Gravitational self-force in nonvacuum spacetimes

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The gravitational self-force has thus far been formulated in background spacetimes for which the metric is a solution to the Einstein field equations in vacuum. While this formulation is sufficient to describe the motion of a small object around a black hole, other applications require a more general formulation that allows for a nonvacuum background spacetime. We provide a foundation for such extensions, and carry out a concrete formulation of the gravitational self-force in two specific cases. In the first we consider a particle of mass $m$ and scalar charge $q$ moving in a background spacetime that contains a background scalar field. In the second we consider a particle of mass $m$ and electric charge $e$ moving in an electrovac spacetime. The self-force incorporates all couplings between the gravitational perturbations and those of the scalar or electromagnetic fields. It is expressed as a sum of local terms involving tensors defined in the background spacetime and evaluated at the current position of the particle, as well as tail integrals that depend on the past history of the particle. Because such an expression is rarely a useful starting point for an explicit evaluation of the self-force, we also provide covariant expressions for the singular potentials, expressed as local expansions near the world line; these can be involved in the construction of effective extended sources for the regular potentials, or in the computation of regularization parameters when the self-force is computed as a sum over spherical-harmonic modes.

I. INTRODUCTION AND OVERVIEW

The prospect of measuring low-frequency gravitational waves generated by a solar-mass compact object spiraling toward a supermassive black hole has motivated a large effort to describe the motion of such a body beyond the test-mass approximation. In this treatment [1, 2], the body’s gravitational influence is taken into account, and the motion is no longer geodesic in the background spacetime of the large black hole. The motion is instead accelerated, the perturbation created by the small body giving rise to a gravitational self-force. It is this self-force that is responsible for the body’s inspiraling motion, implied by the loss of orbital energy and angular momentum to gravitational waves. To date the gravitational self-force was formulated rigorously [3–5], it was computed and implemented in orbital evolutions around nonrotating black holes [6], it was implicated in an improved calculation of the innermost circular orbit of a Kerr black hole [7], and it was extended to second order in perturbation theory [8–12]. The consequences of the gravitational self-force have been compared to the predictions of high-order post-Newtonian theory [13] and numerical relativity [14]. Other achievements of the gravitational self-force program are reviewed in Refs. [15, 16].

The gravitational self-force has thus far been formulated for bodies moving in vacuum spacetimes, and it is indeed an important restriction of the formulation that the background metric be everywhere a vacuum solution to the Einstein field equations. For the applications reviewed in the preceding paragraph, the background spacetime is produced by an isolated black hole, and the vacuum formulation of the gravitational self-force is perfectly adequate. Other applications, however, may require an extension to nonvacuum spacetimes. For example, one might wish to compute the self-force acting on a satellite of a material body, and this would require a formulation that allows for the presence of matter somewhere in the spacetime.

Another application that requires such an extension is the elucidation of the role played by the self-force in scenarios that aim to produce a counter-example to cosmic censorship by overcharging a near-extremal Reissner-Nordström black hole. Back in 1999, Hubeny [17] showed that a charged black hole near the extremal state can absorb a particle of such charge, mass, and energy that the final configuration cannot be a black hole, because its charge-to-mass ratio exceeds the extremal bound. A variation on this theme was explored by Jacobson and Sotiriou [18], who revealed that a near-extremal Kerr black hole can absorb a particle and be driven toward a final state with too much angular momentum to be a black hole. These works treated the particle as a test particle in the black-hole spacetime, and it was soon realized that self-force and radiative effects can play an important role in these overcharging and overspinning scenarios. In fact, Hubeny incorporated approximate self-force effects in her original analysis, and Barausse, Cardoso, and Khanna [19] took into account the gravitational radiation emitted by the particle on its way to overspin a Kerr black hole.

These partial attempts to incorporate (conservative and radiative) self-force effects could not rule out all overcharging and overspinning scenarios. Another partial attempt [20], based on a calculation of the electromagnetic self-force acting on a charged particle falling toward a Reissner-Nordström black hole, was more successful: a thorough sampling of the parameter space failed to produce a single instance of an overcharged final state. This work bolstered the case that the self-force acts as a cosmic censor in these scenarios, but the analysis was still an incomplete one: the calcu-
lation included the electromagnetic self-force but neglected the gravitational self-force, and it did not account for the coupling between electromagnetic and gravitational perturbations in a background Reissner-Nordström spacetime. A complete account of self-force effects in overcharging scenarios will have to overcome these limitations.

Our purpose in this paper is to provide a foundation for performing self-force calculations in nonvacuum background spacetimes. (Other foundational elements have been provided by Gralla [21, 22].) We have in mind the situation described previously: a particle of mass $m$ and electric charge $e$ moves in an electrovac spacetime with metric $g_{\alpha\beta}$ and electromagnetic field $F_{\alpha\beta}$, solutions to the Einstein-Maxwell equations. The particle produces a gravitational perturbation $h_{\alpha\beta}$, an electromagnetic perturbation $f_{\alpha\beta}$, and we wish to calculate the complete self-force acting on the particle. This includes terms originating in $h_{\alpha\beta}$ and scaling as $m^2$, terms originating in $f_{\alpha\beta}$ and scaling as $e^2$, but there are also terms originating in the coupling between gravitational and electromagnetic perturbations and scaling as $em$.

It is this coupling that gives rise to the most challenging aspects of the formulation. The physical origin of the coupling is easy to identify. The gravitational perturbation produced by the particle gives rise to a shift in the background electromagnetic field, and the electromagnetic perturbation gives rise to a shift in the field’s energy-momentum tensor, which in turn produces a shift in the background metric. To first order in perturbation theory, $h_{\alpha\beta}$ is the sum of the gravitational perturbation generated by the particle and the electromagnetic shift in the background metric, and $f_{\alpha\beta}$ is the sum of the electromagnetic perturbation sourced by the particle and the gravitational shift in the background electromagnetic field.

The coupling is manifested in the field equations satisfied by the perturbations. To display this it is convenient to introduce a “trace-reversed” gravitational perturbation $\gamma_{\alpha\beta} := h_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} h_{\mu\nu}$, a vector potential $b_\alpha$ such that $f_{\alpha\beta} = \nabla_\alpha b_\beta - \nabla_\beta b_\alpha$, and to collect the perturbations into a “meta-potential” $\psi^A := \{\gamma^{\alpha\beta}, b^\alpha\}$. Adopting the Lorenz gauge conditions $\nabla_\beta \gamma^{\alpha\beta} = 0$ and $\nabla_\alpha b^\alpha = 0$, we find that the perturbation equations take the form of the coupled wave equations

$$\Box \psi^A + M^A_{\, B\mu} \nabla^\mu \psi^B + N^A_{\, B} \psi^B = -4\pi \mu^A,$$  \hspace{1cm} (1.1)

where $\Box := g^{\alpha\beta} \nabla_\alpha \nabla_\beta$ is the covariant wave operator in the background spacetime, $M^A_{\, B\mu}$ and $N^A_{\, B}$ are tensors in the background spacetime, and $\mu^A$ collects the source terms — the particle’s energy-momentum tensor and current density. Summation over a repeated meta-index $B$ is understood, and the coupling between gravitational and electromagnetic perturbations is revealed by the fact that $M^A_{\, B\mu}$ and $N^A_{\, B}$ possess off-diagonal components — terms that involve either a pair of tensorial indices for $A$ and a single tensorial index for $B$, or a single index for $A$ and a pair of indices for $B$.

While the coupling introduces a level of complexity not experienced before in self-force calculations, the perturbation equations are of the same mathematical type — hyperbolic equations — as those encountered in previous formulations. As such they can be handled in exactly the same way, and the starting point is to integrate Eq. (1.1) with the help of a retarded Green’s function $G^A_{\, B}(x, x')$ that satisfies a special case of Eq. (1.1) with a Dirac distribution on the right-hand side. The solution is

$$\psi^A(x) = \int G^A_{\, B}(x, x') \mu^B(x') \, dV',$$  \hspace{1cm} (1.2)

where $dV'$ is the invariant volume element in the background spacetime. The Green’s function possesses both diagonal and off-diagonal components, and the right-hand side of Eq. (1.2) includes a contribution from the particle’s energy-momentum tensor and a contribution from its current density.

The potential $\psi^A$ is singular on the world line, and so is $\nabla_{\mu} \psi^A$, which enters the equations of motion satisfied by the particle. These singularities are now well understood and easily tamed, and because Eq. (1.1) has the same mathematical structure as the perturbation equations examined in the past, they can be handled with the same regularization techniques. Here we follow the approach advocated by Detweiler and Whiting [23]: we decompose the retarded potential $\psi^A$ into a precisely-defined singular potential $\psi^A_R$ and a regular remainder $\psi^A_{\text{reg}}$, and assert that only $\psi^A_R$ appears in the equations of motion. The manipulations associated with the decomposition can all be performed at the level of Eqs. (1.1) and (1.2), and the behavior of the potentials near the world line can also be obtained directly from these equations. For these purposes there is no need to specify the precise identity of the potentials $\psi^A$, and there is no need to identify the tensors $M^A_{\, B\mu}$ and $N^A_{\, B}$; the only important aspect is that $\psi^A$ is a solution to Eq. (1.1).

The methods developed here to handle coupled perturbations in nonvacuum spacetimes are therefore quite general, and they apply well beyond the specific context of gravitational and electromagnetic perturbations of an electrovac background spacetime. We do apply the formalism to this specific situation, but we also consider the case of a particle with scalar charge $q$ moving in a background spacetime that contains a background scalar field. (We name “scalarvac spacetime” such a solution to the Einstein-scalar field equations.) Other applications, for example, a computation of the gravitational self-force in a scalar-tensor theory of gravity, could also benefit from the techniques developed here.
We begin in Sec. II with a derivation of the perturbation equations in the case of a scalarvac spacetime perturbed by a particle of mass \( m \) and scalar charge \( q \). In Sec. III we turn to the perturbation equations for an electrovac spacetime perturbed by a particle of mass \( m \) and electric charge \( e \). In Sec. IV we cast the perturbation equations in the general form of Eq. (1.1) and provide additional details regarding the meta-index notation. In Sec. V we introduce the Green’s function \( G_\mu^\alpha(x,x') \), and construct its local Hadamard expansion when \( x \) is sufficiently close to \( x' \). In Sec. VI we examine the retarded, singular, and regular potentials near the world line, and collect the essential ingredients required in the computation of the self-force. The self-force acting on a particle of mass \( m \) and scalar charge \( q \) moving in a background scalarvac spacetime is calculated in Sec. VII; our final result is displayed in Eqs. (7.20) and (7.21), where \( \mathbf{F} \) is the force exerted by the background field in the background spacetime, \( \mathbf{F}_{\text{local}} \) is the local piece of the self-force, which depends on tensors defined in the background spacetime and evaluated at the current position of the particle, and \( \mathbf{F}_{\text{tail}} \) is the tail piece of the self-force, which depends on the past history of the particle. Because a formal expression in terms of local and tail pieces is rarely a useful starting point for the explicit evaluation of the self-force, we also provide covariant expressions for the singular potential \( \psi^\alpha \) expressed as a local expansion near the world line. These expressions can be found in Secs. VII C VIII D and VIII E. The singular potential can be involved in the construction of an effective extended source for the regular potential \( \psi^\alpha \), or in the computation of regularization parameters \( \{27–29\} \) when the self-force is computed as an infinite sum over spherical-harmonic modes. We do not pursue such computations here, but refer the reader to an independent effort to calculate the singular part of the self-force acting on a charged particle moving in an electrovac spacetime.

As mentioned previously, our derivation of the self-force in nonvacuum spacetimes rests on the Detweiler-Whiting prescription to regularize the potentials of a point particle, which diverge on the world line. Another set of methods, based on effective field theory \( \{30, 31\} \), can also be adopted, and this shall be explored in a forthcoming publication.

Our developments in this paper rely heavily on the general theory of bitensors in curved spacetime, the theory of Green’s functions and their Hadamard representation, the description of a neighborhood of a world line in terms of Fermi normal coordinates, and a host of other techniques that have become standard fare in the self-force literature. For all this we refer the reader to the comprehensive review of Poisson, Pound, and Vega \[16\], to which we repeatedly refer as PPV.

II. PERTURBED SCALARVAC SPACETIMES

We consider a background spacetime whose metric \( g_{\alpha\beta} \) is a solution to the Einstein field equations in the presence of a scalar field \( \Phi \). With a suitable normalization for the scalar field, the action functional for the system is

\[
S = \frac{1}{16\pi} \int R \, dV - \frac{1}{8\pi} \int \left( \frac{1}{2} g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi + F \right) \, dV,
\]

where \( R \) is the Ricci scalar, \( dV := \sqrt{-g} \, d^4x \) is the invariant volume element, and \( F(\Phi) \) is a potential for the scalar field. Variation of the action with respect to the metric yields the field equations

\[
G_{\alpha\beta} = 8\pi T_{\alpha\beta} = \nabla_\alpha \Phi \nabla_\beta \Phi - \left( \frac{1}{2} \nabla^\mu \Phi \nabla_\mu \Phi + F \right) g_{\alpha\beta}.
\]

They imply

\[
R_{\alpha\beta} = \nabla_\alpha \Phi \nabla_\beta \Phi + F g_{\alpha\beta}, \quad R = \nabla^\alpha \Phi \nabla_\alpha \Phi + 4F.
\]

Variation of the action with respect to the scalar field produces the wave equation

\[
\Box \Phi = F'.
\]
for the scalar field, where $\Box := g^{\alpha\beta} \nabla_\alpha \nabla_\beta$ is the covariant wave operator and $F' := dF/d\Phi$.

The background spacetime is perturbed by a point particle carrying a scalar charge $q$ and a mass $m$; the particle moves on a word line $\gamma$ described by the parametric relations $z^\mu(\tau)$, in which $\tau$ is proper time in the background spacetime. This is achieved by adding

$$S_{\text{pert}} = -m \int_\gamma d\tau + q \int_\gamma \Phi(z(\tau)) \, d\tau = - \int_\gamma (m - q\Phi) \, d\tau$$  \hspace{1cm} (2.5)$$


to the background action; the first term is the action of a free particle in a curved spacetime, and the second term accounts for its interaction with the scalar field. The perturbed metric is $g_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}$, where $h_{\alpha\beta}$ is the metric perturbation, and the perturbed scalar field is $\Phi = \Phi + \phi$, where $\phi$ is the perturbation. We introduce

$$\gamma_{\alpha\beta} := h_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} h$$  \hspace{1cm} (2.6)$$
as a “trace-reversed” perturbation, where $h := g^{\alpha\beta} h_{\alpha\beta}$. It is understood that indices on $h_{\alpha\beta}$ and $\gamma_{\alpha\beta}$ are raised with the inverse background metric $g^{\alpha\beta}$.

The perturbation is sourced by the particle’s energy-momentum tensor $t^{\alpha\beta}$ and its scalar-charge density $j$. The energy-momentum tensor is obtained by varying $S_{\text{pert}}$ with respect to the metric, and we get

$$t^{\alpha\beta}(x) = \int_\gamma (m - q\Phi) g^\alpha_{\mu}(x, z) g^\beta_{\nu}(x, z) u^\mu u^\nu \delta_4(x, z) \, d\tau,$$  \hspace{1cm} (2.7)$$

where $\Phi := \Phi(x)$, $u^\mu := dz^\mu/d\tau$ is the particle’s velocity vector, $g^\alpha_{\mu}(x, z)$ is the parallel propagator from $z(\tau)$ to $x$, and $\delta_4(x, z)$ is a scalarized Dirac distribution. The scalar-charge density is obtained by varying $S_{\text{pert}}$ with respect to $\Phi$, and we get

$$j(x) = q \int_\gamma \delta_4(x, z) \, d\tau.$$  \hspace{1cm} (2.8)$$

We denote $G_{\alpha\beta}$ the Einstein tensor of the perturbed spacetime. Its perturbation $\delta G_{\alpha\beta} = G_{\alpha\beta} - G_{\alpha\beta}$ relative to the background Einstein tensor is given by

$$\delta G_{\alpha\beta} = \frac{1}{2} \left( -\Box g_{\alpha\beta} + \nabla_\alpha \nabla_\beta \gamma_{\mu}^\gamma + \nabla_\beta \nabla_\gamma \gamma_{\mu}^\alpha - g_{\alpha\beta} \nabla_\mu \nabla_\nu \gamma_{\gamma}^\mu - 2 R_{\alpha \beta \gamma}^\mu \gamma_{\mu}^\nu + R_{\alpha \mu} \gamma_{\beta}^\gamma + R_{\beta \mu} \gamma_{\alpha}^\gamma + g_{\alpha \beta} R_{\mu}^{\gamma \mu} \gamma_{\nu}^\nu - R_{\gamma \alpha \beta} \right),$$  \hspace{1cm} (2.9)$$

in which $\nabla_\alpha$ indicates covariant differentiation in the background spacetime. Similarly, we denote $T_{\alpha\beta}$ the energy-momentum tensor of the perturbed scalar field in the perturbed spacetime. Its perturbation $\delta T_{\alpha\beta} = T_{\alpha\beta} - T_{\alpha\beta}$ is given by

$$8\pi \delta T_{\alpha\beta} = \nabla_\alpha \Phi \nabla_\beta \phi + \nabla_\beta \Phi \nabla_\alpha \phi - g_{\alpha\beta} \nabla^\mu \Phi \nabla_\mu \phi - g_{\alpha\beta} F^\phi \phi + \frac{1}{2} g_{\alpha\beta} \nabla_\mu \Phi \nabla_\mu \phi - \frac{1}{2} \nabla_\mu \Phi \nabla_\mu \phi + F \gamma_{\alpha\beta},$$  \hspace{1cm} (2.10)$$

in which $F$ and $F'$ are evaluated at the background field $\Phi$.

The Einstein field equations $G_{\alpha\beta} = 8\pi(T_{\alpha\beta} + t_{\alpha\beta})$ become $\delta G_{\alpha\beta} = -8\pi \delta T_{\alpha\beta} = -8\pi t_{\alpha\beta}$ after linearization. The equations simplify (and become manifestly hyperbolic when the gravitational perturbation is required to satisfy the one-parameter family of gauge conditions

$$\nabla_\beta \gamma_{\alpha\beta} = 2\lambda \phi \nabla_\alpha \Phi,$$  \hspace{1cm} (2.11)$$
in which $\lambda$ is a free (dimensionless) parameter. Making the substitutions from Eqs. (2.9) and (2.10) and exploiting Eqs. (2.8) returns the wave equation

$$\Box \gamma_{\alpha\beta} + M_{\alpha\beta} \nabla^\mu \phi + N_{\alpha\beta} \gamma_{\gamma\delta} + N_{\alpha\beta} \gamma_{\phi} = -16\pi t_{\alpha\beta},$$  \hspace{1cm} (2.12)$$

for the gravitational perturbation. We have introduced the tensors

$$M_{\alpha\beta} := 2(1 - \lambda) \left( \delta_{\alpha\beta} \nabla^\mu \phi + \delta_{\alpha\beta} \nabla^\alpha \phi - g^{\alpha\beta} \nabla_\mu \phi \right),$$  \hspace{1cm} (2.13a)$$

$$N_{\alpha\beta} := 2R_{\gamma\delta} \gamma_{\alpha\beta} - \delta_{\alpha\beta} (\gamma \nabla^\beta \phi \nabla_\delta \phi) - \delta_{\alpha\beta} (\gamma \nabla^\alpha \phi \nabla_\delta \phi),$$  \hspace{1cm} (2.13b)$$
\[ N^\alpha\beta_{\mu\nu} = -2 \left[ (1 - \lambda) g^\alpha\beta F' + 2\lambda \nabla^\alpha \nabla^\beta \Phi \right]. \] (2.13c)

The notation for the \( M \) and \( N \) tensors, involving vertical stokes and dots, will be explained fully below. Notice that \( M^\alpha\beta_{\mu\nu} = 0 \) when \( \lambda = 1 \); in this gauge the wave equation does not involve first-derivative terms in \( \phi \).

The scalar field equation \( g^\alpha\beta D_\alpha D_\beta \Phi = F'(\Phi) - 4\pi j \), in which \( D_\alpha \) indicates covariant differentiation in the perturbed spacetime, becomes
\[
\Box \phi + N_{\alpha\beta\gamma} D_\alpha D_\beta \Phi = -4\pi j
\] (2.14)
after linearization. Here
\[
N_{\alpha\beta\gamma} = -\left( \nabla_\alpha \nabla_\beta \Phi - \frac{1}{2} F' \delta_{\alpha\beta} \right),
\] (2.15a)
\[
N_{\alpha} = -2\lambda \nabla^\gamma \Phi \nabla_\gamma \Phi + F''.
\] (2.15b)

This equation does not feature a coupling to \( \nabla^\mu \gamma^\alpha \beta \), irrespective of the choice of gauge.

### III. PERTURBED ELECTROVAC SPACETIMES

In this section the background spacetime contains an electromagnetic field \( F_{\alpha\beta} \) instead of a scalar field \( \Phi \). The action functional is now
\[
S = \frac{1}{16\pi} \int R dV - \frac{1}{16\pi} \int F_{\alpha\beta} F^{\alpha\beta} dV,
\] (3.1)
and the Einstein field equations are
\[
G_{\alpha\beta} = 8\pi T_{\alpha\beta} = 2F_\gamma F_{\beta\gamma} - \frac{1}{2} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta};
\] (3.2)
they imply \( R_{\alpha\beta} = G_{\alpha\beta}, \; R = 0 \). Maxwell’s equations are
\[
\nabla_\beta F^{\alpha\beta} = 0, \quad \nabla_\alpha F_{\beta\gamma} + \nabla_\gamma F_{\alpha\beta} + \nabla_\beta F_{\gamma\alpha} = 0;
\] (3.3)
we specifically assume that there is no source for the background electromagnetic field.

The background spacetime is perturbed by a point particle carrying an electric charge \( e \) and a mass \( m \); the particle moves on a word line \( \gamma \) described by the parametric relations \( z^\mu(\tau) \), in which \( \tau \) is proper time in the background spacetime. This is achieved by adding
\[
S_{\text{pert}} = -m \int_\gamma d\tau + e \int_\gamma A_\mu u^\mu \delta^4(x,z) d\tau,
\] (3.4)
to the action functional, where \( A_\mu \) is a vector potential for the electromagnetic field. The metric perturbation was introduced previously, and the perturbed electromagnetic field is \( F_{\alpha\beta} = F_{\alpha\beta} + f_{\alpha\beta} \), with \( f_{\alpha\beta} \) denoting the perturbation. It is useful to introduce a second vector potential \( b_\alpha \) such that
\[
f_{\alpha\beta} = \nabla_\alpha b_\beta - \nabla_\beta b_\alpha.
\] (3.5)
It is understood that indices on \( b_\alpha \) are raised with the inverse background metric \( g^{\alpha\beta} \).

The perturbation is sourced by the particle’s energy-momentum tensor \( t^{\alpha\beta} \), which is now given by
\[
t^{\alpha\beta}(x) = m \int_\gamma g^\alpha_\mu(x,z) g^\beta_\mu(x,z) u^\mu u^\nu \delta_4(x,z) d\tau,
\] (3.6)
and by the current density
\[
j^\alpha(x) = e \int_\gamma g^\alpha_\mu(x,z) u^\mu \delta_4(x,z) d\tau.
\] (3.7)
The perturbation of the Einstein tensor is still given by Eq. (2.9), and the perturbation of the electromagnetic energy-momentum tensor is

\[ 4\pi \delta T_{\alpha \beta} = F_\alpha^\mu f_{\beta \mu} + F_\beta^\mu f_{\alpha \mu} - \frac{1}{2} g_{\alpha \beta} F^{\mu \nu} f_{\mu \nu} - F_{\alpha \mu} F_{\beta \nu} \gamma^{\mu \nu} + \frac{1}{2} g_{\alpha \beta} F_\lambda^\mu F_{\nu \lambda} \gamma^{\mu \nu} - \frac{1}{4} F^{\mu \nu} F_{\mu \nu} \gamma_{\alpha \beta} + (2\pi T_{\alpha \beta}) \gamma. \]  

(3.8)

As in the previous section we have that the linearized Einstein field equations are \( \delta G_{\alpha \beta} = 8\pi \delta T_{\alpha \beta} = 8\pi t_{\alpha \beta} \), and these lead to a set of differential equations for the metric perturbation \( \gamma_{\alpha \beta} \). Maxwell’s equations \( D_\beta F_{\alpha \beta} = 4\pi j_\alpha \) and \( D_\alpha F_{\beta \gamma} + D_\gamma F_{\alpha \beta} + D_\beta F_{\gamma \alpha} = 0 \) can similarly be cast in the form of differential equations for \( b^\alpha \). To simplify the linearized field equations we impose a Lorenz-gauge condition on both \( b_\alpha \) and \( \gamma_{\alpha \beta} \):

\[ \nabla_\alpha b^\alpha = 0, \quad \nabla_\beta \gamma^{\alpha \beta} = 0. \]  

(3.9)

After some manipulations we find that the perturbation equations become

\[ \Box \gamma^{\alpha \beta} + M^{\alpha \beta}_{\gamma |\mu} \nabla^\mu b^\gamma + N^{\alpha \beta}_{\gamma |\delta} \nabla^\delta b^\gamma = -16\pi l^{\alpha \beta}. \]  

(3.10)

and

\[ \Box b^\alpha + M^\alpha_{\beta |\gamma \mu} \nabla^\mu \gamma^{\beta \gamma} + N^\alpha_{\gamma |\delta} \nabla^\delta \gamma^{\beta \gamma} + N_b^{\alpha |\beta} = -4\pi j^\alpha, \]  

(3.11)

with

\[ M^{\alpha \beta}_{\gamma |\mu} = 16\delta^{(\alpha}_{[\mu} F^{\beta]}_{\gamma \mu}) - 4g^{\alpha \beta} F^{\mu \nu \gamma} \]  

(3.12a)

\[ N^{\alpha \beta}_{\gamma |\delta} = 2(R^{\alpha \beta}_{\gamma |\delta} - \delta^{\alpha}_{(\gamma} F^{\beta \delta) \gamma - \delta^{\alpha}_{(\gamma} F^{\delta \alpha \gamma} - 2F^{\alpha}_{(\gamma} F^{\beta}_{\delta)} - g_{\beta \gamma} F^{\alpha \gamma} F^{\beta}_{\delta}) \]  

(3.12b)

and

\[ M^\alpha_{\beta |\gamma \mu} = \delta^\alpha_{(\beta} F^{\gamma}_{\mu \gamma)}, \]  

(3.13a)

\[ N^\alpha_{\beta |\gamma} = \nabla^{(\beta} F^{\alpha \gamma)}, \]  

(3.13b)

\[ N^\alpha_{|\beta} = -R^\alpha_{\beta}. \]  

(3.13c)

There is no choice of gauge that permits the elimination of \( M^{\alpha \beta}_{\beta |\gamma \mu} \) and \( M^\alpha_{\beta |\gamma \mu} \) from the perturbation equations; these necessarily contain first-derivative terms in both \( \gamma_{\alpha \beta} \) and \( b_\alpha \).

**IV. CONDENSED INDEX NOTATION**

To integrate the linearized field equations displayed in the preceding sections, it is convenient to handle them all at once by exploiting a condensed index notation for the various fields. We thus introduce a meta-index \( A \) which can stand for any index, \( A = \alpha \beta \), which is understood to be symmetrized. Second, \( A \) can stand for a single index, \( A = \alpha \). And third, \( A \) can stand for an absence of index, \( A = \cdot \). In this way we can collect the tensor field \( \gamma^{\alpha \beta} \), the vector field \( b^\alpha \), and the scalar field \( \phi \) into the single meta-object \( \psi^A \).

In a similar way we collect the source terms \( 4\pi t^{\alpha \beta} \), \( j^\alpha \), and \( j \) into the meta-object \( \mu^A \). The perturbation equations can then be expressed as

\[ \Box \psi^A + M^A_{B \mu} \nabla^\mu \psi^B + N^A_{B \cdot} \psi^B = -4\pi l^A, \]  

(4.1)

where summation over repeated meta-indices is understood. Such sums include all possible combinations of indices; for example, when Eq. (4.1) represents Eq. (3.11) and \( A \) stands for \( \alpha \), the summation over \( B \) in \( N^A_{B \cdot} \psi^B \) stands explicitly for \( N^\alpha_{\beta |\gamma} \gamma^{\beta \gamma} + N^\alpha_{|\beta} \cdot \). The various tensor fields \( M^A_{B \mu} \) and \( N^A_{B \cdot} \) can be read off from Eqs. (2.13), (2.15), (3.12), and (3.13).

In the more explicit notation used in these equations, a vertical stroke separates the indices collected in \( A \) from those collected in \( B \); for example, \( N^A_{B \cdot} \) is denoted \( N^{\alpha |\beta} \cdot \) when \( A \) stands for \( \alpha \beta \) and \( B \) stands for \( \cdot \).

We next introduce other useful meta-objects. We set

\[ q^A := \begin{cases} 4(m - q\Phi)u^\alpha u^\beta \\ eu^\alpha \\ q \end{cases}, \]  

(4.2)
\[
\hat{q}^A := \begin{cases} 
8(m - q\Phi)a^{(\alpha}u^{\beta)} - 4q\Phi u^\alpha u^\beta \\
\frac{e^{a\alpha}}{c} \\
0
\end{cases},
\]

(4.3)

\[
\hat{q}^A := \begin{cases} 
8(m - q\Phi)\tilde{a}^{(\alpha}u^{\beta)} + a^{(\alpha}a^{\beta)} - 16q\Phi a^{(\alpha}u^{\beta)} - 4q\Phi u^\alpha u^\beta \\
\frac{e^{a\alpha}}{c} \\
0
\end{cases},
\]

(4.4)

\[
\delta^A_{B'} := \begin{cases} 
\delta^{(\alpha}_{\hat{\alpha}} \delta^{\beta)} \\
\delta^{\alpha}_{\hat{\beta}} \\
1
\end{cases},
\]

(4.5)

\[
g^A_{B'}(x, x') := \begin{cases} 
g_{\gamma}^{\alpha}(x, x')g^\beta_{\hat{\gamma}}(x, x') \\
g_{\gamma}^{\alpha}(x, x') \\
1
\end{cases}.
\]

(4.6)

With \(\delta^A_{B'}\) and \(g^A_{B'}\), it is understood that the object is zero whenever \(A\) and \(B\) (or \(B'\)) stand for different types of indices; for example \(\delta^\alpha_\gamma = 0\) and \(g^{\alpha\beta}|_\gamma = 0\). We have introduced \(a^\alpha := Du^\alpha/d\tau\) as the particle’s acceleration vector, and \(\dot{a}^\alpha := Da^\alpha/d\tau\) as its rate of change along the world line. We also denote \(\dot{\Phi} := u^\alpha \nabla_\mu \Phi, \ddot{\Phi} := u^\nu \nabla_\mu (u^\alpha \nabla_\nu \Phi)\).

With these objects we find that Eqs. (2.7), (2.8), and (3.7) are all contained in the meta-expression

\[
\mu^A(x) = \int_\gamma g^A_M(x, z)q^M(\tau)\delta_4(x, z) d\tau.
\]

(4.7)

Another meta-object we shall need in further developments is

\[
R^A_{B'\mu'\nu'} := \begin{cases} 
g^{(\alpha}_{\gamma}g^\beta_{\hat{\gamma}}R^\mu'_{\delta\mu'\nu'} + g^{(\alpha}_{\gamma}g^\beta_{\hat{\gamma}}R^\mu'_{\delta\mu'\nu'} \\
g^{(\alpha}_{\gamma}R^\mu'_{\delta\mu'\nu'} \\
0
\end{cases}.
\]

(4.8)

in which the Riemann tensor is evaluated at \(x'\), and the parallel propagators refer to both \(x\) and \(x'\). Here also it is understood that \(R^A_{B'\mu'\nu'}\) is zero when \(A\) and \(B'\) stand for different types of indices.

V. GREEN’S FUNCTION AND HADAMARD EXPANSION

To integrate Eq. (4.1) we introduce a Green’s function \(G^A_{B'}(x, x')\) that satisfies

\[
\Box^A_{B'} + M^A_{B'\mu'}\nabla^\mu G^B_{B'} + N^A_{B'\mu'}G^B_{B'} = -4\pi g^A_{B'} \delta_4(x, x')
\]

(5.1)

together with suitable boundary conditions; we shall be exclusively concerned with the retarded Green’s function, which vanishes when \(x\) is in the past of \(x'\). The solution to the wave equation can then be expressed as

\[
\psi^A(x) = \int G^A_{B'}(x, x')\mu^B(x') dV',
\]

(5.2)

where \(dV' := \sqrt{-g'} d^4x'\) is the element of spacetime volume at \(x'\). With the source term of Eq. (4.1) this becomes

\[
\psi^A(x) = \int G^A_M(x, z)q^M(\tau) d\tau,
\]

(5.3)

in which the Green’s function is evaluated at \(x' = z(\tau)\). It must be remembered that this equation contains a summation over the repeated meta-index \(M\). For example, when Eq. (5.3) represents the solution to the gravity-scalar wave equation (2.12), its explicit expression is

\[
\gamma_{\alpha\beta}(x) = 4\int_\gamma (m - q\Phi)G_{\hat{\nu}\nu}(x, z)u^\mu u^\nu d\tau + q\int_\gamma G_{\hat{\gamma}\gamma}(x, z) d\tau.
\]

(5.4)
When $x$ is within the normal convex neighborhood of $x'$, the retarded Green’s function can be cast in the Hadamard form

$$G^A_{B'}(x, x') = U^A_{B'}(x, x')\delta_+(\sigma) + V^A_{B'}(x, x')\Theta_-(\sigma),$$

(5.5)
in which $\sigma(x, x')$ is Synge’s world function, $\delta_+(\sigma)$ and $\Theta_-(\sigma)$ are respectively the Dirac and Heaviside distributions restricted to the future of $x'$, and $U^A_{B'}(x, x')$ and $V^A_{B'}(x, x')$ are smooth bitensors. Following the steps detailed in Sec. 14.2 of PPV [16], we find that $U^A_{B'}$ satisfies the differential equation

$$2\sigma^\mu\nabla_\mu U^A_{B'} + \sigma^\mu M^A_{B\mu} U^B_{B'} + (\sigma^\mu - 4) U^A_{B'} = 0$$

(5.6)

together with the boundary conditions

$$U^A_{B'}(x', x') = g^A_{B'}(x', x') = \delta^A_{B'}.\quad (5.7)$$

We also find that $V^A_{B'}$ satisfies

$$2\sigma^\mu\nabla_\mu V^A_{B'} + \sigma^\mu M^A_{B\mu} V^B_{B'} + (\sigma^\mu - 2) V^A_{B'} = \square U^A_{B'} + M^A_{B\mu} \nabla^\mu U^B_{B'} + N^A_{B} U^B_{B'}$$

(5.8)
on the light cone $\sigma(x, x') = 0$, as well as the wave equation

$$\square V^A_{B'} + M^A_{B\mu} \nabla^\mu V^B_{B'} + N^A_{B} V^B_{B'} = 0$$

(5.9)
everywhere. In these equations, $\sigma^\mu$ stands for a partial derivative of $\sigma(x, x')$ with respect to $x^\mu$, and $\sigma^\mu\nu := \nabla_\mu \nabla_\nu \sigma(x, x')$ — here also the derivatives are taken with respect to $x$.

Equations (5.6) and (5.7) allow us to construct $U^A_{B'}(x, x')$ as a covariant expansion in powers of $\sigma^\mu\nu$ about $x'$. We write

$$U^A_{B'}(x, x') = g^A_{B'} \left[\delta^A_{B'} + U^A_{B'\mu'} \sigma^{\mu'} + \frac{1}{2} U^A_{B'\mu'\nu'} \sigma^{\mu'} \sigma^{\nu'} + O(\epsilon^2)\right]$$

(5.10)
in which $U^A_{B'\mu'}$ and $U^A_{B'\mu'\nu'}$ are ordinary tensors at $x'$, and $\epsilon$ is a measure of distance between $x$ and $x'$. By inserting this expression in Eq. (5.6) and solving order-by-order in $\sigma^{\mu'}$, we arrive at

$$U^A_{B'\mu'} = \frac{1}{2} M^A_{B'\mu'},\quad (5.11a)$$

$$U^A_{B'\mu'\nu'} = -\frac{1}{2} \nabla_{(\mu'} M^A_{B'\nu')\nu'} + \frac{1}{4} M^A_{B'\nu_1'\nu_2'} M^C_{B'\nu_1'\nu_2'} C^{\nu_1'\nu_2'} + \frac{1}{6} \delta^A_{B'} R_{\nu_1'\nu_2'},\quad (5.11b)$$

where $M^A_{B'\mu'}$ is the tensor field $M^A_{B\mu}$ evaluated at $x'$. In the expression for $U^A_{B'\mu'\nu'}$ the indices contained in $B'$ are excluded from the symmetrization over the $\mu'$ and $\nu'$ indices, and summation over $C'$ involves all possible combinations of indices. The manipulations leading to Eqs. (5.11a) and (5.11b) rely on standard identities among bitensors, such as $\sigma^{\mu'} = -g^{\alpha'\beta'} \sigma_{\alpha'\beta'}, \sigma^{\mu'} \nabla_\mu g^\alpha_{\alpha'} = 0$ (PPV Sec. 5.4), and the expansion $\sigma^{\mu'} = 4 - \frac{1}{3} R_{\nu'\nu'} \sigma^{\mu'} \sigma^{\nu'} + O(\epsilon^3)$ (PPV Sec. 6.2).

From Eq. (5.10) we may compute derivatives of $U^A_{B'}$. For this purpose we use the expansions (PPV Sec. 6.2)

$$\nabla_\mu g^A_{B'} = \frac{1}{2} g^\alpha_{\alpha'} R^B_{\alpha'\beta'} g^\beta_{\mu'} \sigma^{\nu'} + O(\epsilon^2), \quad \nabla_\mu \nabla_\nu g^A_{B'} = \frac{1}{2} g^\alpha_{\alpha'} g^{\mu'} R^B_{\alpha'\beta'} g^\beta_{\nu'} \sigma^{\nu'} + O(\epsilon^2)$$

(5.12)
to show that with $g^A_{B'}$ defined by Eq. (4.16) and $R^A_{B'\mu'\nu'}$ defined by Eq. (4.28),

$$\nabla_\mu g^A_{B'} = \frac{1}{2} R^A_{B'\mu'\nu'} \sigma^{\nu'} + O(\epsilon^2), \quad \nabla_\mu \nabla_\nu g^A_{B'} = \frac{1}{2} g^\mu_{\mu'} R^A_{B'\mu'\nu'} \sigma^{\nu'} + O(\epsilon^2).$$

(5.13)

With this and the standard expansions for $\sigma^{\nu'}$, displayed in Sec. 6.2 of PPV, we obtain

$$\nabla_\mu U^A_{B'} = g^A_A \left[U^A_{B'\mu'} + (\nabla_\mu U^A_{B'\nu'}) \sigma^{\nu'}\right] + \frac{1}{2} R^A_{B'\mu'\nu'} \sigma^{\nu'} + O(\epsilon^2)$$

(5.14)
and

$$\nabla_\mu U^A_{B'} = -g^\mu_{\mu'} g^A_A \left[U^A_{B'\mu'} + U^A_{B'\mu'\nu'} \sigma^{\nu'}\right] + \frac{1}{2} g^\mu_{\mu'} R^A_{B'\mu'\nu'} \sigma^{\nu'} + O(\epsilon^2).$$

(5.15)
Additional differentiations produce
\begin{align}
\nabla_{\nu'}\nabla_{\nu''}U_{B'}^{A'} &= g_{\lambda'}^A \left( \nabla_{\nu'}U_{B'}^{A'} + \nabla_{\nu''}U_{B'}^{A'} + U_{B'}^{A'\nu''} \right) - \frac{1}{2} R_{B'\nu''}^A + O(\epsilon), \\
\nabla_{\mu}\nabla_{\nu'}U_{B'}^{A'} &= -g_{\mu\nu'}^A \left( \nabla_{\nu'}U_{B'}^{A'} + U_{B'}^{A'\nu'} \right) + \frac{1}{2} g_{\mu\nu'} R_{B'\nu'}^A + O(\epsilon), \\
\nabla_{\mu}\nabla_{\nu'}U_{B'}^{A'} &= g_{\mu\nu'}^A g_{A'}^{B'} U_{B'}^{A'\nu''} + \frac{1}{2} g_{\mu\nu'} R_{B'\nu''}^A + O(\epsilon).
\end{align}

The leading term of an expansion of $V_{B'}^A$, in powers of $\sigma^{\mu}$ can readily be obtained by inserting
\begin{equation}
V_{B'}^A(x,x') = g_{A'}^A \left[ V_{B'}^A + O(\epsilon) \right]
\end{equation}
on the left-hand side of Eq. (5.8) and substituting Eqs. (5.10), (5.15), and (5.16) on the right-hand side. After involvement of Eq. (6.11) and some simplification, we arrive at
\begin{equation}
V_{B'}^{A'} = -\frac{1}{4} \nabla_{\nu''} M_{B'}^{A'\nu''} - \frac{1}{8} M_{C'}^{A'} M_{B'}^{C'\nu''} + \frac{1}{2} N_{B'}^{A'} + \frac{1}{12} R_{B'}^{A'} R',
\end{equation}
in which $N_{B'}^{A'}$ is the tensor field $N_{B'}^{A}$ evaluated at $x'$, and $R'$ is the Ricci scalar at $x'$.

VI. POTENTIALS NEAR THE WORLD LINE

A. Retarded, singular, and regular potentials

With $\psi^A = \{\gamma^\alpha, b^\alpha, \phi\}$, the solutions to Eqs. (2.12), (2.14), and (3.11) are all given by the expression of Eq. (6.3),
\begin{equation}
\psi^A(x) = \int G_M^A(x,z)q^M(\tau) d\tau,
\end{equation}
in which $q^M = \{4(m - q\Phi)u^\mu u^{\nu'}, ew^\mu, q\}$; summation over the repeated meta-index includes all individual indices (or absence of index) contained in $M$.

Following techniques detailed in Sec. 17.2 of PPV, we find that the retarded solution to the perturbation equation can be expressed as
\begin{equation}
\psi^A(x) = \frac{1}{r} U_{B'}^{A'}(x,x') q^{B'}(u) + \int_{\tau<}^u V_{M}^{A'}(x,z) q^M(\tau) d\tau + \int_{-\infty}^{\tau<} G_{M}^{A'}(x,z) q^M(\tau) d\tau
\end{equation}
when $x$ is close to the world line $z(\tau)$. Here $x' = z(u)$ denotes the retarded point associated with $x$, defined by the null condition $\sigma(x, x') = 0$, $r := \sigma_{\alpha'}(x, x') u^\alpha'$ is the retarded distance to the world line, and $\tau <$ is the proper time at which the world line enters the normal convex neighborhood of the field point $x$.

Following techniques detailed in Sec. 17.5 of PPV, we identify the Detweiler-Whiting singular potential as
\begin{equation}
\psi_S^A(x) = \frac{1}{2\tau} U_{B'}^{A'}(x,x') q^{B'}(u) + \frac{1}{2r_{adv}} U_{B'}^{A'}(x,x'') q^{B''}(v) - \frac{1}{2} \int_u^{v} V_{M}^{A'}(x,z) q^M(\tau) d\tau,
\end{equation}
where $x'' = z(v)$ is the advanced point associated with $x$, and $r_{adv} := -\sigma_{\alpha''} u^{\alpha''}$ is the advanced distance to the world line. This also is a solution to the inhomogeneous wave equation, as written in Eq. (4.1). The Detweiler-Whiting regular potential is then $\psi_R^A := \psi^A - \psi_S^A$, or
\begin{equation}
\psi_R^A(x) = \frac{1}{2\tau} U_{B'}^{A'}(x,x') q^{B'}(u) - \frac{1}{2r_{adv}} U_{B'}^{A'}(x,x'') q^{B''}(v) + \int_{\tau<}^u V_{M}^{A'}(x,z) q^M(\tau) d\tau
\end{equation}
The regular potential $\psi^A_R$ will be implicated in the equations of motion satisfied by the particle, and for the purpose of deriving these equations it is convenient to obtain explicit expressions in terms of Fermi coordinates. The construction of these coordinates is detailed in Sec. 9 of PPV.

The Fermi coordinates $(t, x^a)$ refer to a point $\bar{x} = z(t)$ on the world line which is simultaneous to $x$, in the sense that $x$ and $\bar{x}$ are linked by a spacelike geodesic that is orthogonal to the world line. The precise condition is $\sigma_a u^a = 0$, and the geodesic distance $s$ from $\bar{x}$ to $x$ is given by $s^2 = 2\sigma(\bar{x}, x)$. The spatial Fermi coordinates are defined by $x^a := -\sigma^a e^a_\alpha$, in which $e^a_\alpha$ is a triad of Fermi-Walker transported spatial vectors on the world line (triad indices are raised with $\delta^{ab}$). From this it follows that $\sigma_a = -x^a e^a_\alpha$.

We must relate the retarded point $x' = z(u)$ and the advanced point $x'' = z(v)$ to the simultaneous point $\bar{x} = z(t)$. To achieve this we rely on expansion techniques developed in Sec. 11 of PPV. We introduce $\Delta := t - u$, $\Delta' := v - t$, as well as the scalar function

$$\sigma(\tau) := \sigma(x, z(\tau))$$

(6.5)
on the world line; $x$ is taken to be fixed on the right-hand side of this relation. In this notation we have that $\sigma(u) = 0$, $\sigma(t) = \frac{1}{2}s^2$, and $\sigma(v) = 0$. To obtain $\Delta$ we write $0 = \sigma(u) = \sigma(t - \Delta)$, expand the right-hand side in powers of $\Delta$, and solve for $\Delta$ expressed as an expansion in powers of $s$. To obtain $\Delta'$ we start instead with $0 = \sigma(v) = \sigma(t + \Delta')$.

In this way we obtain

$$\Delta = s \left\{ 1 - \frac{1}{2} a_a x^a + \frac{3}{8} (a_a x^a)^2 + \frac{1}{24} \dot{a}_t s^2 - \frac{1}{6} \dot{a}_a x^a - \frac{1}{6} R_{tattt} x^a x^b + O(s^3) \right\},$$

(6.6a)

$$\Delta' = s \left\{ 1 - \frac{1}{2} a_a x^a + \frac{3}{8} (a_a x^a)^2 + \frac{1}{24} \dot{a}_t s^2 - \frac{1}{6} \dot{a}_a x^a - \frac{1}{6} R_{tattt} x^a x^b + O(s^3) \right\},$$

(6.6b)

where $a_a := a_a e^a_\alpha$, $R_{tattt} := R_{\alpha\beta\gamma\delta} u^\alpha e^\beta_{\alpha} e^\gamma_{\beta} e^\delta_{\gamma}$, and so on are components of tensors in Fermi coordinates, evaluated at $\bar{x}$.

To relate $r$ to $s$ we notice that $r = \sigma(u) = \sigma(t - \Delta)$, which can be expanded in powers of $\Delta$. Similarly we have that $r_{\text{adv}} = -\sigma(v) = -\sigma(t + \Delta')$, which can be expanded in powers of $\Delta'$. With the expressions provided in Eq. 6.6 we eventually obtain

$$r = s \left\{ 1 - \frac{1}{2} a_a x^a + \frac{1}{8} (a_a x^a)^2 - \frac{1}{8} \dot{a}_t s^2 + \frac{1}{3} \dot{a}_a x^a + \frac{1}{6} R_{tattt} x^a x^b + O(s^3) \right\},$$

(6.7a)

$$r_{\text{adv}} = s \left\{ 1 - \frac{1}{2} a_a x^a + \frac{1}{8} (a_a x^a)^2 - \frac{1}{8} \dot{a}_t s^2 + \frac{1}{3} \dot{a}_a x^a + \frac{1}{6} R_{tattt} x^a x^b + O(s^3) \right\}.$$  

(6.7b)

From all this we can form the combinations

$$\frac{1}{2} \left( \frac{1}{r} - \frac{1}{r_{\text{adv}}} \right) = \frac{1}{3} a_t x^t + O(s^2),$$

(6.8a)

$$\frac{1}{2} \left( \frac{\Delta}{r} + \frac{\Delta'}{r_{\text{adv}}} \right) = 1 - a_t x^t + O(s^2),$$

(6.8b)

$$\frac{1}{2} \left( \frac{\Delta^2}{r} - \frac{\Delta'^2}{r_{\text{adv}}} \right) = O(s^3),$$

(6.8c)

which will be required in a moment.

To express the regular potential in Fermi coordinates we must relate $U^A_B q^B$ and $U^A_B q^B'$ to quantities defined at the simultaneous point $\bar{x}$. To achieve this we define

$$U^A(\tau) := U^A_M(x, z(\tau))q^M(\tau),$$

(6.9)

write $U^A_B q^B = U^A(u) = U^A(t - \Delta)$ and $U^A_B q^B' = U^A(v) = U^A(t + \Delta')$, and expand in powers of $\Delta$ or $\Delta'$. The derivatives of $U^A(\tau)$ can be calculated with the help of Eqs. (6.13) and (6.10). We have

$$U^A = g^A_B \left[ q^A + q^B U^A_B \sigma^\rho + \frac{1}{2} q^B U^A_B \sigma^\rho \sigma^\varphi \right] + O(s^3),$$

(6.10a)

$$U^A = g^A_B \left[ q^A + q^B U^A_B u^\mu + \left( q^B U^A_B + q^B U^A_B + q^B U^A_B \delta^\rho_{\delta^\varphi_{\mu}} \right) \sigma^\varphi \right] + \frac{1}{2} q^B R^A_B \sigma^\rho u^\rho.$$  

(6.10b)
\[ U^A = g^A_A [\dot{q}^A + 2q^B U^A_{B\bar{\mu}} \dot{u}^\bar{\mu} + 2q^B U^A_{B\mu} u^\mu + q^B U^A_{B\bar{\mu}} \dot{u}^\bar{\mu} + q^B U^A_{B\mu} u^\mu] + O(s), \] 

(6.10c)

where \( \dot{q}^A \) and \( \dot{q}^A \) were introduced in Eqs. (4.3) and (4.4), respectively, and \( \dot{U}^M_{N\mu} := u^\nu \nabla_\nu U^M_{N\mu} \); the barred tensorial indices indicate that the expressions are evaluated at \( \tau = t \).

Defining

\[ V^A(\tau) := V^A_M(x, z(\tau)) q^M(\tau) \] 

(6.11)

the integrals involving \( V^A_M q^M \) in Eq. (6.14) can be written as

\[ \int_\tau^t U^A_M q^M d\tau + \frac{1}{2} \int_\tau^t V^A_M q^M d\tau = \int_\tau^t V^A d\tau - \int_\tau^t V^A d\tau + \frac{1}{2} \int_\tau^t V^A d\tau, \] 

(6.12)

and close to the world line the last two terms evaluate to \(-\frac{1}{2}(\Delta - \Delta')V^A(t) = O(s^2)\). For future reference we note that

\[ V^A(t) = g^A_A V^A_B \dot{q}^B + O(s). \] 

(6.13)

Making the substitutions in Eq. (6.14) produces

\[ \psi^A_R(t, x^\alpha) = -(1 - a_c x^c) \dot{U}^A(t) + \frac{1}{3} \dot{U}^A(t) \dot{a}_c x^c + \psi^A[\text{tail}] + O(s^2) \] 

(6.14a)

\[ = -g^A_A \left( \dot{\bar{U}}^A + q^B U^A_{B\bar{c}} \right) (1 - a_c x^c) + g^A_A \left( \frac{1}{3} \dot{\bar{U}}^A - q^B U^A_{B\bar{c}} + q^B U^A_{B\bar{c}} + q^B U^A_{B\bar{c}} \right) x^c \]

\[ + \frac{1}{2} q^B R^A_{B\bar{c} x^c} + \psi^A[\text{tail}] + O(s^2), \] 

(6.14b)

where

\[ \psi^A[\text{tail}](x) := \int_\tau^t V^A_M(x, z) q^M(\tau) d\tau + \int_\tau^t G^A_M(x, z) q^M(\tau) d\tau \] 

(6.15a)

\[ = \int_{-\infty}^t G^A_M(x, z) q^M(\tau) d\tau \] 

(6.15b)

is the “tail part” of the potential; in the second form the integration is cut short at \( \tau = t^- := t - 0^+ \) to avoid the singular behavior of the Green’s function when the limit \( x \to \hat{x} \) is eventually taken.

A complete listing of the potentials \( \psi^A_R = \{ \gamma^R_{\alpha\beta}, b^R_{\alpha}, \varphi_R \} \) can now be obtained from Eq. (6.14b). The calculation requires explicit expressions for the tensors \( U^A_{B\bar{\mu}}, U^A_{B\mu}, \) and \( V^A_B \), and these will be computed at a later stage. It also requires the components of the parallel propagator \( g^\alpha_{\beta} \) in Fermi coordinates. These are (PPV Sec. 9.4)

\[ g^t_t = 1 - a_a x^a + O(s^2), \quad g^t_a = O(s^2), \quad g^a_t = O(s^2), \quad g^a_b = \delta^a_b + O(s^2). \] 

(6.16)

C. Singular potential in covariant form

The singular potentials \( \psi^A_R \) are often implicated in the calculation of regularization parameters or regularized sources for self-force computations. The starting point of such calculations is a covariant expression for the singular potentials, which can then be evaluated in any convenient coordinate system and involved in computations of effective sources or regularization parameters. We provide such a covariant expression here, using methods first devised by Haas and Poisson (2006).

The singular potential is defined by Eq. (6.3), and we wish to consolidate its dependence on the world line from the two points \( x' \) and \( x'' \) related to \( x \) by a null condition, to a single point \( \hat{x} \) that has no particular relationship to \( x \); the points \( x \) and \( \hat{x} \) are assumed to be in a spacelike relation, but otherwise \( \hat{x} \) is a fixed arbitrary point on the world line. We let \( \hat{x} = z(\hat{\tau}), \Delta^- := u - \frac{\hat{\tau}}{c}, \Delta^+ := v - \frac{\hat{\tau}}{c}, \) and

\[ \hat{\tau} := \sigma_{\alpha}(x, \hat{x}) u^\alpha(\hat{\tau}), \quad \rho^2 := (\dot{\sigma}^{\alpha\beta} + u^{\alpha} u^{\beta}) \sigma_{\alpha}(x, \hat{x}) \sigma_{\beta}(x, \hat{x}). \] 

(6.17)
To express $\psi^A$ in terms of $\hat{\tau}$ we apply the same methods as in the preceding subsection. First, $\Delta_\pm$ is determined by writing $0 = \sigma(u) = \sigma(\hat{\tau} + \Delta_-)$ or $0 = \sigma(v) = \sigma(\hat{\tau} + \Delta_+)$ and expanding in powers of $\Delta_\pm$; here $\sigma(\tau) := \sigma(x, z(\tau))$. With $\sigma(\hat{\tau}) = \frac{1}{2}(\rho^2 - \hat{\tau}^2)$, $\hat{\sigma}(\hat{\tau}) = \hat{\tau}$, and so on, we obtain an expansion of the form

$$
\Delta_\pm = (\hat{\tau} \pm \rho) \pm \frac{(\hat{\tau} \pm \rho)^2}{2\rho} a_{\hat{\sigma}} \hat{\sigma} + \cdots,
$$

(6.18)
in which the first group of terms is of order $\rho$, the second group is of order $\rho^2$, and a third group of terms at order $\rho^3$ was calculated but is too large to be displayed here. Second, $r$ and $r_{\text{adv}}$ are related to $\rho$ through the relations $r = \sigma(u) = \sigma(\hat{\tau} + \Delta_-)$ and $r_{\text{adv}} = -\hat{\sigma}(v) = -\hat{\sigma}(\hat{\tau} + \Delta_+)$, which are also expanded in powers of $\Delta_\pm$. Third, $U^A B^\mu U^A(u)$ and $U^A B^\mu = U^A(v)$ are expressed in a similar way, and fourth, the integral of $V^A := V^A_M q^M$ is evaluated as $-\frac{1}{2}(\Delta_+ - \Delta_-) V^A(\hat{\tau})$.

Collecting results, we obtain the covariant expression

$$
\psi^A = \frac{1}{\rho} \left[ \gamma_1 U^A(\hat{\tau}) + \hat{\tau} \gamma_2 \hat{U}^A(\hat{\tau}) + \frac{1}{2}(\rho^2 + \hat{\tau}^2) \hat{U}^A(\hat{\tau}) - \rho^2 V^A(\hat{\tau}) + O(\rho^3) \right]
$$

(6.19)
for the singular potential. We have introduced

$$
\gamma_1 := \frac{\rho}{2} \left( \frac{1}{r} + \frac{1}{r_{\text{adv}}} \right)
$$

(6.20a)

$$
= 1 + \frac{\rho^2 - \hat{\tau}^2}{2\rho^2} a_{\hat{\rho}} a_{\hat{\mu}} - \frac{3\rho^4 + 6\rho^2 \hat{\tau}^2 - \hat{\tau}^4}{24\rho^2} a^2 + \frac{3(\rho^2 - \hat{\tau}^2)^2}{8\rho^4} (a_{\hat{\rho}} a_{\hat{\mu}})^2 + \frac{\hat{\tau} (3\rho^2 - \hat{\tau}^2)}{6\rho^2} a_{\hat{\rho}} a_{\hat{\mu}},
$$

(6.20b)

where $a^2 := a_{\hat{\mu}} a_{\hat{\mu}}$, and

$$
\gamma_2 := \frac{\rho}{2\hat{\tau}} \left( \frac{\Delta_\pm}{r} + \frac{\Delta_\mp}{r_{\text{adv}}} \right) = 1 + \frac{3\rho^2 - \hat{\tau}^2}{2\rho^2} a_{\hat{\rho}} a_{\hat{\mu}} + O(\rho^2).
$$

(6.21)

The tensors $U^A, \hat{U}^A, \hat{U}^A$ can be imported from Eq. (6.10), and $V^A$ can be obtained from Eq. (6.13), with the understanding that these objects are now evaluated at $\tau = \hat{\tau}$ instead of $\tau = t$, so that the expressions involve tensors with hatted indices instead of barred indices. Explicit expressions will be given below.

**VII. SELF-FORCE IN SCALARVAC SPACETIMES**

In this section we compute the self-force acting on a particle of mass $m$ and scalar charge $q$ in a scalarvac spacetime; the metric of the background spacetime is $g_{\alpha\beta}$, and the background scalar field is $\Phi$. To simplify the calculations we adopt the gauge of Eq. (2.11) with $\lambda = 1$. A glance at Eq. (2.13) reveals that this choice of gauge eliminates the term $M^A_{\beta\mu} \nabla^B \psi^B$ from the perturbation equation, and this considerably simplifies the structure of the Hadamard Green’s function.

**A. Equations of motion**

On a formal level, the particle’s equations of motion are formulated in the perturbed spacetime. We have

$$
m \frac{Du^\mu}{dt} = q(g^{\mu\nu} + u^\mu u^\nu) D_\mu \Phi,
$$

(7.1)

where $m := m - q\Phi$ is an inertial mass parameter that satisfies

$$
\frac{dm}{dt} = -q u^\mu D_\mu \Phi.
$$

(7.2)

We have introduced $t$ as proper time in the perturbed spacetime, $u^\mu = dz^\mu/dt$, and $D/dt$ indicates covariant differentiation (in the perturbed spacetime) along the world line.
Substitution of \( g_{\alpha \beta} = g_{\alpha \beta} + h_{\alpha \beta} \), \( \Phi = \Phi + \phi \), \( d\tau / dt = 1 + \frac{1}{2} h_{\mu \nu} u^\mu u^\nu \), and linearization with respect to all perturbations produce

\[
ma^\mu = (g^{\mu \nu} + u^\mu u^\nu) \left[ q \nabla_\nu \Phi - qu^\nu h_{\lambda \rho} \nabla_\nu \Phi - qh_\lambda^\nu \nabla_\lambda \Phi + q \nabla_\nu \phi - \frac{1}{2} m(2 \nabla_\rho h_{\nu \lambda} - \nabla_\nu h_{\lambda \rho}) u^\lambda u^\rho \right]
\]  

(7.3)

and

\[
\frac{dm}{d\tau} = -qu^\mu \nabla_\mu \Phi - qu^\mu \nabla_\mu \phi,
\]

(7.4)

in which all quantities refer to the background spacetime.

As was mentioned, these equations have formal validity only, because the retarded potentials \( h_{\alpha \beta} \) and \( \phi \) are singular on the world line. To make sense of these equations we follow the Detweiler-Whiting prescription [23], which asserts that the motion of the particle is driven entirely by the regular piece of the potentials, obtained after removal of the singular piece: \( h_{\alpha \beta}^R = h_{\alpha \beta} - h_{\alpha \beta}^S \) and \( \phi_R = \phi - \phi_S \). The equations of motion, therefore, are written as in Eqs. (7.3) and (7.4), but with the regular potentials standing in for the retarded potentials.

In Fermi coordinates we have

\[
ma_a = q \nabla_a \Phi - qh_{tt} \nabla_a \Phi - qh_{t a} \nabla_f \Phi - qh_{b a} \nabla_b \Phi + q \nabla_a \phi - \frac{1}{2} m(2 \nabla_s h_{t a} - \nabla_a h_{tt})
\]  

(7.5)

and

\[
\frac{dm}{dt} = -q \nabla_t \Phi - q \nabla_t \phi,
\]

(7.6)

where we suppress the label “R” on the potentials \( h_{\alpha \beta} \) and \( \phi \).

**B. Regular potentials**

Importing \( M^A_{B \mu} \) and \( N^A_{B \nu} \) from Sec. 11 and inserting them within Eqs. (6.11) and (5.18) reveals that the tensors that appear in the Hadamard Green’s function are given explicitly by

\[
U^A_{B \mu} = 0,
\]

(7.7a)

\[
U^{\alpha \beta}_{|\gamma \delta \mu \nu} = \frac{1}{6} \delta^{(\alpha \beta)}_{(\gamma \delta)} R_{\mu \nu},
\]

(7.7b)

\[
U^\alpha_{|\mu \nu} = \frac{1}{6} R_{\mu \nu},
\]

(7.7c)

\[
V^{\alpha \beta}_{|\gamma \delta} = R^{\alpha \beta}_{\gamma \delta} - \delta^\alpha_{(\gamma} \delta^\beta_{\delta)} \Phi \nabla^\gamma \nabla^\delta \Phi + \frac{1}{12} \delta^\alpha_{(\gamma} \delta^\beta_{\delta)} R,
\]

(7.7d)

\[
V^{\alpha \beta}_{|\gamma} = -2 \nabla^\alpha \nabla^\beta \Phi,
\]

(7.7e)

\[
V_{|\alpha \beta} = -\frac{1}{2} \left( \nabla_\alpha \nabla_\beta \Phi - \frac{1}{2} F' g_{\alpha \beta} \right),
\]

(7.7f)

\[
V_{|} = -\frac{1}{2} \left( 2 \nabla_\mu \Phi \nabla^\mu \Phi + F'' \right) + \frac{1}{12} R,
\]

(7.7g)

where we omit the primes on indices to keep the notation uncluttered.

Making the substitutions in Eq. (6.11), we obtain

\[
\gamma_{R | t} = 4 q \Phi (1 - 3 \gamma_{a c}) + \frac{2}{3} (m - q \Phi) (2 \dot{a}_c + R_{t c}) x^c + \gamma_{[tail]} + O(s^2),
\]

(7.8a)

\[
\gamma_{R | a} = -4 (m - q \Phi) a^a + 2 (m - q \Phi) (4 a^a a_c - R^a_{t c}) x^c + \gamma_{[tail]} + O(s^2),
\]

(7.8b)

\[
\gamma_{R | a} = \gamma_{[tail]} + O(s^2)
\]

(7.8c)

for the gravitational potentials, and

\[
\phi_R = \frac{1}{6} q (2 \dot{a}_a + R_{a a}) x^c + \phi_{[tail]} + O(s^2)
\]

(7.9)
for the scalar perturbation. The tail terms are

\[ \gamma^{\alpha\beta}[\text{tail}] := 4 \int_{-\infty}^{t} (m - q\dot{\Phi}) G^{\alpha\beta}_{\mu\nu}(x, z) u^\mu u^\nu \, d\tau + q \int_{-\infty}^{t} G^{\alpha\beta}_{\mu\nu}(x, z) \, d\tau \]  

(7.10)

and

\[ \phi[\text{tail}] := 4 \int_{-\infty}^{t} (m - q\dot{\Phi}) G^{\nu}_{\mu\nu}(x, z) u^\mu u^\nu \, d\tau + q \int_{-\infty}^{t} G^{\nu}_{\mu\nu}(x, z) \, d\tau. \]  

(7.11)

After trace reversal and lowering the indices using the metric \( g_{tt} = -1 - 2a_c x^c + O(s^2) \), \( g_{ta} = O(s^2) \), \( g_{ab} = \delta_{ab} + O(s^2) \), the gravitational potentials become

\[ h_{tt}^R = 2q\dot{\Phi}(1 + a_c x^c) + \frac{1}{3} (m - q\dot{\Phi})(2\dot{a}_c + R_{tc}) x^c + h_{tt}[\text{tail}] + O(s^2), \]  

(7.12a)

\[ h_{ta}^R = 4(m - q\dot{\Phi})a_a + 2(m - q\dot{\Phi})R_{atc} x^c + h_{ta}[\text{tail}] + O(s^2), \]  

(7.12b)

\[ h_{ab}^R = 2q\dot{\Phi}\delta_{ab}(1 - a_c x^c) + \frac{1}{3} (m - q\dot{\Phi})\delta_{ab}(2\dot{a}_c + R_{tc}) x^c + h_{ab}[\text{tail}] + O(s^2), \]  

(7.12c)

with

\[ h_{\alpha\beta}[\text{tail}] := 4 \int_{-\infty}^{t} (m - q\dot{\Phi}) G_{\alpha\beta\mu\nu}(x, z) u^\mu u^\nu \, d\tau + q \int_{-\infty}^{t} G_{\alpha\beta\mu\nu}(x, z) \, d\tau, \]  

(7.13)

where the overbar indicates the operation of trace reversal.

The relevant covariant derivatives of the potentials are

\[ \nabla_t h_{ta} = 4(m - q\dot{\Phi})(\dot{a}_a + \dot{V}_t a_{[t]} - q\dot{\Phi}a_a + q\dot{V}_t a_{[t]} + h_{tat}[\text{tail}] + O(s), \]  

(7.14a)

\[ \nabla_a h_{tt} = \frac{1}{3} (m - q\dot{\Phi})(2\dot{a}_a + R_{ta}) + h_{tat}[\text{tail}] + O(s), \]  

(7.14b)

\[ \nabla_t \phi = q\dot{V}_t + 4(m - q\dot{\Phi})V_{tt} + \phi_{t}[\text{tail}] + O(s), \]  

(7.14c)

\[ \nabla_a \phi = \frac{1}{6} q(2\dot{a}_a + R_{ta}) + \phi_a[\text{tail}] + O(s), \]  

(7.14d)

where

\[ h_{\alpha\beta\gamma}[\text{tail}] := 4 \int_{-\infty}^{t} (m - q\dot{\Phi}) \nabla_\gamma \bar{G}_{\alpha\beta\mu\nu}(x, z) u^\mu u^\nu \, d\tau + q \int_{-\infty}^{t} \nabla_\gamma \bar{G}_{\alpha\beta\mu\nu}(x, z) \, d\tau, \]  

(7.15a)

\[ \phi_{a}[\text{tail}] := 4 \int_{-\infty}^{t} (m - q\dot{\Phi}) \nabla_a \bar{G}_{\mu\nu}(x, z) u^\mu(\tau) u^\nu(\tau) \, d\tau + q \int_{-\infty}^{t} \nabla_a \bar{G}_{\mu\nu}(x, z) \, d\tau. \]  

(7.15b)

To arrive at these results we relied on the fact that in Fermi coordinates, the relevant Christoffel symbols are \( \Gamma_{ta} = a_a + O(s) \) and \( \Gamma_{tt} = a^o + O(s) \); all other symbols are \( O(s) \) and not required in this computation.

C. Explicit form of the equations of motion

Making the substitutions in Eqs. (7.5) and (7.6), we obtain

\[ \frac{dm}{dt} = -q\dot{V}_t + q^2 \left( \nabla_\mu \Phi \nabla^\mu \Phi + \frac{1}{2} F'' - \frac{1}{12} R \right) + \frac{1}{2} m(2h_{tat}[\text{tail}] - h_{ta}[\text{tail}]). \]  

(7.16)

and

\[ \frac{dm}{dt} = q^2 \left( \nabla_\mu \Phi \nabla^\mu \Phi + \frac{1}{2} F'' - \frac{1}{12} R \right) + \frac{1}{2} m(2\dot{\Phi}_t + \Phi'' - q\phi_{t}[\text{tail}]). \]  

(7.17)
In the perturbation terms we no longer distinguish between \( m - q \Phi \) and \( \mathbf{m} := m - q \Phi - q \phi \), where \( \phi \) is now identified with \( \phi_R = \phi_{\text{tail}} \). These equations can be simplified by inserting the background equation of motion, \( (m - q \Phi) a_a = q \nabla_a \Phi \), on the right-hand side; taking into account the variation of \( \Phi \) on the world line, the equation implies \( \dot{a}_a = (q/m) \nabla_a \Phi + 2(q/m)^2 \nabla_t \Phi \nabla_a \Phi \). The equations of motion can also be simplified by inserting Eq. (2.3) for the background Ricci tensor. The end result is

\[
ma_a = q \nabla_a \Phi + \frac{1}{6} q^2 \left( 5 + \frac{4 q^2}{m^2} - \frac{11 m^2}{q^2} \right) \nabla_t \Phi \nabla_a \Phi - \frac{3}{4} m \left( 5 - \frac{q^2}{m^2} \right) \nabla_t \nabla_a \Phi - q h_{\mu \nu \lambda \rho} [\text{tail}] \nabla_\mu \Phi - q h_{\lambda \rho}^{\text{tail}} \nabla_\nu \Phi - q \phi_a [\text{tail}] - \frac{1}{2} m \left( 2 h_{\mu \nu \lambda \rho} [\text{tail}] - h_{\mu \nu \lambda \rho} [\text{tail}] \right)
\]

\[ (7.18) \]

and

\[
\frac{dm}{d\tau} = -q \nabla_t \Phi + q^2 \left( \frac{11}{12} \nabla_\mu \Phi \nabla^{\mu} \Phi - \frac{1}{3} F + \frac{1}{2} F'' \right) + m \left( 2 \nabla_t \nabla_\nu \Phi + F' \right) - q \phi_t [\text{tail}].
\]

\[ (7.19) \]

At this stage it is a simple matter to express the equations of motion in covariant form. We have

\[
ma^{\mu} = \left( g^{\mu \nu} + u^{\mu} u^{\nu} \right) \left( q \nabla_\nu \Phi + \frac{1}{6} q^2 \left( 5 + \frac{4 q^2}{m^2} - \frac{11 m^2}{q^2} \right) u^{\lambda} \nabla_\lambda \Phi \nabla_\nu \Phi - \frac{3}{4} m \left( 5 - \frac{q^2}{m^2} \right) u^{\lambda} \nabla_\lambda \nabla_\nu \Phi - q u^{\lambda} u^{\rho} h_{\lambda \rho} [\text{tail}] \nabla_\nu \Phi - q h_{\nu}^{\lambda} [\text{tail}] \nabla_\lambda \Phi + q \phi_\nu [\text{tail}] - \frac{1}{2} m \left( 2 h_{\nu \lambda \rho} [\text{tail}] - h_{\lambda \rho}^{\nu} [\text{tail}] \right) u^{\lambda} u^{\rho} \right)
\]

\[ (7.20) \]

and

\[
\frac{dm}{d\tau} = -q u^{\mu} \nabla_\mu \Phi + q^2 \left( \frac{11}{12} \nabla^{\mu} \Phi \nabla_\mu \Phi - \frac{1}{3} F + \frac{1}{2} F'' \right) + m \left( 2 u^{\mu} u^{\nu} \nabla_\nu \Phi + F' \right) - q u^{\mu} \phi_\mu [\text{tail}].
\]

\[ (7.21) \]

The tail terms are still given by the equations displayed previously, except that the dependence on \( t^- \) can now be replaced by a dependence on \( \tau^- := \tau - 0^+ \); the variable of integration should then be replaced by \( \tau' \).

### D. Singular potentials

The equations of motion displayed in Eq. (7.19) and (7.20) are typically not the most useful starting point to calculate the motion of a point particle. A more practical formulation is based instead on the form

\[
ma^{\mu} = \left( g^{\mu \nu} + u^{\mu} u^{\nu} \right) \left[ q \nabla_\nu \Phi - q u^{\lambda} u^{\rho} h_{\lambda \rho}^{\text{R}} \nabla_\nu \Phi - q h_{\nu}^{\lambda \rho} \nabla_\lambda \Phi + q \nabla_\nu \phi^{\text{R}} - \frac{1}{2} m \left( 2 \nabla_\mu h_{\nu \lambda}^{\text{R}} - \nabla_\nu h_{\lambda \mu}^{\text{R}} \right) u^{\lambda} u^{\rho} \right]
\]

\[ (7.22) \]

and

\[
\frac{dm}{d\tau} = -q u^{\mu} \nabla_\mu \Phi - q u^{\mu} \nabla_\mu \phi^{\text{R}},
\]

\[ (7.23) \]

in which the singular potentials \( h_{\alpha \beta}^{\text{S}} \) and \( \phi^{\text{S}} \) were explicitly removed from the retarded potentials \( h_{\alpha \beta} \) and \( \phi \). This subtraction can be implemented by formulating field equations for the regular potentials in terms of extended effective sources, or by obtaining regularization parameters when the self-force is computed as a sum over spherical-harmonic modes.

The starting point of such computations is the singular potentials of Eq. (6.19), in which we insert Eqs. (6.10). With the results displayed in Eqs. (7.7), we have

\[
U^{\alpha \beta} = g^{(\alpha \beta)}_{\text{a}} g^{(\beta)}_{\text{b}} \left\{ 4 m u^a u^b \left( 1 + \frac{1}{6} R_{\mu \nu \sigma \tau} \sigma^\mu \tau^\nu \right) + O(\rho^3) \right\},
\]

\[ (7.24a) \]

\[
\bar{U}^{\alpha \beta} = g^{(\alpha \beta)}_{\text{a}} g^{(\beta)}_{\text{b}} \left\{ 4 m \left( 2 u^a u^b + \frac{1}{6} u^a u^b R_{\mu \nu \sigma \tau} \sigma^\mu \tau^\nu \right) + R_{\gamma \mu \nu \rho} u^\gamma u^b u^a u^\rho - 4 q \Phi u^a u^b + O(\rho^2) \right\},
\]

\[ (7.24b) \]

\[
\bar{U}^{\alpha \beta} = g^{(\alpha \beta)}_{\text{a}} g^{(\beta)}_{\text{b}} \left\{ 4 m \left( 2 u^a u^b + 2 u^a u^b + \frac{1}{6} u^a u^b R_{\mu \nu \sigma \tau} \sigma^\mu \tau^\nu \right) - 16 q \Phi u^a u^b - 4 \Phi u^a u^b + O(\rho) \right\},
\]

\[ (7.24c) \]
\[ V^\alpha_\beta = g^\alpha_{\hat{\alpha}} g^\beta_{\hat{\beta}} \left\{ 4m \left( R_{\hat{\gamma}}^{\hat{\beta}} \hat{\delta} u^\gamma u^\delta - u^\alpha u^\gamma \nabla^\alpha \Phi \nabla^\gamma \Phi + \frac{1}{12} u^\alpha u^\beta R \right) - 2q \nabla^\alpha \nabla^\beta \Phi + O(\rho) \right\} \] (7.24d)

and

\[ U^\alpha = q \left( 1 + \frac{1}{12} R_{\mu\nu\sigma\rho} \sigma^\mu \sigma^\nu \right) + O(\rho^3), \] (7.25a)

\[ \dot{U}^\mu = \frac{1}{6} q R_{\mu\nu\rho\sigma} u^\nu u^\rho + O(\rho^2), \] (7.25b)

\[ \ddot{U}^\mu = \frac{1}{6} q R_{\mu\nu\rho\sigma} u^\nu u^\rho + O(\rho), \] (7.25c)

\[ V^\mu = -\frac{1}{2} q \left( 2 \nabla^\alpha \Phi \nabla^\beta \Phi + F^\alpha \right) - 2m \left( u^\alpha u^\beta \nabla^\gamma \Phi + \frac{1}{2} F^\beta \right) + O(\rho). \] (7.25d)

**VIII. SELF-FORCE IN ELECTROVAC SPACETIMES**

In this section we compute the self-force acting on a particle of mass \( m \) and electric charge \( e \) in an electrovac spacetime; the metric of the background spacetime is \( g_{\alpha\beta} \), and the background electromagnetic field is \( F_{\alpha\beta} \).

**A. Equations of motion**

On a formal level, the motion of the particle is governed by the Lorentz-force equation

\[ m \frac{Du^\mu}{dt} = eF^\mu_\nu u^\nu, \] (8.1)

in which \( t \) is proper time in the perturbed spacetime, \( u^\mu = ds^\mu/dt \), and \( D/dt \) indicates covariant differentiation (in the perturbed spacetime) along the world line. Substitution of \( g_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta}, F_{\alpha\beta} = F_{\alpha\beta} + f_{\alpha\beta} \), and linearization with respect to all perturbations produce

\[ ma^\mu = eF^\mu_\nu u^\nu - \frac{1}{2} e u^\lambda u^\rho h_{\lambda\rho} F^\mu_\nu u^\nu - e \left( g^{\mu\nu} + u^\mu u^\nu \right) h_{\nu\lambda} F^\lambda_\rho u^\rho + e f^\mu_\nu u^\nu - \frac{1}{2} m \left( g^{\mu\nu} + u^\mu u^\nu \right) \left( 2 \nabla_\nu h_{\lambda\rho} - \nabla_\nu h_{\lambda\rho} \right) u^\lambda u^\rho, \] (8.2)

in which all quantities now refer to the background spacetime. To make sense of these equations we continue to follow the Detweiler-Whiting prescription to remove the singular piece of all potentials. The equations of motion continue to be given by Eq. (8.2), but with the regular potentials standing in for the retarded potentials.

In Fermi coordinates we have

\[ ma_a = eF_{at} - \frac{1}{2} e h_{at} F^b_{\beta} + f_{at} - \frac{1}{2} m \left( 2 \nabla_t h_{ta} - \nabla_a h_t \right), \] (8.3)

where we suppress the label “R” on the potentials.

**B. Regular potentials**

Importing \( M^A B^\mu \) and \( N^A B^\mu \) from Sec. [III] and inserting them within Eqs. [5.11] and [5.18] reveals that the tensors that appear in the Hadamard Green’s function are given explicitly by

\[ U_{\gamma\mu}^{\alpha\beta} = 8 \delta_{\alpha\beta} F_{\gamma\mu} + 2 \sigma_{\alpha\beta} F_{\gamma\mu}, \] (8.4a)

\[ U_{\beta\gamma}^{\alpha\beta} = \frac{1}{2} \delta_{\alpha\beta} F_{\gamma\mu} - \frac{1}{4} g_{\beta\gamma} F_{\mu}, \] (8.4b)

\[ U_{\gamma\delta\mu\nu}^{\alpha\beta} = \delta_{\mu\nu} F_{(\gamma} F_{\delta)\rho} + \delta_{\nu} F_{(\gamma} F_{\delta)(\rho} F_{\mu)} - \delta_{(\gamma} F_{\delta)(\rho} F_{\mu)} - \delta_{(\gamma} F_{\delta)(\rho} F_{\mu)} - \delta_{(\gamma} F_{\delta)(\rho} F_{\mu)} - \frac{1}{2} g_{\gamma\delta} F_{\mu} + g_{\gamma\delta} F_{\mu} - \frac{1}{2} g_{\gamma\delta} F_{\mu} F_{\nu} + \frac{1}{2} g_{\gamma\delta} F_{\mu} F_{\nu} + \frac{1}{2} g_{\gamma\delta} F_{\mu} F_{\nu}, \] (8.4c)

\[ U_{\gamma\mu\nu}^{\alpha\beta} = -4 \delta_{(\mu} F_{\nu)} F_{\gamma} + \delta_{\gamma} F_{(\mu} F_{\nu)} + 2 g_{\gamma\delta} F_{(\mu} F_{\nu)} F_{\delta), \] (8.4d)
where we omit the primes on indices to keep the notation uncluttered.

Making the substitutions in Eq. \[6.14\b\], we obtain

\[
\gamma_{\mu}^{\nu} = 2 \frac{m}{3} (2a_{c} + R_{tc}) x^c + e \left( -\nabla_{t} F_{ct} + 2 a^{d} F_{cd} \right) x^c + \gamma_{\mu}^{\nu}[\text{tail}] + O(s^2),
\]

\[
\gamma_{R}^{a} = -4ma^{a} + m(8a_{a} a_{c} - 2R_{tc} + F_{a}^{c} F_{c} + \delta_{a}^{c} F_{ct} F_{t}^{e}) x^c + e \left( -\nabla_{t} F_{ac}^{c} - 2a^{d} F_{cd} \right) x^c + \gamma_{a}^{\mu}[\text{tail}] + O(s^2),
\]

\[
\gamma_{R}^{ab} = m(2\delta^{(a}_{c} F_{b) c} - 4F^{a}_{c} F_{b} + 2a^{d} F_{cd} F_{a}^{e} x^c + e (2\delta^{(a}_{c} \nabla_{t} F_{b) c} - \delta_{ab} \nabla_{t} F_{ct} + 4a^{d} \delta_{c}^{(a}_{e} F_{b) c} - 2a^{d} a^{e} F_{cc}) x^c + \gamma_{ab}^{\mu}[\text{tail}] + O(s^2)
\]

for the gravitational potentials, and

\[
b_{R}^{a} = \frac{1}{2} e \left( 2a_{c} + 3R_{tc} - 2F_{ac} F_{tc} + 2a^{d} F_{cd} \right) x^c + e \left( -\nabla_{t} F_{ac}^{c} + 2a^{d} F_{cd} \right) x^c + b^{a}[\text{tail}] + O(s^2), \]

\[
b_{R}^{ab} = -2a^{d} F_{cd} + \nabla_{t} F_{ac}^{c} + \delta_{a}^{c} F_{ct} F_{t}^{e} x^c + e (2\delta^{(a}_{c} \nabla_{t} F_{b) c} - \delta_{ab} \nabla_{t} F_{ct} + 4a^{d} \delta_{c}^{(a}_{e} F_{b) c} - 2a^{d} a^{e} F_{cc}) x^c + \gamma_{ab}^{\mu}[\text{tail}] + O(s^2)
\]

for the electromagnetic potentials. The tail terms are

\[
\gamma_{\alpha\beta}[\text{tail}] := 4m \int_{-\infty}^{t} G_{\alpha\beta|\mu}(x, z) u^\mu u^\nu d\tau + e \int_{-\infty}^{t} G_{\alpha\beta|\mu}(x, z) u^\mu d\tau
\]

and

\[
b^{a}[\text{tail}] := 4m \int_{-\infty}^{t} G_{a|\mu}(x, z) u^\mu u^\nu d\tau + e \int_{-\infty}^{t} G_{a|\mu}(x, z) u^\mu d\tau.
\]

After trace reversal and lowering the indices using the metric \(g_{tt} = -1 - 2a_{c} x^c + O(s^2), g_{ta} = O(s^2), g_{ab} = \delta_{ab} + O(s^2),\) the gravitational potentials become

\[
h_{\mu}^{t} = \frac{1}{3} m (2a_{c} + R_{tc}) x^c + e \left( -\nabla_{t} F_{ct} + 2 a^{d} F_{cd} \right) x^c + h_{\mu}[\text{tail}] + O(s^2),
\]

\[
h_{R}^{a} = 4ma^{a} + m(2R_{tc} - 2a_{c} F_{tc} + \delta_{a}^{c} F_{ct} F_{t}^{e} - 2a^{d} F_{cd} ) x^c + e \left( -\nabla_{t} F_{ac}^{c} + 2a^{d} F_{cd} \right) x^c + h_{a}[\text{tail}] + O(s^2),
\]

\[
h_{R}^{ab} = \frac{1}{3} m(2a_{c} + R_{tc}) x^c + m(2\delta^{(a}_{c} F_{b) c} F_{c}^{c} + 4F_{t}^{a} F_{b} - 2a_{a} F_{c}^{c} + 2a^{d} F_{cd} F_{t}^{e} ) x^c + e (2\delta^{(a}_{c} \nabla_{t} F_{b) c} + \delta_{a}^{c} F_{ct} + 4a^{d} \delta_{c}^{(a}_{e} F_{b) c} - 2a^{d} a^{e} F_{cc}) x^c + h_{ab}[\text{tail}] + O(s^2),
\]

with tail terms now given by

\[
h_{\alpha\beta}[\text{tail}] := 4m \int_{-\infty}^{t} \tilde{G}_{\alpha\beta|\mu}(x, z) u^\mu u^\nu d\tau + e \int_{-\infty}^{t} \tilde{G}_{\alpha\beta|\mu}(x, z) u^\mu d\tau,
\]
where the overbar indicates the operation of trace reversal. The electromagnetic potentials become

$$\begin{align*}
b^R &= -\frac{1}{6}e(2\dot{a}_c + 3F_{ce}F^c_t + R_{ct})x^c + \frac{1}{2}m(\nabla_t F_{ct} + 4a^dF_{cd})x^c + b_t[\text{tail}] + O(s^2), \\
b^R_a &= -ea_a - mF_{at} + \frac{1}{2}e(2a_c a_a - R_{act} + \delta_{ac}F_{ct}F^c_t)x^c \\
&\quad + \frac{1}{2}m(2a_c F_{at} - 4a_a F_{ct} + \nabla_t F_{ac} - \nabla_c F_{at})x^c + b_a[\text{tail}] + O(s^2)
\end{align*}$$

(8.11a)

(8.11b)

after lowering the indices.

The relevant covariant derivatives of the potentials are

$$\begin{align*}
\nabla_t h_{ta} &= 4m(\dot{a}_a + \dot{V}_{ta[tr]} + e\dot{V}_{ta[t]} + h_{tat[tail]} + O(s)), \\
\nabla_a h_{tt} &= \frac{1}{3}m(2\dot{a}_a + R_{ta}) + e(-\nabla_t F_{at} + 2a^cF_{ac}) + h_{tt[ta][tail]} + O(s), \\
\nabla_{\dot{t}} b_a &= e(-\dot{a}_a + V_{a[t]} + m(-\nabla_t F_{at} - a^cF_{ac} + 4V_{a[ta]} + b_{at[ta]} + O(s), \\
\nabla_a b_t &= -\frac{1}{6}e(2\dot{a}_a + 3F_{ac}F^c_t + R_{ta}) + \frac{1}{2}m(\nabla_t F_{at} + 4a^cF_{ac}) + b_{ta[ta]} + O(s),
\end{align*}$$

(8.12a)

(8.12b)

(8.12c)

(8.12d)

where

$$\begin{align*}
h_{\alpha\beta\gamma}[\text{tail}] &:= 4m \int_{-\infty}^{t^-} \nabla_\gamma G_{\alpha\beta\mu\nu}(x,z)u^\mu u^\nu d\tau + e \int_{-\infty}^{t^-} \nabla_\gamma G_{\alpha\beta\mu\nu}(x,z)u^\mu d\tau, \\
b_{\alpha\beta}[\text{tail}] &:= 4m \int_{-\infty}^{t^-} \nabla_\beta G_{\alpha\beta\mu\nu}(x,z)u^\mu u^\nu d\tau + e \int_{-\infty}^{t^-} \nabla_\beta G_{\alpha\beta\mu\nu}(x,z)u^\mu d\tau.
\end{align*}$$

(8.13a)

(8.13b)

C. Explicit form of the equations of motion

Making the substitutions in Eq. [8.23], we obtain

$$\begin{align*}
ma_a &= eF_{at} + m^2 \left(\frac{11}{3}\dot{a}_a + \frac{1}{6}R_{ta} + 2F_{ac}F^c_t\right) + e^2 \left(\frac{2}{3}\dot{a}_a + \frac{1}{3}R_{ta} - F_{ac}F^c_t\right) + 4meer^cF_{ac} \\
&\quad - \frac{1}{2}e\dot{h}_{tt}[ta][F_{at} - eh_{ab}[tail]F^b_t] + e(b_{ta}[ta] - b_{at[ta]}) - \frac{1}{2}m(2h_{t[a][tail]} - h_{ta[ta]}).
\end{align*}$$

(8.14)

This is simplified by inserting the background equation of motion, $ma_a = eF_{at}$, on the right-hand side; the equation implies $\dot{a}_a = (e/m)\nabla_t F_{at} + (e/m)^2 F_{ac}F^c_t$. It is also simplified by inserting $R_{ta} = -2F_{ac}F^c_t$ for the background Ricci tensor. The end result is

$$\begin{align*}
ma_a &= eF_{at} - \frac{1}{3}me \left(11 - \frac{2e^2}{m^2}\right)\nabla_t F_{at} - \frac{1}{3}e^2 \left(4 - \frac{2e^2}{m^2} - \frac{5m^2}{e^2}\right) F_{ac}F^c_t \\
&\quad - \frac{1}{2}eh_{tt}[ta][F_{at} - eh_{ab}[tail]F^b_t] + e(b_{ta}[ta] - b_{at[ta]}) - \frac{1}{2}m(2h_{t[a][tail]} - h_{ta[ta]}).
\end{align*}$$

(8.15)

The covariant form of the equations of motion is

$$\begin{align*}
ma^\mu &= eF^\mu_{\nu\nu}u^\nu + (g^\mu\nu + u^\nu u^\mu) \left\{ -\frac{1}{3}me \left(11 - \frac{2e^2}{m^2}\right)u^\lambda u^\rho \nabla_\lambda F_{\nu\lambda} - \frac{1}{3}e^2 \left(4 - \frac{2e^2}{m^2} - \frac{5m^2}{e^2}\right)F_{\nu\lambda}F^\lambda_{\rho\nu}u^\rho \\
&\quad - \frac{1}{2}eu^\lambda u^\rho h_{\lambda\rho}[tail]F_{\nu\tau}u^\tau - eh_{\nu\lambda}[tail]F^\lambda_{\rho\nu}u^\rho + e(b_{\lambda\nu}[tail] - b_{\nu\lambda}[tail])u^\lambda \\
&\quad - \frac{1}{2}m(2h_{\nu\lambda\rho}[tail] - h_{\lambda\nu\rho}[tail])u^\lambda u^\rho \right\}.
\end{align*}$$

(8.16)

The tail terms are still given by the equations displayed previously, except that the dependence on $t^-$ can now be replaced by a dependence on $\tau^- := \tau - 0^+$; the variable of integration should then be replaced by $\tau'$. 
D. Singular potentials

The equation of motion displayed in Eq. (8.10) is typically not the most useful starting point to calculate the motion of a point particle. A more practical formulation is based instead on the form

\[ ma^\mu = e F^\mu_\nu u^\nu - \frac{1}{2} \varepsilon_{\alpha \beta \gamma \delta} h^\alpha_\nu h^\beta_\rho F^\gamma_\rho u^\nu + e f^{\alpha \beta}_R u^\gamma h^\alpha_\nu h^\beta_\rho F^\gamma_\rho u^\nu + \frac{1}{2} m (2 \nabla_\rho h^R_\gamma - \nabla_\nu h^R_\rho) u^\lambda u^\nu, \]  

(8.17)

in which the singular potentials \( h^\alpha_\nu \) and \( b^\alpha_\nu \) were explicitly removed from the retarded potentials \( h_\alpha^\beta \) and \( b_\alpha^\mu \). This subtraction can be implemented by formulating field equations for the regular potentials in terms of extended effective sources, or by obtaining regularization parameters when the self-force is computed as a sum over spherical-harmonic modes.

The starting point of such computations is the singular potentials of Eq. (6.19), in which we insert Eqs. (6.10). The explicit expressions are

\[ U^{\alpha \beta} = g^{(a \beta)}_\alpha \left\{ 4 m \left( u^\alpha u^\beta + \frac{1}{2} U^{\alpha \beta} \right) + e \left( U^{\alpha \beta} + \frac{1}{2} U^{\beta \alpha} \right) + O(\rho^3) \right\}, \]  

(8.18a)

\[ \dot{U}^{\alpha \beta} = g^{(a \beta)}_\alpha \left\{ 4 m \left( 2 a^\alpha u^\beta + U^{\alpha \beta} \right) + e \left( U^{\alpha \beta} + \frac{1}{2} U^{\beta \alpha} \right) + O(\rho^3) \right\}, \]  

(8.18b)

\[ \ddot{U}^{\alpha \beta} = g^{(a \beta)}_\alpha \left\{ 4 m \left( 2 a^\alpha u^\beta + U^{\alpha \beta} \right) + e \left( U^{\alpha \beta} + \frac{1}{2} U^{\beta \alpha} \right) + O(\rho^3) \right\}, \]  

(8.18c)

\[ V^{\alpha \beta} = g^{(a \beta)}_\alpha \left\{ 4 m V^{\alpha \beta} + e V^{\alpha \beta} + O(\rho) \right\}, \]  

(8.18d)

and

\[ U^{\alpha} = g^{\alpha \beta}_a \left\{ e \left( \dot{u}^\alpha + \frac{1}{2} \dot{U}^{\alpha \beta} u^\beta + \frac{1}{2} U^{\alpha \beta} u^\beta + \frac{1}{2} U^{\alpha \beta} \right) + O(\rho^3) \right\}, \]  

(8.19a)

\[ \dot{U}^{\alpha} = g^{\alpha \beta}_a \left\{ e \left( \dot{u}^\alpha + \dot{U}^{\alpha \beta} u^\beta + \frac{1}{2} \dot{U}^{\alpha \beta} u^\beta + \frac{1}{2} U^{\alpha \beta} \right) + O(\rho^3) \right\}, \]  

(8.19b)

\[ \ddot{U}^{\alpha} = g^{\alpha \beta}_a \left\{ e \left( \dot{u}^\alpha + \dot{U}^{\alpha \beta} u^\beta + \frac{1}{2} \dot{U}^{\alpha \beta} u^\beta + \frac{1}{2} U^{\alpha \beta} \right) + O(\rho^3) \right\}, \]  

(8.19c)

\[ V^{\alpha} = g^{\alpha \beta}_a \left\{ e V^{\alpha \beta} + 4 m V^{\alpha \beta} + O(\rho) \right\}. \]  

(8.19d)

The various tensors that appear in these expressions were displayed in Eqs. (8.4).

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[1] Y. Mino, M. Sasaki, and T. Tanaka, *Gravitational radiation reaction to a particle motion*, Phys. Rev. D 55, 3457 (1997), arXiv:gr-qc/9606018.
