INEQUALITIES OF HERMITE-HADAMARD TYPE FOR EXTENDED s-CONVEX FUNCTIONS AND APPLICATIONS TO MEANS

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Abstract. In the paper, the authors introduce a new concept “extended s-convex functions”, establish some new integral inequalities of Hermite-Hadamard type for this kind of functions, and apply these inequalities to derive some inequalities of special means.

1. Introduction

Throughout this paper, we use the following notation:

\begin{align}
\mathbb{R} &= (-\infty, \infty), \quad \mathbb{R}_0 = [0, \infty), \quad \text{and} \quad \mathbb{R}_+ = (0, \infty).
\end{align}

The following definitions are well known in the literature.

Definition 1.1. A function \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) is said to be convex if

\begin{align}
f(\lambda x + (1 - \lambda)y) &\leq \lambda f(x) + (1 - \lambda)f(y)
\end{align}

holds for all \( x, y \in I \) and \( \lambda \in [0, 1] \).

Definition 1.2 ([5]). A function \( f : I \subseteq \mathbb{R} \to \mathbb{R}_0 \) is said to be \( P \)-convex if

\begin{align}
f(\lambda x + (1 - \lambda)y) &\leq f(x) + f(y)
\end{align}

holds for all \( x, y \in I \) and \( \lambda \in [0, 1] \).

Definition 1.3 ([6]). A function \( f : I \subseteq \mathbb{R} \to \mathbb{R}_0 \) is said to be a Godunova-Levin function if \( f \) is nonnegative and

\begin{align}
f(\lambda x + (1 - \lambda)y) &\leq \frac{f(x)}{\lambda} + \frac{f(y)}{1 - \lambda}
\end{align}

holds for all \( x, y \in I \) and \( \lambda \in (0, 1) \).

Definition 1.4 ([7]). Let \( s \in (0, 1] \) be a real number. A function \( f : \mathbb{R}_0 \to \mathbb{R}_0 \) is said to be \( s \)-convex (in the second sense) if

\begin{align}
f(\lambda x + (1 - \lambda)y) &\leq \lambda^s f(x) + (1 - \lambda)^s f(y)
\end{align}

holds for all \( x, y \in I \) and \( \lambda \in [0, 1] \).

In recent decades, a lot of inequalities of Hermite-Hadamard type for various kinds of convex functions have been established. Some of them may be recited as follows.

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Theorem 1.1 ([4]). Let \( f : I^o \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^o \) and \( a, b \in I^o \) with \( a < b \). If \( |f'(x)| \) is convex on \([a, b]\), then

\[
|f(a) + f(b)| - \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{(b - a)(|f'(a)| + |f'(b)|)}{8}.
\]

Theorem 1.2 ([9]). Let \( f : I \subseteq \mathbb{R}_0 \to \mathbb{R} \) be differentiable on \( I^o \) and \( a, b \in I \) with \( a < b \). If \( |f'(x)|^q \) is s-convex on \([a, b]\) for some fixed \( s \in (0, 1) \) and \( q \geq 1 \), then

\[
|f(a) + f(b)| - \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{b - a}{2} \left( \frac{1}{2} \right)^{1/q} \left[ \frac{2 + 1/2^s}{(s + 1)(s + 2)} \right]^{1/q} \left[ |f'(a)|^q + |f'(b)|^q \right]^{1/q}.
\]

Theorem 1.3 ([8]). Let \( f : I \subseteq \mathbb{R}_0 \to \mathbb{R} \) be differentiable on \( I^o \), \( a, b \in I \) with \( a < b \), and \( f' \in L[a, b] \). If \( |f'(x)|^q \) is s-convex on \([a, b]\) for some fixed \( s \in (0, 1) \) and \( q > 1 \), then

\[
|f(a + b) - f(a) - f(b)| - \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{b - a}{4} \left[ \frac{1}{(s + 1)(s + 2)} \right]^{1/q} \left( \frac{1}{2} \right)^{1/p} \times \left\{ \left[ |f'(a)|^q + (s + 1) \left| f'(\frac{a + b}{2}) \right|^q \right]^{1/q} + \left[ |f'(b)|^q + (s + 1) \left| f'(\frac{a + b}{2}) \right|^q \right]^{1/q} \right\},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

Theorem 1.4 ([12]). Let \( f : I \subseteq \mathbb{R}_0 \to \mathbb{R} \) be differentiable on \( I^o \), \( a, b \in I \) with \( a < b \), and \( f' \in L[a, b] \). If \( |f'(x)| \) is s-convex on \([a, b]\) for some \( s \in (0, 1) \), then

\[
\left| \frac{1}{6} \left[ f(a) + f(b) + 4f\left( \frac{a + b}{2} \right) \right] - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{(s - 4)6^{s+1} + 2 \times 5^{s+2} - 2 \times 3^{s+2} + 2}{6^{s+2}(s + 1)(s + 2)} \frac{(b - a)(|f'(a)| + |f'(b)|)}{6},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

Some inequalities of Hermite-Hadamard type were also obtained in [1, 2, 3, 10, 11, 13, 14, 15, 16, 17, 18] and related references therein.

In this paper, we will introduce a new concept “extended s-convex functions”, establish some new integral inequalities of Hermite-Hadamard type for extended s-convex functions, and apply these newly established integral inequalities to derive some inequalities of special means. These results generalize inequalities stated in Theorems 1.1 to 1.4.

2. Definition and lemmas

We first define the concept “extended s-convex functions” and establish an integral identity.

Definition 2.1. For some \( s \in [-1, 1] \), a function \( f : I \subseteq \mathbb{R}_0 \to \mathbb{R}_0 \) is said to be extended s-convex if

\[
f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)
\]

holds for all \( x, y \in I \) and \( \lambda \in (0, 1) \).
It is obvious that the extended 1-convex function, 0-convex function, and \(-1\)-convex function are just the usually convex function in Definition 1.1, the \(P\)-convex functions in Definition 1.2, and Godunova-Levin convex function in Definition 1.3, respectively. It is also clear that Definition 2.1 extends Definition 1.4.

For establishing new integral inequalities of Hermite-Hadamard type for extended \(s\)-convex functions, we need the following integral identity.

**Lemma 2.1.** Let \(f : I \subseteq \mathbb{R} \to \mathbb{R}\) be differentiable on \(I^\circ\) and \(a, b \in I\) with \(a < b\). If \(f' \in L[a,b]\) and \(\lambda, \mu \in \mathbb{R}\), then

\[
\frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a + b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx = \frac{b-a}{4} \int_0^1 \left( (1 - \lambda - t)f'\left( t a + (1-t)\frac{a+b}{2} \right) + (\mu - t)f'\left( t a + (1-t)\frac{a+b}{2} + (1-t)b \right) \right) \, dt.
\]

**Proof.** Integrating by parts and changing variable of definite integral yield

\[
\int_0^1 (1 - \lambda - t)f'\left( t a + (1-t)\frac{a+b}{2} \right) \, dt
\]

\[
= - \frac{2}{b-a} \left[ (1 - \lambda - t)f\left( t a + (1-t)\frac{a+b}{2} \right) \right]_0^1 + \int_0^1 f\left( t a + (1-t)\frac{a+b}{2} \right) \, dt
\]

\[
= \frac{2}{b-a} \left[ \lambda f(a) + (1 - \lambda)f\left( \frac{a+b}{2} \right) \right] - \frac{4}{(b-a)^2} \int_a^{(a+b)/2} f(x) \, dx
\]

and

\[
\int_0^1 (\mu - t)f'\left( t a + (1-t)b \right) \, dt
\]

\[
= - \frac{2}{b-a} \left[ (\mu - t)f\left( t a + (1-t)b \right) \right]_0^1 + \int_0^1 f\left( t a + (1-t)b \right) \, dt
\]

\[
= \frac{2}{b-a} \left[ (1 - \mu)f\left( \frac{a+b}{2} \right) + \mu f(b) \right] - \frac{4}{(b-a)^2} \int_a^{b} f(x) \, dx.
\]

Adding these two equations leads to Lemma 2.1. \(\square\)

**Lemma 2.2.** Let \(s > -1\), \(0 \leq \xi \leq 1\), \(\omega \in \mathbb{R} \setminus \{0\}\), \(\eta \geq 0\), and \(\omega + \eta \geq 0\). Then

\[
(2.2) \quad \int_0^1 |\xi - t| (\omega t + \eta)^s \, dt = 2(\omega \xi + \eta)^{s+2} - [\eta + (s+2)\omega \xi] \eta^{s+1} - [2\omega \xi + \eta + s\omega (\xi - 1) - \omega](\omega + \eta)^{s+1}.
\]

In particular, if \((\omega, \eta)\) = \((1, 0), (1, 1), (-1, 1), \) or \((-1, 2)\) respectively, then

\[
\int_0^1 |\xi - t| t^s \, dt = \frac{2\xi^{s+2} - (s+2)\xi + s + 1}{(s+1)(s+2)},
\]

\[
\int_0^1 |\xi - t|(1 + t)^s \, dt = \frac{2(\xi + 1)^{s+2} - [(s+2)\xi - s]^{s+1} - (s+2)\xi - 1}{(s+1)(s+2)}.
\]
\[
\int_0^1 |\xi - t|(1-t)^s \, dt = \frac{2(1-\xi)^{s+2} + (s+2)\xi - 1}{(s+1)(s+2)},
\]
\[
\int_0^1 |\xi - t|(2-t)^s \, dt = \frac{2(2-\xi)^{s+2} + [(s+2)\xi - 2]2^{s+1} + (s+2)\eta - s - 3}{(s+1)(s+2)}.
\]

**Proof.** These follow from straightforward computation of definite integrals. \qed

## 3. Some Integral Inequalities of Hermite-Hadamard Type

Now we are in a position to establish some new integral inequalities of Hermite-Hadamard type for differentiable extended s-convex functions.

**Theorem 3.1.** Let \( f : I \subseteq \mathbb{R}_+ \to \mathbb{R} \) be differentiable on \( I^s \), \( a, b \in I \) with \( a < b \), \( f' \in L[a,b] \), and \( 0 \leq \lambda, \mu \leq 1 \). If \( |f'(x)|^q \) for \( q \geq 1 \) is an extended s-convex function on \([a,b]\) for some fixed \( s \in [-1, 1] \), then

1. when \( -1 < s \leq 1 \), we have

\[
\frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left( \frac{a + b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{b - a}{2^{s/q+2}} \left[ \frac{1}{(s+1)(s+2)} \right]^{1/q} \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1-1/q} \left[ \{2 - \lambda\}^{s+2} + (s+2)\lambda - s - 3 \right] |f'(a)|^q + \{2\lambda^{s+2} - (s+2)\lambda + s + 1\} |f'(b)|^q \right. \\
\left. + \{2(2 - \mu)^{s+2} + [(s+2)\mu - 2]^{s+1} + (s+2)\mu - s - 3\} |f'(b)|^q \right\}^{1/q};
\]

2. when \( s = -1 \), we have

\[
\left| f\left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{2^{3-2/q}} \left\{ \left( \frac{2\ln 2 - 1}{|f'(a)|^q} + |f'(b)|^q \right)^{1/q} + \left[ |f'(a)|^q + (2\ln 2 - 1)|f'(b)|^q \right]^{1/q} \right\}. 
\]

**Proof.** For \( -1 < s \leq 1 \), since \( |f'(x)|^q \) is extended s-convex on \([a,b]\), by Lemmas 2.1 and 2.2 and by Hölder integral inequality, we have

\[
\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left( \frac{a + b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{4} \left[ \int_0^1 |1 - \lambda - t| \left| f'\left( ta + (1-t)\frac{a+b}{2} \right) \right| \, dt + \int_0^1 |\mu - t| \left| f'\left( \frac{a+b}{2} + (1-t)b \right) \right| \, dt \right] \\
= \frac{b - a}{4} \left[ \int_0^1 |1 - \lambda - t| \left| f'\left( \frac{1 + t}{2}a + \frac{1-t}{2}b \right) \right| \, dt + \int_0^1 |\mu - t| \left| f'\left( \frac{a}{2} + \frac{2-t}{2}b \right) \right| \, dt \right] \\
\leq \frac{b - a}{2^{s/q+2}} \left( \int_0^1 |1 - \lambda - t| \, dt \right)^{1-1/q} \left[ \int_0^1 |1 - \lambda - t| \left( (1 + t)^s |f'(a)|^q + (1-t)^s |f'(b)|^q \right) \, dt \right]^{1/q};
\]

\[
\left| f\left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{2^{3-2/q}} \left\{ \left( \frac{2\ln 2 - 1}{|f'(a)|^q} + |f'(b)|^q \right)^{1/q} + \left[ |f'(a)|^q + (2\ln 2 - 1)|f'(b)|^q \right]^{1/q} \right\}.
\]
Corollary 3.1.2. Under conditions of Theorem 3.1, we have
\begin{align*}
&+ \left( \int_{0}^{1} |\mu - t| \, dt \right)^{1-1/q} \left[ \int_{0}^{1} |\mu - t|(t^{s}|f'(a)|^{q} + (2 - t)^{s}|f'(b)|^{q}) \, dt \right]^{1/q} \\
&= \frac{b-a}{2s+q+2} \left\{ \left( \frac{2}{2} - \lambda + \lambda^{2} \right)^{1-1/q} \left[ \frac{1}{(s+1)(s+2)} \left[ (2(2-\lambda)^{s+2} + ((s+2)\lambda - 2)2^{s+1} + (s+2)\lambda - 3)|f'(a)|^{q} + (2\lambda^{s+2} + s + 1 - (s+2)\lambda)|f'(b)|^{q} \right]^{1/q} \\
&+ \left( \frac{1}{2} - \mu + \mu^{2} \right)^{1-1/q} \left[ \frac{1}{(s+1)(s+2)} \left[ (2\mu^{s+2} + s + 1 - (s+2)\mu)|f'(a)|^{q} + (s+2)\mu - s - 3)|f'(a)|^{q} \right]^{1/q} \right\}.
\end{align*}

For $s = -1$, since $|f'(x)|^{q}$ is extended $-1$-convex on $[a, b]$, by Lemma 2.1 and Hölder integral inequality, we have
\begin{align*}
&\left| f\left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \\
&\leq \frac{b-a}{4} \left[ \int_{0}^{1} \left| f'\left( ta + (1-t) \frac{a+b}{2} \right) \right| (1-t) \, dt + \int_{0}^{1} t \left| f'\left( \frac{a+b}{2} + (1-t)b \right) \right| \, dt \right] \\
&\leq \frac{b-a}{2^{2-s/2}} \left\{ \left[ \int_{0}^{1} (1-t) \, dt \right]^{1-1/q} \left[ \int_{0}^{1} (1-t) \left( (1+t)^{-1} |f'(a)|^{q} + (1-t)^{-1} |f'(b)|^{q} \right) \, dt \right]^{1/q} \\
&+ \left( \int_{0}^{1} t \, dt \right)^{1-1/q} \left[ \int_{0}^{1} t^{-1} |f'(a)|^{q} + (2-t)^{-1} |f'(b)|^{q} \, dt \right]^{1/q} \right\}
\end{align*}

Theorem 3.1 is proved. \hfill \Box

Corollary 3.1.1. Under conditions of Theorem 3.1,

1. if $q = 1$ and $-1 < s \leq 1$, we have

\begin{equation}
\left| \lambda f(a) + \mu f(b) \right| + \frac{2 - \lambda - \mu}{2} - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{b-a}{2s+2(s+1)(s+2)} \left[ (2(2-\lambda)^{s+2} + 2\mu^{s+2} + ((s+2)\lambda - 2)2^{s+1} + (s+2)(\lambda - \mu) - 2)2^{s+1} + (s+2)\mu - 2s + 1 + (s+2)(\mu - \lambda) - 2)2^{s+1} \right];
\end{equation}

2. if $q = 1$ and $s = -1$, we have

\begin{equation}
\left| f\left( \frac{a+b}{2} \right) - \frac{1}{b-a} \int_{a}^{b} f(x) \, dx \right| \leq (b-a)(\ln 2) \left[ |f'(a)| + |f'(b)| \right].
\end{equation}

Corollary 3.1.2. Under conditions of Theorem 3.1,
(1) when \( \lambda = \mu \) and \(-1 < s \leq 1 \), we have
\[
\left| \frac{\lambda f(a) + f(b)}{2} + (1 - \lambda) f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right|
\leq \frac{b - a}{2^{2/q + 2}} \left[ \frac{1}{(s + 1)(s + 2)} \right]^{1/q} \left[ \frac{1}{2 - \lambda + \lambda^2} \right]^{1 - 1/q} \left\{ \left[ (2(2 - \lambda)^s + 1) + (s + 2s)\lambda = 2 \lambda^s + 2 + (s + 2s)\lambda - 3 \right] f'(a)^q + (2\lambda^{s+2} + s + 1) f'(a)^q \right. \\
- \left. (s + 2s)\lambda f'(b)^q \right\}^{1/q} + \left[ (2\lambda^{s+2} - (s + 2s)\lambda + s + 1) f'(a)^q \right.
+ \left. (2 - \lambda)^s + 1) f'(b)^q \right]^{1/q} ;
\]

(2) when \( \lambda = \mu = 1 \), \(-1 < s \leq 1 \), and \( q = 1 \),
\[
\left| \frac{\lambda f(a) + f(b)}{2} + (1 - \lambda) f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right|
\leq \frac{b - a}{2^{2/q + 2}} \left[ \frac{2}{(s + 1)(s + 2)} \right]^{1/q} \left\{ \left[ \left( \frac{2s + 1}{2s} \right) f'(a)^q + \frac{f'(b)^q}{2s} \right]^{1/q} \right.
+ \left. \left[ \frac{2s + 1}{2s} f'(b)^q \right]^{1/q} \right\}^{1/q} \frac{4 + (1/2)^s - 1}{(s + 1)(s + 2)}^{1/q} .
\]

(3) when \( \lambda = \mu = 1 \), \(-1 < s \leq 1 \), we have
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) \, dx \right|
\leq \left( \frac{b - a}{8} \right) \left[ \frac{2}{(s + 1)(s + 2)} \right]^{1/q} \left\{ \left[ \left( \frac{2s + 1}{2s} \right) f'(a)^q + \frac{f'(b)^q}{2s} \right]^{1/q} \right.
+ \left. \left[ \frac{2s + 1}{2s} f'(b)^q \right]^{1/q} \right\}^{1/q} \frac{4 + (1/2)^s - 1}{(s + 1)(s + 2)}^{1/q} .
\]

Remark 3.1. The inequality (1.7) is a special case of (3.5) applied to \( 0 < s \leq 1 \). The inequality (1.9) can be deduced from (3.4) applied to \( \lambda = \mu = 1/2 \) and \( 0 < s \leq 1 \). These show that Theorem 3.1 and its corollaries generalize some main results obtained in [9, 12].

Corollary 3.1.3. Under conditions of Theorem 3.1,
(1) when \( s = 1 \), we have
\[
\left| \frac{\lambda f(a) + \mu f(b)}{2} + 2 - \lambda - \mu f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right|
\leq \frac{b - a}{2^{1/q + 2}} \left( \frac{1}{6} \right)^{1/q} \times \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1 - 1/q} \right.
\left[ \left( 4 - 9\lambda + 12\lambda^2 - 2\lambda^3 \right) f'(a)^q + \left( 2 - 3\lambda + 2\lambda^2 \right) f'(b)^q \right]^{1/q} \right.
\left. + \left( \frac{1}{2} - \mu + \mu^2 \right)^{1 - 1/q} \left[ \left( 2 - 3\mu + 2\mu^2 \right) f'(a)^q + \left( 4 - 9\mu + 12\mu^2 - 2\mu^3 \right) f'(b)^q \right]^{1/q} \right\} ;
\]

(2) when \( s = 1 \) and \( q = 1 \), we have
Theorem 3.2.\[
\left| \frac{\lambda f(a) + \mu f(b)}{2} + 2 - \frac{\lambda - \mu}{2} f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{48} \left\{ (6 - 9\lambda + 12\lambda^2 - 2\lambda^3 - 3\mu + 2\mu^3)|f'(a)| + (6 + 3\lambda + 2\lambda^3 - 9\mu + 12\mu^2 - 2\mu^3)|f'(b)| \right\};
\]
(3) when $s = 1$ and $\lambda = \mu$,
\[
\left| \frac{\lambda f(a) + f(b)}{2} + (1 - \lambda)f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{4} \left( \frac{1}{12} \right)^{1/q} \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1 - 1/q} \times \left\{ \left[ (4 - 9\lambda + 12\lambda^2 - 2\lambda^3)|f'(a)|^q + (2 - 3\lambda + 2\lambda^3)|f'(b)|^q \right]^{1/q} + \left[ (2 - 3\lambda + 2\lambda^3)|f'(a)|^q + (4 - 9\lambda + 12\lambda^2 - 2\lambda^3)|f'(b)|^q \right]^{1/q} \right\};
\]
(4) when $s = 1$, $q = 1$, and $\lambda = \mu$, we have
\[
(3.6) \quad \left| \frac{\lambda f(a) + f(b)}{2} + (1 - \lambda)f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{8} \left( 1 - 2\lambda + 2\lambda^2 \right) |f'(a) + f'(b)|.
\]

Remark 3.2. Letting $\lambda = 1$ in (3.6) yields the inequality (1.6) in [4].

Corollary 3.1.4. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on $I^o$, $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'(x)|^q$ is convex on $[a, b]$ for $q \geq 1$, then
\[
\left| \frac{1}{2} \left[ f(a) + f(b) + f \left( \frac{a + b}{2} \right) \right] - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{16} \left\{ \left[ \frac{3|f'(a)|^q + |f'(b)|^q}{4} \right]^{1/q} + \left[ \frac{|f'(a)|^q + 3|f'(b)|^q}{4} \right]^{1/q} \right\},
\]
\[
\left| \frac{1}{3} \left[ f(a) + f(b) + f \left( \frac{a + b}{2} \right) \right] - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{5(b - a)}{72} \left\{ \left[ \frac{37|f'(a)|^q + 8|f'(b)|^q}{45} \right]^{1/q} + \left[ \frac{8|f'(a)|^q + 37|f'(b)|^q}{45} \right]^{1/q} \right\},
\]
\[
\left| \frac{1}{6} \left[ f(a) + f(b) + 4f \left( \frac{a + b}{2} \right) \right] - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{5(b - a)}{72} \left\{ \left[ \frac{61|f'(a)|^q + 29|f'(b)|^q}{90} \right]^{1/q} + \left[ \frac{29|f'(a)|^q + 61|f'(b)|^q}{90} \right]^{1/q} \right\}.
\]

Theorem 3.2. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on $I^o$, $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'(x)|^q$ for $q \geq 1$ is an extended $s$-convex function on $[a, b]$, then for $s \in (-1, 1)$ and $0 \leq \lambda, \mu \leq 1$,
\[
\left| \frac{\lambda f(a) + \mu f(b)}{2} + 2 - \frac{\lambda - \mu}{2} f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{4} \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1 - 1/q} \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1 - 1/q} \left[ (2(1 - \lambda)^{s+2} + (s + 2)\lambda - 1)|f'(a)|^q + (2\lambda^{s+2} + s + 1) \right] \right\}.
\]
Proof. By similar arguments as in the proof of Theorem 3.1 and by the extended $s$-convexity of the function $|f'(x)|^q$, we have

\[
\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \\
\leq \frac{b - a}{4} \left\{ \left( \int_0^1 |1 - \lambda - t| \, dt \right)^{1-1/q} \left[ \int_0^1 |1 - \lambda - t| \left( \int_0^t \left| f\left( \frac{a + b}{2} \right) \right|^q + t^s |f'(a)|^q \right) \, dt \right]^{1/q} \\
+ \left( \int_0^1 |\mu - t| \, dt \right)^{1-1/q} \left[ \int_0^1 |\mu - t| \left( t^s |f\left( \frac{a + b}{2} \right)|^q + (1 - t)^s |f'(b)|^q \right) \, dt \right]^{1/q} \right\} \\
- \left( \frac{2 - \lambda + \lambda^2}{2} \right)^{1-1/q} \left[ \frac{1}{(s + 1)(s + 2)} \left( (2(1 - \lambda)^{s+2} + (s + 2)\lambda - 1)|f'(a)|^q \\
+ (2\lambda^{s+2} - (s + 2)\lambda + s + 1)|f'\left( \frac{a + b}{2} \right)|^q \right)^{1/q} + \left( \frac{1}{2} - \mu + \mu^2 \right)^{1-1/q} \left[ \frac{1}{(s + 1)(s + 2)} \\
\times \left( 2\mu^{s+2} + s + 1 - (s + 2)\mu \right)|f'\left( \frac{a + b}{2} \right)|^q + (2(1 - \mu)^{s+2} + (s + 2)\mu - 1)|f'(a)|^q \right)^{1/q} \right\}. \\
\]

Combining this with

\[
|f'\left( \frac{a + b}{2} \right)|^q \leq \left( \frac{1}{2} \right)^s \left[ |f'(a)|^q + |f'(b)|^q \right]
\]

leads to Theorem 3.2. \hfill \square

Corollary 3.2.1. Under conditions of Theorem 3.2, when $q = 1$, we have

\[
\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \\
\leq \frac{b - a}{4(s + 1)(s + 2)} \left\{ (2(1 - \lambda)^{s+2} + (s + 2)\lambda - 1)|f'(a)| + [2\lambda^{s+2} + 2\mu^{s+2} \\
+ (s + 2)(1 - \lambda - \mu) + s] |f'\left( \frac{a + b}{2} \right)| + [2(1 - \mu)^{s+2} + (s + 2)\mu - 1]|f'(b)| \right\} \\
\leq \frac{b - a}{2s^2(s + 1)(s + 2)} \left\{ (1 - \lambda)^{s+2} + 2\lambda^{s+2} + 2\mu^{s+2} + ((s + 2)\lambda - 1)2^s
\right\}. \\
\]
Corollary 3.2.2. Under conditions of Theorem 3.2, if \( \lambda = \mu \), then

\[
\frac{|\lambda f(a) + f(b)|}{2} + (1 - \lambda)2f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{b - a}{4} \left[ \frac{1}{(s + 1)(s + 2)} \right]^{1/q} \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1 - 1/q} \left\{ \left[ (2(1 - \lambda)^s + (s + 2)\lambda - 1) |f'(a)|^q + (2\lambda^s + s + 1 - (s + 2)\lambda) |f'(a)|^q \right]^{1/q} + \left[ (2\mu^s + s + 1 - (s + 2)\lambda) |f'(a)|^q + (2\lambda^{s+1} + (1 - \lambda)^s + 2s + 1 + ((s + 2)\lambda - 1) 2^s - (s + 2)\lambda + s + 1) |f'(b)|^q \right]^{1/q} \right. 
\]

\[
\left. \left\{ (2\lambda^{s+1} - (s + 2)\lambda + s + 1) |f'(a)|^q + (2\lambda^{s+2} - (s + 2)\lambda + s + 1) |f'(b)|^q \right\}^{1/q} \right) \}
\]

Remark 3.3. The inequality (1.8) can be deduced from letting \( \lambda = \mu = 0 \) in Corollary 3.2.2.

Corollary 3.2.3. Under conditions of Theorem 3.2, when \( s = 1 \), we have

\[
\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{4} \left( \frac{1}{6} \right)^{1/q} \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1 - 1/q} \left[ (1 - 3\lambda + 6\lambda^2 - 2\lambda^3) |f'(a)|^q \right.
\]

\[
\left. + (2\lambda^3 - 3\lambda + 3) |f'(a)|^q \right\}^{1/q} + \left( \frac{1}{2} - \mu + \mu^2 \right)^{1 - 1/q} \left[ (2\mu^3 - 3\mu + 2) |f'(a)|^q \right.
\]

\[
\left. + (1 - 3\mu + 6\mu^2 - 2\mu^3) |f'(b)|^q \right\}^{1/q} \right) \}
\]

\[
\leq \frac{b - a}{2^{1/q + 2}} \left( \frac{1}{6} \right)^{1/q} \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right)^{1 - 1/q} \left[ (4 - 9\lambda + 12\lambda^2 - 2\lambda^3) |f'(a)|^q \right.
\]

\[
\left. + (2\lambda^3 - 3\lambda + 2) |f'(a)|^q \right\}^{1/q} + \left( \frac{1}{2} - \mu + \mu^2 \right)^{1 - 1/q} \left[ (2\mu^3 - 3\mu + 2) |f'(a)|^q \right.
\]

\[
\left. + (4 - 9\mu + 12\mu^2 - 2\mu^3) |f'(b)|^q \right\}^{1/q} \right). 
\]

Theorem 3.3. Let \( f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R} \) be differentiable on \( I^c \), \( a, b \in I \) with \( a < b \), and \( f' \in L[a, b] \). If \( |f'(x)|^q \) for \( q \geq 1 \) is an extended s-convex function on \( [a, b] \), then for \( s \in (-1, 1) \) and \( 0 \leq \lambda, \mu \leq 1 \),

(1) when \( q = 1 \), we have
A straightforward computation gives
$$\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a + b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{2s+2(s+1)}.$$

Substituting the last four equalities into the first inequality and simplifying establish the inequality (3.7).

(2) when $q > 1$, we have
$$\begin{align*}
\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a + b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| &\leq \frac{b-a}{2s/q + 2} \left( \frac{q-1}{2q-1} \right)^{1-1/q} \\
\times &\left\{ \left( \frac{1}{s+1} \right)^{1/q} \left[ (1 - \lambda)^{(2q-1)/(q-1)} + \lambda^{(2q-1)/(q-1)} \right]^{1-1/q} \left[ (2^{s+1} - 1) |f'(a)|^q + |f'(b)|^q \right]^{1/q} \\
+ &\left[ (1 - \mu)^{(2q-1)/(q-1)} + (1 - \mu)^{2q-1} \right]^{1-1/q} \left[ |f'(a)|^q + (2^{s+1} - 1) |f'(b)|^q \right]^{1/q} \right\}.
\end{align*}$$

Proof. For $q > 1$, by the extended $s$-convexity of $|f'(x)|^q$ on $[a, b]$, Lemma 2.1, and Hölder integral inequality, we have
$$\begin{align*}
\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a + b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| &\leq \frac{b-a}{4} \left\{ \int_0^1 |1 - \lambda - t| f\left(\frac{ta + (1-t)\frac{a+b}{2}}{2}\right) dt + \int_0^1 |\mu - t| f\left(\frac{ta + (1-t)\frac{a+b}{2}}{2}\right) dt \right\} \\
&\leq \frac{b-a}{2s/q + 2} \left\{ \left( \int_0^1 |1 - \lambda - t|^{q/(q-1)} dt \right)^{1-1/q} \left\{ \int_0^1 \left[ (1 + t)^s |f'(a)|^q + (1 - t)^s |f'(b)|^q \right] dt \right\}^{1/q} \\
+ &\left( \int_0^1 |\mu - t|^{q/(q-1)} dt \right)^{1-1/q} \left\{ \int_0^1 \left( (2-t)^s |f'(a)|^q + (2-t)^s |f'(b)|^q \right) dt \right\}^{1/q} \right\}.
\end{align*}$$

In virtue of Lemma 2.2, a direct calculation yields
$$\begin{align*}
\int_0^1 |1 - \lambda - t|^{q/(q-1)} dt &= \frac{q-1}{2q-1} \left[ (\lambda^{(2q-1)/(q-1)} + (1 - \lambda)^{(2q-1)/(q-1)} \right], \\
\int_0^1 |\mu - t|^{q/(q-1)} dt &= \frac{q-1}{2q-1} \left[ (\mu^{(2q-1)/(q-1)} + (1 - \mu)^{(2q-1)/(q-1)} \right].
\end{align*}$$

A straightforward computation gives
$$\begin{align*}
\int_0^1 \left[ (1 + t)^s |f'(a)|^q + (1 - t)^s |f'(b)|^q \right] dt &= \frac{(2^{s+1} - 1) |f'(a)|^q + |f'(b)|^q}{s+1}, \\
\int_0^1 \left[ (2-t)^s |f'(a)|^q + (2-t)^s |f'(b)|^q \right] dt &= \frac{|f'(a)|^q + (2^{s+1} - 1) |f'(b)|^q}{s+1}.
\end{align*}$$

Substituting the last four equalities into the first inequality and simplifying establish the inequality (3.7).

For $q = 1$, utilizing the extended $s$-convexity of $|f'(x)|^q$ on $[a, b]$, Lemma 2.1, and Hölder integral inequality, we have
$$\begin{align*}
\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a + b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| &\leq \frac{b-a}{4} \left\{ \int_0^1 |1 - \lambda - t| f\left(\frac{ta + (1-t)\frac{a+b}{2}}{2}\right) dt + \int_0^1 |\mu - t| f\left(\frac{ta + (1-t)\frac{a+b}{2}}{2}\right) dt \right\}
\end{align*}$$
Corollary 3.3.1. Under conditions of Theorem 3.3, 

(1) when $\lambda = \mu$ and $q > 1$, we have

$$\left| \frac{\lambda f(a) + f(b)}{2} + (1 - \lambda)f \left( \frac{a + b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{2s+2} \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right) \left[ |f'(a)| + (2s+1)|f'(b)| \right] \right. $$

and

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{2s+2} \left( \frac{q - 1}{2q - 1} \right)^{1-1/q} \left( \frac{1}{s + 1} \right)^{1/q} \times \left\{ \left[ (2s+1-1)|f'(a)|^q + |f'(b)|^q \right]^{1/q} + \left[ |f'(a)|^q + (2s+1-1)|f'(b)|^q \right]^{1/q} \right\};$$

(2) when $\lambda = \mu = 0$, $a > 1$ and $s > 1$, we have

$$\left| \frac{\lambda f(a) + f(b)}{2} + \frac{2 - \lambda - \mu}{2} f \left( \frac{a + b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{16} \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right) \left[ |f'(a)| + 3|f'(b)| \right] + \left( \frac{1}{2} - \mu + \mu^2 \right) \left[ 3|f'(a)| + |f'(b)| \right] \right\};$$

(2) when $q > 1$ and $s = 1$, we have

$$\left| \frac{\lambda f(a) + f(b)}{2} + \frac{2 - \lambda - \mu}{2} f \left( \frac{a + b}{2} \right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{2^{2q+2}} \left( \frac{q - 1}{2q - 1} \right)^{1-1/q} \times \left\{ \lambda^{(2q-1)/(q-1)} + (1 - \lambda)^{(2q-1)/(q-1)} \right\}^{1-1/q} \left[ 3|f'(a)|^q + |f'(b)|^q \right]^{1/q} \times \left[ (2q-1)/(q-1) + (1 - \mu)^{(2q-1)/(q-1)} \right]^{1-1/q} \left[ |f'(a)|^q + 3|f'(b)|^q \right]^{1/q};$$
When $q = 1, \lambda = \mu, \text{and } s = 1$, we have

$$\left| \frac{\lambda[f(a) + f(b)]}{2} + (1 - \lambda)f\left(\frac{a + b}{2}\right) - \frac{1}{b - a}\int_a^b f(x) \, dx \right| \leq \frac{b - a}{4}\left\{ (1 - 2\lambda + 2\lambda^2)[|f'(a)| + |f'(b)|] \right\};$$

when $q > 1, \lambda = \mu, \text{and } s = 1$, we have

$$\left| \frac{\lambda[f(a) + f(b)]}{2} + (1 - \lambda)f\left(\frac{a + b}{2}\right) - \frac{1}{b - a}\int_a^b f(x) \, dx \right| \leq \left( \frac{q - 1}{2q - 1} \right)^{1-1/q}\frac{b - a}{2^{2/q}} \times \left[ \lambda^{(2q-1)/(q-1)} + (1 - \lambda)^{(2q-1)/(q-1)} \right]^{1-1/q}[|f'(a)|^q + |f'(b)|^q]^{1/q}.$$
Proof. For $q > 1$, since $|f'(x)|^q$ is extended $s$-convex on $[a, b]$, by Lemma 2.1 and Hölder integral inequality, we have
\[
\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \\
\leq \frac{b - a}{4} \left\{ \left( \int_0^1 \left| 1 - \lambda - t \right| \left| f'(t a + (1 - t) \frac{a + b}{2}) \right| \, dt \right) + \int_0^1 \left| \mu - t \right| \left| f'(t \frac{a + b}{2} + (1 - t) b) \right| \, dt \right\} \\
\leq \frac{b - a}{4} \left\{ \left( \int_0^1 \left| 1 - \lambda - t \right|^{q/(q - 1)} \, dt \right)^{1 - 1/q} \left[ \int_0^1 \left| t^s |f'(a)|^q + (1 - t)^s |f'(\frac{a + b}{2})|^q \right| \, dt \right]^{1/q} \\
+ \left( \int_0^1 \left| \mu - t \right|^{q/(q - 1)} \, dt \right)^{1 - 1/q} \left[ \int_0^1 \left( t^s \left| f'(\frac{a + b}{2}) \right|^q \right) \, dt \right]^{1/q} \right\}.
\]
If $q = 1$, we have
\[
\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \\
\leq \frac{b - a}{4} \left\{ \left( \int_0^1 \left| 1 - \lambda - t \right| \left| f'(t a + (1 - t) \frac{a + b}{2}) \right| \, dt \right) + \int_0^1 \left| \mu - t \right| \left| f'(t \frac{a + b}{2} + (1 - t) b) \right| \, dt \right\} \\
\leq \frac{b - a}{4} \left\{ \left( \int_0^1 \left| 1 - \lambda - t \right| \, dt \right)^{1 - 1/q} \left[ \int_0^1 \left| t^s |f'(a)| + (1 - t)^s |f'(\frac{a + b}{2})| \right| \, dt \right]^{1/q} \\
+ \left( \int_0^1 \left| \mu - t \right| \, dt \right)^{1 - 1/q} \left[ \int_0^1 \left( t^s \left| f'(\frac{a + b}{2}) \right| \right) \, dt \right]^{1/q} \right\}.
\]
Theorem 3.4 is thus proved. \qed

Corollary 3.4.1. Under conditions of Theorem 3.4,
(1) when $q = 1$ and $\lambda = \mu$, we have
\begin{align*}
(3.10) \quad & \left| \frac{\lambda [f(a) + f(b)]}{2} + (1 - \lambda) f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \\
& \leq \frac{b - a}{4(2s + 1)} \left( \frac{1}{2} \right) \left[ |f'(a)| + 2 \left| f'(\frac{a + b}{2}) \right| + |f'(b)| \right] \\
& \leq \frac{b - a}{2s + 1} \left( \frac{1}{2} - \lambda + \lambda^2 \right) (2s + 1 + 1)(|f'(a)| + |f'(b)|);
\end{align*}
(2) when $q > 1$ and $\lambda = \mu$, we have
\begin{align*}
\left| \frac{\lambda [f(a) + f(b)]}{2} + (1 - \lambda) f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \\
& \leq \frac{b - a}{4} \left( \frac{q - 1}{2q - 1} \right)^{1 - 1/q} \left( \frac{1}{s + 1} \right)^{1/q} \left[ \lambda^{(2q - 1)/(q - 1)} + (1 - \lambda)^{(2q - 1)/(q - 1)} \right]^{1 - 1/q} \\
& \times \left\{ \left[ |f'(a)|^q + \left| f'(\frac{a + b}{2}) \right|^q \right]^{1/q} + \left[ \left| f'(\frac{a + b}{2}) \right|^q + |f'(b)|^q \right]^{1/q} \right\} \\
& \leq \frac{b - a}{2s + 2} \left( \frac{q - 1}{2q - 1} \right)^{1 - 1/q} \left( \frac{1}{s + 1} \right)^{1/q} \left[ \lambda^{(2q - 1)/(q - 1)} + (1 - \lambda)^{(2q - 1)/(q - 1)} \right]^{1 - 1/q}
\end{align*}
\[
\times \left\{ \left[ (2^s + 1) |f'(a)|^q + |f'(b)|^q \right]^{1/q} + \left[ |f'(a)|^q + (2^s + 1) |f'(b)|^q \right]^{1/q} \right\}.
\]

**Corollary 3.4.2.** Under conditions of Theorem 3.4,

1. when \( q = 1 \) and \( s = 1 \), we have
\[
\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \\
\leq \frac{b - a}{4(s + 1)} \left\{ \left( \frac{1}{2} - \lambda + \lambda^2 \right) \left| f'(a) \right| + \left| f' \left( \frac{a + b}{2} \right) \right| \right\} + \left( \frac{1}{2} - \mu + \mu^2 \right) \left| f' \left( \frac{a + b}{2} \right) \right| + \left| f'(b) \right| \right\}
\]

2. when \( q > 1 \) and \( s = 1 \), we have
\[
\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f \left( \frac{a + b}{2} \right) - \frac{1}{b - a} \int_a^b f(x) \, dx \right| \\
\leq \frac{b - a}{2^{1/q + 2}} \left( \frac{q - 1}{2q - 1} \right)^{1 - 1/q} \left\{ \left[ \lambda^{(2q-1)/(q-1)}(1 - \lambda)^{(2q-1)/(q-1)} \right]^{1 - 1/q} \left[ \left| f' \left( \frac{a + b}{2} \right) \right| + \left| f'(b) \right| \right]^{1/q} \right\}
\]

4. Applications to means

Finally, we apply some inequalities of Hermite-Hadamard type for extended s-convex functions to construct some inequalities for means.

For two positive numbers \( a > 0 \) and \( b > 0 \), let
\[
A(a, b) = \frac{a + b}{2} \quad \text{and} \quad L_s(a, b) = \begin{cases} 
\left( \frac{b^{s+1} - a^{s+1}}{(s + 1)(b - a)} \right)^{1/s}, & \text{if } s \neq 0, -1 \text{ and } a \neq b, \\
\ln b - \ln a, & \text{if } s = -1 \text{ and } a \neq b, \\
\frac{1}{e} \left( \frac{b^b}{a^a} \right)^{1/(b-a)}, & \text{if } s = 0 \text{ and } a \neq b, \\
ap, & \text{if } a = b.
\end{cases}
\]

They are called the arithmetic and generalized logarithmic means of two positive numbers \( a \) and \( b \) respectively.

Let \( f(x) = x^s \) for \( x > 0 \), \( s > 0 \), and \( q \geq 1 \). If \( 0 \leq (s - 1)q \leq 1 \) and \( 0 \leq s - 1 \leq 1 \), then
\[
|f'(tx + (1 - t)y)|^q \leq s^q \left[ f^{(s-1)q}x^{(s-1)q} + (1 - t)^{(s-1)q}y^{(s-1)q} \right] \\
\leq t^{s-1} |f'(x)|^q + (1 - t)^{s-1} |f'(y)|^q
\]
for \( x, y > 0 \) and \( t \in (0, 1) \). If \(-1 < (s - 1)q \leq 0 \) and \(-1 < s - 1 \leq 0 \), then
\[
|f'(tx + (1 - t)y)|^q \leq t^{s-1} |f'(x)|^q + (1 - t)^{s-1} |f'(y)|^q
\]
for \( x, y > 0 \) and \( t \in (0, 1) \). If \(-1 < (s - 1)q \leq 1 \) and \(-1 < s - 1 \leq 1 \), then \(|f'(x)|^q = |s|^{q(x-1)}q\) is an extended \((s - 1)\)-convex function on \([a, b]\).

Applying Corollary 3.1.2 to \(|s|^{q(x-1)}q\) yields the following theorem.

**Theorem 4.1.** Let \( b > a > 0 \), \( q \geq 1 \), \( 0 < s \leq 2 \), \(-1 < (s - 1)q \leq 1 \), and \( 0 \leq \lambda \leq 1 \). Then

\[
|\lambda A(a^s, b^s) + (1 - \lambda)A^s(a, b) - L_s^*(a, b)| \leq \frac{(b - a)s}{2s^2 - 1} \frac{1}{(s + 1)^2} \left( \frac{1}{2} \lambda + \lambda^2 \right)^{1-1/q} \times \left\{ \left( (2(2 - \lambda)s)^{s+1} + 2^s ((s + 1)\lambda - 2) + (s + 1)\lambda - s - 2 \right)A^{(s-1)q} + \lambda^s + 2^s ((s + 1)\lambda - 2) + (s + 1)\lambda - s - 2 \right\}^{1/q}.
\]

Specially, if \( q = 1 \), then

\[
|\lambda A(a^s, b^s) + (1 - \lambda)A^s(a, b) - L_s^*(a, b)| \leq \frac{(b - a)s}{2s - 1} \frac{1}{s(s + 1)} \left( 2^s - 1 \right)A(a^{s-1}, b^{s-1}).
\]

Taking \( f(x) = x^s \) for \( x > 0 \) and \( s > 0 \) in Corollary 3.2.2 derives the following inequalities for means.

**Theorem 4.2.** Let \( b > a > 0 \), \( q \geq 1 \), \( 0 < s \leq 2 \), \(-1 < (s - 1)q \leq 1 \), and \( 0 \leq \lambda \leq 1 \). Then

\[
|\lambda A(a^s, b^s) + (1 - \lambda)A^s(a, b) - L_s^*(a, b)| \leq \frac{(b - a)s}{2s^2 - 1} \frac{1}{s(s + 1)} \left( \frac{1}{2} \lambda + \lambda^2 \right)^{1-1/q} \times \left\{ \left( (2(1 - \lambda)s)^{s+1} + (s + 1)\lambda - 1 \right)A^{(s-1)q} + \lambda a^s + 2^s (2s - 1)A^{(s-1)q} + \lambda^s + 2^s ((s + 1)\lambda - 2) + (s + 1)\lambda - s - 2 \right\}^{1/q}.
\]

In particular, if \( q = 1 \), then

\[
|\lambda A(a^s, b^s) + (1 - \lambda)A^s(a, b) - L_s^*(a, b)| \leq \frac{(b - a)s}{2s} \frac{1}{s(s + 1)} \left( 2^s - 1 \right)A(a^{s-1}, b^{s-1}) + \lambda A^{s-1} + 2^s (2s - 1)A^{(s-1)q} + \lambda^s + 2^s ((s + 1)\lambda - 2) + (s + 1)\lambda - s - 2 \right\}^{1/q}.
\]

Letting \( f(x) = x^s \) for \( x > 0 \) and \( s > 0 \) in Corollary 3.3.1 generates inequalities below.

**Theorem 4.3.** Let \( b > a > 0 \), \( q \geq 1 \), \( 0 < s \leq 2 \), and \( 0 \leq \lambda \leq 1 \).

1. If \( q > 1 \), then

\[
|\lambda A(a^s, b^s) + (1 - \lambda)A^s(a, b) - L_s^*(a, b)| \leq \frac{(b - a)s}{2s^2 - 1} \frac{1}{2^{s(q-1)}} \left( \frac{q - 1}{2^s - 1} \right)^{1/q} \left( \frac{1}{2} \lambda + \lambda^2 \right)^{1/q} \left[ \lambda^{(2q-1)(q-1)} + 2^s \right]^{1/q} + \left( (2s - 1)A^{(s-1)q} + \lambda^s + 2^s \right)^{1/q}.
\]

2. If \( q = 1 \), then

\[
|\lambda A(a^s, b^s) + (1 - \lambda)A^s(a, b) - L_s^*(a, b)| \leq \frac{(b - a)s}{s + 1} \left( \frac{1}{2} \lambda + \lambda^2 \right) A(a^{s-1}, b^{s-1}).
\]
From Corollary 3.4.1, it follows that

**Theorem 4.4.** Let \( b > a > 0, q \geq 1, 0 < s \leq 2, \) and \( 0 \leq \lambda \leq 1. \)

(1) If \( q > 1 \) and \(-1 < (s-1)q \leq 1,\) then

\[
\begin{align*}
&\left| \lambda A(a^s, b^s) + (1 - \lambda) A^s(a, b) - L^s(a, b) \right| \\
&\leq \frac{(b-a)s}{4} \left( \frac{q-1}{2q-1} \right)^{1-1/q} \left( \frac{1}{s+1} \right)^{1/q} \left[ \lambda \frac{(2q-1)/(q-1)}{s+1} \right] \\
&+ (1 - \lambda) \frac{(2q-1)/(q-1)}{s+1} \\
&\times \left\{ \left[ A(s-1)q(a, b) + b(s-1)q \right]^{1/q} + \left[ A(s-1)q(a, b) + b(s-1)q \right]^{1/q} \right\}.
\end{align*}
\]

(2) If \( q = 1,\) then

\[
\begin{align*}
(4.5) \quad &\left| \lambda A(a^s, b^s) + (1 - \lambda) A^s(a, b) - L^s(a, b) \right| \\
&\leq \frac{(b-a)s}{2(s+1)} \left( 1 - 2 \lambda^2 \right) A(a^{s-1}, b^{s-1}) + A^{s-1}(a, b).
\end{align*}
\]

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**References**

[1] R.-F. Bai, F. Qi, and B.-Y. Xi, **Hermite-Hadamard type inequalities for the m- and \((a, m)\)-logarithmically convex functions**, Filomat 27 (2013), no. 1, 1–7; Available online at http://dx.doi.org/10.2298/FIL1301001B.

[2] S.-P. Bai, S.-H. Wang, and F. Qi, **Some Hermite-Hadamard type inequalities for n-time differentiable \((a, m)\)-convex functions**, J. Inequal. Appl. 2012, 2012:267, 11 pages; Available online at http://dx.doi.org/10.1186/1029-242X-2012-267.

[3] L. Chun and F. Qi, **Integral inequalities of Hermite-Hadamard type for functions whose third derivatives are convex**, J. Inequal. Appl. 2013, 2013:451, 10 pages; Available online at http://dx.doi.org/10.1186/1029-242X-2013-451.

[4] S. S. Dragomir and R. P. Agarwal, **Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula**, Appl. Math. Lett. 11 (1998), no. 5, 91–95; Available online at http://dx.doi.org/10.1016/S0893-9659(98)00086-X.

[5] S. S. Dragomir, J. Pečarić, and L. E. Persson, **Some inequalities of Hadamard type**, Soochow J. Math. 21 (1995), no. 3, 335–341.

[6] E. K. Godunova and V. I. Levin, **Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions**, Numerical Mathematics and Mathematical Physics, 138–142, 166, Moskov. Gos. Ped. Inst., Moscow, 1985. (Russian) 1

[7] H. Hudzik and L. Maligranda, **Some remarks on s-convex functions**, Aequationes Math. 48 (1994), no. 1, 100–111; Available online at http://dx.doi.org/10.1007/BF01837981.

[8] S. Hussain, M. I. Bhatti, and M. Iqbal, **Hadamard-type inequalities for s-convex functions, I**, Punjab Univ. J. Math. (Lahore) 41 (2009), 51–60.

[9] U. S. Kirmaci, M. Klaričić Bakula, M. E. Özdemir, and J. Pečarić, **Hadamard-type inequalities for s-convex functions**, Appl. Math. Comput. 193 (2007), no. 1, 26–35; Available online at http://dx.doi.org/10.1016/j.amc.2007.03.030.

[10] F. Qi, Z.-L. Wei, and Q. Yang, **Generalizations and refinements of Hermite-Hadamard’s inequality**, Rocky Mountain J. Math. 35 (2005), no. 1, 235–251; Available online at http://dx.doi.org/10.1216/rmjm/1181069779.

[11] F. Qi and B.-Y. Xi, **Some integral inequalities of Simpson type for GA-ϕ-convex functions**, Georgian Math. J. 20 (2013), no. 4, 775–788; Available online at http://dx.doi.org/10.1515/gmj-2013-0043.
[12] M. Z. Sarikaya, E. Set, and M. E. Özdemir, *On new inequalities of Simpson’s type for s-convex functions*, Comput. Math. Appl. 60 (2010), no. 8, 2191–2199; Available online at http://dx.doi.org/10.1016/j.camwa.2010.07.033.

[13] Y. Shuang, H.-P. Yin, and F. Qi, *Hermite-Hadamard type integral inequalities for geometric-arithmetically s-convex functions*, Analysis (Munich) 33 (2013), no. 2, 197–208; Available online at http://dx.doi.org/10.1524/anly.2013.1192.

[14] S.-H. Wang, B.-Y. Xi, and F. Qi, *Some new inequalities of Hermite-Hadamard type for n-time differentiable functions which are m-convex*, Analysis (Munich) 32 (2012), no. 3, 247–262; Available online at http://dx.doi.org/10.1524/anly.2012.1167.

[15] B.-Y. Xi, R.-F. Bai, and F. Qi, *Hermite-Hadamard type inequalities for the m- and (α, m)-geometrically convex functions*, Aequationes Math. 84 (2012), no. 3, 261–269; Available online at http://dx.doi.org/10.1007/s00010-011-0114-x.

[16] B.-Y. Xi and F. Qi, *Some Hermite-Hadamard type inequalities for differentiable convex functions and applications*, Hacet. J. Math. Stat. 42 (2013), no. 3, 243–257.

[17] B.-Y. Xi and F. Qi, *Some integral inequalities of Hermite-Hadamard type for convex functions with applications to means*, J. Funct. Spaces Appl. 2012 (2012), Article ID 980438, 14 pages; Available online at http://dx.doi.org/10.1155/2012/980438.

[18] T.-Y. Zhang, A.-P. Ji, and F. Qi, *Some inequalities of Hermite-Hadamard type for GA-convex functions with applications to means*, Matematiche (Catania) 68 (2013), no. 1, 229–239; Available online at http://dx.doi.org/10.4418/2013.68.1.17.

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