Heat content asymptotics with singular data

M van den Berg and P Gilkey

1 School of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, UK
2 Mathematics Department, University of Oregon, Eugene, OR 97403, USA

E-mail: M.vandenBerg@bris.ac.uk and gilkey@uoregon.edu

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Abstract
We study the asymptotic behaviour of the heat content on a compact Riemannian manifold with boundary and with singular specific heat and singular initial temperature distributions. Assuming the existence of a complete asymptotic series, we determine the first three terms in that series. In addition to the general setting, the interval is studied in detail.

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1. Introduction
Let $M$ be a compact Riemannian manifold with smooth boundary $\partial M$, and let $\delta$ denote the geodesic distance to the boundary. Let $\psi_1$ and $\psi_2$ be smooth functions on the interior of $M$. Then $\psi_1$ will represent the initial temperature of $M$ and $\psi_2$ will represent the specific heat of $M$. Since $M$ is compact and $\partial M$ is smooth, the distance function is smooth near $\partial M$. We have to assume that $\delta^{\alpha_1} \psi_1$ and $\delta^{\alpha_2} \psi_2$ are smooth on a closed collared neighbourhood of $\partial M$. The parameters $\alpha_1$ and $\alpha_2$ control the growth or decay of $\psi_1$ and $\psi_2$ near $\partial M$. Let $D$ be an operator of Laplace type on $M$. Impose Dirichlet boundary conditions to define the realization of $D$. Let $\mathrm{e}^{-t D}$ be the fundamental solution of the heat equation for the Dirichlet Laplacian. Then,

$$u_1(\cdot; t) := \mathrm{e}^{-t D} \psi_1 = \int_M p_M(\cdot, x; t) \psi_1(x) \, dx$$

represents the temperature of the manifold for $t > 0$. The heat content $Q$ is defined by

$$Q(\psi_1, \psi_2, D)(t) := \int_M u_1(x; t) \psi_2(x) \, dx$$

$$= \int_M \int_M p_M(x_1, x_2; t) \psi_1(x_1) \psi_2(x_2) \, dx_1 \, dx_2,$$
Conjecture 1. Let $\alpha_1 + \alpha_2 \notin \mathbb{Z}$, $\alpha_1 < 2$, $\alpha_2 < 2$. There is a complete asymptotic series as $t \downarrow 0$

$$Q(\psi_1, \psi_2, D)(t) \sim \sum_{n=0}^{\infty} t^n \beta_n^M + \sum_{j=0}^{\infty} t^{(1+j-\alpha_1-\alpha_2)/2} \beta_j^M,$$

(1)

where the $\beta_n^M$, $n = 0, 1, \ldots$ are regularized integrals of local invariants over $M$ and where the $\beta_j^M$: $j = 0, 1, \ldots$ are integrals of local invariants over the boundary.

Remark 1. It is convenient to let $\alpha_1$ and $\alpha_2$ be complex as we may then use analytic continuation. For $\Im(\alpha_1) \ll 0$ and $\Im(\alpha_2) \ll 0$,

$$\beta_n^M = (-1)^n \frac{1}{n!} \int_M \psi_2(x) D^n \psi_1(x) \mathrm{d}x.$$

The values of $\beta_n^M$ for more general values of $\alpha_1$ and $\alpha_2$ may then be obtained as regularized integrals as discussed in [11]. We omit the technical details concerning the requisite regularizations in the interests of brevity as they will play no role in our analysis.

The heat content has obvious physical relevance and the invariants $\beta_j^M$, which reflect the asymptotic behaviour as $t \downarrow 0$, relate the geometry of $M$ to the underlying physical properties of $M$. Much of the previous work in the field has been devoted to the computation of the invariants $\beta_j^M$ in the smooth setting ($\alpha_1 = 0$, $\alpha_2 = 0$). They were originally studied for the scalar Laplacian with $\psi_1 = \psi_2 = 1$ [4, 6, 12]. Subsequently, general initial temperatures and specific heats were investigated—see [7, 11, 14, 20–23] and the references contained therein. Other boundary conditions (Neumann, Zaremba, etc) have been considered [5, 10]. The growth of the coefficients $\beta_j^M$ has also been of interest [2, 9, 24]—see also [13] for related work on the heat trace asymptotics. The case where $\psi_1$ is singular ($\alpha_1 > 0$) but $\psi_2$ is smooth ($\alpha_2 = 0$) was studied previously [3, 11]. The current paper is devoted to the study of the invariants $\beta_j^M$ in the doubly singular case.

The special case of a ball of radius $r$ in $\mathbb{R}^3$ is well understood. The following result was proved in [8].

Theorem 1. Let $B_r = \{x \in \mathbb{R}^3 : |x| \leq r\}$, and $D$ be the Dirichlet Laplacian acting in $L^2(B_r)$. If $\alpha_1 < 2$, $\alpha_2 < 2$, $\alpha_1 + \alpha_2 > 3$, $J \in \mathbb{N}$, then there exist coefficients $b_0, b_1, \ldots$ depending on $\alpha_1$, $\alpha_2$ only such that for $t \downarrow 0$

$$Q(\delta^{-\alpha_1}, \delta^{-\alpha_2}, D)(t) = 4\pi c_{\alpha_1, \alpha_2} r^{2(1-\alpha_1-\alpha_2)/2} - 4\pi (c_{\alpha_1-1, \alpha_2} + c_{\alpha_1, \alpha_2-1}) r^{(2-\alpha_1-\alpha_2)/2}$$

$$+ 4\pi c_{\alpha_1-1, \alpha_2-1} t^{(3-\alpha_1-\alpha_2)/2} + \sum_{j=0}^{J} \frac{b_j t^{3-\alpha_1-\alpha_2+j}/2 + O(t^{(J+1)/2})},$$

(2)

where

$$c_{\alpha_1, \alpha_2} = 2^{-\alpha_1-\alpha_2} \pi^{-1/2} \Gamma((2 - \alpha_1 - \alpha_2)/2)$$

$$\times \int_0^{\frac{1}{\sqrt{2}}} (\rho^{\alpha_1} + \rho^{\alpha_2}) \left((1 - \rho)^{\alpha_1+\alpha_2-2} - (1 + \rho)^{\alpha_1+\alpha_2-2}\right) \mathrm{d}\rho,$$

(3)

and

$$b_0 = -8\pi ((\alpha_1 + \alpha_2 - 1)(\alpha_1 + \alpha_2 - 2)(\alpha_1 + \alpha_2 - 3))^{-1}, \quad b_1 = 0,$$

$$b_2 = 8\pi \alpha_1 \alpha_2 ((\alpha_1 + \alpha_2 + 1)(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_2 - 1))^{-1}, \quad b_3 = 0.$$

We see that in the special case of a ball in $\mathbb{R}^3$ there are three non-zero terms $t^{(1+j-\alpha_1-\alpha_2)/2}$, $j = 0, 1, 2$. So $\beta_j^M = 0$ for $j > 2$ in equation (1). The fact that $b_1 = b_2 = 0$ in equation (2) suggests that $b_j = 0$, $j = 1, 3, 5, \ldots$. This would imply the presence of only three
non-zero boundary terms in the case of a ball in $\mathbb{R}^3$. For balls in other dimensions, we do not have such detailed formulae. However, the first three boundary terms can be computed with the help of theorem 3. We have little insight into the case where $\alpha_1 + \alpha_2 \in \mathbb{Z}$. We conjecture that if $\alpha_1 + \alpha_2 = j$, $j = 1, 2, 3$ the $j$th boundary term in equation (2) will involve a log $t$ term. This is supported by the special case calculation of a ball in $\mathbb{R}^3$. See theorem 4 and the comments in the preceding paragraph.

It is convenient to use a standard formalism to describe the invariants $\beta_j^M$ in the general setting (see remark 2). Let $D$ be an operator of Laplace type acting on the space of smooth sections to some vector bundle $V$ over a Riemannian manifold $(M, g)$. Choose a local system of coordinates $(x^1, \ldots, x^n)$ for $M$ and a local frame for $V$. We adopt the Einstein convention and sum over repeated indices. Let $\mathrm{d}x^2 = g_{\mu\nu} \mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu}$ define the Riemannian metric and $g^{\mu\nu}$ be the inverse matrix where $1 \leq \mu, \nu \leq m$. We may then express

$$D = -(g^{\mu\nu} \partial_\mu \partial_\nu + A^\mu \partial_\mu + B),$$

for suitably chosen endomorphisms $A^\mu$ and $B$ of $V$. If $\nabla$ is a connection on $V$, we use $\nabla$ and the Levi-Civita connection to covariantly differentiate tensors of all types and let ':' denote multiple covariant differentiation. If $\psi_1$ is a section to $V$ which is smooth on $\text{int}(M)$, then let $\psi_{1,\mu}$ be the components of $\nabla^2 \psi_1$. If $E$ is an auxiliary endomorphism of $V$, we define the associated modified Bochner Laplacian by setting

$$D(g, \nabla, E) \psi_1 := -g^{\mu\nu} \psi_{1,\mu\nu} - E \psi_1.$$ 

Let $\Gamma^\mu_{\nu\sigma}$ and $\Gamma^\mu_{\nu\sigma}$ be the Christoffel symbols of the Levi-Civita connection. Then (see, for example, the discussion in [7]) we have the following.

**Lemma 2.** If $D$ is an operator of Laplace type, then there exists a unique connection $\nabla$ on $V$ and a unique endomorphism $E$ of $V$ so that $D = D(g, \nabla, E)$. The connection 1-form $\omega$ of $\nabla$ and the endomorphism $E$ are given by

$$\omega_\mu = \frac{1}{2} (g_{\mu\nu} A^\nu + g^{\sigma\nu} \Gamma^\mu_{\sigma\nu} \text{Id}),$$

$$E = B - g^{\mu\nu} \left( \partial_\nu \omega_\mu + \omega_\nu \omega_\mu - \omega_\mu \Gamma^\mu_{\nu\sigma} \right).$$

The specific heat $\psi_2$ is a section to the dual vector bundle $\tilde{V}$. We use the dual connection on $\tilde{V}$ to covariantly differentiate $\psi_2$. Note that the connection 1-form $\tilde{\omega}_\nu$ for $\tilde{V}$ is the dual of $-\omega_\nu$. Thus

$$\tilde{\nabla}_{\partial_\nu} = \partial_\nu - \frac{1}{2} (g_{\mu\nu} A^\nu + g^{\sigma\nu} \Gamma^\mu_{\nu\sigma} \text{Id}).$$

Near the boundary, choose an orthonormal frame $\{e_1, \ldots, e_m\}$ for the tangent bundle of $M$ so that $e_m$ is the inward unit geodesic normal. Let indices $a, b$ range from 1 to $m - 1$ and index the induced orthonormal frame $\{e_1, \ldots, e_{m-1}\}$ for the tangent bundle of the boundary. We let ':' and ':' denote the components of tangential covariant differentiation defined by $\nabla$ and the Levi-Civita connection of the boundary. Let $L_{ab} := g(\nabla_{e_a} e_b, e_m) = \Gamma^m_{ab}$ be the components of the second fundamental form. The difference between ':' and ':' is then measured by $L$. Let $\tilde{D}$ be the dual operator of Laplace type on $\tilde{V}$. The following relations will be useful subsequently:

$$D \psi_1 = -(\psi_{1,ab} + \psi_{1,mm} - L_{ab} \psi_{1,m} + E \psi_1),$$

$$\tilde{D} \psi_2 = -(\psi_{2,ab} + \psi_{2,mm} - L_{ab} \psi_{2,m} + \tilde{E} \psi_2).$$

We expand $\psi_1$ and $\psi_2$ near the boundary of $M$ in the form

$$\psi_1(y, \delta) \sim \delta^{-\alpha_1} \sum_{j=0}^{\infty} \psi_1^j \delta^j, \quad \psi_2(y, \delta) \sim \delta^{-\alpha_2} \sum_{j=0}^{\infty} \psi_2^j \delta^j,$$
where $\nabla_{\alpha} \psi_1^j = 0$ and $\nabla_{\alpha} \psi_2^j = 0$. We shall usually be working with scalar operators and can choose local sections $s$ and $\tilde{s}$ so that $\nabla_{\alpha} \epsilon s = 0$ and $\nabla_{\alpha} \tilde{s} = 0$. We may then express $\psi_1 = \psi_1 s$, $\psi_1' = \psi_1's$, $\psi_2 = \psi_2 \tilde{s}$ and $\psi_2' = \psi_2' \tilde{s}$, where $\psi_1'$ and $\psi_2'$ are smooth functions defined on the boundary so that we have the modified Taylor series

$$\delta^a \psi_j(y, \tilde{s}) \sim \sum_{j=0}^{\infty} \psi_j(y) \delta^j \quad \text{for} \quad i = 1, 2.$$ 

**Remark 2.** For the Laplacian, the bundles and connections under consideration are trivial so this formalism is unnecessary. However, for more general operators, the connections in question are not flat and this formalism is essential. We shall be using the method of ‘universal examples’ in what follows. It is a peculiar feature of this method that even if we were only interested in the scalar Laplacian for a smooth bounded domain in $\mathbb{R}^m$, it would be necessary to deal with quite general operators, as we shall see presently while proving lemma 11 in section 3. Although we are only interested in the scalar setting, it is useful to introduce the language of vector bundles to emphasize that there is no canonical section or gauge for expressing the operator; we shall in fact be using different gauges in what follows.

Let $\text{Ric}$ be the Ricci tensor of $M$, let $\tau$ be the scalar curvature of $M$ and $\text{dy}$ be the Riemannian measure of $\partial M$. Section 3 is devoted to the proof of the following.

**Theorem 3.** Let $\alpha_1 + \alpha_2 \notin \mathbb{Z}$, $\alpha_1 < 2$, $\alpha_2 < 2$. Assume that conjecture 1 holds. Let $c_{\alpha_1, \alpha_2}$ be as given in equation (3). Then

$$\beta^{3M}_0 = \int_{\partial M} c_{\alpha_1, \alpha_2} \psi_1^0 \psi_2^0 \text{dy},$$

$$\beta^{3M}_1 = \int_{\partial M} \left\{ c_{\alpha_1-1, \alpha_2} \psi_1^1 \psi_2^0 - \frac{1}{2} \left( c_{\alpha_1-1, \alpha_2} + c_{\alpha_1, \alpha_2-1} \right) \psi_1^0 \psi_2^0 L_{\alpha_1} + c_{\alpha_1, \alpha_2-1} \psi_1^0 \psi_2^1 \right\} \text{dy},$$

$$\beta^{3M}_2 = \int_{\partial M} \left\{ c_{\alpha_1-2, \alpha_2} \psi_1^1 \psi_2^0 - \frac{1}{2} \left( c_{\alpha_1-2, \alpha_2} + c_{\alpha_1-1, \alpha_2-1} \right) L_{\alpha_1} \psi_1^0 \psi_2^0 \right.$$ 

$$+ c_{\alpha_1, \alpha_2} E \psi_1^0 \psi_2^0 + c_{\alpha_1, \alpha_2-2} \psi_1^0 \psi_2^0 - \frac{1}{2} \left( c_{\alpha_1-1, \alpha_2-1} + c_{\alpha_1, \alpha_2-2} \right) L_{\alpha_1} \psi_1^0 \psi_2^1$$

$$+ \left( -\frac{1}{4} c_{\alpha_1-2, \alpha_2} - \frac{1}{4} c_{\alpha_1, \alpha_2-2} + \frac{1}{2} c_{\alpha_1, \alpha_2} \right) \left( L_{\alpha_1} L_{\alpha_2} + \text{Ric}_{mm} \right) \psi_1^0 \psi_2^0$$

$$- c_{\alpha_1, \alpha_2} \psi_1^0 \psi_1^0 \psi_2^0 + 0 r \psi_1^0 \psi_1^0 + c_{\alpha_1-1, \alpha_2-1} \psi_1^1 \psi_2^1$$

$$+ \left( \frac{1}{8} c_{\alpha_1-2, \alpha_2} + \frac{1}{8} c_{\alpha_1, \alpha_2-2} + \frac{1}{4} c_{\alpha_1-1, \alpha_2-1} - \frac{1}{4} c_{\alpha_1, \alpha_2} \right) L_{\alpha_1} L_{\alpha_2} \psi_1^0 \psi_2^0 \right\} \text{dy}.$$

For the ball $B_r$ in $\mathbb{R}^3$, $L_{\alpha_1} = 2r^{-1}$, $L_{\alpha_2} L_{\alpha_3} = 4r^{-2}$ and $L_{\alpha_2} L_{\alpha_3} = 2r^{-2}$. Hence

$$\beta^{3B}_1 = -4\pi r (c_{\alpha_1, \alpha_2} + c_{\alpha_1, \alpha_2-1}),$$

and the coefficient of $r^{1/2-\alpha_1-\alpha_2}$ agrees with equation (2) in theorem 3. Similarly, the next term in the series given by $\beta^{3B}_2$ is consistent with theorem 3.

We observe that the $\Gamma$-function in the expression for $c_{\alpha_1, \alpha_2}$ which is given in equation (3) has simple poles for $\alpha_1 + \alpha_2 \in \{2, 4, 6, \ldots\}$. Furthermore, the integrand with respect to $\rho$ equals 0 for $\alpha_1 + \alpha_2 = 2$. It is easily seen that this singularity is removable. On the other hand, the integral with respect to $\rho$ is finite only for $\alpha_1 < 2$, $\alpha_2 < 2$ and $\alpha_1 + \alpha_2 > 1$. This suggests that the $j$th term ($j = 1, 2, 3$) in equation (2) will take a different form for $\alpha_1 + \alpha_2 = j$. This
is indeed the case for an interval in \( \mathbb{R} \). Let \( a > 0 \), and let \( \chi_1, \chi_2 \) be non-negative \( C^\infty \) functions on \( \mathbb{R}^+ \) defined by

\[
\chi_{1,2}(x) = \begin{cases} 
1 & \text{if } 0 \leq x \leq \epsilon_{1,2}, \\
0 & \text{if } x > \epsilon_{3,4}, 
\end{cases}
\]

where \( 0 < \epsilon_1 < \epsilon_3 < a/2 \) and \( 0 < \epsilon_2 < \epsilon_4 < a/2 \). In section 2, we shall prove the following.

**Theorem 4.** Let \( \alpha_1 < 2, \alpha_2 < 2, \alpha_1 + \alpha_2 = 1 \). If \( t \downarrow 0 \), then

\[
\int \int_{[0,a]^2} p_{[0,a]}(x_1, x_2; t) \chi_1(\delta(x_1)) \chi_2(\delta(x_2)) \delta(x_1)^{-\alpha_1} \delta(x_2)^{-\alpha_2} \, dx_1 \, dx_2 = \log(\epsilon^2/t) + \gamma + 4 \log(2^{1/2} - 1) + 4 \log 2 \\
+ 2 \int_{[\epsilon, a/2]} \chi_1(x) \chi_2(x) x^{-1} \, dx + \int_{[0,1]} dq \, (1 + q^2)^{-1} \left( (1 + q)/(1 - q)^{\alpha_1} + (1 - q)/(1 + q)^{\alpha_2}\right) \\
- 2(1 - q)(1 + q^{2-1/2}) + O(\epsilon^{1/2} \log t),
\]

where \( p_{[0,a]}(x_1, x_2; t) \), \( x_1 \in [0, a], x_2 \in [0, a], t > 0 \), is the Dirichlet heat kernel for the interval \([0, a]\). \( \gamma \) is Euler’s constant and \( \epsilon = \min(\epsilon_1, \epsilon_2) \).

We note that the \( \epsilon \)-dependence in the right-hand side of equation (4) is fictitious. Since \( \chi_1(x) = \chi_2(x) = 1 \) for \( 0 < x \leq \epsilon \), we have that

\[
\log(\epsilon^2/t) + 2 \int_{[\epsilon, a/2]} \chi_1(x) \chi_2(x) x^{-1} \, dx = 2 \int_{[\sqrt{t}, a/2]} \chi_1(x) \chi_2(x) x^{-1} \, dx,
\]

which is independent of \( \epsilon \) for \( 0 < t < \epsilon^2 \). We also note that the leading term in theorem 4 jibes with theorem 1.4 (2) in [11], since the volume of the boundary of the interval \([0, a]\) is equal to 2. This supports the following.

**Conjecture 2.** Let \( M \) be a compact Riemannian manifold with smooth boundary \( \partial M \) and let \( \delta \) denote the distance to the boundary. Let \( \alpha_1 < 2, \alpha_2 < 2, \alpha_1 + \alpha_2 = 1, \) and \( \chi_1 \) and \( \chi_2 \) be smooth functions on \( \mathbb{R}^+ \) with support contained in an interval \([0, b]\), and equal to 1 in a neighbourhood of 0, and where \( b \) is such that \( \delta \) is smooth on the collar \( \partial M \times [0, b] \). If \( t \downarrow 0 \) then

\[
Q(\delta^{-\alpha_1} \chi_1(\delta), \delta^{-\alpha_2} \chi_2(\delta), D)(t) = 2^{-1} \int_{\partial M} \, dy \log t + o(\log t).
\]

We note that theorem 4 and conjecture 2 include the cases where either \( 1 < \alpha_1 < 2 \) or \( 1 < \alpha_2 < 2 \). This requires more care in the proof of theorem 4 than the case where both \( \alpha_1 < 1 \) and \( \alpha_2 < 1 \).

2. The proof of theorem 4

The first step in the proof of theorem 4 is to reduce the calculation on the interval \([0, a]\) to a calculation on the half-line \( \mathbb{R}^+ \). The main motivation for this is that the heat kernel for a half-space is a very simple expression in terms of its variables, while the heat kernel for an interval is given in terms of eigenfunction expansion involving the full spectral resolution of the Laplacian with Dirichlet boundary conditions. A straightforward application of the principle of not feeling the boundary (see [1]) implies that if \( x_1 \in [0, a] \) and \( x_2 \in [0, a] \), then

\[
p_{\mathbb{R}^+}(x_1, x_2; t) \geq p_{[0,a]}(x_1, x_2; t) \\
\geq p_{\mathbb{R}^+}(x_1, x_2; t) - (4\pi t)^{-1/2} e^{-((2a-x-y)^2)/(4at)}.
\]

(5)
This shows for example that if $0 < x_1 < a/2$ and $0 < x_2 < a/2$ the error of approximating the heat kernel for an interval by the one for a half-space is at most $t^{-1/2}e^{-\alpha^2/(4t)}$. This remainder, however, is not integrable when weighted with initial data and specific heat which are not in $L^1([0, a])$. The main part of the proof of lemma 5 is to refine the probabilistic argument which led to equation (5). Lemma 6 deals with the calculation of the doubly weighted half-space integral, and its proof completes the proof of theorem 4.

**Lemma 5.** Let $\alpha_1 < 2$, $\alpha_2 < 2$, $\alpha_1 + \alpha_2 = 1$. If $t \downarrow 0$, then

$$\int \int_{[0,a]^2} p_{[0,a]}(x_1, x_2; t) \chi_1(\delta(x_1)) \chi_2(\delta(x_2)) \delta(x_1)^{-\alpha_1} \delta(x_2)^{-\alpha_2} \, dx_1 \, dx_2$$

$$= 2 \int \int_{\mathbb{R}^2} p_{[0,a]}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_2) x_1^{-\alpha_1} x_2^{-\alpha_2} \, dx_1 \, dx_2 + O(e^{-(\eta_0)c^2/(4t)}),$$

(6)

where $\kappa = a - \epsilon_3 - \epsilon_4$ and $\eta = \max(\alpha_1/2, \alpha_2/2)$.

**Proof.** Without loss of generality, we may assume that $\alpha_1 \geq \alpha_2$. We partition the region of integration $[0, a]^2 = \bigcup_{i=1}^5 A_i$, where

$$A_1 = [0, \epsilon_3] \times [0, \epsilon_4], \quad A_2 = [0, \epsilon_3] \times [a - \epsilon_4, a],$$

$$A_3 = [a - \epsilon_3, a] \times [0, \epsilon_4], \quad A_4 = [a - \epsilon_3, a] \times [a - \epsilon_4, a],$$

$$A_5 = A \setminus \bigcup_{i=1}^4 A_i.$$  

The integrand on the left-hand side of equation (6) is identically equal to 0 on $A_5$, and this set does not contribute to the integral. Since $p_{[0,a]}(x_1, x_2; t) = p_{[0,a]}(a - x_1, a - x_2; t)$ and $p_{[0,a]}(x_1, a - x_2; t) = p_{[0,a]}(a - x_1, x_2; t)$, the contributions of $A_1$ and $A_2$ to the integral on the left-hand side of equation (6) equal the contributions of $A_4$ and $A_3$, respectively. Since $|x_1 - x_2| \geq \kappa$ for $(x_1, x_2) \in A_2$, we have by monotonicity of the Dirichlet heat kernel that

$$p_{[0,a]}(x_1, x_2; t) \leq p_{[0,a]}(x_1, x_2; t) = (4\pi t)^{-1/2} e^{-\eta(x_1-x_2)^2/(4t)} - e^{-x_1^2-x_2^2/(4t)}$$

$$= (4\pi t)^{-1/2} e^{-(x_1-x_2)^2/(4t)}(1 - e^{-x_1^2-x_2^2/(4t)})$$

$$\leq t^{-3/2} x_1 x_2 e^{-\kappa^2/(4t)}.$$  

(7)

Hence, the contribution from $A_2$ to the integral on the left-hand side of equation (6) is bounded from above by

$$t^{-3/2} e^{-\kappa^2/(4t)} \int \int_{A_1} \chi_1(x_1) \chi_2(x_2) x_1^{-\alpha_1} x_2^{-\alpha_2} \, dx_1 \, dx_2 = O(e^{-\kappa^2/(4t)}).$$

The contribution from $A_1$ to the integral on the left-hand side of equation (6) is bounded from above by

$$\int \int_{A_1} p_{[0,a]}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_2) x_1^{-\alpha_1} x_2^{-\alpha_2} \, dx_1 \, dx_2$$

$$= \int \int_{\mathbb{R}^2} p_{[0,a]}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_2) x_1^{-\alpha_1} x_2^{-\alpha_2} \, dx_1 \, dx_2.$$  

This completes the proof of the upper bound.

To establish the lower bound, we note that

$$\int \int_{[0,a]^2} p_{[0,a]}(x_1, x_2; t) \chi_1(\delta(x_1)) \chi_2(\delta(x_2)) \delta(x_1)^{-\alpha_1} \delta(x_2)^{-\alpha_2} \, dx_1 \, dx_2$$

$$\geq 2 \int \int_{A_1} p_{[0,a]}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_2) x_1^{-\alpha_1} x_2^{-\alpha_2} \, dx_1 \, dx_2.$$
It is well known that the Dirichlet heat kernel for an open set \( \Omega \subset \mathbb{R}^m \) has the following probabilistic representation:

\[
p_{\Omega}(x_1, x_2; t) = \frac{1}{|\Omega|} \int_{\Omega} e^{-|x-y|^2/4t} \, dy
\]

where \( (B(s), 0 \leq s \leq t) \) is a Brownian bridge on \( \mathbb{R}^m \) with \( B(0) = x_1, B(t) = x_2 \) and \( \mathbf{P}_{x_1, x_2} \) is the associated conditional probability. For \( \Omega = [0, a] \subset \mathbb{R} \) and for \( x \in [0, a], y \in [0, a] \), we have that

\[
p_{[0,a]}(x_1, x_2; t) = \frac{1}{a} \int_{[0,a]} e^{-|x-y|^2/4t} \, dy
\]

By Hölder’s inequality, we have for \( \eta \in (0, 1) \)

\[
p_{[0,a]}(x_1, x_2; t) \leq \frac{1}{a} \int_{[0,a]} |x-y|^\eta \, dy \leq \frac{1}{a} \int_{[0,a]} (x-y)^2 \, dy = \frac{1}{a} \int_{[0,a]} |x-y|^2 \, dy
\]

By the last inequality in equation (7),

\[
(p_{[0,a]}(x_1, x_2; t))^\eta \leq \frac{1}{a} \int_{[0,a]} |x-y|^2 \, dy
\]

By the last inequality in equation (7),

\[
(p_{[0,a]}(x_1, x_2; t))^\eta \leq \frac{1}{a} \int_{[0,a]} (x-y)^2 \, dy = O(e^{-\eta |a-\frac{a}{2}|(x-y)^2/(4t)})
\]

Integrating the above right-hand side with respect to \( \chi_1(x_1) \chi_2(x_2) x_1^{-\alpha_1} x_2^{-\alpha_2} \, dx_1 \, dx_2 \) yields a bound

\[
t^{-\frac{\eta}{2}} \int_{[0,a]} \chi_1(x_1) x_1^{-\alpha_1} \, dx_1 \int_{[0,a]} \chi_2(x_2) x_2^{-\alpha_2} \, dx_2 e^{-\eta |x-y|^2/(4t)}
\]

\[
\leq t^{-\frac{\eta}{2}} e^{-\eta |a-\frac{a}{2}|(x-y)^2/(4t)} \int_{[0,a]} \chi_1(x_1) x_1^{-\alpha_1} \, dx_1 \int_{[0,a]} \chi_2(x_2) x_2^{-\alpha_2} \, dx_2
\]

Note that since \( 2 > \alpha_1 \) and \( \eta = \alpha_1/2, x_1^{-\alpha_1} = x_1^{\alpha_1/2} \) is integrable at 0. Since \( 1 = \alpha_1 + \alpha_2 \leq 2a_1 \), we have that \( \alpha_1 \geq 1/2 > 0 \). Hence, \( x_2^{\alpha_2} = x_2^{1+(3a_1/2)} \) is also integrable at 0. This completes the proof of the lower bound. \( \square \)

In order to prove theorem 4, it clearly suffices to prove the following.

**Lemma 6.** Let \( \alpha_2 \leq \alpha_1 < 2, \alpha_1 + \alpha_2 = 1 \). If \( t \downarrow 0 \), then

\[
\int_{\mathbb{R}^2} p_{[0,a]}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_2) x_1^{\alpha_1} x_2^{\alpha_2} \, dx_1 \, dx_2
\]

\[
= 2^{-1} \log(e^2/t) + 2^{-1} r + 2 \log(2^{1/2} - 1) + 2 \log 2
\]

\[
+ \int_{[0,a]} \chi_1(x) \chi_2(x) x^{-1} \, dx + 2^{-1} \int_{[0,1]} dq (1 + q^2)
\]

\[
\times ((1 + q)/(1 - q))^{\alpha - 1} + ((1 - q)/(1 + q))^{\alpha' - 1}
\]

\[
- 2(1 - q)(1 + q^2)^{-1/2} + O(t^{1/2} \log t).
\]
Proof. Define
\[ C = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1^2 + x_2^2 \geq \epsilon^2, 0 \leq x_1 \leq \epsilon, 0 \leq x_2 \leq \epsilon_4 \}, \]
\[ C_1 = \{(x_1, x_2) \in C : |x_1 - x_2| \leq \sigma \}, \]
where \( \sigma \in (0, \epsilon/5) \) will be chosen later on. The left-hand side of equation (11) can be written as \( B_1 + B_2 \), where
\[
B_1 = \int_{C \cap \{ |x_1 - x_2| < \epsilon^2 \}} p_{\mathbb{R}^+}(x_1, x_2; t) x_1^{-\alpha_1} x_2^{-\alpha_2} \, dx_1 \, dx_2,
\]
\[
B_2 = \int_{C} p_{\mathbb{R}^+}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_2) x_1^{-\alpha_1} x_2^{-\alpha_2} \, dx_1 \, dx_2.
\] (12)

To estimate \( B_2 \), we first consider the contribution from the set \( C \setminus C_1 \). We have by equation (7) that
\[
p_{\mathbb{R}^+}(x_1, x_2; t) \leq \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} p_{\mathbb{R}^+}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_2) x_1^{-\alpha_1} x_2^{-\alpha_2} \, dx_1 \, dx_2 \leq K \tau^{-1/2} e^{-\sigma^2/4t}, \]
where
\[ K = \int_{C} \chi_1(x_1) \chi_2(x_2) x_1^{-\alpha_1} x_2^{-\alpha_2} \, dx_1 \, dx_2. \] (13)

On \( C \cap \{ |x_1 - x_2| \leq \epsilon/5 \} \), we have that \( x_2 \rightarrow \chi_2(x_2) x_2^{-\alpha_2} \) is \( C^\infty \). Hence, there exists \( L \) depending on \( \alpha_2 \) and \( C \) such that \( |\chi_2(x_2) x_2^{-\alpha_2} - \chi_2(x_2) x_1^{-\alpha_2}| \leq L |x_1 - x_2| \). It is easily seen that both \( x_1 \geq \epsilon/2 \) and \( x_2 \geq \epsilon/2 \) on \( C \cap \{ |x_1 - x_2| \leq \epsilon/5 \} \). Since the Dirichlet heat kernel on \( \mathbb{R}^+ \) is bounded from above by \( t^{-1/2} \), we have that
\[
\int_{C_1} p_{\mathbb{R}^+}(x_1, x_2; t) \chi_1(x_1) x_1^{-\alpha_1} L |x_1 - x_2| \, dx_1 \, dx_2 \leq L (2/\epsilon) t^{-1/2} \int_{C_1} |x_1 - x_2| \, dx_1 \, dx_2 \leq 2aL (2/\epsilon) t^{-1/2} \sigma^2. \] (14)

We now choose \( \sigma^2 \) so as to minimize \( t^{-3/2} e^{-\sigma^2/4t} + t^{-1/2} \sigma^2 \), i.e.
\[ \sigma^2 = 4t \log(t^{-1/2}). \]

This gives that for \( t \) sufficiently small the right-hand sides of equations (13) and (14) are \( O(t^{1/2}) \) and \( O(t^{1/2} \log(t^{-1})) \), respectively. We conclude that
\[
B_2 = \int_{C_1} p_{\mathbb{R}^+}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_2) x_1^{-\alpha_1} \, dx_1 \, dx_2 + O(t^{1/2} \log(t^{-1})). \] (15)

We now write
\[ C_1 = (C_1 \cap \{ x_1^2 \geq \epsilon^2/2 \}) \cup (C_1 \cap \{ x_1^2 < \epsilon^2/2 \}) = C_2 \cup C_3. \]

Since \( x_1 \geq \epsilon/2 \) on \( C_1 \), we have that the integrand in the first term on the right-hand side of equation (15) is bounded by \( 2e^{-t} t^{-1/2} \). Hence
\[
\int_{C_1} p_{\mathbb{R}^+}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_2) x_1^{-\alpha_1} \, dx_1 \, dx_2 \leq 2e^{-t} t^{-1/2} |C_3|, \]
where \( |\cdot| \) denotes the Lebesgue measure. It is easily seen that \( |C_3| \leq \sigma^2/2 \). Consequently,
\[ 0 \leq \int_{C_1} p_{\mathbb{R}^+}(x_1, x_2; t) \chi_1(x_1) \chi_2(x_2) x_1^{-\alpha_1} \, dx_1 \, dx_2 \leq e^{-t} t^{-1/2} \sigma^2, \]
and so the contribution from $C_3$ to the integral in equation (15) is $O(t^{1/2} \log(t^{-1}))$. Furthermore by monotonicity of the Dirichlet heat kernel $p_{\mathbb{R}^2}(x_1, x_2; t) \leq p_{\mathbb{R}}(x_1, x_2; t)$. Hence,

\[
\int_{C_2} p_{\mathbb{R}^2}(x_1, x_2; t) \chi_1(x_1)\chi_2(x_1)\chi_1^{-1} \, dx_1 \, dx_2 \leq \int_{[t^2 \geq \sigma^2]} p_{\mathbb{R}}(x_1, x_2; t) \chi_1(x_1)\chi_2(x_1)\chi_1^{-1} \, dx_1 \, dx_2
\]

\[
= \int_{[t^2 \geq \sigma^2]} \chi_1(x_1)\chi_2(x_1)\chi_1^{-1} \, dx_1.
\]

To obtain a lower bound for the contribution from $C_2$ to the integral in equation (15), we first observe that $(4\pi t)^{-1/2}e^{-(x_1^2 + x_2^2)/(4t)} \leq t^{-1/2}e^{-x^2/(4t)}$ and $x \geq \epsilon/2$ for $(x_1, x_2) \in C_2$. Therefore,

\[
0 \leq \int_{C_2} (4\pi t)^{-1/2}e^{-(x_1^2 + x_2^2)/(4t)} \chi_1(x_1)\chi_2(x_1)\chi_1^{-1} \, dx_1 \, dx_2
\]

\[
\leq 2e^{-1/2}t^{-1/2}e^{-x^2/(4t)} |C_2| \leq 2\pi^2 e^{-1/2}t^{-1/2}e^{-x^2/(4t)} = O(e^{-x^2/(5t)}).
\]

Finally,

\[
\int_{C_2} p_{\mathbb{R}^2}(x_1, x_2; t) \chi_1(x_1)\chi_2(x_1)\chi_1^{-1} \, dx_1 \, dx_2 \geq \int_{[t^2 \geq \sigma^2]} p_{\mathbb{R}}(x_1, x_2; t) \chi_1(x_1)\chi_2(x_1)\chi_1^{-1} \, dx_1 \, dx_2
\]

\[
- \int_{[|x_1-x_2| \geq \sigma, t^2 \geq \sigma^2]} p_{\mathbb{R}}(x_1, x_2; t) \chi_1(x_1)\chi_2(x_1)\chi_1^{-1} \, dx_1 \, dx_2
\]

\[
= \int_{[t^2 \geq \sigma^2]} \chi_1(x_1)\chi_2(x_1)\chi_1^{-1} \, dx_1
\]

\[
- \int_{[|x_1-x_2| \geq \sigma, t^2 \geq \sigma^2]} p_{\mathbb{R}}(x_1, x_2; t) \chi_1(x_1)\chi_2(x_1)\chi_1^{-1} \, dx_1 \, dx_2.
\]

Moreover,

\[
\int_{[|x_1-x_2| \geq \sigma]} p_{\mathbb{R}}(x_1, x_2; t) \, dx_2 \leq \int_{[|x_1-x_2| \geq \sigma]} (4\pi t)^{-1/2}e^{-|x_1-x_2|^2/(4t)} \, dx_2
\]

\[
= 4\pi^{-1/2}t^{-1/2}e^{-\sigma^2/(4t)} = O(t^2).
\]

Putting all this together gives that

\[
B_2 = \int_{[t^2 \geq \sigma^2]} \chi_1(x)\chi_2(x)x^{-1} \, dx + O(t^{1/2} \log(t^{-1}))
\]

\[
= \int_{[t^2 \geq \sigma^2]} \chi_1(x)\chi_2(x)x^{-1} \, dx + 2^{-1} \log 2 + O(t^{1/2} \log(t^{-1}))
\]

since $\chi_1(x)\chi_2(x)x^{-1} = x^{-1}$ for $0 < x \leq \epsilon/2$.

In order to obtain the asymptotic behaviour of $B_1$ in equation (12), we introduce polar coordinates $x = (4t)^{1/2}\rho \cos \theta$, $y = (4t)^{1/2}\rho \sin \theta$ to find that

\[
B_1 = \pi^{-1/2} \int_{[0, \pi/2]} d\theta (\cos \theta)^{-\alpha} (\sin \theta)^{\beta-1}
\]

\[
\times \int_{[0, \epsilon/(4t)^{1/2}]} d\rho (e^{-\rho^2(1-\sin(2\theta))} - e^{-\rho^2(1+\sin(2\theta))})
\]

\[
= \pi^{-1/2} \int_{[0, \pi/2]} d\theta (\cos \theta)^{-\alpha} (\sin \theta)^{\beta-1}
\]

\[
\times \int_{[0, \epsilon/(4t)^{1/2}]} d\rho (e^{-\rho^2(1-\sin(2\theta))} - e^{-\rho^2(1+\sin(2\theta))})
\]
A further change of variable \( \theta = \phi + \pi/4 \) yields that
\[
B_1 = (2/\pi)^{1/2} \int_{0,\pi/4} d\phi \left( \frac{\cos \phi + \sin \phi}{\cos \phi - \sin \phi} \right)^{\alpha - 1} \left( \frac{\cos \phi - \sin \phi}{\cos \phi + \sin \phi} \right)^{\alpha - 1} \\
\times \int_{0,\epsilon/(4\pi^{1/2})} d\rho (e^{-2\rho^2} - e^{-2\rho^2}) = B_3 + B_4 + B_5,
\]
where
\[
B_3 = (2/\pi)^{1/2} \int_{0,\pi/4} d\phi \left( \frac{\cos \phi + \sin \phi}{\cos \phi - \sin \phi} \right)^{\alpha - 1} \left( \frac{\cos \phi - \sin \phi}{\cos \phi + \sin \phi} \right)^{\alpha - 1} - 2
\]
\[
\times \int_{0,\infty} d\rho (e^{-2\rho^2} - e^{-2\rho^2}) = 2^{-1} \int_{0,\epsilon/(4\pi^{1/2})} d\phi (\cos \phi)^{-1} (\sin \phi)^{-1} \\
\times \left( \frac{(\cos \phi + \sin \phi)^{\alpha - 1}}{(\cos \phi - \sin \phi)^{\alpha - 1}} + \frac{(\cos \phi - \sin \phi)^{\alpha - 1}}{(\cos \phi + \sin \phi)^{\alpha - 1}} - 2(\cos \phi - \sin \phi) \right)
\]
\[
= 2^{-1} \int_{0,\epsilon/(4\pi^{1/2})} d\phi q^{-1}(1 + q^2) \times [(1 + q)/(1 - q)]^{\alpha - 1} \\
+ ((1 - q)/(1 + q))^{\alpha - 1} - 2(1 - q)(1 + q^{-1})^{1/2},
\]
(16)
\[
B_4 = -(2/\pi)^{1/2} \int_{0,\pi/4} d\phi \int_{0,\epsilon/(4\pi^{1/2})} d\rho (e^{-2\rho^2} - e^{-2\rho^2})
\]
\[
\times \left( \frac{(\cos \phi + \sin \phi)^{\alpha - 1}}{(\cos \phi - \sin \phi)^{\alpha - 1}} + \frac{(\cos \phi - \sin \phi)^{\alpha - 1}}{(\cos \phi + \sin \phi)^{\alpha - 1}} - 2 \right)
\]
and
\[
B_5 = (8/\pi)^{1/2} \int_{0,\pi} d\phi \int_{0,\epsilon/(4\pi^{1/2})} d\rho (e^{-2\rho^2} - e^{-2\rho^2})
\]
\[
(17)
\]
We have used the standard change of variables \( \tan \phi = q \) to obtain the last identity in equation (16).

In order to find the asymptotic behaviour of \( B_4 \) as \( t \downarrow 0 \) we first consider the contribution of the second term in the integrand with respect to \( \rho \) in equation (17), and write
\[
-(8/\pi)^{1/2} \int_{0,\pi/4} d\phi \int_{0,\epsilon/(4\pi^{1/2})} d\rho e^{-2\rho^2} \cos \phi^2
\]
\[
= -\int_{0,\pi/4} d\phi (\cos \phi)^{-1} + (8/\pi)^{1/2} \int_{0,\pi/4} d\phi \int_{\epsilon/(4\pi^{1/2})} d\rho e^{-2\rho^2} \cos \phi^2
\]
\[
= \log(2^{1/2} - 1) + O(e^{-\epsilon^2/(5\pi)}).
\]
The contribution of the first term in the integrand with respect to \( \rho \) in equation (17) is calculated as follows:
\[
(8/\pi)^{1/2} \int_{0,\pi/4} d\phi \int_{0,\epsilon/(4\pi^{1/2})} d\rho e^{-2\rho^2} \sin \phi^2
\]
\[
= (8/\pi)^{1/2} \int_{0,\pi/4} d\phi \int_{0,\epsilon/(4\pi^{1/2})} d\rho e^{-2\rho^2} \phi^2 + \int_{0,\pi/4} d\phi ((\sin \phi)^{-1} - \phi^{-1})
\]
\[
+ (8/\pi)^{1/2} \int_{0,\pi/4} d\phi \int_{\epsilon/(4\pi^{1/2})} d\rho (e^{-2\rho^2} - e^{-2\rho^2})
\]
(18)
The third term on the right-hand side of equation (18) is \( O(\epsilon^{-7/5}) \). The second term on the right-hand side of equation (18) is equal to \( \log(2^{1/2} - 1) + 3 \log 2 - \log \pi \). The first term on the right-hand side of equation (18) equals
\[
\frac{4}{\pi} \int_{[0, \pi \epsilon/(32 \gamma)^2]} d\phi \int_{[0, \phi]} d\rho e^{-\rho^2}
= (4/\pi)^{1/2} \int_{[0, \pi \epsilon/(32 \gamma)^2]} d\phi e^{-\rho^2}
\]

Furthermore, for \( \phi \in [0, \pi / 4] \),
\[
0 \leq \int_{[\epsilon / (4t)^{1/2}, \infty)} d\rho (e^{-2\rho^2} (\sin \phi)^2 - e^{-2\rho^2} (\cos \phi)^2)
\]

Hence,
\[
|B_4| \leq \pi^{-1/2} \frac{1}{\epsilon^{1/2}} \epsilon^{-1} \int_{[0, \pi / 4]} d\phi (\sin \phi)^{-2} (\cos \phi)^{-2}
\times \left| \frac{(\cos \phi + \sin \phi)^{\alpha}}{(\cos \phi - \sin \phi)^{\alpha}} + \frac{(\cos \phi - \sin \phi)^{\alpha}}{(\cos \phi + \sin \phi)^{\alpha-1}} - 2((\cos \phi)^2 - (\sin \phi)^2) \right|.
\]

We see that the integral with respect to \( \phi \) converges both at \( \phi = 0 \) and \( \phi = \pi / 4 \). We conclude that \( B_4 = O(\epsilon^{1/2}) \).

3. The proof of theorem 3

We shall assume that \( \Re(\alpha_1) < 0 \) and \( \Re(\alpha_2) < 0 \) and then apply analytic continuation to establish the general case. We shall also assume that \( \alpha_1 + \alpha_2 \notin \mathbb{Z} \) to ensure that the interior and the boundary terms do not interact. The invariants \( \beta^j_M \) are given by local formulae. Standard
arguments using dimensional analysis yield the following result; as these arguments are by
now standard (see, for example, the discussion in [7]), we omit details in the interests of
brevity.

**Lemma 7.** There exist universal constants $\varepsilon_{i_1,i_2}^0$ so that
\[ \beta_{0M}^i = \int_M \varepsilon_{i_1,i_2}^0 \langle \psi_1^0, \psi_2^0 \rangle \, dy \]
\[ \beta_{1M}^i = \int_M \left\{ \varepsilon_{i_1,i_2}^1 \langle \psi_1^0, \psi_2^0 \rangle + \varepsilon_{i_1,i_2}^2 \{ L_{\omega_0} \psi_1^0, \psi_2^0 \} + \varepsilon_{i_1,i_2}^3 \langle \psi_1^1, \psi_2^1 \rangle \right\} \, dy, \]
\[ \beta_{2M}^i = \int_M \left\{ \varepsilon_{i_1,i_2}^4 \langle \psi_1^1, \psi_2^0 \rangle + \varepsilon_{i_1,i_2}^5 \{ L_{\omega_0} \psi_1^1, \psi_2^0 \} + \varepsilon_{i_1,i_2}^6 \langle E \psi_1^0, \psi_2^0 \rangle + \varepsilon_{i_1,i_2}^7 \langle \psi_1^1, \psi_2^1 \rangle \right\} \, dy, \]
\[ + \varepsilon_{i_1,i_2}^8 \{ L_{\omega_0} \psi_1^0, \psi_2^0 \} + \varepsilon_{i_1,i_2}^9 \{ R_{\kappa_{\omega_0}} \psi_1^0, \psi_2^0 \} + \varepsilon_{i_1,i_2}^{10} \{ L_{\omega_0} \psi_1^0, \psi_2^0 \} + \varepsilon_{i_1,i_2}^{11} \{ \psi_1^0, \psi_2^0 \} + \varepsilon_{i_1,i_2}^{12} \{ \psi_1^1, \psi_2^1 \} + \varepsilon_{i_1,i_2}^{13} \{ \psi_1^0, \psi_2^0 \} + \varepsilon_{i_1,i_2}^{14} \{ \psi_1^1, \psi_2^1 \} \right\} \, dy. \]

**Remark 3.** We note that $\varepsilon_{i_1,i_2}^0 = \varepsilon_{i_1,i_2}^1$, is given by equation (3).

There is a basic symmetry which is useful. Let $e^{-tD}$ denote the fundamental solution of
the Dirichlet Laplacian and let $\tilde{D}$ be the dual operator on the dual vector bundle $V$. The lemma
below follows immediately from the identity
\[ Q(\psi_1, \psi_2, D)(t) = \int_M (e^{-tD} \psi_1, \psi_2) \, dx = \int_M (\psi_1, e^{-t\tilde{D}} \psi_2) \, dx = Q(\psi_2, \psi_1, \tilde{D})(t). \]

**Lemma 8.** Adopt the notation of lemma 7.
\[ \varepsilon_{i_1,i_2}^0 = \varepsilon_{i_1,i_2}^0, \quad \varepsilon_{i_1,i_2}^1 = \varepsilon_{i_1,i_2}^1, \quad \varepsilon_{i_1,i_2}^2 = \varepsilon_{i_1,i_2}^2, \quad \varepsilon_{i_1,i_2}^3 = \varepsilon_{i_1,i_2}^3, \quad \varepsilon_{i_1,i_2}^4 = \varepsilon_{i_1,i_2}^4, \quad \varepsilon_{i_1,i_2}^5 = \varepsilon_{i_1,i_2}^5, \quad \varepsilon_{i_1,i_2}^6 = \varepsilon_{i_1,i_2}^6, \quad \varepsilon_{i_1,i_2}^7 = \varepsilon_{i_1,i_2}^7, \quad \varepsilon_{i_1,i_2}^8 = \varepsilon_{i_1,i_2}^8, \quad \varepsilon_{i_1,i_2}^9 = \varepsilon_{i_1,i_2}^9, \quad \varepsilon_{i_1,i_2}^{10} = \varepsilon_{i_1,i_2}^{10}, \quad \varepsilon_{i_1,i_2}^{11} = \varepsilon_{i_1,i_2}^{11}, \quad \varepsilon_{i_1,i_2}^{12} = \varepsilon_{i_1,i_2}^{12}, \quad \varepsilon_{i_1,i_2}^{13} = \varepsilon_{i_1,i_2}^{13}, \quad \varepsilon_{i_1,i_2}^{14} = \varepsilon_{i_1,i_2}^{14}. \]

Next, we consider some product formulae.

**Lemma 9.** Suppose that $M = M_1 \times M_2$, that $g_M = g_{M_1} + g_{M_2}$, that $\partial M_1 = \emptyset$ and that $D_M = D_{M_1} + D_{M_2}$ where $D_{M_1}$ and $D_{M_2}$ are scalar operators of Laplace type on $M_1$ and on $M_2$, respectively. Suppose that $\psi_1^M = \psi_1^{M_1} \psi_1^{M_2}$ and $\psi_2^M = \psi_2^{M_1} \psi_2^{M_2}$ decompose similarly. Then,
\[ (a) \quad \beta(\psi_1^M, \psi_2^M, D_M)(t) = \beta(\psi_1^{M_1}, \psi_2^{M_1}, D_{M_1})(t) \beta(\psi_1^{M_2}, \psi_2^{M_2}, D_{M_2})(t). \]
\[ (b) \quad \int_M \beta_{k,0,0}^M(\psi_1^M, \psi_2^M, D_M) \, dy = \sum_{\lambda_1 + \lambda_2 = k} \frac{(-1)^{\lambda_1}}{\lambda_1 ! \lambda_2 !} \int_M \beta_{\lambda_1,0}^{M_1}(\psi_1^{M_1}, (\tilde{D}_{M_1})^\lambda \psi_1^{M_1}) \, d\lambda_1 \int_M \beta_{\lambda_2,0}^{M_2}(\psi_2^{M_2}, (\tilde{D}_{M_2})^\lambda \psi_2^{M_2}) \, d\lambda_2 \cdot \int_M \beta_{0,0,0}^{M_1}(\psi_1^{M_1}, \psi_2^{M_1}, D_{M_1}) \, d\lambda_1 \int_M \beta_{0,0,0}^{M_2}(\psi_1^{M_2}, \psi_2^{M_2}, D_{M_2}) \, d\lambda_2. \]
\[ (c) \quad \text{The universal constants} \varepsilon_{i_1,i_2}^M \text{are dimension free}. \]
\[ (d) \quad \varepsilon_{i_1,i_2}^0 = \varepsilon_{i_1,i_2}^0, \quad \varepsilon_{i_1,i_2}^1 = \varepsilon_{i_1,i_2}^1, \quad \varepsilon_{i_1,i_2}^2 = \varepsilon_{i_1,i_2}^2, \quad \varepsilon_{i_1,i_2}^3 = \varepsilon_{i_1,i_2}^3, \quad \varepsilon_{i_1,i_2}^4 = \varepsilon_{i_1,i_2}^4, \quad \varepsilon_{i_1,i_2}^5 = \varepsilon_{i_1,i_2}^5, \quad \varepsilon_{i_1,i_2}^6 = \varepsilon_{i_1,i_2}^6, \quad \varepsilon_{i_1,i_2}^7 = \varepsilon_{i_1,i_2}^7, \quad \varepsilon_{i_1,i_2}^8 = \varepsilon_{i_1,i_2}^8, \quad \varepsilon_{i_1,i_2}^9 = \varepsilon_{i_1,i_2}^9, \quad \varepsilon_{i_1,i_2}^{10} = \varepsilon_{i_1,i_2}^{10}, \quad \varepsilon_{i_1,i_2}^{11} = \varepsilon_{i_1,i_2}^{11}, \quad \varepsilon_{i_1,i_2}^{12} = \varepsilon_{i_1,i_2}^{12}, \quad \varepsilon_{i_1,i_2}^{13} = \varepsilon_{i_1,i_2}^{13}, \quad \varepsilon_{i_1,i_2}^{14} = \varepsilon_{i_1,i_2}^{14}. \]

**Proof.** Assertion (a) follows from the identity $e^{-tD_M} = e^{-tD_{M_1}} e^{-tD_{M_2}}$ and assertion (b) follows from assertion (a). If we take $M_1 = S^1$, $D_{M_1} = -\partial_{r^2}$, $\psi_1^{M_1} = 1$ and $\psi_2^{M_1} = 1$, we have that
\[ \beta(\psi_1^{M_1}, \psi_2^{M_1}, D_{M_1})(t) = 2\pi. \]
This then yields the identity
\[ \int_M \beta_{k,0,0}^M(\psi_1^{M_1}, \psi_2^{M_1}, D) \, dy = 2\pi \int_M \beta_{k,0,0}^{M_1}(\psi_1^{M_1}, \psi_2^{M_1}, D_{M_1}) \, dy. \]

Assertion (c) now follows. We take $M_2 = [0, 1]$ and $D_2 = -\partial_{r^2}$. We take
\[ \psi_1^{M_2} = \psi_2^{M_2} = 0 \quad \text{near} \quad r = 1, \]
\[ \psi_1^{M_2} = r^{-a_2} \quad \text{and} \quad \psi_2^{M_2} = r^{b_2} \quad \text{near} \quad r = 0. \]
Since the structures on $M_2$ are flat, we have $\psi_1^k = \psi_2^k = 0$ for $k > 0$ while

$$\psi_2^0 = \psi_1^0 = \begin{cases} 0 & \text{at } r = 1 \\ 1 & \text{at } r = 0 \end{cases}.$$ 

Consequently,

$$\beta^{M_2}_k(\psi_1^{M_1}, \psi_2^{M_1}, D_{M_2})(r) = \begin{cases} 0 & \text{if } r = 1 \text{ and } k \geq 0 \\ 0 & \text{if } r = 0 \text{ and } k > 0 \\ \epsilon_{a_1a_2} & \text{if } r = 0 \text{ and } k = 0 \end{cases}. $$

As the second fundamental form vanishes, the distinction between ‘;’ and ‘’ disappears, and we have $D_1\psi_2^{M_1} = - (\psi_2^{2,\alpha} + \tilde{E}\psi_2)$. Calculating on the interior then implies that

$$\beta_2(\psi_1^{M_1}, \psi_2^{M_1}, D_{M_1}) = \int_{M_1} \left( \psi_1^{M_1}, \psi_2^{M_1} + \tilde{E}\psi_2^{M_1} \right) \ dx_1.$$ 

We may therefore use assertion (b) to derive the following identity from which assertion (e) will follow:

$$\int_{\partial M} \beta^{M_2}_{2,a_1a_2}(\psi_1^{M_1}, \psi_2^{M_1}, D_{M_2}) \ dy = \psi_2^{0, a_1a_2} \int_{M_1} \left( \psi_1^{M_1}, \psi_2^{M_1} + \tilde{E}\psi_2^{M_1} \right) \ dx_1. \quad \Box$$

We continue our study by index shifting.

**Lemma 10.**

$$\epsilon_{a_1a_2} = \begin{cases} \epsilon_0 & a_1 \neq a_2 \text{ and } a_1 \neq 1 \\ \epsilon_0 & a_1 = a_2 \text{ and } a_1 \neq 1 \\ \epsilon_0 & a_1 = a_2 \text{ and } a_1 = 1 \\ \epsilon_0 & a_1 \neq a_2 \text{ and } a_1 = 1 \\ \epsilon_0 & a_1 = a_2 \text{ and } a_1 = 1 \end{cases}.$$ 

**Proof.** We assume $\psi_1$ and $\psi_2$ have compact support near the boundary of $M$. We set $\tilde{\psi}_1 := (\delta^n \psi)_\delta^{-a_1 - a_2}$ and $\tilde{\psi}_2 := (\delta^n \psi)_\delta^{-a_2 - a_2}$ for $n \in \mathbb{N}$. We compute

$$\sum_k \xi^{(1-k-n_1-n_2-a_2)/2} \int_{\partial M} \beta_{k,a_1+a_2+a_2}(\tilde{\psi}_1, \tilde{\psi}_2, D) \ dy = \sum_k \xi^{(1-c-a_2)/2} \int_{\partial M} \beta_{k,a_1+a_2}(\psi_1, \psi_2, D) \ dy.$$

We set $k = \ell + n_1 + n_2$, and equate powers of $t$ to see

$$\beta_{\ell+n_1+n_2+a_1+a_2}(\tilde{\psi}_1, \tilde{\psi}_2, D) = \beta_{\ell,a_1,a_2}(\psi_1, \psi_2, D).$$ 

Note that $\tilde{\psi}_1^{\mu+n_1} = \tilde{\psi}_1^\mu$ and $\tilde{\psi}_2^{\mu+n_2} = \tilde{\psi}_2^\mu$. The desired result now follows by taking $(n_1, n_2) = (1, 0), (0, 1), (2, 0), (1, 1)$ and $(0, 2). \quad \Box$

**Lemma 11.** Let $\mathbb{T}^{m-1}$ denote the torus with periodic parameters $(y_1, \ldots, y_{m-1})$ and $M := \mathbb{T}^{m-1} \times [0, 1]$. Let $f_\alpha \in C^\infty([0, 1])$ have compact support near $r = 0$ with $f_\alpha(0) = 0$. Let $\Theta(r) \in C^\infty([0, 1])$ have compact support near $r = 0$ with $\Theta \equiv 1$. Let $\delta_\alpha \in \mathbb{R}$. Set

$$d\Sigma = \sum_{\alpha} e^{-2f_\alpha} \ dy_2 \circ dy_0 + dr \circ dr, \quad \psi_2 := \Theta(r)e^{-\sum_{\alpha} f_\alpha}r^{-a_2}, \quad M_{\delta_\alpha} := \sum_{\alpha} e^{-2f_\alpha}(\delta_{y_2} + \delta_\alpha \partial_{y_0}) - \delta^2, \quad \psi_1 := \Theta(r)r^{-a_1}.$$ 

**(a)** If $k > 0$, then $\int_{\partial M} \beta_{k,a_1a_2}^{M_2}(\psi_1, \psi_2, D_M) \ dy = 0.$

**(b)** $-\frac{1}{2} \epsilon_{a_1a_2} - \epsilon_{a_1a_2}^2 - \frac{1}{2} \epsilon_{a_1a_2}^3 = 0.$

**(c)** $-\frac{1}{2} (\epsilon_{a_1a_2} + \epsilon_{a_1a_2}^2) = 0.$

**(d)** $\frac{1}{2} \epsilon_{a_1a_2} + \frac{1}{2} \epsilon_{a_1a_2} + \frac{1}{2} \epsilon_{a_1a_2}^2 = 0.$

**(e)** $\epsilon_{a_1a_2} + \frac{1}{2} \epsilon_{a_1a_2}^2 + \frac{1}{2} \epsilon_{a_1a_2}^3 = \epsilon_{a_1a_2}^4 + \epsilon_{a_1a_2}^5 + \epsilon_{a_1a_2}^6 + \epsilon_{a_1a_2}^7 = 0.$
The structures are flat on \([0, 1]\) and \(D_M\) on \(M\). Since \(\Theta\) vanishes near \(r = 1\), this boundary component plays no role. Let \(u(r, t)\) be the solution of the heat equation on \([0, 1]\) with Dirichlet boundary conditions and initial temperature \(\psi_1\). The parameter \(r\) is the geodesic distance to the boundary near \(r = 0\). Since the problem decouples, \(u(r, t)\) is also the solution of the heat equation on \(M\) with Dirichlet boundary conditions. The Riemannian measure
\[
dx = \sqrt{\det g_{ij}} \, dy \, dr = e^{\sum_i f_i} \, dy \, dr.
\]
As \(\psi_2 = \Theta e^{-\sum_i f_i} r^{-\alpha_i}\), \(\psi_2 \, dx = \dTheta \, dx \, dr\). Since \(\text{vol}(\mathbb{T}^{m-1}) = (2\pi)^{m-1}\),
\[
Q(\psi_1, \psi_2, D)(t) = \int u(r, t) \psi_2 \, dx = (2\pi)^{m-1} \int_0^1 u(r, t) \Theta(r) r^{-\alpha_i} \, dr
\]
\[
= (2\pi)^{m-1} \beta(\Theta^{-\alpha_i}, \Theta r^{-\alpha_i} - \delta^2_{\gamma_i})(t).
\]
The structures are flat on \([0, 1]\). Since \(\Theta\) vanishes identically near \(r = 1\) and \(\Theta\) is identically 1 near \(r = 0\), only the term \(\beta_0\) is relevant in computing the boundary terms; the \(\beta_k, \alpha_i, \alpha_j\) vanish for \(k \geq 1\).

To apply assertion (a), we must determine the relevant tensors. We have
\[
\Gamma_{\alpha\beta\gamma} = -\int_0^1 f_a^{\beta\gamma} \, e^\sigma \, f_a, \quad \Gamma^{\alpha\beta\gamma} = -\int_0^1 f_a^{\alpha\sigma} \, e^\sigma \, f_a, \quad L_{ab} = \Gamma^{\alpha\beta\gamma} |_{ab} = -\int_0^1 f_a^{\alpha\beta} \, e^\sigma \, f_a, \quad \omega_a = \int_0^1 \frac{1}{2} \epsilon^{\alpha\beta\gamma} \, e^\sigma \, f_a, \quad \omega_m = \int_0^1 \frac{1}{2} \sum_a f_a^\alpha.
\]
Consequently,
\[
R_{\alpha\beta\gamma\delta} = g((\nabla_\alpha \nabla_\delta - \nabla_\delta \nabla_\alpha) e_\beta, e_\gamma) = \Gamma_{\alpha\beta\gamma}^m \Gamma_{\mu\nu}^m - \partial_m \Gamma_{\alpha\beta\gamma}^m = -\frac{1}{2} (f_a^\alpha f_a^\beta + f_a^\beta f_a^\alpha) \, e^{\sigma\delta} \delta_{ab},
\]
\[
\text{Ric}_{\alpha\beta} = -\sum_a \left\{ f_a^\alpha f_a^\beta - f_a^\alpha f_a^\beta \right\},
\]
\[
|_{ab} = -\partial_m \omega_m - \omega_a^2 - \omega_m^2 + \omega_m \Gamma_{\alpha\beta\gamma}^m = \frac{1}{2} \sum_a f_a^\alpha - \frac{1}{4} \sum_a \delta_a^2 - \frac{1}{2} \sum_a f_a f_a + \frac{1}{2} \sum_a f_a f_a = \frac{1}{2} \sum_a f_a^\alpha - \frac{1}{4} \sum_a \delta_a^2 + \frac{1}{2} \sum a f_a f_a.
\]

We compute
\[
\psi_0^0 = 1, \quad \psi_1^0 = (\nabla_\alpha (r^\mu \psi_1)) \big|_{\mathbb{T}} = \left\{ \left( \partial_\alpha - \frac{1}{2} \sum_a f_a^\alpha \right) (1) \right\} \big|_{\mathbb{T}} = -\frac{1}{2} \sum_a f_a,
\]
\[
\psi_1^1 = \frac{1}{2} \left\{ (\nabla_\alpha)^2 (r^\mu \psi_1) \right\} \big|_{\mathbb{T}} = \frac{1}{2} \left\{ \left( \partial_\alpha - \frac{1}{2} \sum_a f_a^\alpha \right)^2 (1) \right\} \big|_{\mathbb{T}} = \frac{1}{8} \sum_{a,b} f_a f_b - \frac{1}{4} \sum_a f_a^\alpha.
\]
We must now simply trace through the logic train. We have computed that
\[
\psi_2^0 = 1,
\]
\[
\psi_2^1 = \left\{ \nabla_{\alpha_2}(\psi_2) \right\}_{\partial M} = \left\{ \left( \partial_\tau + \frac{1}{2} \sum_a f_a' (e^{- \sum_{\tau} f_a}) \right) \right\}_{\partial M} = -\frac{1}{2} \sum_a f_a',
\]
\[
\psi_2^2 = \frac{1}{2} \left\{ (\nabla_{\alpha_2})^2 \psi_2 \right\}_{\partial M} = \frac{1}{2} \left\{ \left( \partial_\tau + \frac{1}{2} \sum_a f_a' \right)^2 (e^{- \sum_{\tau} f_a}) \right\}_{\partial M} = \frac{1}{8} \sum_a f_a' f_a - \frac{1}{4} \sum_a f_a''.
\]

Considering the terms \( \sum_a f_a' \) in \( \beta_{1,\alpha}^{BM} \) yields assertion (b), \( \sum_a \delta_a^2 \) in \( \beta_{2,\alpha}^{BM} \) yields assertion (c), \( \sum_a f_a'' \) in \( \beta_{3,\alpha}^{BM} \) yields assertion (d), \( \sum_{a,b} f_a f_b \) in \( \beta_{2,\alpha}^{BM} \) yields assertion (e) and \( \sum_a (f_a')^2 \) in \( \beta_{3,\alpha}^{BM} \) yields assertion (f).

### 3.1. The proof of theorem 3

We must now simply trace through the logic train. We have computed that
\[
e_{a_1,a_2}^0 = c_{a_1,a_2}, \quad \text{and} \quad e_{a_1,a_2}^1 = c_{a_1-1,a_2} + c_{a_1,a_2-1},
\]
\[
e_{a_1,a_2}^2 = -\frac{1}{2} (e_{a_1,a_2}^1 + e_{a_1,a_2}^3) = -\frac{1}{2} (c_{a_1-1,a_2} + c_{a_1,a_2-1}),
\]
\[
e_{a_1,a_2}^3 = c_{a_1-2,a_2} \quad \text{and} \quad e_{a_1,a_2}^4 = c_{a_1,a_2-2},
\]
\[
e_{a_1,a_2}^4 = c_{a_1,a_2} \quad \text{and} \quad e_{a_1,a_2}^5 = c_{a_1-1,a_2-1},
\]
\[
e_{a_1,a_2}^5 = -c_{a_1,a_2}, \quad \text{and} \quad e_{a_1,a_2}^6 = e_{a_1-1,a_2-1} - c_{a_1,a_2-2},
\]
\[
e_{a_1,a_2}^6 = e_{a_1-1,a_2} = -\frac{1}{2} (c_{a_1-2,a_2} + c_{a_1-1,a_2-1}),
\]
\[
e_{a_1,a_2}^7 = e_{a_1,a_2-1} = -\frac{1}{2} (c_{a_1-1,a_2-1} + c_{a_1,a_2-2}),
\]
\[
e_{a_1,a_2}^8 = e_{a_1-1,a_2-2} = -\frac{1}{2} \left( \frac{1}{4} e_{a_1-2,a_2} + \frac{1}{4} e_{a_1,a_2-2} + \frac{1}{2} e_{a_1-1,a_2} \right)
\]
\[
= -\frac{1}{4} c_{a_1-2,a_2} - \frac{1}{4} c_{a_1,a_2-2} + \frac{1}{2} c_{a_1,a_2},
\]
\[
e_{a_1,a_2}^9 = \frac{1}{2} e_{a_1,a_2} + \frac{1}{2} e_{a_1-1,a_2+1} + \frac{1}{2} e_{a_1-1,a_2+1} + \frac{1}{2} e_{a_1,a_2} + \frac{1}{2} e_{a_1,a_2}
\]
\[
= \left\{ \frac{1}{4} e_{a_1-2,a_2} + \frac{1}{4} c_{a_1-2,a_2} + \frac{1}{4} c_{a_1-1,a_2-1} + \frac{1}{4} c_{a_1,a_2} + \frac{1}{4} c_{a_1,a_2-2} + \frac{1}{2} c_{a_1-1,a_2} + \frac{1}{4} c_{a_1,a_2-2} + \frac{1}{2} c_{a_1-1,a_2-1} \right\}
\]
\[
= \left\{ \frac{1}{8} c_{a_1,a_2} - \frac{1}{4} c_{a_1-1,a_2-1} + \frac{1}{4} c_{a_1,a_2-2} - \frac{1}{2} c_{a_1,a_2} \right\}.
\]

### 4. Conclusion

In this paper, we have continued our investigation of the heat content asymptotics with singular initial temperature and singular specific heat distributions. In theorem 3, we have given formulae for the first three invariants assuming the existence of the series in general. This gives useful information on the structure of these coefficients and jibes with previous computations on the ball (theorem 1). In theorem 4, we obtained the first two terms of an asymptotic series of the heat content for an interval in the case \( \alpha_1 + \alpha_2 = 1 \). This is an instance where a log \( t \) term occurs, and will be a crucial special case calculation.

Much remains to be done in this area. In particular, further investigations into the nature of the possible logarithmic singularities when \( \alpha_1 + \alpha_2 \in \mathbb{Z} \) are appropriate. A variety of boundary conditions such as Neumann, Robin, transmission, transfer, Zaremba and oblique are likely to be important in physical problems. Variable geometries, inhomogeneous boundary conditions, spectral boundary conditions and non-minimal operators are also likely to play an important role.
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