The Additive Structure of Natural Numbers with the lower Wythoff Sequence

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Abstract

We have provided a model-theoretic proof for the decidability of the additive structure of natural numbers together with the function $f$ mapping $x$ to $\lfloor \varphi x \rfloor$ where $\varphi$ is the golden ratio.

Introduction

While the theory of the structure $(\mathbb{N}, +, \cdot)$ is famously undecidable, tame reducts of this structure have been subject of various literature, see for example \cite{1} \cite{12} \cite{13} \cite{9}. A classical result in this direction, is the decidability of the structure $(\mathbb{N}, +, \{p_n\}_{n \in \mathbb{N}}, 0, 1)$, which is known as Presburger arithmetic (note that Presburger arithmetic is also referred to the theory of the structure $(\mathbb{Z}, +, <, \{p_n\}_{n \in \mathbb{N}}, 0, 1)$, but in this paper by the term Presburger arithmetic we mean the former structure). In the mentioned structure, multiplication in $\mathbb{N}$ is replaced by infinitely-many unary predicates $p_n$, where $p_n(x)$ holds if $x \equiv 0$. More recent relevant results are, for example, that there are no intermediate structures between the group of integers and Presburger arithmetic (Conant in \cite{4}); and that the theory of integers with a predicate for prime numbers is decidable (Kaplan and Shelah in \cite{9}).

In this paper we prove the decidability of the structure $(\mathbb{N}, +, f, 0, 1)$ where $f(x) = \lfloor \varphi x \rfloor$, and $\varphi$ is the golden ratio. We are already aware that this follows from the decidability of the theory of $(\mathbb{R}, \mathbb{N}, \alpha \mathbb{N}, +, <)$ for a quadratic
irrational number $\alpha$, as proved by Hieronymi in \cite{12}. His proof relies on the continued fractions and Ostrowski representations and the fact that the structure in question is interpretable in the structure $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \in, s_n)$. Since the latter structure is decidable by a classical result of Büchi \cite{10}, so is the former. We have also been later informed by the referee that the decidability of our structure can also be obtained as a consequence of two papers of Shallit et al., \cite{6,1} where they even propose an automata-based decision algorithms for Fibonacci words.

Nevertheless, we rely on facts on Fibonacci words and the lower Wythoff sequence to explain the properties of the function $f$, and then our approach is pure model-theoretic and based on a quantifier-elimination result in a suitable language. It was indeed suggested by Hieronymi, that a model-theoretic treatment of the properties of Beatty sequences would lead to the decidability of our structure.

Recall that any sequence of the form $B_r = (\lfloor rn \rfloor)_{n \in \mathbb{N}}$ for a positive irrational $r$, is called a Beatty sequence. For $r > 1$ and $s = r/(r - 1)$, $(B_r, B_s)$ form a so-called pair of complementary Beatty sequences; that is $B_r \cup B_s = \mathbb{N}$ and $B_r \cap B_s = \emptyset$ (see \cite{7,8} for more on Beatty sequences). The Beatty sequence $B_r$, in the special case that $r = \varphi$, is the golden ratio, is called the lower Wythoff sequence.

We augment the language of Presburger arithmetic by the unary function symbol $f$ and denote the obtained language by $\mathcal{L}$. This choice of the language suggests that in order to have a chance for quantifier-elimination, we need to deal with the systems of equations involving congruence relations and the function $f$. It turns out that the solvability of such systems is closely related to a classical theorem of Kronecker (see Fact 8) that the set of decimal parts of elements of the form $\varphi n$, for $n \in \mathbb{N}$, is dense in the unit interval $(0, 1)$. We will deploy this connection as a major means for our axiomatization.

We will show that in spite of the fact that the decimal parts are not included in the language $\mathcal{L}$, their order is definable in this language. That is there is an $\mathcal{L}$-formula $R(x, y)$ such that $R(n, m)$ holds for two natural numbers $n, m$ if and only if the decimal part of $\varphi n$ is smaller than the decimal part of $\varphi m$. Hence we add a binary predicate $R(x, y)$ to $\mathcal{L}$ to obtain the language $\mathcal{L}^*$ (see Notation 2), and our main theorem is the following.

**Theorem.** The structure $(\mathbb{N}, +, f, R, \{p_n\}_{n \in \mathbb{N}}, 0, 1)$ admits elimination of quantifiers.

The paper is structured as follows. Basic facts about the properties of the function $f$ are gathered in Section 1. In section 2, some auxiliary lemmas are proved to be used in Section 3 as the basis of our axiomatization. The
quantifier-elimination result and the decidability that follows immediately from it are established in Section 4.

1 Preliminaries on the function $f$

By properties of the floor function, it is clear that for natural numbers $m$ and $n$, we have either $f(m + n) = f(m) + f(n)$, or $f(m) + f(n) + 1$. Hence, for each natural number $k$, there is $0 \leq \ell \leq k - 1$ such that $f(kn) = kf(n) + \ell$. Of course $\ell$ is the unique number such that $f(kn) \equiv k \ell$.

Lemma 1. For every $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that either $m = f(n)$ or $m = f(n) + n$.

Proof. As $\frac{\varphi}{\varphi - 1} = \varphi + 1$, $(\mathcal{B}_\varphi, \mathcal{B}_{\varphi + 1})$ is a complementary pair of Beatty sequences, which clearly means that each natural number $m$ is either equal to $f(n)$ or $f(n) + n$ for some natural number $n$.

Depending on whether $m$ belongs to the image of $f$ or $f + id$, where id denoted the identity function, and by the properties of the Beatty sequences one obtains a recursive definition for the function $f$ as in the following lemma.

Lemma 2. $f(0) = 0$, $f(1) = 1$, and for each natural number $n > 1$, $f(f(n) + n) = 2f(n) + n$ and $f(f(n)) = f(n) + n - 1$.

Proof. The fact that $f(f(n) + n)) = 2f(n) + n$ follows from [7, Theorem 1]. Now $f(f(n) + n)$ equals either to $f(f(n)) + f(n)$ or $f(f(n)) + f(n) + 1$. The former cannot occur since the images of $f$ and $f + id$ are disjoint. So $f(f(n) + n) = 2f(n) + n = f(f(n)) + f(n) + 1$, and the result follows.

The lemma above implies in particular that for every $n \in \mathbb{N}$, $f(n) = \min \mathbb{N} \setminus \{f(i), f(i) + i : i < n\}$, hence in particular the function $f$ is strictly increasing.

We now need to establish the connection between the function $f$ and the Fibonacci sequence $F_n$ with $F_0 = F_1 = 1$. This connection will be the essence of our proofs in this section of the properties of $f$ in natural numbers.

Consider the sequence $(c_n)_{n \in \mathbb{N}}$, where $c_n = 1$ if $n$ is in the image of $f$ and $0$ otherwise. By the properties of the floor function and the fact that $\varphi > 1$ it is easy to check that there are no successive zeros in $(c_n)_{n \in \mathbb{N}}$. A curious way to obtain $(c_n)_{n \in \mathbb{N}}$ is to start with the word $10$ (of course of length 2) and then replace 1 with 10 and 0 with 1 (to obtain a word of length 3), and apply the same change to the word obtained (to obtain a word of length 5), and continue the same way. So the length of each such word is a Fibonacci
number and the last digit alternates between 0 and 1. So \( c_{F_{2n+1}} = 1 \) and \( c_{F_{2n}} = 0 \) for each \( n \in \mathbb{N} \), and \( c_{F_{n}} = c_i \) for each \( 1 \leq i \leq F_{n-1} \).

Meanwhile, note that each natural number \( n \) has a unique Fibonacci representation. To see this one needs to find the largest Fibonacci number \( F_{i_1} < n \), and write \( n = F_{i_1} + G_1 \). Now let \( F_{i_2} \) be the largest Fibonacci number less than \( G_1 \) and write \( n = F_{i_1} + F_{i_2} + G_2 \) and continue with this procedure to end up with a Fibonacci number (see also [3]). The explanation above on the sequence \((c_n)_{n \in \mathbb{N}}\) with this representation yields the following fact.

**Fact 3.** For each \( n \), the smallest index appearing in the unique Fibonacci representation of \( n \) determines \( c_n \), where \( c_n = 1 \) if and only if this index is odd.

But this in turn provides us with a concrete rule for \( f \) as follows.

**Fact 4.**

1. \( f(F_i) = F_{i+1} \) if \( i \) is even, and \( f(F_i) = F_{i+1} - 1 \), otherwise.
2. If \( m \) has the Fibonacci representation \( m = F_{i_1} + F_{i_2} + \ldots + F_{i_\ell} \) with \( i_1 > i_2 > \ldots > i_\ell \), then \( f(m) = F_{i_1+1} + F_{i_2+1} + \ldots + F_{i_\ell+1} \) if \( i_\ell \) is even, and \( f(m) = F_{i_1+1} + F_{i_2+1} + \ldots + F_{i_\ell+1} - 1 \), otherwise.

**Proof.** Note that \( f(m) \) is equal to the index of the \( m \)th occurrence of 1 in the sequence \((c_n)_{n \in \mathbb{N}}\). Now, by construction, the number of 1’s in the sequence \((c_n)_{n \leq F_{i_1}}\) is \( F_i \). So, \( f(F_i) = F_{i+1} \) if \( i \) is even and \( f(F_i) = F_{i+1} - 1 \) if \( i \) is odd. Similarly the number of 1’s in \((c_n)_{n \leq N}\) with \( N = F_{i_1+1} + F_{i_2+1} + \ldots + F_{i_\ell+1} \), is \( F_{i_1} + F_{i_2} + \ldots + F_{i_\ell} \), and this implies the second item.

Note that our sequence \((c_n)_{n \in \mathbb{N}}\) is indeed the complement of the so-called Fibonacci word, given by \( 2 + \lfloor \varphi n \rfloor - \lfloor \varphi (n+1) \rfloor \).

We end this section by a simple, and yet key fact (for a proof see [2]).

**Fact 5.** For any positive integer \( k \) the sequence \((F_i \mod k)_{i \in \mathbb{N}}\) is periodic beginning with 0, 1.

## 2 Auxiliary Lemmas for the Axiomatization

The lemmas in this section are all about the natural numbers and some of them correspond to the axioms we present for our structure in the next section. In the first lemma in this sequel, we show that the image of \( f \) contains elements in any congruence class.
Lemma 6. For each $k, m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $f(n)^k \equiv m$.

Proof. By Fact 3 if the smallest index in the Fibonacci representation of $m$ is odd, then $m$ is in the image of $f$ and the equation is solved instantly, hence assume for the rest of the proof that this index is even.

Since by Fact 5 the Fibonacci sequence modulo $k$ is periodic, and the first two elements of this period are 0 and 1, there is a Fibonacci number $F_i$, with $i$ larger than the largest index in the Fibonacci representation of $m$, such that $F_i^k \equiv 1$. That is there is $u \in \mathbb{N}$ such that $F_i - 1 = ku$. Because the index of the smallest Fibonacci number in the representation of $m$ is even, i.e. $c_m = 0$, we have $c_{F_i-1+m} = c_{m-1} = 1$. Hence $F_i - 1 + m$ is in the image of $f$ and $F_i - 1 + m^k \equiv m$.

The following lemma generalizes the above in the way that $n$ is also in a desirable congruence class.

Lemma 7. The system

$$\begin{cases} x^k \equiv m \\ f(x)^{k'} \equiv m' \end{cases}$$

has a solution in $\mathbb{N}$, for any $k, m, k', m' \in \mathbb{N}$.

Proof. It is obvious that $m$ is a solution of the equation $x^k \equiv m$. Let $F_{i_1} + F_{i_2} + \ldots + F_{i_\ell}$ be the Fibonacci representation of $m$ with $i_1 > i_2 > \ldots > i_\ell$. Suppose that $m''$ is such that $f(m)^{k'} \equiv m''$. If $m'' \neq m'$, by Fact 5 we can find a Fibonacci number $F_j$ such that $F_j^k \equiv 0$, $F_j^{k'} \equiv 0$, $f(F_j) = F_{j+1}^{k'} \equiv 1$, $j > i_1 + 1$, and $j$ is an even number. Now since $F_j^k \equiv 0$, we have $m + F_j^k \equiv m$.

Also because $j > i_1 + 1$ is even, by Fact 4 $f(m + F_j) = f(m) + f(F_j)^{k'} \equiv m'' + 1$. This procedure gives, after finitely many steps, a natural number $n$ such that $n^k \equiv m$ and $f(n)^{k'} \equiv m'$.

As it turns out, the lemma above has a close connection to the following fact about the distribution in the interval $(0, 1)$ of the decimal parts of $\varphi n$, for all $n \in \mathbb{N}$. Before mentioning the fact, let us fix some notation for the decimal parts.

Notation 1. We denote the decimal part of $\varphi n$ by $[\varphi n]$.

Fact 8. (Kronecker [5, Theorem 439]) If $r$ is irrational, then $([rn])_{n \in \mathbb{N}}$ is dense in $(0, 1)$. 

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To explain the connection in question we need yet another lemma.

**Lemma 9.** For each $k$ and $0 \leq i < k$, $f(n) \equiv k$ if and only if $\left(\frac{n}{k}\right) \in \left(\frac{i}{k}, \frac{i+1}{k}\right)$.

**Proof.** Write $\left(\frac{n}{k}\right) = u + r$, for some $u$ and $0 < r < 1$. So $f(n) \equiv k$ if and only if $\varphi n = ku + kr$, with $i < kr < i + 1$, that is $\frac{i}{k} < r < \frac{i+1}{k}$. Therefore $f(n) \equiv k$ if and only if $\left(\frac{n}{k}\right) = u + r$ and $\frac{i}{k} < r < \frac{i+1}{k}$.

**Remark 1.** By the lemma above, $[\varphi n] \in \left(\frac{i}{k}, \frac{i+1}{k}\right)$ if and only if $f(kn) \equiv k$.

Hence to find a natural number $n$ such that $[\varphi n]$ is in a desired small subinterval $\left(\frac{i}{k}, \frac{i+1}{k}\right)$ of $(0, 1)$ one needs to solve the following system of congruence-relation equations:

$$\begin{cases} x \equiv 0 \\ f(x) \equiv i. \end{cases}$$

This means that Lemma 7 is indeed an equivalent expression to the Kronecker’s theorem that $\{[\varphi n]\}_{n \in \mathbb{N}}$ is dense in $(0, 1)$.

The remark above is interesting, because it suggests that although in our language $\mathcal{L}$ it is not possible to have any symbol to refer to the decimal part of $\varphi x$, we are capable of finding an equivalent way of expressing some facts about it. Indeed many expressions about the function $f$ has an equivalent in terms of the decimal parts. For example, $f(n + m) = f(n) + f(m)$ means that $[\varphi n] + [\varphi m] < 1$, and similarly $f(n + m) = f(n) + f(m) + 1$ means that $[\varphi n] + [\varphi m] > 1$. Yet, as we see in the following lemma, much more can be said about the decimal parts in the language $\mathcal{L}$.

**Lemma 10.** There is a formula $R(x, y)$ such that for all $n, m \in \mathbb{N}$, $(\mathbb{N}, +, f, 0, 1) \models R(n, m)$ if and only if $[\varphi n] < [\varphi m]$.

**Proof.** Simply let $R(x, y)$ be the following formula

$$\forall z \left(f(x + z) = f(x) + f(z) + 1 \rightarrow f(y + z) = f(y) + f(z) + 1\right). \quad (1)$$

We first show that if $[\varphi n] < [\varphi m]$, then $(\mathbb{N}, +, f, 0, 1) \models R(n, m)$. Note that when $[\varphi n] < [\varphi m]$ then for each $r \in \mathbb{N}$, $[\varphi n] + [\varphi r] < [\varphi m] + [\varphi r]$. Hence if $[\varphi n] + [\varphi r] > 1$ then $[\varphi m] + [\varphi r] > 1$. But this is exactly what the formula $R(n, m)$ implies.

For the other direction, note that if $[\varphi n] > [\varphi m]$, then $1 - [\varphi n] < 1 - [\varphi m]$. Hence by Kronecker’s theorem (Fact 8), there is a natural number $r$ such that $1 - [\varphi n] < [\varphi r] < 1 - [\varphi m]$. But this means that $f(n + r) = f(n) + f(r) + 1$ and $f(m + r) = f(m) + f(r)$, which means that the negation of $R(n, m)$ holds in $(\mathbb{N}, +, f, 0, 1)$.

\[\square\]
By the above lemma, the order of the decimal parts is definable in \((\mathbb{N}, +, f, 0, 1)\). We now expand our language by the binary predicate \(R(x, y)\) which is interpreted by formula (1) above.

**Notation 2.** We denote by \(\mathcal{L}^*\) the language \(\mathcal{L} \cup \{R\}\).

**Remark 2.** We interpret \(R\) in \(\mathbb{N}\) as in Lemma 10. We will later add to our axioms that \(R\) is indeed an “order relation”.

As mentioned in the introduction, proving quantifier-elimination would involve solving systems of equations in the language. Note that in \(\mathcal{L}\) such systems involve, in particular, equations of the form \(f(rx + sf(x) + a) = b\) or \(f(rx + sf(x) + a) = rf(x) + sf^2(x) + f(a) + j\). This is as hard as it gets, and we do not have more involved equations, because the powers of \(f\) reduce to one simply by noting that \(f^2(x) = f(x) + x - 1\). The lemmas in the rest of this section are to provide tools for solving such systems by dealing with the intervals in which the decimal parts of \(\varphi x\) and \(\varphi f(x)\) lie. The next lemma shows that even these two are related.

**Lemma 11.** The decimal part of \(\varphi f(n)\), for a natural number \(n\) is determined by the decimal part of \(\varphi n\) as in the following:

\[
[\varphi f(n)] = \begin{cases} 
(1 - \varphi)[\varphi n] + 1 & [\varphi n] < \frac{1}{\varphi - 1} \\
(1 - \varphi)[\varphi n] + 2 & [\varphi n] > \frac{1}{\varphi - 1}
\end{cases}
\]

(2)

*Proof.* Note that \(\varphi f(n) = \varphi(\varphi n - [\varphi n]) = \varphi^2 n - \varphi[\varphi n] = \varphi n + n - \varphi[\varphi n]\), where the latter is the case because \(\varphi^2 = \varphi + 1\). Hence, as \(n\) is a natural number, \([\varphi f(n)] = [\varphi n - \varphi[\varphi n]]\).

If \([\varphi n] < \frac{1}{\varphi}\), then obviously \([\varphi[\varphi n]] = \varphi[\varphi n]\). Also it is clear that \([\varphi n] < \varphi[\varphi n]\), hence:

\([\varphi f(n)] = [\varphi n] - \varphi[\varphi n] + 1\).

If \(\frac{1}{\varphi} < [\varphi n] < \frac{1}{\varphi - 1}\), then \([\varphi[\varphi n]] = \varphi[\varphi n] - 1\). In addition, because \([\varphi n] < \frac{1}{\varphi - 1}\), \(\varphi[\varphi n] - 1 < [\varphi n]\). So

\([\varphi f(n)] = [\varphi n] - \varphi[\varphi n] + 1\).

Finally if \([\varphi n] > \frac{1}{\varphi - 1}\), then \([\varphi[\varphi n]] = \varphi[\varphi n] - 1\) and also \(\varphi[\varphi n] - 1 > [\varphi n]\), therefore

\([\varphi f(n)] = [\varphi n] - \varphi[\varphi n] + 2\)

and this exhausts all possibilities. \(\Box\)
In Corollary 13 we will prove that solvability in $\mathbb{N}$ of a system of equations involving symbols of $\mathcal{L}^*$ is expressible by a quantifier-free $\mathcal{L}^*$-formula. To make the required argument simpler, we first deal with an easier but essential case in the following lemma.

**Lemma 12.** There is a quantifier-free formula $\Phi(y_1, y_2)$ in the language $\mathcal{L}^*$ such that for all natural numbers $n_1, n_2$, $(\mathbb{N}, +, f, R, 0, 1) \models \Phi(n_1, n_2)$ if and only if the following system of equations has a solution in $\mathbb{N}$.

\[
\begin{aligned}
  &f(r_1 x + s_1 f(x) + n_1) = r_1 f(x) + s_1 f^2(x) + f(n_1) + j_1 \\
  &f(r_2 x + s_2 f(x) + n_2) = r_2 f(x) + s_2 f^2(x) + f(n_2) + j_2
\end{aligned}
\]  

(3)

where $r_1, s_1, j_1, r_2, s_2, j_2$ are fixed natural numbers.

Note that the formula required in lemma above depends on $r_1, s_1, j_1, r_2, s_2, j_2$, but for simplicity we have not reflected this dependence in the notation.

**Proof.** The equations in (3) can be rewritten in terms of the decimal parts as follows:

\[
\begin{aligned}
  &j_1 - [\varphi n_1] < r_1 [\varphi x] + s_1 [\varphi f(x)] < j_1 + 1 - [\varphi n_1] \\
  &j_2 - [\varphi n_2] < r_2 [\varphi x] + s_2 [\varphi f(x)] < j_2 + 1 - [\varphi n_2].
\end{aligned}
\]

By Lemma 11 depending on whether $(\varphi - 1)[\varphi x] < 1$ or $(\varphi - 1)[\varphi x] > 1$ we may write $[\varphi f(x)]$ in terms of $[\varphi x]$ and add another equation to the above system. So let us without loss of generality consider the following system:

\[
\begin{aligned}
  &j_1 - [\varphi n_1] - s_1 < (r_1 + s_1 - s_1 \varphi)[\varphi x] < j_1 + 1 - [\varphi n_1] - s_1, \\
  &j_2 - [\varphi n_2] - s_2 < (r_2 + s_2 - s_2 \varphi)[\varphi x] < j_2 + 1 - [\varphi n_2] - s_2, \\
  &((\varphi - 1)[\varphi x] < 1.
\end{aligned}
\]

(4)

Replacing $[\varphi x]$ with a new variable, say $z \in (0, 1)$, checking the solvability of such a system in terms of $z$ in $\mathbb{R}$ is rather easy. In each row, we have an equation of the form $Az \in (B, C)$. Each line $w = Az$ passes through the origin and such lines are clearly intersecting. The system is therefore solvable, unless two such lines are the same and the endpoints do not match.

To be precise, assume that the coefficients of $z$ in the first and the second equation are equal, so that the mentioned equations are written as

\[
\begin{aligned}
  &Az \in (j_1 - [\varphi n_1] - s_1, j_1 + 1 - [\varphi n_1] - s_1), \\
  &Az \in (j_2 - [\varphi n_2] - s_2, j_2 + 1 - [\varphi n_2] - s_2).
\end{aligned}
\]

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The intervals above have intersection if and only if one endpoint of one such interval belongs to the other one. This clearly happens when we have $[\varphi n_1] < [\varphi n_2] + \ell$ for some $\ell \in \mathbb{Z}$ determined by the $j_i$ and $s_i$'s. Now, if $\ell \geq 1$ this is always the case, if $\ell \leq -1$, it never happens, and if $\ell = 0$ then it is the case if and only if $R(n_1, n_2)$ holds. In either case, thanks to the predicate $R$ one can easily write the formula that expresses the solvability of the system. 

Now the same strategy (that is writing the equations in terms of the decimal parts and analyzing the obtained linear equations in terms of parallel or non-parallel lines in $\mathbb{R}$) leads to the following corollary. Note that in the following corollary, two other types of equations of the form $R(x, n)$ and $R(f(x), n)$ and two congruence relation equations of the form $x \overset{r}{\equiv} j$ and $f(x) \overset{r'}{\equiv} j'$ are also added, but this does not change the way we need to treat the system. It is because equations $R(x, n)$ and $R(f(x), n)$ are written in terms of the decimal parts as $[\varphi x] \in (a, b)$ for suitable $a, b$. Also by Remark 1 congruence equations of the $x \overset{r}{\equiv} j$ and $f(x) \overset{r'}{\equiv} j'$ are also equivalent to an equation $[\varphi x] \in (a', b')$.

**Corollary 13.** There is a quantifier free $\mathcal{L}^*$-formula $\theta(y_1, y_2, \ldots, y_k)$ such that for all $n_1, \ldots, n_k \in \mathbb{N}$ we have $(\mathbb{N}, +, f, R, 0, 1) \models \theta(n_1, \ldots, n_k)$ if and only if the following system of equations has a solution in $\mathbb{N}$:

$$
\begin{cases}
  f(r_1x + s_1f(x) + n_1) = r_1f(x) + s_1f^2(x) + f(n_1) + j_1 \\
  \vdots \\
  f(r_{k-2}x + s_{k-2}f(x) + n_{k-2}) = r_{k-2}f(x) + s_{k-2}f^2(x) + f(n_{k-2}) + j_{k-2} \\
  R(x, n_{k-1}) \\
  R(f(x), n_k) \\
  x \overset{r_{k-1}}{\equiv} j_{k-1} \\
  f(x) \overset{r_k}{\equiv} j_k
\end{cases}
$$

(5)

where $r_i, s_i, j_i \in \mathbb{N}$.

### 3 Axiomatization

In this section, we present an axiomatization $T$ for our structure, in the language $\mathcal{L}^*$ as in Notation 2. These axioms are based on what we developed in the previous two sections. More precisely, the axiom-scheme $(T1)$ below expresses the basic properties of $(\mathbb{N}, +, \{p_n\}_{n \in \mathbb{N}}, 0, 1)$ as a model of Presburger.
arithmetic. \((T2)\) and \((T3)\) express the main properties of the function \(f\) based on Lemma \(^1\) and Lemma \(^2\). \((T4)\) asserts that \(R(x, y)\) is an order relation, which, based on Remark \(^2\) can be naturally thought of as the “order of the decimal parts”. \((T5)\) expresses the Kronecker’s theorem on the distribution of the decimal parts, based on Remark \(^1\) that is it says that if \(R(a, b)\) holds (which can be thought of as \(\lfloor \varphi a \rfloor < \lfloor \varphi b \rfloor\)) then there is \(c\) such that \(R(a, c)\) and \(R(c, b)\) hold (which again can be thought of as \(\lfloor \varphi a \rfloor < \lfloor \varphi c \rfloor < \lfloor \varphi b \rfloor\)). Finally in the light of Corollary \(^13\) the axiom-scheme \((T6)\) expresses when a given system of equations has a solution.

**Definition 14.** Let \(T\) be the theory obtained by the axioms expressing the following.

\((T1)\) Presburger arithmetic,

\begin{align*}
\forall x &\exists y ((x = f(y)) \lor (x = f(y) + y)) \land \forall x, y (f(x + y) = f(x) + f(y) \lor f(x + y) = f(x) + f(y) + 1), \\
&T2) \forall x \exists y ((x = f(y)) \lor (x = f(y) + y)) \land \forall x, y (f(x + y) = f(x) + f(y) \lor f(x + y) = f(x) + f(y) + 1),
\end{align*}

\((T3)\) \((f(0) = 0) \land (f(1) = 1) \land \forall x (f(f(x)) = f(x) + x - 1 \land f(f(x) + x) = 2f(x) + x),

\((T4)\)

\begin{itemize}
  \item \(\forall x \neg R(x, x)\),
  \item \(\forall x, y (R(x, y) \rightarrow \neg R(y, x))\),
  \item \(\forall x, y, z (R(x, z) \land R(z, y) \rightarrow R(x, y))\),
\end{itemize}

\((T5)\) \(\forall x, y \left(R(x, y) \rightarrow \exists z \left(R(x, z) \land R(z, y)\right)\right)\),

\((T6)\) \(\forall y_1, \ldots, y_k \left(\exists x \left( \bigwedge_{i=1}^{k-2} (f(r_i x + s_i f(x) + y_i) = r_i f(x) + s_i f^2(x) + f(y_i) + j_i) \land R(x, y_{k-1}) \land R(f(x), y_k) \land p_{r_{k-1}}(x - j_{k-1}) \land p_{r_k}(f(x) - j_k)\right) \Leftrightarrow \theta(y_1, \ldots, y_k)\right)\).

Note that \((T1)\) and \((T6)\) are actually axiom schemes.

**Theorem 15.** \((\mathbb{N}, +, f, R, \{p_n\}_{n \in \mathbb{N}}, 0, 1)\) is a model of \(T\).

The proof of the theorem above is clear by the way we have obtained the axioms (and the explanation before Definition \(^{14}\)). Now it is easy to verify that \(T\) is recursively enumerable. In the next section we have proved that \(T\) eliminates quantifiers and this leads to the fact that \(T\) is complete and decidable.
4 Quantifier-Elimination and Decidability

For the rest of the paper, let $\mathcal{M}_1$ and $\mathcal{M}_2$ be models of $\mathbf{T}$ and $\mathcal{M}$ be a common substructure. We assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ are $|\mathcal{M}|$-saturated. We aim to present a back-and-forth system of isomorphisms between substructures of $\mathcal{M}_1$ and $\mathcal{M}_2$ containing $\mathcal{M}$.

Lemma 16. There is $\mathcal{M}' \models (T1), (T2)$ such that $\mathcal{M} \subseteq \mathcal{M}'$ and $\mathcal{M}' \subseteq \mathcal{M}_1 \cap \mathcal{M}_2$.

Proof. Put $\mathcal{M}' = \{x \mid x \in \mathcal{M}, \mathcal{M}_1, \mathcal{M}_2 \models p_n(x)\}$. Indeed $\mathcal{M}'$ is the algebraically-prime model of the Presburger arithmetic containing $\mathcal{M}$.

We claim that $\mathcal{M}'$ is closed under the function $f$, and hence bears an $\mathcal{L}$-structure. Suppose that $t = \frac{a}{n} \in \mathcal{M}'$. Then $a = nt$ and $a \in M$. By properties of the function $f$, we have $\mathcal{M}_1 \models f(a) = nf(t) + \ell$, where $\ell$ is the remainder of the division of $f(a)$ by $n$. So $\mathcal{M}_1 \models p_n(f(a) - \ell)$, and $f(a) - \ell \in M$. Therefore $f(t) = \frac{f(a) - \ell}{n} \in \mathcal{M}'$.

To prove Axiom (T2), let $a$ be an arbitrary element in $\mathcal{M}'$. Since $\mathcal{M}_1$ is a model of $\mathbf{T}$, there is $b \in M_1$ such that $\mathcal{M}_1 \models a = f(b) \lor a = f(b) + b$. We will show that $b \in M'$.

If $\mathcal{M}_1 \models a = f(b)$, then by (T3), $\mathcal{M}_1 \models f(a) = a + b - 1$ and hence $\mathcal{M}_1 \models b = f(a) - a + 1$. It is clear that $b \in M'$.

If $\mathcal{M}_1 \models a = f(b) + b$ then by (T3), $\mathcal{M}_1 \models f(a) = f(f(b) + b) = 2f(b) + b = a + f(b)$. Therefore $\mathcal{M}_1 \models f(b) = f(a) - a$. On the other hand, $a \in M'$ and $f(a) \in M'$ so $f(b) = f(a) - a \in M'$. Now since $f(b) \in M'$, by the above argument we have $b \in M'$.

The second part of axiom (T2) is clearly inherited from $\mathcal{M}_1, \mathcal{M}_2$. 

Theorem 17. The theory $\mathbf{T}$ admits elimination of quantifiers.

Proof. According to Lemma 16, we add to the assumptions at the beginning of this section that $\mathcal{M} \models (T1), (T2), (T3), (T4)$. Note that (T3) and (T4) come for free because they are universal and hence are inherited from $\mathcal{M}_1$ and $\mathcal{M}_2$. We will show that for all $a \in M_1 - M$, there is $a' \in M_2 - M$ such that a substructure of $\mathcal{M}_1$ containing $a$ and $f(a)$ is isomorphic to a substructure of $\mathcal{M}_2$ containing $a'$ and $f(a')$ with an isomorphism which sends $a$ to $a'$ and $f(a)$ to $f(a')$.

To achieve this we need to prove that any system consisting of finitely-many equations (with parameters in $M$) satisfied by $a$ is solvable in $M_2$ by an element $b$. This together with the assumption of saturation of $\mathcal{M}_2$ implies the existence of $a'$ as above.

The most general system of equations to consider for $a$ consists of finitely-many equations, each of which of one of the following forms, with parameters
c, d, e, e′, g, and g′ in M and coefficients m, n, m′, r, s, t, u, j in \( \mathbb{N} \). Note that the negation of each of the following equations (except for the last one) has the same format as itself.

\[
\begin{cases}
x \overset{n}{=} m \\ f(x) \overset{n′}{=} m′ \\ f(rx + sf(x) + c) = rf(x) + sf^2(x) + f(c) + j \\
R(x, e) \\
R(e′, x) \\
R(f(x), g) \\
R(g′, f(x)) \\
tf(x) = ux + d
\end{cases}
\]

Also notice that, we do not get more involved equations (say in terms of the powers of \( f \)) simply because the powers of \( f \) reduce to one by Axiom (T3).

We first claim that if the system above actually contains an equations of the last form \( tf(x) = ux + d \), then \( a \) is already in \( M \), hence \( b \in M_2 \) can be taken to be \( a \) itself. Indeed such an equation has an “algebraic nature” where the rest of the equations, which only concern with the decimal parts can be thought of being “non-algebraic” (see [11]).

**Claim 1.** If for \( d \in M \), \( M_1 \models f(a) = \frac{u}{t}a + d \) then \( a \in M \).

**Proof.** Suppose that \( M_1 \models f(a) = \frac{u}{t}a + d \), so \( M_1 \models f(f(a)) = f(\frac{u}{t}a + d) \).

By axiom (T3), \( M_1 \models f(f(a)) = f(a) + a - 1 \), hence similar to the proof of the previous lemma,

\[
M_1 \models f(a) + a - 1 = \frac{uf(a) + \ell}{t} + f(d) + j
\]

for some integer \( \ell \) and natural number \( j \). Replacing \( f(a) \) in both sides of the above formula with \( \frac{u}{t}a + d \) we get a linear equation in terms of \( a \). This forces that \( a \in M \), because the linear equation gives \( a \) by the divisibility relation and \( M \) is a model of Presburger arithmetic.

By the above claim, if there is an equation of the form \( tf(x) = ux + b \) in system (6), then the solution of this system is already in \( M \). Hence, in the rest we drop the last equation from the system.

Meanwhile by the Chinese remainder theorem one reduces the congruence equation relations for \( f(x) \) and \( x \) to a single one. Similarly by the properties of a linear order, one can assume that there is only one equation of each form \( R(x, e) \) and \( R(f(x), g) \) in the system. Now if follows from axiom (T6) that
the system has a solution in $M_2$. This is because the quantifier-free formula in the mentioned axiom is satisfied in $M$, and this is because the system clearly possesses a solution, that is $a$, in $M_1$.

**Corollary 18.** The theory $T$ is complete and hence equivalent to $\text{Th}(\mathbb{N}, +, f, R, \{p_n\}_{n \in \mathbb{N}}, 0, 1)$.

And finally since $T$ is recursively enumerable, we get the main corollary.

**Corollary 19.** The theory $T$ is decidable.

**Remarks**

1. In our earlier versions we claimed that the structure $(\mathbb{Z}, +, <, f, \{p_n\}_{n \in \mathbb{N}}, 0, 1)$ eliminates quantifiers, but with the help of the referee’s comments we found out that the proof was flawed. In this version we have removed the order, and have considered $\mathbb{N}$ instead of $\mathbb{Z}$. Note that the order in $\mathbb{N}$ is definable by the formula $\exists z (y = x + z)$ for $x < y$. Regarding our previous proof, two interesting questions should be addressed here. We assume that the answer to both would be positive, but needs some work.

   (a) Does the structure $(\mathbb{Z}, +, <, f, R, \{p_n\}_{n \in \mathbb{N}}, 0, 1)$ eliminate quantifiers?

   (b) Is the relation $R(x, y)$ equivalent to a quantifier-free formula in $L$, so that we have quantifier-elimination without even the predicate $R$?

2. The proofs provided in the last version of this paper, are inspired by [11], where a much more difficult situation is dealt with in a similar manner. We decided to adopt the same technology here, as it made the proofs neater compared to our original proof.

3. The formula $R(x, y)$ suggests that the structure $(\mathbb{N}, +, f, 0, 1)$ has the so-called “order property, which determines its place in terms of the model-theoretic classification of theories.

4. We think that $\varphi$ can be replaced by any algebraic number, and the proofs will be essentially similar. But this needs to be checked. As mentioned, in [11] a much more general case is treated.
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