Quantum kappa-deformed differential geometry and field theory

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I introduce in $\kappa$-Minkowski noncommutative spacetime the basic tools of quantum differential geometry, namely bicovariant differential calculus, Lie and inner derivatives, the integral, the Hodge-$*$ and the metric. I show the relevance of these tools for field theory with an application to complex scalar field, for which I am able to identify a vector-valued four-form which generalizes the energy-momentum tensor. Its closedness is proved, expressing in a covariant form the conservation of energy-momentum.

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1. INTRODUCTION

The $\kappa$-deformation of the Poincaré algebra, $U(so(3,1))\rtimes A^*$ was introduced by Lukierski and collaborators [1, 2]. As the Poincaré algebra could not be deformed using the Drinfeld–Jimbo scheme [3–5], applicable only to simple Cartan–Lie algebras, the starting point was the $q$-deformation of the 5-dimensional anti-de Sitter algebra $o(3,2) \to U_q(o(3,2))$. Dimensionalizing the generator through the introduction of a de Sitter radius $R$, one can perform the the Inönü–Wigner contraction of that algebra, $R \to \infty$ together with the $q \to 0$ limit, in such a way that the dimensionful quantity

$$\kappa^{-1} = R \log q,$$

remains constant. This contraction scheme was first introduced by Celeghini and collaborators [6] and it is useful to introduce a dimensionful deformation parameter. Later Zakrzewski [7] used the same r-matrix implied by the $\kappa$-Poincaré algebra to generate a deformed Poisson structure on the Poincaré group, and its quantization led to the $\kappa$-Poincaré quantum group $A\rtimes \mathbb{C}[SO(3,1)]$. This quantum group was then proven to be dual to the $\kappa$-Poincaré algebra by Kosinski and Maslanka. Majid and Ruegg clarified the bicrossproduct structure of $\kappa$-Poincaré, consisting of a semidirect product of the classical Lorentz algebra $so(1,3)$ acting in a deformed way on the translation sector $A^*$, and a backreaction of the momentum sector on the Lorentz transformations, which renders also the coalgebra semidirect. This work allowed to introduce in a consistent way an homogeneous space of the $\kappa$-deformed symmetry $A$, as the quotient Hopf algebra of the $\kappa$-Poincaré group with the Lorentz group $A\rtimes \mathbb{C}[SO(3,1)]/\mathbb{C}[SO(3,1)]$. The result is a noncommutative Hopf algebra with primitive coproduct, antipode and counit, which can be interpreted, in a noncommutative-geometrical fashion, as the algebra of functions over a noncommutative space-time, called $\kappa$-Minkowski. Symbolically $A \sim \mathbb{C}_q[\mathbb{R}^{3,1}]$. Differential calculus is a fundamental tool which is necessary to study field theory over $\kappa$-Minkowski. Several differential structures can be defined on a noncommutative space, and the requirement of bicovariance is a particularly selective one. Still, there are several bicovariant differential calculi, but in the case of $\kappa$-Minkowski it makes sense to ask covariance under the symmetries of this space, which are encoded in the $\kappa$-Poincaré group. In particular, as Sitarz [9] proved, there are no 4-dimensional bicovariant differential calculi that are also Lorentz-covariant. The simplest calculus that achieve this is 5-dimensional. This phenomenon of the natural emergence of higher-dimensional calculi is a common feature of several noncommutative spaces, as noticed by Majid [10, 11].
In this paper, I introduce a series of concepts which represent the basis to do differential geometry in a noncommutative setting. This allows to study field theory over \( \kappa \)-Minkowski, and to construct manifestly Lorentz-covariant theories. Particularly relevant, for this, have been the results of Radko and Vladimirov \[12\] and Brzezinski \[13\], which are extensively used in this paper. Several results are also based on the star-product introduced by Sitarz and Durhuss \[14\], and the twisted graded trace introduced by the author with Sitarz \[15\].

In Section II I develop the 5-dimensional differential calculus introduced by Sitarz beyond the one-forms, defining the entire differential complex up to 5-forms, which are commutative and, being isomorphic to 0-forms, close the complex. Section III is devoted to the Lie derivative and the inner derivative, exploiting the graded Hopf algebra construction for the differential complex and its dual introduced in \[12\], from which a natural concept of Lie and inner derivative emerge. In Section IV I review the construction of the integral, that has been defined in \[15\], and I derive some useful properties. The last structure that I introduce is the Hodge-\( \ast \). It is defined axiomatically and it is then explicitly constructed. This is probably the most relevant contribution of this paper, and is contained in Section V. Section VI shows an application in field theory of all of the structures I introduced, the differential complex, the Lie and inner derivatives, and the Hodge-\( \ast \). The application is the construction of a vector-valued 4-form, which is the noncommutative analogous of the vector-valued three-form whose components are those of the energy-momentum tensor of a scalar field in Minkowski space. The closedness of this form express the energy and momentum conservation. This reformulation in terms of differential forms of the conservation law allows to express it in a manifestly covariant way, a feat that, without the language of differential forms is problematic in a noncommutative spacetime. The last Section contains the conclusions.

**Notation**

Einstein’s convention for the sum over repeated indices is assumed. Greex indices \( \mu, \nu, \ldots \) go from 0 to 3. Latin beginning-of-alphabet letters \( a, b, \ldots \) refer to indices going from 0 to 4. Latin letters following the \( i \) \( (j, k, l, \ldots) \) refer to spatial indices, 1, 2, 3.

Boldface symbols like \( \omega \) refer to \( n \)-forms, \( n > 0 \), while regular ones \( (i.e. \, f) \) refer to functions, or 0-forms. With an overline \( (\bar{z}, \, z \in \mathbb{C}) \) we indicate complex conjugation. The involution is represented with a dagger \( (x^\dagger, \, x \in \Gamma^\wedge \, \text{or} \, x \in U(so(3,1))\mathbb{A}^*) \).

Symmetrization and antisymmetrization of indices are indicated by curly and square brackets:

\[
\omega_{\{123\}} = \omega_{123} + \omega_{213} + \omega_{231} + \omega_{321} + \omega_{312} + \omega_{132}, \quad \rho_{[ab]} = \rho_{ab} - \rho_{ba}.
\]
II. DIFFERENTIAL CALCULUS OVER $\kappa$-MINKOWSKI

A. $\kappa$-Minkowski and $\kappa$-Poincaré algebras

The $\kappa$-Minkowski space was introduced by Majid and Ruegg [16], as a homogeneous space of $\kappa$-deformed Poincaré symmetries. Majid and Ruegg identified the bicrossproduct structure of the $\kappa$-Poincaré algebra introduced by Lukierski, Nowicki and Ruegg [1, 2], and this in turn allowed to correctly identify the homogeneous space as a noncommutative space, dual to the translation subalgebra.

The $\kappa$-Minkowski algebra $A$, understood as an Hopf $*$-algebra (the involution is represented with the dagger ($\cdot$)$^\dagger$ operation) is generated by $x^\mu$, $\mu = 0, \ldots, 3$,

\[
[x_j, x_0] = \frac{i}{\kappa} x_j, \quad [x_j, x_k] = 0,
\]

\[
\Delta x^\mu = x^\mu \otimes 1 + 1 \otimes x^\mu , \quad \varepsilon(x^\mu) = 0, \quad S(x^\mu) = -x^\mu ,
\]

\[
(x^\mu)^\dagger = x^\mu ,
\]

where $\kappa$ is a real deformation parameter. Commutative Minkowski spacetime is obtained through the limit $\kappa \to 0$. The translation algebra $A^*$ is the dual Hopf $*$-algebra to $\kappa$-Minkowski,

\[
[P_\mu, P_\nu] = 0 ,
\]

\[
\Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0 , \quad \Delta P_j = P_j \otimes 1 + e^{-P_0/\kappa} \otimes P_j , \quad \varepsilon(P_\mu) = 0, \quad S(P_0) = -P_0 , \quad S(P_j) = -e^{P_0/\kappa} P_j ,
\]

\[
(P_\mu)^\dagger = P_\mu .
\]

The $\kappa$-Poincaré Hopf $*$-algebra $U(so(3,1)|\mathbf{\mathbb{C}}) \rtimes A^*$ is the bicrossproduct generated by $N_j, R_k \in U(so(3,1))$, $P_\mu \in A^*$, defined by the following additional relations

\[
[N_j, P_k] = i \delta_{jk} \left( \frac{\kappa}{2}(1 - e^{-2P_0/\kappa}) + \frac{1}{2\kappa} |\vec{P}|^2 \right) - \frac{i}{\kappa} P_j P_k ,
\]

\[
[N_j, P_0] = i P_j , \quad [R_j, P_k] = i \epsilon_{jkl} P_l ,
\]

\[
\Delta N_k = N_k \otimes 1 + e^{-P_0/\kappa} \otimes N_k + \frac{i}{\kappa} \epsilon_{klm} P_l \otimes R_m , \quad \Delta R_j = R_j \otimes 1 + 1 \otimes R_j , \quad \varepsilon(N_j) = 0 , \quad \varepsilon(R_k) = 0 , \quad S(N_j) = -e^{P_0/\kappa} N_j + \frac{i}{\kappa} \epsilon_{jkl} e^{\lambda P_0} P_k R_l , \quad S(R_k) = -R_k ,
\]

\[
(N_j)^\dagger = N_j , \quad (R_k)^\dagger = R_k .
\]

The translation algebra acts covariantly from the left on $\kappa$-Minkowski,

\[
t \triangleright x = x_{(1)} \langle t, x_{(2)} \rangle , \quad t \in A^* , f \in A .
\]
and since the bicrossproduct construction involve a right action of $U(so(3,1))$ on $\mathcal{A}^*$, which is encoded into the commutators (3):

$$t \lhd h = [h, t], \quad t \in \mathcal{A}^*, h \in U(so(3,1)),$$  \hspace{1em} (5)

the $U(so(3,1))$ acts too from the left on $\mathcal{A}$, by dualizing the right-action on $\mathcal{A}^*$:

$$\langle t, h \triangleright x \rangle = \langle t \lhd h, x \rangle, \quad t \in \mathcal{A}^*, x \in \mathcal{A}, h \in U(so(3,1)).$$  \hspace{1em} (6)

Then there is a left covariant action of the whole $\kappa$-Poincaré algebra $U(so(3,1)) \triangleright \triangleright \mathcal{A}^*$ on $\mathcal{A}$, which can be obtained by the action on the coordinate base $x^\mu$:

$$P_0 \triangleright x_0 = i, \quad P_j \triangleright x_j = 0, \quad P_j \triangleright x_0 = 0, \quad P_j \triangleright x_k = -i \delta_{jk},$$  \hspace{1em} (7)

$$R_j \triangleright x_0 = 0, \quad R_j \triangleright x_k = \epsilon_{jkl} x_l, \quad N_j \triangleright x_0 = x_j, \quad N_j \triangleright x_k = \delta_{jk} x_0,$$

and extending it on products of coordinates through the coproducts of $U(so(3,1)) \triangleright \triangleright \mathcal{A}^*$.

The left action of $\kappa$-Poincaré over $\kappa$-Minkowski is covariant under involution, in the sense that

$$(h \triangleright x)^\dagger = S(h) \triangleright x^\dagger, \quad h \in U(so(3,1)) \triangleright \triangleright \mathcal{A}^*, \quad x \in \mathcal{A}.$$  \hspace{1em} (8)

### B. Poincaré invariant differential calculus

In [9] a 5-dimensional bicovariant, Poincaré invariant differential calculus over $\mathcal{A}$ is introduced. We refer to it as $\Gamma$. It is generated by $e^\mu = dx^\mu$ and $e^4$, and it is an $\mathcal{A}$-$\star$-bimodule\(^1\) defined by the following commutation relations

$$[x_j, dx_k] = \frac{i}{\kappa} \delta_{jk} (dt - e^4), \quad [x_j, dt] = \frac{i}{\kappa} dx_j,$$

$$[t, dx_j] = 0, \quad [t, dt] = \frac{i}{\kappa} e^4,$$

$$[x_j, e^4] = \frac{i}{\kappa} dx_j, \quad [t, e^4] = \frac{i}{\kappa} dt,$$

$$\quad (dx^\mu)^\dagger = dx^\mu, \quad (e^4)^\dagger = e^4,$$

which can be written in a more compact form as

$$[x^\mu, e^\nu] = \frac{i}{\kappa} (\eta^{\mu\nu} e^0 - \eta^{0\nu} e^\mu - \eta^{\nu\mu} e^4), \quad [x^\mu, e^4] = \frac{i}{\kappa} e^\mu.$$  \hspace{1em} (10)

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\(^1\) We can’t make out of $\Gamma$ alone an Hopf algebra, because it cannot be closed under coproduct - the identity does not belong to $\Gamma$ (there is no such thing as the identity one-form).
The rules above can be obtained from those calculated in [9] with the substitutions
\[
t \rightarrow -it, \quad dt \rightarrow i dt, \\
x \rightarrow ix, \quad dx_j \rightarrow -idx_j, \\
\phi \rightarrow i\kappa e^4.
\]

(11)

The differential is a map \( d: A \rightarrow \Gamma \) satisfying the Leibniz rule
\[
d(fg) = (df)g + f(dg),
\]

(12)

the commutation relations between functions and differential forms can be written as [12]
\[
e^a f = (\lambda^b \triangleright f) e^b,
\]

(13)

where \( \lambda^a_b \in A^* \), and the differential map can be written as
\[
d f = (i \xi_a \triangleright f) e^a
\]

(14)

where, again, \( \xi_a \in A^* \). Then this, and the Leibniz rule for the differential imply that the coproduct of \( \xi_a \) is
\[
\Delta(\xi_a) = \xi_b \otimes \lambda^b_a + 1 \otimes \xi_a,
\]

(15)

and its antipode and counit are
\[
S(\xi_a) = -\xi_b S(\lambda^b_a) , \quad \epsilon(\xi_a) = 0 ,
\]

(16)

and the associativity of the product between forms and functions \( (\omega(fg) = (\omega f)g, \omega \in \Gamma \) and \( f, g, \in A \)) imply
\[
\Delta(\lambda^a_b) = \lambda^d_c \otimes \lambda^c_b.
\]

(17)

From the formulas above the following additional properties can be derived [12]:
\[
\epsilon(\lambda^a_b) = \delta^a_b, \quad S(\lambda^a_c) \lambda^c_b = \delta^a_b.
\]

(18)

**Proposition 1.** From the relation [9] one deduces the following expressions for \( \lambda^a_b \),
\[
\lambda^a_b = \begin{pmatrix}
\cosh \frac{P_0}{\kappa} + \frac{1}{2\kappa^2} e^{P_0/\kappa} |\vec{P}|^2 & \frac{1}{\kappa} \vec{P} & -\sinh \frac{P_0}{\kappa} - \frac{1}{2\kappa^2} e^{P_0/\kappa} |\vec{P}|^2 \\
\frac{1}{\kappa} e^{P_0/\kappa} \vec{P} & I & -\frac{1}{\kappa} e^{P_0/\kappa} \vec{P} \\
-\sinh \frac{P_0}{\kappa} + \frac{1}{2\kappa^2} e^{P_0/\kappa} |\vec{P}|^2 & \frac{1}{\kappa} \vec{P} & \cosh \frac{P_0}{\kappa} - \frac{1}{2\kappa^2} e^{P_0/\kappa} |\vec{P}|^2
\end{pmatrix},
\]

(19)

where \( I \) is the 3 \times 3 identity matrix, and for \( \xi_a \),
\[
\xi_a = \left\{-\kappa \sinh \frac{P_0}{\kappa} + \frac{1}{2\kappa} e^{P_0/\kappa} |\vec{P}|^2, \vec{P}, \kappa \cosh \frac{P_0}{\kappa} - \frac{1}{2\kappa} e^{P_0/\kappa} |\vec{P}|^2 - \kappa \right\}.
\]

(20)
The elements $\lambda^a_b$ form a matrix which is an element of the 5-dimensional Lorentz group, $SO(4,1)$. In fact
\[ \eta^{cd} \lambda^a_c \lambda^b_d = \eta^{ab} , \] (21)
where $\eta^{ab} = \text{diag}\{-1,1,1,1,1\}$, as one can compute easily.

One could have adopted the opposite convention for the commutation relations between functions and differential forms,
\[ f e^a = e^b (\sigma^a_b \triangleright f) , \] (22)
in this case, of course,
\[ \sigma^a_b = S(\lambda^a_b) . \] (23)
Analogous relations hold for the differential
\[ df = e^a (i \chi_a \triangleright f) , \] (24)
where
\[ \chi_a = -S(\xi_a) , \] (25)
the coproduct of $\chi_a$ is then
\[ \Delta \chi_a = \chi_a \otimes 1 + \sigma_a^b \otimes \chi_b . \] (26)

Now the elements $\chi_a$ are explicitly Lorentz covariant, in the sense that, introducing a natural action of the $\kappa$-Poincaré algebra $U(so(3,1))\triangleright \blacktriangleleft \mathcal{A}^*$ over one-forms in this way [9],
\[ h \triangleright (f dg) = (h^{(1)} \triangleright f)d(h^{(2)} \triangleright g) , \quad h \triangleright (df g) = d(h^{(1)} \triangleright f)(h^{(2)} \triangleright g) , \] (27)
then
\[ R_j \triangleright e_0 = 0 , \quad R_j \triangleright e_k = \epsilon_{jkl} e_l , \quad N_j \triangleright e_0 = e_j , \quad N_j \triangleright e_k = \delta_{jk} e_0 , \] (28)
and the commutation relations between $\chi_a$ and $N_j,R_k$ are
\[ [M_{\mu\nu}, \chi_\rho] = i(\eta_{\mu\rho}\chi_\nu - \eta_{\nu\rho}\chi_\mu) , \quad [M_{\mu\nu}, \chi_4] = 0 , \] (29)
where $M_{0j} = N_j$ and $M_{jk} = \epsilon_{jkl} R_l$. Put in other way, $\chi_\mu$ transform like a 4-vector, while $\chi_4$ transforms like a scalar.

As a last remark for this section let’s notice that both $\chi_a$ and $\xi_a$, when squared with the metric
\[ \eta^{ab} = \text{diag}\{-1,1,1,1,1\} , \] (22)
generate the mass Casimir of $\kappa$-Poincaré [2],
\[ \eta^{ab} \xi_a \xi_b = \eta^{ab} \chi_a \chi_b = \Box_{\kappa} , \] (30)
which id invariant under antipode $S(\Box_\kappa) = \Box_\kappa$ and is a central element of $U(so(3,1))\triangleright \blacktriangleleft \mathcal{A}^*$.
C. The differential complex (forms of degree higher than one)

In [9] it is shown that $\Gamma^2$ is generated by $e^\mu \wedge e^\nu = -e^\nu \wedge e^\mu$ and $e^\mu \wedge e^4 = -e^4 \wedge e^\mu$, with the additional relations

$$e^j \wedge e^j = -e^0 \wedge e^0, \quad de^4 = i \kappa (e^j \wedge e^j - e^0 \wedge e^0),$$  \hspace{1cm} (31)$$

$\Gamma^2$ is another $A$-*-bimodule, and the Jacobi identities applied to mixed products of the kind $x^\mu e^a \wedge e^b$ imply that

$$e^0 \wedge e^0 = e^1 \wedge e^1 = e^2 \wedge e^2 = e^3 \wedge e^3 = e^4 \wedge e^4 = 0, \quad de^4 = 0;$$  \hspace{1cm} (32)$$
to make $\Gamma^2$ into an Hopf *-bimodule, we add the involution as

$$(e^\mu \wedge e^\nu)^\dagger = -e^\mu \wedge e^\nu, \quad (e^\mu \wedge e^4)^\dagger = -e^\mu \wedge e^4.$$  \hspace{1cm} (33)$$

The commutation relations of all the $\Gamma^n$s can be found through the associative property

$$[x^\mu, e^{a_1} \wedge \cdots \wedge e^{a_n}] = [x^\mu, e^{a_1}] \wedge e^{a_2} \wedge \cdots \wedge e^{a_n} + \cdots + e^{a_1} \wedge e^{a_2} \wedge \cdots \wedge [x^\mu, e^{a_n}],$$  \hspace{1cm} (34)$$
and under involution the basic forms of $\Gamma^n$ behave as

$$(e^{a_1} \wedge \cdots \wedge e^{a_n})^\dagger = (-1)^{n(n-1)/2} e^{a_1} \wedge \cdots \wedge e^{a_n}.$$  \hspace{1cm} (35)$$

Due to the graded-commutativity of the wedge product of base forms, $\Gamma^5$ is one-dimensional and is generated only by the (penta-) volume form $\text{vol}^5 = e^0 \wedge e^1 \wedge e^2 \wedge e^3 \wedge e^4$, which is self-adjoint

$$(\text{vol}^5)^\dagger = \text{vol}^5,$$  \hspace{1cm} (36)$$
and commutes with $A$,

$$[x^\mu, \text{vol}^5] = 0,$$  \hspace{1cm} (37)$$
as can be easily proved by direct calculation.

In [13] it is shown that the entire exterior algebra $\Gamma^\wedge = A \oplus \Gamma \oplus \Gamma^2 \oplus \Gamma^3 \oplus \Gamma^4 \oplus \Gamma^5$ can be made into a graded Hopf *-algebra, with coproduct

$$\Delta(e^a) = e^a \otimes 1 + 1 \otimes e^a,$$  \hspace{1cm} (38)$$

These relations were left as matters of choice in [9].
antipode and counit

\[ S(e^a) = -e^a, \quad \epsilon(e^a) = 0, \quad (39) \]

where the extension of the multiplication to the tensor product is nontrivial, and satisfy the rule

\[ (\omega \otimes \rho) \wedge (\omega' \otimes \rho') = (-1)^{nm} (\omega \wedge \omega') \otimes (\rho \otimes \rho'), \quad (40) \]

where \( \omega, \rho, \omega', \rho' \in \Gamma^\wedge \) and \( \rho, \omega' \) are homogeneous forms of degree, respectively, \( n \) and \( m \).

Also the differential map can be extended to \( \Gamma^\wedge \): it is a map \( d : \Gamma^n \rightarrow \Gamma^{n+1} \) obeying the graded Leibniz rule

\[ d(\omega \wedge \rho) = (d\omega) \wedge \rho + (-1)^n \omega \wedge (d\rho), \quad (41) \]

for \( \omega \) homogeneous \((\omega \in \Gamma^n)\) and any \( \rho \in \Gamma^\wedge \), and the nilpotency condition

\[ d \circ d = 0. \quad (42) \]

The extension of \( d \) to the tensor product \( \Gamma^\wedge \otimes \Gamma^\wedge \) is trivial

\[ d(\omega \otimes \rho) = (d\omega) \otimes \rho + (-1)^n \omega \otimes (d\rho) \quad (43) \]

for any \( \rho \in \Gamma^\wedge \) and \( \omega \in \Gamma^n \); the equation above implies that the coproduct of \( \Gamma^\wedge \) and \( d \) commute:

\[ \Delta \circ d = d \circ \Delta. \quad (44) \]

From the covariance of the action of \( A^* \) under involution \( \delta \), we deduce the covariance of the differential

\[ d(\omega^\dagger) = (-1)^n (d\omega)^\dagger. \quad (45) \]

III. THE \( \kappa \)-DEFORMED LIE AND INNER DERIVATIVES

In \cite{12} a (graded) Hopf algebra is built from \( \Gamma^\wedge \) and \( (\Gamma^\wedge)^* = A^* \oplus \Gamma^* \oplus \Gamma^2 \oplus (\Gamma^3)^* \oplus (\Gamma^4)^* \oplus (\Gamma^5)^* \) as the cross product \( \Gamma^\wedge \times (\Gamma^\wedge)^* \). The duality relations between \( \Gamma^\wedge \) and \( (\Gamma^\wedge)^* \) are such that

\[ \langle \xi, \omega \rangle = 0 \iff \xi \in (\Gamma^n)^*, \omega \notin \Gamma^n, \quad (46) \]

and \( \langle \xi, \omega \rangle \) reduces to the duality relation between \( (\Gamma^n)^* \) and \( \Gamma^n \) when \( \xi \in (\Gamma^n)^* \) and \( \omega \in \Gamma^n \).
A. Lie derivative

The algebra $A^*$ in [12] is interpreted as the space of left-invariant vector fields on $A$, and the Lie derivative along an element $h$ of $A^*$ is defined as the adjoint action of $A^*$ over $\Gamma^\wedge \times (\Gamma^\wedge)^*$:

$$\mathcal{L}_h := h \triangleright \text{ad} ,$$

that, on $\Gamma^\wedge$, reduces to

$$\mathcal{L}_h \triangleright \omega = \omega(1) \langle h, \omega(2) \rangle , \quad \forall \omega \in \Gamma^\wedge .$$

Then the Lie derivative of forms along the vector field $h \in A^*$ can be defined as a map $\mathcal{L}_h : \Gamma^n \to \Gamma^n$ such that

$$\mathcal{L}_h(\omega) = (h \triangleright \omega_{a_1...a_n}) e^{a_1} \wedge \cdots \wedge e^{a_n} ,$$

for all $\omega = \omega_{a_1...a_n} e^{a_1} \wedge \cdots \wedge e^{a_n} \in \Gamma^n$. The Lie derivative of products of forms depend on the coproduct of the vector field $h$:

$$\mathcal{L}_h(\omega \wedge \rho) = \mathcal{L}_h(1)(\omega) \wedge \mathcal{L}_h(2)(\rho) ,$$

the coproduct of $h$ is in general non-primitive, with the exception of $P_0$ (the dual element to $x^0$), so in general $\mathcal{L}_h$ does not satisfy the (graded) Leibniz rule. We conclude this subsection with the observation, reported in [12], that the Lie derivative commutes with the differential,

$$\mathcal{L}_h \circ d = d \circ \mathcal{L}_h .$$

B. Inner derivative

The authors of [12] propose to relate inner derivations with elements of $\Gamma^*$. Starting from the base $\{\theta_0, \theta_1, \theta_2, \theta_3, \theta_4\}$ of $\Gamma^*$ which is dual to the base $\{e^0, e^1, e^2, e^3, e^4\}$ of $\Gamma$ we define:

$$\langle \theta_a, e^b f \rangle = \epsilon(f) \delta^b_a , \quad f \in \mathcal{A} ,$$

and has zero bracket with the other elements of $\Gamma^\wedge$

$$\langle \theta_a, \omega \rangle = 0 , \quad \omega \in \mathcal{A}, \Gamma^2, \Gamma^3, \Gamma^4, \Gamma^5 ,$$

3 The definition of the dual base in [12] was different: $\langle \theta_a, f e^b \rangle = \epsilon(f) \delta^b_a$. Here we need to put $f$ on the right to enforce Lorentz covariance (see below).
one defines a base of inner derivations $i_a : \Gamma^n \to \Gamma^{n-1}$ in this way:

$$i_a := \theta_a \triangleright_{\text{ad}} .$$

(54)

The inner derivation of functions is then zero,

$$i_a(f) = f^{(1)} \langle \theta_a, f^{(2)} \rangle = 0 , \quad \forall f \in \mathcal{A} ,$$

(55)

and that of one-forms is

$$i_a(\omega) = \omega_a , \quad \forall \omega = e^a \omega_a \in \Gamma ,$$

(56)

while that of two-forms is

$$i_a(\omega) = e^b (\omega_{ab} - \omega_{ba}) , \quad \forall \omega = e^a \wedge e^b \omega_{ab} \in \Gamma^2 .$$

(57)

In general the inner derivative of an $n$-form can be written as

$$i_a \omega = \delta^{[a_0} e^{b_1} \wedge \cdots \wedge e^{b_{n-1}]} \omega_{b_1 \ldots b_n} , \quad \omega = e^{a_1} \wedge \cdots \wedge e^{a_n} \omega_{a_1 \ldots a_n} \in \Gamma^n .$$

(58)

The inner derivative does not satisfy the graded Leibniz rule like in the commutative case: in fact, for example, the wedge product of two 2-forms is

$$\omega \wedge \rho = e^a \omega_a \wedge e^b \rho_b = e^a \wedge e^b (\sigma^c \triangleright \omega) \rho_c$$

(59)

so that the inner derivative of $\omega \wedge \rho$ is

$$i_a(\omega \wedge \rho) = [(\sigma^c \triangleright \omega) \rho_c - (\sigma^a \triangleright \omega) \rho_a] e^b .$$

(60)

In [12] it is shown that the Cartan identity for the Lie, inner and exterior derivatives holds without modifications,

$$\mathcal{L}_{\chi_a} = d \circ i_a + i_a \circ d ,$$

(61)

and we see that our choice for the duality brackets defining the inner derivative [52] selects $\mathcal{L}_{\chi_a}$, which is Lorentz-covariant in the sense that

$$\mathcal{L}_{\chi_a}(M_{\rho \sigma} \triangleright f) = M_{\rho \sigma} \triangleright \mathcal{L}_{\chi_a}(f) - i \eta_{\rho | \rho} \mathcal{L}_{\chi_a}(f) , \quad f \in \mathcal{A} .$$

(62)
IV. TWISTED-CYCLIC INTEGRAL

This section summarizes the results contained in \[15\] about the twisted graded trace. In addition, we remark the compatibility of the trace with the involution, and we extend the results to the \(3 + 1\)-dimensional case.

Exploiting the commutativity between \(\Gamma^5\) and \(\mathcal{A}\) we introduce a left-invariant integral as in \[15\]. The integral is a linear map

\[
\int : \Gamma^5 \to \mathbb{C}
\]

which respects the involution\(^4\)

\[
\left(\int f\right)^\dagger = \int f^\dagger,
\]

and is invariant under the left action of the whole \(\kappa\)-Poincaré algebra \(U(so(3,1))\rhd \mathcal{A}^*\)

\[
\int h \rhd \rho = \varepsilon(h) \int \rho, \quad \forall h \in U(so(3,1))\rhd \mathcal{A}^*
\]

where the action of \(\kappa\)-Poincaré algebra \(U(so(3,1))\rhd \mathcal{A}^*\) on 5-forms is trivially induced from the action on \(\mathcal{A}\):

\[
\rho = \rho \operatorname{vol}^5 \quad \Rightarrow \quad h \rhd \rho = (h \rhd \rho) \operatorname{vol}^5.
\]

**Proposition 2.** The integral is closed, in the sense that

\[
\int d\omega = 0, \quad \forall \omega \in \Gamma^4,
\]

**Proof.** The closedness follows from the left-invariance under the action of \(\mathcal{A}^*\), in fact for every 4-form \(\omega\):

\[
d\omega = (\xi_a \rhd \omega_{b_1 b_2 b_3 b_4}) e^a \wedge e^{b_1} \wedge \cdots \wedge e^{b_4} = \xi_{[0} \rhd \omega_{1234]} \operatorname{vol}^5,
\]

then

\[
\int d\omega = \varepsilon(\xi_{[0}) \int \omega_{1234]} \operatorname{vol}^5 = 0.
\]

\[\]  

\[^4\] This property is a straightforward consequence of the definition given in \[13\]. The integral is introduced there as the standard Lebesgue integral over \(\mathbb{R}^2\) (generalization to 4 dimensions is straightforward), applied to the functions which give a realization of the \(\kappa\)-Poincaré algebra through the \(*\)-product introduced in \[14\]. The involution too has a realization in terms of these functions, and the property \[63\] can be deduced from it.
The integral, however, is not cyclic with respect to the product of 5-forms with elements of $\mathcal{A}$. Instead, it satisfies a twisted cyclic property \cite{13, 15}:

$$\int fg \text{vol}^5 = \int g(T \circlearrowright f) \text{vol}^5,$$

where $T \in \mathcal{A}^*$ is an automorphism of $\mathcal{A}$:

$$\Delta(T) = T \otimes T, \quad S(T)T = TS(T) = 1, \quad \epsilon(T) = 1.$$ \hfill (71)

The explicit expression of $T$ is

$$T = e^{3P_0/\kappa},$$ \hfill (72)

which is the same result obtained in \cite{15}, but elevated to the third power.

**Proposition 3.** The twisted cyclicity property holds also for products of forms.

**Proof.** If $\omega \in \Gamma^n$, $\rho \in \Gamma^{5-n}$

$$\int \omega \wedge \rho = \int \omega_{a_1 \ldots a_n} e^{a_1} \wedge \cdots \wedge e^{a_n} \wedge \rho_{a_{n+1} \ldots a_5} e^{a_{n+1}} \wedge \cdots \wedge e^{a_5}$$

$$= \int \omega_{a_1 \ldots a_n} (\lambda^{a_1}_{b_1} \ldots \lambda^{a_n}_{b_n} \circlearrowright \rho_{a_{n+1} \ldots a_5}) e^{b_1} \wedge \cdots \wedge e^{b_n} \wedge e^{a_{n+1}} \wedge \cdots \wedge e^{a_5}$$

$$= \frac{1}{5!} \epsilon^{b_1 \ldots b_n a_{n+1} \ldots a_5} \int \omega_{a_1 \ldots a_n} (\lambda^{a_1}_{b_1} \ldots \lambda^{a_n}_{b_n} \circlearrowright \rho_{a_{n+1} \ldots a_5}),$$

now the $\lambda^{a}_b$s are $SO(4,1)$ matrices, then they leave the 5-dimensional Levi-Civita symbol invariant, and the following is true:

$$\lambda^{a_1}_{b_1} \ldots \lambda^{a_n}_{b_n} e^{b_1 \ldots b_n a_{n+1} \ldots a_5} = S(\lambda^{a_{n+1}}_{b_{n+1}}) \ldots S(\lambda^{a_5}_{b_5}) \epsilon^{a_1 \ldots a_n b_{n+1} \ldots b_5},$$ \hfill (73)

then

$$\int \omega \wedge \rho = \frac{1}{5!} \epsilon^{a_1 \ldots a_n b_{n+1} \ldots b_5} \int \omega_{a_1 \ldots a_n} (S(\lambda^{a_{n+1}}_{b_{n+1}}) \ldots S(\lambda^{a_5}_{b_5}) \circlearrowright \rho_{a_{n+1} \ldots a_5}),$$

$$= \frac{1}{5!} \epsilon^{a_1 \ldots a_n b_{n+1} \ldots b_5} \int (S(\lambda^{a_{n+1}}_{b_{n+1}}) \ldots S(\lambda^{a_5}_{b_5}) \circlearrowright \rho_{a_{n+1} \ldots a_5}) (T \circlearrowright \omega_{a_1 \ldots a_n}),$$

$$= \int \rho_{a_{n+1} \ldots a_5} (\lambda^{a_{n+1}}_{b_{n+1}} \ldots \lambda^{a_5}_{b_5} T \circlearrowright \omega_{a_1 \ldots a_n}) e^{a_1} \wedge \cdots \wedge e^{a_n} \wedge e^{b_{n+1}} \wedge \cdots \wedge e^{b_5},$$

$$= \int \rho_{a_{n+1} \ldots a_5} e^{a_{n+1}} \wedge \cdots \wedge e^{a_5} (T \circlearrowright \omega_{a_1 \ldots a_n}) e^{a_1} \wedge \cdots \wedge e^{a_n},$$

$$= \int \rho \wedge (T \circlearrowright \omega).$$
V. HODGE-∗ AND METRIC

We now introduce for the first time a metric structure in \( \kappa \)-Minkowski, through the Hodge-∗.

**Definition 1.** The Hodge-∗ is an involutive map

\[
* : \Gamma^n \to \Gamma^{5-n} , \quad * \circ * = (-1)^{n(5-n)} id ,
\]

which is left and right \( \mathcal{A} \)-linear:

\[
*(f \omega) = f * (\omega) , \quad *(\omega f) = * (\omega) f ,
\]

such that the following sesquilinear form (symplectic form)

\[
(\omega, \rho) = \int \omega^\dagger \wedge * \rho , \quad \omega, \rho \in \Gamma^n ,
\]

is a nondegenerate (indefinite) inner product between forms of the same degree, that is,

\[
(\omega, \rho) = (\rho, \omega) .
\]

**Proposition 4.** The Hodge-∗ is defined by the following rules

\[
\begin{cases}
* (1) = \text{vol}^5 \\
* (e^a) = \frac{1}{3!} \eta^{ab} \epsilon_{bcdef} e^c \wedge e^d \wedge e^e \wedge e^f , \\
* (e^a \wedge e^b) = \frac{1}{3!} \eta^{ac} \eta^{bd} \epsilon_{cdefg} e^c \wedge e^f \wedge e^g ,
\end{cases}
\]

where \( \eta^{ab} = \text{diag}\{-1, 1, 1, 1, 1\} \), and \( \epsilon_{abcde} \) is the 5-dimensional Levi-Civita symbol.

**Proof.** For the left and right \( \mathcal{A} \)-linearity It is sufficient to prove the compatibility of the commutation rules \([10]\), which means

\[
[x^\mu, *(e^\nu)] = \frac{i}{\kappa} (\eta^{\mu\nu} *(e^0) - \eta^0\nu * (e^\mu) - \eta^{\mu\nu} * (e^4)) , \quad [x^\mu, *(e^4)] = \frac{i}{\kappa} *(e^\mu) ,
\]

this can be verified by direct calculation.

It remains to show that the inner product is well-defined. For 0-forms it trivially descends from the compatibility of the integral with the involution. For \( n \)-forms we have

\[
\int \omega^\dagger \wedge * \rho = \frac{1}{(5-n)!} \eta^{a_1b_1} \ldots \eta^{a_nb_n} \epsilon_{c_1 \ldots c_5} \int e^{a_1} \wedge \ldots \wedge e^{a_n} \left( \omega^\dagger_{a_1 \ldots a_n} \rho_{b_1 \ldots b_n} \right) e^{c_{n+1}} \wedge \ldots \wedge e^{c_5}
\]

\[
= \frac{1}{(5-n)!} \eta^{a_1b_1} \ldots \eta^{a_nb_n} \epsilon_{c_1 \ldots c_5} \left( \int \omega^\dagger_{a_1 \ldots a_n} \rho_{b_1 \ldots b_n} \right) \int \omega^\dagger_{a_1 \ldots a_n} \rho_{b_1 \ldots b_n}
\]

\[
= \frac{1}{(5-n)!} \eta^{a_1b_1} \ldots \eta^{a_nb_n} \left( \int \rho^\dagger_{b_1 \ldots b_n} \omega^\dagger_{a_1 \ldots a_n} \right) = \left( \int \rho^\dagger \wedge * \omega \right) .
\]
The nondegeneracy

\[(\omega, \cdot) = 0 \iff \omega = 0 .\]  

(81)

can be proven by considering the following object:

\[(\omega, \tilde{\omega}) = \sum_{a,b,\ldots} \int (\omega^{ab\ldots})^\dagger \omega^{ab\ldots}\]  

(82)

where \(\tilde{\omega}^{ab\ldots} = -\omega^{ab\ldots}\) if the number of indices \(ab\ldots\) which are zero is odd, and \(\tilde{\omega}^{ab\ldots} = \omega^{ab\ldots}\) otherwise. Then \((\omega, \tilde{\omega})\) is a positive sum of terms of the type \(\int f^\dagger f\), and the following chain of implications follow: \((\omega, \rho) = 0 \forall \rho \Rightarrow (\omega, \tilde{\omega}) = 0 \Rightarrow \int (\omega^{ab\ldots})^\dagger \omega^{ab\ldots} = 0 \forall a, b,\ldots\).

We have then reduced the nondegeneracy of the inner product to the nondegeneracy of the norm of 0-forms.

\[\int f^\dagger f = 0 \iff f = 0 ,\]  

(83)

I will prove this in the 2-dimensional case (3-dimensional differential calculus), to exploit the results of [15]. Generalization to 4 dimensions pose no difficulties. From Eq. (2.2) and (2.3) of [15] we can represent the \(f^\dagger f\) through the star product as

\[(f^\dagger f)(\alpha, \beta) = \frac{1}{(2\pi)^2} \int dvdu \int dwdz \tilde{f}(\alpha + u + w, e^{-z/\kappa} \beta) f(\alpha, e^{-v/\kappa} \beta) e^{-i(uv+wz)} ,\]  

(84)

the trace is represented as the ordinary Lebesgue integral over \(\mathbb{R}^2\), so upon a simple change of variable we get

\[\int (f^\dagger f)(\alpha, \beta) = \frac{1}{(2\pi)^2} \int d\alpha d\beta \tilde{F}(\alpha, \beta) F(\alpha, \beta) ,\]  

(85)

where \(F(\alpha, \beta) = \frac{1}{2\pi} \int dvdu e^{iw^\dagger f}(\alpha + u, e^{-v/\kappa} \beta) = f^\dagger(-\alpha, \beta)\), so that \(\int (f^\dagger f)(\alpha, \beta) = 0 \iff F(\alpha, \beta) = 0\). The proof is concluded by the observation that \(F(\alpha, \beta) = 0 \Rightarrow f = 0\), which is trivial.

\[\square\]

The Hodge-\(*\) defined in this way is covariant under the action of the \(\kappa\)-Poincaré algebra \(U(so(3,1))\blacktriangleleft A^*\), in the sense that

\[\ast(h \triangleright \omega) = h \triangleright \ast(\omega) , \quad h \in U(so(3,1))\blacktriangleleft A^* , \]  

(86)

in fact covariance under translations is trivial, because they have null action on basic forms

\[P_\mu \triangleright e^{a_1} \wedge \cdots \wedge e^{a_n} = 0 ,\]
and the left and right $\mathcal{A}$-linearity of the Hodge-$\ast$ implies eq. [86] for $h \in \mathcal{A}^*$. Lorentz covariance is no less straightforward. The action of both boost $N_j$ and rotation $R_k$ generators on basic forms is primitive, in the sense that on products of $e^a$ they act with the Leibniz rule, e.g.:

$$
N_j \triangleright (e^a \wedge e^b) = (N_j^{(1)} \triangleright e^a) \wedge (N_j^{(2)} \triangleright e^a) =
$$

$$
= (N_j \triangleright e^a) \wedge e^b + (e^{-\lambda P_0} \triangleright e^a) \wedge (N_j \triangleright e^b) + \epsilon_{jkl}(P_k \triangleright e^a) \wedge (R_l \triangleright e^b)
$$

$$
= (N_j \triangleright e^a) \wedge e^b + e^a \wedge (N_j \triangleright e^b),
$$

and they both have classical action over a single one-form [9]:

$$
N_j \triangleright e^k = i \delta^k_j e^0, \quad N_j \triangleright e^0 = -i e^j, \quad N_j \triangleright e^4 = 0, \quad (87)
$$

$$
R_j \triangleright e^k = i \epsilon_{jkl} e^l, \quad R_j \triangleright e^0 = 0, \quad R_j \triangleright e^4 = 0, \quad (88)
$$

so it’s easy to see that the rules [78] are covariant. Then the covariance for general forms is proven through the left and right $\mathcal{A}$-linearity.

The $\kappa$-Hodge-$\ast$ induces a metric, understood as a sesquilinear map of one-forms $g : \Gamma \otimes \Gamma \to \mathcal{A}$

$$
g(\omega, \rho) = \ast(\omega^\dagger \wedge * \rho), \quad (89)
$$

which is hermitian

$$
g(\omega, \rho) = g(\rho, \omega)^\dagger. \quad (90)
$$

If applied to the basis forms the metric gives its components

$$
g(e^a, e^b) = \eta^{ab}. \quad (91)
VI. CLASSICAL FIELD THEORY

With $\kappa$-deformed classical field theory I mean any theory which substitutes elements of $\mathcal{A}$ or $\Gamma^\wedge$ to scalar or tensor fields, and which is based on a variational principle or simply on equations of motion, which identify some subset of $\mathcal{A}$ (or $\Gamma^\wedge$) as the space of solutions. A $\kappa$-deformed quantum field theory should be based on an appropriately defined measure over $\mathcal{A}$, and an associated partition function, allowing to perform a path integral. The understanding of classical field theory should prelude the study of quantum field theory over $\kappa$-Minkowski, as is the case also in the commutative Minkowski space.

There is a very strong physical motivation for the study of $\kappa$-deformed field theory, coming from 2+1-dimensional background-independent quantum gravity [18]. In 2+1 dimensions, Einstein gravity reduces to a topological field theory which is solvable, and quantizable with a path integral through spin-foam techniques. Coupling this theory to a scalar field, and integrating out the gravitational degrees of freedom (which corresponds to taking the “no-gravity” limit $G \to 0$) one ends up with an effective partition function for the scalar field, in which the field is valued in $\mathcal{A}$, the 2+1 dimensional $\kappa$-Minkowski space. This gives an indication that $\kappa$-Minkowski may be the fundamental state of quantum gravity, and its noncommutativity could be a manifestation of the non-local correlations induced on fields by the quantum gravitational degrees of freedom. This calculation cannot be performed in 3+1 dimension where quantum gravity is not understood, but is nevertheless one of the most significant results in quantum gravity, pointing out that its fundamental state is likely not to be a classical spacetime.

Therefore the study of field theory over $\kappa$-Minkowski is very relevant for physics, as it may provide the interface between quantum gravity, noncommutative geometry and their observable manifestations. Today there is a fairly large literature on $\kappa$-deformed field theory [19–25]. However, until now, it was hard to build a field theory which is manifestly Lorentz covariant, as the only tool at disposal to define “vectors” was Sitarz’ differential calculus, and one needs much more: at least the higher-degree forms, but also an Hodge-$*$ to create maximal degree forms out of them, and an integral to form an action. I introduced all of these structure with the explicit purpose of making this possible, as I’ll show in this section.

Some authors have considered the problem of establishing an analogue of the Noether theorem in these theories, associating conserved charges to the symmetries of $\kappa$-Minkowski [21, 23, 26]. However, the Lorentz covariance of the conserved charges that were found was never considered, and the meaning of such conservation laws remained obscure. I present a geometrical way of under-
standing the conservation laws, which is allowed by the differential-geometrical tools I developed in the previous sections. A conservation law is expressed as the closure of a current vector-valued 4-form. This form is the energy-momentum tensor expressed in the language of differential forms. In the commutative Minkowski space its analogue is a vector-valued 3-form, but here we need a 4-form due to the additional dimension of the differential calculus. To calculate this current 4-form, I will need also the Lie and inner derivative, and all of the new structure I introduced in this paper will then find an application in field theory.

A. Scalar field

As action for a complex scalar field we take

\[
S = \frac{1}{2} \int \left\{ (d\phi)^\dagger \star (d\phi) + m^2 \phi^\dagger \star (\phi) \right\},
\]

(92)

with some calculations we can see that this action is the same as that used in [22]

\[
S = \frac{1}{2} \int \left\{ \eta^{bc}(\xi_a \triangleright \phi^\dagger) (\lambda^a \xi_b \triangleright \phi) + m^2 \phi^\dagger \phi \right\} \text{vol}^5,
\]

(93)

To calculate the equations of motion we make use of a variational procedure, which, written in Fourier transform following the techniques shown in [14, 15] gives the same results and makes perfectly sense

\[
\delta S = \frac{1}{2} \int \left\{ d(\delta \phi^\dagger) \star (d\phi) + d(\phi^\dagger) \star (d\delta \phi) + m^2 \delta \phi^\dagger \star (\phi) + m^2 \phi^\dagger \star (\delta \phi) \right\}
\]

\[
= \frac{1}{2} \int \left\{ \delta \phi^\dagger \left[ m^2 \star (\phi) - d \star d (\phi) \right] + d(\phi^\dagger) \star (d\delta \phi) + m^2 \phi^\dagger \star (\delta \phi) \right\}
\]

(94)

imposing the minimization of the action functional we end up with the following equations of motion

\[
\delta S = 0 \quad \Rightarrow \quad \star d \star d \phi - m^2 \phi = 0, \quad \star d \star d \phi^\dagger - m^2 \phi^\dagger = 0.
\]

(95)
We easily see that the map $\ast d \ast d$ is identical to the action of the mass casimir $\Box_\kappa$ on scalar fields

$$\ast d \ast d = \ast d [\xi_a \triangleright \phi \ast (e^a)] = \frac{1}{5!} \eta^{ab} \varepsilon_{bcdef} \ast d [\xi_a \triangleright \phi \ast d e^c \wedge e^d \wedge e^e \wedge e^f] =$$

$$\frac{1}{5!} \eta^{ab} \varepsilon_{bcdef} \xi_g \xi_a \triangleright \phi \ast (e^c \wedge e^d \wedge e^e \wedge e^f) =$$

$$\eta^{ab} \varepsilon_{bcdef} \xi_g \xi_a \triangleright \phi \ast (vol^5) = \eta^{ab} \xi_b \xi_a \triangleright \phi = \Box_\kappa \triangleright \phi \ .$$

### B. Noether theorem and energy-Momentum tensor

The following current vector-valued four-form:

$$j_a = \frac{1}{2} \left\{ (\chi_a \triangleright \phi^\dagger) \ast d \phi + \ast d (\sigma^b_a \triangleright \phi^\dagger) \wedge (\chi_b \triangleright \phi) \right\} - i_a (L) , \quad (97)$$

where $L = L \ vol^5 = \frac{1}{2} \left\{ -\phi^\dagger (\Box_\kappa \triangleright \phi) + m^2 \phi^\dagger \phi \right\} vol^5$, is conserved on-shell, in the sense that it is a closed form when $\phi$ and $\phi^\dagger$ minimize the action. Let’s prove it:

$$d j_a = \frac{1}{2} \left\{ d (\chi_a \triangleright \phi^\dagger) \wedge d \phi + d \ast d (\sigma^b_a \triangleright \phi^\dagger) \wedge (\chi_b \triangleright \phi) \right.$$  

$$+ (\chi_a \triangleright \phi^\dagger) \wedge d \ast d \phi + d \ast d (\sigma^b_a \triangleright \phi^\dagger) \wedge d (\chi_b \triangleright \phi) \right\} - L_{\chi a} (L) , \quad (98)$$

using the equations of motion

$$d \ast d (\phi) = m^2 \ast (\phi) , \quad d \ast d (\phi^\dagger) = m^2 \ast (\phi^\dagger) ,$$

$$d j_a = \frac{1}{2} \left\{ d (\chi_a \triangleright \phi^\dagger) \wedge d \phi + m^2 \ast (\sigma^b_a \triangleright \phi^\dagger) \wedge (\chi_b \triangleright \phi) \right.$$  

$$+ m^2 (\chi_a \triangleright \phi^\dagger) \wedge \ast (\phi) + d \ast d (\sigma^b_a \triangleright \phi^\dagger) \wedge d (\chi_b \triangleright \phi) \right\} - L_{\chi a} (L) , \quad (99)$$

it’s trivial to prove the identities

$$\ast d (\sigma^b_a \triangleright \phi^\dagger) \wedge d (\chi_b \triangleright \phi) = d (\sigma^b_a \triangleright \phi^\dagger) \wedge \ast d (\chi_b \triangleright \phi) ,$$

$$m^2 \ast (\sigma^b_a \triangleright \phi^\dagger) \wedge (\chi_b \triangleright \phi) = m^2 (\sigma^b_a \triangleright \phi^\dagger) \wedge \ast (\chi_b \triangleright \phi) ,$$

then the first term becomes equal to the action of the Lie derivative over the Lagrangian

$$d j_a = \frac{1}{2} L_{\chi a} \left\{ d (\phi^\dagger) \wedge \ast d (\phi) + m^2 \phi^\dagger \wedge \ast (\phi) \right\} - L_{\chi a} (L) = 0 . \quad (100)$$

The components of the current form are the components of the energy-momentum tensor:

$$j_a = \ast (e^b) T_{ab} , \quad (101)$$
these components are
\[ T_{ab} = \frac{1}{2} \left\{ \left( \sigma^b \chi_a \triangleright \phi^\dagger \right) \left( \chi_c \triangleright \phi \right) + \left( \sigma^c \chi_b \triangleright \phi^\dagger \right) \left( \chi_c \triangleright \phi \right) \right\} - \eta_{ab} \mathcal{L} , \]  
if we take a solution of the equations of motion (the order of the \( x \)s is relevant),
\[ \phi = e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 x_0} , \quad \eta^{ab} \chi_a(k) \chi_b(k) = m^2 , \]  
which is at the same time an eigenfunction of the \( \xi_a \) vector fields:
\[ \chi_a \triangleright \phi = \chi_a(k) \phi , \quad \chi_a \triangleright \phi^\dagger = \xi_b(k) \phi^\dagger , \]  
where \( \chi_a(k) = \left\{ -\kappa \sinh \frac{k_0}{\kappa} - \frac{1}{2\kappa} e^{k_0/\kappa} |\vec{k}|^2, e^{\lambda k_0/\kappa} \right\} \) and similarly for \( \xi_a(k) \), and evaluate the energy-momentum tensor over this solution we get
\[ T_{ab} = -\frac{1}{2} \left[ \xi_a(k) \xi_b(k) + \xi_b(k) \xi_a(k) \right] \phi^\dagger \phi . \]  
We are also able to identify a current 3-form associated to the symmetry under global phase transformations \( \phi' = e^{i\alpha} \phi \),
\[ j = *(\phi^\dagger d\phi - d\phi^\dagger \phi) , \]  
which is conserved on-shell
\[ d \cdot j = d\phi^\dagger \ast (d\phi) - \ast (d\phi^\dagger) d\phi + \phi^\dagger (d \ast d\phi) - (d \ast d\phi) \phi = 0 . \]  

VII. CONCLUSIONS

I defined constructively most of the structures that are needed to do differential geometry on \( \kappa \)-Minkowski. These structures are all covariant under the symmetries of this noncommutative spacetime. They allow two kind of future developments: one is the study of field theory over \( \kappa \)-Minkowski, for which now we are equipped with all of the necessary tools to construct covariant field theories. We can define vector and tensor fields with the differential forms of various degrees, we can multiply these forms thanks to the Hodge-\( \ast \) and the integral, which allow to associate scalar numbers to every field, as required by an action principle. We can act with a vector field on forms through the Lie derivative and reduce their degree with the internal derivative, and this is sufficiently powerful to construct the conserved currents associated to the symmetries of the spacetime. We are now armed with sufficient tools to start doing serious quantum field theory on \( \kappa \)-Minkowski, exploiting its symmetries in the correct way. Another strand of studies which can
take this paper as starting point is a more geometrical study of the properties of $\kappa$-Minkowski, which exploits the differential-form structures I defined to build actual differential-geometric objects on it, like Cartan’s frame fields, a connection, torsion and curvature, and so on. This, in the long term, might even lead to a proposal for a perturbative construction of quantum gravity as a noncommutative field theory, if $\kappa$-Minkowski proves to be a good fundamental state for quantum gravity.

An interesting recent development in quantum gravity is the so-called “Relative Locality” proposal, which is a framework for interpreting the classical remnants of quantum gravity effects in terms of a curved momentum space [27–29]. I have shown in a paper [30] with G. Gubitosi that the $\kappa$-Poincaré quantum group fits perfectly in the framework of this theory, and all the Hopf-algebra structures of this quantum group are necessary to identify a coherent Relative Locality model. This suggest the following interpretative scheme: the “classical” field theory considered in this paper should only be understood as preparatory to a quantum field theory expressed in terms of a path integral. Its classical limit $\hbar \to 0$ should eliminate the noncommutativity of $\mathcal{A}$, but should leave a trace into the symplectic structure of the phase space of particles. Moreover, this phase space in the classical limit should tend to the cotangent bundle of a curved momentum space, which I described in [30].

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[6] E. Celeghini, R. Ciachetti, E. Sorace, and M. Tarlini, “The three-dimensional euclidean group $E(3)_q$ and its R-matrix,” *J. Math. Phys.* **32** (1991) 1159.

[7] S. Zakrzewski, “Quantum Poincare group related to the kappa-Poincare algebra,” *J. Phys.* **A27** no. 6, (1994) 2075.

[8] S. L. Woronowicz, “Differential calculus on compact matrix pseudogroups (quantum groups),” *Commun. Math. Phys.* **122** (1989) 125–170.

[9] A. Sitarz, “Noncommutative differential calculus on the kappa Minkowski space,” *Phys. Lett.* **B349** (1995) 42–48, hep-th/9409014.

[10] S. Majid, “Algebraic approach to quantum gravity II: noncommutative spacetime,” hep-th/0604130v1.

[11] S. Majid, *Foundations of Quantum Group Theory*. Cambridge University Press, Cambridge, United Kingdom, paperback ed., 2000.

[12] O.V. Radko and A.A. Vladimirov, “On the algebraic structure of differential calculus on quantum groups,” *J. Math. Phys.* **38** (1997) 5434–5446, q-alg/9702020v1.

[13] T. Brzeziński, “Remarks on bicovariant differential calculi and exterior Hopf algebras,” *Lett. Math. Phys.* **27** (1993) 287–300.

[14] B. Durhuus and A. Sitarz, “Star product realizations of kappa-Minkowski space,” arXiv:1104.0208.

[15] F. Mercati and A. Sitarz, “$\kappa$-Minkowski differential calculi and star product,” *Proc. Sci. CNCFG2010* (2011) 030, arXiv:1105.1599 [math-ph].

[16] S. Majid and H. Ruegg, “Bicrossproduct structure of $\kappa$-Poincare group and non-commutative geometry,” *Phys. Lett.* **B334** (1994) 348–354, hep-th/9405107v2.

[17] J. Kustermans, G. J. Murphy, and L. Tuset, “Differential calculi over quantum groups and twisted cyclic cocycles,” *J. Geom. Phys.* **44** (2003) 570–594.

[18] L. Freidel and E. Livine, “3d Quantum Gravity and Effective Non-Commutative Quantum Field Theory,” *Phys. Rev. Lett.* **96** (2006) 221301, hep-th/0512113v2.

[19] P. Kosiniński, J. Lukierski, and P. Maślanka, “Local D=4 Field Theory on $\kappa$–Deformed Minkowski Space,” *Phys. Rev.* **D62** (1999) 025004, PTUV/99-05 and IFIC/99-05 January 27, 1999, hep-th/9902037v2.

[20] G. Amelino-Camelia and S. Majid, “Waves on noncommutative spacetime and gamma-ray bursts,” *Int. J. Mod. Phys.* **A15** (2000) 4301–4324, hep-th/9907110.

[21] A. Agostini, G. Amelino-Camelia, M. Arzano, A. Marciano, and R. A. Tacchi, “Generalizing the Noether theorem for Hopf-algebra spacetime symmetries,” *Mod. Phys. Lett.* **A22** (2007) 1779–1786, hep-th/0607221.

[22] G. Amelino-Camelia, G. Gubitosi, A. Marciano, P. Martinetti, and F. Mercati, “A no-pure-boost uncertainty principle from spacetime noncommutativity,” *Phys. Lett.* **B671** (2009) 298–302, arXiv:0707.1863 [hep-th].

[23] G. Amelino-Camelia, A. Marciano, and D. Pranzetti, “On the 5D differential calculus and translation
transformations in 4D kappa-Minkowski noncommutative spacetime,” *Int. J. Mod. Phys.* A24 (2009) 5445–5463, arXiv:0709.2063

[24] J. Kowalski-Glikman and S. Nowak, “Doubly special relativity and de Sitter space,” *Class. Quant. Grav.* 20 (2003) 4799–4816, hep-th/0304101

[25] M. Arzano, J. Kowalski-Glikman, and A. Walkus, “Lorentz invariant field theory on kappa-Minkowski space,” *Class. Quant. Grav.* 27 (2010) 025012, arXiv:0908.1974

[26] L. Freidel and J. Kowalski-Glikman, “kappa-Minkowski space, scalar field, and the issue of Lorentz invariance,” arXiv:0710.2886

[27] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, and L. Smolin, “Relative locality: A deepening of the relativity principle,” arXiv:1106.0313 [hep-th]

[28] G. Amelino-Camelia, M. Matassa, G. Rosati, and F. Mercati, “Taming Nonlocality in Theories with Planck-Scale Deformed Lorentz Symmetry,” *Phys. Rev. Lett.* 106 (2011) 071301, arXiv:1006.2126 [gr-qc]

[29] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, and L. Smolin, “The principle of relative locality,” arXiv:1101.0931 [hep-th]

[30] G. Gubitosi and F. Mercati, “Relative Locality in κ-Poincaré,” arXiv:1106.5710