On Two Unsolved Problems Concerning Matching Covered Graphs

Dedicated to the memory of Professor W.T.Tutte on the occasion of the centennial of his birth

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Abstract

A cut $C := \partial(X)$ of a matching covered graph $G$ is a separating cut if both its $C$-contractions $G/X$ and $G/\overline{X}$ are also matching covered. A brick is solid if it is free of nontrivial separating cuts. We (Carvalho, Lucchesi and Murty) showed in [6] that the perfect matching polytope of a brick may be described without recourse to odd set constraints if and only if it is solid, and in [8] we proved that the only simple planar solid bricks are the odd wheels. The problem of characterizing nonplanar solid bricks remains unsolved.

A bi-subdivision of a graph $J$ is a graph obtained from $J$ by replacing each of its edges by paths of odd length. A matching covered graph $J$ is a conformal minor of a matching covered graph $G$ if there exists a bi-subdivision $H$ of $J$ which is a subgraph of $G$ such that $G - V(H)$ has a perfect matching. For a fixed matching covered graph $J$, a matching covered graph $G$ is $J$-based if $J$ is a conformal minor of $G$ and, otherwise, $G$ is $J$-free. A basic result due to Lovász [14] states that every nonbipartite matching covered graph is either $K_4$-based or is $\overline{C_6}$-based or both, where $\overline{C_6}$ is the triangular prism. In [13], we (Kothari and Murty) showed that, for any cubic brick $J$, a matching covered graph $G$ is $J$-free if and only if each of its bricks is $J$-free. We
also found characterizations of planar bricks which are $K_4$-free and those which are $C_6$-free. Each of these problems remains unsolved in the nonplanar case.

In this paper we show that the seemingly unrelated problems of characterizing nonplanar solid bricks on the one hand, and on the other of characterizing nonplanar $C_6$-free bricks are essentially the same. We do this by establishing that a simple nonplanar brick, other than the Petersen graph, is solid if and only if it is $C_6$-free. In order to prove this, we first show that any nonsolid brick has one of the four graphs $C_6$, the bicorn, the tricorn and the Petersen graph (depicted in Figure 1) as a conformal minor. Then, using a powerful theorem due to Norine and Thomas [18], we show that the bicorn, the tricorn and the Petersen graph are dead-ends in the sense that any simple nonplanar nonsolid brick which contains any one of these three graphs as a proper conformal minor also contains $C_6$ as a conformal minor.

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1 Background and Preliminaries

1.1 Matching Covered Graphs

For graph theoretical terminology and notation, we essentially follow the book by Bondy and Murty [1]. All graphs considered in this paper are loopless.

For a subset $S$ of the vertex set of a graph $G$, we denote the number of odd components of $G - S$ by $o(G - S)$. Tutte [19] established the following fundamental theorem.

**Theorem 1.1 (Tutte’s Theorem)** A graph $G$ has a perfect matching if and only if

$$o(G - S) \leq |S|$$

for any subset $S$ of $V(G)$.

An edge $e$ of a graph $G$ is **admissible** if there is some perfect matching of $G$ that contains it. A **matching covered** graph is a connected graph of order at least two in which every edge is admissible. A simple argument shows that a matching covered graph cannot have a cut vertex. Tutte [19] used Theorem 1.1 to strengthen a classical theorem of J. Petersen (1891) by showing that every 2-connected cubic graph is matching covered.
Let $G$ be a graph that has a perfect matching. A subset $B$ of $V(G)$ is a barrier if $o(G - B) = |B|$. Using Tutte’s Theorem, one may easily deduce the following characterization of inadmissible edges:

**Proposition 1.2** Let $G$ be a graph that has a perfect matching. An edge $e$ is inadmissible if and only if there exists a barrier that contains both ends of $e$.

This yields the following characterization of matching covered graphs:

**Corollary 1.3** Let $G$ be a connected graph that has a perfect matching. Then $G$ is matching covered if and only if every barrier of $G$ is a stable set.

There is an extensive theory of matching covered graphs and its applications. In the book by Lovász and Plummer [16], matching covered graphs are referred to as 1-extendable graphs. The terminology we use here was introduced by Lovász in his seminal work [15] and in the follow-up work by three of us in [4]. This work relies on a number of notions introduced and results proved by us and, among others, Lovász, and Norine and Thomas. For the benefit of the readers, we shall describe them and provide references. For uniformity, we have found it necessary, in some cases, to change the notation and terminology used in the original sources.

A number of cubic graphs play special roles in this theory. They include the complete graph $K_4$, and the four graphs shown in Figure 1, namely the triangular prism which is denoted by $C_6$ because it is the complement of the 6-cycle, the bicorn and the tricorn (as they resemble, in our imagination, the two-cornered and three-cornered hats worn by pirates), and the ubiquitous Petersen graph which we denote by $P$.

### 1.2 Bi-subdivisions

A bi-subdivision of an edge $e$ of a graph $J$ consists of subdividing it by inserting an even number of vertices. A graph $H$ obtained from $J$ by bi subdividing each edge, in any subset of the edges, is called a bi-subdivision of $J$. (The term ‘bi-subdivision’ is due to McCuaig [17]. The same notion has been called an ‘even subdivision’ by some authors, and ‘odd subdivision’ by some others). If $J$ is a matching covered graph, then any bi-subdivision $H$ of $J$ is also matching covered; in fact, there is a one-to-one correspondence
between the sets of perfect matchings of $J$ and of $H$. Figure 2 shows bi-subdivisions of the complete graph $K_4$ and of the triangular prism $\overline{C_6}$.

Figure 1: (a) $\overline{C_6}$, (b) the bicorn, (c) the tricorn, and (d) $\mathbb{P}$

Figure 2: (a) A bi-subdivision of $K_4$; (b) a bi-subdivision of $\overline{C_6}$
1.3 Splicing and Separation

1.3.1 The operation of splicing

Let $G_1$ with a specified vertex $u$, and $G_2$ with a specified vertex $v$, be two disjoint graphs. Suppose that the degree of $u$ in $G_1$ and the degree of $v$ in $G_2$ are the same, and that $\pi$ is a bijection between the set $\partial_1(u)$ of edges of $G_1$ incident with $u$, and the set $\partial_2(v)$ of edges of $G_2$ incident with $v$. We denote by $(G_1 \circ G_2)_{u,v,\pi}$ the graph obtained from the union of $G_1 - u$ and $G_2 - v$ by joining, for edge $e$ in $\partial_1(u)$, the end of $e$ in $G_1 - u$ to the end of $\pi(e)$ in $G_2 - v$, and refer to it as the graph obtained by splicing $G_1$ at $u$ with $G_2$ at $v$ with respect to the bijection $\pi$. The proof of following proposition is straightforward:

**Proposition 1.4** The graph $(G_1 \circ G_2)_{u,v,\pi}$ obtained by splicing two matching covered graphs $G_1$ and $G_2$ is also matching covered. \[\square\]

In general, the result of splicing two graphs $G_1$ and $G_2$ depends on the choices of $u$, $v$, $\pi$. (Both the pentagonal prism and the Petersen graph can be realized as splicings of two copies of the 5-wheel at their hubs.) However, if $H$ is a vertex-transitive cubic graph, then the result of splicing $G_1 = K_4$ with $G_2 = H$ does not depend, up to isomorphism, on the choices of $u$, $v$, and $\pi$, and we denote it simply by $K_4 \circ H$. More generally, for any cubic graph $H$, the result of splicing $K_4$ and $H$ depends, up to isomorphism, only on the orbit of the automorphism group of $H$ to which $v$ belongs (and the choices of $u$ and $\pi$ are immaterial); and we denote it simply by $(K_4 \circ H)_v$.

For example, since both $K_4$ and $C_6$ are vertex-transitive, there is only one way of splicing $K_4$ with itself or with $C_6$. Thus $K_4 \circ K_4 = C_6$, and $K_4 \circ C_6$ is the bicorn. But the automorphism group of the bicorn has three orbits and, consequently, three different graphs (one of which is the tricorn) can be produced by splicing $K_4$ with the bicorn (Figure 3). The automorphism group of the tricorn also has three orbits and splicing $K_4$ with the tricorn yields three different graphs (Figure 4).

1.3.2 Cuts and cut-contractions

For a subset $X$ of the vertex set $V(G)$ of a graph $G$, we denote the set of edges of $G$ which have exactly one end in $X$ by $\partial(X)$ and refer to it as the cut of $X$. (For a vertex $v$ of $G$, we simplify the notation $\partial(\{v\})$ to $\partial(v)$.) If
Figure 3: Cases of splicing $K_4$ and the bicorn

Figure 4: Cases of splicing $K_4$ and the tricorn

$G$ is connected and $C := \partial(X) = \partial(Y)$, then $Y = X$ or $Y = \overline{X} = V - X$, and we refer to $X$ and $\overline{X}$ as the shores of $C$.

For a cut $C$ of a matching covered graph, the parities of the cardinalities of the two shores are the same. Here, we shall only be concerned with those cuts that have shores of odd cardinality. A cut is trivial if either shore has just one vertex, and is nontrivial otherwise.

Given any cut $C := \partial(X)$ of a graph $G$, one can obtain a graph by shrinking $X$ to a single vertex $x$ (and deleting any resulting loops); we denote it by $G/(X \to x)$ and refer to the vertex $x$ as its contraction vertex. The two graphs $G/(X \to x)$ and $G/((\overline{X} \to \overline{x})$ are the two $C$-contractions of $G$. When the names of the contraction vertices are irrelevant we shall denote the two
C-contractions of $G$ simply by $G/X$ and $G/\overline{X}$.

### 1.3.3 Separating cuts

A cut $C := \partial(X)$ of a matching covered graph $G$ is _separating_ if both the $C$-contractions of $G$ are also matching covered. All trivial cuts are clearly separating cuts. Figures 5(a) and (b) show examples of separating cuts, but the cut indicated in Figure 5(c) is not a separating cut.

Figure 5: (a) and (b) are separating cuts, but (c) is a cut that is not

The following proposition provides a necessary and sufficient condition under which a cut in a matching covered graph is a separating cut, and is easily proved.

**Proposition 1.5 (H, Lemma 2.19)** A cut $C$ of a matching covered graph $G$ is a separating cut if and only if, given any edge $e$, there is a perfect matching $M_e$ of $G$ such that $e \in M_e$ and $|C \cap M_e| = 1$. □

Let $G_1$ and $G_2$ be two disjoint matching covered graphs. Then, as noted before, any graph $G = (G_1 \odot G_2)_{u,v,\pi}$ obtained by splicing $G_1$ and $G_2$ is also matching covered. Clearly the cut $C := \partial(V(G_1) - u) = \partial(V(G_2) - v)$, which we refer to as the _splicing cut_, is a separating cut of $G$, and $G_1$ and $G_2$ are the two $C$-contractions of $G$. Conversely, if $C := \partial(X)$ is a separating cut of a matching covered graph $G$, then $G$ can be recovered from its two $C$-contractions $G_1 := G/(X \rightarrow \overline{X})$ and $G_2 := G/(X \rightarrow x)$, by splicing them at the contraction vertices with respect to the identity mapping between $\partial_1(\overline{X})$, which is equal to $C$, and $\partial_2(x)$, which is also equal to $C$. Thus, a matching covered graph $G$ has a nontrivial separating cut if and only if it can be obtained by splicing two smaller matching covered graphs $G_1$ and $G_2$. 
1.3.4 Separating cut decompositions

Suppose that $G$ is a matching covered graph with a nontrivial separating cut $C$. Then the two $C$-contractions of $G$ provide a decomposition of $G$ into two smaller matching covered graphs. If either $G_1$ or $G_2$ has nontrivial separating cuts, then that graph can be decomposed into even smaller matching covered graphs. By applying this procedure repeatedly, any matching covered graph may be decomposed into a list of matching covered graphs which are free of nontrivial separating cuts. However, depending on the choice of cuts used in the decomposition procedure, the results of two 'separating cut decompositions' may result in entirely different lists of graphs which are free of separating cuts. For example, consider the matching covered graph shown in Figure 6 with the four indicated separating cuts $C_1$, $C_2$, $C_3$, and $C_4$. By first considering cut-contractions with respect to $C_1$, we obtain a $K_4$ (with multiple edges) which is free of nontrivial separating cuts, and a graph in which $C_2$ is a nontrivial separating cut. Of the two $C_2$-contractions of this second graph, one is a $K_4$ (with multiple edges) and the other is a graph in which $C_3$ is a nontrivial separating cut. The $C_3$-contractions of this third graph are two copies of $K_4$ (with multiple edges). Thus, this sequence of separating cut contractions yields a list of four copies of $K_4$ (with multiple edges). However, just one application of the decomposition procedure which involves the cut $C_4$, yields two copies of the 5-wheel (with multiple edges) which happen to be free of nontrivial separating cuts.

1.4 Bricks and Braces

1.4.1 Tight cuts, bricks and braces

A cut $C$ in a matching covered graph $G$ is a tight cut of $G$ if $|C \cap M| = 1$ for every perfect matching $M$ of $G$. It follows from Proposition 1.5 that every tight cut of $G$ is also a separating cut of $G$. However, the converse does not always hold. For example, the cut shown in Figure 5(b) is a separating cut, but it is not a tight cut.

A matching covered graph, which is free of nontrivial tight cuts, is a brace if it is bipartite and is a brick if it is nonbipartite.
1.4.2 Tight cut decompositions

A tight cut decomposition of a matching covered graph consists of applying the previously described separating cut decomposition procedure where we restrict ourselves to cut-contractions with respect to nontrivial tight cuts. Clearly, any application of the tight cut decomposition procedure on a given matching covered graph produces a list of bricks and braces. In striking contrast to the separating cut decomposition procedure, the tight cut decomposition procedure has the following significant property established by Lovász [15].

**Theorem 1.6 (Uniqueness of the Tight Cut Decomposition)**

Any two applications of the tight cut decomposition procedure on a matching covered graph yield the same list of bricks and braces (up to multiple edges).

In particular, any two applications of the tight cut decomposition procedure on a matching covered graph $G$ yield the same number of bricks; we denote this invariant by $b(G)$ and refer to it as the number of bricks of $G$. 

![Figure 6: A matching covered graph that admits two different separating cut decompositions](image-url)
1.4.3 Barrier cuts and 2-separation cuts

Let $G$ be a matching covered graph. If $B$ is a barrier of $G$ then, for any perfect matching $M$ of $G$ and any odd component $K$ of $G - B$, a simple counting argument shows that $|M \cap \partial(V(K))| = 1$ (and also that $G - B$ has no even components). Consequently, $\partial(V(K))$ is a tight cut of $G$ for any component $K$ of $G - B$. Tight cuts of $G$ which arise in this manner are called barrier cuts associated with the barrier $B$ (see Figure 5(a)).

We shall refer to a vertex cut $\{u, v\}$ of $G$ which is not a barrier as a 2-separation of $G$. When $\{u, v\}$ is a 2-separation of $G$, the fact that $\{u, v\}$ is not a barrier implies that each component of the disconnected graph $G - u - v$ is even. Let $S$ denote the vertex set of the union of a nonempty proper subset of the components of $G - u - v$. It can then be verified that both $C := \partial(S \cup \{u\})$ and $D := \partial(S \cup \{v\})$ are tight cuts of $G$. Tight cuts which arise in this manner are called 2-separation cuts. See Figure 7.

![Figure 7: Two 2-separation cuts in a matching covered graph](image)

An ELP-cut in a matching covered graph is a tight cut which is either a barrier cut or is a 2-separation cut. A theorem due to Edmonds, Lovász and Pulleyblank [11] states that if a matching covered graph has nontrivial tight cuts, then it has an ELP-cut. The following characterization of bricks is a consequence of that basic result.

**Theorem 1.7 (The ELP Theorem)** A matching covered graph is a brick if and only if it is 3-connected and is free of nontrivial barriers.
Characterization of braces can be found in [15] and [16].

1.4.4 Six families of bricks and braces

We now describe six families of graphs that are of particular interest in this work.

Odd Wheels

Let \( C_k \) be an odd cycle of length at least three. Then, the *odd wheel* \( W_k \) is defined to be the join of \( C_k \) and \( K_1 \). The smallest odd wheel is \( W_3 \cong K_4 \). For \( k \geq 5 \), \( W_k \) has one vertex of degree \( k \), called its *hub*; the remaining \( k \) vertices lie on a cycle which is referred to as the *rim*. Every odd wheel is a *brick*.

Biwheels

Let \( C_{2k} \) be an even cycle of length six or more with bipartition \((X, X')\), and let \( h \) and \( h' \) be two vertices (hubs) not on that cycle. The graph obtained by joining \( h \) to each vertex in \( X \), and \( h' \) to each vertex in \( X' \), is known as a *biwheel* with \( h \) and \( h' \) as its hubs. We shall denote it by \( B_{2k} \). Figure 8(a) shows a biwheel on eight vertices (the two half-edges labelled \( e \) are to be identified to complete the rim); it is isomorphic to the cube. Every biwheel is a *brace*.

Truncated biwheels

Let \((v_1, v_2, \ldots, v_{2k})\) be a path of odd length, where \( k \geq 2 \), and let \( h \) and \( h' \) be two vertices (hubs) not on that path. We shall refer to the graph obtained by joining \( h \) to vertices in \( \{v_1, v_3, \ldots, v_{2k-1}\} \cup \{v_{2k}\} \), and joining \( h' \) to vertices in \( \{v_1\} \cup \{v_2, v_4, \ldots, v_{2k}\} \) as a *truncated biwheel*. We shall denote it by \( T_{2k} \). The smallest truncated biwheel is isomorphic to \( C_6 \). Figure 8(b) shows a truncated biwheel on eight vertices. Every truncated biwheel is a *brace*.

Norine and Thomas [18] refer to truncated biwheels as *lower prismoids*.

Prisms

Let \((u_1, u_2, \ldots, u_k, u_1)\) and \((v_1, v_2, \ldots, v_k, v_1)\) be two disjoint cycles of length at least three. The graph on \( 2k \) vertices obtained from the union of these two
cycles by joining $u_i$ to $v_i$, for $1 \leq i \leq k$ is the $k$-prism. (In other words, the $k$-prism is the Cartesian product of the $k$-cycle $C_k$ and the complete graph $K_2$.) We shall denote the $k$-prism by $P_{2k}$. The 3-prism $P_6$, commonly known as the triangular prism, is isomorphic to $C_6$, and the 4-prism is isomorphic to the cube. The graph shown in Figure 8(b) is the 5-prism, commonly known as the pentagonal prism. (For every odd $k$, the $k$-prism is a brick and for every even $k$, the $k$-prism is a brace.)

**Möbius Ladders**

Let $(u_1, u_2, \ldots, u_k)$ and $(v_1, v_2, \ldots, v_k)$ be two disjoint paths of length at least two. The graph obtained from the union of these two paths by joining $u_i$ to $v_i$, for $1 \leq i \leq k$, and, in addition, joining $u_1$ to $v_k$, and $u_k$ to $v_1$, is known as the Möbius ladder of order $2k$. The Möbius ladder of order six is isomorphic to $K_{3,3}$, and the Möbius ladder of order eight is shown in Figure 8(a) (this drawing is to be taken as an embedding on the Möbius strip). When $k$ is odd, the Möbius ladder of order $2k$ is a brace and, when $k$ is even, the Möbius ladder of order $2k$ is a brick.

**Staircases**

Let $(u_1, u_2, \ldots, u_k)$ and $(v_1, v_2, \ldots, v_k)$ be two disjoint paths of length at least two. The graph obtained from the union of these two paths by adjoining two new vertices $x$ and $y$, and joining $u_i$ to $v_i$, for $1 \leq i \leq k$, and, in addition joining $x$ to $u_1$ and $v_1$, $y$ to $u_k$ and $v_k$, and $x$ and $y$ to each other, is referred to as a staircase by Norine and Thomas [18]. The staircase on six vertices is isomorphic to the triangular prism. The graph shown in Figure 8(b) is the staircase on eight vertices. Every staircase is a brick.
1.4.5 Near-bricks

Let $G$ be a matching covered graph and let $C := \partial(X)$ be a separating cut of $G$ such that the subgraph $G[X]$ induced by $X$ is bipartite. As $C$ is a separating cut, by definition, the $C$-contraction $G_1 := G/(X \to \overline{X})$ is matching covered and thus $|X|$ is odd. So, one of the colour classes of $G[X]$ is larger than the other. We denote the larger colour class by $X_+$ and the smaller colour class by $X_-$ and refer to them, respectively, as majority and minority parts of $X$. If the contraction vertex $\overline{X}$ were joined by an edge to a vertex $v$ in the minority part, then the graph $G_1 - \{\overline{X}, v\}$ would be a bipartite graph with colour classes of different cardinalities, implying that there is no perfect matching of $G_1$ which contains the edge $\{\overline{X}, v\}$. This is impossible because $G_1$ is matching covered. The following results may be easily deduced from this observation.

**Proposition 1.8** Let $C := \partial(X)$ be a separating cut of a matching covered graph $G$ such that the subgraph $G[X]$ is bipartite. Then majority part $X_+$ of $X$ is a barrier of $G$, and $C$ is a tight cut associated with this barrier. 

**Corollary 1.9** A cut of a bipartite matching covered graph is separating if and only if it is tight.

**Corollary 1.10** A matching covered graph $G$ is bipartite if and only if $b(G) = 0$.

**Corollary 1.11** Let $G$ be a matching covered graph with $b(G) = 1$, and let $C$ be a tight cut of $G$. Then one of the $C$-contractions of $G$ is bipartite, the majority part of that shore is a barrier of $G$, and $C$ is a barrier cut associated with that barrier.

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Figure 9: (a) a Möbius ladder, (b) the bicorn (a staircase)
We refer to a matching covered graph $G$ with $b(G) = 1$ as a *near-brick*. Properties of near-bricks are in many ways akin to those of bricks and, in trying to prove statements concerning bricks by induction, it is often convenient to try to prove the corresponding statements for near-bricks.

### 1.4.6 Bi-contractions and retracts

Suppose that $v_0$ is a vertex of degree two in a matching covered graph $G$ of order four or more, and let $v_1$ and $v_2$ denote the two neighbours of $v_0$. Then $\partial(X)$, where $X := \{v_0, v_1, v_2\}$, is a tight cut of $G$. The graph $G/X$ is a brace on four vertices, and $G/X$ is a matching covered graph on $|V(G)| - 2$ vertices, and is said to be obtained by *bi-contracting* the vertex $v_0$ in $G$.

Let $G$ be any matching covered graph which has order four or more and is not an even cycle. Then one can obtain a sequence $(G_1, G_2, \ldots, G_r)$ of graphs such that (i) $G_1 = G$, (ii) $G_i$ has no vertices of degree two, and (iii) for $2 \leq i \leq r$, the graph $G_i$ is obtained from $G_{i-1}$ by bi-contracting some vertex of degree two in it. Then, up to isomorphism, the graph $G_r$ does not depend on the sequence of bi-contractions performed (see [7] Proposition 3.11). We denote it by $\tilde{G}$ and refer to it as the *retract* of $G$. The retracts of the two graphs shown in Figure 2 are, respectively, $K_4$ and $C_6$. More generally, the retract of a bi-subdivision $H$ of a brick $J$ is $J$ itself.

Now we define the operation of bi-splitting a vertex which may be regarded as the converse of the above defined operation of bi-contraction. Given any vertex $v$ of a matching covered graph $H$, we first split $v$ into two new vertices $v_1$ and $v_2$ (called *outer vertices*), add a third new vertex $v_0$ (called the *inner vertex*), join $v_0$ to both $v_1$ and $v_2$, and then distribute the edges of $H$ incident to $v$ among $v_1$ and $v_2$ in such a way that both $v_1$ and $v_2$ have at least two distinct neighbours. We denote the resulting graph by $H\{v \rightarrow (v_1, v_0, v_2)\}$ and say that it is obtained from $H$ by *bi-splitting* the vertex $v$ (see Figure 11). It is easy to see that $H\{v \rightarrow (v_1, v_0, v_2)\}$ is a matching covered graph and that it has two more vertices and two more edges than $H$. Note that $H$ can be recovered from $H\{v \rightarrow (v_1, v_0, v_2)\}$ by bi-contracting the vertex $v_0$.

### 1.5 Solid bricks

A matching covered graph is *solid* if every separating cut of $G$ is a tight cut. In particular, any bipartite matching covered graph is solid. That is,
Figure 10: Bi-splitting a vertex in a matching covered graph

in a bipartite matching covered graph, every separating cut is also a tight cut (Corollary 1.9). However, nonbipartite graphs, even bricks, may have separating cuts which are not tight. (For example, see Figure 5(b).) Solid bricks are precisely those bricks which are free of nontrivial separating cuts. It can be verified that the graph shown in Figure 5(c) is a solid brick. The following theorem is a consequence of Corollary 2.26 in [4].

**Theorem 1.12** A matching covered graph \( G \) is solid if and only if each of its cut-contractions with respect to any tight cut is also solid. (In particular, \( G \) is solid if and only if each of its bricks is solid.)

The notion of solid matching covered graphs was introduced in [4] by three of us (CLM – Carvalho, Lucchesi and Murty). We noted there that certain special properties that are enjoyed by bipartite graphs are shared by the more general class of solid matching covered graphs, and exploited these properties in establishing the validity of a conjecture due to Lovász.

In a later paper [6], we (CLM) showed that bipartite matching covered graphs and solid near-bricks share the property that their perfect matching polytopes may be defined without using the odd set inequalities.

The problem of recognizing solid bricks is in co-\( \mathcal{NP} \), since any nontrivial separating cut serves as a certificate for demonstrating that a brick is nonsolid. In the same paper mentioned above, we (CLM) presented a proof of the following (unpublished) theorem due to two friends that provides another succinct certificate for demonstrating that a brick is nonsolid:
Theorem 1.13 (B. A. Reed and Y. Wakabayashi)

A brick $G$ has a nontrivial separating cut if and only if it has two disjoint odd cycles $C_1$ and $C_2$ such that $G - (V(C_1) \cup V(C_2))$ has a perfect matching. □

We showed in [8] that the only simple planar solid bricks are the odd wheels. The solid-brick-recognition problem remains unsolved for nonplanar graphs.

Unsolved Problem 1.14 Characterize solid bricks. (Is the problem of deciding whether or not a given brick is solid in the complexity class $\mathcal{NP}$? Is it in $\mathcal{P}$?)

As stated in the abstract, the objective of this paper is to establish a connection between this unsolved problem and another basic problem (Problem 1.31) concerning matching covered graphs.

A graph is odd-intercyclic if any two odd cycles in it have a vertex in common. (Odd wheels and Möbius ladder of order $4k$, $k \geq 1$, are examples of odd-intercyclic bricks.) It follows from Theorem 1.13 that every odd-intercyclic brick is solid.

Kawarabayashi and Ozeki [12] showed that an internally 4-connected graph $G$ is odd-intercyclic if and only if it satisfies one of the following conditions: (i) $G - v$ is bipartite for some $v \in V$, (ii) $G - \{e_1, e_2, e_3\}$ is bipartite for some three edges $e_1, e_2$ and $e_3$ which constitute the edges of a triangle of $G$, (iii) $|V| \leq 5$, or (iv) $G$ can be embedded in the projective plane so that each face boundary has even length.

The above result leads to a polynomial-time algorithm for recognizing odd-intercyclic bricks. However, not all solid bricks are odd-intercyclic; the graph shown in Figure 5(c) is a solid brick which is not odd-intercyclic. We have not been able to find a cubic solid brick that is not odd-intercyclic.

Conjecture 1.15 Every cubic solid brick is odd-intercyclic.

We conclude this section by defining an important parameter related to each nontrivial separating cut $C$ of a nonsolid brick $G$. Since $G$ is free of nontrivial tight cuts, it follows that some perfect matching $M$ meets $C$ in at least three edges. We define the characteristic of $C$ to be the minimum value of $|M \cap C|$, where the minimum is taken over all perfect matchings $M$ of $G$ that meet $C$ in at least three edges. (In particular, the characteristic of a nontrivial separating cut is at least three.)
1.6 Removable Classes

1.6.1 Removable edges and doubletons

Let $G$ be a matching covered graph and let $e$ and $f$ be two edges of $G$. We say that $e$ depends on $f$, and write $e \Rightarrow f$, if every perfect matching of $G$ that contains $e$ also contains $f$. Edges $e$ and $f$ are mutually dependent if $e \Rightarrow f$ and $f \Rightarrow e$, and we write $e \Leftrightarrow f$ to signify this. It is easy to see that $\Leftrightarrow$ is an equivalence relation on the edge set $E(G)$ of $G$. In general, the cardinality of an equivalence class may be arbitrarily large. (For example, $C_{2k}$, the cycle of length $2k$, has two equivalence classes of size $k$ each.) However, in a brick, an equivalence classes has at most two edges (see Theorem 1.17).

The relation $\Rightarrow$ may be visualized by means of the directed graph on the edge set $E(G)$ of $G$, where there is an arc with $e$ as tail and $f$ as head whenever $e \Rightarrow f$. From this digraph we obtain a new digraph, denoted by $D(G)$, by identifying equivalence classes under the relation $\Leftrightarrow$. Clearly $D(G)$ is acyclic. We refer to the equivalence classes that correspond to the sources of $D(G)$ as minimal classes. For any edge $e$ of $G$, a source $Q$ of $D$ that contains an edge $f$ that depends on $e$ is said to be a minimal class induced by $e$. (Here we admit the possibility that $e$ and $f$ may be the same.)

If $R$ is a minimal class of $G$, then every edge of $G - R$ is admissible. Moreover, if $G - R$ happens to be connected then $G - R$ is matching covered; in this case, we shall say that $R$ is a removable class.

An edge $e$ of a matching covered graph $G$ is a removable edge if $G - e$ is matching covered, and a pair $\{e, f\}$ of edges of $G$ is a removable doubleton if neither $e$ nor $f$ is individually removable, but the graph $G - \{e, f\}$ is matching covered. In the former case, $\{e\}$ is a minimal class, and in the latter, $\{e, f\}$ is a minimal class.

The result below concerning braces will prove to be useful.

**Theorem 1.16** ([3], Lemma 3.2) Every edge in a brace of order six or more is removable. 

1.6.2 Removable classes in bricks

A matching covered graph $G$ is near-bipartite if it has a removable doubleton $R$ such $G - R$ is a bipartite matching covered graph.
Theorem 1.17 ([3, Lemma 2.3], [15, Lemma 3.4]) Any equivalence class \( R \) in a brick \( G \) has cardinality at most two. Moreover, if \( |R| = 2 \), say \( R = \{e, f\} \), then \( G - e - f \) is a bipartite graph, both ends of \( e \) are in one part of the bipartition of \( G - e - f \) and both ends of \( f \) are in the other part.

In particular, every removable class of a brick is either a removable edge or is a removable doubleton. It follows from the above theorem that every brick with a removable doubleton is indeed near-bipartite. Truncated biwhewels, prisms of order 2 (modulo 4), Möbius ladders of order 0 (modulo 4), and staircases are examples of near-bipartite bricks. The bicorn (Figure 1(b)) has two removable doubletons, and also a unique removable edge. The two bricks \( K_4 \) and \( C_6 \) have three removable doubletons each, but have no removable edges; the following was established by Lovász [15]:

Theorem 1.18 Every brick distinct from \( K_4 \) and \( C_6 \) has a removable edge.

There is an extensive discussion of removable edges in bricks in our paper [10]. We now present a technical result which will turn out to be useful in the proof of the Main Theorem (2.1) in Section 2.

For a fixed vertex \( v_0 \) of a matching covered graph \( G \), a subset \( M \) of the edges of \( G \) is a \( v_0 \)-matching if \( |M \cap \partial(v)| = 1 \) for each vertex \( v \) distinct from \( v_0 \), and if \( |M \cap \partial(v_0)| > 1 \). A simple counting argument shows that \( |M \cap \partial(v_0)| \) is odd. The following may also be easily verified:

Proposition 1.19 Let \( G[A, B] \) be a bipartite graph such that \( |A| = |B| \). Then \( G \) does not have a \( v_0 \)-matching for any vertex \( v_0 \).

Lemma 1.20 Let \( G \) be a brick, \( v_0 \) be a vertex of \( G \), and \( M \) be a \( v_0 \)-matching. Let \( e \) be an edge in \( \partial(v_0) - M \) and let \( Q \) be a minimal class of \( G \) induced by \( e \). Then, \( Q \) is a singleton which is disjoint from \( M \).

Proof: If \( e \) is the only member of \( Q \) then there is nothing to prove. Let \( f \) be an edge of \( Q \) such that \( f \neq e \). Then, \( f \Rightarrow e \) in \( G \). As \( G - e \) has a perfect matching, and \( f \) is inadmissible, by Proposition 1.2, \( G - e \) has a barrier \( B \) containing both ends of \( f \), and \( G - e - B \) has exactly \( |B| \) odd components, two of which contain the ends of \( e \). In particular, \( v_0 \) lies in an odd component \( K \) of \( G - e - B \).

As \( |V(G)| \) is even, \( |M \cap \partial(v_0)| \) is odd and so, \( |M \cap \partial(V(K))| \) is also odd. Moreover, \( |M \cap \partial(V(K'))| \geq 1 \) for any other odd component \( K' \) of \( G - e - B \). By simple counting, and taking into account that \( e \notin M \), we conclude that
$|M \cap \partial(V(K'))| = 1$, for each odd component $K'$ of $G - e - B$, and that each vertex of $B$ is matched by $M$ with a vertex in an odd component of $G - e - B$. Thus, $f \notin M$. As $f$ is an arbitrary edge of $Q - \{e\}$, and since $e \notin M$, we conclude that the minimal class $Q$ does not meet $M$.

It remains to argue that $|Q| = 1$. By Theorem 1.17 $Q$ has at most two edges. Suppose that $|Q| = 2$. By Theorem 1.17 $G - Q$ is a bipartite matching covered graph. Since $Q \cap M$ is empty, $M$ is a $v_0$-matching of the bipartite graph $G - Q$, and this contradicts Proposition 1.19. Thus, $|Q| = 1$.  

\[\square\]

1.6.3 Removable classes in solid graphs

Here we state some useful results regarding the properties and existence of removable edges in solid graphs.

**Theorem 1.21 ([4, Theorem 2.2.8])** For any removable edge $e$ of a solid matching covered graph $G$, the graph $G - e$ is also solid.

We now proceed to prove a result which we shall refer to as the Lemma on Odd Wheels (1.23) which will play a crucial role in the proof of the main theorem of Section 2. (A weaker version of this result appeared in [5].) The proof of this lemma relies on the following:

**Theorem 1.22 ([10, Theorem 6.11])** Let $G$ be a solid brick, let $v$ be a vertex of $G$, let $n$ be the number of neighbours of $v$, and let $d$ be the degree of $v$. Enumerate the $d$ edges of $\partial(v)$ as $e_i := vv_i$, for removable $v$. Enumerate the $d$ edges of $\partial(v)$ as $e_1, e_2, \ldots, e_d$, where $e_i$ joins $v$ to $v_i$, for $i = 1, 2, \ldots, d$. Assume that neither $e_1$ nor $e_2$ is removable in $G$. Then, $n = 3$ and, for $i = 1, 2$, there exists an equipartition $(B_i, I_i)$ of $V(G)$ such that

(i) $e_i$ is the only edge of $G$ that has both ends in $I_i$,

(ii) every edge that has both ends in $B_i$ is incident with $v_3$, and

(iii) the subgraph $H_i$ of $G$, obtained by the removal of $e_i$ and each edge having both ends in $B_i$, is matching covered and bipartite, with bipartition $(B_i, I_i)$.

Moreover, $B_1 = (I_2 - v) \cup \{v_3\}$ and $B_2 = (I_1 - v) \cup \{v_3\}$. (See Figure 11 for an illustration.)
We say that $G$ is a $v_0$-wheel if $G$ is a wheel having $v_0$ as a hub.

**Lemma 1.23 (Lemma on Odd Wheels)** Let $G$ be a simple solid brick, $v_0$ be a vertex of $G$, and $M_0$ be a $v_0$-matching. Then either $G$ is a $v_0$-wheel or $G$ has a removable edge $e \notin M_0 \cup \partial(v_0)$.

**Proof:**

**Case 1** The brick $G$ has a vertex $v \neq v_0$ that has degree four or more in $G$.

As $G$ is simple, at least two edges, $e_1$ and $e_2$, are not in $M_0 \cup \partial(v_0)$ but are incident with $v$. By Theorem 1.22, one of $e_1$ and $e_2$ is removable in $G$.

We may thus assume that every vertex $v \neq v_0$ has degree three in $G$.

**Case 2** Every vertex of $G - v_0$ is adjacent to $v_0$.

Since every vertex $v \neq v_0$ has degree three in $G$ and is adjacent to $v_0$, every vertex distinct from $v_0$ has degree two in $G - v_0$. Then $G - v_0$ is a collection of cycles. By the 3-connectivity of $G$, it follows that $G - v_0$ is a cycle and, consequently, $G$ is a $v_0$-wheel.

**Case 3** The previous cases are not applicable.

Every vertex $v \neq v_0$ of $G$ has degree three in $G$. Moreover, $G$ has a vertex, $v \neq v_0$, that is not adjacent to $v_0$. Let $e_i := vv_i$, $i = 1, 2, 3$, be the three edges incident with $v$. Adjust notation so that $e_3 \in M_0$.

1.23.1 One of the edges $e_1$ and $e_2$ is removable in $G$. 

Figure 11: Graphs $G$, $G - e_1$ and $G - e_2$
**Proof:** Assume the contrary. By Theorem 1.22, \( G \) has an equipartition \((B_1, I_1)\) such that \( e_1 \) is the only edge having both ends in \( I_1 \) and every edge having both ends in \( B_1 \) is incident with \( v_3 \). Moreover, the bipartite graph \( H \) obtained from \( G \) by the removal of \( e_1 \) and each edge having both ends in \( B_1 \) is matching covered. Vertex \( v_3 \), a vertex adjacent to \( v \), is distinct from \( v_0 \). Thus, \( v_3 \) has degree three. As \( H \) is matching covered, precisely one edge of \( \partial(v_3) \), say \( f \), has both ends in \( B_1 \). But \( e_3 \) is the only edge of \( M_0 \) incident with \( v \) and its end \( v \) is in \( I_1 \). Thus, \( f \not\in M_0 \). In particular, \( M_0 \) is a \( v_0 \)-matching of \( H \), and this contradicts Proposition 1.19. \( \square \)

The proof of the Lemma on Odd Wheels is complete. \( \square \)

### 1.7 Ear Decompositions

#### 1.7.1 Deletions and additions of ears

A path \( P := v_0v_1 \ldots v_\ell \) of odd length in a graph \( G \) is a single ear in \( G \) if each of its internal vertices \( v_1, v_2, \ldots, v_{\ell-1} \) has degree two in \( G \). If \( P_1 \) and \( P_2 \) are two vertex-disjoint single ears in \( G \), then \( \{P_1, P_2\} \) is a double ear with \( P_1 \) and \( P_2 \) as its constituent single ears. The deletion of a single ear \( P \) from \( G \) consists of deleting all the internal vertices of \( P \), and the graph obtained by deleting \( P \) from \( G \) is denoted by \( G - P \). Likewise, the deletion of a double ear \( \{P_1, P_2\} \) consists of deleting each of its constituent single ears \( P_1 \) and \( P_2 \).

A single ear \( P \) in a matching covered graph \( G \) is removable if the graph \( G - P \) obtained by deleting \( P \) from \( G \) is also matching covered. If \( P_1 \) and \( P_2 \) are two vertex-disjoint single ears neither of which is removable, but the graph \( G - P_1 - P_2 \) is matching covered, then the double ear \( \{P_1, P_2\} \) is removable. When the length of a single ear is one, then we identify it with its only edge.

The following basic result concerning ear decompositions was proved by Lovász and Plummer [16]:

**Theorem 1.24 (The Two-Ear Theorem)** Given any matching covered graph \( G \), there exists a sequence \((G_1, G_2, \ldots, G_r)\) of matching covered subgraphs of \( G \) such that:

1. \( G_1 = K_2 \) and \( G_r = G \); and
2. for \( 2 \leq i \leq r \), the graph \( G_{i-1} \) is obtained from \( G_i \) by the deletion of either a removable single ear or of a removable double ear.
1.7.2 Conformal subgraphs

A matching covered subgraph $H$ of a matching covered graph $G$ is conformal
if the graph $G - V(H)$ has a perfect matching. It is easily seen that this
notion obeys transitivity:

**Proposition 1.25** Any conformal subgraph of a conformal subgraph of a
matching covered graph $G$ is also a conformal subgraph of $G$. ■

Conformal subgraphs have been referred to by various other names (‘nice’
sugraphs, ‘central’ subgraphs and ‘well-fitted’ subgraphs) in the literature.
The following result is due to Lovász and Plummer [16].

**Theorem 1.26** A matching covered subgraph $H$ of a matching covered
graph $G$ is conformal if and only if there is some ear decomposition $G :=
(G_1, G_2, \ldots, G_r)$ of $G$ such that $H$ is one of the graphs in $G$. ■

It follows from the above theorem that if $H$ is a conformal matching
covered subgraph of a matching covered graph $G$, then $H$ can be obtained
from $G$ by a sequence of deletions of removable ears (single or double). But
the deletion of an ear amounts to first reducing that ear to one of length
one by means of bi-contractions, and then deleting the only edge of that ear.
This observation implies the following:

**Corollary 1.27** A matching covered subgraph $H$ of a matching covered
graph $G$ is conformal if and only if it can be obtained from $G$ by bi-contractive,
ations of vertices of degree two and deletions of removable classes. ■

Every ear decomposition of a bipartite matching covered graph involves
only single ear additions. However, in an ear decomposition of a nonbipartite
matching covered graph there must be at least one double ear addition.
Lovász established the following fundamental result concerning nonbipartite
graphs:

**Theorem 1.28** ([14]) Every nonbipartite matching covered graph has an
ear decomposition such that either the third graph is a bi-subdivision of $K_4$
or the fourth graph is a bi-subdivision of $C_6$.

This theorem gives rise to natural questions which are described in terms
of special types of minors of matching covered graphs which we now proceed
to discuss.
1.7.3 Conformal minors and matching minors

Let $G$ be a matching covered graph. A matching covered graph $J$ is a conformal minor of $G$ if some bi-subdivision $H$ of $J$ is a conformal subgraph of $G$. Figure 12(a) shows that $K_4$ is a conformal minor of $P$, the Petersen graph. It is not too difficult to show that $C_6$ is not a conformal minor of $P$. However, if $P + e$ is any graph obtained by adding an edge $e$ to $P$ joining two nonadjacent vertices, then $C_6$ is a conformal minor of $P + e$ as illustrated in Figure 12(b), and also of $K_4 \odot P$ as illustrated in Figure 12(c).

![Figure 12: Conformal minors: (a) $K_4$ of $P$; (b) $C_6$ of $P + e$; (c) $C_6$ of $K_4 \odot P$](image)

Given a fixed matching covered graph $J$, we say that a matching covered graph $G$ is $J$-based if $J$ is a conformal minor of $G$, and, otherwise $G$ is $J$-free. For example, the Petersen graph $P$ is $K_4$-based but is $C_6$-free, and $P + e$ depicted in Figure 12(b) is both $K_4$-based and $C_6$-based.

Theorem 1.28 implies that every nonbipartite matching covered graph is either $K_4$-based or is $C_6$-based (or both), and it raises two natural problems: characterize those matching covered graph that are $K_4$-free, and those that are $C_6$-free. Two of us (KM – Kothari and Murty) showed that it suffices to solve these problems for bricks by establishing the following result concerning cubic bricks.

**Theorem 1.29** ([13]) Suppose that $J$ is a cubic brick and that $C$ is a tight cut of a matching covered graph $G$. Then $G$ is $J$-free if and only if each $C$-contraction of $G$ is $J$-free. (In particular, $G$ is $J$-free if and only if each brick of $G$ is $J$-free.)
The restriction that $J$ be a cubic brick is crucial for the validity of the above statement. (Curiously, it is not valid even for cubic braces. For example, consider the graph $G := K_4 \odot K_{3,3}$. If $C$ denotes the unique nontrivial tight cut in $G$, one of the $C$-contractions of $G$ is the brace $K_{3,3}$. However, $K_{3,3}$ is not a conformal minor of $G$.)

In light of Theorem 1.29, it suffices to solve the following problems:

**Unsolved Problem 1.30** Characterize $K_4$-free bricks.

**Unsolved Problem 1.31** Characterize $C_6$-free bricks.

Using the brick generation theorem of Norine and Thomas, which will be described later on, we (KM) were able to resolve Problems 1.30 and 1.31 in the special case of planar bricks by proving the following results. (By a well-known theorem of Whitney (1933), every simple 3-connected planar graph has a unique embedding in the plane (Theorem 10.28 in [1].)

**Theorem 1.32 ([13])** A simple planar brick is $K_4$-free if and only if its (unique) planar embedding has precisely two odd faces.

**Theorem 1.33 ([13])** The only simple planar $C_6$-free bricks are the odd wheels, staircases of order $4k$, and the tricorn.

In the case of nonplanar bricks, Problems 1.30 and 1.31 remain unsolved.

Norine and Thomas [18] call a matching covered graph $J$ a matching minor of a matching covered graph $G$ if $J$ can be obtained from a conformal subgraph $H$ of $G$ by means of bi-contractions. By Corollary 1.27, any conformal subgraph $H$ of $G$ can be obtained from $G$ by means of bi-contractions and deletions of removable classes. And, if $H$ were a bi-subdivision of $J$, then $J$ can clearly be obtained from $H$ by means of bi-contractions. (A restricted bi-contraction is a bi-contraction of a vertex of degree two — one of whose neighbours also has degree two. If $H$ is a bi-subdivision of $J$, then $J$ can in fact be obtained from $H$ by restricted bi-contractions.)

It follows from the above observations that every conformal minor of $G$ is also a matching minor of $G$. But the converse is not true in general (due to the fact unrestricted bi-contractions are permissible in obtaining a matching minor of $G$.) For example, the wheel $W_5$ is a matching minor of the graph $G$ shown in Figure [13] because $W_5$ can be obtained from $G$ by first deleting the edge $e$ and then bi-contracting the vertex $v$ in the resulting graph. But
it is not a conformal minor of $G$ for the simple reason that $G$ has no vertices of degree greater than four.

However, it is easily seen that if a cubic matching covered graph $J$ is obtained from a matching covered graph $H$ by means of bi-contractions, then $H$ must be a bi-subdivision of $J$. Consequently, we have the following:

**Corollary 1.34** A cubic matching covered graph $J$ is a matching minor of a matching covered graph $G$ if and only if $J$ is a conformal minor of $G$.

![Figure 13: The wheel $W_5$ is a matching minor of $G$](image)

We conclude this section by noting that Theorems 1.12 and 1.21 together imply the following:

**Corollary 1.35** Every conformal minor of a solid matching covered graph is a solid matching covered graph.

However, not every conformal minor of a nonsolid graph is nonsolid. For example, the bicorn is nonsolid, but $K_4$, which is solid, is a conformal minor of the bicorn.

Since $C_6$ is nonsolid, we have the following consequence:

**Corollary 1.36** Every solid matching covered graph is $C_6$-free.
1.8 Robust cuts in bricks

1.8.1 $b$-invariant edges

Recall that, for a matching covered graph $G$, the symbol $b(G)$ represents the number of bricks in any tight cut decomposition of $G$.

A removable edge $e$ of a brick $G$ is $b$-invariant if $b(G - e) = b(G) = 1$. Motivated by his work on the matching lattice, Lovász [15] had conjectured that every brick distinct from $K_4$, $C_6$ and $P$ has a $b$-invariant edge. All bricks other than $K_4$ and $C_6$ have removable edges; in fact, every edge of the Petersen graph is removable. What is striking about Lovász’s conjecture is that it asserts that among bricks which have removable edges, Petersen graph is the only brick which has no $b$-invariant edges. In [5], three of us presented a proof of a strengthening of Lovász’s conjecture. We showed:

**Theorem 1.37** ([5]) Every brick distinct from $K_4$, $C_6$, the bicorn, and the Petersen graph $P$ has two $b$-invariant edges.

Many of the notions and results used in the proofs of the new results in this paper arose in that context of our proof of the above theorem. We record below only those definitions and results that are essential for proving the main result of this paper.

1.8.2 Existence of robust cuts

The starting point of our attempt to resolve Lovász’s conjecture was to investigate the implications of the existence of a removable edge in a brick which fails to be $b$-invariant. This led us to the serendipitous discovery of solid bricks. We were able to show:

**Theorem 1.38** ([4]) Any brick which has a removable edge that is not $b$-invariant has a nontrivial separating cut. (Consequently, every removable edge in a solid brick is $b$-invariant.)

It follows from the above theorem that it suffices to prove Lovász’s conjecture for nonsolid bricks. As every nonsolid brick $G$ has nontrivial separating cuts, it was natural to try to show that $G$ has a $b$-invariant edge inductively by considering cut-contractions of $G$ with respect to a suitable separating cut. This idea led us to the notion of a robust cut.
A nontrivial separating cut $C := \partial(X)$ of a nonsolid brick $G$ is robust if both the $C$-contractions $G/X$ and $G/\overline{X}$ of $G$ are near-bricks. We say that a robust cut $C$ is $k$-robust if $C$ has characteristic $k$. We were able to prove the following fundamental result.

**Theorem 1.39** ([5, Theorem 4.1]) Every simple brick distinct from the Petersen graph has a 3-robust cut. The Petersen graph has only 5-robust cuts.\(^1\)

Both the cut $C$ and $D$ in the brick depicted in Figure 14 are separating cuts, but only one of them, namely $C$, is a robust cut.

![Figure 14: Cut C is robust, but D is not](image)

**2 Conformal Minors of Nonsolid Bricks**

We shall refer to $\overline{C_6}$, the bicorn, the tricorn and the Petersen graph as the basic nonsolid bricks.

**Theorem 2.1 (Main Theorem)** Every nonsolid matching covered graph contains a basic nonsolid brick as a conformal minor.

\(^1\)In fact, in every simple brick distinct from the Petersen graph, every robust cut is 3-robust [2].
**Proof:** Let \( G \) be any nonsolid matching covered graph. We shall prove the validity of the assertion by induction on the number of edges of \( G \).

It follows from Theorem 1.28 that the smallest nonsolid matching covered graph is \( C_6 \), which is a basic nonsolid brick. For the general case, we adopt as the inductive hypothesis that every nonsolid matching covered graph with fewer edges than \( G \) has one of the four basic nonsolid bricks as a conformal minor.

**Case 1** \( G \) contains a proper conformal subgraph \( H \) that is a nonsolid matching covered graph

By the induction hypothesis, \( H \) contains a basic nonsolid brick as a conformal minor. Hence, by Proposition 1.25, \( G \) also contains a basic nonsolid brick as a conformal minor.

Note that this case applies when \( G \) has multiple edges.

**Case 2** Graph \( G \) has a nontrivial tight cut \( C \).

By Theorem 1.12 \( G \) has a \( C \)-contraction, \( G_1 \), that is nonsolid. By the induction hypothesis, \( G_1 \) contains a basic nonsolid brick as a conformal minor. Since each of the basic nonsolid bricks is cubic, it follows from Theorem 1.29 that \( G \) also contains a basic nonsolid brick as a conformal minor.

**Case 3** Previous cases do not apply.

The graph \( G \) is free of nontrivial tight cuts, hence \( G \) is either a brick or a brace. Every bipartite graph is solid. Thus, \( G \) is a brick. In fact, \( G \) is a simple nonsolid brick, free of nonsolid conformal minors. In sum,

**Lemma 2.2** Let \( R \) be a nonempty set of edges of \( G \). If \( G - R \) is matching covered then it is solid.

We shall prove that \( G \) is one of the four basic nonsolid bricks. If \( G \) is the Petersen graph then we are done. We may thus assume that \( G \) is not the Petersen graph. We shall prove that \( G \) is either \( C_6 \), the bicorn or the tricorn.

As \( G \) is not the Petersen graph, then, by Theorem 1.39 \( G \) has 3-robust cuts. Let \( C := \partial(X) \) be a 3-robust cut of \( G \) and let \( M_0 \) be a perfect matching of \( G \) such that \( |M_0 \cap C| = 3 \). Let \( G_1 := G/(X \rightarrow \bar{x}) \) and \( G_2 := G/(X \rightarrow x) \) be the two \( C \)-contractions of \( G \) obtained by contracting \( \bar{X} \) and \( X \) to single vertices \( \bar{x} \) and \( x \), respectively. As \( C \) is robust, the graphs \( G_1 \) and \( G_2 \) are near-bricks.
Lemma 2.3 Let $R$ be a nonempty set of edges of $G$. If $G_1 - R$ and $G_2 - R$ are both matching covered then the graphs $G_1 - R$ and $G_2 - R$ are both solid and $C - R$ is tight in $G - R$.

Proof: Suppose that $G_1 - R$ and $G_2 - R$ are both matching covered. Then, $G - R$ is matching covered, and the cut $C - R$ is separating in $G - R$. By Lemma 2.2, the graph $G - R$ is solid. Thus, $C - R$ must be tight in $G - R$. Moreover, by Theorem 1.12, both $(C - R)$-contractions of $G - R$ must be solid. That is, $G_1 - R$ and $G_2 - R$ are both solid. \end{proof}

Corollary 2.4 Let $e$ be an edge of $G$. If $G_1 - e$ and $G_2 - e$ are both matching covered then $e \in M_0$ and $G_1 - e$ and $G_2 - e$ are both solid.

Proof: Suppose that the graphs $G_1 - e$ and $G_2 - e$ are both matching covered. By Lemma 2.3, the cut $C - e$ is tight in $G - e$. Thus, $M_0$ is not a perfect matching of $G - e$, hence $e \in M_0$. Moreover, also by Lemma 2.3, the graphs $G_1 - e$ and $G_2 - e$ are both solid. \end{proof}

Lemma 2.5 The graphs $G_1$ and $G_2$ are bricks.

Proof: Suppose that $G_1$ is not a brick. As $G_1$ is a near-brick that is not a brick, it has nontrivial tight cuts. Moreover, by Corollary 1.11, for any tight cut $D$ of $G_1$, one of the $D$-contractions of $G_1$ must be bipartite, the other must be a near-brick.

2.5.1 Let $D$ be a nontrivial tight cut of $G_1$, and let $Y$ be its nonbipartite shore. The vertices $\overline{x}$ and $y$ lie in distinct parts of the bipartition of $H := G_1/(Y \rightarrow y)$.

Proof: The graph $H$ is bipartite and matching covered. Any part of $H$ that is disjoint with $\{\overline{x}, y\}$ is a nontrivial barrier of $G$. Thus, not only is $\overline{x}$ a vertex of $H$, but also it lies in the part of $H$ that does not contain vertex $y$. \end{proof}

Consider now a tight cut decomposition of $G_1$. For any tight cut $D$ of $G_1$, every tight cut in some $D$-contraction of $G_1$ is also a tight cut of $G_1$. Using this observation and applying (2.5.1) repeatedly, we conclude that there exists a nested sequence

$$X_0 \subset X_1 \subset \cdots \subset X_r = X \quad (r \geq 1)$$

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of subsets of $X$ such that the $r$ cuts $\partial(X_i)$, $0 \leq i < r$ are the tight cuts used in the tight cut decomposition of $G_1$, and, for $1 \leq i \leq r$, the graph $H_i := (G/(X_{i-1} \to x_{i-1}))/((X_i \to \overline{x_i})$ is a brace of order four or more. Moreover, $G_0 := G/X_0$ is a brick.

Let us analyze the situation in the brace $H_1$. See Figure 15. By (2.5.1), the vertices $x_0$ and $\overline{x_1}$ lie in distinct parts of $H_1$. Let $v$ denote a vertex in the same part that contains $\overline{x_1}$, but distinct from $\overline{x_0}$. Then, no edge incident with $v$ is in $C$.

Consider first the case in which $v$ is adjacent to at most one vertex of $X_0$. As $G$ is a brick, then $v$ is adjacent to three or more vertices of the brace $H_1$. Thus, $H_1$ has six or more vertices. By Theorem 1.6 all the edges of $H_1$ are removable. In particular, one of the edges of $H_1$ incident with $v$ is not in $M_0 \cup \partial(X_0)$. Thus, $G_1$ has a removable edge that does not lie in $M_0 \cup C$. This is a contradiction to Corollary 2.4.

Alternatively, suppose that $v$ is adjacent to two vertices of $X_0$, say, $w_1$ and $w_2$. The edges $vw_1$ and $vw_2$ are multiple edges in $G/X_0$, hence removable in $G_1/X_0$. At least one of the edges $vw_1$ and $vw_2$ is not in $M_0$. Adjust notation so that $vw_1 \notin M_0$. By Lemma 1.20 either $vw_1$ is removable in the brick $G_0$ or $G_0$ has an edge that is removable and does not lie in $M_0 \cup \partial(X_0)$. In both cases, $G_1$ has a removable edge that does not lie in $M_0 \cup C$, a contradiction to Corollary 2.4.

In all cases considered, we derived a contradiction. We deduce that $G_1$ is
a brick. Likewise, a similar argument may be used to prove that $G_2$ is also a brick. 

\textbf{Lemma 2.6} $C \subseteq M_0$.

\textbf{Proof}: Suppose, to the contrary, that $C - M_0$ contains an edge, $e$. By Corollary \ref{corollary:2.3} at least one of the graphs $G_1 - e$ and $G_2 - e$ is not matching covered. Adjust notation so that $G_1 - e$ is not matching covered. That is, the edge $e$ is not removable in the brick $G_1$. By Lemma \ref{lemma:1.20} $G_1$ has a removable edge, $f$, that does not lie in $M_0 \cup C$. Thus, $G_1 - f$ and $G_2 - f = G_2$ are both matching covered, a contradiction to Corollary \ref{corollary:2.4}. Indeed, $C \subseteq M_0$. 

\textbf{Lemma 2.7} If a $C$-contraction $H$ of $G$ is solid then $H = K_4$.

\textbf{Proof}: Adjust notation so that $G_1$ is solid. Assume, to the contrary, that $G_1 \neq K_4$. The cut $C$ consists only of three edges in $M_0$ and $G_1$ is a brick. Thus, $G_1$ is simple but is not a wheel having $\overline{x}$ as a hub. By the Lemma on Wheels, $G_1$ has a removable edge, $e$ that does not lie in $M_0 \cup C$. Thus, $G_1 - e$ and $G_2 - e = G_2$ are both matching covered and $e \notin M_0$. This is a contradiction to Corollary \ref{corollary:2.4}. Indeed, $G_1 = K_4$, as asserted. 

\textbf{Lemma 2.8} If a $C$-contraction $H$ of $G$ is not solid then $H$ is one of the four basic nonsolid bricks.

\textbf{Proof}: Suppose that a $C$-contraction of $G$ is not solid. Adjust notation so that $G_1$ is not solid. By induction, $G_1$ has a conformal minor $J$ that is one of the four basic nonsolid bricks. Thus, some bisubdivision $H$ of $J$ is a conformal subgraph of $G_1$. Assume that $G_1$ is not $J$. As $G_1$ is a brick, if $G_1 = H$ then $G_1 = J$. In this case, the assertion holds.

We may thus assume that $H$ is a proper subgraph of $G_1$. By Theorem \ref{theorem:1.26} $G_1$ has a removable ear $R$ such that $H$ is a conformal subgraph of $G_1 - R$. As $G_1$ is a brick, it follows that the edges of $R$ constitute either a removable edge or a removable doubleton. \footnote{It can be shown that $R$ is a singleton, but that is not necessary in this argument.}

If $R$ and $C$ are disjoint then $G_1 - R$ is matching covered and nonsolid (by Corollary \ref{corollary:1.35}), and $G_2 - R = G_2$ is matching covered. This is a contradiction to Lemma \ref{lemma:2.3} Thus, $R$ contains an edge, $e$, in $C$. Clearly, $e$ is the only edge of $R$ in $C$. Let $S$ be a minimal class of the dependence relation in $G_2$ induced
by edge \( e \). As \( G_2 \) is a brick, \( G_2 - S \) is matching covered. If \( e \in S \) then \( G_1 - (R \cup S) \) is \( G_1 - R \), a nonsolid matching covered graph, and \( G_2 - (R \cup S) \) is \( G_2 - S \), a matching covered graph. Alternatively, if \( e \notin S \) then \( G_1 - S = G_1 \) is nonsolid and \( G_2 - S \) is matching covered. In both alternatives, we derive a contradiction to Lemma 2.3. We deduce that \( G_1 \) is one of the four basic nonsolid bricks.

Let us denote the bicorn by \( R_8 \) and the tricorn by \( R_{10} \). We now know that \( G_1 \) and \( G_2 \) are both in the set \( \{ K_4, C_6, R_8, R_{10}, \mathbb{P} \} \). We now begin by showing that in fact no \( C \)-contraction of \( G \) is the Petersen graph and no \( C \)-contraction of \( G \) is the tricorn.

**Lemma 2.9** Neither \( G_1 \) nor \( G_2 \) is in \( \{ \mathbb{P}, R_{10} \} \).

**Proof:** Assume, to the contrary, that \( G_2 \) is the Petersen graph. Every one of the 15 edges of \( \mathbb{P} \) is removable and 9 of them do not lie in \( M_0 \cup C \), a contradiction to Corollary 2.4. As asserted, \( G_2 \neq \mathbb{P} \).

Suppose now, to the contrary, that \( G_2 \) is the tricorn. The tricorn has three removable edges, \( e_i, i = 1, 2, 3 \). The three removable edges of the tricorn lie in a hexagon, \( H \), together with the edges \( f_i, i = 1, 2, 3 \). See Figure 16.

![Figure 16: The three removable edges of the tricorn: \( e_1, e_2 \) and \( e_3 \)](image-url)

If \( M_0 \) does not contain at least one of these three edges then again we get a contradiction to Corollary 2.4. We may thus assume that \( \{ e_1, e_2, e_3 \} \subset M_0 \). In this case, the contraction vertex \( x \) of \( G_2 \) cannot be in \( V(H) \), hence \( H \) is an
$M_0$-alternating cycle. We may then replace $M_0$ by its symmetric difference with $E(H)$, and again obtain a contradiction to Corollary 2.4.

We conclude that $G_2 \notin \{R_{10}, \mathbb{F}\}$. The same conclusion holds for $G_1$.

**Lemma 2.10** At least one $C$-contraction of $G$ is solid.

**Proof:** Assume the contrary. By Lemma 2.8 both $C$-contractions of $G$ are basic nonsolid bricks. By Lemma 2.9 $G_1$ and $G_2$ are both in $\{C_6, R_8\}$.

Assume that $G_2 = C_6$. The brick $C_6$ has three removable doubletons $R_i := \{e_i, f_i\}$, $i = 1, 2, 3$. See Figure 17.

![Figure 17: The three removable doubletons of $C_6$: $\{e_i, f_i\}$, $i = 1, 2, 3$](image17)

No vertex of $C_6$ is incident to edges of all three doubletons. Thus, $G_2$ has a removable doubleton $R$ disjoint with $C$, hence $G_1 - R = G_1$ is nonsolid and $G_2 - R$ is matching covered. This is a contradiction to Lemma 2.3.

Alternatively, assume that $G_2 = R_8$. The graph $R_8$ has three removable classes, the two doubletons $\{e_i, f_i\}$, $i = 1, 2$ and the edge $e$. See Figure 18.

![Figure 18: The three removable classes of $R_8$: the doubletons $\{e_i, f_i\}$, $i = 1, 2$ and the edge $e$](image18)
The edge $e$ and $\{e_1, f_1\}$ are disjoint. Thus, $G_2$ has a removable class $R$ disjoint with $C$, hence $G_1 - R = G_1$ is nonsolid and $G_2 - R$ is matching covered. This is a contradiction to Lemma 2.3.

In all cases considered, we derived a contradiction. Indeed, at least one $C$-contraction of $G$ is solid.

If $G_1$ and $G_2$ are both solid then, by Lemma 2.7, $G_1$ and $G_2$ are both $K_4$, therefore $G$ is $\overline{C}_6$. The assertion holds in this case. We may thus assume that at least one of $G_1$ and $G_2$ is nonsolid. Adjust notation so that $G_2$ is nonsolid. By Lemma 2.10, the brick $G_1$ is solid. By Lemmas 2.7, 2.8 and 2.9, the brick $G_1$ is $K_4$ and the brick $G_2$ is either $\overline{C}_6$ or the bicorn.

If $G_2$ is $\overline{C}_6$ then $G$ is the bicorn. Consider next the case in which $G_2$ is the bicorn. If $x$ is not incident with the only removable edge $e$ of $G_2$ but $e \in M_0$ (see Figure 18), then there is only one possibility, up to automorphisms. The edge $e$ lies in an $M_0$-alternating quadrilateral $Q$ and we may replace $M_0$ by its symmetric difference with $E(Q)$, in contradiction to Corollary 2.4. Thus, $x$ is an end of $e$. In that case, $G$ is the tricorn.

Indeed, if $G$ is not the Petersen graph then $G$ is either $\overline{C}_6$, the bicorn or the tricorn. The proof of Theorem 2.1 is complete.

\section{3 Equivalence of Problems 1.14 and 1.31}

\subsection{3.1 Thin and strictly thin edges}

Motivated by the problem of recursively generating bricks, we were led to the notion of thin edges. An edge $e$ of a brick $G$ is thin if the retract of $G - e$ is also a brick. (Our definition of a thin edge in [8] was phrased in terms of sizes of barriers, but is equivalent to the one given here.) Using Theorem 1.37, we proved in [8] the following assertion.

\textbf{Theorem 3.1} Every brick distinct from $K_4$, $\overline{C}_6$ and $\mathbb{P}$ (the Petersen graph) has a thin edge.\hfill\Box

The above theorem implies the following corollary which has the flavour of Theorem 1.24.

\textbf{Corollary 3.2} Given any brick $G$, there exists a sequence $(G_1, G_2, \ldots, G_r)$ of bricks such that:
(i) $G_r = G$ and $G_1 \in \{K_4, \overline{C_6}, \mathbb{P}\}$; and

(ii) for $1 < i \leq r$, the brick $G_i$ has a thin edge $e_i$ such that $G_{i-1}$ is the retract of $G_i - e_i$.

This corollary is the basis of a recursive procedure for generating bricks described in [8]. We showed that there exist four elementary ‘expansion operations’ which can be used to build any brick starting from one of $K_4$, $\overline{C_6}$, and $\mathbb{P}$. (The simplest of these operations consists of just adding an edge joining two distinct vertices of a given brick $G$. The other three involve bi-splitting vertices and adding edges.)

We associate with each thin edge a number called its index, as defined below. Let $G$ be a brick and let $e$ be a thin edge of $G$. Then the retract of $G - e$, by definition, is a brick. The index of $e$ is:

- **zero**, if both ends of $e$ have degree four or more in $G$;
- **one**, if exactly one end of $e$ has degree three in $G$;
- **two**, if both ends of $e$ have degree three in $G$ and edge $e$ does not lie in a triangle;
- **three**, if both ends of $e$ have degree three in $G$ and edge $e$ lies in a triangle.

Examples of thin edges of indices one, two, and three are indicated by solid lines in the three bricks, respectively, shown in Figure 19.

![Figure 19](image)

Figure 19: (a) The wheel $W_5$; (b) the pentagonal prism; (c) the tricorn

The following consequence of Theorem 129 will be useful later.
Proposition 3.3 \((\text{[13]}))\) Let \(G\) be a brick and \(e\) be a thin edge of \(G\). For any cubic brick \(J\), if the retract of \(G - e\) is \(J\)-based then \(G\) is also \(J\)-based. \(\square\)

In order to establish recursive procedures for generating simple bricks, one needs the notion of a strictly thin edge. An edge \(e\) of a simple brick \(G\) is strictly thin if \(e\) is thin and the retract of \(G - e\) is simple. There are five infinite families of bricks that are free of strictly thin edges; these are (i) odd wheels, (ii) prisms of order 2(modulo 4), (iii) Möbius ladders of order 0(modulo 4), (iv) staircases, and (v) truncated biwheels. We refer to bricks in these five families together with the Petersen graph as Norine-Thomas bricks. (Note that \(K_4\) is the Möbius ladder of order four, and \(C_6\) is the prism of order six.) For brevity, we shall denote the family of all Norine-Thomas bricks by \(\mathcal{NT}\).

Norine and Thomas established the following strengthening of Theorem 3.1.

Theorem 3.4 \((\text{[13]}))\) Every simple brick \(G\) which is not a Norine-Thomas brick has a strictly thin edge. \(\square\)

The work of Norine and Thomas is independent of our work and uses entirely different methods. After learning about the statement of their result, we were able to show that it is possible to derive Theorem 3.4 from our Theorem 3.1. Our proof appears in an unpublished report \([9]\). As an immediate consequence of the above theorem, we have:

Corollary 3.5 Given any simple brick \(G\), there exists a sequence 
\((G_1, G_2, \ldots, G_r)\)
of simple bricks such that:

(i) \(G_r = G\) and \(G_1\) is in \(\mathcal{NT}\); and

(ii) for \(1 < i \leq r\), the brick \(G_i\) has a strictly thin edge \(e_i\) such that \(G_{i-1}\) is the retract of \(G_i - e_i\). \(\square\)

In the same paper \([13]\), Norine and Thomas have also proved the following powerful generalization of Theorem 3.4; it belongs to a class of theorems in structural graph theory known as ‘splitter theorems’. To state this generalization, we need to define a new class of graphs. The graph \(T_{2k}^+\) is obtained from the truncated biwheel \(T_{2k}\) by joining its hubs. The extended Norine-Thomas family \(\mathcal{NT}^+\) is the union of \(\mathcal{NT}\) and \(\{T_{2k}^+ : k \in \mathbb{Z}, k \geq 3\}\).
Theorem 3.6 Let $G$ be a simple brick which is not in $\mathcal{N}T^+$ and let $J$ be a simple brick that is distinct from $K_4$ and $\overline{C_6}$. If $J$ is a matching minor of $G$ then there exists a sequence $G_1, G_2, \ldots, G_r$ of simple bricks such that:

(i) $G_r = G$ and $G_1 = J$, and

(ii) for $1 < i \leq r$, the brick $G_i$ has a strictly thin edge $e_i$ such that $G_{i-1}$ is the retract of $G_i - e_i$.

Since any cubic brick which is a conformal minor of $G$ is also a matching minor of $G$, the above theorem is applicable to the case in which $J$ is a cubic brick, distinct from $K_4$ and $\overline{C_6}$, that happens to be conformal minor of $G$.

3.2 Proof of the equivalence

Let us recall that every solid brick is $\overline{C_6}$-free. The only simple planar solid bricks are the odd wheels. The only simple planar $\overline{C_6}$-free bricks are staircases of order 0 (modulo 4), the tricorn, and of course the odd wheels. We now show that apart from the staircases of order 0 (modulo 4), the tricorn, and the ubiquitous Petersen graph, a simple brick is solid if and only if it is $\overline{C_6}$-free. It suffices to prove the following:

Theorem 3.7 Any simple nonplanar nonsolid brick $G$, distinct from the Petersen graph, is $\overline{C_6}$-based.

Proof: As the only nonplanar members of the family $\mathcal{N}T^+$ are the Möbius ladders of order 0 (modulo 4) and the Petersen graph, it follows that $G \notin \mathcal{N}T^+$. Also, by Theorem 2.1 $G$ contains one of the four basic nonsolid bricks as a conformal minor. To complete the proof, it suffices to show that if $G$ has either the bicorn, the tricorn, or the Petersen graph as a conformal minor, then it also has $\overline{C_6}$ as a conformal minor. Towards this end, let $J$ be one of the above-mentioned bricks (that is, bicorn, tricorn or the Petersen graph) such that $J$ is a conformal minor of $G$.

By Theorem 3.6 there exists a sequence $G_1, G_2, \ldots, G_r$ of simple bricks such that (i) $G_r = G$ and $G_1 = J$, and (ii) for $1 < i \leq r$, the brick $G_i$ has a strictly thin edge $e_i$ such that $G_{i-1}$ is the retract of $G_i - e_i$. In particular, $G_1 = J$ is the retract of $G_2 - e_2$. Since $J$ is a cubic brick, it follows that $e_2$ is a strictly thin edge of index zero and, hence, that $J = G_2 - e_2$. In other words, $G_2$ is obtained from $J$ by joining two nonadjacent vertices by an edge. By Proposition 3.3 every cubic brick that is a conformal minor of $G_2$ is also
a conformal minor of $G$. Thus, in order to complete the proof, all we need to do is to show that any brick obtained from either the bicorn, the tricorn, or the Petersen graph contains a bi-subdivision of $\overline{C_6}$ as a conformal subgraph. This is routine. 

\[\blacksquare\]

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