 14** (Bivalent tree). A tree $\mathcal{T}$ is bivalent if and only if it has a perfect matching.

**Proof.** First note that a tree is bipartite and that a 1-regular graph is a perfect matching.
Assume $T$ be a bivalent tree. Then there exists an eigenvector $v$ with entries solely in \{1, -1\} built from a $d$-regular bipartite graph by adding edges between nodes of equal values. Since a tree always has leaves (nodes of degree 1), $d$ must be equal to 1, the subgraph is 1-regular hence a perfect matching.

Inversely, if a tree has a perfect matching, it is easy to construct a bivalent eigenvector by taking opposite values in each edge of the matching, as there are no cycles in a tree, this can be done by Breadth-First Search (BFS) or Depth-First Search (DFS) algorithms.

The authors in [12] proved Th.14 (Corollary 2.12).

For a general graph, the existence of a perfect matching is not a sufficient condition to be bivalent. As examples, we show in Fig.10 two asymmetric graphs i.e. which have no symmetries.

![Two asymmetric graphs of 6 nodes. They have a perfect matching but are not bivalent.](image)

4. Trivalent graphs

**Theorem 15 (Trivalent graphs).** Trivalent graphs are obtained from soft regular graphs by applying on the same trivalent eigenvector the transformations:

- add link between two equal nodes,
- extension of soft nodes
- add/delete an alternate perfect matching,
- replace an edge by a soft square.
Proof. Let $G$ be a graph having a trivalent eigenvector $v$ affording $\lambda$.

We reduce $G$ by deleting all the edges between equal nodes Th.9 and deleting soft nodes that are not adjacent to non-soft nodes Th.10, thus obtaining a graph where edges only connect nodes with different values in $\{1, -1, 0\}$. This is a tripartite graph.

For soft nodes $j$, the eigenvector condition

\[ (d_j)(0) + \sum_{i \sim j, v_i = 1} (-1)(1) + \sum_{i \sim j, v_i = -1} (-1)(-1) = (\lambda)(0) = 0, \]

requires that

\[ \text{card}\{i \sim j, v_i = +1\} = \text{card}\{i \sim j, v_i = -1\}. \]

The eigenvector condition for nodes $j$ such that $v_j = 1$,

\[ (d_j)(1) + \sum_{i \sim j, v_i \neq 0} (-1)(-1) + \sum_{i \sim j, v_i = 0} (-1)(0) = (\lambda)(1). \]

Similarly for nodes $j$ such that $v_j = -1$. Thus,

\[ \lambda = d_j + \tilde{d}_j = 2d_j - s_j, \quad \forall j \in S^c, \tag{4} \]

where $S = \{k, v_k = 0\}$ the set of the soft nodes, $S^c = \{1, \ldots, N\} \backslash S$ the complement of $S$ i.e. the set of the non-soft nodes, $\tilde{d}_j = \text{card}\{i \sim j, v_i \neq 0\}$ the number of the non-soft neighbors of $j$ and $s_j = \text{card}\{i \sim j, v_i = 0\}$ the number of the soft neighbors of $j$.

The eigenvalue formula (4) is satisfied for $G$ being soft regular for $v$. For trivalent graphs $G$ that are not soft regular (an example is shown in the left of Fig.11), one can transform $G$ to soft regular graphs by applying Th.11 several times and replacing each edge between nodes of opposite values by a square of two soft nodes (as shown in the right of Fig.11).

Reciprocally, a soft regular tripartite graph $G$ satisfies the eigenvalue condition (4) so that $G$ is trivalent.

\[ \square \]
Below we give a classification by eigenvalues of the smallest trivalent graphs. Then, the transformations connecting the elements within each class generate trivalent graphs.

The smallest trivalent graph having eigenvalue $\lambda = d_j + \tilde{d}_j = 1$ (where $j$ is non-soft vertex) satisfies $d_j = 1$, $\tilde{d}_j = 0$. That is the path on 3 nodes shown in Fig.12.

Trivalent trees are constructed from trivalent path on 3 vertices $(1, 0, -1)^T$ by adding nodes between two equal-valued vertices Th.9 and extension of soft nodes Th.10. A characterization of all trees that have 1 as the third smallest Laplacian eigenvalue is presented in [13].

The smallest trivalent graphs having eigenvalue $\lambda = d_j + \tilde{d}_j = 2$ (where $j$ is non-soft vertex) satisfy:

- $d_j = 2$, $\tilde{d}_j = 0$. That is the cycle 4 shown in the left of Fig.13
- $d_j = \tilde{d}_j = 1$. That is the 1-regular bivalent graph.
The smallest trivalent graphs having eigenvalue $\lambda = d_j + \tilde{d}_j = 3$ (where $j$ is non-soft vertex) satisfy:

- $d_j = 3$, $\tilde{d}_j = 0$. That is the graph shown in the left of Fig. 14.
- $d_j = 2$, $\tilde{d}_j = 1$. That is the graph shown in the right of Fig. 14.

The smallest trivalent graphs having eigenvalue $\lambda = d_j + \tilde{d}_j = 4$ (where $j$ is non-soft vertex) satisfy:

- $d_j = 4$, $\tilde{d}_j = 0$. That is the graph shown in the left of Fig. 15.
- $d_j = 3$, $\tilde{d}_j = 1$. That is the graph shown in the middle of Fig. 15.
- $d_j = \tilde{d}_j = 2$. That is the 2-regular bivalent graph (right of Fig. 15).
Figure 15: The smallest trivalent graphs affording $\lambda = 4$.

The smallest trivalent graphs having eigenvalue $\lambda = d_j + \tilde{d}_j = 5$ (where $j$ is non-soft vertex) satisfy:

- $d_j = 5$, $\tilde{d}_j = 0$. That is the graph shown in the left of Fig.16.
- $d_j = 4$, $\tilde{d}_j = 1$. That is the graph shown in the middle of Fig.16.
- $d_j = 3$, $\tilde{d}_j = 2$. That is the graph shown in the right of Fig.16.

Figure 16: The smallest trivalent graphs affording $\lambda = 5$.

The smallest trivalent graphs having eigenvalue $\lambda = d_j + \tilde{d}_j = 6$ (where $j$ is non-soft vertex) satisfy:

- $d_j = 6$, $\tilde{d}_j = 0$. That is the first graph in Fig.17.
- $d_j = 5$, $\tilde{d}_j = 1$. That is the second graph in Fig.17.
- $d_j = 4$, $\tilde{d}_j = 2$. That is the third graph in Fig.17.
- $d_j = \tilde{d}_j = 3$. That is the 3-regular bivalent graph (right of Fig.17).
5. Conclusion

We have characterized bivalent and trivalent graphs by applying Laplacian eigenvector transformations; these are links between two equal nodes, replacing an edge by a soft square, and adding or deleting an alternate perfect matching. We show that bivalent graphs are the regular bipartite graphs and their extensions obtained by adding edges between two equal nodes. We define a soft regular graph as having a Laplacian eigenvector with soft nodes such that each non-soft node has the same degree. Trivalent graphs are shown to be the soft regular graphs and their extensions. However, the question of whether a given graph is bivalent or trivalent, is difficult and remains open. The exploration of these graphs is important for nonlinear dynamical systems on networks.

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