1. Introduction

Let $p$ be a prime number, $E$ a number field and

$$\varphi : \text{Gal}(\overline{E}/E) \longrightarrow \text{GL}_n(\mathbb{F}_p),$$

a continuous (i.e., which factors through a finite extension) representation. It was shown by Mazur that the set of deformations of $\varphi$, with minor technical conditions, with values in finite extensions of $\mathbb{Q}_p$, can be arranged in a natural rigid space $X_{\varphi}$. Conjecturally, and now in many known cases ([CHLN11, CHI3, Shil14, HLTT16, Sch15]), we can use automorphic representations to construct Galois representations, and in particular points in $X_{\varphi}$ that we call of automorphic nature, or just automorphic. It is then natural to wonder which structure has this set of automorphic points in $X_{\varphi}$: is it an algebraic subspace? a closed one? is it Zariski dense?

The first example beyond the case of characters was studied by Gouvêa and Mazur. In this situation $E = \mathbb{Q}$ and $n = 2$, and $\varphi$ is irreducible, modular and unobstructed, so that $X_{\varphi}$ is a 3-dimensional open ball. In that case, automorphic points are related to modular forms. In [GM98], Gouvêa and Mazur show that automorphic points are Zariski dense in $X_{\varphi}$ using the so called infinite fern. Let us explain this name: up to twisting by powers of the cyclotomic character, we can replace $X_{\varphi}$ by a two-dimensional open ball. Modular forms (of finite slope) can be interpolated by a geometric object, the Coleman–Mazur Eigencurve $\mathcal{E}$ ([CM98]), which is a rigid-analytic curve, whose points are refined $p$-adic modular forms of finite slope. Generically, a classical modular forms has two refinements, thus gives rise to two distinct points in the Eigencurve. Moreover the points corresponding to refined classical modular form are Zariski-dense in the Eigencurve. By $p$-adic interpolation, it is possible to associate a 2-dimensional $p$-adic representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to a point of $\mathcal{E}$. The points giving rise to deformations of $\varphi$ form a union $\mathcal{E}(\varphi)$ of connected components of $\mathcal{E}$ and the universal property of $X_{\varphi}$ implies the existence of a map

$$\mathcal{E}(\varphi) \longrightarrow X_{\varphi}.$$
Generically, each modular point \( f \) in \( X_\mathcal{E} \) has two preimages in \( E \), giving rise to two distinct small curves around those preimages, whose image in \( X_\mathcal{E} \) meet only at \( f \). By density, each of these two small curves has a Zariski-dense set of modular points, and for each of these points there is another small curve passing through, and so on, giving a fractal-like object which we picture as follows:

![Diagram showing small curves forming a fractal-like object](image)

giving a justification for the name of the infinite fern.

This article deals with a generalization of this result to more general number fields and greater values of \( n \). First, we need to assume that the number field \( E \) is a CM field, with totally real field \( F \), in order to be able to associate Galois representations to automorphic representations. Second, it is expected that for general \( n \) the automorphic points are not Zariski dense in \( X_\mathcal{E} \), thus we reduce to the case of \( \chi \)-polarized Galois representations, for a character \( \chi : G_E \longrightarrow \mathbb{Q}_p^\times \), i.e. continuous group homomorphisms \( \rho : \text{Gal}(\overline{E}/E) \longrightarrow \text{GL}_n(\mathbb{Q}_p) \) such that

\[
\rho^\vee \simeq \rho^\vee \otimes \chi^{n-1}, \quad \text{where } \rho^\vee := \rho(c \cdot c^{-1}),
\]

where \( \varepsilon \) is the cyclotomic character, and \( c \in \text{Gal}(\overline{E}/F) \) is a lift of the unique non trivial element of \( \text{Gal}(E/F) \). Fix \( S \) a finite set of primes of \( E \) containing all primes above \( p \). In this situation, assume that \( \overline{\mathfrak{p}} \) is \( \chi \)-polarized, absolutely irreducible (for simplicity) and unramified away from \( S \). Let \( R_{\mathfrak{p}}^{\chi-\text{pol}} \) be the complete noetherian local algebra parametrizing deformations of \( \overline{\mathfrak{p}} \) which are \( \chi \)-polarized and unramified away from \( S \). Its rigid fiber \( X_{\mathfrak{p}}^{\chi-\text{pol}} \) is a rigid space of dimension at least \( [F : \mathbb{Q}][n(n+1) / 2] \). A natural source of automorphic points in \( X_{\mathfrak{p}}^{\chi-\text{pol}} \) is given by the regular, algebraic, essentially polarized, cuspidal automorphic representations of \( \text{GL}_n(\mathbb{A}_E) \), by work of many authors ([HT01, CHT08, CHLN11, CH13, Shi14] for example). In this paper, we make the following hypothesis,

**Hypothesis 1.1.** — 
1. \( \overline{\mathfrak{p}} \) is conveniently modular (see Definition 6.1),
2. All primes above \( p \) in \( F \) are unramified, and split in \( E \),
3. the character \( \chi \) satisfies \( \chi = \chi^c \) and satisfies a sign condition (see Hypothesis 5.2 and section 4)

Under the previous hypothesis, we prove the following result:
Theorem 1.2. — The Zariski-closure of automorphic points contains a (non empty) union of irreducible components of $\Lambda_{\mathfrak{p}}^{\chi-\text{pol}}$, each of which are of dimension $[F : \mathbb{Q}] n(n+1)/2$.

We say that our deformation problem is unobstructed if $H^2(G_{F,S}, \text{ad}(\overline{\mathfrak{p}})) = \{0\}$ where $\tau$ is some extension of $\mathfrak{p}$ to $G_{F,S}$ the Galois group of the maximal unramified extension of $F$ (see section 5). In this situation, we know that $\Lambda_{\mathfrak{p}}^{\chi-\text{pol}}$ is a rigid open unit ball in $[F : \mathbb{Q}] n(n+1)/2$-variables.

Corollary 1.3. — Under the previous hypothesis, if moreover $\overline{\mathfrak{p}}$ is unobstructed, then automorphic points are Zariski dense in $\Lambda_{\mathfrak{p}}^{\chi-\text{pol}}$.

Remark 1.4. — In [Gui20] Giraud proved that if $\pi$ is an extremely regular automorphic representation of $\text{GL}_n(A_E)$ (see [BLGGT14a] section 2.1), then there exists a density 1 set of primes $\lambda$ of $E$ such that $\rho_{\pi,\lambda}$ is unobstructed. As we have assumed $\mathfrak{p}$ to be conveniently modular, we can actually find some extremely regular $\pi$, so that $\overline{\mathfrak{p}} = \overline{\mathfrak{p}}_{\pi,\lambda}$ for $\lambda | p$, and thus using [Gui20], up to change $\lambda$ in a density 1 set, we can assume unobstructedness. In particular, under our assumption, $\overline{\mathfrak{p}}$ is part of a compatible system for which in a density 1 set of primes, we have Zariski density of automorphic points in the associated deformation spaces.

Before explaining the strategy of proof, let us say what was known. The first case was the non-polarized case, $n = 2$ and $E = F = \mathbb{Q}$, when unobstructed, which was proven by Gouvêa–Mazur [GM98], and generalised by Böckle [Bö1]. The non-polarised case for $n = 2$ and totally real fields $E = F$, and the polarised case of $n = 3$ (and general CM fields $E/F$) was proved by Chenevier ([Che11]). A generalisation for greater $n$; but under restrictive hypothesis (of Taylor–Wiles type) on $E$ and $\mathfrak{p}$, was proven recently by Hellmann–Margarin–Schnraen ([HMS22]). All of these proofs use the analogue (in higher dimensions) of the infinite fern. Now we explain our strategy together with the relation to the previous works.

The Galois representations that we study can be viewed as $p$-adic ($L-$ or $C-$) parameters of a reducive group: $\text{GL}_2/\mathbb{Q}$ in the situation of Gouvêa–Mazur and $\text{U}_{E/F}(n)$, or $\text{GU}_{E/F}(n)$, or one of its inner forms, a (similitude) unitary group in $n$-variables, for polarized deformation problems. A natural source of automorphic points is given by automorphic representations of these groups. It turns out that these groups give rise to Shimura data, that we can use to construct $p$-adic refined families of automorphic forms, that is $p$-adic automorphic eigenforms together with the extra data of a refinement. These families generalize the Eigencurve of Coleman–Mazur and are called Eigenvarieties (see [CM98, Che04, Urb01, Eme06, AIP15, Her22] for example). Let us assume for simplicity $\chi = 1$ for the rest of the introduction.

For general $n$, a given automorphic form $f$ has at most $n![F : \mathbb{Q}]$ refinements $f_1$, and generically exactly $n![F : \mathbb{Q}]$ refinements. Moreover the Eigenvariety $\mathcal{E}$ has equidimension $n[F : \mathbb{Q}]$, $\Lambda_{\mathfrak{p}}^{\chi-\text{pol}}$ has dimension at least (but conjecturally exactly) $n(n+1)/2[F : \mathbb{Q}]$ and there is a map $\mathcal{E}(\overline{\mathfrak{p}}) \to \Lambda_{\mathfrak{p}}^{\chi-\text{pol}}$ which forgets the refinement.

(1) an open-closed subvariety of $\mathcal{E}$
Definition 1.5. — The image of the map $\mathcal{E}(\overline{\rho}) \longrightarrow X_{\overline{\rho}}^{\text{pol}}$ is called the infinite fern and is denoted by $\mathcal{F}(\overline{\rho})$.

Actually we can make our main theorem more precise:

Theorem 1.6. — Under the previous hypothesis, the Zariski closure of the infinite fern $\mathcal{F}(\overline{\rho})$ in $X_{\overline{\rho}}^{\text{pol}}$ is a non-empty union of irreducible components, each of which are of dimension $\frac{n(n+1)}{2}[F : \mathbb{Q}]$.

Let us comment the various hypothesis we made. The main advantage compared to [HMS22] is that we don’t have any hypothesis on $\rho$, $E/F$ or $n$, and moreover we don’t need to assume $\overline{\rho}$ absolutely irreducible (or has big image) if we use Chenevier’s determinants, which we do (see section 5). The hypothesis of being conveniently modular is necessary to expect the infinite fern to be non-empty, and is in practice very close to the usual modularity hypothesis which is anyway necessary. The hypothesis on the splitting of primes above $p$ is technical, and we hope to come back on this question soon.

Following the strategy of Chenevier, the main goal is to prove that for a Zariski dense set of points $\rho$ in the infinite fern, the part of the tangent space at $\rho$ in $X_{\overline{\rho}}^{\text{pol}}$ coming from $\mathcal{E}$ has dimension at least $\frac{n(n+1)}{2}[F : \mathbb{Q}]$. This will imply that the closure of the infinite fern $\mathcal{F}(\overline{\rho})$, has dimension at least $\frac{n(n+1)}{2}[F : \mathbb{Q}]$. As, by construction, automorphic points are Zariski-dense in $\mathcal{E}$, thus in $\mathcal{F}(\overline{\rho})$, this will prove that the Zariski closure of automorphic points has dimension at least $\frac{n(n+1)}{2}[F : \mathbb{Q}]$. Thus to prove the assertion on the tangent space, we need to show that the tangent spaces $T_f \mathcal{E}$ are “transverse”, more precisely that the sum of the images of the tangent spaces $T_f \mathcal{E}$ in $T_{\overline{\rho}} X_{\overline{\rho}}^{\text{pol}}$, for well chosen automorphic forms $f$, has large enough dimension. But, clearly, as soon as $n \geq 3$ it is not sufficient that these tangent spaces are pairwise distinct, and this is the main difficulty to extend the proof of Gouvêa and Mazur. To overcome this problem, Chenevier suggested a strategy which he applied successfully when $n = 3$ and which can be sketched as follows:

1. find a good Zariski-dense subset $D$ of the infinite fern $\mathcal{F}(\overline{\rho})$;
2. show that the analogue of the transversality on the tangent spaces of points in $D$ but for local deformation rings is valid;
3. show that the Global situation “embeds well” in the local situation, and thus gives the result.

For the first part, Chenevier suggests to look at automorphic points $\rho$ which he calls generic: they are crystalline at $\rho$ and all their refinements are non-critical. More precisely, if $\rho$ is crystalline at a place $v \mid p$, its restriction $\rho_v$ to a decomposition group at $v$ is characterized by a $n$-dimensional vector space $V = D_{\text{cris}}(\rho|_{\text{Gal}(\overline{\mathbb{F}}_p/E_v)})$ with its Hodge-Tate filtration $\mathcal{F}_{HT}$ (a complete flag) and Frobenius operator $\varphi$. The refinements of $\rho$ correspond to the complete flags of $V$ stable by $\varphi$. We say that a refinement of $\rho$ is non critical it is opposite to $\mathcal{F}_{HT}$. Actually, Chenevier proved that the second step works for crystalline points which have $n$ well-positioned non-critical flags and call those points weakly-generic. He moreover proved that weakly generic points (with some extra but harmless conditions) are Zariski dense in the infinite fern when $n = 3$ and uses those as the subset $D$. 


Concerning the second point, the weakly generic condition is used to carry an induction in the local situation and prove that tangent spaces of local, refined, deformations problem spawn the tangent space of the full local deformation ring. This is where the definition of well positioned refinements comes from.

For the last point, actually it is enough to embed the situation at the level of tangent spaces. Chenevier proves that for all preimages of points in $D$, the map $E \longrightarrow W$ to the weight space is étale, and deduces that some Selmer group vanishes at those points, allowing to embed infinitesimally the global situation into the local one, transversally to the crystalline locus, and thus deduce the result. This last argument is classical in the Taylor-Wiles method.

The main issue to generalise Chenevier’s strategy in higher dimensions is that it is completely unclear that weakly generic points satisfies some density assumptions when $n \geq 4$ (see remark 7.2).

The strategy of [HMS22] is different but shares some similarities: for the first point they choose points which are crystalline with some genericity assumption(2) which are less restrictive than being generic or weakly-generic. Their set $D$ is then automatically Zariski dense. For the second point, they use a local model for the local deformations spaces, which is of purely geometric nature, and a rather evolved but completely elementary argument allows to conclude in the second point, using not only all refinements but also companion points, which are extra-points appearing when the refinement is critical (whose existence is proved in [BHS19]) and relies on the Taylor-Wiles hypothesis.

Then the third point is the most delicate one and is proved by Taylor-Wiles-Kisin method via “patched eigenvarieties” (see [BHS19]). The second and third point relies deeply on the Taylor-Wiles hypothesis, in particular it crucially needs $\rho$ to have adequate image.

In this article we use a strategy closer to Chenevier’s, but using the local model of [BHS19] as in [HMS22], but we never need any Taylor-Wiles assumption. Namely, using the local model and a careful study of its geometry, we first prove the second point without using companion points but rather generalizing Chenevier’s transversality result at critical refinements (see section 3). For the first point, we show that setting for $D$ the set of crystalline points satisfying genericity conditions as in [HMS22] and which have moreover enormous image are actually Zariski dense in the infinite fern; we call those points almost-generic (see Definition 7.4) because they will replace Chenevier’s generic points in our argument. The density of these points is far from being automatic and the argument is originally due to Bellaïche–Chenevier and Taïbi (see section 7). Then, for the third point, we show that using the enormous image, and a result of Newton–Thorne, we have the vanishing of the expected Selmer group at points of $D$. We then show that this can be used to relate the global situation to the local one. As a byproduct, we obtain that our Eigenvariety is smooth at those points, as it was the case in other situations (see [BHS19, Ber20]) (see section 8). Then, a local calculation which was previously carried out in [Alli16], we show that our almost generic points are smooth points of $\Lambda_{p}^{\frac{1}{p}-\text{pol}}$ of the expected dimension.

(2) precisely on the Frobenius eigenvalues and Hodge-Tate weights
The results of Chenevier were combined with those by Allen ([All19]) (who proves that under some hypothesis every component contains an automorphic point) to prove full density of the infinite fern when \( n \leq 3 \), without assuming unobstructedness. We can adapt this generalisation also here,

**Corollary 1.7 (Allen).** — Assume hypothesis 8.12, then the infinite fern is Zariski dense in \( \mathfrak{g}^{\chi_{\text{pol}}} \).

The only thing we need to check for this corollary is that we use classical points which are automorphic representations for a similitude unitary group, which moreover contributes to the coherent \( H^0 \), whereas Allen’s proof a priori only provides an (essentially) polarised automorphic representation of \( \text{GL}_n \).

**Acknowledgements:** We would like to warmly thank Gaëtan Chenevier for very helpful suggestions concerning this work. We would also like to thank George Boxer, Laurent Clozel and Olivier Schiffmann for many interesting discussions. Finally, we would like to thank Anne vau gon and Marc Mezzarobba for their help for using SAGE to compute the local tangent spaces.

## 2. A lemma on Borel enveloppes

In this section we prove a technical, but essential, generalization of the linear algebra result [HMS22, Thm. 2.3]. The main result of this section is Lemma 2.1.

Fix \( n \) an integer, \( k \) a field, \( G = \text{GL}_n/k \), \( B \) the upper triangular Borel, \( T \) its diagonal torus, \( g,b,t \) their respective Lie algebras. We also set \( u \) the nilpotent radical of \( b \). Let \( W \) be the Weyl group of the pair \( (G,T) \) that we identify to \( S_n \). Let \( w_0 \in W = S_n \) be the longest element for the order given by \( b \). For \( g,h \in G(k) \), we set \( b_g = g^{-1}bg \) and \( (b_g \cap b_h)^{gr = w_0} \) the \( k \)-linear subspace of \( M \in b_g \cap b_h \) such that, in \( t \),

\[
hMh^{-1} \pmod{u} = \text{Ad}(w_0)(gMg^{-1} \pmod{u}) \in t.
\]

Let \( \mathfrak{c}_n \) be the set of “full” cycles:

\[
\mathfrak{c}_n := \{ c_{i,j} := (i, i - 1, \ldots, j + 1, j) \in S_n \mid i \geq j \}.
\]

**Lemma 2.1.** — For every Borel subalgebra \( b' \subset g \), we have

\[
b' = \sum_{w \in \mathfrak{c}_n w'} (b' \cap b_{w})^{gr = w_0},
\]

for some element \( w' \in S_n \) depending on \( b' \). Moreover if \( b' = b_{w_0b} \) for some \( b \in B(k) \), we can choose \( w' = 1 \).

**Proof.** — We can write \( b' = b_g \) for \( g \in G(k) \). Let \( g = uls \), \( u \in b, l \in b_{w_0} \) lower triangular i.e. \( l = w_0b_{w_0} \) with \( b \in B(k) \), and \( s \in W \). Thus \( b_g = (b_{w_0})_gq \), for \( q = w_0s \in W \). Up to conjugate by \( bg \), we check at once that it is enough to show

\[
b_{w_0} = \sum_{w \in \mathfrak{c}_n} (b_{w_0} \cap b_{w_0})^{gr = w_0}.
\]

Thus we reduce to show the following lemma:
Lemma 2.2. — For all \( i \geq j \), there exists \( x_{\ell} \in k \) such that

\[
a^{i,j} := \delta_{i,j} + \sum_{\ell=j+1}^{i} x_{\ell}\delta_{\ell,j} \in \langle b_{w_0} \cap b_{c_{i,j}b^{-1}} \rangle^{gr-u_0}.
\]

Proof. — We follow the proof of [HMS22]. For \( i \geq j \), let \( a^{i,j} \) be the element constructed at the beginning of the proof of Lemma 2.1 in loc. cit. For the convenience of the reader we recall its construction. Let \( e_1, \ldots, e_n \) be the standard basis of \( k^n \) and let \( V_* \) be the standard flag of \( k^n \). Let \( B \) be the basis

\[
b(e_1), b(e_2), \ldots, b(e_{j-1}), e_j, b(e_{j+1}), \ldots, b(e_i), e_{i+1}, \ldots, e_n
\]

of \( k^n \). Then \( a^{i,j} \) is the matrix, in the standard basis, of the endomorphism \( \pi \) of \( k^n \) defined by \( \pi(x) = 0 \) if \( x \in B \langle e_j \rangle \) and \( \pi(e_j) = e_i \). As in loc. cit. we check that

(i) \( e_{\ell} \in \ker(\pi) \) if \( \ell < j \) or \( \ell > i \);
(ii) \( \text{Im}(\pi) \subset k e_i \) and \( \pi(e_j) = e_i \);
(iii) the endomorphism \( b^{-1} \pi b \) stabilizes the flag \( c_{i,j}^{-1}V_* \).

The first two points are checked in loc. cit. The third point follows from the fact that

\[
(1) \quad b^{-1} \pi b(c_{i,j}^{-1} V_\ell) = \begin{cases} 0 & \text{if } \ell = 1, \ldots, i-1 \\ kb^{-1}(e_i) & \text{if } \ell \geq i \end{cases}
\]

and \( kb^{-1}(e_i) \in V_* = c_{i,j}^{-1}V_i \). This shows that the matrix \( a^{i,j} \) is an element of \( b_{w_0} \cap b_{c_{i,j}b^{-1}} \) and has the form

\[
(2) \quad \delta_{i,j} + \sum_{\ell=j+1}^{i} x_{\ell}\delta_{\ell,k}
\]

for some \( x_{\ell} \in k \).

It remains to check that \( a^{i,j} \in \langle b_{w_0} \cap b_{c_{i,j}b^{-1}} \rangle^{gr-u_0} \) which is equivalent to check that the diagonal elements of \( a^{i,j} \) and \( \text{Ad}(c_{i,j}b^{-1})a^{i,j} \) are the same. It follows from (1) that

\[
(\text{Ad}(c_{i,j}b^{-1})\pi)(e_\ell) = \begin{cases} 0 & \text{if } k < i \\ c_{i,j}^{-1}b^{-1}(e_i) & \text{if } k = i \end{cases}
\]

and \( (\text{Ad}(c_{i,j}b^{-1})\pi)(e_i) \in V_\ell \) if \( \ell > i \). Therefore the diagonal of \( \text{Ad}(c_{i,j}b^{-1})a^{i,j} \) is zero except for the coefficient \((i,i)\). On the other hand, we see by (2) that the diagonal entries of \( a^{i,j} \) are all zero except \((i,i)\). As the matrices \( a^{i,j} \) and \( \text{Ad}(c_{i,j}b^{-1})a^{i,j} \) are conjugated they have the same trace and thus the same diagonal. \( \square \)

3. Local deformation rings

The aim of this section is to prove Proposition 3.7 which says that the tangent space of the deformation ring of a crystalline \( \varphi \)-regular and Hodge-Tate regular \( (\varphi, \Gamma) \)-module is generated by the tangent space of some quasi-trianguline deformation subspaces.

Let \( k \) be a field of characteristic 0. Let \( G \) be a split reductive group over \( k \), \( B \subset G \) a Borel subgroup of \( G \), \( T \subset B \) a maximal split torus of \( G \), \( U \) the unipotent radical of \( B \)
and $U^-$ the unipotent radical of the opposite Borel subgroup to $B$ with respect to $T$ (in particular $U^- \cap B = \{1\}$). Let $W := N_G(T)/T$ be the Weyl group of $(G,T)$. We denote $w_0$ the longest element of $W$ with respect to the Bruhat order induced by the choice of $B$. Let $\frak{g}$, $\frak{b}$, $t$, $u$, $u^-$ be the respective Lie algebras of $G$, $B$, $T$, $U$, $U^-$. Let $\tilde{\frak{g}} \subset \frak{g} \times G/B$ be the Grothendieck simultaneous resolution of $\frak{g}$ and $X := \tilde{\frak{g}} \times \tilde{\frak{g}}$. We recall that $X$ has irreducible components $X_w$ which are indexed by the elements of the Weyl group $W$ (see [BHS19, Def. 2.2.3]). The map $\tilde{\frak{g}} \to t$ sending $(\psi, gB)$ to the projection of $\text{Ad}(g)^{-1} \psi$ on $t$ via $b/u \simeq t$ gives rise to two different maps $\kappa_1, \kappa_2 : X \to t$ corresponding to the two projections $X \to \tilde{\frak{g}}$ and to a map $\kappa := (\kappa_1, \kappa_2) : X \to t \times_t \mathfrak{t}$. If $w \in W$, we let $t^w \subset t$ be the subspace of elements fixed by $w$ and $T_w \subset t \times_t \mathfrak{t}$ be the irreducible component

$$T_w := \{(x_1, x_2) \in t \times t \mid x_1 = \text{Ad}(w)x_2\}.$$

The space $X$ has a partition by locally closed subschemes $V_w$ defined as inverse images of the Bruhat strata $U_w \subset G/B \times G/B$ by the map $\pi : X \to G/B \times G/B$ and $X_w = \overline{V_w}$. We have an inclusion $\kappa(X_w) \subset T_w$ ([BHS19, Lem. 2.5.1]).

**Proposition 3.1.** — Let $x = (g_1 B, 0, g_2 B) \in X_{w_0}(k) \subset G(k)/B(k) \times \frak{g} \times G(k)/B(k)$ be a $k$-point. Let $w \in W$ be such that $x \in V_w$ and assume that $w_0 w^{-1}$ is a product of distinct simple reflections. Then we have an equality of $k$-vector spaces

$$T_x X_{w_0} = T_x \kappa^{-1}(T_{w_0}).$$

**Proof.** — The inclusion $X_{w_0} \subset \kappa^{-1}(T_{w_0})$ induces an inclusion $T_x X_{w_0} \subset T_x \kappa^{-1}(T_{w_0})$. We will prove that these two $k$-vector spaces have the same dimension.

Let $k[\varepsilon] := k[X]/(X^2)$. The tangent space $T_x(\kappa^{-1}(t_{w_0}))$ is the set of $k[\varepsilon]$-points $(\tilde{g}_1 B, \varepsilon A, \tilde{g}_2 B)$ of $X$ specialising to $x$ such that moreover

$$\text{Ad}(\tilde{g}_1)^{-1}(\varepsilon A) = \text{Ad}(w_0) \text{Ad}(\tilde{g}_2)^{-1}(\varepsilon A)$$

in $t \otimes_{k} k[\varepsilon]$. Let $\tilde{x} = (\tilde{g}_1 B, \varepsilon A, \tilde{g}_2 B)$ be such a point. We can write $\tilde{g}_i = g_i(1 + \varepsilon h_i)$, where $h_i \in u^-$. Using $\varepsilon^2 = 0$, the condition $\tilde{x} \in X(k[\varepsilon])$ is equivalent to $\text{Ad}(g_i^{-1})A \in \frak{b}$ for $i \in \{1, 2\}$. The condition (3) is then equivalent to $\text{Ad}(g_1^{-1})A = \text{Ad}(w_0) \text{Ad}(g_2^{-1})A$ in $t$. Note that, up to changing $x$ by a point of its $G(k)$-orbit, we can assume, without changing the dimensions of the tangent spaces, that $g_1 = 1$ and $g_2 = w$. The conditions above are then equivalent to

$$A \in t^{w_0 w^{-1}} + (u \cap \text{Ad}(w)u)$$

which is a $k$-vector space of dimension $\dim_k t^{w_0 w^{-1}} + \lg(w_0 w^{-1})$. As $w_0 w^{-1}$ is a product of distinct simple reflections, we have ([Car72, Lem. 2 & 3])

$$\dim_k t^{w_0 w^{-1}} = \dim_k t - \lg(w_0 w^{-1}).$$

Namely, we have a $W$-equivariant isomorphism $t \simeq \text{Hom}(X^*(T), k)$, so that it is sufficient to prove that

$$\dim_{\mathbb{R}}(X^*(T) \otimes \mathbb{R})^w = \dim_{\mathbb{R}}(X^*(T) \otimes \mathbb{R}) - \lg(w')$$

when $w'$ is a product of simple reflections. Let $V$ be the subspace of $X^*(T) \otimes \mathbb{R}$ generated by the roots of $(G,T)$. It is stable under $W$ and has a direct summand on which $W$ is
acting trivially. It is therefore sufficient to prove that
\[ \dim_k V' \leq \dim_k V - \lg(w'). \]
As \( W \) is a finite group, \( \dim_k V - \dim_k V' = \dim_k V = \dim_k V' \) is equal to the number of eigenvalues of \( w' \) acting on \( V \) which are different from 1. By [Car72, Lem. 2], this number is equal to the number \( l(w') \) of loc. cit. (which is a priori not \( \lg(w') \)). By [Car72, Lem. 3], we have \( l(w) = \lg(w') \) since the set of simple roots is a set of linearly independent vectors of \( V \).

Finally we deduce that
\[ \dim_k T \kappa^{-1}(T_{w_0}) \leq \dim_k (G/B \times G/B) + \dim_k t = \dim G. \]
On the other hand, we know that \( X_{w_0} \) is irreducible of dimension \( G \). Consequently we have
\[ \dim G \leq \dim_k T \kappa^{-1}(T_{w_0}) \leq \dim_k T \kappa^{-1}(T_{w_0}) \leq \dim G \]
so that \( T \kappa^{-1}(T_{w_0}) \).

**Lemma 3.2.** — Let \( w \in W \) and \( b \in B \). Then the point \( bwB \) is in the closure of the \( T \)-orbit of \( bwB \) in \( G/B \).

**Proof.** — Let \( \nu \) be a cocharacter of \( T \) such that \( \langle \nu, \alpha \rangle > 0 \) for all positive root \( \alpha \) of \( (G, B, T) \). Then the map \( \mathbb{G}_m \to G \) defined by \( u \nu u^{-1} \) for \( u \) in the unipotent radical of \( B \) extends to a map \( \mathbb{A}^1 \to G \) sending 0 to 1, thus as does the map \( \nu b \nu^{-1} \). Consequently, as \( w \) normalises \( T \), \( bwB \) is in the closure of the image of the map \( t \mapsto \nu(t)bwB = \nu(t)bw^{-1}B \).

**Lemma 3.3.** — Let \((w_1, w_2) \in W^2 \) and \( b \in B \). If \((w_1 B, bw_2 B) \in U_w \subset G/B \times G/B \), we have \( w_1 w_2 \leq w \) in the Bruhat order.

**Proof.** — If \( t \in T \), we have \( tw_1 B = w_1 B \) in \( G/B \) so that \((w_1 B, tw_2 B) \in U_w \). It follows from Lemma 3.2 that \( tw_2 B \) is in the closure of the set \( \{tbw_2 B \mid t \in T \} \) so that \( (w_1 B, w_2 B) \) is in the closure if \( U_w \). As the closure of \( U_w \) is the union of the \( U'_w \) \( w' \leq w \) and \((w_1 B, w_2 B) \in U'_w \) we obtain the result.

From now on we consider \( K/\mathbb{Q}_p \) a finite extension, and denote \( \Gamma = \text{Gal}(K(\zeta_{p^n})/K) \). We fix \( L \) a finite extension of \( \mathbb{Q}_p \) that splits \( K \), i.e.
\[ L \otimes_{\mathbb{Q}_p} K \simeq L[K/\mathbb{Q}_p] \]
and we denote by \( k_L \) its residue field. We follow to the notations of [KPX14] concerning \((\varphi, \Gamma)\)-modules over Robba rings. Let \( \mathcal{R}(\pi_K) \) be the Robba ring for \( K \) (see [KPX14] definition 2.2.2). We define \( t \in \mathcal{R}(\pi_K) \) by \( t = \log(1 + \pi_K). \) Let \( C_L \) be the category local artinian \( O_L \)-algebra \( A \) with maximal ideal \( m_A \) such that the natural map \( k_L \to A/m_A \) is an isomorphism. If \( A \) is an object of \( C_L \), we denote \( \mathcal{R}_A(\pi_K) := A \otimes_{O_L} \mathcal{R}(\pi_K) \). We refer to [KPX14, Def. 2.2.12] for the notion of \((\varphi, \Gamma)\)-module over \( \mathcal{R}_A(\pi_K) \). Let \( D \) be a \((\varphi, \Gamma)\)-module over \( \mathcal{R}_L(\pi_K) \). We denote
\[ \chi_D : C_L \to \text{Sets} \]
the deformation functor of \( D \), i.e. for an object \( A \) of \( C_L \), \( \chi_D(A) \) is the set of isomorphism classes of pairs \((D_A, i_A)\) where \( D_A \) is a \((\varphi, \Gamma)\)-modules over \( \mathcal{R}_A(\pi_K) \) and \( i_A : L \otimes_{\mathbb{Q}_p} D_A \simeq \)
$D$ is an isomorphism of $(\varphi, \Gamma)$-modules. If $(\rho, V)$ is a continuous representation of $G_K$ on a finite dimensional $L$-vector space, the functor $\mathcal{D}_{rig}$ of [Ber02] induces an isomorphism of deformation functors (see [HMS22, §3.6] for details)

$$\mathcal{D}_{rig} : \mathcal{X}_V \longrightarrow \mathcal{X}_{\mathcal{D}_{rig}(V)}.$$ 

Let $\mathcal{F} = (0 = \mathcal{F}_0 \subseteq \cdots \subseteq \mathcal{F}_n = D[t^{-1}])$ be a complete filtration of $D[t^{-1}]$ by sub-(\varphi, \Gamma)$-modules over $\mathcal{R}_L(\pi_K)[t^{-1}]$ which are direct factors as $\mathcal{R}(\pi_K)[t^{-1}]$-modules (we will call such a filtration *aquasi-triangulation*). We define similarly

$$\mathcal{X}_{D, \mathcal{F}} : \mathcal{C}_L \longrightarrow \text{Sets}$$

the deformation functor of the pair $(D, \mathcal{F})$, i.e. for $\mathcal{A}$ in $\mathcal{C}_L$, the set $\mathcal{X}_{D, \mathcal{F}}(\mathcal{A})$ is the set of isomorphism classes of triples $(D_A, \mathcal{F}_A, i_A)$ where $(D_A, i_A) \in \mathcal{X}_D(\mathcal{A})$ and $\mathcal{F}_A$ is a filtration of $D_A[t^{-1}]$ by $(\varphi, \Gamma)$-stable $\mathcal{R}_A(\pi_K)[t^{-1}]$-submodules which are direct factors of $D_A[t^{-1}]$ in the category of $\mathcal{R}_A(\pi_K)[t^{-1}]$-modules and such that $i_A(L \otimes_A \mathcal{F}_A) = \mathcal{F}_i$ for all $i \in \mathbb{Z}$.

We recall some notations of [BHS19] section 3 and we refer the reader to loc. cit. for more precisions. Let $W$ be an $L \otimes_{\mathbb{Q}_p} B_{dR}^+$-representation of $G_K$ which is almost de Rham. Let $W^+$ be a $G_K$-stable $L \otimes_{\mathbb{Q}_p} B_{dR}^+$-lattice of $W$. Let $\mathcal{X}_W : \mathcal{C}_L \longrightarrow \text{Sets}$ be the deformation functor of $W$, which means that $\mathcal{X}_W(\mathcal{A})$ is the set of isomorphism classes of pairs $(W_A^+, i_A)$ such that $W_A^+$ is a finite free $A \otimes_{\mathbb{Q}_p} B_{dR}^+$-module endowed with a continuous semilinear action of $G_K$ and $i_A$ is a $G_K$-equivariant isomorphism $L \otimes_A W_A^+ \cong W^+$ of $L \otimes_{\mathbb{Q}_p} B_{dR}^+$-modules. If we fix an $L \otimes_{\mathbb{Q}_p} K$-linear isomorphism $\alpha : (L \otimes_{\mathbb{Q}_p} K)^n \xrightarrow{\sim} D_{dR}(W)$ we can define $\mathcal{X}_{W^+, \alpha} : \mathcal{C}_L \longrightarrow \text{Sets}$ the deformation functor of the pair $(W^+, \alpha)$. Let $F_i$ be a $G_K$-stable flag of $L \otimes_{\mathbb{Q}_p} B_{dR}$-submodules of $W$, we define $\mathcal{X}_{W^+, F_i}$ the deformation functor of the pair $(W^+, F_i)$ and $\mathcal{X}_{W^+, F_i, \alpha}$ the deformation functor of the triple $(W^+, F_i, \alpha)$.

**Remark 3.4.** — Assume that $D$ is a crystalline, $\varphi$-generic, $(\varphi, \Gamma)$-module (see [HMS22, §3.3]). In this case, it can be convenient to choose $\alpha$ compatible with the Frobenius structure. The isocrystal $(D_{cris}(D), \varphi)$ has exactly $n!$-refinements, i.e. complete flags stable by $\varphi$, which are in natural bijection with the orderings of the eigenvalues of the linearized Frobenius $\varphi^f$. By $\varphi$-genericity, there exists an isomorphism $\alpha_0 : (L \otimes_{\mathbb{Q}_p} K_0)^n \cong L \otimes_{\mathbb{Q}_p} D_{cris}(D)$ sending the canonical basis on an eigenbasis of $\varphi$. Then $\alpha_0$ induces a bijection between the complete flags of $(L \otimes_{\mathbb{Q}_p} K_0)^n$ stable under the group of diagonal matrices and the set of refinements of $D_{cris}(D)$. Denote $\alpha = \alpha_0 \otimes_{K_0} \text{Id}_K : (L \otimes_{\mathbb{Q}_p} K)^n \cong D_{dR}(D) = D_{dR}(D) \otimes_{\mathbb{Q}_p} K$. By [HMS22] Lemma 3.7, the map $\mathcal{F} \mapsto D_{cris}(\mathcal{F}[1/\ell])$ induces a bijection between the set of triangulations of $D$ and the set of $n!$ refinements of $D_{cris}(D)$. If $\mathcal{F}$ is a triangulation of $D$ and $w \in \mathfrak{S}_n \subset \text{GL}_n(L \otimes_{\mathbb{Q}_p} K)$ we denote $w \cdot \mathcal{F}$ the unique triangulation of $D$ such that $w^{-1}(D_{dR}(\mathcal{F}[1/\ell])) = \alpha^{-1}(D_{dR}(w \cdot \mathcal{F}[1/\ell]))$. This defines a simply transitive action of the group $\mathfrak{S}_n$ on the set of triangulations of $D$ (which depends on the choice of $\alpha$).

Now we fix $G = \text{GL}_{n, K}$, $B \subset G$ the Borel subgroup of upper triangular matrices and $T \subset B$ the maximal torus of diagonal matrices. We recall that $\mathfrak{g}$ is the $K$-Lie algebra of $G$ and $X = \mathfrak{g} \times \mathfrak{g}$ $\mathfrak{g}$. We also note $X_{K/\mathbb{Q}_p}$ and $\mathfrak{g}_{K/\mathbb{Q}_p}$ their Weil restrictions from $K$ to $\mathbb{Q}_p$. If $A$ is an object of $\mathcal{C}_L$ and $(W_A^+, \mathcal{F}_A, \alpha_A)$ is an element of...
Let $Zariski$ closure of $K_p$ be its image in $X_{cris}^\square (A)$. As $A \in \mathcal{C}_L$ and let $(D_A, \alpha_A) \in X_{cris}^\square (A)$ and let $(D_A, \alpha_A, F_\bullet A)$ be its image in $X_{cris}^\square (A)$.

Now we remark that the schematic inverse image of $K_0$ by the natural map $X_{cris}^\square (A)$ is contained in the irreducible component $X_{cris}^\square (A)$. Namely it is sufficient to prove the inverse image $Z$ of $\mathfrak{g}$ by the natural map $K_0$-schemes $X \to \mathfrak{g}$ is contained in $X_{w_0}$. But $Z$ is $G/B \times \{0\} \times G/B$ which is the Zariski closure of $V_\mathfrak{w}_0 \cap (G/B \times \{0\} \times G/B)$, so that $Z \subset X_{w_0}$.

This implies the that the image of $(D_A, \alpha_A, F_\bullet A)$ in $X_{cris}^\square (A)$ is contained in $X_{cris}^\square (A)$ and finally that $X_{cris}^\square (A)$ is contained in $X_{cris}^\square (A)$.

**Definition 3.6.** — Let $D$ be a crystalline $\varphi$-generic and HT regular $(\varphi, \Gamma)$-module over $R_\ell(\pi_K)$. Let $F_\bullet$ be a triangulation of $D$. Let $w(D, F_\bullet) \in \mathbb{S}^{[K:Q_p]}$ be the element measuring the relative position of $F_\bullet$ and the Hodge filtration of $D$ (see [BHS19] before Proposition 3.6.4). We say that $(D, F_\bullet)$ is associated to a product of distinct transpositions if $w(D, F_\bullet)$ is a product of distinct simple transpositions. Moreover we say that the triangulation $F_\bullet$ is non-critical if $w(D, F_\bullet) = 0$.
We can now prove the main result of this section.

**Proposition 3.7.** — Let \( D \) be a \( \varphi \)-generic, regular, crystalline, \( (\varphi, \Gamma) \)-module over \( R_L(\pi_K) \). Denote \( \text{Tri}(D) \) the set of triangulations of \( D \).

(i) The following \( L \)-linear map is surjective

\[
\bigoplus_{\mathcal{F} \in \text{Tri}(D)} T^w_{D,\mathcal{F}[1/t]} \longrightarrow T^\triangledown_X D.
\]

Assume moreover that there exists \( \mathcal{F}_{\text{nc}} \) a non-critical triangulation of \( D \).

(ii) For any \( c \in \mathbb{C}_n \), the pair \( (D, c \cdot \mathcal{F}_{\text{nc}}) \) is associated to a product of simple transpositions.

(iii) The following \( L \)-linear map is surjective:

\[
\bigoplus_{c \in \mathbb{C}_n} T^w_{D,\mathcal{F}[1/t]} \longrightarrow T^\triangledown_X D.
\]

**Proof.** — Let \( \Sigma \) be the set of embeddings \( \tau : K \to L \). Let \( U \) be the kernel of the map

\[
T^\triangledown_X D \to T^w_{\text{W}_{\text{dR}}(D),\text{W}_{\text{dR}}(\mathcal{F}[1/t])}. \]

It follows from Lemma 3.5 as in [HMS22, Cor. 3.13] that the following sequence is exact for any triangulation \( \mathcal{F} \) of \( D \):

\[
0 \longrightarrow U \longrightarrow T^w_{D,\mathcal{F}[1/t]} \longrightarrow T^w_{\text{W}_{\text{dR}}(D),\text{W}_{\text{dR}}(\mathcal{F}[1/t])} \longrightarrow 0.
\]

Therefore, if \( S \subset \text{Tri}(D) \) is any subset, we have the following commutative diagram

\[
\begin{array}{cccc}
0 & & 0 & \\
\bigoplus_{\mathcal{F} \in S} U & \xrightarrow{\Sigma} & U & \to 0 \\
\downarrow & & \downarrow & \\
\bigoplus_{\mathcal{F} \in S} T^w_{D,\mathcal{F}[1/t]} & \longrightarrow & T^\triangledown_X D & \xrightarrow{W_{\text{dR}}} \\
\downarrow & & \downarrow & \\
\bigoplus_{\mathcal{F} \in S} T^w_{\text{W}_{\text{dR}}(D),\text{W}_{\text{dR}}(\mathcal{F}[1/t])} & \longrightarrow & T^\triangledown_X \text{W}_{\text{dR}}(D) & \\
\end{array}
\]

Thus to prove that the middle horizontal arrow is surjective, it is sufficient to prove that the bottom horizontal arrow is surjective. As \( \mathfrak{X}_D^\triangledown \to \mathfrak{X}_D \) is formally smooth, it is sufficient to prove that the map

\[
\bigoplus_{\mathcal{F} \in S} T^w_{\text{W}_{\text{dR}}(D),\text{W}_{\text{dR}}(\mathcal{F}[1/t])} \longrightarrow T^\triangledown_X \text{W}_{\text{dR}}(D)
\]

is surjective. Fix \( \alpha : (L \otimes_{\mathbb{Q}_p} K)^n \simeq D_{\text{dR}}(D) \) an isomorphism compatible with Frobenius as in Remark 3.4. Let \( \mathcal{F} \) the triangulation of \( D \) such that \( \alpha^{-1}(D_{\text{dR}}(\mathcal{F}[1/t])) \) is the standard flag of \( (L \otimes_{\mathbb{Q}_p} K)^n \) and let \( F_{\ast} := D_{\text{dR}}(\mathcal{F}[1/t]) \). We denote \( x_{\mathcal{F}} := (\alpha^{-1}(F_{\ast}), 0, \alpha^{-1}(F_{\ast}^{\text{ur}})) \in X_K/\mathbb{Q}_p(L) \). It follows from [BHS19, Thm. 3.2.5 & Cor. 3.5.9] that the vertical arrows in the following commutative diagram are isomorphisms.
\[ X_{W^+}^{\square,F} \quad \xrightarrow{=} \quad X_{W^+}^{\square} \]

\[ \pi_2 : \widehat{\mathbb{G}}_\tau \times \mathbb{G}_\tau \to \mathbb{G} \text{ is the second projection}. \]

Recall that we have a decomposition \( X_{K/Q_p,L} \cong \prod_{\tau \in \Sigma} X_\tau \) where \( X_\tau \cong L \times_{K,\tau} X \) and \( \widehat{\mathbb{G}}_{K/Q_p,L} \cong \prod_{\tau \in \Sigma} \widehat{\mathbb{G}}_\tau \). and the map \( \pi_2 \) is of the form \( (\pi_2)_\tau \) with \( \pi_2, \tau \) the base change of the second projection \( X \to \widehat{\mathbb{G}} \). Moreover the irreducible component of \( X_{K/Q_p,L} \) corresponding to the longest element is isomorphic to \( \prod_{\tau \in \Sigma} X_{w_0,\tau} \) with \( w_0 \) the longest element of \( \mathfrak{S}_n \).

Therefore we have to prove that the map

\[ \bigoplus_{\tau \in \Sigma} \tau \xrightarrow{=} \bigoplus_{\tau \in \Sigma} \pi_2, \tau \xrightarrow{=} \widehat{\mathbb{G}}, \tau \xrightarrow{=} \pi_2, \tau \]

is surjective at the level of tangent spaces. As the formation of tangent spaces commutes with finite products, it is sufficient to prove that for a fixed embedding \( \tau \in \Sigma \), the following map is surjective

\[ \bigoplus_{\tau \in \Sigma} \tau \xrightarrow{=} \bigoplus_{\tau \in \Sigma} \pi_2, \tau \xrightarrow{=} \widehat{\mathbb{G}}, \tau \xrightarrow{=} \pi_2, \tau \]

Let \( g_\tau, h_\tau \in \text{GL}_{n}(L) \) such that \( x_{\tau} = (g_\tau B(L), 0, h_\tau B(L)) \). By the choice of \( \alpha \), we have \( g_\tau = 1 \), for all \( \tau \).

The desired surjectivity is equivalent to the following equality

\[ \text{Im} \left( \sum_{w \in S} T_{(w B(L), 0, h_\tau B(L))} X_{w_0,\tau} \to T_{(0, h_\tau B(L))} \right) \]

for a well chosen \( S = \{ w \in \mathfrak{S}_n \mid w \tau \in \mathcal{S} \} \). By Proposition 3.1, we deduce that, for any \( w \in \mathfrak{S}_n \),

\[ T_{(w B(L), 0, h_\tau B(L))} X_{w_0,\tau} = T_{(w B(L), 0, h_\tau B(L))} (T_{w_0}) \]

\[ = T_{w B(L)} G/B \bigoplus (w b w^{-1}_\tau \cap h_\tau b h^{-1}_\tau | g r w_0, T_{h_\tau B(L)} G/B) \]

By Lemma 2.1 there exists an element \( w_\tau \in \mathfrak{S}_n \) such that

\[ \text{Im} \left( \sum_{c \in \mathcal{C}_n} T_{(c w_\tau B(L), 0, h_\tau B(L))} X_{w_0,\tau} \to T_{(0, h_\tau B(L))} \right) \]

\[ = \left( \sum_{c \in \mathcal{C}_n} (c w_\tau b w^{-1} b h^{-1} | g r w_0, T_{h_\tau B(L)} G/B) = h_\tau b h^{-1} \right) \oplus T_{h_\tau B(L)} G/B = T_{(0, h_\tau B(L))} \]

This implies the surjectivity of the map \( (5) \) when \( S = \{ c w_\tau \mathcal{F} \mid c \in \mathcal{C}_n \} \), and in particular when \( S = \text{Tr}y(D) \) (which is independent of \( \tau \)). This implies the surjectivity of the map \( (4) \) and our first statement.

Now we assume that \( \mathcal{F} \) is non critical. By non criticality of \( \mathcal{F}, h_\tau \in B(L) w_0 \). Let \( c \in \mathcal{C}_n \) and let \( \sigma \in \mathfrak{S}_n \) be such that \( (c B(L), 0, h_\tau B(L)) \in U_\tau \). We claim that \( w_0 \sigma^{-1} \) is a product
of distinct simple reflections. Namely it follows from Lemma 3.3 that $e^{-1}w_0 \leq \sigma$ so that $w_0 \sigma^{-1} \leq e$ and, as $e$ is a product of simple reflections, so is $w_0 \sigma^{-1}$. This proves part (i).

Moreover as $h = B(L)w_0$, we can choose $w_0 = 1$ by Lemma 2.1. This implies that the surjectivity of (5) is true with $S = \{cF \mid c \in \mathcal{C}_n\}$ for all $\tau \in \Sigma$. This proves the surjectivity of (4) for this $S$ and part (ii).

4. A remark on signs

By a theorem of Artin, all elements of order 2 are conjugate in $G_{\mathbb{Q}}$. Let $C_{\mathbb{Q}} \subset G_{\mathbb{Q}}$ be their conjugacy class and let $H \subset G_{\mathbb{Q}}$ be the closed subgroup generated by $C_{\mathbb{Q}}$. There is a unique continuous morphism $\varepsilon : H \rightarrow \{\pm 1\}$ such that $\varepsilon(c) = -1$ for all $c \in C_{\mathbb{Q}}$. Let $K$ be a number field. Then $K$ is totally real if and only if $H \subset G_{K}$. Let $E$ be a CM field with totally real subfield $F$, we have $H_1 := \ker \varepsilon \subset G_E$ and $G_E = G_E H$. Let $c \in C_{\mathbb{Q}}$. We can consider the action of $c$ on $G_E$ by conjugacy, and we have $G_E = G_E \rtimes \{1, c\}$. If $\rho$ is a morphism of $G_E$ in some group and $c \in C_{\mathbb{Q}}$, we set $\rho^c = \rho(c^{-1} \cdot c) = \rho(c \cdot c)$. The $c$-conjugacy induces an automorphism $c$ of $G_E^{ab}$. As $H_1 \subset G_E$, this automorphism does not depend on the choice of $c \in C_{\mathbb{Q}}$.

**Lemma 4.1** — Let $\chi : G_F \rightarrow \overline{\mathbb{Q}}_p^\times$ be a continuous character. Then $\chi|_{G_E} = (\chi|_{G_E})^c$. If there exists a continuous character $\psi : G_E \rightarrow \overline{\mathbb{Q}}_p^\times$ such that $\psi^c = \chi|_{G_E}$, then the element $\chi(c)$ for $c \in C_{\mathbb{Q}}$ does not depend on the choice of $c \in C_{\mathbb{Q}}$. Conversely if we assume that $\chi$ is algebraic and that $\chi(c)$ does not depend on the choice of $c \in C_{\mathbb{Q}}$, then there exists an algebraic continuous character $\psi : G_E \rightarrow \overline{\mathbb{Q}}_p^\times$ such that $\psi^c = \chi|_{G_E}$. If moreover $\chi$ is unramified outside a finite set $S$ of places of $F$, we can assume that $\psi$ is unramified outside $S$.

**Proof.** — The first statement is clear as $\chi(G_F)$ is abelian. Assume that $\chi|_{G_E} = \psi^c$ for some $\psi$ and let $c_1$ and $c_2$ be two elements of $C_{\mathbb{Q}}$. Then

$$
\chi(c_1 c_2^{-1}) = \chi(c_1 c_2) = \psi(c_1 c_2^2) \psi(c_1 c_2 c) = \psi(c_1 c_2) \psi(c_1 c_2 c) = \psi^{-1}(c_1 c_2) \psi(c_1 c_2 c) = 1.
$$

The last statements are then direct consequences of [CHT08, Lem. 4.1.1 & 4.1.4].

Note that a continuous morphism $\chi : G_E \rightarrow G$ extends to a morphism $G_F \rightarrow G$ if and only if $\chi = \chi^c$. The fact that the extension of $\chi$ to $G_F$ satisfies the assumptions of Lemma 4.1 depends only on the restriction of $\chi$ to $G_E$ and is equivalent to the fact that $\chi|_{H_1}$ is trivial.

Let $\chi : G_F \rightarrow \overline{\mathbb{Q}}_p^\times$ be a continuous character. We say that a continuous representation $\rho : G_F \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_p)$ is polarized by $\chi$ if

$$
\rho^c \simeq \rho^c \otimes (\chi|_{G_E} e^{n-1}).
$$

If $\rho$ is irreducible then we can define its sign, with respect to $\chi|_{G_E} e^{n-1}$, $\lambda \in \{\pm 1\}$ as in [BCII, 1.1]. This is the sign of the pairing appearing in (6).

Fix an isomorphism $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_p$. Let $\rho : G_E \rightarrow \text{GL}_n(\mathbb{Q}_p)$ be a semi-simple continuous representation. Assume that there exists a cuspidal regular algebraic automorphic representation $\Pi$ of $\text{GL}_n(\mathbb{A}_E)$ such that $\rho$ is strongly associated with $\Pi$ (see Definition A.1).
with respect to \( \iota \). We will say that \( \Pi \) is polarized by \( \chi \) if the pair \((\Pi, \chi \circ \text{Art}_F)\) is polarized in the sense of [BLGGT14b, 2.1], which means \( \Pi^c \simeq \Pi^c \otimes (\iota^{-1} \circ \chi|_{G_E} \circ \text{Art}_F) \). Then \( \rho \) is polarized by \( \chi \) and \( \rho \) satisfies the properties of [BGGT14, Thm. 2.1].

The following result is essentially [BCII, Thm. 1.2].

\[ \text{Theorem 4.2 (Bellaïche-Chenevier).} \quad \text{Assume that} \quad \Pi \quad \text{is conjugate self-dual and regular algebraic and that} \quad \chi|_{H_1} \quad \text{is trivial. Let} \quad \psi : G_E \longrightarrow \overline{\mathbb{Q}}_p \quad \text{be a continuous character such that} \quad \psi \psi^c = \chi|_{G_E}. \quad \text{Then every irreducible constituent} \quad \rho \quad \text{satisfies the properties of [BGGT14, Thm. 2.1].} \]

\[ \text{Proof.} \quad \text{If} \quad \psi = 1 \quad \text{this is [BCII] Theorem 1.2. We deduce the general case from [BCII] Lemma 2.1.} \]

5. Deformation spaces

Denote by \( k \) a topological field and \( O \) a complete noetherian local \( \mathbb{Z}_p \)-algebra with residue field \( k \).

Fix \( E \) a totally imaginary CM number field with maximal totally real subfield \( F \) and fix \( S \) a finite set of finite places of \( E \) containing the places above \( p \), and the ramified places of \( E \). Denote

\[ G_{E,S} = \text{Gal}(E_S/E), \]

the Galois group of the maximal unramified outside \( S \) extension \( E_S/E \).

Suppose given

\[ \overline{\rho} : G_{E,S} \longrightarrow \text{GL}_n(k), \]

a continuous semi-simple Galois representation. From now on we choose \( c \in G_F \setminus G_E \) such that \( c^2 = 1 \) a complex conjugation, denoting similarly its image in \( \text{Gal}(E/F) \). As \( O \) is a \( \mathbb{Z}_p \)-algebra, we have a continuous ring homomorphism \( \mathbb{Z}_p \longrightarrow k \) and we denote \( \overline{\chi} \) its composition with the cyclotomic character. We also assume that \( \overline{\rho} \) is polarized by \( \overline{\chi} \), i.e.

\[ \overline{\rho}^c = \overline{\rho} \otimes (\overline{\iota}^{-1} \overline{\chi}), \]

for some continuous character \( \chi : G_{E,S} \longrightarrow k^\times \) satisfying \( \chi^c = \chi \). We denote \( C_O \), or \( C \) if the context is clear, the category of artinian local \( O \)-algebras with residue field \( k \).

\[ \text{Hypothesis 5.1.} \quad \text{From now on in this section, we suppose that we are in either one of the two situations:} \]

- \( k \subset \mathbb{F}_p \) with the discrete topology, \( O \) a finite totally ramified extension of \( W(k) \)
- \( k \subset \mathbb{Q}_p \) a finite extension of \( \mathbb{Q}_p \) with its \( p \)-adic topology and in this case we set \( O = k \).

In the second case \( \overline{\iota} = \varepsilon \) is just the \( \mathbb{Z}_p^\times \)-valued cyclotomic character.

Denote by \( \text{tr} \overline{\rho} \) the Determinant (in the sense of Chenevier [Che14, Définition 1.5]) of \( \overline{\rho} \). As \( \overline{\rho} \) is semi-simple it is completely determined by \( \text{tr} \overline{\rho} \) (by [Che14, Cor. 2.13]). We fix once and for all a continuous character \( \chi : G_{E,S} \longrightarrow O^\times \) lifting \( \overline{\chi} \) and such that \( \chi^c = \chi \). We will need sometimes to assume the following hypothesis on \( \chi \).

\[ \text{Hypothesis 5.2.} \quad \text{The character} \quad \chi \quad \text{is algebraic at} \quad p \quad \text{and} \quad \chi|_{H_1} \quad \text{is trivial.} \]
Definition 5.3. — We denote by $\mathcal{F}^{\chi}_{\overline{p}}$ the functor that associates to any object $A$ of $\mathcal{C}$ the set of continuous determinants $D$ lifting $\text{tr} \overline{p}$ such that $D^\vee = D^\vee \otimes \chi^{e_n-1}$. It is pro-representable by a ring $\mathcal{R}^{\chi}_{\overline{p}}$ \cite[Prop. 3.3]{Che14}. We denote the associated formal scheme $\mathcal{X}^{\chi}_{\overline{p}} = \text{Spf}(\mathcal{R}^{\chi}_{\overline{p}})$. When $k$ is a finite field of characteristic $p$, we denote the generic fibre of $\mathcal{X}^{\chi}_{\overline{p}}$ by

$$\mathcal{X}^{\chi}_{\overline{p}} := \text{Spf}(\mathcal{R}^{\chi}_{\overline{p}})^{\text{rig}}.$$ 

If $\overline{p}$ is absolutely irreducible, this coincides with the rigid fiber of the polarized-by-$\chi$ deformation space of $\overline{p}$.

Recall that we fixed an isomorphism $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$.

Definition 5.4. — A point of $x \in \mathcal{X}^{\chi}_{\overline{p}}(\mathbb{Q}_p)$ is $GL_n$-modular\(^{(4)}\) if there exists a polarized by $\chi$ automorphic representation $\Pi$ of $\text{GL}_n(\mathbb{A}_E)$ such that $\rho_{\Pi,x} = \rho_x$.

It follows from Lemma 4.1 that there exists a finite extension $\mathcal{O}' / \mathcal{O}$ of discrete valuation rings and a continuous algebraic character $\psi_0 : G_{E,S} \longrightarrow (\mathcal{O}')^\times$. We fix such a character $\psi_0$.

We start with the following reduction to lighten slightly the notations in the rest of the text.

Lemma 5.5. — Assume $\chi$ is as before, and $\overline{p}$ satisfies $\overline{p}^\vee \simeq \overline{p} \otimes \overline{\mathbb{F}}_p^{e_n-1}$. Then there is an isomorphism, which identifies modular points,

$$\mathcal{X}^{\chi-\text{pol}}_{1-p} \otimes_{\mathcal{O}[1/p]} \mathcal{O}'[1/p] \xrightarrow{\sim} \mathcal{X}^{\chi-\text{pol}}_{\psi_0} \otimes_{\mathcal{O}[1/p]} \mathcal{O}'[1/p].$$

In particular it is enough to prove theorem 1.6 for $\chi = 1$.

Proof. — The character $\psi_0 \circ \text{Art}_E$ is automorphic as $\psi_0 : G_{E,S} \longrightarrow (\mathcal{O}')^\times$ is algebraic. Moreover, the isomorphism is given by

$$\rho \longmapsto \rho_{\psi_0}^{-1}.$$ 

This is obviously an isomorphism, and because $\psi_0 \circ \text{Art}_E$ is automorphic it identifies $(\text{GL}_n, \cdot)$-modular points on both sides. 

Our goal is to understand the geometry of the set of modular points in $\mathcal{X}^{\chi}_{\overline{p}}$ when $k \subset \mathbb{F}_p$, $\mathcal{O} = \mathcal{O}_K$, $K / \mathbb{Q}_p$ finite. However we will need some result concerning the situation where $k = \mathcal{O}$ is a finite extension of $\mathbb{Q}_p$ but for completeness we prove them in the two situations of Hypothesis 5.1. Following \cite{cht08} we introduce

$$\mathcal{G}_n = \langle \text{GL}_n \times \text{GL}_1 \rangle \rtimes \text{Gal}(E/F),$$

where $c \in \text{Gal}(E/F)$ acts on $(g,x) \in \text{GL}_n \times \text{GL}_1$ via $(x'g^{-1}, x)$. We denote $\nu$ the homomorphism $\mathcal{G}_n \longrightarrow \text{GL}_1$ sending $(g,x)$ to $x$ and $c$ to $-1$.

Finally we assume until the end of this section that $\overline{p}$ is absolutely irreducible and denote $\lambda$ its sign (as defined in section 4). Then we can extend $\chi$, which satisfies $\chi^e = \chi$, to $G_E \cong G_E \rtimes \text{Gal}(E/F)$ by setting $\chi(c) := (-1)^e \lambda$ and we extend compatibly $\overline{\chi}$ to $G_F$.

\(^{(3)}\)for $R_{\overline{p}}$ and then $R_{\overline{p}}^{\chi}=R_{\overline{p}}/I$ with $I = (D_{\text{univ, } \nu}(g) - D_{\text{univ, } \nu}(g)\chi^{e_n-1}(g), g \in G)$

\(^{(4)}\)Compare with Definition 6.1.
so that \( \mu := \chi e^{n-1} \) satisfies \( \mu(c) = -\lambda \). By [CHT08] Lemma 2.1.1 we can thus extend \( \mathfrak{p} \) to a continuous homomorphism

\[
\mathfrak{p} : G_{F,S} \rightarrow G_n(k),
\]

such that \( c \in G_F \) is sent to \( c \in \text{Gal}(E/F) \) via \( \mathfrak{p} \) and projection and \( \nu \circ \mathfrak{p} = \chi^{-1} e^{1-n} \) (as extended before to \( G_F \)).

**Definition 5.6.** — Let \( \text{Def}^\chi_n(R) \) be the functor that associates to any object \( R \) of \( \mathcal{C} \) the set \( \text{Def}^\chi_n(R) \) of lift \( r : G_{F,S} \rightarrow G_n(R) \) of \( \mathfrak{p} \) such that \( \nu \circ r = \chi^{-1} e^{1-n} \) considered up to \( 1 + m_R M_n(R) \)-conjugation. As in [CHT08, Prop. 2.2.9](5), this functor is pro-represented by a local complete noetherian \( \mathcal{O} \)-algebra \( R^\chi_n \). When \( k \) is a finite field of characteristic \( p \), we denote by \( X^\chi_n \) the generic fiber of the formal scheme \( X^\chi_n = \text{Spf}(R^\chi_n) \).

In the following, all cohomology groups are continuous cohomology groups.

**Proposition 5.7.** — Assume that \( \mathfrak{p} \) is absolutely irreducible, \( \text{char } k \neq 2 \), \( (\chi') = (-1)^n \) for all complex conjugacy \( c' \) and, if \( k \) is of characteristic \( p \), \( R^\chi_n[1/p] \neq 0 \). Then

\[
\dim(R^\chi_n[1/p]) \geq \dim_k H^1(G_{F,S}, \text{ad}(\mathfrak{p})) - \dim_k H^2(G_{F,S}, \text{ad}(\mathfrak{p})) = \frac{n(n+1)}{2} [F : \mathbb{Q}].
\]

Moreover the topological \( \mathcal{O}[1/p] \)-algebra \( R^\chi_n[1/p] \) is formally smooth of relative dimension \( \frac{n(n+1)}{2} [F : \mathbb{Q}] \) if \( H^2(G_{F,S}, \text{ad}(\mathfrak{p})) = 0 \).

**Proof.** — This appeared already in [CHT08, All16], let us give the argument. As \( \mathfrak{p} \) is absolutely irreducible, \( \text{ad}(\mathfrak{p})^c = H^0(G_F, \text{ad}(\mathfrak{p})) = 0 \) by [CHT08, Lem. 2.1.7(3)]. For each place \( v|\mathfrak{c} \), we have [[CHT08, Lem. 2.1.3]]

\[
\dim_k H^0(G_{F_v}, \text{ad}(\mathfrak{p})) = \frac{n(n+1)(c_v)}{2} = \frac{n(n-1)}{2}
\]

(where \( c_v \) is the complex conjugation in \( F_v \)).

Now the equality

\[
\dim_k H^1(G_{F,S}, \text{ad}(\mathfrak{p})) - \dim_k H^2(G_{F,S}, \text{ad}(\mathfrak{p})) = \frac{n(n+1)}{2} [F : \mathbb{Q}]
\]

follows from [CHT08, Lem. 2.3.3](6) when \( k \) is a finite field and from [All16, Lem. 1.3.4] when \( k \) is a finite extension of \( \mathbb{Q}_p \).

When \( k \) is a finite extension of \( \mathbb{Q}_p \), the result follows from the analogue of [CHT08, Cor. 2.2.12] (but without the \( +1 \) since here \( \mathcal{O} = k \)). When \( k \) is a finite field, it follows from [CHT08, Cor. 2.2.12] that

\[
\dim(R^\chi_n) \geq 1 + \frac{n(n+1)}{2} [F : \mathbb{Q}].
\]

---

(5) There the field \( k \) is finite, but we can check that everything carries over in our setting, as already remarked in [Kis09b]

(6) Note that in [CHT08, 2.3], it is supposed that the places of \( S \) are split in \( E \) but this is not used in their Lemma 2.3.3.
Let \( x \in \text{Spec}(R_F^\chi[1/p]) \) be a closed point, \( p_x \) the corresponding prime ideal and \( r_x : G_F \to G_n(k(x)) \) the corresponding representation. It follows from [All16, Prop. 1.3.11(1)] that the localization-completion of \( R_x^\chi \) at \( p_x \) is isomorphic to \( R_{x_q}^\chi \). It follows that
\[
\dim(R^\chi_F[1/p]) \geq \dim(R^\chi_x) \geq \frac{n(n+1)}{2}[F : Q]
\]
using the case where \( k \) has characteristic 0.

The assertion concerning the formal smoothness follows from [CHT08, Cor. 2.2.12] and [All16, Prop. 1.3.11]

**Remark 5.8.** — The hypothesis on the sign of \( \chi \) in Proposition 5.7 will be satisfied in the rest of the text where we choose \( \chi = \psi \psi_0^\epsilon \) for some \( \psi_0 : G_E \to \hat{\mathbb{Q}}_p^\times \), actually because of the sign theorem 4.2. Indeed, by [CHT08] Lemma 2.1.1 we need to extend \( \mu = \chi^\epsilon \) to \( G_{E,S} \), for an absolutely irreducible \( \rho \), by sending \( c \) to \(-\lambda \). Thus the previous hypothesis is equivalent to \( \lambda = 1 \).

**Proposition 5.9.** — Denote \( \bar{\rho} \) as before. Suppose it is absolutely irreducible, and denote \( \bar{\tau} \) the chosen \( G_n \)-extension as before. Suppose \( \text{Char}(k) \neq 2 \). Then the natural map
\[
R^\chi_F \longrightarrow R^\chi_{\bar{\tau}}^{\text{pol}}
\]
is an isomorphism.

**Proof.** — This is also [All19, Prop. 2.2.3]. Denote \( \rho, \rho' \) resp. \( R, R' \) valued points of \( F^{\chi-\text{pol}}_R \), with \( R' \to R \), and \( \rho' \otimes R = \rho \). Suppose we have a fixed pairing \( <,> : \rho \otimes \rho^c \to \chi^{-1} \epsilon^{-1-n} \) inducing \( r : G_{E,S} \to G_n(R) \) by [CHT08] Lemma 2.1.1. Choose any pairing fixing \( <,>_{\rho'} \) for \( \rho' \). Then reducing to \( R \) this gives a pairing for \( \rho \), but as \( \bar{\rho} \) is absolutely irreducible, \( \rho \) is also and thus there is only one pairing up to scalar for \( \rho \), i.e. \( <,>_{\rho'} \otimes R = \alpha <,> \) for some \( \alpha \in R^\times \). Choose a lift \( \beta \) of \( \alpha^{-1} \), then set \( <,>':= \beta <,>_{\rho'} \), then \( <,> ' \) reduces to \( <,> \) and to \( (\rho', \chi, <,>) \) is associated by [CHT08] Lemma 2.1.1 an \( r' : G_{E,S} \to G_n(R') \), reducing to \( r \). Let \( r'' \) another point over \( R' \), inducing \( \rho' \) and reducing to \( r' \), then it corresponds to \( \gamma <,> ' \) with \( \gamma \equiv 1 \pmod{m_{R'}} \), thus writing \( \gamma = 1 + m \) with \( m \in m_{R'} \) we have \( \gamma = (1 + \frac{1}{2} m)^2 \pmod{m_{R'}^2} \) and as \( R'' \) is artinian, a direct induction shows that \( \gamma \) is a square in \( R'' \), thus \( r' = r'' \).

The same argument with \( \tau \) for \( r \) and \( r' \) for \( r'' \) shows that we can actually choose \( r \) inside \( X^\chi_{\bar{\tau}} \) (and thus automatically for any \( r' \) above) and thus proves etaleness, and surjectivity. As the map is an isomorphism in special fiber, this is an isomorphism.

By [CHT08, 2.1], the adjoint action \( ad(r) \) for some \( r : G_F \to G_n(k) \) coincides with the extension of \( ad(\rho) \) to \( G_F \) as in [NT19] (when \( \chi = 1 \), when \( \rho \) corresponds to \( r \) by [CHT08, Lemma 2.1.1]). Thus we will often denote abusively \( H^1(G_F, ad(\rho)) \) instead of \( H^1(G_F, ad(r)) \).

---

\[\text{(7) [CHT08]}\] is written over a field, but their proof applies here.
6. Eigenvarieties and the infinite fern

There are at least two ways to define Eigenvarieties as explained in [BC09] which, at least, in our case of interest end up to be the same. By the Lemma 5.5, we can assume \( \chi = 1 \).

In this section \( (\mathcal{O}, k) \) are as in the first case of Hypothesis 5.1. We fix \( \overline{\mathcal{O}} : G_E \rightarrow \text{GL}_n(k) \) a semi-simple polarised-by-1 continuous representation, i.e. \( \overline{\mathcal{O}}^\vee \cong \overline{\mathcal{O}} \otimes \pi^{n-1} \). Let \( \mathcal{X}_\mathcal{O}^{\text{pol}} := \mathcal{X}_\mathcal{O}^{1-\text{pol}} \) be its polarised pseudodeformation space. Let \( G \) be the quasi-split similitude unitary group of dimension \( n \) over \( \mathbb{Q} \) whose \( R \)-points, for \( R \) a \( \mathbb{Q} \)-algebra, are:

\[
G(R) = \{(g, \nu) \in \text{GL}_n(R \otimes \mathbb{Q} E) \times R^\times \mid \nu(c(g)J g = \nu J)\}
\]

where \( J \) is the \( n \times n \) matrix \( \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix} \). Moreover let \( G_1 \) be the kernel of the morphism \( \nu : G \rightarrow \mathbb{G}_m \). As \( p \) is unramified in \( E \), we also fix a reductive model \( \mathbb{G}_m \) over \( \mathbb{Z}_p \) of \( G \) defined by the similar formula (replacing \( R \otimes \mathbb{Q} E \) by \( R \otimes \mathbb{Z}_p \mathcal{O}_E \)).

We fix embeddings \( \overline{\mathbb{Q}} \hookrightarrow \mathbb{C} \) and \( \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \) that we use to identify the embeddings of \( E \) (resp. \( F \)) in \( \overline{\mathbb{Q}}_p \) with the set \( \Sigma_E \) (resp. \( \Sigma_F \)) of embeddings of \( E \) (resp. \( F \)) in \( \mathbb{C} \). We fix a CM type \( \Phi \) for \( E \). For \( \sigma \in \Sigma_E \), we use the notation \( \sigma = \sigma \circ c \). If \( \tau \in \Sigma_F \), let \( \sigma_\tau \in \Sigma_E \) be the unique element such that \( \tau = \sigma_\tau |_F \) and \( \sigma_\tau \in \Phi \).

We fix a PEL datum \((E, c, V, \langle \cdot, \cdot \rangle, h)\) for the previous group \( G \) and denote its signature \( (p_{\sigma_\tau}, q_{\sigma_\tau})_{\tau \in \Sigma_F} \) at infinity\(^{(8)}\). In particular we have that \( p_{\sigma_\tau} + q_{\sigma_\tau} = n \) doesn’t depend on \( \tau \). We define more generally \( (p_{\sigma}, q_{\sigma})_{\tau \in \Sigma_F} \) by \( p_{\sigma} = p_{\sigma_\tau} \) if \( \sigma = \sigma_\tau \) and \( p_{\sigma} = q_{\sigma_\tau} = n - p_{\sigma_\tau} \) if \( \sigma = \sigma_\tau \). We also sometimes abuse notation and write \( p_{\tau}, q_{\tau} \) for \( p_{\sigma_\tau}, q_{\sigma_\tau} \). Let \((G, h)\) be a Shimura datum associated to \( G \). We let \( \mathcal{S} = (S_K)_K \) be the tower of Shimura varieties for \((G, h)\) ([Lan13] or [Her22] which we will use later). Let \( \mu : \mathbb{G}_m \rightarrow \mathbb{G}_C \) be the cocharacter associated to \( h \) and let \( \overline{P} \) be the parabolic subgroup fixing the Hodge filtration associated to \( \mu \). Let \( M \) be the Levi subgroup of \( \overline{P} \) fixing the Hodge decomposition of \( V_C \) (defined over some extension \( L \) of the reflex field). Let \( \mathfrak{p} \) be the Lie algebra of \( \overline{P} \) and let \( K_\mathfrak{p} \) be the centralizer of \( \mathfrak{h}(i) \) in \( G_1(\mathbb{R}) \).

**Definition 6.1.** We say that a polarised-by-1 representation

\[
\rho : G_E \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_p),
\]

is **modular** if \( \rho \) is (strongly essentially) associated to a cuspidal algebraic automorphic representation \( \pi \) for \( G \) as in Definition A.3. We say that \( \rho \) is **holomorphically modular** if its Hecke eigensystem appears in the space of cuspidal sections of some coherent automorphic sheaf on some Shimura variety of \( \mathcal{S} \). This is equivalent to the fact that \( \pi \) is cuspidal and holomorphic at infinity; i.e. \( H^0(p, K_\mathfrak{p}, \pi_{\mathfrak{p}} \otimes \sigma) \neq 0 \) for some algebraic representation \( \sigma \) of \( K_\mathfrak{p} \)\(^{(9)}\) by [Har90b] Proposition 5.4.2.

\(^{(8)}\)Because \( G \) is quasi-split these integers are explicit, but we keep the slightly general notation as we think it is a bit clearer.

\(^{(9)}\)These hypothesis are here to assure a concrete (= computable) way to verify if our \( \pi \) "appears" in an Eigenvariety. We could introduce the notion of \( p \)-adically modular for which we ask for a Hecke eigensystem appearing in the considered Eigenvariety \( \mathcal{E} \) whose associated trace is \( tr \rho \). It is enough to assume \( p \)-adic modularity for \( \overline{\mathcal{O}} \) to get Theorem 1.6.
We say that \( \mathfrak{p} \) is modular if it admits a lift \( \rho \) which is modular. We say that \( \mathfrak{p} \) is \textit{conveniently modular} if it has a lift \( \rho \) associated to a cuspidal automorphic representation \( \pi \) which can be chosen unramified at \( p \) and outside \( S \) and its Hecke eigensystem appears in \( i \)-th interior coherent cohomology group on some Shimura variety of \( S \), with values in some coherent automorphic sheaf, for some \( i \geq 0 \).

If \( K^p \) is a compact open open subgroup of \( G(\mathbb{A}^p, \infty) \), we say that \( \mathfrak{p} \) is \textit{conveniently modular of tame level} \( K^p \) if \( \pi \) can be moreover chosen such that \( \pi^{K^p} \neq 0 \).

\textbf{Hypothesis 6.2.} — For the rest of the article, we assume that every \( v \mid p \) in \( F \) is unramified, and splits in \( E \). Moreover we assume that \( \mathfrak{p} \) is conveniently modular.

In particular, if \( v \) is a place of \( F \) dividing \( p \), among the two places \( w, \overline{w} \) of \( E \) above \( v \) only one, say \( w \), corresponds to an element of \( \Phi \). We fix this choice, which allows us to identify \( E_w \) with \( F_v \). Choose a sufficiently large \( p \)-adic field \( L \) such that \( M \) and \( P \) are defined over \( L \) and \( L \) splits \( E \), i.e. \( E \otimes_{\mathbb{Q}_p} L = \prod_{v \mid p \in \mathcal{E}} \mathbb{Q}_v \). Let \( T \) be the rigid space over \( L \) given by \( \prod_{v \mid p} \text{Hom}((F_v^\times)_w, G_m) \), and \( W = \prod_{v \mid p} \text{Hom}((O_{F_v})^\times, G_m) \) the weight space. There is thus a restriction map \( \mathcal{T} \rightarrow W \).

From now on we fix a finite field \( k \) of characteristic \( p \) and a continuous semisimple representation \( \mathfrak{p} : G_E \to \text{GL}_n(k) \) which polarized by \( \chi^{p-1} \) and is conveniently modular. We fix a tame level outside of \( p \), \( K^p \), which is hyperspecial outside \( S \) and deep enough so that \( \mathfrak{p} \) is conveniently modular of tame level \( K^p \).

Let \( Z_{K^p} \subset \mathcal{X}_\mathfrak{p}(\mathbb{Q}_p) \times T(\mathbb{Q}_p) \) the set of pairs \( (D, \delta) \) where \( D \) is the determinant associated to (the Galois representation of) a cuspidal, regular, algebraic, unramified at \( p \) automorphic form \( \pi \) for \( G \) appearing in degree 0 coherent cohomology \(^{(10)}\) by Corollary A.9, of level \( K^p \) outside \( p \), of Hodge-Tate weights \( k_{v, \tau, 1} > k_{v, \tau, 2} > \ldots > k_{v, \tau, n} \) \(^{(11)}\) for each \( v \mid p \) in \( F, \tau \), and \( \delta \) such that for all \( v, i \), \( \delta_{v, \tau, i} \) coincides on \( O_{F_v}^\times \) with \( \prod_{\tau} \tau^{k_{v, \tau, i}} \) and sends \( p \) to \( \phi_v \), where \( \phi_v, 1, \ldots, \phi_v, n \) is an admissible refinement for \( \pi_v \) (and obviously such that \( D \) lifts \( \mathfrak{p} \)).

\textbf{Remark 6.3.} — By [Box15] Theorem D, [PS16] or [GK15] Theorem I.3.1, we have that, under the hypothesis 6.2, \( Z_{K^p} \) is non empty, i.e. we can choose a lift of \( \mathfrak{p} \) that is holomorphically modular.

\textbf{Definition 6.4.} — The Eigenvariety for \( G, \mathfrak{p}, \chi = 1 \) and \( K^p \) is the Zariski closure \( \mathcal{E}_{K^p}(\mathfrak{p}) \subset \mathcal{X}_\mathfrak{p}(\mathbb{Q}_p) \times T \), of \( Z_{K^p} \). The infinite fern \( \mathcal{F}_{K^p}(\mathfrak{p}) \) is the image of \( \mathcal{E}_{K^p}(\mathfrak{p}) \) by the first projection.

As \( G \) is a unitary similitude group with similitude in \( \mathbb{Q}_p \), thus giving rise to a PEL Shimura datum, and \( p \) is unramified in \( E \), we also constructed in [Her22] an Eigenvariety

\(^{(10)}\)this means that \( H^0(K_\infty, \pi_v \otimes V) \neq 0 \) for a finite dimensionnal representation of \( K_\infty \). In particular \( D \) is holomorphically modular.

\(^{(11)}\)We choose the convention for which the cyclotomic character has Hodge-Tate weight +1

\(^{(12)}\)for these local data at \( p \) we have used the implicit choice of \( w/v \)
for $G$, for any type $K^p$ outside $p$. Actually these two constructions compare, and allow us to deduce the following proposition.

**Remark 6.5.** — Actually we could take $G$ any similitude unitary group with similitude factor in $\mathbb{Q}$ instead of the quasi-split one. Indeed, as long as $p$ is unramified for $G$ the construction of [Her22] applies and we get the following proposition. In particular, if we have a result analogous to [Her22] for ramified primes (i.e. primes $v|p$ in $F$ which are ramified, but still assuming $v = w\mathfrak{m}$ in $E$) then all the methods of this article applies (see [BP20]). For the moment, we still need our $p$-adic group to be (a product of) $GL_n$ to use results on the trianguline variety, but we hope to come back on this question in the future.

**Proposition 6.6.** — The rigid space $\mathcal{E}_{K^p}(\overline{\rho})$ is equidimensional of dimension $n[F: \mathbb{Q}]$. The map

$$h : \mathcal{E}_{K^p}(\overline{\rho}) \rightarrow \mathcal{W},$$

is locally, on the goal and the source, finite. In particular the image of any irreducible component of $\mathcal{E}_{K^p}(\overline{\rho})$ is open in $\mathcal{W}$. Moreover, for all $C > 0$, if $\mathcal{Z}_C \subset \mathcal{Z}_{K^p}$ consists of classical points, crystalline at $p$, which are moreover $C$-very regular (i.e. its Hodge-Tate weights satisfies $k_{v,\tau,i} > k_{v,\tau,i+1} + C$ for all $v, \tau, i$), then $\mathcal{Z}_C$ is Zariski dense in $\mathcal{E}_{K^p}(\overline{\rho})$ and accumulation at every point of $\mathcal{Z}_C$.

We need to introduce a few notations. Let $T$ be the diagonal torus of $G_{\mathbb{Z}_p}$. Its group of $\mathbb{Q}_p$-points has the following description,

$$T(\mathbb{Q}_p) = \left\{ \begin{pmatrix} a_{1,w} & \cdots & a_{n,w} \\ \vdots & \ddots & \vdots \\ a_{w,1} & \cdots & a_{w,n} \end{pmatrix} \in \prod_{w|p \in E} GL_n(E_w), \exists r \in \mathbb{Q}_p^\times, a_{i,w}a_{n-i+1,w} = r, \forall i, w \right\} .$$

Denote by $T^1$ the subtorus with trivial similitude character (i.e. $r = 1$). We identify $T$ with the space of characters of $T^1(\mathbb{Q}_p)$ using the isomorphism $(F_v^\times)^n \cong T^1(\mathbb{Q}_p)$ sending $(a_{1,v}, \ldots, a_{n,v})$ to the diagonal matrix of $GL_n(E_v)$ with diagonal $(a_{1,v}, \ldots, a_{n,v})$, via the identification $E_v \cong F_v$ where $w \mid v$ and $w \in \Phi$. $T(\mathbb{Q}_p)$ (resp. $T^1(\mathbb{Q}_p)$) can be identified also with a subgroup of the $L$-points of the torus (resp. subtorus of $r = 1$ elements) of $M \cong G_m \times \prod_{v|p \in F} \prod_{w \mid v} GL_{p_{\sigma}} \times GL_{q_{\tau}}$, using,

$$\begin{pmatrix} a_{1,w} \\ \vdots \\ a_{w,n} \end{pmatrix}_w \mapsto \left( r, \begin{pmatrix} \tau(a_{w,1})^{-1} \\ \vdots \\ \tau(a_{w,n})^{-1} \end{pmatrix}, \begin{pmatrix} \tau(a_{1,w})^{-1} \\ \vdots \\ \tau(a_{m,w})^{-1} \end{pmatrix} \right) .$$

**Definition 6.7.** — Let $\kappa = (\kappa_{\sigma,i})_{\sigma \in \Sigma_E, 1 \leq i \leq p_{\sigma}} \in \mathbb{Z}_n[F: \mathbb{Q}]$. We say that a character $\chi \in \mathcal{W}(\mathbb{C}_p)$ is algebraic of coherent weight $\kappa$ if

$$\forall z = (z_{v,i}) \in \prod_{v|p} (O_{F_v}^\times)^n, \quad \chi(z) = \prod_{v|p} \prod_{i=1}^{p_{\sigma} - p_{\sigma_{\tau}}} \sigma_\tau(z_{v,i})^{k_{\sigma,i}} \prod_{i=1}^{q_{\tau} - p_{\sigma_{\tau}}} \sigma_\tau(z_{v,i+1})^{-k_{\sigma,i+1}} .$$

We say that an algebraic character of coherent weight $\kappa$ is $M$-dominant if $k_{\sigma,i} \geq k_{\sigma,i+1}$ for $\sigma \in \Sigma_E$ and $1 \leq i \leq p_{\sigma} - 1$. 


This corresponds to the choice of the upper triangular Borel for \( M \), in the sense that if we have character \( \kappa \) of \( M \) which is dominant for the upper Borel of \( M \), then its restriction to \( T^{-1}(Q_p) \) via the previous embedding gives a \( M \)-dominant \( \chi \) in the previous sense. Suppose \( \chi \) is algebraic for some coherent weight \( \kappa \). For \( h = (h_{r,i})_{r \in \Sigma_E, 1 \leq i \leq n} \in \mathbb{Z}^n[F : \mathbb{Q}] \), if

\[
\chi(z) = \prod_{\tau \in \Sigma_E} \sigma_\tau(z)^{h_{r,i}},
\]

then we say that \( \chi \) is of infinitesimal weight \( h \). We say that such a \( \chi \) is dominant (or \( G \)-dominant) if \( h_{r,1} \geq h_{r,2} \geq \cdots \geq h_{r,n} \) for all \( \tau \).

**Proof.** — Let \( \mathcal{E}' \) together with a locally finite map \( w : \mathcal{E}' \to \mathcal{W} \) the Eigenvariety for \( G \) of tame level \( K^p \) constructed in [Her22]. It is an equidimensional rigid space of dimension \( n[F : \mathbb{Q}] \).

Let \( S \) be a finite set of places of \( F \) containing the places dividing \( p \) and the places where \( K^p \) is not hyperspecial. For \( v \notin S \), let \( \mathcal{H}_v \) be the spherical Hecke algebra \( \mathbb{Z}[G(F_v)/K_v^p] \) of \( G \) and let \( \mathcal{H}_S = \bigotimes_{v \in S} \mathcal{H}_v^e \). For \( v \mid p \), let \( \mathcal{A}_v \) be the (commutative) \( \mathbb{Z} \)-algebra generated by \( T_v = (F_v) \) and their inverses with \( T_v \) the diagonal torus of \( GL_n \), and \( T_v^v = (F_v) \) the subgroup of matrices \( \text{Diag}(p^{a_1}, \ldots, p^{a_n}) \) with \( a_i \geq a_{i+1} \). Let \( \mathcal{A}(p) = \bigotimes_{v \mid p} \mathcal{A}(v) \) be the Atkin-Lehner algebra. It follows from [Her22, §7] \(^{14} \) that there exists homomorphism \( \lambda : \mathcal{H}_S \to \Gamma(\mathcal{E}', \mathcal{O}_{\mathcal{E}}) \) and \( \mathcal{A}(p) \to \Gamma(\mathcal{E}', \mathcal{O}_{\mathcal{E}}) \), sending \( \text{Diag}(1, \ldots, 1, p^{-1}, \ldots, p^{-1}) \) to a Hecke operator \( U_{v,i} \) at \( v \), such that, if \( z \in \mathcal{E}'(\mathbb{C}_p) \), the evaluation of these morphisms at \( z \) induces a non-zero eigenspace in \( H^0(S_{K^p}^t(v), \omega^\kappa(z)^{-1}(-D)) \), with \( K = K^p I \), and \( I \) a Iwahori subgroup at \( p \), which is a space of overconvergent cuspidal forms, defined in [Her22] Definition 6.12. Moreover if \( \kappa \in \mathcal{W}(\mathbb{C}_p) \) is an algebraic character of \( M \)-dominant coherent weight, then the action of \( \mathcal{H}_S \) preserves the subspace of classical forms \( H^0(S_{K^p}^t, \omega^\kappa(-D)) \) and coincides with the “usual” action of \( \mathcal{H}_S^e \) on \( H^0(S_{K^p}^t, \omega^\kappa(-D)) \).

We remind now that \( \mathcal{E}' \) contains an accumulation and Zariski dense subspace of automorphic points that we will call very regular small slope classical points.

Let \( Z \subset \mathcal{E}'(\mathbb{C}_p) \) be the set of points \( z \) satisfying [Her22] Proposition 8.2, a slightly stronger form of Theorem 8.3, namely

\[
\max_{\tau} (n_{\tau} + v_p(\alpha_\tau), 0) < \inf_{\tau} (k_{\sigma, p_r} + k_{\sigma, q_r}), \quad \forall \tau \in \Sigma_E
\]

where \( w(z) \) is (thus) a \( G \)-dominant algebraic character of coherent weight \( (k_{\sigma, p_r}, k_{\sigma, q_r}) \) and \( n_\tau \) is the eigenvalue for the operator \( U_{v, \min(p_r, q_r)} \), and \( n_\tau \) is a normalisation constant depending only on \( (p_r, q_r) \), asking that \( w(z) \) is moreover far from the walls as in [Har90a] Lemma 3.6.1. For \( C > 0 \), we define \( Z_C \subset Z \) adding the condition \( k_{\sigma, i} - k_{\sigma, i+1} > C \) for all \( i \). For \( C \gg 0 \), these points give rise to crystalline representations at \( p \) as we will see. Each of the sets \( Z_C \) is accumulation in \( Z \), we can thus prove the claim with \( Z_C \) replaced by \( Z \).

By [Bij16] (see [Her22, Thm 9.4]), if \( z \in Z \), the system of eigenvalues corresponding to \( z \) has an eigenvector in \( H^0(S_{K^p}^t, \omega^\kappa(-D)) \). This implies that actually \( z \in \mathcal{E}'(\overline{\mathbb{Q}}_p) \). It follows

\(^{13}\) This just means that \( k_{\sigma, i} = h_{r,i}, i \leq p_r \) and \( -k_{\sigma, i} = h_{r,n+1-i} \) for \( i > p_r \).

\(^{14}\) There \( \mathcal{H}_S^e \) is denoted \( \mathcal{H}_{K^p} \). See also remark 7.12 of [Her22]
from [SU02], or [Har90a], that there exists a cuspidal automorphic form $\pi$ of $G(\mathbb{A}_F)$ such that $\pi_{\mathfrak{f}^*} \neq 0$, the Satake parameter of $\pi_v$, $v \mid \mathfrak{f}$, corresponds to $\lambda|_{\mathcal{H}_v} \otimes k(z)$ and $\pi_\mathfrak{f}$ is tempered of weight $((k_{\mathfrak{f},\mathfrak{p}_1}, \ldots, k_{\mathfrak{f},\mathfrak{p}_r}, k_{\mathfrak{f},\mathfrak{q}_1}, \ldots, k_{\mathfrak{f},\mathfrak{q}_s}))_{\mathfrak{f} \in \Sigma_F} - \rho_G = -\tau_0 M \rho_G$, with $\rho_G$ the half-sum of positive roots, as we will explain.

To be able to clearly label the weights, let $P_{\mathfrak{h}}$(15) be the parabolic corresponding to $\mathfrak{p}$, and choose a Borel, equivalently a set $\Phi^+$ of positive roots such that if $\Phi_\mathfrak{h}^+$ is the set of positive roots contained in the Levi of $\mathfrak{p}$, then $\Phi_{\mathfrak{h}^c}^+ := \Phi^+ \setminus \Phi_{\mathfrak{h}}^+$ is chosen to be included in $\mathfrak{g}_C / \mathfrak{p}$. Equivalently $B \subset P_{\mathfrak{h}}^{\mathfrak{pp}} = P_{\mathfrak{h}}$, the parabolic opposite to $P_{\mathfrak{h}}$. This allows us to label similarly (classical, dominant) weights of representations of $K_{\mathbb{C}}$ (with respect to $\Phi_\mathfrak{h}^+$) and of $G$ (with respect to $\Phi^+$). Let us be more precise for these choices. Let $(\mathcal{G}_\mathfrak{p})_0$ our unitary group, thus given by the hermitian form $\langle \cdot , \cdot \rangle$, see as the Levi for half-sum of positive roots in $\mathfrak{g}_{\mathbb{C}}$. We choose the diagonal torus and upper Borel for $\mathfrak{g}$, is constructed using the previous upper triangular Borel, then the algebraic representation of highest weight $\rho_{\mathfrak{h}}(\mathfrak{p})$ of coherent weight $\rho_{\mathfrak{h}}$, if and only if $k_{\mathfrak{f},\mathfrak{p}_i} \geq -k_{\mathfrak{f},\mathfrak{q}_j}$.

Let $z \in \mathbb{Z}$, which corresponds to a classical automorphic form (itself giving an automorphic representation $\pi$) appearing in $H^0(S^G_{\mathfrak{h}}(\mathfrak{g}_{\mathbb{C}}^*), \omega^\kappa)$ for $\kappa = (k_{\mathfrak{f},i})_{\mathfrak{f} \in \Sigma_F, 1 \leq i \leq \mathfrak{r}_F}$, as before, which is a classical (and $M$-dominant) weight. Then $\chi = w(z) \in \mathcal{W}$ is the algebraic character of weight $\kappa$, i.e. is $w_0 M \kappa \circ \iota$. This sheaf $\omega^\kappa$ coincides with the coherent sheaf $V_\kappa$ over $\mathbb{C}$ defined by Harris ([Har90b]), associated to the highest weight $s$ representation of $M = \prod_{\mathfrak{f} \in \Phi} GL_{\mathfrak{p}_i} \times GL_{\mathfrak{q}_j}$, with $s = (-k_{\mathfrak{f},\mathfrak{p}_1}, \ldots, -k_{\mathfrak{f},\mathfrak{p}_r}, k_{\mathfrak{f},\mathfrak{q}_1}, \ldots, k_{\mathfrak{f},\mathfrak{q}_s})$ with the previous identifications. This calculation is the one done in [FP19] section 7.4, based on [Gol14].

(15) Also called $P_{\mathfrak{h}}^{\mathfrak{ppd}}$ in other references.

(16) This is actually the subgroup of $M$ of element with similitude factor 1, but in all this discussion we ignore the similitude factor to simplify the notations.
Remark that if \( \chi \) is algebraic of weight \( \kappa \) and dominant, then the dominant representative of \(-s\) is given by \( wt(\chi) = (k_{\sigma,1}, \ldots, k_{\sigma,p}, -k_{\pi,q}, \ldots, -k_{\pi,1}) \). In particular as the Hecke eigensystem corresponding to \( z \) appears in \( H^0(S^1_{tor}, \omega^s) \) thus in \( H^0(S^1_{tor}(\mathbb{C}), V_z) \) this means that
\[
H^0(p, K_{z}, \pi_Z \otimes V_z) \neq \{0\},
\]
i.e. that the infinitesimal character of \( \pi_Z \) is \( -s - \rho_G \) (up to reordering) by e.g. [Har90b] Proposition 4.3.2 (see also [BP20] Proposition 5.37). But if \( z \in \mathcal{Z} \) and \( \pi \) an automorphic representation corresponding to its system of eigenvalues \( \lambda(z) \) of \( \mathcal{H}^Z \), then using that \( w(z) \) is far from the walls, by Corollary A.9 there is a semisimple representation \( \rho^u = \rho^u_z : G_E \to GL_n(\overline{\mathbb{Q}}_p) \) such that \( \rho^u(\text{Frob}_v) \) is associated to the semi-simple conjugacy class at \( v \) determined by \( \lambda \), for all \( v \notin S \) and satisfying moreover
\[
(\rho^u)^\vee \simeq (\rho^u)^c \otimes \epsilon^{n-1}.
\]
By for example [BGGT14], the previous calculation of the infinitesimal weight means that \( \rho^u \), associated to \( \pi \), has Hodge-Tate weights given by \( -s - \rho_G - \frac{n-1}{2} \) i.e. the \( \upsilon, \tau \) Hodge-Tate weights of \( \rho^u \) are (up to order)
\[
(k_{\upsilon,\tau,p} + 1 - n, k_{\upsilon,\tau,p} + 2 - n, \ldots, k_{\upsilon,\tau,1} + p_{\tau} - n - k_{\upsilon,\tau,1} + p_{\tau} - n + 1, \ldots, -k_{\upsilon,\tau,q} )
\]
which we can reorder to be dominant for very regular \( z \in \mathcal{Z} \),
\[
(k_{\upsilon,\tau,1} + (p_{\tau} - n), k_{\upsilon,\tau,2} + (p_{\tau} - n - 1), \ldots, k_{\upsilon,\tau,p} + 1 - n - k_{\upsilon,\tau,1} + (p_{\tau} - n + 1)).
\]
We first construct a union of connected components of \( \mathcal{E}' \) and a map from this subspace to \( \chi^\vee_{\mathbb{F}} \). As in [Che04], we construct a determinant
\[
D : G_E \longrightarrow \mathcal{O}_{\mathcal{E}'}.
\]
Let \( z \in \mathcal{Z} \) and \( \pi \) an automorphic representation corresponding to its system of eigenvalues \( \lambda(z) \) of \( \mathcal{H}^Z \), as we have seen, by Corollary A.9 there is a semisimple representation \( \rho^u : G_E \to GL_n(\overline{\mathbb{Q}}_p) \) associated to \( \lambda(z) \) such that
\[
(\rho^u)^\vee \simeq (\rho^u)^c \otimes \epsilon^{n-1}.
\]
Let \( D_z \) the pseudo-representation of \( \rho^u_z \). The continuous map \( (D_z)_{z \in \mathcal{Z}} \) from \( G_E \) to \( \prod_{z \in \mathcal{Z}} \mathcal{K}(x) \) factors actually through \( \Gamma(\mathcal{E}', \mathcal{O}_{\mathcal{E}}^\vee) \) and gives rise to a pseudo deformation \( D \) on \( \Gamma(\mathcal{E}', \mathcal{O}_{\mathcal{E}}^\vee) \). By continuity, we have \( D^{\vee} \simeq D^{\vee} \otimes \epsilon^{n-1} \).

As there is only a finite number of possible reductions modulo \( p \) of \( D \), there is \( \mathcal{E}'(\mathfrak{p}) \) an open and closed subset of \( \mathcal{E}' \) of points whose reduction of \( D \) is \( (tv)\mathfrak{p} \). This is non empty by Hypothesis 6.2. In particular the restriction of the previous \( D \) to \( \mathcal{E}'(\mathfrak{p}) \) induces a morphism of rigid analytic spaces
\[
\mathcal{E}'(\mathfrak{p}) \longrightarrow \chi^\vee_{\mathbb{F}}.
\]

Now we construct a rigid analytic map \( \mathcal{E}' \to \mathcal{T} \).
Denote \( w = (w_{v,1}, \ldots, w_{v,n}) \) the universal character of \( ((\mathcal{O}_F \otimes \mathbb{Z}_p)^\times)^n \). In [Her22, Section 7.2.2] we constructed Hecke operators which are in \( \mathcal{O}(\mathcal{E}')^\times \), denoted by \( U_{v,i} \) for \( v \mid p \).
in $F$ and $i = 0, \ldots, n$. The operator $U_{v, i}$ coincides up to normalisation (this normalisation is made in order to vary in family) with the double class,

$$\left( \begin{array}{cc} I_i & \sigma_{v = v'} I_{n-i} \\ p^{-\delta_{x = x'}} & p^{-\delta_{x = x'}} I_i \end{array} \right) \in G(\mathbb{Q}_p) \subset \prod_{w = v' \mid F} \text{GL}_n(E_w) \times \text{GL}_n(E_{\pi}).$$

If $\mathcal{U}_{\text{class}}$ denotes the action of the (classical, i.e. non normalised) Hecke operator corresponding to the previous Iwahori double class acting on global sections of the classical automorphic sheaf, for (fixed) algebraic weight $w \in \mathcal{W}$, then the normalisation is

$$U_{v, i} = \tilde{w}_v \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & p^{-1} \\ & & & \ddots & p^{-1} \end{pmatrix} \mathcal{U}_{\text{class}}^{-1}.$$

where $\tilde{w}_v$ is the (unique) algebraic extension of $w_v$ as a character of $\tilde{F}$, and there is $i$ times 1 (and $n - i$ times $p^{-1}$) appearing in the matrix (see before and remark 7.5, together with remark 8.3 of [Her22]). For all $i \in \{1, \ldots, n\}$, we set

$$F_{v, i} := U_{v, i}^{-1} U_{v, i-1}.$$

It corresponds, up to normalisation, to the Hecke operator in $A(p)$,

$$F_{v, i}^{cl} := \begin{pmatrix} I_{i-1} & \sigma_{v = v'} I_{n-i} \\ p^{\delta_{x = x'}} & I_{n-i} \end{pmatrix} \in G(\mathbb{Q}_p),$$

and the normalisation is the following, for $w$ algebraic of infinitesimal weight $h = (h_{\tau, i})_{\tau, i}$

$$F_{v, i} = \tilde{w}_v \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & p^{h_{1, i}} F_{v, i}^{cl} \\ & & & \ddots & \end{pmatrix} = p^{\sum_{\tau} h_{\tau, i}} F_{v, i}^{cl},$$

where $p$ is in position $i$. The all point is that $F_{v, i}^{cl} \notin \mathcal{O}(\mathcal{E})$, i.e. they don’t interpolate, whereas $F_{v, i} \in \mathcal{O}(\mathcal{E}')$. We construct characters $\delta_{v, i}^{0} : F_{v, i}^{\times} \longrightarrow \mathcal{O}(\mathcal{E}')^{\times}$ by setting

$$\delta_{v, i}^{0}(p) := F_{v, i}$$

and

$$\delta_{v, i}^{0}(w_{v, i}) = w_{v, i}.$$
and we finally set
\( \delta_{v,i} := \delta^0_{v,i} \times \prod_{\tau} x_{\tau}^{s_{\tau}(i)} \times |i|^{\frac{1}{2n}}, \)
where for \( \tau : F_v \hookrightarrow \mathbb{C}_{p^\varpi}, x_{\tau}P_{\mathcal{O}_{K_v}} = \tau \mathcal{O}_{K_v} \) and \( x_{\tau}(p) = 1, \) and \( s_{\tau}(i) = \frac{1-n}{2} + p_{\tau} - i \) if \( 1 \leq i \leq p_{\tau}, \) and \( s_{\tau}(i) = \frac{n-1}{2} - (i - p_{\tau} - 1) \) if \( i > p_{\tau} \).

Thus the characters \( \delta_{v,i}, \chi_{v,i}, \) related to the eigenvalues of \( F^\text{cris}_{v,i} \tau \) and \( \chi \), are in the image of the previous map
\[ E' \longrightarrow \mathcal{T}. \]

Still denote \( Z \) for \( Z \cap E'(\mathcal{P}) \), which is Zariski dense and accumulation. Now we are reduced to prove that the two constructed maps \( E'(\mathcal{P}) \longrightarrow \mathcal{X}_{K^\text{pol}}^\text{pol} \) and \( E'(\mathcal{P}) \longrightarrow \mathcal{T} \) are compatible, in the sense that for \( z \in Z \) the second map is the parameter of a triangulation for the image of \( z \) via the first map. By local global compatibility at \( v \) for \( \pi \) and \( \rho^\text{pol}_\mathcal{P} \), we have that, using \( \pi^\text{pol}_{\mathcal{P}} \neq \{0\} \) by construction of \( E' \), that \( \pi_v \) is a subquotient of the Borel induction of an unramified character \( \chi \) of \( (F^\text{pol}_v)^n \) (e.g. [BC09] Proposition 6.4.3 and 6.4.4) with \( \chi \) related to the eigenvalues of \( F^\text{cris}_{v,i} \tau \) at \( z \) \( \chi = (\varphi_1, \ldots, \varphi_n) = (F^\text{pol}_{v,1}, \ldots, F^\text{pol}_{v,n})^{1/2} \).

But \( F^\text{pol}_{v,i} \) has a locally constant valuation (thus not \( F^\text{pol}_{v,i} \)), so up to choose another point of \( Z \) close to \( z \), we can assume that this induced representation is irreducible, and thus unramified. By local global compatibility this proves that there is an accumulation subset of \( Z \), which accumulates at any point of \( E \) with algebraic weight, consisting of points \( z \) with representation \( \rho_z \) semi simple corresponding to \( D_z \), crystalline at every \( v|p \) and such that \( D^\text{cris}(\rho_z) \) has all its refinement, one of which is given by \( (D^\text{cris}(\delta_{v,i}))_{1 \leq i \leq \pi_{\mathcal{P}}} \).

Moreover, the calculation for \( z \in Z \) we did in equation 9, together with the definition of the weight of \( \delta \) in equation 10 implies that the Hodge-Tate weights of \( \rho_z \) are given by \( \delta_{(\mathcal{O}_F \otimes \mathbb{Z}_p)^*} \), in the right order! This means that the map
\[ D \times \delta : E(\mathcal{P}) \longrightarrow \mathcal{X}^\text{pol}_{\mathcal{P}} \times \mathcal{T}, \]
sends a dense subset of \( Z \) (namely the previous one where points are crystalline) into \( Z^\prime_{K^\text{pol}} \), but conversely by construction of \( E' \), all points of \( Z^\prime_{K^\text{pol}} \) are in the image of the previous map.

Moreover \( D \times \delta \) is a closed immersion. Indeed, by construction \( E' \) is (locally) constructed as the image of \( \mathcal{H} \otimes \mathcal{O}_W = \mathcal{H}^S \otimes \mathcal{O}_T \) where \( \mathcal{H} = \mathcal{H}^S \otimes \mathcal{A}(p) \) acting on some space of overconvergent, locally analytic modular forms (of finite slope). Let \( U \subset \mathcal{X}^\text{pol}_{\mathcal{P}} \times \mathcal{T} \) be an affineoid, in particular it is quasi-compact thus the slopes of \( \mathcal{A}(p) \) on \( U \) are bounded (say by \( \alpha \)). Thus \( (D \times \delta^{-1})(U) \) is included in \( E_{\mathcal{X}_{K^\text{pol}}^\text{pol}}^\mathcal{P} \) for some \( v, w \) (see [Her22]) and then, by local-global compatibility, it is clear that \( (D \times \delta)^{-1}(U) \longrightarrow U \) is a closed immersion. As explained, \( Z \) accumulates to any point with classical weight of \( E' \), thus to \( Z^\prime \). If we denote by \( h \) the composite of the map
\[ E \longrightarrow \mathcal{T} \longrightarrow \mathcal{W}, \]
it coincides with a map
\[ E \overset{w}{\longrightarrow} \mathcal{W} \overset{\phi}{\longrightarrow} \mathcal{W}, \]

\([19]\) Remark that actually in our quasi split situation we have \( p_{r}, q_{r} \) which doesn't depend on \( r \in \Phi \). In any case \( (s_{r}(1), \ldots, s_{r}(n))_{r} = w_{0, M} (0, \ldots, n-1)_{r} + \frac{1-n}{2} = -w_{0, M} p_{G}, \) with \( p_{G} \) defined in the next paragraph.
where φ is the isomorphism of W given by the definition (10). The properties of the map \( h \) thus comes from the analogous one for \( w \), proven in \[\text{[Her22]}\] Theorem 9.5 (see also \[\text{[Che04]}\] which was the first to prove those properties).

\[\text{Corollary 6.8.} \quad \text{If } \pi \text{ is a cuspidal, algebraic, regular automorphic form (of tame level } K^p), \text{ which appears in degree 0 coherent cohomology and which is finite slope at } p, \text{ then } \rho_\pi \text{ appear in } \mathcal{F}_{K^p}(\mathcal{G}_\pi). \text{ If moreover } \delta \text{ is an accessible refinement for } \rho_\pi, \text{ then } \{\rho_\pi, \delta\} \in \mathcal{E}_{K^p}(\mathcal{G}_\pi). \text{ In particular, } \mathcal{E}_{K^p}(\mathcal{G}_\pi) \text{ is the closure in } \mathcal{E}_{K^p} \times T \text{ of homomorphically modular (for a cuspidal, algebraic, regular automorphic representation of tame level } K^p), \text{ finite slope at } p, \text{ representations } p, \text{ together with an accessible refinement.}

\[\text{Proof.} \quad \text{As any finite slope automorphic form has an accessible refinement, it is enough to prove the second part of the statement. This is a direct consequence of the fact that } \mathcal{E}(\mathcal{G}_\pi) \text{ coincides with the Eigenvariety } \mathcal{E}(\mathcal{G}_\pi) \text{ constructed in } \text{[Her22]} \text{. We take the notation of } \text{[Her22]} \text{ section 6. Let } \pi \text{ be an automorphic representation as in the statement and assume given } n \text{ and } f \in H^0(\mathcal{X}_1(p^n)_{\text{tor}}, \omega^\kappa) \text{ be a cuspidal holomorphic modular form corresponding to } \pi \text{ which is a finite slope eigenvector at } p, \text{ where } \mathcal{X}_1(p^n)_{\text{tor}} \text{ is a toroidal compactification of the Shimura variety of level at } p,

\[\Gamma_1(p^n) = \{ M \in G(\mathbb{Z}_p) \mid M \in U(\mathbb{Z}_p) \pmod{p^n}\},\]

with \( U \subset B \subset G_{\mathbb{Z}_p} \) are unipotent and Borel subgroup. Then there is an open \( \mathcal{X}_1(p^n)_{\text{tor}}(v) \subset \mathcal{X}_1(p^n)_{\text{tor}} \), for \( v \) small enough (a strict neighborhood of the ordinary locus), and for the Shimura variety \( \mathcal{X}_0(p^n)_{\text{tor}} \) of level

\[\Gamma_0(p^n) = \{ M \in G(\mathbb{Z}_p) \mid M \in B(\mathbb{Z}_p) \pmod{p^n}\}.\]

we have a corresponding open \( \mathcal{X}_0(p^n)_{\text{tor}}(v) \), such that over \( \mathcal{X}_0(p^n)_{\text{tor}}(v) \) there exists the sheaf \( \omega^\kappa_n(-D) \) of \( w \)-analytic cuspidal overconvergent modular forms of (any \( n \)-analytic) \( p \)-adic weight \( \chi \). By definition, finite slope (at \( p \)) sections of \( H^0(\mathcal{X}_0(p^n)_{\text{tor}}(v), \omega^\kappa_n(-D)) \) which are eigenvectors for the Hecke operators do appear in the Eigenvariety \( \mathcal{E}_{K^p} \). By construction, there exists a rigid open \( \mathcal{I}W^+_{w}(v) \subset \mathcal{T}^+ / U \) in the torsor of trivialisations of \( \omega \) above \( \mathcal{X}_1(p^n)_{\text{tor}}(v) \), so that \( H^0(\mathcal{X}_0(p^n)_{\text{tor}}(v), \omega^\kappa_n(-D)) \) is the set of sections \( H^0(\mathcal{I}W^+_{w}(v), \mathcal{O}_{\mathcal{I}W^+_{w}}[\chi](-D)) \) which are \( \chi \)-equivariant for the action of \( T(\mathbb{Z}_p)(20) \) on \( \mathcal{I}W^+_{w}(v) \) above \( \mathcal{X}_0(p^n)_{\text{tor}}(v) \) (see \[\text{[Her22]}\] Definition 6.10). Let \( \varepsilon \) be a finite order character of \( T(\mathbb{Z}_p) \) trivial on \( T(1 + p^n\mathbb{Z}) \), i.e. a character of \( \Gamma_0(p^n) / \Gamma_1(p^n) \). We claim that there is an injection, for any algebraic \( \kappa \)

\[H^0(\mathcal{X}_1(p^n)_{\text{tor}}, \omega^\kappa(-D))(\varepsilon) \hookrightarrow H^0(\mathcal{I}W^+_{w}(v), \mathcal{O}_{\mathcal{I}W^+_{w}}[\chi\varepsilon](-D)).\]

But this comes from the identification (see \[\text{[Her22], Definition 3.1]}\]

\[H^0(\mathcal{X}_1(p^n)_{\text{tor}}, \omega^\kappa) = H^0(\mathcal{T}^+ / U, \mathcal{O}[\kappa^-]),\]

the fact that the restriction to the open \( \mathcal{X}_1(p^n)_{\text{tor}}(v) \) is injective by analytic continuation, and that the restriction from \( \mathcal{T}^+ / U \times \mathcal{X}_1(p^n)_{\text{tor}} \mathcal{X}_1(p^n)_{\text{tor}}(v) \) to \( \mathcal{I}W^+_{w}(v) \) is also injective again by analytic continuation. Moreover all those maps are equivariant for the action of the Atkin-Lehner algebra \( \mathcal{A}(p) \). More precisely, the action of \( \mathcal{A}(p) \), i.e. of matrices of the

(20) A warning: this action is twisted compared to the one on \( T^+ / U \), see \[\text{[Her22]}\] see Proposition 6.16 and its proof.
form $\text{Diag}(p^{\alpha_1}, \ldots, p^{\alpha_n})$, on $\mathcal{E}$, together with the map $T(\mathbb{Z}_p) \rightarrow \Gamma(W, \mathcal{O}_W)$, thus giving a map $T(\mathbb{Z}_p) \rightarrow \Gamma(\mathcal{E}, \mathcal{O}_\mathcal{E})$, can be packaged together as an map $T(\mathbb{Q}_p) \rightarrow \Gamma(\mathcal{E}, \mathcal{O}_\mathcal{E})$. Then the previous injection is equivariant for this action of $T(\mathbb{Q}_p)$. But we can assume that $f$ is an eigenvector for $\Gamma(\mathbb{Q}_p)$ of some character $\varepsilon$ as before. Thus the associated refinement at $p$ of $f$ gives a character $\delta$ of $T(\mathbb{Q}_p)$ depending on the refinement, such that $\delta_{T(\mathbb{Z}_p)} = \kappa \varepsilon$. Then the previous injection assures that $f$ with this $\delta$ appears in $\mathcal{E}'_{K,\rho}$. For the last part of the statement, since any crystalline point at $p$ is indeed finite slope, and finite slope holomorphic points are already in $\mathcal{E}'_{K,\rho}(\overline{\mathcal{P}})$ as we just proved, we get the result.

From now on, to lighten notations denote $\mathcal{E}(\overline{\mathcal{P}}) := \mathcal{E}_{K,\rho}(\overline{\mathcal{P}})$, $F(\overline{\mathcal{P}}) := F_{K,\rho}(\overline{\mathcal{P}})$ $\mathbb{Z}' := \mathbb{Z}'_{K,\rho}$ accordingly.$^{(21)}$

7. Automorphic forms, infinite fern and big image

Let $K$ be finite extension of $\mathbb{Q}_p$ and $\overline{K}$ an algebraic closure of $K$. Denote $v_p$ the $p$-adic absolute value of $K$ such that $v_p(p) = 1$. Let $K_0 \subset K$ be the maximal unramified extension of $\mathbb{Q}_p$ with Frobenius operator $\sigma$ and set $f := [K_0 : \mathbb{Q}_p]$. Let $L$ be a finite extension of $\mathbb{Q}_p$ such that $K \otimes_{\mathbb{Q}_p} L \cong L[K_{\mathbb{Q}_p}]$. If $(\rho, V)$ is a crystalline representation of $\text{Gal}(\overline{K}/K)$, we denote $(D_{\text{cris}}(V), \varphi)$ its associated $\varphi$-module. It is a finite dimensional free $K_0 \otimes_{\mathbb{Q}_p} L$-module of rank $n = \dim_L V$ with a $\sigma \otimes \text{Id}_L$-linear automorphism $\varphi$. Its de Rham module $D_{\text{dR}}(V)$ is a filtered finite free $K \otimes_{\mathbb{Q}_p} L$-module. The Hodge-Tate type of $V$ is the $[K : \mathbb{Q}_p]$-uple $(k_1, \tau \geq \cdots \geq k_{n,\tau})_{\tau : K \rightarrow L}$ where the $k_{i,\tau}$ are the integers $m$ such that $\text{gr}^{-m}(D_{\text{dR}}(V) \otimes_{K,\tau} L) \neq 0$ counted with multiplicity.

Definition 7.1. — We say that a crystalline Galois representation $(\rho, V)$ over $L$ is Hodge-Tate regular (or simply HT-regular) if, for all $\tau : K \rightarrow L$, the integers $k_{i,\tau}$ are pairwise distinct. It is said to be $\varphi$-generic if the linear endomorphism $\varphi^f$ of the finite free $K_0 \otimes_{\mathbb{Q}_p} L$-module $D_{\text{cris}}(V)$ is split semisimple regular i.e has $\dim_L V$ pairwise distinct eigenvalues $\varphi_i$ in $L$ such that $\varphi_i(1/p^j) \notin \{1, p^j\}$ (note that these eigenvalues are in $L$ since $\varphi^f$ commutes to the semilinear action of $\text{Gal}(K_0/\mathbb{Q}_p)$ on $D_{\text{cris}}(V)$ given by $\varphi$).

Remark 7.2. — In [Chell, §3] Chenevier introduced the notion of weakly generic crystalline $(\varphi, \Gamma)$-module, i.e. crystalline $(\varphi, \Gamma)$-modules for which all refinements are non-critical, and weakly generic crystalline $(\varphi, \Gamma)$-module for which $n$ (the rank of the $(\varphi, \Gamma)$-module) well-positioned refinements are non critical. It is possible to deduce from the results of [Chell] that if the classical points of $\mathcal{E}$ or $\mathcal{X}^\text{Zar}$ which are weakly-generic at $p$ are Zariski-dense, then modular points are dense in $\mathcal{X}^\text{Zar}$ (or the closure has at least the expected dimension). He moreover proves that if $n = 3$, every $\varphi$-generic, HT regular absolutely irreducible $(\varphi, \Gamma)$ is automatically weakly generic. Unfortunately it does not seem to be true anymore even for $n = 4$ that absolutely irreducible points are weakly generic, as shown in the following example, and thus it does not seem any easier to prove that weakly

$^{(21)}$Everything we will say is still dependant of this level $K^p$. At the end of the article, in corollary 8.10, we show that there exists an optimal level $K^p$, but we can’t choose it right away. Compare [Chell] Lemma 2.4.
generic points are dense in $E^{\text{pol}}_n$ when $n \geq 4$ than proving the analogous result for generic points.

**Example 7.3.** — Let $(V, \varphi, \text{Fil}^*)$ be the filtered $\varphi$-module of an irreducible, $\varphi$-generic, HT regular, crystalline 4 dimensional representation of $G_{\overline{\mathbb{Q}}}$, with $L$-coefficients. Choosing $L$ big enough, we can assume that there exists a basis $(f_1, f_2, f_3, f_4)$ of $V$ such that $\varphi(f_i) = \varphi_1 f_i$. Refinements of $V$ are thus given by a permutation $\sigma$ of this basis. By irreducibility and weak-admissibility, we check that it is impossible for $\text{Fil}^k V \neq \{0\}, V$ to be $\varphi$-stable. For example, suppose that the HT weights are $-k_4 < -k_3 < -k_2 < -k_1$ (i.e. the jumps of the filtration on $V$ are $k_1 < k_2 < k_3 < k_4$) with

- $\text{Fil}^{k_1} V = \langle f_2, f_3, f_1 + f_4 \rangle = \langle f_2, f_2 + f_3, f_1 + f_4 \rangle$
- $\text{Fil}^{k_2} V = \langle f_1 + f_4, f_2 + f_3 \rangle$
- $\text{Fil}^{k_3} V = \langle f_4 \rangle$
- $\text{Fil}^{k_4} V = \{0\}$.

We can check that the non critical refinements are given by $\sigma = \text{id}, (23), (14), (14)(23)$, and they don’t form a weakly-nested sequence. Moreover, for generic choices of $k_i$ and $v(\varphi_1)$, the associated $p$-adic representation $V(D)$ will be irreducible. Indeed, if $v(\varphi_1) = v(\varphi_2) = v(\varphi_3) = 16, v(\varphi_4) = 12, k_1 = 0, k_2 = 10, k_3 = 20, k_4 = 30$, then we can check that no non-trivial $\varphi$-stable submodule of $D$ is weakly-admissible. In particular if we do not already know that generic points are Zariski dense, it is not likely to prove that weakly generic ones are.

Moreover we can check that locally the image of the tangent space of those refined points doesn’t cover all the tangent space for the corresponding point in the local deformation ring (i.e. at this point the analogous of proposition 3.7 only for non-critical refinements isn’t true). In the following we will use a replacement of those generic points by the so-called *almost generic* ones, which are Zariski dense in the fern and for which we can apply proposition 3.7 and thus Chevèvre’s strategy. Remark that some irreducible weakly-admissible filtered $\varphi$-modules of dimension 4 which admits a critical refinement are weakly-generic, but we don’t know how to discriminate these from the previous example on the deformation rings (or on the infinite fern).

Recall that $k$ is a finite field of characteristic $p$ and that $\overline{\mathcal{F}}(\overline{\mathcal{P}})$ the Zariski-closure of the image $\overline{\mathcal{F}}(\overline{\mathcal{P}}) \rightarrow E^{\text{pol}}_{\overline{\mathcal{P}}}$, i.e. the Zariski closure of the infinite fern $\overline{\mathcal{F}}(\overline{\mathcal{P}})$.

**Definition 7.4.** — We say that a Galois representation $\rho : G_E \rightarrow \text{GL}_n(\mathbb{Q}_p)$ has enormous image, if $\rho(G_E(\zeta_{x,\infty}))$ is enormous, in the sense of [NT19] Definition 2.27. We say that a point $x$ of $\mathcal{E}(\overline{\mathcal{P}})$ (resp. of $E^{\text{pol}}_{\overline{\mathcal{P}}}$) is *almost generic* if it is in $\mathcal{Z}' = \mathcal{Z}_{K'/p}$ (resp. in the image of $\mathcal{Z}'$ in $E^{\text{pol}}_{\overline{\mathcal{P}}}$), the associated Galois representation $\rho$ has enormous image, and if $\rho|G_{E_v}$ is crystalline, $\varphi$-generic, HT-regular, posses an irreducible refinement and is absolutely irreducible for all $v|p$.

Let $v$ be a place of $E$ dividing $p$. Let $(\rho, V)$ be a continuous finite dimensional representation of $G_{E_v}$ over $L$. It follows from the compactness of $G_{E_v}$ that $\rho(G_{E_v})$ is a closed
subgroup of $GL_L(V)$. The $p$-adic analogue of Cartan’s Theorem shows that $\rho(G_{E_v})$ is a $p$-adic Lie subgroup of $GL_L(V)$ so that we can define $g_\rho := \text{Lie}(\rho(G_{E_v}))$ (see [Ser89, Rem. 1.I.1]) and $\rho_{p,L}$ the $L$-span of $g_\rho$ in $\text{End}_L(V)$. Our goal is to prove that almost generic points are Zariski-dense in the infinite fern $\mathcal{F}(\overline{\mathbb{F}}_p)$. Such a result was proven by Taïbi ([Tai6]) in a slightly different (and more difficult) context (improving results of [BC99]). As the case we consider is easier, and for convenience of the reader, we repeat the argument in our context.

Denote by $K_u/E_v$ the compositum of extensions of degree dividing $n$, this is a finite Galois extension.

**Proposition 7.5.** — Let $(\rho, V)$ be some continuous $n$-dimensional representation of $G_{E_v}$ over $L$ and assume that $(\rho|_{G_{K_u}}, V)$ is absolutely irreducible. Then $V$ is a simple $\rho_{p,L}$-module, $g_{p,L}$ is a reductive Lie algebra and $h_{p,L}$, the semisimple part of $g_{p,L}$, is isomorphic to a sub Lie algebra of $\mathfrak{sl}(V)$ of semisimple rank at most $\dim_L V - 1$.

**Proof.** — Let $K$ be a finite extension of $E_v$, we claim that $\rho|_{G_K}$ is absolutely irreducible. We can assume that $K/E_v$ is Galois. Suppose that $\rho|_{G_K}$ is not absolutely irreducible, then as

$$\rho : G_{E_v} \longrightarrow GL_L(V) \cong GL_n(L)$$

is absolutely irreducible and $G_K$ is normal in $G_{E_v}$, we have, up to enlarge $L$, a decomposition into absolutely irreducible $G_K$-representations

$$\rho|_{G_K} = \bigoplus_{k=1}^r W_k,$$

and $G_{E_v}$ permutes these representations, in particular they have the same dimension. Let $H \subset G_{E_v}$ be the stabiliser of $W_1$, then $G_{E_v}/H$ acts transitively on the $W_k$ and thus $[G_{E_v} : H] \dim(W_1) = n$. In particular $H = G_{K'}$, with $K'$ a finite extension of $E_v$ of degree dividing $n$, thus $K' \subset K_u$, but $\rho|_{G_{K_u}}$ is irreducible, thus as is $\rho|_{G_K}$, and thus $r = 1$ i.e. $\rho|_{G_K}$ is irreducible.

For each open subgroup $H \subset G_{E_v}$, the representation $(\rho|_H, V)$ is irreducible, so that $V$ is a simple $\rho_{p,L}$-module. Thus $\rho_{p,L}$ has a simple faithful module, which implies that $\rho_{p,L}$ is a reductive Lie algebra. Let $\rho_{p,L} = \mathfrak{a}_{p,L} \oplus \mathfrak{h}_{p,L}$ be the decomposition of $\rho_{p,L}$ as direct sum of an abelian and of a semisimple Lie algebras. As $V$ is an absolutely simple $\rho_{p,L}$-module, the Lie algebra $\mathfrak{a}_{p,L}$ acts on $V$ by scalars and $V$ is an absolutely simple $\mathfrak{h}_{p,L}$-module. Then $\mathfrak{a}_{p,L} \subset \text{Lie}_V$ and, as $\mathfrak{h}_{p,L}$ is semisimple, $\mathfrak{h}_{p,L} \subset \mathfrak{g}_{p,L} \cap \mathfrak{sl}(V)$. As a consequence $\mathfrak{h}_{p,L} = \mathfrak{g}_{p,L} \cap \mathfrak{sl}(V)$ is a semisimple Lie algebra and $V$ is an absolutely simple $\mathfrak{h}_{p,L}$-module. As $\mathfrak{sl}(V)$ has rank $\dim_L V - 1$, the rank of $\mathfrak{h}_{p,L}$ is at most $\dim_L V - 1$. □

**Proposition 7.6.** — There exist finitely many nonzero $\mathbb{Q}$-linear forms $\Lambda_1, \ldots, \Lambda_r$ on $\mathbb{Q}^n$ such that the following is true: let $(\rho, V)$ be a crystalline $n$-dimensional representation of $G_{F_v}$ over $L$, with Hodge-Tate weights $(k_1, \sigma \leq \cdots \leq k_n, \sigma)$ such that $(\rho|_{G_{K_u}}, V)$ is absolutely irreducible and such that there exists at least one $\sigma : F_v \to L$ such that for all $1 \leq i \leq r$, $\Lambda_i(k_1, \sigma, k_2, \sigma, \ldots, k_n, \sigma) \neq 0$, then $\rho(G_{E_v})$ contains an open subgroup of $\text{SL}(V)$.

**Proof.** — Let $C$ be some algebraically closed field of characteristic 0. The classification of semisimple Lie algebras and their representations shows that all semisimple Lie algebras
and their finite dimensional simple modules are defined over \( \mathbb{Q} \), that there are finitely many isomorphism classes of semisimple Lie algebras of bounded rank and that each of them has finitely many semisimple modules of bounded rank. Consequently, for a fixed \( n \geq 2 \), there exist a finite number of pairs \((\mathfrak{h}_i, \mathfrak{g}_i)\) where \( \mathfrak{h}_i \) is a semisimple Lie algebra and \( \mathfrak{g}_i \) an embedding of \( \mathfrak{h}_i \) in \( \mathfrak{s}l_n,\mathbb{Q} \) such that for each semisimple Lie subalgebra \( \mathfrak{h} \subseteq \mathfrak{s}l_n,\mathbb{C} \), there exists \( i \) such that \( \mathfrak{h} \cong \mathfrak{h}_i \otimes \mathbb{Q} \mathbb{C} \) and the inclusion is \( GL_n(\mathbb{C}) \)-conjugated to \( \mathfrak{g}_i \otimes \text{Id}_{\mathbb{C}^n} \). As a consequence a Cartan subalgebra of \( \mathfrak{h} \) is conjugated to one of finitely many \( \mathbb{Q} \)-linear subspaces of the space of diagonal matrices in \( \mathfrak{s}l_n,\mathbb{C} \). Moreover it follows from [Bou, VIII.§3 Prop.2.(ii)] and from Borel-Serre Theorem ([Kan01, Thm. 12.1]) that a semisimple Lie subalgebra of \( \mathfrak{s}l_n,\mathbb{C} \) containing \( \mathfrak{h} \) is equal to \( \mathfrak{s}l_n,\mathbb{C} \) or of rank strictly less than \( n - 1 \). Thus there exist finitely many nonzero \( \mathbb{Q} \)-linear forms \( \Lambda_1', \ldots, \Lambda_n' \) on \( \mathbb{Q}^n \) such that if \( \mathfrak{h} \) a semisimple subalgebra of \( \mathfrak{s}l_n,\mathbb{C} \) of rank strictly less than \( n - 1 \) and \( x \in \mathfrak{h} \) is a semisimple element of eigenvalues \( \lambda_1, \ldots, \lambda_n \) (counted with multiplicities), then there exists \( 1 \leq i \leq s \) and \( w \in \mathfrak{s} \mathfrak{e} \mathfrak{n} \mathfrak{a} \mathfrak{n} \mathfrak{t} \) such that \( w(\Lambda_i')(\lambda_1, \ldots, \lambda_n) := \Lambda_i'(\lambda_{w(1)}, \ldots, \lambda_{w(n)}) = 0 \).

We set
\[
\{\Lambda_1, \ldots, \Lambda_r\} = \{w(\Lambda_i') | 1 \leq i \leq s, w \in \mathfrak{s} \mathfrak{e} \mathfrak{n} \mathfrak{a} \mathfrak{n} \mathfrak{t}\}.
\]

Let \( \Theta \in \text{End}_{\mathbb{C}_p \otimes_{\mathbb{Q}_p} L}(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V) \) the Sen operator of \( V \). As \((\rho, V)\) is Hodge-Tate, it follows from [Sen73, Thm. 1] that \( \Theta \) belongs to \( \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathfrak{g}_{\rho,L} \subseteq \mathbb{C}_p \otimes_{\mathbb{Q}_p} \text{End}_{\mathbb{Q}_p} V \).

Suppose \( L \) is big enough so that \( F_v \otimes_{\mathbb{Q}_p} L \cong L(F_v, \mathbb{Q}_p) \). Then
\[
\mathbb{C}_p \otimes_{\mathbb{Q}_p} L = \prod_{\sigma:F_v \rightarrow L} \mathbb{C}_p \otimes_{\mathbb{F}_\sigma} L,
\]
decomposes over all embeddings of \( F_v \) and let \( \Theta_\sigma \) be the \( \mathbb{C}_p \otimes_{\mathbb{F}_\sigma} L \)-linear endomorphism of \( \mathbb{C}_p \otimes_{\mathbb{F}_\sigma} V \) induced by \( \Theta \). The eigenvalues of \( \Theta_\sigma \) are the \( \sigma \)-Hodge-Tate weights \((k_{1,\sigma} \leq \cdots \leq k_{n,\sigma})\) of \((\rho, V)\) (counted by multiplicities).

Assume that \( \Lambda_i(k_{1,\sigma}, \ldots, k_{n,\sigma}) \neq 0 \) for all \( 1 \leq i \leq r \). Then by what preceeds, the element \( \Theta_\sigma \) can’t be contained in a strict semisimple Lie subalgebra of \( \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathfrak{g}(V) \) so that \( \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathfrak{g}_{\rho,L} = \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathfrak{g}(V) \). For dimension reasons, we have \( \mathfrak{g}_{\rho,L} = \mathfrak{g}(V) \). We conclude that \( \rho(G_{F_v}) \) contains an open subgroup of \( \text{SL}(V) \).

**Proposition 7.7.** — The set of points \( x \in \mathcal{F}(p) \) such that \( \text{Tr} \rho_{x}|_{\mathfrak{g}_{K_n}} \) is absolutely irreducible is a Zariski-dense and Zariski-open subset of \( \mathcal{F}(p) \).

**Proof.** — The fact that the absolutely irreducible locus is Zariski-open is a consequence of [Che14, §4.2]. In order to prove that it is Zariski-dense, it is then sufficient to prove that each Zariski-open subset \( U \) of \( \mathcal{F}(p) \) contains a point \( x \) such that \( \text{Tr} \rho_{x}|_{\mathfrak{g}_{K_n}} \) is absolutely irreducible.

Now we follow the strategy of [BC11] and [Tai16]. Let us fix some notation. If \( x = (\rho_x, \delta_x) \in Z' \subseteq \mathcal{E}(p) \) and \( \sigma : K_n \rightarrow \mathbb{Q}_p \), we let \( (k_{\sigma,1}(x) \leq \cdots \leq k_{\sigma,n}(x)) \) the Hodge-Tate weighted at \( \sigma \) of \( \rho_x|_{\mathfrak{g}_{K_n}} \). (\( \phi_1(x), \cdots, \phi_n(x) \in k(x)^n \) the ordered eigenvalues of the linearized Frobenius of \( D_{\text{triv}}(\rho_x)|_{\mathfrak{g}_{K_n}} \)) corresponding to the refinement of \( \rho_x|_{\mathfrak{g}_{K_n}} \) defined by \( \delta_{x,\nu} \). We also set \( k_i(x) = \sum_{\sigma} k_{\sigma,i}(x) \). Let \( e \) be the ramification index of \( K_n/\mathbb{Q}_p \). The functions \( x \mapsto v_p(\phi_i(x)) + e^{-1}k_i(x) \) are therefore locally constant on \( Z' \).

Now fix \( U \) a Zariski open non empty subset of \( \mathcal{F}(p) \). We have \( U \cap \mathcal{F}(p) \neq \emptyset \) so that the inverse image \( V \) of \( U \) in \( \mathcal{E}(p) \) is a non empty Zariski-open subset. Let \( x \in V \cap Z' \). Let
Proof. By Proposition 7.6, there exists an open subset $U \subset \mathcal{E}$ containing $x$ such that there are $\rho_{x,v}$-admissible and non-empty proper sub-\(\phi\)-modules of $\mathcal{E}|_{U}$. Let $C = \max\{1, \max\{v_{\rho}(\phi_{1}(x)) + e^{-1}k_{1}(x) + \cdots + v_{\rho}(\phi_{i}(x)) + e^{-1}k_{i}(x) | i = 2, \ldots, n\}\}$ and $Z_{C}^{\prime}$ the set of points $z \in \mathcal{Z}$ such that $|k_{i,\tau} - k_{i+1,\tau}| > C$ for all $1 \leq i \leq n - 1$ and $\tau \in \Phi$. We claim that any $z \in Z_{C}^{\prime} \cap U$, $z$ has a non-critical refinement at $v$ (in fact even the refinement given by $\mathcal{E}$ at $z$ is non-critical$^{(22)}$). Let $z \in Z_{C}^{\prime} \cap U$ and let $F_{\tau} \subset D(\rho_{x,v})$ be the refinement defined by $z$. We note $F_{\star}$ its $i$-dimensional part. The (linearized) Frobenius eigenvalues on $F_{\tau}$ are $\phi_{1}(z), \ldots, \phi_{i}(z)$. Assume that the refinement $F_{\star}$ is critical. This means that the there exists some $1 \leq i \leq n - 1$ and some $\tau : F_{\tau} \rightarrow \mathbb{Q}_{p}$.

\footnote{(22) These $z$ are numerically non-critical as in [BC09, Remark 2.4.6, (ii)]}
and such that the gaps of the Hodge-Tate filtration on \( \overline{\Omega}_p \otimes_{F_v} \sigma D_{\text{cris}}(\rho_{z,v}) \) are not 
\(-k_{1,\sigma}(z), \ldots, -k_{i,\sigma}(z)\). Choose \( i \) minimal for this property. This implies that the sum of these gaps is greater or equal than 
\(-k_{1,\sigma}(z) + \cdots + k_{i-1,\sigma}(z) + k_{i+1,\sigma}(z)\). Moreover if \( \tau \neq \sigma \), the sum of the gaps of the Hodge-Tate filtration on \( \overline{\Omega}_p \otimes_{F_v} \sigma D_{\text{cris}}(\rho_{z,v}) \) is greater or equal than 
\(-k_{1,\tau}(z) + \cdots + k_{i-1,\tau}(z) + k_{i,\tau}(z)\). As \( F_i \) is \( \varphi \)-stable, the weak admissibility of \( D_{\text{cris}}(\rho_{z,v}) \) implies that

\[
v_p(\phi_1(z)) + \cdots + v_p(\phi_i(z)) \geq -\frac{1}{e} \left( \sum_{\tau \neq \sigma} \left( \sum_{j=1}^{i-1} k_{j,\tau}(z) \right) + \sum_{j=1}^{i-1} k_{j,\sigma}(z) + k_{\sigma,i+1}(z) \right),
\]

thus

\[
\sum_{j=1}^{i} \left( v_p(\phi_j(z)) + \frac{1}{e} k_j(z) \right) \geq \frac{1}{e} (k_{i,\sigma}(z) - k_{i+1,\sigma}(z)) > C,
\]

contradicting \( z \in \mathcal{Z}_C' \). Therefore \( F_* \) is non-critical. \( \square \)

Let \( \mathcal{X}^{\text{mod,ag}} \) be the set of almost generic points in \( \mathcal{F}(\overline{p}) \).

**Theorem 7.9.** — The set of points \( x \) in \( \mathcal{F}(\overline{p}) \) which are in the image of the set \( \mathcal{Z}' \) such that \( \rho_{x,v} \)

is crystalline \( \varphi \)-generic, HT-regular, admits an non-critical refinement, and such that \( \rho_{x}(G_{F_v}) \)

contains an open subgroup of \( \text{SL}(V_x) \) is a Zariski dense accumulation subset. Moreover the set \( \mathcal{X}^{\text{mod,ag}} \) is a Zariski dense accumulation subset in \( \mathcal{F}(\overline{p}) \).

**Proof.** — For any \( v \mid p \), let \( \Lambda_1, \ldots, \Lambda_r \) be nonzero \( \mathbb{Q} \)-linear forms on \( \mathbb{Q}^n \) as in Proposition 7.6. Let \( \sigma \) be some fixed embedding of \( F_v \) into \( K \). The set of classical points \( x \in \mathcal{E}(\overline{p}) \)

which are crystalline and such that the \( \sigma \)-Hodge-Tate weights of the representation \( \rho_x \) are not zeros of all the \( \Lambda_i \), and such that the weights are sufficiently regular in the sense of 7.8, form a Zariski dense accumulation subset in \( \mathcal{E}(\overline{p}) \) (this is a direct consequence of the open image of Proposition 6.6; see [Tal16, Proposition 2.2.6]) by using \( f_i = \delta_{i,v}(\pi_v) \), whose valuation is \( v_p(\phi_i) + \frac{1}{e} k_i \).

As a consequence \( \mathcal{F}(\overline{p}) \) is the Zariski-closure of the images of these points in \( \mathcal{X}^{\text{pol}} \).

By Proposition 7.7 and Proposition 7.8, the subspace of \( \mathcal{X}^{\text{pol}}_\sigma \) where \( \rho_x|_{G_{K_v}} \) is absolutely irreducible is Zariski-open and Zariski-dense in \( \mathcal{F}(\overline{p}) \). We conclude from Proposition 7.6 that the set of classical points \( x \) such that \( \rho_x \) has an open image and possesses a non-critical refinement is Zariski-dense and an accumulation subset in \( \mathcal{F}(\overline{p}) \).

It is enough to prove that classical points such that \( \rho_x(G_{F_v}) \) contains an open subgroup of \( \text{SL}(V_x) \) have enormous image. At such a point \( x \), the Zariski closure of \( \rho_x(G_E) \)

contains \( \text{SL}(V_x) \). As \( E(\zeta_{p^\infty})/E \) is abelian, the derived subgroup \( \rho_x(G_E) \) is included in \( \rho_x(G_{E(\zeta_{p^\infty})}) \). By [Bor91, I §2.1 (e)], the Zariski closure of \( \rho_x(G_{E(\zeta_{p^\infty})}) \) contain the derived subgroup of the Zariski closure of \( \rho_x(G_E) \) and then contains \( \text{SL}(V_x) \). It follows from [NT19] Lemma 2.33, that \( \rho_x(G_{E(\zeta_{p^\infty})}) \) is enormous.

For any \( v' \mid p \), the subset \( U_{v'} \) of \( x \in \mathcal{F}(\overline{p}) \) such that \( \text{Tr} \rho_x|_{G_{F_{v'}}} \) is absolutely irreducible is Zariski open and dense. We easily deduce the second part of the statement. \( \square \)
8. Global and local settings

We come back to our global situation as in section 7, i.e. \( k \) is a finite field of characteristic \( p \) and \( \rho : G_E \to \text{GL}_n(k) \) is a conveniently modular (polarized by \( \chi = 1 \)) semisimple continuous representation.

Let \( x \in \mathcal{X}^{\text{mod.ag}} \). Then \( x \) correspond to a cuspidal automorphic representation \( \pi \) of \( G \). Let \( \Pi \) be the isobaric automorphic representation of \( \text{GL}_n(\mathbb{A}_E) \) associated to \( \pi \) by Theorem A.7.

**Proposition 8.1.** — **The representation** \( \Pi \) **is cuspidal and thus generic.**

**Proof.** — We have

\[
\Pi = \Pi_1 \oplus \Pi_2 \oplus \cdots \oplus \Pi_k
\]

and a character \( \chi_\pi \) where \( \Pi_i \) is a regular algebraic cuspidal automorphic representation of \( \text{GL}_n \) such that \( \Pi_i \otimes \chi_\pi^{-1} \) is self-dual. Thus as \( \rho_x = \rho_\pi \) is up-to-twist by a character given by \( \rho_\Pi = \rho_{\Pi_1} \oplus \cdots \oplus \rho_{\Pi_k} \) but as \( x \) is in \( \mathcal{X}^{\text{mod.ag}} \), \( \rho_\pi \) is irreducible, thus \( k = 1 \) and \( \Pi \) is cuspidal. In particular it is generic by Piatieski-Shapiro, Shalika ([CKM04] Theorem 8.5).

**Corollary 8.2.** — **Let** \( (\rho_x, V_x) \) **be the representation corresponding to a point** \( x \in \mathcal{X}^{\text{mod.ag}} \). **For all** \( v \in S, v \nmid p \), **we have** \( H^0(G_v, \text{ad}(V_x)^*(1)) = \{0\} \), **in particular** \( H^1(G_v, \text{ad}(V_x)) = H^1_f(G_v, \text{ad}(V_x)) \) **(see [Bel09, Notation 2.1]).**

**Proof.** — Let \( \pi \) be the automorphic representation associated to \( x \). Since \( v \) is split in \( E \), the representation \( (\rho_x|_{G_v}, V_x) \) is a twist of the image of \( \pi_v \) by the local Langlands correspondence (see [Carl12]). By Proposition 8.1, the representation \( \pi_v \) is generic thus \( H^0(G_v, \text{ad}(V_x)^*(1)) = \text{Hom}_{G_v}(V_x, V_x(1)) = \{0\} \), thus \( H^1(G_v, \text{ad}(V_x)) \langle H^1_f(G_v, \text{ad}(V_x)) \rangle \) vanishes too (for example [Bel09] Proposition 2.3 (i)).

**Theorem 8.3 (Newton-Thorne).** — **Let** \( x \in \mathcal{X}^{\text{mod.ag}} \) **and let** \( r_x : G_{F,S} \to G_n(\mathbb{Q}_p) \) **be the associated representation. Then** \( H^1_f(G_{F,S}, \text{ad}(r_x)) = \{0\} \).

**Proof.** — This is consequence of [NT19, Thm. A]. Namely as \( x \in \mathcal{X}^{\text{mod.ag}} \), the representation \( r_x \) is associated to an automorphic representation \( \pi \) whose base change to \( \text{GL}_n(\mathbb{A}_E) \) is cuspidal algebraic and regular by Proposition 8.1.

Let \( x \in \mathcal{X}^{\text{mod.ag}} \) and let \( (\rho_x, V_x) \) be the associated representation of \( G_{F,S} \) over a finite extension of \( k(x) \). Our goal is to prove Theorem 8.6, which is invariant by scalar extension, thus we freely extend the base field of \( \mathcal{X}^{\text{pol}}_\mathcal{E} \) so that \( \rho_x \) is defined over \( k(x) \).

Let \( \mathcal{F} \) be a refinement of \( (\rho_x, V_x) \), that is, a family \( (\mathcal{F}_v)_{v \in S_p} \) where \( \mathcal{F}_v \) is a refinement of the crystalline representation \( (\rho_x|_{G_v}, V_x) \). Let \( x_\mathcal{F} \in \mathcal{E} \) be the classical dominant point corresponding to \( \rho_x \) and the refinement \( \mathcal{F} \). In what follows, if \( X \) is a rigid space and \( x \in X \), we set \( X_x := \text{Spec}(\mathcal{O}_{X,x}) \). The projection map \( \mathcal{E} \to \mathcal{X}^{\text{pol}}_\mathcal{F} \) induces a morphism \( \hat{\mathcal{E}}_{x_\mathcal{F}} \to \left( \mathcal{X}^{\text{pol}}_{\mathcal{F}} \right)_x \cong \mathcal{X}^{\text{pol}}_{\rho_x} \). For each \( v \in S_p \), let \( \rho_{x,v} = \rho_x|_{G_v} \), which is irreducible, and
consider the composite map
\[ \mathcal{E}_{x,F} \longrightarrow \left( \chi_{p,v}^{\text{pol}} \right)_x \simeq \chi_{p,v}^{\text{pol}} \longrightarrow \mathcal{X}_{p,v}. \]

Using notation introduced in section 3, we define quasi-trianguline deformation spaces \( \mathcal{X}_{p,v}^{\text{qtri},w} := \mathcal{X}_{D_{\text{rig}}(p_v)\cdot F[1/\ell]}^{w} \) for \( w \in \mathcal{S}_{\mathbb{F}_v, w}^{\mathcal{E}} \). Denote also \( \mathcal{X}_{p,v}^{\text{cris}} := \mathcal{X}_{D_{\text{rig}}(p_v)}^{\text{cris}} \) the crystalline deformation space (see [Chel, page 24] or [HMS22, page 18]).

**Lemma 8.4.** — The map \( \mathcal{E}_{x,F} \rightarrow \mathcal{X}_{p,v} \) factors through \( \mathcal{X}_{p,v}^{\text{qtri},w} \).

**Proof.** — Let \( \overline{p}_v : G_{\mathbb{F}_v} \rightarrow \text{GL}_n(k) \) be the composite of \( \overline{p} \) with \( G_{\mathbb{F}_v} \hookrightarrow G_{E,S} \) and let \( \mathcal{X}_{p,v}^{\square} \) be the framed deformation space of \( \overline{p}_v \). Let \( X_{\text{tr}}(\overline{p}_v) \subset \mathcal{X}_{p,v}^{\text{qtri}} \times \overline{F}_v^n \) be the trianguline variety [BHS17, Def. 2.4]. We choose \( y \in \mathcal{X}_{p,v}^{\square} := \mathcal{X}_{p,v}^{\text{qtri}} \) be a point such that \( \rho_v \) is conjugated to \( \rho_{p,v} \) and let \( y_{\mathbb{F}_v} \) be the dominant point of \( X_{\text{tr}}(\overline{p}_v) \) corresponding to \( y \) and to the refinement \( F_v \). The projection map \( X_{\text{tr}}(\overline{p}_v) \rightarrow \mathcal{X}_{p,v}^{\text{qtri}} \) induces a map \( X_{\text{tr}}(\overline{p}_v)/y_{\mathbb{F}_v} \rightarrow \mathcal{X}_{p,v}^{\square} \) and, by [BHS19, Cor. 3.7.8], this morphism factors through \( \mathcal{X}_{p,v}^{\text{qtri},w} \). As \( \mathcal{X}_{p,v}^{\text{qtri},w} \) is the pullback of \( \mathcal{X}_{p,v}^{\text{qtri}} \) by the formally smooth map \( \mathcal{X}_{p,v}^{\square} \rightarrow \mathcal{X}_{p,v} \), it is sufficient to prove that there exists, locally at \( x \), a factorization
\[
\mathcal{E} \longrightarrow X_{\text{tr}}(\overline{p}_v) \longrightarrow \mathcal{X}_{p,v}^{\square}
\]
sending \( x_{x,F} \) on \( y_{\mathbb{F}_v} \), where \( \mathcal{X}_{p,v}^{\square} \) is the rigid fiber of the pseudo-deformation space, as in Definition 5.3. As \( \rho_{p,v} \) is irreducible, it follows that there exists some affine neighborhood \( U \) of \( x_{x,F} \) in \( \mathcal{E} \) and a continuous morphism \( \rho_{v} : G_{E,S} \rightarrow \text{GL}_n(\mathcal{O}(U)) \) such that \( \text{Tr}(\rho_{v})(z) = \text{Tr}(\rho_{p}) \) for all \( z \in U \). Indeed, by [Che14] Theorem 2.22 there is a representation \( \rho_{v} : G_{E,S} \rightarrow \text{GL}_n(\mathcal{O}\mathcal{E}_{x,F}) \) whose trace is \( D_{\mathcal{O}\mathcal{E}_{x,F}} \). As \( \mathcal{O}\mathcal{E}_{x,F} \) is a direct limit over \( U \), there exists such a \( U \) (see [BC09] Lemma 4.3.7 for a precise argument). This gives us a map \( U \rightarrow \mathcal{X}_{p,v}^{\square} \) and even \( U \rightarrow \mathcal{X}_{p,v}^{\square} \times \overline{F}_v^n \). As the set \( Z' \) is Zariski-dense and accumulation in \( \mathcal{E} \), we can choose \( U \) so that \( U \cap Z' \) is Zariski-dense in \( U \). A point of \( U \cap Z' \) is sent to a point of \( X_{\text{tr}}(\overline{p}) \) by \( U \rightarrow \mathcal{X}_{p,v}^{\square} \times \overline{F}_v^n \) (by definition of \( X_{\text{tr}}(\overline{p}) \), [BHS19] section 3.7) so that we obtain the desired section.

For any representation \( \rho \) of \( G_{E,S} \), we use the notation \( \rho_{p,v} := (\rho_{v})_{v \in \Phi} \) where \( \Phi \) is in our fixed CM type at the beginning of section 6. Then we write \( \mathcal{X}_{p,v}^{\text{qtri},w} := \prod_{v \in \Phi} \mathcal{X}_{p,v}^{\text{qtri},w} \) and \( \mathcal{X}_{p,v}^{\text{cris}} := \prod_{v \in \Phi} \mathcal{X}_{p,v}^{\text{cris}} \).

The following corollary is very similar to [Ber20] and [BHS19].

**Corollary 8.5.** — Let \( x \in \mathcal{X}_{\text{mod},\Phi}^{\text{rig},w} \) and let \( F \) be a refinement such that \( (D_{\text{rig}}(\rho_{x,v})\cdot F_v) \) is associated to a product of distinct transpositions for all \( v \in \Phi \) (see definition 3.6). Then \( x_{x,F} \) is a smooth point of \( \mathcal{E} \) and we have an isomorphism
\[ T_{x,F} \mathcal{E} \simeq T \mathcal{X}_{p,v}^{\text{qtri},w} / T \mathcal{X}_{p,v}^{\text{cris}}. \]
Proof. — Denote by $\mathcal{X}^\text{pol}_{\rho_x}$ the (equicharacteristic) $\chi$-polarised global deformation space of $\text{tr} \rho_x$. It is the completion of $\mathcal{X}^\text{pol}_{\rho}$ at $\rho_x$ by $\text{[Che14, section 4.1]}$. Denote by $\mathcal{X}^\text{qtri}_{\rho_x,F}$ the fiber product $\mathcal{X}^\text{pol}_{\rho_x} \times_{\mathcal{X}^\text{pol}_{\rho_x}} \mathcal{X}^\text{qtri,\{0\}}_{\rho_x,v,F}$. We have a map

$$\tilde{\mathcal{E}}_{x,F} \rightarrow \mathcal{X}^\text{qtri}_{\rho_x,F},$$

induced from the map $\tilde{\mathcal{E}}_{x,F} \rightarrow \mathcal{X}^\text{qtri,\{0\}}_{\rho_x,F}$ and $\tilde{\mathcal{E}}_{x,F} \rightarrow \mathcal{X}^\text{pol}_{\rho_x}$. But then the standard argument that $f : \mathcal{O}_{\mathcal{X}^\text{qtri}_{\rho_x,F}} \rightarrow \mathcal{O}_{\tilde{\mathcal{E}}_{x,F}}$ is surjective comes from the fact that $\mathcal{E}_{x,F}$ is topologically generated by $\mathcal{O}_{\mathcal{X}^\text{pol}_{\rho_x}}$ and $\mathcal{O}_T$ by construction, but $\mathcal{X}^\text{qtri,\{0\}}_{\rho_x,F}$ lies over $T$. Thus we have a closed immersion

$$(11) \quad \tilde{\mathcal{E}}_{x,F} \hookrightarrow \mathcal{X}^\text{qtri}_{\rho_x,F}.$$ 

But the genericity assumption (Corollary 8.2) implies that

$$H^1_f(G_F, ad(\rho_x)) = \ker \left( H^1(G_F, ad(\rho_x)) \rightarrow \prod_{v \mid p} H^1(G_{F_v}, ad(\rho_x))/H^1_f(G_{F_v}, ad(\rho_x)) \right),$$

thus we have the exact sequence

$$(12) \quad 0 \rightarrow H^1_f(G_F, ad(\rho_x)) \cap TX^\text{qtri}_{\rho_x,F} \rightarrow TX^\text{qtri}_{\rho_x,F} \rightarrow \bigoplus_{v \mid p} TX^\text{qtri,\{0\}}_{\rho_x,v,F} / TX^\text{cris}_{\rho_x,v,F}.$$ 

Moreover we have the following inequalities

$$\dim T_{(x,F)} \mathcal{E} \leq \dim TX^\text{qtri}_{\rho_x,F} \leq n[F : Q].$$

The first one is a consequence of (11). The last one is a consequence of the fact that we can compute all the dimensions in the exact sequence (12). Namely the Theorem 8.3 of Newton-Thorne assures that

$$H^1_f(G_F, ad(\rho_x)) = \{0\}.$$ 

The dimension of $TX^\text{qtri,\{0\}}_{\rho_x,v,F}$ comes from $\text{[BHS19]}$ Corollary 3.7.8 and Remark 4.1.6 (i) as $(\rho_x,v,F_v)$ is associated to a product of distinct transpositions for all $v \in \Phi$, and the dimension of $TX^\text{cris}_{\rho_x,v,F}$ comes from $\text{[Kis08]}$ Theorem 3.3.8 (we warn the reader that in these references framed deformation rings are considered, which are formally smooth of relative dimension $n^2 - 1$ over our rings).

As $\mathcal{E}$ is equidimensional of dimension $n[F : Q]$, we have $\dim T_{x,F} \mathcal{E} = n[F : Q]$ and thus $x,F$ is a smooth point of $\mathcal{E}$ and we have

$$\tilde{\mathcal{E}}_{x,F} \sim \mathcal{X}^\text{qtri}_{\rho_x,F},$$

and thus

$$T_{x,F} \mathcal{E} \cong TX^\text{qtri}_{\rho_x,F} \cong \bigoplus_{v \mid p} TX^\text{qtri,\{0\}}_{\rho_x,v,F} / TX^\text{cris}_{\rho_x,v,F}.$$ \hfill $\Box$
Theorem 8.6. — For $x \in \chi^{\text{mod,ag}}$, the image of the natural map
\[
\bigoplus_{\mathcal{F}} T_x, \mathcal{E} \longrightarrow T_x \chi^{\text{pol}}_\mathcal{F},
\]
has dimension at least $\frac{n(n+1)}{2} [F : \mathbb{Q}]$, where $\mathcal{F}$ runs over the $n! [F : \mathbb{Q}]$ refinements of $x$.

Proof. — Let $\mathcal{F}^{\text{nc}} = (\mathcal{F}^{\text{nc}}_v)_{v \in \Phi}$ be a refinement of $\rho_x$ such that $\mathcal{F}^{\text{nc}}_v$ is non critical for any $v \in \Phi$. It exists by definition of $\chi^{\text{mod,ag}}$. For all $(c_v) \in \mathbb{C}_n$, let $\mathcal{F} = (c_v \cdot \mathcal{F}^{\text{nc}}_v)$.

By Proposition 3.7 (ii) the pair $(D_{\text{rig}}(\rho_x, v), c_v \cdot \mathcal{F}^{\text{nc}}_v)$ is associated to a product of simple transposition for all $v \in \Phi$. Thus Corollary 8.5 implies that
\[
T_{x, \mathcal{E}} \cong T_x \chi^{\text{qtr},w_0}_{\rho_x,\mathcal{F}} / T_x \chi^{\text{cris}}_{\rho_x,\mathcal{F}},
\]
and moreover, for all $v|p$, the map
\[
\bigoplus_{c \in \mathbb{C}_n^p} T_{(x,c,\mathcal{F}^{\text{nc}})} \mathcal{E} \longrightarrow T_x \chi^{\text{pol}}_\mathcal{F} = T_x \chi^{\text{pol}}_{\rho_x} \longrightarrow T_x \rho_x / T_x \chi^{\text{cris}}_{\rho_x},
\]
is surjective by Proposition 3.7. In particular, the map
\[
\bigoplus_{c \in \mathbb{C}_n} T_{(x,c,\mathcal{F}^{\text{nc}})} \mathcal{E} \longrightarrow \bigoplus_{v|p} T_{x,v,v,c,c} \mathcal{E} \longrightarrow T_x \chi^{\text{pol}}_\mathcal{F} = T_x \chi^{\text{pol}}_{\rho_x} \longrightarrow T_x \rho_x / T_x \chi^{\text{cris}}_{\rho_x},
\]
is surjective by Corollary 8.5 and Proposition 3.7, and thus has rank at least $\frac{n(n+1)}{2} [F : \mathbb{Q}]$, thus the same is true for the map,
\[
\bigoplus_{\mathcal{F}} T_{(x,\mathcal{F})} \mathcal{E} \longrightarrow T_x \chi^{\text{pol}}_\mathcal{F}.
\]

Remark 8.7. — Note that we don’t actually need all the refinements (for a fixed $v$), only the $1 + \frac{n(n+1)}{2}$ refinements given by $c_v = c_{i,j} := (i, i-1, \ldots, j) \in \mathbb{C}_n$ with $i \geq j$ (starting from a non-critical one). But this is still more than just the $n$ well-positioned refinements for weakly generic points of Chenevier [Chell], even for $n = 3$.

Theorem 8.8. — Let $\overline{\mathcal{F}(\mathcal{P})} \subset \chi^{\text{pol}}_\mathcal{P}$ be the Zariski closure of the image of $\mathcal{E}(\mathcal{P})$. Then $\overline{\mathcal{F}(\mathcal{P})}$ is equidimensional of dimension $\frac{n(n+1)}{2} [F : \mathbb{Q}]$, and is an union of irreducible components of $\chi^{\text{pol}}_\mathcal{P}$.

Proof. — We have already proven that almost generic point are Zariski dense in $\overline{\mathcal{F}(\mathcal{P})}$ (see Theorem 7.9). We will prove, following [All16], that these points are smooth points of $\chi_\mathcal{P}$ whose local ring are $\frac{n(n+1)}{2} [F : \mathbb{Q}]$-dimensional. Let $x$ be such a almost generic point in $\overline{\mathcal{F}(\mathcal{P})}$, thus $\rho_x$ is irreducible and we can consider the polarised deformation space $\chi^{\text{pol}}_{\rho_x}$. Then by an argument of Kisin (see [Kis09a] and [All16, Thm. 1.2.1]),
\[
\chi^{\text{pol}}_{\rho_x} \cong (\chi^{\text{pol}}_{\mathcal{P}})_x.
\]
Thus we need to show that $X_{\rho_x}^{\text{pol}}$ is (formally) smooth of dimension $\frac{n(n+1)}{2} [F : \mathbb{Q}]$. But as $\rho_x$ is absolutely irreducible we can choose a lift $r_x$ to $\mathcal{G}_n$ and by Proposition 5.9 reduce to $X_{r_x}$. Remark here that because of Proposition 8.1 and Theorem 4.2, we can apply Proposition 5.7. Calculations on the dimension of deformation ring made in Proposition 5.7 show that we are thus reduced to show that $h^2(G_{F,S}, \text{ad}(r_x)) = 0$, or what is equivalent $h^1(G_{F,S}, \text{ad}(r_x)) = \frac{n(n+1)}{2} [F : \mathbb{Q}]$. But as $\rho_x$ is generic at $p$ by Proposition 8.1, by Remark 1.2.9 of [All16], we get $H^1_{\text{d}}(G_{F,S}, \text{ad}(\rho_x)) = H^1(F_{F,S}, \text{ad}(\rho_x))$ which vanishes by Brin-Thorne’s Theorem 8.3. Thus the following map is injective,

$$H^1(G_{F,S}, \text{ad}(\rho_x)) \longrightarrow \prod_{v|p} H^1(F_v, \text{ad}(\rho_x))/H^1_q(\text{ad}(\rho_x)).$$

But then we prove exactly as in [All16], Lemma 1.3.5, as our $x$ is HT-regular, that the space $X_{r_x}$ is formally smooth of dimension $\frac{n(n+1)}{2} [F : \mathbb{Q}]$, thus by Proposition 5.7 $X_{\rho_x}^{\text{pol}}$ is formally smooth of dimension $\frac{n(n+1)}{2} [F : \mathbb{Q}]$, but as it contains the local ring of the closure of $\mathcal{F}(\overline{\rho})$ at $\rho_x$, which is of dimension $\geq \frac{n(n+1)}{2} [F : \mathbb{Q}]$ by Theorem 8.6, both local rings are equal (and $\mathcal{F}(\overline{\rho})$ is smooth at these points).

**Remark 8.9.** — Recall that in the previous theorem $\mathcal{E}(\overline{\rho})$ and thus $\overline{\mathcal{F}(\rho)}$ depend on the choice of an auxiliary level $K^p$ outside $p$. We can ask how the closure of the infinite fern depends on $K^p$. If we let $K^p$ appear in the notations, we can at least have an optimal $K^p$.

**Corollary 8.10.** — There exists a level $K^p$ outside $p$, such that for all level $K'^p$ outside $p$, the Zariski closure of the infinite fern of tame level $K^p$, $\overline{\mathcal{F}(K^p)}(\overline{\rho})$, contains the infinite fern of level $K'^p$, $\overline{\mathcal{F}(K'^p)}(\overline{\rho})$.

**Proof.** — As $X_{\rho_x}^{\text{pol}}$ is the generic fiber of a noetherian excellent formal scheme, it has a finite number of connected component (see [Con99, Theorem 2.3.1]). Thus as the number of components in the closure of the infinite fern (by Theorem 8.8) grows with $K^p$, it eventually stabilizes.

**Corollary 8.11.** — Let $\chi : G_F \to \mathbb{O}^*$ be a continuous character satisfying Hypothesis 5.2. Let $\overline{\rho} : G_F \to \text{GL}_n(k)$ be a semi-simple polarized-by-$\chi$ continuously modular continuous representation. The Zariski closure of the set points of $X_{\overline{\rho}}^{\chi}$ which are holomorphically modular and crystalline at $p$ is a union of irreducible components of dimension $\frac{n(n+1)}{2} [F : \mathbb{Q}]$.

**Proof.** — This is a direct consequence of Lemma 5.5, Theorem 8.8 and Corollary 6.8.

We can now deduce the following corollary, which is due to Allen, [All19], for which we need to take care that automorphic points given by [All19] main’s theorem are indeed inside our infinite fern. So we assume the following hypothesis as in [All19].

**Hypothesis 8.12.** —

1. $p > 2$, is unramified in $E$ and every prime $v|p$ in $F$ splits in $E$.
2. $\bar{\varphi}(G_E(\zeta_p))$ is adequate, $\bar{\varphi}$ is polarized by $\chi$ i.e. $\bar{\varphi}^* \simeq \bar{\varphi} \otimes \chi^{p-1}$.
3. There exists a $\text{GL}_n$-automorphic representation $\Pi_v$, which is regular algebraic $\chi$-polarized cuspidal, such that $\rho_{\Pi_v}$ lifts $\bar{\varphi}$ and such that $\rho_{\Pi_v}$ is potentially diagonalisable for all $v|p$, and even ordinary for all $v|p$ if $p|n$. 


4. \( \chi \) satisfies \( \chi = \chi^c \) and satisfies a sign condition (see Hypothesis 5.2 and section 4)
5. \( H^0(G_u, \text{ad}(\overline{\rho})(1)) = 0 \) for all \( v|p \).

We still hope that hypothesis 1. is technical and we hope to be able to remove it, as for Theorem 8.8. It is unknown at the moment if all potentially crystalline representations are potentially diagonalisable, i.e. if we could relax hypothesis 3. to a classical modularity (for \( GL_n \), crystalline at \( p \) say). We hope that hypothesis 2. is unnecessary, but at the moment the main result of [All19] relies on it, and also on 5. We imagine that these could be removed using new results on local deformations rings (e.g. [BIP21]).

Recall that we had constructed the infinite fern inside \( \mathcal{X}_{\overline{\rho}}^{\text{pol}} \), when \( \chi = 1 \). Let \( \overline{\rho} \) as above, and let \( \psi_0 : G_{E,S} \to (\mathcal{O}')^\vee \) as in [CHT08, Lemma 4.1.5] and section 5, so that \( \psi_0 \psi_0^\vee = \chi \). Up to base change by \( \mathcal{O}'[1/p] \), we can and do assume everything is defined over \( \mathcal{O}'[1/p] \). As \( \overline{\rho} \) is \( GL_n \)-automorphic for \( \Pi_0 \) by the above assumption, we have \( \overline{\rho}_0 := \overline{\rho} \psi_0 \) is also \( GL_n \)-automorphic (for \( \Pi_0, \psi_0 \circ \text{Art}_E \)). If we assume that \( \overline{\rho}_0 \) is conveniently modular for our similitude unitary group \( G \) as in section 6, then by Lemma 5.5 we have the following diagram

\[
\begin{array}{ccc}
\mathcal{E}(\overline{\rho}_0) & \xrightarrow{pr_1} & \mathcal{X}_{\overline{\rho}_0}^{\text{pol}} \\
& f \downarrow & \downarrow \psi_0^1 \\
\mathcal{X}_{\overline{\rho}_0}^{\text{pol}} & \xrightarrow{\psi_0^1} & \mathcal{X}_{\overline{\rho}}^{\chi-\text{pol}}
\end{array}
\]

where \( f := \psi_0^{-1} \circ pr_1 \). We call infinite fern, denoted by \( \mathcal{F}(\overline{\rho}) \) the image of \( \mathcal{E}(\overline{\rho}) \) by the diagonal map (Note that it a priori depends on \( \overline{\rho}_0 \), thus \( \psi_0 \)). We have the following

**Corollary 8.13 (Allen).** — Assume the hypothesis 8.12. Then \( \overline{\rho}_0 := \overline{\rho} \psi_0 \) is conveniently modular, the generic fiber of the global deformation ring \( \mathcal{X}_{\overline{\rho}}^{\chi-\text{pol}} \) is equidimensional of dimension \( [F : \mathbb{Q}] \frac{\chi(n+1)}{2} \) and the infinite fern \( \mathcal{F}(\overline{\rho}) \) is Zariski dense in \( \mathcal{X}_{\overline{\rho}}^{\chi-\text{pol}} \) thus in \( \text{Spec}(R_{\overline{\rho}}^{\chi-\text{pol}}) \).

**In particular automorphic points are dense in** \( \text{Spec}(R_{\overline{\rho}}^{\chi-\text{pol}}) \).

**Proof.** — As the set of Hypothesis 8.12 contains strictly the hypothesis of Theorem 8.8, we have that the Zariski closure of \( \mathcal{F}(\overline{\rho}) \) (if non-empty!) is a union of connected components of \( \mathcal{X}_{\overline{\rho}}^{\chi-\text{pol}} \). Thus, it is enough to prove that each component of \( \mathcal{X}_{\overline{\rho}}^{\chi-\text{pol}} \) contains a points in the infinite fern, and by the reduction of Lemma 5.5 and considering \( \Pi_0, \psi_0 \) where \( \psi_0 \) given by [CHT08, Lem.4.1.5] , we can assume \( \chi = 1 \). By [All19], Corollary 5.3.3, we have that \( R_{\overline{\rho}}^{\chi-\text{pol}} \) is \( \mathcal{O} \)-flat, reduced, and complete intersection of the expected dimension, but we still need to check that the automorphic point in all components can be chosen to be in the infinite fern (i.e. holomorphic at infinity automorphic representations for \( GU \)). Let \( \mathcal{C} \) be an irreducible component of \( \mathcal{X}_{\overline{\rho}}^{\chi-\text{pol}} \), which is of the form \( \mathcal{C} = C^{\text{rig}} \) for an irreducible component \( C \) of \( \text{Spec}(R_{\overline{\rho}}^{\chi-\text{pol}}) \) ([All19, Lem. 1.2.3]). By [All19, Thm. 5.3.1 & 5.3.2], there is a \( GL_n \)-automorphic cuspidal point \( \Pi \) in \( C \), which is moreover unramified at places above \( p \), very regular, self dual, and such that \( \Pi \) is a smooth point of \( \mathcal{X}_{\overline{\rho}}^{\chi-\text{pol}} \) (see [All16, Thm. C]). Let \( \Pi_U \) be the global \( A \)-packet associate to \( \Pi \) (see [Mok15, 2.3] for example). Let \( v|\infty \) be an archimedean place of \( F \) and let \( \psi_v \) be the local Arthur parameter at \( v \) associated to \( \psi \).
There exist $a_i, b_i \in \mathbb{C}$ such that $a_i - b_i \in \mathbb{Z}$ for $1 \leq i \leq n$ and $\psi_1|_{\mathbb{C}^n}$ is of the form:

$$z \mapsto (z^{a_1}, z^{a_2}, \ldots, z^{a_n}).$$

As $\Pi$ is regular, algebraic and cuspidal, Clozel’s Purity Lemma [Clo90, Lem. 4.9] implies that there exists $w \in \mathbb{R}$ such that $a_i + b_i = w$ for all $1 \leq i \leq n$. As we have $j \in W_{\mathbb{R}}\mathbb{C}^n$ such that $j z j^{-1} = \overline{z}$, we can check that $\psi_1|_{\mathbb{C}^n}$ is conjugate to $z \mapsto (z^{-b_1} \overline{z}^{-a_1}, \ldots, z^{-b_n} \overline{z}^{-a_n})$.

As the weight is regular this implies that there exists $\sigma \in \mathcal{S}_n$ an involution such that $a_{\sigma(i)} = -b_i$ so that $w = 0$ and $\psi_1$ is tempered. Moreover the regularity of the weight implies also that $\psi_1$ is discrete (see [BC09, Rem. A.11.8]). As $\psi_1$ is tempered, discrete, regular algebraic its associated local A-packet is equal to the L-packet of discrete series. In particular, by [Mok15, Thm. 2.4.2], there exists $\pi_0$ a regular holomorphic algebraic, unramified above $p$ representation of the quasi-split unitary group $U$ in $\Pi_U$. As $\Pi$ is cuspidal, the A-packet $\Pi_\psi$ (i.e. the semisimple class of $\Pi$) is stable and thus $S_\psi = 1$ (see [Rog92, 2.2]), so that $\pi_0$ is also discrete and automorphic. As $\pi_0$ is tempered at infinity, it follows from the main result of [Wal84] that $\pi_0$ is cuspidal.

Choose an algebraic extension of its central character which is unramified at $p$, then by [LS19] there exists an extension of $\pi_0$ to a cuspidal, regular algebraic representation $\pi$ of $GU$. Moreover $\pi_{\psi}$ is also the holomorphic discrete series thus contribute to coherent cohomology in degree 0 and thus gives a point in the Eigenvariety $\mathcal{E}$ (for $GU$), whose Galois representation (given in Corollary A.9) is $\rho$. In particular, $\rho$ is conveniently modular, $F(\pi)$ intersects $\mathcal{C}$, and the corollary is proved.

Appendix A. Similitude Unitary groups, Tori, Base Change and Galois representations

Fix an isomorphism $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_p$. Let $E$ be (complex) CM number field and $F$ its maximal totally real subfield.

Definition A.1. — Let $n \geq 1$ be an integer. Let $\Pi$ be an automorphic representation of $GL_n(\mathbb{A}_E)$. Let $\rho : G_E \rightarrow GL_n(\overline{\mathbb{Q}}_p)$ be a continuous semisimple representation. We say that $\rho$ is strongly (resp. weakly) associated to $\Pi$ if for all finite place $v$ of $E$ (resp. for almost all $v$) not dividing $p$ and such that $\Pi_v$ is unramified, $\rho$ is unramified at $v$ and the semisimple class of $\rho(Frob_v)$ coincides with $\iota Sat_v(\Pi_v)$ where $Sat_v(\Pi_v)$ is the Satake parameter of $\Pi_v$.

By [HLT16] and [Sch15], for any cuspidal regular algebraic automorphic representation $\Pi$, there is a unique $\rho_{1}\Pi$, which is strongly associated to $\Pi$ and for any $\rho$ there is at most one $\Pi$ such that $\rho$ is weakly associated to $\Pi$.

Denote by $G = GU(V)$ a similitude unitary group over $\mathbb{Q}$ (with similitude factor in $\mathbb{Q}^\times$) associated to the CM extension $E/F$, and by $Z \simeq GU(E)$ its center.

Let $\ell$ be a rational prime, unramified in $E$, which is also unramified for $GU(V)$ (i.e. $GU(V)_{\mathbb{Q}_\ell}$ is quasi-split, and split over an unramified extension). Let $\pi$ be a cuspidal automorphic representation of $G$. Assume $\pi$ is unramified at $\ell$, and choose a maximal compact $K$ at $\ell$ for which $\pi$ is unramified. Then $\pi_{\ell}\overline{K}$ is a 1-dimensional representation of $H_{\mathcal{C}}(G(\mathbb{Q}_\ell), K)$, the Hecke algebra of bi-$K$-invariant $\mathbb{C}$-valued functions on $G(\mathbb{Q}_\ell)$ with
compact support. The Satake isomorphism and the unramified local Langlands correspondence ([Bor79]) associate to it an unramified representation with values in the $L$-group of $G$ (actually in the $L$-group of the maximal torus of $G$). Denote $T_U$, $T$ the maximal torus of $U = U(V)$ and $G$ respectively. The natural inclusion $T_U \subset T$, which is compatible with the Galois action and central, gives a map

\[ L_T \longrightarrow L_{T_U}. \]

Proposition A.2. — There is a natural map of $L$-groups

\[ L_{T_U} \longrightarrow L_{T_{GL_n,E}} := \mathbb{G}_m^n \text{Hom}(E, \mathbb{C}) \rtimes W_{Q_{\ell}}, \]

Denote by

\[ r : W_{Q_{\ell}} \longrightarrow L_{T_U}, \]

the unramified Langlands parameter associated to $\pi_\ell$ as above. For all $\lambda| |\ell$ in $F$ and $\lambda'| \lambda$ in $E$, the restriction of $r$ to $W_{E_{\lambda'}}$ followed by the previous map

\[ r : W_{E_{\lambda'}} \longrightarrow L_{T_{GL_n,E}}, \]

induces a well-defined class (up to conjugacy),

\[ r_{\lambda'} : W_{E_{\lambda'}} \longrightarrow GL_n(\mathbb{C}). \]

Proof. — As $\ell$ is unramified for $E$ and $G$ (thus $U$), actually $U_{Q_{\ell}}$ is isomorphic to $U(n)_{E:F,Q_{\ell}}$ for any choice of (unramified) unitary group of rank $n$, so choose the one with anti-diagonal matrix form. With this form, we check that actually the upper triangular Borel is indeed a Borel over $Q_{\ell}$, with maximal torus the diagonal one, given by

\[ T_U = \{ \text{Diag}(a_1, \ldots, a_n) | a_i \in E^n, c(a_i)a_{n+1-i} = 1 \} \subset T_{GL_n,E}, \]

with $c \in \text{Gal}(E/F)$ the complex conjugacy and $T_{GL_n,E}$ the diagonal torus of $\text{Res}_{E/Q} GL_n$. Denote by $\Sigma_E$ the complex embeddings of $E$. We then have that its characters $X^*(T_U)$ are given by the quotient of $(\mathbb{Z}^n)^{\Sigma_E} (= X^*(T_{GL_n,E}))$ by the relation $(\lambda, \sigma_i) = (\lambda_{n+1-i, \sigma_i}, \sigma_i)$. Its cocharacters $X^*_c(T_U) \subset (\mathbb{Z}^n)^{\Sigma_E} = X^*_c(T_{GL_n,E})$ are given by the collections $(\mu_i, \sigma_i)$ satisfying $\mu_i, \sigma_i = \mu^{-1}_{n+1-i, \sigma_i}$. The Galois action of $\sigma \in G_Q$ sends the character $\lambda_{r,i}$ to $\sigma \cdot \lambda_{r,i} = : \lambda_{r,i} \circ \sigma^{-1} = \lambda_{r,i}$. It sends the cocharacter $\mu_{r,i}$ to $\sigma \circ \mu_{r,i} = \mu_{\sigma r,i}$. Thus the dual torus is given by the subtorus of $\prod_{\Sigma_E} \mathbb{G}_m^n$, given by

\[ \widehat{T}_U = \{ (t_1^\sigma, \ldots, t_n^\sigma) | t_i^\sigma t_{n+1-i} = 1 \}. \]

The action of $W_{Q_{\ell}}$ on $\widehat{T}_U$ is given by $s \cdot (h_\sigma) = (h^{-1}_\sigma)$. A priori $E$ is not Galois over $Q$. An analogous computation for the maximal (diagonal) torus of $\text{Res}_{E/Q} GL_n$, $T_{GL_n,E}$, gives

\[ L_{T_{GL_n,E}} = \mathbb{G}_m^{n \Sigma_E} \rtimes W_{Q_{\ell}}, \quad s \cdot (t_i, \sigma) = (t_i, \sigma^{-1}), \]

and we thus have a natural map $L_{T_U} \longrightarrow L_{T_{GL_n,E}}$. As $\pi_\ell$ is unramified, by [Bor79] there is a parameter $r_G : W_{Q_{\ell}} \longrightarrow L_T$, which we can compose to get

\[ r : W_{Q_{\ell}} \longrightarrow L_{T_U}, \]

and by the previous map we get and unramified Langlands parameter $r_{GL_n,E} : W_{Q_{\ell}} \longrightarrow L_{T_{GL_n,E}}$. Restricting this last parameter to $W_{E_{\lambda'}}$, where $W_{E_{\lambda'}} \longrightarrow W_{Q_{\ell}}$ is induced by

This completes the proof of the proposition.
some \( i : E_{\lambda'} \to \overline{\mathbb{Q}_p} \), we get \( W_{E_{\lambda'}} \to L T_{G_{L_{\lambda}}, E} \times W_{E_{\lambda'}} \). Fix an isomorphism \( \phi : \overline{\mathbb{Q}_\ell} \to \mathbb{C} \), so we can identify complex \( \ell \)-adic embeddings of \( E \). But the action of \( W_{E_{\lambda'}} \) fixes the \( \sigma \in \Sigma_E \) over \( \lambda' \), and we can thus project to any such using \( \text{pr}_E : L T_{G_{L_{\lambda}}, E} \to \text{GL}_n \), so choose \( \lambda' = i \circ (E \to E_{\lambda'}) \), the one corresponding to our embedding \( W_{E_{\lambda'}} \to W_{\mathbb{Q}_\ell} \), and denote

\[
\lambda' : W_{E_{\lambda'}} \to L T_{G_{L_{\lambda}}, E} \to \mathbb{C}_p^m, w \mapsto r_{G_{L_{\lambda}}, E}(w) = (h_\sigma) \sigma \mapsto h_{\sigma}. 
\]

Let us show that this is well defined and independent of choices of \( i \) and \( \phi \). Let \( i, j : E_{\lambda'} \to \overline{\mathbb{Q}_\ell} \) two choices. There exists \( s \in W_{\overline{\mathbb{Q}_\ell}} \) such that \( s \circ j = i \). These two maps induce two maps \( W_{E_{\lambda'}} \overset{i, j}{\to} W_{\overline{\mathbb{Q}_\ell}} \), such that \( j_*(-) = s^{-1}i_*(-)s \).

Moreover using the canonical map \( E \to E_{\lambda'} \) this induces two embeddings \( \sigma_{\lambda'}^i, \sigma_{\lambda'}^j : E \to \overline{\mathbb{Q}_\ell} \) above \( \lambda' \) such that \( \sigma_{\lambda'}^j = s \circ \sigma_{\lambda'}^i \). So we compute,

\[
r_{G_{L_{\lambda}}, E}(j_* w) = r_{G_{L_{\lambda}}, E}(s^{-1}i_* w s) = x \times s^{-1}(h_\sigma) \sigma \times w(x \times s^{-1})^{-1} = (x(h_\sigma)s^{-1}ws^{-1} \times w),
\]

which is mapped under projection to the embedding \( \sigma_{\lambda'}^i \) to

\[
x_{\sigma_{\lambda'}^i} h_{\sigma} x_{\sigma_{\lambda'}^i}^{-1} w^{-1} s_{\sigma_{\lambda'}^i}^{-1},
\]

but this is commutative, and \( w^{-1}s_{\sigma_{\lambda'}^i}^{-1} = s_{\sigma_{\lambda'}^i}^{-1} \), as \( w \in W_{E_{\lambda'}} \), thus we get

\[
x_{\sigma_{\lambda'}^i} x_{\sigma_{\lambda'}^j}^{-1} h_{\sigma} x_{\sigma_{\lambda'}^j}^{-1},
\]

i.e. \( \pi_{\sigma_{\lambda'}^i} \circ r_{G_{L_{\lambda}}, E} \circ i_* = \pi_{\sigma_{\lambda'}^j} \circ r_{G_{L_{\lambda}}, E} \circ j_* \) is well defined and independent of the choice of \( i \). Now assume that \( \phi, \phi' \) are different isomorphisms \( \overline{\mathbb{Q}_\ell} \to \mathbb{C} \). So for each \( i : E_{\lambda'} \to \overline{\mathbb{Q}_\ell} \) we get two embeddings of \( E \), namely \( \sigma_{\lambda'}^i \) and \( \sigma_{\lambda'}^j = \sigma_{\lambda'}^i \circ \phi \in G_{\overline{\mathbb{Q}_p}} \). Thus we are reduced to the previous computation with two different embedding above \( \lambda' \). Thus \( r_{\lambda'} = r_{\lambda'} \circ i_{\sigma} \) depends only on the choice of \( \lambda' \) \( \ell \) in \( E \).

Using the previous Proposition, to \( \pi \) we can for all unramified \( \ell \) associate to \( \pi_{\ell} \) a semi-simple conjugacy class in \( L T_{U} \) and for all \( \lambda' \) \( \ell \) in \( E \) a system of semi-simple conjugacy classes \( C_{\lambda'} = r_{\lambda'}(\text{Frob}_{\lambda'}) \) in \( \text{GL}_n \). We denote \( \text{Sat}(\pi_{\ell}) = (\text{Sat}(\lambda_{\pi_{\ell}}))_{\lambda'} = (C_{\lambda'} \setminus \det \pi_{\ell})_{\lambda'} \).

**Definition A.3.** — Fix an isomorphism \( i : \mathbb{C} \cong \overline{\mathbb{Q}_p} \). Let \( \rho : G_{E} \to \text{GL}_n(\overline{\mathbb{Q}_p}) \). We say that \( \rho \) is strongly (resp. weakly) essentially associated to \( \pi \) if for all \( \ell \neq p \) (resp. for almost all \( \ell \neq p \)), unramified in \( E \) and for \( \pi \), for all \( \lambda' \), \( \rho \) is unramified at \( \lambda' \) and the semi simple class of \( \rho(\text{Frob}_{\lambda'}) \) and \( i_{\text{Sat}}(\lambda_{\pi_{\ell}}) \) coincides. We say that \( \rho \) is modular if there exists a cuspidal \( \pi \) as before such that \( \rho \) is strongly essentially associated to \( \pi \).

**Remark A.4.** — 1. This is not the natural definition, it would be more adequate to say essentially modular. The reason is that because we want to work at fixed polarisation character, we have ignored the part of the similitude character for \( \pi \) when looking at \( \text{Sat}(\pi_{\ell}) \). We could do an analogous definition keeping track of the similitude character, but it would be more complicated to describe it, in particular at non split primes when \( E/\mathbb{Q} \) is not Galois.
2. It is enough to check the compatibility with the Satake parameter at \( \ell \) totally split in \( E \), in which case the previous association is easier to describe. Indeed, by Chebotarev density theorem the totally split primes in \( E \) have density 1, thus \( \rho \) is completely determined by the conjugacy class of Frobenius at those primes. Moreover, every \( \lambda = \lambda' \chi | \ell \) is split above \( F \) (with \( \lambda \) is a prime of \( F \). Thus \( GU_{\mathbb{A}_L} \cong (\prod_{\lambda} \mathbb{A}_{\ell, \lambda} \times \mathbb{A}_{\ell, \lambda})^{(23)} \) and the Satake parameter (for \( GU \)) associated to \( \pi_\ell \) has the form
\[
(\text{Diag}(t_1^\lambda, \ldots, t_n^\lambda), x).
\]
Then \( \text{Sat}(\pi_\ell) \) is just the collection
\[
((|\det|^{\frac{n-1}{2}} \text{Diag}(t_1^\lambda, \ldots, t_n^\lambda))_{\lambda}, (|\det|^{\frac{1-n}{2}} \text{Diag}(t_1^{\lambda,\ell}, \ldots, t_n^{\lambda,\ell}))_{\lambda^*}).
\]
3. A modular \( \rho \) is automatically polarized-by-1 (i.e. \( \rho^\vee \cong \rho \otimes \varepsilon_2 \)). Indeed, elements \( t \in T_{\mathbb{F}} \) satisfies \( t^{-1} = w_0 \cdot t^\ell \), where \( w_0 \) is the longest Weyl element of \( GL_n \), thus (because of the twist) \( i \text{Sat}_{\lambda^*}(\pi_\ell) = i \text{Sat}_{\lambda_0}(\pi_\ell) \rho_\ell \). By Chebotarev, this proves the claim.

**Definition A.5.** — We say that a cuspidal automorphic representation \( \pi \) of \( G \) or \( U(V) \) is sufficiently regular if it is a discrete series at infinity and satisfy property (\( \ast \)) of [Lab1] Section 5.1. This is automatic if the parameter at infinity is regular enough (by [Har90a, Lemma 3.6.1] and Mirkovic’s Theorem [Har90a, Theorem 3.5]).

**Remark A.6.** — Because of [Har90a, Lemma 3.6.1] and Mirkovic, there exists \( C > 0 \) such that points of \( Z_C \) (see Proposition 6.6) are sufficiently regular in the previous sense.

**Theorem A.7.** — Let \( \pi \) be a cuspidal automorphic representation of \( G = GU(V) \) which is cohomological and sufficiently regular. There exists \( L \) a Levi subgroup of \( \text{Res}_{E/Q} \mathbb{G}_E \), a cuspidal automorphic representation \( \Pi_L \) of \( L(A) \) together with an automorphic character \( \chi_L \) of \( L(A) \) such that \( \Pi_L \otimes \chi_L^{-1} \) is \( \theta_L \) stable and \( \pi \) and \( \Pi_L \) corresponds to each other at all unramified (for \( \pi \) and \( E \)) finite places. Moreover each factor of \( \Pi_L = \Pi_1 \oplus \Pi_2 \oplus \cdots \Pi_r \) is regular algebraic.

**Proof.** — If \( F = \mathbb{Q} \) this is [Mor1] Corollary 8.5.3, except the last part. But by Shin’s appendix [Gol1](24) Theorem 1.1 (iii), \( \Pi_1 \oplus \Pi_2 \oplus \cdots \Pi_r \) is moreover regular algebraic as \( \pi \) is. Remark that in this case we don’t need \( \pi \) to be sufficiently regular, just cohomological. If \( [F : \mathbb{Q}] \geq 2 \), then we will use [Lab1], thus we need the following lemma. Let us introduce some notation. Let \( Z = \{ x \in E^\times | N_{E/F}(x) \in \mathbb{Q}^\times \} \) and \( Z^1 := \text{Ker}(N_{E/F}) \subset Z \). Then \( Z, Z^1 \) are tori. Moreover we have a maps
\[
0 \longrightarrow Z^1 \longrightarrow Z \times U \longrightarrow G \longrightarrow 0,
\]
and the last map is surjective on geometric points. Note that if \( \ell \) is a prime of \( \mathbb{Q} \), splitting in \( E \), then the sequence (1) is exact on \( \mathbb{Q}_{\ell} \)-points.

**Lemma A.8.** — Let \( \pi \) be an irreducible discrete automorphic representation of \( G \) such that \( \pi, \zeta \) is cohomological for \( \zeta \). Then there exists an automorphic discrete representations \( \psi \otimes \pi_0 \) of \( Z(A) \times U(A) \) such that

(\(23\)) We should choose a CM type to write this isomorphism properly.

(\(24\)) This more generally applies if \( E \) contains an imaginary quadratic field.
1. The restriction of $\psi \otimes \pi_0$ to the image of $Z^1(\mathcal{A})$ is trivial;
2. $\psi = \psi_x$, the restriction of $\pi$ to $Z$;
3. For all place $\ell$ of $\mathbb{Q}$, splitting in $E$, we have $(\pi_\ell)|_{Z(\mathbb{Q}_\ell) \times U(\mathbb{Q}_\ell)} \simeq \psi_\ell \otimes \pi_{0,\ell}$;
4. $\pi_{0,\ell}$ is cohomological for $\xi|^\ell$, thus regular;
5. $\psi_x = \xi_{E^p}^{-1}$.
6. If $\ell$ is a prime which is unramified in $E$, then $\pi_0$ is unramified if $\pi$ is.

**Proof.** — This is analogous to the proof of [HT01] Theorem VI.2.1. Choosing $(g_i)$ in $G(\mathcal{A})$ such that the $\nu(g_i)$ are representatives of the set $\nu(G(\mathcal{A}))/\nu(G(\mathbb{Q}))N_{E/F}(\mathbb{Z}(\mathcal{A}))$ we get as in [HT01],

\[
\mathcal{A}(G(\mathbb{Q})/G(\mathbb{A})) \rightarrow \bigoplus_i \mathcal{A}((Z \times U)(\mathbb{Q})/(Z \times U)(\mathcal{A}))^{Z^1(\mathcal{A})} \\
\xi \mapsto (g_i \cdot \mathcal{A}((Z \times U)(\mathbb{Q})/(Z \times U)(\mathcal{A})))^{Z^1(\mathcal{A})} \xi,
\]

where the $g_i \cdot f$ denotes the right translate of $f$. As a consequence we have an isomorphism of $(Z \times U)(\mathcal{A})$-representations

\[
\mathcal{A}(G(\mathbb{Q})/G(\mathbb{A}))|_{(Z \times U)(\mathcal{A})} \simeq \bigoplus_i \mathcal{A}((Z \times U)(\mathbb{Q})/(Z \times U)(\mathcal{A}))^{Z^1(\mathcal{A})} \xi_i
\]

where the upper script $g_i$ denotes a conjugate action by $g_i$. This shows that, if $\pi$ is an automorphic representation of $G(\mathbb{A})$ and if $\pi'$ is an irreducible subquotient of $\pi|_{(Z \times U)(\mathcal{A})}$, a conjugate of $\pi'$ by one of the $g_i$ is automorphic and trivial on $Z^1(\mathcal{A})$. Let $\psi \otimes \pi_0$ be an automorphic representation of $(Z \times U)(\mathcal{A})$ whose conjugate by one of the $g_i$ is isomorphic to a subquotient of $\pi|_{(Z \times U)(\mathcal{A})}$. Moreover, since $\pi$ is cohomological for $\xi$, there exists an integer $i$ such that $H^i((\text{Lie } G(\mathbb{R})) \hat{\otimes}_{\mathbb{R}} \mathbb{C}, U_{\infty}, \pi_{0,\ell} \otimes \xi^i) \neq 0$ (for $U_{\infty} \subset U(\mathbb{R})$ a maximal compact subgroup of $G(\mathbb{R})$). So we can choose $\psi \otimes \pi_0$ such that $H^i((\text{Lie } (Z \times U)(\mathbb{R})) \hat{\otimes}_{\mathbb{R}} \mathbb{C}, U_{\infty}, \psi \otimes \pi_{0,\ell} \otimes \xi^i|_{(Z \times U)(\mathbb{R})}) \neq 0$. This proves that $\pi_0$ satisfies property 4 of the statement. The property 1 has already been checked and 2 is clear since $Z(\mathcal{A})$ is in the center of $G(\mathbb{A})$. Property 3 is a direct consequence of the fact that if $\ell$ is a prime that splits in $E$, the map $(Z \times U)(\mathbb{Q}_\ell) \rightarrow G(\mathbb{Q}_\ell)$ is surjective on kernel $Z^1(\mathcal{Q}_\ell)$. Now assume that $\ell$ is unramified in $E$. If $\pi$ is unramified at $\ell$, then $\pi$ has non zero fixed vector under an hyperspecial subgroup of $G(\mathbb{Q}_\ell)$. As the image of $Z(\mathbb{Q}_\ell) \times U(\mathbb{Q}_\ell)$ has a finite index in $G(\mathbb{Q}_\ell)$, the restriction of $\pi_{0,\ell}$ to $U(\mathbb{Q}_\ell)$ is isomorphic to a finite direct sum of irreducible representation of $U(\mathbb{Q}_\ell)$ which are conjugated in $G(\mathbb{Q}_\ell)$. As the intersection of an hyperspecial subgroup of $G(\mathbb{Q}_\ell)$ with $U(\mathbb{Q}_\ell)$ is an hyperspecial subgroup of $U(\mathbb{Q}_\ell)$, all irreducible subquotients of $\pi|_{U(\mathbb{Q}_\ell)}$ have nonzero fixed vectors under some hyperspecial subgroup of $U(\mathbb{Q}_\ell)$. This proves property 6. Property 5 is a direct consequence of the equality $\pi|_{E^p} = \xi_{E^p}^{-1}$ following from the fact that $\pi$ is cohomological for $\xi$.

Thus by [Labl1] Cor. 5.3 applied to $\pi_0$, which is sufficiently regular thus satisfies property $(\ast)$, there is a weak base change i.e. $L$ a standard Levi of $\text{Res}_{E/F} \text{GL}_{n,E}$ that is $\theta$-stable, and a $\theta_L$-stable discrete automorphic representation of $L$ $\Pi_L = \Pi'|_L \otimes \cdots \Pi'|_L$ such that $\Pi'_1 \boxtimes \Pi'_2 \boxtimes \cdots \boxtimes \Pi'_r$ is a weak base change for $\pi_0$. As each $\Pi'_i$ is discrete, then by the main theorem of [MW89] we can write $\Pi'_i$ as an automorphic induction of $\tau_i \otimes \text{Sp}(\ell_i)$ for an integer $\ell_i$ and $\tau_i$ a cuspidal automorphic representation of $\text{GL}_{n_i,\ell_i}(\mathbb{A}_E)$. But the proof of [Mor10] 8.5.6 shows that as each $\Pi'_i$ is $\theta_{n_i,\ell_i}$-stable, each $\tau_i$ is $\theta_{n_i,\ell_i}$-stable. In particular, up to reduce $L$, choosing $(\Pi_i)_{j=1,\ldots,r}$ to be the collection $(\tau_i |_{L^{n_i,\ell_i}})$.
for $i = 1, \ldots, s$ and $k = 1, \ldots, \ell$, we get that $\Pi_j$ is cuspidal, and $\Pi_j$ is $\theta$-stable up to twist (by an automorphic character of $L$). But by [Lab11, Cor. 5.3] again we know that the infinitesimal characters of $\pi_0$ and $\Pi'_1 \boxplus \cdots \boxplus \Pi'_r = \Pi_1 \boxplus \cdots \boxplus \Pi_r$ coincides after base change, in particular the latter representation is regular algebraic. Moreover at unramified places this is compatible with the local base change. □

**Corollary A.9.** — Let $\pi$ be a cuspidal automorphic representation of $G = GU(V)$ which is cohomological, sufficiently regular, and unramified outside $S$, which contains ramified places of $E$. Then to $\pi$ is (strongly) essentially associated a unique Galois representation,

$$\rho^\pi : G_{E,S} \longrightarrow GL_n(\mathbb{Q}_p),$$

satisfying,

$$(\rho^\pi)^\vee \simeq (\rho^\pi)^c \epsilon^{n-1}.$$  

In particular, for all prime $\lambda = vF$ of $F$ split in $E$, not in $S$, we have that the semi-simple conjugacy class of $\rho^\pi(Frob_v)$ is equal to the image of the Satake parameter of $\pi_{0,\lambda}(\det |^{1/2})$, seen as a representation of $U(F) \cong GL_n_E$.

**Proof.** — The previous proof allow us to reduce to $\pi_0$ an automorphic representation of $U$ whose weak base change is $\Pi'_1 \boxplus \cdots \boxplus \Pi'_r$; $\theta$-stable, each $\Pi'_i$ being automorphic for $GL_{n_i}$, cuspidal, conjugate self dual up to twist by a character. Thus to $\pi_0$ we can associate by [CH13] again

$$\rho^\pi = \rho_{\Pi'_1} \oplus \cdots \oplus \rho_{\Pi'_r}.$$  

As $\Pi'_1 \boxplus \cdots \boxplus \Pi'_r$ is $\theta$-stable, $\rho^\pi$ satisfies $(\rho^\pi)^\vee \simeq (\rho^\pi)^c \epsilon^{n-1}$. On the other hand, we know the compatibility of the association of $\rho^\pi$ with local Langlands: at all ramified primes $\rho^\pi_v = LL(\pi_{0,v}) |^{1/2}$, i.e. $\rho^\pi$ is strongly associated to $\pi$. □

**References**

[AIP15] F. Andreatta, A. Iovita & V. Pilloni – “$p$-adic families of Siegel modular cuspforms.”, *Ann. Math. (2)* 181 (2015), no. 2, p. 623–697 (English).

[All16] P. Allen – “Deformations of polarized automorphic galois representations and adjoint selmer groups”, *Duke Math. J.* 165 (2016), no. 13, p. 2407 – 2460.

[All19] ________, “On automorphic points in polarized deformation rings”, *Amer. J. Math.* 141 (2019), no. 1, p. 119–167.

[Bö1] G. Böckle – “On the density of modular points in universal deformation spaces”, *Amer. J. Math.* 123 (2001), no. 5, p. 985–1007.

[BC09] J. Bellaïche & G. Chenevier – *Families of Galois representations and Selmer groups*, Paris: Société Mathématique de France, 2009 (English).

[BC11] ________, “The sign of Galois representations attached to automorphic forms for unitary groups”, *Compos. Math.* 147 (2011), no. 5, p. 1337–1352 (English).

[Bel09] J. Bellaïche – “An introduction to the conjecture of Bloch and Kato”, *Lectures at the Clay Mathematical Institute summer School, Honolulu, Hawaii* (2009).

[Ber02] L. Berger – “$p$-adic representations and differential equations”, *Invent. Math.* 148 (2002), no. 2, p. 219–284 (French).

[Ber20] J. Bergdall – “Smoothness of definite unitary eigenvarieties at critical points”, *J. Reine Angew. Math.* 759 (2020), p. 29–60 (English).
[BGGT14] T. Barnet-Lamb, T. Gee, D. Geraghty & R. Taylor – “Compatibilité entre les correspondances de Langlands locales aux places divisant \( l = p \). II.”, *Ann. Sci. Éc. Norm. Supér. (4)* 47 (2014), no. 1, p. 165–179 (English).

[BHS17] C. Breuil, E. Hellmann & B. Schraen – “Une interprétation modulaire de la variété trianguline”, *Math. Ann.* 367 (2017), no. 3-4, p. 1587–1645.

[BHS19] ———, “A local model for the trianguline variety and applications”, *Publ. Math. Inst. Hautes Études Sci.* 130 (2019), p. 299–412.

[Bij6] S. Bijakowski – “Analytic continuation on Shimura varieties with \( \mu \)-ordinary locus.”, *Algebra Number Theory* 10 (2016), no. 4, p. 843–885 (English).

[BIP21] G. Böckle, A. Iyengar & V. Paškunas – “On local galois deformation rings”, *arXiv preprint arXiv:2110.01638* (2021).

[BLGGT14a] T. Barnet-Lamb, T. Gee, D. Geraghty & R. Taylor – “Potential automorphy and change of weight”, *Ann. Math. (2)* 179 (2014), no. 2, p. 501–609 (English).

[BLGGT14b] ———, “Potential automorphy and change of weight”, *Ann. of Math. (2)* 179 (2014), no. 2, p. 501–609.

[Bor79] A. Borel – “Automorphic L-functions”, *Automorphic forms, representations and L-functions*, Proc. Symp. Pure Math. Am. Math. Soc., Corvallis/Oregon 1977, Proc. Symp. Pure Math. 33, 2, 27-61 (1979), 1979.

[Bor91] ———, *Linear algebraic groups. 2nd enlarged ed.*, 2nd enlarged ed. ed., vol. 126, New York etc.: Springer-Verlag, 1991 (English).

[Bou] N. Bourbaki – *Éléments de mathématique. Groupes et algèbres de Lie*, Actualités Scientifiques et Industrielles, No. 1364. Hermann, Paris.

[Box15] G. Boxer – “Torsion in the coherent cohomology of shimura varieties and galois representations”, *arXiv preprint arXiv:1507.05922* (2015).

[BP20] G. Boxer & V. Pilloni – “Higher coleman theory”, (2020).

[Car72] R. W. Carter – “Conjugacy classes in the weyl group”, *Compos. Math.* 25 (1972), no. 1, p. 1–59.

[Carl2] A. Caraiani – “Local-global compatibility and the action of monodromy on nearby cycles”, *Duke Math. J.* 161 (2012), no. 12, p. 2311–2413.

[CH13] G. Chenevier & M. Harris – “Construction of automorphic Galois representations. II.”, *Camb. J. Math.* 1 (2013), no. 1, p. 53–73 (English).

[Che04] G. Chenevier – “Familles \( p \)-adiques de formes automorphes pour \( GL_n \).”, *J. Reine Angew. Math.* 570 (2004), p. 143–217 (French).

[Che11] ———, “Sur la fougère infinie des représentations galoisiennes de type unitaire”, *Ann. Sci. Éc. Norm. Supér. (4)* 44 (2011), no. 6, p. 963–1019 (French).

[Che14] ———, “The \( p \)-adic analytic space of pseudocharacters of a profinite group and pseudorepresentations over arbitrary rings”, in *Automorphic forms and Galois representations. Proceedings of the 94th London Mathematical Society (LMS) – EPSRC Durham symposium, Durham, UK, July 18–28, 2011. Volume 1*, Cambridge: Cambridge University Press, 2014, p. 221–285 (English).

[CHLN11] L. Clozel, M. Harris, J.-P. Labesse & B.-C. Ngô (eds.) – *Stabilization of the trace formula, Shimura varieties, and arithmetic applications. Volume 1: On the stabilization of the trace formula*, Somerville, MA: International Press, 2011 (English).

[CHT08] L. Clozel, M. Harris & R. Taylor – “Automorphy for some \( l \)-adic lifts of automorphic mod \( l \) Galois representations. With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras.”, *Publ. Math., Inst. Hautes Étud. Sci.* 108 (2008), p. 1–181 (English).
Valentin Hernandez, Université Paris-Saclay, CNRS, Laboratoire de mathématiques d'Orsay, 91405, Orsay, France
E-mail: valentin.hernandez@math.cnrs.fr

Benjamin Schraen, Université Paris-Saclay, CNRS, Laboratoire de mathématiques d'Orsay, 91405, Orsay, France
Institut Universitaire de France  ●  E-mail: benjamin.schraen@universite-paris-saclay.fr