METRICS ON SPACES OF SURFACES WHERE HORIZONTALITY EQUALS NORMALITY

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Abstract. In this article, we study metrics on shape space of surfaces that have a particularly simple horizontal bundle. More specifically, we consider reparametrization invariant Sobolev type metrics $G$ on the space $\text{Imm}(M, N)$ of immersions of a compact manifold $M$ in a Riemannian manifold $(N, g)$. The tangent space $T_f \text{Imm}(M, N)$ at each immersion $f$ has two natural splittings: one into components that are tangential/normal to the surface $f$ (with respect to $g$) and another one into vertical/horizontal components (with respect to the projection onto the shape space $B_i(M, N) = \text{Imm}(M, N)/\text{Diff}(M)$ of unparametrized immersions and with respect to the metric $G$). The first splitting can be easily calculated numerically, while the second splitting is important because it mirrors the geometry of shape space and geodesics thereon. Motivated by facilitating the numerical calculation of geodesics on shape space, we characterise all metrics $G$ such that the two splittings coincide. In the special case of planar curves, we show that the regularity of curves in the metric completion can be controlled by choosing a strong enough metric within this class.

1. Introduction

Nowadays, the study of the geometry of the space of all surfaces of a certain type – including the space of plane and space curves – is an active area of research. A strong motivation for this research lies in the applicability to tasks in computational anatomy, shape comparison and image analysis, see e.g. [12, 25, 21, 14, 16, 27, 23, 11, 4]. In this article, we will model surfaces as smooth immersions from a compact $m$-dimensional manifold $M$ into a $n$-dimensional Riemannian manifold $(N, g)$ of bounded geometry, yielding the smooth Fréchet manifold $\text{Imm}(M, N)$, see [15, 13]. We will always assume that $n \geq m$; otherwise $\text{Imm}(M, N)$ is empty. Note that this definition covers in particular the important special case of planar curves.

The notion of shape space we will consider in this article is the space of unparametrized surfaces. This space can be identified with the quotient space

\begin{equation}
B_i(M, N) := \text{Imm}(M, N)/\text{Diff}(M).
\end{equation}

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Here, $\text{Diff}(M)$ denotes the Lie group of all smooth diffeomorphisms on $M$, which acts smoothly on $\text{Imm}(M, N)$ via composition from the right:

$$\text{Imm}(M, N) \times \text{Diff}(M) \to \text{Imm}(M, N), \quad (f, \varphi) \mapsto f \circ \varphi.$$  

The quotient space $B_i(M, N)$ is not a manifold, but only an orbifold with isolated singularities (see [10] for more information).

Given a reparametrization invariant metric $G$ on $\text{Imm}(M, N)$, we can induce a unique Riemannian metric on the quotient space $B_i(M, N)$ such that the projection

$$\pi : \text{Imm}(M, N) \to B_i(M, N) := \text{Imm}(M, N)/\text{Diff}(M)$$

is a Riemannian submersion. A detailed description of this construction is provided in [5, Section 4]. For many metrics, $T \pi$ induces a splitting of the tangent bundle $T\text{Imm}(M, N)$ into a vertical bundle, which is defined as the kernel of $T \pi$, and a horizontal bundle, defined as the $G$-orthogonal complement of the vertical bundle:

$$T\text{Imm}(M, N) = \ker T \pi \oplus (\ker T \pi)^\perp_G =: \text{Ver} \oplus \text{Hor}.$$  

If this is the case, then geodesics on shape space $B_i(M, N)$ correspond exactly to horizontal geodesics on $\text{Imm}(M, N)$, i.e., geodesics whose velocity vector lies in the horizontal bundle. This constitutes an effective way to compute geodesics on shape space provided that the horizontal bundle is not too complicated. A particularly favourable case is when the splitting in equation (4) coincides with the natural splitting into components that are tangential and normal to the immersed surface with respect to the metric $\overline{g}$:

$$T\text{Imm}(M, N) = \text{Tan} \oplus \text{Nor}.$$  

Since $\text{Tan} = \text{Ver}$, this is the case if and only if the notions of horizontality and normality agree. Letting $\pi_N : TN \to N$ denote the projection of a tangent vector onto its foot point, the above bundles are given by

$$T_f\text{Imm}(M, N) = \{ h \in C^\infty(M, TN) : \pi_N \circ h = f \},$$

$$\text{Tan}_f = \{ T f \circ X \mid X \in \mathfrak{X}(M) \},$$

$$\text{Nor}_f = \{ h \in T_f\text{Imm}(M, N) \mid \forall k \in \text{Tan}_f : \overline{g}(h, k) = 0 \}.$$  

One specific example of a metric for which the splittings in equations (4) and (5) coincide is the $L^2$ metric, which is also the simplest and in a way most natural metric on $\text{Imm}(M, N)$. It is defined as

$$G^L_2(f)(h, k) := \int_M \overline{g}(h, k)\text{vol}(g).$$  

Here, $h, k$ are tangent to $f \in \text{Imm}(M, N)$, $g = f^*\overline{g}$ denotes the pullback of the metric $\overline{g}$ along the immersion $f$ and $\text{vol}(g)$ denotes the associated volume form on $M$.

Unfortunately, the $L^2$-metric is unsuited for many applications because the induced geodesic distance on $B_i(M, N)$ and $\text{Imm}(M, N)$ vanishes (see [18, 8]). Namely, any two immersions $f_0, f_1$ can be connected by paths
of arbitrarily short $G^{L^2}$-length. The discovery of the degeneracy of the $L^2$-metric started a quest for stronger and more meaningful metrics. One particular class of metrics that has been introduced are almost local metrics \cite{20, 6, 22, 28}, which are defined as

\begin{equation}
G^\Phi_f(h, k) := \int_M \Phi(f) \overline{g}(h, k) \text{vol}(g),
\end{equation}

where $\Phi: \text{Imm}(M, N) \to \mathbb{R}_{>0}$ is a smooth, reparametrization invariant function. Almost local metrics enjoy the same simple splitting into horizontal and vertical subbundles as the $L^2$-metric, but overcome its degeneracy to some extent. On the positive side, they induce non-vanishing geodesic distance on $B_i(M, N)$. However, on the negative side, well-posedness of the geodesic equation, existence of conjugated points and boundedness of the curvature tensor remain unknown for these metrics.

Another approach to strengthen the $L^2$-metric are Sobolev metrics, which have been studied in a variety of different articles, including \cite{29, 20, 5, 26}. They are defined as

\begin{equation}
G^{L_f}_f(h, k) = \int_M \overline{g}(L_f h, k) \text{vol}(g),
\end{equation}

where for each $f \in \text{Imm}(M, N)$, the operator $L_f$ is an elliptic pseudo-differential operator of order $2l$, which is symmetric and positive with respect to the $L^2$-metric. Almost local metrics are contained in this definition. The standard Sobolev metric of order $n$ is

\begin{equation}
G^{H^l}_f(h, k) = \int_M \overline{g} \left( \sum_{i=0}^{l} \Delta^i h, k \right) \text{vol}(g),
\end{equation}

where the Laplacian depends on the immersion $f$ via the pullback metric $g$.

Under certain assumptions on $L$, well-posedness of the corresponding geodesic equation has been shown (see \cite{20, 5, 8}). From a computational point of view \cite{2}, metrics involving a pseudo-differential operator have the severe disadvantage that the splitting into horizontal and vertical parts is rather complicated: a tangent vector $h$ is horizontal if and only if $L_f h$ is $\overline{g}$-normal to the surface $f$. Therefore, calculating the horizontal component of $h$ involves inverting the operator $L_f$ (or, to be precise, the restriction and projection of the operator $L_f$ to the space $\text{Tan}$, cf. Lemma \[\text{1}\]).

If, on the other hand, the splittings \[\text{(4)}\] and \[\text{(5)}\] coincide, then projecting $h$ onto its horizontal part is easy: one simply takes the normal component of $h$. An immediate application is that one can solve the boundary value problem for geodesics on shape space efficiently by minimising the horizontal energy of paths in the space of immersions. This approach has been used in \cite{19, 20, 6, 7} to calculate geodesics on the spaces $B_i(S^1, \mathbb{R}^2)$ and $B_i(S^2, \mathbb{R}^3)$ under various almost local metrics. However, this approach must be expected to work well for any metric where it is easy to calculate horizontal projections,
e.g., for the class of metrics presented here. We plan to exploit this in similar numerical calculations in future versions of this article.

Our main result is that we classify all Sobolev metrics such that the splitting into horizontal and vertical components coincides with the natural splitting into tangential and normal components. This classification is established for immersions of general manifolds. In the last section, the special case of planar curves \((M = S^1 \text{ and } N = \mathbb{R}^2)\) is investigated. There, it is shown that the class of metrics where horizontality equals normality is rich enough to include metrics which dominate any given standard Sobolev metric. Using a result of [9], it is shown that this allows to control the regularity of curves in the metric completion of shape space. This is important for applications (e.g., stochastics on shape space) where going to the metric completion is indispensable.

2. The decomposition theorem

2.1. Assumptions. Following [5], we now rigorously define the class of metrics that we study in this article. We assume that \(L\) is a smooth section of the bundle

\[
L(T\operatorname{Imm}(M, N); T\operatorname{Imm}(M, N)) \to \operatorname{Imm}(M, N)
\]

such that at every \(f \in \operatorname{Imm}(M, N)\) the operator

\[
L_f : T_f \operatorname{Imm}(M, N) \to T_f \operatorname{Imm}(M, N)
\]

is a pseudo differential operator of order \(2l\), which is symmetric and positive with respect to the \(L^2\)-metric on \(\operatorname{Imm}(M, N)\). Moreover, we assume that \(L\) is invariant under the action of the reparametrization group \(\operatorname{Diff}(M)\) acting on \(\operatorname{Imm}(M, N)\), i.e.

\[
L_{f \circ \phi}(h \circ \phi) = L_f(h) \circ \phi \quad \text{for all } \varphi \in \operatorname{Diff}(M).
\]

These assumptions will remain in place throughout this work. Their immediate use is as follows: being symmetric and positive, \(L\) induces a Sobolev type metric on the manifold of immersions through equation [9]. The \(\operatorname{Diff}(M)\)-invariance of \(L\) implies the \(\operatorname{Diff}(M)\)-invariance of the metric \(G^L\). Therefore, there is a unique metric on \(B_1(M, N)\), such that the projection [3] is a Riemannian submersion, see [5] Thm. 4.7. The resulting geometry of shape space is mirrored by the “horizontal geometry” on the manifold of immersions. For example, there is a one to one correspondence between geodesics on \(B_1(M, N)\) and horizontal geodesics on \(\operatorname{Imm}(M, N)\) (see [5] Section 4). This is advantageous because in many cases, the space of immersions lends itself more easily to theoretical and numerical calculations than shape space itself.
2.2. **Splitting into horizontal and vertical subbundles.** It follows from the definitions that the horizontal and vertical bundles are given by

\[ \begin{align*}
\text{Ver}_f := & \ker(T\pi) = \Tan_f, \\
\text{Hor}_f := & (\text{Ver}_f) \perp^G = \{ h \in T_f\text{Imm}(M, N) : L_fh \in \text{Nor}_f \}.
\end{align*} \tag{14} \]

By the positivity of the metric \( G \), the sum \( \text{Ver}_f \oplus \text{Hor}_f \) is always direct, but it might not span the entire tangent space \( T_f\text{Imm}(M, N) \). In other words, in our infinite dimensional setting, a complement to the vertical bundle might only exist in some completion of the space. In this section, we present sufficient conditions for the existence of a splitting of the tangent space into horizontal and vertical subbundles.

In contrast to the splitting into horizontal and vertical parts, the splitting into tangential and normal parts given in equation (5) always exists. Let \( h^\text{Tan}, h^\text{Nor} \) be the components of a tangent vector \( h \) corresponding to this splitting and let

\[ \begin{align*}
L_f^\text{Tan} : & \Tan_f \to \Tan_f, \ h \mapsto (L_fh)^\text{Tan}, \\
L_f^\text{Nor} : & \text{Nor}_f \to \text{Nor}_f, \ h \mapsto (L_fh)^\text{Nor}.
\end{align*} \tag{15} \]

be projections and restrictions of \( L \) to the corresponding subbundles. These operators allow us to express a sufficient condition for the existence of a horizontal/vertical splitting.

**Lemma 1.** If the operator \( L_f^\text{Tan} \) is invertible, then a splitting (14) into horizontal and vertical parts exists and the corresponding projections are

\[ h^\text{Ver} = (L_f^\text{Tan})^{-1}((L_fh)^\text{Tan}), \quad h^\text{Hor} = h - h^\text{Ver}. \tag{16} \]

**Proof.** It is straight-forward to check that \( h^\text{Ver} \in \text{Ver}_f, h^\text{Nor} \in \text{Nor}_f \) and that these vectors add up to \( h \). \( \square \)

The condition of Lemma 1 is satisfied in many cases, as the following Lemma shows.

**Lemma 2** (Sect. 6.8 in [5]). If the operator \( L_f \) is elliptic, then all of the operators \( L_f, L_f^\text{Tan}, L_f^\text{Nor} \) are elliptic and invertible.

**Proof.** We show the statement for the operator \( L_f^\text{Tan} \). Let \( \sigma^{L_f} \) and \( \sigma^{L_f^\text{Tan}} \) be the principal symbols of \( L_f \) and \( L_f^\text{Tan} \). Then, for any \( x \in M \) and \( \xi \in T^*M \), one has

\[ \begin{align*}
\sigma^{L_f}(\xi) : & T_f(x)N \to T_f(x)N, \quad \sigma^{L_f^\text{Tan}}(\xi) : T_f.T_xM \to T_f.T_xM.
\end{align*} \tag{17} \]

For any \( \xi \neq 0 \), the mapping \( \sigma^{L_f}(\xi) \) is symmetric and positive definite with respect to \( \overline{g} \) because \( L_f \) is elliptic. The relation

\[ \forall h \in T_f.T_xM : \sigma^{L_f^\text{Tan}}(\xi)h = (\sigma^{L_f}(\xi)h)^\text{Tan}. \tag{18} \]

shows that $\sigma^{L_f}_{\text{Tan}}(\xi)$ is also symmetric and positive definite. Therefore, $L_f^{\text{Tan}}$ is elliptic. Moreover, it inherits symmetry and positivity with respect to the scalar product $G_f^L$ from $L_f$:

$$\forall h, k \in \Tan_f : \int_M \varpi(L_f^{\text{Tan}}h, k)\text{vol}(g) = \int_M \varpi(h, L_f^{\text{Tan}}k)\text{vol}(g),$$  

$$\forall h \in \Tan_f : h \neq 0 \Rightarrow \int_M \varpi(L_f^{\text{Tan}}h, h)\text{vol}(g) > 0. \tag{19}$$

For any $j \geq 0$, let $H^j$ be the $j$-th order Sobolev completion of $\Tan_f$. Since $L_f^{\text{Tan}}$ is elliptic and symmetric, it is self-adjoint on $H^j$ and its index as an operator $H^{j+2p} \to H^j$ vanishes. It is injective (since positive) with vanishing index (since self-adjoint elliptic, by [24, theorem 26.2]), hence it is bijective and thus invertible by the open mapping theorem. The inverse $(L_f^{\text{Tan}})^{-1}$ restricts to a continuous linear mapping on the Fréchet space $\Tan_f$. Ellipticity, symmetry, positivity and invertibility of $L_f^{\text{Nor}}$ and $L_f$ are shown in a similar way. □

2.3. On when horizontality equals normality. The vertical and tangential bundles are always the same by definition (c.f. Lemma [1]). Therefore, horizontality equals normality if and only if the horizontal/vertical splitting coincides with the normal/tangential splitting, which is the property we are interested in.

**Theorem 3** (Decomposition theorem). The splittings (4) and (5) coincide under the metric $G_f^L$ if and only if $L$ has a decomposition $L = L_f^{\text{Nor}} \oplus L_f^{\text{Tan}}$ into operators $L_f^{\text{Nor}} : \text{Nor} \to \text{Nor}$ and $L_f^{\text{Tan}} : \text{Tan} \to \text{Tan}$.

**Proof.** Assume that the horizontal and normal bundles coincide. Then, by definition, $L$ maps normal into normal vectors. To see that it also maps tangential into tangential vectors, take a tangential vector $h$ and test $Lh$ against arbitrary normal vectors $k$:

$$\int_M \varpi(Lh, k)\text{vol}(g) = \int_M \varpi(h, Lk)\text{vol}(g) = 0 \tag{20}$$

for all normal vectors $k$, because $L$ is symmetric and $Lk$ is normal. This shows that $Lh$ is tangential. To summarise, we have shown that the normal and tangential bundle are invariant subspaces of $L$. Therefore, (b) holds.

Conversely, assume that $L$ has a decomposition as in the statement of the theorem. Then, by definition, every normal vector is horizontal. To see that every horizontal vector is normal, take a horizontal vector $h$ and split it into its normal and tangential components $h = h^{\text{Nor}} + h^{\text{Tan}}$. By the horizontality of $h$, $Lh$ is normal, which means that $L_f^{\text{Tan}}h^{\text{Tan}} = 0$. Then also

$$G_f^L(h^{\text{Tan}}, h^{\text{Tan}}) = \int_M \varpi(L_f^{\text{Tan}}h^{\text{Tan}}, h^{\text{Tan}})\text{vol}(g) = 0, \tag{21}$$

which implies $h^{\text{Tan}} = 0$ by the non-degeneracy of the metric. Therefore, $h$ is normal. □
2.4. The geodesic equation. If $L = L^{\text{Nor}} \oplus L^{\text{Tan}}$ decomposes as in Theorem 3, then the metric induced by $G^L$ on shape space depends on $L^{\text{Nor}}$, but not on $L^{\text{Tan}}$. Consequently, the geodesic equation on shape space and the horizontal geodesic equation on $\text{Imm}(\text{M}, N)$ should also have this property. This can be verified directly. To this aim, we recall the formula for the horizontal geodesic equation of a $G^L$ metric, see [5, Theorem 6.10]:

$$
\begin{cases}
 f_t \in \text{Hor}, \\
 (\nabla_{\delta_t} f_t)^\text{Hor} = L^{-1} \left( \frac{1}{2} \text{Adj}(\nabla L)(f_t, f_t)^\text{Nor} - \frac{1}{2} \bar{g}(Lf_t, f_t) \text{Tr}^g(S) \\
 - (\nabla_{f_t} L)f_t)^\text{Nor} + \text{Tr}^g(\bar{g}(\nabla f_t, Tf)f)Lf_t \right).
\end{cases}
$$

For this equation to make sense, we need to assume that $L_f$ is invertible and that $\nabla L$ has an adjoint $\text{Adj}(\nabla L)$ defined by

$$
\int \bar{g}(\text{Adj}(\nabla L)(h, k), m)\text{vol} = \int \bar{g}(\nabla_m L)h, k)\text{vol}.
$$

The existence of the adjoint is not guaranteed, but has to be proven for each specific operator $L$, which is usually done by a series of partial integrations. For operators of the form $L = 1 + \Delta^p$, this was done in [5, Section 8].

**Lemma 4.** Let $L = L^{\text{Nor}} \oplus L^{\text{Tan}}$ as in Theorem 3. Then the horizontal geodesic equation (22) does not depend on $L^{\text{Tan}}$.

**Proof.** By linearity, it is sufficient to show that the expression

$$
\frac{1}{2} \text{Adj}(\nabla L)(f_t, f_t)^\text{Nor} - \frac{1}{2} \bar{g}(Lf_t, f_t) \text{Tr}^g(S)
$$

on the right hand side of the geodesic equation vanishes, if $L$ is of the form $L = 0 \oplus L^{\text{Tan}}$, which we assume now. Let $m, h, k$ be vector fields on $\text{Imm}(\text{M}, N)$ taking values in the normal bundle. Then

$$
G^L_f (\text{Adj}(\nabla L)(h, k), m) = G^L_f (\nabla_m L)h, k
$$

$$
= G^L_f \left( \nabla_m (Lh) - L(\nabla_m h), k \right) = G^L_f \left( - \nabla_m h, \frac{Lk}{0} \right) = 0,
$$

which shows that

$$
\text{Adj}(\nabla L)(h, k)^\text{Nor} = 0, \quad (\nabla_m L)h)^\text{Nor} = 0.
$$

Since these expressions are tensorial in $m, h, k$ and $f_t$ is normal, one obtains

$$
\text{Adj}(\nabla L)(f_t, f_t)^\text{Nor} = 0, \quad (\nabla_{f_t} L)f_t)^\text{Nor} = 0.
$$

Since also $Lf_t = 0$, all terms in 24 vanish. \qed
Remark 5. The above Lemma shows that the horizontal geodesic equation (22) is well-defined under the following conditions on \( L_{\text{Nor}} \): the operator \( L_{\text{Nor}} \) is invertible and \( \nabla L_{\text{Nor}} \) restricted to and projected onto the normal bundle has a normal bundle valued adjoint. Note that it is not necessary to impose any conditions on \( L_{\text{Tan}} \), here.

3. The manifold of planar curves

In the previous section, we characterised all metrics such that the horizontal and normal bundles coincide. We argued in the introduction that this property facilitates the numerical calculation of geodesics on shape space. However, there are many additional properties that a metric should satisfy to be useful in applications: e.g., the induced geodesic distance should not vanish, one would like to be able to control the regularity of immersions in the metric completion, and the space should be geodesically complete. We will show in this section that it is indeed possible to meet at least the first two of these requirements simultaneously in the important special case of planar curves. We believe that similar results can be obtained for surfaces instead of curves. However, as of now, metric and geodesic completeness of spaces of surfaces remain unknown, even for standard Sobolev metrics.

In the case of planar curves, the decomposition of tangent vectors \( h \in T_{c}\text{Imm}(S^1, \mathbb{R}^2) \) into tangential and normal components takes the particularly simple form

\[
 h = \langle h, n \rangle n + \langle h, v \rangle v,
\]

where \( v = \partial_{\theta c}/|\partial_{\theta c}|, \ n = iv \).

Here, \( \langle \cdot, \cdot \rangle \) and \( |\cdot| \) denote the Euclidean scalar product and norm on \( \mathbb{R}^2 \) and \( i : \mathbb{R}^2 \to \mathbb{R}^2 \) denotes the counter-clockwise rotation by the angle \( \pi/2 \). This allows us to express a very simple sufficient condition for the equality of the horizontal and normal bundle.

**Lemma 6.** Any metric of the form

\[
 G_c(an + bv, an + bv) = \int_{S^1} \sum_{i=0}^l \left( (A_i)_c(\partial_i^s a)^2 + (B_i)_c(\partial_i^s b)^2 \right) ds
\]

with coefficients

\[
 A_i, B_i : \text{Imm}(S^1, \mathbb{R}^2) \to C^\infty(S^1, \mathbb{R})
\]

has the property that the horizontal and normal bundles coincide.

**Proof.** Using partial integrations, the metric in (29) can be rewritten in the form (30) involving an operator \( L \), which is given by

\[
 L_c(an + bv) = \sum_{i=0}^l (-1)^i \partial_i^s ((A_i)_c(\partial_i^s a))n + \sum_{i=0}^l (-1)^i \partial_i^s ((B_i)_c(\partial_i^s b))v.
\]

Clearly, \( \text{Tan}_c \) and \( \text{Nor}_c \) are invariant subspaces of \( L_c \). The result follows from Theorem 3. \( \square \)
Using Lemma 6, we can easily construct a metric which is stronger than the Sobolev metric of order one and where horizontality equals normality.

**Lemma 7** (Metrics of order one). The metric

\[
G_c(an + bv, an + bv) = \int_{S^1} \left((1 + 2\kappa^2)(a^2 + b^2) + 2(\partial_s a)^2 + 2(\partial_s b)^2\right) ds
\]

dominates the standard Sobolev \(H^1\) metric

\[
G_{H^1}^c(h, h) = \int_{S^1} \left(|h|^2 + |\partial_s h|^2\right) ds
\]

and has the property that the horizontal and normal bundles coincide.

**Proof.** The metric in (32) satisfies the condition of Lemma 6, which implies that the horizontal and normal bundles coincide. It remains to show that the metric dominates the \(H^1\)-metric. To this aim, let \(h = an + bv\) with \(a = \langle h, n \rangle\) and \(b = \langle h, v \rangle\). Then

\[
\partial_s h = (\partial_s a + b\kappa)n + (\partial_s b - a\kappa)v.
\]

Therefore,

\[
G_{H^1}^c(h, h) = \int_{S^1} \left(a^2 + b^2 + (\partial_s a + b\kappa)^2 + (\partial_s b - a\kappa)^2\right) ds.
\]

By the arithmetic-geometric inequality, the estimates

\[
(\partial_s a + b\kappa)^2 \leq 2((\partial_s a)^2 + b^2\kappa^2), \quad (\partial_s b - a\kappa)^2 \leq 2((\partial_s b)^2 + a^2\kappa^2)
\]

hold. Regrouping terms, one obtains that

\[
G_{H^1}^c(h, h) \leq \int_{S^1} \left((1 + 2\kappa^2)(a^2 + b^2) + 2(\partial_s a)^2 + 2(\partial_s b)^2\right) ds. \quad \Box
\]

**Corollary 8.** Then the metric \(G\) from Lemma 7 induces non-vanishing geodesic distance on \(B_i(S^1, \mathbb{R}^2)\), i.e., the infimum of the \(G\)-lengths of paths connecting two non-identical planar curves is strictly greater than zero. Furthermore, all curves in the metric completion of the space \((B_i(S^1, \mathbb{R}^2), G)\) are Lipschitz continuous.

**Proof.** These results follow directly from the corresponding results for the \(H^1\) or \(G^A\) metric. See [5, Section 7.6] for the positivity result and [19, Thm. 3.11] or [17, Thms. 26 and 27] for the metric completion. \(\Box\)

**Remark 9.** Unfortunately, the well-posedness result in [5, Section 6.6] cannot be applied to the metric in Lemma 7. To see this, one has to rewrite the metric in the form (9) involving an operator \(L\). In the present case,

\[
Lh = (1 + 2\kappa^2)h - 2\left(\partial_s^2 \langle h, n \rangle\right)n - 2\left(\partial_s \langle h, v \rangle\right)v.
\]

The problem is that the expression on the right hand side contains third derivatives of \(c\) while \(L\) is only of second order in \(h\). Similar problems arise with higher order metrics and metrics on spaces of surfaces.
Theorem 10. For any \( l \geq 0 \), there is a metric which dominates the Sobolev \( H^l \) metric

\[
G^H(h, h) := \int_{S^1} \sum_{i=0}^{l} |\partial^i_s h|^2 ds
\]

and which has the property that the horizontal and normal bundles coincide.

Proof. The statement for \( l = 0 \) is trivial. For \( l = 1 \), it has been shown in Lemma 7. For \( l = 2 \), it remains to bound the highest order term. An application of Jensen’s inequality yields the estimate

\[
|\partial^2_s (an + bv)|^2 \leq 4(\partial^2_s a)^2 + 16(\partial_s a)^2 \kappa^2 + 4a^2(\partial_s \kappa)^2 + 4a^2 \kappa^4 \\
+ 4(\partial^2_s b)^2 + 16(\partial_s b)^2 \kappa^2 + 4b^2(\partial_s \kappa)^2 + 4b^2 \kappa^4.
\]

Letting \( h = an + bv \) and taking in account also the zero and first order terms, c.f. Lemma 7, one obtains

\[
G^H(h, h) \leq \int_{S^1} \left( 4(\partial^2_s a)^2 + (16\kappa^2 + 2)(\partial_s a)^2 + (4(\partial_s \kappa)^2 + 4\kappa^4 + 2\kappa^2 + 1) a^2 \\
+ 4(\partial^2_s b)^2 + (16\kappa^2 + 2)(\partial_s b)^2 + (4(\partial_s \kappa)^2 + 4\kappa^4 + 2\kappa^2 + 1) b^2 \right) ds.
\]

The expression on the right-hand side defines a metric, which is of the form required in Lemma 6. Therefore, the horizontal bundle equals the normal bundle for this metric. By construction, it dominates the \( H^2 \) metric. For higher order metrics, the proof is similar. \( \square \)

Since we have shown that it is possible to construct arbitrarily strong metrics for which horizontality equals normality, we are able to control the regularity of curves in the metric completion.

Corollary 11. Let \( G \) be a metric which is stronger than the \( H^1 \) metric. Then the metric completion of the space \((\text{Imm}(S^1, \mathbb{R}^2), G)\) is contained in the set \( \text{Imm}^l(S^1, \mathbb{R}^2) \) of all \( H^l \) immersions.

Proof. Every Cauchy sequence with respect to the metric \( G \) is also a Cauchy sequence with respect to the weaker metric \( H^1 \). By Theorem 12, the sequence has a limit in the space \( \text{Imm}^l(S^1, \mathbb{R}^2) \). Therefore, the completion of \((\text{Imm}(S^1, \mathbb{R}^2), G)\) can be seen as a subset of \( \text{Imm}^l(S^1, \mathbb{R}^2) \). \( \square \)

See also [17] for the metric completion of the length-weighted Sobolev \( H^2 \)-metric.

4. Conclusion

We characterised all metrics on spaces of immersions such that the horizontal/vertical splitting coincides with the normal/tangential one (Theorem 3). The first splitting can be easily calculated numerically, while the second
splitting is important both theoretically and in applications because it mirrors the geometry of shape space and geodesics thereon. In future versions of this article, we plan to exploit this to efficiently solve the boundary value problem for geodesics on shape space by minimising the horizontal energy functional on the space of immersions under a wide range of metrics.

To show that the condition that horizontality equals normality is not too stringent, we studied the special case of planar curves in more detail. We proved that in this case, for any Sobolev metric of given order, there is a stronger metric with the property that horizontality equals normality. By a result of [9], this implies that the regularity of curves in the metric completion can be controlled. Therefore, our class of metrics is rich enough for applications (e.g., stochastics on shape space) where working with the metric completion is indispensable.

**Appendix A. Completeness results for Sobolev metrics**

On the manifold of planar curves, the standard reparametrization invariant Sobolev metric is given by

\begin{equation}
G_{c}^{H_l}(h,h) = \int_{S^1} \left( \sum_{i=0}^{l} (-1)^i \partial_s^{2i} h, h \right) ds = \int_{S^1} \sum_{i=0}^{l} |\partial_s^i h|^2 ds.
\end{equation}

Recently, it has been shown in [9] that the $H^l$ metric (42) is geodesically complete for $l \geq 2$. Since the Hopf-Rinow theorem does not hold in our infinite dimensional setting [1], this does not automatically imply metric completeness. Instead, we have the following result:

**Theorem 12.** For $l \geq 2$, the metric completion of the space of smooth immersed curves endowed with the $H^l$ metric (42) is the space of all $H^l$ immersions

\begin{equation}
\text{Imm}^l(S^1, \mathbb{R}^2) := \{ c \in H^l(S^1, \mathbb{R}^2) : |c'| \neq 0 \}.
\end{equation}

In this case, the topology on $\text{Imm}^l(S^1, \mathbb{R}^2)$ induced by the geodesic distance of the metric $G_{c}^{H_l}$ is equal to the standard topology on $H^l(S^1, \mathbb{R}^2)$.

**Proof.** The proof of this theorem is based on a comparison with the flat (non-reparametrization invariant) metric given by

\begin{equation}
G_{c}^{H_l}(d\theta)(h,h) = \int_{S^1} \sum_{i=0}^{l} |\partial_\theta^i h|^2 d\theta.
\end{equation}

Note that in contrast to the reparametrization invariant metric, this metric does not depend on the foot point $c$. To highlight the difference between these metrics, we use the notations $H^l(ds)$ and $H^l(d\theta)$.

According to [9, Lemma 5.1 and Lemma 4.10], the following holds for any $r > 0$ and $c_0 \in \text{Imm}(S^1, \mathbb{R}^2)$.
(a) On any $H^l(ds)$-metric ball $B^d_ic_0$, the $H^l(ds)$-metric is equivalent to the flat $H^l(d\theta)$-metric, with a constant that depends only on the center $c_0$ and the radius $r$.

(b) Moreover, there exists a $C = C(r, c_0) > 0$ such that $|c'| \geq C$ for all $c \in B^d_r(c_0)$.

Let $c_k$ be a Cauchy-sequence in $\operatorname{Imm}(S^1, \mathbb{R}^2)$ with respect to the $H^l(ds)$-geodesic distance. Then, for each $r > 0$ there exists $N > 0$ such that for each $m \geq N$ the curve $c_m$ lies in the $H^l(ds)$-metric ball $B_r(c_N)$. By (a), the sequence $c_k$ is also a Cauchy sequence with respect to the flat metric $H^l(d\theta)$. Therefore, the sequence has a limit $c$ in the Hilbert space $H^l(S^1, \mathbb{R}^2)$. By (b), we conclude that $c$ is still an immersion. Thus, we have shown that the metric completion of $(\operatorname{Imm}(S^1, \mathbb{R}^2), H^l(ds))$ is contained in $(\operatorname{Imm}(S^1, \mathbb{R}^2), H^l(ds))$. In fact, we have equality because any curve in $\operatorname{Imm}(S^1, \mathbb{R}^2)$ can be approximated in the $H^l(ds)$-metric by a sequence of curves in $\operatorname{Imm}(S^1, \mathbb{R}^2)$.

We now prove that the topology on $\operatorname{Imm}(S^1, \mathbb{R}^2)$ induced by $H^l(ds)$ coincides with the topology induced by $H^l(d\theta)$. First, we show that for each $H^l(d\theta)$-ball $B^d_{r_1}(c_0)$, there exists a $H^l(ds)$-ball $B^d_{r_2}(c_0)$ with $B^d_{r_2}(c_0) \subseteq B^d_{r_1}(c_0)$. According to (a), there exists a constant $C$ such that

$$G^H_c(h, h) \leq CG^H_{c}(h, h),$$

for all $c \in B^d_1(c_0)$ and $h \in T_c\operatorname{Imm}(S^1, \mathbb{R}^2)$. Let $r_2 = \min\{1, r_1/\sqrt{C}\}$ and take any $c_1 \in B^d_{r_2}(c_0)$. By the definition of the geodesic distance, there is a path $\gamma$ connecting $c_1$ to the center $c_0$ with $H^l(ds)$-length smaller than $r_2$. Note that $\gamma$ cannot leave the ball because its length would exceed $r_2$, otherwise. Then

$$\text{length}^{d\theta}(\gamma) \leq \sqrt{C} \text{length}^{ds}(\gamma) < \sqrt{C}r_2 \leq r_1,$$

which implies that $c_1 \in B^d_{r_1}(c_0)$. Thus, we have constructed a $H^l(ds)$-ball which is contained in the given $H^l(d\theta)$-ball.

To see that any $H^l(ds)$-ball contains a $H^l(d\theta)$-ball, we use that one can control the first $l$ derivatives of $c$ on $H^l(d\theta)$ balls. Thus, a similar statement as (a) holds also on $H^l(d\theta)$-balls and we can conclude the proof using the same arguments as before.

\[ \square \]

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