\section{Introduction and statement of results}

Let $X$ be a smooth manifold. Let $T_{\mathbb{C}}X$ be the complexification of the tangent bundle $TX$. One defines the Witten bundle on $X$ \cite{15} as follows,

\begin{equation}
\Theta_q(TX) = \bigotimes_{u=1}^{\infty} S_q^u(T\mathbb{C}X - C^{\dim X}) \otimes \bigotimes_{v=1}^{\infty} \Lambda_{-q^{v-\frac{1}{2}}} (T\mathbb{C}X - C^{\dim X}),
\end{equation}

where $S_t(\cdot)$ (resp. $\Lambda_t(\cdot)$) denotes the symmetric (resp. exterior) power and $q = e^{2\pi \sqrt{-1} \tau}$ with $\tau \in \mathbb{H}$, the upper half-plane.

Let $g^{TX}$ be a Riemannian metric on $TX$ and $\nabla^{TX}$ the associated Levi-Civita connection. If we write

\begin{equation}
\Theta_q(TX) = B_0(TX) + B_1(TX)q^{\frac{1}{2}} + B_2(TX)q + \cdots,
\end{equation}

then each $B_i(TX)$ carries a Hermitian metric as well as a Hermitian connection $\nabla^{B_i(TX)}$ canonically induced from $g^{TX}$ and $\nabla^{TX}$. In this way, $\nabla^{TX}$ induces a Hermitian connection $\nabla^{\Theta_q(TX)}$ on the Witten bundle $\Theta_q(TX)$.

Now assume that $X$ is closed, spin and of dimension $4m$. Let $S(TX) = S_+(TX) \oplus S_-(TX)$ be the corresponding Hermitian bundle of spinors. For each $i$, let $D^{B_i(TX)}_{X,+} : \Gamma(S_+(TX) \otimes B_i(TX)) \to \Gamma(S_-(TX) \otimes B_i(TX))$ be the corresponding twisted Dirac operator. It is an important and well-known fact (cf. \cite{16}) that the $q$-series

\begin{equation}
\text{Ind} \left( D^{\Theta_q(TX)}_{X,+} \right) = \sum_{i=0}^{\infty} \text{Ind} \left( D^{B_i(TX)}_{X,+} \right) q^i,
\end{equation}

\abstract{We show that the Atiyah-Patodi-Singer reduced $\eta$-invariant of the twisted Dirac operator on a closed $4m-1$ dimensional spin manifold, with the twisted bundle being the Witten bundle appearing in the theory of elliptic genus, is a meromorphic modular form of weight $2m$ up to an integral $q$-series. We prove this result by combining our construction of certain modular characteristic forms associated to a generalized Witten bundle on spin-c manifolds with a deep topological theorem due to Hopkins.}
which by the Atiyah-Singer index theorem \[2\] equals to the elliptic genus

\[
\int_X \hat{A}(TX, \nabla^{TX}) \ch \left( \Theta_q(TX), \nabla^{\Theta_q(TX)} \right) = \sum_{i=0}^{\infty} q^i \int_X \hat{A}(TX, \nabla^{TX}) \ch \left( B_i(TX), \nabla^{B_i(TX)} \right),
\]

is an integral modular form of weight 2m over \(\Gamma^0(2)\), where \(\Gamma^0(2)\) is the index 2 modular subgroup of \(SL_2(\mathbb{Z})\) defined by

\[
\Gamma^0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid b \equiv 0 \pmod{2} \right\}.
\]

It is natural to look at what would happen if \(X\) is a 4m − 1 dimensional closed spin manifold. In this case, let \(E\) be a Hermitian vector bundle over \(X\) carrying a Hermitian connection \(\nabla^E\). Let \(D^E_X : \Gamma(S(TX) \otimes E) \to \Gamma(S(TX) \otimes E)\) be the associated twisted Dirac operator, which is formally self-adjoint.

Following [1], for any \(\text{Re}(s) >> 0\), set

\[
\eta(D^E_X, s) = \sum_{\lambda \in \text{Spec}(D^E_X) \setminus \{0\}} \frac{\text{Sgn}(\lambda)}{|\lambda|^s}.
\]

By [1], one knows that \(\eta(D^E_X, s)\) is a holomorphic function in \(s\) with \(\text{Re}(s) > \frac{\dim X}{2}\). Moreover, it extends to a meromorphic function over \(\mathbb{C}\), which is holomorphic at \(s = 0\). The \(\eta\) invariant of \(D^E_X\), in the sense of Atiyah-Patodi-Singer [1], is defined by

\[
\eta(D^E_X) = \eta(D^E_X, 0),
\]

while the reduced \(\eta\) invariant is defined and denoted by

\[
\overline{\eta}(D^E_X) = \frac{\dim(\ker D^E_X) + \eta(D^E_X)}{2}.
\]

It is the aim of this paper to study the modularity of the \(q\)-series

\[
\eta \left( D^{\Theta_q(TX)}_X \right) = \sum_{i=0}^{\infty} \eta \left( D^{B_i(TX)}_X \right) q^i,
\]

which is a spectral invariant depending on \(g^{TX}\).

Assume temporarily that \(X\) is the boundary of a 4m dimensional spin manifold \(Y\). Let \(g^{TY}\) be a Riemannian metric on \(TY\) which is of product structure near \(\partial Y = X\) and restricts to \(g^{TX}\) on \(X\). By the Atiyah-Patodi-Singer index theorem established in [1], one has

\[
\int_Y \hat{A}(TY, \nabla^{TY}) \ch \left( \Theta_q(TY), \nabla^{\Theta_q(TY)} \right) - \eta \left( D^{\Theta_q(TY)}_X \right) \in \mathbb{Z}[q^{\frac{1}{2}}].
\]

\[1\] We refer to [11, Section 2.1] and [18, Chapter 1] for the notations of the corresponding characteristic forms appearing below.
The term of integration over \( Y \) in (1.7) is a modular form of weight 2\( m \) over \( \Gamma^0(2) \) (similar to the modularity mentioned above for the elliptic genus in (1.4), c.f. [13]), although it is not necessary to be an integral modular form anymore. Therefore, from (1.7), one sees that if \( X \) is a 4\( m \)-dimensional closed spin manifold. Then the reduced \( \eta \)-invariant can be stated as follows.

(1.8) \[
\int_Y \tilde{A} \left( T\bar{Y}, \nabla^{T\bar{Y}} \right) \text{ch} \left( \Theta_q(T\bar{Y}), \nabla^{\Theta_q(T\bar{Y})} \right) - k \pi \left( D^\Theta^\eta(TX) \right) \in \mathbb{Z}\left[q^{\frac{1}{2}}\right].
\]

From (1.8), one sees that \( \pi(D^\Theta^\eta(TX)) \) is a modular form up to an element in \( \mathbb{Z}[q^{\frac{1}{2}}] \). Thus, the natural classical method gives the conclusion that \( D^\Theta^\eta(TX) \) is a modular form up to an element in \( \mathbb{Q}[q^{\frac{1}{2}}] \) instead of \( \mathbb{Z}[q^{\frac{1}{2}}] \).

On the other hand, if \( \tilde{g} \) is another Riemannian metric on \( TX \) with \( \nabla^{TX} \) being its Levi-Civita connection and \( D^\Theta(TX) \) being the corresponding twisted Dirac operator, then by the variation formula for the reduced \( \eta \) invariant (cf. [1] and [4]), one has

(1.9) \[
\pi\left( D^\Theta^\eta(TX) \right) - \pi\left( D^\Theta^\eta(TX) \right) = \int_X CS_{\Phi}(\nabla^{TX}, \tilde{\nabla}^{TX}, \tau) \mod \mathbb{Z}[q^{\frac{1}{2}}],
\]

where \( CS_{\Phi}(\nabla^{TX}, \tilde{\nabla}^{TX}, \tau) \) is the Chern-Simons transgression form associated to \( \Phi(\nabla^{TX}, \tau) = \left\{ \tilde{A} \left( TX, \nabla^{TX} \right) \text{ch} \left( \Theta_q(TX), \nabla^{\Theta_q(TX)} \right) \right\} \). It is easy to see that \( \int_X CS_{\Phi}(\nabla^{TX}, \tilde{\nabla}^{TX}, \tau) \) is a modular form of weight 2\( m \) over \( \Gamma^0(2) \) (cf. [9]). Thus the variation of \( \pi\left( D^\Theta^\eta(TX) \right) \) has mod \( \mathbb{Z} \) modularity property.

It turns out to be an interesting open problem that whether \( \pi\left( D^\Theta^\eta(TX) \right) \) is by itself a modular form of weight 2\( m \) over \( \Gamma^0(2) \) up to an element in \( \mathbb{Z}[q^{1/2}] \).

The purpose of this short note is to give an answer to this question. Our main result can be stated as follows.

**Theorem 1.1.** Let \( X \) be a 4\( m \)-1 dimensional closed spin Riemannian manifold. Then the reduced \( \eta \)-invariant \( \pi\left( D^\Theta^\eta(TX) \right) \) of the twisted Dirac operator \( D^\Theta(TX) \) is a meromorphic modular form of weight 2\( m \) over \( \Gamma^0(2) \), up to an element in \( \mathbb{Z}[q^{\frac{1}{2}}] \).
Here meromorphic modular form is a weaker notion than modular form without requiring holomorphicity but only meromorphicity on the upper half plane.

To prove Theorem 1.1 instead of using the cobordism result as above, we make use of a result due to Hopkins (cf. [12, Section 8]) which asserts that for any complex vector bundle $V$ over $X$, there is a nonnegative integer $s$ such that $X \times \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1$ ($s$-copies of $\mathbb{C}P^1$) bounds a spin manifold $Y$ and $V \boxtimes H^s$ on $X \times \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1$ extends to $Y$, where $H$ denotes the Hopf hyperplane bundle on $\mathbb{C}P^1$. We then apply the modular characteristic forms, which is associated to a generalized Witten bundle we have constructed in [11], on the bounding manifold, as well as the Atiyah-Patodi-Singer index theorem [1] to get the modularity of the reduced $\eta$-invariant in question.

It remains a challenge to find a purely analytic proof of Theorem 1.1 without using the deep topological results as the above mentioned Hopkins’ theorem.

Theorem 1.1 immediately implies that the quantity in (1.8) is a meromorphic modular form up to an element in $k\mathbb{Z}[[q^{1/2}]]$, where $k$ is the positive integer such that $k$ disjoint copies of $X$ bounds $\widetilde{Y}$ as explained before (1.8). Observe that in (1.8) each $q$-coefficient mod $k$ is a mod $k$ index studied by Freed and Melrose in [10]. It is a topological invariant and the main result in [10] provides a topological interpretation of it. Therefore, as an application of Theorem 1.1, we have

**Corollary 1.1.** Let $Y$ be an $4m$ dimensional spin $\mathbb{Z}/k$-manifold in the sense of Sullivan (cf. [10]). Then the mod $k$ index associated to the Witten bundle $\Theta_q(TY)$ can be represented by a meromorphic modular form of weight $2m$ over $\Gamma^0(2)$.

On the other hand, in view of [7, (25)], which corresponds to the case of $k = 1$ in (1.8) for the category of stable almost complex manifolds, Theorem 1.1 might become a starting point of a kind of tertiary index theory, in the sense of [7, Theorem 4.2], for spin manifolds. Recently Ulrich Bunke informed us that Theorem 1.1 can be given an alternative proof by using the theory of the universal $\eta$ invariant ([6], Lemma 3.1) and a spin version of the $f$-invariant has also been constructed in ([6], Definition 13.2).

For completeness, we would like to point out what happens in dimension $4m+1$. Actually, when $X$ is an $8n+5$ dimensional closed spin manifold, since for each $i$, $\eta \left( D^{B_i}_X(TX) \right) = 0$ and $\dim \left( \ker D^{B_i}_X(TX) \right)$ is even (c.f. [1]), we have $\overline{\eta} \left( D^{\Theta_q}_X(TX) \right) = 0 \mod \mathbb{Z}[[q^{1/2}]]$. In dimension $8n+1$, since $\eta \left( D^{B_i}_X(TX) \right) = 0$ for each $i$ (c.f. [1]), we have $\overline{\eta} \left( D^{\Theta_q}_X(TX) \right) = \frac{\dim \left( \ker D^{\Theta_q}_X(TX) \right)}{2}$. Therefore in view of the Atiyah-Singer mod 2 index theorem, $\overline{\eta} \left( D^{\Theta_q}_X(TX) \right)$ can be identified with Ochanine’s beta invariant $\beta_q(X)$, the modularity of which has been shown in [14].
This paper is organized as follows. In Section 2, we briefly recall our construction (in [11]) of the modular form associated to a generalized Witten bundle involving a complex line bundle. In Section 3, we combine our modular form and the Hopkins boundary theorem to prove Theorem 1.1. In Section 4 we propose a possible refinement of Theorem 1.1 in $8n + 3$ dimension.

2. Complex Line Bundles and Modular Forms

In this section, we briefly review our construction (in [11]) of a modular form, which is associated to a generalized Witten bundle involving a complex line bundle.

Let $M$ be a $4l$ dimensional Riemannian manifold. Let $\nabla^{TM}$ be the associated Levi-Civita connection.

Let $\xi$ be a complex line bundle over $M$. Equivalently, one can view $\xi$ as a rank two real oriented vector bundle over $M$. Let $\xi$ carry a Euclidean metric and also a Euclidean connection $\nabla^{\xi}$, let $c = c(\xi, \nabla^{\xi})$ be the Euler form associated to $\nabla^{\xi}$ (cf. [18, Section 3.4]). Let $\xi_{\mathbb{C}}$ be the complexification of $\xi$.

If $E$ is a complex vector bundle over $M$, set $\tilde{E} = E - \dim E \in K(M)$.

Following [11, (2.5)], set

$$\Theta_q(TM, \xi) = \prod_{u=1}^{\infty} S_{q^u}(\tilde{T}_{\mathbb{C}}M) \otimes \prod_{v=1}^{\infty} \Lambda_{-q^{v-\frac{1}{2}}}(\tilde{T}_{\mathbb{C}}M - 2\tilde{\xi}_{\mathbb{C}}) \otimes \prod_{r=1}^{\infty} \Lambda_{q^{r-\frac{1}{2}}}(\tilde{\xi}_{\mathbb{C}}) \otimes \prod_{t=1}^{\infty} \Lambda_{q^{t+\frac{1}{2}}}(\tilde{\xi}_{\mathbb{C}}),$$

which is an element in $K(M)\llbracket q^{\frac{1}{2}} \rrbracket$. As before, $\nabla^{TM}$ and $\nabla^{\xi}$ induce a Hermitian connection $\nabla^\Theta_q(TM, \xi)$ on $\Theta_q(TM, \xi)$.

Let $P(TM, \xi, \tau) \in \Omega^{4l}(M)$ be the characteristic form defined by

$$P(TM, \xi, \tau) := \left\{ \tilde{A}(TM, \nabla^{TM}) \cosh \left( \frac{c}{2} \right) \text{ch} \left( \Theta_q(TM, \xi), \nabla^\Theta_q(TM, \xi) \right) \right\}^{(4l)}.$$ 

It is shown in [11] that $P(TM, \xi, \tau)$ can be expressed by using the formal Chern roots of $(T_{\mathbb{C}}M, \nabla^{T_{\mathbb{C}}M})$ and $c$ through the Jacobi theta functions, which are defined as follows (cf. [8] and [11, Section 2.3]):

$$\theta(v, \tau) = 2q^{1/8} \sin(\pi v) \prod_{j=1}^{\infty} \left[ (1 - q^j)(1 - e^{2\pi\sqrt{-1}v} q^j)(1 - e^{-2\pi\sqrt{-1}v} q^j) \right],$$

$$\theta_1(v, \tau) = 2q^{1/8} \cos(\pi v) \prod_{j=1}^{\infty} \left[ (1 - q^j)(1 + e^{2\pi\sqrt{-1}v} q^j)(1 + e^{-2\pi\sqrt{-1}v} q^j) \right],$$
\[ \theta_2(v, \tau) = \prod_{j=1}^{\infty} \left[ (1 - q^j)(1 - e^{2\pi \sqrt{-1}v q^j - 1/2})(1 - e^{-2\pi \sqrt{-1}v q^j - 1/2}) \right], \]

\[ \theta_3(v, \tau) = \prod_{j=1}^{\infty} \left[ (1 - q^j)(1 + e^{2\pi \sqrt{-1}v q^j - 1/2})(1 + e^{-2\pi \sqrt{-1}v q^j - 1/2}) \right]. \]

The theta functions are all holomorphic functions for \((v, \tau) \in \mathbb{C} \times \mathbb{H}\), where \(\mathbb{C}\) is the complex plane and \(\mathbb{H}\) is the upper half plane. Let \(\{ \pm 2\pi \sqrt{-1} x_i \}\) be the formal Chern roots for \((T_{\mathbb{C}M}, \nabla_{T_{\mathbb{C}M}})\) and \(c = 2\pi \sqrt{-1} u\), we have (2.3)

\[ P(TM, \xi, \tau) = \left\{ \left( \prod_{i=1}^{2l} \frac{\theta'(0, \tau) \theta_2(x_i, \tau)}{\theta(x_i, \tau) \theta_2(0, \tau)} \right) \theta_1(u, \tau) \theta_2(u, \tau) \theta_3(u, \tau) \right\}^{(4l)}. \]

By using the transformation laws of theta functions (cf. [8] and [11, Section 2.3]), one sees as in [11, Proposition 2.6] that \(P(TM, \xi, \tau)\) is a modular form of weight 2 over \(\Gamma_0(2)\).

3. Proof of the Main Theorem

In this section, we will prove our main result Theorem 1.1.

The topological tool we will use is the following boundary theorem of Hopkins (cf. [12, Section 8]).

**Theorem 3.1** (Hopkins). Let \(X\) be a compact, odd dimensional spin manifold and \(V \to X\) a complex vector bundle over \(X\). Then there is an integer \(s\) such that the vector bundle \(V \boxtimes \bigoplus_{j=1}^s H_j \to X \times (\mathbb{C}P^1)^s\) is a boundary, where \(H_j\) denotes the Hopf hyperplane bundle on the \(j\)-th copy of \(\mathbb{C}P^1\). In other words, there is a spin manifold \(Y\) with a complex vector bundle \(W\) on \(Y\) such that \(W|_{\partial Y} = V \boxtimes \bigoplus_{j=1}^s H_j\).

In what follows, we will combine this Hopkins boundary theorem with the modular characteristic form constructed in Section 2 to give a proof of Theorem 1.1.

**Proof of Theorem 1.1:** Without loss of generality, for the \(4m-1\) dimensional closed spin manifold \(X\), in view of the Hopkins boundary theorem, we take an even integer \(s\) so that the complex line bundle

\[ p^\ast(\bigoplus_{j=1}^s H_j) \to X \times (\mathbb{C}P^1)^s\]

bounds, where \(p : X \times (\mathbb{C}P^1)^s \to (\mathbb{C}P^1)^s\) is the natural projection. This means that there is a spin manifold \(Y\) and a complex line bundle \(\zeta\) over \(Y\) such that \(\partial Y = X \times (\mathbb{C}P^1)^s\) and \(\zeta|_{X \times (\mathbb{C}P^1)^s} = p^\ast(\bigoplus_{j=1}^s H_j)\).

Let \(g^{TX}\) be any Riemannian metric on \(X\). Equip \(\mathbb{C}P^1\)'s with arbitrary Riemannian metrics and the \(H_j\)'s with arbitrary Euclidean metrics and Euclidean connections.
Let \( g^{TY} \) be a metric on \( TY \) such that it is of product structure near \( X \times (\mathbb{CP}^1)^s \) and restricts to the product metric on \( X \times (\mathbb{CP}^1)^s \). Let \( \nabla^{TY} \) be the Levi-Civita connection associated to \( g^{TY} \).

Let \( g^\mathbb{C} \) be an Euclidean metric on \( \zeta \) (viewed as an oriented real plane bundle) such that \( g^\mathbb{C} \) is of product structure near \( X \times (\mathbb{CP}^1)^s \) and restricts to the Euclidean metric on \( p^*(\mathbb{R}^s \oplus H_j) \) on \( X \times (\mathbb{CP}^1)^s \). Let \( \nabla^\mathbb{C} \) be an Euclidean connection of \( g^\mathbb{C} \) which is of product structure near \( X \times (\mathbb{CP}^1)^s \) and restricts to the canonically induced Euclidean connection on \( p^*(\mathbb{R}^s \oplus H_j) \) on \( X \times (\mathbb{CP}^1)^s \).

Let \( c = e(\zeta) \) and \( z_j = \frac{c_j(H_j)}{\pi \sqrt{-1}}, 1 \leq j \leq s \).

By applying the Atiyah-Patodi-Singer index theorem \([1]\) to the twisted Dirac operator \( D_{TY^\mathbb{C}}(TY^2) \), in noting that

\[
(\Theta_\mathbb{C}(TY^2) \otimes \zeta)_{|X \times (\mathbb{CP}^1)^s} = \Theta_\mathbb{C}(T(X \times (\mathbb{CP}^1)^s), (p^*(\mathbb{R}^s \oplus H_j))^j \otimes p^*(\mathbb{R}^s \oplus H_j)),
\]

one finds that there exist integers \( a_i \)'s such that

\[
\begin{align*}
\eta \left( D_{X \times (\mathbb{CP}^1)^s}^{\Theta_\mathbb{C}(T(X \times (\mathbb{CP}^1)^s), (p^*(\mathbb{R}^s \oplus H_j))^j \otimes p^*(\mathbb{R}^s \oplus H_j))} \right) \\
= \int_Y \hat{A}(TY^2, \nabla^{TY^2}) \operatorname{ch} \left( \Theta_\mathbb{C}(TY^2, \zeta^2) \otimes \zeta, \nabla^{\Theta_\mathbb{C}(TY^2, \zeta^2) \otimes \zeta} \right) - \sum_{i=0}^{\infty} a_i q^{\frac{i}{2}} \\
= \int_Y \hat{A}(TY^2, \nabla^{TY^2}) e^c \operatorname{ch} \left( \Theta_\mathbb{C}(TY^2, \zeta^2), \nabla^{\Theta_\mathbb{C}(TY^2, \zeta^2)} \right) - \sum_{i=0}^{\infty} a_i q^{\frac{i}{2}} \\
= \int_Y \hat{A}(TY^2, \nabla^{TY^2}) \cosh(c) \operatorname{ch} \left( \Theta_\mathbb{C}(TY^2, \zeta^2), \nabla^{\Theta_\mathbb{C}(TY^2, \zeta^2)} \right) - \sum_{i=0}^{\infty} a_i q^{\frac{i}{2}},
\end{align*}
\]

where the last equality follows from the fact that \( s \) is an even integer.

Let \( r : X \times (\mathbb{CP}^1)^s \to X \) be the natural projection. For bundles \( E \to X \) and \( F \to (\mathbb{CP}^1)^s \), by separation of variables, we have

\[
\eta \left( D_{X \times (\mathbb{CP}^1)^s}^{(p^*E) \otimes (p^*F)} \right) = \eta(D_E^X) \cdot \operatorname{Ind}(D_{(\mathbb{CP}^1)^s, +}).
\]

So we have

\[
\begin{align*}
\eta \left( D_{X \times (\mathbb{CP}^1)^s}^{(p^*E) \otimes (p^*F)} \right) = \eta(D_E^X) \cdot \operatorname{Ind}(D_{(\mathbb{CP}^1)^s, +}) & + \dim(\ker D_{X}^E) \dim(\ker(D_{(\mathbb{CP}^1)^s, -})) \\
& + \dim(\ker D_{X}^E) \dim(\ker(D_{(\mathbb{CP}^1)^s, -})),
\end{align*}
\]
From the above formula, we can see that there are integers \( b_i \)'s such that

\[
(3.2) \quad \eta \left( D_{X \times (\mathbb{CP}^1)^s}^{\Theta_q(T(X \times (\mathbb{CP}^1)^s), (p^*([\mathbb{Z}_{j=1}^s H_j])^2) \otimes p^*([\mathbb{Z}_{j=1}^s H_j])^2) - \sum_{i=0}^{\infty} b_i q^i \right) \\
= \eta \left( D_{X \times (\mathbb{CP}^1)^s}^{\Theta_q(r^*TX \otimes p^*T(\mathbb{CP}^1)^s, (p^*([\mathbb{Z}_{j=1}^s H_j])^2) \otimes p^*([\mathbb{Z}_{j=1}^s H_j])^2) - \sum_{i=0}^{\infty} b_i q^i \right) \\
= \eta \left( D_{X \times (\mathbb{CP}^1)^s}^{r^* \Theta_q(TX) \otimes p^*(\Theta_q(T(\mathbb{CP}^1)^s), (p^*([\mathbb{Z}_{j=1}^s H_j])^2) \otimes p^*([\mathbb{Z}_{j=1}^s H_j])^2) - \sum_{i=0}^{\infty} b_i q^i \right) \\
= \eta \left( D_{X \times (\mathbb{CP}^1)^s}^{\Theta_q(TX)} \right) \cdot \text{Ind} \left( D_{(\mathbb{CP}^1)^s}^{\Theta_q(T(\mathbb{CP}^1)^s), (p^*([\mathbb{Z}_{j=1}^s H_j])^2) \otimes p^*([\mathbb{Z}_{j=1}^s H_j])^2) - \sum_{i=0}^{\infty} b_i q^i \right) \\
= \eta \left( D_{X \times (\mathbb{CP}^1)^s}^{\Theta_q(TX)} \right) \\
\cdot \int_{(\mathbb{CP}^1)^s} \hat{A}(T(\mathbb{CP}^1)^s, \nabla T(\mathbb{CP}^1)^s) e^{c_1(H_1)+\cdots+c_1(H_s)} \text{ch}(\Theta_q(T(\mathbb{CP}^1)^s, (p^*([\mathbb{Z}_{j=1}^s H_j])^2) \otimes p^*([\mathbb{Z}_{j=1}^s H_j])^2)) \\
= \eta \left( D_{X \times (\mathbb{CP}^1)^s}^{\Theta_q(TX)} \right) \\
\cdot \int_{(\mathbb{CP}^1)^s} \frac{\theta_1(z_1, \tau)}{\theta_1(0, \tau)} \frac{\theta_2(z_1, \tau)}{\theta_2(0, \tau)} \frac{\theta_3(z_1, \tau)}{\theta_3(0, \tau)} \\
\cdot \frac{\theta_1(\sum_{j=1}^s z_j, \tau)}{\theta_1(0, \tau)} \frac{\theta_2(\sum_{j=1}^s z_j, \tau)}{\theta_2(0, \tau)} \frac{\theta_3(\sum_{j=1}^s z_j, \tau)}{\theta_3(0, \tau)} \\
\cdot \frac{\theta_1(z_1, \tau)}{\theta_1(0, \tau)} \frac{\theta_2(z_1, \tau)}{\theta_2(0, \tau)} \frac{\theta_3(z_1, \tau)}{\theta_3(0, \tau)},
\]

where the last equality holds due to the fact that \( \frac{x}{\eta(x, \tau)} \) and \( \theta_2(x, \tau) \) are both even functions about \( x \) and \( \int_{\mathbb{CP}^1} z_n^0 = 0 \) if \( n > 1 \).

Since \( s \) is an even integer, from the knowledge about the modular form \( P(TM, \xi, \tau) \) constructed in Section 2, we know that

\[
\hat{A}(\hat{T}(\mathbb{CP}^1)^s, \nabla \hat{T}(\mathbb{CP}^1)^s) e^{c_1(H_1)+\cdots+c_1(H_s)} \text{ch}(\Theta_q(T(\mathbb{CP}^1)^s, (p^*([\mathbb{Z}_{j=1}^s H_j])^2) \otimes p^*([\mathbb{Z}_{j=1}^s H_j])^2)) = 1,
\]

we see that \( f_s(\tau) \) has constant term 1. Therefore \( f_s^{-1}(\tau) \in \mathbb{Z}[[q^{1/2}]] \).
From (3.1) and (3.2), we have
\[
\overline{\eta}(D_X^{\Theta_\eta TY})
\]
\[
= f_1^{-1}(\tau) \cdot \int_Y \tilde{A}(TY, \nabla TY) \cosh(c) \left( \Theta_q(TY, \zeta^2), \nabla^\Theta_\eta(TY, \zeta^2) \right)
\]
\[
- f_2^{-1}(\tau) \cdot \left( \sum_{i=0}^{\infty} (a_i + b_i) q^i \right).
\]

Still by the modularity of \( P(TM, \xi, \tau) \) constructed in Section 2, we know that
\[
\int_Y \tilde{A}(TY, \nabla TY) \cosh(c) \left( \Theta_q(TY, \zeta^2), \nabla^\Theta_\eta(TY, \zeta^2) \right)
\]
is a modular form of weight \( 2m + s \) over \( \Gamma^0(2) \). So
\[
f_1^{-1}(\tau) \cdot \int_Y \tilde{A}(TY, \nabla TY) \cosh(c) \left( \Theta_q(TY, \zeta^2), \nabla^\Theta_\eta(TY, \zeta^2) \right)
\]
is a meromorphic modular form of weight \( 2m \) over \( \Gamma^0(2) \).

Therefore from (3.3), we see that
\[
\overline{\eta}(D_X^{\Theta_\eta TY}) = f_1^{-1}(\tau) \int_Y \tilde{A}(TY, \nabla TY) \cosh(c) \left( \Theta_q(TY, \zeta^2), \nabla^\Theta_\eta(TY, \zeta^2) \right) \mod \mathbf{Z}[q^{1/2}],
\]
a meromorphic modular form of weight \( 2m \) over \( \Gamma^0(2) \). The proof of Theorem 1.1 is complete.

**Remark 3.1.** The modular form \( f_1(\tau) \) in the above proof can be explicitly expressed by theta functions and their derivatives. For example, we have
\[
f_2(\tau) = -\frac{1}{\pi^2} \left( \frac{\theta_1^4(0, \tau)}{\theta_1(0, \tau)} - 2 \frac{\theta_2^4(0, \tau)}{\theta_2(0, \tau)} + \frac{\theta_3^4(0, \tau)}{\theta_3(0, \tau)} \right)
\]
and
\[
f_1(\tau) = \frac{1}{\pi^2} \left( \frac{\theta_1^{(4)}(0, \tau)}{\theta_1(0, \tau)} - 2 \frac{\theta_2^{(4)}(0, \tau)}{\theta_2(0, \tau)} + \frac{\theta_3^{(4)}(0, \tau)}{\theta_3(0, \tau)} + 18 \left( \frac{\theta_2''(0, \tau)}{\theta_2(0, \tau)} \right)^2 \right.
\]
\[
- \frac{\theta_1''(0, \tau) \theta_2''(0, \tau) \theta_3''(0, \tau)}{\theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau)} - 12 \frac{\theta_2''(0, \tau) \theta_3''(0, \tau) \theta_1''(0, \tau) \theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau)}{\theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau)} + 6 \frac{\theta_1''(0, \tau) \theta_2''(0, \tau) \theta_3''(0, \tau)}{\theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau)} \right).
\]

**Remark 3.2.** Let \( X \) be a compact, odd dimensional spin manifold. Define
\[
H(X) := \{ h \in \mathbf{Z} : \text{the line bundle } p^*(\bigoplus_{j=1}^h H_j) \to X \times (\mathbf{CP}^1)^h \text{ bounds} \},
\]
where \( p : X \times (\mathbf{CP}^1)^h \to (\mathbf{CP}^1)^h \) is the natural projection and \( H_j \) denotes the Hopf hyperplane bundle on the \( j \)-th copy of \( \mathbf{CP}^1 \). Define the Hopkins’ index of \( X \), \( h(X) := \min H(X) \). Obviously, when \( X \) is a boundary by itself, \( h(X) = 0 \). It is clear that \( H(X) = \{ s \in \mathbf{Z} : s \geq h(X) \} \).
In the proof of Theorem 1.1, we may take any even number $s \in H(X)$ and denote the corresponding $Y$ and $\zeta$ by $Y_s$ and $\zeta_s$. Then the proof of Theorem 1.1 tells us that, up to an element in $\mathbb{Z}[[q^{1/2}]]$,

$$
\bar{\eta}(D_X^{\Theta_q(TX)}) = f_s^{-1}(\tau) \int_{Y_s} \hat{A}(TY_s, \nabla TY_s) \cosh(e(\xi_s)) \operatorname{ch} \left( \Theta_q(TY_s, \zeta_s^2), \nabla \Theta_q(TY_s, \zeta_s^2) \right).
$$

Clearly, if $h(X) = 0$, one gets (1.7). Therefore, for every even number $s \geq 2\left[ \frac{h(X)+1}{2} \right]$, one can construct a meromorphic modular form of weight $2m$ over $\Gamma^0(2)$ of above form, that is equal to $\eta(D_X^{\Theta_q(TX)})$ up to an element in $\mathbb{Z}[[q^{1/2}]]$. The poles of these meromorphic modular forms are just the zeros of the modular forms $f_s(\tau)$. We hope that further study of the modular forms $f_s(\tau)$ will bring better understanding of modularity of $\bar{\eta}(D_X^{\Theta_q(TX)})$.

**Remark 3.3.** We refer to [6] for an alternative approach to the modularity of $\bar{\eta}(D_X^{\Theta_q(TX)})$, which is shown to be not only a meromorphic modular form but also a modular form using the theory of universal $\eta$-invariant.

### 4. The cases of dimension $8n+3$

In this section, we discuss the case of dimension $8n+3$. In this dimension, it is known that $\eta(D_X^{\Theta_q(TX)})$ is mod $2\mathbb{Z}[[q^{1/2}]]$ smooth. That is, in the right hand side of (1.9), the term mod $\mathbb{Z}[[q^{1/2}]]$ can be replaced by mod $2\mathbb{Z}[[q^{1/2}]]$. Therefore it is natural to propose the following conjecture whose statement refines Theorem 1.1 in this case.

**Conjecture 4.1.** Let $X$ be an $8n+3$ dimensional closed spin Riemannian manifold. Then the reduced $\eta$-invariant $\bar{\eta}(D_X^{\Theta_q(TX)})$ of the twisted Dirac operator $D_X^{\Theta_q(TX)}$ is a meromorphic modular form of weight $4n+2$ over $\Gamma^0(2)$, up to an element in $2\mathbb{Z}[[q^{1/2}]]$.

Recall that a mod $2k$ refinement of the Freed-Melrose mod $k$ index for real vector bundles over $8n+4$ dimensional manifolds has been defined in [17, Section 3]. In view of this, one can propose a refinement of Corollary 1.1 in the case of dim $Y = 8n+4$, as follows.

**Conjecture 4.2.** Let $Y$ be an $8n+4$ dimensional spin $\mathbb{Z}/k$-manifold in the sense of Sullivan (cf. [10]). Then the mod $2k$ index associated to the Witten bundle $\Theta_q(TY)$ can be represented by a meromorphic modular form of weight $4n+2$ over $\Gamma^0(2)$.

By the method of this paper, in order to prove Conjectures 4.1 and 4.2, one perhaps needs a kind of Hopkins boundary theorem for real vector bundles. Or, one may try to develop a direct analytic approach, which, even for Theorem 1.1, is a challenging problem as we indicated in Section 1.
Acknowledgement This work was started when both of us were participating an international conference at UC Santa Barbara organized by Xianzhe Dai in July, 2012. We would like to thank Xianzhe Dai for the invitation as well as the kind hospitality. We would also like to thank Siye Wu for bringing our attention to [12]. We are grateful to Ulrich Bunke for the discussion on the topic.

The work of F. H. was partially supported by a start-up grant and AcRF R-146-000-163-112 from National University of Singapore. The work of W. Z. was partially supported by NNSFC and MOEC.

References
[1] M. F. Atiyah, V. K. Patodi and I. M. Singer, Spectral asymmetry and Riemannian geometry I. Proc. Camb. Philos. Soc. 77 (1975), 43-69.
[2] M. F. Atiyah and I.M. Singer, The index of elliptic operators, III, Ann. Math. 87 (1968), 546-604.
[3] M. F. Atiyah and I.M. Singer, The index of elliptic operators, V, Ann. Math. 93 (1971), 139-149.
[4] J.-M. Bismut and D.S. Freed, The analysis of elliptic families, II, Comm. Math. Phys. 107, 1986, 103-163.
[5] U. Bunke, On the topological contents of eta invariants, arXiv: 1103.4217.
[6] U. Bunke, The universal eta-invariant for manifolds with boundary, arXiv: 1403.2030.
[7] U. Bunke and N. Naumann, The $f$-invariant and index theory. Manuscripta Math. 132 (2010), 365-397.
[8] K. Chandrasekharan, Elliptic Functions. Springer-Verlag, 1985.
[9] Q. Chen and F. Han, Elliptic genera, transgression and loop space Chern-Simons forms, Comm. Anal. Geom. 17 (2008), 73-106.
[10] D. S. Freed and R. B. Melrose, A mod $k$ index theorem. Invent. Math., 107 (1992), 283-299.
[11] F. Han and W. Zhang, Modular invariance, characteristic numbers and $\eta$ invariants. J. Diff. Geom. 67 (2004), 257-288.
[12] K. R. Klonoff, An Index Theorem in Differential $K$-Theory. Ph. D. Thesis, Univ. Texas at Austin, 2008. Download address: [12]
[13] K. Liu, Modular invariance and characteristic numbers. Commun. Math. Phys. 174 (1995), 29-42.
[14] S. Ochanine, Elliptic genera, modular forms over $KO$, and the Brown-Kervaire invariant. Math. Z. 206 (1991), 277-291.
[15] E. Witten, The index of the Dirac operator in loop space, in P.S. Landweber, ed., Elliptic Curves and Modular Forms in Algebraic Topology (Proceedings, Princeton 1986), Lecture Notes in Math., 1326, pp. 161-181, Springer, 1988.
[16] D. Zagier, Note on the Landweber-Stong elliptic genus, in P.S. Landweber, ed., Elliptic Curves and Modular Forms in Algebraic Topology (Proceedings, Princeton 1986), Lecture Notes in Math., 1326, pp. 216-224, Springer, 1988.
[17] W. Zhang, On the mod $k$ index theorem of Freed and Melrose. J. Diff. Geom. 43 (1996), 198-206.
[18] W. Zhang, Lectures on Chern-Weil Theory and Witten Deformations , Nankai Tracts in Mathematics Vol. 4, World Scientific, Singapore, 2001.
F. Han, Department of Mathematics, National University of Singapore, Block S17, 10 Lower Kent Ridge Road, Singapore 119076 (mathanf@nus.edu.sg)

W. Zhang, Chern Institute of Mathematics & LPMC, Nankai University, Tianjin 300071, P.R. China. (weiping@nankai.edu.cn)