Non-Adaptive Matroid Prophet Inequalities

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Abstract

We investigate non-adaptive algorithms for matroid prophet inequalities. Matroid prophet inequalities have been considered resolved since 2012 when [KW12] introduced thresholds that guarantee a tight 2-approximation to the prophet; however, this algorithm is adaptive. Other approaches of [CHMS10] and [FSZ16] have used non-adaptive thresholds with a feasibility restriction; however, this translates to adaptively changing an item’s threshold to infinity when it cannot be taken with respect to the additional feasibility constraint, hence the algorithm is not truly non-adaptive. A major application of prophet inequalities is in auction design, where non-adaptive prices possess a significant advantage: they convert to order-oblivious posted priceings, and are essential for translating a prophet inequality into a truthful mechanism for multi-dimensional buyers. The existing matroid prophet inequalities do not suffice for this application. We present the first non-adaptive constant-factor prophet inequality for graphic matroids.
1 Introduction

We study the classic prophet inequality problem introduced by Krengel and Sucheston [1977]: \( n \) items arrive online in adversarial order. A gambler observes the value of each item as it arrives, and in that moment, must decide irrevocably whether to take the item or pass on it forever. He can accept at most one item. The gambler knows in advance the (independent) prior distribution of each item’s value. What rule should he use to maximize the value of the item he accepts? In expectation, how does the maximum value that the gambler can guarantee compare to the prophet, who knows all of the realized item values in advance and selects the highest valued one?

The prophet inequality is a standard model for online decision making in a stochastic/Bayesian setting and has many applications, particularly to mechanism design and pricing. Over the last few years many variants of the basic single-item setting have been studied. One natural generalization is to allow the gambler to accept more than one item, subject to a feasibility constraint. Formally, we can represent a feasibility constraint as a collection \( S \) of feasible sets. Then both the gambler and prophet can each select any feasible set of items \( S \in S \); in the single-item setting, the feasible sets are just all singletons. What is the gambler’s best algorithm and guarantee?

A seminal result by Samuel-Cahn [1984] showed that for the basic single-item setting the online algorithm can obtain at least half of the prophet’s value in expectation by determining a single threshold \( T \) and accepting the first item with value exceeding \( T \). Further, this approximation factor is tight: there exist instances where the gambler can do no better than \( \frac{1}{2} \) as well as the prophet. The threshold \( T \) is selected such that the probability that the value of any of the \( n \) items exceeds the threshold is exactly \( \frac{1}{2} \).

In 2012 [1], Kleinberg and Weinberg introduced an alternative approach for setting a single threshold: set \( T = \frac{1}{2} \text{Opt} \). Here, \( \text{Opt} \) is what the prophet can achieve, and this approach guarantees the same \( \frac{1}{2} \)-approximation for a single item. Kleinberg and Weinberg showed that this alternate approach generalizes also to matroid prophet inequalities: where both the gambler and the prophet are restricted to accepting independent sets in a given matroid. In this setting, the approach of Kleinberg and Weinberg still achieves a factor of 2 approximation, matching the single item lower bound.

There is a significant qualitative difference between Samuel-Cahn’s approach for the single item prophet inequality and Kleinberg and Weinberg’s approach for the matroid setting. In particular, the former computes a single threshold that is then used for the entire duration of the algorithm. The latter on the other hand, recomputes thresholds after every decision. The threshold applied to the value of the second item, for example, depends on whether the first item was accepted by the algorithm or not, and in turn on the realized value of the first item. As a consequence, the KW algorithm is more complicated and involves more computation.

In this paper we address a natural problem exposed by this discussion: Can an online algorithm compete against the prophet using static thresholds under a matroid feasibility constraint?

There is an inherent connection between prophet inequalities and Bayesian mechanism design. The original problem by Krengel and Sucheston was formulated as an optimal stopping problem; it was Hajiaghayi et al. [2007] that made the first connection to an economic welfare-maximization problem. Chawla et al. [2010] studied this connection much more deeply, defining a truthful class

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\[1\] Their original result appeared in STOC 2012 [Kleinberg and Weinberg, 2012], but we will cite their journal version from 2019 for the remainder of the paper.
of simple mechanisms called “order-oblivious posted pricings.” They show that one can translate a
prophet inequality for \( n \) items with feasibility constraint \( S \) into an order-oblivious posted pricing for
an \( n \)-unit setting with unit-demand buyers and a service feasibility constraint\(^2\) corresponding to \( S \); this
mechanism is truthful and it yields a revenue guarantee that matches its prophet inequality
guarantee. When \( S \) is a matroid, by Kleinberg and Weinberg [2019], the resulting mechanism
yields \( \frac{1}{2} \)-approximation to the optimal expected revenue.

This reduction from truthful mechanisms to prophet inequalities crucially relies on the buyers
being unit-demand. If we wish instead wish to translate the prophet inequality into a mechanism
for a single constrained-additive buyer subject to feasibility constraint \( S \) over \( n \) heterogeneous
items—that is, the buyer is interested in buying \( more \) than one item—then adaptive thresholds
will not translate to a truthful mechanism. Instead, they correspond to offering each item one-
at-a-time to the buyer in any order, but prices change as a function of previous purchases. This
update will not generally preserve truthfullness; that is, although the buyer may wish to purchase
the first item offered when considered myopically, he may be better off declining, in order to avoid
price increases on later items.

In order to fix this reduction for multi-parameter buyers beyond unit-demand, we must use only
prophet inequalities with non-adaptive thresholds. This is our primary motivation: constructing
non-adaptive prophet inequalities in order to expand the realm of settings where prophet inequalities
can be used for truthful mechanism design (see related work for an understanding of how integral
they are as a tool in this field). However, non-adaptive prophet inequalities possess numerous
other attractive properties as well. For welfare-maximizing mechanisms, non-adaptive prophet
inequalities correspond to prices that are not only order-oblivious, but also \emph{anonymous}, using the
same prices on each item regardless of the buyer. Additionally, since the thresholds (prices) are all
computed before the items arrive and are never updated, there is much less computation required
than for adaptive thresholds.

1.1 Our Contribution and Roadmap

We present the first non-adaptive thresholds that give a constant-factor prophet inequality for
graphic matroids. We finish Section 1 with additional related work and in Section 2 we introduce
mathematical preliminaries. In Section 3 we discuss why extending non-adaptive algorithms to
graphic matroids is such a challenging objective, and why prior methods fail. Section 4 presents
the ex-ante relation to the matroid polytope: a reduction from a given prophet inequality instance
to an alternative setting with convenient properties for designing algorithms. Expert readers can
safely skip this section. Then, in this context, Section 5 presents our construction for non-adaptive
thresholds.

The ex-ante relaxation takes a given item \( i \)’s value distribution and converts it into a Bernoulli
distribution: with probability \( p_i \), item \( i \) is “active,” that is, non-zero, and takes on value \( t_i \). Then,
the threshold for item \( i \) is implicit (just the non-zero value \( t_i \)), and the only remaining questions are
(1) with what probability should our algorithm consider this item, and (2) with what probability
will the item be “unblocked,” or feasible to accept, when the gambler reaches it?

In order to obtain a constant-factor approximation, then the probabilities of both selection and
feasibility must be constant. In a graphic matroid, the elements are the edges and the independent
sets are the forests—that is, any set of edges that does not contain a cycle. Depending on the
\(^2\)A service feasibility constraint \( S \) says that the set of buyers that are served simultaneously must belong to some
set \( S \in S \).
given graph, an edge could be “blocked” by many different edges, and it is unclear how to group elements. Our main idea is to orient the graph to have good properties and then exploit them. For an edge \((u, v)\), suppose it is oriented into vertex \(v\). Notice that if no other edges incident to \(v\) are selected by the algorithm, then \((u, v)\) will certainly be feasible to accept—it cannot possibly form a cycle by taking this edge. We orient the graph such that all edges directed into \(v\) have low enough probability mass such that with probability \(1/2\), none are active. Then, our algorithm decides which edges to consider such that, with constant probability for every \(v\), it will consider the edges into \(v\) and not the edges out of \(v\). Hence, with no edges out of \(v\) and a good chance that no edges into \(v\) will be active, any edge into \(v\) can be accepted with constant probability. Our method for determining which edges to consider is simple: we take a random cut and consider only the edges in one direction across the cut. Since every edge is oriented into some vertex \(v\), it will be both considered and unblocked with constant probability, as desired.

Unfortunately, this approach is quite specific to a graphic matroid. While some properties of the algorithm might extend to other matroids, we know that it cannot generalize to all matroids: Feldman et al. [2019] prove a lower bound of \(\Omega\left(\frac{\log n}{\log \log n}\right)\) for prophet inequalities that use only non-adaptive thresholds for the class of general matroids. Their lower bound example is a gammoid.

We pose the following two remaining open (but likely very difficult) open questions for understanding how far non-adaptive constant-factor approximations reach between graphic matroids and the lower bound of a gammoid.

**Open Problem 1.** What is the boundary within matroids for non-adaptive constant-factor approximations?

**Open Problem 2.** How do approximations decay for non-adaptive thresholds as matroids become more complex?

### 1.2 Additional Related Work

**Non-Adaptive Thresholds.** As mentioned, the two predominant approaches for achieving \(1/2\)-approximation in the single-item setting are both non-adaptive [Samuel-Cahn, 1984; Kleinberg and Weinberg, 2019]. Chawla et al. [2010] provide non-adaptive \(1/2\)-approximations to the prophet for both \(k\)-uniform and partition matroids; they also give a non-adaptive \(O(\log r)\)-approximation for general matroids, where \(r\) is the rank of the matroid. Recent work by Gravin and Wang [2019] gives a non-adaptive algorithm that guarantees a 3-approximation to the prophet for online bipartite matching, which is the intersection of two matroids. Chawla et al. [2020] optimize non-adaptive thresholds for the \(k\)-uniform settings depending on the range that \(k\) is in, improving existing guarantees in the \(k < 20\) regime. No non-adaptive algorithms are known beyond uniform and partition matroids and the special case of bipartite matching.

**Constrained Non-Adaptive.** Another class of algorithms uses non-adaptive thresholds and a restricted feasibility constraint. That is, given a feasibility constraint \(\mathcal{S}\) and the prior distributions for \(n\) items, the algorithms set, prior to the arrival of all items, thresholds \(T_i\) for each item and a restricted feasibility constraint \(\mathcal{S}'\) such that \(\mathcal{S}' \subset \mathcal{S}\). Then, an item is accepted if it exceeds its threshold and is feasible with respect to the items already accepted and the subconstraint \(\mathcal{S}'\). Notice that an item could exceed its threshold, be feasible with respect to previously accepted items and \(\mathcal{S}\), and yet not be accepted because it is not feasible with respect to previously accepted
items and \( S' \). In essence, imposing a subconstraint is equivalent to adaptively changing an item’s threshold to \( T_i = \infty \) if the item is not feasible with respect to the subconstraint.

Why is this different than when the gambler rejects an item that exceeds its threshold but is not feasible with respect to \( S \)? We can interpret the gambler’s value as constrained-additive with respect to \( S \), so the gambler does not have any marginal gain for items that are infeasible with respect to \( S \) and the items he has already accepted. Hence, he has no reason to take items with no positive marginal value to him. This is not a restriction on the algorithm, but rather a result of the gambler’s valuation class.

Chawla et al. [2010] first produced a \( \frac{1}{3} \)-approximation to the prophet for graphic matroids using non-adaptive thresholds with a partition matroid subconstraint. In a very elegant approach, Feldman et al. [2016] produce an Online Contention Resolution Scheme (OCRS) that yields a \( \frac{1}{4} \)-approximation for all matroids using non-adaptive thresholds and a subconstraint built cleverly from the structure of the given matroid.

**Prophet Inequalities Beyond Matroids.** Prophet inequalities are well-studied and the literature is far too broad to cover; see Lucier [2017] for an excellent survey. Note, however, that dynamic algorithms yield good approximations to the prophet in settings reaching beyond matroids. In addition to matroids, the approach of Feldman et al. [2016] also applies to matchings, knapsack constraints, and the intersections of each. Very recent work by Düttig et al. [2020b] gives an algorithm guaranteeing an \( O(\log \log n) \)-approximation for the very general setting of multiple buyers with subadditive valuations.

**Direct Applications to Pricing.** The Chawla et al. [2010] reduction from order-oblivious posted pricings to prophet inequalities was only the first of many pricing applications of prophet inequalities. Feldman et al. [2015] considers the setting where buyers arrive online and face posted prices for items; non-adaptive anonymous prices are posted for each item equal to half its contribution to the optimal welfare. These prices guarantee \( \frac{1}{2} \)-approximation to the optimal welfare for fractionally subadditive valuations. Note that this is a prophet inequality when there is only one item. Düttig et al. [2017, 2020a] connect posted prices and prophet inequalities: they interpret the Kleinberg and Weinberg [2019] thresholds as “balanced prices” and derive an economic intuition for the proof. They extended these balanced prices to more complex settings, including a variety of feasibility constraints and valuation classes. The approach is to prove guarantees in the full information setting, where the realized values are known in advance. Then, via an extension theorem, they prove that the results hold for Bayesian settings too, where distributions are known but values are unknown. Note that their balanced prices result in non-adaptive anonymous prices for all settings they consider except for matroids feasibility constraints, where they remain adaptive and buyer-specific. The recent work of Düttig et al. [2020b] also implements posted prices for buyers with subadditive valuations, but rather than balanced prices, provides a weaker sufficient condition to get a tighter approximation, and shows the existence of such prices through a primal-dual approach.

**More Subtle Applications in Mechanism Design and Analysis.** Beyond direct applications to pricing, prophet inequalities have also been used in to build more complex mechanisms and prove approximation guarantees. Chawla and Miller [2016] design a two-part tariff mechanism to approximate optimal revenue for matroid-constrained buyers. Their benchmark is an ex-ante
relaxation, and they use an OCRS \cite{feldman2016} to achieve a constant fraction of that revenue. \cite{cai2017} prove that the better of a sequential posted price mechanism (where each buyer can only buy one item) and an anonymous sequential posted price mechanism with an entry fee yields a constant-approximation to the optimal revenue for multiple fractionally subadditive buyers (and $O(\log n)$-approximation for fully subadditive). In a specific case of their analysis that analyzes the core of the core (a double core-tail analysis follow the original of \cite{li2013}), they use \cite{feldman2015}. Work by \cite{cai2019} approximates the optimal profit—seller revenue minus cost—for constrained-additive buyers. Like \cite{chawla2016}, they also construct their benchmark using the ex-ante relaxation and use OCRS to bound a term here as well. Recent work by \cite{cai2021} studies gains from trade approximation in a two-sided market with a constrained-additive buyer and single-dimensional sellers—both the single-item prophet inequality of \cite{kleinberg2019} and an OCRS are used to inspire prices for both the buyer and the sellers simultaneously and then show that enough gains from trade will be received to approximate one specific part of their benchmark.

2 Preliminaries

**Definition 1.** A matroid $M = (N, \mathcal{I})$ is defined by a ground set of elements $N$ (with $|N| = n$) and a set of independent sets $\mathcal{I} \subseteq 2^N$. It is a matroid if and only if it satisfies the following two properties:

1. Downward-closed: If $I \subset J$ and $J \in \mathcal{I}$ then $I \in \mathcal{I}$.
2. Matroid-exchange: For $I, J \in \mathcal{I}$, if $|J| > |I|$ then there exists some $i \in J \setminus I$ such that $I \cup \{i\} \in \mathcal{I}$.

We review several standard notions for matroids:

- The **rank** of a set $\text{rank}(S)$ is the size of the largest independent set in $S$: $\max\{|I| \mid I \in \mathcal{I}, I \subseteq S\}$.
- The **span** of a set $\text{span}(S)$ is the largest set that contains $S$ and has the same rank as $S$: $\{i \in N \mid \text{rank}(S \cup \{e\}) = \text{rank}(S)\}$.
- An element $i$ is spanned by a set $S$ when $i \in \text{span}(S)$.

We will informally use the language “blocked” (by a set $S$) to mean that an element is spanned (by the set $S$), and similarly “unblocked” to mean that an element is not spanned (by the set $S$).

For any matroid $M$, we have the **matroid polytope** $\mathcal{P}_M = \{p \in \mathbb{R}^M_{\geq 0} \mid \forall S \in 2^N, \sum_{i \in S} p_i \leq \text{rank}(S)\}$. That is, $\mathcal{P}_M$ is the convex hull of the independent sets $\mathcal{I}$.

**Definition 2.** A Matroid Prophet Inequality instance $(\vec{X}, M)$ is given by a matroid $M = (N, \mathcal{I})$ and distribution of values $\vec{X}$ for the $n$ items that are the ground set $N$. $X_i$ denotes the random variable representing the value for item $i$.

For any given matroid prophet inequality instance, we let $\text{OPT}(\vec{X}, M)$ denote the value of the prophet’s set in expectation of the value of the items. Formally, $\text{OPT}(\vec{X}, M) = \mathbb{E}\left[\max_{I \in \mathcal{I}} \sum_{i \in I} X_i\right]$. We omit the distributions $\vec{X}$ or matroid $M$ when it is obvious from context.
Definition 3. A non-adaptive threshold algorithm is given an instance \((\vec{X}, M)\) and determines thresholds \(\vec{T}\). A threshold \(T_i\) for each item \(i\) is a function only of the random variables \(\vec{X}\) (and, in particular, not as a function of any realizations of \(\vec{X}\) or whether previous items have exceeded thresholds thus far).

For any non-adaptive thresholds \(\vec{T}\), we let \(\text{Alg}(\vec{X}, M, \vec{T})\) denote the expected value obtained by the algorithm. Again, we omit the parameters when they are clear from context.

3 Where Straightforward Extensions Fail

Both of the non-adaptive single-item approaches—the probabilistic approach of Samuel-Cahn [1984] and the \(\frac{1}{2}\text{Opt}\) approach of Kleinberg and Weinberg [2019]—extend to the \(k\)-uniform matroid setting, in which any set of size at most \(k\) is feasible. We first see why these approaches work for \(k\)-uniform matroids yet break down for graphic matroids. Then, we attempt to use an idea for graphic matroids from Chawla et al. [2010] to develop a non-adaptive algorithm, and again highlight where the approach breaks down.

We begin with the two generalizations to \(k\)-uniform methods. Note that we do not claim either as part of our contribution, although to the best of our knowledge, neither approaches’ generalized thresholds and proof is written anywhere.

Formally, a \(k\)-uniform matroid is the matroid where, for any given ground set \(N\), \(\mathcal{I} = \{I \subseteq N : |I| \leq k\}\). Bear in mind that \(k = 1\) returns to the single-item case.

The Probabilistic Approach. (Extension of Samuel-Cahn [1984] single-item algorithm to non-adaptive thresholds for the \(k\)-uniform matroid.) Determine the thresholds \(T\) by setting \(\Pr[\text{< } k \text{ item values exceed } T] = \Pr[\geq 1 \text{ slot empty}] = p = \frac{1}{2}\).

\[
\text{Alg}(\vec{X}, T) \geq \sum_i \Pr[i \text{ not blocked}] \mathbb{E}[(X_i - T)^+] + \Pr[\geq k \text{ above } T] \cdot kT
\]

\[
\geq \Pr[\text{< } k \text{ above } T] \sum_i \mathbb{E}[(X_i - T)^+] + \Pr[\geq k \text{ above } T] \cdot kT
\]

\[
\geq p \mathbb{E} \left[ \max_{S: |S| \leq k} \sum_{i \in S} (X_i - T)^+ \right] + (1 - p)kT
\]

\[
\geq p \mathbb{E} \left[ \sum_{i \in S} X_i - kT \right] + (1 - p)kT
\]

\[
= \frac{1}{2} \left( \mathbb{E} \left[ \max_{S: |S| \leq k} \sum_{i \in S} X_i \right] \right) - \frac{1}{2}kT + \frac{1}{2}kT
\]

\[
= \frac{1}{2} \mathbb{E} \left[ \sum_{i \in S} X_i \right] = \frac{1}{2}\text{Opt}(\vec{X}).
\]

For uniform matroids, a simple characterization based on size exists for sets that do not span any elements that have yet to arrive: they need only be of size strictly less than \(k\). This property does not hold for more complex matroids.
The “Thresholds as Constant-Fraction of Prophet” Approach. (Extension of Kleinberg and Weinberg [2019] single-item algorithm to non-adaptive thresholds for the \( k \)-uniform matroid; almost identical to those in Chawla et al. [2010].) Set \( T = \frac{1}{2k} \mathbb{E} \left[ \max_{S:|S| \leq k} \sum_{i \in S} X_i \right] = \frac{1}{2k} \text{Opt}(\vec{X}) \).

\[
\text{Alg} \geq \sum_i \Pr[i \text{ not blocked}] \mathbb{E}[(X_i - T)^+] + \Pr[\geq k \text{ above } T]kT
\]

\[
\geq \Pr[\geq k \text{ above } T] \mathbb{E} \left[ \sum_i (X_i - T)^+ \right] + \Pr[\geq k \text{ above } T]kT
\]

\[
\geq p \mathbb{E} \left[ \max_{S:|S| \leq k} \sum_{i \in S} (X_i - T)^+ \right] + (1 - p)kT
\]

\[
= p \left( \mathbb{E} \left[ \max_{S:|S| \leq k} \sum_{i \in S} X_i \right] - kT \right) + (1 - p) \frac{1}{2} \mathbb{E} \left[ \max_{S:|S| \leq k} \sum_{i \in S} X_i \right]
\]

\[
= \frac{1}{2} \mathbb{E} \left[ \max_{S:|S| \leq k} \sum_{i \in S} X_i \right] = \frac{1}{2} \text{Opt}(\vec{X}).
\]

In uniform matroids, any element is exchangeable for any other element. Then so long as it contributes enough value, such as at least a constant fraction of the average contribution to the optimal basis, there is no reason not to accept an element. However, this does not hold for more complex matroids. A particular element, even if extremely high value, may cause so many other elements to be spanned that it is not worth taking.

One can imagine more nuanced extensions of either such approach—probabilistic thresholds for \( i \) according to how many elements it might block, or value-based thresholds for \( i \) based on the value of the sets it might block. However, any such extension would require a matroid-specific understanding of the relationship between elements, and element-specific thresholds.

Note that in addition to uniform matroids, both approaches easily extend to partition matroids by applying the approach to thresholds specific to the uniform matroid in each partition.

The Constrained Non-Adaptive Approach. Chawla et al. [2010] construct non-adaptive thresholds for a graphic matroid that work so long as the algorithm can enforce an additional subconstraint. Specifically, they cleverly partition the graph such that, so long as at most one edge is accepted from each partition, then an independent set is guaranteed. Then as items arrive, they are accepted if and only if they exceed their threshold \( T \) and are feasible with respect to the subconstraint—that is, no previous item from its partition has been accepted. This approach guarantees a \( \frac{1}{3} \)-approximation.

As discussed in the introduction, enforcing a subconstraint \emph{is} in fact adaptive. But, we \emph{could}, for example, randomly select one item from each partition in advance, defining our set for consideration \( C \). Then, as items arrive, in each partition, we consider only the item in \( C \), ignoring all other items from each partition. That is, we leave thresholds the same for all items in \( C \) and \emph{a priori} set \( T_i = \infty \) for all \( i \notin C \). This ensures that we only consider a set that complies with our feasibility constraint \emph{without} making any modifications online. Note that we can select items to be in the consideration set \( C \) with whatever probabilities we choose, even in a correlated fashion—as long as we make them prior to items arriving—thus setting all thresholds to \( T_i \) or \( \infty \) in advance. Is there
some clever way that we can implement our feasibility constraint, or any feasibility constraint, yet maintain a constant-factor approximation?

For the approach of CHMS, we might observe that a convenient property that bounds the probability mass of each partition could allow us to form a probability distribution over elements in each partition (i.e. place item $i$ in $C$ with probability $p_i/2$). However, this approach in fact reduces the probability too much, as it combines the probability that the element is active with the probability it is considered, and is no longer constant. If we use a constant probability, it would instead sell to too low of a quantile.

If such an approach were to work, we could convert any non-adaptive matroid prophet inequality to a prophet inequality, as a greedy OCRS exists for all matroids and constructs constrained non-adaptive thresholds all matroids [Feldman et al., 2016]. However, Feldman et al. [2019] also prove a super-constant lower bound of $\Omega\left(\frac{\log n}{\log \log n}\right)$, so guarantees cannot possibly go through for every matroid. Thus, an interesting direction for future work is to characterize when an approach of converting constrained non-adaptive thresholds to a fully non-adaptive algorithm in this way would maintain good guarantees.

4 The Ex-Ante Relaxation to the Matroid Polytope

Reducing a given matroid prophet inequality instance to one with Bernoulli distributions that sits within the matroid polytope is “standard,” and is used in [Feldman et al., 2016]. It’s “just” an ex-ante relaxation to the matroid polytope, and expert readers can safely skip this section. However, we present the reduction in detail for comprehensiveness and ease of reading, as we did not find it elsewhere.

First, given arbitrary independent random variables $X_i$, we reduce the problem to designing an algorithm for independent Bernoulli random variables $X'_i$:

$$X'_i = \begin{cases} 
    t_i & \text{w.p. } p_i \\
    0 & \text{w.p. } 1 - p_i,
\end{cases}$$

where $\bar{p} \in \mathcal{P}_M$.

Reducing to Bernoulli random variables gives two properties which greatly simplify the design of an algorithm:

1. Each element of the ground set is either active or inactive; and
2. There exists a worst-case total ordering of the elements.

The worst-case ordering is the typical greedy ordering. Assume $t_i \leq t_{i+1}$; then greedily selecting elements in order (maintaining independence and according to the rules of our algorithm) results in the lowest weight outcome over all orderings. For the rest of the paper, we assume $t_i \leq t_{i+1}$ for $1 \leq i < n$.

We now state our reduction formally.

**Lemma 1.** Given a matroid $M = (N, \mathcal{I})$ and independent random weights $X_i$, $i \in N$, there exist independent Bernoulli weights $X'_i$, where $X'_i = t_i$ w.p. $p_i$ and $\bar{p} \in \mathcal{P}_M$, such that

$$\text{Opt}(\bar{X}, M) \leq \sum_i p_i t_i.$$
Furthermore, for any algorithm $\text{ALG}$,

$$\text{ALG}(\vec{X}) \geq \text{ALG}(\vec{X'}).$$

**Proof.** First, rewrite the original optimal value as a sum over the ground set:

$$\text{OPT}(\vec{X}, M) = E\left[ \max_{I \in \mathcal{I}} \sum_{i \in I} X_i \right] = \sum_{i \in \mathcal{N}} \Pr[i \in I^*] \cdot E[X_i \mid i \in I^*],$$

where $I^*$ is the maximum weight basis: $I^* = \text{argmax}_{I \in \mathcal{I}} \sum_{i \in I} X_i$. Now let $p_i = \Pr[i \in I^*]$—the ex-ante probability that $i$ is in the prophet’s solution. Since $\vec{p}$ is a convex combination of basis vectors, then $\vec{p} \in \mathcal{P}_M$.

Now, observe that $E[X_i \mid X_i \geq F_i^{-1}(1-p_i)] \geq E[X_i \mid H]$ for any event $H$ with $\Pr[H] = p_i$. Let $t_i = E[X_i \mid X_i \geq F_i^{-1}(1-p_i)]$; then in particular $t_i \geq E[X_i \mid i \in I^*]$. Hence

$$\text{OPT}(\vec{X}, M) \leq \sum_i p_i t_i.$$

Finally, to see that $\text{ALG}(\vec{X}) \geq \text{ALG}(\vec{X'})$, we simply couple $\vec{X}$ and $\vec{X'}$, so that $X_i \geq t_i$ if and only if $X'_i = t_i$. For any ordering of the elements, the algorithm applied to the original instance selects the same items as the algorithm applied to the Bernoulli instance. \hfill \blacksquare

## 5 A Constant-Factor Approximation for Graphic Matroids

Given a Bernoulli instance from the matroid polytope, we show how to utilize it to obtain a constant-factor non-adaptive algorithm for graphic matroids.

A graphic matroid is defined by an undirected graph $G$ with vertices $V$ and edges $E$. The edges of the graph form the ground set, and the independent sets $\mathcal{I}$ are forests, i.e., cycle-free sets of edges: $\mathcal{I} = \{I \subseteq E : I$ contains no cycles$\}$; every spanning tree is a basis. In light of Lemma 1, each edge $i \in E$ has an associated weight $t_i$ and is active (non-zero) with probability $p_i$, where $\vec{p} \in \mathcal{P}_G$, the matroid polytope for the graphic matroid $G$. The objective is then to select a maximum weight spanning tree. As discussed in the previous section, we assume the edges arrive in order with $t_i \leq t_{i+1}$ for all $1 \leq i \leq n - 1$; this order obtains the worst-case performance.

Our approach works by considering only a subset of the edges which has the properties that (1) a significant fraction of the prophet’s benchmark is accounted for and yet (2) with constant probability, elements selected earlier in the ordering do not block later elements.

Specifically, we do this in two steps. First, we show there exists a way to direct the edges such that every edge has at most a constant probability of being spanned by edges except for those leaving the vertex into which it is directed. Then, we take a random cut in the graph and allow our algorithm to select only edges crossing the cut in one direction, ensuring that for every vertex, the edges entering it are considered while the edges leaving it are not with constant probability.

**Notation.** We use $b_i(S)$ to denote the probability that element $i$ is “blocked” or spanned by the active elements in a set $S$ with respective to active probabilities $\vec{p} \in \mathcal{P}_M$. For $\vec{p} \in \mathcal{P}_M$, let $R_{\vec{p}}(S)$
be the random set containing \(i \in S\) independently with probability \(p_i\). We call this the “active” set. Formally, \(b_i(S) = \Pr[i \in \text{span}(R_{\vec{p}}(S \setminus \{i\}))]\). Notice that even if \(i \in S\), we do not worry that it would span itself.

One convenience of using the ex-ante relaxation is that, so long as each element is unblocked with constant probability, that is, \(1 - b_i(S) \geq c\), we obtain a constant-factor approximation.

5.1 Directing the Graph

**Lemma 2.** For \(p \in \frac{1}{4}\mathcal{P}_G\), there exists a way to orient the edges of \(G\) such that for each vertex the total probability mass of incoming edges is at most \(1/2\).

**Proof.** Any vector from the graphic matroid polytope \(\mathcal{P}_G\) is a convex combination bases, or spanning trees. The average vertex degree in any spanning tree is at most 2, so the average fractional degree in a convex combination of spanning trees is at most 2, and hence the average fractional degree under the scaled \(\vec{p} \in \frac{1}{4}\mathcal{P}_G\) is at most \(\frac{1}{2}\).

Let \(\text{in-deg}(v)\) denote the fractional in-degree of \(v\) in the constructed directed graph. That is, the sum of the “active” probabilities for the edges directed into \(v\). We can find an orientation of the edges in the graph given probabilities \(\vec{p}\) such that \(\text{in-deg}(v) \leq \frac{1}{2}\) for all vertices \(v\): because the average degree is at most \(\frac{1}{2}\), there exists some vertex \(v\) with degree at most \(\frac{1}{2}\). Orient all of the edges incident to \(v\) toward \(v\), as \(\text{in-deg}(v) \leq \frac{1}{2}\), and then recurse on the graph among the remaining vertices.

**Corollary 1.** Given a graph as guaranteed by Lemma 2, let \(\text{in}(v)\) be the set of incoming edges to vertex \(v\) and let \(\text{out}(v)\) be the outgoing edges. For any \(i\), let \(v\) be the vertex such that \(i \in \text{in}(v)\). Then for any \(S \subseteq E\),

\[
b_i(S \setminus \text{out}(v)) \leq \frac{1}{2}.
\]

**Proof.** Observe that for \(i \in \text{in}(v)\), \(v\) cannot be spanned by a set that contains no other edges incident to \(v\). Then in order for \(i\) to be spanned in \(S \setminus \text{out}(v)\), at least one edge in \(\text{in}(v)\) other than \(i\) must be active. By construction, \(\sum_{i \in \text{in}(v)} p_i \leq \frac{1}{2}\). So the probability that no edges are active is at least \(\frac{1}{2}\) by the union bound.

5.2 Random Cut

Assume \(\vec{p} \in \frac{1}{4}\mathcal{P}_G\), and direct the graph as described above. Let \(A \subseteq V\) be a random set of vertices such that each vertex is included in \(A\) independently with probability \(1/2\), and let \(\hat{A} = B = V \setminus A\). Let \(\hat{S}\) be the set of directed edges across the cut from \(A\) to \(B\), formally, \(\hat{S} = \{i : i \in \text{out}(u) \cap \text{in}(v), u \in A, v \in B\}\).

**Claim 1.**

\[
\mathbb{E}_{\hat{S}} \left[ \sum_{i \in \hat{S}} p_i t_i (1 - b_i(\hat{S})) \right] \geq \frac{1}{8} \sum_{i \in E} p_i t_i.
\]
Proof.

\[
E_{\hat{S}} \left[ \sum_{i \in \hat{S}} p_i t_i (1 - b_i(\hat{S})) \right] = \sum_{(u,v) \in E} p_{uv} t_{uv} \Pr[(u,v) \in \hat{S}] E \left[ 1 - b_{uv}(\hat{S}) \big| (u,v) \in \hat{S} \right]
\]
\[
= \sum_{(u,v) \in E} p_{uv} t_{uv} \Pr[u \in A] \Pr[v \in B] E \left[ 1 - b_{uv}(\hat{S}) \big| u \in A, v \in B \right]
\]
\[
= \frac{1}{4} \sum_{(u,v) \in E} p_{uv} t_{uv} E \left[ 1 - b_{uv}(\hat{S}) \big| u \in A, v \in B \right]
\]
\[
\geq \frac{1}{8} \sum_{(u,v) \in E} p_{uv} t_{uv}
\]

where the last inequality follows from Corollary \[ \square \]

5.3 Final Algorithm

For discrete random variables $\vec{X}$, our algorithm is constructive, albeit not efficient, because we can compute $\vec{p}$ and $\vec{t}$ as guaranteed by Lemma \[ \square \] (Of course, we can discretize continuous random variables to arbitrary approximation.)

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1: Compute $\vec{p}$ and $\vec{t}$ as guaranteed by Lemma \[ \square \]
2: Direct the graph as outlined in Lemma \[ \square \]
3: Choose a cut $(A, B)$ uniformly at random; let $\hat{S} = \{ i : i \in \text{out}(u) \cap \text{in}(v), u \in A, v \in B \}$.
4: For all edges $i \in \hat{S}$, set $T_i = t_i$.
5: For all edges $i \not\in \hat{S}$, set $T_i = \infty$.

Step 3 can be derandomized using the standard Max-Cut derandomization. Our main result is that this algorithm gives a $\frac{1}{32}$-approximation.

**Theorem 2.** Let $G$ be a graphic matroid with independent edge weights $\vec{X}$. Then

$$32 E[\text{ALG}(G, \vec{X})] \geq \text{OPT}(G, \vec{X}).$$

**Proof.** Let $\vec{p} \in \mathcal{P}_G$ and $\vec{t}$ be the probabilities and values guaranteed by Lemma \[ \square \]. Let $p'_i = \frac{1}{4} p_i$. Then our algorithm obtains $\text{ALG} = E_{\hat{S}} \left[ \sum_{i \in \hat{S}} p'_i t_i (1 - b_i(\hat{S})) \right]$, which by our construction of $\hat{S}$ and Claim \[ \square \] gives

$$\text{ALG} \geq \frac{1}{8} \sum_{(u,v) \in E} p'_{uv} t_{uv} = \frac{1}{32} \sum_{i \in E} p_i t_i.$$ 

$$\square$$

Our approximation factor is, of course, a factor of 16 worse than the dynamic thresholds of \[ \text{Kleinberg and Weinberg, 2019} \] and a 10.67-factor worse than the constrained non-adaptive thresholds of \[ \text{Chawla et al., 2010} \]. However, our guarantee holds for fully non-adaptive thresholds, and thus will guarantee truthful mechanisms in multi-parameter mechanism design applications.
References

Yang Cai and Mingfei Zhao. Simple Mechanisms for Subadditive Buyers via Duality. In Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2017, pages 170–183, New York, NY, USA, 2017. ACM. ISBN 978-1-4503-4528-6. doi: 10.1145/3055399.3055465. URL http://doi.acm.org/10.1145/3055399.3055465.

Yang Cai and Mingfei Zhao. Simple mechanisms for profit maximization in multi-item auctions. In Proceedings of the 2019 ACM Conference on Economics and Computation, EC ’19, page 217?236, New York, NY, USA, 2019. Association for Computing Machinery. ISBN 9781450367929. doi: 10.1145/3328526.3329616. URL https://doi.org/10.1145/3328526.3329616.

Yang Cai, Kira Goldner, Steven Ma, and Mingfei Zhao. On Multi-Dimensional Gains from Trade Maximization. In ACM-SIAM Symposium on Discrete Algorithms (SODA21), 2021.

Shuchi Chawla and J. Benjamin Miller. Mechanism design for subadditive agents via an ex ante relaxation. In Proceedings of the 2016 ACM Conference on Economics and Computation, EC ’16, pages 579–596, New York, NY, USA, 2016. ACM. ISBN 978-1-4503-3936-0. doi: 10.1145/2940716.2940756. URL http://doi.acm.org/10.1145/2940716.2940756.

Shuchi Chawla, Jason D Hartline, David L Malec, and Balasubramanian Sivan. Multi-parameter mechanism design and sequential posted pricing. In Proceedings of the forty-second ACM symposium on Theory of computing, pages 311–320. ACM, 2010. URL http://dl.acm.org/citation.cfm?id=1806733.

Shuchi Chawla, Nikhil Devanur, and Thodoris Lykouris. Static pricing for multi-unit prophet inequalities, 2020.

Paul Dütting, Michal Feldman, Thomas Kesselheim, and Brendan Lucier. Prophet inequalities made easy: Stochastic optimization by pricing non-stochastic inputs. In Chris Umans, editor, 58th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2017, Berkeley, CA, USA, October 15-17, 2017, pages 540–551. IEEE Computer Society, 2017. doi: 10.1109/FOCS.2017.56. URL https://doi.org/10.1109/FOCS.2017.56.

Paul Dütting, Michal Feldman, Thomas Kesselheim, and Brendan Lucier. Prophet inequalities made easy: Stochastic optimization by pricing nonstochastic inputs. SIAM J. Comput., 49(3): 540–582, 2020a. doi: 10.1137/20M1323850. URL https://doi.org/10.1137/20M1323850.

Paul Dütting, Thomas Kesselheim, and Brendan Lucier. An o(log log m) prophet inequality for subadditive combinatorial auctions. In 61st IEEE Annual Symposium on Foundations of Computer Science, FOCS 2020, 2020b. URL https://arxiv.org/abs/2004.09784.

Michal Feldman, Nick Gravin, and Brendan Lucier. Combinatorial auctions via posted prices. In Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 123–135. SIAM, 2015.

Moran Feldman, Ola Svensson, and Rico Zenklusen. Online contention resolution schemes. In Robert Krauthgamer, editor, Proceedings of the Twenty-Seventh Annual ACM-SIAM
Moran Feldman, Ola Svensson, and Rico Zenklusen. Online contention resolution schemes, 2019.

Nikolai Gravin and Hongao Wang. Prophet inequality for bipartite matching: Merits of being simple and non-adaptive. In Proceedings of the 2019 ACM Conference on Economics and Computation, EC ’19, pages 93–109, New York, NY, USA, 2019. Association for Computing Machinery. ISBN 9781450367929. doi: 10.1145/3328526.3329604. URL https://doi.org/10.1145/3328526.3329604.

Mohammad Taghi Hajiaghayi, Robert Kleinberg, and Tuomas Sandholm. Automated online mechanism design and prophet inequalities. In Proceedings of the 22nd National Conference on Artificial Intelligence - Volume 1, AAAI'07, page 58?65. AAAI Press, 2007. ISBN 9781577353232.

Robert Kleinberg and S. Matthew Weinberg. Matroid prophet inequalities. In Howard J. Karloff and Toniann Pitassi, editors, Proceedings of the 44th Symposium on Theory of Computing Conference, STOC 2012, New York, NY, USA, May 19 - 22, 2012, pages 123–136. ACM, 2012. doi: 10.1145/2213977.2213991. URL https://doi.org/10.1145/2213977.2213991.

Robert Kleinberg and S. Matthew Weinberg. Matroid prophet inequalities and applications to multi-dimensional mechanism design. Games Econ. Behav., 113:97–115, 2019. doi: 10.1016/j.jegb.2014.11.002. URL https://doi.org/10.1016/j.jegb.2014.11.002.

Ulrich Krengel and Louis Sucheston. Semiamarts and finite values. Bulletin of the American Mathematical Society, 83(4):745–747, 1977.

Xinye Li and Andrew Chi-Chih Yao. On revenue maximization for selling multiple independently distributed items. Proceedings of the National Academy of Sciences, 110(28):11232–11237, 2013. URL http://dx.doi.org/10.1073/pnas.1309533110.

Brendan Lucier. An economic view of prophet inequalities. SIGecom Exch., 16(1):24?47, September 2017. doi: 10.1145/3144722.3144725. URL https://doi.org/10.1145/3144722.3144725.

Ester Samuel-Cahn. Comparison of threshold stop rules and maximum for independent nonnegative random variables. The Annals of Probability, pages 1213–1216, 1984.