CERTAIN UNIFIED INTEGRAL FORMULAS INVOLVING
THE GENERALIZED MODIFIED $k$-BESSEL FUNCTION
OF FIRST KIND

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Abstract. Generalized integral formulas involving the generalized modified $k$-Bessel function $J_{\gamma,\lambda}^{c,\gamma,\lambda}(z)$ of first kind are expressed in terms generalized Wright functions. Some interesting special cases of the main results are also discussed.

1. Introduction

The integral formula involving various special functions have been studied by many researchers ([3, 7]). In 1888, Pincherle gave the integrals involving product of Gamma functions along vertical lines (see [16, 18, 19]). Barnes [2] and Cahen [4] extended some of these integrals in the study of Riemann zeta function and other Drichlet’s series. The integral representation of Fox $H$-functions and hypergeometric $2F_1$ functions studied by [11] and [1] respectively. Also, the integral representation of Bessel functions are given in many recent works (see [3, 5, 6, 22]).

In [21], the $k$-Bessel function of the first kind defined by the following series

$$J_{k,v}^{c,\nu}(z) := \sum_{n=0}^{\infty} \frac{\Gamma_k(\lambda n + \nu + 1)}{\Gamma_k(\lambda n + \nu + 1)} \frac{(-1)^n (z/2)^n}{(n!)^2},$$

where $k \in \mathbb{R}$; $\alpha, \lambda, \gamma, \nu \in \mathbb{C}$; $\Re(\lambda) > 0$ and $\Re(\nu) > 0$. Here $(\gamma)_n$ refer the well-known $k$–Pochhammer symbol defined as (see [8])

$$\Gamma_k(\gamma)_n = \left\{ \begin{array}{ll} \frac{\Gamma_k(\gamma + nk)}{\Gamma_k(\gamma)} & (k \in \mathbb{R};\gamma \in \mathbb{C} \setminus \{0\}) \\ \gamma(\gamma + k) \cdots (\gamma + (n - 1)k) & (n \in \mathbb{N};\gamma \in \mathbb{C}) \end{array} \right.$$

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while \( \Gamma_k(z) \) denotes the \( k \)-gamma function defined by (see [8])
\[
(1.3) \quad \Gamma_k(z) = \int_0^\infty e^{-t^2} t^{z-1} dt, \quad \Re(z) > 0, \quad k > 0.
\]
Clearly, \( \Gamma_k(z) \) reduces to the classical \( \Gamma(z) \) function for \( k = 1 \).

In this paper, we introduce a generalization of the \( k \)-Bessel functions defined as:

\[
(1.4) \quad J_{b,c,\gamma,\sigma}^{k,v}(z) = \sum_{n=0}^{\infty} \frac{(c)^n (\gamma)_{n,k} \Gamma_k(\sigma n + v + \frac{b+1}{2})}{(n!)^2} \left(\frac{z}{2}\right)^{v+2n},
\]
where to note that \( J_{1,1,\gamma,\sigma}^{1,v} \) is the \( k \)-Bessel function and \( J_{1,1,\gamma,\sigma}^{1,v} \) denotes the modified \( k \)-Bessel function.

Here our aim is to establish two generalized integral formulas, which are expressed in terms of generalized \( k \)-Wright functions, by inserting newly generalized modified \( k \)-Bessel function.

The generalized Wright hypergeometric function \( p\Psi_q(z) \) is given by the series
\[
(1.5) \quad p\Psi_q(z) = \prod_{i=1}^{p} \Gamma(\alpha_i)^{1/p} \prod_{j=1}^{q} \Gamma(\beta_j)^{1/q} \sum_{n=0}^{\infty} \frac{(\alpha_i)_{n,k} \Gamma(\sigma n + v + \frac{b+1}{2})}{(n!)^2} \left(\frac{z}{2}\right)^{v+2n},
\]

where \( \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1 \)

was found in the work of [23, 24]. Properties of this generalized Wright function were investigated in [13], (see also [14, 15]). In particular, it was proved [13] that \( p\Psi_q(z) \), \( z \in \mathbb{C} \) is an entire function under the condition (1.6).

The generalized hypergeometric function represented as follows [20]:
\[
(1.7) \quad _pF_q^\left( \begin{array}{c} (\alpha_p) \cr (\beta_q) \end{array} \right) \left( \begin{array}{c} z \\ n \end{array} \right) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} (\alpha_i)_n \beta_j n!}{\prod_{j=1}^{q} (\beta_j)_n n!},
\]
provided \( p \leq q; p = q + 1 \) and \( |z| < 1 \), where \( (\lambda)_n \) is well known Pochhammer symbol defined for \( \lambda \in \mathbb{C} \) (see [20])

\[
(1.8) \quad (\lambda)_n := \begin{cases} \frac{1}{\lambda(\lambda + 1)\cdots(\lambda + n - 1)} & (n = 0) \\ \lambda(n + \lambda + 1)\cdots(n + \lambda + n - 1) & (n \in \mathbb{N} := \{1, 2, 3, \ldots\}) \end{cases}
\]

\[
(1.9) \quad (\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-),
\]
where \( \mathbb{Z}_0^- \) is the set of non positive integers.
If we put \(\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1\) in (1.5), then (1.7) is a special case of the generalized Wright function:
\[
p \Psi_q(z) = p \Psi_q \left( (\alpha_1, 1), \ldots, (\alpha_p, 1); z \right) = \prod_{j=1}^p \Gamma(\beta_j) \prod_{i=1}^p \Gamma(\alpha_i) \quad {}_pF_q \left[ \begin{array}{c} \alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q; \\ z \end{array} \right].
\]

For the present investigation, we need the following result of Oberhettinger [17]
\[
(1.11) \quad \int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} dx = 2 \left( \frac{a}{2} \right) \Gamma (2\mu) \Gamma (\lambda - \mu) \Gamma (1 + \lambda + \mu)
\]
provided \(0 < \Re (\mu) < \Re (\lambda)\).

Also, we need the following relation of \(\Gamma_k\) with the classical gamma Euler function (see [21]):
\[
(1.12) \quad \Gamma_k(z + k) = z \Gamma_k(z), \\
(1.13) \quad \Gamma_k(z) = k^{\frac{z}{k} - 1} \Gamma \left( \frac{z}{k} \right), \\
(1.14) \quad (\gamma)_{n,k} = k^n \left( \frac{z}{k} \right)_n.
\]

2. Main results

Two generalized integral formulas established here, which expressed in terms of generalized Wright functions (1.10) by inserting the generalized modified \(k\)-Bessel function of the first kind (1.4) with the suitable argument in the integrand of (1.11).

**Theorem 1.** For \(\lambda, \mu, \nu, c, \sigma \in \mathbb{C}, \Re (\lambda + \nu + 2) > \Re (\mu) > 0\) and \(x > 0\). Then the following formula holds true:
\[
I = \int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} J_{b,c,\gamma,\sigma}^{k,\nu,\kappa,\omega} \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) dx = \frac{2^{1-\nu} y^{\nu} k^{1-\frac{\nu}{2} - \kappa}}{\Gamma (2\mu)} \times \Psi_4 \left[ \begin{array}{c} \frac{\nu + 1}{2}, (\lambda + \nu + 2), (\lambda + \nu - \mu, 2); \\ +\frac{\kappa}{2}, (\lambda + \nu, 2), (1 + \lambda + \nu + \mu, 2), (1, 1); \\ \frac{ck^{1-\frac{\nu}{2}} y^2}{4a^2} \end{array} \right].
\]

*Proof.* By applying (1.4) to the LHS of (2.1) and interchanging the order of integration and summation, which is verified by uniform convergence of the involved series under the given conditions, we obtain

\[
I = \int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} J_{b,c,\gamma,\sigma}^{k,\nu,\kappa,\omega} \left( \frac{y}{x + a + \sqrt{x^2 + 2ax}} \right) dx
= \int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda}
\]
Now, using (1.13) and (1.14), we have

\[
I = \sum_{n=0}^{\infty} \frac{(c)^n k^n \Gamma \left( \frac{n}{\gamma} + \frac{1}{\gamma} \right)}{\Gamma \left( \frac{n}{2} + \frac{1}{2} \right)} \left( \frac{y}{2} \right)^{\nu+2n} \times \int_0^\infty x^{\nu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\nu-2n} dx.
\]

In view of the conditions given in Theorem 1, since \( \Re(\lambda + \nu) > \Re(\mu) > 0 \), \( k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).

Applying (1.11) to the integrand of (2.1) and obtain the following expression:

\[
I = \sum_{n=0}^{\infty} \frac{(c)^n k^n \Gamma \left( \frac{n}{\gamma} + \frac{1}{\gamma} \right)}{\Gamma \left( \frac{n}{2} + \frac{1}{2} \right)} \left( \frac{y}{2} \right)^{\nu+2n} \times 2 \nu y^\nu k^{1-\frac{1}{\gamma} - \frac{n}{\nu+2n}} \frac{\Gamma(2\mu)}{\Gamma(1+\lambda+\nu+\mu+2n)} \Gamma(\lambda + \nu + 2n - \nu)
\]

Corollary 2.1 ([6]). Let the conditions of theorem 1 be satisfied and let \( k = \gamma = \sigma \) and \( c = -c \) in (2.1). Then the following integral formula holds:

\[
\int_0^{\infty} x^{\nu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\nu} J_{\nu}^\mu \left( \frac{y}{x+a+\sqrt{x^2+2ax}} \right) dx = 2^1 - \nu y^\nu \Gamma(2\mu) \quad (2.2)
\]

Theorem 2. For \( \sigma, \mu, \nu, c, b \in \mathbb{C}, 0 < \Re(\mu + \nu + 2) < \Re(\lambda + \nu + 2) \) and \( x > 0 \), then the following integral formula holds true:

\[
\int_0^{\infty} x^{\nu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\nu} J_{\nu}^\mu \left( \frac{xy}{x+a+\sqrt{x^2+2ax}} \right) dx = 2^1 - 2\nu y^\nu \Gamma(\lambda - \mu) \quad (2.3)
\]
Proof. By applying (1.4) to the LHS of (2.3) and interchanging the order of integration and summation, which is verified by uniform convergence of the involved series under the given conditions, we obtain

\[ I_1 = \int_0^\infty x^{n-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} J_{\nu, \gamma}^{b, c, \sigma} \left( \frac{x y}{x + a + \sqrt{x^2 + 2ax}} \right) dx \]

\[ = \int_0^\infty x^{n-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} \times \sum_{n=0}^\infty \frac{(c)^n (\gamma)_{n,k}}{\Gamma_k (\sigma n + \nu + \frac{b+1}{2})} \Gamma (n+\nu+2n+1) \frac{1}{(n!)^2} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\nu-2n} \frac{(x y)^{\nu+2n}}{(n!)^2} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-(\nu+2n)} dx. \]

Using the (1.13) and (1.14), we have

\[ I_1 = \sum_{n=0}^\infty \frac{(c)^n k^n \Gamma \left( \frac{\sigma}{k} + n \right)}{\Gamma \left( \frac{\sigma}{k} \right) \Gamma \left( \frac{\nu}{k} + \frac{\nu + 1}{2} + \frac{b+1}{2k} \right)} \frac{1}{(n!)^2} \times \int_0^\infty x^{(\mu+\nu+2n)-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-(\lambda+\nu+2n)} dx. \]

Applying (1.11) to the integrand of (2.3), we obtain the following expression:

\[ I_1 = \sum_{n=0}^\infty \frac{(c)^n k^n \Gamma \left( \frac{\sigma}{k} + n \right)}{\Gamma \left( \frac{\sigma}{k} \right) \Gamma \left( \frac{\nu}{k} + \frac{\nu + 1}{2} + \frac{b+1}{2k} \right)} \frac{1}{(n!)^2} \times 2 (\lambda + \nu + 2n) a^{-(\lambda+\nu+2n)} \gamma \frac{\mu+\nu+2n}{\lambda} \Gamma (2\mu + 2\nu + 4n) \Gamma (\lambda - \mu) \Gamma (1 + \lambda + 2\nu + \mu + 4n). \]

By making the use of (1.12) it follows that

\[ I_1 = \frac{2^{1-2\nu-\mu} y^3 a^{\mu-\lambda} \Gamma (\lambda - \mu)}{k^{\frac{\nu + 1}{2}} \frac{1}{2} \Gamma \left( \frac{\nu}{k} \right)} \sum_{n=0}^\infty \frac{\Gamma \left( \frac{\sigma}{k} + n \right) \Gamma (\lambda + \nu + 1 + 2n) \Gamma (\lambda + \nu + 2n)}{n! \Gamma (\frac{\sigma}{k} + \frac{\nu}{k} + \frac{b+1}{2k}) \Gamma (\lambda + \nu + 2n) \Gamma (\lambda + 2\nu + \mu + 1 + 4n) \Gamma (n+1) \frac{1}{4}} \gamma^n, \]

and the desired result follows from the definition (1.5).

Corollary 2.2 (6). Let the conditions given in theorem 2 satisfied and set \( k = \sigma = \gamma = 1 \) and \( c = -e \), Theorem 2 reduces to the following form

\[ \int_0^\infty x^{\mu-1} \left( x + a + \sqrt{x^2 + 2ax} \right)^{-\lambda} J_{\nu, \gamma}^{b, c, \sigma} \left( \frac{x y}{x + a + \sqrt{x^2 + 2ax}} \right) dx \]

\[ = 2^{1-2\nu-\mu} y^3 a^{\mu-\lambda} \Gamma (\lambda - \mu) \times 2 \Psi_3 \left[ \left( \lambda + \nu + 1, 2 \right), (2\mu + 2\nu, 4); (\nu + \frac{b+1}{2}, 1), (\nu + \lambda, 2), (\lambda + \mu + 2\nu + 1, 4); -\frac{cy^2}{4} \right]. \]

Remark 2.1. Setting \( b = c = 1 \) in (2.1) and (2.3) with some appropriate parameter replacements, we respectively get two integral formula for the Bessel function \( J_\nu (z) \) given by Choi and Agarwal [5].
3. Conclusion

The integral formulas for generalized modified $k$-Bessel function of first kind is derived and the results expressed in term of Fox-Wright function. Some of interesting special cases also derived from the main results. Using some suitable parametric replacement, theorem 1 and theorem 2 gives the unified integral representation of Bessel function when $c = -1 = -b$ and integral representation of modified Bessel function by putting $b = c = 1$.

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