Optical Vortices and Vortex Solitons

BY

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Abstract

Optical vortices are phase singularities nested in electromagnetic waves that constitute a fascinating source of phenomena in the physics of light and display deep similarities to their close relatives, quantized vortices in superfluids and Bose-Einstein condensates. We present a brief overview of the major advances in the study of optical vortices in different types of nonlinear media, with emphasis on the properties of vortex solitons. Self-focusing nonlinearity leads, in general, to the azimuthal instability of a vortex-carrying beam, but it can also support novel types of stable or meta-stable self-trapped beams carrying nonzero angular momentum, such as ring-like solitons, necklace beams, and soliton clusters. We describe vortex solitons created by multi-component beams, by parametrically coupled beams in quadratic nonlinear media, and in partially incoherent light, as well as discrete vortex solitons in periodic photonic lattices.
## CONTENTS

1 Introduction 3

2 Self-trapped vortices in Kerr-type media 5
   2.1 Vortices in defocusing nonlinear media 5
   2.2 Ring-like beams in focusing nonlinear media 8
   2.3 Azimuthal modulational instability 10

3 Composite spatial solitons with phase dislocations 12
   3.1 Soliton-induced waveguides 12
   3.2 Higher-order vector solitons 14
   3.3 Multi-component vortex solitons 16
   3.4 Partially coherent vortices 18

4 Multi-color vortex solitons 19
   4.1 Model 20
   4.2 Frequency doubling with vortex beams 21
   4.3 Families of the vortex solitons 21
   4.4 Spontaneous break-up: azimuthal instability 23
   4.5 Induced break-up: soliton algebra 24
   4.6 Dark multi-color vortex solitons 25

5 Stabilization of vortex solitons 26
   5.1 Cubic-quintic nonlinearity 26
   5.2 Quadratic-cubic nonlinearity 28
   5.3 Spatiotemporal spinning solitons 28

6 Other optical beams carrying angular momentum 30
   6.1 Soliton spiraling 30
   6.2 Optical necklace beams 31
   6.3 Soliton clusters 33

7 Discrete vortices in two-dimensional lattices 36

8 Links to vortices in other fields 38
   8.1 Vortices in dissipative optical systems 39
   8.2 Vortices in matter waves 40
   8.3 Optical vortices and quantum information 43

9 Concluding remarks 44

10 Acknowledgements 45

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§1. Introduction

In physics, wave propagation is traditionally analyzed by means of regular solutions of wave equations. However, solutions of wave equations in two and three dimensions often possess singularities, the points or lines in space at which mathematical quantities that describe physical properties of waves become infinite or change abruptly (Berry [2000]). For example, at the point of phase singularity, the phase of the wave is undefined and wave intensity vanishes. Phase singularities are recognized as important features common to all waves. They were first discussed in depth in a seminal paper by Nye and Berry [1974]. However, the earliest known scientific description of phase singularity was made in the 1830’s by Whewell, as discussed by Berry [2000]. While Whewell studied the ocean tides, he came to the extraordinary conclusion that rotary systems of tidal waves possess a singular point at which all cotidal lines meet and at which tide height vanishes. Waves that possess a phase singularity and a rotational flow around the singular point are called vortices. They can be found in physical systems of different nature and scale, ranging from water whirlpools and atmospheric tornadoes to quantized vortices in superfluids and quantized lines of magnetic flux in superconductors (Pismen [1999]).

In a light wave, the phase singularity is known to form an optical vortex: The energy flow rotates around the vortex core in a given direction; at the center, the velocity of this rotation would be infinite and thus the light intensity must vanish. The study of optical vortices and associated localized objects is important from the viewpoint of both fundamental and applied physics. The unique nature of vortex fields is expected to lead to applications in many areas that include optical data storage, distribution, and processing. Optical vortices propagating, e.g., in air, have been suggested also for the establishment of optical interconnects between electronic chips and boards (Scheuer and Orenstein [1999]), as well as for free-space communication links (Gibson, Courtial, Padgett, Vasnetsov, Pas’ko, Barnett, and Franke-Arnold [2004]; Bouchal and Celyevsky [2004]), based on the multidimensional alphabets afforded by the corresponding angular momentum states (Molina-Terriza, Torres, and Torner [2002]). The ability to use light vortices to create reconfigurable patterns of complex intensity in an optical medium could aid optical trapping of particles in a vortex field (Gahagan and Swartzlander [1999]), and could enable light to be guided by the light itself, or in other words by the waveguides created by optical vortices (Truscott, Friese, Heckenberg, and Rubinsztein-Dunlop [1999]; Law, Zhang, and Swartzlander [2000]; Carlsson, Malmberg, Anderson, Lisak, Ostrovskaya, Alexander, and Kivshar [2000]; Salgueiro, Carlsson, Ostrovskaya, and Kivshar [2004]). Thus, singular optics, the study of wave singularities in optics (Nye and Berry [1974]; Vasnetsov and Staliunas [1999]; Soskin and Vasnetsov [2001]), is now emerging as a new discipline (for an extended list of references, see http://www.u.arizona.edu/grovers/SO/so.html).

In a broad perspective, the study of optical vortices brings inspiring similarities between different and seemingly disparate fields of physics: the comparison of singularities of optical and other origins leads to theories that transcend the confines of specific fields. Vortices play an important role in many branches of physics, even those not directly related to wave propagation. An example is the Kosterlitz-Thouless phase transition (Kosterlitz and Thouless [1973]) in solid-state physics models, characterized by creation of tightly bound pairs of point-like vortices that restore the quasi-long-range order of a two-dimensional model at low temperatures. Such vortex-induced phase transitions can be observed in superfluid helium films, thin superconducting films, and surfaces of solids, as well as in models of interest to particle physicists and cosmologists.

The Bose-Einstein condensate (BEC), a state of matter in which a macroscopic number of particles share the same quantum state, constitutes a well-researched example of a superfluid in which topological defects with a circulating persistent current are observed. Nearly 75 years ago, Bose and Einstein introduced the idea of condensate of a dilute gas at temperatures close to absolute zero. The BEC was experimentally created in 1995 by the JILA group (Anderson, Ensher, Matthews, Wieman, and Cornell [1995]), who trapped thousands (later, millions) of alkali $^{87}$Rb atoms in a 10-μm cloud and then cooled them to a millionth of a degree above absolute zero. The extensive study of vortices in BEC (Williams and Holland [1999]; Matthews, Anderson, Haljan, Hall, Wieman, and Cornell [1999]; Madison, Chevy, Wohlleben, and Dalibard [2000]; Raman, Abo-Shaeer, Vogels, Xu, and Ketterle [2001]) promises a deeper understanding of deep links between the physics of superfluidity, condensation, and nonlinear singular optics.

To introduce the notion of optical vortices, we recall that a light wave can be represented by a complex scalar function $\psi$ (e.g., an envelope of an electric field), which varies smoothly in space and/or time. Phase singularities of the wave function $\psi$ appear at the points (or lines in space) at which its modulus vanishes, i.e., when $\text{Re } \psi = \text{Im } \psi = 0$. Such points are referred to as wave-front screw dislocations or optical vortices, because the surface of constant phase structurally resembles a screw dislocation in a crystal lattice, and because the phase gradient direction swirls around the singular line much like fluid in a whirlpool. Optical vortices are associated with zeros in light intensity (black spots) and can be recognized by a specific helical wave front. If the complex wave function is presented as, $\psi(r, t) = \rho(r, t) \exp\{i\theta(r, t)\}$, in terms of its real modulus $\rho(r, t)$ and phase $\theta(r, t)$, the dislocation strength (sometimes referred to as the vortex topological charge) is defined by the circulation of the phase gradient around the singularity,

$$S = \frac{1}{2\pi} \oint \nabla \theta dr.$$  \hspace{1cm} (1.1)
1. INTRODUCTION

Figure 1: Propagation of the Gaussian beam with a phase dislocation generated by the input beam $E(r, \varphi, z = 0) = 2r \exp \left(-r^2/4 + i\varphi \right)$, in (a) linear medium, $\chi^{(3)} = 0$, and (b) self-focusing Kerr medium, $\chi^{(3)} > 0$. Shown is the field intensity. Note the difference in the intensity scales.

The result is an integer because the phase changes by a multiple of $2\pi$. Under appropriate conditions, it also measures an orbital angular momentum of the vortex associated with the helical wave-front structure. If a light wave is characterized by an extra parameter, e.g., the wave polarization, its mathematical representation is no longer a scalar but a vector field. In vector fields, several types of line singularity exist; for example, those analogous to disclinations in liquid crystals, which could be edge type, screw type, or mixed edge-screw type, that could move relative to background wave fronts and could interact in several different ways (Nye and Berry [1974]; Soskin and Vasnetsov [2001]). In the linear theory of waves, each wave dislocation could be understood as a simple consequence of destructive wave interference. In this review we mostly address screw phase singularities existing in scalar wave fields and thus we concentrate our analysis in the corresponding vortices. However, other types of singularities whose analysis falls beyond the scope of this review, such as polarization singularities (Freund [2004a,b]), do exist and exhibit fascinating properties.

A laser beam with a phase singularity generally has a doughnut-like shape and diffracts when it propagates in a free space. However, when the vortex-bearing beam propagates in a nonlinear medium, a variety of interesting effects can be observed. Nonlinear optical media are characterized by the electromagnetic response that depends on the strength of the propagating light. The polarization of such a medium can be described as $P = \chi^{(1)}E + \chi^{(2)}E^2 + \chi^{(3)}E^3$, where $E$ is the amplitude of the light wave’s electric field, and the coefficients characterize both the linear and the nonlinear response of the medium (Shen [1984]; Butcher and Cotter [1992]; Boyd [1992]). The $\chi^{(1)}$ coefficient describes the linear refractive index of the medium. When $\chi^{(2)}$ vanishes (as happens in the case of centro-symmetric media), the main nonlinear effect is produced by the third term that can be presented as an intensity-induced change of the refractive index proportional to $\chi^{(3)}E^3$. An important consequence of such intensity-dependent nonlinearity is the spontaneous focusing of a beam that is due to the lensing property of a self-focusing medium (i.e. when $\chi^{(3)} > 0$). This focusing action of a nonlinear medium can precisely balance the diffraction of a laser beam, resulting in the creation of optical solitons, which are self-trapped light beams that do not change shape during propagation (Kivshar and Luther-Davies [1998]).

A stable bright spatial soliton is radially symmetric, and it has no nodes in its intensity profile. If, however, a beam with elaborate geometry carries a topological charge and propagates in a self-focusing nonlinear medium, it has a doughnut like structure. However, such a doughnut beam is unstable, and it decays into a number of more fundamental bright spatial solitons, such an example is shown in Fig. 1. The resulting field distribution does not preserve the radial symmetry, and the vortex beam decays into several solitons that repel and twist around one another as they propagate. This rotation is due to the angular momentum of the vortex beam transferred to the splinters.

Remarkably, the behavior of a laser beam in a self-defocusing nonlinear medium (i.e. when $\chi^{(3)} < 0$) is distinctly different, see an example in Fig. 2. Such a medium cannot produce a lensing effect and therefore cannot support bright solitons. Nevertheless, a negative change of the refractive index can compensate for spreading in light intensity of the dip, thus creating a dark soliton (Kivshar and Luther-Davies [1998]), a self-trapped, localized low-intensity state (a dark hole) in a uniformly illuminated background. Vortices of single and multiple topological charges can be created in both linear and nonlinear media by use of, e.g., computer-generated holograms or spatial light modulators. Propagating through a nonlinear self-defocusing medium, such as a vortex-carrying beam, creates a self-trapped state, a dark vortex soliton. Dark vortex solitons have been observed experimentally in different materials with self-defocusing nonlinearity, such as slightly absorbent...
2. Self-trapped vortices in Kerr-type media

In this section, we describe the conventional \textit{scalar} optical vortices in Kerr-like nonlinear media. In a defocusing medium, a diffracting core of an optical vortex may get self-trapped and the resulting beam with a singular core should be classified as a vortex soliton. In contrast, as discussed above, a vortex-carrying beam itself becomes self-trapped in a focusing nonlinear medium, and it is known to suffer the azimuthal modulational instability.

2.1. VORTICES IN DEFOCUSING NONLINEAR MEDIA

In a self-defocusing nonlinear medium, a screw dislocation in the wave phase can create a stationary beam structure with a phase singularity resulting in a self-trapped vortex beam or a vortex soliton. To describe the major properties of vortex solitons, we consider the propagation of a continuous wave (CW) beam in a bulk self-defocusing medium governed by a (2 + 1)-dimensional NLS equation. In the specific case of the Kerr nonlinearity, this equation can be written in the normalized form

\[
\frac{i}{2} \frac{\partial u}{\partial z} + \frac{1}{2} \nabla^2 u - |u|^2 u = 0.
\]  

(2.1)

We can eliminate the background of constant amplitude \(u_0\) through the transformation, \(z' = u_0^2 z\), \(x' = u_0 x\), \(y' = u_0 y\), \(u = u_0 \psi \exp(-iu_0^2 z)\), and obtain the following equation for the normalized field \(\psi\):

\[
\frac{i}{2} \frac{\partial \psi}{\partial z} + \frac{1}{2} \nabla^2 \psi + (1 - |\psi|^2) \psi = 0,
\]

(2.2)

where we have dropped the primes for simplicity of notation. The dimensionless field should satisfy the boundary conditions \(|\psi| \to 1\) as \(x\) and \(y \to \pm \infty\).

The existence of vortex solutions for the \((2 + 1)\)-dimensional cubic NLS equation (2.2) can be established using an analogy between optics and fluid dynamics. Using the so-called Madelung transformation (Spiegel [1980]; Donnelly [1991]; Nore, Brachet, and Fauve [1993]),

\[
\psi(\mathbf{r}, z) = \chi(\mathbf{r}, z) \exp[\varphi(\mathbf{r}, z)],
\]

(2.3)
2. VORTICES IN χ(3) MEDIA

Figure 3: (a) Schematic of the intensity distribution in an optical beam carrying a vortex (mesh) and its helical wave front (gray surface). After Kivshar and Ostrovskaya [2001]. (b) Vortex profiles in a self-defocusing Kerr medium for four first values of the integer vortex charge m (Neu [1990]).

where \( r \) is a two-dimensional vector with coordinates \( x \) and \( y \), we can transform the NLS equation (2.2) into the following set of two coupled equations:

\[
\frac{\partial \chi}{\partial z} + \nabla \cdot (\chi^2 \nabla \phi) = 0, \\
\frac{\partial \phi}{\partial z} + \frac{1}{2} (\nabla \phi)^2 = 1 - \chi^2 + \frac{\nabla^2 \chi}{2 \chi}.
\]

These equations can be viewed as the equations governing the conservation of mass and momentum for a compressible inviscid fluid of density \( \rho = \chi^2 \) and velocity \( \mathbf{v} = \nabla \phi \), with the effective pressure defined as \( p = \rho^2 / 2 \). More importantly, this kind of analogy between optics and fluid mechanics remains valid even for the generalized NLS equation with an arbitrary form of \( g(|u|^2) \), provided the effective pressure is defined as

\[
p(\rho) = \int \rho \frac{dg(\rho)}{d\rho} \, d\rho.
\]

The analogy, however, is not exact because, in addition to the standard pressure, Eq. (2.5) includes a second term that has no analog in fluid mechanics. This term results from the so-called quantum-mechanical pressure in the context of superfluids.

The Madelung transformation is singular at the points where \( \chi = 0 \). Around such points on the plane \((x, y)\), the circulation of \( \mathbf{v} \) is not zero but equals \( 2\pi \). These points present topological defects of the scalar field, and they are called vortices. To find the stationary solution corresponding to a vortex soliton, also called a dark soliton with circular symmetry, see Fig. 3(a), we look for solutions of the cubic NLS equation (2.2) in the polar coordinates \( r \) and \( \theta \),

\[
\psi(r, \theta; z) = U(r)e^{im\theta},
\]

where the integer \( m \) is the so-called vortex winding number, sometimes also called the vortex charge, and the real function \( U(r) \) satisfies the amplitude equation,

\[
\frac{d^2 U}{dr^2} + \frac{1}{r} \frac{dU}{dr} + \frac{m^2}{r^2} U + (1 - U^2) U = 0,
\]

with the boundary conditions

\[
U(0) = 0, \quad U(\infty) = 1.
\]

The continuity of \( U \) at \( r = 0 \) forces the first condition, while \( U(\infty) = 1 \) is consistent with a uniform background of intensity \( U_0^2 \) as \( r \rightarrow \infty \).

Equation (2.8) can be solved numerically to find the shape \( U(r) \) of the vortex soliton for different values of \( m \), shown in Fig. 3(b) (Neu [1990]). Alternatively, the approximate envelopes converge to the stationary states in the numerical simulation of the full Eq. (2.2) (Velchev, Dreischuh, Neshev, and Dinev [1997]). The region in the vicinity of \( r = 0 \), where \( U(r) \) is significantly less than 1, is called the vortex core. The functional form of \( U(r) \) near \( r = 0 \) and \( r = \infty \) can be established directly from Eq. (2.8) by taking the appropriate limit and is found to be

\[
U(r) \sim \begin{cases} 
ar^{|m|} + O(r^{|m|+2}) & \text{as } r \rightarrow 0, \\
1 - \frac{m^2}{2r^2} + O(1/r^4) & \text{as } r \rightarrow \infty.
\end{cases}
\]
Figure 4: Intensity distributions at the output of a nonlinear medium (a Rubidium vapor). The position of the vortex in (a) and (b) was controlled by translation of the vortex mask. (c) The intensity profile at the cross-section made through the vortex core in the cases (a) (solid) and (b) (dashed), respectively (Christou, Tikhonenko, Kivshar, and Luther-Davies [1996]).

The structure of the vortex soliton for an arbitrary form of the nonlinearity can be found using the same method and solving numerically for the amplitude function $U(r)$. No qualitatively new features are found when the nonlinearity is allowed to saturate (Chen and Atai [1992]). However, the effective diameter of the vortex core increases almost linearly with the saturation parameter $s = I_0 / I_s$, where $I_s$ and $I_0$ are the saturation and background intensities, respectively (Tikhonenko, Kivshar, Steblina, and Zozulya [1998]).

Stability of vortex solitons associated with the generalized NLS equation has not yet been fully addressed. However, it is believed that vortices with the winding numbers $m = \pm 1$ are topologically stable but that those with larger values of $|m|$ are unstable and should decay into $|m|$ single-charge vortices. In the context of superfluids, the multi-charged vortex solitons are found to survive for a relatively long time (Aranson and Steinberg [1996]). A similar behavior is found to occur in optics (Dreischuh, Paulus, Zacher, Grasbon, Neshev, and Walther [1999]). For this reason, multi-charged vortices are usually classified as metastable. As an example, an intentional perturbation of a triply charged vortex leads to its incomplete decay to a long-lived doubly charged vortex and a singly charged vortex (Dreischuh, Paulus, Zacher, Grasbon, and Walther [1999]), confirming that saturation of nonlinearity can effectively suppress the instability. We should, however, stress that multi-charged vortices are strongly unstable in anisotropic nonlinear media (Mamaev, Saffman, and Zozulya [1997]).

Experimentally, a vortex soliton appears as a dark region that maintains its shape on a diffracting background beam displaying a nontrivial dynamical behavior (Swartzlander and Law [1992]; Tikhonenko and Akhmediev [1996]). To generate the vortex input beam, one images the waist of the Gaussian beam onto the surface of a singly charged phase mask with a telescope. The first diffracted order of this mask is then imaged onto the plane of the nonlinear cells input window, providing an initial condition consisting of a singly charged vortex nested centrally at the waist of a Gaussian beam. The position of the vortex in the initial field is controlled by translation of the phase mask across the beam.

Figure 4(a) shows a typical intensity distribution at the output with the vortex nested at the approximate center of the beam. Figure 4(b) shows the output under the same conditions, apart from a translation of the phase mask. There is little change in the beam away from the core of the vortex, as is seen in Fig. 4(c), upon which cross sections on a line through the vortex cores of Figs. 4(a) and 4(b) are overlaid. To measure the size of the output background, we should remove the vortex from the profile and calculated the average $1/e^2$ radius of undisturbed background. Phenomena such as the rotation and radial drift of the vortex relative to the background CW beam are often observed experimentally, even though they cannot always be predicted from a casual analysis of the stationary solution of the NLS equation. A proper theoretical description of these effects requires analytical techniques capable of analyzing the vortex motion (Christou, Tikhonenko, Kivshar, and Luther-Davies [1996]; Kivshar, Christou, Tikhonenko, Luther-Davies, and Pismen [1998]). Physically, specific dynamical features such as vortex rotation and drift result from a nonuniform intensity profile of the background field which is typically a Gaussian beam.
The rotation rate of optical vortices can be controlled by introducing a modulated phase gradient of the background beam, when the slope of the helical front is not uniform in the azimuthal direction (Kim, Lee, Kim, and Suk [2003]). Phase profile determines not only the dynamics of a single vortex but also interaction between two vortices, such as attraction and repulsion of counter- and co-rotating vortices, respectively (Luther-Davies, Powles, and Tikhonenko [1994]; Velchev, Dreischuh, Neshev, and Dinev [1996]). A pure phase modulation, obtained by using computer-synthesized holograms, was used to create dark ring solitary waves (Kivshar and Yang [1994]; Baluschev, Dreischuh, Velchev, Dinev, and Marazov [1995]; Dreischuh, Fliesser, Velchev, Dinev, and Windholz [1996]; Dreischuh, Kamenov, and Dinev [1996]; Kamenov, Dreischuh, and Dinev [1997]; Neshev, Dreischuh, Kamenov, Stefanov, Dinev, Fliesser, and Windholz [1997]; Dreischuh, Neshev, Paulus, Grasbon, and Walther [2002]). Similar examples of phase patterns with singularities include vortex arrays and lattices in self-defocusing Kerr type media studied by Neshev, Dreischuh, Assa, and Dinev [1998]; Kim, Jeon, Noh, Ko, Moon, Lee, and Chang [1998]; Kim, Jeon, Noh, Ko, and Moon [1998]; Dreischuh, Chervenkov, Neshev, Paulus, and Walther [2002].

2.2. RING-LIKE BEAMS IN FOCUSING NONLINEAR MEDIA

In an isotropic optical medium with a local nonlinear response the propagation of a paraxial light beam is described by the well-known generalized nonlinear Schrödinger (NLS) equation (Kivshar and Agrawal [2003]). In the dimensionless form, the generalized NLS equation has the form (cf. Eq. (2.1)),

$$i \frac{\partial E}{\partial z} + \Delta_\perp E + F(I)E = 0,$$

where $E$ is the complex envelope of the electric field, $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the transverse Laplacian, and $z$ is the propagation distance measured in the units of the diffraction length $L_D$. Function $F(I)$ describes the nonlinear properties of an optical medium, and it is assumed to depend on the total beam intensity, $I \equiv |E|^2$. Examples of these Kerr-like materials include two-level model of resonant gases, $F = I$, or pure Kerr nonlinearity; the so-called saturable nonlinearity, $F = I(1 + \alpha I)^{-1}$, or its low-intensity expansion, the cubic-quintic model, $F = I - \alpha I^2$, etc. Here the parameter $\alpha$ defines the nonlinearity saturation.

In a self-focusing medium, i.e. $F(I) \geq 0$, the diffraction of a light beam can be compensated by the nonlinearity and the balance between these two counter-acting “forces” corresponds to the stationary state. Spatial optical solitons are stationary spatially localized solutions of the NLS equation (2.11) which do not change their intensity profile during propagation (Segev [1998]; Stegeman and Segev [1999]; Kivshar and Stegeman [2002]). Such a definition covers many different types of stationary beams with a finite power and, in general, the spatial solitons can be found in a generic form,

$$E(x, y, z) = U(x, y) \exp \left[ i k z + i \phi(x, y) \right],$$

where the real functions $U$ and $\phi$ are the soliton amplitude and phase, respectively, and $k$ is the soliton propagation constant. Substituting Eq. (2.12) into Eq. (2.11), we arrive at the system of coupled equations for the soliton amplitude and phase,

$$\Delta_\perp U - k U - (\nabla \phi)^2 U + F(U^2) U = 0,$nabla $\phi + 2 \nabla \phi \nabla \ln U = 0.$$

We start with the solutions of the system (2.13), (2.14) with a constant phase, taking $\phi = 0$ without restriction of generality. In this case, it can be shown that the only type of a structure localized in both transverse dimensions should possess a radial symmetry, i.e. $U(x, y) = U(r)$, where $r = \sqrt{x^2 + y^2}$. Solutions of this type include the fundamental (bell-shaped) soliton [see Fig. 5(b) for $m = 0$] and higher-order modes with several rings surrounding the central peak (Haus [1966]; Yananskas [1966]; Soto-Crespo, Heatley, Wright, and Akhmediev [1991]; Edmundon [1997]). The number of radial nodes, defined by the index $n$, distinguishes the higher-order radially-symmetric spatial solitons. The main parameter characterizing the spatial soliton is its power

$$P = \int |E|^2 \, dr = \int U^2 \, dr,$$

being the integral of motion associated with phase invariance of a solution to Eq. (2.11). Radial modes with $n$-rings in the intensity profile, $U_n$, belong to the different branches of the dependance $P(k)$, i.e. they form a discrete set of soliton families, $P_n(k)$. The generalization of each soliton family includes transversely moving solitons, obtained by applying the Galilean transformation, $r \rightarrow r - 2qz$ and $\phi \rightarrow \phi + q(r - qz)$, where $v = 2q$ is the soliton transverse velocity. Such moving solitons can be characterized by the soliton linear momentum

$$L = \text{Im} \int E^* \nabla E \, dr = \int \nabla \phi \ U^2 \, dr.$$
which is defined for the fundamental solitons as \( \mathbf{L} = \mathbf{q} P \).

A novel class of spatially localized beams in self-focusing nonlinear media, associated with the rotation of the field phase, was introduced by Kruglov and Vlasov [1985]. The beam phase has a spiral structure with a singularity at the origin, as the one shown in Fig. 3(a), representing a phase dislocation of the wave front in the form of an optical vortex (Kivshar and Ostrovskaya [2001]). The intensity of such a beam vanishes at the beam center, and, at the same time, the beam remains localized (i.e. its intensity decays at infinity) propagating in the form of a ring-like beam, see Fig. 5(a,b). The existence of the ring-profile solitary waves can be explained intuitively. Indeed, quasi-one-dimensional solitary waves, i.e. the (1+1)-dimensional solitary waves embedded into a (2+1)-dimensional bulk medium, undergo the transverse modulational instability (Kivshar and Pelinovsky [2000]). One of the possible ways to suppress this instability is to consider a ring-profile structure created from a (1+1)-dimensional soliton stripe wrapped around its tail (Lomdahl, Olsen, and Christiansen [1980]; Afanasjev [1995]; Anastassion, Pigier, Segev, Kip, Eugenieva, and Christodoulides [2001]). The beams studied by Kruglov, Logvin, and Volkov [1992] provide another example of nonstationary ring-profile solitary waves.

Introduced by Kruglov and Vlasov [1985] ring-profile vortex solitons represent the first example of a spatial soliton with the field dependence on the azimuthal coordinate \( \varphi = \tan^{-1}(y/x) \). They can be found as the solutions to Eqs. (2.13), (2.14) with a rotating spiral phase \( \phi \) in the form of a linear function of the polar angle \( \phi = m \varphi \). Substituting this expression into Eqs. (2.13), (2.14) we obtain

\[
\frac{d^2 U}{dr^2} + \frac{1}{r} \frac{dU}{dr} - \frac{m^2}{r^2} U - kU + F(U^2)U = 0, \tag{2.17}
\]

where the radially symmetric amplitude \( U(r) \) vanishes at the center, \( U \sim r^{|m|} \) at \( r \to 0 \). The rotation velocity should be quantized by the condition of the field univocality Eq. (1.1), and therefore \( m \) is integer [see Fig. 5(b,c)]. The index \( m \) stands for a phase twist around the intensity ring, and it is usually called winding number, topological index or topological charge of the solitary wave. Such winding number distinguishes azimuthal higher-order stationary states, in addition to the radial modes, so that the full set of radially-symmetric spatial soliton families can be denoted by \( P_{n,m}(k) \), with radial and azimuthal quantum numbers \( n \) and \( m \). Figure 5(b) illustrates the envelopes \( U(r) \) for single ring (\( n = 0 \)) vortex solitons with different topological charges \( m \). Corresponding families are divided by the minimal threshold power, necessary for the formation of corresponding higher-order states, the dependencies \( P_{0,m}(k) \) for a fixed nonzero saturation \( \alpha = 0.2 \) are shown in Fig. 5(c). The threshold power is defined in the limit of zero saturation \( \alpha = 0 \), i.e. in pure Kerr medium \( F(I) = I \), where soliton constant \( k \) plays a role of a scaling parameter, see horizontal lines in Fig. 5(c). Subsequently, vortex solitons were re-discovered in other studies by Kruglov, Logvin, and Volkov [1992]; Atai, Chen, and Sotocrespo [1994]; Afanasjev [1995] and for other types of nonlinear optical media, including quadratic nonlinear media (Torner and Petrov [1997a,b]; Firth and Skryabin [1997]; Skryabin and Firth [1998a]), the latter nonlinear model is discussed in Sect. 4.

The important integral of motion associated with this type of the solitary waves is the beam angular momentum,

\[
M \mathbf{e}_z = \text{Im} \int E^*(r \times \nabla E)dr,
\]
which can be expressed through the soliton amplitude and phase,

\[ M = \int \frac{\partial \phi}{\partial \varphi} U^2 \, dr. \] \hspace{1cm} (2.18)

It is important to point out that the nonvanishing angular momentum is an overall property of the light beam, not necessarily directly connected to the quantum properties at the single photon level. Similarly, the angular momentum is not necessarily associated with optical singularities although in practice the two phenomena may occur together (Allen [2002]; Berry, Dennis, and Soskin [2004]; Padgett, Courtial, and Allen [2004]). The angular momentum of a paraxial light beam can be separated into spin and orbital parts (see, e.g., Cohen-Tannoudji et al. [1989]; Barnett [2002]), where the spin momentum is associated with the polarization structure of the light, and the orbital momentum is associated with the spatial structure of the beam, in particular, the beam carrying optical vortices. Therefore, the angular momentum of a scalar vortex soliton defined by Eq. (2.18) should be identified as an orbital angular momentum. However, as we describe below, the ring-profile vortex optical beams experience the azimuthal instability in nonlinear media, and they decay into a number of moving fundamental solitons. Because the input beams carry the overall angular momentum given by Eq. (2.18), the splinters fly off the ring along the tangential trajectories. Therefore, in the soliton community it has become customary to refer to the soliton spin angular momentum, and thus to spinning solitons, and to use orbital angular momentum in the case of several interacting solitons. This notion leads to the description of the breakup of a vortex due to modulational instability in terms of the transformation of the initial soliton spin angular momentum to the net orbital angular momentum of moving splinters (Firth and Skryabin [1997]; Skryabin and Firth [1998a]). The ratio of the soliton angular momentum to its power can be identified with the soliton spin, \( S = M/P \), and for the vortex solitons the spin is equal to its nonzero topological charge \( S = m \) [cf. Eq. (1.1)]. We notice, however, that such a notation might be confusing when it is used in other areas where the concepts of spin angular momentum and orbital angular momentum are employed in the rigorous, proper sense.

2.3. AZIMUTHAL MODULATIONAL INSTABILITY

As was shown in many numerical and analytical studies, the ring-like vortex beams in self-focusing nonlinear media are subject to the azimuthal symmetry-breaking modulational instability, a specific type of transverse modulational instability similar to one which is responsible for filamentation of the beams and generation of trains of optical solitons (Kivshar and Pelinovsky [2000]). This effect should not be mixed with the well-known collapse instability (Berge [1998]) which is eliminated in the nonlinear media with saturable and quadratic nonlinearity. As a result, stable fundamental solitons have been found in two and three spatial dimensions and the instability criterion was established by Vakhitov and Kolokolov [1973]. It states that the principal mode of the non-linear wave equation (2.11) is stable if its integrated intensity Eq. (2.15) has a positive slope, \( \partial P/\partial k > 0 \), and the latter is possible if the nonlinearity growth monotonically with intensity, i.e. for \( dF/dI \geq 0 \). Both these conditions are satisfied for the vortex solitons in, e.g., self-focusing media with saturation, see Fig. 5(c). However, this property does not guarantee the vortex stability against azimuthal perturbations.

So far, no universal criterion, similar to the Vakhitov-Kolokolov stability criterion for the fundamental optical solitons, has been suggested for the stability of vortex solitons, and the vortex stability should be addressed separately for different models. Below, we present the stability analysis developed by Soto-Crespo, Wright, and Akhmediev [1992]; Firth and Skryabin [1997]; Torres, Soto-Crespo, Torner, and Petrov [1998a].

We assume that there exists a stationary solution \( E = E_0 \) solving Eq. (2.11). The linear stability of this solution can be established by the behavior of an additional small perturbation \( |p| \ll |E_0| \). Substituting the perturbed solution \( E = E_0 + p \) into Eq. (2.11) and linearizing it with respect to a small perturbation \( p \), we obtain

\[ \frac{\partial p}{\partial z} + \Delta_{\perp} p + \left( F_0 + |E_0|^2 F_0^* \right) p + E_0^2 F_0^* p^* = 0, \] \hspace{1cm} (2.19)

where \( F_0 = F(|E_0|^2) \) and \( F_0^* = (dF/dI)|_{I=|E_0|^2} \). Equation (2.19) describes the evolution of initially small perturbation \( p \) corresponding to the solution \( E_0 \): if \( p \) does not grow with the beam propagation, the stationary solution \( E_0 \) is linearly stable.

The linear equation (2.19) can be solved by the separation of variables, depending on the geometry of underlying stationary point \( E_0 \). For the case of radially symmetric vortices determined by (2.17), i.e. \( E_0 = U(r) \exp(i m \varphi + ik z) \), the perturbation \( p \) should posses an azimuthal periodicity and, therefore, it can be represented as a Fourier series

\[ p(r, \varphi, z) = \sum_{n = -\infty}^{\infty} p_n(r, z) \exp(in \varphi). \] \hspace{1cm} (2.20)

Substituting this decomposition into Eq. (2.19) and matching terms with equal angular dependence, we obtain the infinite set of equations for complex modal functions \( p_n \). However, for any integer \( s \), only two modes \( p_{m+s} \)
Figure 6: Growth rate, \( \text{Re} \ gamma_s(k) \), of the instability modes for (a) single- and (b) double-charged vortex solitons in saturable medium, \( F(I) = I/(1 + I) \). Corresponding values of perturbation index \( s \) are shown next to the curves. After Skryabin and Firth [1998a]. Break-up of the vortex solitons with the maximum growth rate and the charges (c) \( m = 1 \) and (d) \( m = 2 \). Dashed curves at \( z = 50 \) show the peak intensity of the initial rings and trajectories of the solitons flying away after the decay.

and \( p_{m-s} \) are actually coupled and build a closed system:

\[
\begin{align*}
\left\{ i \frac{\partial}{\partial z} + \hat{L}^\pm \right\} p_{m \pm s} + \exp(i2kz)Ap^*_{m \mp s} &= 0, \\
\end{align*}
\]

where \( A \equiv U^2F_0^2 \) and \( \hat{L}^\pm \equiv d^2/dr^2 + r^{-1}d/dr - (m \pm s)^2r^{-2} + F_0 + A \). Solution to these equations is given by \( p_{m+s}(r, z) = u_s(r)\exp(ikz + \gamma_s z) \) and \( p_{m-s}(r, z) = v_s^*(r)\exp(-ikz + \gamma_s^*z) \), with complex perturbation wave number \( \gamma_s \) and the modes \( u_s \) and \( v_s \) that solve an eigenvalue problem

\[
\begin{pmatrix}
i \gamma_s & u_s \\
v_s & i \gamma_s
\end{pmatrix}
= 
\begin{pmatrix}
k - \hat{L}^+ & -A \\
A & -k + \hat{L}^-
\end{pmatrix}
\begin{pmatrix}
u_s \\
v_s
\end{pmatrix},
\]

In these notations, if the eigenvalue \( \gamma_s \) has a positive real part for some \( s \), the perturbation modes \( p_{m \pm s} \) grow exponentially with the growth rate \( \text{Re} \ gamma_s > 0 \); and such modes are called instability modes. The equivalent representation of perturbation \( p(r, \varphi, z) = \exp(im\varphi + ikz) \left\{ u_s(r)\exp(is\varphi + \gamma_s z) + v_s^*(r)\exp(-is\varphi + \gamma_s^*z) \right\} \) guarantees, due to the completeness of the basis of azimuthal harmonics Eq. (2.20), that all possible perturbations are taken into account.

The instability growth rate depends on the vortex power, as is shown in Figs. 6(a,b) for a particular case of spatial vortex solitons with the topological charges \( m = 1 \) and \( m = 2 \) in a saturable medium with \( \alpha = 1 \). For any \( s \), the growth rate vanishes in the linear limit \( k \to 0 \), and also in the opposite limit of infinite power, \( k \to 1/\alpha \), but at least one mode has a nonzero positive value in the whole domain of the vortex soliton existence, \( k \in [0, 1/\alpha] \). Thus, all vortex solitons are linearly unstable in saturable media. Similar results hold for the parametric interaction in a quadratic medium, as we discuss in detail in Sect. 4.

The index of the modes with the highest growth rate depends on the model and the mode topological charge. For example, a single-charged vortex always has the \( s = 2 \) mode growing faster than other modes in a saturable medium, while this can be the \( s = 3 \) mode in a quadratic medium, and, as shown in Fig. 1(b), also in a pure Kerr medium with \( \alpha = 0 \). In general, the higher-charge vortices allow for competition between different modes with close values of the growth rate, such as the modes with \( s = 3 \) and \( s = 4 \) for \( m = 2 \), see Fig. 6(b).

The symmetry-breaking instability of the ring-like vortex beams has been observed experimentally in saturable vapors (Tikhonenko, Christou, and Lutherdaves [1995]; Tikhonenko, Christou, and Luther-Davies [1996]; Bigelow, Zerom, and Boyd [2004]a), photorefractive (Chen, Shih, Segev, Wilson, Muller, and Maker [1997]) and quadratic (Petrov, Torner, Martorell, Vilaseca, Torres, and Cojocariu [1998]) nonlinear media. In all such cases, the generation of different numbers of fundamental solitons due to the ring instability was observed. Figure 6 shows a numerical example of the breakup scenario of the ring-like vortex solitons, when the initial stationary ring-profile structure decays, under the action of a numerical noise, into two [the case \( m = 1 \], see Fig. 6(a)] or four [the case \( m = 2 \], see Fig. 6(b)] fundamental solitons. The number of splinters coincides exactly with the topological index of the instability mode with highest growth rate, see Fig. 6(a,b), thus the predictions of a linear stability analysis are in an excellent agreement with the numerical solution of the full system. The detailed stability analysis of solitary waves with central phase dislocation were reported by Skryabin and Firth [1998a] for both saturable self-focusing and quadratic nonlinear media, and for the latter case also by Torres,
Soto-Crespo, Torner, and Petrov [1998a]. In the experiments, the excitation of the ring-profile vortex beams is conducted by pumping the nonlinear media by suitable approximations of Laguerre-Gaussian modes, and it was shown earlier that not only the spontaneous symmetry-breaking instability of the vortex solitons could be observed in that geometry but also that such excitation conditions offer additional possibilities by inducing suitable instabilities (Torner and Petrov [1997a,b]), as we discuss in detail in Sect. 4.

An analytical approach to the study of the filament dynamics after the breakup, based on the conservation of the beam angular momentum and Hamiltonian, was developed by Skryabin and Firth [1998a]. Given initial values of the conserved quantities, it is possible to predict the features of the filament trajectories and estimate their number. Two different analytical expressions for the velocity of the filaments in the transverse plane were derived, both formulas giving a reasonably good quantitative predictions for the velocities. The formula based on angular momentum being particularly simple and effective: the escape velocity can be estimated as \( v \approx \frac{m}{R} \), where \( R \) is the initial radius of the vortex ring. The important conclusion, giving an insight into the underlying physics of the beam breakup, is that when filaments move out along tangents to the initial ring, they carry away its angular momentum. In our notations we can describe this effect as the transformation of initial spin angular momentum of the vortex soliton to the angular momentum of the splinters spiraling out.

Stabilization of coherent vortex solitons against the azimuthal instabilities remains a major challenge in the physics of spatial vortex solitons. Several theoretical models were suggested which support stable vortex solitons, including the formation of vortex solitons in the presence of competing nonlinearities (see Sect. 5), nonlocal nonlinear media (Yakimenko, Zaliznyak, and Kivshar [2004]; Breidis, Petersen, Edmundson, Krolikowski, and Bang [2005]), or Bessel photonic lattices (Kartashov, Vysloukh, and Torner [2005b]), without experimental observations so far.

\section{3. Composite spatial solitons with phase dislocations}

In this section we describe optical vortices (and other closely related higher-order spatial solitons) composed of several beams (components) interacting via cross-phase modulation (XPM). Because all of those components contribute to the total beam intensity, the refractive index change is usually refereed as commonly induced waveguide, and such composite solitons can be thought as corresponding guided modes. In the simplest case of two interacting beams, e.g. two orthogonally polarized components of a vector soliton, this system posses radially-symmetric vector vortices as well as azimuthally modulated multipole and necklace-ring vector solitons. For large number of mutually incoherent components the composite solitons are closely related to partially coherent self-trapped beams and vortices.

\subsection{3.1. SOLITON-INDUCED WAVEGUIDES}

Spatial solitons may be understood as the modes of the effective waveguides they induce in a nonlinear medium (Kivshar and Agrawal [2003]). A natural extension of this concept is to assume that the waveguide induced by relatively powerful soliton beam may guide and control another weak beam. This concept may be applicable to bright (Delafuente, Barthelmy, and Froehly [1991]) as well as dark (Luther-Davies and Xiaoping [1992]; Luther-Davies and Yang [1992]) spatial solitons.

The possibility of effective waveguiding of a weak probe (or signal) beam via the XPM type interaction can be analyzed within the coupled system of NLS equations for two wave envelopes \( E_{1,2} \),

\begin{align}
\frac{i}{2} \frac{\partial E_1}{\partial z} + \Delta_1 E_1 + \sigma \left( c_{11} |E_1|^2 + c_{12} |E_2|^2 \right) E_1 &= 0, \\
\frac{i}{2} \frac{\partial E_2}{\partial z} + \Delta_2 E_2 + \sigma \left( c_{21} |E_1|^2 + c_{22} |E_2|^2 \right) E_2 &= 0,
\end{align}

where the coefficient \( \sigma = \pm 1 \) defines a focusing and defocusing nonlinear medium, respectively. In a general case with different carrier frequencies and polarization states of two interacting beams, the contributions \( c_{nn} \) from self-phase modulation (SPM) and the XPM coefficients \( c_{nj} \) are all different, \( c_{nj} \neq c_{jn} \) (\( n, j = 1, 2 \) and \( j \neq n \)). If the two waves have the same carrier frequency but different polarization states, the interaction is symmetric, \( c_{nj} = c_{jn} \) and \( c_{jj} = c_{nn} \), and one of them can be scaled away, e.g. SPM coefficients \( c_{nn} = 1 \).

In the latter case, the XPM coefficients \( c_{nj} = 2 / 3 \), for linearly polarized components, \( c_{nj} = 2 \), for circular polarized components, and in the case of elliptically polarized components this parameter satisfied the relation: \( 2/3 < c_{nj} < 2 \). Transition to the Manakov system \( c_{nj} = 1 \), completely integrable in 1D geometry (Manakov [1974]), occurs at ellipticity angle \( 35^\circ \) (Menyuk [1989]).

When the signal beam, e.g. \( E_2 \), is much weaker than the soliton beam \( |E_2| \ll |E_1| \), the system (3.1),(3.2) can be linearized with respect to \( E_2 \). Then the equation (3.1) transforms to the NLS equation (2.11) with the nonlinear potential \( F(I) = \sigma c_{11} |E_1|^2 \) which supports stationary soliton solutions, while the signal wave \( E_2 \) propagates in effectively linear regime in the waveguide with the refractive index \( \sigma c_{21} |E_1|^2 \). Depending of the actual shape of the soliton beam \( E_1 \) and its peak intensity, the induced waveguide can be single-moded, i.e.
Figure 7: (a) Higher-order modes guided by a fundamental soliton in the self-focusing photorefractive medium: plot shows the calculated distribution of the refractive index induced by the fundamental soliton and experimental photos demonstrate the intensity of three first modes of a guided red probe beam (Petter, Denz, Stepken, and Kaiser [2002]). (b) Radially symmetric fundamental (top) and vortex (bottom) modes localized in the waveguide induced by the vortex soliton (thick lines) in self-defocusing medium (Carlsson, Malmberg, Anderson, Lisak, Ostrovskaya, Alexander, and Kivshar [2000]).

In focusing nonlinear media, such as anisotropic photorefractive crystals, the fundamental soliton has a bell-shaped form (Segev, Crosignani, and Yariv [1992]). Similar to the 1D case (Morin, Duric, Salamo, and Segev [1995]; Shih, Chen, Mitchell, Segev, Lee, Feigelson, and Wilde [1997]), the (grin) light beam from a frequency-doubled Nd:YAG laser creates a two-dimensional spatial soliton and induces an effective waveguide for the probe beam at less photosensitive wavelength; in SBN crystal it can be taken from the HeNe laser (red beam). The effective contribution to the refractive index from the red beam is negligible, i.e. both the SPM $c_{22} \sim 0$ and XPM coefficient $c_{12} \sim 0$ in Eqs. (3.1),(3.2). At the same time the guidance of a probe beam by a single two-dimensional soliton as well as by the pair of interacting solitons is indeed possible, it was demonstrated by Petter and Denz [2001]. Furthermore, because of the anisotropy of photorefractive screening nonlinearity (Zozulya, Anderson, Mamaev, and Saffman [1998]), the effective waveguide is highly anisotropic and may support higher-order modes. Figure 7(a) shows the induced refractive index profile and corresponding higher-order TEM guided modes, generated experimentally by Petter, Denz, Stepken, and Kaiser [2002].

In defocusing nonlinear media, spatial solitons are associated with a nontrivial phase structure, and a two-dimensional soliton is a dark vortex, as discussed in Sect. 2.1. In this case, the spatial profile of the induced refractive index (“induced fiber”) allows to support radially symmetric spatially localized modes (Snyder, Poladian, and Mitchell [1992]). The waveguiding properties of dark optical vortex solitons in self-defocusing Kerr media have been analyzed by Sheppard and Haelterman [1994]; Law, Zhang, and Swartzlander [2000]; Carlsson, Malmberg, Anderson, Lisak, Ostrovskaya, Alexander, and Kivshar [2000]. It was shown that these properties depend crucially on the relative strength of the cross- and self-phase modulation effects. Families of composite solitons formed by a vortex and its guided mode with or without a topological charge have been identified. Examples of these modes are presented in Fig. 7(b).

A soliton-waveguiding experiment has been conducted by Truscott, Friese, Heckenberg, and Rubinsztein-Dunlop [1999] in an atomic vapor. A weak probe beam tuned near one atomic resonance is guided through a waveguide written by an intense pump vortex beam at a different atomic resonance. As the pump beam is tuned close to resonance, it creates a nonlinear refractive index profile in the atomic vapor with which the weak probe beam interacts. The efficiency of the guiding is found to depend strongly on the power and frequency of the guiding beam. Moreover, since the guiding takes place in an atomic vapor, it is possible to tune to both sides of the atomic resonance. This has the distinct advantage that it allows the guiding of light into either bright or dark regions of the guiding beam. The theory of waveguides electromagnetically induced in Rb vapors was developed by Kapoor and Agarwal [2004]. Their density matrix approach was based on the three-level V-system, and it was generalized latter to the five-level model by Andersen, Friese, Truscott, Ficek, Drummond, Heckenberg, and Rubinsztein-Dunlop [2001], who took into account the hyperfine structure of the D-line of rubidium as well as the presence of the two major isotopes. The results allow one to deduce which frequency combinations are likely to give successful guiding.

Systematic analysis of the waveguiding properties of the vortex solitons and vortex-mode vector solitons in saturable nonlinear media, for both self-defocusing and self-focusing nonlinearities, was performed recently...
Figure 8: Examples of constituents of (a) dipole-mode \((s = 0.3)\) and (b) vortex-mode \((s = 0.65)\) two-component vector solitons (top row: \(|E_1|^2\), middle row: \(|E_2|^2\)) shown with the phase distributions needed to generate the \(E_2\)-modes experimentally (bottom row), after Krolikowski, Ostrovskaya, Weilnau, Geisser, McCarthy, Kivshar, Denz, and Luther-Davies \([2000]\). Delayed-action interaction between the vortex-mode vector solitons and formation of spiraling dipole (Musslimani, Soljacic, Segev, and Christodoulides \([2001a]\)).

by Salgueiro and Kivshar \([2004b]\). Following the earlier analysis of Carlsson, Malmberg, Anderson, Lisak, Ostrovskaya, Alexander, and Kivshar \([2000]\), the authors examine two major regimes of the vortex waveguiding. The most interesting nonlinear regime corresponds to large intensities of the guided beam, and it gives rise to composite (or vector) solitons, that have been identified and analyzed numerically.

In quadratic media, the simultaneous guidance of both fundamental and second-harmonic waves by an optical vortex soliton has been analyzed by Salgueiro, Carlsson, Ostrovskaya, and Kivshar \([2004]\). These authors describe novel types of three-component vector soliton created by a vortex beam together with both fundamental and second-harmonic parametrically coupled localized modes and determine conditions for a potential enhancement of the conversion efficiency.

3.2. HIGHER-ORDER VECTOR SOLITONS

When the guided beam is weak, i.e. in the linear waveguiding regime, the analysis of the soliton waveguiding properties can be carried out by using approximate analytical methods, and reducing the problem to the well-known analysis of the linear guided-wave theory (for example, see Law, Zhang, and Swartzlander \([2000]\) and also the theory of optical vortices in optical fibers by Volyar and Fadeeva \([1998, 1999]\)). For a finite-amplitude probe beam, the linearization in Eqs. \((3.1), (3.2)\) is no longer valid, and the nonlinear theory of soliton-induced waveguides should be developed (Ostrovskaya and Kivshar \([1998]\)); this theory takes into account mutual effect of the interacting waves on each other.

Several light beams generated by a coherent source can be combined together to produce a vector soliton with a complex internal structure. Properties of two-component vector solitons have been extensively studied in both self-focusing (Manakov \([1974]\); Christodoulides and Joseph \([1988]\); Snyder, Hewlett, and Mitchell \([1994]\); Christodoulides, Singh, Carvalho, and Segev \([1996]\); Krolikowski, Akhmediev, and Luther-Davies \([1996]\)) and self-defocusing (Kivshar and Turitsyn \([1993]\); Haelterman and Sheppard \([1994]\)) nonlinear media. The structure of multi-component vector soliton may become rather complicated and, for example, the soliton intensity profile may display several peaks. These so-called “multi-hump” solitons propagate as the corresponding higher-order modes of the soliton-induced waveguides. The first experimental demonstration of such multi-component solitons has been reported by the Princeton group (Mitchell, Segev, and Christodoulides \([1998]\)), who observed both single and multi-peak spatial solitons. Linear stability of the two-component vector solitons has been studied by Ostrovskaya, Kivshar, Skryabin, and Firth \([1999]\), and it has been shown that the two-peak structures can be stable in propagation while three-peak soliton structures are unstable.

Recently, the concept of multi-hump spatial solitons has been extended to two transverse dimensions. First, Musslimani, Segev, Christodoulides, and Soljacic \([2000]\); Musslimani, Segev, and Christodoulides \([2000]\) studied stationary propagation of the vortex-mode vector soliton which has a nodeless shape in one component and a vortex in the other component, see Fig. 8(b). However, it appears that such a radially symmetric, ring-like vector soliton (which is analogous to the Laguerre-Gaussian modes of cylindrical waveguide) may undergo a symmetry-breaking instability (Garcia-Ripoll, Perez-Garcia, Ostrovskaya, and Kivshar \([2000]\); Malmberg,
Dipole-mode vector solitons have been observed experimentally in photorefractive media by Krolikowski, Ostrovskaya, Weilmann, Geisser, McCarthy, Kivshar, Denz, and Luther-Davies [2000]. Two different methods have been used, one is based on a phase imprinting technique, and another uses the symmetry-breaking instability of vortex-mode soliton. In the latter case, the initial angular momentum of vortex component is transformed to the dipole and leads to its rotation during propagation after the break-up. Skryabin, McSloy, and Firth [2002] showed that rotational velocity provides an additional parametrization for the dipole-soliton family, and, with the help of a generalized Vakhitov-Kolokolov stability criterion, they predicted stability thresholds for spiraling solutions. Rotating dipole-mode soliton can be viewed as an optical “propeller” because of the mutually tilted phase fronts of the dipole lobes (Carmon, Uzdin, Pigier, Musslimani, Segev, and Nepomnyashchy [2001]). In anisotropic photorefractive media, however, the rotation of the dipole is limited because of preferable direction for its orientation (Neshev, McCarthy, Krolikowski, Ostrovskaya, Kivshar, Calvo, and Agullo-Lopez [2001]; Motzke, Stepken, Kaiser, Belić, Ahles, Weilmann, and Denz [2001]). It appears that the interaction of composite solitons, determined by the effective interaction potential (Malomed [1998]), depends significantly on their angular momenta. Musslimani, Soljacic, Segev, and Christodoulides [2001a] demonstrated numerically that collisions of vortex-mode solitons with opposite topological charges (“spins”) can lead to mutual trapping of two composite beams and the formation of bound state with a prolonged lifetime of about 35 diffraction lengths. The metastable bound state eventually disintegrates giving rise to new vector solitons, the process is characterized as the “delayed action interaction”. If both colliding vortex-mode solitons have the same topological charge, the resulting object is a stable rotating dipole-mode soliton, as is shown in Fig. 8(c). These authors draw an analogy with spin-orbit coupling in interaction of soliton as effective “particles”. The comprehensive study of collisions between vortex-mode solitons was reported by Musslimani, Soljacic, Segev, and Christodoulides [2001b], and out-of-plane scattering of the dipole-mode solitons was studied by Pigier, Uzdin, Carmon, Segev, Nepomnyashchy, and Musslimani [2001]; Krolikowski, McCarthy, Kivshar, Weilmann, Denz, Garcia-Ripoll, and Perez-Garcia [2003].

The dipole-mode vector soliton is the first example of azimuthally-modulated spatial solitons. As we noted above, close to the bifurcation line this composite structure can be described as a linear mode guided by a scalar fundamental soliton. Natural extension of this approach is to search for higher-order modes such as multipole-mode vector solitons. Indeed, the vortex-mode soliton can be described as a coherent superposition of two dipole components twisted in space and shifted in phase by $\pi/2$ with respect to each other. Similarly, the higher-charge optical vortex might be constructed from multi-poles, e.g. double charged vortex consists of two quadrupoles. This analogy with linear waveguide theory suggests the general ansatz for the azimuthally modulated component of a composite spatial soliton,

$$E_2(x, y, z) = U(r) \{\cos m\varphi + ip\sin m\varphi\} \exp(ikz),$$

where the parameter $p$ is real, $m$ is integer, and $k$ is the propagation constant. For $p = 1$, Eq. (3.3) describes a vortex-mode soliton with the charge $m$ and the radially symmetric intensity $U^2(r)$. This ansatz has been used by Desyatnikov, Neshev, Ostrovskaya, Kivshar, Krolikowski, Luther-Davies, Garcia-Ripoll, and Perez-Garcia [2001] as a variational approximation to the modes [$E_2$-component in equations similar to Eqs. (3.1), (3.2)] of the waveguide, induced by fundamental soliton in the $E_1$ component. Experimental results of the generation of quadrupole and hexapole vector solitons are shown in Fig. 9. The corresponding parameters in the ansatz Eq. (3.3) are $m = 2$ and $m = 3$ correspondingly, with $p = 0$ in both cases. For $p \neq 0$, the angular momentum of the azimuthally modulated solitons is nonzero and this leads to the soliton rotation.

Incoherent interaction between the components of a composite (or vector) ring-like beam allows to compensate for repulsion of beamlets, creating a new type of quasi-stationary self-trapped structure exhibiting the properties of the necklace-ring beams and ring vortex solitons. The physical mechanism for creating such composite vector ringlike solitons is somewhat similar to the mechanism responsible for the formation of the so-called soliton gluons (Ostrovskaya, Kivshar, Chen, and Segev [1999]) and multi-hump vector solitary waves (Ostrovskaya, Kivshar, Skryabin, and Firth [1999]), and it is explained by a balance of the interaction forces acting between the coherent and incoherent components of a composite soliton. In that case, the mutual repulsion of out-of-phase beamlets in the $E_2$-component is balanced by the incoherent attraction of the mutually coupled $E_1$ component.
3.3. MULTI-COMPONENT VORTEX SOLITONS

The effective optical waveguide induced by the fundamental soliton or dark vortex soliton in a self-focusing or self-defocusing nonlinear media, respectively, has a relatively simple bell-like shape whose modes are well known from the linear guided-wave theory. In contrast, the doughnut shape of a bright vortex soliton does not allow simple predictions about its guided modes, and, consequently, the possible structure of multi-component vortex solitons. Moreover, similar to the one-dimensional multi-hump vector solitons, their two-dimensional counterparts have a complex non-monotonous radial envelope ([Musslimani, Segev, Christodoulides, and Soljacic [2000]]). For example, in addition to the well-understood “first” bifurcation from the scalar fundamental soliton, giving rise to the vortex-mode solitons, the corresponding stationary solutions have the second bifurcation point where the “guided” component transforms to the scalar vortex ([Desyatnikov, Neshev, Ostrovskaya, Kivshar, McCarthy, Krolikowski, and Luther-Davies [2002]]). This situation can also be regarded as guiding of a simple bell-shaped beam by a ring-like waveguide. Very recently, the comparison between the so-called single- and double-vortex solitons has been carried out by [Salgueiro and Kivshar [2004a]]. Here the notation ”single-vortex” is used to distinguish a vortex-mode soliton, which consists of a strong fundamental component and a weak guided vortex (see Fig. 8(b)), from the other case (”second” bifurcation) with a strong vortex component and a small fundamental beam. At the same time, the ring vortex waveguide also supports double-vortex (or ”vortex-vortex”) vector solitons. Several types of the double-vortex solitons are shown in the Fig. 10(a).

Extension of the concept of two-dimensional multi-hump vector solitons to the case of larger number of mutually incoherent components was proposed by [Musslimani, Segev, and Christodoulides [2000]], who studied a radially symmetric potential commonly created by a strong fundamental soliton and several vortex beams. Similar idea applied to the ring vortex beams result in the so-called “necklace-ring” vector solitons ([Desyatnikov and Kivshar [2001]]). In general, the interaction of N paraxial beams via XPM can be described by the system of coupled NLS equations for the envelopes $E_n(x,y,z) (n = 1, 2, \ldots, N)$,

$$i\frac{\partial E_n}{\partial z} + \nabla^2 E_n + F(I)E_n = 0,$$

where, similar to Eq. (2.11), the function $F(I)$ describes the nonlinear refractive index, and the total beam intensity is defined as $I = \sum_{n=1}^{N} |E_n|^2$. In contrast to the system (3.1),(3.2), all the SPM and XPM contributions are taken to be equal here for simplicity. In addition to a variety of radially symmetric solutions, such as those displayed in Fig. 10(a), there exists a special class of azimuthally modulated stationary states. The simplest route to find this class of spatial solitons is to search for the modes of a radially symmetric potential with $I = U^2(r)$. This is possible if all components have the same radial envelope $U(r)$, i.e. we are looking for the stationary solution in the form

$$E_n(x,y,z) = U(r)\Phi_n(\varphi) \exp(ikz),$$

with the complex azimuthal envelopes $\Phi_n(\varphi) = a_n \cos m\varphi + b_n \sin m\varphi$ selected in such a way that $\sum_{n=1}^{N} |\Phi_n|^2 = 1$. The latter condition requires $\sum \Re(a_n b_n^*) = 0$ and $\sum |a_n|^2 = \sum |b_n|^2 = 1$. These equations define exact solutions to the system (3.5) for any $N$ and, in the particular case $N = 1$, they describe a scalar vortex of the charge $m$ with $a = 1$ and $b = i$. Radially symmetric multi-component vortices correspond to the symmetric case, $b_n = \pm ia_n$; in this case the total soliton spin is integer $m$ or zero, for the counter-rotating vortices. In a general case, the components resemble azimuthally modulated optical necklaces, introduced by [Soljacic, Sears, and Segev [1998]] (see Sect. 6.2), and the total spin may take a fractional value.

The simplest two-component solution of the necklace-ring type with $m = 1$ is given by $a_1 = b_2 = 1$, $a_2 = b_1 = 0$, and it represents two crossed dipoles, shown in Fig. 10(b). Although such a structure has no vorticity and
carries zero angular momentum, the total intensity has a profile of a single-charge scalar vortex soliton. This perfect symmetry helps to find such solutions, but it is not crucial for their existence. Indeed, similar solutions have been found numerically and generated experimentally by Ahles, Motzek, Stepken, Kaiser, Weilnau, and Denz [2002] in anisotropic photorefractive medium, which posses no radially symmetry. The main outcome of these studies, presented by Desyatnikov and Kivshar [2001], is that the vectorial interaction allows for additional stabilization of otherwise nonstationary beams, such as expanding necklace beams. Nevertheless, no linearly stable necklace-ring solitons have been found so far.

Similar stabilization of counter-rotating vector vortices was reported for the case of a self-focusing saturable medium by Bigelow, Park, and Boyd [2002], and was demonstrated experimentally in a self-defocusing photorefractive medium by Mamaev, Saffman, and Zozulya [2004]. The total angular momentum for interacting vortices with opposite topological charges is less than that of co-rotating vortices, and this was found to be the main reason for long lifetimes. Theoretically, the maximal growth rate of the azimuthal instability is significantly smaller for zero-spin vector solitons (Ye, Wang, Dong, and Li [2004]). The azimuthal instability can be eliminated completely in the so-called cubic-quintic (CQ) model, as discussed in Sect. 5.1, and there exists also the stability domain for the co-rotating vortices in presence of four-wave mixing (Mihalache, Mazilu, Towers, Malomed, and Lederer [2002]). The counter-rotating vortices with zero total angular momentum have smaller stability domain and exhibit an interesting “internal” instability dynamics in CQ medium (Desyatnikov, Mihalache, Mazilu, Malomed, Denz, and Lederer [2004]). Due to the exchange of the angular momentum between interacting components, the soliton slowly reverse the topological charges, keeping the zero total angular momentum and perfectly stable total intensity. This phenomenon is related to “charge-flipping” effect predicted by Alexander, Sukhorukov, and Kivshar [2004] to occur for the discrete vortices in two-dimensional optical lattices (see Sect. 7).

Very recently, Park and Eberly [2004] showed that the necklace-type solutions exist for the model of two-component Bose-Einstein condensates where the symmetry between the SPM and XPM contributions to the nonlinear interaction is broken. These nontopological vortices exhibit “spin” dynamical behavior and may accumulate the Berry phase under an adiabatic change of external fields that control the trapping potential (see also Sect. 8.2).

The generalization of the concept of necklace-ring vector solitons include the temporal effects in dispersive media, where the three-dimensional spatiotemporal vortex solitons, or spinning light bullets, have been predicted to exist (Desyatnikov, Maimistov, and Malomed [2000]). Corresponding solutions were found by Andersen and Kovachev [2002]; Kovachev [2004a] for nonlinear Maxwell equations and by Kovachev [2004b] for Maxwell-Dirac equations. The internal structure of the three-dimensional vector vortices with radially symmetric total intensity is described in this case by spherical harmonics.

Finally, combining the counter-rotating vortices with strong guiding from the fundamental soliton leads to a possibility to stabilize completely the necklace-ring type solutions, as was shown by Desyatnikov, Kivshar, Motzek, Kaiser, Weilnau, and Denz [2002]; Motzek, Kaiser, Weilnau, Denz, McCarthy, Krolikowski, Desyatnikov, and Kivshar [2002] in the case of three-component composite solitons. With further increasing the number of interacting components, the composite beams possess a complex internal structure, and they can serve as the modal approximation for spatially incoherent and partially coherent beams. The latter however posses the distinctive features which we describe below.
3. VECTOR VORTICES

Figure 11: Numerical and experimental results showing the stabilization of the vortex with growing incoherence: (a) input intensity, (b) vortex after 9mm of propagation for the coherent case, (c) vortex after 9mm for the partially incoherent case, $\theta_0 = 0.14$ (less coherent), and (d) vortex after 9mm for the partially incoherent case, $\theta_0 = 0.29$ (least coherent). The incoherence stabilize the vortex soliton when voltage of 2.5 kV is applied.

3.4. PARTIALLY COHERENT VORTICES

As was demonstrated in all examples presented above, optical vortices occur in coherent systems having a vanishing intensity at the vortex position and well-defined phase front topology being associated with the circulation of momentum around the helix axis. If a vortex-carrying beam is partially incoherent, the phase front topology is not well defined, and statistics are required to quantify the phase. In the incoherent limit neither the helical phase nor the characteristic zero intensity at the vortex center is observable. However, several recent studies have shed light on the question how phase singularities can develop in incoherent light fields and how these phase singularities can be unveiled (Gbur and Visser [2003]; Schouten, Gbur, Visser, and Wolf [2003]; Palacios, Maleev, Marathay, and Swartzlander [2004]). In particular, Palacios, Maleev, Marathay, and Swartzlander [2004] used experimental and numerical techniques to explore how a beam transmitted through a vortex phase mask changes as the transverse coherence length at the input of the mask is changed. Assuming a quasi-monochromatic, statistically stationary light source and ignoring temporal coherence effects, they demonstrated that robust attributes of the vortex remain in the beam, most prominently in the form of a ring dislocation in the cross-correlation function.

Propagating in nonlinear coherent systems, optical vortices become highly unstable when the nonlinear medium is self-focusing, see Sect. 2.3. However, when spatial incoherence of light exceeds a certain threshold, the stable propagation of optical vortices in self-focusing nonlinear media is possible and has been recently demonstrated in experiments conducted with a biased photorefractive SBN crystal. Jeng, Shih, Motzek, and Kivshar [2004] generated partially incoherent vortices and vortex solitons, and then inspected their stability. First, a cw laser light beam (at 488 nm) of the extraordinary polarization was made partially incoherent by passing it through a lens and then through a rotating diffuser. The rotating diffuser introduced random-varying phase and amplitude on the light beam every 1µs, which is much shorter than the response time (about 1 s) of photorefractive crystal. By adjusting the position of the diffuser to near (away from) the focal point of the lens in front the diffuser, it is possible to increase (decrease) the degree of the light coherence, and collect the light after the rotating diffuser by a second lens and then pass is through a computer-generated hologram to imprint a vortex phase. Since the partially incoherent light beam can be considered as a superposition of many mutually-incoherent light beams, the first-order diffracted light beam after the hologram becomes a superposition of many mutually-incoherent vortex beams. Then, the partially coherent vortex beam was launched into the SBN crystal along its a-axis. The total power of the vortex beam is of 0.17µW, which results in the nonlinearity of the photorefractive crystal falling into the Kerr region for the peak intensity of the vortex beam to the background intensity is much less than unity. A lens was used to project the images at the input and output faces onto a CCD camera.

When the diffuser is removed from the experimental setup, the vortex beam at the input face of the crystal is shown as Fig. 11(a). While a 2.5 kV biasing voltage is applied on the photorefractive crystal creating a Kerr-type self-focusing nonlinear medium, the vortex beam breaks up into two pieces [Fig. 11(b)]. This vortex break-up is due to the azimuthal instability. As the rotating diffuser is used, Fig. 11(c) clearly shows that the vortex light beam is stabilized by the reduction of the degree of coherence though two very unclear bright spots
still can be seen on the opposite sides (top and bottom) of the ring-like intensity distribution. The rotating
diffuser has been than moved further away from the focal point of the lens to make the light more incoherent,
Fig. 11(d) shows the generated stable partially incoherent vortex soliton at the output face of the crystal.

The propagation of partially incoherent optical vortices in a photorefractive medium has been studied
numerically by Jeng, Shih, Motzek, and Kivshar [2004] using the coherent density approach developed
by Christodoulides, Coskun, Mitchell, and Segev [1997]; Anastassiou, Soljacic, Segev, Eugenieva,
Christodoulides, Kip, Musslimani, and Torres [2000]. The coherent density approach is based on the fact
that partially incoherent light can be described by a superposition of mutually incoherent light beams that are
tilted with respect to the z-axis at different angles. One thus makes the ansatz that the partially incoherent light
consists of many coherent, but mutually incoherent light beams \( E_j \): \( I = \sum_j |E_j|^2 \). By setting \( |E_j|^2 = G(j\theta)I \),
where \( G(\theta) = (1/\sqrt{\pi\theta_0}) \exp(-\theta^2/\theta_0^2) \) is the angular power spectrum, one obtains a partially incoherent light
beam whose coherence is determined by the parameter \( \theta_0 \), i.e. less coherence means larger \( \theta_0 \). Here, \( j\theta \) is the angle at which the \( j \)-th beam is tilted with respect to the z-axis. A set of 1681 mutually incoherent vortices
was used in simulations, all initially tilted at different angles.

The top row in Figs. 11(a-d) shows numerical results for the propagation of an input Gaussian beam carrying
a phase dislocation [(a)] after the total propagation (9 mm) in a nonlinear medium for the coherent light [(b)]
and two different partially incoherent beams [(c,d)], corresponding to the values \( \theta_0 = 0.14 \) and \( \theta_0 = 0.29 \),
respectively. The most obvious difference to the scenario of the propagation the a coherent vortex is that the vortex
decay undergoes a visible delay when the degree of incoherence grows. Furthermore, in the incoherent
case the vortex changes its profile only very slowly as it propagates and thus can be considered as being in a
transition stage between the decay and stabilization.

Spatial coherence properties of optical vortices created in partially coherent light were studied by Motzek,
Kivshar, and Swartzlander Jr. [2004]; Motzek, Kivshar, Shih, and Swartzlander Jr. [2004], who revealed
the existence of phase singularities in the spatial coherence function of a vortex field that can characterize the
stable propagation of vortices through nonlinear media. Thus, the phase singularities of the spatial coherence
function predicted to exist in incoherent vortices propagating in linear media (Palacios, Maleev, Marthahay,
and Swartzlander [2004]) also survive the propagation through nonlinear media. The intensity distribution in the
far field shows a local minimum in the center of the beam, contrary to what one would obtain if the vortex was
propagating through a linear medium, and also in contrast to the result we would obtain if we were propagating
a light beam without topological charge. This emphasizes the importance of the interaction between coherence
and nonlinearity. Not only the phase structure, but also the intensity distribution strongly depends on the
initial form of the coherence function of the light beam.

The interaction of a coherent vortex beam with partially coherent fundamental soliton, similar to the vortex-
mode soliton discussed in Sect. 3.2, was considered recently by Motzek, Kaiser, Salgueiro, Kivshar, and Denz
[2004]. Strong destabilization and enhancement of azimuthal instability of vortex component is observed
for a low-amplitude incoherent beam. In the opposite limit, vortex can be stabilized by a large-amplitude
fundamental beam with the value of its incoherence above a certain threshold. These results are consistent with
the stabilization dynamics of a coherent vortex- and dipole-mode solitons (Yang and Pelinovsky [2003]).

§ 4. Multi-color vortex solitons

Similar to Kerr-type (or \( \chi^{(3)} \) nonlinear media, self-induced trapping of light occurs in quadratic (\( \chi^{(2)} \) nonlinear
media (Karamzin and Sukhorukov [1974, 1975]; Kanashov and Rubenchik [1981]). In this case, both spatial
and temporal multi-color solitons form through the mutual focusing and trapping of the waves parametrically
interacting in the nonlinear medium. Occurrence of self-focusing effects in quadratic nonlinear processes were
sporadically suggested under specific conditions, namely when the parametric interaction is weak resulting
in an effective third-order effect for the pump wave (Ostrovskii [1967]; Flytzanis and Bloembergen [1976]).
However, it took two decades before such effective nonlinearity-induced phase shift was identified experimentally
(Belashenkov, Gagarsky, and Inochkin [1989]; DeSalvo, Hagan, Sheik-Bahae, Stegeman, Vanstryland, and
Vanherzeele [1992]), and until the importance of the associated, so-called cascaded nonlinearities was properly
appreciated by Stegeman, Sheik-Bahae, Vanstryland, and Assanto [1993] (for a review, see Stegeman, Hagan,
and Torner [1996]). Since then, formation of spatial and temporal multi-color solitons has been observed experimentally
in a variety of physical settings (Torner and Stegeman [2001]; Buryak, Di Trapani, Skryabin, and Trillo [2002])
after the pioneering observations in the case of second-harmonic generation (SHG) by Torruellas, Wang, Hagan,
Vanstryland, Stegeman, Torner, and Menyuk [1995] in a crystal of potassium titanium phosphate (KTP), and by Schiek, Baek, and Stegeman [1996] in a planar waveguide made of lithium niobate (LiNbO3).

Cascaded nonlinearities can be modelled by the standard theory of the \( \chi^{(2)} \)-mediated three-wave mixing
described in detail in several books on nonlinear optics (Shen [1984]; Butcher and Cotter [1992]; Boyd [1992]),
including more complex multistep parametric processes in quadratic media (Saltiel, Sukhorukov, and Kivshar
[2004]). A comprehensive review on optical quadratic solitons was published by Buryak, Di Trapani, Skryabin,
by introducing the angles \( \rho \)

Here the Laplace operator \( \Delta \) acts on a transverse coordinates (\( x, y \))

\( \alpha \)

\( \omega \)

\( \Delta \)

\( E \)

\( j \)

\( \sum \)

\( \exp(ik_j z - i\omega_j t) + c.c. \)

\( \omega_1 + \omega_2 = \omega_3 \)

\( \rho_j \ll 1 \)

\( k \)

\( \omega \)

\( \omega_1 \)

\( \omega_2 \)

\( \omega_3 \)

\( \delta \)

\( j \)

\( a \)

\( \beta \)

\( 1 \)

\( 2 \)

\( 3 \)

\( \Delta k = k_1(\omega_1) + k_2(\omega_2) - k_3(\omega_3) \)

\( \Delta \)

\( r \)

\( \pi/\Delta k \)

\( \rho_j \)

\( \omega \)

\( |a|^2 \sim 10 \)

\( k_{1r} \)

\( \delta_j \)

\( k_j \)

\( \alpha_j \)

\( 15 \mu m \)

\( 2.5 \mathrm{mm} \)

\( \beta \pm 3 \)

\( \pi/|\Delta k| \sim 2.5 \mathrm{mm} \)

\( |a|^2 \sim 10 \)

\( 1 - 10 \mathrm{GW/cm}^2 \)

\( \mathrm{KTP} \)

\( \omega \)

\( 2\omega \)

\( \beta \ll 1 \)

\( \beta \gg 1 \)

\( \beta \)

\( \Delta \)

\( 2\omega \)

\( a_1 = a_2 = u \)

\( a_3 \exp(i\beta z) \)

\( \Delta \Delta u + u^* v = 0 \)

\( \Delta \bar{v} - i\delta \Delta \bar{v} - \beta v + \frac{1}{4} \Delta^2 u + u^2 = 0 \)
\[ \frac{1}{\beta} \frac{\partial u}{\partial z} + \frac{1}{2} \Delta u + \frac{1}{\beta} |u|^2 u \simeq 0, \]

which poses well-known stable soliton solutions in one-dimensional geometries. However, it is worth stressing that most quadratic solitons occur under conditions where the above reduction does not hold. One obvious example is the case analyzed here of ring-profile vortex solitons, which exist in two-dimensional geometries. Similarly, the above derivation does not hold near phase-matching, whenever the second-harmonic waves are intense, and in general with high enough light intensities. However, these are the conditions where most quadratic solitons are generated in practice. Therefore, the analogy indicated by Eq. (4.6) must be used with the proper understanding of its important limitations in the interpretation of most experiments.

4.2. FREQUENCY DOUBLING WITH VORTEX BEAMS

Second-harmonic generation is a particular case of frequency conversion processes associated with the energy transfer between several waves propagating in a nonlinear medium. The nonlinear wave-mixing obeys conservation of energy, linear momentum, and, under proper conditions, angular momentum. In the general case of frequency mixing in which two fields of optical frequencies \( \omega_1 \) and \( \omega_2 \) combine to produce a third field of frequency \( \omega_3 \), conservation of energy requires the condition \( \omega_1 + \omega_2 = \omega_3 \). Conservation of the linear momentum leads to the phase-matching requirement \( k_1 + k_2 = k_3 \).

Under proper conditions and definitions, the angular momentum carried by the light beam must also be conserved. Conservation of spin angular momentum imposes constraints in the polarization of the input and output light beams. In the case of co-linear interaction of paraxial light beams in a quadratic medium, conservation of the paraxial beam orbital angular momentum holds too, and affects the spatial shape of the generated beams. The simplest situation occurs in the case of Type I frequency doubling of single Laguerre-Gaussian (LG) pump modes (Basistiy, Bazhenov, Soskin, and Vasnetsov [1993]; Dholakia, Simpson, Padgett, and Allen [1996]; Soskin and Vasnetsov [1998]; Allen, Padgett, and Babiker [1999]). In this case, the winding number of the LG pump modes must double, because of the azimuthal symmetry of the governing equations. In the general case of Type II phase-matching or frequency-mixing of single LG modes with winding numbers \( m_j, j = 1, 2, 3 \), the symmetry of the equations leads to \( m_1 + m_2 = m_3 \), as it is observed experimentally by Berzinskis, Matijosius, Piskarskas, Smilgevicius, and Stabinis [1997].

However, conservation of orbital angular momentum in parametric processes with arbitrary input beams and general phase-matching geometries does not necessarily translate into simple algebraic rules between the winding numbers of the vortices present in the beam. Illustrative cases are multi-mode and complex pump beams (Petrov and Torner [1998]; Berzinski, Matijosius, Piskarskas, Smilgevicius, and Stabinis [1998]; Beržanski, Piskarskas, Smilgevičius, Stabinis, and Di Trapani [1999]; Petrov, Molina-Terriza, and Torner [1999]; Molina-Terriza and Torner [2000]; Jarutis, Matijosius, Smilgevicius, and Stabinis [2000]; Stabinis, Orlov, and Jarutis [2001]), or the generation of vortex-streets by the presence of Poynting-vector walk-off (Molina-Terriza, Torner, and Petrov [1999]; Molina-Terriza, Petrov, Recolons, and Torner [2002]). Actually, the orbital angular momentum of a light beam is not necessarily directly related to the properties of the vortices that it contains.

Notice also that, in general, the angular momentum at the classical level is an overall property of the light beam, and must be clearly distinguished from the quantum angular momentum at the single photon level. That in the latter case the conservation of orbital angular momentum in parametric processes is not necessarily given by simple algebraic rules is most clearly illustrated in non-collinear geometries (Molina-Terriza, Torres, and Torner [2003]), or in the so-called transverse-emitting processes (Torres, Osorio, and Torner [2004]).

In what follows we concentrate in vortex solitons, which are intense light beams carrying energies which might exceed tens of \( \mu J \) at visible or near-infrared wavelengths, thus far from any single-photon effects. In the next subsections, we discuss the families and stability properties of quadratic vortex solitons — nonlinear optical beams with strong energy exchange between its constituents.

4.3. FAMILIES OF THE VORTEX SOLITONS

Because the Type II phase-matching process involves three beams, there is a wider variety of different solutions than in the case of the Type I SHG geometries, which we will regard as a limit of degeneracy of a former model when both fundamental frequency (FF) beams are of the same polarization, see Sect. 4.1. The Type II model described by Eqs. (4.1)–(4.3) defines an infinite-dimensional Hamiltonian system with a conserved Hamiltonian which in the absence of walk-off \( (\delta_j = 0) \) is given by

\[ H = \frac{1}{2} \int \left( \frac{1}{2} \sum \alpha_j |\nabla_\perp A_j|^2 + \beta |A_3|^2 - A_1^* A_2 A_3 - A_1 A_2 A_3 \right) dr_\perp, \]
Figure 12: (a)-(d) Typical solutions for the vortex solitary waves with different combinations of topological charges, \( m_3 = m_1 + m_2 \). Solid line: ordinary-polarized fundamental beam; dashed line: extraordinary-polarized fundamental beam; dotted line: second-harmonic field. Parameters are: \( \beta = 3 \), \( \kappa_1 = 2 \), and \( I_u = 0 \). Notice that in (c) the curves corresponding to two fundamental beams are identical. (e) Families of the lowest-order vortex solitons presented through the energy flow-Hamiltonian diagram. Plot (f) is a zoom of the corresponding region of plot (e). Numbers in parentheses stand for the topological charges of the two fundamental beams, i.e., \((m_1, m_2)\). The curve labelled (0,0) corresponds to the family of lowest-order, vorticity-less bright solitons Torres, Soto-Crespo, Torner, and Petrov [1998b].

where \( A_{1,2} = a_{1,2} \), and \( A_3 = a_3 \exp(-i\beta z) \). We use two additional conserved quantities of the beam evolution, namely the total beam power or energy flow \( I \),

\[
I = \sum I_j = \int \left( \frac{1}{2} |A_1|^2 + \frac{1}{2} |A_2|^2 + |A_3|^2 \right) dr, \tag{4.8}
\]

and the energy imbalancing \( I_i \),

\[
I_i = \frac{1}{2} \int (|A_1|^2 - |A_2|^2) dr. \tag{4.9}
\]

The conservation of \( I_i \) means that the energy transfer between the fundamental waves and the second-harmonic wave cannot favor any of the two orthogonal polarizations that compose the fundamental beam. Notice that strictly speaking the above expressions hold only for continuous-wave light propagation, and that the temporal effects on pulsed light might introduce important new features in the imbalancing of the quadratic solitons (Minardi, Yu, Blasi, Varanavicius, Valiulis, Berzanskis, Piskarskas, and Di Trapani [2003]). In the absence of the Poynting vector walk-off, the total beam orbital angular momentum, defined as

\[
M = \frac{1}{4i} \int \left\{ \left[ r_\perp \times \sum (A_j^* \nabla_\perp A_j - A_j \nabla_\perp A_j^*) \right] e_z \right\} dr, \tag{4.10}
\]

is also conserved during the beam evolution. For our purposes, it is convenient to investigate configurations without walk-off, and we hereafter set \( \delta_j = 0 \).

Spatial optical solitons are optical beams with constant transverse profile along the propagation direction, defined as stationary solutions of the corresponding propagation equations Eqs. (4.1)–(4.3). From this definition, it follows that we may look for the soliton solutions in a form of the generic ansatz, \( a_j = V_j(x,y) \exp(i\kappa_j z) \), where \( \kappa_j \) are the nonlinear corrections to the corresponding propagation constants.

Stationary propagation of the multi-color solitons requires vanishing power exchange between the fundamental and second-harmonic waves; to avoid this exchange the condition \( \kappa_3 = \kappa_1 + \kappa_2 + \beta \) applies. Radially symmetrical solutions are found by separation of variables in the form

\[
V_j(x,y) = U_j(\rho) \exp(im_j \varphi), \tag{4.11}
\]

with real amplitudes \( U_j \) depending on the polar radius \( \rho = \sqrt{x^2 + y^2} \) and phases being linear functions of the azimuthal coordinate \( \varphi = \arctan(y/x) \). Using this ansatz and the algebraic constraint to topological charges
\[ m_1 + m_2 = m_3, \]

we obtain the \( z \)-independent (stationary) version of Eqs. (4.1)–(4.3):

\[ \alpha_j \frac{d^2}{dp^2} + \frac{1}{\rho} \frac{d}{dp} - \frac{m_j^2}{\rho^2} \right) U_j + U_p U_q = \kappa_j U_j, \]

(4.12)

with \( j, p, q \in \{1, 2, 3\} \) and \( j \neq p \neq q \). For a fixed set of the topological charges, solutions depend on two parameters (e.g. \( \kappa_1 \) and \( \kappa_2 \)), which correspond to different total and relative (imbalancing) energy flows between the three interacting waves (Buryak, Kivshar, and Trillo [1996]; Buryak and Kivshar [1997]; Peschel, Etrich, Lederer, and Malomed [1997]).

Properties of the fundamental (bell-shaped) quadratic spatial solitons have been described in a number of studies (see, e.g., the review paper by Buryak, Di Trapani, Skryabin, and Trillo [2002]), and instabilities of higher-order vorticityless modes with central peak and one or more surrounding rings are also known (Skryabin and Firth [1998b]). We do not consider these issues here and focus on the “doughnut”–shaped vortex solitons.

Families of vortex solitons as solutions of Eqs. (4.12), for different combinations of the topological charges, wave vector mismatches, and both zero and nonzero imbalancing \( I_i \), have been found by Firth and Skryabin [1997]; Torres, Soto-Crespo, Torner, and Petrov [1998b]; Skryabin and Firth [1998a]; Molina-Terriza, Torres, Torner, and Soto-Crespo [1998]. Figure 12 shows typical shapes of vortex solitary waves with different combinations of topological charges. We choose a particular case of zero imbalancing \( I_i = 0 \) because it also covers the solutions with equal amplitudes for both FF beams, such as that shown in Fig. 12(c). This particular branch corresponds to stationary solutions for the Type I geometry Eqs. (4.4)–(4.5), consisting of only two components. The latter ones have been studied in detail by Firth and Skryabin [1997]; Torres, Soto-Crespo, Torner, and Petrov [1998a]; Skryabin and Firth [1998a].

Useful information about the soliton families is given by integrals of motion defined above; for the stationary solutions under consideration the angular momentum and Hamiltonian can be expressed in terms of the beam powers,

\[ M = \frac{1}{2} \{(m_1 + m_2)I + (m_1 - m_2)I_1\}, \]

(4.13)

\[ H = -\frac{1}{2} \{(\kappa_1 + \kappa_2)I + (\kappa_1 - \kappa_2)I - \beta I_3\}. \]

(4.14)

Note that the vectorial nature of three-wave multi-color vortex solitons allows combinations including vorticityless beam, e.g. \( m_1 = 1 \) and \( m_2 = 0 \), or having zero total angular momentum \( M = 0 \) (e.g. \( m_1 = -m_2 \) and \( I_1 = 0 \)), similar to their \( \chi^3 \) counterparts described in Sect. 3. However, the two-component solutions in the Type I model are always limited by the constraints \( \kappa_1,2 \equiv \kappa, m_3 = 2m \) with \( m_1,2 \equiv m \), and \( \beta = 0 \), therefore \( M = mI \), similar to the scalar \( \chi^3 \) spatial soliton. Figures 12 (e)-(f) show some examples of the Hamiltonian dependencies Eq. (4.14) in the case \( I_i = 0 \), the similar plots for the nonzero imbalancing are available in Molina-Terriza, Torres, Torner, and Soto-Crespo [1998].

4.4. SPONTANEOUS BREAK-UP: AZIMUTHAL INSTABILITY

Generation of different vortex patterns due to the frequency conversion, described in Sect. 4.2, occurs for the input powers of the FF pump beam below some threshold. For higher input powers, which are sufficient for the soliton formation, the generation of sets of simple fundamental solitons was predicted numerically for SHG by Petrov and Torner [1997]; Torner and Petrov [1997a,b]. Such phenomenon is related to the azimuthal modulational instability of the corresponding stationary states – ring-shaped vortex solitons. The linear stability analysis of the Type I vortex solitons was performed by Firth and Skryabin [1997]; Skryabin and Firth [1998a] and Torres, Soto-Crespo, Torner, and Petrov [1998a]. The experimental confirmation of the spontaneous break-up of optical vortex solitons in quadratic crystal was reported by Petrov, Torner, Martorell, Vilaseca, Torres, and Cojocaru [1998].

To outline briefly the main steps of these calculations, we examine the stability of the vortex solitary waves against azimuthal perturbations and seek the perturbed solutions of the form

\[ a_j = \{U_j(r) + \epsilon [f_{j,s}(r, z) \exp(is\varphi) + g_{j,s}(r, z) \exp(-is\varphi)]\} \exp(ik_j z + im_j \varphi), \]

(4.15)

where \( s, f_{j,s}, \) and \( g_{j,s} \) stand for the azimuthal index and the envelopes of the perturbation eigenfunctions, respectively. Inserting Eq. (4.15) into Eqs. (4.1)–(4.3) and linearizing the equations in respect to small perturbations, we obtain a set of six coupled linear partial differential equations for \( f \) and \( g \) at a given value of \( s \). Such equations have many different solutions; some of them, the so-called instability modes, display exponential growth along the propagation direction. To obtain such solutions, one can use the method of averaging the growth rate of perturbation over the propagation direction described by Soto-Crespo, Heatley, Wright, and Akhmediev [1991]; Soto-Crespo, Wright, and Akhmediev [1992], or further reduce the problem by setting \( \{f; g\} = \exp(\Gamma z)\{\tilde{f}(r); \tilde{g}(r)\} \) and solving the corresponding boundary value problem for \( \tilde{f} \) and \( \tilde{g} \) (see also the
4. VORTICES IN $\chi(2)$ MEDIA

Figure 13: (a)-(d) Instability growth rate for perturbations with different azimuthal indices as a function of the nonlinear wave number shift $\kappa_1$, for the various families of vortex solitary waves shown in Fig. 12 (Torres, Soto-Crespo, Torner, and Petrov [1998b]). (e)-(f) Stroboscopic view of the decay of an exact vortex solitary wave solution in the presence of the corresponding exact azimuthal perturbation of a given index. The plots show the light patterns of the ordinary polarized fundamental beams calculated at $z=4, 8, 12, 16$ propagation units, when the input is the field of Fig. 12 plus the corresponding symmetry-breaking perturbation with azimuthal index $s=3$. Amplitude of the added perturbation: in (e) $\epsilon = 10^{-2}$ and in (f) $\epsilon = 10^{-3}$. The extraordinary polarized fundamental and the second-harmonic beams exhibit similar features and thus are not shown (Torner, Torres, Petrov, and Soto-Crespo [1998]).

4.5. INDUCED BREAK-UP: SOLITON ALGEBRA

The process of vortex beam break-up by azimuthal modulational instabilities is spontaneous, and thus governed by the perturbations that happen to have the highest growth rate. This process produces beautiful patterns of solitons flying off the input vortex ring, but leaves little control over the number of spots present in such soliton pattern. Vortex ring-shaped solitons might be broken in a controllable manner by inducing their splitting in a way that favors predetermined azimuthal symmetries.

One way to favor a given azimuthal symmetry is to impose a specific phase-pattern to the input beam. In the case of parametric interactions in quadratic media, such goal can also be accomplished by seeding the SHG process with a vortex in the SH frequency: The azimuthal phase-varying relation between the interacting beams generates a prescribed azimuthal symmetry of energy exchange between the beams that produces the desired soliton pattern (Torner, Torres, Petrov, and Soto-Crespo [1998]).

Such a process was termed soliton algebra, and is implemented by changing the topological charge of a weak, co-linear input SH seed beam. If this topological charge $m_3$ satisfies the condition $m_3 = m_1 + m_2$, where $m_{1,2}$ are the charges of two orthogonally polarized FF pump beams, the generation and subsequent spontaneous azimuthal instability of three-wave vortex solitons occurs. However, if the condition above is violated, $m_3 \neq m_1 + m_2$, the beam break-up into solitons is induced by the local, azimuthally-varying phase difference that exists between
4. VORTICES IN $\chi(2)$ MEDIA

Figure 14: Results of the induced break-up of the input beams containing topological charges: (a) [0,0,1], (b) [1,1,0], (c) [1,1,-1], (d) [2,2,0], (e) [2,2,-1], and (f) [2,2,-2]. Input energy flows: at the fundamental frequency $I_1 = I_2 = 36\pi$ for (a)-(c) and $I_1 = I_2 = 128\pi$ for (d)-(f); at the second harmonic $I_3 = 2\pi$. The plots show the ordinary-polarized beams at $z = 10$ (Torner, Torres, Petrov, and Soto-Crespo [1998]).

the pump and the seed signals and hence by the initial local direction of the energy flow between the FF and SH waves. The number of solitons formed in each portion depends on the input light conditions, such as the total energy flow and SH seed intensity. As a result, the information coded in the value of the input array $(m_1, m_2, m_3)$ is transformed into a certain number of output soliton spots (Torner, Torres, Petrov, and Soto-Crespo [1998]). Different typical output patterns are presented in Fig. 14.

The experimental demonstration of the concept of soliton algebra were performed in a Type I SHG geometry by Minardi, Molina-Terriza, Di Trapani, Torres, and Torner [2001]. Under properly chosen operating conditions, the number of generated solitons $n$ was shown to follow the rule $n = |2m_{FF} - m_{SH}|$, with $m_{FF}$ and $m_{SH}$ being the topological charges of the input FF and SH beams, respectively. When $2m_{FF} = m_{SH}$, the beam break-up is known to occur through spontaneous azimuthal modulation instability, therefore the input was designed in such a way that $2m_{FF} \neq m_{SH}$. Both processes, spontaneous or induced, require features similar to those needed for the generation of solitons with Gaussian-like beams and thereby is found to be very robust.

In the context of soliton control by the presence of phase dislocations in the input beams, we notice the observation of the deflection of multi-color solitons generated by edge-like topological amplitude and phase dislocations reported by Petrov, Carrasco, Molina-Terriza, and Torner [2003]. The experiments were conducted near phase-matching in a bulk potassium titanium phosphate crystal pumped with picosecond light pulses at 1064 nm, and the angular deflection of the solitons was found to be controllable through the position of the edge dislocation.

4.6. DARK MULTI-COLOR VORTEX SOLITONS

Existence of multi-color dark vortex solitons has been discussed by Alexander, Buryak, and Kivshar [1998] who analyzed also some basic properties of such beams which were found to be highly unstable against modulational instabilities of their nonvanishing background (Buryak, Di Trapani, Skryabin, and Trillo [2002]). Such instability is known to be suppressed by a strong effect of competing nonlinearities (Alexander, Buryak, and Kivshar [1998]; Alexander, Kivshar, Buryak, and Sammut [2000]), but such prediction did not find yet experimental verification. Nevertheless, significant efforts have been put to overcome modulational instability due to parametric wave interaction and to generate dark vortex-carrying beams. The most important advance was reported by Di Trapani, Chinaglia, Minardi, Piskarskas, and Valiulis [2000], who used a large walk-off between the components to quench the modulational instabilities. In the reported experimental observations, the SH beam broke-up and formed many spikes, having an energy content much larger than the rest of the SH beam carrying a phase dislocation. Because of the large walk-off, the spikes propagated out of the beam rapidly. The generation of such spikes was claimed to be essential for self-quenching of the instability process. The experiment was performed with negative large phase-mismatch, where at moderate powers the cascading nonlinearities leads to an effective defocusing Kerr nonlinearity (Eq. (4.6)), known to support stable dark vor-
Di Trapani, Chinaglia, Minardi, Piskarskas, and Valiulis [2000], the generated SH vortex dislocation exhibited a size much smaller than the host diffracting beam, indicating at least a transient trapping effect. The corresponding numerical simulations performed by Di Trapani, Chinaglia, Minardi, and Valiulis [1999] confirmed such transient self-trapping and the propagation of the dislocation in the fundamental frequency beam for a distance of the order of several diffraction lengths without core spreading. In contrast, the linear diffraction of the beam results in strong spreading of the vortex core. The vortex beams observed in the experiments exhibited some features expected from a dark vortex soliton, but a comprehensive investigation of these important observations is not yet fully developed. Thus, the existence of true stable dark vortices in a quadratic nonlinear medium remains an open problem.

§ 5. Stabilization of vortex solitons

In this section we discuss several theoretical predictions of the stabilization of bright vortex solitons in nonlinear media. Several models supporting stable vortex solitons were suggested, e.g. the Kerr media made of alternating self-focusing and self-defocusing layers (Towers and Malomed [2002]; Montesinos, Perez-Garcia, and Michinel [2004]; Montesinos, Perez-Garcia, Michinel, and Salgueiro [2004]; Adhikari [2004]) and nonlocal self-focusing medium (Yakimenko, Zaliznyak, and Kivshar [2004]; Breidis, Petersen, Edmundson, Krolikowski, and Bang [2005]). In particular, we discuss here two distinct models with so-called competing nonlinearities. The first model includes self-focusing cubic and self-defocusing quintic terms in the power-law Kerr-type nonlinearity, whereas the second model includes phase-dependent quadratic and self-defocusing cubic nonlinear interaction. We also summarize numerical results demonstrating stable spatiotemporal vortex solitons in the $(3 + 1)$-dimensional geometry, the so-called spinning light bullets.

5.1. CUBIC-QUINTIC NONLINEARITY

Nonlinear models discussed in previous sections correspond to the lowest order nonlinearities available, namely to the first two nonlinear terms in expansion of optical medium polarization $P = \chi^{(1)}E + \chi^{(2)}E^2 + \chi^{(3)}E^3 + \ldots$. For the centro-symmetric media all even terms vanish and taking into account higher-order terms one can represent the refractive index as a power-law Kerr-type nonlinearity, $n = n_0 + n_2 I + n_4 I^2 + \ldots$, here the intensity $I \equiv |E|^2$. Obviously, if nonlinear coefficients $n_2$ and $n_4$ have the same signs, corresponding models exhibit simple increasing of strength of nonlinear self-action, self-focusing for $n_2, n_4 > 0$ or self-defocusing for $n_2, n_4 < 0$. More interesting situation occurs when two nonlinear contributions have opposite signs, $n_2 n_4 < 0$, this case usually refereed as “competing” cubic-quintic (CQ) nonlinearity: corresponding nonlinear terms in propagation equation being of the third and fifth orders:

\[
\frac{i}{\partial z} \frac{\partial E}{\partial z} + \Delta E + n_2 |E|^2 E + n_4 |E|^4 E = 0. \tag{5.1}
\]

Let $n_4$ be of self-focusing type, $n_4 > 0$. Then, for any sign of Kerr contribution $n_2$, there will be a threshold power of (transversely two-dimensional) light beam $E$, when higher-order self-focusing will predominate both the linear diffraction and $n_2$ contribution. In this case the beam will collapse, similar to the pure Kerr case with $n_2 > 0$ and $n_4 = 0$ (see recent review by Berge [1998]). However, if $n_4 < 0$, than the collapse can be stopped, because the parts of light beam with high enough intensity $I > I_{th}$ experience effectively self-defocusing environment, $dn/\partial I < 0$. Here the threshold intensity is given by $I_{th} = -n_2/(2n_4)$. Furthermore, the CQ nonlinearity can be regarded as a power-law expansion for any collapse-free nonlinearity with saturation, for example the phenomenological one discussed above, $n = n_0 + n_2 I/(1 + s I)$. In this case $n_4 = -s n_2$, and the CQ medium is refereed also as a saturable one.

Saturable nonlinearity offered a great advantage over the conventional cubic one because it supports stable solitons, free of collapse instability. That is why its simplest version, the CQ nonlinearity, attracts significant attention of theoreticians from the early days of nonlinear optics (Zakharov, Sobolev, and Synakh [1971]). Numerical studies have confirmed the stability of fundamental spatial solitons in this system (Wright, Lawrence, Torruellas, and Stegeman [1995]; Dimitrevski, Reinhult, Svensson, Ohgren, Anderson, Berntson, Lisak, and Quiroga-Teixeiro [1998]; Quiroga-Teixeiro, Berntson, and Michinel [1999]). Experimentally a CQ nonlinear dielectric response with positive cubic and negative quintic contributions has been observed in chalcogenide glasses (Smektal, Quemard, Couderc, and Barthelemy [2000]; Boudebs, Cherukulappurath, Leblond, Troles, Smektala, and Sanchez [2003]), and in organic materials (Zhan, Zhang, Zhu, Wang, Li, Li, Lu, Zhao, and Nie [2002]). However, in all these cases the quintic nonlinearity is accompanied by significant higher-order multiphoton processes such as two-photon absorption, therefore the validity of the CQ models to light propagation in these materials requires additional explorations. Very recently, the criteria for the experimental
Figure 15: Stationary solutions to the Eq. (5.1) for different topological charges $m$ and soliton parameter $\kappa$ (shown next to the corresponding curves). For $m = 1$, we also show the example of two types of optical vortices, localized (bright) and non-localized (dark), to coexist in media with competing nonlinearities. Similar solutions exist for any number $m$. Note that the slopes of envelopes for both types, i.e. the size of the vortex core, is sufficiently different for the case of $\kappa = 0.1$, and almost exactly the same for $\kappa = 0.18$, indicating the transformation of bright to dark vortex with increase of power, see text for the details.

observation of multidimensional solitons in CQ type saturable media were developed by Chen, Beckwitt, Wise, and Malomed [2004].

The “auto-waveguide” propagation of the “spiral beams” with nonzero topological charge has been predicted by Kruglov and Vlasov [1985] for pure Kerr model. As early as 1988, it was found that saturation in the form of CQ nonlinearity stabilize optical vortices against collapse, and “the data from the computer experiment show that these beams are stable” (after Kruglov, Volkov, Vlasov, and Drits [1988]). Further study, however, reveal that the azimuthal instability of CQ vortices may take place (Kruglov, Logvin, and Volkov [1992]). A key insight was put forward by QuirogaTeixeiro and Michinel [1997], who found by numerical simulations that the vortex solitons with charge $m = 1$ could be stable provided that the power of the beam is over some critical value. Fig. 15 shows several examples of the bright and dark vortex soliton solutions to Eqs. (5.1) coexisting in the CQ model (Berezhiani, Skarka, and Aleksic [2001]). Note that with increase of power the amplitude of the solutions saturates and starting from some value of the soliton parameter (or, equivalently, some value of soliton power), the slope of the bright vortex soliton coincide with the one for dark vortex. That may indicate the transition from unstable vortices in self-focusing regime to the stable ones in effectively self-defocusing regime. The critical power for this transition has been found analytically by Michinel, Campo-Taboas, Quiroga-Teixeiro, Salgueiro, and Garcia-Fernandez [2001], it was shown to exceed four times the threshold power of the generation for vortex soliton with charge $m = 1$.

The issue of the stability of vortex solitons in CQ model was put on more solid mathematical grounds by calculating the linear stability spectrum by Towers, Buryak, Sammut, Malomed, Crasovan, and Mihalache [2001] and Skarka, Aleksic, and Berezhiani [2001]. The growth rate of small azimuthal perturbations was found to be nonzero for a limited domain $0 < \kappa < \kappa_{\text{stab}}$, with linearly stable solutions above the critical value $\kappa > \kappa_{\text{stab}}$. In addition, very small instability with respect to shift of the dislocation core (central dark spot) was found by Towers, Buryak, Sammut, Malomed, Crasovan, and Mihalache [2001]; Malomed, Crasovan, and Mihalache [2002]. Nevertheless, the supercritical vortex solitons appear to be strong attractors, the gaussian beam with nested phase dislocation may initially break to several splitters but then restores radially symmetric vortex shape (Skarka, Aleksic, and Berezhiani [2001]). A mathematically rigorous stability analysis was performed by Pego and Warchall [2002], who predicted the stability of higher-order vortex solitons with $m > 1$ as well, this issue was also addressed recently by Davydova and Yakimenko [2004]. A detailed study of the stability of (2+1) dimensional vortex solitons in both conservative and dissipative CQ models can be found in Crasovan, Malomed, and Mihalache [2001b], the related issue of the stability of spatio-temporal (3+1)-dimensional spinning light bullets we discuss in Sect. 5.3.

Formal analogies between the CQ vortex solitons and quantum fluids have been discussed by Michinel, Campo-Taboas, Garcia-Fernandez, Salgueiro, and Quiroga-Teixeiro [2002] (see also comments by Coffey [2002]; Weiss [2003]). In particular, analogies have been drawn between the collisional dynamics of vortex solitons and surface tension properties by Paz-Alonso, Olivieri, Michinel, and Salgueiro [2004]. Such features can be accurately explained by the internal oscillations of spatial solitons in the domain of their stability (Dong, Ye, Wang, Cai, and Li [2004]).

To conclude this section we note that competing nonlinearities of different kinds have been suggested. Examples include thermal mechanism, studied by Kruglov, Logvin, and Volkov [1992], or nonlinearity of the form $n = 1 + n_2 I - n_K K^2$ (Skarka, Aleksic, and Berezhiani [2003]). In the latter case, it was suggested that for intense laser pulses in air the parameter $K$ might be as high as $K = 20$, albeit propagation of intense pulses in air usually involves a variety of strong multiphoton processes not captured by the above reduced model.
5.2. QUADRATIC-CUBIC NONLINEARITY

As discussed previously, bright doughnut-shaped vortex solitons in pure $\chi^{(2)}$ media are unstable against azimuthal symmetry-breaking perturbations, similar to their $\chi^{(3)}$ counterparts in self-focusing media. This property might be a generic feature of bright vortices in the nonlinear models where the balance between counteracting self-focusing type nonlinearity and repulsive diffraction “forces” allows stationary radially-symmetric states, but it is not sufficient to damp the azimuthal modulational instability along the ring. The dark vortices in self-defocusing $\chi^{(3)}$ medium, however, are the stable entities provided their nonzero constant background, the stationary plane wave solutions, is stable against small modulation.

The competing $\chi^{(3)} - \chi^{(5)}$ nonlinearities, discussed in Sect. 5.1, were shown to possess both, the ability to localize the bright solitons, and to guarantee additional stabilization in certain parameters region. Expanding the analogy to the case of parametric solitons, one may expect stable vortex solitons to exist in $\chi^{(2)}$-mediated wave mixing, if there will be possible additional nonlinearity to compete, for example of the $\chi^{(3)}$ self-defocusing type.

To start with, the dynamical equations that govern the interaction between a weakly modulated plane wave and its second harmonic for materials with asymmetric crystal structure, in which the effects of both the quadratic and the cubic nonlinear susceptibility tensors must be considered, were derived by Bang [1997]. Following these derivations and taking into account the diffraction in two transverse dimensions and paraxial approximation, one can describe soliton-like propagation of narrow beams:

\begin{align*}
    i \frac{\partial u}{\partial z} + \Delta u - \kappa_1 u + u^* v - \left( \frac{|u|^2}{4} + 2|v|^2 \right) u &= 0, \\
    2i \frac{\partial v}{\partial z} + \Delta v - \kappa v + u^2/2 - \left( 4|u|^2 + 2|v|^2 \right) u &= 0,
\end{align*}

(5.2) \hspace{1cm} (5.3)

where $\kappa_1$ is the nonlinear contribution to the propagation constant for the fundamental wave $u$, and parameter $\kappa$ combines it with the phase-mismatch $\Delta k$, $\kappa = 2(\Delta k + 2\kappa_1)$.

Stable fully localized $(2 + 1)$-dimensional ring solitons with intrinsic vorticity in optical media with competing quadratic and self-defocusing cubic nonlinearities have been found by Towers, Buryak, Sammut, and Malomed [2001]. It is noteworthy that properties of the stationary solutions to Eqs. (5.2)-(5.3) are very similar to those shown in Fig. 15: with increasing of power, the amplitude saturates and soliton width diverges to the dark state. Stability windows for sufficiently broad ring solitons with the spin $m = 1$ and $2$ have been found, both in direct dynamical simulations and analyzing eigenvalues of the linearized equations. Similar to their $\chi^{(3)} - \chi^{(5)}$ counterparts, stable two-color vortex solitons survive strong perturbations such as collisions, as it was shown by Malomed, Peng, Chu, Towers, Buryak, and Sammut [2001].

It is necessary to say that conventional nonlinear materials with strong $\chi^{(2)}$ nonlinearity do not satisfy the requirement of the model to have a negative $\chi^{(3)}$ coefficient at both the fundamental and second-harmonic frequencies. Different possibilities to create a necessary effective $\chi^{(3)}$ nonlinearity have been proposed. For example, Malomed, Peng, Chu, Towers, Buryak, and Sammut [2001] suggested by creating a layered medium in which layers providing for the $\chi^{(2)}$ nonlinearity periodically alternate with others that account for the self-defocusing Kerr nonlinearity. Engineering $\chi^{(2)}$ quasi-phase-matched gratings (Bang, Clausen, Christiansen, and Torner [1999]) also produces effective $\chi^{(3)}$ nonlinearities. However, notice that in this case the higher-order nonlinearities are induced on average over all the Fourier components associated to the phase-matching modulation, thus the models only hold under proper conditions when averaging is justified and thus these models might not be able to stabilize otherwise unstable solitons.

A modulationally stable branch of a plane two-wave solutions to the system Eqs. (5.2)-(5.3) and the corresponding stable dark vortex solitons were found by Alexander, Buryak, and Kivshar [1998]. Latter on, Alexander, Kivshar, Buryak, and Sammut [2000] reported existence of novel vortex states on infinite background, the so-called “halo-vortex” and “ring-vortex”. It is interesting to note that in a similar model there exist a kind of bright vortex states, localized in transverse plane by additional harmonic trapping potential; such system might be perhaps used to describe some features of hybrid atomic-molecular Bose-Einstein condensates (Alexander, Ostrovskaya, Kivshar, and Julienne [2002]).

5.3. SPATIOTEMPORAL SPINNING SOLITONS

Three-dimensional optical spatiotemporal solitons, the so-called “light bullets” (LB, this term was introduced by Silberberg [1990]), attract a growing interest, as they represent a new fundamental physical object. They have been suggested to implement ultra-fast all-optical switching in bulk media (McLeod, Wagner, and Blair [1995]; Liu, Beckwitt, and Wise [2000]; Wise and Di Trapani [2002]). Physical content of this new object lies in the spatiotemporal analogy, which allows one to consider in the same way both, the temporal dispersion of the short light pulse, and the diffraction (or “spatial dispersion”) of the narrow beam (see, e.g., paper by Kanashov and Rubenchik [1981]). In the presence of group-velocity dispersion, the evolution (along propagation direction
z) of slowly-varying envelope of the electromagnetic field \( E(x, y, z; t) \) is described by the paraxial equations similar to the Eq. (5.1) and Eqs. (5.2)-(5.3), but with time-dependent Laplacian \( \Delta = \nabla_x^2 + k DE \tau T \). Here \( k \) is the propagation constant (wave number), \( D = -\partial^2 \kappa / \partial \omega^2 > 0 \) is the coefficient of the temporal dispersion assumed anomalous, \( T \equiv t - z/v_g \) (\( v_g \) being the group velocity of the carrier wave) is the “reduced time”, and \( \nabla_x^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 \) represents spatial diffraction. Normalizing reduced time \( \tau = T/\sqrt{\kappa T} \), one obtains spatio-temporal Laplacian \( \Delta = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial r^2 \), completely symmetrical with respect to the spatial coordinates \((x, y)\) and reduced time \( \tau \).

In nonlinear media, the self-focusing may balance out both, the temporal and the spatial broadenings of sufficiently short pulses of light. The combination of these two effects, responsible for the formation of \((1+1)\) temporal and \((2+1)\) spatial solitons, supports stationary states known as \((3+1)\)-dimensional solitons, or light bullets. For the details of physics of LB, the review on theoretical and experimental progress in this field, as well as for the description of fundamental (bell-shaped) three-dimensional solitons, we refer to the recent paper by Malomed, Mihalache, Wise, and Torner [2004]. Here we consider only the higher-order LB with phase dislocations, or optical vortices in spatiotemporal domain. Similar to the \((2+1)\)-dimensional case, stationary solutions corresponding to the fundamental \((3+1)D\) soliton can be obtained using radially-symmetrical ansatz of the form \( E = U(r) \exp(ikz) \), where \( k \) is a soliton parameter and the radius is given by \( r^2 = x^2 + y^2 + \tau^2 \). The same ansatz describes higher-order spherically-symmetrical modes consisting of several concentric shells surrounding inner core (Edmundson [1997]). Angle-dependent higher-order states, however, do not allow simple separation of variables, such as \( E = U(r) \exp(\im \theta + ikz) \), used for two-dimensional vortex solitons. Therefore, the search for higher-order LB with phase dislocations requires solving the full multidimensional stationary equation, quite a nontrivial task. The so-called “three-dimensional spinning solitons”, introduced by Desyatnikov, Maimistov, and Malomed [2000], is only known example of LB with phase dislocation, and it may represent much broader class of possible (and yet not known) spatiotemporal optical vortices. In other systems, such example include the “smoke rings” of vortex lines in non-degenerate optical parametric oscillator (Weiss, Vaupel, Staliunas, Slekys, and Taranenko [1999]), and the “parallel vortex rings” in matter waves trapped by the three-dimensional external potential (Crasovan, Perez-Garcia, Danaila, Mihalache, and Torner [2004]).

The structure of the spinning LB can be well understood by using the approximate (e.g. variational) solutions. We suppose that stationary optical pulse \( E = A(x, y, \tau) \exp(\im k z) \) (localized in time \( \tau \), so that \( A \rightarrow 0 \) for \( \tau \rightarrow \pm \infty \)) has a phase-dislocation located in the transverse plane, \( A(x, y, \tau) = V(x, y, \tau) \exp(\im \theta) \) with azimuthal angle \( \varphi = \tan^{-1}(y/x) \), similar to the two-dimensional CW vortex beams. Because of this dislocation, field should vanish in the origin and produce a “doughnut” shape in transverse plane, \( V \rightarrow 0 \) for \( \rho \rightarrow 0 \), here \( \rho \) is a polar radius, \( \rho^2 = x^2 + y^2 \). Thus spinning LB can be thought as a CW beam modulated in time by an additional multiplier, e.g. of the \( \text{sech}(\tau) \) shape,

\[
V = U_{\text{cyl}}(\rho) \text{sech}(\tau), \tag{5.4}
\]

and we note that this ansatz offers the separation of variables in cylindrical coordinates, which does not satisfy the nonlinear equation of course. Physically, the ansatz Eq. (5.4) can follow as a result of temporal (longitudinal) modulational instability of initially continuous (CW) optical vortex beam, responsible for generation of soliton trains (this so-called “neck”- instability in the spatial domain was experimentally observed by Fuerst, Baboiu, Lawrence, Torruellas, Stegeman, Trillo, and Wabnitz [1997]). Similar shape can be also modelled by a spherical harmonic,

\[
V = U_{\text{sph}}(r) \cos(\theta), \tag{5.5}
\]

with spherical coordinates \((x, y, \tau) \rightarrow (r, \varphi, \theta)\). Two model envelopes Eqs. (5.4) and (5.5) can be used as the trial functions for variational method. The advantage of this method is that it actually allows one to separate the variables in corresponding nonlinear equation and greatly simplifies the problem (variational methods in optics were reviewed recently by Malomed [2002]). Partial-differential equation for stationary envelope \( V(x, y, \tau) \) is than reduced to ordinary differential equations for the envelope \( U_{\text{cyl}}(\rho) \) or \( U_{\text{sph}}(r) \). Solutions to these equations and their analysis for the model with cubic-quintic nonlinearity show that radial envelopes \( U_{\text{cyl}}(\rho) \) and \( U_{\text{sph}}(r) \) are qualitatively similar to the ones shown in Figs. 15 for two-dimensional case, and they define very close values for the parameters of stationary spinning LB solution, for example the minimal threshold energy of soliton formation (Desyatnikov, Maimistov, and Malomed [2000]). However, variational solutions do not provide a full information about stationary states family and their stability, and direct numerical modelling is necessary.

Exact numerical solutions for spinning LD were obtained by Mihalache, Mazilu, Crasovan, Malomed, and Lederer [2000a] in the Type I SHG model, and by Mihalache, Mazilu, Crasovan, Malomed, and Lederer [2000b] for the CQ medium. Authors tested the stability of solutions in numerical propagation and observed azimuthal modulational instability which leads to the breakup of doughnut solitons into several fragments, each being a stable moving zero-spin soliton (see Fig. 16). The general conclusion based on direct numerical simulations was that the spinning LBs are always unstable against azimuthal perturbations (Crasovan, Malomed, and Mihalache [2001b]). Later on, however, more accurate study of the associated linear stability problem in the CQ model...
6. MULTI-SOLITON SPIRALING

Figure 16: Examples of stable and unstable spinning light bullets in CQ model for topological charges \( m = 1 \) in (a), (b), and \( m = 2 \) in (c), (d). Top row – soliton solutions (perturbed in (b)) at \( z = 0 \). Bottom row – after propagation of \( z = 60 \) in (a); \( z = 100 \) in (b); \( z = 50 \) in (c), and \( z = 90 \) in (d). After Mihalache, Mazilu, Towers, Malomed, and Lederer [2003].

revealed first ever found completely stable spatiotemporal vortex soliton (Mihalache, Mazilu, Crasovan, Towers, Buryak, Malomed, Torner, Torres, and Lederer [2002]). Example of stable propagation of initially strongly perturbed spinning LB is shown in Fig. 16 (b). The reason for stabilization of spinning LBs was found in the competition between nonlinearities. Following these predictions, stable spinning LBs were found in the model with competing quadratic and cubic nonlinearities (Mihalache, Mazilu, Crasovan, Towers, Malomed, Buryak, Torner, and Lederer [2002]) and in two-component vectorial CQ system (Mihalache, Mazilu, Towers, Malomed, and Lederer [2003]).

6. OTHER OPTICAL BEAMS CARRYING ANGULAR MOMENTUM

In this section we summarize some theoretical and experimental results on the study of self-trapped optical beams carrying angular momentum, which differ from the optical vortex beams discussed above. In general, such beams do not necessarily correspond to the stationary states, and their angular momentum manifests itself in the complex interaction of simple spatial solitons and leads to their spiraling. Multi-soliton complexes with an imposed angular momentum, such as necklaces and soliton clusters, can also be regarded as multi-soliton spiraling beams.

6.1. SOLITON SPIRALING

Spiraling of two spatial solitons was suggested theoretically by Poladian, Snyder, and Mitchell [1991]. This should occur when two fundamental solitons collide with trajectories that are not lying in a single plane, so that they form a two-body system with nonzero orbital angular momentum. Then, if the mutual interaction is attractive, the centrifugal repulsive force can be balanced out, and two solitons orbit about each other in a double-helix structure, as illustrated in Fig. 17. It is interesting to note, that similar predictions concerning three-dimensional solitons, or light bullets, were made by Edmundson and Enns [1993], based on the particle-like nature of soliton mutual interaction (Edmundson and Enns [1995]). In parallel, the experimental and numerical study of azimuthal instability of vortex solitons, described in Sect. 2.3, revealed the spiraling behavior of splitters, first demonstrated experimentally in rubidium vapors by Tikhonenko, Christou, and LutherDavies [1995]; Tikhonenko, Christou, and LutherDavies [1996].

In quadratic media, soliton spiraling was predicted and studied in detail theoretically by Steblina, Kivshar, and Buryak [1998]; Buryak and Steblina [1999]. The mechanical model, based on potential of soliton interaction (Malomed [1998]) has been derived, with extremal points of effective potential corresponding to spiraling bound states. Similar to any phase-sensitive soliton interaction the effective mass corresponding to the phase degree of freedom was shown to be always negative. As a result, corresponding stationary points are of saddle type, i.e. spiraling bound states are unstable. Depending on initial soliton states, such as soliton velocities, relative
6. MULTI-SOLITON SPIRALING

Figure 17: (Left) An illustration of the soliton spiraling process. The arrows indicate the initial direction of the two soliton beams. After Shih, Segev, and Salamo [1997]. (Right) Spiraling of solitons with initially skewed trajectories in photorefractive crystal. (a),(c) Initial position of the beams; cross and circle denote output positions of solitons A1 and A2, respectively, during individual propagation. (b),(d) Output positions of the solitons during simultaneous propagation. After Stepken, Belic, Kaiser, Krolikowski, and Luther-Davies [1999].

phase, and the impact parameter, soliton can reflect, spiral, and fuse. More recently, experimental observation of related phenomena has been reported by Simos, Couvrec, and Barthelemy [2002].

The spiraling of mutually incoherent spatial solitons was observed experimentally (Shih, Segev, and Salamo [1997]) and studied theoretically (Desyatnikov and Maimistov [1998]; Schjodt-Eriksen, Schmidt, Rasmussen, Christiansen, Gaididei, and Berge [1998]) as a possible scenario for a dynamically stable two-soliton bound state formed when two solitons are launched with initially twisted trajectories. Mutual incoherent solitons always attract each other in isotropic Kerr-type medium, independently of their relative phase, and effective potential minimum corresponds to stable bound state. Note the very similar results obtained by Ren, Hemker, Fonseca, Duda, and Mori [2000] for “braided light” in plasmas. As a matter of fact, the soliton spiraling due to an effectively vectorial beam interaction is associated with large-amplitude oscillations of a dipole-mode vector state generated by the interaction of two initially mutually incoherent optical beams (Skryabin, McSloy, and Firth [2002]).

In anisotropic photorefractive medium, however, the mutually incoherent solitons demonstrate much complicated anomalous interaction (Krolikowski, Saffman, Luther-Davies, and Denz [1998]; Stepken, Kaiser, Belic, and Krolikowski [1998]; Krolikowski, Denz, Stepken, Saffman, and Luther-Davies [1998]). Anomalous interaction results in complex trajectories which typically show partial mutual spiraling, followed by damped oscillations and the fusion of solitons. The rotation can be propelled to prolonged spiraling by the skewed launching of beams. This nontrivial behavior is caused by the anisotropy of the nonlinear refractive index change in the crystal, as was shown by Stepken, Belic, Kaiser, Krolikowski, and Luther-Davies [1999]; Belic, Stepken, and Kaiser [1999] and summarized by Krolikowski, Luther-Davies, Denz, Petter, Weihnau, Stepken, and Belic [1999]; Denz, Krolikowski, Petter, Weihnau, Tschudi, Belic, Kaiser, and Stepken [1999]. Nevertheless, the fascinating analogy between spiraling solitons and mechanical two-body system is applicable if one takes into account the anisotropic nature of spatial screening solitons interacting in photorefractive medium (Belic, Stepken, and Kaiser [2000]), the comparison of the above model to the isotropic one was published recently by Belic, Vujic, Stepken, Kaiser, Calvo, Agullo-Lopez, and Carrascosa [2002].

6.2. OPTICAL NECKLACE BEAMS

Since the decay of the ring-profile vortex solitons is associated with the growing azimuthal modulation of their intensity and the symmetry-breaking instability, one may try to stabilize the ring structure by imposing the initial intensity and phase modulation. The azimuthally modulated rings resemble “optical necklaces”; they are closely related to the higher-order guided modes, as we discussed in Sect. 3.1, and also to suitable superpositions of Laguerre-Gaussian beams.

In Kerr media, the first experimental results on the self-trapping of necklace-type beams were reported by Barthelemy, Froehly, Shalaby, Domnat, Paye, and Migus [1993]. Figure 18(a) shows the experimental data for the case in which an input beam in the form of a higher-order Laguerre-Gaussian mode was launched at the input of a Cs2 cell. The beam diameter was 260 µm, with a petal thickness of about 80 µm. The output pattern after 5 cm of propagation is shown when the intensity was low enough that the nonlinear effects were negligible (middle). As expected, the beam diameter increased to 265 µm, because of diffraction, while the petal thickness remained close to 80 µm. As the beam power was gradually increased, the petal thickness decreased, because of self-focusing. The petal size reduced to 30 µm at an intensity level of 5 × 107 W/cm2 (Barthelemy, Froehly, and Shalaby [1994]). Such self-trapped structures are remarkably stable and allow one to transport optical
beams with powers several times the critical power at which a Gaussian beam would otherwise collapse because of self-focusing; they disintegrate for input intensities lower than the self-trapping intensity.

The concept of necklace beams was developed by Soljacic, Sears, and Segev [1998], who studied numerically the propagation of azimuthally modulated bright rings in Kerr medium. These authors demonstrated that, in contrast to all previous studies in this model, the self-trapped beams localized in transverse plain can preserve their shape during the propagation and escape the collapse instability. It is possible if such beams are constructed as a “necklace” of the out-of-phase “pearls”, each carrying the power less than critical power of catastrophic self-focusing. Each petal thus slowly diffracts, if it propagates alone, and this diffraction is greatly suppressed within the ring, because of collective self-trapping of the ring as a whole. Due to the repulsion between neighboring petals the ring expands self-similarly.

The shape of the necklace can be approximated by the ansatz similar to Eq. (3.3) with $p = 0$. To slow down the expansion of the ring, its radius should be taken as large as possible, so that the radial envelope $U(r)$ in Eq. (3.3) can be approximated by the corresponding 1D soliton of sech shape, which is a bright stripe in two-dimensional spatial domain. Extensive numerical simulations and semi-analytical analysis performed by Soljacic and Segev [2000] showed that the dynamics of the necklace can be controlled and reduced to the quasi-stationary if the radius of the ring, its width, the amplitude, and the order of azimuthal modulation (the winding number $m$ in Eq. (3.3)), minimize corresponding action integral. These “quasi-solitons” have a shape of a thin modulated stripe wrapped to a large ring, and they propagate stably over several tenths of diffraction lengths.

Necklaces with additional phase modulation, introducing a nonzero angular momentum, exhibit a series of phenomena typically associated with rotation of rigid bodies and centrifugal force effects (Soljacic and Segev [2001]). The simplest way to explore these novel features is to consider the ansatz Eq. (3.3) with $p \neq 0$, which, from one hand, describes a vortex with the topological charge $m$ at $p = 1$, and, from the other hand, it includes the varying modulation parameter $p (0 < p < 1)$. The spin $S$ of this nonstationary structure is defined as

$$S = \frac{2mp}{1 + p^2},$$

and it vanishes for $p \to 0$ (see the definition of spin after Eq. (2.18)). When the ring vortex is only slightly modulated ($p \approx 1$), it decays into a complex structure of filaments because of a competition between different instability modes of the corresponding vortex soliton (Desyatnikov and Kivshar [2002b]).

When the modulation becomes deeper, e.g. for $p \approx 0.5$, the initial vortex transforms into a necklace-like structure [see Fig. 18(b)], and its dynamics is modified dramatically. The modulated ring-profile structure does not decay but, instead, it expands with small rotation, the rotation is much weaker because the initial angular
6. MULTI-SOLITON SPIRALING

Figure 19: Left: Intensity and phase distribution for a four-soliton cluster in a saturable medium. Note that in terms of the azimuthal coordinate \( \phi = \tan^{-1}(y/x) \), the vortex phase is given as a linear function \( m\phi \) with integer \( m \), while the staircase-like phase of the cluster is a nonlinear phase dislocation (Desyatnikov, Denz, and Kivshar [2004]). Right: Cluster composed of six spatiotemporal two-color solitons. The topological charge \( m \) of the soliton cluster is equal to one. (a) The fundamental frequency field and (b) the second harmonic field. (c) The phase distribution at fundamental frequency and (d) the phase distribution at the second harmonic (Crasovan, Kartashov, Mihalache, Torner, Kivshar, and Perez-Garcia [2003]).

momentum (spin) is much smaller than that in the case of \( p \approx 1 \). These ringlike structures were introduced by Soljacic and Segev [2001] as the first example of optical beams with fractional spin [see Eq. (6.1)] and rotating intensity. Note, that the anticlockwise direction of the rotation is determined by the gradient of phase, which grows anticlockwise for \( m > 0 \), see Fig. 18(b). The angular velocity vanishes as the ringlike structure expands, in analogy with Newtonian mechanics and “scatter on ice” effect.

6.3. SOLITON CLUSTERS

In saturable media, the fundamental solitons are stable and demonstrate the particle-like robust interaction, e.g. spiraling out of the initial vortex ring after it breaks, see Sect. 2.3. The analogy with particles and forces between them applied to spatial solitons allows one to search for the bound states of several solitons corresponding to the balance between all acting forces, as in classical mechanics. In order to create non-expanding configurations of \( N \) solitons in a bulk medium, first we recall the basic physics of coherent interaction of two spatial solitons.

It is well known (Kivshar and Agrawal [2003]) that such an interaction depends crucially on the relative soliton phase, say \( \theta \), so that two solitons attract each other for \( \theta = 0 \), and repel each other for \( \theta = \pi \). For the intermediate values of the soliton phase, \( 0 < \theta < \pi \), the solitons undergo an energy exchange and inelastic interaction.

Here we follow the original paper by Desyatnikov and Kivshar [2002a] and analyze possible stationary configurations of \( N \) coherently interacting solitons for a ringlike geometry. It is easy to understand that such a ringlike configuration will be radially unstable due to an effective tension induced by bending of the soliton array. Thus, a ring of \( N \) solitons will collapse, if the mutual interaction between the neighboring solitons is attractive, or expand otherwise, resembling the expansion of the necklace beams. Nevertheless, a simple physical mechanism will provide stabilization of the ringlike configuration of \( N \) solitons, if we introduce an additional phase on the scalar field that twists by \( 2\pi m \) along the soliton ring. This phase introduces an effective centrifugal force that can balance out the tension effect and stabilize the ringlike soliton cluster. Due to a net angular momentum induced by such a phase distribution, the soliton clusters will rotate with an angular velocity which depends on the number of solitons and phase charge \( m \).

To describe the soliton clusters analytically, we consider a coherent superposition of \( N \) solitons with the envelopes \( G_n(x, y, z), n = 1, 2, \ldots N \), propagating in a self-focusing bulk nonlinear medium. The equation for the slowly varying field envelope \( E = \sum G_n \) can be written in the form of the NLS equation (2.11). For a ring of identical weakly overlapping solitons launched in parallel, we can calculate the integrals of motion employing a Gaussian ansatz for a single beam \( G_n \),

\[
G_n = A \exp \left( - \frac{|r - r_n|^2}{2a^2} + i\alpha_n \right),
\]

where \( r_n = (x_n; y_n) \) describes the soliton location, and \( \alpha_n \) is the phase of the \( n \)-th beam. We assume that the beams \( G_n \) are arranged in a ring-shaped array of radius \( R \), i.e. \( r_n = \{ R \cos \varphi_n; R \sin \varphi_n \} \) with \( \varphi_n = 2\pi n/N \).
Analyzing many-soliton clusters, we remove the motion of the center of the mass and put $L = 0$, here the linear momentum is given by Eq. (2.16). Applying this constraint, we find the conditions for the soliton phases, $\alpha_{i+n} - \alpha_i = \alpha_{i+n} - \alpha_k$, which are satisfied provided the phase $\alpha_n$ has a linear dependence on $n$, i.e. $\alpha_n = \theta n$, where $\theta$ is the relative phase between two neighboring solitons in the ring. Then, we employ the periodicity condition in the form $\alpha_{n+N} = \alpha_n + 2\pi m$, and find:

$$\theta = \frac{2\pi m}{N}. \quad (6.3)$$

In terms of the field theory, Eq. (6.3) gives the condition of the vanishing energy flow $L = 0$, because the linear momentum $L = \int \mathbf{j} \cdot d\mathbf{r}$ can be presented through the local current $\mathbf{j} = \text{Im}(E^* \nabla E)$. Therefore, Eq. (6.3) determines a nontrivial phase distribution for the effectively elastic soliton interaction in the ring. In particular, for the well-known case of two solitons ($N = 2$), this condition gives only two states with the zero energy exchange, when $m$ is even ($\theta = 0$, mutual attraction) and when $m$ is odd ($\theta = \pi$, mutual repulsion).

For a given $N > 2$, the condition (6.3) predicts the existence of a discrete set of allowed states corresponding to a set of the values $\theta = \theta^{(m)}$ with $m = 0, \pm 1, \ldots, \pm (N - 1)$. Here, two states $\theta^{(\pm m)}$ differ by the sign of the angular momentum, similar to the case of vortex solitons. Moreover, for any positive (negative) $m_+$ within the domain $\pi < |\theta| < 2\pi$, one can find the corresponding negative (positive) value $m_-$ within the domain $0 < |\theta| < \pi$, so that both $m_+$ and $m_-$ describe the same stationary state. For example, in the case $N = 3$, three states with zero energy exchange are possible: $\theta^{(0)} = 0$, $\theta^{(1)} = 2\pi/3$, and $\theta^{(2)} = 4\pi/3$, and the correspondence is $\theta^{(\pm 1)} \leftrightarrow \theta^{(2)}$. The number $m$ determines the full phase twist around the ring, and it plays a role of the topological charge of the corresponding phase dislocation, see Fig. 19.

Applying the effective-particle approach, Desyatnikov and Kivshar [2002a] derived an effective interaction energy for the soliton cluster and have shown that it can be classified in a simple way by its extremal points. The existence of a minimum point suggests that such a configuration describes a stable or long-lived ring-like cluster of a particular number of solitons. This prediction was verified by a series of numerical simulations for different $N$-soliton rings and their propagation in a saturable medium.

For example, the effective potential is always attractive for $m = 0$, and thus the ring of in-phase solitons exhibit oscillations and fusion. Another scenario of the mutual soliton interaction corresponds to the repulsive potential, e.g., for the case $\theta > \pi/2$. In the numerical simulations corresponding to this case, the ringlike soliton array expands with the slowing down rotation, similar to the rotating necklace beams, see Fig. 18(b).

Finally, the stationary soliton bound state that corresponds to a minimum of the effective potential is shown in Fig. 19 for particular case $N = 4$ and $m = 1$. Here the angular momentum is nonzero, and it produces a repulsive centrifugal force that balances out an effective attraction of $\pi/2$-out-of-phase solitons. The general rule, generalizing a two-soliton phase sensitive interaction, predicts the existence of a bound state of $N$ solitons if the nonzero phase step $\theta$ is equal or less $\pi/2$, i.e. the interaction is attractive and can be balanced out by the net centrifugal force for $0 < \theta \leq \pi/2$. Therefore, soliton cluster with topological charge $m$ can be quasi-stationary only for $N \geq 4m$, see Eq. (6.3). In addition, there possible “excited” states with the radius of the cluster oscillating near the minimum of interaction potential during the propagation.

The concept of soliton clusters was extended to the case of light propagation in quadratically nonlinear media by Kartashov, Molina-Terriza, and Torner [2002], where it is intimately related to the concept of induced beam break-up or soliton algebra discussed above. Due to the phase relation between the fundamental wave and the second harmonic beam (see Sect. 4), the topological charge in the second-harmonic field is double the fundamental wave charge. Therefore, the corresponding phase jump between neighboring solitons Eq. (6.3) is doubled, see Fig. 20. Different regimes of cluster propagation were found to be possible.

Metastable, robust propagation of clusters in media with competing quadratic and self-defocusing nonlinearities was reported by Kartashov, Crasovan, Mihalache, and Torner [2002]. These results were followed by similar findings in the cubic-quintic model as well by Mihalache, Mazilu, Crasovan, Malomed, Lederer, and Torner [2003]. Similar to the extension of the concept of two-dimensional vortex solitons into spatiotemporal domain (see paper by Desyatnikov, Maimistov, and Malomed [2000] and Sect. 5.3), the bound states of three-dimensional solitons, or light bullets, can be constructed using the approach described above for two-dimensional cw beams. Tree-dimensional, robust “soliton molecules” were introduced by Crasovan, Kartashov, Mihalache, Torner, Kivshar, and Perez-Garcia [2003]; Mihalache, Mazilu, Crasovan, Malomed, Lederer, and Torner [2004] using this concept. In Fig. 20 such a molecule is shown, constructed from six two-color “atoms”.

It turns out that the special initial condition for $N$ solitons in the ring, namely the “inverse” picture of the vortex splitting Fig. 6, allows one to reconstruct the vortex soliton. Desyatnikov, Denz, and Kivshar [2004] demonstrated how several initially well separated solitons, being launched toward the target ring in the absence of perturbations, can excite a metastable vortex ring. Mutual trapping of several solitons on the collision can be regarded as a synthesis of soliton molecules, and it corresponds to a transfer of an initial angular momentum of a system of solitons to angular momentum stored by the optical vortex. Similar results were obtained by Mihalache, Mazilu, Crasovan, Malomed, Lederer, and Torner [2004] for 3D clusters of light bullets, launched with additional azimuthal tilts which mimics the vortex phase.
Figure 20: (a) Rotating vector cluster consisting of four fundamental solitons. The angle of rotation, corresponding to the distance $z = 225L_D$, is $\sim 11.25\pi$. Note that the direction of rotation is opposite to the one of necklace in Fig. 18(b), despite the fact that in both cases the angular momentum is positive. After Desyatnikov and Kivshar [2002b]. (b) Iso-surface plot of two counter-propagating vortices. Breakup into three rotating beamlets is visible. Because of the topological charge, the beamlets start to spiral, which is only weakly visible due to the short propagation distance. The beamlets rotate in the direction indicated by the arrow on the right (Motzek, Jander, Desyatnikov, Belic, Denz, and Kaiser [2003]).

The analogy with particles and their bound states, employed to develop the concept of scalar soliton clusters, can be applied to the composite solitons, constructed from incoherently coupled beams, see Sect. 3. In this case, two mutually incoherent beams can be regarded as atoms of different sorts, always attracting each other (see, e.g., papers by Segev [1998]; Stegeman and Segev [1999]). Now, the combination of three types of forces acting between solitons allows to construct a great variety of bound states, or quasi-solitons. These forces are of different origin, namely the coherent phase-dependant attraction/repulsion, incoherent attraction, and the repulsive centrifugal force in clusters with angular momentum. The important issue of the effective interaction potential was investigated by Malomed [1998]; Maimistov, Malomed, and Desyatnikov [1999], and conservation of angular momentum for interacting two-dimensional and three-dimensional solitons discussed by Desyatnikov and Maimistov [2000].

Combining the ringlike clusters of $N$ fundamental solitons with the staircase phase distribution and the stabilizing effect of the vectorial beam interaction, Desyatnikov and Kivshar [2002b] introduced a concept of the vector soliton clusters. Indeed, if we take a scalar cluster of solitons and combine it with the other beam that interacts incoherently with the primary beam supporting the cluster structure, the resulting vector soliton will demonstrate the surprising long-lived rotational dynamics, as clearly seen in Fig. 20(a). Moreover, the vectorial interaction can allow to trap not only a large number of solitons with $N \geq 4$, but also three and even two solitons. In the latter case, this structure is, as a matter of fact, the rotating dipole-mode vector soliton or optical propeller (Krollikowski, Ostrovskaya, Weimann, Geisser, McCarthy, Kivshar, Denz, and Luther-Davies [2000]; Carmon, Uzdin, Pigier, Musslimani, Segev, and Nepomnyashchy [2001]), where two out-of-phase solitons are trapped by the other beam interacting incoherently. Therefore, the rotating structure presented in Fig. 20(a) is somewhat similar to the four-soliton propeller.

Similar ideas are applicable in other field, such as light in plasmas (Ren, Hemker, Fonseca, Duda, and Mori [2000]; Berezhiani, Mahajan, Yoshida, and Pekker [2002]) or atomic mixtures of Bose-Einstein condensates (Perez-Garcia and Vekslerchik [2003]). The latter case of trapped matter-waves is especially interesting because of recent breakthroughs in experimental realization of vortices, we discuss it in Sect. 8.2. Recent results of interaction of counter-propagating vortices presented by Motzek, Jander, Desyatnikov, Belic, Denz, and Kaiser [2003], also point out to the existence of robust stationary points resembling soliton clusters, see Fig. 20(b). This particular system is known to be a reach source of the dynamic instabilities and chaos, thus the clustering of vortices resulted from this instability may indicate the presence of “islands” of order attracting unstable system. Finally, we mention here clusters of dissipative solitons in lasers and externally driven cavities (Vladimirov,
§ 7. Discrete vortices in two-dimensional lattices

The optical vortices discussed above propagate in homogeneous nonlinear media. When refractive index is periodically modulated, it modifies the wave diffraction properties, and it can affect strongly both nonlinear propagation and localization of light (Kivshar and Agrawal [2003]; Christodoulides, Lederer, and Silberberg [2003]). As a result, periodic photonic structures and photonic crystals recently attracted a lot of interest due to the unique ways they offer for controlling light propagation. In particular, many nonlinear effects, including formation of lattice solitons, have been demonstrated experimentally for one- and two-dimensional optically-induced photonic lattices (Fleischer, Carmon, Segev, Efremidis, and Christodoulides [2003]; Neshev, Ostrovskaya, Kivshar, and Krolikowski [2003]; Fleischer, Segev, Efremidis, and Christodoulides [2003]). The concept of optically-induced lattices arises from the possibility to modify the refractive index of a nonlinear medium with periodic optical patterns, and use a weaker probe beam to study scattering of light from the resulting periodic photonic structure. Current experiments employ photorefractive crystals with strong electro-optic anisotropy to create a linear optically-induced lattice with a polarization orthogonal to that of a probe beam, which also eliminates the nonlinear interaction between the beam and the lattice.

A vortex beam propagating in an optical lattice can be stabilized by the effective lattice discreteness in a self-focusing nonlinear media creating a two-dimensional discrete vortex soliton. This has been shown in several theoretical studies of the discrete (Johansson, Aubry, Gaididei, Christiansen, and Rasmussen [1998]; Malomed and Kevrekidis [2001]; Kevrekidis, Malomed, Bishop, and Frantzeskakis [2002]; Kevrekidis, Malomed, and Gaididei [2002]; Kevrekidis, Malomed, Chen, and Frantzeskakis [2004]) and continuous models with an external periodic potential (Yang and Musslimani [2003]; Baizakov, Malomed, and Salerno [2003]; Yang [2004]), and such vortices have been also generated experimentally by Fleischer, Neshev, Bartal, Alexander, Cohen, Ostrovskaya, Manela, Martin, Hudock, Makasyuk, Chen, Christodoulides, Kivshar, and Segev [2004]. Below, we discuss some of the basic properties of the discrete vortices and summarize the major experimental observations.

We consider two-dimensional optically-induced lattices created in photorefractive crystals. In this case, the evolution of a laser beam can be described by the generalized nonlinear Schrödinger-type equation,

\[ i \frac{\partial \Psi}{\partial z} + D \left( \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right) - G(x, y, |\Psi|^2) \Psi = 0, \]

(7.1)

where \( \Psi(x, y, z) \) is the normalized envelope of the electric field, the transverse coordinates \( x, y \) and the propagation coordinate \( z \) are normalized to the characteristic values \( x_0 \) and \( z_0 \), respectively, \( D = z_0 \lambda/(4\pi n_0 x_0^2) \) is the beam diffraction coefficient, where \( n_0 \) is the average medium refractive index and \( \lambda \) is the vacuum wavelength. The function \( G(x, y, |\Psi|^2) \) accounts for both lattice potential and nonlinear beam self-action effects,

\[ G = \gamma \left\{ I_b + I_0 \sin^2(\pi x/d) \sin^2(\pi y/d) + |\Psi|^2 \right\}^{-1}, \]

(7.2)

where \( \gamma \) is proportional to the external biasing field, \( I_b \) is the dark irradiance, and \( I_0 \) is the intensity of interfering beams that induce a square lattice of period \( d \) through the photorefractive effect (see details by Fleischer, Segev, Efremidis, and Christodoulides [2003]; Neshev, Alexander, Ostrovskaya, Kivshar, Martin, Makasyuk, and Chen [2004]; Fleischer, Bartal, Cohen, Manela, Segev, Hudock, and Christodoulides [2004]). Similar mathematical models appear for describing the self-action effects in nonlinear photonic crystals (Mingaleev and Kivshar [2001]), and the nonlinear dynamics of atomic BEC in optical lattices (Ostrovskaya and Kivshar [2003]).

Self-focusing nonlinearity in Eq. (7.1) can compensate for the diffraction-induced beam spreading in the transverse directions, leading to the formation of stationary structures in the form of spatial solitons, \( \Psi(x, y, z) = \psi(x, y) e^{i\beta z}, \) where \( \psi(x, y) \) is the soliton envelope, and \( \beta \) is a soliton parameter, a nonlinear shift of the beam propagation constant. In order to analyze the vortex-like structures in a periodic potential, we present the field envelope in the form, \( \psi(x, y) = |\psi(x, y)| \exp[i\varphi(x, y)] \), and assume that the accumulation of the phase \( \varphi \) around a singular point (at \( \psi = 0 \)) is \( 2\pi M \), where the integer \( M \) is a topological charge of the phase singularity. We consider spatially localized structures in the form of vortex-like bright solitons with the envelopes decaying at infinity. Such structures may exist when the soliton eigenvalue \( \beta \) is inside a gap of the linear Floquet-Bloch spectrum of the periodic structure (Mingaleev and Kivshar [2001]; Ostrovskaya and Kivshar [2003]). More importantly, a self-induced waveguide created by the vortex soliton is double-degenerated, and it supports simultaneously two modes, \( |\psi(x, y)| \cos \varphi \) and \( |\psi(x, y)| \sin \varphi \), for the same value of \( \beta \). For symmetric vortex-like configurations, i.e. those possessing a 90° rotational symmetry, this is always the case.

The profiles of stable symmetric vortex solitons, mentioned above, resemble closely a ring-like structure of the soliton clusters in homogeneous media, see Sect. 6.3. Using the reduced Hamiltonian approach developed by Desyatnikov and Kivshar [2002a] in homogeneous media, Alexander, Sukhorukov, and Kivshar [2004] have
LATTICE VORTICES

The symmetric modes can be represented as the angular Bloch modes due to specific properties of the coupling coefficients calculated for realistic periodic structures. Note also that shown in Fig. 21, where (b,c) symmetric square vortex solitons; (d) rectangular structure that can only have a trivial phase profile; (e) rhomboid configuration with a topological charge +1. Shown are the intensity profiles (top) and phase structure (bottom). After Alexander, Sukhorukov, and Kivshar [2004].

generalized it to construct the discrete vortex solitons as a superposition of a finite number of the fundamental (no nodes) solitons, similar to Eq. (6.2). Here, in contrast to the case of a homogeneous medium, the positions of individual solitons are fixed by the lattice potential, provided the lattice is sufficiently strong. This approximation is valid when the overlapping integrals between solitons with numbers \( n \) and \( m \) are the small parameter, \( c_{n \neq m} \ll 1 \). It was rigorously demonstrated by MacKay and Aubry [1994]; Aubry [1997] that under such conditions the soliton amplitudes are slightly perturbed due to their interaction, and one can seek stationary solutions using the perturbation approach. In the first order a simple constraint for the soliton phases \( \alpha_n \) have been obtained, cf. Eq. (6.3),

\[
\sum_{m=0}^{N-1} c_{nm} \sin(\alpha_m - \alpha_n) = 0. \tag{7.3}
\]

In the sum (7.3), each term defines the energy flow between the solitons with numbers \( n \) and \( m \), so that the equations (7.3) introduce a condition for a balance of energy flows which is a necessary condition for stable propagation of a soliton cluster and the vortex-soliton formation. These conditions are satisfied trivially when all the solitons are in- or out-of-phase. The nontrivial solutions of Eqs. (7.3) correspond to the vortex-like soliton clusters have been analyzed only for symmetric configurations (Eilbeck, Lomdahl, and Scott [1985]; Eilbeck and Johansson [2003]), and even then some important solutions have been missed. Using this approach, Alexander, Sukhorukov, and Kivshar [2004] introduced different novel types of asymmetric vortex solitons, some of them are shown in Fig. 21. Moreover, the existence properties of asymmetric vortex-like solutions are highly nontrivial, due to specific properties of the coupling coefficients calculated for realistic periodic structures. Note also that the symmetric modes can be represented as the angular Bloch modes (Ferrando [2004]).

In a number of recent publications, authors develop the concept of higher-order lattice solitons, such as dipole solitons (Yang, Makasyuk, Bezryadina, and Chen [2004]), “quasivortices” (Kevrekidis, Malomed, Chen, and Frantzeskakis [2004]), or even periodic soliton trains (Chen, Martin, Eugenieva, Xu, and Bezryadina [2004]; Kartashov, Vysloukh, and Torner [2004b]). These multi-humped states can be constructed as the bound states of individual lowest order lattice solitons (Kevrekidis, Malomed, and Bishop [2001]). Though the simplest stationary solutions with in-phase solitons also exist, usually the stable configuration requires the edge-type \( \pi \)-phase dislocations between neighboring sites (Kartashov, Egorov, Torner, and Christodoulides [2004]). The approach developed by Alexander, Sukhorukov, and Kivshar [2004] includes into this rich family the higher order solitons with nested screw phase dislocations and nonzero angular momentum.

To provide evidence for the existence of the discrete vortex solitons two separate groups performed independent experimental investigations (Neshev, Alexander, Ostrovskaya, Kivshar, Martin, Makasyuk, and Chen [2004]; Fleischer, Bartal, Cohen, Manela, Segev, Hudock, and Christodoulides [2004]). Both relied on optical induction to create a nonlinear lattice in a photosensitive (photorefractive) material (Efremidis, Sears, Christodoulides, Fleischer, and Segev [2002]). In this technique ordinarily polarized light is periodically modulated (by interference or by imaging a mask) to induce a 2D array of waveguides in an anisotropic photorefractive crystal. A separate probe beam of extraordinary polarization acquires a vortex structure by passing through a phase mask and is then launched into the array. The degree of nonlinearity is controlled by applying a voltage across the \( c \)-axis of the crystal (photorefractive screening nonlinearity) and controlling the intensity of the probe beam.

Typical experimental results are summarized in Figs. 22. A two-dimensional square lattice was first created, with its principal axes oriented in the diagonal directions shown in the top panel of Fig. 22(a). The resulting periodic structure acts as a square array of optically induced waveguides for the probe beam. The vortex beam,
Figure 22: Experimental observation of an optical vortex propagating with (top panel) and without (bottom panel) an optically-induced lattice. Middle panel - three dimensional representation. (a) input intensity pattern of the lattice (top) and input (middle) and linear output vortex beam (bottom). (b-d) intensity patterns of the vortex beam at crystal output with a bias field of 600, 1200, and 3000 V/cm, respectively. The bright ring in the lattice pattern (a) indicates the location of the input vortex. After Neshev, Alexander, Ostrovskaya, Kivshar, Martin, Makasyuk, and Chen [2004].

shown in the bottom panel of Fig. 22(a), was then launched straight into the middle of the lattice “cell” of four waveguides, as indicated by a bright ring in the lattice pattern. Due to the coupling between closely spaced waveguides of the lattice, the vortex beam exhibits discrete diffraction when the nonlinearity is low [Fig. 22(b)], whereas it forms a discrete vortex soliton at an appropriate level of higher nonlinearity [see Fig. 22(c,d), the top-middle panel]. As predicted, the observed discrete diffraction and discrete self-trapping of the vortex beam in the photonic lattice is remarkably different from that in a homogeneous medium [see Fig. 22(bottom row)].

In addition, we mention that the concept of lattice solitons has been recently explored theoretically in lattices induced by Bessel beams Kartashov, Vysloukh, and Torner [2004a]; Kartashov, Egorov, Vysloukh, and Torner [2004a]. In this case the optically induced potential possesses a cylindrical symmetry and support stable soliton complexes in the form of ring-shaped multipoles and necklaces (Kartashov, Egorov, Vysloukh, and Torner [2004b]). Remarkably, such lattices allow stable ring vortex soliton to exist even in self-defocusing medium (Kartashov, Vysloukh, and Torner [2005b]).

Further generalization of the concept of lattice vortex solitons include higher-band vortex solitons (Manela, Cohen, Bartal, Fleischer, and Segev [2004]), made of two components from different bands, composite vortices in Bessel lattices (Kartashov, Vysloukh, and Torner [2005a]), and in conventional honeycomb lattices made in quadratic nonlinear media (Xu, Kartashov, Crasovan, Mihalache, and Torner [2004]). We also notice recent investigations of higher-order antiguiding modes (Yan and Shum [2004]) and optical vortices (Ferrando, Zacares, de Cordoba, Binosi, and Monsoriu [2004]) in photonic crystal fibers which feature interesting similarities with higher-order solitons and vortices in optically-induced lattices.

§ 8. Links to vortices in other fields

Many diverse concepts in physics, ranging from the vortex clusters in quantum dots (Saarikoski, Harju, Puska, and Nieminen [2004]) to the data vortex switch architecture in the networks of optical waveguides (Yang and Bergman [2002]), may find their analogies with the physics of optical vortices. The concepts of linear singular optics can be expanded to the femtosecond regime (Bezuhanov, Dreischuh, Paulus, Schatzel, and Walther [2004]) as well as to apply in the entirely new wavelength domain, e.g., the X-ray regime (Peele, McMahon, Paterson, Tran, Mancuso, Nugent, Hayes, Harvey, Lai, and McNulty [2002]; Peele and Nugent [2003]; Turner, Dhal, Hayes, Mancuso, Nugent, Paterson, Scholten, Tran, and Peele [2004]). The fundamental physical concept of phase singularities finds many promising applications, such as extensively studied manipulation of micro-objects by optical tweezers and spanners (Simpson, Allen, and Padgett [1996]; Friese, Nieminen, Heckenberg, and Rubinsztain-Dumlop [1998]; Gahagan and Swartzlander [1999]; Kounura, Zijlstra, van Delden, Harada, and Feringa [1999]; Paterson, MacDonald, Arlt, Sibbett, Bryant, and Dholakia [2001]; MacDonald, Paterson, Volke-Sepulveda, Arlt, Sibbett, and Dholakia [2002]; Grier [2003]). Moreover, in many cases, the physics of optical
8.1. VORTICES IN DISSIPATIVE OPTICAL SYSTEMS

Above, we described the vortex solitons and related phenomena in optical systems with the help of the conservative NLS-type equations. The NLS equation can be linked to a more general dissipative Ginzburg-Landau (GL) model as its conservative limit. The theoretical approach of Ginzburg and Landau [1950] was introduced to describe the phenomena of superconductivity (Cyrot [1973]). Abrikosov [1957] developed further this approach for the type II superconductors, more common in nature, and showed that the flux penetrates the superconductor in the form of a regular array of flux tubes or vortices (Abrikosov vortices). In two dimensions, the complex GL equation (CGL) admits extensively studied quantized vortices (Cross and Hohenberg [1978]; Pismen [1999]), the stationary limit of the latter equation is also known as the stationary Gross-Pitaevskii equation. A large number of publications is devoted to the study of vortices in boson condensates (such as superconductors and superfluids), described by the CGL equation (see, e.g., Ovchinnikov and Sigal [1997, 1998a, b, 2002], and references therein). In the context of nonlinear optics, the latter model corresponds to the stationary NLS with self-defocusing type nonlinearity, discussed above in Sect. 2.1. It is interesting to note that related integrable complex sine-Gordon model admits the continuous families of the non-radially symmetric dark vortex solitons (Barashenkov, Shchesnovich, and Adams [2002]), while their existence remains an open question for the nonintegrable CGL equation (Ovchinnikov and Sigal [2000]).

The analogy between superfluids and laser optics was recognized as early as 1970 (Graham and Haken [1970]). In particular, the vortex solutions to laser equations were found by Coullet, Gil, and Rocca [1989]; Tamm and Weiss [1990] and intensively studied latter, both theoretically and experimentally (Weiss, Vaupel, Stalumias, Sleksys, and Taranenko [1999]). The concept of vortices in lasers is connected to the dissipative optical solitons (DOS), or auto-solitons. This kind of solitons was initially predicted for wide-aperture nonlinear interferometers excited by external radiation (Rozanov and Khodova [1988]) and for laser systems with saturable absorbers (Rozanov and Fedorov [1992]). If the material relaxation time is much smaller than that for the field in optical resonator (the so-called class-A laser), the master Maxwell-Bloch equations can be reduced to a CGL equation (Mandel [1997])

\[
\frac{\partial E}{\partial \zeta} = (\delta + i)\Delta_d E + f(|E|^2)E + E_i. \tag{8.1}
\]

Here the evolution variable \( \zeta \) stands for time, in the resonator schemes, or the propagation coordinate \( z \), in a bulk medium, \( \delta \) is the effective diffusion coefficient, and the diffraction operator \( \Delta_d \) is acting in \( d = 1,2,3 \) “transverse” coordinates. The parameter \( E_i \) represents an external driving plane-wave field, and the nonlinearity \( f(|E|^2) \) is a complex function, so that the conservative limit Eq. (2.11) is given by \( \delta = E_i = 0 \) and \( f(|E|^2) = iF(|E|^2) \).

The wide-aperture interferometers, filled with passive or active nonlinear media (optical cavities) and excited by external radiation \( E_i \), exhibit optical bistability (Gibbs [1985]). Due to the spatial hysteresis (Rozanov...
there possible domain walls connecting two stable and otherwise spatially homogeneous transmitted waves. Domain walls are usually identified with switching waves, and they represent building blocks for DOS’s — the localized solutions emerging as bound states of switching waves. This implies the discreteness of the DOS spectrum in contrast to the continuous “families” of conservative solitons. The external radiation $E_i$ determines the frequency and the phase of DOS’s, or “cavity solitons”; they exist on the nonzero background and have oscillating tales. Several DOSs may interact (Afanasyev, Malomed, and Chu [1997]; Ramazza, Benkler, Bortolozzo, Boccaletti, Ducci, and Arecchi [2002]; Schapers, Feldmann, Ackemann, and Lange [2000]; Schapers, Ackemann, and Lange [2003]; Tlidi, Vladimirov, and Mandel [2003]) and form bound states or clusters of cavity solitons (Vladimirov, McSloy, Skryabin, and Firth [2002]). In contrast to the conservative model discussed in Sect. 6.3, the clusters may appear due to the oscillating effective interaction potential (Malomed [1991, 1998]) with a discrete set of the equilibrium distances between cavity solitons. A large number of publications is devoted to the study of cavity solitons and their link to spontaneous pattern formation, as also discussed in the recent review papers (Arecchi, Boccaletti, and Ramazza, [1999]; Rozanov [2000]; Firth and Weiss [2002]; Peschel, Michaelis, and Weiss [2003]; Lugato [2003]), and reflected in the comprehensive list of references prepared by Mandel and Tlidi [2004].

Interesting applications of vortices such as the pump beams in externally driven cavities was suggested for degenerate optical parametric oscillator, as discussed by Oppo, Scroggie, and Firth [2001], and in the vertical-cavity surface-emitting lasers. In the former case, the stable domain walls appear as being trapped in the beam, while in the latter case the cavity solitons perform a uniform rotary motion along the crater of a doughnut-shaped holding beam (Barland, Branbilla, Columbo, Furfaro, Giudici, Hachair, Kheradmand, Lugato, Maggipinto, Tissoni, and Tredicce [2003]). Micro-cavities offer novel possibilities for the cavity soliton generation and control (Barland, Tredicce, Branbilla, Lugato, Balles, Giudici, Maggipinto, Spinelli, Tissoni, Knöll, Miller, and Jäger [2002]; Debernardi, Bava, di Sopra, and Willemesen [2003]; Maggipinto, Brambilla, and Firth [2003]; Vahala [2003]), and the examples include the spontaneous generation of the “optical vortex crystals” (Scheuer and Orenstein [1999]).

In lasers, the phase of the field is free ($E_i = 0$ in Eq. (8.1)) and the topological solitons are possible (Firth and Weiss [2002]). The comprehensive overview of the theoretical studies and the experimental generation of laser vortices can be found in Weiss, Vaupel, Staliunas, Slekeys, and Taranenko [1999]; Weiss, Staliunas, Vaupel, Taranenko, Slekeys, and Larionova [2003]. In lasers with saturable absorbers, i.e. when nonlinearity in Eq. (8.1) is given by $f(E^2) = -1 + g_0/(1 + |E|^2) - a_0/(1 + b|E|^2)$, novel types of solitons may appear, such as transversely asymmetric and rotating structures without phase dislocations and radially symmetric vortices with higher-order topological charges (Rozanov, Fedorov, Fedorov, and Khodova [1995]; Fedorov, Rosanov, Shatsev, Veretenov, and Vladimirov [2003]). Furthermore, bright dissipative vortex solitons can form strongly coupled “vortex clusters”, as shown in Fig. 23(a,b), as well as weakly coupled bound states (Rozanov, Fedorov, and Shatsev [2004]). The latter exhibit spontaneous rotation during the evolution, see Fig. 23(c).

Stabilization of dissipative vortex solitons in the cubic-quintic CGL and new types of radially symmetric solitons, such as erupting, flat-top, and composite vortices, were reported recently by Crasovan, Malomed, and Mihalache [2001a,c]. In the same model, the existence of stable clusters of dissipative solitons rotating around a central vortex core was predicted by Skryabin and Vladimirov [2002]. This extends the results presented in Sect. 6.3 to the case of dissipative CGL systems.

Another example of the dissipative optical systems supporting topological spatial solitons is given by the nonlinear interferometer formed by a liquid crystal light valve with a feedback. The so-called “triangular solitons” with the rich structure of phase singularities were found by Ramazza, Bortolozzo, and Pastur [2004]; Bortolozzo, Pastur, Ramazza, Tlidi, and Kozyreff [2004] in this system.

The vectorial generalization of the CGL model and corresponding patterns of phase dislocations studied by Hernandez-Garcia, Hoyuelos, Colet, and Miguel [2000]; Hoyuelos, Hernandez-Garcia, Colet, and San Miguel [2003]. Similar polarization patterns and vectorial defects were observed in numerical simulations of the three-wave optical parametric oscillators by Santagiustina, Hernandez-Garcia, San-Miguel, Scroggie, and Oppo [2002].

Finally, we mention the three-dimensional generalization of cavity solitons, namely the “bubbles with a dark skin” and 3D Turing structures in synchronously pumped degenerate, and 3D vortex rings in non-degenerate optical parametric oscillators (Weiss, Vaupel, Staliunas, Slekeys, and Taranenko [1999]).

### 8.2. Vortices in Matter Waves

The concept of optical vortices can take us away from optics itself and emphasize the relevance of optical solitons and optical vortices to other fields of nonlinear physics. In particular, the study of vortices is an important research topic in the rapidly developing field of coherent matter waves and nonlinear atom optics. In particular, vortices appear in the nonlinear dynamics of the Bose–Einstein condensates and they provide a close link between self-focusing of light in nonlinear optics and the nonlinear dynamics of matter waves.

The phenomenon known as Bose–Einstein condensation (BEC) was actually predicted in 1924 for systems...
8. LINKS TO VORTICES IN OTHER FIELDS

Figure 24: Generation of vortices and vortex lattices in a rotating $^{87}\text{Rb}$ condensate. (Courtesy P. Engels).

whose particles obey the Bose statistics and whose total particle number is conserved. It was shown that there exists a critical temperature below which a finite fraction of all particles condenses into the same quantum state. Since 1995, the BEC phenomenon has been observed using several different types of atoms, confined by a magnetic trap and cooled down to extremely low temperatures (Anderson, Ensher, Matthews, Wieman, and Cornell [1995]; Bradley, Sackett, Tollett, and Hulet [1995]; Davis, Mewes, Andrews, Vandruten, Durfee, Kurn, and Ketterle [1995]; Fried, Killian, Willmann, Landhuis, Moss, Kleppner, and Greytak [1998]).

From a mathematical point of view, the dynamics of BEC wave function can be described by an effective mean-field equation known as the Gross–Pitaevskii (GP) equation (Dalfovo, Giorgini, Pitaevskii, and Stringari [1999]). This is a classical nonlinear equation that takes into account the effects of particle interaction through an effective mean field. As a matter of fact, the complete theoretical description of a BEC requires a quantum many-body approach (Dalfovo, Giorgini, Pitaevskii, and Stringari [1999]). The many-body Hamiltonian describing $N$ interacting bosons is expressed through the boson field operators $\hat{\Phi}(\mathbf{r})$ and $\hat{\Phi}^\dagger(\mathbf{r})$ that, respectively, annihilate and create a particle at the position $\mathbf{r}$. A mean-field approach is commonly used for the interacting systems to overcome the problem of solving exactly the full many-body Schrödinger equation. Apart from the convenience of avoiding heavy numerical work, mean-field theories allow one to understand the behavior of a system in terms of a set of parameters that have a clear physical meaning. Actually, most of the experimental results show that the mean-field approach is very effective in providing both qualitative and quantitative predictions for the static and dynamic properties of the trapped ultracold gases.

Because of the similarities between the GP equation in the BEC theory and the NLS equation in nonlinear optics, many of the phenomena predicted and observed in nonlinear optics are expected to occur for the BEC macroscopic quantum states, even though the underlying physics can be quite dissimilar. In particular, this includes the dynamics of BEC vortices (Williams and Holland [1999]; Garcia-Ripoll and Perez-Garcia [2000]) recently reviewed by Fetter and Svidzinsky [2001].

Historically, quantum vortices in trapped atomic gases were first observed in 1999, using two-component condensates (Matthews, Anderson, Haljan, Hall, Wieman, and Cornell [1999]). The possibility of trapping more than one BEC component arises from the hyperfine atomic structure. Atoms in internal states with different total angular momentum may coexist in the BEC fraction, and it is possible to induce transitions between their different states. To form a vortex soliton, a phase gradient was imprinted in one of the BEC components, which caused it to rotate. The system was stabilized at a configuration in which the nonrotating component was localized at the center of the trap acting as an effective potential on the rotating component, which resided in the outer region.

The main properties of a two-component BEC can be described using a system of two incoherently coupled GP equations, similar to the two coupled NLS equations that describe optical vector solitons, except for the presence of a trapping potential that prevents the condensate with repulsive interaction from spreading. The two-component GP equation has solutions with remarkable properties (Garcia-Ripoll and Perez-Garcia [2000]). In a situation where the two components overlap considerably, the creation of a vortex in just one of the components is not dynamically stable. The reason is that the angular momentum can be transferred after some
time from one component to the other one, initiating a cyclic process. If one only monitors the density profile of the two atomic species, it may seem that the vortex disappears and eventually reappears in a periodic manner. If a nearly two-dimensional trap is made to rotate, the situation changes qualitatively. In the rotating frame, the Coriolis force manifests itself through an additional term in the Hamiltonian, $-\Omega L_z$, where $L_z$ is a component of the angular momentum operator $\mathbf{L}$ and $\Omega$ is the angular frequency (Fetter and Svidzinsky [2001]). This additional force produces the centrifugal barrier proportional to $L_z^2 z$, where $L_z = m \hbar$ is the angular momentum per particle. Thus, it is always energetically costly to have a high angular momentum. However, for nonzero values of $\Omega$, it may be energetically favorable to develop small positive values of $L_z$. If $\Omega$ is sufficiently large, solutions with $L_z$ greater than $\hbar$ may have the lowest energy; a vortex is created in this situation.

For $\Omega$ greater than a critical frequency $\Omega_c = 0.22 \omega_0$, where $\omega_0$ is the trap frequency, the lowest energy corresponds to the state with $L_z = \hbar$. As the frequency increases, states with increasingly high angular momentum become the effective ground state in the rotating frame, thereby creating vortices with a topological charge $m > 1$. Such configurations are unstable and decompose into several single-charge vortices (Butts and Rokhsar [1999]). The interaction between vortices is generally believed to be repulsive. Thus, at rotating frequencies high enough to generate many stable vortices, vortices tend to move apart and drift toward the borders of the condensate, where they would disappear if their existence were not favored by the rotation. The result is that, at very high angular frequencies, vortices tend to form a regular array, as also observed experimentally. Figure 24 shows examples of vortices for a rotating Rb-vapor BEC. The array formation is akin to what has been long known for type II superconductors, where it is the presence of a magnetic field that forms a triangular vortex crystal—the so-called Abrikosov lattice (Abrikosov [1957]).

In another experiment by Madison, Chevy, Wohlleben, and Dalibard [2000], the formation of a regular vortex array was observed in a $^{87}$Rb BEC as the number of stable vortices raises from 0 to 4 by increasing the angular frequency. The formation of a triangular vortex lattice with as many as 130 vortices was observed in an experiment where BEC was obtained using sodium atoms (Abo-Shaeer, Raman, Vogels, and Ketterle [2001]). The optimum configuration in terms of size and regularity is achieved after 500 ms. For times shorter than that, regular order is not completely established, and a blurry structure is formed because of the misalignment of some vortices with respect to the rotation axis. For times much longer than 500 ms, inelastic collisions induce atom losses and a decrease in the number of vortices. The spin texture of various vortex-lattice states at higher rotation rates and in the presence of an external magnetic field has been presented recently by Mizushima, Kobayashi, and Machida [2004], and intriguing reshaping from the triangular to a square lattice has been observed by Schweikhard, Coddington, Engels, Tung, and Cornell [2004].

Several experiments have studied the nucleation of vortices in a BEC stirred by a laser beam. In the experiment by Raman, Abo-Shaeer, Vogels, Xu, and Ketterle [2001], vortices were generated in a BEC cloud stirred by a laser beam and observed with time-of-flight absorption imaging. Depending on the stirrer size, either discrete resonances or a broad response was visible as the stir frequency was varied. Stirring beams that were small compared to the condensate size generated vortices below the critical rotation frequency for the nucleation of surface modes, suggesting a local mechanism of vortex generation. In addition, it was observed that the centrifugal distortion of the condensate induced by a rotating vortex lattice led to bending of the vortex lines.

Recent developments in the topic of vortices in BEC were summarized by Kevrekidis, Carretero-Gonzalez, Frantzeskakis, and Kevrekidis [2005], including discrete vortices in periodic lattices. Corresponding continuous model is very similar to the optically induced photonic lattices, described in Sect. 7, and supports the “gap vortex soliton” identified by Ostrovskaya and Kivshar [2004]. However, there are examples of the phenomena which have no analogy in optics, for example the structural transitions of the lattice of vortices in the rapidly rotating periodic potential, reported by Pu, Baksanat, Yi, and Bigelow [2003]. Similarly, even the properties of a single vortex soliton in BEC and self-defocusing Kerr media are very similar, the stable vortex dipoles in nonrotating BEC, introduced by Crasovan, Vekslerchik, Perez-Garcia, Torres, Mihalache, and Torner [2003], can not exist in optical system without the external trapping potential, see Sect. 2.1.

Different from singular vortices, the nonsingular topologically nontrivial states, or skyrmions, have attracted an attention since the creation of BEC spinors (Matthews, Anderson, Haljan, Hall, Wieman, and Cornell [1999]). Two-dimensional (2D) particlelike solitons of this kind are sometimes referred as the coreless vortices. Skyrmions can be created out of the ground state, in which all the spins are aligned, by reversing the average spin in a finite region of space (Al Khawaja and Stoof [2001]). Than the Skyrmion is characterized by the winding number, the analog of a topological charge. The half-charge skyrmion has been successfully generated in experiment with three-component BEC by Leanhardt, Shin, Kielpinski, Pritchard, and Ketterle [2003], and possible spin textures for anti- and ferromagnetic interactions in this system were compared by Mueller [2002].

The stability of 3D (Al Khawaja and Stoof [2001]) skyrmions, composed of two coaxial tori, and 2D (Zhai, Chen, Xu, and Chang [2003]) single-charged skyrmions has been studied in two-component BEC. The general conclusion been that skyrmions can exist as a metastable state only, though the stabilizing mechanisms were suggested for single- and multiply-quantized skyrmions by Savage and Ruostekoski [2003]; Ruostekoski [2004]. Skyrmions have important applications in nuclear physics and quantum-Hall effect (Makhankov et al. [1993]),
8. LINKS TO VORTICES IN OTHER FIELDS

After Torres, Deyanova, Torner, and Molina-Terriza [2003], it is expected that observation of these structures in BEC would enable a direct comparison between theory and experiment.

8.3. OPTICAL VORTICES AND QUANTUM INFORMATION

During the recent years a new fascinating avenue for applications of optical vortices in the area of quantum optics has been identified. As discussed extensively throughout this chapter, light beams with nested optical vortices carry orbital angular momentum, a property that holds as well for the mode functions that describe the photon quantum states, including states corresponding to single photons or to entangled pairs. The quantum angular momentum of light contains a spin and an orbital contribution, and in general only the total angular momentum is an observable quantity (Cohen-Tannoudji et al. [1989]). However, within the paraxial regime, both contributions can be measured and manipulated separately (Van Enk and Nienhuis [1994b,a]; Simpson, Dholakia, Allen, and Padgett [1997]; Barnett [2002]; Neil, MacVicar, Allen, and Padgett [2002]; Leach, Courtial, Skeldon, Barnett, Franke-Arnold, and Padgett [2004]). The spin contribution is described by a two-dimensional state, thus can be employed to generate qubits, whereas the orbital contribution can generate multi-dimensional quantum entangled states, or qudits, with an arbitrarily large number of entanglement dimensions. While the spin angular momentum is a workhorse of quantum optics and quantum information (Bouwmeester et al. [2000]), only recently the orbital angular momentum has been added to the toolkit (Arnaut and Barbosa [2000]; Mair, Vaziri, Weihs, and Zeilinger [2001]; Molina-Terriza, Torres, and Torner [2002]).

Allen, Beijersbergen, Spreeuw, and Woerdman [1992] showed a decade ago that paraxial Laguerre-Gaussian laser beams, with a nested vortex, carry a well-defined orbital angular momentum associated to their spiral wave fronts (Allen, Padgett, and Babiker [1999]). The formal analogy between paraxial optics and quantum mechanics implies that such modes are the eigenmodes of the quantum mechanical angular momentum operator. The Laguerre-Gaussian modes form a complete Hilbert set and can thus be used to represent the quantum photon states within the paraxial regime of light propagation. The quantum angular momentum number carried by the photon is then represented by the topological charge, or winding number \( m \), of the corresponding mode, and each mode carries an orbital angular momentum of \( m\hbar \) per photon. Multi-dimensional vector photon states can be constructed with controllable projections into modes with well-defined winding numbers, thus providing higher-dimensional alphabets Molina-Terriza, Torres, and Torner [2002]. In particular, mode functions in the form of vortex-pancakes allow the manipulation, including the addition and removal, of specific projections of the vector states. It is worth stressing that beams without nested vortices, or alternatively with a complex topological structure, can also carry orbital angular momentum. A beautiful illustrative example was shown by Santamato, Sasso, Piccirillo, and Vella [2002] in the classical regime, by studying the optical angular momentum transfer to transparent particles using light beams carrying zero average angular momentum. The concept applies as well to the quantum regime in a variety of beams shapes and geometries. However, because Laguerre-Gaussian modes carrying nested optical vortices are eigenstates of the quantum orbital angular momentum operator, vortices play a central role in the area.

The generation of quantum states entangled in orbital angular momentum relies in the process of spontaneous...
parametric down conversion (Klyshko [1967]; Arnaut and Barbosa [2000]; Mair, Vaziri, Weihs, and Zeilinger [2001]; Franke-Arnold, Barnett, Padgett, and Allen [2002]; Padgett, Courtial, Allen, Franke-Arnold, and Barnett [2002]). The generated entangled two-photon states can be prepared in desired states by making use of transverse engineering of quasi-phase-matched geometries, Torres, Alexandrescu, Carrasco, and Torner [2004] or by appropriate tailoring of the spatial characteristics of the pump beam. The latter can be accomplished by a variety ways, including by pumping the nonlinear crystal with several nested optical vortices (Torres, Deyanova, Torner, and Molina-Terriza [2003]). An illustrative example of the potential engineering of quantum entangled states with pump beams with nested vortex pancake is shown in Fig. 25. The characterization of the entangled photon pairs in terms of eigenstates of the orbital angular momentum operator yields the concept of quantum spiral bandwidth, which was found to depend on the shape of the beam that pumps the down-converting crystal, and on the material properties and length on the crystal (Torres, Alexandrescu, and Torner [2003]). All these schemes are awaiting experimental demonstration, albeit the related concept of entanglement concentration put forward by Torres, Deyanova, Torner, and Molina-Terriza [2003] has been observed experimentally using an alternative approach, as mentioned below.

A fundamental question that arises is the conservation of OAM in the process of photon down-conversion (Arnaut and Barbosa [2000]; Eliei, Dutra, Nienhuis, and Woerdman [2001]; Visser, Eliei, and Nienhuis [2002]; Barbosa and Arnaut [2002]; Caetano, Almeida, Souto Ribeiro, Huguenin, Coutinho dos Santos, and Khoury [2002]). In collinear down-conversion, the two-photon entangled state constituted by the signal and idler photons is described by a transverse mode function that is globally paraxial. Therefore, the orbital angular momentum of all the involved photons is a well-defined quantity that in the absence of momentum transfer between light and matter must be conserved, a feature that within the experimental accuracy is consistent with the observations by Mair, Vaziri, Weihs, and Zeilinger [2001], in the quasi-collinear geometry used. In non-collinear geometries the relation between the orbital angular momentum and the vorticity of the mode function is not necessarily given by simple algebraic rules, as most clearly illustrated in highly non-collinear settings (Molina-Terriza, Torres, and Torner [2003]; Torres, Osorio, and Torner [2004]).

The quantum applications of optical vortices holds promise for exciting developments in the near future, in particular to the proof-of-principle demonstration of quantum algorithms and to explore fundamental quantum features in higher-dimensional Hilbert spaces. Significant advances along this direction have been already achieved during the last years after the observation of OAM entanglement by Mair, Vaziri, Weihs, and Zeilinger [2001]. Important steps include the development of interferometric schemes that might be used to sort out single photons according to their OAM (Vasnetsov, Slyusar, and Soskin [2001]; Leach, Padgett, Barnett, Franke-Arnold, and Courtial [2002]), the generation of qutrits encoded in OAM and their use to observe violation of Bell inequalities in three-dimensional Hilbert spaces (Vaziri, Weihs, and Zeilinger [2002]), the demonstration of concentration of higher-dimensional entanglement (Vaziri, Pan, Jennewein, Weihs, and Zeilinger [2003]), the triggered production, transmission and reconstruction of qutrits for different quantum communication protocols (Molina-Terriza, Vaziri, Rehacek, Hradil, and Zeilinger [2004]), the use of qutrits for quantum bit commitment (Langford, Dalton, Harvey, O’Brien, Pryde, Gilchrist, Bartlett, and White [2004]), the proposal of innovative set-ups to efficiently measure high-dimensional entanglement (Oemrawsingh, Aiello, Eliei, Nienhuis, and Woerdman [2004]), and the demonstration of coin-tossing algorithms based in qutrits (Molina-Terriza, Vaziri, Ursin, and Zeilinger [2004]). Vortices are also being used to explore fundamental quantum features, like the uncertainty principle for angular position and angular momentum (Franke-Arnold, Barnett, Yao, Leach, Courtial, and Padgett [2004]), or the effects induced by the quantum vacuum on the perfect zero of a classical field (the vortex core) (Berry and Dennis [2004]). Much more is expected to come soon, as this area of research is at its infancy.

§ 9. Concluding remarks

We have presented a comprehensive overview of exciting research in the field of nonlinear singular optics that studies the propagation of optical vortices and optical beams carrying an angular momentum in nonlinear media. Understanding and controlling the properties of optical vortices could lead to applications in the near future, ranging from optical communications and data storage to the trapping, control, and manipulation of particles and cold atoms. Indeed, optical vortices provide an efficient way to control light by creating reconfigurable waveguides in bulk media. The study of phase singularities in optical parametric processes not only suggests novel directions of fundamental research in optics but also provides links to other branches of physics. For example, the recent discovery of a rich variety of exotic topological defects in unconventional superfluids (such as $^3$He-A) and superconductors points to the likelihood that deep analogies exist between vortices in complex superfluids and singularities in light waves.
Figure 26: The number of papers cited in this review vs. the publication year.

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