The Newton polygons of overconvergent $F$-crystals

Kiran S. Kedlaya

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Abstract

It is conjectured that every overconvergent $(F, \nabla)$-crystal over $k((t))$ is potentially semistable (equivalently, quasi-unipotent), and so has “generic” and “special” Newton polygons. It is easy to construct a Newton polygon for an arbitrary overconvergent $F$-crystal that coincides with the generic Newton polygon for potentially semistable crystals. We give an analogous construction for the special Newton polygon. In a subsequent preprint, we use this construction to prove the aforementioned conjecture.

Crew [1] asked whether every overconvergent $(F, \nabla)$-crystal over $k((t))$ is quasi-unipotent. An equivalent conjecture [6, Conjecture 4.12] is that every such crystal is potentially semistable, which is to say, isomorphic to a log-crystal over $k[[u]]$ for some finite extension $k((u))$ of $k((t))$. (A related conjecture is the “$p$-adic monodromy” conjecture of Fontaine [3], that every de Rham representation is potentially semistable.) Under this conjecture, every overconvergent $(F, \nabla)$-crystal over $k((t))$ has two well-defined Newton polygons, corresponding to the fibres of the crystal over the generic and special points of $k[[u]]$. By Grothendieck’s specialization theorem, these polygons have the same endpoints and the special polygon lies on or above the generic polygon.

One difficulty in proving this conjecture has been the lack of an a priori description of the special Newton polygon. The generic Newton polygon coincides with the Newton polygon as a crystal over $k((t))$. The special Newton polygon is only uniquely determined from this data alone when the generic slopes are all equal, in which case the generic and special Newton polygons must coincide. In fact, the conjecture was previously known to hold when the generic slopes are all equal, by a result of Tsuzuki [9].

In this paper, we describe a construction of a Newton polygon for an arbitrary overconvergent $F$-crystal over $k((t))$ which coincides with the special Newton polygon for potentially semistable crystals and has some of the expected formal properties of the special Newton polygon. In a subsequent preprint [8], we use this construction to prove Crew’s conjecture. (Note that Yves André and Zoghman Mebkhout have each independently announced proofs of this conjecture, at least in the case where the residue field is the algebraic closure of a finite field.)
1 Notations

We maintain the notations of [3], recalled below. (Note: wherever it appears in the table, * represents an unspecified decoration.)

- $k$: An algebraically closed field of characteristic $p > 0$
- $\mathcal{O}$: A finite extension of the Witt ring $W(k)$
- $\sigma$ on $\mathcal{O}$: An automorphism of $\mathcal{O}$ lifting the absolute Frobenius on $k$
- $\mathcal{O}_0$: The elements of $\mathcal{O}$ fixed by $\sigma$
- $|\cdot|$: The valuation of $\mathcal{O}$ normalized so that $|p| = p^{-1}$
- $K$: The field of formal power series over $k$
- $K^\text{perf}$: The perfect closure of $K$
- $K^\text{sep}$: The separable closure of $K$
- $K^\text{alg}$: The algebraic closure of $K$
- $K^\text{imm}$: The ring of series $x = \sum_{i\in I} x_i t^i$ over $k$ with $I \subseteq \mathbb{Q}$ well-ordered
- $\Omega_t$ ($= \Omega$): The power series ring $\mathcal{O}[t]$.
- $\Gamma$: The $p$-adic completion of $\Omega[t^{-1}]$
- $\sigma$ on $\Gamma$: An endomorphism lifting $x \mapsto x^p$ compatible with $\sigma$ on $\mathcal{O}$
- $\sigma_t$: The endomorphism of $\Gamma$ with $t \mapsto t^p$
- $\Gamma^L$: The $p$-adically complete extension of $\Gamma$ with residue field $L$
- $\Gamma^*$: Equal to $\Gamma^{K^*}$ for $* \in \{\text{perf}, \text{sep}, \text{alg}, \text{imm}\}$
- $\Gamma^{\text{alg}(c)}$: The subring of $x = \sum_i x_i t^i \in \Gamma^{\text{alg}}$ for which for each $n \geq 0$, there exists $r_n$ such that $|x^{s^n} - r_n| < p^{-cn}$
- $\Gamma_{\text{con}}^*$: The ring of $x = \sum_{i=\infty}^{i=-\infty} x_i t^i \in \Gamma^*$ with $x_i \in \mathcal{O}$ and $\lim_{i\to\infty} v_p(x_{-i})/i > 0$
- $\Gamma_{\text{an,con}}^*$: The ring of $x = \sum_{i=\infty}^{i=-\infty} x_i t^i$ with $x_i \in \mathcal{O}[\frac{1}{p}]$, $\lim_{i\to\infty} v_p(x_{-i})/i > 0$, and $\lim_{i\to+\infty} v_p(x_i)/i \geq 0$

We also mention two rings that are not explicitly defined in [3]. The ring $\Omega_{\text{imm}}$ consists of those elements $x = \sum x_i t^i$ of $\Gamma^{\text{imm}}$ with $x_i = 0$ for $i < 0$. The ring $\Omega^{\text{alg}}$ is the intersection of $\Omega^{\text{imm}}$ with $\Gamma^{\text{alg}}$.

2 Analytic rings revisited

In this section, we make a more careful study of some of the rings introduced in [3] but not used extensively there. In particular, we need a careful analysis of the rings $\Gamma_{\text{an,con}}$ and its extensions.

Recall that the ring $\Gamma_{\text{an,con}}$ is defined as the set of series $x = \sum_{n=-\infty}^{n=\infty} x_n t^n$ with $x_n \in \mathcal{O}[-\frac{1}{p}]$ satisfying $v_p(x_n) \geq -cn$ for $n$ sufficiently negative (and some choice of $c$) and $v_p(x_n) = o(n)$ for $n$ sufficiently positive. Also recall that the ring $\Gamma^{\text{imm}}$ is defined as the set of series $x = \sum_{n\in\mathbb{Q}} x_n t^n$ with $x_n \in \mathcal{O}$ such that for each $r > 0$, the set of $n$ with $|x_n| > p^{-r}$ is a well-ordered subset of $\mathbb{Q}$. To put these together, we define the ring $\Gamma_{\text{an,con}}^{\text{imm}}$ as the set of series $x = \sum_{n\in\mathbb{Q}} x_n t^n$, with $x_n \in \mathcal{O}[\frac{1}{p}]$, satisfying the following conditions:
1. For each $r \in \mathbb{R}$, the set of $n \in \mathbb{Q}$ with $|x_n| > p^r$ is well-ordered.

2. There exists a constant $c > 0$ such that for $n$ sufficiently negative, $|x_n| < p^c$.

3. For every constant $d > 0$, we have $|x_n| > p^{dn}$ for $n$ sufficiently positive.

This ring has an unusual property not shared by any of the rings introduced so far: it contains nonzero solutions of the equation $x^\sigma = \lambda x$ for $\lambda \in \mathbb{O}$ not a unit. For example, if $\lambda^\sigma = \lambda$, then $x = \sum_{n=-\infty}^{\infty} \lambda^{-n} t^n$ is a solution.

Notice that in $\Gamma^{\text{imm}}_{\text{alg}, \text{con}}$, the equation $x^\sigma = \mu x$ with $\mu \in \mathbb{O}$ has nontrivial solutions whenever $|\mu| \leq 1$. (By contrast, in $\Gamma^{\text{imm}}_{\text{con}, \text{con}}$, there are no solutions unless $|\mu| = 1$.) In fact, it is easy to write down all such $x$: they are given as

$$x = \sum_{n=-\infty}^{\infty} \mu^ny^{-n}; \quad y = \sum_{i \in [a,ap]} y_it^i,$$

where $a$ is a fixed positive rational.

Recall that we view $\Gamma^{\text{alg}}_{\text{con}, \text{con}}$ as a subring of $\Gamma^{\text{imm}}_{\text{con}, \text{con}}$. Within $\Gamma^{\text{imm}}_{\text{alg}, \text{con}}$, we can correspondingly identify the subring $\Gamma^{\text{alg}}_{\text{con}, \text{con}}$ consisting of elements $x = \sum x_it^i$ such that for each $j$, $\sum_{i<j} x_it^i \in \Gamma^{\text{alg}}_{\text{con}, \text{p}}$. Alternatively, for each $j$, there exists $y \in \Gamma^{\text{alg}}_{\text{con}, \text{p}}$ with $x_i = y_i$ for $j < i$; the equivalence of these two conditions reduces to the fact that the truncation of an element of $\Gamma^{\text{alg}}_{\text{con}, \text{con}}$ is still in $\Gamma^{\text{alg}}_{\text{con}, \text{con}}$. This statement can be proved directly or deduced from the classification of algebraic generalized power series [3 Theorem 8].

**Lemma 2.1.** Suppose $m$ is a positive integer, $\lambda, \mu \in \mathbb{O}$ are nonzero and $x = \sum x_it^i \in R$, for $R = \Gamma^{\text{alg}}_{\text{con}, \text{con}}$ or $R = \Gamma^{\text{imm}}_{\text{con}, \text{con}}$.

(a) If $|\lambda| \geq |\mu|$, then there exists $y \in R$ such that $\lambda y^m - \mu y = x$.

(b) If $|\lambda| \leq |\mu|$ and $x_i = 0$ for $i < 0$, then there exists $y \in R$ such that $\lambda y^m - \mu y = x$.

(c) If $x \in \Gamma^{\text{imm}}_{\text{con}, \text{con}}, |\lambda| \leq |\mu|$ and there exists $y \in R$ such that $\lambda y^m - \mu y = x$, then $\mu y \in \Gamma^{\text{imm}}_{\text{con}, \text{con}}$.

**Proof.** There is no loss of generality in assuming $\lambda, \mu \in \mathbb{O}_0$.

(a) First suppose $|\lambda| = |\mu|$; without loss of generality, we may assume $\lambda = \mu = 1$. Write $x = a + x_0 + b$ with $a = \sum_{i<0} x_it^i$ and $b = \sum_{i>0} x_it^i$. Set

$$c = \sum_{i<0} \sum_{n=1}^{\infty} x_{ip-\text{mn}}^{\sigma-\text{mn}}, \quad d = \sum_{i>0} \sum_{n=0}^{\infty} x_{ip-\text{mn}}^{\sigma-\text{mn}}.$$

The first inner sum converges because $v_p(x_i) \geq O(-i)$ for $i \rightarrow -\infty$, while the second inner sum converges because $|x_i| \rightarrow 0$ as $i$ runs over any decreasing sequence. It is easily checked that $\lambda c^m - \mu c = b$ and that $\lambda d^m - \mu d = b$. Let $y_0 \in \mathbb{O}$ be a solution of $y_0^m - y_0 = x_0$; then we may set $y = c + d + y_0$ to obtain a solution of $\lambda y^m - \mu y = x$. 


Next, suppose $|\lambda| > |\mu|$. Write $x = a + b$ with $a = \sum_{i < 1} x_i t^i$ and $b = \sum_{i \geq 1} x_i t^i$. (The index 1 could be replaced by any positive index.) We wish to set $\Gamma$ of an element of $j/p$.

The first sum converges $p$-adically in $\Gamma_{\text{con}}^{\text{alg}}[\frac{1}{p^n}]$ or $\Gamma_{\text{con}}^{\text{imm}}[\frac{1}{p^n}]$ to a solution of $\lambda c^{\sigma_m} - \mu c = a$. The second sum converges $t$-adically (since $b^{\sigma_m}$ has no coefficients of index less than $p^{mn}$) to a solution of $\lambda d^{\sigma_m} - \mu d = b$. Thus we may set $y = c + d$ to obtain a solution of $\lambda y^{\sigma_m} - \mu y = x$.

(b) Set
\[ y = -\sum_{i \geq 0} \sum_{n=0}^{\infty} x_{ip^{-mn}} \frac{\lambda^n}{\mu^{n+1}}, \]
the inner sum converges because $x_{ip^{-mn}}$ is bounded and $(\lambda/\mu)^n \to 0$. Then it is easily verified that $\lambda y^{\sigma_m} - \mu y = x$.

(c) Suppose $y \notin \Gamma_{\text{con}}^{\text{imm}}$: put $y = \sum_j y_j t^j$ and let $j$ be the smallest index such that $|\mu y_j| > 1$. First suppose $j \geq 0$. Comparing the coefficients of $t^j$ in the equation $\lambda y^{\sigma_m} - \mu y = x$, we have $\lambda y_j^{\sigma_m/p^n} - \mu y_j = x_j$. In this equation, $|\mu y_j| > 1$ but $|x_j| \leq 1$, so $|\lambda y_j^{\sigma_m/p^n}| = |\mu y_j| > 1$. On the other hand, $|\lambda y_j^{\sigma_m/p^n}| \leq |\mu y_j/p^n| \leq 1$ since $j/p^n < j$, contradiction.

Now suppose $j < 0$. In this case, the above argument gives $|y_j/p^n| = |y_j \mu/\lambda|$. By comparing the coefficients of $t^j/p^n, t^{j/p^n}, \ldots$ as well, we obtain $|y_j/p^n| = |y_j \mu^i/\lambda|$ by induction on $l$. But this conclusion contradicts the fact that $|y_i|$ is bounded for $i < 0$. □

## 3 Factorization theorems

In this section, we pick up the thread begun in [3, Section 4], and prove some additional factorization theorems for analytic rings. Our goal is to prove that finitely generated ideals in analytic rings are principal (Lemma 3.15), a fact which will be crucial in the proof of the main theorem.

We begin with a lemma that will allow us to focus on $\Omega_{\text{an}}$ and its analogues instead of on $\Gamma_{\text{an,con}}$ and its analogues. In fact, the corresponding statement for $\Gamma_{\text{an,con}}$ and $\Omega_{\text{an}}$ is [3, Lemma 4.7], and the proof here is simply a careful generalization of the proof there.

**Lemma 3.1.** Every nonzero element of $\Gamma_{\text{an,con}}^{\text{imm}}$ (resp. $\Gamma_{\text{an,con}}^{\text{alg}}$) can be factored as the product of an element of $\Gamma_{\text{con}}^{\text{imm}}$ and an element of $\Omega_{\text{an}}^{\text{imm}}$ (resp. an element of $\Gamma_{\text{con}}^{\text{alg}}$ and an element of $\Omega_{\text{an}}^{\text{alg}}$), the latter having nonzero constant coefficient.

**Proof.** Let $x = \sum_i x_i t^i$ be an element of $\Gamma_{\text{an,con}}^{\text{imm}}$. (The proof for $\Gamma_{\text{an,con}}^{\text{alg}}$ is analogous, so we omit reference to it hereafter.) Let $c$ be an irrational number, so that $\min_i \{v_p(x_i) + ci\}$ occurs for a unique value of $i$. Without loss of generality, we may assume that value is $i = 0$.
and that \( x_0 = 1 \). (Otherwise, we can multiply by a suitable constant times a suitable power of \( t \).) Put
\[
 r = \min_{i < 0} \left\{ \frac{v_p(x_i)}{i} \right\}, \quad s = \min_{i > 0} \left\{ \frac{-v_p(x_i)}{i} \right\};
\]
then by construction, \( r < c < s \).

Now define a sequence \( \{y_n\}_{n=0}^{\infty} \) as follows. Let \( k \) be the smallest index such that \( v_p(x_k) < 0 \). (If no such \( k \) exists, then \( x \in \Gamma_{\text{con}}^{\text{imm}} \) and there is nothing to prove.) Set \( y_0 = 0 \). To define \( y_{n+1} \) from \( y_n \), set \( (1 + y_n)x = \sum_i y_{n,i} t^i \) and let
\[
 y_{n+1} = -1 + (1 + y_n) \left( \sum_{i < k} y_{n,i} t^i \right)^{-1}.
\]
We will show that \( \{y_n\} \) converges, in a suitable sense, to a limit \( y \) and that \( yx \in \Omega_{\text{an}}^{\text{imm}} \).

Suppose \( d \geq 0 \) satisfies \( v_p(y_{n,i}) \geq -ri + d \) for all \( i < k \). Let \( a_n = -1 + \sum_{i < k} y_{n,i} t^i \), \( b_n = \sum_{i \geq k} y_{n,i} t^i \) and \( c_n = (1 + a_n)^{-1} - 1 \). Then \( c_n = \sum_{j=1}^{\infty} (-a_n)^j \); if we put \( c_n = \sum_{i} c_{n,i} t^i \), it follows that \( v_p(c_{n,i}) \geq -ri + d \) for all \( i < k \). Then
\[
(1 + y_{n+1})x = (1 + y_n)x(1 + b_n) = (1 + a_n + b_n)(1 + c_n) = 1 + b_n(1 + c_n).
\]
Therefore for \( j < k \) nonzero,
\[
v_p(y_{n+1,j}) = v_p\left( \sum_{i \geq k} y_{n,i} c_{n,j-i} \right) \\
\geq \min_{i \geq k} \{ v_p(y_{n,i}) + v_p(c_{n,j-i}) \} \\
\geq \min_{i \geq k} \{ -si - r(j - i) + d \} \\
= \min_{i \geq k} \{ -rj + d + (r - s)j \} \\
= -rj + d + (r - s)k.
\]
For \( n = 0 \), the initial inequality holds for \( d = 0 \). Therefore for \( j < k \), \( v_p(y_{n,j}) \geq -rj + n(r - s)k \) by induction on \( n \), and the same inequality holds for the \( c_{n,j} \) as noted above.

Let \( R \) be the subring of \( \Gamma_{\text{con}}^{\text{imm}} \) consisting of series \( z = \sum_i z_i t^i \) such that \( v_p(z_i) \geq -ri \) for all \( i < 0 \). Then \( R \) carries a valuation \( v' \) defined by \( v'(z) = \min_i \{ v_p(z_i) + ri \} \). With respect to \( v' \), the product \( \prod_{i=1}^{\infty} (1 + c_n) \) converges; we call the limit \( y \). Moreover, we can extend \( v' \) to the subring of \( \Gamma_{\text{an,con}}^{\text{imm}} \) defined by \( v_p(z_i) \geq -ri \) for all \( i \), in which it is clear that \( y_n x \rightarrow yx \).
In particular, for \( j < k \), \( (yx)_j = \lim_{n \rightarrow \infty} y_{n,j} = 0 \). Thus \( yx \in \Omega_{\text{an}}^{\text{imm}} \), and we may factor \( x \) as \( y^{-1}(yx) \) with \( y^{-1} \in \Gamma_{\text{con}}^{\text{imm}} \) and \( yx \in \Omega_{\text{an}}^{\text{imm}} \), as desired. \( \square \)
Unless otherwise specified, throughout this section \( R \) will denote one of \( \Omega_{\text{an}}, \Omega_{\text{alg}}, \) or \( \Omega_{\text{an}}^{\text{imm}} \). For each \( r > 0 \), define the norm \( w_r \) as follows:

\[
w_r(x) = \max_{i \geq 0} \{|x_i| p^{-ri}\} \quad x \neq 0.
\]

We can use these norms to endow \( R \) with a Fréchet topology. Recall that this means a sequence \( \{x_n\} \) is Cauchy if and only if for each \( r > 0 \) and \( \epsilon > 0 \), there exists \( N \) such that \( w_r(x_m - x_n) < \epsilon \) for \( m, n \geq N \). Note that each \( w_r \) extends to a certain subring of \( \Gamma_{\text{an,con}}^{\text{imm}} \), but not to the whole ring.

**Proposition 3.2.** The ring \( R \) is complete for the Fréchet topology.

**Proof.** Let \( \{x_n\} \) be a Cauchy sequence in the Fréchet topology, and put \( x_n = \sum_i x_{n,i} t^i \). Then by hypothesis, for \( i \geq 0, \epsilon > 0 \) and \( r > 1 \), there exists \( N \) such that for \( m, n \geq N, |x_{m,i} - x_{n,i}| r^i < \epsilon \). In particular, for each fixed \( i \), \( \{x_{n,i}\} \) is a Cauchy sequence in \( O \), so it has a limit \( y_i \).

Now put \( y = \sum_i y_i t^i \). For any given \( \epsilon > 0 \) and \( r > 0 \), choose \( N \) so that for \( m, n \geq N \), \( w_r(x_m - x_n) < \epsilon \). Then \( |x_{m,i} - x_{n,i}| < \epsilon p^{-ri} \) for all \( m, n \geq N \). Since the absolute value on \( O \) is nonarchimedean, we also have \( |x_{n,i} - y_i| < \epsilon p^{-ri} \), so \( w_r(x_n - y) < \epsilon \). We conclude that \( x_n \to y \).

We associate to each nonzero element \( x \) of \( R \) a Newton polygon as follows. For each index \( i \) with \( x_i \neq 0 \), plot the point \((i, v_p(x_i))\) in the coordinate plane. We define the Newton polygon of \( x \) to be the lower convex hull of these points and the slopes of \( x \) to be the negations of the slopes of its Newton polygon (ignoring 0 if it occurs). We define the multiplicity of a slope to be the difference between the \( x \)-coordinates of the endpoints of the segment of the Newton polygon with that slope (or 0 if there is no such segment). We say \( x \in R \) is pure if \( x \) has nonzero constant coefficient and exactly one slope; in particular, this implies \( x \in \Omega_{\text{an}}^{\text{imm}}[\frac{1}{p}] \).

We single out a special class of pure elements that can be treated like ordinary polynomials. We say \( x = \sum x_i t^i \) is truncated if \( x \) is pure of some slope \( s \) with multiplicity \( m \), and \( x_i = 0 \) for \( i > m \). As for polynomials, there is a division lemma for truncated elements.

**Lemma 3.3 (Division lemma).** Let \( x \in R \) be truncated of slope \( s \) and multiplicity \( m \). Then for any \( y \in R \), there exists a unique pair \( q, r \) of elements of \( R \) such that \( y = qx + r \) and \( r = \sum r_i t^i \) satisfies \( r_i = 0 \) for \( i > m \). Moreover, \( w_s(r) \leq w_s(y) \) and \( w_s(q) \leq w_s(y) \).

We will refer to \( r \) as \( y \) mod \( x \).

**Proof.** Since \( x \in \Gamma_{\text{con}}^{\text{imm}}[\frac{1}{p}] \), \( x \) is invertible in that ring, and \( x^{-1} \) can be written as \( t^m b \), where \( b = \sum_i b_i t^i \) satisfies \( b_i = 0 \) for \( i > 0 \). Now write \( x^{-1} y = \sum_i z_i t^i \), and set \( q = \sum_{i \geq 0} z_i t^i \). Then \( x^{-1} y - q \) has no coefficients of positive index, and so \( r = y - qx = x(x^{-1} y - q) \) has no coefficients of index \( m \) or greater.

Note that \( w_s(x^{-1}) \) is well-defined; since \( w_s(x) = 1 \), we must have \( w_s(x^{-1}) = 1/w_s(x) = 1 \). Thus \( w_s(x^{-1} y) = w_s(y) \). Replacing some coefficients of a series with zeroes cannot increase
its norm, so \( w_s(q) \leq w_s(x^{-1}y) = w_s(y) \) and \( w_s(r) = w_s(x^{-1}r) = w_s(x^{-1}y - q) \leq w_s(x^{-1}y) \leq w_s(y) \).

A \textit{slope factorization} of a nonzero element \( x \) of \( R \) is a product \( x = t^j \prod_{i=1}^N y_i \) for \( N \) either a nonnegative integer or \( \infty \), convergent (if \( N = \infty \)) in the Fréchet topology, such that:

(a) \( y_i \) is pure of slope \( s_i \);

(b) the sequence \( s_1, s_2, \ldots \) is strictly decreasing;

(c) if \( N = \infty \), then \( s_i \to 0 \).

In our next few propositions, we show that these factorizations exist and are unique up to units, and use these factorizations to establish some structural properties of \( R \).

**Proposition 3.4.** Let \( x \) and \( y \) be elements of \( R \) with nonzero constant coefficient. Then the Newton polygon of \( xy \) is the sum of the Newton polygons of \( x \) and \( y \). That is, the multiplicity of a slope of \( xy \) equals the sum of the multiplicities of the corresponding slope of \( x \) and of \( y \).

**Proof.** Fix a slope \( s \). Suppose the Newton polygon of \( x \) intersects its support line of slope \(-s\) from \((i, v_p(x_i))\) to \((j, v_p(x_j))\), and the Newton polygon of \( y \) intersects its support line of slope \(-s\) from \((k, v_p(y_k))\) to \((l, v_p(y_l))\), with \( i \leq j \) and \( k \leq l \). We claim the Newton polygon of \( uv \) intersects its support line of slope \(-s\) at \((i + k, v_p(x_i) + v_p(y_k))\) and \((j + l, v_p(x_j) + v_p(y_l))\), which would imply the statement of the lemma.

To verify the claim, we first note that

\[
v_p((xy)_{i+k}) = v_p \left( \sum_{m} x_{i+m} y_{k-m} \right) \geq \min_{m \in \mathbb{Q}} \{ v_p(x_{i+m}) + v_p(y_{k+m}) \},
\]

with equality if the minimum occurs for a single value of \( m \). For \( m < 0 \), we have \( v_p(x_{i+m}) > v_p(x_i) - ms \) and \( v_p(y_{k-m}) \geq v_p(y_k) + ms \), so \( v_p(x_{i+m}) + v_p(y_{k-m}) \geq v_p(x_i) + v_p(y_k) \). Similarly, if \( m > 0 \), we have \( v_p(x_{i+m}) \geq v_p(x_i) - ms \) and \( v_p(y_{k-m}) \geq v_p(y_k) + ms \), so again \( v_p(x_{i+m}) + v_p(y_{k-m}) \geq v_p(x_i) + v_p(y_k) \). Thus the minimum is achieved only for \( m = 0 \), and so \( v_p((xy)_{i+k}) = v_p(x_i) + v_p(y_k) \). Likewise, \( v_p((xy)_{j+l}) = v_p(x_j) + v_p(y_l) \).

By a similar argument, we also have that for \( n > 0 \), \( v_p((xy)_{i+k-n}) > v_p(x_i) + v_p(y_k) + ns \) and \( v_p((xy)_{j+l-n}) > v_p(x_j) + v_p(y_l) + ns \). Namely, \( v_p((xy)_{i+k-n}) \geq \min_{m \in \mathbb{Q}} \{ v_p(x_{i+m}) + v_p(y_{k-m-n}) \} \)

\[
\begin{align*}
v_p(x_{i+m}) &\geq v_p(x_i) - ms \\
v_p(y_{k-m-n}) &\geq v_p(y_k) + (m + n)s,
\end{align*}
\]

the second inequality is strict for \( m < 0 \), and the third is strict for \( m > -n \), so \( v_p(x_{i+m}) + v_p(y_{k-m-n}) > v_p(x_i) + v_p(y_k) + ns \) and the inequality follows; the other inequality follows.
analogously. (The minimum really is a minimum, not an infimum, so termwise strict inequality implies strict inequality for the minimum.) Thus the Newton polygon of $xy$ intersects its support line of slope $-s$ from $(i+k, v_p(x_i) + v_p(y_k))$ to $(j+l, v_p(x_j) + v_p(y_l))$, respectively. □

**Corollary 3.5.** The invertible elements of $\Gamma_{\text{an,con}}$ (resp. $\Gamma_{\text{an,con}}^{\text{alg}}$, $\Gamma_{\text{an,con}}^{\text{imm}}$) are precisely the nonzero elements of $\Gamma_{\text{con}}[1/p]$ (resp. $\Gamma_{\text{con}}^{\text{alg}}[1/p]$, $\Gamma_{\text{con}}^{\text{imm}}[1/p]$).

**Lemma 3.6.** Let $x$ be an element of $R$ which has nonzero constant coefficient 1. Let $s$ be the first slope of $x$ and $m$ its multiplicity. Then $x$ can be factored as $yz$ where $y$ is truncated of slope equal to the first slope of $x$, with the same multiplicity.

*Proof.* We construct a sequence of elements $y_1, y_2, \ldots$ of $R$ supported on $[0, sm]$, having constant coefficient 1, convergent under the norm $w_s$, such that $w_s(x \mod y_n) \to 0$; this will imply that $\{y_n\}$ and $\{x \mod y_n\}$ converge in the Fréchet topology as well, and that if $y_n \to 0$, then $x \mod y = 0$. Specifically, we set

\[
y_1 = \sum_{0 \leq i \leq m} x_it^i \quad y_{n+1} = y_n - (x \mod y_n) \quad (n > 1).
\]

Let $c = w_r(x - y_1)$; by construction, $c < 1$. We prove by induction that $w_r(x \mod y_n) \leq c^n$. This holds by design for $n = 1$. Now suppose it holds up to some $n$. Then we can write $x = a_n y_n + b_n$ with $b_n = x \mod y_n$, and

\[
x \mod y_{n+1} = (a_n y_n + b_n) \mod y_{n+1} = (a_n(y_{n+1} - b_n) + b_n) \mod y_{n+1} = b_n(1 - a_n) \mod y_{n+1}.
\]

Since $w_r(y_n - y_1) \leq c$ by the induction hypothesis, and $w_r(y_1) = 1$, we have $w_r(y_n) = 1$. Thus

\[
w_r(1 - a_n) = w_r(y_n - a_n y_n) = w_r(y_n - x + b_n) \leq \max\{w_r(y_n - y_1), w_r(y_1 - x), w_r(b_n)\} \leq \max\{c, c^n\} = c.
\]

We conclude that $w_r(x \mod y_{n+1}) \leq w_r(b_n(1 - a_n)) \leq c^{n+1}$, completing the induction.

By the induction, we have that $\{y_n\}$ is Cauchy, hence convergent, and that $\{x \mod y_n\}$ converges to 0. Thus the limit $y$ of $\{y_n\}$ satisfies $x \mod y = 0$, as desired. □

**Corollary 3.7.** Every pure element of $R$ factors as a truncated element times a unit.
Proposition 3.8. If $x$ has nonzero constant coefficient, then $(x, t)$ is the unit ideal.

Proof. Let $x = \sum_i x_i t^i$. Without loss of generality, assume $x_0 = 1$, and let $k$ be the smallest index such that $v_p(x_k) < 0$. Let $y = x(\sum_{0 \leq i < k} x_i t^i)^{-1}$; if we put $y = \sum_i y_i t^i$, then $y_i = 0$ for $0 < i < k$. That is, $y \equiv 1 \pmod{t^k}$, so $(y, t^k)$ is the unit ideal, as then is $(x, t)$. 

Lemma 3.9. (a) Let $x$ be a pure element of $R$ of slope $s$ and $y$ an element of $R$ with nonzero constant coefficient whose Newton polygon has all slopes greater than $s$. Then $(x, y)$ is the unit ideal.

(b) Let $x$ and $y$ be pure elements of $R$ of slope $s$. Then $(x, y)$ is generated by a pure element of slope $s$.

Proof. (a) Without loss of generality, we may assume $x$ is truncated and $y$ has constant coefficient 1. Put $c = w_r(1 - y)$; then $c < 1$, and $w_r((1 - y)^n) = c^n$. Thus if we put $z_n = (1 - y)^n \mod x$, we also have $w_r(z_n) \leq c^n$, so the series $\sum_{n=0}^{\infty} z_n$ converges in the Fréchet topology to a limit $z$. Likewise, $\sum_{n=0}^{\infty} y(z_n \mod x)$ converges to 1, so $zy - 1$ is divisible by $x$, and $(x, y)$ is thus the unit ideal.

(b) Let $m$ and $n$ be the multiplicities of $s$ as a slope of $x$ and $y$. Then $m$ and $n$ are integral multiples of $v/s$, where $v$ is the smallest positive valuation of $\mathcal{O}$. Thus we can induct on $m + n$. Assume without loss of generality that $m \leq n$, and that $x$ is truncated. Let $z = y \mod x$, so that $(x, y) = (x, z)$ and it suffices to show that $(x, z)$ is generated by a pure element of slope $s$.

If $z = 0$, then $(x, y) = x$ and we are done, so assume $z \neq 0$. Otherwise, let $z = \sum_i z_i t_i$ and choose $j$ to minimize $v_p(z_j) + rj$. Since $(x, t)$ is the unit ideal by the previous lemma, we can find $a$ such that $az_j t_j \equiv 1 \pmod x$, and $(x, z) = (x, az)$. Let $b = az \mod x = \sum_i b_i t_i$; then $b_0 = 1$ and all slopes of $b$ are at least $s$. Moreover, the multiplicity of $s$ as a slope of $b$ is strictly less than $m$. By Lemma 3.8, we can factor $b$ as $cd$ with $c$ pure of slope $s$ with the same multiplicity, and $d$ having all slopes greater than $s$. By (a), $(x, d)$ is the unit ideal, so $(x, z)$, which is equal to $(x, cd)$, is also equal to $(x, c)$. By the induction hypothesis, $(x, c)$ is generated by a pure element of slope $s$, as desired.

Proposition 3.10. For $x, y \in R$ such that $x$ admits a slope factorization, $x$ divides $y$ if and only if each factor in a slope factorization of $x$ divides $y$.

Proof. If $x$ divides $y$, then obviously any factor of $x$ divides $y$. Conversely, suppose $ct^j \prod y_i$ is a slope factorization of $x$. Put $z_i = y/(ct^j y_1 \cdots y_i)$; then the sequence $\{z_i\}$ is Cauchy, so has a limit $z$, and clearly $zx \to y$.

Proposition 3.11. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of pure elements of $R$ whose slopes are strictly increasing and tend to 0. Then there exists $x \in R$ admitting a slope factorization $ct^j \prod y_n$ such that $y_n$ and $z_n$ generate the same ideal for all $n$. 

9
Proof. Let $s_n$ be the slope of $z_n$, and let $v$ be the smallest positive valuation of $\mathcal{O}$. Put $z_n = \sum_i z_{n,i} t^i$; then $z_{n,i} \in \mathcal{O}$ for $i < v/s_n$. In particular, $\sum_{i < v/s_n} z_{n,i} t^i$ is a unit in $R$; let $u_n$ be its inverse and put $y_n = u_n z_n$. If we put $y_n = \sum_i y_{n,i} t^i$, then $y_{n,i} = 0$ for $0 < i < v/s_n$ by construction and $v_p(y_{n,i}) \geq -s_n i$ for $i \geq v/s_n$. In particular, we have $w_r(y_n - 1) \leq p^{-v-rv/s_n}$ for $r \geq s_n$. Therefore for any fixed $r > 1$, eventually $w_r(y_n - 1) \leq p^{-v-rv/s_n}$; since $s_n \to 0$, $p^{-rv/s_n} \to 0$. We conclude that $\prod_n y_n$ converges in the Fréchet topology, and we may take the limit as our desired $x$.

Lemma 3.12. Every nonzero element $x$ of $R$ has a slope factorization.

Proof. Without loss of generality, assume $x$ has nonzero constant coefficient. Factor $x$ as $y_1 x_2$ as in Lemma 3.6 so that $y_1$ is pure of slope equal to the first slope of $x$, with the same multiplicities. Then by Proposition 3.11, the Newton polygon of $x_2$ is equal to that of $x$ with its first segment removed. Then factor $x_2$ as $y_2 x_3$ in the same fashion, and so on.

If the Newton polygon of $x$ has finitely many slopes, then this process eventually represents $x$ as a product of pure elements, as desired. If the Newton polygon of $x$ is infinite, Proposition 3.11 allows us to construct an element $z$ with a slope factorization whose slope factors are unit multiples of the $y_i$. In particular, $x$ and $z$ have the same Newton polygon. Since $x$ is divisible by each $y_i$, by Proposition 3.10 $x$ is divisible by $z$. Moreover, the Newton polygon of $x/z$ is empty, so $x/z$ is a unit. We can modify the slope factorization of $z$ by multiplying its first factor by $x/z$ to obtain the desired slope factorization of $x$.

The next proposition is a rigid analytic version of the Chinese remainder theorem.

Proposition 3.13 (Chinese remainder theorem). Let $(x_n)_{n=1}^\infty$ be a sequence of pure elements of $R$ whose slopes are strictly increasing and tend to 0, and let $(y_n)_{n=1}^\infty$ be elements of $R$. Then there exists $y \in R$ such that $y \equiv y_n \pmod{x_n}$.

Proof. Let $s_n$ be the slope of $x_n$, and let $v$ be the smallest positive valuation in $\mathcal{O}$. By imitating the proof of Proposition 3.11, we may assume that $x_n = \sum_i x_{n,i} t^i$ is such that $x_0 = 1$ and $x_{n,i} = 0$ for $0 < i < v/s_n$. In particular, this implies that $\prod x_n$ converges to a limit $x$; let $u_n = x/x_n$.

We construct a sequence $(z_n)_{n=1}^\infty$ such that $u_n z_n \equiv y_n \pmod{x_n}$ and $\sum u_n z_n$ converges; then we may set $y = \sum u_n z_n$ and be done. First, choose $v_n$ with $u_n v_n \equiv y_n \pmod{x_n}$, which exists because $u_n$ and $x_n$ are relatively prime.

It suffices to show that for any $\epsilon > 0$, there exists $c$ such that for all $r \geq p^{s_n-1}$, $w_r(1 + cx_r) < \epsilon$; for example, one can then choose such an $\epsilon$ with $|w_r(u_n v_n)| < 1/n$ for all $r \geq s_n - 1$, and set $z_n = v_n(1 + cx_n)$.

Note that $w_r(1 - x_n) < 1$ for $r > p^{s_n}$, so in those norms, the sequence $c_n = -1 - (1 - x_n) - \cdots - (1 - x_n)^m$ is Cauchy. In particular, there exists $m$ such that $w_r(u_n v_n(1 + c x_n)) < \epsilon$ for all $r \geq s_n - 1$. Set $z_n = v_n(1 + c x_n)$; it is now clear that the series $\sum u_n z_n$ is Cauchy for all of the $w_r$, and thus convergent. We then take $y = \sum u_n z_n$ and the proof is complete.
Proposition 3.14. The slope factorization of $x$ is unique up to units.

Proof. The constant coefficient and power of $t$ are clearly uniquely determined by $x$, so we may suppose $\prod y_i$ and $\prod z_j$ are slope factorizations of $x$. It suffices to prove that each $z_j$ divides one of the $y_i$ (and vice versa). Taking the greatest common divisor of $z_j$ with the $y_i$ of the same slope, if it exists, and dividing out that divisor, we may reduce to the case where each $y_i$ is relatively prime to $z_j$.

In this case, for each $i$, there exists $w_i$ such that $w_i z_j \equiv 1 \pmod{y_i}$. By Proposition 3.13, there exists $w$ such that $wz_j \equiv 1 \pmod{y_i}$ for each $i$, so by Proposition 3.10 $wz_j - 1$ is divisible by $x$. But then $wz_j$ and $wz_j - 1$ are both divisible by $z_j$, a contradiction since $z_j$ is not a unit. Thus $z_j$ divides one of the $y_i$, as desired. 

Finally, we use slope factorization to prove a structure theorem about ideals in analytic rings.

Lemma 3.15. Every finitely generated ideal in $\Gamma_{\text{an,con}}$ (resp. $\Gamma_{\text{an,con},\text{alg}}$, $\Gamma_{\text{an,con},\text{imm}}$) is principal.

Proof. It suffices to show that for $x, y \in \Gamma_{\text{an,con}}$ (resp. $x, y \in \Gamma_{\text{an,con},\text{imm}}$), there exist $r, s$ such that $rx + sy$ generates the ideal $(x, y)$. By Lemma 3.11 we may reduce to the case where $x, y \in \Omega_{\text{an}}$ (resp. $\Omega_{\text{an},\text{alg}}$, $\Omega_{\text{an},\text{imm}}$) and have nonzero constant coefficients. By Proposition 3.11, there exists $z$ whose slope factorization consists of the greatest common divisors of the slope factors of $x$ and $y$, and by Proposition 3.10, $x$ and $y$ are divisible by $z$. Dividing off $z$, we reduce to the case where $x, y$ have no common slope factors.

Let $x = c \prod y_i$ be the slope factorization of $x$. By assumption, $y$ is not divisible by $y_i$, so there exists $z_i$ such that $yz_i \equiv 1 \pmod{y_i}$. By Proposition 3.13, there exists $z$ such that $z \equiv z_i \pmod{y_i}$; then $yz - 1$ is divisible by all of the $y_i$. Therefore there exists $w$ such that $yz - 1 = wx$, and $1 \in (x, y)$ as desired. \qed

4 A direct calculation

The critical non-formal step in the proof of the main theorem is a direct computation to establish the existence of certain eigenvectors. This computation takes the form of two related lemmas.

Lemma 4.1. Let $M$ be an $F$-crystal over $R = \Gamma_{\text{an,con}}$ or $R = \Gamma_{\text{an,con},\text{imm}}$, and let $\lambda \in \mathcal{O}_0$ be a uniformizer. Suppose $M$ admits a basis $v_1, \ldots, v_n, w$ such that for some $c_i \in R$,

\[
Fv_i = v_{i+1} \quad (i = 1, \ldots, n-1) \\
Fv_n = \lambda^{n+1}v_1 \\
Fw = w + \sum_{i=1}^n c_i v_i.
\]

Then $M$ has an eigenvector $y$ with $Fy = \lambda y$. 

11
Proof. Observe that for \( j = 1, \ldots, n - 1, \)
\[
F(w + cv_j) = (w + cv_j) + cv_{j+1} - v_j + \sum_{i=1}^{n} c_i v_i.
\]

Thus we can modify \( w \) by a suitable linear combination of \( v_1, \ldots, v_n \) to obtain \( x \) such that \( Fx = x + yv_1 \) for some \( y \in R \).

Suppose \( y = ax + b_1 v_1 + \cdots + b_n v_n + bx \) satisfies \( Fy = \lambda y \).
Comparing coefficients in this equation, we have \( a^\sigma = \lambda a, b_i^\sigma = \lambda b_{i+1} \) for \( i = 1, \ldots, n - 1 \), and \( \lambda^{n+1}b_n^\sigma + ax = b_1 \).

If \( a \) and \( b \) satisfy the equations
\[
a^\sigma = \lambda a, \quad \lambda b^\sigma = b - \lambda^{-1} ax, \tag{1}
\]
then setting \( b_1 = b \) and \( b_i = b_i^{\sigma-1} \rho^{-i+1} \) for \( i = 2, \ldots, n \) and \( y = ax + b_1 v_1 + \cdots + b_n v_n + bx \) satisfies \( Fy = \lambda y \).
Thus it suffices to show that (1) has a solution.

Notice that replacing \( x \) by \( x + \lambda^{n+1} y^\sigma - y \) does not alter whether a solution exists: for any \( a \) such that \( a^\sigma = \lambda a, \)
\[
a(\lambda^{n+1} y^\sigma - y) = \lambda^{-n}[\lambda^{n+1}(ay)^\sigma - \lambda^n ay].
\]
If we write \( x = c + d \) with \( c = \sum_{i<0} x_i t^i \) and \( d = \sum_{i\geq 0} x_i t^i \), then the equation \( \lambda^{n+1} y^\sigma - y = d \)
has a solution by Lemma 2.1. Thus without loss of generality, we may assume \( x_i = 0 \) for \( i \geq 0 \). In particular, we now have \( x \in \Gamma_{\text{con}}^{[1,1/p]} \).

We next reduce to the case where \( x \) is supported on \([-1, -1/p^n]\). Set
\[
y = \sum_{j=1}^{\infty} \sum_{k=1}^{j} \sum_{i \in [-1, -1/p^n]} \lambda^{(n+1)(k-1)} x_{ip-n-j}^\sigma + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \sum_{i \in [-1, -1/p^n]} \lambda^{-(n+1)k} x_{ip-n-j}^\sigma.
\]
The first sum is evidently convergent in \( R \). As for the second, recall that there exists constants \( c, d \) such that \( v_p(x_{i-n}) \geq cn - d \) for \( n < 0 \). Thus
\[
v_p(\lambda^{-i} x_{i-p}) \geq (i) c p^i - d - j(n+1)v_p(\lambda),
\]
and the right side grows exponentially in \( j \), so the second sum is also convergent. Thus we can replace \( x \) by
\[
x - \lambda^{n+1} y^\sigma + y = \sum_{i \in [-1, -1/p^n]} \sum_{j=-\infty}^{\infty} x_{ip-n-j}^\sigma \lambda^{(n+1)j},
\]
which is supported on \([-1, -1/p^n]\).

We now assume \( x \) is supported on \([-1, -1/p^n]\). If \( x = 0 \), then \( M \) has an eigenvector \( x \)
with \( Fx = x \), which can be multiplied by a suitable scalar to produce the desired \( y \), so we assume \( x \neq 0 \); indeed, we may assume \( |x| = 1 \). Define \( c_i \) for all \( i < 0 \) by setting \( c_i = x_i \) for
\(i \in [-1, -1/p^n]\) and extending by the rule \(c_{ip^n} = \lambda^{n+1} c_i^n\). We say an index \(i\) is a corner of \(c\) if \(|c_j| < |c_i|\) for all \(j < i\); then all corners are of the form \(jp^{-nk}\) for \(j\) in a finite set and \(k\) an arbitrary integer.

The solutions of \(a^\sigma = \lambda a\) have the form

\[
a = \sum_{i \in [1, p)} \sum_{j = -\infty}^{\infty} a_i^{\sigma j} \lambda^{-j} i^j p^j \tag{1}
\]

for \(\sum_{i \in [1, p)} a_i t^i \in R\). For such \(a\), define

\[
f(a) = \sum_{i \in [1, p^{-n}l]} t^i \sum_{j = -\infty}^{\infty} (xa)_{ip^{-n}j}^{\sigma j} \lambda^j
\]

\[
= \sum_{i \in [1, p^{-n}l]} t^i \sum_{j = -\infty}^{\infty} \sum_{k \in [-1, -1/p^n]} x_k^{\sigma j} a_{ip^{-n}j - k} \lambda^j
\]

\[
= \sum_{i \in [1, p^{-n}l]} t^i \sum_{j = -\infty}^{\infty} c_k a_{ip^{-n}j - k}
\]

Then \(f\) is additive, and \(f(a) = 0\) if and only if the equation \(\lambda b^{\sigma n} = b - \lambda^{-1} a x\) has a solution.

For \(i > 0\) and \(\alpha \in \mathcal{O}\), put \(a(\alpha, i) = \sum_{j = -\infty}^{\infty} \alpha^{\sigma j} \lambda^{-j} i^j p^j\). Write

\[
f(a(\alpha, i)) = \sum_{s \in [1, p^{-n}l]} t^s \sum_{j = -\infty}^{\infty} \lambda^{-j} c_{s - ip} a^{\sigma j}.
\]

Let \(j(i)\) be the smallest \(j\) which achieves maximum \(\sum_{s \in [1, p^{-n}l]} |\lambda^{-j} c_{s - ip}|\) for some \(s\), let \(s(i)\) be the smallest such \(s\), and let \(k(i) = s(i) - ip^{j(i)}\). Then \(k(i)\) is a corner of \(c\) if \(s(i) \neq l\). On the other hand, if \(s(i) = l\) and \(k(i)\) were not a corner, we could find \(k' < k\) with \(|c_{k'}| \geq |c_k|\). Let \(m > 0\) be the unique integer such that \(p^{-mn} (k' - ip^{j(i)}) = \lfloor l, pl \rfloor\). Then

\[
|\lambda^{-j(i) + mn} c_{p^{-mn}k' - ip^{j(i)-mn}}| = |\lambda^{-j(i) - m} c_{k' - ip^{j(i)}}| > |\lambda^{-j(i)} c_{k' - ip^{j(i)}}|,
\]

contradiction. Thus \(k(i)\) is also a corner if \(s(i) = l\), so \(k(i)\) is piecewise constant, as is \(j(i)\); therefore \(s(i)\) is piecewise linear and increasing in \(i\). Also, clearly \(s(ip) = s(i)\) and \(j(ip) = j(i) - 1\).

We claim that if \(\lim_{i' \to i^-} s(i')\) and \(\lim_{i' \to i^+} s(i')\) lie in \((l, p^{-n}l)\) for some \(i\), then \(s\) is continuous at \(i\). Let \(k_0\) and \(k_1\) be the value of \(k(i - \epsilon)\) and \(k(i + \epsilon)\), respectively, for small \(\epsilon > 0\); define \(j_0\) and \(j_1\) analogously. If \(|\lambda^{-ja} c_{k_0}| < |\lambda^{-ja} c_{k_1}|\), then \(|\lambda^{-ja} c_{s - (i+j)\epsilon p^0}| < |\lambda^{-ja} c_{k_1}|\) for \(s = k_0 + (i + \epsilon)p^0\), contradicting the definition of \(j(i + \epsilon)\). A similar contradiction arises if \(|\lambda^{-ja} c_{k_0}| < |\lambda^{-ja} c_{k_1}|\). Hence \(\lambda^{-ja} c_{k_0}\) and \(\lambda^{-ja} c_{k_1}\) have the same norm. Now by the definition
of $s$, we have $k_0 + (i - \epsilon)p^{j_0} \leq k_1 + (i - \epsilon)p^{j_1}$ and $k_0 + (i + \epsilon)p^{j_0} \geq k_1 + (i + \epsilon)p^{j_1}$. Thus $k_0 + ip^{j_0} = k_1 + ip^{j_1}$ and $s$ is continuous at $i$.

We can also show that $\lim_{i \to i+} s(i') = l$ if and only if $\lim_{i \to i-} s(i') = p^{-n}l$, but the argument is more delicate. Define $k_0, k_1, j_0, j_1$ as above. If $\lim_{i \to i-} s(i') = p^{-n}l$ but $\lim_{i \to i+} s(i') = s_1 > l$, then for $i' = i - \epsilon$ we have $|\lambda^{-j}c_{k_1}| = |\lambda^{-j}c_{s_1-ip^{j_1}}| < |\lambda^{-j}c_{k_0}|$. On the other hand, for $i' = i + \epsilon$, we can take $s = p^n(k_0 + i'p^{j_0}) < s_1$ and obtain

$$|\lambda||\lambda^{-j}c_{k_0}| = |\lambda^{-j}n_{c_{k_0}p^n}| < |\lambda^{-j}c_{k_1}|.$$  

Thus $|\lambda^{-j}c_{k_1}|$ is sandwiched between $|\lambda||\lambda^{-j}c_{k_0}|$ and $|\lambda^{-j}c_{k_0}|$, but there is no norm between these two because $\lambda$ is a uniformizer, contradiction. If $\lim_{i \to i+} s(i') = l$ but $\lim_{i \to i-} s(i) = s_0 < p^{-n}l$, then $|\lambda^{-j}c_{k_0}| = |\lambda^{-j}c_{s_0-ip^{j_0}}| \leq c_{k_1}|\lambda^{-j}|$. On the other hand, for $i' = i - \epsilon$, we can take $s = p^n(k_0 + (i - \epsilon)p^{j_0}) > s_0$ and obtain

$$|\lambda^{-1}||\lambda^{-j}c_{k_1}| = |\lambda^{-j}c_{k_1}p^{-n}| \leq |\lambda^{-j}c_{k_0}|$$

but $|\lambda^{-1}| > 1$, so again we obtain a contradiction.

Since $s(ip) = s(i)$ and $s$ is increasing and continuous except for jumps between $p^{-n}l$ and $l$, it follows that $s$ maps $[1, p)$ onto $[l, p^{-n}l)$ one or more times. Choose $r \in [1, p)$ such that $s(r) = l$; then $s$ also maps $[r, pr)$ onto $[l, p^{-n}l)$ one or more times. It follows that for any $y$ supported on $[l, p^{-n}l)$ with $|y| \leq 1$, there exists $a$ with $a^\sigma = \lambda a$ with $|a_i| \leq 1$ for $i \in [r, pr)$ such that $|y - f(a)| < 1$. This $a$ can be constructed by a transfinite recursion: find $i \in [r, pr)$ and $a \in R$ such that $f(a(\alpha, i))$ has leading coefficient congruent to the leading coefficient of $y$ modulo $\lambda$, then subtract off and repeat.

Additionally, note that $s$ must change slope at some point in $[r, pr)$ (possibly equal to $r$), since $j(r) \neq j(pr)$. If $s$ changes slope at $i$, then $f(a(\alpha, i))$ has at least two distinct terms with minimal norm. Namely, if again $k_0$ and $j_0$ (resp. $k_1$ and $j_1$) are the values of $k(i - \epsilon)$ and $j(i - \epsilon)$ (resp. $k(i + \epsilon)$ and $j(i + \epsilon)$) for $\epsilon > 0$ small, then the terms $c_{k_0}\lambda^{-j}a_{j_0}$ and $c_{k_1}\lambda^{-j}a_{j_1}$ have the same minimal norm. Note that there exists $\alpha \in \mathcal{O}$ such that the sum of the terms of minimal norm has norm less than 1 (because the residue field of $\mathcal{O}$ is algebraically closed). Thus in the transfinite recursion of the previous paragraph, there is more than one choice that can be made at $i$. In particular, there exists $a$ with $a^\sigma = \lambda a$, $|f(a)| < 1$ and $|a_i| = 1$ for some $i \in [r, pr)$.

From this analysis, we can construct a nonzero solution of $[\square]$. Start with $a^{(0)}$ such that $(a^{(0)})^\sigma = \lambda a^{(0)}$, $|f(a^{(0)})| < 1$ and $|a_1^{(0)}| = 1$ for some $i \in [r, pr)$. Now construct a sequence $(a^{(m)})_{m=0}^\infty$ such that

(a) $(a^{(m)})^\sigma = \lambda a^{(m)}$ for all $m$;

(b) $f(a^{(m)}) < |\lambda^m|$;

(c) $|a_i^{(m)} - a_i^{(m+1)}| < \lambda^m$ for $i \in [r, pr)$.

Specifically, given $a^{(m)}$, let $y = f(a^{(m)})/\lambda^m$, find $a$ such that $a^\sigma = \lambda a$ with $a_i \leq 1$ for $i \in [r, pr)$ and $|y - f(a)| < 1$, then set $a^{(m+1)} = a^{(m)} + \lambda^m a$. Then $(a^{(m)})$ converges in
the Fréchet topology to a nonzero $a$ with $a^\sigma = \lambda a$ and $f(a) = 0$. Thus (1) has a nonzero solution, and the proof is complete.

\[\Box\]

**Lemma 4.2.** Let $M$ be an $F$-crystal over $R = \Gamma_{\text{an,con}}^{\text{alg}}$ or $R = \Gamma_{\text{an,con}}^{\text{imm}}$ admitting a basis $v, w$ such that

\[Fv = \pi^m v\]
\[Fw = w + cv\]

for $\pi \in O_0$ a uniformizer, $c \in R$, and $m \geq 2$. Then there exists an eigenvector $x$ of $M$ such that $F^n x = \pi^{-m-1}x$.

**Proof.** We imitate the previous proof restricted to $n = 1$. For starters, given $x \in R$ supported on $[-1, -1/p]$ with $|x| = 1$, it again suffices to show that the equations

\[a^\sigma = \pi^{m-1}a, \quad \pi \sigma = b - \pi^{-1}ax\]

have a solution with $a, b \in R$ not both zero (the case $x = 0$ is self-evident). Define $c_i$ for all $i < 0$ by setting $c_i = x_i$ for $i \in [-1, -1/p^m)$ and extending by the rule $c_{ip^m} = \lambda^{n+1}c_i^\sigma$. Again, we say an index $i$ is a corner of $c$ if $|c_j| < |c_i|$ for all $j < i$. For a solution of $a^\sigma = \pi^{m-1}a$, define

\[f(a) = \sum_{i \in [l, l/p)} t^i \sum_{k < 0} c_k a_{i-k};\]

it suffices to exhibit $a$ such that $f(a) = 0$.

Continuing to imitate the previous proof, for $i > 0$, define $j(i), k(i), s(i)$ by taking $j(i)$ as the smallest $j$ for which $\max_{s,j}\{|\pi^{-(m-1)}c_{s-ijp^m}\}$ is achieved, $s(i)$ as the smallest $s$ for which the maximum is achieved with $j = j(i)$, and $k(i) = s(i) - ip^j(i)$. Then as before, $k(i)$ and $j(i)$ are piecewise constant and so $s(i)$ is increasing and piecewise linear.

We again prove that $\lim_{i' \to i^+} s(i')$ and $\lim_{i' \to i^-} s(i')$ are either equal, or equal to $l$ and $l/p$, respectively. The proof that if both limits lie in $(l, l/p)$, then they are equal, carries through as before, as does the proof that we cannot have $\lim_{i' \to i^+} s(i') = l$ and $\lim_{i' \to i^-} s(i') < l/p$. Now suppose $\lim_{i' \to i^-} s(i') = l/p$ but $\lim_{i' \to i^+} s(i') = s_1 > l$. For $i' = i - \epsilon$ we have $|\pi^{-(m-1)i}c_{k_1}| = |\pi^{-(m-1)j}c_{s_1-ip^j}| < |\pi^{-(m-1)j}c_{k_0}|$. On the other hand, for $i' = i + \epsilon$, we can take $s = p(k_0 + i'p^j) < s_1$ and obtain

\[|\pi||\pi^{-(m-1)j}c_{k_0}| = |\pi^{-(m-1)j}c_{k_0}| < |\pi^{-(m-1)j}c_{k_1}|.\]

Thus $|\pi^{-(m-1)j}c_{k_1}|$ is sandwiched between $|\pi||\pi^{-(m-1)j}c_{k_0}|$ and $|\pi^{-(m-1)j}c_{k_0}|$, but there is no norm between these two because $\pi$ is a uniformizer, contradiction.

Given the results of the previous paragraph, the rest of the proof proceeds as in the previous lemma. Namely, a transfinite recursion can be used to generate a solution of $f(a) = 0$, which completes the proof.

\[\Box\]
5 Construction of the special Newton polygon

This section is devoted to the proof of the following theorem, the main result of this paper. For \( n \) a positive integer and \( s \) a rational number such that \( sn \) is a valuation of an element \( \lambda \in \mathcal{O} \), let \( M_{n,s} \) denote the crystal over \( \Omega \) whose action of Frobenius on a basis \( v_1, \ldots, v_n \) is given by \( Fv_i = v_{i+1} \) for \( i = 1, \ldots, n - 1 \) and \( Fv_n = \lambda v_1 \). We say a crystal is standard if it is principal, so we may choose \( \lambda \in \mathcal{O} \) and \( s \in \mathbb{Q} \) with \( \lambda \in \mathcal{O} \), and \( s \) an element of an \( \mathcal{O} \)-crystal over a ring \( \mathcal{O} \). The analogous statement for \( (\rho, \lambda) \)-crystals follows by a similar argument; therefore the slopes of \( \mathcal{O} \)-crystals over \( \mathcal{O} \) are the same up to a permutation.

Corollary 5.2. Let \( M \) be an \( F \)-crystal over \( R \) spanned by eigenvectors. Then \( M \) splits as a direct sum of standard subcrystals.

By Lemma 5.3 below, the slopes of the eigenvectors that form a basis of \( M \) do not depend on the choice of basis. We call these the special slopes of \( M \), and we define the special Newton polygon of \( M \) as the convex polygon whose slopes are the special slopes of \( M \); we will catalog its basic properties in the next section.

Lemma 5.3. Suppose \( D \) and \( E \) are diagonal matrices over \( \mathcal{O} \) and \( U \) is an invertible matrix over \( \Gamma_{\text{an,con}}^{\text{imm}} \) such that \( U^{-1}DU^\sigma = E \). Then the slopes (valuations of the diagonal entries) of \( D \) and \( E \) are the same up to a permutation.

Proof. Since \( U \) has nonzero determinant, we can find a permutation matrix \( V \) such that \( UV \) has nonzero diagonal entries. Put \( W = UV \) and \( F = V^{-1}EV \); then \( F \) is diagonal and its entries are a permutation of those of \( E \). From \( DV^\sigma = VF \) we have \( D_{ii}V^\sigma_{ii} = V_{ii}F_{ii} \) for each \( i \); since \( V_{ii} \) is nonzero, this implies \( |D_{ii}| \leq |F_{ii}| \). In particular, the \( k \)-th largest slope of \( D \) is greater than or equal to the \( k \)-th largest slope of \( E \) for each \( k \). The analogous statement with \( D \) and \( E \) reversed follows by a similar argument; therefore the slopes of \( D \) and \( E \) are equal up to permutation.

For \( v \) an element of an \( F \)-crystal over a ring \( R \), the ideal generated by the coordinates of \( v \) in some basis is independent of the choice of basis; we call this ideal the coordinate ideal of \( v \). We say \( v \) is primitive if its coordinate ideal is the trivial ideal. Equivalently, \( v \) is primitive if and only if it can be extended to a basis of \( R \).

Lemma 5.4. Let \( M \) be an \( F \)-crystal over \( \Gamma_{\text{an,con}} \) for \( R = \Gamma_{\text{alg}} \) or \( R = \Gamma_{\text{imm}} \) and \( v \in M \) a nonzero element such that \( Fv = \mu v \). Then \( v \) is a multiple of a primitive eigenvector of \( M \). In particular, if \( M \) has no eigenvectors of slope less than \( v_p(\mu) \), then \( v \) is primitive.

Proof. Let \( I \) be the coordinate ideal of \( v \). By Lemma 3.13, \( I \) is principal, so we may choose a generator \( r \). Since \( Fv = \mu v \), the ideal \( I \) is invariant under \( \sigma \) and \( \sigma^{-1} \), so \( r^\sigma = cr \) for \( c \) a unit in \( R \). By Lemma 3.1 we can write \( c = \lambda d \) with \( \lambda \in \mathcal{O} \) and \( d \) a unit in \( R \). The equation \( t^\sigma = dt \) has a solution with \( t \in R \), and \( s = r/t \) satisfies \( s^\sigma = \lambda s \) and generates \( I \). Therefore there exists \( w \in M \) with \( v = sw \), \( w \) primitive, and \( Fw = (\mu/\lambda)w \), as desired.

16
Beware that there may be primitive eigenvectors which cannot be extended to a basis consisting solely of eigenvectors. For example, there is always an eigenvector of slope equal to the largest generic slope, which usually does not extend to a basis of eigenvectors. However, it will turn out that the eigenvector of minimum slope will always extend to a basis of eigenvectors.

Proof of Theorem 5.1. We proceed by induction on \( n = \dim M \). For \( n = 1 \), \( F \) acts on a basis vector by an invertible scalar, i.e. an element of \( \Gamma_{\text{con}}^\text{imm} [\frac{1}{p}] \). Without loss of generality, we may assume this scalar is in \( \Gamma_{\text{con}}^\text{imm} \) and has norm 1, in which case the result follows from Lemma 2.1.

Now suppose \( n > 1 \). We are done if \( M \) is isomorphic to \( M_{n,d/n} \), so we assume that this does not occur. Let \( d \) be the slope of \( \wedge^n M \). For \( r \) a rational number, define the \( \mathcal{O} \)-index of \( r \) as the smallest positive integer \( s \) such that \( rs \) is a valuation of \( \mathcal{O} \).

Since the set of rationals of \( \mathcal{O} \)-index less than \( n \) is discrete, there is a smallest such rational that occurs as the slope of an eigenvector of \( M \) over a suitable extension of \( \mathcal{O} \); call this number \( r \). Let \( m \) be the \( \mathcal{O} \)-index of \( r \), and let \( \lambda \in \mathcal{O}_0 \) have valuation \( rm \). Let \( v \) be an eigenvector of \( M \) over \( R[\lambda^{1/m}] \) with \( Fv = \lambda^{1/m}v \). Write \( v = \sum_{i=0}^{m-1} w_i\lambda^{-i/m} \), so that each \( w_i \) is an element of \( M \) over \( R \) with \( F^m w_i = \lambda w_i \), and let \( M_1 \) be the span of \( w_0, \ldots, w_{m-1} \) within \( M \).

Since \( \dim M_1 \leq m < n \), we may apply the induction hypothesis to \( M_1 \). If \( M_1 \) has more than one standard summand, it has an eigenvector of slope less than or equal to \( r \) with \( \mathcal{O} \)-index strictly less than \( m \). Thus \( M \) has an eigenvector of slope strictly less than \( r \) with \( \mathcal{O} \)-index less than \( n \), contradiction. Thus \( M_1 \) itself is standard; specifically, it must be isomorphic to \( M_{m,r} \) (in particular, \( \dim M_1 \) must equal \( m \)). Likewise, \( M/M_1 \) has a direct sum decomposition \( N_2 \oplus \cdots \oplus N_k \) of the specified type.

Let \( P_i \) be the preimage of \( N_i \) in \( M \); to complete the proof, it suffices to show that the exact sequence \( 0 \to M_1 \to P_i \to N_i \to 0 \) splits for \( i = 2, \ldots, k \). First suppose \( k > 2 \). Then the dimension of each \( P_i \) is less than \( n \), so we can apply the induction hypothesis to \( P_i \). If the slope of \( N_i \) were less than that of \( M_1 \), then the induction hypothesis would imply that \( P_i \) has a slope less than \( r \) of \( \mathcal{O} \)-index less than or equal to \( \dim P_i \leq n \), contradicting the minimality of \( r \). Thus the slope of \( N_i \) is greater than or equal to that of \( M_1 \). In this case, Lemma 2.1 can be used to show that the exact sequence splits. Namely, by imitating the argument at the beginning of the proof of Lemma 1.1, we can reduce this splitting to the existence of solutions of equations of the form \( \lambda a^d - \mu a = x \), where \( d \) is the least common multiple of \( \dim M_1 \) and \( \dim N_1 \), and \( |\lambda| > |\mu| \). Then Lemma 2.1 implies that each of these equations has a solution.

The case \( k = 2 \) requires special scrutiny, as \( P_2 = M \) and the induction hypothesis does not apply. Let \( s \) be the slope of \( N_2 \). As above, the exact sequence splits if \( r \leq s \), so assume on the contrary that \( r > s \); this implies in particular that \( r > d/n \). If \( n = 2 \), we immediately obtain a contradiction from Lemma 1.2, so we may assume \( n > 2 \).

We show that \( M \) has an eigenvector of slope less than or equal to \( d/n \) over some finite extension of \( \mathcal{O} \). Pick an eigenvector \( v \) of slope \( r \), and apply the induction hypothesis to the
quotation of \( M \) by the span of this eigenvector. This yields an eigenvector \( w \) of the quotient of slope at most \((d - r)/(n - 1)\). Applying the induction hypothesis again, this time to the preimage of \( w \), gives an eigenvector of \( M \) of some slope \( r' \leq (r + (d - r)/(n - 1))/2 \). Let \( O^1 \) be a finite extension of \( O \) whose value group contains \( r' \), and let \( r_1 \) be the smallest rational of \( O^1 \)-index less than \( n \) that occurs as the slope of an eigenvector. Again, take the quotient of \( M \) by the span of an eigenvector of slope \( r_1 \), this time over \( O^1 \) and apply the induction hypothesis. If its slopes are not all equal, we deduce that \( M \) has the desired splitting over \( O^1 \), and in particular has an eigenvector of slope less than or equal to \( d/n \) over \( O^1 \). Otherwise, we can repeat the process to produce an extension \( O^2 \) of \( O \), and the smallest rational \( r_2 \) of \( O^2 \)-index less than \( n \) occurring as the slope of an eigenvector will be at most \((r_1 + (d - r_1)/(n - 1))/2 \), and so on.

The existence of an eigenvector of slope at most \( d/n \) is assured if the above process ever terminates, so assume it continues indefinitely. Then the sequence \( \{r_n\} \) converges to \( d/n \), as it is sandwiched between \( d/n \) and a sequence obtained by iterating \( r \mapsto (r + (d - r)/(n - 1))/2 \), and the latter converges to \( d/n \); therefore, there exist eigenvectors of \( M \) of every rational slope greater than \( d/n \). Let \( O' \) be an extension of \( O \) whose value group contains \( d/n \), and let \( v_0 \) be its minimum positive valuation. Take an eigenvector \( v \) of \( M \) of slope \( d/n + v_0/(n - 1) \) over an extension of \( O' \) of degree \( n - 1 \). The span of \( v \) over \( O \) has dimension at most \( n - 1 \) and sum of slopes at most \((n - 1)d/n + v_0 \). Apply the induction hypothesis to the span; if the sum of slopes is not equal to \((n - 1)d/n + v_0 \), then it is at most \((n - 1)d/n \), and the span contains an eigenvector of slope at most \( d/n \). If the sum of slopes is equal to \((n - 1)d/n + v_0 \) but the slopes are not all equal, and \( t \) is the least slope and its multiplicity is \( m \), then \( mt < md/n + mv_0/(n - 1) \); since \( mt \) and \( md/n \) are integral multiples of \( v_0 \), we deduce \( mt \leq md/n \) and the span again contains an eigenvector of slope at most \( d/n \). Finally, if all of the slopes of the span are equal to \( d/n + v_0/(n - 1) \), then Lemma 4.1 implies that \( M \) has an eigenvector of slope \( d/n \). Thus in all cases, \( M \) has an eigenvector of slope less than or equal to \( d/n \).

Let \( \lambda \) be an element of a finite extension of \( O \) of valuation \( d/n \). Because \( M \) has an eigenvector of slope less than or equal to \( d/n \), it must also have one of slope equal to \( d/n \), over some finite extension of \( O[\lambda] \). In fact, from a solution of \( Fv = \lambda v \) over a finite extension of \( O[\lambda] \), we can obtain a solution over \( O[\lambda] \): decompose the solution over a basis of the finite extension over \( O[\lambda] \), and choose any nonzero component. Thus we may assume \( v \) is defined over \( O[\lambda] \).

Let \( m \) be the \( O \)-index of \( d/n \); then we can write \( v = \sum_{i=0}^{m-1} \lambda^{-i} w_i \). Let \( N \) be the span of \( w_0, \ldots, w_{m-1} \). If \( \dim N < n \), then we can apply the induction hypothesis to \( N \) to express it as a direct sum of standard subcrystals. The projection of \( w_0 \) onto at least one of these subcrystals must be nonzero, and since \( F^m w_0 = \lambda^m w_0 \), the same equation holds for the projection of \( w_0 \). Thus this subcrystal has an eigenvector of slope \( d/n \); its slope must then be less than \( d/n \). Since this slope has \( O \)-index at most \( m \leq n \), we conclude \( r \leq d/n \), contradiction. On the other hand, if \( \dim N = n \), then \( N \) is isomorphic to \( M_{n,d/n} \). Moreover, \( w_0 \wedge \cdots \wedge w_{n-1} \) is an eigenvector of \( \wedge^n M \) of slope \( d \), so must be primitive. Thus \( N = M \) is isomorphic to \( M_{n,d/n} \), contrary to an earlier assumption. We conclude that the assumption
$r > s$ leads to a contradiction in all cases, so we must have $r \leq s$ and so the exact sequence $0 \rightarrow M_1 \rightarrow P_2 \rightarrow N_2 \rightarrow 0$ splits.

In summary, we have that each $P_i$ splits as a direct sum of $M_1$ with another summand; now $M$ splits as a direct sum of these other summands with $M_1$, as desired.

We do not know whether or not an arbitrary $F$-crystal over $\Gamma_{\text{an,con}}^{\text{imm}}$ is spanned by eigenvectors, and hence has a basis of eigenvectors. Indeed, if it were known that every $F$-crystal over $\Gamma_{\text{an,con}}^{\text{imm}}$ has a nonzero eigenvector, it would follow by induction that every $F$-crystal is spanned by eigenvectors.

6 Properties of the Newton polygons

Unless otherwise specified throughout this section, let $M$ be an $F$-crystal over $\Gamma_{\text{con}}^{\text{imm}}$. In this section, we establish that the special and generic Newton polygons satisfy some relations that one would expect from the case of potentially semistable $(F, \nabla)$-crystals over $\Gamma_{\text{con}}^{\text{imm}}$.

**Proposition 6.1.** Let $M$ be an $F$-crystal over $\Omega$. Then the special Newton polygon of $M$ equals the Newton polygon of the reduction of $M$ modulo $t$.

This follows immediately from Dwork’s trick [6, Lemma 4.3]. Beware that the natural generalization of this proposition to $\Omega$ is false.

**Proposition 6.2.** If $\ell_1 \leq \cdots \leq \ell_n$ are the special slopes of $M$, the special slopes of $\wedge^k M$ (for $k = 0, \ldots, m$) are given by $\ell_{i_1} + \cdots + \ell_{i_k}$ for $1 \leq i_1 < \cdots < i_k \leq m$.

**Proof.** If $v_1, \ldots, v_n$ form a basis of eigenvectors of $M \otimes_{\Gamma_{\text{con}}^{\text{imm}}}^{\Gamma_{\text{an,con}}^{\text{imm}}} \Gamma_{\text{an,con}}^{\text{imm}}$ with $Fv_i = \lambda_i v_i$, then

$$F(v_{i_1} \wedge \cdots \wedge v_{i_k}) = \lambda_{i_1} \cdots \lambda_{i_k} v_{i_1} \wedge \cdots \wedge v_{i_k},$$

so $v_{i_1} \wedge \cdots \wedge v_{i_k}$ is an eigenvector of slope $v_p(\lambda_{i_1}) + \cdots + v_p(\lambda_{i_k})$.

Similarly, the special slopes of $M_1 \oplus M_2$ are the union of the special slopes of $M_1$ and $M_2$, and the special slopes of $M_1 \otimes M_2$ are the products of the special slopes of $M_1$ and $M_2$.

**Proposition 6.3.** The special Newton polygon lies above the generic Newton polygon and has the same endpoints.

**Proof.** By the previous proposition, it suffices to prove that the highest special slope is no greater than the highest generic slope. Let $v_1, \ldots, v_n$ be a basis of eigenvectors of $M \otimes_{\Gamma_{\text{an,con}}^{\text{imm}}}^{\Gamma_{\text{an,con}}^{\text{imm}}} \Gamma_{\text{an,con}}^{\text{imm}}$ and let $w$ be an eigenvector of $M$ of highest slope over $\Gamma_{\text{an,con}}^{\text{imm}}$, which exists by [7, Proposition 2.1]. Write $w = \sum c_i v_i$ with $c_i \in \Gamma_{\text{an,con}}^{\text{imm}}$. If $Fw = \lambda w$ and $Fv_i = \mu_i v_i$, then for $i = 1, \ldots, n$, we have $c_i^2 \mu_i = c \lambda$, so $c_i = (\lambda/\mu_i)^{1/2}$. The latter equation only has solutions in $\Gamma_{\text{an,con}}^{\text{imm}}$ if $|\lambda/\mu_i| \leq 1$. Thus each special slope is less than or equal to the highest generic slope.
Proposition 6.4. The vertices of the special Newton polygon are of the form \((i, j)\), where \(i\) is an integer and \(j\) is an integral multiple of the smallest positive valuation in \(\mathcal{O}\).

Proof. This follows immediately from Theorem 5.1. \(
\)

Proposition 6.5. Let \(M\) be an \(F\)-crystal over \(\Gamma_{\text{con}}\) whose generic and special Newton polygons coincide. Then \(M\) becomes unipotent over \(\Gamma_{\text{sep}} \otimes \mathcal{O}'\) for an extension \(\mathcal{O}'\) of \(\mathcal{O}\) whose value group contains the slopes of \(M\).

Proof. It suffices to show that the eigenvectors of \(M\) of lowest slope are defined over \(\Gamma_{\text{sep}}\). Let \(v_1, \ldots, v_n\) be eigenvectors of \(M\) over \(\Gamma_{\text{alg}}\), with \(Fv_i = \lambda_i v_i\) for \(\lambda_i \in \mathcal{O}_0\) such that \(|\lambda_1| \leq \cdots \leq |\lambda_n|\). By the descending slope filtration for \(F\)-crystals [7, Lemma 2.1], we can find elements \(w_1, \ldots, w_n\) of \(M\) over \(\Gamma_{\text{alg}}\) such that \(Fw_i = \lambda_i w_i + \sum_{j<i} A_{ij} w_j\) for some \(A_{ij} \in \Gamma_{\text{alg}}\).

Write \(v_n = \sum_i b_i w_i\) with \(b_i \in \Gamma_{\text{alg}}\), then applying \(F\) to both sides, we have \(\lambda_i b_i = \lambda_i b_i^\sigma + \sum_{j<i} b_j^\sigma A_{ji}\) for \(i = 1, \ldots, n\). For \(i = n\), this implies \(b_n \in \mathcal{O}_0\). By Lemma 2.1 and descending induction on \(i\), we obtain \(b_i \in \Gamma_{\text{alg}}\) for \(i = n-1, \ldots, 1\), and so \(v_n\) is defined over \(\Gamma_{\text{alg}}\). On the other hand, by the ascending slope filtration [7, Proposition 2.2], the eigenvectors \(v\) of \(M\) over \(\Gamma_{\text{alg}}\) satisfying \(Fv = \lambda_n v\) are defined over \(\Gamma_{\text{sep}}\). Thus \(v_n\) is defined over \(\Gamma_{\text{alg}} \cap \Gamma_{\text{sep}} = \Gamma_{\text{sep}}\). Since \(v_n\) could have been taken to be any eigenvector over \(\Gamma_{\text{alg}}\) of lowest slope, this proves the claim. \(\square\)

Corollary 6.6. Let \(M\) be an \(F\)-crystal over \(\Gamma_{\text{con}}\) whose generic and special Newton polygons coincide. Then \(M\) has a filtration \(0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_l = M\) by subcrystals such that \(M_i/M_{i-1}\) is isoclinic for \(i = 1, \ldots, n\).

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