THE RESONANCE METHOD FOR LARGE CHARACTER SUMS

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ABSTRACT. We consider the size of large character sums, proving new lower bounds for the quantity \( \Delta(N, q) = \sup_{\chi \neq \chi_0 \text{ mod } q} \left| \sum_{n \leq N} \chi(n) \right| \) for almost all ranges of \( N \). Our results improve those of Granville and Soundararajan \[2\], and are typically stronger than corresponding bounds known for real character sums. The results are proven using the resonance method and saddle point analysis.

1. INTRODUCTION

In \[2\] Granville and Soundararajan have made an extensive study of large character sums. Varying the Dirichlet character \( \chi \) among non-trivial characters to a large modulus \( q \) they establish (among other results) new lower bounds for the quantity

\[
\Delta(N, q) = \sup_{\chi \neq \chi_0 \text{ mod } q} \left| \sum_{n \leq N} \chi(n) \right| ;
\]

varying \( D \) among squarefree numbers \( q \leq |D| \leq 2q \) they also consider large values of real character sums, giving lower bounds for the quantity\[1\]

\[
\Delta_R(N, q) = \sup_{q \leq |D| \leq 2q} \left| \sum_{n \leq N} \left( \frac{D}{n} \right) \right| .
\]

In both cases their results have been the best known for essentially all ranges of \( N \).

Omega results of these types are interesting because even assuming powerful (conjectural) analytic tools like the Generalized Riemann Hypothesis and bounds derived from Random Matrix Theory for the associated \( L \)-functions, the upper bounds that are available for character sums are not clearly sharp for many ranges of \( N \). Pólya and Vinogradov proved the classical unconditional bound

\[
\left| \sum_{n \leq N} \chi(n) \right| \leq \sqrt{q} \log q
\]

and Montgomery and Vaughan \[5\] improved this to \( \ll \sqrt{q} \log \log q \) under assumption of the GRH for \( L(s, \chi) \). This bound is sharp up to constants, because Paley \[6\] proved that there exist quadratic characters with character sum of size \( \gg \sqrt{q} \log \log q \), and Granville and Soundararajan have shown the same result for non-quadratic characters in \[3\]. Nonetheless, if we believe in roughly square root cancellation in long character sums, then the

\[1\] \( (\mathbb{Z}) \) is the Legendre symbol.
GRH bound appears good only if \( N \) is of size \( q^{1-\epsilon} \). Recently Farmer, Gonek and Hughes \cite{FGH} have conjectured that

\[
\left| L\left(\frac{1}{2} + it; \chi \right) \right| \ll \exp \left( (1 + o(1)) \sqrt{\frac{1}{2} \log(q + |t|) \log \log(q + |t|)} \right)
\]

on the basis of a calculation involving large unitary random matrices. This would lead to a bound for smoothed character sums of

\[
\sum_{n \leq N} \chi(n) \phi\left(\frac{n}{N}\right) \ll \sqrt{N} \exp \left( (1 + o(1)) \sqrt{\frac{1}{2} \log q \log q} \right), \quad \phi \geq 0 \in C_c^\infty(\mathbb{R}^+),
\]

improving the GRH bound for \( N < q^{1-\epsilon} \). If the bound \( (3) \) is true then we might speculate that for \( N \) a power of \( q \), \( N = q^\theta \), \( 0 < \theta < 1 \) that the resulting bound for \( \sum \chi(n) \phi\left(\frac{n}{N}\right) \) is nearly sharp; one reason for this belief is that in our Theorem 1.6 below, we demonstrate that for any such \( q, N \) there are many non-principal \( \chi \) modulo \( q \) for which

\[
\left| \sum_{n \leq N} \chi(n) \right| \geq \sqrt{N} \exp \left( (1 + o(1)) \sqrt{1 - \theta} \log q \log q \right).
\]

When \( N < \exp \left( \sqrt{2 \log q \log \log q} \right) \) even the random matrix bound is trivial. One insight into large character sums for such small \( N \) is available through a conjecture of Granville and Soundararajan. For arithmetic function \( f \), write as in \cite{GS98}

\[
\Psi(x, y; f) = \sum_{n \leq x, p|n \Rightarrow p \leq y} f(n)
\]

for the sumatory function of \( f \) restricted to numbers having their largest prime factor at most \( y \), and also set \( \Psi(x, y) := \Psi(x, y; 1) \) for the number of such '\( y \)-smooth numbers' less than \( x \). Granville and Soundararajan have proposed the following conjecture.

**Conjecture 1.1** \((\text{\cite{GS98} Conjecture 1})\). There exists a constant \( A > 0 \) such that

\[
\sum_{n \leq N} \chi(n) = \Psi(N, y; \chi) + o(\Psi(N, y; \chi_0))
\]

for \( y = (\log q + \log^2 N) \log_2 A^4 q \).

In particular, this conjecture implies the upper bound

\[
\left| \sum_{n \leq N} \chi(n) \right| \ll \Psi(N, (\log q + \log^2 N) \log_2 A^4 q).
\]

Note that in the range \( \log N > \sqrt{\log q} \) where the \( \log^2 N \) term dominates, \( \Psi(N, \log^2 N) \) is of size \( \sqrt{N} \exp\left( \frac{\log N}{\log_2 N} \right) \), which is already much larger than the random matrix bound if \( N \) is a small power of \( q \); there is not wide-spread consensus, however, regarding the conjectured bound \( (3) \), and so it is plausible that even this larger bound for character sums is near the truth. When \( \log N < \sqrt{\log q} \), by modifying an argument of Granville and Soundararajan (their Theorem 2) one can show\footnote{Although we do not do so here.} that the upper bound \( (4) \) with, say, \( A = 3 \) follows from...
Conversely, in our Theorem 1.2 we establish that if \( q \) is prime and \( \log N = (\log q)^{1/2 - \varepsilon} \) then there is non-principal character \( \chi \) modulo \( q \) with
\[
\left| \sum_{n \leq N} \chi(n) \right| \geq \Psi(N, \log q \log_2^{1-o(1)} q);
\]
thus if Conjecture 1.1 is to hold, one must take \( A \geq 1 \).

There is some heuristic reason to think that the constant \( A = 1 \) may be the right one: given roughly \( q \) characters modulo \( q \), we might expect that the first \( \log q \) values of \( \chi(p) \) may be correlated towards 1 for some character \( \chi \), which would produce a positive bias on the \( \log q \log_2 q \)-smooth numbers. We should point out, however, that our argument in Theorem 1.2 does not condition the first \( \log q \) primes, but rather finds a smaller bias among many primes that are larger than \( \log q \log_2 q \). Thus there may be reason to believe that the large values of \( \sum_{n \leq N} \chi(n) \) are even larger than \( \Psi(N, \log q \log_2 q) \).

1.1. Discussion of previous work. Surprisingly, the methods used in [2] in treating general characters and real characters are not at all related; the lower bounds for \( \Delta(N,q) \) are produced by taking high moments of the character sums\(^3\), while for \( \Delta_R(N,q) \) the argument appeals to quadratic reciprocity and Dirichlet’s theorem on primes in arithmetic progressions to first restrict to prime \( |D| \) for which \( \left( \frac{D}{p} \right) = 1 \) for all small \( p \ll \log q \). This produces a large positive bias in the sum coming from the smooth numbers that have all of their prime factors less than \( \log q \).

When \( N \) is relatively small compared to \( q \), in the range \( \log N < \sqrt{\log q} \), the two methods perform roughly equally although the results for real characters are slightly stronger. For larger \( N \) this difference becomes more pronounced, essentially because the first method is most effective when the moment taken is quite large. Note, however, that the two methods are not directly comparable in [2] because the results for large real character sums are produced for conductors \( D \) that are prime, whereas the results for general characters are stated primarily for the worst case when \( q \) is the product of many distinct small primes; when the conductor \( q \) of a character sum is highly composite in this way it greatly reduces the size of the large character sums. To give a trivial example, it may transpire that \( \Delta(q, \log q) = 0 \) since it is possible that for all \( n \leq \log q, (n, q) > 1 \).

In this paper we adapt the ‘resonance method’ introduced by Soundararajan in [7] to prove new bounds for \( \Delta(N,q) \). We consider separately the bounds that this obtains for \( q \) prime and for any \( q \). For general \( q \) we improve the bounds for \( \Delta(N,q) \) in [2] for all \( N \) larger than a fixed power of \( \log q \). For prime \( q \) our bounds are stronger than those obtained in [2] for \( \Delta_R(N,q) \) for all \( N \) in the range \( \exp((\log \log q) \log_2 q) \ll N \ll q \exp(-((\log \log q)^2)) \); outside this range the bounds given by [2] were already (at least conjecturally) best possible. Moreover, for large \( N \), \( \exp(\sqrt{\log q}) \ll N \ll q^{1-\varepsilon} \) our improvements over the previous bounds are substantial.

\(^3\)Essentially the \( 2k \)th moment where \( k = \left\lfloor \frac{\log q}{\log N} \right\rfloor \).
We will describe our results in greater detail in the next section, but we first pause to explain the resonance method in brief, and its conceptual advantage over the two methods previously developed in [2]; there is a sense in which this method generalizes each of the earlier approaches. The starting point is the simple inequality

\[
\min\{x_1, \ldots, x_n\} \leq \frac{w_1 x_1 + \ldots + w_n x_n}{w_1 + \ldots + w_n} \leq \max\{x_1, \ldots, x_n\},
\]

which is valid for any non-negative weights \(w_1, \ldots, w_n\) with some \(w_i \neq 0\). In the resonance method for character sums, the indices are characters, the variables \(x_\chi = \sum_{n \leq N} \chi(n)\) or \(x_\chi = \left|\sum_{n \leq N} \chi(n)\right|^2\) are character sums, and the weights are (squared norms of) Dirichlet polynomials:

\[
w_\chi = \left|\sum_{n \leq x} r(n) \chi(n)\right|^2, \quad x \leq \frac{q}{N},
\]

Here the coefficients \(r(n)\) are fixed, non-negative, and multiplicative, and are chosen so as to maximize the ratio in (5). When \(N\) is small, a good (but not optimal) choice for the weight \(w_\chi\) is

\[
w_\chi = \left|\left(\sum_{n \leq N} \chi(n)\right)^{k-1}\right|^2,
\]

and with this choice of weights, the resonance method is seen to contain the first method of [2]. On the other hand, we are free to choose a weight of the form

\[
w_\chi = \left|\prod_{p < P} (1 + \chi(p))\right|^2,
\]

which has the effect of placing much more emphasis on those \(\chi\) for which \(\chi(p) \approx 1\) for many small primes. Thus the resonance method can be interpreted as taking a conditional expectation of character sums with optimal conditioning, which, at least philosophically, extends the second method of [2].

### 1.2. Precise statement of results.

Our lower bounds for \(\Delta(N, q)\) come in three forms: we give lower bounds for \(\Delta(N, q)\) that hold when \(q\) is prime, and for any \(q\). When \(q\) is prime we also consider the dual problem of ‘long’ character sums, and give lower bounds for \(\Delta(N, q)\). Recall that the Erdős-Kac Theorem says that a ‘typical’ number of size \(x\) has \(\sim \log \log x\) distinct prime factors; our lower bounds for \(\Delta(N, q)\), \(q\) prime in fact apply equally well if \(q\) is typical, and even if \(q\) has \(\log^{1-\epsilon} q\) prime factors, but we restrict to the case that \(q\) is prime to ease the exposition. For the dual problem we rely on a ‘Fourier expansion’ of character sums, due to Pólya, that is only valid for primitive \(\chi\); in this case the restriction to prime \(q\) seems to be necessary to the method.

In our first two theorems we consider the range \(\log N < \sqrt{\log q}\).

**Theorem 1.2.** Let \(q\) be a large prime and let \(\log N < \sqrt{\log q}\) and define functions

\[
G(\sigma) = \frac{\Gamma(1 - \frac{1}{2\sigma})\Gamma(\frac{3}{2} - 1)}{\Gamma(\frac{3}{2\sigma})}, \quad \kappa(\sigma) = 2^{1-\sigma} G(\sigma)^{\frac{\sigma}{2\sigma}}.
\]
Let $\sigma > \frac{1}{2}$ solve $\log N = \frac{\log q}{2(1-\sigma)} G(\sigma)^{\sigma}$. Then
$$\Delta(N, q) \geq \Psi(N, (1 + o(1)) \kappa(\sigma) \log q).$$

Let $\sigma' > \frac{1}{2}$ solve $\log N = \frac{\log q}{2^2(1-\sigma')} G(\sigma')^{\sigma'}$. Then
$$\Delta\left(\frac{q}{N}, q\right) \gg \sqrt{q} \frac{\log \log q}{\log N} \Psi(N, \log q).$$

Furthermore, if $\log N < \log^2 q \log^{-10} q$ then
$$\Delta(N, q) \gg \sqrt{q} \frac{\log \log q}{\log N} \Psi(N, \log q).$$

The function $\kappa(\sigma)$ is decreasing on $(\frac{1}{2}, 1)$. It satisfies $\lim_{\sigma \to 1} \kappa(\sigma) = \frac{8}{e^3}$ and $\kappa(\sigma) \sim \frac{1}{4\sigma - 8}$ as $\sigma \downarrow \frac{1}{2}$.

We can deduce the following corollary.

**Corollary 1.3.** Suppose $\log N = (\log q)^{o(1)}$. Then
$$\Delta(N, q) \geq \Psi(N, (\frac{8}{e^3} + o(1)) \log q), \quad \frac{8}{e^3} = 0.3982965...$$

If $\frac{1}{2} < \sigma < 1$ is fixed and $\log N = (\log q)^{1-\sigma}$ then
$$\Delta(N, q) \geq \Psi(N, (\kappa(\sigma) + o(1)) \log q).$$

Finally, if instead $\log N = \sqrt{\log q} \exp(- (\log q)^{o(1)})$ then
$$\Delta(N, q) \geq \Psi(N, \log q \log^2 q^{1-o(1)}) q).$$

For $q$ prime, in [2] Theorem 3 the bound $\Delta(N, q) \geq \Psi(N, \log q)$ was proved for $\log N < \frac{\log q}{\log^2 q}$. In this range we have $\Psi(N, \log q) \sim \Psi(N, \frac{8}{e^3} \log q)$ and so our theorem extends this result to $\log N < \sqrt{\log q}$. A more direct comparison is to [2] Theorem 9, where in the range $\log N < \sqrt{\log q}$ they prove give a bound for real characters of
$$\Delta_R(N, q) \geq \Psi(N, \frac{1}{3} \log q).$$

In the range $\log N = (\log q)^{o(1)}$ the bound from Theorem [1,2] is already superior, and as $\log N$ increases, the ratio $\frac{\kappa(\sigma) \log q}{\frac{1}{3} \log q}$ in the number of smooth numbers taken, tends to $\infty$. Regarding the dual statement, [2] Theorem 8 previously gave the bound
$$\max_{\nu \leq T} \max_{\chi \neq \chi_0 mod q} \left| \sum_{n \leq T} \chi(n) \right| \gg \sqrt{q} \frac{1}{N} \Psi(N, (\log q^{\frac{1}{(\log q)^{10}}});$$
our theorem thus removes the need for a second maximum over $t$, and improves the bound. The gain over the previous result for real characters ([2] Theorem 11) is comparable to the improvement for $\Delta(N, q)$.

We next state our result for general moduli $q$. 
Theorem 1.4. Let \( q \) be any large integer and let \( N \) be such that \( \log^B q < N < \exp(\sqrt{\log q}) \) for a sufficiently large fixed constant \( B \). In the range \( \log N < \log_2^3 q \log_3 q \), there exists a parameter \( \eta = \eta(N, q) = (1 + o(1)) \log_2 q \) such that,
\[
\Delta(N, q) \geq \frac{N^{\frac{1}{2} + \frac{|u'\log N|}{2\log N}} \left( \frac{e}{\sqrt{2 + o(1)}} \right)^{|u'|}}{\log N^{u'}} , \quad u' = \frac{\eta \log N}{\eta + 1 \log_2 q}.
\]

In the wider range \( \log N \gg \log_2^3 q \log_3 q \), set \( \log N = (\log q)^{1-\sigma} \). We have
\[
\Delta(N, q) \geq \frac{N}{\log N^u} \left( \frac{e - o(1)}{\sqrt{2(2\sigma - 1)}} \right)^u , \quad u = \frac{\log N}{\log_2 q}.
\]

This theorem should be compared with [2] Theorem 4; for all \( N \) larger than a fixed power of \( \log q \) we obtain an improvement which is at least exponential in \( u \); as \( \log N \) increases to \( \sqrt{\log q} \) the improvement is larger than any fixed exponential in \( u \).

Next we consider the range \( \log \log N = (\frac{1}{2} + o(1)) \log \log q \).

Theorem 1.5. Let \( q \) be any large integer and let \( \log N = \tau \sqrt{\log q \log_2^2 q} \) with \( \tau = (\log_2 q)^{O(1)} \). Define \( A \) and \( \tau' \) by solving
\[
\tau = \int_A^\infty e^{-x} \frac{dx}{x} , \quad \tau' = \int_A^\infty e^{-x} \frac{dx}{x^2}.
\]

Then
\[
\Delta(N, q) \geq \sqrt{N} \exp \left( (1 + o(1))A(\tau + \tau') \sqrt{\frac{\log q}{\log_2 q}} \right).
\]

If \( q \) is prime then instead define \( A \) and \( \tau' \) by
\[
\sqrt{2\tau} = \int_A^\infty e^{-x} \frac{dx}{x} , \quad \tau = \int_A^\infty e^{-x} \frac{dx}{x^2}.
\]

With this new definition we have
\[
\Delta\left(\frac{q}{N}, q\right) \geq \sqrt{\frac{q}{N}} \exp \left( (1 + o(1))A \left( \tau + \frac{\tau'}{\sqrt{2}} \right) \sqrt{\frac{\log q}{\log_2 q}} \right).
\]

Note that as \( \tau \to \infty, A\tau \to 0 \) and \( A\tau' \to 1 \). Meanwhile, as \( \tau \to 0, A \to \infty \) and \( \tau \sim \frac{\sqrt{A}}{\tau'}, \tau' \sim \frac{\sqrt{A}}{A\tau} \). This theorem should be compared to [2] Theorems 5 and 8; a direct comparison is difficult because their statement is not explicit, but asymptotically our result is stronger as \( \tau \to 0 \) and \( \tau \to \infty \).

Finally we consider longer character sums.

Theorem 1.6. Let \( q \) be a large integer and let \( 4\sqrt{\log q \log_2 q \log_3 q} < \log N \) and \( N = q^\theta \) with \( \theta < 1 - \epsilon \). Then
\[
\Delta(N, q) \geq \sqrt{N} \exp \left( (1 + o(1)) \sqrt{\frac{(1 - \theta) \log q}{\log_2 q}} \right).
\]
If in addition \( q \) is prime then

\[
\Delta\left(\frac{q}{N}, q\right) \geq \sqrt{\frac{q}{N}} \exp \left( (1 + o(1)) \sqrt{\frac{(1 - \theta) \log q}{2 \log_2 q}} \right).
\]

These bounds are substantially larger than those proved in [2] Theorems 6, 7, and 8 for \( \Delta(N, q) \) in the corresponding range. The improvement is most noticeable when \( N = q^\theta \) is a power of \( q \); for this range [2] Theorem 7 gives only \( \Delta(N, q) > \sqrt{N} (\log q)^{O(\theta^{-1})} \). Our bound is also larger than the one for real characters in [2] Theorem 10:

\[
\Delta_R(N, q) \geq \sqrt{N} \exp \left( (1 + o(1)) \frac{\sqrt{\log q}}{\log_2 q} \right).
\]

2. The basic propositions and outline of proofs

The basic proposition of the resonance method is the following.

**Proposition 2.1** (Fundamental Proposition). Let \( N < \sqrt{q} \) and \( x = \frac{o(q)}{N} \). Let \( r(n) \) be a completely multiplicative function satisfying \( r(p) \geq 0 \) and \( p|q \Rightarrow r(p) = 0 \). Set

\[
B = \frac{\sum_{n=1}^{\infty} r(n)^2}{\sum_{n \leq \frac{x}{N}} r(n)^2}.
\]

Then

\[
\Delta(N, q) \geq O(1) + \frac{1}{B} \sum_{n \leq N} r(n).
\]

Furthermore, let \( M \) be minimal such that \( \sum_{p \leq M} \log p > \log q \); the bound remains valid if the restriction \( p|q \Rightarrow r(p) = 0 \) is replaced with \( p \leq M \Rightarrow r(p) = 0 \).

Remark. The following proof will show (essentially) that

\[
\Delta(N, q) \geq \sum_{n \leq N} r(n) \left[ \frac{\sum_{m \leq x \frac{z}{n}} r(m)^2}{\sum_{m \leq x} r(m)^2} \right]
\]

for all non-negative multiplicative functions \( r \). In practice we will apply Proposition [2.1] with \( B = 1 + o(1) \) as \( q \to \infty \) so that we aim to solve the optimization problem

\[
\text{Maximize: } \sum_{n \leq N} r(n)
\]

\[
\text{Subject to: } \sum_{n \leq \frac{x}{N}} r(n)^2 = (1 + o(1)) \sum_{n} r(n)^2
\]

Here the constraint condition (8) is closely related to the condition at primes

\[
\sum_{p} \log p \frac{r(p)^2}{1 - r(p)^2} < \log x
\]
via the saddle point method. If we assume that the optimal choice for \( r \) in (7) satisfies 
\[
\sum_{n \leq N} r(n) = N^{\theta},
\]
then 
\[
\sum_{n \leq N} r(n) \sim \left( \frac{N}{\theta} \right)^{\theta}.
\]
Thus maximization of (7) with respect to (8) via Lagrange multipliers leads to the heuristic solution 
\[
\frac{r(p)}{(1 - r(p)^2)^2} = \lambda \frac{\log p}{p^\theta},
\]
which guides our choice of resonator functions.

One might reasonably wonder whether the imposed condition \( B = 1 + o(1) \) is superfluous; we could instead suppose that 
\[
\sum_{n \leq x} r(n)^2 \sim x^\alpha.
\]
Then 
\[
\sum_{n \leq x} r(n)^2 \sim \left( \frac{x}{n} \right)^{\alpha},
\]
so that we would instead obtain the optimization problem

\[
\begin{align*}
\text{Maximize:} & \quad \sum_{n \leq N} r(n)n^{-\alpha} \\
\text{Subject to:} & \quad \sum_{n \leq x} r(n)^2 \sim x^\alpha.
\end{align*}
\]

Replacing \( r \) with \( \tilde{r}(n) = \frac{r(n)}{n^\alpha} \), this is is subsumed in the previous optimization problem.

**Proof.** Define ‘resonator’ \( R(\chi) = \frac{1}{\sqrt{\phi(q)}} \sum_{n \leq x} r(n)\chi(n) \). Plainly

\[
\Delta(N, q) \geq \sum_{\chi \neq \chi_0} |R(\chi)|^2 \sum_{n \leq N} \chi(n) / \sum_{\chi} |R(\chi)|^2.
\]

By orthogonality of characters, the denominator is \( \sum_{n \leq x} r(n)^2 \). Meanwhile the numerator is

\[
\sum_{\chi} |R(\chi)|^2 \sum_{n \leq N} \chi(N) - |R(\chi_0)|^2 \sum_{n \leq N} \chi_0(n) = \sum_{n \leq N} \sum_{m \leq \frac{x}{n}} r(m)r(mn) - O \left( \frac{xN}{\phi(q)} \sum_{n \leq x} r(n)^2 \right) \geq \left( \frac{1}{B} \sum_{n \leq N} r(n) - O(1) \right) \sum_{n \leq x} r(n)^2,
\]

which proves the first statement in the proposition.

To prove the second statement, let \( r \) be any non-negative completely multiplicative function supported on primes larger than \( M \), and set \( B = \frac{\sum_n r(n)^2}{\sum_{n \leq N} r(n)^2} \). Enumerate

\[
\{ p > M : p | q \} = \{ q_1, \ldots, q_R \}, \quad \{ p \leq M : p \nmid q \} = \{ p_1, \ldots, p_S \}.
\]

Note that \( S \geq R \). Now swapping the values \( r(p_1), \ldots, r(p_R) \) with \( r(q_1), \ldots, r(q_R) \) we define a new completely multiplicative function \( \tilde{r} \). Obviously \( \tilde{r} \) is supported on primes not dividing \( q \), and also

\[
\sum_{n} \tilde{r}(n)^2 = \sum_{n} r(n)^2, \quad \sum_{n \leq \frac{N}{n}} \tilde{r}(n)^2 \geq \sum_{n \leq \frac{N}{n}} r(n)^2 \quad \Rightarrow \quad \frac{\sum_{n \leq \frac{N}{n}} \tilde{r}(n)^2}{\sum_{n \leq \frac{N}{n}} r(n)^2} \leq B
\]

\[
\sum_{n \leq N} \tilde{r}(n) \geq \sum_{n \leq N} r(n)
\]
The first part of the proposition applied to $\tilde{r}$ thus gives
\[
\Delta(N, q) \geq O(1) + \frac{1}{B} \sum_{n \leq N} r(n).
\]

For primitive characters $\chi$ Pólya proved the Fourier expansion\(^4\) (see [5] Lemma 1)
\[
\sum_{n \leq q} \chi(n) = \frac{\tau(\chi)}{2\pi i} \sum_{|h| \leq H \atop h \neq 0} \overline{\chi(h)} \frac{1}{h} (1 - e(-\frac{h}{N})) + O(1 + qH^{-1} \log q).
\]
Choose $H = \sqrt{Nq} \log q$ and define $S(\chi) = \sum_{|h| \leq H \atop h \neq 0} \overline{\chi(h)} (1 - c(\frac{h}{N}))$. If $\chi$ is even ($\chi(1) = 1$) this vanishes, but for odd primitive characters $\chi$ we have
\[
\left| \sum_{n \leq N} \chi(n) \right| = \frac{\sqrt{q}}{2\pi} |S(\chi)| + O(\sqrt{\frac{q}{N}}).
\]
Using this relation we now prove a dual version of our main proposition.

**Proposition 2.2** (Fundamental Proposition, dual version). Let $q$ prime, $N < \sqrt{q}$, $H = \sqrt{Nq} \log q$, $x = \frac{q}{2H}$ and $N' < H$ a parameter. Let $r$ be a non-negative completely multiplicative function and put
\[
B_{N'} = \frac{\sum_{n \leq N'} r(n)^2}{\sum_{n \leq N} r(n)^2}.
\]
We have
\[
\Delta(\frac{q}{N}, q) \geq \frac{\sqrt{q}}{\pi} \frac{1}{B_{N'}} \sum_{0 < n \leq N'} \frac{r(n)}{n} (1 - c(\frac{n}{N})) + O(\sqrt{\frac{q}{N}}).
\]
Specializing to $N' = N/2$ we obtain
\[
\Delta(\frac{q}{N}, q) \gg \frac{\sqrt{q}}{N^2} \frac{1}{B_{N/2}} \sum_{n \leq N} nr(n) + O(\sqrt{\frac{q}{N}}).
\]

**Proof.** Define, as before, ‘resonator’ $R(\chi) = \frac{1}{\sqrt{q-1}} \sum_{n \leq x} r(n) \chi(n)$. Then
\[
\Delta(N, q) + O(\sqrt{\frac{q}{N}}) \geq \frac{\sqrt{q}}{\pi} \sup_{\chi} |S(\chi)| \geq \frac{\sqrt{q}}{\pi} \sum_{\chi} |R(\chi)|^2 S(\chi) \left/ \sum_{\chi} |R(\chi)|^2 \right.
\]
The denominator is $\sum_{n \leq x} r(n)^2$. Since $x = \frac{q}{2H}$, the sum in the numerator is
\[
\sum_{0 \neq |h| \leq H} \frac{(1 - c(\frac{h}{N}))}{h} \sum_{n_1, n_2 \leq x \atop n_1 \equiv n_2 \mod q} r(n_1) r(n_2) = \sum_{0 < h \leq H} \frac{r(h)(1 - c(\frac{h}{N}))}{h} \sum_{n \leq \frac{N}{2}} r(n)^2.
\]
Since all terms in the numerator are positive, we can discard those $h > M$ to obtain the result. \qed

\(^{4}\text{We use the notation } e(x) = e^{2\pi ix} \text{ and } c(x) = \cos(2\pi x).\)
2.1. **Outline of proofs, and lemmas.** After introduction of a ‘resonator’ multiplicative function \( r(n) \), the proofs of Theorems 1.2-1.6 proceed in the same three steps.

A. Let \( x \) be the length of the resonator and \( N \) the length of the sum. We check that
\[
\sum_{n \leq x} r(n)^2 = (1 + o(1)) \prod_p (1 - r(p)^2)^{-1}
\]
so that \( B \) in Proposition 2.1 may be taken as \( 1 + o(1) \).

B. From the Fundamental Proposition and part A it follows \( \Delta(N, q) \sim \sum_{n \leq N} r(n) \). We determine the asymptotic shape of this sum via the Perron integral
\[
\sum_{n \leq N} r(n) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} R(s) N^s \frac{ds}{s}, \quad R(s) = \prod_p (1 - r(p)p^{-s})^{-1}
\]
and a saddle point calculation.

C. We analyze the various implicitly defined parameters that arise in the saddle point calculation of part B in order to obtain explicit lower bounds for \( \Delta(N, q) \).

The bound in the first step (A) is accomplished by an appeal to the following simple lemma.

**Lemma 2.3.** Let \( f_i(n) \) be a sequence of non-negative, completely multiplicative functions satisfying \( f_i(n) < 1 \), and let \( y_i \to \infty \) be a growing sequence of parameters. Define \( \alpha_i = \frac{\log \log y_i}{\log y_i} \).

Suppose
\[
\sum_p \log p \frac{f_i(p)}{1 - f_i(p)} < \log y_i - \frac{\log y_i}{\log \log y_i}
\]
and
\[
\sum_{k \log p > \log y_i} f(p_i)^k p_i^{k\alpha_i} = o(1), \quad i \to \infty.
\]

Then
\[
\sum_{n \leq y_i} f_i(n) = (1 + o(1)) \sum_{n = 1}^{\infty} f_i(n), \quad i \to \infty.
\]

**Proof.** By ‘Rankin’s trick,’
\[
\sum_{n \leq y_i} f_i(n) = \sum_{n = 1}^{\infty} f_i(n) + O \left( y_i^{\alpha_i} \sum_{n = 1}^{\infty} f_i(n)n^{\alpha_i} \right).
\]
The logarithm of the ratio between the error term and the main term is
\[
(11) \quad - \alpha_i \log y_i + \log \prod_p \left( \frac{1 - f_i(p)}{1 - f_i(p)p^{\alpha_i}} \right).
\]
The logarithm of the infinite product is
\[
\sum_{p,k} \frac{f_i(p)^k}{k} \left[ p^{\alpha_i} - 1 \right] \leq \left( 1 + O(\log \log^{-2} y_i) \right) \alpha_i \sum_p \log p \frac{f_i(p)}{1 - f_i(p)} + \sum_{k \log p > \log y_i} \frac{f_i(p)p^{\alpha_i}}{k} \leq \alpha_i \left( \log y_i - \frac{\log y_i}{\log \log y_i} + O\left( \frac{\log y_i}{(\log \log y_i)^2} \right) \right) + o(1);
\]
which proves that the quantity in (11) tends to $-\infty$ as $i \to \infty$. □

The analysis in the second step (B) closely follows the corresponding analysis of smooth numbers contained in [4]. We briefly recall this theory and quote the results from it that we will need.

Recall that we set \( \Psi(x, y) = \#\{n \leq x : p|n \Rightarrow p \leq y\} \) for the number of \( y \)-smooth numbers less than \( x \) and set also
\[
\zeta(s, y) = \prod_{p<y} (1 - p^{-s})^{-1}
\]
for the corresponding generating function. Analysis of \( \Psi(x, y) \) depends on the behavior of the logarithm of \( \zeta(s, y) \) and its first few derivatives,
\[
\psi(s, y) = \log \zeta(s, y), \quad \psi_j(s, y) = (-1)^j \frac{d^j}{ds^j} \psi(s, y), \quad j = 0, 1, 2, ...
\]
for \( s \) near the saddle point \( \alpha = \alpha(x, y) > 0 \), solving \( \psi_1(\alpha, y) = \log x \). The basic result is the following.

**Theorem 2.4.** Uniformly in the range \( x \geq y \geq 2 \), we have
\[
\Psi(x, y) = \frac{x^\alpha \zeta(\alpha, y)}{\alpha \sqrt{2\pi \psi_2(\alpha, y)}} \left\{ 1 + O\left( \frac{\log y}{\log x} + \frac{\log y}{y} \right) \right\}.
\]

**Proof.** This is [4] Theorem 1.6. □

The first stage in the proof of Theorem 2.4 is to write
\[
\Psi(x, y) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \zeta(s, y)x^s \frac{ds}{s}
\]
and to truncate the integral at some height \( T \). The following lemma, which we use, is essentially the one given in [4] to bound the error from truncation.

**Lemma 2.5.** Let \( f(n) \) be a bounded, non-negative arithmetic function with Dirichlet series \( F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \), which converges absolutely at \( s = \sigma_0 > 0 \). Uniformly in \( x \geq 2, T \geq 1, \sigma \geq \sigma_0 \) and \( 0 < \tau < T \) we have
\[
\sum_{n \leq x} f(n) = \frac{1}{2\pi i} \int_{\sigma-i\tau}^{\sigma+i\tau} F(s) \frac{x^s}{s} ds + R
\]
where $R$ is bounded by

$$R \ll 1 + x^\sigma F(\sigma) \left(T^{-\frac{1}{2}} + \sup_{\tau \leq t \leq T} \left| \frac{F(\sigma + it)}{F(\sigma)} \right| \log T \right).$$

Proof. See [4] Lemma 3.3.

For the purpose of making comparisons in Theorem 1.2 it will be sufficient for us to know the asymptotic behavior of $\log \Psi(x, y)$. In this case, the behavior is well understood in a wide range of $x$ and $y$. Set $u = \frac{\log x}{\log y}$ and let $\rho(u)$ denote the Dickman-de Bruijn function.

**Theorem 2.6.** For any fixed $\epsilon > 0$ we have

$$\log(\Psi(x, y)/x) = \left\{ 1 + O(\exp\{-\log u^{3/5-\epsilon}\}) \right\} \log \rho(u)$$

uniformly in the range

$$y \geq 2, \quad 1 \leq u \leq y^{1-\epsilon}.$$

Proof. This is [4] Theorem 1.2.

**Theorem 2.7.** For $u \geq 1$ we have

$$\rho(u) = \exp \left\{ -u \left( \log u + \log_2(u+2) - 1 + O\left( \frac{\log_2(u+2)}{\log u} \right) \right) \right\}.$$

Proof. This is [4] Corollary 2.3.

In particular we have the following simple lemma.

**Lemma 2.8.** Let $u = \frac{\log x}{\log y}$ as above. When $u < \sqrt{y}$ and for $|\kappa| < 1$ we have

$$\log \frac{\Psi(x, e^\kappa y)}{\Psi(x, y)} = \left( \kappa + O(\log^{-1} u) \right) u(\log u + \log_2(u+2)).$$

3. **Short character sums to prime moduli, Proof of Theorem 1.2**

We dispose of quickly the case of small $N$, $\log N < \log_2^2 q \log_3^{-10} q$ for both $\Delta(N, q)$ and $\Delta(\frac{N}{q}, q)$. The main work of Theorem 1.2 will then be to consider the range $\log_2^2 q \log_3^{-10} q < \log N = o(\sqrt{\log q} \log_2 q)$.

3.1. **Case of small $N$.** When $\log N < \log_2^2 q \log_3^{-10} q$ notice that for all $n \leq N \log q$, $d(n) \ll \log_2^2 q \log_3^{-11} q$. Thus choosing

$$r(p) = \begin{cases} 1 - (\log_2 q)^{-2} & 1 < p \leq \frac{\log q}{\log_2 q} \\ 0 & \text{otherwise} \end{cases},$$

where $r(p)$ is the number of prime divisors of $p$. setting
we verify
\[
\sum_p \log p \frac{r(p)^2}{1 - r(p)^2} \leq \log_2^4 q \sum_{p < \frac{\log q}{\log_2^2 q}} \log p \ll \frac{\log q}{\log_2 q}
\]
and
\[
\sum_{k \log p > \frac{\log q}{\log_2 q}} r(p)^2 k \frac{\log_2^2 q}{p^{\frac{1}{2} \log q}} \ll \log q \left(1 - \frac{\log_2 q}{\log q}\right) \frac{2 \log q}{\log_2 q} = o(1),
\]
so that by Lemma 2.3, \(\sum_{n \leq q}^n r(n)^2 = (1 + o(1)) \prod_p (1 - r(p)^2)^{-1}\). Furthermore, for all \(n < N \log q\) such that \(p | n \Rightarrow p < \frac{\log q}{\log_2 q}\) we have \(r(n) \gg 1\).

Choosing \(x = \frac{q}{N}\), we have \(\sum_{n \leq \frac{q}{N}} r(n)^2 = (1 + o(1)) \sum_n r(n)^2\), and so by Proposition 2.2 we have
\[
\Delta(N, q) \geq (1 + o(1)) \sum_{n < N} r(n) \geq (1 + o(1)) \Psi(N, \frac{\log q}{\log_2 q}) \geq (1 + o(1)) \Psi(N, \log q).
\]

Choosing \(x = \frac{1}{\log q} \sqrt{N}\) and setting \(N' = N \log q\), we have \(\sum_{n \leq \frac{q}{N'}} r(n)^2 = (1 + o(1)) \sum_n r(n)^2\), and so by Proposition 2.2 we have
\[
\Delta\left(\frac{q}{N}, q\right) \gg \sqrt{q} \sum_{n \leq N} \frac{r(n)}{n} \left(1 - c\left(\frac{n}{N}\right)\right)
\]
\[
\gg \sqrt{q} \sum_{A=1}^{\log q} \sum_{\frac{N}{A} < h < \frac{3N}{A}} \frac{r(AN + h)}{AN + h} \gg \sqrt{q} \sum_{A=1}^{\log q} \frac{1}{A} \sum_{\frac{N}{A} < h < \frac{3N}{A}} r(AN + h).
\]
It now follows as in [2], (‘Proof of Theorem 11’, p. 394) that
\[
\Delta\left(\frac{q}{N}, q\right) \gg \frac{\sqrt{q}}{N} \Psi(N, \log q) \frac{\log \log q}{\log \left(\frac{\log N}{\log \log q}\right)}.
\]
This completes the proof of Theorem 1.2 in the case \(\log N < \log_2^2 q \log_3^{-10} q\).

3.2. Main case. Henceforth we assume that \(\log_2^2 q \log_3^{-10} q < \log N = o(\sqrt{\log q \log_2 q})\). We are going to describe the analysis of \(\Delta(N, q)\) in detail. Afterwards we will sketch the necessary modifications in order to handle the dual case of \(\Delta\left(\frac{q}{N}, q\right)\).

Throughout the treatment of \(\Delta(N, q)\) we fix \(x = \frac{q}{N}\). Let \(\epsilon = \epsilon(q) > 0\) be a parameter tending to 0 as \(q \rightarrow \infty\) and set \(M = (1 - \epsilon) \log q\). We let \(\sigma = \sigma(N, q), \frac{1}{2} + \frac{1}{\log_2 q} < \sigma < 1\) be another parameter which will eventually be the location of the relevant saddle point. We define completely multiplicative ‘resonator’ function \(r_\sigma(n)\) by
\[
r_\sigma(p) = f_\sigma\left(\frac{p}{M}\right)
\]
where \(0 < f_\sigma(x) < 1\) is the unique continuous solution to the equations
\[
\frac{f_\sigma(x)}{(1 - f_\sigma(x)^2)^2} = \left(\frac{c_\sigma}{x}\right)^\sigma, \quad \int_0^\infty \frac{f_\sigma(x)^2}{1 - f_\sigma(x)^2} dx = 1.
\]
Note that the second equation implicitly defines the constant $c_\sigma$. The following basic properties of the function $f_\sigma$ may be established with a little calculus.

**Lemma 3.1.** For each $\sigma \in (\frac{1}{2}, 1)$, the function $f_\sigma$ satisfies the following properties.

1. $f_\sigma$ is smooth, decreasing, and a bijection $(0, \infty) \to (0, 1)$.
2. $f_\sigma(x) \leq \left(\frac{\sigma}{c_\sigma}\right)^x$.
3. $\min\left(\frac{1}{2}, \frac{1}{4} \left(\frac{x}{c_\sigma}\right)^{\frac{1}{2}}\right) \leq 1 - f_\sigma(x) \leq \left(\frac{x}{c_\sigma}\right)^{\frac{1}{2}}$.
4. The value of $c_\sigma$ is
   \[ c_\sigma = \frac{\Gamma\left(\frac{3}{2\sigma}\right)}{\Gamma\left(1 - \frac{1}{2\sigma}\right)\Gamma\left(\frac{3}{\sigma} - 1\right)} \]
   In particular, $c_\sigma < 1$, $c_\sigma \to \frac{1}{2}$ as $\sigma \uparrow 1$ and $c_\sigma \sim (2\sigma - 1)$ as $\sigma \downarrow \frac{1}{2}$.
5. The Mellin transform of $f_\sigma$ is
   \[ \hat{f}_\sigma(s) = \int_0^\infty f_\sigma(x)x^{s-1}dx = \frac{c_\sigma^s \Gamma\left(\frac{s}{2\sigma} - \frac{s}{2(2\sigma - 1)}\right)\Gamma\left(\frac{3s}{2} - 1\right)}{\Gamma\left(\frac{3s}{2}\right)} \Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{3}{2\sigma}\right) \]
   which converges absolutely in $0 < \Re(s) < \sigma$. In particular
   \[ \hat{f}_\sigma(1 - \sigma) = \frac{1}{2(1 - \sigma)} e^{-\sigma}. \]
6. The Mellin transform of $g_\sigma(x) = \frac{f_\sigma(x)^2}{1 - f_\sigma(x)^2}$ is
   \[ \hat{g}(s) = \int_0^\infty g(x)x^{s-1}dx = \frac{c_\sigma^s \Gamma\left(1 - \frac{s}{2\sigma}\right)\Gamma\left(\frac{3s}{2} - 1\right)}{\Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{3s}{2}\right)} \Gamma\left(\frac{3}{2\sigma}\right) \]
   This integral converges absolutely in $\frac{s}{2} < \Re(s) < 2\sigma$.

**Proof.** The various integrals may be computed by substituting $\frac{dx}{x} = -\left[\frac{4f}{1-f^2} + \frac{1}{f}\right] df$. □

**Lemma 3.2.** Uniformly in $\frac{1}{2} < \sigma < 1$ there is a constant $c > 0$ such that
\[ \left| \frac{\hat{f}_\sigma(1 - \sigma + it)}{\hat{f}_\sigma(1 - \sigma)} \right| \leq \left\{ \begin{array}{ll}
\left(1 + \frac{c^2}{(1 - \sigma)^2}\right)^{\frac{1}{2}} & |t| < \frac{1}{4} \\
\frac{1}{1 - c} & |t| \geq \frac{1}{4}.
\end{array} \right. \]

**Proof.** This follows on Taylor expanding $\hat{f}(s)$ about $s = 1 - \sigma$. □

3.3. **Bound of tail for sum of squares, Proof of part A of Theorem 1.2.** We apply Lemma 2.3 with $y_i = y_q = \frac{x}{N}$. Recall that we set $\alpha = \frac{\log y}{\log \log y}$ and that we assume $\sigma > \frac{1}{2} + \frac{1}{\log_2 q}$. 

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For \( \epsilon > \frac{C}{\log \log q} \), \( C \) fixed but sufficiently large, we have uniformly in \( \frac{1}{2} < \sigma < 1 \),
\[
\sum_p \log p \frac{r(p)^2}{1 - r(p)^2} \leq \sum_n \Lambda(n) \frac{f_\sigma(\frac{n}{M})^2}{1 - f_\sigma(\frac{n}{M})^2} = \frac{1}{2\pi i} \int_{\frac{1}{2} + \sigma} \frac{-C'(s)M^s \hat{g}(s)}{\zeta(s)} ds \leq M(1 + O(\exp(-\sqrt{\log M}))) \leq \log \frac{x}{N} - \log \frac{x}{\log \log q}
\]
by shifting the contour into the standard zero-free region for \( \zeta \) and passing the pole at 1 of \( -\zeta' \zeta \). Thus the first condition of the lemma is satisfied. Meanwhile,
\[
\sum_{\sigma > \frac{1}{2} + \frac{1}{\log_2 q}} \log p \sum_{p > \exp(\log^3 M)} r(p)^2 p^{2k\alpha} \ll \sum_{p < \exp(\log^3 M)} \frac{(1 - (1 - f_\sigma(\frac{p}{M})^2))^2}{1 - f_\sigma(\frac{p}{M})^2} + \sum_{p > \exp(\log^3 M)} \frac{f_\sigma(\frac{p}{M})^2 p^{\alpha}}{1 - f_\sigma(\frac{p}{M})^2 p^{\alpha}}.
\]
Substituting the lower bound [(3) of Lemma 3.1] for \( 1 - f_\sigma \) in the first sum and the upper bound [(2) of Lemma 3.1] for \( f_\sigma \) in the second sum proves that each sum is \( o(1) \), uniformly in \( \sigma > \frac{1}{2} + \frac{1}{\log_2 q} \). This verifies the second condition of Lemma 2.3. It follows that
\[
\sum_{n \leq N} r_\sigma(n)^2 = (1 + o(1)) \prod_p (1 - r_\sigma(p)^2)^{-1}, \quad q \to \infty,
\]
uniformly for \( \sigma \in [\frac{1}{2} + \frac{1}{\log_2 q}, 1] \).

3.4. Saddle point asymptotics, Proof of part B of Theorem 1.2. We introduce the generating Dirichlet series
\[
R_\sigma(s) = \prod_p \left( 1 - \frac{r_\sigma(p)}{p^s} \right)^{-1}, \quad \Re(s) \geq 1 - \sigma
\]
Define its logarithm and logarithmic derivatives by
\[
\phi_{0,\sigma}(s) = \log R_\sigma(s), \quad \phi_{j,\sigma}(s) = (-1)^j \frac{d^j}{ds^j} \phi_0(s), \quad j > 0.
\]
Explicitly,
\[
\phi_{j,\sigma}(s) = \sum_p \log^j p \sum_{n=1}^{\infty} n^{j-1} \left( \frac{r_\sigma(p)}{p^s} \right)^n, \quad (j \geq 0).
\]
We prove the following proposition, which asymptotically evaluates \( \sum_{n \leq N} r_\sigma(n) \).

**Proposition 3.3.** Let \( \frac{1}{2} < \sigma < 1 \) solve \( \log N = \phi_{1,\sigma}(\sigma) \). Then
\[
\sum_{n \leq N} r_\sigma(n) \sim \frac{N^\sigma e^{\phi_{0,\sigma}(\sigma)}}{\sigma \sqrt{2\pi \phi_{2,\sigma}(\sigma)}}.
\]
In order to establish this proposition, we first require some bounds on \( R_{\sigma}(s) \) away from the real axis. These are very similar to the bounds established in [4] for \( \zeta(s, y) \).

It will be convenient to work with the ‘non-multiplicative’ approximations

\[
\tilde{\phi}_{j,\sigma}(s) = \sum_{n=1}^{\infty} \Lambda(n) f_{\sigma}(\frac{nM}{q}) \log^{j-1} n.
\]

Our first lemma demonstrates that \( \tilde{\phi}_j \) is in fact a strong approximation to \( \phi_j \).

**Lemma 3.4.** Let \( s = \sigma + it \). Uniformly in \( \frac{1}{2} + \frac{1}{\log_2 q} \leq \sigma \leq 1 - \frac{1}{\log_2 q} \) we have

\[
\| \phi_{j,\sigma}(s) - \tilde{\phi}_{j,\sigma}(s) \| \ll \frac{\log^2 M}{M^{\min\left(\frac{1}{2}, \sigma - \frac{1}{2}\right)}}.
\]

**Proof.** We have

\[
\| \phi_{j,\sigma}(s) - \tilde{\phi}_{j,\sigma}(s) \| = \left| \sum_p \log^j p \sum_{n=2}^{\infty} n^{j-1} f_{\sigma}(\frac{nM}{q})^n - f_{\sigma}(\frac{p^n}{M}) \right|
\]

\[
\leq \sum_p \log^j p \sum_{n=2}^{\infty} n^{j-1} \left| 1 - f_{\sigma}(\frac{nM}{q})^n \right| + \left| 1 - f_{\sigma}(\frac{p^n}{M}) \right|
\]

\[
\leq \sum_p \log^j p \left| 1 - f_{\sigma}(\frac{p}{M}) \right| \left( \sum_{n=2}^{\infty} n^{j-1} \frac{p^n}{p^{n\sigma}} \right) + \sum_p \log^j p \sum_{n=2}^{\infty} n^{j-1} \left| 1 - f_{\sigma}(\frac{p^n}{M}) \right|
\]

and the claimed bound follows on substituting the upper bound in (3) of Lemma 3.1. \( \square \)

Our next lemma allows us to make explicit the relationship between \( N \) and \( \sigma \) by evaluating \( \phi_{1,\sigma}(\sigma) \).

**Lemma 3.5.** We have uniformly in \( \frac{1}{2} + \frac{2}{\log_2 q} < \sigma < 1 \), and \( |t| < \log_2 q \)

\[
\phi_{1,\sigma}(\sigma + it) = \hat{f}_{\sigma}(1 - \sigma - it) M^{1-\sigma - it} - \frac{\zeta'}{\zeta}(\sigma + it) + O(M^{1-\sigma} \exp(-\sqrt{\log M}))
\]

In particular, for \( \sigma \) solving \( \phi_{1,\sigma}(\sigma) = \log N \), the bounds \( \log_q \log_2 q^{-10} < \log N < \sqrt{\log q} \) imply \((1 - \sigma) \gg \frac{\log_q q}{\log_2 q} \) and \((\sigma - \frac{1}{2}) \gg \frac{\log_q q}{\log_2 q} \) as \( q \to \infty \).

**Proof.** It suffices to prove the corresponding result for \( \tilde{\phi}_1 \) since the previous lemma implies that the resulting error is contained in the error term. To prove this lemma, write

\[
\tilde{\phi}_1(\sigma + it) = \frac{1}{2\pi i} \int_{(\frac{1}{2})} \left( -\frac{\zeta'}{\zeta}(s + \sigma + it) \right) M^s \hat{f}_{\sigma}(s) ds,
\]

shift contours, and use the standard zero-free region. \( \square \)

**Lemma 3.6.** Suppose that \( \sigma \) varies with \( q \) in such a way that \( (1-\sigma)\log_2 q \to \infty \), \( 2(\sigma-1)\log_2 q \to \infty \) as \( q \to \infty \). Then

\[
\phi_{j,\sigma}(\sigma) \sim_j \phi_{0,\sigma}(\sigma) \log^j q.
\]
Proof. Fix \( Y \) such that \( \log Y = \log M + \sqrt{\frac{\log M}{2\sigma - 1}} \). Then
\[
\tilde{\phi}_{j,\sigma}(\sigma) = \sum_{n} \frac{\Lambda(n) f_{\sigma}(\frac{n}{M}) \log^{j-1}(n)}{n^\sigma} \leq \tilde{\phi}_{0,\sigma}(\sigma) \log^{j} Y - \sum_{n > Y} \frac{\Lambda(n) f_{\sigma}(\frac{n}{M}) \log^{j-1}(n)}{n^\sigma}.
\]
Since \( f_{\sigma}(\frac{n}{M}) \leq (\frac{M c_\sigma}{n})^\sigma \) and \( \log Y \geq \frac{1}{2\sigma - 1} \), the negative term is bounded by
\[
\frac{(M c_\sigma)^\sigma}{(2\sigma - 1)Y^{2\sigma - 1}} \log^{j-1} Y \lesssim \frac{\log Y}{(1 - \sigma) \log M} \left( \frac{1}{(M/Y)^{2\sigma - 1}} \right) \lesssim \frac{\log^{j} Y \tilde{\phi}_{0,\sigma}(\sigma)}{M} \left( \frac{M}{Y} \right)^{2\sigma - 1},
\]
and this is \( o(\tilde{\phi}_{0,\sigma}(\sigma)) \). In particular, \( \tilde{\phi}_{0,\sigma}(\sigma) \gg \frac{1}{\log M} \tilde{\phi}_{1,\sigma}(\sigma) \).

Now choose \( Z \) so that \( \log Z = \log M - \sqrt{\frac{\log M}{1 - \sigma}} \). Then
\[
\tilde{\phi}_{j,\sigma}(\sigma) \geq \log^{j} Z \left[ \tilde{\phi}_{0,\sigma}(\sigma) - \sum_{n < Z} \frac{\Lambda(n) f_{\sigma}(\frac{n}{M})}{n^\sigma \log n} \right].
\]
Bounding \( f_{\sigma} < 1 \) and using \( \log Z \geq \frac{1}{1 - \sigma} \), we have
\[
\sum_{n < Z} \frac{\Lambda(n) f_{\sigma}(\frac{n}{M})}{n^\sigma \log n} \asymp Z^{1-\sigma} \lesssim \frac{Z^{1-\sigma}}{(1 - \sigma) \log Z} \lesssim \frac{M^{1-\sigma}}{\log M} \left( \frac{Z}{M} \right)^{1-\sigma} \asymp \frac{\tilde{\phi}_{1,\sigma}(\sigma)}{\log M} \left( \frac{Z}{M} \right)^{1-\sigma} \ll \tilde{\phi}_{0,\sigma}(\sigma) \left( \frac{Z}{M} \right)^{1-\sigma} = o(\tilde{\phi}_{0,\sigma}(\sigma)).
\]

Thus \( \tilde{\phi}_{j,\sigma}(\sigma) \sim_{j} \tilde{\phi}_{0,\sigma}(\sigma) \log^{j} M \). But then applying Lemmas \( 3.4 \) and \( 3.5 \), \( \tilde{\phi}_{j,\sigma}(\sigma) \sim \tilde{\phi}_{j,\sigma}(\sigma) \) for each \( j \).

Lemma 3.7. Let \( \phi_{1,\sigma}(\sigma) = \log N \). Then uniformly for \( N \) varying in the range \( \log^{2} q \log^{3-10} q < \log N < \sqrt{\log q} \) we have
\[
\Re \left[ \tilde{\phi}_{0,\sigma}(\sigma) - \tilde{\phi}_{0,\sigma}(\sigma + it) \right] \gg \begin{cases} t^2 \phi_{2,\sigma}(\sigma), & |t| < (\log Y)^{-1} \\ \phi_{1,\sigma}(\sigma) \min \left( \frac{t^2}{\log^{2} M}, \frac{1}{2\sigma - 1} \right), & (\log Y)^{-1} < |t| < M \end{cases},
\]
where \( \log Y = \log M + \sqrt{\frac{\log M}{2\sigma - 1}} \).

Proof. For all \( t \) we have
\[
\Re \left[ \tilde{\phi}_{0,\sigma}(\sigma) - \tilde{\phi}_{0,\sigma}(\sigma + it) \right] = \sum_{n} \frac{\Lambda(n) f_{\sigma}(\frac{n}{M})}{n^\sigma \log n} (1 - \cos(t \log n)).
\]
For \( |t| < (\log Y)^{-1} \) this is
\[
\gg t^2 \tilde{\phi}_{2,\sigma}(\sigma) - O(t^2 \sum_{n > Y} \frac{\Lambda(n) f_{\sigma}(\frac{n}{M}) \log n}{n^\sigma}) \gg t^2 \tilde{\phi}_{2,\sigma}(\sigma) \left( 1 - O \left( \left( \frac{M}{Y} \right)^{2\sigma - 1} \right) \right),
\]
by bounding the tail as in the previous lemma. This is \( \gg t^2 \tilde{\phi}_{2,\sigma}(\sigma) \) since \( (2\sigma - 1) \log M \to \infty \).
For \((\log Y)\) we have

\[
\left(14\right) \geq \frac{1}{\log Y} \Re \left\{ \sum_n \frac{\Lambda(n) f_{\sigma}(\frac{n}{M})}{n^\sigma} (1 - n^{-it}) - \sum_{n > Y} \frac{\Lambda(n) f_{\sigma}(\frac{n}{M})}{n^\sigma} \right\}
\]

\[
\geq \frac{1}{\log Y} \Re \left\{ \hat{f}_\sigma(1 - \sigma) M^{1-\sigma} \left( 1 - \frac{\hat{f}_\sigma(1 - \sigma - it) M^{-it}}{\hat{f}_\sigma(1 - \sigma)} \right) + \frac{\zeta'}{\zeta} (\sigma + it) - \frac{\zeta'}{\zeta} (\sigma) + O \left( \frac{M^{1-\sigma}}{\exp(-\sqrt{\log M})} \right) \right\}.
\]

Since \(\hat{f}_\sigma(1 - \sigma) \gg (1 - \sigma)^{-1}\) we obtain

\[
\phi_1(\sigma) \min \left( \frac{t^2}{(1-\sigma)^2}, 1 \right),
\]

by applying Lemma 3.2.

Proof of Proposition 3.3: Choose \(T = \log N \log^2 M, \delta = 1 - \sigma\) and apply Lemma 2.5 and the second bound of Lemma 4.4 to deduce that

\[
\sum_{n \leq N} r(n) = N^\sigma e^{\phi_{0,\sigma}(\delta)} \left\{ \frac{1}{2\pi} \int_{-(1-\sigma)}^{1-\sigma} \exp(it \log N + \phi_{0,\sigma}(\sigma + it) - \phi_{0,\sigma}(\sigma)) \frac{dt}{\sigma + it} \right\} + O \left( \frac{\log M \sqrt{\log N}}{\log Y} \right) + O \left( \frac{e^{-\frac{c \log N}{3 \log^2 M}}}{\log \log N} \right).
\]

Since \(\phi_{2,\sigma}(\sigma) \sim \log N \log M\), these are genuine error terms. Now \(|\phi_{3,\sigma}(\sigma + it)| \leq \phi_{3,\sigma}(\sigma) \sim \log^2 M \log N\) holds for all \(t\). Splitting the integral accordingly at \(|t| = \log^{-2/3} M \log^{-1/3} N\), and \(|t| = \frac{1}{\log Y}\) with \(Y\) as in Lemma 4.4 we obtain the main term by Taylor expanding \(\phi_0\) on the interval around 0, and error terms in the remaining part of the integral.

3.5. Comparison to smooth number asymptotics, proof of part C of Theorem 1.2. In this section we complete the proof of Theorem 1.2 in the case \(\log^2 q \log^3 q < \log N < \sqrt{\log q}\) by comparing \(\sum_{n \leq N} r_{\sigma}(n)\) to \(\Psi(N, \kappa(\sigma) M)\). via the following proposition.

Proposition 3.8. Let \(\kappa(\sigma)\) be defined by \(\kappa^{1-\sigma} = (1 - \sigma) \hat{f}_\sigma(1 - \sigma)\). Let, as before, \(\log N = \phi_{1,\sigma}(\sigma)\). We have

\[
\left| \log \left( \frac{\sum_{n \leq N} r_{\sigma}(n)}{\Psi(N, \kappa(\sigma) M)} \right) \right| = O \left( \frac{\log N}{(2\sigma - 1) \log^2 M} \right).
\]

By Lemma 2.8 so long as \(|\theta| < 1\) and \(|\theta| \log_3 q \to \infty\) as \(q \to \infty\),

\[
\log \frac{\Psi(N, e^{\theta \kappa(\sigma) M})}{\Psi(N, \kappa(\sigma) M)} \sim \frac{\theta}{\log M} \log N \log \frac{\log N}{\log M}.
\]

Therefore, since

\[
(2\sigma - 1) \log \left( \frac{\log N}{\log M} \right) \to \infty
\]
Proposition 3.8 establishes that there is some $\kappa' = (1 + o(1))\kappa$ for which $\Delta(N, q) \geq \Psi(N, (1 + o(1))\kappa \log q)$, which completes the proof of Theorem 1.2 for $\Delta(N, q)$. Therefore, it suffices to prove Proposition 3.8.

Recall that we defined $\psi_j(s; y) = \sum_{p < y} \log^{j-1} p \sum_{n=1}^{\infty} \frac{1}{n^{s-1}} n^s$. The following essential lemma allows us to establish the approximation $\tilde{\phi}_{j, \sigma}(\sigma) \approx \tilde{\psi}_{j}(\sigma, \kappa M)$, $j = 1, 2$.

**Lemma 3.9.** We have, uniformly in $\frac{1}{2} + \frac{1}{\log M} < \sigma < 1$,

1. $\phi_{1, \sigma}(\sigma) - \psi_{1}(\sigma, \kappa M) = O(M^{1-\sigma} \exp(-\sqrt{\log M}))$.

2. $\phi_{0, \sigma}(\sigma) - \psi_{0}(\sigma, \kappa M) = O\left(\frac{\phi_{1, \sigma}(\sigma)}{(2\sigma - 1) \log^{2} M}\right)$

**Proof.** In analogy with (13) we introduce

$$\tilde{\psi}_{j}(s, \kappa M) = \sum_{n \leq \kappa M} \frac{\Lambda(n) \log^{j-1} n}{n^{s}}.$$  

In the range $\frac{1}{2} + \frac{1}{\log M} \leq \sigma \leq 1 - \frac{1}{\log M}$, the uniform bound

$$\left| \psi_{j}(\sigma, \kappa M) - \tilde{\psi}_{j}(\sigma, \kappa M) \right| \ll M^{\frac{1}{2} - \sigma} \log^{j} M$$

is straightforward to establish along the lines of Lemma 3.4. Thus it suffices to prove the corresponding statements of this lemma for $\tilde{\phi}_{i, \sigma}$ and $\tilde{\psi}_{i}$.

To prove the first statement, write

$$\tilde{\phi}_{1, \sigma}(\sigma) - \tilde{\psi}_{1}(\sigma, \kappa M) = \frac{1}{2\pi i} \int_{(\frac{1}{2})} \left( -\frac{\zeta'(s) + \sigma}{\zeta(s)} \right) M^{s} \left[ f_{\sigma}(s) - \frac{\kappa^{s}}{s} \right] ds$$

and note that the two poles, at $s = 1 - \sigma$ and at $s = 0$, are nullified by the difference. Shift contours.

For the second, we have

$$\tilde{\phi}_{0, \sigma}(\sigma) - \tilde{\psi}_{0}(\sigma, \kappa M) = \sum_{n \leq \kappa M} \frac{\Lambda(n) (f_{\sigma}(\frac{n}{M}) - 1)}{n^{\sigma} \log n} + \sum_{n \geq \kappa M} \frac{\Lambda(n) f_{\sigma}(\frac{n}{M})}{n^{\sigma} \log n}$$

$$= \frac{1}{\log M} \left[ \sum_{n \leq \kappa M} \frac{\Lambda(n) (f_{\sigma}(\frac{n}{M}) - 1)}{n^{\sigma}} + \sum_{n \geq \kappa M} \frac{\Lambda(n) f_{\sigma}(\frac{n}{M})}{n^{\sigma}} \right]$$

$$+ O\left( \sum_{n \leq \kappa M} \frac{\Lambda(n) |1 - f_{\sigma}(\frac{n}{M})|}{n^{\sigma}} \left| \log^{-1} n - \log^{-1} M \right| \right)$$

$$+ O\left( \sum_{n \geq \kappa M} \frac{\Lambda(n) f_{\sigma}(\frac{n}{M})}{n^{\sigma}} |\log^{-1} n - \log^{-1} M| \right)$$
The bracketed term is \( \frac{1}{\log M}[\tilde{\phi}_1,\sigma(\sigma) - \tilde{\psi}_1(\sigma, \kappa M)] = O(M^{1-\sigma} \exp(-\sqrt{\log M})) \), which is permissible since \( \phi_1,\sigma(\sigma) \sim \psi_1(\sigma, \kappa M) \sim \frac{(\kappa M)^{1-\sigma} - 1}{1-\sigma} \). In the first error term, we use \(|1 - f_\sigma(\frac{n}{M})| \ll \left(\frac{n}{\kappa M}\right)^{\frac{\sigma}{2}}\) for \( n < \kappa M \), and bound trivially \(|1 - f_\sigma(\frac{n}{M})| < 1\) for \( n > \kappa M \) to obtain

\[
\ll \frac{(\kappa M)^{1-\sigma}}{1-\frac{\sigma}{2}} \left| \log c_\sigma \right| + 1 + \frac{(\kappa M)^{1-\sigma}}{1-\sigma} \left| \log \kappa \right| + 1 \frac{1}{\log^2 M}.\]

In the second error term we bound \( f_\sigma(\frac{n}{M}) \ll \left(\frac{\kappa M}{n}\right)^{\sigma} \) to obtain

\[
\ll \frac{(\kappa M)^{1-\sigma}}{2\sigma - 1} \left(\frac{\kappa M}{n}\right)^{\sigma} \left[ \left| \log \kappa \right| + 1 \frac{1}{\log^2 M} + \frac{1}{(2\sigma - 1) \log^2 M} \right].\]

Since \( \frac{1}{2\sigma - 1} \left(\frac{\kappa M}{n}\right)^{\sigma} \) is bounded as \( \sigma \downarrow \frac{1}{2} \) and \( |\log c_\sigma|, |\log \kappa| \ll \frac{1}{2\sigma - 1} \), these errors are also permissible.

We also need the following analogy of Lemma 3.6 for the \( \psi_j(\sigma, \kappa M) \).

**Lemma 3.10.** Assume that as \( q \to \infty \), \( \sigma \) satisfies \( \frac{1}{2} < \sigma < 1 \) and \( (\sigma - \frac{1}{2}) \log M \to \infty \), \( (1 - \sigma) \log M \to \infty \). Then

\[
\psi_j(\sigma, \kappa M) \sim_j \log^j M \psi_0(\sigma, \kappa M).\]

**Proof.** Observe

\[
\tilde{\psi}_j(\sigma, \kappa M) = \sum_{n \leq \kappa M} \frac{\Lambda(n) \log^{j-1} n}{n^\sigma} \sim \frac{(\kappa M)^{1-\sigma} \log^{j-1} M}{1-\sigma},
\]

so that \( \tilde{\psi}_j(\sigma, \kappa M) \sim \log^j M \tilde{\psi}_0(\sigma, \kappa M) \). The statement follows, since \( \psi_j(\sigma, \kappa M) \sim \tilde{\psi}_j(\sigma, \kappa M) \) for all \( j \).

**Proof of Proposition 3.8** Define \( \tilde{N} \) by \( \log \tilde{N} = \psi_1(\sigma, \kappa M) \). Then

\[
\left| \log \left( \frac{\sum_{n \leq N} r_\sigma(n)}{\Psi(\tilde{N}, \kappa M)} \right) \right| \leq \left| \log \left( \frac{\sum_{n \leq N} r(n)}{\Psi(\tilde{N}, \kappa M)} \right) \right| + \left| \log \left( \frac{\Psi(\tilde{N}, \kappa M)}{\Psi(N, \kappa M)} \right) \right| = I + II.
\]

We first consider \( I \). By Proposition 3.3 applied to \( \sum_{n \leq N} r_\sigma(n) \) and and Theorem 2.4 applied to \( \Psi(\tilde{N}, \kappa M) \) we have

\[
I = \left| \sigma(\phi_{1,\sigma}(\sigma) - \psi_1(\sigma, \kappa M)) + \phi_{0,\sigma}(\sigma) - \psi_0(\sigma, \kappa M) - \frac{1}{2} \left| \log \phi_{2,\sigma}(\sigma) - \log \psi_2(\sigma, \kappa M) \right| \right| + o(1).
\]

Since \( \phi_{2,\sigma}(\sigma) \sim \psi_2(\sigma, \kappa M) \sim \log N \log M \), substituting the bounds of Lemma 3.9 gives

\[
I = O \left( \frac{\log N \log \left( \frac{\log N}{\log M} \right)}{(2\sigma - 1) \log^2 M} \right).
\]
For $II$, let $\alpha$ solve $\psi_1(\alpha, \kappa M) = \log N$ so that, by Theorem 2.4 applied to both $\Psi(\tilde{N}, \kappa M)$ and $\Psi(N, \kappa M)$,

$$II = \sigma \log \tilde{N} - \alpha \log N + \psi_0(\sigma, \kappa M) - \psi_0(\alpha, \kappa M) - \frac{1}{2} [\log \psi_2(\sigma, \kappa M) - \log \psi_2(\alpha, \kappa M)] + o(1).$$

Now by the Mean Value Theorem

$$|\log \tilde{N} - \log N| = |\psi_1(\sigma, \kappa M) - \psi_1(\alpha, \kappa M)| = |\sigma - \alpha| \cdot \psi_2(\gamma, \kappa M),$$

$$|\psi_0(\sigma, \kappa M) - \psi_0(\alpha, \kappa M)| = |\sigma - \alpha| \cdot \psi_1(\gamma', \kappa M),$$

for some $\gamma, \gamma'$ between $\alpha$ and $\sigma$. Moreover,

$$\log N \sim \log \tilde{N}$$

so

$$\psi_2(\gamma, \kappa M) \sim \psi_2(\alpha, \kappa M) \sim \psi_2(\sigma, \kappa M) \sim \log N \log M$$

and therefore

$$|\sigma - \alpha| \ll \frac{|\phi_1(\sigma) - \psi_1(\sigma, \kappa M)|}{\log N \log M} \ll \exp(-\sqrt{\log M}).$$

Combining these estimates, we find

$$II \ll \sigma |\log \tilde{N} - \log N| + |\sigma - \alpha| \log N + o(1) = O\left(\frac{\log N}{\exp(-\sqrt{\log M})}\right),$$

completing the proof.

$\Box$

3.6. Dual case $\Delta(\frac{q}{N}, q)$. In this section we take $x = \frac{1}{\log q} \sqrt{\frac{q}{N}}$, $M = (\frac{1}{2} - \epsilon) \log q$ and define completely multiplicative function $r_\sigma(n)$ by

$$r_\sigma(p) = f_\sigma(\frac{p}{M}).$$

A calculation that is exactly analogous to the one in Section 3.3 proves that for $\epsilon$ tending sufficiently slowly to 0,

$$\sum_{n \leq \frac{q}{N}} r_\sigma(n)^2 = (1 + o(1)) \prod_p (1 - r_\sigma(p)^2)^{-1},$$

so that, by the second part of Proposition 2.2

$$\Delta(\frac{q}{N}, q) \gg \frac{q}{N^2} \sum_{n \leq \frac{N}{2}} nr_\sigma(n).$$

The evaluation of $\sum_{n \leq \frac{N}{2}} nr_\sigma(n)$ by a Perron integral yields

$$\frac{N}{2} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{N}{2}\right)^{s} R_\sigma(s) \frac{ds}{s + 1};$$

the only difference between this integral and the one evaluated in Proposition 3.3 is the factor of $2^{-s}$ and the fact that $\frac{1}{s}$ has been replaced by $\frac{1}{s+1}$ in the denominator. Exactly the same method as there yields, for $\sigma$ solving $\phi_{1, \sigma}(\sigma) = \log N - \log 2$,

$$\sum_{n \leq \frac{N}{2}} nr_\sigma(n) \sim \frac{(N/2)^{1+\sigma} e^{\phi_{0, \sigma}(\sigma)}}{(1 + \sigma) \sqrt{2\pi \phi_{2, \sigma}(\sigma)}}.$$
It then follows that for \( \kappa'(\sigma) \) solving \( \kappa'^{1-\sigma} = (1 - \sigma) \hat{f}(1 - \sigma) \),

\[
\Delta \left( \frac{q}{N}, q \right) \gg \frac{\sqrt{q}}{N} \Psi \left( N, (1 + o(1))\kappa'(\sigma) M \right).
\]

Since \( M = (1 - o(1)) \frac{\log q}{2} \), therefore

\[
\Delta \left( \frac{q}{N}, q \right) \gg \sqrt{q} N \Psi \left( N, \left( \frac{1}{2} + o(1) \right) \kappa(\sigma) \log q \right).
\]

### 4. Short sums to composite moduli, Proof of Theorem 1.4

Throughout let \( M \) be minimal such that \( \sum_{p \leq M} \log p > \log q \), and define \( u = \frac{\log N}{\log M} \). Also, in this section \( x = \frac{o(q)}{N} \). When \( q \) is the product of many small distinct primes the behavior of \( \Delta(N, q) \) changes near where \( \log N = \log_3 q \). Informally this is explained by the fact that for \( N < \exp(\log_3^2 q) \), most numbers less than \( N \) but having all prime factors larger than \( \log q \) are composed of the same number of prime factors for larger \( N \) this is no longer the case.

#### 4.1. Case \( \log N = o(\log_2 q \log_3 q) \)

Let \( P > M \), \( \log P \sim \log M \) be a parameter to be chosen, and let \( \epsilon = \frac{C \log q}{\log_2 q} \) for a fixed sufficiently large constant \( C \). Set \( L = \sqrt{MP} \frac{1}{(1+\epsilon)\sqrt{2}} \). We define completely multiplicative \( r(n) \) by

\[
 r(p) = \begin{cases} 
 \frac{L \log p}{p}, & P < p < P^2 \\
 0, & \text{otherwise} 
\end{cases}
\]

**Remark.** Intuitively we can understand the parameter \( P \) as follows. In order to maximize \( \sum_{n \leq N} r(n) \) we would like to have \( r(n) \) be non-zero for as many values \( n \) as possible, but we would also like its value to be as large as possible. By increasing the starting point \( P \) of the resonator we decrease the number of \( n \) for which \( r(n) \) is non-zero, but for the remaining \( n \) we increase the value of \( r(n) \). \( P \) will ultimately be chosen as a compromise between these two competing factors.

#### 4.1.1. Bound for tail of squares, Proof of part A of Theorem 1.4

We are going to apply Lemma 2.3 to prove \( \sum_{n \leq x} r(n)^2 = (1 + o(1)) \sum_n r(n)^2 \). Thus \( y = \frac{x}{N} \) and \( \alpha = \frac{\log_2 y}{\log y} \).

We have

\[
\sum_{P < p < P^2} \log p \frac{r(p)^2}{1 - r(p)^2} \leq \frac{L^2 \log^2 p}{1 - L^2 \log^2 p} \sum_{P < p < P^2} \frac{\log^3 p}{p^2}
\]

\[
\leq (1 + O(\log^{-1} P)) P \frac{L^2 \log^2 P}{P^2 - L^2 \log^2 P}
\]

\[
\leq (1 + O(\log^{-1} M) - \epsilon + O(\epsilon^2)) M
\]

\footnote{This number could be thought of as \( \lfloor u \rfloor \), but this is not quite accurate.}
For $\epsilon = \frac{C}{\log 2}$ with $C$ sufficiently large, this is $\leq \log \frac{x}{N}(1 - \log^{-1} \frac{x}{N})$, so that $r(n)$ satisfies the first condition of the Lemma. Meanwhile, since $r(p) \leq \frac{1}{\sqrt{p}}$, there is some fixed $c > 0$ such that

$$\sum_{P < p < P^2} \sum_{k \log p \geq \log y} r(p)^{2k} p^k \alpha \ll P^2 e^{-c \frac{\log y}{\log 2 y}} = o(1).$$

Thus $r(n)$ also satisfies the second condition of Lemma 2.3.

4.1.2. Evaluation of sum, Proof of part B of Theorem 1.4. Before we evaluate the sum $\sum_{n \leq N} r(n)$ we introduce two more parameters. Let $\sigma(> \frac{1}{2})$ be the location of the saddle point in the resulting Perron integral and set

$$u = \frac{1}{(1 + \epsilon)\sqrt{2}} \sqrt{\frac{M}{P} P^{1-\sigma}} + \frac{\eta e^\eta}{1+\eta} \log P;$$

ultimately this will be chosen so that $u \sim u$. The following Proposition characterizes our choice of $P, u, \sigma$; they are taken to by any simultaneous solution to the following system.

**Proposition 4.1.** There exists a simultaneous solution $P, u, \eta = \sigma \log P$ to the system of equations

1. $u = \frac{(MP)^{\frac{1}{2}}}{\eta e^\eta} \cdot \frac{1}{1+\eta}$
2. $u = \left\lfloor u \cdot \eta \right\rfloor$ + $\omega$, for some $0 \leq \omega < \frac{2}{\log P}$,
3. $P > M, \log P \sim \log M$.

Moreover, any solution to this system has $\eta = (1 + o(1)) \log M$.

**Proof.** Suppose b and c are satisfied. Then

$$\frac{\log N}{\log P} = \left\lfloor \log N \frac{\eta}{\log M \eta + 1} \right\rfloor \frac{\eta + 1}{\eta} \frac{\eta}{\eta} - \frac{\eta + 1}{\eta} \omega \leq \frac{\log N}{\log M},$$

so in fact, the first part of condition d is redundant. Recall $u = O(\log^2 M \log \log M)$. Combining a and b,

$$MP = \eta^2 e^{2\eta} 2(1 + \epsilon)^2 \left\lfloor u \cdot \frac{\eta}{1+\eta} \right\rfloor^2 \Rightarrow M \ll \eta e^\eta \log^2 M \log_2 M$$

and so $\eta \geq \log M - 4 \log_2 M$. Thus b and c now imply $\log P \sim \log M$, so condition d may be completely discarded. Furthermore, this guarantees that at a solution $\eta \sim \log M$.

A solution may now be found as follows: beginning from $\eta = \eta_0 = \log M - 4 \log_2 M$, increase the value of $\eta$ while defining $P = P(\eta)$ by requiring

$$\left\lfloor u \cdot \frac{\eta}{1+\eta} \right\rfloor = \frac{(MP)^{\frac{1}{2}}}{\eta e^\eta} \cdot \frac{1}{(1 + \epsilon)\sqrt{2}}.$$

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Clearly \( P(\eta) \to \infty \) as \( \eta \to \infty \) and \( \frac{\log N}{\log P} \cdot \frac{\eta}{\eta + 1} \to 0 \). Since \( \frac{\log N}{\log P} \cdot \frac{\eta_0}{\eta_0 + 1} > \left\lfloor \frac{u\eta_0}{1 + \eta_0} \right\rfloor \), the proof of existence is completed by checking that \( \frac{\log N}{\log P} \) jumps by at most \( 2 + o(1) \) at discontinuities of the floor function. □

We need two specific consequences of Proposition 4.1.

i. \[ \frac{\log N}{\log P} - \pi \cdot \left( 1 + \frac{1}{\sigma \log P} \right) = o \left( \sqrt{\frac{\pi}{\log \pi}} \right) \]

ii. \[ \frac{\log N}{\log P} - \frac{\pi}{\sigma \log P} \in \mathbb{Z} + o \left( \sqrt{\frac{\pi}{\log \pi}} \log^{-1} P \right) \]

Note also that \( \log P \sim \log M \) and \( \pi \sim u = O(\log^2 M \log_2 M) \) implies \( \sigma = 1 - o(1) \).

For the above choice of \( P \) we are going to give the following evaluation for \( \sum_{n \leq N} r(n) \).

**Proposition 4.2.** We have

\[
\sum_{n \leq N} r(n) \geq (1 + o(1)) \frac{N^\sigma e^{\pi u}}{2\pi \sigma u}, \quad u \to \infty.
\]

Before proceeding to the proof we introduce the generating function

\[ R(s) = \prod_p (1 - r(p)p^{-s})^{-1} \]

and its logarithm

\[ \log R(s) = \phi_0(s) = \sum_p \sum_{n=1}^\infty \frac{1}{n} \left( \frac{r(p)}{p^s} \right)^n \]

and record several simple properties.

**Lemma 4.3.** Let \( s = \sigma + it \). Uniformly in \( \frac{1}{2} \leq \sigma \leq 1 \) and \( |t| < M \),

\[
\phi_0(s) = \pi \left\{ \frac{P^{-it}}{1 + \frac{it}{\sigma}} + O(\exp(-\sqrt{\log M})) \right\} - \frac{P^{-it}}{1 + \frac{it}{\sigma}} + o(1).
\]

**Proof.** This follows from the Prime Number Theorem. □

**Lemma 4.4.** Let \( \frac{1}{2} < \sigma < 1 \) and set \( \tau = 10 \left( \frac{\log \pi}{\pi} \right)^2 \).

1. \[
\Re \left[ 1 - \frac{P^{-it}}{1 + \frac{it}{\sigma}} \right] \geq \begin{cases} \frac{t^2}{20}, & |t| < \frac{\pi}{2} \\ \frac{2t^2}{20}, & |t| \geq \frac{\pi}{2} \end{cases}
\]

2. For \( t = t_0 + \frac{2\pi j}{\log P} \), \( j \in \mathbb{Z} \), \( |j| < \frac{r \log P}{\log \pi} \) and \( \frac{100 r \log P}{\log \pi} < |t_0| < \frac{\pi}{\log P} \) we have

\[
\Re \left[ 1 - \frac{P^{-it}}{1 + \frac{it}{\sigma}} \right] \geq \frac{t_0^2 \log^2 P}{20}.
\]
Proof of Proposition \[\text{4.2} \] Applying Lemma \[\text{2.5} \] with \( T = \pi^t \) we find
\[
\sum_{n \leq N} r(n) = \frac{N^\sigma e^{\phi_0(\sigma)}}{2\pi \sigma} \left\{ \int_{\frac{1}{2}}^{1} \exp \left( it \log N + \phi_0(\sigma + it) - \phi_0(\sigma) \right) \frac{dt}{1 + \frac{it}{\sigma}} \right. \\
+ O(u^{-3/2}) + O \left( \sup_{\frac{1}{2} < |t| < \pi^3} |\exp(\phi_0(\sigma + it) - \phi_0(\sigma))| \log u \right) \right\}
\]

On this range, Lemma \[\text{4.3} \] gives \( \phi_0(\sigma + it) = \pi^t \frac{P^{-it}}{1 - \frac{it}{\sigma}} + o(1) \), and so the bound of Lemma \[\text{4.4} \] gives that the second error term is \( O(e^{-\pi \log P}) \), so that both errors are permissible. The main integral is thus
\[
\int_{|t| < \tau + \frac{\pi}{2} \log P} \exp \left( it \log N + \frac{P^{-it}}{1 + \frac{it}{\sigma}} - 1 \right) (1 + o(1)) dt + O \left( \int_{\tau < |t| < \frac{1}{2}} \exp(-u t^2) dt \right).
\]

Again the error is permissible. Partitioning the integral into short intervals we obtain
\[
\sum_{|j| < \frac{T \log P}{2\pi \log P}} \int_{0 < \frac{T \log P}{2\pi \log P}}^{\frac{T \log P}{2\pi \log P}} \exp \left( i \log N \left( \frac{2\pi j}{\log P} + t_0 \right) + \frac{P^{-it}}{1 + \frac{it}{\sigma}} - 1 \right) (1 + o(1)) dt = 1 + o(1)
\]

with an error of \( o(u^{-1}) \) from truncating the integral (apply Lemma \[\text{4.4} \] (2)).

We Taylor expand the inner bracket as
\[
\begin{align*}
&\left( 1 - it_0 - \frac{t_0^2}{2} + O(\tau^3) \right) \left( 1 - \frac{i(2\pi j + t_0)}{\sigma \log P} - \left( \frac{2\pi j + t_0}{\sigma \log P} \right)^2 + O(\tau^3) \right) - 1 \\
&= -it_0 - \frac{2\pi j + t_0}{\sigma \log P} - \frac{t_0^2}{2} - t_0 \left( \frac{2\pi j + t_0}{\sigma \log P} \right) - \left( \frac{2\pi j + t_0}{\sigma \log P} \right)^2 + O(\tau^3)
\end{align*}
\]

The conditions (i) and (ii) on \( P, \sigma, \pi \) have been specifically made so that the linear oscillatory phases in both the sum and the integral are now \( o(1) \) throughout the range of integration. Thus we obtain
\[
\sum_{|j| < \frac{T \log P}{2\pi \log P}} e^{-\pi \left( \frac{2\pi j}{\sigma \log P} \right)^2} \int_{|t_0| < 100\tau} e^{-\pi \left( t_0^2 \left( \frac{1}{\sigma \log P} \right) - \frac{t_0^2 \left( \frac{2\pi j}{\sigma \log P} + \frac{2\pi j}{\sigma \log P} \right)}{2} \right)} dt_0.
\]

Estimating the integral, completing the sum to an infinite one, and then applying Poisson summation yields that this is \( \geq \frac{1-o(1)}{u} \).
4.1.3. Evaluation of saddle point asymptotics, Proof of part C of Theorem 1.4. Recall the notation \( u = \frac{\log N}{\log M}, \ u' = \frac{\eta}{1 + \eta}u, \) and that we set \( \bar{u} = \lfloor u' \rfloor. \)

To prove the first part of Theorem 2 it only remains to estimate \( N^\sigma. \) From Proposition 4.1 we have [recall \( \eta = \sigma \log P \)]

\[
P^{1-\sigma} \frac{1}{\sigma \log P} \left( \frac{M}{P} \right)^{\frac{1}{2}} \frac{1}{(1+\epsilon)^{\sqrt{2}}} = \log N \frac{\eta}{\log P} \cdot \frac{1}{1 + \eta} + \omega
\]

with \( 0 \leq \omega < \frac{2}{\log P}. \) Put \( \delta = 1 - \sigma. \) Then

\[
\delta \log P = \frac{1}{2} \left( \log P - \log M \right) + \log((1+\epsilon)^{\sqrt{2}}) + \log(1 - \delta) + \log \log N + \log \frac{\eta}{\eta + 1} + O(\log^{-1} N)
\]

from which it follows

\[
\delta \log N \leq o(1) + \frac{1}{2} \log N \left( 1 - \frac{\log M}{\log P} \right) + \log N \left( \log N + \log((1+\epsilon)^{\sqrt{2}}) \right)
\].

Write

\[
\frac{\log M}{\log P} = \frac{\log N}{\log P} \cdot \frac{\log M}{\log N} = \frac{\eta + \omega}{\eta} - \frac{\lfloor u' \rfloor}{u'} - O(\log^{-1} N).
\]

Furthermore,

\[
\frac{\log N}{\log P} \leq (1 + O(\log^{-1} M)) \bar{u}.
\]

Putting this together we deduce

\[
N^\sigma e^{\bar{u}} \gg \frac{N^\frac{1}{2} + \frac{\lfloor u' \rfloor}{u'}}{(\log N)^{\lfloor u' \rfloor}} \left( \frac{e}{\sqrt{2} + o(1)} \right)^{\lfloor u' \rfloor}
\]

which proves the Theorem for small \( N. \)

4.2. Case \( \log N \gg \log_2 q \log_3 q. \) Recall that we set \( M \) to be minimal such that \( \sum_{p < M} \log p > \log q, \) and that we choose \( x = \frac{\phi(q)}{N}. \)

Let \( \sigma > \frac{1}{2} + \frac{1}{\log_2 q} \) be a parameter and define completely multiplicative function \( r_\sigma(n) \) by

\[
r_\sigma(p) = \begin{cases} \frac{\log p}{p^\sigma}, & M < p < M^2 \\ 0, & \text{otherwise} \end{cases} \quad \lambda = (1 - \epsilon) \sqrt{2(\sigma - 1)} M^\sigma,
\]

where \( \epsilon = \frac{C}{\log_2 q} \) for a sufficiently large constant \( C. \)

4.2.1. Bound for tail of squares, Proof of part A of Theorem 1.4. As we have previously, we set \( y = \frac{x}{N}, \) \( \alpha = \log^{-1} y \log_2 y \) and apply seek to apply Lemma 2.3.

For \( \epsilon = \frac{C}{\log_2 q} \) with \( C \) sufficiently large,

\[
\sum_{M < p < M^2} \log p \frac{r_\sigma(p)^2}{1 - r_\sigma(p)^2} \leq (1 - \epsilon)(2\sigma - 1) \sum_{M < p < M^2} \frac{\log p}{p^{2\sigma}} < \log y(1 - \log_2^{-1} y),
\]

\[
\sum_{M^2 < p < M^3} \log p \frac{r_\sigma(p)^2}{1 - r_\sigma(p)^2} \leq (1 - \epsilon)(2\sigma - 1) \sum_{M^2 < p < M^3} \frac{\log p}{p^{2\sigma}} < \log y(1 - \log_2^{-1} y),
\]

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so that the first condition of the lemma is satisfied. Meanwhile, for some constant $c > 0$, 

$$
\sum_{M < p < M^2} \sum_{n > \log u \log y} \log y \log p \log 2 M^2 e^{-c \log y} = o(1),
$$

so that the second condition is also satisfied. Thus Lemma 2.3 gives (uniformly in $\sigma$) that 

$$
\sum_{n \leq N} r_\sigma(n)^2 = (1 + o(1)) \sum_n r_\sigma(n)^2
$$

so that $\Delta(N, q) \geq (1 + o(1)) \sum_{n \leq N} r_\sigma(n)$.

### 4.2.2. Asymptotic analysis of sum, Proof of part B of Theorem 1.4

Introduce the generating function 

$$
R(s) = R_\sigma(s) = \prod (1 - r_\sigma(p) p^{-s})^{-1}
$$

and its logarithm and derivatives 

$$
\phi_0(s) = \log R(s), \quad \phi_j(s) = (-1)^j \frac{d^j}{ds^j} \phi_0(j), \quad j \geq 0.
$$

We prove the following evaluation of $\sum_{n \leq N} r_\sigma(n)$.

**Proposition 4.5.** Let $N$ satisfy $\log^3 q \log_2 q \ll \log N < \sqrt{\log q \log_2^{-1}}$ and define $\sigma$ by $\phi_1(\sigma) = \log N$. We have 

$$
\sum_{n \leq N} r_\sigma(n) \sim \frac{N^\sigma \exp(\phi_0(\sigma))}{\sigma \sqrt{2\pi \phi_2(\sigma)}}.
$$

In order to prove this Proposition by the saddle point method we need the following estimates.

**Lemma 4.6.** Let $\sigma$ satisfy $2\sigma - 1 > \log^3 M / \log M$. Then uniformly in $\sigma$,

1. $\phi_j(\sigma) = (1 + O(M^{-1/2})) \sum_{M < p < M^2} r_\sigma(p) \log^j p$.
2. $\phi_j(\sigma) = (1 + o(1)) \log^j M \phi_0(\sigma)$
3. $|\phi_3(\sigma + it)| \leq \phi_3(\sigma)$
4. For $|t| < \frac{1}{2 \log M}$,  
   \[ \Re[\phi_0(\sigma) - \phi_0(\sigma + it)] \gg t^2 \phi_2(\sigma) \]
5. For $\frac{1}{2 \log M} < |t| < M$,  
   \[ \Re[\phi_0(\sigma) - \phi_0(\sigma + it)] \gg \phi_0(\sigma) \min\left(\frac{t^2}{(2\sigma - 1)^2}, 1\right). \]

**Proof.** The first three items are straightforward. For (4) note 

$$
\Re[\phi_0(\sigma) - \phi_0(\sigma + it)] \geq \sum_{M < p < M^2} \frac{r_\sigma(p)}{p^\sigma} (1 - \cos(t \log p)) \gg t^2 \sum_{M < p < M^2} \frac{r_\sigma(p) \log^2 p}{p^\sigma} \gg t^2 \phi_2(\sigma).
$$

\[^6\text{Note that the function } \phi_1 \text{ itself implicitly depends upon the parameter } \sigma \text{ through } R_\sigma(s). \]
For (5), write
\[ \Re[\phi_0(\sigma) - \phi_0(\sigma + it)] \geq \frac{\lambda}{2 \log M} \sum_{M < p < M^2} \log p \frac{p^{2\sigma}}{p^{2\sigma}} (1 - p^{-it}) \]
\[ \geq \frac{\lambda}{2 \log M} \sum_{M < n < M^2} \frac{n(\sigma)}{n^{2\sigma}} (1 - n^{-it}) \] + \mathcal{O}(M^{1/2 - \sigma})
\[ \geq \frac{\lambda}{2 \log M} \left( \frac{M^{1-2\sigma}}{2\sigma - 1} - \frac{M^{2-4\sigma}}{2\sigma - 1} \right) - \left( \frac{M^{1-2\sigma + it}}{2\sigma - 1 + it} - \frac{M^{2-4\sigma + 2it}}{2\sigma - 1 + it} \right) \]
\[ + \mathcal{O}(\phi_1(\sigma) \exp(-\sqrt{\log M})) \]
Now the bracket is
\[ \geq \frac{M^{1-2\sigma}}{2\sigma - 1} \left\{ 1 - \frac{1 + M^{1-2\sigma}}{1 - M^{1-2\sigma}} \left| \frac{1}{1 + \frac{it}{2\sigma - 1}} \right| \right\} \]
from which we deduce that the above is
\[ \geq \frac{\phi_1(\sigma)}{2 \log M} \left\{ \min(t^2, (\sigma - \frac{1}{2})^2) \right\} + \mathcal{O}(M^{1-2\sigma}) + \mathcal{O}(\exp(-\sqrt{\log M/2})) \]
\[ \gg \phi_0(\sigma) \min\left( \frac{t^2}{(2\sigma - 1)^2}, 1 \right), \quad |t| > \frac{1}{2 \log M}. \]

**Proof of Proposition 4.5**  Apply Lemma 2.5 with \( T = \log N \log^2 M \) and \( \delta = 2\sigma - 1 \) to obtain
\[ \sum_{n \leq N} r_n(n) = \frac{N^\sigma \exp(\phi_0(\sigma))}{\sigma} \left\{ \frac{1}{2\pi} \int_{-\delta}^{\delta} \exp(it \log N + \phi_0(\sigma + it) - \phi_0(\sigma)) \frac{dt}{1 + \frac{it}{\sigma}} \right\} \]
\[ + \mathcal{O}(\log^2 N \log^{-1} M) + \mathcal{O}(\log q \exp(-c\phi_0(\sigma))) \]
Since \( \phi_0(\sigma) \sim \frac{\log N}{\log M} \) while \( \phi_2(\sigma) \sim \log N \log M \), both error terms are permissible.

Split the remaining integral at \(|t| = \phi_3(\sigma) \frac{1}{\delta} \sim \log^\frac{1}{2} N \log^\frac{1}{2} M \) and at \(|t| = \frac{1}{2 \log M} \). In the interval near \( t = 0 \), Taylor expand \( \phi_0(\sigma + it) \) to obtain the main term of size \( (2\pi \phi_2(\sigma))^{-\frac{1}{2}} \).

For \( \phi_3(\sigma) \frac{1}{\delta} < |t| < \frac{1}{2 \log M} \) use the bound (4) of the previous lemma to obtain an error term. On the remaining interval \( \frac{1}{2 \log M} < |t| < \frac{1}{2\sigma - 1} \) use the bound (5); for \( \log N > C \log q \log_{\log_2 q} q \) for a sufficiently large fixed constant \( C \) this also produces an error term.

4.2.3. **Analysis of implicit parameters, Proof of part C of Theorem 1.4**  By hypothesis, \( \phi_1(\sigma) = \log N \). By Lemma 4.6 we have \( \phi_0(\sigma) \sim \frac{\log N}{\log_2 q} \) and \( \phi_2(\sigma) \sim \log N \log_2 q \) so it remains to determine the quantity \( N^\sigma \). Again by Lemma 4.6
\[ \phi_1(\sigma) = (1 + O(M^{-1/2})) \lambda \sum_{M < p < M^2} \log p \frac{p^{2\sigma}}{p^{2\sigma}} = (1 + O(\log^{-1} M)) \frac{M^{1-\sigma}}{\sqrt{2\sigma(2\sigma - 1)}}. \]
Set $A = (2\sigma - 1) \log M$. Then $A$ satisfies

$$(1 + O(\log^{-1} M)) A e^A = \frac{1}{2\sigma} \log q \log_2 q \log^2 N = \frac{1}{2\sigma^2},$$

where we have set $\log N = \tau \sqrt{\log q \log_2 q}$ with $\tau < \frac{1}{\sqrt{\log_2 q}}$. Hence $A = -\log(2\sigma) + 2 \log \tau^{-1} - \log \log \tau^{-1} - 2 + o(1)$ and therefore

$$N^\sigma = \sqrt{N} \exp \left( u \log \tau^{-1} - \frac{u}{2} \log \log \tau^{-1} - u \log 2 - \frac{1}{2} u \log \sigma + u + o(u) \right).$$

Since $e^{\phi_0(\sigma)} = e^{u + o(u)}$ we obtain the final asymptotic

$$\Delta(N, q) \geq N^{\frac{1}{2}} \exp \left( u \log \tau^{-1} - \frac{1}{2} \log \log \tau^{-1} - u \log 2 - \frac{1}{2} \log \sigma + o(1) \right).$$

Set $\log N = (\log q)^{1 - \sigma'} < \frac{1}{2} < \sigma' < 1$. Then we have $\tau^{-1} = \log^{\sigma' - \frac{1}{2}} q \log_2^\frac{1}{2} q$ and so $\log \tau^{-1} = (\sigma' - \frac{1}{2}) \log_2 q + \frac{1}{2} \log_3 q$ and $\log_2 \tau^{-1} = \log_3 q + \log(\sigma' - \frac{1}{2})$. In particular, $\sigma = \sigma' + o(1)$.

Thus

$$\Delta(N, q) \geq N^{\sigma'} \exp \left( -u \left( \log 2 - 1 + \frac{1}{2} \log(\sigma' - \frac{1}{2}) + \frac{1}{2} \log \sigma' + o(1) \right) \right)$$

$$= \frac{N}{(\log N)^u} \left( \frac{e + o(1)}{\sqrt{2(2\sigma' - 1)(\sigma')}} \right)^u.$$

5. THE RANGE $\log \log N \sim \frac{1}{2} \log \log q$, PROOF OF THEOREM 1.5

In this section we set $\log N = \tau \log^\frac{1}{2} q \log_2^\frac{1}{2} q$. We assume that $\tau^{-1} \leq \log_2^{O(1)} q$ and we may assume $\tau \ll \log_3 q$ since the case of larger $\tau$ is contained in Theorem 1.6. We handle the main case $\Delta(N, q)$ and the dual case $\Delta(\frac{q}{N}, q)$ ($q$ prime) simultaneously. In the first place we put $x = \frac{\phi(q)}{N}$ and in the dual case we choose $x = \sqrt{\frac{q}{N}} \log^{-1} q$.

In either case, set $\lambda = \log^\frac{1}{2} x \log^\frac{1}{2} q$ and define completely multiplicative function $r(n)$ by

$$r(p) = \begin{cases} \frac{\lambda}{p^2 \log p} & \lambda^2 < p < \exp(\log^2 \lambda) \\ 0 & \text{otherwise} \end{cases}.$$

5.1. Bound for tail of sum of squares, Proof of part A of Theorem 1.5. We apply Lemma 2.3 with $y_i = \frac{x}{N}$. Recall $\alpha = \frac{\log_2 y_i}{\log y_i}$.

$$\sum_{\lambda^2 < p < \exp(\log^2 \lambda)} \log p \frac{r(p)^2}{1 - r(p)^2} \leq (1 + O(\log_2^2 q)) \lambda^2 \sum_{\lambda^2 < p < \exp(\log^2 \lambda)} \frac{1}{p \log p}$$

$$\leq (1 + O(\log_2^2 q)) \lambda^2 \left( 2 \log \lambda^{-1} - \log^{-2} \lambda \right)$$

$$\leq \log x - (1 - o(1)) \frac{\log x \log_3 x}{\log_2 x}.$$
Since \( \log N = O(\log^{1/2+\varepsilon} x) \), the first condition in Lemma 2.3 is satisfied.

Meanwhile, for some \( c > 0 \),

\[
\sum_{\lambda^2 < p < \exp(\log^2 \lambda)} \sum_{k > \frac{\log y}{\log p \log^2 y}} r(p)^{2k} p^\alpha k \ll \exp(\log^2 \lambda)(\log \lambda)^{-\frac{\log y}{\log^2 y}} = o(1)
\]

so that the second condition is also satisfied. Thus by Lemma 2.3,

\[
\sum_{n \leq \frac{N}{\lambda}} r(n)^2 = (1 + o(1)) \sum_{n} r(n)^2
\]

and therefore

\[
\Delta(N, q) \geq (1 + o(1)) \sum_{n \leq N} r(n), \quad \Delta\left(\frac{q}{N}, q\right) \gg \sqrt{q} \sum_{n \leq \frac{N}{\lambda}} nr(n).
\]

5.2. Saddle point asymptotics, Proof of part B of Theorem 1.5. Define for \( \Re(s) > 0 \),

\[
R(s) = \prod_p \left(1 - \frac{r(p)}{p^s}\right)^{-1}
\]

and the logarithm and derivatives

\[
\phi_0(s) = \log R(s), \quad \phi_j(s) = (-1)^j \frac{d^j}{ds^j} \phi_0(s).
\]

We have the following asymptotic expansions of \( \sum_{n \leq N} r(n), \sum_{n \leq N} nr(n) \).

**Proposition 5.1.** Let \( \sigma > 0 \) be the unique solution to \( \phi_1(\sigma) = \log N \). We have

\[
\sum_{n \leq N} r(n) \sim \frac{N^\sigma e^{\phi_0(\sigma)}}{\sqrt{2\pi \phi_2(\sigma)}}, \quad \sum_{n \leq N} nr(n) \sim \frac{N^{1+\sigma} e^{\phi_0(\sigma)}}{(1 + \sigma) \sqrt{2\pi \phi_2(\sigma)}}.
\]

The proof of Proposition 4.5 is easily adapted to this case, so we simply record the necessary estimates.

**Lemma 5.2.** Uniformly in \( \frac{1}{4} < \sigma < \frac{3}{5} \) We have the following estimates regarding the functions \( \phi_j \).

1. \( |\phi_3(\sigma + it)| \leq \phi_3(\sigma) \)

2. \( \phi_2(\sigma) \ll \log N \log^2 \lambda \)

3. For \( |t| < \log^{-2} \lambda \),

\[
\Re[\phi_0(\sigma) - \phi_0(\sigma + it)] \gg t^2 \phi_2(\sigma).
\]

4. For \( \log^{-2} \lambda < |t| < \lambda \),

\[
\Re[\phi_0(\sigma) - \phi_0(\sigma + it)] \gg \min(t^2, 1)\lambda^{1/2}.
\]
Proof. The first three items are straightforward, so we show the proof of (4). We have

\[ \mathbb{R}[\phi_0(\sigma) - \phi_0(\sigma + it)] \geq \sum_{\lambda^2 < p < \exp(\log^2 \lambda)} \frac{\lambda}{p^{2\sigma} \log p} [1 - \cos(t \log p)] \]

\[ \geq \mathbb{R} \left\{ \frac{\lambda}{3 \log^2 \lambda} \sum_{\lambda^2 < p < \lambda^3} \frac{\log p}{p^\gamma} [1 - p^{-it}] \right\} \]

\[ \gg \frac{\lambda^3}{\log^2 \lambda} \left[ 1 - \left| \frac{1}{5} + it \right|^{-1} \right] \gg \lambda^3 \min(t^2, 1). \]

\[ \square \]

We point out one simple consequence of Proposition 5.1.

Corollary 5.3. We have

\[ \sum_{n \leq \frac{N}{2}} nr(n) \asymp \sum_{n \leq N} nr(n). \]

Proof. Let \( \sigma' \) solve \( \phi_1(\sigma') = \log N - \log 2 \). It evidently suffices to check that \( \phi_0(\sigma) - \phi_0(\sigma') = O(1) \). But by the Mean Value Theorem,

\[ \phi_0(\sigma) - \phi_0(\sigma') = (\sigma' - \sigma) \phi_1(\gamma) \]

for some \( \gamma \in [\sigma, \sigma'] \) and

\[ \sigma' - \sigma = \frac{\phi_1(\sigma) - \phi_1(\sigma')}{\phi_2(\gamma')} = \frac{\log 2}{\phi_2(\gamma')} \]

for some \( \gamma' \in [\sigma, \sigma'] \). The claim now follows because \( \phi_2(\gamma') \geq \phi_2(\sigma') \geq \phi_1(\sigma') \sim \phi_1(\gamma) \sim \log N \).

\[ \square \]

5.3. Evaluation of parameters, Proof of part C of Theorem 1.5. By (2) of Lemma 5.2, \( \phi_2(\sigma) = \log^{O(1)} q \), and therefore Proposition 5.1 and its Corollary imply that for \( \sigma > 0 \) solving \( \phi_1(\sigma) = \log N \) we have

\[ \Delta(N, q) = N^\sigma \exp(\phi_0(\sigma) + O(\log_2 q)) \]

and for \( q \) prime,

\[ \Delta(q \frac{N}{q}, q) \gg \sqrt{\frac{q}{N}} N^\sigma \exp(\phi_0(\sigma) + O(\log_2 q)). \]

The error term will be negligible, so it suffices to determine \( N^\sigma \) and \( \phi_0(\sigma) \).

We first show that for \( N \) such that \( \sigma < \frac{1}{2} + \frac{1}{\log_2 q \log_5 q} \)

\[ (15) \]

\[ \Delta(N, q) \geq \sqrt{N} \exp \left( (1 + o(1)) \sqrt{\frac{\log q}{\log_2 q}} \right), \]

\[ \Delta(q \frac{N}{q}, q) \geq \sqrt{\frac{q}{N}} \exp \left( (1 + o(1)) \sqrt{\frac{\log q}{2 \log_2 q}} \right), \]

(\( q \) prime).
Note that for $\sigma = 1/2 + \frac{1}{\log_2 q \log_3 q}$,

$$\log N = \phi_1(\sigma) \geq \lambda \sum_{\lambda^2 < p < \exp(\log^2 \lambda)} \frac{1}{p^{1 + \log^{-1} q \log^{-2} q}} \gg \sqrt{\log q \log_2 q \log_4 q}$$

so that $\tau \gg \log_4 q$. Since $A\tau \to 0$ and $A\tau' \to 1$ as $\tau \to \infty$, the bounds (15) verify the theorem in this range.

When $\sigma < 1/2$, since $\log N = \phi_1(\sigma) \ll \sqrt{\log q \log_2 q \log_3 q}$ and $\phi_2(\alpha)$ is decreasing in $\alpha > 0$ we have

$$\frac{1}{2} - \sigma \ll \frac{\sqrt{\log q \log_2 q \log_3 q}}{\phi_2(1/2)}.$$

Meanwhile

$$\phi_2(1/2) \geq \sum_{\lambda^2 < p < \exp(\log^2 \lambda)} \frac{\lambda \log p}{p} \gg \lambda \log^2 \lambda.$$

It follows that $\frac{1}{2} - \sigma \ll \frac{\log_3 q}{\log_5 q}$ and therefore

$$(\frac{1}{2} - \sigma) \log N \ll \frac{\log_3 q}{\log_2 q} \sqrt{\frac{\log x}{\log_2 x}}.$$

Meanwhile,

$$\phi_0(\sigma) \geq \phi_0(1/2) \geq \lambda \sum_{\lambda^2 \leq p < \exp(\log^2 \lambda)} \frac{1}{p \log p} \geq (1 + o(1)) \frac{\lambda}{2 \log \lambda} = (1 + o(1)) \frac{\sqrt{\log x}}{\log_2 x}.$$ 

Combining these estimates we obtain (15) for the case $\sigma < 1/2$.

In the range $1/2 \leq \sigma < 1/2 + \log_2^{-1} q \log_3^{-2} q$ we have $\phi_0(1/2) - \phi_0(\sigma) \leq (\sigma - 1/2) \phi_1(1/2)$. Now

$$\phi_1(1/2) \sim \lambda \sum_{\lambda^2 < p < \exp(\log^2 \lambda)} \frac{1}{p} = O(\lambda \log_2 \lambda)$$

and therefore

$$\phi_0(1/2) - \phi_0(\sigma) \ll \frac{\log_3 q}{\log_2 q} \sqrt{\frac{\log x}{\log_2 x}}.$$

Thus we also have (15) for $\sigma < 1/2 + \frac{1}{\log_2 q \log_3 q}$.

We now consider the case $\log_2^{-1} q \log_3^{-2} q < \sigma - 1/2 < \log_2^{-1} q \log_3^2 q$. In this range we have

$$\log N = \phi_1(\sigma) = (1 + O(\lambda^{-1})) \lambda \sum_{\lambda^2 < p < \exp(\log^2 \lambda)} \frac{1}{p^{2 - \sigma}}.$$ 

By the prime number theorem and partial summation, this is

$$(1 + O(\exp(-\sqrt{\log \lambda}))) \int_{2(\sigma - 1/2) \log \lambda}^{(\sigma - 1/2) \log \lambda} e^{-x} \frac{dx}{x} = (1 + O(\exp(-\sqrt{\log \lambda}))) \lambda \int_{(2 \sigma - 1) \log \lambda}^\infty e^{-x} \frac{dx}{x}.$$ 

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Since $\tau \sim \frac{\log N}{\lambda}$ in the case of $\Delta(N, q)$ and $\tau \sim \frac{\log N}{\sqrt{2}\lambda}$ in the dual case of $\Delta(\frac{q}{N}, q)$, the range $\log_2^1 q \log_3^2 q < \sigma - \frac{1}{2} < \log_2^1 q \log_3^2 q$ corresponds to $\tau$ within the range $e^{-O(\log^3 q)} \leq \tau \ll \log_4 q$, so the range of $\sigma$ considered is sufficient for the theorem.

Let $A$ solve $\log N = \int_{A}^{\infty} e^{-x} \frac{dx}{x}$ and let $A' = (2\sigma - 1) \log \lambda$. It is readily seen that $|A - A'| \leq e^{-O(\log \lambda)}$ and therefore

$$\sigma \log N = \frac{1}{2} \log N + \frac{A \log N}{2 \log \lambda} + O(\log N \exp(-\frac{1}{2} \sqrt{\log \lambda})).$$

Meanwhile, applying the prime number theorem a second time

$$\phi_0(\sigma) = (1 + o(1)) \lambda \sum_{\lambda^2 < p < \exp(\log^2 \lambda)} \frac{1}{p^{\frac{1}{2}+\sigma} \log p} = (1 + o(1)) \lambda \int_{2 \log \lambda}^{\log^2 \lambda} e^{(\frac{1}{2} - \sigma)y} dy \frac{dy}{y^2}$$

and after a change of variables, this is

$$(1 + o(1))(\sigma - \frac{1}{2}) \lambda \int_{2(\sigma - \frac{1}{2}) \log \lambda}^{\infty} e^{-x} \frac{dx}{x^2} = (1 + o(1))(\sigma - \frac{1}{2}) \lambda \int_{A}^{\infty} e^{-x} \frac{dx}{x^2}.$$ 

Since $A = (1 + o(1))(\sigma - \frac{1}{2}) 2 \log \lambda$ and recalling that we set $\tau' = \int_{A}^{\infty} e^{-x} \frac{dx}{x^2}$ we obtain

$$\phi_0(\sigma) = (1 + o(1)) \frac{A \sigma \lambda}{2 \log \lambda}.$$

Thus

$$N^\sigma e^{\phi_0(\sigma)} = N^{\frac{1}{2}} \exp \left( (1 + o(1)) \left( \frac{A \log N}{2 \log \lambda} + \frac{A \tau' \lambda}{2 \log \lambda} \right) \right),$$

and therefore,

$$\Delta(N, q) \geq \sqrt{N} \exp \left( (1 + o(1)) \left( A(\tau + \tau') \sqrt{\frac{\log q}{\log \log q}} \right) \right)$$

$$\Delta(\frac{q}{N}, q) \gg \sqrt{\frac{q}{N}} \exp \left( (1 + o(1)) \left( A \left( \tau + \frac{\tau'}{\sqrt{2}} \right) \sqrt{\frac{\log q}{\log_2 q}} \right) \right), \quad (q \text{ prime}).$$

### 6. Long Character Sums, Theorem 1.6

When $\log N \geq 4 \sqrt{\log q \log_2 q \log_3 q}$ we use a ‘second moment’ version of the resonance method, which avoids saddle point analysis. The situation now becomes similar to that in the original paper [7].

The second moment version of the Fundamental Proposition is as follows.

**Proposition 6.1** (Fundamental Proposition, second moment version). Let $\log^4 q < N < q$ and set $x = \frac{q}{N}$. Let $r(n)$ be a non-negative multiplicative function supported on squarefree $n$ and
such that \( p|q \Rightarrow r(p) = 0 \). Then for any parameter \( z < \frac{N}{\log^4 q} \) we have

\[
\Delta(N, q)^2 + O \left( \frac{\phi(q)}{q} N \right) \geq \frac{\phi(q)}{q} N \sum_{n_1, n_2 \leq z} \frac{r(n_1) r(n_2)}{\max(n_1, n_2)} \left[ \sum_{g \leq \max(n_1, n_2)} r(g)^2 / \prod_p (1 + r(p))^2 \right].
\]

Moreover, let \( M \) be minimal such that \( \sum_{p \leq M} \log p > \log q \). The conclusion remains valid if the condition \( p|q \Rightarrow r(p) = 0 \) is replaced with \( p \leq M \Rightarrow r(p) = 0 \).

**Proof.** Define ‘resonator’ \( R(\chi) = \frac{1}{\sqrt{\phi(q)}} \sum_{n \leq x} r(n) \chi(n) \). Plainly

\[
\Delta(n, q)^2 \geq \sum_{\chi \neq \chi_0} |R(\chi)|^2 \left| \sum_{n \leq N} \chi(n) \right|^2 / \sum_{\chi} |R(\chi)|^2.
\]

By orthogonality of characters, the denominator is \( \sum_{n \leq x} r(n)^2 \leq \prod_p (1 + r(p)^2) \). Meanwhile, the numerator is

\[
\sum_{m_1, m_2 \leq x} r(m_1) r(m_2) \sum_{n_1, n_2 \leq N} \frac{r(n_1) r(n_2)}{\max(n_1, n_2)} - |R(\chi_0)|^2 \left| \sum_{n \leq N} \chi_0(n) \right|^2.
\]

By Cauchy-Schwartz, \( |R(\chi_0)|^2 \leq \frac{\phi(q)}{q} \sum_{m \leq x} r(m)^2 \) and since \( N > \log^4 q \), \( \sum_{n \leq N} \chi_0(n) \sim \frac{\phi(q)}{q} N \). Therefore the negative term contributes \( O(\frac{\phi(q)}{q} N) \) to the ratio (16). In the main term, let \( g = (m_1, m_2), h = (n_1, n_2) \) and replace \( m_i := \frac{m_i}{g} \) to obtain

\[
\sum_{m_1, m_2 \leq x} r(m_1) r(m_2) \sum_{g \leq \max(m_1, m_2)} \frac{r(g)^2}{(g, m_1, m_2) = 1} \sum_{h \leq x} r(h)^2.
\]

Discarding some non-negative terms, this is

\[
\geq \frac{\phi(q)}{q} N \sum_{m_1, m_2 \leq x} \frac{r(m_1) r(m_2)}{\max(m_1, m_2)} \sum_{g \leq \max(m_1, m_2)} \frac{r(g)^2}{(g, m_1, m_2) = 1},
\]

which proves the first part of the Proposition.

For the second statement, let \( r \) be any non-negative multiplicative function supported on squarefree numbers and satisfying \( p \leq M \Rightarrow r(p) = 0 \). Enumerate \( \{q_1, ..., q_r\} \) the set of primes greater than \( M \) that divide \( q \), and \( \{p_1, ..., p_s\} \) the set of primes at most \( M \) that do not divide \( q \). Then \( s \geq r \) so we may define a new multiplicative function \( \tilde{r} \), supported on squarefrees and satisfying \( p|q \Rightarrow \tilde{r}(p) = 0 \), by exchanging the values of \( r(p_i) \) and \( r(q_i) \) for \( 1 \leq i \leq r \). Evidently

\[
\prod_p (1 + \tilde{r}(p)^2) = \prod_p (1 + r(p)^2)
\]
and also, for any $z$,

$$\sum_{m_1, m_2 \leq z \atop \text{max}(m_1, m_2) = 1} \frac{r(m_1) r(m_2)}{\max(m_1, m_2)} \sum_{g \leq \max(m_1, m_2) \atop \text{max}(m_1, m_2) = 1} r(g)^2 \leq \sum_{m_1, m_2 \leq z \atop \text{max}(m_1, m_2) = 1} \frac{\tilde{r}(m_1) \tilde{r}(m_2)}{\max(m_1, m_2)} \sum_{g \leq \max(m_1, m_2) \atop \text{max}(m_1, m_2) = 1} \tilde{r}(g)^2,$$

which reduces the second statement in the Proposition to the first case.

In the case $q$ is prime we also have a dual version of the Second Moment Proposition. Recall that we set

$$S(\chi) = \sum_{|h| \leq H, h \neq 0} \frac{\chi(h)}{h} (1 - c(\frac{h}{N})), \quad H = \sqrt{Nq} \log q,$$

and from Pólya’s Fourier expansion,

$$\left| \sum_{n \leq N} \chi(n) \right| \geq \frac{\sqrt{q}}{2\pi} |S(\chi)| + O\left(\frac{q}{N}\right),$$

also $\chi(1) = 1 \Rightarrow S(\chi) = 0$ so that, in particular, $S(\chi_0) = 0$.

**Proposition 6.2** (Fundamental Proposition, dual second moment version). Let $q$ be prime, $N \leq \frac{q}{\log^2 q}$ and set $x = \frac{1}{2 \log q} \sqrt{\frac{q}{N}}$. Let $r(n)$ be a multiplicative function supported on squarefree numbers not divisible by $q$. Then

$$\sup_{\chi \neq \chi_0} |S(\chi)|^2 \gg \frac{1}{N} \sum_{m_1, m_2 \leq \min(x, \frac{N}{2}) \atop \text{max}(m_1, m_2) = 1} \frac{r(m_1) r(m_2) m_1 m_2}{\max(m_1, m_2)^3} \sum_{g \leq \max(m_1, m_2) \atop \text{max}(m_1, m_2) = 1} \frac{r(g)^2}{\prod_p (1 + r(p)^2)}.$$

**Proof.** Set $R(\chi) = \frac{1}{\phi(q)} \sum_{m < x} r(m) \chi(m)$. Plainly

$$\sup_{\chi \neq \chi_0} |S(\chi)|^2 \geq \sum_{\chi} |R(\chi) S(\chi)|^2 \left/ \sum_{\chi} |R(\chi)|^2 \right..$$

The denominator is bounded by $\prod_p (1 + r(p)^2)$ while the numerator is equal to

$$\sum_{m_1, m_2 \leq x} r(m_1) r(m_2) \sum_{0 \neq |n_1|, |n_2| \leq H \atop m_1 n_2 \equiv m_2 n_1 \mod q} \frac{(1 - c(\frac{n_1}{N}))(1 - c(\frac{n_2}{N}))}{n_1 n_2}.$$
The congruence modulo \( q \) is possible only if \( n_1 \) and \( n_2 \) have the sign. Discarding those (positive) terms with \( \max(n_1, n_2) > \frac{N}{2} \), the numerator is
\[
\gg N^{-4} \sum_{m_1, m_2 \leq x} r(m_1) r(m_2) \sum_{0 < n_1, n_2 \leq \frac{N}{m_1 m_2} = n_2 m_1} n_1 n_2
\]
\[
= N^{-4} \sum_{m_1, m_2 \leq x} r(m_1) r(m_2) m_1 m_2 \sum_{g \leq \max(m_1, m_2)} r(g)^2 \sum_{h \leq \frac{N}{2 \max(m_1, m_2)}} h^2
\]
\[
\gg \frac{1}{N} \sum_{m_1, m_2 \leq \min(x, \frac{N}{2})} r(m_1) r(m_2) m_1 m_2 \max(m_1, m_2)^3 \sum_{g \leq \max(m_1, m_2)} r(g)^2.
\]

\[\blacklozenge\]

6.1. Choice of resonator, and some lemmas. Let either \( x = \frac{q}{N} \) in the main case of \( \Delta(N, q) \) or \( x = \frac{1}{2 \log q} \sqrt{\frac{N}{x}} \) in the dual case of \( \Delta(N, q, q) \). In either case, set \( \lambda = \sqrt{\log x \log \log x} \) and as in [7], define multiplicative function \( r(n) \) at prime powers by
\[
r(p) = \begin{cases} \frac{\lambda}{\sqrt{p \log p}} & \lambda^2 \leq p \leq \exp((\log \lambda)^2) \\ 0 & \text{otherwise} \end{cases}
\]
\[
r(p^n) = 0, \quad n \geq 2.
\]
We also define multiplicative function \( t \) by \( t(p^n) = \frac{r(p^n)}{1 + r(p^n)} \).

The following two estimates are extrapolated from those used in the proof of [7] Theorem 2.1.

**Lemma 6.3.** Assume \( z > \exp(3 \lambda \log \log \lambda) \). As \( x \to \infty \) we have
\[
\sum_{m \leq z} \frac{t(m)}{\sqrt{m}} \sim \prod_p \left( 1 + \frac{t(p)}{\sqrt{p}} \right) = \exp \left( (1 + o(1)) \frac{\lambda}{2 \log \lambda} \right).
\]
Also, for \( \alpha = \frac{1}{(\log \lambda)^2} \),
\[
x^{-\alpha} \sum_{m_1, m_2 \leq z} \frac{r(m_1) r(m_2)}{(m_1 m_2)^{\frac{1}{2} - \alpha}} \sum_{(d, m_1 m_2)} r(d)^2 d^\alpha \left( \sum_{m \leq z} \frac{t(m)}{\sqrt{m}} \right)^2 \sum_d r(d)^2
\]
\[
\leq \exp \left( -(1 + o(1)) \frac{32 \log x}{(\log \log x)^4} \right).
\]

**Proof.** First (17): By ‘Rankin’s trick’, for any \( \alpha > 0 \) the sum is
\[
\prod_p \left( 1 + \frac{t(p)}{\sqrt{p}} \right) + O \left( z^{-\alpha} \prod_p \left( 1 + \frac{t(p)}{p^{\frac{1}{2} - \alpha}} \right) \right).
\]
The main term is of the desired size ([7], top of page 6) so it suffices to bound the error. Choose \( \alpha = \frac{1}{(\log \lambda)^2} \). Then the ratio of the error term to the main term is

\[
\ll z^{-\alpha} \prod_p \left( 1 + \frac{t(p)}{p^{\frac{1}{2} - \alpha}} \right) \leq \frac{z^{-\alpha} \exp \left( \sum_{p} t(p) \sqrt{p} (p^\alpha - 1) \right)}{1 + t(p) p^{1/2 - \alpha} \prod_p (1 + t(p)) p^{\alpha}},
\]

and since \( \alpha \log p \leq \frac{1}{\log \lambda} \) for all \( p \) with \( t(p) \neq 0 \), this last is bounded by

\[
\ll \exp \left( \alpha \left( -\log z + 2 \sum_{p} \frac{t(p) \log p}{\sqrt{p}} \right) \right) \leq \exp \left( \alpha \left( -\log z + 2 \sum_{p < \exp((\log \lambda)^2)} \frac{1}{p} \right) \right) = o(1).
\]

Now (18): By our calculation for (17), the denominator of (18) is

\[
(1 + o(1)) \prod_p \left( 1 + \frac{t(p)}{\sqrt{p}} \right) \prod_p (1 + r(p)^2).
\]

Meanwhile, the double sum in the numerator is bounded by

\[
\leq \sum_{m_1, m_2 \leq z} \frac{t(m_1)t(m_2)}{(m_1m_2)^{2 - \alpha}} \sum_d r(d)^2 d^{\alpha} \leq \left( \prod_p \left( 1 + \frac{t(p)}{p^{\frac{1}{2} - \alpha}} \right) \right)^2 \prod_p (1 + r(p)^2 d^{\alpha}).
\]

Therefore the ratio in (18) is bounded by

\[
\ll x^{-\alpha} \prod_p \left( \frac{1 + t(p) p^{1 - \alpha}}{1 + t(p) p^{\frac{1}{2} - \alpha}} \right)^2 \prod_p \left( \frac{1 + r(p)^2 p^\alpha}{1 + r(p)^2} \right)
\ll \exp \left( -\alpha \log x + 2\alpha \sum_{\lambda^2 \leq p \leq \exp((\log \lambda)^2)} \frac{r(p) \log p}{\sqrt{p}} + \alpha \sum_{\lambda^2 \leq p \leq \exp((\log \lambda)^2)} r(p)^2 \log p \right).
\]

Substituting the definition of \( r(p) \) and \( \alpha \), and using the prime number theorem, the last expression is bounded by \( \exp \left( -\frac{\lambda^2 (1+o(1))}{\log^2 \lambda} \right) \). \( \square \)

We record one more elementary estimate.

**Lemma 6.4.** Uniformly in \( y \geq 1 \) and for any \( k > 0 \),

\[
(19) \quad \sum_{d \leq y \atop (d,k)=1} \frac{\mu(d)t(d)^2}{d} = 1 + o(1), \quad (x \to \infty).
\]

**Proof.** The sum is

\[
1 + \sum_{\lambda^2 < d \leq y \atop (d,k)=1} \frac{\mu(d)t(d)^2}{d} = 1 + O \left( \lambda^2 \sum_{d > \lambda^2} \frac{1}{d^2 (\log d)^2} \right) = 1 + o(1).
\] \( \square \)
Combining the above two lemmas we obtain our basic estimate.

**Lemma 6.5.** *Uniformly in $z \geq 1$,*

$$
\sum_{m_1, m_2 \leq z} \frac{t(m_1)t(m_2)m_1m_2}{\max(m_1, m_2)^3} \gg \frac{1}{\log z} \left( \sum_{m \leq z} \frac{t(m)}{\sqrt{m}} \right)^2.
$$

In particular, for $\log z > 3\lambda \log \log \lambda$ we have

$$
\sum_{m_1, m_2 \leq z} \frac{t(m_1)t(m_2)m_1m_2}{\max(m_1, m_2)^3} \geq \exp \left( (1 + o(1)) \frac{\lambda}{\log \lambda} \right).
$$

**Proof.** The left hand side is equal to

(20)

$$
\sum_{m_1, m_2 \leq z} \frac{t(m_1)t(m_2)m_1m_2}{\max(m_1, m_2)^3} \sum_d \mu(d) = \sum_{m_1, m_2 \leq z} \frac{t(m_1)t(m_2)m_1m_2}{\max(m_1, m_2)^3} \sum_{d \leq \max(m_1, m_2)} \frac{\mu(d)t(d)^2}{d}
$$

(21)

$$
= (1 + o(1)) \sum_{m_1, m_2 \leq z} \frac{t(m_1)t(m_2)m_1m_2}{\max(m_1, m_2)^3}
$$

by (19) of Lemma 6.4. This last sum is

$$
\gg \sum_{0 \leq k < \log z} \left( \sum_{\frac{k}{z+1} < m \leq \frac{k}{z}} \frac{t(m)}{\sqrt{m}} \right)^2 \geq \frac{1}{\log z} \left( \sum_{m \leq z} \frac{t(m)}{\sqrt{m}} \right)^2
$$

by Cauchy-Schwartz. \qed

### 6.2. Proof of Theorem 1.6

Recall that we set either $x = \frac{q}{N}$ for $\Delta(N, q)$, or $x = \frac{1}{2 \log q} \sqrt{N}$ for $\Delta(\frac{q}{N}, q)$ (q prime) and in either case $\lambda = \sqrt{\log x \log_2 x}$. Note that in the case of $\Delta(N, q)$, the condition $N < q \exp \left( -\frac{2 \log q}{\log_2 q} \right)$ guarantees $\lambda^2 > 2 \log q$; in particular if $M$ is minimal such that $\sum_{p \leq M} \log p > \log q$ as in Proposition 6.1, then the function $r$ is supported on primes greater than $2 \log q > M$ so that $r$ satisfies the conditions of that Proposition.

Let $z = \min(N, x)^\frac{1}{2}$. Since we assume $\log N \geq 4 \sqrt{\log q \log_2 q \log_3 q}$ this guarantees that $z \geq \exp(3\lambda \log_2 \lambda)$. By Propositions 6.1 and 6.2 both the bound for $\Delta(N, q)$ and $\Delta(\frac{q}{N}, q)$ follow from the estimate

(22) $$
\sum_{n_1, n_2 \leq z} \frac{r(n_1)r(n_2)n_1n_2}{\max(n_1, n_2)^3} \sum_{g \leq \max(n_1, n_2)} \frac{r(g)^2}{\prod_p (1 + r(p))^2} \geq \exp \left( (2 + o(1)) \sqrt{\frac{\log x}{\log_2 x}} \right).
$$
Applying Rankin’s trick to the sum over $g$ with $\alpha = \frac{1}{\log \lambda}$, we obtain the desired main term of

$$\sum_{m_1, m_2 \leq z} \frac{t(m_1) t(m_2) m_1 m_2}{\max(m_1, m_2)^3} \gg \frac{1}{\log z} \left( \sum_{m < z} \frac{t(m)}{\sqrt{m}} \right)^2 \geq \exp \left( (1 + o(1)) \frac{\lambda}{\log \lambda} \right)$$

with an error term of

$$\left( \prod_p (1 + r(p)^2) \right)^{-1} x^{-\alpha} \sum_{m_1, m_2 \leq z} \frac{r(m_1) r(m_2) (m_1 m_2)^{1+\alpha}}{\max(m_1, m_2)^3} \sum_{(g, m_1 m_2) = 1} r(g)^2 g^\alpha.$$

By Lemmas [6,3] and [6.5] the ratio of this error term to the main term is bounded by $\log z$ times the expression in (18), and thus this ratio is $o(1)$.

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