In this paper we continue the study (initiated in [8]) of the semi-classical behavior of the scattering data of a non-self-adjoint Dirac operator with a real, positive, fairly smooth but not necessarily analytic potential decaying at infinity; in this paper we allow this potential to have several local maxima and minima. We provide the rigorous semiclassical analysis of the Bohr-Sommerfeld condition for the location of the eigenvalues, the norming constants, and the reflection coefficient.

1. Introduction

Consider the initial value problem (IVP) for the one-dimensional focusing non-linear Schrödinger equation with cubic nonlinearity (focusing NLS) for the complex field $u(x,t)$, i.e.

\[\begin{align*}
  i\hbar \partial_t u + \frac{\hbar^2}{2} \partial_x^2 u + |u|^2 u &= 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R} \\
  u(x,0) &= A(x), \quad x \in \mathbb{R}
\end{align*}\]  

(1.1)

in which $A$ is a real valued function and $\hbar$ is a fixed (at first) positive number; it is a measure of the ratio of the effect of dispersion to the effect of non-linearity.

A problem like (1.1) has attracted much interest due to the wide applicability of the NLS equation. Indeed, the NLS equation has been derived in many diverse fields of study, governing a plethora of phenomena. Just to name a few, it has applications

- to the propagation of light in nonlinear optical fibers (cf. [2])
- to Bose–Einstein condensates (see [20])
- to Langmuir waves in hot plasma physics (cf. [15])
- to superconductivity (NLS arises from the Ginzburg-Landau equation as a simplified $(1+1)$-dimensional form, see [5])
- to hydrodynamics, for example in small amplitude gravity waves on the surface of deep inviscid (zero-viscosity) water (cf. [3],[1]).

One can claim that the focusing cubic NLS equation is one of the basic canonical non-linear partial differential equations.

Zakharov and Shabat in [23] have proved back in 1972 that (1.1) is integrable via the Inverse Scattering Method. A crucial step of the method is the analysis of the following eigenvalue (EV) problem

\[\mathcal{D}_\hbar |u| = \lambda u\]  

(1.2)

where
\( \mathcal{D}_\hbar \) is the Dirac (or Zakharov-Shabat) operator

\[
\mathcal{D}_\hbar = \begin{bmatrix}
i\hbar \partial_x & -iA \\
-iA & -i\hbar \partial_x
\end{bmatrix}
\]  

(1.3)

- \( u = [u_1 \ u_2]^T \) is a function from \( \mathbb{R} \) to \( \mathbb{C}^2 \)
- \( \lambda \in \mathbb{C} \) is a “spectral” parameter.

If the solutions \( u \) are in \( L^2(\mathbb{R}; \mathbb{C}^2) \) the corresponding \( \lambda \)'s are eigenvalues. The EVs of this problem are related to coherent structures (e.g. solitons and breathers) for the IVP (1.1) (see [15]). The real part of such an EV represents the speed of the soliton while the imaginary part is related to its amplitude. On the other hand the continuous spectrum corresponds to bounded (but not \( L^2 \) “generalized” eigenfunctions \( u \); in our case it is the real line.

In fact the method of Zakharov-Shabat solves (1.1) by first studying and characterizing appropriate “scattering data” for the potential \( A \), then following the (trivial) evolution of such data with respect to time (when we let the potential of the Dirac operator evolve according to the NLS equation) and finally using an inverse scattering procedure to recover the actual solution of (1.1). The scattering data for the Dirac operator consist of
- eigenvalues
- “norming constants” related to the \( L^2 \)-norms of the corresponding eigenfunctions \( u \) and
- the so-called “reflection” coefficient defined on the continuous spectrum.

Now let us suppose that \( \hbar \) is small compared to the magnitudes of \( x, t \) that we are interested in. We are led to the mathematical question: what is the behavior of solutions of the IVP (1.1) as \( \hbar \downarrow 0 \)? Because of the work of Zakharov and Shabat, the first step in the study of this IVP in the semiclassical limit \( \hbar \downarrow 0 \) has to be the asymptotic spectral analysis of the scattering problem (1.2) as \( \hbar \downarrow 0 \), keeping the function \( A \) fixed. This is our main object here.

The rigorous analysis of this direct scattering problem was initiated in [7] (in the case of real analytic data) and more generally in [8] for data which is only required to be somewhat smooth. The rigorous analysis of the inverse scattering problem was initiated much earlier in [10] by use of an ansatz which was justified later in [11]. The eigenvalue problem (1.2) is not and cannot be written as an EV problem for a self-adjoint operator. What we study here is a semiclassical WKB problem (or LG problem) for the corresponding non-self-adjoint Dirac operator with potential \( A \).

This work complements our previous work [8] where the potential is considered to be a positive, smooth and even bell-shaped function, in which we employed Olver’s theory. Working on the same lines, we now discard the evenness assumption and additionally let the potential have multiple “humps” (instead of just a single assumed in [8]). We should point out that our methods are necessarily different from the ones found in [7] and [9]. Those works use the exact WKB method which requires analyticity. Our ideas here are rather influenced by the paper [22] of D. R. Yafaev where an analogous problem is treated for the self-adjoint Schrödinger operator; which in turn relies on the work [17] of F. W. J. Olver. We rely heavily on [16] instead.

\(^1\)Olver’s work draws upon the studies of N. D. Kazarinoff, R. E. Langer and R.W. McKelvey (see the references in [10]).
Under the hypothesis that the EVs of $\mathcal{O}_h$ are purely imaginary (at least for small enough values of $\hbar$, see Hypothesis 4.3), the EV problem (1.2) under consideration becomes a single linear differential equation of second order

$$\frac{d^2y}{dx^2} = \left[\hbar^{-2}f(x, \mu) + g(x, \mu)\right]y \quad (1.4)$$

where $y$ is related to $u$ while $\mu \in \mathbb{R}_+$ is a parameter that substitutes $\lambda$. The functions $f, g$ are given by the following formulae

$$f(x, \mu) = \mu^2 - A(x)^2$$

and

$$g(x, \mu) = \frac{3}{4} \left[\frac{A'(x)}{A(x) + \mu}\right]^2 - \frac{1}{2} \frac{A''(x)}{A(x) + \mu}.$$

As the zeros of $f$ play a crucial role in the study of the solutions of (1.4), we give the following definition.

**Definition 1.1.** Consider a differential equation of the form (1.4) in which $\mu > 0$ is a parameter and $x \in \Delta \subseteq \mathbb{R}$ an interval. The zeros in $\Delta$ (with respect to $x$) of the function $f(x, \mu)$ are called the **turning points** (or **transition points**) of the above differential equation.

The presentation of our work in the forthcoming sections will be as follows. In section §2 we shall deal with approximate solutions of (1.4) when $A$ behaves as a single hump (or a single lobe facing upwards in Klaus-and-Shaw’s terminology found in [13]) in some open neighborhood in $\mathbb{R}$. In this section we generalize results obtained in [8]. But we dispense with the eveness assumption considered in [8] and account for all possible cases for the open neighborhood in which $A$ behaves like a single hump. More precisely, in §2.1 we apply the Liouville transform to change equation (1.4) to a new one of the form

$$\frac{d^2X}{d\zeta^2} = \left[\hbar^{-2}(\zeta^2 - \alpha^2) + \psi(\zeta, \alpha)\right]X \quad (1.5)$$

for some new variables $\zeta, X$ and a function $\psi$, along the lines first discussed in [16]; here the role of the spectral parameter is assumed by the new variable $\alpha$. In §2.2 we prove a useful lemma concerning the continuity of $\psi$ which we use in §2.3 to prove Theorem 2.13 about approximate solutions to (1.5) for $\zeta \geq 0$. In this case, the approximate solutions are expressed in terms of Parabolic Cylinder Functions (PCFs).

Next, in §2.4 we compute asymptotics for the solutions constructed previously and in §2.5 we “connect” the approximants for $\zeta \geq 0$ to approximants for $\zeta \leq 0$ using the so-called connection coefficients. Finally, in subsection §2.6 we combine the tools assembled in this section so far, which results in some theorems concerning action integrals and quantization conditions.

The presentation of the material in section §3 follows the same manner of that in §2. The main difference now is that $A$ behaves locally as a single basin (or bowl; a single lobe facing downwards using Klaus-Shaw terminology). If we apply now the Liouville transform to (1.4) we end up having an equation of the form

$$\frac{d^2X}{d\zeta^2} = \left[\hbar^{-2}(\beta^2 - \zeta^2) + \psi(\zeta, \beta)\right]X \quad (1.6)$$
for the same variables $\zeta, X$ as in (1.5) and a function $\overline{\psi}$ (here the bar does not denote complex conjugation); the spectral parameter is played now by $\beta$. Again, $\overline{\psi}$ can be proven to be continuous; we do this in §3.2. In paragraph §3.3 (cf. Theorem 3.9) we construct approximants to (1.6) for $\zeta \geq 0$ expressed in terms of modified Parabolic Cylinder Functions (mPCFs). After finding their asymptotic behavior in §3.4 we “bridge” them with the approximate solutions for $\zeta \leq 0$ and obtain their relevant connection formulas. The final subsection, namely §3.6, is the place where a “fixing behavior” is observed giving rise to a definition of fixing conditions (along the lines of Yafaev; see Definition 5.7 in [22]).

**Remark 1.2.** Let us denote by $\mathcal{R}_A \subset \mathbb{R}^+$ the range of the potential function $A: \mathbb{R} \to \mathbb{R}^+$ and take an $\mu \in \mathcal{R}_A$. Assuming that equation $A(x) = \mu$ has a finite number of solutions, these divide the domain $\mathbb{R}$ of $A$ to a finite number of intervals where $A(x) > \mu$ and to (finitely many) intervals where $A(x) < \mu$ (see Figure 1). We call the former “barriers” and the latter “wells”. When an interval giving rise to a barrier (well) is bounded, we say that we have a barrier (well) of finite width or simply a finite barrier (well). Correspondingly, when we have unbounded intervals, we are in the presence of infinite barriers (or wells), i.e. barriers (wells) of infinite width.

![A(x)](#A(x)

**Figure 1.** Barriers and wells for a potential $A$ at a specific energy level $\mu$.

Next, in paragraph §5 we study the semiclassical spectrum of our operator with multiple potential humps. After the introduction of the necessary notation in §4.1 we show in paragraphs §§4.2 - 4.4 how our problem can be transformed to one where Olver’s theory (as adapted in sections 2 and 3) can be applied. The results about
the EVs and their corresponding quantization conditions are presented in §5.1. We show that for each EV there exists at least one barrier for which an associated Bohr-Sommerfeld quantization condition can be obtained, essentially in the same way as for the one barrier problem. Also, we establish a one-to-one correspondence between the EVs of the Dirac operator $D_{\hbar}$ lying in $i\mathbb{R}$ (i.e. imaginary axis) and their WKB approximations.

The last component of the semiclassical scattering data is the reflection coefficient. This has been studied semiclassically in [8]; for completeness we present it briefly in paragraph §6 (the reflection coefficient away from zero is presented in §6.1 while the behavior closer to zero is found in §6.2). Since our motivation comes from the application to semiclassical NLS, we discuss the effect of our direct scattering estimates to the inverse scattering problem in section 7. It turns out that the asymptotic analysis of the inverse problem already conducted for the bell-shaped case in [10] and [11] is still relevant. The main change affects the new density of eigenvalues, which fortunately still retains its nice properties that enable the asymptotic analysis of the associated Riemann-Hilbert factorization problem.

For the sake of the reader, as the approximate solutions to our problems involve Airy, Parabolic Cylinder Functions and modified Parabolic Cylinder Functions, we present all the necessary results concerning these functions in sections A and B of the appendix. Finally, in section C of the appendix we present a theorem concerning integral equations that is the backbone of the theory that we use in order to arrive at our results.

Before we start our main exposition, we specify some notation used throughout our work.

- Complex conjugation is denoted with a star superscript, “$*$”; i.e. $z^*$ is the complex conjugate of $z$ (we emphasize that a bar over a number, does not indicate its complex conjugate).
- The letters $c, C$ denote generically positive constants (unless specified otherwise), appearing mainly in estimates.
- For the Wronskian of two functions $f, g$ we use the symbol $W[f, g]$.
- The notation $f^2(x)$ denotes the square of the value of the function $f$ at $x$; hence, the symbols $f^2(x)$ and $f(x)^2$ are used interchangeably and are not to be confused with the composition $f \circ f$.
- The transpose of a matrix $M$ is denoted by $M^T$.
- For the complement of a set $B$ we write $\overline{B}$.
- For a set $\Sigma \subseteq \mathbb{R}$, when we write $i\Sigma$ we mean the set $\{i\kappa \mid \kappa \in \Sigma\} \subset \mathbb{C}$. Also, for $z_1, z_2 \in \mathbb{C}$ the set $[z_1, z_2] \subset \mathbb{C}$ denotes the closed line segment starting at $z_1$ and ending at $z_2$.
- For the restriction of a function $f : \mathbb{R} \to \mathbb{R}$ to the interval $\Delta \subseteq \mathbb{R}$ we have $f|_{\Delta} : \Delta \to \mathbb{R}$.
- The closure of a set $\Sigma \subseteq \mathbb{R}$ is denoted by $\text{clos}(\Sigma)$.
- Take a set $\Sigma \subseteq \mathbb{R}$ and consider a function $f : \Sigma \to \mathbb{R}$. The set $\mathcal{R}_f = \{f(x) \mid x \in \Sigma\} \subseteq \mathbb{R}$ represents its range.
- Unless otherwise specified, $f^{-1}$ shall always denote the inverse of an invertible function $f$.
- If $T$ is an operator, then $\sigma(T)$, $\sigma_{\text{ess}}(T)$ and $\sigma_p(T)$ denote its spectrum, essential (continuous) spectrum and point spectrum respectively.
2. Passage through a Potential Barrier

We start the investigation for the behavior of solutions of equation
\[
\frac{d^2y}{dx^2} = \left\{ \hbar^{-2} \mu^2 - A^2(x) + 3 \left[ \frac{A'(x)}{A(x) + \mu} \right]^2 - \frac{1}{2} A''(x) \right\} y
\] (2.1)
where \( \mu > 0 \) and the potential function \( A : \mathbb{R} \to \mathbb{R}_+ \) is characterized by a barrier of finite width (also called finite barrier) in some bounded interval in \( \mathbb{R} \). Precisely, we consider one of the following assumptions for \( A \).

For the case where the finite barrier lies between two wells of finite width (called finite wells), we assume the following.

Assumption 2.1. (finite barrier & finite wells) The function \( A \) is positive on some bounded interval \([x_1, x_2]\) in \( \mathbb{R} \) and has a unique extremum in \((x_1, x_2)\); particularly, a maximum \( A_{\text{max}} > 0 \) at \( b_0 \in (x_1, x_2) \). Also \( A \) is assumed to be \( C^4 \) in \((x_1, x_2)\) and of class \( C^5 \) in a neighborhood of \( b_0 \). Additionally, \( A'(x) > 0 \) for \( x \in (x_1, b_0) \) and \( A'(x) < 0 \) for \( x \in (b_0, x_2) \). At \( b_0 \) we have \( A'(b_0) = 0 \) and \( A''(b_0) < 0 \). Furthermore, if we let \( A_* = \max\{A(x_1), A(x_2)\} \) and take \( \mu \in (A_*, A_{\text{max}}) \subset \mathbb{R}_+ \), the equation \( A(x) = \mu \) has two solutions \( b_-(\mu), b_+(\mu) \) in \((x_1, x_2)\). These satisfy \( b_- < b_+, A(x) > \mu \) for \( x \in (b_-, b_+) \) and \( A(x) < \mu \) for \( x \in (x_1, b_-) \cup (b_+, x_2) \) (the above imply \( \pm A'(b_\pm) < 0 \)). Finally, when \( \mu = A_{\text{max}} \) the two points \( b_-, b_+ \) coalesce into one double root at \( b_0 \).

On the other hand, if the finite barrier is surrounded by one or two infinite wells, we have the following variants of Assumption 2.1. In these cases, we need to put some additional decay assumptions on \( A, A' \) and \( A'' \) at the infinite ends. Hence we have one of the following.

Assumption 2.2. (finite barrier & left infinite well) The function \( A \) is positive on some bounded interval \([x_1, x_2]\) in \( \mathbb{R} \) and has a unique extremum in \((x_1, x_2)\); particularly, a maximum \( A_{\text{max}} > 0 \) at \( b_0 \in (x_1, x_2) \). Also \( A \) is assumed to be...
$C^4$ in $(-\infty, x_2)$ and of class $C^5$ in a neighborhood of $b_0$. Additionally, $A'(x) > 0$ for $x \in (-\infty, b_0)$ and $A'(x) < 0$ for $x \in (b_0, x_2)$. At $b_0$ we have $A'(b_0) = 0$ and $A''(b_0) < 0$. Furthermore, if we take $\mu \in (A(x_2), A_{max}) \subset \mathbb{R}_+$, the equation $A(x) = \mu$ has two solutions $b_-(\mu), b_+(\mu)$ in $(-\infty, x_2)$. These satisfy $b_- < b_+, A(x) > \mu$ for $x \in (b_-, b_+)$ and $A(x) < \mu$ for $x \in (-\infty, b_-) \cup (b_+, x_2)$ (the above imply $\pm A'(b_{\pm}) < 0$). When $\mu = A_{max}$ the two points $b_-, b_+$ coalesce into one double root at $b_0$. Finally, there exists a number $\tau > 0$ so that

\[
A(x) = \mathcal{O}\left(\frac{1}{|x|^{1+\tau}}\right) \text{ as } x \downarrow -\infty \\
A'(x) = \mathcal{O}\left(\frac{1}{|x|^{1+\tau}}\right) \text{ as } x \downarrow -\infty \\
A''(x) = \mathcal{O}\left(\frac{1}{|x|^{1+\tau}}\right) \text{ as } x \downarrow -\infty
\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{An example of a finite potential barrier accompanied by an infinite well on the left and a finite well on the right.}
\end{figure}

**Assumption 2.3. (finite barrier & right infinite well)** The function $A$ is positive on some interval $[x_1, +\infty)$ in $\mathbb{R}$ and $\lim_{x \uparrow +\infty} A(x) = 0$. It has a unique extremum in $(x_1, +\infty)$; particularly, a maximum $A_{max} > 0$ at $b_0 \in (x_1, +\infty)$. Also $A$ is assumed to be $C^4$ in $(x_1, +\infty)$ and of class $C^5$ in a neighborhood of $b_0$. Additionally, $A'(x) > 0$ for $x \in (x_1, b_0)$ and $A'(x) < 0$ for $x \in (b_0, +\infty)$. At $b_0$ we have $A'(b_0) = 0$ and $A''(b_0) < 0$. Furthermore, if we take $\mu \in (A(x_1), A_{max}) \subset \mathbb{R}_+$, the equation $A(x) = \mu$ has two solutions $b_-(\mu), b_+(\mu)$ in $(x_1, +\infty)$. These satisfy $b_- < b_+, A(x) > \mu$ for $x \in (b_-, b_+)$ and $A(x) < \mu$ for $x \in (x_1, b_-) \cup (b_+, +\infty)$ (the above imply $\pm A'(b_{\pm}) < 0$). When $\mu = A_{max}$ the two points $b_-, b_+$ coalesce into one double root at $b_0$. Finally, there exists a number $\tau > 0$ so that

\[
A(x) = \mathcal{O}\left(\frac{1}{|x|^{1+\tau}}\right) \text{ as } x \uparrow +\infty \\
A'(x) = \mathcal{O}\left(\frac{1}{|x|^{1+\tau}}\right) \text{ as } x \uparrow +\infty \\
A''(x) = \mathcal{O}\left(\frac{1}{|x|^{1+\tau}}\right) \text{ as } x \uparrow +\infty
\]
Assumption 2.4. (finite barrier & two infinite wells) The function $A$ is positive on $\mathbb{R}$ and $\lim_{x \to \pm \infty} A(x) = 0$. It has a unique extremum in $(-\infty, +\infty)$; particularly, a maximum $A_{\text{max}} > 0$ at $b_0 \in (-\infty, +\infty)$. Also $A$ is assumed to be $C^4$ in $(-\infty, +\infty)$ and of class $C^5$ in a neighborhood of $b_0$. Additionally, $A'(x) > 0$ for $x \in (-\infty, b_0)$ and $A'(x) < 0$ for $x \in (b_0, +\infty)$. At $b_0$ we have $A'(b_0) = 0$ and $A''(b_0) < 0$. Furthermore, if we take $\mu \in (0, A_{\text{max}}) \subset \mathbb{R}_+$, the equation $A(x) = \mu$ has two solutions $b_-(\mu), b_+(\mu)$ in $(-\infty, +\infty)$. These satisfy $b_- < b_+$, $A(x) > \mu$ for $x \in (b_-, b_+)$ and $A(x) < \mu$ for $x \in (-\infty, b_-) \cup (b_+, +\infty)$ (the above imply $\pm A'(b_{\pm}) < 0$). When $\mu = A_{\text{max}}$ the two points $b_-, b_+$ coalesce into one double root at $b_0$. Finally, there exists a number $\tau > 0$ so that

$$A(x) = O\left(\frac{1}{|x|^{1+\tau}}\right) \text{ as } x \to \pm \infty$$

$$A'(x) = O\left(\frac{1}{|x|^{2+\tau}}\right) \text{ as } x \to \pm \infty$$

$$A''(x) = O\left(\frac{1}{|x|^{3+\tau}}\right) \text{ as } x \to \pm \infty$$
Remark 2.5. The case where $A$ is an even function, satisfying Assumption 2.4 is treated in complete detail in [8].

2.1. The Liouville transform for a barrier. We start by assuming the first one of the assumptions above. All of them can be treated in a similar manner. Assume 2.1 (see Figure 2), with $\mu = A(b_-) = A(b_+)$. We temporarily drop the subscript and set

$$b_+ \equiv b$$

$$I^- = (x_1, b_0), \quad I^+ = (b_0, x_2)$$

and define

$$G^\pm = \left( A_{|\text{clos}(I^\pm)} \right)^{-1}. \quad (2.2)$$

Take an arbitrary $\mu_1 \in (A_, A_{max})$ and consider the $b_1 \in (b_0, G^+(A_+))$ so that $A(b_1) = \mu_1$ (cf. Assumption 2.1 and 2.2); then $\mu \in [\mu_1, A_{max}]$ implies $b \in [b_0, b_1]$. For every $\hbar > 0$, equation (2.1) reads

$$\frac{d^2 y}{dx^2} = \hbar^{-2} f(x, b) + g(x, b) y, \quad (x, b) \in (x_1, x_2) \times [b_0, b_1] \quad (2.3)$$

in which the functions $f$ and $g$ satisfy

$$f(x, b) = A^2(b) - A^2(x) \quad (2.4)$$

and

$$g(x, b) = \frac{3}{4} \left[ \frac{A'(x)}{A(x) + A(b)} \right] - \frac{1}{2} \frac{A''(x)}{A(x) + A(b)}. \quad (2.5)$$

We see that our equation (2.3) has two turning points (cf. Definition 1.1) at $x = b_{\pm}$ when $b \in (b_0, b_1)$ coalescing into one double at $x = b_0$; then $b$ becomes $b_0$.

Next, we introduce new variables $X$ and $\zeta$ according to the Liouville transform

$$X = \dot{x}^{-\frac{1}{2}} y$$

Figure 5. An example of a finite potential barrier accompanied by two infinite wells.
where the dot signifies differentiation with respect to $\zeta$. Equation (2.3) becomes
\[
\frac{d^2X}{d\zeta^2} = \left[ h^{-2} \dot{x}^2 f(x, b) + \dot{x}^2 g(x, b) + \dot{x}^\frac{3}{2} \frac{d^2}{d\zeta^2}(\dot{x}^{-\frac{1}{2}}) \right] X. \tag{2.6}
\]

Let us treat the noncritical case $\mu \in [\mu_1, A_{\max})$ first; two turning points $b_\pm$ being present. In this case $f(\cdot, b)$ is negative in $(b_-, b_+)$ and positive in $(x_1, b_-) \cup (b_+, x_2)$. Hence we prescribe
\[
\dot{x}^2 f(x, b) = \zeta^2 - \alpha^2
\] where $\alpha > 0$ is chosen in such a way that $x = b_-$ corresponds to $\zeta = -\alpha$ and $x = b_+$ to $\zeta = \alpha$ accordingly.

After integration, (2.7) yields
\[
\int_{b_-}^{2\pi} [-f(t, b)]^\frac{1}{2} dt = \int_{-\alpha}^{\alpha} (\alpha^2 - \tau^2)^\frac{1}{2} d\tau \tag{2.8}
\]
provided that $b_- \leq x \leq b_+$ (notice that by taking these integration limits, $b_-$ corresponds to $-\alpha$). For the remaining correspondence we require
\[
\int_{b_-}^{b_+} [-f(t, b)]^\frac{1}{2} dt = \int_{-\alpha}^{\alpha} (\alpha^2 - \tau^2)^\frac{1}{2} d\tau
\]
and hence
\[
\alpha^2(\mu) = \frac{2}{\pi} \int_{b_-}^{b_+}(\mu) \sqrt{A^2(t) - \mu^2} dt. \tag{2.9}
\]

For every fixed value of $\hbar$, relation (2.9) defines $\alpha$ as a continuous and decreasing function of $\mu$ which vanishes as $\mu \uparrow A_{\max}$. Set
\[
\alpha_1 = \alpha(\mu_1) > 0. \tag{2.10}
\]
Then $\mu \in [\mu_1, A_{\max})$ implies $\alpha \in (0, \alpha_1]$.

Next, from (2.8) we find
\[
\int_{b_-}^{b_+} [-f(t, b)]^\frac{1}{2} dt = \frac{1}{2} \alpha^2 \arccos \left( -\frac{\zeta}{\alpha} \right) + \frac{1}{2} \frac{\zeta}{\alpha} \left( \zeta^2 - \alpha^2 \right)^{\frac{1}{2}} \quad \text{for} \quad b_- \leq x \leq b_+ \tag{2.11}
\]
with the principal value choice for the inverse cosine taking values in $[0, \pi]$. For the remaining $x$-intervals, we integrate (2.7) to obtain
\[
\int_{x_1}^{b_-} f(t, b)^{\frac{1}{2}} dt = -\frac{1}{2} \alpha^2 \arccosh \left( -\frac{\zeta}{\alpha} \right) - \frac{1}{2} \frac{\zeta}{\alpha} \left( \zeta^2 - \alpha^2 \right)^{\frac{1}{2}} \quad \text{for} \quad x_1 < x \leq b_- \tag{2.12}
\]
and
\[
\int_{b_+}^{x_2} f(t, b)^{\frac{1}{2}} dt = -\frac{1}{2} \alpha^2 \arccosh \left( \frac{\zeta}{\alpha} \right) + \frac{1}{2} \frac{\zeta}{\alpha} \left( \zeta^2 - \alpha^2 \right)^{\frac{1}{2}} \quad \text{for} \quad b_+ \leq x < x_2 \tag{2.13}
\]
with $\arccosh(x) = \ln (x + \sqrt{x^2 - 1})$ for $x \geq 1$.

Equations (2.11), (2.12) and (2.13) show that $\zeta$ is a continuous and increasing function of $x$ which shows that there is a one-to-one correspondence between these two variables. Thus, if we set
\[
\zeta_j = \lim_{x \to x_j} \zeta(x) \quad \text{for} \quad j = 1, 2 \tag{2.14}
\]
then $(x_1, x_2)$ is mapped by $\zeta$ to $(\zeta_1, \zeta_2)$. Notice that $-\infty < \zeta_1 < 0 < \zeta_2 < +\infty$ since both $x_1, x_2$ are finite by Assumption 2.1 (if $x_1 = -\infty$ then $\zeta_1 = -\infty$ and if $x_2 = +\infty$ then $\zeta_2 = +\infty$).
Remark 2.6. In the critical case in which the two (simple) turning points coalesce into one (double) point, we get a limit of the above transformation with \( b = b_0 \). In this case, the analogous relations to (2.11), (2.12), (2.13) are

\[
\int_{x}^{b_0} f(t, b_0) \frac{1}{2} dt = \frac{1}{2} \zeta^2 \quad \text{for} \quad x_1 < x \leq b_0
\]

(2.15)

\[
\int_{b_0}^{x} f(t, b_0) \frac{1}{2} dt = \frac{1}{2} \zeta^2 \quad \text{for} \quad b_0 \leq x < x_2
\]

(2.16)

and \( \alpha = 0 \).

Finally, having in mind Remark 2.6, we substitute (2.7) in (2.6) and obtain the following proposition.

Proposition 2.7. For every \( \hbar > 0 \) equation

\[
d^2y/dx^2 = [\hbar^2 f(x, b) + g(x, b)]y, \quad (x, b) \in (x_1, x_2) \times [b_0, b_1]
\]

where \( f, g \) as in (2.4), (2.5) respectively, is transformed to the equation

\[
d^2X/d\zeta^2 = [\hbar^{-2}(\zeta^2 - \alpha^2) + \psi(\zeta, \alpha)]X, \quad (\zeta, \alpha) \in (\zeta_1, \zeta_2) \times [0, \alpha_1]
\]

(2.17)

in which \( \zeta \) is given by the Liouville transform (2.7), \( \alpha \) is given by (2.9), \( \zeta_j, j = 1, 2 \) are given by (2.14), \( \alpha_1 \) as in (2.10) and the function \( \psi(\zeta, \alpha) \) is given by the formula

\[
\psi(\zeta, \alpha) = \frac{i}{4} \frac{d^2}{d\zeta^2}(\dot{x}^2 - \frac{1}{2} x^2)g(x, b) + \frac{i}{4} \frac{d}{d\zeta^2}(\dot{x}^2 - \frac{1}{2} x^2).
\]

(2.18)

Since in the following paragraphs we shall be interested in approximate solutions of equation (2.17), we have the following.

Definition 2.8. The function \( \psi \) found in the differential equation (2.17) shall be called the error term of this equation.

For the error term we have the following proposition.

Proposition 2.9. The error term \( \psi \) can be written equivalently as

\[
\psi(\zeta, \alpha) = \frac{1}{4} \frac{3\zeta^2 + 2\alpha^2}{(\zeta^2 - \alpha^2)^2} + \frac{1}{16} \frac{\zeta^2 - \alpha^2}{f(x, b)} \left\{ 4f(x, b)f''(x, b) - 5[f'(x, b)]^2 \right\}
\]

\[
+ (\zeta^2 - \alpha^2) \frac{g(x, b)}{f(x, b)}
\]

(2.19)

where prime denotes differentiation with respect to \( x \). The same formula can be used in the critical case of one double turning point simply by setting \( b = b_0 \) and \( \alpha = 0 \).

Proof. Using (2.18), (2.5) and (2.7), simple algebraic manipulations shown that \( \psi \) takes the desired form. \( \square \)
2.2. Continuity of the error term. In this subsection we prove a lemma concerning the continuity of the function \( \psi(\zeta, \alpha) \) defined in (2.18) or (2.19). This fact will be used subsequently in §2.3 to prove the existence of approximate solutions of equation (2.17). We state it explicitly.

**Lemma 2.10.** The function \( \psi(\zeta, \alpha) \) defined in (2.18), is continuous in \( \zeta \) and \( \alpha \) in the region \( (\zeta_1, \zeta_2) \times [0, \alpha_1] \) of the \( (\zeta, \alpha) \)-plane.

**Proof.** For \( x \in (x_1, x_2) \), \( \mu \in [\mu_1, A_{\text{max}}] \) and \( b \in [b_0, b_1] \) we introduce an auxiliary function \( p \) by setting

\[
 f(x, b) = (x - b_+)(x - b)p(x, b).
\]

(2.20)

Having in mind that \( A(b) = \mu \), we see that for \( \mu \in [\mu_1, A_{\text{max}}] \)

\[
 p(b_\pm, b) = \mp \frac{2\mu}{b - b_-} A'(b_\pm) > 0
\]

while for \( \mu = A_{\text{max}} \)

\[
 p(b_0, b_0) = -A_{\text{max}} A''(b_0) > 0.
\]

Our functions \( f \), \( g \) and \( p \) defined by (2.4), (2.5) and (2.20) respectively satisfy the following properties

(i) \( p \), \( \frac{\partial p}{\partial x} \), \( \frac{\partial^2 p}{\partial x^2} \) and \( g \) are continuous functions of \( x \) and \( b \) (this means in \( x \) and \( b \) simultaneously and not separately) in the region \( (x_1, x_2) \times [b_0, b_1] \)

(ii) \( p \) is positive throughout the same region

(iii) \( |\frac{\partial^3 p}{\partial x^3}| \) is bounded in a neighborhood of the point \( (x, b) = (b_0, b_0) \) in the same region and

(iv) \( f \) is a non-increasing function of \( b \in [b_0, b_1] \) when \( x \in [b_-, b] \).

Indeed, (i) and (iii) follow from (2.4), (2.5), (2.20) and the fact that \( A \) is in \( C^4 \) and of class \( C^5 \) in some neighborhood of \( b_0 \) (see Assumption 2.1). For (ii), use the definition (2.20) of \( p \) and recall the sign of \( f \) using (2.4). Finally (iv) is a consequence of (2.4) and the monotonicity of \( A \) in \( [b_0, x_2] \) (again cf. Assumption 2.1). By Lemma I in Olver’s paper [16], the function \( \psi \) defined by (2.18) is continuous in the corresponding region of the \( (\zeta, \alpha) \)-plane. \( \square \)

2.3. Approximate solutions in the barrier case. We return to equation (2.17) and state an existence theorem concerning its approximate solutions. To this goal, we need a way to assess the error. We do this by introducing an error-control function \( H \) along with a balancing function \( \Omega \).

**Definition 2.11.** Define the balancing function \( \Omega \) by

\[
 \Omega(x) = 1 + |x|^\frac{1}{2}.
\]

(2.21)

As an error-control function \( H(\zeta, \alpha, \hbar) \) of equation (2.17) we consider any primitive of the function

\[
 \frac{\psi(\zeta, \alpha)}{\Omega(\zeta \sqrt{2h^{-1}})}.
\]

Furthermore, we need the notion of the variation of the error-control function \( H \) in a given interval. We have the following.
Definition 2.12. Take \((\gamma, \delta) \subseteq (\zeta_1, \zeta_2) \subseteq \mathbb{R}\) (cf. (2.14)). The variation \(V_{\gamma, \delta}[H]\) in the interval \((\gamma, \delta)\) of the error-control function \(H\) of equation (2.17) is defined by

\[
V_{\gamma, \delta}[H](\alpha, h) = \int_{\gamma}^{\delta} \frac{|\psi(t, \alpha)|}{\Omega(t\sqrt{2h^{-1}})} dt.
\]

Finally, for any \(c \leq 0\) set

\[
l_1(c) = \sup_{x \in (0, +\infty)} \left\{ \frac{\Omega(x)M(x, c)^2}{\Gamma(\frac{1}{2} - c)} \right\} \tag{2.22}
\]

where \(M\) is a function defined in terms of Parabolic Cylinder Functions in section B.1 of the appendix and \(\Gamma\) denotes the Gamma function. We note that the above supremum is finite for each value of \(c\). This fact is a consequence of (2.21) and the first relation in (B.9). Furthermore, because the relations (B.9) hold uniformly in compact intervals of \((-\infty, 0]\), the function \(l_1\) is continuous.

We are now ready for the main theorem of this paragraph.

Theorem 2.13. For each value of \(h > 0\) the equation

\[
d^2X/d\zeta^2 = \left[h^{-2}(\zeta^2 - \alpha^2) + \psi(\zeta, \alpha)\right]X
\]

has in the region \([0, \zeta_2] \times [0, \alpha_1]\) of the \((\zeta, \alpha)\)-plane, two solutions \(Y_+\) and \(Z_+\) satisfying

\[
Y_+(\zeta, \alpha, h) = U(\zeta \sqrt{2h^{-1}}, -\frac{1}{2}h^{-1}\alpha^2) + \varepsilon_1(\zeta, \alpha, h) \tag{2.23}
\]

\[
Z_+(\zeta, \alpha, h) = \overline{U}(\zeta \sqrt{2h^{-1}}, -\frac{1}{2}h^{-1}\alpha^2) + \varepsilon_2(\zeta, \alpha, h) \tag{2.24}
\]

where \(U, \overline{U}\) are the PCFs defined in appendix B.1. These two solutions \(Y_+, Z_+\) are continuous and have continuous first and second partial \(\zeta\)-derivatives. The errors \(\varepsilon_1, \varepsilon_2\) in the relations above satisfy the estimates

\[
\left|\varepsilon_1(\zeta, \alpha, h)\right| \leq \frac{1}{E(\zeta \sqrt{2h^{-1}}, -\frac{1}{2}h^{-1}\alpha^2)} \left( \exp \left\{ \frac{1}{2}(\pi h)^{\frac{1}{2}}l_1(-\frac{1}{2}h^{-1}\alpha^2)\nu_{\zeta, \zeta_2}[H](\alpha, h) \right\} - 1 \right) \tag{2.25}
\]

and

\[
\left|\varepsilon_2(\zeta, \alpha, h)\right| \leq \frac{1}{E(\zeta \sqrt{2h^{-1}}, -\frac{1}{2}h^{-1}\alpha^2)} \left( \exp \left\{ \frac{1}{2}(\pi h)^{\frac{1}{2}}l_1(-\frac{1}{2}h^{-1}\alpha^2)\nu_{0, \zeta}[H](\alpha, h) \right\} - 1 \right). \tag{2.26}
\]

Proof. In order to prove this theorem, we rely on Theorem I in [10]. There, it is stated that it suffices to prove two things. First that the function \(\psi\) is continuous in the region \([0, \zeta_2] \times [0, \alpha_1]\), a fact that has already been proven in (2.12) and second that the integral

\[
\nu_{0, \zeta_2}[H](\alpha, h) = \int_0^{\zeta_2} \frac{|\psi(t, \alpha)|}{\Omega(t\sqrt{2h^{-1}})} dt \tag{2.27}
\]

\footnote{The functions \(E, M\) and \(N\) are related with the PCF theory found in appendix B.1}
converges uniformly in $\alpha$. But this is obvious since $\zeta_2 < +\infty$. \hfill $\square$

**Remark 2.14.** If we were assuming either Assumption 2.3 or Assumption 2.4, we would have $\zeta_2 = +\infty$. In such a case, Theorem 2.13 would still be true. To obtain it, we have to argue as in the proof of Theorem 6.1 in [8].

### 2.4. Asymptotics of the approximate solutions for the barrier

In order to extract the asymptotic behavior of the solutions $Y_+(\zeta, \alpha, h)$, $Z_+(\zeta, \alpha, h)$ when $h \downarrow 0$, we need to determine the asymptotic form of the error bounds (2.25), (2.26) examining closely $l_1(-\frac{1}{2}h^{-1}\alpha^2)$ and $V_{0,\zeta_2}[H](\alpha, h)$ as $h \downarrow 0$.

Let us deal with the noncritical case $\alpha \in (0, \alpha_1]$ first. By applying the same analysis found in §8 of [8] we obtain

$$
l_1(-\frac{1}{2}h^{-1}\alpha^2) = \mathcal{O}(1) \quad \text{as} \quad h \downarrow 0. \tag{2.28}
$$

Next, we examine $V_{0,\zeta_2}[H](\alpha, h)$. Again in §8 of [8] it is shown that

$$
V_{0,\zeta_2}[H](\alpha, h) = \int_0^{\zeta_2} \frac{|\psi(t, \alpha)|}{1 + (t\sqrt{2h^{-1}})^\frac{3}{2}} dt = \mathcal{O}(h^{1/6}) \quad \text{as} \quad h \downarrow 0 \tag{2.29}
$$

when $\zeta_2 = +\infty$. Clearly the same asymptotics hold in the case when $\zeta_2 < +\infty$ too.

The last two relations applied to (2.25) and (2.26) supply us with the desired results as $h \downarrow 0$

$$
\varepsilon_1(\zeta, \alpha, h) = \frac{M(\zeta\sqrt{2h^{-1}} - \frac{3}{2}h^{-1}\alpha^2)}{E(\zeta\sqrt{2h^{-1}} - \frac{3}{2}h^{-1}\alpha^2)} \mathcal{O}(h^{\frac{1}{6}}) \tag{2.30}
$$

\[
\varepsilon_2(\zeta, \alpha, h) = E(\zeta\sqrt{2h^{-1}} - \frac{3}{2}h^{-1}\alpha^2)M(\zeta\sqrt{2h^{-1}} - \frac{3}{2}h^{-1}\alpha^2) \mathcal{O}(h^{\frac{1}{6}})
\]

\[
\frac{\partial \varepsilon_1}{\partial \zeta}(\zeta, \alpha, h) = \frac{N(\zeta\sqrt{2h^{-1}} - \frac{3}{2}h^{-1}\alpha^2)}{E(\zeta\sqrt{2h^{-1}} - \frac{3}{2}h^{-1}\alpha^2)} \mathcal{O}(h^{\frac{1}{6}})
\]

\[
\frac{\partial \varepsilon_2}{\partial \zeta}(\zeta, \alpha, h) = E(\zeta\sqrt{2h^{-1}} - \frac{3}{2}h^{-1}\alpha^2)N(\zeta\sqrt{2h^{-1}} - \frac{3}{2}h^{-1}\alpha^2) \mathcal{O}(h^{\frac{1}{6}})
\]

uniformly for $\zeta \in [0, \zeta_2]$ and $\alpha \in (0, \alpha_1]$.

**Remark 2.15.** In the special case $\alpha = 0$ (i.e. when equation (2.17) has a double turning point at $\zeta = 0$), $l_1(0)$ is independent of $h$. Using the definition (2.21) of $\Omega$, we see that we have a similar estimate to (2.29); namely $V_{0,\zeta_2}[H](0, h) = \mathcal{O}(h^{\frac{1}{6}})$ as $h \downarrow 0$. Hence the results above about the errors, hold for the case $\alpha = 0$ too.

### 2.5. Connection formulae for a barrier

We can determine the asymptotic behavior of $Y_-, Z_-$ for small $h > 0$ and $\zeta < 0$ by establishing appropriate connection formulae. We can replace $\zeta$ by $-\zeta$ in Theorem 2.13 to ensure two more solutions $Y_-, Z_-$ of equation (2.17) satisfying as $h \downarrow 0$

$$
Y_-(\zeta, \alpha, h) = U(-\zeta\sqrt{2h^{-1}} - \frac{3}{2}h^{-1}\alpha^2) + \frac{M(-\zeta\sqrt{2h^{-1}} - \frac{3}{2}h^{-1}\alpha^2)}{E(-\zeta\sqrt{2h^{-1}} - \frac{3}{2}h^{-1}\alpha^2)} \mathcal{O}(h^{\frac{1}{6}}) \tag{2.31}
$$

\[
Z_-(\zeta, \alpha, h) = U(-\zeta\sqrt{2h^{-1}} - \frac{3}{2}h^{-1}\alpha^2) + E(-\zeta\sqrt{2h^{-1}} - \frac{3}{2}h^{-1}\alpha^2)M(-\zeta\sqrt{2h^{-1}} - \frac{3}{2}h^{-1}\alpha^2) \mathcal{O}(h^{\frac{1}{6}})
\]

uniformly for $\zeta \in (\zeta_1, 0]$ and $\alpha \in [0, \alpha_1]$. 

---

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Remark 2.16. The two sets \( \{Y_+, Z_+\} \) and \( \{Y_-, Z_-\} \) consist of two linearly independent functions. This can be seen by their Wronskians. For example, using B.3 we have
\[
\mathcal{W}[U(\cdot, -\frac{1}{2}h^{-1}\alpha^2), \overline{U}(\cdot, -\frac{1}{2}h^{-1}\alpha^2)] = \sqrt{\frac{2}{\pi}} \Gamma \left( \frac{1}{2} + \frac{1}{2}h^{-1}\alpha^2 \right)
\]
Using this and (2.23), (2.24) we see that \( \mathcal{W}[Y_+, Z_+] \neq 0 \). Similarly, we have \( \mathcal{W}[Y_-, Z_-] \neq 0 \) as well.

We express \( Y_+, Z_+ \) in terms of \( Y_-, Z_- \). So for \( (\zeta, \alpha) \in (\zeta_1, 0) \times [0, \alpha_1] \) we may write
\[
Y_+(\zeta, \alpha, h) = \sigma_{11}(\alpha, h) Y_-(\zeta, \alpha, h) + \sigma_{12}(\alpha, h) Z_-(\zeta, \alpha, h) \quad (2.32)
\]
\[
Z_+(\zeta, \alpha, h) = \sigma_{21}(\alpha, h) Y_-(\zeta, \alpha, h) + \sigma_{22}(\alpha, h) Z_-(\zeta, \alpha, h). \quad (2.33)
\]
The connection will become clear once we find approximations for the coefficients \( \sigma_{ij}, i, j = 1, 2 \) in the linear relations (2.32) and (2.33). We evaluate at \( \zeta = 0 \) equations (2.32), (2.33) and their derivatives. After algebraic manipulations we obtain
\[
\sigma_{11}(\alpha, h) = \frac{\mathcal{W}[Y_+(\cdot, \alpha, h), Z_-(\cdot, \alpha, h)](0)}{\mathcal{W}[Y_-(\cdot, \alpha, h), Z_-(\cdot, \alpha, h)](0)}
\]
\[
\sigma_{12}(\alpha, h) = -\frac{\mathcal{W}[Y_+(\cdot, \alpha, h), Y_-(\cdot, \alpha, h)](0)}{\mathcal{W}[Y_-(\cdot, \alpha, h), Z_-(\cdot, \alpha, h)](0)}
\]
\[
\sigma_{21}(\alpha, h) = \frac{\mathcal{W}[Z_+(\cdot, \alpha, h), Z_-(\cdot, \alpha, h)](0)}{\mathcal{W}[Y_-(\cdot, \alpha, h), Z_-(\cdot, \alpha, h)](0)}
\]
\[
\sigma_{22}(\alpha, h) = -\frac{\mathcal{W}[Z_+(\cdot, \alpha, h), Y_-(\cdot, \alpha, h)](0)}{\mathcal{W}[Y_-(\cdot, \alpha, h), Z_-(\cdot, \alpha, h)](0)}.
\]
Now set
\[\varphi(\alpha, h) = (1 + h^{-1}\alpha^2) \frac{\pi}{4} \]
By using the results and properties of Parabolic Cylinder Functions and their auxiliary functions from section B.1 in the appendix, we find that as \( h \downarrow 0 \)
\[
Y_1(0, \alpha, h) = M(0)[\sin \varphi(\alpha, h) + \mathcal{O}(h^{\frac{3}{2}})]
\]
\[
Y_2(0, \alpha, h) = M(0)[\cos \varphi(\alpha, h) + \mathcal{O}(h^{\frac{3}{2}})]
\]
\[
Y_3(0, \alpha, h) = M(0)[\sin \varphi(\alpha, h) + \mathcal{O}(h^{\frac{3}{2}})]
\]
\[
Y_4(0, \alpha, h) = M(0)[\cos \varphi(\alpha, h) + \mathcal{O}(h^{\frac{3}{2}})]
\]
\[
\dot{Y}_1(0, \alpha, h) = -\sqrt{2h^{-1}}N(0)[\cos \varphi(\alpha, h) + \mathcal{O}(h^{\frac{3}{2}})]
\]
\[
\dot{Y}_2(0, \alpha, h) = \sqrt{2h^{-1}}N(0)[\sin \varphi(\alpha, h) + \mathcal{O}(h^{\frac{3}{2}})]
\]
\[
\dot{Y}_3(0, \alpha, h) = \sqrt{2h^{-1}}N(0)[\cos \varphi(\alpha, h) + \mathcal{O}(h^{\frac{3}{2}})]
\]
\[
\dot{Y}_4(0, \alpha, h) = -\sqrt{2h^{-1}}N(0)[\sin \varphi(\alpha, h) + \mathcal{O}(h^{\frac{3}{2}})].
\]
Recall that the dot denotes differentiation with respect to $\zeta$. Finally, using these estimates we obtain as $\hbar \downarrow 0$

\begin{align*}
\sigma_{11}(\alpha, \hbar) &= \sin\left(\frac{1}{2}\pi \hbar^{-1} \alpha^2\right) + O(\hbar^{3/2}) \\
\sigma_{12}(\alpha, \hbar) &= \cos\left(\frac{1}{2}\pi \hbar^{-1} \alpha^2\right) + O(\hbar^{3/2}) \\
\sigma_{21}(\alpha, \hbar) &= \cos\left(\frac{1}{2}\pi \hbar^{-1} \alpha^2\right) + O(\hbar^{3/2}) \\
\sigma_{22}(\alpha, \hbar) &= -\sin\left(\frac{1}{2}\pi \hbar^{-1} \alpha^2\right) + O(\hbar^{3/2})
\end{align*}

(2.34)

uniformly for $\alpha \in [0, \alpha_1]$.

2.6. Applications in the barrier case. We are assuming (2.1) (similar arguments hold for the other cases as well). Recalling (2.9), we define the following.

**Definition 2.17.** If we assume that equation (2.3) describes the motion of a system (e.g. a particle), then the function

\[ \Phi(\mu) = \frac{\pi}{2} \alpha^2(\mu) = \int_{b_-(\mu)}^{b_+(\mu)} \sqrt{A(x)^2 - \mu^2} \, dx. \]

(2.35)

is called the abbreviated action of this motion.

It is easily checked that $\Phi$ is of class $C^1$. Differentiating relation (2.35) while using $A(b_\pm) = \mu$, we obtain

\[ \frac{d\Phi(\mu)}{d\mu} = -2\mu \int_{b_-(\mu)}^{b_+(\mu)} [A(x)^2 - \mu^2]^{-\frac{1}{2}} \, dx < 0. \]

(2.36)

The asymptotic behavior as $\hbar \downarrow 0$ of an arbitrary non-trivial real solution $X$ of equation (2.17) on the $\zeta$-interval corresponding to the finite $x$-barrier $(b_-, b_+)$ of $A$, can be examined through the functions $Y_+, Z_+$ and $Y_-, Z_-$. Since $\{Y_+, Z_+\}$ and $\{Y_-, Z_-\}$ are two sets of linearly independent functions (cf. Remark 2.16), for $X$ we can write

\[ X(\zeta, \alpha, \hbar) = \gamma_+(\alpha, \hbar) Y_+(\zeta, \alpha, \hbar) + \delta_+(\alpha, \hbar) Z_+(\zeta, \alpha, \hbar) = \gamma_-(\alpha, \hbar) Y_-(\zeta, \alpha, \hbar) + \delta_-(\alpha, \hbar) Z_-(\zeta, \alpha, \hbar) \]

(2.37)

for some $\gamma_\pm(\alpha, \hbar), \delta_\pm(\alpha, \hbar) \in \mathbb{R}$. We put

\[ v_\pm(\alpha, \hbar) = \sqrt{\gamma_\pm(\alpha, \hbar)^2 + \delta_\pm(\alpha, \hbar)^2} \]

\[ \gamma_\pm(\alpha, \hbar) = v_\pm(\alpha, \hbar) \cos \xi_\pm(\alpha, \hbar) \]

\[ \delta_\pm(\alpha, \hbar) = v_\pm(\alpha, \hbar) \sin \xi_\pm(\alpha, \hbar) \]

(2.38)

and define

\[ \xi(\mu, \hbar) = \xi_+(\alpha(\mu), \hbar) + \xi_-(\alpha(\mu), \hbar). \]

(2.39)

Recall that $\alpha$ is function of $\mu$. Whence we can see that $\xi_\pm$, $\xi$ depend on $\mu$. Sometimes we shall simply write $\xi_\pm(\mu, \hbar)$ meaning $\xi_\pm(\alpha(\mu), \hbar)$.

The ideas that follow are essentially the same as those used in the derivation of the Bohr-Sommerfeld quantization condition found in §10 of [8]. We start with a theorem.
Theorem 2.18. Under Assumption 2.1, there is a non-negative integer \( n = n(\mu, \hbar) \) (i.e. that depends on \( \mu \) and \( \hbar \)) such that the functions \( \Phi \) and \( \xi \) in (2.35) and (2.39) respectively, satisfy the formula

\[
\Phi(\mu) = \left(2n + 1\right)\frac{\pi}{2} - \xi(\mu, \hbar) \hbar + O(\hbar^{\frac{5}{3}}) \quad \text{as} \quad \hbar \downarrow 0. \tag{2.40}
\]

Proof. Using (2.37) and (2.38) we have

\[
0 = W[X, X] = W[\gamma_+ Z_+ + \delta_+ Y_+, \gamma_- Z_- + \delta_- Y_-] = v_+ v_- W[Y_-, Z_-] (-\sigma_{12} \cos \xi_+ \cos \xi_- + \sigma_{11} \cos \xi_+ \sin \xi_- - \sigma_{22} \sin \xi_+ \cos \xi_- + \sigma_{21} \sin \xi_+ \sin \xi_-)
\]

(we have suppressed the dependence on \( \alpha \) and \( \hbar \) for notational simplicity). From (2.38), (2.34) and (2.39)

\[
\cos \left[ h^{-1} \Phi(\mu) + \xi(\mu, \hbar) \right] = O(h^{\frac{5}{3}}) \quad \text{as} \quad h \downarrow 0
\]

from which the result follows. \( \square \)

If \( \xi(\mu, \hbar) = 0 \) (mod \( \pi \)), then relation (2.40) reduces to the Bohr-Sommerfeld quantization condition. In particular, this is true if \( \xi_\pm(\mu, \hbar) = 0 \) (mod \( \pi \)) at both turning points \( b_\pm(\mu) \). We state this explicitly.

Theorem 2.19. Under Assumption 2.1, suppose that a non-trivial real solution of (2.17) fulfills (2.37) and (2.38) with \( \xi_\pm(\mu, \hbar) = 0 \) (mod \( \pi \)). Then function \( \Phi \) in (2.35) satisfies the condition

\[
\cos \left[ h^{-1} \Phi(\mu) \right] = O(h^{\frac{5}{3}}) \quad \text{as} \quad h \downarrow 0 \tag{2.41}
\]

whence

\[
\Phi(\mu) = \pi \left( n + \frac{1}{2} \right) \hbar + O(\hbar^{\frac{5}{3}}) \quad \text{as} \quad h \downarrow 0 \tag{2.42}
\]

for some non-negative integer \( n = n(\mu, \hbar) \).

Remark 2.20. It is possible that \( \xi_\pm(\mu, \hbar) = 0 \) (mod \( \pi \)) only for \( \hbar \in \Sigma \subset \mathbb{R}_+ \) such that \( 0 \in \text{clos}(\Sigma) \). Then conditions (2.41), (2.42) are also satisfied for \( \hbar \in \Sigma \).

What follows is a result converse to Theorem 2.19.

Theorem 2.21. Under Assumption 2.1, suppose that for some non-negative integer \( n \), the point \( \pi \left( n + \frac{1}{2} \right) \hbar \) lies in \( (0, \frac{\pi}{2} \alpha_\gamma^2) \). Then there exists a value \( \tilde{\mu} = \tilde{\mu}(n, \hbar) \) such that

\[
\Phi(\tilde{\mu}) = \pi \left( n + \frac{1}{2} \right) \hbar + O(\hbar^{\frac{5}{3}}) \quad \text{as} \quad h \downarrow 0
\]

and

\[
Y_+(\zeta, \alpha(\tilde{\mu}), \hbar) = \sigma_{11}(\alpha(\tilde{\mu}), \hbar) Y_-(\zeta, \alpha(\tilde{\mu}), \hbar)
\]

where

\[
\sigma_{11}(\alpha(\tilde{\mu}), \hbar) = (-1)^n + O(h^{\frac{5}{3}}) \quad \text{as} \quad h \downarrow 0.
\]

Proof. Recall the connection coefficients \( \sigma_{ij} \), \( i, j = 1, 2 \) from (2.43) and define the function

\[
\sigma(\mu, \hbar) = \sigma_{12}(\alpha(\mu), \hbar). \tag{2.43}
\]
From (2.32), it is enough to show that \( \sigma \) vanishes for some \( \tilde{\mu} = \tilde{\mu}(n, h) \) satisfying

\[
|\Phi(\tilde{\mu}) - \pi(n + \frac{1}{2})| \leq C h^{\frac{3}{2}}
\]

where \( C \) does not depend neither on \( n \) nor on \( h \). Then, the rest follow from the first asymptotic relation in (2.34).

From (2.36) we know that \( \Phi \) maps a neighborhood of \( \tilde{\mu} \) in a one-to-one way onto a neighborhood of \( \Phi(\tilde{\mu}) \). Let \( X = \Phi(\mu) \) and set

\[
\chi(X, h) = \sigma(\Phi^{-1}(X), h) - \cos(h^{-1}X).
\]

By definition (2.43) of \( \sigma \) and the second relation in (2.34) we have

\[
|\chi(X, h)| \leq C h^{\frac{3}{2}}
\]

for a constant \( C \) independent of \( h \) and \( X \). With the above definitions, our equation now reads

\[
0 = \sigma(\mu, h) = \chi(X, h) + \cos(h^{-1}X).
\]

So this equation has to have a solution \( \tilde{X} = \tilde{X}(n, h) = \Phi(\tilde{\mu}(n, h)) \) satisfying the estimate

\[
|\tilde{X} - \pi(n + \frac{1}{2})| \leq C h^{\frac{3}{2}}.
\]

A change of variables \( s = h^{-1}X \) transforms our problem to the equivalent assertion that equation

\[
\chi(hs, h) + \cos s = 0
\]

has to have a solution with respect to \( s \), namely \( \tilde{s} = \tilde{s}(n, h) = h^{-1}\tilde{X} \), such that

\[
|\tilde{s} - \pi(n + \frac{1}{2})| \leq C h^{\frac{3}{2}}.
\]

But this is true because

\[
\chi(hs, h) = O(h^{\frac{3}{2}}) \quad \text{as} \quad h \downarrow 0.
\]

\( \square \)

3. The Case of One Potential Well

In this section we are interested in the solutions of equation

\[
\frac{d^2y}{dx^2} = \left( h^{-2}[\mu^2 - A^2(x)] + \frac{3}{4} \left[ \frac{A'(x)}{A(x) + \mu} \right]^2 - \frac{1}{2} \frac{A''(x)}{A(x) + \mu} \right)y
\]

(3.1)

where \( \mu > 0 \) and the potential function \( A : \mathbb{R} \to \mathbb{R}_+ \) behaves as a finite well (a well of finite width) in some bounded interval in \( \mathbb{R} \). We assume the following (see Figure 6).

**Assumption 3.1.** The function \( A \) is positive on some bounded interval \([x_1, x_2]\) in \( \mathbb{R} \) and has a unique extremum in \((x_1, x_2)\); particularly, a minimum \( A_{\text{min}} > 0 \) at \( w_0 \in (x_1, x_2) \). Also \( A \) is assumed to be \( C^4 \) in \((x_1, x_2)\) and of class \( C^5 \) in a neighborhood of \( w_0 \). Additionally, \( A'(x) < 0 \) for \( x \in (x_1, w_0) \) and \( A'(x) > 0 \) for \( x \in (w_0, x_2) \). At \( w_0 \), we have \( A'(w_0) = 0 \) and \( A''(w_0) > 0 \). Furthermore, if we let \( A_*= \min \{ A(x_1), A(x_2) \} \) and take \( \mu \in (A_{\text{min}}, A_*) \subset \mathbb{R}_+ \), the equation \( A(x) = \mu \) has two solutions \( w_-(\mu), w_+(\mu) \) in \((x_1, x_2)\). These satisfy \( w_- < w_+ \), \( A(x) < \mu \) for \( x \in (w_-, w_+) \) and \( A(x) > \mu \) for \( x \in (x_1, w_-) \cup (w_+, x_2) \) (the above imply
Finally, when $\mu = A_{\text{min}}$ the two points $w_-, w_+$ coalesce into one double root at $w_0$.

3.1. The Liouville transform for the case of a well. Let us first fix some notation. We set $w_- \equiv w$,

$$J^- = (x_1, w_0), \quad J^+ = (w_0, x_2)$$

and define

$$G^\pm = \left|A_{\text{clos}}(J^\pm)\right|^{-1}. \quad (3.2)$$

We take an arbitrary $\mu_1 \in (A_{\text{min}}, A_{\ast})$ and consider the $w_1 \in (G^-(A_{\ast}), w_0)$ such that $A(w_1) = \mu_1$; then $\mu \in [A_{\text{min}}, \mu_1]$ implies $w \in [w_1, w_0]$. For every $\hbar > 0$ our equation (3.1) reads

$$\frac{d^2 y}{dx^2} = \left[\hbar - 2 A''(w) + \frac{4}{A(x)} A'(x)\right] y, \quad (x, w) \in (x_1, x_2) \times [w_1, w_0] \quad (3.3)$$

in which the functions $f$ and $g$ satisfy

$$f(x, w) = A^2(w) - A^2(x) \quad (3.4)$$

and

$$g(x, w) = \frac{3}{4} \left[\frac{A'(x)}{A(x)} + \frac{A''(x)}{A(x)}\right]^2 - \frac{1}{2} \frac{A''(x)}{A(x)} \quad (3.5)$$

Observe that our equation possesses two simple turning points at $x = w_\pm$ when $w \in [w_1, w_0]$ which combine into one double at $x = w_0$ when $w$ equals $w_0$.

We introduce new variables $X$ and $\zeta$ according to the Liouville transform

$$X = \dot{x}^{-\frac{3}{2}} y$$

where the dot denotes differentiation with respect to $\zeta$. Equation (3.3) becomes

$$\frac{d^2 X}{d\zeta^2} = \left[\hbar^{-2} \dot{x}^2 f(x, w) + \dot{x}^2 g(x, w) + \dot{x}^2 \left(\ddot{x}^{-\frac{1}{2}}\right)^2\right] X. \quad (3.6)$$
We begin with the noncritical case \( \mu \in (A_{\min}, \mu_1] \) with two turning points \( w_{\pm} \). In this case \( f(\cdot, w) \) is positive in \((w_-, w_+)\) and negative in \((x_1, w_-) \cup (w_+, x_2)\). Hence we prescribe

\[
\dot{x}^2 f(x, w) = \beta^2 - \zeta^2 \tag{3.7}
\]

where \( \beta > 0 \) is chosen in such a way that \( x = w_- \) corresponds to \( \zeta = -\beta \) and \( x = w_+ \) to \( \zeta = \beta \) accordingly.

The integration of (3.7) yields

\[
\int_{w_-}^{x} f(t, w)^{\frac{1}{2}} dt = \int_{-\beta}^{\zeta} (\beta^2 - \tau^2)^{\frac{1}{2}} d\tau \tag{3.8}
\]

provided that \( w_- \leq x \leq w_+ \) (notice that by taking these integration limits, \( w_- \) corresponds to \( -\beta \)). For the remaining correspondence we require

\[
\int_{w_-}^{w_+} f(t, w)^{\frac{1}{2}} dt = \int_{-\beta}^{\beta} (\beta^2 - \tau^2)^{\frac{1}{2}} d\tau
\]

yielding

\[
\beta^2(\mu) = \frac{2}{\pi} \int_{w_-}^{w_+} \sqrt{\mu^2 - A^2(t)} dt. \tag{3.9}
\]

For every fixed value of \( \hbar \), relation (3.9) defines \( \beta \) as a continuous and increasing function of \( \mu \) which vanishes as \( \mu \downarrow A_{\min} \). Set

\[
\beta_1 = \beta(w_-) > 0. \tag{3.10}
\]

Then \( \mu \in (A_{\min}, \mu_1) \) implies \( \beta \in (0, \beta_1] \).

Next, from (3.8) we find

\[
\int_{w_-}^{x} f(t, w)^{\frac{1}{2}} dt = \frac{1}{2} \alpha^2 \arccos \left( -\frac{\zeta}{\alpha} \right) + \frac{1}{2} \zeta (\alpha^2 - \zeta^2)^{\frac{1}{2}} \quad \text{for} \quad w_- \leq x \leq w_+ \tag{3.11}
\]

with the principal value choice for the inverse cosine taking values in \([0, \pi]\). For the remaining \( x \)-intervals, we integrate (3.7) to obtain

\[
\int_{x}^{w_-} [-f(t, w)]^{\frac{1}{2}} dt = -\frac{1}{2} \alpha^2 \arccosh \left( -\frac{\zeta}{\alpha} \right) - \frac{1}{2} \zeta (\zeta^2 - \alpha^2)^{\frac{1}{2}} \quad \text{for} \quad x_1 < x \leq w_- \tag{3.12}
\]

and

\[
\int_{w_+}^{x} [-f(t, w)]^{\frac{1}{2}} dt = -\frac{1}{2} \alpha^2 \arccosh \left( \frac{\zeta}{\alpha} \right) + \frac{1}{2} \zeta (\zeta^2 - \alpha^2)^{\frac{1}{2}} \quad \text{for} \quad w_+ \leq x < x_2 \tag{3.13}
\]

with \( \text{arcosh}(x) = \ln \left( x + \sqrt{x^2 - 1} \right) \) for \( x \geq 1 \).

Equations (3.11), (3.12) and (3.13) show that \( \zeta \) is a continuous and increasing function of \( x \) which shows that there is a one-to-one correspondence between these two variables. Thus, if we set

\[
\zeta_j = \lim_{x \to x_j} \zeta(x) \quad \text{for} \quad j = 1, 2 \tag{3.14}
\]

then \( (x_1, x_2) \) is mapped by \( \zeta \) to \((\zeta_1, \zeta_2)\). Notice that \(-\infty < \zeta_1 < 0 < \zeta_2 < +\infty\).

**Remark 3.2.** In the critical case in which the two (simple) turning points coalesce into one (double) point, we get a limit of the above transformation with \( w = w_0 \). In this case, the relevant relations to (3.11), (3.12), (3.13) are

\[
\int_{x}^{w_0} [-f(t, w_0)]^{\frac{1}{2}} dt = \frac{1}{2} \zeta^2 \quad \text{for} \quad x_1 < x \leq w_0 \tag{3.15}
\]
\[ \int_{w_0}^{x} \left[ -f(t, w_0) \right] \frac{1}{2} dt = \frac{1}{2} \zeta^2 \quad \text{for} \quad w_0 \leq x < x_2 \]  

(3.16)

and \( \beta = 0 \).

Consequently, noticing Remark 2.6 we substitute (3.7) in (3.6) and obtain the following proposition.

**Proposition 3.3.** For every \( \hbar > 0 \) equation

\[ \frac{d^2 y}{dx^2} = \left[ \hbar^{-2} f(x, w) + g(x, w) \right] y, \quad (x, w) \in (x_1, x_2) \times [w_{-1}, w_0] \]

where \( f, g \) as in (3.4), (3.5) respectively, is transformed to the equation

\[ \frac{d^2 X}{d\zeta^2} = \left[ \hbar^{-2}(\beta^2 - \zeta^2) + \overline{\psi}(\zeta, \beta) \right] X, \quad (\zeta, \beta) \in (\zeta_1, \zeta_2) \times [0, \beta_1] \]  

(3.17)

in which \( \zeta \) is given by the Liouville transform (3.7), \( \beta \) is given by (3.9), \( \zeta_j, j = 1, 2 \) are given by (3.14), \( \beta_1 \) as in (3.10) and the function \( \overline{\psi}(\zeta, \beta) \) is given by the formula

\[ \overline{\psi}(\zeta, \beta) = x^2 g(x, w) + \frac{1}{4} \frac{d^2}{d\zeta^2}(x^{-\frac{1}{2}}). \]  

(3.18)

In the following paragraphs we shall be interested in approximate solutions of equation (3.17), so we introduce the following terminology.

**Definition 3.4.** The function \( \overline{\psi} \) found in the differential equation (3.17) shall be called the **error term** of this equation.

For the error term we have the following proposition.

**Proposition 3.5.** The error term \( \overline{\psi} \) can be written equivalently as

\[ \overline{\psi}(\zeta, \beta) = \frac{1}{4} \frac{3\zeta^2 + 2\beta^2}{(\beta^2 - \zeta^2)^2} + \frac{1}{16} \frac{\beta^2 - \zeta^2}{f^3(x, w)} \left\{ 4f(x, w)f''(x, w) - 5[f'(x, w)]^2 \right\} \]

\[ + (\beta^2 - \zeta^2) \frac{g(x, w)}{f(x, w)} \]  

(3.19)

where prime denotes differentiation with respect to \( x \). The same formula can be used in the critical case of one double turning point simply by setting \( w = w_0 \) and \( \beta = 0 \).

**Proof.** Using (3.18), (3.5) and (3.7), simple algebraic manipulations shown that \( \overline{\psi} \) takes the desired form. \( \square \)

### 3.2. Continuity of the error term in the case of a well.

In this subsection we prove that the function \( \overline{\psi}(\zeta, \beta) \) resulting from the Liouville transformation defined above, is continuous in \( \zeta \) and \( \beta \). This will be used subsequently to prove the existence of approximate solutions of equation (3.17). We have the following.

**Lemma 3.6.** The function \( \overline{\psi}(\zeta, \beta) \) defined in (3.18), is continuous in \( \zeta \) and \( \beta \) in the region \((\zeta_1, \zeta_2) \times [0, \beta_1]\) of the \((\zeta, \beta)\)-plane.

**Proof.** For \( x \in (x_1, x_2) \), \( \mu \in [A_{\min}, \mu_1] \) and \( w \in [w_{-1}, w_0] \) we introduce an auxiliary function \( q \) by setting

\[ f(x, w) = (w - x)(w_+ - x)q(x, w). \]  

(3.20)
Having in mind that \( A(w) = A(w_+) = \mu \), we see that for \( \mu \in (A_{\text{min}}, \mu_1] \)

\[
q(w_\pm, w) = \pm \frac{2\mu}{w - w_+} A'(w_\pm) < 0
\]

while for \( \mu = A_{\text{min}} \)

\[
q(w_0, w_0) = -A_{\text{min}} A''(w_0) < 0.
\]

Our functions \( f, g \) and \( q \) defined by (3.4), (3.5) and (3.20) respectively satisfy the following properties

(i) \( q, \frac{\partial q}{\partial x}, \frac{\partial^2 q}{\partial x^2} \) and \( g \) are continuous functions of \( x \) and \( w \) in the region \((x_1, x_2) \times [w_-, w_0]\)

(ii) \( q \) is negative throughout the same region

(iii) \( |\frac{\partial^3 q}{\partial x^3}| \) is bounded in a neighborhood of the point \((x, w) = (w_0, w_0)\) in the same region and

(iv) \( f \) is a non-increasing function of \( w \in [w_-, w_0] \) when \( x \in [w, w_+] \).

As in §2.2 these relations follow directly from (3.4), (3.5), (3.20) and Assumption 3.1. By referring again to Lemma I in [16] (actually a slight variant of it properly defined for case III treated in Olver’s [16]), the function \( \psi \) defined by (3.18) (or (3.19)) is continuous in the corresponding region of the \((\zeta, \beta)\)-plane.

3.3. Approximate solutions in the case of a well. Here we state a theorem concerning approximate solutions of equation (3.17). First we define a balancing function \( \Omega \) as in the barrier case using (2.21). Now we define an error-control function which will provide us with a way to assess the error.

**Definition 3.7.** As an error-control function \( \Pi(\zeta, \beta, h) \) of equation (3.17) we consider any primitive of the function

\[
\frac{\overline{\psi}(\zeta, \beta)}{\Omega(\zeta \sqrt{2h^{-1}})}.
\]

As in §2.3 we define the variation of \( \Pi \) in an interval \((\gamma, \delta) \subseteq (\zeta_1, \zeta_2) \subset \mathbb{R} \) (cf. (3.14)).

**Definition 3.8.** The variation \( V_{\gamma, \delta} \left[ \Pi \right] \) in the interval \((\gamma, \delta) \) of the error-control function \( \Pi \) of equation (3.17) is defined by

\[
V_{\gamma, \delta} \left[ \Pi \right](\beta, h) = \int_{\gamma}^{\delta} \frac{\overline{\psi}(t, \beta)}{\Omega(t \sqrt{2h^{-1}})} dt.
\]

Finally, for any \( c \geq 0 \) set

\[
l_2(c) = \sup_{x \in (0, +\infty)} \left\{ \Omega(x) \overline{M}(x, c)^2 \right\}
\] (3.21)

where \( \overline{M} \) is a function defined in terms of modified Parabolic Cylinder Functions in section 3.2 of the appendix. We note that the above supremum is finite for each value of \( c \). This fact is a consequence of (2.21) and the first relation in (3.21). Furthermore, because the relations (3.21) hold uniformly in compact intervals of the parameter \( c \), the function \( l_2 \) is continuous.

Now the existence of approximate solutions is guaranted by the following.
Theorem 3.9. For each value of $\hbar > 0$, equation
\[
\frac{d^2 X}{d\zeta^2} = [h^{-2}(\beta^2 - \zeta^2) + \psi(\zeta, \beta)] X
\]
has in the region $[0, \zeta_2] \times [0, \beta_1]$ of the $(\zeta, \beta)$-plane, two solutions $Y_+$ and $Z_+$. They satisfy
\[
Y_+(\zeta, \beta) = k (\frac{1}{2} \hbar^{-1} \beta^2)^{\frac{1}{2}} W(-\zeta \sqrt{2\hbar^{-1}}, \frac{1}{2} \hbar^{-1} \beta^2) + \tau_1(\zeta, \beta, \hbar) \tag{3.22}
\]
\[
Z_+(\zeta, \beta) = k (\frac{1}{2} \hbar^{-1} \beta^2)^{-\frac{1}{2}} W(\zeta \sqrt{2\hbar^{-1}}, \frac{1}{2} \hbar^{-1} \beta^2) + \tau_2(\zeta, \beta, \hbar) \tag{3.23}
\]
where $k$, $W$ are functions found in appendix B.2 about modified PCFs. These $Y_+$, $Z_+$ are continuous and have continuous first and second partial $\zeta$-derivatives. The errors $\tau_1, \tau_2$ satisfy
\[
\frac{|\tau_1(\zeta, \beta, \hbar)|}{M(\zeta \sqrt{2\hbar^{-1}}, \frac{1}{2} \hbar^{-1} \beta^2)} \leq \mathbb{E}(\zeta \sqrt{2\hbar^{-1}}, \frac{1}{2} \hbar^{-1} \beta^2) \left(\exp \left\{ \frac{l_2(\frac{1}{2} \hbar^{-1} \beta^2)}{\sqrt{2\hbar^{-1}}} \mathcal{V}_{0, \zeta} \mathcal{P}(\beta, \hbar) \right\} - 1 \right) \tag{3.24}
\]
and
\[
\frac{|\tau_2(\zeta, \beta, \hbar)|}{M(\zeta \sqrt{2\hbar^{-1}}, \frac{1}{2} \hbar^{-1} \beta^2)} \leq \mathbb{E}(\zeta \sqrt{2\hbar^{-1}}, \frac{1}{2} \hbar^{-1} \beta^2) \left(\exp \left\{ \frac{l_2(\frac{1}{2} \hbar^{-1} \beta^2)}{\sqrt{2\hbar^{-1}}} \mathcal{V}_{0, \zeta} \mathcal{P}(\beta, \hbar) \right\} - 1 \right). \tag{3.25}
\]

Proof. The proof is similar to that of Theorem 2.13 in §2.3 and details need not be recorded.

3.4. Asymptotics of the approximate solutions for the well. As in the case of the function $l_1$ in §2.4, we find that $l_2$ is continuous in $[0, +\infty)$. Using (2.21) and an analysis similar to that mentioned in §2.4, we find
\[
l_2(\frac{1}{2} \hbar^{-1} \beta^2) = \mathcal{O}(1) \quad \text{as} \quad \hbar \downarrow 0. \tag{3.26}
\]

Next, $\mathcal{V}_{0, \zeta} \mathcal{P}(\beta, \hbar)$ can be examined as in §§ of §8. We find that
\[
\mathcal{V}_{0, \zeta} \mathcal{P}(\beta, \hbar) = \int_0^{\zeta_2} \frac{|\psi(t, \beta)|}{1 + (t\sqrt{2\hbar^{-1}})^2} dt = \mathcal{O}(\hbar^{1/6}) \quad \text{as} \quad \hbar \downarrow 0. \tag{3.27}
\]

The last two relations applied to (3.24) and (3.25) return as $\hbar \downarrow 0$
\[
\tau_1(\zeta, \beta, \hbar) = \mathbb{E}(\zeta \sqrt{2\hbar^{-1}}, \frac{1}{2} \hbar^{-1} \beta^2) \mathcal{M}(\zeta \sqrt{2\hbar^{-1}}, \frac{1}{2} \hbar^{-1} \beta^2) \mathcal{O}(\hbar^{\frac{1}{2}}) \tag{3.28}
\]
\[
\tau_2(\zeta, \beta, \hbar) = \mathbb{E}(\zeta \sqrt{2\hbar^{-1}}, \frac{1}{2} \hbar^{-1} \beta^2) \mathcal{M}(\zeta \sqrt{2\hbar^{-1}}, \frac{1}{2} \hbar^{-1} \beta^2) \mathcal{O}(\hbar^{\frac{1}{2}})
\]
\[
\frac{\partial \tau_1}{\partial \zeta}(\zeta, \beta, \hbar) = \mathbb{E}(\zeta \sqrt{2\hbar^{-1}}, \frac{1}{2} \hbar^{-1} \beta^2) \mathcal{M}(\zeta \sqrt{2\hbar^{-1}}, \frac{1}{2} \hbar^{-1} \beta^2) \mathcal{O}(\hbar^{\frac{1}{2}})
\]
\[
\frac{\partial \tau_2}{\partial \zeta}(\zeta, \beta, \hbar) = \mathbb{E}(\zeta \sqrt{2\hbar^{-1}}, \frac{1}{2} \hbar^{-1} \beta^2) \mathcal{M}(\zeta \sqrt{2\hbar^{-1}}, \frac{1}{2} \hbar^{-1} \beta^2) \mathcal{O}(\hbar^{\frac{1}{2}})
\]
uniformly for $\zeta \in [0, \zeta_2]$ and $\beta \in (0, \beta_1]$. 

Remark 3.10. In the special case $\beta = 0$ (i.e., when equation (3.17) has a double turning point at $\zeta = 0$), $l_2(0)$ is independent of $h$. Using the definition (2.21) of $\Omega$, we see that we have a similar estimate to (2.29); namely $V_{0,\zeta} \left[ F \right](0, h) = O(h^{\frac{1}{2}})$ as $h \downarrow 0$. Hence the results about the errors above hold for the case $\beta = 0$ too.

3.5. Connection formulae for a well. Here, we determine the asymptotic behavior of $Y_+, Z_+$ for small $h > 0$ and $\zeta < 0$ by establishing appropriate connection formulae. We can replace $\zeta$ by $-\zeta$ in Theorem 3.9 to ensure two more solutions $Y_-, Z_-$ of equation (3.17) satisfying as $h \downarrow 0$

$$Y_-(\zeta, \alpha, h) = k(\frac{1}{2} h^{-1} \beta^2)^\frac{1}{2} W(\zeta \sqrt{2h^{-1}}, \frac{1}{2} h^{-1} \beta^2) + \mathcal{E}(\zeta \sqrt{2h^{-1}}, \frac{1}{2} h^{-1} \beta^2) \mathcal{M}(\zeta \sqrt{2h^{-1}}, \frac{1}{2} h^{-1} \beta^2) O(h^{\frac{1}{2}})$$

$$Z_-(\zeta, \alpha, h) = k(\frac{1}{2} h^{-1} \beta^2)^{-\frac{1}{2}} W(-\zeta \sqrt{2h^{-1}}, \frac{1}{2} h^{-1} \beta^2) + \mathcal{M}(\zeta \sqrt{2h^{-1}}, \frac{1}{2} h^{-1} \beta^2) \mathcal{E}(\zeta \sqrt{2h^{-1}}, \frac{1}{2} h^{-1} \beta^2) O(h^{\frac{1}{2}})$$

uniformly for $\zeta \in (\zeta_1, 0] \text{ and } \beta \in [0, \beta_1]$.

Remark 3.11. The two sets $\{Y_+, Z_+\}$ and $\{Y_-, Z_-\}$ consist of two linearly independent functions. This can be seen by their Wronskians. For example, using (3.14) we have

$$\mathcal{W}[W(-, \frac{1}{2} h^{-1} \beta^2), W(\cdot, \frac{1}{2} h^{-1} \beta^2)] = 1$$

Using this and (3.22), (3.23) we see that $\mathcal{W}[Y_+, Z_+] \neq 0$. Similarly, we have $\mathcal{W}[Y_-, Z_-] \neq 0$ as well.

We express $Y_+, Z_+$ in terms of $Y_-, Z_-$. So for $(\zeta, \beta) \in (\zeta_1, 0) \times [0, \beta_1]$ we write

$$Y_+(\zeta, \beta, h) = \tau_{11}(\beta, h) Y_-(\zeta, \beta, h) + \tau_{12}(\beta, h) Z_-(\zeta, \beta, h)$$

$$Z_+(\zeta, \beta, h) = \tau_{21}(\beta, h) Y_-(\zeta, \beta, h) + \tau_{22}(\beta, h) Z_-(\zeta, \beta, h).$$

As in (3.25) we find approximations for the coefficients $\tau_{ij}$, $i, j = 1, 2$ in the linear relations (3.29) and (3.30). We take equations (3.29), (3.30) along with their derivatives and evaluate them at $\zeta = 0$. We obtain

$$\tau_{11}(\beta, h) = \frac{\mathcal{W}[Y_+(\cdot, \beta, h), Z_-(\cdot, \beta, h)](0)}{\mathcal{W}[Y_-(\cdot, \beta, h), Z_-(\cdot, \beta, h)](0)}$$

$$\tau_{12}(\beta, h) = -\frac{\mathcal{W}[Y_+(\cdot, \beta, h), Z_-\cdot(\cdot, \beta, h)](0)}{\mathcal{W}[Y_-(\cdot, \beta, h), Z_-(\cdot, \beta, h)](0)}$$

$$\tau_{21}(\beta, h) = \frac{\mathcal{W}[Z_+(\cdot, \beta, h), Z_-(\cdot, \beta, h)](0)}{\mathcal{W}[Y_-(\cdot, \beta, h), Z_-(\cdot, \beta, h)](0)}$$

$$\tau_{22}(\beta, h) = -\frac{\mathcal{W}[Z_+(\cdot, \beta, h), Y_-(\cdot, \beta, h)](0)}{\mathcal{W}[Y_-(\cdot, \beta, h), Z_-(\cdot, \beta, h)](0)}.$$
The estimate \( \xi(3.17) \) is given by the formulae (3.35), where the phases \( X \) in independent functions (cf. Remark 2.16), for using (3.31), a straightforward calculation yields Remark 3.13.

Proof. Simply use formulae (3.32).

**Lemma 3.12.** The matrix \( \tau \) formed by the connection coefficients in (3.29), (3.30) satisfies

\[
\det \tau = \det \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix} = -1 + O(h^\frac{3}{2}) \quad \text{as} \quad h \downarrow 0. \tag{3.33}
\]

**Proof.** Theorem 3.14. Under Assumption 3.1, an arbitrary real solution \( X \) of equation (3.17) on the \( \zeta \)-interval corresponding to the finite \( x \)-well \((w_-, w_+)\) of \( A \), can be examined through the functions \( Y_+, Z_+ \) and \( Y_-, Z_- \). Since \( \{Y_+, Z_+\} \) and \( \{Y_-, Z_-\} \) are two sets of linearly independent functions (cf. Remark 2.16), for \( X \) we can write

\[
X(\zeta, \beta, h) = \gamma_1(\beta, h)Y_+(\zeta, \beta, h) + \delta_1(\beta, h)Z_+(\zeta, \beta, h)
= \gamma_2(\beta, h)Y_-(\zeta, \beta, h) + \delta_2(\beta, h)Z_-(\zeta, \beta, h) \tag{3.35}
\]

for some \( \gamma_j(\beta, h), \delta_j(\beta, h) \in \mathbb{R}, j = 1, 2 \). For \( j = 1, 2 \) we put

\[
\begin{align*}
v_j(\beta, h) &= \sqrt{\gamma_j(\beta, h)^2 + \delta_j(\beta, h)^2} \\
\gamma_j(\beta, h) &= v_j(\beta, h) \cos \xi_j(\beta, h) \\
\delta_j(\beta, h) &= v_j(\beta, h) \sin \xi_j(\beta, h) \tag{3.36}
\end{align*}
\]

We start with a theorem.

**Theorem 3.14.** Under Assumption 3.1, an arbitrary real solution \( X \) of equation (3.17) is given by the formulae (3.35), where the phases \( \xi_j, j = 1, 2 \) satisfy the estimate

\[
\sin \xi_1(\beta, h) \sin \xi_2(\beta, h) = O(h^\frac{3}{2}) \quad \text{as} \quad h \downarrow 0. \tag{3.37}
\]

**Proof.** We start with (3.35), i.e.

\[
\gamma_1 Y_+ + \delta_1 Z_+ = \gamma_2 Y_- + \delta_2 Z_-
\]

where we do not mention the dependence on \( \beta, h \) for simplicity and take the Wronskian of both sides with \( Y_+ \). Using (3.36), (3.31) and (3.34) we see that

\[
\det \tau \cdot v_1 \cdot \sin \xi_1 = v_2 \cdot (\tau_{11} \sin \xi_2 - \tau_{12} \cos \xi_2).
\]

Finally, relying on (3.33), (3.32) and (B.15) we obtain

\[
v_1(\beta, h) \sin \xi_1(\beta, h) = v_2(\beta, h)O(h^\frac{3}{2}) \quad \text{as} \quad h \downarrow 0.
\]

Similarly, one has

\[
v_2(\beta, h) \sin \xi_2(\beta, h) = v_1(\beta, h)O(h^\frac{3}{2}) \quad \text{as} \quad h \downarrow 0.
\]
Multiplying the last two equations and neglecting the common factor \( v_1(\beta, \hbar)v_2(\beta, \hbar) \) we arrive at the desired result. \( \Box \)

The theorem above gives rise to the next corollary, the proof of which is straightforward.

**Corollary 3.15.** For every \( \hbar > 0 \), at least one of the phases \( \xi_j, j = 1, 2 \) satisfies the condition

\[
\sin \xi_j(\beta, \hbar) = \mathcal{O}(\hbar^{2/3}) \quad \text{as} \quad \hbar \downarrow 0. \tag{3.38}
\]

The results above can be reformulated in the following theorem.

**Theorem 3.16.** Under Assumption 3.1, an arbitrary real solution \( X \) of equation (3.17) admits

\[
X(\zeta, \beta, \hbar) = \gamma_2(\beta, \hbar)Y_-(\zeta, \beta, \hbar) + \delta_2(\beta, \hbar)Z_-(\zeta, \beta, \hbar) \quad \text{for} \quad \zeta \in (\zeta_1, \zeta(w_-)) \tag{3.39}
\]

and

\[
X(\zeta, \beta, \hbar) = \gamma_1(\beta, \hbar)Y_+(\zeta, \beta, \hbar) + \delta_1(\beta, \hbar)Z_+(\zeta, \beta, \hbar) \quad \text{for} \quad \zeta \in (\zeta(w_+), \zeta_2) \tag{3.40}
\]

where for the phases in (3.36) we have

\[
\sin \xi_j(\beta, \hbar) = \mathcal{O}(\hbar^{2/3}), \quad \text{as} \quad \hbar \downarrow 0 \tag{3.41}
\]

at least for one \( j = 1, 2 \). We call (3.41) a **fixing condition**.

4. **Using the Liouville Transform for our Problem**

In this section we show how our initial problem can be shaped to one for which the Liouville transform (i.e. Olver’s theory) can be applied. After some preparatory notational comments, we state the problem explicitly and transform it to a new one relevant with that ones on paragraphs §§2, 3. The main assumption that shall be used for the potential of our Dirac operator is the following.

**Assumption 4.1.** The function \( A : \mathbb{R} \to \mathbb{R} \) is positive, of class \( C^4(\mathbb{R}) \) and such that \( \lim_{x \to \pm \infty} A(x) = 0 \). It has finitely many local extrema and a maximum denoted by \( A_{\max} \). Furthermore, in some neighborhoods of these extrema it is of class \( C^5 \). Additionally, at these extreme points \( A' \) vanishes, while \( A'' \) is either positive (leading to local minima) or negative (for local maxima and maximum). Also, for \( \mu > 0 \), equation \( A(x) = \mu \) has only finitely many solutions. Finally, there exists a number \( \tau > 0 \) such that as \( |x| \to +\infty \) we have

\[
A(x) = \mathcal{O}\left(\frac{1}{|x|^{1+\tau}}\right) \\
A'(x) = \mathcal{O}\left(\frac{1}{|x|^{3+\tau}}\right) \\
A''(x) = \mathcal{O}\left(\frac{1}{|x|^{3+\tau}}\right).
\]

\(^3\)This maximum can be realized in more than one points simultaneously.
4.1. **Notation.** Let us begin by fixing some notation so that we can use it for our purposes. The zeros of equation $A(x) = \mu$ for $\mu > 0$ can either be simple or double (when they hit an extreme point). Let us first deal with the (non-critical) case where all the zeros of this equation are simple. In such a case, there is a number $L \in \mathbb{N}$ so that we can set $x_{\ell}^- = x_{\ell}^+(\mu)$, $\ell = 1, \ldots, L$, for these solutions. We enumerate them as follows (see Figure 7)

$$x_1^- < x_1^+ < x_2^- < x_2^+ < \cdots < x_L^- < x_L^+.$$ 

Obviously, the number $L$ counts the number of finite barriers that are present. Hence, this yields $L$ barriers $\mathcal{B}_\ell(\mu) = (x_{\ell}^-(\mu), x_{\ell}^+(\mu))$, $\ell = 1, \ldots, L$ of finite width (finite barriers) separated by $L-1$ wells $\mathcal{W}_\ell(\mu) = (x_{\ell}^+(\mu), x_{\ell+1}^-(\mu))$, $\ell = 1, \ldots, L-1$ of finite width (finite wells). We also have two infinite wells (i.e. wells of infinite width) $\mathcal{W}_0(\mu) = (-\infty, x_1^{-}(\mu))$ and $\mathcal{W}_L(\mu) = (x_L^{+}(\mu), +\infty)$. Observe that $\pm A'(x_{\ell}^{\pm}(\mu)) < 0$ for all $\ell = 1, \ldots, L$. Also, let $b_{\ell}^0(\mu) \in \mathcal{B}_\ell(\mu)$, $\ell = 1, \ldots, L$ and $w_{\ell}^0(\mu) \in \mathcal{W}_\ell(\mu)$, $\ell = 1, \ldots, L-1$ denote the points where $A$ has its extremes.

Using this notation, we define for $\ell = 1, \ldots, L$ the intervals

$$I_{\ell}^- = (x_1^- (\mu), b_{\ell}^0(\mu)) \quad \text{and} \quad I_{\ell}^+ = (b_{\ell}^0(\mu), x_1^+(\mu))$$ 

and for $\ell = 1, \ldots, L-1$ the intervals

$$J_{\ell}^- = (x_{\ell}^+(\mu), w_{\ell}^0(\mu)) \quad \text{and} \quad J_{\ell}^+ = (w_{\ell}^0(\mu), x_{\ell+1}^-(\mu))$$

Having done this, we define for $\ell = 1, \ldots, L$ the functions

$$F_{\ell}^\pm = \left( A|_{\text{clos}(I_{\ell}^\pm)} \right)^{-1}$$
and for \( \ell = 1, \ldots, L - 1 \) the functions

\[
G^\pm_\ell = \left( A|_{\text{clos}(J^\pm_\ell)} \right)^{-1}.
\]

Lastly, for each such barrier, we introduce the function

\[
\Phi_\ell(\mu) = \int_{x^-_\ell(\mu)}^{x^+_\ell(\mu)} \sqrt{A(t)^2 - \mu^2} \, dt.
\]

(4.1)

It is easy to check that \( \Phi_\ell \) is \( C^1 \). Moreover, differentiating (4.1) and using the relations \( A(x^+_\ell) = \mu \), we obtain

\[
\frac{d\Phi_\ell}{d\mu}(\mu) = -2\mu \int_{x^-_\ell(\mu)}^{x^+_\ell(\mu)} \left[ A(t)^2 - \mu^2 \right]^{-1/2} \, dt < 0.
\]

(4.2)

Thus, \( \Phi_\ell \) is a one-to-one mapping.

Let us now pass to the case of double zeros. In such a case, we hit local minima and/or local maxima. Without any loss of generality and for clarity and simplicity of notation, we shall deal with the case of a potential function with two humps presented in Figure 8. In this situation we have a potential \( A \) that attains a single local minimum \( m_1 \) and two local maxima \( M_1 < M_2 \), the biggest of which is the total maximum. Let us examine in detail the two (critical) situations of hitting either a local minimum or a local maximum.

- **Hitting a local minimum**
  
  When \( 0 < \mu < m_1 \) (cf. subfigure 8a) we have only one finite barrier \( (x^-_1(\mu), x^+_1(\mu)) \). When \( \mu \) grows to reach \( m_1 \), equation \( A(x) = m_1 \) has now three zeros; two simple at \( x^+_1(m_1) \) and one double at \( x^0_1(m_1) \) (see subfigure 8b). Observe that in such a case, there emerges a new point \( x^0_1 \) between \( x^-_1 < x^+_1 \) that previously (i.e. when \( \mu < m_1 \)) defined the barrier.

- **Hitting a local maximum**
  
  When \( M_1 < \mu < M_2 \) (cf. subfigure 8c) we see that we again have only one finite barrier \( (x^-_1(\mu), x^+_1(\mu)) \). When \( \mu \) grows to reach \( M_2 \), equation \( A(x) = M_2 \) has now only one zero; a double one at \( x_1 \). (see subfigure 8d). Observe that in such a case, the two points that previously (i.e. when \( M_1 < \mu < M_2 \)) defined a barrier coalesce to a single point \( x_1 \). The same behavior is observed in subfigures 8e and 8f when \( \mu = M_1 \). In this latter case we are left with a double zero \( x_1(M_1) \) and a finite barrier having as endpoints the simple zeros \( x^-_2(M_1) \) and \( x^+_2(M_1) \).

The general case follows exactly by arguing along the same lines of the observations just made. In short, when we hit a local minimum, a new point is being created in a barrier, while when we hit a local maximum, a barrier is suppressed to a point.
4.2. Statement of the problem. We study the problem

$$D_h[u] = \lambda u$$

(4.3)
where $\mathcal{D}_h$ is the following Dirac (or Zakharov-Shabat) operator

$$
\mathcal{D}_h = \begin{bmatrix}
i h \partial_x & -iA \\
iA & -ih \partial_x
\end{bmatrix}
$$

with $h$ a positive parameter, $A$ a function satisfying Assumption 4.1 and $u = [u_1 \ u_2]^T$ a function from $\mathbb{R}$ to $\mathbb{C}^2$. As usual, $\lambda \in \mathbb{C}$ plays the role of the spectral parameter.

To be more precise, we treat $\mathcal{D}_h$ as a densely defined operator on $L^2(\mathbb{R}; \mathbb{C}^2)$ and want to investigate the EVs of problem (4.3) as $h \downarrow 0$. So, first we explain what we mean when we talk about eigenvalues of this equation.

**Definition 4.2.** For a fixed $h > 0$, we say that $\lambda \in \mathbb{C}$ is an eigenvalue (EV) of the operator $\mathcal{D}_h$ in (4.4), if equation (4.3) -with this value of $\lambda$- has a non-trivial solution $u = [u_1 \ u_2]^T \in L^2(\mathbb{R}; \mathbb{C}^2)$; that is

$$0 < \int_{-\infty}^{+\infty} \left[ |u_1(x)|^2 + |u_2(x)|^2 \right] dx < +\infty.$$

In general, a non-self-adjoint operator like $\mathcal{D}_h$ has complex EVs. For such an operator (with a potential $A$ satisfying Assumption 4.1), we know the following about its spectrum (see article [13] by Klaus and Shaw and [9] by Hirota and Wittsten).

- If $\mathcal{D}_h$ has EVs, then there is a purely imaginary EV whose imaginary part is strictly larger than the imaginary part of any other EV.
- The EV formation threshold is

$$h^{-1} \|A\|_{L^1(\mathbb{R})} > \frac{\pi}{2}$$

and is hence always achieved for sufficiently small $h$.
- Let $N$ be the largest nonnegative integer such that

$$h^{-1} \|A\|_{L^1(\mathbb{R})} > (2N - 1)\frac{\pi}{2}.$$

Then there are at least $N$ purely imaginary EVs.
- The spectrum of $\mathcal{D}_h$ is symmetric with respect to reflection in $\mathbb{R}$.
- The continuous (essential) spectrum consists of the entire real line $\mathbb{R}$, i.e.

$$\sigma_{ess}(\mathcal{D}_h) = \mathbb{R}.$$

- Apart from the origin, there are no real EVs.

We proceed by supposing the following for our Dirac operator.

**Hypothesis 4.3.** There exists a positive number $h_0$, such that for every $0 < h < h_0$, $\mathcal{D}_h$ has only purely imaginary EVs. Equivalently stated, for the point spectrum of $\mathcal{D}_h$ we suppose that

$$\exists h_0 > 0 \text{ such that } \forall 0 < h < h_0, \quad \sigma_p(\mathcal{D}_h) \subset i[-A_{\text{max}}, A_{\text{max}}].$$

We conjecture that the forementioned hypothesis is always true for all potentials $A$ satisfying Assumption 4.1. It has been proved in the case of two lobes [9] and it looks probable that the argument can be extended to the multi-hump case. Hence, from now on we always assume that $0 < h < h_0$. 


4.3. **Trasforming spectral parameter & changing variables.** From now on, we assume that the Hypothesis 4.3 is satisfied. Recall that the spectrum of $D_\hbar$ is symmetric with respect to reflection in $\mathbb{R}$. Hence, we start by changing the spectral parameter $\lambda \in i(0, A_{\text{max}}]$ to $\mu \in \mathbb{R}_+$ by setting

$$\lambda = i\mu.$$  \hspace{1cm} (4.5)

Hence, (4.3) is written as

$$\hbar \begin{bmatrix} u'_1(x, \mu, \hbar) \\ u'_2(x, \mu, \hbar) \end{bmatrix} = \begin{bmatrix} \mu & A(x) \\ -A(x) & -\mu \end{bmatrix} \begin{bmatrix} u_1(x, \mu, \hbar) \\ u_2(x, \mu, \hbar) \end{bmatrix}, \quad x \in \mathbb{R}. \tag{4.6}$$

Under the change of variables (cf. equation (4) in [19])

$$y_{\pm} = \frac{u_2 \pm u_1}{\sqrt{A \mp \mu}},$$ \hspace{1cm} (4.7)

system (4.6) is equivalent to the following two independent eigenvalue equations

$$y''_{\pm}(x, \mu, \hbar) = \left\{ \hbar^{-2}[\mu^2 - A^2(x)] + \frac{3}{4} A'(x) \right\} y_{\pm}(x, \mu, \hbar), \quad x \in \mathbb{R}. \tag{4.8}$$

Since $A(x) \in \mathbb{R}_+, \quad x \in \mathbb{R}$, we will only consider the “minus” case for the lower index in (4.8) and thus work with the equation

$$\frac{d^2y}{dx^2} = \left\{ \hbar^{-2}[\mu^2 - A^2(x)] + \frac{3}{4} A'(x) \right\} y, \quad x \in \mathbb{R}. \tag{4.9}$$

Observe that the change of variables (4.7) does not alter the discrete spectrum. Hence we are led to the following important fact.

**Proposition 4.4.** Under Assumption 4.1 and Hypothesis 4.3, finding the discrete spectrum of $D_\hbar$ in (4.4), is equivalent to finding the values $\mu \in (0, A_{\text{max}}]$ for which (4.9) has an $L^2(\mathbb{R}; \mathbb{C})$ solution.

4.4. **Reformulating the equation.** We introduce another symbol for the spectral parameter in order to rely on the results from sections §2 and §3. Recall that we started with $\lambda$ and then changed to $\mu$ using (4.5).

In a neighborhood of a finite (noncritical) barrier $B_\ell = (x_\ell^-, x_\ell^+)$, $\ell = 1, \ldots, L$, equation (4.9) can be written as

$$\frac{d^2y}{dx^2} = [\hbar^{-2}f(x, x_\ell^+) + g(x, x_\ell^+)]y$$

where $f$ and $g$ satisfy

$$f(x, x_\ell^+) = A^2(x_\ell^+) - A^2(x)$$

and

$$g(x, x_\ell^+) = \frac{3}{4} A'(x_\ell^+) - \frac{1}{2} \frac{A''(x_\ell^+)}{A(x_\ell^+) + A(x_\ell^+)}. \tag{4.10}$$

This simply says that in a neighborhood of a barrier, we can use the results obtained in §2.

Similarly, in a neighborhood of a finite (noncritical) well $W_\ell = (x_\ell^+, x_{\ell+1}^-)$, $\ell = 1, \ldots, L - 1$, equation (4.9) can be put in the form

$$\frac{d^2y}{dx^2} = [\hbar^{-2}f(x, x_{\ell+1}^-) + g(x, x_{\ell+1}^-)]y$$

and
where $f$ and $g$ satisfy
\[ f(x, x_L^+) = A^2(x_L^+) - A^2(x) \]
and
\[ g(x, x_L^+) = \frac{3}{4} \left[ \frac{A'(x)}{A(x) + A(x_L^-)} \right]^2 - \frac{1}{2} \frac{A''(x)}{A(x) + A(x_L^-)}. \]

But as before, this guarantees us that in a neighborhood of a well, all the results of §3 can be used freely.

From paragraphs §2 and §3 we know that after applying the Liouville transform, the above differential equations are transformed correspondingly to another of the form
\[ \frac{d^2 X}{d\zeta^2} = \left[ \pm \hbar^{-2}(\zeta^2 - \gamma^2) + \varphi(\zeta, \gamma) \right] X \quad (4.10) \]
where for the “+” sign, $\gamma = \alpha$ and $\varphi = \psi$ (cf. §2.1) while for the “−” case, $\gamma = \beta$ and $\varphi = \bar{\psi}$ (see §3.1). Hence having in mind Proposition 4.4 we are led to the following.

**Proposition 4.5.** Under Assumption 4.1 and Hypothesis 4.3, finding the discrete spectrum of $\mathcal{D}_h$ in (4.4) is equivalent to referring to equation (4.10) for the “+” sign with $\gamma = \alpha$, $\varphi = \psi$ and finding the values $\alpha \geq 0$ for which it possesses an $L^2(\mathbb{R}; \mathbb{C})$ solution.

5. Semiclassical Spectral Results for Multiple Barriers

In this section, we use the results from paragraphs §2 and §3 to study the EVs and their corresponding norming constants of a Dirac operator with potential $A_h$. Here, we let this potential have multiple humps (see Figure 7). To be precise, we assume the following.

5.1. Quantization conditions for the EVs. In this subsection, using Assumption 4.1, Hypothesis 4.3 and what we have gathered so far, we present the results for the EVs of the Dirac operator $\mathcal{D}_h$.

**Theorem 5.1.** Consider a potential $A$ of the Dirac operator $\mathcal{D}_h$ in (4.4) that satisfies Assumption 4.1. Also assume Hypothesis 4.3 and take $0 < \mu_1 < \mu_2 \leq A_{\text{max}}$. Suppose that $\lambda = i \mu \in i[\mu_1, \mu_2]$ (where $\lambda = \lambda(h)$ and $\mu = \mu(h)$) is an EV of $\mathcal{D}_h$. Then using the notation from §4.4, at least for one $\ell = \ell(h) \in \{1, 2, \ldots, L\}$, there is a non-negative integer $n = n(\mu, \ell, h)$ such that
\[ \Phi_{\ell}(\mu) = \pi \left( n + \frac{1}{2} \right) \hbar + O(\hbar^2) \quad \text{as} \quad \hbar \downarrow 0. \quad (5.1) \]

**Proof.** From Theorem 3.16 we see that each well $\mathcal{W}_\ell(\mu)$, $\ell = 1, \ldots, L - 1$ yields at least one fixing condition (cf. §4.11). Moreover, the asymptotic form of $Y_+(\zeta, \alpha(\mu), h)$ as $\zeta \to +\infty$ and the asymptotics for $Y_-(\zeta, \alpha(\mu), h)$ and $Z_-(\zeta, \alpha(\mu), h)$ as $\zeta \to -\infty$ (see §2.30 and §2.31) show that in the presence of an EV, the coefficient $\sigma_{12}$ in equation (2.32) has to be zero. But this is translated to the fact that the fixing conditions are fulfilled at the right point $x_+^\ell(\mu)$ of the well $\mathcal{W}_\ell(\mu)$ and at the left point $x_-^\ell(\mu)$ of the well $\mathcal{W}_\ell(\mu)$. Thus for every $0 < h < h_0$, we have at least $L + 1$ fixing conditions for $L$ barriers. Hence, there exists a barrier $\mathcal{B}_\ell(\mu, h)$ (depending on $h$ as well) for which a fixing condition is satisfied on each of its two ends. Finally, we refer to Theorem 2.19 which gives us the desired results. □
Earlier, from formula (4.2) we saw that $\Phi_\ell$ is a one-to-one mapping. Whence there exists the inverse $\Phi_\ell^{-1}$. Using this remark, we can write (5.1) equivalently as

$$\mu(h) = \Phi_\ell^{-1}\left[\pi\left(n + \frac{1}{2}\right)h\right] + \mathcal{O}(h^{\frac{5}{3}}) \quad \text{as} \quad h \downarrow 0. \quad (5.2)$$

This formula (5.2) leads to the following definition about WKB EVs. In a sense, a WKB EV approximates one actual EV of our operator.

**Definition 5.2.** If $\lambda(h) = i\mu(h)$ is an EV of $D_h$, then from Theorem 5.1 there exists at least one $\ell \in \{1, 2, \ldots, L\}$ so that formula (5.2) is true for some $n \in \mathbb{N}$. Then the number

$$\lambda_{\ell,n}^{WKB}(h) = i\mu_{\ell,n}^{WKB}(h) = i\Phi_\ell^{-1}\left[\pi\left(n + \frac{1}{2}\right)h\right] \quad (5.3)$$

shall be defined to be a WKB eigenvalue related to the actual EV $\lambda(h) = i\mu(h)$.

Consider now the intervals

$$\Delta_{\ell,n}(h) = \left(\mu_{\ell,n}^{WKB}(h) - ch^{\frac{5}{3}}, \mu_{\ell,n}^{WKB}(h) + Ch^{\frac{5}{3}}\right) \quad (5.4)$$

for some arbitrary $h$-independent constants $c, C > 0$. The lengths of these intervals are of order $\mathcal{O}(h^{\frac{5}{3}})$ while for different $m, n \in \mathbb{N}$ the distances between the points $\mu_{\ell,m}^{WKB}(h)$ and $\mu_{\ell,n}^{WKB}(h)$ are of order $\mathcal{O}(h)$. This says that for sufficiently small $h$

$$\Delta_{\ell,m}(h) \cap \Delta_{\ell,n}(h) = \emptyset \quad \text{for} \quad m \neq n.$$

However if we consider different $k, \ell \in \{1, 2, \ldots, L\}$, it may occur that

$$\Delta_{k,n}(h) \cap \Delta_{\ell,n}(h) \neq \emptyset \quad \text{for} \quad k \neq \ell.$$

Theorem 5.1 says that each EV of $D_h$ belongs to one of the intervals $i\Delta_{\ell,n}(h)$. Hence for sufficiently small $h$, equivalently stated this can be written as

$$\sigma_p(D_h) \cap i(\mu_1, \mu_2) \subset \bigcup_{\ell=1}^{L} \bigcup_{n} i\Delta_{\ell,n}(h)$$

or

$$\text{dist}\left\{\sigma_p(D_h) \cap i(\mu_1, \mu_2), \bigcup_{\ell=1}^{L} \bigcup_{n} \{\lambda_{\ell,n}^{WKB}(h)|n \in \mathbb{N}\}\right\} = \mathcal{O}(h^{\frac{5}{3}}) \quad \text{as} \quad h \downarrow 0.$$

Now, we shall be concerned with the converse; namely the existence of an actual EV for our operator in (4.4). We have the following theorem.

**Theorem 5.3.** Let Assumption 4.1 be satisfied by $A$ and assume Hypothesis 4.3 for $D_h$. Then for every $\ell \in \{1, 2, \ldots, L\}$ and every non-negative integer $n$ such that

$$\Phi_\ell^{-1}\left[\pi\left(n + \frac{1}{2}\right)h\right] \in (\mu_1, A_{\max}) \quad (5.5)$$

there exists an EV of $D_h$, namely $\lambda = i\mu$ (where $\lambda = \lambda(\ell, n, h)$ and $\mu = \mu(\ell, n, h)$), that satisfies

$$\lambda = \lambda_{\ell,n}^{WKB}(h) + \mathcal{O}(h^{\frac{5}{3}}) \quad \text{as} \quad h \downarrow 0.$$
Proof. Fix some $\ell \in \{1, 2, \ldots, L\}$ and some non-negative integer $n$ so that (5.5) is true. By Theorem 2.21 there exists $\mu = \mu(\ell, n, \hbar)$ obeying
\[
\mu = \Phi^{-1}_\ell \left[ \pi \left( n + \frac{1}{2} \right) \hbar \right] + \mathcal{O}(\hbar^{3/2}) \quad \text{as} \quad \hbar \downarrow 0
\] (5.6)
and such that
\[
Y_-(\zeta, \alpha_\ell(\mu), \hbar) = \sigma(\alpha_\ell(\mu), \hbar) Y_+ (\zeta, \alpha_\ell(\mu), \hbar)
\]
where
\[
\alpha_\ell(\mu) = \sqrt{\frac{2}{\pi}} \Phi_\ell(\mu) \quad \text{and}
\]
\[
\sigma(\alpha_\ell(\mu), \hbar) = (-1)^n + \mathcal{O}(\hbar^{3/2}) \quad \text{as} \quad \hbar \downarrow 0.
\]

Fix this value of $\mu$. Let a cut-off function $\chi_\ell \in C^\infty_0(\mathbb{R})$ be such that $\chi_\ell(\zeta) = 1$ in some neighborhood of the interval $\zeta(\mathcal{W}_\ell(\mu))$ (recall that $\zeta$ is a continuous and increasing function of $x$) and $\chi_\ell(\zeta) = 0$ outside of some larger neighborhood of that interval. Particularly, then we have $\chi_\ell(\zeta) = 0$ on all other intervals $\zeta(\mathcal{W}_k(\mu))$ with $k \neq \ell$. We set
\[
f_{\ell, n}(\zeta, \hbar) = Y_-(\zeta, \alpha_\ell(\mu), \hbar) \chi_\ell(\zeta).
\]
Observe that $f_{\ell, n}(\cdot, \hbar) \in C^2_0(\mathbb{R})$. Since the function $Y_-(\zeta, \alpha_\ell(\mu), \hbar)$ satisfies
\[
d^2 Y_- \quad \text{for} \quad f_{\ell, n}
\]
we have
\[
\frac{d^2 f_{\ell, n}}{d\zeta^2} = \left[ h^{-2} (\zeta^2 - \alpha_\ell^2(\mu)) + \psi(\zeta, \alpha_\ell(\mu)) \right] Y_-
\]
for $f_{\ell, n}$ we have
\[
2 \frac{dY_-}{d\zeta} \frac{d\chi_\ell}{d\zeta} + Y_- \frac{d^2 \chi_\ell}{d\zeta^2} = \sigma(\alpha_\ell(\mu), \hbar) \left( 2 \frac{dY_+}{d\zeta} \frac{d\chi_\ell}{d\zeta} + Y_+ \frac{d^2 \chi_\ell}{d\zeta^2} \right).
\]

**Figure 9.** Barriers and wells in $x$-space and Liouville space.

Due to the derivatives of $\chi_\ell$, the expression above differs from zero only on compact subsets of the intervals $\zeta(\mathcal{W}_{\ell-1}(\mu))$ and $\zeta(\mathcal{W}_\ell(\mu))$ (see Figure 9). But the definitions of $Y_-$ and $\sigma$ along with (5.2) show that this expression tends to zero as $\hbar \downarrow 0$ on both $\zeta(\mathcal{W}_{\ell-1}(\mu))$ and $\zeta(\mathcal{W}_\ell(\mu))$. This shows (cf. Proposition 4.5) that
\( \lambda = i\mu \) is the desired EV. Finally, using the definition (5.3) and (5.6) we find that \( \lambda \) satisfies the specified asymptotics as \( \hbar \downarrow 0 \). \( \square \)

**Remark 5.4.** We cannot exclude the possibility that EVs coming from different barriers (i.e. different values of \( \ell \)) get too close (closer than \( O(\hbar^{5/3}) \)) or even coincide. So we could even have double EVs. But this does not affect the applications to NLS. For the semiclassical analysis of the inverse scattering, the important fact is that we have different sets of EVs from different barriers, each set with a different density.

### 5.2. Norming constants.

A straightforward application of Theorem 5.1 above, allows us to express the norming constants of the Dirac operator \( \mathcal{D}_\hbar \). In particular we see that the asymptotics obtained, agree with the ones for a bell-shaped even potential (see Chapter 3 of [10] or Corollary 10.5 in [8]). We have the following corollary.

**Corollary 5.5.** Consider the Dirac operator \( \mathcal{D}_\hbar \) as in (4.4) satisfying Assumption 4.1 and Hypothesis 4.3. Suppose that \( \lambda(\hbar) \) is an EV of this operator. Then there is a non-negative integer \( n \) (depending both on \( \lambda \) and \( \hbar \)) such that the corresponding norming constant has asymptotics

\[
(-1)^n + O(\hbar^{5/3}) \quad \text{as} \quad \hbar \downarrow 0.
\]

**Proof.** Since \( \lambda(\hbar) = i\mu(\hbar) \) is an EV, by Theorem 5.1 we arrive at (5.2) for some \( \ell \in \{1, \ldots, L\} \) and \( n \in \mathbb{N} \), i.e.

\[
\mu(\hbar) = \Phi_{\ell}^{-1} \left[ \pi \left( n + \frac{1}{2} \right) \hbar \right] + O(\hbar^{5/3}) \quad \text{as} \quad \hbar \downarrow 0.
\]

But since \( \lambda(\hbar) = i\mu(\hbar) \) is an EV, from (2.32) and (2.34) we obtain

\[
Y_-(\zeta, \alpha_\ell(\mu(\hbar)), \hbar) = \sigma_{11}(\alpha_\ell(\mu(\hbar)), \hbar) Y_+(\zeta, \alpha_\ell(\mu(\hbar)), \hbar)
\]

where

\[
\alpha_\ell(\mu(\hbar)) = \sqrt{\frac{2}{\pi}} \Phi_{\ell}(\mu(\hbar)) \quad \text{and}
\]

\[
\sigma_{11}(\alpha_\ell(\mu(\hbar)), \hbar) = \sin \left( \frac{1}{\pi} \hbar^{-1} \alpha_\ell(\mu(\hbar))^2 \right) + O(\hbar^{5/3}) \quad \text{as} \quad \hbar \downarrow 0.
\]

But from the asymptotics for \( \mu\hbar \) above (or equivalently (5.2)) we find

\[
\sigma_{11}(\alpha_\ell(\mu(\hbar)), \hbar) = (-1)^n + O(\hbar^{5/3}) \quad \text{as} \quad \hbar \downarrow 0.
\]

And this completes the proof. \( \square \)

### 5.3. Eigenvalues near zero.

For the applications to the semiclassical theory of the focusing NLS equation, it is important to understand the behavior of the EVs near 0. The whole investigation is essentially the same to the one we exploited in [8]. We repeat once more the steps here since we wish to generalize the results we presented in that work of ours.

We begin with a potential function \( A \) that satisfies Assumption 4.1. Such a function has finitely many local minima, say \( N \in \mathbb{N}_0 \), accounting for the case of a function having none \((N = 0)\); if there are \( N \in \mathbb{N} \) local minima, we denote them by \( m_j, j = 1, \ldots, N \). We set \( \tilde{m} \) to be

\[
\tilde{m} = \begin{cases} 
A_{\text{max}}, & \text{if } N = 0 \\
\min_{j \in \{1, \ldots, N\}} m_j, & \text{if } N \in \mathbb{N}.
\end{cases}
\]  (5.7)
So, in this paragraph, we would like to investigate the (semiclassical) behavior of the EVs of $D_\hbar$ (with potential $A$) that lie in $i(0, \tilde{m})$.

We start by considering $\mu(\hbar) \in (0, \tilde{m})$ so that $\mu(\hbar) \downarrow 0$ as $\hbar \downarrow 0$. We emphasize that we are in the presence of only one (finite) barrier (see Figure 10). In this setting, using (2.4), (2.5), (2.20) and having in mind that $A(b_{\pm}) = \mu$ (for the notation, consult § 2.1), we define

$$\bar{f}(x, \hbar) = f(x, b_{+}(\mu(\hbar))) = \mu(\hbar)^2 - A^2(x)$$

$$\bar{g}(x, \hbar) = g(x, b_{+}(\mu(\hbar))) = 3\left[\frac{A'(x)}{2A(x) + \mu(\hbar)}\right]^2 - \frac{1}{2} \frac{A''(x)}{A(x) + \mu(\hbar)}$$

and

$$\bar{f}(x, \hbar) = (x - b_{-}(\mu(\hbar)))(x - b_{+}(\mu(\hbar)))\bar{p}(x, \hbar).$$

where

$$\bar{p}(x, \hbar) = p(x, b_{+}(\mu(\hbar))).$$

![Figure 10](image_url). The potential barrier in a case of near zero EVs.

We apply the Liouville transform once again (as in § 2.1), and arrive at the following proposition (cf. Proposition 2.7).

**Proposition 5.6.** For every $\hbar > 0$, equation

$$\frac{d^2y}{dx^2} = [\hbar^2 \bar{f}(x, \hbar) + \bar{g}(x, \hbar)]y, \quad x \in \mathbb{R}$$

is transformed to equation

$$\frac{d^2X}{d\zeta^2} = [\hbar^{-2}(\zeta^2 - \alpha(\mu(\hbar))) + \psi(\zeta, \alpha(\mu(\hbar)))]X, \quad \zeta \in \mathbb{R}$$

in which $\zeta$ is given by the Liouville transform (2.7), $\alpha$ is given by (2.9) and the function $\psi(\zeta, \alpha(\mu(\hbar)))$ is given by the formula

$$\psi(\zeta, \alpha(\mu(\hbar))) = \frac{1}{4} \frac{3\zeta^2 + 2\alpha(\mu(\hbar))^2}{[\zeta^2 - \alpha(\mu(\hbar))^2]^2} + \frac{1}{16} \frac{\zeta^2 - \alpha(\mu(\hbar))^2}{f(x, \hbar)^3} \cdot [4\bar{f}(x, \hbar)f''(x, \hbar) - 5f'(x, \hbar)^2] \frac{\bar{g}(x, \hbar)}{f(x, \hbar)}$$
where prime denotes differentiation with respect to $x$.

**Remark 5.7.** By recalling the definition of $\alpha$ in (2.9), and the fact that $b_{\pm}(\mu(h)) \to \pm \infty$, as $h \downarrow 0$, we obtain

$$\alpha(\mu(h)) \uparrow \sqrt{\frac{2}{\pi}} ||A||_{L^1(\mathbb{R})} \text{ as } h \downarrow 0. \quad (5.10)$$

It is easy to see that for each value of $h$, the functions $\tilde{f}, g$ and $\tilde{p}$ satisfy properties $(i)$ through $(iv)$ in the proof of Lemma 2.10 in §2.2. This in turn implies -again with the help of Lemma I in [16]- that for each $h$ the function

$$\psi(\zeta, \alpha(\mu(h))) = \frac{1}{4} \frac{3\zeta^2 + 2\alpha(\mu(h))^2 - \alpha(\mu(h))^2}{f(x,h)} + \frac{1}{16} f(x,h)^3$$

$$\cdot \left[ 4f(x,h)f''(x,h) - 5f'(x,h)^2 \right] + \left[ \zeta^2 - \alpha(\mu(h))^2 \right] \frac{\tilde{g}(x,h)}{f(x,h)} \quad (5.11)$$

is continuous in the corresponding region of the $(\zeta, \alpha)$-plane.

So in order to have a conclusion such as Theorem 2.13 and eventually results like Theorem 2.19 and Theorem 2.21 we need to investigate the convergence of the integral in (2.27) (cf. proof of Theorem 2.13 or proof of Theorem 6.3 in §6 of [8]), i.e.

$$\int_0^{+\infty} \frac{\psi(t, \alpha(\mu(h))))}{\Omega(t\sqrt{2h-1})} dt. \quad (5.12)$$

Here we need to place an additional assumption on the behavior of the potential $A$ at $\pm \infty$.

**Assumption 5.8.** Suppose there are real positive numbers $1 < r^+ \leq s^+$, so that

$$\frac{C_1^+(x)}{|x|^{s^+}} \leq A(x) \leq \frac{C_2^+(x)}{|x|^{r^+}} \text{ for } x > 0$$

where $C_1^+, C_2^+$ are bounded functions and $2r^+ - s^+ > \frac{1}{3}$; and there are real positive numbers $1 < r^- \leq s^-$, so that

$$\frac{C_1^-(x)}{|x|^s} \leq A(x) \leq \frac{C_2^-(x)}{|x|^r} \text{ for } x < 0$$

where $C_1^-, C_2^-$ are bounded functions and $2r^- - s^- > \frac{1}{3}$. Alternatively, suppose there are real positive numbers $0 < r \leq s$ so that

$$C_1(x)e^{-|x|^r} \leq A(x) \leq C_2(x)e^{-|x|^r}, \quad x \in \mathbb{R}$$

where $C_1, C_2$ are bounded functions.

Finally, recall (2.13) where now $x_2 = +\infty$. It shows that $x \uparrow +\infty$ as $\zeta \uparrow +\infty$. The lemma below deals with the asymptotic behavior of $x$ as $\zeta \uparrow +\infty$. It shall be used to allow us understand the nature of $\psi$ for “big” $\zeta$.

**Lemma 5.9.** Considering $x$ as a function of $\zeta$ we see that

$$x = \frac{\zeta^2}{2\mu} \left[ 1 + O \left( \frac{\log \zeta}{\zeta^2} \right) \right] \text{ as } \zeta \uparrow +\infty \quad (5.13)$$

uniformly with respect to $\mu = A(b_{\pm}) \in (0, \tilde{m})$.

**Proof.** See Lemma 5.2 in §5 of [8]. □
It is now straightforward to check that Olver’s theory is uniformly applicable all the way to $\mu = 0$. For example, consider first the case where $A$ is rational:

$$A(x) = \frac{1}{|x|^r} \quad \text{for} \quad |x| \geq 1 \quad (5.14)$$

where $r > 1$ (clearly satisfying Assumption 2.4 and Assumption 5.8). In this case, using (5.13) we get

$$x = \frac{\zeta^2}{2\mu(h)}\left[1 + \mathcal{O}\left(\frac{\log \zeta}{\zeta^r}\right)\right] \quad \text{as} \quad \zeta \uparrow +\infty \quad (5.15)$$

while using (5.8), (5.9), (5.11), (5.13), (5.14) and (5.15) we arrive at

$$\psi(\zeta, \alpha(\mu(h))) = \psi_1(\zeta, \alpha(\mu(h)))\left[1 + \mathcal{O}\left(\frac{\log \zeta}{\zeta^r}\right)\right] \quad \text{as} \quad \zeta \uparrow +\infty \quad (5.16)$$

uniformly in $\alpha$ and consequently in $\hbar$, where

$$\psi_1(\zeta, \alpha(\mu(h))) = \frac{1}{4} \frac{3\zeta^2 + 2\alpha(\mu(h))}{\zeta^2 - \alpha(\mu(h))^2} \cdot \left[\frac{\zeta^{4r} + \frac{r^2 - 1}{2r + 1} \mu(\hbar)^{2r - 2} \zeta^{4r - 4} - \alpha(\mu(h))^2}{\zeta^{4r} - 2r^2 \mu(\hbar)^{2r - 2}}\right]$$

$$- r(2r + 1)2^{2r + 1} \mu(\hbar)^{2r - 2} \zeta^{4r - 4} \left[\frac{\zeta^2 - \alpha(\mu(h))^2}{\zeta^2 - \alpha(\mu(h))^2}\right]$$

$$- r(r + 1)2^{r + 1} \mu(\hbar)^{r - 1} \zeta^{4r - 4} \left[\frac{\zeta^2 - \alpha(\mu(h))^2}{\zeta^2 - \alpha(\mu(h))^2}\right]$$

$$- \frac{\zeta^2 - \alpha(\mu(h))^2}{\zeta^2 - \alpha(\mu(h))^2} \left[\frac{\zeta^2 - \alpha(\mu(h))^2}{\zeta^2 - \alpha(\mu(h))^2}\right]$$

Consider now the case where $A$ is exponentially decreasing:

$$A(x) = e^{-|x|^r} \quad \text{for} \quad |x| \geq 1 \quad (5.17)$$

where $r > 0$; it clearly satisfies Assumption 2.4 and Assumption 5.8. Using (5.8), (5.9), (5.11), (5.13), (5.14) and (5.15) we arrive at

$$\psi(\zeta, \alpha(\mu(h))) = \psi_2(\zeta, \alpha(\mu(h)))\left[1 + \mathcal{O}\left(\frac{\log \zeta}{\zeta^r}\right)\right] \quad \text{as} \quad \zeta \uparrow +\infty \quad (5.18)$$
uniformly in $\alpha$ and consequently in $\hbar$, where

$$
\psi_2(\zeta, \alpha(\mu(h))) = \frac{1}{4} \frac{3\zeta^2 + 2\alpha(\mu(h))^2}{\zeta^2 - \alpha(\mu(h))^2} \left[ \exp \left\{ - \frac{\zeta^{2r}}{2r\mu(h)} \right\} \right.
$$

$$
+ \frac{r}{2^r} \frac{\zeta^2 - \alpha(\mu(h))^2}{\mu(h)^{r-2}} \frac{\zeta^{2r-4}}{\mu(h)^{r-2}} \left[ \exp \left\{ - \frac{\zeta^{2r}}{2r\mu(h)} \right\} - \mu(h)^2 \right]^{3/2}.
$$

$$
\left[ \frac{r}{2^r} \frac{\zeta^{2r}}{\mu(h)^r} \exp \left\{ - \frac{\zeta^{2r}}{2r\mu(h)^r} \right\} + 2(r-1) \exp \left\{ - \frac{\zeta^{2r}}{2r-1\mu(h)^r} \right\} \right.
$$

$$
+ \frac{r}{2^r-2} \frac{\zeta^{2r}}{\mu(h)^{r-2}} - 2(r-1) \frac{\mu(h)}{2r-1}
$$

$$
\left. \right\} \right. \left. \mu(h) \right]^{3/2}.
$$

These asymptotics imply that for each $h > 0$, the integral in (5.12) converges; furthermore, this convergence is uniform in $\alpha$. Now similar computations can be easily performed for any $A$ satisfying Assumption 2.4 and Assumption 5.8. The result still remains the same. The integral in (5.12) converges uniformly in $\alpha$. A variation of Theorem 2.13 can be applied to guarantee the existence of approximate solutions in these cases too. Hence, we arrive at the following theorem.

**Theorem 5.10.** For every $h > 0$, equation

$$
\frac{d^2Y}{d\zeta^2} = \left[ \hbar^{-2} \left( \zeta^2 - \alpha(\mu(h))^2 \right) + \psi(\zeta, \alpha(\mu(h))) \right] Y
$$

(5.19)

has in the region $[0, +\infty) \times [0, \alpha(\mu(h))]$ of the $(\zeta, \alpha)$-plane solutions $Y_+$ and $Z_+$ which are continuous, have continuous first and second partial $\zeta$-derivatives, and are given by

$$
Y_+(\zeta, \alpha(\mu(h)), h) = U(\zeta \sqrt{2h^{-1}}, -\frac{1}{2}h^{-1}\alpha(\mu(h))^2) + \varepsilon(\zeta, \alpha(\mu(h)), h)
$$

$$
Z_+(\zeta, \alpha(\mu(h)), h) = U(\zeta \sqrt{2h^{-1}}, -\frac{1}{2}h^{-1}\alpha(\mu(h))^2) + \varepsilon(\zeta, \alpha(\mu(h)), h)
$$

(cf. (2.23), (2.24)) where for the remainders we have the relations

$$
\frac{\varepsilon(\zeta, \alpha(\mu(h)), h)}{M(\zeta \sqrt{2h^{-1}}, -\frac{1}{2}h^{-1}\alpha(\mu(h))^2)} \leq \frac{\varepsilon(\zeta, \alpha(\mu(h)), h)}{\sqrt{2h^{-1}N(\zeta \sqrt{2h^{-1}}, -\frac{1}{2}h^{-1}\alpha(\mu(h))^2)}}
$$

$$
\leq \frac{1}{E(\zeta \sqrt{2h^{-1}}, -\frac{1}{2}h^{-1}\alpha(\mu(h))^2)} \left( \exp \left\{ \frac{1}{2}(\pi h)^{\frac{3}{2}} l_1(-\frac{1}{2}h^{-1}\alpha(\mu(h))^2) \right\} - 1 \right)
$$
and

\[
\frac{\left|\varpi(\zeta, \alpha(\mu(h)), h)\right|}{M(\zeta \sqrt{2h^{-1}}, -\frac{1}{2} h^{-1} \alpha(\mu(h))^2)} \leq E(\zeta \sqrt{2h^{-1}}, -\frac{1}{2} h^{-1} \alpha(\mu(h))^2) \left(\exp \left\{ \frac{1}{2}(\pi h)^{\frac{1}{4}} l_1 (-\frac{1}{2} h^{-1} \alpha(\mu(h))^2) V_{0, \zeta}[H](\alpha(\mu(h)), h)\right\} - 1\right)
\]

(analogous to (2.25), (2.26)).

**Proof.** The proof follows exactly the lines of that for Theorem 2.13. One has only to observe that Theorem 2.2 comes into play and ensures that everything remains unchanged. \(\square\)

Additionally, \(l_1\) and \(V_{0, +\infty}[H]\) satisfy the same asymptotics as before (cf. (2.28), (2.29)) and consequently one obtains the same asymptotic behavior of solutions as in §2.4, namely

\[
\varepsilon(\zeta, \alpha(\mu(h)), h) = \frac{M(\zeta \sqrt{2h^{-1}}, -\frac{1}{2} h^{-1} \alpha(\mu(h))^2)}{E(\zeta \sqrt{2h^{-1}}, -\frac{1}{2} h^{-1} \alpha(\mu(h))^2)} \mathcal{O}(h^{\frac{3}{2}})
\]

\[
\varpi(\zeta, \alpha(\mu(h)), h) = E(\zeta \sqrt{2h^{-1}}, -\frac{1}{2} h^{-1} \alpha(\mu(h))^2) M(\zeta \sqrt{2h^{-1}}, -\frac{1}{2} h^{-1} \alpha(\mu(h))^2) \mathcal{O}(h^{\frac{3}{2}})
\]

\[
\frac{\partial \varpi}{\partial \zeta}(\zeta, \alpha(\mu(h)), h) = \frac{N(\zeta \sqrt{2h^{-1}}, -\frac{1}{2} h^{-1} \alpha(\mu(h))^2) M(\zeta \sqrt{2h^{-1}}, -\frac{1}{2} h^{-1} \alpha(\mu(h))^2)}{E(\zeta \sqrt{2h^{-1}}, -\frac{1}{2} h^{-1} \alpha(\mu(h))^2)} \mathcal{O}(h^{\frac{3}{2}})
\]

\[
\frac{\partial \varepsilon}{\partial \zeta}(\zeta, \alpha(\mu(h)), h) = E(\zeta \sqrt{2h^{-1}}, -\frac{1}{2} h^{-1} \alpha(\mu(h))^2) N(\zeta \sqrt{2h^{-1}}, -\frac{1}{2} h^{-1} \alpha(\mu(h))^2) \mathcal{O}(h^{\frac{3}{2}})
\]

as \(h \downarrow 0\) uniformly for \(\zeta \geq 0\) and \(\alpha\).

Arguing as in (2.5), we obtain two more solutions of (5.19), namely \(Y_-\) and \(Z_-\), satisfying

\[
Y_-(\zeta, \alpha(\mu(h)), h) = U(-\zeta \sqrt{2h^{-1}}, -\frac{1}{2} h^{-1} \alpha(\mu(h))^2) + \frac{M(-\zeta \sqrt{2h^{-1}}, -\frac{1}{2} h^{-1} \alpha(\mu(h))^2)}{E(-\zeta \sqrt{2h^{-1}}, -\frac{1}{2} h^{-1} \alpha(\mu(h))^2)} \mathcal{O}(h^{\frac{3}{2}})
\]

\[
Z_-(\zeta, \alpha(\mu(h)), h) = U(-\zeta \sqrt{2h^{-1}}, -\frac{1}{2} h^{-1} \alpha(\mu(h))^2) + E(-\zeta \sqrt{2h^{-1}}, -\frac{1}{2} h^{-1} \alpha(\mu(h))^2) M(-\zeta \sqrt{2h^{-1}}, -\frac{1}{2} h^{-1} \alpha(\mu(h))^2) \mathcal{O}(h^{\frac{3}{2}})
\]

as \(h \downarrow 0\) uniformly for \(\zeta \leq 0\) and \(\alpha\).

Consequently we have the same connection formulae (all the results of §2.5 are not altered at all). Indeed, expressing \(Y_+, Z_+\) in terms of \(Y_-, Z_-\) and writing

\[
Y_+(\zeta, \alpha(\mu(h)), h) = \sigma_{11}(\alpha(\mu(h)), h) Y_-(\zeta, \alpha(\mu(h)), h) + \sigma_{12}(\alpha(\mu(h)), h) Z_-(\zeta, \alpha(\mu(h)), h)
\]

\[
Z_+(\zeta, \alpha(\mu(h)), h) = \sigma_{21}(\alpha(\mu(h)), h) Y_-(\zeta, \alpha(\mu(h)), h) + \sigma_{22}(\alpha(\mu(h)), h) Z_-(\zeta, \alpha(\mu(h)), h)
\]

(confer (2.32), (2.33) in the same way we find that

\[
\sigma_{11}(\alpha(\mu(h)), h) = \sin\left(\frac{1}{2} \pi h^{-1} \alpha(\mu(h))^2\right) + \mathcal{O}(h^{\frac{3}{2}})
\]

\[
\sigma_{12}(\alpha(\mu(h)), h) = \cos\left(\frac{1}{2} \pi h^{-1} \alpha(\mu(h))^2\right) + \mathcal{O}(h^{\frac{3}{2}})
\]

\[
\sigma_{21}(\alpha(\mu(h)), h) = \cos\left(\frac{1}{2} \pi h^{-1} \alpha(\mu(h))^2\right) + \mathcal{O}(h^{\frac{3}{2}})
\]

\[
\sigma_{22}(\alpha(\mu(h)), h) = -\sin\left(\frac{1}{2} \pi h^{-1} \alpha(\mu(h))^2\right) + \mathcal{O}(h^{\frac{3}{2}})
\]

(like (2.34)) as \(h \downarrow 0\) uniformly for \(\alpha\).
Eventually, this means that the results of §2.6 for the EVs remain the same. But before we state this result, let us remind the reader of the function $\Phi$ in (2.35), namely

$$\Phi(\mu) = \frac{\pi}{2} \alpha(\mu)^2 = \int_{b_-(\mu)}^{b_+(\mu)} \sqrt{A(x)^2 - \mu^2} \, dx.$$  

(5.20)

where $A(b_\pm) = \mu$. We have seen that $\Phi$ is a $C^1$ one-to-one mapping satisfying

$$\frac{d\Phi}{d\mu}(\mu) = -2\mu \int_{b_-(\mu)}^{b_+(\mu)} [A(x)^2 - \mu^2]^{-1/2} \, dx < 0.$$

Now we are ready to state the main result of this section. Combining Theorem 2.19 and Theorem 2.21, we arrive at the following.

**Theorem 5.11.** Let the potential function $A$ satisfy Assumption 4.1, Assumption 5.8 and set $\tilde{m}$ as in (5.7). Suppose that $\lambda(h) = i\mu(h) \in i(0, \tilde{m})$ is an EV of the operator $\mathfrak{D}_h$ (see (4.4)). Then there exists a non-negative integer $n$ for which

$$\Phi(\mu(h)) = \pi \left( n + \frac{1}{2} \right) h + O(h^\frac{5}{3}) \quad \text{as} \quad h \downarrow 0.$$  

(5.21)

Conversely, for every non-negative integer $n$ such that $\pi(n + \frac{1}{2}) h \in (\Phi(\tilde{m}), \|A\|_{L^1(\mathbb{R})})$ (recall (5.10), (5.20)) there exists a unique EV of $\mathfrak{D}_h$, namely $\lambda_n(h) = i\mu_n(h)$, so that

$$\left| \Phi(\mu_n(h)) - \pi \left( n + \frac{1}{2} \right) h \right| \leq C h^\frac{5}{3}$$

with a constant $C$ depending neither on $n$ nor on $h$.

**Proof.** The proof of this theorem is essentially the same to the proof of Theorem 10.1 in §10 of §8. \hfill \square

Next, in the spirit of Definition 5.2, we have the following definition.

**Definition 5.12.** If $\lambda(h) = i\mu(h)$ is an EV of $\mathfrak{D}_h$, then from Theorem 5.11 there exists some $n \in \mathbb{N}$ so that formula (5.21) is true. We call the number

$$\lambda_n^{WKB}(h) = i\mu_n^{WKB}(h) = i\Phi^{-1} \left[ \pi \left( n + \frac{1}{2} \right) h \right]$$  

(5.22)

a WKB eigenvalue related to the actual EV $\lambda(h) = i\mu(h)$.

So, we have arrived at the following corollary which explains the behavior of the EVs of $\mathfrak{D}_h$ that lie near zero.

**Corollary 5.13.** Consider a function $A$ satisfying Assumption 4.1 and Assumption 5.8. Also, set $\tilde{m}$ as in (5.7). Then for every non-negative integer $n$ such that $\pi(n + \frac{1}{2}) h \in (\Phi(\tilde{m}), \|A\|_{L^1(\mathbb{R})})$, there exists a unique EV of $\mathfrak{D}_h$, namely $\lambda_n(h)$ satisfying

$$|\lambda_n(h) - \lambda_n^{WKB}(h)| = O(h^{\frac{5}{3}}) \quad \text{as} \quad h \downarrow 0$$

uniformly for $\lambda_n(h)$ in $i(0, \tilde{m})$. 
6. Reflection Coefficient

In this paragraph we consider the behavior of the reflection coefficient for our Dirac operator (4.4). This completes the investigation of the set of (semiclassical) scattering data for our operator. The results in this section were actually obtained rigorously in [8]. For completeness sake, we briefly present them here as well without proof. We remind the reader that the continuous spectrum of such a Dirac operator with a potential \( A \) satisfying the asymptotics of Assumption 4.1 at \( \pm \infty \), is the whole real line.

6.1. Reflection away from zero. Let us begin in this subsection by considering a \( \lambda \in \mathbb{R} \) that is independent of \( \hbar \). Under the change of variables

\[ y \pm = \frac{u_2 \pm u_1}{\sqrt{A \pm i\lambda}} \]

equation (4.3) -with the help of (4.4) - is transformed to the following two independent equations

\[ y''_\pm (x, \lambda, \hbar) = \left\{ \hbar^{-2}[-A^2(x) - \lambda^2] + \frac{3}{4} \left[ \frac{A'(x)}{A(x) \pm i\lambda} \right]^2 - \frac{1}{2} \frac{A''(x)}{A(x) \pm i\lambda} \right\} y_\pm(x, \lambda, \hbar). \]

Again we only consider the lower index and work with the equation

\[ -\hbar^2 \frac{d^2 y}{dx^2} = [-\hbar^{-2} \tilde{f}(x, \lambda) + \tilde{g}(x, \lambda)]y \quad (6.1) \]

where \( \tilde{f} \) and \( \tilde{g} \) satisfy

\[ \tilde{f}(x, \lambda) = A^2(x) + \lambda^2 \]

and

\[ \tilde{g}(x, \lambda) = \frac{3}{4} \frac{A'(x)}{A(x) - i\lambda}^2 - \frac{1}{2} \frac{A''(x)}{A(x) - i\lambda}. \]

Next we define the Jost solutions. Equation (6.1) can be put in the form

\[ -\hbar^2 \frac{d^2 y}{dx^2} + [-A^2(x) + \hbar^2 \tilde{g}(x, \lambda)]y = \lambda^2 y. \]

This is the Schrödinger equation with a complex potential. The Jost solutions are defined as the components of the bases \( \{ J^L_-, J^L_+ \} \) and \( \{ J^R_-, J^R_+ \} \) of the two-dimensional linear space of solutions of equation (6.1), which satisfy the asymptotic conditions

\[ J^L_\pm(x, \lambda) \sim \exp \left\{ \pm \frac{i\lambda}{\hbar} x \right\} \quad \text{as} \quad x \to -\infty \]

\[ J^R_\pm(x, \lambda) \sim \exp \left\{ \pm \frac{i\lambda}{\hbar} x \right\} \quad \text{as} \quad x \to +\infty. \]

From scattering theory, we know that the reflection coefficient \( R(\lambda, \hbar) \) for the waves incident on the potential from the right, can be expressed in terms of Wronskians of the Jost solutions. More precisely, we have

\[ R(\lambda, \hbar) = \frac{\mathcal{W}[J^L_-, J^R_-]}{\mathcal{W}[J^L_+, J^R_+]} \quad (6.2) \]

Examination of the behavior of \( R(\lambda, \hbar) \) can be achieved using the same techniques as in [8]. More precisely, for \( \lambda \in \mathbb{R} \) with \( |\lambda| \geq \delta > 0 \), we have the following theorem (the reader seeking more information and proofs, is advised to look at §12 of the aforementioned work).
Theorem 6.1. Let $A$ satisfy Assumption 4.1. The reflection coefficient of equation (6.1) as defined by (6.2), satisfies

$$R(\lambda, h) = O(h) \quad \text{as} \quad h \downarrow 0$$

uniformly for $|\lambda| \geq \delta > 0$.

6.2. Reflection close to zero. Now we turn to the case where $\lambda \in \mathbb{R}$ depends on $h$ ($\lambda = \lambda(h)$) and particularly we let $\lambda$ approach 0 like $h^b$ for an $h$-independent positive constant $b$. Arguing along the same lines as before, we arrive at the following theorem (again, for the proof see §12 in [8]).

Theorem 6.2. Let $A$ satisfy Assumption 4.1. Consider $b, s > 0$ (independent of $h$). Then the reflection coefficient of equation (6.1) as defined by (6.2), satisfies

$$R(\lambda(h), h) = O\left(h^{1-sb}\right) \quad \text{as} \quad h \downarrow 0$$

uniformly for $\lambda(h)$ in any closed interval of $[h^b, +\infty)$.

Remark 6.3. We can ensure that $b$ is as large as we want by letting $s$ very small if we are happy with a weak error estimate $O(h^c)$ for small positive $c$, as $h \downarrow 0$. We can at best guarantee asymptotics of order $O(h^{1-s})$ for small positive $s$, if we are allowed to accept a small $b$.

7. INVERSE SCATTERING AND SEMICLASSICAL NLS

According to the so-called finite gap ansatz (or more properly hypothesis) the solution $\psi(x, t)$ of (1.1) is asymptotically ($h \downarrow 0$) described (locally) as a slowly modulated $G + 1$ phase wavetrain. Setting $x = x_0 + h\hat{x}$ and $t = t_0 + \hat{t}$, so that $x_0, t_0$ are “slow” variables while $\hat{x}, \hat{t}$ are “fast” variables, there exist parameters

- $a$
- $U = (U_0, U_1, \ldots, U_G)^T$
- $k = (k_0, k_1, \ldots, k_G)^T$
- $w = (w_0, w_1, \ldots, w_G)^T$
- $Y = (Y_0, Y_1, \ldots, Y_G)^T$
- $Z = (Z_0, Z_1, \ldots, Z_G)^T$

depending on the slow variables $x_0$ and $t_0$ (but not on $\hat{x}, \hat{t}$) such that generically $\psi(x, t) = \psi(x = x_0 + h\hat{x}, t = t_0 + \hat{t})$ has the following leading order asymptotics as $h \downarrow 0$:

$$\psi(x, t) \sim a(x_0, t_0)e^{i\int_{t_0}^{t_0} \frac{U(x_0, t_0)}{h} \, dt_0}e^{i\left(k_0(x_0, t_0)\hat{x} - w_0(x_0, t_0)\hat{t}\right)}$$

$$\cdot \Theta\left(Y(x_0, t_0) + i\frac{U(x_0, t_0)}{h} + i\left(k(x_0, t_0)\hat{x} - w(x_0, t_0)\hat{t}\right)\right)$$

$$\cdot \Theta\left(Z(x_0, t_0) + i\frac{U(x_0, t_0)}{h} + i\left(k(x_0, t_0)\hat{x} - w(x_0, t_0)\hat{t}\right)\right). \quad (7.1)$$

All parameters can be defined in terms of an underlying Riemann surface $X$ which depends solely on $x_0, t_0$. The moduli of $X$ vary with $x, t$, i.e. they depend on $x_0, t_0$ but not on $h, \hat{x}, \hat{t}$. $\Theta$ is the $G$-dimensional Jacobi theta function associated with $X$. The genus of $X$ can vary with $x_0, t_0$. In fact, the $x, t$-plane is divided into open regions in each of which $G$ is constant. On the boundaries of such regions (sometimes called “caustics”; they are unions of analytic arcs), some degeneracies appear in the mathematical analysis (we may have “pinching” of the
surfaces $X$ for example) and interesting physical phenomena can appear (like the famous Peregrine rogue wave [3]). The above formulae give asymptotics which are uniform in compact $(x, t)$-sets not containing points on the caustics.

For the exact formulae for the parameters as well as the definition of the theta functions we refer to [10] or [11]. Near the caustics the correct interpretation of (1.4) requires some more work. For an analysis of the somewhat more delicate behaviour (especially for higher order terms in $\hbar$) near the first caustic see [3].

In [10] we have been able to prove the finite gap hypothesis under some technical assumptions that enabled us to proceed with the semiclassical asymptotic analysis of the inverse scattering transform (more precisely the equivalent Riemann-Hilbert formulation). Such technical assumptions were justified in [11]. In both works we assumed the possibility of an analytic extension of a function $\rho$ a priori defined on an imaginary interval, that gives the density of eigenvalues of the Dirac operator (accumulating on a compact interval on the imaginary axis). Eventually (see [7]) it was realized that the analyticity assumption could be discarded by use of a simple auxiliary scalar Riemann-Hilbert problem.

However, the above proofs have assumed that the reflection coefficient for the related Dirac operator is identically zero and that one can safely replace the actual eigenvalues by their WKB-approximants. Strictly speaking, this assumption is not true. But the results in the previous sections enable us to show that the resulting error is only $o(1)$-small as $\hbar \downarrow 0$.

In §5 we have established a 1-1 correspondence between WKB-approximants (coming from different wells and barriers) and actual eigenvalues. Furthermore the WKB-approximants are uniformly $O(h^{5/3})$-close to the actual eigenvalues. This is an analogous result to our "single-lobe" result in [8], although we should underline the fact that while in the single-lobe case it is known that eigenvalues are purely imaginary, here we state this as a hypothesis, at least for small $h$. In fact, we conjecture that the eigenvalues are always imaginary for our general multi-humped potentials as long as $h$ is small enough.

The crucial quantities considered in the analysis [10] are the “Blaschke” products

$$\prod_{n=0}^{N-1} \frac{\lambda - \lambda_n^*}{\lambda - \lambda_n}$$

where $\lambda_n$ runs over either the actual eigenvalues in the upper half-plane, or respectively their WKB approximations $\lambda_n^{WKB}$. Here $\lambda$ lies on a union of contours encircling $[-iA_{max}, iA_{max}]$ and only touching it at the point 0, transversally. It follows easily that if $|\lambda_n(h) - \lambda_n^{WKB}(h)| = O(h^{5/3})$ then

$$\frac{\lambda - \lambda_n^{WKB}}{\lambda - \lambda_n^{WKB}} = \frac{\lambda - \lambda_n^*}{\lambda - \lambda_n} + O(h^{5/3}|\lambda|)$$

and hence the two corresponding Blaschke products are $1 + O(h^{2/3}/|\lambda|)$-close (since from §5 the total number of EVs $N$ is of order $O(h^{-1/2})$), which is good enough if $\lambda$ is not too close to zero. For the somewhat intricate details concerning what happens near zero, we refer to [10]; see also the discussion of the reflection coefficient below.

In the previous section we have also shown that the reflection coefficient can be ignored as long as we are at a distance $h^b$ from 0, with any $b > 1$. On the other hand, it is worth recalling that the Jost functions and hence the reflection coefficient are defined via asymptotics of the form $\exp\{i(\lambda x + \lambda^2 t)/h\}$ as $x \to \pm \infty$. 

This shows that the Jost functions are bounded uniformly in \( \hbar \) in the region \( \lambda \hbar < 1 \). Apart from possible poles at 0 (to be discussed later), the same thing holds for the reflection coefficient.

It easily follows from the so-called “Schwarz reflection” symmetry conditions (Appendix A in [10]) that the relevant “parametrix” Riemann-Hilbert problem coming from the non-triviality of the reflection coefficient is solvable and in fact its solution is \( o(1) \) as \( \hbar \downarrow 0 \).

More precisely, for the existence of the solution of the Riemann-Hilbert factorization problem that involves only the reflection coefficient \( R \) near 0 and ignores the eigenvalues we have the following result.

**Theorem 7.1.** Let \( b > 1 \). Define a Riemann-Hilbert factorization problem as follows. Find a \( 2 \times 2 \) matrix \( m \) so that

- its entries are analytic in \( \mathbb{C} \setminus [-\hbar^b, \hbar^b] \)
- \( m_+(\lambda) = m_-(\lambda)v(\lambda) \) for \( \lambda \in [-\hbar^b, \hbar^b] \) where \( v \) is the matrix
  \[
  v(\lambda) = \begin{bmatrix}
  1 & R(\lambda) \exp\left\{-\frac{2i\lambda}{\hbar}(x + \lambda t)\right\} \\
  R^*(\lambda) \exp\left\{\frac{2i\lambda}{\hbar}(x + \lambda t)\right\} & 1 + |R(\lambda)|^2
  \end{bmatrix}
  \]
  and \( m_{\pm} \) denote the limiting values of \( m \) from above (+) and below (−)
- \( m \to I \) as \( z \to \infty \).

Then this Riemann-Hilbert factorization problem has a unique solution. The same holds if the discontinuity contour is taken to be the whole real line.

**Proof.** Follows directly from the Schwarz reflection symmetry of the contour and the jump matrix as well as the fact that \( \Re(v + v^*) \) is positive definite; see (A.6) and Theorem A.1.2 of [10]. Uniqueness follows from the fact that the determinant of \( v \) is 1. \( \square \)

The fact that the contribution from the above Riemann-Hilbert problem (with jump contour \( [-\hbar^b, \hbar^b] \)) is \( o(1) \) as \( \hbar \downarrow 0 \) comes from the uniform boundedness of the resolvent of the related singular integral operator (because of the uniform boundedness of the Jost functions and the reflection coefficient) and the \( \hbar \)-small size of the contour. This is standard Riemann-Hilbert asymptotic theory, for example see Theorem 7.103 and Corollary 7.108 in [6]. Similarly, we can now extend our result to the Riemann-Hilbert factorization problem defined on the whole real line and with the same jump as above. The crucial fact is that the jump matrix in \( \mathbb{R} \setminus [-\hbar^b, \hbar^b] \) is \( o(1) \)-close to the identity in the uniform sense; again see the proof of Corollary 7.108 in [6].

Finally, it is easy to combine the contributions of the above Riemann-Hilbert problem on the whole line and the “pure soliton” Riemann-Hilbert problem (determined by setting \( R = 0 \) but not disallowing the poles at the eigenvalues) by, say, taking the product of the two separate Riemann-Hilbert problem solutions. The fact that the solution of that with jump on the real line is \( o(1) \)-small implies that the solution of the full problem (EVs + real spectrum) is \( o(1) \)-close to the “soliton ensembles” Riemann-Hilbert problem.

It can happen (non-generically, for isolated values of \( \hbar \)) that the reflection coefficient actually has a pole singularity at 0. In other words there may be a spectral singularity at 0. In such a case one can amend the analysis by considering a small circle around 0 say of radius \( \mathcal{O}(\hbar) \) and removing the singularity exactly in the same way we have removed the poles due to the eigenvalues in [10].
The reflection coefficient of course is not analytically extensible in general but one can simply extract the singular part of the reflection coefficient which is of course rational. The main result is not affected.

Having estimated the error of the WKB approximation at the level of the scattering data, this error can be built into the Riemann-Hilbert analysis of [10] and [11] as another layer of approximation and it does not affect the final finite-gap asymptotics. The only remaining change in the inverse scattering analysis for a multi-humped potential $A$ is that the density function $\rho$ gets to be somewhat more complicated.

**Theorem 7.2.** Consider $A_{\text{max}} = \max_{x \in \mathbb{R}} A(x)$. Given $\lambda \in [0, iA_{\text{max}}]$ let

$$x_1^-(\lambda) \leq x_1^+(\lambda) \leq x_2^-(\lambda) \leq x_2^+(\lambda) \leq \cdots \leq x_L^-(\lambda) \leq x_L^+(\lambda), \quad L \in \mathbb{N}$$

(for the notation, cf. §4.4) be the real solutions of the equation $A(x)^2 + \lambda^2 = 0$ (allowing for the non-generic limiting cases $x_{l}^-(\lambda) = x_{l}^+(\lambda)$ and $x_{l+1}^-(\lambda) = x_{l}^+(\lambda)$ for some $l \in \{1, \ldots, L\}$). Also, let $\Lambda$ be the set

$$\Lambda = \{ \lambda \in (0, iA_{\text{max}}] \mid \lambda \text{ is an EV of } \mathcal{D}_\hbar \}$$

and consider the signed measure

$$d\mu_\hbar = \hbar \sum_{\lambda \in \Lambda} (\delta_{\lambda^*} - \delta_{\lambda})$$

where $\delta_x$ denotes the Dirac measure centered at $x$. Then, as $\hbar \downarrow 0$, $d\mu_\hbar$ converges to a continuous measure in the weak-* sense. More precisely

$$d\mu_\hbar \xrightarrow{\text{weak-}* \ h \downarrow 0} \rho(\lambda) \chi_{[0, iA_{\text{max}}]} d\lambda + \rho(\lambda^*) \chi_{[-iA_{\text{max}}, 0]} d\lambda$$

where the density $\rho(\lambda)$ satisfies

$$\rho(\lambda) = \frac{\lambda}{\pi} \sum_{l=1}^{L} \int_{x_l^-}^{x_l^+} \frac{dx}{(A(x)^2 + \lambda^2)^{1/2}}$$

(7.2)

**Proof.** The proof of the theorem follows directly from the results in section §5. □

It is thus clear that $\rho$ is a continuous function on $[-iA_{\text{max}}, iA_{\text{max}}]$ (our discussion in [10] shows that it is even piecewise analytic). Analyticity of $\rho$ was crucial in the proofs of [10] and [11]. But as we have shown in [7] continuity will suffice; indeed the proofs of [10] actually become more “natural” by solving an auxiliary scalar Riemann-Hilbert problem with jump across $[-iA_{\text{max}}, iA_{\text{max}}]$, so continuity is more than enough.

We can finally conclude that, at least under the extra hypothesis that the eigenvalues of the Dirac operator are imaginary, and of course the Assumption 4.1 and Assumption 5.8, the finite gap property is generically valid in the sense described above.

**Appendix A. Airy Functions**

In this section, some basic properties of Airy functions are presented. For further reading one may consult [17].
Consider the Airy equation
\[-\frac{d^2 w}{dt^2} + tw = 0, \quad t \in \mathbb{R}\]

We denote by \(Ai\) and \(Bi\) its two linearly independent solutions having the asymptotics
\[
Ai(t) = \frac{1}{2 \sqrt{\pi}} t^{-\frac{1}{4}} \exp\left(-\frac{2}{3} t^{\frac{3}{2}}\right) \left[1 + O(t^{-\frac{3}{2}})\right] \quad \text{as} \quad t \to +\infty \tag{A.1}
\]
and
\[
Bi(t) = -\frac{1}{\sqrt{\pi}} t^\frac{1}{4} \sin\left(\frac{2}{3} |t|^\frac{3}{2} - \frac{\pi}{4}\right) + O(|t|^{-\frac{7}{4}}) \quad \text{as} \quad t \to -\infty \tag{A.2}
\]

Their behavior on the opposite side of the real line is known to be
\[
Ai(t) = \frac{1}{\sqrt{\pi}} |t|^{-\frac{1}{4}} \sin\left(\frac{2}{3} |t|^\frac{3}{2} + \frac{\pi}{4}\right) + O(|t|^{-\frac{5}{4}}) \quad \text{as} \quad t \to -\infty \tag{A.3}
\]
and
\[
Bi(t) \leq C(1 + t)^{-\frac{1}{4}} \exp\left(\frac{2}{3} t^\frac{3}{2}\right), \quad t \geq 0
\]
where \(C\) is a positive constant. Observe that as \(t \to -\infty\), \(Ai\) and \(Bi\) only differ by a phase shift. Also \(Ai(t), Bi(t) > 0\) for all \(t \geq 0\). Note that all asymptotic relations (A.1), (A.2) and (A.3) can be differentiated in \(t\); for example
\[
Ai'(t) = -\frac{1}{\sqrt{\pi}} |t|^\frac{1}{4} \cos\left(\frac{2}{3} |t|^\frac{3}{2} + \frac{\pi}{4}\right) + O(|t|^{-\frac{8}{4}}) \quad \text{as} \quad t \to -\infty
\]
and
\[
Ai'(t) = -\frac{1}{2 \sqrt{\pi}} t^\frac{1}{4} \exp\left(-\frac{2}{3} t^{\frac{3}{2}}\right) \left[1 + O(t^{-\frac{3}{2}})\right] \quad \text{as} \quad t \to +\infty.
\]
Another property says that
\[ |Ai(t)| \leq C(1 + |t|)^{-\frac{1}{4}}, \quad t \in \mathbb{R} \]
where \( C \) is a positive constant. The wronskian of \( Ai, Bi \) satisfies
\[ W[Ai, Bi](t) := Ai(t)Bi'(t) - Ai'(t)Bi(t) = \frac{1}{\pi}, \quad t \in \mathbb{R}. \]

In order to have a convenient way of assessing the magnitudes of \( Ai \) and \( Bi \) we introduce a modulus function \( M \), a phase function \( \vartheta \) and a weight function \( E \) related by
\[ E(x)Ai(x) = M(x)\sin \vartheta(x), \quad \frac{1}{E(x)}Bi(x) = M(x)\cos \vartheta(x), \quad x \in \mathbb{R}. \]

Actually, we choose \( E \) as follows. Denote by \( c_\ast \) the biggest negative root of the equation \( Ai(x) = Bi(x) \) (numerical calculations show that \( c_\ast = -0.36605 \) correct up to five decimal places); then define
\[ E(x) = \begin{cases} 1, & x \leq c_\ast \\ \sqrt{\frac{Ai^2(x) + Bi^2(x)}{2Ai(x)Bi(x)}}, & x > c_\ast \end{cases} \]
with this choice in mind, \( M, \vartheta \) become
\[ M(x) = \begin{cases} \sqrt{Ai^2(x) + Bi^2(x)}, & x \leq c_\ast \\ \sqrt{2Ai(x)Bi(x)}, & x > c_\ast \end{cases} \text{ and } \vartheta(x) = \begin{cases} \arctan \left[ \frac{Ai(x)}{Bi(x)} \right], & x \leq c_\ast \\ \frac{\pi}{4}, & x > c_\ast \end{cases} \]
where the branch of the inverse tangent is continuous and equal to \( \frac{\pi}{4} \) at \( x = c_\ast \).

For these functions the asymptotics for large \(|x|\) read
\[ E(x) \sim \begin{cases} 1, & x \to -\infty \\ \sqrt{2} \exp \left\{ \frac{2}{3} t^\frac{3}{2} \right\}, & x \to +\infty \end{cases} \]
\[ M(x) \sim \frac{1}{\sqrt{\pi}} |x|^{-\frac{1}{4}}, \quad |x| \to +\infty \]
\[ \vartheta(x) = \begin{cases} \frac{2}{3} |x|^\frac{3}{2} + \frac{\pi}{4} + O \left( \frac{3}{2} |x|^{-\frac{3}{2}} \right), & x \to -\infty \\ \frac{\pi}{4}, & x \to +\infty \end{cases} \]

APPENDIX B. PARABOLIC CYLINDER FUNCTIONS & MODIFIED PARABOLIC CYLINDER FUNCTIONS

The results of the main theorems about existence of approximate solutions of the differential equations treated in the main text involve PCFs and modified PCFs (cf. [1]). So in this section we state a few properties which will be in heavy use, especially about their asymptotic character, wronskians and zeros. We prove none of them. For a rigorous exposition on PCFs and mPCFs one may consult §5 of [10] or §12 of [18] and the references therein.

B.1. PCFs. Consider Weber’s equation
\[ \frac{d^2w}{dx^2} = (\frac{1}{4} x^2 + b)w. \] (B.1)

The behavior of the solutions depends on the sign of \( b \). When \( b \) is negative then there exist two turning points \( \pm 2\sqrt{-b} \). The solutions are of oscillatory type in the interval between these points but not in the exterior intervals. When \( b > 0 \) there
are no real turning points and there are no oscillations at all. Since only the case 
\(b \leq 0\) will be of interest to us, from now on we seldom mention properties having to do with the other case.

Standard solutions of (B.1) are \(U(\pm x, b)\) and \(\overline{U}(\pm x, b)\) defined by

\[
U(\pm x, b) = \frac{\pi^{\frac{1}{2}}2^{-\frac{1}{4}(2b+1)}}{\Gamma(\frac{1}{4} + \frac{1}{2}b)} e^{-\frac{1}{4}x^2} \frac{1}{\Gamma\left(\frac{3}{4} + \frac{1}{2}b\right)} x e^{\frac{1}{4}x^2} F_1\left(\frac{3}{4} + \frac{1}{2}b; \frac{3}{2}; \frac{1}{2}x^2\right)
\]

\[
\overline{U}(\pm x, b) = \frac{\pi^{\frac{1}{2}}2^{-\frac{1}{4}(2b+1)}}{\Gamma(\frac{1}{4} + \frac{1}{2}b)} e^{-\frac{1}{4}x^2} \frac{1}{\Gamma\left(\frac{3}{4} + \frac{1}{2}b\right)} x e^{\frac{1}{4}x^2} F_1\left(\frac{3}{4} + \frac{1}{2}b; \frac{3}{2}; \frac{1}{2}x^2\right)
\]

where \(F_1\) denotes the confluent hypergeometric function (again cf. [1]). The pair \(U(x, b), \overline{U}(x, b)\) is a numerically satisfactory set of solutions (in the sense of [12]) when \(x \geq 0\) and \(b \leq 0\); both are continuous in \(x\) and \(b\) in this region.

For \(b \in \mathbb{R}\), their values at \(x = 0\) obey

\[
U(0, b) = \frac{\pi^{\frac{1}{2}}2^{-\frac{1}{4}(2b+1)}}{\Gamma(\frac{1}{4} + \frac{1}{2}b)} \sin\left(\frac{\pi}{4} - \frac{1}{2}b\pi\right)
\]

\[
U'(0, b) = -\pi^{\frac{1}{2}}2^{-\frac{1}{4}(2b-1)} \Gamma(\frac{3}{4} + \frac{1}{2}b) \sin\left(\frac{3\pi}{4} - \frac{1}{2}b\pi\right)
\]

\[
\overline{U}(0, b) = \frac{\pi^{\frac{1}{2}}2^{-\frac{1}{4}(2b+1)}}{\Gamma(\frac{1}{4} + \frac{1}{2}b)} \sin\left(\frac{\pi}{4} - \frac{1}{2}b\pi\right)
\]

\[
\overline{U}'(0, b) = -\pi^{\frac{1}{2}}2^{-\frac{1}{4}(2b-1)} \Gamma(\frac{3}{4} + \frac{1}{2}b) \sin\left(\frac{3\pi}{4} - \frac{1}{2}b\pi\right).
\]

Those values of \(b\) that make the Gamma functions in the definitions of \(U\) and \(\overline{U}\) infinite (the Gamma function has simple poles at the non-positive integers), are called exceptional values. For a fixed \(b \in \mathbb{R}\) other than an exceptional value, the behaviors of \(U\) and \(\overline{U}\) as \(x \to +\infty\) satisfy

\[
U(x, b) \sim x^{-b - \frac{1}{2}} e^{-\frac{1}{4}x^2}
\]

\[
U'(x, b) \sim -\frac{1}{2} x^{-b + \frac{1}{2}} e^{-\frac{1}{4}x^2}
\]

\[
\overline{U}(x, b) \sim \sqrt{\frac{2}{\pi}} \Gamma\left(\frac{1}{2} - b\right) x^{-\frac{3}{2}} e^{\frac{1}{2}x^2}
\]

\[
\overline{U}'(x, b) \sim (2\pi)^{-\frac{1}{2}} \Gamma\left(\frac{1}{2} - b\right) x^{b + \frac{3}{2}} e^{\frac{1}{2}x^2}.
\]

These estimates are uniform in \(b\) when \(b\) takes values over a fixed compact interval not containing exceptional values.

For the wronskian of \(U(\cdot, b), \overline{U}(\cdot, b)\) we have

\[
\mathcal{W}[U(\cdot, b), \overline{U}(\cdot, b)](x) = 2^{\frac{1}{2}} \pi^{-\frac{1}{2}} \Gamma\left(\frac{1}{2} - b\right), \quad x \in \mathbb{R}.
\]

When \(b = 0\) the standard solutions of equation (B.1) are related to the modified Bessel functions \(K_{\frac{1}{4}}\) and \(I_{\frac{1}{4}}\) in the following way. For \(x \geq 0\) we have

\[
U(x, 0) = (2\pi)^{-\frac{1}{2}} x^{\frac{1}{2}} K_{\frac{1}{4}}\left(\frac{1}{4}x^2\right)
\]

\[
\overline{U}(x, 0) = (\pi x)^{\frac{1}{2}} I_{\frac{1}{4}}\left(\frac{1}{4}x^2\right) + (2\pi x)^{-\frac{1}{2}} x^{\frac{1}{2}} K_{\frac{1}{4}}\left(\frac{1}{4}x^2\right).
\]
In order to express the character of these standard solutions for large negative $b$, we need some preparations first. Take $\nu \gg 1$ to be a large positive number and set $b = -\frac{1}{2} \nu^2$ and $x = \nu y \sqrt{2}$ where $y \geq 0$. If we consider the function $\eta$ to be

$$\eta(y) = \begin{cases} -\left[\frac{3}{2} \int_{-\infty}^{0} (1 - s^2)^{1/2} ds \right]^{\frac{3}{2}}, & 0 \leq y \leq 1 \\ \frac{3}{2} \int_{0}^{y} (s^2 - 1)^{1/2} ds \right]^{\frac{3}{2}}, & y \geq 1 \end{cases}$$

then as $\nu \to +\infty$ we have

$$U(\nu y \sqrt{2}, -\frac{1}{2} \nu^2) = \frac{2^{\frac{1}{3}} \pi^{\frac{1}{3}}}{{\nu}^{\frac{2}{3}}} \left( \frac{\eta}{y^2 - 1} \right)^{\frac{1}{3}} \left[ \text{Ai}(\nu^{\frac{2}{3}} \eta) + \frac{M(\nu^{\frac{2}{3}} \eta)\mathcal{O}(\nu^{-2})}{E(\nu^{\frac{2}{3}} \eta)} \right]$$

$$U(\nu y \sqrt{2}, -\frac{1}{2} \nu^2) = \frac{2^{\frac{1}{3}} \pi^{\frac{1}{3}}}{{\nu}^{\frac{2}{3}}} \left( \frac{\eta}{y^2 - 1} \right)^{\frac{1}{3}} \left[ \text{Bi}(\nu^{\frac{2}{3}} \eta) + M(\nu^{\frac{2}{3}} \eta)E(\nu^{\frac{2}{3}} \eta)\mathcal{O}(\nu^{-2}) \right]$$

where $\text{Ai}$, $\text{Bi}$, $E$ and $M$ are the standard Airy functions’ terminology (cf. section [A] in the appendix).

For $b \leq 0$, the number of zeros of $U(\cdot, b)$ in the interval $[0, +\infty)$ is $\left[ \frac{1}{3} - \frac{1}{2} b \right]$ while $\bar{U}(\cdot, b)$ has $\left[ \frac{1}{3} - \frac{1}{2} b \right]$, zeros in $[0, +\infty)$. Actually, the zeros of $U(\cdot, b)$ and $\bar{U}(\cdot, b)$ do not cross each other. They interlace, with the largest one belonging to $\bar{U}(\cdot, b)$. For sufficiently large $|b|$, all the real zeros of these two functions lie to the left of $2\sqrt{-b}$, the positive turning point of Weber’s equation$^4$.

To express the errors for the approximations of our problem, we need to define some auxiliary functions having to do with the nature of $U(\cdot, b)$ and $\bar{U}(\cdot, b)$ for negative $b$. In this case the character of each is partly oscillatory and partly exponential, so we introduce one weight function $\mathcal{E}$, two modulus functions $\mathcal{M}$ and $\mathcal{N}$, and finally two phase functions $\theta$ and $\omega$.

We denote by $\rho(b)$ the largest real root of the equation

$$U(x, b) = \bar{U}(x, b).$$

We know (cf. §13 of [18] and the references therein) that $\rho(0) = 0$ and $\rho(b) > 0$ for $b < 0$. Also, $\rho$ is continuous when $b \in (-\infty, 0]$. An asymptotic estimate for large negative $b$ is

$$\rho(b) = 2(-b)^{1/2} + c_*(-b)^{-\frac{1}{2}} + \mathcal{O}(b^{-\frac{3}{2}}) \quad \text{as} \quad b \to -\infty$$

where $c_*(\approx -0.36605)$ is the smallest in absolute value root of the equation $\text{Ai}(x) = \text{Bi}(x)$.

For $b \leq 0$ we define

$$\mathcal{E}(x, b) = \begin{cases} 1, & 0 \leq x \leq \rho(b) \\ \sqrt{\frac{E(x, b)}{U(x, b)}}^{1/2}, & x > \rho(b). \end{cases}$$

It is seen that $\mathcal{E}$ is continuous in the region $[0, +\infty) \times (-\infty, 0]$ of the $(x, b)$-plane and for $b \leq 0$ fixed, $\mathcal{E}(\cdot, b)$ is non-decreasing in the interval $[0, +\infty)$. Again for $b \leq 0$ and $x \geq 0$ we set

$$U(x, b) = \frac{1}{\mathcal{E}(x, b)} M(x, b) \sin \theta(x, b), \quad \bar{U}(x, b) = \mathcal{E}(x, b) M(x, b) \cos \theta(x, b)$$

$^4$For $U(\cdot, b)$, this result holds for all $b \leq 0$.  

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and
\[ U'(x, b) = \frac{1}{E(x, b)} N(x, b) \sin \omega(x, b), \quad U'(x, b) = E(x, b) N(x, b) \cos \omega(x, b). \]

Thus
\[
M(x, b) = \begin{cases} [U(x, b)^2 + U'(x, b)^2]^{1/2}, & 0 \leq x \leq \rho(b) \\ [2U(x, b)U'(x, b)]^{1/2}, & x > \rho(b) \end{cases}
\] (B.8)

and
\[
\theta(x, b) = \begin{cases} \arctan \left[ \frac{U(x, b)}{U'(x, b)} \right], & 0 \leq x \leq \rho(b) \\ \frac{\pi}{4}, & x > \rho(b) \end{cases}
\]

where the branch of the inverse tangent is continuous and equal to \( \frac{\pi}{4} \) at \( x = \rho(b) \).

Similarly
\[
N(x, b) = \begin{cases} [U'(x, b)^2 + U''(x, b)^2]^{1/2}, & 0 \leq x \leq \rho(b) \\ \left[ \frac{U'(x, b)^2 U(x, b) + U''(x, b)^2 U(x, b)^2}{U(x, b)U''(x, b)} \right]^{1/2}, & x > \rho(b) \end{cases}
\]

and
\[
\omega(x, b) = \begin{cases} \arctan \left[ \frac{U'(x, b)}{U''(x, b)} \right], & 0 \leq x \leq \rho(b) \\ \arctan \left[ \frac{U'(x, b)}{U(x, b)U'(x, b)} \right], & x > \rho(b) \end{cases}
\]

where the branches of the inverse tangents are chosen to be continuous and fixed by \( \omega(x, b) \to -\frac{\pi}{4} \) as \( x \to +\infty \).

For large \( x \) we have
\[
E(x, b) \sim \left( \frac{2}{\pi} \right)^{1/4} \Gamma \left( \frac{1}{2} - b \right)^{1/2} x^b e^{-\frac{1}{2}x^2}
\]

and
\[
M(x, b) \sim \left( \frac{8}{\pi} \right)^{1/4} \Gamma \left( \frac{1}{2} - b \right)^{1/2} x^b, \quad N(x, b) \sim \frac{\Gamma \left( \frac{1}{2} - b \right)^{1/2}}{(2\pi)^{1/4}} x^b. \] (B.9)

Both of these hold for fixed \( b \) and are also uniform for \( b \) ranging over any compact interval in \( (-\infty, 0] \).

B.2. mPCFs. Consider the equation
\[
\frac{d^2w}{dx^2} = (b - \frac{1}{2}x^2)w. \] (B.10)

When \( b > 0 \) there exist two turning points \( \pm 2\sqrt{b} \). The solutions are monotonic in the interval between these points and oscillate in the two exterior intervals. When \( b \leq 0 \) there are no real turning points and the entire real axis is an interval of oscillation. Only the case \( b \geq 0 \) will be of interest to us.
Standard solutions of $(B.10)$ are $W(\pm x, b)$ defined by

$$W(\pm x, b) = 2^{-\frac{1}{4}} \left| \frac{\Gamma\left(\frac{1}{4} + \frac{i}{2}b\right)}{\Gamma\left(\frac{3}{4} + \frac{i}{2}b\right)} \right|^\frac{1}{2} e^{\frac{i}{2}ix^2} {}_1F_1\left(\frac{1}{4} + \frac{i}{2}b; \frac{1}{2}; -\frac{i}{2}x^2\right)$$

$$\pm 2^{-\frac{1}{4}} \left| \frac{\Gamma\left(\frac{3}{4} + \frac{i}{2}b\right)}{\Gamma\left(\frac{1}{4} + \frac{i}{2}b\right)} \right|^\frac{1}{2} xe^{\frac{i}{2}ix^2} {}_1F_1\left(\frac{3}{4} + \frac{i}{2}b; \frac{3}{2}; -\frac{i}{2}x^2\right)$$

where as in $(B.1)$, $F_1$ denotes the confluent hypergeometric function (cf. [1]). A numerically satisfactory set of solutions is obtained by taking appropriate multiples of $W(\pm x, b)$. Both of them are real and continuous for all real values of $x$ and $b$.

Before presenting their basic properties that are useful to us, we fix some notation first. We set

$$k(b) = (1 + e^{2\pi b})^{\frac{1}{2}} - e^{\pi b}$$

(B.11)

and

$$\phi(b) = \frac{\pi}{4} + \frac{1}{2} \text{ph} \left\{ \Gamma\left(\frac{1}{2} + ib\right) \right\}$$

(B.12)

where it is being understood that the phase of $\Gamma\left(\frac{1}{2} + ib\right)$ in $(B.12)$ is continuous and vanishes for $b = 0$. Also we know that as $b$ increases from $-\infty$ to $+\infty$, $k(b)$ decreases monotonically from 1 to 0.

For $b \in \mathbb{R}$ and $x = 0$ we have

$$W(0, b) = 2^{-\frac{1}{4}} \left| \frac{\Gamma\left(\frac{1}{4} + \frac{i}{2}b\right)}{\Gamma\left(\frac{3}{4} + \frac{i}{2}b\right)} \right|^\frac{1}{2}$$

$$W'(0, b) = -2^{-\frac{1}{4}} \left| \frac{\Gamma\left(\frac{3}{4} + \frac{i}{2}b\right)}{\Gamma\left(\frac{1}{4} + \frac{i}{2}b\right)} \right|^\frac{1}{2} .$$

For a fixed $b \in \mathbb{R}$ the behavior of $W(\pm b)$ and $W'(\pm b)$ as $x \to +\infty$ satisfy

$$W(x, b) = \sqrt{\frac{2k(b)}{x}} \cos \left[ \frac{1}{4}x^2 - b \ln x + \phi(b) \right] + O(x^{-\frac{1}{2}})$$

(B.13)

$$W'(x, b) = -\sqrt{\frac{k(b)x}{2}} \sin \left[ \frac{1}{4}x^2 - b \ln x + \phi(b) \right] + O(x^{-\frac{1}{2}})$$

$$W(-x, b) = \sqrt{\frac{2}{k(b)x}} \sin \left[ \frac{1}{4}x^2 - b \ln x + \phi(b) \right] + O(x^{-\frac{1}{2}})$$

$$W'(-x, b) = -\sqrt{\frac{x}{2k(b)}} \cos \left[ \frac{1}{4}x^2 - b \ln x + \phi(b) \right] + O(x^{-\frac{1}{2}}).$$

These estimates are uniform in $b$ lying in any fixed compact interval.

For the wronskian of $W(\cdot, b)$, $W(-\cdot, b)$ we have

$$\mathcal{W}[W(\cdot, b), W(-\cdot, b)](x) = 1, \quad x \in \mathbb{R}.$$  

(B.14)

When $b = 0$ the standard solutions of equation $(B.10)$ are related to the Bessel functions $J_{\pm \frac{1}{4}}$ and $J_{\pm \frac{1}{4} + 2}$ in the following way. Since $k(0) = \sqrt{2} - 1$ by $(B.11)$ and
\[ \phi(0) = \frac{\pi}{4} \] by \([B.12]\), for \( x \geq 0 \) we have
\[ W(\pm x, 0) = 2^{-\frac{\pi}{4}}(\pi x)^{\frac{3}{2}}[J_{-\frac{1}{4}}(\frac{1}{4}x^2) \mp J_{\frac{1}{4}}(\frac{1}{4}x^2)] \]
\[ W'(\pm x, 0) = 2^{-\frac{\pi}{4}}\pi^2x^2[\mp J_{\frac{1}{4}}(\frac{1}{4}x^2) - J_{-\frac{1}{4}}(\frac{1}{4}x^2)]. \]

The behavior of these standard solutions for large positive \( b \) can be seen by setting \( b = \frac{1}{2}\nu^2 \) and \( x = \nu y \sqrt{2} \) where \( \nu \gg 1 \) is a large positive number and \( y \geq 0 \).

Then as \( \nu \to +\infty \) we have
\[ k\left(\frac{1}{2}\nu^2\right) = 1 + O(\nu^{-2}) \quad \text{as} \quad \nu \to +\infty \]
\[ \phi\left(\frac{1}{2}\nu^2\right) = \frac{1}{\nu^2} \ln \left(\frac{1}{2} \nu^2\right) - \frac{1}{4} \nu^2 + \frac{\pi}{4} + O(\nu^{-2}) \]
\[ k\left(\frac{1}{2}\nu^2\right)^{\frac{1}{4}} W(\nu y \sqrt{2}, \frac{1}{2} \nu^2) = \frac{2 \pi \nu}{\nu^2} \left(\frac{\nu}{y^2 - 1}\right) \left[ B(\nu^2 \eta) + M(\nu^2 \eta) \nu^2 \right] \]
\[ k\left(\frac{1}{2}\nu^2\right)^{\frac{1}{4}} W(-\nu y \sqrt{2}, \frac{1}{2} \nu^2) = \frac{2 \pi \nu}{\nu^2} \left(\frac{\nu}{y^2 - 1}\right) \left[ A(\nu^2 \eta) + M(\nu^2 \eta) \nu^2 \right] \]

where \( A, B, E \) and \( M \) are the standard Airy functions’ terminology (cf. section \([A] \) in the appendix) and \( \eta \) is as in \([B.4]\). In the last two relations, the \( O \)-terms are uniformly valid in any \( y \)-interval that includes \([0, +\infty)\).

To express the errors for the approximations in Theorem \([3.9]\) we need to define some auxiliary functions having to do with the nature of \( k(b)^{-\frac{1}{4}} W(\cdot, b) \) and \( k(b)^{\frac{1}{4}} W(\cdot, b) \) for positive \( b \). As in the case of the PCFs in \([B.1]\) we introduce one weight function \( \mathcal{E} \), two modulus functions \( \overline{M} \) and \( \overline{N} \), and finally two phase functions \( \overline{\eta} \) and \( \overline{\varphi} \).

Take \( b \geq 0 \) and denote by \( \overline{\eta}(b) \) the smallest real root in \([0, +\infty)\) of the equation
\[ k(b)^{-\frac{1}{4}} W(x, b) = k(b)^{\frac{1}{4}} W(-x, b). \]
We know (cf. \S13 of \([18]\) and the references therein) that
\[ k(b)^{-\frac{1}{4}} W(x, b) > k(b)^{\frac{1}{4}} W(-x, b) > 0, \quad 0 \leq x < \overline{\eta}(b). \]
Also, \( \overline{\eta} \) is continuous when \( b \in [0, +\infty) \).

An asymptotic estimate for large positive \( b \) is
\[ \overline{\eta}(b) = 2b^{\frac{1}{2}} - c_+ b^{-\frac{1}{4}} + O\left(b^{-\frac{3}{4}}\right) \quad \text{as} \quad b \to +\infty \]
where as in \([B.1]\) \( c_+ \approx -0.36605 \) is the smallest in absolute value root of the equation \( A(\eta) = B(\eta) \).

So for \( b \geq 0 \) we define
\[ \mathcal{E}(x, b) = \begin{cases} \mathcal{E}(x, b), & x < 0 \\ \left[\frac{k(b)W(x, b)}{W(x, b)}\right]^{1/2}, & 0 \leq x \leq \overline{\eta}(b) \\ 1, & x > \overline{\eta}(b). \end{cases} \]
It is seen that \( \mathcal{E} \) is continuous in the region \((-\infty, +\infty) \times [0, +\infty)\) of the \((x, b)\)-plane and for \( b \geq 0 \) fixed, \( \mathcal{E}(\cdot, b) \) is non-decreasing in the interval \([0, +\infty)\). Again for \( b \leq 0 \) and \( x \geq 0 \) we have
\[ k(b)^{\frac{1}{4}} \leq \mathcal{E}(x, b) \leq 1 \]
For \( b \geq 0 \) and \( x \geq 0 \), modulus and phase functions are defined by
\[
k(b)^{-\frac{1}{2}} W(x, b) = \frac{\overline{M}(x, b)}{\overline{E}(x, b)} \sin \overline{\theta}(x, b), \quad k(b)^{\frac{1}{2}} W(-x, b) = \frac{\overline{M}(x, b)}{\overline{E}(x, b)} \cos \overline{\theta}(x, b)
\]
and
\[
k(b)^{-\frac{1}{2}} W'(x, b) = \frac{\overline{N}(x, b)}{\overline{E}(x, b)} \sin \overline{\omega}(x, b), \quad k(b)^{\frac{1}{2}} W'(-x, b) = -\frac{\overline{N}(x, b)}{\overline{E}(x, b)} \cos \overline{\omega}(x, b).
\]

Thus
\[
\overline{M}(x, b) = \begin{cases} 
\left[2W(x, b)W(-x, b)\right]^{1/2}, & 0 \leq x \leq \overline{\rho}(b) \\
\left[k(b)^{-1}W(x, b)^2 + k(b)W(-x, b)^2\right]^{1/2}, & x > \overline{\rho}(b)
\end{cases}
\]
and
\[
\overline{\theta}(x, b) = \begin{cases} 
\frac{\pi}{2}, & 0 \leq x \leq \overline{\rho}(b) \\
\arctan \left[k(b)^{-1}W(x, b)\right], & x > \overline{\rho}(b)
\end{cases}
\]
where the branch of the inverse tangent is continuous and equal to \( \frac{\pi}{2} \) at \( x = \overline{\rho}(b) \).

Similarly
\[
\overline{N}(x, b) = \begin{cases} 
\left[W'(x, b)^2W(-x, b)^2 + W'(-x, b)^2W(x, b)^2\right]^{1/2}, & 0 \leq x \leq \overline{\rho}(b) \\
\left[k(b)^{-1}W'(x, b)^2 + k(b)W'(-x, b)^2\right]^{1/2}, & x > \overline{\rho}(b)
\end{cases}
\]
and
\[
\overline{\omega}(x, b) = \begin{cases} 
-\arctan \left[W'(x, b)W(-x, b)\right], & 0 \leq x \leq \overline{\rho}(b) \\
-\arctan \left[k(b)^{-1}W'(x, b)\right], & x > \overline{\rho}(b)
\end{cases}
\]
where the branches of the inverse tangents are chosen to be continuous and fixed by \( \overline{\omega}(x, b) \to -\frac{\pi}{2} \) as \( x \to +\infty \).

For large \( |x| \) we have
\[
\overline{M}(x, b) \sim \frac{2}{x} \left| \frac{1}{2} \right|^\frac{1}{2}, \quad \overline{N}(x, b) \sim \frac{x}{2} \left| \frac{1}{2} \right|^\frac{1}{2}.
\]
Both of these hold for fixed \( b \) and are also uniform for \( b \) ranging over any compact interval in \( [0, +\infty) \).

**Appendix C. A Theorem on Integral Equations**

The proofs of theorems about WKB approximation when there is an absence of turning points (like Theorems 2.1 and 2.2 in chapter 6 of [17]), may be adapted to other types of approximate solutions of linear differential equations where turning points may be present. For second-order equations the basic steps consist of

(i) construction of a Volterra integral equation for the error term -say \( h \) of the solution, by the method of variation of parameters

(ii) construction of the Liouville-Neumann expansion (a uniformly convergent series) for the solution \( h \) of the integral equation in (i) by Picard’s method of successive approximations
(iii) confirmation that $h$ is twice differentiable by construction of similar series for $h'$ and $h''$
(iv) production of bounds for $h$ and $h'$ by majoring the Liouville-Neumann expansion.

It would be tedious to carry out all these steps in every case. But we have the following general theorem which automatically provides (ii), (iii) and (iv) in problems relevant to us.

**Theorem C.1.** Consider the equation

$$h(\zeta) = \int_\beta^\zeta K(\zeta, t)\phi(t)\{J(t) + h(t)\} dt \quad (C.1)$$

for the function $h$ accompanied by the following assumptions

- the “path” of integration consists of a segment $[\beta, \zeta]$ of the real axis, finite or infinite where $\beta \leq t \leq \zeta \leq \gamma$
- the real functions $J$ and $\phi$ are continuous in $(\beta, \gamma)$ except for a finite number of discontinuities and infinities
- the real kernel $K$ and its first two partial derivatives with respect to $\zeta$ are continuous functions of both variables when $\zeta, t \in (\beta, \gamma)$
- $K(\zeta, \zeta) = 0$, $\zeta \in (\beta, \gamma)$
- when $\zeta \in (\beta, \gamma)$ and $t \in (\beta, \zeta]$ we have

$$|K(\zeta, t)| \leq P_0(\zeta)Q(t), \quad \left| \frac{\partial K(\zeta, t)}{\partial \zeta} \right| \leq P_1(\zeta)Q(t), \quad \left| \frac{\partial^2 K(\zeta, t)}{\partial \zeta^2} \right| \leq P_2(\zeta)Q(t)$$

where the $P_j, j = 0, 1, 2$ and $Q$ are continuous real functions, the $P_j, j = 0, 1, 2$ being positive.
- when $\zeta \in (\beta, \gamma)$, the integral

$$\Phi(\zeta) = \int_\beta^\zeta |\phi(t)| dt$$

converges and the following suprema

$$\kappa = \sup_{\zeta \in (\beta, \gamma)} \{Q(\zeta)|J(\zeta)|\}, \quad \kappa_0 = \sup_{\zeta \in (\beta, \gamma)} \{P_0(\zeta)Q(\zeta)\}$$

are finite.

Under these assumptions, equation $(C.1)$ has a unique solution $h$ which is continuously differentiable in $(\beta, \gamma)$ and satisfies

$$\frac{h(\zeta)}{P_0(\zeta)} \to 0, \quad \frac{h'(\zeta)}{P_1(\zeta)} \to 0 \quad \text{as} \quad \zeta \downarrow \beta.$$ 

Furthermore,

$$\frac{|h(\zeta)|}{P_0(\zeta)} \leq \frac{\kappa}{\kappa_0} \frac{|h'(\zeta)|}{P_1(\zeta)} \leq \frac{\kappa}{\kappa_0} \left[ \exp\left\{\kappa_0 \Phi(\zeta)\right\} - 1\right]$$

and $h''$ is continuous except at the discontinuities -if any- of $\phi, J$.

**Proof.** The proof is a slight variation of that for Theorem 10.1 of chapter 6 in [17].

\[\square\]
We are going to use this theorem to prove the existence and behavior of approximate solutions of the equation
\[
\frac{d^2Y}{d\zeta^2} = [h^{-2}(\zeta^2 - \alpha^2) + \psi(\zeta, h, \alpha)]Y. \tag{C.2}
\]
We have the following

**Theorem C.2.** For each value of $h$, assume that the function $\psi(\zeta, h, \alpha)$ is continuous in the region $[0, Z] \times [0, \delta]$ of the $(\zeta, \alpha)$-plane\(^6\) take $\Omega$ as in (2.21) and consider that

\[
\mathcal{V}_{0, Z}[H](\alpha, h) = \int_0^Z \frac{|\psi(t, \alpha)|}{\Omega(t\sqrt{2h^{-1}})} dt
\]
converges uniformly with respect to $\alpha$. Then in this region, equation (C.2) has solutions $Y_1$ and $Y_2$ which are continuous, have continuous first and second partial $\zeta$-derivatives and are given by

\[
Y_1(\zeta, \alpha, h) = U(\zeta\sqrt{2h^{-1}} - \frac{1}{2}h^{-1}\alpha^2) + e_1(\zeta, \alpha, h)
\]
\[
Y_2(\zeta, \alpha, h) = U(\zeta\sqrt{2h^{-1}} - \frac{1}{2}h^{-1}\alpha^2) + e_2(\zeta, \alpha, h)
\]

where

\[
\frac{|e_1(\zeta, \alpha, h)|}{M(\zeta\sqrt{2h^{-1}} - \frac{1}{2}h^{-1}\alpha^2)^{1/2}} \leq \frac{1}{E(\zeta\sqrt{2h^{-1}} - \frac{1}{2}h^{-1}\alpha^2)} \left( \exp \left\{ \frac{1}{2}(\pi h)^{1/2} l(-\frac{1}{2}h^{-1}\alpha^2) \mathcal{V}_{0, Z}[H](\alpha, h) \right\} - 1 \right) \tag{C.3}
\]

and

\[
\frac{|e_2(\zeta, \alpha, h)|}{M(\zeta\sqrt{2h^{-1}} - \frac{1}{2}h^{-1}\alpha^2)^{1/2}} \leq \frac{1}{E(\zeta\sqrt{2h^{-1}} - \frac{1}{2}h^{-1}\alpha^2)} \left( \exp \left\{ \frac{1}{2}(\pi h)^{1/2} l(-\frac{1}{2}h^{-1}\alpha^2) \mathcal{V}_{0, Z}[H](\alpha, h) \right\} - 1 \right) \tag{C.4}
\]

**Proof.** We will prove the theorem only for the first solution since the proof for the second follows mutatis mutandis. Observe that the approximating function $U(\zeta\sqrt{2h^{-1}} - \frac{1}{2}h^{-1}\alpha^2)$ satisfies $\frac{d^2U}{d\zeta^2} = h^{-2}(\zeta^2 - \alpha^2)U$. If we subtract this from (C.2) we obtain the following differential equation for the error term

\[
\frac{d^2e_1}{d\zeta^2} - h^{-2}(\zeta^2 - \alpha^2)e_1 = \psi(\zeta, \alpha, h)\left[ e_1 + U(\zeta\sqrt{2h^{-1}} - \frac{1}{2}h^{-1}\alpha^2) \right].
\]

By use of the method of variation of parameters and also (B.3) one arrives at the integral equation

\[
e_1(\zeta, \alpha, h) = \frac{1}{2\Gamma(\frac{1}{2} + \frac{1}{2}h^{-1}\alpha^2)} \int_0^Z K(\zeta, t)\psi(t, \alpha, h)\left[ e_1(t, \alpha, h) + U(t\sqrt{2h^{-1}} - \frac{1}{2}h^{-1}\alpha^2) \right] dt
\]

\(^6\)Here $Z$ is always positive and may depend continuously on $\alpha$, or be infinite. Also, $\delta$ is a positive finite constant.
in which
\[ K(\zeta, t) = U(\zeta \sqrt{2\hbar^{-1}}, -\frac{1}{2} \hbar^{-1} \alpha^2) \overline{U}(t \sqrt{2\hbar^{-1}}, -\frac{1}{2} \hbar^{-1} \alpha^2) - U(t \sqrt{2\hbar^{-1}}, -\frac{1}{2} \hbar^{-1} \alpha^2) \overline{U}(\zeta \sqrt{2\hbar^{-1}}, -\frac{1}{2} \hbar^{-1} \alpha^2). \]

Bounds for the kernel \( K \) and its first two partial derivatives (with respect to \( \zeta \)) are expressible in terms of the auxiliary functions \( E, M \) and \( N \). We have
\[ |K(\zeta, t)| \leq \frac{E(t \sqrt{2\hbar^{-1}}, -\frac{1}{2} \hbar^{-1} \alpha^2)}{E(\zeta \sqrt{2\hbar^{-1}}, -\frac{1}{2} \hbar^{-1} \alpha^2)} M(\zeta \sqrt{2\hbar^{-1}}, -\frac{1}{2} \hbar^{-1} \alpha^2) M(t \sqrt{2\hbar^{-1}}, -\frac{1}{2} \hbar^{-1} \alpha^2) \]
\[ \left| \frac{\partial K}{\partial \zeta}(\zeta, t) \right| \leq \sqrt{2\hbar^{-1}} \frac{E(t \sqrt{2\hbar^{-1}}, -\frac{1}{2} \hbar^{-1} \alpha^2)}{E(\zeta \sqrt{2\hbar^{-1}}, -\frac{1}{2} \hbar^{-1} \alpha^2)} N(\zeta \sqrt{2\hbar^{-1}}, -\frac{1}{2} \hbar^{-1} \alpha^2) M(t \sqrt{2\hbar^{-1}}, -\frac{1}{2} \hbar^{-1} \alpha^2) \]
and similarly
\[ \frac{\partial^2 K}{\partial \zeta^2}(\zeta, t) = (2\hbar^{-1})^{\frac{3}{2}} \zeta K(\zeta, t). \]

All these estimates allow us to solve the equation (C.2) by applying Theorem [C.1].

Using the notation of that theorem we have
\[ \phi(t) = \frac{\psi(\zeta, \alpha, \hbar)}{\Omega(\zeta \sqrt{2\hbar^{-1}})} \]
\[ \psi_1(t) = 0 \]
\[ J(t) = U(t \sqrt{2\hbar^{-1}}, -\frac{1}{2} \hbar^{-1} \alpha^2) \]
\[ K(\zeta, t) = -\frac{1}{2} \frac{(\pi \hbar)^{\frac{1}{2}}}{\Gamma(\frac{1}{2} + \frac{1}{2} \hbar^{-1} \alpha^2)} \Omega(t \sqrt{2\hbar^{-1}}) K(\zeta, t) \]
\[ Q(t) = \frac{1}{2} \frac{(\pi \hbar)^{\frac{1}{2}}}{\Gamma(\frac{1}{2} + \frac{1}{2} \hbar^{-1} \alpha^2)} \Omega(t \sqrt{2\hbar^{-1}}) E(t \sqrt{2\hbar^{-1}}, -\frac{1}{2} \hbar^{-1} \alpha^2) M(t \sqrt{2\hbar^{-1}}, -\frac{1}{2} \hbar^{-1} \alpha^2) \]
\[ P_0(\zeta) = \frac{M(\zeta \sqrt{2\hbar^{-1}}, -\frac{1}{2} \hbar^{-1} \alpha^2)}{E(\zeta \sqrt{2\hbar^{-1}}, -\frac{1}{2} \hbar^{-1} \alpha^2)} \]
\[ P_1(\zeta) = \frac{\sqrt{2\hbar^{-1}} N(\zeta \sqrt{2\hbar^{-1}}, -\frac{1}{2} \hbar^{-1} \alpha^2)}{E(\zeta \sqrt{2\hbar^{-1}}, -\frac{1}{2} \hbar^{-1} \alpha^2)} \]
\[ \Phi(\zeta) = \mathcal{V}_{0, \zeta}[\mathcal{H}](\alpha, \hbar) \]
\[ \kappa_0 \leq \frac{1}{2} (\pi \hbar)^{\frac{1}{2}} l(-\frac{1}{2} \hbar^{-1} \alpha^2) \]

where the role of \( \beta \) is played here by \( Z \) and \( \kappa \) is replaced for simplicity by the upper bound \( \kappa_0 \). Then the bounds (C.3) and (C.4) follow from Theorem [C.1].

Finally, observe that all the integrals which occur in the analysis above, converge uniformly when \( \alpha \in [0, \delta] \) and \( \zeta \) lies in any compact interval of \( [0, Z] \); allowing us to state that \( \epsilon_1 \) and its first two partial \( \zeta \)-derivatives are continuous in \( \alpha \) and \( \zeta \). Consequently, the same stands for \( \mathcal{Y}_1 \) which signifies the end of the proof. \( \square \)

**Data Availability**

Data sharing is not applicable to this article as no new data were created or analyzed in this study.
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