A Tight Lower Bound for Clock Synchronization in Odd-Ary M-Toroids

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Abstract

Synchronizing clocks in a distributed system in which processes communicate through messages with uncertain delays is subject to inherent errors. Prior work has shown upper and lower bounds on the best synchronization achievable in a variety of network topologies and assumptions about the uncertainty on the message delays. However, until now there has not been a tight closed-form expression for the optimal synchronization in \(k\)-ary \(m\)-cubes with wraparound, where \(k\) is odd. In this paper, we prove a lower bound of \(\frac{1}{4}umk\) \((k - \frac{1}{2})\), where \(k\) is the (odd) number of processes in each of the \(m\) dimensions, and \(u\) is the uncertainty in delay on every link. Our lower bound matches the previously known upper bound.

1 Introduction

Synchronizing clocks in a distributed system in which processes communicate through messages with uncertain delays is subject to inherent errors. A body of work has sought bounds on how closely the clocks can be synchronized when there is no drift in the hardware clocks and there are no failures. Prior work has shown upper and lower bounds on the best synchronization achievable in a variety of network topologies and assumptions about the uncertainty on the message delays.

Lundelius and Lynch \([4]\) showed that, in an \(n\)-process clique with the same uncertainty \(u\) on every link, the best synchronization possible is \(u(1 - \frac{1}{n})\). Subsequently, Halpern et al. \([3]\) considered arbitrary topologies in which each link may have a different uncertainty, and they showed that the optimal clock synchronization is the solution of an optimization problem; however, no general closed-form expression was given. Biaz and Welch \([2]\) gave a collection of closed-form upper and lower bounds on the optimal clock synchronization for \(k\)-ary \(m\)-cubes \((m\)-dimensional hypercubes with \(k\) processes in every dimension\), both with and without wraparound, in which every link has the same uncertainty, \(u\). When there is no wraparound, the tight bound is \(\frac{1}{4}um(k - 1)\). When there is wraparound and \(k\) is even, the tight bound is \(\frac{1}{4}umk\). However, when there is wraparound and \(k\) is odd, there is a gap between the upper bound of \(\frac{1}{4}um(k - \frac{1}{2})\) and the lower bound of \(\frac{1}{4}um(k - 1)\).

In this paper, we consider \(k\)-ary \(m\)-cubes with wraparound ("\(m\)-toroids") and odd \(k\). We show a lower bound of \(\frac{1}{4}um(k - \frac{1}{2})\), which matches the previously known upper bound. We use the same shifting technique from previous lower bounds for clock synchronization (e.g., \([4, 3, 2]\)). The key insight in our improved lower bound is to exploit the fact that the graph is a collection of rings in each dimension and to use multiple shifted executions instead of just one.

2 Preliminaries

We first present our model and problem statement (following \([4, 1, 2]\)). We consider a graph of \(k^m\) processes, where \(k \geq 3\) is odd and \(m \geq 1\), in which each process \(i\) is a tuple \(<p_0, p_1, ..., p_{m-1}>\) where each \(p_i \in \{0, 1, ..., k - 1\}\). There are links in both directions between any two processes \(p\) and \(q\) if and only if their ids differ in exactly one component, say the \(i\)-th, such that \(p_i = q_i + 1\) (addition on process indices is modulo \(k\) throughout). Each process \(p\) has a hardware clock modeled as a function \(HC_p\) from reals (real time) to reals (clock time). We assume there is no drift, so \(HC_p(t) = t + c_p\) for some constant \(c_p\). Each process is modeled as a state machine whose transition function takes as input the current state, current value of the hardware clock, and current event (receipt of a message or some internal occurrence), and produces a new state and a message to send over each incident link.

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A history of process $\vec{p}$ is a sequence of alternating states and pairs of the form (event, hardware clock value), beginning with $\vec{p}$’s initial state. Each state must follow correctly from the previous one according to $\vec{p}$’s transition function and the hardware clock values must increase. A timed history of $\vec{p}$ is a history together with an assignment of a real time $t$ to each pair $(e, T)$ in the history such that $HC_\vec{p}(t) = T$. An execution is a set of $k^m$ timed histories, one per process, with a bijection for each link between the set of messages sent over the link and the set of messages received over the link. The delay of a message is the difference between the real time when it is received and the real time when it is sent. An execution is admissible if every message has delay in $[0, u]$ where $u$ is a fixed value called the uniform uncertainty.

We assume each process $\vec{p}$ has a local variable $adj_\vec{p}$ as part of its state and we define its adjusted clock $AC_\vec{p}(t)$ to be equal to $HC_\vec{p}(t) + adj_\vec{p}(t)$. An execution has terminated once all processes have stopped changing their adjusted clock variables. We say the algorithm achieves $\epsilon$-synchronized clocks if every admissible execution eventually terminates with $|AC_\vec{p}(t) - AC_\vec{q}(t)| \leq \epsilon$ for all processes $\vec{p}$ and $\vec{q}$ and all times $t$ after termination.

“Shifting” an execution changes the real times at which events occur [4]. Let $x$ be an $m$-dimensional matrix of real numbers with $k$ elements in each dimension, which we call a shift matrix; elements of $x$ are indexed by process ids. Define $shift(\alpha, x)$ be the result of adding $x_{\vec{p}}$ to the real time associated with each event in $\vec{p}$’s timed history in $\alpha$. Shifting changes the hardware clocks and message delays as follows [4]:

**Lemma 1.** Let $\alpha$ be an execution with hardware clocks $HC_\vec{p}$ and let $x$ be a shift matrix. Then $shift(\alpha, x)$ is a (not necessarily admissible) execution in which

(a) the hardware clock of each $\vec{p}$, denoted $HC_\vec{p}'(t)$, equals $HC_\vec{p}(t) + x_{\vec{p}}$ and

(b) every message from $\vec{p}$ to $\vec{q}$ has delay $\delta = x_{\vec{p}} + x_{\vec{q}}$, where $\delta$ is the message’s delay in $\alpha$.

## 3 Lower Bound

**Theorem 1.** For any algorithm that achieves $\epsilon$-synchronized clocks in a $k$-ary $m$-toroid with uniform uncertainty $u$, where $k$ is odd, it must be that $\epsilon \geq \frac{\ln(2)}{2}$.

**Proof.** Let $A$ be any algorithm that achieves $\epsilon$-synchronized clocks in a $k$-ary $m$-toroid with uniform uncertainty $u$, where $k = 2r + 1$ for some integer $r \geq 1$. Let $\alpha$ be the admissible execution of $A$ in which $HC_\vec{p}(t) = t$ for each process $\vec{p}$, every message from $\vec{p}$ to $\vec{q}$, where $\vec{q}$ is $\vec{p}$’s neighbor in the $h$-th dimension such that $q_h = p_h + 1$, has the same fixed delay $\delta_{\vec{p}, \vec{q}}$, which is $0$ if $0 \leq p_h < r$ and is $u$ if $r \leq p_h < k$, and every message from $\vec{q}$ to $\vec{p}$ has the same fixed delay $\delta_{\vec{q}, \vec{p}} = -\delta_{\vec{p}, \vec{q}}$.

For $0 \leq i < k$, define $\alpha^i = shift(\alpha, x^i)$, where the $\vec{p}$-th element of the shift matrix $x^i$, denoted $x^i_{\vec{p}}$, is defined as $\sum_{j=0}^{m-1} W^i_{j, \vec{p}}$, where $W$ is defined as follows:

| range of $i$ | $0 \leq i < r$ | $r \leq i < k$ |
|--------------|-----------------|-----------------|
| range of $p_j$ | $0 \leq p_j \leq i$ | $0 \leq p_j \leq i - r$ |
| $i < p_j \leq r$ | $(p_j - i)u$ | $(i - r)u$ |
| $r < p_j \leq r + i + 1$ | $(r - i)u$ | $(i - p_j)u$ |
| $r + i + 1 < p_j \leq 2r$ | $(2r - p_j + 1)u$ | $0$ |

The idea behind the shift amounts in $W$ is to cause two processes that are farthest apart in the graph to be shifted as far apart in real time as possible—thus achieving a large skew between their adjusted clocks—while maintaining valid message delays between all neighbors. By considering multiple shifted executions, we can cancel out terms involving adjusted clocks, leaving behind only terms that involve the system parameters $\epsilon$ and $u$, and the graph parameters $k$ and $m$.

As an example, consider the case when $k = 5$ and $m = 1$, that is, a 5-element ring (cf. Figure 1). We will denote the process with id $(i)$ by $p_i$. 


Figure 1: Graph of a 5-ary 1-toroid

Figure 2: Delay Pattern in \( \alpha \) for 5-ary 1-toroid

Figure 2 depicts the pattern of message delays in \( \alpha \) for the 5-element ring, with \( p_0 \) occurring twice for convenience in representing the wrap-around. The interpretation is that every message, if any, sent from \( p_0 \) to \( p_1 \) has delay 0, every message sent from \( p_1 \) to \( p_0 \) has delay \( u \), etc. We make no assumption about when or if such messages are sent, as that depends on the algorithm.

Figure 3: Shifted Delay Pattern for \( \alpha^1 \)

Figure 4: Shifted Delay Pattern for \( \alpha^4 \)

Now we consider two of the five shifts for this special case. The shift matrix defined by \( W \) for \( \alpha^1 \) is \([0, 0, u, u, u]\) and that for \( \alpha^4 \) is \([0, u, 2u, u, 0]\). Figures 3 and 4 depict the pattern of message delays in \( \alpha^1 \) and \( \alpha^4 \) respectively, reflecting the changes indicated by Lemma 1(b). Visual inspection shows that the delays are still in the valid range and thus the shifted executions are admissible.

Admissibility and Lemma 1(a) imply that \( AC'_1 - AC'_4 = AC_1 - AC_4 + u \leq \epsilon \) and \( AC'_4 - AC'_2 = AC_4 - AC_2 + 2u \leq \epsilon \). Similarly, one can check that \( \alpha^0, \alpha^2, \) and \( \alpha^3 \) are admissible and then get similar inequalities. Summing the five inequalities results in \( 6u \leq 5\epsilon \), or \( \epsilon \geq 6u/5 \) which agrees with Theorem 1.

We now show that all shifted executions are admissible.

**Lemma 2.** For all \( i, 0 \leq i < k \), \( \alpha^i \) is admissible.

**Proof.** Fix \( i \) with \( 0 \leq i < k \). We must show that all message delays are in \([0, u]\). Let \( \bar{p} \) and \( \bar{q} \) be two
neighbors that differ in the h-th dimension such that \( q_j = p_j + 1 \) and \( q_j = p_j \) for all \( j \neq h \). Denote the (fixed) delay of messages from \( \vec{p} \) to \( \vec{q} \) in \( \alpha \) by \( \delta_{\vec{p},\vec{q}}^\alpha \). By Lemma 1(b), \( \delta_{\vec{p},\vec{q}}^\alpha = \delta_{\vec{q},\vec{p}}^\alpha + \Delta_{\vec{p},\vec{q}}^\alpha \) where \( \Delta_{\vec{p},\vec{q}}^\alpha \) denotes \(-x_{\vec{p}}^h + x_{\vec{q}}^h\). Observe that \( \Delta_{\vec{p},\vec{q}}^\alpha = -\Delta_{\vec{q},\vec{p}}^\alpha \).

\[
\Delta_{\vec{p},\vec{q}}^\alpha = - \sum_{j=0}^{m-1} W_{p_j}^i + \sum_{j=0}^{m-1} W_{q_j}^i \quad \text{by definition of shift vector } x^i \text{ for } \alpha' \\
= -W_i^{p_0} + W_i^{q_0} \quad \text{since } \vec{p} \text{ and } \vec{q} \text{ only differ in the } h\text{-th dimension} \\
= -W_i^{p_0} + W_i^{q_0+1} \quad \text{by definition of } \vec{q}.
\]

Referring to the table defining \( W \), we get the following values for \( \Delta_{\vec{p},\vec{q}}^\alpha \):

| range of \( i \) | range of \( p_h \) | \( \Delta_{\vec{p},\vec{q}}^\alpha \) | range of \( p_h \) | \( \Delta_{\vec{p},\vec{q}}^\alpha \) |
|-------------------|------------------|------------------|------------------|
| \( 0 \leq i < r \) | \( 0 \leq p_h < i \) | \( 0 \) | \( 0 \leq p_h < i - 1 \) | \( u \) |
| \( i \leq p_h < r \) | \( i - r \leq p_h < r \) | \( -u \) | \( r \leq p_h < r + i + 1 \) | \( 0 \) |
| \( r + 1 \leq p_h \leq 2r \) | \( i \leq p_h < 2r \) | \( 0 \) | \( i \leq p_h < 2r \) | \( 0 \) |

To gain an intuition for why \( \alpha' \) is admissible, consider how the delays chosen for \( \alpha \) relate to \( \Delta_{\vec{p},\vec{q}}^\alpha \). Recall that \( \vec{p} \) and \( \vec{q} \) are neighbors in dimension \( h \). If \( \vec{p} \) occurs before index \( r \) in dimension \( h \), then \( \delta_{\vec{p},\vec{q}}^\alpha \), the delay from \( \vec{p} \) to \( \vec{q} \) in \( \alpha \), is chosen such that \( \Delta_{\vec{p},\vec{q}}^\alpha \) can be maximized; otherwise it is chosen so that \( \Delta_{\vec{p},\vec{q}}^\alpha \) can be minimized. To keep the shifted message delays in the valid range, \( \Delta_{\vec{p},\vec{q}}^\alpha \) must be between \( u \) and \(-u \). In particular, the delay in \( \alpha \) when \( \vec{p} \) occurs before index \( r \) is chosen so that \( \Delta_{\vec{p},\vec{q}}^\alpha \) can be \( u \); otherwise it is chosen so that \( \Delta_{\vec{p},\vec{q}}^\alpha \) can be \(-u \). Below we formalize these ideas.

Since \( \delta_{\vec{p},\vec{q}}^\alpha \) is in \([0, u]\), so is \( \delta_{\vec{p},\vec{q}}^\alpha \) for all table entries where \( \Delta_{\vec{p},\vec{q}}^\alpha = 0 \). For all table entries where \( \Delta_{\vec{p},\vec{q}}^\alpha = u \), the definition of \( \alpha \) states that \( \delta_{\vec{p},\vec{q}}^\alpha = 0 \), and thus \( \delta_{\vec{p},\vec{q}}^\alpha = u + u = 2u \). For all table entries where \( \Delta_{\vec{p},\vec{q}}^\alpha = -u \), the definition of \( \alpha \) states that \( \delta_{\vec{p},\vec{q}}^\alpha = u \), and thus \( \delta_{\vec{p},\vec{q}}^\alpha = u + (-u) = 0 \). In all cases \( \delta_{\vec{p},\vec{q}}^\alpha \) is in \([0, u]\).

Since \( \delta_{\vec{p},\vec{q}}^\alpha \) is defined in \( \alpha \) to be \( u - \delta_{\vec{p},\vec{q}}^\alpha \) and \( \Delta_{\vec{p},\vec{q}}^\alpha = -\Delta_{\vec{q},\vec{p}}^\alpha \), it follows that \( \delta_{\vec{q},\vec{p}}^\alpha = u - \delta_{\vec{p},\vec{q}}^\alpha \). Since we just showed that \( \delta_{\vec{p},\vec{q}}^\alpha \) is in \([0, u]\), the same is true of \( \delta_{\vec{q},\vec{p}}^\alpha \). Thus \( \alpha' \) is admissible.

Fix any \( i \) with \( 0 \leq i < r \). We focus on two processes that are maximally far away from each other. Since \( \alpha' \) is admissible by Lemma 2, A must ensure that \( AC_{\{(i,...,i)\}} - AC_{\{(i+r+1,...,i+r+1)\}} \leq \epsilon \), where \( AC_{\alpha}^{\alpha'} \) denotes the adjusted clock of process \( \vec{p} \) after termination in \( \alpha' \). By definition of \( \alpha' \) and Lemma 1(a), \( AC_{\{(i,...,i)\}} = AC_{\{(i,...,i)\}}^{\alpha'} = AC_{\{(i+r+1,...,i+r+1)\}} - m(r - i)u \). Thus by substituting we get:

\[
AC_{\{(i,...,i)\}} - AC_{\{(i+r+1,...,i+r+1)\}} \leq -m(r - i)u + \epsilon \quad \text{for } 0 \leq i < r \quad \text{(1)}
\]

Similarly, we can show:

\[
AC_{\{(i,...,i)\}} - AC_{\{(i-r,...,i-r)\}} \leq -m(i - r)u + \epsilon \quad \text{for } r \leq i < k \quad \text{(2)}
\]

Adding together the \( r \) inequalities from (1) and the \( k - r \) inequalities from (2) gives

\[
\sum_{i=0}^{r-1} AC_{\{(i,...,i)\}} - \sum_{i=0}^{r-1} AC_{\{(i-r,...,i-r)\}} + \sum_{i=r}^{k-1} AC_{\{(i,...,i)\}} - \sum_{i=r}^{k-1} AC_{\{(i-r,...,i-r)\}} \leq -um \left[ \sum_{i=0}^{r-1} (r - i) + \sum_{i=r}^{k-1} (i - r) \right] + k\epsilon \quad \text{(3)}
\]

The left-hand side of (3) resolves to 0 and the expression in square brackets equals \((k^2 - 1)/4\), and thus \( \epsilon \geq 4um (k - \frac{1}{2}) \).

4 Conclusion

We have closed the gap between the best previously-known closed-form upper and lower bounds on the optimal clock synchronization for \( k \)-ary \( m \)-toroids when \( k \) is odd and the uncertainty on each link is the same. By applying a more involved set of shifts than those in the prior work [2] and exploiting the specific network topology, we achieved a lower bound that equals the upper bound due to the algorithm in [2].
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