Model-Free Statistical Inference on High-Dimensional Data *

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Abstract: This paper aims to develop an effective model-free inference procedure for high-dimensional data. We first reformulate the hypothesis testing problem via sufficient dimension reduction framework. With the aid of new reformulation, we propose a new test statistic and show that its asymptotic distribution is χ² distribution whose degree of freedom does not depend on the unknown population distribution. We further conduct power analysis under local alternative hypotheses. In addition, we study how to control the false discovery rate of the proposed χ² tests, which are correlated, to identify important predictors under a model-free framework. To this end, we propose a multiple testing procedure and establish its theoretical guarantees. Monte Carlo simulation studies are conducted to assess the performance of the proposed tests and an empirical analysis of a real-world data set is used to illustrate the proposed methodology.

Keywords: False discovery rate control; Marginal coordinate hypothesis; Orthogonality; Sufficient dimension reduction.

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1 Introduction

High-dimensional data are frequently collected in a large variety of areas such as biomedical imaging, functional magnetic resonance imaging, tomography, tumor classifications, and finance. Many regularization methods have been proposed for simultaneous estimation and variable selection (Fan et al., 2020a). Statistical inference for high-dimensional data receives considerable attention in the recent literature. Zhang and Zhang (2014) proposed the debiased Lasso, a low-dimensional projection approach to construct confidence interval and perform hypothesis testing, for linear regression models. Van de Geer et al. (2014) and Javanmard and Montanari (2014) proposed the desparsified Lasso under the setting of generalized linear models, and Ning and Liu (2017) proposed a decorrelated score method that can be applied to a more general family of penalized M-estimators. The debiased Lasso, desparsified Lasso and the decorrelated score method share the same spirit. See, for example, Chapter 7 of Fan et al. (2020a) for detailed discussion. Fang et al. (2017) generalized the decorrelation method for the low dimensional hypothesis testing approach in high-dimensional proportional hazards models. Fang et al. (2020) extended the decorrelated score method to longitudinal data with ultrahigh-dimensional predictors. Shi et al. (2019) developed the constrained partial regularization method for testing linear hypotheses in high-dimensional generalized linear models. Sun and Zhang (2021) proposed a modified profile likelihood-based statistic for hypothesis testing without penalizing the parameters of interest. Aforementioned statistical inference procedures were developed for a specific model such as linear model, generalized linear model or Cox’s model. This work aims to develop a model-free statistical inference procedure for high-dimensional data.

In the initial stage of high-dimensional data modeling, it may be quite challenging in correctly specifying a regression model, if not impossible. Furthermore there is lack of effective procedures to validate specified model assumption in the high-dimensional setting. Thus, it is of great interest to develop statistical inference procedures for high-dimensional
data without a pre-specified parametric model. To be precise, let \( Y \in \mathbb{R} \) be the response variable along with predictor vector \( \mathbf{X} = (X_1, \cdots, X_p)\top \in \mathbb{R}^p \). Denote \( F(Y|\mathbf{X}) \) to be the conditional distribution of \( Y \) given \( \mathbf{X} \). The active set \( \mathcal{A} \) is defined as

\[ \mathcal{A} = \{ j : F(Y|\mathbf{X}) \text{ functionally depends on } X_j \} \]

Then \( \mathcal{A}^c \), the complement of \( \mathcal{A} \), is the index set consisting of all irrelevant predictors. Let \( \mathbf{X}_A = \{ X_j, j \in \mathcal{A} \} \) denote the vector containing all the active predictors. Thus, \( Y \) is independent of \( \mathbf{X} \) given \( \mathbf{X}_A \), denote it by \( Y \perp \perp \mathbf{X}|\mathbf{X}_A \). It is of interest to test whether a predictor \( X_j \) contributes to the response \( Y \) or not given the other predictors. This can be formulated as testing the hypothesis whether \( H_{0j} : j \in \mathcal{A}^c \).

In this paper, we propose a new model-free test for \( H_{0j} \) in the presence of ultrahigh dimensional predictors. We first develop a new reformulation of \( H_{0j} \) by using statistical framework of sufficient dimension reduction (Li, 2018). Based on this reformulation, we further propose a test statistic for \( H_{0j} \) and show that it converges to \( \chi^2 \) distribution whose degree of freedom does not depend on unknown population quantities. We also show that the newly proposed test statistic can retain power under local alternative hypotheses.

Under the model-free framework, we study the false discovery rate (FDR) control in the large scale simultaneous testing problem: \( H_{0j} : j \in \mathcal{A}^c \) for all \( j = 1, \cdots, p \). Classical FDR control procedures including Benjamini-Hochberg (BH) procedure (Benjamini and Hochberg, 1995) and Storey procedure (Storey, 2002) are typically built on the independence assumption. Other FDR control procedures such as Benjamini-Yekutieli (BY) procedure (Benjamini and Yekutieli, 2001) require positive dependence structure and tend to be overly conservative (Cai et al., 2019). These procedures may fail to control the FDR in the presence of complex and strong dependence structure in practice. Liu (2013) developed FDR control procedures for multiple \( z \)-tests under Gaussian graphical model setting by assuming the number of pairs of strongly correlated tests bounded. This method can be viewed as a truncated version of BH method, but it cannot be directly extended to the underlying setting.
of this paper because of complicated dependence structure. We propose a new FDR control procedure by using the maximum correlation coefficient to measure the dependence of two tests, and show that the proposed procedure asymptotically controls the FDR at nominal level under mild conditions.

The paper is organized as follows. In section 2, we present a new reformulation. In section 3, we propose a new test procedure and investigate its asymptotic properties under the null and local alternative hypotheses. In section 4, we investigate the FDR control procedure which is important to identify active predictors. We present numerical studies in section 5 and conclusions in section 6. Proofs are given in the Appendix. Technical lemmas and their proofs are given in the supplementary material of this paper.

2 A reformulation via sufficient dimension reduction

We now reformulate the hypothesis $H_{0j} : j \in \mathcal{A}^c$ by using concepts related to sufficient dimension reduction. Note that $H_{0j}$ is called marginal coordinate hypothesis in the seminal work of Cook (2004), in which the author evaluated the predictor’s effect in a model-free setting by adopting the sufficient dimension reduction framework (Li, 2018). Following Cook (2004), several authors have developed test procedures for $H_{0j}$ (Shao et al., 2007; Yu and Dong, 2016; Dong et al., 2016) when the dimension $p$ is fixed. These tests have a major limitation that the asymptotic distributions of these test statistics are sums of weighted $\chi^2$ distributions with unknown weights. This makes their practical implementation challenging.

Define the central subspace $S_{Y \mid X}$ to be the minimum subspace $\mathcal{S}$ given by the column space of a $p \times d$ matrix $B = (\beta_1, \cdots, \beta_d)$ with $d < p$ such that $Y \perp \perp X \mid B^\top X$. Under some mild conditions, the central subspace uniquely exists and contains all information of $F(Y \mid X)$ (Li, 2018). Throughout this paper, it is assumed that $S_{Y \mid X}$ exists and its dimension $d$ is fixed.
Based on the concepts of the central subspace, we can reformulate $H_{0j}$. To this end, we firstly consider the estimation of the central subspace $S_{Y|X}$ because the basis matrix $B$ is unknown. Without loss of generality, assume that $\mathbb{E}(X) = 0$, and $\Sigma = \text{Var}(X) > 0$. Following the literature of sufficient dimension reduction, we impose linearity condition (LC):

$$\mathbb{E}[X|B^\top X]$$ is a linear function of $B^\top X$.

The LC is satisfied when $X$ follows an elliptical distribution, but can be more general in asymptotical sense (Hall and Li, 1993). Note that this condition is imposed only on the distribution of $X$, rather than the conditional distribution $F(Y|X)$. In fact, the relationship between $Y$ and $X$ is unspecified, and hence is model-free.

Under the linearity condition, it follows by Yin and Cook (2002) that for any function $f(Y)$,

$$\Sigma^{-1}\text{Cov}(X, f(Y)) \in S_{Y|X}.$$  

This enables us to choose a series of transformations on $Y$: $f_1(Y), \cdots, f_h(Y)$, with pre-specified $h > d$. We will discuss how to choose $f_k(\cdot)$ and $h$ in Section 5. Define

$$\beta^0_k = \arg\min_{\beta_k} \mathbb{E}[(f_k(Y) - X^\top \beta_k)^2], \quad \text{for } k = 1, \cdots, h.$$  

Denote $B_0 = (\beta_1^0, \cdots, \beta_h^0)$. Then $\text{Span}(B_0) \subseteq S_{Y|X}$. Following the literature of sufficient dimension reduction, we take one step further by assuming the coverage condition (CC)

$$\text{Span}(B_0) = S_{Y|X},$$  

whenever $\text{Span}(B_0) \subseteq S_{Y|X}$ so that the subspace spanned by $B_0$ coincides with the central subspace. This condition is reasonable. Actually the larger $h$ is, the $\text{Span}(B_0)$ is larger. Thus when $h$ is large, $\text{Span}(B_0)$ could be very close to $S_{Y|X}$.

For $j = 1, \cdots, p$, let $b_j^\top = (\beta_{1j}^0, \cdots, \beta_{hj}^0)$ be the $j$th row of $B_0$, where $\beta_{kj}^0$ is the $j$th element of $\beta_k^0 \in \mathbb{R}^p$, $k = 1, \cdots, h$. Note that if $j \in A^c$, then any of the $h$ linear combinations $X^\top \beta_k^0, k = 1, \cdots, h$ must not involve $X_j$. Thus, $Y \perp X|X_A$ and $Y \perp X|B_0^\top X$ imply that
\[ \sum_{k=1}^{h} |\beta_{kj}^0| > 0 \text{ for } j \in A, \text{ and } \sum_{k=1}^{h} |\beta_{kj}^0| = 0 \text{ for } j \in A^c. \] Thus, the following proposition are valid.

**Proposition 1.** Suppose that the LC and CC conditions hold, we have: \( H_{0j} : j \in A^c \) if and only if \( H_{0j}' : b_j = 0. \)

This reformulation converts the original hypothesis testing problem into a parametric hypothesis without knowing the specific form of the conditional distribution of \( Y \) given \( X \).

We next construct a score-type statistic for testing \( H_{0j}' \).

## 3 A new test statistic and its limiting distribution

In this section, we develop a test statistic for \( H_{0j}' \) to achieve the goal of testing \( H_{0j} \). To get the insight of proposed test for \( H_{0j}' \), we first consider the low dimensional setting in which \( p \) is fixed. By the definition of \( \beta_k^0 \), we may consider using \( \mathbb{E}[X_j \{f_k(Y) - X^\top \beta_k^0\}] \) (i.e., the derivative of \( \mathbb{E}[\{f_k(Y) - X^\top \beta_k^0\}^2] \) with respect to \( \beta_{kj}^0 \), for \( k = 1, \cdots, h \) to construct a test statistic. This is similar to the score test in likelihood setting. Under \( H_{0j}' \),

\[ \mathbb{E}[X_j \{f_k(Y) - X^\top \beta_k^0\}] = \mathbb{E}[X_j \{f_k(Y) - Z_j^\top \gamma_{kj}\}], \]

where \( Z_j \) is the subvector of \( X \) without \( X_j \), and \( \gamma_{kj} \) is the subvector of \( \beta_k^0 \) without \( \beta_{kj}^0 \).

Suppose that \( \{X_i, Y_i\}, i = 1, \cdots, n, \) are random samples generated from \( \{X, Y\} \). Similarly, we denote \( Z_{ij} \) for the sample of \( Z_j \). By the definitions of \( \beta_k^0 \), we then obtain the least squares estimate of \( \gamma_{kj}, \tilde{\gamma}_{kj} \), by a linear regression of \( f_k(Y) \) on \( Z_j \). Then it is natural to consider the score-type test statistic:

\[ T_{nk}^j = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_{ij} (f_k(Y_i) - Z_{ij}^\top \tilde{\gamma}_{kj}). \]

It can be shown that under the null hypothesis, \( (T_{n1}^j, \cdots, T_{nh}^j) \) jointly converges to a multivariate normal distribution with zero mean.
We now consider high-dimensional setting. The test procedure does not work in high-dimensional setting since the least squares estimate is not well defined. However, we may consider the score type test $T_{nk}^j$ by replacing the least squares estimate with the corresponding penalized estimate $\hat{\gamma}_{kj}$ of $\gamma_{kj}$. In high-dimensional setting, it is not uncommon to impose sparsity assumption on regression coefficients. That is, only a small subset of the predictors are significant to the response variable. We estimate $\gamma_{kj}$ by its penalized least squares estimate

$$\hat{\gamma}_{kj} = \arg \min \frac{1}{2n} \sum_{i=1}^{n} \{f_k(Y_i) - Z_{ij}^\top \gamma_{kj}\}^2 + \sum_{l=1}^{p-1} p_{\lambda_Y}(|\gamma_{kj,l}|), \ k = 1, \cdots, h, \quad (3.1)$$

where $p_{\lambda_Y}(\cdot)$ is a penalty function with a tuning parameter $\lambda_Y$.

**Remark 1.** For multiple testing problem to be studied in next section, we need to calculating all $\hat{\gamma}_{kj}$, $j = 1, \cdots, p$ and $k = 1, \cdots, h$. This requires to solve $hp$ regularized optimization problems, and leads to expensive computational cost. For multiple testing problem, we recommend to estimate $\beta^0_k$ by using

$$\hat{\beta}_k = \arg \min \frac{1}{2n} \sum_{i=1}^{n} \{f_k(Y_i) - X_{ij}^\top \beta_k\}^2 + \sum_{l=1}^{p} p_{\lambda_Y}(|\beta_{kl}|), \ k = 1, \cdots, h. \quad (3.2)$$

Then set $\hat{\gamma}_{kj}$ as the subvector of $\hat{\beta}_k$ without $\hat{\beta}_{kj}$. This can save computational cost dramatically.

The test statistic $T_{nk}^j$ can be further refined since it could fail in some situation. To see this, note that

$$T_{nk}^j = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_{ij}(f_k(Y_i) - Z_{ij}^\top \hat{\gamma}_{kj})$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_{ij}(f_k(Y_i) - Z_{ij}^\top \gamma_{kj}) + \frac{(\gamma_{kj} - \hat{\gamma}_{kj})^\top}{\sqrt{n}} \sum_{i=1}^{n} Z_{ij}X_{ij}.$$

Although under mild conditions, the first term converges to a normal distribution, the second term may not be asymptotically normal. Indeed, for the penalized estimate of $\gamma_{kj}$, it may not be always valid without strong condition on $\gamma_{kj}$, such as minimal signal condition.
to ensure that \(\sqrt{n}(\hat{\gamma}_{kj} - \gamma_{kj})\) converges to normal distribution. Moreover, the Euclidean norm of the term \(n^{-1}\sum_{i=1}^{n} Z_{ij} X_{ij}\) can diverge since its dimension is \(p\). Thus it is of importance to refine the test statistic \(T_{nk}^j\) so that it performs well in general settings.

We next employ the idea of orthogonalization to refine \(T_{nk}^j\). The orthogonalization plays a critical role in reducing the bias from the estimators of high-dimensional nuisance parameters (Belloni et al., 2015; Ning and Liu, 2017; Belloni et al., 2018).

Instead of employing \(E[X_j \{f_k(Y) - Z_j^\top \hat{\gamma}_{kj}\}]\) in high-dimensional setting, we propose using

\[
E \left[ (X_j - Z_j^\top \theta_j^*) \{f_k(Y) - Z_j^\top \gamma_{kj}\} \right] = \mathbb{E} \left[ Z_j (X_j - Z_j^\top \theta_j^*) \right] = 0
\]

(3.3)

under some conditions.

In order to reduce the influence of nuisance parameters, we estimate \(\theta_j^*\) by a penalized least squares estimate

\[
\hat{\theta}_j = \arg \min \frac{1}{2n} \sum_{i=1}^{n} (X_{ij} - Z_{ij}^\top \theta_j)^2 + \sum_{l=1}^{p-1} p_{\lambda_X}(|\theta_{j,l}|),
\]

(3.4)

where \(p_{\lambda_X}(\cdot)\) is a penalty function with a tuning parameter \(\lambda_X\).

Define \(S_i^j = (S_{i1}^j, \ldots, S_{ih}^j)^\top\) with \(S_{ik}^j = (X_{ij} - Z_{ij}^\top \hat{\theta}_j)(f_k(Y_i) - Z_{ij}^\top \hat{\gamma}_{kj})\) for each \(k\). By the orthogonality property of \(E \left[ (X_j - Z_j^\top \theta_j^* \right) \{f_k(Y) - Z_j^\top \gamma_{kj}\}]\), it is natural to construct a test statistic based on

\[
\bar{S}_i^j = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} S_i^j.
\]

That is, \(\bar{S}_n^j = (\bar{S}_{n1}^j, \ldots, \bar{S}_{nh}^j)^\top\) with \(\bar{S}_{nk}^j = n^{-1/2} \sum_{i=1}^{n} (X_{ij} - Z_{ij}^\top \hat{\theta}_j)(f_k(Y_i) - Z_{ij}^\top \hat{\gamma}_{kj})\).

As shown in the Appendix, \(\bar{S}_n^j\) asymptotically follows a normal distribution with mean 0 and covariance matrix \(\Omega_j\), where the \((l, k)\)-element of \(\Omega_j\) is \(E[\eta_l^j \eta_k^j (X_j - Z_j^\top \theta_j^*)^2]\) for
$l, k = 1, \cdots, h$, where $\eta^i_k = f_k(Y) - Z_j^\top \gamma_{kj}, k = 1, \cdots, h$. Thus, we propose the following test statistic for $H_{0j}^i$

$$W_{nj} = \tilde{S}_n^\top \tilde{\Omega}^{-1}_j \tilde{S}_n^j,$$

where $\tilde{\Omega}_j$ is a consistent estimate of $\Omega_j$ with $(k, l)$-elements $\tilde{\omega}_{kl}$ being

$$\frac{1}{n} \sum_{i=1}^n (f_i(Y_i) - Z_{ij}^\top \tilde{\gamma}_{lj})(f_k(Y_i) - Z_{ij}^\top \tilde{\gamma}_{kj})(X_{ij} - Z_{ij}^\top \tilde{\theta}_j)^2.$$

Let $s_Y$ be the sparsity level for $\gamma_{kj}, k = 1, \cdots, h$. That is, $s_Y = \max_{1 \leq j \leq p, 1 \leq k \leq h} \|\gamma_{kj}\|_0$. Also denote the sparsity level for the parameter $\theta^*_j$ to be $s_X = \max_{1 \leq j \leq p} \|\theta^*_j\|_0$. Before we introduce technical conditions, let us introduce some new notations related to sub-Gaussian and sub-exponential random variables.

**Definition 1.** (i) A random variable $X$ is sub-Gaussian if its tail probability satisfies

$$\Pr\{|X| \geq t\} \leq 2 \exp\left(-t^2/K_1^2 \right)$$

for all $t \geq 0$. $\|X\|_{\psi_2} = \inf\{t > 0 : \mathbb{E}\exp(X^2/t^2) \leq 2\}$ is the sub-Gaussian norm of $X$, and is the smallest $K_1$.

(ii) A random variable $X$ is sub-exponential if its tail probability satisfies

$$\Pr\{|X| \geq t\} \leq 2 \exp\left(-t/K_2 \right)$$

for all $t \geq 0$. $\|X\|_{\psi_1} = \inf\{t > 0 : \mathbb{E}\exp(|X|/t) \leq 2\}$ is the sub-exponential norm of $X$, and is the smallest $K_2$.

The following technical conditions are imposed to establish the asymptotical null distribution of $W_{nj}$, although they may not be the weakest conditions.

(B1) $\|\hat{\gamma}_{kj} - \gamma_{kj}\|_1 = O_p(\lambda_Y s_Y), \|\hat{\theta}_j - \theta^*_j\|_1 = O_p(\lambda_X s_X)$.

(B2) $\eta^i_k, k = 1, \cdots, h$, and $X_j, j = 1, \cdots, p$ are all sub-Gaussian random variables with finite expectation. $\|\eta^i_k\|_{\psi_2}$ and $\|X_j\|_{\psi_2}$ are uniformly bounded.

(B3) $\max\{s_Y, s_X\} = o(\sqrt{n}/\log p), \max\{\lambda_X, \lambda_Y\} = C \sqrt{\log p}/n$.

(B4) $\Sigma = \operatorname{Cov}(X) > 0$, and $\Omega_j > 0$. 

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These conditions are commonly used in the literature. Condition (B1) is for error bound of the penalized least squares estimators in linear regression models. The penalized least squares estimates with the Lasso, SCAD and MCP penalties satisfy this condition (Fan et al., 2020a). Condition (B2) is imposed for establishing the $L_\infty$ norm of $n^{-1/2} \sum_{i=1}^n n_i Z_{ij}$. Sub-Gaussian assumption is a commonly used condition in the literature (Fan et al., 2020a). Furthermore, the dimension of $p$ is allowed to be an exponential order of the sample size $n$ according to Conditions (B2) and (B3). The rate for the sparsity levels in Condition (B3) is commonly imposed, see for instance Ning and Liu (2017). Condition (B4) is also a mild condition.

The limiting distribution of $W_{nj}$ under the null hypothesis and local alternative hypotheses are given in the following theorem, whose proof is given in the Appendix.

**Theorem 1.** Suppose that the LC and CC conditions, and Conditions (B1)–(B4) are valid.

(i) Under the null hypothesis $H_{0j}$, it follows that $W_{nj} \to \chi^2_h$ in distribution, where $\chi^2_h$ stands for the chi-square distribution with $h$ degrees of freedom;

(ii) Under the alternative hypothesis $H_{1n} : \sqrt{n}b_j \to b \neq 0$, it follows that

$$W_{nj} \to \chi^2_h (\delta_j^2 b^\top \Omega_j^{-1} b)$$

in distribution, where $\chi^2_h(a)$ is the non-central chi-square distribution with $h$ degrees of freedom and noncentrality parameter $a$, and $\delta_j = E(X_j^2) - E(X_j Z_j^\top) E(Z_j Z_j^\top)^{-1} E(Z_j X_j)$.

Theorem 1(a) implies that we are able to test the hypothesis whether $F(Y|X)$ functionally depends on the $j$th predictor or not in ultrahigh dimensional setting without imposing a specific functional form on regression model. This desirable feature is significantly different from the existing works based on specific parametric model such as linear model or generalized linear model.

Theorem 1(b) implies that the proposed test $W_{nj}$ can detect the alternative hypotheses which converge to the null at the rate of $n^{-1/2}$. From the proof of Theorem 1(b), we can
show that \( W_{nj} \to \infty \) for fixed alternative hypothesis \( H_1 : b_j \neq 0 \). Thus, \( W_n \) has asymptotic power 1 for any fixed alternative hypothesis.

4 False discovery rate control

In Section 3, we propose a test of significance for an individual predictor. In practice, it is of interest to test simultaneously all \( p \) hypotheses. That is, \( H_{0j} : j \in A^c \) for all \( j = 1, \cdots, p \). This is potentially useful to identify important features in the high-dimensional data. To avoid spurious discoveries, the FDR control is commonly adopted. Recently, Barber and Candès (2015) introduced a novel knockoff framework to control the FDR under the linear model. Candes et al. (2018) further developed a model-X knockoff procedure that can control FDR without assuming a specific regression model. However, the model-X knockoff procedure requires knowing the joint distribution of the predictors. This makes the knockoff method challenging in practical implementation. See also Fan et al. (2020b). This section aims to develop a FDR control procedure under a model-free framework. The new procedure does not require the knowledge of the joint distribution of the predictors.

For a series of individual hypothesis \( H_{0j} \), we can construct the corresponding test statistics \( W_{nj} \). As shown in the Theorem 1, under null hypothesis \( W_{nj} \) follows an asymptotically chi-squared distribution with degrees of freedom \( h \).

A FDR controlling procedure is to find a threshold \( t \) to control the multiple testing effect. For any threshold \( t > 0 \), let \( R_0(t) = \sum_{j \in A^c} I(W_{nj} \geq t) \), and \( R(t) = \sum_{j=1}^p I(W_{nj} \geq t) \) be the total number of false discoveries and the total number of discoveries associated with \( t \), respectively.

Define false discovery proportion (FDP) and FDR as follows:

\[
FDP(t) = \frac{R_0(t)}{R(t)} \vee 1, \quad \text{FDR}(t) = \mathbb{E}\{\text{FDP}(t)\}.
\]
Under a pre-specified FDR level $\alpha$, the ideal choice of the threshold $t$ would be

$$t_0 = \inf \{ t \in \mathbb{R} : \text{FDP}(t) \leq \alpha \}.$$ 

In practice, we need to estimate $R_0(t)$ since the inactive set is unknown. Intuitively, $R_0(t)$ can be reasonably estimated by $|\mathcal{A}^c|G(t)$, where $G(t) = \Pr(\chi^2_h \geq t)$. As a result, FDP($t$) can be estimated by

$$\hat{\text{FDP}}(t) = \frac{|\mathcal{A}^c|G(t)}{R(t) \lor 1}.$$ 

Thus we propose to estimate the threshold level $t$ by

$$\hat{t} = \inf \left\{ 0 \leq t \leq b_p : \hat{\text{FDP}}(t) \leq \alpha \right\}.$$ 

and reject the null hypothesis $H_{0j}$ if $W_{nj} \geq \hat{t}$, where $b_p = 2 \log p + 2d_0 \log \log p$ with $2d_0 < h - 4$. If there is no $t \in [0, b_p]$ such that $\hat{\text{FDP}}(t) \leq \alpha$, we will directly set $\hat{t} = 2 \log p + (h - 1) \log \log p$.

In above, we estimate $R_0(t)$ by $|\mathcal{A}^c|G(t)$ when $t \in [0, b_p]$. In fact under the restriction $2d_0 < h - 4$ and other mild conditions, we will show

$$\sup_{0 \leq t \leq b_p} \left| \frac{\sum_{j \in \mathcal{A}^c} I(W_{nj} \geq t)}{|\mathcal{A}^c|G(t)} - 1 \right| \to 0, \quad (4.1)$$ 

in probability. In other words, it implies that the total number of false discoveries $R_0(t)$ can be consistently estimated by $|\mathcal{A}^c|G(t)$ in this pre-specified region. In practice, $|\mathcal{A}^c|$ is not a prior knowledge, we could approximate $|\mathcal{A}^c|$ by $p$ since $|\mathcal{A}| = o(p)$.

**Remark 2.** The choice of $d_0$ depends on the value of $h$. Since $h - 4$ is the boundary value to guarantee the theoretical results, we suggest choosing a slightly smaller value $d_0 = h - 4 - \epsilon_h$ in practice, where $\epsilon_h$ is a small positive constant. It is noteworthy that $d_0$ is allowed to be negative if $h \leq 4$. In [Xia et al. (2018)](https://example.com), they studied the case of asymptotic normal test statistic which corresponds to $h = 1$. They set the search region to be $[0, \sqrt{2 \log p - 2 \log \log p}]$, where the right boundary point is smaller than $\sqrt{2 \log p}$. 

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It is crucial to characterize the dependence structure across the tests in order to establish (4.1). Denote
\[ \tilde{S}_{jn} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \eta_i \left( X_{ij} - Z^r_{ij} \theta^*_j \right), \quad \tilde{W}_{nj} = \tilde{S}_{jn}^\top \Omega^{-1}_j \tilde{S}_{jn}. \]
Further denote \( \tilde{K}^i_n = \Omega^{-1/2}_j \tilde{S}_{jn} \). Define the maximum correlation coefficient
\[ \rho^*_{ij} = \max_{\|a\|=1,\|b\|=1} \left| \text{corr} \left( a^\top \tilde{K}^i_n, b^\top \tilde{K}^j_n \right) \right| \]
which quantifies the dependence between \( \tilde{W}_{ni} \) and \( \tilde{W}_{nj} \). For \( i, j \in A^c \), define
\[ B_1 := \{(i, j) : i \neq j, \rho^*_{ij} \geq (\log p)^{-2-\epsilon_0}\} \]
which contains the pairs \((i, j)\) with high correlation.

In order to ensure the estimated FDR could be controlled at pre-specified level, the difference between constructed tests \( W_{nj} \) and \( \tilde{W}_{nj} \) must be uniformly bounded. The following proposition implies that this is valid under mild assumptions.

**Proposition 2.** Suppose that the LC and CC conditions, and conditions B1, B2 and B4 are valid. Assume \( \lambda_X, \lambda_Y = O(\sqrt{\log p}) \), \( \max(s_X, s_Y) = s \). It follows that
\[ (i) \quad \| S^i_n - \tilde{S}^i_n \|_1 = O_p(F(n, p, s)) = O_p(\frac{\log p}{\sqrt{n}} s), \]
\[ (ii) \quad \| \hat{\Omega}_j - \Omega_j \|_2 = O_p(G(n, p, s)), \]
hold uniformly for \( j \in A^c \), where
\[ G(n, p, s) = \sqrt{\frac{(\log np)}{n}} \vee \frac{s \log p \log(np)}{n} \vee s^2 \log(np) \left( \frac{\log p}{n} \right)^2 \vee s^3 \log(np) \left( \frac{\log p}{n} \right)^2. \]
Furthermore, it follows that
\[ \max_{j \in A^c} | W_{nj} - \tilde{W}_{nj} | = O_p \left( \max \left( \frac{(s \log p)^2}{n}, \frac{s(\log p)^2}{\sqrt{n}}, G(n, p, s) \log p \right) \right). \]

The proof of this proposition is given in the supplementary material of this paper. We next show the FDR control result of the proposed procedure in Theorem 2 below. The following technical conditions are imposed to facilitate its technical proof.
(D1) There exists a positive constant $\epsilon_1$ such that $|B_1| = O(\log p)^{h-2d_0-2-\epsilon_1}$.

(D2) $\max (F(n,p,s), G(n,p,s) \log p) = o(1)$.

(D3) $\lambda_{\min}(\Omega_j^{-1}) \geq \lambda_0 > 0$ and $\lambda_{\max}(\Omega_j^{-1}) \leq \lambda_1$ uniformly in $j = 1, \cdots, p$.

(D4) $\delta_j \geq c > 0$ uniformly for $j$ where $\delta_j = \mathbb{E}(X_j^2) - \mathbb{E}(X_jZ_j^\top)\mathbb{E}(Z_jZ_j^\top)^{-1}\mathbb{E}(Z_jX_j)$.

These conditions are mild and commonly used in the literature of FDR. See, e.g., Liu (2013), Xia et al. (2018) and references therein. Condition D1 requires a bounded number of strongly correlated pairs which correspond to tests in inactive set. The purpose is to control the variance of $R_0(t)$. Condition D2 assumes the difference between constructed test and its ideal version is uniformly bounded with rate $o(1)$, and therefore to ensure the consistency of $\sum_{j \in A^c} I(W_{nj} \geq t)$ to $|A^c|G(t)$. Condition D3 assumes the smallest and the largest eigenvalues of $\Omega_j$ are uniformly bounded.

**Theorem 2.** Under Conditions in Proposition 2 and Conditions D1–D4 with $p \leq cn^r$ for some $c > 0$ and $r > 0$. In addition, assume $|A| = o(p)$. It follows that

$$\limsup_{(n,p) \to \infty} \text{FDR}(\hat{t}) \leq \alpha$$

with probability 1.

In Theorem 1, the dimension $p$ is allowed to grow exponentially fast in $n$. However, in order to control FDR, it requires $p = O(n^r)$ to apply the moderate deviation result in Liu (2013).

**5 Numerical studies**

In this section, we conduct numerical studies to assess the finite sample performance of the proposed procedures. To implement the proposed test, we need to determine the transformation functions $f_k(\cdot)$. In our simulation, we take the $f_k(\cdot), k = 1, \cdots, h$ to be a set of linear
B-spline bases with \( h - 2 \) inner knots, which yields a set of \( h \) linear B-spline bases. In all simulation studies, we set \( p = 2000 \), and all simulation results are based on 1000 replications. We consider the Lasso \cite{Tibshirani1996} and the SCAD \cite{FanLi2001} in the penalized least squares estimate introduced in Section 3. We first study the impact of \( h \) on the performance of the proposed test.

### 5.1 The choice of \( h \)

One needs to choose the value of \( h \) to implement the proposed procedure. We investigate this issue by Monte Carlo study. To this end, we generate the data from the following three models.

- **Model I:** \( Y = X_1 + X_2 + \epsilon \).
- **Model II:** \( Y = (X_1 + X_2) / \{0.5 + (1.5 + X_3 + X_4)^2\} + 0.1\epsilon \).
- **Model III:** \( Y = 3\sin(X_1) + 3\sin(X_p) + \exp(-2X_3)\epsilon \).

These three models correspond to linear model, nonlinear model and heteroscedastic model, respectively. The dimension of central subspace, \( d \), is 1, 2, and 3, respectively. The covariate vector \( \mathbf{X} = (X_1, \ldots, X_p) \) is generated from \( N(0, \Sigma) \) with the \((i, j)\)-element of \( \Sigma \) being \( \sigma_{ij} = 0.5^{|i - j|} \) for \( 1 \leq i, j \leq p \). The error \( \epsilon \) is generated from \( N(0, 1) \) and independent of \( \mathbf{X} \). The active sets \( \mathcal{A} \) for these models are \( \{1, 2\} \), \( \{1, 2, 3, 4\} \), and \( \{1, 3, p\} \), respectively. We consider sample size \( n = 200 \) and 400.

To examine the impact of \( h \) on the performance of \( W_{nj} \), we set \( h = 1, 2, \ldots, 20 \) and plot the power curve as a function of \( h \) for several \( j \)s. This enables us to see the overall trend of the power function as \( h \) increases. Moreover, we study the impact of choice of penalty function on the performance of test statistics.

Figure 1 presents the empirical power functions with respect to \( h \) for several \( j \)s when the penalty in the penalized least squares is taken to be the Lasso penalty. The results for the SCAD penalty are similar to those for the Lasso penalty, and are depicted in Figure S.1 in
Figure 1: The plot of empirical power with respect to $h$ when the Lasso penalty is used.

As shown in Figure 1, it can be seen that for truly active variables, the empirical power has a sharp increase as $h$ increases if $h \leq d + 1$, and remains stable in a wide range of $h$. It is also observed that the empirical power decreases slowly as $h$ increases when $h$ is much larger than $d$ and $n = 200$. It can also be seen from Figure 1 that power of some active variables in Model II does not approach to 1. In particular, the empirical power
of $X_3$ in Model II achieves its maximum around $h = 5$, but its value is slightly less than 1. This phenomenon disappears when the sample size increases from 200 to 400. Figure I also implies that the empirical powers stand firm around 1 when $h > d + 1$ for $n = 400$. For inactive variables, the empirical size always stays around nominal level $\alpha = 0.05$. Figure I and S.1 seems to imply that $h = 5$ is a good choice for Models I, II and III. Thus, we set $h = 5$ in Sections 5.2, 5.3 and 5.4.

5.2 Comparison with existing methods

This subsection is devoted to comparing the performance of the proposed procedure with two popular methods: the LDPE method (Zhang and Zhang, 2014) and the decorrelated score method (Ning and Liu, 2017) by pretending all the observations are characterized by a linear model. Let $T^{ZZ}$ and $T^{NL}$ stand for these two test procedures, respectively. We compare the performance of the three procedures in terms of empirical size and power. The proposed test uses linear B-spline transformation with $h = 5$.

The data were generated from Models I, II and III described in Section 5.1. We summarize the simulation results in terms of rejection rate, which is defined to be the proportion of $p$-values being smaller than the nominal level $\alpha = 0.05$ based on 1000 replications. In each simulation, we test $H_{0j}$ for $j = 1, \cdots, p$. For inactive predictors (i.e., $X_j$ with $j \in A^c$), it is expected that the proportion is close to the nominal level, while for active predictors, it is expected to reject $H_{0j}$ with a large probability, and the rejection rate is the estimated power of the underlying test.

The results are summarized in Tables 1 and S.1 when the Lasso penalty and the SCAD penalty are used, respectively. These two tables only display the rejection rates for predictors $X_1$-$X_5$ and $X_{1996}$-$X_{2000}$ to save space. From these two tables, it can be seen that for linear model (Model I), the newly proposed test performs as well as the other two tests. All three tests control the size well and have good empirical power. For nonlinear model (Model II)
Table 1: Empirical rejection rate of $H_{0j}$ with $\alpha = 5\%$ when the Lasso penalty is used.

| $n$ | Method | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_{1996}$ | $X_{1997}$ | $X_{1998}$ | $X_{1999}$ | $X_{2000}$ |
|-----|--------|-------|-------|-------|-------|-------|-------------|-------------|-------------|-------------|-------------|
| 200 | $W_n$-Lasso | 1.000 | 1.000 | 0.077 | 0.053 | 0.058 | 0.038 | 0.052 | 0.052 | 0.047 | 0.047 |
|     | $T^{NL}$-Lasso | 1.000 | 1.000 | 0.057 | 0.040 | 0.032 | 0.030 | 0.030 | 0.032 | 0.037 | 0.040 |
|     | $T^{ZZ}$ | 1.000 | 1.000 | 0.141 | 0.070 | 0.054 | 0.043 | 0.049 | 0.042 | 0.053 | 0.056 |
| 400 | $W_n$-Lasso | 1.000 | 1.000 | 0.059 | 0.067 | 0.048 | 0.052 | 0.046 | 0.042 | 0.048 | 0.059 |
|     | $T^{NL}$-Lasso | 1.000 | 1.000 | 0.045 | 0.039 | 0.043 | 0.041 | 0.039 | 0.033 | 0.035 | 0.042 |
|     | $T^{ZZ}$ | 1.000 | 1.000 | 0.093 | 0.063 | 0.060 | 0.056 | 0.055 | 0.048 | 0.043 | 0.052 |
|     | Model II: $X_1, X_2, X_3$, and $X_4$ are active predictors |
| 200 | $W_n$-Lasso | 1.000 | 1.000 | 0.976 | 0.975 | 0.074 | 0.058 | 0.036 | 0.041 | 0.043 | 0.053 |
|     | $T^{NL}$-Lasso | 1.000 | 0.999 | 0.027 | 0.046 | 0.027 | 0.031 | 0.032 | 0.017 | 0.039 | 0.025 |
|     | $T^{ZZ}$ | 1.000 | 1.000 | 0.041 | 0.038 | 0.045 | 0.049 | 0.055 | 0.042 | 0.057 | 0.039 |
| 400 | $W_n$-Lasso | 1.000 | 1.000 | 0.999 | 1.000 | 0.047 | 0.047 | 0.057 | 0.037 | 0.055 | 0.046 |
|     | $T^{NL}$-Lasso | 1.000 | 1.000 | 0.064 | 0.101 | 0.037 | 0.033 | 0.035 | 0.034 | 0.033 | 0.031 |
|     | $T^{ZZ}$ | 1.000 | 1.000 | 0.029 | 0.082 | 0.040 | 0.047 | 0.048 | 0.047 | 0.045 | 0.046 |
|     | Model III: $X_1, X_3$, and $X_p$ are active predictors |
| 200 | $W_n$-Lasso | 1.000 | 0.074 | 0.919 | 0.078 | 0.059 | 0.043 | 0.041 | 0.051 | 0.080 | 0.999 |
|     | $T^{NL}$-Lasso | 0.240 | 0.051 | 0.440 | 0.049 | 0.032 | 0.027 | 0.015 | 0.035 | 0.036 | 0.255 |
|     | $T^{ZZ}$ | 0.285 | 0.078 | 0.499 | 0.083 | 0.052 | 0.046 | 0.034 | 0.055 | 0.052 | 0.294 |
| 400 | $W_n$-Lasso | 1.000 | 0.066 | 0.999 | 0.042 | 0.044 | 0.048 | 0.029 | 0.058 | 0.066 | 1.000 |
|     | $T^{NL}$-Lasso | 0.350 | 0.049 | 0.461 | 0.053 | 0.032 | 0.040 | 0.025 | 0.040 | 0.037 | 0.329 |
|     | $T^{ZZ}$ | 0.358 | 0.063 | 0.496 | 0.064 | 0.042 | 0.054 | 0.040 | 0.047 | 0.050 | 0.343 |

and heteroscedastic error model (Model III), all three tests also control Type I error at the desired nominal level. But the newly proposed test has much higher power than the other two tests. In particular, the power of the proposed test statistic $W_{nj}$ for Model II is very close to 1 under $H_{03}: 3 \in \mathcal{A}^c$, while the empirical powers of $T^{ZZ}$ and $T^{NL}$ are around 0.05.
even when $n = 400$.

### 5.3 Comparison of FDR control procedures

We next investigate the finite sample performance of the proposed FDR control procedures by simultaneously testing $H_{0j} : j \in \mathcal{A}^c$ for all $j = 1, \cdots, p$. This simulation study is designed to compare the newly proposed FDR control procedure with model-X knockoff procedures (Candes et al., 2018), for which we consider the Lasso coefficient-difference (LCD) statistic.

To understand the impact of sparsity level on FDR control procedures, we consider the following two models, which are modified from Models I and II.

**Model IV:**
\[ Y = \mathbf{X}^\top \mathbf{\beta}_1 + \epsilon. \]

**Model V:**
\[ Y = \left( \mathbf{X}^\top \mathbf{\beta}_2 / \left(0.5 + (1.5 + X_{p-1} + X_{p-2})^2\right) \right) + 0.1\epsilon. \]

In our simulation, we set $\mathbf{\beta}_1 = (1, \cdots, 1, 0, \cdots, 0)$ and $\mathbf{\beta}_2 = (1, \cdots, 1, 0, \cdots, 0)$. The sparsity levels of Model IV and V are $s_1$ and $s_2 + 2$, respectively. The sparsity levels of each model are set to be 4, 6, and 8. The distribution of $\mathbf{X}$ and $\epsilon$ are the same as those for Models I and II. In this simulation study, $p = 2000$ and the sample size $n = 200$ and 400. For the newly proposed FDR control procedure, the Lasso is also used, and we set $h = 5$. We search the $\hat{t}$ over $[0, 2 \log p + 0.75 \log \log p]$ and set $\hat{t} = 2 \log p + 4 \log \log p$ if $\hat{t}$ does not exist over the interval.

The empirical FDR and empirical power over 1000 replications are presented in Table 2 at the significance level $\alpha = 0.1$ or 0.2, where $\text{Power} = |\hat{\mathcal{A}} \cap \mathcal{A}|/|\mathcal{A}|$ with $\hat{\mathcal{A}} = \{ j : W_{nj} \geq \hat{t} \}$.

From Table 2, it can be seen that for Models VI and V, the proposed procedure and the model-X knockoff procedures can control the FDR well. For nonlinear model (model V), the proposed method generally has larger powers than the model-X knockoff procedures. The Knockoff+, a more conservative knockoff procedure, has much conservative FDR and therefore has very low power under the setting of Model V.
Table 2: Empirical size and power of FDR control procedures at nominal level $\alpha = 0.1, 0.2$.

| Sparsity | Method          | Model IV $\alpha = 0.1$ | Model IV $\alpha = 0.2$ | Model V $\alpha = 0.1$ | Model V $\alpha = 0.2$ |
|----------|-----------------|--------------------------|-------------------------|------------------------|------------------------|
|          | FDR Power       | FDR Power                | FDR Power               | FDR Power              | FDR Power              |
| n = 200  |                 |                          |                         |                        |                        |
| 4        | Proposed        | 0.089 0.954 0.104 0.954  | 0.086 0.954 0.118 0.979 |                        |                        |
|          | LCD-Knockoff    | 0.061 1.000 0.154 1.000  | 0.062 0.500 0.129 0.500 |                        |                        |
|          | LCD-Knockoff+   | 0.008 0.012 0.153 0.480  | 0.002 0.002 0.065 0.049 |                        |                        |
| 6        | Proposed        | 0.083 0.787 0.123 0.800  | 0.089 0.860 0.148 0.869 |                        |                        |
|          | LCD-Knockoff    | 0.065 1.000 0.155 1.000  | 0.063 0.638 0.158 0.646 |                        |                        |
|          | LCD-Knockoff+   | 0.023 0.051 0.137 1.000  | 0.007 0.007 0.152 0.314 |                        |                        |
| 8        | Proposed        | 0.097 0.600 0.151 0.654  | 0.087 0.693 0.163 0.745 |                        |                        |
|          | LCD-Knockoff    | 0.080 1.000 0.164 1.000  | 0.068 0.628 0.160 0.646 |                        |                        |
|          | LCD-Knockoff+   | 0.063 0.236 0.158 1.000  | 0.017 0.026 0.151 0.548 |                        |                        |
| n = 400  |                 |                          |                         |                        |                        |
| 4        | Proposed        | 0.083 1.000 0.094 1.000  | 0.064 0.970 0.110 0.970 |                        |                        |
|          | LCD-Knockoff    | 0.061 1.000 0.151 1.000  | 0.060 0.500 0.133 0.500 |                        |                        |
|          | LCD-Knockoff+   | 0.016 0.026 0.153 0.482  | 0.0 0 0.059 0.046       |                        |                        |
| 6        | Proposed        | 0.066 0.999 0.139 0.999  | 0.076 0.992 0.152 0.993 |                        |                        |
|          | LCD-Knockoff    | 0.067 1.000 0.151 1.000  | 0.058 0.667 0.151 0.667 |                        |                        |
|          | LCD-Knockoff+   | 0.026 0.060 0.141 1.000  | 0.004 0.004 0.147 0.321 |                        |                        |
| 8        | Proposed        | 0.056 0.987 0.173 0.994  | 0.064 0.987 0.183 0.993 |                        |                        |
|          | LCD-Knockoff    | 0.078 1.000 0.158 1.000  | 0.061 0.739 0.151 0.744 |                        |                        |
|          | LCD-Knockoff+   | 0.065 0.257 0.154 1.000  | 0.016 0.028 0.132 0.735 |                        |                        |

5.4 A real data example

We illustrate the proposed procedures by an empirical analysis of the data studied in Scheetz et al. (2006), Huang et al. (2010) and Li et al. (2017). This data set contains 120 twelve-week old male rats from which over 31042 different probes (genes) from eye tissue were measured. The intensity values were normalized using the RMA (robust multi-chip averaging, Bolstad et al. (2003)) method to obtain summary expression values for each probe. We are interested in finding the probes that could significantly affect the gene TRIM32. This gene was reported in Chiang et al. (2006) and was believed to cause Bandet-Biedl syndrome which is a genetically heterogeneous disease of multiple organ systems including the retina.

Based on the procedure in Scheetz et al. (2006), we first exclude probes that were not
expressed in the eye or that lacked sufficient variation. As a result, a total of 18976 probes were considered “sufficient variables”. Among these 18976 probes, we further apply the procedure reported in Huang et al. (2008) to select 3000 probes with large variances. It is expected that only a few genes are related to TRIM32.

We apply the newly proposed procedures to test whether the $j$th gene is associated with TRIM32. The proposed FDR control procedure identifies 19 probes at the nominal level 0.05. The IDs of identified probes and their p-values are listed in Table 3.

Among these 19 probes, 1383110 at is also identified by Huang et al. (2008, 2010); Wang et al. (2012); Li et al. (2017). Other probes such as 1379597 at, 1382263 at and 1390401 at are repeatedly detected by other selection methods (Wang et al., 2012; Huang et al., 2008). Due to the complexity of gene microarray data, it is not surprised that different approaches detect different important gene sets. Based on these repeated findings, we are more confident that the probes selected by the proposed procedures provide useful information for further biological study.

Table 3: Probe ID selected.

| Probe ID      | P-value    | Probe ID      | P-value    |
|---------------|------------|---------------|------------|
| 1383110 at    | 3.04 x 10^{-6} | 1383983 at    | 8.28 x 10^{-5} |
| 1379597 at    | 1.17 x 10^{-5} | 1393021 at    | 8.86 x 10^{-5} |
| 1382263 at    | 1.30 x 10^{-5} | 1371081 at    | 1.19 x 10^{-4} |
| 1380978 at    | 2.30 x 10^{-5} | 1370551 at    | 1.24 x 10^{-4} |
| 1389460 at    | 2.34 x 10^{-5} | 1382517 at    | 1.39 x 10^{-4} |
| 1390401 at    | 3.34 x 10^{-5} | 1393684 at    | 1.41 x 10^{-4} |
| 1369453 at    | 4.07 x 10^{-5} | 1381515 at    | 1.56 x 10^{-4} |
| 1376829 at    | 4.21 x 10^{-5} | 1389955 at    | 1.59 x 10^{-4} |
| 1379818 at    | 5.32 x 10^{-5} | 1373177 at    | 1.65 x 10^{-4} |
| 1382743 at    | 5.94 x 10^{-5} |
Conclusions and discussions

In this paper, we have developed model-free inference procedures for high-dimensional data by using a new reformation within sufficient dimension reduction framework and the orthogonalization technique. The proposed test statistic is shown to asymptotically follow $\chi^2$ distribution under the null hypothesis, and a non-central $\chi^2$ distribution under local alternative hypotheses. The FDR control for a large-scale multiple testing problem using the proposed test to identify important predictors is also studied. We introduce the maximum correlation coefficient to quantify the dependence, and develop a truncated BH method to control the FDR. We show that the procedure is valid for controlling FDR under some mild conditions.

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Appendix

A.1 Proof of Theorem 1. Let us start the proof of Part (a). Note that $\bar{S}_{nk} = S_{nk}^{(1)} + S_{nk}^{(2)}$, where

$$S_{nk}^{(1)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \eta_{ik}(X_{ij} - Z_{ij}^\top \hat{\theta}_j) \quad \text{and} \quad S_{nk}^{(2)} = \frac{(\gamma_{kj} - \hat{\gamma}_{kj})^\top}{\sqrt{n}} \sum_{i=1}^{n} Z_{ij}(X_{ij} - Z_{ij}^\top \hat{\theta}_j).$$

We first show that $S_{nk}^{(2)}$ is asymptotically negligible. Note that

$$|S_{nk}^{(2)}| \leq \sqrt{n} \|\hat{\gamma}_{kj} - \gamma_{kj}\|_1 \frac{1}{n} \sum_{i=1}^{n} \|Z_{ij}(X_{ij} - Z_{ij}^\top \hat{\theta}_j)\|_\infty.$$
It follows by Condition (B1) that \( \|\hat{\gamma}_{kj} - \gamma_{kj}\|_1 = O_p(\lambda_Y s_Y) \). Furthermore, it can be shown that \( \|\frac{1}{n} \sum_{i=1}^{n} Z_{ij}(X_{ij} - Z_{ij}' \hat{\theta}_j)\|_\infty \leq C\lambda_X \) by using the Karush-Kuhn-Tucker condition and similar arguments to those in Loh and Wainwright (2015). Thus it follows that \( |S_{nk}^{(2)}| = O_p(\sqrt{n}\lambda_Y \lambda_X s_Y) = o_p(1) \) since \( s_Y = o(\sqrt{n}/\log p) \) and \( \lambda_X, \lambda_Y = O(\sqrt{\log p/n}) \) in Condition (B3).

We next deal with \( S_{nk}^{(1)} \). Note that \( S_{nk}^{(1)} = S_{11} + S_{12} \), where

\[
S_{11} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \eta_{ik}^j (X_{ij} - Z_{ij}' \theta_j^*), \quad \text{and} \quad S_{12} = \frac{(\theta_j^* - \hat{\theta}_j)'}{\sqrt{n}} \sum_{i=1}^{n} \eta_{ik}^j Z_{ij}.
\]

It can be shown by using the central limit theorem that \( S_{11} \) converges to a normal distribution. Thus, we focus on \( S_{12} \). Let \( Z_{jl} \) be the \( l \)th element of \( Z_j \). Under Condition (B2) and using Lemma 1.7.2 in Vershynin (2018), \( \eta_{ik}^j Z_{jl} \) is sub-exponential. Then it follows from Corollary 2.8.3 in Vershynin (2018) that

\[
\Pr\left( \left| \frac{1}{n} \sum_{i=1}^{n} \eta_{ik}^j Z_{ij,l} \right| \geq t \right) \leq 2 \exp \left( -Cn \min \left( \frac{t^2}{K^2}, \frac{t}{K} \right) \right),
\]

where \( K_i = \| \eta_k Z_{jl} \|_{\varphi_1} \). Then it follows that

\[
\Pr \left( \frac{1}{n} \left| \sum_{i=1}^{n} \eta_{ik}^j Z_{ij} \right|_{\infty} > t \right) \leq 2p \max_{1 \leq l \leq p} \exp \left( -Cn \min \left( \frac{t^2}{K^2}, \frac{t}{K} \right) \right) \leq 2p \exp \left( -Cn \min \left( \frac{t^2}{K^2}, \frac{t}{K} \right) \right),
\]

where \( K = \max_{l} K_l \), which is finite by Condition (B2).

Setting \( t = C' \sqrt{\log p/n} \) and \( C' \) be a large enough constant, it leads to

\[
\left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \eta_{ik}^j Z_{ij} \right\|_{\infty} = O_p(\sqrt{\log p}).
\]

On the other hand, under Condition (B1), \( \|\theta_j^* - \hat{\theta}_j\|_1 = O_p(\lambda_X s_X) \). Thus when \( s_X = o(\sqrt{n}/\log p) \), it follows that \( |S_{12}| \leq \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \eta_{ik}^j Z_{ij} \right|_{\infty} \| \theta_j^* - \hat{\theta}_j \|_1 = O_p(\lambda_X s_X \sqrt{\log p}) = o_p(1) \).

In summary, \( S_{nk}^{(j)} = S_{11} + o_p(1) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \eta_{ik}^j (X_{ij} - Z_{ij}' \theta_j^*) + o_p(1) \). Denote \( \Omega_j \) to be a \( h \times h \) matrix with \((i, k)\)-element being \( E[\eta_{ik}^j (X_{ij} - Z_{ij}' \theta_j^*)^2] \). By definition of \( S_{nk}^{(j)} \), it follows that \( S_{nk}^{(j)} \to N_h(0, \Omega_j) \).
As a result, we have that $S_n^{ij\top} \Omega_j^{-1} S_n^{ij} \to \chi^2_h$. In practice, $\Omega_j$ is generally unknown, but can be consistently estimated by $\hat{\Omega}_j = \frac{1}{n} \sum_{i=1}^n S_i^{ij\top} S_i^{ij}$, whose $(l,k)$-element is

$$\hat{\omega}_{lk} = \frac{1}{n} \sum_{i=1}^n (f_l(Y_i) - Z_{ij}^\top \tilde{\gamma}_i)(f_k(Y_i) - Z_{ij}^\top \tilde{\gamma}_k)(X_{ij} - Z_{ij}^\top \hat{\theta}_j)^2.$$ 

The consistency of $\hat{\Omega}_j$ to $\Omega_j$ will follow from the consistency of $Z_{ij}^\top \tilde{\gamma}_i, Z_{ij}^\top \tilde{\gamma}_k, Z_{ij}^\top \hat{\theta}_j$ to $Z_{ij}^\top \gamma_i, Z_{ij}^\top \gamma_k, Z_{ij}^\top \theta_j^*$, respectively. Thus, it follows that $\max_{1 \leq i \leq n} |Z_{ij}^\top (\hat{\theta}_j - \theta_j^*)| = o_p(1)$.

Other consistency results can be established by similar arguments. Note that

$$Z_{ij}^\top (\hat{\theta}_j - \theta_j^*) = (Z_{ij} - \mu_{Z_j})^\top (\hat{\theta}_j - \theta_j^*) + \mu_{Z_j}^\top (\hat{\theta}_j - \theta_j^*);$$

$$|\mu_{Z_j}^\top (\hat{\theta}_j - \theta_j^*)| \leq \|\mu_{Z_j}\|_\infty \|\hat{\theta}_j - \theta_j^*\|_1 = o_p(1).$$

To control the term $(Z_{ij} - \mu_{Z_j})^\top (\hat{\theta}_j - \theta_j^*)$, we first derive the order of the term $\max_{1 \leq i \leq n} \|Z_{ij} - \mu_{Z_j}\|_\infty$. Under Condition (B2), it follows that

$$\Pr(\max_{1 \leq i \leq n} \|Z_{ij} - \mu_{Z_j}\|_\infty > t) \leq 2np \max_{1 \leq l \leq p} \exp \left(-\frac{C t^2}{J_l}\right) \leq 2np \exp \left(-\frac{C t^2}{J}\right),$$

where $J = \max_i J_i, J_l = \|Z_{j,l} - \mu_{Z_{j,l}}\|_{\psi_2}^2$. Letting $t = C'' \sqrt{\log n + \log p}$ with $C''$ being a large constant, it follows that

$$\max_{1 \leq i \leq n} \|Z_{ij} - \mu_{Z_j}\|_\infty = O_p(\sqrt{\log n + \log p}).$$

Thus, $\max_{1 \leq i \leq n} |Z_{ij}^\top (\hat{\theta}_j - \theta_j^*)| = o_p(1)$. Therefore, under condition both $s_X$ and $s_Y$ is of order $o(\sqrt{n}/\log p)$, $W_{nj} = S_n^{ij\top}\hat{\Omega}_j^{-1} S_n^{ij} \to \chi^2_h$ in distribution.

We next prove Part (b). Consider $\bar{S}_{nk}^j$. Similar to the proof of Part (a), we have

$$\bar{S}_{nk}^j = \frac{1}{\sqrt{n}} \sum_{i=1}^n \eta_k^j (X_{ij} - Z_{ij}^\top \theta_j^*) + o_p(1),$$

where $\eta_k^j = f_k(Y) - Z_{ij}^\top \gamma_k, \gamma_k = \mathbb{E} [Z_j Z_j^\top]^{-1} \mathbb{E} [Z_j f_k(Y)]$ and $\theta_j^* = \mathbb{E} [Z_j Z_j^\top]^{-1} \mathbb{E} [Z_j X_j]$. Under local alternative hypotheses $H_{1n}; f_k(Y) = X^\top \beta_k^0 + \epsilon_k$, with $\mathbb{E} [\epsilon_k X] = 0$. Thus,

$$\eta_k^j = \epsilon_k + X_{ij} \beta_{jk}^0 - \beta_{jk}^0 \theta_j^* Z_{ij}.$$
Since \( E \left[ \theta_j^\top Z_j (X_j - Z_j^\top \theta_j^*) \right] = 0 \), we have

\[
S_{nk}^j = \frac{1}{\sqrt{n}} \sum_{i=1}^n \epsilon_{ik} (X_{ij} - Z_{ij}^\top \theta_j^*) + \frac{\sqrt{n} \beta_{jk}}{n} \sum_{i=1}^n X_{ij} (X_{ij} - Z_{ij}^\top \theta_j^*)
\]

\[
- \frac{\sqrt{n} \beta_{jk}}{n} \sum_{i=1}^n \theta_j^\top Z_{ij} (X_{ij} - Z_{ij}^\top \theta_j^*) + o_p(1)
\]

\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^n \epsilon_{ik} (X_{ij} - Z_{ij}^\top \theta_j^*) + \frac{\sqrt{n} \beta_{jk}}{n} \sum_{i=1}^n X_{ij} (X_{ij} - Z_{ij}^\top \theta_j^*) + o_p(1).
\]

By definition of \( S_{nk}^j \), we have \( S_{nk}^j \to N_h(\delta_j, \Omega_j) \), where \( \delta_j = E(X_j^2) - E(X_j Z_j^\top) E(Z_j Z_j^\top)^{-1} E(Z_j X_j) \).

Thus, \( W_{nj} \to \chi_h^2(\delta_j^2 b^\top \Omega_j^{-1} b) \). This completes the proof of Theorem 1.

### A.2 Proof of Theorem 2

Set \( b_p = 2 \log p + 2d_0 \log \log p \) and \( a_p = 2 \log p + (h - 1) \log \log p \).

If \( \tilde{\tau} \) does not exist in \([0, b_p]\), we consider the event \( J_0 = \left\{ \sum_{j \in \mathcal{A}} I(W_{nj} \geq a_p) \geq 1 \right\} \). By Proposition 2, it holds

\[
\Pr(\sum_{j \in \mathcal{A}} I(W_{nj} \geq a_p) \geq 1) = \Pr(\exists j \in \mathcal{A}, \tilde{W}_{nj} + W_{nj} - \tilde{W}_{nj} \geq a_p)
\]

\[
\leq \Pr(\exists j \in \mathcal{A}, \tilde{W}_{nj} \geq a_p - Co(1)) + \Pr(\exists j \in \mathcal{A}, W_{nj} - \tilde{W}_{nj} \geq Co(1))
\]

\[
\leq p \max_{j \in \mathcal{A}} \Pr(W_{nj} \geq a_p) + o(1)
\]

\[
= p \max_{j \in \mathcal{A}} \Pr(\chi_h^2 \geq a_p)(1 + o(1)) + o(1)
\]

\[
= O\left(\frac{1}{\sqrt{\log p}}\right) + o(1) = o(1).
\]

The second to last equality follows by Theorem 3.1 and Lemma 3.2 in [Cai et al. (2019)] which implies that \( \max_{j \in \mathcal{A}} \left| \frac{\Pr(W_{nj} \geq t)}{G(t)} - 1 \right| = O\left((\log p)^{-3/2}\right) \) uniformly in \( t \in [0, a_p] \). The last equality comes from (S.1) in Lemma S.4. Hence, \( \text{FDP}(a_p) \to 0 \) with probability 1.

Now consider the case when \( 0 \leq \tilde{\tau} \leq b_p \) holds. We have

\[
\text{FDP}(\tilde{\tau}) = \frac{\sum_{j \in \mathcal{A}} I \{ W_{nj} \geq \tilde{\tau} \}}{\max \left\{ \sum_{j=1}^p I \{ W_{nj} \geq \tilde{\tau} \}, 1 \right\}} \leq \frac{|\mathcal{A}| \cdot G(\tilde{\tau})}{\max \left\{ \sum_{j=1}^p I \{ W_{nj} \geq \tilde{\tau} \}, 1 \right\}} (1 + A_p),
\]

where \( A_p = \sup_{0 \leq \tilde{\tau} \leq b_p} \left| \frac{\sum_{j \in \mathcal{A}} I \{ W_{nj} \geq \tilde{\tau} \}}{|\mathcal{A}| \cdot G(\tilde{\tau})} - 1 \right| \). Note that by definition of \( \tilde{\tau} \),

\[
\frac{|\mathcal{A}| \cdot G(\tilde{\tau})}{\max \left\{ \sum_{j=1}^p I \{ W_{nj} \geq \tilde{\tau} \}, 1 \right\}} \leq a.
\]

The proof is complete if \( A_p \to 0 \) in probability.
We next show
\[
\sup_{0 \leq t \leq b_p} \left| \frac{\sum_{j \in \mathcal{A}^c} I \{W_{nj} \geq t\}}{|\mathcal{A}^c|G(t)} - 1 \right| \to 0. \tag{A.1}
\]

Recall the Condition D2 that \(|\hat{W}_{nj} - W_{nj}| = o_p(1)\) uniformly for all \(j\), based on Lemma S.5 we have \(G(t + o(1))/G(t) = 1 + o(1)\) uniformly for \(t \in [0, b_p]\). Thus, to show (A.1), it is sufficient to show that
\[
\sup_{0 \leq t \leq b_p} \left| \frac{\sum_{j \in \mathcal{A}^c} I \{\hat{W}_{nj} \geq t\}}{|\mathcal{A}^c|G(t)} - 1 \right| \to 0.
\]

To this aim, it suffices to show that
\[
\sup_{0 \leq t \leq b_p} \left| \frac{\sum_{j \in \mathcal{A}^c} \left\{ I \left( \hat{W}_{nj} \geq t \right) - G(t) \right\}}{pG(t)} \right| \to 0
\]
in probability since \(|\mathcal{A}^c|/p \to 1\). Define \(v_p = 1/\log \log p\) and \(t_l = lv_p\), for \(l = 1, \ldots, L - 1\), and \(t_0 = 0, t_L = b_p\). Thus for any \(t \in [0, b_p]\), there exists \(l \in \{1, \ldots, L - 1\}\) such that \(t_l \leq t \leq t_{l+1}\). Then we have
\[
\sum_{j \in \mathcal{A}^c} I \left( \left| \hat{W}_{nj} \right| \geq t_{l+1} \right) \frac{G(t_{l+1})}{G(t_l)} \leq \sum_{j \in \mathcal{A}^c} I \left( \left| \hat{W}_{nj} \right| \geq t \right) \frac{G(t_l)}{pG(t)} \leq \sum_{j \in \mathcal{A}^c} I \left( \left| \hat{W}_{nj} \right| \geq t_l \right) \frac{G(t_l)}{G(t_{l+1})},
\]

Since \(\frac{G(t_{l+1})}{G(t_l)} \to 1\) uniformly over \(l \in \{1, \ldots, L - 1\}\), according to the Lemma 6.3 in Liu (2013), it is sufficient to show that
\[
\int_0^{b_p} \Pr \left[ \left| \frac{\sum_{j \in \mathcal{A}^c} \left\{ I \left( \left| \hat{W}_{nj} \right| \geq t \right) - \Pr \left( \left| \hat{W}_{nj} \right| \geq t \right) \right\}}{pG(t)} \right| \geq \epsilon \right] dt = o(v_p) \tag{A.2}
\]
\[
\sum_{l=0}^{L-1} \Pr \left[ \left| \frac{\sum_{j \in \mathcal{A}^c} \left\{ I \left( \hat{W}_{nj} \geq t_l \right) - G(t_l) \right\}}{pG(t_l)} \right| \geq \epsilon \right] = o(1). \tag{A.3}
\]

We will show only (A.2) since (A.3) can be proved in a similar way. We next show (A.2).
By Markov inequality, it is enough to show that
\[
\int_0^{b_p} \mathbb{E} \left| \sum_{j \in A^c} \left\{ I \left( |\bar{W}_{nj}| \geq t \right) - \Pr ( \bar{W}_{nj} \geq t ) \right\} \right|^2 \frac{dt}{pG(t)} = o(v_p).
\]

Note that
\[
\int_0^{b_p} \mathbb{E} \left| \sum_{j \in A^c} \left\{ I \left( |\bar{W}_{nj}| \geq t \right) - \Pr ( \bar{W}_{nj} \geq t ) \right\} \right|^2 \frac{dt}{pG(t)}
\]
\[
= \int_0^{b_p} \sum_{i,j \in B_0} \left\{ \Pr ( \bar{W}_{ni} \geq t, \bar{W}_{nj} \geq t ) - \Pr ( \bar{W}_{ni} \geq t ) \Pr ( \bar{W}_{nj} \geq t ) \right\} \frac{pt^2G^2(t)}{pG(t)} dt
\]
\[
+ \int_0^{b_p} \sum_{i,j \in B_1} \left\{ \Pr ( \bar{W}_{ni} \geq t, \bar{W}_{nj} \geq t ) - \Pr ( \bar{W}_{ni} \geq t ) \Pr ( \bar{W}_{nj} \geq t ) \right\} \frac{pt^2G^2(t)}{pG(t)} dt
\]
\[
+ \int_0^{b_p} \sum_{j \in A^c} \left\{ \Pr ( \bar{W}_{nj} \geq t ) - \Pr ( \bar{W}_{nj} \geq t ) \right\} \frac{pt^2G^2(t)}{pG(t)} dt
\]
\[=: I_0 + I_1 + I_2
\]
where \( B_0 = \{ i, j \in A^c; \rho_{ij}^* \leq (\log p)^{-2-\epsilon_0} \} \) and \( B_1 = \{ i, j \in A^c; \rho_{ij}^* \geq (\log p)^{-2-\epsilon_0} \} \).

We first show that \( I_2 = o(v_p) \).
\[
I_2 \leq \int_0^{b_p} \sum_{j \in A^c} \left\{ \Pr ( \bar{W}_{nj} \geq t ) \right\} \frac{pt^2G^2(t)}{pG(t)} dt \leq \int_0^{b_p} \frac{1}{pG(t)(1 + o_p(1))} dt \leq C \int_0^{b_p} \frac{1}{pG(t)} dt \leq C \frac{b_p}{pG(b_p)}.
\]

Note that \( G(b_p) \geq C_h p^{-1} \log p \) by (5.2) and \( b_p = O(\log p) \). We can obtain that
\[
\frac{b_p}{pG(b_p)} \leq C \log p^{2d_0 - h/2}. \]

Under the condition \( 2d_0 < h - 4 \), we have \( I_2 = o(v_p) \).

As to \( I_0 \), it follows by Lemma 3 of Cai et al. (2019) that
\[
I_0 \leq C \int_0^{b_p} (\log p)^{-\frac{h}{2}} dt = O((\log p)^{-\frac{h}{2}}) = o(v_p).
\]

As to \( I_1 \), it follows by Lemma 3 of Cai et al. (2019) that for any \( \delta > 0 \),
\[
I_1 \leq \int_0^{b_p} \sum_{i,j \in B_1} \left\{ \Pr ( \bar{W}_{ni} \geq t, \bar{W}_{nj} \geq t ) \right\} \frac{pt^2G^2(t)}{pG(t)} dt
\]
\[
\leq C \int_0^{b_p} \sum_{i,j \in B_1} \{ (t + 1)^{-1} \exp(-t/(1 + \rho_{ij}^* + \delta)) \} \frac{pt^2G^2(t)}{pG(t)} dt.
\]
Note that
\[ I_1 \leq C \sum_{i,j \in B_1} \int_0^{b_p} \frac{\exp(-t/(1 + \rho_{i,j}^* + \delta))}{p^2 G(t)^2} dt. \]

Note \(G(t)\) is a strictly decreasing function,\[ I_1 \leq C \frac{1}{p^2 G(b_p)^2} \sum_{i,j \in B_1} \int_0^{b_p} \exp(-t/(1 + \rho_{i,j}^* + \delta)) dt \]
\[ \leq C \frac{1}{p^2 G(b_p)^2} \sum_{i,j \in B_1} (1 + \rho_{i,j}^* + \delta)[1 - \exp(-\frac{b_p}{1 + \rho_{i,j}^* + \delta})] \]
\[ \leq C \frac{1 + \rho_{i,j}^* + \delta}{p^2 G(b_p)^2} |B_1| = O\left(\frac{|B_1|}{[\log p]^{h-2d_0-2}}\right). \]

By Condition (D1), we have \(I_1 = o(v_p)\). As a result, \((A.2)\) is valid. We can show \((A.3)\) in a similar proof to that for \((A.2)\). Then, it completes the proof of Theorem 2.

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Supplement to
Model-Free Statistical Inference on High-Dimensional Data

Abstract

Section S.1 consists of some technical lemmas which are used in the proof of Theorem 1, Theorem 2, and Proposition 2. Section S.2 provides the proof of Proposition 2. Section S.3 presents additional simulation results.

We first present assumptions on penalty functions used in the penalized least squares in Section 3.

Assumption (P):

(P1) Assume that the penalty function \( p_\lambda(t) \) is symmetric around zero, increasing on the nonnegative real line, and has a continuous derivative \( p'_\lambda(t) \) with \( p'_\lambda(0^+) = \lambda L > 0 \).

(P2) For \( t > 0 \), the function \( p_\lambda(t)/t \) is non-increasing in \( t \).

(P3) There exists a constant \( \gamma > 0 \) such that \( p_{\lambda,\gamma}(t) = p_\lambda(t) + \gamma t^2/2 \) is convex.

Assumption (P) corresponds to Assumption 1 in Loh and Wainwright (2015) and is very mild. Many commonly-used penalty functions including the Lasso penalty (Tibshirani, 1996) and SCAD penalty (Fan and Li, 2001) satisfy Assumption (P).

S.1 Some useful lemmas

Lemma S.1. (Loh and Wainwright, 2015; Ning and Liu, 2017) Suppose assumption (P), conditions (B2)-(B4) are valid. \( \hat{\theta}_j \) defined in (3.4) satisfies

\[
\|\hat{\theta}_j - \theta^*_j\|_1 \leq c \lambda_X s_X,
\]

\[
\frac{1}{n} \sum_{i=1}^{n} [Z_{ij}^{T}(\hat{\theta}_j - \theta_j^*)]^{2} \leq C \lambda^2_X s_X,
\]

with probability at least \( 1 - \frac{1}{p^{1+\epsilon_2}} \) where \( \epsilon_2 \) is a small constant. Here \( c, C \) are universal constants.
Lemma S.2. Suppose assumption (P), conditions (B2)-(B4) are valid. With probability goes to 1, \( \hat{\gamma}_{kj} \) defined in (3.1) or (3.2) both satisfy
\[
\| \hat{\gamma}_{kj} - \gamma_{kj} \|_1 \leq c_Y s_Y,
\]
\[
\frac{1}{n} \sum_{i=1}^{n} |Z_{ij}^\top (\hat{\gamma}_{kj} - \gamma_{kj})|^2 \leq C_Y^2 s_Y,
\]
uniformly for \( j = 1, \ldots, p \).

Proof. If \( \hat{\gamma}_{kj} \) is given by (3.1), the corresponding statements hold by lemma 1 and theorems 1 and 2 in Loh and Wainwright (2015). The uniformly property can be derived by union bound inequality.

If \( \hat{\gamma}_{kj} \) is given by (3.2), we observe that
\[
\| \hat{\gamma}_{kj} - \gamma_{kj} \|_1 \leq \| \hat{\beta}_k - \beta_0^k \|_1 + \| \hat{\beta}_{kj} - \beta_{0kj} \|_1 \leq C' Y s_Y,
\]
and
\[
\frac{1}{n} \sum_{i=1}^{n} (Z_{ij}^\top (\hat{\gamma}_{kj} - \gamma_{kj}))^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i^\top (\hat{\beta}_k - \beta_0^k) - X_{ij} (\hat{\beta}_{kj} - \beta_{0kj}))^2
\]
\[
\leq \frac{1}{n} \sum_{i=1}^{n} (X_i^\top (\hat{\beta}_k - \beta_0^k))^2 + \frac{1}{n} \sum_{i=1}^{n} (X_{ij} (\hat{\beta}_{kj} - \beta_{0kj}))^2
\]
\[
\leq C_Y^2 s_Y + (\hat{\beta}_{kj} - \beta_{0kj})^2 \frac{1}{n} \sum_{i=1}^{n} X_{ij}^2
\]
\[
= C_Y^2 s_Y,
\]
where the last inequality uses the subgaussian property of \( X_{ij} \) with uniform norm.

Lemma S.3. Let \( X_1, \ldots, X_n \) be i.i.d mean 0 random variables. If there exist constants \( L_1 \) and \( L_2 \), such that \( \Pr (|X_i| \geq x) \leq L_1 \exp \left( -L_2 x^r \right) \), for some \( r > 0 \), then for \( x \geq \sqrt{8 \mathbb{E}(X_i^2)/n} \)
\[
\Pr \left( \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right| \geq x \right) \leq 4 \exp \left( -\frac{1}{8} n^{r/(2+r)} x^{2r/(2+r)} \right)
\]
\[
+ 4nL_1 \exp \left( \frac{L_2 n^{r/(2+r)} x^{2r/(2+r)}}{2^r} \right).
\]

This lemma builds a general concentration inequality of univariate random variable. Details of proof are referred to Ning and Liu (2017).
Lemma S.4. Suppose $X$ follows a chi-square distribution with $h$ degrees of freedom where $h \geq 2$. Denote $b_p = 2 \log p + 2d_0 \log \log p$. The tail probability $\Pr(X \geq b_p)$ satisfies

\begin{align*}
\Pr(X \geq b_p) &\leq C_1 \frac{(\log p)^{h/2-d_0-1}}{p}, \\
\Pr(X \geq b_p) &\geq C_2 \frac{(\log p)^{h/2-d_0-1}}{p},
\end{align*}

with $p \to \infty$. Here $C_1$ and $C_2$ are two positive constants.

Proof. By Lemma 1 in Inglot and Ledwina (2006), we have

\begin{align*}
\Pr(X \geq t) &\leq \frac{1}{\sqrt{\pi}} \frac{t}{t - h + 2} \mathcal{E}_h(t),
\end{align*}

where $\mathcal{E}_h(t) = \exp \left\{ -\frac{1}{2} (t - h - (h - 2) \log(t/h) + \log h) \right\}$. Since we consider $h$ as a fixed number and $p \to \infty$, we have

\begin{align*}
\Pr(X \geq b_p) &\leq C \frac{\exp(h/2)}{\sqrt{h}} \frac{b_p}{h} \frac{1}{h/2} \exp \left( -\frac{1}{2} b_p \right) \\
&\leq C C_h \frac{\{ \log[p(\log p)^{d_0}] \}}{p(\log p)^{d_0}}^{\frac{h}{2}-1}.
\end{align*}

Thus we have

\begin{align*}
G(b_p) &\leq C C_h \frac{(\log p)^{h/2-d_0-1}}{p}
\end{align*}

for some positive constant $C$ with $p$ goes to $\infty$.

The proof of inequality (S.2) follows from the proposition 3.1 in Inglot (2010):

\begin{align*}
\Pr(X \geq t) &\geq \frac{1 - e^{-2}}{2} \frac{t}{t - h + 2} \mathcal{E}_h(t).
\end{align*}

By substituting the formula of $b_p$, we have

\begin{align*}
\mathcal{E}_h(b_p) &\leq \frac{\exp(h/2)}{\sqrt{h}} \frac{b_p}{h} \frac{1}{h/2} \exp \left( -\frac{1}{2} b_p \right) \\
&= C_h \frac{\{ \log[p(\log p)^{d_0}] \}}{p(\log p)^{d_0}}^{\frac{h}{2}-1}
\end{align*}

where $C_h = \frac{\exp(h/2)}{h^{(h-1)/2}}$. After simple calculation, we can obtain

\begin{align*}
\Pr(X \geq b_p) &\geq C_h \frac{(\log p)^{\frac{h}{2}-d_0-1}}{p}
\end{align*}

for some constant $C_h$ with $p$ goes $\infty$. 

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Lemma S.5. Suppose $X$ follows a chi-square distribution with $h$ degrees of freedom where $h \geq 2$. Let $G(t) = \Pr(X \geq t)$. For any $t \in [0, K \log p]$, the following equation
\[
G(t + \delta) = G(t)(1 + o(1))
\]
holds uniformly if $\delta = o(1)$.

Proof. Note
\[
\frac{G(t + \delta)}{G(t)} = 1 - \frac{\int_t^{t+\delta} g(x)dx}{G(t)} = 1 - \frac{g(t')\delta}{G(t)} t' \in [t, t + \delta]
\]
(S.3)

Firstly, we consider the case $t \geq h - 1$. From the proposition 3.1 in Inglot (2010):
\[
G(t) \geq 1 - e^{-2} \frac{t}{2} - \frac{t}{t - h + 2\sqrt{h}} E_h(t)
\]
where $E_h(t) = \exp \{-\frac{1}{2}(t - h - (h - 2) \log (t/h) + \log h)\}$. Thus we have
\[
\frac{g(t')\delta}{G(t)} \leq C_h \frac{g(t')\delta}{E(t)} \leq C_h (t')^{h/2-1} \exp(-t'/2) \exp(t/2) \exp(t - h/2 - h) \leq C_h \frac{(t')^{h/2-1} \exp(t - t'/2)}{E(t)} = C_h \delta(1 + o(1))
\]
where $C_h$ only depends on $h$. If $t < h - 1$, $G(t) \geq G(h - 1) > 0$. The result still holds by noting
\[
\frac{G(t + \delta)}{G(t)} \geq 1 - \frac{g(t')\delta}{G(h - 1)} \geq 1 - \frac{\max_{t \in [0,h-1]} g(t)}{G(h - 1)} \delta = 1 + o(1).
\]
The reverse direction holds by
\[
\frac{G(t + \delta)}{G(t)} \leq 1 - \frac{g(t')\delta}{G(0)} \leq 1 - \min_{t \in [0,K \log p]} g(t)\delta = 1 + o(1).
\]
S.2 Proof of Proposition 2

Due to the similarity between two procedures in (3.1) and (3.2) of estimating $\gamma_{kj}$, we only present the proofs that $\hat{\gamma}_{kj}$ is given by (3.1).

From the proof of Theorem 1(a), it follows that

$$
\max_{j \in A} \|\bar{S}_n^j - \tilde{S}_n^j\|_1 = O_p\{F(n, p, s_X, s_Y)\} = O_p\left(\max\{s_X, s_Y\} \log p/\sqrt{n}\right).
$$

Under Condition (B3), $\max\{s_X, s_Y\} = o(\sqrt{n}/\log p)$. Thus, $\|\bar{S}_n^j - \tilde{S}_n^j\|_1 = o_P(1)$. This will be used to prove $|W_{nj} - \tilde{W}_{nj}| = o_P(1)$. Although this condition is enough to show $W_{nj} \to \chi_h^2$ for the fixed $j \in A^c$, it is not enough to control the false discovery rate at pre-specified level. In the proof of this proposition, we establish its convergence rate with respect to $(n, p, s_X, s_Y)$. Here we set $\max\{s_X, s_Y\} = s$.

We begin with estimation of the rate of $\|\hat{\Omega}_j - \Omega_j\|_2$. Recall that $\hat{\Omega}_j = \frac{1}{n} \sum_{i=1}^n S_i^j S_i^j\top$, we define $\tilde{\Omega}_j = \frac{1}{n} \sum_{i=1}^n \tilde{S}_i^j \tilde{S}_i^j\top$, where the $k$-th component of $\tilde{S}_i^j$ is $\tilde{S}_{ik}^j = \eta_{ik}(X_{ij} - Z_{ij}\theta_j^*)$, $\eta_{ik} = f_k(Y_i - Z_{ij}\gamma_{kj})$ for $k \in \{1, \cdots, h\}$. Denote $I_{1j} = \|\hat{\Omega}_j - \tilde{\Omega}_j\|_2$ and $I_{2j} = \|\tilde{\Omega}_j - \Omega_j\|_2$. It follows by the triangle inequality that $\|\hat{\Omega}_j - \Omega_j\|_2 \leq I_{1j} + I_{2j}$.

Since $\|\tilde{\Omega}_j - \Omega_j\|_2 \leq h\|\tilde{\Omega}_j - \Omega_j\|_{\text{max}}$ and $h$ is a fixed number by assumption, $I_{2j}$ has the same convergence rate as $\|\tilde{\Omega}_j - \Omega_j\|_{\text{max}}$. Denote $d_{kl}^j$ to be the $(k, l)$-element of $\tilde{\Omega}_j - \Omega_j$. $d_{kl}^j = \frac{1}{n} \sum_{i=1}^n \{\eta_{ik}\eta_{il}(X_{ij} - Z_{ij}\theta_j^*)^2 - \omega_{kl}\}$, where $\omega_{kl}$ is the $(k, l)$-element of $\Omega_j$. By union bound inequality, we have $\Pr(|\tilde{\Omega}_j - \Omega_j|_{\text{max}} \geq t) \leq h^2 \max_{1 \leq k, l \leq h} \Pr(|d_{kl}^j| \geq t)$.

Note that

$$
\Pr(|\tilde{\eta}_{ik}\tilde{\eta}_{il}(X_{ij} - Z_{ij}\theta_j^*)^2 - \omega_{kl}| \geq t)
\leq \Pr(|\tilde{\eta}_{ik}\tilde{\eta}_{il}(X_{ij} - Z_{ij}\theta_j^*)^2| \geq t - |\omega_{kl}|)
\leq \Pr\left(|\tilde{\eta}_{ik}\tilde{\eta}_{il}| \geq t - |\omega_{kl}|\right) + \Pr((X_{ij} - Z_{ij}\theta_j^*)^4 \geq t - |\omega_{kl}|)
= \Pr(|\tilde{\eta}_{ik}\tilde{\eta}_{il}| \geq \sqrt{t - |\omega_{kl}|}) + \Pr((X_{ij} - Z_{ij}\theta_j^*)^2 \geq \sqrt{t - |\omega_{kl}|})
$$

while $t \geq |\omega_{kl}|$. Under Condition (B2), $\tilde{\eta}_{ik}\tilde{\eta}_{il}$ and $(X_{ij} - Z_{ij}\theta_j^*)^2$ are sub-exponential random variables with uniform bounded sub-exponential norm. Thus we can find some constants.
$L_{1kl}, L_{2kl}$ such that $\Pr(|\eta_{ik}^j \eta_{kl}^j| \geq \sqrt{t - |\omega_{kl}|}) \leq L_{1kl} \exp(-L_{2kl} \sqrt{t})$. While $t < |\omega_{kl}|$, we can select $L_{3kl}$ which satisfies $L_{3kl} \exp(-L_{2kl} \sqrt{t}) \geq 1$. Similar result holds for $(X_{ij} - Z_{ij}^\top \theta_j)^2$.

Hence, we verified the condition of Lemma S.3.

It follows by Lemma S.3 that for some $t \geq C\sqrt{(\log np)^5/n}$ with large constant $C$,

$$\Pr(\|\hat{\Omega}_j - \Omega_j\|_{max} \geq t) \leq 4h^2 \exp\left(-\frac{1}{8}n^{1/5}t^{2/5}\right) + 4nh^2C_1 \exp\left(-C_2n^{1/5}t^{2/5}\right) = o\left(\frac{1}{np}\right)$$

for some constants $C_1$ and $C_2$.

Thus, $\|\hat{\Omega}_j - \Omega_j\|_{max} = O_p(\sqrt{(\log np)^5/n})$. Since $\Pr(\|\hat{\Omega}_j - \Omega_j\|_{max} \geq t) = o(1/np)$ for fixed $j$, it is easy to show max $\|\hat{\Omega}_j - \Omega_j\|_{max} = O_p(\sqrt{(\log np)^5/n})$ holds uniformly for $j \in \mathcal{A}^c$ by the union bound inequality.

As to $I_{1j}$, it suffices to derive the rate of $\|\hat{\Omega}_j - \Omega_j\|_{max}$. By definitions,

$$\|\hat{\Omega}_j - \Omega_j\|_{max} \leq 2h\|\frac{1}{n} \sum_{i=1}^{n} [S_i^j - \tilde{S}_i^j][\tilde{S}_i^j]^\top\|_{max} + h\|\frac{1}{n} \sum_{i=1}^{n} [\tilde{S}_i^j - S_i^j][\tilde{S}_i^j - S_i^j]^\top\|_{max}$$

Denote $I_{1j1} = \frac{1}{n} \sum_{i=1}^{n} [S_i^j - \tilde{S}_i^j][\tilde{S}_i^j]^\top$ and $I_{1j2} = \frac{1}{n} \sum_{i=1}^{n} [\tilde{S}_i^j - S_i^j][\tilde{S}_i^j - S_i^j]^\top$. Decomposing $I_{1j2}$, it follows that

$$\|I_{1j2}\|_{max} \leq C\sum_{v=1}^{6} \|I_{1j2v}\|_{max}$$

where the $(k, l)$-element of $I_{1j2v}$ is given by

$$\{I_{1j21}\}_{kl} = \frac{1}{n} \sum_{i=1}^{n} \eta_{ik}^j \eta_{il}^j [Z_{ij}^\top (\theta_j^* - \hat{\theta}_j)]^2$$

$$\{I_{1j22}\}_{kl} = \frac{1}{n} \sum_{i=1}^{n} \eta_{ik}^j (X_{ij} - Z_{ij}^\top \theta_j^*) (\hat{\theta}_j - \theta_j^*)^\top Z_{ij}^\top (\hat{\gamma}_{lj} - \gamma_{lj})$$

$$\{I_{1j23}\}_{kl} = \frac{1}{n} \sum_{i=1}^{n} \eta_{ik}^j Z_{ij}^\top (\hat{\gamma}_{lj} - \gamma_{lj}) [Z_{ij}^\top (\hat{\theta}_j - \theta_j^*)]^2$$

$$\{I_{1j24}\}_{kl} = \frac{1}{n} \sum_{i=1}^{n} (X_{ij} - Z_{ij}^\top \theta_j^*)^2 (\hat{\gamma}_{lj} - \gamma_{lj})^\top Z_{ij} Z_{ij}^\top (\hat{\gamma}_{kj} - \gamma_{kj})$$

$$\{I_{1j25}\}_{kl} = \frac{1}{n} \sum_{i=1}^{n} (X_{ij} - Z_{ij}^\top \theta_j^*) [Z_{ij}^\top (\theta_j^* - \hat{\theta}_j)] (\hat{\gamma}_{lj} - \gamma_{lj})^\top Z_{ij} Z_{ij}^\top (\hat{\gamma}_{kj} - \gamma_{kj})$$

$$\{I_{1j26}\}_{kl} = \frac{1}{n} \sum_{i=1}^{n} [Z_{ij}^\top (\hat{\theta}_j - \theta_j^*)]^2 (\hat{\gamma}_{lj} - \gamma_{lj})^\top Z_{ij} Z_{ij}^\top (\hat{\gamma}_{kj} - \gamma_{kj})$$

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For ease of presentation, we abbreviate $1 \leq i \leq n, 1 \leq j \leq p, 1 \leq k, l \leq h$ as $i, j, k, l$ in the rest of proof procedure, respectively.

For $I_{1j21}$, we obtain $\max_{1 \leq j \leq p} \|I_{1j21}\|_{\max} \leq \max_{i,j,k,l} |\eta^j_{ik} \eta^j_{jl}| \times \max_{1 \leq j \leq p} \frac{1}{n} \sum_{i=1}^{n} [Z_{ij}^\top (\theta^*_j - \hat{\theta}_j)]^2$. Under Condition (B2), we have

$$\Pr(\max_{j,i,k} |\eta^j_{ik} - \mu^j_k| > t) \leq nph \max_{1 \leq k \leq h} \Pr(|\eta^j_{ik} - \mu^j_k| > t) \leq 2nph \exp\left(-C \frac{t^2}{J}\right),$$

where $\mu^j_k = \mathbb{E}(\eta^j_{ik})$, $J = \max_{j,k} \|\eta^j_{ik} - \mu^j_k\|_2^2$. Setting $t = C' \sqrt{\log np}$ with $C'$ being a large constant, we have

$$\max_{j,i,k} |\eta^j_{ik} \eta^j_{jl}| \leq C \log np.$$

with probability goes to 1.

By Lemma, it indicates that $\frac{1}{n} \sum_{i=1}^{n} [Z_{ij}^\top (\theta^*_j - \hat{\theta}_j)]^2 = O_p(\lambda^2 \log s_X) = O_p(\frac{\log p}{n} s)$ uniformly holds for all $j \in (1, \cdots, p)$. Hence, we obtain

$$\max_{1 \leq j \leq p} \|I_{1j21}\|_{\max} = O_p(n^{-1} \log p \log(np)s).$$

By the property of sub-Gaussian variables and Cauchy-Schwartz inequality, it holds that

$$\max_{1 \leq j \leq p} \|I_{1j22}\|_{\max} \leq \max_{i,j,k} |\eta^j_{ik} (X_{ij} - Z_{ij}^\top \theta^*_j)| \max_{j,l} \frac{1}{n} \sum_{i=1}^{n} (\hat{\theta}_j - \theta^*_j)^\top Z_{ij} Z_{ij}^\top (\hat{\gamma}_{lj} - \gamma_{lj})$$

$$\leq \max_{i,j,k} |\eta^j_{ik} (X_{ij} - Z_{ij}^\top \theta^*_j)| \max_{j,l} \sqrt{\frac{1}{n} \sum_{i=1}^{n} (Z_{ij}^\top (\hat{\gamma}_{lj} - \gamma_{lj}))^2} \sqrt{\frac{1}{n} \sum_{i=1}^{n} (Z_{ij}^\top (\hat{\theta}_j - \theta^*_j))^2}$$

$$= O_p(\frac{\log p \log(np)}{n} s).$$
Similarly, we have

\[
\max_{1 \leq j \leq p} \|I_{1j23}\|_{\infty} \leq \max_{i,j,k} \left| \eta_{ik}^j \right| \max_{i,j} \frac{1}{n} \sum_{i=1}^{n} Z_{ij}^\top (\hat{\gamma}_{ij} - \gamma_{ij}) Z_{ij}^\top (\hat{\theta}_j - \theta_j^*)^2 \\
\leq \max_{i,j,k} \left| \eta_{ik}^j \right| \max_{j,l} \frac{1}{n} \sum_{i=1}^{n} (\hat{\gamma}_{ij} - \gamma_{ij}) (\hat{\theta}_j - \theta_j^*)^2 \\
\leq \max_{i,j,k} \left| \eta_{ik}^j \right| \|Z_{ij}\|_{\infty} \max_{j,l} \|\hat{\gamma}_{ij} - \gamma_{ij}\| \frac{1}{n} \sum_{i=1}^{n} (\hat{\theta}_j - \theta_j^*)^2 \\
= O_p \left( s^2 \log(np) \left( \frac{\log p}{n} \right)^{\frac{3}{2}} \right),
\]

and

\[
\max_j \|I_{1j24}\|_{\infty} \leq \max_{i,j} (X_{ij} - Z_{ij}^\top \theta_j^*)^2 \max_{j,k,l} \frac{1}{n} \sum_{i=1}^{n} (\hat{\gamma}_{ij} - \gamma_{ij}) Z_{ij}^\top Z_{ij}^\top (\hat{\gamma}_{kj} - \gamma_{kj}) \\
\leq \max_{i,j} (X_{ij} - Z_{ij}^\top \theta_j^*)^2 \max_{j,k,l} \frac{1}{n} \sum_{i=1}^{n} (Z_{ij}^\top (\hat{\gamma}_{ij} - \gamma_{ij}))^2 \frac{1}{n} \sum_{i=1}^{n} (\hat{\theta}_j - \theta_j^*)^2 \\
= O_p \left( \frac{\log p \log(np)}{n} s \right).
\]

Moreover,

\[
\max_{1 \leq j \leq p} \|I_{1j25}\|_{\infty} = O_p \left( \log(np) \left( \frac{\log p}{n} \right)^{\frac{3}{2}} \right) \\
\max_{1 \leq j \leq p} \|I_{1j26}\|_{\infty} = O_p \left( s^3 \log(np) \left( \frac{\log p}{n} \right)^2 \right).
\]

In summary, we have that

\[
\max_{1 \leq j \leq p} \|I_{1j1}\|_{\infty} = O_p(r(n, p, s)),
\]

where

\[
\begin{align*}
\frac{s \log p \log(np)}{n} \vee s^2 \log(np) \left( \frac{\log p}{n} \right)^{\frac{3}{2}} \vee s^3 \log(np) \left( \frac{\log p}{n} \right)^2.
\end{align*}
\]

Similarly, we can show that

\[
\max_{1 \leq j \leq p} \|I_{1j1}\|_{\infty} = O_p(\sqrt{r(n, p, s)}),
\]

it follows Lemma S.1.16 of Fang et al. (2020) that

\[
\frac{1}{n} \sum_{i=1}^{n} \left| S_i^j - \tilde{S}_i^j \right| \left| S_i^j - \tilde{S}_i^j \right|^\top \leq \frac{1}{n} \sum_{i=1}^{n} \left( S_i^j - \tilde{S}_i^j \right) \left( S_i^j - \tilde{S}_i^j \right)^\top \frac{1}{2} \frac{1}{n} \sum_{i=1}^{n} \tilde{S}_i^j \tilde{S}_i^j \frac{1}{2}.
\]

Thus, we have obtained the rate of \( \max_{1 \leq j \leq p} \|\tilde{\Omega}_j - \Omega_j\|_{\infty} \) as follows

\[
O_p \left( \sqrt{\frac{(\log np)^5}{n}} \vee \frac{s \log p \log(np)}{n} \vee s^2 \log(np) \left( \frac{\log p}{n} \right)^{\frac{3}{2}} \vee s^3 \log(np) \left( \frac{\log p}{n} \right)^2 \right).
\]
Hence \( \max_{1 \leq j \leq p} \| \tilde{\Omega}_j - \Omega_j \|_2 \leq h \max_{1 \leq j \leq p} \| \tilde{\Omega}_j - \Omega_j \|_{\text{max}} = O_p(G(n, p, s)). \)

We next derive the rate of \( |\tilde{S}_n^j \tilde{\Omega}_j^{-1} \tilde{S}_n^j - \tilde{S}_n^j \tilde{\Omega}_j^{-1} \tilde{S}_n^j | \) under the null hypothesis.

Firstly, we show that \( \max_{j \in \mathcal{A}^c} \| \tilde{S}_n^j \|_2^2 = O_p(\log p) \). It follows
\[
\Pr \left( \frac{\| \tilde{S}_n^j \|_2}{\sqrt{n}} \geq C \sqrt{\frac{\log p}{n}} \right) = o\left( \frac{1}{p} \right).
\]

where \( j \in \mathcal{A}^c \). The last equation holds due to the sub-Gaussian assumption in Condition (B2).

Furthermore, \( \| \tilde{\Omega}_j^{-1} \tilde{S}_n^j \|_\infty \leq \| \tilde{\Omega}_j^{-1} \tilde{S}_n^j \|_2 \leq \| \Omega_j^{-1} \tilde{S}_n^j \|_2 + \| (\tilde{\Omega}_j^{-1} - \Omega_j^{-1}) \tilde{S}_n^j \|_2 = O_p(\log p) \) by condition (D3). Therefore,
\[
|\tilde{S}_n^j \tilde{\Omega}_j^{-1} \tilde{S}_n^j - \tilde{S}_n^j \tilde{\Omega}_j^{-1} \tilde{S}_n^j | \\
\leq \| \tilde{\Omega}_j^{-1} \|_{\text{max}} \| \tilde{S}_n^j - \tilde{S}_n^j \|_1^2 + 2 \| \tilde{\Omega}_j^{-1} \tilde{S}_n^j \|_\infty \| \tilde{S}_n^j - \tilde{S}_n^j \|_1 + \| \Omega_j^{-1} - \tilde{\Omega}_j^{-1} \|_2 \| \tilde{S}_n^j \|_1^2 \\
= O_p(\| \tilde{S}_n^j - \tilde{S}_n^j \|_1^2 \vee \| \tilde{S}_n^j - \tilde{S}_n^j \|_1 \log p \vee \| \Omega_j^{-1} - \tilde{\Omega}_j^{-1} \|_2 \log p) \\
= O_p \left( \max \left( \frac{(s \log p)^2}{n}, \frac{s(\log p)^2}{\sqrt{n}}, G \log p \right) \right),
\]

since
\[
\| \tilde{\Omega}_j^{-1} - \Omega_j^{-1} \|_2 \leq \| \tilde{\Omega}_j^{-1} \|_2 \| \Omega_j^{-1} \|_2 \| \Omega_j - \Omega_j \|_2 = O_p(G(n, p, s)).
\]
uniformly in \( j \in \mathcal{A}^c \). The last equality is due to the Weyl’s inequality \( \| \tilde{\Omega}_j^{-1} \|_2 \leq \{ \lambda_{\min}(\Omega_j) - \| \tilde{\Omega}_j - \Omega_j \|_2 \}^{-1} = O_p(1) \).

S.3 Additional simulation results

This section presents some simulation results when the SCAD penalty function is used in the penalized least squares procedure proposed in Section 3.
Figure S.1: The plot of empirical power with respect to $h$ when the SCAD penalty is used.
Table S.1: Empirical rejection rate of $H_{0j}$ with $\alpha = 5\%$ when the SCAD penalty is used.

| $n$ | Method    | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_{1996}$ | $X_{1997}$ | $X_{1998}$ | $X_{1999}$ | $X_{2000}$ |
|-----|-----------|-------|-------|-------|-------|-------|------------|------------|------------|------------|------------|
| 200 | $W_n$-SCAD| 1.000 | 1.000 | 0.05  | 0.040 | 0.055 | 0.037      | 0.05       | 0.049      | 0.047      | 0.05       |
|     | $T^{NL}$-SCAD| 1.000 | 1.000 | 0.036 | 0.036 | 0.038 | 0.030      | 0.038      | 0.028      | 0.039      | 0.042      |
| 400 | $W_n$-SCAD| 1.000 | 1.000 | 0.044 | 0.056 | 0.052 | 0.050      | 0.051      | 0.045      | 0.043      | 0.056      |
|     | $T^{NL}$-SCAD| 1.000 | 1.000 | 0.042 | 0.040 | 0.050 | 0.046      | 0.044      | 0.039      | 0.038      | 0.042      |
| 200 | $W_n$-SCAD| 1.000 | 1.000 | 0.909 | 0.908 | 0.072 | 0.057      | 0.042      | 0.045      | 0.042      | 0.053      |
|     | $T^{NL}$-SCAD| 1.000 | 1.000 | 0.062 | 0.077 | 0.031 | 0.029      | 0.038      | 0.027      | 0.043      | 0.036      |
| 400 | $W_n$-SCAD| 1.000 | 1.000 | 0.998 | 0.997 | 0.047 | 0.046      | 0.060      | 0.036      | 0.054      | 0.035      |
|     | $T^{NL}$-SCAD| 1.000 | 1.000 | 0.130 | 0.136 | 0.039 | 0.040      | 0.050      | 0.038      | 0.049      | 0.039      |
| 200 | $W_n$-SCAD| 1.000 | 0.064 | 0.843 | 0.065 | 0.054 | 0.043      | 0.049      | 0.045      | 0.057      | 0.999      |
|     | $T^{NL}$-SCAD| 0.238 | 0.043 | 0.399 | 0.043 | 0.031 | 0.031      | 0.017      | 0.035      | 0.033      | 0.242      |
| 400 | $W_n$-SCAD| 1.000 | 0.031 | 0.991 | 0.041 | 0.040 | 0.040      | 0.032      | 0.049      | 0.051      | 1.000      |
|     | $T^{NL}$-SCAD| 0.335 | 0.044 | 0.436 | 0.042 | 0.042 | 0.049      | 0.037      | 0.043      | 0.039      | 0.316      |