Kontsevich product and gauge invariance

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We analyze the question of $U_\mu(1)$ gauge invariance in a flat non-commutative space where the parameter of non-commutativity, $\theta^{\mu\nu}(x)$, is a local function satisfying Jacobi identity (and thereby leading to an associative Kontsevich product). We show that in this case, both gauge transformations as well as the definitions of covariant derivatives have to modify so as to have a gauge invariant action. We work out the gauge invariant actions for the matter fields in the fundamental and the adjoint representations up to order $\theta^2$ while we discuss the gauge invariant Maxwell theory up to order $\theta$. We show that despite the modifications in the gauge transformations, the covariant derivative and the field strength, Seiberg-Witten map continues to hold for this theory. In this theory, translations do not form a subgroup of the gauge transformations (unlike in the case when $\theta^{\mu\nu}$ is a constant) which is reflected in the stress tensor not being conserved.

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I. INTRODUCTION

Non-commutativity of space-time coordinates have been by now studied exhaustively from various points of view. It arises naturally in the quantization of open strings and membranes attached to D-branes in the presence of background fields. The non-commutativity is related to the background fields so that if the background fields are constant, one obtains the more familiar case where the parameter of non-commutativity $\theta^{\mu\nu}$ is a constant. In such a case, the standard multiplication of functions is replaced by the Grœnewold-Moyal product. On the other hand, if the background fields depend on space-time coordinates, then, one expects the parameter of non-commutativity to be a local function. Theories with a local parameter of non-commutativity have not been studied as vigorously. In this case, the multiplication of functions is replaced by the Kontsevich product and there are two natural cases that can arise. Namely, the parameter of non-commutativity $\theta^{\mu\nu}(x)$ satisfies the Jacobi identity in which case the Kontsevich product is associative. The second possibility is that $\theta^{\mu\nu}(x)$ does not satisfy the Jacobi identity leading to a non-associative Kontsevich product. The first case corresponds to the embedding of a curved D-brane in a flat background while the second arises for a curved brane embedded in a curved background (with the non-associativity related to the curvature of the background). In the case where $\theta^{\mu\nu}$ is local and satisfies Jacobi identity, there exists so far only a single analysis of a model, namely, the Cattaneo-Felder model which involves the study of a boundary conformal field theory. In this paper, we would like to extend such a possibility to the case of gauge theories.

This is also important from the point of view of studying the properties of non-commutative field theories (independent of their origin). By now, non-commutative field theories in flat space-time with $\theta^{\mu\nu}$ constant have been studied extensively and various interesting properties have been noted. However, eventually one would like to study the properties of such theories in a curved background (possibly including non-commutative gravity). In such a case, it is clear that $\theta^{\mu\nu}$ can no longer be considered a constant. As a first attempt at studying such theories, it would be interesting to study the behavior of a field theory in a flat non-commutative space-time where $\theta^{\mu\nu}$ is a local function. In fact, it is even more interesting to study the question of gauge invariance in such a case. With that in mind, we have chosen to study a $U_\mu(1)$ gauge theory with matter in the fundamental as well as in the adjoint representations and we find many interesting features from our analysis of such theories.

Let us recall that when $\theta^{\mu\nu}$ is local, the star product is given by the Kontsevich product:

\[
\star \, f \star g = fg + \frac{i}{2} \theta^{\mu\nu} \partial_\mu f \partial_\nu g - \frac{1}{8} \theta^{\mu\nu} \theta^{\lambda\rho} \partial_\mu \partial_\lambda f \partial_\rho \partial_\nu g - \frac{1}{12} \theta^{\lambda\nu} \partial_\nu \theta^{\mu\rho} (\partial_\lambda \partial_\rho f \partial_\nu g - \partial_\rho f \partial_\lambda \partial_\nu g) + \cdots ,
\]

where $\cdots$ represent higher order terms in $\theta$. It is clear from (1) that we can identify

\[
[x^\mu, x^\nu] = x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}(x) ,
\]

and if $\theta^{\mu\nu}(x)$ satisfies the Jacobi identity,

\[
\theta^{\mu\nu} \partial_\nu \theta^{\lambda\rho} + \theta^{\nu\rho} \partial_\nu \theta^{\lambda\mu} + \theta^{\lambda\mu} \partial_\nu \theta^{\rho\nu} = 0 ,
\]
then, the product \( [\mathbf{1}] \) is associative. Throughout this paper, we will restrict ourselves to such a case. It is also clear from \( [\mathbf{1}] \) that we can represent

\[
i\theta^{\mu\nu}(x)\partial_\nu f(x) = [x^\mu, f] .
\]

(4)

If \( \theta^{\mu\nu}(x) \) has an inverse, this can even be inverted to write

\[
\partial_\mu f(x) = -i(\theta^{-1})_{\mu\nu}(x) [x^\nu, f] \neq [-i(\theta^{-1})_{\mu\nu} x^\nu, f] ,
\]

(5)

where the last relation holds only for the case when \( \theta^{\mu\nu} \) is a constant.

Unlike the more studied case of constant \( \theta^{\mu\nu} \), the products inside an integral no longer satisfy cyclicity when the parameter of non-commutativity is a local function and as a result, the analysis of such theories is a bit more involved. We note here that when \( \theta^{\mu\nu} \) is a local function, it can be thought of as a genuine Lorentz tensor. On the other hand, we do know that unitarity in a non-commutative theory is violated unless \( \theta^0 = 0 \). Presumably, one can make such a choice by going to a particular reference frame. However, for purposes of studying gauge invariance properties of the action at the tree level, we do not concern ourselves with this question in this paper. This certainly is an important question which deserves further study and we will report on this in the future. In this paper, we assume that \( \theta^{\mu\nu}(x) \) is a genuine tensor (although that really does not enter into our analysis at all).

The organization of our paper is as follows. In section \( \text{II} \), we discuss how matter (scalar fields) in the fundamental as well as in the adjoint representations can be coupled to the photon in a \( U_\star(1) \) invariant manner. This necessitates a modification of the covariant derivative as well as the gauge transformation for the gauge field. We demonstrate how this can be achieved systematically up to order \( \theta^2 \) in the matter sector. The definition of the field strength also changes as a consequence and we determine the gauge invariant action for the Maxwell theory up to order \( \theta \) (since it is much more involved than the matter sector because of the Lorentz indices, but the procedure is clear). One of the surprising outcomes of this analysis is that \( \theta^{\mu\nu}(x) \) does not transform under the gauge transformation (although, naively, one would have expected it to transform in the adjoint representation). We note that all the modifications that we find vanish when \( \theta^{\mu\nu} \) is a constant reducing to the conventional \( U_\star(1) \) gauge invariant studied in the literature \( \text{[15]} \).

In section \( \text{III} \), we show that in spite of these modifications, the Seiberg-Witten map \( \text{[4]} \) between the non-commutative and the commutative theories continues to hold. This is surprising and suggests some deeper meaning of the map that we have not studied further. We show the equivalence of the equations of motion (non-commutative and commutative) as well as the stress tensors of the Maxwell theory under the map. Furthermore, we show that in the present case (unlike in the case of a constant \( \theta^{\mu\nu} \)), the stress tensor is not conserved. This is traced to the fact that when \( \theta^{\mu\nu}(x) \) is a local function, it can be thought of as an external field which violates translation invariance. In fact, since \( \theta^{\mu\nu}(x) \) is inert under a gauge transformation, while translation invariance requires it to transform, it follows that in this case, translations do not form a subgroup of the \( U_\star(1) \) gauge transformations (as is the case for constant \( \theta^{\mu\nu} \)).

We close with a brief summary in section \( \text{IV} \).

\section{II. GAUGE INVARIANT ACTIONS}

In this section, we will construct actions invariant under \( U_\star(1) \) gauge transformations. Let us start with the action for a complex scalar field (which would represent matter in the fundamental representation), which conventionally has the form

\[
S_{\text{fund}} = \int dx \left( (D_\mu \phi)^* \ast (D^\mu \phi) - m^2 \phi^* \ast \phi \right) .
\]

(6)

We have left the dimensionality of space-time arbitrary since that does not enter into our analysis. In the more familiar case of a constant \( \theta^{\mu\nu} \), the covariant derivative has the form \( \text{[15]} \)

\[
D_\mu \phi = \partial_\mu \phi - iA_\mu \ast \phi ,
\]

(7)

and the action is invariant under the infinitesimal gauge transformations

\[
\delta \phi = i\epsilon \ast \phi , \quad \delta A_\mu = \partial_\mu \epsilon - i [A_\mu, \epsilon] ,
\]

(8)

where \( \epsilon(x) \) represents the infinitesimal parameter of gauge transformations. When \( \theta^{\mu\nu} \) becomes a local function, the action in \( \text{[6]} \) with \( \text{[7]} \) is no longer invariant under the gauge transformations \( \text{[8]} \). In this case, we have to systematically determine the modifications necessary in the definitions of the covariant derivative and the gauge transformations under which the action \( \text{[3]} \) will be invariant. We will demonstrate how this can be done up to order \( \theta^2 \) in this theory.
To begin with, let us note that when $\theta^{\mu\nu}$ is local,

$$
\int dx \ A(x) \ast B(x) \neq \int dx \ B(x) \ast A(x).
$$

(9)

However, form the definition of the product in (11), we note that

$$
(A(x) \ast B(x))^* = B^*(x) \ast A^*(x),
$$

(10)

so that if we can find a definition of the covariant derivative as well as gauge transformations such that

$$
\delta \phi = i \alpha(x) \ast \phi, \quad \delta (D_\mu \phi) = i \beta(x) \ast (D_\mu \phi),
$$

(11)

for real functions $\alpha(x), \beta(x)$, then, action (6) will be gauge invariant.

To systematically determine these modifications, let us represent

$$
\delta \phi = i \epsilon(x) \ast \phi(x) + P(x),
$$

$$
\delta A_\mu = \partial_\mu \epsilon(x) - i [A_\mu, \epsilon] + Y_\mu(x),
$$

$$
D_\mu \phi(x) = \partial_\mu \phi(x) - i A_\mu \ast \phi + Z_\mu(x),
$$

(12)

where the modifications $P(x), Y_\mu(x)$ and $Z_\mu(x)$ in (12) are assumed to be of order $\theta$ or higher and such that (11) holds. Furthermore, since $\theta^{\mu\nu}(x)$ is a local function, we have to allow for the possibility that it may transform under a gauge transformation and recognize that, in such a case,

$$
\delta (A \ast B) \neq (\delta A) \ast B + A \ast (\delta B).
$$

(13)

The analysis is tedious but can be carried out systematically and we find, up to order $\theta^2$, that with

$$
D_\mu \phi = \partial_\mu \phi - i A_\mu \ast \phi + \frac{1}{2} \partial_\mu \theta^{\lambda\rho} (A_\lambda \ast \partial_\rho \phi)
$$

$$
- \frac{i}{12} \partial_\mu (\theta^{\sigma\tau} \partial_\sigma \theta^{\lambda\rho}) (\partial_\tau A_\lambda \partial_\rho \phi - A_\lambda \partial_\rho \partial_\tau \phi) + \frac{1}{12} (2 \theta^{\sigma\tau} \partial_\sigma \partial_\rho \theta^{\lambda\rho} - \partial_\mu \theta^{\sigma\tau} \partial_\sigma \theta^{\lambda\rho}) A_\lambda A_\tau \partial_\rho \phi,
$$

$$
\delta \theta^{\mu\nu} = 0;
$$

$$
\delta \phi = i \epsilon \ast \phi,
$$

$$
\delta A_\mu = \partial_\mu \epsilon(x) - i [A_\mu, \epsilon] + \frac{1}{2} \partial_\mu \theta^{\lambda\rho} (A_\lambda \ast \partial_\rho \epsilon)
$$

$$
- \frac{i}{12} \partial_\mu (\theta^{\sigma\tau} \partial_\sigma \theta^{\lambda\rho}) (\partial_\tau A_\lambda \partial_\rho \epsilon - A_\lambda \partial_\rho \partial_\tau \epsilon) + \frac{1}{12} (2 \theta^{\sigma\tau} \partial_\sigma \partial_\rho \theta^{\lambda\rho} - \partial_\mu \theta^{\sigma\tau} \partial_\sigma \theta^{\lambda\rho}) A_\lambda A_\tau \partial_\rho \epsilon,
$$

$$
\delta (D_\mu \phi) = i \epsilon \ast (D_\mu \phi),
$$

(14)

the action in (6) is invariant. In this case, the modified covariant derivative transforms covariantly. There are two main features worth noting here. First, all the modifications vanish when $\theta^{\mu\nu}$ is a constant so that (14) reduces to (15). The more surprising aspect is that even though $\theta^{\mu\nu}(x)$ is a local function (and one would naively expect it to transform in the adjoint representation), it does not transform under a gauge transformation.

For a real scalar field (matter in the adjoint representation), the conventional action has the form

$$
S_{\text{adj}} = \int dx \ \left( \frac{1}{2} (D_\mu \phi) \ast (D^\mu \phi) - \frac{m^2}{2} \phi \ast \phi \right).
$$

(15)

When $\theta^{\mu\nu}$ is constant, with

$$
D_\mu \phi = \partial_\mu - i [A_\mu, \phi],
$$

(16)

the gauge transformations

$$
\delta \phi = - i [\phi, \epsilon], \quad \delta A_\mu = \partial_\mu \epsilon(x) - i [A_\mu, \epsilon],
$$

(17)
define an invariance of $(15)$. In generalizing the covariant derivative and the gauge transformation to the case when $\theta^{\mu\nu}$ is a local function, we note that $(10)$ is no longer useful because the field variable is real and, as a result, even if the covariant derivative transforms covariantly, it does not help and we have to analyze the invariance of the action as a whole.

We can modify the covariant derivative as well as the transformation laws along the lines of $(12)$ and write

$$
\delta \phi = -i [\phi, \epsilon] + P(x), \quad D_\mu \phi = \partial_\mu \phi(x) - i [A_\mu, \phi] + Z_\mu,
$$

with $\delta \theta^{\mu\nu}, \delta A_\mu$ already determined in $(14)$. The analysis of the invariance of the action then determines

$$
\delta \phi = -i [\phi, \epsilon],
$$

$$
D_\mu \phi = \partial_\mu \phi - i [A_\mu, \phi] + \partial_\mu \theta^{\lambda\rho} A_\lambda \partial_\rho \phi
$$

$$
- \frac{1}{2} \theta^{\sigma\tau} \partial_\sigma \theta^{\lambda\rho} \partial_\rho A_\lambda A_\tau + \frac{1}{2} \left( \theta^{\sigma\tau} \partial_\sigma \theta^{\lambda\rho} - \partial_\sigma \theta^{\lambda\rho} \theta^{\sigma\tau} \right) A_\lambda A_\tau \partial_\rho \phi.
$$

It is worth noting here that in determining the invariance of the action for the scalar field in the adjoint representation, we require that the parameter of anti-commutativity be divergenceless, namely,

$$
\partial_\mu \theta^{\mu\nu}(x) = 0, \tag{20}
$$

in addition to satisfying the Jacobi identity $(8)$. This is a sufficient condition for the Jacobi identity and is essential in the discussion of the Seiberg-Witten map in the next section. We point out here that it is possible, in principle, to have an invariant action involving modified covariant derivatives and transformations without using $(20)$, but such modified quantities become highly non-local as the order of $\theta$ increases and we do not find that very appealing. We also note that all the modifications in $(19)$ vanish when $\theta^{\mu\nu}$ is constant.

With the construction of the invariant actions for the matter fields, let us next construct the gauge invariant action for the Maxwell theory. Conventionally, the invariant action has the form

$$
S_{\text{Maxwell}} = -\frac{1}{4} \int \, dx \, F_{\mu\nu} \star F^{\mu\nu}. \tag{21}
$$

When $\theta^{\mu\nu}$ is constant, the field strength tensor is given by

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu] = [D_\mu, D_\nu], \tag{22}
$$

where $D_\mu$ is the covariant derivative $(9)$ in the fundamental representation. When $\theta^{\mu\nu}$ is local, the commutator (in the star product sense) of the covariant derivative in $(13)$ does not even give a multiplicative operator so that the definition of the field strength as well as the analysis of the invariance of the action $(21)$ need to be done independently. This is lot more tedious than that for the action for the matter fields because of the Lorentz structures and we will present an invariant action up to order $\theta$ although the procedure can be carried out to any order in $\theta$ in principle.

Here, as in the case of the actions for the matter fields, the idea is to modify the field strength $(22)$ as

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu] + X_{\mu\nu}(x), \tag{23}
$$

where $X_{\mu\nu} = -X_{\nu\mu}$ is at least of order $\theta$ and is determined so that the action $(21)$ is invariant under the gauge transformations (for the $\theta^{\mu\nu}, A_\mu$) determined in $(14)$. To order $\theta$, this is easily carried out and if we do not assume $(20)$, the field strength becomes non-local (more so with increasing order of $\theta$). Therefore, we assume $(20)$ in which case the modified field strength that leads to an invariant action has the form

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu] + \frac{1}{2} \left( \partial_\mu \theta^{\lambda\rho} A_\lambda \partial_\rho A_\nu + F_{\mu\nu}) - (\mu \leftrightarrow \nu) \right). \tag{24}
$$

Once again, we see that the modification is such that it vanishes when $\theta^{\mu\nu}$ is constant. This demonstrates that there is a systematic procedure for determining an action (both for matter as well as gauge fields) which is invariant under $U_*(1)$ transformations when $\theta^{\mu\nu}$ is a local function.

III. SEIBERG-WITTEN MAP

As in the case when $\theta^{\mu\nu}$ is constant, here we can also ask if there is a Seiberg-Witten map $(6)$ that would take the gauge theory on the non-commutative manifold to a theory on a commutative space. If we denote quantities on the non-commutative manifold with “hats”, then we wish to determine if there exist functions

$$
\hat{\epsilon} = \hat{\epsilon}(\epsilon, A), \quad \hat{A}_\mu = \hat{A}_\mu(A), \tag{25}
$$
such that
\[ \hat{A}_\mu(A + \delta A) = \hat{A}_\mu(A) + \hat{\delta}_\epsilon \hat{A}_\mu(A), \] 
(26)
where \( \delta A \) represents the usual \( U(1) \) gauge transformation in a commutative manifold. Namely, we wish to determine \( \hat{A}_\mu \) in powers of \( \theta \) such that the gauge field in the non-commutative manifold is (gauge) equivalent to the one on the commutative manifold. We will determine this to linear order in \( \theta \) as is also done in the case when \( \theta^{\mu\nu} \) is constant [17], but the procedure can be carried out to any order in \( \theta \).

We recognize that \( A_\mu \) denotes a \( U(1) \) gauge field in a commutative space so that
\[ \delta A_\mu = \partial_\mu \epsilon(x). \] 
(27)
Therefore, using the transformation for the gauge field in (14) and (27) in (26) (up to order one on the commutative manifold. We will determine this to linear order in \( \theta \) as is also done in the case when \( \theta^{\mu\nu} \) is constant [17], but the procedure can be carried out to any order in \( \theta \).

Let us assume that
\[ \hat{A}_\mu(A) = A_\mu + A'_\mu(A), \] 
(29)
where the primed quantities are (at least) of order \( \theta \). Then, (28) can be easily solved to determine
\[ \hat{A}_\mu = A_\mu - \frac{1}{2} \theta^{\rho\lambda} A_\lambda (\partial_\rho A_\mu + F_{\rho\mu}) - \frac{1}{2} \theta^{\rho\lambda} A_\lambda (\partial_\rho \hat{A}_\mu + \hat{F}_{\rho\mu}) - \frac{1}{2} \theta^{\rho\lambda} A_\lambda (\partial_\rho \hat{A}_\mu + \hat{F}_{\rho\mu}) = 0. \] 
(30)
We recognize this to be exactly the Seiberg-Witten map for the case when \( \theta^{\mu\nu} \) is constant and it continues to hold even in the case when \( \theta^{\mu\nu} \) is a local function. It is interesting that the extra modifications depending on derivatives of \( \theta \) do identically cancel out so that the usual Seiberg-Witten map holds. In fact, what is even more interesting is that the field strength tensor defined in (24) goes over under this map to (up to order \( \theta \))
\[ \hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i [\hat{A}_\mu, \hat{A}_\nu] + \frac{1}{2} \left( \partial_\mu \theta^{\rho\lambda} \hat{A}_\lambda (\partial_\rho A_\nu + \hat{F}_{\rho\nu}) - (\mu \leftrightarrow \nu) \right). \] 
(31)
which represents the same map of the field strength as in the case of constant \( \theta^{\mu\nu} \), even though the field strength in the non-commutative space in the present case has a much more complicated structure (24). This, therefore, suggests a deeper meaning underlying the Seiberg-Witten map that deserves further study. We wish to point out here that all of this works only when \( \theta^{\mu\nu} \) holds.

With the Seiberg-Witten maps determined, we can now easily show to linear order in \( \theta \) that the action for the gauge field (24) goes over to
\[ \hat{S}_{\text{Maxwell}} = -\frac{1}{4} \int dx \hat{F}_{\mu\nu} \wedge \hat{F}^{\mu\nu} = -\frac{1}{4} \int dx \left( 1 - \frac{1}{2} \theta^{\lambda\rho} F_{\lambda\rho} \right) F_{\mu\nu} F^{\mu\nu} + 2 \text{Tr} F^3 - \partial_\rho \left( \theta^{\rho\lambda} A_\lambda F_{\mu\nu} \right). \] 
(32)
where we have used an obvious matrix notation in writing the action. In Eq. (32), we have kept a total divergence which does not contribute to the equations of motion, but is essential for the definition of the stress tensor of the theory [17, 18]. The equation of motion following from the non-commutative theory (to linear order in \( \theta \)) has the form
\[ \partial_\mu \hat{F}^{\mu\nu} + \theta^{\rho\mu} \partial_\lambda \hat{A}_\rho \partial_\nu \hat{F}^{\mu\nu} - \frac{1}{2} \partial_\nu \theta^{\rho\mu} (\partial_\rho \hat{A}_\lambda + \hat{F}_{\rho\lambda}) \hat{F}^{\mu\nu} + \partial_\rho \left( \partial_\mu \theta^{\rho\lambda} \hat{A}_\lambda \hat{F}^{\mu\nu} \right) - \frac{1}{2} \partial_\nu \left( \partial_\rho \theta^{\rho\lambda} \hat{A}_\lambda \hat{F}^{\mu\nu} \right) = 0. \] 
(33)
On the other hand, the equation of motion following from the mapped theory in (30) leads to
\[ \partial_\mu \left[ 1 - \frac{1}{2} \theta^{\lambda\rho} F_{\lambda\rho} \right] F^{\mu\nu} - (F^2 \theta + \theta F + \theta F^2)^{\mu\nu} - \frac{1}{4} \theta^{\mu\nu} F_{\lambda\rho} F^{\lambda\rho} = 0. \] 
(34)
At first sight, (33) does not seem to map into (34) under (30) and (31). However, as is the case for the constant \( \theta^{\mu\nu} \) case [17], the two equations are identical under the map if we use the identity (which holds to order \( \theta \))
\[ \partial_\mu \left[ (F^2 \theta)^{\mu\nu} + \frac{1}{4} \theta^{\mu\nu} F_{\lambda\rho} F^{\lambda\rho} \right] - \partial_\mu \theta^{\lambda\nu} (F^2)^{\lambda\nu} = 0. \] 
(35)
In a similar manner, we can determine the stress tensor from the theory in the non-commutative space as well as from the theory transformed under the Seiberg-Witten map. With the total divergence in (32), it is straightforward to show that they coincide and have the form

\[
T^{\mu\nu} = \left(1 - \frac{1}{2} \partial^\rho F_\rho \right) \left( (F^2)^{\mu\nu} + \frac{1}{4} \eta^{\mu\nu} F_\lambda F^{\lambda\rho} \right) - \left( (F^2 \theta F)^{\mu\nu} + (F \theta F^2)^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} Tr \theta F^3 \right) - \partial_\nu \left( \theta^{\sigma\tau} A_\sigma \left( (F^2)^{\mu\nu} + \frac{1}{4} \eta^{\mu\nu} F_\lambda F^{\lambda\rho} \right) \right).
\]

The stress tensor is manifestly symmetric and traceless. Using the equation of motion (33) (or (34)), it can be checked that this is not, however, conserved as is also the case when \(\theta^{\mu\nu}\) is constant. In that case, we can define a modified stress tensor that is neither symmetric nor traceless, but conserved [17]. In contrast, in the present case we find that even a modified stress tensor such as

\[
\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\rho \left( \theta^{\lambda\rho} A_\lambda T^{(0)}{^\mu}_\nu \right) - (\theta F T^{(0)})^{\mu\nu},
\]

is no longer conserved. Here \(T^{(0)}\) denotes the stress tensor independent of \(\theta\). In fact, the divergence of the modified stress tensor leads to

\[
\partial_\mu \hat{T}^{\mu\nu} = -\frac{1}{2} \partial^\rho \theta^{\lambda\rho} \left( FT^{(0)} \right)_{\lambda\nu} = -\partial^\rho \theta^{\lambda\rho} \frac{\delta S_{\text{Maxwell}}}{\delta \theta^{\lambda\rho}}.
\]

We note from (38) that for a constant \(\theta^{\mu\nu}\), the modified stress tensor would be conserved (even though it is neither symmetric nor traceless).

The non-conservation of the stress tensor is not hard to understand. When \(\theta^{\mu\nu}\) is a local function, we can think of the action as representing the interaction (in addition to the self-interactions) of the Maxwell field with an external field \(\theta^{\mu\nu}(x)\) and as a result, our system cannot be thought of as a closed system. It is, of course, not necessary for a system that is not closed to have conservation of energy. If we have a complete theory where \(\theta^{\mu\nu}\) is a fundamental dynamical field (one may speculate that such a situation may arise in a gravitational theory with a dynamical \(\theta^{\mu\nu}\) as is the case in string theory) leading to a closed system, then, the complete energy including that of the dynamical field \(\theta^{\mu\nu}\) has to be conserved. The non-conservation can be understood yet in a different manner. We recall that conservation of stress tensor follows from translation invariance of a system. In the presence of an external field, translation invariance does not hold (since the external field does not transform). This is manifest in the last equality in (38). This brings out a very interesting feature that contrasts with the case of constant \(\theta^{\mu\nu}\). Namely, it is well known for constant \(\theta^{\mu\nu}\) that translations form a subgroup of the \(U_1(1)\) transformation group [4]. In contrast, when \(\theta^{\mu\nu}\) is local, we have noted in (14) that the parameter of non-commutativity does not transform under a gauge transformation. On the other hand, under transformations \(\theta^{\mu\nu}\) has to transform so that we conclude translations do not form a subgroup of the \(U_1(1)\) gauge transformations when the parameter of non-commutativity is a local function. In retrospect, in view of the inequality in (5), it is clear that when \(\theta^{\mu\nu}\) is a local function, we can no longer represent a gauge transformation to include translations [4].

IV. CONCLUSION

In this paper, we have analyzed systematically the question of \(U_1(1)\) gauge invariance in a flat non-commutative manifold where the parameter of non-commutativity is a local function satisfying the Jacobi identity (and, therefore, leading to an associative Kontsevich product). We have shown that in this case, the definitions for both the covariant derivative as well as the gauge transformation have to modify in order to have an invariant action. The modifications can be systematically determined. We have demonstrated this up to order \(\theta^2\) in the matter sector (for both fundamental and adjoint representations) and up to order \(\theta\) in the Maxwell theory. One of the surprising features of this analysis is that \(\theta^{\mu\nu}(x)\) does not transform under a gauge transformation. The modifications in other variables vanish in the case when \(\theta^{\mu\nu}\) is constant. We have shown that when \(\theta^{\mu\nu}\) is a local function, there exists a Seiberg-Witten map, which surprisingly coincides with the one for the case when \(\theta^{\mu\nu}\) is constant. We have shown that the equations of motion as well as the stress tensor in the non-commutative theory go over under the map to the ones derived from the theory on the commutative manifold. The stress tensor in the present case is not conserved and this is traced to non-invariance under translations in such a theory. We have shown that, unlike the case when \(\theta^{\mu\nu}\) is constant, in the present case translations do not form a subgroup of the \(U_1(1)\) gauge group.
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