THE MODULI OF ANNULI IN RANDOM CONFORMAL GEOMETRY

MORRIS ANG, GUILLAUME REMY, AND XIN SUN

Abstract. We obtain exact formulae for three basic quantities in random conformal geometry that depend on the modulus of an annulus. The first is for the law of the modulus of the Brownian annulus describing the scaling limit of uniformly sampled planar maps with annular topology, which is as predicted from the ghost partition function in bosonic string theory. The second is for the law of the modulus of the annulus bounded by a loop of a simple conformal loop ensemble (CLE) on a disk and the disk boundary. The formula is as conjectured from the partition function of the $O(n)$ loop model on the annulus derived by Saleur-Bauer (1989) and Cardy (2006). The third is for the annulus partition function of the SLE$_{8/3}$ loop introduced by Werner (2008), confirming another prediction of Cardy (2006). The physics principle underlying our proofs is that 2D quantum gravity coupled with conformal matters can be decomposed into three conformal field theories (CFT): the matter CFT, the Liouville CFT, and the ghost CFT. At the technical level, we rely on two types of integrability in Liouville quantum gravity, one from the scaling limit of random planar maps, the other from the Liouville CFT. We expect our method to be applicable to a variety of questions related to the random moduli of non-simply-connected random surfaces.

1. Introduction

Over the last two decades, tremendous progress has been made in understanding random surfaces of simply connected topology. By [Le 13, Mie13, BM17] and their extensions, the uniformly sampled random planar maps on the sphere and the disk converge as metric-measure spaces in the scaling limit. The limiting surfaces are known as the Brownian sphere and disk, respectively. Moreover, as proved in various senses [MS20, MS21a, MS21b, GMS20, HSar], once these surfaces are conformally embedded in the complex plane, the embedded random geometry is given by Liouville quantum gravity (LQG) [DS11, DDDF20, GM21b] with $\gamma = \sqrt{8/3}$. Finally, as shown in [AHS17, Cer21, AHS21], the law of the random field inducing the random geometry is given by the Liouville conformal field theory on the sphere [DKRV16] and the disk [HRV18].

For random planar maps decorated with statistical physics models, the scaling limit is described by LQG surfaces decorated with Schramm Loewner evolution (SLE) and conformal loop ensemble (CLE). Although there are still major open questions concerning the scaling limit, the picture in the continuum is well understood, thanks to the theory of the quantum zipper [She16] and the mating-of-trees [DMS21]. For more background on this subject, see the surveys [GHS19, Gwy20, She22] and the notes [LGM12, Var17, BP].

![Figure 1. Conformal embedding of the Brownian annulus to the annulus $\mathbb{A}_\tau = \{ z : |z| \in (e^{-2\pi \tau}, 1) \}$ with (random) modulus $\tau$.](image-url)

For non-simply-connected surfaces, the key new feature is the non-uniqueness of the conformal structure. For example, if we conformally embed the Brownian annulus to the complex plane (Figure 1), the modulus...
of the planar annulus is random. Precise descriptions of these surfaces, including the laws of the random moduli, were conjectured in [DRV16] for the torus, in [GRV19] for higher genus surfaces, and in [Rem18] by the second named author for the annulus. The conjecture is based on a fundamental principle that dates back to Polyakov’s seminal work [Pol81] on bosonic string theory. Namely, 2D quantum gravity coupled with conformal matter can be decomposed into three conformal field theories (CFT): the matter CFT, the Liouville CFT, and the ghost CFT. In this paper, we prove this conjecture for the annulus in the case of pure gravity which corresponds to the Brownian annulus, and the cases where the matter corresponds to the simple CLE [She09, SW12] (i.e. CLEκ with κ ∈ (8/3, 4]) or Werner’s SLEκ/3 loop [Wer08].

We also derive some exact formulae for SLE/CLE conjectured from physics. Since the discovery of SLE [Sch00], it has been proved or widely conjectured that SLE/CLE describe the scaling limits of interfaces in 2D lattice models such as percolation [Smi01, CN08] and the Ising model [CDCH+14, CS12, BH19]. Many scaling exponents/dimensions of such models predicted by CFT [BPZ84] were derived rigorously from SLE/CLE (e.g. [LSW02, MSW14]). Moreover, some exact formulae predicted via boundary CFT, such as Cardy’s formula for percolation crossing, were rigorously derived from SLE (see e.g. [Wer04]). Saleur-Bauer [SB89] and Cardy [Car06] derived a formula for the partition function of the $O(n)$ loop model on the annulus, which provides crucial information on the CFT description of its scaling limit. Based on the conjectural relation between CLE and the $O(n)$ loop model (see e.g. [She09]), this partition function corresponds to an exact formula for the loop statistics of CLE. Viewing the self avoiding loop as the $O(3)$ model, Cardy [Car06] further conjectured a formula for the annulus partition function of the SLEκ/3 loop. Deriving these formulae for SLE/CLE has been an open question. In this paper we prove them for simple CLE and the SLEκ/3 loop.

The key to our proofs is an explicit relation between the partition function of LQG surfaces and the law of their moduli. This is obtained from the conformal bootstrap of Liouville CFT on the annulus due to Wu [Wu22]. It can be viewed as a Knizhnik-Polyakov-Zamolodchikov (KPZ) relation [KPZ88] between the quantum and the Euclidean geometry at the partition function level, whereas the traditional KPZ relation established in probability [DS11, BGRV16, GHM15, GP19] is at the level of scaling exponents/dimensions. Our KPZ relation reduces the computation of an annular quantity depending on the modulus to the same quantity for LQG surfaces, which is easier to obtain from the random planar map perspective.

Our work represents the first step towards understanding the relation between LQG in the random geometry framework and measures on the moduli space of Riemann surfaces, although quantum gravity intuition has inspired important developments of the latter subject since 1980s [BK86, Wit91, Kon92, Mir07]. Our work is also a further step towards uncovering the CFT content of CLE, after the proof of the imaginary DOZZ formula by the first and third named authors [AS21]. It continues to demonstrate the rich interplay between various kinds of integrability in conformal probability, a theme recently explored in [AHS21, ARS21, AS21]. We expect that methods in our paper together with the remarkable developments in the integrability of Liouville CFT [KRV20, GKR20, GKR21] will result in further progress in this direction.

In the rest of the introduction, we first state our result on the modulus of the Brownian annulus in Section 1.1. Then we present the matter-Liouville-ghost decomposition for the Brownian annulus and the CLE decorated quantum annulus in Section 1.2. We state our KPZ relation for annulus partition functions in Section 1.3 and results for simple CLE and the SLEκ/3 loop in Sections 1.4 and 1.5, respectively. In Section 1.6 we discuss some subsequent projects and open questions concerning other models or topologies.

**Notation for the modulus.** Let $\mathbb{C}$ be the complex plane. For $\tau \in (0, \infty)$, let $A_\tau = \{z \in \mathbb{C} : |z| \in (e^{-2\pi \tau}, 1)\}$. For a Riemann surface $A$ of the annular topology, there exists a unique $\tau$ such that $A$ and $A_\tau$ are conformally equivalent. We call $\tau$ the **modulus** of $A$.

### 1.1. Random moduli for the Brownian annulus.

There are various equivalent ways of introducing the Brownian annulus. For the convenience of describing its conformal structure, we will introduce it via the Brownian disk. For $a > 0$, the **Brownian disk with boundary length $a$** is a random metric-measure space with the disk topology, which is the Gromov-Hausdorff-Prokhorov (GHP) scaling limit of random quadrangulations or triangulations of a large polygon under the critical Boltzmann weight [BM17, GM19, AHS20a, BCFS21]; see Proposition B.2 for a precise scaling limit result. If we reweight the law of a Brownian
disk by its total area and add an interior point according to the area measure, then we get a Brownian disk with one interior marked point which we denote by \((D, p, d, \mu)\). Here \(p \in D\) is the marked point, \(d\) is the metric and \(\mu\) is the area measure. For \(0 < r < d(p, \partial D)\), let \(B(p, r) = \{x \in D : d(p, x) \leq r\}\) be the metric ball of radius \(r\) centered at \(p\). Then each connected component of \(D \setminus B_r(p, r)\) is locally absolutely continuous to a Brownian disk near the boundary, hence has a notion of boundary length; see [LG19, Theorem 3].

**Definition 1.1.** Fix \(a > 0\) and \(b > 0\). Let \((D, p, d, \mu)\) be a Brownian disk with boundary length \(a\) and an interior marked point. On the event \(d(p, \partial D) > 1\), let \(A\) be the (annular) connected component of \(D \setminus B(p, 1)\) whose boundary contains \(\partial D\). Let \(L\) be the length of the boundary component of \(\partial A\) other than \(\partial D\). Let the probability measure \(BA(a, b)^\#\) be the conditional law of the metric-measure space \((A, d, \mu)\) conditioning on \(L = b\). We call a sample from \(BA(a, b)^\#\) a Brownian annulus of boundary lengths \(a\) and \(b\).

We give the full details for Definition 1.1 in Section 6.1. Although we define \(BA(a, b)^\#\) through conditional probability, which is only specified for almost every \(b\), there are various ways to remove this ambiguity. For example, using techniques in [BM17, GM19, AHS20a], it is possible to show that \(BA(a, b)^\#\) is the \(n \to \infty\) GHP scaling limit of the critical Boltzmann weighted quadrangulation or triangulation of an annulus with \([an]\) and \([bn]\) edges on the two boundaries, respectively. Moreover, it is possible to describe \(BA(a, b)^\#\) directly in the continuum by some conditioned variants of the Brownian snake as in the sphere [Le 13, Mie13] and disk [BM17] cases. One can also give a mating-of-trees [DMS21] description of \(BA(a, b)^\#\) based on the bijection in [BHS18]. Since our main curiosity lies in understanding the law of the modulus, we will not elaborate on these aspects of the Brownian annulus.

According to [MS20, MS21a] which we recall in Section 6.1, given \((D, p, d, \mu)\) in Definition 1.1, we can construct a random metric \(d_{\mathbb{H}}\) and a random measure \(\mu_{\mathbb{H}}\) on the upper half plane \(\mathbb{H}\) using \(\sqrt{3}/\)LQG such that viewed as a (marked) metric-measure space, \((\mathbb{H}, i, d_{\mathbb{H}}, \mu_{\mathbb{H}})\) equals \((D, p, d, \mu)\). We call \((\mathbb{H}, i, d_{\mathbb{H}}, \mu_{\mathbb{H}})\) a conformally embedded Brownian disk with boundary length \(a\) and an interior marked point \(i\). On the event \(d(p, \partial D) > 1\), let \(B_{d_{\mathbb{H}}}(i, 1) = \{x : d_{\mathbb{H}}(i, x) \leq 1\}\) and let \(A\) be the connected component of \(\mathbb{H} \setminus B_{d_{\mathbb{H}}}(i, 1)\) with \(\partial \mathbb{H}\) on its boundary. Then \((A, d_{\mathbb{H}}, \mu_{\mathbb{H}})\) is a conformal embedding of \((A, d, \mu)\). Let Mod\((A)\) be the modulus of the annulus \(A\). We call Mod\((A)\) the modulus of \((A, d, \mu)\). By [MS21b, GMS20], \((d_{\mathbb{H}}, \mu_{\mathbb{H}})\) is determined by \((D, p, d, \mu)\) modulo conformal symmetries. From this we can see that Mod\((A)\) is determined by the metric-measure space \((A, d, \mu)\); see Lemma 6.7. Our theorem below describes the law of Mod\((A)\) under \(BA(a, b)^\#\).

**Theorem 1.2.** For \(\tau > 0\), let \(X_\tau\) be a random variable such that \(\mathbb{E}[X_\tau^t] = \frac{2 \tau e^{-\frac{2 \pi \tau}{3 \sinh(\frac{2 \pi}{3})}}}{3 \sinh(\frac{2 \pi}{3})}\) for \(t \in \mathbb{R}\). Let \(\rho_\tau(\cdot)\) be the probability density function of \(X_\tau\). Then the law of Mod\((A)\) under \(BA(a, b)^\#\) is the probability measure on \((0, \infty)\) proportional to \(1_{\tau > 0} \eta(2\tau) \rho_\tau(\frac{\tau}{a}) d\tau\), where \(\eta(\tau) = e^{-\frac{\pi}{12} \frac{\tau}{3}} \prod_{k=1}^{\infty} (1 - e^{-2\pi k \tau})\) is the Dedekind eta function.

Theorem 1.2 yields statements on the scaling limit of the conformal structure of uniformly random planar maps on the annulus. For example, the law of Mod\((A)\) should describe the limiting distribution of the effective resistance of the critical Boltzmann triangulation with the given boundary lengths, viewed as an electric network. Proving this rigorously amounts to establishing the quenched scaling limit of random walks on such maps. Analogous statements should hold for other quantities that encode the modulus of the annulus, such as the annulus crossing probability for the critical percolation or Ising model. The quenched scaling limit needed to prove such statements is only known for site percolation on triangulations [Hsar].

Theorem 1.2 is a consequence of the matter-Liouville-ghost decomposition for the Brownian annulus (Theorem 1.3), as we will explain in the next subsection.

1.2. Matter-Liouville-ghost decomposition for 2D quantum gravity. The decomposition of 2D quantum gravity coupled with conformal matter was originally due to Polyakov [Pol81] where the matter is the free boson. David [Dav88] and Distler-Kawai [DK89] generalized it to conformal matter with central charge \(c_M \leq 1\). The ghost part of this decomposition only manifests itself for non-simply connected surfaces. In the probabilistic framework of LQG, this was first explained in [DRV16] for the torus case. The annulus case was treated by the second named author [Rem18] in parallel to the torus case. For higher genus closed surfaces, it was shown in [GRV19] that for free boson matter this decomposition indeed produces
a convergent theory envisioned in [Pol81]. Based on the belief that random planar maps decorated with statistical physics models are discretizations of matter-coupled 2D quantum gravity, precise scaling limit conjectures were formulated in each of the three papers [DRV16, GRV19, Rem18]. In particular, applying the decomposition to the pure gravity on the annulus, a precise LQG description of the Brownian annulus was provided in [Rem18, Section 4.3], following the physics paper [Mar03]. We now recall this conjecture, which we prove as Theorem 1.3 below.

We first introduce the free Brownian annulus with no constraint on the boundary lengths. Let

\begin{equation}
BA = \int_0^\infty \frac{1}{2\sqrt{ab(a+b)}} \text{BA}(a,b)^\# \, da \, db.
\end{equation}

We call a sample from the infinite measure BA a free Brownian annulus. Here the factor \(1/\sqrt{ab(a+b)}\) is according to enumeration results for annular planar maps [BF18]; see Lemma B.3. From the relation between \(\sqrt[8]{\text{3-LQG}}\) and Brownian surfaces established in [MS20, MS21a, MS21b], a free Brownian annulus can be represented as \((\phi, \tau)\) where \(\tau \in (0, \infty)\) gives the modulus of the annulus where the Brownian annulus is conformally embedded, and \(\phi\) describes the variant of Gaussian free field on the annulus that determines the \(\sqrt[8]{\text{3-LQG}}\) geometry.

The first part of the conjecture in [Rem18] is that the law of \(\phi\) given \(\tau\) can be described by the Liouville conformal field theory. To make the statement as clean as possible, we work on the horizontal cylinder \(\mathbb{C}_\tau := [0, \tau] \times [0, 1] / \sim\) of unit circumference and length \(\tau\), where \(\sim\) means identifying \([0, \tau] \times \{0\}\) with \([0, \tau] \times \{1\}\); see Figure 2 (left). This way, the modulus of \(\mathbb{C}_\tau\) is \(\tau\).\(^1\) Let \(\mathbb{P}_\tau\) be the law of the free boundary Gaussian free field on \(\mathbb{C}_\tau\) defined as in [She07]; see Section 2.1. The Liouville field measure \(\text{LF}_\tau\) on \(\mathbb{C}_\tau\) is the pushforward of \(\mathbb{P}_\tau \times dc\) under \((h, c) \mapsto h + c\), where \(dc\) is the Lebesgue measure on \(\mathbb{R}\). The first part of the conjecture can be stated as follows. There exists a measure \(m(\text{d} \tau)\) on \((0, \infty)\) such that \((\phi, \tau)\) sampled from \(\text{LF}_\tau(\text{d} \phi)m(\text{d} \tau)\) gives a free Brownian annulus conformally embedded on a horizontal cylinder.

The second and more mysterious part of the conjecture is the exact formula for \(m(\text{d} \tau)\). According to matter-Liouville-ghost decomposition of pure gravity, the free Brownian annulus can be written as

\begin{equation}
Z_{\text{GFF}}(\tau) \text{LF}_\tau(\text{d} \phi) \times Z_{\text{ghost}}(\tau) \text{d} \tau, \quad \text{with } Z_{\text{GFF}}(\tau) = \frac{1}{\sqrt{2\eta(2i\tau)}} \text{ and } Z_{\text{ghost}}(\tau) = \eta(2i\tau)^2.
\end{equation}

The measure \(Z_{\text{GFF}}(\tau) \text{LF}_\tau(\text{d} \phi)\) is the base measure for the path integral of the Liouville action on \(\mathbb{C}_\tau\), where \(Z_{\text{GFF}}(\tau)\) is the free field partition function coming from this path integral. The function \(Z_{\text{ghost}}(\tau)\) is the partition function on \(\mathbb{C}_\tau\) of the ghost field for bosonic string theory, which is the non-physical conformal field theory with central charge \(c_{\text{ghost}} = -26\) coming from conformal gauge fixing. See [Rem18, Section 5.2] for more background on \(Z_{\text{GFF}}(\tau)\) and \(Z_{\text{ghost}}(\tau)\). Putting the two parts of the conjecture together we get:

**Conjecture 1** ([Rem18]). We have \(BA = 1_{\tau > 0} 2^{-1/2} \eta(2i\tau) \cdot \text{LF}_\tau(\text{d} \phi) \text{d} \tau\) in the sense that a pair \((\phi, \tau)\) sampled from (1.2) gives a free Brownian annulus conformally embedded on \(\mathbb{C}_\tau\).

Our formulation of Conjecture 1 is an identity on two infinite measures while the formulation in [Rem18] is in terms of convergent functional integrals over these measures, where the convergence is achieved by introducing cosmological constants. In this paper we prove the following theorem.

**Theorem 1.3.** Conjecture 1 holds.

Once we couple 2D quantum gravity with a conformal matter, the influence on the random geometry can be intuitively understood from the scaling limit of random planar maps. In this context, a conformal matter corresponds to a statistical physics model \(M\) with a conformal invariant scaling limit. The 2D quantum gravity coupled with \(M\) can be approximated by random planar maps decorated by model \(M\), where the law of the planar map is the uniform measure weighted by the partition function of \(M\). Now the conjectural description of the conformally embedded geometry for the scaling limit with annular topology becomes

\begin{equation}
Z_M(\tau) \times Z_{\text{GFF}}(\tau) \text{LF}_\tau(\text{d} \phi) \times Z_{\text{ghost}}(\tau) \text{d} \tau,
\end{equation}

\(^1\)Our modular parameter \(\tau\) is denoted by \(\ell\) in [Rem18].
where $Z_M(\tau)$ is the partition function of the matter $M$ on $\mathcal{C}_\tau$. As explained in [DRV16, GRV19, Rem18], suppose the CFT describing the scaling limit of $M$ has central charge $c_M \leq 1$. Then the random geometry is induced by the Liouville CFT with central charge $c_L = -c_{\text{ghost}} - c_M = 26 - c_M$. Here $c_L$ is related to the background charge $Q$ and the coupling parameter $\gamma$ via $c_L = 1 + 6Q^2$ and $Q = \frac{7}{2} + \frac{2}{\gamma}$.

For each $\gamma \in (\sqrt{8}/3, 2)$ (i.e. $c_M \in (0, 1)$), a quantum surface of annular topology called the quantum annulus was introduced by the first and third named authors in [AS21]. The definition is purely in the continuum in terms of $\gamma$-LQG coupled with CLE$_{2\gamma}$ as we recall in Section 4. Intuitively, the quantum annulus describes the scaling limit of the random planar maps of annular topology decorated with a particular matter $M$. For example, for $\gamma = \sqrt{3}$, the matter $M$ is the Ising model on the annulus with boundary condition on both sides and the extra condition that there is a + arm crossing the annulus. Using the relation between the Ising model and CLE$_{\gamma}$ [BH19], the scaling limit of $M$ is the CLE$_{\gamma}$ on the annulus without non-contractible loops. Similarly, for $\gamma \in (\sqrt{8}/3, 2)$, the quantum annulus QA$_\gamma$ describes the natural LQG surface of annular topology decorated by CLE$_{2\gamma}$ without non-contractible loops.

Assuming the relation between CLE and the scaling limit of O($n$)-loop model, the conjectural formula from [SB89, Car06] for the O($n$)-loop model (see (1.12) and Corollary 1.10) yields a formula for the annulus partition function $Z_M(\tau)$. Combined with (1.3) we arrive at a conjectural description of QA$_\gamma$, which we prove as the next theorem.

**Theorem 1.4.** In the same sense as in Conjecture 1, for $\gamma \in (\sqrt{8}/3, 2)$, QA$_\gamma$ is given by

$$Q\alpha^\gamma = 1_{\tau > 0} \cdot \cos(\pi \left( \frac{4}{\gamma^2} - 1 \right)) \cdot \frac{\gamma}{2\pi} \theta_1(\frac{\gamma^2}{8} i\tau) \Lambda(\varphi) \text{d}\tau$$

where $\theta_1(z, i\tau) := -ie^{-\pi/4} \sum_{n=-\infty}^{\infty} (-1)^n e^{-n(n+1)i\tau} e^{(2n+1)i\pi z}$ is the Jacobi theta function.

Theorems 1.3 and 1.4 are proved using the same and widely applicable method as we outline below.

### 1.3. A KPZ relation for annulus partition functions

Fix $\tau > 0$. Let $h$ be the free boundary Gaussian free field (see e.g. [She07]) on the cylinder $\mathcal{C}_\tau$. As an example of Gaussian multiplicative chaos [Kah85, DS11, RV14], for $\gamma \in (0, 2)$ the $\gamma$-LQG length measure on $\partial \mathcal{C}_\tau$ is defined by $L_h^\gamma = \lim_{\epsilon \to 0} \epsilon^2 e^{-\frac{\gamma}{2} h_{\epsilon}(z)} dz$, where $dz$ is the Lebesgue measure on $\partial \mathcal{C}_\tau$, and $h_\epsilon(z)$ is the average of $h$ over $\{w \in \mathcal{C}_\tau : |w - z| = \epsilon\}$.

The $\gamma$-LQG area measure on $\mathcal{C}_\tau$ is $\mathcal{A}_\gamma^\tau = \lim_{\epsilon \to 0} \epsilon^2 e^{\frac{\gamma}{2} h_{\epsilon}(z)} dz$. Let $\partial_0 \mathcal{C}_\tau = \{0\} \times [0, 1]/\sim$ and $\partial_1 \mathcal{C}_\tau = \{\tau\} \times [0, 1]/\sim$.

In our previous work [ARS21], based on the conformal bootstrap of boundary Liouville CFT, we gave a conjecture ([ARS21, Conjecture 1.4]) on the joint distribution of $\mathcal{L}_h^\gamma(\partial_0 \mathcal{C}_\tau)$, $\mathcal{L}_h^\gamma(\partial_1 \mathcal{C}_\tau)$ and $\mathcal{A}_\gamma^\tau(\mathcal{C}_\tau)$ under the Liouville field measure LF$_c$. This conjecture was recently proved by Wu [Wu22] based on the rigorous bootstrap framework for Liouville CFT developed in [GKR20, GKR21]. The starting point of our proof of Theorems 1.3 and 1.4 is a corollary of Wu’s theorem, which is interesting in its own right. It gives a simple description of the law of $\frac{\mathcal{L}_h^\gamma(\partial_0 \mathcal{C}_\tau)}{\mathcal{L}_h^\gamma(\partial_1 \mathcal{C}_\tau)}$ where $h$ is a free boundary GFF. (Although the definition of free boundary GFF involves an additive constant as we recall in Section 2, the ratio $\frac{\mathcal{L}_h^\gamma(\partial_0 \mathcal{C}_\tau)}{\mathcal{L}_h^\gamma(\partial_1 \mathcal{C}_\tau)}$ is canonically defined.)

**Theorem 1.5.** Let $\gamma \in (0, 2)$ and $\tau > 0$, the characteristic function of $\log \mathcal{L}_h^\gamma(\partial_1 \mathcal{C}_\tau) - \log \mathcal{L}_h^\gamma(\partial_0 \mathcal{C}_\tau)$ is

$$\mathbb{E}[e^{ix(\log \mathcal{L}_h^\gamma(\partial_1 \mathcal{C}_\tau) - \log \mathcal{L}_h^\gamma(\partial_0 \mathcal{C}_\tau))}] = \frac{\pi x e^{-\frac{\pi^2 x^2}{4}}}{4 \sinh(\frac{\pi x}{2})} \text{ for } x \in \mathbb{R}.$$
Given Theorem 1.6, our proofs of Theorems 1.3 and 1.4 follow the same two-step strategy:

1. Show that BA and QA\(\gamma\) can be written as LF\(_r\)(\(d\phi\))\(m(\tau)\) for some measure \(m(\tau)\). We achieve this by proving a characterization of LF\(_r\) in Section 2 and verifying this property for BA and QA\(\gamma\).
2. The integral \(\left< L_1 e^{-L_0 L_0^{\gamma}} \right>\) for BA and QA\(\gamma\) can be easily extracted from the existing literature. Then computing \(m(\tau)\) from (1.6) is accomplished via an inverse Laplace transform.

We view Theorem 1.6 as a KPZ relation in the following sense. For many natural random surfaces such as BA and QA\(\gamma\), \(\left< L_1 e^{-L_0 L_0^{\gamma}} \right>\) is easy to compute from the perspective that LQG describes the scaling limit of random planar maps. From this we can compute \(m(\tau)\) by (1.6). This reduces the computation of a difficult quantity depending on the conformal structure to an easier quantity in LQG. This is the essence of the derivation of the scaling exponents/dimensions of random fractals using the traditional KPZ relation \([DS11, \text{GHM15}, \text{BGRV16}, \text{GP19}]\). For example, for QA\(\gamma\), the \(\tau \to \infty\) asymptotic of \(m(\tau)\) given in (1.4) encodes the dimension of the CLE carpet \([SSW09]\). Therefore, using (1.6) to determine the asymptotic of \(m(\tau)\) can be viewed as applying the traditional KPZ relation to determine the CLE carpet dimension. In this sense Theorem 1.6 upgrades the KPZ relation from the exponents/dimensions level to the annular partition function level.

In principle our KPZ relation (1.6) allows one to obtain the annulus partition \(Z_M(\tau)\) of any conformal matter \(M\) from its quantum counterpart. In physics \(Z_M(\tau)\) is usually obtained as \(\lim_{\delta \to 0} Z_{M}^{\delta}(\tau)\) where \(Z_{M}^{\delta}(\tau)\) is the partition function of \(M\) on a lattice of mesh size \(\delta\) that approximates \(C_{\kappa}\). When the scaling limit of \(M\) is described by SLE/CLE, the function \(Z_M(\tau)\) is supposed to describe a natural SLE/CLE quantity that depends on the modular parameter. For example, for QA\(3\), the corresponding \(Z_M(\tau)\) describes the law of the modulus of the annulus bounded by an outermost loop of a CLE\(_{\kappa}\) on a disk and the disk boundary; see Theorem 1.7. However, in many cases the convergence of \(M\) to SLE/CLE is an outstanding problem. Theorem 1.6 allows us to bypass this difficulty and obtain \(Z_M(\tau)\) from SLE/CLE using quantum gravity.

In Section 1.4 we summarize our results for simple CLE based on the KPZ approach. In Section 1.5 we apply the same approach to the case when \(M\) is the self avoiding loop.

1.4. Random Moduli for simple CLE loops. CLE\(_\kappa\) with \(\kappa \in (8/3, 8)\) was first defined in \([\text{She09}]\) as a canonical conformally invariant probability measure on infinite collections of non-crossing loops, where each loop looks like an SLE\(_\kappa\) curve. For \(\kappa \in (8/3, 4]\), the loops are simple and CLE\(_\kappa\) can be characterized by a domain Markov property and conformal invariance \([\text{SW12}]\). CLE has been proved or conjectured to be the scaling limit of many important 2D loop models such as percolation, the Ising model, the \(O(n)\)-loop model, and the random cluster model; see e.g. \([\text{CN08, BH19, KS16}]\). LQG coupled with CLE \([\text{MSW20, MSW21}]\) are proved or conjectured to describe the scaling limit of random planar maps decorated with these models. Recently, this coupling was used in \([\text{AS21, Mil21, MS22}]\) to derive results for CLE itself. As we recall in Section 4, the quantum annulus QA\(\gamma\) is defined by a symmetric modification of the law of the annulus on a quantum disk bounded by an outermost loop and the disk boundary. In fact choosing any loop in a CLE gives a natural LQG surface of annular topology. Using the KPZ relation (1.6) we get an analog of Theorem 1.4 for these surfaces. This yields exact formulae for the moduli of the annuli bounded by CLE loops, as we summarize below.

Let \(\mathbb{D}\) be the unit disk. For a simple loop \(\eta \subset \mathbb{D}\), let \(A_\eta\) be the annulus bounded by \(\eta\) and \(\partial \mathbb{D}\) and let Mod(\(\eta\)) be the modulus of \(A_\eta\). For \(\kappa \in (\frac{8}{3}, 4]\), let \(\Gamma\) be a CLE\(_\kappa\) on \(\mathbb{D}\). Let \(\{\eta_j\}_{j \geq 1}\) be the sequence of loops in \(\Gamma\) that surrounds the origin, ordered such that \(\eta_{j+1}\) is surrounded by \(\eta_j\); see Figure 2 (middle). Our first result is for the moment generating function of Mod(\(\eta_j\)).

**Theorem 1.7.** For \(j \geq 1\) and \(\lambda > \frac{3\kappa}{8\pi} + \frac{2}{\kappa} - 1\), we have

\[
E[e^{-2\pi\lambda\text{Mod}(\eta_j)}] = \frac{(\frac{4}{\kappa} - 1) \cos^j(\pi(\frac{4}{\kappa} - 1))}{\sin^j(\pi(1 - \frac{4}{\kappa}))} \times \frac{\sin(\frac{\pi}{2}\sqrt{(\frac{4}{\kappa} - 1)^2 - \frac{8\kappa}{\kappa}})}{\sqrt{(\frac{4}{\kappa} - 1)^2 - \frac{8\kappa}{\kappa}} \cos^j(\pi\sqrt{(\frac{4}{\kappa} - 1)^2 - \frac{8\kappa}{\kappa}})}.
\]
Let $\text{CR}(\eta_j)$ be the conformal radius of $\eta_j$ viewed from 0. Namely, $\text{CR}(\eta_j) = |\psi'(0)|$ where $\psi$ is a conformal map from the region surrounded by $\eta_j$ to $\mathbb{D}$ that fixes the origin. The law of $\text{CR}(\eta_j)$ is obtained in [SSW09]:

\begin{equation}
\mathbb{E}[\text{CR}(\eta_j)^k] = \left(\frac{-\cos(4\pi\kappa)}{\cos(\pi\sqrt{(\frac{1}{2} - 1)^2 - \frac{8\lambda}{\kappa}})}\right)^j \quad \text{for } \lambda > \frac{3\kappa}{32} + \frac{2}{\kappa} - 1.
\end{equation}

Our next theorem gives the joint law of $\text{Mod}(\chi_1, \kappa, \chi)$ and $\text{CR}(\eta_j)$ for $\kappa \in (\frac{4}{3}, 4]$ and $j \geq 1$.

**Theorem 1.8.** For $j \geq 1$, $\text{Mod}(\eta_j)$ and $e^{2\pi \text{Mod}(\eta_j)} \text{CR}(\eta_j)$ are independent.

The case $\kappa = 4, j = 1$ for Theorems 1.7 and 1.8 was obtained in [ALS20] via the level set coupling of CLE$_4$ and the Gaussian free field. As shown in [ALS20, SSW09], the law of $\text{Mod}(\eta_1)$ for $\kappa = 4$ and $\text{CR}(\eta_1)$ for general $\kappa$ can both be described in terms of hitting times of Brownian motion. This is also possible for $\text{Mod}(\eta_j)$ for general $\kappa$ and $m$; see Remark 5.6. The independence between $\text{Mod}(\eta_j)$ and $e^{2\pi \text{Mod}(\eta_j)} \text{CR}(\eta_j)$ does not appear to be a simple consequence of the domain Markov property of CLE. Our proof relies on the method in [AHS21, ARS21, AS21] which solves the law of conformal radii of SLE/CLE loops using LCFT.

For a fixed $n > 0$, the $O(n)$ loop model on a graph is a statistical physics model where a configuration $\mathcal{L}$ is a collection of disjoint loops (i.e., simple cycles). Given the temperature $T > 0$, the weight of each configuration is $T^{-|\mathcal{L}|} e^{V(\mathcal{L})}$, where $V(\mathcal{L})$ is the total number of vertices occupied by all loops in $\mathcal{L}$ and $|\mathcal{L}|$ is the total number of loops in $\mathcal{L}$. When $n \in (0, 2]$, for the $O(n)$ loop model on a planar lattice, there exists a critical temperature $T_c > 0$ such that at $T_c$ the scaling limit is conjectured to be CLE$_\kappa$ for $\kappa \in (\frac{4}{3}, 4]$; see [She09] for more background. The parameters $\kappa$ and $n$ related as follows:

\begin{equation}
g = \frac{4}{\kappa} \in [1, \frac{3}{2}], \quad \chi = (1 - g)\pi \in (-\frac{\pi}{2}, 0], \quad \text{and} \quad n = 2 \cos \chi \in (0, 2].
\end{equation}

The central charge of the CFT describing the scaling limit is given by:

\begin{equation}
c = 1 - \frac{6(1-g)^2}{g} = 1 - 6\left(\frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}\right)^2 \in (0, 1].
\end{equation}

Consider the $O(n)$ loop model on a lattice approximation of the cylinder $\mathcal{C}_\tau$. Let

\begin{equation}
q = e^{-\pi/\tau}, \quad \tilde{q} = e^{-2\pi\tau}, \quad \text{and} \quad n' = 2 \cos \chi' \quad \text{for } \chi' \in \mathbb{R}.
\end{equation}

Now suppose each contractible loop is assigned weight $n$ while each non-contractible loop is assigned weight $n'$ instead. Then as predicted in [SB89, Car06], in the continuum limit at $T_c$ the partition function is

\begin{equation}
Z(\tau, \kappa, \chi') = q^{\frac{\tau}{g}} \prod_{r=1}^{\infty} (1 - q^{2r})^{-1} \sum_{p \in \mathbb{Z}} \frac{\sin(p + 1)\chi'}{\sin \chi'} q^{\frac{8p^2 - (1-g)p}{4}}
\end{equation}

\begin{equation}
= \left(\frac{2}{g}\right)^{1/2} q^{\frac{\tau}{g}} \prod_{r=1}^{\infty} (1 - q^{2r})^{-1} \sum_{m \in \mathbb{Z}} \frac{\sin((\chi' + 2m\pi)/g)}{\sin \chi'} \frac{(x' + 2zm)^2 - (1-g)^2}{2x'y}.
\end{equation}
Equations (1.12) and (1.13) were derived in [SB89] via the quantum group method, and derived in [Car06] via the Coulomb gas method. (See also the lecture notes [Jac09, Section 14.7.2]). As pointed out in [Car06], it would be interesting to derive $Z(\tau, \kappa, \chi')$ rigorously in the SLE framework. As a consequence of Theorem 1.8, we solve this problem.

**Theorem 1.9.** For any non-negative measurable function $f$ on $(0, \infty)$, we have

$$
(1.14) \quad E \left[ \sum_{j=1}^{\infty} \left( \frac{n'}{n} \right)^{j-1} f(\text{Mod}(\eta_j)) \right] = \frac{\sqrt{N(\pi - 1)} \cos(\pi(j-1))}{\sqrt{2} \sin(\pi(1 - \frac{x}{\pi}))} \int_{0}^{\infty} f(\tau) e^{\frac{\tau}{4\pi}(\pi - 1)^2} Z(\tau, \kappa, \chi') \eta(2i\tau) d\tau,
$$

where $\eta(i\tau) = e^{-\frac{\pi}{\tau}} \prod_{k=1}^{\infty} (1 - e^{-2\pi k\tau})$ is the Dedekind eta function.

A more transparent way to see the geometric meaning of $Z(\tau, \kappa, \chi')$ is to consider CLE on the annulus. Given $\Gamma$, we choose $\eta$ from the counting measure of $\{\eta_j\}_{j \geq 1}$ and let $A_\eta$ be the annulus bounded by $\eta$ and $\partial \mathbb{D}$. Let $\text{CLE}_\kappa^\tau$ be the conditional law of the loops of $\Gamma$ inside $A_\eta$ conditioning on $\text{Mod}(\eta) = \tau$. Then we use conformal invariance to extend the definition of $\text{CLE}_\kappa^\tau$ to any annulus of modulus $\tau$. This gives one natural definition of $\text{CLE}_\kappa$ on an annulus that allows non-contractible loops; see Figure 2 (right). In fact, if $\eta = \eta_j$, then the number of non-contractible loops is $j - 1$. Therefore Theorem 1.9 has the following corollary.

**Corollary 1.10.** For $\tau > 0$, let $N$ be the number of non-contractible loops of a $\text{CLE}_\kappa^\tau$. Then

$$
(1.15) \quad E \left[ \left( \frac{n'}{n} \right)^N \right] = \frac{Z(\tau, \kappa, \chi')}{Z(\tau, \kappa, \chi)}
$$

where $\chi = (1 - \frac{x}{\pi}) \pi$, $n = 2 \cos \chi$, and $n' = 2 \cos \chi'$ for $\chi' \in \mathbb{R}$.

Annulus partition functions are fundamental in the CFT framework of 2D statistical physics. The annulus partition function of a CFT in term of the $q$ variable is of the form $q^{-c/24} \sum_{s \in \text{bdy}} c(s) K_s(q)$ as in (1.12), where $c$ is the central charge and $\text{S}_{\text{bdy}}$ is the so called boundary spectrum, and $K_s(q)$ is a universal s-dependent function called the Virasoro character. This summation is called the conformal bootstrap in the open string channel. When $\text{S}_{\text{bdy}}$ is finite, the CFT is called rational. For example, the Ising model partition function equals $Z(\tau, \kappa, \chi)$ with $n = 1$. It can be written as $Z_{++} + Z_{+-}$, where $Z_{++}$ (resp., $Z_{+-}$) is the annulus partition function with $++$ (resp., $+-$) boundary condition. Both $Z_{++}$ and $Z_{+-}$ are examples of Virasoro characters. In terms of the $q$ variable, annulus partition functions are of the form $\tilde{q}^{-c/12} \sum_{s \in \text{bdy}} |c(s)|^2 K_s(\tilde{q}^2)$. The set $\text{S}_{\text{bdy}}$ is called the bulk spectrum and the summation is called the conformal bootstrap in the closed string channel. The equality between (1.12) and (1.13) is called modular invariance or open-closed duality. Conformal bootstrap formulas were also derived for torus partition functions of various models; see e.g. [DFSZ87]. The spectrum is given by $\text{S}_{\text{bdy}}$ and a similar modular invariance holds. As a cornerstone for 2D CFT, modular invariance imposes strong constraints on these theories, which is crucial to the classification of an important class of rational CFTs called the minimal models. The annulus and torus partition functions can often be derived for lattice models by methods such as algebraic CFT principle, Coulomb gas techniques, and transfer matrix, and along the way one obtains $c$, $\text{S}_{\text{bdy}}$, and $\text{S}_{\text{bdy}}$. See [ISZ88, DFMS97] for more background on CFT and 2D statistical physics.

The spectra $S_{\text{bdy}}$ and $S_{\text{bdy}}$ determine the allowed boundary and bulk scaling dimensions for the corresponding model. Many scaling dimensions were later rigorously derived in the SLE framework as the exponents of certain arm events, or as the dimensions of certain fractals. Our paper provides a method to derive the full annulus partition function, strengthening the connection between the SLE and the CFT approaches to 2D statistical physics. Although the connection between CFT and SLE has been studied intensively from the martingale observable approach (e.g. [FW03, BB03, KM21]), it appears challenging to derive results for CLE using this approach. In the next subsection we demonstrate how our method applies to another model: the self avoiding loop.

### 1.5. Annulus partition function of the $\text{SLE}_{8/3}$ loop

Werner [Wer08] constructed an infinite conformally invariant measure on simple loops on each Riemann surface which we call the $\text{SLE}_{8/3}$ loop measure. Modulo a multiplicative constant, the $\text{SLE}_{8/3}$ loop measure is uniquely characterized by the conformal restriction property. It describes the conjectural scaling limit of self avoiding polygons [Wer08, Conjecture 1].
A planar map version of this conjecture was proved in [AHS22] based on [GM21a]. The construction of the SLE$_{8/3}$ loop measure in [Wer08] is based on the Brownian loop measure. Let $\mu_C$ be an SLE$_{8/3}$ loop measure on the complex plane $\mathbb{C}$. In Section 7 we will recall a construction of $\mu_C$ due to Zhan [Zha21].

Given a domain $D \subset \mathbb{C}$, let $\mu_D$ be the restriction of $\mu_C$ to the set of loops that are contained in $D$. Let $\psi$ be a conformal map between two domains $D$ and $D'$. Then the conformal restriction property of the SLE$_{8/3}$ loop measure asserts that $\mu_D$ is the pushforward of $\mu_{D'}$ under $\psi$; see Theorem 7.3. Given an annulus $A \subset \mathbb{C}$ of modulus $\tau$, let $Z_{8/3}(\tau)$ be the $\mu_C$-mass of loops in $A$ that are not contractible. Then the conformal restriction property implies that $Z_{8/3}(\tau)$ depends on $A$ through its modulus $\tau$. We call $Z_{8/3}(\tau)$ the annulus partition function of the SLE$_{8/3}$ loop. It is supposed to be the scaling limit of $\sum \nu_c$-loop length where the summation runs over non-contractible loops of a lattice approximation of the cylinder $\mathbb{C}_\tau$, and $\nu_c$ is the so-called connectivity constant of the lattice. (For the honeycomb lattice $\nu_c = \sqrt{2 + \sqrt{2}$ [DCS12].)

The sharp asymptotic of $Z_{8/3}(\tau)$ was obtained in [Wer08, Section 7], where deriving an exact formula for $Z_{8/3}(\tau)$ was left as an open question. A formula was conjectured by Cardy [Car06, Eq (5)] based on the perspective that self-avoiding loop can be viewed as $O(n)$ loop model with $n = 0$. We rigorously prove Cardy’s formula for $Z_{8/3}(\tau)$ as the next theorem.

**Theorem 1.11.** There exists a constant $C > 0$ such that $Z_{8/3}(\tau) = CZ_{\text{Cardy}}(\tau)$ where

$$Z_{\text{Cardy}}(\tau) = \prod_{r=1}^{\infty} (1 - q^r)^{-1} \sum_{k \in \mathbb{Z}} k(-1)^{k-1} q^{3k^2 - k + 1} \quad \text{for } \tau > 0 \text{ and } q = e^{-\pi/\tau}.$$  

Since $\mu_C$ is characterized up to a multiplicative constant, the constant $C$ in Theorem 1.11 is not canonical. We prove Theorem 1.11 in Section 7. The starting point of our proof is the following fact proved in [AHS22]: gluing together two Brownian disks with equal boundary lengths gives a Brownian sphere decorated with an SLE$_{8/3}$ loop. From this we can construct the SLE$_{8/3}$-loop-decorated quantum annulus by gluing together two Brownian annuli. Then the two-step strategy outlined below (1.6) gives the law of its modulus. Finally, the conformal restriction property of the SLE$_{8/3}$ loop allows us to derive $Z_{8/3}(\tau)$ from the law of the modulus.

**1.6. Perspectives and outlook.** We discuss a few directions that a subset of the authors and their collaborators are planning to investigate in subsequent works, and some open questions.

1. Our KPZ method for proving Theorems 1.4 and 1.11 can extend to other models, such as uniform spanning tree and percolation. In a subsequent work we will carry this out for CLE$_{\kappa} \in (4, 8)$, which will extend results in Section 1.4 to $\kappa \in (4, 8)$ and confirm conjectures in this regime from [Car06]. We will also prove Cardy’s crossing formula for percolation on the annulus from [Car02]; see also [Dub04].

2. We can consider variants of (1.6) where we have one boundary marked point on the annulus. Combining with the conformal bootstrap results for Liouville CFT in [Wa22], this will give interesting relations between the 1-point torus conformal block [GRS20] and SLE/CLE observables.

3. We can also consider variants of (1.6) where the underlying domain is a disk with four boundary marked points or with one bulk and two boundary points. This will give access to SLE observables such as crossing probabilities and the one-point Green’s function.

4. We will consider the variant of (1.6) in the pair of pants case, where we expect to obtain the analogous results for the Brownian pair of pants and for CLE on a disk with two holes. As a limiting case, we expect to obtain an exact formula for certain two-point correlation functions for CLE on the disk.

5. At the high level, our two-step strategy below (1.6) can extend to closed surfaces such as the torus. But in the second step, we would need to find other observables than the boundary length to obtain the law of the modulus. This could still be tractable for the torus, but we consider it as a major open question to extend our results to arbitrary Riemann surfaces. Essential new ideas are required to understand the much more complicated moduli space and the integrability of Liouville CFT on these surfaces. See [GRV19, Section 5] for precise conjectures in the closed surface case.

6. Determining the full CFT content of CLE and related discrete models is an active topic in both mathematics and physics. See [JPS20, Section 4.2] for a summary of the state of the art in physics for the Q-Potts model. In mathematics, the so-called imaginary DOZZ formula conjectured in [IJS19]
was proved in [AS21], and the annulus partition function is proved in this paper. It would be a breakthrough to carry out a functional analytic conformal bootstrap program for CLE similar to the Liouville case [GKRV20, GKR21]. It would already be very interesting to give a purely CLE-based proof of results in our Section 1.4, as for the $\kappa = 4$ case in [ALS20]. Such a proof might be extended to prove the torus counterpart of Corollary 1.10 as predicted by [DFSZ87, Pin94]. If knowledge on the conformal matter can be obtained without LQG, then it can be used as a substitute for the boundary length observable in (1.6) to determine the law of the modulus.

(7) It would be interesting to understand the coupling of the ghost field with the matter CFT and Liouville CFT at the probabilistic level, which goes beyond the factorization of partition functions. See [LZ91, BMP92] for related work at the representation theoretic level.

**Organization of the paper.** In Section 2 we give a characterization of $\LF_{\tau}$. In Section 3 we prove Theorems 1.5 and 1.6. In Section 4 we carry out the two-step strategy outlined below (1.6) to prove Theorem 1.4. In Section 5 we prove results for CLE from Section 1.4. In Section 6 we prove Theorem 1.3 using the same two-step strategy, which implies Theorem 1.2. In Section 7 we prove Theorem 1.11.

**Acknowledgements.** We thank Baojun Wu for communicating with us his concurrent work [Wu22]. We thank Jesper Jacobsen and Hubert Saleur for very helpful feedback on an earlier version of this paper. M.A. was partially supported by NSF grant DMS-1712862. G.R. was supported by an NSF mathematical sciences postdoctoral research fellowship, NSF Grant DMS-1902804. X.S. was supported by the NSF grant DMS-2027986 and the NSF Career award 2046514.

2. Liouville fields and their domain Markov property

In this section we introduce Liouville fields on the annulus and the disk, and prove their domain Markov property (Propositions 2.4 and 2.9) that we will use throughout the paper. We then give a characterization of the Liouville field on the annulus (Proposition 2.13). In the rest of the paper, we let $D = \{ z \in \mathbb{C} : 3z > 0 \}$ be the upper half plane. For $x < y$, let $C(x, y) = [x, y] \times [0, 1] / \sim$ be a finite horizontal cylinder so that $C_{\tau} = C(0, \tau)$. Let $\partial_0 C(x, y) = \{ x \} \times [0, 1] / \sim$ and $\partial_1 C(x, y) = \{ y \} \times [0, 1] / \sim$. For infinite measures, we will still adopt the probabilistic terminologies such as random variable and law.

2.1. Gaussian free field. We first gather some facts about two dimensional Gaussian free field (GFF). See e.g. [She07, Dub09] for more background. For concreteness, let $D$ be a planar domain that is conformally equivalent of a disk or an annulus. Throughout this section we assume that

\begin{equation}
\rho(dx) \text{ is a compactly supported probability measure such that } \int \log |x - y| \rho(dx) \rho(dy) < \infty.
\end{equation}

Our discussion could extend to a broader class of measures but (2.1) is sufficient for us. For a measure $\rho$ as in (2.1) supported on $D$, let $H(D; \rho)$ be the Hilbert closure of $\{ f \in C^\infty(D) : \int_D f(x) \rho(dx) = 0 \}$ under the inner product $(f, g)_\rho = (2\pi)^{-1} \int_D (\nabla f \cdot \nabla g) \, dx$. Let $(f_n)_{n \geq 1}$ be an orthonormal basis for $H(D; \rho)$ and $(\alpha_n)_{n \geq 1}$ be a sequence of independent standard normal random variables. Then the random series $\sum_{n=1}^\infty \alpha_n f_n$ converges almost surely as a random generalized function. We call a random generalized function on $D$ with the law of $\sum_{n=1}^\infty \alpha_n f_n$ a free boundary Gaussian free field on $D$ normalized to have average zero over $\rho(dx)$.

Now suppose $A$ is conformally equivalent to an annulus, with the two boundary components denoted by $\partial_1 A$ and $\partial_2 A$. Let $H(A)$ be the Hilbert closure of $\{ f \in C^\infty(A) : \text{supp}(f) \cap \partial_1 A = \emptyset \}$ under the inner product $(f, g)_\rho = (2\pi)^{-1} \int_A (\nabla f \cdot \nabla g) \, dx$. Here $\text{supp}(f)$ means the support of $f$. Let $(g_n)_{n \geq 1}$ be an orthonormal basis for $H(A)$. Let $(\alpha_n)_{n \geq 1}$ be a sequence of independent standard normal random variables. Then $\sum_{n=1}^\infty \alpha_n g_n$ converges almost surely as a random generalized function. We call a random generalized function on $A$ with the law of $\sum_{n=1}^\infty \alpha_n g_n$ a Gaussian free field on $A$ with zero-boundary condition on $\partial_1 A$ and free boundary condition on $\partial_2 A$.

We will need an instance of the domain Markov property for the GFF. Let $D$ be a domain conformally equivalent to an annulus or disk and such that $\partial D$ is smooth. Let $\eta$ be a Jordan curve (namely simple closed curve) inside $D$ such that we can find a connected component $A$ of $D \setminus \eta$ that is conformally equivalent to an annulus. We let $\partial_1 A$ be the boundary component $\eta$ and $\partial_2 A$ be the other boundary component of $A$. 

Let \( \rho(dx) \) be a probability measure on \( D \setminus A \) satisfying \((2.1)\). Let \( \text{Harm}(D; \rho; A) \) be the space of functions in \( H(D; \rho) \) that are harmonic on \( A \) and have normal derivative zero on \( \partial_2 A \). Then we have the orthogonal decomposition
\[
H(D, \rho) = H(A) \oplus \text{Harm}(D; \rho; A),
\]
which yields the following domain Markov property of the free boundary GFF.

**Lemma 2.1.** With \( D, A, \rho \) defined in the paragraph right above, let \( h \) be a free boundary GFF on \( D \) normalized by \( \int h(x) \rho(dx) = 0 \). Let \( h^{\text{har}} \) be the harmonic extension of \( h|_{D \setminus A} \) onto \( A \) with zero normal derivative on \( \partial_2 A \). Let \( h_A = h - h^{\text{har}} \). Then \( h_A \) is a GFF on \( A \) with zero boundary condition on \( \partial_1 A \) (namely \( \eta \)), and free boundary condition on \( \partial_2 A \). Moreover, \( h_A \) is independent of \( h^{\text{har}} \).

**Proof.** Since \( h^{\text{har}} \) is the projection of \( h \) onto \( \text{Harm}(D; \rho; A) \), we obtain Lemma 2.1 from \((2.2)\) via the standard argument from [She07]. \( \square \)

### 2.2. Liouville fields on the annulus and the disk

We first define Liouville fields on annuli. For convenience we focus on the horizontal cylinder \( C = C(0, \tau) = [0, \tau] \times [0, 1]/\sim \).

**Definition 2.2.** For \( \tau > 0 \), let \( \rho \) be a probability measure on \( C \) satisfying \((2.1)\) and \( \mathbb{P}_{\tau, \rho} \) be the law of the free boundary Gaussian free field on \( C \) with \( \int_C h(x) \rho(dx) = 0 \). Sample \( (h, c) \) from \( \mathbb{P}_{\tau, \rho} \times dc \) and set \( \phi = h + c \). We write \( \text{LF}_\tau \) as the law of \( \phi \), and call \( \phi \) a Liouville field on \( C \).

**Lemma 2.3.** The measure \( \text{LF}_\tau \) does not depend on the choice of the measure \( \rho \).

**Proof.** This follows from the translation invariance of the Lebesgue measure \( dc \). \( \square \)

Lemma 2.1 and 2.3 yield a domain Markov property for \( \text{LF}_\tau \). To state it, we recall that \( \partial_0 C = \{0\} \times [0, 1]/\sim \) and \( \partial_1 C = \{\tau\} \times [0, 1]/\sim \) are the two boundary components of \( C \).

**Proposition 2.4.** Let \( \eta \subset C \) be a non-contractible Jordan curve so that both components of \( C \setminus \eta \) are of annular topology. Let \( A_\eta \) be the component bounded by \( \eta \) and \( \partial_0 C \). Let \( \phi \) be a Liouville field on \( C \). Let \( \phi|_{C \setminus A_\eta} \) be the harmonic extension of \( \phi|_{C \setminus A_\eta} \) onto \( A_\eta \) with zero normal derivative on \( \partial_0 C \). Then conditioning on \( \phi|_{C \setminus A_\eta} \), the conditional law of \( \phi_{A_\eta} \) is a GFF on \( A_\eta \) with zero boundary condition on \( \eta \) and free boundary condition on \( \partial_0 C \). If \( A_\eta \) is bounded by \( \eta \) and \( \partial_1 C \) instead, the same holds with \( \partial_1 C \) in place of \( \partial_0 C \).

Since \( \text{LF}_\tau \) is an infinite measure, the word “conditioning” above requires a proper interpretation.

**Definition 2.5.** Suppose \((\Omega, \mathcal{F}) \) and \((\Omega', \mathcal{F}') \) are two measurable spaces. We say \( \Lambda : \Omega \times \mathcal{F}' \rightarrow [0, 1] \) is a Markov kernel if \( \Lambda(\omega, \cdot) \) is a probability measure on \((\Omega', \mathcal{F}') \) for each \( \omega \in \Omega \) and \( \Lambda(\cdot, A) \) is \( \mathcal{F} \)-measurable for each \( A \in \mathcal{F}' \). If \((X, Y) \in \Omega \times \Omega' \) is a sample from \( \Lambda(\omega, d\omega') \mu(d\omega) \) for some measure \( \mu \) on \((\Omega, \mathcal{F}) \), we say that the conditional law of \( Y \) given \( X \) is \( \Lambda(X, \cdot) \).

**Proof of Proposition 2.4.** We focus on the case where \( A_\eta \) is bounded by \( \eta \) and \( \partial_0 C \) as the other case follows by symmetry. Let \( \rho \) be a probability measure on the connected component of \( C \setminus \eta \) other than \( A_\eta \) satisfying \((2.1)\). Let \( \mathbb{P}_{\tau, \rho} \) be the law of the free boundary GFF on \( C \) with \( \int_C h(x) \rho(dx) = 0 \). Sample \( (h, c) \) from \( \mathbb{P}_{\tau, \rho} \times dc \) and set \( \phi = h + c \). Then \( \phi \) is a Liouville field on \( C \) by definition.

Let \( h^{\text{har}} \) be the harmonic extension of \( h|_{C \setminus A_\eta} \) onto \( A_\eta \) with zero normal derivative on \( \partial_0 C \). Let \( h_{A_\eta} = h - h^{\text{har}} \). Then \( \phi^{\text{har}} = h^{\text{har}} + c \) and \( \phi_{A_\eta} = h_{A_\eta} \). By Lemma 2.1, conditioning on \( (h|_{C \setminus A_\eta}, c) \), the conditional law of \( h_{A_\eta} \) is a GFF on \( A_\eta \) with zero boundary condition on \( \eta \) and free boundary condition on \( \partial_0 C \). This gives the desired conditional law of \( \phi_{A_\eta} \) given \( \phi|_{C \setminus A_\eta} \). \( \square \)

We now recall the Liouville field on the disk. We focus on the upper half plane \( \mathbb{H} \).

**Definition 2.6.** Fix \( \gamma \in (0, 2) \) and \( Q = \frac{\gamma}{2} + \frac{\gamma}{2} \). Let \( \text{LF}_{\mathbb{H}} \) be the law of the free boundary Gaussian free field on \( \mathbb{H} \) with average zero over the uniform measure on the semi-circle \( \{z \in \mathbb{H} : |z| = 1\} \). Sample \( (h, c) \) from \( \mathbb{P}_{\mathbb{H}} \times e^{-Qc} dc \) and set \( \phi(z) = h(z) + c - 2Q \log |z|_+, \) where \( |z|_+ = \max\{|z|, 1\} \). We write \( \text{LF}_{\mathbb{H}} \) as the law of \( \phi \), and call \( \phi \) a Liouville field on \( \mathbb{H} \).
Here we omit the dependence of LF_H in γ because it will be a globally fixed parameter whenever we consider LF_H. We recall Liouville fields on H with one bulk insertion, following [ARS21].

**Definition 2.7.** Fix α ∈ ℝ and z₀ ∈ ℍ. Sample (h, c) from \((2\text{Im}z₀)^{-α^2/2}|z₀|^\alpha \mathbb{P}_\mathbb{H} \times e^{(α-Q)c} dc\). Set \(\phi(z) = h(z) + c - 2Q\log|z| + αG_0(z, z₀)\) where \(G_0(z, w) = -\log(|z - w||z - \bar{w}|) + 2\log|z| + 2\log|w|\). We write \(\text{LF}_\mathbb{H}^{(α,z₀)}\) as the law of \(\phi\), and call \(\phi\) a Liouville field on \(H\) with an \(α\)-insertion at \(z₀\).

Unlike the case of the annulus, the choice of normalization of the GFF \(h\) affects the multiplicative constant of the Liouville field on the disk, a phenomenon called the Weyl anomaly [DKRV16, HRV18]. The following computation was carried out in [HRV18, Section 3.3]; we rephrase it here and give a quick proof.

**Proposition 2.8.** Let \(ρ\) be a probability measure on \(H\) satisfying (2.1). Set
\[
Z_ρ := \exp\left(\frac{Q^2}{2} \int (-\log|z - w|-\log|z - \bar{w}|)ρ(dz)ρ(dw)\right).
\]
Let \(P_ρ\) denote the law of the GFF \(\phi\) on \(H\) normalized so \((h, ρ) = 0\). Sample \((h, c)\) from \(P_ρ \times [Z_ρe^{-Qc} dc]\). Then the law of \(h + c - QG_ρ(\cdot, \infty)\) is LF_H, where \(G_ρ(\cdot, \cdot)\) is the covariance kernel for \(P_ρ\).

**Proof.** In Definition 2.6, the covariance kernel for \(P_ρ\) is \(G_ρ(\cdot, \infty) = 2\log|z|\). Therefore
\[
G_ρ(z, w) = G_ρ(z, w) - \int G_ρ(z, x)ρ(dx) - \int G_ρ(y, y)ρ(dy) + \int G_ρ(x, y)ρ(dx)ρ(dy)
\]
(2.3) \(G_ρ(z, ∞) = G_ρ(z, ∞) - \int G_ρ(z, x)ρ(dx) - \int G_ρ(y, ∞)ρ(dy) + \int G_ρ(x, y)ρ(dx)ρ(dy).
\]
Sample \((h, c)\) from \(P_ρ(dh)e^{-Qc} dc\). Let \(h = h + c - QG_ρ(\cdot, ∞)\) so that the law of \(ϕ\) is LF_H. Let \(h_0 = h - Q \int G_ρ(\cdot, x)ρ(dx)\) and \(c = (h_0, ρ) - Q \int G_ρ(y, ∞)ρ(dy)\). Then \(ϕ = h - (h_0, ρ) + c - QG_ρ(\cdot, ∞)\). By Girsanov’s theorem the law of \((h₀, c')\) is \(Z_ρP_ρ \times e^{-Qc'} dc'\), where we used that \(Z_ρ\) defined above equals \(E_{P_ρ}[e^{Q(h, ρ)}e^{-Q^2\int G_ρ(y, ∞)ρ(dy)}]\). Since the law of \((h₀ - (h₀, ρ), c')\) is \(P_ρ \times [Z_ρe^{-Qc'} dc']\), we are done. 

We have the following analog of Proposition 2.4.

**Proposition 2.9.** Let \(η ∊ \mathbb{H}\) be a Jordan curve. Let \(A_η\) be the component of \(H\setminus η\) bounded by \(η\) and \(∂H = ℝ\). Let \(ϕ\) be a Liouville field on \(H\). Let \(ϕ^{\text{bar}}\) be the harmonic extension of \(ϕ\)|\(H\setminus A_η\) onto \(A_η\) with zero normal derivative on ∂H and with \(\lim_{z→∞}ϕ^{\text{bar}}(z)/|z| = -2Q\). Let \(ϕ_{A_η} = ϕ - ϕ^{\text{bar}}\). Then conditioning on \(ϕ_{H\setminus A_η}\), the conditional law of \(ϕ_{A_η}\) is a GFF on \(A_η\) with zero boundary condition on \(η\) and free boundary condition on \(∂H\).

**Proof.** The same argument for the proof of Proposition 2.4 works here. Let \(ρ\) be a probability measure supported in \(H\setminus A_η\) satisfying (2.1). Write \(ϕ = h + c - QG_ρ(\cdot, ∞)\) as in Proposition 2.8. Let \(h^{\text{bar}}\) be the harmonic extension of \(h|H\setminus A_η\) onto \(A_η\) with zero normal derivative on ∂H. Let \(h_{A_η} = h - h^{\text{bar}}\). By (2.3), \(G_η(z, ∞) - G_ρ(z, ∞) = o(\log|z|)\) as \(z→∞\). Since \(G_η(z, ∞) = 2\log|z|\), we have \(\lim_{z→∞}G_η(z, ∞)/\log|z| = 2\). Therefore \(ϕ^{\text{bar}} = h^{\text{bar}} + c - QG_ρ(\cdot, ∞)\) and \(ϕ_{A_η} = h_{A_η}\). By Lemma 2.1, conditioning on \((h|H\setminus A_η, c)\), the conditional law of \(h_{A_η}\) is a GFF on \(A_η\) with zero boundary condition on \(η\) and free boundary condition on \(∂H\). This gives the desired conditional law of \(ϕ_{A_η}\) given \(ϕ_{H\setminus A_η}\). 

**Lemma 2.10.** Fix α ∈ ℝ and \(z₀ ∈ ℍ\). Let η ⊂ ℍ be a Jordan curve surrounding \(z₀\). Then the statement of Proposition 2.9 holds with LF_H in place of LF_H.

**Proof.** By [ARS21, Lemma 2.2], we have \(\text{LF}_H^{(α,z₀)}(dϕ) = \lim_{ε→0}ε^{α^2/2}e^{αϕ_{c}(z)}\text{LF}_H(dϕ)\) where \(ϕ_{c}\) is the average of \(ϕ\) over \(\{z ∈ ℍ : |z - z₀| = ε\}\). Since \(η\) surrounds \(\{z ∈ ℍ : |z - z₀| = ε\}\) for small enough \(ε\), Lemma 2.10 follows from Proposition 2.9.

We need the domain Markov property for LF_H to some local sets in the sense of [SS13]. We restrict ourselves to two concrete cases needed later; see [SS13, Section 3] for the general background.

**Lemma 2.11.** Fix α ∈ ℝ and \(z₀ ∈ ℍ\). Suppose \((φ, η)\) satisfies one of the following conditions:
The law of \((\phi, \eta)\) is a product measure \(\text{LF}^{(\alpha, z_0)}_H \times \mathbb{P}\) where \(\mathbb{P}\) is a probability measure on Jordan curves in \(H\) surrounding \(z_0\).

The law of \(\phi\) is \(\text{LF}^{(\alpha, z_0)}_H\), and \(\eta \subset H\) is a Jordan curve surrounding \(z_0\) such that for each deterministic open set \(U \subset H\), the event \(\{D \subset U\}\) is measurable w.r.t. \(\phi|_{U}\), where \(D\) is the bounded simply-connected component of \(H \setminus \eta\).

Then with \(\phi_A\) defined as in Proposition 2.9, conditioning on \((\phi|_{H \setminus A}, \eta)\), the conditional law of \(\phi_A\) is the GFF on \(A\) with zero boundary condition on \(\eta\) and free boundary condition on \(\partial H\).

Proof. Both of the two cases are examples of local sets as defined in [SS13, Section 3]. Using the standard \(\phi\) values on \(H\) and \(\eta\) as defined in \([\text{SS13, Lemma 3.9}]\) that extends the Markov property to the strong Markov property, we obtain Lemma 2.11 from Lemma 2.10.

2.3. A resampling characterization of \(\text{LF}_\tau\). Let \(H^1(C_\tau)\) be the Sobolev space on \(C_\tau\) with inner product \(\int_{C_\tau} (fg + \nabla f \cdot \nabla g) \, d\tau\). Let \(H^{-1}(C_\tau)\) be the dual of \(H^1(C_\tau)\). Then both the GFF on \(C_\tau\) and \(\text{LF}_\tau\) can be viewed as measures on \(H^{-1}(C_\tau)\) endowed with the Borel \(\sigma\)-algebra. Let \(\phi_{\text{bar}}\) be the harmonic extension of \(\phi|_{C(0, \frac{2\tau}{\gamma})}\) onto \(C(0, \frac{2\tau}{\gamma})\) with zero normal derivative on \(\partial_0 C_\tau\). Let \(\Lambda(\phi, \cdot)\) be the law of \(h + \phi_{\text{bar}}\) where \(h\) is a GFF on \(C(0, \frac{2\tau}{\gamma})\) with zero boundary condition on \(\{\frac{2\tau}{\gamma}\} \times [0,1] / \sim_\gamma\) and free boundary condition on \(\partial_0 C_\tau\). By Proposition 2.4 \(\text{LF}_\tau\) is an invariant measure of the Markov kernel \(\Lambda\), namely \(\text{LF}_\tau = \int \Lambda(\phi, \cdot) \text{LF}_\tau(d\phi)\).

Let \(\Lambda\) be the Markov kernel defined in the same way as \(\Lambda\) in the flipped direction. More precisely, let \(\phi_{\text{bar}}\) be the harmonic extension of \(\phi|_{C(0, \frac{\tau}{\gamma})}\) onto \(C(0, \frac{\tau}{\gamma})\) with zero normal derivative on \(\partial_0 C_\tau\). Let \(\Lambda(\phi, \cdot)\) be the law of \(h + \phi_{\text{bar}}\) where \(h\) is a GFF on \(C(0, \frac{\tau}{\gamma})\) with zero boundary condition on \(\{\frac{\tau}{\gamma}\} \times [0,1] / \sim_\gamma\) and free boundary condition on \(\partial_0 C_\tau\). Then by symmetry \(\text{LF}_\tau\) is an invariant measure of \(\Lambda\) as well. We now explain that modulo a multiplicative constant \(\text{LF}_\tau\) is the unique measure that is invariant under both \(\Lambda\) and \(\Lambda\). For both its proof and applications, it will be convenient for us to truncate \(\text{LF}_\tau\) by the quantum length.

We first recall the notion of quantum length. Let \(h\) be a free boundary GFF on a domain \(D\) which contains a straight line segment \(L\). Fix \(\gamma \in (0, 2)\). Then the \(\gamma\)-LQG length measure on \(L\) is defined by \(L^\gamma_h = \lim_{\epsilon \to 0} \epsilon^\gamma \tau e^{\frac{\phi_{\text{bar}}}{\gamma}(z)} \, dz\), where \(dz\) is the Lebesgue measure on \(L\), and \(h_\epsilon(z)\) is the average of \(h\) over \(\{w \in D : |w - z| = \epsilon\}\). This is an example of Gaussian multiplicative chaos (GMC); see e.g. [RV14], the limit exists in probability in the weak topology of measures on \(L\). Suppose \(\phi\) is a Liouville field on \(C_\tau\), we can define \(L^\gamma_h = \lim_{\epsilon \to 0} \epsilon^\gamma \tau e^{\frac{\phi_{\text{bar}}}{\gamma}(z)} \, dz\) as well. Let \(I\) be a finite interval in \((0, \infty)\). Let \(\Omega_I \subset H^{-1}(C_\tau)\) be the event that \(L^\gamma_h(\partial_0 C_\tau) \subset I\) and \(L^\gamma_h(\partial_0 C_\tau) \subset I\). By the following lemma, \(\text{LF}_\tau(\Omega_I) < \infty\).

Lemma 2.12. Fix \(\gamma \in (0, 2)\) and \(\tau > 0\). Suppose we are in the setting of Definition 2.2 where \(h\) is a free boundary GFF on the horizontal cylinder \(C_\tau\) with a certain normalization. Then for non-negative measurable functions \(f\) and \(g\) on \((0, \infty)\), we have

\[
\int \text{LF}_\tau [f(L^\gamma_h(\partial_0 C_\tau))g(L^\gamma_h(\partial_1 C_\tau))] = \frac{2}{\gamma} \int_0^\infty \ell^{-1} f(\ell) \mathbb{E} \left[ \int_0^\infty \ell^{-1} g(\ell L^\gamma_h(\partial_0 C_\tau)) \, d\ell \right] \, d\ell.
\]

Note that the ratio \(\frac{L^\gamma_h(\partial_0 C_\tau)}{L^\gamma_h(\partial_1 C_\tau)}\) does not depend on normalization of \(h\).

Proof. By definition \(\text{LF}_\tau [f(L^\gamma_h(\partial_0 C_\tau))g(L^\gamma_h(\partial_1 C_\tau))] = \int_0^\infty \mathbb{E}[f(e^{\frac{\phi}{\gamma}} L^\gamma_h(\partial_0 C_\tau))g(e^{\frac{\phi}{\gamma}} L^\gamma_h(\partial_1 C_\tau))] \, d\ell\). Setting \(\ell = e^{\frac{\phi}{\gamma}} L^\gamma_h(\partial_0 C_\tau)\) and use \(dc = \frac{2}{\gamma} \ell^{-1} \, d\ell\) we conclude the proof.

Proposition 2.13. Let \(I \subset (0, \infty)\) be a compact interval. Let \(\text{LF}_{\tau, I}\) be the probability measure proportional to the restriction of \(\text{LF}_\tau\) to \(\Omega_I\). For \(\phi \in \Omega\) and \(A \subset \Omega\), let \(\Lambda_I(\phi, A) = \Lambda(\phi, A) / \Lambda(\phi, \Omega_I)\) and \(\Lambda_I(\phi, A) = \Lambda(\phi, \Omega_I) / \Lambda(\phi, \Omega_I)\). Then \(\text{LF}_{\tau, I}\) is the unique probability measure on \(\Omega_I\) that is invariant under both \(\Lambda_I\) and \(\Lambda_I\).

Proof. We say that a Markov kernel \(K\) is irreducible if there exists a positive measure \(\rho\) on the state space such that the following holds. For every \(x\) and \(\rho(A) > 0\) there exists a positive integer \(n\) (which may depend on \(x\) and \(A\)) such that \(K^n(x, A) > 0\). It is known (from e.g. [MT09, Propositions 4.2.1 and 10.1.1, and Theorem 10.0.1]) that if an irreducible Markov kernel has an invariant probability measure, then it has only one invariant probability measure.
Then there is a positive constant \( - (3.2) \) constant, which was derived in [Rem20]. The unspecified constant \( U \) bootstrap formula for the joint law of quantum length and area but we do not need it. The expression for \( x \) if \( h = \Lambda \),.

Starting from (3.1), by using the above formula for \( \tau \) in (3.1), we see that the KPZ relation in Theorem 1.5.

Proof of Theorem 1.5. By Proposition 2.4, we see that LF, is a GFF in \( C(\tau) \) such that

\[
\Gamma(\tau, \lambda) = \text{GFF in } C(\tau),
\]

\[
\tau = \text{GFF in } C(\tau),
\]

\[
\tau / \gamma = \Gamma(\tau, \lambda) \text{ with zero boundary conditions on } C(\tau).
\]

Moreover, the annulus boundary partition function for Liouville CFT

\[
\text{for } \gamma > 0, \text{ for all } \gamma > 0, \text{ and setting } \gamma = 1 \text{ we have:}
\]

\[
\text{with } C(\tau) \text{ as in Theorem 3.1. By Lemma 2.12:}
\]

\[
\text{Using furthermore } \Gamma(1 - ix) \Gamma(1 - ix), \text{ we obtain:}
\]

\[
\text{Using the Fourier inversion formula } f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\omega} f(u) du, \text{ we obtain:}
\]

\[
\text{Setting } x = 0 \text{ we get } C(\tau) = 1 \text{ for all } \tau > 0 \text{ as desired.}
\]
Remark 3.2. We can decompose $h$ into $h = R + Y$ where $R$ is the radial part of the field given by $R(x) = \int_0^1 h(x + iy)\,dy$, and $Y = h - R$ has zero average along the $y$-axis for every $x$. It is well known that $Y$ and $R$ are independent. From here it is straightforward to decompose our functional of interest as
\[
E[(\mathcal{L}_h^\gamma(\partial_1 C_\gamma))^x_1(z) (\mathcal{L}_h^\gamma(\partial_0 C_\gamma))^{-i\bar{z}}] = E[e^{\frac{\pi \gamma^2}{4}\left((R_\gamma - R(0))^2\right)}]E[(\mathcal{L}_h^\gamma(\partial_1 C_\gamma))^x_1(z) (\mathcal{L}_h^\gamma(\partial_0 C_\gamma))^{-i\bar{z}}],
\]
where here $\mathcal{L}_h^\gamma(\partial_0 C_\gamma)$ has the same definition as $\mathcal{L}_h^\gamma(\partial_0 C_\gamma)$ with $h$ replaced by $Y$, and similarly for $\mathcal{L}_h^\gamma(\partial_1 C_\gamma)$. By direct computation on the covariance of $R$, one can obtain $E[e^{\frac{\pi \gamma^2}{4}\left((R_\gamma - R(0))^2\right)}] = e^{-\frac{\pi \gamma^2}{4}}$. By Theorem 1.5, we have $E[(\mathcal{L}_h^\gamma(\partial_1 C_\gamma))^x_1(z) (\mathcal{L}_h^\gamma(\partial_0 C_\gamma))^{-i\bar{z}}] = \frac{\pi \gamma^2}{4\sinh(\frac{i\pi}{4})}$. Therefore, the law of $\mathcal{L}_h^\gamma(\partial_1 C_\gamma)/\mathcal{L}_h^\gamma(\partial_0 C_\gamma)$ is a log-logistic distribution. We observe that interestingly the law of this ratio does not depend on $\tau$. Moreover, as $\tau \to \infty$, $\mathcal{L}_h^\gamma(\partial_1 C_\gamma)$ and $\mathcal{L}_h^\gamma(\partial_0 C_\gamma)$ become independent and both of them converge in law to the total mass of the Gaussian multiplicative chaos on the circle explicitly solved in [Rem20]. This provides an alternative route to proving Theorem 1.5 without relying on Theorem 3.1 but we do not pursue it in this paper.

Proof of Theorem 1.6. By (3.2) and Theorem 1.5, we have
\[
(3.4) \quad \text{LF}_{\gamma} \left[ L_1 e^{-L_1} \right] = 2 \gamma \int_0^\infty \int_0^{\infty} \sqrt{d\ell \xi e^{-\ell}} \xi \left[ \frac{L_1^\gamma}{L_0^\gamma} \right] = \frac{\pi \gamma^2}{2\sinh(\frac{i\pi}{4})} \Gamma(1 + iz).
\]
Since $\int_0^\infty \int_0^{\infty} \sqrt{d\ell \xi e^{-\ell}} \xi \left[ \frac{L_1^\gamma}{L_0^\gamma} \right] = \text{LF}_{\gamma} \left[ L_1 e^{-L_1} \right]$ by definition, we get (1.6). \qed

4. Quantum annulus: Liouville field description and the random modulus

The quantum annulus is a quantum surface introduced in [AS21] in the context of the quantum disk coupled with CLE. In this section we clarify its relation with Liouville CFT and determine the law of its modulus.

We first recall the definition of quantum surfaces. Fix $\gamma \in (0, 2)$ and $Q = \frac{2}{\gamma} + \frac{2}{\alpha}$. Consider pairs of the form $(D, h)$ where $D$ is a planar domain and $h$ is a generalized function on $D$. We say that $(D, h) \sim_\gamma (\hat{D}, \hat{h})$ if there exists a conformal map $\psi : D \to \hat{D}$ such that $h = h \circ \psi + Q \log |\psi'|$. A quantum surface in $\gamma$-LQG is an equivalence class under $\sim_\gamma$. We write $(D, h, \gamma) / \sim_\gamma$ as the quantum surface corresponding to $(D, h)$. An embedding of a quantum surface is a choice of its representative. We extend $\sim_\gamma$ to quantum surfaces with decorations/markings. For example, for triples $(D, h, z)$ where $z \in D$, we further require $z = \psi(\hat{z})$ when defining $(D, h, z) \sim_\gamma (\hat{D}, \hat{h}, \hat{z})$ and call $(D, h, z) / \sim_\gamma$ a quantum surface with an interior marked point. We can similarly define quantum surfaces decorated by curves.

The notion of quantum length is intrinsic to quantum surfaces. Suppose $(D, h) \sim_\gamma (\hat{D}, \hat{h})$ through the conformal map $\psi : D \to \hat{D}$. Recall the GMC definition of quantum length above Lemma 2.12. If both $D$ and $\hat{D}$ contain a line segments which are related by $\psi$, then $L_0^\gamma$ is the pushforward of $L_0^\gamma$ under $\psi$. Therefore, we can use this coordinate change to consistently define $L_0^\gamma$ on $\partial D$ even if it does not contain a line segment. We now recall the Liouville field description of the quantum disk with an alpha bulk insertion defined in [ARS21].

Definition 4.1. For $\gamma \in (0, 2)$ and $\alpha > \frac{\gamma}{2}$, let $\mathcal{M}_{\text{disk}}(\alpha)$ be the law of the quantum surface $(\mathbb{H}, \phi, i) \sim_\gamma$, where $\phi$ is sampled from $\text{LF}_{\mathbb{H}}^{(\alpha)}$. Let $\text{LF}_{\mathbb{H}}^{(\alpha)}(\ell)$ be the disintegration $\text{LF}_{\mathbb{H}}^{(\alpha)}(\ell) = \int_0^\ell \text{LF}_{\mathbb{H}}^{(\alpha)}(\ell') d\ell'$ such that samples from $\text{LF}_{\mathbb{H}}^{(\alpha)}(\ell)$ have boundary length $\ell$. Let $\mathcal{M}_{\text{disk}}(\alpha; \ell)$ be the law of $(\mathbb{H}, \phi, i) \sim_\gamma$ where $\phi$ is sampled from $\text{LF}_{\mathbb{H}}^{(\alpha)}(\ell)$.

In this section we only use $\mathcal{M}_{\text{disk}}(\gamma; \ell)$ but we will use $\mathcal{M}_{\text{disk}}(\alpha; \ell)$ in the next one. By [ARS21, Theorem 3.4], $\mathcal{M}_{\text{disk}}(\gamma; \ell)$ is the law of a quantum disk with boundary length $\ell$ and one interior marked point as defined in [DMS21]. The reason we require $\alpha > \frac{\gamma}{2}$ is that the total mass $|\text{LF}_{\mathbb{H}}^{(\alpha)}(\ell)|$ of $\text{LF}_{\mathbb{H}}^{(\alpha)}(\ell)$ (hence also $\mathcal{M}_{\text{disk}}(\alpha; \ell)$) is finite in this case. The precise value of $|\text{LF}_{\mathbb{H}}^{(\alpha)}(\ell)|$ was computed in [Rem20] as explained in [ARS21, Lemma 2.7 and Proposition 2.8]:
\[
(4.1) \quad |\text{LF}_{\mathbb{H}}^{(\alpha)}(\ell)| = \frac{2}{\gamma} \frac{\pi^2}{e^{-\alpha}} Q \cdot \left( \frac{2\pi}{\Gamma(1 - \frac{\alpha}{2})} \right)^2 \left( \frac{Q - \gamma^2}{4} \right) \Gamma \left( \frac{\alpha - \gamma^2}{4} \right) \times \ell^{\frac{\alpha}{4}(Q - \gamma^2)} \quad \text{for} \quad \alpha > \frac{\gamma}{2}.
\]
Let $\kappa \in (\frac{4}{3}, 4)$ and $\gamma = \sqrt{\kappa}$. Let $\Gamma$ be a CLE$_\kappa$ on $\mathbb{H}$ and $h$ be a free boundary GFF on $\mathbb{H}$ independent of $\Gamma$. Let $\eta$ be a loop in $\Gamma$. Let $A_\eta$ and $D_\eta$ be the two connected components $\mathbb{H} \setminus \eta$ where $D_\eta$ is simply connected and $A_\eta$ is the annulus. By the theory of quantum zipper [She16], the quantum length measures of the quantum surfaces $(A_\eta, h)/\sim_\gamma$ and $(D_\eta, h)/\sim_\gamma$ agree on $\eta$. This defines a quantum length measure on $\eta$. If $h$ is replaced by a Liouville field, the quantum length measure on $\eta$ can also be defined. The following theorem is proved in Sections 2 and 3 of [AS21].

**Theorem 4.2.** For $a > 0$, let $(\phi, \Gamma)$ be a sample from $LF^{\gamma,i}_H(a) \times \text{CLE}^{\mathbb{H}}_{\kappa}$ where CLE$^{\mathbb{H}}_{\kappa}$ is the law of a CLE$_\kappa$ on $\mathbb{H}$. Let $\eta$ be the outermost loop of $\Gamma$ surrounding $i$. Then the law of the quantum boundary length $L^\gamma_\phi(\eta)$ of $\eta$ is given by

\[
(4.2) \quad LF^{\gamma,i}_H(a) \times \text{CLE}^{\mathbb{H}}_{\kappa}[L^\gamma_\phi(\eta) \in I] = \int_I \frac{\cos(\pi(\frac{a}{2} - 1))}{\pi} \cdot \frac{b|LF^{\gamma,i}_H(b)|}{\sqrt{ab(a+b)}} db \quad \text{for } I \subset (0, \infty).
\]

Moreover, conditioning on $(A_\eta, h)/\sim_\gamma$, the condition law of $(D_\eta, \phi)/\sim_\gamma$ is $M^\text{disk}_{\Gamma}(\gamma; L^\gamma_\phi(\eta))$.

We now recall the definition of the quantum annulus.

**Definition 4.3.** Consider the setting of Theorem 4.2. For $b > 0$, let $\text{QA}^\gamma(a, b)^\#$ be the conditional law of the quantum surface $(A_\eta, \phi)/\sim_\gamma$ conditioning on $L^\gamma_\phi(\eta) = b$, which is a probability measure. Let

\[
(4.3) \quad \text{QA}^\gamma(a, b) = \frac{\cos(\pi(\frac{a}{2} - 1))}{\pi} \cdot \frac{\text{QA}^\gamma(a, b)^\#}{\sqrt{ab(a+b)}}.
\]

Let $\text{QA}^\gamma = \int_0^\infty b\text{QA}^\gamma(a, b) db$. We call a sample of $\text{QA}^\gamma$ a quantum annulus.

Here the conditioning $\{L^\gamma_\phi(\eta) = b\}$ is understood in the sense of disintegration, which is uniquely defined for almost every $b \in (0, \infty)$. This ambiguity does not affect the definition of $\text{QA}^\gamma$ and the proof of Theorem 1.4 below. After we prove Theorem 1.4 $\text{QA}^\gamma(a, b)$ can be canonically defined for every $(a, b)$. Therefore we omit this temporary ambiguity. The definition of $\text{QA}^\gamma$ is such that by Theorem 4.2, the joint law of $(A_\eta, \phi)/\sim_\gamma$ and $(D_\eta, \phi)/\sim_\gamma$ is

\[
(4.4) \quad \int_0^\infty b\text{QA}^\gamma(a, b) \times M^\text{disk}_{\Gamma}(\gamma; b) db.
\]

The intuitive meaning of the factor $b$ in (4.4) is that the number of ways of gluing together $\text{QA}^\gamma(a, b)$ and $M^\text{disk}_{\Gamma}(\gamma; b)$ is proportional to the boundary length $b$ of their interface; see [AS21, Sections 2.4].

Our main goal of this section is to prove Theorem 1.4 which describes the matter-Liouville-ghost decomposition of $\text{QA}^\gamma$. Namely, the law of the quantum surface $(\mathcal{C}_\tau, \phi)/\sim_\gamma$ is $\text{QA}^\gamma$ if we sample $(\tau, \phi)$ from

\[
(4.5) \quad 1_{\tau > 0} \cdot \cos(\pi(\frac{4}{\gamma^2} - 1)) \cdot \frac{\gamma}{2\pi} \theta_1(\gamma^2 \frac{\tau^2}{4} + \frac{\gamma^2}{4} i\tau) LF_\tau(d\phi) d\tau.
\]

In Section 4.1 we prove that $\text{QA}^\gamma$ can be written in the form of $LF_\tau(d\phi)m(d\tau)$ for some $m(d\tau)$ . This is based on the domain Markov property of Liouville fields from Section 2. Then in Section 4.2 we identify $m(d\tau)$ as given in Theorem 1.4 using the KPZ relation Theorem 1.6. In Section 5, we will prove the CLE results stated in Section 1.4 using Theorems 1.4 and 4.2.

### 4.1. Identification of the Liouville field via the domain Markov property.

**Proposition 4.4.** There exists a measure $m(d\tau)$ on $(0, \infty)$ such that if we sample $(\tau, \phi)$ from $LF_\tau(d\phi)m(d\tau)$, then the law of the quantum surface $(\mathcal{C}_\tau, \phi)/\sim_\gamma$ is $\text{QA}^\gamma$.

Our proof of Proposition 4.4 based on the domain Markov property will be used multiple times to prove similar statements in later sections. We recall the notions in Section 2. Let $\mathcal{C}(x, y)$ be the cylinder $(x, y) \times [0, 1]/\sim$ where $\sim$ is the identification of $[x, y] \times \{0\}$ and $[x, y] \times \{1\}$. We will write the circle $(z) \times [0, 1]/\sim$ as $\{z\} \times [0, 1]$ for simplicity. Let $\mathcal{C}_\tau = \mathcal{C}(0, \tau)$ for each $\tau > 0$. We prove the needed domain Markov property through the following two lemmas.
Lemma 4.5. Let \((\phi, \Gamma)\) be a sample from \(\text{LF}^{\gamma,i}_{H} \times \text{CLE}^{\bar{H}}\). Let \(\eta\) be the outermost loop of \(\Gamma\) surrounding \(i\) and \(A_\eta\) be the component of \(\mathbb{H} \setminus \eta\) bounded by \(\eta\) and \(\partial \mathbb{H}\). Let \(\hat{\tau}\) be such that \(\mathfrak{A}_\eta\) is conformally equivalent to \(C_\eta\). Fix \(x \in \{0\} \times [0, 1]\) and let \(\psi : C_\eta \to A_\eta\) be the conformal map such that \(\psi(x) = 0\). Let \(\hat{\phi}(z) = \phi \circ \psi(z) + Q \log |\psi'(z)|\) for \(z \in C_\eta\). Let \(\phi^\text{bar}\) be the harmonic extension of \(\hat{\phi}|_{C(\frac{2\hat{\tau}}{3})}\) onto \(C(0, \frac{2\hat{\tau}}{3})\) with zero normal derivative on \(\{0\} \times [0, 1]\). Then conditioning on \(\hat{\tau}\) and \(\hat{\phi}|_{C(\frac{2\hat{\tau}}{3})}\), the conditional law of \(\phi - \phi^\text{bar}\) is a GFF on \(C(0, \frac{2\hat{\tau}}{3})\) with zero boundary condition on \(\{\frac{2\hat{\tau}}{3}\} \times [0, 1]\) and free boundary condition on \(\{0\} \times [0, 1]\).

Proof. Let \(\hat{\eta} \subset \mathbb{H}\) be the image of \(\{\frac{2\hat{\tau}}{3}\} \times [0, 1]\) under \(\psi\). Then the law of \((\phi, \hat{\eta})\) is a product measure hence satisfies the first condition in Lemma 2.11. Let \(D_\hat{\eta}\) and \(A_\hat{\eta}\) be the disk and annulus component of \(\mathbb{H} \setminus \hat{\eta}\), respectively. Let \(\phi^\text{bar}\) be the harmonic extension of \(\phi|_{D_\hat{\eta}}\) onto \(A_\hat{\eta}\) with zero normal derivative on \(\partial \mathbb{H}\) and satisfying \(\lim_{|z| \to \infty} \phi^\text{bar}(z)/\log |z| = -2Q\). Let \(\phi_{A_\hat{\eta}} = \phi - \phi^\text{bar}\). Then by Lemma 2.11, conditioning on \(\hat{\eta}\) and \(\phi|_{D_\hat{\eta}}\), the conditional law of \(\phi_{A_\hat{\eta}}\) is a GFF on \(A_\hat{\eta}\) with zero boundary condition on \(\hat{\eta}\) and free boundary condition on \(\partial \mathbb{H}\). Since \((\hat{\tau}, \hat{\phi}|_{C(\frac{2\hat{\tau}}{3})})\) is determined by \((\hat{\eta}, \phi|_{D_\hat{\eta}})\), by the conformal invariance of GFF, conditioning on \((\hat{\tau}, \hat{\phi}|_{C(\frac{2\hat{\tau}}{3})})\) the conditional law of \(\phi_{A_\hat{\eta}} \circ \psi\) is a GFF on \(C(0, \frac{2\hat{\tau}}{3})\) with zero boundary condition on \(\{\frac{2\hat{\tau}}{3}\} \times [0, 1]\) and free boundary condition on \(\{0\} \times [0, 1]\).

Note that \(\phi - (\phi^\text{bar} \circ \psi + Q \log |\psi'|) = \phi \circ \psi - \phi^\text{bar} \circ \psi = \phi_{A_\hat{\eta}} \circ \psi\), which is zero on \(C(0, \frac{2\hat{\tau}}{3})\). We just need to show that \(\phi^\text{bar} = \phi^\text{bar} \circ \psi + Q \log |\psi'|\). To see this, note that \(\phi^\text{bar} \circ \psi + Q \log |\psi'|\) is harmonic inside \(C(0, \frac{2\hat{\tau}}{3})\) since \(\psi\) is conformal. It remains to show that \(\phi^\text{bar} \circ \psi + Q \log |\psi'|\) has the desired boundary condition on \(\{0\} \times [0, 1]\). Let \(\psi_1(z) = \frac{z - i}{i(z - 1)}\) which maps the unit disk \(\mathbb{D}\) to \(\mathbb{H}\). Then the normal derivative of \(\phi_1^\text{bar} := \phi_{A_\hat{\eta}} \circ \psi_1 + Q \log |\psi_1'|\) on \(\partial \mathbb{D}\) is zero everywhere. For \(w \in C_\eta\), let \(\psi_2(w) = \psi_1^{-1}(\psi(w))\) so that \(\psi = \psi_1 \circ \psi_2\). Then \(\phi_{A_\hat{\eta}} \circ Q \log |\psi'| = \phi_1^\text{bar} \circ \psi_2 + Q \log |\psi_2'|\). Since \(\psi_2\) maps \(\partial \mathbb{D}\) to \(\{0\} \times [0, 1]\), the normal derivative of \(\phi_1^\text{bar} \circ \psi + Q \log |\psi'|\) is zero everywhere on \(\{0\} \times [0, 1]\) as desired.

\[\text{Figure 3. Illustration of objects in the proof of Lemma 4.5.}\]

Lemma 4.6. For a sample \((\phi, \Gamma)\) from \(\text{LF}^{\gamma,i}_{H} \times \text{CLE}^{\bar{H}}\), let \(\mathcal{L}^\gamma_\eta(\eta)\) be the quantum length of the outermost loop \(\eta\) surrounding \(i\). Let \(f(b) = b|\text{LF}^{\gamma,i}_{H}(b)|\) and consider the reweighted measure \(M = \frac{1}{f(\mathcal{L}^\gamma_\eta(\eta))} \cdot \text{LF}^{\gamma,i}_{H} \times \text{CLE}^{\bar{H}}\). Let \(A_\eta\) be the component of \(\mathbb{H} \setminus \eta\) bounded by \(\eta\) and \(\partial \mathbb{H}\). Then the \(M\)-law of \((A_\eta, \phi)\) is QA\(^\gamma\). Moreover, the statement in Lemma 4.5 on the conditional law of \(\phi - \phi^\text{bar}\) given \((\hat{\tau}, \hat{\phi}|_{C(\frac{2\hat{\tau}}{3})})\) still holds under \(M\).

Proof. By (4.4), the \(M\)-law of \((A_\eta, \phi)\) is QA\(^\gamma\). For the assertion on the conditional law of \(\phi - \phi^\text{bar}\), we note that the most crucial point in the proof of Lemma 4.5 is the following. Conditioning on \(\hat{\eta}\) and \(\phi|_{D_\hat{\eta}}\), the conditional law of \(\phi_{A_\hat{\eta}}\) is a mixed boundary GFF on \(A_\hat{\eta}\). Since \(\psi\) and \(\hat{\eta}\) determine each other, the quantum length \(\mathcal{L}^\gamma(\eta)\) of \(\eta\) is determined by \((\hat{\eta}, \phi|_{D_\hat{\eta}})\). Therefore under the reweighted measure \(M\), the conditional law of \(\phi_{A_\hat{\eta}}\) given \((\hat{\eta}, \phi|_{D_\hat{\eta}})\) does not change. The rest of the proof of Lemma 4.5 now gives the last assertion in Lemma 4.6.

We recall the symmetry the quantum annulus proved as [AS21, Remark 3.12]. This will allow us to flip the sides when applying the domain Markov property to prove Proposition 4.4.

Lemma 4.7 ([AS21]). For each \(a, b > 0\), we have QA\(^\gamma\)(a, b) = QA\(^\gamma\)(b, a).
Proof of Proposition 4.4. Throughout this proof, we let $(\hat{\phi}, \hat{\tau})$ be an embedding of $\text{QA}^\gamma$ such that the law of $\hat{\phi}$ is invariant under rotations against the axis of the cylinder $C(0, \hat{\tau})$. Since rotations are the only conformal automorphisms of a finite cylinder, the law of $(\hat{\phi}, \hat{\tau})$ is unique. We call $(\hat{\phi}, \hat{\tau})$ the rotational invariant embedding of $\text{QA}^\gamma$. We will show that the law of $(\hat{\phi}, \hat{\tau})$ is $\text{LF}(d\phi)m(d\tau)$ for some $m(d\tau)$, which will prove Proposition 4.4.

Let $I$ be a finite interval on $(0, \infty)$. Given a quantum surface of annular topology, let $E_I$ be the event that both boundary lengths are in $I$. Since $|\text{QA}^\gamma(a,b)|$ is proportional to $\frac{1}{\sqrt{ab(a+b)}}$, we have $\text{QA}^\gamma(E_I) = \int_{I \times I} |\text{QA}^\gamma(a,b)| da db < \infty$. Let $\mathbb{P}_I$ be the probability measure describing the law of $(\hat{\phi}, \hat{\tau})$ conditioning on $E_I$. For $\tau > 0$, let $P_{I,\tau}$ be the regular conditional probability of $\hat{\phi}$ under $\mathbb{P}_I$ given $\{\hat{\tau} = \tau\}$.

Note that the domain Markov property in Lemmas 4.5 and 4.6 hold for each fixed $x \in \{0\} \times [0, 1]$ in the definition of the conformal map $\psi$ from $C_\tau$ to $A_\eta$. Therefore Lemmas 4.5 and 4.6 still hold if we choose $x$ uniformly at random on $\{0\} \times [0, 1]$. This way, under the measure $M$ in Lemma 4.6 $(\hat{\phi}, \hat{\tau})$ defined there is a rotational invariant embedding of $\text{QA}^\gamma$, hence equal in law to $(\hat{\phi}, \hat{\tau})$. Therefore by the domain Markov property in Lemma 4.6, $P_{I,\tau}$ is an invariant probability measure of the kernel $A_I$ in Proposition 2.13.

By the symmetry in Lemma 4.7, $P_{I,\tau}$ is also an invariant probability measure of the kernel $A_I$. Hence the uniqueness of invariant measure in Proposition 2.13 gives $P_{I,\tau} = \text{LF}_{I,\tau}$. More precisely, for each $I$, $\text{LF}_{I,\tau}$ is a version of the regular conditional probability of $\hat{\phi}$ given $\{\hat{\tau} = \tau\}$. We now write the assertion of Proposition 4.4 as $\text{QA}^\gamma = \int \text{LF}_I m(d\tau)$ for simplicity. Then $P_{I,\tau} = \text{LF}_{I,\tau}$ implies that restricted to $E_I$, we have $\text{QA}^\gamma = \int \text{LF}_I m_I(d\tau)$ for some measure $m_I(d\tau)$ on $(0, \infty)$. Setting $m(d\tau) = m_I(d\tau)$ on each interval $I$ we get a well-defined measure $m(d\tau)$ on $(0, \infty)$. Then $\text{QA}^\gamma = \int \text{LF}_I m(d\tau)$.

4.2. Identification of the law of the modulus by comparing the boundary lengths. By Theorem 1.6 and Proposition 4.4, the measure $m(d\tau)$ satisfies the following: for $x \in \mathbb{R}$

\begin{equation}
(4.6) \quad \int_0^\infty e^{-\pi x^2/\tau^2} m(d\tau) = \frac{2\sinh(\pi x/2)}{\pi x \Gamma(1 + ix)} \text{QA}^\gamma[L_1 e^{-L_1} L_0^{ix}].
\end{equation}

Here we abuse notion and use $L_0$ and $L_1$ to represent the quantum lengths of the two boundaries of a sample from $\text{QA}^\gamma$. We now identify the measure $m$ hence proving Theorem 1.4.

Lemma 4.8. For $\gamma \in (\sqrt{2}, 2)$, we have

\begin{equation}
(4.7) \quad \text{QA}^\gamma[L_1 e^{-L_1} L_0^{ix}] = \cos(\pi(\frac{4}{\gamma^2} - 1)) \cdot \frac{\Gamma(1 + ix)}{\cosh(\pi x)} \quad \text{for } x \in \mathbb{R}.
\end{equation}

Proof. By the definition of $\text{QA}^\gamma$, the total mass $|\text{QA}^\gamma(a,b)|$ of $\text{QA}^\gamma(a,b)$ is given by

\begin{equation}
(4.8) \quad |\text{QA}^\gamma(a,b)| = \frac{\cos(\pi(\frac{4}{\gamma^2} - 1))}{\pi} \cdot \frac{1}{\sqrt{ab(a+b)}} \quad \text{for } a > 0 \text{ and } b > 0.
\end{equation}

Therefore

\begin{equation}
\text{QA}^\gamma[L_1 e^{-L_1} L_0^{ix}] = \int_0^\infty ae^{-a b^{ix}} |\text{QA}^\gamma(a,b)| da db = \frac{\cos(\pi(\frac{4}{\gamma^2} - 1))}{\pi} \int_0^\infty \frac{ae^{-a b^{ix}}}{\sqrt{ab(a+b)}} da db.
\end{equation}

Setting $b = at$ and using $\int_0^\infty t^\delta e^{-a t} dt = \frac{\Gamma(1 + ix)}{\cosh(\pi x)}$, we get

\begin{equation}
(4.9) \quad \int_0^\infty \frac{ae^{-a b^{ix}}}{\sqrt{ab(a+b)}} da db = \int_0^\infty \frac{ae^{-a (at)^{ix}}}{a \sqrt{t(1+t)}} da dt = \int_0^\infty e^{-a t^{ix}} da \int_0^\infty \frac{t^{ix} - \frac{1}{2}}{1 + t} dt = \frac{\pi \Gamma(1 + ix)}{\cosh(\pi x)}.
\end{equation}

This gives (4.7).

Proof of Theorem 1.4. By (4.6) and Lemma 4.8 we have

\begin{equation}
(4.10) \quad \int_0^\infty e^{-\pi x^2/\tau^2} m(d\tau) = \cos(\pi(\frac{4}{\gamma^2} - 1)) \cdot \frac{2\sinh(\pi x)}{\pi \gamma x \cosh(\pi x)} \quad \text{for } x \in \mathbb{R}.
\end{equation}
By (A.3) in Appendix A, we have
\begin{equation}
\int_{0}^{\infty} \theta_{1}\left(\frac{\gamma^{2}}{\kappa}, \frac{\gamma^{2}}{4} i \tau\right) e^{-\frac{\pi \gamma^{2}}{4} x} \, d\tau = \frac{4 \sinh\left(\frac{\pi}{4} \tau\right)}{\gamma^{2} \cosh\left(\pi \tau\right)} \quad \text{for } x \in \mathbb{R}.
\end{equation}
Taking the inverse Laplace transform, we get
\begin{equation*}
M(\tau) = \cos\left(\pi\left(\frac{4}{\kappa} - 1\right)\right) \cdot \frac{\gamma^{2}}{8} \theta_{1}\left(\frac{\gamma^{2}}{8}, \frac{\gamma^{2}}{4} i \tau\right) \, d\tau \quad \text{on } (0, \infty).
\end{equation*}

5. The random moduli for CLE loops

In this section, we prove the three results on CLE$_{\kappa}$ presented in Section 1.4. We will work with \((\mathbb{H}, i)\) instead of \((\mathbb{D}, 0)\). Namely, let \(\kappa \in (8/3, 4)\). We assume that \(\Gamma\) is a CLE$_{\kappa}$ on the upper half plane \(\mathbb{H}\). Given \(\Gamma\), we let \(\eta_{j} \geq 1\) be the sequence of loops surrounding the point \(i\) ordered such that \(\eta_{j+1}\) is surrounded by \(\eta_{j}\). Moreover, let \(D_{j}\) be the Jordan domain bounded by \(\eta_{j}\), and \(A_{j}\) be the annulus bounded by \(\eta_{j}\) and \(\partial \mathbb{H}\). Finally, let \(\text{Mod}(\eta_{j})\) be the modulus of \(A_{j}\).

5.1. Random modulus of a single loop: proof of Theorem 1.7

We first prove the \(j = 1\) case of Theorem 1.7 in the case \(\kappa \in (\frac{8}{3}, 4)\).

**Proposition 5.1.** For \(\kappa \in (\frac{8}{3}, 4)\) and \(\gamma = \sqrt{\kappa}\), let \(m(d\tau) = 1_{\tau > 0} \cdot \cos\left(\pi\left(\frac{4}{\kappa} - 1\right)\right) \cdot \frac{\gamma^{2}}{2} \theta_{1}\left(\frac{\gamma^{2}}{8}, \frac{\gamma^{2}}{4} i \tau\right) \, d\tau\). Then the law of \(\text{Mod}(\eta_{1})\) has a density proportional to \(e^{\frac{2\pi}{\gamma} (\frac{4}{\kappa} - 1)^{2}} \, m(d\tau)\). Moreover,
\begin{equation}
\mathbb{E}[e^{-2\pi \lambda \text{Mod}(\eta_{1})}] = \left(\frac{4}{\pi} \right) \cdot \cos\left(\pi\left(\frac{4}{\kappa} - 1\right)\right) \cdot \frac{\gamma^{2}}{2} \theta_{1}\left(\frac{\gamma^{2}}{8}, \frac{\gamma^{2}}{4} i \tau\right) \, d\tau
\end{equation}
for \(\lambda > \frac{3\pi}{32} + \frac{2}{\kappa} - 1\).

**Proof.** For \(\tau > 0\), define \(LF_{\tau}(a, b)\) by the disintegration
\begin{equation}
LF_{\tau} = \int_{0}^{\infty} LF_{\tau}(a, b) \, dadb
\end{equation}
where \(LF_{\tau}(a, b)\) is supported on the event that the quantum boundary lengths of \(\{0\} \times [0, 1]\) and \(\{\tau\} \times [0, 1]\) are \(a\) and \(b\), respectively. Let \(C_{\gamma} \in (0, \infty)\) be a \(\gamma\)-dependent constant that can vary from line to line. Let \(f(\ell) = \ell |LF_{\tau}^{(\gamma, i)}(\ell)|\), which equals \(C_{\gamma} e^{1 - \frac{\tau}{\kappa}}\) for some \(C_{\gamma}\). By (4.4) and Theorem 1.4, for a non-negative measurable function \(\rho\) on \((0, \infty)\) we have
\begin{equation}
LF_{\tau}^{(\gamma, i)}(a) \times \text{CLE}_{\kappa}[\rho(\text{Mod}(\eta_{1}))] = \int_{0}^{\infty} \int_{0}^{\infty} \rho(\tau) |LF_{\tau}(a, b)| f(b) \, db \, d\tau.
\end{equation}
For a sample from \(LF_{\tau}^{(\gamma, i)}\) let \(L\) be the quantum boundary length of \(\partial \mathbb{H}\). Integrating over \(ae^{-\alpha}da\), we have
\begin{equation}
LF_{\tau}^{(\gamma, i)}[Le^{-L}] \times \text{CLE}_{\kappa}[\rho(\text{Mod}(\eta_{1}))] = \int_{0}^{\infty} \rho(\tau) LF_{\tau}[L_{1} e^{-L_{1}} f(L_{0})] \, d\tau.
\end{equation}
Recall from (3.4) that \(LF_{\tau}[L_{1} e^{-L_{1}} L_{0}] = \frac{\pi \sigma_{\kappa} e^{(1 + \alpha) x}}{2 \sinh(\frac{\pi}{4} \tau)} e^{-\frac{\pi \gamma^{2}}{4} x^{2}}\) for \(x \in \mathbb{R}\). This gives
\begin{equation}
\frac{LF_{\tau}[L_{1} e^{-L_{1}} f(L_{0})]}{LF_{\tau}^{(\gamma, i)}[Le^{-L}]} = C_{\gamma} e^{\frac{\pi \gamma^{2}}{8} (\frac{4}{\kappa} - 1)^{2}} = C_{\gamma} e^{\frac{2\pi}{\gamma} (\frac{4}{\kappa} - 1)^{2}}.
\end{equation}
Therefore the law of \(\text{Mod}(\eta_{1})\) is \(C_{\gamma} e^{\frac{2\pi}{\gamma} (\frac{4}{\kappa} - 1)^{2}} \, m(d\tau)\) as desired. Consequently,
\begin{equation}
\mathbb{E}[e^{-2\pi \lambda \text{Mod}(\eta_{1})}] = C_{\gamma} \int_{0}^{\infty} e^{-2\pi \lambda \tau} e^{\frac{2\pi}{\gamma} (\frac{4}{\kappa} - 1)^{2}} \, m(d\tau).
\end{equation}
By (A.2) the function \(\tau \mapsto \theta_{1}\left(\frac{\gamma^{2}}{8}, \frac{\gamma^{2}}{4} i \tau\right)\) decays as \(O(e^{-\frac{2\pi}{\gamma} \tau})\) as \(\tau \to \infty\). Therefore \(\mathbb{E}[e^{-2\pi \lambda \text{Mod}(\eta_{1})}]\) is finite and analytic in \(\lambda\) on a complex neighborhood of \((\frac{3\pi}{32} + \frac{2}{\kappa} - 1, \infty)\). By (5.4) and (A.3), for \(\lambda\) sufficiently large,
we can evaluate the integral on the right side of (5.5) to get
\[
E[e^{-2\pi i \Lambda \text{Mod}(\eta_j)}] = \frac{C_\gamma \sin\left(\frac{\pi}{2} \sqrt{\frac{2}{\kappa} - \frac{8\pi}{\kappa} \cos(\pi \sqrt{\frac{2}{\kappa} - \frac{8\pi}{\kappa}})}\right)}{\sqrt{\frac{2}{\kappa} - \frac{8\pi}{\kappa} \cos(\pi \sqrt{\frac{2}{\kappa} - \frac{8\pi}{\kappa}})}}.
\]
By the analyticity in \(\lambda\) (5.5) holds for all \(\lambda \in (\frac{3\pi}{2}, \frac{\kappa}{2} - 1, \infty)\). Finally, setting \(\lambda = 0\) we see that the constant \(C_\gamma\) in (5.4) equals \(\left(\frac{1}{\sin(x1)}\right)\), completing the proof. \(\square\)

Our derivation for \(\eta_j\) works exactly the same way for \(\eta_j\) with \(j \geq 2\). We first introduce the analog of QA\(\gamma\) following Section 4.

**Definition 5.2.** Fix \(j \geq 2\). For \(a > 0\), let \((\phi, \Gamma)\) be a sample from LF\(\gamma\)\(i\)(\(a\)) \(\times\) CLE\(\gamma\) and let \(\eta_j\) be the \(j\)-th outermost loop of \(\Gamma\) surrounding \(i\). For \(b > 0\), let QA\(\gamma\)^\((a,b)\) be the conditional law of the quantum surface \((A_j, \phi)/\sim_\gamma\) conditioning on \(L_0(\eta) = b\).

**Proposition 5.3.** Let QA\(\gamma\)^\((a)\) = QA\(\gamma\). For \(j \geq 2\), let QA\(\gamma\)^\((a,b)\) = QA\(\gamma\)^\((a)\)\(\times\)QA\(\gamma\)^\((b)\) where the total mass QA\(\gamma\)^\((a,b)\) of QA\(\gamma\)^\((a,b)\) is given by
\[
|QA\(\gamma\)^\((a,b)\)| = \int_0^\infty \ell|QA\(\gamma\)^\((a,\ell)\)||QA\(\gamma\)^\((\ell, b)\)|d\(\ell\).
\]
Then the joint law of \((A_j, h)/\sim_\gamma\) and \((D_j, \phi, 0)/\sim_\gamma\) is
\[
\int_0^\infty bQA\(\gamma\)^\((a,b)\) \times M^\text{dis}(\gamma; b)db \text{ for } j \geq 1.
\]
**Proof.** By the Markov property of CLE\(\gamma\), conditioning on \(\eta_j\), the law of the loops of inside \(D_j\) is a CLE\(\gamma\) on \(D_j\). Iteratively applying (4.4) we get (5.7). \(\square\)

We now extend Theorem 1.4 to QA\(\gamma\).

**Proposition 5.4.** Fix \(j \geq 2\). Let QA\(\gamma\)^\((a)\) = \(\int_0^\infty\) QA\(\gamma\)^\((a,b)\)dadb. Let \(m_j(\text{d}\tau)\) be the unique measure on \((0, \infty)\) such that
\[
\int_0^\infty e^{-\pi x^2 + \frac{\tau}{4}} m_j(\text{d}\tau) = \cos^2(\pi(\frac{1}{\gamma} - 1)) \cdot \frac{2 \sinh(\frac{1}{\pi} \pi x)}{\pi \gamma x \cosh^2(\pi x)} \text{ for } x \in \mathbb{R}.
\]
Sample \((\tau, \phi)\) from LF\(\tau\)\((\text{d}\phi)\)m\(j\)(\(\text{d}\tau\)). Then the law of the quantum surface \((C, \phi)/\sim_\gamma\) is QA\(\gamma\).

To prove Proposition 5.4, we first prove the analog of Lemma 4.8.

**Lemma 5.5.** For \(j \geq 2\), we have
\[
QA\(\gamma\)^\((L_1 e^{-L_1 L_{0}}) = \frac{\cos(\pi(\frac{1}{\gamma} - 1))}{\cosh(\pi x)} QA\(\gamma\)^\((L_1 e^{-L_1 L_{0}})\) \text{ for } x \in \mathbb{R}.
\]
**Proof.** By definition, QA\(\gamma\)^\((L_1 e^{-L_1 L_{0}})\) = \(\int_0^\infty\) ae\(-a\|b\|\|\ell\|\)QA\(\gamma\)^\((L_{0} - (a, \ell))\)||QA\(\gamma\)^\((b, \ell)\)||d\ell dadb, which equals
\[
\cos(\pi(\frac{1}{\gamma} - 1)) \int_0^\infty ae^{-\|b\|\|\ell\|} d\ell dadb.
\]
Setting \(b = \ell t\) and using \(\int_0^\infty \frac{\ell^t}{\sqrt{t(1 + t)}} dt = \frac{\pi}{\cosh(\pi x)}\), we get
\[
\int_0^\infty \frac{ae^{-\|b\|\|\ell\|} QA\(\gamma\)^\((L_{0} - (a, \ell))\)||d\ell dadb = \int_0^\infty \frac{ae^{-a(\ell t)^{\|\ell\|} QA\(\gamma\)^\((L_{0} - (a, \ell))\)||d\ell dadb = \frac{\pi}{\cosh(\pi x)} QA\(\gamma\)^\((L_{0})\). \(\square\)
Proof of Proposition 5.4. By the identical argument as for Proposition 4.4, there exists a measure \( m_j(d\tau) \) on \((0, \infty)\) such that if we sample \((\tau, \phi)\) from \( LF_\alpha(d\phi)m_j(d\tau)\), then the law of the quantum surface \((\mathcal{A}_\tau, \phi)/\sim_\gamma\) is \( QA_\gamma^\alpha\). Similar as for the \( j = 1 \) case in (4.6), by Theorem 1.6 we have

\[
(5.10) \quad \int_0^\infty e^{-\frac{s^2+\phi^2}{r}} m_j(d\tau) = \frac{2\sinh\left(\frac{s^2}{r}\right)}{\pi r x(1+ix)} QA_\gamma^\alpha[L_i e^{-L_1 L_0}] \quad \text{for } x \in \mathbb{R}.
\]

By Lemmas 4.8 and 5.5, we conclude the proof. \(\square\)

Proof of Theorem 1.7. First fix \( \kappa = \gamma^2 \in (\frac{8}{3}, 4) \). By the same calculation as for (5.4) we have

\[
(5.11) \quad \mathbb{E}[e^{-2\pi \lambda \text{Mod}(\eta_j)}] = C_\gamma \int_0^\infty e^{-2\pi \lambda \tau} e^{2\pi \kappa (\frac{8}{3} - 1)\tau m_j(d\tau)},
\]

for some \( \gamma \)-dependent constant \( C_\gamma \). By Proposition 5.4 and the argument for (5.4), we have

\[
(5.12) \quad \mathbb{E}[e^{-2\pi \lambda \text{Mod}(\eta_j)}] = \frac{C_\gamma \sin\left(\frac{4\pi}{\kappa}\right) \sqrt{(\frac{4}{\kappa} - 1)^2 - \frac{8\pi}{\kappa}}}{\sqrt{(\frac{4}{\kappa} - 1)^2 - \frac{8\pi}{\kappa}}} \quad \text{for } \lambda > \frac{3\kappa}{32} + \frac{2}{\kappa} - 1
\]

for some constant \( C_\gamma \). Setting \( \lambda = 0 \) we get \( C_\gamma = \frac{(\frac{8}{3} - 1) \cos(\pi(\frac{8}{3} - 1))}{\sin(\pi(\frac{8}{3} - 1))} \) as desired. This proves Theorem 1.7 for \( \kappa \in (\frac{8}{3}, 4) \). The \( \kappa = 4 \) case follows from sending \( \kappa \to 4 \). See [AS21, Appendix A] for the needed continuity in \( \kappa \). \(\square\)

Remark 5.6 (Relation to Brownian motion). For \( a \geq 0 \), let \( B^a \) be a Brownian motion starting from \( a \) and \( Y_a = \inf\{t : |B^a_t| = 1\} \) and \( T_0 = \sup\{t < Y_0 : B^a_t = 0\} \). Then as explained in, e.g., [BPY01], we have

\[
\mathbb{E}[e^{-\frac{a}{2} Y_a}] = \frac{\cosh(a\theta)}{\cosh \theta} \quad \text{and} \quad \mathbb{E}[e^{-\frac{a^2}{4} T_0}] = \frac{\tanh \theta}{\theta} \quad \text{for } \theta \in \mathbb{R}.
\]

For \( j \geq 1 \) let \( m_j(d\tau) \) be the measure such that \( \mathbb{E}[e^{-2\pi \lambda \text{Mod}(\eta_j)}] = C_\gamma \int_0^\infty e^{-2\pi \lambda \tau} e^{2\pi \kappa (\frac{8}{3} - 1)\tau m_j(d\tau)} \) as in (5.11). Let \( M_j \) is sampled from the probability measure proportional to \( m_j(d\tau) \). Then by (4.10) we have the equality \( 2\pi M_1 \overset{d}{=} \frac{\alpha^2}{4} T_0 + \frac{\alpha^2}{3} Y_a \) in law. For \( \kappa = 4 \), since \( Y_1 = 0 \) we have \( M_1 \overset{d}{=} \frac{\kappa}{2} T_1 \) as shown in [ALS20]. Since \( \mathbb{E}[e^{-\frac{\alpha^2}{4} Y_0}] = \frac{1}{\cosh \theta} \), by (5.11) we have \( 2\pi M_1 \overset{d}{=} 2\pi M_1 + \sum_{m=1}^{j-1} X_m \) where \( X_m \)'s are independent copies of \( Y_0 \) that are independent with \( M_1 \).

5.2. Joint law with the conformal radius; proof of Theorem 1.8. Let \( \text{Loop}_\kappa \) be the law of \( \eta_1 \), which is the outermost loop surrounding \( i \) of a CLE_\kappa on \( \mathbb{H} \). For a sample \( \eta \) from \( \text{Loop}_\kappa \), let \( \psi : \mathbb{H} \to D_\eta(i) \) be the unique conformal map with \( \psi(i) = i \) and \( \psi'(i) > 0 \). For \( \alpha \in \mathbb{R} \), let \( \text{Loop}_\alpha^\kappa \) be the measure defined by reweighting \( \text{Loop}_\kappa \) as follows:

\[
(5.13) \quad \frac{d\text{Loop}_\kappa^\alpha}{d\text{Loop}_\kappa}(\eta) = |\psi'_\eta(i)|^{2\Delta_\alpha - 2} \quad \text{where } \Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2}).
\]

Recall \( M_1^\text{disc}(\alpha) \) from Definition 4.1. We use the following proposition to prove Theorem 1.8.

Proposition 5.7. For \( \kappa \in (\frac{8}{3}, 4) \), \( \gamma = \sqrt{\kappa} \) and \( \alpha > 2 \), let \( (\phi, \eta) \) be sampled from \( LF_\kappa^\alpha \times \text{Loop}_\kappa^\alpha \). Let \( A_1 \) be the annulus bounded by \( \eta_1 \) and \( \partial \mathbb{H} \), and \( D_1 \) be the Jordan domain bounded by \( \eta_1 \). Then the joint law of \( (A_1, \phi)/\sim_\gamma \) and \( (D_1, \phi, 0)/\sim_\gamma \) is

\[
(5.14) \quad \int_0^\infty bQA^\gamma(a, b) \times M_1^\text{disc}(\alpha; b) db.
\]

Proof. When \( \alpha = \gamma \), this is simply the \( j = 1 \) case of Proposition 5.3. The general \( \alpha \) case can be obtained from the \( \alpha = \gamma \) case via a reweighting argument as in [ARS21, Section 4.2]. There the loop touches \( \partial \mathbb{H} \) but the exact same argument works here. \(\square\)
Proof of Theorem 1.8. Let $f_\alpha(\ell) = |M^{\text{disk}}(\alpha; \ell)|| = |L \Gamma_{\alpha}^j(\ell)| = C_{\alpha, \gamma} \tau^j(\alpha - Q)$ for some $(\alpha, \gamma)$-dependent constant $C_{\alpha, \gamma}$. By Proposition 5.7, the identity (5.3) holds with $\gamma$-insertion replaced by a generic $\alpha$. Namely
\begin{equation}
(5.15) \quad \text{Loop}_\alpha^\#[\rho(\text{Mod}(\eta_1))] = \int_0^\infty \rho(\tau)\text{Loops}_\alpha[\tau L_1 e^{-\Delta f_\alpha(L_0)}] \mu(\tau).
\end{equation}
where $m(\tau) = 1_{\tau > 0} \cdot \cos(\pi(\frac{\tau}{2} - 1)) \cdot \frac{2}{\pi} \theta_1(\frac{\tau}{2}, \frac{\tau}{2} + i \tau) \mu(\tau)$ and $\rho$ is a non-negative measurable function.

Let $E$ be the expectation for $C_{\alpha, \gamma}^\#$. Let $X = e^{2\pi \text{Mod}(\eta_1)}|\psi_\alpha^j(i)|$. Then for $\alpha > \frac{\pi}{2}$ we have
\begin{equation}
(5.16) \quad E[X(2\Delta_n - 2)^2] = \text{Loops}_\alpha[e^{2\pi \text{Mod}(\eta_1)(2\Delta_n - 2)} \rho(\text{Mod}(\eta_1))] = \int_0^\infty \rho(\tau)g(\tau, \alpha)m(\tau)
\end{equation}
where $g(\tau, \alpha) = \frac{e^{2\pi(2\Delta_n - 2)\tau} L_1 e^{-\Delta f_\alpha(L_0)} L_1}{\text{Loops}_\alpha^\#(1 - \tau)}$. Since
\begin{equation}
\text{Loops}_\alpha[\tau L_1 e^{-\Delta f_\alpha(L_0)}] = \frac{\pi \gamma \pi \Gamma(1 + ix)}{2 \sin(\frac{\pi}{\alpha} \pi x)} e^{-\frac{\pi}{\alpha}^2 \pi^2 x} \quad \text{and} \quad e^{\frac{\pi}{\alpha}^2 \pi^2 (\frac{\pi}{\alpha}^2 (\alpha - Q))} e^{2\pi(2\Delta_n - 2)\tau} = e^{\pi \tau Q^2 - 4},
\end{equation}
for a fixed $\gamma$, the function $(\alpha, \tau) \mapsto e^{2\pi(2\Delta_n - 2)\tau} \text{Loops}_\alpha[\tau L_1 e^{-\Delta f_\alpha(L_0)}]$ can be factorized as a function of $\alpha$ times a function of $\tau$. Therefore $g(\alpha, \tau)$ can be factorized as a function of $\alpha$ times a function of $\tau$ as well. Therefore $X$ is independent of $\text{Mod}(\eta_1)$. Since $X = e^{2\pi \text{Mod}(\eta_1)}|\psi_\alpha^j(i)|$, by the definition of conformal radius we obtain Theorem 1.8 with $j = 1$ and $\kappa \in (8/3, 4)$. For $j \geq 2$, the identity (5.16) holds with $m(\tau)$ replaced by $m_\tau(\tau)$ from Proposition 5.4. This gives Theorem 1.8 for $\kappa \in (8/3, 4)$. The $\kappa = 4$ case follows by sending $\kappa \uparrow 4$ as in the proof of Theorem 1.7.

5.3. Nesting statistics: proof of Theorem 1.9. By Theorem 1.7, we have for
\begin{equation}
(5.17) \quad \sum_{j \geq 1} \beta_j^{-1}E[e^{-2\pi \lambda \text{Mod}(\eta_j)}] = \left(\frac{\pi}{\alpha} - 1\right) \frac{\sqrt{\frac{\pi}{\alpha} \pi}}{2 \sin(\pi(\frac{\pi}{\alpha} - 1))} \frac{\cos(\pi(\frac{\pi}{\alpha} - 1))}{\cos(\pi(\frac{\pi}{\alpha} - 1))}. \sin(\pi(1 - \frac{\pi}{\alpha})), \sqrt{\frac{\pi}{\alpha} \pi} \left(\frac{\pi}{\alpha} - 1\right) \frac{\cos(\pi(\frac{\pi}{\alpha} - 1))}{\cos(\pi(\frac{\pi}{\alpha} - 1))}.
\end{equation}
Suppose $t = 2\pi \lambda - \frac{\sqrt{\pi}}{4} \left(\frac{\pi}{\alpha} - 1\right)^2 > 0$. Set $C_\kappa = \sqrt{\frac{\pi}{\alpha} \pi} \cos(\pi(\frac{\pi}{\alpha} - 1))$. By (5.17), we have
\begin{equation}
(5.18) \quad \sum_{j \geq 1} \beta_j^{-1}E[e^{-t \text{Mod}(\eta_j)} e^{-\frac{\pi}{\alpha} \pi^2 (\frac{\pi}{\alpha} - 1)^2 \text{Mod}(\eta_j)}] = \frac{\sinh(\frac{\pi}{\alpha} \pi \frac{4t}{\alpha} \pi)}{\sqrt{\frac{4t}{\alpha} \pi \cos(\pi(\frac{\pi}{\alpha} - 1))}}.
\end{equation}
Let $n = 2 \cos(\pi(1 - \frac{\pi}{\alpha}))$. For $n' \in [-2, 2]$, let $\beta = \frac{n'}{n}$ so that $n' / 2 = \beta \cos(\pi(1 - \frac{\pi}{\alpha}))$. Let $\chi' = -\arccos(\frac{n'}{2})$. Then for $f(\tau) = e^{-\tau} e^{-\frac{\pi}{\alpha} \pi^2 (\frac{\pi}{\alpha} - 1)^2 \tau}$, we have
\begin{equation}
\text{E} \left[ \sum_{j \geq 1} \left(\frac{n'}{n}\right)^{j-1} f(\text{Mod}(\eta_j)) \right] = \frac{C_\kappa \sinh(\sqrt{\frac{\pi}{\alpha} \pi t/4})}{\sqrt{t}(\cosh(\sqrt{4\pi t/\kappa - \cos \chi'})}. \end{equation}
Now Theorem 1.9 is immediate from the following fact on the function $Z(\tau, \kappa, \chi')$ in (1.12).

Lemma 5.8. Let $L[f(\tau)](t) = \int_0^\infty e^{-\tau t} f(\tau) d\tau$ be the Laplace transform. For $\kappa \in (\frac{\pi}{4}, 4]$ and $\chi' \in \mathbb{R}$,
\begin{equation}
(5.19) \quad L[Z(\tau, \kappa, \chi')(2i\tau)](t) = \frac{\sqrt{\pi} \sinh(\sqrt{\kappa \pi t/4})}{2t(\cosh(\sqrt{4\pi t/\kappa - \cos \chi'}))} \quad \text{for } t > 0.
\end{equation}
Proof. By the modular property $\eta(i/\tau) = \sqrt{\tau} \eta(i\tau)$ of the eta function, we have
\begin{equation}
(5.20) \quad \eta(2i\tau) = (2\tau)^{-1/2} \eta(\frac{i}{2\tau}) = (2\tau)^{-1/2} q^L \prod_{r=1}^\infty (1 - q^r) \quad \text{with } q = e^{-\tau}.\end{equation}
By the representation (1.13) of \( Z(\tau, \kappa, \chi') \), for \( g = \frac{1}{\kappa} \) we have

\[
Z(\tau, \kappa, \chi') \eta(2i\tau) = \frac{1}{\sqrt{2\pi}} \sum_{p \in \mathbb{Z}} \frac{\sin(p + 1)\chi'}{\sin \chi'} e^{\sqrt{2\pi}q(p-\frac{1+2}{2})^2}.
\]

Since \( \mathcal{L}[\frac{1}{\sqrt{\pi}} e^{-\frac{\tau}{\sqrt{2}}}](t) = \frac{\sqrt{\pi e^{-\sqrt{\pi t}}}}{\sqrt{t}} \) for each \( a > 0 \), we have \( \mathcal{L}[\frac{1}{\sqrt{\pi}} q^2(p-\frac{1+2}{2})^2](t) = \frac{\sqrt{\pi e^{-|p+1-g-1|\sqrt{\pi t}}}}{\sqrt{t}} \) hence

\[
\mathcal{L}[Z(\tau, \kappa, \chi') \eta(2i\tau)](t) = \frac{1}{\sqrt{2\pi}} \sum_{p \in \mathbb{Z}} \frac{\sin(p + 1)\chi'}{\sin \chi'} e^{-|p+1-g-1|\sqrt{\pi t}}.
\]

Shifting \( p + 1 \) and dividing the sum according to \( p > 0 \) or \( p < 0 \), we get

\[
\sum_{p \in \mathbb{Z}} \frac{\sin(p + 1)\chi'}{\sin \chi'} e^{-|p+1-g-1|\sqrt{\pi t}} = \sum_{p \geq 1} \frac{\sin(p\chi')}{\sqrt{t} \sin \chi'} (e^{-(p-g-1)\sqrt{\pi t}} - e^{-(p+g-1)\sqrt{\pi t}})
\]

\[
= \sum_{p \geq 1} \frac{2\sin(p\chi') \sinh(\sqrt{\pi t}/g)}{\sqrt{t} \sin \chi'} e^{-p\sqrt{\pi t}} = \frac{\sin(\sqrt{\pi t}/g)}{\sqrt{t} (\cosh g\pi t - \cos \chi')},
\]

where the last step uses

\[
\sum_{p \geq 1} 2\sin(p\chi') e^{-p\sqrt{\pi t}} = \frac{\sin \chi'}{\cosh g\pi t - \cos \chi'}.
\]

Now (5.19) follows from (5.22). \( \Box \)

Proof of Theorem 1.9. Comparing (5.18) and (5.19), we get

\[
\mathbb{E} \left[ \sum_{j=1}^{\infty} \left( \frac{n'}{n} \right)^{j-1} f(\text{Mod}(\eta_j)) \right] = \frac{\sqrt{2\pi} C_{\kappa}}{\sqrt{\pi}} \mathcal{L}[Z(\tau, \kappa, \chi') \eta(2i\tau)](t) \quad \text{for} \quad t > 0,
\]

where \( C_{\kappa} = \frac{2\sin(\pi(\frac{\kappa}{2} - 1))}{\pi} \) and \( f(\tau) = e^{-\tau r} e^{-\frac{\pi}{\sqrt{\pi t}}(\frac{\kappa}{2} - 1)^2 r}. \) This gives (1.14) for this choice of \( f \). Varying \( t \) we see that (1.14) holds if \( f \) is an arbitrary non-negative measurable function. \( \Box \)

6. The random modulus for the Brownian annulus

In this section we prove Theorem 1.2. In Section 6.1 we give a precise construction of the conformally embedded Brownian disk and Brownian annulus as quantum surfaces in \( \sqrt{8/3} \)-LQG, with more background on Brownian surfaces provided in Appendix B. In Section 6.2 we use the strategy in Section 4 based on the KPZ relation (Theorem 1.6) to complete the proof.

6.1. Preliminaries on Brownian surfaces. The Brownian sphere is a random metric-measure space obtained in [Le 13, Mie13] as the Gromov-Hausdorff-Prokhorov scaling limit of uniform quadrangulations, under the name Brownian map. We write \( \text{BS}_2(1)^\# \) as the law of the unit-area Brownian sphere with two marked points. For \( A, L > 0 \), the Brownian disk with area \( A \) and boundary length \( L \) is a random metric-measure space obtained in [BM17, Section 2.3] as the scaling limit of random quadrangulations with boundary. We denote its law by \( \text{BD}_{0,1}(L; A)^\# \). We use the subscripts \( \{0, 1\} \) because a sample from \( \text{BD}_{0,1}(L; A)^\# \) has no interior marked points and one boundary marked point. For \( L > 0 \), define

\[
\text{BD}_{0,1}(L)^\# = \int_0^\infty \frac{L^3}{\sqrt{2\pi A^5}} e^{-\frac{L^2}{4A}} \text{BD}_{0,1}(L; A)^\# \, dA.
\]

Then \( \text{BD}_{0,1}(L)^\# \) is the probability measure for the law of the Brownian disk with boundary length \( L \) and free area. See Theorems B.1 and B.2 in Appendix B for precise scaling limit results concerning \( \text{BS}_2(1)^\# \) and \( \text{BD}_{0,1}(L)^\# \). Both the Brownian sphere and the Brownian disk can be defined purely in the continuum using the Brownian snake as done in [Le 13, Mie13, BM17], but this construction is not explicitly needed for Sections 6 and 7. We now introduce a few variants of \( \text{BS}_2(1)^\# \) and \( \text{BD}_{0,1}(L)^\# \) which are more convenient for our purpose.
**Definition 6.1.** Given a sample of $BS_2(1)^#$, if its area measure is rescaled by some $A > 0$, we write $BS_2(A)^#$ for the law of this new metric-measure space of area $A$ with two marked points. Let

$$BS_2 = \int_0^\infty A^{-3/2}BS_2(A)^# dA.$$  

We call a sample from $BS_2$ a free Brownian sphere. For $L > 0$ and $A > 0$, let

$$BD_{0,1}(L) = L^{-\frac{1}{2}}BD_{0,1}(L)^# \quad \text{and} \quad BD_{0,1}(L; A) = \frac{L^\frac{1}{2}}{\sqrt{2\pi A^3}} e^{-\frac{L^2}{2A}}BD_{0,1}(L; A)^#.$$ 

Given a sample from $\int_0^\infty \frac{4}{\pi}BD_{0,1}(L; A) dA$, forget the marked point on the boundary and add an interior marked point according to its area measure. We denote the law of the resulting metric-measure space with one interior marked point by $BD_{1,0}(L)$, and the probability measure proportional to $BD_{1,0}(L)$ by $BD_{1,0}(L)^#$.

The following Proposition 6.2 and Lemma 6.3 allow us to find Brownian disks and annuli inside a Brownian sphere. Here whenever a subset of a metric space is viewed as a new metric space, we will use the internal metric on the subset.

**Proposition 6.2 ([LG19, Theorem 3]).** Sample $(S, x, y)$ from $BS_2$. On the event $d(x, y) > 2$ let $D_y$ be the connected component of $S \setminus B(x, 1)$ containing $y$, where $B(x, 1)$ is the metric ball of radius 1 around $x$. Conditioned on $d(x, y) > 2$ and on the boundary length $\mathcal{L}_y$ of $D_y$, the conditional law of $(D_y, y)$ is given by $BD_{1,0}(L_y)^#$ conditioned on the event that the distance between the interior marked point and the disk boundary is greater than 1.

In the setting of Proposition 6.2, we write $B^*(x, 1) = S \setminus D_y$ and called it a filled metric ball of radius 1 around $x$. We define $B^*(y, 1)$ in the same way with the role of $x$ and $y$ swapped.

**Lemma 6.3.** Sample $(S, x, y)$ from $BS_2$. Conditioned on $d(x, y) > 2$ and on the boundary lengths $\mathcal{L}_x$ of $B^*(x, 1)$ and $\mathcal{L}_y$ of $B^*(y, 1)$, the conditional law of $S \setminus (B^*(x, 1) \cup B^*(y, 1))$ is $BA(\mathcal{L}_x, \mathcal{L}_y)^#$.

**Proof.** This follows from Definition 1.1 of $BA(a, b)^#$ and Proposition 6.2. □

**Lemma 6.4.** As measures on metric-measure spaces, we have $BA(a,b)^# = BA(b,a)^#$ for $a, b > 0$.

**Proof.** For $(S, x, y)$ sampled from $BS_2$, the law of $(S, y, x)$ is also $BS_2$. Now Lemma 6.4 follows from Lemma 6.3. □

Miller and Sheffield [MS20, MS21a, MS21b] gave the $\sqrt{8/3}$-LQG description of the conformally embedded Brownian sphere and the Brownian disk. In this section we only need their disk result, which we recall now. The sphere result will be recalled as Theorem 7.1 in Section 7. Fix $\gamma = \sqrt{8/3}$. For $a > 0$, recall that the measure $LF^{(\gamma,i)}_H(a)$ is defined by the disintegration $LF^{(\gamma,i)}_H(a) = \int_0^\infty LF^{(\gamma,i)}_H(a) da$ such that samples from $LF^{(\gamma,i)}_H(a)$ have boundary length $a$. By (4.1) one has $|LF^{(\gamma,i)}_H(a)| < \infty$. Let $LF^{(\gamma,i)}_H(a)^#$ be the probability measure proportional to $LF^{(\gamma,i)}_H(a)$. Now sample $\phi$ from $LF^{(\gamma,i)}_H(a)^#$. Let $\mu_\phi = e^{\gamma\phi(x)} dz$ be the quantum area measure [DS11] defined via the Gaussian multiplicative chaos. By [DDDF20, GM21b], we can find a smooth regularization $\phi_\epsilon$ and a normalizing sequence $z_\epsilon$ such that as $\epsilon \to 0$ the Riemannian metric on $H$ induced by metric tensor $z_\epsilon^{-1}e^{4\gamma\phi_\epsilon}(dx^2 + dy^2)$ converges in probability to a metric $d_\phi$ on $H$, which is called the $\sqrt{8/3}$-LQG metric of $\phi$. Moreover, modulo a multiplicative constant, $d_\phi$ agrees with the metric constructed earlier in [MS20, MS21a] via the quantum Loewner evolution. Here the exponent $\frac{1}{4}$ is the Hausdorff dimension of a Brownian surface. The following theorem summarizes the relation between $LF^{(\gamma,i)}_H(a)^#$ and the Brownian disk.

**Theorem 6.5.** There exist constants $c_1$ and $c_2$ such that the following holds.\(^3\) Suppose $\phi$ is sampled from $LF^{(\gamma,i)}_H(a)^#$ for some $a > 0$. Let $(D, p, d, \mu)$ be the (marked) metric-measure space given by $(H, i, c_1d_\phi, c_2\mu_\phi)$.\(^3\)

\(^3\)The metric constant $c_1$ can be set to 1 by choosing the normalizing sequence in the definition of $d_\phi$ appropriately. The area constant $c_2$ is canonically defined due to the different definitions of area for Brownian surfaces and $\sqrt{8/3}$-LQG surfaces. As explained in [GHS21, Remark 3.12], $c_2 = \sqrt{3}$. 

Then $(D, p, d, \mu)$ is a Brownian disk with boundary length $a$ and an interior marked point; namely, its law is $\text{BD}_{1,0}(a)$. Moreover, the notion of boundary length for the Brownian disk agrees with the $\sqrt{8/3}$-LQG length. Finally, $(D, p, d, \mu)$ and the quantum surface $(\mathbb{H}, \phi, i)/\sim_\gamma$ are measurable with respect to each other.

**Proof.** By [ARS21, Theorem 3.4], $(\mathbb{H}, \phi, i)/\sim_\gamma$ is a quantum disk with boundary length $a$ and an interior marked point as defined in the mating-of-trees framework [DMS21]. By [MS20, MS21a] together with [LG19, Section 6], it is possible to require the two notions of boundary lengths agree and find scaling constants $c_1$ and $c_2$ such that the law of $(D, p, d, \mu)$ is $\text{BD}_{1,0}(a)^\#$. The measurability result was first proved in [MS21b] and a more constructive proof was given in [GMS20]. \hfill \Box

**Remark 6.6.** By (4.1) with $\gamma = \sqrt{8/3}$, we have

$$|\text{LF}_{H}^{\gamma,i}(a)| = Ca^{-\frac{\gamma}{2}} = Ca^{-\frac{3}{4}} = C|\text{BD}_{1,0}(a)|$$

for some constant $C > 0$. Therefore Theorem 6.5 still holds with $\text{LF}_{H}^{\gamma,i}(a)$ and $C \cdot \text{BD}_{1,0}(a)$ in place of $\text{LF}_{H}^{\gamma,i}(a)^\#$ and $\text{BD}_{1,0}(a)^\#$.

Let $d_H = c_1d_{\phi}$ and $\mu_H = c_2\mu_{\phi}$. Then $(H, d_H, \mu_H)$ is a conformal embedding of $(D, p, d, \mu)$ as discussed below Definition 1.1 of the Brownian annulus. On the event that $d(p, \partial D) > 1$, let $A$ be the (annular) connected component of $D \setminus B_d(p, 1)$ whose boundary contains $\partial D$. Let $B^*(p, 1) = D \setminus A$, which we call the filled metric ball of radius 1 around $p$. Let $B_d(i, 1) = \{x \in H : d_H(i, x) \leq 1\}$ and $A$ the annular component of $H \setminus B_d(i, 0, 1)$. Then $(A, d_H, \mu_H)$ is a conformal embedding of $(A, d, \mu)$. Recall that $\text{Mod}(A)$ is the modulus of the planar annulus $A$.

**Lemma 6.7.** The metric-measure space $(A, d, \mu)$ and quantum surface $(A, \phi)/\sim_\gamma$ are measurable with respect to each other. In particular, $\text{Mod}(A)$ is measurable with respect to $(A, d, \mu)$.

Lemma 6.7 follows from the measurability results in [MS21b, GMS20] for the Brownian sphere and disk. See Appendix B for a proof. Our next lemma specifies the law of $L$ in Definition 1.1 of $\text{BA}(a, b)^\#$. It follows from several known facts on the enumerations and scaling limits of quadrangulations. We also give its proof in Appendix B. To draw the analogy with Section 4 we introduce the function

$$(6.2) \quad |\text{Ball}_1(\ell)| = e^{-\frac{\mu_2}{2}} \quad \text{for } \ell > 0.$$ 

It plays the same role as $|\text{LF}_{H}^{\gamma,i}(\ell)|$ in the description of $L^\gamma_\gamma(\eta)$ in Theorem 4.2. In light of Proposition B.4, $|\text{Ball}_1(\ell)|$ can be thought of as the partition function of the filled metric ball with radius 1 and perimeter $\ell$.

**Lemma 6.8.** Fix $a > 0$. Let $(D, p, d, \mu)$ be a Brownian disk sampled from $\text{BD}_{1,0}(a)^\#$. On the event that $d(p, \partial D) > 1$, let $A$ be the annulus in Definition 1.1 and let $L$ be the length of $\partial A \setminus \partial D$, which is also the boundary length of the filled metric ball $B^*(p, 1)$. Then the law of $L$ conditioned on $d(p, \partial D) > 1$ is

$$(6.3) \quad \frac{1}{Z(a)} \frac{1}{\sqrt{ab(a+b)}} |\text{Ball}_1(\ell)| \, db \quad \text{where } Z(a) = \int_0^\infty \frac{a^{3/2}b|\text{Ball}_1(\ell)| \, db}{\sqrt{ab(a+b)}}.$$ 

Moreover, $\text{BD}_{1,0}(a)^\#|d(p, \partial D) > 1| = Z(a)/Z(\infty)$ where $Z(\infty) = \int_0^\infty b^{3/2} |\text{Ball}_1(\ell)| \, db$. Finally, conditioning on $B^*(p, 1)$, the conditional law of $A$ only depends on $\ell$.

We record the following Brownian sphere analog of Lemma 6.8, which we need in Section 7. See Appendix B for the proof.

**Lemma 6.9.** For $(S, x, y)$ sampled from $\text{BS}_2$, let $L_x$ and $L_y$ be the boundary lengths of $B^*(x, 1)$ and $B^*(y, 1)$, respectively. Then under the restriction of $\text{BS}_2$ to the event $d(x, y) > 2$, the joint law of $(L_x, L_y)$ has density

$$C \cdot 1_{a > 0, b > 0} \frac{\sqrt{ab}|\text{Ball}_1(a)||\text{Ball}_1(b)|}{a+b} \, da \, db \quad \text{for some constant } C \in (0, \infty).$$
6.2. Liouville field description and the law of the modulus. Let
\begin{equation}
BA(a, b) = \frac{BA(a, b)^\#}{2\sqrt{ab(a + b)}} \quad \text{and} \quad BA = \int_0^\infty BA(a, b) \, da \, db.
\end{equation}

In light of Lemma 6.7, we can view the measures $BA$ and $BA(a, b)$ as laws of quantum surfaces. The proof of Theorem 6.12 is parallel to that of Theorem 1.4. We first give the counterpart of Theorem 4.2. This is essentially a reformulation of Lemma 6.8 where we describe Brownian disks in terms of Liouville fields.

**Proposition 6.10.** Set $\gamma = \sqrt{8/3}$. Sample $\phi$ from $LF_{\mathbb{H}}^{\gamma,i}$. On the event $E = \{d_{\mathbb{H}}(i, \partial \mathbb{H}) > 1\}$, let $A_\eta$ be the annular connected component of $\mathbb{H} \setminus B_{d_{\eta}}(i, 1)$. Let $\eta$ be the loop corresponding to $\partial A_\eta \setminus \partial \mathbb{H}$ and $L_\mathbb{H}^{\gamma}(\eta)$ be its quantum length. Let $D_\eta$ be the Jordan domain bounded by $\eta$. Let $f(b) = |Ball_1(b)|$. Then under the measure $M = \frac{1}{f(L_\mathbb{H}^{\gamma}(\eta))} \cdot LF_{\mathbb{H}}^{\gamma,i}$, the law of $(A_\eta, \phi)/\sim_{\gamma}$ is $C \cdot BA$ for some constant $C > 0$. Moreover, conditioning on $(D_\eta, \phi)/\sim_{\gamma}$, the conditional law of $(A_\eta, \phi)/\sim_{\gamma}$ only depends on $L_\mathbb{H}^{\gamma}(\eta)$.

**Proof.** In this proof $C$ is a positive constant that could change from line to line. By Theorem 6.5 and Remark 6.6, under $LF_{\mathbb{H}}^{\gamma,i}$ the law of $(\mathbb{H}, \phi, i)/\sim_{\gamma}$ is $C \cdot BD_{1,0}$. Note that
\begin{equation}
\frac{a^{3/2}b|Ball_1(b)|}{\sqrt{ab(a + b)}} = C \frac{|BA(a, b)| |Ball_1(b)|}{|BD_{1,0}(a)|}.
\end{equation}

By Lemma 6.8, under $M$ the joint law of $L_{\mathbb{H}}^{\gamma}(\partial \mathbb{H})$ and $L_{\mathbb{H}}^{\gamma}(\eta)$ is $C_{1,a>0,b>0} |BA(a, b)| dadb$. By the definition of $BA(a, b)^\#$, under $M$ the law of $(A_\eta, \phi)/\sim_{\gamma}$ is $C_{1,a>0,b>0} |BA(a, b)| BA(a, b)^\# = C \cdot BA$ as desired. Moreover, the last assertion of Proposition 6.10 follows from the last assertion of Lemma 6.8.

**Lemma 6.11.** Set $\gamma = \sqrt{8/3}$. There exists a measure $m(d\tau)$ on $(0, \infty)$ such that if we sample $(\tau, \phi)$ from $LF_{\mathbb{H}}(d\phi)m(d\tau)$, then the law of the quantum surface $(C_\tau, \phi)/\sim_{\gamma} = BA$.

**Proof.** The argument is almost identical to that of Proposition 4.4 so we only point out the differences. Suppose we are in the setting of Proposition 6.10 where $M$ is the reweighted measure. Let $\hat{\tau}$ be such that $A_\eta$ is conformally equivalent to $C_{\hat{\tau}}$, and let $\psi : C_{\hat{\tau}} \rightarrow A_\eta$ be the conformal map such that $\psi(0) = 0$. Let $\hat{\phi}(z) = \frac{1}{z} \cdot \psi(z) + Q \log |\psi(z)|$ for $z \in C_{\hat{\tau}}$. Let $\hat{\phi}_{\text{har}}$ be the harmonic extension of $\hat{\phi}$ onto $C(0, \frac{2\pi^2}{3})$ with zero normal derivative on $\{0\} \times [0, 1]$. By Proposition 6.10, under $M$, the law of $(A_\eta, \hat{\phi})/\sim_{\gamma} = C \cdot BA$. Therefore, under $M$ the law of $(\hat{\phi}, \hat{\tau})$ is a rotational invariant embedding of $C \cdot BA$. Now the statement of Lemma 4.6 still holds for $(\phi, \eta)$ under $M$ with the same argument except the following modification: when we apply the domain Markov property from Lemma 2.11, we are in the second scenario of that lemma since the metric $d_{\mathbb{H}}$ is determined by the field $\phi$ locally. This means that conditioning on $\hat{\tau}$ and $\hat{\phi}_{\text{har}}$, the conditional law of $\hat{\phi} - \hat{\phi}_{\text{har}}$ is a GFF on $C(0, \frac{2\pi^2}{3})$ with the appropriate boundary condition.

Another key ingredient for the proof of Proposition 4.4 is the symmetry of $QA$ under flipping the sides, namely Lemma 4.7. By Lemma 6.4, we have $BA(a, b)^\# = BA(b, a)^\#$ which gives the desired symmetry for $BA$. The rest of the argument based on Proposition 2.13 is identical to that of Proposition 4.4.

**Theorem 6.12.** The measure $m(d\tau)$ in Lemma 6.11 equals $1_{\tau > 0} 2^{-1/2} \eta(2\pi \tau) d\tau$.

**Proof.** The argument is essentially the same as that of Theorem 1.4. By (6.4)
\begin{equation}
BA[L_1 e^{-L_1} L_{0}^{ix}] = \frac{\pi \Gamma(1+ix)}{2 \cosh(\pi x)} |BA(a, b)| dadb = \frac{1}{2} \int_0^\infty \frac{ae^{-a|\gamma x^2|}}{\sqrt{a(a+b)}} \, da \, db.
\end{equation}

By (4.9) we have $BA[L_1 e^{-L_1} L_{0}^{ix}] = \frac{\pi \Gamma(1+ix)}{2 \cosh(\pi x)}$. Now the KPZ relation (Theorem 1.6) yields that
\begin{equation}
\int_0^\infty e^{-\pi \gamma^2 x^2 \tau/4} m(d\tau) = \frac{2\sinh(\frac{\pi x}{2})}{\pi \gamma x \Gamma(1+ix)} BA[L_1 e^{-L_1} L_{0}^{ix}] = \frac{\sinh(\frac{\pi x}{2})}{\gamma x \cosh(\pi x)} \quad \text{for} \quad x \in \mathbb{R}.
\end{equation}

By (A.4) in Appendix A, we have
\begin{equation}
\int_0^\infty e^{-\pi \gamma^2 x^2 \tau} \eta(2\pi \tau) d\tau = \frac{\sqrt{3} \sinh(\frac{2\pi x}{3})}{2 \cosh(\pi x)} \quad \text{for} \quad x \in \mathbb{R}.
\end{equation}
Since $\gamma = \sqrt{8/3}$, comparing the last two equations we get $m(dr) = 1_{\tau > 0}2^{-1/2}\eta(2i\tau)dr$. \[ \square \]

**Proof of Theorems 1.2 and 1.3.** By Theorem 6.12 we get Theorem 1.3. For Theorem 1.2, note that $X_\tau$ agrees in law with $\mathcal{L}_{\gamma}^3(\partial_0C_\tau^s)\mathcal{L}_{\gamma}^3(\partial_1C_\tau^s)$ in Theorem 1.5 with $\gamma = \sqrt{8/3}$. By Lemma 2.12, when $\gamma = \sqrt{8/3}$ we have

$$
\text{LF}_{\tau}[f(\mathcal{L}_\gamma^3(\partial_0C_\tau^s))g(\mathcal{L}_\gamma^3(\partial_1C_\tau^s))] = \frac{2}{\gamma} \int_0^\infty \ell^{-1} f(\ell) g(\ell r) \rho_{\tau}(r) dr d\ell = \frac{2}{\gamma} \int_0^\infty a^{-2} f(a) g(b) \rho_{\tau}(\frac{b}{a}) db da.
$$

Therefore the joint density of the two boundary lengths $(L_0, L_1)$ under LF$_\tau$ is $\frac{2}{\gamma}a^{-2}\rho_{\tau}(\frac{b}{a}) db da$. Now by Theorem 1.3, the law of $\text{Mod}(A)$ under BA$(a,b)$ is proportional to $1_{\tau > 0} \rho_{\tau}(\frac{b}{a})\eta(2i\tau)dr$. \[ \square \]

**Remark 6.13** (BA as the limit of QA). Note that $\lim_{\gamma \to \sqrt{8/3}} 2\cos(\frac{\pi}{4} + \frac{\pi}{2\gamma}) = |\text{BA}(a,b)|$ for QA$\gamma$ from Section 4.1. In fact Theorems 1.3 and 1.4 show that the law of the modulus and the field under the C$\gamma$ embedding converge to BA$(a,b)$ at the quantum surface level, since both the law of the modulus and the field under the C$\gamma$ embedding converge.

7. The annulus partition function of the SLE$_{8/3}$ loop

In this section we prove Theorem 1.11. In Section 7.1 we recall the coupling between the SLE$_{8/3}$ loop and the conformally embedded Brownian sphere. In Section 7.2 we use this coupling and the KPZ relation from Theorem 1.6 to complete the proof.

**7.1. The SLE$_{8/3}$ loop zipper on the Brownian sphere.** We first recall the $\sqrt{8/3}$ LQG construction of Miller and Sheffield [MS20, MS21a, MS21b] for the conformally embedded Brownian sphere, and Zhan’s construction of the SLE$_{8/3}$ loop measure [Zha21]. These two constructions (i.e. Theorem 7.1 and Definition 7.2) are included mainly for concreteness. The only input we need is the conformal welding result from [AHS22] for their coupling, which we recall as Theorem 1.6.

**Theorem 7.1** ([MS20, MS21a, MS21b]). Fix $\gamma = \sqrt{8/3}$ and $Q = \frac{2}{7} + \frac{7}{7}$. Let $(B_s)_{s \geq 0}$ be a standard Brownian motion conditioned on $B_s - (Q - \gamma)s < 0$ for all $s > 0$, and let $(B_s)_{s \geq 0}$ be an independent copy of $(B_s)_{s \geq 0}$. Let

$$
Y_t = \begin{cases} 
B_t - (Q - \gamma)t & \text{if } t \geq 0 \\
B_{-t} + (Q - \gamma)t & \text{if } t < 0 
\end{cases}
$$

Let $C = \mathbb{R} \times [0, 2\pi] / \sim$ be a horizontal cylinder. Let $h^1(z) = Y_{\text{rez}}$ for each $z \in C$. Let $h^2_C$ be independent of $h^1$ and have the law of the lateral component of the Gaussian free field on $C$. Let $\hbar = h^1 + h^2_C$. Let $c \in \mathbb{R}$ be sampled from $\frac{2}{\gamma}e^{2(\gamma - Q)c}$ independently of $\hbar$ and set $\hbar = \hbar + c$. Let $(S, x, y)$ be the (marked) metric-measure space given by $(C, \hbar, h^1 - c, h^2_C)$, with the constants $c_3$ and $c_4$ from Theorem 6.5. Then there exists a constant $C \in (0, \infty)$ such that the law of $(S, x, y)$ is $C \cdot \text{BS}_2$. Moreover, $(S, x, y)$ and the quantum surface $(C, \hbar, + \infty, -\infty) / \sim$ are measurable with respect to each other.

The quantum surface $(C, \hbar, +\infty, -\infty) / \sim$ is the two-pointed quantum sphere defined in [DMS21]. For more details of its definition such as how to take the zero-probability conditioning when defining $Y$, and of the definition of the lateral component of the GFF on $C$, we refer to [DMS21, Section 4]. See also the preliminary sections of [AHS20b, AHS21, AS21]. For Zhan’s construction of the SLE$_{8/3}$ loop measure, we first recall that for $\kappa \in (0, 8)$ and two distinct points $p, q \in \mathbb{C}$, the two-sided whole plane SLE$_\kappa$ is the probability measure on pairs of curves $(\eta_1, \eta_2)$ on $\mathbb{C}$ connecting $p$ and $q$ where $\eta_1$ is a so-called whole-plane SLE$_\kappa(2)$ from $p$ to $q$, and conditioning on $\eta_1$, the curve $\eta_2$ is a chordal SLE$_\kappa$ on the complement of $\eta_1$. We can view SLE$_\kappa^{p=q}$ as a measure on oriented loops by reversing the direction of $\eta_2$. Given a loop $\eta$ sampled from SLE$_\kappa^{p=q}$, with probability 1 its $d_\kappa$-dimensional Minkowski content $\text{Cont}(\eta)$ exists [LR15], where $d_\kappa := 1 + \frac{\kappa}{8}$.

**Definition 7.2.** Fix $\kappa = \frac{8}{7}$. Let SLE$_\kappa^{p=q}$ be two-sided whole plane SLE$_\kappa$ between $p, q$. Zhan’s SLE$_{8/3}$ loop measure is the infinite measure on unmarked oriented loops defined by

$$
\mu_C(\text{d}u) = \text{Cont}(\eta)^{-2} \int_{C \times C} |p - q|^{-2(2 - d_\kappa)} \text{SLE}_\kappa^{p=q}(\text{d}u) dp dq.
$$

For $\tau > 0$, let $Z_{8/3}(\tau) = \mu_C[E_\tau]$ where $E_\tau$ is the set of non-contractible loops in $A_\tau = \{e^{-2\pi \tau} < |z| < 1\}$. 

\[ \square \]
Zhan’s construction works for all $\kappa \in (0, 8)$ and it was shown in [Zha21] that modulo a multiplicative constant, the $\kappa = \frac{8}{3}$ case as in Definition 7.2 gives the loop measure defined by Werner [Wer08]; see [Zha21, Remark 5.3]. In particular, it has the following conformal restriction property.

**Theorem 7.3 ([Wer08, Zha21]).** For each domain $\Omega \subset \mathbb{C}$, let $\mu_\Omega$ be the restriction of $\mu_C$ to loops in $\Omega$. Then for any conformal map $f : \Omega \to \Omega'$ between two domains, we have $f \circ \mu_\Omega = \mu_{\Omega'}$. Namely, for $\eta$ sampled from $\mu_\Omega$, the law of $f \circ \eta$ is $\mu_{\Omega'}$.

We now recall a conformal welding result for $\mu_C$ proved in [AHS22]. This particular variant was proved as [AS21, Proposition 2.20] based on [AHS22].

**Theorem 7.4.** Fix $\gamma = \sqrt{8/3}$. Let $\mathbb{F}_C$ be the law of the field $\phi = h \circ \log - Q \log | \cdot |$ on $\mathbb{C}$ where $h$ as in Theorem 7.1 and $\log$ is the conformal map from $\mathbb{C}$ to $\mathcal{C}$. Let $\mu_C$ be the restriction of $\mu_C$ to the set of loops separating 0 and $\infty$. Now sample $(\phi, \eta)$ from $\mathbb{F}_C \times \mu_C$. Let $D_0$ and $D_\infty$ be the two connected components of $\mathbb{C} \setminus \eta$ such that $0 \in D_0$. Then the joint law of $(D_0, \phi, 0)/\sim_\gamma$ and $(D_\infty, \phi, \infty)/\sim_\gamma$ is

$$C \int_0^\infty \ell \cdot \text{BD}_0(\ell) \times \text{BD}_0(\ell) \, d\ell \quad \text{for some constant } C \in (0, \infty).$$

Here $\text{BD}_0(\ell)$ is identified as a law of quantum surface in $\sqrt{8/3}$-LQG using Theorem 6.5.

### 7.2. The annulus partition function from the KPZ relation

The idea behind our proof of Theorem 1.11 is that if we glue together two samples of BA, this gives a sample of BA decorated by a non-contractible $\text{SLE}_{8/3}$ loop, where the law of the modulus of the annulus is reweighted by the partition function $Z_{8/3}$. To make it rigorous, we realize the Brownian annulus as a subset of the Brownian sphere and apply Theorem 7.4.

**Lemma 7.5.** Let $\phi$ be a sample from $\mathbb{F}_C$ in Theorem 7.4. Let $E$ be the event that $d(0, \infty) > 2$, where $d = c_1 \ell_\phi$ is the Brownian metric. On the event $E$, let $B^*(0, 1)$ and $B^*(\infty, 1)$ be the filled metric ball of radius 1 around 0 and $\infty$, respectively. Let $\mathcal{L}_0$ and $\mathcal{L}_\infty$ be the boundary lengths of $B^*(0, 1)$ and $B^*(\infty, 1)$, respectively. Let $A = \mathbb{C} \setminus (B^*(0, 1) \cup B^*(\infty, 1))$. Let $f(b, \ell) = b |\text{Ball}_1(b)|$ as in Proposition 6.10. Let $M$ be $\frac{1}{f(\mathcal{L}_0) / \mathcal{L}_\infty} \mathbb{F}$ restricted to $E$. Then under $M$ the law of $(A, \phi)/\sim_\gamma$ is $C \cdot \text{BA}$ for some constant $C > 0$.

**Proof.** The result follows from Lemmas 6.3 and 6.9 via the same argument as in Proposition 6.10. \(\square\)

Recall from Lemma 6.11 and Theorem 6.12 that if we sample $(\phi, \tau)$ from $\text{LF}_\tau(\phi, m(d\tau))$ with

$$m(d\tau) = 1_{\tau > 0} 2^{-1/2} \eta(2i\tau) d\tau,$$

then the law of the quantum surface $(C, \tau, \phi)/\sim_\gamma$ is $\text{BA}$.

**Lemma 7.6.** Let $(\phi, \eta)$ be a sample from $\mathbb{F}_C \times \mu_C$ in Theorem 7.4. Let $E = \{d(0, \infty) > 2\}$, $f(\ell) = \ell |\text{Ball}_1(\ell)|$, and $A = \mathbb{C} \setminus (B^*(0, 1) \cup B^*(\infty, 1))$ as in Lemma 7.5. Let $F$ be the event that $\eta \subset A$ and $\eta$ is non-contractible within $A$. Let $\hat{M}$ be the measure $\frac{1}{f(\mathcal{L}_0) / \mathcal{L}_\infty} \mathbb{F} \times \mu_C$ restricted to $E \cap F$. Then under $\hat{M}$ the law of $(A, \phi)/\sim_\gamma$ is that of $(C, \tau, \phi)/\sim_\gamma$ with $(\phi, \tau)$ sampled from $\text{CLF}_\tau(\phi) Z_{8/3}(\tau) m(d\tau)$ for some constant $C > 0$.

**Proof.** By the conformal restriction property Theorem 7.3, given $A$, the $\hat{\mu}_C$-mass of the event $F$ is $Z_{8/3}(\text{Mod}(A))$, where $\text{Mod}(A)$ is the modulus of $A$. By Lemma 7.5, we obtain the desired marginal law of $(A, \phi)/\sim_\gamma$ under $\hat{M}$ after integrating over $\eta$. \(\square\)

Similar to $|QA^2_\gamma(a, b)|$ in (5.6), for $a > 0$ and $b > 0$ we let

$$|\text{BA}_2(a, b)| = \int_0^\infty |\text{BA}(a, \ell) \ell |\text{BA}(\ell, b)| d\ell.$$

Now the following lemma together with Lemma 7.6 will allow us to apply the KPZ relation in Theorem 1.6.

**Lemma 7.7.** In the setting of Lemma 7.6, the joint law of $(\mathcal{L}_0, \mathcal{L}_\infty)$ under $\hat{M}$ has density

$$C \cdot 1_{a > 0, b > 0} |\text{BA}_2(a, b)| d\alpha d\beta \quad \text{for some constant } C \in (0, \infty).$$
Proof. By Theorem 7.4 and Lemma 6.8, under $\mathbb{F} \times \hat{\mu}_C$ restricted to $E \cap F$ the joint law of $(\mathcal{L}_0, \mathcal{L}_\infty)$ is

$$C \cdot \mathbf{1}_{a > 0, b > 0} |\text{Ball}_1(a)| |\text{Ball}_2(a, b)| |\text{Ball}_1(b)| \, da \, db$$

for some constant $C \in (0, \infty)$.

Since $f(\ell) = |\text{Ball}_1(\ell)|$, we get the desired law of $(\mathcal{L}_0, \mathcal{L}_\infty)$ under $\hat{M}$. \hfill $\Box$

Proof of Theorem 1.11. By Theorem 1.6 and Lemmas 7.6–7.7, we have

$$\int_0^\infty e^{-\frac{\sqrt{3}}{2}q^{3/2}r} Z_{8/3}(r) m(dr) = \frac{2 \sinh(\frac{\sqrt{3}}{2}q) \pi x}{\pi x 4 \Gamma(1 + i\tau)} \int_0^\infty ae^{-a^2/2} |\text{Ball}_2(a, b)| \, da \, db \quad \text{for} \ x \in \mathbb{R},$$

where $\gamma = \sqrt{8/3}$ and $m(dr) = 1_{r > 2} 2^{-1/2} \eta(2i\tau) d\tau$ is as in (7.2). By the same calculation in the proof of Lemma 5.5 we get

$$\int_0^\infty e^{-a^2/2} |\text{Ball}_2(a, b)| \, da \, db = \frac{C_1}{\cosh(\pi x)} \text{BA}[L_1 e^{-L_1} L_0^r] \quad \text{for some constant} \ C_1 > 0.$$

Recall from the proof of Theorem 6.12 that $\text{BA}[L_1 e^{-L_1} L_0^r] = \frac{\pi \Gamma(1 + i\tau)}{\pi \cosh(\pi x)}$. Therefore

$$\int_0^\infty e^{-a^2/2} \eta(2i\tau) Z_{8/3}(r) \, dr = C_2 \frac{\sinh(\frac{\sqrt{3}}{2}q \pi x)}{\pi x} \quad \text{for} \ x \in \mathbb{R} \text{ and } C_2 = \gamma^{-1} C_1.$$

Now we take the inverse Laplace transform to compute $Z_{8/3}(\tau)$. Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace transform and its inverse. Let $F(s) := \frac{\sinh(\sqrt{3}q \pi x)}{\sqrt{3} \cosh(\sqrt{3}q \pi x)}$ with $\alpha = \frac{2}{3}$. Since $\frac{1}{\cosh(\alpha \pi x)} = \sum_{n=1}^\infty (-1)^{n-1} 4n e^{-2n \pi x}$, we have

$$F(s) = \frac{e^{\alpha \sqrt{3} q \pi x} - e^{-\alpha \sqrt{3} q \pi x}}{2 \sqrt{3} \cosh(\sqrt{3}q \pi x)} = 2 \sum_{n=1}^\infty (-1)^{n-1} n \frac{e^{-(2n-\alpha) \sqrt{3} q \pi x} - e^{-(2n+\alpha) \sqrt{3} q \pi x}}{\sqrt{3} q \pi x}.$$

Since $\mathcal{L}^{-1}[e^{-a^2/2\sqrt{3} q \pi x}](t) = \frac{t^{a^2/2\sqrt{3} q \pi x}}{\Gamma(a^2/2\sqrt{3} q \pi x)}$ for $a \geq 0$, we have

$$\mathcal{L}^{-1}[F(s)](t) = 2 \sum_{n=1}^\infty (-1)^{n-1} n \frac{e^{-(2n-\alpha) \sqrt{3} q \pi x} - e^{-(2n+\alpha) \sqrt{3} q \pi x}}{\sqrt{3} q \pi x} = \frac{2}{\sqrt{3} q \pi x} \sum_{k \in \mathbb{Z}} k(-1)^k e^{-\frac{a^2}{2\sqrt{3} q \pi x}(k - \frac{1}{2})^2}.$$

By (7.5) we have $\mathcal{L}[\eta(2i\tau) Z_{8/3}(\tau)](\frac{2}{3} \pi x) = C_2 F(s)$. Therefore

$$\eta(2i\tau) Z_{8/3}(\tau) = \frac{2C_2}{3 \pi} \mathcal{L}^{-1}[F(s)](\frac{2}{3} \pi x) = \frac{4C_2}{\pi \sqrt{3}} \cdot (2\tau)^{-1/2} \sum_{k \in \mathbb{Z}} k(-1)^k e^{-\frac{3\sqrt{3}}{2\pi \tau} (k - \frac{1}{2})^2}.$$

On the other hand, recall from (5.20) that $\eta(2i\tau) = (2\tau)^{-1/2} q^{\frac{3\pi}{4}} \prod_{r=1}^\infty (1 - q^r)$ with $q = e^{-\pi / \tau}$. Therefore

$$Z_{\text{Cardy}}(q) \eta(2i\tau) = (2\tau)^{-1/2} q^{\frac{3\pi}{4}} \sum_{k \in \mathbb{Z}} k(-1)^k e^{-\frac{3\sqrt{3}}{2\pi \tau} (k - \frac{1}{2})^2} = (2\tau)^{-1/2} \sum_{k \in \mathbb{Z}} k(-1)^k e^{\frac{3\sqrt{3}}{2\pi \tau} (k - \frac{1}{2})^2}.$$

Comparing (7.6) and (7.7), we see that $Z_{8/3} = C Z_{\text{Cardy}}$ where $C = \frac{4C_2}{\pi \sqrt{3}}$. \hfill $\Box$

APPENDIX A. BACKGROUND ON SPECIAL FUNCTIONS

The Dedekind eta function with imaginary argument is given by

$$\eta(i\tau) = e^{-\frac{\pi i}{12}} \prod_{k=1}^\infty (1 - e^{-2\pi k \tau}) \quad \text{with} \ \tau > 0.$$

We use the following convention for the Jacobi theta function with imaginary argument:

$$\theta_1(x, i\tau) := -ie^{-\pi / 4} \sum_{n = -\infty}^\infty (-1)^n e^{-n(\tau + \pi i)\pi x} \, e^{(2n+1)\pi i x} \quad \text{for} \ x \in \mathbb{R} \text{ and } \tau \in (0, \infty).$$

By Euler’s pentagonal identity $\prod_{k=1}^\infty (1 - x^k) = \sum_{n \in \mathbb{Z}} (-1)^n x^{n(n-1)/2}$, we have $\theta_1(\frac{1}{3}, i\tau) = \sqrt{3}\eta(3i\tau)$. 

THE MODULI OF ANNULI IN RANDOM CONFORMAL GEOMETRY 29
By e.g. [BS02, Section 11, Appendix 2], we have the following Laplace transforms:

\[(A.3)\] \[
\int_0^\infty e^{-\alpha \theta} \delta_1(x, i\nu \tau) \, d\tau = \frac{\pi}{\eta} \frac{\sinh(2\pi \sqrt{\frac{2a}{b}})}{\cosh(\sqrt{\pi^2 \frac{b}{a}})} \quad \text{for } a > 0, b > 0 \quad \text{and} \quad x \in [-1/2, 1/2].
\]

\[(A.4)\] \[
\int_0^\infty e^{-\alpha \eta(i\nu \tau)} \, d\tau = \frac{\pi}{\eta} \frac{\sinh(2\pi \sqrt{\frac{\nu}{\eta}})}{\cosh(\sqrt{\pi \eta})} \quad \text{for } a > 0.
\]

**Appendix B. Background on Brownian surfaces**

In this appendix we provide more background on Brownian surfaces and give proofs for Lemmas 6.7–6.9. Some of these lemmas may already exist in the literature, but we cannot locate them hence include a proof for completeness. We first recall a scaling limit result for the Brownian sphere due to [Le 13, Mie13].

**Theorem B.1** ([Le 13, Mie13]). Let \( S_n \) be uniformly sampled from the set of quadrangulations of the sphere with \( n \) faces and two distinguished directed edges. View \( S_n \) as a two-pointed metric-measure space as follows. The metric is the graph distance on the vertex set rescaled by \( (\frac{\pi}{2})^{1/4}n^{-1} \). The measure is the probability measure on its vertex set where the weight of a vertex is proportional to its degree. Then \( S_n \) converges in law to the Gromov-Hausdorff-Prokhorov topology as compact metric-measure spaces with two marked points.

We now recall a scaling limit result for \( BD_{0,1}(L)\# \) proved by Gwynne and Miller [GM19]; see [AHS20a] for the triangulation case. The topology of convergence is the natural one for compact metric-measure spaces decorated with a continuous curve, which is called the Gromov-Hausdorff-Prokhorov-uniform (GHPU) topology; see [GM19, Section 1.2.3] for its definition. To state the result, for a positive integer \( a \) we let \( Q_{0,1}^{\text{disk}}(2a) \) be the set of quadrangulations with simple boundary, having boundary length \( 2a \), and having a marked directed boundary edge. For later use we also let \( Q_{1,1}^{\text{disk}}(2a) \) be the set of such quadrangulations where we further mark a directed interior edge (i.e. an edge whose endpoints are both interior vertices).

**Theorem B.2** ([GM19]). Let \( L > 0 \) and let \((a_n)\) be a sequence of integers satisfying \( \lim_{n \to \infty} \frac{2a_n}{3\pi} = L \). Let \( BD_{0,1}^{a_n}(2a)\# \) be the probability measure on \( Q_{0,1}^{\text{disk}}(2a_n) \) where each map with \( v \) vertices is assigned mass proportional to \( \rho^v \) with \( \rho = \frac{1}{2L} \). Let \( (D_n, e_n) \) be a sample from \( BD_{0,1}^{a_n}(2a)\# \) where \( e_n \) is the boundary marked edge of \( D_n \). View \( D_n \) as a metric measure space where the metric is graph distance rescaled by \( n^{-1} \), and the mass at each vertex \( v \) with degree \( \deg(v) \) is \( \frac{2}{3}n^{-4}\deg(v) \). We also view the boundary of \( D_n \) as a continuous curve starting and ending at \( e_n \). Then with respect to the GHPU topology, as \( n \to \infty \), the law of \((D_n, e_n)\) converges weakly to \( BD_{0,1}(L)\# \) defined in (6.1).

Let \( Q_{\text{ann}}(2a, 2b) \) be the set of annular quadrangulations with labelled boundaries, where the number of edges on the two boundaries are \( 2a \) and \( 2b \), respectively. The weak limit of the critical Boltzmann measure on \( Q_{\text{ann}}(a, b) \) should be \( \text{BA}(a, b) \). Although this scaling limit result is not needed for our paper, we will use the following enumeration asymptotic, which explains why we set \(|\text{BA}(a, b)|\) to be proportional to \( \frac{1}{\sqrt{ab(a+b)}} \).

**Lemma B.3.** Let \( \rho = \frac{1}{2L} \), \( \theta = \frac{1}{3\pi} \). Let \( Q_{1,1}^{\text{disk}}(2a; v) \subset Q_{1,1}^{\text{disk}}(2a) \) be the subset of maps with \( v \) interior vertices. Then

\[
\sum_{v \geq 2} \# Q_{1,1}^{\text{disk}}(2a; v) \rho^v \theta^a \sim \frac{1}{\sqrt{2\pi a}}, \quad \text{where} \sim \text{indicates the ratio tends to 1 as } a \to \infty.
\]

Likewise, let \( Q_{\text{ann}}^2(2a, 2b) \) be the set of such quadrangulations in \( Q_{\text{ann}}(2a, 2b) \) where each boundary has a marked edge. Let \( Q_{\text{ann}}(2a, 2b; v) \subset Q_{\text{ann}}(2a, 2b) \) and \( Q_{\text{ann}}^2(2a, 2b; v) \subset Q_{\text{ann}}^2(2a, 2b) \) be the subset of maps with \( v \) interior vertices, respectively. Then with \( \sim \) indicating the ratio tends to 1 as \( a, b \to \infty \), we have

\[
\sum_{v \geq 0} \# Q_{\text{ann}}(2a, 2b; v) \rho^v \theta^{a+b} \sim \frac{1}{\sqrt{\pi ab(a+b)}}, \quad \text{and} \quad \sum_{v \geq 0} \# Q_{\text{ann}}^2(2a, 2b; v) \rho^v \theta^{a+b} \sim \frac{\sqrt{ab}}{\pi(a+b)}.
\]
Proof. The key is the following enumeration from [BF18, Theorem 1.2]:

\[(B.1) \quad \# Q_{\text{ann}}^2(2a, 2b; v) = \frac{3^{v}(3(a+b) + 2v - 1)!}{v!(3(a+b) + v)!} \cdot 4ab\binom{3a}{a}\binom{3b}{b}.\]

We claim that

\[\sum_{v \geq 0} \# Q_{\text{ann}}^2(2a, 2b; v)\rho^v = \frac{4ab}{3(a+b)}\binom{3a}{a}\binom{3b}{b}, \quad \text{and} \quad \sum_{v \geq 1} \# Q_{1,1}^\text{disk}(2a; v)\rho^v = \frac{2a}{9(a+1)}\binom{3a}{a}.\]

The first identity follows from (B.1) and [GKP94, (5.68), (5.70)]. The second is obtained by identifying the interior marked edge as a boundary of length 2, then using the first identity with \(b = 1\). Stirling’s approximation then gives both asymptotics. Finally the asymptotic for \(\# Q_{\text{ann}}\) follow from that of \(\# Q_{\text{ann}}^2\) by dropping the boundary marked edges.

We now turn to the proof of Lemmas 6.8 and 6.9. We first state the asymptotic of quadrangulation metric balls in Proposition 6.4 below. For positive integers \(r, a\) we let \(Q_{\text{ball}}(r; 2a)\) be the set of quadrangulations with the disk topology with boundary length \(2a\), with a distinguished oriented edge called the root edge, such that the four vertices of every quadrangle sharing an edge with the boundary are at distances \(r-1, r, r+1, r\) from the root vertex, respectively.

**Proposition 6.4 ([CLG17]).** Let \(\rho = \frac{1}{\sqrt{2}}\) and \(\theta = \frac{1}{\sqrt{2}}\) as in Lemma 6.3. Let \(P_{\text{ball}}(n)\) be the probability measure on \(\bigcup_{a=1}^\infty Q_{\text{ball}}(n; 2a)\) where each element is assigned weight proportional to \(\rho^V(\rho/\theta)^L\), where \(V\) is the number of internal vertices and \(L\) is the number of boundary vertices. Let \(L_n\) be the number of boundary vertices of a sample from \(P_{\text{ball}}(n)\). Then \((3n^2)^{-1}L_n\) converges in law as \(n \to \infty\) to the random variable with density proportional to \(1_{\ell > 0}[\text{Ball}_1(\ell)]d\ell\), where \([\text{Ball}_1(\ell)]\) is \(e^{-\frac{\pi}{\ell^2}}\) as defined in (6.2).

**Proof.** Recall that the Uniform Infinite Planar Quadrangulation (UIPQ) is the local limit of uniform quadrangulations; see e.g. [CLG17]. Let \(P_{\text{UIPQ}}(n)\) be the law of the filled metric ball around the root at radius \(n\) on a UIPQ. Here a filled metric ball is the union of the ball itself and all its bounded complementary components. Let \(Q_{\text{disk}}^\text{disk}(2a; v)\) be the set of quadrangulations with simple boundary, \(2a\) boundary vertices and \(v\) internal vertices, and a marked boundary edge. As explained in [CLG17, Section 6.2],

\[\# Q_{0,1}^\text{disk}(2a; v) = \frac{3^{v-1}(3a)!}{v!(2a-1)!} \frac{(3a-3+2v)!}{(a-1)!}, \quad \text{and} \quad \# Q_{0,1}^\text{disk}(2a; v) \sim C^{(a)}(\rho)^{-v^5/2} \text{ as } v \to \infty\]

for \(C^{(a)}(\rho) = \frac{a^{-a-1}(3a)!}{3^{3a}a!(2a-1)!}\). Therefore, by the definition of UIPQ, the probability measure \(P_{\text{UIPQ}}(n)\) is proportional to \(\theta^{L_n/2}C^{(a)}(\ell_n/2)^{\rho_{\text{ball}}}(n)\). By the main result of [CLG17], the law of \((3n^2)^{-1}L_n\) under \(P_{\text{UIPQ}}(n)\) weakly converges to the probability measure with density proportional to \(1_{\ell > 0}e^{-\ell^2/2}\), which is the law of the boundary length of the filled metric ball of radius 1 around the root on the Brownian plane. This statement for triangulations (namely UIPT) was proved as Theorem 2 in [CLG17] and then extended to the UIPQ case in Section 6.2 there. Since \(C^{(a)}(\rho) \sim \frac{1}{8\sqrt{3\pi}}\theta^{-a}\sqrt{a}\), the law of \((3n^2)^{-1}L_n\) under \(\rho_{\text{ball}}(n)\) weakly converges to the probability measure with density proportional to \(1_{\ell > 0}e^{-\ell^2/2}\). \(1_{\ell > 0}\text{Ball}_1(\ell)]d\ell\). \(\square\)

**Proof of Lemma 6.8.** Let \(\rho = \frac{1}{\sqrt{2}}\) and \(\theta = \frac{1}{\sqrt{2}}\) as in Lemma 6.3 and Proposition 6.4. Let \(a_n = \frac{1}{2}n^2a\) and \(\text{BD}_{1,1}^n(2a_n)\) be the probability measure on \(Q_{1,1}^\text{disk}(2a_n)\) where each element is assigned weight proportional to \(\rho^V\), with \(V\) being the number of internal vertices. Let \((D_n, p_n, e_n)\) be a sample from \(\text{BD}_{1,1}^n(2a_n)\), where \(p_n\) is the interior marked edge and \(e_n\) is the boundary one. Let \(E^n\) be the event that the graph distance between \(p_n\) and \(\partial D_n\) is at least \(n\). On the event \(E^n\), let \(B_1^n(p_n)\) be the filled metric ball of \(D_n\) around \(p_n\) of radius \(n\). Here a filled metric ball is the union of the ball itself and all its bounded complementary components except the one touching the boundary. Let \(L_n\) be the number of edges on the boundary of \(B_1^n(p_n)\). Let \(M_{\text{ball}}(n; 2a_n)\) be the (non-probability) measure describing the law of \(B_1^n(p_n)\) under the restriction of \(\text{BD}_{1,1}^n(2a_n)\) to \(E^n\). Let \(Z_n = \sum_{x=1}^\infty \rho^V(\rho/\theta)^L\) where the summation is over \(\bigcup_{x=1}^\infty Q_{\text{ball}}(n; 2x)\) as in the definition of \(P_{\text{ball}}(n)\) in
Proposition B.4. Then with $Q^\alpha_{\text{ann}}(2a_n, \mathcal{L}_n; v)$ and $Q^\text{disk}(2a_n; v)$ defined in Lemma B.3, we have
\begin{equation}
M_n^\text{ball}(2a_n) = \sum_v \#Q^\alpha_{\text{ann}}(2a_n, \mathcal{L}_n; v) \rho^v \#Q^\text{disk}(2a_n; v) \rho^v Z_n \epsilon^\text{ball}(n).
\end{equation}
By definition the probability $BD^\alpha_{\text{ann}}(2a_n)^\#(E^n)$ is the total mass $|M_n^\text{ball}(2a_n)|$ of $M_n^\text{ball}(2a_n)$. By Theorem B.2, $BD^\alpha_{\text{ann}}(2a_n)^\#(E^n)$ converges to $BD_{1,1}(a)^\#[d(p, \partial D) > 1]$. Recall that $|BA(a, b)| = \frac{1}{2\sqrt{ab(a+b)}}$ and $|BD_{1,1}(a)| = a^{-1/2}$. By the asymptotics in Lemma B.3 and Proposition B.4, we have
\[
|M_n^\text{ball}(2a_n)| = C_1(1 + o_n(1))Z_n \int_0^\infty \frac{ab|BA(a, b)||\text{Ball}(1)(b)|}{|BD_{1,1}(a)|} \, db
\]
for some constant $C_1 > 0$. Therefore $Z_n$ converges to a constant $C_2 > 0$ such that
\[
BD_{1,1}(a)^\#[d(p, \partial D) > 1] = C_1 C_2 \int_0^\infty \frac{ab|BA(a, b)||\text{Ball}(1)(b)|}{|BD_{1,1}(a)|} \, db = CZ(a)
\]
for some constant $C > 0$ not depending on $a$ and $Z(a)$ as defined in Lemma 6.8. Therefore
\[
BD_{1,0}(a)^\#[d(p, \partial D) > 1] = BD_{1,0}(a)^\#[d(p, \partial D) > 1] = CZ(a).
\]
Sending $a \to \infty$ we get $C = Z(\infty)^{-1}$ as desired. Now combining (B.2), the asymptotics in Lemma B.3 and Proposition B.4, and that $Z_n \to C_2$, we see that the law of $(3n^2)^{-1} \mathcal{L}_n$ under the probability measure proportional to $M_n^\text{ball}(2a_n)$ weakly converges to the law of $\mathcal{L}$ described in Lemma 6.8. Finally, the conditional independence of the annulus $\mathcal{A}$ and the filled metric ball $\mathcal{B}(p, 1)$ is an instance of the domain Markov property of the peeling process; see e.g. [MS20, MS21a] for this property from the QLE perspective.

Proof of Lemma 6.9. We claim that under BS$_2$ restricted to $d(x, y) > 1$, the law of $\mathcal{L}_x$ has density
\[
1_{\ell > 0} |BD_{1,0}(\ell)| |\text{Ball}(1)(\ell)| d\ell.
\]
This follows from the same argument as in the proof of Lemma 6.8 with $a_n$ being 1 for all $n$ instead. We omit the detailed proof. Now Lemma 6.9 follows from Proposition 6.2 and Lemma 6.8.

We finally explain how the measurability result Lemma 6.7 follows from known results.

Proof of Lemma 6.7. Suppose $(S, x, y)$ is a Brownian sphere with two marked points conditioned on the event that $d(x, y) > 2$. Sample a point $q$ on $\partial \mathcal{B}(x, 1)$ according to the boundary length measure. We claim that the filled metric ball $(\mathcal{B}(x, 1), q)$ and the Brownian disk $(S \setminus \mathcal{B}(x, 1), y, q)$ as marked metric-measure spaces together determine $(S, x, y, q)$. To see this, from the Brownian snake construction of the Brownian map, $(S, x)$ is determined by the geodesic tree $T_x$ rooted at $x$ and its dual tree $\tilde{T}_x$ mated together as a pair of rooted planar trees. Here $\tilde{T}_x$ is a Brownian continuum random tree (CRT) and the contour function of $T_x$ is a Brownian snake indexed by the CRT; see [LP08] for more on this mating-of-trees perspective for the Brownian map. For each point $z \in S$, if $z \in \mathcal{B}(x, 1)$, then the geodesic from $z$ to $x$ must be inside $\mathcal{B}(x, 1)$. Otherwise, the geodesic from $z$ to $x$ is a concatenation of two internal geodesics, one in $S \setminus \mathcal{B}(x, 1)$ and the other in $\mathcal{B}(x, 1)$. Therefore, knowing $(\mathcal{B}(x, 1), q)$ and $(S \setminus \mathcal{B}(x, 1), q)$, we can determine how $T_x$ and $\tilde{T}_x$ are mated together to form $(S, x)$. The point $q$ is added is to fix how boundary points on $\mathcal{B}(x, 1)$ and $S \setminus \mathcal{B}(x, 1)$ are identified.

The Brownian disk $\mathcal{D}$ with an interior point $p$ can also be viewed as the mating of the geodesic tree rooted at $p$ and its dual, which is a planar forest rooted at $\partial D$; see [LG19]. On the event $d(p, \partial D) > 1$, we let $\mathcal{A}$ be the annulus bounded by $\partial \mathcal{B}(p, 1)$ and $\partial D$. Sample a point $q$ on $\partial \mathcal{B}(p, 1)$ from its length measure. Then the same argument as above shows that $(\mathcal{A}, q)$ and $(\partial \mathcal{B}(p, 1), q)$ together determine $(\mathcal{D}, p, q)$. This further determines the conformal structure of $(\mathcal{D}, p, q)$ and hence that of $(\mathcal{A}, q)$. On the other hand, let $\mathcal{L}$ be the boundary length of $(\mathcal{B}(p, 1), q)$. By the last assertion of Lemma 6.8, once we rescale the metric and measure of $(\mathcal{B}(p, 1), q)$ by $\mathcal{L}^{-1/2}$ and $\mathcal{L}^{-2}$ respectively, then the rescaled metric measure space is independent of the conformal structure of $(\mathcal{A}, q)$. Therefore the conformal structure of $(\mathcal{A}, q)$ is determined by the metric-measure structure of $(\mathcal{A}, q)$ alone. Forgetting the extra marked point $q$ we are done.
References

[AHS17] Juhan Aru, Yichao Huang, and Xin Sun. Two perspectives of the 2D unit area quantum sphere and their equivalence. Comm. Math. Phys., 356(1):261–283, 2017.

[AHS20a] Marie Albenque, Nina Holden, and Xin Sun. Scaling limit of triangulations of polygons. Electronic Journal of Probability, 25(none):1 – 43, 2020.

[AHS20b] Morris Ang, Nina Holden, and Xin Sun. Conformal welding of quantum disks. arXiv preprint arXiv:2009.08899, 2020.

[AHS21] Morris Ang, Nina Holden, and Xin Sun. Integrability of SLE via conformal welding of random surfaces. ArXiv e-prints, Apr 2021.

[AHS22] Morris Ang, Nina Holden, and Xin Sun. The SLE loop via conformal welding of quantum disks. arXiv e-prints, page arXiv:2205.05074, May 2022.

[ALS20] Juhan Aru, Titus Lupu, and Avelio Sepúlveda. Extremal distance and conformal radius of a CLE_4 loop. arXiv e-prints, page arXiv:2004.04788, April 2020.

[ARS21] Morris Ang, Guillaume Remy, and Xin Sun. FZZ formula of boundary Liouville CFT via conformal welding. arXiv e-prints, page arXiv:2104.09478, April 2021.

[AS21] Morris Ang and Xin Sun. Integrability of the conformal loop ensemble. arXiv e-prints, page arXiv:2107.01788, July 2021.

[BB03] Michel Bauer and Denis Bernard. Conformal field theories of stochastic Loewner evolutions. Commun. Math. Phys., 239:493–521, 2003.

[BCFS21] Jérémie Bettinelli, Nicolas Curien, Luis Freydes, and Avelio Sepúlveda. Nonbijective scaling limit of maps via restriction. arXiv preprint arXiv:2104.12716, 2021.

[BF18] Olivier Bernardi and Éric Fusy. Bijections for planar maps with boundaries. Journal of Combinatorial Theory, Series A, 158:176–227, 2018.

[BGRV16] Nathanaël Berestycki, Christophe Garban, Rémi Rhodes, and Vincent Vargas. KPZ formula derived from Liouville heat kernel. J. Lond. Math. Soc. (2), 94(1):186–208, 2016.

[BH19] Stephane Benoist and Clement Hongler. The scaling limit of critical Ising interfaces is CLE_3. Ann. Probab., 47(4):2049–2086, 2019.

[BHS18] Jérémie Bettinelli and Grégory Miermont. Compact Brownian surfaces I: Brownian disks. Probab. Theory Related Fields, 167(3-4):555–614, 2017.

[BMP92] Peter Bouwknegt, Jim McCarthy, and Krzysztof Pilch. BRST analysis of physical states for 2D gravity coupled to c < 1 matter. Comm. Math. Phys., 145(3):541–560, 1992.

[BB68] A. A. Belavin and V. G. Knizhnik. Algebraic geometry and the geometry of quantum strings. Phys. Lett. B, 168(3):201–206, 1986.

[BM17] Jérémie Bettinelli and Grégory Miermont. Compact Brownian surfaces I: Brownian disks. Probab. Theory Related Fields, 167(3-4):555–614, 2017.

[BMP92] Peter Bouwknegt, Jim McCarthy, and Krzysztof Pilch. BRST analysis of physical states for 2D gravity coupled to c < 1 matter. Comm. Math. Phys., 145(3):541–560, 1992.

[BP] Nathanaël Berestycki and Ellen Powell. Introduction to the Gaussian Free Field and Liouville Quantum Gravity. Available at https://homepage.univie.ac.at/nathanael.berestycki/articles.html.

[BPY01] Philippe Biane, Jim Pitman, and Marc Yor. Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions. Bull. Amer. Math. Soc. (N.S.), 38(4):435–465, 2001.

[BPZ84] A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. Nuclear Physics B, 241(2):333–380, 1984.

[BS02] Andrei N. Borodin and Paavo Salminen. handbook of Brownian motion—facts and formulae. Probability and its Applications. Birkhäuser Verlag, Basel, second edition, 2002.

[Car06] John Cardy. Crossing formulae for critical percolation in an annulus. Journal of Physics A: Mathematical and General, 35(41):L565, 2002.

[Car06] John Cardy. The o(n) model on the annulus. J. Stat. Phys., 125(1):1–21, 2006.

[CDC+08] Dmitry Chelkak, Hugo Duminil-Copin, Clément Hongler, Antti Kemppainen, and Stanislav Smirnov. Convergence of Ising interfaces to Schramm’s SLE curves. C. R. Math. Acad. Sci. Paris, 352(2):157–161, 2014.

[Cer21] Baptiste Cerclé. Unit boundary length quantum disk: a study of two different perspectives and their equivalence. ESAIM Probab. Stat., 25:433–459, 2021.

[CLG17] Nicolas Curien and Jean-François Le Gall. Scaling limits for the peeling process on random maps. Ann. Inst. Henri Poincaré Probab. Stat., 53(1):322–357, 2017.

[CN08] Federico Camia and Charles M. Newman. SLE_6 and CLE_6 from critical percolation. In Probability, geometry and integrable systems, volume 55 of Math. Sci. Res. Inst. Publ., pages 103–130. Cambridge Univ. Press, Cambridge, 2008.

[CS12] Dmitry Chelkak and Stanislav Smirnov. Universality in the 2D Ising model and conformal invariance of fermionic observables. Invent. Math., 189(3):515–580, 2012.

[DCS12] Hugo Duminil-Copin and Stanislav Smirnov. The connective constant of the honeycomb lattice equals $\sqrt{2 + \sqrt{2}}$. Ann. of Math. (2), 175(3):1653–1665, 2012.
Scott Sheffield. Gaussian free fields for mathematicians. *Probab. Theory Related Fields*, 139(3-4):521–541, 2007.

Scott Sheffield. Exploration trees and conformal loop ensembles. *Duke Math. J.*, 147(1):79–129, 2009.

Scott Sheffield. Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *Ann. Probab.*, 44(5):3474–3545, 2016.

Scott Sheffield. What is a random surface? *arXiv e-prints*, page arXiv:2203.02470, March 2022.

Stanislav Smirnov. Critical percolation in the plane: conformal invariance, Cardy’s formula, scaling limits. *C. R. Acad. Sci. Paris Sér. I Math.*, 333(3):239–244, 2001.

Oded Schramm and Scott Sheffield. A contour line of the continuum Gaussian free field. *Probab. Theory Related Fields*, 157(1-2):47–80, 2013.

Oded Schramm, Scott Sheffield, and David B. Wilson. Conformal radii for conformal loop ensembles. *Comm. Math. Phys.*, 288(1):43–53, 2009.

Scott Sheffield and Wendelin Werner. Conformal loop ensembles: the Markovian characterization and the loop-soup construction. *Ann. of Math. (2)*, 176(3):1827–1917, 2012.

Scott Sheffield, Samuel S Watson, and Hao Wu. Simple CLE in doubly connected domains. *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, 53(2):594–615, 2017.

Vincent Vargas. Lecture notes on Liouville theory and the DOZZ formula. *ArXiv e-prints*, Dec 2017.

Wendelin Werner. Random planar curves and Schramm-Loewner evolutions. In *Lectures on probability theory and statistics*, volume 1840 of *Lecture Notes in Math.*, pages 107–195. Springer, Berlin, 2004.

Wendelin Werner. The conformally invariant measure on self-avoiding loops. *J. Amer. Math. Soc.*, 21(1):137–169, 2008.

Edward Witten. Two-dimensional gravity and intersection theory on moduli space. In *Surveys in differential geometry (Cambridge, MA, 1990)*, pages 243–310. Lehigh Univ., Bethlehem, PA, 1991.

Baojun Wu. Conformal Bootstrap on the Annulus in Liouville CFT. *arXiv e-prints*, page arXiv:2203.11830, March 2022.

Dapeng Zhan. SLE loop measures. *Probab. Theory Related Fields*, 179(1-2):345–406, 2021.