On form-factor expansions for the XXZ chain in the massive regime

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Abstract

We study the large-volume-$L$ limit of form factors of the longitudinal spin operators for the XXZ spin-$1/2$ chain in the massive regime. We find that the individual form factors decay as $L^{-n}$, $n$ being an even integer counting the number of physical excitations – the holes – that constitute the excited state. Our expression allows us to derive the form-factor expansion of two-point spin-spin correlation functions in the thermodynamic limit $L \to +\infty$. The staggered magnetisation appears naturally as the first term in this expansion. We show that all other contributions to the two-point correlation function are exponentially small in the large-distance regime.

Introduction

The study of form factors in integrable massive quantum field theories goes back to the late ‘70s when the bootstrap program was proposed by Karowski and Weiss [22]. It was then supplemented with an additional axiom by Kirillov and Smirnov in [23]. The resolution of this bootstrap program within the off-shell Bethe Ansatz resulted in multiple-integral based representations for the densities of form factors in numerous models (see [45] and references therein). These led to series of multiple integrals for the two- and multi-point correlation functions which appeared to be efficient tools for extracting the exponentially decaying long-distance asymptotic behaviour of multi-point correlators.

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Later, in the mid ‘90s, Jimbo and Miwa proposed a representation-theoretic approach to the calculation of form factors of the spin-$\frac{1}{2}$ XXZ chain in its massive regime \[20\]. They employed $q$-vertex operators in the diagonalisation of the infinite XXZ chain and applied them to the evaluation of the form factors of the operators $\sigma^z$ and $\sigma^\pm$. The $q$-vertex operator approach has been used to calculate form factors and correlation functions of higher-rank and higher-spin lattice models and was also successfully applied to the calculation of form factors of the XYZ chain \[37\][38]. The latter result enabled the use of form factors in the calculation of correlation functions of critical lattice models \[10\][31]. So far the $q$-vertex operator approach is limited in that it does not allow one to calculate form factors and correlation functions at finite magnetic fields or finite lengths or temperatures.

The latter is possible within the algebraic Bethe Ansatz approach. A key element of this approach is the Slavnov formula \[43\] for the scalar product between an on-shell and an off-shell Bethe vector. It paved the way for the computation of form factors in quantum integrable lattice models. It was this formula that allowed Slavnov to derive a representation for the form factors of the current operator in the non-linear Schrödinger model \[44\]. This formula was used to derive a representation for the form factors of the current operator in the non-linear Schrödinger model. It was this formula that allowed Slavnov to derive a representation for the form factors of the current operator in the non-linear Schrödinger model.

A first calculation of form factors of a quantum integrable lattice model by means of the algebraic Bethe Ansatz approach became possible after the resolution of the so-called quantum-inverse scattering problem by Kitanine, Maillet and Terras \[32\] in 1999. These authors derived finite-size determinant representations for the form factors of local operators in the finite-length spin-$\frac{1}{2}$ XXZ chain. In the same year Izergin, Kitanine, Maillet and Terras \[19\] were able to analyse the large-$L$ limit of a special form factor representing the spontaneous staggered magnetisation \[4\][20] of the XXZ spin-$\frac{1}{2}$ chain in the massive regime. More recently form factors for the SU(3)-invariant spin chain \[5\] and for the cyclic solid-on-solid model \[39\] were calculated within the algebraic Bethe Ansatz approach.

The interest in form factors of quantum integrable lattice models was recently renewed after the authors of \[27\][29] managed to extract the large-volume behaviour of the particle-hole form factors of the XXZ spin-$1/2$ chain in the massless regime. These formulae, along with so-called restricted sum summation, led to a novel approach to the large-distance and long-time asymptotic behaviour of correlation functions in massless quantum integrable models \[28\][30][31]. The same method works in infinite volume but at finite temperatures \[15\][16][36].

In the present work we address the problem of the calculation of the large-$L$ behaviour of the form factors of local operators in the XXZ spin-$\frac{1}{2}$ chain in its massive regime by the algebraic Bethe Ansatz approach. We utilize a finite-determinant representation of the form factors of the finite chain which was derived in \[26\] for the purpose of the large-$L$ analysis of form factors in the massless regime. This representation follows from \[32\] and turns out to be useful in the massive regime as well.

Recall that the ground state (G.S.) of the XXZ spin-$\frac{1}{2}$ chain in the massive regime belongs to the zero magnetisation sector \[49\]. In the large $L$ limit, the excited states $|\{\chi_0\}^m_1;\{\nu_0\}^m_1\rangle$ having a finite excitation energy above the ground state are parametrised by ‘hole’-parameters $\{\nu_0\}^m_1$ and complex roots $\{\chi_0\}^m_1$ which solve a set of higher-level Bethe Ansatz equations \[2\][46] of the form

$$\mathcal{Y}_0(\chi_a; \{\chi_0\}^{m_a}_1;\{\nu_0\}^{m_a}_1) = 0, \; a = 1, \ldots, n_y.$$  (0.1)

Note that the complex roots $\{\chi_0\}^{m_a}_1$ arise as a reduced number of variables parametrising the string-like and wide-pair solutions of the original Bethe equations \[6\]. As a result of our asymptotic analysis of the determinant expressions for the form factors we obtain the leading large-$L$ behaviour of form factors of spin operators. The
We shall assume that the length of the XXZ spin-1 model is described by the Hamiltonian

\[ H = J \sum_{n=1}^{L} \left( \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z \right) - \frac{h}{2} \sum_{n=1}^{L} \sigma_n^z. \]  

Here \( J > 0 \) is a coupling constant measuring the strength of the exchange interaction, \( \Delta \) is the longitudinal anisotropy in the couplings, \( h \) is an external magnetic field and the \( \sigma_n^a \) correspond to Pauli matrices understood as acting non-trivially on the \( n \)-th quantum space \( V_n \cong \mathbb{C}^2 \) in the tensor product decomposition \( \otimes_{n=1}^{L} V_n \) of the Hilbert space on which \( H \) acts. We will focus on the massive regime of the chain \( \Delta > 1 \) and \( 0 < h < h_c \) with \( h_c \) given by \( \Delta = 20 \) and corresponding to the critical value of the magnetic field above which the model becomes massless. We shall assume that the length \( L \) of the chain is even. In this way we avoid having to distinguish ground states of total spin projection 1/2 and -1/2.

The eigenvectors of this Hamiltonian were first constructed within the coordinate Bethe Ansatz \( [6, 42] \) and later also by means of the algebraic Bethe Ansatz \( [2, 17] \).

The XXZ spin-1/2 model is described by the Hamiltonian

\[ H = J \sum_{n=1}^{L} \left( \sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \Delta \sigma_n^z \sigma_{n+1}^z \right) - \frac{h}{2} \sum_{n=1}^{L} \sigma_n^z. \]  

The eigenvectors of this Hamiltonian were first constructed within the coordinate Bethe Ansatz \( [6, 42] \) and later also by means of the algebraic Bethe Ansatz \( [2, 17] \).
1.1 The Bethe equations of the massive XXZ chain and the counting function

Within the Bethe Ansatz approach, the eigenvectors $|\psi(\{\mu_i\}_N^N)\rangle$ of the XXZ Hamiltonian are parametrised by a set of $N$ roots $\{\mu_i\}_N^N$. The integer $N$ is related to the spin sector to which the eigenvector belongs. It is well known that these roots satisfy a set of algebraic equations, that are now referred to as the Bethe Ansatz equations (1.2). In the massive regime of the XXZ spin-1/2 chain, $\Delta = \cosh(\eta) > 1$ with $\eta > 0$, these equations take the form

$$-e^{2i\alpha} = \left(\frac{\sin(\mu_a - i\eta/2)}{\sin(\mu_a + i\eta)}\right)^L \prod_{b=1}^N \frac{\sin(\mu_a - \mu_b + i\eta)}{\sin(\mu_a - \mu_b - i\eta)}.$$  \hspace{1cm} (1.2)

The equation (1.2), per se, contains the so-called twist parameter $e^{2i\alpha}$. This system of equations is often referred to as the system of $\alpha$-twisted Bethe Ansatz equations. The additional parameter $\alpha$ is zero for the original problem. However, it will turn out to be useful for the calculation of form factors. It has been established in (47) that, for $\alpha \in \mathbb{R}$, any solution $\{\mu_i\}_N^N$ to the Bethe Ansatz equations (1.2) is invariant under complex conjugation, viz. $\{\bar{\mu}_a\}_N^N = \{\mu_a\}_N^N$. Also, in the following, we shall assume that the roots $\mu_a$ are always pairwise distinct. This property is known to hold in integrable models with repulsive interactions (see, e.g. (7)) such as the $\delta$-function Bose gas (40) in the repulsive regime. Still, to the best of our knowledge, it remains an open question whether such property holds in the case of the XXZ chain.

Given a set of Bethe roots $\{\mu_i\}_N^N$ satisfying (1.2), it is convenient to introduce its associated counting function

$$\tilde{\xi}_\mu(\omega) = \frac{p_0(\omega)}{2\pi} - \frac{1}{2i\pi L} \sum_{k=1}^N \theta(\omega - \mu_k) + \frac{1 - 2\alpha}{2L}.$$  \hspace{1cm} (1.3)

The expression for $\tilde{\xi}_\mu$ involves two auxiliary functions: the bare phase $\theta$ and the bare momentum $p_0$ whose definitions read

$$\theta(\lambda) = 2i\pi \int_{-\pi/2}^\lambda K(\mu) \cdot d\mu \quad \text{and} \quad p_0(\lambda) = \int_{-\pi/2}^\lambda p_0'(\mu) \cdot d\mu.$$  \hspace{1cm} (1.4)

The integration in the definition of $\theta$ and $p_0$ runs along the oriented segments

$$[-\pi/2; i\Im(\lambda) - \pi/2] \cup [i\Im(\lambda) - \pi/2; \lambda]$$  \hspace{1cm} (1.5)

and the integrands are given by

$$K(\mu) = \frac{\sinh(2\eta)}{2\pi \sin(\mu + i\eta) \sin(\mu - i\eta)} = \frac{\cot(\mu - i\eta) - \cot(\mu + i\eta)}{2i\pi},$$

$$p_0'(\mu) = \frac{\sinh(\eta)}{\sin(\mu + i\eta/2) \sin(\mu - i\eta/2)} = \frac{\cot(\mu - i\eta/2) - \cot(\mu + i\eta/2)}{i}.$$  \hspace{1cm} (1.6)

A different choice of the integration contour may change the values of $\theta$ and $p_0$ by multiples of $2i\pi$. Still, the above definitions of $\theta$ and $p_0$ do imply that

- for $-\eta < \Im(\lambda) < \eta$ (resp. $-\eta/2 < \Im(\lambda) < \eta/2$) the function $\theta$ (resp. $p_0$) is quasi-periodic $\theta(\lambda + \pi) = \theta(\lambda) + 2i\pi$ and quasi-odd $\theta(-\lambda) = 2i\pi - \theta(\lambda)$ (resp. $p_0(\lambda + \pi) = p_0(\lambda) + 2\pi$ and $p_0(-\lambda) = 2\pi - p_0(\lambda)$);

- for $|\Im(\lambda)| > \eta$ (resp. $|\Im(\lambda)| > \eta/2$), the function $\theta$ (resp. $p_0$) becomes periodic $\theta(\lambda + \pi) = \theta(\lambda)$ and odd $\theta(-\lambda) = -\theta(\lambda)$ (resp. $p_0(\lambda + \pi) = p_0(\lambda)$ and $p_0(-\lambda) = -p_0(\lambda)$).
The quasi-periodicity properties of the bare phase and bare momentum ensure that, for any \( x \in \mathbb{R} \),

\[
\hat{\xi}_\mu(x + \pi/2) - \hat{\xi}_\mu(x - \pi/2) = \frac{N + n_w}{L} \tag{1.8}
\]

where we restricted ourselves to the zero total longitudinal spin sector corresponding to \( N = L/2 \). The number of Bethe roots \( \mu_a \) such that \( |\Im(\mu_a)| > \eta \) is denoted by \( n_w \).

The main advantage of the counting function is that it enables us to recast the Bethe Ansatz equations (1.2) in a very simple form

\[
e^{2i\pi L \hat{\xi}_\mu(\mu_a)} = 1. \tag{1.9}
\]

The important feature is that, on the level of (1.9), one deals with an equation in one variable. Hence, once a small but \( L \)-independent neighbourhood \( \hat{\xi}_\mu \) can be easily solved in the \( L \to +\infty \) limit. In particular, its structure determines the large-\( L \) form of the distribution of the roots \( \{\mu_a\}_1^N \).

### 1.2 Large-\( L \) behaviour of the counting function

We shall carry out the large-\( L \) asymptotic analysis of \( \hat{\xi}_\mu \) under the following hypotheses:

- the set \( \{\mu_a\}_1^N \) contains precisely \( n \), \( n \) being fixed and independent of \( L \), complex roots \( \{z_a\}_1^N \) with non-zero imaginary part;
- the counting function \( \hat{\xi}_\mu \) given in (1.3) is strictly increasing on \( [-\pi/2; \pi/2] \) and its derivative \( \hat{\xi}_\mu' \) is bounded from below by an \( L \)-independent constant \( \kappa \), \( \hat{\xi}_\mu'(\lambda) > \kappa > 0 \) for any \( \lambda \in [-\pi/2; \pi/2] \).

The first hypothesis simply expresses the fact that we restrict the analysis to some subset of solutions to the Bethe Ansatz equations. The second hypothesis ensures that the real roots can be unambiguously parametrised in terms of integers. It also guarantees that there exists a small but \( L \)-independent neighbourhood \( U \) of \( \mathbb{R} \) in \( \mathbb{C} \) such that \( \hat{\xi}_\mu \) maps \( U \cap \mathbb{H}^+ \) into \( \mathbb{H}^+ \), hence guaranteeing that there is no complex roots \( z_a \) in \( U \). In other words, there exists an \( L \)-independent constant \( \tau \) such that \( \min_{a} |\Im(z_a)| > \tau > 0 \). We conjecture that, in fact, the second hypothesis is a consequence of the first one. We have not been able to prove such a statement. We have checked, however, that the above hypotheses hold \textit{a posteriori}, on the level of the answer we obtain.

The quasi-periodicity of \( \hat{\xi}_\mu \) adjoined to its strict increase on \( [-\pi/2; \pi/2] \) implies that the counting function is strictly increasing on \( \mathbb{R} \). As a consequence, for every \( \ell \in \mathbb{Z} \) there exists a unique \( x_\mu \) such that

\[
\hat{\xi}_\mu(x_\mu - \pi/2) = \frac{1 - 2\ell}{2L}. \tag{1.10}
\]

Likewise, it follows from the above hypotheses and from the quasi-periodicity \( (1.8) \) that the equation \( e^{2i\pi L \hat{\xi}_\mu(\nu_a)} = 1 \) admits \( N + n_w \) real roots \( \{\nu_a\}_1^{N+n_w} \) belonging to the interval \([x_\mu - \pi/2; x_\mu + \pi/2]\) of length \( \pi \), where \( \nu_a \) is the unique real solution to

\[
\hat{\xi}_\mu(\nu_a) = \frac{a - \ell}{L}. \tag{1.11}
\]

We fix \( \ell \) uniquely by demanding that \( \{\nu_a\}_1^{N+n_w} \subset [x_\mu - \pi/2; x_\mu + \pi/2] \cap [-\pi/2; \pi/2] \). The Bethe Ansatz equations (1.2) determine the Bethe roots \( \mu_a \) only modulo \( \pi \). For this reason we may assume that \( \Re(\mu_a) \in [-\pi/2; \pi/2] \),
\( a = 1, \ldots, N \). Since the roots \( \{ \mu_a \}^N_1 \) are pairwise distinct, it follows that \( N - n \) among the roots \( \{ \nu_{a}^{N+n_{w}} \) coincide with the real roots which arise in the solution \( \{ \mu_a \}^N_1 \). Namely, there exists integers \( h_1 < \cdots < h_n, h_i \in [1 : N+n_{w}] \) and \( n_h = n + n_w \) such that
\[
\{ \mu_a \}^N_1 = \{ z_a \}^n_1 \cup \{ \nu_{a}^{N+n_{w}} \setminus \{ \nu_{h_{a}}^{n_{h}} \} \}.
\] (1.12)

In the following it will appear convenient to partition the set of complex roots \( \{ z_a \}^n_1 \) in two sub-sets:

- one built of close roots \( \{ z_a \}^n_1 \) viz. those satisfying \( |\Im(z_a)| < \eta \);
- one built of the wide roots \( \{ z_a \}^n_1 \) viz. those satisfying \( |\Im(z_a)| > \eta \).

In order to state the large-\( L \) behaviour of the counting function, we still need to introduce two auxiliary functions, namely, the dressed phase \( \phi \) and the dressed momentum \( p \). They are defined as the analytic continuations, starting from \([-\pi/2 : \pi/2] \), of the unique solutions to the linear integral equations
\[
(I + K)[\phi(s, z)](\omega) = \theta(\omega - z)
\] (1.13)
and
\[
(I + K)[p](\omega) = \frac{\rho_0(\omega)}{2\pi} + \frac{1}{2\pi}\{ p(-\frac{\pi}{2}) \cdot \theta(\omega + \frac{\pi}{2}) - p(\frac{\pi}{2}) \cdot \theta(\omega - \frac{\pi}{2}) \}.
\] (1.14)

Here we agree that \( I \) is the identity operator while the operator \( K \) acts as
\[
K[f](\omega) = \int_{-\pi/2}^{\pi/2} K(\omega - s)f(s) \cdot ds.
\] (1.15)

Notice that in (1.13), \* indicates the running variable on which \( I + K \) acts. The functions \( \phi \) and \( p \) can be explicitly represented in terms of their Fourier series or, respectively, in terms of ratios of \( q \)-Gamma functions or Jacobi Theta functions (see Appendix A.2 and A.4 for more details). It seems convenient in the following to use a homogenised version of the dressed phase defined as
\[
\varphi(\omega, z) = \phi(\omega, z) - \frac{1}{2} \cdot 1_{|\Im(z)|<\eta} \cdot \phi(\omega, \frac{\pi}{2})
\] (1.16)
where we introduced the notation
\[
1_{\text{condition}} = \begin{cases} 1 & \text{if ‘condition’ is satisfied} \\ 0 & \text{otherwise} \end{cases}.
\] (1.17)

We call \( \varphi(\omega, z) \) a homogenised version of the dressed phase since it solely depends on the difference \( \omega - z \), cf. (A.32), (A.36) - (A.37).

In order to close the listing of solutions to linear integral equations that will be of use to our study, we introduce the dressed energy \( \varepsilon \) as the solution to the linear integral equation
\[
(I + K)[\varepsilon](\omega) = \varepsilon_0(\omega) \quad \text{where} \quad \varepsilon_0(\lambda) = \hbar - 2J \sinh(\eta) \rho'(\lambda).
\] (1.18)

The dressed energy enters in the description of the excitation energy above the ground state and can be expressed as
\[
\varepsilon(\lambda) = \frac{\hbar}{2} - 4\pi J \sinh(\eta) \cdot p'(\lambda).
\] (1.19)

We are now in position to describe the large-\( L \) asymptotic behaviour of the counting function for arguments lying uniformly away from the lines \( \Im(z) = \pm \eta \).
Then, the counting function \( \xi^{(\infty)}(\omega) \). It is readily seen that, to leading order in
\( L \in [12] \) but already implicitly present in various earlier works.

By taking into account the absent roots \( \alpha \) in (1.10). Setting
\[ L \in \{ \text{values} \} \]
\[ \Gamma \in \{ \text{domain} \} \]

The above large-volume asymptotic expansions yield the large-
volume asymptotic expansion of the parameter \( \mu \) in
(1.10). It is readily seen that, to leading order in \( L \), one has
\[ x = \frac{1}{2\pi Lp^2(\pi/2)} \left\{ \text{terms} \right\} + O(L^{-2}) \]

It is easy to check a posteriori that all of the hypotheses stated at the beginning of Sub-Section [12] are indeed fulfilled.

Proof —
Let \( \Gamma = \Gamma^{(1)} \cup \Gamma^{(1)} \) correspond to the loop around \( x - \pi/2 \) \( x + \pi/2 \) depicted in Fig. [1] where
\[ \left\{ \begin{array}{l}
\Gamma^{(1)} = \{ x + \pi/2 \in \mathbb{R} \} \\
\Gamma^{(2)} = \{ x - \pi/2 \in \mathbb{R} \}
\end{array} \right. \]

By taking into account the absent roots \( \nu_{h1}, \ldots, \nu_{hn} \), the representation (1.3) for \( \xi \) can be transformed as
\[ \tilde{\xi}(\omega) = \frac{p_0(\omega)}{2\pi} + \frac{1}{2\pi L} \left\{ \sum_{a=1}^{n} \theta(\omega - \nu_{h_a}) - \sum_{a=1}^{n} \theta(\omega - z_{a}) \right\} + \frac{1 - 2\alpha}{2L} - \int_{\Gamma(\omega)} \frac{\theta(\omega - s)\tilde{\xi}(s)}{\omega - 2\pi i \tilde{\xi}(s)} \frac{ds}{2\pi i}. \]

This expression holds for any \( \omega \) such that \( \Im(\omega) \neq \eta \), as one can always pick \( \tau \) small enough so that the cuts of \( \theta \) lie away of the integration domain. By hypothesis, the function \( \tilde{\xi} \) is real analytic in a neighbourhood of \( \mathbb{R} \) and strictly increasing along the real axis. Setting \( x = \Re(\omega), y = \Im(\omega) \) and using the Cauchy-Riemann equations we find that, for all \( \lambda \in \mathbb{R} \),
\[ \frac{\partial}{\partial y} \Re(\tilde{\xi}) \mid_{\omega = \lambda} = \frac{\partial}{\partial x} \Im(\tilde{\xi}) \mid_{\omega = \lambda} > 0. \]

Proposition 1.1 Let the set of Bethe roots \( \{ \mu \}_{1}^{N} \) satisfy the above stated hypotheses and set
\[ \xi^{(\infty)}(\omega) = p(\omega) + \frac{1}{2\pi L} \left\{ \sum_{a=1}^{n} \varphi(\omega, \nu_{h_a}) - \sum_{a=1}^{n} \varphi(\omega, z_{a}) \right\} + \frac{1 - \alpha - \iota}{2L}. \]
Thus, by $\pi$-periodicity of its derivative, $\hat{\xi}_\mu$ has a strictly positive imaginary part in a finite strip above the real axis and a strictly negative imaginary part in a finite strip below the real axis. This ensures that, close to the real axis, $\hat{\xi}_\mu(\omega)$ has no other zeros than the $\nu_j$, $j = 1, \ldots, N + n_\omega$, or these numbers shifted by integer multiples of $\pi$.

Observe that the function $\hat{u}_\mu(s) = \left\{ \begin{array}{ll} -2i\pi L\hat{\xi}_\mu(s) + \hat{\xi}_\mu^+(s) & s \in \Gamma^{(1)}_\mu \\ \hat{\xi}_\mu^-(s) & s \in \Gamma^{(2)}_\mu \end{array} \right.$ where $\hat{\xi}_\mu^+(s) = \ln\left(1 - e^{2i\pi L\hat{\xi}_\mu(s)}\right)$ defines an anti-derivative of the counting function part of the integrand in (1.23). Here, the logarithm is defined by its principal determination, i.e. $\arg \in (-\pi; \pi]$. This ensures that the functions

\begin{equation}
\hat{u}_\mu(s) = \left\{ \begin{array}{ll} -2i\pi L\hat{\xi}_\mu(s) + \hat{\xi}_\mu^+(s) & s \in \Gamma^{(1)}_\mu \\ \hat{\xi}_\mu^-(s) & s \in \Gamma^{(2)}_\mu \end{array} \right. \quad \text{where} \quad \hat{\xi}_\mu^+(s) = \ln\left(1 - e^{2i\pi L\hat{\xi}_\mu(s)}\right) \tag{1.25}
\end{equation}

are $\pi$-periodic on $\mathbb{H}_\pm \cap V$, where $V$ is some sufficiently small open neighbourhood of $[-\pi/2; \pi/2]$. As a consequence, their $\pm$-boundary values on $V \cap \mathbb{R}$ are $\pi$-periodic as well. In particular, one has

\begin{equation}
\lim_{\varsigma \to 0^+} \left\{ \ln\left(1 - e^{2i\pi L\hat{\xi}_\mu(x_\mu + \pi/2 + i\varsigma)}\right) \right\} = \lim_{\varsigma \to 0^+} \left\{ \ln\left(1 - e^{2i\pi L\hat{\xi}_\mu(x_\mu - \pi/2 + i\varsigma)}\right) \right\} = \ln 2. \tag{1.27}
\end{equation}

Then, upon an integration by parts, one obtains

\begin{equation}
- \oint_{\Gamma^{(1)}_\mu} \frac{\theta(\omega - s)\hat{\xi}_\mu^+(s)}{e^{2i\pi L\hat{\xi}_\mu(s)} - 1} \cdot \frac{ds}{2i\pi} = - \oint_{\Gamma^{(1)}_\mu} K(\omega - s)\hat{\xi}_\mu(s) \cdot \frac{ds}{2i\pi L} + \frac{1}{2i\pi} \left\{ \hat{\xi}_\mu(x_\mu - \pi/2) \cdot \theta(\omega + \pi/2 - x_\mu) - \hat{\xi}_\mu(x_\mu + \pi/2) \cdot \theta(\omega - \pi/2 - x_\mu) \right\}. \tag{1.28}
\end{equation}

Note that in (1.28) above, the boundary terms which involve the logarithmic part of $\hat{u}_\mu$ have canceled out due to (1.27). It solely remains to squeeze down to $[x_\mu - \pi/2; x_\mu + \pi/2]$ the part of the integral on the $rhs$ of (1.28) that
involve $-2i\pi L\xi_\mu$ and then re-centre it on $[-\pi/2; \pi/2]$. All in all, one arrives at the non-linear integral equation

$$\tilde{\xi}_\mu(\omega) + \int_{-\pi/2}^{\pi/2} K(\omega - s) \cdot \tilde{\xi}_\mu(s) \cdot ds = \frac{p_0(\omega)}{2\pi} - \frac{1}{4i\pi} \theta(\omega - \frac{\pi}{2}) + \frac{1}{2i\pi L} \left\{ \sum_{a=1}^{n+n_\nu} \theta(\omega - \nu_a) - \sum_{a=1}^{n} \theta(\omega - z_a) \right\} + \frac{1 - 2\alpha}{2L} \int_{[\omega]<\eta} + \frac{1 - 2\alpha}{2L} - \frac{n_\nu}{2i\pi L} \theta(\omega - \frac{\pi}{2}) + \nu_{\xi_\mu}(\omega)$$

where we agree upon

$$\nu_{\xi_\mu}(\omega) = -\sum_{\ell=\pm} \frac{1}{2\pi L} \int_{\Gamma^{(\ell)}} K(\omega - s) \cdot \tilde{\xi}_\mu'(s) \cdot ds \quad \text{with} \quad \begin{cases} \Gamma^{(+)} = [\pi/2; -\pi/2] + i\tau \\ \Gamma^{(-)} = [-\pi/2; \pi/2] - i\tau \end{cases}$$

In order to obtain this representation, we have invoked the $\pi$-periodicity of the integrand in (1.30) so as to cancel out the contribution of the lines parallel to $i$ and shift the integration along the upper/lower part of $\Gamma_\mu$ up to $\Gamma^{(\pm)}$. Due to the remark that follows (1.23) and by its very construction $\nu_{\xi_\mu}(\omega) = O(L^{-\infty})$. Therefore, when acting with $(I + K)^{-1}$ on (1.29) one obtains the asymptotic expansion in the region $|\Im(\omega)| < \eta$.

In order to obtain the asymptotic expansion of $\tilde{\xi}_\mu(\omega)$ outside of the strip $|\Im(\omega)| < \eta$, we introduce the functions

$$\begin{align*}
\text{for } \Im(\omega) > \eta & \quad \kappa_1(\omega) \\
\text{for } \Im(\omega) < \eta & \quad \kappa_c(\omega) \\
\text{for } \Im(\omega) < -\eta & \quad \kappa_1(\omega)
\end{align*}$$

Then, it is immediately seen that, for $|\Im(\omega)| > \eta$, $\tilde{\xi}_\mu(\omega)$ is given by

$$\tilde{\xi}_\mu(\omega) = -\kappa_1(\omega) + \frac{p_0(\omega)}{2\pi} - \frac{1}{4i\pi} \theta(\omega - \frac{\pi}{2}) + \frac{1}{2i\pi L} \left\{ \sum_{a=1}^{n+n_\nu} \theta(\omega - \nu_a) - \sum_{a=1}^{n} \theta(\omega - z_a) \right\} + \frac{1 - 2\alpha}{2L} - \frac{n_\nu}{2i\pi L} \theta(\omega - \frac{\pi}{2}) - \int_{-\pi/2}^{\pi/2} K(\omega - s) \cdot (I + K)^{-1} \nu_{\xi_\mu}(s) \cdot ds \quad (1.32)$$

The functions $\kappa_1$ and $\kappa_c$ satisfy the system of jump conditions

$$\begin{align*}
\kappa_{1,+}(\omega) - \kappa_{1,-}(\omega) &= -\xi_\mu(\omega - i\eta) \quad \text{for } \omega \in [-\pi/2; \pi/2] + i\eta \\
\kappa_{1,-}(\omega) - \kappa_{1,+}(\omega) &= -\xi_\mu(\omega + i\eta) \quad \text{for } \omega \in [-\pi/2; \pi/2] - i\eta
\end{align*}$$

Here and hereafter, we denote for an arbitrary function $g_{\pm}(\omega) = \lim_{\epsilon \to 0} g(\omega \pm i\epsilon)$. Furthermore, the integral equation satisfied by $\xi_\mu^{\infty}$ ensures that

$$\kappa_c(\omega) = -\xi_\mu^{\infty}(\omega) + \frac{p_0(\omega)}{2\pi} - \frac{1}{4i\pi} \theta(\omega - \frac{\pi}{2}) + \frac{1}{2i\pi L} \left\{ \sum_{a=1}^{n+n_\nu} \theta(\omega - \nu_a) - \sum_{a=1}^{n} \theta(\omega - z_a) - n_\nu \theta(\omega - \frac{\pi}{2}) \right\} + \frac{1 - \alpha - \ell}{L}$$

This provides us with the sought analytic continuation of $\kappa_1$ and yields the claimed form of the large-$L$ asymptotics of $\tilde{\xi}_\mu$ in the whole complex plane.
1.3 The higher-level Bethe Ansatz equations

We are now prepared to derive the so-called higher-level Bethe Ansatz equations that, in the limit of large system size, determine the positions of the complex roots \( \{z_a\}^n \) as functions of the holes \( \{\gamma_a\}^n \) which turn into free parameters in this limit. The concept of higher-level Bethe Ansatz equations emerged from the long struggle to characterise the structure of the complex solutions to the Bethe Ansatz equations describing the low-lying excited states. Indeed, starting from Bethe’s seminal work and until the early ‘80s it was widely accepted that such complex solutions organise into strings, with an exponential precision in \( L \). The counting of strings, which is an important consistency test for the string-based thermodynamics, was even mistakenly claimed to prove the completeness of the Bethe ansatz. The break-through came in 1982 with the pioneering analysis of Destri and Lowenstein of the structure of the complex solutions to the Bethe equations describing the chiral invariant Gross-Neveu model. It was shown in [13] that the complex solutions to the Bethe Ansatz equations describing the low-lying excited states form two-strings, quartets and wide pairs, but do not form general, larger strings. The work of Destri and Lowenstein was followed a few months later by two independent papers on the XXZ chain [2, 48]. Woynarovich [46] studied the massless regime at anisotropy \( 1 > \Delta > 0 \), while the analysis by Babelon, de Vega and Viallet [2] dealt with all real values of the anisotropy parameter. In 1984, the derivation of the higher-level Bethe Ansatz equations for the massive regime of the XXZ chain was reconsidered by Virosztek and Woynarovich [46] who confirmed again the picture of two-strings, quartets and wide pairs, but obtained a slightly different form of the higher-level Bethe Ansatz equations. As we shall see, our analysis of the previous sub-section reconfirms the latter set of equations.

**Proposition 1.2** Let \( \{\mu_a\}^N\) be a solution of the \( \alpha \)-twisted Bethe equations (1.2) satisfying the hypotheses stated in Sub-Section 1.2. Then, in the large-\( L \) limit, the real numbers \( \{\eta_a\}^{N+n} \) densely fill the interval \( [-\pi/2, \pi/2] \). Furthermore, in the \( L \to +\infty \) limit, the complex roots \( \{z_a\}^n \) organise into

- wide pairs \( \{y_a, \gamma_{a}\}^{\infty}, \mathcal{S}(y_a) > 0 \);
- strings \( \{w_a, w_a - i\eta + \delta_a\}^{\infty}, \mathcal{S}(w_a) > 0 \) with \( \delta_a \), the string-deviation parameter, satisfying \( \delta_a = O(L^{-\infty}) \).

The two types of limiting complex roots solve the following sets of equations

\[
1 = -e^{-2i\pi\alpha} \prod_{b=1}^{n} \frac{\sin(w_a - v_{b} - i\eta)}{\sin(w_a - v_{b})} \prod_{b=1}^{n/2} \frac{\sin(w_a - w_{b} + i\eta)}{\sin(w_a - w_{b})} \prod_{b=1}^{n/2} \frac{\sin(w_a - y_{b} + i\eta)\sin(w_a - \gamma_{b})}{\sin(w_a - y_{b} - i\eta)\sin(w_a - \gamma_{b} - 2i\eta)} (1.36)
\]

in what concerns the close roots, and

\[
1 = -e^{-2i\pi\alpha} \prod_{b=1}^{n+n_{w}} \frac{\sin(y_a - v_{b} - i\eta)}{\sin(y_a - v_{b})} \prod_{b=1}^{n/2} \frac{\sin(y_a - w_{b} + i\eta)}{\sin(y_a - w_{b})} \prod_{b=1}^{n/2} \frac{\sin(y_a - y_{b} + i\eta)\sin(y_a - \gamma_{b})}{\sin(y_a - y_{b} - i\eta)\sin(y_a - \gamma_{b} - 2i\eta)} (1.37)
\]

\[
1 = -e^{-2i\pi\alpha} \prod_{b=1}^{n+n_{w}} \frac{\sin(\gamma_{a} - v_{b})}{\sin(\gamma_{a} - v_{b} + i\eta)} \prod_{b=1}^{n/2} \frac{\sin(\gamma_{a} - w_{b} + 2i\eta)}{\sin(\gamma_{a} - w_{b})} \prod_{b=1}^{n/2} \frac{\sin(\gamma_{a} - y_{b} + 2i\eta)\sin(\gamma_{a} - \gamma_{b} + i\eta)}{\sin(\gamma_{a} - y_{b})\sin(\gamma_{a} - \gamma_{b} - i\eta)} (1.38)
\]

in what concerns the wide roots.

For \( \mathcal{S}(w_a) \neq i\eta/2 \), the close roots organise in quartets \( \{w_a, w_a - i\eta + \delta_a, \overline{w_a}, \overline{w_a} + i\eta + \delta_a\} \) or two-strings \( \{w_a, w_a - i\eta + \delta_a\} \) if \( w_a = \overline{w_a} + i\eta + \delta_a \), since the complex roots always appear in complex conjugated pairs. We

---

3In our terminology here, such excited states have an energy above the ground state which stays bounded in the large-\( L \) limit.
The new parameters \( \chi_a, a = 1, \ldots, n_x \), now satisfy
\[
e^{2i\pi\alpha} = -\prod_{b=1}^{2n_h} \frac{\sin(\chi_a - \nu_{h_b} - i\eta/2)}{\sin(\chi_a - \nu_{h_b} + i\eta/2)} \cdot \prod_{b=1}^{n_w} \frac{\sin(\chi_a - \nu_{w_b} + i\eta)}{\sin(\chi_a - \nu_{w_b} - i\eta)}.
\] (1.40)

The higher-level Bethe Ansatz equations written in the above form look similar to the Bethe Ansatz equations associated with an inhomogeneous XXZ chain of length \( 2n_y \).

There is a discrepancy between the above higher-level Bethe Ansatz equations and those obtained in [2]. The higher-level Bethe Ansatz equations in [2] contain an additional factor \( e^{\delta} \) in the equations for the close roots. The additional constant \( e^{\delta} \) takes the form
\[
e^{\delta} = \exp\left\{-2i\sum_{a=1}^{2n_h} \nu_{h_a} \right\} \prod_{a=1}^{n_x} \left\{ \frac{\cos^2(\chi_a - i\eta/2)}{\cos^2(\chi_a + i\eta/2)} \right\} \cdot \prod_{r=0}^{+\infty} \left( \frac{\cos^2(\nu_{h_a} + i(2r + 1)\eta) \cos^2(\nu_{h_a} - i2(r + 1)\eta)}{\cos^2(\nu_{h_a} - i(2r + 1)\eta) \cos^2(\nu_{h_a} + i2(r + 1)\eta)} \right) .
\] (1.41)

The factor \( e^{\delta} \) appeared in [2] since certain \( O(1/L) \) terms which would have produced counter-terms were disregarded. This fact becomes manifest within our approach. By using the functional equation satisfied by the homogenised dressed phase \( \phi(\omega, z) \), it is seen straightforwardly that
\[
delta = 2 \cdot 2i\pi L \cdot (\tilde{\xi}_{\mu}(-\pi/2) - \tilde{\xi}_{\mu}(\pi/2)) + O(L^{-\infty}) .
\] (1.42)

Thus, \( \delta \) is related to the small \( O(L^{-1}) \) shift \( \chi_{\mu} \) of the ‘Fermi-zone’ where the roots \( \{\nu_{a}\}_{a=1}^{N_x \times n_y} \) condense.

The absence of the \( e^{\delta} \) term was already observed in [46], where the authors tried to explain it with the claim that \( e^{\delta} = 1 \) when evaluated at solutions to the higher level Bethe equations. This can be seen to be incorrect by solving explicitly the higher-level Bethe Ansatz equations with \( n_x = 1 \) and \( \alpha = 0 \). In case of one complex root (and thus two holes) the only solutions, up to \( \pi \)-periodicity, read
\[
\chi_1 = \frac{\nu_{h_1} + \nu_{h_2}}{2}, \quad \chi_2 = \frac{\nu_{h_1} + \nu_{h_2} + \pi}{2} .
\] (1.43)

Inserting e.g. \( \chi_1 \) into (1.41) one concludes that \( e^{\delta} \neq 1 \), unless \( \nu_{h_1} = -\nu_{h_2} \) which is clearly not a generic distribution of hole parameters.

**Proof of Proposition 7.2 —**

Up to corrections of the order \( O(L^{-\infty}) \) the counting function \( \tilde{\xi}_{\mu} \) is determined by \( \tilde{\xi}_{\mu}^{(\infty)} \) (see Proposition 1.1). Using Proposition 1.1 in equation (1.9), inserting \( z_a \) and \( \nu_{h_a} \) for \( \mu_a \) and neglecting the \( O(L^{-\infty}) \) corrections we obtain a closed system of finitely many equations that determine the holes and the close and wide roots. For the holes we obtain
\[
\tilde{\xi}_{\mu}^{(\infty)}(\nu_{h_a}) = \frac{h_a - \ell}{L} + O(L^{-\infty}) .
\] (1.44)
From this equation and the explicit form of the function \( \hat{\xi}_\mu^{(\infty)}(\nu_a) \) (see (1.20)) we infer that the holes become free real parameters for \( L \to \infty \).

We now justify the organisation of the close roots into strings. Expressing \( \hat{\xi}_\mu \) by means of the non-linear integral equation it satisfies, one obtains the representation

\[
e^{2i\pi \phi(\omega)} = -e^{-i\pi(a+1)} e^{2i\pi L_p(\omega)} \prod_{b=1}^{n} \frac{\sin(\omega - z_b + i\eta)}{\sin(\omega - z_b - i\eta)} \cdot \prod_{b=1}^{n+n_r} \frac{\sin(\omega - \nu_b - i\eta)}{\sin(\omega - \nu_b + i\eta)} \cdot e^{\phi(\omega)} \cdot \left(1 + O(L^{-\infty})\right) \quad (1.45)
\]

in which the \( O(L^{-\infty}) \) remainder is regular in the strip \( |\Im(\omega)| < \eta \). The function

\[
\psi(\omega) = \sum_{a=1}^{n} K[\varphi(\omega, z_a)](\omega) - \sum_{a=1}^{n+n_r} K[\varphi(\omega, \nu_a)](\omega)
\]

arising above is already regular for \( \omega \) belonging to the strip \( |\Im(\omega)| < \eta \).

Let \( z_f^1, 0 < \Im(z_f^1) < \eta \), be a close root. Since, \( |e^{2i\pi \phi(\omega)}| < 1 \) for \( 0 < \Im(\omega) < \eta \), cf. (A.18), it follows that, for \( e^{2i\pi \phi(\omega)} = 1 \) to hold, \( z_f^1 \) has to approach, up to some exponentially small correction in \( L \), \( z_f^1 + i\eta \), where \( z_f^1 \) is another close root such that \( -\eta < \Im(z_f^1) < 0 \). As a consequence, one can partition the roots as

\[
[z_a]^n = [y_a, \tau_a]_{(1)}^{\infty} \bigcup [w_a, w_a - i\eta + \delta_a]_{(1)}^{\infty}, \quad (1.47)
\]

where \( \Im(y_a) > 0, \Im(w_a) > 0 \) and \( \delta_a \) is some exponentially small correction in \( L \), \( \delta_a = O(L^{-\infty}) \). In order to obtain an equation for \( w_a \) that is free from divergent terms, we multiply the equations satisfied by \( w_a \) and \( w_a - i\eta + \delta_a \). The evaluation of most terms is straightforward with the help of the identities (A.30), (A.36) and (A.37) whereas, in what concerns the singular ones, it is enough to use the identity

\[
\lim_{\delta \to 0} \left[ \exp \left[ \varphi(\lambda, t) + \varphi(t - i\eta + \delta, t) + \varphi(\lambda, t - i\eta + \delta) + \varphi(\lambda - i\eta + \delta, t - i\eta + \delta) \right] \right]_{|t| = \infty} = 1 . \quad (1.48)
\]

This leads precisely to equation (1.36).

Finally, the form of the equations satisfied by the wide pairs follows straightforwardly from the large-\( L \) asymptotic behaviour of \( \hat{\xi}_p(\omega) \) when \( |\Im(\omega)| > \eta \) provided that one uses the reduction properties of the dressed phase (A.30), (A.36) and (A.37).

\[
\text{\textbf{1.4 Energy and momentum of excited states}}
\]

The energy and momentum of an excited state parametrised by the roots \( [\mu_a]^N_1 \) of the \( \alpha \)-twisted Bethe equations (1.2) take the form

\[
E_{\text{ex}} = \sum_{a=1}^{N} (\varepsilon_0(\mu_a) - \varepsilon_0(\lambda_a)) \quad \text{and} \quad P_{\text{ex}} = \sum_{a=1}^{N} (p_0(\mu_a) - p_0(\lambda_a)). \quad (1.49)
\]

Here \( [\mu_a]^N_1 \) denotes the Bethe Ansatz roots of the ground state which solve (1.2) with \( \alpha = 0 \). The bare excitation energy \( \varepsilon_0 \) is defined in (1.13) while the momentum \( p_0 \) of bare excitations is given in (1.4). The large-\( L \) behaviour of the counting functions associated with the systems of Bethe roots \( [\mu_a]^N_1 \) and \( [\lambda_a]^N_1 \) allows one to compute both, the excitation energy and the momentum of an \( \alpha \)-twisted excitation, up to \( O(L^{-\infty}) \) corrections.
Proposition 1.3 Under the hypothesis of Sub-Section 1.2 the quantities defined in (1.49) admit the large-$L$ asymptotic behaviour

$$E_{\text{ex}} = -\sum_{a=1}^{2n} e^{(0)}(v_h_a) + O(L^{-\infty}) \quad \text{and} \quad \mathcal{P}_{\text{ex}} = (\alpha + \iota)\pi - 2\pi \sum_{a=1}^{2n} p(v_h_a) + O(L^{-\infty})$$

where $e^{(0)}$ corresponds to the dressed energy at zero magnetic field:

$$e^{(0)}(\lambda) = -4\pi J \sinh(\eta)p'(\lambda).$$

The result stated above can be obtained by direct calculation of Fourier coefficients, cf. [2], or by the dressed function trick which relies on a direct handling of the linear integral equations. It was first obtained by Johnson, Krinsky and McCoy [21] based on results for the transfer matrix of the eight-vertex model.

Here we propose a different proof, based solely on analytic continuation. The technique we develop is useful for characterising many other relations among solutions to linear integral equations in the massive regime, in particular those involving the resolvent.

Proof —

We first estimate the excitation energy:

$$E_{\text{ex}} = \sum_{a=1}^{n} e_0(z_a) - \sum_{a=1}^{n+n_u} e_0(v_h_a) + E^{(v)} - E^{(A)}$$

with

$$E^{(v)} = \sum_{a=1}^{N+n_u} e_0(v_a) \quad \text{and} \quad E^{(A)} = \sum_{a=1}^{N} e_0(\lambda_a).$$

By applying an argument similar to the one in Section 1.2, the latter two terms have representations

$$E^{(v/A)} = L \int_{-\pi/2}^{\pi/2} \tilde{\xi}^{(v/A)}(s) \cdot e_0(s) \cdot ds - \sum_{\epsilon \in \Gamma(\omega)} \int_{-\pi/2}^{\pi/2} \tilde{\xi}^{(v/A)}(s) \cdot \tilde{\xi}^{(\epsilon)}_{\mu}(s) \cdot \frac{ds}{2i\pi},$$

where obvious notations are employed: $\tilde{\xi}^{(v/A)}(s)$ denotes the counting function that is obtained from (1.3) upon replacing $\mu_k$ by $\lambda_k$ and setting $\alpha = 0$, while $\tilde{\xi}^{(\epsilon)}_{\mu}(s)$ is obtained from (1.25) by replacing $\tilde{\xi}_{\mu}(s)$ by $\tilde{\xi}^{(v/A)}(s)$.

Since $e_0$ is a periodic function, one obtains

$$E_{\text{ex}} = \sum_{a=1}^{n} e_0(z_a) - \sum_{a=1}^{n+n_u} e_0(v_h_a) - \int_{-\pi/2}^{\pi/2} \tilde{F}(s) e_0(s) \cdot ds + O(L^{-\infty}),$$

where

$$\tilde{F}(\omega) = L \cdot (\tilde{\xi}^{(v)}(\omega) - \tilde{\xi}_{\mu}(\omega))$$

is the shift function associated with the states parametrised by $[\mu_a]^N$ and $[\lambda_a]^N$. Its large-$L$ expansion is readily deduced from Proposition 1.1

$$\tilde{F}(s) = \frac{1}{2i\pi} \left\{ \sum_{a=1}^{n} \varphi(s, z_a) - \sum_{a=1}^{n+n_u} \varphi(s, v_h_a) \right\} + \frac{\alpha + \iota}{2} + O(L^{-\infty}).$$
Here we have anticipated that $\epsilon = 0$ for the ground state (see section 1.5). The large-$L$ asymptotics of the function $\tilde{F}^j(s)$ can be expressed in terms of the resolvent kernel, cf. Appendix A.5. More precisely, it follows immediately from the integral equation satisfied by $\phi(s, z)$ and the quasi-periodicity properties of $\theta$ that

$$\phi(s + \pi/2, z) - \phi(s - \pi/2, z) = i\pi \cdot 1_{|\Im z|<\eta}$$  \hspace{1cm} (1.57)

As a consequence, $\partial_s \phi(s, z)$ solves the linear integral equation

$$(I + K)[\partial_s \phi(s, z)](s) = 2i\pi K(s-z).$$  \hspace{1cm} (1.58)

Applying the inverse $I-R$ to the operator $I+K$ and invoking the integral equation (A.21) satisfied by the resolvent kernel $R(s)$ leads to

$$\frac{1}{2i\pi} \partial_s \phi(s, z) = K(s-z) - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R(s-w)K(w-z)dw.$$  \hspace{1cm} (1.59)

The expression for the integral term depends on the domain in the complex plane where $z$ belongs to. The integral can, in fact, be estimated by using a reasoning similar to that invoked when implementing the analytic continuations of the functions $\kappa_{1/1}$ and $\kappa_c$. One obtains

$$\frac{1}{2i\pi} \partial_s \phi(s, z) = \begin{cases} R(s-z) & |\Im(z)| < \eta \\ R(s-z) + R(s-z \pm i\eta) & \pm \Im(z) > \eta \end{cases}.$$  \hspace{1cm} (1.60)

Now introduce the function

$$R_{\pm}[\epsilon_0](z) \quad \text{for} \quad \Im(z) > \eta$$

$$R_c[\epsilon_0](z) \quad \text{for} \quad |\Im(z)| < \eta$$

$$R_{\pm}[\epsilon_0](z) \quad \text{for} \quad \Im(z) < -\eta$$

It follows from

$$R(s-z) \sim \frac{\pi}{s-n} \cot(s-t) \quad \text{with} \quad z = t \pm i\eta$$  \hspace{1cm} (1.62)

that the three functions above satisfy the jump relations

$$R_{\pm}[\epsilon_0](x+i\eta) - R_{c-}[\epsilon_0](x+i\eta) = -\epsilon_0(x) \quad \text{and} \quad R_{\pm}[\epsilon_0](x-i\eta) - R_{c+}[\epsilon_0](x-i\eta) = -\epsilon_0(x).$$  \hspace{1cm} (1.63)

with $x \in \mathbb{R}$. Thanks to these, we are able to estimate $R_{1/1}[\epsilon_0](z)$ by analytic continuation based on $R_c[\epsilon_0](z) = \epsilon_0(z) - \epsilon(z)$:

$$R_{\pm}[\epsilon_0](z) = -\epsilon_0(z-i\eta) + \epsilon_0(z) - \epsilon(z) \quad \text{and} \quad R_{\pm}[\epsilon_0](z) = -\epsilon_0(z+i\eta) + \epsilon_0(z) - \epsilon(z).$$  \hspace{1cm} (1.64)

In the above formulae, $\epsilon$ refers to the dressed energy that has been defined in (1.18).

We have just proven that

$$\epsilon_0(z) - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \partial_s \phi(s, z)\epsilon_0(s) \cdot \frac{ds}{2i\pi} = \epsilon(z) + \epsilon(z \mp i\eta) \quad \text{for} \quad \pm \Im(z) > \eta$$  \hspace{1cm} (1.65)

$$\epsilon_0(z) - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \partial_s \phi(s, z)\epsilon_0(s) \cdot \frac{ds}{2i\pi} = \epsilon(z) \quad \text{for} \quad |\Im(z)| < \eta.$$  \hspace{1cm} (1.66)
By utilising (1.54) and the asymptotic form (1.56) one obtains
\[ E_{\text{ex}} = \sum_{a=1}^{n_w} \{ \varepsilon(y_a) + \varepsilon(y_a - i\eta) + \varepsilon(y_a - i\eta) \} + \sum_{a=1}^{n_w} \{ \varepsilon(w_a) + \varepsilon(w_a - i\eta) + \varepsilon(w_a - i\eta) \} - \sum_{a=1}^{n+na} \varepsilon(v_{h_a}) + O(L^{-\infty}). \] (1.67)

Using the \( i\eta \) anti-periodicity of \( p' \), one arrives at the expression
\[ E_{\text{ex}} = n_{\lambda}h - \sum_{a=1}^{2n_w} \varepsilon(v_{h_a}). \] (1.68)

It is then enough to use the explicit expression for \( \varepsilon \) so as to see that the \( h \)-dependent terms cancel out, hence leading to (1.50).

We now pass on to the estimation of the \( P_{\text{ex}} \). A handling similar to the previous steps yields
\[ \sum_{a=1}^{N+na} p_0(v_a) = -L \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tilde{\xi}_\mu(s)p'_0(s) \cdot ds + p_0(\frac{\pi}{2}) \cdot (N + n_w + \frac{1}{2}) - \sum_{\varepsilon=\pm} \int_{\Gamma_{\text{ex}}} p'_0(s)\tilde{\mu}_{\varepsilon}(s) \cdot \frac{ds}{2\pi} \]
\[ + L \int_{0}^{2\pi \frac{\varepsilon}{2}} \tilde{\xi}_\mu(s + \frac{\pi}{2}) \cdot \left[ \tilde{\xi}_\mu(s + \frac{\pi}{2}) - \tilde{\xi}_\mu(s + \frac{\pi}{2}) \right] ds. \] (1.69)

Therefore,
\[ P_{\text{ex}} = \frac{\alpha + \iota}{2} p_0(\frac{\pi}{2}) + \sum_{a=1}^{n} \tilde{p}(z_a) - \sum_{a=1}^{n+na} \tilde{p}(v_{h_a}) + n_{\lambda}p_0(\frac{\pi}{2}) + O(L^{-\infty}) \] (1.70)

where we have set
\[ \tilde{p}(z) = p_0(z) + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \phi(s,z)p'_0(s) \cdot \frac{ds}{2\pi}. \] (1.71)

It follows from the integral equation satisfied by the dressed momentum and from the fact that \( z \mapsto \phi(s,z) \) is analytic on \( |\Im(z)| < \eta \) uniformly in \( s \in [-\pi/2, \pi/2] \) that \( \tilde{p}(z) = 2\pi p(z) \) for \( |\Im(z)| < \eta \). Its value in other regions of the complex plane can be obtained by computing the jumps on the lines \( \Im(z) = \pm\eta \) of the functions
\[ \Phi_{\pm}[p_0](z) \quad \text{for} \quad \Im(z) > \eta \]
\[ \Phi_{\pm}[p_0](z) \quad \text{for} \quad |\Im(z)| < \eta \]
\[ \Phi_{\pm}[p_0](z) \quad \text{for} \quad \Im(z) < -\eta \]
\[ = \frac{\pi}{2} \int_{-\pi/2}^{\pi/2} \phi(s,z)p'_0(s) \cdot \frac{ds}{2\pi}. \] (1.72)

We start from the jump equations
\[ \theta(\lambda - x \pm i\eta + i0^+) = \theta(\lambda - x \pm i\eta - i0^+) = -2i\pi \delta_{-\varepsilon} \] (1.73)

and introduce an convenient function \( g_{\pm} \) defined by
\[ g_{\pm}(\lambda, x) = \phi(\lambda, x \pm i\eta - i0^+) - \phi(\lambda, x \pm i\eta + i0^+). \] (1.74)
Immediately seen, it solves the integral equation
\[(I + K)[g(\cdot, x)](\lambda) = \mp 2i\pi \mathbf{1}_{\Delta \in [-\pi/2, \pi/2]} \quad \text{viz.} \quad g_{\pm}(\lambda, x) = \mp 2i\pi \mathbf{1}_{\Delta \in [-\pi/2, \pi/2]} \mp \phi(\lambda, \frac{\pi}{2}) \pm \phi(\lambda, x) \, . \quad (1.75)\]

This leads to the jump conditions
\[
\Phi_{\pm, e}[p_0](x + i\eta) - \Phi_{\pm, c}[p_0](x + i\eta) = p(x) - p\left(\frac{\pi}{2}\right) \quad \text{and} \quad \Phi_{\pm, c}[p_0](x + i\eta) - \Phi_{\pm, e}[p_0](x + i\eta) = p(x) - p\left(\frac{\pi}{2}\right) .
\]

Thus, for \(\pm \Im(z) > \eta\) but close to the line \(\Im(z) = \pm \eta\) one has that
\[
\tilde{p}(z) = p_0(z) + \left(p(z) + p(z \mp i\eta) - p_0(z) - p_0\left(\frac{\pi}{2}\right)\right) = -p_0\left(\frac{\pi}{2}\right) \quad (1.76)
\]
This formula holds, in fact, in the whole domain \(|\Im(z)| > \eta\) by analytic continuation. All-in-all, this leads to \((1.50)\). \n
\section{1.5 General parametrisation of the excited states for large \(L\)}

To lay a firm ground for the calculation of form factors, we have reconsidered the analysis of the excitations of the XXZ chain in the massive regime in the large-\(L\) limit, originally performed in \([2, 46]\). We have obtained, in particular, the large-\(L\) limits of the counting function and the shift function \((1.55)\) which will play a prominent role below. It remains to summarize our results and to interpret the parameter \(\iota\) introduced in \((1.10)\) and \((1.11)\). A similar parameter was first introduced in \([46]\). As we shall see, it distinguishes states that are degenerate in the thermodynamic limit \(L \to \infty\). It is clear that such a degeneracy must exist for the ground state sector in the Ising limit \(\Delta \to \infty\), where the symmetric and antisymmetric combinations of the two Néel states are the degenerate ground states of the model. As was already observed by Orbach \([42]\) such type of degeneracy persists for any finite \(\Delta > 1\) in the limit \(L \to \infty\). We shall see that the corresponding states are distinguished by different values of \(\iota\).

Let us start with the discussion of the ground state. It follows from the arguments brought up in \([49]\) that the Bethe roots \(\{\lambda_n\}_1^N\) pertaining to the ground state are all real and solve the logarithmic Bethe Ansatz equations,
\[
\frac{p_0(\lambda_n)}{2\pi} - \sum_{k=1}^{N} \frac{\theta(\lambda_n - \lambda_k)}{2i\pi L} = a - 1/2 \quad \text{with} \quad a = 1, \ldots, N \, . \quad (1.77)
\]
Consequentially, they can be deduced from the counting function \((1.3)\) with \(\alpha = 0\) and \(n = n_w = n_h = 0\).

There is another state sharing these properties. The real Bethe roots \(\{\tilde{\lambda}_n\}_1^N\) describing this state correspond to the solution to the logarithmic Bethe Ansatz equations
\[
\frac{p_0(\tilde{\lambda}_n)}{2\pi} - \sum_{k=1}^{N} \frac{\theta(\tilde{\lambda}_n - \tilde{\lambda}_k)}{2i\pi L} = a - 3/2 \quad \text{with} \quad a = 1, \ldots, N \, . \quad (1.78)
\]
Again \(n = n_w = n_h = 0\). Comparing \((1.3)\), \((1.77)\), \((1.78)\) and \((1.11)\) we infer that \(\iota = 1\) for this state, while \(\iota = 0\) for the ground state. It follows from Proposition \((1.3)\) that the excitation energy associated with this state is \(E_{\text{ex}} = O(L^{-\infty})\). Thus, in the thermodynamic limit it is degenerate with the ground state. For this reason we shall call it the quasi ground state. Again from Proposition \((1.3)\) it follows that the momentum of the quasi ground state is \(P_{\text{ex}} = \pi + O(L^{-\infty})\). This situation is familiar from the Ising limit, where we have two degenerate ground states with momenta 0 and \(\pi\).

\[\text{In fact, it follows directly from (1.77), (1.78) that } P_{\text{ex}} = \pi \text{ for arbitrary } L.\]
The logarithmic Bethe equations (1.77), (1.78) are distinguished by an overall shift of the parameters \( a \) by one. Note that any other overall shift would reduce the solution of the corresponding logarithmic Bethe equations to one of the two solutions \( \{ \lambda_{\alpha} \}^N \) or \( \{ \hat{\lambda}_{\alpha} \}^N \) up to shifts by multiples of \( \pi \) of the Bethe roots. This can be seen as follows. Let \( \{ \hat{\lambda}_{\alpha} \}^N \) be the solution to (1.77) and define a set of roots

\[
\{ v_{\alpha} \}^N = \{ v_1, \ldots, v_N \} = \{ \lambda_{k+1}, \ldots, \lambda_N, \lambda_1 + \pi, \ldots, \lambda_k + \pi \}.
\] (1.79)

It is then easy to see, by using the quasi-periodicity properties of the bare phase and momentum, that

\[
\frac{\rho_0(v_{\alpha})}{2\pi} = \frac{\sum_{h=1}^N \theta(v_{\alpha} - v_h)}{2\pi L} = \frac{a + 2k - 1/2}{L}.
\] (1.80)

A similar statement holds with respect to the roots \( \{ \hat{\lambda}_{\alpha} \}^N \), replacing \( 2k \leftrightarrow 2k - 1 \).

The result of this section is that, in the large-\( L \) limit, we can associate with every solution \( \{ \mu_{\alpha} \}^N_1 \) of the \( \alpha \)-twisted Bethe equations (1.2) a unique counting function \( \xi_\mu^{(\infty)} \) characterized by a set of ‘macroscopic data’: \( \iota \in \mathbb{Z} \); \( n, n_\nu \in \mathbb{Z}_+ \), and \( \{ v_{\alpha,1} \}^N_{n+n_\nu} \), \( \{ v_{\nu,1} \}^N_{n+n_\nu} \) determined by (1.40) and (1.44). This counting function also determines the shift function (1.55) and the energy and momentum of excitations through (1.54) and (1.69). Thus, all information about the excited states at large \( L \) is encoded in \( \xi_\mu^{(\infty)} \) and its parameters. They provide a general parametrisation of the excited states which we will use below in order to describe the form factors of the local operator \( O \).

It appears convenient for our further handlings to choose a slightly different parametrisation of the excited states. Originally, we fix \( \iota \in \mathbb{Z} \) uniquely by demanding that \( \{ v_{\alpha,1} \}^N_{n+n_\nu} \subset [-\pi/2 : \pi/2] \). However, changing \( \iota \) by an even integer leaves \( e^{2\pi i \iota} \) unchanged while Bethe roots and hole parameters change at most by multiples of \( \pi \). In such a situation they are not, in general, inside of the interval \([-\pi/2 : \pi/2]\) anymore. Still, such a shift does not change the expressions of normalised Bethe states and form factors. Therefore, we will parametrise the excited states solely by the values \( \iota \in [0, 1) \) while allowing some of the parameters \( \{ v_{\alpha,1} \}^N_{n+n_\nu} \) to be located outside of \([-\pi/2 : \pi/2]\). In this situation, some of the Bethe roots \( \{ \mu_{\alpha} \}^N_1 \) may as well move out of \( \Re(\mu_{\alpha}) \in [-\pi/2 : \pi/2] \). With this choice of parametrisation of the states, they are now defined through (1.12).

As we have seen with the example of the ground state and the pseudo ground state above, states with counting functions which differ by the values of \( \iota \) are generally inequivalent. Still, if we admit \( \iota = 0 \) and \( \iota = 1 \) for any choice of \( n, n_\nu \) we may encounter the situation where two different sets of hole parameters \( \{ v_{\alpha,1} \}^N_{n+n_\nu} \), with \( \iota = 0, 1 \) are congruent modulo \( \pi \). Since \( x_\mu^{(0)} - x_\mu^{(1)} = O(1/L) \), where we denoted the corresponding boundary parameters by \( x_\mu^{(i)} \), we expect that only holes near the boundaries \( \pm \pi/2 \) may be connected by shifts by \( \pm \pi \). Consider a hole \( v_{h,j}^{(1)} \) near the left boundary. Expanding (1.11) with \( v_a = v_{h,j} \) around \( x_\mu^{(1)} - \pi/2 \) and using (1.10) and (1.20) we obtain

\[
v_{h,j}^{(1)} + \pi = x_\mu^{(1)} + \pi - \frac{h_j^{(1)} - 1/2}{Lp(-\pi/2)} + O(1/L^2).
\] (1.81)

If this is equal to a root \( v_{h_j}^{(0)} \), then the right hand side must be smaller than \( x_\mu^{(0)} + \pi/2 \). Assuming that there is only a single such hole and using (1.21) we conclude that

\[
\frac{h_j^{(1)} - 1/2}{Lp(-\pi/2)} < x_\mu^{(0)} - x_\mu^{(1)} = \frac{1}{Lp(-\pi/2)} + O(1/L^2).
\] (1.82)

This can only be true if \( h_j^{(1)} = 1 \). It follows from (1.20) that the set of holes \( \{ v_1, v_{h_2}, \ldots, v_{h_n} \} \) with \( \iota = 1 \) and \( \{ v_{h_2}, \ldots, v_{h_n}, v_1 + \pi \} \) with \( \iota = 0 \) belong indeed to the same counting functions \( \xi_\mu^{(1)} = \xi_\mu^{(0)} \).
This means that if we take all solutions of \((1.44)\) and \((1.40)\) with \(\ell = 0, 1\) we double-count a certain number of equivalent solutions. For a fixed number \(n + n_w\) of holes there are \(\binom{N+n_w-1}{n+n_w}\) sets of holes with \(h_1 = 1\). This implies that the relative number of such double-counted states is
\[
\frac{(N + n_w - 1)}{(n + n_w - 1)}\frac{N + n_w}{n + n_w} = \frac{n + n_w}{N + n_w} = O(1/L)
\]
(1.83)

Using a similar argument as above one can easily show that equivalent sets of holes with different values of \(\ell = 0, 1\) can at most differ by one element, in such a way that \(\psi_1^{(1)} + \pi = \psi_1^{(0)}\). Hence, the case considered above is already the most general case of double-counting. Since the relative number of double-counted states vanishes in the thermodynamic limit, we shall assume that \(\ell\) can be arbitrarily chosen from \((0, 1)\).

To summarize, in the large-\(L\) limit, the excited states having an excitation energy with respect to the ground state which is finite in \(L\) can be parametrised by

- a choice of \(\ell \in \{0, 1\}\);
- a choice of \(n_h = n + n_w\) hole-integers \(1 < h_1 < \cdots < h_{n_h} < N + n_w\) which give rise to a set of \(n_h\) hole parameters \(\{v_{h_1}\}_{h_1}^{n_h}\) such that \(v_{h_k} = \gamma_{h_k} + O(L^{-1})\) with \(\gamma_k\) being the unique solution to \(p(\gamma_k) = k/L, k \in \mathbb{Z}\);
- a set of complex roots \(\{\gamma\}_{n}^{\alpha_1}, n_k = (n + n_w)/2\), which corresponds to a solution to the higher-level Bethe Ansatz equations \((1.40)\).

2 The form factors of the spin operator \(\sigma^z\)

In the present section we shall estimate the large-\(L\) behaviour of the quantity

\[
T^{(\ell)}_m(\{\mu\}_{1}^{N}; \{\lambda\}_{1}^{N}) \equiv \frac{\langle \psi(\{\lambda\}_{1}^{N})|\sigma^z|\psi(\{\mu\}_{1}^{N})\rangle \langle \psi(\{\mu\}_{1}^{N})|\sigma^z|\psi(\{\lambda\}_{1}^{N})\rangle}{||\psi(\{\lambda\}_{1}^{N})||^2 ||\psi(\{\mu\}_{1}^{N})||^2}
\]

(2.1)

where \(\{\mu\}_{1}^{N}\) is a set of Bethe roots solving the Bethe equations \((1.2)\) at \(\alpha = 0\) and satisfying the hypotheses of Subsection \(1.2\) while \(\{\lambda\}_{1}^{N}\) is the set of Bethe roots which characterize the ground state. Note that \(T^{(\ell)}_m(\{\mu\}_{1}^{N}; \{\lambda\}_{1}^{N})\) is the square of the ground state magnetization and therefore vanishes. For this reason and in order to avoid to distinguish cases we assume in the following that \(\{\mu\}_{1}^{N} \neq \{\lambda\}_{1}^{N}\).

2.1 Formulae at finite \(L\)

The form factor \(T^{(\ell)}_m(\{\mu\}_{1}^{N}; \{\lambda\}_{1}^{N})\) is most easily accessed by means of its generating function (see e.g. \(27\) and references therein). More precisely, one has

\[
T^{(\ell)}_m(\{\mu\}_{1}^{N}; \{\lambda\}_{1}^{N}) = \frac{2}{\pi^2} \sin^2 \left(\frac{\mathcal{P}_{ex}}{2}\right) \cdot e^{im\mathcal{P}_{ex}} \cdot \frac{\partial^2}{\partial \alpha^2} S^{(\alpha)}_N(\{\mu\}_{1}^{N}; \{\lambda\}_{1}^{N})|_{\alpha=0}
\]

(2.2)

In this formula, the momentum \(\mathcal{P}_{ex}\) and the \(\alpha\)-twisted scalar product \(S^{(\alpha)}_N\) on the right-hand side are both evaluated at a solution \(\{\mu\}_{1}^{N}\) to the \(\alpha\)-twisted Bethe equation \((1.2)\) such that \(\mu_{|\alpha=0} = \tilde{\mu}_a\).

In order to present the expression for \(S^{(\alpha)}_N\), we first need to introduce some notation. Let \(I + \tilde{U}_{\theta}\) be an operator acting on the space of functions supported on the loop \(\mathbb{I}^\ell_{\theta} = \Gamma^+(\theta) \cup \Gamma^-(\theta)\), cf. \((1.30)\), whose integral kernel \(\tilde{U}_{\theta}(\omega, \omega')\)

\[\text{By choosing this particular loop we have made the assumption that } \omega \mapsto 1 - e^{2i\omega \tilde{F}(\theta)} \text{ has no zeroes inside. This, however, is not a restriction. Indeed, the assumption always holds for } S(\alpha) > 0 \text{ and large enough. The expressions that are given below should then be understood as analytic continuations from the domain } S(\alpha) > 0 \text{ and large enough to } S(\alpha) = 0. \text{ See } [44] \text{ for more details.}\]
We have set in the above

can recast as well as of the products takes the form

Finally, we need to introduce the shorthand notation for a double product of interest:

Note that the di

\[ \Xi(S^\omega) = \Phi(S^\omega) \]

\[ \Xi(\sigma) = \phi(\sigma) \]

We have set in the above

\[ K_{a}(\omega) = \frac{-1}{2i\pi} \left\{ \cot(\omega + i\eta) - e^{2i\pi\omega} \cot(\omega - i\eta) \right\} \text{ implying } K_{0}(\omega) = K(\omega). \] (2.4)

Finally, we need to introduce the shorthand notation for a double product of interest:

\[ \mathcal{W}(x_{a}^{m}, y_{a}^{n}) = \prod_{a,b=1}^{N} \left\{ \frac{1}{\sin(\lambda_{a} - \mu_{b})} \cdot \left\{ \frac{e^{2i\pi\theta} - 1}{2} \right\} \right\} \cdot \prod_{a=1}^{N} \left\{ \frac{\det \left[ \Xi^{\mu} \right] \cdot \det \left[ \Xi^{\nu} \right]}{\left[ 1 - e^{2i\pi\theta} \right]} \right\} \] (2.6)

and involves determinants of the matrices

\[ \Xi^{(d)}_{ab} = \delta_{ab} + \frac{K(\lambda_{a} - \lambda_{b})}{L_{J}(\lambda_{b})} \quad \text{and} \quad \Xi^{(\mu)}_{ab} = \delta_{ab} + \frac{K(\mu_{a} - \mu_{b})}{L_{J}(\mu_{b})}. \] (2.7)

Note that the difference between the representation (2.6) for \( S_{N}^{(\omega)} \) and the one that appeared in [27] stems from the fact that we have already implemented the rotation of Bethe roots by \( e^{\pi b} \) which is appropriate for treating the massive regime of the chain. The representation (2.6) involves two parameters \( \theta_{1} \) and \( \theta_{2} \). They can be chosen arbitrarily in that the representation (2.6), taken as a whole, does not depend on the \( \theta \)'s, see [26] for more details.

The initial expression for \( S_{N}^{(\omega)} \) is not convenient for taking the large-\( N \) limit. After some algebra, however, one can recast \( S_{N}^{(\omega)} \) into a form which is more convenient for the large-\( N \) analysis:

\[ S_{N}^{(\omega)}(\mu_{a}^{N}; \lambda_{a}^{N}) = \mathcal{D}_{1} \cdot \mathcal{D}_{2} \cdot \mathcal{A}_{\text{reg}} \cdot \mathcal{A}_{\text{sing}}. \] (2.8)

The coefficients \( \mathcal{D}_{k} \), \( k = 1, 2 \) are built out of the auxiliary function

\[ \mathcal{D}(x_{a}^{m}; y_{a}^{n}) = \prod_{a,b=1}^{m} \frac{\sin(x_{a} - x_{b}) \cdot \prod_{a,b=1}^{n} \sin(y_{a} - y_{b})}{\prod_{a=1}^{m} \prod_{b=1}^{n} \sin(x_{a} - y_{b}) \sin(y_{b} - x_{a})} \] (2.9)

as well as of the products

\[ V(\omega) = \prod_{a=1}^{N} \frac{\sin(\omega - \lambda_{a})}{\prod_{a=1}^{N} \sin(\omega - \nu_{a})} \quad \text{and} \quad V_{s}(\omega) = \prod_{a=1}^{N} \frac{\sin(\omega - \lambda_{a})}{\prod_{a=1}^{N} \sin(\omega - \nu_{a})} \] (2.10)
Indeed, the coefficient $\tilde{D}_1$ is expressed as

$$
\tilde{D}_1 = (-1)^N (i)^{n_y} (2i)^{n_a} \prod_{a=1}^N \left\{ \frac{e^{2i\pi \xi_a} - 1}{2\pi \xi_a} \right\} \cdot \prod_{a=1}^{N+n_a} \left\{ \frac{e^{-2i\pi \xi_a} - 1}{2\pi \xi_a} \right\} \cdot \tilde{D}\left(\{v_a\}^{N+n_a}; \{\lambda_a\}^N\right). \tag{2.11}
$$

On this occasion, we recall that the $\{v_a\}^{N+n_a}$ correspond to the roots of $e^{2i\pi \xi_a} - 1$ located in the interval $[x_{\mu} - \pi/2; x_{\mu} + \pi/2]$. In its turn, the coefficient $\tilde{D}_2$ reads

$$
\tilde{D}_2 = \tilde{D}\left(\{v_{h_a}\}_1^{n_a}; \{z_a\}_1^n\right) \cdot \prod_{a=1}^{n_a} \left\{ \frac{e^{2i\pi \xi_a} (v_{h_a}) - V^{-1}(v_{h_a} + i\eta) \cdot V^{-1}(v_{h_a} - i\eta)}{V^2(z_a) \cdot V^{-1}(z_a + i\eta) \cdot V^{-1}(z_a - i\eta)} \right\} \tag{2.12}
$$

The factor $\tilde{A}_{\text{reg}}$ contains all the terms that have a regular behaviour in the $L \to +\infty$ limit:

$$
\tilde{A}_{\text{reg}} = \mathcal{W}\left(\{v_a\}^{N+n_a}; \{\lambda_a\}^N\right) \cdot \mathcal{W}_{\text{reg}}\left(\{v_{h_a}\}_1^{n_a}; \{z_a\}_1^n\right) \cdot \frac{(i)^{n_a} \cdot (2i)^{n_a} \cdot (-1)^n \cdot (1 - e^{2i\pi \xi_a})^2}{\det_{N+n_a} [\Xi^{(v)}] \cdot \det_N [\Xi^{(v)}]} \times \prod_{a=1}^{N+n_a} \left\{ \frac{e^{2i\pi \xi_a} (v_{h_a}) - 1}{e^{2i\pi \xi_a} (v_{h_a}) - 1} \right\} \cdot \prod_{\rho=1}^n \left\{ \frac{\det \left[ I + \tilde{U}_{\rho} \right] V(\theta_{\rho} + i\eta)}{1 - e^{2i\pi \xi_a} (v_{h_a})} \right\} \tag{2.13}
$$

Above, we agree upon

$$
\mathcal{W}_{\text{reg}}\left(\{v_{h_a}\}_1^{n_a}; \{z_a\}_1^n\right) = \prod_{a=1}^{n_a/2} \left\{ \sin(-\delta_a) \right\} \cdot \mathcal{W}\left(\{v_{h_a}\}_1^{n_a}; \{z_a\}_1^n\right). \tag{2.14}
$$

In other words, $\mathcal{W}_{\text{reg}}\left(\{v_{h_a}\}_1^{n_a}; \{z_a\}_1^n\right)$ corresponds to the regular part of $\mathcal{W}\left(\{v_{h_a}\}_1^{n_a}; \{z_a\}_1^n\right)$, i.e. the one which admits a finite value for its large-$L$ asymptotics. We have also introduced the matrix

$$
\Xi^{(v)}_{ab} = \delta_{ab} + \frac{K(v_a - v_b)}{L \xi_a (v_b)}. \tag{2.15}
$$

Finally, it remains to describe $\tilde{A}_{\text{sing}}$. The latter term contains products of factors which, taken individually, exhibit strong singularities due to the formation of strings:

$$
\tilde{A}_{\text{sing}} = \prod_{a=1}^{n_a/2} \left\{ \frac{-1}{\sin(\delta_a)} \right\} \cdot \prod_{a=1}^n \left\{ \frac{1}{2\pi \xi_a (z_a)} \right\} \cdot \frac{\det_{N+n_a} [\Xi^{(v)}]}{\det_N [\Xi^{(v)}]} \tag{2.16}
$$

### 2.2 The large-$L$ behaviour of the form factor: the result

Each term in (2.8) needs a careful analysis to estimate the large-$L$ behaviour. We therefore first present the final expression, then describe the details of the analysis of each component in the subsequent sections. We need some notation for the statement.

The thermodynamic limit of the shift function reads

$$
F_i(s | \{v_a\}_1^{n_a}; \{\lambda_a\}_1^N) = F_i(s) = \frac{i}{2} + \frac{1}{2i\pi} \left\{ \sum_{a=1}^n \varphi(s, z_a) - \sum_{a=1}^{n_a} \varphi(s, v_a) \right\} \tag{2.17}
$$
The particular combination of 
so that the higher-level Bethe Ansatz equations take the form 
then introduce \( \det \frac{e^{-2ixF_i(s)}}{\prod_{a=1}^{n_i} \sin(s - \tilde{\mu}_a^{(i)})} \).

For our convenience we introduce a \( \pi \)-periodic Cauchy transform by 

\[
C[F](\omega) = \int_{-\pi/2}^{\pi/2} \frac{F(s)}{s - \omega} \cdot \frac{ds}{2i\pi} 
\]

A particular combination of \( \pi \)-periodic Cauchy transforms will be denoted by \( \mathcal{L}[F](\omega) \).

\[
\mathcal{L}[F](\omega) = 2\pi[C[F](\omega + i\eta) + C[F](\omega - i\eta)] - C_+ [F](\omega) - C_- [F](\omega) \]

The \( \pm \)-boundary values of the Cauchy transform only matter in the case when \( \omega \in \mathbb{R} \). We denote by \( \det [I + K] \) and by \( \det \left[ I + U_{\eta}[F_i] \right] \) the Fredholm determinants associated to the thermodynamic limit of \( \det, \mathcal{N}[\Xi^{(i)}] \) and \( \det \left[ I + \tilde{U}_{\theta} \right] \). For explicit forms see Proposition 2.3.

Recall the reparametrisation of the complex roots \( \{z_a^n\} \) in terms of the roots \( \{\chi_a\} \) as introduced in (1.39). We then introduce 

\[
\mathcal{Y}_a \left( x_a \mid \{y_b^{(n_a)} \} ; \{\nu_b^{(n_a)} \} \right) = 1 + e^{-2ix\zeta} \prod_{b=1}^{2n_t} e^{ip_b(x_a - y_b)} . \prod_{b=1}^{n_t} e^{-\Theta(x_a - y_b)}
\]

so that the higher-level Bethe Ansatz equations take the form \( \mathcal{Y}_a \left( \chi_a \mid \{\mu_b^{(n_a)} \} ; \nu_b^{(n_a)} \right) = 0 \). We recall that \( n_x = n_w + n_c/2 = (n + n_w)/2 \).

**Theorem 2.1** Let \( \{\mu_a^{(N)} \} \neq \{\chi_a^{(N)} \} \) be a solution of the Bethe Ansatz equations such that its associated counting function satisfies the hypotheses stated in Sub-Section 1.2. Let \( \{z_a^{n_a}\} \) be the complex roots for this state and \( \{\nu_b^{(n_a)}\} \) the positions of holes characterising this state. Then, the \( 2n_x \)-particle form factor \( \mathcal{F}_m^{(x)} (\{\mu_a^{(N)} \} ; \{\chi_a^{(N)} \} ) \) exhibits the large-volume asymptotic behaviour 

\[
\mathcal{F}_m^{(x)} (\{\mu_a^{(N)} \} ; \{\chi_a^{(N)} \} ) = e^{im\pi} \prod_{a=1}^{n_w} \left\{ \frac{e^{-2i\pi m p(u_{b_a})} \left( F^{(x)} (\{\nu_b^{(n_a)} ; \chi_a^{(n)} \} ) \right)^2}{\det \frac{\partial}{\partial u_b} H_0 (u_a \mid \{\chi_a^{(n)} \} ; \nu_b^{(n_a)} \} )_{u_b = \chi_a} \right\} \cdot (1 + O(L^{-1}) ) .
\]

\[\text{(2.23)}\]
Here the normalized squared form factor of the spin operator takes the form

\[
\left( F_{\ell}^{(\ell)}(\{v_{\alpha}\}_{1}^{n_{1}}; \{x_{\alpha}\}_{1}^{n_{1}}) \right)^{2} = -16 \sin^{2} \left( \frac{\pi \tau}{2} + \sum_{a=1}^{n_{a}} \pi p(v_{\alpha}) \right) \times W_{\text{reg}}(\{v_{\alpha}\}_{1}^{n_{1}}; \{x_{\alpha}\}_{1}^{n_{1}}) \times D(\{v_{\alpha}\}_{1}^{n_{1}}; \{x_{\alpha}\}_{1}^{n_{1}})
\]

\[
\times \exp \left\{ \int_{-\pi/2}^{\pi/2} F_{\text{per}}(s)F_{\ell}^{(\ell)}(w) \frac{dwdw}{2 \tan(w - s)} - \int_{-\pi/2}^{\pi/2} F_{\text{per}}(s)F_{\ell}^{(\ell)}(w) \frac{dwdw}{\tan(w - s - i\eta)} - \int F_{\text{per}}(s) \ln' \left[ e^{-2\pi F_{\ell}(s)} - 1 \right] ds \right\}
\]

\[
\times \prod_{a=1}^{n_{a}} \left\{ 4 \sin^{2} \left[ \pi F_{\ell}(v_{\alpha}) \right] \cdot e^{\ell[F_{\text{per}}(v_{\alpha})]} \right\} \cdot n \left\{ e^{-\ell[F_{\text{per}}(x_{\alpha})]} \right\} \cdot \exp \left\{ -\frac{n_{w}}{\pi} \int_{-\pi/2}^{\pi/2} G_{\ell}(s) \cdot ds + i\pi \sum_{\ell=1}^{n_{\ell}} F_{\ell}(\theta_{\ell}) \right\}
\]

\[
\times \frac{(i)^{n_{2}}e^{n_{w}2\pi i a_{n}}}{\det^{2}[I + K]} \cdot e^{\eta_{\text{reg}}(2\pi a_{n})}
\]

\[
\ell_{1} + U_{\theta_{\mu}}[F_{1}] \left[ 1 - e^{2\pi i \theta_{\mu}(\theta_{\mu} + i\eta)} \right] \prod_{a=1}^{n_{a}} \sin(\theta_{\mu} - v_{\alpha} + i\eta) \left[ \cos(\theta_{\mu} + i\eta) \right]^{n_{\ell}} \right\}
\]

\[
. (2.24)
\]

The above theorem is the main result of this paper. We will present supplemental arguments in Section 3, which show the usefulness of the above formula. Especially the comparison against the vertex-operator approach for the two-spinon case will be discussed in detail. In the rest of this section, as promised above, the individual treatment of the four components in (2.8) is described in detail.

The large-\(L\) behaviour of the coefficients \(\hat{D}_{1}, \hat{D}_{2}\) is obtained in Subsection (2.3), Propositions (2.122). The one of the coefficients \(\hat{A}_{\text{reg}}, \hat{A}_{\text{sing}}\) is obtained in Subsection (2.4), Propositions (2.324). All these asymptotic behaviours are uniform in \(a\) bounded. Gathering together these results and then applying (2.2) leads to Theorem 2.1.

2.3 Large-\(L\) behaviour of the coefficients \(\hat{D}_{k}\)

In order to transform the expressions for the coefficients \(\hat{D}_{k}\) into a form which allows one to take the large-\(L\) limit easily, we first need to establish a technical lemma. This lemma will involve functions defined with the help of the determination ‘\(\ln\)’ of the logarithm defined below:

\[
\ln[\sin(z)] \equiv -i \cdot \text{sgn}(\Im(z)) \cdot \left( z - \frac{\pi}{2} \right) - \ln 2 + \ln \left[ 1 - e^{2\pi \text{sgn}(\Im(z))} \right] \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R}.
\]

We stress that the function ‘\(\ln\)’ appearing in the rhs of (2.25) corresponds to the principal branch of the logarithm.

**Lemma 2.1** Let \(\hat{\xi}_{\lambda/\mu}\) be a strictly increasing counting function

- having \(N_{\lambda/\mu}\) roots \(\lambda_{a}/[v_{a}]\) in the interval \([x_{\lambda/\mu} - \frac{1}{2}; x_{\lambda/\mu} + \frac{1}{2}]\);

- satisfying the quasi-periodicity requirement

\[
\hat{\xi}_{\lambda/\mu}(\omega + \pi) = N_{\lambda/\mu} \cdot L^{-1} + \hat{\xi}_{\lambda/\mu}(\omega);
\]

- being analytic in some open neighbourhood of the real axis.

\^8\text{We recall on this occasion that } x_{\lambda}, x_{\mu} \text{ are the unique solutions on } \mathbb{R} \text{ to: } L \cdot \hat{\xi}_{\lambda}(x_{\lambda} - \pi/2) = 1/2, L \cdot \hat{\xi}_{\mu}(x_{\mu} - \pi/2) = 1/2 - t.
Let $\tilde{\tau}_{\lambda/\mu}$ be the periodised form of $\xi_{\lambda/\mu}$ that vanishes on the boundary of $x_{\lambda/\mu} + [-\pi/2 : \pi/2]$.

$$\tilde{\tau}_{\lambda/\mu}(\omega) = \tilde{\xi}_{\lambda/\mu}(\omega) - \frac{N_{\lambda/\mu}}{L}(\omega - x_{\lambda/\mu} + \frac{\pi}{2}) + \tilde{C}_{\lambda/\mu} \text{ with } \begin{cases} C_\mu = -\frac{1 - 2t}{2L} \\ C_\lambda = -\frac{1}{2L} \end{cases} . \quad (2.27)$$

Then, the functions

$$f_\varsigma([\nu_a], [\lambda_a]) = \sum_{a=1}^{N_\nu} \sum_{b=1}^{N_\lambda} \ln \left[ \sin(\nu_a - \lambda_b - i\varsigma) \right] \quad \text{and} \quad f(\omega \mid [\lambda_a]) = \sum_{b=1}^{N_\lambda} \ln \left[ \sin(\omega - \lambda_b) \right] , \quad (2.28)$$

where $\varsigma > 0$, can be recast as

$$f_\varsigma([\nu_a], [\lambda_a]) = \frac{\pi}{2} \int_{-\pi/2}^{\pi/2} \frac{L \tilde{\tau}_\lambda(s) \cdot \tilde{\xi}_\mu(w)}{\tan(\omega - s - i\varsigma)} \cdot ds \, dw - iL \int_{-\pi/2}^{\pi/2} \tilde{\tau}_\mu(s) \cdot ds + \sum_{a=1}^{N_\nu} \sum_{b=1}^{N_\lambda} u^{(\varsigma)}_{ab}(\nu_a - i\varsigma)$$

$$+ \left( \varsigma - \ln 2 - i\left( \frac{\pi}{2} + \lambda_a - \lambda_b \right) \right) \cdot N_\nu N_\mu + \tau_\varsigma([\nu_a], [\lambda_a]) . \quad (2.29)$$

and

$$f(\omega \mid [\lambda_a]) = \int_{-\pi/2}^{\pi/2} \frac{L \tilde{\tau}_\lambda(s)}{\tan(\omega - s)} \cdot ds \left( -\text{sgn}(\Im(\omega)) \cdot (\omega - x_\lambda - \frac{\pi}{2}) - \ln 2 \right) N_\lambda + \sum_{\epsilon = \pm} \delta_{\omega} \cdot u^{(\epsilon)}(\omega) + \tau(\omega \mid [\lambda_a]) . \quad (2.30)$$

The indicator function $\chi_{0 < x < \tau}$ is non-zero only if $0 < x < \tau$, where $\tau$ corresponds to the distance of the integration contours $\Gamma^{(s)}$ to $\mathbb{R}$, cf. (1.30). $\tau_\varsigma([\nu_a], [\lambda_a])$ and $\tau(\omega \mid [\lambda_a])$ are remainder terms. One has

$$\tau_\varsigma([\nu_a], [\lambda_a]) = \sum_{\epsilon = \pm} \int_{\Gamma^{(s)}} ds \int_{\Gamma^{(s)}} dw \frac{u^{(\epsilon)}_{\lambda}(s) \cdot \tilde{\xi}_\mu(w)}{2i\pi} \cdot \frac{\tilde{\tau}_\mu(w)}{\tan(\omega - s - i\varsigma)}$$

$$+ \sum_{\epsilon = \pm} L \int_{\Gamma^{(s)}} \int_{\Gamma^{(s)}} \frac{\tilde{\tau}_\lambda(s) \cdot \tilde{u}^{(\epsilon)}_{\lambda}(w)}{2i\pi} \cdot \frac{\tilde{\tau}_\mu(w)}{\tan(\omega - s - i\varsigma)} - i \sum_{\epsilon = \pm} \int_{\Gamma^{(s)}} \tilde{u}^{(\epsilon)}_{\lambda}(s) \cdot \frac{ds}{2i\pi} . \quad (2.31)$$

Here the subscript ‘in’ occurring in $\Gamma^{(s)} \equiv \mathbb{R}^{(s)} \cup \mathbb{R}^{(-)}$ indicates that the latter contour is contained inside of the outer contour $\Gamma^{(s)}$, i.e. $\max[|\Im(s)| : s \in \Gamma^{(s)} \cup \Gamma^{(-)}] < \tau$ and does not surround the pole at $w = s - i\varsigma$. Analogously,

$$\tau(\omega \mid [\lambda_a]) = \sum_{\epsilon = \pm} \int_{\Gamma^{(s)}} \frac{\tilde{u}^{(\epsilon)}_{\lambda}(s)}{\tan(\omega - s)} \cdot \frac{ds}{2i\pi} . \quad (2.32)$$

Finally, the functions $\tilde{u}_{\lambda/\mu}$ and $\tilde{u}^{(\epsilon)}_{\lambda/\mu}$ occurring above are as defined in (1.25).

Proof —
It is easily seen by integrating by parts that

\[ f_s((v_a), \{\lambda_a\}) = \sum_{a=1}^{N_\nu} \int_{\Gamma_{(\epsilon)}} \frac{\bar{u}_s(s)}{\tan(v_a - s - i\epsilon)} \cdot \frac{ds}{2i\pi} + 1_{0 < \epsilon < \tau} \sum_{a=1}^{N_\nu} \bar{u}_s(\bar{v}_a - i\epsilon) \]

\[ - \sum_{\epsilon = \pm} L \tilde{\xi}_\lambda(x_\lambda - \epsilon\pi/2) \int_{\Gamma_{(\epsilon)}} \ln[\sin(v_a - x_\lambda + \epsilon\pi/2 - i\epsilon)] \] (2.33)

where the contour \( \Gamma_{(\epsilon)} \) is as defined by (1.22). The roots \( \{\bar{v}_a\} \) are translates by \( \pi \) of the roots \( \{v_a\} \) so that the condition \( \bar{v}_a \in [x_\lambda - \pi/2 : x_\lambda + \pi/2] \) holds. In other words, \( \bar{v}_a = v_a + p\pi \), with \( p \in \{1, 0, -1\} \) chosen in such a way that the condition holds.

At this point, we split the integration vs. \( \bar{u}_s \). Due to \( \pi \)-periodicity, the parts involving the logarithms can be reduced to integrations along the lines \( \Gamma_{(\epsilon)} \). Further, in the part involving \( \tilde{\xi}_\lambda \), we reconstruct the function \( \tilde{\tau}_\lambda \). This induces additional terms which cancel out the boundary contributions obtained above. Finally, observe that since \( \bar{u}_s^{-1} \) is \( \pi \)-periodic in some open neighbourhood of \( \mathbb{R} \) lying in the lower half-plane, one has \( \bar{u}_s^{-1}(v_a - i\epsilon) = \bar{u}_s^{-1}(v_a - i\epsilon) \). Thus, the following expression holds,

\[ f_s((v_a), \{\lambda_a\}) = \int_{x_\lambda - \pi/2}^{x_\lambda + \pi/2} ds \int_{\Gamma_{(\epsilon)}} \frac{L^2 \cdot \tilde{\tau}_\lambda(s) \cdot \tilde{\xi}_\mu(w)}{\tan(w - s - i\epsilon)} \cdot \int_{\Gamma_{(\epsilon)}} \ln[\sin(v_a - s - i\epsilon)] \cdot ds \]

\[ + 1_{0 < \epsilon < \tau} \sum_{a=1}^{N_\nu} \bar{u}_s(\bar{v}_a - i\epsilon) + \sum_{\epsilon = \pm} \int_{x_\lambda - \pi/2}^{x_\lambda + \pi/2} ds \int_{\Gamma_{(\epsilon)}} \frac{L \cdot \tilde{\tau}_\lambda(s) \cdot (\bar{u}_s(\bar{v}_a - i\epsilon))'}{2i\pi \tan(w - s - i\epsilon)} \]

\[ + \sum_{\epsilon = \pm} \int_{\Gamma_{(\epsilon)}} \int_{r_{\mu}^{\epsilon} \cup r_{\mu}^{\epsilon} \cap \Gamma_{(\epsilon)}} \frac{d\bar{u}_s(\bar{v}_a - i\epsilon)}{2i\pi} \tan(w - s - i\epsilon). \] (2.34)

Above, the contour \( \tilde{\Gamma}_{(\epsilon)} \) stands for a variant of the contour \( \Gamma_{(\epsilon)} \) where the parameter \( \tau \) is chosen such that \( 0 < \tau < \varsigma \). Since \( \tilde{\tau}_\lambda \) is \( \pi \)-periodic and holomorphic in an open neighbourhood of \( \mathbb{R} \) lying in the lower half-plane, one can deform the \( s \)-contours in the 4th term in (2.34) to \( [-\pi/2 : \pi/2] + 3i\pi/2 \) and then deform the \( w \)-contour from \( \tilde{\Gamma}_{(\epsilon)} \) to \( \Gamma_{(\epsilon)} \).

To conclude, it still remains to estimate

\[ \frac{N_\lambda}{\pi} \prod_{a=1}^{N_\nu} \int_{x_\lambda - \pi/2}^{x_\lambda + \pi/2} ds \int_{\Gamma_{(\epsilon)}} \ln[\sin(v_a - s - i\epsilon)] \cdot ds = N_\lambda N_\mu(\varsigma - \ln 2 - i(x_\lambda + \pi/2)) + iN_\lambda \int_{\Gamma_{(\epsilon)}} s \tilde{\tau}_\lambda(s) \cdot ds = iN_\lambda \int_{\Gamma_{(\epsilon)}} s \tilde{\tau}_\lambda(s) \cdot ds. \] (2.35)

Then, a set of straightforward manipulations leads to (2.29). The rewriting of \( f(\omega \mid \{\lambda_a\}) \) follows the same steps, so that we leave the details to the reader.

The above lemma already yields the representation of \( \tilde{D}_1 \) in a compact form on the level of which, moreover, the \( L \to +\infty \) limit is transparent. Namely, we have the

**Proposition 2.1** \( \tilde{D}_1 \) introduced in (2.11) admits the representation

\[ \tilde{D}_1 = \exp \left\{ \int_{-\pi/2}^{\pi/2} \frac{F_{pe}(s) F'(w) - F_{pe}(w) F'(s)}{2 \tan(w - s)} dw \right\} \cdot e^{i\nu_1} \] (2.36)
where \( \nu_{D_1} \) is a remainder such that \( \nu_{D_1} = O(L^{-\infty}) \) whereas \( \tilde{D}_{\text{per}} \) is the periodised shift function:

\[
\tilde{D}_{\text{per}}(\lambda) = \tilde{F}(\lambda) + \frac{n_w}{\pi} \cdot (\lambda + \frac{\pi}{2}) + \frac{x_D N_1 - x_D N_2}{\pi} - r.
\]  

(2.37)

Proof —

The singular factor \( \tilde{D} \) admits the representation

\[
\tilde{D}([v_d]^N_{1}; n_o; \lambda_o | V) = \lim_{\varsigma \to 0^+} \left\{ (-i\varsigma)^{-2N-n_o} \prod_{a,b=1}^{N+n_o} \sin(v_a - v_b - i\varsigma) \cdot \prod_{a,b=1}^{N} \sin(\lambda_a - \lambda_b - i\varsigma) \right\}.
\]

(2.38)

One can use the first part of Lemma 2.1 to compute the different products. Note that, in the intermediate calculations, one is free to choose any determination of the logarithm, and \( \ln \) in particular. This yields

\[
\tilde{D}([v_d]^N_{1}; n_o; \lambda_o | V) \times (2i)^{-n_o^2} \exp \left\{ \int_{-\pi/2}^{\pi/2} \tilde{F}(\lambda) d\lambda \right\} \cdot \exp \left\{ \int_{-\pi/2}^{\pi/2} \tilde{F}(\lambda) d\lambda \right\} \cdot e^{i\nu_{D_1}}.
\]

(2.39)

We have made use of the Plemelj formula at an intermediate stage. Also, \( \nu_{D_1} \) is the remainder whose explicit expression can be obtained by combining the ones provided in Lemma 2.1. It is clear that \( \nu_{D_1} = O(L^{-\infty}) \), so we shall not dwell any longer on this quantity. It remains to observe that

\[
\int_{-\pi/2}^{\pi/2} \tilde{F}(\lambda) d\lambda = \frac{1}{2} \left[ \tilde{F}^2_{\text{per}}(\pi/2) - \tilde{F}^2_{\text{per}}(-\pi/2) \right] - \frac{n_w}{\pi} \int_{-\pi/2}^{\pi/2} \tilde{F}_{\text{per}}(\lambda) d\lambda.
\]

(2.40)

Finally, the integral representation (2.36) follows upon symmetrising the principal-value integral.

In order to obtain a similar type of expression for \( \tilde{D}_2 \) some more work is necessary with the additional factors.

Still, one obtains the

Proposition 2.2

The coefficient \( \tilde{D}_2 \) introduced in (2.12) admits the representation

\[
\tilde{D}_2 = \tilde{D}([v_h_a]^n_b; [\tilde{e}_{a_1}]^n_1) \cdot e^{2\nu_{D_2}} \cdot \prod_{a=1}^{n_h} \left\{ \frac{4 \sin^2 \frac{\pi}{\mu} \tilde{F}(v_{h_b})}{2\pi \tilde{L}_{h}^a(v_{h_b})} \cdot e^{L[\tilde{F}_{\text{per}}](v_{h_a})} \right\} \cdot \prod_{a=1}^{n} \left\{ e^{-L[\tilde{F}_{\text{per}}]}(n) \right\} \cdot e^{i\nu_{D_2}}
\]

(2.41)

where \( \nu_{D_2} = O(L^{-\infty}) \) is a remainder and \( \mathcal{L} \) is defined in (2.21).

Proof —

On the basis of arguments similar to the ones invoked in the course of the proof of Proposition 2.1, one establishes that, for any \( \omega \) away from the real axis (\( \text{viz.} \) one can choose \( \tau \) such that \( |\Omega(\omega)| > \tau \)),

\[
V(\omega) = 2^{n_w} \exp \left\{ \text{sgn}(\Omega(\omega)) \left( n_w(\omega - \frac{\pi}{2}) + N x_D - (N + n_w) x_D \right) \right\} \cdot e^{-2\pi\mathcal{L}[\tilde{F}_{\text{per}}](\omega)} \cdot e^{i\nu(\omega)}
\]

(2.42)
whereas, one has,

\[ V_{\nu_h}(\nu_h) = (2.4) - 2\sin \frac{\pi F(\nu_h)}{2\pi L_E'(\nu_h)} \exp \left\{ -i\pi (C_-[\tilde{F}_{\text{per}}](\nu_h) + C_+[\tilde{F}_{\text{per}}](\nu_h)) \right\} \cdot e^{i\nu_0(\nu_h)}. \] (2.43)

In the two cases above, the remainder reads

\[ r_V(\omega) = \sum_{\epsilon = \pm} \int_{\Gamma(\epsilon)} \tilde{u}_d^{(\epsilon)}(s) - \tilde{u}_u^{(\epsilon)}(s) \tan(\omega - s) \cdot ds \cdot 2i\pi. \] (2.44)

Upon putting the various bits together, one obtains the desired representation for \( \tilde{D}_2 \). In particular, the remainder \( r_{\tilde{D}_2} \) is then expressed in terms of \( r_V(\omega) \) evaluated at the hole \( \nu_h \) or complex root \( z_a \).

### 2.4 The \( \tilde{A} \)-coefficients

#### 2.4.1 The regular factor \( \tilde{A}_{\text{reg}} \)

Observe that the shift function satisfies the quasi-periodicity property

\[ \tilde{F}(x + \pi/2) - \tilde{F}(x - \pi/2) = -n_w. \] (2.45)

Since it is real valued on \( -\pi/2, \pi/2 \] and continuous, the function

\[ s \mapsto e^{-2i\pi\tilde{F}(s)} - 1 \] (2.46)

has at least \( n_w \) zeroes on \( -\pi/2, \pi/2 \]. We denote these zeroes by \( \tilde{z}_1, \ldots, \tilde{z}_{n_w} \). As a consequence, the function

\[ G(s) = \ln \left[ \frac{e^{-2i\pi\tilde{F}(s)} - 1}{\prod_{a=1}^{n_w} \sin(s - \tilde{z}_a)} \right] \] (2.47)

is holomorphic in some open neighbourhood of the real axis.

#### Proposition 2.3

The regular coefficient \( \tilde{A}_{\text{reg}} \) defined in (2.13) admits the representation

\[
\tilde{A}_{\text{reg}} = W_{\text{reg}}(\{\nu_{h_a}\}_{a=1}^{n_w}; [z_a]) \cdot \exp \left\{ -\frac{\pi^2}{2} \int_{-\pi/2}^{\pi/2} \tilde{F}_{\text{per}}(s) \tilde{F}_{\text{per}}'(w) \tan(w - s - i\eta) \, ds \cdot dw - \int_{-\pi/2}^{\pi/2} \tilde{F}_{\text{per}}(s) \ln' \left[ e^{-2i\pi\tilde{F}(s)} - 1 \right] \cdot ds \right\}
\]

\[
\times (i)^{n_w} (-1)^{n/2} e^{-n_w^2} \cdot \exp \left\{ -\frac{\pi^2}{2} \int_{-\pi/2}^{\pi/2} G(s) \cdot ds + i\pi \sum_{\ell=1}^{n_w} F(3\ell) \right\} \cdot \frac{(1 - e^{2i\pi\theta})^2}{\det^2 [I + K]}
\]

\[
\times \prod_{p=1}^{2} \left\{ \text{det} \left[ I + U_{\theta_p} [\tilde{F}] \right] \frac{e^{-2i\pi c_{\theta_p + i\eta}}}{\cos(\theta_p + i\eta)} \prod_{a=1}^{n_w} \sin(\theta_p - \nu_{h_a} + i\eta) \prod_{a=1}^{n_w} \sin(\theta_p - z_a + i\eta) \right\} \cdot e^{i\xi_{\text{reg}}} \] (2.48)
Above, \( \det [I + K] \) is the Fredholm determinant of the integral operator \( I + K \) acting on \( L^2((-\pi/2; \pi/2)) \) with the integral kernel \( K(\lambda - \mu) \). The Cauchy transform \( C \) is as defined in (2.20). Finally, the integral kernel \( U_\theta[F] \) reads

\[
U_\theta[F](\omega, \omega') = e^{-n_w \eta} \cdot \exp \left\{ 2i\pi \cdot \left( C[F_{\text{per}}](\omega) - C[F_{\text{per}}](\omega + i\eta) \right) \right\} \cdot \frac{K_\alpha(\omega - \omega') - K_\alpha(\theta - \omega')}{1 - e^{2i\pi F(\omega)}}
\]

\[
\times \prod_{a=1}^{n_h} \left\{ \frac{\sin(\omega - \nu_{ha} + i\eta)}{\sin(\omega - \nu_{ha})} \right\} \cdot \prod_{a=1}^{n} \left\{ \frac{\sin(\omega - \zeta_a)}{\sin(\omega - \zeta_a + i\eta)} \right\}
\]

\[
\times \exp \left\{ i(1 - \text{sgn}(\Im(\omega))(n_w(\omega - \frac{\pi}{2}) + N\chi_{a} - (N + n_w)\chi_{a}) \right\} . \quad (2.49)
\]

Thanks to the identity

\[
e^{2i\pi C[F_{\text{per}}](\omega)} \cdot e^{-isgn(3(\omega))(n_w(\omega - \frac{\pi}{2}) + N\chi_{a} - (N + n_w)\chi_{a})} = [2 \cos(\omega)]^{n_w} \cdot e^{2i\pi C[F](\omega)}, \quad (2.50)
\]

one derives the following expression for \( \tilde{U}_\theta[F](\omega, \omega') \)

\[
\tilde{U}_\theta[F](\omega, \omega') = \exp \left\{ 2i\pi \cdot \left( C[F](\omega) - C[F](\omega + i\eta) \right) \right\} \cdot \frac{\cos(\omega)}{\cos(\omega + i\eta)}^{n_w}
\]

\[
\times \prod_{a=1}^{n_h} \left\{ \frac{\sin(\omega - \nu_{ha} + i\eta)}{\sin(\omega - \nu_{ha})} \right\} \cdot \prod_{a=1}^{n} \left\{ \frac{\sin(\omega - \zeta_a)}{\sin(\omega - \zeta_a + i\eta)} \right\} \cdot \frac{K_\alpha(\omega - \omega') - K_\alpha(\theta - \omega')}{1 - e^{2i\pi F(\omega)}} . \quad (2.51)
\]

Proof —

The double product \( \mathcal{W} \) depending on \( N \) as well as parts of the integral kernel \( \tilde{U}_\theta \) that involve \( N \)-dependent products are recast by means of Lemma (2.4) leading to:

\[
\mathcal{W}(\{\nu_{a1}\}^{N+n_h}; \{\lambda_{a1}\}^{N}) = (2ie^{-\eta})^{n_w} \cdot \exp \left\{ - \int_{-\pi/2}^{\pi/2} \frac{\tilde{F}_{\text{per}}(s)\tilde{F}_{\text{per}}(w)}{\tan(w - s - i\eta)} \text{d}sw \right\} \cdot e^{iw} \quad (2.52)
\]

and the representation (2.49) for \( \tilde{U}_\theta[F] \), up to \( 1 + O(L^{-\infty}) \) corrections. Further, the two determinants appearing in the denominator of \( \mathcal{A}_{\text{reg}} \) have representations in terms of Fredholm determinants,

\[
\det_{N+n_w}[\Xi^{(v)}] = \det_{\Gamma}[I + \mathcal{K}^{(v)}] \quad \text{and} \quad \det_{\Gamma}[\Xi^{(d)}] = \det_{\Gamma}[I + \mathcal{K}^{(d)}], \quad (2.53)
\]

where \( I + \mathcal{K}^{(v)} \), resp. \( I + \mathcal{K}^{(d)} \), is an integral operator acting on functions supported on the loop \( \Gamma \) whose integral kernel reads

\[
\mathcal{K}^{(v)}(\omega, \omega') = \frac{K(\omega - \omega')}{e^{2i\pi \xi_{\omega}(\omega') - 1}} \quad \text{and resp.} \quad \mathcal{K}^{(d)}(\omega, \omega') = \frac{K(\omega - \omega')}{e^{2i\pi \xi_{\omega}(\omega') - 1}} . \quad (2.54)
\]

It is then enough to decompose the kernels as, e.g.,

\[
\mathcal{K}^{(v/d)}(\omega, \omega') = K(\omega - \omega') \cdot \left\{ - 1_{3(\omega') > 0} + \sum_{\epsilon = \pm} \frac{-\epsilon \cdot 1_{3(\omega') > 0}}{e^{2i\epsilon \pi L_{\xi_{\omega}(\omega')}} - 1} \right\} , \quad (2.55)
\]

deform the contour of the action of the first term up to \( [\pi/2; -\pi/2] \) and finally apply standard continuity theorems with respect to the kernel for Fredholm determinants [13] so as to drop the exponentially small in \( L \) terms. This yields

\[
\det_{N+n_w}[\Xi^{(v)}] \cdot \det_{\Gamma}[\Xi^{(d)}] = \det^2[I + K] \cdot (1 + O(L^{-\infty})). \quad (2.56)
\]
Hence, the only factor left to evaluate is the product

\[
\mathcal{P} = \prod_{a=1}^{N+n_w} e^{-2i\pi \tilde{F}(\lambda_a)} - 1.
\]  

(2.57)

In order to rewrite the product in a form that is convenient for taking the \( L \to +\infty \) limit, we extract explicitly the product over the real zeroes of \( e^{-2i\pi \tilde{F}(\omega)} - 1 \) and treat these separately:

\[
e^{-2i\pi \tilde{F}(\omega)} - 1 = e^{G(\omega)} \prod_{\ell=1}^{n_{1}} \sin(\omega - \tilde{z}_\ell).
\]  

(2.58)

The function \( G \) is already holomorphic on some open neighbourhood of \( \mathbb{R} \), hence leading to

\[
\sum_{a=1}^{N} G(\lambda_a) = -L \int_{-\pi/2}^{\pi/2} G'(s) \tilde{F}_a(s) \cdot ds + \frac{N_1}{\pi} \int_{-\pi/2}^{\pi/2} G(s) \cdot ds - \sum_{\varepsilon = \pm} \int_{\Gamma(\varepsilon)} G'(s) \frac{\hat{u}_\varepsilon(s)}{2\pi} \cdot ds.
\]  

(2.59)

The product involving the real roots \( \tilde{z}_\ell \) is readily estimated by using the results gathered in Lemma 2.1. Altogether, one obtains

\[
\mathcal{P} = 2^{n_{1}n_{w}} e^{-\frac{n_{w}}{2} \int_{-\pi/2}^{\pi/2} G(s) \cdot ds + \sum_{\ell=1}^{n_{1}} \int_{-\pi/2}^{\pi/2} G'(s) \frac{\hat{u}_\ell(s)}{2\pi} \cdot ds} \cdot e^{\tau_p}
\]  

(2.60)

in which the remainder term reads

\[
\tau_p = - \sum_{\varepsilon = \pm} \int_{\Gamma(\varepsilon)} [\hat{u}_\varepsilon(s) - \hat{u}_\mu(s)] \cdot \ln' \left[ e^{-2i\pi \tilde{F}(s)} - 1 \right] \cdot \frac{ds}{2\pi}.
\]  

(2.61)

It now only remains to add up all of the results together.

\[\Box\]

2.4.2 The singular factor \( \tilde{A}_{\text{sing}} \)

**Proposition 2.4** The singular coefficient \( \tilde{A}_{\text{sing}} \) defined in (2.16) admits the large-\( L \) asymptotic expansion

\[
\tilde{A}_{\text{sing}} = \frac{(-i)^n \cdot e^{\tau_{A_{\text{sing}}}}}{\det_{\tau_{A_{\text{sing}}}} \left[ \partial_{\alpha_b} Y_a(u_a | \{ u_b \}_{1}^{r}, \{ \nu_{b_1} \}_{1}^{2n_1}) \right]_{u_a = \chi_a}},
\]  

(2.62)

where \( \tau_{A_{\text{sing}}} = O(L^{-1}) \) is a remainder.

The Jacobian in (2.62) can be thought of as the higher-level norm formula for an excited state. Indeed, it arises in a way similar to the higher-level Bethe Ansatz equation; namely by factoring out from the norm formula for an excited state the contribution of the ‘bulk’ roots \( \{ \nu_{a} \}_{1}^{N+n_w} \) that form a dense distribution on \( [ -\pi/2 ; \pi/2 ] \).

**Proof —**
We start by explicitly dividing out the determinant of the matrix \( \Xi^{(\nu)} \) out of the matrix \( \Xi^{(\mu)} \). We introduce the integral kernel
\[
K^{(\mu)}(s, w) = \frac{K(s - w)}{\text{e}^{2\pi i L_{\nu}(w)} - 1}.
\]
(2.63)

Let \( V = \{y_1, \ldots, y_n\} \), \( Z = \{z_1, \ldots, z_m\} \) and \( \mathcal{D}_{\text{tot}} = \{\Gamma_j \setminus \bigcup_{a=1}^{n} \partial D_{y_\nu} \} \cup \{\bigcup_{a=1}^{n} \partial D_{z_a} \} \). Here, \( \partial D_{z_a} \) stands for the boundary (oriented counterclockwise) of the disk of radius \( \epsilon \) centred at \( z \). Then, one has
\[
\det \mathcal{D}_{\text{tot}} [I + K^{(\mu)}] = \sum_{n \geq 0} \frac{1}{n!} \prod_{a=1}^{n} \left\{ \int_{\Gamma_a} \frac{d\mu_a}{\text{e}^{2\pi i L_{\nu}(\lambda_a)} - 1} - \sum_{\lambda \in V} \frac{1}{L_{\nu}(\lambda_a)} + \sum_{\lambda \in Z} \frac{1}{L_{\nu}(\lambda_a)} \right\} \cdot \det_n [K(\lambda_a - \lambda_b)].
\]
(2.64)

One can thus interpret the determinant of the operator \( I + K^{(\mu)} \) understood as acting on functions supported on \( \mathcal{D}_{\text{tot}} \) as the one of an operator \( I + \hat{K}^{(\mu)} \) acting on the space of functions supported on \( \hat{x} = \Gamma_j \cup V \cup Z \). The operator \( \hat{K}^{(\mu)} \) has the matrix decomposition with respect to such a partition of the space \( \hat{x} \):
\[
\hat{K}^{(\mu)} = \begin{pmatrix}
K^{(\mu)}(s, w) & -K(s - y_\nu) \cdot [L_{\nu}^\prime(y_\nu)]^{-1} & K(s - z_b) \cdot [L_{\nu}^\prime(z_b)]^{-1} \\
K^{(\mu)}(v_\nu, w) & -K(v_\nu - y_\nu) \cdot [L_{\nu}^\prime(y_\nu)]^{-1} & K(v_\nu - z_b) \cdot [L_{\nu}^\prime(z_b)]^{-1} \\
K^{(\mu)}(z_a, w) & -K(z_a - y_\nu) \cdot [L_{\nu}^\prime(y_\nu)]^{-1} & K(z_a - z_b) \cdot [L_{\nu}^\prime(z_b)]^{-1}
\end{pmatrix}.
\]
(2.65)

Then, evaluating explicitly the contour integrals corresponding to the support \( \Gamma_j \), one obtains that
\[
\det_n \left[ \Xi^{(\mu)} \right] = \det_{n+2(n+n_\nu)} \left[ \Xi^{(\nu)} \right],
\]
(2.66)

where the \([N + 2(n + n_\nu)] 	imes [N + 2(n + n_\nu)]\) matrix \( \Xi^{(\mu)} \) reads
\[
\Xi^{(\mu)} = \begin{pmatrix}
\Xi^{(\nu)}_{ab} & -K(v_\nu - y_\nu) \cdot [L_{\nu}^\prime(y_\nu)]^{-1} & K(v_\nu - z_b) \cdot [L_{\nu}^\prime(z_b)]^{-1} \\
K(v_\nu - y_\nu) \cdot [L_{\nu}^\prime(y_\nu)]^{-1} & \delta_{h,b} \cdot K(y_\nu - y_\nu) \cdot [L_{\nu}^\prime(y_\nu)]^{-1} & K(y_\nu - z_b) \cdot [L_{\nu}^\prime(z_b)]^{-1} \\
K(z_a - y_\nu) \cdot [L_{\nu}^\prime(y_\nu)]^{-1} & -K(z_a - y_\nu) \cdot [L_{\nu}^\prime(y_\nu)]^{-1} & \delta_{ab} + K(z_a - z_b) \cdot [L_{\nu}^\prime(z_b)]^{-1}
\end{pmatrix}.
\]
(2.67)

The inverse matrix to \( \Xi^{(\nu)}_{ab} \) can be represented with the help of the so-called discrete resolvent \( \hat{R} \). The latter is defined as the unique solution to the equation
\[
\hat{R}(v_j, z) = K(v_j - z) - \sum_{\ell=1}^{N+n_\nu} \frac{\hat{R}(v_j, v_\ell) K(v_\ell - z)}{L_{\nu}^\prime(v_\ell)}
\]
(2.68)
in which \( z \in \mathbb{C} \). The discrete resolvent evaluated at two arbitrary complex numbers \( (z, z') \) is then defined through the formula
\[
\hat{R}(z, z') = K(z - z') - \sum_{\ell=1}^{N+n_\nu} \frac{K(z - v_\ell) \hat{R}(v_\ell, z')}{L_{\nu}^\prime(v_\ell)}.
\]
(2.69)

With this object at hand, one readily checks that\(^8\)
\[
[(\Xi^{(\nu)})^{-1}]_{ab} = \delta_{ab} - \frac{\hat{R}(v_\nu, v_\nu)}{L_{\nu}^\prime(v_\nu)}.
\]
(2.70)

\(^8\)It follows from the large-\( L \) behaviour given in (2.56) and from \( \det[I + K] > 0 \) that \( \Xi^{(\nu)} \) is indeed invertible.
By using the factorisation of determinants of block matrices
\[
\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det[A] \cdot \det[D - C \cdot A^{-1} \cdot B]
\]  
(2.71)

with \( A = \Xi^{(v)} \), we are able to recast the determinant \( \det_N[\Xi^{(v)}] \) as
\[
\det_N[\Xi^{(v)}] = \det_{N+n_v}[\Xi^{(v)}] \cdot \prod_{a=1}^n \left( \frac{1}{L_{\xi_{\mu}}(z_a)} \right) \cdot \det_{2n+n_v}[\Upsilon].
\]  
(2.72)

The matrix \( \Upsilon \) appearing above takes the form
\[
\Upsilon = \begin{pmatrix}
\delta_{ab} - \bar{R}(v_{ha},v_{hb}) \cdot (L_{\xi_{\mu}}(v_{ha}))^{-1} & \bar{R}(v_{ha},z_b) \\
-\bar{R}(z_a,v_{hb}) \cdot (L_{\xi_{\mu}}(v_{ha}))^{-1} & \delta_{ab} L_{\xi_{\mu}}(z_b) + \bar{R}(z_a,z_b)
\end{pmatrix}
\]  
(2.73)

In order to proceed further, we need to recall a technical result established in [24]. Let \( P_M = (p_1, \ldots, p_M) \), \( X \) be an \( M \times M \) symmetric matrix and set
\[
\Delta_M \left( P_M ; \{X_{ab}\}_1^M \right) = \det_M \left[ \delta_{ab} \cdot (p_a - \sum_{k=1}^M X_{a,k}) + X_{a,b} \right].
\]  
(2.74)

Then, in the \( X_{M-2(s-1),M-2s+1} \to \infty \) limit, \( s = 1, \ldots, n_c/2 \), \( \Delta_M \) admits the large-\( M \) asymptotic behaviour
\[
\Delta_M \left( P_M ; \{X_{ab}\}_1^M \right) = \prod_{s=1}^{n_c/2} \left[ -X_{M-2(s-1),M-2s+1} \right] \cdot \Delta_{M-n_c/2} \left( P_{M-n_c/2}^{(n_c/2)\to M-n_c/2} ; \{X_{a,b}^{(n_c/2)}\}_1^{M-n_c/2} \right)
\]
\[
\times \left( 1 + O(\max_s|X_{M-2(s-1),M-2s+1}|^{-1}) \right)
\]  
(2.75)

where
\[
P_{M-n_c/2}^{(n_c/2)} = (p_1, \ldots, p_{M-n_c}, p_{M-n_c+1} + p_{M-n_c+2}, \ldots, p_{M-1} + p_M)
\]  
(2.76)

and, for \( 1 \leq a, b \leq M - n_c \) and \( 1 \leq p, \ell \leq n_c/2 \)
\[
X_{a,b}^{(n_c/2)} = X_{a,b}, \quad X_{a,M-n_c+p}^{(n_c/2)} = X_{a,M-n_c+p,a}^{(n_c/2)} = X_{a,M-n_c+2(p-1)} + X_{a,M-n_c+2p-1}
\]  
(2.77)

whereas
\[
X_{M-n_c+\ell,M-n_c+p}^{(n_c/2)} = X_{M-n_c+2(\ell-1),M-n_c+2(p-1)} + X_{M-n_c+2(\ell-1),M-n_c+2p-1} + X_{M-n_c+2(\ell-1),M-n_c+2p-1}.
\]  
(2.78)

The determinant of \( \Upsilon \) is indeed of the type (2.74). Thus, one obtains the reduction
\[
\det_{2n+n_v}[\Upsilon] = \prod_{s=1}^{n_c/2} \left( -K(\delta_s - i\eta) \right) \cdot \det_{2n+n_v} [\tilde{\Upsilon}] \cdot \left( 1 + O(L^{-\infty}) \right)
\]  
(2.79)
The entries \( y^{(ab)} \) of the block matrix \( Y \) read

\[
y^{(11)} = \begin{pmatrix}
\delta_{ab} - \hat{R}(v_{h_a}, v_{h_b}) \\
-\hat{R}(v_{h_a}, v_{h_b}) \cdot \{L \xi'_\mu(v_{h_b})\}^{-1} \\
\end{pmatrix}
\begin{pmatrix}
\hat{R}(v_{h_a}, y_b) \\
\hat{R}(\nu_{a}, y_b) \\
\end{pmatrix}
\]

\[
y^{(12)} = \begin{pmatrix}
\hat{R}(v_{h_a}, w_b) + \hat{R}(v_{h_a} - i \eta + \delta_{a}, v_{h_b}) \\
L \xi'_\mu(v_{h_b}) \\
\end{pmatrix}
\begin{pmatrix}
\hat{R}(v_{h_a}, y_b) \\
\hat{R}(\nu_{a}, y_b) \\
\end{pmatrix}
\]

\[
y^{(21)} = \begin{pmatrix}
\hat{R}(w_{a}, y_b) + \hat{R}(v_{h_a}, w_{\ell} - i \eta + \delta_{\ell}) \\
R(\nu_{a}, w_{\ell} - i \eta + \delta_{\ell}) \\
\end{pmatrix}
\begin{pmatrix}
\hat{R}(v_{h_a}, y_b) \\
\hat{R}(\nu_{a}, y_b) \\
\end{pmatrix}
\]

and

\[
y^{(22)} = \delta_{pf} \cdot L \xi'_\mu(w_{\ell}) + \xi'_\mu(w_{\ell} - i \eta + \delta_{\ell}) + \hat{R}(w_{p}, w_{\ell}) + \hat{R}(w_{p} - i \eta + \delta_{p}, w_{\ell}) + \hat{R}(w_{p}, w_{\ell} - i \eta + \delta_{\ell}) + \hat{R}(w_{p} - i \eta + \delta_{p}, w_{\ell} - i \eta + \delta_{\ell}).
\]

The expression for the entries of the various blocks defining the matrix \( Y \) can be further simplified. The counting function can be expressed, in the large-\( L \) limit, by means of Proposition 1.1 and use of the functional equation satisfied by the homogenised dressed phase. Furthermore, the large-\( L \) behaviour of the discrete resolvent \( \hat{R}(z, z') \) can be characterised by means of the representation

\[
\hat{R}(z, z') = K(z - z') - \int_{-\pi/2}^{\pi/2} K(z - s) \cdot \hat{R}(s, z') \cdot ds + O(L^{-\infty}).
\]

For \( |\Im(z)| < \eta \) and \( |\Im(z')| < \eta \), the equation can be solved elementarily. The leading in \( L \) expression for \( \hat{R}(z, z') \) when \( |\Im(z)| > \eta \) or \( |\Im(z')| > \eta \) is then obtained by analytic continuation on the basis of the method that has been used in the proof of Proposition 1.3. All-in-all, one obtains

if \( |\Im(z)|, |\Im(z')| < \eta \) then \( \hat{R}(z, z') = R(z - z') + O(L^{-\infty}) \)

if \( |\Im(z)| < \eta \) and \( \Im(z') > \eta \) then \( \hat{R}(z, z') = R(z - z') + R(z - z' \pm i \eta) + O(L^{-\infty}) \)

if \( \Im(z) > \eta \) and \( \Im(z') > \eta \) then \( \hat{R}(z, z') = 2R(z - z') + R(z - z' - i \eta) + R(z - z' + i \eta) + O(L^{-\infty}) \)

if \( \Im(z) > \eta \) and \( \Im(z') < -\eta \) then \( \hat{R}(z, z') = 2R(z - z' - i \eta) + R(z - z' - 2i \eta) + R(z - z') + O(L^{-\infty}) \)

where \( R \) is the resolvent operator to \( I + K \) understood as acting on functions supported on \( [-\pi/2, \pi/2] \), see Appendix A.3. All other instances of the parameters \( z, z' \) are obtained from the symmetry \( \hat{R}(z, z') = \hat{R}(z', z) \) and the reflection property \( \hat{R}(z, z') = \hat{R}(z, z) \).
By using the above asymptotic expression for $\hat{R}$, the shift recurrence relation satisfied by the resolvent (A.24) and the $\chi$-reparametrisation of the complex roots $\{z_{a}\}_{1}^{n}$, one obtains that

$$Y_{ab} = \delta_{ab} \frac{1}{2\pi} \sum_{x=1}^{2n_{r}} p_{0}(x_{a} - v_{h_{a}}) - \sum_{x=1}^{n_{r}} K(x_{a} - \chi_{b}) + K(\chi_{a} - \chi_{b}) + O(L^{-\infty}).$$

(2.87)

Now observe that the first block column of the matrix $\hat{Y}$ is of the form $\delta_{ab} + O(L^{-1})$. Therefore, since $\det_{n_{r}}[Y] \neq 0$, up to $O(L^{-1})$ corrections, the determinant of the matrix $\hat{Y}$ reduces to the one of the matrix $Y$. As a consequence, one finds that

$$\det_{n_{r} + n_{h} + n_{w}}[\hat{Y}] = (-2i\pi)^{-n_{r}} \cdot \det_{n_{r}} \left[ \frac{\partial}{\partial u_{b}} \mathcal{Y}_{a}(u_{a} \mid \{u_{c}\}_{1}^{n_{r}} \mid \{v_{h_{c}}\}_{1}^{2n_{r}}) \right]_{|u_{a} = \chi_{a}} \cdot \left(1 + O(L^{-1})\right).$$

(2.88)

The claim then follows.

\[ \square \]

3 The form-factor series

3.1 The form-factor series

In this section we build on the large-volume behaviour of individual form factors so as to write down the form-factor series expansion for the spin-spin correlation function in the massive regime. The first term in this series corresponds to the staggered magnetisation. In the large-distance limit, the $\sigma^{z}\sigma^{z}$ correlator approaches the staggered magnetisation exponentially fast.

Recall that the temporal evolution of an operator takes the form

$$\sigma^{z}_{k}(t) = e^{itH} \sigma^{z}_{k} e^{-itH}.$$  \hspace{1cm} (3.1)

Within such a convention for the temporal evolution, the space- and time-dependent spin-spin correlation function in the limit $L \to +\infty$ admits the form-factor expansion

$$\langle \sigma^{z}_{1}(0) \cdot \sigma^{z}_{m+1}(t) \rangle = (-1)^{m} \prod_{n \geq 1} \left(1 - \frac{e^{-2m\epsilon}}{1 + e^{-2m\epsilon}}\right)$$

$$+ \sum_{i=0}^{m} \sum_{n_{k} \in 2\mathbb{N}} \frac{(-1)^{m}}{(n_{k})!} \int_{-\pi/2}^{\pi/2} \frac{d^{n_{k}}y_{\nu}}{(2\pi)^{n_{k}}} \prod_{a=1}^{n_{k}} \left(e^{i\nu_{a}h_{a}} - 2\pi m p_{a}(h_{a})\right) \cdot \mathcal{F}^{(c)}(\{v_{a}\}_{m+1}^{n_{a}}).$$  \hspace{1cm} (3.2)

The function $e^{(0)}$ stands for the dressed energy at zero magnetic field (see Proposition 1.3). The non-trivial part of the integrand is defined in terms of a multi-dimensional residue

$$\mathcal{F}^{(c)}(\{v_{a}\}_{1}^{n_{a}}) = \frac{1}{n_{k}!} \int_{\Gamma_{c}(\{v_{a}\})} \left(\mathcal{F}^{(c)}(\{v_{a}\}_{1}^{n_{a}}; \{\psi_{c}\}_{1}^{n_{a}})\right)^{2} \cdot \frac{d^{n_{k}}\psi}{(2\pi)^{n_{k}}}.$$  \hspace{1cm} (3.3)

More precisely, the $n$-dimensional integral runs through the skeleton associated with the higher-level Bethe Ansatz equations subordinate to the choice of holes $\{v_{a}\}_{1}^{n_{a}}$.

$$\Gamma_{c}(\{v_{a}\}) = \left\{ \psi \in \mathbb{C}^{n_{a}} \mid |\mathcal{Y}_{\nu}(\psi_{a} \mid \{\psi_{c}\}_{1}^{n_{a}})\{v_{a}\}_{1}^{n_{a}}| = \epsilon \quad a = 1, \ldots, n_{c} \right\}.$$  \hspace{1cm} (3.4)
with $\epsilon > 0$ but small enough. Upon computing the integral, it produces a summation over all solutions $\{\chi_a\}_1^{n_h}$ to the higher-level Bethe Ansatz equations

$$\mathcal{Y}_0(\chi_a \mid \{\chi_a\}_1^{n_h}) = 0 \quad a = 1, \ldots, n_x ,$$

(3.5)

(see e.g. [11] for more details).

Proof —

The above series expansion for the spin-spin two-point function can be obtained as follows. We first focus on excited states (i.e. those containing a non-zero number of holes). In this case, the contribution to the form-factor expansion originating from the sector with $n_h = n + n_w$ hole excitations takes the form

$$\langle \sigma^z_{m+1}(0) \sigma^z_m(t) \rangle_{n_h} = \sum_{h_1, \ldots, h_{n_h}} \sum_{\{\chi_a\}_{HBAE}} \sum_{\nu} \frac{e^{i\nu\pi} \cdot \prod_{a=1}^{n_h} \left[ e^{i\nu\pi(\chi_a)} - 2\pi i m p(\nu_a) \right] \cdot \mathcal{F}^{(c)}(\nu_a) - \nu_a \cdot \mathcal{Y}_0(\nu_a \mid \{\nu_a\}_1^{n_h})}{\prod_{a=1}^{n_h} \left[ 2\pi L p'(\nu_a) \right] \cdot \det_{n_h} \left[ \frac{\partial}{\partial \nu_a} \mathcal{Y}_0(\nu_a \mid \{\nu_a\}_1^{n_h}) \right]_{\nu_a = \chi_a}} (1 + O(L^{-\infty})).$$

(3.6)

The first sum goes through the two possible choices of the $\iota$ parameter which allows us to distinguish between the excitations above the ground state or the quasi-ground state. The second sum runs over all the possible choices of integers $1 \leq h_1 < \cdots < h_{n_h} \leq N + n_w$ which determine the configuration $\{\nu_a\}_1^{n_h}$ of hole positions in the given excited state. Finally, the last summation symbol runs through all solutions to the higher-level Bethe Ansatz equations subordinate to the choice $\{\nu_a\}_1^{n_h}$ of the hole parameters, namely:

$$\mathcal{Y}_0(\chi_a \mid \{\chi_a\}_1^{n_h}) = 0 \quad a = 1, \ldots, n_x .$$

(3.7)

Clearly, one can drop the exponentially small in $L$ corrections. Furthermore, the sum over all solutions to the higher-level Bethe Ansatz equation can be recast as a multi-dimensional residue integral

$$\sum_{\{\chi_a\}_{sols} \text{HBAE}} \frac{\mathcal{F}^{(c)}(\nu_a)}{\det_{n_h} \left[ \frac{\partial}{\partial \nu_a} \mathcal{Y}_0(\nu_a \mid \{\nu_a\}_1^{n_h}) \right]_{\nu_a = \chi_a}} = \mathcal{F}^{(c)}(\nu_a)$$

(3.8)

in which $\mathcal{F}^{(c)}(\nu_a)$ is as it has been defined in (3.3). Note that this function is analytic in some open neighbourhood of $[-\pi/2; \pi/2]$.

Finally, it follows from Proposition 1.1 that the hole parameters satisfy the equation

$$p(\nu_a) = \frac{h_u}{L} + O(L^{-1})$$

(3.9)

with a remainder that is uniform in respect to $h_u$. Thus, in the $L \to +\infty$ limit, the sum over the hole parameters turns into an integral over $[-\pi/2; \pi/2]$ as a Riemann sum:

$$\lim_{L \to +\infty} \sum_{h_1, \ldots, h_{n_h}} \prod_{a=1}^{n_h} \frac{e^{i\nu\pi(\nu_a) - 2\pi i m p(\nu_a)}}{2\pi L p'(\nu_a)} \cdot \mathcal{F}^{(c)}(\nu_a) = \int_{-\pi/2}^{\pi/2} \prod_{a=1}^{n_h} \frac{e^{i\nu\pi(\nu_a) - 2\pi i m p(\nu_a)}}{2\pi L p'(\nu_a)} \cdot \mathcal{F}^{(c)}(\nu_a) \cdot \frac{d^n \nu}{(2\pi)^n} .$$

(3.10)
We now focus on the case when there are no holes and \( \ell \in \{1, 0\} \). When \( \ell = 0 \), it follows from Theorem 2.1 that the form factor is simply zero. Since we are in a situation when \( \ell = 1 \) and \( \{v_\alpha\} = \{z_\alpha\} = \{0\} \) most of the terms in (2.24) simplify. Denoting by \( \{\tilde{\lambda}_1^N\} \) the solution of the Bethe Ansatz equations describing the quasi-ground state, one has

\[
\mathcal{T}^{(c)}_m([\tilde{\lambda}_1^N; \{\lambda_1^N\}) = -4(-1)^m \exp \left\{ - \int_{-\pi/2}^{\pi/2} \frac{ds}{\tan(s - \theta - i\eta)} \right\} \frac{\det^2 \left[ I + U_\theta[F_1(\cdot | \{0\}; \{0\})] \right]}{\det^2 [I + K]} (1 + O(L^{-\infty})).
\]  

(3.11)

Above, we have already specified the two arbitrary parameters \( \theta_1 \) and \( \theta_2 \) to take the same value \( \theta \in [-\pi/2; \pi/2] \) and made use of the fact that

\[
F_1(\cdot | \{0\}; \{0\}) = \frac{1}{2}.
\]  

(3.12)

The integral in the exponent can be computed explicitly:

\[
\int_{-\pi/2}^{\pi/2} \frac{ds}{\tan(s - \theta - i\eta)} = i\pi.
\]  

(3.13)

Furthermore, the integral kernel of the operator \( U_\theta[1/2] \) takes the simple form

\[
U_\theta[1/2](\omega, \omega') = \frac{K(\omega - \omega') - K(\theta - \omega')}{2} \cdot \exp \left\{ \frac{i\pi}{2} \left( \text{sgn}(\Im(\omega)) - 1 \right) \right\}.
\]  

(3.14)

Thus, squeezing the contour \( \Gamma \) to \( [-\pi/2; \pi/2] \), one obtains

\[
\det \left[ I + U_\theta[F_1(\cdot | \{0\}; \{0\})] \right] = \det \left[ I + V \right] \quad \text{with} \quad V(\omega, \omega') = -[K(\omega - \omega') - K(\theta - \omega')] \cdot \exp \left\{ \frac{i\pi}{2} \left( \text{sgn}(\Im(\omega)) - 1 \right) \right\}.
\]  

(3.15)

The determinants of \( I + K \) and \( I + V \) can be computed explicitly. Indeed, with respect to the orthonormal basis \( \{e^{2\pi i m/\pi}\}_{m \in \mathbb{Z}} \), one has

\[
K[e^{2\pi i m/\pi}](\lambda) = e^{-2\pi i \eta} \cdot \frac{e^{2\pi i m/\pi}}{\pi} \quad \text{and} \quad - \int_{-\pi/2}^{\pi/2} K(\theta - \lambda)e^{2\pi i m/\pi} \frac{d\lambda}{\pi} = \frac{-1}{\pi} e^{-2\pi i \theta - 2\pi i \eta}.
\]  

(3.16)

As a consequence of (3.16), one has

\[
\det[I + K] = \prod_{m \in \mathbb{Z}} \left( 1 + e^{-2\pi i \eta} \right) \quad \text{and} \quad \det[I + V] = \prod_{m \in \mathbb{Z}} \left( 1 - e^{-2\pi i \eta} \right).
\]  

(3.17)

Thus, one arrives at the representation

\[
\mathcal{T}^{(c)}_m([\tilde{\lambda}_1^N; \{\lambda_1^N\}) = (-1)^m \prod_{n \geq 1} \left( \frac{1 - e^{-2\pi i \eta}}{1 + e^{-2\pi i \eta}} \right)^4 \cdot (1 + O(L^{-\infty})). \quad (3.18)
\]

It then solely remains to put all the partial results together, which leads to the form-factor series (3.2) upon the hypothesis of its convergence.
3.2 Comparison with the vertex-operator approach

Multiple-integral representations for the form factors of the spin operators in the XXZ spin-$\frac{1}{2}$ chain in the massive regime were computed by means of the vertex-operator formalism in [20]. The results of [20] allowed for an analysis of the density structure factor in the massive regime of the chain [9]. This work was later generalized by Lashkevich to the XYZ case [37] whose result was then used to conjecture the form factors of local fields in the sine-Gordon model [41] and also to compute the longitudinal structure factor for the XXZ model in the massless regime at zero magnetic field [10]. For the moment, we are unable to make a direct and general connection between our formulae and those obtained within the vertex-operator approach. Still, in the subsequent analysis, we find agreement in the case of two-hole excitations.

It is interesting to note that the formula by Jimbo and Miwa for the two-particle form factor of $\sigma^z$ involves a non-trivial contour integral, whereas the expression found by Lashkevich does not contain any integrals. Therefore, we prefer to compare our result with Lashkevich’s result [37]. Taking the limit to the massive XXZ model, we obtain for the two-particle amplitude

$$\frac{1}{4} \left| \langle \text{vac}|\sigma^z|v_1, v_2 \rangle^{(0)} \pm \langle \text{vac}|\sigma^z|v_1, v_2 \rangle^{(1)} \right|^2 = |f_\pm(v_1, v_2)|^2$$

(3.19)

where the indices (0) and (1) label the ground states and $\epsilon \in \{+,-\}$ is the spin index. The functions $f_\pm$ are defined by

$$f_\pm(v_1, v_2) = f_\pm(v_1 + \pi, v_2) = \frac{\pi}{\eta} G(v_1 - v_2) \theta_1 \left( \frac{\pi v_1 + v_2}{2\eta} | e^{-\pi^2/\eta} \right) \frac{i \theta_1(0) e^{-\pi^2/\eta}}{\sin \left( \frac{v_1 - v_2}{2} + \frac{\eta}{2} \right)} ,$$

(3.20)

wherein

$$G(x) = \mathrm{q}^{2(x+\pi i)2} \left( \frac{q^4; q^4, q^4}{q^2; q^4, q^4} \right)_\infty \left( \frac{e^{-2ix}; q^4, q^4}{e^{-2ix}; q^4, q^4} \right)_\infty \left( \frac{e^{2ix}; q^4, q^4}{e^{2ix}; q^4, q^4} \right)_\infty .$$

(3.21)

Here we have introduced the notation

$$(x; p_1, p_2)_\infty = \prod_{k,l=0}^{\infty} \left( 1 - x p_1^k p_2^l \right) .$$

(3.22)

For our conventions for Jacobi-theta functions we refer to (A.13).

Let us consider now our result, Theorem 2.1, in the case of an arbitrary two-hole excitation. This implies that we have a single higher-level Bethe equation (for one root $\chi$) whose solutions were given in (1.43). Inserting the solution $\chi$ in Theorem 2.1 we obtain the amplitude

$$T^{(\mathcal{N})}_m(\tilde{\mu}_{a_1}^N; \lambda_a^N) = \sin^2 \left( \frac{(\pi/2 + \pi p(v_1) + \pi p(v_2))}{2} \right) \frac{e^{-2\pi i m (p(v_1) + p(v_2))}}{(2\pi L)^2 (v_1 + v_2)^2} \left( -1 \right)^m$$

$$\times 128 \frac{\sin(\pi F(v_1)) \sin(\pi F(v_2))}{\sin^2(\eta)} \frac{D}{\prod_{n \in \mathbb{Z}} (1 + q^{2m})^2}$$

$$\times (q^2; q^4)_\infty^8 \left( \frac{q^4; q^4, q^4}{q^2; q^4, q^4} \right)_\infty \prod_{\epsilon = \pm} \left( \frac{q^4 e^{2\pi i \epsilon v_1}; q^4, q^4}{q^4 e^{2\pi i \epsilon v_2}; q^4, q^4} \right)_\infty^4 .$$

(3.23)
where \( \epsilon \in \{0, 1\} \) and \( \nu_1 = \nu_1 - \nu_2 \). The infinite product is defined by
\[
(x; p) = \prod_{k=0}^{\infty} (1 - x p^k) .
\] (3.24)
The factor \( D \) is given by Fredholm determinants,
\[
D = \det(\mathcal{I} + \mathcal{U}_{\nu_1}) \cdot \mathcal{R}_{\nu_1} \cdot \det(\mathcal{I} + \mathcal{U}_{\nu_2}) \cdot \mathcal{R}_{\nu_2} .
\] (3.25)
The determinants involve operators acting on the interval \([-\pi/2, \pi/2]\) shifted in the lower half plane by \( 0 < \epsilon < \eta \). The kernel is given by
\[
\mathcal{U}_\theta(x, y) = e^{-i\epsilon} \frac{\partial_4(x - \nu_1 | q^2)\partial_4(x - \nu_2 | q^2)}{\partial_1(x - \nu_1 | q^2)\partial_1(x - \nu_2 | q^2)} [K(x - y) - K(\theta - y)] ,
\] (3.26)
where \( \epsilon \in \{0, 1\} \). Finally, the factor \( \mathcal{R}_{\nu_1} \) is defined by
\[
\mathcal{R}_{\nu_1} = 1 - \frac{2\pi e^{-i\epsilon n}}{e^{-2\pi i K(\nu_1)} - 1} \frac{\partial_4(0 | q^2)\partial_4(\nu_1 - \nu_2 | q^2)}{\partial_1(0 | q^2)\partial_1(\nu_1 - \nu_2 | q^2)} \times \left( K(0) - K(\nu_1 - \nu_2) - \int_{-\frac{\pi}{2} - i\epsilon}^{\frac{\pi}{2} - i\epsilon} dz \mathcal{R}_{\nu_2}(v_1, z) [K(v_1 - z) - K(v_2 - z)] \right) ,
\] (3.27)
The corresponding equation for \( \mathcal{R}_{\nu_2} \) is obtained by interchanging \( \nu_1 \) and \( \nu_2 \). The function \( \mathcal{R}_\theta(x, y) \) is the resolvent of the operator \( \mathcal{U}_\theta \) defined as the solution to the linear integral equation
\[
\mathcal{R}_\theta(x, y) = \mathcal{U}_\theta(y, x) - \int_{-\frac{\pi}{2} - i\epsilon}^{\frac{\pi}{2} - i\epsilon} dz \mathcal{R}_\theta(z, y) \mathcal{U}_\theta(z, x) .
\] (3.28)
Note that the amplitude for the second solution \( \chi_2 \) of the higher-level equations can be obtained by replacing \( (\nu_1, \nu_2) \leftrightarrow (\nu_1 + \pi, \nu_2) \) in (3.24).

The quantity we should compare with the vertex-operator formula (3.19) is the form-factor density
\[
A_2(\nu_1, \nu_2 | \ell) = \mathcal{F}_{\nu_1, \nu_2}^{(\ell)}(\mu, \nu_1, \nu_2 | A_1^{(\ell)} N) \cdot \left( (2\pi R)^2 p'(\nu_1) p'(\nu_2) \right) .
\] (3.29)
For the moment, we do not know how to calculate the Fredholm determinant part \( D \) analytically. However, we can compute it numerically and then compare with (3.19). Remarkably, we find from a numerical calculation that \( \mathcal{R}_{\nu_2} = 0 \) for the cases \( \kappa = \chi_1, \ell = 0 \) and \( \kappa = \chi_2, \ell = 1 \). The remaining two non-zero amplitudes are connected by a shift \( \nu_1 \leftrightarrow \nu_1 + \pi \). Thus, our results agree with those from the vertex-operator approach if
\[
A_2(\nu_1, \nu_2 | \ell = 1) = 2 |f(\nu_1, \nu_2)|^2
\] (3.30)
holds.\textsuperscript{10} Our numerical calculation shows that our conjecture (3.30) is indeed correct (cf. Figure 3). Note that (3.30) implies a non-trivial identity for the Fredholm determinant part \( D \) defined in (3.25). It would be interesting to have a direct proof for this identity.

We have shown the equivalence of our expressions to those from the vertex-operator approach in the case of two-hole excitations. We expect that the equality holds, in fact, in each \( 2n_1 \)-hole excitation sector, hence leading to highly non-trivial identities between multiple integrals. We plan to explore this question in a separate publication.

\textsuperscript{10}The factor 2 is due to the summation over the spin index \( \epsilon \) in (3.19).
3.3 The large-\(m\) asymptotic expansion

**Proposition 3.1** Assume that the form-factor series (3.2) is convergent. Then, one has the large-distance asymptotic expansion

\[
\langle \sigma^z_{1}(0) \cdot \sigma^z_{m+1}(0) \rangle = (-1)^m \prod_{n=1}^{\infty} \left( \frac{1 - e^{-2\eta \nu}}{1 + e^{-2\eta \nu}} \right)^4 + O(e^{-cm})
\]

(3.31)

for some \(c > 0\).

**Proof**

The sole complication in getting the result stems from the justification of the possibility to deform contours from \([-\pi/2; \pi/2]\) up to \([-\pi/2; \pi/2] - i\tau\), with \(\tau > 0\) but small enough. In deforming the contours, one will, in principle get the contributions of the boundaries \([\pi/2; \pi/2] - i\tau\) ∪ \([-\pi/2 - i\tau; -\pi/2]\). These do not cancel out directly since the integrand is not \(\pi\)-periodic with respect to the parameters \(\{\nu_a\}_n\).

Indeed, by using the explicit expression for the thermodynamic limit of the counting function, one obtains

\[
F_i(s | \nu_a + \pi \delta_{ab} \nu_1^n : \chi_{v_1}^{n_1} ) = F_i(s | \nu_a^{n_1} : \chi_{v_1}^{n_1} ) + \frac{1}{2}.
\]

(3.32)

This property along with straightforward manipulations implies that

\[
\mathbb{F}^0_{\nu}((\nu_a + \pi \delta_{ab})_1^n) = \mathbb{F}^0_{\nu}(\nu_1^{n_1}) \text{ and } \mathbb{F}^0_{\nu}(\nu_a + \pi \delta_{ab})_1^n = \mathbb{F}^0_{\nu}(\nu_1^{n_1}).
\]

(3.33)

Likewise, the quasi-periodicity of the dressed momentum \(p(s + \pi) = p(s) + 1/2\) ensures that

\[
(-1)^m \prod_{a=1}^{n_b} e^{-2\pi \text{amp}(\nu_a)} \xrightarrow{\nu_a \mapsto \nu_a + \pi \delta_{ab}} (-1)^{(s+1)m} \prod_{a=1}^{n_b} e^{-2\pi \text{amp}(\nu_a)}
\]

(3.34)
Hence, all-in-all, one has that the function
\[
\{v_a\}_1^{n_a} \mapsto \sum_{i=0}^{1} (-1)^m \prod_{a=1}^{n_a} e^{-2\pi i m p(v_a)} \cdot \mathcal{G}_i^{(c)}(\{v_a\}_1^{n_a})
\] (3.35)
is \pi\text{-periodic} in each of the variables \(v_a\). Furthermore, the integrands in each term of the form-factor series are holomorphic in some open neighbourhood of \([-\pi/2; \pi/2]\). We thus deform the original contour to the lower-half plane. Note that, due to the \(\pi\text{-periodicity}\), the contributions issuing from an integration on \([\pi/2; \pi/2 - i\tau]\) cancel out with those issuing from an integration on \([-\pi/2 - i\tau; -\pi/2]\).

The lack of a better insight into the analytic structure of the integrand (3.3) does not allow us, for the moment, to obtain a better estimation of the constant \(c > 0\) in (3.31) directly from Theorem 2.1. However, if we assume that

- conjecture (3.30) is correct,
- the higher-spinon excitations in the form-factor expansion give rise to a sub-dominant large-\(m\) asymptotic behaviour relative to the two-spinon excitations,

we can calculate the next term in the asymptotic expansion (3.31). The second hypothesis might seem trivial on first thought. However, the features of a large-parameter asymptotic behaviour of multi-dimensional deformations of a Fredholm determinant – which is basically the case of all series of multiple integrals of representations for the correlation functions in integrable models away from their free fermion point – can go quite far outside the scheme of classical asymptotic analysis. See, e.g. [35], where it was shown that the large-distance asymptotic behaviour of the generating function of density-to-density correlation functions in the non-linear Schrödinger model gives rise to a tower of correlation lengths that is quite different from the one that could be expected on the basis of a ‘classical’ term-by-term analysis of the individual integrals building up the series.

**Proposition 3.2** Under the above assumptions, the asymptotic expansion below holds
\[
\langle \sigma_i^z(0) \cdot \sigma_{m+1}^z(0) \rangle = (-1)^m \prod_{n \geq 1} \left( \frac{1 - e^{-2\eta q}}{1 + e^{-2\eta q}} \right)^4 + A \cdot \frac{k(q^2)^m}{m^2} \left( (-1)^m - \tanh^2 \left( \frac{\eta}{2} \right) \frac{(q; q^2)\!_{\infty}^4}{(-q; q^2)\!_{\infty}^4} \right) \left( 1 + O \left( m^{-1} \right) \right) \] (3.36)
where
\[
k(q^2) = \frac{\theta_2^2(0 | q^2)}{\theta_3^2(0 | q^2)}, \quad A = \frac{1}{\pi \sinh^2 \left( \frac{\eta}{2} \right)} \frac{(q^2; q^2)_\infty^4}{(q^4; q^4)_\infty^4} \left( \frac{(q^2; q^2)_\infty^4}{(q^4; q^4)_\infty^4} \right)^8.
\] (3.37)

**Proof**

Suppose that (3.30) is correct. Then the two-hole contribution to the form-factor series (3.2) is given by
\[
I_2(m) = \int_{-\pi}^{0} \frac{dv_2}{2\pi} e^{2\pi i m p(v_2)} \int_{-\pi}^{0} \frac{dv_1}{2\pi} e^{2\pi i m p(v_1)} \left( (-1)^m \cdot |f_-(v_1, v_2)|^2 + |f_+(v_1, v_2)|^2 \right) \] (3.38)
Since the integrand is \(\pi\text{-periodic and holomorphic}^{[1]}\) in the strip \(0 \leq \Im(v) \leq \pi/2\), we may shift the contour by \(\pi/2\) in the upper half-plane (note that \(e^{2\pi i m p(v)}\) vanishes at \(-\pi + i\eta/2, i\eta/2\) hence compensating the poles stemming from the form factor density). The exponent becomes real and negative on this contour with a single maximum at

---

\(^{[1]}\) Of course, we have to evaluate the modulus for real arguments and then continue analytically to the upper half-plane.
\[ \langle \sigma_z^1(0) \cdot \sigma_z^2(0) \rangle^{2-\text{spinon}} = - \prod_{n \geq 1} \left( 1 - \frac{e^{-2n\eta}}{1 + e^{-2n\eta}} \right)^4 + I_2(1) \]

Figure 3: Comparison of the exact nearest-neighbour correlator \cite{23} (black line) and the 2-spinon approximation (red line). The relative difference between both curves is of the order $10^{-3}$ for $\Delta > 2$. Note that both functions remain finite in the limit $\Delta \to 1$ with the known ratio of ca. 73\% (cf. \cite{8}).

\[ \nu = -\pi/2 + i\eta/2. \] A saddle-point analysis then leads to \eqref{3.36}. Note that a similar analysis was performed in \cite{21} to determine the correlation lengths in the eight-vertex model.

A similar method can be used to study the limit $m, t \to \infty$ (with fixed $m/t = \nu$) of the dynamical correlation functions. We plan to study this problem in future work.

We would like to add that the integral $I_2(m)$ can also be computed numerically (using the shifted contour) for short distances $m$ where the asymptotic expansion is not efficient. Remarkably, away from the isotropic point $\Delta = 1$, the 2-spinon contribution

\[ \langle \sigma_z^1(0) \cdot \sigma_z^2(0) \rangle^{2-\text{spinon}} = - \prod_{n \geq 1} \left( 1 - \frac{e^{-2n\eta}}{1 + e^{-2n\eta}} \right)^4 + I_2(1) \]

(3.39)

to the nearest-neighbour correlation functions approximates the exact result \cite{23} with high precision (cf. Figure 3).

4 Summary and outlook

We revisited the problem of the evaluation of form factors of the spin-$\frac{1}{2}$ XXZ model in the massive regime from the algebraic Bethe Ansatz perspective. We started from a determinant expression of the form factor of the operator $\sigma_z^i$ in the finite volume \cite{32, 26} and performed a careful analysis of its large-$L$ behaviour. For this purpose we re-analysed the Bethe Ansatz equations using a non-linear integral equation for the counting function. This allowed us to resolve a certain controversial fine-point in the older literature \cite{2, 46}. From the non-linear integral equations we obtained the higher-level Bethe Ansatz equations \cite{46} that determine the complex Bethe roots pertaining to the low-lying excitations of the model in the thermodynamic limit. We managed to analyse form factors parametrised by such complex roots in the large-$L$ limit. Our main result is their explicit characterization in Theorem \ref{2.1}.
We would like to emphasise that, at least for a small number of holes, the formulae in Theorem 2.1 are efficient for a numerical calculation of form factors and amplitudes. We exemplified this in Section 3, where we considered the form-factor expansion of the $\sigma^z - \sigma^z$ two-point function. Assuming convergence of the form-factor series we could show that the first term is given by the staggered magnetization of Baxter [4, 20], while the remainder decays exponentially fast with the distance. We further compared our result for the two-spinon case with the formula obtained by Lashkevich [37] in the context of the XYZ model and found numerical agreement. We naturally expect agreement in general, which implies non-trivial identities among multiple-contour integrals and Fredholm determinants. We hope to clarify this point in near future. Using the form of the two-spinon amplitude implied by Lashkevich’s form-factor expression, we obtained an explicit result for the first decaying correction to the zeroth order formula given by the staggered magnetization. This result appears to be heretofore unknown and generalizes an old formula of Johnson, Krinsky and McCoy [21].

We have also started to analyse the temperature correlation functions of the XXZ model in the massive regime by means of the quantum transfer matrix approach at low temperatures. This provides a different view on the two-point functions and again structurally different formulae which we plan to publish separately. In separate work we shall also work out the asymptotics of the dynamical two-point functions that follows from the form-factor series expansion. It would be interesting to compare, in this case, the results with those issuing from the analysis of the density structure factor [9].

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A Solutions to linear integral equations

A.1 The Fourier coefficients

The linear integral equations driven by the integral operator $I + K$ can be solved by means of Fourier series expansion of $\pi$-periodic functions

$$f(\lambda) = \sum_{n \in \mathbb{Z}} c_n[f] \cdot e^{2i n \lambda} \quad \text{with} \quad c_n[f] = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} f(\lambda)e^{-2i n \lambda} \cdot \frac{d\lambda}{\pi}.$$  

(A.1)

It is easy to check that

$$c_n[K] = \frac{e^{-2i n \eta}}{\pi} \quad \text{and} \quad \int_{-\pi/2}^{\pi/2} f(\lambda - \mu)g(\mu) \cdot d\mu = \pi \sum_n c_n[f] \cdot c_n[g] \cdot e^{2i n \lambda}.$$  

(A.2)

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Further, defining
\[ b_n(t) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \theta(\lambda - t)e^{-2i n \lambda} \frac{d\lambda}{\pi} \]  \hspace{1cm} (A.3)
onumber
one obtains that

- when \(|\Im(t)| > \eta\),
  \[ b_0(t) = 2\eta \text{sgn}(\Im(t)) \quad \text{and} \quad b_n(t) = \frac{2}{n} \sinh(2n\eta)e^{-2i \eta \cdot \delta_n < 0} \left( \delta_{n<0} 1_{\Im(t) > \eta} - \delta_{n>0} 1_{\Im(t) < -\eta} \right) \] \hspace{1cm} (A.4)
onumber
for \(n \neq 0\);

- and, when \(|\Im(t)| < \eta\), one has
  \[ b_0(t) = i(\pi - 2t) \quad \text{and} \quad b_n(t) = \frac{(-1)^n + 1}{n} + \frac{1}{n} e^{-2i nt - 2i n \eta} \] \hspace{1cm} (A.5)
onumber
for \(n \neq 0\).

The coefficients \(b_n(t)\), for \(n \neq 0\), are readily obtained through an integration by parts. In order to compute \(b_0(t)\), it remains to observe that \(t \mapsto b_0(t)\) is analytic in the regions \(|\Im(t)| > \eta\) and \(|\Im(t)| < \eta\), and that
\[ b'_0(t) = \begin{cases} 0 & |\Im(t)| > \eta \\ -2i & |\Im(t)| < \eta \end{cases} \] \hspace{1cm} (A.6)
onumber
Further, the function \(\theta\) has jumps on \([-\pi/2 ; \pi/2] \pm i \eta\)
\[ \theta(\lambda - x \pm i \eta + i0^+) - \theta(\lambda - x \pm i \eta - i0^+) = \pm 2i\pi 1_{\lambda \in \{x \pm i \eta\}}. \] \hspace{1cm} (A.7)
onumber
The asymptotics
\[ \lim_{y \to \pm \infty} \theta(\lambda - iy) = \pm 2\eta \quad \text{and the jump} \quad b_0(x \pm i \eta - i0^+) - b_0(x \pm i \eta + i0^+) = \pm 2i(\pi/2 - x) \] \hspace{1cm} (A.8)
fix the values of \(b_0(t)\) in each of the regions.

### A.2 The dressed momentum and energy

The integral equation (1.14) can be solved explicitly by means of Fourier transformations. One finds, for \(\lambda \in [-\pi/2 ; \pi/2]\),
\[ p(\lambda) = \frac{1}{2} p\left(\frac{\pi}{2}\right) + \frac{\lambda + \pi/2}{2\pi} + \frac{1}{2\pi} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{e^{2in\lambda}}{2in \cosh(n\eta)}. \] \hspace{1cm} (A.9)
Thus
\[ p\left(\frac{\pi}{2}\right) = 0 \quad \text{and} \quad p\left(\frac{\pi}{2}\right) = \frac{1}{2} \quad \text{viz.} \quad p(z) = \int_{-\pi/2}^{z} p'(s) \cdot ds. \] \hspace{1cm} (A.10)
As a consequence

\[ p(\lambda) = \frac{\lambda + \pi/2}{2\pi} + \frac{1}{2\pi} \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{e^{2in\lambda}}{2in \cosh(n\eta)}. \]  

(A.11)

In fact, the derivative \( p' \) can be expressed in terms of elliptic functions as

\[ p'(\lambda) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \frac{e^{2in\lambda}}{\cosh(n\eta)} = \sum_{n \in \mathbb{Z}} \frac{1}{2\eta \cosh \left( \frac{\pi}{\eta} (n\pi - \lambda) \right)} = \frac{1}{2\pi} \prod_{n \geq 1} \left( \frac{1 - q^{2n}}{1 + q^{2n}} \right)^2 \cdot \frac{\vartheta_3(\lambda \mid q)}{\vartheta_4(\lambda \mid q)}. \]  

(A.12)

The second representation for \( p' \) ensures that \( p \) is strictly increasing on \([-\pi/2 ; \pi/2]\). Furthermore, it is also easy to deduce from it that \( p' \) is \( \pi \)-periodic and \( i\eta \) anti-periodic. Above, we used the following convention for the \( \vartheta \) functions of nome \( q = e^{-\eta}\)

\[ \vartheta_3(\lambda \mid q) = \prod_{n \geq 1} \left( (1 - q^{2n}) \cdot (1 + 2q^{2n-1} \cos(2\lambda) + q^{4n-2}) \right) \]

\[ \vartheta_3(\lambda \mid q) = \vartheta_3(\lambda - \pi/2 \mid q), \quad \vartheta_1(\lambda \mid q) = -ie^{i\lambda/\eta} \vartheta_4(\lambda + i\eta/2 \mid q), \quad \vartheta_2(\lambda \mid q) = \vartheta_1(\lambda + \pi/2 \mid q). \]  

(A.13)

We stress that the second equality in (A.12) follows from the Poisson summation formula whereas the third one follows from the fact that \( \lambda \mapsto p'(\lambda) \) is a \( \pi \)-periodic, \( i\eta \) anti-periodic function whose only simple pole in the fundamental simplex is located at \( \eta/2 \) and has residue \( 2i\pi \).

The function \( p \) can be explicitly computed in terms of the semi-infinite product

\[ (z)_{\infty} = \prod_{n \geq 0} (1 - q^{4n}) \quad \text{and} \quad \left( \frac{\{a_k\}_1^p}{\{b_k\}_1^q} \right)_{\infty} = \frac{\prod_{k=1}^{p} (a_k)_{\infty}}{\prod_{k=1}^{q} (b_k)_{\infty}} \]  

(A.14)

as

\[ p(\lambda) = \frac{\lambda + \pi/2}{2\pi} + \frac{1}{2i\pi} \ln \left( \frac{qe^{-2i\lambda}; q^3 e^{2i\lambda}}{q^{2i\lambda}; q^3 e^{-2i\lambda}} \right)_{\infty} = \frac{\lambda + \pi/2}{2\pi} + \frac{1}{2i\pi} \ln \left( \frac{\vartheta_4(\lambda + i\eta/2 \mid q^2)}{\vartheta_4(\lambda - i\eta/2 \mid q^2)} \right). \]  

(A.15)

The last expression represents the dressed momentum as a \( q \)-deformation of \( p_0 \).

It follows either from the integral representation for \( p \) or from the above representation, that it is an \( i\eta \) anti-periodic function with values in \( \mathbb{C}/(2i\pi\mathbb{Z}) \). For instance, one checks that all terms cancel out in the expression for

\[ \exp \left[ 2i\pi [p(\lambda) + p(\lambda - i\eta)] \right] = 1. \]  

(A.16)

Finally, a direct integration shows that, given \( z = x + iy \),

\[ \Re[2i\pi p(z)] = 2\eta \ln \left[ \prod_{n \in \mathbb{Z}} \frac{\cosh \left[ \frac{x}{\eta} (x - n\pi) \right] - \sin \left[ \frac{y}{\eta} (x - n\pi) \right]}{\cosh \left[ \frac{x}{\eta} (x - n\pi) \right] + \sin \left[ \frac{y}{\eta} (x - n\pi) \right]} \right]. \]  

(A.17)

Hence,

\[ 0 < \Im(z) < \eta \Rightarrow \Re[2i\pi p(z)] < 0 \quad \text{and} \quad -\eta < \Im(z) < 0 \Rightarrow \Re[2i\pi p(z)] > 0. \]  

(A.18)
A direct manipulation of the linear integral equation driving the dressed energy shows that the latter can be recast as

\[ \varepsilon(\lambda) = \frac{\hbar}{2} - 4\pi J \sinh(\eta) \cdot p'(\lambda). \]  

(A.19)

It follows from this representation that the model is massive (viz. \( \varepsilon(\lambda) < 0 \) on \([-\pi/2; \pi/2]\)) if the magnetic field satisfies \( 0 \leq h < h_c \), with the critical field \( h_c \) being given by

\[ h_c = 4J \sinh(\eta) \prod_{n \geq 1} \left( \frac{1 - q^n}{1 + q^n} \right)^2. \]  

(A.20)

A.3 The resolvent

The resolvent kernel is defined as the solution to the linear integral equation

\[ R(\lambda - \mu) + \int_{-\pi/2}^{\pi/2} K(\lambda - \nu) R(\nu - \mu) \cdot d\nu = K(\lambda - \mu). \]  

(A.21)

The equation can be solved explicitly in terms of Fourier expansion, which yields

\[ R(\lambda) = \sum_{n \in \mathbb{Z}} c_n[R] e^{2i\mu \lambda} \quad \text{with} \quad c_n[R] = \frac{e^{-2i\mu \eta}}{\pi (1 + e^{-2i\mu \eta})}. \]  

(A.22)

One can recast the Fourier expansion of \( R \) in a form that is more suited to the study of the analytic properties of \( R \). Namely, one has

\[
R(\lambda) = \sum_{n \in \mathbb{Z}} \sum_{\ell \geq 1} \frac{(-1)^{\ell-1}}{\pi} e^{-2i\mu(\ell\eta + 2n\lambda)} = \frac{1}{2\pi} + \sum_{\ell \geq 1} \left( -\frac{1}{\pi} \sum_{n \geq 1} \{ e^{(2i\lambda - 2i\eta)\ell n} + e^{-(2i\lambda + 2i\eta)\ell n} \} \right) \\
= \frac{1}{2\pi} + \frac{1}{\pi} \sum_{\ell \geq 1} \left( e^{2i\lambda - 2i\eta} \frac{1}{1 - e^{-2i\lambda - 2i\eta}} + \frac{e^{-2i\lambda - 2i\eta}}{1 - e^{2i\lambda - 2i\eta}} \right). \]  

(A.23)

The above representation clearly shows that \( R \) admits a meromorphic extension to \( \mathbb{C} \) which satisfies the first order finite difference equation

\[ R(\lambda + i\eta) + R(\lambda) = \frac{1}{\pi} \left( \frac{1}{1 - e^{-2i\lambda}} - \frac{1}{1 - e^{-2i\lambda + 2i\eta}} \right). \]  

(A.24)

A.4 The dressed phase and its homogenised version

A.4.1 The case of ‘close’ auxiliary argument \(|\Im(z)| < \eta\)

When \(|\Im(z)| < \eta\), it is easily checked that the dressed phase has its Fourier coefficients given by

\[ c_n[\phi(\ast, z)] = \frac{\delta_{n,0}}{2} \left( 1 - \frac{\pi \cdot 2z}{2} \right) + \frac{(1 - \delta_{n,0}) \cdot e^{-2i\mu z - 2i\eta z}}{\pi \cdot (1 + e^{-2i\mu \eta})}. \]  

(A.25)

Above, \( \ast \) refers to the variable in respect to which the Fourier coefficients are computed.
The Fourier series for $\phi(\lambda, z)$ can, in fact, be re-summed and re-cast as

$$\phi(\lambda, z) = \frac{\pi + 4\lambda - 2z}{2} - \ln \left( \frac{-q^2 e^{2i\lambda}; -q^2 e^{-2i\lambda}; q^4 e^{2i(\lambda - z)}; q^4 e^{-2i(\lambda - z)} }{ -q^2 e^{2i\lambda}; -q^4 e^{2i(\lambda - z)}; q^2 e^{2i(\lambda - z)} } \right)_{\infty}. \quad (A.26)$$

Indeed, by using that

$$\lambda = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{(-1)^{n+1}}{2in} e^{2in\lambda} \quad \text{on} \quad [-\pi/2; \pi/2], \quad (A.27)$$

one can rewrite $\phi(\lambda, z)$ as

$$\phi(\lambda, z) = \frac{\pi + 4\lambda - 2z}{2} + \sum_{\epsilon = \pm 1} \sum_{n \geq 1} \frac{(-1)^n + e^{-2in\epsilon}}{n(1 + e^{-2n\epsilon})} e^{-2n \epsilon + 2in\lambda}. \quad (A.28)$$

At this stage it solely remains to observe that, for $|\Im(z)| < \eta$ and $\lambda$ real, one has

$$\sum_{\epsilon = \pm 1} \sum_{n \geq 1} \frac{e^{-2in\lambda}}{n(1 + e^{-2n\lambda})} e^{2in(\lambda - z)e} = \sum_{\epsilon = \pm 1} \sum_{k \geq 1} \sum_{n \geq 1} \frac{(-1)^{k-1}}{n} e^{2in(\lambda - z)e} e^{-2nk\epsilon}$$

$$= \sum_{\epsilon = \pm 1} \sum_{n \geq 1} \sum_{p \geq 1} \ln \left( \frac{1 - q^4 p e^{2i(\lambda - z)\epsilon}}{1 - q^4 p e^{-2i(\lambda - z)\epsilon}} \right) = \ln \left( \frac{q^2 e^{-2i(\lambda - z)}; q^4 e^{2i(\lambda - z)} }{q^4 e^{2i(\lambda - z)}; q^2 e^{2i(\lambda - z)} } \right)_{\infty} \quad (A.29)$$

and then add up the expressions for $z$ general and for $z = \pi/2$.

The representation (A.26) immediately leads to the identity

$$e^{\phi(\lambda, z) + \phi(\lambda + i\eta, z)} = \frac{\cos(\lambda) \sin(\lambda - z)}{\cos(\lambda + i\eta) \sin(\lambda + i\eta)} \quad \text{and} \quad e^{\phi(\lambda, z) + \phi(\lambda - i\eta, z)} = \frac{\cos(\lambda - i\eta) \sin(\lambda - z - i\eta)}{\cos(\lambda) \sin(\lambda - z)}. \quad (A.30)$$

The homogenised counterpart of the dressed phase admits the Fourier series expansion

$$\varphi(x, z) = \left( \frac{\pi}{2} + x - z \right) + 2i \sum_{n = 1}^{\infty} \frac{\sin \left[ 2n(x - z) \right]}{n(1 + e^{-2n\lambda})} \quad (A.31)$$

By using the method described above, the Fourier series can be re-summed as

$$\varphi(x, z) = i \left( \frac{\pi}{2} + x - z \right) + \ln \left( \frac{\Gamma_{q^4} \left( \frac{1}{2} - i \frac{(x - z)}{2q} \right) \cdot \Gamma_{q^4} \left( 1 + i \frac{(x - z)}{2q} \right) }{\Gamma_{q^4} \left( \frac{1}{2} + i \frac{(x - z)}{2q} \right) \cdot \Gamma_{q^4} \left( 1 - i \frac{(x - z)}{2q} \right) } \right) \quad (A.32)$$

where $\Gamma_{q}$ is the $q$-Gamma function defined in the whole complex plane by its product representation

$$\Gamma_{q}(x) = (1 - q)^{-1} \prod_{n=1}^{\infty} \frac{1 - q^n}{1 - q^{n+x-1}}. \quad (A.33)$$

Using the fundamental functional relation of the $q$-Gamma function, $\Gamma_{q}(x + 1) = [x]_{q} \Gamma_{q}(x)$, where $[x]_{q} = (1 - q^x)/(1 - q)$, we obtain the following functional relation for the periodised dressed phase,

$$e^{\varphi(x, z) + \varphi(x \pm i\eta, z)} = \left( \frac{\sin(x - z)}{\sin(x \pm z \pm i\eta)} \right)^{\pm 1} \quad (A.34)$$
A.4.2 The case of ‘wide’ auxiliary argument $|\Im(z)| > \eta$

When $|\Im(z)| > \eta$, the dressed phase and its periodised version coincide. It is readily seen that the dressed phase has its Fourier coefficients given by

$$c_n[\phi(\ast, z)] = \eta \text{sgn}(\Im(z)) \cdot \delta_{n,0} + (1 - \delta_{n,0}) \cdot \frac{2 \sinh(2\eta n)e^{-2mn}}{n \cdot (1 + e^{-2\eta n})} (\delta_{n<0} \mathbb{1}_{\Im(z) > \eta} - \delta_{n>0} \mathbb{1}_{\Im(z) < -\eta}). \quad (A.35)$$

The Fourier series for $\phi(\lambda, z)$ and $\phi(\lambda, \overline{z})$ can, in fact, be re-summed for $\Im(z) > \eta$, as

$$\phi(\lambda, z) = \varphi(\lambda, z) = \ln \left( \begin{array}{c} e^{-2i(\lambda - z)}; q^2 e^{-2i(\lambda - z)} \\ q^4 e^{-2i(\lambda - z)}; q^{-2} e^{-2i(\lambda - z)} \end{array} \right)_\infty + \eta = \ln \left( \frac{\sin(\lambda - z)}{\sin(\lambda - z + i\eta)} \right), \quad (A.36)$$

while, for $\Im(z) < -\eta$, it can be re-summed as

$$\phi(\lambda, z) = \varphi(\lambda, z) = \ln \left( \begin{array}{c} q^{-2} e^{2i(\lambda - z)}; q^4 e^{2i(\lambda - z)} \\ q^2 e^{2i(\lambda - z)}; e^{2i(\lambda - z)} \end{array} \right)_\infty - \eta = \ln \left( \frac{\sin(\lambda - z - i\eta)}{\sin(\lambda - z)} \right). \quad (A.37)$$

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