Elements of the analytic structure of anomalous scaling and intermittency are described. We focus here on the structure functions of velocity differences that satisfy inertial range scaling laws \( S_n(R) \sim R^{\zeta_n} \), and the correlation of energy dissipation \( K_{\epsilon\epsilon}(R) \sim R^{-\eta} \). The goal is to understand the exponents \( \zeta_n \) and \( \mu \) from first principles. In paper II of this series it was shown that the existence of an ultraviolet scale (the dissipation scale \( \eta \)) is associated with a spectrum of anomalous exponents that characterize the ultraviolet divergences of correlations of gradient fields. The leading scaling exponent in this family was denoted \( \Delta \). The exact resummation of ladder diagrams resulted in the calculation of \( \Delta \) which satisfies the scaling relation \( \Delta = 2 - \zeta_2 \). In this paper we continue our analysis and show that nonperturbative effects may introduce multiscaling (i.e. \( \zeta_n \) not linear in \( n \)) with the renormalization scale being the infrared outer scale of turbulence \( L \). It is shown that deviations from K41 scaling of \( S_n(R) \) (\( \zeta_n \neq n/3 \)) must appear if the correlation of dissipation is mixing (i.e. \( \mu > 0 \)). We derive an exact scaling relation \( \mu = 2\zeta_2 - \zeta_4 \). We present analytic expressions for \( \zeta_n \) for all \( n \) and discuss their relation to experimental data. One surprising prediction is that the time decay constant \( \tau_n(R) \propto R^{\zeta_n} \) of \( S_n(R) \) scales independently of \( n \): the dynamic scaling exponent \( \zeta_n \) is the same for all \( n \)-order quantities, \( \zeta_n = \zeta_2 \).

\section{1. Introduction}

This paper is the last in a series of four papers. The aim of these four papers is to lay out the analytic basis for the description of scaling properties in fully developed hydrodynamic turbulence. We are concerned here with the statistical properties of turbulence in terms of averages over fields of the fluid. The fundamental field in hydrodynamics is the fluid’s Eulerian velocity, denoted as \( \mathbf{u}(\mathbf{r}, t) \) where \( \mathbf{r} \) is a point in \( d \)-dimensional space (usually \( d = 2 \) or \( 3 \)) and \( t \) is the time. The statistical quantities that have attracted decades of experimental and theoretical attention (see for example \cite{4,5,6,7,8}) are the structure functions of velocity differences, denoted as \( S_n(R) \)

\begin{equation}
S_n(R) = \langle |u(\mathbf{r} + \mathbf{R}, t) - u(\mathbf{r}, t)|^n \rangle
\end{equation}

where \( \langle \ldots \rangle \) stands for a suitably defined ensemble average. It has been asserted for a long time that the structure functions depend on \( R \) as power laws:

\begin{equation}
S_n(R) \propto R^{\zeta_n},
\end{equation}

when \( R \) is inside the so called “inertial range”, i.e. \( \eta \ll R \ll L \). Here \( \eta \) and \( L \) are respectively the inner (viscous) and outer (energy containing) scale of turbulence. One of the major questions in fundamental turbulence research is whether the scaling exponents \( \zeta_n \) are correctly predicted by the classical Kolmogorov 41 theory \( \hat{\eta} \) (known as K41) in which \( \zeta_n = n/3 \), or whether these exponents deviate from \( n/3 \) as has been indicated by experiments. In particular we want to know whether the exponents \( \zeta_n \) may manifest the phenomenon of “multiscaling” with \( \zeta_n \) being a nonlinear function of \( n \).

Experimental research has not confined itself to the measurement of the structure functions of velocity differences. Gradient fields have featured as well. For example the correlation function of the energy dissipation field has been studied extensively. The dissipation field \( \epsilon(\mathbf{r}, t) \) is defined as

\begin{equation}
\epsilon(\mathbf{r}, t) \equiv \frac{\nu}{2} |\nabla_\alpha u_\beta(\mathbf{r}, t) + \nabla_\beta u_\alpha(\mathbf{r}, t)|^2
\end{equation}

with \( \nu \) being the kinematic viscosity. The correlation function of the dissipation field \( K_{\epsilon\epsilon}(R) \) is

\begin{equation}
K_{\epsilon\epsilon}(R) = \langle \tilde{\epsilon}(\mathbf{r} + \mathbf{R}, t)\tilde{\epsilon}(\mathbf{r}, t) \rangle,
\end{equation}

where \( \tilde{\epsilon}(\mathbf{r}, t) = \epsilon(\mathbf{r}, t) - \bar{\epsilon} \). Here and below \( \bar{\epsilon} \equiv \langle \epsilon \rangle \). Experiments appeared to show \( \hat{\eta} \) that \( K_{\epsilon\epsilon}(R) \) decays according to a power law,

\begin{equation}
K_{\epsilon\epsilon}(R) \sim R^{-\eta}, \quad \eta \ll R \ll L
\end{equation}

with \( \mu \) having a numerical value of 0.2 – 0.3. The analytic derivation of this law from the equations of fluid mechanics and the calculation of the numerical value of
the scaling exponent $\mu$ have been among the elusive goals of theoretical research.

In paper 0 of this series (subtitled “Line-Resummed Diagrammatic Perturbation Theory”) we reviewed the literature ([1][2][3][4] with the aim of introducing the student to the available techniques. That paper did not present any new results. In paper I of this series [3] (subtitled “The Ball of Locality and Normal Scaling”) we dealt with the perturbative theory of the correlation, response and structure functions of hydrodynamic turbulence. The main result of that paper (and see also [2]) was that after appropriate resumptions and renormalizations the perturbative theory for these quantities is finite order by order. All the integrals appearing in the theory are convergent both in the ultraviolet and in the infrared limits. This means that there is no perturbative mechanism to introduce a length scale into the theory of the structure functions. In turn this result indicated that as far as the perturbative theory is concerned there is no mechanism to shift the exponents $\zeta_n$ away from their K41 values. Of course, nonperturbative effects may furnish such a mechanism and therefore in paper II [4] (subtitled “A Ladder to Anomalous Scaling”) we turned to the analysis of nonperturbative effects. It was shown there that the renormalized perturbation theory for correlation functions that include velocity derivatives (to second or higher power) exhibit in their perturbation expansion a logarithmic dependence on the viscous scale $\eta$ [13][14]. In this way the inner scale of turbulence appears explicitly in the analytic theory. The perturbative series could be resummed to obtain integrodifferential equations for some many-point objects of the theory. These equations have also non-perturbative scale-invariant solutions that may be represented as power laws of $\eta$ to some exponents $\Delta$. We argued [15] that if $\Delta < \Delta_c$ where

$$\Delta_c = 2 - \zeta_2$$  \hspace{1cm} (1.6)

(a situation referred to as the “subcritical scenario”), then K41 scaling is asymptotically exact for infinite $Re$. In this case $K_{\epsilon\epsilon}(R) \sim \tilde{\epsilon}^2(\eta/R)^{2(4/3-\Delta)}$, and the exponent $\mu$ is identified with $2(4/3-\Delta)$. The renormalization length is then the inner length $\eta$.

In paper II it was shown that the exponent $\Delta$ takes on exactly the critical value $\Delta = \Delta_c$ given by (1.6), and the subcritical scenario is not realized. In the present paper it will be shown that the correlation function of the energy dissipation field has the dependence:

$$K_{\epsilon\epsilon}(R) \sim \tilde{\epsilon}^2(L/R)^{2\zeta_2-\zeta_4}(R/\eta)^{2(\Delta-\Delta_c)}.$$  \hspace{1cm} (1.7)

As a result of the criticality of $\Delta$ the inner scale $\eta$ disappears from the correlation $K_{\epsilon\epsilon}(R)$ which finally takes on the form

$$K_{\epsilon\epsilon}(R) \simeq \tilde{\epsilon}^2(L/R)^{2\zeta_2-\zeta_4}.$$  \hspace{1cm} (1.8)

Note that this implies that for K41 scaling (in which $2\zeta_2 = \zeta_4$), $K_{\epsilon\epsilon}(R)$ does not decay as a function of $R$ (i.e. the correlation is not mixing). This is hardly physical for a random field. On the other hand if there exists anomalous scaling in the sense that $\zeta_4 < 2\zeta_2$, then mixing is regained. Thus we will argue that the critical scenario $\Delta = \Delta_c$ goes hand in hand with anomalous scaling if $K_{\epsilon\epsilon}(R)$ is mixing, and then $\mu$ is identified with $\zeta_4 - 2\zeta_2$,

$$\mu = \zeta_4 - 2\zeta_2 .$$  \hspace{1cm} (1.9)

This is one of the exact scaling relations that are derived below.

The main purpose of the present paper is to elucidate the possible mechanisms for anomalous (non-K41) scaling of the structure functions in the theory of turbulence in fluids whose dynamics is described by the Navier-Stokes equations. We show that the analysis of the statistical theory of Navier Stokes dynamics leads to the possibility of multiscale with nonlinear dependence of $\zeta_n$ on $n$. For such a scenario we offer an evaluation of the scaling exponents $\zeta_n$ from first principles. We will show that the mechanisms for anomalous scaling are wholly nonperturbative. Order by order considerations always lead to K41 scaling. We show that a useful analysis of anomalous scaling can be developed on the basis of the so-called “balance equations” which are non-perturbative. In section 2 we derive the balance equations (2.24, 2.25) by writing the equations of motion for the structure function $S_n(R)$ and of related quantities. These equations, in the stationary state, exhibit a balance between a convective (interaction) term which is denoted as $D_n(R)$ and a dissipative term, denoted by $J_n(R)$, which is the cross-correlation between the energy dissipation $\tilde{\epsilon}$ and $(n-2)$ velocity differences across a scale $R$. Both terms are expressed as many-point correlation functions that depend on many coordinates, but some of these coordinates are the same. This last fact leads to the main strategy of this paper, which is to understand the fusion rules which describe the scaling of many-point correlation functions when some of their coordinates fuse together. When this happens the distance between the coordinates crosses the dissipative scale, and at that very moment the ultraviolet divergences are picked up. Since we learned in paper II how to compute these exactly, we have an important part of the fusion rules at hand. Following this strategy we succeed to compute exactly the dissipative term $J_n(R)$ and the correlation function $K_{\epsilon\epsilon}(R)$.

The fusion rules are developed in sections 3,4 and 5. In section 3 we show that the exponent $\Delta$ of paper II is indeed the relevant exponent for describing pair coalescence. In section 4 we discuss four-point correlations and their 2-point fusion rules. This leads to the calculation of the correlation $K_{\epsilon\epsilon}(R)$ and the dissipative term $J_4$. In section 5 we discuss the fusion rules of $n$-point correlation functions, leading to the exact evaluation of $J_n(R)$. A consequence of this calculation is the derivation of the dynamical scaling exponents $\zeta_n$ describing the characteristic decay time $\tau_n(R) \sim R^{2\zeta_n}$ of $n$-point, $n$-time correlation functions of BL-velocity differences. We ar-
gue at the end of section 5 that all the exponents \( \zeta_n \) are independent of \( n \) and that they equal \( \zeta_2 \).

In section 6 we turn to the evaluation of the interaction term \( D_n(R) \). Unfortunately we have not yet succeeded to evaluate \( D_n \) exactly. Accordingly we describe in sections 6 and 7 various possible evaluations of \( D_n \) and the resulting implications on \( \zeta_n \). It is shown in section 6 that the naive evaluation of \( D_n \) leads to the scaling exponents of the \( \beta \)-model. We give arguments however leading to the conjecture that the naive evaluation of \( D_n \) is an overestimate. We propose that there may exist a delicate cancellation in this evaluation, and the next order evaluation leads unequivocally to multiscale. In order to compute the scaling exponents \( \zeta_n \) we need to know \( D_n \) very precisely, coefficients and all.

Not having an exact theory for \( D_n \) we resort in section 7 to modeling. Guided by some diagrammatics and some common sense we conjecture an analytic form for \( D_n \) which conforms with our expectations about eddy-viscosity. This part of the paper is not rigorous, and the results of section 7 should be considered therefore as tentative. The advantage of the approximation based on eddy viscosity is that we can offer analytic estimates of all the exponents \( \zeta_n \). We discuss these estimates and compare them with experimental findings in Table 1 of section 7B2. Section 8 is a summary and some discussion of the road ahead.

Throughout the text we may refer to equations appearing in papers 0, I or II. When we do so we denote them as Eqs. (0–n), (I–n), (II–p) etc.

2. DERIVATION OF THE BALANCE EQUATIONS

Some of the most important nonperturbative constraints on the statistical theory of turbulence are the balance equations (also known as the moment equations) for the structure functions. These equations are derived in this section and used in later sections to deduce the scenarios for multiscale in the structure functions. Before we derive the equations we introduce the statistical objects that appear naturally in the discussion.

A. Structure functions and related quantities

In this subsection we define quantities that are related to the structure functions \( S_n(R) \). It is common in experiments to measure only the longitudinal structure functions \( S_1(R) \). For theoretical treatment these quantities are not necessarily the most convenient since they are not invariant to rotations. It is useful therefore to introduce some related objects that have simple transformation properties under rotations and inversions. The first one is the scalar quantity which is appropriate for even orders of \( S_n \). To keep in mind its scalar nature we will denote it as \( S_{2m}^\circ(R) \) and define it in Eulerian terms as

\[
S_{2m}^\circ(R) \equiv \langle \delta u(r_0|R, t)^{2m} \rangle, \quad R \equiv r - r_0, \tag{2.1}
\]

where

\[
\delta u(r_0|R, t) \equiv u(r_0 + R, t) - u(r_0, t) \tag{2.2}
\]

is a simultaneous Eulerian velocity difference. The quantity \( S_{2m}^\circ(R) \) is analytic. For odd order structure functions we introduce a vector object \( S_{2m+1}(R) \) according to

\[
S_{2m+1}^\circ(R) \equiv \langle \delta u_\alpha(r_0|R, t)\delta u_\beta(r_0|R, t)|\delta u(r_0|R, t)|^{2m} \rangle. \tag{2.3}
\]

Here and below we will use Greek indices to indicate vector and tensor components, and Roman indices to indicate the order of the quantity. The placement of indices as subscripts or superscripts has no meaning, and is chosen for convenience.

For isotropic turbulence the vector \( S_{2m+1}^\circ(R) \) can only be oriented along \( R \). This allows us the introduction of a scalar quantity \( S_{2m+1}(R) \) which depends on the magnitude of \( R \):

\[
S_{2m+1}^\circ(R) = \frac{R_\alpha}{R} S_{2m+1}(R). \tag{2.4}
\]

Lastly we will need also the tensor objects \( S_{2m+2}^{\alpha\beta}(R) \):

\[
S_{2m+2}^{\alpha\beta}(R) \equiv \langle \delta u_\alpha(r_0|R, t)\delta u_\beta(r_0|R, t)|\delta u(r_0|R, t)|^{2m} \rangle. \tag{2.5}
\]

Note that the objects introduced in (2.1, 2.3, 2.5) involve an arbitrary number of velocity fields but only two spatial points separated by \( R \). We refer to them loosely as "two-point" correlation functions of velocity differences. The theory below calls for the introduction of three-point functions as well. It is best to define them using the four-point functions

\[
T_{2m+2}^{\alpha\beta}(R_1, R_2, R) \equiv \langle \delta u_\alpha(r_0|R_1, t)\delta u_\beta(r_0|R_2, t) \times|\delta u(r_0|R, t)|^{2m} \rangle, \tag{2.6}
\]

\[
T_{2m+3}^{\alpha\beta\gamma}(R_1, R_2, R) \equiv \langle \delta u_\alpha(r_0|R_1, t)\delta u_\beta(r_0|R_2, t) \times|\delta u_\gamma(r_0|R, t)|^{2m} \rangle, \tag{2.7}
\]

\[
T_{2m+4}^{\alpha\beta\gamma\delta}(R_1, R_2, R) \equiv \langle \delta u_\alpha(r_0|R_1, t)\delta u_\beta(r_0|R_2, t) \times|\delta u_\gamma(r_0|R, t)|^{2m} \rangle. \tag{2.8}
\]

We will see that the theory produces expressions involving these quantities with two of the arguments being identical (i.e. \( R_1 = R_2 \), etc.) This fact will lead to the study of fusion rules in later sections.
B. The Balance Equations

In this subsection we derive the balance equation that relates structure functions to correlation functions involving the dissipation field. We start by deriving some equations of motion for the quantities defined above.

1. Equations of Motion

Our starting point is the Navier-Stokes equations for an incompressible fluid:

\[
\frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial t} + \mathbf{u}(\mathbf{r}, t) \cdot \nabla \mathbf{u}(\mathbf{r}, t) - \nu \nabla^2 \mathbf{u}(\mathbf{r}, t) - \nabla p(\mathbf{r}, t) = \mathbf{f}(\mathbf{r}, t), \quad \nabla \cdot \mathbf{u} = 0 .
\]

(2.9)

For simplicity we will choose the forcing such that \( \nabla \cdot \mathbf{f} = 0 \). This equation can be rewritten in terms of the Belinicher-L'vov velocities \( \mathbf{v}(\mathbf{r}_0, \mathbf{r}, t) \) defined by Eq. (1-2.2) as follows:

\[
\frac{\partial \mathbf{v}(\mathbf{r}_0|\mathbf{r}, t)}{\partial t} + [\mathbf{w}(\mathbf{r}_0|\mathbf{r}, t) \cdot \nabla] \mathbf{w}(\mathbf{r}_0|\mathbf{r}, t) - \nu \nabla^2 \mathbf{w}(\mathbf{r}_0|\mathbf{r}, t) - \nabla \bar{p}(\mathbf{r}_0|\mathbf{r}, t) = \bar{\mathbf{f}}(\mathbf{r}_0|\mathbf{r}, t) ,
\]

(2.10)

where we also have the incompressibility condition \( \nabla \cdot \mathbf{w} = 0 \). Here \( \mathbf{w} \) is the BL velocity difference

\[
\mathbf{w}(\mathbf{r}_0|\mathbf{r}, t) \equiv \mathbf{v}(\mathbf{r}_0|\mathbf{r}, t) - \mathbf{v}(\mathbf{r}_0|\mathbf{r}_0, t) ,
\]

(2.11)

and \( \bar{p}(\mathbf{r}_0|\mathbf{r}, t), \bar{\mathbf{f}}(\mathbf{r}_0|\mathbf{r}, t) \) are BL-transformed pressure and forcing (for more detail, see Section 3A of paper I). Applying the transverse projector \( \mathbf{p} \) this equation takes on the form

\[
\frac{\partial \mathbf{v}(\mathbf{r}_0|\mathbf{r}, t)}{\partial t} + \mathbf{p}[\mathbf{w}(\mathbf{r}_0|\mathbf{r}, t) \cdot \nabla] \mathbf{w}(\mathbf{r}_0|\mathbf{r}, t) - \nu \nabla^2 \mathbf{w}(\mathbf{r}_0|\mathbf{r}, t) = \bar{\mathbf{f}}(\mathbf{r}_0|\mathbf{r}, t) .
\]

(2.12)

Evaluate now the scalar product obtained from Eq. (2.12) at two spatial points \( \mathbf{r} \) and \( \mathbf{r}_0 \). Subtracting the two equations we get

\[
\frac{\partial w_{\alpha}(\mathbf{r}_0|\mathbf{r}, t)}{\partial t} + \int d\mathbf{r}' [P_{\alpha\beta}(\mathbf{r} - \mathbf{r}' - \mathbf{r}_0|\mathbf{r}, t) - P_{\alpha\beta}(\mathbf{r}_0|\mathbf{r}, t)] \cdot \nabla_{\mathbf{r}'} w_{\beta}(\mathbf{r}_0|\mathbf{r}, t) = \nu [\nabla^2 + \nabla_{\mathbf{r}_0}^2] w_{\alpha}(\mathbf{r}_0|\mathbf{r}, t) + \bar{f}_\alpha(\mathbf{r}_0|\mathbf{r}, t) - \bar{\bar{f}}_\alpha(\mathbf{r}_0|\mathbf{r}, t) ,
\]

(2.15)

where \( \nabla_{\mathbf{r}_0} \equiv \partial / \partial \mathbf{r}_0 \). Introduce now the shorthand notation \( w^2 \) and \( w^{2m} \) in situations in which the arguments are \( (\mathbf{r}_0|\mathbf{r}, t) \):

\[
|\mathbf{w}(\mathbf{r}_0|\mathbf{r}, t)|^2 \equiv w^2 , \quad |\mathbf{w}(\mathbf{r}_0|\mathbf{r}, t)|^{2m} \equiv w^{2m} .
\]

(2.16)

When other arguments appear they will be displayed explicitly. In term of these quantities the structure functions (2.1)-(2.3) are written as

\[
\bar{S}_{2m}(R) \equiv \langle w^{2m} \rangle ,
\]

(2.17)

\[
S_{2m+1}(R) \equiv \langle w^{2m} \rangle .
\]

(2.18)

where again \( \mathbf{R} = \mathbf{r} - \mathbf{r}_0 \) and \( \mathbf{w} \) without arguments means \( \mathbf{w}(\mathbf{r}_0|\mathbf{r}, t) \). Next observe that

\[
\frac{\partial w^{2m}}{\partial t} = 2mw^{2m-1} \mathbf{w} \cdot \frac{\partial \mathbf{w}}{\partial t} .
\]

(2.19)

Evaluate now the scalar product obtained from Eq. (2.15) by multiplying it on the left by \( 2mw^{2m-1} \mathbf{w} \). Using Eq. (2.19) this is written exactly as

\[
\frac{\partial S_{2m+1}(R)}{\partial t} + D_{2m}(R) = J_{2m}(R) + Q_{2m}(R) ,
\]

(2.20)

where we denoted

\[
D_{2m}(R) \equiv 2m \int d\mathbf{r}_1 P_{\alpha\beta}(\mathbf{r}_1) \frac{\partial}{\partial \mathbf{r}_1} \langle w^{2m-1} w_{\alpha} \rangle \times \left[ w_{\gamma}(\mathbf{r}_1|\mathbf{r}_1 + \mathbf{r}_0, t) w_{\beta}(\mathbf{r}_0|\mathbf{r} + \mathbf{r}_1, t) - w_{\gamma} w_{\beta}(\mathbf{r}_0|\mathbf{r} + \mathbf{r}_1, t) \right] ,
\]

(2.21)

\[
J_{2m}(R) \equiv 2m \nu \left[ w^{2m-1} w_{\alpha} \left[ \nabla_{\mathbf{r}}^2 + \nabla_{\mathbf{r}_0}^2 \right] w_{\alpha} \right] ,
\]

(2.22)

\[
Q_{2m}(R) \equiv 2m \left[ w^{2m-1} w_{\alpha} \left[ f_{\alpha}(\mathbf{r}_0|\mathbf{r}, t) - f_{\alpha}(\mathbf{r}_0|\mathbf{r}_0, t) \right] \right] .
\]

(2.23)

In deriving the equation for \( D_{2m} \) we used the incompressibility condition (which is not available in the Burgers’ equation) and performed changes of variables in the integrals according to \( \mathbf{r}' = \mathbf{r} \) and \( \mathbf{r}' - \mathbf{r}_0 = \mathbf{r}_1 \). In the stationary state \( \partial S_{2m}(R)/\partial t = 0 \), and we can write the scalar balance equation

\[
D_{2m}(R) = J_{2m}(R) + Q_{2m}(R) .
\]

(2.24)

Next we need to derive vectorial balance equations for \( S_{2m+1} \). Repeating similar steps we end up with

\[
D_{2m+1}(R) = J_{2m+1}(R) + Q_{2m+1}(R) ,
\]

(2.25)

where

\[
D_{2m+1}(R) \equiv 2m \int d\mathbf{r}_1 P_{\alpha\beta}(\mathbf{r}_1) \frac{\partial}{\partial \mathbf{r}_1} \langle w^{2m-1} w_{\alpha} \rangle \times \left[ w_{\gamma}(\mathbf{r}_1|\mathbf{r}_1 + \mathbf{r}_0, t) w_{\beta}(\mathbf{r}_0|\mathbf{r} + \mathbf{r}_1, t) \right] .
\]
obvious scalar counterparts, similar to the one introduced for isotropic turbulence all these vector quantities have equation, since there is no pressure in that case. On the other hand the Burgers equation describes a compressible flow, and this fact introduces the coefficient \((n + 1)/n\) in the RHS of this expression. For the Navier-Stokes equations we can only use the expression for a rough evaluation of \(D_{2m}\). Notwithstanding, it is exact for the case \(m = 1\), where it takes the form

\[
D_2(R) = \frac{\partial}{\partial R_\alpha} S^\gamma_\alpha(R) .
\] (2.35)

The deep reason for this simplification in the case \(m = 1\) is the conservation law (in the absence of viscosity) of the quadratic invariant of the BL equation of motion (2.10), i.e. \(\int d\mathbf{r} w^2 |\mathbf{r}|, t)\). \(D_2(R)\) is simply the rate of change of \(w^2 |\mathbf{r}|, t\) and therefore must be a pure divergence. None of the higher powers of \(w|\mathbf{r}|, t\) forms an invariant, and as a consequence none of the higher orders \(D_n(R)\) can be written as a pure divergence. On the other hand we see from Eq. (2.22) that with the full kernel we have two terms that can be expressed in terms of the \(T\) tensors:

\[
D_{2m}(R) = \frac{\partial}{\partial R_\alpha} \int d\mathbf{r} P_{\alpha \beta}(r_1) (2.36)
\]

\[
\times \frac{\partial}{\partial r_\gamma} \left[ T^{\alpha \beta \gamma}_{2m+1}(\mathbf{r} + \mathbf{r}_1, \mathbf{R} + \mathbf{r}_1, \mathbf{R}) - T^{\alpha \beta \gamma}_{2m+1}(\mathbf{r}_1, \mathbf{r}_1, \mathbf{R}) \right] .
\]

This is the final form of \(D_{2m}(R)\).

Following the same steps of analysis we find the final expression for \(D_{2m+1}(R)\).

\[
D_{2m+1}(R) = 2m \int d\mathbf{r} P_{\alpha \beta}(r_1) (2.37)
\]

\[
\times \frac{\partial}{\partial r_\gamma} \left[ T^{\alpha \beta \gamma}_{2m+2}(\mathbf{r} + \mathbf{r}_1, \mathbf{R} + \mathbf{r}_1, \mathbf{R}) - T^{\alpha \beta \gamma}_{2m+2}(\mathbf{r}_1, \mathbf{r}_1, \mathbf{R}) \right]
\]

\[
+ \int d\mathbf{r} P_{\alpha \beta}(r_1) \frac{\partial}{\partial r_\gamma} \left[ T^{\beta \gamma}_{2m+2}(\mathbf{R} + \mathbf{r}_1, \mathbf{R} + \mathbf{r}_1, \mathbf{R}) - T^{\beta \gamma}_{2m+2}(\mathbf{r}_1, \mathbf{r}_1, \mathbf{R}) \right] .
\]

The naive evaluation of every term in Eq. (2.36) is given by (2.34). However this evaluation is only acceptable under two assumptions: (i) that the integral converges, and (ii) that there are no cancellations between the terms with opposite signs in Eq. (2.37). The analysis of convergence requires understanding the asymptotic properties of the \(T\) correlators. Similar properties will determine the evaluation of \(J_{2m}\) as will be seen next. For that reason we will devote the next sections to questions of asymptotics and fusion rules for \(n\)-point correlation function of velocity differences. We will learn that the integral in Eq. (2.36) indeed converges, but on the other hand cancellations are not excluded and are the possible source of multiscaling in turbulence. In fact, such a cancellation is the only mechanism of multiscaling that we can identify.
2. The Dissipation Term

Starting with Eq. (2.22) we first use the fact that for simultaneous correlations we can use the Eulerian differences $\delta \mathbf{u}$ instead of the BL-velocity differences. Secondly we use the symmetry with respect to exchange of $\mathbf{r}$ and $\mathbf{r}_0$ to write the expression with twice the Laplacian with respect to $\mathbf{r}_0$ only, instead of the sum of Laplacians. Lastly we use transential invariance to move one of the gradients around. We find

$$J_{2m}(R) \equiv -4m\nu \left\{ \left[ \delta \mathbf{u}(\mathbf{r}_0|R) \right]^{2(2m-1)} \frac{\partial u_\alpha(\mathbf{r}_0)}{\partial r_\beta} \right\}^2,$$

$$+ 2(m-1) \left\{ \left[ \delta \mathbf{u}(\mathbf{r}_0|R) \right]^{2(2m-2)} \delta u_\alpha(\mathbf{r}_0|R) \delta u_\gamma(\mathbf{r}_0|R) \times \frac{\partial}{\partial r_\beta} u_\gamma(\mathbf{r}_0) \frac{\partial}{\partial r_\beta} u_\alpha(\mathbf{r}_0) \right\}. \tag{2.38}$$

To relate this quantity to the tensors $T$ defined in Eqs. (2.23, 2.8) we write the derivative as the limit of velocity differences:

$$\frac{\partial u_\alpha(\mathbf{r}_0)}{\partial r_\beta} \equiv \lim_{d_\beta \to 0} \frac{u_\alpha(\mathbf{r}_0 + d_\beta \hat{e}_\beta) - u_\alpha(\mathbf{r}_0)}{d_\beta}, \tag{2.39}$$

where $\hat{e}_\beta$ is the unit vector in the $\beta$ direction. With this in mind the expression for $J_{2m}$ is

$$J_{2m}(R) = -4m\nu \lim_{d_\beta \to 0} \frac{1}{d_\beta} \left[ T^{2m}_{2m} (d_\beta \hat{e}_\beta, d_\beta \hat{e}_\beta, R) \right.\right.$$  
\left. + 2(m-1) T^{2m}_{2m} (d_\beta \hat{e}_\beta, d_\beta \hat{e}_\beta, R) \right]. \tag{2.40}

One should understand that a summation over $\alpha, \beta$ and $\gamma$ is implied as usual. Note that for $d_\beta \ll \eta$ the velocity field is expected to be smooth, and both $T$ functions become proportionnal to $d_\beta^2$, cancelling this factor in the denominator. We will not have a dependence of the limit on the direction of the vector $\mathbf{d} \equiv d_\beta \hat{e}_\beta$.

It can be seen that the evaluation of $J_{2m}(R)$ again requires elucidation of the asymptotic properties of the correlations functions including the rules of coalescence of groups of points. In paper I we proved the property of locality that means that the coalescence itself is regular. However, we have here differential operators on coalescing points and we need to consider not only leading behaviours but also next order properties. This is done in the section 5.

The quantities $T^{2m}_{2m+1}(R)$ can be expressed in terms of the third and fifth rank $T$ tensors, but we avoid displaying the result since we do not use it explicitly.

Lastly we need to discuss the forcing terms $Q_{2m}(R)$ and $Q_{2m+1}(R)$. This is the topic of the next subsection.

3. The Forcing Term

In this subsection we show that the forcing term can be neglected in the inertial range of scales. The reader who finds this statement believable can skip this subsection.

To discuss the forcing terms it is useful to introduce a higher order Green’s function according to

$$\mathcal{G}^{\alpha\beta}_{2m-1,1}(\mathbf{r}_0|x, x') \equiv \left\langle \frac{\delta w^{(m-1)}(t)_{\alpha}(\mathbf{r}_0)}{\delta f_{\beta}(\mathbf{r}_0|x')} \right\rangle \tag{2.41}$$

We remind the reader that $x = (\mathbf{r}, t)$ and that according to our convention we can omit the argument $(\mathbf{r}_0|x)$ in the function $w$.

When the statistics of the random forcing is Gaussian we can write $Q_{2m}(R)$ of Eq. (2.22) as

$$Q_{2m}(R) = \int \! dt \! dt' \mathcal{G}^{\alpha\beta}_{2m-1,1}(\mathbf{r}_0|x, x') \left[ D_{\alpha\beta}(\mathbf{r}_0|\mathbf{r}', t', t - t') - D_{\alpha\beta}(\mathbf{r}_0|\mathbf{r}', 0) \right]. \tag{2.42}$$

Here, as usual, $R = \mathbf{r} - \mathbf{r}_0$ and $D_{\alpha\beta}(\mathbf{r}_0|x, x')$ is the correlator of $\tilde{f}_{\alpha}(x)$ and $\tilde{f}_{\beta}(x)$.

We are going to estimate $Q_{2m}(R)$ by taking first a delta-correlated forcing,

$$D_{\alpha\beta}(\mathbf{r}_0|x, x') = D_{\alpha\beta}(\mathbf{r} - \mathbf{r}') \delta(t - t'). \tag{2.43}$$

The consideration of the effect of finite correlation time is deferred to the end. Using (2.44) Eq. (2.42) simplifies to

$$Q_{2m}(R) = \int \! dt' \mathcal{G}^{\alpha\beta}_{2m-1,1}(\mathbf{r}_0|\mathbf{r}', 0) \left[ D_{\alpha\beta}(\mathbf{r}' - \mathbf{r}) - D_{\alpha\beta}(\mathbf{r}' - \mathbf{r}_0) \right]. \tag{2.44}$$

Because of the delta functions we have lost the time integration and we have the zero-time Green’s function, which is not dressed by the interaction (and see paper I for an explicit demonstration). Using the chain rule of differentiation in (2.41) we get

$$\mathcal{G}^{\alpha\beta}_{2m-1,1}(\mathbf{r}_0|x, x') = \left\langle \frac{\delta w^{(m-1)}(t)_{\alpha}(\mathbf{r}_0)}{\delta f_{\beta}(\mathbf{r}_0|x')} \right\rangle \tag{2.45}$$

$$+ 2(m-1) \left\langle \frac{\delta w^{(m-2)}_{\alpha\beta}(\mathbf{r}_0|x)}{\delta f_{\gamma}(\mathbf{r}_0|x')} \right\rangle. \tag{2.46}$$

Next we recall that at time $t = 0$ the unaveraged response $\delta w/\delta \tilde{f}$ is uncorrelated with the velocity field since any interaction involves a vertex with time integration between 0 and $t$. Accordingly we can decouple the response in (2.45) and write

$$\mathcal{G}^{\alpha\beta}_{2m-1,1}(\mathbf{r}_0|x, x') = \mathcal{G}_{\gamma\delta}^{\alpha\beta}(\mathbf{r}_0|\mathbf{r}, \mathbf{r}') \mathcal{G}_{2m-1,1}(R) + 2(m-1) \mathcal{G}_{\gamma\delta}^{\alpha\beta}(\mathbf{r}_0|\mathbf{r}, \mathbf{r}') \mathcal{G}_{2m-1,1}(R), \tag{2.46}$$

where $\mathcal{G}^{\alpha\beta}$ is the bare Green’s function which was defined and computed in paper I sections 2, 3. We proceed to evaluate the order of magnitude of $Q_{2m}$ by substituting
the last equation in (2.44). Suppressing vector indices we evaluate
\[ Q_{2m}(R) \sim m(2m-1)S_{2(m-1)}(R)Q_2(R), \quad (2.47) \]
where
\[ Q_2(R) = \int dr' G^0(r_0|r, r', 0) \left[ D(r' - r) - D(r' - r_0) \right]. \quad (2.48) \]

We can estimate \( Q_2 \) by recalling that \( D(R) \) has a characteristic scale \( L \) (which is the outer scale of turbulence). For \( R \ll L \) we expand \( D(R) \) in \( R \):
\[ D(R) = D(0) + D'' R^2 / 2 , \quad D'' \simeq D(0)/L^2. \quad (2.49) \]

In its turn \( D(0) \) is about the rate of energy input, and for stationary turbulence \( D(0) \simeq \bar{\epsilon} \), the rate of energy dissipation. Finally we estimate
\[ D(R) - D(0) \simeq \bar{\epsilon}(R/L)^2. \quad (2.50) \]

Together with the evaluation of \( G^0 \) that was presented in paper I Section 3 B we evaluate \( Q_2 \) as
\[ Q_2(R) \sim \bar{\epsilon} \left( \frac{R}{L} \right)^2 \ln \left( \frac{R}{\eta} \right). \quad (2.51) \]

The integral can be evaluated by introducing the ultraviolet cutoff at the viscous scale \( \eta \):
\[ Q_2 \sim \bar{\epsilon} \left( \frac{R}{L} \right)^2 \ln \left( \frac{R}{\eta} \right). \quad (2.52) \]

Substituting in (2.47) and omitting numerical factors we end up with
\[ Q_{2m}(R) \sim \bar{\epsilon} \left( \frac{R}{L} \right)^2 S_{2(m-1)}(R) \ln \left( \frac{R}{\eta} \right). \quad (2.53) \]

We are now in a position to compare \( Q_{2m} \) with the interaction term \( D_{2m} \) given in (2.33). For K41 scaling and delta-correlated forcing we have
\[ \frac{Q_{2m}}{D_{2m}} \sim \left( \frac{R}{L} \right)^2 \ln \left( \frac{R}{\eta} \right). \quad (2.54) \]

It should be noted that the logarithm in this equation is a direct result of our use of delta correlated forcing. This choice brought in the zero-time Green’s function with its inverse cubic \( R \) dependence, which led to the logarithm. Any realistic forcing would be integrated against the finite time response which is less singular than the zero time function. We thus assert at this point that a more detailed analysis should reveal that \( Q_{2m} \) is negligible compared to \( D_{2m} \) in the core of the inertial interval. The power law \( (R/L)^2 \) for the ratio of these quantities is appropriate for K41 scaling, and the exponent may be slightly different for non-K41 exponents. At any rate we are satisfied that for \( R \ll L \) the forcing term is negligible compared to the transfer term.

3. THE ANOMALOUS EXPONENT ASSOCIATED WITH PAIR COALESCENCE

In this section we begin the exploration of the ultraviolet properties of \( n \)-point correlation functions with the aim of computing \( J_n \) and \( D_n \) of the previous section. To achieve this we examine first a second-order Green’s function of the type introduced in paper II and whose ultraviolet properties are characterized by an anomalous exponent \( \Delta \). We will prove that these ultraviolet properties are shared by the second order structure function, and that \( \Delta \) attains the critical value \( \Delta = 2 - \zeta_2 \). Although this statement was made already in paper II section 5.B, we discuss it here in full detail because of its crucial importance for the structure of the theory of turbulence, and in particular in determining also the ultraviolet properties of the quantities appearing in the balance equations (2.24) and (2.25). We begin with the 4-point Green’s functions. The analysis of this section will allow us already to evaluate the correlation function of the dissipation field (3.4) and to derive the important scaling relation (3.5). The strategy of this section is to first introduce the needed many-point Green’s functions, then to examine and classify their diagrammatic representation, and finally to resum the diagrammatic series to obtain exact integral equations for these quantities. The exact equations will allow us to find non-perturbative solutions.

A. Many-Point Green’s Functions

1. Definitions

Consider the nonlinear Green’s functions \( G_{m,n} \) which are the response of the product of \( m \) BL-velocity differences to \( n \) perturbations. In particular
\[ G_{2,1}^{\alpha\beta\gamma}(r_0|x_1, x_2, x_3) = \left\langle \frac{\delta[w_\alpha(r_0)x_1]w_\beta(r_0|x_2)]}{\delta h_\gamma(x_3)} \right\rangle, \quad (3.1) \]
\[ G_{1,2}^{\alpha\beta\gamma}(r_0|x_1, x_2, x_3) = \left\langle \frac{\delta^2 w_\alpha(r_0|x_1)}{\delta h_\beta(r_0|x_2)\delta h_\gamma(x_3)} \right\rangle, \quad (3.2) \]
\[ G_{2,2}^{\alpha\beta\gamma\delta}(r_0|x_1, x_2, x_3, x_4) = \left\langle \frac{\delta^2[w_\alpha(r_0|x_1)w_\beta(r_0|x_2)]}{\delta h_\gamma(x_3)\delta h_\delta(x_4)} \right\rangle, \quad (3.3) \]
\[ G_{3,1}^{\alpha\beta\gamma\delta}(r_0|x_1, x_2, x_3, x_4) = \left\langle \frac{\delta[w_\alpha(r_0|x_1)w_\beta(r_0|x_2)w_\gamma(r_0|x_3)]}{\delta h_\delta(x_4)} \right\rangle. \quad (3.4) \]

Note that these are different from the nonlinear Greens’ function \( G_2 \) that was dealt with extensively in paper II,
\[ G_{2}^{\alpha\beta\gamma\delta}(r_0|x_1, x_2, x_3, x_4) = \left\langle \frac{\delta w_\alpha(r_0|x_1)}{\delta h_\gamma(x_3)} \frac{\delta w_\beta(r_0|x_2)}{\delta h_\delta(x_4)} \right\rangle. \quad (3.5) \]
One can see that the relation between these 4-point Green’s functions follows from the chain rule of differentiation and is

\[ G_{2;2}^{\alpha\beta\gamma\delta}(r_0|x_1, x_2, x_3, x_4) = G_{2}^{\alpha\beta\gamma\delta}(r_0|x_1, x_2, x_3, x_4) + G_{2}^{\alpha\beta\gamma\delta}(r_0|x_1, x_2, x_4, x_3) \]

\[ + \left( \frac{w_x(r_0|x_1)}{\delta h_{x_3} \delta h_{x_4}} \right) + \left( \frac{w_x(r_0|x_2)}{\delta h_{x_3} \delta h_{x_4}} \right). \]  

(3.6)

In order to analyze the properties of the functions \( G_{m,n} \) we will explore their diagrammatic representation.

2. Diagrammatic Representation

In section 3A of paper II it was explained in detail how to produce the diagrammatic representation of \( G_{2} \). Very similar steps lead to the diagrams for \( G_{2;2} \), which are shown in Fig. 1. Of course, due to (3.4) all the diagrams of \( G_{2} \) appear also in the representation of \( G_{2;2} \). These common diagrams all have two principal paths made of Green’s functions as explained in paper II. In \( G_{2;2} \) there are contributions coming from the last two terms in (3.6).

To understand the structure of these new diagram note that every diagram representing \( \Sigma^2 \) in the principal paths has a principal path of Green’s functions that starts with coordinates \( x_1 \) or \( x_2 \) which splits at some point into two principal paths made of Green’s functions ending up with the coordinates \( x_3, x_4 \). The field \( w_2 \) itself is another tree of the type appearing in Fig. 3 of paper II. Upon multiplying these trees and averaging over the Gaussian ensemble of forces we get diagrams that are ready to be resummed. One set of diagrams that appear are those that have a bridge made of a single Green’s function connecting the “left” part of the diagram (coordinates \( r_1, r_2 \)) to the “right” part of the diagram (coordinates \( r_3, r_4 \)). By necessity this Green’s function belongs to the principal path of the diagram which is made of Green’s functions. We note that a bridge made of a single propagator may appear only once. A second singly propagator bridge makes the diagram “one-eddy reducible”, and such diagrams have already been line-resummed; see paper 0 for more details. Every decoration to the left of the bridge may be resummed into a dressed vertex on the left, and every decoration on the right of the bridge may be resummed into a dressed vertex on the right. The fully resummed series of weakly linked (via one-propagator bridge) diagrams which we denote as \( G_{2;2}^w \) contains exactly the two diagrams that are shown in Fig. 1, panels b-d.

The reader’s attention should be drawn to the weakly linked diagrams which have appeared here for the first time. It will turn out that these diagrams play a very important role in the mechanism that we propose for anomalous scaling in turbulence. For that reason we pause for a moment to discuss the topological structure of these diagrams in more detail. By construction the bridge between points (1,2) and (3,4) in the diagrams for \( G_{2;2}^w(r_0|x_1, x_2, x_3, x_4) \) can consist of a Green’s function but not of a correlator. In panel c of Fig. 1 the coordinates of the Green’s function are denoted by \( x_a, x_b \). On the two sides of the \((a-b)\) bridge there exist three-point objects. These objects have an exact resummed form in terms of the dressed vertices \( A \) and \( B \) which were briefly introduced in paper I. Note that we have a freedom in the diagrammatic representation as to whether to include the bridge itself together with the object on the right or on the left. In the former case the resulting object on the right is \( G_{1;2}(r_0|x_a, x_3, x_4) \) which was defined in (3.2). In the latter case, which is the convention that we choose, see panel b, the resulting object on the left side is \( G_{2;1}(r_0|x_1, x_2, x_b) \) which is defined in (3.1). In order to display the bridge \((a-b)\) explicitly we introduce in panel c a new object denoted as \( D(r_0|x_1, x_2, x_a) \). This object has two “entries” \( 1 \) and \( 2 \) starting with propagators and one “exit” denoted \( x_a \) which ends with a vertex, panel d.

We will use an empty small circle to denote the position of a vertex. This is done to distinguish a vertex from a propagator leg.

Next, as explained in paper II, we need to identify two-eddy reducible and two-eddy irreducible diagrams. The first type are diagrams that can be split into a left and a right part by cutting two propagators. For \( G_{2} \) these propagators can only be Green’s functions. This is the cross section denoted by (a) in Fig. 1. In the present case we can also have cross sections of type (b) with one Green’s function and one propagator. There cannot be cross sections with two correlators because we always have at least one principal path of Green’s functions that connects the left part of the diagrams to its right part. Finally we can have a cross section of type (c) in which two Green’s functions appear in opposite orientations.

In summary, the structure of the diagrammatic series can be described as follows: we get ladder diagrams that have alternating Green’s functions and double-correlators in any possible order, as long as there is at least one Green’s function between two rungs which carries the principal path. The ladder diagrams can have alternating rungs of three types, depending on the type of propagators preceding the rung. The three types of rungs are sums of two-eddy irreducible diagrams that are denoted \( \Sigma_{(1,3)} \), \( \Sigma_{(2,2)} \) and \( \Sigma_{(3,1)} \) respectively. Graphically they are represented by a dashed bar, empty bar and doubly dashed bar respectively. The first one has three straight tails and one wavy tail, the second one has two straight and two wavy tails and the last has three wavy and one straight tail. The series for \( \Sigma_{(n,m)} \) are shown in Fig. 8 and Fig. 10. Fig. 2 shows the diagrams for \( \Sigma_{(2,2)} \). This series is composed of all the diagrams in Fig. 10a of paper II (and diagrams 1, 2 and 5 in Fig. 8 are examples of those) in addition to new diagrams like diagrams 3 and 4 in Fig. 8. All the old diagrams have a horizontal principal cross section that cuts through correlators only. The new diagrams contain the split in the principal path, and the principal cross section through correlators turns up...
or down by 90°. The series for the two other two-eddy irreducible mass operator Σ(1,3) and Σ(3,1) are shown in Fig. 5b. The diagrams in these series have no principal cross section that cuts through correlators only.

3. Resummation of the strongly linked contributions

As in the case of the ladder diagrams of $G_2$ we can resum also in the case of $G_{2,2}$ all the ladder diagrams into integral equations. The resummed ladders are shown in Fig. 3. The diagrammatic notation of $G_{2,2}$ is an empty circle with two wavy and two straight lines. It has three different types of contributions. First come the reducible contributions that we denote by $G_{2,2}^{(0)}$ and are represented by the unlinked diagrams (1) and (2) of Fig. 2. The first two ladders (3) and (4) in Fig. 3 are identical to the RHS of the resummed equation for $G_2$. The next term on the RHS in Fig. 3 are new. The resummation of the ladders is shown in Figs. 3. On the RHS of the diagrammatic equation we find a Green’s function which is defined in Fig. 3 as $G_{s,1}$ and whose graphic notation has a crossed circle with three wavy and one straight tails. This object is again resummed in terms of itself and $G_{2,2}$ as shown in Fig. 5b.

In conclusion we learn that the 4-point Green’s functions $G_{m,n}$ satisfy equations that contains linear operators acting on $G_{m,n}$ and inhomogeneous terms that are products of $G$‘s and the weakly linked contributions. To be sure, the linear operators are themselves functionals of $G_{m,n}$, so that our equations are in fact nonlinear. If we expand the solutions of these inhomogeneous nonlinear equations around the inhomogeneous terms we generate back the initial diagrammatic expansion. However, now we can also explore the “homogeneous” solutions of these equation that are obtained by discarding the inhomogeneous terms. Such solutions, if they exist, are manifestly nonperturbative effects. We will need to show a posteriori that these nonperturbative solutions are much larger than the solutions that can be found from perturbative analysis. Indeed, in paper II it was shown in detail that $G_2$ has such a nonperturbative homogeneous solution with the following property: when the first two coordinates $r_1$ and $r_2$ are of the same order (say $r$) and much smaller than the last two coordinates $r_3$ and $r_4$ (which are of the order of $R$) then

$$\nabla^2 G_2(r, r, R, R) \sim r^{-\Delta} \quad \text{for} \quad R \gg r \quad . \quad (3.7)$$

Since the same terms appear in the equation for $G_{2,2}$ we know that such a divergence (with $r \to 0$) must appear also in all $G_{a,m}$. (This of course hinges on the assumption that the new terms in Fig. 3 do not contribute an exact cancellation; in view of their different nature we judge this possibility unlikely). The new terms may have a stronger or a weaker divergence, and we turn now therefore to an exact calculation of the value of $\Delta$.

B. Calculation of $\Delta$

In order to evaluate $\Delta$ we establish a fundamental identity which is

$$\frac{\delta F_{\alpha\beta}(r_0|x_1,x_2)}{\delta D_{\gamma\delta}(x_3, x_4)} = G_{\alpha\beta}^{\gamma\delta}(r_0|x_1,x_2,x_3,x_4) \quad , \quad (3.8)$$

where the covariance $D$ is the correlation of the perturbations,

$$D_{\gamma\delta}(x_3, x_4) = \langle h_\gamma(x_3) h_\delta(x_4) \rangle \quad . \quad (3.9)$$

The identity is proved most easily using the path integral formulation as reviewed in paper I. In terms of the functional $Z(l,m)$ of Eq. (I–3.12) the second order Green’s function is

$$G^{\alpha\beta\gamma\delta}_{2,2}(r_0|x_1,x_2,x_3,x_4) = - \langle w_\alpha(r_0|x_1) w_\beta(r_0|x_2) p_\gamma(x_3) p_\delta(x_4) \rangle \quad . \quad (3.10)$$

On the other hand we see from Eq. (3.14) that the derivative with respect to $D_{\gamma\delta}$ brings down $ip_\alpha p_\beta$:

$$\frac{\delta I_0}{\delta D_{\gamma\delta}(x-y)} = ip_\gamma(x)p_\delta(y) \quad . \quad (3.11)$$

This means that the functional derivative of $\langle w_\alpha w_\beta \rangle$ with respect to $D_{\gamma\delta}$ is precisely the RHS of (3.10). This is proof of the identity.

Rewrite now the identity in the form

$$\delta F_{\alpha\beta}(r_0|x_1,x_2) = \int dx_3 dx_4 G_{\alpha\beta}(r_0|x_1,x_2,x_3,x_4) \delta D_{\gamma\delta}(x_3, x_4) \quad . \quad (3.12)$$

In this form this is a relation of the response $\delta F$ in the velocity correlator $F$ to a variation in the correlator of the random forcing $\delta D$. It is clear now that if the random forcing is limited to scales $r_1, r_2 \gg r_1, r_2$, the existence of flux equilibrium with a scaling solution for $F_{\alpha\beta}(r_0|x_1,x_2)$ means that the variation $\delta F_{\alpha\beta}(r_0|x_1,x_2)$ must be proportional to $F_{\alpha\beta}(r_0|x_1,x_2)$ itself. We can have a change in the amplitude but not in the functional form:

$$\delta F_{\alpha\beta}(r_0|x_1,x_2) \propto F_{\alpha\beta}(r_0|x_1,x_2) \quad \text{for} \quad r_1, r_2 \ll r_3, r_4 \quad . \quad (3.13)$$

To understand what are the requirements of the variation $\delta D_{\gamma\delta}(x_3, x_4)$ that guarantee the validity of the universal behaviour (3.13) let us write again the Wyld equation (II–2.9). We will use economic notation, such that the vector index carries implicitly also the position coordinate. Repeated indices must be summed upon and the convention is that this sum also requires integration over the intermediate position coordinates. This convention allows us to write the Wyld equation as

$$F_{\alpha\beta} = G_{\alpha\gamma}[D_{\gamma\delta} + \Phi_{\gamma\delta}]G_{\beta\delta} \quad . \quad (3.14)$$
The variation $\delta D_{\gamma\delta}$ causes a variation in $F$. In our convention
\[ F_{\alpha\beta} + \delta F_{\alpha\beta} = G_{\alpha\gamma}[D_{\gamma\delta} + \delta D_{\gamma\delta} + \Phi_{\gamma\delta}]G_{\beta\delta}. \] (3.15)
Next note that $\Phi_{\gamma\delta}(r_0|x_a, x_b)$ can be exactly expressed as a second derivative of the 4-point correlation function $F_4(x_a, x_a, x_b, x_b)$, cf. Eq. (I-4.5). Consequently
\[ \Phi_{\gamma\delta}(r_0|x_a, x_b) \sim r_{ab}^{\zeta_4 - 2}. \] (3.16)
We know that $\zeta_4$ is expected to be considerably smaller than 2. (The K41 estimate is $\zeta_4 = 4/3$, whereas experimentally one finds $\zeta_4 \sim 1.2$.) Thus $\Phi_{\gamma\delta}(r_0|x_a, x_b)$ is growing when the coordinates become smaller, whereas $\delta D_{\gamma\delta}$ is restricted to the large scales and is decaying for smaller coordinates. We expect therefore that $\delta F$ will be proportional to $F$ and the constant of proportionality is determined by the boundary conditions at large scales where both $D$ and $\delta D$ are not negligible.

For future reference we should note at this point that the proportionality of $\delta F_{\alpha\beta}$ and $F_{\alpha\beta}$ is not restricted only to their scaling exponents. In fact the two quantities share the same tensor structure. In other words they are the same function of $r_1 - r_0$ and $r_2 - r_0$ up to constants that are independent of the tensor indices. To complete the argument choose now $t_1 = t_2$. Next notice the fact that
\[ \nabla_1 \cdot \nabla_2 F_{\alpha\beta}(r_0|r_1, r_2) \propto r_{12}^{\zeta_2 - 2}. \] (3.17)
Now restrict $\delta D_{\gamma\delta}(x_3, x_4)$ to $r_3, r_4 \sim R \gg r_1, r_2$. Applying the operator $\nabla_1 \cdot \nabla_2$ to (3.14) we conclude that
\[ \nabla_1 \cdot \nabla_2 G_{\alpha\beta\gamma\delta}(r_0|r_1, r_2, x_3, x_4) \propto r_{12}^{\zeta_2 - 2}. \] (3.18)
Note that the RHS is a function of $r_{12}$ only; the reason for this is that under the derivitives the dependence on $r_0$ disappears. Accordingly the restriction of validity of this result is not necessarily $r_1, r_2 \ll R$ but just $r_{12} \ll R$. Comparing with (3.17) we reach the central result of this section:
\[ \Delta = 2 - \zeta_2. \] (3.19)
Finally we can argue that the application of the operator $\nabla_1 \cdot \nabla_2$ to $G_{3,1}$ will give rise to the same exponent $\Delta$ as in (3.18). To see this apply the operator to the two equations in Fig.5. Suppose that the divergence associated with $G_{3,1}$ is stronger than $\Delta$. This will immediately force the divergence of the LHS of Fig.5a to be stronger than $\Delta$, in contradiction with our exact result. In the same manner the divergence of $\nabla_1 \cdot \nabla_2 G_{3,1}$ cannot be weaker than $\Delta$ due to the equation in Fig.5b. We thus conclude that also
\[ \nabla_1 \cdot \nabla_2 G_{3,1}(r_0|r_1, r_2, x_3, x_4) \propto r_{12}^{\zeta_2 - 2}. \] (3.20)
C. Three Point Objects and the Weakly Linked inhomogeneous Contributions

Equation (3.19) was derived by neglecting the inhomogeneous part of the equation for $G$. In order to be sure that the inhomogeneous part is not important we need to evaluate now the weakly linked contributions and compare them with (3.18) and (3.20). We are not going to perform the evaluation with the same care as we did in the computation of $\Delta$. All that we need is to show that these terms are negligibly small. We will show this by assuming that the diagrams have the property of rigidity which was demonstrated in paper II order by order. This will be sufficient since we will be able to show that there exists a large gap in the value of the scaling exponents of the homogeneous and the inhomogeneous terms. Such a large gap is not expected to be swamped by nonperturbative effects.

To prepare for this evaluation we need first the diagrammatic representation of three-point objects. These are the 3-point correlator $G_{3,2}$ and the Green’s function $G_{2,1}$ and $G_{1,2}$. All the three point objects can be represented with the help of the tensor of dressed vertices which were denoted in paper II as A, B and C respectively, see paper II Fig. 7. We remind the reader that vertex A is a junction of one straight and two wavy lines, vertex B is the junction of two straight and one wavy line, whereas vertex C is the junction of three straight lines. The diagrammatic representation of the triple correlation function $G_3$ in terms of these vertices is shown in Fig. 7 of paper II, and is reproduced in more compact form in Fig. 8 panels b,c. In the same manner the Green’s function $G_{2,1}$ is represented here in Fig. 8 panel c.

Consider now the weakly linked diagrams for $G_{2,2}$ which are resummed into the form shown in the diagram in Fig. 8 panel b. In the limit of $r_1 \sim r_2 \sim r \rightarrow 0$ and $r_3 \sim r_4 \sim R$, rigidity means that the integral over $r_6$ contributes mostly in the region $r_0 \sim R$. We thus need to understand the $r$ dependence of $G_{2,1}(r_0|x_1, x_2, x_3)$ in the asymptotic situation $r_1, r_2 \ll R$. Looking at Fig. 8 panels c, d we see that due to rigidity the vertices A and B will contribute mostly in the region $r_0 \sim r$. In K41 scaling the exponent of $r$ is found by noticing that the correlator contributes $r^{\zeta_2}$, and the Green’s function together with the integration over $x_a$ contribute $r^2$. The Green’s function of the bridge $a - b$ in Fig. 8 panel c contributes $r/R^4$, or $r^4$. The vertex A itself in diagrams 1 and 2 in Fig. 8 panel d gives $r^{-1}$, and in total we find $r^{\zeta_2 + 2}$ for these contributions. The same arguments give the same asymptotic behaviour of diagram 3 shown in Fig. 8 panel d. This behaviour should be compared with $r^{\zeta_2}$ for the homogeneous term, justifying the neglect of the inhomogeneous contribution. Note that small corrections to the exponents are not expected to erase the large gap $\zeta \simeq 2/3$. 
4. 2-POINT FUSION RULE FOR 4-POINT CORRELATION FUNCTIONS: THE EQUATION FOR THE EXPONENT $\mu$

In this section we use the fact that we know exactly the ultraviolet exponent $\Delta$ to derive the fusion rule for 4-point correlations. One immediate consequence of these rules will be a derivation of the scaling law (4.3). The $n$-point simultaneous correlation function of Eulerian velocity differences is defined as

$$F_{\alpha_1...\alpha_n}(r_1, r_2, \ldots, r_n) \equiv \langle \delta u_\alpha(r_0) R_{1, \alpha} \delta u_\beta(r_0) R_{2, \beta} \cdots \delta u_\omega(r_0) R_{n, \omega} \rangle .$$

(4.1)

where $\delta u$ was defined by (2.2) such that $R_j = r_j - r_0$. By “fusion rules” we mean the scaling structure of $F_n$ when two or more of the vector separations $R_j$ tend to zero or to each other. In this section we discuss 2-point fusion rules. Without loss of generality we can denote the coalescing coordinates as $r_1$ and $r_2$, and we will consider all distances of all the other coordinates $r_3, \ldots, r_n$ from $r_0$ to be of the same order $O(R) \gg r_1, r_2$. To understand the fusion rules we will use diagrammatic language; at the end of the discussion one can recast the results into formal operator algebra of a new type without reference to diagrams. This operator algebra (which allows multi-scaling) is an exciting subject that will be taken up fully in a separate publication [17].

A. Classification of the Diagrams for the 4-Point Correlator

In this section we will analyze the 4-point correlator and the dissipation correlation function $K_{cc}$. In particular we derive the scaling relation (4.3) for the exponent that governs the power law decay of $K_{cc}$ according to the (4.3). The derivation begins with the examination of the 4-point correlator $F_4^{\alpha_1...\alpha_3}(r_1, r_2, r_3, r_4)$. This quantity is represented as an object with 4 wavy tails, each of which represents one of the velocity differences with coordinates $r_1, \ldots, r_4$. The trivial contributions to $F_4$, which we denote as $F_4^{(0)}$, are the three different products of $F_2$ which graphically are the “unlinked” (or “reducible”) diagrams which stem from the Gaussian decomposition. The next set of diagrams that contributes to the 4-point correlator is shown in Fig. 7. These are all the weakly linked diagrams in which the coordinates $x_1, x_2$ are linked to the coordinates $x_3, x_4$ via one propagator. As in the case of $G_{n,m}$ we cannot have repeated one-propagator bridges since such contributions are resummed in the Dyson-Wyld line resummation. There are two possible types of bridges: (i) the bridge ends with a straight line and (ii) the bridge ends with a wavy line. In case (i) the diagrams can be resummed into diagram 1 of Fig. 8 panel a. In case (ii) the diagrams resum into diagram 2. The left part of diagram 1 is the Green’s function $G_{2,1}$. The diagrammatic presentation of this object is shown in Fig. 8 panels c and d. The right part of diagram 1 is the three point object $A$ shown in panel b of Fig. 8. It is the resummed part of the dressed vertices A,B and C, and the dressed propagators. The left part of diagram 2 in Fig. 8 is the third order correlation function $F_3$. Its diagrammatic representation in terms of dressed vertices and propagators is given by Fig. 7 of paper II, and in a more compact form in Fig. 8 panel c. We find again in this case the same three-point objects $A$ and $D$ that appeared already in panel b of this figure and in panel d of Fig. 8.

In addition to these weakly linked diagrams we have the strongly linked diagrams appearing in Fig. 7 panel b. All these are diagrams whose resummed parts are connected via two-propagator bridge. The strongly linked diagrams are constructed as follows: locate the principal cross section of a diagram in the infinite expansion of $F_4$. Move from the principal cross section to the right and to the left until you find the first two-eddy reducible link, which is going to form the two-propagator bridge. The object between these two bridges is a contribution towards one of the resummed central objects on the right. To understand the fusion rules we will use diagrammatic language; at the end of the discussion one can recast the results into formal operator algebra of a new type without reference to diagrams. This operator algebra (which allows multi-scaling) is an exciting subject that will be taken up fully in a separate publication [17].

B. Asymptotics

1. Strongly Linked Diagrams

Assume that $r_1, r_2 \sim r$ are much smaller than $r_3, r_4 \sim R$ but still in the inertial interval. Denote the structure whose outer coordinates are $r_a, r_b, r_3, r_4$ as $\Psi_m(r_a, r_b, r_3, r_4)$ with $m = 1 \ldots 4$ according to the enumeration of the diagrams in Fig. 7. Diagram (1–4) can now be written analytically as

$$(1) = \int dx_0 dx_b G_{2,2}(r_0|x_1, x_2, x_a, x_b) \Psi_1(r_a, r_b, r_3, r_4) ,$$

(4.2)

$$(2) = \int dx_0 dx_b G_{3,1}(r_0|x_1, x_2, x_a, x_b) \Psi_2(r_a, r_b, r_3, r_4) ,$$

(4.3)

$$(3) = \int dx_0 dx_b G_{2,2}(r_0|x_1, x_2, x_a, x_b) \Psi_3(r_a, r_b, r_3, r_4) ,$$

(4.4)

$$(4) = \int dx_0 dx_b G_{3,1}(r_0|x_1, x_2, x_a, x_b) \Psi_4(r_a, r_b, r_3, r_4) .$$

(4.5)
Suppose that we can argue that all the functions \( \Psi_3(r_a, r_b, r_3, r_4) \) contribute at \( r_a, r_b \) smaller contributions than those of \( \Phi(r_a, r_b) \) in Eq. (3.13). If so, we can think of these \( \Psi \) functions as generalized perturbations \( \delta D \) that leave the \( r_{12} \) dependence universal. According to (3.12) and (3.13) the main \( r_{12} \) dependence comes from diagrams (1) and (3) and it is \( r_{12}^{\alpha} \).

To demonstrate that this is indeed the case we can look at a typical diagrammatic contribution to any of the diagrams in Fig. 3. For example, the first contribution to \( \Psi_1(r_a, r_b, r_3, r_4) \) is shown in Fig. 3. This particular example comes from diagram (1) in Fig. 3. Panel b upon taking diagram (1) for the rung in Fig. 2. Take \( r_a \sim r_b \) and examine the dependence on \( r_a \). The two vertices contribute \( r_{12}^{-2} \). The correlator in between is worth \( r_{12}^{\delta} \). The two stretched correlators connecting to larger coordinates contribute \( r_{12}^{2\delta} \). The two time integrals in the vertices \( c \) and \( d \) contribute \( r_{12}^{2\delta} \) where \( z \) is the dynamic scaling exponent, \( \tau(r) \propto r^z \) (and see details in paper I). In total we have \( r_{12}^{\delta} \) with \( \delta = 3\zeta_2 + 2z - 2 \). This has to be compared with the exponent \( \alpha \) of \( \Phi(r_a, r_b) \) which is \( \alpha = \zeta_2 - 2 \) according to (3.10). In K41 evaluation \( \beta = 4/3 \) whereas \( \alpha = -2/3 \). The gap is so large that small corrections to K41 cannot change the fact that the exponent of \( \Phi \) is dominant, as we want. Note that this demonstration is perturbative (order by order with only the lowest order done explicitly), but again the magnitude of the gap is sufficient to validate the claim.

Exactly the same arguments hold for the contribution (3.14). The contributions (4.3) and (4.13) relate to the asymptotic behaviour of the other Green's function \( G_{2.1} \). It is clear however that due to the result in (3.24) the exponents are the same.

### 2. Weakly Linked Diagrams

The analysis of the weakly linked diagrams in this case follows closely the discussion of section 3C. The diagrams can be assumed as shown in Fig. 3, panel b. We have two contributions; in diagram (a) we have on the left \( F_3 \) and on the right the object defined in the second line of the figure. In diagram (2) we have \( G_{2.1} \) (shown in Fig. 3) on the left and the object defined in the third line of Fig. 3, panel b. In the limit \( r_1, r_2 \sim r \ll r_3, r_4 \sim R \) the integrals over \( r_3 \) and \( r_4 \) in both diagrams contribute mainly in the region \( r_3 \sim R \). Accordingly in diagram (2) we have the same situation that was discussed in section 3C, and see Fig. 3, panel b. Diagram (1) on the other hand may be bounded from above by the asymptotics of \( F_3 \) with one large and two small coordinates. Under the property of rigidity one can see that this correlator is independent of the large coordinate, and is therefore proportional to \( r^4 \). Again we have a gap, and this gap can only increase if we take into account the time integral of the \( x_b \) vertex. Thus the weakly linked diagrams can be again disregarded.

### C. 2-Point Correlation of the Energy Dissipation

At this point we can employ our results to evaluate the correlation function (4.4). We first write this quantity in a way that makes its relation to \( F_4 \) clear:

\[
K_{\epsilon \ell}(R) = \nu^2 \lim_{r_{12}, r_{34} \to 0} \lim_{r_{13} \to R} \nabla_1 \nabla_2 \nabla_3 \nabla_4 \left[ F_4^{\alpha' \beta' \gamma' \delta'}(r_1, r_2, r_3, r_4) - F_4^{\alpha' \beta'}(r_1, r_2) F_4^{\gamma' \delta'}(r_3, r_4) \right] \\
\times \left[ \delta_{\alpha \beta} \delta_{\alpha' \beta'} + \delta_{\alpha \beta'} \delta_{\alpha' \beta} \right] \left[ \delta_{\gamma \delta} \delta_{\gamma' \delta'} + \delta_{\gamma \delta'} \delta_{\gamma' \delta} \right]. \tag{4.6}
\]

This is a generalization of the situation discussed above in the sense that we have two pairs of coalescing points, \( r_1, r_2 \) and \( r_3, r_4 \). By examining Fig. 5a we can see that the symmetry allows us to treat each coalescing pair separately. In the limit \( r_{12} \) and \( r_{34} \) being much smaller than \( R \) each pair of derivatives \( \nabla_1 \nabla_2 \) and \( \nabla_3 \nabla_4 \) will contribute a divergent term proportional to \( r_{12}^{-\Delta} \) and \( r_{34}^{-\Delta} \) respectively. The dependence on \( R \) can be found knowing that the overall exponent of \( F_4 \) is \( \zeta_4 \). Thus the overall scaling exponent of the quantity in the RHS (4.6) must be \( \zeta_4 - 4 \).

Using the fact that \( \Delta = 2 - \zeta_4 \) we conclude that

\[
\nabla_1 \nabla_2 \nabla_3 \nabla_4 F_4(r_1, r_2, r_3, r_4) \sim \xi^{4/3} r_{12}^{-2\Delta} R^{4-2\zeta_4} \xi^{4/3-\zeta_4}, \tag{4.7}
\]

where for simplicity we suppressed the vector indices and we made the relation dimensionally correct by introducing some renormalization scale \( \ell \). Note that for K41 scaling exponents the factor \( \xi^{4/3} \) fixes the correct dimensionality of the RHS. For anomalous exponents one needs a renormalization scale to take care of the difference between the K41 and the actual value of the exponent \( \zeta_4 \). It will be shown later that this renormalization scale must the outer scale \( L \). At this point it does not matter. The divergence in the limit indicated in (4.6) should be understood in light of the full theory for \( F_4 \), in which the \( \nu \)-diffusive terms are explicit. The role of these terms is precisely to truncate the divergence that is implied by (4.7).

As a consequence the divergence is only applicable in the inertial range with \( r_{12}, r_{34} > \eta \), whereas in the dissipative regime the divergence disappears. Thus for evaluating \( K_{\epsilon \ell}(R) \) via inertial range values we must replace the limit \( r_{12}, r_{34} \to 0 \) by \( r_{12} = r_{34} = \eta \). Thus we can write

\[
K_{\epsilon \ell}(R) \sim \xi^{4/3} \eta^{-2\Delta} R^{4-2\zeta_4} \xi^{4/3-\zeta_4}. \tag{4.8}
\]

Finally, we need to evaluate the viscous scale \( \eta \). This is done from the definition of \( \xi \) which is

\[
\xi \sim \nu \lim_{r_{12} \to \eta} \nabla_1 \nabla_2 S_2(r_{12}). \tag{4.9}
\]

In the same spirit we can write the structure function \( S_2(R) \) as

\[
S_2(R) \sim \xi^{2/3} R^{2\zeta_4} \xi^{2/3-\zeta_2}. \tag{4.10}
\]
where again the renormalization scale $\ell$ fixes the dimensions. From the last two equations one can compute
\[ \hat{c}^{1/3} \sim \nu^{-\Delta} \ell^{2/3-\zeta_2}. \tag{4.11} \]
Substituting this in (4.8) we get finally
\[ K_{nn}(R) \sim \hat{c}^2 \left( \frac{\ell}{R} \right)^{2\zeta_2-\zeta_4}, \tag{4.12} \]
which is the scaling law (1.3) that we wanted to derive. It will turn out that the renormalization scale $\ell$ is the outer scale of turbulence $L$. From this we get Eq. (1.9)
\[ \mu = 2\zeta_2 - \zeta_4. \tag{4.13} \]

5. 2-POINT FUSION RULE FOR MANY-POINT CORRELATION FUNCTIONS: THE DERIVATION OF $J_n$

In this section we evaluate the scaling exponent of the dissipative terms $J_n$ (2.40) of the balance equations. The strategy is to expose the divergence with the ultraviolet cutoff $\eta$ for which we have an exact evaluation of the exponent $\Delta$. Together with the overall scaling which is determined by $\zeta_n$ we will be able to compute the $R$ dependence of $J_n(R)$. Having that we will use an argument due to Kraichnan to find the coefficient of the power law. We believe that our result for $J_n(R)$ is exact, even though some of steps taken are not rigorous. Afterwards we will evaluate the interaction term $D_n(R)$ and use the balance equation as a non-perturbative constraint to deduce the scaling exponents $\zeta_n$.

A. Classification of the Diagrams as Weakly Linked or Strongly Linked

The $n$-point correlation function $F_n$ can be represented symbolically as an object with $n$ wavy tails, each one representing one of the $n$ coordinates $r_j$ associated with a velocity difference $\delta u(r_0|R_j)$. We remind the reader that for simultaneous correlation functions one can use either the Eulerian differences (2.2) or the BL-velocity differences (2.11), since they are the same at $t = 0$. In the diagrammatic expansion for the simultaneous $F_n$ there appear time dependent correlations, and there the theory calls for the use of BL-velocity differences.

Having two special coordinates $r_a$ and $r_b$ we can ask how the part of the diagram containing these coordinates is linked to the rest of the diagram. This part can be connected via one propagator, see Fig. 10 panel a, via two propagators, see Fig. 10 panel b, or via three or more. As in the case of the 4-point correlator and 4-point Green’s function a one-propagator bridge cannot appear again between the legs carrying the designation $a, b$ and the body of the diagram. In total one has $n(n-1)/2$ weakly-linked contributions in each of which the role of the weakly linked pair is played by one of the available pairs of legs. In addition one has also double weakly-linked contributions with two bridges made of a single propagator which connects different pairs, etc, and see diagram 3 in Fig. 10 panel a. Such diagrams do not play an important role in our analysis. In displaying the diagrams in Fig.10a we have included the bridge (which is a correlator or a Green’s function) in the object on the left of the bridge. For this reason the object on the right of the bridge begins with a vertex $x_e$. According to our convention this vertex is denoted by a small empty circle. The diagrams appearing in Fig. 10 panel a should already look familiar. In diagram 1 we again have a $F_3$ correlator on the left, integrated over $x_e$, with an $(n-1)$-point object on the right. This is the generalization of the diagram 2 in Fig. 10 panel a. In diagram 2 we have the 3-point Green’s function $G_{3,1}$ of Eq.(3.1) on the left, again integrated over $x_e$ with a different $(n-1)$-point object. This can be compared with diagram 1 in Fig. 10 panel a. For our considerations the precise nature of the object on the right is irrelevant.

Similarly, all the contributions with two propagators serving as links can be resummed into the objects shown in Fig. 10 panel b. Again we included the two propagators that form the bridges in the objects appearing to the left of the bridges. Again this results in the $n$-point objects on the right being attached to the left by two vertices $x_a$ and $x_b$. Their interpretation is as follows: diagram 1 is a fourth order correlator linked via integrals over $x_a$ and $x_b$ to the rest of the diagram. In diagram 2 the linked object is $G_{3,1}$, (3.4). In diagram 3 the linked object is precisely our familiar second order Green’s function $G_{2,2}$.

Links with three or more propagators have been already taken into account in this presentation. This classification of the diagrams is based on starting with the two special coordinates $x_1, x_2$ on the left, and then moving to the right and stopping when the first one-propagator bridge appears. All these diagrams are resummed exactly into one of the contributions in Fig.10a. If there is no one propagator bridge, we start again from the left and monitor all the two-propagator bridges, ending with the last pair. All such diagrams are resummed into one of the contributions in Fig. 10 panel b.

B. Asymptotic Tensor Structure and Fusion Rules

In this section we find the tensor structure of $F_n^{\alpha\beta...\omega}$ when the two coordinates $r_1, r_2$ are much smaller than the rest. When all the coordinates $r_3, \ldots r_n$ are of the order of $R \gg r$, whereas $r_1, r_2$ are small, the property of rigidity that was demonstrated in paper II requires that the main contribution to the integrations over $r_a$ and $r_b$ in the diagrams of Fig. 10 panel b come from the
region \( r_a \sim r_b \sim R \). In this case we have a very similar situation to the one discussed before in the context of Fig. [6] panel b. Accordingly the three objects on the left of the double bridge of the diagrams in Fig. [6] panel b are familiar and have the same scaling with respect to \( r \), i.e. \( r^2 \). The weakly linked contributions shown in Fig. [4] panel a have the same objects on the left of the bridge as those shown in Fig. [6] panel a for \( F_2 \), and they are irrelevant for the same reasons.

Limiting our attention to strongly-linked diagrams with two propagator links we examine now the tensor structure of \( F_n^{\alpha \beta \ldots \omega} \). We need to keep in mind equation [4.13] which means that when \( r_1, r_2 \to 0 \) \( \delta F^{\alpha \beta} \propto F^{\alpha \beta} \). In light of Fig. [4] the objects \( G_{2,2}^{\alpha \gamma \delta} \) and \( G_{4,1}^{\alpha \gamma \delta} \) have the same asymptotics. We have found before that they are proportional to \( F^{\alpha \beta}(r_1, r_2) \). Diagram 1 is again proportional to the same quantity in light of Fig. [6] panel b. We can therefore write the following fusion rule:

\[
\lim_{r_1, r_2 \to 0} F^{\alpha \beta \gamma \ldots \omega}_n (r_1, r_2, R_3 \ldots R_n) = F^{\alpha \beta}(r_1, r_2) \Psi_{n-2}^{\gamma \ldots \omega}(R_3 \ldots R_n). \tag{5.1}
\]

Here \( \Psi_{n-2} \) is a homogeneous function of its arguments when they are all in the inertial range. The scaling exponent of \( \Psi_{n-2} \) is \( \zeta_n - \zeta_2 \). The reason for that is clear: the scaling exponent of \( F_n \) is \( \zeta_n \), but one \( \zeta_2 \) is already carried by \( F^{\alpha \beta} \). The tensor structure of \( \Psi_{n-2} \) is not known in detail at this point, except that it conforms with incompressibility and isotropy. In other words, this quantity is independent of the vectors \( r_1, r_2 \).

C. The Evaluation of \( J_n(R) \)

At this point we are ready to evaluate the dissipative terms in the balance equation. The equation to consider is (2.40). As was done in the context of the evaluation of \( K_n \) we evaluate the quantity in the inertial interval, and then replace the limits \( d_3 \to 0 \) with the ultraviolet cutoff \( d_3 = \eta \). In the inertial interval we write

\[
T^{\alpha \alpha}_{2m}(d_3 e_\beta, d_3 e_\beta, \eta a, R) = F^{\alpha \alpha}(d_3 e_\beta, d_3 e_\beta) \Psi_{2m-2}^{(1)}(R), \tag{5.2}
\]

\[
T^{\alpha \gamma \alpha \gamma}_{2m}(d_3 e_\beta, d_3 e_\beta, R) = F^{\alpha \gamma}(d_3 e_\beta, d_3 e_\beta) \Psi_{2m-2}^{(2)}(R), \tag{5.3}
\]

where the fusion rule (5.1) has been used. The two functions \( \Psi \) satisfy

\[
\Psi_{2m-2}^{(1)}(R) = A_1 R^{\zeta_2 - \zeta_2}, \tag{5.4}
\]

\[
\Psi_{2m-2}^{(2)}(R) = A_2 R^{\zeta_2 - \zeta_2} \left[ \delta_{\alpha \gamma} - a_m \frac{R_\alpha R_\gamma}{R^2} \right], \tag{5.5}
\]

with \( A_1 \) and \( A_2 \) being constants. The reason for these forms stems again from the fusion rule. \( \Psi^{(1)} \) and \( \Psi^{(2)} \) are scalar and 2-tensor respectively, and we wrote their general forms for isotropic conditions. Incompressibility dictates the value of \( a_m \) but we do not need to compute it for our purposes. Lastly we note that

\[
F^{\alpha \gamma}(d_3 e_\beta, d_3 e_\beta) = S^{\alpha \gamma}_2 (d_3) \propto d_3^2 \left[ \delta_{\alpha \gamma} - a_1 \frac{R_\alpha R_\gamma}{R^2} \right]. \tag{5.6}
\]

Incompressibility requires the relation \( a_1 = \zeta_2/(1 + \zeta_2) \). Presently we can substitute all this knowledge into Eq. (2.40). One should note that in the inertial range the velocity field is not smooth, \( \zeta_2 < 2 \) and we may run into the danger that the quantity computed depends on the angle between the vectors \( d, R \). However the procedure implied requires taking the limit such that all \( d_3 = \eta \). This can be checked explicitly by introducing the tensor structure to all the quantities appearing in the limit that indeed the condition that all the components \( d_3 \) are the same guarantees that the limit is independent of the angle. The result of the substitution is

\[
J_{2m}(R) = 2m C_{2m} \eta^{1-\Delta} R^{2m-\zeta_2} \delta^{2m/3 - \zeta_2}. \tag{5.7}
\]

We again use a renormalization scale \( \ell \) to fix the dimensions. It will turn out that this renormalization scale is the outer scale of turbulence \( L \).

We will see that this form of \( J_n \) is sufficient for the calculation of the exponents \( \zeta_n \) through the use of the balance equation only if the dependence of \( \zeta_n \) on \( n \) is linear (i.e. in the \( \beta \)-model). If the dependence is nonlinear (multiscaling) we need to determine the coefficients \( C_{2m} \) exactly. To do so we rewrite now Eq. (5.7) in terms of the structure functions \( S_n \). This way of writing is compelling only when the scaling exponents are a nonlinear function of \( n \), as will be clear in a moment. We write \( S_n \) in the form

\[
S_{2m}(R) \sim \ell^{2m/3} R^{2m-\zeta_2}. \tag{5.8}
\]

This form is the generalization of (4.10) and is the most general form that conforms with scaling and is dimensionally correct. Using Eqs. (4.10) and (4.11) in (5.8) we find the convenient representation

\[
J_{2m}(R) = m C_{2m} C_2 J_{2m}(R) \frac{S_{2m}(R)}{S_2(R)}. \tag{5.9}
\]

We stress that this result is valid only when \( R \gg \eta \), since we used the asymptotics, and only when \( \zeta_n \) is nonlinear in \( n \). If the scaling exponents are linear in \( n \) we can have other contributions like \( S_{2m+1}/S_3 \) or any other ratio whose scaling exponent is \( \zeta_n - \zeta_2 \). In particular (5.9) is not applicable to Burgers turbulence.

Finally, we will employ an idea that is due to R.H. Kraichnan [17] to argue that in the multiscaling case the coefficient \( C_{2m} \) is \( m \)-independent. Begin with Eq. (2.23) which is rewritten as an integral over the distribution function \( P(w) \):
Here \( \langle \nabla_r^2 + \nabla_{r_0}^2 \rangle \) is the conditional average of \( \nabla_r^2 + \nabla_{r_0}^2 \) conditioned on a given value of \( \textbf{w}(\textbf{r}_0) + \textbf{R}, t \). The point to observe now is that the only way to recover our result (5.9) when \( \zeta_n \) is a nonlinear function of \( n \) is to demand that the conditional average satisfies

\[
\langle \nabla_r^2 + \nabla_{r_0}^2 \rangle |_{\textbf{w}(\textbf{r}_0) + \textbf{R}, t} = C \frac{w_n(\textbf{r}_0) + \textbf{R}, t}{S_2(R)} ,
\]

where \( C \) is some coefficient which is evidently independent of \( m \). It follows that \( C_{2m} \) is independent of \( m \).

Note that this result for the conditional average is only valid in the inertial range of scales, since it has been derived using Eq. (5.3) which is only valid there. Notwithstanding we can employ (5.11) right away to compute the vector quantity \( J_{2m+1}^0(R) \). Writing (2.27) again as an integral over the distribution function \( P(\textbf{w}) \) and substituting (5.11) we find

\[
J_{2m+1}^0(R) = \frac{(2m+1)}{2} J_2 \frac{S_{2m+1}(R)}{S_2(R)} .
\]

We can thus summarize this section with a result that is valid for both odd and even \( n \) by using the scalar counterpart of the vector quantities:

\[
J_n(R) = J_2 \frac{n S_n(R)}{2 S_2(R)} .
\]

This is the final result of this section. We note that such a scaling formula for \( J_{2m} \) was suggested by Kraichnan in the context of passive scalar advection ([17]), and was derived in ([15]).

6. THE INTERACTION TERM IN THE BALANCE EQUATION

In this section we present the analysis of the interaction term \( D_n \), and see (2.34) and (2.37). The question that was left at the end of sec 2C 1 is whether the integral over \( \textbf{r}_1 \) converges. Order by order analysis of the type presented in paper I indicates that the answer is yes. However we need now to consider the nonperturbative answer using what we have learned so far.

A. Locality of the integral in the interaction term

In order to do this we need further asymptotic properties of the functions \( T_n \) which appear in the integral. For brevity we will suppress the tensor indices of these objects, and to consider even and odd \( n \) in the same way. The convergence of the integrals depend on \( T_n(\textbf{R}+\textbf{r}_1, \textbf{R}+\textbf{r}_1, \textbf{R}) \) and \( T_n(\textbf{r}_1, \textbf{r}_1, \textbf{R}) \) when \( r_1 \gg R \) and when \( r_1 \ll R \). So far we have only analyzed \( T_n(\textbf{r}_1, \textbf{r}_1, \textbf{R}) \) when \( r_1 \ll R \). The full analysis of the two unknown asymptotics is as involved as the one presented above, and we will present them in a separate publication ([16]). Here we will simply employ the results that we need for the present analysis.

Consider first \( T_n(\textbf{R}+\textbf{r}_1, \textbf{R}+\textbf{r}_1, \textbf{R}) \) for \( r_1 \) small. The analysis in ([16]) shows that for \( r_1 \ll R \)

\[
z_n = \zeta_2 \quad \text{for all } n .
\]

This is a non-trivial prediction that to our knowledge has never been tested either in experiments or simulations. It appears to be exact. It is interesting to notice that independently of the question of multiscaling in the spatial scale, the temporal scaling is simple.

It is amusing to try understand (5.13) intuitively. In doing so we want to separately understand why \( z_2 = \zeta_2 \) and then why all \( z_n \) are the same. The first finding seems to contradict the naive dimensional evaluation of \( \tau_2(R) \) as \( R/\sqrt{S_2(R)} \), which is the “turn-over” time of \( R \)-eddies with characteristic velocity \( \sqrt{S_2(R)} \). This evaluation would lead to \( z_2 = 1 - \zeta_2/2 \) which is wrong.

Another way of thinking that leads to the right result is to estimate the rate of energy dissipation as the ratio of energy of \( R \)-motions, which is \( S_2(R) \), by the time scale \( \tau_2(R) \). Since the rate of energy dissipation is \( R \)-independent (being \( \tilde{c} \)), this fixes \( \tau_2(R) \) to scale as \( R^{d-2} \).

The \( n \)-independence of \( z_n \) is more subtle, and we postpone its discussion to the forthcoming paper ([16]). Here we just want to point out that this result entails a prediction about the measurement of dimensionless ratios of structure functions, like \( S_3/S_2^{2/3}, S_4/S_2^{3/2} \) etc. in decaying turbulence. The prediction is that such relations are \( R \)-dependent but not time dependent. We believe that this is not in contradiction with what is known about decaying turbulence behind a grid.
\[ T_n(R + r_1, R + r_1, R) - S_n(R) \propto S_2(r_1) \propto r_1^{\xi_2}. \] (6.1)

Next consider \( T_n(r_1, r_1, R) \) in the limit \( r_1 \gg R \). The analysis in Eq. (6.2) leads to
\[ T_n(r_1, r_1, R) \propto R^{\zeta_n-2} r_1^{\zeta_n-\zeta_{n-2}}. \] (6.2)

These results can be employed now in the integral for \( D_{2m} \). In this integral we have the projection operator \( \mathbf{P}_u \), which has a delta function and a longitudinal part. It was demonstrated in section 2 that the delta function leads to the expression (2.37), as if there were no pressure. The longitudinal part of \( \mathbf{P}(r_1) \) is proportional to \( 1/r_1^3 \). The integral \( \int dr_1 \mathbf{P}(r_1) \) by itself diverges logarithmically. The rest of the integrand (i.e. \( \partial T_n/\partial r_1 \)) behaves like \( r_1^{\Delta r_1} \). Simple power counting indicates that the integral diverges on the whole in the ultraviolet region. In fact, this power counting is misleading, since the integration over the angles vanishes. The projection operator is an even function under the inversion of \( r_1 \), whereas the leading term of the rest of the integrand is odd. The next term in the expansion of \( \partial T_n/\partial r_1 \) is even under the inversion of \( r_1 \), and is of the order of \( r_1^{\xi_2}. \) The resulting integral \( \int dr_1 r_1^{\xi_2-1} \) converges in the ultraviolet.

Note that this analysis indicates that each of the two terms in the integral for \( D_n \), converge independently. In fact, we see from Eqs. (6.1) and (5.2) that the two terms have precisely the same asymptotics, and they may exactly cancel in the limit. In addition we see from (2.34) that the leading asymptotics of the two terms cancels exactly also in the infrared. These facts are important: we will argue below that the leading scaling behaviour for \( D_n \) which is naively calculated from each term separately may cancel, and the actual scaling is determined by the next order contribution. This will be a mechanism for multiscaling.

Notwithstanding the exact cancellation of the leading infrared behavior we need to examine the infrared convergence of the integral. Each one of the terms in the integrand of \( D_{2m} \) has the asymptotic form
\[ \int (dr_1/r_1) \partial r_1^{\zeta_n-\zeta_{n-2}}/dr_1 \] which converges separately in the infrared. The difference should converge even faster.

In summary, we argued here that the proof of locality of the integral for \( D_{2m} \) extends beyond order by order considerations. Similar arguments allow reaching the same conclusion for the integrals in \( D_{2m+1} \).

**B. Rough Evaluation of \( D_n \)**

The conclusion of the last subsection is that the main contribution to the integrals appearing in \( D_n \) comes from the region \( r_1 \sim R \). Accordingly the integral can be evaluated as discussed in (6.3):
\[ D_n(R) \sim \frac{dS_{n+1}(R)}{dR}. \] (6.3)

This is exactly of the form computed in Eq. (2.34) for the case without pressure. We will see next that if this evaluation is to be trusted, its unavoidable consequence is that the scaling exponents \( \zeta_n \) are a linear function of \( n \). We will see that this allows only K41 scaling or \( \beta \)-model type scaling (9). It is possible however that (6.3) is an overestimate; we argued above that the interaction term as shown in Eqs. (2.34) and (2.37) may have a cancellation of the leading scaling behavior which is valid for every one of the terms in the integrals separately. We will therefore also study now the next order term that will be the proper evaluation of \( D_n \) if the leading evaluation indeed cancels. We will see that the resulting evaluation culminates in multiscaling in close agreement with experimental observations. Moreover, we will argue that as far as we can see within the new approach developed in this series of papers, the scenario that involves the cancellation of the leading scaling behaviour of \( D_n \) is the only scenario that allows multiscaling in turbulence. It is possible that there exists a symmetry or a sum rule that leads to such a cancellation but we do not have at the present time a theoretical proof or even a good argument as to why and how it happens.

To evaluate the next order scaling contribution of \( D_n \) we need to return to the diagrammatic expansion of \( F_n \), Fig. 10. In the discussion in section 5 we explained that in the asymptotic regime of two small coordinates the weakly linked diagrams are negligible compared with the two-propagator bridged contributions, even in K41 scaling. Now however we are interested in these diagrams when all the coordinates are of the same order, and it is evident that K41-wise they all have the same scaling with \( R \). In fact, the unlinked contributions to \( F_n \) which are obtained from the Gaussian decomposition (i.e. all the contributions \( F_pF_q \) with \( p + q = n \)) also have the same K41 evaluation. In addition we have a set of weakly linked contributions such as the ones displayed in Fig. 10 panel a. Again they have the same scaling in K41. Accordingly we need to think which contributions are dominant when the leading scaling \( (R/L)^{\zeta_n} \) cancels in the evaluation of \( D_n \). (Since we are going to show that anomalous scaling requires the normalization scale to be \( L \), we assume this in the present discussion without further ado).

The estimate of the scaling exponents of all these various contributions is facilitated by the fact that we are interested now in the “local” situation when all the coordinates are of the order of \( R \). Thus for example the diagrams 2 in Fig. 10 panel a are redrawn in Fig. 14. The objects in the left grey ellipse in diagrams 1 and 2 are representation of \( F_3 \) shown in panel c of 13. The object in the right ellipse of diagram 2 belongs to \( F_{n-1} \). One can see this by taking \( n = 3 \) and looking back to panel c of Fig. 3. However in diagram 2 we counted the \( F_2 \) bridge twice. The overall scaling exponent is therefore \( \xi_3 + \zeta_{n-1} - \xi_2 \). Using Fig. 3 panel b it can be also seen that the diagram 1 in Fig. 13 has the same scaling exponent. Similarly we can analyze the diagram 1 in
Fig. [10] panel a and argue that its scaling exponent is \( \zeta_3 + \zeta_{n-1} - \zeta_2 \), etc.

Now we need to understand which of these contributions will take the lead if the main scaling \((R/L)^{\zeta_3}\) cancels. To guide our thinking we will assume that the scaling exponents are neither K41 nor \( \beta \)-model, but are non-linear functions of \( n \). Hölder inequalities then require that the increments between \( \zeta_n \) and \( \zeta_{n-1} \) will be non-increasing functions of \( n \), i.e.

\[
\zeta_{n+1} - \zeta_n \leq \zeta_n - \zeta_{n-1} .
\]

(6.4)

With these inequalities one sees that the unlinked contributions which scale with \((R/L)^{\zeta_n+\zeta_4}\) are always smaller than the weakly linked contributions, and that of all the weakly linked contributions the leading one is the one which we singled out in Fig. [10] panel a, with the scaling exponent \( \zeta_3 + \zeta_{n-1} - \zeta_2 \).

The meaning of this result is that instead of evaluating \( T_n \) in the integrals for \( D_n \) as \( S_n(R) \) we need to evaluate it as \( S_{n-1}(R)S_3(R)/S_2(R) \). Correspondingly the evaluation (6.3) changes to

\[
D_n(R) = d_n(\zeta_n) \frac{S_n(R)S_3(R)}{RS_2(R)} ,
\]

(6.5)

where \( d_n(\zeta_n) \) is a coefficient which depends on the numerical value of the scaling exponent.

7. ANALYSIS OF THE BALANCE EQUATION AS A NONPERTURBATIVE CONSTRAINT

At this happy moment we can use all the knowledge accumulated so far to go back to the balance equations (2.24) and (2.25) that we rewrite in the form

\[
D_n(R) = J_n(R) ,
\]

(7.1)

where the forcing term is not displayed because it is negligible, cf. section 2C3. The evaluation of \( J_n(R) \) is given in (5.13). The evaluation of \( D_n \) is either (6.3) or (6.5), depending whether there is or is not cancellation in the leading scaling behaviour of \( D_n \). We will show now that option (6.3) leads inevitably to linear scaling, whereas option (6.5) leads to multiscaling.

A. Scenario of Linear Scaling: Burgers Turbulence and the \( \beta \)-model

The evaluation (6.3) is exactly correct only when there is no pressure term and the projection operator is a delta function. This is the situation for example in Burgers turbulence [21]. It may or may not be a proper evaluation of \( D_n \) also in the case of Navier-Stokes turbulence, as discussed above. We examine now the consequences of this evaluation when substituted in the balance equation. Substituting (6.3) and (5.13) in (7.1) we find

\[
\frac{S_{n+1}(R)}{R} \sim \zeta \frac{S_n(R)}{S_2(R)} .
\]

(7.2)

For \( n = 2 \) we recover the known result that \( S_3(R) \sim \epsilon R \). Accordingly we can rewrite (7.2) as

\[
S_{n+1}(R)S_2(R) \sim S_n(R)S_3(R) .
\]

(7.3)

In terms of the scaling exponents this results reads

\[
\zeta_{n+1} + \zeta_2 \sim \zeta_n + \zeta_3 .
\]

(7.4)

The only solution of this equation is the linear law \( \zeta_n = a + bn \), where \( a \) and \( b \) are some constants. Knowing that \( \zeta_3 = 1 \) and using our scaling law (1.9) which is \( \mu = 2\zeta_2 - \zeta_4 \) we find that the only solution is

\[
\zeta_n = \frac{n}{3} - \mu \frac{(n-3)}{3} .
\]

(7.5)

It is interesting to note that this result is identical to the prediction of the \( \beta \)-model [13], coefficients and all. This should not surprise us too much. After all, once we have a linear dependence two constraints fix the linear law completely. Note that this law includes as a special case the exponents of Burgers turbulence which are \( \zeta_n = 1 \) for all \( n \) [24]. This is obtained from (7.3) when \( \mu = 1 \). Nevertheless the full analysis of the Burgers equation using the techniques developed in this series of papers needs special attention due to the importance of the incompressibility constraint in so many of our calculations. The Burgers case deviates so strongly from K41 that the issues of locality and rigidity of the various diagrams needs to be assessed separately [21].

We can also show now that (7.3) implies that the renormalization scale is the outer scale of turbulence \( L \) as claimed before. Eq. (7.3) was derived by asserting that the Gaussian contribution to \( J_n \) is negligible compared to the connected ladder contributions which led to (5.13). The Gaussian contributions are dominated by \( \epsilon S_{n-2} \) whose scaling exponent is \( \zeta_2 \). If we used this contribution as the leading one in the balance equation (7.1) we would have obtained the scaling relation \( \zeta_{n+1} = \zeta_3 + \zeta_{n-2} \) with the obvious boundary condition \( \zeta_0 = 0 \). The solution of this recursion relation is K41 scaling with \( \zeta_n = n/3 \). If this is to be rejected in favor of (7.3) the Gaussian contributions must be indeed smaller than the ones we kept. The conclusion is that

\[
S_2(R)S_{n-2}(R) < S_n(R) .
\]

(7.6)

In turn this inequality implies that

\[
\left( \frac{R}{\ell} \right)^{\zeta_{n-2}} < \left( \frac{R}{\ell} \right)^{\zeta_n - \zeta_2} .
\]

(7.7)

Substituting the scaling exponents from (7.3) we conclude that \( (R/\ell)^m < 1 \), which can only happen if \( R < \ell \) for any \( R \) in the inertial interval. This identifies \( \ell \) with \( L \).
B. Scenario of Multiscaling

Substituting (6.3) and (5.13) in (7.4) we find
\[
2d_n(\zeta_n) = n .
\]
(7.9)

To aim the rough evaluation of \(D_n\) in section 6B is not sufficient; we need to be much more precise in order to compute the \(\zeta_n\) dependence of \(d_n\).

Clearly, the computation of coefficients is exceedingly hard. Only when we have exact form for the functional dependence of the many point functions we can hope to compute the coefficient. We did have an exact form for \(J_n\) because we understood how to resum its diagrams; consequently we believe that we have computed the coefficient of \(J_n\) exactly. \(D_n\) is a different matter; at present we do not have an exact functional form for it. Order by order considerations are not helpful for this issue, and we still do not know how to exactly resum the diagrammatics for \(D_n\). We will therefore try to guess the \(\zeta_n\) dependence of the coefficient of \(D_n\).

1. The Eddy-Viscosity Approximation of \(D_n(R)\)

To guide our thinking we recall some results from the theory of passive scalar advection [17]. In that problem \(D_n\) had the form of an eddy-diffusivity operator:
\[
D_n^{\text{passive}}(R) = \frac{1}{R^2} \frac{d}{dR} R^2 h(R) \frac{d}{dR} S_n(R) ,
\]
(7.10)

where \(h(R)\) is the eddy diffusivity which scales with \(R\) as a power law \(R^{\zeta_n}\). Can we use this to guess a form for \(D_n(R)\) in the present case? On the face of it the answer is no. Our evaluation (6.3) indicated that if we have a differential operator it operates on \(S_{n+1}\) rather than on \(S_n\). On the other hand, once we assume that the leading order evaluation cancels in \(D_n\), the next order is again in terms of \(S_n\) as in the case of passive scalar. In fact, it is not difficult to see (Appendix A) that the topology of the weakly linked diagrams for \(D_n\) is identical (after the cancellation of the leading order) to the topology of the leading contributions for \(D_n\) in the case of the passive scalar. We thus guess that for the aim of evaluation of the coefficient we can write \(D_n(R)\) with an eddy-viscosity similar to (7.10) in which \(h(R)\) is found by comparing (7.10) and (6.3):
\[
D_n(R) = b \frac{1}{R^2} \frac{d}{dR} R^2 S_3(R) \frac{d}{dR} S_n(R) ,
\]
(7.11)

with \(b\) being now an \(n\)-independent coefficient. The physical meaning of this guess is that the \(R\)-dependent eddy viscosity \(h(R)\) takes here the form
\[
h(R) = \frac{b RS_3(R)}{S_3(R)} .
\]
(7.12)

Note that the eddy diffusivity which is introduced here scales like \(R^3-\zeta = R^\lambda\).

2. Scaling Exponents in the Eddy-Viscosity Approximation

Using (7.11) we compute
\[
d_n(\zeta_n) = b \zeta_n(3 + \zeta_n - \zeta) ,
\]
(7.13)

where we used the fact that \(\zeta_3 = 1\). Together with (7.9) we find a quadratic equation for \(\zeta_n\):
\[
2b\zeta_n(3 + \zeta_n - \zeta) = n .
\]
(7.14)

We remind the reader that this equation is valid for \(n > 2\) since there is no cancellation of the leading scaling order in \(D_2\). Using again the fact that \(\zeta_3 = 1\) we find the constraint
\[
2b(4 - \zeta) = 3 .
\]
(7.15)

This leaves us in (7.14) with only one unknown number which we take as \(\zeta_2\). Solving for \(\zeta_n\) we find
\[
\zeta_n = \frac{3 - \zeta_2}{2} \left[ -1 + \sqrt{1 + \frac{4n(4 - \zeta_2)}{3(3 - \zeta_2)^2}} \right] .
\]
(7.16)

Not having a theoretical value of \(\zeta_2\) we can take for it any of the following values: (i) The K41 value \(\zeta_2 = 2/3\). (ii) The experimentally “accepted” value \(\zeta_2 = 0.70\). (iii) We can assume that Eq. (7.14) is correct also for \(n = 2\). This gives \(\zeta_2 = 8/11 \approx 0.727\). These different choices lead to the numerical values of \(\zeta_n\) shown in Table 1 under the rows labelled “model (i), (ii) and (iii)”. In the same table we also show the result of three available experiments. It should be noted that all these experiments refer to Reynolds number \(Re_\lambda < 10^3\), and thus as discussed above, our asymptotic predictions may not apply. Corrections to our asymptotic behaviour are still too large to be ignored. This may be the reason for the deviations between the theoretical predictions and the experimental numbers.

8. CONCLUDING REMARKS FOR PAPERS I-III AND THE ROAD AHEAD

We believe that the theory that was presented in papers I-III contains novel elements that are likely to remain as cornerstones in the theory of the fine structure of
hydrodynamic turbulence. Since the approach is highly technical, we attempt in these concluding remarks to summarize first what are the essential elements of the analysis both from the point of view of technique and of physics.

One fundamental issue that separates our approach from the traditional K41 approach is the explicit appearance of anomalously sensitive with the ultraviolet cutoff length as the renormalization scale. It is intuitively clear that gradient fields cannot be insensitive to the resolution length as the renormalization scale. In reality Δ = Δ_{cr} exactly equals its critical value Δ_{cr} = 2 - ζ₂. This fact is of crucial importance for the nature of anomalous scaling in hydrodynamic turbulence. In the hypothetical case Δ < Δ_{cr} one expects K41 scaling of the structure functions in the limit Re→ ∞. There can be only subcritical corrections to this, and such corrections have the viscous scale η as the renormalization scale. In reality Δ = Δ_{cr} and this opens up the possibility for the destruction of K41 for all values of Re.

The analysis of the possible mechanisms for anomalous (non-K41) scaling of the structure function is the main topic of Paper III. This analysis is based on the exact and nonperturbative balance equations D_n(R) = J_n(R) which follow from the equation of motion for the structure functions S_n(R). These equations are a direct consequence of the Navier-Stokes equations in the statistical stationary state in which ∂S_n(R, t)/∂t = 0. We succeeded to compute (in the limit Re→ ∞) the viscous term J_n(R) exactly: J_n(R) = n J_2 S_n(R)/2 S_2(R). This result is the leading term among a few terms, and is the contribution of strongly linked diagrams. It is proportional to R^{-n-ζ₂} as a direct consequence of the criticality of the theory in the sense that Δ = 2 - ζ₂. This contribution is much larger than the contribution of non-linked diagrams, which are the result of Gaussian decompositions. The largest of these next order terms is proportional to R^{-n+ζ₂} in the hypothetical subcritical case when Δ < 2 - ζ₂ the latter term dominates the strongly linked terms and classical K41 scaling is then unavoidable.

Because of the importance of this point we reiterate: because Δ attains its critical value 2 - ζ₂ the dominant contribution to the viscous term J_n(R) is proportional to S_n(R)/S_2(R) and not to S_{n-2}(R). This is one of three
findings which open the way to multiscaling in the theory of hydrodynamical turbulence. The second one is less solid: it is the assumption that the leading (strongly linked) contributions to the interaction term \( D_n(R) \) cancel exactly. We have only some arguments as to why this may happen. The detailed analysis of such a possibility and the clarification of the possible relation of the cancellation with (hidden) symmetries of fluid mechanics are important aspects of future research. What we propose now is that this cancellation is the only visible mechanism to induce multiscaling in the theory of the structure functions. It is very possible that this is the only mechanism for multiscaling, and if we accept the opinion of the majority of the workers in the field of turbulence that experiments indicate the existence of multiscaling, we are forced to accept that there is a cancellation of the strongly linked contribution to the interaction term \( D_n(R) \). Lastly, the third finding leading to multiscaling is the understanding of the important role of weakly linked contributions to \( S_n(R) \) which scale like \( R^{n-1+\Omega_1^{-1}G_3^{-1}} \). In a multiscaling situation this term dominates over the leading irreducible contribution which is proportional to \( R^{n+2+\Omega_1^{-1}G_3^{-1}} \). If one misses the weakly linked terms, K41 scaling reappears again.

Besides issues of principle, there remains the actual calculation of the scaling exponents \( \zeta_n \). To accomplish such a calculation we need to know the coefficient in the evaluation of \( D_n \). From the “eddy viscosity” approximation for \( D_n \) we were led to the exponents shown in Table 1. We cannot say that the agreement with the experimental data is amazing. The discrepancies may stem from various sources. Firstly, it is possible that the “eddy viscosity” approximation misses something important in the \( n \) dependence of the coefficient of \( D_n \). This is possible, but may be not the main reason for the discrepancy. After all, in our theoretical developments we used very strongly the limit \( \text{Re} \to \infty \). This limit was assumed for example in neglecting the disconnected contributions to \( D_n \) in favor of the weakly linked contributions. The difference in the \( R \) dependence between these two types of terms (in the case of \( D_4 \) for example) is of the order of \( R^{\Omega_4^{-1}} \). We thus need at least 3-4 decades of inertial range to justify this procedure. It is very possible that the experimental results are still suffering from the effects of subleading contributions, and only future analysis that takes such contributions into account may shed further light on the issue.

One particularly pressing subject for future research is the asymptotic behaviour of \( n \)-order correlation functions. We found in the present paper that when two of the coordinates of \( F_n(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n) \) (say \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \)) coalesce, the correlation function is proportional to \( S_2(\mathbf{r}_1) \). This is just one example of the asymptotic properties that are summarized under the term “fusion rules”. We need to understand the deep structure of the theory that is responsible for this fusion rule, but that will also allow us to predict what happens, say, when three or more points coalesce. We guess that the \( n \)-point correlation function in that case will be proportional to \( F_n \), etc. Formally we need to develop the operator algebra that will automatically furnish all the needed fusion rules, and will be also compatible with multiscaling. We plan to propose elements of such a theory in a forthcoming publication [4].

Finally, we comment on the relation and differences between scaling in turbulence and scaling in better understood subjects like critical phenomena. Because of the superficial similarities (many body problems with strong interaction and scale invariance) there were many attempts to apply formal schemes in the wake of critical phenomena to understand turbulence: renormalization groups, \( \epsilon \)-expansion, \( 1/d \)-expansion, \( 1/N \)-expansions, and what not. If the approach taken in this series of papers turns out to be correct, this will mean that the theory of turbulence is significantly different from critical phenomena. Some elements reappear: sums of ladder diagrams contribute anomalous exponents, fusion rules are needed, and the smell of operator algebra is there.

However that are at least two major differences: there exists flux equilibrium instead of thermodynamic equilibrium, and the interaction in the theory of turbulence is highly non-local because of the effects of pressure. In contrast, in critical phenomena it is sufficient to have local interactions that build up to global criticality because of the cancellation of energetic and entropic contributions to the free energy. In turbulence, notwithstanding the nonlocality of the interaction it turns out that the diagrammatic theory in BL variables is finite order by order. In contrast, the perturbative analysis of critical phenomena leads to divergences that result (after renormalization) in anomalous scaling. Thus the mechanism for anomalous scaling in turbulence must be different.

Due to the flux equilibrium there is a global connection between the largest and smallest scales in the problem. A deep consequence of the flux equilibrium is the 2-point fusion rule that was discussed above as one of the cornerstones for multiscaling. In addition, flux equilibrium and the need to satisfy boundary conditions at the two ends of the energy cascade introduces the possibility of having the outer scale as the renormalization length without infrared divergences in order by order expansions for the structure functions.

As a result of all these difference we do not have the fixed point structure with a small number of unstable directions that is so typical of critical phenomena. In some sense, the independence of the perturbative terms from a typical scale means that we have infinite number of marginal operators. The resummation of the perturbative theory results in a possibility of dressing these marginal operators, and there can be infinitely many independent exponents. Whether such a concept can be turned into a computational scheme is a question for the future.
9. APPENDIX: WEAKLY LINKED CONTRIBUTIONS TO THE INTERACTION TERM

In this appendix we discuss the weakly linked contributions to the interaction term $D_n$. In Eqs. (2.36) and (2.37) the integrals depend on the $n + 1$-order correlation function $T_{n+1}(r,r,R)$ in which the two first coordinates $r$ are special (they are either $r_1$ or $R + r_1$ and there is a derivative with respect to $r_1$). In its turn every weakly linked contribution to $F_{n+1}(r_1, r_2, \ldots, r_{n+1})$ (or to $T_{n+1}(r,r,R) = F_{n+1}(r,r,R, R, \ldots, R)$) has two weakly linked legs (denoted in Fig.10a as $x_a, x_b$), connected to the body of the diagram via a one-propagator bridge. There are $C_n^2 = n(n+1)/2$ weakly linked contributions to $F_{n+1}$ in which the role of weakly linked legs is played by each pair taken from the $n + 1$ total number of legs. The $C_n^2$ contributions to $D_n$ can be subdivided into three groups. The first group consists of just one term in which two special coordinates in $T_{n+1}$ are exactly the coordinates of the two weakly linked legs ($r = r_a = r_b$). The second group of terms in $D_n$ has $2(n - 1)$ terms in which just one of the special coordinates in $T_{n+1}$ (but only one of the two) is associated with a weakly linked leg. The second special coordinate is free to be associated with any of the $(n - 1)$ body of the weakly linked diagram for $F_{n+1}$. This body is an $n$-point object (see Fig.10a) in which one leg is used to create the bridge. The last (third) group of terms in $D_n$ has $C_n^2 - 1 = (n - 1)(n - 2)/2$ terms in which two special $T$-coordinates may be chosen from the coordinates of any $(n - 1)$ free legs in the body of the weakly linked diagram for $F_{n+1}$.

The first two groups of terms correspond exactly to the topology of the diagrammatic representation of the interaction term $D_n$ in the problem of turbulent advection of a passive scalar field $T(r,t)$, and cf. section VB in [2] and section II B2 in [1]. Consider Fig.10 of [2]. The dashed lines in this figure represent 2-point velocity correlators, and these are replaced in our case by wavy correlator lines. The wavy lines in the passive scalar case represent two point scalar correlators, and they remain as wavy lines in the present case. The fragment of the diagram in this figure which is placed to the right of the legs denoted by $q_2, q_3, q_4$ and to the right of the vertex between $k$ and $q_1$ must be interpreted now as a contribution to the strongly-linked four-point velocity correlator. The last serves a a four-point body of a weakly linked contribution to a 5-point velocity correlator $F_5$. We thus conclude that the topology of the diagrams for $D_n$ in the case of turbulent passive advection and the first two groups of weakly linked diagrams for $D_n$ in the case of Navier-Stokes turbulence is the same.

It appears that the third group of $C_n^2$ terms which we described above forms a major difference between the problems of turbulent advection and Navier Stokes turbulence. In fact this group does not contribute. In the case of passive scalar this group is absent because of the zero value of the $(Tv^1)$ correlator. It is remarkable that in the present case of Navier-Stokes turbulence this group cancels under the same condition of cancellation of the leading (strongly linked) contributions to $D_n$. These terms may be considered (after severing the bridge to the weakly linked fragment) as strongly linked contributions to $D_{n-1}$. They must cancel if the scenario leading to multiscaling is assumed.

The conclusion of this appendix is far from being trivial, and in some sense is very surprising. It says that although the passive advection problem is linear and local, whereas Navier Stokes in nonlinear and non-local (pressure!), it appears that if multiscaling is expected, the topology of the diagrams for $D_n$ is very similar in the two cases. If this is correct, it must be related to some deep symmetry. If so, the Eddy Viscosity Approximation used in section 7 may contain some essential aspects of the truth.

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| source | n=2 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|--------|-----|---|---|---|---|---|---|----|----|----|----|----|----|----|----|
| expt. [23]| 0.70 | 1.28 | 1.54 | 1.78 | 2.00 | 2.23 | - | - | - | - | - | - | - | - |
| expt. [24]| 0.70 | 1.20 | 1.52 | 1.62 | 1.96 | 2.00 | 2.36 | 2.36 | 2.70 | 2.68 | 3.08 | 3.02 | 3.48 | - | - |
| expt. [25]| 0.71 | 1.30 | 1.57 | 1.82 | 2.06 | 2.25 | 2.46 | 2.60 | 2.80 | 2.92 | 3.08 | 3.19 | 3.35 | 3.45 | 3.58 |
| model (i) | 0.667 | 1.243 | 1.463 | 1.667 | 1.856 | 2.035 | 2.204 | 2.365 | 2.519 | 2.667 | 2.809 | 2.946 | 3.079 | 3.208 | 3.333 |
| model (ii) | 0.700 | 1.242 | 1.462 | 1.666 | 1.864 | 2.032 | 2.200 | 2.360 | 2.514 | 2.661 | 2.803 | 2.939 | 3.072 | 3.200 | 3.320 |
| model (iii) | 0.727 | 1.242 | 1.461 | 1.663 | 1.851 | 2.029 | 2.197 | 2.357 | 2.509 | 2.656 | 2.797 | 2.933 | 3.065 | 3.193 | 3.317 |
FIG. 1. Diagrammatic representation of the 4-point Green’s function $G_{2,2}$ defined by Eq. (3.3). Panel (a): $G_{2,2}$ as a sum of weakly linked contributions $G_{2,2}^{wl}$ (shown in Panels (b–d)) and strongly linked contribution $G_{2,2}^{sl}$ (shown on Fig. 2). Panel (b): $G_{2,2}^{wl}$ is presented in terms of the three-point Green’s function $G_{2,1}$, defined by Eq. (3.1), and the dressed vertex $A$. Panels (c,d) give the diagrammatic representation of $G_{2,1}$ in terms of the propagators $G$ and $F_2$ and the dressed vertices $A$ and $B$. For the diagrammatic expansion of the dressed vertices $A$, $B$, and $C$ – see Fig. 7 of paper II. Note the new notation of an empty little circle in the object $D$. This circle designates a vertex that can be either bare or dressed.

FIG. 2. The diagrammatic presentation of the strongly linked contribution $G_{2,2}^{sl}$ to the four-point Green’s function $G_{2,2}$, (see Fig. 1). Diagrams 1 and 2 are the Gaussian decomposition of $G_{2,2}^{sl}$, diagrams 3 and 5 are ladders with one rung (of two types), and diagrams 3 and 6 are the ladders with two and three rungs respectively. In contrast to the expansion in Fig. 9 of paper II for $G_2$, here one has rungs of three types. These are $\Sigma_{3,1}$ like in diagram 5, $\Sigma_{2,2}$ as in diagrams 3 and 4, and rung $\Sigma_{1,3}$ as the first rung in diagram 6. The diagrammatic expansion of $\Sigma_{2,2}$ is shown in Fig. 3, and the expansion of $\Sigma_{3,1}$ and $\Sigma_{1,3}$ in Fig. 4. The exact resummation of this series is shown in Fig. 5.

FIG. 3. Diagrammatic expansion for the mass operator $\Sigma_{2,2}$ which is found in the expansion of $G_{2,2}^{sl}$ shown in Fig. 2. This mass operator is an important element in the system of equations for $G_{2,2}^{sl}$ and $G_{2,1}^{sl}$ shown in Fig. 5. The principal cross sections are shown as dashed lines.

FIG. 4. Diagrammatic expansions of the mass operator $\Sigma_{3,1}$ (panel a) and $\Sigma_{1,3}$ (panel b) which appear in the expansion of $G_{2,2}^{sl}$ shown in Fig. 2. These mass operators are the essential elements in the equations shown in Fig. 5.

FIG. 5. The exact system of equations for the strongly linked contributions to $G_{2,2}$ (panel a) and $G_{3,1}$ (panel b) in terms of the propagators $G$ and $F_2$ and the mass operators $\Sigma_{3,1}$, $\Sigma_{2,2}$ and $\Sigma_{1,3}$. The first terms in the expansion of $\Sigma_{m,n}$ are shown in Figs. 3, 4. This system results from the exact resummation of ladder diagrams.

FIG. 6. Diagrammatic representation of the weakly linked contributions to the four-point correlation function $F_4$. Panel a: a weakly linked contribution to $F_4(r_{0}|x_1, x_2, x_3, x_4)$ in which a one-propagator bridge is placed between (1,2) pair and the (3,4) pairs of legs. One has two more similar weakly-linked contributions with the bridge connecting the pairs (1,3) with (2,4) and (1,4) with (2,3). On the left of the diagrams 1 and 2 one find the objects $G_{2,1}$ and $F_1$ respectively. These objects are presented diagrammatically in Fig. 1c and panel b of the present figure, respectively. On the right side of diagrams 1 and 2 one has the three-point objects that were designated as $A$ and $D$. These objects can be found in Fig. 1d and in panel b of the present figure.

FIG. 7. Panel a: diagrammatic representation of the irreducible four-point correlation function as a sum of weakly and strongly linked parts. Diagram 1 is the weakly linked part $F_4^{wl}$ and diagram 2 is the strongly linked part $F_4^{sl}$. $F_4^{sl}$ is presented in Fig. 6. Panel b: Exact presentation of $F_4^{sl}$. The elements appearing in this presentation are the four-point Green’s functions $G_{2,2}$ and $G_{4,1}$ on the left side and on the right side of the diagrams, and three different new four-point objects. The notation used to distinguish the three new objects is with zero, one or two horizontal lines inside. The diagrammatic representation of the new objects is shown in Fig. 8.

FIG. 8. Diagrammatic representation for the three different central blocks which appeared in Fig. 7. These are infinite order expansions in terms of contributions with increasing numbers of correlators in their principal cross section (i.e. 2, 3, 4 and more). The four-point objects in diagram 1 in each panel are the mass operators $\Sigma_{2,2}$ and $\Sigma_{3,1}$ which are the dressed rungs of the ladders whose presentation is shown in Figs. 3 and 4.

FIG. 9. An example of a fragment contribution to the right hand part of diagram 1 in Fig. 7.

FIG. 10. Diagrammatic representation of n-point correlation functions. Panel a: weakly linked contributions. Diagrams 1 and 2 are the generalization of diagrams 1 and 2 for $F_4$ shown in Fig. 6 panel a. Diagrams 3 is an example of a weakly linked contribution with two one-propagator bridges. Such contributions do not appear in the case of $F_4$. Panel b: Strongly linked contributions to $F_n$, considering the legs designated as 1 and 2 special. In diagram 1 one has $F_1$ on the left; in diagrams 2 and 3 one has $G_{3,1}$ and $G_{2,2}$, the same objects that appeared in panel b of Fig. 7 for $F_4$. 

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FIG. 11. Diagrammatic representation of the weakly linked contribution for $F_n$ shown as diagram 2 in Fig. 10, panel a. For the 3-point correlator $F_3(\mathbf{r}_0 | x_a, x_b, x_c, x_d)$ we used the representation of Fig. 6, panel c and placed it here in the dashed ellipse which is denoted “left”. On the right we have an $m$-point object $D_m$ which is a generalization for the case $m > 3$ of $D_3$ which appeared in Fig.1 panel d.
(a)  
\[ \begin{align*} 
\text{Diagram 1} &= \text{(1)} + \text{(2)} \text{ (4)} \\
\text{Diagram 2} &= \text{(1)} + \text{(2)} + \text{(3)} + \text{(4)} + \text{(5)} + \text{(6)} 
\end{align*} \]
\[(a)\]

\[
\begin{array}{ccc}
1 & 3 \\
2 & 4 \\
\end{array} =
\begin{array}{ccc}
1 & 3 \\
2 & 4 \\
\end{array} - 
\begin{array}{ccc}
4 \\
\end{array} + 
\begin{array}{ccc}
4 \\
\end{array}
\]

\[(b)\]

\[
\begin{array}{ccc}
4 \\
2 & 4 \\
\end{array} =
\begin{array}{ccc}
1 & a & c \\
2 & b & d \\
\end{array} + 
\begin{array}{ccc}
1 & a & c \\
2 & b & d \\
\end{array} + 
\begin{array}{ccc}
1 & a & c \\
2 & b & d \\
\end{array} + 
\begin{array}{ccc}
1 & a & c \\
2 & b & d \\
\end{array}
\]
(a) \[ \begin{array}{c}
\text{[Diagram]} \quad = \quad \text{[Diagram]} + \text{[Diagram]} + \ldots
\end{array} \]

(b) \[ \begin{array}{c}
\text{[Diagram]} \quad = \quad \text{[Diagram]} + \text{[Diagram]} + \ldots
\end{array} \]

(c) \[ \begin{array}{c}
\text{[Diagram]} \quad = \quad \text{[Diagram]} + \text{[Diagram]} + \ldots
\end{array} \]
(a) \[ \begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{c}
\end{array} + \begin{array}{c}
\text{a} \\
\text{b}
\end{array}
\end{array} + (3)
\begin{array}{c}
\text{c}
\end{array} + \ldots \]
