The power of clockings

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Abstract
We investigate the expressive power of a Turing-complete logic based on game-theoretic semantics. By defining suitable fragments and variants of the logic, we obtain a range of natural characterizations for some fundamental families of model classes.

1 Introduction
We investigate the expressive power of the Turing-complete logic defined originally in [8]. The logic is based on game-theoretic semantics and has constructors for looping and modification of relations. The logic is a particularly natural extension of standard first-order logic FO.

In particular, we make use of fragments and variants of the Turing-complete logic T, see the preliminaries for the specification. The point is to give characterizations for some of the higher complexity classes. In addition to using the features already available in T, we make use of clocking terms. These are syntactic operators that can be used to limit the time and space required in semantic games.

There exist several works that make use of clockings in the literature. Much of the work in [2] and [3] and [5], including the background motivations for those studies, relates to investigating the properties of T with clockings. The paper [2] provides a game-theoretic semantics for ATL with clock values declared on the fly. The paper [3] has similar motivations, but investigates the more involved case of ATL+. The article [5] concentrates on the μ-calculus. The clockings are part of the semantics and help control the semantic game. Also non-standard clockings (i.e., ones that lead to semantic variants of the original logics) are studied with the aim of defining interesting variants of the original systems. Defining logics with different kinds of bounding constructors has of course been done also elsewhere, with different kinds of motivations. For some interesting examples, see, e.g., [1] and [7].
The clocking terms in this article are syntactic arithmetic expressions (e.g., $2^n$) that directly limit the duration of (or space used in) the semantic game. For example, if an operator is coupled with $2^n$, this means that it should not be encountered in the semantic game for more than $2^n$ times, where $n$ is the size of the domain of the input model (that is, the model from where the semantic game begins). By such natural and simple additions to the syntax of $T$, we provide characterizations for $k$-fold exponential time and space classes for all $k$. These are capture results in the usual sense of descriptive complexity [10], [6].

2 Preliminaries

We denote models by $A$, $B$, $M$, et cetera. The corresponding domain is denoted by $A$, $B$, $M$, et cetera. Models are assumed finite with a finite relational vocabulary. We note that we often use the same symbol $R$ to denote both a relation $R^M$ and the underlying relation symbol $R$. This is for the sake of simplicity. Now, suppose we have a linear order $<$ over the domain $M$ of a model $M$. Suppose we have also ordered the set of relation symbols in the vocabulary of $M$. Then we let $\text{enc}(M)$ denote the binary encoding of $M$ as defined in [10] (please see the full details there).

Basically, the encoding $\text{enc}(M)$ first lists the bit 1 for $|M|$ times, followed by 0. Then the relations are encoded so that the relation encodings become concatenated one at a time in the order that the ordering of the vocabulary requires. A $k$-ary relation is encoded as a binary string $s$ of length $M^k$ with bit 1 at the position $j$ of $s$ indicating that the $j$th tuple of $M^k$ (with respect to the lexicographic order given by $<$) is in the relation.

We assume that all relational vocabularies always have a canonical order associated with them, so we will only have to make sure a suitable linear is present when defining encodings of models. We note that, in the elaborations below, we shall in fact mainly use a successor relation (of the linear order) rather than a linear order itself. This obviously does not change the encoding and is not a matter of substance any way.

In this article we study the Turing-complete logic defined in [8]. The syntax and semantics of the logic is carefully specified in that article, so we shall not reiterate all the formalities here. The main point is to add two classes of constructs to first-order logic $\text{FO}$, namely, operators that allow to modify the underlying model and constructs that enable looping in semantic games when evaluating formulae. The operators that modify the model are as follows.

1. The operator $Ix$ adds a new point into the model domain and names it with the variable $x$. So, when encountering a formula $Ix\varphi$ in a semantic game, the following happens.

   (a) The model is extended with a new element (keeping all relations as they are).

   (b) The new element is called $x$, i.e., the current assignment function $g$ is modified so that it sends the variable symbol $x$ to the new element.
2. The operator \( I(R(x_1,\ldots,x_k)) \) adds a tuple to the \( k \)-ary relation \( R \) and lets \( x_1,\ldots,x_k \) denote the elements of that tuple. That is, the assignment \( g \) is modified such that \( x_1,\ldots,x_k \) map to the elements of the new tuple. When encountering a formula \( I(R(x_1,\ldots,x_k))\varphi \), the verifying player first does the addition and then the game continues from \( \varphi \). Note that this step does not involve adding any new domain points to the model. Note also that \( R \) is not necessarily genuinely extended: the verifier picks a tuple \( r \) from the model and then \( R \) is updated to \( R \cup \{r\} \). This may of course leave \( R \) as it was, which happens if we already had \( r \in R \).

3. The operator \( D(R(x_1,\ldots,x_k)) \) deletes a tuple \( (m_1,\ldots,m_k) \) from the \( k \)-ary relation \( R \) and then the deleted tuple \( (m_1,\ldots,m_k) \) is marked by the tuple \( (x_1,\ldots,x_k) \) of variables. That is, the assignment \( g \) is modified such that \( x_1,\ldots,x_k \) map to the elements of the new tuple. Therefore, when encountering a formula \( D(R(x_1,\ldots,x_k))\varphi \), the verifying player first does the modifications and then the game continues from \( \varphi \). We note that the verifier is not required to actually delete a tuple from \( R \), instead the verifier appoints a tuple \( (m_1,\ldots,m_k) \) and then this tuple is deleted from \( R \) if \( R \) has that tuple. Otherwise \( R \) stays as it is. Note that the domain does not become modified.

4. The looping operator \( C \) allows self-reference, i.e., formulae can refer to themselves. The operator acts as an atomic formula as well as a labelling operator. A formula \( C\varphi \) has the label symbol \( C \) in front of it. Intuitively, the symbol \( C \) names \( \varphi \) to be the formula called “\( C \)”. In a semantic game, from positions with \( C\varphi \), we simply move on to the position with \( \varphi \). Now, \( C \) can also be an atomic formula inside \( \varphi \). When encountering the atom \( C \), the game jumps back to the position \( C\varphi \). When \( C \) is an atom, we may refer to it as a looping atom. For example, in the formula \( \varphi(x) := C(Px \lor \exists y(Rxy \land \exists x(y = x \land C)) \) the first occurrence of \( C \) is a label symbol and the second one a looping atom. We note that non-looping-atoms are also called first-order atoms.

The logic also enables the use of tape predicates. These are ordinary relation symbols, with the difference that they are not included in the vocabulary of the models under investigation. Thus they can also be regarded as relation variables. The interpretation of each tape predicate is the empty relation in the beginning of the semantic game. During the game, also tape predicates \( X \) can of course be modified by the operators \( D(X(x_1,\ldots,x_k)) \) and \( I(X(x_1,\ldots,x_k)) \).

Now, why are the predicates \( X \) that are not part of the input signature called tape predicates? The analogy with tape symbols here is that tape predicates are auxiliary objects used in the semantic game rather than relations in the underlying vocabulary of the models considered.

The arity of a relation (or the relation symbol) \( R \) is denoted by \( \text{ar}(R) \). The same convention holds for tape predicate symbols and the related relations.
In our logic, it is also possible to define a deletion operator $Dx$ that removes a point already labelled by $x$. Then the assignment is modified accordingly, see [9]. Whether or not we include this operator in our base logic does not affect the results below. They are the same.

The semantic games end in a position with a first-order atom (e.g., $R(x_1, \ldots, x_k)$ or $x = y$). The verifier wins the play if the atom holds, and otherwise the falsifier wins. If some required move cannot be made, the game ends with neither player winning. For example, if a position with a looping atom $C$ is encountered and there does not exist a corresponding label symbol $C$ in the formula, then the game ends with neither of the players winning. As another example, in a first-order atom that has a variable that does not appear in the domain of the current assignment, the game ends with neither player winning.

The semantic game is played between Eloise and Abelard. Eloise begins as the verifier (and negation changes the role of Eloise from verifier to falsifier and vice versa). We write $\mathfrak{M}, g \models \varphi$ and consider $\varphi$ true in $\varphi$ if Eloise has a winning strategy in the game involving $\mathfrak{M}, g$ and $\varphi$. We may write $\mathfrak{M} \models \varphi$ if $g$ is the empty assignment or otherwise irrelevant. We note that winning strategies are assumed positional. However, this makes no difference due to the positional determinacy of reachability games (which holds even on infinite models) [4].

Recall the formula

$$\varphi(x) := C(Px \lor \exists y(Rxy \land \exists x(y = x \land C)))$$

from above. We have $\mathfrak{M}, \{(x, m)\} \models \varphi(x)$ iff we can reach from $m$ an element satisfying $P$. Note that we can make the semantic game always terminate in finite models by considering instead the formula $C(Px \lor \exists y(Rxy \land x \neq y \land Dx \exists x(y = x \land C)))$.

We call the so defined logic T. Formally, we let T be the logic $L$ as defined in [8]. As discussed above, the following rules hold.

1. If a first-order atom with a variable $x$ is reached, and the current assignment gives no interpretation to $x$, then the play of the game ends. Neither player wins that play of the game.

2. If a position with a looping atom $C$ is encountered, and there is no subformula $C\varphi$ in the main formula, then the play of the game ends. Neither players wins that play.

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1. However, in the empty model, if a position with $\exists x, I(R(x_1, \ldots, x_k))$ or $D(R(x_1, \ldots, x_k))$ is encountered, then the game ends and the current verifier loses. This is because we cannot label anything with the variable symbols. Yet further, in the empty model, $I(R)$ and $D(R)$ for a nullary symbol $R$ are fine and can be performed: we can add or remove the empty tuple when considering a nullary relation. So there the game will continue. However, there is no genuinely interesting reason to consider the empty model from the point of view of this article.

2. It may be more intuitive to consider $\varphi$ verifiable rather than true. Technically this makes no difference.

3. We note that in [8], this rule lead to the verifier losing. This alternative convention would not affect any of the below proofs or results.
Such positions are pathological and not really needed for the results below. They could be avoided by defining a suitable notion of a strongly closed formula and then limiting to such formulae the study below.

We note that $T$ does not contain $Dx$, but it makes no difference concerning the results and proofs below whether or not we include the operator $Dx$ or not. We also note that the results and proofs fo through as such for the logic $L^*$ precisely as defined in [3], and whether of not we add $Dx$ to $L^*$ also makes no difference.

The formula from where a semantic game begins is often referred to as the original formula or input formula. The model in the beginning of a semantic game is the original model or input model. Sometimes even the terms original input formula and original input model are used.

3 Characterizations of higher classes

In this section we provide characterizations for $k$-EXPSPACE for all $k \in \mathbb{N}$ by defining suitable restrictions for the looping operators $C$ and model extension operators $Ix$. The extensions essentially use a term $t(n)$, depending on the model domain size $n$, that restricts how many times the operator instance can be used in a play of the semantic game. We note that for these characterizations, it makes no difference whether we consider ordered models or not.

Before introducing the related restriction constructs, we begin by characterizing EXPTIME as the fragment of $T$ that forbids the use of $Ix$. Let $T[-Ix]$ denote the restriction of $T$ to the syntax that does not allow the use of the operator $Ix$ (for any variable $x$). Note that the logic does allow the operators $I$ that insert tuples into relations; only the model domain extension capacity is restricted.

**Theorem 3.1.** $T[-Ix]$ captures EXPTIME.

**Proof.** Fix a formula $\varphi$ of $T[-Ix]$. We show how to design an EXPTIME model checking procedure for recognizing models of the fixed formula $\varphi$.

Now, suppose $\varphi$ has $m$ tape predicates of arities $k_1, \ldots, k_m$. Then a position in the semantic game is fully encoded by a tuple of type $(\mathfrak{M}, g, \psi, \#)$ specified as follows.

1. $\mathfrak{M}$ is the underlying model at the current stage of the game.
2. $g$ is an assignment that interprets first order variables in the domain $M$ of $\mathfrak{M}$. Furthermore, $g$ interprets the tape predicates $X_i$ as relations $X \subseteq M^{ar(X)}$.
3. $\psi$ is the current subformula being played.
4. $\# \in \{+, -\}$ indicates whether Eloise is currently the verifying player (+) or falsifying player (−).

For the fixed formula $\varphi$, the descriptions of positions $(\mathfrak{M}, g, \psi, \#)$ in the semantic game are polynomial in the size of (the description of) the original input model. Indeed, the number of tape predicates $X$ and relation symbols $R$ in the vocabulary is constant.
for the fixed formula $\varphi$, and thus the sizes of encodings of the corresponding relations $X \subseteq M^{ar}(X)$ and $R \subseteq M^{ar}(R)$ is not a problem with regard to polynomiality. Also, the number of first-order variables that need to be encoded is constant, and so is the number of subformulae of the original input formula.

As we need only polynomial amount of memory to encode and arbitrary position in the semantic game, it is clear that we can model the semantic games for $\varphi$ (for all input models) by an alternating polynomial space Turing machine $TM_{\varphi}$. Indeed, the machine simply keeps track of the current position, and positions where Eloise moves correspond to existential states while Abelard’s positions correspond to universal states.

As $\text{APSpace}$ equals $\text{ExpTime}$, we have found the required Turing machine corresponding to $\varphi$.

Assume then that we have a Turing machine $TM$ running in $\text{APSpace}$. We will translate the alternating polynomial space machine $TM$ to a corresponding sentence $\varphi_{TM}$. Let the space required by the machine $TM$ to be bounded from above by the polynomial $p(x)$ of order $k$. Note that there are $|M|^{k+1}$ tuples of arity $k+1$ in a model $M$, so we can encode the tape cells required by $TM$ into $(k+1)$-tuples of the model. However, of course we may have $p(x) > x^{k+1}$ for some small enough $x$, but we can deal with the those finitely many small models by a first-order sentence $\chi$ that accepts precisely those small models that $TM$ will, rejecting the remaining small models. For greater sizes, we will write a separate sentence. Thus, without loss of generality, we ignore the issue with small models and thereby assume that the running time of $TM$ is everywhere bounded from above by $|M|^{k+1}$.

Now, to construct $\varphi_{TM}$, we fix a new tape predicate $S$ which will be built into a successor relation over the input model. We also define a new $(k+1)$-ary predicate $Y_q$ for each state $q$ of $TM$. The computation is encoded such that when the read-write head of $TM$ is in the cell $j$ and the current state is $q$, then the predicate $Y_q$ holds in the tuple $(m_1, \ldots, m_{k+1}) \in M^{k+1}$ that is lexicographically (with respect to the successor relation $S$) the $j$th tuple. During that computation step, $Y_q$ holds nowhere else, and for each $q' \neq q$, the predicate $Y_{q'}$ does not hold anywhere. When modifying such a predicate $Y_q$ to simulate the computation, we can

1. use an extra $(k+1)$-ary tape predicate $Y$ to encode where $Y_q$ currently holds,
2. then delete $Y_q$ (i.e., to make it hold nowhere), and
3. then use $Y$ to modify the predicate $Y_{q_{\text{new}}}$ so that it holds at the right slot.

Note that the reason we use the temporary storage predicate $Y$ is that if $q = q_{\text{new}}$, the predicate $Y$ will help distinguishing between the old and the new locations of $Y_q$. There are other ways around this problem, but using the store predicate $Y$ is a particularly easy solution suitable for our purposes.

We also define a $(k+1)$-ary predicate $X_A$ for each (tape or input) symbol $A$ of $TM$. These are modified to hold in those cells (i.e., $(k+1)$-tuples) where they would hold during the computation.
In the beginning of computation, after creating the successor relation \( S \), the sentence \( \varphi_{TM} \) will make sure that the binary encoding \( \text{enc}(\mathfrak{M}) \) will be written into the \((k + 1)\)tuples of \( M^{k+1} \) using the predicates \( X_A \). This is easy to do, using further auxiliary tape predicates \( Z_i \) to gain easier control on the specification.

The computation of \( TM \) itself is simulated in the natural way. Consider a state \( q \) and a symbol \( A \). Suppose \( TM \) has the following \( m \) allowed transition instructions from the state \( q \) when scanning the symbol \( A \):

\[
(q, A) \mapsto (B_1, D_1, q_1) \\
\vdots \\
(q, A) \mapsto (B_m, D_m, q_m)
\]

where each \( B_i \) denotes a new symbol to be written to the current cell; \( D_i \in \{\text{left, right}\} \) denotes the direction where \( TM \) is to move; and \( q_i \) denotes the new state. For each \( i \), let \( \psi(B_i, D_i, q_m) \) denote the formula stating that Eloise should modify the model as follows.

1. The tape predicate \( X_{B_i} \) should be made to hold in the tuple which is at the position indicated by the store predicate \( Y \) (which points at the current position of the read-write head). Furthermore, the tape predicate \( X_A \) should be modified so that it does no longer hold at the position indicated by \( Y \) (unless we have \( A = B_i \)). All this amounts to the symbol \( A \) being erased from the current cell and \( B_i \) being written to that cell instead.

2. The tape predicate \( Y_q \) should be deleted from the current cell and the tape predicate \( Y_{q_i} \) should be made to hold in the adjacent cell which is in the direction \( D_i \in \{\text{left, right}\} \) from the current cell.\(^4\)

3. The predicate \( Y \) should be updated to hold precisely at the position of \( Y_{q_i} \) and Eloise should enter the atom \( C_{\text{loop}} \). However, if the new state \( q_i \) is an accepting state, then, instead of \( C_{\text{loop}} \), we have the atom \( \top \) where Eloise can win the play of the semantic game. Similarly, if \( q_i \) is a rejecting state, then we have \( \bot \) instead of \( \top \).

Now, if \( q \) is an existential state, then let \( \varphi_{(q, A)} \) denote the formula

\[
\varphi'_{q, A} \land (\psi_{q_1} \lor \ldots \lor \psi_{q_m})
\]

where \( \varphi'_{q, A} \) states that the current cell where \( Y_q \) and \( Y \) hold has the symbol \( A \) in it, i.e., also \( X_A \) holds in that cell. On the other hand, if \( q \) is a universal state, then we let \( \varphi_{q, A} \) be the formula

\[
\varphi'_{q, A} \land \psi_{q_1} \land \ldots \land \psi_{q_m}.
\]

\(^4\)Note that all kinds of fringe effects are straightforward to deal with. For example, it is trivial to alter the Turing machine so that it never attempts to go left from the leftmost cell.
Now, let $I$ list the state-symbol pairs $(q,A)$ that act as inputs to the transition relation that specifies $TM$. Define then the formula

$$\varphi_1 := \bigvee_{(q,A) \in I} \varphi_{q,A}$$

where we recall that the formulae $\varphi_{q,A}$ loop (when they loop) via the looping atom $C_{\text{loop}}$.

Let $\alpha_{\text{input}}$ be the formula

$$I(X_{A_1}(x_1, \ldots, x_{k+1}))C_{\text{enc}}$$

$$\lor$$

$$\vdots$$

$$I(X_{A_{m'}}(x_1, \ldots, x_{k+1}))C_{\text{enc}}$$

$$\lor$$

$$I(Z_1(x_1, \ldots, x_{\text{ar}(Z_1)}))C_{\text{enc}}$$

$$\lor$$

$$\vdots$$

$$I(Z_{\ell}(x_1, \ldots, x_{\text{ar}(Z_{\ell})}))C_{\text{enc}}$$

which allows the verifier to choose one of the tape predicates $X_{A_1}, \ldots, X_{A_{m'}}, Z_1, \ldots, Z_{\ell}$ and add a tuple to it, after which the game proceeds to the looping atom $C_{\text{enc}}$. Here we assume that the collection of input symbols of $TM$ is $\{A_1, \ldots, A_{m'}\}$. The predicates $Z_i$ are the auxiliary predicates that help in making sure the encoding of the input model becomes modelled correctly on the successor relation over the $(k+1)$-tuples of the input model $M$. Note that the successor relation over the $(k+1)$-tuples is defined lexicographically based on the binary successor relation $S$ over the domain of the input model. It is a relation of arity $2(k+1)$. The encoding is written to a prefix of the related successor relation (the cells of computation).

Now, based on $\alpha_{\text{input}}$, define the formula

$$\gamma := C_{\text{enc}}(\neg\chi_{\text{enc}} \land \alpha_{\text{input}}) \lor (\chi_{\text{enc}} \land \varphi_1)$$

where $\chi_{\text{enc}}$ states that the binary encoding of the input model $M$ is encoded in a prefix of the successor relation over $M^{(k+1)}$ in the correct way. Recall that $\varphi_1$ is the formula written already above.

Define then, based on $\gamma$, the formula

$$\varphi_{TM} := C_{\text{succ}}(\neg\chi_{\text{succ}} \land \alpha(C_{\text{succ}}) \lor (\chi_{\text{succ}} \land \gamma))$$

where the following conditions hold.
1. $\alpha(C_{\text{succ}})$ requires Eloise to construct the auxiliary predicates $S$ and $S'$ so that $S'$ is a linear order and $S$ the corresponding binary successor relation over the input model.

2. $\chi_{\text{succ}}$ is a first-order formula that states that $S'$ is a linear order over the model and $S$ the corresponding successor. (Including the linear order makes it possible to express in first-order logic that $S$ is indeed a successor.)

Note that the main computational part of the formula is $\varphi_1$. The other parts relate to initial constructions creating the binary input encoding and the successor to enable that encoding.

We then define a generalization of the syntax of the logic $T$. We begin by defining a collection of suitable terms that relate to complexity classes with $k$-fold exponential limitations on resources.

Let $\text{Pol}$ denote syntactic terms of the form

$$c_m n^m + c_{m-1} n^{m-1} + \cdots + c_1 n + c_0$$

where

1. $n$ is variable symbol (distinct from the usual logic variable symbols $x$, $y$, etc.).
2. $c_i \in \mathbb{N}$ are binary strings denoting natural numbers.
3. The exponents $m - j$ are, likewise, binary strings denoting natural numbers.

The set $\text{Pol}$ thus contains (all) terms denoting polynomials in the variable $n$ and with integer coefficients.

Now consider the set of terms of type

$$\uparrow(k, t)$$

where $k$ is a binary string denoting a number in $\mathbb{N}$ and $t$ a term in $\text{Pol}$. Now, if $t$ denotes the polynomial $p(n)$, then $\uparrow(k, t)$ denotes

$$2^{2^{p(n)}}$$

where the tower (excluding $p(n)$) is of height $k$, i.e., the number 2 is written $k$ times (ignoring the possible occurrences of 2 in $p(n)$). Let $\text{all-ExpTerm}$ denote the set of terms of type

$$c \cdot \uparrow(k, t) + d$$

where $c$, $k$ and $d$ are binary strings denoting numbers in $\mathbb{N}$ and $t$ is a term in $\text{Pol}$. If $x$ and $y$ denote the natural numbers encoded by $c$ and $d$ and if $t$ corresponds to $p(n)$, then the term $c \cdot \uparrow(k, t) + d$ denotes

$$x \cdot 2^{2^{p(n)}} + y.$$
Note that functions in
\[ O\left(2^{2 \cdot 2^{p(n)}}\right) \]
are naturally bounded by functions that can be expressed in the form
\[ x \cdot 2^{2 \cdot 2^{p(n)}} + y \]
for different values of \( x \) and \( y \) in \( \mathbb{N} \) and with the input variable being \( n \) (the tower is of course assumed to of the same height in both cases). Thus we are encoding \( k \)-fold exponential functions. We let \( k\text{-ExpTerm} \) denote the set of terms of type
\[ c \cdot \uparrow (\ell, t) + d \]
where \( \ell \) denotes a number in \( \{0, \ldots, k\} \). In the case \( \ell = 0 \), the term \( \uparrow (0, t) \) just outputs the term \( t \). We note that more general sets of terms could serve our purposes quite well in the below elaborations, but the related generalizations are not difficult to investigate, and the one we use here does the job well enough from the point of view of the current work. Generalizations are left for the future.

Suppose that \( k \geq 1 \). We let \( T[\text{Ix} \mid k\text{Exp}] \) denote the restriction of \( T \) where each operator \( \text{Ix}, \text{Iy}, \text{etc.} \), must be written in the form \( \text{Ix}(t), \text{Iy}(t), \text{etc.} \), where \( t \) is a \( k\text{Exp} \)-term. On the semantic side, a node \( \text{Ix}(t)\phi \) in the syntax tree of the original formula can be visited only \( t \) times, i.e., the number \( \ell \in \mathbb{N} \) of times that the term \( t \) refers to. If the semantic game proceeds to that we visit such a node for one more time, the play of the game then ends there and neither of the players win that play.

A natural way to think of this definition is provided by clockings. We add a clock function \( c \) to the semantic game. The function \( c \) is similar to the assignment function \( g \) but instead gives some value \( p \in \mathbb{N} \) for each node of type \( \text{Ix}(t)\psi \). Initially that value is the (binary representation of the) number that \( t \) encodes. After that, we decrease the value for node \( \text{Ix}\psi \) each time that the node \( \text{Ix}\psi \) is visited. If we enter the node when the clock value is already 0, then the play of the game ends and indeed neither player wins that play.

**Theorem 3.2.** \( T[\text{Ix} \mid k\text{Exp}] \) captures \((k + 1)\)-ExpTime.

**Proof.** Let \( \varphi \) be a formula of \( T[\text{Ix} \mid k\text{Exp}] \). Let us find the required Turing machine.

Positions of the semantic game are of type \( (\mathcal{M}, g, c, \# \psi) \) such that the following conditions hold.

1. \( \mathcal{M} \) is model which can be dynamically modified during the game.
2. \( g \) interprets first-order variables and tape predicates in the model.
3. \( c \) gives the clock values for the (nodes of the original formula) that have element insertion operators \( \text{Ix}, \text{Iy}, \text{etc.} \).
4. \( \# \in \{+, -\} \) indicates whether Eloise is the verifier (+) or the falsifier (−).
5. $\psi$ is a subformula of the original formula.

Since the use of the element insertion operator is limited by a $k$-fold exponential term, the size of the (description of the) position $(M, g, c, \#, \psi)$ is, likewise, limited to $k$-fold exponential with respect to the description of the original model. We shall do as in the proof of Theorem 3.1: that is, we use alternating turing machines. Namely, we use an alternating Turing machine running in alternating $k$-exponential space to simulate the evaluation game. As alternating $k$-ExpSpace equals $(k + 1)$-Exptime, the translation from the formula $\varphi$ to the required machine is clear. We simply simulate the game with an alternating machine, and the space allowed for the alternating machine suffices.

For the converse translation, suppose we have an alternating $k$-ExpSpace machine $TM$. Let the space required be bounded above by the $k$-ExpSpace function $f(n)$ corresponding to a $k$-ExpTerm. Simulating $TM$ by a formula $\varphi_{TM}$ is based on the following steps.

1. The formula $\varphi_{TM}$ begins with $CI(t)x$ where $t$ is a syntactic counting term for $f(n)$. In fact, we choose from $k$-ExpTerm a sufficiently large term term $t$ so that $f$ is in the big $O$ class for $t$, and we can deal with possible minor fringe effects that become realized in our below construction.

The formula $\varphi_{TM}$ enables Eloise to first add new points to the model, the number of them bounded by $t$.

2. The new points are labelled by a unary tape predicate $N$ as ‘new points’ not part of the original domain of the input model. This labelling is done simultaneously to adding the points. The formula $\varphi_{TM}$ required is of the form

$$C_{\text{build}} I(t)x I(t)(Ny)((C_{\text{build}} \land y = x) \lor \chi)$$

where $\chi$ first builds a successor relation $S$ over the new elements and then deals with the rest of the computation. (Obviously $\chi$ does not contain looping atoms $C_{\text{build}}$, but instead uses other looping atoms.)

3. Then a prefix (with respect to the new successor relation $S$ over the new elements) is labelled by the string $enc(M)$, where $M$ is the original input model. Note that the encoding requires another successor order $S'$ to be built over $M$. The labelling of the prefix with $enc(M)$ is done with unary tape predicates $P_0$ and $P_1$ for the bits 0 and 1. For this step, we use auxiliary predicates (of sufficient arity) that scan and connect the original input model $M$ to the new part with the successor relation $S$. Having all of first-order logic together with the looping capacity and a supply of auxiliary tape predicates in our logic, this is straightforward to do. A similar step was already done in the proof of Theorem 3.1, but this time we have a clearly separate new part of the model where $enc(M)$ is encoded. Furthermore, this time the bits of the binary input can simply be unary predicates, $P_0$ and $P_1$.

4. After $enc(M)$ is recorded, the formula forces Eloise and Abelard to simulate the computation of $TM$ on the new part of the model. This is done in the same way.
as in the proof of Theorem 3.1. The main difference is that now the tape contents are encoded simply by unary predicates.

The formula \( \varphi_{TM} \) thus indeed simulates \( TM \).

We then investigate the restriction of \( T \) to the case where the use of looping symbols is restricted to \( k\text{-Exp} \) terms. That is, label symbols \( C \) must be written in the form \( C(t) \) where \( t \) is a \( k\text{-Exp}\)-term. We denote this logic by \( T[k\text{Exp}] \). Note that in this logic, the use of constructs \( Ix \) is trivially limited to the \( k\text{Exp} \) case because in fact all looping is limited.

Let \( k \geq 1 \). We next observe that \( T[k\text{Exp}] \) captures \( k\text{-ExpSpace} \).

**Theorem 3.3.** \( T[k\text{Exp}] \) captures \( k\text{-ExpSpace} \).

*Proof.* The proof is almost identical to the proof of Theorem 3.2. The principal difference is that this time we limit the number of loopings (playing time) in formulae and use the fact that alternating \( k\text{-ExpTime} \) equals \( k\text{-ExpSpace} \).

In general, it is interesting and worth it to use clocking terms to build custom-made yet natural logics for capturing complexity classes. Characterizations with polynomial clocks (terms in \( \text{Pol} \)) are interesting of course. We characterize \( \text{PSPACE} \) by the restriction of \( T[\neg Ix] \) where the looping construct \( C \) is limited similarly to the one in \( T[k\text{Exp}] \), but this time with the terms \( t \) in \( \text{Pol} \). We call this logic \( T[\text{Pol}] \).

**Theorem 3.4.** \( T[\text{Pol}] \) captures \( \text{PSPACE} \).

*Proof.* The proof is almost identical to the proof of Theorem 3.1. This time we use the fact that alternating polynomial time equals \( \text{PSPACE} \). Indeed, positions in the semantic game require a polynomial amount of memory, and now also playing time is limited polynomially. Thus we can simulate semantic games with an alternating polynomial time machine.

Also the other direction is similar. The limitation of the label symbols is not a problem when simulating an alternating machine, as the machine is now limited in running time by a polynomial.

We note that we obtain a characterization of \( \text{ELEMENTARY} \) for free. The logic \( T[\text{allExp}] \) does the job. (The logic is as \( T[k\text{Exp}] \) but allows all terms from \( \text{all-ExpTerm} \) as opposed to only the ones in \( k\text{-ExpTerm} \).) All kinds of clockings are of future interest. These include term families pointing to logarithmic (and lower) functions as well as functions that grow extremely fast.

We also note that the above proofs do not need tape predicates, with the exception of the characterizations of \( \text{PSPACE} \) and \( \text{ExpTime} \). This is because we can use different

\footnote{We are not attaching clock terms to looping atoms, just the corresponding label symbols. Thus it is important that in the game, when we transition from a looping atom \( C \), we jump to a corresponding position \( C(t)\varphi \) with the label \( C \) (rather than directly to \( \varphi \)). Then we reduce the corresponding clock value by one. The clock value is also lowered when we first come to the node \( C(t)\varphi \) (not necessarily from a looping atom).}
kinds of gadgets to encode new relation symbols. However, for this to hold, the underlying vocabulary must include at least one binary (or higher-arity) relation. Gadgets using only at most unary relation symbols do not suffice.

### 3.1 Extensions

We will then consider some extensions of the logic $T$. In this section, when discussing the logic $T$, the reader may also consider the extension of $T$ with all the deletion operators. The results here are not sensitive to whether we restrict to $T$ itself or consider some of its very close variants.

To define extensions of the logic $T$, consider first the formulae of $T$ itself. It is easy to see that formulae with the recursion construct $C$ can be unraveled by replacing an atom $C$ by the corresponding reference formula $\varphi$. The reference formula $\varphi$ is the formula in $C\varphi$ where we use $C$ as a naming symbol to name $\varphi$. Notice that here we assume that each atom $C$ has a unique reference formula. This can be assumed without loss of generality. Now, we can do the unraveling to all atoms $C$ repeatedly, ultimately ending up with an infinitely deep formula equivalent to the original formula we started with.

Thus formulae of $T$ can be seen as finitary encodings of infinitarily deep formulas. Let us define a related infinitary language $T_{\infty}$. For this purpose, consider first the syntax trees of formulae of $T$. Let $V$ denote the operators that can occur in non-leaf positions of the syntax trees of formulae of $T$. Thus for example $\neg$ and $\lor$ belong to $V$, as do $Ix$ and $\forall x$, and so on. Similarly, let $U$ denote the labels of leaf positions. Thus for example $Rxy$ and $C$ are in $U$. Now, let $V_0 \subseteq V$ and $U_0 \subseteq U$ be the restrictions of $V$ and $U$ with no looping symbols $C$, meaning that we remove all such symbols $C_1$, $C_2$, and so on. The formulas of $T_{\infty}$ can now be defined as follows.

1. The formulae are possibly infinite trees $t$ with maximum branch length $\omega$. Like a syntax tree, the tree is directed and has a unique root node.
2. The non-leaf nodes are labeled with operators from $V_0$.
3. The leaf nodes are labeled with atoms from $U_0$.
4. Nodes labeled with $\land$ or $\lor$ have two children, and other nodes with a label from $V_0$ have one child.

The point is that formulae of $T_{\infty}$ are like those of $T$ without looping constructs, but can have infinitely deep branches. The game-theoretic semantics for $T$ extends to this logic $T_{\infty}$ directly.

**Proposition 3.5.** Formulae of $T$ translate to formulae of $T_{\infty}$.

**Proof.** The unraveling translation turns formulae in $T$ to ones in $T_{\infty}$. If there are atoms $C$ without reference formulae, these can also be dealt with using suitable gadgets. Under the semantics where a non-referring looping atom is considered a tie (in the game), we can create an infinite branch with repeated, say, negations. Under the semantics where
it is a loss to one of the players, we can replace it with a suitable formula $\top$ or $\bot$, depending on the negations above the position.

Now, back to finitary logics, notice that the logic $T$ can of course be extended by a classical negation(s). Let $\sim$ denote the classical negation and extend the game-theoretic semantics of $T$ by defining the semantics for $\sim$ in the way described next.

The game trees are as before, the only addition being the novel positions where the formula begins with $\sim$. Consider (cf. [8]) a position

$$(\mathfrak{B}, g, \#, \sim \varphi),$$

where $\mathfrak{B}$ is the current model, $g$ the assignment, $\# \in \{+,-\}$ an indicator giving the current verifier, and $\sim \varphi$ the current formula. Then the next position in the game is

$$(\mathfrak{B}, g, \#, \varphi).$$

This means that during a play of the evaluation game, we simply remove $\sim$ and continue to the next position. This does not yet determine the semantics of the extended logic. We shall define the semantics next, and this will cover also the semantics of $T_\infty(\sim)$, that is, the extension of $T_\infty$ with the possibility of using $\sim$ in non-leaf positions.

Consider a formula $\varphi$, a model $\mathfrak{M}$, an assignment $g$ and the induced game-tree beginning from the root position $(\mathfrak{M}, g, +, \varphi)$. Intuitively, the positions with a formula $\sim \psi$ are at this stage irrelevant since nothing crucial happens in them. Thus we are essentially considering a game for $T$ or $T_\infty$. Now, in the game-tree, label each position in the tree by $\text{win}(\exists)$ if Eloise has a winning strategy in the subtree starting from that position. Similarly, label each position by $\text{win}(\forall)$ if Abelard has a winning strategy from there. Note that the game is a reachability game for both players, so positional strategies suffice to cover what can be done with general strategies.

Now, as the next step, consider each node with a main connective $\sim$ such that the node does not have any ancestor node with the main connective $\sim$, that is, consider those nodes with $\sim$ that can be reached from the root without going through any earlier node with $\sim$. The nodes to be considered can be called commencing $\sim$-nodes. In the game tree, replace each subtree whose root is a commencing $\sim$-node according to the following rules.

- Suppose the node position is $(\mathfrak{M}', f, +, \sim \psi)$.
  
  1. If the node is labeled $\text{win}(\exists)$, then replace the subtree beginning with that node (including the node itself) by a leaf node with a position $(\mathfrak{M}', f, +, \bot)$. This means that Eloise immediately loses in such a node.
  2. If the node is not labeled $\text{win}(\exists)$, then replace the subtree beginning with that node (including the node itself) by a leaf node with a position $(\mathfrak{M}', f, +, \top)$. This means that Eloise immediately wins in such a node.

In the same scenario, we can define an alternative classical negation $\dot{\sim}$ as follows.
• Suppose the node position is \((M', f, +, \sim \psi)\).

1. If the node is labeled \(\text{win}(\forall)\), then replace the subtree beginning with that node (including the node itself) by a leaf node with a position \((M', f, +, \top)\). This means that Eloise immediately wins in such a node.

2. Otherwise replace the subtree beginning with that node (including the node itself) by a leaf node with a position \((M', f, +, \bot)\). This means that Eloise immediately loses in such a node.

The first classical negation has the reading “is not true” and the second one “is false.” (Alternatively, we can read these as “is not verifiable” and “is falsifiable.”) Note that we do not consider here game-trees where \(\neg\) can occur before \(\sim\) or \(\dot{\sim}\).

To define truth (or a related concept), we write \(M \models + \varphi\) if Eloise has a winning strategy in the game involving \(\varphi\). If the formula involves \(\sim\) or \(\dot{\sim}\), then the game is of course the two-stage game where we first consider all strategies, then label nodes, and then consider playing in the new, modified game-tree. We also write \(\models + \varphi\) if we have \(M \models + \varphi\) for all models \(M\).

Now, consider the sentences \(C \neg C\) and \(C \sim C\) and also \(C \dot{\sim} C\). These give the liar paradox, or formalizations of it. Let us consider, in particular, \(C \sim C\). Now, let us indeed give this the above semantics. We first observe that neither \(\exists\) nor \(\forall\) has a winning strategy in the game. Thus, according to the above semantics, the commencing \(\sim\)-position does not become labeled with either \(\text{win}(\exists)\) or \(\text{win}(\forall)\). Therefore we have \(\models + C \sim C\). Note also that we have \(M \not\models + C \sim C\) for all models \(M\).

Now, consider \(C \neg C\) again. This formula is indeterminate, neither player has a winning strategy. This means that we do not get a well-founded truth for \(C \neg C\). Here, a well-founded truth means Eloise winning the standard evaluation game for \(\varphi\). This is the notion of truth (or verifiability) in \(T\) (cf. [9]). Under that notion, truth means that a well-founded procedure exists for reducing the truth of \(\varphi\) to truths of first-order atomic formulas. A winning strategy gives a well-founded subtree of the full game-tree where every path leads in a finite number of steps to an obvious atomic truth. This subtree can be seen as a proof of \(\varphi\) in the relevant model. In general, this gives a natural pre-theoretic concept for truth: it needs to be based on a well-founded, reductionist and finite (or somehow finitary) process that ends with atomic facts whose truth value is self-evident. The key is reduction to atomic literals with obvious truth values. This could be described as a coinductive reduction process.

Concerning general intuitions about truth, we often have such a well-founded procedure in mind. In the liar sentence \(C \neg C\) and truth teller \(CC\), the related procedure fails (cf. [9]). The attempt to find the firm ground by working towards atomic truths fails, and instead the process seems to run infinitely long. Thus, for a suggested resolution of the paradox, we can consider the following.

---

6 We remark that in finite models, even winning strategies are finite in \(T\) (due to König’s lemma).
Firstly, let \( LS \) denote the liar sentence, which we do not have to fix syntactically or semantically here. Nevertheless, \( C \neg C \) gives one possibility, but of course not the only one. Similarly, let \( TT \) denote the truth teller.

Now, as a potential strategy for explaining the paradox, we first accept that neither \( LS \) nor \( TT \) has a well-founded truth value (or a truth value based on a well-founded process) in the reductionist sense described above. We equate such well-founded truth values with the standard, desired and unquestioned truth. We could characterize it as first-level truth.

As the next step, we may require that some truth value must be found for \( LS \) (or \( TT \) or both). This steps seems questionable, as it rests upon requiring bivalence. However, if we in any case wish to force a truth value for \( LS \) or \( TT \), we can do so without any problems. This is because what we are after is not forcing a first-level truth. Instead, we seek a second-level one. There is, of course, a conceptual similarity here with Russell’s type theoretic hierarchies. Note that this move does not depend on the fact that we chose our first-level truth to be the well-founded truth. We could choose different notions of truth as our first-level truths, or even simply not point out any. The important issue is to declare that whatever the first-level truth is, we are now doing something beyond that. Indeed, it is an important point that while our first-level truth here relates to well-foundedness, we could do the argument without specifying what first-level truth is.

This approach clarifies things. We are not forcing the same, first-level truth notions on \( LS \) and \( TT \), we are choosing higher-level ones. They need not be comparable with the first-level ones. Furthermore, we can choose them in any arbitrary way without being inconsistent. This is not to say that all ways are equally natural, but they are consistent.

Which values to choose then? This can be done arbitrarily, as the second-level truth does not interfere with the first-level one. This indeed does not mean that all choices are equally natural in every possible way, but it does mean that no choice is inconsistent. A short path to the same conclusion goes as follows. Firstly, \( LS \) seems to give an infinitely flipping sequence of (first-level) truths, true-false-true, and so on. Therefore the (second-level) truth-value of \( LS \) is “infinitely flipping (first-level) truth values.”

To summarize, the problem was to expect a first-level reductionist truth value. That process did not stop. The next step was to try to force a truth or falsity with the direct reading of the sentence. This seemed to lead to a flipping truth value. The next step was to force a second-level truth value, which ever one. This is analogous to adding imaginary numbers to the reals. In fact, paraconsistent truth values are also easy to accept with a similar abstraction, considering them just new abstract entities with a new kind of an interpretation that expands the old paradigm.

Now, back to the formal semantics. Notice that there we also found different truth values for the paradoxical sentence “\( C \neg C \)”, one for the case where \( \sim \) is the negation and another one for \( \bar{\sim} \). However, a perhaps even more natural option would simply be to give it the new truth value “infinitely flipping” which does not try to get associated with true or false too directly, and which also describes what happens with the attempt to get reductionist first-level truth values.
Finally, the logic \( T \) obtains a natural compositional semantics as a corollary of the game-theoretic one. However, the semantics has an interesting issue. For the first-order connectives, the semantics goes as follows.

\[
\begin{align*}
\mathfrak{A}, g \models^+ \varphi \land \psi &\iff \mathfrak{A}, g \models^+ \varphi \text{ and } \mathfrak{A}, g \models^+ \psi \\
\mathfrak{A}, g \models^+ \varphi \lor \psi &\iff \mathfrak{A}, g \models^+ \varphi \text{ or } \mathfrak{A}, g \models^+ \psi \\
\mathfrak{A}, g \models^+ \neg \varphi &\iff \mathfrak{A}, g \models^+ \varphi \\
\mathfrak{A}, g \models^+ \exists x \varphi &\iff \mathfrak{A}, g[a/x] \models^+ \varphi \text{ for some } a \in A \\
\mathfrak{A}, g \models^+ \forall x \varphi &\iff \mathfrak{A}, g[a/x] \models^+ \varphi \text{ for all } a \in A \\
\mathfrak{A}, g \models^+ \neg \varphi &\iff \mathfrak{A}, g \models^+ \varphi \\
\mathfrak{A}, g \models^+ \exists x \varphi &\iff \mathfrak{A}, g[a/x] \models^+ \varphi \text{ for some } a \in A \\
\mathfrak{A}, g \models^+ \forall x \varphi &\iff \mathfrak{A}, g[a/x] \models^+ \varphi \text{ for all } a \in A
\end{align*}
\]

The clauses for first-order atomic formulae are as usual. This semantic system extends to \( T \) easily. First note that we have the following.

\[
\begin{align*}
\mathfrak{A}, g \models^+ Ix\varphi &\iff (\mathfrak{A} + a), g[a/x] \models^+ \varphi \\
\mathfrak{A}, g \models^+ I(Rx_1...x_k)\varphi &\iff \mathfrak{A}, g[(R \cup (a_1, ..., a_k))/R][(a_1, ..., a_k)/(x_1, ..., x_k)] \models^+ \varphi \\
&\text{for some } a_1, ..., a_k \in A \\
\mathfrak{A}, g \models^+ Ix\varphi &\iff (\mathfrak{A} + a), g[a/x] \models^+ \varphi \\
\mathfrak{A}, g \models^+ I(Rx_1...x_k)\varphi &\iff \mathfrak{A}, g[(R \cup (a_1, ..., a_k))/R][(a_1, ..., a_k)/(x_1, ..., x_k)] \models^+ \varphi \\
&\text{for all } a_1, ..., a_k \in A
\end{align*}
\]

where \( \mathfrak{A} + a \) means \( \mathfrak{A} \) expanded with a fresh isolated element \( a \). The clauses for the deletion operators are similar and easily understood, so we skip them. The clauses for \( C \) are as follows.

\[
\begin{align*}
\mathfrak{A}, g \models^+ C\varphi &\iff \mathfrak{A}, g \models^+ \varphi \\
\mathfrak{A}, g \models^+ C &\iff \mathfrak{A}, g \models^+ C\varphi \\
\mathfrak{A}, g \models^+ C\varphi &\iff \mathfrak{A}, g \models^+ \varphi \\
\mathfrak{A}, g \models^+ C &\iff \mathfrak{A}, g \models^+ C\varphi
\end{align*}
\]

where we assume \( C \) has a unique reference formula. If not, we need to take into account many reference formulas, which is also easy to formulate.

We note that these clauses (the compositional semantics) is, in some sense, a corollary of the game-theoretic semantics. Also, we note that we will not get a clear evaluation of (for example) the sentence \( CC \) by using this compositional semantics. The game-theoretic semantics does tell that the sentence is indeterminate, but the compositional clauses we deduced here do not say anything. They simply keep referring to each other. Nevertheless, the above compositional semantic equivalences are true. The possible circularity is not really an issue, and of course circularity does not imply inconsistency anyway, just that the circular point is underdetermined.
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