ASYMPTOTICS TO ALL ORDERS OF THE EULER–DARBOUX EQUATION IN A TRIANGLE

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Abstract. In Einstein’s theory of relativity, the interaction of two collinearly polarized plane gravitational waves can be described by a Goursat problem for the Euler-Darboux equation in a triangular domain. In this paper, using a representation of the solution in terms of Abel integrals, we give a full asymptotic expansion of the solution near the diagonal of the triangle. The expansion is related to the formation of a curvature singularity of spacetime. Our framework allows for boundary data with derivatives which are singular at the corners. This level of generality is crucial for the application to gravitational waves.

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1. Introduction

The collision of two collinearly polarized plane gravitational waves in Einstein’s theory of relativity can be described mathematically [13] by a Goursat problem for the Euler-Darboux equation [20]

\[
V_{xy} - \frac{V_x + V_y}{2(1 - x - y)} = 0, \quad (x, y) \in D,
\]

(1.1)
in the triangular region \(D\) defined by (see Figure 1)

\[
D = \{(x, y) \in \mathbb{R}^2 | x \geq 0, \ y \geq 0, \ x + y < 1\}.
\]

(1.2)

Besides its importance for collinearly polarized plane waves, equation (1.1) can also be seen as the linear limit of the nonlinear hyperbolic Ernst equation [2] describing the collision of two (not necessarily collinearly polarized) plane gravitational waves. The
hyperbolic Ernst equation is a reduction of the vacuum Einstein field equations which is similar to the elliptic version of the same equation discovered by F. J. Ernst [6]. While many exact solutions of the hyperbolic Ernst equation have been found (see [5, 8, 10, 21]), it seems that the problem of determining the solution from given data has only been treated in the series of papers [15–18] and the papers [1, 11] as well as the recent work [22]. In [15–18] the authors relate the Goursat problem to the solution of a homogeneous Hilbert problem whereas in [1, 11, 22] the problem is studied by means of inverse scattering. However, both in [11] and [15], the solution for the linear limit/collinear polarization (1.1) is represented in terms of Abel integrals, which allows for applying more classical techniques.

A similar reduction as the one yielding Ernst’s equation can be deduced in case of the Einstein-Maxwell equations for electromagnetic fields [3, 7]. The resulting Goursat problem for two coupled nonlinear PDEs describes the interaction of two colliding electromagnetic waves. In the linear limit, the interaction reduces again to (1.1).

Of particular interest is the behavior of the solution of the Goursat problem for (1.1) near the triangle’s diagonal edge \( x + y = 1 \), because this behavior is related to the formation of a curvature singularity of the spacetime by the mutual focusing of the colliding waves. In [23], it was observed that the solution must behave like \( \ln(1 - x - y) \) as \( x + y \to 1 \). In this paper, using an Abel integral representation for the solution, we derive an asymptotic expansion to all orders as \( x + y \to 1 \).

Our first result (Theorem 1) provides a mathematically precise formulation of the classical Abel integral representation (cf. e.g. [11,14,15]) of the solution of the Goursat problem for (1.1) with given boundary data. We discuss regularity, the behavior at the boundary of \( D \), and uniqueness of the solution under “reasonable” assumptions for the boundary data. Here “reasonable” means that our results can be applied to the common examples for collinear solutions (cf. e.g. [9,19,23]), which particularly includes boundary data that has singular derivatives at the corners of \( D \). As a consequence, our representation of the Solution of (1.1) contains singular integrands, which is the main difficulty in the analysis of the Goursat problem.

Our second result (Theorem 2) gives the asymptotic behavior of the solution near the diagonal of \( D \). We show that \( V(x, 1 - x - \epsilon) \) admits an asymptotic expansion to all orders...
of the form
\[ V(x, 1 - x - \epsilon) = \sum_{j=0}^{J} f_j(x) \epsilon^j \ln(\epsilon) + \sum_{j=0}^{J} g_j(x) \epsilon^j + O(\epsilon^{J+1} \ln(\epsilon)), \quad \epsilon \downarrow 0, \]
where the coefficients \( f_j, g_j \) are given explicitly in terms of the boundary data \( V_0(x) = V(x, 0) \) and \( V_1(y) = V(0, y) \), and where the error term is uniform on compact subsets of \((0, 1)\). The first few terms of the asymptotic formula are given by
\[
V(x, 1 - x - \epsilon) = -\frac{1}{\pi} (h_0(x) + h_1(x)) \ln(\epsilon) \\
+ \frac{1}{\pi} \frac{d}{dx} \left( \int_{1-x}^{x} h_0(k) \ln(4(x-k)) dk - \int_{x}^{1} \ln(4(k-x)) h_1(k) dk \right) \\
- \frac{1}{2\pi} (h_0'(x) + h_1'(x)) \epsilon \ln(\epsilon) \\
+ \frac{1}{2\pi} \frac{d^2}{dx^2} \left( \int_{1-x}^{x} h_0(k) \ln(4(x-k)) dk - \int_{x}^{1} \ln(4(k-x)) h_1(k) dk \right) \epsilon \\
+ O(\epsilon^2 \ln(\epsilon)), \quad \epsilon \downarrow 0, \tag{1.3}
\]
where
\[ h_0(k) = \sqrt{1-k} \int_{0}^{k} \frac{V_0(x')}{\sqrt{k-x'}} dx', \quad h_1(k) = \sqrt{k} \int_{0}^{1-k} \frac{V_1(y')}{\sqrt{1-k-y'}} dy'. \tag{1.4} \]

The two main results are presented in Section 2, and their proofs are given in Section 3 and Section 4 respectively. In Section 5 we apply our results to collinearly polarized colliding gravitational waves and give full asymptotic expansions for the components of the Weyl tensor.

2. Main results

Since (1.1) is a linear equation, we can, without loss of generality, assume that \( V \) is real-valued and that \( V(0, 0) = 0 \). We introduce a notion of \( C^\alpha \)-solution of the Goursat problem for (1.1) as follows.

Definition 2.1. Let \( V_0(x), x \in [0, 1), \) and \( V_1(y), y \in [0, 1), \) be real-valued functions. A function \( V : D \to \mathbb{R} \) is called a \( C^\alpha \)-solution of the Goursat problem for (1.1) in \( D \) with data \( \{V_0, V_1\} \) if there exists an \( \alpha \in [0, 1) \) such that
\[
\begin{cases}
V \in C(D) \cap C^\alpha(\text{int}(D)), \\
V(x, y) \text{ satisfies the Euler-Darboux equation (1.1) in int}(D), \\
x^\alpha V_x, y^\alpha V_y, x^\alpha y^\alpha V_{xy} \in C(D), \\
V(x, 0) = V_0(x) \text{ for } x \in [0, 1), \\
V(0, y) = V_1(y) \text{ for } y \in [0, 1).
\end{cases}
\]

The following theorem solves the Goursat problem for (1.1) in \( D \) by providing a representation for the solution in terms of the boundary data.

Theorem 1 (Solution of the Euler-Darboux equation in a triangle). Let \( n \geq 2 \) be an integer and suppose \( \alpha \in [0, 1) \). Let \( V_0(x), x \in [0, 1), \) and \( V_1(y), y \in [0, 1), \) be two real-valued functions such that
\[
\begin{cases}
V_0, V_1 \in C([0, 1)) \cap C^\alpha((0, 1)), \\
x^\alpha V_0_x, y^\alpha V_1_y, x^{\alpha+1} V_0_{xx}, y^{\alpha+1} V_1_{yy} \in C([0, 1)), \\
V_0(0) = V_1(0) = 0.
\end{cases} \tag{2.1}
\]
Then
\begin{equation}
V(x, y) = \frac{1}{\pi} \int_0^x \frac{\sqrt{1-k}}{\sqrt{(1-y-k)(x-k)}} \left( \int_0^k \frac{V_{0x}(x')}{\sqrt{k-x'}} dx' \right) dk \\
+ \frac{1}{\pi} \int_{1-y}^1 \frac{\sqrt{k}}{\sqrt{(k-(1-y))(k-x)}} \left( \int_0^{1-k} \frac{V_{1y}(y')}{\sqrt{1-y'-k}} dy' \right) dk,
\end{equation}
(2.2)
is a $C^{n-1}$-solution of the Goursat problem for (1.1) in $D$ with data $\{V_0, V_1\}$ (and with the same $\alpha$). Furthermore, $V$ is the only solution in the sense of Definition 2.1.

**Remark 2.2.** In the case $n = 2$, Theorem 1 states that $V$ is at least in $C^1$. However, the derivative $V_{xy}$ always exists and is continuous in $\text{int} D$. Furthermore, $x^n y^n V_{xy}$ is still in $C(D)$ in this case as it is demanded in Definition 2.1.

**Remark 2.3.** An alternative integral representation for the solution of the Goursat problem for the Euler-Darboux equation was derived already in [23] using Riemann’s classical method [4, 12]. The representation (2.2) relies on Abel integrals and is analogous to formulas derived in [15], whereas the formula of [23] involves the Legendre function $P_{-1/2}$ of order $-1/2$.

The representation (2.2) can be used to study the behavior of the solution $V(x, y)$ near the diagonal $x + y = 1$, i.e. the behavior of $V(x, 1-x-\epsilon)$ as $\epsilon \downarrow 0$. Letting $y = 1-x-\epsilon$ in our representation formula (2.2), we find
\begin{equation}
V(x, 1-x-\epsilon) = \frac{1}{\pi} \int_0^x \frac{\sqrt{1-k}}{\sqrt{(x-k+\epsilon)(x-k)}} \left( \int_0^k \frac{V_{0x}(x')}{\sqrt{k-x'}} dx' \right) dk \\
+ \frac{1}{\pi} \int_{x+\epsilon}^1 \frac{\sqrt{k}}{\sqrt{(k-x-\epsilon)(k-x)}} \left( \int_0^{1-k} \frac{V_{1y}(y')}{\sqrt{1-y'-k}} dy' \right) dk
=: X_1 + X_2.
\end{equation}
(2.3)

We define $h_0, h_1$ by (1.4) and the kernels
\begin{equation}
K_0(u, t) = \frac{1}{\sqrt{u\sqrt{u+t}}}, \quad K_1(u, t) = \frac{1}{\sqrt{u\sqrt{u-t}}},
\end{equation}
(2.4)
for $u > 0$ and $-u < t < u$. Furthermore, we define for $j \geq 0$
\begin{equation}
c_j = \frac{1}{2^j} \left( \prod_{l=0}^{j-1}(2l+1) \right) = (-1)^j \frac{\partial^j}{\partial t^j} K_0(1, 0) = \frac{\partial^j}{\partial t^j} K_1(1, 0)
\end{equation}
(2.5)
and
\begin{equation}
H_0^j(x, k) = h_0(k) - \sum_{l=0}^j \frac{h_0^{(l)}(x)(-1)^l}{l!} (x-k)^l, \quad H_1^j(x, k) = h_1(k) - \sum_{l=0}^j \frac{h_1^{(l)}(x)}{l!} (k-x)^l.
\end{equation}
(2.6)

Using this notation, we can give the asymptotics of $V(x, 1-x-\epsilon)$ as $\epsilon \downarrow 0$ up to all orders.

**Theorem 2** (Asymptotic expansion to all orders). Let $J \geq 0$ be an integer. Suppose that $V_0, V_1 \in C^{J+2}((0,1))$ satisfy the conditions (2.1). Then the unique solution $V(x, y)$
of the Goursat problem for \([1.1]\) with data \(\{V_0, V_1\}\) enjoys the asymptotic expansion

\[
V(x, 1 - x - \epsilon) = -\frac{1}{\pi} \sum_{j=0}^{J} \frac{c_j}{(j!)^2} (h_0^{(j)}(x) + h_1^{(j)}(x)) e^j \ln(\epsilon)
+ \frac{1}{\pi} \sum_{j=0}^{J} ((-1)^j A_j(x) + B_j(x)) e^j
+ O(\epsilon^{\delta+1} \ln(\epsilon)), \quad \epsilon \downarrow 0, \quad 0 < x < 1,
\]

where the error term is uniform with respect to \(x\) in compact subsets of \((0, 1)\),

\[
A_j(x) = \frac{c_j}{j!} \left( \int_{0}^{x} \frac{H_0^{(j)}(x, k)}{(x - k)^{j+1}} dk + \sum_{l=0}^{j-1} \frac{(-1)^l}{l!(l-j)} h_0^{(l)}(x)x^{l-j} + \frac{(-1)^j h_0^{(j)}(x)}{j!} \ln(x) \right)
+ \frac{h_0^{(j)}(x)}{j!} \left( \int_{0}^{1} \frac{v^j}{\sqrt{v + 1}} dv + \int_{1}^{\infty} \frac{v^j}{\sqrt{v + 1}} \left( \frac{1}{\sqrt{v + 1}} - \sum_{l=0}^{j-1} \frac{1}{l!} (-1)^j c_l v^{l-1} \right) dv \right)
+ \frac{h_0^{(j)}(x)}{j!} \sum_{l=0}^{j-1} (-1)^j \frac{c_l}{l!(j-l)},
\]

\[
B_j(x) = \frac{c_j}{j!} \left( \int_{x}^{1} \frac{H_1^{(j)}(x, k)}{(k - x)^{j+1}} dk + \sum_{l=0}^{j-1} \frac{1}{l!(l-j)} h_1^{(l)}(x)(1-x)^{l-j} + \frac{h_1^{(j)}(x)}{j!} \ln(1-x) \right)
+ \frac{h_1^{(j)}(x)}{j!} \left( \int_{1}^{\infty} \frac{v^j}{\sqrt{v - 1}} \left( \frac{1}{\sqrt{v - 1}} - \sum_{l=0}^{j-1} \frac{c_l}{l!} v^{l-1} \right) dv + \sum_{l=0}^{j-1} \frac{c_l}{l!(l-j)} \right),
\]

and \(h_0, h_1, H_0^j, H_1^j\) are defined by \([1.4]\) and \([2.6]\). The first few terms can also be represented as in \([1.3]\) (see Remark \([4.1]\)).

3. PROOF OF THEOREM \([1]\)

In oder to prove Theorem \([1]\) we study some properties of Abel integrals.

**Definition 3.1.** Let \(\alpha \in [0, 1)\) and suppose \(k^{\alpha} h \in C([0, 1))\). Then the Abel transform \(Ah\) of the function \(h\) is defined by

\[
Ah(x) = \int_{0}^{x} \frac{h(k)}{\sqrt{x - k}} dk, \quad x \in [0, 1).
\]

The following lemma shows that the Abel transform is well-defined and gives some basic properties.

**Lemma 3.2.** Let \(n \geq 0\) and \(\alpha \in [0, 1)\). Suppose \(h \in C^n((0, 1))\) and \(k^{\alpha+j} h^{(j)} \in C([0, 1))\) for \(0 \leq j \leq n\). Then the Abel transform

\[
f(x) = Ah(x) = \int_{0}^{x} \frac{h(k)}{\sqrt{x - k}} dk
\]

of \(h\) is in \(C^n((0, 1))\). Furthermore, we have \(x^{\alpha-1/2+j} f^{(j)} \in C([0, 1))\) for all \(0 \leq j \leq n\).

**Proof.** First, we show that the function \(f\) is well-defined. Fix some \(x \in (0, 1)\). Then we can define the constant \(M = \sup_{k \in [0, x]} |h(k)k^\alpha|\) and get

\[
|f(x)| \leq \int_{0}^{x} \left| \frac{h(k)}{\sqrt{x - k}} \right| dk \leq M \int_{0}^{x} \frac{1}{k^{\alpha} \sqrt{x - k}} dk.
\]
The transformation \( u = k/x \) gives
\[
\int_0^x \frac{1}{k^\alpha \sqrt{x-k}} dk = \int_0^1 \frac{x}{(ux)\alpha \sqrt{x-ux}} du = x^{1/2-\alpha} \int_0^1 \frac{1}{u^{\alpha} \sqrt{1-u}} du = x^{1/2-\alpha} B \left( 1-\alpha, \frac{1}{2} \right),
\]
where \( B \) denotes the Beta function. By the identity \( B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y) \), \( \Gamma \) denoting the Gamma function, this yields
\[
\int_0^x \frac{h(k)}{\sqrt{x-k}} dk \leq M x^{1/2-\alpha} \sqrt{\pi} \frac{\Gamma(1-\alpha)}{\Gamma(3/2-\alpha)}.
\]
(3.1)
Hence \( f \) is well defined for all \( x \in (0,1) \).

Now we show that \( f \) is continuous on \((0,1)\). We apply the transformation \( u = k/x \) again and get
\[
f(x) = \int_0^x \frac{h(k)}{\sqrt{x-k}} dk = \sqrt{x} \int_0^1 \frac{h(ux)}{\sqrt{1-u}} du.
\]
(3.2)
Then it follows easily by Lebeugue’s dominated convergence theorem that \( f \) is in \( C((0,1)) \).

To show that \( x^{\gamma} f \in C([0,1]) \) for \( \gamma = \alpha - 1/2 \) we show that \( \lim_{x\to 0} x^{\gamma} f(x) \) exists. Note that analogously to (3.1), we obtain
\[
\inf_{k \in [0,x]} (k^\alpha h(k)) x^{1/2-\alpha} \sqrt{\pi} \frac{\Gamma(1-\alpha)}{\Gamma(3/2-\alpha)} \leq f(x) \leq \sup_{k \in [0,x]} (k^\alpha h(k)) x^{1/2-\alpha} \sqrt{\pi} \frac{\Gamma(1-\alpha)}{\Gamma(3/2-\alpha)}
\]
for all \( x \in (0,1) \). Hence, by multiplying the inequality above with \( x^{\gamma} \), we get
\[
\lim_{x\to 0} x^{\alpha-1/2} f(x) = \lim_{k\to 0} (k^\alpha h(k)) \sqrt{\pi} \frac{\Gamma(1-\alpha)}{\Gamma(3/2-\alpha)}.
\]
In particular, the function \( x^{\gamma} f \) can be continuously extended onto \([0,1]\).

To show differentiability of \( f \), we consider formula (3.2) again. By applying Lebeugue’s dominated convergence theorem to the difference quotient, and by using that \( k^{1+\alpha} h' \in C([0,1]) \), we get that
\[
\frac{d}{dx} Ah(x) = \frac{1}{2\sqrt{x}} \int_0^1 \frac{h(ux)}{\sqrt{1-u}} du + \sqrt{x} \int_0^1 \frac{h'(ux)}{\sqrt{1-u}} du
\]
\[
= \frac{1}{2x} Ah(x) + \frac{1}{x} A(kh')(x).
\]
(3.3)
Since \( kh' \) also fulfills the assumptions of the Lemma (but with reduced \( n \)), all remaining statements follow. □

**Remark 3.3.** We will also use the following formula for the derivative of the Abel transform, given by
\[
\frac{d}{dx} \left( \int_0^x \frac{h(k)}{\sqrt{x-k}} dk \right) = \frac{h(\delta)}{\sqrt{x-\delta}} - \frac{1}{2} \int_0^\delta \frac{h(k)}{(x-k)^{3/2}} dk + \int_\delta^x \frac{h'(k)}{\sqrt{x-k}} dk
\]
(3.4)
for an arbitrary \( 0 < \delta < x \). This is obtained by splitting the integral, integrating by parts, and differentiating.

**Remark 3.4.** Note that the proof of Lemma 3.2 would not change if we replaced the segment \([0,1]\) in the lemma by a segment \([0,b]\), where \( 0 < b < 1 \).

The following lemma will be useful in the proof of Theorem 1 since the Abel integrals in the integral representation (2.2) depend on two variables.
Lemma 3.5. Let \( h \in C((0,1)) \) and \( x^\alpha h \in C([0,1)) \) for some \( \alpha < 1 \). Suppose that \( g \in C(D) \) and that there exists a number \( \beta \geq 0 \) and a constant \( C > 0 \) such that
\[
|g(y, k)| \leq \frac{C}{(1 - x - y)^\beta}, \quad \text{for all} \ (x, y) \in D \text{ and } k \in [0, x].
\]
Then the function \( f \) defined by
\[
f(x, y) = \int_0^x g(y, k) \frac{h(k)}{\sqrt{x - k}} dk
\]
is in \( C(\text{int} \ D) \). Furthermore, we have \( x^{\alpha-1/2} f \in C(D) \).

Proof. Let \( U \) be an open set in \( D \setminus \{ x = 0 \} \) such that \( \overline{U} \subset D \setminus \{ x = 0 \} \), where \( \overline{U} \) denotes the closure of \( U \). Then we have by the substitution \( u = k/x \) and the assumptions
\[
\int_0^x \left| g(y, k) \frac{h(k)}{\sqrt{x - k}} \right| \ du = \frac{C}{\sqrt{x}} \int_0^1 \left| g(y, ux) \frac{h(ux)}{\sqrt{1 - u}} \right| \ du \leq \frac{C}{(1 - x - y)^\beta} \int_0^1 \frac{|h(ux)|}{\sqrt{1 - u}} \ du.
\]
The integrand \( |h(ux)|/\sqrt{1 - u} \) is bounded by an \( L^1((0,1)) \)-function independent of \( x \) and \( (1 - x - y)^{-\beta} \) is bounded by a constant on \( U \). This shows that \( f \) is in \( C(D \setminus \{ x = 0 \}) \) by again applying Lebegue’s dominate convergence theorem.

To prove the last statement of the lemma, note that
\[
\int_0^1 x^\alpha |h(ux)| \sqrt{1 - u} \ du \leq \sup_{k \in [0, x]} (k^\alpha |h(k)|) \int_0^1 \frac{1}{u^{\alpha} \sqrt{1 - u}} \ du.
\]
Hence, if we fix a neighborhood \( W \) around a point \( (0, y) \in D \) such that \( \overline{W} \subset D \), we can bound
\[
x^{\alpha-1/2} \sqrt{x} \int_0^1 \left| g(y, ux) \frac{h(ux)}{\sqrt{1 - u}} \right| \ du
\]
individually of \( x \) and \( y \). This shows that \( x^{\alpha-1/2} f \in C(D) \). \( \square \)

The next lemma gives a well-known inversion formula for Abel integrals which we need for the boundary conditions \( V(x, 0) = V_0(x) \) and \( V(0, y) = V_1(y) \).

Lemma 3.6. Let \( h, f \) be as in Lemma 3.2. Then we have
\[
h(k) = \frac{1}{\pi} \frac{d}{dk} \left( \int_0^k \frac{f(x)}{\sqrt{k - x}} \ dx \right), \quad k \in (0, 1).
\]

Proof. Due to Lemma 3.2, the integral
\[
\int_0^k \frac{f(x)}{\sqrt{k - x}} \ dx
\]
exists and is differentiable on \( (0, 1) \). By the definition of \( f \), we have
\[
\int_0^k \frac{f(x)}{\sqrt{k - x}} \ dx = \int_0^k \int_0^x \frac{h(\kappa)}{\sqrt{x - \kappa \sqrt{k - x}} \kappa} \ dx d\kappa = \int_0^k h(\kappa) \int_\kappa^k \frac{1}{\sqrt{x - \kappa \sqrt{k - x}}} \ dx d\kappa.
\]
We apply the transformation \( y = (k - x)/(k - \kappa) \) and get
\[
\int_0^k \frac{f(x)}{\sqrt{k - x}} \ dx = \int_0^k h(\kappa) \int_0^1 \frac{dy}{\sqrt{y} \sqrt{1 - y}} d\kappa = \pi \int_0^k h(\kappa) d\kappa,
\]
giving the desired result. \( \square \)
Proof of Theorem 4. It is an immediate consequence of Lemma 3.5 with
\[ g(y, k) = \sqrt{1 - k}/\sqrt{1 - y - k} \]
that \( V \) is in \( C(D) \) with \( V(0, 0) = 0 \). Furthermore, letting
\[ G(y, k) = \frac{\sqrt{1 - k}}{\sqrt{1 - y - k}} AV_0(x)(k) \]
we have
\[
V_x(x, y) = \frac{1}{\pi} \frac{\partial}{\partial x} \int_0^x \frac{G(y, k)}{\sqrt{x - k}} dk \\
+ \frac{1}{\pi} \frac{\partial}{\partial x} \left( \int_1^1 \frac{\sqrt{k}}{\sqrt{(k - (1 - y))(k - x)}} \left( \int_0^{1 - k} \frac{V_1(y')}{\sqrt{1 - y' - k}} dy' \right) dk \right)
\] (3.5)

Note that, due to Lemma 3.2, \( x^{\alpha-1/2} AV_{0x}(x) \) is in \( C([0, 1]) \) and hence all assumptions in the lemma are satisfied in the sense of Remark 3.4. For the second integral in (3.5), it is clear, that we can interchanged the derivative and the integral sign. Hence, by using (3.3), we find
\[
V_x(x, y) = \frac{1}{2\pi^2} A(G(y, \cdot))(x) + \frac{1}{x\pi} A(kG_k(y, \cdot))(x) \\
+ \frac{1}{2\pi} A \left( \frac{\sqrt{1 - (\cdot)}}{(1 - x - (\cdot))^{3/2}} AV_{1y}(\cdot) \right)(y).
\] (3.6) (3.7)

Since Lemma 3.5 can be applied to all three terms (with different choices of \( g, h \) and in the third case with \( x \) and \( y \) interchanged), we obtain that \( V_x \in C(\text{int } D) \) and \( x^\alpha V_x \in C(D) \). The cases \( V_y \) and \( V_{xy} \) are analogous and inductively we get \( V \in C^{n-1}(\text{int } D) \).

To prove that \( V \) satisfies the equation (1.1), we use the alternative formula (3.4) to represent the derivative. We only prove that the first term
\[ X_1(x, y) = \frac{1}{\pi} \int_0^x \frac{\sqrt{1 - k}}{(1 - x - k)(x - k)} \left( \int_0^k \frac{V_0(x')}{\sqrt{k - x'}} dx' \right) dk \]
of (2.2) satisfies the equation, since the other one is analogous. We fix some \((x, y) \in \text{int } D\) and denote again
\[ G(y, k) = \frac{\sqrt{1 - k}}{\sqrt{1 - k - y}} AV_0(x)(k), \quad (y, k) \in D. \]

By (3.4) we have for any fixed \( \delta \in (0, x) \)
\[ \pi X_{1x}(x, y) = \frac{G(y, \delta)}{\sqrt{x - \delta}} - \frac{1}{2} \int_0^\delta \frac{G(y, k)}{(x - k)^{3/2}} dk + \int_\delta^x \frac{G_k(y, k)}{\sqrt{x - k}} dk, \]
\[ \pi X_{1y}(x, y) = \int_0^x \frac{G_y(y, k)}{\sqrt{x - k}} dk, \]
\[ \pi X_{1xy}(x, y) = \frac{G_y(y, \delta)}{\sqrt{x - \delta}} - \frac{1}{2} \int_0^\delta \frac{G_y(y, k)}{(x - k)^{3/2}} dk + \int_\delta^x \frac{G_{yk}(y, k)}{\sqrt{x - k}} dk. \]

In particular,
\[ \pi X_{1x}(x, y) = \lim_{\epsilon \to 0} \left( \frac{G(y, x)}{\sqrt{\epsilon}} - \frac{1}{2} \int_0^{x-\epsilon} \frac{G(y, k)}{(x - k)^{3/2}} dk \right), \]
\[ \pi X_{1xy}(x, y) = \lim_{\epsilon \to 0} \left( \frac{G_y(y, x)}{\sqrt{\epsilon}} - \frac{1}{2} \int_0^{x-\epsilon} \frac{G_y(y, k)}{(x - k)^{3/2}} dk \right). \]
Summing up the first partial derivatives and using \( G_y(y, k) = (2(1 - y - k))^{-1}G(y, k) \) yields

\[
\pi X_{1x}(x, y) + \pi X_{1y}(x, y) = \lim_{\epsilon \to 0} \left( \frac{G(y, x)}{\sqrt{\epsilon}} - \frac{1}{2} \int_0^{x-\epsilon} \frac{G(y, k)(1 - y - k) - G(y, k)(x - k)}{(1 - y - k)(x - k)^{3/2}} dk \right) = 2\pi(1 - x - y)X_{1xy}(x, y).
\]

Since the other term of (2.2) is analogous, this proves that \( V \) satisfies (1.1).

Next we prove that \( V \) satisfies the boundary conditions. We have

\[
V(x, 0) = \frac{1}{\pi}A(AV_0)(x).
\]

Furthermore, Lemma 3.6 (with \( h = V_0 \) and \( f = AV_0 \)) gives

\[
V_0(x) = \frac{1}{\pi} \frac{d}{dx}A(AV_0)(x) = V_x(x, 0)
\]

for all \( x \in (0, 1) \) and hence \( V(x, 0) = V_0(x) + c \) for a constant \( c \in \mathbb{R} \). But \( 0 = V(0, 0) = V_0(0) \). Thus we have \( c = 0 \) and \( V(x, 0) = V_0(x) \). The case \( V(0, y) \) is completely analogous.

Now it only remains to show that \( V \) is the only solution. For this, it suffices to show that a function \( V \) satisfying (1.1) with \( V(x, 0) = 0 = V(0, y) \) must be zero in \( D \). Let \( D_T = \{(x, y) \in D : x + y < T\} \) for \( 0 < T < 1 \). Then the coefficients

\[
\frac{1}{2(1 - x - y)}
\]

of (1.1) are bounded on \( D_T \). It follows by the method of successive approximations [12 pp. 136-137] that \( V \) is unique on \( D_T \) and hence vanishes on \( D_T \). Since \( T \) was arbitrary in \( (0, 1) \), the solution \( V \) is zero on \( D \). This completes the proof. \( \square \)

4. PROOF OF THEOREM 2

We define \( h_0, h_1, K_0, K_1, H_0^j, H_1^j, X_1, X_2, \) and \( c_j \) by (1.4), (2.3), (2.4), (2.5), and (2.6). Note that, due to Lemma 3.2, \( h_0 \) and \( h_1 \) have the same order of regularity as \( V_0 \) and \( V_0y \), respectively, and that \( k^{a-1/2}h_0(k) \) and \( k^{a-1/2}h_1(k) \) are in \( C((0, 1)) \). Furthermore, the kernels satisfy \( K_i(au, at) = a^{-1}K_i(u, t) \) for \( i = 0, 1 \) and \( a > 0 \).

**Proof of Theorem 2** Fix some compact subset \( I \subset (0, 1) \). Consider \( X_1 \) first. Then, for fixed \( x \in I \), we have

\[
X_1 = \frac{1}{\pi} \int_0^x \frac{H_0^j(x, k)}{\sqrt{\epsilon - k\sqrt{\epsilon - k + \epsilon}}} dk + \frac{1}{\pi} \sum_{j=0}^\infty \frac{h_0^{(j)}(x)(-1)^j}{j!} \int_0^x K_0(x - k, \epsilon)(x - k)^j dk.
\]

After the substitution \( u = x - k \), the rightmost integral in the expression above can be written as

\[
\int_0^x u^j K_0(u, \epsilon) du = \int_0^\epsilon u^j K_0(u, \epsilon) du + \int_\epsilon^x u^j K_0(u, \epsilon) du = \epsilon^j \int_0^1 u^j K_0(v, 1) dv + \int_\epsilon^x u^j K_0(u, \epsilon) du,
\]

where we used the transformation \( u = \epsilon v \). Denoting

\[
R_{K_0, j}(u, \epsilon) = K_0(u, \epsilon) - \sum_{l=0}^J \frac{\epsilon^l}{l!} (-1)^l c_l u^{-l-1},
\]

\[
X_1 = \frac{1}{\pi} \int_0^x \frac{H_0^j(x, k)}{\sqrt{\epsilon - k\sqrt{\epsilon - k + \epsilon}}} dk + \frac{1}{\pi} \sum_{j=0}^\infty \frac{h_0^{(j)}(x)(-1)^j}{j!} \int_0^x K_0(x - k, \epsilon)(x - k)^j dk.
\]
we find
\[
\int_\epsilon^x u^j K_0(u, \epsilon) du = \int_\epsilon^x u^j \left( K_0(u, \epsilon) - \sum_{l=0}^j \frac{\epsilon^l}{l!} (-1)^l c_l u^{-l-1} \right) du + \sum_{l=0}^j \frac{\epsilon^l}{l!} (-1)^l c_l \int_\epsilon^x u^{j-l-1} du \\
= \epsilon^j \int_1^\infty u^j \left( K_0(v, 1) - \sum_{l=0}^j \frac{1}{l!} (-1)^l c_l v^{-l-1} \right) dv - \int_x^\infty u^j R_{K_0, J}(u, \epsilon) du \\
+ \sum_{l=0}^j \frac{\epsilon^l}{l!} (-1)^l c_l \frac{1}{j-l} (x^{j-l} - \epsilon^{j-l}) + \frac{\epsilon^l}{j!} (-1)^l c_l (\ln(x) - \ln(\epsilon)),
\]
(4.1)
where we again substituted \( u = \epsilon v \). Note that we can split the sum \( \sum_{l=0}^j \frac{1}{l!} (-1)^l c_l v^{-l-1} \)
at \( j \) and get
\[
\int_\epsilon^x u^j K_0(u, \epsilon) du = \epsilon^j \int_1^\infty u^j \left( K_0(v, 1) - \sum_{l=0}^j \frac{1}{l!} (-1)^l c_l v^{-l-1} \right) dv - \int_x^\infty u^j R_{K_0, J}(u, \epsilon) du \\
+ C_j(x) - \sum_{l=0}^j \frac{\epsilon^l}{l!} (-1)^l c_l \frac{1}{j-l} x^{j-l} + \frac{\epsilon^l}{j!} (-1)^l c_l (\ln(x) - \ln(\epsilon)),
\]
where
\[
C_j(x) = \sum_{l=0}^{j-1} \frac{\epsilon^l}{l!} (-1)^l c_l \frac{1}{j-l} x^{j-l}.
\]
In the first term of \( X_1 \) we can expand \( K_0 \), since \( H_0^J(x, k) \sim (x - k)^{J+1} \), and get
\[
\int_0^x \frac{H_0^J(x, k)}{\sqrt{x-k} \sqrt{x-k+\epsilon}} dk = \sum_{j=0}^J \frac{\epsilon^j}{j!} (-1)^j c_j \int_0^x H_0^J(x, k) (x-k)^{-j-1} dk \\
+ \int_0^x H_0^J(x, k) R_{K_0, J}(x-k, \epsilon) dk.
\]
In the integrals \( \int_0^x H_0^J(x, k) (x-k)^{-j-1} dk \), we write
\[
H_0^J(x, k) = h(k) - \sum_{l=0}^j \frac{h_0^{(l)}(x)(-1)^l}{l!} (x-k)^l - \sum_{l=j+1}^J \frac{h_0^{(l)}(x)(-1)^l}{l!} (x-k)^l.
\]
Then the sums \( \sum_{l=j+1}^J (\cdots) \) vanish together with the terms
\[
\sum_{j=0}^J \frac{h_0^{(j)}(x)(-1)^j}{j!} C_j(x).
\]
Now it remains to show that the remainders are in \( O(\epsilon^{J+1} \ln(\epsilon)) \). We write
\[
R_{K_0, J}(u, \epsilon) = \frac{\partial^{J+1} K_0(u, \xi_\epsilon)}{\partial \xi^{J+1}} \frac{\epsilon^{J+1}}{(J+1)!} = \epsilon^{J+1} \frac{c_{J+1}}{(J+1)!} \frac{(-1)^{J+1}}{(\sqrt{u(\sqrt{\xi_\epsilon})})^{2J+3}},
\]
for some \( \xi_\epsilon \in [0, \epsilon] \). Then
\[
|R_{K_0, J}(u, \epsilon)| \leq \epsilon^{J+1} \frac{C_J}{u^{J+2}}
\]
for some constant $C_J > 0$. Hence
\[
\left| \int_x^\infty u^j R_{K_0,J}(u,\epsilon)du \right| \leq \epsilon^{J+1} C_J \int_x^\infty u^{J-2}du = O(\epsilon^{J+1}), \quad \epsilon \downarrow 0,
\]
uniformly on $I$, since $C_J$ does not depend on $x$. For the second remainder, we denote $m = \min I > 0$ and $M = \max I < 1$ and we split the integral
\[
\int_0^x H_0^J(x,k)R_{K_0,J}(x-k,\epsilon)dk = \int_0^{x/2} H_0^J(x,k)R_{K_0,J}(x-k,\epsilon)dk + \int_{x/2}^x H_0^J(x,k)R_{K_0,J}(x-k,\epsilon)dk.
\]
Since $H_0^J(x,k)k^\gamma$, where $\gamma = \alpha - 1/2$, is bounded by a constant independently of $x$ on $[0,M/2]$, there exists a constant $C > 0$, such that
\[
\left| \int_0^{x/2} H_0^J(x,k)R_{K_0,J}(x-k,\epsilon)dk \right| \leq C \epsilon^{J+1} \int_0^{x/2} \frac{1}{k^{(x-k)J+2}} dk
\]
uniformly on $I$. For the other integral, we observe
\[
|H_0^J(x,k)| = \left| \int_k^x \frac{h_0^{(J+1)}(t)}{J!} (k-t)^J dt \right| \leq \sup_{t \in [m/2,M]} |h_0^{(J+1)}(t)| \frac{(x-k)^{J+1}}{J!}
\]
for $k \in [x/2,x]$ and hence there exist constants $C,C' > 0$, independent of $x$, such that
\[
\left| \int_{x/2}^x H_0^J(x,k)R_{K_0,J}(x-k,\epsilon)dk \right| \leq C \int_{x/2}^x (x-k)^{J+1} \int_0^\epsilon \left| \frac{\partial^{J+1}}{\partial t^{J+1}} K_0(x-k,t) \right| (\epsilon - t)^J dt dk
\]
\[
= C' \int_{x/2}^x \int_0^\epsilon \frac{(x-k)^{J+1}}{\sqrt{x-k}(\sqrt{x-k}+t)^{2J+3}} (\epsilon - t)^J dt dk
\]
\[
\leq C' \epsilon^{J+1} \int_{x/2}^x \int_0^1 \frac{1}{(x-k)+\epsilon s} ds dk
\]
\[
= C' \epsilon^{J+1} \left( (x/2 + \epsilon) \ln(x/2 + \epsilon) - (x/2) \ln(x/2) - \epsilon \ln(\epsilon) \right).
\]
Now note that
\[
\frac{(x/2 + \epsilon) \ln(x/2 + \epsilon) - (x/2) \ln(x/2)}{\epsilon} = \ln(\xi_\epsilon) + 1
\]
for some $\xi_\epsilon \in [x/2,x/2+\epsilon]$ and the quotient is hence bounded on $I$. This yields that
\[
\int_{x/2}^x H_0^J(x,k)R_{K_0,J}(x-k,\epsilon)dk = O(\epsilon^{J+1} \ln(\epsilon)), \quad \epsilon \downarrow 0,
\]
uniformly on $I$. A similar procedure for $X_2$ completes the proof. \hfill \Box

**Remark 4.1.** The first few terms of the asymptotic expression found in Theorem 2 can also be written as in \([1,3]\), where $x \in (0,1)$ and where we assume that $V_0, V_1 \in C^3((0,1))$, satisfy the conditions \(2.1\). Indeed, consider the case of $X_1$; the case of $X_2$ is similar. By splitting
\[
\pi X_1 = \int_0^x h_0(k) K_0(x-k,\epsilon)dk = \int_0^{x/2} h_0(k) K_0(x-k,\epsilon)dk + \int_{x/2}^x h_0(k) K_0(x-k,\epsilon)dk,
\]
If we directly compute the asymptotic behavior of \( V \) which is consistent with (4.2).

As \( \epsilon \downarrow 0 \). Since \( h'_0 \) is bounded on \([x/2, x]\), this gives

\[
\pi X_1 = \int_0^{x/2} \frac{h_0(k)}{x-k} dk - h_0(x) \ln(\epsilon) + h_0(x/2) \ln(2x) \\
+ 2 \int_0^{x/2} \ln(\sqrt{x-k} + \sqrt{x-k + \epsilon}) h'_0(k) dk + O(\epsilon),
\]

as \( \epsilon \downarrow 0 \). Since \( h'_0 \) is bounded on \([x/2, x]\), this gives

\[
\pi X_1 = \int_0^{x/2} \frac{h_0(k)}{x-k} dk - h_0(x) \ln(\epsilon) + h_0(x/2) \ln(2x) \\
+ \int_{x/2}^x \ln(4(x-k)) h'_0(k) dk + O(\ln(\epsilon)) \epsilon \downarrow 0.
\]

Now note that similarly to (3.4), we have

\[
\frac{d}{dx} \left( \int_0^x h_0(k) \ln(4(x-k)) dk \right) = \int_0^{x/2} \frac{h_0(k)}{x-k} dk + h_0(x/2) \ln(2x) + \int_{x/2}^x \ln(4(x-k)) h'_0(k) dk.
\]

The following coefficients can be computed similarly by integrating by parts iteratively.

Alternatively, (1.3) can be obtained directly from Theorem 2 by writing

\[
\frac{d}{dx} \left( \int_0^x h_0(k) \ln(4(x-k)) dk \right) = \frac{d}{dx} \left( \int_0^x H_0^0(x,k) \ln(4(x-k)) dk + \int_0^x h_0(x) \ln(4(x-k)) dk \right) \\
= \int_0^x H_0^0(x,k) \frac{h_0(k)}{x-k} dk - h'_0(x) \int_0^x \ln(4(x-k)) dk \\
+ \frac{d}{dx} \left( h_0(x) \int_0^x \ln(4(x-k)) dk \right) \\
= \int_0^x H_0^0(x,k) \frac{h_0(k)}{x-k} dk + h_0(x) \ln(4) + h_0(x) \ln(x),
\]

which is equal to \( A_0^0(x) \) in Theorem 2. This works similarly in the \( h_1 \)-case and for the next coefficients.

**Example 4.2** (Solution of Khan and Penrose). The solution of Khan and Penrose (cf. [13, 19]) is given by

\[
V(x, y) = - \ln \left( \frac{1 + \sqrt{x} \sqrt{1-y} + \sqrt{y} \sqrt{1-x}}{1 - \sqrt{x} \sqrt{1-y} - \sqrt{y} \sqrt{1-x}} \right).
\]

If we directly compute the asymptotic behavior of \( V \) as \( x + y \to 1 \) we get

\[
V(x, 1-x - \epsilon) = 2 \ln(\epsilon) - \ln(16(1-x)x) - \frac{1 - 2x}{2(1-x)x} \epsilon + \frac{3(1 - 2x + 2x^2)}{16(1-x)^2 x^2} \epsilon^2 + O(\epsilon^3).
\]

(4.2)

If we compute the asymptotics by using Theorem 2 we get \( h_0 = -\pi = h_1 \) and

\[
A_0(x) = -\pi(\ln(x) + \ln(4)), \\
A_1(x) = \frac{\pi}{2x}, \\
A_2(x) = \frac{3\pi}{16x^2},
\]

\[
B_0(x) = -\pi(\ln(1-x) + \ln(4)) \\
B_1(x) = \frac{\pi}{2(1-x)}, \\
B_2(x) = \frac{3\pi}{16(1-x)^2},
\]

which is consistent with (4.2).
5. Application to Gravitational Waves

It is shown in Eq. (10.2) of [13] that the colliding gravitational wave problem for collinearly polarized plane waves reduces to solving the equation

\[ V_{fg} + \frac{V_f + V_g}{2(f + g)} = 0 \] (5.1)

in the triangular region

\[ \{ (f, g) \in \mathbb{R}^2 | f \leq \frac{1}{2}, g \leq \frac{1}{2}, f + g > 0 \}. \]

The change of variables \( x = \frac{1}{2} - g, y = \frac{1}{2} - f \) transforms (5.1) into (1.1). The components of the Weyl tensor (cf. Eq. (10.4) in [13]) are given by

\[
\begin{align*}
\Psi_0 &= -\frac{g'(v)^2}{4} \left( 2V_{gg} + \frac{3}{f + g} V_g - (f + g)V_g^3 \right) \\
&= -\frac{g'(v)^2}{4} \left( 2V_{xx} - \frac{3}{1 - x - y} V_x + (1 - x - y)V_x^3 \right), \\
\Psi_2 &= f'(u)g'(v) \left( V_f V_g - \frac{1}{(f + g)^2} \right) \\
&= f'(u)g'(v) \left( V_x V_y - \frac{1}{(1 - x - y)^2} \right), \\
\Psi_4 &= -\frac{f'(u)^2}{4} \left( 2V_{ff} + \frac{3}{f + g} V_f - (f + g)V_f^3 \right) \\
&= -\frac{f'(u)^2}{4} \left( 2V_{yy} - \frac{3}{1 - x - y} V_y + (1 - x - y)V_y^3 \right),
\end{align*}
\]

where \( u, v \geq 0 \) are suitable null coordinates in the interaction region [13, Chapter 6]. Using that \( f(u) \) and \( g(v) \) have the form (see Eq. (7.6) in [13])

\[
\begin{align*}
f(u) &= \frac{1}{2} - (c_1 u)^{n_1}, \quad g(v) = \frac{1}{2} - (c_2 v)^{n_2}, \\
f'(u) &= -c_1 n_1 (c_1 u)^{n_1 - 1} = -\frac{n_1}{u}(1 - f) = -\frac{n_1}{u} y = -c_1 n_1 y^{1 - \frac{1}{n_1}}, \\
g'(v) &= -c_2 n_2 (c_2 v)^{n_2 - 1} = -c_2 n_2 x^{1 - \frac{1}{n_2}},
\end{align*}
\]

with some constants \( c_1, c_2, n_1, n_2, \) we find

\[
\begin{align*}
\Psi_0 &= \frac{(c_2 n_2 x^{1 - \frac{1}{n_2}})^2}{4} \left( 2V_{xx} - \frac{3}{1 - x - y} V_x + (1 - x - y)V_x^3 \right), \\
\Psi_2 &= c_1 n_1 y^{1 - \frac{1}{n_1}} c_2 n_2 x^{1 - \frac{1}{n_2}} \left( V_x V_y - \frac{1}{(1 - x - y)^2} \right), \\
\Psi_4 &= \frac{(c_1 n_1 y^{1 - \frac{1}{n_1}})^2}{4} \left( 2V_{yy} - \frac{3}{1 - x - y} V_y + (1 - x - y)V_y^3 \right). \quad (5.2)
\end{align*}
\]

Theorem 2 implies

\[
V(x, y) = \sum_{j=0}^{N} f_j(x) e^j \ln(\epsilon) + \sum_{j=0}^{N} g_j(x) e^j + O(\epsilon^{N+1} \ln(\epsilon))
\]
as \( \epsilon = \epsilon(x, y) = 1 - x - y \to 0 \) uniformly for \( x, y \) in some neighborhood of the diagonal away from the corners, where

\[
f_j(x) = \frac{-c_j}{\pi(j!)^2} (h_0^{(j)}(x) + h_1^{(j)}(x)), \quad g_j(x) = \frac{1}{\pi} ((-1)^j A_j(x) + B_j(x)).
\]

Due to symmetry of the representation \( \Delta \), we also have

\[
V(x, y) = \sum_{j=0}^{N} f_j(1 - y)\epsilon^j \ln(\epsilon) + \sum_{j=0}^{N} \tilde{g}_j(1 - y)\epsilon^j + O(\epsilon^{N+1} \ln(\epsilon))
\]

as \( \epsilon = 1 - x - y \to 0 \), where \( \tilde{g}_j \) is defined by interchanging \( h_0 \) and \( h_1 \) in the definition of \( g_j \). It is easy to see from the proof of Theorem \( \Delta \) that the \( y \)-derivative of the remainder is \( O((1 - x - y)^N \ln(1 - x - y)) \) uniformly. Letting \( \epsilon = \epsilon(x, y) = 1 - x - y \), this gives

\[
V_y(x, y) = -\sum_{j=0}^{N} j f_j(x)\epsilon^{j-1} \ln(\epsilon) - \sum_{j=0}^{N} \gamma_j(x)\epsilon^{j-1} + O((\epsilon^N \ln(\epsilon))), \quad \epsilon \downarrow 0,
\]

and

\[
V_{yy}(x, y) = \sum_{j=0}^{N} (j - 1) j f_j(x)\epsilon^{j-2} \ln(\epsilon) + \sum_{j=0}^{N} G_j(x)\epsilon^{j-2} + O(\epsilon^{N-1} \ln(\epsilon)), \quad \epsilon \downarrow 0,
\]

where \( \gamma_j = j g_j + f_j \) and \( G_j = (j - 1) \tilde{g}_j + j f_j \). In the same way we get

\[
V_x(x, y) = -\sum_{j=0}^{N} j f_j(1 - y)\epsilon^{j-1} \ln(\epsilon) - \sum_{j=0}^{N} \tilde{\gamma}_j(1 - y)\epsilon^{j-1} + O(\epsilon^N \ln(\epsilon)), \quad \epsilon \downarrow 0,
\]

\[
V_{xx}(x, y) = \sum_{j=0}^{N} (j - 1) j f_j(1 - y)\epsilon^{j-2} \ln(\epsilon) + \sum_{j=0}^{N} \tilde{G}_j(1 - y)\epsilon^{j-2} + O(\epsilon^{N-1} \ln(\epsilon)), \quad \epsilon \downarrow 0,
\]

where \( \tilde{\gamma}_j = j \tilde{g}_j + f_j \) and \( \tilde{G}_j = (j - 1) \tilde{\gamma}_j + j f_j \). Together with \( \Delta \) this leads to full expansions of the components of the Weyl tensor near the diagonal. We have shown the following corollary of Theorem \( \Delta \)

**Corollary 5.1** (Asymptotics of the Weyl tensor). Let \( N \geq 1 \) be and integer. Suppose \( V_0, V_1 \in C^{N+2}((0, 1)) \) satisfy \( \Delta \). Then the components of the Weyl tensor associated to the solution \( V(x, y) \) of the Goursat problem for \( \Delta \) with data \( \{V_0, V_1\} \) have the following asymptotic behaviour as \( \epsilon = \epsilon(x, y) = 1 - x - y \to 0 \):

\[
\Psi_0^\epsilon(x, y) = \frac{(c_2 n x)^{1 - \frac{1}{n_2}}}{4} \left[ \sum_{j=0}^{N} (2 \tilde{G}_j(1 - y) + 3 \tilde{\gamma}_j(1 - y))\epsilon^{j-2}
\right.

\[
+ \sum_{j=0}^{N} (j + 2) j f_j(1 - y)\epsilon^{j-2} \ln(\epsilon)
\]

\[
- \epsilon \left( \sum_{j=0}^{N} j f_j(1 - y)\epsilon^{j-1} \ln(\epsilon) + \sum_{j=0}^{N} \tilde{\gamma}_j(1 - y)\epsilon^{j-1} \right)^3 + O(\epsilon^{N-1} \ln(\epsilon)^3) \right],
\]
\[
\Psi_2^o(x, y) = c_1 n_1 c_2 n_2 y^{1 - \frac{1}{n_1}} x^{1 - \frac{1}{n_2}} \left[ \sum_{j=0}^{N} \left( \sum_{k+l=j} \gamma_k(x) \tilde{\gamma}_l(1 - y) \right) e^{j - \frac{1}{\epsilon^2}} \right] \\
+ \sum_{j=0}^{N} \left( \sum_{k+l=j} k f_k(x) \tilde{\gamma}_l(1 - y) + k f_k(1 - y) \gamma_l(x) \right) e^{j - 2 \ln(\epsilon)} \\
+ \sum_{j=0}^{N} \left( k l f_k(x) f_l(1 - y) \right) e^{j - 2 \ln(\epsilon)} \\
+ O(\epsilon^N - \frac{1}{\ln(\epsilon)}),
\]

\[
\Psi_4^o(x, y) = \left( c_1 n_1 y^{1 - \frac{1}{n_1}} \right)^2 \left[ \sum_{j=0}^{N} (2G_j(x) + 3\gamma_j(x)) e^{j - 2 \ln(\epsilon)} \right] \\
+ \sum_{j=0}^{N} (j + 2) j f_j(x) e^{j - 2 \ln(\epsilon)} \\
- \epsilon \left( \sum_{j=0}^{N} j f_j(x) e^{j - 1 \ln(\epsilon)} + \sum_{j=0}^{N} \gamma_j(x) e^{j - 1} \right)^3 + O(\epsilon^N - \frac{1}{\ln(\epsilon)^3}),
\]

where the error terms are uniform for \((x, y) \in D\) bounded away from the corners of \(D\).

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