PROPAGATION OF REGULARITY FOR THE MHD SYSTEM IN OPTIMAL SOBOLEV SPACE

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Abstract. We study the problem of propagation of regularity of solutions to the incompressible viscous non-resistive magneto-hydrodynamics system. According to scaling, the Sobolev space \( H^{s-1/2}(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \) is critical for the system. We show that if a weak solution \((u(t), b(t))\) is in \( H^s(\mathbb{R}^n) \times H^{s+1}(\mathbb{R}^n) \) with \( s > \frac{n}{2} - 1 \) at a certain time \( t_0 \), then it will stay in the space for a short time, provided the initial velocity \( u(0) \) is in \( H^s(\mathbb{R}^n) \). In the case that the uniqueness of weak solution in \( H^s(\mathbb{R}^n) \times H^{s+1}(\mathbb{R}^n) \) is known, the assumption of \( u(0) \) being in \( H^s(\mathbb{R}^n) \) is not necessary.

KEY WORDS: Magneto-hydrodynamics; propagation of regularity; optimal Sobolev spaces.

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1. Introduction

In this paper we consider the incompressible viscous non-resistive magneto-hydrodynamics (MHD) system:

\[
\begin{align*}
u & u_t + u \cdot \nabla u - b \cdot \nabla b + \nabla p = \nu \Delta u, \\
b_t + u \cdot \nabla b - b \cdot \nabla u = 0, \\
\nabla \cdot u &= 0, \\
\nabla \cdot b &= 0,
\end{align*}
\]

with the initial conditions

\[
\begin{align*}
u u(x, 0) &= u_0(x), \\
b(x, 0) &= b_0(x), \\
\nabla \cdot u_0 &= \nabla \cdot b_0 = 0,
\end{align*}
\]

where \( x \in \mathbb{R}^n \) with \( n \geq 2 \), \( t \geq 0 \), \( u \) is the fluid velocity, \( p \) is the fluid pressure, \( b \) is the magnetic field, and \( \nu > 0 \) is the viscosity coefficient. System (1.1) describes the dynamics of magnetic field in electrically conducting fluid, for instance, plasmas and salt water. It has been extensively investigated by mathematicians in the last few decades. The quantitative properties of solutions in critical spaces (with respect to the scaling) have arisen great interest. It is known that system (1.1) has the following scaling,

\[
u u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), \quad b_\lambda(x, t) = \lambda b(\lambda x, \lambda^2 t), \quad p_\lambda(x, t) = \lambda^2 p(\lambda x, \lambda^2 t)
\]

solves the system if \((u(x, t), b(x, t), p(x, t))\) does so, with accordingly scaled initial data. For the Navier-Stokes equation in (1.1), \( H^{s-1}(\mathbb{R}^n) \) is scaling invariant (also called being critical). For the magnetic field equation in (1.1), one would expect that \( H_\lambda^{s+1}(\mathbb{R}^n) \) is critical, since it is analogous with the Euler equation. However, the local well-posedness of (1.1) in \( H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \) for \( s > \frac{n}{2} \) established by Fefferman, McCormick, Robinson and Rodrigo suggests that \( H_\lambda^{s+1}(\mathbb{R}^n) \), rather than \( H_\lambda^{s+1}(\mathbb{R}^n) \), is critical for the magnetic field equation. This “inconsistency” with Euler equation may be explained by the fact that the magnetic field equation...
is linear in \(b\), while the Euler equation is nonlinear in \(u\). Based on the analysis, one can assert that \(H^{\frac{n}{2}}(\mathbb{R}^n) \times H^{\frac{n}{2}}(\mathbb{R}^n)\) is critical for the non-resistive MHD system \((1.1)\).

With insight from the scaling argument, it is natural to seek the optimal space for local well-posedness, which would be \(H^s(\mathbb{R}^n) \times H^{s+1}(\mathbb{R}^n)\) for \(s > \frac{n}{2} - 1\). Indeed, Fefferman, McCormick, Robinson and Rodrigo \(5\) first showed that system \((1.1)\) is locally well-posed in \(H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)\) for \(s > \frac{n}{2}\). Later, the same authors \(9\) improved the local well-posedness space to \(H^{s+\varepsilon}(\mathbb{R}^n) \times H^{s+1}(\mathbb{R}^n)\) for \(s > \frac{n}{2} - 1\) and a small enough constant \(\varepsilon > 0\). The reason \(\varepsilon\) cannot be taken 0 is that the maximal regularity estimate for the Stokes equation from \(H^s\) to \(L^1(0, T; H^{s+2})\) cannot be obtained. On the other hand, the authors of \(2\) established local existence for the system in the critical Besov space \(B_{2,1}^{\frac{n}{2}+1} \times B_{2,1}^{\frac{n}{2}}\).

This paper concerns the problem of propagation of regularity of solutions to the non-resistive MHD system \((1.1)\) in the optimal Sobolev spaces \(H^s(\mathbb{R}^n) \times H^{s+1}(\mathbb{R}^n)\) for \(s > \frac{n}{2} - 1\). The problem of propagation of regularity for the Navier-Stokes equation (NSE) was first studied by Leray in \(10\). Leray showed that if a weak solution of the NSE is regular at certain time \(t_0\), the solution will stay regular for a short time on \((t_0, T)\); and an estimate on \(T - t_0\) was obtained. Since the global regularity remains open, such finding regarding regularity propagation is of great interest. Back to the non-resistive MHD system, we will show that if a weak solution is in \(H^s(\mathbb{R}^n) \times H^{s+1}(\mathbb{R}^n)\) for \(s > \frac{n}{2} - 1\) at some time \(t_0 > 0\), then it will stay in the same space for a short time. The main ingredient to achieve the goal is the type of maximal regularity estimate for the Stokes equation established in Lemma 2.3. This lemma also has its own interest, for it explains the failure to obtain a solution \(u\) of the Stokes equation in \(L^1(0, T; H^{s+2})\) provided the initial data \(u_0 \in H^s(\mathbb{R}^n)\), which is the essential obstacle to remove \(\varepsilon\) in the local well-posedness of \(2\). The estimate obtained in Lemma 2.3 reveals that the obstruction to gain two derivatives in \(L^1(0, T)\) is at the initial time. Precisely, the norm of the solution in \(L^1(0, T; H^{s+2})\) may blow up on the time interval \((\tau, T)\) like \(\log(T/\tau)\) as \(\tau \to 0\).

Our main result states as follows.

**Theorem 1.1.** Let \((u, b)\) be a Leray-Hopf weak solution of \((1.1)-(1.2)\). Assume \(u_0 \in H^s(\mathbb{R}^n)\) with \(s > \frac{n}{2} - 1\) and \(b_0 \in L^2(\mathbb{R}^n)\). In addition, assume that \(u(t_0) \in H^s(\mathbb{R}^n)\) and \(b(t_0) \in H^{s+1}(\mathbb{R}^n)\) for a time \(t_0 > 0\). Then there exists a time \(T = T(||u_0||_{H^s}, ||u(t_0)||_{H^s}, ||b(t_0)||_{H^{s+1}}) > t_0\) such that

\[
    u \in L^\infty(t_0, T; H^s(\mathbb{R}^n)) \cap L^2(t_0, T; H^{s+1}(\mathbb{R}^n)),
\]

\[
b \in L^\infty(t_0, T; H^{s+1}(\mathbb{R}^n)).
\]

In the case that the Leray-Hopf weak solution is unique in the space \(H^s(\mathbb{R}^n) \times H^{s+1}(\mathbb{R}^n)\) with \(s > \frac{n}{2} - 1\), the assumption of \(u_0 \in H^s(\mathbb{R}^n)\) in Theorem 1.1 is not necessary. Namely, we can show the result below.

**Corollary 1.2.** Assume \((u_0, b_0) \in L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\). Let \((u, b)\) be the unique Leray-Hopf weak solution of \((1.1)-(1.2)\) in \(H^s(\mathbb{R}^n) \times H^{s+1}(\mathbb{R}^n)\) with \(s > \frac{n}{2} - 1\). Assume that \(u(t_0) \in H^s(\mathbb{R}^n)\) and \(b(t_0) \in H^{s+1}(\mathbb{R}^n)\) for a time \(t_0 > 0\). Then there exists a time \(T = T(||u_0||_{H^s}, ||u(t_0)||_{H^s}, ||b(t_0)||_{H^{s+1}}) > t_0\) such that

\[
    u \in L^\infty(t_0, T; H^s(\mathbb{R}^n)) \cap L^2(t_0, T; H^{s+1}(\mathbb{R}^n)),
\]

\[
b \in L^\infty(t_0, T; H^{s+1}(\mathbb{R}^n)).
\]
The justification of Corollary 1.2 follows from Theorem 1.1 and the uniqueness immediately. Indeed, let \((u, b)\) be the solution of (1.1)-(1.2) satisfying assumptions of Corollary 1.2. The basic energy estimate (see Definition 2.4) guarantees that \(u \in L^2(0, \infty; H^1(\mathbb{R}^n))\) which implies \(u(t) \in H^1(\mathbb{R}^n)\) for almost all \(t > 0\). Thus, one can pick up a time \(\tau_0\) close enough to the initial time 0 such that \(u(\tau_0) \in H^s(\mathbb{R}^n)\). If \(\frac{n}{2} - 1 < s \leq 1\), \(u(\tau_0) \in H^s(\mathbb{R}^n)\) also holds by embedding. Then the uniqueness of weak solution in \(H^s(\mathbb{R}^n) \times H^{s+1}(\mathbb{R}^n)\) allows us to apply Theorem 1.1 by considering \(\tau_0\) as the initial time, and the conclusion of propagation of regularity follows right away.

**Remark 1.3.** Notice that \(H^{\frac{n}{2} - 1}\) is a critical space for the Navier-Stokes equation. In the high regularity space \(H^s(\mathbb{R}^n) \times H^{s+1}(\mathbb{R}^n)\) with \(s > \frac{n}{2} - 1\) for the MHD system (1.1), there is a good chance that the “weak-strong” type of uniqueness holds. In that case, the assumption of \(u_0 \in H^s(\mathbb{R}^n)\) in Theorem 1.1 could be dropped. This issue of uniqueness will be addressed in future work.

## 2. Preliminaries

### 2.1. Notation.
We denote by \(A \lesssim B\) an estimate of the form \(A \leq CB\) with some absolute constant \(C\), and by \(A \sim B\) an estimate of the form \(C_1B \leq A \leq C_2B\) with some absolute constants \(C_1, C_2\). We also write \(\| \cdot \|_p = \| \cdot \|_{L^p}\), and \((\cdot, \cdot)\) stands for the \(L^2\)-inner product.

### 2.2. Littlewood-Paley decomposition.
The techniques presented in this paper rely strongly on the frequency localization approach and paradifferential calculus. Thus we recall the Littlewood-Paley decomposition theory briefly. For a more detailed description on this theory we refer the readers to the books by Bahouri, Chemin and Danchin [1] and Grafakos [7].

Let \(\mathcal{F}\) and \(\mathcal{F}^{-1}\) denote the Fourier transform and inverse Fourier transform, respectively. Define \(\lambda_q = 2^q\) for integers \(q\). A nonnegative radial function \(\chi \in C_0^\infty(\mathbb{R}^n)\) is chosen such that

\[
\chi(\xi) = \begin{cases} 
1, & \text{for } |\xi| \leq \frac{3}{4} \\
0, & \text{for } |\xi| \geq 1.
\end{cases}
\]

Let

\[\varphi(\xi) = \chi\left(\frac{\xi}{2}\right) - \chi(\xi)\]

and

\[\varphi_q(\xi) = \begin{cases} 
\varphi(\lambda_q^{-1}\xi), & \text{for } q \geq 0, \\
\chi(\xi), & \text{for } q = -1.
\end{cases}\]

For a tempered distribution vector field \(u\) we define the Littlewood-Paley projection

\[
h = \mathcal{F}^{-1}\varphi, \quad \tilde{h} = \mathcal{F}^{-1}\chi, \quad u_q := \Delta_qu = \mathcal{F}^{-1}(\varphi(\lambda_q^{-1}\xi)\mathcal{F}u) = \lambda_q^n \int h(\lambda_qy)u(x-y)dy, \quad \text{for } q \geq 0, \quad u_{-1} = \mathcal{F}^{-1}(\chi(\xi)\mathcal{F}u) = \int \tilde{h}(y)u(x-y)dy.
\]
By the Littlewood-Paley theory, the following identity

\[ u = \sum_{q=-1}^{\infty} u_q \]

holds in the distribution sense. Essentially the sequence of the smooth functions \( \varphi_q \) forms a dyadic partition of the unit. To simplify the notation, we denote

\[ u \leq Q = \sum_{q=-1}^{Q} u_q, \quad u_{(Q,N]} = \sum_{p=Q+1}^{N} u_p, \quad \tilde{u}_q = \sum_{|p-q| \leq 1} u_p. \]

**Definition 2.1.** A tempered distribution \( u \) belongs to the Besov space \( B_{p,\infty}^s \) if and only if

\[ \|u\|_{B_{p,\infty}^s} = \sup_{q \geq -1} \lambda_q^s \|u_q\|_p < \infty. \]

We also note that,

\[ \|u\|_{\dot{H}^s} \lesssim \left( \sum_{q=-1}^{\infty} \lambda_q^{2s} \|u_q\|_2^2 \right)^{1/2} \]

for each \( u \in \dot{H}^s \) and \( s \in \mathbb{R} \).

We recall Bernstein’s inequality for the dyadic blocks of the Littlewood-Paley decomposition in the following.

**Lemma 2.2.** (See [9].) Let \( n \) be the space dimension and \( r \geq s \geq 1 \). Then for all tempered distributions \( u \),

\[ \|u_q\|_r \lesssim \lambda_q^{n\left(\frac{1}{s} - \frac{1}{r}\right)} \|u_q\|_s. \]

**2.3. Bony’s paraproduct and commutator.** Bony’s paraproduct formula

\[ \Delta_q (u \cdot \nabla v) = \sum_{|q-p| \leq 2} \Delta_q (u \leq p-2 \cdot \nabla v_p) + \sum_{|q-p| \leq 2} \Delta_q (u_p \cdot \nabla v \leq p-2) \]

\[ + \sum_{p \geq q-2} \Delta_q (\tilde{u}_p \cdot \nabla v_p), \]

will be used constantly to decompose the nonlinear terms in energy estimate. We will also use the notation of the commutator

\[ [\Delta_q, u \leq p-2 \cdot \nabla] v_p := \Delta_q (u \leq p-2 \cdot \nabla v_p) - u \leq p-2 \cdot \nabla \Delta_q v_p. \]

**Lemma 2.3.** The commutator satisfies the following estimate, for any \( 1 < r < \infty \)

\[ \|[\Delta_q, u \leq p-2 \cdot \nabla] v_p\|_r \lesssim \|\nabla u \leq p-2\|_{\infty} \|v_p\|_r. \]

**2.4. Definition of solutions.** We recall some classical definitions of weak and regular solutions.

**Definition 2.4.** A weak solution of (1.1) on \([0, T]\) (or \([0, \infty)\) if \( T = \infty \)) is a pair of functions \((u, b)\) in the class

\[ u, b \in C_w([0, T]; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)), \]
with \( u(0) = u_0, b(0) = b_0 \), satisfying (1.1) in the distribution sense; moreover, the following energy inequality

\[
\|u(t)\|_2^2 + \|b(t)\|_2^2 + 2\nu \int_{t_0}^t \|\nabla u(s)\|_2^2 ds \leq \|u(t_0)\|_2^2 + \|b(t_0)\|_2^2
\]

is satisfied for almost all \( t_0 \in (0, T) \) and all \( t \in (t_0, T] \).

2.5. Estimate for the Stokes equation. To provide a general result, we consider the Stokes equation with fractional Laplacian

\[
u \frac{d}{dt} u + (\Delta)^\alpha u + \nabla p = f, \quad \nabla \cdot u = 0, \quad u(0) = u_0
\]

with \( \alpha > 0 \). We will prove a type of maximal regularity result in \( L^p_t \) with delay of an arbitrarily short time.

**Lemma 2.5.** Assume \( f \in L^r(0, T; H^s(\mathbb{R}^n)), u_0 \in H^s(\mathbb{R}^n) \) and \( \nabla \cdot u_0 = 0 \) for \( 1 < r < \infty \) and \( s > 0 \). Let \( \varepsilon \in (0, T) \). Then the solution \( u \) of the Stokes equation (2.6) satisfies

\[
\int_{\varepsilon}^{T} \| \nabla^\alpha u(t) \|_{H^{r-s}} dt \leq C\nu^{-1} \left( \log \frac{T}{\varepsilon} \right) \| u_0 \|_{H^s} + C\nu^{-1} T^{1-\frac{4}{s}} \| f \|_{L^r(0, T; H^s)}
\]

for some absolute constant \( C \).

**Proof:** The validation of the Littlewood-Paley projection of a function being a test function is discussed in [3]. We multiply (2.6) by \( \Delta_q (\lambda_q^{2s+4\alpha} u_q) \) and integrate on \( \mathbb{R}^n \) to arrive

\[
\frac{1}{2} \frac{d}{dt} \lambda_q^{2s+4\alpha} \| u_q \|_2^2 + \nu \lambda_q^{2s+4\alpha} \| \nabla^\alpha u_q \|_2^2 \leq \lambda_q^{2s+4\alpha} \int_{\mathbb{R}^n} f_q \cdot u_q \, dx.
\]

Applying Hölder’s and Young’s inequalities to the flux integral yields

\[
\left| \int_{\mathbb{R}^n} f_q \cdot u_q \, dx \right| \leq \| f_q \|_2 \| u_q \|_2 \leq \frac{\nu}{2} \lambda_q^{2s} \| u_q \|_2^2 + \frac{1}{\nu \lambda_q^{2\alpha}} \| f_q \|_2^2.
\]

It follows from the last two inequalities that

\[
\frac{d}{dt} \lambda_q^{2s+4\alpha} \| u_q \|_2^2 + \nu \lambda_q^{2s+4\alpha} \| \nabla^\alpha u_q \|_2^2 \leq \frac{1}{\nu} \lambda_q^{2s+2\alpha} \| f_q \|_2^2.
\]

We then apply Duhamel’s principle over \( [0, t] \) to get

\[
\lambda_q^{2s+4\alpha} \| u_q(t) \|_2^2 \leq \lambda_q^{2s+4\alpha} \| u_q(0) \|_2^2 e^{-\nu \lambda_q^{2\alpha} t} + \frac{1}{\nu} \lambda_q^{2s+2\alpha} \int_0^t \| f_q(s) \|_2^2 e^{-\nu \lambda_q^{2\alpha} (t-s)} \, ds.
\]

Adding the inequality above for all \( q \geq -1 \), taking square root and integrating over \( (\varepsilon, T] \) gives rise to

\[
\int_{\varepsilon}^{T} \left( \sum_{q \geq -1} \lambda_q^{2s+4\alpha} \| u_q(t) \|_2^2 \right)^{\frac{1}{2}} \, dt
\]

\[
\leq \int_{\varepsilon}^{T} \left( \sum_{q \geq -1} \lambda_q^{2s+4\alpha} \| u_q(0) \|_2^2 e^{-\nu \lambda_q^{2\alpha} t} \right)^{\frac{1}{2}} \, dt
\]

\[
+ \int_{\varepsilon}^{T} \left( \sum_{q \geq -1} \frac{1}{\nu} \lambda_q^{2s+2\alpha} \int_0^t \| f_q(s) \|_2^2 e^{-\nu \lambda_q^{2\alpha} (t-s)} \, ds \right)^{\frac{1}{2}} \, dt.
\]
The estimates of the two terms on the right hand side of (2.8) are shown in the following. We have by some fundamental inequalities that

\[
\int_T^\infty \left( \sum_{q \geq -1} \lambda_q^{2s+4}\|u_q(0)\|^2_2 e^{-\nu\lambda_q^2 t} \right)^{\frac{1}{2}} \, dt \\
\leq \int_T^\infty \left( \sum_{q \geq -1} \lambda_q^{2s} \|u_q(0)\|^2_2 \right)^{\frac{1}{2}} \left( \sum_{q \geq -1} \lambda_q^{2s} e^{-\nu\lambda_q^2 t} \right)^{\frac{1}{2}} \, dt \\
\leq \|u_0\|_{H^s} \int_T^\infty \sum_{q \geq -1} \lambda_q^{2s} e^{-\frac{1}{2}\nu\lambda_q^2 t} \, dt \\
\leq \|u_0\|_{H^s} \int_T^\infty \left( \sum_{q \geq -1} \lambda_q^{2s} e^{-\frac{1}{4}\nu\lambda_q^2 t} \right)^2 \, dt.
\]

By Lemma 4.1 from [4], we have

\[
\sum_{q \geq -1} \lambda_q^{2s} e^{-\frac{1}{4}\nu\lambda_q^2 t} \lesssim \nu^{-\frac{1}{2}} t^{-\frac{1}{2}}.
\]

Therefore, we have

\[
\int_T^\infty \left( \sum_{q \geq -1} \lambda_q^{2s+4}\|u_q(0)\|^2_2 e^{-\nu\lambda_q^2 t} \right)^{\frac{1}{2}} \, dt \lesssim \|u_0\|_{H^s} \int_T^\infty \nu^{-1} t^{-1} \, dt \lesssim \nu^{-1} \|u_0\|_{H^s} \log \frac{T}{\varepsilon}.
\]

To handle the second term on the right hand side of (2.8), we first apply Hölder’s inequality, and then change the order of integration

\[
\int_T^\infty \left( \sum_{q \geq -1} \frac{1}{\nu^{2s+2}} \int_T^\infty \|f_q(\tau)\|^2_2 e^{-\nu\lambda_q^2 (t-\tau)} \, d\tau \right)^{\frac{1}{2}} \, dt \\
\leq (T-\varepsilon)^{\frac{1}{2}} \left( \int_T^\infty \sum_{q \geq -1} \frac{1}{\nu^{2s+2}} \int_T^\infty \|f_q(\tau)\|^2_2 e^{-\nu\lambda_q^2 (t-\tau)} \, d\tau \, dt \right)^{\frac{1}{2}} \\
\leq (T-\varepsilon)^{\frac{1}{2}} \left( \int_T^\infty \int_T^\infty \sum_{q \geq -1} \frac{1}{\nu^{2s+2}} \|f_q(\tau)\|^2_2 e^{-\nu\lambda_q^2 (t-\tau)} \, d\tau \, dt \right)^{\frac{1}{2}} \\
\leq (T-\varepsilon)^{\frac{1}{2}} \left( \int_T^\infty \sum_{q \geq -1} \frac{1}{\nu^{2s+2}} \|f_q(\tau)\|^2_2 [1 - e^{-\nu\lambda_q^2 (T-\tau)}] \, d\tau \right)^{\frac{1}{2}} \\
\leq (T-\varepsilon)^{\frac{1}{2}} \left( \int_T^\infty \frac{1}{\nu^2} \|f(\tau)\|^2_{H^s} \, d\tau \right)^{\frac{1}{2}} \\
\lesssim \frac{1}{\nu} T^{1-\frac{1}{p}} \left( \int_T^\infty \|f(\tau)\|^2_{H^s} \, d\tau \right)^{\frac{1}{2}}.
\]
where \( r \geq 2 \) is required for the last step. In fact, for \( 1 < r \leq 2 \), the estimate can be obtained by duality, see [8].

Therefore, (2.7) is obtained by combining the above estimates.

\[ \square \]

3. PROOF OF THE MAIN RESULT

We proceed to prove Theorem 1.1 in this section. We will only show a priori estimates satisfied by smooth solutions. A rigorous analysis relies on performing estimates on the Galerkin approximations and then the passage to the limit.

Formally, multiplying the first equation in (1.1) by \( \lambda^2 q \Delta^2 u \) and the second one by \( \lambda^2 q \Delta^2 b \), and adding up for all \( q \geq -1 \) we obtain

\begin{equation}
\frac{1}{2} \frac{d}{dt} \sum_{q \geq -1} \lambda^2 q \| u_q \|_2^2 + \nu \sum_{q \geq -1} \lambda^{2q+2} \| u_q \|_2^2 \leq -I_1 - I_2,
\end{equation}

(3.9)

\begin{equation}
\frac{1}{2} \frac{d}{dt} \sum_{q \geq -1} \lambda^{2r} ||b_q||_2^2 \leq -I_3 - I_4,
\end{equation}

(3.10)

with

\[ I_1 = \sum_{q \geq -1} \lambda^2 q \int_{\mathbb{R}^3} \Delta_q (u \cdot \nabla u) \cdot u_q \, dx, \quad I_2 = -\sum_{q \geq -1} \lambda^2 q \int_{\mathbb{R}^3} \Delta_q (b \cdot \nabla b) \cdot u_q \, dx, \]

\[ I_3 = \sum_{q \geq -1} \lambda^{2r} \int_{\mathbb{R}^3} \Delta_q (u \cdot \nabla b) \cdot b_q \, dx, \quad I_4 = -\sum_{q \geq -1} \lambda^2 q \int_{\mathbb{R}^3} \Delta_q (b \cdot \nabla u) \cdot b_q \, dx, \]

Using Bony’s paraproduct and the commutator notation, \( I_1 \) is decomposed as

\[ I_1 = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda^2 q \int_{\mathbb{R}^3} \Delta_q (u_{\leq p-2} \cdot \nabla u_{p}) u_q \, dx \]

\[ + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda^2 q \int_{\mathbb{R}^3} \Delta_q (u_{p} \cdot \nabla u_{\leq p-2}) u_q \, dx \]

\[ + \sum_{q \geq -1} \sum_{p \geq -2} \lambda^2 q \int_{\mathbb{R}^3} \Delta_q (u_p \cdot \nabla u_p) u_q \, dx \]

\[ = I_{11} + I_{12} + I_{13}, \]

with

\[ I_{11} = \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda^2 q \int_{\mathbb{R}^3} \Delta_q, u_{\leq p-2} \cdot \nabla |u_{p} u_q \, dx \]

\[ + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda^2 q \int_{\mathbb{R}^3} u_{\leq p-2} \cdot \nabla |\Delta q | u_{p} u_q \, dx \]

\[ + \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda^2 q \int_{\mathbb{R}^3} (u_{\leq p-2} - u_{\leq q-2}) \cdot \nabla |\Delta q | u_{p} u_q \, dx \]

\[ = I_{111} + I_{112} + I_{113}. \]
One can see that $I_{112} = 0$ due to the fact $\sum_{|p-q|\leq 2} \Delta_q u_p = u_q$ and $\nabla \cdot u_{\leq q-2} = 0$. By the commutator estimate, we obtain

$$|I_{111}| \leq \sum_{q \geq 1} \sum_{|p-q| \leq 2} \lambda_q^{2s} \|\nabla u_{\leq p-2}\|_\infty \|u_p\|_2 \|u_q\|_2$$

$$\lesssim \sum_{q \geq 1} \lambda_q^{2s} \|u_q\|_2^2 \sum_{p \leq q} \lambda_p^{\frac{s+1}{2}} \|u_p\|_2$$

$$\lesssim \sum_{q \geq 1} \lambda_q^{(s+1)\theta} \|u_q\|_2 \lambda_q^{(2-\theta)} \|u_q\|_2^{2-\theta} \sum_{p \leq q} \lambda_p^{(s+1)\delta} \|u_p\|_2 \lambda_p^{\delta(1-\delta)} \|u_p\|_2^{1-\delta} \left( \lambda_q^{-\theta} \lambda_p^{\frac{s+1}{2} - s - \delta} \right)$$

$$\lesssim \sum_{q \geq 1} \lambda_q^{(s+1)\theta} \|u_q\|_2 \lambda_q^{(2-\theta)} \|u_q\|_2^{2-\theta} \sum_{p \leq q} \lambda_p^{(s+1)\delta} \|u_p\|_2 \lambda_p^{\delta(1-\delta)} \|u_p\|_2^{1-\delta} \lambda_p^{\theta - q}$$

for parameters $\theta$ and $\delta$ satisfying $0 < \theta < 2$, $0 < \delta < 1$ and

$$s \geq \frac{n}{2} + 1 - \theta - \delta. \tag{3.11}$$

It then follows from Young’s inequality with $(r_1, r_2, r_3, r_4) \in (1, \infty)^4$ satisfying

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = 1, \quad r_1 = \frac{2}{\theta}, \quad r_3 = \frac{2}{\delta}$$

such that for some $\theta_1 > 0, \theta_2 > 0$

$$|I_{111}| \leq \frac{\nu}{64} \sum_{q \geq 1} \lambda_q^{2s+2} \|u_q\|_2^2 + C_\nu \sum_{q \geq 1} (\lambda_q^{2s} \|u_q\|_2 \|u_q\|_2^{(2-\theta)r})$$

$$+ \frac{\nu}{64} \sum_{q \geq 1} \sum_{p \leq q} \lambda_p^{2s+2} \|u_p\|_2 \lambda_p^{\theta_2} + C_\nu \sum_{q \geq 1} \sum_{p \leq q} (\lambda_q^{2s} \|u_q\|_2 \|u_q\|_2^{(1-\delta)r})$$

$$\lesssim \frac{\nu}{32} \sum_{q \geq 1} \lambda_q^{2s+2} \|u_q\|_2^2 + C_\nu \left( \sum_{q \geq 1} \lambda_q^{2s} \|u_q\|_2 \right)^{2-\theta} + C_\nu \left( \sum_{q \geq 1} \lambda_q^{2s} \|u_q\|_2 \right)^{(1-\delta)r}$$

Notice that (3.11) and (3.12) imply that $s > \frac{\theta}{2} - 1$.

While

$$|I_{113}| \leq \sum_{q \geq 1} \sum_{|p-q| \leq 2} \lambda_q^{2s} \|u_{\leq p-2} - u_{\leq q-2}\|_2 \|\nabla u_p\|_\infty \|u_q\|_2$$

$$\lesssim \sum_{q \geq 1} \lambda_q^{2s+\frac{\theta}{2}+1} \|u_q\|_2^3$$

$$\lesssim \sum_{q \geq 1} \lambda_q^{(s+1)\theta} \|u_q\|_2 \lambda_q^{(3-\theta)} \|u_q\|_2^{3-\theta} \lambda_q^{\frac{s+1}{2} - s - \theta}$$

$$\lesssim \frac{\nu}{32} \sum_{q \geq 1} \lambda_q^{2s+2} \|u_q\|_2^2 + C_\nu \left( \sum_{q \geq 1} \lambda_q^{2s} \|u_q\|_2 \right)^{\frac{2-\theta}{4}}$$

for $s \geq \frac{\theta}{2} + 1 - \theta$ and $0 < \theta < 2$. Notice that $I_{12}$ and $I_{13}$ can be estimated similarly as $I_{111}$ and $I_{113}$, respectively. Thus

$$I_1 \leq \frac{\nu}{8} \|\nabla u\|_{H^s}^2 + C_\nu \|u\|_{H^s}^{2+\gamma_1} + C_\nu \|u\|_{H^s}^{2+\gamma_2}$$

for $s > \frac{\theta}{2} - 1$ and some $\gamma_1, \gamma_2 > 0$.\tag{3.13}
Using Bony’s paraproduct and the commutator notation, $I_2$ is decomposed as

$$I_2 = -\sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b_{\leq p} \cdot \nabla b_p) u_q \, dx$$

$$- \sum_{q \geq -1} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b_p \cdot \nabla b_{\leq p}) u_q \, dx$$

$$- \sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b_p \cdot \nabla b_p) u_q \, dx$$

$$= I_{21} + I_{22} + I_{23}.$$ $I_{21}$ can be estimated as

$$|I_{21}| \leq \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda_q^{2s+1} \| b_{\leq p} \|_\infty \| b_p \|_2 \| u_q \|_2$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{2s+1} \| b_q \|_2 \| u_q \|_2 \sum_{p \geq q} \lambda_p \| b_p \|_2$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{s+1} \| u_q \|_2 \lambda_q \| b_q \|_2 \sum_{p \geq q} \lambda_p \| b_p \|_2 \lambda_p^{s-r} \lambda_q^{2r+s-2r}$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{s+1} \| u_q \|_2 \lambda_q \| b_q \|_2 \sum_{p \geq q} \lambda_p \| b_p \|_2 \lambda_q^{s-r}$$

for $\frac{q}{2} + s - 2r \leq 0$ and $s < r$. It follows from Young’s and Jensen’s inequalities that

$$|I_{21}| \leq \frac{\nu}{16} \sum_{q \geq -1} \lambda_q^{2s+2} \| u_q \|_2^2 + C_\nu \sum_{q \geq -1} \lambda_q \| b_q \|_2 \left( \sum_{p \geq q} \lambda_p \| b_p \|_2 \lambda_p^{s-r} \right)^{2}$$

$$\leq \frac{\nu}{16} \sum_{q \geq -1} \lambda_q^{2s+2} \| u_q \|_2^2 + C_\nu \left( \sum_{q \geq -1} \lambda_q \| b_q \|_2 \right)^{2+\gamma}$$

for some $\gamma > 0$. We observe

$$|I_{22}| \lesssim \sum_{q \geq -1} \lambda_q^{2s} \| u_q \|_2 \sum_{p \geq q} \lambda_p^{s+1} \| b_p \|_2 \lesssim |I_{21}|.$$ 

Hence $I_{22}$ shares the same estimate as $I_{21}$. To handle $I_{23}$, integration by parts followed by Hölder’s and Bernstein’s inequalities leads to

$$|I_{23}| = \left| \sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (b_p \otimes \bar{b}_p) \cdot \nabla u_q \, dx \right|$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{s+1} \| u_q \|_2 \sum_{p \geq q-4} \| b_p \|_2 \| b_p \|_\infty$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{s+1} \| u_q \|_2 \sum_{p \geq q-4} \lambda_p^{s} \| b_p \|_2$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{s+1} \| u_q \|_2 \sum_{p \geq q-4} \lambda_p^{2r} \| b_p \|_2 \lambda_p^{s-r} \lambda_q^{2r+s-2r}$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{s+1} \| u_q \|_2 \sum_{p \geq q-4} \lambda_p^{2r} \| b_p \|_2 \lambda_q^{s-r}$$
for \( \frac{n}{2} + s - 2r \leq 0 \). It then follows from Young’s inequality that

\[
|I_{23}| \leq \frac{\nu}{16} \sum_{q \geq -1} \lambda^2 s + 2 \|u_q\|^2 + C_{\nu} \sum_{q \geq -4} \left( \sum_{p \geq q} \lambda^2 p \|b_p\|^2 \lambda^2 p - q \right)^2
\]

\[
\leq \frac{\nu}{16} \sum_{q \geq -1} \lambda^2 s + 2 \|u_q\|^2 + C_{\nu} \left( \sum_{q \geq -1} \lambda^2 q \|b_q\|^2 \right)^2.
\]

To conclude, we obtain

\[
(3.14) \quad |I_2| \leq \frac{3\nu}{16} \|\nabla u\|^2_{H^s} + C_{\nu} \|b\|^4_{H^r} + C_{\nu} \|b\|^{2+\gamma}_{H^r}
\]

for \( \frac{n}{2} + s - 2r \leq 0 \) and \( s < r \).

Now we estimate \( I_3 \) by first decomposing it as

\[
I_3 = \sum_{q \geq -1} \sum_{|p-q| \leq 2} \lambda^2 q \int_{\mathbb{R}^3} \Delta_q (u_{\leq p-2} \cdot \nabla b_p) b_q \, dx
\]

\[
+ \sum_{q \geq -1} \sum_{|p-q| \leq 2} \lambda^2 q \int_{\mathbb{R}^3} \Delta_q (u_p \cdot \nabla b_{\leq p-2}) b_q \, dx
\]

\[
+ \sum_{q \geq -1} \sum_{p \geq q-2} \lambda^2 q \int_{\mathbb{R}^3} \Delta_q (u_p \cdot \nabla \bar{b}_p) b_q \, dx
\]

\[
= I_{31} + I_{32} + I_{33},
\]

with

\[
I_{31} = - \sum_{q \geq -1} \sum_{|p-q| \leq 2} \lambda^2 q \int_{\mathbb{R}^3} \Delta_q, u_{\leq p-2} \cdot \nabla |b_p| b_q \, dx
\]

\[
- \sum_{q \geq -1} \sum_{|p-q| \leq 2} \lambda^2 q \int_{\mathbb{R}^3} u_{\leq q-2} \cdot \nabla \Delta_q b_p b_q \, dx
\]

\[
- \sum_{q \geq -1} \sum_{|p-q| \leq 2} \lambda^2 q \int_{\mathbb{R}^3} (u_{\leq p-2} - u_{\leq q-2}) \cdot \nabla \Delta_q b_p b_q \, dx
\]

\[
= I_{311} + I_{312} + I_{313}.
\]

One can see that \( I_{312} = 0 \) due to the fact \( \sum_{|p-q| \leq 2} \Delta_q b_p = b_q \) and \( \nabla \cdot u_{\leq q-2} = 0 \).

By the commutator estimate and Hölder’s inequality, we obtain

\[
|I_{311}| \leq \sum_{q \geq -1} \sum_{|p-q| \leq 2} \lambda^2 q \|\nabla u_{\leq p-2}\|_{\infty} \|b_p\|_2 \|b_q\|_2
\]

\[
\lesssim \|\nabla u\|_{H^s} \sum_{q \geq -1} \lambda^2 q \|b_q\|^2_2
\]

\[
\lesssim \|\nabla u\|_{H^{s+1}} \sum_{q \geq -1} \lambda^2 q \|b_q\|^2_2
\]
Similarly, Hölder’s and Bernstein’s inequalities applied to $I_{313}$ is estimated as

$$|I_{313}| \leq \sum_{q \geq -1} \sum_{|p-q| \leq 2} \lambda_q^{2r} \|u_{p-1} - u_{q-1}\|_2 \|
abla b_q\|_2$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{2r+1} \|u_q\| \|b_q\|_2^2$$

$$\lesssim \|\nabla u\|_\infty \sum_{q \geq -1} \lambda_q^{2r} \|b_q\|_2^2$$

$$\lesssim \|\nabla u\|_{H^{r+1}} \sum_{q \geq -1} \lambda_q^{2r} \|b_q\|_2^2.$$  

Similarly, Hölder’s and Bernstein’s inequalities applied to $I_{32}$ gives

$$|I_{32}| = \left| \sum_{q \geq -1} \sum_{|p-q| \leq 2} \lambda_q^{2r} \int_{\mathbb{R}^3} \Delta_q (u_p \cdot \nabla b_{p-2}) b_q dx \right|$$

$$\leq \sum_{q \geq -1} \sum_{|p-q| \leq 2} \lambda_q^{2r} \|u_q\|_2 \|
abla b_{p-2}\|_\infty \|b_q\|_2$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{2r} \|u_q\|_2 \|b_q\|_2 \sum_{p \leq q} \lambda_p^{q+1} \|b_p\|_2$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{s+2} \|u_q\|_2 \lambda_q^r \|b_q\|_2 \sum_{p \leq q} \lambda_p^r \|b_p\|_2 \lambda_q^{r-s-1}$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{s+2} \|u_q\|_2 \lambda_q^r \|b_q\|_2 \sum_{p \leq q} \lambda_p^r \|b_p\|_2 \lambda_q^{r-s-2}$$

since $s > \frac{n}{2} - 1$. Again it follows from Young’s and Jensen’s inequalities that

$$|I_{32}| \lesssim \|\nabla u\|_{H^{r+1}} \sum_{q \geq -1} \lambda_q^r \|b_q\|_2 \sum_{p \leq q} \lambda_p^r \|b_p\|_2 \lambda_q^{r-s-2}$$

$$\lesssim \|\nabla u\|_{H^{r+1}} \sum_{q \geq -1} \lambda_q^{2r} \|b_q\|_2^2$$

for $r < s + 2$. Integrating by parts for $I_{33}$, we have

$$|I_{33}| \leq \sum_{q \geq -1} \sum_{p \geq q-2} \lambda_q^{2r} \|u_p\|_2 \|\nabla b_q\|_\infty$$

$$\lesssim \sum_{q \geq -1} \lambda_q^{2r+1} \|b_q\|_2 \sum_{p \geq q} \|u_p\|_2 \|b_p\|_2$$

$$\lesssim \sum_{q \geq -1} \lambda_q^r \|b_q\|_2 \sum_{p \geq q} \lambda_p^{q+2} \|u_p\|_2 \lambda_p^r \|b_p\|_2 \lambda_q^{q-r-1}$$

$$\lesssim \sum_{q \geq -1} \lambda_q^r \|b_q\|_2 \sum_{p \geq q} \lambda_p^{q+2} \|u_p\|_2 \lambda_p^r \|b_p\|_2 \lambda_q^{q-r-1}$$

$$\lesssim \|\nabla u\|_{H^{r+1}} \sum_{q \geq -1} \lambda_q^r \|b_q\|_2 \sum_{p \geq q} \lambda_p^r \|b_p\|_2 \lambda_q^{q-r-1}$$
Thus, we obtain for $s > \frac{n}{2} - 1$. The same routine of applying Young’s and Jensen’s inequalities gives

$$|I_{33}| \lesssim \|\nabla u\|_{H^{s+1}} \sum_{q \geq -1} \lambda^2 r \|b_q\|^2.$$ 

Using Bony’s paraproduct and the commutator notation, $I_4$ can be written as

$$I_4 = - \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda^2 r \int_{\mathbb{R}^3} \Delta_q (b_{\leq p-2} \cdot \nabla u_p) b_q \, dx$$

$$- \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda^2 r \int_{\mathbb{R}^3} \Delta_q (b_p \cdot \nabla u_{\leq p-2}) b_q \, dx$$

$$- \sum_{q \geq -1} \sum_{p \geq -2} \lambda^2 r \int_{\mathbb{R}^3} \Delta_q (b_p \cdot \nabla u_p) b_q \, dx$$

$$= I_{41} + I_{42} + I_{43}.$$ 

One can observe that $I_{12}$ and $I_{43}$ can be estimated in an analogous way as for $I_{311}$ and $I_{33}$, respectively. Thus we only show the estimate of $I_{41}$,

$$|I_{41}| \leq \sum_{q \geq -1} \sum_{|q-p| \leq 2} \lambda^2 r \|b_{\leq p-2}\|_\infty \|\nabla u_p\|_2 \|b_q\|_2$$

$$\lesssim \|b\|_\infty \sum_{q \geq -1} \lambda_q^r \|b_q\|_2 \lambda^2 r \|\nabla u_p\|_2 \lambda^2 r \gamma - 1$$

$$\lesssim \|\nabla u\|_{H^{s+1}} \|b\|_\infty \sum_{q \geq -1} \lambda_q^r \|b_q\|_2$$

$$\lesssim \|\nabla u\|_{H^{s+1}} \|b_q\|^2_{\dot{H}^r},$$

for $\frac{n}{2} < r \leq s + 1$. Thus we have for $\frac{n}{2} < r \leq s + 1$ that

$$|I_4| \lesssim \|\nabla u\|_{H^{s+1}} \|b_q\|^2_{\dot{H}^r}.$$

Inequality (3.13) along with estimates (3.12) and (3.14) implies that, there exist various constants $C_\nu$ depending on $\nu$ such that

$$\frac{d}{dt} \|u\|^2_{\dot{H}^r} + \nu \|\nabla u\|^2_{\dot{H}^r}$$

$$\leq C_\nu \|u\|^{2+\gamma_1}_{\dot{H}^r} + C_\nu \|u\|^{2+\gamma_2}_{\dot{H}^r} + C_\nu \|b\|^{2+\gamma_3}_{\dot{H}^r} + C_\nu \|b\|^{4}_{\dot{H}^r},$$

with parameters satisfying

$$\frac{n}{2} + s - 2r \leq 0, \quad \frac{n}{2} - 1 < s < r$$

and some constants $\gamma_1, \gamma_2, \gamma_3 > 0$. Combining estimates (3.10), (3.15) and (3.16) gives rise to

$$\frac{d}{dt} \|b\|^2_{\dot{H}^r} \leq \nu \|\nabla u\|_{\dot{H}^r} + C_0 \|\nabla u\|_{H^{s+1}} \|b\|^2_{\dot{H}^r} + C_\nu \|u\|^{2+\gamma_1}_{\dot{H}^r} + C_\nu \|b\|^{2+\gamma_2}_{\dot{H}^r},$$

with

$$\frac{n}{2} < r \leq s + 1.$$
Adding the last two energy inequalities leads to, by dropping similar terms for the sake of simplification

\[
\frac{d}{dt} \left( \|u\|_{H^s}^2 + \|b\|_{H^s}^2 \right) + \frac{\nu}{2} \|\nabla u\|_{H^s}^2 \\
\leq C_{\nu} \left( \|u\|_{H^{s+1}}^2 + \|b\|_{H^{s+1}}^2 \right)^{1+\gamma_1} + C_{\nu} \left( \|u\|_{H^{s+1}}^2 + \|b\|_{H^{s+1}}^2 \right)^{1+\gamma_2} \\
+ C_0 \|\nabla u\|_{H^{s+1}} \left( \|u\|_{H^s}^2 + \|b\|_{H^s}^2 \right)
\]

with parameters \( r \) and \( s \) satisfying (3.18) and (3.19). Indeed, we can choose \( r = s+1 - \delta \) for any \( \delta \in \left[ 0, \frac{1}{2}(s - \frac{n}{2} + 1) \right] \). For simplicity, we take \( r = s+1 \) from now on.

Now we pause to estimate \( \int_{t_0}^{t} \|\nabla u(\tau)\|_{H^{s+1}} \, d\tau \) which will appear on the right hand side of (3.20) after integration over the time interval \([t_0, t]\). First it follows from (2.7) that for \( t \leq t_0 + 1 \) and any \( \beta > 1 \)

\[
\int_{t_0}^{t} \|\nabla u(\tau)\|_{H^{s+1}} \, d\tau \leq C_{\nu}^{-1} \left( \log \frac{t}{t_0} \right)^{\beta} \|u_0\|_{H^s} + C_{\nu}^{-1} t^{1-\frac{\delta}{\beta}} \|f\|_{L^\beta(0, t; H^s)}
\]

with \( f := -(u \cdot \nabla)u + (b \cdot \nabla)b \). Notice that \( s+1 > \frac{n}{2} \) and hence \( H^{s+1} \) is an algebra, we deduce

\[
\| (b \cdot \nabla)b \|_{H^s} = \| \nabla \cdot (b \otimes b) \|_{H^s} \lesssim \| b \otimes b \|_{H^{s+1}} \lesssim \| b \|_{H^{s+1}}^2 \lesssim \| b \|_{H^{s+1}}^2.
\]

While for the term with \( u \), we have

\[
\| (u \cdot \nabla)u \|_{H^s} \lesssim \| u \|_{H^{s+1}} \| u \|_{H^{s+1}} \| u \|_{H^{s+1}}.
\]

Agmon’s inequality gives rise to

\[
\| u \|_{H^2} \lesssim \| u \|_{H^{s+1}} \| u \|_{H^{s+1}}^\frac{1-\theta_1}{\theta_1 n_2 + \epsilon_0}
\]

with \( \epsilon_0 = s - \frac{n}{2} + 1 > 0 \), \( \theta_1 \in (0, 1) \) and \( s_1 \) satisfying

\[
\frac{n}{2} = \theta_1 s_1 + (1 - \theta_1) \left( \frac{n}{2} + \epsilon_0 \right).
\]

Notice that \( s_1 = \frac{n}{2} + \epsilon_0 (1 - \frac{n}{2}) < \frac{n}{2} \). Then by Gagliardo-Nirenberg’s interpolation inequality we have

\[
\| u \|_{H^{s+1}} \lesssim \| u \|_{L^2}^{\theta_2} \| u \|_{H^{s+1}}^{1-\theta_2}
\]

with \( \theta_2 \in (0, 1) \) satisfying

\[
\frac{1}{2} = \frac{s_1}{n} + \left( \frac{1}{2} - \frac{s + 1}{n} \right) (1 - \theta_2) + \frac{\theta_2}{2}.
\]

Putting the last three inequalities together yields

\[
\| (u \cdot \nabla)u \|_{H^s} \lesssim \| u \|_{L^2}^{\theta_2} \| u \|_{H^{s+1}}^{1-\theta_2} \| u \|_{H^{s+1}}^{2-\theta_2} \| u \|_{H^{s+1}}^{2-\theta_2} \| u \|_{H^{s+1}}^{2-\theta_2} \| u \|_{H^{s+1}}^{2-\theta_2}
\]

with \( \theta_1 \theta_2 = 1 - \frac{n}{2(n+1)} \in (0, 1) \) since \( s > \frac{n}{2} - 1 \). As a consequence of (3.22) and (3.23) we have for \( \beta = \frac{r}{2-\theta_1\theta_2} > 1 \) that

\[
\| f \|_{L^\beta(0, t; H^s)} \lesssim \int_0^t \left( \| b(\tau) \|_{H^{s+1}}^2 + \| u(\tau) \|_{L^2}^{\theta_1\theta_2} \| u(\tau) \|_{H^{s+1}}^{2-\theta_1\theta_2} \right) \, d\tau
\]

(3.24)
Therefore, it follows from (3.21) and (3.24) that

\[
\int_{t_0}^{t} \| \nabla u(\tau) \|_{H^{s+1}} \, d\tau \leq C_{\nu} \left( \log \frac{T}{t_0} \right) \| u_0 \|_{H^{s}} \tag{3.25}
\]

\[+ C_{\nu} (t - t_0)^{1 - \frac{1}{\beta}} \left( \int_{t_0}^{t} \| b(\tau) \|_{H^{s+1}}^{2\beta} + \| u(\tau) \|_{H^{s+1}}^{\beta} \| u(\tau) \|_{H^{s+1}}^{\beta} \, d\tau \right)^{\frac{1}{\beta}} \]

with constant \( C_{\nu} \) depending only on \( \nu \), \( 2\beta = \frac{8(s+1)}{2(s+1)+n} \), and \( \theta \theta_2 \beta = \frac{4(s+1)-2n}{2(s+1)+n} \).

In the following, we will proceed a delicate analysis based on (3.17), (3.20), (3.26) and a contradiction argument to close the proof of the theorem. We claim that there exists a time \( T > t_0 \) such that

\[
\| u(t) \|_{H^{s+1}}^2 + \| b(t) \|_{H^{s+1}}^2 
\leq 4(\| u(t_0) \|_{H^{s+1}}^2 + \| b(t_0) \|_{H^{s+1}}^2), \quad \text{for all } \ t \in [t_0, T].
\tag{3.26}
\]

The following notations are adapted:

\[A(t) = \| u(t) \|_{H^{s+1}}^2 + \| b(t) \|_{H^{s+1}}^2, \quad A_0 = A(t_0),\]

\[M_0 = \| u_0 \|_{H^{s+1}}^2 + \| b_0 \|_{H^{s+1}}^2,\]

\[M_1 = C_{\nu} (4A_0)^{1+\gamma} + C_{\nu} (4A_0)^{1+\gamma_2} + C_{\nu} (4A_0)^{1+\gamma_3} + C_{\nu} (4A_0)^2,\]

\[F(T, A_0, M_0, M_1, \nu, t_0) = C_{\nu} \left( \log \frac{T}{t_0} \right) \| u_0 \|_{H^{s+1}}^2 \]

\[+ C_{\nu} (T - t_0)^{1 - \frac{1}{\beta}} \left( A_0^{\beta} T + \nu^{-1} M_0^{\beta_1} \| u_0 \|_{H^{s+1}}^{\beta_2} (A_0 + M_1 T) \right)^{\frac{1}{\beta}}.\]

Since \( \beta > 1 \), \( F(T, A_0, M_0, M_1, \nu, t_0) \) is increasing in \( T \) and \( F(t_0, A_0, M_0, M_1, \nu, t_0) = 0 \). Thus, \( F \) can be arbitrarily small provided \( T \) is arbitrarily close to \( t_0 \). Indeed, the time \( T \) can be chosen as small as that

\[e^{F(T, A_0, M_0, M_1, t_0)} < 2, \quad \text{and} \quad 2M_1 (T - t_0)/A_0 < 1. \tag{3.27}
\]

Take \( T_1 = \sup \{ \tau \in [t_0, T] : A(t) \leq 4A_0 \text{ for all } t \in [t_0, \tau] \} \).

Suppose \( T_1 < T \). Inequality (3.17) implies that

\[\int_{t_0}^{T_1} \| \nabla u(t) \|_{H^{s+1}}^2 \, dt \leq \nu^{-1} (A_0 + M_1 (T_1 - t_0)).\]

It then follows from (3.26) that

\[\int_{t_0}^{T_1} \| \nabla u(\tau) \|_{H^{s+1}} \, d\tau \leq C_{\nu} \left( \log \frac{T_1}{t_0} \right) A_0^\frac{1}{\beta} + C_{\nu} (T_1 - t_0)^{1 - \frac{1}{\beta}} \left( \int_{t_0}^{T_1} A_0^\beta + M_0^{\beta_1 \theta_2} \| u(\tau) \|_{H^{s+1}}^{\beta_2} \, d\tau \right)^{\frac{1}{\beta}} \leq C_{\nu} \left( \log \frac{T_1}{t_0} \right) A_0^\frac{1}{\beta} + C_{\nu} (T_1 - t_0)^{1 - \frac{1}{\beta}} \left( A_0^\beta T_1 + \nu^{-1} M_0^{\beta_1 \theta_2} (A_0 + M_1 T_1) \right)^{\frac{1}{\beta}} =: F(T_1, A_0, M_0, M_1, \nu, t_0).\]
Then energy estimate (3.20) together with the inequality above implies
\begin{align*}
A(T_1) \leq & A(t_0) + M_1(T_1 - t_0) + C_0 \int_{t_0}^{T_1} \| \nabla u(\tau) \|_{H^{s+1}} A(\tau) \, d\tau \\
\leq & (A_0 + M_1(T_1 - t_0)) \exp \left( \int_{t_0}^{T_1} \| \nabla u(\tau) \|_{H^{s+1}} \, d\tau \right) \\
\leq & (A_0 + M_1(T_1 - t_0)) e^{F(T_1, A_0, M_0, M_1, \nu, t_0)} \\
\leq & 2A_0 + 2M_1(T_1 - t_0) < 3A_0,
\end{align*}
where the last two steps follow from the choice of time $T$ as in (3.27). However, the consequence $A(T_1) < 3A_0$ contradicts the definition of $T_1$ and the assumption of $T_1 < T$. Therefore $T_1 = T$ and (3.26) is justified. In the end, it follows from (3.20), (3.25) and (3.26) that
\begin{align*}
\int_{t_0}^{T} \| \nabla u(t) \|_{H^{s+1}}^2 \, dt & \leq C \left( \| u(t_0) \|_{H^s}, \| b(t_0) \|_{H^{s+1}}, \nu, t_0, n, T \right), \\
\int_{t_0}^{T} \| \nabla u(t) \|_{H^{s+1}} \, dt & \leq C \left( \| u(t_0) \|_{H^s}, \| b(t_0) \|_{H^{s+1}}, \nu, t_0, n, T \right),
\end{align*}
for various constants $C$ depending on $\| u(t_0) \|_{H^s}, \| b(t_0) \|_{H^{s+1}}, \nu, t_0, n,$ and $T$. It concludes the proof of Theorem 1.1. 

\[\blacksquare\]

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