Exceptional and Rigid Sheaves on Surfaces with Anticanonical Class without Base Components.

S. Kuleshov.

November 2, 2018

Abstract

The paper consists of three parts. In the first of them different kinds stability are discussed. In particular, the stability concept with respect to nef divisor is introduced. A structure of rigid and superrigid vector bundles on smooth projective surfaces with nef anticanonical class is studied in the second part. We prove that any superrigid bundle has a unique exceptional filtration. In the last part we give constructible description of exceptional bundles on these surfaces.

Introduction.

This paper contains a generalisation of the theory of rigid (Ext\(^1\)(E, E) = 0) and exceptional (Ext\(^0\)(E, E) = \(\mathbb{C}\), Ext\(^i\)(E, E) = 0 for \(i > 0\)) sheaves on Del Pezzo surfaces, which was described in [12]. The objects of this paper are rigid and exceptional sheaves on smooth projective surfaces \(S\) such that \(-K_S\) is nef.

If \(K^2_S > 0\) then these surfaces may be obtained from \(\mathbb{P}^2\) by successively blowing up at most 8 points and are natural extension of Del Pezzo surface class.

At first the exceptional sheaves appeared in [3] for the description the possible Chern classes which a stable bundle on \(\mathbb{P}^2\). Besides in [5] was proved that any rigid bundle on projective plane is a direct sum of exceptional ones.

I proved the same fact for all Del Pezzo surfaces ([12]). But if \(-K_S\) is nef then there exists indecomposable and not simple rigid bundles, though a structure of them is described in terms of exceptional collections. The proof of this statement is a goal of the second part. The information about superrigid bundles gives convenient method for researching exceptional sheaves.

The theory of exceptional bundles on Del Pezzo surfaces uses stability with respect to anticanonical class. In this paper surfaces have the nef anticanonical class. A question is arose: is there a sufficient stability notion with respect to nef divisor, and which slope axioms are sufficient for constructing the having meaning stability theory? For example, when does Garder–Narasimhan filtration exist? An answer is the first part’s subject.

Finally in the last part of this work we prove a constructibility of exceptional bundles on smooth projective surfaces \(S\) over \(\mathbb{C}\) with nef anticanonical class and \(K^2_S > 0\). Here the constructibility means that any exceptional bundle can be obtained from finite fixed collection of exceptional sheaves by finite procedure.
1 Axioms of Stability.

1.1 Definitions and Simple Properties.

The Gieseker and Mumford–Takemoto stabilities are well known. Recently the vector stability with respect to a collection of polarizations notion arisen ([22]). All these theories have a slope and similar properties of stable and semistable sheaves. In this section we introduce some slope axioms and obtain basic properties of stable sheaves.

Definition. Let $\gamma$ be a function from the set of torsion free sheaves on a complete algebraic manifold $X$ over $\mathbb{C}$ to $\mathbb{R}^n$ with the lexicographic order. Assume $\gamma$ satisfy the axioms:

| $k$ | $\text{Ext}^k(F, A)$ | $\text{Ext}^k(F, B)$ | $\text{Ext}^k(F, C)$ |
|-----|---------------------|---------------------|---------------------|
| 0   | $*$                 | $?$                 | $*$                 |
| 1   | 0                   | $?$                 | 0                   |
| 2   | $*$                 | $?$                 | $*$                 |

This table calculates $\text{Ext}^1(F, B)$ . From the table it follows that $\text{Ext}^1(F, B) = 0$. 

The author is grateful to professor A. N. Rudakov and to contestants of his seminaire for useful discussions. Besides, the author is grateful to Sorros fond for a finance support.
1.1 Definitions and Simple Properties.

SLOPE.1. For any exact triple of torsion free sheaves

\[ 0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0 \]

the following equivalences hold true:

\[ \gamma(F) < \gamma(E) \iff \gamma(E) < \gamma(G), \]
\[ \gamma(F) > \gamma(E) \iff \gamma(E) > \gamma(G), \]
\[ \gamma(F) = \gamma(E) \iff \gamma(E) = \gamma(G); \]

SLOPE.2. For any two sheaves without torsion \( F \subset E \) from \( r(F) = r(E) \) it follows that

\[ \gamma(F) \leq \gamma(E). \]

Then we say that \( \gamma \) is a slope function and \( \gamma(E) \) is \( \gamma \)-slope of \( E \) or simply slope of \( E \) if no confusion is likely. If \( \gamma(E) \in \mathbb{R}^n \) then \( \gamma \) is called a vector slope function.

Definition. A torsion free sheaf \( E \) on algebraic manifold \( X \) is said to be \( \gamma \)-(semi)stable or simply (semi)stable if for any its subsheaf \( F \) with \( r(F) < r(E) \) the following inequality holds true

\[ \gamma(F) < \gamma(E) \]

\((\gamma(F) \leq \gamma(E) \text{ for semistable}).\)

A subsheaf, which contradicts (semi)stability is called destabilizing.

1.1.1 Remark. 1. If rank of torsion free sheaves equals 1 then they are stable with respect to any slope, because they have not rank zero subsheaves.

2. By virtue of the lexicographic order on \( \mathbb{R}^n \) the function \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \) is the slope iff any \( \gamma_i \) satisfies the slope axioms.

3. For slopes \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \) and \( \gamma' = (\gamma_1, \gamma_2, \ldots, \gamma_n, \gamma_{n+1}, \ldots, \gamma_m) \) the following statements are true
   a) a \( \gamma \)-stable sheaf is \( \gamma' \)-stable;
   b) a \( \gamma' \)-semistable sheaf is \( \gamma \)-semistable.

1.1.2 Lemma A torsion free sheaf \( E \) on a manifold \( X \) is (semi)stable if for any its subsheaf \( F \) such that \( E/F \) has not torsion

\[ \gamma(F) < \gamma(E) \quad (\gamma(F) \leq \gamma(E)). \]

Proof. Let \( F \) be a subsheaf of \( E \) with \( r(F) < r(E) \). Denote by \( T \) the torsion of the
sheaf $G = E/F$. Consider the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & F' & \rightarrow & E & \rightarrow & G' & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \rightarrow & F & \rightarrow & E & \rightarrow & G & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & & & & & & & & 0 \\
\end{array}
\]

Here $G'$ is a nonzero torsion free sheaf and $r(F) = r(F')$.

Since $E$ has not torsion, we see that $F'$ is also a torsion free sheaf. Therefore it follows from SLOPE.2. that $\gamma(F) \leq \gamma(F')$. By the lemma condition, $\gamma(F') < \gamma(E)$. Thus $\gamma(F) < \gamma(E)$. This inequality proves the fact that $E$ is stable.

1.1.3 Lemma. A torsion free sheaf $E$ is stable (semistable) if and only if the slope of any its torsion free quotient $G$ satisfy the inequality:

$$\gamma(E) < \gamma(G) \quad (\gamma(E) \leq \gamma(G)).$$

The proof follows from 1.1.2 and SLOPE.1.

For studying stability properties, we need the following:

1.1.4 Remark. Taking into account the definition of stability and SLOPE.2, we obtain that for any pair of torsion free sheaves $F \subset E$, from semistability of $E$ follows the inequality $\gamma(F) \leq \gamma(E)$ (without a condition on the ranks). If $E$ is stable then

$$\gamma(F) = \gamma(E) \quad \Rightarrow \quad r(E) = r(F).$$

Similarly, if $G$ is a quotient of a semistable sheaf $E$ then $\gamma(E) \leq \gamma(G)$. If $E$ is stable then

$$\gamma(G) = \gamma(E) \quad \Rightarrow \quad E = G.$$

1.1.5 Lemma. Let $E$ and $F$ be semistable sheaves. Suppose $\gamma(E) > \gamma(F)$; then

$$\text{Hom}(E, F) = 0.$$

Proof. Assume that there exists a nonzero morphism $\varphi : E \rightarrow F$. Denote $\text{im}\varphi$ by $I$. Since $I$ is a quotient of $E$ and a subsheaf of $F$, it follows from 1.1.4 that

$$\gamma(E) \leq \gamma(I); \quad \gamma(I) \leq \gamma(F).$$
But by assumption, $\gamma(E) > \gamma(F)$. This contradiction proves the lemma.

1.1.6 Lemma. Let $E$ and $F$ be semistable sheaves with $\gamma(E) = \gamma(F)$, let $\varphi : E \to F$ be a nonzero morphism. Then

- a) $E$ is stable $\iff$ $\varphi$ is an injection;
- b) $F$ is stable $\iff$ $\varphi$ is an epimorphism in general point.

Proof. By definition, put $I = \text{im } \varphi$. As above, $\gamma(E) \leq \gamma(I)$; $\gamma(I) \leq \gamma(F)$, and by lemma condition, $\gamma(E) = \gamma(I) = \gamma(F)$.

a) Assume that $E$ is stable. Using [1.1.4], we get $E = I$, i.e. $\varphi$ is a monomorphism.
b) If $F$ is stable then $r(I) = r(F)$, i.e. $\text{coker } \varphi$ is a torsion sheaf.

1.1.7 Lemma. A stable sheaf is simple, that is any its endomorphism has the form $\lambda \cdot \text{id}$.

Proof. Let us take any nonzero $\varphi \in \text{End}(E)$. Using the lemma [1.1.6], we obtain that $\varphi$ is isomorphism on an open set. Let $\lambda$ denote a eigenvalue of this isomorphism over some point $x \in X$. If the map $(\varphi - \lambda \text{id}_E) \in \text{End}(E)$ is nontrivial then it is injection. Thus there exists the monomorphism of determinants $\Lambda^{r(E)}(\varphi - \lambda \text{id}_E) : \text{det}(E) \to \text{det}(E)$.

Since the map $(\varphi - \lambda \text{id}_E)$ degenerates at the point $x$, the corresponding determinant map at $x$ is equal to zero. Hence, $\Lambda^{r(E)}(\varphi - \lambda \text{id}_E) \equiv 0$. Therefore $(\varphi - \lambda \text{id}_E)$ has a nontrivial kernel. It is impossible, i.e. $(\varphi - \lambda \text{id}_E) \equiv 0$. This completes the proof.

1.1.8 Lemma. Let

$$0 \to E \to i \to G \to F \to 0,$$

be an exact sequence of torsion free sheaves such that $\gamma(E) = \gamma(G) = \gamma(F)$. Then $G$ is semistable if and only if both $E$ and $F$ are semistable. In particular, for any finitedimensional vector space $V$ over $\mathbb{C}$ and a divisor $D$ on $X$ the bundle $V \otimes \mathcal{O}_X(D)$ is semistable.

Proof. Assume that $G$ is semistable. Since $E \subset G$, we see that any subsheaf $E'$ of $E$ is also a subsheaf of $G$. Hence, $\gamma(E') \leq \gamma(G) = \gamma(E)$, i.e. the sheaf $E$ is also semistable. Using (1.1.2), it can similarly by checked that the sheaf $F$ is also semistable.

Now suppose the sheaves $E$ and $F$ are semistable and $G$ has torsion free quotient $\varphi : G \to Q \to 0$ such that $r(Q) < r(G)$ and $\gamma(G) > \gamma(Q)$ [1.1.3]. If $Q$ is not semistable then consider its quotient $Q'$ such that $r(Q') < r(Q)$ and $\gamma(Q')$, etc... Taking into account [1.1.1], we can assume without loss of generality that $Q$ is semistable. Thus it follows from

$$\gamma(E) = \gamma(G) > \gamma(Q)$$
that \( \varphi \circ i \) is the zero map (see \([1.1.5]\)). On the other hand, for the same reason,
\[
\text{Hom}(F, Q) = 0
\]

This contradiction concludes the proof.

### 1.2 Harder-Narasimhan Filtration.

The aim of this section is the construction for a torsion free sheaf the well known canonical filtration which becomes trivial when the sheaf is semistable. Let us remember the main definition and notations.

The record \( \text{Gr}(E) = (G_n, G_{n-1}, \ldots, G_1) \) means that the sheaf \( E \) has a filtration:
\[
0 = E_{n+1} \subset E_n \subset \cdots \subset E_2 \subset E_1 = E
\]
and \( E_i/E_{i+1} = G_i \). The sheaves \( E_i \) are called terms of filtration and \( G_i \) are quotients of filtration. Note that \( G_n = E_n \) (since \( E_{n+1} = 0 \)).

**Definition.** A filtration of a torsion free sheaf \( E \)
\[
\text{Gr}(E) = (G_n, G_{n-1}, \ldots, G_1)
\]
is called **Harder-Narasimhan filtration** if all quotients \( G_i \) are semistable and their slopes satisfy inequalities:
\[
\gamma(G_{i+1}) > \gamma(G_i) \quad (i = 1, 2, \ldots, n - 1).
\]

To construct this filtration, we need another one slope axiom and several lemmas.

#### 1.2.1 Lemma. Let \( E \) be a sheaf without torsion on \( X \) and \( \mathcal{G} \) the set of all torsion free quotients of \( E \). Then there is \( \gamma_0 \) such that \( \gamma(G) \geq \gamma_0 \) for each \( G \in \mathcal{G} \).

**Proof.** Let us choose an ample divisor \( A \) on \( X \). Then it follows from the Serre theorem ([23]) that there exists a natural number \( n \) such that the sheaf \( E(nA) \) is generated by global sections. Hence we have the exact sequence:
\[
0 \longrightarrow F \longrightarrow H^0(E(nA)) \otimes \mathcal{O} \longrightarrow E(nA) \longrightarrow 0.
\]

Therefore \( E \) is a quotient of the semistable bundle
\[
H^0(E(nA)) \otimes \mathcal{O}(-nA).
\]
If \( G \) is a torsion free quotient of \( E \) then there exists an epimorphism
\[
H^0(E(nA)) \otimes \mathcal{O}(-nA) \longrightarrow G \longrightarrow 0.
\]

Now the lemma follows from \([1.1.3]\).
1.2.2 **Lemma.** Suppose a slope function $\gamma$ satisfies the axiom:

**SLOPE.3.** Let $\gamma_0$ be a value of the function $\gamma$ and $M = \{G_1, G_2, G_3, \ldots\}$ an ordered set of sheaves without torsion with $r(G_i) \leq r$ for any $i$. Then the condition

$$\gamma(G_i) > \gamma(G_{i+1}) \geq \gamma_0 \quad \forall i = 1, 2, 3, \ldots$$

implies that $M$ is finite.

Then each torsion free sheaf $E$ has the quotient $G$ with the minimal slope $\gamma(G)$.

That is, for another torsion free quotient $Q$ of $E$, we have: $\gamma(Q) \geq \gamma(G)$.

The proof follows from SLOPE.3 and the previous lemma.

1.2.3 **Proposition.** If a slope function $\gamma$ satisfies the axioms SLOPE.1 - SLOPE.3 then any torsion free sheaf $E$ has the Harder-Narasimhan filtration

$$Gr(E) = (G_n, G_{n-1}, \ldots, G_1).$$

Moreover, if $Gr(E) = (G'_m, G'_{m-1}, \ldots, G'_1)$ is another filtration with semistable quotients and the inequalities $\gamma(G'_i) > \gamma(G'_{i-1})$ for each $i = 2, 3, \ldots, m$ then $m = n$ and $G'_i = G_i$ $\forall i$.

**Proof of existence.** A semistable sheaf has trivial filtration. Suppose $E$ is not semistable. Denote by $G_1$ the torsion free quotient of $E_1 = E$ with the minimal $\gamma$-slope and the maximal rank. That is for another quotient $Q$ of $E$ we have $\gamma(Q) \geq \gamma(G_1)$, and the equality $\gamma(Q) = \gamma(G_1)$ implies $r(Q) \leq r(G_1)$. Let $E_2$ be the corresponding subsheaf in $E$:

$$0 \rightarrow E_2 \rightarrow E_1 \rightarrow G_1 \rightarrow 0.$$

If $E_2$ is not semistable then let us choose the torsion free quotient $G_2$ of $E_2$ with the minimal $\gamma$-slope and the maximal rank. Denote by $E_3$ the corresponding subsheaf in $E_2$, etc. Note that all $G_i$ are semistable by construction. Let us check the inequality $\gamma(G_i) < \gamma(G_{i+1})$ with aid of the following commutative diagram:

$\begin{array}{ccc}
G_{i+1} & \rightarrow & 0 \\
\uparrow & & \\
E_{i+1} & \rightarrow & E_i \\
\uparrow & & \\
0 & \rightarrow & G_i \\
\uparrow & & \\
E_{i+2} & \rightarrow & E_i \\
\uparrow & & \\
0 & \rightarrow & Q \\
\uparrow & & \\
0 & \rightarrow & G_{i+1} \\
\uparrow & & \\
0 & \rightarrow & 0
\end{array}$

We have that $Q$ is torsion free quotient of $E_i$. It follows from $r(Q) > r(G_i)$ that $\gamma(Q) > \gamma(G_i)$. Finally, if we recall the axiom SLOPE.1, we get $\gamma(G_{i+1}) > \gamma(G_i)$. This concludes the proof of existence.
1.2.4 Lemma. Let $E, F$ be sheaves on $X$ and $Gr(E) = (G_n, G_{n-1}, \ldots, G_1)$ a filtration of $E$. Then

a) $\text{Ext}^k(G_i, F) = 0 \quad \forall i \quad \implies \quad \text{Ext}^k(E, F) = 0$;

b) $\text{Ext}^k(F, G_i) = 0 \quad \forall i \quad \implies \quad \text{Ext}^k(F, E) = 0$.

Proof. Let us prove the second statement. The first one can be checked similarly. Consider the exact sequences:

$$0 \rightarrow E_{i+1} \rightarrow E_i \rightarrow G_i \rightarrow 0,$$

where $E_i$ are the terms of the filtration. It follows from the corresponding long cohomology sequences that the triples

$$\text{Ext}^k(F, E_{i+1}) \rightarrow \text{Ext}^k(F, E_i) \rightarrow \text{Ext}^k(F, G_i)$$

are exact $\forall i = n-1, \ldots, 1$. Since $E_n = G_n$, by lemma conditions we have

$$\text{Ext}^k(F, E_n) = \text{Ext}^k(F, E_{n-1}) = \cdots = \text{Ext}^k(F, E_1) = \text{Ext}^k(F, E) = 0.$$

This completes the proof.

1.2.5 Corollary. Let $Gr(E) = (G_n, G_{n-1}, \ldots, G_1)$ be the Harder-Narasimhan filtration of a sheaf $E$ and let $F$ be a semistable sheaf. Then

a) $\gamma(F) < \gamma(G_1) \quad \implies \quad \text{Hom}(E, F) = 0$;

b) $\gamma(F) > \gamma(G_n) \quad \implies \quad \text{Hom}(F, E) = 0$.

The proof follows easily from the lemmas [1.2.4, 1.1.5], and the definition of Harder-Narasimhan filtration.

1.2.6 Lemma. If a sheaf $E$ has a filtration $Gr(E) = (G_n, G_{n-1}, \ldots, G_1)$ then $G_n$ is a subsheaf of $E$ and $Gr(E/G_n) = (G_{n-1}, \ldots, G_1)$.

Proof. Since the last quotient of the filtration consists with its last term, we get $G_n \subset E$. Now this lemma is immediate if we consider the following commutative diagram:

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\uparrow & \uparrow & \uparrow & \\
0 \rightarrow E_{i+1}/G_n & \rightarrow E_i/G_n & \rightarrow G_i & \rightarrow 0 \\
\uparrow & \uparrow & \uparrow & \\
0 \rightarrow E_{i+1} & \rightarrow E_i & \rightarrow G_i & \rightarrow 0 \\
\uparrow & \uparrow & \uparrow & \\
0 \rightarrow G_n & \rightarrow G_n & \rightarrow 0 \\
\uparrow & \uparrow & \\
0 & 0 & 
\end{array}
$$

Proof of the uniqueness of Harder-Narasimhan filtration.

Let

$$Gr(E) = (G_n, G_{n-1}, \ldots, G_1) = (G'_m, G'_{m-1}, \ldots, G_1)$$

be two Harder-Narasimhan filtrations. Suppose $\gamma(G_1) \neq \gamma(G'_1)$. For example, $\gamma(G_1) > \gamma(G'_1)$. Then it follows from the corollary and semistability of $G'_1$ that $\text{Hom}(E, G'_1) =$
0. This contradicts an existence of an epimorphism: \( E \to G'_1 \to 0 \). In the same way, the equality \( \gamma(G_n) = \gamma(G'_m) \) is proved.

Denote by \( E'_i \) the terms of the second filtration. Let us show by induction on \( i \) that \( G_n \) is a subsheaf in \( E'_i \). For \( i = 1 \), there is nothing to prove.

By the induction hypothesis, we have the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \to & E'_{i+1} & \to & E'_i & \to & G'_i & \to & 0 \\
\uparrow & & \uparrow \varphi_i & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & G_n & \to & G'_n & \to & 0 .
\end{array}
\]

By the lemma about a snake, \( \ker \varphi_i \subset E'_{i+1} \). On the other hand, the slopes of semistable sheaves \( G_n, G'_m \) and \( G'_i \) satisfy conditions: \( \gamma(G_n) = \gamma(G'_m) > \gamma(G'_i) \) if \( i < m \). Hence, \( \varphi_i = 0 \) and \( \ker \varphi_i = G_n \).

Thus, \( G_n \subset E'_i \) for \( i < m \). In particular, \( G_n \subset G'_m \).

In the same way, we obtain that \( G'_m \subset G_n \). Therefore, \( G'_m = G_n \).

It follows from the lemma [1.2.6] that

\[ \text{Gr}(E/G_n) = (G_{n-1}, \ldots, G_1) = (G'_{m-1}, \ldots, G'_1). \]

Moreover, these are the Harder-Narasimhan filtrations of \( E/G_n \). Now the uniqueness of the Harder-Narasimhan filtration follows easily by induction on a rank of \( E \).

### 1.3 Examples of Slopes and Kinds of Stability.

The reason of slope axioms are the following well known slopes.

The slope of bundles on a curve: \( \mu(E) = \frac{\deg E}{r(E)} \), where \( \deg E \) is the degree of the bundle’s determinant;

the Mumford-Takemoto slope with respect to an ample divisor \( A \) on \( n \)-dimensional manifold \( X \): \( \mu_A(E) = \frac{c_1(E) \cdot A^{n-1}}{r(E)} \);

the Gieseker slope relative to an ample divisor \( A \): \( \gamma_A(E,n) = \frac{\chi(E(nA))}{r(E)} \).

Let us check that these slopes and the slope \( \mu_H(E) = \frac{c_1(E) \cdot H^{n-1}}{r(E)} \), where \( H \) is nef, really satisfy the slope axioms. By definition, a divisor \( A \) is nef if the number \( D \cdot A^{n-1} \) is greater then or equals 0 for any effective divisor \( D \) on \( X \).

We see that all examples of slopes, except for \( \gamma_A \), have the form \( \gamma = d/r \), where \( d \) is an additive function of \( K_0(X) \) to \( \mathbb{Z} \) and \( r \) is the rank function.

#### 1.3.1 Lemma. The slope function \( \gamma = d/r \) defined before satisfies the axioms SLOPE.1 and SLOPE.3.

**Proof.** For any exact triple of torsion free sheaves

\[ 0 \to F \to E \to G \to 0 \]
we have that $\gamma(E) = \frac{d(F) + d(G)}{r(F) + r(G)}$. Note that the sign of the determinant

$$\begin{vmatrix}
  d(F) & d(G) \\
  r(F) & r(G)
\end{vmatrix}$$

corresponds to the comparison sign between the fractions: $\frac{d(F)}{r(F)} \pm \frac{d(G)}{r(G)}$. Besides,

$$\begin{vmatrix}
  d(F) & d(G) \\
  r(F) & r(G)
\end{vmatrix} = \begin{vmatrix}
  (d(F) + d(G)) & d(G) \\
  (r(F) + r(G)) & r(G)
\end{vmatrix}$$

This implies that $\gamma$ satisfies the first axiom.

For checking the axiom SLOPE.3, note that $|\gamma(G_1) - \gamma(G_2)| \geq 1/r^2$ if the ranks of torsion free sheaves $G_1$ and $G_2$ are less than or equal to $r$.

1.3.2 Corollary. Let $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ be a vector function of $K_0(X)$ such that each $\gamma_i$ has the form $d_i/r$, where $d_i$ is an additive function of $K_0(X)$ to $\mathbb{Z}$ and $r$ is the rank function. If values of $\gamma$ are lexicographic compared then $\gamma$ satisfies the axioms SLOPE.1 and SLOPE.3.

As for the Gieseker slope $\gamma_A$, it is a polynomial of the degree $\dim X$ with rational coefficients. So far as the inequality $\gamma_A(E, n) > \gamma_A(F, n)$ holds true if it holds for sufficiently large $n$, then the comparison $\gamma_A$-slopes is equivalent to lexicographic ordering of the coefficients of the polynomials.

The Hilbert polynomial $\chi(E(nA))$ is an additive function. Hence the Geaseker slope satisfies the axiom SLOPE.1 (see the proof of lemma 1.3.1).

For checking SLOPE.3 note that by Hirzebruch-Riemann-Roch theorem (see. [23]), the Euler characteristic of sheaf on a smooth manifold can be calculated in the following way:

$$\chi(E) = \deg(ch(E) \cdot td(T_X))_n, \quad (1)$$

where

$\deg(...)_n$ means a degree $n$ component in the cohomology ring of $X (H^*(X, \mathbb{Q}))$;

$T_X$ is the tangent bundle of $X$;

$$ch(E) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \frac{1}{24}(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4) + \cdots;$$

$$td(E) = 1 + c_1/2 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}(c_1c_2) - \frac{1}{120}(c_1^4 - 4c_1^2c_2 - 3c_2^2 - c_2c_3 + c_4) + \cdots.$$

($c_i$ are the Chern classes of a sheaf $E$).

This yields that the denominators of the coefficients of the Hilbert polynomial $\chi(E(nA))$ do not depend of $E$. After some proof modifications of the lemma [1.3.1], it is easily shown that the Gieseker slope $\gamma_A(E)$ satisfies the axiom SLOPE.3.

All examples of slopes satisfy the axiom SLOPE.2 in the different degree.
1.3 Examples of Slopes and Kinds of Stability.

1.3.3 Lemma. a) For any pair of torsion free sheaves $F \subset E$ with the same rank on a manifold $X$ and any nef divisor $H$ the following inequality holds $\mu_H(F) \leq \mu_H(E)$. Moreover, in this case the equality $\mu_H(F) = \mu_H(E)$ is possible only if

$$\text{codim supp}(E/F) \geq 1.$$ 

Provided a slope function satisfies this reduction of axiom SLOPE.2, we shall call it the weak slope;

b) for any pair of torsion free sheaves $F \subset E$ with the same rank on a manifold $X$ and any ample divisor $A$ the following inequality holds $\mu_A(F) \leq \mu_A(E)$. Moreover, in this case the equality is possible only if

$$\text{codim supp}(E/F) \geq 2.$$ 

Provided a slope function satisfies this reduction of axiom SLOPE.2, we shall call it the Mumford-Takemoto slope;

c) for any pair of torsion free sheaves $F \subset E$ with the same rank on a manifold $X$ and any ample divisor $A$ the following inequality holds $\gamma_A(F) \leq \gamma_A(E)$. Moreover, in this case the equality of slopes is equivalent to $E = F$.

Provided a slope function satisfies this reduction of axiom SLOPE.2, we shall call it the Gieseker slope;

d) the slope $\mu$ of bundles on a curve is the Gieseker slope.

Proof. The number $c_1(E) \cdot D^{n-1}$, which is determined by a sheaf $E$ on a $n$-dimensional manifold and a divisor $D$, is called the degree of sheaf with respect to $D$ and is denoted by $\text{deg}_D(E)$.

Since the ranks of sheaves $E$ and $F$ coincide, we see that the comparison of their slopes is equivalent to the comparison of the degree $\text{deg}_D$ and the quotient $Q = E/F$ has the zero rank. Hence $c_1(Q) = c_1(E) - c_1(F)$ is effective or zero divisor.

By the definition of nef divisor, $\text{deg}_H(Q) \geq 0$. This proves the first statement of lemma.

If $A$ is ample and $c_1(Q) \neq 0$ then the Nakai-Moishezon criterion (23) implies that $\text{deg}_A(Q) > 0$. This yields the second statement of lemma.

If $A$ is ample then by the Serre theorem (23) for any nonzero sheaf $Q$ and for sufficiently large $n \chi(Q(nA)) > 0$. Therefore the third point of lemma also holds.

Finally, the degree of the effective divisor $c_1(Q)$ on a curve is nonnegative and it is equal to zero only if $c_1(Q) = 0$. This completes the proof.

The more precise conditions of SLOPE.2 allow to formulate the following statement that is stronger the lemma 1.1.6

1.3.4 Lemma. a) Let $E$ and $F$ be a semistable sheaves with respect to Mumford-Takemoto slope $\gamma$, $\gamma(E) = \gamma(F)$, and $F$ stable. Then the cokernal of any nonzero morphism $\varphi : E \longrightarrow F$ has a support $C$ such that $\text{codim} C \geq 2$. In particular, $\varphi$ is an epimorphism if $E$ is locally free.

b) Let $E$ and $F$ be semistable sheaves with respect to Gieseker slope $\gamma$, $\gamma(E) = \gamma(F)$ and $F$ stable. Then any nonzero map of $E$ to $F$ is an epimorphism.
This lemma can be proved in the same way as \[1.1.6\]. Nevertheless, let us recall that $\Ext^1(Q, E) = 0$ if $E$ is locally free and $\text{codim supp}(Q) \geq 2$.

Let us remark that the slope $\gamma = \mu_A(E) = (c_1(E) \cdot A^{n-1})/r(E)$ has the following property:

SLOPE.4. For any torsion free sheaf $E$ and a divisor $D$ the equalities

$$\gamma(E^*) = -\gamma(E), \quad \gamma(E(D)) = \gamma(E) + \gamma(\mathcal{O}(D))$$

are true.

1.3.5 Lemma. Assume that slope function $\gamma$ satisfies the axiom SLOPE.4; then a torsion free sheaf $E$ is (semi)stable if and only if $E(D)$ is (semi)stable; and the $\gamma$-(semi)stability of a reflexive sheaf $E$ ($E^\vee = E$) is equivalent to the $\gamma$-semistability of the dual sheaf $E^*$.

Proof. Consider any subsheaf $F$ in $E(D)$. Hence, $F(-D) \subset E$ and $\gamma(F(-D)) \leq \gamma(E)$ if $E$ is semistable. Therefore,

$$\gamma(F) = \gamma(F(-D)) + \gamma(\mathcal{O}(D)) \leq \gamma(E) + \gamma(\mathcal{O}(D)) = \gamma(E(D)).$$

That is, the sheaf $E(D)$ is semistable.

Now suppose that $E$ is reflexive. It is sufficient to prove that the semistability of $E^*$ implies the semistability of $E$. Denote by $G$ any torsion free quotient of $E$. Hence, $G^* \subset E^*$. Therefore, $\gamma(G^*) \leq \gamma(E^*)$. Using SLOPE.4, we obtain that $\gamma(G) \geq \gamma(E)$. This implies the semistability of $E$.

It is useful to remark that besides the canonical Harder-Narasimhan filtration, each semistable sheaf has Jordan-Holder filtration:

1.3.6 Proposition. Any $\gamma$-semistable sheaf $E$ has the filtration

$$Gr(E) = (G_n, G_{n-1}, \ldots, G_1)$$

with stable quotients and the equalities: $\gamma(G_i) = \gamma(E)$.

We do not prove this proposition. But we constructs more exact filtration.

1.3.7 Proposition. Any sheaf $E$ semistable with respect to Gieseker slope (see \[1.2.3\]) has the filtration with isotypic quotients:

$$Gr(E) = (G_n, G_{n-1}, \ldots, G_1),$$

where each of $G_i$ has the filtration with isomorphic quotients:

$$Gr(G_i) = (Q_i, Q_i, \ldots, Q_i) \quad (\gamma(Q_i) = \gamma(G_i) = \gamma(E)).$$

Moreover, this filtration can be constructed in the such way that

$$\Hom(E_i, G_{i-1}) = \Hom(G_i, G_{i-1}) = \Hom(G_{i-1}, G_i) = 0,$$
where $E_i$ are the filtration terms.

**Proof.** For a stable sheaf this filtration is trivial. If a sheaf $E = E_1$ is semistable then it has destabilizing torsion free quotient $Q$ ($\gamma(Q) = \gamma(E_1)$). From all such quotients let us choose a sheaf $Q_1$ with the minimal rank. By choice, it is stable. Let $E_1^1$ be the corresponding subsheaf. It follows from the exact sequence:

$$0 \to E_1^1 \to E_1 \to Q_1 \to 0,$$

and the equality $\gamma(Q_1) = \gamma(E_1)$ that $E_1^1$ is semistable and $\gamma(E_1^1) = \gamma(E_1)$ (see SLOPE.1, 1.1.8). If Hom$(E_1^1, Q_1) = 0$ then $E_2 = E_1^1$ is the second term of filtration and $G_1 = Q_1$ is the first quotient of it.

Conversely, there exists an epimorphism; $E_1^1 \to Q_1 \to 0$ (1.3.4). Denote by $E_1^2$ the kernel of this epimorphism.

Continuing this procedure, we obtain the semistable subsheaf $E_1^k$ such that Hom$(E_1^k, Q_1) = 0$.

By definition, put $E_2 = E_1^k$ and $G_1 = E_1/E_2$. From this construction it follows that $G_1$ and $E_2$ are semistable, $\gamma(E_2) = \gamma(G_1) = \gamma(E_1)$, $Gr(G_1) = (Q_1, Q_1, \ldots, Q_1)$ and Hom$(E_2, Q_1) = 0$. Now using the lemma 1.2.4 we obtain, Hom$(E_2, G_1) = 0$.

By the induction hypothesis, we can assume that $E_2$ has the filtration with isotypic quotients: $Gr(E_2) = (G_n, G_{n-1}, \ldots, G_2)$. Let us show that the filtration

$$Gr(E) = (G_n, G_{n-1}, \ldots, G_2, G_1)$$

satisfies the proposition conditions.

It remains to check that Hom$(G_2, G_1) = Hom(G_1, G_2) = 0$. Since Hom$(E_2, G_1) = 0$ and there exists an epimorphism $E_2 \to G_2 \to 0$, the equality Hom$(G_2, G_1) = 0$ is trivial.

Suppose there exists a nonzero morphism $G_1 \to G_2$. Let us recall that $Gr(G_i) = (Q_i, Q_i, \ldots, Q_i)$ Therefore by 1.2.4 Hom$(Q_1, G_2) \neq 0$. Hence, there is a nonzero map $\varphi : Q_1 \to Q_2$. It follows from (1.1.6 and 1.3.4) that $\varphi$ is an isomorphism. It implies that Hom$(Q_2, G_1) \neq 0$. But $Q_2$ is a quotient of $G_2$, and $Q_1$ is a subsheaf of $G_1$. Thus, Hom$(G_2, G_1) \neq 0$. This contradiction concludes the proof.

2 Rigid Sheaves.

2.1 Preliminary information.

We shall study sheaves on a smooth projective surface $S$ over $\mathbb{C}$ such that $h^1(O_S) = 0$ and the anticanonical class $H = -K_S$ has not base components. Note that from the last condition it follows that $H$ is nef. In reality, suppose the cup product $H \cdot C$ is negative for some curve $C$; then $H$ and $C$ have a common base component.

It is known, if $S$ is a smooth projective surface over an algebraically closed field with nef anticanonical class then we have one of the following cases:

1. $K_S = 0$;

2. $S \cong \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(2))$;
3. $S \cong \mathbb{P}(F)$, where $F$ is a rank 2 vector bundle on an elliptic curve which is an extension of degree zero line bundles;

4. $S \cong \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$;

5. $S$ is obtained from $\mathbb{P}^2$ by successively blowing up at most nine points.

I want to note that this class of surfaces contains the surfaces such that they are obtained from the singular Fano surfaces (taken from the notes of Batyrev in the paper [2]) by blowing up singular points.

Let us recall the general facts, which will be needed in this text.

2.1.1 Theorem. (The Riemann-Roch formula for surfaces.) The Euler characteristic of two coherent sheaves $E$ and $F$ on a smooth projective surface $X$ is calculated by the following formula:

$$\chi(E, F) = r(E)r(F)(\chi(O_X) + \frac{1}{2}(\mu_H(F) - \mu_H(E))) + q(F) + q(E) - \frac{(c_1(E) \cdot c_1(F))}{r(E)r(F)},$$

where $\mu_H(E) = \frac{1}{r(E)}(-K_X \cdot c_1(E))$, $q(E) = \frac{c_2(E) - 2c_2(E)}{2r(E)}$.

This theorem can be proved by direct calculation. (see (I)).

Note that in our case we have $\chi(O_S) = 1$.

2.1.2 Corollary. Let $E, F$ be a sheaves on a smooth projective regular ($h^1(O_S) = 0$) surface with $\chi(E, E) = \chi(F, F) = 1$; then

$$q(E) = \frac{1}{2}\left(\frac{c_2(E)}{r^2(E)} + 1\right) - 1$$

and

$$\chi(E, F) = \frac{r(E)r(F)}{2}(\mu_H(F) - \mu_H(E) + \frac{1}{r^2(E)} + \frac{1}{r^2(F)} + \frac{(c_1(F) - c_1(E))^2}{r(E)}).$$

2.1.3 Theorem. (The Serre duality.) For any coherent sheaves $E$ and $F$ on a smooth projective surface $X$ the following equality

$$\text{Ext}^k(E, F)^* \cong \text{Ext}^{2-k}(F, E(K_X))$$

holds.

The proof is contained in [23].

2.1.4 Lemma. (Mukai.) Let $X$ be a smooth projective surface.

1. For any torsion free sheaf $E$ on $X$ we have

$$h^1(E, E) \geq h^1(E^{**}, E^{**}) + 2\text{length}(E^{**}/E).$$
2.2 Exceptional sheaves.

2.a) Suppose the sheaves $G_1$ and $G_2$ on $X$ from the exact sequence

$$0 \to G_2 \to E \to G_1 \to 0,$$

satisfy the conditions: $\text{Hom}(G_2, G_1) = \text{Ext}^2(G_1, G_2) = 0$; then

$$h^1(E, E) \geq h^1(G_1, G_1) + h^1(G_2, G_2).$$

b) If besides, $h^1(E, E) = 0$; then

$$h^0(E, E) = h^0(G_1, G_1) + h^0(G_2, G_2) + \chi(G_1, G_2)$$

$$h^2(E, E) = h^2(G_1, G_1) + h^2(G_2, G_2) + \chi(G_2, G_1).$$

This lemma follows from the spectral sequence associated with the exact triple. Besides, the proof is contained in [15] and [12].

2.2 Exceptional sheaves.

**Definition.** A sheaf $E$ on a manifold $X$ is called *rigid* whenever

$$\text{Ext}^1(E, E) = 0.$$

The most trivial rigid sheaves are exceptional.

**Definition.** A sheaf $E$ on a manifold is called *exceptional*, provided $\text{Ext}^0(E, E) = \mathbb{C}$ and $\text{Ext}^i(E, E) = 0 \quad \forall i > 0$.

Using methods of S. Mukai ([15]), A. Gorodentsev ([1]), D. Orlov ([12]) and S. Zube ([9]) we get the starting information about a structure of rigid and exceptional sheaves.

2.2.1 **Lemma.** A rigid sheaf without torsion on a smooth projective surface is locally free.

This lemma follows from Mukai lemma (2.1.4).

Let us recall that $S$ is a smooth projective surface over $\mathbb{C}$ with the anticanonical class $H = -K_S$ without base components

Let $G$ be a sheaf on $S$. Denote by $TG$ its torsion subsheaf and by $T^0G$ the subsheaf in $TG$ such that $T^1G = TG/T^0G$ has not a torsion with 0-dimensional support.

2.2.2 **Lemma.** (Gorodentsev-Orlov.) Any sheaves $G$ and $F$ on the surface $S$ satisfy the following conditions:

a) the inequality

$$h^0(F, G) \geq h^2(G, F)$$

holds whenever the support of $T^0G$ has not common points with the base set of anticanonical linear system $|H|$;

b) the inequality

$$h^0(G, G) > h^2(G, G)$$
holds provided there exists a curve \( D \in |H| \) such that \( D \cap \text{supp}G \neq \emptyset \). In particular, this inequality is satisfied whenever \( r(G) > 0 \).

**Proof.** a) For some section \( \varphi \in H^0(\mathcal{O}_S(H)) \) let us consider the exact sequence:

\[
0 \rightarrow \mathcal{O}_S \xrightarrow{\varphi} \mathcal{O}_S(H) \rightarrow \mathcal{O}_S(H)|_D \rightarrow 0.
\]

We apply the functor \( \mathcal{H}om(\cdot, G) \) to it to obtain

\[
0 \rightarrow \mathcal{H}om(\mathcal{O}_S(H)|_D, G) \rightarrow G(K_S) \rightarrow G \rightarrow \mathcal{E}xt^1(\mathcal{O}_S(H)|_D, G) \rightarrow 0
\]

\((\mathcal{H}om(\mathcal{O}_S(H), G) \cong \mathcal{O}_S(H)^* \otimes G = G(-H) \) and \( H = -K_S \).

Since the quotient \( G' = G/TG \) has not torsion, we get \( \mathcal{H}om(\mathcal{O}_S(H)|_D, G') = 0 \). Hence it follows from the exact sequence

\[
0 \rightarrow TG \rightarrow G \rightarrow G' \rightarrow 0
\]

that

\[
\mathcal{H}om(\mathcal{O}_S(H)|_D, TG) = \mathcal{H}om(\mathcal{O}_S(H)|_D, G).
\]

The support of \( T^1G \) is a curve. By the lemma conditions, the support of \( T^0G \) has the empty intersection with the base set of \( |H| \). On the other hand, \( |H| \) has not base component. Thus there exists a curve \( D \in |H|, \) such that

\[
\dim(D \cap \text{supp}T^1G) < 1, \quad D \cap \text{supp}T^0G = \emptyset.
\]

Therefore,

\[
\mathcal{H}om(\mathcal{O}_S(H)|_D, T^1G) = \mathcal{H}om(\mathcal{O}_S(H)|_D, T^0G) = 0.
\]

From the exact sequence

\[
0 \rightarrow T^0G \rightarrow TG \rightarrow T^1G \rightarrow 0
\]

and the previous equalities it follows that

\[
\mathcal{H}om(\mathcal{O}_S(H)|_D, G) = \mathcal{H}om(\mathcal{O}_S(H)|_D, TG) = 0,
\]

that is we have the exact triple:

\[
0 \rightarrow G(K_S) \rightarrow G \rightarrow \mathcal{E}xt^1(\mathcal{O}_S(H)|_D, G) \rightarrow 0.
\]

It implies that \( \text{Hom}(F, G(K_S)) \subset \text{Hom}(F, G) \) and \( h^0(F, G(K_S)) \leq h^0(F, G) \).

Now the first lemma statement follows from the Serre duality.

b) Note that \( \mathcal{E}xt^1(\mathcal{O}(H)|_D, G) \cong G \otimes \mathcal{O}_D \). Hence the previous exact sequence has the form:

\[
0 \rightarrow G(K_S) \rightarrow G \rightarrow G \otimes \mathcal{O}_D \rightarrow 0.
\]

By assumption, there exists a curve \( D \) such that \( G \otimes \mathcal{O}_D \neq 0 \). Therefore, the map \( \text{Hom}(G, G) \rightarrow \text{Hom}(G, G \otimes \mathcal{O}_D) \) is nonzero. Thus we obtain the strong inequality:

\[
h^0(G, G) > h^2(G, G).
\]
2.2 Exceptional sheaves.

2.2.3 Corollary. Let $G$ be a rigid sheaf on $S$; then its torsion subsheaf $TG$ and torsion free quotient $G' = G/TG$ are rigid sheaves. Moreover, $T^0G = 0$.

Proof. Obviously, $\text{Hom}(TG, G') = 0$ and $T^0G' = 0$. By the previous lemma, $h^0(TG, G') \geq h^2(G', TG)$. We apply the Mukai lemma (2.1.4) to the exact triple

$$0 \to TG \to G \to G' \to 0$$

to obtain

$$h^1(G, G) \geq h^1(TG, TG) + h^1(G', G'),$$

i.e. $TG$ and $G'$ are rigid as well.

Suppose $T^0G \neq 0$. Applying the Mukai lemma to the exact sequence:

$$0 \to T^0G \to TG \to T^1G \to 0,$$

we see that $T^0G$ is rigid. This contradicts the following equality:

$$h^1(T^0G, T^0G) = 2\text{length}(T^0G) \neq 0.$$

2.2.4 Lemma. Suppose $E$ is an exceptional torsion sheaf on $S$; then $c_1^2(E) = -1$. Furthermore,

- either $E = \mathcal{O}_e(d)$, where $e$ is some irreducible rational curve with $e^2 = -1$
- or one of the components of the support of $E$ has zero cup product with $K_S$.

Proof. Using the Riemann-Roch formula for surfaces (2.1.1), we get

$$\chi(E, E) = r^2 + (r - 1)c_1^2 - 2rc_2.$$

On the other hand, since $E$ is exceptional, we have $\chi(E, E) = 1$. Besides, from the lemma conditions it follows that $r(E) = 0$. Therefore, $c_1^2(E) = -1$.

Denote by $C$ the support of the sheaf $E$. Since $E$ is rigid, we obtain $T^0E = 0$ (2.2.3). Suppose $C$ is irreducible; then $E$ is a vector bundle on it. Let us denote the rank of this bundle by $r_C$. Hence, $c_1(E) = r_CC$ and $c_1(E)^2 = r_C^2C^2 = -1$. Thus, $r_C = 1$.

By adjunction $(2g - 2 = C \cdot (C + K_S))$,

$$2g = C \cdot K_S + 1 \leq 1$$

($-K_S$ is nef). That is $C$ is a rational curve $e$ with $e^2 = -1$, and $E = \mathcal{O}_e(d)$ (as a line bundle on a projective line).

Now suppose that $C$ is reducible and $C_0$ is one of its irreducible component. Consider the sequence of restriction to $C_0$:

$$0 \to E_1 \to E \to E_0 \to 0,$$

where $\text{supp}E_0 = C_0$ and $\text{supp}E_1 = C_1$. Since $T^0E = 0$, we get $T^0E_0 = T^0E_1 = 0$. Hence, $\text{Hom}(E_1, E_0) = \text{Ext}^2(E_0, E_1) = 0$.

Now the application of the Mukai lemma (2.1.4) yields

$$h^0(E, E) = h^0(E_0, E_0) + h^0(E_1, E_1) + \chi(E_0, E_1),$$
\[ h^2(E, E) = h^2(E_0, E_0) + h^2(E_1, E_1) + \chi(E_1, E_0). \]

Since \( E \) is exceptional, we get
\[ 1 = h^0(E_0, E_0) - h^2(E_0, E_0) + \chi(E_0, E_1) + h^0(E_1, E_1) - h^2(E_1, E_1) - \chi(E_1, E_0). \]

By the Riemann-Roch theorem for exceptional sheaves (2.1.2),
\[ \chi(E_0, E_1) - \chi(E_1, E_0) = H \cdot \left( r(E_0)c_1(E_1) - r(E_1)c_1(E_0) \right) = 0 \]
(\( r(E_0) = r(E_1) = 0 \)). Finally, we obtain
\[ 1 = h^0(E_0, E_0) - h^2(E_0, E_0) + h^0(E_1, E_1) - h^2(E_1, E_1). \]

To conclude the proof, it remains to note that the last equality is possible only if either \( C_0 \) or \( C_1 \) has the zero cup product with \( H \).

2.2.5 Lemma. Suppose \( E \) is an exceptional sheaf on \( S \); then the support of its torsion has zero cup product with \( K_S \).

Proof. It follows from corollary 2.2.3 that \( TE \) and \( E' = E/TE \) are rigid as well. The application Mukai lemma to the exact sequence
\[ 0 \to TE \to E \to E' \to 0 \]
yields
\[ h^0(E, E) = h^0(TE, TE) + h^0(E', E') + \chi(E', TE), \]
\[ h^2(E, E) = h^2(TE, TE) + h^2(E', E') + \chi(TE, E'). \]

Using the equality \( h^0(E, E) - h^2(E, E) = 1 \), we obtain
\[ 1 = h^0(TE, TE) - h^2(TE, TE) + h^0(E', E') - h^2(E', E') + \chi(E', TE) - \chi(TE, E'). \]

By lemma 2.2.2, \( h^0(E', E') - h^2(E', E') \geq 1 \) and \( h^0(TE, TE) - h^2(TE, TE) \geq 0 \). Thus we have
\[ 0 \geq \chi(E', TE) - \chi(TE, E') = H \cdot (r(E')c_1(TE) - r(TE)c_1(E')) = r(E')H \cdot c_1(TE). \]

On the other hand, since \( H \) is nef, we get \( H \cdot c_1(TE) \geq 0 \). This completes the proof.

Combining 2.2.1, 2.2.4 and 2.2.5, we can formulate the following proposition:

2.2.6 Proposition. Suppose \( E \) is an exceptional sheaf on \( S \); then we have one of the following cases:
1) \( E \) is locally free;
2) \( E \) has a torsion subsheaf such that \( (\text{supp}TE) \cdot K_S = 0 \);
3) \( E \cong \mathcal{O}_e(d) \) for some rational curve \( e \) with \( e^2 = -1 \);
4) \( r(E) = 0 \) and the support of \( E \) contains an irreducible component \( C_0 \) such that \( C_0 \cdot K_S = 0 \).
2.3 Exceptional Collections.

2.2.7 Corollary. (Orlov.) If $-K_S$ is ample ($S$ is the Del Pezzo surface) then exceptional sheaf on $S$ either is locally free or has the form $\mathcal{O}_e(d)$ for some rational curve $e$ with $e^2 = -1$.

Now let us prove the stability of exceptional bundles on $S$ with respect to the anticanonical class $H = -K_S$.

2.2.8 Lemma. (S. Zube) Let $D$ be a smooth elliptic curve from $|H|$ and let $E$ be an exceptional bundle on $S$. Then the restriction of $E$ to $D$ ($E' = E|_D$) is a simple bundle, i.e. $\text{Ext}^0(E', E') = \mathbb{C}$.

Proof. Consider the exact sequence

$$0 \rightarrow E^* \otimes E(K_S) \rightarrow E^* \otimes E \rightarrow (E^* \otimes E)|_D \rightarrow 0.$$  

By Serre duality,

$$\text{Ext}^k(E, E(K_S))^* \cong \text{Ext}^{2-k}(E, E).$$  

Since $E$ is exceptional, we obtain

$$\text{Ext}^0(E, E) = \mathbb{C}, \quad \text{Ext}^1(E, E) = \text{Ext}^2(E, E) = 0.$$  

Therefore, the cohomology table associated with the exact sequence has the form:

| $k$ | $\text{Ext}^k(E, E(K_S))$ | $\text{Ext}^k(E, E)$ | $\text{Ext}^k(E', E')$ |
|-----|-----------------|-----------------|-----------------|
| 0   | 0               | $\mathbb{C}$    | ?               |
| 1   | 0               | 0               | ?               |
| 2   | $\mathbb{C}$   | 0               | ?               |

It implies that $\text{Ext}^0(E', E') = \mathbb{C}$.

2.2.9 Corollary. Any exceptional bundle $E$ on $S$ is stable with respect to the slope $\mu_H = (H \cdot c_1(E))/r(E)$, where $H = -K_S$.

Proof. By the Zube lemma the restriction of $E$ to an elliptic curve $D \in |-K_S|$ is simple. It is known that simple bundles on an elliptic curve are stable with respect to the slope $\mu(E) = \frac{\deg(E)}{r(E)}$. Besides, $\mu_H(E) = \mu(E')$, where $E' = E|_D$. Now suppose $F$ is a subsheaf of $E$ such that $r(F) < r(E)$ and $\mu_H(F) > \mu_H(E)$. Without loss of generality we can assume that $F' = F|_D$ is locally free. Thus, $\mu(F') > \mu(E')$. This contradicts stability of $E'$. The corollary is proved.

2.3 Exceptional Collections.

The main results about rigid and superrigid sheaves are formulated in terms of exceptional collections. The aim of this section is to study these collections on the surface $S$.

Definition. An ordered collection $(E_1, E_2, \ldots, E_n)$ of exceptional sheaves is called exceptional whenever

$$\text{Ext}^k(E_i, E_j) = 0 \quad \text{for} \ i > j \quad \text{and} \quad \forall k = 0, 1, 2.$$
An exceptional collection \((E, F)\) is an *exceptional pair*.

By definition, an ordered collection is exceptional if and only if each its pair is exceptional. Thus we shall study exceptional pairs on \(S\).

Suppose \((E, F)\) is an exceptional pair on Del Pezzo surface. It is known that then we have one of the following cases:

1. a pair \((E, F)\) has the type *hom* (or in other words \((E, F)\) is a *hom*-pair), that is
   \[
   \Ext^i(E, F) = 0 \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad \Hom(E, F) \neq 0;
   \]
2. a pair \((E, F)\) has the type *ext* (or in other words \((E, F)\) is a *ext*-pair), that is
   \[
   \Ext^i(E, F) = 0 \quad \text{for} \quad i = 0, 2 \quad \text{and} \quad \Ext^1(E, F) \neq 0;
   \]
3. a pair \((E, F)\) has the type *zero* (or in other words \((E, F)\) is a zero-pair), that is
   \[
   \Ext^i(E, F) = 0 \quad \text{for} \quad i = 0, 1, 2.
   \]

There appear exceptional pairs of a new type in our surfaces.

**Definition.** An exceptional pair \((E, F)\) is called *singular* if
\[
\Ext^i(E, F) \neq 0 \quad \text{for} \quad i = 0, 1 \quad \text{and} \quad \Ext^2(E, F) = 0.
\]

**2.3.1 Proposition.** Let \((E, F)\) be an exceptional pair of bundles on the surface \(S\); then we have one of the following cases:

a) \((E, F)\) is the *hom*-pair \(\iff\) \(\mu_H(E) < \mu_H(F)\);

b) \((E, F)\) is the *ext*-pair \(\iff\) \(\mu_H(E) > \mu_H(F)\);

c) \((E, F)\) is singular or the zero-pair \(\iff\) \(\mu_H(E) = \mu_H(F)\).

**Proof.** Consider the restriction sequence to a smooth elliptic curve:

\[
0 \to E^* \otimes F(K_S) \to E^* \otimes F \to (E^* \otimes F)|_D \to 0.
\]

Denote \(E|_D\) and \(F|_D\) by \(E'\) and \(F'\). Combining Serre duality and the definition of exceptional pairs, we get \(\Ext^i(E, F(K_S)) = \Ext^{2-i}(F, E) = 0\). Hence the cohomology sequence associated with this exact sequence has the form:

\[
\begin{array}{cccc}
 k & \Ext^k(E, F(K_S)) & \to & \Ext^k(E, F) \\
 0 & 0 & * & * \\
 1 & 0 & * & * \\
 2 & 0 & * & *
\end{array}
\]

That is,
\[
\Ext^i(E, F) \cong \Ext^i(E', F') \quad \forall i.
\]

Since \(E'\) and \(F'\) are stable bundles on the elliptic curve (see the proof of lemma 2.2.8), we obtain that only one of the spaces \(\Ext^0(E', F')\) and \(\Ext^1(E', F')\) is nonzero whenever
\[
\mu_H(E') \neq \mu_H(F') \quad \text{and} \quad (\mu(E') = \frac{\deg(E')}{r(E')} = \mu_H(E)).
\]
Moreover, $\text{Ext}^0(E', F') \neq 0$ iff $\chi(E', F') > 0$ and $\text{Ext}^1(E', F') \neq 0$ iff $\chi(E', F') < 0$. In this case $\chi(E', F')$ is the Euler characteristic of two sheaves on an elliptic curve ([11]):

$$\chi(E', F') = r(E')r(F')(\mu(F') - \mu(E')).$$

Finally, in any case we have $\text{Ext}^2(E', F') = 0$. This completes the proof.

### Lemma

Let $(E, F)$ be an exceptional pair of bundles on $S$ with $\mu_H(E) = \mu_H(F)$. Let $C$ be $c_1(F) - c_1(E)$. Then:

1. $r(E) = r(F)$.
2. $C^2 = -2$ and $K_S \cdot C = 0$.
3. Suppose $(E, F)$ is a singular pair; then
   
   (a) $C$ is a connected curve;
   
   (b) $\text{Ext}^0(E, F) = \text{Ext}^1(E, F) = C$;
   
   (c) there exists an exact sequence

   $$0 \rightarrow E \rightarrow F \rightarrow Q \rightarrow 0,$$

   where $Q$ is a torsion sheaf with $c_1(Q) = C$.

**Proof.** By the definition of an exceptional pair, $\chi(F, E) = 0$. Substituting the discrete invariants of $E$ and $F$ in the Riemann-Roch formula for exceptional sheaves ([2.1.2]), we get

$$0 = \frac{1}{r^2(E)} + \frac{1}{r^2(F)} + \left(\frac{c_1(F)}{r(F)} - \frac{c_1(E)}{r(E)}\right)^2.$$

From lemma 2.2.8 it follows that the restriction of an exceptional bundle to the elliptic curve $D \in |H|$ is a simple bundle. Moreover,

$$\mu_H(E) = \mu(E|_D).$$

If $L$ is a simple bundle on an elliptic curve; then $r(L)$ and $\deg(L)$ are coprime ([1]). Hence it follows from the equality $\mu(E|_D) = \mu(F|_D)$ that $r(E) = r(F) = r$.

Whence we obtain

$$0 = \frac{2}{r^2} + \frac{1}{r^2} \left(c_1(F) - c_1(E)\right)^2.$$

This means that

$$C^2 = (c_1(F) - c_1(E))^2 = -2.$$

On the other hand,

$$\mu_H(E) = \frac{c_1(E) \cdot H}{r(E)} = \mu_H(F) = \frac{c_1(F) \cdot H}{r(F)}.$$

Therefore, $C \cdot H = 0$, i.e., $C \cdot K_S = 0$. This concludes the proof of the first and the second lemma statements.
3. Let \((E, F)\) be a singular pair, i.e., there exists a nonzero map \(\varphi : E \longrightarrow F\). Since exceptional bundles on \(S\) are \(\mu_H\)-stable, it follows from lemma \([1.1.6]\) that \(\varphi\) is an injection. Moreover, the cokernel of \(\varphi\) has the zero rank. By definition, put \(Q = \text{coker}\varphi\). Since the first Chern class is an additive function, we get \(c_1(Q) = c_1(F) - c_1(E) = C\).

Consider the restriction sequence:

\[
0 \longrightarrow E^* \otimes F(K_S) \longrightarrow E^* \otimes F \longrightarrow (E^* \otimes F)|_D \longrightarrow 0.
\]

We have the following isomorphisms:

\[
\text{Hom}(E, F) \cong \text{Hom}(E', F'); \quad \text{Ext}^1(E, F) \cong \text{Ext}^1(E', F'); \quad \text{Ext}^2(E, F) = 0,
\]

where \(E' = E|_D\) and \(F' = F|_D\).

By assumption, \(\text{Hom}(E, F) \neq 0\). Hence there exists a nonzero map \(\varphi' : E' \longrightarrow F'\). Since \(E'\) and \(F'\) are stable bundles on a curve and \(\mu(E') = \mu(F')\), we see that \(\varphi'\) is an isomorphism. Further, stable bundles are simple \([1.1.7]\), and the canonical class of an elliptic curve is trivial. It follows from Serre duality that \(\text{Ext}^1(E', F') = \mathbb{C}\). Thus we have \(\text{Ext}^1(E, F) \cong \text{Ext}^0(E, F) = \mathbb{C}\).

Now we show that \(Q\) is simple. Let us write cohomology tables associated with the exact sequence

\[
0 \longrightarrow E \longrightarrow F \longrightarrow Q \longrightarrow 0.
\]

| \(k\) | \(\text{Ext}^k(E, E)\) | \(\text{Ext}^k(E, F)\) | \(\text{Ext}^k(E, Q)\) | \(\text{Ext}^k(F, E)\) | \(\text{Ext}^k(F, F)\) | \(\text{Ext}^k(F, Q)\) | \(\text{Ext}^k(Q, Q)\) | \(\text{Ext}^k(Q, F)\) | \(\text{Ext}^k(Q, E)\) |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0   | \(\mathbb{C}\)  | \(\mathbb{C}\)  | \(?\)           | \(0\)           | \(?\)           | \(?\)           | \(?\)           | \(0\)           | \(?\)           |
| 1   | 0               | \(\mathbb{C}\)  | \(?\)           | \(0\)           | \(?\)           | \(?\)           | \(0\)           | \(?\)           | \(0\)           |
| 2   | 0               | 0               | \(?\)           | \(0\)           | \(?\)           | \(?\)           | \(0\)           | \(?\)           | \(0\)           |

From the last table it follows that the quotient \(Q\) is simple. Hence \(C = \text{supp}Q\) is connected. In reality, the endomorphism group of \(Q\) contains projectors unless \(\text{supp}Q\) is connected.

The following lemma completes the classification of exceptional pairs of bundles on \(S\).

2.3.3 LEMMA. Suppose \((E, F)\) is an exceptional pair of bundles on \(S\) with \(\mu_H(E) = \mu_H(F)\) and \(C = c_1(F) - c_1(E)\) is a connected -2-curve \((C^2 = -2)\); then \((E, F)\) is singular.

To prove this statement, we need the following remark.

2.3.4 REMARK. Let \(C\) be an irreducible smooth curve on the surface \(S\) with \(C^2 = -2\) and \(C \cdot H = 0\). Then \(\text{supp}C\) is rational and there are \(x, y \in \mathbb{N}\) and \(d \in \mathbb{Z}\) such that \(E' = E|_C = xO_C(d) \oplus yO_C(d + 1)\)
2.3 Exceptional Collections.

**Proof.** Let us show that
\[ \text{Ext}^2(E, E(-C)) = 0. \]

By Serre duality, \( \text{Ext}^2(E, E(-C))^\ast \cong \text{Hom}(E, E(K_S + C)). \)

Suppose, \( K_S^2 > 0; \) then \( \mu_H(E) > \mu_H(E(K_S + C)). \) Since exceptional bundles on \( S \) are \( \mu_H \)-stable, it follows from 1.1.5 that \( \text{Hom}(E, E(K_S + C)) = 0. \)

Suppose, \( K_S^2 = 0; \) then \( \mu_H(E) = \mu_H(E(K_S + C)). \) This implies that any nonzero map \( \varphi : E \rightarrow E(K_S + C) \) is injective. Denote by \( Q \) the cokernel of \( \varphi. \) It is clear that the support of \( Q \) is a curve and \( c_1(Q) = r \cdot (K_S + C), \) where \( r = r(E). \) Since the linear system \( |C| \) is one-dimensional and \( |-K_S| \) has not base components, the divisor \( r \cdot (K_S + C) \) cannot be effective.

Thus, \( \text{Ext}^2(E, E(-C)) = 0. \) Consider the restriction sequence to \( C: \)
\[ 0 \rightarrow E^\ast \otimes E(-C) \rightarrow E^\ast \otimes E \rightarrow (E^\ast \otimes E)|_C \rightarrow 0. \]

We have,
\[ \text{Ext}^1(E, E) \rightarrow \text{Ext}^1(E', E') \rightarrow 0. \]

By the definition of exceptional sheaves, \( \text{Ext}^1(E, E) = 0. \) This yields that \( E' \) is a rigid bundle on the curve \( C. \) Finally, by adjunction,
\[ 2g - 2 = C \cdot (C + K_S). \]

Therefore the curve \( C \) is rational and \( E' \) is a direct sum of line bundles. Now the remark follows from Bott formula (4).

**Proof of lemma 2.3.3** It is known that any irreducible component of a -2-curve \( C \) on \( S \) is a smooth rational -2-curve provided \( C \cdot K_S = 0. \) It follows from the lemma condition, the definition of a singular pair and 2.3.1 that it is sufficient to show that there exists a nonzero map \( E \rightarrow F. \) Hence, without loss of generality we can assume that \( C \) is an irreducible smooth rational curve.

Let us denote \( r = r(F), \) \( D = c_1(F) \) and
\[ \Delta G = c_1^2(G) - 2c_2(G), \tag{2} \]
where \( G \) is any coherent sheaf on \( S. \)

Summing 2.3.2 and the lemma conditions, we get
\[ r(E) = r, \quad c_1(E) = D - C. \]

Since the sheaves \( E \) and \( F \) are exceptional, it follows from 2.1.2 that
\[ \Delta F = \frac{D^2 + 1}{r} - r^2 \quad \text{and} \quad \Delta E = \frac{(D - C)^2 + 1}{r} - r^2, \]
that is
\[ \Delta E = \Delta F - \frac{2}{r}(D \cdot C + 1) \tag{3} \]

Substituting (2) in this equality, we obtain
\[ \frac{D \cdot C + 1}{r} = c_2(E) - c_2(F) + D \cdot (D - C) + 1. \]
Since the right part of the last equality is integer, we get
\[ D \cdot C \equiv -1 \mod{r}. \]  
(4)

Remark 2.3.4 implies that \( F' = F|_C = x\mathcal{O}_C(d) \oplus y\mathcal{O}_C(d + 1) \), where \( x + y = r \) and \( xd + y(d + 1) = D \cdot C \). Hence by (4),
\[ F' = \mathcal{O}_C(d) \oplus (r - 1)\mathcal{O}_C(d + 1). \]

Besides,
\[ d + 1 = \frac{D \cdot C + 1}{r}. \]
(5)

We see that there exists an exact sequence:
\[ 0 \rightarrow G \rightarrow F \rightarrow \mathcal{O}_C(d) \rightarrow 0. \]
(6)

Using this triple, let us calculate the discrete invariants of \( G \). It is easily shown that
\[ r(G) = r(F) = r(E), \quad c_1(G) = c_1(E), \quad \Delta G = \Delta F - \Delta \mathcal{O}_C(d). \]
(7)

For calculating \( \Delta \mathcal{O}_C(d) \) let us use Riemann-Roch formula (2.1.1):
\[ \chi(\mathcal{O}, \mathcal{O}_C(d)) = r(\mathcal{O}_C(d)) + (H \cdot c_1(\mathcal{O}_C(d)))/2 + (\Delta \mathcal{O}_C(d))/2. \]

Since \( r(\mathcal{O}_C(d)) = 0 \) and \( c_1(\mathcal{O}_C(d)) \cdot H = C \cdot H = 0 \), we get
\[ \chi(\mathcal{O}, \mathcal{O}_C(d)) = \frac{1}{2} \Delta \mathcal{O}_C(d). \]

On the other hand,
\[ \chi(\mathcal{O}, \mathcal{O}_C(d)) = \chi(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(d)) = d + 1, \]
whereby
\[ \Delta \mathcal{O}_C(d) = 2(d + 1). \]

Combining (3), (7) and the last equality, we obtain
\[ \Delta G = \Delta F - \frac{2}{r}(D \cdot C + 1). \]

Therefore, \( \Delta G = \Delta E \) (see (3)).

We see that the sheaf \( G \) has the same discrete invariants as the exceptional bundle \( E \).
Hence \( \chi(E, G) = \chi(E, E) = 1 \), i.e.,
\[ h^0(E, G) + h^2(E, G) \geq 1. \]

Suppose, \( h^2(E, G) \neq 0 \); then by Serre duality, \( h^0(G, E(K_S)) \neq 0 \). That is there exists a nonzero map:
\[ G \rightarrow E(K_S). \]
(8)

Since the bundle \( F \) is \( \mu_H \)-stable, it follows from (3) that \( G \) is also \( \mu_H \)-stable and \( \mu_H(G) = \mu_H(F) = \mu_H(E) \).
2.4 Structure of Rigid Sheaves.

In the paper [12] it was proved that any rigid bundle on Del Pezzo surface is a direct sum of exceptional bundles. But there exist indecomposable rigid bundles $E$ with $\text{Hom}(E,E) \neq \mathbb{C}$ provided $H = -K_S$ is nef. For example, consider a -2-curve $C$ on $S$ with $C \cdot K_S = 0$. It is easily shown that $(\mathcal{O}_S, \mathcal{O}_S(C))$ is an exceptional singular pair. Denote by $E$ a nontrivial extension of $\mathcal{O}_S$ by $\mathcal{O}_S(C)$:

$$0 \rightarrow \mathcal{O}(C) \rightarrow E \rightarrow \mathcal{O} \rightarrow 0.$$ 

It can be proved that $E$ is rigid and $\text{Hom}(E,E) \cong \mathbb{C}^2$.

In this section we prove that any rigid bundle on the surface $S$ with nef anticanonical class and $K_S^2 > 0$ has similar structure. Unfortunately, a structure of rigid bundles on $S$ with $K_S^2 = 0$ is not known. Further, we shall assume that $K_S^2 > 0$.

**DEFINITION.** We say that a torsion free sheaf $F$ has an **exceptional filtration** whenever there exists a filtration of $F$

$$\text{Gr}(F) = (x_n E_n, x_{n-1} E_{n-1}, \ldots, x_1 E_1),$$

where $(E_1, E_2, \ldots, E_n)$ is an exceptional collection of bundles such that $\mu_H(E_i) \leq \mu_H(E_{i+1})$ for $i = 1, 2, \ldots, n - 1$.

The aim of this section is to prove the following theorem:

**2.4.1 Theorem.** Let $S$ be a smooth projective surface over $\mathbb{C}$ with the anticanonical class without base components and $K_S^2 > 0$. Then

1. Any torsion free rigid sheaf on $S$ is a direct sum of $\mu_H$-semistable rigid bundles.
2. Any indecomposable rigid sheaf without torsion on $S$ is $\mu_H$-semistable.
3. Any $\mu_H$-semistable rigid sheaf has an exceptional filtration. Moreover, all pairs of associated exceptional collection have zero or singular type.

We shall use the vector slope

$$\bar{\gamma}(E) = (\mu_H(E), \mu_A(E), \frac{c_2^2(E) - 2c_2(E)}{r(E)}),$$
where \( A \) is an ample divisor, \( H \) is the anticanonical class of \( S \), and \( \mu_D(E) = \frac{c_1(E) \cdot D}{r(E)} \) with \( D = H \) or \( A \).

It can easily be checked that the stability with respect to this slope is the Gieseker stability \([1.3.3]\). In particular, any \( \bar{\gamma} \)-semistable sheaf has the filtration with isotypic quotients \([1.3.7]\).

The slope \( \mu_H(E) \) has Mumford-Takemoto type and satisfy the axiom SLOPE.4.

2.4.2 Lemma. Let \( F \) be \( \bar{\gamma} \)-semistable rigid sheaf on \( S \) with \( K_S^2 > 0 \). Suppose \( F \) has a filtration with \( \bar{\gamma} \)-stable isomorphic one to another quotients:

\[
Gr(F) = (G_n, G_{n-1}, \ldots, G_1) \quad (G_i \cong E \quad \forall i);
\]

Then \( F \) is multiple of the exceptional bundle \( E \), i.e \( F = nE \).

Proof. Consider the spectral sequence associated with the filtration of \( F \), which converge to the groups \( \text{Ext}^k(F,F) \). Its \( E_1 \)-term has the form:

\[
E_1^{pq} = \bigoplus_i \text{Ext}^{p+q}(G_i, G_{p+i}).
\]

Since the quotients \( G_i \cong E \) are \( \bar{\gamma} \)-stable, we see that they are \( \mu_H \)-semistable (see remark \([1.1.1]\)). Hence from lemma \([1.3.5]\) it follows that the sheaf \( E(K_S) \) is also \( \mu_H \)-semistable. On the other hand, the square of the canonical class of our surface is positive. Thus,

\[
\mu_H(E(K_S)) = \mu_H(E) + K_S \cdot H < \mu_H(E).
\]

Now, using Serre duality and lemma \([1.1.5]\) we have

\[
\text{Ext}^2(E, E) = 0.
\]

Thus, \( E_1 \)-term of the spectral sequence has the form:
2.4 Structure of Rigid Sheaves.

This yields that $\text{Ext}^1(G_n,G_1) = E_1^{1-n,n} = E_\infty^{1-n,n}$ . On the other hand,

$$E_\infty^{1-n,n} \subset \text{Ext}^1(F,F) = 0.$$  

But, $G_i \cong E \quad \forall i$. Consequently $E$ is a torsion free rigid sheaf. From lemma 2.2.1 it follows that $E$ is locally free. Besides, since $E$ is $\tilde{\gamma}$-stable, we see that it is simple and $\text{Ext}^2(E,E) = 0$, whereby $E$ is an exceptional bundle.

Finally, since $G_i \cong E \quad \forall i$ and $E$ is exceptional, we have

$$\text{Ext}^1(G_i,G_j) = 0 \quad \forall i, j.$$  

This implies the equality:

$$F = \bigoplus G_i = nE.$$

2.4.3 Lemma. Suppose $F$ is a rigid $\tilde{\gamma}$-semistable sheaf on $S$; then $F$ is a direct sum of exceptional bundles.

Proof. From proposition 1.3.7 it follows that $F$ has a filtration with isotypic quotients:

$$0 = F_{n+1} \subset F_n \subset \cdots \subset F_2 \subset F_1 = F,$$

where $G_i = F_i/F_{i+1}$ are $\tilde{\gamma}$-semistable and they have filtrations with isomorphic one to another $\tilde{\gamma}$-stable quotients. Besides,

$$\forall i : \quad \tilde{\gamma}(G_i) = \tilde{\gamma}(F), \quad \text{Hom}(F_{i+1},G_i) = 0.$$

Let us apply Mukai lemma (2.1.4) to the exact sequence:

$$0 \longrightarrow F_{i+1} \longrightarrow F_i \longrightarrow G_i \longrightarrow 0.$$  

It can be proved by induction on $i$ that $G_i$ and $F_{i+1}$ are torsion free rigid sheaves.

Note that each $G_i$ satisfies assumption of the previous lemma. Therefore we have $G_i = x_iE_i$, where $E_i$ are exceptional bundles.

Since all $G_i$ are $\tilde{\gamma}$-semistable, we see that they are $\mu_H$-semistable (1.1.1). Moreover,

$$\forall i : \quad \tilde{\gamma}(G_i) = \tilde{\gamma}(F), \quad \implies \quad \mu_H(G_i) = \mu_H(F).$$  

Whence, by the same argument as before, we have $\text{Ext}^2(G_i,G_j) = 0 \quad \forall i, j$. Thus the $E_1$-term of the spectral sequence associated with the filtration of $F$ has the same form as in the proof of lemma 2.4.2. Therefore,

$$\text{Ext}^1(G_n,G_1) = 0.$$  

But in this case, the filtration quotients of $F$ are different. To complete the proof we need an information about the groups $\text{Ext}^1(G_i,G_j)$ for $i < j$.

Let us remember that $G_i = x_iE_i$, where $E_i$ are the exceptional bundles. Hence,

$$\text{Ext}^1(G_n,G_1) \implies \text{Ext}^1(E_n,E_1) = 0. \quad (9)$$
By construction, the bundles $E_i$ are $\bar{\gamma}$-stable and $\bar{\gamma}(E_i) = \bar{\gamma}(F)$ \(\forall i\). Since $E_i$ are $\bar{\gamma}$-stable, it follows from lemma 1.3.4 that $E_n \cong E_1$ provided $\text{Hom}(E_n, E_1) \neq 0$ or $\text{Hom}(E_1, E_n) \neq 0$.

Suppose $E_n \cong E_1$; then from (3) we have $\text{Ext}^1(E_1, E_n) = 0$.

Assume that $E_n \not\cong E_1$; then we obtain

$$\text{Ext}^0(E_n, E_1) = \text{Ext}^0(E_1, E_n) = 0. \tag{10}$$

Since $\bar{\gamma}(E_1) = \bar{\gamma}(E_n)$, we get $\mu_H(E_1) = \mu_H(E_n)$.

Combining $\mu_H$-stability of exceptional bundles on $S$, Serre duality and the inequality $K^2_S > 0$, we obtain

$$\text{Ext}^2(E_n, E_1) = \text{Ext}^2(E_1, E_n) = 0.$$  (10)

Combining this with (10), we get

$$\chi(E_1, E_n) = -h^1(E_1, E_n); \quad \chi(E_n, E_1) = -h^1(E_n, E_1) = 0.$$

On the other hand, it follows from Riemann-Roch theorem for exceptional sheaves (2.1.2) and the equality $\mu_H(E_1) = \mu_H(E_n)$ that $\chi(E_1, E_n) = \chi(E_n, E_1)$. That is,

$$h^1(E_1, E_n) = h^1(E_n, E_1) = 0.$$

Thus we proved that $\text{Ext}^1(E_1, E_n) = 0$. This yields that $\text{Ext}^1(G_1, G_n) = 0$.

Let us remark that the second term of filtration (i.e. $F_2$) satisfies the assumptions of our lemma. But

$$\text{Gr}(F_2) = (G_n, G_{n-1}, \ldots, G_2).$$

By the inductive hypothesis, it can be assumed that

$$F_2 = \bigoplus_{i=2}^n G_i.$$

That is the sheaf $F$ is included in the exact sequence:

$$0 \rightarrow \bigoplus_{i=2}^n G_i \rightarrow F \rightarrow G_1 \rightarrow 0.$$

Since $\text{Ext}^1(G_1, G_n) = 0$, we obtain $F = \tilde{F} \oplus G_n$, where $\tilde{F}$ is $\bar{\gamma}$-semistable rigid sheaf with

$$\text{Gr}(\tilde{F}) = (G_{n-1}, G_{n-2}, \ldots, G_1).$$

Using the inductive hypothesis again, we have $\tilde{F} = \bigoplus_{i=1}^{n-1} G_i$. That is,

$$F = \bigoplus_{i=1}^n G_i = \bigoplus_{i=1}^n x_i E_i.$$

This completes the proof.

2.4.4 Lemma. Any rigid $\mu_H$-semistable sheaf $F$ on the surface $S$ has an exceptional filtration:

$$\text{Gr}(F) = (x_m E_m, x_{m-1} E_{m-1}, \ldots, x_1 E_1)$$
such that all exceptional pairs of collection \((E_1, E_2, \ldots, E_n)\) are singular or zero-pairs.

**Proof.** Suppose \(F\) is \(\bar{\gamma}\)-semistable; then by the previous lemma, \(F = \bigoplus_i x_i E_i\), where \(E_i\) are \(\bar{\gamma}\)-stable exceptional bundles with equal one to another \(\bar{\gamma}\)-slopes. Without loss of generality it can be assumed that \(E_i \not\cong E_j\) for \(i \neq j\). Using lemma [1.3.4], we have

\[
\text{Ext}^0(E_i, E_j) = 0 \quad \forall i, j. \tag{11}
\]

On the other hand the equality \(\bar{\gamma}(E_i) = \bar{\gamma}(E_j)\) yields that \(\mu_H(E_i) = \mu_H(E_j)\). Now the equality

\[
\text{Ext}^2(E_i, E_j) = 0 \quad \forall i, j \tag{12}
\]

can be proved by the standard method on the surface \(S\) with \(K^2_S > 0\).

Finally, since \(F\) is rigid, we have

\[
0 = \text{Ext}^1(F, F) = \text{Ext}^1(\bigoplus_i x_i E_i, \bigoplus_i x_j E_j) = \bigoplus_{i,j} \text{Ext}^1(E_i, E_j),
\]
i.e., \(\text{Ext}^1(E_i, E_j) = 0\).

Combining this with equalities (11) and (12), we see that each pair of bundles in the collection \((E_1, E_2, \ldots, E_n)\) is an exceptional zero-pair.

Now we suppose that \(F\) is not \(\bar{\gamma}\)-semistable. Consider its Harder-Narasimhan filtration:

\[
\text{Gr}(F) = (G_n, G_{n-1}, \ldots, G_1)
\]

(see proposition [1.2.3]).

Since \(G_i\) are \(\bar{\gamma}\)-semistable and \(\bar{\gamma}(G_i) > \bar{\gamma}(G_{i-1})\) for all \(i\), we get

\[
\text{Ext}^0(G_i, G_j) = 0 \quad \forall i > j. \tag{13}
\]

Note that lemma [1.1.8] and \(\mu_H\)-semistability of the sheaf \(F\) imply \(\mu_H\)-semistability of the quotients \(G_i\) and the equality \(\mu_H(G_i) = \mu_H(F)\). Therefore, as before,

\[
\text{Ext}^2(G_i, G_j) = 0 \quad \forall i, j. \tag{14}
\]

Combining (13) and (14), we see that the \(E_1\)-term of the spectral sequence associated with Harder-Narasimhan filtration of \(F\) has the form:
This spectral sequence is convergent to the groups Ext\(^k(F,F)\) of the rigid sheaf. Hence,
\[
\text{Ext}^1(G_i, G_j) = 0 \quad \forall i \geq j.
\] (15)

In particular, G\(_i\) are rigid \(\tilde{\gamma}\)-semistable sheaves. By the previous lemma, G\(_i\) = \(\bigoplus\_k x\_ik E\_ik\), where E\(_{ik}\) are exceptional bundles. Besides, any pair \((E\_ik, E\_is)\) has the zero type.

Combining (13), (14) and (13), we obtain that Ext\(^d(E\_ik, E\_js)\) = 0 for \(i > j\) and \(d = 0, 1, 2\). In other words, the set of all bundles E\(_{ik}\) can be numerated such that the collection \((E\_1, E\_2, \ldots, E\_m)\) will be exceptional. It remains to remark that all bundles E\(_i\) have the \(\mu_H\)-slope coincided with \(\mu_H(F)\). Thus it follows from proposition 2.3.1 that each pair of this collection is either singular or a zero-pair.

The proof plan of the main theorem is clear. We consider the spectral sequence associated with Harder-Narasimhan filtration of a rigid torsion free sheaf for obtaining the information about the groups Ext\(^1(G\_i, G\_j)\), where G\(_i\) are quotients of this filtration. For this we need the following last statement.

2.4.5 Lemma. Let G\(_1\) and G\(_2\) be \(\mu_H\)-semistable rigid sheaves on the surface S. Suppose \(\mu_H(G\_2) > \mu_H(G\_1)\); then the equality Ext\(^1(G\_2, G\_1) = 0\) implies Ext\(^1(G\_1, G\_2) = 0\).

Proof. It follows from lemma 2.4.4 that each of G\(_i\) has the exceptional filtration:
\[
Gr(G\_1) = (x\_1 E\_1, \ldots, x\_11 E\_11); \quad Gr(G\_2) = (x\_2 E\_m, \ldots, x\_12 E\_12).
\]
Moreover, \(\mu_H\)-slopes of E\(_{ij}\) do not depend on the first index, i.e, \(\mu_H(E\_i1) = \mu_H(G\_1)\) and \(\mu_H(E\_j2) = \mu_H(G\_2)\).

Denote by G\(_i\)' the restriction of the sheaves G\(_i\) to an elliptic curve D \(\in | - K\_S|\). It is obvious that the sheaves G\(_i\)' have the filtrations:
\[
Gr(G\_1\') = (x\_1 E\_1\', \ldots, x\_11 E\_11\'); \quad Gr(G\_2\') = (x\_2 E\_m, \ldots, x\_12 E\_12),
\]
where E\(_{ki}\) = E\(_{ki}\)|\(_D\). Further, since E\(_{ki}\) are exceptional bundles, we see that E\(_{ki}\)' are stable with respect to the standard slope \(\mu\) on a curve (see lemma 2.2.8). Moreover,
\[
\mu(E\_ki\') = \mu_H(E\_ki) = \mu_H(G\_i) = \mu(G\_i').
\]
Now it follows from lemma 1.1.8 that G\(_i\)' are \(\mu\)-semistable and \(\mu(G\_2\') > \mu(G\_1')\). Thus from lemma 1.1.5 it follows that Hom\((G\_2\', G\_1\') = 0\).

Using the last equality and the long cohomology sequence associated with the exact triple
\[
0 \rightarrow G\_2\' \otimes G\_1(K\_S) \rightarrow G\_2\' \otimes G\_1 \rightarrow (G\_2\' \otimes G\_1)|\(_D\) \rightarrow 0,
\]
we obtain
\[
\text{Ext}^1(G\_2, G\_1(K\_S)) \subset \text{Ext}^1(G\_2, G\_1).
\]
Now the proof follows from Serre duality.

Proof of Theorem 2.4.1. Let F be any torsion free rigid sheaf on S. Consider its Harder-Narasimhan filtration with \(\mu_H\)-semistable quotients
\[
Gr(F) = (G\_n, G\_n\_1, \ldots, G\_1).
\]
It follows from the inequalities $\mu_H(G_i) > \mu_H(G_j)$ for $i > j$ that

$\text{Hom}(G_i, G_j) = 0$ for $i > j$ and $\text{Ext}^2(G_j, G_i) = 0$ for $i \geq j$.

Therefore $E_1$-term of the spectral sequence associated with this filtration has the form:

\[
\begin{array}{c}
q \\
\ast & \ast & \ast \\
0 & \ast & \ast \\
0 & 0 & 0 \\
\ast & \ast & 0 \\
\ast & \ast \\
\ast
\end{array}
\]

$p$

Since the sequence is convergent to the groups $\text{Ext}^i(F, F)$ of the rigid sheaf, we obtain

\[
0 = E_{\infty}^{-1,2} = E_1^{-1,2} = \bigoplus_i \text{Ext}^1(G_i, G_{i-1}),
\]

\[
0 = E_{\infty}^{0,1} = E_1^{0,1} = \bigoplus_i \text{Ext}^1(G_i, G_i),
\]

That is $G_i$ are rigid $\mu_H$-semistable sheaves and $\text{Ext}^1(G_i, G_{i-1}) = 0$.

By the previous lemma, the groups $\text{Ext}^1(G_{i-1}, G_i)$ are also trivial. In particular,

$\text{Ext}^1(G_1, G_2) = 0$.

Let $F_2$ be the first term of the filtration $Gr(F)$, i.e.,

\[
0 \longrightarrow F_2 \longrightarrow F \longrightarrow G_1 \longrightarrow 0.
\] (16)

Note that $Gr(F_2) = (G_n, G_{n-1}, \ldots, G_2)$ is also Harder-Narasimhan filtration and $\mu_H(G_2) > \mu_H(G_1)$. Taking into account corollary 1.2.5, we obtain $\text{Hom}(F_2, G_1) = 0$. In addition, the sheaves $F_2$ and $G_1$ have not torsion. Hence we can apply to these sheaves lemma 2.2.2.

That is, $\text{Ext}^2(G_1, F_2) = 0$.

Now we apply Mukai lemma to (16) to obtain

$\text{Ext}^1(F, F) \geq \text{Ext}^1(F_2, F_2) + \text{Ext}^1(G_1, G_1)$.

That is the sheaf $F_2$ is also rigid. The number of its Harder-Narasimhan filtration quotients is less then $n$. Hence by the inductive hypothesis we have $F_2 = \bigoplus_{i=2}^n G_i$, and

\[
0 \longrightarrow \bigoplus_{i=2}^n G_i \longrightarrow F \longrightarrow G_1 \longrightarrow 0.
\]
Let us remember that $\text{Ext}^1(G_1, G_2) = 0$. Therefore, $F = \tilde{F} \oplus G_2$, where $\tilde{F}$ is a torsion free rigid sheaf. We apply again the inductive hypothesis to the sheaf $\tilde{F}$ to obtain

$$F = \bigoplus_{i=1}^n G_i.$$ 

Thus any rigid sheaf without torsion on $S$ is a direct sum of $\mu_H$-semistable rigid ones. They are locally free by virtue of 2.2.1. In particular, if $F$ is indecomposable then it is $\mu_H$-semistable. This concludes the proof of the first and the second theorem statements. The last one is equivalent to lemma 2.4.4.

2.4.6 Corollary. Any torsion free rigid sheaf on Del Pezzo surface $X$ is a direct sum of exceptional bundles.

Proof. Since the anticanonical class of Del Pezzo surface is ample, we see that an exceptional pairs on $X$ cannot be singular (see 2.3.2). On the other hand, we have proved that any indecomposable torsion free rigid sheaf on $S$ (in particular, on $X$) has an exceptional filtration. Besides, all pairs in associated exceptional collection are singular or zero pairs. Thus the quotients of the exceptional filtration of any torsion free rigid sheaf on $X$ are its direct summands.

2.5 Structure of superrigid sheaves.

In the previous section we have proved that any $\mu_H$-semistable sheaf has the exceptional filtration and any torsion free rigid sheaf is a direct sum of $\mu_H$-semistable rigid bundles. Whereby for classifying rigid bundles we need a description of exceptional bundles and collections of ones. This description is the subject-matter of the next part. But for it we need the following theorem.

2.5.1 Theorem. Let $S$ be a smooth projective surface over $\mathbb{C}$ with the anticanonical class $H$ without base components and $H^2 > 0$. Then the following statements hold true.

1. For any exceptional collection of bundles $(E_1, E_2, \ldots, E_n)$ on $S$ such that $\forall i \mu_H(E_i) \leq \mu_H(E_{i+1})$ there exists a superrigid bundle $E$ such that

   $$(\text{Ext}^1(E, E) = \text{Ext}^2(E, E) = 0)$$

   such that $Gr(E) = (x_nE_n, x_{n-1}E_{n-1}, \ldots, x_1E_1)$. We say that this bundle is associated with the exceptional collection.

2. Any superrigid torsion free sheaf $E$ has the exceptional filtration

   $$Gr(E) = (x_nE_n, x_{n-1}E_{n-1}, \ldots, x_1E_1),$$

   i.e. the collection $(E_1, E_2, \ldots, E_n)$ is exceptional and the $\mu_H$-slopes of bundles $E_i$ satisfy the inequalities: $\mu_H(E_i) \leq \mu_H(E_{i+1})$ $\forall i$.

3. Suppose a superrigid torsion free sheaf $E$ has two exceptional filtrations:

   $$Gr(E) = (x_nE_n, x_{n-1}E_{n-1}, \ldots, x_1E_1) = (y_mE_m, y_{m-1}E_{m-1}, \ldots, y_1E_1);$$
then \( m = n \) and the exceptional collection \((F_1, F_2, \ldots, F_m)\) can be obtained from \((E_1, E_2, \ldots, E_n)\) by mutations of neighboring zero-pairs \((E_i, E_{i+1})\).

Note that this theorem is obvious provided \( S \) is Del Pezzo surface (see corollary \( \text{[2.4.6]} \)). But if \(-K_S\) is nef then this theorem is interesting and its proof is difficult.

Let us state and prove several lemmas.

### 2.5.2 Remark. Suppose a sheaf \( F \) has a filtration \( \text{Gr}(F) = (G_n, G_{n-1}, \ldots, G_1) \) such that \( \text{Gr}(G_i) = (E_{ik}, \ldots, E_{i1}) \); then there exists the filtration

\[
\text{Gr}(F) = (E_{nk}, \ldots, E_{n1}, \ldots, E_{1k}, \ldots, E_{11}).
\]

And back to front, the neighboring quotients can be "join".

### 2.5.3 Remark. Suppose \( \text{Gr}(F) = (G_n, G_{n-1}, \ldots, G_1) \) is a filtration of a sheaf \( F \) such that \( \text{Ext}^1(G_i, G_{i+1}) = \text{Ext}^1(G_{i+1}, G_i) = 0 \) for same \( i \); then the sheaf \( F \) has the filtration

\[
\text{Gr}(F) = (G_n, G_{n-1}, \ldots, G_i, G_{i+1}, \ldots, G_1).
\]

### 2.5.4 Lemma. Suppose \( F \) is an indecomposable rigid bundle on \( S \) with \( K_S^2 > 0 \); then \( F \) has the following filtrations:

a) \( \text{Gr}_R(F) = (Q_n, Q_{n-1}, \ldots, Q_1) \) such that \( \forall i \quad Q_i = \bigoplus y_{is} E_{is}; E_{is} \) are exceptional bundles, the collection \((E_{i1}, \ldots, E_{im}, \ldots, E_{n1}, \ldots, E_{nm})\) is exceptional and for each bundle \( E_{is} \quad (i = 1, \ldots, n - 1) \) there is \( E_{i+1,k} \) such that the pair \((E_{is}, E_{i+1,k})\) is singular.

b) \( \text{Gr}_L(F) = (G_n, G_{n-1}, \ldots, G_1) \) such that \( \forall i \quad G_i = \bigoplus x_{is} E_{is}; E_{is} \) are exceptional bundles, the collection \((E_{i1}, \ldots, E_{ik}, \ldots, E_{n1}, \ldots, E_{nk})\) is exceptional and for each bundle \( E_{is} \quad (i = 2, \ldots, n) \) there is \( E_{i-1,l} \) such that the pair \((E_{i-1,l}, E_{is})\) is singular.

**Proof.** Let us construct the first filtration. Similarly the second one is constructed.

By the theorem about rigid bundles \( \text{[2.4.1]} \) the sheaf \( F \) has an exceptional filtration

\[
\text{Gr}(F) = (x_n E_n, x_{n-1} E_{n-1}, \ldots, x_1 E_1).
\]

Let us partition the exceptional collection

\[
(E_1, E_2, \ldots, E_n),
\]

associated with this filtration into subcollections

\[
(E_{i_{s-1}+1}, E_{i_{s-1}+2}, \ldots, E_{is}),
\]

where \( i_0 = 0 \) such that the following conditions hold.

**PART.1:** Any pair of each subcollection has the zero type whenever this subcollection has greater than one bundle.

**PART.2:** For last bundle \( E_{is} \) of each subcollection there is \( E_j \) in the next subcollection such that the pair \((E_{is}, E_j)\) is singular.

There exists at most one singular pair in this collection because the bundle \( F \) is indecomposable. This implies that this partition exists.
Denote by $Q_s$ the direct sum
\[ \bigoplus_{j=i_{s-1}+1}^{i_s} x_j E_j \]
of exceptional bundles from the subcollection with index $s$. It follows from remark 2.5.2 that $F$ has filtration $Gr(F) = (Q_k, Q_{k-1}, \ldots, Q_1)$, where $k$ is the quantity of all subcollections. Note that this filtration consists with $Gr_R$ if and only if the collection decomposed into subcollections satisfies the conditions PART.1 and the following

PART.2R: for any bundle $E_i$ of each subcollection there is a bundle $E_j$ in the next subcollection such that the pair $(E_i, E_j)$ is singular.

For constructing the required collection we shall intermix the bundles of subcollections and move sometimes bundles from subcollection to next one.

Suppose there is a bundle $E_\alpha$ in the first subcollection such that for all bundles $E_\beta$ in the second one $(E_\alpha, E_\beta)$ are zero-pairs. Let us move $E_\alpha$ to the second subcollection. Since this moving can be realized by permutations of members in neighboring zero-pairs, we see that the obtained collection is exceptional. Besides, it satisfy the conditions PART.1 and PART.2. It is clear that after a finite number of such movings we get the exceptional collection decomposed into subcollections such that for each bundle $E_\alpha$ of the first subcollection there is $E_\beta$ in the second one such that $(E_\alpha, E_\beta)$ is singular pair. Let us mention, that one can do the some thing with an arbitrary pair of neighboring subcollections. Let this process be called the *displacement*.

Let us do the displacement with each pair of the neighboring subcollections, beginning from the first one. The number of the subcollections cannot be changed during the process. The number of the bundles in the first subcollection can lessen only. Two latter subcollections will satisfy the condition PART.2.R. But since we moved the bundles from the left to the right, now one can find two neighboring subcollections (with the numbers $s$ and $s + 1$, for example) satisfying the following conditions. Any pair $(E_i, E_j)$ with $E_i$ belonging to the $s$-th subcollection and $E_j$ belonging to the $(s + 1)$-th subcollection has the type zero. Moreover, one can warrant that two latter subcollection satisfy the condition PART.2R only.

Let us join, if it is necessary, the neighboring subcollections to satisfy the conditions PART.1 and PART.2.

Let us do the displacement with each pair of neighboring subcollection and join all what is possible to join, ets...

This process cannot be repeated ad infinitum. Really, there exists $k_0 \in \mathbb{N}$ such that for any $k > k_0$ the number of the subcollections will not change after doing the $k$-th step — "the displacement and the join". After some successive step the number of bundles in the first subcollection will not change, ets... Thus, the number of the subcollections and the number of the bundles in each subcollection will not change since some moment. That means, any bundle does not go from one subcollection to another. Hence we are done.

I should like to remark that the last lemma is trivial provided each pair of the exceptional collection associated with a rigid bundle is singular. But it is not the case. On the surface $S$ there is an exceptional collection of bundles of equal $\mu_H$-slope consisted of both singular and zero-pairs. Besides, the superrigid bundle associated with this collection is indecomposable. The following example was found by Yu. B. Zuev.

Let $X_1$ be a surface obtained from $\mathbb{P}^2$ by blowing up a point $\sigma_1 : X_1 \to \mathbb{P}^2$. 
Denote by $e$ the preimage of $x_1$ ($e = \sigma_1^{-1}(x_1)$). Let us choose two points $x_2$ and $x_3$ on $e$. Suppose $X$ is obtained from $X_1$ by blowing up $x_2$ and $x_3$: $\sigma_2 : X \to X_1$. By definition, put
\[ e_1 = \sigma_2^{-1}(e), \quad e_2 = \sigma_2^{-1}(x_2), \quad e_3 = \sigma_2^{-1}(x_3). \]
We see that the curves $e_1$, $e_2$, and $e_3$ are exceptional, that is $e_i^2 = -1$ for $i = 1, 2, 3$. Besides, $(e_1 - e_2)$ and $(e_1 - e_3)$ are connected $-2$-curves. It can easily be checked that the collection of line bundles
\[ (\mathcal{O}_X, \mathcal{O}_X(e_1 - e_2), \mathcal{O}_X(e_1 - e_3)) \]
is exceptional. Moreover, the pairs $\mathcal{(O}_X, \mathcal{O}_X(e_1 - e_2))$ and $\mathcal{(O}_X, \mathcal{O}_X(e_1 - e_3))$ are singular and $\mathcal{(O}_X(e_1 - e_2), \mathcal{O}_X(e_1 - e_3))$ is a zero-pair.

2.5.5 Lemma. Let $F$ be an indecomposable torsion free rigid sheaf on the surface $S$. Assume that
\[ Gr(F) = (x_nE_n, x_{n-1}E_{n-1}, \ldots, x_1E_1) = (y_nF_n, y_{n-1}F_{n-1}, \ldots, y_1F_1) \]
are two exceptional filtrations of $F$, i.e. the collections
\[ (E_1, E_2, \ldots, E_n) \quad \text{and} \quad (F_1, F_2, \ldots, F_m) \]
are exceptional. Suppose
\[ \mu_H(E_i) = \mu_H(F_j) = \mu_H(F) \quad \forall i, j; \quad (17) \]
then $m = n$ and the collection $(E_1, E_2, \ldots, E_n)$ can be obtained from $(F_1, F_2, \ldots, F_m)$ by mutations of neighboring zero-pairs.

Proof. It follows from proposition 2.3.1 that each pair of these collections has zero or singular type. Let us show that any such collection can be ordered by $\bar{\gamma}$-slope by virtue of permutations of neighboring zero-pairs members only. In this case the lemma follows from the uniqueness of Harder-Narasimhan filtration (1.2.3).

The possibility of such ordering is obtained by induction on the number of collection members from the following arguments.

Suppose $(E, F)$ is a singular pair; then it follows from lemma 2.3.2 that ranks of the sheaves $E$ and $F$ are equal and $c_1(F) - c_1(E) = C$ is an effective $-2$-divisor. (Recall that the $\bar{\gamma}$-slope is the vector
\[ (\mu_H, \mu_A, \frac{c_1^2 - 2c_2}{r}), \]
where $\mu_A = \frac{c_1A}{r}$, and $A$ is an ample divisor.) Since $A$ is ample, we get $\mu_A(E) < \mu_A(F)$. By assumption we have $\mu_H(E) = \mu_H(F)$. Therefore, $\bar{\gamma}(E) < \bar{\gamma}(F)$.

2.5.6 Lemma. Suppose $(E, F)$ is an exceptional singular pair on $S$ and $G$ is a torsion free sheaf; then the following implications hold true.
\[ a) \text{ Ext}^2(G, E) = 0 \implies \text{ Ext}^2(G, F) = 0; \]
\[ b) \text{ Ext}^0(G, F) = 0 \implies \text{ Ext}^0(G, E) = 0; \]
\[ c) \text{ Ext}^0(E, G) = 0 \implies \text{ Ext}^0(F, G) = 0; \]
d) \( \text{Ext}^2(F, G) = 0 \implies \text{Ext}^2(E, G) = 0; \)

**Proof.** The lemma follows from the cohomology tables associated with the exact triple

\[ 0 \to E \to F \to Q \to 0, \]

where \( Q \) is a torsion sheaf. Besides, since \( Q \) has zero rank and \( G \) is torsion free, we see that \( \text{Hom}(Q, G) = 0 \). Moreover, using Serre duality, we have \( \text{Ext}^2(G, Q) = 0. \)

2.5.7 **Lemma.** Let \( F \) be a \( \mu_H \)-semistable rigid bundle. Let

\[ Gr(F) = (x_n E_n, x_{n-1} E_{n-1}, \ldots, x_1 E_1) \]

be its exceptional filtration and \( G \) sheaf without torsion. Then

a) \( \text{Ext}^i(G, F) = 0 \forall i \iff \text{Ext}^i(G, E_k) = 0 \forall i, k; \)

b) \( \text{Ext}^i(F, G) = 0 \forall i \iff \text{Ext}^i(E_k, G) = 0 \forall i, k. \)

**Proof.** Without loss of generality it can be assumed that \( F \) is indecomposable. Consider its filtration \( Gr_R(F) = (Q_m, Q_{m-1}, \ldots, Q_1) \) from lemma 2.5.4. By 2.5.5, we can assume without loss of generality that

\[ Q_j = \bigoplus_{i=s_j}^{s_{j+1}-1} y_i E_i \quad 1 = s_1 < s_2 < \cdots < s_m < s_{m+1} = n + 1. \]

To prove the first statement of our lemma it is sufficient to check the following implication

\[ \text{Ext}^i(G, F) = 0 \forall i \implies \text{Ext}^i(G, E_k) = 0 \forall i, k. \]

(The another implication follows from 1.2.4.) Let us apply the functor \( \text{Ext}^i(G, \cdot) \) to each of the sequences:

\[ 0 \to F_{j+1} \to F_j \to Q_j \to 0, \]

where \( F_j \) are terms of the filtration \( Gr_R(F) \), \( F_1 = F \) and \( F_n = Q_n \). We see that for \( j = 2, 3, \ldots, n \) \( \text{Hom}(G, F_j) = 0 \). In particular, \( \text{Hom}(G, Q_n) = 0. \)

**Step 1.** Let us show that \( \text{Hom}(G, E_i) = 0 \) for all \( i. \)

It follows from the equality \( \text{Hom}(G, Q_n) = 0 \) that \( \text{Hom}(G, E_i) = 0 \) for \( s_m \leq i \leq n. \)

By the construction of the filtration \( Gr_R(F) \), for each direct summand \( E_\alpha \) of the bundle \( Q_{n-1} \) there exists a bundle \( E_\beta \) with \( s_m \leq \beta \leq n \) such that \( (E_\alpha, E_\beta) \) is a singular pair. The application of lemma 2.5.6 to this pair yields, \( \text{Hom}(G, E_\alpha) = 0. \)

In the same way, using the properties of the filtration \( Gr_R(F) \) and 2.5.6, we conclude the first step.

**Step 2.** Let us check that \( \text{Ext}^2(G, E_i) = 0 \) for all \( i. \)

Now let us intermix the bundles in collection \( (E_1, E_2, \ldots, E_n) \) to obtain the filtration \( Gr_L(F) \). Without loss of generality we can assume that

\[ G_j = \bigoplus_{i=s_j}^{s_{j+1}-1} y_i E_i \quad 1 = s_1 < s_2 < \cdots < s_m < s_{m+1} = n + 1. \]

are quotients of \( Gr_L(F) \).
Applying the functor $\text{Ext}(G, \cdot)$ to the exact triple

$$0 \to F'_2 \to F \to G_1 \to 0,$$

where $F'_2$ is the first term of the filtration $Gr_L(F)$, we get, $\text{Ext}^2(G, G_1) = 0$. This means that $\text{Ext}^2(G, E_i) = 0$ for any direct summand of the bundle $G_1$.

Using the properties of the filtration $Gr_L(F)$ and lemma 2.5.6 as before, we have

$$\text{Ext}^2(G, E_i) = 0 \quad \forall i.$$ 

Thus it was proved that for any quotient $E_i$ of the exceptional filtration of the bundle $F$ the groups $\text{Ext}^0(G, E_i)$ and $\text{Ext}^2(G, E_i)$ are trivial. Hence, $\chi(G, E_i) \leq 0 \quad \forall i$. Since Euler characteristic of sheaves is additive function, we have

$$\sum_{i=1}^{n} x_i \chi(G, E_i) = \chi(G, F) = 0.$$ 

Moreover, all $x_i$ are natural numbers. Thus, $\chi(G, E_i) = 0 \quad \forall i$. The first statement of the lemma was proved. Similarly the second one is proved.

**Proof of theorem 2.5.1**

1. At first assume that all pairs of the collection $(E_1, E_2, \ldots, E_n)$ have zero or singular type. The proof is by induction on the number $n$ of bundles in the collection. For $n = 1$, there is nothing to prove. 

By induction hypothesis there exists a superrigid bundle $E'$ such that 

$$Gr(E') = (E_n, E_{n-1}, \ldots, E_2).$$ 

Suppose the pair $(E_1, E_i)$ has the zero type for any $i$; then $E = E' \oplus E_1$ is a superrigid bundle (see 1.2.4).

Suppose there is an index $i$ such that $(E_1, E_i)$ is singular; then $\text{Ext}^k(E_1, E_i) = \mathbb{C}$ for $k = 0, 1$ and

$$\text{Ext}^2(E_1, E_j) = \text{Ext}^k(E_j, E_1) = 0 \quad \forall j, k.$$ 

Therefore, $\text{Ext}^k(E', E_1) = 0 \quad \forall k$ and 

$$\text{Ext}^0(E_1, E') = V \neq 0, \quad \text{Ext}^1(E_1, E') = W \neq 0, \quad \text{Ext}^2(E_1, E') = 0.$$ 

Consider the universal extension:

$$0 \to E' \to E \to W \otimes E_1 \to 0.$$ 

By means of cohomology tables let us show that $E$ is superrigid. The first table has the form:

| $k$ | $\text{Ext}^k(E_1, E')$ | $\text{Ext}^k(E_1, E)$ | $W \otimes \text{Ext}^k(E_1, E_1)$ |
|-----|----------------|----------------|---------------------------------|
| 0   | $V$            | ?              | $W$                             |
| 1   | $W$            | ?              | 0                               |
| 2   | 0              | ?              | 0                               |

Since the extension is universal, we see that the coboundary homomorphism 

$$W \to \text{Ext}^1(E_1, E')$$
CONSTRUCTIBILITY OF EXCEPTIONAL BUNDLES.

is isomorphism. Hence

\[ \text{Ext}^1(E_1, E) = \text{Ext}^2(E_1, E) = 0. \]

The next tables have the form:

| \[ k \] | \[ \text{Ext}^k(E', E') \rightarrow \text{Ext}^k(E', E) \rightarrow W \otimes \text{Ext}^k(E', E_1) \] |
|--------|-----------------------------------------------|
| 0      | \*                                            |
| 1      | 0 ?                                            |
| 2      | 0 ?                                            |

| \[ k \] | \[ W^* \otimes \text{Ext}^k(E_1, E) \rightarrow \text{Ext}^k(E, E) \rightarrow \text{Ext}^k(E', E') \] |
|--------|-----------------------------------------------|
| 0      | \*                                            |
| 1      | 0 ?                                            |
| 2      | 0 ?                                            |

Thus, \( E \) is a superrigid bundle.

Now assume that \( (E_1, E_2, \ldots, E_n) \) is an arbitrary exceptional collection of bundles such that

\[ \mu_H(E_1) \leq \mu_H(E_2) \leq \cdots \leq \mu_H(E_n). \]

Let us partition it into subcollections of bundles with equal \( \mu_H \)-slopes. Since all pairs in obtained subcollections are singular or zero-pairs, we see that there exists superrigid bundles \( G_1, G_2, \ldots, G_k \) constructed by these subcollections. Moreover, \( \mu_H(G_i) < \mu_H(G_{i+1}) \). Now let us remember that a pair \((E_i, E_j)\) of bundles has the type \( \text{hom} \) provided \( \mu_H(E_i) < \mu_H(E_j) \), i.e \( \text{Ext}^k(E_i, E_j) = 0 \) for \( k = 1, 2 \) and \( \text{Ext}^k(E_j, E_i) = 0 \) for \( k = 0, 1, 2 \). This yields that the bundle \( E_i \oplus E_j \) is superrigid. Thus, \( \bigoplus G_i \) is the required bundle.

2. It follows from theorem 2.4.1 that a torsion free superrigid sheaf is a direct sum of \( \mu_H \)-semistable rigid bundles \( F = \bigoplus_{j=1}^{m} F_j \). Without loss of generality we can assume that

\[ \mu_H(F_j) < \mu_H(F_{j+1}) \quad \forall i. \]

Since \( F \) is superrigid, we see that \( \text{Ext}^k(F_i, F_j) = 0 \) for \( k = 1, 2 \) and for any pair \( i, j \). Besides, it follows from \( \mu_H \)-semistability of \( F_i \) and the last inequality that \( \text{Hom}(F_j, F_i) = 0 \) for \( j > i \).

Taking into account theorem 2.4.1, we obtain that each \( F_j \) has the exceptional filtration

\[ \text{Gr}(F_j) = (x_{s_j-1}E_{s_j-1}, \ldots, x_{s_{j-1}}E_{s_{j-1}}). \]

Using the previous lemma and the proved fact \( \text{Ext}^k(F_j, F_i) = 0 \) for \( j > i \) and \( k = 0, 1, 2 \), we see that the collection of the direct summands of all bundles \( F_j \) (with the preservation of the order) is exceptional. This concludes the proof of the second theorem statement.

3. Since any torsion free rigid sheaf is locally free and direct summands are unambiguous determined, we see that it is sufficient for proving the third theorem statement in the case \( F \) is an indecomposable superrigid bundle. But this case is proved in 2.5.5.

This completes the proof of the theorem.

3 Constructibility of Exceptional bundles.
3.1 Introduction to the Helix Theory.

In this section, following [24], [4], [3], and [3], we tell about the general concepts and facts connected with exceptional sheaves on manifolds (see the definition of exceptional sheaves in 2.2) and exceptional objects in a derived category.

The notion of exceptional bundles was introduced in the paper [7]. The main result of that paper is a description of Chern classes of semistable bundles on $\mathbb{P}^2$. Exceptional bundles appeared there as some kind of boundary points.

Further the exceptional bundles and the exceptional objects in a derived category of sheaves were researched on Rudakov’s seminar in Moscow. It became clear that the exceptional objects (sheaves) organized as exceptional collections can generate the derived category of sheaves. Therefore, there exists a spectral sequence of Beilinson type associated with an exceptional collection. Let us remark that firstly a spectral sequence of such type on $\mathbb{P}^2$ appeared in [8]. But the general result independently on $\mathbb{P}^2$ was obtained by A. L. Gorodentsev ([6]).

The existence of the spectral sequence is the weighty reason for studying the exceptional sheaves. Besides, the exceptional bundles are interesting as bundles with a zero-dimensional moduli space.

The next application of the exceptional bundles is a description of moduli spaces of semistable bundles. There exists such description for the case of projective plane ([7]) and of smooth 2-dimensional quadric ([22]).

The helix theory is connected with number theory. Namely, A. A. Markov, in particular, studied solutions of the following Diophantus equation:

$$x^2 + y^2 + z^2 = 3xyz.$$  \hspace{1cm} (18)

(Now this equation is called the Markov equation and its solutions are called the Markov numbers.) It was proved that the Markov numbers coincide with ranks of the exceptional bundles on $\mathbb{P}^2$ which form a foundation of a helix.

A. A. Markov formulated the following hypothesis:

Any triple of natural solutions of the equation (18) is uniquely determined by its maximal element.

This hypothesis can be formulated in terms of the exceptional bundles in the following way.

Suppose $E$ and $F$ are exceptional bundles on $\mathbb{P}^2$ with equal ranks; then either $E = F(n)$ or $E^* = F(n)$ for some natural $n$.

More detail can be found in [19].

Now let us pass to the helix theory. The definition of a helix and the first results about helices appeared in [19]. The definition of the helix is due to A. L. Gorodentsev and A. N. Rudakov. The word ”helix” and the idea of considering a helix as an infinite system of bundles with some form of periodicity is due to W. N. Danilov.

Below following [20], we tell about axioms of helix theory.

We shall consider pairs of objects of a category $\mathfrak{U}$ or elements of a set $\mathfrak{U}$.

DEFINITION. A pair $(A, B)$ is called left admissible if a certain pair $(L_A B, A)$ is defined. The pair $(L_A B, A)$ is called a left mutation of $(A, B)$ and the object $L_A B$ is called a left shift of $B$. Similarly, a pair $(A, B)$ is right admissible if a certain pair $(B, R_B A)$ is defined.
The pair \((B, R_B A)\) in this case is called a right mutation of \((A, B)\) and the object \(R_B A\) is a right shift of \(A\).

The axioms are following.

(1L) If \((A, B)\) is left admissible then \((L_A B, A)\) is right admissible and \(R_A L_A B = B\).

(1R) If \((A, B)\) is right admissible then \((B, R_B A)\) is left admissible and \(L_B R_B A = A\).

(2L) Let \((A, B, C)\) be such that the pairs \((B, C), (A, L_B C)\) and \((A, B)\) are left admissible. Then the pairs \((A, C), (B', L_A C)\) are left admissible, where \(B' = L_A B\) and \(L_A L_B C = L_B' L_A C\).

(2R) Let \((A, B, C)\) be such that the pairs \((B, C), (A, B)\) and \((R_B A, C)\) are right admissible. Then the pairs \((A, C), (R_C A, B')\) are right admissible, where \(B' = R_C B\) and \(R_C R_B A = R_B R_C A\).

The equalities in the axioms (2L) and (2R) are usually called the triangle equations.

It will be convenient to denote the object \(L_A L_B C\), which appeared in (2L) by \(L^{(2)} C\) and also to set \(R^{(2)} A = R_C R_B A\). In the same way, if \((A_0, A_1, A_2, \ldots, A_n)\) is a system of objects we put \(L^{(0)} A_s = A_s, L^{(1)} A_s = L_{A_{s-1}} A_s, \ldots, L^{(i)} A_s = L_{A_{s-i}} L^{(i-1)} A_s\), with the condition that the resulting pairs are left admissible. Analogous notation will be used for right mutations.

**Definition.** The collection \(\{A_i| i \in \mathbb{Z}\}\) will be called a helix of period \(n\) if for all \(s\) the following condition holds

\[\text{HEL: The pairs } (A_{s-1}, A_s), (A_{s-2}, L^{(1)} A_s), \ldots, (A_{s-n+1}, L^{(n-2)} A_s) \text{ are left admissible and } L^{(n-1)} A_s = A_{s-n}.\]

Further we shall assume that (1L), (1R), (2L) and (2R) are satisfied. Then HEL is equivalent to

\[\text{HEL': The pairs } (A_{s-n}, A_{s-n+1}), (R^{(1)} A_{s-n}, A_{s-n+2}), \ldots, (R^{(n-2)} A_{s-n}, A_s) \text{ are right admissible and } R^{(n-1)} A_{s-n} = A_{s}.\]

Each collection of the form \(A_i, A_{i+1}, \ldots, A_{i+n-1}\) is called a foundation of the helix \(\{A_i\}\). Note that a helix is uniquely determined by any of its foundations.

A collection \(\{B_i| i \in \mathbb{Z}\}\) with

\[
B_i = L A_{i+1} \quad \text{for } i \equiv m - 1 (mod \ n), \\
B_i = A_{i-1} \quad \text{for } i \equiv m (mod \ n), \\
B_i = A_i \quad \text{for } i \not\equiv m, m - 1 (mod \ n),
\]
is called a left mutation of the helix at \(A_m\) and is denoted by \(L_m\).

A collection \(\{C_i| i \in \mathbb{Z}\}\) with

\[
C_i = R A_{i-1} \quad \text{for } i \equiv m + 1 (mod \ n), \\
C_i = A_{i+1} \quad \text{for } i \equiv m (mod \ n), \\
C_i = A_i \quad \text{for } i \not\equiv m, m + 1 (mod \ n),
\]
is called a right mutation of the helix at \(A_m\) and is denoted by \(R_m\).

The basic fact about helices is the following.

3.1.1 **Theorem.** A right or left mutation of a helix is again a helix.

All applications of helices are based on this theorem.

If we recall the triangle equations, we see that the mutations of helices define an action of the braid group on the set of all helices. One of the main questions of the helix theory is the question about an orbit number of this action.
3.1 Introduction to the Helix Theory.

Let us return to exceptional sheaves on surfaces and define mutations of exceptional pair of sheaves. (The definition of exceptional pairs and their types can be found in section 2.3.)

**Lemma-definition.** 1. Let \((E, F)\) be an exceptional hom-pair of sheaves. Consider the canonical homomorphisms

\[
\text{Hom}(E, F) \otimes E \xrightarrow{lcan} F \quad \text{and} \quad E \xrightarrow{rcan} \text{Hom}(E, F)^* \otimes F.
\]

If \(lcan\) is an epimorphism then the pair \((E, F)\) is left admissible and

\[L_E F = \ker lcan.\]

Besides, the sheaf \(L_E F\) is exceptional and the pair \((L_E F, E)\) is also exceptional.

The pair \((E, F)\) is right admissible provided \(rcan\) is a monomorphism. In this case,

\[R_F E = \operatorname{coker} rcan.\]

In addition, the sheaf \(R_F E\) and the pair \((F, R_F E)\) are also exceptional.

In both these cases the mutation of the pair \((E, F)\) is called regular.

Suppose \(lcan\) is a monomorphism; then the left shift of \(F\) is defined as \(L_E F = \operatorname{coker} lcan.\) (The pair \((L_E F, E)\) is exceptional as well.)

The right shift of \(E\) is defined as \(R_F E = \ker rcan\) whenever \(rcan\) is an epimorphism. (The pair \((F, R_F E)\) is exceptional in this case also.)

2. The ext-pair \((E, F)\) is both left and right admissible. The following universal extensions define the mutations of the ext-pair.

\[
0 \longrightarrow F \longrightarrow L_E F \longrightarrow \text{Ext}^1(E, F) \otimes E \longrightarrow 0,
\]

\[
0 \longrightarrow \text{Ext}^1(E, F)^* \otimes F \longrightarrow R_F E \longrightarrow E \longrightarrow 0.
\]

In this case, as above, \(L_E F\) and \(R_F E\) are exceptional and \((L_E F, E), (F, R_F E)\) are hom-pairs.

3. Both a left and a right mutation of a zero-pair is permutation of pair terms.

It follows from this lemma that there are cases when left or right mutation of a hom-pair is not defined. Moreover, there are not mutations of a singular pair of sheaves.

For overcoming of these limitations, following \([3]\), let us pass to the bounded derived category of sheaves on the surface \(S = (D^b(S)).\) Exceptional objects and collections in this category are defined in the same way as in the basic category of sheaves.

**Lemma-definition.** Let \((E, F)\) be an exceptional pair in \(D^b(S).\) Objects \(L_E F\) and \(R_F E,\) which complete the canonical morphisms

\[
L_E F \longrightarrow R \text{Hom}(E, F) \otimes E \longrightarrow F \quad \text{and} \quad E \longrightarrow R \text{Hom}(E, F)^* \otimes F \longrightarrow R_F E
\]

up to the distinguished triangles, just as the pairs \((L_E F, E), (F, R_F E)\), are exceptional.

The category of sheaves is imbedded into \(D^b(S)\) by morphism \(\delta.\) Any exceptional sheaf remains exceptional under this imbedding. Mutations in basic and in derived category are
connected in the following way. If an exceptional pair of sheaves \((E, F)\) is left admissible then the left shift of \(\delta(F)\) in the derived category is quasiisomorphic to \(\delta(L_E F)\). That is, it is a complex with a unique nonzero cohomology coincided with \(L_E F\), and vice versa. The similar statement holds true in the case of a right mutation. Thus we can reckon that any exceptional pair of sheaves is both left and right admissible.

3.1.2 Theorem. (Gorodentsev-Orlov.) Any exceptional object of \(D^b(S)\) is quasiisomorphic to an exceptional sheaf provided \(S\) is Del Pezzo surface. That is, all mutations of exceptional pairs of sheaves belong to the basic category.

3.1.3 Theorem. (Rudakov-Gorodentsev.)
1. The above defined mutations of sheaves and exceptional objects of derived category satisfy the axioms \((1L),(2L),(1R)\) and \((2R)\).
2. An exceptional collection remains exceptional whenever some its pair of neighboring sheaves is replaced by a mutation of this pair. This procedure is called a mutation of the collection.

Definition. Let \(\sigma = (E_1, E_2, \ldots, E_n)\) be an exceptional collection of sheaves or objects of \(D^b(S)\). It is full provided \(D^b(S)\) is generated by \(\sigma\), i.e. the set of all objects of \(D^b(S)\) can be obtained from the members of \(\sigma\) by means of direct summing, tensoring and forming cones of all possible morphisms.

For example, the following collection of line bundles on \(\mathbb{P}^2\)
\[(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2))\]
is a full exceptional collection.

3.1.4 Theorem. (Bondal.) Let \(\sigma = (E_1, E_2, \ldots, E_n)\) be an exceptional collection of sheaves or objects of derived category on a manifold \(X\). Then the following statements are true.
1. If \(\sigma\) is full then its left and right mutations are full collections.
2. The collection of the form
\[
\sigma_\infty = \{E_i| i \in \mathbb{Z}, E_{i+sn} = E_i(-sK_X)\}
\]
is a helix of period \(n\) if and only if \(\sigma\) is full.

We see that full collections are closely connected with helices.

For writing the spectral sequence mentioned at the beginning of this section define dual collections.

Let \(\sigma = (E_1, E_2, \ldots, E_k)\) be an exceptional collection. A collection \((^\vee E_{-k}, \ldots, ^\vee E_{-1}, ^\vee E_0)\), where
\[
^\vee E_0 = E_0, ^\vee E_{-1} = L E_1, ^\vee E_{-2} = L(2) E_2, \ldots, ^\vee E_{-k} = L(k) E_k
\]
is called left dual to \(\sigma\). A collection \((E_k^\vee, E_{k-1}^\vee, \ldots, E_0^\vee)\), where
\[
E_0^\vee = R(k) E_0, E_1^\vee = R(k-1) E_1, E_2^\vee = R(k-2) E_2, \ldots, E_k^\vee = E_k
\]
3.1 Introduction to the Helix Theory.

is called right dual to \( \sigma \).

In this notation the following theorem holds true.

3.1.5 Theorem. (Gorodentsev.) Let \( Q \) be an exceptional object belonging to the subcategory generated by an exceptional collection \((E_0, E_1, E_2, \ldots, E_k)\). Then there exists a spectral sequence

\[
E^{p,q} \implies H^{p+q}(Q),
\]

in which the \( E_1 \) term has the form

\[
E_1^{p,q} = \bigoplus_{r+s=q} \text{Hom}^r_{D^b(S)}(E_{k-p}, Q) \otimes H^s(\vee E_{k-p}).
\]

In this case we say that the spectral sequence is associated with the left dual collection.

Similarly, a spectral sequence associated with right dual collection can be written.

3.1.6 Corollary. Let \((E_0, E_1, E_2, \ldots, E_k)\) be an exceptional collection of sheaves on the surface \( S \). Suppose the left dual collection belongs to the basic category, i.e. each member of the left dual collection is a sheaf; then for any sheaf \( Q \) belonging to the category generated by this collection there exists a spectral sequence \( E^{p,q} \), with the \( E_1 \)-term of the form

\[
E_1^{p,q} = \text{Ext}^{q-\Delta_p}(E_{-p}, Q) \otimes \vee E_{-p},
\]

where \( \Delta_p \) is the number of nonregular mutations needed for constructing the sheaf \( \vee E_{-p} \). Besides, there exists a spectral sequence \( E^{p,q} \) with \( E_1 \)-term

\[
E_1^{p,q} = \text{Ext}^{k-q-\Delta_p}(Q, E_{-p})^* \otimes E_\vee_{-p},
\]

where \( \Delta_p \) is the number of nonregular mutations needed for constructing the sheaf \( E_\vee_{-p} \).

Both these sequences converge to \( Q \) on the principal diagonal, i.e. \( E^{p,q}_\infty = 0 \) for \( p + q \neq 0 \) and

\[
Gr(Q) = (E_{-n,n}^0, E_{-n,n}^{-1,1}, \ldots, E_{-n,n}^{-n,n}).
\]

The helix theory has the following problems.

1. Are there full exceptional collections on a given manifold?
2. How many orbits has the braid group under the action on the set of all helices?
We say that all helices (full exceptional collections) are constructible provided the orbits is unique.
3. Does an arbitrary exceptional collection belong to a foundation of a helix? In other words, is there full exceptional collection containing a given exceptional collection? We say that the exceptional sheaves are constructible whenever the solutions of the second and the last problems are positive.
4. We can consider an action of braid group on the set of exceptional collections generated one and the same derived subcategory of \( D^b(X) \). How many orbits has this action?
5. Description of stable subgroups of braid group action.

Full collections were found on $\mathbb{P}^n$, Del Pezzo surfaces, $G(2, 4)$. Besides, the following theorem proved by D. Orlov in [18].

3.1.7 Theorem. (Orlov.) 1. Let $\mathbb{P}(E) \to M$ be a projectivisation of a vector bundle on a manifold $X$. Suppose there is a full exceptional collection on $X$; then there exists such collection on $\mathbb{P}(E)$.

2. Let $\tilde{X}$ be obtained from $X$ by blowing up a smooth regular submanifold $Y$. Suppose there are full exceptional collections on $X$ and on $Y$; then there is a full exceptional collection on $\tilde{X}$ as well.

In the papers [19], [21], [12] it was proved that all exceptional sheaves and all helices on Del Pezzo surfaces are constructible. The constructibility of helices on ruled surfaces with the rational base and on $\mathbb{P}^3$ was proved in ([10]).

In the last part of our paper we shall prove the following theorem.

3.1.8 Theorem. 1. Let $\sigma$ be an exceptional collection of bundles on a smooth projective surfaces $S$ with anticanonical class without base components and $K_S^2 > 0$. Suppose rank of each bundle of this collection is greater then 1; then there is a full exceptional collection $\tau$ such that $\sigma$ is a subcollection of $\tau$. Moreover, $\tau$ can be obtained by mutations from the basic full collection. In other words, all helices on $S$ are constructible.

2. The condition about ranks can be rejected provided $K_S^2 > 1$.

3.2 Restriction of Superrigid Bundles to an Exceptional Curve.

Let us remember that we deal with the surface $S$ with anticanonical class $H = -K_S$ without base components. This means that $H$ is nef.

In the beginning of section 2.4 we limited class of considered surfaces by the condition: $K_S^2 > 0$. Using the description of surfaces with numerically effective anticanonical class from section 2.1, we see that the surfaces satisfying such condition are the following: $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ or surfaces obtained from $\mathbb{P}^2$ by blowing up at most 8 points. Among them only the quadric has not blowing down onto $\mathbb{P}^2$.

The helix theory on the quadric is known ([21], [10]). Further we can assume that the surface $S$ satisfy the following conditions.

1. $K_S^2 > 0$.
2. There is a blowing down of $S$ onto $\mathbb{P}^2$.

Suppose $S$ is obtained from $\mathbb{P}^2$ in the following way

$$S \xrightarrow{\sigma_d} S_{d-1} \xrightarrow{\sigma_{d-1}} \cdots \xrightarrow{\sigma_2} S_1 \xrightarrow{\sigma_1} S_0 = \mathbb{P}^2,$$

where $\sigma_i$ is blow up a point $p_{i-1} \in S_{i-1}$ and $d \leq 8$. By definition, put $e_i = \sigma_i^{-1}(p_{i-1})$.

It is clear that $e_i$ are exceptional $-1$-curves for all $i$ and $e_d$ is irreducible. We see that $e_d$ is a smooth rational curve.
3.2 Restriction of Superrigid Bundles to an Exceptional Curve.

It is known, that the divisors \( h, e_1, \ldots, e_d \) generate the group \( \text{Pic}(S) \) (here \( h \) is the preimage of a line on \( \mathbb{P}^2 \)). Besides,

\[
he_i = e_ie_j = 0 \quad \text{for} \quad i \neq j \quad \text{and} \quad e_i^2 = -1.
\]

\[
K_S = -3h + \sum_{i=1}^{d} e_i.
\]

3.2.1 Remark. The divisor \( h \) is numerically effective.

Proof. In reality, a line on \( \mathbb{P}^2 \) has not base components. Hence, its preimage is base set free as well. Therefore, a cup product \( h \) with any curve on \( S \) is nonnegative.

3.2.2 Lemma. Let \( E \) and \( F \) be exceptional bundles on \( S \) with equal \( \mu_H \)-slopes and let \( e = e_d \) be an irreducible exceptional curve. Suppose \( E = F \) or \( c_1(E) - c_1(F) = C \) is -2-curve; then

- either \( \text{Ext}^2(E, F(-e)) = 0 \)
- or \( K_S^2 = 1 \) and \( (E, F) \) is exceptional pair of the form \( (\mathcal{O}_S(D), \mathcal{O}_S(D + e + K_S)) \),

where \( D \) is some divisor of \( \text{Pic}(S) \).

Proof. By Serre duality theorem,

\[
\text{Ext}^2(E, F(-e))^* \cong \text{Hom}(F, E(e + K_S)).
\]

Suppose \( K_S^2 > 1 \); then

\[
\mu_H(E(e + K_S)) = \mu_H(E) - K_S \cdot e - K_S^2 < \mu_H(E) = \mu_H(F).
\]

Hence the equality \( \text{Hom}(F, E(e + K_S)) = 0 \) follows from \( \mu_H \)-stability of exceptional bundles on \( S \) and \[1.1.5\].

Now suppose \( K_S^2 = 1 \) and \( \text{Hom}(F, E(e + K_S)) \neq 0 \). It follows from the equality \( \mu_H(E) = \mu_H(E(e + K_S)) \) and \[1.1.6\] that there exists the exact triple:

\[
0 \longrightarrow F \longrightarrow E(e + K_S) \longrightarrow Q \longrightarrow 0, \tag{19}
\]

where \( Q \) is a torsion sheaf. Denote by \( r \) the rank of bundles \( E \) and \( F \). (Let us remember that \( r(E) = r(F) \)). We get

\[
c_1(Q) = c_1(E) - c_1(F) + r(e + K_S) = C + r(e + K_S).
\]

Recall that the first Chern class of a torsion sheaf must be ”nonnegative”, i.e. either effective or trivial. Assume that \( E = F \); then \( c_1(Q) = r(e + K_S) \). It is not possible, since this divisor is ineffective.

Assume that \( E \neq F \). Then, by the lemma conditions, \( C = c_1(E) - c_1(F) \) is -2-divisor such that \( C \cdot K_S = 0 \) (recall that \( \mu_H(E) = \mu_H(F) \) and \( r(E) = r(F) \)). Such divisors were described by Yu. I. Manin in \[13\]. Using his results, we can state that if \( C = ah - \sum_{i=1}^{d} b_ie_i \), then \( |a| \leq 3 \). Moreover, \( C = 3h - e_j - \sum_{i=1}^{d} e_i \) whenever \( a = 3 \).
We assume that sequence (19) exists. In this case the divisor \( C + r(e + K_S) \) is effective. Whereby, the cup product \( h \cdot (C + r(e + K_S)) \) is nonnegative (3.2.1). Thus, \( C = 3h - e_j - \sum_{i=1}^{d} e_i \) and \( r = 1 \).

We have \( C + r(e + K_S) = 2e_d - 2e_j \) (recall that \( e = e_d \)). The curve \( e_d \) is irreducible. Hence, \( 2e_d - 2e_j \) is ineffective when \( j \neq d \). Therefore, \( e_d = e_j \) and \( C = -K_S - e \). Thus the pair \( (E, F) \) is equal to

\[
(O_S(D), O_S(D + e + K_S)).
\]

This concludes the proof.

3.2.3 Corollary of the proof. Suppose \( C \) is -2-divisor with \( C \cdot K_S = 0 \) and \( e = e_d \); then the divisor \( C + e + K_S \) is nonpositive.

3.2.4 Lemma. Let \( (E, F) \) be an exceptional pair of bundles on \( S \) with \( \mu_H(E) < \mu_H(F) \) and \( e = e_d \) the irreducible exceptional curve. Then

\[
\text{Ext}^2(E, F(-e)) = 0.
\]

Proof. By Serre duality theorem we have

\[
\text{Ext}^2(E, F(-e))^* \cong \text{Hom}(F, E(e + K_S)).
\]

But,

\[
\mu_H(E(e + K_S)) = \mu_H(E) + 1 - K_S^2 \leq \mu_H(E) < \mu_H(F)
\]

and the proof follows from \( \mu_H \)-stability of exceptional bundles on \( S \) and lemma 1.1.5.

3.2.5 Lemma. Suppose \( E \) and \( F \) are rigid \( \mu_H \)-semistable bundles on \( S \). Assume that they have exceptional filtrations

\[
Gr(E) = (x_nE_n, x_{n-1}E_{n-1}, \ldots, x_1E_1), \quad Gr(F) = (y_mF_m, y_{m-1}F_{m-1}, \ldots, y_1F_1).
\]

In addition we assume that the following conditions hold.

1. \( \text{Ext}^k(F, E) = 0 \) \( \forall k = 0, 1, 2 \).
2. \( \mu_H(E) < \mu_H(F) < \mu_H(E) + K_S^2 \).
3. Provided \( K_S^2 = 1 \) the exceptional collections \((E_1, E_2, \ldots, E_n), (F_1, F_2, \ldots, F_m)\) have not pairs of the form \((O_S(D), O_S(D + e + K_S))\), where \( D \in \text{Pic}(S) \), and \( e = e_d \) is irreducible exceptional curve.

Then the restrictions of \( E \) and \( F \) to \( e \) have the form

\[
E' = E|_e = \alpha O_e(s - 1) \oplus \beta O_e(s), \quad F' = F|_e = \gamma O_e(s - 1) \oplus \delta O_e(s) \oplus \epsilon O_e(s + 1),
\]

where \( \alpha, \beta, \gamma, \delta, \epsilon \) are nonnegative integer numbers with \( \gamma \epsilon = 0 \).

Proof. It follows from the lemma conditions and 2.4.1 that all pairs \((E_i, E_j)\) and \((F_i, F_j)\) for \( i < j \) are exceptional singular or zero-pairs. By proposition 2.3.2 we have

\[
\mu_H(E_i) = \mu_H(E_j), \quad \mu_H(F_i) = \mu_H(F_j)
\]
3.2 Restriction of Superrigid Bundles to an Exceptional Curve.

and remainders of the first Chern classes

\[ c_1(E_j) - c_1(E_i) \text{ and } c_1(F_j) - c_1(F_i) \]

are \(-2\)-divisors. Since among these pairs there are not the pairs of the kind

\[ (\mathcal{O}_S(D), \mathcal{O}_S(D + e + K_S)) \]

(in the case \(K_S^2 = 1\)), it follows from lemma 3.2.2 that

\[ \text{Ext}^2(E_i, E_j(-e)) = \text{Ext}^2(E_i, E_i(-e)) = \text{Ext}^2(F_i, F_i(-e)) = \text{Ext}^2(F_i, F_j(-e)) = 0 \]

for any pair \(i, j\). Thus the equalities

\[ \text{Ext}^2(E, E(-e)) = \text{Ext}^2(F, F(-e)) = 0 \]

follow from 3.2.2

Since the bundles \(E\) and \(F\) are rigid, using the exact triples

\[ 0 \rightarrow E^{*} \otimes E(-e) \rightarrow E^{*} \otimes E \rightarrow (E^{*} \otimes E)|_e \rightarrow 0, \]

\[ 0 \rightarrow F^{*} \otimes F(-e) \rightarrow F^{*} \otimes F \rightarrow (F^{*} \otimes F)|_e \rightarrow 0 \]

we get, \(\text{Ext}^1(E', E') = \text{Ext}^1(F', F') = 0\).

By Grothendieck theorem [17], any bundle on a projective line (in particular, \(E'\) and \(F'\) on \(e\)) is a direct sum of line bundles. From rigidity of \(E'\) and \(F'\) and Bott formula, which calculates cohomologies of line bundles on the projective line ([4]) we obtain

\[ E' = E|_e = \alpha \mathcal{O}_e(s - 1) \oplus \beta \mathcal{O}_e(s), \quad F' = F|_e = \gamma \mathcal{O}_e(s' - 1) \oplus \delta \mathcal{O}_e(s'). \]

Using the first and the second conditions of the lemma, let us show that

\[ s \leq s' \leq s + 1. \]

Note that from condition 1 and proposition 2.3.1 it follows that each pair \((E_i, F_j)\) is exceptional. Besides,

\[ \mu_H(E_i) < \mu_H(F_j) < \mu_H(E_i) + K_S^2. \]

Applying lemma 3.2.4 to the pairs \((E_i, F_j)\), we get \(\text{Ext}^2(E_i, F_j(-e)) = 0\). This means that \(\text{Ext}^2(E, F(-e)) = 0\).

By virtue of the inequalities on \(\mu_H\)-slopes and 2.3.1, the pairs \((E_i, F_j)\) have the type \(\text{hom}\). In particular, \(\text{Ext}^4(E_i, F_j) = 0\). Hence we have \(\text{Ext}^4(E, F) = 0\). Now it follows from a long exact cohomology sequence associated with the restriction sequence to the exceptional curve

\[ 0 \rightarrow E^{*} \otimes F(-e) \rightarrow E^{*} \otimes F \rightarrow (E^{*} \otimes F)|_e \rightarrow 0 \]

that \(\text{Ext}^4(E', F') = 0\).

By Serre duality and the first lemma condition we get

\[ \text{Ext}^k(E, F(K_S)) = 0 \quad \text{for} \quad k = 0, 1, 2. \]
The second lemma condition yields the inequality
\[ \mu_H(F(K_S)) < \mu_H(E) < \mu_H(F(K_S)) + K_S^2. \]
Repeating the reasoning for rigid bundles \( F(K_S) \) and \( E \), we see that \( \text{Ext}^1(F(K_S)|_e, E') = 0 \).

Note that \( F(K_S)|_e = F'(-1) \), i.e. \( \text{Ext}^1(F'(-1), E') = 0 \). Now, the inequality \((s \leq s' \leq s + 1)\) follows from Bott formula. This completes the proof.

3.2.6 Corollary. Assume that an ordered collection of \( \mu_H \)-semistable rigid bundles \((E_1, E_2, \ldots, E_m)\) satisfies the following conditions.

1. \( \text{Ext}^k(E_i, E_j) = 0 \) for \( j > i, k = 0, 1, 2 \).
2. \( \mu_H(E_1) < \mu_H(E_2) < \cdots < \mu_H(E_m) < \mu_H(E_1) + K_S^2 \).
3. Provided \( K_S^2 = 1 \) the exceptional collections corresponding to the exceptional filtrations of all \( E_i \) have not pairs of the form \((\mathcal{O}_S(D), \mathcal{O}_S(D + e + K_S))\), where \( e = e_d \) is the irreducible exceptional curve.

Then there is a number \( i \) such that
\[ (E_i \oplus \cdots \oplus E_m \oplus E_1(-K_S) \oplus \cdots \oplus E_{i-1}(-K_S))_{|_e} = \alpha \mathcal{O}_e(s) \oplus \beta \mathcal{O}_e(s+1). \]

**Proof.** We shall say that an ordered pair of rigid \( \mu_H \)-semistable bundles \((E, F)\) on \( S \) has a zero type of decomposition whenever
\[ (E \oplus F)_{|_e} = \alpha \mathcal{O}_e(s) \oplus \beta \mathcal{O}_e(s+1). \]
It has a first type of decomposition whenever
\[ E_{|_e} = \alpha \mathcal{O}_e(s) \oplus \beta \mathcal{O}_e(s+1), \quad F_{|_e} = \gamma \mathcal{O}_e(s+1) \oplus \delta \mathcal{O}_e(s+2), \]
with \( \alpha \cdot \delta \neq 0. \)

From the previous lemma it follows that each pair from our collection has either the zero or the first type of decomposition.

We see that the lemma statement holds true provided the pair \((E_1, E_i)\) has the zero type of decomposition for all \( i \).

In the converse case, denote by \( i \) the minimal number such that the pair \((E_1, E_i)\) has the first type of decomposition. Note that, in this case, \( \forall j < i \leq k \) the pair \((E_j, E_k)\) has the first type of decomposition and the pair \((E_s, E_l)\) has the zero type of decomposition whenever \( i \leq s < l \) or \( s < l \leq i \). Besides, if a pair \((E, F)\) has the first type of decomposition then the pair \((F, E(-K_S))\) has the zero type of decomposition.

Thus, each pair of the collection
\[ (E_i \oplus \cdots \oplus E_m \oplus E_1(-K_S) \oplus \cdots \oplus E_{i-1}(-K_S)) \]
has the zero type of a decomposition. This completes the proof.
3.3 Equivalence of Collections and the Key Exact Sequence.

Definition. We shall say that an exceptional collection \( \sigma = (E_1, E_2, \ldots, E_k) \) (of sheaves or of objects in \( D^b(S) \)) on \( S \) is constructible whenever there is a full exceptional collection \((E_1, \ldots, E_k, E_{k+1}, \ldots, E_n)\) containing \( \sigma \) such that it is obtained from the basic collection

\[
\sigma_0 = (\mathcal{O}_S, \mathcal{O}_S(h), \mathcal{O}_S(2h), \mathcal{O}_{e_1}(-1), \ldots, \mathcal{O}_{e_d}(-1))
\]

by mutations. Here \( h \) is the preimage of a line on \( \mathbb{P}^2 \) and \( e_i \) are the blow up divisors. (It follows from [18] that the basic collection is exceptional and full.)

We say that an exceptional collection \( \sigma \) is equivalent to an exceptional collection \( \tau \) whenever the following condition holds true. The collection \( \sigma \) is constructible if and only if \( \tau \) is constructible.

3.3.1 Lemma. a) Suppose an exceptional collection \( \sigma \) is obtained from an exceptional collection \( \tau \) by mutations; then these collections are equivalent.

b) An exceptional collection \((E_1, E_2, \ldots, E_k)\) is equivalent to the following collections:

\[
(E_k(K_S), E_1, \ldots, E_{k-1}) \quad \text{and} \quad (E_2, \ldots, E_k, E_1(-K_S)).
\]

Proof. a) Assume that an exceptional collection \( \sigma = (E_1, E_2, \ldots, E_k) \) is obtained from \( \tau = (F_1, F_2, \ldots, F_k) \) by mutations. Since all mutations of collections are invertible (see the axioms of the helix theory), we can assume that \( \tau \) is also obtained from \( \sigma \) by mutations. Suppose \( \sigma \) is constructible, i.e. there is a full exceptional collection \( \sigma' = (E_1, \ldots, E_k, E_{k+1}, \ldots, E_n) \) obtained from the basic collection by mutations. Then the exceptional collection \( \tau' = (F_1, \ldots, F_k, E_{k+1}, \ldots, E_n) \) is also obtained from the basic collection by mutations. Therefore, \( \tau' \) is full (3.1.4). Besides, the basic collection and \( \tau' \) belong to one and the same orbit of the braid group action. Thus \( \tau \) is also constructible.

b) For proving the second lemma statement it is sufficient to check that the collections \( \sigma = (E_1, E_2, \ldots, E_k) \) and \((E_2, \ldots, E_k, E_1(-K_S))\) are equivalent. Suppose \( \sigma \) is constructible and \( \sigma_1 = (E_1, \ldots, E_k, E_{k+1}, \ldots, E_n) \) is a full exceptional collection corresponding to it. By theorem 3.1.4, if follows from full of the collection \( \sigma_1 \) that it is a foundation of a helix and \( E_1(-K_S) = R^{n-1} E_1 \). That is, the collection

\[
\sigma_2 = (E_2, \ldots, E_k, E_{k+1}, \ldots, E_n, E_1(-K_S))
\]

is equivalent to \( \sigma_1 \). Now we shift each of the sheaves \( E_n, E_{n-1}, \ldots, E_{k+1} \) to the right over \( E_1(-K_S) \) to obtain the full collection

\[
\tau_1 = (E_2, \ldots, E_k, E_1(-K_S), RE_{k+1}, RE_{k+2}, \ldots, RE_n),
\]

equivalent to \( \sigma_2 \). Thus \( \tau \) is constructible as well.

Since all operations are invertible, we see that the collection \( \sigma \) is equivalent to \( \tau \).

Notation. Let \( \sigma = (E_1, E_2, \ldots, E_k) \) be an exceptional collection of bundles. By definition, put

\[
\mu_-(\sigma) = \min \{\mu_H(E_i)\}, \quad \mu_+(\sigma) = \max \{\mu_H(E_i)\}.
\]
3.3.2 Lemma. For any exceptional collection of bundles $\sigma = (E_1, E_2, \ldots, E_k)$ there exists an exceptional collection of bundles $\tau = (F_1, F_2, \ldots, F_k)$ equivalent to $\sigma$ such that
\[
\mu_-(\sigma) \leq \mu_-(\tau) = \mu_H(F_1) \leq \cdots \leq \mu_H(F_k) = \mu_+ (\tau) \leq \mu_+ (\sigma).
\]

Further we shall say that the exceptional collection of bundles $(F_1, F_2, \ldots, F_n)$ is hom-collection whenever
\[
\mu_H(F_1) \leq \mu_H(F_2) \leq \cdots \leq \mu_H(F_n).
\]

Proof. Let $s$ be the minimal number such that $\mu_H(E_s) > \mu_H(E_{s+1})$. It follows from proposition 2.3.1 that the exceptional pair $(E_s, E_{s+1})$ has the type $ext$. Consider the left mutation of this pair
\[
0 \longrightarrow E_{s+1} \longrightarrow L_{E_s} E_{s+1} \longrightarrow E_s \otimes \text{Ext}^1 (E_s, E_{s+1}) \longrightarrow 0.
\]

Since the sheaves $E_s$ and $E_{s+1}$ are locally free, we see that $L_{E_s} E_{s+1}$ is locally free as well. In view of $\mu_H$-stability of exceptional bundles, we have
\[
\mu_H(E_{s+1}) < \mu_H(L_{E_s} E_{s+1}) < \mu_H(E_s).
\]

Now suppose that $\mu_H(L_{E_s} E_{s+1}) < \mu_H(E_{s-1})$; then we do the left mutation of this $ext$-pair, et cetera...

It is clear that after a finite number of the mutations we shall obtain an exceptional collection $\sigma'$ equivalent to the original one such that
\[
\mu_-(\sigma) \leq \mu_-(\sigma') < \mu_+(\sigma') \leq \mu_+(\sigma).
\]

Moreover, if we denote by $s'$ the minimal number such that
\[
\mu_H(E_{s'}) > \mu_H(E_{s'+1})
\]
then $s' > s$. This implies the lemma statement.

3.3.3 Lemma. For any exceptional collection of bundles $\sigma = (E_1, E_2, \ldots, E_k)$ there exists an exceptional hom-collection of bundles $\tau = (F_1, F_2, \ldots, F_k)$ equivalent to $\sigma$ such that
\[
\mu_+(\tau) - \mu_-(\tau) < K^2_S.
\]

Proof. By definition, put $\Delta \mu (\sigma) = \mu_+(\sigma) - \mu_-(\sigma)$. Assume that $\Delta \mu (\sigma) > K^2_S$. By lemma 3.3.2 without loss of generality it can be assumed that $\sigma$ is a hom-collection. We have
\[
\mu_-(\sigma) = \mu_H(E_1), \quad \mu_+(\sigma) = \mu_H(E_n).
\]
Since $\Delta \mu (\sigma) > K^2_S$ and $\mu_H(E_1(-K_S)) = \mu_H(E_1) + K^2_S$, we see that there is a number $s$ such that
\[
\mu_H(E_{s-1}) \leq \mu_H(E_1(-K_S)) < \mu_H(E_s).
\]
The collection $\sigma_1 = (E_s, \ldots, E_n, E_1(-K_S), \ldots, E_{s-1}(-K_S))$ is equivalent to $\sigma$ and it has the following $\mu_H$-slope limits:
\[
\mu_-(\sigma_1) = \mu_H(E_1(-K_S)) = \mu_H(E_1) + K^2_S,
\]
3.3 Equivalence of Collections and the Key Exact Sequence.

\[ \mu_+(\sigma_1) = \max\{\mu_H(E_{s-1}(-K_S)), \mu_H(E_s)\}. \]

Suppose \( \mu_+(\sigma_1) = \mu_H(E_{s-1}(-K_S)) \); then \( s > 1 \) and

\[ \Delta \mu_+(\sigma_1) = \mu_H(E_{s-1}) - \mu_H(E_1) \leq K_S^2. \]

Hence, ordering the collection \( \sigma_1 \) as in [3.3.2], we obtain the hom-collection \( \sigma_2 \) equivalent to the original one such that \( \Delta \mu(\sigma_2) \leq K_S^2 \).

Suppose \( \mu_+(\sigma_1) = \mu_H(E_n) = \mu_+(\sigma) \); then ordering the collection \( \sigma_1 \) by \( \mu_H \)-slopes, we obtain the hom-collection \( \sigma_3 \) equivalent to the original one such that \( \Delta \mu(\sigma_3) \leq \Delta \mu(\sigma) - K_S^2 \).

Doing this operation for several times we all the same obtain a hom-collection equivalent to \( \sigma \) with \( \Delta \mu \leq K_S^2 \).

Now let us assume that \( \Delta \mu(\sigma) = K_S^2 \). Denote by \( s \) the minimal number such that \( \mu_H(E_s) < \mu_H(E_{s+1}) \). Consider the equivalent collection

\[ \tau = (E_{s+1}, \ldots, E_n, E_1(-K_S), \ldots, E_s(-K_S)). \]

By the choice of \( s \) we have

\[ \mu_+(\tau) = \mu_H(E_n) = \mu_H(E_1(-K_S)) = \ldots = \mu_H(E_s(-K_S)) = \mu_+(\sigma), \]

and \( \mu_-(\tau) = \mu_H(E_{s+1}) > \mu_-(\sigma) \).

In other words, \( \tau \) is the hom-collection with \( \Delta \mu(\tau) < K_S^2 \). This completes the proof.

3.3.4 Lemma. Let \( \sigma = (E_1, E_2, \ldots, E_k) \) be an exceptional collection of bundles on the surface \( S \) with \( K_S^2 \geq 1 \). In addition assume that in the case \( K_S^2 = 1 \) this collection has not pairs of the form \((\mathcal{O}_S(D), \mathcal{O}_S(D + e + K_S))\), where \( D \) is some divisor and \( e = e_d \) is the exceptional rational curve.

Then there is an exceptional hom-collection \((F_1, F_2, \ldots, F_k)\) equivalent to \( \sigma \) such that the associated with it superrigid bundle \( F \left( Gr(F) = (x_n F_n, \ldots, x_1 F_1) \right) \) is included in the exact sequence

\[ 0 \rightarrow G \rightarrow F \rightarrow \text{Hom}(F, \mathcal{O}_e(-1))^* \otimes \mathcal{O}_e(-1) \rightarrow 0, \quad (20) \]

where \( G \) is a superrigid bundle with

\[ \text{Ext}^k(G, \mathcal{O}_e(-1)) = 0 \quad \forall k. \]

Proof. By lemma [3.3.3] there is a hom-collection \( \tau = (E_1', E_2', \ldots, E_k') \) equivalent to the original one such that \( \mu_+(\tau) - \mu_-(\tau) < K_S^2 \).

We shall separate this collection into groups of bundles with equal \( \mu_H \)-slopes. We shall construct from these groups superrigid \( \mu_H \)-semistable bundles \( \mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_m \) (see theorem [2.5.1]).

We have \( \text{Ext}^k(\mathcal{E}_j, \mathcal{E}_i) = 0 \) for \( j > i, \quad k = 0, 1, 2 \); and

\[ \mu_H(\mathcal{E}_1) < \mu_H(\mathcal{E}_2) < \cdots < \mu_H(\mathcal{E}_m) < \mu_H(\mathcal{E}_1) + K_S^2. \]

In view of corollary [3.2.6] there is a number \( i \) such that

\[ (\mathcal{E}_i \oplus \cdots \oplus \mathcal{E}_m \oplus \mathcal{E}_1(-K_S) \oplus \cdots \mathcal{E}_{i-1}(-K_S))|_e = \alpha \mathcal{O}_e(d-1) \oplus \beta \mathcal{O}_e(d). \]
Hence there is a number \( j \) such that the superrigid bundle \( \bar{F} \) associated with the \( \text{hom} \)-collection

\[
\tau' = (E'_j, \ldots, E'_{k}, E'_1(-K_S), \ldots, E'_{j-1}(-K_S))
\]
satisfy the condition

\[
\bar{F}|_e = \alpha \mathcal{O}_e(d - 1) \oplus \beta \mathcal{O}_e(d).
\]

Thus the superrigid bundle \( F \) constructed from the exceptional \( \text{hom} \)-collection

\[
\tau'' = \tau'(dK_S) = (E'_j(dK_S), \ldots, E'_{k}(dK_S), E'_1((d - 1)K_S), \ldots, E'_{j-1}((d - 1)K_S))
\]
restricts to the curve \( e \) in the following way:

\[
F|_e = \alpha \mathcal{O}_e(-1) \oplus \beta \mathcal{O}_e.
\]

It is easily shown that the collection \( \tau'' \) is equivalent to the original one \( \sigma \).

The following equalities can be obtained by direct calculations.

\[
h^i(F, \mathcal{O}_e(-1)) = \begin{cases} 
\alpha & \text{for } i = 0 \\
0 & \text{for } i > 0
\end{cases},
\]

\[
h^i(\mathcal{O}_e(-1), F) = \begin{cases} 
\beta & \text{for } i = 1 \\
0 & \text{for } i \neq 1
\end{cases}.
\]

Consider the canonical map:

\[
F \to \text{Hom}(F, \mathcal{O}_e(-1))^* \otimes \mathcal{O}_e(-1).
\]

Since the restriction of this map to the curve \( e \) is an epimorphism, we see that exact sequence (20) holds true.

The sheaf \( G \) from this sequence, as a subsheaf of a bundle, has not torsion. For calculating its cohomologies let us consider cohomology tables corresponding to exact sequence (20). For this, denote by \( L \) the torsion sheaf \( \mathcal{O}_e(-1) \).

| \( k \) | \( \text{Hom}(F, \mathcal{L}) \otimes \text{Ext}^k(\mathcal{L}, \mathcal{L}) \rightarrow \text{Ext}^k(F, \mathcal{L}) \rightarrow \text{Ext}^k(G, \mathcal{L}) \) |
|---|---|---|
| 0 | \( \text{Hom}(F, \mathcal{L}) \otimes \mathbb{C} \) | \( \text{Hom}(F, \mathcal{L}) \) | ? |
| 1 | 0 | 0 | ? |
| 2 | 0 | 0 | ? |

| \( k \) | \( \text{Hom}(F, \mathcal{L}) \otimes \text{Ext}^k(\mathcal{L}, F) \rightarrow \text{Ext}^k(F, F) \rightarrow \text{Ext}^k(G, F) \) |
|---|---|---|
| 0 | 0 | \( * \) | ? |
| 1 | \( * \) | 0 | ? |
| 2 | 0 | 0 | ? |

| \( k \) | \( \text{Ext}^k(G, G) \rightarrow \text{Ext}^k(G, F) \rightarrow \text{Hom}(F, \mathcal{L})^* \otimes \text{Ext}^k(G, \mathcal{L}) \) |
|---|---|---|
| 0 | ? | \( * \) | 0 |
| 1 | ? | 0 | 0 |
| 2 | ? | 0 | 0 |

This concludes the lemma proof.

I want to remark that the idea of the construction of exact sequence (20) on Del Pezzo surface with an exceptional bundle as \( F \) pertains to D. O. Orlov ([12]).
3.4 Category Generated by a Pair.

In the previous section we constructed from an exceptional collection $\sigma$ of bundles the hom-collection $\tau = (F_1, F_2, \ldots, F_k)$ equivalent to $\sigma$ such that the superrigid bundle associated with $\tau$ is included in exact sequence (20). In the next section using double induction, we shall show that from this sequence the constructibility of the collection $\tau$ follows.

Here we check a base of one of the inductions. Namely we prove the following proposition.

3.4.1 Proposition. Suppose a superrigid sheaf $F$ on the surface $S$ is included in the exact sequence

$$0 \rightarrow y_1G_1 \rightarrow F \rightarrow y_0G_0 \rightarrow 0,$$

where $(G_0, G_1)$ is an exceptional pair, $y_i$ are positive integer and $G_1$ is a bundle; then

1. Assume that $G_0$ is locally free; then $F$ has a unique (to within permutations of quotients) exceptional filtration

$$Gr(F) = (x_1F_1, x_0F_0) \quad (x_i \geq 0).$$

Moreover,

(a) the pair $(F_0, F_1)$ is obtained from the pair $(G_0, G_1)$ by mutations,

(b) $r(F_0) + r(F_1) \geq r(G_0) + r(G_1)$ whenever $x_0 \cdot x_1 \neq 0$,

(c) $r(F_0) \geq r(G_0) + r(G_1)$ provided $x_1 = 0$,

(d) the equality of the sums of ranks is achieved if and only if $F_i = G_i$ for $i = 1, 2$.

2. Assume that $G_0 = O_e(-1)$ for the exceptional rational curve $e = e_d$; then

$$F = x_1F_1 \oplus x_0F_0 \quad (x_i \geq 0).$$

Moreover,

(a) the exceptional pair $(F_0, F_1)$ is obtained from the pair $(G_0, G_1)$ by mutations,

(b) $r(F_0) + r(F_1) \geq r(G_0) + r(G_1)$, whenever $x_0 \cdot x_1 \neq 0$,

(c) $r(F_0) \geq r(G_0) + r(G_1)$ provided $x_1 = 0$,

(d) $F_0$ is locally free.

To prove this proposition, we need several lemmas.

3.4.2 Lemma. let $A$ and $B$ be sheaves on a manifold $X$ and let $\varphi : V \otimes A \rightarrow W \otimes B$ be a morphism of sheaves. Then

1. The canonical map $\text{lcan} : \text{Hom}(A, B) \otimes A \rightarrow B$ is an epimorphism provided $\varphi$ is also an epimorphism.

2. The canonical map $\text{rcan} : A \rightarrow \text{Hom}(A, B)^* \otimes B$ is a monomorphism provided $\varphi$ is a monomorphism as well.
3 CONSTRUCTIBILITY OF EXCEPTIONAL BUNDLES.

PROOF. In view of symmetry of statements it is sufficient to check the first of them. At first consider the case of the one-dimensional space $W$, i.e.

$$\varphi : V \otimes A \rightarrow B \rightarrow 0.$$  

Recall that the canonical map $lcan$ is determined by the element of $\text{Hom}(A, B)^* \otimes \text{Hom}(A, B)$ corresponding to the identical morphism $\text{Hom}(A, B) \rightarrow \text{Hom}(A, B)$. Denote by $lcan$ this element as well. Let us define a line map $\psi : V \rightarrow \text{Hom}(A, B)$ so that

$$\psi^* \otimes \text{id}_{\text{Hom}(A, B)} : lcan \mapsto \varphi.$$  

Thus the following commutative diagram is arisen:

$$\begin{array}{ccc}
\text{Hom}(A, B) \otimes A & \xrightarrow{lcan} & B \\
\psi \otimes \text{id}_A \uparrow & & \varphi \downarrow \\
V \otimes A & \xrightarrow{id_B} & B \rightarrow 0
\end{array}$$

From this diagram it follows that $lcan$ is an epimorphism.

Now suppose,

$$\varphi : V \otimes A \rightarrow W \otimes B \rightarrow 0.$$  

Then

$$\text{Hom}(A, W \otimes B) \otimes A \xrightarrow{lcan} W \otimes B \rightarrow 0.$$  

We see that there is a commutative diagram

$$\begin{array}{ccc}
0 & 0 \\
\uparrow & \uparrow \\
\text{Hom}(A, B) \otimes A & \xrightarrow{lcan} & B \rightarrow 0 \\
\pi \circ \text{id}_A \uparrow & & \pi \uparrow \\
\text{Hom}(A, W \otimes B) \otimes A & \xrightarrow{lcan} & W \otimes B \rightarrow 0
\end{array}$$  

where $\pi$ is a projection $W \otimes B \rightarrow B \rightarrow 0$.

3.4.3 Lemma. Let $F$ be a rigid sheaf and $(A, B)$ an exceptional hom-pair of sheaves on the surface $S$. Then the following statements hold.

1. If the sequence

$$0 \rightarrow F \rightarrow xA \rightarrow yB \rightarrow 0 \quad (21)$$

is exact for positive integer $x$ and $y$ then

(a) the left mutation of pair $(A, B)$ belongs to the basic category and it is regular;  
(b) either $F = wA \oplus zL_A B$ or there exists an exact sequence

$$0 \rightarrow F \rightarrow zL_A B \rightarrow wA \rightarrow 0$$

for some nonnegative integer $z$ and $w$.  

2. If the sequence
\[ 0 \rightarrow xA \rightarrow yB \rightarrow F \rightarrow 0 \]
(22)
is exact for positive integer \( x \) and \( y \) then

(a) the right mutation of the pair \((A, B)\) belongs to the basic category and it is regular;

(b) either \( F = wB \oplus zR_B A \) or there exists an exact sequence
\[ 0 \rightarrow wB \rightarrow zR_B A \rightarrow F \rightarrow 0 \]
for some nonnegative integer \( z \) and \( w \).

**Proof.** In view of duality of the lemma statements it is sufficient to prove the first of them.

The regularity of the left mutation of the \( \text{hom} \)-pair \((A, B)\) follows from its definition, sequence (21) and lemma 3.4.2. Note that in this case the pair \((L_AB, A)\) has also the type \( \text{hom} \).

Sequence (21) yields that the sheaf \( F \) belongs to category generated by the pair \((A, B)\). Therefore there exists a spectral sequence \( E^{p,q} \) (3.1.6) convergent to \( F \) on the principal diagonal. Its \( E_1 \) term has the form:

\[
\begin{align*}
E_1^{-1,1} &= \text{Ext}^1(B, F) \otimes L_AB \\
E_1^{-1,0} &= \text{Ext}^0(B, F) \otimes L_AB
\end{align*}
\]

\[
\begin{align*}
d &\rightarrow E_1^{0,1} = \text{Ext}^1(A, F) \otimes A \\
d &\rightarrow E_1^{0,0} = \text{Ext}^0(A, F) \otimes A
\end{align*}
\]

It follows from exact sequence (21) and the fact that the pair \((A, B)\) is exceptional that the group \( \text{Ext}^0(B, F) \) is trivial. Hence the spectral sequence degenerates into two exact triples:

\[ 0 \rightarrow C \rightarrow \text{Ext}^1(B, F) \otimes L_AB \rightarrow \text{Ext}^1(A, F) \otimes A \rightarrow 0, \]

\[ 0 \rightarrow \text{Hom}(A, F) \otimes A \rightarrow F \rightarrow C \rightarrow 0. \]

Assume that \( \text{Hom}(A, F) \neq 0 \). Consider the cohomology table corresponding to the first of these triples:

| \( k \) | \( V \otimes \text{Ext}^k(A, A) \rightarrow W \otimes \text{Ext}^k(L_AB, A) \rightarrow \text{Ext}^k(C, A) \) |
|---|---|---|
| 0 | * | * | ? |
| 1 | 0 | 0 | ? |
| 2 | 0 | 0 | ? |

where \( \text{Ext}^1(A, F)^* = V \), \( \text{Ext}^1(B, F)^* = W \). The first and the second columns are filled with using the properties of the pair \((L_AB, A)\).

From the table the equality \( \text{Ext}^1(C, A) = 0 \) follows. This means that

\[ F = C \oplus \text{Hom}(A, F) \otimes A. \]

Since \( F \) is rigid, we get \( \text{Ext}^1(A, C) = 0 \) and \( \text{Ext}^1(A, A) = 0 \). Hence, \( \text{Ext}^1(A, F) = 0 \). Thus,

\[ C = \text{Ext}^1(B, F) \otimes L_AB \]
and $F$ is a direct sum of multiplicities of the sheaves $A$ and $L_AB$.

Assume that $\text{Hom}(A, F) = 0$; then the spectral sequence degenerates into the exact triple

$$0 \rightarrow F \rightarrow \text{Ext}^1(B, F) \otimes L_AB \rightarrow \text{Ext}^1(A, F) \otimes A \rightarrow 0.$$  

This concludes the lemma proof.

3.4.4 Lemma. Let $(E_0, E_1)$ be an exceptional $\text{ext}$-pair of sheaves on a manifold $X$ with $\chi(E_0, E_1) < -1$. In addition, assume that for each natural $n$ the following sheaves are determined:

$$E_{n+1} = R_{E_n}E_{n-1}, \quad E_{-(n+1)} = L_{E_{-n}}E_{1-n}.$$  

Suppose that for a given sheaf $F$ and for any positive integer $n$ there are natural numbers $x_n, y_n, z_n, w_n$ such that the following exact sequences take place:

$$0 \rightarrow F \rightarrow x_nE_{-(n+1)} \rightarrow y_nE_{-n} \rightarrow 0,$$

$$0 \rightarrow z_nE_n \rightarrow w_nE_{n+1} \rightarrow F \rightarrow 0$$

Then the Euler characteristic $\chi(F, F)$ is nonpositive.

Proof. Denote by $e_n$ the images of $E_n$ in $K_0(X)$. The modulus $K_0(X)$ inherits the bilinear form $\chi(\cdot, \cdot)$. Denote it by $(\cdot, \cdot)$.

Using the lemma conditions we have

$$(e_0, e_0) = (e_1, e_1) = 1, \quad (e_1, e_0) = 0, \quad (e_0, e_1) = -h < -1.$$  

By the definition of mutations of an $\text{ext}$-pair, we get

$$e_{-1} = e_1 + he_0, \quad e_2 = he_1 + e_0.$$  

It follows from the exact sequences and the lemma assumptions that all pairs $(E_n, E_{n+1})$ for $n \in \mathbb{Z}^+$ have the type $\text{hom}$ and both mutations of these pairs (except for the left mutation of $(E_1, E_2)$ and the right one of $(E_{-1}, E_0)$) are regular (3.4.2).

The following formulae are easily obtained from the definition of mutations of $\text{ext}$ and $\text{hom}$-pairs.

$$e_{-n} = he_{1-n} - e_{2-n} \quad (n > 1),$$

$$e_n = he_{n-1} - e_{n-2} \quad (n > 2),$$

$$\forall n \in \mathbb{Z} : \quad (e_n, e_n) = 1, \quad (e_{n+1}, e_n) = 0$$

and for $n \neq 0$ $$(e_n, e_{n+1}) = h.$$  

Denote by $x_n$ and $x_{n-1}$ coordinates of vector $e_n$ $(n > 0)$ with respect to the basis $\{e_1, e_0\}$: $e_n = x_ne_1 + x_{n-1}e_0$. The recurrence relations

$$x_0 = 0, \quad x_1 = 1, \quad x_{n+1} = hx_n - x_{n-1}$$

are proved by induction on $n$.  


3.4 Category Generated by a Pair.

Note that the vectors \( e_{-n} \) \((n > 0)\) are expressed by means of the same numbers:

\[ e_{-n} = x_{n-1}e_1 + x_n e_0. \]

Let \( V \) be a 2-dimensional vector space over \( \mathbb{Q} \) generated by \( e_0, e_1 \). Let us choose an affine map \( U \) in \( \mathbb{P}(V) \) containing the image of \( e_0 \) as origin.

\[ xe_1 + ye_0 \sim \frac{x}{y} e_1 + e_0. \]

We preserve the notations for the images of \( e_n \) on \( U \). Let us calculate coordinates \( l_+ \) and \( l_- \) of limit points \( e_{+\infty} = \lim_{n \to \infty} e_n, \quad e_{-\infty} = \lim_{n \to \infty} e_{-n} \) on \( U \).

\[ l_+ = \lim_{n \to \infty} \frac{x_n}{x_{n-1}} = h - \lim_{n \to \infty} \frac{x_{n-2}}{x_{n-1}} = h - l_- = h - 1/l_. \]

Hence, \( l_+ \) and \( l_- \) are roots of the equation \( l^2 - hl + 1 = 0 \). That is,

\[ l_\pm = \frac{h \pm \sqrt{h^2 - 4}}{2} \]

(by assumption, \( h \geq 2 \)). Taking into account the exact triples from the lemma condition, we see that the point \( f \) on \( U \) corresponding the sheaf \( F \) has a coordinate \( x \in [l_-, l_+] \).

On the other hand, a sign of \( \chi(F,F) \) is determined by a sign of \( (e_0 + xe_1)^2 = x^2 - hx + 1 \). Now the lemma proof follows from the inequality

\[ x^2 - hx + 1 \leq 0 \quad \text{for} \quad x \in [l_-, l_+]. \]

3.4.5 Corollary of the proof. Under the conditions of the previous lemma, we have \( r(E_n) \geq r(E_0) + r(E_1) \) (for \( n \neq 0 \) and \( n \neq 1 \) ). Moreover, \( r(E_n) > r(E_0) + r(E_1) \) for \( n \neq 0 \) and \( n \neq 1 \) whenever both \( E_0 \) and \( E_1 \) has a positive rank.

Proof. In reality, we see that the image of the sheaf \( E_n \) in \( K_0(X) \) has the form: \( e_n = ae_0 + be_1 \) for some natural \( a \) and \( b \). Thus our statement follows from the additivity of rank function.

Proof of proposition 3.4.1 Suppose \( G_i \) are locally free. If the pair \( (G_0, G_1) \) has zero or hom type then \( h^1(G_0, G_1) = 0 \) and \( F = y_0 G_0 \oplus y_0 G_1 \).

If the pair \( (G_0, G_1) \) is singular then \( \mu_H(G_0) = \mu_H(G_1) \) and the proof follows from the uniqueness of an exceptional filtration \( (2.5.1) \).

Now, suppose \( G_0 \) is a torsion sheaf and \( G_1 \) is a bundle; then the pair \( (G_0, G_1) \) is necessarily an ext-pair.

Thus, let \( (G_0, G_1) \) be an ext-pair. Following traditions, put

\[ G_{n+1} = R_{G_0} G_{n-1} \quad \text{and} \quad G_{-n} = L_{G_{1-n}} G_{2-n}. \]

Step 1. One of the following possibilities holds true:

\[ F = x_1 G_1 \oplus x_2 G_2, \]
3 CONSTRUCTIBILITY OF EXCEPTIONAL BUNDLES.

0 → x_1G_1 → x_2G_2 → F → 0.

Consider the spectral sequence convergent to F, which is constructed by the right dual collection \((G_1', G_0')\) (3.1.6). (Recall that \(G_1' = G_1\) and \(G_0' = R_{G_1}G_0 = G_2\).) Since the right mutation of the pair \((G_0, G_1)\) is nonregular, we get

\[ \Delta_0 = 1 \quad \text{and} \quad E_1^{0,q} = \text{Ext}^{-q}(F, G_0)^* \otimes G_2. \]

In addition, we do not mutations to obtain the sheaf \(G_1'\). Hence, \(\Delta_1 = 0\) and \(E_1^{-1,q} = \text{Ext}^{-q}(F, G_1)^* \otimes G_1\).

Thus we see that \(E_1\) term of the spectral sequence has the form

\[
\begin{array}{c|c|c|c|c}
& y_0\text{Ext}^k(G_0, G_1) & \rightarrow & \text{Ext}^k(F, G_1) & \rightarrow & y_1\text{Ext}^k(G_1, G_1) \\
\hline
k & 0 & \rightarrow & \rightarrow & \rightarrow & \\
0 & ? & \rightarrow & \rightarrow & \rightarrow & 0 \\
* & ? & \rightarrow & \rightarrow & \rightarrow & 0 \\
0 & ? & \rightarrow & \rightarrow & \rightarrow & 0 \\
\end{array}
\]

Whereby, the spectral sequence degenerates into two exact triples:

\[
0 \rightarrow \text{Ext}^1(F, G_1)^* \otimes G_1 \rightarrow \text{Ext}^0(F, G_0)^* \otimes G_2 \rightarrow C \rightarrow 0,
\]

\[
0 \rightarrow C \rightarrow F \rightarrow \text{Ext}^0(F, G_1)^* \otimes G_1 \rightarrow 0.
\]

Now, as in the proof of lemma 3.4.3, using the first of these triples, it is easily shown that \(\text{Ext}^1(G_1, C) = 0\). Therefore, if \(\text{Ext}^0(F, G_1) \neq 0\) then \(F\) is a direct sum. In the converse case, the sheaf \(F\) is included in the exact sequence.

**Step 2.** One of the following possibilities holds true:

\[
F = x_{-1}G_{-1} \oplus x_0G_0,
\]

\[
0 \rightarrow F \rightarrow x_{-1}G_{-1} \rightarrow x_0G_0 \rightarrow 0.
\]
This step is checked in the same way as the first one with using the spectral sequence associated with the left dual collection \((G_{-1}, G_0)\).

**STEP 3.** The sheaf \(F\) is decomposed into the direct sum:

\[
F = x_{n-1}G_{n-1} \oplus x_nG_n
\]

for some \(n \in \mathbb{Z}\) and nonnegative integer \(x_{n-1}, x_n\). (That is \(F_0 = G_{n-1}\) and \(F_1 = G_n\) in the formulation of proposition.)

Using the first two steps and lemma 3.4.3, it can be stated that for any \(n > 0\) the following exact triples

\[
0 \rightarrow x_nG_n \rightarrow x_{n+1}G_{n+1} \rightarrow F \rightarrow 0,
\]

\[
0 \rightarrow F \rightarrow x_{-n}G_{-n} \rightarrow x_{1-n}G_{1-n} \rightarrow 0
\]

hold unless \(F = x_{n-1}G_{n-1} \oplus x_nG_n\).

Let us show that these triples contradict the proposition conditions.

Suppose \(h^1(G_0, G_1) > 1\); then it follows from these sequences and lemma 3.4.4 that \(\chi(F, F) \leq 0\). This contradicts the fact that \(F\) is rigid.

Suppose \(h^1(G_0, G_1) = 1\); then the series of the exceptional sheaves \(G_n\) is formed by \(G_0, G_1, G_2\). In reality, in this case both the right and the left mutation of the ext-pair \((G_0, G_1)\) is described by the sequence

\[
0 \rightarrow G_1 \rightarrow G_2 \rightarrow G_0 \rightarrow 0.
\]

Whence, \(L_{G_0}G_1 = G_2\) and \(R_{G_2}G_1 = G_0\). Hence there are exact triples:

\[
0 \rightarrow y_1G_1 \rightarrow F \rightarrow y_0G_0 \rightarrow 0,
\]

\[
0 \rightarrow x_2G_2 \rightarrow x_0G_0 \rightarrow F \rightarrow 0.
\]

Since \(G_0\) is indecomposable, it follows from the second sequence that \(h^1(F, G_2) \neq 0\). We apply the functor \(\text{Ext}^1(\cdot, G_2)\) to the first triple to obtain

\[
y_0\text{Ext}^1(G_0, G_2) \rightarrow \text{Ext}^1(F, G_2) \rightarrow y_1\text{Ext}^1(G_1, G_2).
\]

Since the pair \((G_2, G_0)\) is exceptional, we get \(\text{Ext}^1(G_0, G_2) = 0\). Besides, since \((G_1, G_2)\) is a hom-pair, we obtain \(\text{Ext}^1(G_1, G_2) = 0\). Thus, \(h^1(F, G_2) = 0\). This contradiction proves the 3-th step.

**STEP 4.** Suppose \(G_1\) is a bundle and \(G_0 = \mathcal{O}_e(-1)\); then \(F\) is locally free or \(F_0\) is a bundle and \(F_1 = \mathcal{O}_e(-1)\).

By the proposition assumption, the sheaf \(F\) is included in the exact triple:

\[
0 \rightarrow y_1G_1 \rightarrow F \rightarrow y_0\mathcal{O}_e(-1) \rightarrow 0.
\]

Since \(F\) is rigid, we see that \(F\) is locally free whenever \(F\) has not torsion (2.2.1). Therefore its direct summands are locally free as well.
Assume that $F$ has a torsion $TF$. Since $G_1$ is locally free, we obtain the following commutative diagram:

\[
\begin{array}{cccccc}
0 & 0 & 0 \\
\uparrow & \uparrow & \uparrow \\
0 & y_1G_1 & F' & Q & 0 \\
\uparrow & \uparrow & \uparrow & \varphi & \uparrow \\
0 & y_1G_1 & F & y_0O_e(-1) & 0, \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & TF & TF & 0 \\
\end{array}
\]

where $F'$ is torsion free. Since $TF$ as a subsheaf of $y_0O_e(-1)$ and the curve $e$ is isomorphic to the projective line, we get

\[
TF \cong \bigoplus z_i O_e(s_i).
\]

Hence,

\[
Q \cong \left[ \bigoplus w_j O_e(d_j) \right] \oplus T^0,
\]

where $T^0$ is a torsion sheaf with a zero-dimensional support.

Consider the upper row of the diagram. Assume that $T^0 \neq 0$. Since $G_1$ is locally free and the support of $T^0$ is zero-dimensional, we get $\text{Ext}^1(T^0, y_1G_1) = 0$. Hence $T^0$ is a direct summand of $F'$. But this contradicts the fact that $F'$ has not torsion. For the same reason,

\[
\text{Ext}^1(O_e(d_j), G_1) \neq 0 \quad \forall j.
\]

Let us show that this yields the inequality $d_j \leq -1$.

In reality, by assumption, $(O_e(-1), G_1)$ is an exceptional pair. Therefore, it easily follows from a calculation of cohomologies that $(G_1)_e = r(G_1)O_e$. Thus,

\[
\text{Ext}^1(O_e(d_j), G_1)^* \cong \text{Ext}^1(G_1, O_e(d_j) \otimes K_S) = r(G_1)\text{Ext}^1(O_e, O_e(d_j - 1)) \neq 0
\]

and the inequalities $d_j \leq -1$ hold for all $j$.

On the other hand, $Q$ is a quotient of $y_0O_e(-1)$. Hence, $d_j \geq -1 \quad \forall j$. From these inequalities it follows that $d_j = -1 \quad \forall j$ and $Q = wO_e(-1)$.

We see that $TF = zO_e(-1)$.

Now, note that from the exact sequence

\[
0 \longrightarrow y_1G_1 \longrightarrow F' \longrightarrow wO_e(-1) \longrightarrow 0
\]

it follows that $\text{Ext}^1(F', O_e(-1)) = 0$, i.e. $F = F' \oplus zO_e(-1)$.

By the previous step, $F = x_{n-1}G_{n-1} \oplus x_nG_n$. Therefore, $F = x_{-1}G_{-1} \oplus x_0O_e(-1)$ or $F = x_{-1}G_{-1} \oplus x_0O_e(-1) \oplus x_1G_1$. Since $(O_e(-1), G_1)$ is the ext-pair and $F$ is superrigid sheaf, we see that the last inequality is impossible. On the other hand, $F'$ is locally free, as rigid sheaf without torsion (2.2.1). Thus the sheaf $x_0F_0 = x_{-1}G_{-1} = F'$ is locally free as well.
3.5 Proof of the Main Theorem.

It follows from lemma 3.3.4 that for any exceptional collection of bundles on the surface \( S \) satisfying the conditions of the main theorem there is a \( hom \)-collection

\[
\tau = (F_0, F_1, F_2, \ldots, F_k)
\]
equivalent to the original one such that the superrigid bundle \( F \) associated with \( \tau \) is included in exact sequence (20):

\[
0 \longrightarrow G \longrightarrow F \longrightarrow \text{Hom}(F, \mathcal{O}_e(-1))^* \otimes \mathcal{O}_e(-1) \longrightarrow 0,
\]
where \( G \) is a superrigid bundle with \( \text{Ext}^k(G, \mathcal{O}_e(-1)) = 0 \quad \forall k = 0, 1, 2 \). (Further we shall denote by \( \mathcal{B}(F_0, F_1, F_2, \ldots, F_k) \) the superrigid bundle associated with a \( hom \)-collection \( (F_0, F_1, F_2, \ldots, F_k) \).

In particular, we see that \( G|_e = s\mathcal{O}_e \). Therefore there exists a superrigid bundle \( G' \) on the surface \( S' \) obtained from \( S \) by blowing down the curve \( e \) \( (\sigma : S \longrightarrow S') \) such that \( \sigma^*(G') = G \).

Since \( G' \) is superrigid, we see that there is an exceptional filtration of it:

\[
Gr(G') = (y_nG'_n, y_{n-1}G'_{n-1}, \ldots, y_1G'_{1}).
\]

Using the induction on the number of blow up divisors on \( S \), we can assume that the exceptional collection of bundles \( (G'_1, G'_2, \ldots, G'_n) \) is constructible. That is it included in a full exceptional collection obtained from the basic collection

\[
\left( \mathcal{O}_S, \mathcal{O}_S(h), \mathcal{O}_S(2h), \mathcal{O}_{e_1}(-1), \ldots, \mathcal{O}_{e_{d-1}}(-1) \right)
\]
by mutations. (Note that \( K_S^2 = K_S^2 + 1 > 1 \). Therefore the constructibility of the collection \( (G'_1, G'_2, \ldots, G'_n) \) does not depend on ranks of the sheaves \( G'_j \) (see theorem 3.1.8)).

Let us remember that the base of the induction, i.e. the case of the projective plane, had been checked in the paper [13].

Since \( \sigma^*(G') = G \), we obtain that the bundle \( G \) has the exceptional filtration

\[
Gr(G) = (y_nG_n, y_{n-1}G_{n-1}, \ldots, y_1G_1),
\]
where \( G_i = \sigma^*(G'_i) \). Moreover, the collection \( \tau' = (\mathcal{O}_e(-1), G_1, \ldots, G_n) \) is exceptional (the triviality of the groups \( \text{Ext}^k(G_i, \mathcal{O}_e(-1)) \) follows from the fact that \( G|_{e} = s_i\mathcal{O}_e \)). Furthermore, the constructibility of the collection \( (G'_1, G'_2, \ldots, G'_n) \) implies the constructibility of \( \tau' \).
Now to prove theorem 3.1.8 it is sufficient to show that the collection $\tau$ is included in an exceptional collection obtained from $\tau'$ by mutations.

Let us illustrate the procedure of that inclusion in the projectivisation of $K_0(S) \otimes \mathbb{Q} = K$. To each sheaf $E$ on $S$ assign a vector $[E]$ in $K_0(S) \otimes \mathbb{Q}$, which corresponds to Euler characteristic of sheaves $\chi(E, F)$. Since all exceptional sheaves satisfy the equation $\chi(E, E) = 1$, we see that the corresponding vectors are not proportional. Let us pass to the projectivisation of $K_0(S)$. In this case, vectors corresponding to sheaves of an exceptional collection are projected to vertexes of some simplex.

The key exact sequence implies that the vector $[F]$ gets into the simplex with the vertexes $[\mathcal{O}_e(-1)], [G_1], ..., [G_n]$.

Let us project the point $[F]$ to the edge $([\mathcal{O}_e(-1)], [G_1])$. Note that this projection corresponds to a superrigid sheaf, and the exceptional pair $(G'_0, G'_1)$ associated with it obtained by mutations of the pair $(\mathcal{O}_e(-1), G_1)$. As a result, we get a lesser simplex containing $[F]$. Next let us project $[F]$ to the face $([G'_1], [G_2], ..., [G_n])$, etc... It remains to show that this process is finite.

Let us prove two lemmas about projections.

3.5.1 Lemma. Let

$$ 0 \to G \to F \to E \to 0 $$

be an exact sequence of superrigid sheaves on the surface $S$. Let

$$ Gr(E) = (y_k E_k, y_{k-1} E_{k-1}, ..., y_1 E_1), $$

$$ Gr(G) = (y_m G_m, y_{m-1} G_{m-1}, ..., y_{k+1} G_{k+1}) $$

be exceptional filtrations of $E$ and $G$ such that the collection

$$(E_1, ..., E_k, G_{k+1}, ..., G_m)$$
is exceptional. Let us divide the filtration of the sheaf $G$ into two groups

$$0 \to G' \to G \to G'' \to 0,$$

(24)

where $G'$ and $G''$ are the sheaves with the exceptional filtrations

$$Gr(G') = (y_m G_m, y_{m-1} G_{m-1}, \ldots, y_{s+1} G_{s+1}),$$

$$Gr(G'') = (y_s G_s, y_{s-1} G_{s-1}, \ldots, y_{k+1} G_{k+1}).$$

Then

1. $G'$ and $G''$ are superrigid;
2. $\text{End}(G') \cong \text{Hom}(G', F)$;
3. $\text{Ext}^i(G', F) = 0$ for $i > 0$;
4. $\text{Ext}^2(F, G') = 0$;
5. there is an exact sequence:

$$0 \to G' \to F \to E' \to 0,$$

(25)

where $E'$ is a superrigid sheaf included in the exact triple

$$0 \to G'' \to E' \to E \to 0.$$

(26)

Besides, $\text{Ext}^i(G', E') = 0 \ \forall i$.

**Proof.** By the definition of an exceptional collection, $\text{Ext}^k(G_j, G_i) = 0$ for $j > i$ and all $k$. Therefore, $\forall k : \text{Ext}^k(G', G'') = 0$ (1.2.4). Hence it follows from lemma 2.2.2 that $\text{Ext}^2(G'', G') = 0$. We apply the Mukai lemma to exact sequence (24) to obtain that $G'$ and $G''$ are rigid. Since the collection $(G_{k+1}, \ldots, G_m)$ is exceptional, we see that $\text{Ext}^2(G_i, G_j) = 0$ for any pair $i, j$. This implies that $\text{Ext}^2(G', G') = \text{Ext}^2(G'', G'') = 0$. Thus the first lemma statement holds.

We saw that $\text{Ext}^k(G', G'') = 0 \ \forall k$. Whence, using exact triple (24) and the fact that $G'$ is superrigid, we have

$$\text{Hom}(G', G) \cong \text{End}(G') \quad \text{and} \quad \text{Ext}^i(G', G) = 0 \ \text{for} \ i > 0.$$

Besides, in view of the definition of the sheaf $G'$ and the fact that the collection

$$(E_1, \ldots, E_k, G_{k+1}, \ldots, G_m)$$

is exceptional the following inequalities are valid.

$$\text{Ext}^i(G', E) = 0 \ \forall i; \quad \text{Ext}^2(E, G') = \text{Ext}^2(G, G') = 0.$$

Consider two cohomology tables corresponding to sequence (24).

| $k$ | $\text{Ext}^k(G', G')$ | $\text{Ext}^k(G', F)$ | $\text{Ext}^k(G', E)$ |
|-----|----------------------|----------------------|----------------------|
| $\text{End}(G')$ | $?$ | 0 |
| 0 | ? | 0 |
| 0 | ? | 0 |
The lemma statements 2, 3 and 4 follow from this tables. Exact triples (23) and (24) give the following commutative diagram:

\[
\begin{array}{cccc}
0 & \uparrow & & \\
G'' & 0 & 0 & \\
& \uparrow & \uparrow & \uparrow \\
0 & \rightarrow & G & \rightarrow & F & \rightarrow & E & \rightarrow & 0 \\
& \uparrow & \uparrow & \uparrow & \\
0 & \rightarrow & G' & \rightarrow & F & \rightarrow & E' & \rightarrow & 0 \\
& \uparrow & \uparrow & \uparrow & \\
0 & 0 & G'' & \\
& \uparrow & \\
0 & \\
\end{array}
\]

It yields exact sequences (25) and (26).

Now to prove the lemma it remains to check that the sheaf \( E' \) is superrigid and for all \( i \) \( \text{Ext}^i(G', E') = 0 \). All these facts follow from the following cohomology tables associated with sequence (23)

| \( k \) | \( \text{Ext}^k(G', G') \) | \( \rightarrow \) | \( \text{Ext}^k(G', F) \) | \( \rightarrow \) | \( \text{Ext}^k(G', E') \) |
|-------|-----------------|-----|----------------|-----|----------------|
| \( * \) | \( ? \) | \( \rightarrow \) | \( * \) | \( ? \) | \( \rightarrow \) | \( * \) | \( ? \) | \( \rightarrow \) | \( * \) | \( ? \) | \( \rightarrow \) | \( * \) | \( ? \) | \( \rightarrow \) | \( * \) | \( ? \) | \( \rightarrow \) | \( * \) | \( ? \) | \( \rightarrow \) | \( * \) | \( ? \) | \( \rightarrow \) | \( * \) | \( ? \) | \( \rightarrow \) | \( * \) | \( ? \) | \( \rightarrow \) | \( * \) | \( ? \) | \( \rightarrow \) |

This completes the proof.

By the same argument the dual statement can be proved.

3.5.2 **Lemma.** Under the conditions of the previous lemma let us divide the filtration of the sheaf \( E \) into two groups:

\[
0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0,
\]
where $E'$ and $E''$ are sheaves with the exceptional filtrations
\[ Gr(E') = (y_k E_k, y_{k-1} E_{k-1}, \ldots, y_{s+1} E_{s+1}), \quad Gr(E'') = (y_s E_s, y_{s-1} E_{s-1}, \ldots, y_1 E_1). \]

Then
1. $E'$ and $E''$ are superrigid;
2. $\text{End}(E'') \cong \text{Hom}(F, E'')$;
3. $\text{Ext}^i(F, E'') = 0$ for $i > 0$;
4. $\text{Ext}^2(E'', F) = 0$;
5. There is an exact sequence:
\[ 0 \rightarrow G' \rightarrow F \rightarrow E'' \rightarrow 0, \]
where $G'$ is a superrigid sheaf included in the exact triple:
\[ 0 \rightarrow G \rightarrow G' \rightarrow E' \rightarrow 0. \]

Besides, $\text{Ext}^i(G', E'') = 0 \ \forall i.$

3.5.3 Remark.
1. Lemma 3.5.1 is also valid provided $E = y_1 O_e(-1)$ for the exceptional rational curve $e = e_d$;
2. Lemma 3.5.2 is correct as well provided $E = y_1 E_1 \oplus y_2 E_2$, where $E_1$ is an exceptional bundle and $E_2 = O_e(-1)$;
3. The procedure described in 3.5.1 is called the transfer of the collection $(G_{k+1}, \ldots, G_s)$ to the right, and the similar procedure from 3.5.2 is the transfer of the collection $(E_{s+1}, \ldots, E_k)$ to the left.

Now let us prove a proposition concluding the proof of the main theorem.

3.5.4 Proposition. Suppose a superrigid bundle $F = B(F_0, F_1, F_2, \ldots, F_k)$ on the surface $S$ with $K_S^2 > 0$ is included in the exact sequence
\[ 0 \rightarrow G \rightarrow F \rightarrow E \rightarrow 0, \] (27)
where $G$ is a superrigid bundle with an exceptional filtration
\[ Gr(G) = (y_n G_n, y_{n-1} G_{n-1}, \ldots, y_s G_s), \]
and $E$ is a superrigid sheaf. In addition we assume that $E$ is either locally free and
\[ Gr(E) = (y_{s-1} G_{s-1}, y_{s-2} G_{s-2}, \ldots, y_0 G_0) \]
or $E = y_0 G_0 = y_0 O_e(-1)$; but the collection $(G_0, G_1, G_2, \ldots, G_n)$ is exceptional in any case. Then
1. \( k \leq n; \)

2. the collection \((F_0, F_1, F_2, \ldots, F_k)\) is included in an exceptional collection obtained from \((G_0, G_1, G_2, \ldots, G_n)\) by mutations;

3. \( \sum_{i=0}^{k} r(F_i) \geq \sum_{j=0}^{n} r(G_j) ; \)

4. If \( E \) is locally free then the equality \( \sum_{i=0}^{k} r(F_i) = \sum_{j=0}^{n} r(G_j) ; \) yields the equality \( k = n . \)

Moreover, in this case we have \( F_i = G_i \) after some mutations of neighboring zero-pairs.

**Proof.** The proof is by induction on number of sheaves in the collection 
\[ (G_0, G_1, G_2, \ldots, G_n). \]

The case \( n = 1 \) had been checked in the previous section.

**Statement.** Without loss of generality it can be assumed that \( E \) and \( G \) is locally free.

Proof. Suppose \( E = y_0 G_0 = y_0 O_\epsilon(-1). \) Following remark \[ 3.5.3 \] , let us do the transfer of \( G_1 \) to the right. Namely, let us denote by \( G' \) the bundle \( B(G_2, G_3, \ldots, G_n) \) and let us consider the exact sequences
\[ 0 \rightarrow G' \rightarrow F \rightarrow E' \rightarrow 0, \]
\[ 0 \rightarrow y_1 G_1 \rightarrow E' \rightarrow y_0 G_0 \rightarrow 0. \]

Taking into account lemma \[ 3.5.1 \] and proposition \[ 3.4.1 \] , we obtain that \( E' \) is a superrigid bundle such that \( E' = x_0 E'_0 \oplus x_1 E'_1 \) (or \( E' = x_0 E'_0 \)), where the exceptional pair \((E'_0, E'_1)\) (or \( E'_0 \)) is obtained by mutations of the pair \((G_0, G_1)\). Moreover, \( E'_0 \) is locally free and
\[ r(E'_0) + r(E'_1) \geq r(G_0) + r(G_1) \quad (r(E'_0) \geq r(G_0) + r(G_1)). \]

Let us show that the collection \((E'_0, E'_1, G_2, \ldots, G_n)\) is exceptional. From lemma \[ 3.5.1 \] it follows that \( \text{Ext}^k(G', E') = 0 \) \forall k. But, \( E' = x_0 E'_0 \oplus x_1 E'_1 \) and \( G' = B(G_2, G_3, \ldots, G_n) \).

Provided \( E'_i \) is locally free, the triviality of the groups \( \text{Ext}^k(G_j, E'_i) \) for \( j = 2, \ldots, n \) follows from lemma \[ 2.5.7 \] . Let us check this property for the case \( E'_1 = O_\epsilon(-1). \) Since \( \text{Ext}^k(G', O_\epsilon(-1)) = 0 \) \forall k, we see that the restriction of \( G' \) to the curve \( e \) is trivial. Therefore there exists a superrigid bundle \( L \) on the surface \( S' \) obtained from \( S \) by blowing down the curve \( e \) \( (\sigma : S \rightarrow S') \) such that \( \sigma^*(L) = G' \).

Since \( L \) is superrigid, we see that it has the exceptional filtration
\[ Gr(L) = (z_m L_m, z_{m-1} L_{m-1}, \ldots, z_2 L_2). \]

Besides, \( Gr(G') = (z_m \sigma^*(L_m), z_{m-1} \sigma^*(L_{m-1}), \ldots, z_2 \sigma^*(L_2)) \) is the exceptional filtration of the bundle \( G' \). Now, by theorem \[ 2.5.1 \] \( m = n \) and \( G_i = \sigma^*(L_i) \). Thus the collection \((E'_0, E'_1, G_2, \ldots, G_n)\) is exceptional.

Our statement is correct in the case \( E' = y_0 E'_0 \).

Assume that \( E' = x_0 E'_0 \oplus x_1 E'_1 \) with positive \( x_0, x_1 \). Let us do the transfer of \( E'_1 \) to the left:
\[ 0 \rightarrow \tilde{G} \rightarrow F \rightarrow x_0 E'_0 \rightarrow 0, \]
3.5 Proof of the Main Theorem.

\[ 0 \rightarrow G' \xrightarrow{} \tilde{G} \xrightarrow{} x_1 E'_1 \xrightarrow{} 0. \]

Using lemma \[8.5.2\] and the induction hypothesis, we obtain that \( \tilde{G} \) is a superrigid sheaf with an exceptional filtration \( Gr(\tilde{G}) = (x'_m G'_m, x'_{m-1} G'_{m-1}, \ldots, x'_1 G'_1) \). In addition, the collection \( (G'_1, G'_2, \ldots, G'_m) \) is included in an exceptional collection obtained from \( (E'_1, G_2, \ldots, G_n) \) by mutations and

\[ \sum r(G'_i) \geq \sum r(G_i) + r(E'_1). \]

Note that the sheaf \( \tilde{G} \) has no torsion, as a subsheaf of a bundle. Since \( \tilde{G} \) is rigid, we see that it is locally free. It can be checked as above that the collection \( (E'_0, G'_1, \ldots, G'_m) \) is exceptional. This completes the statement proof.

We shall name \textit{bounding} the collection \( (G_0, G_1, G_2, \ldots, G_n) \) from the formulation of our proposition and all collections obtained from it by mutations.

Now, consider exact sequence (27). We shall do the transfer of the bundle \( G_s \) to the right and to the left. Recall that in this procedure the sum of ranks of the bounding collections is not decreased.

Since the sum of ranks of the bounding collections is less than or equals rank of the bundle \( F \), we see that this process cannot be continued ad infinitum. Hence beginning with some moment the sum of ranks is a constant. Let us study this moment in the following statement.

**Statement.** Assume that under the conditions of our proposition the sum of ranks of the bounding collection bundles does not change after the transfers of the bundle \( G_s \) to the right and to the left; then \( k = n \) and

\[ (F_0, F_1, F_2, \ldots, F_k) = (G_0, G_1, G_2, \ldots, G_n). \]

to within mutations of neighboring zero-pairs.

**Proof.** After the transfer of the bundle \( G_s \) to the right two exact sequence appear:

\[ 0 \rightarrow \mathcal{B}(G_{s+1}, \ldots, G_n) \rightarrow F \rightarrow \mathcal{B}(G'_0, G'_1, G'_2, \ldots, G'_l) \rightarrow 0, \]

\[ 0 \rightarrow y_s G_s \rightarrow \mathcal{B}(G'_0, G'_1, G'_2, \ldots, G'_l) \rightarrow \mathcal{B}(G_0, G_1, G_2, \ldots, G_{s-1}) \rightarrow 0. \]

Since \( G_0, \ldots, G_{s-1}, G_s \) are locally free, we obtain that by the induction hypothesis it follows from the equality

\[ \sum_{i=0}^{l} r(G'_i) = \sum_{i=0}^{s} r(G_i) \]

that \( l = s \) and \( G_i = G'_i \) (to within mutations of neighboring zero-pairs). Therefore there is an exact sequence

\[ 0 \rightarrow \mathcal{B}(G_{s+1}, \ldots, G_n) \rightarrow F \rightarrow \mathcal{B}(G_0, G_1, G_2, \ldots, G_s) \rightarrow 0. \]

Moreover, \( (G_0, G_1, G_2, \ldots, G_s) \) is the \textit{hom}-collection. Whereby, \( \mu_H(G_i) \geq \mu_H(G_j) \) for \( s \geq i > j \).

Now let us do the transfer of the bundle \( G_s \) to the left (by assumption, the sum of ranks does not change as well):

\[ 0 \rightarrow \mathcal{B}(G''_s, \ldots, G''_m) \rightarrow F \rightarrow \mathcal{B}(G_0, G_1, G_2, \ldots, G_{s-1}) \rightarrow 0, \]
As before, by the induction hypothesis, we obtain that the collection \((G''_s, \ldots, G''_m)\) consists with the collection \((G_s, \ldots, G_n)\) to within mutations of neighboring zero-pairs. Hence \((G_s, \ldots, G_n)\) is the hom-collection and \(\mu_H(G_j) \leq \mu_H(G_i)\) for \(s \leq j < i\).

As a result we obtain that the all bounding collections \((G_0, G_1, G_2, \ldots, G_n)\) is the hom-collection. Thus we can construct the exceptional filtrations of the bundle

\[ F = \mathcal{B}(F_0, F_1, F_2, \ldots, F_k) \]

from the exceptional filtrations of the bundles \(E\) and \(G\) in sequence (27).

Now the proof following from the uniqueness of the exceptional filtration.

References

[1] M. F. Atiyah: Vector Bundles Over an Elliptic Curve.// Proc. Lond. Math. Soc., VII (1957), 414-452.

[2] V. V. Batyrev: Dual Polyhedra and Mirror Symmetry for Calabi–Yau Hypersurfaces in Toric Varieties.// Univ.–GH–Essen, Fachbereich 6, Math. Univ. 3, Posttach 103764, D–4300. Essen 1, FRG (1992).

[3] A. I. Bondal: Helixes, Representations of Quivers and Koszul Algebras.// Helixes and Vector Bundles, London Math. Soc., Lecture Noute Series 148. Cambridge Univ. Press, p.75-96.

[4] R. Bott: Homogeneous vector bundles.// Ann. of Math., v.66, p.203-248.

[5] A. L. Gorodentsev: Exceptional bundles on a surface with a moving anticanonical class.// Math. USSR Izv.33 (1989) 740-755.

[6] A. L. Gorodentsev: Exceptional Objects and Mutations in Derived Categories.// Helixes and Vector Bundles, London Math. Soc., Lecture Note Series 148. Cambridge Univ. Press, p.57-74.

[7] J.-M. Drezet and J.Le Potier: Fibres stables et fibres exceptionnels sur \(\mathbb{P}_2\).// Ann. Sci. ENS(4)18(1985),193-243.

[8] J.-M. Drezet: Fibres exceptionnels et suite spectrale de Beilinson generalisee sur \(\mathbb{P}_2(\mathbb{C})\).// Math. Ann. 275,(1) (1986), 25-48.

[9] S. K. Zube, D. Yu. Nogin: Computing Invariants of Exceptional Bundles on a Quadric.// Helixes and Vector Bundles, London Math. Soc., Lecture Note Series 148. Cambridge Univ. Press, p.23-32.

[10] S. Yu. Zyuzina. Constructibility of exceptional pairs of vector bundles on a quadric.// (Russian) Akad.Nauk SSSR Ser. Mat. 57 (1993), no 1, 183-191.

[11] S. A. Kuleshov: Construction of Bundles on an Elliptic Curve.// Helixes and Vector Bundles, London Math. Soc., Lecture Note Series 148. Cambridge Univ. Press, p.7-22.
REFERENCES

[12] S. A. Kuleshov, D. O. Orlov: Exceptional sheaves on Del Pezzo surfaces. // (Russian) Izv. Akad. Nauk Russia Ser. Mat. 58 (1994), no. 3, 59–93.

[13] Yu. I. Manin: Cubic forms.// M.: Nauka. 1972.

[14] A. A. Markov: About binary quadratic forms of positive definition.// SPb., 1880, p.44.

[15] S. Mukai: On the Moduli Spaces of Bundles on K3 Surfaces, I// in Vector Bundles ed. Atiyah et al, Oxford Univ. Press, Bombay, (1986) P. 67-83.

[16] D. Yu. Nogin: Helices of period four and Markov-type equations.// Math. USSR – Izv. 37 (1991), no. 1, 209–226.

[17] C. Okonek, M. Schneider, H. Spindler: Vector Bundles on Complex Projective Spaces.// Birkhauser, Boston, 1980.

[18] D. O. Orlov: Projective bundles, monoidal transformations and derived categories of coherent sheaves.// (Russian) Izv. Akad. Nauk USSR Ser. Mat. 56 (1992), no 4, 852-862.

[19] A. N. Rudakov: Markov numbers and exceptional bundles on $\mathbb{P}^2$.// Math. USSR – Izv. 32 (1989), no. 1, 99–112.

[20] A. N. Rudakov: Exceptional Collections, Mutations and Helixes.// Helixes and Vector Bundles, London Math. Soc., Lecture Noute Series 148. Cambridge Univ. Press, p.1-6.

[21] A. N. Rudakov: Exceptional vector bundles on a quadric.// Math. USSR – Izv. 33 (1989), no. 1, 115–138.

[22] A. N. Rudakov: A description of Chern classes of semistable sheaves on a quadric surface.// Schriftenriehe. Hett Nr.88. Forschungsschwerpunkt Komplexe Mannigfattigkeiten. Erlangen(1990).

[23] R. Hartshorne: Algebraic Geometry.// Springer – Verlag New York Heidelberg Berlin, 1977.
Contents

Introduction.  
Notations.  

1 Axioms of Stability.  
  1.1 Definitions and Simple Properties.  
  1.2 Harder-Narasimhan Filtration.  
  1.3 Examples of Slopes and Kinds of Stability.  

2 Rigid Sheaves.  
  2.1 Preliminary information.  
  2.2 Exceptional sheaves.  
  2.3 Exceptional Collections.  
  2.4 Structure of Rigid Sheaves.  
  2.5 Structure of superrigid sheaves.  

3 Constructibility of Exceptional bundles.  
  3.1 Introduction to the Helix Theory.  
  3.2 Restriction of Superrigid Bundles to an Exceptional Curve.  
  3.3 Equivalence of Collections and the Key Exact Sequence.  
  3.4 Category Generated by a Pair.  
  3.5 Proof of the Main Theorem.  