CONVERGENCE OF SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS TO POWER-TYPE FUNCTIONS

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Abstract. In this article we study the asymptotic behavior of solutions of some fractional differential equations. We prove convergence to power type functions under some assumptions on the nonlinearities. Our results extend and generalize some existing well-known results on solutions of ordinary differential equations. Appropriate estimations and lemmas such as a fractional version of L'Hopital's rule are used.

1. Introduction

We consider the initial value problems
\[
(C^\alpha D_0^\alpha x)'(\tau) = f(\tau, x(\tau), C^\beta D_0^\beta x(\tau)), \quad 0 < \beta < \alpha < 1, \tau > 0, \quad C^\alpha D_0^\alpha x(\tau)|_{\tau=0} = b_2, \quad C^\beta D_0^\beta x(\tau)|_{\tau=0} = b_1, \quad b_1, b_2 \in \mathbb{R},
\]
and
\[
C^\alpha D_0^\alpha x(\tau) = f(\tau, x(\tau), C^\beta D_0^\beta x(\tau)), \quad 0 \leq \beta < \alpha < 1, \tau > 0, \quad x(\tau)|_{\tau=0} = b,
\]
where \(C^\alpha D_0^\alpha\) is the Caputo fractional derivative. The definition of the Caputo fractional derivative is given in the next section. We prove that the solutions of (1.1) approach power type functions and the solutions of (1.2) are bounded. To this end, the fractional differential problems (1.1) and (1.2) are first transformed into equivalent integral equations in appropriate underlying spaces. Various appropriate estimates, comparison theorems and lemmas are used. Moreover, we prove a Caputo fractional version of L'Hopital's rule. Our arguments here are quite different from those used so far in the literature.

The behavior of solutions of various classes of ODEs (ordinary differential equations) has been discussed in fairly a large number of papers in the literature. For example the equation
\[
x''(\tau) + f(\tau, x(\tau)) = 0,
\]
has been studied in [17, 18, 31] and other papers. The authors proved that, under various conditions, all solutions of (1.3) are asymptotic to \(ct + b\) as \(\tau \to \infty\), \(c, b \in \mathbb{R}\). For the equation
\[
x''(\tau) + f(\tau, x(\tau), x'(\tau)) = 0,
\]
2010 Mathematics Subject Classification. 34E10, 26A33, 34A08.
Key words and phrases. Asymptotic behavior; boundedness; fractional differential equation; Caputo fractional derivative; Riemann-Liouville fractional derivative.
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Submitted September 15, 2020. Published November 4, 2020.
see, for instance [8, 10, 19, 22, 23, 25, 27, 28]. It is proved that all solutions of (1.4) can be expressed asymptotically as \( \c + \beta \) as \( \tau \to \infty \), \( c, b \in \mathbb{R} \).

Medved and Pekárková [23] studied the one-dimensional p-Laplacian equation
\[
(\lvert x^p \rvert^{p-1} x')' = f(\tau, x, x'), \quad p > 1.
\] (1.5)
They demonstrated that any solution of (1.5) behaves asymptotically as \( b + c \tau \) as \( \tau \to \infty \) for some real numbers \( b, c \).

In [22], the initial value problem
\[
(\Phi_p(x'))' + f(\tau, x, x') = 0, \quad 1 < p < 2,
\]
\[
x(\tau_0) = x_0, \quad x'(\tau_0) = x_1, \quad \tau_0 \geq 1,
\] (1.6)
was studied, where \( \Phi_p(u) = \lvert u \rvert^{p-2} u \) and \( \Psi(\tau) \) is a continuous positive function. Sufficient conditions under which all solutions of (1.3) obey the asymptotic expansion \( x(\tau) = b + c \tau \) are established.

In contrast, the fractional case of equations (1.3) and (1.4) have been studied by comparatively a only few researchers; see, for instance [1, 2, 3, 4, 5, 6, 11, 12, 13, 14, 15, 20, 21, 24]. In 2009, Băleanu and Mustafa [1] studied the nonlinear fractional differential equation
\[
C^{\alpha}_0 x(\tau) = f(\tau, x(\tau)), \quad 0 < \alpha < 1, \quad \tau > 0.
\] (1.7)
They showed that the solutions of (1.7) are asymptotic to \( o(\tau^\alpha) \) as \( \tau \to \infty \), for some \( c \).

In 2012, Medved [20] studied the problem
\[
C^{\alpha+1}_a x(\tau) = f(\tau, x(\tau)), \quad 0 < \alpha < 1, \quad \tau \geq a > 1
\]
\[
x(a) = c_1, \quad x'(a) = c_2.
\] (1.8)
He demonstrated that any solution of (1.8) has the asymptotic property \( x(\tau) = b + c \tau \) as \( \tau \to \infty \), for some \( c, b \in \mathbb{R} \).

Also, in 2013, Medved [21] discussed the equation
\[
C^{\alpha+1}_a x(\tau) = f(\tau, x(\tau), x'(\tau)), \quad 0 < \alpha < 1, \quad \tau \geq a > 1
\] (1.9)
and proved that every solution of (1.9) can be expressed asymptotically as \( b + c \tau \) as \( \tau \to \infty \), for some \( c, b \in \mathbb{R} \).

Brestovanská and Medved [6] studied the problem
\[
x''(\tau) + f(\tau, x(\tau), x'(\tau)) + \sum_{i=1}^m r_i(\tau) \int_0^\tau (\tau-s)^{\alpha_i-1} f_i(s, x(s), x'(s))ds = 0
\]
\[
x(1) = b_1, \quad x'(1) = b_2, \quad 0 < \alpha_i < 1, \quad i = 1, 2, \ldots, m.
\] (1.10)
They showed that any solution enjoys the asymptotic expansion \( x(\tau) = b + c \tau \) as \( \tau \to \infty \), for some \( c, b \in \mathbb{R} \).

In 2015, Medved and Pospíšil [24] considered the equation
\[
C^\alpha_a x(\tau) = f(\tau, x(\tau), C_0^\beta x(\tau)), \quad \tau > a.
\] (1.11)
They proved that any solution \( x(\tau) \) with \( 0 < \beta < \alpha < 1 \), has the asymptotic property \( x(\tau) = c \tau^\beta + o(\tau^\beta) \) as \( \tau \to \infty \), for some \( c \in \mathbb{R} \). Also they proved that any solution \( x(\tau) \) of (1.11), \( 0 < \beta < 1 < \alpha < 2 \), has the asymptotic property \( x(\tau) = c \tau + o(\tau) \) as \( \tau \to \infty \), for some \( c \in \mathbb{R} \). Moreover, they proved that there
exists a constant $c \in \mathbb{R}$ such that any global solution $x(\tau)$ of the initial value problem

\[ C^{\alpha}_a x(\tau) = f(\tau, x(\tau), x'(\tau), \ldots, x^{(n-1)}(\tau), C^{\beta}_a x(\tau), \ldots, C^{\beta_n}_a x(\tau)), \quad \tau > a, x^{(n-1)}(a) = c_i, \quad i = 0, 1, \ldots, n - 1, \]

has the asymptotic property $x(\tau) = c\tau^k + o(\tau^k)$ as $\tau \to \infty$, where $k \in \max\{n - 1, \beta_m\}, 0 < \beta_1 < \ldots < \beta_m < \alpha < n$, and $n, m \in \mathbb{N}$.

In Sections 2 and 3, we prepare some material which will be needed later in our proofs. Sections 4, 5 and 6 are devoted to the main results on the asymptotic behavior results and boundedness of solutions for non-fractional and fractional source terms, respectively.

2. Preliminaries

In this section, we introduce some basic definitions, notation, properties and lemmas to be used in our results. We refer the reader to cite Kilbas, Podl-1999, Samko for more details.

**Definition 2.1** ([16]). We introduce the space

\[ C_\eta[a, b] = \{ h : (a, b] \to \mathbb{R} : h(\tau - a)^\eta \in C[a, b]) \}, \quad 1 > \eta \geq 0. \]

**Definition 2.2** ([16]). The left-sided Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by

\[ \mathcal{I}^\alpha_a f(\tau) := \frac{1}{\Gamma(\alpha)} \int_a^\tau \frac{f(s)}{(\tau - s)^{1-\alpha}} ds, \quad \tau > a, \]

provided that the right hand side exists.

**Definition 2.3** ([16]). The left-sided Riemann-Liouville fractional derivative of order $\alpha \geq 0$, $n - 1 \leq \alpha < n$, $n = [-\alpha]$, is defined by

\[ D^\alpha_a f(\tau) = D^n \mathcal{I}^{n-\alpha}_a f(\tau) = \left( \frac{d}{d\tau} \right)^n \mathcal{I}^{n-\alpha}_a f(\tau) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{d\tau} \right)^n \int_a^\tau \frac{f(s)}{(\tau - s)^{n-\alpha+1}} ds, \quad \tau > a. \]

In particular, when $\alpha = n$ we have $D^n_a f = D^n f$, and when $\alpha = 0$, $D^0_a f = f$.

**Definition 2.4** ([16]). The left-sided Caputo fractional derivative of order $\alpha \geq 0$, $n - 1 \leq \alpha < n$, $n = [-\alpha]$, is defined by

\[ C^{\alpha}_a f(\tau) = \mathcal{I}^{n-\alpha}_a f^{(n)}(\tau), \quad \tau > a. \]

The fractional integral and fractional derivative of power functions have the same effect as the integer-order integral and derivative. Namely,

**Lemma 2.5** ([16]). If $\beta > 0$ and $\alpha \geq 0$, then

\[ \mathcal{I}^\alpha_a (\tau - a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta + \alpha)} (\tau - a)^{\alpha + \beta + 1}, \quad \alpha > 0, \tau > a, \]

\[ C^{\alpha}_a (\tau - a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} (\tau - a)^{\beta - \alpha - 1}, \quad \alpha \geq 0, \tau > a. \]

If $\beta = 1$, then $(C^{\alpha}_a 1)(\tau) = 0$, $\tau > a$. 
The Riemann-Liouville fractional integral (Definition 2.2) satisfies the following semigroup property.

**Lemma 2.6.** [16] Let $0 \leq \eta < 1$, $\alpha > 0$ and $\beta > 0$. If $h \in C_{\eta}[a, b]$, then

$$\mathcal{I}_{a}^{\beta} \mathcal{I}_{a}^{\alpha} h(\tau) = \mathcal{I}_{a}^{\beta + \alpha} h(\tau), \quad \tau > a.$$  

The following result provides another composition of the fractional integration operator $\mathcal{I}_{a}^{\alpha}$ with the fractional differentiation operator $\mathcal{D}_{a}^{\beta}$.

**Lemma 2.7** (16). Let $\alpha > 0$, $0 \leq \eta < 1$, $n = [\alpha]$. If $h \in C_{\eta}[a, b]$ and $\mathcal{I}_{a}^{n-\alpha} h \in C_{\eta}^{n}[a, b]$, then

$$\mathcal{I}_{a}^{\alpha} \mathcal{D}_{a}^{\alpha} h(\tau) = h(\tau) - \sum_{i=1}^{n} \frac{(\mathcal{D}_{a}^{\eta} \mathcal{I}_{a}^{\alpha} h)(a)}{\Gamma(\alpha - i + 1)} (\tau - a)^{\alpha - i}, \quad \tau > a.$$  

**Lemma 2.8** (16). Let $\alpha > 0$, $n = [\alpha]$. If $h \in C^{n}[a, b]$ or $h \in AC^{n}[a, b]$, then

$$\mathcal{I}_{a}^{\alpha} \mathcal{D}_{a}^{\alpha} h(\tau) = h(\tau) - \sum_{k=0}^{n-1} \frac{h^{(k)}(a)}{k!} (\tau - a)^{k}, \quad \tau > a.$$  

**Lemma 2.9.** Let $0 < \beta \leq \alpha < 1$. If $h \in AC[a, b]$, then

$$\mathcal{C} \mathcal{D}_{a}^{\beta} h = \mathcal{I}_{0}^{\alpha - \beta} \mathcal{D}_{a}^{\alpha} h.$$  

**Proof.** From Definition 2.3 and Lemma 2.6 we have

$$\mathcal{C} \mathcal{D}_{a}^{\beta} h = \mathcal{I}_{0}^{1-\beta} h' = \mathcal{I}_{0}^{1-\beta} \mathcal{I}_{0}^{1-\alpha} h' = \mathcal{I}_{0}^{\alpha - \beta} \mathcal{C} \mathcal{D}_{a}^{\alpha} h.$$  

**Lemma 2.10** ([12]). Let $f \in L_{1}(0, \infty)$. Then

$$\lim_{\tau \to \infty} \frac{1}{\tau^\alpha} \mathcal{I}_{a}^{\alpha+1} f(\tau) = \frac{1}{\Gamma(\alpha + 1)} \int_{0}^{\infty} f(s) ds = \frac{1}{\Gamma(\alpha + 1)} \mathcal{I}_{0}^{\alpha} f(\infty), \quad \alpha > 0.$$  

**Lemma 2.11.** Let $0 < \alpha < 1$ and $0 \leq \eta < 1$. Assume that $x \in AC[0, \infty)$ and $\mathcal{I}_{0}^{1-\alpha} x' \in C_{\eta}^{1}[0, \infty)$. Then

$$\lim_{\tau \to \infty} \frac{x(\tau)}{\tau^\alpha} = \lim_{\tau \to \infty} \frac{\mathcal{C} \mathcal{D}_{0}^{\alpha} x(\tau)}{\Gamma(1 + \alpha)}.$$  

**Proof.** Since $x \in AC[0, \infty) \subset C_{\eta}[0, \infty)$ and $\mathcal{I}_{0}^{1-\alpha} x' \in C_{\eta}^{1}[0, \infty)$, we can use Lemma 2.7 to obtain

$$\mathcal{I}_{0}^{\alpha} \mathcal{D}_{0}^{\alpha} x'(\tau) = x'(\tau) - \frac{\mathcal{I}_{0}^{1-\alpha} x'(0)}{\Gamma(\alpha)} \tau^{\alpha - 1}, \quad \tau > 0.$$  

Applying $\mathcal{I}_{0}^{\beta}$ to both sides of (2.2), using Lemma 2.5 (with $\beta = \alpha$) and Lemma 2.6, we obtain

$$x(\tau) = x(0) + \frac{\mathcal{I}_{0}^{1-\alpha} x'(0)}{\Gamma(\alpha + 1)} \tau^{\alpha} + \mathcal{I}_{0}^{1+\alpha} \mathcal{D}_{0}^{\alpha} x'(\tau), \quad \tau > 0.$$  

Dividing both sides of (2.3) by $\tau^\alpha$, we obtain

$$\frac{x(\tau)}{\tau^\alpha} = \frac{x(0)}{\tau^\alpha} + \frac{\mathcal{I}_{0}^{1-\alpha} x'(0)}{\Gamma(\alpha + 1)} + \frac{1}{\tau^\alpha} \mathcal{I}_{0}^{1+\alpha} \mathcal{D}_{0}^{\alpha} x'(\tau), \quad \tau > 0.$$  

Next, we take the limit as $\tau \to \infty$, we arrive at

$$\lim_{\tau \to \infty} \frac{x(\tau)}{\tau^\alpha} = \frac{\mathcal{I}_{0}^{1-\alpha} x'(0)}{\Gamma(1 + \alpha)} + \frac{1}{\Gamma(1 + \alpha)} \lim_{\tau \to \infty} \mathcal{I}_{0}^{1} \mathcal{D}_{0}^{0} x'(\tau),$$  

(2.4)
where we used Lemma 2.10. Moreover, we conclude that
\[ \mathcal{J}_0^1 \mathcal{D}_0^{-a} x'(\tau) = \mathcal{J}_0^1 D_0^{1-a} x'(\tau) = \mathcal{J}_0^{1-a} x'(\tau) - \mathcal{J}_0^{1-a} x'(0) = C \mathcal{D}_0^a x(\tau) - C \mathcal{D}_0^a x(0), \quad \tau > 0, \]
and (2.1) follows directly from (2.4) and (2.5).

\[ \square \]

3. SOME USEFUL INEQUALITIES

First, we define the following special classes of functions
\[ H_k = \{ h \in L_1(0, \infty) : h \text{ is positive and } s^k h \in L_1(1, \infty), \ k > -1 \}, \quad (3.1) \]
\[ M = \{ F : (0, \infty) \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ where } 0 \leq F(\tau, s) - F(\tau, r) \leq H(\tau)(s - r), \quad (3.2) \]
for some continuous function \( H \) on \( \mathbb{R}_+ \), \( s \geq r \geq 0 \) and \( \tau > 0 \} \).
\[ \Phi = \{ \varphi \in C(0, \infty) : \varphi \text{ is nondecreasing and positive on } (0, \infty), \]
\[ \frac{1}{v} \varphi(w) \leq \varphi\left( \frac{w}{v} \right), \quad w > 0, \ v \geq 1 \}. \quad (3.3) \]
The proofs of the following lemmas are based on an application of the Bihari inequality which is a generalization of the Gronwall inequality.

**Lemma 3.1 ([12]).** Let \( g(\tau) \) and \( z(\tau) \) be nonnegative continuous functions defined for \( \tau \geq 0 \), \( \varphi \in \Phi \) and \( c_i \in \mathbb{R}, \ i = 1, 2, 3 \). Then
\[ z(\tau) \leq c_1 + c_2 \tau^\gamma + c_3 \tau^\gamma \int_0^\tau g(s)\varphi(z(s))ds, \quad \tau, \ \gamma \geq 0, \quad (3.4) \]
implies
\[ z(\tau) \leq \begin{cases} E^{-1}(E(|c_1| + |c_2|) + |c_3| \int_0^\tau g(s)ds), & 0 \leq \tau < 1 \\ \tau^\gamma E^{-1}(E(A) + |c_3| \int_1^\tau s^\gamma g(s)ds), & \tau \geq 1, \end{cases} \quad (3.5) \]
where
\[ A = |c_1| + |c_2| + |c_3| \varphi(E^{-1}(C)) \int_0^1 g(s)ds, \]
\[ C = E(|c_1| + |c_2|) + |c_3| \int_0^1 g(s)ds < \infty, \]
and \( E^{-1} \) is the inverse function of
\[ E(\xi) = \int_{\xi_0}^\xi \frac{ds}{\varphi(s)}. \]

**Lemma 3.2 ([12]).** Let \( z(\tau) \) satisfy
\[ z(\tau) \leq c_1 \tau^\gamma + c_2 \tau^\gamma \int_0^\tau [F_1(s, z(s) + c_3) + F_2(s, z(s) + c_4) + h(s)]ds, \quad \tau \geq 0, \quad (3.6) \]
where \( h : C[\mathbb{R}_+, \mathbb{R}_+] \), \( F_j \in M, j = 1, 2 \) and \( \gamma, c_i > 0, i = 1, 2, 3, 4 \). Then
\[ z(\tau) \leq \tau^\gamma f(\tau), \quad \tau > 0, \quad (3.7) \]
where
\[ f(\tau) = \left( c_1 + c_2 \int_0^\tau [F_1(s, c_3) + F_2(s, c_4) + h(s)]ds \right) \times \exp \left( c_2 \int_0^\tau s^\gamma [N_1(s) + N_2(s)]ds \right), \quad \tau > 0, \quad (3.8) \]
Remark 3.3. If $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then for $\alpha > 0$, $p(\alpha - 1) + 1 > 0 \iff q\alpha > 1$.

Lemma 3.4 \cite{[13]}.

If $v, \lambda + 1 > 1/r$, for some $r > 1$, and $g$ is a nonnegative continuous function defined on $[0, \infty)$, then
\[
\int_0^\tau (\tau - s)^{v-1} s^\lambda g(s) ds \leq C \tau^{v+\lambda-1/r} \left( \int_0^\tau g^r(s) ds \right)^{1/r}, \quad \tau > 0, \tag{3.9}
\]
where
\[
C = K_{p(v-1)+1,p\lambda} = \frac{\Gamma(p(-1)+1)}{\Gamma(p+v-1+2)}, \quad \frac{1}{p} + \frac{1}{r} = 1.
\]

Lemma 3.5 \cite{[13]}.

Let $h$ and $z$ be nonnegative continuous functions defined on $[0, \infty)$. Let $\varphi_i(z) > 0$ on $(0, \infty)$, $i = 1, 2$, and $\varphi_i(z)$ are continuous nondecreasing functions defined on $(0, \infty)$. If
\[
z(\tau) \leq K_1 + K_2 \left( \int_0^\tau h^q(s) \varphi_1^q(z(s)) \varphi_2^q(z(s)) ds \right)^{1/q}, \quad q > 1, \tau > 0, \tag{3.10}
\]
where $K_i \in \mathbb{R}^+$, $i = 1, 2$, then
\[
z(\tau) \leq \left[ E^{-1} \left( E(2\tau^{-1}K_1) + 2\tau^{-1}K_2 \int_0^\tau h^q(s) ds \right) \right]^{1/q}, \quad \tau > 0,
\]
where $E^{-1}$ is the inverse of
\[
E(\xi) = \int_{\xi_0}^\xi \frac{ds}{\varphi_1^q(s^{1/q}) \varphi_2^q(s^{1/q})}, \quad \xi > \xi_0, \quad 0 > \xi_0.
\]

4. Source without fractional derivatives

We consider the asymptotic behavior of solutions of the equation
\[
(C\mathcal{D}_0^\alpha x)'(\tau) = f(\tau, x(\tau)), \quad 0 < \alpha < 1, \tau \geq 0, \tag{4.1}
\]
subject to
\[
x(0) = b_1, \quad (C\mathcal{D}_0^\alpha x)(\tau = 0) = b_2, \tag{4.2}
\]
in the space
\[
C_1^{-\alpha}[0, \infty) = \{ x \in AC[0, \infty), (C\mathcal{D}_0^\alpha x)' \in C_{1-\alpha}[0, \infty) \}. \tag{4.3}
\]

We assume the following conditions:
(C1) $f(\tau, x) \in C([0, \infty) \times \mathbb{R}, \mathbb{R})$ is such that $f(\cdot, x(\cdot)) \in C_{1-\alpha}[0, \infty)$ for any $x \in AC[0, \infty]$.

(C2) There are continuous functions $P, \varphi : [0, \infty) \to [0, \infty)$ such that
\[
|f(\tau, x(\tau))| \leq \varphi(|x(\tau)|) P(\tau), \quad \tau \geq 0, \tag{4.4}
\]
where
\[
\int_1^\infty s^\alpha P(s) ds < \infty, \tag{4.5}
\]
and $\varphi \in \Phi$.

Theorem 4.1. Suppose $f$ satisfies (C1), (C2) and $x \in AC[0, \infty)$ is a solution of (4.1)-(4.2). Then
\[
\lim_{\tau \to \infty} \frac{x(\tau)}{\tau^\alpha} = a \in \mathbb{R}, \quad \text{as } \tau \to \infty.
\]
Proof. Integrating both sides of (4.1), we find
\[ C D^\alpha_0 x(\tau) = b_2 + \int_0^\tau f(s, x(s))ds = b_2 + \mathcal{J}_0^\alpha f(\tau, x(\tau)). \tag{4.6} \]
Applying \( \mathcal{J}_0^\alpha \) to both sides of (4.6), and using Lemmas 2.8 and 2.9, we arrive at
\[ x(\tau) = b_1 + \frac{b_2}{\Gamma(\alpha + 1)} \tau^\alpha + \frac{1}{\Gamma(\alpha + 1)} \int_0^\tau (\tau - s)^\alpha f(s, x(s))ds, \quad \tau \geq 0. \tag{4.7} \]
By using (4.9) we obtain
\[ |x(\tau)| \leq |b_1| + \frac{|b_2|}{\Gamma(\alpha + 1)} \tau^\alpha + \frac{1}{\Gamma(\alpha + 1)} \tau^\alpha \int_0^\tau P(s)\phi(|x(s)|)ds, \quad \tau \geq 0. \tag{4.8} \]
Applying Lemma 3.1 to (4.8) we obtain
\[ |x(\tau)| \leq \begin{cases} E^{-1}(E(|b_1| + \frac{|b_2|}{\Gamma(\alpha + 1)}) + \frac{1}{\Gamma(\alpha + 1)} \int_0^\tau P(s)ds), & 0 \leq \tau < 1 \\ \tau^\alpha E^{-1}(E(A) + \frac{1}{\Gamma(\alpha + 1)} \int_1^\tau s^\alpha P(s)ds), & \tau \geq 1, \end{cases} \tag{4.9} \]
where
\[ A = |b_1| + \frac{|b_2|}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\alpha + 1)} \phi(E^{-1}(K)) \int_0^1 P(s)ds, \]
\[ K = E(|b_1| + \frac{|b_2|}{\Gamma(\alpha + 1)}) + \frac{1}{\Gamma(\alpha + 1)} \int_0^1 P(s)ds < \infty. \]
From (4.5) and the continuity of \( P \) on \( \mathbb{R}_+ \), we see that
\[ |x(\tau)| \leq \begin{cases} C_1, & 0 \leq \tau < 1 \\ \tau^\alpha C_2, & \tau \geq 1, \end{cases} \tag{4.10} \]
with
\[ C_1 = E^{-1}\left(E(|b_1| + \frac{|b_2|}{\Gamma(\alpha + 1)}) + \frac{1}{\Gamma(\alpha + 1)} \int_0^1 P(s)ds \right) < \infty, \]
\[ C_2 = E^{-1}\left(E(A) + \frac{1}{\Gamma(\alpha + 1)} \int_1^\infty s^\alpha P(s)ds \right) < \infty. \]
Next, it is clear that
\[ \int_0^\tau |f(s, x(s))|ds \leq \int_0^\tau P(s)\phi(|x(s)|)ds \]
\[ \leq \frac{1}{\Gamma(\alpha + 1)} \int_0^1 P(s)\phi(|x(s)|)ds + \int_1^\tau P(s)\phi(|x(s)|)ds \]
\[ \leq \frac{1}{\Gamma(\alpha + 1)} \int_0^1 P(s)\phi(|x(s)|)ds + \int_1^\tau s^\alpha P(s)\phi\left(\frac{|x(s)|}{s^\alpha}\right)ds, \quad \tau > 0. \tag{4.11} \]
By (4.10) and (4.11), we have
\[ \lim_{\tau \to \infty} \int_0^\tau f(s, x(s))ds < \infty. \]
On the other hand, integrating (4.11) yields
\[ C D^\alpha_0 x(\tau) = b_2 + \int_0^\tau f(s, x(s))ds, \quad \tau > 0. \tag{4.12} \]
From (4.11) and (4.12), we conclude
\[ \lim_{\tau \to \infty} C^\alpha_0 x(\tau) = c, \quad c \in \mathbb{R}. \]
Further, by Lemma 2.11, we can write
\[ \lim_{\tau \to \infty} \frac{x(\tau)}{\tau^\alpha} = \lim_{\tau \to \infty} \frac{C^\alpha_0 x(\tau)}{\Gamma(\alpha + 1)} = a, \]
for some real number \( a \). □

**Example 4.2.** All solutions of
\[ (C^\alpha_0 x(\tau))' = e^{-\tau} x^r(\tau), \quad 0 < \alpha, r \leq 1, \tau > 0. \] (4.13)
satisfy \( \lim_{\tau \to \infty} \frac{x(\tau)}{\tau^\alpha} = a, \) as \( \tau \to \infty \), for some real number \( a \).

To prove this claim, let \( \varphi(\tau) = \tau^r \) and \( P(\tau) = e^{-\tau} \). Then
\[ \int_0^\infty s^\alpha P(s)ds \leq \int_0^\infty s^\alpha e^{-s}ds = \Gamma(\alpha + 1) < \infty. \]
Obviously \( \varphi \) is a nondecreasing and positive function with
\[ w\varphi(v) = uv^r \leq (vu)^r = \varphi(vu), \quad u \geq 1, \ v > 0, \]
and
\[ \int_0^\infty \frac{ds}{\varphi(s)} = \int_0^\infty \frac{ds}{s^r} = \infty. \]
Then \( \varphi \in \Phi \). All the conditions of Theorem 4.1 are satisfied, therefore every solution \( x \) of (4.13) has satisfy \( \lim_{\tau \to \infty} \frac{x(\tau)}{\tau^\alpha} = a, \) \( a \in \mathbb{R} \), as \( \tau \to \infty \).

5. EQUATIONS WITH FRACTIONAL SOURCE TERMS

We study problem (1.1) in the space \( C^{\alpha, 1}_{1-\alpha}[0, \infty) \) defined in (4.3) with the following assumptions:

(C3) \( f(\tau, v, w) : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R} \) is so that \( f(\cdot, v(\cdot), w(\cdot)) \in C_{1-\alpha}[0, \infty) \) for every \( v, w \in AC[0, \infty) \).

(C4) \[ |f(\tau, u(\tau), v(\tau))| \leq F_1(\tau, |u(\tau)|) + F_2(\tau, \tau^\beta|v(\tau)|), \quad \tau \geq 0, \] (5.1)
where \( F_i \in M, \ i = 1, 2. \)

**Lemma 5.1.** Suppose that \( f \) satisfies (C3), (4) and \( x \in AC[0, \infty) \) is a solution of (1.1). Then
\[ \max \{|x(\tau)|, \tau^\beta |C^\beta_0 x(\tau)|\} \leq |b_1| + z(\tau), \quad \tau > 0, \] (5.2)
where
\[ z(\tau) = C_2 \tau^\alpha + C_3 \tau^\alpha \int_0^\tau \left[ F_1(s, |x(s)|) + F_2(s, \tau^\beta |C^\beta_0 x(s)|) \right] ds, \quad \tau > 0, \] (5.3)
with
\[ C_3 = \max \left\{ \frac{1}{\Gamma(\alpha + 1)}, \frac{1}{\Gamma(\alpha - \beta + 1)} \right\}, \quad C_2 = |b_2| C_3. \]
Then, every solution

Then from this and (5.1) we obtain the bound

Next, we apply \(I_0^\alpha\) to both sides of (5.4), using Lemmas 2.6, 2.8 and 2.9, we find

for \(\tau > 0\). Thus, from (5.5) and (5.1) we have

for \(\tau > 0\). By Lemma 2.9 we see that

Let us insert the expression (5.4) into (5.7), using Lemmas 2.6 and 2.5, we have

for \(\tau > 0\). Thus, from (5.5) and (5.1) we have

for \(\tau > 0\). By Lemma 2.9 we see that

Let us insert the expression (5.4) into (5.7), using Lemmas 2.6 and 2.5, we have

Then from this and (5.1) we obtain the bound

Relation (5.2) follows directly from (5.3), (5.6) and (5.8).

**Theorem 5.2.** Suppose that \(f\) satisfies (C3)-(C4) and

Then, every solution \(x(\tau)\) of problem (1.1) has the following property

\[
\lim_{\tau \to \infty} \frac{x(\tau)}{\tau^\alpha} = a, \quad a \in \mathbb{R}.
\]
Proof. By using Lemma 5.1 we have
\[ F_1(\tau, |x(\tau)|) \leq F_1(\tau, |b_1| + z(\tau)), \quad \tau > 0, \quad (5.10) \]
\[ F_2(\tau, \tau^{\beta}|D_0^\beta x(\tau)|) \leq F_2(\tau, z(\tau) + |b_1|), \quad \tau > 0. \quad (5.11) \]
Taking into account (5.3), (5.10) and (5.11) we arrive at
\[ z(\tau) \leq C_2 \tau^\alpha + C_3 \tau^\alpha \int_0^\tau |F_1(s, |b_1|) + z(s)| + F_2(s, z(\tau) + |b_1|)ds, \quad \tau > 0. \]
Then, by Lemma 5.2, we find that
\[ z(\tau) \leq C \tau^\alpha, \quad \tau > 0 \quad (5.12) \]
where
\[ C = \left( C_2 + C_3 \int_0^\infty [F_1(s, |b_1|) + F_2(s, |b_1|)]ds \right) \times \exp(C_3 \int_0^\infty s^\gamma |N_1(s) + N_2(s)|ds) < \infty. \]
It follows from Lemma 5.1 and (5.12) that
\[ |x(\tau)| \leq |b_1| + C \tau^\alpha, \quad \tau^\beta|D_0^\beta x(\tau)| \leq |b_1| + C \tau^\alpha, \quad \tau > 0. \quad (5.13) \]
Again by hypothesis (5.1) we have
\[ \left| \int_0^\tau f(s, x(s), D_0^\beta x(s))ds \right| \leq \int_0^\tau |f(s, x(s), D_0^\beta x(s))|ds \leq \int_0^\tau [F_1(s, |x(s)|) + F_2(s, s^\beta|D_0^\beta x(s)|)]ds, \quad \tau > 0. \]
for \( \tau > 0 \). From this inequality and (5.13), we obtain
\[ \left| \int_0^\tau f(s, x(s), D_0^\beta x(s))ds \right| \leq \int_0^\tau [F_1(s, |b_1| + Cs^\alpha) + F_2(s, |b_1| + Cs^\alpha)]ds \]
\[ = \int_0^\tau \left\{ F_1(s, |b_1| + Cs^\alpha) - F_1(s, |b_1|) + F_1(s, |b_1|) \right\}ds, \quad \tau > 0. \]
As the functions \( F_i, i = 1, 2 \) are in \( M \), we obtain
\[ \left| \int_0^\tau f(s, x(s), D_0^\beta x(s))ds \right| \leq C \int_0^\tau s^\alpha |N_1(s) + N_2(s)|ds + \int_0^\tau |F_1(s, |b_1|) + F_2(s, |b_1|)|ds < \infty, \quad (5.14) \]
where we have used (5.9). Then
\[ \lim_{\tau \to \infty} \int_0^\tau f(s, x(s), D_0^\beta x(s))ds < \infty. \]
By (5.4) we conclude that there is \( b \in \mathbb{R} \) such that \( \lim_{\tau \to \infty} C D_0^\alpha x(\tau) = b \). Further, by Lemma 2.11 we deduce that
\[ \lim_{\tau \to \infty} \frac{x(\tau)}{\tau^\alpha} = \lim_{\tau \to \infty} \frac{C D_0^\alpha x(\tau)}{\Gamma(\alpha + 1)} = a, \]
and the proof is now complete. □

6. Boundedness

We consider the fractional differential problem (1.2) in the space
\[ C^\alpha[0, \infty) = \{ x \in AC[0, \infty) : C^\alpha_0x \in C[0, \infty) \} \] (6.1)

We assume the following conditions:

(C5) \( f : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R} \) is so that \( f(\cdot, v(\cdot), w(\cdot)) \in C[0, \infty) \) for every \( v, w \) in \( C[0, \infty) \).

(C6) \[ |f(\tau, u, v)| \leq \tau^\gamma h(\tau)\varphi_1(|u(\tau)|)\varphi_2(|v(\tau)|), \quad \tau > 0, \] (6.2)

where \( h, \varphi_1, \varphi_2 : \mathbb{R}_+ \to \mathbb{R}_+ \) are continuous functions with \( \varphi_i, i = 1, 2, \) are nondecreasing functions and \( h \in L_q(0, \infty) \) for some \( q > \frac{1}{\alpha-\beta}, \gamma = \frac{1}{q} - \alpha \).

Lemma 6.1.: Suppose that \( f \) satisfies (C5), (C6) and \( x \in AC[0, \infty) \) is a solution of (1.2). Then
\[ \max \{ |x(\tau)|, |C^\alpha_0x(\tau)| \} \leq z(\tau), \quad \tau \geq \tau_0 > 0, \] (6.3)

where
\[ z(\tau) = |b| + K_1 \left( \int_0^\tau h^q(s)\varphi_1^q(|x(s)|)\varphi_2^q(|C^\alpha_0x(s)|)ds \right)^{1/q}, \quad \tau > 0, \] (6.4)

and
\[ K_1 = \max \left\{ \frac{K_1^{p(\alpha-1),p\gamma}}{\Gamma(\alpha)}, \frac{K_1^{\alpha-1,p\gamma}}{\Gamma(\alpha-\beta)\tau_0^\beta} \right\}, \]
\[ K_{\alpha,\beta} = \frac{1}{\Gamma(\alpha+\beta+1)} \left( \frac{\Gamma(\beta+1)}{\Gamma(\alpha)} \right), \quad \frac{1}{p} + \frac{1}{q} = 1. \]

Proof. Applying \( \mathfrak{D}_0^\alpha \) to (1.2) and taking into account Lemma 2.8 we have
\[ x(\tau) = b + \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau-s)^{\alpha-1}f(s,x(s),C^\alpha_0x(s))ds, \quad \tau > 0. \] (6.5)

Using the inequality (6.2), we obtain
\[ |x(\tau)| \leq |b| + \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau-s)^{\alpha-1}s^\gamma h(s)\varphi_1(|x(s)|)\varphi_2(|C^\alpha_0x(s)|)ds, \] (6.6)

for \( \tau > 0 \). It follows from the assumptions \( \beta < \alpha, q > \frac{1}{\alpha-\beta} \) and \( \gamma = \frac{1}{q} - \alpha \) that \( p(\alpha-1) + 1 \geq p(\alpha-\beta-1) + 1 > 0 \) and \( p\gamma + 1 = p(\frac{1}{q} - \alpha) + 1 = p(1 - \alpha) > 0 \). Then, we apply Lemma 3.3 to obtain
\[ |x(\tau)| \leq |b| + K_1 \left( \int_0^\tau h^q(s)\varphi_1^q(|x(s)|)\varphi_2^q(|C^\alpha_0x(s)|)ds \right)^{1/q}, \quad \tau > 0. \] (6.7)
Therefore, Lemma 3.5 implies for some positive constant $\tau > h$ because of (3.9) we have

In view of Lemma 6.1 we have Example 6.3. Also, by Lemma 2.9, we conclude that

Theorem 6.2. Assume that $f$ satisfies (C5), (C6). Then, any solution $x$ of (1.2) satisfies

for $\tau > 0$. Again, from Lemma 3.3, we find

Therefore (6.3) follows from (6.4), (6.7) and (6.9).

**Theorem 6.2.** Assume that $f$ satisfies (C5), (C6). Then, any solution $x$ of (1.2) satisfies

$$|x(\tau)| \leq C, \quad |C\mathcal{D}_0^\beta x(\tau)| \leq C,$$

for some positive constant $C$, $\tau > 0$, provided that

$$\int_0^\infty \frac{ds}{\varphi_1(s^{1/q})\varphi_2(s^{1/q})} = \infty, \quad \xi_0 > 0.$$

**Proof.** In view of Lemma 6.1 we have

$$\varphi_1(|x(\tau)|) \leq \varphi_1(z(\tau)), \quad 2(|C\mathcal{D}_0^\beta x(\tau)|) \leq \varphi_2(z(\tau)), \quad \tau > 0. \quad (6.10)$$

From this inequality and (6.4), we obtain

$$z(\tau) \leq |b| + K_1 \left( \int_0^\tau h^q(s)\varphi_1^q(z(s))\varphi_2^q(z(s)) ds \right)^{1/q}, \quad \tau > 0. \quad (6.11)$$

Therefore, Lemma 3.3 implies

$$z(\tau) \leq \left[ E^{-1} \left( E(2^{q-1}K_1) + 2^{q-1}K_2 \int_0^\tau h^q(s) ds \right) \right]^{1/q} < \infty,$$

because $h \in L_q(0, \infty)$. This completes the proof.

**Example 6.3.** Consider the problem

$$C\mathcal{D}_0^{2/3} x(\tau) = \tau^{1/q-2/3} e^{-\lambda\tau} (x(\tau))^{3/5} \left( C\mathcal{D}_0^{1/3} x(\tau) \right)^{1/3} \left( \cos(C\mathcal{D}_0^{1/3} x) \right) \quad \tau > 0,$$

$$x(0) = b, \quad q > 3, \quad \lambda > 0. \quad (6.12)$$
Let $\varphi_1(\tau) = \tau^{3/5}$, $\varphi_2(\tau) = \tau^{1/3}$ and $h(\tau) = e^{-\lambda \tau}$, $\gamma = 1/q - 2/3$. Then $h \in L_q(0, \infty)$ and
\[
\int_{\xi_0}^{\infty} \frac{ds}{\varphi_1^q(s^{1/q}) \varphi_2^q(s^{1/q})} = \int_{\xi_0}^{\infty} \frac{ds}{s^{3/5} s^{1/3}} = \int_{\xi_0}^{\infty} \frac{ds}{s^{14/15}} = \infty.
\]
Then, by Theorem 6.2, we deduce that any solution $x$ of (6.12) satisfies
\[
| x(\tau) | \leq C, \quad |C D^\beta_0 x(\tau)| < C,
\]
for $\alpha = 2/3$, $\beta = 1/3$, and $\tau > 0$.

**Acknowledgements.** M. D. Kassim wants to thank Imam Abdulrahman Bin Faisal University for its support and facilities. N. E. Tatar is grateful for the financial support and the facilities provided by King Fahd University of Petroleum and Minerals through project number IN181008.

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