The stable and the real rank of $\mathcal{Z}$-absorbing $C^*$-algebras

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Abstract
Suppose that $A$ is a $C^*$-algebra for which $A \cong A \otimes \mathcal{Z}$, where $\mathcal{Z}$ is the Jiang–Su algebra: a unital, simple, stably finite, separable, nuclear, infinite dimensional $C^*$-algebra with the same Elliott invariant as the complex numbers. We show that:

(i) The Cuntz semigroup $W(A)$ of equivalence classes of positive elements in matrix algebras over $A$ is almost unperforated\(^1\).

(ii) If $A$ is exact, then $A$ is purely infinite if and only if $A$ is traceless.

(iii) If $A$ is separable and nuclear, then $A \cong A \otimes O_\infty$ if and only if $A$ is traceless.

(iv) If $A$ is simple and unital, then the stable rank of $A$ is one if and only if $A$ is finite.

We also characterise when $A$ is of real rank zero.

1 Introduction
Jiang and Su gave in their paper [12] a classification of simple inductive limits of direct sums of dimension drop $C^*$-algebras. (A dimension drop $C^*$-algebra is a certain sub-$C^*$-algebra of $M_n(C([0, 1]))$, a precise definition of which is given in the next section.) They prove that inside this class there exists a unital, simple, infinite dimensional $C^*$-algebra $\mathcal{Z}$ whose Elliott invariant is isomorphic to the Elliott invariant of the complex numbers, that is,

\[ (K_0(\mathcal{Z}), K_0(\mathcal{Z})^+, [1]) \cong (\mathbb{Z}, \mathbb{Z}^+, 1), \quad K_1(\mathcal{Z}) = 0, \quad T(\mathcal{Z}) = \{\tau\}, \]

\(^1\)Almost perforation is a natural extension of the notion of weak unperforation for simple ordered abelian (semi-)groups, see Section 3.
where $\tau$ is the unique tracial state on $\mathcal{Z}$. They proved that $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z} \cong \bigotimes_{n=1}^{\infty} \mathcal{Z}$, and that $A \otimes \mathcal{Z} \cong A$ if $A$ is a simple, unital, infinite dimensional AF-algebra or if $A$ is a unital Kirchberg algebra. Toms and Winter have in a paper currently under preparation extended the latter result by showing that $A \otimes \mathcal{Z} \cong A$ for all approximately divisible $C^*$-algebras. Toms and Winter note that upon combining results from [12] with [16, Theorem 8.2] one obtains that a separable $C^*$-algebra $A$ is $\mathcal{Z}$-absorbing if and only if there is a unital embedding of $\mathcal{Z}$ into the relative commutant $\mathcal{M}(A)_{\omega} \cap A'$, where $\mathcal{M}(A)_{\omega}$ is the ultrapower, associated with a free filter $\omega$ on $\mathbb{N}$, of the multiplier algebra, $\mathcal{M}(A)$, of $A$. This provides a partial answer to the question raised by Gong, Jiang, and Su in [9] if one can give an intrinsic description of which (separable, nuclear) $C^*$-algebras absorb $\mathcal{Z}$.

Gong, Jiang, and Su prove in [9] that $(K_0(A), K_0(A)^+) \cong (K_0(A \otimes \mathcal{Z}), K_0(A \otimes \mathcal{Z})^+)$ if and only if $K_0(A)$ is weakly unperforated as an ordered group, when $A$ is a simple $C^*$-algebra; and hence that $A$ and $A \otimes \mathcal{Z}$ have isomorphic Elliott invariant if $A$ is simple with weakly unperforated $K_0$-group. This result indicates that $A \cong A \otimes \mathcal{Z}$ whenever $A$ is “classifiable” in the sense of Elliott (see Elliott, [8], or [23] by the author).

The results quoted above show on the one hand that surprisingly many $C^*$-algebras, including for example the irrational rotation $C^*$-algebras, absorb the Jiang–Su algebra, but on the other hand that not all simple, unital, nuclear, separable $C^*$-algebras are $\mathcal{Z}$-absorbing. Villadsen’s example from [27] of a simple, unital AH-algebra whose $K_0$-group is not weakly unperforated cannot absorb $\mathcal{Z}$. The example by the author in [21] of a simple, unital, nuclear, separable $C^*$-algebra with a finite and an infinite projection is prime (i.e., is not the tensor product of two non type I $C^*$-algebras), and is hence not $\mathcal{Z}$-absorbing. Toms gave in [25] an example of a simple, unital ASH-algebra which is not $\mathcal{Z}$-absorbing, but which has weakly unperforated $K_0$-group. The latter two examples (by the author and by Toms) have the same Elliott invariant as, but are not isomorphic to, their $\mathcal{Z}$-absorbing counterparts; and so they serve as counterexamples to the classification conjecture of Elliott (as it is formulated in [23, Section 2.2]).

It appears plausible that the Elliott conjecture holds for all simple, unital, nuclear, separable $\mathcal{Z}$-absorbing $C^*$-algebras.

In the present paper we begin by showing that the Cuntz semigroup of equivalence classes of positive elements in a $\mathcal{Z}$-absorbing $C^*$-algebra is almost unperforated (a property that for simple ordered abelian (semi-)groups coincides with the weak unperforation property, see Section 3). We use this to show that the semigroup $V(A)$ of Murray–von Neumann equivalence classes of projections in a $\mathcal{Z}$-absorbing $C^*$-algebra $A$, and in some cases also $K_0(A)$, is almost (or weakly) unperforated. We show that the stable rank of $A$ is one if $A$ is a simple, finite, unital $\mathcal{Z}$-absorbing $C^*$-algebra, thus answering in the affirmative
2 Preliminary facts about the Jiang–Su algebra $\mathcal{Z}$

We establish a couple of results that more or less follow directly from Jiang and Su’s paper \cite{12} on their $C^*$-algebra $\mathcal{Z}$.

For each triple of natural numbers $n, n_0, n_1$, for which $n_0$ and $n_1$ divides $n$, the dimension drop $C^*$-algebra $I(n_0, n, n_1)$ is the sub-$C^*$-algebra of $C^*([0,1], M_n)$ consisting of all functions $f$ such that $f(0) \in \varphi_0(M_{n_0})$ and $f(1) \in \varphi_1(M_{n_1})$, where $\varphi_j: M_{n_j} \to M_n$, $j = 0, 1$, are fixed unital *-homomorphisms. (The $C^*$-algebra $I(n_0, n, n_1)$ is—up to *-isomorphism— independent on the choice of the *-homomorphisms $\varphi_j$.) The dimension drop $C^*$-algebra $I(n_0, n, n_1)$ is said to be prime if $n_0$ and $n_1$ are relatively prime, or equivalently, if $I(n_0, n, n_1)$ has no projections other than the two trivial ones: 0 and 1, cf. \cite{12}.

The $C^*$-algebra $I(n, nm, m)$ can, and will in this paper, be realized as the sub-$C^*$-algebra of $C^*([0,1], M_n \otimes M_m)$ consisting of those functions $f$ for which $f(0) \in M_n \otimes \mathbb{C}$ and $f(1) \in \mathbb{C} \otimes M_m$.

A unital *-homomorphism $\psi: I(n_0, n, n_1) \to \mathcal{Z}$ will here be said to be standard, if

$$\tau(\psi(f)) = \int_0^1 \text{tr}(f(t)) \, dm(t), \quad f \in I(n_0, n, n_1), \quad (2.1)$$

where $\tau$ is the unique trace on $\mathcal{Z}$, and where $\text{tr}$ is the normalised trace on $M_n$.

The following theorem is essentially contained in Jiang and Su’s paper (\cite{12}).

\textbf{Theorem 2.1 (Jiang–Su)} Let $n, n_0, n_1$ be a triple of natural numbers where $n_0$ and $n_1$ divide $n$, and where $n_0$ and $n_1$ are relatively prime. As above, let $\tau$ denote the unique trace on $\mathcal{Z}$.

(i) For each faithful tracial state $\tau_0$ on $I(n_0, n, n_1)$ there exists a unital embedding $\psi: I(n_0, n, n_1) \to \mathcal{Z}$ such that $\tau \circ \psi = \tau_0$. In particular, there is a standard unital embedding of $I(n_0, n, n_1)$ into $\mathcal{Z}$.

(ii) Two unital embeddings $\psi_1, \psi_2: I(n_0, n, n_1) \to \mathcal{Z}$ are approximately unitarily equivalent if and only if $\tau \circ \psi_1 = \tau \circ \psi_2$. In particular, $\psi_1$ and $\psi_2$ are approximately unitarily equivalent if they both are standard.

\textbf{Proof:} For brevity, denote the prime dimension drop $C^*$-algebra $I(n_0, n, n_1)$ by $I$. 

Find an increasing sequence $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$ of sub-$C^*$-algebras of $\mathcal{Z}$ such that each $B_k$ is (isomorphic to) a prime dimension drop algebra of the form $I(n_0(k), n(k), n_1(k))$, and such that $\bigcup_{k=1}^{\infty} B_k$ is dense in $\mathcal{Z}$. Simplicity of $\mathcal{Z}$ ensures that $n_0(k), n(k),$ and $n_1(k)$ all tend to infinity as $k$ tends to infinity.

It is shown in \cite{12} Lemma 2.3 that $K_0(I)$ and $K_0(B_k)$ are infinite cyclic groups each generated by the class of the unit in the corresponding algebra, and $K_1(I)$ and $K_1(B_k)$ are both trivial. This entails that $KK(\psi_1) = KK(\psi_2)$ for any pair of unital $^*$-homomorphisms $\psi_1, \psi_2: I \to B_k$.

In both parts of the proof we shall apply the uniqueness theorem, \cite{12} Corollary 5.6, in Jiang and Su’s paper. For each $x \in [0, 1]$ consider the extremal tracial state $\tau_x$ on a dimension drop $C^*$-algebra $I(m_0, m, m_1)$ given by $\tau_x(f) = \text{tr}(f(x))$ (where tr is the normalised trace on $M_m$). Each self-adjoint element $f$ in a dimension drop $C^*$-algebra $I(m_0, m, m_1)$ gives rise to a function $\hat{f} \in C_\mathbb{R}([0, 1])$ defined by $\hat{f}(x) = \tau_x(f)$. If $f$ is a self-adjoint element in the center of $I(m_0, m, m_1)$, then $\hat{f} = f$. Every $^*$-homomorphism $\psi: I(m_0, m, m_1) \to I(m_0', m', m_1')$ between two dimension drop $C^*$-algebras induces a positive linear mapping $\psi_*: C_\mathbb{R}([0, 1]) \to C_\mathbb{R}([0, 1])$ given by $\psi_*(f) = \hat{\psi(f)}$ (when we identify $C_\mathbb{R}([0, 1])$ with the self-adjoint part of the center of $I(m_0, m, m_1)$). Let $h_{D,d} \in C_\mathbb{R}([0, 1])$, $d = 1, 2, \ldots, D$, be the test functions defined in \cite{12} (5.5) (and previously considered by Elliott).

(i). Let $\tau_0$ be a faithful trace on $I = I(n_0, n, n_1)$. Let $F_1 \subseteq F_2 \subseteq \cdots$ be an increasing sequence of finite subsets of $I$ with dense union. By a one-sided approximate intertwining argument (after Elliott, see eg. \cite{17} Theorem 1.10.14) it suffices to find a sequence $1 \leq m(1) < m(2) < m(3) < \cdots$ of integers, a sequence $\psi_j: I \to B_{m(j)}$ of unital $^*$-homomorphisms, and unitaries $u_j \in B_{m(j)}$ such that

$$\|u_{j+1}^* \psi_j(f) u_{j+1} - \psi_{j+1}(f)\| \leq 2^{-j}, \quad |\tau(\psi_j(f)) - \tau_0(f)| \leq 1/j, \quad f \in F_j,$$

for all $j \in \mathbb{N}$. It will then follow that there exist a $^*$-homomorphism $\psi: I \to \mathcal{Z}$ and unitaries $v_j \in \mathcal{Z}$ such that $\|v_j^* \psi(f)v_j - \psi(f)\|$ tends to zero as $j$ tends to infinity for all $f \in I$. This will imply that $\tau \circ \psi = \tau_0$.

For each $j$ choose a natural number $D_j$ such that $\|f(s) - f(t)\| \leq 2^{-j}$ for all $f \in F_j$ and for all $s, t \in [0, 1]$ with $|s - t| \leq 6/D_j$. Let $G_j$ be the finite set that contains $F_j$ and the test functions $h_{D_j,d}$, $d = 1, \ldots, D_j$. Put

$$c_j = \frac{2}{3} \min \{\tau_0(h_{D_j,d-1} - h_{D_j,d}) \mid d = 2, 3, \ldots, D_j\} > 0.$$
By [12, Corollary 4.4] — if $m(j)$ are chosen large enough — there exists for each $j$ a unital $^*$-homomorphism $\psi_j : I \to B_{m(j)}$ such that $|\tau'(\psi_j(f)) - \tau_0(f)| < \min\{1/j, c_j/2, c_{j-1}/2\}$ for all tracial states $\tau'$ on $B_{m(j)}$ and for all $f \in G_j$. In particular, $|\tau(\psi_j(f)) - \tau_0(f)| < 1/j$ for $f \in F_j$, and

$$(\psi_j)_*(h_{D_j,d-1} - h_{D_j,d}) \geq c_j,$$

$$(\psi_{j+1})_*(h_{D_j,d-1} - h_{D_j,d}) \geq c_j,$$

$$\|(\psi_{j+1})_*(h_{D_j,d}) - (\psi_j)_*(h_{D_j,d})\|_\infty < c_j,$$

for $d = (1), 2, 3, \ldots, D_j$. It now follows from [12, Corollary 5.6] that there exists a unitary $u_{j+1}$ in $B_{m(j+1)}$ such that $\|u_{j+1}^*\psi_j(f)u_{j+1} - \psi_{j+1}(f)\| \leq 2^{-j}$.

(ii). The “only if” part is trivial. Assume that $\tau \circ \psi_1 = \tau \circ \psi_2$. Take a finite subset $F$ of $I$ and let $\varepsilon > 0$. It is shown in [12] (and in [6]) that the dimension drop $C^*$-algebra $I = I(n_0, n, n_1)$ is semiprojective. We can therefore, for some large enough $k_0$, find unital $^*$-homomorphisms $\psi_1^{(k)}, \psi_2^{(k)} : I \to B_k$ for each $k \geq k_0$ such that

$$\lim_{k \to \infty} \|\psi_j(f) - \psi_j^{(k)}(f)\| = 0, \quad f \in I, \ j = 1, 2. \quad (2.2)$$

We assert that

$$\lim_{k \to \infty} \|(\psi_j^{(k)})_*(h) - (\tau \circ \psi_j)(h)1\|_\infty = 0, \quad j = 1, 2, \quad h \in C_\mathbb{R}([0, 1]), \quad (2.3)$$

when we identify $C_\mathbb{R}([0, 1])$ with the self-adjoint portion of the center of $I = I(n_0, n, n_1)$. Indeed, because $\tau$ is the unique trace on $\mathcal{Z}$, the quantity

$$\sup_{\tau' \in T(B_\ell)} |\tau'(b) - \tau(b)|, \quad b \in B_\ell,$$

tends to zero as $k$ tends to infinity (with $k \geq \ell$). Hence, if we let $\iota_{k,\ell}$ denote the inclusion mapping $B_\ell \to B_k$, then $\|(\iota_{k,\ell})_*(h) - \tau(h)1\|_\infty$ tends to zero as $k$ tends to infinity ($k \geq \ell$). It follows from (2.2) that $\|(\iota_{k,\ell} \circ \psi_j^{(k)})(h) - (\psi_j^{(k)})_*(h)\|_\infty$ is small if $\ell$ is large (and $k \geq \ell$). The claim in (2.3) follows from these facts and the identity $$(\iota_{k,\ell} \circ \psi_j^{(k)})_* = (\iota_{k,\ell})_* \circ (\psi_j^{(k)})_*.$$ Choose an integer $D$ such that $\|f(s) - f(t)\| < \varepsilon/9$ for all $f \in F$ and for all $s, t \in [0, 1]$ with $|s - t| \leq 6/D$. Each $\psi_j(h_{D,d-1} - h_{D,d})$ is a non-zero and positive element in $\mathcal{Z}$, and we can therefore find $c > 0$ such that $\|\psi_j(h_{D,d-1} - h_{D,d})\| \geq 2c$ for $d = 2, 3, \ldots, D$ and $j = 1, 2$. Use (2.2) and the assumption $\tau \circ \psi_1 = \tau \circ \psi_2$ to find $k \geq k_0$ such that
\[ \| \psi_j(f) - \psi_j^{(k)}(f) \| < \varepsilon/3 \text{ and } \\
(\psi_j^{(k)})_*(h_{D,d-1} - h_{D,d}) \geq c, \quad \|(\psi_1^{(k)})_*(h_{D,d}) - (\psi_2^{(k)})_*(h_{D,d})\|_{\infty} < c, \]

for all \( f \in F \), for \( d = (1), 2, \ldots, D \), and for \( j = 1, 2 \). It then follows from [12, Corollary 5.6] that there is a unitary element \( u \) in \( B_k \) such that \( \|\psi_2^{(k)}(f) - u^*\psi_1^{(k)}(f)u\| \leq \varepsilon/3 \) for all \( f \in F \); whence \( \|\psi_2(f) - u^*\psi_1(f)u\| \leq \varepsilon \) for all \( f \in F \). This proves that \( \psi_1 \) and \( \psi_2 \) are approximately unitarily equivalent. \( \square \)

For any natural numbers \( n \) and \( m \) let \( E(n,m) \) be the \( C^* \)-algebra that consists of all functions \( f \) in \( C([0, 1], M_{n,\infty} \otimes M_{m,\infty}) \) for which \( f(0) \in M_{n,\infty} \otimes \mathbb{C} \) and \( f(1) \in \mathbb{C} \otimes M_{m,\infty} \).

**Proposition 2.2** There is a unital embedding of \( E(n,m) \) into \( \mathcal{Z} \) for every pair of natural numbers \( n, m \) that are relatively prime.

**Proof:** For each \( k \) there is a unital embedding \( \sigma_k : M_{n,k} \otimes M_{m,k} \to M_{n,k+1} \otimes M_{m,k+1} \) which satisfies

\[ \sigma_k(M_{n,k} \otimes \mathbb{C}) \subseteq M_{n,k+1} \otimes \mathbb{C}, \quad \sigma_k(\mathbb{C} \otimes M_{m,k}) \subseteq \mathbb{C} \otimes M_{m,k+1}. \]

Thus \( f \mapsto \sigma_k \circ f \) defines a \(*\)-homomorphism \( \rho_k : I(n^k, n^km^k, m^k) \to I(n^{k+1}, n^{k+1}m^{k+1}, m^{k+1}) \), and \( E(n,m) \) is the inductive limit of the sequence

\[ I(n, nm, m) \xrightarrow{\rho_1} I(n^2, n^2m^2, m^2) \xrightarrow{\rho_2} I(n^3, n^3m^3, m^3) \xrightarrow{\rho_3} \cdots \to E(n, m). \]

Take standard unital embeddings \( \psi_k : I(n^k, n^km^k, m^k) \to \mathcal{Z} \) (cf. Theorem 2.1 (i)). Then \( \psi_k \) and \( \psi_{k+1} \circ \rho_k \) are both standard unital embedding of \( I(n^k, n^km^k, m^k) \) into \( \mathcal{Z} \), so they are approximately unitarily equivalent by Theorem 2.1 (ii). We obtain the desired embedding of \( E(n,m) \) into \( \mathcal{Z} \) from this fact combined with a one-sided approximate intertwining (after Elliott), see for example [17, Theorem 1.10.14]. \( \square \)

### 3 Almost unperforation

Consider an ordered abelian semigroup \( (W, +, \leq) \). An element \( x \in W \) is called **positive** if \( y + x \geq y \) for all \( y \in W \), and \( W \) is said to be positive if all elements in \( W \) are positive. If \( W \) has a zero-element 0, then \( W \) is positive if and only if \( 0 \leq x \) for all \( x \in W \). An abelian semigroup equipped with the **algebraic order:** \( x \leq y \) iff \( y = x + z \) for some \( z \in W \), is positive.
Definition 3.1 A positive ordered abelian semigroup $W$ is said to be almost unperforated if for all $x,y \in W$ and all $n,m \in \mathbb{N}$, with $nx \leq my$ and $n > m$, one has $x \leq y$.

Let $W$ be a positive ordered abelian semigroup. Write $x \propto y$ if $x,y$ are elements in $W$ and $x \leq ny$ for some natural number $n$ (i.e., $x$ belongs to the ideal in $W$ generated by the element $y$). The element $y$ is said to be an order unit for $W$ if $x \propto y$ for all $x \in W$.

For each positive element $x$ in $W$ let $S(W,x)$ be the set of order preserving additive maps $f: W \to [0,\infty]$ such that $f(x) = 1$. Although we shall not use this fact, we mention that $S(W,x)$ is non-empty if and only if for all natural numbers $n$ and $m$, with $nx \leq mx$, one has $n \leq m$. This follows from \cite{10} Corollary 2.7 and the following observation that also will be used in the proof of the proposition below. For any element $x \in W$ the set $W_0 = \{z \in W \mid z \propto x\}$ is an order ideal in $W$, and $x$ is an order unit for $W_0$. Moreover, any state $f$ in $S(W_0,x)$ extends to a state $\overline{f}$ in $S(W,x)$ by setting $\overline{f}(z) = \infty$ for $z \in W \setminus W_0$.

Proposition 3.2 Let $W$ be a positive ordered abelian semigroup. Then $W$ is almost unperforated if and only if the following condition holds: For all elements $x,y$ in $W$, with $x \propto y$ and $f(x) < f(y)$ for all $f \in S(W,y)$, one has $x \leq y$.

Proof: Following the argument above we can—if necessary by passing to an order ideal of $W$—assume that $y$ is an order unit for $W$. The “only if” part now follows from \cite{22} Proposition 3.1], which again uses Goodearl and Handelman’s extension result \cite{10} Lemma 4.1].

To prove the “if” part, take elements $x,y \in W$ and $n \in \mathbb{N}$ such that $(n+1)x \leq ny$. Then $x \propto y$ because $x \leq (n+1)x \leq ny$; and $f(x) \leq n(n+1)^{-1} < 1 = f(y)$ for all $f \in S(W,y)$, whence $x \leq y$. \hfill \Box

Definition 3.3 An ordered abelian group $(G,G^\circ)$ is said to be almost unperforated if for all $g \in G$ and for all $n \in \mathbb{N}$, with $ng,(n+1)g \in G^\circ$, one has $g \in G^\circ$.

Lemma 3.4 Let $(G,G^\circ)$ be an ordered abelian group. Then $G$ is almost unperforated if and only if the positive semigroup $G^\circ$ is almost unperforated.

Proof: Suppose that $G$ is almost unperforated and that $x,y \in G^\circ$ satisfy $(n+1)x \leq ny$ for some natural number $n$. Then $n(y-x) \geq x \geq 0$ and $(n+1)(y-x) \geq y \geq 0$, whence $y-x \geq 0$ and $y \geq x$. Conversely, suppose that $G^\circ$ is almost unperforated and that $ng,(n+1)g \in G^\circ$ for some $n \in \mathbb{N}$. Since $(n+1)ng = n(n+1)g$, we get $ng \leq (n+1)g$, which implies that $g = (n+1)g - ng$ belongs to $G^\circ$. \hfill \Box

A simple ordered abelian group is almost unperforated if and only if it is weakly unperforated. Indeed, if $n \in \mathbb{N}$ and $g \in G$ are such that $ng \in G^\circ \setminus \{0\}$, then, by simplicity of
there is a natural number \( k \) such that \( kn g \geq g \). Thus \( (kn - 1)g \) and \( kn g \) are positive, so \( g \) is positive if \( G \) is almost unperforated (cf. Lemma 3.2). Conversely, if \( G \) is weakly unperforated and \( ng, (n + 1)g \in G^+ \), then \( g \in G^+ \) if \( ng \neq 0 \), and \( g = (n + 1)g \in G^+ \) if \( ng = 0 \).

Elliott considered in [7] a notion of what he called weak unperforation of (non-simple) ordered abelian groups with torsion. (We have refrained from using the term “weak unperforation” in Definition 3.3 to avoid conflict with Elliott’s definition.) A torsion free group is weakly unperforated in the sense of Elliott if and only if it is unperforated: \( (2, -2) \in G^+ \) but \( (1, -1) \notin G^+ \), so it is not weakly unperforated in the sense of Elliott. However, the group \((G, G^+)\) is almost unperforated, as the reader can verify.

In the converse direction, any weakly unperforated group is almost unperforated. Indeed, if \( G \) is weakly unperforated and \( g \in G \) and \( n \in \mathbb{N} \) are such that \( ng, (n + 1)g \) are positive, then \( g \) is positive modulo torsion, i.e., \( g + t \) is positive for some \( t \in G_{\text{tor}} \). Let \( k \in \mathbb{N} \) be the order of \( t \), find natural numbers \( \ell_1, \ell_2 \) such that \( N = \ell_1 n + \ell_2(n + 1) \) is congruent with -1 modulo \( k \). Then \( Ng = Ng + (N + 1)t \) is positive, whence \( -t \leq N(g + t) \), which by the hypothesis of weak unperforation implies that \( g = (g + t) + (-t) \) is positive.

4 Weak and almost unperforation of \( \mathcal{Z} \)-absorbing \( C^* \)-algebras

Cuntz associates in [5] to each \( C^* \)-algebra \( A \) a positive ordered abelian semigroup \( W(A) \) as follows. Let \( M_*(A)^+ \) denote the (disjoint) union \( \bigcup_{n=1}^{\infty} M_n(A)^+ \). For \( a \in M_n(A)^+ \) and \( b \in M_m(A)^+ \) set \( a \oplus b = \text{diag}(a, b) \in M_{n+m}(A)^+ \), and write \( a \preceq b \) if there is a sequence \( \{x_k\} \) in \( M_{m,n}(A) \) such that \( x_k^* b x_k \to a \). Write \( a \sim b \) if \( a \preceq b \) and \( b \preceq a \). Put \( W(A) = M_*(A)^+/\sim \), and let \( \langle a \rangle \in W(A) \) be the equivalence class containing \( a \) (so that \( W(A) = \{ \langle a \rangle \mid a \in M_*(A)^+ \} \)). Then \( W(A) \) is a positive ordered abelian semigroup when equipped with the relations:

\[
\langle a \rangle + \langle b \rangle = \langle a \oplus b \rangle, \quad \langle a \rangle \leq \langle b \rangle \iff a \preceq b, \quad a, b \in M_*(A)^+.
\]

Following the standard convention, for each positive element \( a \in A \) and for each \( \varepsilon \geq 0 \), write \( (a - \varepsilon)_+ \) for the positive element in \( A \) given by \( h_\varepsilon(a) \), where \( h_\varepsilon(t) = \max\{t - \varepsilon, 0\} \). We recall below some facts about the comparison of two positive elements \( a, b \) in a \( C^* \)-algebra.
A (see [5] Proposition 1.1 and [22] Section 2):

(a) \( a \preceq b \) if and only if \( (a - \varepsilon)_+ \preceq b \) for all \( \varepsilon > 0 \).

(b) \( a \preceq b \) if and only if for each \( \varepsilon > 0 \) there exists \( x \in A \) such that \( x^* bx = (a - \varepsilon)_+ \).

(c) If \( \|a - b\| < \varepsilon \), then \( (a - \varepsilon)_+ \preceq b \).

(d) \( ((a - \varepsilon_1)_+ - \varepsilon_2)_+ = (a - (\varepsilon_1 + \varepsilon_2))_+ \).

(e) \( a + b \preceq a \oplus b \); and if \( a \perp b \), then \( a + b \sim a \oplus b \).

If \( a \) belongs to the closed two-sided ideal, \( \overline{AbA} \), generated by \( b \), then \( (a - \varepsilon)_+ \) belongs to the algebraic two-sided ideal, \( AbA \), generated by \( b \) for all \( \varepsilon > 0 \), in which case \( (a - \varepsilon)_+ = \sum_{i=1}^n x_i^* bx_i \) for some \( n \in \mathbb{N} \) and some \( x_i \in A \). This shows that

\[
a \in \overline{AbA} \iff \forall \varepsilon > 0 \exists n \in \mathbb{N} : \langle (a - \varepsilon)_+ \rangle \leq n \langle b \rangle.
\]  

(4.1)

**Lemma 4.1** Let \( A \) and \( B \) be two \( C^* \)-algebras, let \( a, a' \in A \) and \( b, b' \in B \) be positive elements, and let \( n,m \) be natural numbers.

(i) If \( n \langle a \rangle \leq m \langle a' \rangle \) in \( W(A) \), then \( n \langle a \otimes b \rangle \leq m \langle a' \otimes b' \rangle \) in \( W(A \otimes B) \).

(ii) If \( n \langle b \rangle \leq m \langle b' \rangle \) in \( W(B) \), then \( n \langle a \otimes b \rangle \leq m \langle a \otimes b' \rangle \) in \( W(A \otimes B) \).

**Proof:** (i). Assume that \( n \langle a \rangle \leq m \langle a' \rangle \) in \( W(A) \). Then there is a sequence \( x_k = \{x_k(i,j)\} \) in \( M_{m,n}(A) \) such that \( x_k^* (a' \otimes 1_m) x_k \to a \otimes 1_n \) (or, equivalently, such that \( \sum_{i=1}^n x_{k}(l,i)^* a' x_k(l,j) \to \delta_{ij} a \) for all \( i,j = 1, \ldots, n \)). Let \( \{e_k\} \) be a sequence of positive contractions in \( B \) such that \( e_k b e_k \to b \). Put \( y_k(i,j) = x_k(i,j) \otimes e_k \in A \otimes B \), and put \( y_k = \{y_k(i,j)\} \in M_{m,n}(A \otimes B) \). Then \( y_k^* ((a' \otimes b) \otimes 1_m) y_k \to (a \otimes b) \otimes 1_n \) (or, equivalently, \( \sum_{l=1}^m y_k(l,i)^* (a' \otimes b) y_k(l,j) \to \delta_{ij} (a \otimes b) \) for all \( i,j = 1, \ldots, n \)). This shows that \( n \langle a \otimes b \rangle \leq m \langle a' \otimes b' \rangle \).

(ii) follows from (i) by symmetry. \(\square\)

**Lemma 4.2** For all natural numbers \( n \) there exists a positive element \( e_n \) in \( \mathcal{Z} \) such that \( n \langle e_n \rangle \leq \langle 1_\mathcal{Z} \rangle \leq (n + 1) \langle e_n \rangle \).

**Proof:** By Theorem 2.1 (a fact which follows easily from from Jiang and Su’s paper [12]) the \( C^* \)-algebra \( I = I(n, n(n + 1), n + 1) \) admits a unital embedding into \( \mathcal{Z} \), so it suffices to find a positive element \( e_n \) in \( I \) such that \( n \langle e_n \rangle \leq \langle 1_I \rangle \leq (n + 1) \langle e_n \rangle \) in \( W(I) \).

The idea of the proof is simple (but verifying the details requires some effort): There are positive functions \( f_1, f_2, \ldots, f_{n+1} \) in \( I \) such that

9
(i) \[ f_i(0) = \begin{cases} \begin{array}{ll} e_{ii}^{(n)} \otimes 1, & i = 1, \ldots, n, \\ 0, & i = n + 1, \end{array} \end{cases} \quad f_i(t) = 1 \otimes e_{ii}^{(n+1)}, \quad t \in [1/2, 1], \]

(where \( \{e_{ij}^{(m)}\}_{i,j=1}^m \) denotes the canonical set of matrix units for \( M_m(\mathbb{C}) \)),

(ii) \( f_1, f_2, \ldots, f_n \) are pairwise orthogonal,

(iii) \( \sum_{i=1}^{n+1} f_i = 1 \), and

(iv) \( f_{n+1} \preceq f_1 \sim f_2 \sim \cdots \sim f_n \).

It will follow from (iii) and (e) that \( \sum_{i=1}^{n+1} \langle f_i \rangle \geq \langle 1 \rangle \); and (ii) and (e) imply that \( \sum_{i=1}^{n} \langle f_i \rangle \leq \langle 1 \rangle \). It therefore follows from (iv) that \( e_n = f_1 \) has the desired property.

We proceed to construct the functions \( f_1, \ldots, f_{n+1} \). Put

\[ W = \sum_{i,j=1}^{n} e_{ij}^{(n)} \otimes e_{ji}^{(n+1)} + 1 \otimes e_{n+1,n+1}^{(n+1)}. \tag{4.2} \]

Then \( W \) is a self-adjoint unitary element in \( M_n \otimes M_{n+1} \), and

\[ W(1 \otimes e_{ii}^{(n+1)})W^* = e_{ii}^{(n)} \otimes (1 - e_{n+1,n+1}^{(n+1)}) \leq e_{ii}^{(n)} \otimes 1, \tag{4.3} \]

for \( 1 \leq i \leq n \). Choose a continuous path of unitaries \( t \mapsto V_t \) in \( M_n \otimes M_{n+1}, t \in [0, 1] \), such that \( V_0 = 1 \) and \( V_t = W \) for \( t \in [1/2, 1] \). Put \( V_t = V_t W \). Choose a continuous path \( t \mapsto \gamma_t \in [0, 1] \) such that \( \gamma_0 = 0 \) and \( \gamma_t = 1 \) for \( t \in [1/2, 1] \). Define \( f_i : [0, 1] \to M_n \otimes M_{n+1} \) by

\[ f_i(t) = \gamma_t W_t(1 \otimes e_{ii}^{(n+1)})W_t^* + (1 - \gamma_t) V_t(e_{ii}^{(n)} \otimes 1)V_t^*, \quad i = 1, \ldots, n, \]

\[ f_{n+1}(t) = \gamma_t W_t(1 \otimes e_{n+1,n+1}^{(n+1)})W_t^*, \]

where \( t \in [0, 1] \). It is easy to check that (i) and (iii) above hold. From (i) we see that all \( f_i \) belong to \( I \). Use (4.3) to see that \( f_i(t) \leq V_t(e_{ii}^{(n)} \otimes 1)V_t^* \) for all \( t \) and all \( i = 1, \ldots, n \), and use again this to see that (ii) holds.

We proceed to show that (iv) holds. Let \( S \in M_n \) and \( T \in M_{n+1} \) be the permutation unitaries for which \( S^{i-1} e_{11}^{(n)} S^{-(i-1)} = e_{ii}^{(n)} \) and \( T^{i-1} e_{11}^{(n+1)} T^{-(i-1)} = e_{ii}^{(n+1)} \) for \( i = 1, \ldots, n, (n+1) \). Put

\[ R_i(t) = V_t(S^{i-1} \otimes 1)V_t^*, \quad t \in [0, 1/2], \quad i = 1, \ldots, n. \]
Brief calculations show that \( R_t(0) = S_t^{-1} \otimes 1 \in M_n \otimes \mathbb{C} \), \( f_i(t) = R_t(i)f_i(t)R_t(t)^* \) for \( t \in [0, 1/2] \), and

\[
R_t(1/2)(1 \otimes e_{11}^{(n+1)}) R_t(1/2)^* \quad = \quad R_t(1/2)f_i(1/2) R_t(1/2)^* = f_i(1/2) = 1 \otimes e_{ii}^{(n+1)}
\]

\[
= (1 \otimes T(i-1))(1 \otimes e_{11}^{(n+1)})(1 \otimes T^{-i-1}),
\]

for \( i = 1, \ldots, n \). The unitary group of the relative commutant \( M_n \otimes M_{n+1} \cap \{ 1 \otimes e_{11}^{(n+1)} \}' \) is connected, so we can extend the paths \( t \mapsto R_t(t), t \in [0, 1/2] \), to continuous path \( t \mapsto R_t(t), t \in [0, 1] \), such that \( R_t(1) = 1 \otimes T^{-i-1} \in \mathbb{C} \otimes M_{n+1} \) and \( R_t(t)f_i(t)R_t(t)^* = 1 \otimes e_{ii}^{(n+1)} = f_i(t) \) for \( t \in [1/2, 1] \) and for \( i = 2, \ldots, n \). Thus \( R_t \) is a unitary element in \( I \) and \( R_t f_1 R_t^* = f_i \) for \( i = 2, \ldots, n \). This proves that \( f_1 \sim f_2 \sim \cdots \sim f_n \).

We must also show that \( f_{n+1} \not\sim f_1 \). Let \( g_i \in I \) be given by \( g_i(t) = \gamma_i W_t(1 \otimes e_{ii}^{(n+1)}) W_t^* \) for \( i = 1, \ldots, n+1 \) (so that \( f_{n+1} = g_{n+1} \)). A calculation then shows that \( R_t g_i R_t^* = g_i \) so that \( g_1, \ldots, g_n \) are unitarily equivalent. By symmetry, \( f_{n+1} = g_{n+1} \) is unitarily equivalent to \( g_1 \), and as \( g_1 \leq f_1 \), we conclude that \( f_{n+1} \not\sim f_1 \).

**Lemma 4.3** Let \( A \) be any \( C^* \)-algebra, and let \( a, a' \) be positive elements in \( A \) for which \( (n+1)a \leq n(a') \) in \( W(A) \) for some natural number \( n \). Then \( a \otimes 1_Z \leq a' \otimes 1_Z \) in \( W(A \otimes Z) \).

**Proof:** Take \( e_n \) in \( Z \) as in Lemma 4.2. Then, by Lemma 4.1

\[
(a \otimes 1_Z) \leq (n+1)(a \otimes e_n) \leq n(a' \otimes e_n) \leq (a' \otimes 1_Z)
\]

in \( W(A \otimes Z) \).

**Lemma 4.4** Let \( A \) be a \( Z \)-absorbing \( C^* \)-algebra. Then there is a sequence of isomorphisms \( \sigma_n : A \otimes Z \to A \) such that

\[
\lim_{n \to \infty} \| \sigma_n(a \otimes 1) - a \| = 0, \quad a \in A.
\]

**Proof:** It is shown in [12] that \( Z \) is isomorphic to \( \bigotimes_{k=1}^\infty Z \). We may therefore identify \( A \) with \( A \otimes (\bigotimes_{k=1}^\infty Z) \). With this identification we define \( \sigma_n : A \otimes (\bigotimes_{k=1}^\infty Z) \otimes Z \to A \otimes (\bigotimes_{k=1}^\infty Z) \) to be the isomorphism that fixes \( A \) and the first \( n \) copies of \( Z \), which sends the last copy of \( Z \) to the copy of \( Z \) at position “\( n+1 \)”, and which shifts the remaining copies of \( Z \) one place to the right.

**Theorem 4.5** Let \( A \) be a \( Z \)-absorbing \( C^* \)-algebra. Then \( W(A) \) is almost unperforated.
Proof: Let $a, a'$ be positive elements in $M_\infty(A)$ for which $(n+1)a \leq n(a')$. Upon replacing $A$ by a matrix algebra over $A$ (which still is $\mathcal{Z}$-absorbing) we may assume that $a$ and $a'$ both belong to $A$. Let $\varepsilon > 0$. It follows from Lemma 4.3 that $a \otimes 1 \preceq a' \otimes 1$ in $A \otimes \mathcal{Z}$, so there exists $x \in A \otimes \mathcal{Z}$ with $\|x^*(a' \otimes 1)x - a \otimes 1\| < \varepsilon$. Let $\sigma_k: A \otimes \mathcal{Z} \to A$ be as in Lemma 4.4 and put $x_k = \sigma_k(x)$. Then $\|x_k^*\sigma_k(a' \otimes 1)x_k - \sigma_k(a \otimes 1)\| < \varepsilon$, whence $\|x_k^*a'x_k - a\| < \varepsilon$ if $k$ is chosen large enough. This shows that $\langle a \rangle \leq \langle a' \rangle$ in $W(A)$. □

A $C^*$-algebra $A$ where $W(A)$ is almost unperforated has nice comparability properties, as we shall proceed to illustrate in the remaining part of this section, and in the later sections of this paper.

We recall a few facts about dimension function, introduced by Cuntz in [5]. A dimension function on a $C^*$-algebra $A$ is an additive order preserving function $d: W(A) \to [0, \infty]$. (We can also regard $d$ as a function $M_\infty(A)^+ \to [0, \infty]$ that respects the rules $d(a \oplus b) = d(a) + d(b)$ and $a \preceq b \Rightarrow d(a) \leq d(b)$ for all $a, b \in M_\infty(A)^+$.) A dimension function $d$ is said to be lower semi-continuous if $\overline{d} = \overline{d}$, where

$$\overline{d}(a) \overset{\text{def}}{=} \lim_{\varepsilon \to 0^+} d((a-\varepsilon)_+), \quad a \in M_\infty(A)^+. \tag{4.4}$$

Moreover, $\overline{d}$ is a lower semi-continuous dimension function on $A$ for each dimension function $d$, cf. [22, Proposition 4.1]. Note that $d((a-\varepsilon)_+) \leq \overline{d}(a) \leq d(a)$ for every dimension function $d$ and for every $\varepsilon > 0$, and that $\overline{d}(p) = d(p)$ for every projection $p$.

By an extended trace on a $C^*$-algebra $A$ we shall mean a function $\tau: A^+ \to [0, \infty]$ which is additive, homogeneous, and has the trace property: $\tau(x^*x) = \tau(xx^*)$ for all $x \in A$. If $\tau$ is an extended trace on $A$, then

$$d_\tau(a) \overset{\text{def}}{=} \lim_{\varepsilon \to 0^+} \tau(f_\varepsilon(a)) \left(= \lim_{n \to \infty} \tau(a^{1/n})\right), \quad a \in M_\infty(A)^+, \tag{4.5}$$

where $f_\varepsilon: \mathbb{R}^+ \to \mathbb{R}^+$ is given by $f_\varepsilon(t) = \min\{\varepsilon^{-1}t, 1\}$, defines a lower semi-continuous dimension function on $A$. If $A$ is exact, then every lower semi-continuous dimension function on $A$ is of the form $d_\tau$ for some extended trace $\tau$ on $A$. (This follows from Blackadar and Handelman, [2, Theorem II.2.2], who show that one can lift $d$ to a quasitrace, and from Haagerup, [11], and Kirchberg, [14], who show that quasitraces are traces on exact $C^*$-algebras).

With the characterization of lower semi-continuous dimension functions above and with Theorem 4.5 at hand one can apply the proof of [22, Theorem 5.2] to obtain:
Corollary 4.6 Let $A$ be a $C^*$-algebra for which $W(A)$ is almost unperforated (in particular, $A$ could be a $\mathbb{Z}$-absorbing $C^*$-algebra), and suppose in addition that $A$ is exact, simple and unital. Let $a, b$ be positive elements in $A$. If $d_\tau(a) < d_\tau(b)$ for every tracial state $\tau$ on $A$, then $a \preceq b$.

We also have the following “non-simple” version of the result above.

Corollary 4.7 Let $A$ be a $C^*$-algebra for which $W(A)$ is almost unperforated (in particular, $A$ could be a $\mathbb{Z}$-absorbing $C^*$-algebra). Let $a, b$ be positive elements in $A$. Suppose that $a$ belongs to $AbA$ and that $d(a) < d(b)$ for every dimension function $d$ on $A$ with $d(b) = 1$. Then $a \preceq b$.

Proof: It follows from (4.1) that $\langle (a - \varepsilon)_+ \rangle \propto \langle b \rangle$ in $W(A)$ for each $\varepsilon > 0$; and by assumption, $d(\langle (a - \varepsilon)_+ \rangle) \leq d(\langle a \rangle) < d(\langle b \rangle)$ for every $d \in S(W(A), \langle b \rangle)$. Thus $\langle (a - \varepsilon)_+ \rangle \leq \langle b \rangle$ by Proposition 3.2 and this proves the corollary as $\varepsilon > 0$ was arbitrary. □

Gong, Jiang and Su proved in [9] that the $K_0$-group of a simple unital $\mathbb{Z}$-absorbing $C^*$-algebra is weakly unperforated. At the level of semigroups, we can extend this result to the non-simple case, as explained below.

Let $V(A)$ denote the semigroup of Murray-von Neumann equivalence classes of projections in matrix algebras over $A$ equipped with the algebraic order: $x \leq y$ if there exists $z$ such that $y = x + z$. The relation “$\preceq$”, defined in the beginning of this section, agrees with the usual comparison relation when applied to projections $p$ and $q$, i.e., $p \preceq q$ if and only if $p$ is equivalent to a subprojection of $q$. The corollary below is thus an immediate consequence of Theorem 4.5.

Corollary 4.8 The semigroup $V(A)$ is almost unperforated for every $\mathbb{Z}$-absorbing $C^*$-algebra $A$.

It follows from Lemma 3.2 that if $A$ is a stably finite $C^*$-algebra with an approximate unit consisting of projections, then $K_0(A)$ is almost unperforated if and only if $K_0(A)^+$ is almost unperforated. It seems plausible that $K_0(A)^+$ is almost unperforated whenever $V(A)$ is almost unperforated; and this is trivially the case when $V(A)$ has the cancellation property. This implication also holds when $V(A)$ is simple. Indeed, let $\gamma: V(A) \to K_0(A)$ be the Grothendieck map, so that $K_0(A)^+ = \gamma(V(A))$. Take $x, y \in V(A)$ and $n \in \mathbb{N}$ such that $(n + 1)\gamma(x) \leq n\gamma(y)$. Then $(n + 1)x + u \leq ny + u$ for some $u \in V(A)$. Repeated use of this inequality yields $N(n + 1)x + u \leq Nny + u$ for all natural numbers $N$. That $V(A)$ is simple means that every non-zero element, and hence $y$, is an order unit for $V(A)$, so there is a
natural number \( k \) with \( u \leq ky \). Now, \( N(n+1)x \leq N(n+1)x+u \leq Nny+u \leq (Nn+k)y \), which for \( N \geq k+1 \) yields \( x \leq y \) and hence \( \gamma(x) \leq \gamma(y) \).

We thus have the following result, that slightly extends [9, Theorem 1].

**Corollary 4.9** Let \( A \) be a stably finite \( \mathcal{Z} \)-absorbing \( C^* \)-algebra with an approximate unit consisting of projections. If \( V(A) \) has the cancellation property or if \( V(A) \) is simple, then \( K_0(A) \) is almost unperforated.

**Corollary 4.10** Let \( A \) be an exact \( C^* \)-algebra for which \( W(A) \) is almost unperforated (in particular, \( A \) could be an exact \( \mathcal{Z} \)-absorbing \( C^* \)-algebra). Let \( p \) and \( q \) be projections in \( A \) such that \( p \) belongs to \( \overline{AqA} \). Suppose that \( \tau(p) < \tau(q) \) for every extended trace \( \tau \) on \( A \) with \( \tau(q) = 1 \). Then \( p \preceq q \).

**Proof:** We show that \( d(p) < d(q) \) for every dimension function \( d \) on \( A \) with \( d(q) = 1 \), and the result will then follow from Corollary 4.7. Let \( \overline{d} \) be the lower semi-continuous dimension function associated with \( d \) in (4.4). As remarked above, by Haagerup’s theorem on quasitraces, \( \overline{d} = d_\tau \) for some extended trace \( \tau \), cf. (4.5). Because \( d, \overline{d} \) and \( \tau \) agree on projections, we have \( \tau(q) = d(q) = 1 \) and \( d(p) = \tau(p) < \tau(q) = d(q) \) as desired. \( \square \)

## 5 Applications to purely infinite \( C^* \)-algebras

In this short section we derive two results that say when a \( \mathcal{Z} \)-absorbing \( C^* \)-algebra is purely infinite and \( \mathcal{O}_\infty \)-absorbing. Similar results were obtained in [16] for approximately divisible \( C^* \)-algebras. An exact \( C^* \)-algebra is said to be *traceless* when it admits no extended trace (see Section 3) that takes values other than 0 and \( \infty \).

**Corollary 5.1** Let \( A \) be an exact \( C^* \)-algebra for which \( W(A) \) is almost unperforated (in particular, \( A \) could be an exact \( \mathcal{Z} \)-absorbing \( C^* \)-algebra). Then \( A \) is purely infinite if and only if \( A \) is traceless.

**Proof:** Note first that \( A \), being traceless, can have no abelian quotients. Each lower semi-continuous dimension function on \( A \) arises from an extended trace on \( A \) (as remarked above Corollary 4.6), and must therefore take values in \( \{0, \infty \} \), again because \( A \) is traceless.

Take positive elements \( a, b \) in \( A \) such that \( a \) belongs to \( \overline{AbA} \). We must show that \( a \preceq b \) (cf. [15]), and it suffices to show that \( (a - \varepsilon)_+ \preceq b \) for all \( \varepsilon > 0 \). Take \( \varepsilon > 0 \). Let \( d \) be a dimension function on \( A \) such that \( d(b) = 1 \) (if such a dimension function exists), and let \( \overline{d} \) be its associated lower semi-continuous dimension function, cf. (4.4). Then \( \overline{d}(b) = 0 \) (as
remarked above). Use \((4.1)\) to see that 
\( \bar{d}((a - \varepsilon/2)_+ ) = 0\), and hence that 
\( d((a - \varepsilon)_+) = 0\). Thus \( (a - \varepsilon)_+ \not\succeq b \) by Corollary \(4.7\) □

**Theorem 5.2** Let \( A \) be a nuclear separable \( \mathcal{Z} \)-absorbing \( C^* \)-algebra. Then \( A \) absorbs \( \mathcal{O}_\infty \) (i.e., \( A \cong A \otimes \mathcal{O}_\infty \)) if and only if \( A \) is traceless.

**Proof:** If \( A \) absorbs \( \mathcal{O}_\infty \), then \( A \) is traceless (see [16, Theorem 9.1]). Suppose conversely that \( A \) is traceless. We then know from Corollary \(5.1\) that \( A \) is purely infinite. It follows from [16, Theorem 9.1] (in the unital or the stable case) and from [13, Corollary 8.1] (in the general case) that \( A \) absorbs \( \mathcal{O}_\infty \) if \( A \) is strongly purely infinite, cf. [16, Definition 5.1].

We need therefore only show that any \( \mathcal{Z} \)-absorbing purely infinite \( C^* \)-algebra is strongly purely infinite. Let
\[
\begin{pmatrix} a & x \\ x^* & b \end{pmatrix} \in M_2(A)^+,
\]
and let \( \varepsilon > 0 \) be given. Take pairwise orthogonal non-zero positive contractions \( h_1, h_2 \) in the simple \( C^* \)-algebra \( \mathcal{Z} \). Then
\[
a \otimes 1 \in (A \otimes \mathcal{Z})(a \otimes h_1)(A \otimes \mathcal{Z}), \quad b \otimes 1 \in (A \otimes \mathcal{Z})(b \otimes h_2)(A \otimes \mathcal{Z}).
\]

Since \( A \otimes \mathcal{Z} \cong A \) is purely infinite there are elements \( c_1, c_2 \) in \( A \otimes \mathcal{Z} \) such that
\[
\| c_1^*(a \otimes h_1)c_1 - a \otimes 1 \| < \varepsilon, \quad \| c_2^*(b \otimes h_2)c_2 - b \otimes 1 \| < \varepsilon.
\]

Let \( \sigma_n : A \otimes \mathcal{Z} \to A \) be as in Lemma \(4.4\) and put \( d_{1,n} = \sigma_n((1 \otimes h_1^{1/2})c_1) \) and \( d_{2,n} = \sigma_n((1 \otimes h_2^{1/2})c_2) \). Then
\[
\| d_{1,n}^* \sigma_n(a \otimes 1)d_{1,n} - \sigma_n(a \otimes 1) \| < \varepsilon, \quad \| d_{2,n}^* \sigma_n(b \otimes 1)d_{2,n} - \sigma_n(b \otimes 1) \| < \varepsilon,
\]
\[
d_{2,n}^* \sigma_n(x \otimes 1)d_{1,n} = 0.
\]

The norm of \( d_{j,n} \) does not dependent on \( n \). Thus, if we take \( d_1 = d_{1,n} \) and \( d_2 = d_{2,n} \) for some large enough \( n \), then we obtain the desired estimates: \( \| d_1^* ad_1 - a \| < \varepsilon, \| d_2^* bd_2 - b \| < \varepsilon \), and \( \| d_2^* xd_1 \| < \varepsilon \). □

### 6 The stable rank of \( \mathcal{Z} \)-absorbing \( C^* \)-algebras

We shall in this section show that simple, finite \( \mathcal{Z} \)-absorbing \( C^* \)-algebras have stable rank one.
**Definition 6.1** A unital $C^*$-algebra $A$ is said to be *strongly $K_1$-surjective* if the canonical mapping $\mathcal{U}(A_0 + \mathbb{C}1_A) \to K_1(A)$ is surjective for every full hereditary sub-$C^*$-algebra $A_0$ of $A$. If the canonical mapping $\mathcal{U}(A_0 + \mathbb{C}1_A)/\mathcal{U}_0(A_0 + \mathbb{C}1_A) \to K_1(A)$ is injective for every full hereditary sub-$C^*$-algebra $A_0$, then we say that $A$ is *strongly $K_1$-injective*.

Note that we do not assume simplicity in the two next lemmas.

**Lemma 6.2** Every full hereditary sub-$C^*$-algebra in a unital approximately divisible $C^*$-algebra contains a full projection.

**Proof:** Let $B$ be a full hereditary sub-$C^*$-algebra of a unital approximately divisible $C^*$-algebra $A$. Take a full positive element $b$ in $B$. Then $n(b) \geq \langle 1_A \rangle$ in $W(A)$ for some natural number $n$ (by (4.1)). Since $A$ is approximately divisible, there is a unital embedding of $M_{n+1} \oplus M_{n+2}$ into $A$, and, as shown in [26], $A$ is $\mathcal{Z}$-absorbing. (We shall only apply this lemma in the case where $A$ is the tensor product of a unital $C^*$-algebra with a UHF-algebra, and in this case we can conclude that $A$ is $\mathcal{Z}$-absorbing by the result in [12] that any non-elementary simple AF-algebra, and in particular, every UHF-algebra, is $\mathcal{Z}$-absorbing.)

Let $e$ and $f$ be one-dimensional projections in $M_{n+1}$ and $M_{n+2}$, respectively, and let $p \in A$ be the image of $(e, f)$ under the inclusion mapping $M_{n+1} \oplus M_{n+2} \to A$. Then $p$ is a full projection that satisfies $(n + 1)\langle p \rangle \leq \langle 1 \rangle \leq n\langle b \rangle$. Hence, by Theorem 4.3, $\langle p \rangle \leq \langle b \rangle$, i.e., $p \precsim b$. It follows that $p = x^*bx$ for some $x \in A$. Put $v = b^{1/2}x$. Then $p = v^*v$ and $p \sim vv^* = b^{1/2}x^*xb^{1/2} \in B$, so $vv^*$ is a full projection in $B$. □

**Lemma 6.3** Every unital approximately divisible $C^*$-algebra is strongly $K_1$-surjective.

**Proof:** Let $B$ be a full hereditary sub-$C^*$-algebra of $A$. We must show that the canonical map $\mathcal{U}(B + \mathbb{C}1_A) \to K_1(A)$ is surjective. Use Lemma 6.2 to find a full projection $p$ in $B$. It suffices to show that the canonical map $\mathcal{U}(pAp + \mathbb{C}(1_A - p)) \to K_1(A)$ is surjective. Take an element $g$ in $K_1(A)$, and represent $g$ as the class of a unitary element $u$ in $M_n(A)$ for some large enough natural number $n$. Upon replacing $M_n(A)$ by $A$ we can assume that $n = 1$.

Let $\mathcal{P}$ be the set of projections $q \in A$ such that there exists a unitary element $v \in qAq$ for which $g = [v + (1_A - q)]_1$ in $K_1(A)$. We must show that $\mathcal{P}$ contains all full projections in $A$. Note first that if $q_1, q_2$ are projections in $A$ with $q_1 \precsim q_2$ and $q_1 \in \mathcal{P}$, then $q_2 \in \mathcal{P}$. Indeed, if $v_1 \in q_1Aq_1$ is unitary with $g = [v_1 + (1_A - q_1)]_1$ and if $s^*s = q_1$, $ss^* \preceq q_2$, then $v_2$ given by $sv_1s^*+(q_2-ss^*)$ is a unitary element in $q_2Aq_2$, and $[v_2 + (1_A - q_2)]_1 = [v_2 + (1_A - q_2)]_1$.

Let $p \in A$ be a full projection. Then $(n - 1)\langle p \rangle \geq \langle 1_A \rangle$ in $W(A)$ for some large enough natural number $n$, cf. (4.1). By approximate divisibility of $A$ there is a unitary element
Proposition 6.5
Given a pull-back diagram

\[ \begin{array}{ccc}
\varphi_1 & \mathbf{A} & \varphi_2 \\
A_1 \searrow & \pi & \swarrow A_2 \\
\psi_1 & B & \psi_2
\end{array} \]

(6.1)

Proof: The first claim follows from the fact that \( x a_\varepsilon = 0 \) for every \( x \in A^0 \). Suppose that \( w \) is a unitary element in \( A \) with \( w a_{\varepsilon'} = a_{\varepsilon'} \). Then \( a_{\varepsilon'} w = a_{\varepsilon'} \), so \( w - 1_A \) is orthogonal to \( a_{\varepsilon'} \). But the orthogonal complement of \( a_{\varepsilon'} \) is contained in \( A^0 \). □

Lemma 6.4 Let \( A \) be a unital \( C^* \)-algebra, let \( a \) be a positive element in \( A \), let \( 0 < \varepsilon' < \varepsilon \) be given, and set \( A^0 = g_\varepsilon(a) A g_\varepsilon(a) \), where \( g_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+ \) is given by \( g_\varepsilon(t) = \max \{ 1 - t/\varepsilon, 0 \} \). Then \( w a_\varepsilon = a_{\varepsilon'} \) for every \( w \in A^0 + 1_A \); and if \( w \) is a unitary element in \( A \) that satisfies \( w a_{\varepsilon'} = a_{\varepsilon'} \), then \( w \) belongs to \( A^0 + 1_A \).

Proof: The first claim follows from the fact that \( x a_\varepsilon = 0 \) for every \( x \in A^0 \). Suppose that \( w \) is a unitary element in \( A \) with \( w a_{\varepsilon'} = a_{\varepsilon'} \). Then \( a_{\varepsilon'} w = a_{\varepsilon'} \), so \( w - 1_A \) is orthogonal to \( a_{\varepsilon'} \). But the orthogonal complement of \( a_{\varepsilon'} \) is contained in \( A^0 \). □

Rieffel proved in [19] that if \( A \) is a unital \( C^* \)-algebra of stable rank one, then the canonical map \( \mathcal{U}(A) / \mathcal{U}_0(A) \to K_1(A) \) is an isomorphism, and hence injective. Rieffel also showed for any such \( C^* \)-algebra \( A \) and any hereditary sub-\( C^* \)-algebra \( B \) of \( A \) (full or not) that the stable rank of \( B + \mathbb{C}1_A \) is one.

Take now a full hereditary sub-\( C^* \)-algebra \( B \) of \( A \), where \( A \) is unital and of stable rank one. Then

\[ \mathcal{U}(B + \mathbb{C}1_A) / \mathcal{U}_0(B + \mathbb{C}1_A) \to K_1(B) \to K_1(A) \]

is an isomorphism (the second map is an isomorphism by Brown’s theorem, which guarantees that \( A \otimes \mathcal{K} \cong B \otimes \mathcal{K} \)).

This shows that any unital \( C^* \)-algebra of stable rank one is strongly \( K_1 \)-injective.

For each element \( x \) in a \( C^* \)-algebra \( A \) we can write \( x = v|\overline{x}|_+ \), where \( v \) is a partial isometry in \( A^{**} \). The element \( x_\varepsilon \overset{\text{def}}{=} v(|\overline{x}| - \varepsilon)_+ \) belongs to \( A \) for every \( \varepsilon > 0 \), and \( \| x - x_\varepsilon \| \leq \varepsilon \). If \( x \) is positive, then \( x_\varepsilon = (x - \varepsilon)_+ \).

Let \( A \) be a unital \( C^* \)-algebra, let \( a \) be a positive element in \( A \), let \( 0 < \varepsilon' < \varepsilon \) be given, and set \( A^0 = g_\varepsilon(a) A g_\varepsilon(a) \), where \( g_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+ \) is given by \( g_\varepsilon(t) = \max \{ 1 - t/\varepsilon, 0 \} \). Then \( w a_\varepsilon = a_{\varepsilon'} \) for every \( w \in A^0 + 1_A \); and if \( w \) is a unitary element in \( A \) that satisfies \( w a_{\varepsilon'} = a_{\varepsilon'} \), then \( w \) belongs to \( A^0 + 1_A \).
with surjective *-homomorphisms $\psi_1$ and $\psi_2$. Suppose that $A, A_1, A_2$ and $B$ are unital $C^*$-algebras and that $a \in A$ are such that

(i) $A_1$ and $A_2$ are strongly $K_1$-surjective,

(ii) $B$ is strongly $K_1$-injective,

(iii) $\text{Im}(K_1(\psi_1)) + \text{Im}(K_1(\psi_2)) = K_1(B)$,

(iv) $a^*a$ is non-invertible in every non-zero quotient of $A$.

Then $a$ belongs to the closure of $\text{GL}(A)$ if and only if $\varphi_j(a)$ belongs to the closure of $\text{GL}(A_j)$ for $j = 1, 2$.

The pull-back diagram (6.1) can, given $\psi : A_j \to B$, $j = 1, 2$, be realized with $A = \{(a_1, a_2) \in A_1 \oplus A_2 \mid \psi_1(a_1) = \psi_2(a_2)\}$ and with $\varphi_j(a_1, a_2) = a_j$.

**Proof:** The “only if” part is trivial. Assume now that $\varphi_j(a)$ belongs to the closure of $\text{GL}(A_j)$ for $j = 1, 2$. Let $\varepsilon > 0$ be given. It then follows from [20, Theorem 2.2] that there are unitary elements $u_j$ in $A_j$ such that $\varphi_j(a_{\varepsilon/2}) = u_j|\varphi_j(a_{\varepsilon/2})|$ for $j = 1, 2$. We show below that there are unitary elements $v_j$ in $A_j$, $j = 1, 2$, such that $\varphi_j(a_{\varepsilon}) = v_j|\varphi_j(a_{\varepsilon})|$, $j = 1, 2$, and $\psi_1(v_1) = \psi_2(v_2)$. It follows that $v = (v_1, v_2)$ is a unitary element in $A$ and that $a_{\varepsilon} = v|a_{\varepsilon}|$. This shows that $a$ belongs to the closure of the invertibles in $A$ (because $\varepsilon > 0$ was arbitrary).

Let $g_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ be as in Lemma 6.4 and put $A_0^0 = g_\varepsilon(|a|)A_0g_\varepsilon(|a|)$. Put

$$A_j^0 = \varphi_j(A_0^0) = g_\varepsilon(|\varphi_j(a)|)A_0g_\varepsilon(|\varphi_j(a)|), \quad B_0^0 = \pi(A_0^0) = g_\varepsilon(|\pi(a)|)B_0g_\varepsilon(|\pi(a)|).$$

Assumption (iv) implies that $A_0^0$ is full in $A$. It follows that the hereditary subalgebras $A_1^0, A_2^0, B_0^0$ are full in $A_1, A_2$ and $B$, respectively.

It follows from the identity

$$\pi(a_{\varepsilon/2}) = \psi_1(u_1)|\pi(a_{\varepsilon/2})| = \psi_2(u_2)|\pi(a_{\varepsilon/2})|,$$

that $\psi_2(u_2)^*\psi_1(u_1)|\pi(a_{\varepsilon/2})| = |\pi(a_{\varepsilon/2})|$, and so $z \overset{\text{def}}{=} \psi_2(u_2)^*\psi_1(u_1)$ belongs to $B_0^0 + 1_B$ (cf. Lemma 6.4). We show below that $z = \psi_2(w_2)\psi_1(w_1^*)$ for some unitaries $w_j$ in $A_0^0 + \mathbb{C}1_{A_j}$, $j = 1, 2$.

Use conditions (i) and (iii) to find unitaries $y_j \in A_j^0 + \mathbb{C}1_{A_j}$ such that $[\psi_2(y_2)\psi_1(y_1)^*]_1 = [z]_1$ in $K_1(B)$. By condition (ii), the unitary element $(z_0 = ) z\psi_1(y_1)\psi_2(y_2^*)$ is homotopic to
1 in the unitary group of $B^0 + C1_B$. Hence $z_0 = \psi_2(y_0)$ for some unitary $y_0$ in $A^0_2 + C1_{A_2}$. Now, $w_1 = y_1$ and $w_2 = y_0y_2$ are as desired.

Upon replacing $w_1$ and $w_2$ by $\lambda w_1$ and $\lambda w_2$ for a suitable $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, we can assume that $w_j \in A^0_j + 1_A$. Then, by Lemma 6.4, $w_j|\varphi_j(a_\varepsilon)| = |\varphi_j(a_\varepsilon)|$, $j = 1, 2$. It follows that $(v_j =) u_jw_j$ is a unitary in $A_j$, that $v_j|\varphi_j(a_\varepsilon)| = u_j|\varphi_j(a_\varepsilon)| = \varphi_j(a_\varepsilon)$ for $j = 1, 2$, and $\psi_1(v_1) = \psi_1(u_1)\psi_1(w_1) = \psi_2(u_2)\psi_2(w_2) = \psi_2(v_2)$, as desired. □

**Lemma 6.6** Let $A$ be a simple, unital, finite $C^*$-algebra. Then $a \otimes 1$ belongs to the closure of the invertibles in $A \otimes Z$ for every $a \in A$.

**Proof:** Let $E_{2,3}$ be the $C^*$-algebra which in Proposition 6.4 is shown to have a unital embedding into $Z$. It suffices to show that $a \otimes 1$ belongs to the invertibles in $A \otimes E_{2,3}$. If $a^*a \otimes 1$ is invertible in some non-zero quotient of $A \otimes E_{2,3}$, then $a^*a$ is invertible in $A$ by simplicity of $A$, which again implies that $a$ is invertible, because $A$ is finite. The claim of the lemma is trivial in this case. Suppose now that there is no non-zero quotient of $A \otimes E_{2,3}$ in which $a^*a \otimes 1$ is invertible.

Identify $A \otimes E_{2,3}$ with

$$\{ f \in C([0, 1], A \otimes B) \mid f(0) \in A \otimes B_1, f(1) \in A \otimes B_2 \},$$

where $B, B_1, B_2$ are UHF algebras of type $6^\infty$, $2^\infty$, and $3^\infty$, respectively, with $B_j \subseteq B$. We have a pull-back diagram

$$\begin{array}{ccc}
D_1 & \xrightarrow{\text{ev}_1} & D_2 \\
\varphi_1 & \downarrow & \varphi_2 \\
A \otimes B & \xrightarrow{\text{ev}_1} & A \otimes E_{2,3} \\
\varphi_1 & \downarrow & \varphi_2 \\
D_1 & \xleftarrow{\text{ev}_1} & D_2 \\
\end{array}$$

(6.2)

where

$D_1 = \{ f \in C([0, 1], A \otimes B) \mid f(0) \in A \otimes B_1 \}, \quad D_2 = \{ f \in C([1/2, 1], A \otimes B) \mid f(1) \in A \otimes B_2 \},$

and where $\varphi_1$ and $\varphi_2$ are the restriction mappings.

We shall now use Proposition 6.5 to prove that $a \otimes 1$ belongs to the closure of the invertibles in $A \otimes E_{2,3}$. Condition (iv) of Proposition 6.5 is satisfied by the assumption on $a$ made in the first paragraph of the proof. The $C^*$-algebras $D_1$ and $D_2$ are approximately
divisible because $D_j \cong D_j \otimes B_j$. It thus follows from Lemma 6.3 that condition (i) of Proposition 6.5 is satisfied. The $C^*$-algebra $A \otimes B$ is of stable rank one (cf. [21]), whence $A \otimes B$ is strongly $K_1$-injective (cf. the remarks below Lemma 6.3).

We have natural inclusions $A \otimes B_j \subseteq D_j$ (identifying an element in $A \otimes B_j$ with a constant function), and the composition $A \otimes B_j \to D_j \to A \otimes B$ is the inclusion mapping. Hence, to prove that (iii) of Proposition 6.5 is satisfied, it suffices to show that $K_1(A \otimes B)$ is generated by the images of the two mappings $K_1(A \otimes B_j) \to K_1(A \otimes B)$, $j = 1, 2$. We have natural identifications:

$$K_1(A \otimes B_1) = K_1(A) \otimes \mathbb{Z}[1/2], \quad K_1(A \otimes B_2) = K_1(A) \otimes \mathbb{Z}[1/3],$$

$$K_1(A \otimes B) = K_1(A) \otimes \mathbb{Z}[1/6].$$

The desired identity now follows from the elementary fact that $\mathbb{Z}[1/2] + \mathbb{Z}[1/3] = \mathbb{Z}[1/6].$

Retaining the inclusion $A \otimes B_j \subseteq D_j$ from the previous paragraph, $\varphi_j(a \otimes 1_{E_{2,3}}) = a \otimes 1_{B_j}$. Following [21], $a \otimes 1_{B_j}$ belongs to the closure of the invertibles in $A \otimes B_j$ (and hence to the closure of the invertibles in $D_j$) if (and only if) $\alpha_s(a) = 0$; and $\alpha_s(a) = 0$ for every element $a$ in any unital, finite, simple $C^*$-algebra $A$. It thus follows that $\varphi_j(a)$ belongs to the closure of $\text{GL}(D_j)$, $j = 1, 2$. (If we had assumed that $A$ is stably finite, then we could have used [21, Corollary 6.6] to conclude that the stable rank of $A \otimes B_j$ is one, which would have given us a more direct route to the conclusion above.)

**Theorem 6.7** Every simple, unital, finite $\mathcal{Z}$-absorbing $C^*$-algebra has stable rank one.

**Proof:** Let $A$ be a simple, unital, finite $C^*$-algebra such that $A$ is isomorphic to $A \otimes \mathcal{Z}$. Let $a \in A$ and $\varepsilon > 0$ be given. It follows from Lemma 6.6 that there is an invertible element $b \in A \otimes \mathcal{Z}$ such that $\|a \otimes 1 - b\| < \varepsilon/2$. Let $\sigma_n : A \otimes \mathcal{Z} \to A$ be as in Lemma 4.3 and choose $n$ such that $\|\sigma_n(a \otimes 1) - a\| < \varepsilon/2$. Then $\|a - \sigma_n(b)\| < \varepsilon$, and $\sigma_n(b)$ is an invertible element in $A$. 

**7 The real rank of $\mathcal{Z}$-absorbing $C^*$-algebras**

We conclude this paper with a result that describes when a simple $\mathcal{Z}$-absorbing $C^*$-algebra is of real rank zero. A simple approximately divisible $C^*$-algebra is of real rank zero if and only if projections separate quasitraces, as shown in [3] (and, as remarked earlier, each quasitrace on an exact $C^*$-algebra is a trace by [11] and [14]). It is not true that any $\mathcal{Z}$-absorbing $C^*$-algebra, where quasitraces are being separated by projections, is of real
rank zero. The Jiang–Su \( Z \) itself is a counterexample. We must therefore require some further properties, for example that the \( K_0 \)-group is \emph{weakly divisible}: for each \( g \in K_0^+ \) and for each \( n \in \mathbb{N} \) there are \( h_1, h_2 \in K_0^+ \) such that \( g = nh_1 + (n+1)h_2 \).

Let \( A \) be a unital \( C^* \)-algebra. Let \( T(A) \) denote the simplex of tracial states on \( A \), and let \( \text{Aff}(T(A)) \) denote the normed space of real valued affine continuous functions on \( T(A) \). Let \( \rho : K_0(A) \to \text{Aff}(T(A)) \) be the canonical map defined by \( \rho(g)(\tau) = K_0(\tau)(g) \).

The result below is essentially contained in [22, Theorem 7.2] (see also [3]).

**Proposition 7.1** Let \( A \) be an exact unital simple \( C^* \)-algebra of stable rank one for which \( W(A) \) is almost unperforated. Then \( A \) is of real rank zero if and only if \( \rho(K_0(A)) \) is uniformly dense in the normed space \( \text{Aff}(T(A)) \).

The proof of [22, Theorem 7.2] applies almost verbatim. (At the point where we have a positive element \( x \in A \) and \( \delta > 0 \) such that \( d_{\tau}(\langle f_{\delta/2}(x) \rangle) < d_{\tau}(\langle f_{\delta/4}(x) \rangle) \) for all \( \tau \in T(A) \), then, because \( \tau \mapsto d_{\tau}(y) \) defines an element in \( \text{Aff}(T(A)) \) for every \( y \in A^+ \), it follows by density of \( \rho(K_0(A)) \) in \( \text{Aff}(T(A)) \) that there is an element \( g \in K_0(A) \) such that \( d_{\tau}(\langle f_{\delta/2}(x) \rangle) < K_0(\tau)(g) < d_{\tau}(\langle f_{\delta/4}(x) \rangle) \) for all \( \tau \in T(A) \). One next uses weak unperforation of \( K_0(A) \) ([9] or Corollary 4.9) to conclude that \( g \) is positive, i.e., that \( g = [q] \) for some projection \( q \) in a matrix algebra over \( A \).)

**Theorem 7.2** The following conditions are equivalent for each unital, simple, exact, finite, \( Z \)-absorbing \( C^* \)-algebra \( A \).

(i) \( \text{RR}(A) = 0 \),

(ii) \( \rho(K_0(A)) \) is uniformly dense in \( \text{Aff}(T(A)) \),

(iii) \( K_0(A) \) is weakly divisible and projections in \( A \) separate traces on \( A \).

If \( A \) is a simple, \emph{infinite}, \( Z \)-absorbing \( C^* \)-algebra, then \( A \) is purely infinite by [9]; and purely infinite \( C^* \)-algebras are of real rank zero by [28].

**Proof:** It follows from Theorem 6.7 that \( W(A) \) is almost unperforated, and from Theorem 6.1 that the stable rank of \( A \) is one. We therefore get (ii) \( \Rightarrow \) (i) from Proposition 7.1

If (iii) holds, then for each element \( g \) in \( K_0(A)^+ \) and for each natural number \( n \) there exists \( h \in K_0(A)^+ \) such that \( nh \leq g \leq (n+1)h \). This implies that the uniform closure of \( \rho(K_0(A)) \) in \( \text{Aff}(T(A)) \) is a closed subspace which separates points. Thus, by Kadison’s Representation Theorem (see [1, II.1.8]), \( \rho(K_0(A)) \) is uniformly dense in \( \text{Aff}(T(A)) \), so (ii) holds. If (i) holds, then \( A \) is weakly divisible by [18] and traces on \( A \) are separated by projections. \( \square \)
A $C^*$-algebra is said to have property (SP) ("small projections") if each non-zero hereditary sub-$C^*$-algebra contains a non-zero projection.

**Corollary 7.3** Let $A$ be a simple, unital, exact $\mathbb{Z}$-absorbing $C^*$-algebra with a unique trace $\tau$. Then the following conditions are equivalent:

(i) $\text{RR}(A) = 0$,

(ii) $K_0(\tau)(K_0(A))$ is dense in $\mathbb{R}$,

(iii) $K_0(A)$ is weakly divisible,

(iv) $A$ has property (SP).

**Proof:** The equivalence of (i), (ii), and (iii) follows immediately from Theorem 7.2. The implication (i) $\Rightarrow$ (iv) is trivial, and one easily sees that (iv) implies (ii). \hfill $\Box$

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