Bremsstrahlung corrections to the decay $b \to s\gamma$

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Abstract

We calculate the $O(\alpha_s)$ gluon Bremsstrahlung corrections to the inclusive decay $b \to s\gamma$, involving the full operator basis $\hat{O}_1 - \hat{O}_8$. Confirming and extending earlier calculations of Ali and Greub, we give formulas for the total decay width as well as the perturbative photon spectrum, regarding the former as a necessary part of the forthcoming complete NLO analysis. We explore in detail the renormalization scale dependence of our results and find it considerably increased.
1 Introduction

If we define rare weak decays of hadrons as weak decays which are rare because they are loop-induced (as opposed to CKM-suppressed), the analysis of these decays may shed light on at least three important topics: (i) Since they can proceed even at leading order only through diagrams with a loop of virtual particles, these decays do test the standard model of weak interactions as quantum field theory. (ii) Since they are decays of hadrons, they are of course strongly affected by QCD effects. If, however, one naively calculated these corrections, due to the massiveness of the vector bosons in the loop one would have to cope with large logarithms \( \ln \frac{M_W}{m_q} \), where \( M_W \) is the electroweak scale and \( m_q \) the usual hadronic scale. These large logarithms spoil perturbation theory: one has to resum them invoking the powerful techniques of the renormalization group. Especially in the case \( b \to s\gamma \), where the strong interaction is known to enhance the decay rate by a factor of 2-3, we have hence an ideal testing ground for those frequently employed concepts of partial resummation of the perturbative series. (iii) Since they are rare even in the standard model, they are sensitive to the effects of new physics: plainly spoken, new heavy particles running in the loop will contribute to the decay if their masses are not much larger than the electroweak scale, as expected for supersymmetry, left-right-symmetric models, technicolor and so on. Of course also the couplings of these particles must not be too small: so if we can find no positive sign of new physics, i.e. a deviation from the Standard Model (SM) prediction, some more or less sharp restrictions on the parameter space are all we can hope for.

Among all rare weak decays \( b \to s\gamma \) takes a special role: it is of order \( G_F^2\alpha \) (not \( G_F^4 \) or \( G_F^2\alpha^2 \)), and accordingly the corresponding branching ratio is much larger than that of most other rare decays. In fact, within the field of B physics, \( b \to s\gamma \) is the only one which has been measured experimentally, and this only recently. Following the first observation of the exclusive mode \( B \to K^*\gamma \) in 1993 [1], the CLEO collaboration reported by now measurements [2] of the inclusive branching ratio, \( \text{BR}[B \to X_s\gamma] = (2.32\pm 0.67) \times 10^{-4} \) as well as of the photon spectrum. Here \( X_s \) denotes an arbitrary state of total strangeness \(-1\), and experimentally some lower cutoff in the photon energy has to be imposed in order to exclude the tree-level channel \( b \to c\bar{c}s\gamma \). As it is well known, the inclusive rate is of much more theoretical interest than the exclusive modes, because within the framework of Heavy Quark Effective Theory (HQET) one can show that it is given by the free quark decay model plus perturbative QCD, up to calculable corrections of order \( 1/m_b^2 \) [3]. The same is surprisingly true for suitable defined moments of the photon spectrum [4]. Recall that a continuous photon spectrum in \( b \to s\gamma \) arises by reason of two effects: perturbatively, by emission of gluon bremsstrahlung (the topic of our paper), which
results in a long tail of the spectrum below the kinematical endpoint, and non-perturbatively by the Fermi motion of the b quark inside the B meson, which leads to a symmetric smearing around the endpoint. Combining these two effects is a quite non-trivial task, primarily because one has to make a consistent separation between the perturbative and the non-perturbative region. The analysis of the spectrum has been the subject of some recent papers [5, 6, 7, 8], and it will thus not be the central point of our work; instead we will give complete results for the bremsstrahlung contribution to the total decay width. To our knowledge, there exists only one calculation of Ali and Greub [9] on this subject, which for a long time was in need of an independent check. In view of the increasing desire for precise predictions of BR[B → Xsγ] we have performed this check, confirming and extending Ref. [9].

Our paper is organized as follows: In section 2 we give the effective Hamiltonian which we used in our calculations, and discuss very briefly the structure of a complete Next-to-Leading-Order (NLO) analysis and the place of this paper within that greater task. In section 3 we give the amplitude \( b \to s\gamma g \) using this Hamiltonian, and we also calculate those \( O(\alpha_s) \) corrections to the amplitude \( b \to s\gamma \) which are needed for cancellation of infrared divergences. In section 4 we compute photon spectrum and total decay width from this amplitude, giving some details about the phase space integration in \( D = 4 - 2\epsilon \) dimensions. Subsequently, in section 5 we analyze these results numerically, exploring their range of applicability and discussing their dependence on the renormalization scale as well as on the other input parameters. The paper closes with a short summary and outlook in section 6.

2 The effective Hamiltonian

In order to make use of the renormalization group techniques for calculation of short-distance QCD effects, we work within the framework of an effective five quark theory where the W boson and the top quark have been removed as explicit dynamical degrees of freedom. Neglecting contributions with smaller CKM parameters (\(|V_{ub}V_{us}|/|V_{tb}V_{ts}| < 0.02\)), the relevant Hamiltonian for the process \( b \to s\gamma, s\gamma g \) is in leading order of the operator product expansion given by

\[
\hat{H}_{\text{eff}} = -\frac{4G_F}{\sqrt{2}} V_{tb}V_{ts}^* \sum_{i=1}^{8} C_i(\mu) \hat{O}_i.
\] (1)

Here, \( G_F \) is the Fermi coupling constant, \( V_{ij} \) are the CKM matrix elements, \( C_i(\mu) \) denote the Wilson coefficients evaluated at the scale \( \mu \), and the operator basis reads

\[
\hat{O}_1 = (\bar{c}_L\gamma^\mu b_L)(\bar{s}_L\gamma_\mu c_L),
\]
\[
\hat{O}_2 = (\bar{c}_L\gamma^\mu b_L)(\bar{s}_L\gamma_\mu c_L),
\]
\[ \hat{O}_3 = (\bar{s}_{\alpha \gamma \mu} b_{\alpha}) \sum_{q=u,d,s,c,b} (\bar{q}_{\beta \gamma \mu} q_{\beta}), \]
\[ \hat{O}_4 = (\bar{s}_{\alpha \gamma \mu} b_{\alpha}) \sum_{q=u,d,s,c,b} (\bar{q}_{\beta \gamma \mu} q_{\alpha}), \]
\[ \hat{O}_5 = (\bar{s}_{\alpha \gamma \mu} b_{\alpha}) \sum_{q=u,d,s,c,b} (\bar{q}_{\beta \gamma \mu} q_{\beta}), \]
\[ \hat{O}_6 = (\bar{s}_{\alpha \gamma \mu} b_{\beta}) \sum_{q=u,d,s,c,b} (\bar{q}_{\beta \gamma \mu} q_{\alpha}), \]
\[ \hat{O}_7 = \frac{e}{16\pi^2} \bar{s}_{\alpha} \sigma^{\mu\nu}(m_R + m_L) b_{\alpha} F_{\mu\nu}, \]
\[ \hat{O}_8 = \frac{g_s}{16\pi^2} \bar{s}_{\beta} \sigma^{\mu\nu}(m_R + m_L) T_{\beta\alpha} b_{\alpha} G_{\mu\nu}, \]

with \( e \) and \( g_s \) denoting the electromagnetic and strong coupling, respectively, and the projection operators \( R, L = \frac{1}{2}(1 \pm \gamma_5) \). Some caution is necessary if we apply this result, which was originally obtained only for the process \( b \to s \gamma \), to the process \( b \to s \gamma g \). This is, because potentially new diagrams like the one shown in Fig. 1 could lead to new operators containing

\[ \text{Figure 1: A standard model Feynman diagram contributing to the decay } b \to s \gamma g. \text{ Diagrams of this type lead to low energy effective operators which are not contained in the Hamiltonian (1), which was originally derived for the process } b \to s \gamma \text{ only. However, these operators are of dimension 8 or higher and hence are suppressed by powers of } m_b^2/m_W^2. \]

two quark fields, the photon and the gluon field simultaneously. However, a detailed analysis following the lines of [10], i.e. using gauge invariance and the equations of motion, shows that all these operators must necessarily be of dimension 8 and hence are suppressed by powers of \( m_b^2/m_W^2 \).

The inclusive decay rate was calculated in the approximation of leading logarithms, i.e. holding terms of order \( (\alpha_s \ln M_W/m_b)^n, n = 0, \ldots, \infty \). An exhaustive discussion of these results can be found in Ref. [11]. In this reference, there is also analyzed the general structure of next-to-leading calculations, which hold terms of order \( \alpha_s(\alpha_s \ln M_W/m_b)^n, n = 0, \ldots, \infty \). The main points we would like to stress here are: (i) The dominant theoretical uncertainty in the leading
order result arises from the renormalization scale dependence of the Wilson coefficient $C_7(\mu)$. It amounts to about $\pm 25\% \,[12]$, and it can hopefully be reduced by the full NLO calculation to about $\pm 3\%$. (ii) The full NLO calculation requires three steps: $O(\alpha_s)$ matching of the effective theory to the SM matrix elements at $\mu = M_W$ which gives $C_i(\mu = M_W)$, evolution of these Wilson coefficients down to $\mu \simeq m_b$ with the help of the NLO anomalous dimension matrix, calculation of $O(\alpha_s)$ corrections to the matrix elements of the effective theory at the low scale.

From these three steps the first has been tackled in Ref. [13]. Certain parts of the second have also been calculated, see Ref. [14] for a summary of the present state of the art. Unfortunately, the most difficult (and presumably fairly significant) ones, involving finite parts of two-loop and divergent parts of three-loop diagrams, are not yet known. The last step splits into two pieces: $O(\alpha_s)$ corrections to the $b \to s\gamma$ amplitude, and the bremsstrahlung amplitude $b \to s\gamma g$ to $O(\alpha_s)$. The latter has to be considered, because the final state $s+\text{gluon}$ contributes to the measured state $X_s$ as well as a single strange quark. In the following sections, we calculate these bremsstrahlung corrections using the full operator basis (2). We would like to emphasize that the contributions of the penguin operators $\hat{O}_3 - \hat{O}_6$, which were neglected in [3], should not be considered numerically negligible to begin with although their Wilson coefficients are small: in leading order, their effect on the decay width amounts to about $15\%$ (in the NDR scheme), hence a consistent calculation certainly should include all their contributions beyond leading order, too.

We now proceed to give a somewhat detailed calculation of the above mentioned bremsstrahlung contributions, closing with some numerical results at the end of section 5.

### 3 The amplitude $b \to s\gamma g$

Using the effective Hamiltonian (1) we will calculate in this section the complete transition amplitude for the decay

$$b_\alpha(p) \to s_\beta(p') + \gamma(q, \epsilon) + g_a(r, \eta),$$

where $\alpha, \beta$ and $a$ denote colour indices and $\epsilon$ and $\eta$ are the four dimensional polarization vectors of photon and gluon, respectively. The amplitude will have the form

$$M^{\text{brems}} = \sum_{i=1}^{8} M_i = T^a_{\beta \alpha} V \sum_{i=1}^{8} \tilde{C}_i(\mu) \bar{u}(p') T_i u(p),$$

---

1) Unless noted otherwise, by scheme we mean throughout this paper the regularization scheme used for the treatment of $\gamma_5$ in $D$ dimensions. NDR stands for Naive Dimensional Regularization, and HV for the 't Hooft-Veltman scheme.
Figure 2: The tree-level Feynman diagrams which contribute within the effective theory (1) to the decay $b \to s\gamma g$.

where we have extracted the couplings in a common factor

$$V = \frac{-iG_F}{\sqrt{2\pi}} V_{tb} V_{ts} g_s e,$$

(4)

$T^n$ denotes as usually the Gell-Mann matrices, $T_i$ are some Dirac structures depending on $p,q,r$, $\epsilon$ and $\eta$, and $\tilde{C}_i$ are the so-called effective Wilson coefficients [11] which are linear combinations of the usual Wilson coefficients (1), defined (in the NDR scheme) by

$$\tilde{C}_{1...6} = C_{1...6}, \quad \tilde{C}_7 = C_7 - \frac{1}{3} C_5 - C_6, \quad \tilde{C}_8 = C_8 + C_5.$$  

(5)

In the HV scheme, $\tilde{C}_i = C_i$. The reason for introducing these quantities will be explained after calculating the matrix elements of the penguin operators $\hat{O}_3 - \hat{O}_6$ in Sec. 3.3.

Throughout this section, we work (where necessary) in $D = 4 - 2\epsilon$ dimensions in order to regularize UV divergences, and we treat $\gamma_5$ in $D$ dimensions according to the NDR scheme.

### 3.1 Magnetic penguins: $\hat{O}_7$ and $\hat{O}_8$

The magnetic penguin operators $\hat{O}_7$ and $\hat{O}_8$ contribute only through the tree-level diagrams of Fig. 2 which yield trivially

$$T_7 = \frac{1}{2} \left[ \frac{1}{-2pr} (m_b R + m_s L) \hat{g}(\gamma - \bar{\beta} + m_b) \hat{g} + \frac{1}{2p'r} \hat{\gamma}(\bar{\beta} - \bar{\alpha} + m_s) \hat{g}(m_b R + m_s L) \right],$$

(6)

$$T_8 = \frac{Q_d}{2} \left[ \frac{1}{-2pq} (m_b R + m_s L) \hat{g}(\gamma - \bar{\beta} + m_b) \hat{g} + \frac{1}{2p'q} \hat{\gamma}(\bar{\beta} - \bar{\alpha} + m_s) \hat{g}(m_b R + m_s L) \right],$$

(7)

with $Q_d = -1/3$.

Now these parts of the bremsstrahlung amplitude stemming from the magnetic penguins are affected by infrared divergences. That is, after phase space integration $|M_7|^2$ and $|M_8|^2$ yield no finite decay rate. Therefore the first step is to regularize these divergences: for the sake of simplicity, we choose dimensional regularization [13] so that the phase space integration is to be done in $D = 4 - 2\epsilon$ dimensions and therefore becomes a little bit involved. The second
step is to eliminate the divergences. According to the well-known Bloch-Nordsieck mechanism \[16\], this can be achieved by considering the appropriate diagrams with virtual instead of real gluons. Since the infrared divergences of \( \hat{O}_8 \) occur only at the low energy endpoint of the photon spectrum which is experimentally not accessible, they are only of minor interest and need not to be considered here. We are left with the one diagram of Fig. 3f which yields (together with the \( O(\alpha_s^0) \) contribution) in the Feynman gauge and before renormalization:

\[
M_{\text{virt}}^7 = \frac{ieG_F}{2\sqrt{2\pi^2}}V_{tb}V_{ts}^*C_7(\mu)\delta_{\alpha\beta}\bar{u}(p')T_{\text{virt}}^{(0)}u(p),
\]  

(8)

with

\[
T_{\text{virt}}^{(0)} = (m_b R + m_s L)\slashed{q}(1 + K_g^{(0)}),
\]

\[
K_g^{(0)} = \frac{\alpha_s}{3\pi}(4\pi)^\epsilon\Gamma(1 + \epsilon)\frac{1 + \rho}{1 - \rho} \left[ \frac{-1}{\epsilon} \ln \rho + \frac{1}{2} \ln^2 \frac{m^2}{\mu^2} - \frac{1}{2} \ln \frac{m_b^2}{\mu^2} - 2 \ln \rho \right],
\]

\[
\rho = \frac{m_s^2}{m_b^2}.
\]  

(9)

Note that this amplitude is UV finite: the \( 1/\epsilon \) poles correspond to infrared singularities. If we wish to calculate the transition amplitude, we also must make allowance for diagrams with quark self energy parts. To this end, remember that according to the LSZ reduction formula the transition amplitude is given by the corresponding truncated Green function times the residua of the propagators of the external particles to the same order in \( \alpha_s \). Now let us employ the on-shell renormalization condition

\[
\frac{\partial \Sigma(p)}{\partial \bar{q}}|_{\bar{q}=m} = 0
\]  

(10)

for determining the finite parts of the wave function renormalization constant \( Z_\psi \), which yields

\[
Z_\psi(m) = 1 - \frac{\alpha_s}{3\pi} \left( \frac{3}{\epsilon} - 3\gamma + 3 \ln \frac{4\pi \mu^2}{m^2} + 4 \right)
\]  

(11)

in Feynman gauge (here the singularity contains an \( 1/\epsilon \) UV and an \( 2/\epsilon \) IR contribution). With this choice, the residuum of the renormalized quark propagator equals 1 (to one loop), and the transition amplitude is simply given by the renormalized truncated Green function, i.e. by the amplitude (8) given above after including renormalization. To achieve the latter, we have to add the counterterm amplitude \((Z_{77}Z_mZ_\psi^{1/2}(m_b)Z_\psi^{1/2}(m_s) - 1)M_{\text{virt}}^7\), where we choose the operator and mass renormalization constants \( Z_{77} \) and \( Z_m \) according to the \( \overline{\text{MS}} \) scheme \[17\]. At this place, employing the \( \overline{\text{MS}} \) scheme is of course necessary, because the whole calculation of the Wilson coefficients has been done in this scheme. The result of this renormalization procedure amounts to replacing \( T_{\text{virt}}^{(0)} \) in (8) by

\[
T_{\text{virt}}^{(R)} = (\overline{m}_b(\mu)R + \overline{m}_s(\mu)L)\slashed{q}(1 + K_g^{(R)}),
\]  

(12)
Figure 3: Examples of one-loop Feynman diagrams which contribute within the effective theory (1) to the decay $b \to s\gamma g$. (a), (b) and (c) depict insertions of the current-current operators, whereas (d) and (e) illustrate the two distinct types of insertions of penguin operators. (f) displays the virtual gluon correction to the $b \to s\gamma$ matrix element of the magnetic penguin operator $\hat{O}_7$. This graph is needed for cancellation of infrared divergences at the high energy endpoint of the photon spectrum in $b \to s\gamma g$.

with

$$K_g^{(R)} = \frac{\alpha_s}{3\pi} (4\pi)^\epsilon \Gamma(1 + \epsilon) \left\{ -\frac{1}{\epsilon} \left[ \frac{1 + \rho}{1 - \rho} \ln \rho + 2 \right] + \frac{1 + \rho}{1 - \rho} \left[ \frac{1}{2} \ln^2 \frac{m_s^2}{\mu^2} - \frac{1}{2} \ln^2 \frac{m_b^2}{\mu^2} - 2 \ln \rho \right] + \frac{3}{2} \ln \frac{m_s^2}{\mu^2} + \frac{3}{2} \ln \frac{m_b^2}{\mu^2} - 4 \right\}$$

(13)

and $\overline{m}_q(\mu)$ the running quark masses in the $\overline{MS}$ scheme. Again, this is UV finite.

### 3.2 Current-current operators: $\hat{O}_1$ and $\hat{O}_2$

The current-current operators contribute through diagrams of the type shown in Fig. 3a. Note that diagrams as Fig. 3b in which only one particle is radiated from the loop vanish on-shell, as
one may easily verify through explicit calculation. From chirality, the same is true for diagrams as Fig. 3c in which there is no particle at all radiated from the loop.

Since their colour structure entails a factor \( \text{tr} T^a \), all diagrams with an insertion of \( \hat{O}_1 \) vanish:

\[
T_1 = 0.
\]  

Using some easily calculable one-loop three-point functions, one obtains for the diagrams involving \( \hat{O}_2 \)

\[
T_2 = Q_u \kappa \left( \frac{m_b^2}{m_c^2} s \right) W_2,
\]

with

\[
W_2 = \left\{ \frac{1}{q^2} \left[ (\eta q) \bar{q} \bar{q} - (\epsilon r) \bar{q} \bar{q} - (\epsilon r) (\eta q) (\bar{q} - \bar{r}) + \bar{q} (\bar{q} - \bar{r}) - (\epsilon q) (\bar{q} - \bar{r}) + 2(\eta q) \bar{r} \right] \right\} L,
\]

\[
\kappa(s) = \frac{1}{2} + \frac{Q_0(s)}{s},
\]

\[
s = \frac{2qr}{m_b^2},
\]

\( Q_u = 2/3 \) and the function \( Q_0(s) \) defined by

\[
Q_0(s) = \int_0^1 \frac{dx}{x} \ln(1 - sx + sx^2 - i\delta),
\]

with \( \delta \) positive infinitesimal. The integration in \( 19 \) can be performed analytically, yielding \( Q_0(s) \) as given in eq. (9) of Ref. \[9\]. Note that all UV divergences cancel to yield an UV-finit result. Note also that (6), (7) and (15) are in complete agreement with Ref. \[9\] after some rearrangements in the Dirac structure using the Dirac equation.

### 3.3 Penguin operators: \( \hat{O}_3 - \hat{O}_6 \)

Evaluating diagrams containing penguin operators, one must take into account two types of possible insertions: the somewhat natural one of Fig. 3d, but also the one that leads to diagrams like Fig. 3e. The latter stems from terms in the second current of the operator containing the \( b \) or the \( s \) quark. But as it turns out, only one insertion for every operator survives due to colour structure.

In a calculation analogous to the previous one and without making a Fierz transformation we obtain:

\[
T_3 = Q_d [\kappa_b + \kappa_s] W_2,
\]

\[
T_4 = \left\{ \frac{1}{6} + Q_d [\kappa_b + \kappa_s] + Q_u \kappa_c \right\} W_2,
\]

\[
T_5 = Q_d \left( m_b W_b^b R + m_s W_s^s L \right),
\]

\[
T_6 = -T_4.
\]
where we have used the abbreviation $\kappa_q \equiv \kappa \left( \frac{m_b^2}{m_q} s \right)$. $W_2$, $\kappa(s)$, and $s$ are defined in (16), (17) and (18), respectively, and

$$W_5^q = \frac{1}{m_q^2} \left[ \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma + (q r) \gamma^\mu \gamma^\nu - (\eta \epsilon) \gamma^\rho \gamma^\sigma - (\epsilon r) \gamma^\mu \gamma^\nu - (\gamma q) \gamma^\rho \gamma^\sigma \right] (1 - 2 \kappa_q)$$

$$+ 4 \left[ (\eta \epsilon) - \frac{(\epsilon r) (\eta q)}{q r} \right] \kappa_q.$$ (24)

There is one more point to be considered: in the case of the penguin operators $\hat{O}_5$ and $\hat{O}_6$, there are also non-vanishing diagrams in which at least one photon (or gluon) is radiated from an external leg. But as in $b \to s \gamma$, $sg$ the resulting amplitudes are proportional to either $T_7$ or $T_8$. Thus the same redefinition of the Wilson coefficients as in $b \to s \gamma$, $sg$ absorbs these contributions into the tree-level amplitudes $T_7$, $T_8$. This is why we have introduced the quantities $\tilde{C}_i$ as defined in (5)

4 Spectrum and decay rate

We now proceed to calculate some physical observables from the amplitude $M^{\text{brems}}$ combined with $M^{\text{virt}}$.

4.1 Squared and summed amplitude

Summing the squared amplitude $|M^{\text{brems}}|^2$ over spins, polarizations and colours is a straightforward procedure. First of all, we can write

$$|M^{\text{brems}}|^2 = \frac{1}{6} \sum_{\text{spin,pol.,col.}} |M^{\text{brems}}|^2 = \frac{2}{3} m_b^4 |V|^2 \sum_{i,j=1}^{8} \tilde{C}_i(\mu) \tilde{C}_j(\mu) M_{ij},$$ (25)

where the factor $1/6$ stems from averaging over spin and colour of the incoming $b$ quark and $V$ is given in (4). Introducing the kinematical variables

$$s = \frac{2qr}{m_b^2}, \quad t = \frac{2pr}{m_b^2}, \quad u = \frac{2pq}{m_b^2}$$ (26)

and again using the abbreviation $\kappa_q \equiv \kappa \left( \frac{m_b^2}{m_q} s \right)$, the contributions from the various operators can be expressed in the limit $m_s = 0$ as

2) Strictly speaking, these diagrams contribute only in the NDR scheme; in the HV scheme they vanish and therefore in the latter $\tilde{C}_i = C_i$. This difference is compensated by a corresponding scheme dependence of the leading order anomalous dimension matrix.
\[ M_{22} = \frac{8}{9} |\kappa_c|^2 (1 - s), \]
\[ M_{23} = -\frac{8}{9} \text{Re} \left[ \kappa_c^* \left( \frac{1}{2} + \kappa_b \right) \right] (1 - s), \]
\[ M_{24} = -M_{26} = \frac{8}{9} \text{Re} \left[ \kappa_c^* (2\kappa_c - \kappa_b) \right] (1 - s), \]
\[ M_{25} = -\frac{4}{9} \text{Re} \left[ \kappa_c^* (1 - 2\kappa_b) \right] (1 - s)s, \]
\[ M_{27} = -3M_{28} = -\frac{4}{3} \text{Re}(\kappa_c)s, \]
\[ M_{33} = \frac{1}{9} \left( \frac{1}{2} + \kappa_b \right)^2 (1 - s), \]
\[ M_{34} = -M_{36} = -\frac{4}{9} \text{Re} \left[ \left( \frac{1}{2} + \kappa_b \right) (2\kappa_c - \kappa_b) \right] (1 - s), \]
\[ M_{35} = \frac{2}{9} \text{Re} \left[ \left( \frac{1}{2} + \kappa_b \right) (1 - 2\kappa_b^*) \right] (1 - s)s, \]
\[ M_{37} = -3M_{38} = \frac{2}{3} \text{Re} \left( \frac{1}{2} + \kappa_b \right)s, \]
\[ M_{44} = M_{66} = \frac{1}{9} |2\kappa_c - \kappa_b|^2 (1 - s), \]
\[ M_{45} = -M_{56} = -\frac{2}{9} \text{Re} \left[ (2\kappa_c - \kappa_b)(1 - 2\kappa_b^*) \right] (1 - s)s, \]
\[ M_{47} = -M_{67} = -3M_{48} = 3M_{68} = -\frac{2}{3} \text{Re}(2\kappa_c - \kappa_b)s, \]
\[ M_{55} = \frac{1}{9} \left[ 16|\kappa_b|^2 + |(1 - 2\kappa_b)s + 4\kappa_b|^2 \right] (1 - s), \]
\[ M_{57} = -3M_{58} = \frac{8}{3} \text{Re}(\kappa_b)s, \]
\[ M_{78} = \frac{1}{3} \left( 1 + \frac{2}{tu} \right)s. \quad (27) \]

For \( m_s \neq 0 \) the corresponding expressions become quite lengthy. The complete formulas for this case can be found in the appendix.

There are still two contributions missing. \( M_{77} \) is the one which will by far dominate the photon spectrum (at least near the endpoint), so in this case it may be interesting to know the full formula (i.e. \( m_s \neq 0 \)), which reads

\[ M_{77} = (1 + \rho) \left\{ (1 - \rho) M_{77}^{(1)} - 2M_{77}^{(2)} \right\}, \]
\[ M_{77}^{(1)} = \frac{1}{t} \left[ 1 + \bar{u} + \frac{2\bar{u}(\bar{u} - 2)}{1 - \bar{u}} \bar{t} + \frac{2\bar{u} - 1}{1 - \bar{u}} \bar{t}^2 \right], \]
\[ M_{77}^{(2)} = \frac{1}{t^2} \left[ 1 - \frac{1 + \rho}{1 - \bar{u}} \bar{t} + \frac{\rho}{(1 - \bar{u})^2} \bar{t}^2 \right]. \quad (28) \]
where we have introduced the rescaled kinematic variables $\bar{t} = t/(1 - \rho)$, $\bar{u} = u/(1 - \rho)$, and $\rho = m^2_s/m_b^2$ from (1). In writing (28), we have split $M_{77}$ into an infrared safe part $M_{77}^{(1)}$ and an infrared divergent part $M_{77}^{(2)}$. From the amplitude (3) it should then be obvious that the quantity $M_{88}/Q_d^2$ is given by the above expression for $M_{77}$ if one interchanges $\bar{u}$ with $\bar{t}$.

4.2 The infrared finite parts

As stated above, in the region of experimental interest all contributions $M_{ij}$ to the squared amplitude yield a IR finite spectrum and decay rate, with exception of $M_{77}$. For those finite contributions, we therefore can do all phase space integrations in 4 dimensions. Note that from conservation of energy-momentum we have the relation $u + t - s = 1 - \rho$ and the kinematic boundaries $u \in [0, 1 - \rho]$, $t \in [1 - u - \rho, 1 - \rho]$. Furthermore, in the rest frame of the decaying $b$ quark the variables $u$ and $t$ become the energy of the emitted photon and gluon, respectively, measured in units of $m_b/2$. Splitting the spectrum into an infrared finite and an infrared singular part,

$$\frac{d\Gamma^{\text{brems}}}{du} = \frac{d\Gamma_F}{du} + \frac{d\Gamma^{\text{brems}}_{7}}{du},$$

one therefore obtains trivially for the former

$$\frac{d\Gamma_F}{du} = \sum_{j \neq 77} \frac{d\Gamma_{ij}}{du} = \frac{2\alpha_s}{3\pi} \Gamma_0 \sum_{j \neq 77} \tilde{C}_i(\mu) \tilde{C}_j(\mu) \int_{t_{\text{min}}}^{t_{\text{max}}} dt M_{ij}(t, u),$$

with

$$\Gamma_0 = \frac{|V_{tb}V_{ts}^\ast|^2 G_F^2 \alpha_s}{32\pi^4} m_b^5.$$

Regrettably, due to the presence of the functions $\kappa(s)$ in the expressions (27) for $M_{ij}$ the integrations in (29) cannot be done analytically, so we postpone the numerical analysis of (29) to sections 5.1 (spectrum) and 5.2 (decay width).

4.3 The infrared divergent part

Integration of $|M_7|^2$ yields a photon spectrum which contains a non-integrable singularity as the photon energy reaches the kinematical endpoint, $E_\gamma \rightarrow \frac{m_b}{2}(1 - \rho)$. To regularize this divergence, we will do the whole phase space integration in $D = 4 - 2\epsilon$ dimensions. The singularity will then manifest itself in a part of the decay width $\Gamma^{\text{brems}}_7 = \int d\Gamma^{\text{brems}}_7$ proportional to $1/\epsilon$. This singularity will cancel if we add the equivalent singular contribution $\Gamma^{\text{virt}}_7$ stemming from the virtual gluons, leaving us with an unambiguous finite result $\Gamma_7 = \Gamma^{\text{brems}}_7 + \Gamma^{\text{virt}}_7$. Let us now calculate $\Gamma_7$. 

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In $D$ Dimensions, the Lorentz Invariant Phase Space (LIPS) is given by

$$d\Phi(p) = \frac{d^{D-1}p}{(2\pi)^{D-1}2p_0},$$

and accordingly the differential decay width for an $n$-body decay with transition amplitude $M$ reads

$$d\Gamma = \frac{1}{2m}(2\pi)^D\delta^D(p - \sum_{i=1}^n p_i) |M|^2 \prod_{i=1}^n d\Phi(p_i).$$

(32)

Specializing this to the case $b \to s\gamma g$ and performing the delta function as well as the angular integrations (note that our squared and summed amplitude $|M^{\text{brems}}|^2$ only depends on the energies of the particles in the final state), one obtains

$$d\Gamma^{\text{brems}} = (1 - \rho)^2 \bar{C}_T^2(\mu) \frac{2\alpha_s}{3\pi} \Gamma_0 \frac{1}{\Gamma(2 - 2\epsilon)} \left[\bar{u} \left(1 - z(\bar{t}, \bar{u})^2\right)\right]^{-2\epsilon} M_{7\gamma}(\bar{t}, \bar{u}) \, d\bar{t} \, d\bar{u},$$

(33)

where we have defined the function

$$z(\bar{t}, \bar{u}) = \frac{2}{1 - r} \left(\frac{\bar{u} + \bar{t} - 1}{\bar{u} \bar{t}}\right) - 1,$$

(35)

which is actually the cosine of the angle between $\vec{q}$ and $\vec{r}$ in the rest frame of the decaying $b$ quark. The crucial point to be observed now is that even in the first integration over $\bar{t}$ (which yields no divergences) one has to hold terms of order $\epsilon$, because they may multiply later on with $1/\epsilon$ poles of the $\bar{u}$-integration and so give finite contributions to the total decay width.

After carrying out the integration over $\bar{t}$, we obtain the following expression for the photon spectrum to $O(\epsilon)$:

$$\frac{d\Gamma^{\text{brems}}}{d\bar{u}} = (1 + \rho)(1 - \rho)^3 \bar{C}_T^2(\mu) \frac{2\alpha_s}{3\pi} \Gamma_0 C_\epsilon \left[\frac{S^{(1)}}{(1 - \bar{u})^{1+2\epsilon}} - \frac{2S^{(2)}}{(1 - \bar{u})^{1+2\epsilon}}\right],$$

(36)

with

$$C_\epsilon = \frac{(1 - \rho)^{-4\epsilon} \left(\frac{4\pi a_s^2}{m_b^2}\right)^{2\epsilon}}{\Gamma(2 - 2\epsilon)} \bar{u}^{-2\epsilon},$$

$$S^{(1)} = \frac{1}{2} \frac{(1 - \rho) \bar{u}(1 - \bar{u})(2\bar{u} - 1)}{(1 - u)^2} + \frac{1}{2} \frac{(1 - \rho) \bar{u}(2\bar{u}^2 - 5\bar{u} - 1)}{1 - u} - (1 + \bar{u}) \ln(1 - u),$$

$$S^{(2)} = S_a^{(2)} + \epsilon S_b^{(2)},$$

$$S_a^{(2)} = \left(1 + \frac{\rho}{1 - u}\right) \bar{u} + \frac{1 + \rho}{1 - \rho} \ln(1 - u),$$

$$S_b^{(2)} = \frac{\rho}{1 - u} \left[2 + \ln(1 - u)\right] \bar{u} - \frac{1 - u}{1 - \rho} \ln(1 - u) + \frac{1 + \rho}{1 - \rho} \left[\frac{1}{2} \ln^2(1 - u) - 2 \text{Li}_2(u)\right],$$

(37)
\[ \rho = m_s^2 / m_b^2 \] and \[ \text{Li}_2(x) = -\int_0^1 \frac{dt}{t} \ln(1 - xt) \] is the Spence function (or dilogarithm). From this, we can now accomplish our final goal, the integration over \( \bar{u} \). Since our results agree entirely with (29) and (30) of Ref. [9], we will not give them here explicitly.

Then one has to go through the same steps with \( M_7^{\text{virt}} \): square it, sum over spins and polarizations, do the phase space integration in \( D \) dimensions. The result will contain a singular term that cancels the one of \( \Gamma_7^{\text{brems}} \). In fact these calculations are much simpler than the one sketched above, because we only have to struggle with a two-body decay. After some work, we arrive at the following final result\(^3\) for the total decay width via \( \hat{O}_7 \):

\[ \Gamma_7 = \Gamma_7^{(0)} + \delta \Gamma_7, \quad (38) \]

with

\[ \Gamma_7^{(0)} = (1 + \rho)(1 - \rho)^2 C_7(\mu)^2 \left( \frac{m_b(m_b)}{m_b^2} \right)^2 \Gamma_0, \quad (39) \]

and

\[ \delta \Gamma_7 = \frac{2\alpha_s}{3\pi} \left\{ \frac{4 \pi^2}{3} \frac{7 - 8 \rho + 5 \rho^2}{(1 - \rho)^3} \ln \rho - 4 \ln(1 - \rho) \right. \]
\[ + \left. \frac{1 + \rho}{1 - \rho} \left[ 4 \text{Li}_2(\rho) - \frac{32}{3} \pi^2 + 2 \ln \rho \ln(1 - \rho) \right] + 4 \ln \left( \frac{m_b^2}{\mu^2} \right) \right\} \Gamma_7^{(0)}. \quad (40) \]

In deriving (38) – (40) we have expanded the running quark mass \( \overline{m}_b(\mu) \), which occurs from (12) in the matrix element \( M_7^{\text{virt}} \), around \( \mu = m_b \),

\[ \overline{m}_b(\mu) = \overline{m}_b(m_b) \left( 1 + \frac{2\alpha_s(m_b)}{\pi} \ln \frac{m_b}{\mu} \right), \quad (41) \]

and we have also made use of the equivalence of \( \overline{m}_b(m_b)^2 \) and the squared pole mass \( m_b^2 \) up to corrections of order \( \alpha_s(m_b) \). In the limit \( m_s = 0 \) (38) looks much nicer:

\[ \Gamma_7 = \tilde{C}_7(\mu)^2 \left[ 1 + \frac{2\alpha_s}{3\pi} \left( \frac{8 - 2\pi^2}{3} + 4 \ln \left( \frac{m_b^2}{\mu^2} \right) \right) \right] \left( \frac{\overline{m}_b(m_b)}{m_b} \right)^2 \Gamma_0. \quad (42) \]

We now hasten to undertake a critical discussion of all these results.

\(^3\)The attentive reader will perhaps feel uneasy about the fact that in order to obtain the following result, we apparently have calculated \( \Gamma_7^{\text{virt}} \) with the effective Wilson coefficient \( \tilde{C}_7 \) instead of \( C_7 \). Of course the full two-loop matrix elements of \( \hat{O}_5 \) and \( \hat{O}_6 \) are not proportional to \( M_7^{\text{virt}} \). Nevertheless, due to cancellation of infrared divergences they must contain at least a part proportional to \( M_7^{\text{virt}} \). The other parts have to be considered in the full calculation of two-loop \( b \to s \gamma \) matrix elements, which is not the aim of this paper.
5 Numerical analysis

5.1 Photon spectrum

If we neglect all non-perturbative effects, what does the photon spectrum due to bremsstrahlung actually look like? The answer depends on the experimental setup: Let us assume the resolution of the photon detector is $\frac{m_b}{2}\Delta$ so that in the endpoint region we cannot discriminate photons within the energy interval $E_{\gamma}^{\text{max}} - \frac{m_b}{2}\Delta < E_{\gamma} < E_{\gamma}^{\text{max}}$, with $E_{\gamma}^{\text{max}} = \frac{m^2 - m_s^2}{2m_b}$ the maximum energy of the emitted photon. If we now define the function

$$\Gamma_{\text{tot}}(u_0) = \Gamma_7 - \int_0^{u_0} \frac{d\Gamma_{\text{brems}}}{du} du + \int_{u_0}^{1-\rho} \frac{d\Gamma_F}{du} du,$$

(43)

which is the total decay width including all photons with an energy $E_{\gamma} \geq \frac{m_b}{2}u_0$, then the photon energy spectrum is given by

$$\frac{d\Gamma_{\Delta}}{du} = \begin{cases} \frac{d\Gamma_{\text{brems}}}{du}, & u < 1 - \rho - \Delta \\ \frac{1}{\Delta}\Gamma_{\text{tot}}(1 - \rho - \Delta), & 1 - \rho - \Delta < u < 1 - \rho \\ 0, & u > 1 - \rho \end{cases},$$

(44)

Here, $d\Gamma_{\text{brems}}/du$ is the perturbative bremsstrahlung spectrum which we calculated in Sec. 4, cf. eqs. (29), (30) and (36). We have plotted the resulting photon spectrum in Fig. 4, where we have used $m_s = 200$ MeV, $\Delta = 25$ MeV and the Wilson coefficients $\tilde{C}_i(m_b)$ as given and discussed in the following section. Though we will not dwell upon any further details, a few comments may be useful:

First of all, if we consider the limit $\Delta \to 0$, up to what minimal resolution $\Delta_{\text{crit}}$ the spectrum (44) can be trusted? Clearly, this is a question of the reliability of perturbation theory. We can estimate $\Delta_{\text{crit}}$ if we consider the case where the pure $O(\alpha_s)$ bremsstrahlung contributions from the region $u < 1 - \rho - \Delta$ become of the same order of magnitude as the contributions from the peak in the region $1 - \rho - \Delta < u < 1 - \rho$:

$$\int_0^{1-\rho-\Delta_{\text{crit}}} \frac{d\Gamma_{\text{brems}}}{du} du \simeq \Gamma_{\text{tot}}(1 - \rho - \Delta_{\text{crit}}).$$

(45)

This is, because if the l.h.s. of (45) becomes greater than the r.h.s., then the $O(\alpha_s)$ bremsstrahlung corrections exceed the two-body decay peak so that perturbation theory is no longer valid. If one wishes to explore the case $\Delta < \Delta_{\text{crit}}$, one will have to resort to some Sudakov-type partial resummation of the most singular terms (“exponentiation”) in order to obtain a reliable spectrum. However, numerically we get from (45), with $m_s = 200$ MeV, $\Delta_{\text{crit}} \simeq 0.015$, so that
Figure 4: Two different views on the $B \to X_s \gamma$ photon energy spectrum due to gluon bremsstrahlung, eq. (44). The spectrum is given in units of $|\bar{C}_7(m_b)|^2 \Gamma_0$, the leading order decay width with $m_s = 0$. In both plots, the solid curve corresponds to the full operator basis and $m_s = 200$ MeV. (a) shows the region $0.6 < u < 0.95$, where $u$ is the energy of the emitted photon in units of $m_b/2$. The dotted curve represents the contribution from the operator $\hat{O}_7$ alone, which is obviously the dominant one. (b) takes a closer look at the endpoint region, where the spectrum is sensitive to the mass of the strange quark. Here, the dashed curve depicts the case $m_s = 500$ MeV.

an infrared improvement remains unquestionably necessary only in the last, say, 30–40 MeV near the kinematical endpoint. This is well below any experimental resolution, and moreover it will presumably be of no importance for the lower moments of the spectrum [7]. Therefore, we suggest to take (44) with $\Delta \equiv 0.015$ as the perturbative photon spectrum and as the input for the inclusion of non-perturbative effects.

From Fig. 4a one sees that the spectrum is completely dominated by the contribution stemming from the Operator $\hat{O}_7$ (solid vs. dotted line). We can express this observation in a more quantitative way, if we define the (normalized) moments of (44) as

$$M_n(u_0, \Delta) = \frac{1}{\Gamma_{tot}(u_0)} \int_{u_0}^{\infty} u^n \frac{d\Gamma_{\Delta}}{du} du, \quad n \in \mathbb{N}. \quad (46)$$

For instance, we obtain thereby for the first moment $M_1(0.6, 0.015) = 0.961$ if we include contributions from all operators, and $M_1(0.6, 0.015) = 0.958$ with $C_i(\mu) \equiv 0$ for $i \neq 7$. These dominance of $\hat{O}_7$ is a quite welcome feature, because $d\Gamma_{\Delta}^{\text{brems}}/du$ is the only contribution which we can calculate analytically (without any numerical integration).

The moments (46) are also useful to illustrate the dependence of the spectrum on the mass of the strange quark. Varying $m_s$ between 200 and 500 MeV (solid vs. dashed line in Fig. 4b), $M_1(0.6, 0.015)$ changes from 0.961 to 0.952. So the moments are decreased by increasing values
of $m_s$, in contrary to the result found in Ref. [7]. This is primarily due to our slightly different normalization of the moments, eq. (46), which we have chosen because the so-defined moments can in principle be extracted from experiment without knowledge of $m_s$. On the other hand, it is also due to our different treatment of the infrared divergent endpoint region: whereas Kapustin and Ligeti trusted the perturbative bremsstrahlung spectrum up to arbitrary small $\Delta$, with (44) we rely on first order perturbation theory only where it is clearly valid. Since the $m_s$-dependence of the spectrum manifests itself mainly in the endpoint region, we regard our method as the more appropriate one.

5.2 Decay width

Taking into account the discussion of Sec. 2 it should be perfectly clear that from our calculations we cannot present any definite numbers for the total decay width (or, equivalently, the branching ratio). What we can do is, however, to list the complete bremsstrahlung contributions to the decay rate. When we will know all the remaining parts of the NLO analysis, especially the two-loop $b \rightarrow s \gamma$ matrix elements of the four-quark operators, we may then simply put together all these numbers to get some final answer.

We will give our numerical results in units of $|\tilde{C}_7(m_b)|^2 \Gamma_0$, which is the leading order decay width if one neglects the mass of the strange quark. Therefore, our only free parameter is $\rho = m_s^2/m_b^2$. Additionally, we have to assume some value for $E_{\gamma}^{\min}$, the minimum photon energy we can see with the detector. For some choices of $\rho$ and $E_{\gamma}^{\min}$, the resulting bremsstrahlung contributions to the total decay width calculated from (30) and (38) are shown in Table 1.

In our analysis we have used the leading order effective Wilson coefficients $\tilde{C}_i(\mu)$, see e.g. [11] for explicit formulas, as well as the two-loop running coupling $\alpha_s(\mu)$, and we set $\mu = m_b$ with $m_b = 4.8\text{ GeV}$ being the pole mass of the $b$ quark. The resulting numerical values for the Wilson coefficients $\tilde{C}_i(m_b)$ are displayed in Table 2, where they have been computed with the parameters $M_W = 80.2\text{ GeV}$, $m_t(m_t) = 170\text{ GeV}$ and $\alpha_s(M_Z) = 0.117$.

Let us now discuss some interesting features of Table 1:

(i) Contributions from different operators

The dominant contribution to $\delta\Gamma_{\text{tot}} = \Gamma_{\text{tot}} - \Gamma_{\text{tot}}^{(0)}$ comes from the operator $\hat{O}_7$ alone, all other contributions amount to only 15% of this one. However, this statement has to be taken with a grain of salt, because $\hat{O}_7$ is the only operator where we have included the virtual corrections. Contributions from the penguin operators $\hat{O}_3 - \hat{O}_6$ (last but one column) are completely negligible as far as they are not included in the definitions of the effective Wilson coefficients. Whereas
Table 1: Different contributions to the total decay width $B \to X_s \gamma$ due to gluon bremsstrahlung, in percent of $|\tilde{C}_7(m_b)|^2 \Gamma_0$, the leading order decay width with $m_s = 0$, as a function of the parameters $\rho = m_s^2/m_b^2$ and $E_{\gamma}^{\text{min}}$. We have used the leading order effective Wilson coefficients from Table 2, $m_c = 1.4\text{ GeV}$, $m_b = 4.8\text{ GeV}$ and $\overline{m}_b(m_b) = 4.4\text{ GeV}$. The last column shows the sum of all the different contributions.

| $\rho$ | $E_{\gamma}^{\text{min}}$ | $\Gamma_7$ | $\Gamma_{22}$ | $\Gamma_{27}$ | $\Gamma_{28}$ | $\Gamma_{78}$ | $\Gamma_{88}$ | $\sum_{\text{all others}}$ | $\Gamma_{\text{tot}}$ |
|--------|--------------------------|-----------|------------|------------|------------|------------|------------|----------------|-----------------|
| 0.01   | 1.92 GeV                 | 66.92     | 0.94       | -0.09      | 0.03       | 0.16       | 0.02       | 0.00           | 67.98           |
|        | 1.44 GeV                 | 68.83     | 1.85       | -0.27      | 0.08       | 0.32       | 0.09       | 0.01           | 70.91           |
| 0 GeV  | 0.92 GeV                 | 69.16     | 2.48       | -0.48      | 0.15       | 0.48       | -           | 0.02           | -               |
| 0.002  | 1.92 GeV                 | 67.54     | 1.14       | -0.08      | 0.02       | 0.20       | 0.06       | 0.01           | 68.89           |
|        | 1.44 GeV                 | 69.09     | 2.12       | -0.27      | 0.06       | 0.37       | 0.18       | 0.01           | 71.52           |
| 0 GeV  | 0.92 GeV                 | 69.40     | 2.80       | -0.48      | 0.10       | 0.54       | -           | 0.02           | -               |

Table 2: Effective Wilson coefficients $\tilde{C}_i(\mu)$ at the scale $\mu = m_b = 4.8\text{ GeV}$, in the approximation of leading logarithms. These coefficients were evaluated with the two-loop $\alpha_s(\mu)$ and the parameters $M_W = 80.2\text{ GeV}$, $\overline{m}_t(m_t) = 170\text{ GeV}$, $\alpha_s(M_Z) = 0.117$.

| $\tilde{C}_1$ | $\tilde{C}_2$ | $\tilde{C}_3$ | $\tilde{C}_4$ | $\tilde{C}_5$ | $\tilde{C}_6$ | $\tilde{C}_7$ | $\tilde{C}_8$ |
|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| -0.242        | 1.104         | 0.011         | -0.025        | 0.007         | -0.030        | -0.309        | -0.147        |

The virtual corrections lower the decay rate significantly, bremsstrahlung tends to enhance it slightly.

(ii) Dependence on $m_s$ and $E_{\gamma}^{\text{min}}$

Since $m_s^2 \ll m_b^2$, from the outset we did not expect the mass of the strange quark to make any sizeable effect in $\Gamma_{\text{tot}}$, and we used $m_s$ therefore mainly as regulator of collinear divergences. Indeed it turns out that varying $m_s$ between 0 and 500 MeV leads to a change of only 0.9% in $\Gamma_{\text{tot}}$. Here, we should note in particular that $\Gamma_7 = \Gamma_7^{\text{brems}} + \Gamma_7^{\text{virt}}$ and all bremsstrahlung contributions $\Gamma_{ij}$ with $ij \neq 77$ are well defined in the limit $m_s \to 0$, except for $\Gamma_{88}$. This divergent behaviour of $\Gamma_{88}$ has recently been discussed in Ref. [8], but since, from Table 1, $\Gamma_{88}$ is in the relevant part of the spectrum and for sensible values of $m_s$ of no numerical importance, this observation is of no great relevance here.

The dependence on $E_{\gamma}^{\text{min}}$ can be interpolated from Table 1. Note that values $E_{\gamma}^{\text{min}} < \rho$.
1.4 GeV are not realistic. Above that, if one is interested in the limit $E_{\gamma}^{\text{min}} \to 0$, one has to deal with the infrared (soft photon) divergence due to $\hat{O}_8$, cf. Ref. [6, 8].

(iii) Dependence on the renormalization scale $\mu$

As mentioned in Sec. 2, an important offspring of the complete NLO calculation should be a considerably reduced renormalization scale dependence and thus a considerably improved theoretical uncertainty. From theory we know that in order to perform the correct resummation of large logarithms the renormalization scale $\mu$ should be fixed to a value of order $m_b$, but that on the other hand it is arbitrary provided that $\ln \frac{m_b}{\mu}$ remains a small parameter. Thus this uncertainty can be estimated if one varies $\mu$ between, say, $m_b/2$ and $2m_b$. Performing this exercise with the expression (42) for $\Gamma_7$, one obtains a startling large relative change of $\pm 69\%$ $-38\%$. This is an even larger $\mu$-dependence than the $+28\%$ $-20\%$ of the pure leading order result. What lesson can we learn from this peculiar effect? Of course a very elementary one: namely, that it is a mistake to expect any improved predictive power from partial NLO calculations. Let us explain this in detail.

The $\mu$-dependence of the Wilson coefficients $\tilde{C}_i(\mu)$ is governed by the renormalization group equation [18]

$$\mu \frac{d}{d\mu} \tilde{C}_i(\mu) = \tilde{\gamma}_{ij}(\alpha_s) \tilde{C}_j(\mu),$$

(47)

where the NLO anomalous dimension matrix $\tilde{\gamma}_{ij}$ is given by

$$\tilde{\gamma}(\alpha_s) = \tilde{\gamma}^{(0)}(\alpha_s) \frac{\alpha_s(\mu)}{4\pi} + \tilde{\gamma}^{(1)} \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2.$$

(48)

Here, $\tilde{\gamma}^{(0)}$ represents the leading order (LO) contribution, and $\tilde{\gamma}^{(1)}$ the NLO correction to it. $\tilde{\gamma}^{(0)}$ has been calculated by various authors and is given e.g. in Ref. [11], and the calculation of $\gamma^{(1)}$ is not yet completed. From (47), (48) we can infer that the dominant $\mu$-dependence of $\tilde{C}_7(\mu)$ near $\mu = m_b$ is given by

$$\tilde{C}_7(\mu) = \tilde{C}_7(m_b) + \frac{\alpha_s(m_b) \gamma_{ij}^{(0)} \tilde{C}_i(m_b)}{4\pi} \ln \frac{\mu}{m_b} + O \left( \frac{\alpha_s^2 \ln \frac{\mu}{m_b}}{m_b}, \frac{\alpha_s^2 \ln^2 \frac{\mu}{m_b}}{m_b} \right).$$

(49)

The second term in (49) is exactly the one that after a complete NLO calculation should be cancelled by the $\mu$-dependence of the $b \to s\gamma$ matrix elements $\langle \hat{O}_i(\mu) \rangle$, since such a calculation should be free of renormalization scale uncertainties at $O(\alpha_s)$. From these general grounds we therefore conclude that the transition amplitude $b \to s\gamma$ at the NLO-level,

$$M(b \to s\gamma) = -\frac{4G_F}{\sqrt{2}} V_{tb} V_{ts}^* \sum_{i=1}^{8} C_i(\mu)^{NLO} \langle \hat{O}_i(\mu) \rangle,$$

(50)
can be written as
\[
M(b \to s\gamma) \sim \left[ \tilde{C}_7(\mu) + \frac{\alpha_s(m_b)}{4\pi} \left( \gamma_{77}^{(0)} \tilde{C}_7(m_b) \ln \frac{m_b}{\mu} + r_7 \tilde{C}_7(m_b) \right) \right] \langle \hat{O}_7 \rangle_{\text{tree}},
\]
(51)
where \( r_7 \) are functions of the external momenta, spins and polarizations that contain no \( \mu \)-dependence, and the last factor denotes the tree-level matrix element of \( \hat{O}_7 \). In (51), the first summand \( \tilde{C}_7(\mu) \) has to be taken at the NLO level, whereas for the other Wilson coefficients LO is sufficient, since they are multiplied already with \( \alpha_s \). In order to obtain a prediction for the \( \mu \)-dependence of the quantity \( \Gamma_7 \) which we calculated explicitly in this paper, cf. eq. (42), we square (51), set \( \tilde{C}_i(\mu) \equiv 0 \) for \( i \neq 7 \) and get
\[
\Gamma_7 \sim \tilde{C}_7(\mu)^2 \left[ 1 + \frac{\alpha_s(m_b)}{4\pi} \left( \gamma_{77}^{(0)} \ln \frac{m_b^2}{\mu^2} + \tilde{r}_7 \right) \right],
\]
(52)
where \( \tilde{r}_7 \) is an unpredicted number. With \( \gamma_{77}^{(0)} = \frac{32}{3} \) from [11], we indeed arrive at the same coefficient of \( \ln \frac{m_b^2}{\mu^2} \) as found in eq. (42), and cancellation with the \( \mu \)-dependence of \( \tilde{C}_7(\mu) \) should work. Obviously, it didn’t. This is for the following reason, which illustrates once more the importance of operator mixing under renormalization: since we neglected all the \( b \to s\gamma \) matrix elements \( \langle \hat{O}_i(\mu) \rangle \) with \( i \neq 7 \), only that \( \mu \)-dependence of \( C_7(\mu) \) which is proportional to \( \gamma_{77}^{(0)} \), cf. eq. (49), could be cancelled within our calculation. But \( \gamma_{77}^{(0)} \tilde{C}_7(m_b) < 0 \) and \( \sum_{i \neq 7} \gamma_{77}^{(0)} \tilde{C}_i(m_b) > |\gamma_{77}^{(0)} \tilde{C}_7(m_b)| \), so actually \( \tilde{C}_7(\mu) < 0 \) is a monotonous growing function of \( \mu \). Consequently, by cancellation of the \( \mu \)-dependence due to the anomalous dimension \( \gamma_{77}^{(0)} \), this monotonous growth is even more enlarged instead of being reduced. Such a reduction will only take place after including the \( O(\alpha_s) \) corrections to all \( b \to s\gamma \) matrix elements \( \langle \hat{O}_i(\mu) \rangle \).

6 Summary

In summary, we have calculated in this paper the complete \( O(\alpha_s) \) gluon bremsstrahlung corrections to the decay \( B \to X_s\gamma \). We confirmed all the results of Ali and Greub, and additionally we gave in Table 1 a compilation of the complete bremsstrahlung contributions to the total decay width. We also presented formulas for the photon energy spectrum and made a simple suggestion how to use this bremsstrahlung spectrum as an input for the inclusion of non-perturbative effects. We found that in both cases contributions from the QCD penguin operators \( \hat{O}_3 - \hat{O}_6 \) are entirely negligible, except for those parts of them which can be taken into account by the use of the effective Wilson coefficients (3) instead of the usual ones. We investigated the considerably increased renormalization scale dependence of our results and showed that a reduction of this large theoretical error will not be possible unless all \( b \to s\gamma \) matrix elements \( \langle \hat{O}_i(\mu) \rangle \) will
have been calculated up to $O(\alpha_s)$. Very recently, the first calculation of such two-loop matrix elements was indeed performed \[19\]; as it should be, the results allow for a drastically reduced renormalization scale error.

Finally, we would like to emphasize the urgent demand for the calculation of the still outstanding parts of the complete NLO analysis. Without these calculations, a \textit{consistently} improved prediction for the branching ratio will not be possible, so we do not include such a prediction in our paper. We hope that these improvements in theory will be forthcoming soon, and that it will be possible to compare them with even more improved inclusive measurements, too.

\section*{Acknowledgements}

I would like to thank Ulrich Nierste and Manfred Münz for numerous helpful discussions, comments and suggestions. I am also very grateful to Andrzej J. Buras and Mikolaj Misiak for critically reading the manuscript.

\section*{Appendix}

Below we explain how to obtain from eqs. (27) the complete expressions (i.e. the case $m_s \neq 0$) for the various contributions $M_{ij}$ to the squared and summed bremsstrahlung amplitude (25). First of all, in every factor $1/2 + \kappa_b$ and $2\kappa_c - \kappa_b$ in (27) replace $\kappa_b$ by $\kappa_b + \kappa_s - 1/2$.

Additionally, if both $i$ and $j \in \{2, 3, 4, 6\}$: make in the formulas of (27) the replacement

\begin{equation}
(1 - s) \rightarrow (1 - \rho)^2 - (1 + \rho)s.
\end{equation}

If $i \neq 5$ and $j = 7$: make in the formulas of (27) the replacement

\begin{equation}
s \rightarrow (1 + \rho)s + \frac{2\rho s^2}{(s - t)\ell}.
\end{equation}

If $i \neq 5, 7$ and $j = 8$: make in the formulas of (27) the replacement

\begin{equation}
s \rightarrow (1 + \rho)s + \frac{2\rho s^2}{(s - u)u}.
\end{equation}

Finally, in the expressions for $M_{25}, M_{35}, M_{45}$ and $M_{56}$ make the replacement

\begin{equation}
(1 - 2\kappa_b)(1 - s)s \rightarrow [(1 - 2\kappa_b)(1 - \rho - s) - (1 - 2\kappa_s)(1 - \rho + s)] s
\end{equation}
The remaining four expressions read explicitly

\begin{align}
M_{78} &= \frac{s}{3t\bar{u}} \left\{ (1 + \rho)(2 + t\bar{u}) + \frac{\rho}{1 - \rho} \left[ 1 - \frac{2(1 - \rho) - \bar{t}\bar{u}}{(1 - t)(1 - \bar{u})} \right] \bar{s} \right\}, \\
M_{55} &= \left\{ 16|\kappa_b|^2 + |(1 - 2\kappa_b)s + 4\kappa_b|^2 + \rho \left[ 16|\kappa_s|^2 + \frac{1}{\rho}(1 - 2\kappa_s)s + 4\kappa_s^2 \right] \right\} (1 - s) \\
&\quad + 16 \Re \left\{ 8\rho\kappa_b\kappa_s^* + s [\kappa_b - 2(1 + \rho)\kappa_b\kappa_s^* + \rho\kappa_s^*] \right\}, \\
M_{57} &= -\frac{2}{3} \Re \left\{ 4(\kappa_b + \rho\kappa_s)s - [4\rho(\kappa_b + \kappa_s) + s((1 - 2\kappa_s) + \rho(1 - 2\kappa_b)))] \frac{s^2}{t(s - t)} \right\}, \\
M_{58} &= \frac{2}{9} \Re \left\{ 4(\kappa_b + \rho\kappa_s)s - [4\rho(\kappa_b + \kappa_s) + s((1 - 2\kappa_s) + \rho(1 - 2\kappa_b)))] \frac{s^2}{u(s - u)} \right\},
\end{align}

Here, \(s\), \(t\) and \(u\) are the kinematical variables defined in \((23)\), \(\bar{x} = x/(1 - \rho)\) with \(x = s, t, u\), \(\rho = m_s^2/m_b^2\) and \(\kappa_q \equiv \kappa \left( \frac{m_q^2}{m_s^2} \right)\) with \(\kappa(s)\) from \((17)\). \(M_{77}\) and \(M_{88}\) are already given in \((28)\), and all other contributions vanish.

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