A CO-CHAIN MAP FOR THE G-IN Variant DE RHAM COMPLEX.
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1. Introduction

In this note we characterize the Lie group actions for which there exists, at least locally, an evaluation map that defines a cochain map from the differential complex of invariant forms on a manifold to the De Rham complex for the quotient. This problem is motivated by the principle of symmetric criticality [4].

Before giving any specific definitions we would like to illustrate the notion of such an evaluation map with a simple example. Consider the two dimensional Abelian Lie group \( G = \mathbb{R}^2 \) with coordinates \((a, b)\) acting on \( \mathbb{R}^3 \) by
\[
(a, b) \ast (x, y, z) = (x, y + a, z + b).
\]
If \( \alpha \in \Omega^2(\mathbb{R}^3)^G \) and \( \nu \in \Omega^3(\mathbb{R}^3)^G \), where we use the convention that a group superscript denotes the invariants of the group, then \( \alpha \) and \( \nu \) are necessarily of the form
\[
\alpha = a(x)dx \wedge dy + b(x)dx \wedge dz + c(x)dy \wedge dz \quad \text{and} \quad \nu = A(x)dx \wedge dy \wedge dz.
\]
The Lie algebra of infinitesimal generators of this action of \( G \) is generated by \( \partial_y, \partial_z \) and it is easy to check that evaluation on the generators
\[
\alpha(\partial_y, \partial_z) = c(x), \quad \nu(\partial_y, \partial_z, -) = A(x)dx
\]
defines a cochain map from \( \Omega^*(\mathbb{R}^3)^G \) to \( \Omega^{*-2}(\mathbb{R}) \), that is,
\[
(d\alpha)(\partial_y, \partial_z, -) = d(\alpha(\partial_y, \partial_z)) = c(x)'dx.
\]
As we shall see, not all group actions admit cochain evaluation maps.

2. Lie group actions and invariant vector fields

Let \( G \) be a \( p \)-dimensional Lie group which acts effectively on an \( n \)-dimensional manifold \( M \) with multiplication map \( \mu : G \times M \to M \). We write \( gx \) instead of \( \mu(g, x) \). For \( x \in M \) and \( g \in G \), we define \( \mu_x : G \to M \) and \( \mu_g : M \to M \) to be the maps
\[
\mu_x(g) = \mu_g(x) = gx.
\]
For any \( g \in G \), \( \mu_g \) is a diffeomorphism of \( M \). We let \( G_x \) denote the isotropy subgroup of \( G \) at \( x \),
\[
G_x = \{ g \in G \mid gx = x \}.
\]
For each \( x \in M \), the map \( \tilde{\mu}_x : G/G_x \to M \) given by \( \tilde{\mu}_x([g]) = gx \) is a one-to-one immersion which is also \( G \) equivariant with respect to the canonical action of \( G \) on the coset space \( G/G_x \).

The Lie algebra \( g \) of the Lie group \( G \) is the Lie algebra of right invariant vector fields on \( G \). The action of \( G \) on \( M \) induces a Lie algebra homomorphism
\( r : g \to X(M) \) of \( g \) to the vector fields on \( M \) whose image is the Lie algebra of the infinitesimal generators of the action of \( G \) on \( M \) \cite{2}. We write \( \Gamma = r(g) \). Because the action of \( G \) on \( M \) is assumed effective, the map \( r \) is injective. Let \( \Gamma \subset TM \) denote the (integrable) distribution generated by \( \Gamma \).

The action of the Lie group \( G \) on \( M \) is said to be \textit{regular} if the space of orbits is a manifold \( \overline{M} = M/G \) such that the quotient map
\[
q : M \to \overline{M}
\]
is a submersion. We will assume from here on that all actions are regular. For regular actions the orbits all have the same dimension which we assume to be \( q \) and so the isotropy subgroup \( G_x \), for any \( x \in M \), will have dimension \( p - q \). Let \( \text{Vert} M \to M \) be the sub-bundle of \( q \) vertical vectors in \( TM \), so \( \text{Vert} M = \ker q_* = \Gamma \). We also have the important property \( (\tilde{\mu}_g)_*(T_eG/G_x) = \text{Vert} x M \).

The action of \( G \) on \( M \) defines an action of \( G \) on \( TM \) using the differential
\[
(\mu_g): TM \to TM.
\]
For each \( g \in G_x \), equation (2.1) gives 
\[
(\mu_g)_*: T_xM \to T_xM
\]
which defines the linear isotropy representation of \( G_x \) on the tangent space \( T_xM \).

Suppose now that \( X \) is a \( G \) invariant vector field, that is, 
\[
(\mu_g)_*X_x = X_{gx}.
\]
If \( g \in G_x \), then equation (2.2) implies that
\[
X_x \in (T_xM)^{G_x}.
\]
This observation leads us to define the following subset of \( TM \),
\[
\kappa(TM) = \bigcup_{x \in M} \kappa(T_xM), \quad \kappa(T_xM) = (T_xM)^{G_x}.
\]
Equation (2.3) implies that every \( G \) invariant vector field \( X \) takes values in the subset \( \kappa(TM) \subset TM \).

Since \( q \circ \mu_g = q \), the action of \( G \) on \( TM \) restricts to an action on \( \text{Vert} M \) and the linear isotropy representation of \( G_x \) also restricts to a representation on vertical vectors
\[
(\mu_g)_*: \text{Vert} x M \to \text{Vert} x M, \quad g \in G_x.
\]
Thus a \( G \) invariant vertical vector field takes values in the set
\[
\kappa(\text{Vert} M) = \bigcup_{x \in M} \kappa(\text{Vert} x M), \quad \kappa(\text{Vert} x M) = (\text{Vert} x M)^{G_x}.
\]
In the next theorem we give conditions which guarantee the existence of invariant vector fields. This is a special case of the general construction given in [2] or on p. 657 in [3].

\textbf{Theorem 2.1}. If \( \kappa(TM) \subset TM \) is a vector sub-bundle, then for each \( x \in M \) and \( Y \in \kappa(T_xM) \) there exists a \( G \) invariant vector field \( X \) on \( M \) such that \( X_x = Y \). The analogous statement holds for \( G \) invariant vertical vector fields if \( \kappa(\text{Vert} M) \subset \text{Vert} M \) is a vector sub-bundle.

\textbf{Remark 2.1} For the rest of this article we assume that all group actions are regular and that \( \kappa(TM) \), and \( \kappa(\text{Vert} M) \) are bundles.
3. Lie algebra cohomology

Given a Lie group $G$ and a Lie subgroup $K \subset G$, with corresponding Lie algebras $\mathfrak{g} \subset \mathfrak{g}$, define the vector space of $K$ relative forms on $\mathfrak{g}$ by

$$A^r(\mathfrak{g}, K) = \{ \alpha \in A^r(\mathfrak{g}) \mid \iota_v \alpha = 0, \forall v \in \mathfrak{k} \text{ and } \text{Ad}^*_g \cdot \alpha = \alpha, \forall g \in K \},$$

where $A^r(\mathfrak{g})$ are the alternating $r$-forms on $\mathfrak{g}$ and $\text{Ad}^*$ denotes the co-adjoint representation of $G$ on $A^r(\mathfrak{g})$.

The usual exterior derivative $d$ on $A^*(\mathfrak{g})$ restricts to make $A^*(\mathfrak{g}, K)$ a differential complex whose cohomology is denoted by $H^*(\mathfrak{g}, K)$, the Lie algebra cohomology of $\mathfrak{g}$ relative to the subgroup $K$.

If $K \subset G$ is a closed Lie subgroup, let $H^*(\mathfrak{g}, K)^G$ be the $d$-cohomology of the $G$ invariant forms on $G/K$.

**Lemma 3.1.** If $K \subset G$ is closed, then $\Omega^r(G/K)^G \simeq A^r(\mathfrak{g}, K)$ and $H^*(\Omega^*(G/K)^G) \simeq H^*(A^*(\mathfrak{g}, K)).$

See Theorem 13.1 in [6] for a proof of this Lemma. It is well-known [8], that if $G$ is connected and compact and $K$ closed, then $H^*(\mathfrak{g}, K)$ computes the De Rham cohomology of the homogeneous space $G/K$.

It is useful to note that if $K_2 = gK_1g^{-1}$ are conjugate subgroups of $G$ then $\text{Ad}(g)$ induces an isomorphism

$$A^*(\mathfrak{g}, K_1) = A^*(\mathfrak{g}, K_2).$$

**Example:** Consider the two sphere $S^2$ and the projective plane $\mathbb{R}P^2$ as the homogeneous spaces $SO(3)/SO(2)$ and $SO(3)/O(2)$. Letting $X_1, X_2, X_3$ be a basis for $so(3)$ with $X_3$ the basis for $so(2)$ (which particular $so(2)$ is actually irrelevant because of (3.1)) and letting $\alpha^1, \alpha^2, \alpha^3$ be the dual basis, we find

$$A^1(so(3), SO(2)) = \{0\} \quad \text{and} \quad A^2(so(3), SO(2)) = \{\alpha^1 \wedge \alpha^2\}.$$ 

Therefore $H^2(so(3), SO(2))$ is generated by $\alpha_1 \wedge \alpha_2$. On the other hand, there is a reflection in $O(2)$ which maps $X_1$ to $-X_1$ and $X_2$ to $X_2$ so that

$$A^1(so(3), O(2)) = \{0\} \quad \text{and} \quad A^2(so(3), O(2)) = \{0\}$$

and therefore $H^2(so(3), O(2)) = 0$. Of course, these computations reflect the fact that $S^2$ is orientable whereas $\mathbb{R}P^2$ is not.

4. A map on the $G$ invariant De Rham complex

In this section we generalize the evaluation map from the introduction by studying the problem of defining a map

$$\rho^k_{\lambda} : \Omega^k(M)^G \to \Omega^{k-q}(\mathcal{M})$$

which shifts form degree by the orbit dimension $q$ of $G$ on $M$. To begin, we define $\Lambda_q(\text{Vert} M) \to M$ to be the vector bundle of vertical $q$-chains on $\text{Vert} M$ (alternatively, the bundle of vertical multi-vectors of degree $q$). Given that the orbits of $G$ have dimension $q$ it follows that about each point $x \in M$ there exists an open set $U$ and vector fields $X_1, X_2, \ldots, X_q$ in $\Gamma$ which define a local frame for $\Gamma|_U = \text{Vert} U$. Consequently if $\chi$ is a section of $\Lambda_q(\text{Vert} M)$ then $\chi|_U$ can written as

$$\chi|_U = J X_1 \wedge X_2 \wedge \cdots \wedge X_q,$$
where $J \in C^\infty(U)$. The action of $G$ on $\text{Vert} M$ described in section 2, induces an action of $G$ on $\Lambda_q(\text{Vert} M)$.

Given a $G$ invariant q-chain $\chi : M \to \Lambda_q(\text{Vert} M)$, we now define a map $\iota \chi : \Omega^k(M) \to \Omega^{k-q}_{\text{sb}}(M)$ where
\[
\Omega^{k}_{\text{sb}}(M) = \{ \omega \in \Omega^k(M) \mid \iota \chi \omega = 0 \quad \text{for all} \ X \in \Gamma \}
\]
are the $q$ semi-basic forms on $M$. The map $\iota \chi$ is defined by setting
\[
(\iota \chi \omega)_x(Y_1, Y_2, \ldots, Y_{k-q}) = \omega_x(\chi_x, Y_1, Y_2, \ldots, Y_{k-q})
\]
for $\omega \in \Omega^k(M)$ and $Y_i \in T_x M$. If $\omega \in \Omega^k(M)^G$ then $\iota \chi \omega$ is $q$ semi-basic, and since $\chi$ is $G$ invariant, $\iota \chi \omega$ is $G$ invariant and so $G$ basic. By this last statement $\iota \chi \omega \in \Omega^{k-q}_{\text{sb}}(M)^G$, and therefore by Lemma A.3 in [1], we find there exists a unique $(k-q)$-form $\iota \chi \omega$ on $M$ satisfying $q^*(\iota \chi \omega) = \iota \chi \omega$. The sought after evaluation map $\rho \chi$ is then defined by
\[
(4.1) \quad \rho^k_{\chi}(\omega) = (-1)^{(n-k)q} \iota \chi \omega.
\]
Note that for each invariant $\chi$ we have a map $\rho \chi$.

**Theorem 4.1.** If there exists a non-vanishing $G$ invariant vertical q-chain $\chi$ on $M$, then
\[
(4.2) \quad A^q(\mathfrak{g}, G_x) \neq 0 \quad \text{for all} \ x \in M.
\]
Conversely, if for each $x \in M$, $A^q(\mathfrak{g}, G_x) \neq 0$ then about each $x_0 \in M$ there exists a $G$ invariant open set $U$ and non-vanishing $G$ invariant vertical q-chain $\chi$ on $U$.

**Proof.** Let $\chi$ be a non-vanishing $G$ invariant vertical q-chain. Let $x \in M$ and let $\tilde{\chi}$ be the restriction of $\chi$ to $G/G_x$, so that $(\tilde{\mu}_x)_* \tilde{\chi} = \chi$. By the equivariance property of $\tilde{\mu}_x$ the q-chain $\tilde{\chi}$ is $G$ invariant. Now let $\alpha \in \Omega^q(M)$ satisfy $\alpha(\chi) = 1$. The form $\alpha$ is not unique, and it is not necessarily invariant. We claim the form $\tilde{\mu}_x^* \alpha$ defines a non-zero element of $\Omega^q(G/G_x)^G$. We compute
\[
(4.1) \quad (g^* \tilde{\mu}_x^* \alpha)(\tilde{\chi}) = \alpha((\tilde{\mu}_x)_* g_* \tilde{\chi}) = \alpha((\tilde{\mu}_x)_* \tilde{\chi}) = \alpha(\chi) = 1.
\]
Thus $\tilde{\mu}_x^* \alpha$ is a non-vanishing $G$ invariant form of top degree on $G/G_x$ and so, by Lemma 3.1, $A^q(\mathfrak{g}, G_x) \neq 0$.

We now prove the converse part of the theorem. Let
\[
\kappa(\Lambda_q(\text{Vert} M)) = \bigcup_{x \in M} \kappa(\Lambda_q(\text{Vert}_x M)),
\]
where $\kappa(\Lambda_q(\text{Vert}_x M)) = (\Lambda_q(\text{Vert}_x M))^{G_x}$. We shall show that $A^q(\mathfrak{g}, G_x) \neq 0$ implies $\kappa(\Lambda_q(\text{Vert} M))$ is a line bundle. Then the existence of a $G$ invariant q-chain is guaranteed (in a similar manner to Theorem 2.1) by Theorem 1.2 in [2].

If $A^q(\mathfrak{g}, G_x) \neq 0$ then by Lemma 3.1 there exists a non-vanishing $\tilde{\alpha} \in \Omega^q(G/G_x)^G$. Let $\tilde{\chi}$ be the invariant q-chain defined by $\tilde{\alpha}(\tilde{\chi}) = 1$. Then $\chi_x = (\mu_x)_* \tilde{\chi} |_{G_x} \in \Lambda_q(\text{Vert}_x M)^{G_x}$ by the equivariance of $\mu_x$, and is non-zero. Thus $\Lambda_q(\text{Vert}_x M)^{G_x} = \Lambda_q(\text{Vert}_x M)$ and so $\kappa(\Lambda_q(\text{Vert} M)) = \Lambda_q(\text{Vert} M)$ is a line bundle. \qed
5. THE COCHAIN CONDITION

In this section we find necessary and sufficient conditions on the action of $G$ on $M$ that determine whether we can choose a $G$ invariant $q$-chain $\chi$ so that the map $\rho\chi : \Omega^*(M)^G \to \Omega^{*-q}(M)$ defined in (4.1) is a cochain map, that is,

$$\rho\chi(d\omega) = d\rho\chi(\omega).$$

Granted that the action of $G$ on $M$ satisfies the conditions in Remark 2.1, the solution to this problem is given by the following theorem.

**Theorem 5.1.** If there exists a non-vanishing invariant $q$-chain $\chi$ such that the map $\rho\chi$ in (4.1) defines a cochain map, then $H^q(g, G_x) \neq 0$ for all $x \in M$. Conversely, if $H^q(g, G_x) \neq 0$ for all $x \in M$ then about each $x_0 \in M$ there exists a $G$ invariant open set $U$ and a non-vanishing $G$ invariant vertical $q$-chain $\chi$ on $U$ such that $\rho\chi : \Omega^*(U)^G \to \Omega^{*-q}(U/G)$ is a cochain map.

In order to prove this theorem, we need a number of preliminary results. The first of these is the important observation that the cochain condition (5.1), which is a condition that involves the quotient manifold $\overline{M}$, can be expressed as a condition entirely on $M$.

**Lemma 5.2.** A $G$ invariant, vertical $q$-chain $\chi$ defines a cochain map $\rho\chi$ if and only if

$$\iota\chi d\omega = (-1)^q d(\iota\chi \omega)$$

for all $\omega \in \Omega^*(M)^G$.

**Proof.** If $\eta$ is any $G$ basic form, then $d\eta$ is also $G$ basic. Let $\bar{\eta}$ be the unique form on $\overline{M}$ such that $q^*(\bar{\eta}) = \eta$. Then, since

$$q^*(d\bar{\eta}) = dq^*(\bar{\eta}) = d\eta$$

the two forms $d\bar{\eta}$ and $dq^*(\bar{\eta})$ pullback by $q$ to the same form and must therefore be equal. Since $\chi$ and $\omega$ are both $G$ invariant, we can apply this observation to the $G$ basic form $\iota\chi \omega$ to deduce that

$$d(\iota\chi \omega) = d(\iota\chi \omega).$$

The cochain condition (5.1) can therefore be expressed as

$$(-1)^q \iota\chi d(\omega) = d(\iota\chi \omega).$$

But two $G$ basic forms on $M$ are equal if and only if the corresponding forms on $\overline{M}$ are equal and so (5.2) proves the equivalence of (5.1) with (5.3). □

**Lemma 5.3.** If (5.2) holds for all $G$ invariant $(n-1)$-forms, then (5.3) holds for all $G$ invariant $r$-forms, $r \geq q$.

**Proof.** Suppose (5.2) holds true for all $G$ invariant $(n-1)$-forms. Let $\omega$ be a $G$ invariant $r$-form, where $q \leq r < n - 1$. Then, if $\alpha$ is any $G$ basic $(n-r-1)$-form, $\omega \wedge \alpha$ is a $G$ invariant $(n-1)$-form and therefore we can use (5.2) to write

$$\iota\chi d(\omega \wedge \alpha) = (-1)^q d(\iota\chi (\omega \wedge \alpha)).$$

Because $\alpha$ (and hence $d\alpha$) is $G$ basic, the expansion of both sides of this equation gives

$$\iota\chi d\omega \wedge \alpha = (-1)^q d(\iota\chi \omega) \wedge \alpha.$$
Since $\alpha$ is an arbitrary $G$ basic form and $\iota \chi d\omega$ and $d(\iota \chi \omega)$ are both $G$ basic this implies

$$
\iota \chi d\omega = (-1)^q d(\iota \chi \omega).
$$

\[\square\]

**Lemma 5.4.** Let $\mu$ be a $G$ basic $(n - q)$-form on a $G$ invariant open set $U$. Let $\chi$ be a non-vanishing, $G$ invariant, vertical $q$-chain on $U$ and let $\alpha$ be any $q$-form such that $\alpha(\chi) = 1$. Then

$$
\nu = \alpha \wedge \mu
$$
is a $G$ invariant $n$-form on $U$.

**Proof.** For any $g \in G$, we compute

$$
[\mu^*_g(\alpha)](\chi) = \alpha((\mu_g)_*(\chi)) = \alpha(\chi) = 1
$$
and therefore

$$
\iota \chi [\mu^*_g(\nu)] = \iota \chi [\mu^*_g(\alpha) \wedge \mu] = \mu = \iota \chi \nu.
$$

This suffices to prove that $\mu^*_g(\nu) = \nu$. \[\square\]

**Lemma 5.5.** If $\chi$ is non-vanishing vertical $q$-chain and $R$ is a $G$ invariant vector field then

$$
\mathcal{L}_R \chi = \lambda_R \chi,
$$
where $\lambda_R$ is a $G$ invariant function.

**Proof.** Let $X_1, \ldots, X_q$ be vector fields in $\Gamma$ which form a local basis for $\text{Vert} M$ in some neighborhood about the point $x$. Then $\chi = J X_1 \wedge X_2 \wedge \cdots \wedge X_q$ and, since $[R, X_i] = 0$,

$$
\mathcal{L}_R \chi = R(J) X_1 \wedge X_2 \wedge \cdots \wedge X_q = \frac{R(J)}{J} \chi.
$$

\[\square\]

**Theorem 5.6.** If $\chi$ is a non-vanishing, $G$ invariant, vertical $q$-chain, then the map

$$
\rho_\chi : \Omega^*(M)^G \to \Omega^{n-q}(M)
$$
is a cochain map if and only if

$$
(5.4)
\mathcal{L}_R \chi = 0
$$
for all $G$ invariant vector fields $R$ on $M$.

**Proof.** We start by assuming (5.4). Then by Lemma 5.3 it suffices to prove (5.4) for $G$ invariant $(n - 1)$-forms.

Given the non-vanishing $G$ invariant vertical $q$-chain $\chi$, let $x \in M$ and use Lemma 5.4 to construct a non-vanishing $G$ invariant $n$-form $\nu = \alpha \wedge \mu$ on an invariant open set $U$ about $x$. Let $\omega \in \Omega^{n-1}(M)^G$, then restricted to $U$ there exists a unique $G$ invariant vector field $S$ on $U$ such that

$$
(5.5)
\omega_U = \iota_S \nu.
$$

Let $R$ be a $G$ invariant vector field on $M$ which agrees with $S$ on an invariant open set $V \subset U$ of $x$ so that

$$
(5.6)
\omega_V = \iota_R \nu.
$$

With $\omega_V$ given by (5.6), we compute on $V$

$$
[d(\iota \chi \omega)]_V = d(\iota \chi \iota_R \nu) = (-1)^q d(\iota_R \chi \nu) = (-1)^q d(\iota_R \mu) \quad \text{and}
$$

$$
[i \chi d\omega]_V = i \chi d(\iota_R \nu) = i \chi \mathcal{L}_R(\nu) = \mathcal{L}_R(\mu) - i \mathcal{L}_R(\chi) \nu.
$$
But it is easy to check that if \( \mu \) is a \( G \) basic \((n-q)\)-form, then \( d\mu = 0 \) and therefore

\[
eq 0.
\]

Evaluating (5.7) at \( x \in V \) shows that if (5.4) holds at \( x \) then (5.2) holds at \( x \) for all \( G \) invariant \((n-1)\)-forms \( \omega \). Since our original point \( x \in M \) was arbitrary, equation (5.2) holds on \( M \).

To prove that (5.2) implies (5.4) we reverse the argument above. Let \( R \) be a \( G \) invariant vector field on \( M \) and let \( x \in M \). Choose a \( G \) basic \((n-q)\)-form \( \mu \) which doesn’t vanish at \( x \). Then the form \( \omega = \iota_R \nu \), where \( \nu = \alpha \land \mu \) with \( \alpha(\chi) = 1 \), is a \( G \) invariant \( n-1 \) form on \( M \). Equation (5.4) (evaluated at \( x \)) coupled with Lemma 6.3 shows that (5.2) implies (5.4) at \( x \). But \( x \) was arbitrary so (5.4) holds on \( M \).

We are now in a position to prove Theorem 5.1.

Proof. We begin the proof by first noting that the condition \( H^q(G, G_x) \neq 0 \) is equivalent to the following:

i) For each \( x \in M \) there exists a non-vanishing \( \tilde{\alpha} \in \Omega^q(G/G_x)^G \); and

ii) for all \( \tilde{\eta} \in \Omega^{q-1}(G/G_x)^G \), \( d\tilde{\eta} = 0 \).

Suppose there exists a non-vanishing \( G \) invariant \( q \)-chain \( \chi \) such that \( \rho_\chi \) is a cochain map. Let \( \alpha \in \Omega^q(M) \) with \( \alpha(\chi) = 1 \). Then as was shown in Theorem 4.1, given any \( x \in M \), \( \mu_x^\ast \alpha \in \Omega^q(G/G_x)^G \) and is non-vanishing. Thus condition i) is true.

Let \( \tilde{\eta} \in \Omega^{q-1}(G/G_x)^G \). Then \( \tilde{\eta} \) can be written \( \tilde{\eta} = \iota_Y \mu_x^\ast \alpha \) where \( Y \) is a \( G \) invariant vector field on \( G/G_x \). Now \( (\mu_x)_x Y_x \in \kappa(Vert_x M) \) and by the hypothesis on invariant vector fields (Theorem 2.1), there exists a \( G \) invariant vector field \( Y \) on \( M \) such that \( Y_x = (\mu_x)_x Y_x \). Thus \( \tilde{\eta} = \mu_x^\ast (\iota_Y \alpha) \).

In order to calculate \( d\tilde{\eta} \) we let \( \mu \) be a \( G \) basic \((n-q)\)-form which doesn’t vanish at \( x \) so that by Lemma 5.4 \( \alpha \land \mu \) is \( G \) invariant \( n \)-form which doesn’t vanish at \( x \). It is simple to check that \( \iota_Y [d(\iota_Y \alpha)]_x = 0 \) if and only if \( \iota_Y [d(\iota_Y \alpha) \land \mu]_x = 0 \).

By using the fact \( \mu \) is \( \delta \)-closed and by applying equation (5.2) to the invariant one-form \( \iota_Y (\alpha \land \mu) = (\iota_Y \alpha) \land \mu \), we find

\[
\iota_Y [d(\iota_Y \alpha) \land \mu]_x = \iota_Y [d(\iota_Y \alpha) \land \mu]_x = (-1)^q [d(\iota_Y \alpha) \land \mu]_x = 0.
\]

Thus \( \tilde{\eta} [d(\iota_Y \alpha)]_x = 0 \). Now computing

\[
\tilde{\eta} (\tilde{\chi})_x = [\mu_x^\ast (\iota_Y \alpha)(\tilde{\chi})]_x = [\mu_x^\ast d(\iota_Y \alpha)(\tilde{\chi})]_x = [d(\iota_Y \alpha)(\tilde{\chi})]_x = 0
\]

and using the invariance of \( \tilde{\eta} \) we get \( d\tilde{\eta} = 0 \). This proves ii) and therefore \( H^q(G, G_x) \neq 0 \).

To prove the converse, choose \( x_0 \in M \). Then by Theorem 4.1 the hypothesis \( A^q(G, G_x) \neq 0 \) implies there exists a non-vanishing \( G \) invariant \( q \)-chain \( X_0 \) on an invariant open neighbourhood \( U \) of \( x_0 \). Suppose that the rank of \( \kappa(Vert M) \) is \( s \) and that the rank of \( \kappa(TM) \) is \( r \). Let \( Y_a, a = 1, \ldots s \) be a local frame about \( x_0 \) for \( \kappa(Vert M) \) consisting of invariant vector fields. Choose invariant vector fields \( Z_t, t = s + 1, \ldots, r \) which together with \( Y_a \) form a local frame about \( x_0 \) for \( \kappa(TM) \). Refine \( U \) so all these objects exist on an invariant open set which we again call \( U \).

We now show that if \( H^q(G, G_x) \neq 0 \) then \( \mathcal{L}_{Y_a} X_0 = 0 \). First we compute

\[
(\mathcal{L}_{Y_a} \alpha)(X_0) = (\alpha(X_0)) - \alpha(\mathcal{L}_{Y_a} X_0) = -\alpha(\mathcal{L}_{Y_a} X_0).
\]
Expanding out the left side of this equation we get
\[(5.9) \quad d(\iota_\alpha Y_\alpha) + \iota_\alpha d\alpha \mid (\chi_0) = -\alpha(\mathcal{L}_{Y_\alpha}\chi_0).\]
Immediately \(\iota_\chi_0\iota_\alpha Y_\alpha d\alpha = 0\), because \(Y_\alpha\) is vertical, while condition ii) implies by the argument used above that \(d(\iota_\alpha Y_\alpha) \mid (\chi_0) = 0\), and so \((5.9)\) along with Lemma 5.4 implies \(\mathcal{L}_{Y_\alpha}\chi_0 = 0\).
To finish the proof of the theorem we now show there exists an invariant \(K\) which doesn’t vanish at \(x_0\) so that \(\chi = K\chi_0\) satisfies equation \((5.4)\) for \(Y_\alpha, Z_t\). Using the fact that \(\mathcal{L}_{Y_\alpha}\chi_0 = 0\), it is easy to check \(\mathcal{L}_{Y_\alpha}(K\chi_0) = 0\). The conditions \(\mathcal{L}_{Z_t}(K\chi_0) = 0\) leads to the differential equations for \(K\)
\[(5.10) \quad \tilde{Z}_t(K) + \tilde{K}\lambda_{Z_t} = 0.\]
The integrability conditions for \(K\) or \(\tilde{K}\) can be easily verified by a computation using Lemma 5.5. Therefore there exists an open neighbourhood \(\tilde{V}\) of \(\tilde{x}_0\) and a non-vanishing \(\tilde{K}\) which is a solution to \((5.10)\) on \(\tilde{q}^{-1}(\tilde{V})\).

6. Examples

Example 1. As our first example consider the two dimensional solvable group \(G = \mathbb{R}^* \times \mathbb{R}\) with coordinates \((a, b)\) acting on \(\mathbb{R} \times \mathbb{R}^* \times \mathbb{R}\) by
\[(a, b) \ast (x, y, z) = (ax + b, ay, z).\]
This is a free action and so \(H^2(\mathfrak{g}, G_x) = H^2(\mathfrak{g})\) and one easily computes \(H^2(\mathfrak{g}) = 0\).
We proceed to check Theorem 5.6 for this example. The most general \(G\) invariant vertical 2-chain \(\chi\) is of the form
\[\chi = K(z)y^2\partial_x \wedge \partial_y.\]
The invariant vector fields are
\[R = f(z)y\partial_x + g(z)y\partial_y + h(z)\partial_z.\]
Computing \(\mathcal{L}_y \partial_y \chi\) we get
\[\mathcal{L}_y \partial_y \chi = K(z)y^2\partial_x \wedge \partial_y\]
and so, consistent with Theorem 5.6 and Theorem 5.1 there is no choice of non-zero \(K(z)\) so that \((5.4)\) is satisfied, and no cochain exists.

Example 2. Consider the action of the two dimensional Abelian group \(G = \mathbb{R}^2\) with coordinates \((a, b)\) on \(\mathbb{R}^2\) given by
\[(a, b) \ast (x, y) = (x + ay + b, y).\]
The fact the group is Abelian implies \(H^1(\mathfrak{g}, G_x) \neq 0\), for all \(x \in \mathbb{R}^2\), so a cochain map exists by Theorem 5.1. The \(G\) invariant vertical 1-cochains are given by
\[(6.1) \quad \chi = K(y)\partial_x,\]
and the invariant vector fields are
\[R = a(y)\partial_x.\]
So every cochain $\chi$ in \((6.1)\) satisfies \((5.4)\). This examples demonstrates the fact that the $q$-chain may not be unique, and by a further simple computation, that the cochain map $\rho\chi$ may not be surjective.

As a final remark we state a theorem on the surjectivity of $\rho\chi$.

**Theorem 6.1.** Let $\chi$ be a $G$ invariant vertical $q$-chain such that $\rho\chi$ defines a cochain map. Then $\rho\chi$ is surjective if and only if there exists $\alpha \in \Omega^q(M)$ such that $\alpha(\chi) = 1$ and $\alpha$ is $G$ invariant.

See [4] and [5] for other examples.

**References**

[1] I.M. Anderson and M.E. Fels, *Exterior Differential Systems with Symmetry*, submitted, Acta. Appl. Math., SPT. 2004.
[2] I.M. Anderson and M.E. Fels, *Topology and its Applications*, **123**, 2002, pp. 443-459
[3] I.M. Anderson and M.E. Fels, *Commun. Math. Phys.*, **212**, 2000, pp. 653–686
[4] I.M. Anderson and M.E. Fels, *Amer. Jour. Math.*, **119**, 1997, pp. 609–670
[5] M.E. Fels, C.G. Torre, *Class. Quantum Grav.*, **19**, 2002, pp. 641–675.
[6] C. Chevalley, S. Eilenberg, *Trans. Amer. Math. Soc.*, **63**, 1948, pp. 85–124.
[7] P.J. Olver, *Applications of Lie groups to differential equations*, Springer-Verlag, 1993
[8] M. Spivak, *A comprehensive introduction to differential geometry Vol. 5*, Publish or Perish, 1979

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