Uniform estimates for oscillatory integrals with homogeneous polynomial phases of degree 4

Michael Ruzhansky¹,² · Akbar R. Safarov³,⁴ · Gafurjan A. Khasanov⁴

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Abstract
In this paper we consider the uniform estimates for oscillatory integrals with homogeneous polynomial phases of degree 4 in two variables. The obtained estimate is sharp and the result is an analogue of the more general theorem of Karpushkin (Proc I.G.Petrovsky Seminar 9:3–39, 1983) for sufficiently smooth functions, thus, in particular, removing the analyticity assumption.

Keywords Oscillatory integral · Phase function · Amplitude

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Akbar R. Safarov
safarov-akbar@mail.ru

Michael Ruzhansky
michael.ruzhansky@ugent.be

Gafurjan A. Khasanov
khasanov-g75@mail.ru

¹ Department of Mathematics Analysis, Logic and Discrete Mathematics, Ghent University, Krijgslaan 281, Ghent, Belgium
² School of Mathematical Sciences, Queen Mary University of London, London, UK
³ Institute of Mathematics named after V. I. Romanovskiy at the Academy of Sciences of the Republic of Uzbekistan, Olmazor district University 46, Tashkent, Uzbekistan
⁴ Department Mathematics, Samarkand State University, 15 University Boulevard, Samarkand 140104, Uzbekistan
1 Introduction

In this paper we continue the study of oscillatory integrals with smooth phase functions initiated in [8], in the case when the phase function is a homogeneous polynomial of degree four in two variables. The issue of whether integrals of real-analytic functions remain finite under small deformations was considered in [7]. An approach based on uniform estimates for certain classes of one-dimensional integrals is introduced. It is strong enough to recover the stability properties of real integrals in two dimensions which follow from the work of V. N. Karpushkin [20, 21], and to obtain new results in higher dimensions. In dimension three, the new stability results are sharp, as shown by the well-known example of Varchenko [1].

Definition 1.1 An oscillatory integral with phase \( f \) and amplitude \( a \) is an integral of the form

\[
J(\lambda, f, a) = \int_{\mathbb{R}^n} a(x) e^{i\lambda f(x)} \, dx,
\]

where \( a \in C_0^\infty(\mathbb{R}^n) \) and \( \lambda \in \mathbb{R} \).

If the support of \( a \) lies in a sufficiently small neighborhood of the origin and \( f \) is an analytic function at \( x = 0 \), then for \( \lambda \to \infty \) the following asymptotic expansion holds ([19]):

\[
J(\lambda, f, a) \approx e^{i\lambda f(0)} \sum_s \sum_{k=0}^{n-1} b_{s,k}(a) \lambda^s (\ln \lambda)^k,
\]

where \( s \) belongs to a finite number of arithmetic progressions, independent of \( a \), composed of negative rational numbers.

Definition 1.2 The oscillation exponent \( \beta(f) \) at the point 0 is the number \( \beta(f) \), maximum among all numbers \( s \) possessing the following property: for any neighborhood of the point 0 there exists a function \( a \) supported in this neighborhood for which in the decomposition (1.2) there is \( k \) such that \( b_{s,k}(a) \neq 0 \).

Definition 1.3 The multiplicity \( p(f) \) of the oscillation exponent \( \beta(f) \) is the maximal \( p \) with the property: for any neighborhood of the point 0 there exists \( a \) with support in it, for which \( b_{\beta(f), p(f)}(a) \neq 0 \). The pair \((\beta(f), p(f))\) is denoted by \( O(f) \).
Let $U \subset V \subset \mathbb{R}^2$ be bounded neighborhoods of the origin, $\overline{U}(\overline{V})$ the closure of $U(V)$, respectively. Suppose that the function $f : \overline{V} \rightarrow \mathbb{R}$ (where $f \in C^N(\overline{V})$, $(N \geq 8)$) has the form (for the proof of Theorem 1.6 we need $C^8$ smoothness of the function $f$):

$$f(x_1, x_2) = f_\pi(x_1, x_2) + g(x_1, x_2), \quad (1.3)$$

where $f_\pi(x_1, x_2)$ is a homogeneous polynomial of degree 4 having the root $b(0, 0)$ of multiplicity at most 2 (e.g. polynomial $f_\pi(x)$ of order 4 has at most two roots of multiplicity two on the unit circle in $\mathbb{R}^2$ centered at the origin), and $g \in C^N(V)$ is such that $D^\alpha g(0, 0) = 0$ for all $\alpha_1 + \alpha_2 \leq 4$, where $D^\alpha$ is $D^\alpha = \frac{\partial^{\alpha}|g|}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$.

$\alpha := (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ is a multi-index, $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$ are the non-negative integers.

**Definition 1.4** Let $F \in C^N(\overline{V})$ be a function such that $\|F\|_{C^N(\overline{V})} < \varepsilon$, where $N$ is a natural number and $\varepsilon$ is a sufficiently small positive number, where $\|F\|_{C^N(\overline{V})} = \max_{\overline{V}} \sum_{|\alpha| \leq N} \left| \frac{\partial^{\alpha}|F|}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \right|$. Then the function $f + F$ is called to be a deformation of $f$ (see [20]).

**Definition 1.5** ([20]) Let $f : V \rightarrow \mathbb{R}$ be a $C^N(\overline{V})$ function. We say that the oscillatory integral with phase $f$ has the uniform estimate $(\beta_u, p_u)$, where $\beta_u \leq 0$, $p_u \geq 0$, $p_u$ is an integer, if there are $C_V > 0$, $\varepsilon_V > 0$ and a neighborhood $U$ of $0 \in \mathbb{R}^2$, such that $U \subset \subset V$ and for all $\lambda \geq 2$, $\|F\|_{C^N(U)} < \varepsilon$, $a \in C^2(U)$, we have

$$|J(\lambda, f + F, a)| \leq C_V \lambda^{\beta_u} (\ln \lambda)^{p_u} \|a\|_{C^2}.$$  

The main result of this work is the following.

**Theorem 1.6** Let $f \in C^8(\overline{V})$ have the form (1.3). Then there exists a positive number $\varepsilon$ and a neighborhood $U \subset V$ of the origin such that for any functions $a \in C^1_0(U)$ and $F \in C^8(\overline{V})$, $\|F\|_{C^8(\overline{V})} < \varepsilon$, the following estimate holds:

$$\left| \int_U e^{i\lambda(f + F)}a(x) \, dx \right| \leq \frac{C \|a\|_{C^1} \ln(2 + |\lambda|)}{|\lambda|^2}. \quad (1.4)$$

(1) In paper [21], V. N. Karpushkin showed that, if $f$ is analytic at the origin, $df|_0 = 0$, and $d^2f|_0$ has corank 2, then the result of this theorem holds. Here, we show that an analog of V. N. Karpushkin’s result [21] (and also see [1]) still holds true for sufficiently smooth phase functions.

(2) Note that if $g \equiv 0$, $F \equiv 0$, and $a(0) \neq 0$, then we have ([9])

$$\lim_{\lambda \to +\infty} \frac{\lambda^{\frac{1}{2}}}{\ln \lambda} \int_{\mathbb{R}^2} e^{if(x)}a(x) \, dx = ca(0)$$

with $c \neq 0$. 

In the proof we combine the method of the partition of unity used in several places of the book [4] and an idea of the paper by Karpushkin [21]. V. N. Karpushkin considered oscillatory integrals with analytic phase function. It turns out that the implicit function theorem for smooth functions allows one to get analogous results for sufficiently smooth functions. We do not concentrate our attention on the optimal regularity properties of the phase functions. The optimality of smoothness of the phase function will be considered elsewhere.

2 Auxiliary statements

We first present some simple auxiliary definitions.

**Definition 2.1** ([17]) Consider the space $R^n$ with fixed coordinates $x_1, x_2, \ldots, x_n$. A function $f : R^n \setminus \{0\} \rightarrow R$ is said to be a quasi-homogeneous function of degree $d$ (with $d \in R$) with exponents $\alpha_1, \alpha_2, \ldots, \alpha_n$ (where $\alpha_j (j = 1, \ldots, n)$ are non-negative rational numbers), if for all $\lambda > 0$ and $x \neq 0$ we have $f (\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, \ldots, \lambda^{\alpha_n} x_n) = \lambda^d f (x_1, x_2, \ldots, x_n)$. The powers $\alpha_j$ are called weights of the variables $x_j$.

**Definition 2.2** Following [20], the coordinate subspace $B \subset E_1$ is said to be a versal subspace if $(I_{f_\pi} \cap E_1) \oplus B = E_1$, that is,

$$I_{f_\pi} \cap E_1 \cap B = \{0\}$$

and $(I_{f_\pi} \cap E_1) + B = E_1$.
It is easy to show that $B = \text{Span}\{1, x_1, x_2, x_1^2, x_2^2, x_1x_2, x_1^3, x_2^3\}$ is a versal subspace for the function $f_{\pi}(x_1, x_2)$ (see Lemma 2.4).

Let $\pi_d(F) = \sum_{m_1 + m_2 < d} s_{m_1} x_1^{m_1} x_2^{m_2}$ be the Taylor polynomial at the point 0 of the function $F$. Thus, $\pi_1$ defines a mapping of the space $C^N(V)$ onto the space $E_d$, where $d \leq N/4$.

**Lemma 2.4** Let $f_{\pi}$ be given in one of the following forms: $x_1^4 + \mu x_2^2 x_1^2 + x_2^4$ or $x_1^2(x_1^2 \pm x_2^2)$. Then $B = \text{Span}\{1, x_1, x_2, x_1^2, x_2^2, x_1x_2, x_1^3, x_2^3\}$ is a versal subspace for $f_{\pi}$.

**Proof** We have that $\partial_1 f_{\pi}$ and $\partial_2 f_{\pi}$ are linearly independent (see [20] Lemma 20, page 33) and $x_1^3$ and $x_2^3$ do not belong to the space spaned by $\partial_1 f_{\pi}$ and $\partial_2 f_{\pi}$. Moreover, in the considered cases $\partial_1 f_{\pi}$ and $\partial_2 f_{\pi}$ are linearly independent and spaces spanned by $\langle \partial_1 f_{\pi}, \partial_2 f_{\pi} \rangle$ and $\langle x_1^3, x_2^3 \rangle$ have only trivial intersection, that is $\langle \partial_1 f_{\pi}, \partial_2 f_{\pi} \rangle \cap \langle x_1^3, x_2^3 \rangle = \{0\}$. Consequently $B$ is a versal subspace of $E_1$ corresponding to $f_{\pi}$. Lemma 2.4 is proved.

Let denote

$$\vartheta_{C^8(V)}(\varepsilon) := \left\{ F \in C^8(V), \left\| F \right\|_{C^8(V)} < \varepsilon \right\}.$$

The next proposition on the possibility of smoothly choosing coordinates is an analogue of the versality theorem. An analogue of the deformation of the function of $f$ is $f + F$, where $F \in \vartheta_{C^8(V)}(\varepsilon)$ (deformation with an infinite number of parameters). An analogue of the versal deformation of $f$ is $f + F$, where $F \in \vartheta_{C^8(V)}(\varepsilon)$, $\pi_1(F) \in B$.

Here $B$ is a versal subspace ([20]).

Let $f_{\pi} + P$ be a deformation, where $P \in B \subset E_1$.

**Definition 2.5** We call $f_{\pi} + P$ a versal deformation of $f$, if for any deformation $F$, there exists a vector $z(F) \in \mathbb{R}^2$ such that the projection $\pi_1 : (f + F)(z) \rightarrow P(F)$ is well defined, where $P(F)$ is the segment of the Taylor series (see [17], page 37).

Now we consider the function $f + F$ and we represent it as

$$f + F = s_{10} x_1 + s_{01} x_2 + s_{20} x_1^2 + 2s_{11} x_1 x_2 + s_{02} x_2^2 + s_{30} x_1^3 + s_{21} x_1^2 x_2 + s_{12} x_1 x_2^2 + s_{31} x_1 x_2 \left[ x_1^4 + s_{40} x_1^4 + s_{30} x_1 x_2 \right] x_1^3 x_2 + s_{22} x_1 x_2 x_1^2 x_2 + s_{13}(x_1, x_2) x_1 x_2^3 + s_{04}(x_1, x_2) x_2^4,$$

where $s_{10} = \frac{\partial F(0, 0)}{\partial x_1}$, $s_{01} = \frac{\partial F(0, 0)}{\partial x_2}$, $s_{20} = \frac{1}{2} \frac{\partial^2 F(0, 0)}{\partial x_1^2}$, $s_{11} = \frac{1}{2} \frac{\partial^2 F(0, 0)}{\partial x_1 \partial x_2}$, $s_{02} = \frac{1}{2} \frac{\partial^2 F(0, 0)}{\partial x_2^2}$, $s_{30} = \frac{1}{3} \frac{\partial^3 F(0, 0)}{\partial x_1^3}$, $s_{21} = \frac{1}{3} \frac{\partial^3 F(0, 0)}{\partial x_1^2 \partial x_2}$, $s_{12} = \frac{1}{3} \frac{\partial^3 F(0, 0)}{\partial x_1 \partial x_2^2}$, $s_{31} = \frac{1}{3} \frac{\partial^3 F(0, 0)}{\partial x_2^3}$, $s_{03} = \frac{1}{3} \frac{\partial^3 F(0, 0)}{\partial x_1^2}$, $s_{k_1 k_2}(x_1, x_2) := s_{k_1 k_2} = \int_0^1 (1 - u)^2 \frac{\partial^4 (F + g)(ax_1, ax_2)}{\partial x_1^3 \partial x_2^2} du$, $k_1 + k_2 = 4$.

We change variables $x_1 - z_1(F) = y_1, x_2 - z_2(F) = y_2$, and expanding the function $(F + g)(y + z(F))$ by the Taylor formula at the point $(y_1, y_2) = (0, 0)$ we have
\((f + F)(y + z(F)) = \alpha_{00}(y, F) + \alpha_{10}(y, F)y_1 + \alpha_{01}(y, F)y_2 \\
+ \alpha_{20}(y, F)y_1^2 + \alpha_{11}(y, F)y_1y_2 + \alpha_{02}(y, F)y_2^2 \\
+ \alpha_{30}(y, F)y_1^3 + \alpha_{21}(y, F)y_1^2y_2 \\
+ \alpha_{12}(y, F)y_1y_2^2 + \alpha_{03}(y, F)y_2^3 + \sum_{i_1 + i_2 = 4} \alpha_{i_1i_2}(y, F)y_1^{i_1}y_2^{i_2} + f_\pi(y_1, y_2),\)

where \(\alpha_{00}(z) = f(0) + F(0)\) and \(\alpha_{i_0}(y, F), \alpha_{01}(y, F), \alpha_{20}(y, F), \alpha_{11}(y, F), \alpha_{02}(y, F), \alpha_{30}(y, F), \alpha_{03}(y, F), \alpha_{21}(y, F), \alpha_{12}(y, F)\) are functionals of \(F\). Functionals \(\alpha_{21}(y, F), \alpha_{12}(y, F)\) are satisfying the conditions \(\alpha_{21}(0, 0) = 0, \alpha_{12}(0, 0) = 0\).

**Proposition 2.6** There exists a positive number \(\varepsilon > 0\) such that for any \(\|F\|_{C^8(\overline{V})} < \varepsilon\) there exists a mapping \((z_1, z_2) := (z_1(F), z_2(F)) \in C^4(U \to \mathbb{R}^2)\), defined in some neighborhood of \(U\) for which the following equality holds

\[\pi_1 \left( f(y_1 + z_1, y_2 + z_2) + F(y_1 + z_1, y_2 + z_2) \right) = \tilde{c}_0(F) + \tilde{c}_1(F)y_1 \\
+ \tilde{c}_2(F)y_2 + \tilde{c}_3(F)y_1^2 + \tilde{c}_4(F)y_2^2 + \tilde{c}_5(F)y_1y_2 + \tilde{c}_6(F)y_1^3 + \tilde{c}_7(F)y_2^3,\]

where \(\pi_1(\cdot)\) is the projection mapping from the space \(C^4(V)\) to the space \(E_1\).

Now consider the following functional equations for \((z_1, z_2)\):

\[\Phi_1(y, F, z) := \alpha_{12}(z) = 0, \quad \Phi_2(y, F, z) := \alpha_{21}(z) = 0. \tag{2.1}\]

**Remark 2.7** We need \(z\) to be a \(C^4\) function and it depends on \(4^{th}\) derivatives of the function \(F\). Hence in order to apply the implicit function Theorem 1.6 we need \(C^8\) smoothness of the function \(F\).

Let us give an auxiliary lemma.

**Lemma 2.8** Let \(F \in C^8(\overline{V})\). For \(C^4\) functionals \(\Phi_1(y, F, z), \Phi_2(y, F, z)\) in the space \(U_1 \times C^4(V_1) \times U_2\) with \(U_1 \subset \mathbb{R}^2, U_2 \subset \mathbb{R}^2\), there exists a partial derivative with respect to \(z\).

**Proof** For the sake of definiteness, let us show the existence of partial derivatives with respect to \(z\) and differentiability of the operator \(\Phi_1(y, F, z)\); for the function \(\Phi_2(y, F, z)\) the proof is analogous.

Since \(F \in C^8(\overline{V})\) and \(g \in C^8(\overline{V})\), this implies the differentiability of the operator \(\Phi_1(y, F, z)\). The existence of derivatives of the mapping \(\Phi_2(y, F, z)\) is considered similarly. \(\Box\)

**Lemma 2.9** Operators \(\Phi_1(y, F, z), \Phi_2(y, F, z)\) satisfy \(\Phi_1(y, 0, 0) \equiv 0, \Phi_2(y, 0, 0) \equiv 0\), and

\[
\begin{vmatrix}
\frac{\partial \Phi_1}{\partial z_1} & \frac{\partial \Phi_1}{\partial z_2} \\
\frac{\partial \Phi_2}{\partial z_1} & \frac{\partial \Phi_2}{\partial z_2}
\end{vmatrix} \neq 0.
\]

**Proof** From the explicit form of the operators \(\Phi_1, \Phi_2\) it follows that we have \(\Phi_1(y, 0, 0) \equiv 0\) and \(\Phi_2(y, 0, 0) \equiv 0\). We also have, that

\[
\left| \frac{\partial \Phi_1(0)}{\partial z_1} \frac{\partial \Phi_2(0)}{\partial z_2} - \frac{\partial \Phi_1(0)}{\partial z_2} \frac{\partial \Phi_2(0)}{\partial z_1} \right| = 4 \neq 0
\]

and hence the last statement is also true. Lemma 2.9 is proved. \(\Box\)
We proceed to the proof of Proposition 2.6. Since the operators $\Phi_1(y, F, z)$, $\Phi_2(y, F, z)$ by Lemma 2.8 satisfy the conditions of the implicit mapping theorem, then there is a solution $z_1 = z_1(y_1, y_2, F(y_1))$, $z_2 = z_2(y_1, y_2, F(y))$ of the equation (2.1), which are smooth functions depending on the smoothness of the mapping $F$.

Let us assume that the function $\phi$ is a fixed function of the form

$$\phi = x_1^3 b(x) + x_1 Q(x_2, x_3, ..., x_n),$$  \hspace{1cm} (2.2)

where $b$ is a smooth function with $b(0) \neq 0$. Suppose in (2.2) that $Q(x_2, x_3, ..., x_n)$ is an analytic function satisfying the conditions $Q(0) = 0$, $\nabla Q(0) = 0$. Suppose that $Q \neq 0$ and the pair $(\beta(Q), p(Q))$ is the type of asymptotics of the oscillatory integral with phase $Q$ (see [20]).

**Lemma 2.10** Let $Q$ be an analytic function satisfying the condition $\beta(Q) \leq \frac{1}{4}$. Then for the uniform type oscillatory integral with phase $f$, the following relations hold:

$$\beta_u(\phi) = \beta(\phi) \quad \text{and} \quad p_u(\phi) = \begin{cases} p(Q), & \text{if } \beta(Q) < \frac{1}{4}, \\ p(Q) + 1, & \text{if } \beta(Q) = \frac{1}{4}. \end{cases}$$

We present the following lemma.

**Lemma 2.11** Let

$$J_A(\lambda, \sigma, a) = \int_\mathbb{R} e^{i\lambda(x^3 b(x, \sigma) + \sigma x)} a(x) dx,$$  \hspace{1cm} (2.3)

where $b(0, 0) \neq 0$, and $a(x)$ has bounded variation and is supported in a sufficiently small neighborhood of the origin. Then there exists a constant $c$ such that for the integral (2.3) the following inequality holds uniformly for all parameters $\sigma$ and $\lambda$:

$$|J_A(\lambda, \sigma, a)| \leq \frac{c \|a\|_V}{|\lambda|^{\frac{3}{2}} + |\lambda|^{\frac{1}{2}} |\sigma|^{\frac{1}{4}}},$$

where $\|a\|_V = |a(-\infty)| + V_R[a]$ and $V_R[a]$ is total variation, for all parameter $\sigma \in \mathbb{R}$.

For proof of Lemma 2.11 see [10], also [3, 5, 14].

**Proof** As we have the function $\phi = x_1^3 b(x) + x_1 Q + \varphi$, where $|\varphi| < \epsilon$, then we will change the variables $x_1 = x_1(y_1, x_2, ..., x_n)$ so that we have

$$\phi = y_1^3 + \Sigma(x_2, ..., x_n) y_1 + \psi(x_2, ..., x_n).$$

Moreover, $\Sigma$ has the form $\Sigma = Q \Sigma_1(x_2, ..., x_n)$, where $\Sigma_1$ is an analytic function satisfying the condition $\Sigma_1(0) \neq 0$. Then we have

$$\phi = y_1^3 + Q \Sigma_1 y_1 + \psi,$$
and integral (1.1) has the form
\[
J(\lambda, f, a) = \int_{\mathbb{R}^n} e^{i\lambda \phi} a(y_1) dy_1 dx_2 \cdots dx_n.
\] (2.4)

We consider the following cases:

Case 1. \[|\frac{\lambda}{2} Q \Sigma_1| \leq M,\] where \(M\) is sufficiently large and fixed. Using Theorem 6.1 in the paper [22] [page 438] and Lemma 2.11 we have:
\[
|J| \leq \mu \left( \left| \frac{\lambda}{2} Q \Sigma_1 \right| \leq M \right) \leq c \frac{\lambda^{-\frac{3}{2}} (Q \Sigma_1)(\ln \lambda)^{p(Q)}}{\lambda^{\frac{1}{2} + \frac{1}{2} \beta(Q)}}.
\]

Case 2. \[|\frac{\lambda}{2} Q \Sigma_1| > M.\] Consider the sets
\[
A_k = \left\{ 2^k \leq \frac{\lambda}{2} |Q \Sigma_1| \leq 2^{k+1} \right\}.
\]
As \(\Sigma_1(0) \neq 0\), without loss of generality we can assume that \(\Sigma_1(0) = 1\) at \(|x| < \epsilon\) and \(\frac{1}{2} \leq \Sigma_1(x) \leq 2\). Hence
\[
A_k \subset \left\{ \frac{2^{k-1}}{\lambda^{\frac{3}{2}}} \leq |Q| \leq \frac{2^{k+2}}{\lambda^{\frac{3}{2}}} \right\}.
\]
For a measure of a set of smaller values we use Lemma 1' in the paper [20] and we have:
\[
\mu \left( |Q| \leq \frac{2^{k+2}}{\lambda^{\frac{3}{2}}}, x \in U \right) \leq \left( \frac{2^{k+2}}{\lambda^{\frac{3}{2}}} \right)^{\beta} \left( \ln \left| \frac{\lambda^{\frac{3}{2}}}{2^{k+2}} \right| \right)^p.
\]

Due to Lemma 2.11 for the integral
\[
J_k = \int_{A_k} e^{i\lambda (y_1^3 + y_1 Q \Sigma_1)} a(y_1, x) dy_1,
\]
we find the following estimate:
\[
|J_k| = \int_{A_k} e^{i\lambda (y_1^3 + y_1 Q \Sigma_1)} a(y_1, x) dy_1 \leq \frac{\mu(A_k)}{\lambda^{\frac{1}{2}} |Q \Sigma_1|^\frac{1}{4}} \leq \left( \frac{2^{k+2}}{\lambda^{\frac{3}{2}}} \right)^{\beta} \left( \ln \left| \frac{\lambda^{\frac{3}{2}}}{2^{k+2}} \right| \right)^p.
\]

From here we find the sum \(J_k\) and, by estimating the integral \(J\), we find the required estimates. Lemma 2.10 is proved. \(\square\)
3 A partition of unity

The oscillatory integral is estimated using the partition of unity below.

Let \( k = (k_1, k_2) \) and \( r > 0 \) be fixed. We consider the mapping \( \delta^k_r : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by the formula:

\[
\delta^k_r(x) = \left( r^{\frac{1}{4}}x_1, r^{\frac{1}{4}}x_2 \right).
\]

Let us introduce a function \( \beta(x) \), satisfying the conditions:

1) \( \beta \in C^\infty (\mathbb{R}^2) \),
2) \( 0 \leq \beta(x) \leq 1 \) for all \( x \in \mathbb{R}^2 \),
3) \( \beta(x) = \begin{cases} 1, & \text{when } |x| \leq 1, \\ 0, & \text{when } |\delta_2^{-1}(x)| \geq 1. \end{cases} \)

The existence of such a function was proved in [18] (see also [4]).

Let us denote

\[
\chi(x) = \beta(x) - \beta(\delta^k_2(x)).
\]

The main properties of the function \( \chi(x) \) are contained in the following lemma.

**Lemma 3.1** The function \( \chi(x) \) satisfies the following conditions:

1. For any natural number \( v_0 \geq 1 \) and for an arbitrary fixed \( x \), we have

\[
\beta(\delta_2^{-v_0}(x)) + \sum_{v=v_0}^{\infty} \chi(\delta_2^{-v}(x)) = 1.
\]

2. For any \( x \neq 0 \) there exists \( v_0(x) \) such that for any \( v \notin [v_0(x), v_0(x) + 4] \),

\[
\chi(\delta_2^v(x)) = 0.
\]

3. For any \( v_0 \) there exists \( \varepsilon > 0 \) such that \( \chi(\delta_2^v(x)) = 0 \) for any \( v < v_0 \) and \( |x| \geq \varepsilon \).

Lemma 3.1 was proved in the paper [4].

4 Proof of the main result (Theorem 1.6)

Since the function has the form \( f(x_1, x_2) = f_\pi(x_1, x_2) + g(x_1, x_2) \), then applying Proposition 2.6 to \( f + F \) we get

\[
f + F = s_{00} + s_{10}y_1 + s_{01}y_2 + s_{20}y_1^2 + s_{02}y_2^2 + s_{11}y_1y_2 + s_{30}y_1^3 + s_{03}y_2^3 + f_\pi \\
+ R_4(y_1, y_2),
\]

(4.1)
where \( R_4(y_1, y_2) \) is the remainder term. Now we estimate the integral \( J \). First, we introduce the “quasi-distance” \( \rho = |s_{10}|^2 + |s_{01}|^2 + |s_{20}|^2 + |s_{02}|^2 + |s_{11}|^2 + |s_{30}|^4 + |s_{03}|^4 \).

First we consider the case \( \lambda \rho \leq 2 \). Then we use change of variables as \( y_1 = \lambda^{-\frac{1}{2}} \tau_1 \) and \( y_2 = \lambda^{-\frac{1}{2}} \tau_2 \). So we have

\[
J(\lambda) = \lambda^{-\frac{1}{2}} \int_{\mathbb{R}^2} a\left( \lambda^{-\frac{1}{2}} \tau_1, \lambda^{-\frac{1}{2}} \tau_2 \right) e^{i \Phi_1} d\tau,
\]

where \( \Phi_1 = \lambda^\frac{3}{2} s_{10} \tau_1 + \lambda^\frac{3}{2} s_{01} \tau_2 + \lambda^\frac{1}{2} s_{20} \tau_1^2 + \lambda^\frac{1}{2} s_{02} \tau_2^2 + \lambda^\frac{1}{2} s_{11} \tau_1 \tau_2 + \lambda^\frac{1}{2} s_{30} \tau_1^3 + \lambda^\frac{1}{2} s_{03} \tau_2^3 + f_\pi + R_4(\lambda^{-\frac{1}{2}} \tau_1, \lambda^{-\frac{1}{2}} \tau_2) \). We now apply Lemma 3.1 for the integral \( J(\lambda) \), to get the decomposition in the form

\[
J(\lambda) = J_0(\lambda) + \sum_{k=k_0}^{\ln \lambda} J_k(\lambda),
\]

where

\[
J_0(\lambda) = \lambda^{-\frac{1}{2}} \int_{\mathbb{R}^2} a\left( \lambda^{-\frac{1}{2}} \tau_1, \lambda^{-\frac{1}{2}} \tau_2 \right) \beta_0(\delta_{2^{-k_0}}(x)) e^{i \rho \Phi_1} d\tau,
\]

\[
J_k(\lambda) = \lambda^{-\frac{1}{2}} \int_{\mathbb{R}^2} a\left( \lambda^{-\frac{1}{2}} \tau_1, \lambda^{-\frac{1}{2}} \tau_2 \right) \chi_0\left( 2^{-\frac{k}{2}} \tau_1, 2^{-\frac{k}{2}} \tau_2 \right) e^{i \rho \Phi_1} d\tau.
\]

First, we estimate the integral \( J_k(\lambda) \). In this integral \( J_k(\lambda) \) we make the change of variables \( 2^{-\frac{k}{2}} \tau_1 = t_1, 2^{-\frac{k}{2}} \tau_2 = t_2 \), so that

\[
J_k(\lambda) = 2^{\frac{k}{2}} \lambda^{-\frac{1}{2}} \int_{\mathbb{R}^2} a\left( 2^{\frac{k}{2}} \lambda^{-\frac{1}{4}} t_1, 2^{\frac{k}{2}} \lambda^{-\frac{1}{4}} t_2 \right) \chi(t_1, t_2) e^{i 2^k \Phi_{1k}(t, s)} d\tau,
\]

where the phase function has the form \( \Phi_{1k} = \lambda^\frac{3}{2} s_{10} t_1 + \lambda^\frac{3}{2} s_{01} t_2 + \lambda^\frac{1}{2} s_{20} t_1^2 + \lambda^\frac{1}{2} s_{02} t_2^2 + \lambda^\frac{1}{2} s_{11} t_1 t_2 + \lambda^\frac{1}{2} s_{30} t_1^3 + \lambda^\frac{1}{2} s_{03} t_2^3 + f_\pi + \frac{1}{2^k} R_4 \left( 2^\frac{k}{2} \lambda^{-\frac{1}{2}} \tau_1, 2^\frac{k}{2} \lambda^{-\frac{1}{2}} \tau_2 \right) \).

Since \( \lambda \rho \leq 2 \) and \( k \) is sufficiently large we may assume that coefficients are small. Then using Van der Corput lemma ( [13]) for the integral \( J_k(\lambda) \), we have

\[
|J_k(\lambda)| \leq \frac{2^\frac{k}{2} \lambda^{-\frac{1}{2}} C}{2^\frac{k}{2}} = \frac{C}{\lambda^\frac{1}{2}}.
\]

(4.2)

Then, summing the integrals \( J_k(\lambda) \), we get

\[
\sum_{k \leq \ln \lambda} J_k(\lambda) \leq \frac{C}{\lambda^\frac{1}{2}} \sum_{k \leq \ln \lambda} 1 \leq \frac{C \ln \lambda}{\lambda^\frac{1}{2}}.
\]
Now we consider the integral $J_0 (\lambda)$. Since amplitude of the integral has compact support the trivial estimate yields

$$|J_0 (\lambda)| \leq \frac{C}{\lambda^\frac{1}{2}}.$$  

In this case we get the estimate (1.4).

Next we consider the case $\lambda, \rho > 2$. We change the variables $y_1 = \rho^\frac{1}{2} \tau_1$, $y_2 = \rho^\frac{1}{2} \tau_2$. Then we have

$$J(\lambda) = \rho^\frac{1}{2} \int_{\mathbb{R}^2} a \left( \rho^\frac{1}{2} \tau_1, \rho^\frac{1}{2} \tau_2 \right) e^{i\lambda\rho \Phi} d\tau,$$

where $\Phi = \frac{s_{10}}{\rho^\frac{1}{2}} \tau_1 + \frac{s_{01}}{\rho^\frac{1}{2}} \tau_2 + \frac{s_{20}}{\rho^\frac{1}{2}} \tau_1^2 + \frac{s_{02}}{\rho^\frac{1}{2}} \tau_2^2 + \frac{s_{11}}{\rho^\frac{1}{2}} \tau_1 \tau_2 + \frac{s_{10}}{\rho^\frac{1}{2}} \tau_1^3 + \frac{s_{01}}{\rho^\frac{1}{2}} \tau_2^3 + f_{\pi} + \frac{1}{\rho} R_4 \left( \rho^\frac{1}{2} \tau_1, \rho^\frac{1}{2} \tau_2 \right)$. We now apply Lemma 3.1 for the integral $J(\lambda)$, to get the decomposition in the form

$$J(\lambda) = J_0(\lambda) + \sum_{k=k_0}^{\infty} J_k (\lambda),$$

where

$$J_k (\lambda) = \rho^\frac{1}{2} \int_{\mathbb{R}^2} a \left( \rho^\frac{1}{2} \tau_1, \rho^\frac{1}{2} \tau_2 \right) \chi \left( 2^{-\frac{k}{2}} \tau_1, 2^{-\frac{k}{2}} \tau_2 \right) e^{i\lambda\rho \Phi} d\tau,$$

$$J_0 (\lambda) = \rho^\frac{1}{2} \int_{\mathbb{R}^2} a \left( \rho^\frac{1}{2} \tau_1, \rho^\frac{1}{2} \tau_2 \right) \beta_0(\delta_{2^{-k_0}}(x)) e^{i\lambda\rho \Phi} d\tau.$$

First, we estimate the integral $J_k (\lambda)$. In this integral $J_k (\lambda)$ we make the change of variables $2^{-\frac{k}{2}} \tau_1 = t_1$, $2^{-\frac{k}{2}} \tau_2 = t_2$, so that

$$J_k (\lambda) = 2^k \rho^\frac{1}{2} \int_{\mathbb{R}^2} a \left( 2^\frac{k}{2} \rho^\frac{1}{2} t_1, 2^\frac{k}{2} \rho^\frac{1}{2} t_2 \right) \chi (t_1, t_2) e^{i\lambda^2 k \rho \Phi_k(t,s,\rho)} dt,$$

where the phase function has the form

$$\Phi_k(t,s,\rho) = 2^{-\frac{3k}{4}} \sigma_{10} t_1 + 2^{-\frac{3k}{4}} \sigma_{01} t_2 + 2^{-\frac{k}{4}} \sigma_{20} t_1^2 + 2^{-\frac{k}{4}} \sigma_{02} t_2^2 + 2^{-\frac{k}{4}} \sigma_{11} t_1 t_2 + 2^{-\frac{k}{4}} \sigma_{30} t_1^3 + 2^{-\frac{k}{4}} \sigma_{03} t_2^3 + f_{\pi} + 2^{-k} \rho \left( 2^\frac{k}{2} \rho^\frac{1}{2} t_1, 2^\frac{k}{2} \rho^\frac{1}{2} t_2 \right),$$

where $\sigma_{10} = \frac{s_{10}}{\rho^\frac{3}{4}}$, $\sigma_{01} = \frac{s_{01}}{\rho^\frac{3}{4}}$, $\sigma_{20} = \frac{s_{20}}{\rho^\frac{1}{2}}$, $\sigma_{02} = \frac{s_{02}}{\rho^\frac{1}{2}}$, $\sigma_{11} = \frac{s_{11}}{\rho^\frac{1}{2}}$, $\sigma_{30} = \frac{s_{30}}{\rho^\frac{3}{4}}$, $\sigma_{03} = \frac{s_{03}}{\rho^\frac{3}{4}}$, $R_4 \left( 2^\frac{k}{2} \rho^\frac{1}{2} t_1, 2^\frac{k}{2} \rho^\frac{1}{2} t_2 \right) = \frac{2^k}{6} \left( s_{40} t_1^4 + s_{31} t_1^3 t_2 + s_{22} t_1^2 t_2^2 + s_{13} t_1 t_2^3 + s_{04} t_2^4 \right).$
where \( s_{40}(t, s, \rho) = \frac{1}{6} \int_0^1 (1 - u)^3 \frac{\partial^4 \Phi_k(tu, s, \rho)}{\partial t^4} du \), \( s_{31}(t, s, \rho) = \frac{1}{6} \int_0^1 (1 - u)^3 \frac{\partial^4 \Phi_k(tu, s, \rho)}{\partial t^4} du \), \( s_{22}(t, s, \rho) = \frac{1}{6} \int_0^1 (1 - u)^3 \frac{\partial^4 \Phi_k(tu, s, \rho)}{\partial t^4} du \), \( s_{13}(t, s, \rho) = \frac{1}{6} \int_0^1 (1 - u)^3 \frac{\partial^4 \Phi_k(tu, s, \rho)}{\partial t^4} du \).

We can assume (depending on the support of the amplitude \( \chi_0 \)), by Lemma 3.1, that the number \( k_0 \) is sufficiently large.

We may assume \( 2^t \lambda \rho > L \) and let \( k > k_0 \) be a sufficiently large number. Then, by hypothesis, \( \Phi_k \) can be considered a deformation of the function \( f_\pi(t_1, t_2) \) and \( \chi_0(\tau_1, \tau_2) \in D := \text{supp}(\chi) = \{ \frac{1}{2} \leq |\tau| \leq 2 \} \).

If \( \tau \neq 0 \), using Van der Corput lemma ([13]) (a more general statement is contained in [2, 15]), we have again estimate of the form (4.2).

Since support of the amplitude is on \( 2^t \lambda \frac{\rho}{2} \leq C \), then \( 2^t \lambda \frac{\rho}{2} \leq C \lambda \frac{\rho}{2} \) and \( 2^t \leq C \lambda \frac{\rho}{2} \). So we have \( \sum_{k \leq C \ln \lambda} |J_k| \leq \frac{1}{2^t} \sum_{k \leq C \ln \lambda} 1 \leq \frac{C \ln \lambda}{2^t} \).

Now we consider estimates of the integral \( J_0(\lambda) \). We consider several cases for parameters \( \sigma \).

Let us introduce the quasisphere \( \Omega := \{ |\sigma_{10}|^\frac{1}{2} + |\sigma_{01}|^\frac{1}{2} + |\sigma_{20}|^2 + |\sigma_{02}|^2 + |\sigma_{11}|^2 + |\sigma_{21}|^2 + |\sigma_{30}|^4 + |\sigma_{03}|^4 = 1 \} \) and consider the phase function

\[
\Phi_0(\tau, \sigma, \rho) = \sigma_{10} \tau_1 + \sigma_{01} \tau_2 + \sigma_{20} \tau_1^2 + \sigma_{02} \tau_2^2 + \sigma_{11} \tau_1 \tau_2 + \sigma_{30} \tau_1^3 + \sigma_{03} \tau_1^2 + \sigma_{40} \tau_4 + s_{31} \tau_1^3 \tau_2 + s_{22} \tau_1^2 \tau_2^2 + s_{13} \tau_1^2 \tau_2^3 + s_{04} \tau_4^4 + f_\pi(\tau_1, \tau_2).
\]

We note that on the quasi-sphere we have \( c_1 \leq |\sigma| \leq c_2 \), where \( c_1, c_2 \) are some fixed positive numbers.

Thus, the parameter space and \( \text{supp}(\beta(\delta_{\tau_1, \tau_2}(\cdot))) \) are compact sets. Let, \( \sigma = \sigma^0, |\sigma^0| = c \) be fixed vector and let \( \tau = \tau^0 \) be a fixed point. Then \( \Phi_0(\tau, \sigma, \rho) \) is a sufficiently smooth deformation of the function

\[
\Phi = \sigma_{10} \tau_1 + \sigma_{01} \tau_2 + \sigma_{20} \tau_1^2 + \sigma_{02} \tau_2^2 + \sigma_{11} \tau_1 \tau_2 + \sigma_{30} \tau_1^3 + \sigma_{03} \tau_1^2 + \sigma_{40} \tau_1^4 + s_{31} \tau_1^3 \tau_2 + s_{22} \tau_1^2 \tau_2^2 + s_{13} \tau_1^2 \tau_2^3 + s_{04} \tau_4^4 + f_\pi(\tau_1, \tau_2).
\]

If \( \frac{\partial \Phi_{00}(\tau_1, \tau_2)}{\partial t_2} \neq 0 \) or \( \frac{\partial \Phi_{00}(\tau_1, \tau_2)}{\partial t_1} \neq 0 \), then for \( |\sigma - \sigma_0| < \varepsilon \) and \( |s_{40}| + |s_{31}| + |s_{13}| + |s_{04}| < \varepsilon \) the following estimate holds \( |\nabla \Phi_0(\tau, \sigma, s)| > \delta > 0 \) for a positive number \( \delta \).

Using the integration by parts formula for the integral

\[
J_0^X(\lambda) = \int_{\mathbb{R}^2} \chi(\tau) a(\tau_1, \tau_2) \chi_0(\tau_1, \tau_2) e^{i \lambda \rho \Phi_0(\tau, \sigma, s)} d\tau,
\]

we have

\[
|J_0^X| \leq \frac{c \|a\| c_1}{|\lambda|^\frac{1}{2}},
\]
with $\chi$ a smooth function concentrated in a sufficiently small neighborhood of the point $\tau^0$.

It is enough to consider the case when $\tau^0 = (\tau^0_1, \tau^0_2)$ is a critical point.

Since $\tau^0$ is a critical point, we have the following equalities:

\[
\begin{align*}
\sigma^0_{10} + 2\sigma^0_{20}\tau^0_1 + \sigma^0_{11}\tau^0_2 + 3\sigma^0_{30}(\tau^0_1)^2 + \partial_1 f_\pi(\tau^0_1, \tau^0_2) &= 0, \\
\sigma^0_{01} + 2\sigma^0_{02}\tau^0_2 + \sigma^0_{11}\tau^0_1 + 3\sigma^0_{03}(\tau^0_2)^2 + \partial_2 f_\pi(\tau^0_1, \tau^0_2) &= 0.
\end{align*}
\]

The Hessian of the function $\Phi$ at the point $\tau^0 = (\tau^0_1, \tau^0_2)$ has the form

\[
H = \begin{pmatrix}
2\sigma^0_{20} + 6\sigma^0_{30} + \partial^2_1 f_\pi(\tau^0) & \sigma^0_{11} + \partial_1 \partial_2 f_\pi(\tau^0) \\
\sigma^0_{11} + \partial_1 \partial_2 f_\pi(\tau^0) & 2\sigma^0_{02} + 6\sigma^0_{03}\tau^0_2 + \partial^2_2 f_\pi(\tau^0)
\end{pmatrix}.
\]

We consider two cases for the matrix $H$. If this is a nonzero matrix, then we can use the generalised Van der Corput lemma [2, 15, 16], and we have the required bound.

Indeed, the rank of the Hessian $H$ is at least one. If the rank of matrix is one, then, using the Morse lemma with respect to parameters, for the integral $J_0$ we obtain the following estimate

\[
|J_0| \leq \frac{c \|a\|_{C^1} \rho^{\frac{1}{2}}}{|\lambda|^{\frac{1}{2}}},
\]

Assume now that the matrix $H$ is zero. Then $\sigma^0_{ij} = 0$, $i, j \leq 2$, and $\sigma^0_{30} \neq 0$ or $\sigma^0_{03} \neq 0$.

If $\sigma_{30} \neq 0$ and $\sigma_{03} \neq 0$ then the phase function can be considered as a small perturbation of $D_4$ type singularities in Arnol’d’s notation ([17]). Then we can use results proved by J. J. Duistermaat ([12]), or we can again repeat our arguments and obtain

\[
|J_0| \leq \frac{C \|a\|_{C^2}}{|\lambda|^{\frac{1}{4}}},
\]

which is better than we want. Suppose $\sigma^0_{30} \neq 0$ and $\sigma^0_{03} = 0$ or $\sigma^0_{30} = 0$ and $\sigma^0_{03} \neq 0$.

In this case we will use Lemma 2.10 and obtain

**Lemma 4.1** We have

\[
\left| \int_U e^{i\lambda (x_1^4 b_1(x_2, \sigma) + x_1 g_1(x_2, \varepsilon) + g_2(x_2, \varepsilon))} a(x) \, dx \right| \leq \frac{C \|a\|_{C^1} \ln(2 + |\lambda|)}{|\lambda|^{\frac{1}{2}}},
\]

where $g_1(x_2, \varepsilon)$ is a sufficiently small perturbation of $b_1(x_2, x_2^4)$, where $b_1$ is a smooth function with $b_1(0) \neq 0$. Note that $\beta_d(g_1(x_2, 0)) = \frac{1}{4}$.

Finally, summing up the estimates obtained, we complete at the proof of Theorem 1.6.
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