Relating Friedmann equation to Cardy formula in universes with cosmological constant

Bin Wang\textsuperscript{a,b,1}, Elcio Abdalla\textsuperscript{a,2} and Ru-Keng Su\textsuperscript{c,3}

\textsuperscript{a} Instituto De Fisica, Universidade De Sao Paulo, C.P.66.318, CEP 05315-970, Sao Paulo, Brazil
\textsuperscript{b} Department of Physics, Shanghai Teachers’ University, P. R. China
\textsuperscript{c} CCAST (World Lab), P.O.Box 8730, Beijing, P. R. China
and Department of Physics, Fudan University, Shanghai 200433, P. R. China

Abstract

A relation between the Friedmann equation and the Cardy formula has been found for de Sitter closed and Anti de Sitter flat universes. For the remaining (Anti) de Sitter universes the arguments fail, and we speculate whether the general philosophy of holography can be satisfied in such contents.

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\footnote{1\textsuperscript{e-mail:hinwang@fma.if.usp.br}}\footnote{2\textsuperscript{e-mail:eabdalla@fma.if.usp.br}}\footnote{3\textsuperscript{e-mail:rksu@fudan.ac.cn}}
Twenty years ago, Bekenstein argued that for any isolated physical system of energy $E$ and size $R$, the usual thermodynamic entropy is bounded by $S \leq S_B = 2\pi RE$ [1]. By choosing $R$ to be the particle horizon, he gave a prescription for a cosmological extension of the bound [2]. In view of the example of black hole entropy, an influential holographic principle was put forward recently, which suggests that the maximal entropy is bounded by the area of the spacelike surface enclosing a certain region of space [3]. For systems of limited gravity, Bekenstein’s bound implies the holographic bound. The extension of the holographic bound to cosmological settings was first addressed by Fischler and Susskind (FS)[4]. They have shown that for flat and open Friedmann-Lemaître-Robertson-Walker (FLRW) universes the area of the particle horizon should bound the entropy on the backward-looking light cone. However violation of FS bound was found for closed FLRW universes. Various different modifications of the FS version of the holographic principle have been raised subsequently [5-11]. In addition to the study of holography in homogeneous cosmologies, attempts to generalize the holographic entropy bound to a generic realistic inhomogeneous cosmological setting were carried out in [12,13].

Despite different versions of the holographic entropy bounds constructed in cosmology, the microscopic understanding of these entropy bounds is still lacking. Light on this problem was first shedded in [14] by showing that in the FLRW flat universe the horizon area can be calculated by state counting. The problem was also discussed in an interesting recent paper by Verlinde [15]. He revealed that when a universe-size black hole can be formed, the holographic entropy bound is satisfied and there appears a deep and fundamental connection between the holographic principle, the entropy formula for the conformal field theory (CFT) and the Friedmann equations for a radiation dominated closed universe. This finding is remarkable because it not only relates the entropy bound to the Cardy formula showing the statistical meaning of the bound, but also relates the Friedmann equation describing the bulk to the Cardy formula on the boundary and unifies
the Bekenstein bound [1] and Hubble bound [7] in a cosmological context. The implicit quantum corrections to Cardy formula has also been given in [16]. Verlinde’s generalization of Cardy formula was subsequently challenged by weak-coupling, high temperature CFT calculations [17]. Extending the causal entropy bound (CEB) to arbitrary dimensions and checking it against CFT calculations, Brustein et al [18] claimed that CEB can pass the CFT test and evade the criticism in [17]. They also argued that for a large class of models including high temperature weakly and strongly coupled CFTs with AdS duals, CEB is equivalently to a purely holographic entropy bound proposed in [15]. A very recent study showed that even for free CFTs in dimensions $D = 4, 6$, there are specific cases where Cardy formula can still hold [19], which supports Verlinde’s result.

It is of interest to generalize Verlinde’s discussion to a broader class of universes including a cosmological constant and investigate whether his result is universally true. We will suppose that a de Sitter or an Anti-de Sitter (AdS) universe is occupied by a universe-size black hole. For the AdS universe occupied by a black hole of the universe size, according to the AdS/CFT duality conjecture [20], the system is associated to a strongly coupled CFT residing on the conformal boundary. For the de Sitter spacetime, the close relation between entropy associated with cosmological de Sitter horizon and CFT has been revealed [21]. Recent studies proved that de Sitter spacetime can be embedded in the AdS spacetime and presented the nature of strongly coupled CFT in de Sitter spacetime as well [22]. In this paper we will try to build up the relation between the Friedmann equation and entropy bounds and present an intriguing resemblance of Cardy formula in de Sitter and AdS universes.

A. De Sitter universes.

First we will consider universes with a positive cosmological constant. The Friedmann equation is given by

$$H^2 = \frac{8\pi G}{3} \frac{E}{V} + \frac{\Lambda}{3} - \frac{k}{R^2}$$

(1)
where $H = \dot{R}/R$ is the Hubble parameter, $E/V$ represents the energy density, $\Lambda$ the cosmological constant. $k = 1, -1$ or 0 corresponds to closed, open or flat cosmological models, respectively. We will concentrate our attention on the closed model in the following, which expands at first until it reaches a maximal radius, and subsequently recollapses.

Gravity with a cosmological constant has been studied since long time (see [23]). The general situation, concerning stability and general properties of theory is such that they cannot be excluded (see introduction of [23]). Anywhere inside the event horizon, de Sitter space is stable.

In [24], we have shown from a Geroch process that the Bekenstein bound does not change the form if the background spacetime has a cosmological constant. Thus in the present context, namely that of a closed de Sitter universe with radius $R$ and total energy $E$, the appropriate Bekenstein bound is still

$$S < S_B$$

where the Bekenstein entropy is $S_B = 2\pi ER$.

The Bekenstein bound is only appropriate for systems with limited self-gravity satisfying $HR < 1$. In a strongly self-gravitating universe with $HR > 1$, the possible formation of a black hole has to be considered and the entropy bound has to be modified. Using the idea proposed in [7,8], the total entropy of the strongly gravitating system is bounded by the Hubble bound $S_H = n_H S^H$, where $n_H$ is the number of cosmological horizons within a given comoving volume divided by a volume of a single horizon $n_H = V/|H|^{-3}$; $S^H$ is the entropy within a given horizon $H^{-2}/4G$. The appropriately normalized Hubble bound for $HR > 1$ takes the form [15]

$$S_H = \frac{HV}{2G}.$$  

The holographic principle is based upon the idea that the maximal entropy of a volume $V$ is given by the largest black hole that fits inside $V$ [3].
A black hole in de Sitter space is of the form

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2d\Omega_2^2,$$  \hspace{1cm} (4)$$

where $f(r) = 1 - \frac{2MG}{r} - \frac{r^2}{r_0^2}$, and $r_0 = \sqrt{3/\Lambda}$. There are two positive roots of the cubic equation $f(r) = 0$ corresponding to black hole horizon $r_+$ and cosmological horizon $r_c$. As the mass parameter $M$ increases, the black hole radius $r_+$ increases and the cosmological horizon $r_c$ decreases monotonically. They become equal for maximal value of $M$, $r_+ = r_c = \sqrt{1/\Lambda}$, which is the biggest black hole that can be formed in de Sitter space.

Following directly from the Friedmann equation (1) for closed universe, we find when the universe goes from the weakly to strongly self-gravitating phase, the energy sufficient to form a black hole of the size of the universe, $r_+ = R = \sqrt{1/\Lambda}$, is

$$E_{BH} = \frac{5V}{8\pi GR^2}$$  \hspace{1cm} (5)$$

and the holographic Bekenstein-Hawking entropy of the black hole with universe-size reads

$$S_{BH} = \frac{5V}{4GR}.$$  \hspace{1cm} (6)$$

With Eqs(2,3,6) at hand, we can rewrite the Friedmann equation (1) in closed universe in the form

$$S_H^2 = \frac{4}{43}(5S_BS_{BH} - 2S_{BH}^2).$$  \hspace{1cm} (7)$$

Writing $S_H$ in terms of the energy $E$, the radius $R$ and the Bekenstein-Hawking energy $E_{BH}$, Eq(7) becomes

$$S_H = \frac{2\pi}{5\sqrt{6}} \sqrt{8E_{BH}R(5ER - 2E_{BH}R)}.$$  \hspace{1cm} (8)$$

After making the identifications $L_0 = 5RE, \frac{C}{6} = 8RE_{BH}$, Eq(8) can be changed to

$$S_H = \frac{2\pi}{5\sqrt{6}} \sqrt{\frac{C}{6}(L_0 - \frac{C}{24}).}$$  \hspace{1cm} (9)$$
This is exactly the Cardy formula except for the difference in the numerical pre-factor. Inside the event horizon every term makes sense [23].

Therefore at the turning point $HR = 1$, when the universe develops from a weakly to strongly gravitating system, there is a possibility of forming a largest black hole within the de Sitter closed universe and the correspondence between the Friedmann equation and the Cardy formula holds at this moment. Eq(9) unifies the Bekenstein bound and Hubble bound in the de Sitter cosmological context. Along the evolution of the closed universe it would be no problem to satisfy both Bekenstein bound for weak gravity regime and Hubble bound for strong gravity regime if (9) holds.

The matching of the Friedmann equation to Cardy formula cannot be realized in open and flat de Sitter universes, because those universes always expand and self-gravity of the system is always weak. Only Bekenstein entropy bound can be satisfied in those context. This result agrees to that in [17] where they found that the dual theory does not hold in weakly coupled CFT’s with AdS duals.

B. Anti-De Sitter universes.

Now we start to study the universes with negative cosmological constant. Usually, cosmologists believe that closed universes collapse, whereas open and flat universes expand forever. But the situation is not quite so simple. If there is a negative cosmological constant, the universe collapses independent of whether it is closed, open or flat. The Friedmann equation in AdS universes is

$$H^2 = \frac{8\pi G}{3} \frac{E}{V} - \frac{\Lambda}{3} - \frac{k}{R^2}$$  

(10)

where $H$ the Hubble constant, $E/V$ denotes the energy density, $-\Lambda$ is the value of the negative cosmological constant. $k = 1, -1, 0$ corresponds to closed, open or flat universes, respectively.

From [24] we learnt that the Bekenstein bound still holds as expressed in (2) in AdS universes when the gravitational self-energy is limited ($HR < 1$). When the gravitational
self-energy becomes strong ($HR > 1$), the Bekenstein entropy bound breaks down. We need to introduce the Hubble entropy bound Eq(3) in the spirit of [7,8] in a strongly self-gravitating universe.

In the AdS space, the black hole has the metric (4), where $f(r) = 1 - \frac{2MG}{r} + \frac{r^2}{r_0^2}$ and $r_0 = \sqrt{3/\Lambda}$ is the AdS radius. For a large AdS black hole ($r_0 \ll r_+ < 2MG$ [25]) we find from $f(r) = 0$ that the black hole radius is

$$r_+ = (6MG/\Lambda)^{1/3} - 1/(6GMA^2)^{1/3}.$$  \hfill (11)

The relation between the cosmological constant $\Lambda$ and black hole horizon $r_+$ reads

$$\Lambda = 6GM/r_+^3 - 3/r_+^2.$$  \hfill (12)

For the flat AdS universe ($k = 0$), we suppose the universe is filled with the universe-size AdS black hole ($R = r_+$). From (10) we have $HR = 1$ if we take $E = M, V = 4\pi R^3/3$ and (12). This is exactly the turning point between the weakly and strongly self-gravitating system. According to [15], a Cardy formula can be deduced here to unify the Bekenstein bound and Hubble bound at this turning point. However we cannot directly get the expression of the energy to form a black hole as that obtained in [15], or else in the de Sitter case above, as obtained from the Friedmann equation (10).

Following the idea introduced in [7] by Veneziano, we can write the scale of causal connection instead of the Hubble radius $H^{-1}$, that is,

$$H^{-1} \rightarrow R \int \frac{dt}{R} = R \int_0^R \frac{dR}{RR} \equiv d_H.$$  \hfill (13)

Using (10) and considering the flat AdS universe is occupied by the universe-size AdS hole ($R = r_+$), we get

$$d_H = R^{2/3}(2GE)^{-1/2}(1 - R/2GE)^{-1/6}.$$  \hfill (14)
Substituting (14) into the Hubble bound (3) with $H \to d_H$, we have

$$S_H^2 = \frac{8\pi^2 ER^3}{9G} - \frac{4\pi^2 R^4}{27G^2}.$$  \hfill (15)

Since the holographic Bekenstein-Hawking entropy of the universe-size AdS black hole is $S_{BH} = \frac{A}{4G} = \frac{\pi R^2}{G}$ and the Bekenstein entropy $S_B = 2\pi ER$, (15) can be rewritten as

$$S_H^2 = \frac{4}{9}S_{BH}(S_B - \frac{1}{3}S_{BH}).$$  \hfill (16)

Using the energy $E$ and the Bekenstein-Hawking energy to form the universe-size AdS hole as given by $E_{BH} = R/(2G)$ to replace $S_B$ and $S_{BH}$, (16) becomes

$$S_H = \frac{2\pi}{3\sqrt{3}}\sqrt{4E_{BH}R(3ER - E_{BH}R)}.$$  \hfill (17)

If one takes the Virasoro operator as being $L_0 = 3ER$ and the central charge as $c/6 = 4E_{BH}R$, we get

$$S_H = \frac{2\pi}{3\sqrt{3}}\sqrt{c(L_0 - c/24)/6},$$  \hfill (18)

which is in striking resemblance with the Cardy formula in CFT.

The result of relating the Friedmann equation to the Cardy formula cannot be naively extended to AdS closed and open universes. If one supposes that there forms a universe-size AdS hole satisfying eqs(11,12), one can learn directly from (10) that $H = 0$ and $HR = \sqrt{2}$ for $k = 1$ and $k = -1$ AdS universes respectively. This corresponds to say that if closed and open AdS universes are occupied by universe-size AdS black holes, self-gravity of the system will always be limited or very strong for closed or open AdS universe respectively. It is impossible to unify the Bekenstein bound and Hubble bound in these AdS universes. On the other hand, if one supposes that a turning point $HR = 1$ exists for closed and open AdS universes, (10) will tell us that for a closed AdS universe $R = (6GE/\Lambda)^{1/3} - 2/(6GE\Lambda^2)^{1/3}$, while for the open AdS case $R = (6GE/\Lambda)^{1/3}$, which is smaller or bigger than the AdS black hole radius expressed in (11), respectively. This
means that at the turning point $HR = 1$, neither the closed AdS universe nor the open AdS universe can accommodate a universe-size AdS black hole in it. The general philosophy of the holography which needs the largest black hole to fit inside the space to describe holography cannot be satisfied at the turning point $HR = 1$ of closed and open AdS universes. The fact that it is not possible either to obtain a turning point between weak and strong self gravity, or to form an AdS black hole with the size of the universe implies that there is no connection between the Friedmann equation and the Cardy formula in closed or open AdS cosmologies.

In summary, we have investigated the relationship between Friedmann equation and Cardy formula in de Sitter and AdS universes. We found that at the turning point, when the universe evolves from limited self-gravity to strongly self-gravity system, relation between Friedmann equation and entropy bounds matches strikingly the Cardy formula in de Sitter closed, AdS flat universes. Our result generalized that obtained in closed FLRW universes without cosmological constant [15]. This correspondence further supports holography, since the Friedmann equation describing the bulk can be related to Cardy formula on the boundary directly. The Cardy formula obtained unifies the Bekenstein bound and Hubble bound in de Sitter and AdS cosmological models. It is worth noting that we did not specify the equation of state of matter in our discussions, thus the result derived here is independent of the matter state. The verification of the validity of the generalized Cardy formula from coupled CFT is needed to be carried out in the future.

For de Sitter open or flat universes and AdS closed or open universes, the correspondence between Friedmann equation and Cardy formula has not been found. This is not so surprising if we notice that the largest universe-size black hole cannot be formed in those universes. The general philosophy of the holographic principle cannot be satisfied in those context. The unification of Bekenstein bound and Hubble bound in those cases does not look possible.
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