ON STOCHASTIC MAXIMUM PRINCIPLE FOR RISK-SENSITIVE OF FULLY COUPLED FORWARD-BACKWARD STOCHASTIC CONTROL OF MEAN-FIELD TYPE WITH APPLICATION

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Abstract. In this paper, we are concerned with an optimal control problem where the system is driven by fully coupled forward-backward stochastic differential equation of mean-field type with risk-sensitive performance functional. We study the risk-neutral model for which an optimal solution exists as a preliminary step. This is an extension of the initial stochastic control problem in this type of risk-sensitive performance problem, where an admissible set of controls are convex. We establish necessary as well as sufficient optimality conditions for the risk-sensitive performance functional control problem. Finally, we illustrate our main result of this paper by giving two examples of risk-sensitive control problem under linear stochastic dynamics with exponential quadratic cost function, the second example will be a mean-variance portfolio with a recursive utility functional optimization problem involving optimal control. The explicit expression of the optimal portfolio selection strategy is obtained in the state feedback.

1. Introduction.

1.1. Motivation example. Modeling and controlling cash flow processes of a firm or a project, such as pricing and managing an insurance contract, is a class of problems where forward-backward stochastic differential equations (FBSDEs in short) provide a natural setup and a powerful tool. In this paper, we shall investigate an example of such a situation arising in the pricing of a simple insurance contract. This example is taken from [2].

A policyholder at an insurance company has paid premiums that at time zero have accumulated to the sum \( a \). The money is invested in an asset portfolio with wealth \((x_t)_{t \in [0,T]}\) managed by the insurance company under a time interval \([0,T]\). At each instant \( t \in [0,T]\), the policyholder ought to receive an amount \( \rho_t x_t \). The present value (price) of the cash stream \((\rho, x_s)\), discounted to time \( t \) with a discount

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factor (deflator) \( \exp \left\{ -\int_0^t \zeta_s \, ds \right\} \), where \( \zeta_t \) is assumed nonnegative, bounded, and deterministic, is given by:

\[
y_t = \mathbb{E} \left[ \int_t^T e^{-\int_0^s \zeta_r \rho_t \, ds} \mid \mathcal{F}_t \right].
\]  

(1)

Assume that the portfolio is invested in a simple Black-Scholes market model consisting of a risk-free asset (for example, a bond or a bank account) with a short interest rate \( \rho_t \) assumed bounded and deterministic, and a risky asset evolving as a geometric Brownian motion with rate of return \( \chi_t \) and volatility \( \sigma_t \), both assumed to be bounded and deterministic functions of time, with \( \sigma_t \geq \varepsilon > 0 \) for all \( t \in [0, T] \).

In this market the wealth process \( (x_t)_{t \in [0,T]} \) is governed by the dynamics given by

\[
\begin{aligned}
 dx_t &= (\rho_t x_t + r_t v_t) \, dt + \sigma_t v_t \, dW_t, \\
 x_0 &= a,
\end{aligned}
\]  

(2)

where \( v_t \) is the amount invested in the risky asset and \( r_t = \chi_t - \rho_t \) is the risk premium held for this investment.

The insurance company allocates the amounts \( (v_t) \) in order to come close to the following target at time \( T \): Find the admissible strategies \( (\rho, v) \) which maximize the policyholder’s preferences represented by the utility function \( F \) of the cash streams, under the condition that the total amount to be paid out is equal to the total premium \( a \):

\[
\max_{(\rho, v)} \frac{1}{\theta} \mathbb{E} \left[ F^\theta (x_T) \right].
\]  

(3)

(By selecting an appropriate portfolio choice strategy \( v(\cdot) \), where the exponent \( \theta > 0 \) is called the risk-sensitive parameter. Assume that the policyholder’s utility function is of HARA (hyperbolic absolute risk-aversion) type. That is, \( F(X) = X^\theta / \theta \), where \( \theta \in (0, 1) \).) We need the following definition of admissible strategies suitable for our problem.

An admissible strategy is a pair of \( (\mathcal{F}_t)_{t \geq 0} \)-adapted processes \( (\rho, v) \) such that (2) has a strong solution \( (x_t)_{t \in [0,T]} \) that satisfies

\[
\mathbb{E} \int_0^T |x_t| \, dt < \infty,
\]

and

\[
\mathbb{E} \left[ \int_0^T e^{-\int_0^s \zeta_r \rho_t \, ds} \rho_t \zeta_t \, dt \right]^2 < \infty.
\]

Now, for each admissible strategy \( (\rho, u) \), the \( (\mathcal{F}_t)_{t \geq 0} \)-adapted value process \( (y_t)_{t \geq 0} \) in (1) satisfies the following BSDE:

\[
\begin{aligned}
 dy_t &= -(-\zeta_t y_t + r_t v_t + \rho_t x_t) \, dt + z_t \, dW_t, \\
 y_T &= \xi,
\end{aligned}
\]  

(4)

where \( (z_t)_{t \geq 0} \) is \( (\mathcal{F}_t)_{t \geq 0} \)-adapted and square-integrable with respect to \( dt \times d\mathbb{P} \) over \( [0,T] \times \Omega \).

Hence, (2) and (4) satisfied by \( (x, y, z) \) is a FBSDE.
The purpose of this paper is to study fully coupled mean-field forward-backward stochastic differential equation with risk-sensitive performance, where the set of an admissible control is assumed to be convex, this is a good extension for the system with cash flow, but we need to add the mean-field processes in all the coefficients of FBSDE, this is not a simple or trivial extension, especially with risk-sensitive performance.

To this end we have to give a brief history of what we want to do?

1.2. Brief historic of mean-field problem. We consider stochastic control problems for state processes governed by a fully coupled FBSDE of mean-field type with risk-sensitive performance functional, which is also called McKean-Vlasov type equation, in the sense the coefficients of the fully coupled mean-field FBSDE are allowed to depend on the state of the process as well as on its expected value. More precisely, the system is defined as

\[
\begin{cases}
\begin{aligned}
& dx_t^v = b(t, x_t^v, y_t^v, z_t^v, \mathbb{E}'(x_t^v), \mathbb{E}'(y_t^v), \mathbb{E}'(z_t^v), v_t) \, dt \\
& \quad + \sigma(t, x_t^v, y_t^v, z_t^v, \mathbb{E}'(x_t^v), \mathbb{E}'(y_t^v), \mathbb{E}'(z_t^v), v_t) \, dW_t, \\
& x_0^v = a, \\
& dy_t^v = -f(t, x_t^v, y_t^v, z_t^v, \mathbb{E}'(x_t^v), \mathbb{E}'(y_t^v), \mathbb{E}'(z_t^v), v_t) \, dt + z_t^v \, dW_t, \\
& y_T^v = \xi.
\end{aligned}
\end{cases}
\]

for some functions \(b, \sigma, \) and \(f, \) and \(W\) is a Brownian motion. For every \(t,\) the control \(u_t\) is allowed to take values in some control state space \(U.\)

The mean-field SDE is obtained as the mean-square limit, when \(n \to \infty,\) of a system of interacting particles of the form

\[
\begin{cases}
\begin{aligned}
& dx_t^{v;i,n} = b \left(t, x_t^{v;i,n}, \frac{1}{n} \sum_{j=1}^{j=n} \phi \left(x_t^{v;j,n} \right), v_t \right) \, dt \\
& \quad + \sigma \left(t, x_t^{v;i,n}, \frac{1}{n} \sum_{j=1}^{j=n} \phi \left(x_t^{v;j,n} \right), v_t \right) \, dW_t^i, \\
& x_0^{v;i,n} = a, \\
& \end{aligned}
\end{cases}
\]

where, \((W^i, i \geq 1)\) is a collection of independent Brownian motions (see e.g. Sznitman [21] and the references therein).

Concerning mean-field backward stochastic differential equations (mean-field BSDEs), have been first studied by Buckdahn et al, the interested reader is referred to [5, 4], the paper Carmona and Delarue [6] was to provide an existence result for the solution of fully coupled FBSDE of the mean-field type. Mathematical mean-field approaches play a crucial role in diverse areas, such as physics, chemistry, economics, finance and games theory, see for example Lasry–Lions [14]. Many papers have been studied the problem of mean-field and established stochastic maximum principle, we can cited here some of them, the first work who gave the necessary optimality conditions was Bukdahn et al [3], after this work many authors have generalized this problem into the others fields of applications, as the paper of Anderson et al [1] they have studied the problem of mean-field type of SDE under the assumptions of a convex action space. Beside, the problem of mean-field has been derived also via Malliavin calculus, the authors Meyer-Brandis, Oksendal and Zhou in paper [17] have been obtained the stochastic maximum principle of mean-field, also to the problem of singular mean-field with a good application to finance we can
have the paper of Hu et al. [13]. Li in the paper [15], she has been investigated a large extension which is different from the classical ones to mean-field system with an application to linear quadratic problem.

Our control problem consists in minimizing a cost functional with initial and terminal risk-sensitive performance functional, as follows

$$J^θ (v(.)) = \mathbb{E} \left[ \exp θ \left\{ \Phi \left( x_T^v, \mathbb{E}'(x_T^v) \right) + \Psi \left( y_0^v, \mathbb{E}'(y_0^v) \right) \right\} \right]$$

$$+ \int_0^T l \left( t, x_t^v, y_t^v, z_t^v, \mathbb{E}'(x_t^v), \mathbb{E}'(y_t^v), \mathbb{E}'(z_t^v), v_t \right) dt \right\} \right],$$

for given functions Φ, Ψ and l. This cost functional is also of mean-field type and with exponential expected, as the functions Φ, Ψ and l depend on the marginal law of the state process through its expected value. The pioneering works on the stochastic maximum principle for this kind of problem was first written by Djehiche et al [10], many works have been studied and continue this problem of risk-sensitive, we can mentioned as an example the papers of Chala [7, 8, 12].

In the risk-neutral model, the system is governed by the mean-field FBSDE

$$\begin{cases}
\frac{dx_t^v}{dt} = b \left( t, x_t^v, y_t^v, z_t^v, \mathbb{E}'(x_t^v), \mathbb{E}'(y_t^v), \mathbb{E}'(z_t^v), v_t \right) dt \\
\quad + \sigma \left( t, x_t^v, y_t^v, z_t^v, \mathbb{E}'(x_t^v), \mathbb{E}'(y_t^v), \mathbb{E}'(z_t^v), v_t \right) dW_t,
\end{cases}$$

$$x_0^v = a,$$

$$\begin{cases}
\frac{dy_t^v}{dt} = -f \left( t, x_t^v, y_t^v, z_t^v, \mathbb{E}'(x_t^v), \mathbb{E}'(y_t^v), \mathbb{E}'(z_t^v), v_t \right) dt + z_t^v dW_t,
\end{cases}$$

$$y_T^v = \xi,$$

The expected cost of mean-field type to be minimized, the risk-neutral model i.e that without the risk parameter and without the exponential expected

$$J (v(.)) = \mathbb{E} \left[ \Phi \left( x_T^v, \mathbb{E}'(x_T^v) \right) + \Psi \left( y_0^v, \mathbb{E}'(y_0^v) \right) \right]$$

$$+ \int_0^T l \left( t, x_t^v, y_t^v, z_t^v, \mathbb{E}'(x_t^v), \mathbb{E}'(y_t^v), \mathbb{E}'(z_t^v), v_t \right) dt \right\} \right].$$

A control u is called optimal if

$$J (u) = \inf_{v \in \mathcal{U}} J (v).$$

Existence of an optimal solution for this problem has been solved to achieve the objective of this paper, and establish necessary and sufficient optimality conditions for these two models, see theorem 3.1 below, we proceed as follows. First, we give -without proof- the optimality conditions for risk-neutral controls. The idea is to use the fact a certain auxiliary state process \((m_t^v)_{t \in [0, T]}\) which is a solution of some SDE, see section 3 below, and transfer the system with two equations the first one is a SDE, whereas the second is a backward SDE, into a system governed by three stochastic differential equations, and the set of risk neutral controls is convex. Then, the adjoint equation with respect to these three equations is given, the proof is a combination of the works of Min et al. [18] and those of Yong and Zhou [23, 24], the transformation of the adjoint equations will be used as the best approach to solve the risk sensitive control problem, we suggest this transformation to omit the first adjoint equation (which extended from the first SDE of the process) \((m_t^v)_{t \in [0, T]}\). The
necessary and the sufficient optimality conditions have been established with respect only to the second and the third adjoint equations, by the logarithm transformation method, see El Karoui & Hamedene [11], and by using the fact that the coefficients $b, \sigma, f, l, \Phi$ and $\Psi$ are Lipschitzien with respect to their components, the necessary optimality conditions are obtained directly in the global form.

A stochastic maximum principle (SMP) for risk-sensitive optimal control problems for Markov diffusion processes with an exponential of integral performance functional was obtained in [16] by making the relationship between the SMP and the Dynamic Programming Principle, the authors have used the first order adjoint process as the gradient of the value function of the control problem. This relationship holds only when the value function is smooth (see Assumption B4 in [16]). By using the smoothness assumption the two papers of [19, 20], have extended the approach described above to jump diffusions.

The paper is organized as follows: in Section 2, we formulate precisely our problem, introduce the risk-sensitive model, and state the various assumptions needed throughout this paper. In Section 3, we study our system of fully coupled FB-SDE, the new approach method transformation of the adjoint process is given and studied, stochastic maximum principle for risk-neutral is given. In Section 4, we establish the necessary optimality conditions for risk-sensitive control problem under an additional hypothesis. In Section 5, the sufficient optimality conditions for risk-sensitive performance cost is obtained under the convexity of the Hamiltonian function. In Section 6, we illustrate our main results by two examples; the first is a risk-sensitive control problem under linear stochastic dynamics with exponential quadratic cost function, the second is a financial model of mean-variance with risk-sensitive performance functional. Section 7 concludes the paper.

2. Problem formulation and assumptions. Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}\right)$ be a probability space filtered satisfying the usual conditions, in which a one-dimensional standard Brownian motion $W = (W_t; 0 \leq t \leq T)$ is given. We assume that $\mathcal{F}_t$ is defined by $\forall t \geq 0, \mathcal{F}_t = \sigma(W_r; 0 \leq r \leq t) \lor \mathcal{N}$, where $\mathcal{N}$ denote the totality of $\mathbb{P}$-null sets of $\mathcal{F}$.

Let $\mathcal{M}^2([0, T]; \mathbb{R})$ denote the set of one-dimensional jointly measurable random process \{\(\varphi_t, t \in [0, T]\)\} which satisfy

\[(i) : \|\varphi\|_{\mathcal{M}^2} := \mathbb{E} \left[ \int_0^T |\varphi_t|^2 \, dt \right] < \infty, \quad (ii) : \varphi_t \text{ is } \mathcal{F}_t \text{ measurable, for any } t \in [0, T].\]

We denote similarly by $\mathcal{S}^2([0, T]; \mathbb{R})$ the set of continuous one-dimensional random process which satisfy:

\[(i) : \|\varphi\|_{\mathcal{S}^2} := \mathbb{E} \left[ \sup_{t \in [0, T]} |\varphi_t|^2 \right] < \infty, \quad (ii) : \varphi_t \text{ is } \mathcal{F}_t \text{ measurable, for any } t \in [0, T].\]

Let $T$ be a strictly positive real number and $U$ is a nonempty subset of $\mathbb{R}$, such that $U$ is convex.

Definition 2.1. An admissible control $v$ is a process with valued in $U$ such that $\mathbb{E} \left[ \int_0^T |v_t|^2 \, dt \right] < \infty$. We denote by $\mathcal{U}$ the set of all admissible controls.

For any $v \in \mathcal{U}$, we consider the following fully coupled forward-backward stochastic differential equation of mean-field type control system
where $b : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $\sigma : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $f : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$.

We defined the criterion to be minimized, with initial and terminal risk-sensitive performance functional cost, as follows

\[
J^\theta (v(\cdot)) = \mathbb{E} \left[ \exp \theta \left\{ \Phi \left( x_{T}^v, \mathbb{E}' (x_{T}^v) \right) + \Psi \left( y_{T}^v, \mathbb{E}' (y_{T}^v) \right) \right\} \right] + \int_0^T l \left( t, x_{t}^v, y_{t}^v, z_{t}^v, \mathbb{E}' (x_{t}^v), \mathbb{E}' (y_{t}^v), \mathbb{E}' (z_{t}^v), v_t \right) dt \right],
\]

where $\Phi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $\Psi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $l : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, and $\theta$ is the risk-sensitive index.

The control problem is to minimize the functional $J^\theta$ over $\mathcal{U}$, if $u \in \mathcal{U}$ is an optimal solution, that is

\[
J^\theta (u) = \inf_{v \in \mathcal{U}} J^\theta (v).
\]

We use the Euclidean norm $|.|$ in $\mathbb{R}$, $\top$ is a transpose and $Tr$ is trace of matrix. All the equalities and inequalities, mentioned in this paper, are in the sense of $dt \times d\mathbb{P}$ almost surely on $[0, T] \times \Omega$.

**Notation.** We use the following notations

\[
\Gamma = \begin{pmatrix} x^v \\ y^v \\ z^v \end{pmatrix}, \quad \Gamma' = \begin{pmatrix} \mathbb{E}' (x^v) \\ \mathbb{E}' (y^v) \\ \mathbb{E}' (z^v) \end{pmatrix} \quad \text{and} \quad D \left( t, \Gamma, \Gamma' \right) = \begin{pmatrix} b \\ \sigma \\ -f \end{pmatrix} \left( t, \Gamma, \Gamma' \right).
\]

We assume the following assumptions

**Assumption 2.1.** For each $\Gamma, \Gamma' \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $D \left( t, \Gamma, \Gamma' \right)$ is in

\[\mathcal{M}^2 \left( [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \right), i.e. D \left( t, \Gamma, \Gamma' \right) \text{ is an } \mathcal{F}_t- \text{measurable process defined on } [0, T].\]

**Assumption 2.2.** $D \left( t, \Gamma, \Gamma' \right)$ is uniformly Lipschitz with respect to $\left( \Gamma, \Gamma' \right)$. There exists a constant $k > 0$, such that

\[
\left| D \left( t, \Gamma_1, \Gamma' \right) - D \left( t, \Gamma_2, \Gamma' \right) \right| \leq k \left| \Gamma_1 - \Gamma_2 \right|, \quad \forall \ \Gamma_1, \Gamma_2, \Gamma' \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \ \forall t \in [0, T].
\]

We also need the following monotonic conditions introduced by [18], are the main assumptions in this papers.

**Assumption 2.3.** $\left\langle D \left( t, \Gamma_1, \Gamma' \right) - D \left( t, \Gamma_2, \Gamma' \right), \Gamma_1 - \Gamma_2 \right\rangle \leq \alpha \left| \Gamma_1 - \Gamma_2 \right|^2$, $\forall \ \Gamma_1, \Gamma_2, \Gamma' \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \ \forall t \in [0, T]$ where $\alpha$ is a positive constant.

**Theorem 2.2.** For any given admissible control $v(\cdot)$, and under the above Assumptions 2.1 – 2.3. Then the fully coupled FBSDE of mean-field type control (5) has a unique solution

\[
(x_t^v, y_t^v, z_t^v) \in \mathcal{M}^2 \left( [0, T] ; \mathbb{R} \right) \times \mathcal{M}^2 \left( [0, T] ; \mathbb{R} \right) \times \mathcal{S}^2 \left( [0, T] ; \mathbb{R} \right).
\]
Proof. See [18] Theorem 6 page 3.

A control that solves the problem \{\(5\), \(6\), \(7\)\} is called optimal. Our goal is to establish risk-sensitive necessary as well as sufficient optimality conditions, satisfied by a given optimal control, in the form of mean-field stochastic maximum principle with a risk-sensitive performance functional type.

We also assume that

**Assumption 2.4.**

i) \(b, \sigma, f, l, \Phi \) and \(\Psi\) are continuously differentiable with respect to \((x^v, y^v, z^v, E' (x^v), E (y^v), E' (z^v), v)\).

ii) All the derivatives of \(b, \sigma, f, l\) are bounded by

\[
C \left( 1 + |x^v| + |y^v| + |z^v| + \left| E' (x^v) \right| + \left| E' (y^v) \right| + \left| E' (z^v) \right| + |v| \right).
\]

iii) The derivative of \(\Phi \) and \(\Psi\) is bounded by \(C \left( 1 + |x^v| + \left| E' (x^v) \right| \right)\)

and \(C \left( 1 + |y^v| + \left| E' (y^v) \right| \right)\) respectively.

Under the above assumptions, for every \(v \in \mathcal{U}\) equation (5) has a unique strong solution, and the cost function \(J^v\) is well defined from \(\mathcal{U}\) into \(\mathbb{R}\). For more details of this kind of problem the reader can see paper of Min, Peng and Qin [18].

### 3. Relation between the risk-neutral and risk-sensitive stochastic maximum principle

The proof of our risk-sensitive stochastic maximum principle necessitates a certain an auxiliary state process \(m^v_t\) which is a solution of the following stochastic differential equation of mean-field type control (SDE of mean-field type control), where

\[
dm^v_t = l \left(t, x^v_t, y^v_t, z^v_t, E' (x^v_t), E (y^v_t), E' (z^v_t), v_t \right) dt, \quad m^v_0 = 0.
\]

Our control problem of \{\(5\), \(6\), \(7\)\} and from the above auxiliary process, the fully coupled forward-backward of mean-field type control is equivalent to

\[
\begin{align*}
\inf_{\varphi \in \mathcal{U}} & \mathbb{E} \left[ \exp \theta \left( \Phi (x_T^v, E' (x_T^v)) + \Psi (y_0^v, E (y_0^v)) + m_T^v \right) \right] \\
= & \inf_{\varphi \in \mathcal{U}} \mathbb{E} \left[ \varphi (x_0^v, y_0^v) \right],
\end{align*}
\]

subject to

\[
\begin{align*}
dm^v_t = & \ l \left(t, x^v_t, y^v_t, z^v_t, E' (x^v_t), E (y^v_t), E' (z^v_t), v_t \right) dt, \\
m^v_0 = & \ 0, \\
dx^v_t = & \ b \left(t, x^v_t, y^v_t, z^v_t, E' (x^v_t), E (y^v_t), E' (z^v_t), v_t \right) dt \\
& \ + \ \alpha \left(t, x^v_t, y^v_t, z^v_t, E' (x^v_t), E (y^v_t), E' (z^v_t), v_t \right) dW_t, \\
x^v_0 = & \ a, \\
dy^v_t = & \ -f \left(t, x^v_t, y^v_t, z^v_t, E' (x^v_t), E (y^v_t), E' (z^v_t), v_t \right) dt + z^v_t dW_t, \\
y^v_T = & \ \xi.
\end{align*}
\]

We require the following condition

\[
A^\theta_T := \exp \theta \left( \Phi (x_T^v, E' (x_T^v)) + \Psi (y_0^v, E (y_0^v)) \right) \\
+ \int_0^T l \left(t, x^v_t, y^v_t, z^v_t, E' (x^v_t), E (y^v_t), E' (z^v_t), v_t \right) dt,
\]

where
and we put also
\[
\Theta_T := \Phi \left( x_T^v, E'(x_T^v) \right) + \Psi \left( y_0^v, E'(y_0^v) \right) \\
+ \int_0^T \left\{ t \left( x_t^v, y_t^v, z_t^v, E'(x_t^v), E'(y_t^v), E'(z_t^v), v_t \right) dt, \right.
\]
the risk-sensitive loss of functional is given by
\[
\mathcal{H}(\theta, v) := \frac{1}{\theta} \log \left[ E \left( \exp \theta \left\{ \Phi \left( x_T^v, E'(x_T^v) \right) + \Psi \left( y_0^v, E'(y_0^v) \right) \right\} \right) \right] \\
= \frac{1}{\theta} \log [E (\exp \theta \Theta_T)] .
\]

When the risk-sensitive index \( \theta \) is small, by Chala et al. [9] the loss functional \( \mathcal{H}(\theta, v) \) can be expanded as
\[
E(\Theta_T) + \frac{\theta}{2} Var(\Theta_T) + O(\theta^2),
\]
where, \( Var(\Theta_T) \) denotes the variance of \( \Theta_T \). If \( \theta < 0 \), the variance of \( \Theta_T \), as a measure of risk, improves the performance \( \mathcal{H}(\theta, v) \), in which case the optimizer is called risk seeker. But, when \( \theta > 0 \), the variance of \( \Theta_T \) worsens the performance \( \mathcal{H}(\theta, v) \), in which case the optimizer is called risk averse. The risk-neutral loss functional \( E(\Theta_T) \) can be seen as a limit of risk-sensitive functional \( \mathcal{H}(\theta, v) \) when \( \theta \rightarrow 0 \), for more details the reader can see the papers [10, 11].

In what next let us introduce the following notations.

**Notation.** For convenience, we will use the following notations throughout this paper, for \( \phi \in \{b, \sigma, f, l\} \), respectively, we define
\[
\phi(t) = \phi(t, O^v(t), v_t) , \\
\partial \phi(t) = \phi(t, O^v(t), v_t) - \phi(t, O^u(t), u_t) , \\
\phi_\zeta(t) = \frac{\partial}{\partial \zeta} \phi(t, O^v(t), v_t) ,
\]
and
\[
\bar{H}^\theta(t) = \bar{H}^\theta(t, m_t^v, O^v(t), v_t, \tilde{p}(t), \tilde{q}(t)) , \\
\bar{H}_\zeta^\theta(t) = \frac{\partial}{\partial \zeta} \bar{H}^\theta(t, m_t^v, O^v(t), v_t, \tilde{p}(t), \tilde{q}(t)) ,
\]
and
\[
H^\theta(t) = H^\theta(t, O^v(t), v_t, \tilde{p}_2(t), \tilde{q}_2(t), \tilde{p}_3(t), \tilde{q}_3(t), V^\theta(t), N_3(t)) , \\
H_\zeta^\theta(t) = \frac{\partial}{\partial \zeta} H^\theta(t, O^v(t), v_t, \tilde{p}_2(t), \tilde{q}_2(t), \tilde{p}_3(t), \tilde{q}_3(t), V^\theta(t), N_3(t)) ,
\]
where
\[
O^u(t) = x_t^u, y_t^u, z_t^u, E'(x_t^u), E'(y_t^u), E'(z_t^u) , \\
O^v(t) = x_t^v, y_t^v, z_t^v, E'(x_t^v), E'(y_t^v), E'(z_t^v) , \\
\zeta = x, y, z, \tilde{x}, \tilde{y}, \tilde{z}, v ,
\]
and \( v_t \) is an admissible control from \( U \).
We assume that the Assumptions 2.1 – 2.4 hold, we may apply the SMP for risk-neutral of fully coupled forward-backward of mean-field type control from Min, Peng and Qin [18], and to augmented state dynamics \((m^u, x^u, y^u, z^u)\) to derive the adjoint equation. There exist a unique \(F\) dynamics \((\theta A_T \Phi_x \left( x_T^u, E'(x_T^u) \right), -\Psi_y \left( y_0^u, E'(y_0^u) \right)\) and \((p_2, q_2)\) and \((p_3, q_3)\), which solve the following system matrix of backward SDEs

\[
\begin{align*}
\frac{d\tilde{p}}{dt}(t) &= -A(t) \, dt + R(t) \, dW_t, \\
\begin{pmatrix} p_1(T) \\ p_2(T) \\ p_3(0) \end{pmatrix} &= \theta A_T \begin{pmatrix} \Phi_x \left( x_T^u, E'(x_T^u) \right) \\ -\Psi_y \left( y_0^u, E'(y_0^u) \right) \\ 0 \end{pmatrix} + \theta E' \begin{pmatrix} \Phi_x \left( x_T^u, E'(x_T^u) \right) \\ -\Psi_y \left( y_0^u, E'(y_0^u) \right) \end{pmatrix},
\end{align*}
\]

with

\[
E \left[ \sum_{i=1}^{3} \sup_{t \in [0,T]} |p_i(t)|^2 + \sum_{i=1}^{2} \int_0^T |q_i(t)|^2 \, dt \right] < \infty,
\]

where

\[
A(t) = \begin{pmatrix} 0 & 0 & 0 \\ l_x(t) & b_x(t) & -f_x(t) \\ l_y(t) & b_y(t) & -f_y(t) \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_x(t) & 0 \\ 0 & \sigma_y(t) & 0 \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} + E' \begin{pmatrix} 0 & 0 & 0 \\ l_{\bar{x}}(t) & b_{\bar{x}}(t) & -f_{\bar{x}}(t) \\ l_{\bar{y}}(t) & b_{\bar{y}}(t) & -f_{\bar{y}}(t) \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix}
\]

and

\[
R(t) = \begin{pmatrix} q_1(t) \\ q_2(t) \\ \delta^\theta(t) \end{pmatrix},
\]

such that

\[
\delta^\theta(t) = -Tr \left[ \begin{pmatrix} l_z(t) & b_z(t) \\ \sigma_z(t) & -f_z(t) \end{pmatrix} \begin{pmatrix} p_1(t) & q_2(t) \\ p_2(t) & p_3(t) \end{pmatrix} \right] - Tr \left[ E' \left[ \begin{pmatrix} l_{\bar{z}}(t) & b_{\bar{z}}(t) \\ \sigma_{\bar{z}}(t) & -f_{\bar{z}}(t) \end{pmatrix} \begin{pmatrix} p_1(t) & q_2(t) \\ p_2(t) & p_3(t) \end{pmatrix} \right] \right].
\]

We suppose here that \(\bar{H}^\theta\) be the Hamiltonian associated with the optimal state dynamics \((m^u, x^u, y^u, z^u)\), and the pair of adjoint processes \((\bar{p}(t), \bar{q}(t))\) is given...
investigate the properties of these new processes (\(\tilde{\theta}\)). The following properties of the generic martingale \(V\) sensitive performance functional kind.

Theorem 3.1. Assume that 2.1–2.4 hold. If \((m^u, x^u, y^u, z^u)\) is an optimal solution of the risk-neutral control problem (8), then there is three pairs of \(\mathcal{F}_t\)–adapted processes \((p_1, q_1), (p_2, q_2)\) and \((p_3, q_3)\) that satisfy (9), such that

\[
\tilde{H}^\theta (t) := \tilde{H}^\theta (t, m^u, \mathcal{O}^u (t), u_t, \tilde{p}^t (t), \tilde{q} (t)) \quad (10)
\]

\[
= \begin{pmatrix} l (t) \\ b (t) \\ -f (t) \end{pmatrix} (\tilde{p}^t (t))^\top + \begin{pmatrix} 0 \\ \sigma (t) \\ 0 \end{pmatrix} (\tilde{q} (t))^\top .
\]

4. New adjoint equations and necessary optimality conditions. As we said, the Theorem 3.1 is a good SMP for the risk-neutral of fully coupled forward-backward of mean-field type control problem. We follow the new approach which has been used in [7, 8, 10, 11], and suggest a transformation of the adjoint processes \((p_1, q_1), (p_2, q_2)\) and \((p_3, q_3)\) in such a way to omit the first component \((p_1, q_1)\) in (9), and express the SMP in terms of only the last two adjoint processes, that we denote them by \((\tilde{p}_2, \tilde{q}_2)\) and \((\tilde{p}_3, \tilde{q}_3)\).

Noting that \(dp_1 (t) = q_1 (t) \, dW_t\) and \(p_1 (T) = \theta A^\theta_T\), the explicit solution of this backward SDE is

\[
p_1 (t) = \theta \mathbb{E} \left[ A^\theta_T \mid \mathcal{F}_t \right] = \theta V^\theta_t, \quad (12)
\]

where

\[
V^\theta_t := \mathbb{E} \left[ A^\theta_T \mid \mathcal{F}_t \right], \quad \forall \ 0 \leq t \leq T.
\]

As a good view of (12), it would be natural to choose a transformation of \((\tilde{p}, \tilde{q})\) into an adjoint process \((\tilde{p}, \tilde{q})\), where \(\tilde{p}_1 (t) = \frac{1}{\partial V^\theta_t} p_1 (t) = 1\).

We consider the following transform

\[
\tilde{p} (t) = \begin{pmatrix} \tilde{p}_1 (t) \\ \tilde{p}_2 (t) \\ \tilde{p}_3 (t) \end{pmatrix} := \frac{1}{\partial V^\theta_t} \tilde{p}^t (t), \quad 0 \leq t \leq T. \quad (13)
\]

By using (9) and (13), we have

\[
\tilde{p} (.) := \begin{pmatrix} \tilde{p}_1 (T) \\ \tilde{p}_2 (T) \\ \tilde{p}_3 (0) \end{pmatrix}
\]

\[
= \begin{pmatrix} \Phi_x \left( x^\theta_T, E (x^\theta_T) \right) \\ -\Phi_y \left( y^\theta_0, E (y^\theta_0) \right) \end{pmatrix} + \frac{1}{V^\theta_T} \mathbb{E}' \left( V^\theta_T \begin{pmatrix} 0 \\ \Phi_x \left( x^\theta_T, E (x^\theta_T) \right) \\ -\Phi_y \left( y^\theta_0, E (y^\theta_0) \right) \end{pmatrix} \right).\]

The following properties of the generic martingale \(V^\theta\) are essential in order to investigate the properties of these new processes \((\tilde{p} (t), \tilde{q} (t))\).

In the next, we will state and prove the necessary optimality conditions for the system driven by fully coupled FBSDE of mean-field type control with a risk-sensitive performance functional kind.
As mentioned in the paper of El-Karoui et al. [11], the process $\Lambda^\theta$ is the first component of the $\mathcal{F}_t$-adapted pair of processes $(\Lambda^\theta, N)$ which is the unique solution to the following quadratic backward SDE of mean-field type

\[
\begin{align*}
\frac{d\Lambda^\theta_t}{V^\theta_t} &= -\left(l(t) + \frac{\theta}{2} |N(t)|^2\right) dt + N(t) dW_t, \\
\Lambda^\theta_T &= \Phi\left(x^\theta_T, \mathbb{E}'\left(x^\theta_T\right)\right) + \Psi\left(y^\theta_0, \mathbb{E}'\left(y^\theta_0\right)\right),
\end{align*}
\]

where

\[
\mathbb{E}\left[\int_0^T |N(t)|^2 dt\right] < \infty.
\]

To this end, let us summarize and prove some lemmas that we will use thereafter.

**Lemma 4.1.** Suppose that 2.4 holds. Then

\[
\mathbb{E}\left[\sup_{t \in [0, T]} |\Lambda^\theta_t|\right] \leq C_T,
\]

where, $C_T$ is a positive constant that depends only on $T$ and the boundedness of $l$, $\Phi$ and $\Psi$.

In particular, $V^\theta$ solves the following linear backward SDE

\[
dV^\theta_t = \theta N(t) V^\theta_t dW_t, \quad V^\theta_T = A^\theta_T.
\]

Hence, the process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ by $L^\theta_t$, where

\[
L^\theta_t := \frac{V^\theta_t}{V^{\theta_0}} = \exp\left(\int_0^t \theta N(s) dW_s - \frac{\theta^2}{2} \int_0^t |N(s)|^2 ds\right), \quad \forall 0 \leq t \leq T,
\]

is a uniformly bounded $\mathcal{F}_t$-martingale.

**Proof.** Starting with (14). By Assumption 2.4, the functions $l$, $\Phi$ and $\Psi$ are bounded by constant $C > 0$, we get

\[
0 < e^{-(2+T)C\theta} \leq A^\theta_T \leq e^{(2+T)C\theta}.
\]

Therefore, $V^\theta$ is a uniformly bounded $\mathcal{F}_t$-martingale satisfying

\[
0 < e^{-(2+T)C\theta} \leq V^\theta_t \leq e^{(2+T)C\theta}, \quad \forall 0 \leq t \leq T.
\]

Completed of proof see the paper of [7], Lemma 3.1 pages 405 – 406.

**Proposition 4.1.** The second and the third risk-sensitive adjoint equations for $(\tilde{p}_2, \tilde{q}_2)$, $(\tilde{p}_3, \tilde{q}_3)$ and $(V^\theta, N)$ become

\[
\begin{align*}
d\tilde{p}_2 (t) &= -\left[H^\theta_x(t) + \frac{1}{V^\theta_t} \mathbb{E}'\left[V^\theta_t H^\theta_z(t)\right]\right] dt + [\tilde{q}_2(t) - \theta N(t) \tilde{p}_2(t)] dW^\theta_t, \\
\tilde{p}_2(T) &= \Phi_x\left(x^\theta_T, \mathbb{E}'\left(x^\theta_T\right)\right) + \frac{1}{V^\theta_T} \mathbb{E}'\left[V^\theta_T \Phi_z\left(x^\theta_T, \mathbb{E}'\left(x^\theta_T\right)\right)\right], \\
d\tilde{p}_3 (t) &= -\left[H^\theta_y(t) + \frac{1}{V^\theta_t} \mathbb{E}'\left[V^\theta_t H^\theta_y(t)\right]\right] dt \\
&\quad - \left[H^\theta_z(t) + \frac{1}{V^\theta_t} \mathbb{E}'\left[V^\theta_t H^\theta_z(t)\right]\right] dW^\theta_t, \\
\tilde{p}_3(0) &= -\Psi_y\left(y^\theta_0, \mathbb{E}'\left(y^\theta_0\right)\right) - \frac{1}{V^\theta_T} \mathbb{E}'\left[V^\theta_T \Psi_z\left(y^\theta_0, \mathbb{E}'\left(y^\theta_0\right)\right)\right], \\
dV^\theta_t &= \theta N(t) V^\theta_t dW_t, \\
V^\theta_T &= A^\theta_T.
\end{align*}
\]
The solution \((\tilde{p}, \tilde{q}, V^\theta, N)\) of the system (18) is unique, such that
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} |\tilde{p}(t)|^2 + \sup_{t \in [0,T]} |V^\theta(t)|^2 + \int_0^T \left( |\tilde{q}(t)|^2 + |N(t)|^2 \right) dt \right] < \infty, \quad (19)
\]
where
\[
H^\theta(t) := H^\theta(t, \mathcal{O}^\theta(t), \nu_t, \tilde{p}_2(t), \tilde{q}_2(t), \tilde{p}_3(t), V^\theta(t), N_3(t))
\]
\[
= l(t) + b(t) \tilde{p}_2(t) + \sigma(t) \tilde{q}_2(t) - (f(t) - \theta N_3(t) z^\nu(t)) \tilde{p}_3(t). \quad (20)
\]

Proof. We wish to identify the processes \(\tilde{\alpha}\) and \(\tilde{\beta}\) such that
\[
d\tilde{p}(t) = -\tilde{\alpha}(t) dt + \tilde{\beta}(t) dW_t. \quad (21)
\]
By applying Itô’s formula to the processes \(\tilde{p}(t) = \theta V^\theta(t) \tilde{p}(t)\), and using the expression of \(V^\theta\) in (15), we obtain
\[
d\tilde{p}(t) = -\frac{1}{\theta V^\theta_t} \begin{pmatrix}
0 & 0 & 0 \\
\tilde{l}_x(t) & \tilde{b}_x(t) & -\tilde{f}_x(t) \\
\tilde{l}_y(t) & \tilde{b}_y(t) & -\tilde{f}_y(t)
\end{pmatrix} \begin{pmatrix}
p_1(t) \\
p_2(t) \\
p_3(t)
\end{pmatrix} dt
\]
\[
- \frac{1}{\theta V^\theta_t} \begin{pmatrix}
0 & 0 & 0 \\
\sigma_x(t) & 0 & 0 \\
\sigma_y(t) & 0 & 0
\end{pmatrix} \begin{pmatrix}
q_1(t) \\
q_2(t) \\
q_3(t)
\end{pmatrix} dt
\]
\[
- \frac{1}{\theta V^\theta_t} \mathbb{E}' \begin{pmatrix}
0 & 0 & 0 \\
\tilde{b}_z(t) & \tilde{b}_y(t) & -\tilde{f}_y(t) \\
\tilde{l}_y(t) & \tilde{b}_y(t) & -\tilde{f}_y(t)
\end{pmatrix} \begin{pmatrix}
q_1(t) \\
q_2(t) \\
q_3(t)
\end{pmatrix} dt
\]
\[
+ \frac{1}{\theta V^\theta_t} \begin{pmatrix}
q_1(t) \\
q_2(t) \\
\delta^\theta(t)
\end{pmatrix} dW_t - \theta \begin{pmatrix}
N_1(t) \\
N_2(t) \\
N_3(t)
\end{pmatrix} \tilde{p}(t) dW_t.
\]

By identifying the coefficients the above equation to (21), and using the relation \(\tilde{p}(t) = \frac{1}{\theta V^\theta_t} \tilde{p}(t)\), the diffusion coefficient \(\tilde{\beta}(t)\) it will be written as
\[
\tilde{\beta}(t) = \begin{pmatrix}
\tilde{q}_1(t) \\
\tilde{q}_2(t) \\
\delta^\theta(t)
\end{pmatrix} - \theta \begin{pmatrix}
N_1(t) \\
N_2(t) \\
N_3(t)
\end{pmatrix} \tilde{p}(t), \quad (22)
\]
and the drift coefficient of the process \(\tilde{p}(t)\)
\[
\tilde{\alpha}(t) = \begin{pmatrix}
0 & 0 & 0 \\
\tilde{l}_x(t) & \tilde{b}_x(t) & -\tilde{f}_x(t) \\
\tilde{l}_y(t) & \tilde{b}_y(t) & -\tilde{f}_y(t)
\end{pmatrix} \begin{pmatrix}
\tilde{p}_1(t) \\
\tilde{p}_2(t) \\
\tilde{p}_3(t)
\end{pmatrix}
\]
\[
+ \begin{pmatrix}
0 & 0 & 0 \\
0 & \sigma_x(t) & 0 \\
0 & \sigma_y(t) & 0
\end{pmatrix} \begin{pmatrix}
\tilde{q}_1(t) \\
\tilde{q}_2(t) \\
\tilde{q}_3(t)
\end{pmatrix}
\]
\[
+ \frac{1}{\theta V^\theta_t} \mathbb{E}' \begin{pmatrix}
0 & 0 & 0 \\
\tilde{l}_x(t) & \tilde{b}_x(t) & -\tilde{f}_x(t) \\
\tilde{l}_y(t) & \tilde{b}_y(t) & -\tilde{f}_y(t)
\end{pmatrix} \begin{pmatrix}
\tilde{p}_1(t) \\
\tilde{p}_2(t) \\
\tilde{p}_3(t)
\end{pmatrix}
\]
Finally, we obtain
\[
d\tilde{\rho}(t) = - \left( \begin{array}{ccc}
0 & 0 & 0 \\
l_x(t) & b_x(t) & -f_x(t) \\
l_y(t) & b_y(t) & -f_y(t)
\end{array} \right) \left( \begin{array}{c}
\tilde{\rho}_1(t) \\
\tilde{\rho}_2(t) \\
\tilde{\rho}_3(t)
\end{array} \right) dt
- \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & \sigma_x(t) & 0 \\
0 & \sigma_y(t) & 0
\end{array} \right) \left( \begin{array}{c}
\tilde{q}_1(t) \\
\tilde{q}_2(t) \\
\tilde{q}_3(t)
\end{array} \right) dt
- \theta \left( \begin{array}{c}
N_1(t) \\
N_2(t) \\
N_3(t)
\end{array} \right) \tilde{\beta}(t) dt + \tilde{\beta}(t) dW_t.
\]

It is easily verified that
\[
d\tilde{\rho}_1(t) = \tilde{\beta}_1(t) (dW_t - \theta N_1(t) dt), \quad \tilde{\rho}_1(T) = 1.
\]

In view of (16), we may use Girsanov’s Theorem (see [9], Theorem 2.1 page 115), to claim that
\[
d\tilde{\rho}_1(t) = \tilde{\beta}_1(t) dW^\theta_t, \quad \tilde{\rho}_1(T) = 1,
\]
where \(dW^\theta_t = dW_t - \theta N(t) dt\) is a \(\mathbb{P}^\theta\)—Brownian motion, where,
\[
\frac{d\mathbb{P}^\theta}{d\mathbb{P}}_{\mathcal{F}_t} := L^\theta_t = \exp \left( \int_0^t \theta N(s) dW_s - \frac{\theta^2}{2} \int_0^t |N(s)|^2 ds \right), \quad 0 \leq t \leq T.
\]

In view of (16) and (17), the probability measures \(\mathbb{P}^\theta\) and \(\mathbb{P}\) are in fact equivalent. Hence, noting that \(\tilde{\rho}(t) := \frac{1}{\theta V^\theta_t} \rho(t)\) is square-integrable, we get that \(\tilde{\rho}(t) = \mathbb{E}^\theta[\tilde{\rho}(T) | \mathcal{F}_t] = 1\). Thus, its quadratic variation \(\int_0^T |\tilde{q}_1(t)|^2 dt = 0\). This implies that, for almost every \(0 \leq t \leq T, \tilde{q}_1(t) = 0, \mathbb{P}^\theta\) and \(\mathbb{P}\)–a.s.

\[
d\tilde{\rho}(t) = - \left( \begin{array}{ccc}
0 & 0 & 0 \\
l_x(t) & b_x(t) & -f_x(t) \\
l_y(t) & b_y(t) & -f_y(t)
\end{array} \right) \left( \begin{array}{c}
\tilde{\rho}_1(t) \\
\tilde{\rho}_2(t) \\
\tilde{\rho}_3(t)
\end{array} \right) dt
- \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & \sigma_x(t) & 0 \\
0 & \sigma_y(t) & 0
\end{array} \right) \left( \begin{array}{c}
\tilde{q}_1(t) \\
\tilde{q}_2(t) \\
\tilde{q}_3(t)
\end{array} \right) dt
- \theta \left( \begin{array}{c}
N_1(t) \\
N_2(t) \\
N_3(t)
\end{array} \right) \tilde{\beta}(t) dt + \tilde{\beta}(t) dW_t.
\]
\[-\frac{1}{V_t^\theta} E^t \left[ V_t^\theta \begin{pmatrix} 0 & 0 & 0 \\ \sigma_x(t) & 0 & 0 \\ \sigma_y(t) & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{q}_1(t) \\ \tilde{q}_2(t) \\ \tilde{q}_3(t) \end{pmatrix} \right] \right] dt + \tilde{\beta}(t) dW_t^\theta.\]

Now replacing (22) in the (23), to obtain

\[d\tilde{\rho}(t) = -\left( 0 \quad 0 \quad 0 \\ l_x(t) \quad b_x(t) \quad -f_x(t) \\ l_y(t) \quad b_y(t) \quad -f_y(t) \right) \begin{pmatrix} \tilde{p}_1(t) \\ \tilde{p}_2(t) \\ \tilde{p}_3(t) \end{pmatrix} dt + \frac{1}{V_t^\theta} E^t \left[ V_t^\theta \begin{pmatrix} 0 & 0 & 0 \\ \sigma_x(t) & 0 & 0 \\ \sigma_y(t) & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{q}_1(t) \\ \tilde{q}_2(t) \\ \tilde{q}_3(t) \end{pmatrix} \right] dt\]

\[d\tilde{\rho}(t) = -\left( 0 \quad 0 \quad 0 \\ l_x(t) \quad b_x(t) \quad -f_x(t) \\ l_y(t) \quad b_y(t) \quad -f_y(t) \right) \begin{pmatrix} \tilde{p}_1(t) \\ \tilde{p}_2(t) \\ \tilde{p}_3(t) \end{pmatrix} dt + \frac{1}{V_t^\theta} E^t \left[ V_t^\theta \begin{pmatrix} 0 & 0 & 0 \\ \sigma_x(t) & 0 & 0 \\ \sigma_y(t) & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{q}_1(t) \\ \tilde{q}_2(t) \\ \tilde{q}_3(t) \end{pmatrix} \right] dt\]

\[+ \left( \begin{pmatrix} \tilde{q}_1(t) \\ \tilde{q}_2(t) \\ \tilde{q}_3(t) \end{pmatrix} \right) dW_t^\theta - \theta \begin{pmatrix} N_1(t) \\ N_2(t) \\ N_3(t) \end{pmatrix} \tilde{\rho}(t) dW_t^\theta,\]

where

\[\tilde{\rho}(t) = -Tr \left[ \begin{pmatrix} l_x(t) & b_x(t) \\ \sigma_x(t) & -f_x(t) \end{pmatrix} \begin{pmatrix} \tilde{p}_1(t) \\ \tilde{p}_2(t) \end{pmatrix} \begin{pmatrix} \tilde{q}_1(t) \\ \tilde{q}_2(t) \end{pmatrix} \right] - Tr \left[ \frac{1}{V_t^\theta} E^t \left[ V_t^\theta \begin{pmatrix} l_x(t) & b_x(t) \\ \sigma_x(t) & -f_x(t) \end{pmatrix} \begin{pmatrix} \tilde{p}_1(t) \\ \tilde{p}_2(t) \end{pmatrix} \begin{pmatrix} \tilde{q}_1(t) \\ \tilde{q}_2(t) \end{pmatrix} \right] \right].\]

Therefore, the second and third components of \(\tilde{\rho}_2(t)\) and \(\tilde{\rho}_3(t)\) given by (24), we get

\[d\tilde{\rho}_2(t) = -[l_x(t) + b_x(t) \tilde{\rho}_2(t) - f_x(t) \tilde{\rho}_3(t) + \sigma_x(t) \tilde{q}_2(t)] dt + \frac{1}{V_t^\theta} E^t [V_t^\theta l_x(t) + b_x(t) \tilde{\rho}_2(t) - f_x(t) \tilde{\rho}_3(t) + \sigma_x(t) \tilde{q}_2(t)] dW_t^\theta,\]

and

\[d\tilde{\rho}_3(t) = -[l_y(t) + b_y(t) \tilde{\rho}_2(t) - f_y(t) \tilde{\rho}_3(t) + \sigma_y(t) \tilde{q}_2(t)] dt + \frac{1}{V_t^\theta} E^t [V_t^\theta l_y(t) + b_y(t) \tilde{\rho}_2(t) - f_y(t) \tilde{\rho}_3(t) + \sigma_y(t) \tilde{q}_2(t)] dW_t^\theta.\]

The main risk-sensitive second and third adjoint equations for \((\tilde{\rho}_2, \tilde{\rho}_3)\), \((\tilde{\rho}_3, \tilde{\rho}_3)\), and \((V^\theta, N)\) become
Theorem 4.2. (Necessary optimality conditions for risk-sensitive)
We assume that 2.1 – 2.4 hold. If \((x^u, y^u, z^u, u)\) is an optimal solution of the risk-sensitive control problem \(\{5\}, \{6\}, \{7\}\), then there exist pairs of \(\mathcal{F}_t\)-adapted processes \((V^\theta, N), (\tilde{\rho}, \tilde{q})\) that satisfy (18), (19), such that

\[
H^\theta(t) := H^\theta(t, \mathcal{O}^u(t), u_t, \nu_t, \tilde{\rho}_2(t), \tilde{\rho}_3(t), \tilde{q}_2(t), \tilde{q}_3(t), V^\theta(t), N_3(t))
\]

\[
= l(t) + b(t) \tilde{\rho}_2(t) + \sigma(t) \tilde{q}_2(t) - (f(t) - \theta N_3(t) z^\theta_1) \tilde{\rho}_3(t) .
\]

This finished the proof of Proposition 4.1. \(\Box\)

We can now state the necessary optimality conditions

\[
\begin{align*}
d\tilde{\rho}_2(t) &= - \left[ H^\theta_2(t) + \frac{1}{V^\theta_t} \mathbb{E}' \left[ V^\theta_t H^\theta_2(t) \right] \right] dt + \left[ \tilde{q}_2(t) - \theta N_2(t) \tilde{\rho}_2(t) \right] dW^\theta_t, \\
\tilde{\rho}_3(T) &= \Phi_x \left( x^\theta_f, \mathbb{E}'(x^\theta_f) \right) + \frac{1}{V^\theta_T} \mathbb{E}' \left[ V^\theta_T \Phi_x \left( x^\theta_f, \mathbb{E}'(x^\theta_f) \right) \right], \\
d\tilde{\rho}_3(t) &= - \left[ H^\theta_3(t) + \frac{1}{V^\theta_t} \mathbb{E}' \left[ V^\theta_t H^\theta_3(t) \right] \right] dt \\
&\quad - \left[ H^\theta_2(t) + \frac{1}{V^\theta_t} \mathbb{E}' \left[ V^\theta_t H^\theta_2(t) \right] \right] dW^\theta_t, \\
\tilde{\rho}_3(0) &= - \Psi_y \left( y^\theta_0, \mathbb{E}'(y^\theta_0) \right) - \frac{1}{V^\theta_T} \mathbb{E}' \left[ V^\theta_T \Psi_y \left( y^\theta_0, \mathbb{E}'(y^\theta_0) \right) \right], \\
dV^\theta_t &= \theta N(t) V^\theta_t dW_t, \\
V^\theta_T &= A^\theta_T.
\end{align*}
\]

The solution \((\tilde{\rho}, \tilde{q}, V^\theta, N)\) of the system (18) is unique, such that

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \| \tilde{\rho}(t) \|^2 + \sup_{t \in [0, T]} \| V^\theta(t) \|^2 + \int_0^T \| \tilde{q}(t) \|^2 + |N(t)|^2 \right] dt < \infty,
\]

where

\[
H^\theta(t) := H^\theta(t, \mathcal{O}^u(t), u_t, \tilde{\rho}_2(t), \tilde{q}_2(t), \tilde{\rho}_3(t), V^\theta(t), N_3(t))
\]

\[
= l(t) + b(t) \tilde{\rho}_2(t) + \sigma(t) \tilde{q}_2(t) - (f(t) - \theta N_3(t) z^\theta_1) \tilde{\rho}_3(t) .
\]

for all \(u \in \mathcal{U}\), almost every \(0 \leq t \leq T\) and \(\mathbb{P}\)-almost surely.

Proof. To arrive at a risk-sensitive stochastic maximum principle expressed in terms of the adjoint processes \((\tilde{\rho}_2, \tilde{q}_2), (\tilde{\rho}_3, \tilde{q}_3)\) and \((V^\theta, N)\), which solve (18), where the Hamiltonian \(\tilde{H}^\theta\) associated with (8), given by (10) satisfies

\[
\tilde{H}^\theta(t, m^u_t, \mathcal{O}^u(t), u_t, \tilde{\rho}(t), \tilde{q}(t))
\]

\[
= \{ \theta V^\theta_t \} H^\theta(t, \mathcal{O}^u(t), u_t, \tilde{\rho}_2(t), \tilde{q}_2(t), \tilde{\rho}_3(t), V^\theta(t), N_3(t)) ,
\]

and \(H^\theta\) is the risk-sensitive Hamiltonian given by (20). Hence, since \(V^\theta > 0\), the variational inequality (11) translates into

\[
H^\theta_v(t, \mathcal{O}^u(t), u_t, \tilde{\rho}_2(t), \tilde{q}_2(t), \tilde{\rho}_3(t), V^\theta(t), N_3(t)) (u_t - v_t) \leq 0,
\]

for all \(u \in \mathcal{U}\), almost every \(0 \leq t \leq T\) and \(\mathbb{P}\)-almost surely. This completed the proof of Theorem 4.2. \(\Box\)
5. Sufficient optimality conditions for risk-sensitive performance cost. In this section, we study when the necessary optimality conditions (11) become sufficient. For any \( v \in \mathcal{U} \), we denote by \((x^v, y^v, z^v)\) the solution of equation (5) controlled by \( v \) to state the following result.

**Theorem 5.1. (Sufficient optimality conditions for risk-sensitive)** Assume that the functions \( \Phi, \Psi \) and \((m^v, O^v, v) \to \bar{H}^\theta \) are convex, and for any \( v \in \mathcal{U} \) such that \( \mathbb{E} \left[ \int_0^T |v|^2 \, dt \right] < \infty \). Then, \( u \) is an optimal solution of the control problem \{(5), (6), (7)\}, if it satisfies (11).

**Proof.** Let \( u \) be an arbitrary element of \( \mathcal{U} \) (candidate to be optimal). For any \( v \in \mathcal{U} \), we have

\[
\begin{align*}
J^\theta (v) - J^\theta (u) & = \mathbb{E} \left[ \exp \left( \theta \left\{ \Phi \left( x^v_T, \mathcal{E}^v \left( x^v_T \right) \right) + \Psi \left( y^v_0, \mathcal{E}^v \left( y^v_0 \right) \right) + m^v_T \right\} \right) \right] \\
& \geq \mathbb{E} \left[ \theta A^\theta_T (m^v_T - m^u_T) \right] \\
& \quad + \mathbb{E} \left[ \theta \left( A^\theta_T \Phi_x \left( x^v_T, \mathcal{E}^v \left( x^v_T \right) \right) + \mathcal{E}' \left( A^\theta_T \Phi_x \left( x^v_T, \mathcal{E}^v \left( x^v_T \right) \right) \right) \right) (x^v_T - x^u_T) \right] \\
& \quad + \mathbb{E} \left[ \theta \left( A^\theta_T \Psi_y \left( y^v_0, \mathcal{E}^v \left( y^v_0 \right) \right) + \mathcal{E}' \left( A^\theta_T \Psi_y \left( y^v_0, \mathcal{E}^v \left( y^v_0 \right) \right) \right) \right) (y^v_0 - y^u_0) \right] .
\end{align*}
\]

It follows from (9), we remark that
\[
\begin{align*}
p_1 (T) & = \theta A^\theta_T, \\
p_2 (T) & = \theta \left( A^\theta_T \Phi_x \left( x^v_T, \mathcal{E}^v \left( x^v_T \right) \right) + \mathcal{E}' \left( A^\theta_T \Phi_x \left( x^v_T, \mathcal{E}^v \left( x^v_T \right) \right) \right) \right), \quad \text{and} \\
p_3 (0) & = -\theta \left( A^\theta_T \Psi_y \left( y^v_0, \mathcal{E}^v \left( y^v_0 \right) \right) + \mathcal{E}' \left( A^\theta_T \Psi_y \left( y^v_0, \mathcal{E}^v \left( y^v_0 \right) \right) \right) \right), \quad \text{then we have}
\end{align*}
\]

\[
\begin{align*}
J^\theta (v) - J^\theta (u) & \geq \mathbb{E} [p_1 (T) (m^v_T - m^u_T)] + \mathbb{E} [p_2 (T) (x^v_T - x^u_T)] - \mathbb{E} [p_3 (0) (y^v_0 - y^u_0)].
\end{align*}
\]

By applying Itô’s formula to \( p_1 (t) (m^v_t - m^u_t), p_2 (t) (x^v_t - x^u_t) \) and \( p_3 (t) (y^v_t - y^u_t) \) that lead to

\[
\begin{align*}
\mathbb{E} [p_1 (T) (m^v_T - m^u_T)] & = \mathbb{E} \left[ \int_0^T \left( l \left( t, \mathcal{O}^v \left( t \right), v_t \right) - l \left( t, \mathcal{O}^u \left( t \right), u_t \right) \right) p_1 (t) \, dt \right], \\
\mathbb{E} [p_2 (T) (x^v_T - x^u_T)] & = -\mathbb{E} \left[ \int_0^T \bar{H}^\theta \left( t, m^v_t, \mathcal{O}^v \left( t \right), u_t, \bar{p} \left( t \right), \bar{q} \left( t \right) \right) (x^v_t - x^u_t) \, dt \right] \\
& \quad - \mathbb{E} \left[ \int_0^T \mathcal{E} \left[ \bar{H}^\theta \left( t, m^v_t, \mathcal{O}^v \left( t \right), u_t, \bar{p} \left( t \right), \bar{q} \left( t \right) \right) \right] (x^v_t - x^u_t) \, dt \right] \\
& \quad + \mathbb{E} \left[ \int_0^T (b \left( t, \mathcal{O}^v \left( t \right), v_t \right) - b \left( t, \mathcal{O}^u \left( t \right), u_t \right)) p_2 (t) \, dt \right]
\end{align*}
\]
+ E \left[ \int_0^T (\sigma (t, O^v (t), v_t) - \sigma (t, O^u (t), u_t)) q_2 (t) \, dt \right],

and

\begin{align*}
- E [p_3 (0) (y_0^v - y_0^u)] \\
= - \frac{1}{2} E \left[ \int_0^T \tilde{H}^\theta \left( t, m_{t, t}, O^u (t), u_t, \vec{p} (t), \vec{q} (t) \right) (y_t^v - y_t^u) \, dt \right] \\
- E \left[ \int_0^T \tilde{H}^\theta \left( t, m_{t, t}, O^u (t), u_t, \vec{p} (t), \vec{q} (t) \right) (z_t^v - z_t^u) \, dt \right] \\
- \frac{1}{2} E \left[ \int_0^T \tilde{H}^\theta \left( t, m_{t, t}, O^u (t), u_t, \vec{p} (t), \vec{q} (t) \right) (x_t^v - x_t^u) \, dt \right].
\end{align*}

By replacing (28), (29) and (30) into (27), we get

\begin{align*}
J^\theta (v) - J^\theta (u) \\
\geq \frac{1}{2} E \left[ \int_0^T \tilde{H}^\theta \left( t, m_{t, t}, O^v (t), v_t, \vec{p} (t), \vec{q} (t) \right) \, dt \right] \\
- E \left[ \int_0^T \tilde{H}^\theta \left( t, m_{t, t}, O^u (t), u_t, \vec{p} (t), \vec{q} (t) \right) \, dt \right] \\
- E \left[ \int_0^T \tilde{H}^\theta \left( t, m_{t, t}, O^u (t), u_t, \vec{p} (t), \vec{q} (t) \right) (x_t^v - x_t^u) \, dt \right] \\
- E \left[ \int_0^T E' \left[ \tilde{H}^\theta \left( t, m_{t, t}, O^u (t), u_t, \vec{p} (t), \vec{q} (t) \right) \right] (x_t^v - x_t^u) \, dt \right] \\
- E \left[ \int_0^T \tilde{H}^\theta \left( t, m_{t, t}, O^u (t), u_t, \vec{p} (t), \vec{q} (t) \right) (y_t^v - y_t^u) \, dt \right] \\
- E \left[ \int_0^T E' \left[ \tilde{H}^\theta \left( t, m_{t, t}, O^u (t), u_t, \vec{p} (t), \vec{q} (t) \right) \right] (y_t^v - y_t^u) \, dt \right] \\
- E \left[ \int_0^T \tilde{H}^\theta \left( t, m_{t, t}, O^u (t), u_t, \vec{p} (t), \vec{q} (t) \right) (z_t^v - z_t^u) \, dt \right] \\
- E \left[ \int_0^T E' \left[ \tilde{H}^\theta \left( t, m_{t, t}, O^u (t), u_t, \vec{p} (t), \vec{q} (t) \right) \right] (z_t^v - z_t^u) \, dt \right].
\end{align*}

Since the Hamiltonian $\tilde{H}^\theta$ is convex with respect to $(x, \bar{x}, y, \bar{y}, z, \bar{z}, v)$, we have

\[ E \left[ \int_0^T \tilde{H}^\theta \left( t, m_{t, t}, O^v (t), v_t, \vec{p} (t), \vec{q} (t) \right) \, dt \right]. \]
\[-E \left[ \int_0^T \tilde{H}_0^{\theta} \left( t, m_i^u, O^u (t), u_t, \tilde{p} (t), \tilde{q} (t) \right) dt \right] \]
\[\geq E \left[ \int_0^T \tilde{H}_x^{\theta} \left( t, m_i^u, O^u (t), u_t, \tilde{p} (t), \tilde{q} (t) \right) (x_i^u - x_i^v) dt \right] \]
\[+ E \left[ \int_0^T E' \left[ \tilde{H}_x^{\theta} \left( t, m_i^u, O^u (t), u_t, \tilde{p} (t), \tilde{q} (t) \right) \right] (x_i^u - x_i^v) dt \right] \]
\[+ E \left[ \int_0^T \tilde{H}_y^{\theta} \left( t, m_i^u, O^u (t), u_t, \tilde{p} (t), \tilde{q} (t) \right) (y_i^v - y_i^u) dt \right] \]
\[+ E \left[ \int_0^T E' \left[ \tilde{H}_y^{\theta} \left( t, m_i^u, O^u (t), u_t, \tilde{p} (t), \tilde{q} (t) \right) \right] (y_i^v - y_i^u) dt \right] \]
\[+ E \left[ \int_0^T \tilde{H}_z^{\theta} \left( t, m_i^u, O^u (t), u_t, \tilde{p} (t), \tilde{q} (t) \right) (z_i^v - z_i^u) dt \right] \]
\[+ E \left[ \int_0^T E' \left[ \tilde{H}_z^{\theta} \left( t, m_i^u, O^u (t), u_t, \tilde{p} (t), \tilde{q} (t) \right) \right] (z_i^v - z_i^u) dt \right] \]
\[+ E \left[ \int_0^T \tilde{H}_w^{\theta} \left( t, m_i^u, O^u (t), u_t, \tilde{p} (t), \tilde{q} (t) \right) (v_t - u_t) dt \right] , \]

or equivalently

\[E \left[ \int_0^T \tilde{H}_0^{\theta} \left( t, m_i^u, O^u (t), v_t, \tilde{p} (t), \tilde{q} (t) \right) dt \right] \]
\[- E \left[ \int_0^T \tilde{H}_x^{\theta} \left( t, m_i^u, O^u (t), u_t, \tilde{p} (t), \tilde{q} (t) \right) dt \right] \]
\[- E \left[ \int_0^T \tilde{H}_y^{\theta} \left( t, m_i^u, O^u (t), u_t, \tilde{p} (t), \tilde{q} (t) \right) (x_i^v - x_i^u) dt \right] \]
\[- E \left[ \int_0^T E' \left[ \tilde{H}_x^{\theta} \left( t, m_i^u, O^u (t), u_t, \tilde{p} (t), \tilde{q} (t) \right) \right] (x_i^v - x_i^u) dt \right] \]
\[- E \left[ \int_0^T \tilde{H}_y^{\theta} \left( t, m_i^u, O^u (t), u_t, \tilde{p} (t), \tilde{q} (t) \right) (y_i^v - y_i^u) dt \right] \]
\[- E \left[ \int_0^T E' \left[ \tilde{H}_y^{\theta} \left( t, m_i^u, O^u (t), u_t, \tilde{p} (t), \tilde{q} (t) \right) \right] (y_i^v - y_i^u) dt \right] \]
\[- E \left[ \int_0^T \tilde{H}_z^{\theta} \left( t, m_i^u, O^u (t), u_t, \tilde{p} (t), \tilde{q} (t) \right) (z_i^v - z_i^u) dt \right] \]
\[- E \left[ \int_0^T E' \left[ \tilde{H}_z^{\theta} \left( t, m_i^u, O^u (t), u_t, \tilde{p} (t), \tilde{q} (t) \right) \right] (z_i^v - z_i^u) dt \right] \]
\[- E \left[ \int_0^T \tilde{H}_w^{\theta} \left( t, m_i^u, O^u (t), u_t, \tilde{p} (t), \tilde{q} (t) \right) (v_t - u_t) dt \right] . \]
Remark 5.1. In the last step of proof, and according to (25), we have

\[ J^\theta (v) - J^\theta (u) \geq \mathbb{E} \left[ \int_0^T H^\theta_t \left( t, m^\theta_t, \mathcal{O}^u (t), u_t, \pi_t, p_t, q_t \right) (v_t - u_t) \, dt \right]. \]

In virtue of the necessary optimality conditions (11), then the last inequality implies that \( J^\theta (v) - J^\theta (u) \geq 0 \). Then the theorem is proved. \( \square \)

6. Applications.

6.1. Example 1: Risk-sensitive control applied to the mean-field linear-quadratic. We provide a concrete example of the mean field risk sensitive forward-backward stochastic LQ problem, and we give the explicit optimal control and validate our major theoretical results in Theorem 5.1 (Sufficient optimality conditions for risk-sensitive). First, let the control domain be \( U = [-1, 1] \). Consider the following to the mean-field linear quadratic risk-sensitive control problem

\[
\begin{align*}
\inf_{v \in \mathcal{U}} \mathbb{E} \left[ \exp \left\{ \frac{1}{2} \int_0^T v_t^2 \, dt + \frac{1}{2} (x_T^v)^2 + \frac{1}{2} (y_T^v)^2 \right\} \right] &= \inf_{v \in \mathcal{U}} \mathbb{E} \left[ \varphi (x_T^v, y_T^v) \right], \\
\text{subject to} & \\
\begin{aligned}
\dot{x}_t^v &= A_1 x_t^v + A_2 \mathbb{E}' (x_t^v) + A_3 v_t \, dt + \left( B_1 x_t^v + B_2 \mathbb{E}' (x_t^v) + B_3 v_t \right) \, dW_t, \\
x_0^v &= a, \\
\dot{y}_t^v &= - \left( C_1 x_t^v + C_2 \mathbb{E}' (x_t^v) + C_3 y_t^v + C_4 \mathbb{E}' (y_t^v) + C_5 z_t^v + C_6 \mathbb{E}' (z_t^v) \right) + C_7 v_t \, dt + z_t^v \, dW_t, \\
y_T^v &= \xi,
\end{aligned}
\end{align*}
\]

where \( A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3, C_4, C_5, C_6 \) and \( C_7 \) are constants.

Let \((x_t^v, y_t^v, z_t^v)\) be a solution of (32) associated with \( v_t \). Then, there exist unique \( \mathcal{F}_t \)-adapted three pairs of processes \((p_t, q_t), (p_2, q_2)\) and \((p_3, q_3)\) of the following FBSDE of mean-field type system (called adjoint equations), according to the equations (9). Now we can define these equations as
\[
\begin{align*}
    dp_1 (t) &= q_1 (t) \, dW_t, \\
    p_1 (T) &= \theta A_T^\theta, \\
    dp_2 (t) &= -\left( A_1 p_2 (t) + B_1 q_2 (t) - C_1 p_3 (t) + A_2 E' (p_2 (t)) + B_2 E' (q_2 (t)) \right) dt + q_2 (t) \, dW_t, \\
    p_2 (T) &= \theta x_T^v A_T^\theta, \\
    dp_3 (t) &= \left( C_3 p_3 (t) + C_4 E' (p_3 (t)) \right) dt + \left( C_5 p_3 (t) + C_6 E' (p_3 (t)) \right) dW_t, \\
    p_3 (0) &= -\theta y_0^v A_T^\theta, \\
\end{align*}
\]

where
\[
A_T^\theta := \exp \left\{ \frac{1}{2} \int_0^T v_t^2 \, dt + \frac{1}{2} \left( x_T^v \right)^2 + \frac{1}{2} \left( y_0^v \right)^2 \right\}.
\]

We give the Hamiltonian \( \tilde{H}^\theta \) defined by
\[
\tilde{H}^\theta (t) := \tilde{H}^\theta \left( t, m_t^v, \mathcal{O}^v (t), v_t, \overrightarrow{p} (t), \overrightarrow{q} (t) \right)
\]
\[
= \frac{1}{2} v_t^2 p_1 (t) + \left( A_1 x_t^v + A_2 E' (x_t^v) + A_3 v_t \right) p_2 (t) + \left( B_1 x_t^v + B_2 E' (x_t^v) + B_3 v_t \right) q_2 (t) - \left( C_1 x_t^v + C_2 E' (x_t^v) + C_3 y_t^v + C_4 E' (y_t^v) + C_5 z_t^v + C_6 E' (z_t^v) + C_7 v_t \right) p_3 (t).
\]

We have \( \tilde{H}^\theta (t) = v_t p_1 (t) + A_3 p_2 (t) + B_3 q_2 (t) - C_7 p_3 (t) \). Maximizing the Hamiltonian yields
\[
u_t = (C_7 p_3 (t) - A_3 p_2 (t) - B_3 q_2 (t)) p_1^{-1} (t). \tag{34}
\]

We need only to prove that \( u_t \) of (34) is an optimal control of (32).

**Theorem 6.1.** (Risk-sensitive sufficient optimality conditions for linear quadratic control problem) Suppose that \( u_t \) satisfies (34), where \( \left( \overrightarrow{p}, \overrightarrow{q} \right) \) satisfy (33), then \( u_t \) is the optimal control of the above mean-field forward-backward stochastic differential equation of linear quadratic control problem (32).

**Proof.** From the definition of the cost functional \( J^\theta \), we have
\[
J^\theta (v_t) - J^\theta (u_t) = E \left[ \exp \theta \left\{ \frac{1}{2} \int_0^T v_t^2 \, dt + \frac{1}{2} \left( x_T^v \right)^2 + \frac{1}{2} \left( y_0^v \right)^2 \right\} \right] - E \left[ \exp \theta \left\{ \frac{1}{2} \int_0^T u_t^2 \, dt + \frac{1}{2} \left( x_T^v \right)^2 + \frac{1}{2} \left( y_0^v \right)^2 \right\} \right].
\]

We put \( m_T^v = \frac{1}{2} \int_0^T v_t^2 dt \), and by applying the Taylor’s expansion, we have
\[
J^\theta (v_t) - J^\theta (u_t) = E \left[ p_1 (T) (m_T^v - m_T^v) \right] + E \left[ p_2 (T) (x_T^v - x_T^v) \right] + E \left[ p_3 (0) (y_0^v - y_0^v) \right],
\]

where \( p_1 (T) = \theta A_T^\theta, \ p_2 (T) = \theta x_T^v A_T^\theta \) and \( p_3 (0) = -\theta y_0^v A_T^\theta \).

By applying Itô’s formula to \( p_1 (t) (m_t^v - m_t^v), \ p_2 (t) (x_t^v - x_t^v) \) and \( p_3 (t) (y_t^v - y_t^v) \), and used the explicit forms of the adjoint equations (33), that lead to
\[
E \left[ p_1 (T) (m_T^v - m_T^v) \right] = E \left[ \int_0^T \frac{1}{2} (v_t^2 - u_t^2) p_1 (t) \, dt \right],
\]
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and

$$E[ p_2 (T) (x_T - x_T)]$$

$$= E \left[ \int_0^T C_1 p_3 (t) (x_t - x_t) \, dt \right] + E \left[ \int_0^T C_2 p_3 (t) (x_t - x_t) \, dt \right]$$

$$+ E \left[ \int_0^T A_3 p_2 (t) (v_t - u_t) \, dt \right] + E \left[ \int_0^T B_3 q_2 (t) (v_t - u_t) \, dt \right],$$

and

$$- E[p_3 (0) (y_0 - y_0)]$$

$$= - E \left[ \int_0^T C_1 p_3 (t) (x_t - x_t) \, dt \right] - E \left[ \int_0^T C_2 p_3 (t) (x_t - x_t) \, dt \right]$$

$$- E \left[ \int_0^T p_3 (t) (v_t - u_t) \, dt \right].$$

By replacing the three above formulas into (35), then we get

$$J_\theta (v_t) - J_\theta (u_t)$$

$$= E \left[ \int_0^T \frac{1}{2} (v_t - u_t) (v_t - u_t) p_1 (t) \, dt \right] + E \left[ \int_0^T u_t (v_t - u_t) p_1 (t) \, dt \right]$$

$$- E \left[ \int_0^T p_3 (t) (v_t - u_t) \, dt \right] + E \left[ \int_0^T A_3 p_2 (t) (v_t - u_t) \, dt \right]$$

$$+ E \left[ \int_0^T B_3 q_2 (t) (v_t - u_t) \, dt \right].$$

Then, because of $(v_t - u_t)$ being nonnegative, we have the following result

$$J_\theta (v_t) - J_\theta (u_t)$$

$$\geq E \left[ \int_0^T u_t (v_t - u_t) p_1 (t) \, dt \right] + E \left[ \int_0^T A_3 p_2 (t) (v_t - u_t) \, dt \right]$$

$$+ E \left[ \int_0^T B_3 q_2 (t) (v_t - u_t) \, dt \right] - E \left[ \int_0^T p_3 (t) (v_t - u_t) \, dt \right].$$

Then,

$$J_\theta (v_t) - J_\theta (u_t) \geq E \left[ \int_0^T (u_t p_1 (t) + A_3 p_2 (t) + B_3 q_2 (t) - C_7 p_3 (t)) (v_t - u_t) \, dt \right].$$

By replacing $u_t$ with its value in (36), we obtain

$$J_\theta (v_t) - J_\theta (u_t) \geq 0.$$

Thus, we get

$$J_\theta (v_t) \geq J_\theta (u_t),$$

i.e. $u_t$ is optimal. This proof is finished.
6.2. Example 2: Financial application: Mean-variance risk-sensitive stochastic optimal portfolio problem. Now we return to the problem of optimal stochastic portfolio stated in the motivating example, and deal with the mean-variance risk-sensitive stochastic optimal control problem shown in Section 1, and apply the risk-sensitive necessary optimality condition (Theorem 4.2, we use the results obtained by Tembine [22], Zhou and Li [25] to investigate this example).

Our state dynamics is
\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{dx_t^v}{x_0^v} = (\rho v_t + r x_t^v) \, dt + \sigma v_t \, dW_t, \\
\end{array} \right.
\end{align*}
\]  
(37)

and
\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{dy_t^v}{y_0^v} = -(c x_t^v + \rho v_t - \lambda y_t^v) \, dt + z_t^v \, dW_t, \\
\end{array} \right.
\end{align*}
\]  
(38)

According to Section 3, we conclude
\[
\mathcal{J}^\theta (v(\cdot)) = \frac{1}{\theta} \log \left[ J^\theta (v(\cdot)) \right] = \mathbb{E} (\Theta_T) + \theta Var (\Theta_T) + O (\theta^2),
\]
where
\[
\Theta_T := \Phi \left( x_T^v, \mathcal{E} \left( x_T^v \right) \right) + \Psi \left( y_0^v, \mathcal{E} \left( y_0^v \right) \right) + \int_0^T l (t, O^v (t), v_t) \, dt.
\]

We put \( l(t, O^v (t), v_t) = 0, \Phi \left( x_T^v, \mathcal{E} \left( x_T^v \right) \right) = x_T^v \) and \( \Psi \left( y_0^v, \mathcal{E} \left( y_0^v \right) \right) = -y_0^v \), we get \( \Theta_T := x_T^v - y_0^v \). Then, the cost function is the following
\[
\mathcal{J}^\theta (v(\cdot)) = \frac{1}{\theta} \log \left[ J^\theta (v(\cdot)) \right] = \mathbb{E} \left[ x_T^v - y_0^v + \frac{\theta}{2} \left( x_T^v - y_0^v - \theta \right)^2 \right],
\]
(39)

where \( \theta > 0, \theta \neq 1, \theta = \mathbb{E} (x_T^v - y_0^v) \).

The investor wants to minimize (39) subject to (37) and (38), by taking \( v(\cdot) \) over \( \mathcal{U} \).

The Hamiltonian function (20) gets the form
\[
H^\theta (t) := H^\theta (t, O^v (t), v_t, \tilde{p}_2 (t), \tilde{q}_2 (t), \tilde{p}_3 (t), N_3 (t))
\]  
(40)

\[
= \rho v_t (\tilde{p}_2 (t) - \tilde{p}_3 (t)) + \sigma v_t \tilde{q}_2 (t) + r x_t^v \tilde{p}_2 (t)
\]
\[
+ (\lambda y_t^v - c x_t^v + \theta N_3 (t) z_t^v) \tilde{p}_3 (t).
\]

Let \((x_t^u, y_t^u, z_t^u)\) be an optimal triplet of the system \{(37), (38)\}. The adjoint equations (18) reduce to
\[
\left\{ \begin{array}{l}
d\tilde{p}_2 (t) = c \tilde{p}_3 (t) - r \tilde{p}_2 (t) \, dt + [\tilde{q}_2 (t) - \theta N_2 (t) \tilde{p}_2 (t)] \, dW_t^\theta,
\end{array} \right.
\]
(41)

and
\[
\left\{ \begin{array}{l}
d\tilde{p}_3 (t) = -\lambda \tilde{p}_3 (t) \, dt - \theta N_3 (t) \tilde{p}_3 (t) \, dW_t^\theta,
\end{array} \right.
\]

Minimizing the Hamiltonian (40), we obtain the following result
\[
\rho (\tilde{p}_2 (t) - \tilde{p}_3 (t)) + \sigma \tilde{q}_2 (t) = 0.
\]
(42)

The SDE (37), and the adjoint equation (41) with respect to optimal control, being
\[
\left\{ \begin{array}{l}
\frac{dx_t^u}{x_0^u} = (\rho u_t + r x_t^u) \, dt + \sigma u_t \, dW_t,
\end{array} \right.
\]
(43)
and
\[
\begin{align*}
\begin{cases}
    d\tilde{p}_2^u (t) = & c\tilde{p}_3^u (t) - r\tilde{p}_2^u (t) \, dt + [\tilde{q}_2^u (t) - \theta N_2 (t) \tilde{p}_2^u (t)] \, dW_t^\theta, \\
    \tilde{p}_2^u (T) = & 1 + \theta [x_T^u - y_0^u - \vartheta].
\end{cases}
\end{align*}
\]
Replacing \(dW_t^\theta = dW_t - \theta N_2 (t) \, dt\) in (44), we get
\[
\begin{align*}
\begin{cases}
    d\tilde{p}_2^u (t) = & [c\tilde{p}_3^u (t) - r\tilde{p}_2^u (t) - \theta N_2 (t) \tilde{q}_2^u (t) + \theta^2 N_2^2 (t) \tilde{p}_2^u (t)] \, dt \\
    + & [\tilde{q}_2^u (t) - \theta N_2 (t) \tilde{p}_2^u (t)] \, dW_t, \\
\tilde{p}_2^u (T) = & 1 + \theta [x_T^u - y_0^u - \vartheta].
\end{cases}
\end{align*}
\]
Therefore, an optimal solution \((\tilde{p}_2^u (t), x_1^u, u_t)\) can be obtained by solving the system of FBSDE of mean-field type control (43) and (45). To solve the FBSDE \{(43), (45)\}, we conjecture the solution to (43) and (45) is related by
\[
\tilde{p}_2^u (t) = \varpi (t) x_t^u + \varsigma (t) \mathbb{E}' (x_t^u) + \gamma (t),
\]
for some deterministic differentiable functions \(\varpi (t), \varsigma (t)\) and \(\gamma (t)\), as the best of our acknowledge the term \(\sigma u_t \, dW_t\) is called stochastic integral, so it goes to zero with respect to \(\mathbb{E}'\), we have
\[
\begin{align*}
\begin{cases}
    d\mathbb{E}' (x_t^u) = & \left( \rho \mathbb{E}' (u_t) + r \mathbb{E}' (x_t^u) \right) \, dt, \\
\mathbb{E}' (x_0^u) = & m_0.
\end{cases}
\end{align*}
\]
By applying Itô's formula to (46), we get
\[
\begin{align*}
\begin{cases}
    d\tilde{q}_2^u (t) = & \left[ \left( \tilde{\varpi} (t) + \varpi (t) \, r \right) x_t^u + \left( \tilde{\varsigma} (t) + \varsigma (t) \, r \right) \mathbb{E}' (x_t^u) \right] \, dt \\
    + & \tilde{\gamma} (t) + \varpi (t) \rho u_t + \varsigma (t) \rho \mathbb{E}' (u_t) \, dt + \varpi (t) \sigma u_t \, dW_t, \\
\tilde{p}_2^u (T) = & \varpi (T) x_T^u + \varsigma (T) \mathbb{E}' (x_T^u) + \gamma (T).
\end{cases}
\end{align*}
\]
By equating the coefficients and the terminal conditions of (45) and (47), we have
\[
\begin{align*}
\tilde{q}_2^u (t) = & \varpi (t) \sigma u_t + \theta N_2 (t) \tilde{p}_2^u (t), \\
\varpi (T) = & \theta, \ \varsigma (T) = 0, \ \gamma (T) = 1 - \theta y_0^u - \theta \vartheta,
\end{align*}
\]
and
\[
0 = \left( \tilde{\varpi} (t) + \varpi (t) \, r \right) x_t^u + \left( \tilde{\varsigma} (t) + \varsigma (t) \, r \right) \mathbb{E}' (x_t^u) + \tilde{\gamma} (t) + \varpi (t) \rho u_t + \varsigma (t) \rho \mathbb{E}' (u_t) - c\tilde{p}_3^u (t) + \theta N_2 (t) \tilde{q}_2^u (t) - \theta^2 N_2^2 (t) \tilde{p}_2^u (t).
\]
By substituting (48) into (49), and by using (46), we obtain
\[
0 = \left( \tilde{\varpi} (t) + 2 \varpi (t) \, r \right) x_t^u + \left( \tilde{\varsigma} (t) + 2 \varsigma (t) \, r \right) \mathbb{E}' (x_t^u) + \tilde{\gamma} (t) + r \gamma (t) + \varpi (t) \rho u_t + \varsigma (t) \rho \mathbb{E}' (u_t) - c\tilde{p}_3^u (t) + \theta N_2 (t) \varpi (t) \sigma u_t.
\]
By (50), we deduce that \(\varpi (t), \varsigma (t)\) and \(\gamma (t)\) satisfying the following ordinary differential equations (in short ODEs)
\[
\begin{align*}
\begin{cases}
    \tilde{\varpi} (t) + 2 \varpi (t) \, r = 0, \\
    \varpi (T) = \theta, \\
    \tilde{\varsigma} (t) + 2 \varsigma (t) \, r = 0, \\
    \varsigma (T) = 0, \\
    \tilde{\gamma} (t) + r \gamma (t) + \varpi (t) \rho u_t + \varsigma (t) \rho \mathbb{E}' (u_t) - c\tilde{p}_3^u (t) + \theta N_2 (t) \varpi (t) \sigma u_t = 0, \\
    \gamma (T) = 1 - \theta y_0^u - \theta \vartheta.
\end{cases}
\end{align*}
\]
By solving the first and second ODEs in (51), we get

\[
\begin{align*}
\varpi (t) &= \theta \exp \left( -2 \int_t^T r \, ds \right), \quad (52) \\
\zeta (t) &= 0 \exp \left( -2 \int_t^T r \, ds \right). \quad (53)
\end{align*}
\]

Using the integrating factor method, to solve the third ODE in (51), we know that

\[
\begin{align*}
\begin{cases} 
\dot{\gamma} (t) + r \gamma (t) + \varpi (t) \rho u_t + \zeta (t) \rho \varpi' (u_t) - c \tilde{p}_3^u (t) + \theta N_2 (t) \varpi (t) \sigma u_t = 0, \\
\gamma (T) = 1 - \theta y_0^u - \theta \vartheta.
\end{cases} 
\end{align*} 
\]

We put

\[
\delta (t) = \varpi (t) \rho u_t + \zeta (t) \rho \varpi' (u_t) - c \tilde{p}_3^u (t) + \theta N_2 (t) \varpi (t) \sigma u_t. \quad (55)
\]

We rewrite (54) as follows

\[
\begin{align*}
\begin{cases} 
\dot{\gamma} (t) + r \gamma (t) + \delta (t) = 0, \\
\gamma (T) = 1 - \theta y_0^u - \theta \vartheta.
\end{cases} 
\end{align*} 
\]

The explicit solution of the equation (56) is

\[
\gamma (t) = \left[ 1 - \theta y_0^u - \theta \vartheta - \int_t^T \delta (s) \exp \left( \int_s^T r \, ds \right) \exp \left( - \int_t^T r \, ds \right) \right], \quad (57)
\]

where \( \delta (t) \) is determined by (55).

Finally, we can have the optimal control in the following state feedback form by using (48), we have

\[
u_t = \frac{1}{\varpi (t) \sigma} \tilde{q}_2^u (t) - \frac{1}{\varpi (t) \sigma} \theta N_2 (t) \tilde{p}_3^u (t),
\]

then by replacing the value of \( \tilde{q}_2^u (t) \) from (42), and \( \tilde{p}_3^u (t) \) from (46) into the last expression of \( u_t \) above, we have

\[
u_t = \frac{1}{\sigma^2 \varpi (t)} \varpi (t) x_t^u 
- (\rho + \sigma \theta N_2 (t)) \frac{1}{\sigma^2 \varpi (t)} \zeta (t) \varpi' (x_t^u) 
- (\rho + \sigma \theta N_2 (t)) \frac{1}{\sigma^2 \varpi (t)} \gamma (t) + \frac{\rho}{\sigma^2 \varpi (t)} \tilde{p}_3^u (t),
\]

where \( \varpi (t) \), \( \zeta (t) \) and \( \gamma (t) \) are determined by (52), (53) and (57) respectively.

**Theorem 6.2.** We assume that \( \varpi (t) \), \( \zeta (t) \) and \( \gamma (t) \) have the unique solution given by (52), (53) and (57) respectively. Then the optimal control of the problem \( \{37,39\} \) has the state feedback form (58).

**Remark 6.1.** It’s very important to remark that the solution of the function \( \gamma (t) \) in the expression (54) is depend to the solution of \( \tilde{p}_3^u (t) \). If we put \( \tilde{p}_3^u (t) = E (t) y_t^u + B (t) E' (y_t^u) + \kappa (t) \), for smooth deterministic functions \( E (t) \), \( B (t) \) and \( \kappa (t) \). By using the similar technique as an optimal solution in the last paragraph, to the optimal solution of \( \{ \tilde{p}_3^u (t), y_t^u, u_t \} \), then the solutions of functions \( E (t) \), \( B (t) \) and
Using the integrating factor method, to solve the third ODE in (59), we know that
\[
\begin{align*}
\kappa(t) + (\lambda - \theta^2 N_2^2(t)) \kappa(t) - \mathcal{E}(t) c x_l^u - \mathcal{E}(t) \rho u_t - \mathcal{B}(t) c \mathcal{E}'(x_l^u) \\
- \mathcal{B}(t) \rho \mathcal{E}'(u_t) = 0, \\
\kappa(0) = 1 + \theta x_0^u - \theta \vartheta.
\end{align*}
\]  
(59)

By solving the first and second ODEs in (59), we have
\[
\begin{align*}
\mathcal{E}(t) &= -\theta \exp \left(-2 \int_0^t \left(\lambda - \frac{1}{2} \theta^2 N_2^2(s)\right) ds\right), \\
\mathcal{B}(t) &= 0 \exp \left(-2 \int_0^t \left(\lambda - \frac{1}{2} \theta^2 N_2^2(s)\right) ds\right).
\end{align*}
\]  
(60), (61)

Using the integrating factor method, to solve the third ODE in (59), we know that
\[
\begin{align*}
\kappa(t) + (\lambda - \theta^2 N_2^2(t)) \kappa(t) - \mathcal{E}(t) c x_l^u - \mathcal{E}(t) \rho u_t - \mathcal{B}(t) c \mathcal{E}'(x_l^u) \\
- \mathcal{B}(t) \rho \mathcal{E}'(u_t) = 0, \\
\kappa(0) = 1 - \theta y_0^u - \theta \vartheta.
\end{align*}
\]  
(62)

We put
\[
\begin{align*}
\varrho(t) &= (\lambda - \theta^2 N_2^2(t)), \\
\psi(t) &= -\mathcal{E}(t) c x_l^u - \mathcal{E}(t) \rho u_t - \mathcal{B}(t) c \mathcal{E}'(x_l^u) - \mathcal{B}(t) \rho \mathcal{E}'(u_t).
\end{align*}
\]  
(63)

We rewrite (62) as follows
\[
\begin{align*}
\kappa(t) + \varrho(t) \kappa(t) + \psi(t) = 0, \\
\kappa(0) = 1 + \theta x_0^u - \theta \vartheta.
\end{align*}
\]  
(64)

The explicit solution of equation (64) is
\[
\kappa(t) = \left[1 + \theta x_0^u - \theta \vartheta - \int_0^t \psi(s) \exp \left(\int_0^s \varrho(r) dr\right) ds\right] \exp \left(\int_0^t -\varrho(s) ds\right),
\]  
(65)

where \(\varrho(t)\) and \(\psi(t)\) are determined by (63).

Then by using the expression of \(\hat{p}_3(t)\), the feedback form of the control in (58) can be rewritten as
\[
u_t = -\left(\rho + \sigma \theta N_2(t)\right) \frac{1}{\sigma^2 \mathcal{W}(t)} \mathcal{W}(t) x_l^u
\]  
(66)

Corollary 1. The explicit solution of the first and second ODEs in (59) are given by (60), (61) and the third ODE in (59) has an explicit solution given by (65), where \(\varrho(t)\) and \(\psi(t)\) are determined functions given by (63).

At the end, we can sum up the problem of portfolio \((37), (38), (39)\) for mean-variance with risk-sensitive performance, in the next theorem, as the main result.
Theorem 6.3. We assume that $\varpi(t), \varsigma(t)$ and $\gamma(t)$ have the unique solution given by (52), (53) and (57) respectively, $E(t), B(t)$ and $\kappa(t)$ have the explicit solution given by (60), (61) and (65). Then the optimal control of the problem \{(37), (38), (39)\} has the state feedback form (66), where $\delta(t)$ is determined by (55), $q(t)$ and $\psi(t)$ are given by (63).

7. Conclusion and outlook. This paper contains two main results. The first result (Theorem 4.2), establishes the necessary optimality condition for a system governed by fully coupled FBSDE of mean-field type with risk-sensitive performance, using a scheme almost similar to the one in Chala [7], and Djehiche et al [10]. The second main result, Theorem 5.1, provides sufficient optimality conditions for fully coupled FBSDE of mean-field type with risk-sensitive performance. The proof is based on the convexity conditions of the Hamiltonian function, the initial and terminal terms of the performance function. Note that the risk-sensitive control problems treated by Lim and Zhou [16] are different from ours. We may take as the existing paper which has been established by Djehiche et al [10], and have been generalized into the fully coupled stochastic differential equation with mean-field type, which is motivated by an optimal portfolio choice problem in financial market specially the model of control mean-variance. A problem to be thoroughly addressed in the future, where the system is governed by fully coupled stochastic differential equation of mean-field type with jump diffusion, will be compared with [20]. The maximum principle of risk-neutral obtained by Min et al [18], is similar to ours (Theorem 3.1), but the adjoint equations and maximum conditions depend heavily on the risk-sensitive parameter. If we put $\theta = \lambda = \mu = \kappa = 0$, we can compare our feedback control of (66) with the control obtained by Hafayed et al [12].

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