Inner Product in Quantum Field Theory

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Abstract

In this paper we will analyse the inner product for a general tensor field theory. We will first analyse a generalized inner product for scalar field theories. Then we will use it to construct a inner product for tensor field theories. We will use this inner product to construct the two-point function.

1 Introduction

Now we also know that in nature there are four fundamental forces. Three of those four fundamental forces are described by gauge fields with compact gauge group, and is gravity, which is not a gauge theory with a compact gauge group. But we can also regard gravity as a gauge theory of diffeomorphism \([?]\). In this sense all the forces of nature can be analyzed in the framework of gauge theory \([1]-[13]\). The quantization of any theory with gauge symmetry can be done using the BRST approach \([14]-[23]\). It has been argued that quantization of perturbative gravity in the framework of quantum field theory might not work since gravity is non-renormalizable. However in the light of effective field theory there is no fundamental difference between renormalizable and non-renormalizable theories except the dependence on lower energy scale. As gravity is non-renormalizable, so at any energy scale we will get new arbitrary constants for perturbative quantum gravity. But we can use perturbative quantum gravity safely at those energies where we have measured all these arbitrary constants.

With this approach it makes sense to try to look for properties of quantum gravity by using methods of quantum field theory. To study the behavior of all these forces in the inflationary era it will be essential to study quantum field theory on de Sitter spacetime. If the universe is asymptotically approaches de Sitter spacetime, then we need to study quantum field theory in curved spacetime also. Quantum field theory on curved spacetime is also important in analyzing the behavior of black holes. It is only by using quantum field theory on curved spacetime that Hawking radiation from black holes is predicted. Furthermore, there are many more interesting applications of quantum field theory in curved spacetime \([25]-[73]\). The Unruh effect is a famous application of quantum field theory in curved spacetime. It predicts that an accelerating observer will observe black-body radiation where an inertial observer would observe none. In other words, the background appears to be warm from an accelerating reference
frame. The ground state for an inertial observer is seen as in thermodynamic equilibrium with a non-zero temperature by the uniformly accelerated observer. So, we need to analyse field theory on curved spacetime. However, in order to do that we need to derive a generalized inner product in curved spacetime. This can be used to construct a two-point function in curved spacetime. This is what we will do in this paper.

2 Scalar Field Theory

We start with a scalar field theory with the Lagrangian $L$ and the action $S$. If we minimize this action

$$\delta S = 0,$$

we will get Euler-Lagrange equation of motion as follows:

$$\frac{\partial L}{\partial \phi} - \nabla_c \frac{\partial L}{\partial \nabla_c \phi} = 0.$$

Now if the free scalar field Lagrangian $L$ given by

$$L = \nabla_a \phi \nabla^a \phi + m^2 \phi^2,$$

then the classical equation of motion will given by

$$(\nabla^2 - m^2)\phi(x) = 0.$$

Now we can define a quantity called conjugate momentum current $\pi^c$ as follows:

$$\pi^c = \frac{1}{\sqrt{-g}} \frac{\partial L}{\partial \nabla_c \phi}.$$

Thus we have

$$\pi^c = -\nabla^c \phi.$$

If $\phi_1, \phi_2$ are two solutions of the field equations, and $\pi^c_1, \pi^c_2$ the conjugate momentum currents conjugate to them, then we have

$$\pi^c_1 = -\nabla^c \phi_1,$$

$$\pi^c_2 = -\nabla^c \phi_2.$$

We also define a current $J^c_{(\phi_1, \phi_2)}$ as follows:

$$J^c = i[\phi^*_1 \pi_2^c - \phi_2 \pi_1^c].$$

Now we can write the field equations by using the definition of $\pi^c$ as

$$\nabla_c \pi^c + m^2 \phi^2 = 0.$$

Then $\nabla_c J^c$ can be shown to vanish:

$$\nabla_c J^c = i[\nabla_c [\phi^*_1 \pi_2^c - \phi_2 \pi_1^c]]$$

$$= i[\nabla_c \phi^*_1 \pi_2^c - \nabla_c \phi_2 \pi_1^c + \phi^*_1 \nabla_c \pi_2^c - \phi_2 \nabla_c \pi_1^c]$$

$$= i[\pi^c_{12} \pi^c_2 - \pi^c_1 \pi^c_{2c} + m^2 (\phi^*_1 \phi_2 - \phi^*_2 \phi_1)] = 0.$$
Thus the current $J^c$ is conserved.

We define an inner product on a space-like hyper-surface $\Sigma_c$ as follows:

$$ (\phi_1, \phi_2) = \int d\Sigma_c J^c_{(\phi_1, \phi_2)}. \quad (12) $$

Let us consider the following metric for simplicity

$$ ds^2 = -N^2 dt^2 + \gamma_{ij} dx^i dx^j. \quad (13) $$

If we define $n_c = (N, 0)$ as the past pointing unit normal to the hyper-surface $\Sigma_c$ then, we have

$$ n_c n_c = g^{ab} n_a n_b = \frac{-1}{N^2} N^2 = -1. \quad (14) $$

Now we can write the inner product as follows:

$$ (\phi_1, \phi_2) = \int d^3 x \sqrt{\gamma} n_c J^c $$

$$ = \int d^3 x \sqrt{N} J^0 $$

$$ = \int d^3 x \sqrt{-g} J^0. \quad (15) $$

Now note that

$$ \frac{d}{dt}(\phi_1, \phi_2) = \int d^3 x \partial_0 (\sqrt{-g} J^0). \quad (16) $$

We have shown that $\nabla_c J^c$ vanishes,

$$ \nabla_c J^c = \frac{1}{\sqrt{-g}} [\partial_0 (\sqrt{-g} J^0) + \partial_i (\sqrt{-g} J^i)] = 0. \quad (17) $$

We can also show by Gauss divergence theorem that

$$ \int d^3 x \partial_i (\sqrt{-g} J^i) = 0. \quad (18) $$

So we get

$$ \int d^4 x \partial_0 (\sqrt{-g} J^0) = 0. \quad (19) $$

So this inner product does not vary with time.

Let $\phi_n$ and $\phi_n^*$ be a complete set of solutions to the field equations, then by definition we can expand $\phi$ as follows:

$$ \phi = \sum_n [a_n \phi_n + a_n^* \phi_n^*]. \quad (20) $$

Here the sum is a shorthand notation and may contain integrals as well, for non-compact spacetime.

We also can expand $\pi^c$ in modes as follows:

$$ \pi^c = \sum_n [a_n \pi_n^c + a_n^* \pi_n^c]. \quad (21) $$
Here $\pi^c_n$ and $\pi^{*c}_n$ are given by
\[ \pi^c_n = -\nabla^c \phi_n \] (22)
and
\[ \pi^{*c}_n = -\nabla^c \phi^*_n. \] (23)
So we have
\[ \pi^c = \sum_n \left[ -a_n \nabla^c \phi_n - a^*_n \nabla^c \phi^*_n \right]. \] (24)
We suppose
\[ (\phi_n, \phi^*_m) = 0 \] (25)
and
\[ (\phi_n, \phi_m) = M_{nm}. \] (26)
In quantum field theory when $\phi$ is promoted to an operator $\hat{\phi}$, $a^*_n$ and $a_n$ become creation operators $a^*_n$ and annihilation operators $a_n$, respectively. Thus we have
\[ \hat{\phi} = \sum_n [a_n \phi_n + a^*_n \phi^*_n]. \] (27)
Now as $\pi^c$ is also promoted to an operator $\hat{\pi}^c$, we also have
\[ \hat{\pi}^c = \sum_n \left[ -a_n \nabla^c \phi_n - a^*_n \nabla^c \phi^*_n \right]. \] (28)
Now the state $|0\rangle$ is the state annihilated by $a_n$
\[ a_n |0\rangle = 0. \] (29)
This is called the vacuum state of the theory. Many particle states can be built by repeated action of $a^*_n$ on the vacuum state. It may be noted that as the division between $\phi$ and $\phi^*$ is not unique, there will be non-uniqueness in the definition of the vacuum state also [?].

3 Two-Point Function

Now the two-point function is given by
\[ G(x, x') = \langle 0 | \phi(x) \phi(x') | 0 \rangle. \] (30)
This can be written as
\[
G(x, x') = \sum_{n,m} \langle 0 | (a_n \phi_n + a^*_n \phi^*_n)(a_m \phi_m a^*_m \phi_m | 0 \rangle
= \sum_{n,m} \phi_n \phi^*_m \langle 0 | a_n a^*_m | 0 \rangle
= \sum_{n,m} \phi_n \phi^*_m \langle 0 | [a_n, a^*_m]|0 \rangle \] . (31)
where $[a_n, a^*_m]$ is the commutator and thus given by
\[ [a_n, a^*_m] = a_n a^*_m - a^*_m a_n. \] (32)
To calculate the two-point function explicitly we need to calculate the effect of the commutator of the creation and annihilation operators on vacuum states. To do so we define $C_{nm}$ as follows:

$$C_{nm} = \langle 0 | [a_n, a_m^\dagger] | 0 \rangle. \quad (33)$$

Then we have

$$G(x, x') = \sum_{nm} \phi(x) \phi(x') C_{nm}. \quad (34)$$

Now we can have

$$M_{nm} = (\phi_n, \hat{\phi}_m) = \int d^3x \sqrt{-g} J^0_{(\phi_n, \hat{\phi}_m)}$$

and

$$M_{mn} = (\hat{\phi}_m, \phi_n) = \int d^3x \sqrt{-g} J^0_{(\hat{\phi}_m, \phi_n)} \quad (35, 36)$$

where

$$J^0_{(\phi_n, \hat{\phi}_m)} = i [\phi^*_n \pi^0_m - \phi_m \pi_n^0]. \quad (37)$$

and

$$J^0_{(\hat{\phi}_m, \phi_n)} = i [\hat{\phi}^*_m \pi_n^0 - \phi_n \pi^*_m]. \quad (38)$$

Now as

$$(i[\phi^*_n \pi^0_m - \phi_m \pi_n^0])^* = i[\phi^*_m \pi^0_n - \phi_n \pi^*_m], \quad (39)$$

we have

$$M_{nm} = M^*_{mn}. \quad (40)$$

We also have

$$[(\phi_n, \hat{\phi}), (\hat{\phi}, \phi_m)] = \int d^3x d^3x' \sqrt{-g(x)} \sqrt{-g(x')} [J^0_{(\phi_n, \hat{\phi})}, J^0_{(\hat{\phi}, \phi_m)}], \quad (41)$$

where $J^0_{(\phi_n, \hat{\phi})}$ and $J^0_{(\hat{\phi}, \phi_m)}$ are given by

$$J^0_{(\phi_n, \hat{\phi})} = i [\phi^*_n \pi^0_m - \phi_m \pi_n^0](t, x) \quad (42)$$

and

$$J^0_{(\hat{\phi}, \phi_m)} = i [\hat{\phi}^*_m \pi_n^0 - \phi_n \pi^*_m](t, x'). \quad (43)$$

As $\phi$ and $\pi$ are hermitian, we can write

$$J^0_{(\hat{\phi}, \phi_m)} = i [\hat{\phi} \pi^0_m - \phi_m \pi^*_0](t, x'). \quad (44)$$

Now as

$$[\hat{\phi}(t, x), \hat{\pi}^0(t, x')] = i \delta(x, x') \quad (45)$$

and

$$[\hat{\phi}(t, x), \hat{\pi}^0(t, x')] = [\hat{\pi}(t, x), \hat{\phi}^0(t, x')] = 0, \quad (46)$$

we have

$$[(\phi_n, \hat{\phi}), (\hat{\phi}, \phi_m)] = i \int d^3x \sqrt{-g(x)} [\phi^*_n \pi^0_m - \phi_m \pi^*_n]. \quad (47)$$
Now, as

\[ i[\phi^*_n \pi^0_m - \phi_m \pi^0_n] = J^0_{(\phi_n, \phi_m)}, \quad (48) \]

we get

\[ [(\phi_n, \hat{\phi}), (\hat{\phi}, \phi_m)] = (\phi_n \phi_m) = M_{nm}. \quad (49) \]

Now we have

\[
(\phi_n, \hat{\phi}) = \sum_k (\phi_n, a_k \phi_k) \\
= \sum_k a_k (\phi_n, \phi_k) \\
= \sum_k a_k M_{nk}
\]

and

\[
(\hat{\phi}, \phi_m) = [(\phi_m, \hat{\phi})]^\dagger \\
= \sum_l [(\phi_m, a_l^\dagger \phi_l)]^\dagger \\
= \sum_l a_l^\dagger (\phi_m, \phi_l)^* \\
= \sum_l a_l^\dagger M_{ml} \\
= \sum_l a_l^\dagger M_{lm}. \quad (51)\]

So we get

\[ [[(\phi_n, \hat{\phi}), (\hat{\phi}, \phi_m)] = \sum_{kl} M_{nk} [a_k, a_l^\dagger] M_{lm}. \quad (52)\]

Thus we can write,

\[ \sum_{kl} M_{nk} [a_k, a_l^\dagger] M_{lm} = M_{nm}. \quad (53) \]

So we have,

\[ \sum_{kl} M_{nk} C_{kl} M_{lm} = M_{nm}. \quad (54) \]

We can write this equation in matrix notation as

\[ MCM = M. \quad (55) \]

So we have

\[ C = M^{-1}. \quad (56) \]

Now the two-point function is given by

\[ G(x, x') = \sum_{nm} \psi_n \psi_m' M_{nm}^{-1}. \quad (57) \]
4 Tensor Fields

In this section we will formally generalize what we did for scalar fields to general non-interacting spin fields. Let us denote the tensor field by a shorthand notation $A_{bcde...} = A_I$. The Lagrangian for this field $A_I$ will be a scalar function of $A_I$ and $\nabla_c A_I$. Here again we will not consider higher derivatives as they will again lead to non-unitary quantum field theory. In general, the Lagrangian for higher spin fields might be invariant under some gauge transformation and so we need to add some gauge fixing term. Thus the Lagrangian for a general tensor field can be written as follows:

$$\mathcal{L} = -\sqrt{-g}[\mathcal{L}_1 + \frac{\alpha}{2}\mathcal{L}_2],$$

(58)

where $\mathcal{L}_1$ is the original Lagrangian and $\mathcal{L}_2$ is the contribution coming from the gauge fixing term.

The action $S$ is given by

$$S = \int d^4x \mathcal{L}.$$  

(59)

If we minimize this action we get the equations of motion.

We can define $\pi^{Ic}$ here as we did in the scalar case

$$\pi^{Ic} = \frac{1}{\sqrt{-g}} \frac{\partial \mathcal{L}}{\partial \nabla_c A_I}. $$

(60)

Now if $A_{I1}$ and $A_{I2}$ are two solutions to the field equations then we can define the current $J^c_{(A_{I1}, A_{I2})}$ as follows:

$$J^c = [iA^*_{I1} \pi^{Ic} - A_{I2} \pi^{Ic}]. $$

(61)

This current can again be shown to be conserved by repeating the argument for scalar field:

$$\nabla_c J^c = 0.$$ 

(62)

Now we can define an inner product on a space-like hyper-surface $\Sigma_c$ as follows:

$$(A_1, A_2) = \int d\Sigma_c J^c_{(A_1, A_2)}.$$  

(63)

The inner product here too becomes

$$(A_1, A_2) = \int d^3x \sqrt{-g} J^0.$$  

(64)

We can again show that this inner product does not change with time by following a similar line of argument to what was done in the scalar case. Now if $A_{In}$ and $A^*_{In}$ are a complete set of solutions to the classical equations of motion then we can expand $A_I$ as follows:

$$A_I = \sum_n [a_n A_{In} + a^*_n A^*_{In}].$$

(65)

We also can expand $\pi^{Ic}$ in modes as follows:

$$\pi^{Ic} = \sum_n [a_n \pi^{Ic}_n + a^*_n \pi^{Ic*}_n].$$

(66)
We suppose
\[(A_n, A_n^*) = 0\] (67)
and
\[(A_n, A_m) = M_{nm}.\] (68)

In quantum field theory when \(A_I\) is promoted to an operator \(\hat{A}_I\), \(a_n^*\) and \(a_n\) become creation operators \(a_n^\dagger\) and annihilation operators \(a_n\) respectively. Thus we have
\[
\hat{A}_I = \sum_n [a_n A_{In} + a_n^\dagger A_{I*n}].
\] (69)

Now the two-point function is given by
\[
G_{II'}(x,x') = \langle 0 | A_I(x) A_{I'}(x') | 0 \rangle = \sum_{nm} A_{In}(x) A_{Im}(x') \langle 0 | [a_n, a_m^\dagger] | 0 \rangle.
\] (70)

If we again define \(C_{nm}\) as follows:
\[
C_{nm} = \langle 0 | [a_n, a_m^\dagger] | 0 \rangle.
\] (71)

then following a similar line of argument to the scalar case, we can again show that
\[
[[A_n, \hat{A}](\hat{A}, A_m)] = M_{nm}.
\] (72)

We can also show,
\[
M_{nm} = M_{mn}^*.
\] (73)

Then we can write the above equation in matrix notation as before
\[
MCM = M.
\] (74)

So we have, just like in the scalar case
\[
C = M^{-1}.
\] (75)

Now the two-point function is given by
\[
G(x,x')_{II'} = \sum_{nm} A_{In} A_{Im} M_{nm}^{-1}.
\] (76)

Here the two-point function is expected to split into two parts
\[
G(x,x')_{II'} = P_{II'}(x,x') + Q_{II'}(x,x').
\] (77)

Here \(P_{II'}\) and \(Q_{II'}\) are contributions coming from the physical and pure gauge terms respectively. In this paper we constructed a inner product for a quantum field theory in curved spacetime.
5 Conclusion

In this paper we analysed the quantization of quantum field theory on curved spacetime. We first analysed the quantization of scalar field theory. We thus constructed a inner product for scalar field theory. This inner product was used for constructing a two-point function. We then generalized these results to tensor fields. We constructed a inner product and two-point function for the most general tensor field theory possible. It was observed that the choice of vacuum state was not unique. This is because different vacuum states could be related to each other via a transformation. In fact, it is this transformation that becomes the bases of both the Hawking radiation and Unruh effect. It will be interesting to analyse the Unruh effect for tensor fields using this formalism.

References

[1] L. Fabbri, Int. J. Theor. Phys. 50, 3616, 2011
[2] T. Kugo and I. Ojima, Nucl. Phys. B144, 234, 1978
[3] K. Nishijima and M. Okawa, Prog. Theor. Phys. 60, 272, 1978
[4] N. Nakanishi and I. Ojima, Covariant operator formalism of gauge theories and quantum gravity - World Sci. Lect. Notes. Phys. - , 1990
[5] A. Lesov, arXiv:0911.0058
[6] J. MacDonald and D. J. Mullan, Phys. Rev. D 80, 043507, 2009
[7] A. M. Sinev, arXiv:0806.3212
[8] C. Pagliarone, arXiv:hep-ex/0612037
[9] M. Faizal, Class. Quant. Grav. 29, 035007, 2012
[10] A. Pakman, JHEP 0306, 053, 2003
[11] M. Faizal and A. Higuchi, Phys. Rev. D85: 12402, 2012
[12] K. Izumi and T. Tanaka, Prog. Theor. Phys. 121, 427, 2009
[13] M. Faizal and A. Higuchi, Phys. Rev. D 78, 067502, 2008
[14] M. Faizal, J. Phys. A 44, 402001, 2011
[15] I. A. Batalin and G. A. Vilkovisky, Phys. Lett. B 102, 27, 1981
[16] I. A. Batalin and G. A. Vilkovisky, Phys. Rev. D 28, 2567, 1983
[17] C. Bizdadea and S. O. Saliu, J. Phys. A 31, 8805, 1998
[18] C. Bizdadea, I. Negru and S. O. Saliu, Int. J. Mod. Phys. A 14, 359, 1999
[19] M. Faizal, Found. Phys. 41, 270, 2011
[20] M. Faizal, Phys. Lett. B 705, 120, 2011
[21] J. W. Moffat, Phys. Lett. B 506, 193 , 2001
[22] J. W. Moffat, Phys. Lett. B 491, 345 , 2000
[23] M. Faizal, Mod. Phys. Lett. A27: 1250075, 2012
[24] S. Ahmad, Comm. in Theo.l Phys, 59, 439 , 2013
[25] M. Faizal, Phys. Rev. D 84, 106011 , 2011
[26] M. Faizal and D. J. Smith, Phys. Rev. D85: 105007, 2012
[27] V. Mader, M. Schaden, D. Zwanziger and R. Alkofer, arXiv:1309.0497
[28] M. Faizal, JHEP 1204: 017, 2012
[29] A. Gustavsson, arXiv:1203.5883
[30] M. Faizal, Comm. Theor. Phys. 57, 637 , 2012
[31] M. S. Bianchi, M. Leoni and S. Penati, arXiv:1112.3649
[32] M. Faizal, Europhys. Lett. 98: 31003, 2012
[33] M. Marino and P. Putrov, arXiv:1110.4066
[34] K. Okuyama, arXiv:1110.3555
[35] M. Faizal, JHEP. 1204, 017 , 2012
[36] A. Belhaj, arXiv:1107.2295
[37] M. Faizal, arXiv:1303.5477
[38] D. Zwanziger, AIPConf.Proc.892:121-127, 2007
[39] M. Faizal, Mod. Phys. Lett. A28: 1350034, 2013
[40] D. Zwanziger, Phys. Rev. D76: 125014, 2007
[41] M. Faizal, Int. J. Mod. Phys. A28: 1350012, 2013
[42] M. Golterman, L. Zimmerman, Phys.Rev. D71, 117502 , 2005
[43] M. Faizal, JHEP. 1301: 156, 2013
[44] D. Polyakov, Phys.Lett. B611, 173 , 2005
[45] M. Faizal, Europhys. Lett. 103: 21003, 2013
[46] A. Imaanpur, JHEP 0503 , 030 , 2005
[47] M. Faizal, Nucl. Phys. B. 869: 598, 2013
[48] Jen-Chi Lee, Eur.Phys.J.C1:739-741,1998
[49] M. Faizal,Phys. Rev. D87: 025019, 2013
[50] A. Kapustin, Y. Li, Anton Kapustin, Adv.Theor.Math.Phys. 9 , 559, 2005
[51] M. Faizal, Int. J. Theor. Phys. 52: 392, 2013
[52] Chuan-Tsung Chan, Jen-Chi Lee, Yi Yang, Phys.Rev. D71 086005 , 2005
[53] M. Faizal, Class. Quant. Grav. 29: 215009, 2012
[54] R. P. Malik, arXiv:hep-th/0412333
[55] M. Faizal, Comm. Theor. Phys. 58: 704, 2012
[56] N. Boulanger, J.Math.Phys. 46 , 053508 , 2005
[57] M. Faizal, Mod. Phys. Lett. A27: 1250147, 2012
[58] W. H. Huang, arXiv:1107.2030
[59] M. Faizal and M. Khan, Eur. Phys. J. C 71, 1603 ,2011
[60] J. T. Liu and Z. Zhao, arXiv:1108.5179
[61] M. Fontanini and M. Trodden, Phys. Rev. D 83, 103518 , 2011
[62] V. O. Rivelles, Phys. Lett. B 577, 137 , 2003
[63] B. S. DeWitt, Phys. Rev. 160, 1113 , 1967
[64] M. Faizal, J.Exp.Theor.Phys. 114 , 400, 2012, arXiv:gr-qc/0602094
[65] Y. Ohkuwa, Int. J. Mod. Phys. A 13, 4091 , 1998
[66] M. Faizal, Mod. Phys. Lett. A 27, 1250007 , 2012
[67] I. T. Durham, arXiv:1307.3691
[68] M. Faizal, arXiv:1304.0259
[69] V. Bonzom, Phys.Rev.D84:024009, 2011
[70] M. Faizal, arXiv:1303.5478
[71] Ru-Nan Huang, arXiv:1304.5309
[72] R. M. Wald, Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics- University of Chicago Press- 1994
[73] M. Faizal, arXiv:1301.0224