Global properties of the Skyrme-force-induced nuclear symmetry energy

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Large scale calculations are performed to establish the global mass dependence of the nuclear symmetry energy, \( a_{\text{sym}}(A) \), which in turn depends on two basic ingredients: the mean-level spacing, \( \varepsilon(A) \), and the effective strength of the isovector mean-potential, \( \kappa(A) \). Surprisingly, our results reveal that in modern parameterizations including SLy4, SkO, SkXc, and SkP these two basic ingredients of \( a_{\text{sym}} \) are almost equal after rescaling them linearly by the isoscalar and the isovector effective masses, respectively. This result points toward a new fundamental property of the nuclear interaction that remains to be resolved. In addition, our analysis determines the ratio of the surface-to-volume contributions to \( a_{\text{sym}} \) to be \( \sim 1.6 \), consistent with hydrodynamical estimates for the static dipole polarizability as well as the neutron-skin.

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The knowledge of the nuclear equation of state (EOS) for neutron-rich systems is of fundamental importance for nuclear physics and nuclear astrophysics. The stability of neutron-rich nuclei, the \( r \)-process nucleosynthesis, the structure of neutron star, and the simulations of supernovae-collapse depend sensitively on the EOS and, in particular, on the nuclear symmetry energy (NSE). Our study gives new insight into the two basic ingredients of the symmetry energy, the mean-level spacing, \( \varepsilon(A) \), and the effective strength of the isovector mean-potential, \( \kappa(A) \), revealing the striking property \( m_{\text{sym}}^2 \varepsilon(A) \approx m_{\text{sym}}^2 \kappa(A) \) that holds for velocity-dependent interactions having isoscalar \( m_{\text{sym}}^2 \) and isovector \( m_{\text{sym}}^2 \) effective masses, respectively. This apparently fundamental property of the effective nuclear force remains to be explained.

In asymmetric infinite nuclear matter (INM), in the vicinity of the saturation density, \( \rho_0 \), the EOS (the energy-density per particle) is conveniently parametrized using the following Taylor expansion:

\[
\frac{E_1(\rho)}{A} \approx -a_V + \frac{K_{\infty}}{18\rho_0^3} (\rho - \rho_0)^2 + \ldots
\]

(1)

\[
a_{\text{sym}} + \frac{p}{\rho_0}(\rho - \rho_0) + \frac{\Delta K_{\infty}}{18\rho_0^3} (\rho - \rho_0)^2 + \ldots I^2 + \ldots
\]

where \( I \equiv |N - Z|/A \). The isoscalar INM saturation density \( \rho_0 \), and the values of the volume binding energy \( a_V \), the incompressibility parameter \( K_{\infty} \), and the symmetry energy \( a_{\text{sym}} \) serve as primary constraints for microscopic nuclear models. For modern Skyrme force parameterizations which are subject of the present work \( \rho_0 \approx 0.16 \text{ fm}^{-3} \), \( a_V \approx -15.9 \pm 0.2 \text{ MeV} \), \( a_{\text{sym}} \approx 32 \pm 2 \text{ MeV} \), and \( K_{\infty} \approx 225 \pm 25 \text{ MeV} \). Higher-order curvature corrections to the INM, \( p, \Delta K_{\infty} \), are rather poorly constrained. All these values are derived from the studies of finite nuclei.

Integrating out the \( r \)-dependence from the energy-density leads then to the semi-empirical mass formula (LD) for the energy per particle which is conventionally written as:

\[
\frac{E}{A} = -a_V + \frac{a_s}{A^{1/3}} + b_V \left[ a_{\text{sym}}^{(V)} - a_{\text{sym}}^{(S)} \right] \left( I^2 + \frac{\lambda I}{A} \right) + \ldots,
\]

(2)

where \( a_s \) and \( a_{\text{sym}}^{(S)} \) are coefficients defining contributions from the surface energy and the surface part of the symmetry energy, respectively. There is at present no consensus concerning the magnitude, \( \lambda \), as well as origin of the term linear in \( I \), which is often called the Wigner energy.

Another controversy exists concerning the surface contribution to the NSE. The values of the surface-to-volume ratio \( r_{SV} = a_{\text{sym}}^{(S)}/a_{\text{sym}}^{(V)} \) quoted in the literature vary strongly. For example, Danielewicz estimates it to be \( 2.0 \leq r_{SV} \leq 2.8 \), the mass formula of Ref. yields \( r_{SV} \approx 1.6 \) while the hydrodynamical-type models that include properly polarization of the isovector density predict \( r_{SV} \approx 2 \) which, according to Ref. 4, is consistent within \( 10 \% \) of the Skyrme-force (SF) calculations.

The main objective of this work is to study the symmetry energy within the Skyrme-Hartree-Fock (SHF) model. Notwithstanding, that a deeper understanding of the symmetry energy is crucial in order to reach a consensus on the existing variety of SF parameterizations, or to constrain the coupling constants of a more general local energy density functional (LEDF). Our results point toward a deeper relation between the average level spacing and the strength of the mean isovector potential which has not been addressed hitherto. The calculations presented below also allow us to determine the surface-to-volume ratio of the SHF symmetry energy.

In our previous letter we have demonstrated that the SHF symmetry energy behaves rather unexpectedly according to the formula:

\[
E_{\text{sym}}^{(\text{SHF})} = \frac{1}{2}(A, T_z)T^2 + \frac{1}{2}K(A, T_z)T(T + 1),
\]

(3)
where $\varepsilon(A, T_z) \approx \varepsilon(A)$ and $\kappa(A, T_z) \approx \kappa(A)$ are fairly independent on $T_z$, at least for $T_z \geq 8$, and denote the mean-level spacing at the Fermi energy in iso-symmetric nuclei and effective strength of the isovector mean-potential emerging within the SHF, respectively. More precisely, $\kappa$ is related to the isovector part of the SF induced LEDF (s-LEDF) $\mathcal{H}(r) = \sum_{i=0,1} \mathcal{H}_i(r)$:

$$\mathcal{H}_i = C^s_i \rho^2 + C^t_i \rho \Delta \rho_s + C^\ell_i \rho \tau_\ell + C^s_i \mathbf{J}^2 + C^\tau_i \mathbf{J} \cdot \mathbf{J}.$$  \hspace{1cm} (4)

Definitions of all local densities and currents $\rho, \tau, \mathbf{J}$ as well as the explicit expressions for coupling constants $C_i$ can be found in numerous references and we follow the notation used in Ref. [5]. Due to the isoscalar-density dependence of the SF, the coupling constants $C_i[\rho_0]$ of the s-LEDF are functionals of $\rho_0$, giving rise to the isoscalar rearrangement mean-potential $U_0 = \sum_{i=0,1} \frac{\partial \mathcal{H}_i}{\partial \rho^2}$. Since our procedure of extracting $\varepsilon$ and $\kappa$ involves setting the $C_i \equiv 0$, see Ref. [8], part of the $U_0$ related to the $C_i$ is formally treated as being related to the isovector degrees of freedom. Note that this separation is consistent with the way the symmetry energy constraint is superimposed on the SF. It does not affect the generality of our approach and similar analysis can be performed also within the much wider class of the Hohenberg-Kohn-Sham LEDF theories.

In the present work we focus on the global mass dependence of the SHF values of $\varepsilon(A)$ and $\kappa(A)$ and perform a systematic calculation covering all even-even nuclei having $20 \leq A \leq 128$ from $N = Z$ to almost the neutron drip line. Coulomb and pairing effects are disregarded i.e. the emphasis is on the strong interaction acting in the particle-hole channel. The calculations are performed for a set of different SF parameterizations as the SkP [7], SkXc [8], Sly4 [9], SkO [10], SkM$^+$ [11], and SIII [12], using the SHF code HFODD of Dobaczewski et al. [4,13].

The procedure used to extract $\varepsilon(A, T_z)$ and $\kappa(A, T_z)$ follows exactly the one outlined in Ref. [8]. First, we set all the isovector coupling constants $C_i \equiv 0$ in the s-LEDF [14] and extract $\varepsilon(A, T_z)$ by comparing calculated excitation energy $\Delta E^{(t=0)}_{SHF}(A, T_z) \equiv E^{(t=0)}_{SHF}(A, T_z) - E_{SHF}^{(t=0)}(A, 0)$ to:

$$\Delta E^{(t=0)}_{SHF}(A, T_z) = \frac{1}{2} \varepsilon(A, T_z) T^2.$$  \hspace{1cm} (5)

In the next step, we compute the total SHF binding energy $E_{SHF}(A, T_z)$ and compare:

$$\Delta E_{SHF}(A, T_z) - \Delta E^{(t=0)}_{SHF}(A, T_z) = \frac{1}{2} \kappa(A, T_z) T(T + 1),$$  \hspace{1cm} (6)

in order to determine $\kappa(A, T_z)$.

For each $A$ and small $T_z$, the values of $\varepsilon(A, T_z)$ oscillate quite rapidly. However, they clearly tend to stabilize for $T_z \geq 8$ where $\varepsilon(A, T_z) \approx \varepsilon(A)$. The values of $\kappa(A, T_z)$ appear to stabilize faster and $\kappa(A, T_z) \approx \kappa(A)$ essentially already for $T_z \geq 4$. It should be mentioned that in the case of the SkO parameterization the formula [8] does work only approximately. For this force we observe an enhancement in the linear, $\sim T$. This effect is, however, much weaker than the analogous effect found recently within relativistic mean field [14] where it restores the $E_{sym} \sim T(1 + 1)_{\Delta}$ dependence of the total NSE.

For further quantitative analysis of the mass dependence of the NSE we use the mean values of $\bar{\varepsilon}(A)$ and $\bar{\kappa}(A)$. These averages over $T_z$ at fixed $A$ are calculated using the following restricted set of nuclei: $T_z \geq 4$ for $A = 20$; $T_z \geq 6$ for $A = 24$; and $T_z \geq 8$ for $A \geq 28$. By using a restricted set of nuclei we smooth out both $\varepsilon(A)$ and $\kappa(A)$ curves in order to diminish the possible influence of shell structure.

The global mass dependence of the two components of the symmetry energy, $\bar{\varepsilon}$ and $\bar{\kappa}$ are presented in Fig. 1. Although representative for the SLy4 parameterization, the figure shows several features which appears to be independent of the type of the SF parametrization. These universal features include: (i) strong dependence of $\bar{\varepsilon}(A)$ on kinematics (shell effects); (ii) almost no dependence of $\bar{\kappa}(A)$ on kinematics; (iii) clear surface ($\sim 1/A$) dependence reducing the dominant volume term ($\sim 1/A$) in both $\bar{\varepsilon}(A)$ and $\bar{\kappa}(A)$.

Indeed, the values of $\bar{\varepsilon}(A)$ show characteristic kinks close to double-(semi)magic $A$-numbers. These kinks are magnified when all the calculated nuclei are used (no smoothing) to compute $\bar{\varepsilon}(A)$, but without affecting qualitatively the overall profile of the curve. On the other hand, $\bar{\kappa}(A)$ is almost perfectly smooth with barely visible traces of shell structure. It confirms our earlier conclu-
that the gross features of the Skyrme isovector mean potential can be almost perfectly quantified by a smooth curve parametrized by a small number of global parameters.

In the analysis of a leptonodermous expansion of \( \varepsilon(A) \) and \( \hat{\kappa}(A) \) we consider volume (\( V \)) and surface (\( S \)) terms:

\[
\varepsilon(A) = \frac{\varepsilon_V}{A} - \frac{\varepsilon_S}{A^{4/3}}, \quad \hat{\kappa}(A) = \frac{\kappa_V}{A} - \frac{\kappa_S}{A^{4/3}}. \tag{7}
\]

The values of the isoscalar-effective-mass-scaled expansion coefficients \( \varepsilon^*_V \), \( \varepsilon^*_S \) as well as the values of the isovector-effective-mass-scaled expansion coefficients \( \kappa^*_V \), \( \kappa^*_S \) are collected in Tab. 1. First of all, let us observe that the calculated value of \( \varepsilon^*_V \approx 100\text{MeV} \) corresponds to the pure Fermi gas estimate \( \varepsilon_{FG} \). This result can be understood based on the analytical expression for the Skyrme force NSE coefficient in the limit of symmetric INM, \( a_{sym}^{(\infty)} \), provided that the standard textbook formula is rewritten in the following way:

\[
a_{sym}^{(\infty)} = \frac{1}{8} \bar{\varepsilon}_{FG} \left( \frac{m}{m_0} \right) + \left[ \frac{3\pi^2}{2} C_1^* \rho^{5/3} + C_0^* \rho \right],
\]

\[
\equiv \frac{1}{8} \left[ \varepsilon(\infty) + \kappa(\infty) \right], \tag{8}
\]

where \( C_1^* \) and \( C_0^* \) define the isovector part of the SLDF, see Eq. 4. Eq. 8 clearly separates the contributions from the isovector and the isoscalar part and relates the latter to the single particle energies in INM, \( \varepsilon_p = \frac{\rho^2}{2m} + \Sigma(p, \varepsilon_p) = \frac{\rho^2}{2m_0} \), with a self-energy term, \( \Sigma(p, \varepsilon_p) \), that describes the interaction with the nuclear medium incorporated into the isoscalar effective mass. Hence, Eq. 8 further supports our interpretation of the NSE strength.

The most striking result of our analysis is the near-equality of \( \varepsilon^* \approx \hat{\kappa}^* \) occurring for all modern parameterizations, see Tab. 1 and Fig. 2. Indeed, \( \varepsilon^* \) differs from \( \kappa^* \) only for old parameterizations like the SIII and SkM*. This result confirms the rather loose claims often appearing in textbooks that the kinetic energy \( \varepsilon_{FG} \) and the isovector mean-potential contribute to the \( a_{sym} \) in a similar way. Indeed correct but only after disregarding non-local effects. To our knowledge, it has never been discussed why this apparently independent quantities should be similar.

The relation \( \varepsilon^* \approx \hat{\kappa}^* \) could be accidental. However, given the fact that these modern Skyrme forces have been fitted in a rather different manner, suggests that the near equality is of fundamental nature. To find an explanation of this result will be a challenge for further studies. Note that the analytical relation 3 does not show any explicit scaling due to the presence of the isovector effective mass, \( m_1^* \). This is probably not very surprising since \( m_1^* \) defines the enhancement in the energy weighted sum rule for the translational symmetry violating (finite nucleus) dipole mode. Moreover, the key isovector coupling constant \( C_1^* \) and \( C_2^* \) vary completely erratically from force to force, although we have found that they are linearly correlated for all the SF studied here. It indicates that they are neither well understood nor well constrained.

The advantage of our interpretation of \( \varepsilon(A) \) becomes evident when evaluating \( r_o \) using the semi-classical approximation 15 in straightforward fashion. The appropriate formula which takes into account the diffuseness of the potential, see Ref. 16:

\[
\varepsilon(A) \sim g(\epsilon_F)^{-1} \sim \frac{1}{A} \left( 1 - \frac{\pi}{4k_F} \frac{S}{V_M} + \ldots \right), \tag{9}
\]

where \( g(\epsilon_F) \) is the level density at the Fermi energy, \( k_F \approx 1.36\text{fm}^{-1} \) while \( V_M \) and \( S \) denote volume and surface matter-distribution, respectively. Assuming spherical geometry \( \frac{S}{V_M} \approx \frac{2}{3} \frac{\pi}{4} \rho_o \) and adopting for \( \rho_o \approx 1.14\text{fm}^{-1} \), one obtains \( r_o \approx \frac{\beta}{4k_F} \frac{S}{V_M} \approx 1.52 \) which is indeed very close to the calculated ratios, see Tab. 1.

The shf models yield \( r_{S/V} \approx 1.6 \) in accordance with the LD ratio 2. The static dipole polarizability (SDP) \( \alpha_D [\sigma(\omega)\text{ denotes photo-absorption cross-section}]:

\[
\sigma_{-2} \equiv \int \frac{\sigma(\omega)}{\omega^2} \, d\omega \equiv 2\pi^2 \varepsilon^2 \frac{\alpha_D}{\hbar c}, \tag{10}
\]

provides an independent cross-check of the ratio \( r_{S/V} \). Indeed, using the so called hydrodynamical model simple estimate for \( \alpha_D \) can be derived 3:

\[
\alpha_D \approx \alpha_D^{(M)} \left( 1 + \frac{5}{3} \frac{r_{S/V}}{A^{1/3}} + \ldots \right), \tag{11}
\]

where \( \alpha_D^{(M)} = \frac{2}{3} \frac{\beta}{4k_F} \frac{S}{V_M} \) is the so called Migdal SDP value 17 which is valid for large systems with negli-
expansion coefficients and the

Interestingly, Eq. (12) does not depend on the

\[ r_{\text{exp}} = \frac{2 \kappa_{\text{sym}}}{\kappa_0} \]

and the isovector effective mass scaled \( \bar{\kappa}(A) \); and the isovector effective mass scaled \( \kappa^{(s)}(A) \equiv \frac{\hbar^2}{m} \bar{a}(A) \), respectively; the ratios of volume \( r_{V}^* = \frac{\bar{\kappa}(A)}{\kappa_{\text{sym}}} \) and surface \( r_{S}^* = \frac{\bar{\kappa}(A)}{\kappa_{\text{sym}}} \) expansion coefficients and the INM estimate \( r_{V}^{(\infty)} = \frac{\bar{\kappa}(A)}{\kappa_{\text{sym}}} \); the INM estimate \( a_{\text{sym}}^{(\infty)} \) of the symmetry energy coefficient as defined in (4). The values of \( \bar{\kappa}(A), \kappa_{\text{sym}}^{(s)}, a_{\text{sym}}^{(\infty)}, a_{\text{sym}}^{(s)}, \) and \( a_{\text{sym}}^{(\infty)} \) are given in MeV.

The neutron skin thickness is another quantity which sensitively depends on isovector properties. Neutron rms radii are even today rather poorly known and can therefore not be used to constrain the LEDF or the EOS, see however [13]. For heavy nuclei one can continue to apply the hydrodynamical model of Ref. [8] and evaluate the neutron skin thickness:

\[
\frac{\delta r^2}{\langle r^2 \rangle} \approx \frac{N-Z}{A} \left\{ 1 + \frac{2}{3} \frac{r_{S/V}}{A^{1/3}} - \cdots \right\} \tag{12}
\]

Interestingly, Eq. (12) does not depend on the bulk NSE coefficient but only on the ratio \( r_{S/V} \). Formula (12) taken at \( r_{S/V} \approx 1.65 \) gives again a very consistent results with our microscopic SHF calculations.

In summary, the global mass dependence of the NSE strength \( a_{\text{sym}}(A) \) and its two basic ingredients related to the mean-level spacing, \( \bar{\kappa}(A) \), and to the mean-isovector potential, \( \kappa(A) \) is studied in detail within the SHF theory. Our interpretation of the symmetry energy enables us to unambiguously establish the surface-to-volume ratio of \( a_{\text{sym}}(A), r_{S/V} \approx 1.6 \) in agreement with the LD value of Ref. [2]. This ratio is consistent with simple hydrodynamical estimates for the SDP and neutron skin thickness. The most striking results of our calculations is the near-equality of \( \bar{\kappa} \approx \kappa \) revealing that contribution to \( a_{\text{sym}} \) due to the mean-level spacing and due to the mean-isovector potential are similar but only after disregarding non-local effects. Whether this is a fundamental property of the nuclear mean field is an open question that requires further studies.

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| \( m_0^*/m \) | \( m_1^*/m \) | \( \varepsilon_V^* \) | \( \bar{\kappa}^* \) | \( r_{\text{exp}} \) | \( r_{\text{sym}} \) | \( a_{\text{sym}}^{(\infty)} \) | \( a_{\text{sym}}^{(s)} \) | \( r_{S/V} \) |
|-------------|-------------|----------------|----------------|----------------|----------------|----------------|----------------|----------|
| SLv4        | 0.990       | 0.800          | 94.5           | 147.5          | 1.56           | 94.7           | 137.5          | 1.45      | 1.000      | 1.070      | 1.070      | 32.0      | 31.8      | 48.0      | 1.51     |
| SXe         | 1.000       | 0.792          | 108.6          | 104.3          | 1.34           | 107.6          | 105.2          | 1.54      | 1.010      | 0.999      | 0.888      | 30.1      | 31.4      | 47.9      | 1.53     |
| SHF         | 1.000       | 0.741          | 108.8          | 115.1          | 1.61           | 106.9          | 106.1          | 1.54      | 1.030      | 1.070      | 0.950      | 30.0      | 31.5      | 49.4      | 1.57     |
| SKO         | 0.896       | 0.852          | 107.2          | 106.2          | 1.35           | 106.0          | 116.1          | 1.59      | 0.970      | 0.940      | 0.790      | 32.0      | 31.2      | 49.0      | 1.57     |
| SM*         | 0.789       | 0.653          | 106.3          | 180.1          | 1.70           | 114.7          | 107.3          | 1.50      | 1.490      | 1.490      | 1.370      | 30.0      | 30.5      | 49.2      | 1.61     |
| SHI         | 0.763       | 0.655          | 97.5           | 143.3          | 1.47           | 75.2           | 103.2          | 1.37      | 1.300      | 1.390      | 1.340      | 28.2      | 30.3      | 43.3      | 1.43     |

**TABLE I:** The table includes: the isoscalar, \( m_0^*/m \), and the isovector, \( m_1^*/m \), effective masses; the volume, \( \varepsilon_V^* (\kappa_V^*) \), the surface \( \varepsilon_S^* (\kappa_S^*) \), and the ratios \( r_{\text{exp}} = \varepsilon_S^*/\varepsilon_V^* (r_{\text{sym}} = \varepsilon_S^*/\varepsilon_V^*) \) of the expansion coefficients (4) of the isoscalar effective mass scaled \( \bar{\kappa}(A) \); and the isovector effective mass scaled \( \kappa^{(s)}(A) \equiv \frac{\hbar^2}{m} \bar{a}(A) \), respectively; the ratios of volume \( r_{V}^* = \varepsilon_V^*/\kappa_V^* \) and surface \( r_{S}^* = \varepsilon_S^*/\kappa_S^* \) expansion coefficients and the INM estimate \( r_{V}^{(\infty)} = \varepsilon_V^*/\kappa_V^* \); the INM estimate \( a_{\text{sym}}^{(\infty)} \) of the symmetry energy coefficient as defined in (4). The values of \( \varepsilon_V^*(A), \kappa_V^*(A), a_{\text{sym}}^{(\infty)}, a_{\text{sym}}^{(s)}, \) and \( a_{\text{sym}}^{(\infty)} \) are given in MeV.