Boundedness of Entanglement Entropy, and Split Property of Quantum Spin Chains

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September, 2011

Abstract: We show the boundedness of entanglement entropy for (bipartite) pure states of quantum spin chains implies split property of subsystems. As a corollary the infinite volume ground states for 1-dim spin chains with the spectral gap between the ground state energy and the rest of spectrum have the split property. We see gapless excitation exists for the spinless Fermion on $\mathbb{Z}$ if the ground state is non-trivial and translationally invariant and the $U(1)$ gauge symmetry is unbroken. Here we do not assume uniqueness of ground states for all finite volume Hamiltonians.

Keywords: quantum spin chain, spectral gap, split property, boundedness of entanglement entropy.

AMS subject classification: 82B10
1 Introduction.

In our previous article [20], we considered a relationship between split property and symmetry of translationally invariant pure states for quantum spin chains on an integer lattice $\mathbb{Z}$. The split property is a kind of statistical independence of left and right semi-infinite subsystems. More precisely, we say that a state of a quantum spin chain on an integer lattice $\mathbb{Z}$ has the split property between left and right semi-infinite subsystems if the state is quasi-equivalent to a product state of these infinite subsystems. We have shown that the split property cannot hold for translationally invariant pure states of quantum spin chains if the state is SU(2) invariant and the spin $S$ is half-odd integer. Though this phenomenon looks similar to ground state properties of antiferromagnetic Heisenberg models on the integer lattice $\mathbb{Z}$, no direct connection was established there. The principal purpose of this article is to show that presence of the spectral gap between the ground state energy and the rest of spectrum implies the split property for one-dimensional quantum spin chains. We do not assume translational invariance of infinite volume Hamiltonians and that of states but certain boundedness of the norm of local energy operators.

The key point of proof of the split property is the boundedness of entanglement entropy for bipartite lattice models. More precisely, we consider pure states of infinite volume systems and the von Neumann entropy of the restriction of states to finite systems in a infinite subsystem, say $A$. If the entropy is bounded uniformly in the size of the finite systems, we say the entanglement entropy is bounded. Higher dimensional version of boundedness of the entanglement entropy for bipartite infinite quantum systems is the area law of entanglement entropy. The area law of entanglement entropy has been studied in various context of statistical physics and quantum field theory. See [29] for a overview of the research in this field. In Section 2, we will see that pure states satisfying boundedness of entanglement entropy has the split property between two infinite subsystems. In [14], M.B. Hastings proved the boundedness of entanglement entropy for ground states with a spectral gap and his results implies split property. M.B. Hastings assumed that uniqueness of finite volume Hamiltonians in [14]. However, uniqueness condition of finite volume ground states may not be satisfied for AKLT Hamiltonians for which a pure matrix product state is a ground state. I.Affleck, T.Kennedy, E.Lieb, H.Tasaki proved that the AKLT model of [6] has a unique infinite volume ground state while the dimension of the finite volume ground state is four. Thus it is natural to expect that for any infinite pure ground state with spectral gap, the split property holds without assuming uniqueness of finite volume ground states. To prove this, we adapt the proof of the area law of entanglement entropy due to M.B. Hastings to an infinite dimensional setting suitably and for that purpose. We find that proof of the factorization lemma due to E.Hamza, S.Michalakis, B.Nachtergaele, and R.Sims in [12] is useful. The improved Lieb-Robinson bound is a crucial mathematical tool for proof of the factorization lemma. (See [13], [17], [24], [25], [26].)

As a corollary we will see that a gapless excitation is present in half-odd
integer spin SU(2) invariant quantum spin chains and in $U(1)$ symmetric spinless fermion models on $\mathbb{Z}$ provided that the ground state is non-trivial. At first sight, our result of gapless excitation in infinite systems may seem to follow from known results of [5], [30], [31]. However, the previous works is based on the assumption of uniqueness of finite volume ground states, while we assume only uniqueness of ground states in infinite systems. (Our previous result of [22] is based on stronger assumption.)

Next we describe results precisely. We employ the language of operator algebras and most of definitions and notions we use here can be found in [7] and [8]. We describe our results for quantum spin chains on $\mathbb{Z}$. Boundedness of entanglement entropy is a very restrictive condition for higher dimensional translationally invariant systems on $\mathbb{Z}^n$. We denote the $C^*$-algebra of (quasi)local observables by $\mathfrak{A}$. $\mathfrak{A}$ is the UHF $C^*$-algebra $n^\infty$ (the $C^*$-algebraic completion of the infinite tensor product of n by n matrix algebras):

$$\mathfrak{A} = \bigotimes_{\mathbb{Z}} M_n(\mathbb{C})$$

where $M_n(\mathbb{C})$ is the set of all n by n complex matrices. Each component of the tensor product is specified with a lattice site $j \in \mathbb{Z}$. $\mathfrak{A}$ is the totality of quasi-local observables. We denote by $Q^{(j)}$ the element of $\mathfrak{A}$ with $Q$ in the $j$th component of the tensor product and the identity in any other components:

$$Q^{(j)} = \cdots \otimes 1 \otimes 1 \otimes Q \otimes 1 \otimes 1 \otimes \cdots$$

For a subset $\Lambda$ of $\mathbb{Z}$, $\mathfrak{A}_\Lambda$ is defined as the $C^*$-subalgebra of $\mathfrak{A}$ generated by elements $Q^{(j)}$ with all $j$ in $\Lambda$. We set

$$\mathfrak{A}_{loc} = \bigcup_{\Lambda \subset \mathbb{Z}, |\Lambda| < \infty} \mathfrak{A}_\Lambda$$

where the cardinality of $\Lambda$ is denoted by $|\Lambda|$. We call an element of $\mathfrak{A}_{loc}$ a local observable or a strictly local observable.

By a state $\varphi$ of a quantum spin chain, we mean a normalized positive linear functional on $\mathfrak{A}$ which gives rise to the expectation value of a quantum state. When $\varphi$ is a state of $\mathfrak{A}$, the restriction of $\varphi$ to $\mathfrak{A}_\Lambda$ will be denoted by $\varphi_\Lambda$:

$$\varphi_\Lambda = \varphi|_{\mathfrak{A}_\Lambda}.$$ We set

$$\mathfrak{A}_R = \mathfrak{A}_{[1, \infty)} , \mathfrak{A}_L = \mathfrak{A}_{(-\infty, 0)} , \varphi_R = \varphi_{[1, \infty)} , \varphi_L = \varphi_{(-\infty, 0)}.$$ By $\tau_j$, we denote the automorphism of $\mathfrak{A}$ determined by

$$\tau_j(Q^{(k)}) = Q^{(j+k)}$$
for any \( j \) and \( k \) in \( \mathbb{Z} \). \( \tau_j \) is referred to as the lattice translation of \( \mathfrak{A} \).

Given a state \( \varphi \) of \( \mathfrak{A} \), we denote the GNS representation of \( \mathfrak{A} \) associated with \( \varphi \) by \( \{ \pi_\varphi(\mathfrak{A}), \Omega_\varphi, H_\varphi \} \) where \( \pi_\varphi(\cdot) \) is the representation of \( \mathfrak{A} \) on the GNS Hilbert space \( \mathcal{H}_\varphi \) and \( \Omega_\varphi \) is the GNS cyclic vector satisfying

\[
\varphi(Q) = (\Omega_\varphi, \pi_\varphi(Q) \Omega_\varphi) \quad Q \in \mathfrak{A}.
\]

Let \( \pi \) be a representation of \( \mathfrak{A} \) on a Hilbert space. The von Neumann algebra generated by \( \pi(\mathfrak{A})_\Lambda \) is denoted by \( M_\Lambda \). We set

\[
M_R = M_{(1, \infty)} = \pi(\mathfrak{A}_R)'', \quad M_L = M_{(-\infty, 0]} = \pi(\mathfrak{A}_L)''.
\]

In terms of the above definitions, we introduce the time evolution of infinite volume systems and the ground state in terms of positive linear functionals. By Interaction we mean an assignment \( \{ \Psi(X) \} \) of each finite subset \( X \) of \( \mathbb{Z} \) to a selfadjoint operator \( \Psi(X) \) in \( \mathfrak{A}_X \). We say that an interaction is of finite range if there exists a positive number \( r \) such that \( \Psi(X) = 0 \) if the diameter of \( X \) is larger than \( r \). An interaction is translationally invariant if and only if \( \tau_j(\Psi(X)) = \Psi(X + j) \) for any \( X \subset \mathbb{Z} \) and for any \( j \in \mathbb{Z} \). In what follows, we consider finite range interactions (range = \( r \)), \( \Psi(X) = 0 \) if the diameter of \( X \) is greater than \( r \). If the interaction is not necessarily translationally invariant, we assume the following condition of boundedness:

\[
\sup_{j \in \mathbb{Z}} \sum_{X \ni j} \frac{||\Psi(X)||}{|X|} < \infty, \quad (1.1)
\]

where \( |X| \) is the cardinality of \( X (\subset \mathbb{Z}) \). The infinite volume Hamiltonian \( H \) is an infinite sum of \( \{ \Psi(X) \} \),

\[
H = \sum_{X \subset \mathbb{Z}} \Psi(X).
\]

This sum does not converge in the norm topology, however the following commutator makes sense:

\[
[H, Q] = \lim_{N \to \infty} [H_N, Q] = \sum_{X \subset \mathbb{Z}} [\Psi(X), Q], \quad \lim_{N \to \infty} \frac{e^{itH_N}Qe^{-itH_N}}{N} Q \in \mathfrak{A}_{loc}
\]

where \( H_N = \sum_{X \subset [-N,N]} \Psi(X) \).

Then, the following limit exists for any real \( t \):

\[
\alpha_t(Q) = \lim_{N \to \infty} e^{itH_N}Qe^{-itH_N}
\]

for any element \( Q \) of \( \mathfrak{A} \) in the \( C^* \) norm topology. We call \( \alpha_t(Q) \) the time evolution of \( Q \). It is known that \( \alpha_t(Q) \) as a function of \( t \) has an extension to an entire analytic function \( \alpha_z(Q) \) for any \( Q \in \mathfrak{A}_{loc} \).
Definition 1.1 Suppose the time evolution $\alpha_t(Q)$ associated with an interaction satisfying (1.1) is given. Let $\varphi$ be a state of $\mathfrak{A}$. $\varphi$ is a ground state of $H$ if and only if
\[ \varphi(Q^*[H,Q]) = \frac{1}{i} \frac{d}{dt} \varphi(Q^*\alpha_t(Q)) \geq 0 \] (1.2)
for any $Q$ in $\mathfrak{A}_{\text{loc}}$.

Suppose that $\varphi$ is a ground state for $\alpha_t$. In the GNS representation of $\{\pi_{\varphi}(\mathfrak{A}), \Omega_{\varphi}, \mathfrak{H}_{\varphi}\}$, there exists a positive selfadjoint operator $H_{\varphi} \geq 0$ such that
\[ e^{itH_{\varphi}}\pi_{\varphi}(Q)e^{-itH_{\varphi}} = \pi_{\varphi}(\alpha_t(Q)), \quad e^{itH_{\varphi}}\Omega_{\varphi} = \Omega_{\varphi} \]
for any $Q$ in $\mathfrak{A}$. Roughly speaking, the operator $H_{\varphi}$ is the effective Hamiltonian on the physical Hilbert space $\mathfrak{H}_{\varphi}$ obtained after regularization via subtraction of the vacuum energy.

The spectral gap of an infinite system is that of $H_{\varphi}$. Note that, in principle, a different choice of a ground state gives rise to a different spectrum.

Definition 1.2 We say that $H_{\varphi}$ has a spectral gap if $0$ is a non-degenerate eigenvalue of $H_{\varphi}$ and for a positive $M > 0$, $H_{\varphi}$ has no spectrum in $(0, M)$, i.e. $\text{Spec}(H_{\varphi}) \cap (0, M) = \emptyset$.

It is easy to see that $H_{\varphi}$ has a spectral gap if and only if there exists a positive constant $M$ such that
\[ \varphi(Q^*[H,Q]) \geq M(\varphi(Q^*Q) - |\varphi(Q)|^2). \] (1.3)

Now we state our results on split property.

Definition 1.3 Let $\varphi$ be a state of $\mathfrak{A}$. We say the split property is valid for $\mathfrak{A}_L$ and $\mathfrak{A}_R$ if $\varphi$ is quasi-equivalent to $\psi_L \otimes \psi_R$ where $\psi_L$ is a state of $\mathfrak{A}_L$ and $\psi_R$ is that of $\mathfrak{A}_R$.

Definition 1.4 Let $\varphi$ be a state of $\mathfrak{A}$ and $\rho_N$ be the density matrix of $\varphi_{[-N,N]}$. We consider the entropy $s(\varphi_{[-N,N]}) = -tr_N(\rho_N \ln \rho_N) = -\varphi(\ln \rho_N)$ where the trace $tr$ is normalized as $tr(1) = n^{2N+1}$. We say the boundedness of entanglement entropy holds for $\varphi$ if $s(\varphi_{[M,N]})$ is bounded in $N$, $s(\varphi_{[M,N]}) \leq C$ for any $N$ and $M$ with $M < N$.

Theorem 1.5 Let $\varphi$ be a state of $\mathfrak{A}$ for which the area law of entanglement entropy holds. Then the split property is valid for $\mathfrak{A}_L$ and $\mathfrak{A}_R$.

Corollary 1.6 Let $H$ be a finite range Hamiltonian satisfying the boundedness condition (1.1) and let $\varphi$ be a ground state of $H$ with spectral gap (1.3). Then the split property is valid for $\mathfrak{A}_L$ and $\mathfrak{A}_R$. 

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We combine the above results and those of [20]. We consider half-odd integer spin $SU(2)$ symmetry of quantum spin chains and a $U(1)$ symmetry of spinless Fermion. At this stage we assume translational invariance of Hamiltonians and their ground states.

Let $u(g)$ be the spin $S$ representation of $SU(2)$ and $\gamma_g$ be the infinite product type action $SU(2)$ on $A$ associated with $u(g)$.

\[(\cdots u(g) \otimes u(g) \otimes \cdots)Q(\cdots u(g) \otimes u(g) \otimes \cdots)^{-1} = \gamma_g(Q), \ Q \in A\]

**Theorem 1.7** Consider the quantum spin chain on $\mathbb{Z}$ and the spin at each site is a half-odd integer. Let $H_S$ be a translationally invariant, $SU(2)$ gauge invariant finite range Hamiltonian. Suppose that $\varphi$ is a translationally invariant pure ground state of $H_S$. Assume that $\varphi$ is $SU(2)$ invariant ($\gamma_g$ invariant for any $g$ in $SU(2)$). Then, there exists gapless excitation in the sense that $\text{Spec}(H_\varphi) \cap (0, M) \neq \emptyset$ for any positive $M$.

Next we consider fermions on an integer lattice $\mathbb{Z}$. Due to anti-commutativity we impose parity invariance for states, otherwise the split property cannot be defined. Let $c^*_j$ and $c_j$ be the creation annihilation operators satisfying the standard canonical anti-commutation relations:

\[
\{c_i, c_j\} = 0, \ \{c_i^*, c_j^*\} = 0, \ \{c_i, c_j^*\} = \delta_{ij}1 \quad i, j \in \mathbb{Z}
\]

By $A_F$, we denoted the $C^*$-algebra generated by $c_i^*$ and $c_j$. $A_F$ is referred to as the CAR algebra. The sub-algebras $A_{loc}^F$, $A_\Lambda^F$, $A_L^F$, $A_R^F$ of $A_F$ are defined as before. $\Theta$, $\gamma_\theta^F$, and $\tau_k^F$ are automorphisms of the algebra $A_F$ determined by

\[
\Theta(c_i) = -c_i, \ \Theta(c_i^*) = -c_i^*, \ \gamma_\theta^F(c_i^*) = e^{i\theta}c_i^*, \ \gamma_\theta^F(c_i) = e^{-i\theta}c_i,
\]

\[
\tau_k^F(c_i) = c_{i+k}, \ \tau_k^F(c_i^*) = c_{i+k}^*
\]

$\gamma_\theta^F$ (resp. $\tau_k^F$) is referred to as the $U(1)$ gauge transformation (resp. translation). $\Theta$ will be called parity.

Suppose that $\varphi$ is a $\Theta$ invariant state of $A_F$. A product state $\varphi_\Lambda \otimes \varphi_\Lambda^c$ of $A_F$ specified with

\[
\varphi_\Lambda \otimes_F \varphi_\Lambda^c(Q_1Q_2) = \varphi_\Lambda(Q_1)\varphi_\Lambda^c(Q_2) \quad (Q_1 \in \varphi_\Lambda, \ Q_2 \in \varphi_\Lambda^c)
\]

can be introduced. The split property for fermion systems may be defined as quasi-equivalence of states $\varphi$ and $\varphi_\Lambda \otimes_F \varphi_\Lambda^c$. However for our purpose, the following is convenient.

**Theorem 1.8** Let $\varphi$ be a $\Theta$ invariant pure state of $A_F$ for which the area law of entanglement entropy holds. Then $\pi_\varphi(A_{loc}^F)''$ and $\pi_\varphi(A_L^F)''$ are type I von Neumann algebras.
We consider Hamiltonians of fermion systems satisfying
\[ H^F = \sum_{j=-\infty}^{\infty} h_j \]
\[ h_j \in A^F_{[j-r,j+r]}, \quad \Theta(h_j) = h_j, \quad ||h_j|| \leq C \quad (1.4) \]

**Corollary 1.9** Let \( H^F \) be a finite range Hamiltonian satisfying the boundedness condition (1.1) and let \( \varphi \) be a ground state of \( H^F \) with spectral gap (1.3) . Then \( \pi_{\varphi}(A^F_L)'' \) and \( \pi_{\varphi}(A^F_R)'' \) are type I von Neumann algebras.

By the standard Fock state we mean the state \( \psi_F \) specified by the identity \( \psi_F(c_j^*c_j) = 0 \) for any \( j \) and the standard anti-Fock state is the state \( \psi_{AF} \) specified by the identity \( \psi_{AF}(c_j^*c_j) = 0 \) for any \( j \).

**Theorem 1.10** Consider the spinless Fermion lattice system on \( \mathbb{Z} \). Let \( H_F \) be a translationally invariant , \( U(1) \) gauge invariant finite range Hamiltonian. Suppose that \( \varphi \) is a \( U(1) \) gauge invariant , translationally invariant pure ground state of \( H_F \) and that \( \varphi \neq \psi_F, \varphi \neq \psi_{AF} \). Then, gapless excitation exists between the ground state energy and the rest of the spectrum of the effective Hamiltonian .

Another application of split property is the distillation of infinitely many copies of the maximally entangled pairs in quantum information theory. This was discussed in [15]. We also point out that if the Haag duality holds, Theorem 1.10 and 1.7 can be shown in a different way. The proof of duality in [16] contains a mistake and we are not able to show the duality in the general case at the moment.

In Section 2, we present our proof of split property assuming boundedness of entanglement entropy and as an application, we simplify our previous proof that any Frustration Free ground state is a matrix product state in Section 3. In Section 4, we will see that the Hastings’ factorization lemma implies boundedness of entanglement entropy in infinite dimensional systems. In Section 5, we consider fermionic systems.

## 2 Split Property and Entanglement Entropy

In this section we show that the area law of entanglement entropy implies split property.

First let us recall basic facts of split property or split inclusion of von Neumann algebras. Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be a commuting pair of factors acting on a Hilbert space \( \mathcal{H} \), \( \mathcal{M}_1 \subset \mathcal{M}_2' \). We say the inclusion is split if there exists an intermediate type I factor \( \mathcal{N} \) such that
\[ \mathcal{M}_1 \subset \mathcal{N} \subset \mathcal{M}_2' \subset \mathcal{B}(\mathcal{H}) \quad (2.1) \]
The split inclusion is used for analysis of local QFT and of von Neumann algebras and some general feature of this concept is investigated for abstract von Neumann algebras. by J.von Neumann and later by S.Doplicher and R.Longo in [9]. R.Longo used this notion of splitting for his solution to the factorial Stone-Weierstrass conjecture in [18].

If (2.1) is valid, the inclusion of the type I factors \( N = B(H_1) \subset B(H) \) tells us factorization of the underlying Hilbert spaces and we obtain \( H = H_1 \otimes H_2 \). If (2.2) is valid, there exists a normal conditional expectation (partial states) from the von Neumann algebra \( M_1 \lor M_2 \) generated by \( M_1 \) and \( M_2 \). When \( M_2 \) and \( M_1 \) generate \( B(H) \), the split property of the inclusion \( M_1 \subset M_2 \) is nothing but the condition that \( M_1 \) and hence \( M_2 \) are type I von Neumann algebras due to the relation \( B(H) = M_1 \otimes M_2 \). In the present case, we set \( M_1 = M_\Lambda = \pi_\phi(A_\Lambda)'' \), and \( M_2 = M_\Lambda^c = \pi_\phi(A_\Lambda^c)'' \). By definition, a state \( \varphi \) of \( A \) satisfies the split property if and only if the following inclusion is split: \( M_1 \subset M_2 \). Now we proceed to proof of Theorem 1.5.

The state \( \varphi \) we consider is pure, and if \( \Lambda \) is a finite set of \( \mathbb{Z} \), there exists the tensor splitting of Hilbert spaces;

\[
N_\varphi = H_\Lambda \otimes H_{\Lambda^c}
\]

where the dimension of \( H_\Lambda \) is \( n^{|\Lambda|} \). In this splitting, any unit vector \( \Omega \) can be written as

\[
\Omega = \sum_{j=1}^{l} \sqrt{\lambda_j} \xi_j \otimes \eta_j
\]

where \( 0 < \lambda_{j+1} \leq \lambda_j \leq \cdots \leq \lambda_1 \leq 1 \), \( \sum_{j=1}^{l} \lambda_j = 1 \) and the orthogonality conditions hold:

\[
(\xi_j, \xi_i) = \delta_{ij}, \quad (\eta_j, \eta_i) = \delta_{ij}.
\]

Let \( \varphi \) be a pure state of \( A \) satisfying boundedness of entanglement entropy and \( \Omega_\varphi \) be the GNS cyclic vector associated with \( \varphi \). This factorization (2.4) is referred to as Schmidt decomposition.

We set \( \Lambda = [1, N] \) in (2.3) and (2.4) is now

\[
\Omega_\varphi = \sum_{j=1}^{l(N)} \sqrt{\lambda_j^{(N)}} \xi_j^{(N)} \otimes \eta_j^{(N)}.
\]

Then, in terms of \( \lambda_j^{(N)} \), the entropy of \( s(\varphi[1,N]) \) is given by

\[
s(\varphi[1,N]) = -\sum_j \lambda_j^{(N)} \ln \lambda_j^{(N)}.
\]
Lemma 2.1 We set $S = \sup_N s(\varphi)$. Let $k$ be the integer determined by the following conditions:

\[ \sum_{j=k+1}^{l(N)} \lambda_j^{(N)} < \varepsilon, \quad \sum_{j=k}^{l(N)} \lambda_j^{(N)} \geq \varepsilon. \] (2.6)

Then, the following inequalities are valid:

\[ k \leq \exp\left(\frac{S}{\varepsilon}\right), \quad \exp\left(-\frac{S}{\varepsilon}\right) \leq \lambda_1. \] (2.7)

Proof: We abbreviate $\lambda_j^{(N)}$ and $l(N)$ to $\lambda_j$ and to $l$. As $-\ln \lambda_j \leq -\ln \lambda_j + m$ for $m > 0$, we have

\[ -\varepsilon \ln \lambda_k \leq \sum_{j=k}^{l} -\lambda_j \ln \lambda_k \leq S. \]

Thus $\exp\left(-\frac{S}{\varepsilon}\right) \leq \lambda_k$. On the other hand, $k \lambda_k \leq \sum_{j=1}^{k} \lambda_j \leq 1$. As a consequence, we obtain $k \leq \exp\left(\frac{S}{\varepsilon}\right)$.

Lemma 2.2 Let $\psi_j, [1, N]$ be a state of $A_R$ which is an extension of the vector state of $\xi_j([1, N])$ and let $\varphi_j, [1, N]$ be a state of $A_L$ which is an extension of the vector state of $\eta_j^{(N)}$. We can take a sub-sequence $N(m)$ of natural numbers such that we obtain the following (weak*) convergence for $j = 1, 2, \ldots$:

\[ \psi_{R,j} = \lim_m \psi_{j, [1, N(m)]}, \quad \varphi_{L,j} = \lim_m \varphi_{j, [1, N(m)]}, \quad \lambda_j = \lim_m \lambda_j. \]

If $\lambda_j \neq 0$, $\psi_{R,j}$ is quasi-equivalent to $\varphi_R$ and $\varphi_{L,j}$ is quasi-equivalent to $\varphi_L$.

Proof: By definition, $\varphi_R(Q) = \sum_j \lambda_j \psi_{j, [1, N]}(Q)$ for $Q \in A_{[1, M]}$ if $0 < M < N$. In particular $\varphi_R \geq \lambda_j \psi_{j, [1, N]}$ if these states are restricted on $A_{[1, M]}$. Then, we take the weak* limit $N \to \infty$ and we obtain $\lambda_j \psi_{R,j} \leq \varphi_R$ on $A_R$. As the GNS representation associated with $\varphi_R$ is factor, $\psi_{R,j}$ is quasi-equivalent to $\varphi_R$. The same remark is valid for $\psi_{L,j}$ is quasi-equivalent to $\varphi_L$.

Note that $\frac{\lambda_j}{\varepsilon} \leq \lambda_1$

Proof of Theorem 1.5

We show that $\varphi$ is quasi-equivalent to $\varphi_L \otimes \varphi_R$. Because of Lemma 2.2 it suffices to show that $\varphi$ is quasi-equivalent to $\varphi_{L,1} \otimes \varphi_{R,1}$. We fix a small $\varepsilon$ and $k$ as in Lemma 2.1 and set

\[ \hat{\Omega}(N) = \sum_{j=1}^{k} \sqrt{\lambda_j^{(N)}} \xi_j^{(N)} \otimes \eta_j^{(N)}, \quad \Omega(N) = \frac{\hat{\Omega}(N)}{||\hat{\Omega}(N)||}. \] (2.8)
Then,

\[ 0 < 1 - ||\hat{\Omega}(N)||^2 < \epsilon, \quad 1 - ||\hat{\Omega}(N)|| < \frac{1}{1 + ||\hat{\Omega}(N)||} \epsilon, \quad ||\hat{\Omega}(N) - \Omega_\varphi||^2 < \epsilon. \]

and

\[ ||\Omega(N) - \Omega_\varphi||^2 = \left( \frac{1}{||\Omega(N)||^2} - 1 \right) \left( \sum_{j=k+1}^{k(N)} \lambda_j(N) \right) + \left( \sum_{j=k}^{k(N)} \lambda_j(N) \right) \]

\[ \leq \frac{\epsilon}{1 - \epsilon} + \epsilon < 3\epsilon. \quad (2.9) \]

Let \( \omega_N \) be the vector state associated with \( \Omega(N) \), and let \( \Omega(\infty) \) be any accumulation point of \( \omega_N \) in the weak* topology of the state space when we take \( N \) to \( \infty \). Due to (2.9),

\[ ||\omega_N - \varphi|| \leq 2\sqrt{3}\epsilon, \quad ||\omega_\infty - \varphi|| \leq 2\sqrt{3}\epsilon \]

which shows that if \( \omega_\infty \) is a factor state, \( \omega_\infty \) and \( \varphi \) are quasi-equivalent. On the other hand, by Schwartz inequality, we obtain

\[ \omega_N \leq k \sum_{j=1}^{k_0} \lambda_j \psi_{j,1,N} \otimes \varphi_{j,1,N(m)}, \omega_\infty \leq k \sum_{j=1}^{k_0} \lambda_j \psi_{L,j} \otimes \varphi_{R,j} = C\tilde{\varphi} \quad (2.10) \]

where \( k_0 \) is the number of \( \lambda_j \) which does not vanish. \( C \) is defined by \( C = k \sum_{j=1}^{k_0} \lambda_j \) and \( \tilde{\varphi} \) is the state of \( A \) determined by (2.10). Due to Lemma 2.2, \( \psi_{L,j} \otimes \varphi_{R,j} \) are quasi-equivalent to \( \varphi_L \otimes \varphi_R \) and hence \( \tilde{\varphi} \) is quasi-equivalent to \( \varphi_L \otimes \varphi_R \). As a consequence \( \tilde{\varphi} \) is a factor state. The GNS representation associated with \( \omega_\infty \) is a subrepresentation of that of \( \tilde{\varphi} \) due to (2.10). It turns out that \( \omega_\infty \) is a factor state quasi-equivalent to \( \varphi_L \otimes \varphi_R \), which implies split property of \( \varphi \). End of Proof of Theorem 1.5

**Remark 2.3** In Theorem 1.5 we assumed that boundedness of entanglement entropy for our \( R \) system. For pure states without translational invariance, boundedness of entanglement entropy for the \( L \) system may not follows from that of the \( R \) system. A simplest counter example is a pure product states \( \varphi = \varphi_L \otimes \varphi_R \) with \( \lim_{N \to \infty} s(\varphi|_{-N,-1}) = \infty \). In particular, boundedness of the entanglement entropy for our \( R \) system is not a necessary condition for split property of \( \varphi \). On the other hand for states with translational invariance, boundedness Theorem 1.5 can be extended for factor states with an argument similar to that of Lemma 2 of [2]. Though the proof is very easy we state it as proposition.

**Proposition 2.4** Let \( \varphi \) be a translationally invariant factor state of a quantum spin chain \( A \) on an integer lattice \( Z \) and let \( s \) be the mean entropy of \( \varphi \). Assume that there exists a constant \( C \) satisfying

\[ |s(\varphi|_{0,n-1}) - ns| \leq D \quad (2.11) \]
for any \( n > 0 \). Then, \( \varphi \) and \( \varphi_L \otimes \varphi_R \) are quasi-equivalent.

*Proof:* We use monotonicity of the relative entropy of a full matrix algebra, say \( \mathcal{A} \). Let \( \rho_1 \) and \( \rho_2 \) be density matrices of states \( \varphi_1 \) and \( \varphi_2 \) and let \( s(\varphi_1, \varphi_2) \) be the relative entropy defined by \( s(\varphi_1, \varphi_2) = tr(\rho_1 \ln \rho_1 - \ln \rho_2) \) where we assume that the support of \( \rho_2 \) is smaller than \( \rho_1 \). For any projection \( E \) in \( \mathcal{A} \), due to the monotonicity of \( s(\varphi_1, \varphi_2) \),

\[
\varphi_1(E) \ln \frac{\varphi_1(E)}{\varphi_2(E)} + \varphi_1(1-E) \ln \frac{\varphi_1(1-E)}{\varphi_2(1-E)} \leq s(\varphi_1, \varphi_2) \tag{2.12}
\]

Now we consider a state \( \varphi \) of \( \mathcal{A} \) satisfying the assumption of Proposition 2.1 and set \( \varphi_1 = \varphi_{[-n,n-1]} \) and \( \varphi_2 = \varphi_{[-n,-1]} \otimes \varphi_{[0,n-1]} \). By assumption,

\[
0 \leq s(\varphi_1, \varphi_2) = -s(\varphi_{[-n,n-1]}) + 2s(\varphi_{[0,n-1]}) \leq 3D, \tag{2.13}
\]

If \( \varphi \) and \( \varphi_L \otimes \varphi_R \) are not quasi-equivalent, there exists a projection \( E_\epsilon \) for a sufficient large \( n \) such that \( E_\epsilon \) is localized in \([-n,n-1]\), and

\[
1 - \epsilon \leq \varphi(E) \leq 1, \quad 0 \leq \varphi_L \otimes \varphi_R(E) \leq \epsilon.
\]

Then, due to (2.12), the left-hand side of (2.13). Hence the split property holds.

*End of Proof of Proposition 2.4*

### 3 Frustration Free Ground States

In quantum spin chains, pure states with split property is a generalization of matrix product states (= finitely correlated states =VBS states) . (c.f. [6], [10], [11]) Any matrix product state is a frustration free ground state for a Hamiltonian. More precisely, let \( \varphi \) be a translationally invariant matrix product state. There exists \( h \in \mathfrak{A}_{[0,r]} \) with the following properties:

\[
h = h^* \geq 0, \quad \varphi(h) = \varphi(\tau_j(h)) = 0.
\]

Set

\[
H_{[n,m]} = \sum_{j=n}^{m-r} \tau_j(h) \in \mathfrak{A}_{[n,m]}.
\]

Then, \( \varphi \) is a ground state of \( H_{[n,m]} \) for any \( n, m \) and the dimension of ground states of finite volume ground Hamiltonians \( H_{[n,m]} \) in \( \mathfrak{A}_{[n,m]} \) is finite, bounded uniformly in \( n \) and \( m \) if \( m-n > c_0 \),

\[
1 \leq \dim \ker H_{[n,m]} \leq C
\]

In this section, we consider pure states \( \psi \) of \( \mathfrak{A}_R \) satisfying

\[
\psi(\tau_j(h)) = 0 \tag{3.2}
\]
for any \( j \geq 0 \) or states \( \psi \) of \( \mathfrak{A}_{[n,m]} \) satisfying
\[
\psi(H_{[n,m]}) = 0 \tag{3.3}
\]

The infinite volume ground state \( \varphi \) satisfying the condition (3.2) is called a frustration free ground state. The frustration free ground state was called the zero energy state in our previous paper (c.f. 19) but it seems that the word ‘frustration free ground state’ is frequently used nowadays. In 19 we have shown any frustration free ground state is a matrix product state. We present here a simplified proof of the result in 19.

First we introduce matrix product states. Let \( K \) be a \( n \)-dimensional Hilbert space. Suppose that \( V \) is an isometry from \( K \) to \( C^d \otimes K \). Consider \( E(Q) \) is the linear map from \( M_d(C) \otimes M_n(C) \) to \( M_n(C) \) determined by
\[
E(Q) = V^* Q V \quad \text{for any } Q \text{ in } M_d(C) \otimes M_n(C). \tag{3.4}
\]
Define
\[
E_Q(R) = E(Q \otimes R).
\]
for \( Q \) in \( M_d(C) \) and \( R \) in \( M_n(C) \). As \( V \) is an isometry, the linear map \( E \) and \( E_1 \) defined above is unital (= unit preserving \( E(1) = 1 \), \( E_1(1) = 1 \) CP map .

Suppose that \( \psi \) is a faithful state of \( M_n(C) \) satisfying the invariance condition below:
\[
\psi(R) = \psi(E_1(R)) \tag{3.5}
\]
where \( R \) is any element of \( M_n(C) \). By these data, we can construct a translationally invariant state \( \varphi \) of the UHF algebra \( \mathfrak{A} \) via the following formula:
\[
\varphi(Q_0^{(j)} Q_1^{(j+1)} Q_2^{(j+2)} \ldots Q_L^{(j+L)}) = \psi(E_{Q_0} \circ E_{Q_1} \circ E_{Q_2} \circ \ldots \circ E_{Q_L}(1)). \tag{3.6}
\]
The state \( \varphi \) constructed in this way is called a matrix product state.

**Proposition 3.1.** Suppose that the condition (3.1) is valid. Let \( \varphi \) be a translationally invariant pure ground state. Then the state \( \varphi \) is a matrix product state.

We prove Proposition 3.1 now. Let \( \rho_{[0,N]} \) be the density matrix of the state \( \varphi_{[0,N]} \). As \( \varphi_{[0,N]}(H_{[0,N]}) = 0 \) the rank of \( \rho_{[0,N]} \) is smaller than \( C \) due to the condition (3.1). This implies the boundedness of the entanglement entropy, \( s(\rho_{[0,N]}) \leq \ln C \). As a result, \( \varphi_R \) gives rise to a type I factor representation of \( \mathfrak{A}_R \). Let \( \{\pi_0(\mathfrak{A}_R), \mathfrak{S}_0\} \) be the irreducible representation of \( \mathfrak{A}_R \) quasi-equivalent to the GNS representation associated with \( \varphi_R \). There exists the density matrix \( \rho_R \) for \( \varphi_R \):
\[
\text{tr}_{\mathfrak{S}_0}(\rho_R Q) = \varphi(Q) \quad Q \in \mathfrak{A}_R
\]
where \( \text{tr}_{\mathfrak{S}_0} \) is the trace of \( \mathfrak{S}_0 \).
We claim that the rank of $\rho_R$ is less than or equal to $C$. Suppose that $\rho_R = \sum_j \mu_j p_j$ where $\mu_j$ is an eigenvalue of $\rho_R$ satisfying
$$0 < \mu_{j+1} \leq \mu_j, \quad \sum_{j=1}^C \mu_j = 1,$$
and $\{p_j\}$ are mutually orthogonal rank one projections associated with eigenvectors $\xi_j$. Let $\psi_j$ be the pure vector state of $A_R$ associated with the vector $\xi_j$. $\psi_j$ also satisfies (3.2) due to the inequality $\mu_j \psi_j \leq \varphi_R$. $\psi_j$ restricted to $A_{[0,N]}$ is a stress free ground state satisfying (3.3). For each $N > 0$ there exists a factorization,
$$\psi_j = \sum_k \lambda_k(N,j) \eta_{[0,N]}(k,j) \otimes \eta_{[N,\infty]}(k,j)$$
where $\eta_{[0,N]}(k,j)$ are mutually orthogonal unit vectors of the spin chain on $[0,N]$ and $\eta_{[N,\infty]}(k,j)$ are those on $[N,\infty)$ and $\lambda_k(N,j)$ is a positive number satisfying
$$0 \leq \lambda_{k+1}(N,j) \leq \lambda_k(N,j), \quad \sum_k \lambda_k(N,j) = 1 \quad (3.7)$$
We have
$$\psi_j(Q) = \sum_{k=1}^C \lambda_k(N,j) \eta_{[0,N]}(k,j) \otimes \eta_{[N,\infty]}(k,j) \otimes \eta_{[0,N]}(k,j) \otimes \eta_{[N,\infty]}(k,j) Q \in A_{[0,N]}, \quad (3.8)$$
which shows that vector states associated with $\eta_{[0,N]}(k,j)$ and $\eta_{[N,\infty]}(k,j)$ are stress free ground states as well. It turns out that the number of the summand in (3.7) cannot exceed the dimension of finite volume stress free ground states, and $1 \leq \lambda_1(N,j)$.

**Lemma 3.2** Any weak* accumulation point of the vector state associated with $\eta_{[0,N]}(1,j)$ (when $N \to \infty$) is $\psi_j$.

**Proof:** Let $\pi_j$ be any weak* accumulation point. Due to (3.8) we have $\frac{1}{C} \pi_j \leq \psi_j$. As $\psi_j$ is a pure state of $A_R$ we conclude that $\pi_j = \psi_j$ End of Proof.

The following lemma shows that $\eta_{[0,M]}(1,i)$ are asymptotically orthogonal.

**Lemma 3.3** For any $\epsilon$, there exists $N$ such that for any $M$ with $M \geq N$
$$|(\eta_{[0,M]}(1,i), \eta_{[0,M]}(1,j))| \leq \epsilon \quad (3.9)$$
if $i \neq j \leq C$.

**Proof:** As $p_j$ is in $M_R$, there exists a projection $E_j \in \pi_0(A_{[0,N(1)])}$ such that
$$0 \leq 1 - \psi_j(E_j) < \epsilon, \quad 0 \leq \psi_i(E_j) \leq \epsilon \quad (i \neq j).$$

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We have $N(2)$ such that the following is valid for any $M > N(2)$:

$$0 \leq (\eta_{[0,M]}(1,j), (1 - E_j)\eta_{[0,M]}(1,j)) \epsilon, \quad 0 \leq (\eta_{[0,M]}(1,i), A_j\eta_{[0,M]}(1,i)) < \epsilon$$

for $i \neq j \leq C$. Then,

$$|\langle\eta_{[0,M]}(1,i), \eta_{[0,M]}(1,j)\rangle| \leq |\langle\eta_{[0,M]}(1,i), E_j\eta_{[0,M]}(1,j)\rangle| + \sqrt{\epsilon}$$

$$= (E_j\eta_{[0,M]}(1,i), \eta_{[0,M]}(1,j)) + \sqrt{\epsilon} \leq (\eta_{[0,M]}(1,i), E_j\eta_{[0,M]}(1,i))^{1/2} + \sqrt{\epsilon} \leq 2\sqrt{\epsilon}.$$ 

As the above $\epsilon$ is arbitrary, we obtain $N$ satisfying (3.9). \textit{End of Proof.}

**Lemma 3.4** Suppose that $\{x_1, \cdots, x_L\}$ are unit vectors in a Hilbert space and assume that

$$|\langle x_i, x_j \rangle| < \epsilon \quad \text{for} \quad i \neq j.$$  \hspace{1cm} (3.10)

If $0 < \epsilon < \frac{1}{2}$, $\{x_1, \cdots, x_L\}$ are linearly independent.

**Proof:** We consider complex numbers $c_j$ satisfying $\sum_{j=1}^L c_j x_j = 0$. This equation is written in a matrix form:

$$(1 + B)c = 0, \quad B_{ij} = \langle x_i, x_j \rangle, \quad c = \begin{pmatrix} c_1 \\ \vdots \\ c_L \end{pmatrix}$$

where $B_{ij}$ is the $(i,j)$ component of the hermitian matrix $B$. Due to the condition (3.10) the operator norm of $B$ is less than $(n-1)\epsilon$ and $1 + B$ is strictly positive matrix. Hence, $c = 0$. \textit{End of Proof.}

Lemma 3.3 and Lemma 3.4 tell us that $\{\eta_{[0,M]}(1,i)\}$ are linearly independent and the number of these vectors $\eta_{[0,M]}(1,i)$ cannot exceed the dimension of stress free ground states of $H_{[0,M]}$. As a consequence, the rank of the density matrix $\rho_R$ is finite.

The rest of proof of Proposition 3.1 is easy. As the state $\varphi_R$ is of type I the GNS representation gives rise to a shift of $B(\eta_0)$ associated with the lattice translation $\tau_1$. (c.f. [27]) There exists a representation of the Cuntz algebra $O_d$ with standard generators $S_j$ which implements the shift $\tau_1$:

$$\sum_{j=1}^d S_j \pi_0(Q) S_j^* = \pi_0(\tau_1(Q)) \quad Q \in \mathfrak{A}_R$$

Let $P$ be the support projection of $\varphi_R$ for $\mathfrak{M}_R$. The range of $P$ (in $\mathfrak{M}_0$) is finite dimensional and set $\mathcal{K} = P\mathfrak{M}_0 \quad V_j = S_j^* P = PS_j^* P$ and let $\psi$ be the restriction of $\varphi$ to $\mathcal{B}(\mathcal{K}) = PB(\mathfrak{M}_0)P$. $V$ is an isometry from $\mathcal{K}$ to $C^d \otimes \mathcal{K}$ determined by $V x = (PS_1^* P x, \cdots PS_d^* P x \cdots PS_d^* P)$. With these staffs, it is straight forward to see that $\varphi$ is the matrix product state associated with $\{V, \mathcal{K}, \psi\}$. 

4 Factorization Lemma of M.Hastings

In [14] M.Hastings proved boundedness of entangled entropy for gapped ground states. What M.Hastings proved was estimates of entropy uniformly in sizes of finite volume ground states, which is not exactly same as what we need for split property. We explain here a minor technical difference. The proof below is essentially due to M.Hastings.

Let $H$ be a finite range Hamiltonian with the boundedness condition (1.1) and $\alpha_t$ be the associated time evolution. Suppose $\varphi$ is a ground state of $H$ satisfying the gap condition (1.3). On $H\varphi$ there exists a positive self-adjoint operator $H_\varphi$ satisfying $e^{itH_\varphi}\pi_{\varphi}(Q)e^{-itH_\varphi} = \pi_{\varphi}(\alpha_t(Q))$ and $H_\varphi\Omega_\varphi = 0$. We set $s_n = \sup\{s(\varphi_{[a,j]}| 0 \leq j \leq n)\}$ and our aim is to show $\lim_n s_n < \infty$.

Let $P_0$ be the rank one projection $|\Omega_\varphi><\Omega_\varphi|$ to the ground state vector $\Omega_\varphi$. The following lemma is referred to as Hastings’ Factorization Lemma

Lemma 4.1 Suppose $H_\varphi$ has a spectral gap (specified in (1.3)) For any $n$ and $l(n/8)$ there exist positive constants $C_1, C_2, |||O_B(n,l)O_R(n,l)O_L(n,l) - P_0||| \leq C_1 \exp(-C_2l) \equiv \epsilon(l)$.

(4.1)

where $O_L(n,l), O_R(n,l)$ are projections and $O_B(n,l)$ is a positive selfadjoint operator satisfying

$O_R(n,l) \in \pi_{\varphi}(\mathfrak{A}_{[0,n-1]}), \quad O_L(n,l) \in \pi_{\varphi}(\mathfrak{A}_{[0,n-1]}'), \quad O_B(n,l) \in \pi_{\varphi}(\mathfrak{A}_{(-l,l)\cup(n-l,n+l)}')$, \quad \quad (4.2)

$0 \leq 0 \leq O_B(n,l) \leq 1$ \quad (4.3)

(4.4)

Due to (4.1) $|||O_B(n,l)O_R(n,l)O_L(n,l)||| \leq 2\epsilon(l)$ so by changing constants we may assume

$|||O_B(n,l), O_R(n,l)O_L(n,l)||| \leq \epsilon(l)$. \quad (4.5)

By (4.1), (4.3) and (4.4),

$1 - \epsilon(l) \leq \varphi(O_R(n,l)O_L(n,l)), \quad 1 - \epsilon(l) \leq \varphi(O_B(n,l))$. \quad (4.6)

Boundedness of entanglement entropy follows from Hastings’ factorization lemma. Detail of construction of operators $O_R(n,l), O_L(n,l)$ and $O_B(n,l)$ is not used in the next step of proof. Here we explain an itinerary from Hastings’ Factorization Lemma to boundedness of entanglement entropy.

Proposition 4.2 Suppose that there exist projections $O_R(n,l), O_L(n,l)$ and $O_B(n,l)$ satisfying (4.1), (4.2), (4.3) and (4.4). Then, the entanglement entropy is bounded: $\sup_n s(\varphi_{[1,n]} < \infty$
We set \([0, n] = (-\infty, -1] \cup [n + 1, \infty)\), \(\varphi_{R,n} = \varphi[0,n]\) and \(\varphi_{L,n} = \varphi[0,n]^c\). The density matrix of \(\varphi_{R,n}\) (resp. \(\varphi_{L,n}\)) will be denoted by \(\rho_{R,n}\) (resp. \(\rho_{L,n}\)). The Schmidt decomposition (2.4) shows that the entanglement entropy and the rank of \(\rho_{L,n}\) are equal to those of \(\rho_{R,n}\).

**Lemma 4.3** We define \(p\) via the following equation:

\[
p = (\Omega \varphi, \rho_{L,n} \otimes \rho_{R,n} \Omega \varphi) = \varphi(\rho_{L,n} \otimes \rho_{R,n})
\]

(4.7)

where by abuse of notation we use \(\varphi\) for the normal extension of \(\varphi\) to \(\mathcal{M} = \pi \varphi(\mathcal{A})\). Then,

\[
s(\varphi[0,n]) \leq C_2 \ln(2C_1^2/p) \ln 4d + F
\]

(4.8)

where \(F = (C_2 + 4) \ln 4d + 1 + \ln(d^8 - 1) + \ln(C_2/2 + 1)\).

To show Lemma 4.3 we use the following min-max principle. This should be known, though, as we are not aware of any suitable reference, we include its proof here.

**Lemma 4.4** Let \(\rho\) be a hermitian matrix acting on a \(N\) dimensional space and let \(\rho_k\) be the eigenvalue of \(\rho\) satisfying \(\rho_1 \geq \rho_2 \cdots \rho_k \geq \rho_{k+1} \cdots \geq \rho_N\). Set

\[
\mu_k = \sup \{ \text{tr}(\rho E) \mid E^* = E = E^2, \text{tr}(E) = k \},
\]

i.e. the supremum is taken among projections \(E\) with rank \(k\). Then,

\[
\mu_k = \sum_{i=1}^{k} \rho_i
\]

Proof of Lemma 4.4.

Let \(V_k\) be a \(k\) dimensional subspace. There exists a vector \(\xi \in V_k\) such that \((\rho \xi, \xi) \leq \rho_k\). This is because the \(N - k + 1\) dimensional subspace \(\mathcal{S}\) spanned by eigenvectors with eigenvalues \(\rho_k, \rho_{k+1} \cdots \rho_N\) has non-trivial intersection with \(V_k\). \((V_k \cap \mathcal{S} = \{0\})\) implies that the dimension of the total vector space is \(N + 1\). Now we show our claim by induction of the dimension \(k\). By definition, \(\mu_k \geq \sum_{i=1}^{k} \rho_i\) and we assume that \(\mu_{k-1} = \sum_{i=1}^{k-1} \rho_i\). Let \(E_k\) be a rank \(k\) projection and take a unit vector \(\xi\) in the range of \(E_k\) such that \((\rho \xi, \xi) \leq \rho_k\). For the projection \(F\) to the orthogonal complement of \(\xi\) in the range of \(E_k\), we have \(\text{tr}(\rho F) \leq \sum_{i=1}^{k-1} \rho_i\) and as a consequence, we obtain

\[
\text{tr}(\rho E_k) = \text{tr}(\rho F) + (\rho \xi, \xi) \leq \sum_{i=1}^{k} \rho_i.
\]

End of Proof of Lemma 4.4.
Let $\xi$ be a vector in a tensor product of Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$ and $\{\Psi_k\}$ (resp. $\{\Phi_l\}$) be a CONS of $\mathcal{H}_1$ (resp. $\mathcal{H}_2$). Then $\xi$ can be written as

$$\xi = \sum_{k} c_{kl} \Psi_k \otimes \Phi_l.$$ 

We say $\xi$ has the Schmidt rank $K$ if the rank of the matrix $C$ with entries $c_{kl}$ is $K$. The Schmidt rank of $\xi$ can be determined independent of choice of CONS of $\mathcal{H}_1$ and $\mathcal{H}_2$. For a vector $\xi$ with the Schmidt rank $K$ the Schmidt decomposition is equivalent to the existence of CONS $\{\Psi_k\}$ of $\mathcal{H}_1$ and $\{\Phi_k\}$ of $\mathcal{H}_2$ such that

$$\xi = \sum_{k=1}^{K} c_k \Psi_k \otimes \Phi_k, \quad c_k \geq 0, \quad \sum_{k=1}^{K} c_k^2 = ||\xi||.$$ 

We say that a density matrix $\rho$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ has the Schmidt rank at most $K$ if the Schmidt rank of any eigenvector of $\rho$ is less than or equal to $K$.

**Proof of Lemma 4.3:**

Set $\rho(n, l) = \rho_{L,n} \otimes \rho_{R,n}$ and

$$\tilde{\rho}(n, l) = O_B(n, l)O_R(n, l)O_L(n, l)\rho(n, l)O_R(n, l)O_L(n, l)O_B(n, l).$$

Then, for the norm $||A||_\varphi = ||AP_0||_\varphi = \varphi(A^*A)^{1/2}$, we obtain

$$||\rho(n, l)1/2||_\varphi \leq ||\rho(n, l)1/2 O_R(n, l)O_L(n, l)O_B(n, l)||_\varphi + ||B||_\varphi$$

where $B = \rho(n, l)1/2\{P_0 - O_R(n, l)O_L(n, l)O_B(n, l)\}$. As $||B|| \leq \epsilon$ we have

$$\sqrt{p} - \epsilon \leq \varphi(\tilde{\rho}(n, l))^{1/2}. \quad (4.9)$$

Now we claim

$$1 - 2\frac{\epsilon^2(l)}{p} \leq \frac{\varphi(\tilde{\rho}(n, l))}{tr((\tilde{\rho}(n, l)))}. \quad (4.10)$$

If $\epsilon(l) \geq \sqrt{p}$, the left-hand side of (4.10) is negative. We may assume $0 \leq \sqrt{p} - \epsilon(l)$. Then,

$$1 - \frac{\varphi(\tilde{\rho}(n, l))}{tr((1 - P_0)\tilde{\rho}(n, l))} = \frac{tr((1 - P_0)(\tilde{\rho}(n, l)))}{\varphi(\tilde{\rho}(n, l)) + tr((1 - P_0)(\tilde{\rho}(n, l)))}
\leq \frac{\epsilon^2(l)}{\epsilon^2(l) + \varphi(\tilde{\rho}(n, l))}
\leq \frac{\epsilon^2(l)}{\epsilon^2(l) + (\sqrt{p} - \epsilon(l))^2 = \frac{2(\epsilon(l) - 1/2\sqrt{p})^2 + 1/2 \cdot p}{1/2 \cdot p}. \quad (4.11)$$

Next we consider the Schmidt decomposion of the ground state vector $\Omega_\varphi$ for $\Lambda = [0, n - 1]$ in (2.3)

$$\Omega_\varphi = \sum_{j=1}^{l} \sqrt{\lambda_j} \xi_j \otimes \eta_j, \quad 0 < \lambda_{j+1} \leq \lambda_j \leq \cdots \leq \lambda_1 \leq 1, \quad \sum_{j=1}^{l} \lambda_j = 1.$$
where \( \{ \xi_j \} \) is an orthogonal system of \( \mathfrak{H}[0,n-1] \) and \( \{ \eta_j \} \) is that of \( \mathfrak{H}(-\infty,-1) \cup [n,\infty) \).

We claim that
\[
\sum_{d^{8l-4}+1 \leq j} \lambda_j \leq \frac{2e^2(l)}{p}.
\] (4.12)

Consider the density matrix \( \rho \) defined by
\[
\rho = \tilde{\rho}(n,l) = \sum_{j=1}^{\mu_j} \mu_j |x_j><x_j|,
\]
where \( x_j \) is an eigenvector for the eigenvalue \( \mu_j \) and \( \mu_{j+1} \leq \mu_j \). As the Schmidt rank of \( \rho_1 \otimes \rho_2 \) is one, and as \( O_B(n,l) \) is in the \( d^{8l-4} \) dimensional space \( \mathfrak{H}(-l,l) \cup (n-l,n+l) \), the Schmidt rank of \( x_j \) is at most \( d^{8l-4} \). Set \( M = 8l-4 \).

We may express \( x_j \) in a linear combination of \( \xi_j \otimes \eta_j \) as follows:
\[
x_j = \sum_{kl} c_{kl}^j \xi_k \otimes \eta_l, \quad \sum_{kl} |c_{kl}^j(j)|^2 = 1.
\]

Then,
\[
(\Omega_\varphi, \rho \Omega_\varphi) = \sum_j \mu_j \sum_k \sqrt{\lambda_k} c_{kk}(j)^2.
\] (4.13)

Let \( \Lambda \) and \( C(j) \) be matrices with entries defined by
\[
\Lambda_{kl} = \delta_{kl} \lambda_k, \quad C_{kl}(j) = c_{kl}^j(j).
\]
\( \Lambda \) is a non-negative matrix with \( tr(\Lambda) = 1 \) and the rank of \( C(j) \) is at most \( d^M \) and \( tr((C(j))^* C(j)) = 1 \). By the support projection \( E(j) \) of \( C(j) \) we mean the minimal projection satisfying \( E(j)C(j) = C(j) \), and (4.13) is written as
\[
(\Omega_\varphi, \rho \Omega_\varphi) \leq \sum_j \mu_j |tr(\Lambda^{1/2} E_j C(j))|^2
\]
\[
\leq \sum_j \mu_j tr(\Lambda^{1/2} E_j \Lambda^{1/2}) tr((C(j))^* C(j)) = \sum_j \mu_j tr(\Lambda E_j)
\]

As the rank of \( E(j) \) is at most \( d^M \) Lemma [4.4] implies \( tr(\Lambda E_j) \leq \sum_{k=1}^{d^M} \lambda_k \).

Thus we have
\[
(\Omega_\varphi, \rho \Omega_\varphi) \leq \sum_{k=1}^{d^M} \lambda_k
\]
which shows (4.12).

Next we give the estimate of the entropy (4.8). We use
\[
\sum_{j=1}^{K-1} -x_j \ln x_j \leq \left( \sum_{j=1}^{K-1} x_j \right) \ln K - \left( \sum_{j=1}^{K-1} x_j \right) \left\{ \ln \left( \sum_{j=1}^{K-1} x_j \right) \right\}
\] (4.14)
for any non-increasing sequence of positive numbers \( x_j \). Assuming the conditions (i) \( 0 \leq a_{j+1} \leq a_j \leq \cdots \leq a_1 \leq 1 \) and (ii) \( \sum_{j \geq k} x_j \leq a_k \) for all \( k = 1, 2, \ldots \), we have the following bound:

\[
\sum_{j=1}^{\infty} -x_j \ln x_j \leq \sum_{k=1}^{\infty} -(a_k - a_{k+1}) \ln(a_k - a_{k+1}).
\]  (4.15)

Let \( m' \) be the smallest integer satisfying
\[
2e^2(m'/p) = 2C_1 \exp(-2C_2m')/p < 1.
\]

If \( m' < m \),
\[
\sum_{d^{m-4+1} \leq j} \lambda_j \leq \exp[-2C_2(m - m')].
\]

Due to (4.14) and (4.15), we obtain the following inequalities:

\[
\sum_{j=1}^{d^{m'-1}} -\lambda_j \ln \lambda_j < 8m' \ln d.
\]  (4.16)

\[
\sum_{d^{m-4+1} \leq j} \lambda_j \leq (1 - \exp[-2C_2]) \exp[-2C_2(m - m')](8m - 4) \ln(D^8 - 1)
\]

\[
- (1 - \exp[-2C_2]) \exp[-2C_2(m - m')](\ln(1 - \exp[-2C_2]) - 2C_2(m - m')).
\]  (4.17)

These estimates imply (4.18). End of Proof of Lemma 4.3

Proof of Proposition 4.2

We fix a large number \( S \) and suppose that \( s(\varphi_{[j,i]}) > S \) for \( i \). For any \( k \) satisfying \( k < i \),

\[
s(\varphi_{[j,k]}) \leq s(\varphi_{[j,i]}) + s(\varphi_{[k+1,i]}) \leq s(\varphi_{[j,i]}) + (i - k) \ln d.
\]

Setting \( l_0 = S/(3 \ln d) \), we have \( \frac{2}{3} S \leq s(\varphi_{[j,k]}) \) for \( k \) with \( i - l_0 \leq k \leq i \). Thus, if the entanglement entropy is not bounded, for any large \( S_{\text{cut}} \) there exists \( i \)

\[
S_{\text{cut}} \leq s(\varphi_{[-k,i+k]})
\]  (4.18)

where \( l_0 = S_{\text{cut}}/(2 \ln d) \) and \( 0 \leq k \leq l_0 \).

Due to Lemma 4.3

\[
p \leq 2C^2 \exp[-(S_{\text{cut}} - F)/(C_2 \ln 4d)]
\]  (4.19)

Set \( x = \varphi_{L,i} \otimes \varphi_{R,i}(O_B(i,l)) \) and \( y = \varphi_{L,i} \otimes \varphi_{R,i}(O_L(i,l)O_R(i,l))(\geq 1 - 2\epsilon(l)) \).

For any state \( \psi \), any operators \( E,B \) with \( 0 \leq E, B \leq 1 \), the Schwartz inequality implies

\[
|\psi((E - \psi(E))1)(B - \psi(B))1)| \leq (\psi(E^2) - \psi(E)^2)^{1/2}(\psi(B^2) - \psi(B)^2)^{1/2}
\]

\[
\leq (\psi(E) - \psi(E^2)^{1/2}(\psi(B) - \psi(B)^2)^{1/2}.
\]
Setting \( B = (O_B(i,l), E = (O_L(i,l)O_R(i,l)), \psi = \varphi_{L,i} \otimes \varphi_{R,i} \)
\[
\begin{align*}
xy - |\varphi_{L,i} \otimes \varphi_{R,i}(O_B(i,l)O_L(i,l)O_R(i,l))| & \leq \sqrt{x} - x^2 \sqrt{y} - y^2, \\
x(1 - 2\epsilon(l)) - \sqrt{x} \sqrt{2\epsilon(l)} - \epsilon(l) & \leq |\varphi_{L,i} \otimes \varphi_{R,i}(O_B(i,l)O_L(i,l)O_R(i,l))| - \epsilon(l) \leq p \\
x & \leq \left\{ 2C_1^2 \exp[-(S_{cut} - F)/(C_2 \ln 4d)] + \sqrt{x} \sqrt{2\epsilon(l)} + 2\epsilon(l) \right\} / (1 - 2\epsilon(l)) \quad (4.20)
\end{align*}
\]
We can find \( C_3 \) such that \( x \leq C_3\epsilon(l) < 1 \). We now assume that \( C_2 \ln C_1 \ln 4d + F \leq S_{cut}/2 \) and we obtain
\[
\begin{align*}
l & \leq l_0 \leq (S_{cut} - F)/(C_2 \ln 4d) - C_2 \ln C_1, \\
2C_1^2 \exp[(S_{cut} - F)/(C_2 \ln 4d)] & \leq 2\epsilon(l).
\end{align*}
\]
Due to (4.20),
\[
x \leq \frac{4\epsilon(l) + \sqrt{2x\epsilon(l)}}{1 - 2\epsilon(l)}. \quad (4.21)
\]
which shows that \( x \leq C_4\epsilon(l) \) for a constant \( C_4 \).

On the other hand, due to monotonicity of relative entropy for states \( \varphi_{[-l,t] \cup [i-l,i+t]} \) and \( \varphi_{[-l,-1] \cup [i+1,i+t]} \otimes \varphi_{[0,l] \cup [i-t,i]} \)
\[
\begin{align*}
(1 - 2\epsilon(l)) \ln(1 - 2\epsilon(l)) / x + 2\epsilon(l) \ln 2\epsilon(l) / (1 - x) & \leq \varphi_{[-l,t] \cup [i-l,i+t]} + s(\varphi_{[-l,-1] \cup [i+1,i+t]} + s(\varphi_{[0,l] \cup [i-t,i]})) \quad (4.22)
\end{align*}
\]
implies
\[
- s(\varphi_{[-l,t] \cup [i-l,i+t]} + s(\varphi_{[-l,-1] \cup [i+1,i+t]} + s(\varphi_{[0,l] \cup [i-t,i]})) \geq (1 - 2\epsilon(l)) \ln 1/x - \ln 2.
\]
As a consequence we have a positive constant \( C_5 \) such that
\[
- s(\varphi_{[-l,t] \cup [i-l,i+t]} + s(\varphi_{[-l,-1] \cup [i+1,i+t]} + s(\varphi_{[0,l] \cup [i-t,i]})) \geq (1 - 2\epsilon(l)) \ln 1/\epsilon(l) - C_5.
\]
The above estimate is valid for \( j, l \) if \( j + l \leq i + l_0 \) and \( l \leq l_0 \):
\[
- s(\varphi_{[-l,t] \cup [j-l,j+t]} + s(\varphi_{[-l,-1] \cup [j+1,j+t]} + s(\varphi_{[0,l] \cup [j-t,j]})) \geq (1 - 2\epsilon(l)) \ln 1/\epsilon(l) - C_5. \quad (4.23)
\]
Suppose that \( J \) and \( K \) are any intervals of length \( l \) in \([-l_0, i + l_0]\) and set
\[
S_l = \max \{ s(\varphi_{J \cup K}) \mid J, K \subseteq [-l_0, i + l_0], |K| = |J| = l \}.
\]
By definition \( S_1 \leq \ln 2d \), and due to (4.23)
\[
S_{2l} \leq 2S_l - (1 - 2C_1 \exp [-l/C_2])l/C_2 + \ln C_1 + C_5,
\]
\[
0 \leq S_{2^k} \leq \ln 2d2^k - 2^k k/C_2 + C_0 2^k \quad (4.24)
\]
\[ \sum_{m=0}^{\infty} \frac{2C_1 \exp(-2^m/C_2))l/C_2 + \ln C_1 + C_5}{m} \]

When we take \( k \) satisfying \( 2^k \leq l_0 < 2^{k+1} \),
\[ \ln_2(S_{cut}/2) = \ln_2 l_0 \leq 1 + \ln 2d + C_6 \] (4.25)

Hence, we arrive at the contradiction to the claim that \( S_{cut} \) can be an arbitrary large number.

5 Spinless Fermion

In this section, we consider translationally invariant pure states of spinless Fermion systems on \( \mathbb{Z} \). Let us consider the GNS representation of \( \mathcal{A}_{CAR} \) associated with a translationally invariant pure state \( \psi \) and we show the fermionic version of Haag duality. In general, any translationally invariant factor state \( \psi \) of \( \mathcal{A}_{CAR} \) is \( \Theta \) invariant. (See [4] for basic properties of fermionic systems.)

Suppose that a state \( \psi \) of \( \mathcal{A}_{CAR} \) is \( \Theta \) invariant and let \( \{ \pi_\psi(\mathcal{A}_{CAR}), \Omega_\psi, H_\psi \} \) be the GNS triple associated with \( \psi \). There exists a (unique) selfadjoint unitary \( \Gamma \) on \( H_\psi \) satisfying
\[ \Gamma \pi_\psi(c_j) = \pi_\psi(c_j) \Gamma, \quad \pi_\psi(c_i) = \pi_\psi(c_i) \Gamma, \quad \Gamma^2 = 1, \quad \Gamma = \Gamma^*, \quad \Gamma \Omega_\psi = \Omega_\psi. \] (5.1)

With aid of \( \Gamma \), we introduce another representation \( \pi_\psi \) of \( \mathcal{A}_{CAR} \) via the following equation:
\[ \pi_\psi(c_j) = \pi_\psi(c_j) \Gamma, \quad \pi_\psi(c_i) = \pi_\psi(c_i) \Gamma (j < 0, 0 \leq i) \] (5.2)

for any integer \( j \). Let \( \Lambda \) be a subset of \( \mathbb{Z} \) and \( \psi \) be a state of \( \mathcal{A}_{CAR} \) which is \( \Theta \) invariant. By definition, \( \pi_\psi(\mathcal{A}_{CAR}^{\Lambda^{-}})^{\prime\prime} \subset \pi_\psi(\mathcal{A}_{CAR}^{\Lambda^{+}})^{\prime} \). We say the twisted Haag duality is valid for \( \Lambda \) if and only if
\[ \pi_\psi(\mathcal{A}_{CAR}^{\Lambda^{-}})^{\prime\prime} = \pi_\psi(\mathcal{A}_{CAR}^{\Lambda^{+}})^{\prime} \] (5.3)
holds. To formulate split property of fermion systems, we may consider existence of an intermediate type I factor \( \mathcal{N} \) in \( \pi_\psi(\mathcal{A}_{CAR}^{\Lambda^{-}})^{\prime\prime} \subset \mathcal{N} \subset \pi_\psi(\mathcal{A}_{CAR}^{\Lambda^{+}})^{\prime} \). We note that \( \pi_\psi(\mathcal{A}_{CAR}^{\Lambda^{-}}) \cup \pi_\psi(\mathcal{A}_{CAR}^{\Lambda^{+}}) \) may not act irreducibly on \( \mathcal{H}_\psi \) even if \( \psi \) is pure. It is possible to show the following.

Lemma 5.1 Suppose that \( \psi \) is a \( \Theta \) invariant pure state of \( \mathcal{A}_{CAR} \) and consider the vector state \( \omega_\psi \) associated with \( \Omega_\psi \) of \( \pi_\psi(\mathcal{A}_{CAR}^{R}) \cup \pi_\psi(\mathcal{A}_{CAR}^{L}) \). \( \omega \) is not a pure state if and only if there exists a selfadjoint unitary \( \Gamma_{-} \) satisfying
\[ \Gamma_{-} \pi_\psi(c_j) = -\pi_\psi(c_j) \Gamma_{-}, \quad \Gamma_{-} \pi_\psi(c_i) = \pi_\psi(c_i) \Gamma_{-} (j < 0, 0 \leq i), \]
\[ \Gamma_{-} \Gamma_{-} = -\Gamma_{-} \Gamma_{-}. \] (5.4)

The commutant of \( (\pi_\psi(\mathcal{A}_{CAR}^{R}) \cup \pi_\psi(\mathcal{A}_{CAR}^{L}))^{\prime\prime} \) is generated by \( \Gamma_{-} \).
If we identify $\pi_\psi(\mathfrak{A}_{R}^{CAR})$ with $\pi_\omega(\mathfrak{A}_{L})$ and $(\pi_\psi(\mathfrak{A}_{L}^{CAR}))$ with $\omega_\psi$ is a state of $\mathfrak{A}$. When $\psi$ is $\Theta$ invariant pure state of $\mathfrak{A}^{CAR}$, we might say split property holds if $\pi_\psi(\mathfrak{A}_{R}^{CAR})''$ is of type I. We can show the following theorem in the same way as for the spin systems.

**Theorem 5.2** Let $\psi$ be a translationally invariant pure state of the CAR algebra $\mathfrak{A}^{CAR}$, and let $\{\pi_\psi(\mathfrak{A}_{R}^{CAR}), \Omega_\psi, \Omega_\psi\}$ be the GNS triple for $\psi$. Suppose that the entropy $s(\psi_{[1,L]})$ is bounded uniformly in $L$. Consider the vector state $\omega$ associated with $\Omega_\psi$ of $\mathfrak{A}_{(-\infty,0)}^{CAR} \otimes \mathfrak{A}_{(1,\infty)}^{CAR}$. $\omega$ satisfies the split property for $\mathfrak{A}_{R}$ and $\mathfrak{A}_{L}$. As a consequence, $\pi_\psi(\mathfrak{A}_{R}^{CAR})''$ is of type I and the twisted Haag duality holds for $\Lambda = [1, \infty)$, $\pi_\psi((\mathfrak{A}_{R}^{CAR}))'' = \pi_\psi((\mathfrak{A}_{R}^{CAR}))'$.

To show Theorem 1.10 we employ the Jordan-Wigner transform for infinite systems à la mani`ere de [2], [3]. Fermion systems and quantum spin chains are formally equivalent via the Jordan-Wigner For handling infinite chains, we introduce an automorphism $\Theta_-$ of $\mathfrak{A}_{CAR}$ by the following equations:

$$
\Theta_-(c_j^*) = -c_j^*, \ \Theta_-(c_j) = -c_j \ (j \leq 0),
\Theta_-(c_k^*) = c_k^*, \ \Theta_-(c_k) = c_k \ (k > 0).
$$

Let $\tilde{\mathfrak{A}}$ be the crossed product of $\mathfrak{A}_{CAR}$ by the $\mathbb{Z}_2$ action $\Theta_-$ . $\tilde{\mathfrak{A}}$ is the $C^*$-algebra generated by $\mathfrak{A}_{CAR}$ and a unitary $T$ satisfying

$$
T = T^*, \ T^2 = 1, \ TQT = \Theta_- (Q) \ (Q \in \mathfrak{A}_{CAR}).
$$

Via the following formulae, we regard $\mathfrak{A}$ as a subalgebra of $\tilde{\mathfrak{A}}$:

$$
\sigma_z^{(j)} = 2c_j^*c_j - 1,
\sigma_z^{(j)} = TS_j(c_j + c_j^*)
\sigma_y^{(j)} = iTS_j(c_j - c_j^*).
$$

where

$$
S_n = \begin{cases}
\sigma_z^{(1)} \cdots \sigma_z^{(n-1)} & n > 1 \\
1 & n = 1 \\
\sigma_z^{(0)} \cdots \sigma_z^{(n)} & n < 1.
\end{cases}
$$

We extend the automorphism $\Theta$ of $\mathfrak{A}_{CAR}$ to $\tilde{\mathfrak{A}}$ via the following equations:

$$
\Theta(T) = T, \ \Theta(\sigma_z^{(j)}) = -\sigma_z^{(j)}, \ \Theta(\sigma_y^{(j)}) = -\sigma_y^{(j)}, \ \Theta(\sigma_z^{(j)}) = \sigma_z^{(j)}.
$$

As is the case of the CAR algebra, we set

$$
(\mathfrak{A})_\pm = \{Q \in \mathfrak{A} | \Theta(Q) = \pm Q\}, \ (\mathfrak{A}_\Lambda)_\pm = (\mathfrak{A})_\pm \cap \mathfrak{A}_\Lambda, \ (\mathfrak{A}_{loc})_\pm = (\mathfrak{A})_\pm \cap \mathfrak{A}_{loc}.
$$

Then, it is easy to see that

$$
(\mathfrak{A})_+ = (\mathfrak{A}_{CAR})_+, \ (\mathfrak{A}_\Lambda)_+ = (\mathfrak{A}_\Lambda^{CAR})_+, \ (\mathfrak{A}_{loc})_+ = (\mathfrak{A}_{loc}^{CAR})_+.
$$
Let \( \psi \) be a pure state of \( \mathfrak{A}_{CAR} \) and assume that \( \psi \) is \( \Theta \) invariant. Let \( \psi_+ \) be the restriction of \( \psi \) to \( (\mathfrak{A}_{CAR})_+ = (\mathfrak{A})_+ \). \( \psi_+ \) is extendible to a \( \Theta \) invariant state \( \varphi_0 \) of \( \mathfrak{A} \) via the following formula:

\[
\varphi_0(Q) = \psi_+(Q_+), \quad Q_\pm = \frac{1}{2}(Q \pm \Theta(Q)) \in (\mathfrak{A})_\pm.
\]  

(5.6)

In general, \( \varphi_0 \) may not be a pure state but if \( \varphi \) is a pure state extension of \( \psi_+ \) to \( \mathfrak{A} \), the relation between \( \varphi_0 \) and \( \varphi \) is written as \( \varphi_0(Q) = \varphi(Q_+) \). That \( \varphi_0 \) and \( \varphi \) are identical or not depends on existence of a unitary implementing \( \Theta_- \) on \( \mathfrak{H}_\psi \).

**Proposition 5.3** Let \( \psi \) be a \( \Theta \) invariant pure state of \( \mathfrak{A}_{CAR} \) and \( \psi_+ \) be the restriction of \( \psi \) to \( (\mathfrak{A}_{CAR})_+ \).

(i) Suppose that \( \psi \) and \( \psi \circ \Theta_- \) are not unitarily equivalent. The unique \( \Theta \) invariant extension \( \varphi \) of \( \psi_+ \) to \( \mathfrak{A} \) is a pure state. If \( \psi \) is translationally invariant, \( \varphi \) is translationally invariant as well.

(ii) Suppose that \( \psi \) and \( \psi \circ \Theta_- \) are unitarily equivalent and that \( \psi_+ \) and \( \psi_+ \circ \Theta_- \) are unitarily equivalent as states of \( (\mathfrak{A}_{CAR})_+ \). The unique \( \Theta \) invariant extension \( \varphi \) of \( \psi_+ \) to \( \mathfrak{A} \) is a pure state. If \( \psi \) is translationally invariant, \( \varphi \) is translationally invariant as well.

(iii) Suppose that \( \psi \) and \( \psi \circ \Theta_- \) are unitarily equivalent and that \( \psi_+ \) and \( \psi_+ \circ \Theta_- \) are not unitarily equivalent as states of \( (\mathfrak{A}_{CAR})_+ \). There exists a pure state extension \( \varphi \) of \( \psi_+ \) to \( \mathfrak{A} \) which is not \( \Theta \) invariant. Furthermore, we can identify the GNS Hilbert spaces \( \mathfrak{H}_{\psi_+} \) and \( \mathfrak{H}_\varphi \) and

\[
\pi_\varphi(\mathfrak{A})'' = \pi_\varphi((\mathfrak{A})_+)''.
\]  

(5.7)

If \( \psi \) is translationally invariant, \( \varphi \) is a periodic state with period 2, \( \varphi \circ \tau_2 = \varphi \) and

\[
\pi_\varphi(\mathfrak{A}_L)'' = \pi_\varphi((\mathfrak{A}_L)_+)''", \quad \pi_\varphi(\mathfrak{A}_R)'' = \pi_\varphi((\mathfrak{A}_R)_+)''" \tag{5.8}
\]

where we set \( (\mathfrak{A}_{L,R})_+ = (\mathfrak{A}_{L,R}) \cap (\mathfrak{A})_+ \).

**Proof of Proposition 5.3**

Set \( X_j = c_j + c_j^* \). As \( \psi \) is \( \Theta \) invariant, the GNS space \( \mathfrak{H}_\psi \) is a direct sum of \( \mathfrak{H}_{\psi_+}^{(\pm)} \) where

\[
\mathfrak{H}_{\psi_+}^{(\pm)} = \pi_\varphi((\mathfrak{A})_+^{(\pm)})\Omega, \quad \mathfrak{H}_\varphi^{(\pm)} = \pi_\varphi((\mathfrak{A})_+^{(\pm)}X_j)\Omega.
\]

The representation \( \pi_\varphi((\mathfrak{A})_+) \) of \( (\mathfrak{A})_+ \) on \( \mathfrak{H}_\psi \) is decomposed into mutually disjoint irreducible representations on \( \mathfrak{H}_{\psi_+}^{(\pm)} \).

Let \( \tilde{\psi} \) and \( \psi \) be \( \Theta \) invariant states of \( \mathfrak{A}_{CAR} \). The argument in 2.8 of [28] shows that if \( \psi_+ \) and \( \psi_+ \) of \( (\mathfrak{A})_+ \) are equivalent, \( \psi \) and \( \psi \) are equivalent. Now we show (i). If pure states \( \psi \) and \( \psi \circ \Theta_- \) are not equivalent, \( \psi_+ = \varphi_+ \) is not equivalent to \( (\varphi \circ \Theta_-)_+ \) and \( (\varphi \circ \Theta_- \circ \text{Ad}(X_j))_+ \). Consider the GNS representation.
{πφ(𝒜), Ωφ, ℋφ} of 𝒜. If we restrict πφ to (𝒜)+ it is the direct sum of two irreducible GNS representations associated with ψ+ = φ+ and (φ − Θ − Ad(Xj))+. So we set

\[ ℋ = ℋφ, \quad ℋ_1 = ℋ_2 \oplus ℋ_2, \quad ℋ_1 = ℋφ, \quad ℋ_2 = ℋ(φ − Θ − Ad(Xj))+. \]

Any bounded operator A on ℋ is written in a matrix form,

\[ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad (5.9) \]

where \(a_{11}\) (resp. \(a_{22}\)) is a bounded operator on ℋ1 (resp. ℋ2) and \(a_{12}\) (resp. \(a_{21}\)) is a bounded operator from ℋ2 to ℋ1 (resp. a bounded operator from ℋ1 to ℋ2).

As \(ψ+ = φ+\) is not equivalent to \((φ − Θ − Ad(Xj))_+\),

\[ P = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad (5.10) \]

is an element of \(πφ((𝒜)+)''\) and \(πφ(σ_x^{(j)})\) looks like

\[ πφ(σ_x^{(j)}) = \begin{pmatrix} 0 & d \\ d^* & 0 \end{pmatrix} \quad (5.11) \]

A direct computation shows that an operator A of the matrix form (5.9) commuting with (5.10) and (5.11) is trivial. This shows that the state φ is pure.

The translational invariance of φ follows from translational invariance of ψ and \(φ(Q) = ψ(Q_+)\).

(ii) of Proposition 5.3 can be proved by constructing the representation of 𝒜 on the GNS space of Fermion. By our assumption, \(πψ_+(𝒜)+\) is equivalent to \(πψ_+(Ad(Xj)(𝒜)+)\). Hence \(πψ_+(𝒜)+\) is equivalent to \(πψ_+(Θ−(𝒜)+)\) and \(πψ_+(Ad(Xj)(𝒜)+)\) is equivalent to \(πψ_+(Θ−(Ad(Xj)𝒜)+)\). It turns out that there exists a selfadjoint unitary \(U(Θ−) (U(Θ−))^* = U(Θ−), U(Θ−)^2 = 1\) on \(ℋ_ψ\) such that

\[ U(Θ−)πψ(Q)U(Θ−)^*, \quad U(Θ−) \in πφ((𝒜)+)'' \quad (5.12) \]

for any \(Q\) in \(𝒜^{CAR}\). Any element \(R\) of 𝒜 is written in terms of fermion operators and \(T\) as follows:

\[ R = R_+ + TR_-, \quad (5.13) \]

where

\[ R_+ = \frac{1}{2}(R + Θ(R)) ∈ (𝒜^{CAR})_+, \quad R_- = \frac{1}{2}(TR - TΘ(R)) ∈ (𝒜^{CAR})_-. \]

Using this formula, for any \(R\) in 𝒜, we set

\[ π(R) = πφ(R_+) + U(Θ−)πφ(R_-) \quad (5.14) \]
\( \pi(R) \) gives rise to a representation of \( \mathfrak{A} \) on \( \mathfrak{H}_\psi \) and we set
\[
\varphi(R) = (\Omega_\psi, \pi(R)\Omega_\psi). \tag{5.15}
\]
The representation \( \pi(\mathfrak{A}) \) is irreducible because \( \pi(\mathfrak{A})'' \) contains \( U(\Theta_-) \) and hence \( \pi(\mathfrak{A})'' \) contains \( \pi((\mathfrak{A}^{CAR})_-) \) and \( \pi(\mathfrak{A})'' = \mathfrak{B}(\mathfrak{H}_\psi) \).

As in (i), the translational invariance of \( \varphi \) follows from \( \Theta \) invariance of \( \varphi \) (by construction) and translational invariance of \( \psi \).

To show (iii), we construct an irreducible representation of \( \mathfrak{A} \) on the GNS space \( \mathfrak{H}^+ = \pi_\psi(\mathfrak{A}^{CAR} \cup \mathfrak{A}^{CAR}) \Omega_\psi \).

Now under our assumption there exists a self-adjoint unitary \( V(\Theta_-) \) satisfying
\[
V(\Theta_-)\pi_\psi(Q)V(\Theta_-)^* = \pi_\psi(\Theta(Q)), \quad V(\Theta_-) \in \pi_\psi((\mathfrak{A}^{CAR})_-) \tag{5.16}
\]
for any \( Q \in \mathfrak{A}^{CAR} \).

As in (i), the translational invariance of \( \varphi \) follows from \( \Theta \) invariance of \( \varphi \) (by construction) and translational invariance of \( \psi \).

To show periodicity of the state \( \varphi \), we introduce a unitary \( W \) satisfying
\[
W\Omega_\psi = \Omega_\psi, \quad W\pi_\psi(Q)W^* = \pi_\psi(\Theta_1(Q)), \quad Q \in \mathfrak{A}^{CAR}
\]
The adjoint action of both unitaries \( WV(\Theta_-)W^* \) and \( V(\Theta_-)\pi_\psi(\sigma_1^{(1)}) \) gives rise to the same automorphism on \( \pi_\psi(\mathfrak{A}^{CAR}) \). By irreducibility of the representation \( \pi_\psi(\mathfrak{A}^{CAR}) \), \( WV(\Theta_-)W^* \) and \( V(\Theta_-)\pi_\psi(\sigma_1^{(1)}) \) differ in a phase factor.
\[
WV(\Theta_-)W^* = eV(\Theta_-)\pi_\psi(\sigma_1^{(1)}) \tag{5.18}
\]
where \( e \) is a complex number with \( |e| = 1 \). As both sides in (5.18) are selfadjoint, \( e = \pm 1 \). Then,
\[
W^2V(\Theta_-)(W^2)^* = V(\Theta_-)\pi_\psi(\sigma_1^{(1)}\sigma_2^{(2)}) \tag{5.19}
\]
This implies that the state \( \varphi \) is periodic, \( \varphi \circ \tau_2 = \varphi \).

End of Proof of Proposition 5.3

Remark 5.4 In [21], using expansion technique (but not the exact solution) we have shown the XXZ Hamiltonian \( H_{XXZ} \) with large Ising type anisotropy \( \Delta >> 1 \)
\[
H_{XXZ} = \sum_{j=-\infty}^{\infty} \{ \Delta \sigma_z^{(j)} \sigma_z^{(j+1)} + \sigma_x^{(j)} \sigma_x^{(j+1)} + \sigma_y^{(j)} \sigma_y^{(j+1)} \}
\]
has exactly two pure ground states \( \varphi \) and
\[
\varphi \circ \Theta = \varphi \circ \tau_1 \neq \varphi.
\]
The unique \( \Theta \) invariant ground state \( (1/2 \varphi + \varphi \circ \tau_1) \) is a pure state of \( \mathfrak{A}_+ \). In this example, the phase factor \( c \) of (5.18) is \(-1 \).
To complete our proof of Theorem 1.10, we use a theorem of [20] and Proposition 5.6 below.

**Theorem 5.5** Suppose that the spin $S$ of one site algebra $M_{2S+1}(n = 2S + 1)$ for $\mathfrak{A}$ is $1/2$. Let $\varphi$ be a translationally invariant pure state of $\mathfrak{A}$ such that $\varphi_R$ gives rise to a type I representation of $\mathfrak{A}_R$. Suppose further that $\varphi$ is $U(1)$ gauge invariant, $\varphi \circ \gamma_\theta = \varphi$. Then, $\varphi$ is a product state.

**Proposition 5.6** Let $\psi$ be a translationally invariant pure state of $\mathfrak{A}^{CAR}$.

(i) Suppose further that $\psi$ is $U(1)$ gauge invariant, $\psi \circ \gamma_\theta = \psi$. The $\Theta$ invariant extension of $\psi_L$ to $\mathfrak{A}$ is a translationally invariant pure state.

(ii) Suppose the conditions of (i) and that the von Neumann algebra $\pi_\psi(\mathfrak{A}_L^{CAR})''$ associated with the GNS representation of $\psi_L$ is of type I. Then, either $\psi = \psi_F$ or $\psi = \psi_{AF}$ holds.

**Proof of Proposition 5.6**

To prove Proposition 5.6 (i), we show the case (iii) in Proposition 5.3 is impossible due to assumption of $\gamma_\theta$ invariance. There exists $U(\theta)$ implementing $\gamma_\theta$ on the GNS space of $\psi$. Then

$$U(\theta)V(\Theta_-)U(\theta)^* = c(\theta)V(\Theta_-)$$

as the adjoint action of both unitaries are identical. Moreover these are self-adjoint so $c(\theta) = \pm 1$. Due to continuity in $\theta$ we conclude that $c(\theta) = 1$ and $V(\Theta_-)$ is an even element.

Finally, we consider Proposition 5.6 (ii). Due to (i) of Proposition 5.6 (i), the Fermionic state $\psi$ has a translationally invariant pure state extension $\varphi$ to $\mathfrak{A}$. Then, the split property for Fermion implies that that of the Pauli spin system. It turns out that either $\psi(c_j^*c_j) = \varphi(c_1^{(1)}) = 0$ or $\psi(c_jc_j^*) = \varphi(c_2^{(2)}) = 0$ holds. This completes our proof of Proposition 5.6 (ii).

**End of Proof of Proposition 5.6**

If $\psi$ is a $U(1)$ gauge invariant ground state with spectral gap, the entanglement entropy is bounded and $\pi_\psi(\mathfrak{A}_R^F)''$ is of type I. Thus $\psi$ is trivial, which shows Theorem 1.10.
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