Connections up to homotopy and characteristic classes *

Marius Crainic

Department of Mathematics, Utrecht University, The Netherlands

Introduction

The aim of this note is to clarify the relevance of “connections up to homotopy” [4, 5] to the theory of characteristic classes, and to present an application to the characteristic classes of Lie algebroids [3, 4, 7] (and of Poisson manifolds in particular [6, 8, 13]).

We have already remarked [4] that such connections up to homotopy can be used to compute the classical Chern characters. Here we present a slightly different argument for this, and then proceed with the discussion of the flat characteristic classes. In contrast with [4], we do not only recover the classical characteristic classes (of flat vector bundles), but we also obtain new ones. The reason for this is that ($\mathbb{Z}_2$-graded) non-flat vector bundles may have flat connections up to homotopy. As we shall explain here, in this category fall e.g. the characteristic classes of Poisson manifolds [8, 13].

As already mentioned in [4], one of our motivations is to understand the intrinsic characteristic classes for Poisson manifolds (and Lie algebroids) of [3, 8], and the connection with the characteristic classes of representations [13]. Conjecturally, Fernandes’ intrinsic characteristic classes [7] are the characteristic classes [8] of the “adjoint representation”. The problem is that the adjoint representation is a “representation up to homotopy” only. Applied to Lie algebroids, our construction immediately solves this problem: it extends the characteristic classes of [3, 8] from representations to representations up to homotopy, and shows that the intrinsic characteristic classes [7, 8] are indeed the ones associated to the adjoint representation [5].

I would like to thank J. Stasheff and A. Weinstein for their comments on a preliminary version of this paper.

Non-linear connections

Here we recall some well-known properties of connections on vector bundles. Up to a very slight novelty (we allow non-linear connections), this section is standard [11] and serves to fix the notations.

Let $M$ be a manifold, and let $E = E^0 \oplus E^1$ be a super-vector bundle over $M$. We now consider $\mathbb{R}$-linear operators

$$\mathcal{X}(M) \otimes \Gamma E \longrightarrow \Gamma E, \quad (X, s) \mapsto \nabla_X(s) \quad (1)$$

\*Research supported by NWO
which satisfy
\[
\nabla_X (fs) = f \nabla_X (s) + X(f)s
\]
for all \(X \in \mathcal{X}(M)\), \(s \in \Gamma E\), and \(f \in C^\infty (M)\), and which preserve the grading of \(E\). We say that \(\nabla\) is a non-linear connection if \(\nabla_X (V)\) is local in \(X\). This is just a relaxation of the \(C^\infty (M)\)-linearity in \(X\), when one recovers the standard notion of (linear) connection. The curvature \(k_\nabla\) of a non-linear connection \(\nabla\) is defined by the standard formula
\[
k_\nabla (X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} : \Gamma E \to \Gamma E.
\]
A non-linear differential form\(^1\) on \(M\) is an antisymmetric (\(\mathbb{R}\)-multilinear) map
\[
\omega : \mathcal{X}(M) \times \ldots \times \mathcal{X}(M) \to C^\infty (M)
\]
which is local in the \(X_i\)'s. It is easy to see (and it has been already remarked in \(\llbracket\)) that many of the usual operations on differential forms do not use the \(C^\infty (M)\)-linearity, hence they apply to non-linear forms as well. In particular we obtain the algebra \((A_{nl}(M), d)\) of non-linear forms endowed with De Rham operator. (This defines a contravariant functor from manifolds to dga’s.) Considering \(\Gamma E\)-valued operators instead, we obtain a version with coefficients, denoted \(A_{nl}(M; E)\). Note that a non-linear connection \(\nabla\) can be viewed as an operator \(A^0_{nl}(M; E) \to A^1_{nl}(M; E)\) which has a unique extension to an operator
\[
d_\nabla : A^*_{nl}(M; E) \to A^{*+1}_{nl}(M; E)
\]
satisfying the Leibniz rule. Explicitly,
\[
d_\nabla (\omega)(X_1, \ldots , X_{n+1}) = \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots , \hat{X}_i, \ldots , \hat{X}_j, \ldots , X_{n+1})
\]
\[
+ \sum_{i=1}^{n+1} (-1)^{i+1} \nabla_X \omega(X_1, \ldots , \hat{X}_i, \ldots , X_{n+1}).
\]\n
We now recall the definition of the (non-linear) connection on \(\text{End}(E)\) induced by \(\nabla\). For any \(T \in \Gamma \text{End}(E)\), the operators \([\nabla_X, T]\) acting on \(\Gamma (E)\) are \(C^\infty (M)\)-linear, hence define elements \([\nabla_X, T] \in \Gamma \text{End}(E)\). The desired connection is then \(\nabla_X (T) = [\nabla_X, T]\). Clearly \(k_\nabla \in A^2_{nl}(\text{End}(E))\), and one has Bianchi’s identity \(d_\nabla (k_\nabla) = 0\).

For any \(T \in \Gamma \text{End}(E)\), the operators \([\nabla_X, T]\) acting on \(\Gamma (E)\) are \(C^\infty (M)\)-linear, hence define elements \([\nabla_X, T] \in \Gamma \text{End}(E)\). The desired connection is then \(\nabla_X (T) = [\nabla_X, T]\). Clearly \(k_\nabla \in A^2_{nl}(\text{End}(E))\), and one has Bianchi’s identity \(d_\nabla (k_\nabla) = 0\).

We will use the algebra \(A_{nl}(M; \text{End}(E))\) and its action on \(A_{nl}(M; E)\). The product structure that we consider here is the one which arises from the natural isomorphisms
\[
A_{nl}(M; E) \cong A_{nl}(M) \otimes_{C^\infty (M)} \Gamma (E)
\]
and the usual sign conventions for the tensor products (i.e. \(\omega \otimes x \cdot \eta \otimes y = (-1)^{|x||\eta|} \omega \otimes xy\)). The usual super-trace on \(\text{End}(E)\) induces a super-trace
\[
Tr_s : (A_{nl}(M; \text{End}(E)), d_\nabla) \to (A_{nl}(M), d)
\]
with the property that \(Tr_s d_\nabla = d Tr_s\). We conclude (and this is just a non-linear version of the standard construction of Chern characters \([\llbracket]\)):

\(^1\)as in the case of connections, the non-linearity refers to \(C^\infty (M)\)-non-linearity. As pointed out to me, the terminology might be misleading. Better names would probably be “higher order connections” and “jet-forms”
Lemma 1 If $\nabla$ is a non-linear connection on $E$, then
\[ ch_p(\nabla) = Tr_s(k^p_\nabla) \in A^2_{nl}(M) \] (6)
are closed non-linear forms on $M$.

Up to a boundary, these classes are independent of $\nabla$. This is an instance of the Chern-Simons construction that we now recall. Given $k+1$ non-linear connections $\nabla_i$ on $E$ ($0 \leq i \leq k$) we form their affine combination $\nabla_\text{aff} = (1-t_1-\ldots-t_k)\nabla_0 + t_1\nabla_1 + \ldots + t_k\nabla_k$. This is a non-linear connection on the pullback of $E$ to $\Delta^k \times M$, where $\Delta^k = \{(t_1, \ldots, t_k) : t_i \geq 0, \sum t_i \leq 1\}$ is the standard $k$-simplex. The classical integration along fibers has a non-linear extension
\[ \int_{\Delta^k} : A^*_\text{nl}(M \times \Delta^k) \longrightarrow A^{*+k}_{\text{nl}}(M) \] (7)
given by the explicit formula
\[ (\int_{\Delta^k} \omega)(X_1, \ldots, X_{n-k}) = \int_{\Delta^k} \omega(\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_k}, X_1, \ldots, X_{n-k})dt_1 \ldots dt_k . \]
We then define
\[ cs_p(\nabla_0, \ldots, \nabla_k) = \int_{\Delta^k} ch_p(\nabla_\text{aff}) . \] (8)
Using a version of Stokes’ formula [2] (or integrating by parts repeatedly) we conclude

Lemma 2 The elements (8) satisfy
\[ dcs_p(\nabla_0, \ldots, \nabla_k) = \sum_{i=0}^k (-1)^i cs_p(\nabla_0, \ldots, \nabla_i, \ldots, \nabla_k) . \] (9)

Connections up to homotopy and Chern characters

From now on, $(E, \partial)$ is a super-complex of vector bundles over the manifold $M$,
\[ (E, \partial) : \begin{CD} E^0 @>\partial>> E^1 . \end{CD} \] (10)

We now consider non-linear connections $\nabla$ on $E$ such that $\nabla_X \partial = \partial \nabla_X$ for all $X \in \mathcal{X}(M)$. We say that $\nabla$ is a (linear) connection on $(E, \partial)$ if it also satisfies the identity $\nabla f_X(s) = f \nabla_X(s)$ for all $X \in \mathcal{X}(M)$, $f \in C^\infty(M)$, $s \in \Gamma E$. The notion of connection up to homotopy [4, 5] on $(E, \partial)$ is obtained by relaxing the $C^\infty(M)$-linearity on $X$ to linearity up to homotopy. In other words we require
\[ \nabla f_X(s) = f \nabla_X(s) + [H_\nabla(f, X), \partial] , \]
where $H_\nabla(f, X) \in \Gamma \text{End}(E)$ are odd elements which are $\mathbb{R}$-linear and local in $X$ and $f$.

We say that two non-linear connections $\nabla$ and $\nabla'$ are equivalent (or homotopic) if
\[ \nabla' = \nabla + [\theta(X), \partial] \]
for all $X \in \mathcal{X}(M)$, for some $\theta \in A^1_{\text{nl}}(M; \text{End}(E))$ of even degree. We write $\nabla \sim \nabla'$. 
Lemma 3 A non-linear connection is a connection up to homotopy if and only if it is equivalent to a (linear) connection.

Proof: Assume that $\nabla$ is a connection up to homotopy. Let $U_a$ be the domain of local coordinates $x^k$ for $M$, and put 

$$\nabla^a_X = \nabla_X + [u^a(X), \partial],$$

where $u_a \in \mathcal{A}_{nl}(U_a; \text{End}(E))$ is given by 

$$u_a(\sum f_k \frac{\partial}{\partial x_k}) = -\sum H(\nabla(f_k), \frac{\partial}{\partial x_k}),$$

for all $f_k \in C^\infty(U_a)$. Note that $\nabla_X$ is linear on $X$. Indeed, for any two smooth functions $f, g$ and $X = g \frac{\partial}{\partial x_k}$ we have 

$$\nabla^a_{fX} - f\nabla^a_X = (\nabla_{fX} + [u^a(fX), \partial]) - f(\nabla_X + [u^a(X), \partial]) =$$

$$= (\nabla_{fg} \frac{\partial}{\partial x_k} - [H(\nabla(fg), \frac{\partial}{\partial x_k}), \partial]) + f(\nabla_{g \frac{\partial}{\partial x_k}} - [H(\nabla(g \frac{\partial}{\partial x_k}), \partial)]) =$$

$$= fg \nabla_{\frac{\partial}{\partial x_k}} - fg \nabla_{\frac{\partial}{\partial x_k}} = 0.$$

Next we take $\{\nu_a\}$ to be a partition of unity subordinate to an open cover $\{U_a\}$ by such coordinate domains and put 

$$\nabla' = \sum_a \nu_a \nabla^a; u(X) = \sum_a \nu_a u^a(X).$$

Then $\nabla'$ is a connection equivalent to $\nabla$.

Lemma 4 If $\nabla_0$ and $\nabla_1$ are equivalent, then $ch_p(\nabla_0) = ch_p(\nabla_1)$.

Proof: So, let us assume that $\nabla^1 = \nabla^0 + [\theta, \partial]$. A simple computation shows that 

$$k_{\nabla_1} = k_{\nabla_0} + [d\nabla(\theta) + R, \partial],$$

where $R(X, Y) = [\theta(X), [\theta(Y), \partial]]$. We denote by $Z \subset \mathcal{A}_{nl}(M; \text{End}(E))$ the space of non-linear forms $\omega$ with the property that $[\omega, \partial] = 0$, and by $B \subset Z$ the subspace of element of type $[\eta, \partial]$ for some non-linear form $\eta$. The formula 

$$[\partial, \omega \eta] = [\partial, \omega] \eta + (-1)^{|\omega|} \omega [\partial, \eta]$$

shows that $ZB \subset B$, hence $\mathbf{[14]}$ implies that $k_{\nabla_1}^{p'} \equiv k_{\nabla_0}^{p'}$ modulo $B$. The desired equality follows now from the fact that $Tr_s$ vanishes on $B$.

For (linear) connections $\nabla$ on $(E, \partial)$, $ch_p(\nabla)$ are clearly (linear) differential forms on $M$ whose cohomology classes are (up to a constant) the components of the Chern character $Ch(E) = Ch(E^0) - Ch(E^1)$. Hence an immediate consequence of the previous two lemmas is the following $\mathbf{[3]}$

**Theorem 1** If $\nabla$ is a connection up to homotopy on $(E, \partial)$, then $ch_p(\nabla) = Tr_s(k_{\nabla}^p)$ are closed differential forms on $M$ whose De Rham cohomology classes are (up to a constant) the components of the Chern character $Ch(E)$. 
Flat characteristic classes

As usual, by flatness we mean the vanishing of the curvature forms. Theorem immediately implies

Corollary 1 If \((E, \partial)\) admits a connection up to homotopy which is flat, then \(Ch(E) = 0\).

As usual, such a vanishing result is at the origin of new “secondary” characteristic classes. Let \(\nabla\) be a flat connection up to homotopy. To construct the associated secondary classes we need a metric \(h\) on \(E\). We denote by \(\partial^h\) be the adjoint of \(\partial\) with respect to \(h\). Using the isomorphism \(E^* \cong E\) induced by \(h\) (which is anti-linear if \(E\) is complex), \(\nabla\) induces an adjoint connection \(\nabla^h\) on \((E, \partial^h)\). Explicitly,

\[
L_X h(s, t) = h(\nabla_X s(t), t) + h(s, \nabla_X^h t).
\]

The following describes various possible definitions of the secondary classes, as well as their main properties (note that the role of \(i = \sqrt{-1}\) below is to ensure real classes).

Theorem 2 Let \(\nabla\) be a flat connection up to homotopy on \((E, \partial), p \geq 1\).

(i) For any (linear) connection \(\nabla_0\) on \((E, \partial)\) and any metric \(h\),

\[
i^{p+1}(cs_p(\nabla, \nabla_0) + cs_p(\nabla_0, \nabla^h_0) + cs_p(\nabla^h_0, \nabla^h)) \in A^{2p-1}_{nl}(M)
\]

are differential forms on \(M\) which are real and closed. The induced cohomology classes do not depend on the choice of \(h\) or \(\nabla_0\), and are denoted \(u_{2p-1}(E, \partial, \nabla)\) in cohomology.

(ii) For any connection \(\nabla_0\) equivalent to \(\nabla\), and any metric \(h\),

\[
i^{p+1}cs_p(\nabla_0, \nabla^h_0) \in A^{2p-1}_{nl}(M)
\]

are real and closed, and represent \(u_{2p-1}(E, \partial, \nabla)\) in cohomology.

(iii) If \(\nabla\) is equivalent to a metric connection (i.e. a connection which is compatible with a metric), then all the classes \(u_{2p-1}(E, \partial, \nabla)\) vanish.

(iv) If \(\nabla \sim \nabla'\), then \(u_{2p-1}(E, \partial, \nabla) = u_{2p-1}(E, \partial, \nabla')\).

(v) If \(\nabla\) is a flat connection up to homotopy on both super-complexes \((E, \partial)\) and \((E, \partial')\), then \(u_{2p-1}(E, \partial, \nabla) = u_{2p-1}(E, \partial', \nabla)\).

(vi) Assume that \(E\) is real. If \(p\) is even then \(u_{2p-1}(E, \partial, \nabla) = 0\). If \(p\) is odd, then for any connection \(\nabla_0\) equivalent to \(\nabla\), and any metric connection \(\nabla_m\),

\[
(-1)^{\frac{p+1}{2}} cs_p(\nabla_0, \nabla_m) \in A^{2p-1}_{nl}(M)
\]

are closed differential forms whose cohomology classes equal to \(\frac{1}{2} u_{2p-1}(E, \partial, \nabla)\).

Note the compatibility with the classical flat characteristic classes, which correspond to the case where \(E\) is a graded vector bundle (and \(\partial = 0\)), or, more classically, just a vector bundle over \(M\). As references for this we point out [3] (for the approach in terms of frame bundles and Lie algebra cohomology), and [1] (for an explicit approach which we follow here). For the proof of the theorem we need the following
Lemma 5 Given the non-linear connections \( \nabla, \nabla_0, \nabla_1 \),

(i) If \( \nabla_0 \) and \( \nabla_1 \) are connections up to homotopy then \( cs_p(\nabla_0, \nabla_1) \) are differential forms;
(ii) If \( \nabla_0 \sim \nabla_1 \), then \( cs_p(\nabla_0, \nabla_1) = 0 \);
(iii) For any metric \( h \), \( ch_p(\nabla^h) = (-1)^p ch_p(\nabla) \) and \( cs_p(\nabla^h_0, \nabla^h_1) = (-1)^p cs_p(\nabla_0, \nabla_1) \).

Proof: (i) follows from the fact that Chern characters of connections up to homotopy are differential forms. For (ii) we use Lemma 4. The affine combination \( \nabla \) used in the definition of \( cs_p(\nabla_0, \nabla_1) \) is equivalent to the pull-back \( \nabla_0 \) of \( \nabla_0 \) to \( M \times \Delta^1 \) (because \( \nabla = \nabla_0 + t[\theta, \partial] \)), while \( ch_p(\nabla_0) \) is clearly zero. If \( h \) is a metric on \( E \), a simple computation shows that \( k_{\nabla^h}(X, Y) \) coincides with \(-k_{\nabla}(X, Y)^* \) where * denotes the adjoint (with respect to \( h \)). Then (iii) follows from \( Tr(A^* - Tr(A) \) for any matrix \( A \).

Proof of Theorem 3: (i) Let us denote by \( u(\nabla, \nabla_0, h) \) the forms \( (12) \). Since \( (\nabla_0, \nabla^h_1) \) is a pair of connections on \( E \), and \( (\nabla, \nabla_0), \nabla^h, \nabla^h_0 \) are pairs of connections up to homotopy on \( (E, \partial) \) and \( (E, \partial^h) \), respectively, it follows from (i) of Lemma 3 that \( u(\nabla, \nabla_0, h) \) are differential forms. From Stokes' formula \( (3) \) it immediately follows that they are closed. To prove that they are real we use (iii) of the previous Lemma:
\[
\frac{u(\nabla, \nabla_0, h)}{i^{p+1} d} = (-i)^p \frac{cs_p(\nabla, \nabla_0)}{i} + \frac{cs_p(\nabla_0, \nabla^h_0)}{i} + cs_p(\nabla^h_0, \nabla^h) = \frac{(-i)^p - 1}{i} \frac{cs_p(\nabla^h_0, \nabla^h)}{i} + cs_p(\nabla^h_0, \nabla_0) \frac{cs_p(\nabla_0, \nabla)}{i} = \frac{(-i)^p - 1}{i} \frac{cs_p(\nabla^h_0, \nabla_0)}{i} \frac{cs_p(\nabla_0, \nabla)}{i} = (-i)^p \frac{cs_p(\nabla^h_0, \nabla_0)}{i} \frac{cs_p(\nabla_0, \nabla)}{i} = \frac{cs_p(\nabla^h_0, \nabla_0)}{i} \frac{cs_p(\nabla_0, \nabla)}{i}.
\]

If \( \nabla_1 \) is another connection, using (1) again, it follows that \( u(\nabla, \nabla_0, h) - u(\nabla, \nabla_1, h) = \frac{cs_p(\nabla^h_0, \nabla_0)}{i} \frac{cs_p(\nabla_0, \nabla)}{i} \). (iii) clearly follows from (ii), which in turn follows from (ii) of Lemma 3 and the fact that \( \nabla \sim \nabla_0 \) implies \( \nabla^h \sim \nabla^h_0 \). To see that our classes do not depend on \( h \), it suffices to show that given a linear connection \( \nabla \) on a vector bundle \( F \), \( cs_p(\nabla, \nabla^h) \) is independent of \( h \) up to the boundary of a differential form. Let \( h_0 \) and \( h_1 \) be two metrics. Although the proof below works for general \( \nabla \)'s, simpler formulas are possible when \( \nabla \) is flat. So, let us first assume that (actually we may assume that \( \nabla \) is the canonical connection on a trivial vector bundle).

From Stokes' formula \( (3) \) applied to \( (\nabla, \nabla^h_0, \nabla^h_1) \), it suffices to show that \( cs_p(\nabla^h_0, \nabla^h_1) \) is a closed form. We choose a family \( h_t \) of metrics joining \( h_0 \) and \( h_1 \). One only has to show that \( \frac{\partial}{\partial t} cs_p(\nabla^h_0, \nabla^h_1) \) are closed forms. Writing \( h_t(x, y) = h_0(u_t(x, y), y) \), these Chern-Simons forms are, up to a constant, \( Tr(\omega_t^{2p-1}) \) where
\[
\omega_t = \nabla^h_t - \nabla^h_0 = u_t^{-1} d_{\nabla^h_0}(u_t)
\]
(here is where we use the flatness of \( \nabla \)). A simple computation shows that
\[
\frac{\partial \omega_t}{\partial t} = d_{\nabla^h_0}(v_t) + [\omega_t, v_t],
\]
where \( v_t = u_t^{-1} \frac{\partial u_t}{\partial t} \). Since \( d_{\nabla^h_0}(\omega^2_t) = 0 \), this implies
\[
\frac{\partial \omega_t}{\partial t} \omega_t^{2p-2} = d_{\nabla^h_0}(v_t \omega_t^{2p-2}) + [\omega_t, v_t \omega_t^{2p-2}].
\]
Now, by the properties of the trace it follows that
\[
\frac{\partial}{\partial t} Tr_s(\omega^2_p) = dTr_s(v_1 \omega^{2p-2})
\]
as desired. Assume now that \( \nabla \) is not flat. We choose a vector bundle \( F' \) together with a connection \( \nabla' \) compatible with a metric \( h' \), such that \( F = F \oplus F' \) admits a flat connection \( \nabla_0 \). We put \( \tilde{\nabla} = \nabla \oplus \nabla' \) and, for any metric \( h \) on \( F \), we consider the metric \( \tilde{h} = h \oplus h' \) on \( \tilde{F} \). Clearly \( cs_p(\tilde{\nabla}, \tilde{\nabla}^h) = cs_p(\nabla, \nabla^h) \). Using also (iii) of Lemma 5 and Stokes’ formula, we have:
\[
cs_p(\nabla, \nabla^h) = cs_p(\nabla_0, \tilde{\nabla}) - cs_p(\nabla_0, \nabla) + (-1)^p cs_p(\nabla_0, \tilde{\nabla})
\]
\[
+ d(cs_p(\nabla_0, \tilde{\nabla}) - cs_p(\nabla_0, \tilde{\nabla}^h)).
\]
Hence, by the flat case, \( cs_p(\nabla, \nabla^h) \) modulo exact forms does not depend on \( h \).

For (iv) one uses Stokes’ formula (5) and (ii) of Lemma 5 to conclude that \( cs_p(\nabla', \nabla_0) - cs_p(\nabla, \nabla_0) \) is the differential of the linear form \( cs_p(\nabla, \nabla', \nabla_0) \). To prove (v) we only have to show (see (i)) that there exists a linear connection \( \nabla^0 \) on \( E \) which is compatible with both \( \partial \) and \( \partial' \). For this, one defines \( \nabla^0 \) locally by \( \nabla^0 f = f \nabla^0 \frac{\partial}{\partial \sigma_k} \), and then use a partition of unity argument.

We now assume that \( E \) is real. From Lemma 5,
\[
 cs_p(\nabla_m, \nabla^h_0) = (-1)^p cs_p(\nabla^h_m, \nabla_0) = (-1)^{p+1} cs_p(\nabla_0, \nabla_m).
\]
Combined with Stokes’ formula (5), this implies
\[
 dc_s_p(\nabla_0, \nabla_m, \nabla^h_0) = (1 + (-1)^{p+1}) cs_p(\nabla_0, \nabla_m) - cs_p(\nabla_0, \nabla^h_0),
\]
which proves (vi). \( \square \)

Note that the construction of the flat characteristic classes presented here actually works for \( \nabla \)’s which are “flat up to homotopy”, i.e. whose curvatures are of type \([-,-,\partial]\). Moreover, this notion is stable under equivalence, and the flat characteristic classes only depend on the equivalence class of \( \nabla \) (cf. (iv) of the Theorem). Note also that, as in (ii) (and following (i)), there is a version of our discussion for super-connections up to homotopy. Some of our constructions can then be interpreted in terms of the super-connection \( \partial + \nabla \).

If \( E \) is regular in the sense that \( Ker(\partial) \) and \( Im(\partial) \) are vector bundles, then so is the cohomology \( H(E, \partial) = Ker(\partial)/Im(\partial) \), and any connection up to homotopy \( \nabla \) on \( (E, \partial) \) defines a linear connection \( H(\nabla) \) on \( H(E) \). Moreover, \( H(\nabla) \) is flat if \( \nabla \) is, and the characteristic classes \( u_{2p-1}(E, \partial, \nabla) \) probably coincide with the classical character classes of the flat vector bundle \( H(E, \partial) \). In general, the \( u_{2p-1}(E, \partial, \nabla) \)'s should be viewed as invariants of \( H(E, \partial) \) constructed in such a way that no regularity assumption is required. Let us discuss here an instance of this. We say that \( E \) is \( \mathbb{Z} \)-graded if it comes from a cochain complex
\[
0 \to E(0) \to E(1) \to \ldots \to E(n) \to 0,
\]
In other words, it must be of type \( E = \bigoplus_{k=0}^n E(k) \) with the even/odd \( \mathbb{Z}_2 \)-grading, and with \( \partial(E(k)) \subset E(k+1) \). As usual, we say that \( E \) is acyclic if \( Ker(\nabla) = Im(\nabla) \) (i.e. if (14) is exact).
Proposition 1

(i) If \((E, \partial)\) is acyclic, then any two connections up to homotopy on \((E, \partial)\) are equivalent. Moreover, if \(E\) is \(\mathbb{Z}\)-graded, then \(u_{2p-1}(E, \partial, \nabla) = 0\).

(ii) If \((E^k, \partial^k, \nabla^k)\) are \(\mathbb{Z}\)-graded complexes of vector bundles endowed with flat connections up to homotopy which fit into an exact sequence

\[
0 \to E^0 \xrightarrow{\delta} E^1 \xrightarrow{\delta} \cdots \xrightarrow{\delta} E^n \to 0 \tag{15}
\]

compatible with the structures (i.e. \([\delta, \partial] = [\partial, \nabla] = [\partial, H\nabla] = 0\)), then

\[
\sum_{k=0}^n (-1)^k u_{2p-1}(E^k, \partial^k, \nabla^k) = 0.
\]

Proof: The second part follows from (i) above and (v) of Theorem 2. To see this, we form the super-vector bundle \(E = \oplus_k E^k\) (which is \(\mathbb{Z}\)-graded by the total degree) and the direct sum (non-linear) connection \(\nabla\) acting on \(E\). Then \(\nabla\) is a connection up to homotopy in both \((E, \partial)\) and \((E, \partial + \delta)\). Clearly \(u_{2p-1}(E, \partial, \nabla) = \sum_{k=0}^n (-1)^k u_{2p-1}(E^k, \partial^k, \nabla^k)\), while the exactness of (15) implies that \(\partial + \delta\) is acyclic. Hence we are left with (i). For the first part we remark that the acyclicity assumption implies that \(\partial^* \partial + \partial \partial^*\) is an isomorphism (“Hodge”). Then any operator \(u\) which commutes with \(\partial\) can be written as a commutator \([-v, \partial]\) where

\[
v = ua, \quad a = -(\partial^* \partial + \partial \partial^*)^{-1} \partial^*.
\]

This applies in particular to \(u = \nabla^* - \nabla\) for any two connections up to homotopy on \((E, \partial)\).

Lemma 6 Given a (linear) connection \(\nabla\) on the acyclic cochain complex \((14)\), there exists a super-connection on \(E\) of type

\[
A = \partial + \nabla + A_2 + A_3 + \ldots : \mathcal{A}(M; E) \to \mathcal{A}(M; E),
\]

which is flat and satisfies (i) and (ii) above.

Let us show that this lemma, combined with the result of [1] mentioned above, prove the desired result. Using Stokes’ formula it follows that

\[
cs_p(\nabla, \nabla^h) = cs(A, A^h) + d(cs_p(\nabla, \nabla^h, A^h) - cs_p(\nabla, A, A^h)) +
+cs_p(\nabla, A) - cs_p(\nabla^h, A^h),
\]
and we show that \( cs_p(\nabla, A) = 0 \) (and similarly that \( cs_p(\nabla^h, A^h) = 0 \)). Writing \( \theta = A - \nabla \) and using the definition of the Chern-Simons forms, it suffices to prove that

\[
Tr_s(((1 - t^2)\nabla^2 + (t - t^2)([\nabla, \theta]))^{p-1}\theta) = 0
\]

for any \( t \). Since the only endomorphisms of \( E \) which count are those preserving the degree, we see that the only term which can contribute is \( Tr_s(\nabla^{2(p-2)}[\nabla, \theta][\theta] = Tr_s(\nabla^{2(p-2)}[\nabla, A_2][\theta]) \). But \( \nabla^{2(p-2)}[\nabla, A_2][\theta] \) commutes with \( \partial \) hence its super-trace must vanish (since \( Tr_s \) commutes with taking cohomology).

\[ \square \]

**Proof of Lemma 7:** (Compare with [6]). The flatness of \( A \) gives us certain equations on the \( A_k \)'s that we can solve inductively, using the same trick as in (6) above. For instance, the first equation is \( [\partial, A_2] + \nabla^2 = 0 \). Since \( u_1 = \nabla^2 \) commutes with \( \partial \), this equation will have the solution \( A_2 = u_1a \) (with \( a \) as in (6)). The next equation is \([\partial, A_3] + [A_1, A_2] = 0 \). It is not difficult to see that \( u_2 = [A_1, A_2] \) commutes with \( \partial \), and we put \( A_3 = u_2a \). Continuing this process, at the \( n \)-th level we put \( A_{n+1} = u_na \) where \( u_n = [\nabla, A_n] + [A_1, A_{n-2}] + \ldots \) as they arise from the corresponding equation. We leave to the reader to show that the \( u_n \)'s also satisfy the equations

\[
u_n = u_{n-1}[\nabla, a] + \left( \sum_{i+j=n-1} u_iu_j \right)a^2.
\]

Since \( [\partial, a] = -1 \), \( \partial \) will commute with both \( [\nabla, a] \) and \( a^2 \), hence also with the \( u_n \)'s (induction on \( n \)). It then follows that \( A_{n+1} \) satisfies the desired equation \([\partial, A_{n+1}] = -u_n \).

\[ \square \]

**Application to Lie algebroids**

Recall [6] that a *Lie algebroid* over \( M \) consists of a Lie bracket \([\; , \;]\) defined on the space \( \Gamma g \) of sections of a vector bundle \( g \) over \( M \), together with a morphism of vector bundles \( \rho : g \to TM \) (the anchor of \( g \)) satisfying \([X, fY] = f[X, Y] + \rho(X)(f) \cdot Y\) for all \( X, Y \in \Gamma g \) and \( f \in C^{\infty}(M) \). Important examples are tangent bundles, Lie algebras, foliations, and algebroids associated to Poisson manifolds. It is easy to see (and has already been remarked in many other places [6], [8], [10], etc. etc.) that many of the basic constructions involving vector fields have a straightforward \( g \)-version (just replace \( \mathcal{X}(M) \) by \( \Gamma g \)). Let us briefly point out some of them.

(a) **Cohomology:** the Lie-type formula [6] for the classical De Rham differential makes sense for \( X \in \Gamma g \) and defines a differential \( d \) on the space \( C^*(g) = \Gamma \Lambda^*g^* \), hence a cohomology theory \( H^*(g) \). Particular cases are De Rham cohomology, Lie algebra cohomology, foliated cohomology, and Poisson cohomology.

(b) **Connections and Chern characters:** According to the general philosophy, \( g \)-connections on a vector bundle \( E \) over \( M \) are linear maps \( \Gamma g \times \Gamma E \to \Gamma E \) satisfying the usual identities. Using their curvatures, one obtains \( g \)-Chern classes \( Ch^\theta(E) \in H^*(g) \) independent of the connection.

(c) **Representations:** Motivated by the case of Lie algebras, and also by the relation to groupoids (see e.g. [3]), vector bundles \( E \) over \( M \) together with a flat \( g \)-connection
are called representations of \( g \). This time \( \nabla \) should be viewed as an (infinitesimal) action of \( g \) on \( E \).

(d) **Flat characteristic classes:** The explicit approach to flat characteristic classes (as e.g. in \cite{1}, or as in the previous section) has an obvious \( g \)-version. Hence, if \( E \) is a representation of \( g \), then \( Ch^g(E) = 0 \), and one obtains the secondary characteristic classes \( u_{2p-1}(E) \in H^{2p-1}(g) \). Maybe less obvious is the fact that one can also extend the Chern-Weil type approach, at the level of frame bundles (as e.g. in \cite{9}). This has been explained in \cite{3}, and has certain advantages (e.g. for proving “Morita invariance” of the \( u_{2p-1}(E) \)'s and for relating them to differentiable cohomology).

(e) **Up to homotopy:** All the constructions and results of the previous sections carry over to Lie algebroids without any problem. As above, a representation up to homotopy of \( g \) is a supercomplex of vector bundles over \( M \), together with a flat \( g \)-connection up to homotopy.

(f) **The adjoint representation:** The main reason for working “up to homotopy” is that the adjoint representation of \( g \) only makes sense as a representation up to homotopy \cite{5}. Roughly speaking, it is the formal difference \( g - TM \). The precise definition is:

\[
\text{Ad}(g) : \quad g \xrightarrow{\rho} TM,
\]

with the flat \( g \)-connection up to homotopy \( \nabla^{ad} \) given by \( \nabla^{ad}_X(Y) = [X, Y], \nabla^{ad}_X(V) = [\rho(X), Y] \) (and the homotopies \( H(f, X)(Y) = 0, H(f, X)(V) = V(f, X)\)), for all \( X, Y \in \Gamma g, V \in \mathcal{X}(M) \).

Let us denote by \( u^g_{2p-1} \) the characteristic classes \( u_{2p-1}(\text{Ad}(g)) \) of the adjoint representation. The most useful description from a computational (but not conceptual) point of view is given by (vi) of Theorem 2 (more precisely, its \( g \)-version).

1 **Definition** We call basic \( g \)-connection any \( g \)-connection on \( \text{Ad}(g) \) which is equivalent to the adjoint connection \( \nabla^{ad} \).

It is not difficult to see that any such connection is also basic in sense of \cite{6} (and the two notions are equivalent at least in the regular case). Hence we have the following possible description of the \( u^g_{2p-1} \)'s, which shows the compatibility with Fernandes’ intrinsic characteristic classes \cite{6, 8}:

\[
\begin{cases}
0 & \text{if } p = \text{even} \\
\frac{1}{2} (-1)^{\frac{p+1}{2}} cs_p(\nabla_{\text{bas}}, \nabla_{\text{m}}) & \text{if } p = \text{odd}
\end{cases}
\]

where \( \nabla_{\text{bas}} \) is any basic \( g \)-connection, and \( \nabla_{\text{m}} \) is any metric connection on \( g \oplus TM \). Hence the conclusion of our discussion is the following (which can also be taken as definition of the characteristic classes of \cite{6, 8}).

**Theorem 3** If \( E \) is a representation up to homotopy then \( Ch^g(E) = 0 \), and the secondary characteristic classes \( u_{2p-1}(E) \in H^{2p-1}(g) \) of representations \cite{4} can be extended to such representations up to homotopy. When applied to the adjoint representation \( \text{Ad}(g) \), the resulting classes \( u^g_{2p-1} \) are (up to a constant) the intrinsic characteristic classes of \( g \) \cite{7}. 

More on basic connections: Let us try to shed some light on the notion of basic $g$-connection. In our context these are the linear connections which are equivalent to the adjoint connection, while in [7] they appear as a natural extension of Bott’s basic connections for foliations. Although not flat in general, they are always flat up to homotopy. The existence of such connections is ensured by Lemma 3 and it was also proven in [7]. There is however a very simple and explicit way to produce them out of ordinary connections on the vector bundle $g$.

**Proposition 2** Let $\nabla$ be a connection on the vector bundle $g$. Then the formulas

\[
\nabla^0_X(Y) = [X,Y] + \nabla_{\rho(Y)}(X) \\
\nabla^1_X(V) = [\rho(X),V] + \rho(\nabla_V(X))
\]

$(X,Y \in \Gamma g$, $V \in \Gamma TM)$ define a basic $g$-connection $\tilde{\nabla} = (\nabla^0, \nabla^1)$.

**Proof:** We have $\tilde{\nabla} = \nabla^{ad} + [\theta, \partial]$, where $\theta$ is the (non-linear) End(Ad($g$))-valued form on $g$ given by $\theta(X)(V) = \nabla_V(X)$, $\theta(X)(Y) = 0$.

Depending on the special properties of $g$, there are various other useful basic connections. This happens for instance when $g$ is regular, i.e. when the rank of the anchor $\rho$ is constant. Let us argue that, in this case, the adjoint representation is (up to homotopy) the formal difference $K - \nu$, where $K$ is the kernel of $\rho$, and $\nu$ is the normal bundle $TM/F$ of the foliation $F = \rho(g)$. This time, Bott’s formulas [2] trully make sense on $\nu$ and $K$, making them into honest representations of $g$:

\[
\nabla_X(\bar{Y}) = [X,\bar{Y}], \quad \forall X \in \Gamma g, \ Y \in \Gamma K \\
\nabla_X(Y) = [X,Y], \quad \forall X \in \Gamma g, \ Y \in \Gamma K
\]

(18) (19)

Now, choosing splittings $\alpha : F \rightarrow g$ for $\rho$, and $\beta : TM \rightarrow F$ for the inclusion, we have induced decompositions

\[
g \cong K \oplus F, \quad TM \cong \nu \oplus F.
\]

As mentioned above, the formal difference $K - \nu$ (view it as a graded complex with $K$ in even degree, $\nu$ in odd degree, and zero differential) is a representation of $g$. On the other hand, any $F$-connection $\nabla$ on $F$ defines a $g$-connection on the super-complex

\[
D(F) : F \xrightarrow{\nabla^0} F
\]

(and its homotopy class does not depend on $\nabla$). Hence one has an induced $g$-connection $\nabla^{\alpha,\beta}$ on Ad($g$), so that (Ad($g$), $\nabla^{\alpha,\beta}$) is isomorphic to $(K - \nu) \oplus D(F)$. Explicitly,

\[
\nabla_X^{\alpha,\beta}(Y) = [X,Y - \alpha \rho(Y)] + \alpha \nabla_{\rho(Y)}(\rho X) \\
\nabla_X^{\alpha,\beta}(V) = [\rho(X),V] - \beta [\rho(X),V] + \nabla_{\rho(X)}(\beta(V))
\]

for all $X,Y \in \Gamma g$, $V \in \mathfrak{X}(M)$. Note that the second part of the following proposition can also be derived from (iv) of Proposition 7.
Proposition 3 Assume that $g$ is regular. For any $\mathcal{F}$-connection $\nabla$ on $\mathcal{F}$, and any splittings $\alpha, \beta$ as above, $\nabla^{\alpha,\beta}$ is a basic $g$-connection. In particular

$$u_{2p-1}^g = u_{2p-1}(K) - u_{2p-1}(\nu),$$

where $K$ and $\nu$ are the representations of $g$ defined by Bott’s formulas (18), (19).

Proof: We have $\nabla^{\alpha,\beta} = \nabla^{\text{ad}} + [\theta, \partial]$, where $\theta$ is the $\text{End}(\text{Ad}(g))$-valued non-linear form which is given by

$$\theta(X)(V) = \alpha[\rho(X), \beta(V)] - \alpha\beta[\rho(X), V] - [X, \alpha\beta(V)] + \alpha\nabla^{\rho(X)}\beta(V)$$

for $V \in \Gamma(TM)$, while $\theta(X) = 0$ on $g$ (we leave to the reader to check that the previous formula is $C^\infty(M)$-linear on $V$). \[\square\]

References

[1] J.M. Bismut and J. Lott, Flat vector bundles, direct images and higher real analytic torsion, J. Amer. Math. Soc. 8 (1995), pp. 291-363

[2] R. Bott, Lectures on characteristic classes and foliations, Springer LNM 279, 1-94

[3] M. Crainic, Differentiable and algebroid cohomology, Van Est isomorphisms, and characteristic classes, Preprint 2000

[4] M. Crainic, Chern characters via connections up to homotopy, Preprint 2000 (available as math.DG/0008064)

[5] S. Evens, J.H. Lu and A. Weinstein, Transverse measures, the modular class, and a cohomology pairing for Lie algebroids, Quart. J. Math. Oxford 50 (1999), 417-436

[6] B. Fedosov, A simple geometrical construction of deformation quantization, J. Differential Geometry 40 (1994), 213-238

[7] R.L. Fernandes, Lie Algebroids, Holonomy and Characteristic Classes, preprint DG/0007132

[8] R.L. Fernandes, Connections in Poisson Geometry I: Holonomy and Invariants, to appear in J. of Differential Geometry (available as math.DG/0001129)

[9] F. Kamber and P. Tondeur, Foliated Bundles and Characteristic Classes, Springer LNM 493

[10] K. Mackenzie, Lie groupoids and Lie algebroids in differential geometry, Lecture Notes Series 124, London Mathematical Society, 1987

[11] D. Quillen, Superconnections and the Chern character, Topology 24 89-95 (1985)

[12] I. Vaisman, Lectures on the Geometry of Poisson Manifolds, Birkhauser, 1994
[13] A. Weinstein, *The modular automorphism group of a Poisson manifold*, J. Geom. Phys. 23 (1997), 379-394

Marius Crainic,
Utrecht University, Department of Mathematics,
P.O.Box:80.010,3508 TA Utrecht, The Netherlands,
e-mail: crainic@math.ruu.nl
home-page: [http://www.math.uu.nl/people/crainic/](http://www.math.uu.nl/people/crainic/)