Dense granular flows down inclines

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Dense inclined granular flow

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Steady, fully-developed force balance

Steady \( \frac{\partial}{\partial t} \equiv 0 \), fully-developed \( \frac{\partial}{\partial x} \equiv 0 \) flow

\[
\frac{dS}{dy} = \rho g \sin \alpha
\]

\[
\frac{dN}{dy} = \rho g \cos \alpha
\]

\[
\mu_{\text{eff}} = \frac{S}{N} = \tan \alpha
\]
Newtonian viscous fluid

laminar flow

Constitutive relation

\[ S = -\mu \frac{du}{dy} \]

Force balance

\[ \frac{dS}{dy} = \rho g \sin \alpha \]
\[ \frac{dN}{dy} = \rho g \cos \alpha \]

\[ \mu_{\text{eff}} = \frac{S}{N} = \tan \alpha \]

\[ u = \frac{\rho g \sin \alpha}{2\mu} (h^2 - y^2) \]

\[ \frac{\dot{m}}{W} = \left( \frac{\rho^2 g \sin \alpha}{3\mu} \right) h^3 \]
Newtonian viscous fluid

turbulent flow

\[ S_b = \rho gh \sin \alpha = \text{cst} \times \rho \bar{u}^2 \]

\[ \text{cst} \left( \frac{u^*}{\bar{u}} \right)^2 \approx 3 \times 10^{-3} \]

\[ \frac{\dot{m}}{W} = \rho \bar{u} h = \rho \sqrt{\frac{g \sin \alpha}{\text{cst}}} h^{3/2} \]

George Batchelor
Shallow flows, far sidewalls, bumpy base

\[ \frac{\dot{m}}{W} \propto \text{cst} \, \rho_s \, \bar{v} \, \frac{\sqrt{g \cos \alpha}}{h_{\text{stop}}} \, h^{s/2} \]

\[ \text{Fr} = \frac{\bar{u}}{\sqrt{\frac{N}{\rho}}} = \frac{\bar{u}}{\sqrt{gh \cos \alpha}} \]

Pouliquen, Phys. Fluids 1999

Deboeuf, et al, PRL 2006
Shallow flows, far sidewalls, bumpy base

20° < α < 26°
0.595 > ν > 0.545

\[ \frac{\dot{m}}{W} \propto \text{cst} \rho_s V \sqrt{g \cos \alpha} \frac{h^{5/2}}{h_{\text{stop}}} \]

Leo Silbert
Granular temperature

\[ T = \frac{1}{3} \frac{u'}{u'} \]

“temperature” fluctuation velocity \( u'_i \)
Transport coefficients

viscosity
\[ \mu = \rho_s d \sqrt{T} f_1(\nu; g_{12}) \]

conductivity
\[ \kappa = \rho_s d \sqrt{T} f_2(\nu; g_{12}) \]

dissipation
\[ \gamma = f_3(\nu; g_{12}; e) \rho_s T^{3/2} / d \]
equation of state
\[ N = f_4(\nu; g_{12}) \rho_s T \]
Kinetic theory cartoon

\[
S = \rho v g \sin \alpha = f_1 \rho d \sqrt{T} \frac{du}{dy}
\]

\[
T = \text{“granular temperature”}
\]

\[
N = \rho v g \cos \alpha = f_4 \rho T
\]

\[
u = 2 \left( \frac{\sqrt{g f_4}}{f_1} \right) \frac{\sqrt{g \cos \alpha}}{d} \left( h^{3/2} - y^{3/2} \right)
\]

\[
\frac{m}{W} = 2 \frac{v^{3/2} f_4^{1/2} \rho g^{1/2}}{f_1} \frac{\sin \alpha}{\sqrt{\cos \alpha}} h^{5/2}
\]
Inertial number

\[ I = \frac{|du/\,dy| \, d}{\sqrt{N/\rho_s \, v}} \]

- da Cruz, et al PRE 2005
- Jop, Forterre & Pouliquen Nature 2006
- GDR MiDi, Eur. Phys. J. E 2003
- Chevoir, et al Powder Tech. 2009
Problem solved?

Osborne Reynolds, 1883

\[ \text{Friction coefficient} \]

\[ \text{Re} = \frac{ud}{\mu / \rho} \]
Profile concavity and viscosity

\[ S = \rho_s v g y \sin \alpha = -\mu \frac{du}{dy} \]

\[ \frac{d^2 u}{dy^2} = -\rho_s g y \sin \alpha \left( \frac{v y}{\mu} \right) \left[ \frac{v'}{v} + \frac{1}{y} - \frac{\mu}{\mu} \right] \]

Moderate increase in viscosity with depth for core flows over a bumpy base.

\[ \frac{d^2 u}{dy^2} < 0 \iff \frac{d \ln \mu}{d \ln y} < \frac{d \ln v}{d \ln y} + 1 \]

bumpy base

\[ \frac{d \ln \mu}{d \ln y} = \frac{1}{2} \quad \frac{d \ln v}{d \ln y} = 0 \]
Inverted concavity for a soft base

Sidewall-stabilized heap

\[ \frac{d^2 u}{dy^2} > 0 \iff \frac{d \ln \mu}{d \ln y} > \frac{d \ln \nu}{d \ln y} + 1 \]

Rapid increase in volume fraction and viscosity with depth.

Louge, et al. Powders & Grains 2005
Richard, et al. PRL 2008

Taberlet, et al. PRL 2003
Role of the base

Alexandre Valance

Dissipative, soft base

Far side walls

\[
\frac{\dot{m}}{W} \approx \text{cst} \times \rho_s d \sqrt{gd} \left( \frac{h}{h_{\text{stop}}} - 1 \right)
\]

with inertial number scaling
Role of side walls

Sidewall-stabilized heap

\[ \frac{\dot{m}}{W} \approx \text{cst} \times \rho_s W \sqrt{g W \left( \frac{h}{W} - 1 \right)} \]

\[ \frac{\dot{m}}{W} \approx \text{cst} \times \rho_s d \sqrt{g d \left( \frac{h}{h_{\text{stop}}} - 1 \right)} \]

\[ \tan \alpha = \mu_w \left( \frac{h}{W} \right) + \tan \alpha_{\text{min}} \]

soft base, far side walls

data: Dana Smith, Cornell U.
Simulations: Nicolas Taberlet

Flat, frictional base

Delannay, et al, Nature Mat. 2007

large slip at the bottom

bumpy base

flat, frictional base

same core shear rate

same core volume fraction
Flat, frictional base cartoon

\[ T_0 \sim \frac{\bar{v}}{f_4(\nu_0)} gh \cos \alpha \]

\[ S \sim \rho_sg\bar{v} \sin \alpha \sim f_1(\nu_0) \rho_s d \sqrt{T_0} \frac{\bar{u}}{d} \]

\[ \text{Fr} = \frac{\bar{u}}{\sqrt{gh \cos \alpha}} \sim \left( \frac{\bar{v}^{1/2} f_4^{1/2}}{f_1} \right) \tan \alpha \]

\[ \frac{m}{W} \sim \rho_s g^{1/2} \frac{\bar{v}^{3/2} \sin \alpha \left( f_4(\nu_0) \right)^{1/2}}{\sqrt{\cos \alpha} \left( f_1(\nu_0) \right)^{1/2}} h^{3/2} \]
Flat, frictional base experiments

Louge and Keast, Phys. Fluids 2001
Flat, frictional base; shallow flows

Sustained flows exist at inclinations in the range $15.5^\circ \leq \alpha \leq 20^\circ$.

\[ m^\dagger = \frac{1}{d \sqrt{gd}} \int_0^h uv \, dy \]

\[ H^\dagger = \frac{1}{d} \int_0^h \nu \, dy = \left( \frac{h}{d} \bar{v} \right) \]

Roy Jackson

Louge and Keast, Phys. Fluids 2001
Steady flows require variable friction

For steady, fully-developed flows:

\[ \frac{S}{N} = \mu_{\text{eff}} = \tan \alpha \]

Forces are exerted at contact with the base:

\[ \mu_E = 0.59 \]

\[ \mu_I = 0.14 \]

\[ S = S_I + S_E \]

\[ N = N_I + N_E \]

\[ \mu_{\text{eff}} = \frac{S}{N} \]

Louge and Keast, Phys. Fluids 2001
Effective friction set by basal rolling

\[ \mu_{\text{eff}} = \mu_E - \frac{g_12(v_0)\mu_f^2}{v_0(1 - e)^2}(\mu_E - \mu_f)Fr^2 \]

\[ V_0 \sim \frac{gh\cos\alpha}{u^2}(1 + \frac{\mu_E}{\mu_f}) \]
Fluctuation energy

\[ 0 = -\frac{dq}{dy} + S \frac{du}{dy} - \gamma \]

\[ q = -\kappa \frac{dT}{dy} \]

\[ \kappa = f_2 \rho_s d \sqrt{T} \]

\[ S = f_1 \rho_s d \sqrt{T} \frac{du}{dy} \]

\[ \gamma = f_3 \rho_s T^{3/2} / d \]

- \( q \): heat flux gradient
- \( \kappa \): shear work rate
- \( S \): dissipation
Flux conundrum

\[ N \approx \rho_s \bar{v} \cos \alpha \]

\[ S \approx \rho_s \bar{v} \sin \alpha \]

\[ T \approx \bar{v} \cos \alpha / f_4 \]

\[
\begin{align*}
\frac{f_1}{f_4^{3/2}} \sqrt{y^*} \frac{d}{dy^*} & \left\{ \frac{f_2}{f_4^{3/2}} \sqrt{y^*} \left[ 1 - \left( \frac{d \ln f_4}{d \ln \nu} \right) \left( \frac{d \ln \nu}{d \ln y^*} \right) \right] \right\} + y^* \left[ \tan^2 \alpha - \frac{f_1 f_3}{f_4^2} \right] = 0
\end{align*}
\]

\[
\begin{align*}
\frac{dq}{dy} & = 0 \\
T & \approx \bar{v} \cos \alpha / f_4 \\
q & = -\kappa \frac{dT}{dy}
\end{align*}
\]

\[
\begin{align*}
\frac{d\kappa}{dy} & = 0 \\
0 & = -\frac{dq}{dy} + S \frac{du}{dy}
\end{align*}
\]

Invariant conductivity
Consequence on the volume fraction

Jenkins & Richman 1985
Garzo & Dufty PRE 1999
Kumaran JFM 2006
Phonon conductivity

\[
\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2} + D \frac{\partial^3 \xi}{\partial t \partial x^2}
\]

\[
c^2 = \frac{\beta d^2}{m} \quad D = \frac{d^2 \delta}{m}
\]

\[
\tilde{\kappa} = \frac{3}{2} \frac{\rho_s \nu D}{\sqrt{\epsilon}} \left[1 - \frac{j}{\sqrt{\epsilon}}\right] \quad \epsilon = \frac{D^2 \omega^2}{c^4}
\]

\[
\kappa = \frac{54}{\pi^3} \nu \left(\frac{\ell}{d}\right) \sqrt{\rho_s \beta d} \gg f_c \rho_s d \sqrt{T}
\]

Louge, Gran. Mat. 2011
### Mass flow rate versus flowing depth

| flow type                        | n |
|----------------------------------|---|
| soft, dissipative base           | 1 |
| Newtonian fluid, turbulent       | 1.5 |
| flat, frictional base            | 1.5 |
| core over a bumpy base           | 2.5 |
| Newtonian fluid, laminar         | 3 |

\[ \frac{\dot{m}}{W} \propto H^n \]

channel width $W$, flowing depth $H$
Boundaries matter.

http://grainflowresearch.mae.cornell.edu