Supersymmetric Superconducting Bag as a Core of Kerr Spinning Particle

A. Ya. Burinskii
NSI of Russian Academy of Sciences
B. Tulskaya 52 Moscow 113191 Russia, e-mail:bur@ibrae.ac.ru

The problem of a regular matter source for the Kerr spinning particle is discussed. A class of minimal deformations of the Kerr-Newman solution is considered obeying the conditions of regularity and smoothness for the metric and its matter source.

It is shown that for charged source corresponding matter forms a rotating bag-like core having (A)dS interior and smooth domain wall boundary. Similarly, the requirement of regularity of the Kerr-Newman electromagnetic field leads to superconducting properties of the core.

We further consider the U(I) x U'(I) field model (which was used by Witten to describe cosmic superconducting strings), and we show that it can be adapted for description of superconducting bags having a long range external electromagnetic field and another gauge field confined inside the bag. Supersymmetric version of the Witten field model given by Morris is analyzed, and corresponding BPS domain wall solution interpolating between the outer and internal supersymmetric vacua is considered. The charged bag bounded by this BPS domain wall represents an ‘ultra-extreme’ state with a total mass which is lower than BPS bound of the wall. It is also shown that supergravity suggests the AdS vacuum state inside the bag.

Peculiarities of this model for the rotating bag-like source of the Kerr-Newman geometry are discussed.

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1 Introduction

The Kerr solution is well known as a field of the rotating black hole. However, for the case of a large angular momentum \(|a| = l/m \geq m\) all the horizons of the Kerr metric are absent, and there appears a naked ring-like singularity. This singularity has many unpleasant manifestations and must be regularized by being hidden inside a matter source. The Kerr solution with \(|a| \gg m\) displays some remarkable features indicating a relation to the structure of the spinning elementary particles.

In 1969 Carter [1] observed, that if three parameters of the Kerr-Newman solution are adopted to be \((\hbar=c=1)\)

\[
e^2 \approx 1/137, \quad m \approx 10^{-22}, \quad a \approx 10^{22}, \quad ma = 1/2, \quad (1)
\]

then one obtains a model for the four parameters of the electron: charge \(e\), mass \(m\), spin \(l\) and magnetic moment \(ea\), and the gyromagnetic ratio is automatically the same as that of the Dirac electron. The first treatment of the disc-like source of the Kerr spinning particle was given by Israel [2] in the form of an infinitely thin disc spanned by the Kerr singular ring [1]. Then, some stringy structures were obtained in Kerr geometry [3], and the Israel results where corrected by Hamity [4] showing that the disc is to be in a rigid relativistic rotation. Further, López [5] suggested a regularized model of the source constructing the disc in the form of a (infinitely thin) rotating elliptical shell (bubble) covering the singular ring. Besides the bubble models, the solid disc-like sources were considered, too [6, 7]. The structure of the electromagnetic field near the disc suggested superconducting properties of the material of the source [3, 4, 7], and there was obtained an analogue of the Kerr singular ring with the Nielsen-Olesen [10] and Witten [12] superconducting strings. Since 1992 there has been considerable interest as to black holes in string theory and the point of view that some of black holes can be treated as elementary particles [13, 14, 15, 16]. In particular, Sen [14] has obtained a generalization of the Kerr solution to low energy string theory, and it was shown [15] that near the Kerr singular ring the Kerr-Sen solution acquires a metric similar to the field around a heterotic string.

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1. Disc has the Compton size with radius \(a = l/m = \frac{1}{2\pi c}\).
2. One should also mention a super-version of the Kerr-Newman solution [7, 18] involving fermionic fields which are important for a complete description of spinning particle.
Therefore, it was obtained that the structure of the Kerr source contains some elements of the known extended sources such as strings, membranes, bubbles and domain walls. However, alongside with singular description of such extended sources, there exists a regular soliton-like description based on the nonlinear field models with broken symmetry, and in particular, on the Higgs model of superconductivity.

The aim of this paper is to demonstrate that a smooth bag-like source appears in the core of Kerr geometry as a consequence of simple and natural proposals on the regularity of metric bounded by the Kerr-Schild class, and to represent a field model suitable for the description of the properties of the particle-like source of the Kerr-Newman field. Analyzing the known reasonable smooth sources based on the Higgs models, we show that only the $U(I) \times \tilde{U}(I)$ field model (used by Witten for superconducting cosmic strings [11, 12]) possesses the external long range electromagnetic field which is necessary to describe the Kerr-Newman source. We further consider the supersymmetric version of this field model given by Morris [19] and show that it can be adapted for description of the bag-like superconducting core of the Kerr-Newman field. The supersymmetric vacuum states of this model are analyzed, and an approximate bag-like domain wall solution separating the internal (false) and external (true) vacua is considered. We show that this domain wall represents a BPS saturated state. It is remarkable that the total mass of the charged bag formed from this BPS saturated domain wall turns out to be lower than BPS energy bound for this domain wall (so called ‘ultra-extreme’ state [39]). Therefore, the way to overcome the BPS bound [44] is opened in this model, which is necessary to get the real ratio of parameters [1].

We should note that similar problem of the metric regularization appears in another context as the problem of a regular black hole interior [20, 21, 25, 23, 22, 24]. Many of the results obtained in this direction as well as the methods of treatment resemble those of the problem of the Kerr spinning particle. In particular, in the case of the absence of the horizons the core is treated as a visible semi-closed world, there also appear bubbles with infinitely thin domain walls [24, 27], and a crucial role of the demand of regularity for the Kerr-Schild class of metrics was mentioned [22]. In some treatments, for example in [21], the relation of both of the problems is also discussed. However, when considering the matter field source we apply not to the corrections of stress-energy tensor caused by quantum fields of vac-
uum polarization but rather to the supersymmetric vacuum states caused by a multiplet of Higgs fields in supergravity. As a result, we come to the conclusion that the case of AdS internal space inside the charged core of the Kerr-Newman geometry looks the most plausible.

2 Stress-energy tensor of the Kerr source

The Kerr-Newman metric in the Kerr-Schild form is

$$g_{\mu\nu} = \eta_{\mu\nu} + 2hk_{\mu}k_{\nu},$$

where $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ is an auxiliary Minkowski space, $h = \frac{f(r)}{r^2 + a^2 \cos^2 \theta}$, $k_{\mu}$ is a vector field tangent to the Kerr geodesic and shear free congruence and function $f$ has the Kerr-Newman form $f_{KN}(r) = mr - e^2/2$, where $r$ and $\theta$ are oblate spheroidal coordinates (family of confocal ellipsoids $r = const.$ covers the disc $r = 0$). This metric corresponds to a regular electrovacuum space-time for the exclusion of singular ring at $r = \cos \theta = 0$. This ring is a branch line of space on two sheets corresponding to $r > 0$ and $r < 0$.

Previous attempts to introduce the Kerr source were connected with a truncation of the ‘second’ sheet along the disc $r = 0$. This procedure breaks the analyticity of the space, and there appears a singular distribution of material sources on the disc, as a consequence of the Einstein field equations. The choice of truncation is not unique, and diverse singular models of the Kerr source were considered [2, 5, 6].

Another approach to this problem is connected with a smooth deformation of the Kerr-Newman space in a neighborhood of the Kerr disc [27, 29, 24] keeping the Kerr-Schild form of metric. The deformed metric induces a stress-energy tensor in the right side of the Einstein equations which corresponds to the appearance of some extra matter surrounding the disc $r = 0$, and there appears an image of the source of the Kerr-Newman geometry in the form of a rotating disc with a smooth matter distribution.

On the basis of the results of recent calculations [24] performed in the Kerr-Schild formalism [26], we would like to show here that imposing some minimal conditions of regularity and smoothness on the resulting metric and

\[3\] Its concrete form will not be essential for our consideration in this paper.
energy distribution one can obtain a shape of the source in the form of a rotating bag with a smooth domain wall boundary.

Following [24], we assume that the deformed metric retains the Kerr-Schild form (2) and the form of the Kerr principal null congruence $k_\mu(x)$ while the function $f(r)$ is restricted by the following conditions:

i) the deformed metric has to be regular and smooth for $r \geq 0$ and has to lead to a regular matter distribution in core region;

ii) when out of the core, function $f(r)$ has to take the Kerr-Newman form $f_{KN}(r) = mr - e^2/2$, where $m$ and $e$ are the total mass and charge parameters.

To satisfy the condition i) for small values of $r$, leading term of the function $f(r)$ has to be at least of the order $\sim r^n$ with $n \geq 4$. The analysis of the stress-energy tensor for metrics (2), given in [24], selects some peculiar properties of the case $n = 4$. In this case $f(r) = f_0(r) = \alpha r^4$ in the core region, and in the nonrotational case ($a = 0$) space-time has a constant curvature generated by a homogenous matter distribution with energy density $\rho = \frac{1}{8\pi}6\alpha$ in the core. This fact motivates one more condition:

iii) the internal region of the core has to be described by function $f_0(r) = \alpha r^4$.

As a result of ii) and iii) the boundary of core $r_0$ may be estimated as a point of intersection of $f_0(r)$ and $f_{KN}(r)$ and the resulting smooth function $f(r)$ must be interpolating between functions $f_0(r)$ and $f_{KN}(r)$ near the boundary of the core $r \approx r_0$.

It is surprising that the bag-like structure of the Kerr-Newman source is a direct result of these simple and natural conditions.

The analysis given in [24] shows that metric (2) can be expressed via orthonormal tetrad as follows

$$g_{\mu\nu} = m_\mu m_\nu + n_\mu n_\nu + l_\mu l_\nu - u_\mu u_\nu,$$

and the corresponding stress-energy tensor of the source following from the Einstein equations may be represented in the form

$$T^{(af)}_{\mu\nu} = (8\pi)^{-1}[(D + 2G)g_{\mu\nu} - (D + 4G)(l_\mu l_\nu - u_\mu u_\nu)],$$

where

$$u_\mu = -\sqrt{\Delta} \sum(1, 0, 0, -a\sin^2 \theta)$$
is the unit time-like four-vector,
\[ l_\mu = \sqrt{\Sigma} (0, 1, 0, 0) \]  
(6)
is the unit vector in radial direction, and
\[ n_\mu = \sqrt{\Sigma} (0, 0, 1, 0), \]  
(7)
\[ m_\mu = \frac{\sin \theta}{\sqrt{\Sigma}} (a, 0, 0, -r^2 - a^2) \]  
(8)
are two more space-like vectors. Here \( \Delta = r^2 + a^2 - 2f \) and \( \Sigma = r^2 + a^2 \cos^2 \theta, \)
\[ D = -f''/(r^2 + a^2 \cos^2 \theta), \]  
(9)
\[ G = (f'r - f)/(r^2 + a^2 \cos^2 \theta)^2, \]  
(10)
and the Boyer-Lindquist coordinates \( t, r, \theta, \phi \) are used. The expressions (4), (9), (10) show that the source represents a smooth distribution of rotating confocal ellipsoidal layers \( r = \text{const.} \), and stress-energy tensor has the form corresponding to an anisotropic fluid.

Like to the results for singular (infinitely thin) shell-like source [4, 5], the stress-energy tensor can be diagonalized in a comoving coordinate system showing that the source represents a relativistic rotating disc. However, in this case, the disc is separated into ellipsoidal layers each of which rotates rigidly with its own angular velocity \( \omega(r) = a/(a^2 + r^2) \). In the comoving coordinate system the tensor \( T_{\mu\nu} \) takes the form
\[ T_{\mu\nu} = \frac{1}{8\pi} \begin{pmatrix} 2G & 0 & 0 & 0 \\ 0 & -2G & 0 & 0 \\ 0 & 0 & 2G + D & 0 \\ 0 & 0 & 0 & 2G + D \end{pmatrix}, \]  
(11)
that corresponds to energy density \( \rho = \frac{1}{8\pi} 2G \), radial pressure \( p_{\text{rad}} = -\frac{1}{8\pi} 2G \), and tangential pressure \( p_{\text{tan}} = \frac{1}{8\pi} (D + 2G) \).

\(^4\)Similar interpretation was also given in [27], however the expression for stress-energy tensor is not correct there, it does not contain \( D \) term. The form [1] is close to the expression obtained in [29].
Setting $a = 0$ for the non-rotating case, we obtain $\Sigma = r^2$, the surfaces $r = \text{const.}$ are spheres and we have spherical symmetry for all the above relations. The region described by $f(r) = f_0(r)$ is the region of constant value of the scalar curvature invariant $R = 2D = -2f''_0/r^2 = -24\alpha$, and of constant value of energy density

$$T_{\mu\nu} = \frac{1}{8\pi} 6\alpha \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$ (12)

If we assume that the region of a constant curvature is closely extended to the boundary of source $r_0$ which is determined as a root of the equation $f_0(r_0) = f_{KN}(r_0)$, then, smoothness of the $f(r)$ in a small neighborhood of $r_0$, say $|r - r_0| < \delta$, implies a smooth interpolation for the derivative of the function $f(r)$ between $f'_0(\delta_{r=r_0-\delta})$ and $f'_{KN}(\delta_{r=r_0+\delta})$. Such a smooth interpolation on a small distance $\delta$ shall lead to a shock-like increase of the second derivative $f''(r)$ by $r \approx r_0$. Graphical analysis shows that the result will be different for charged and uncharged cases.

In charged case for $\alpha \leq 0$ there exists only one positive root $r_0$, and second derivative of the smooth function $f''(r)$ is positive near this point. Therefore, there appears an extra tangential stress near $r_0$ caused by the term $D = -f''(r)/(r^2 + a^2 \cos^2 \theta)|_{r=r_0}$ in the expression (11). It can be interpreted as the appearance of an effective shell (or a domain wall) confining the charged ball-like source with a geometry of a constant curvature inside the ball. The case $\alpha = 0$ represents the bubble with a flat interior which has in the limit $\delta \to 0$ an infinitely thin shell. It corresponds to the Dirac electron model in the case of a spherical shell and to the López rotating shell model for a spinning particle [3, 28].

The point $r_e = \frac{\sqrt{2}}{2m}$ corresponding to a “classical size” of electron is a peculiar point as a root of the equation $f_{KN}(r) = 0$. It should be noted that by $\alpha = 0$ the equation (13) yields the root $r_0 = r_e$. For $\alpha < 0$ position of the roots is $r_0 < r_e$, and for $\alpha > 0$ one obtains $r_0 > r_e$. One can also see that $r_0 \to 0$ by $\alpha \to -\infty$. Therefore, there appear four parameters characterizing the function $f(r)$ and the corresponding bag-like core: parameter $\alpha$ characterizing cosmological constant inside the bag $\Lambda_{in} = 6\alpha$, two peculiar points
$r_0$ and $r_e$ characterizing the size of the bag and parameter $\delta$ characterizing the smoothness of the function $f(r)$ or the thickness of the domain wall at the boundary of core.

For $\alpha > 0$ two real roots of (13) can exist. If $r_0^{(1)} < r_0^{(2)}$ then only near the root $r_0^{(1)}$ the second derivative $f''(r)$ is positive, and the confining shell appears. However, if the root $r_0^{(2)}$ is chosen as a boundary of the source then the second derivative $f''(r)$ is negative by $r \approx r_0$, and a tangential pressure appears on the boundary of source instead of the stress.

The internal geometry of the ball is de Sitter one for $\alpha > 0$, anti de Sitter one for $\alpha < 0$ and flat one for $\alpha = 0$. One should note that this effective shell has a more sharp manifestation for AdS interior ($\alpha < 0$), and it can be absent at all for charged source by $\alpha > 0$.

For the uncharged source Eq.(13) has the root $r_0 = 0$, and by $\alpha \leq 0$ there are no positive roots at all. If $\alpha > 0$ there is one positive root $r_0$ and the second derivative $f''(r)$ is negative there leading to the appearance of an extra tangential pressure on the boundary of source instead of the stress. Therefore, the confining shell is absent, and the source is confined by the radial stress coming from the term $G$ that is positive in this case.

Let us consider peculiarities of the rotating Kerr source. In this case $\Sigma = r^2 + a^2 \cos^2 \theta$, and the surfaces $r = \text{const.}$ are ellipsoids described by the equation

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1.$$  \hspace{1cm} (14)

Energy density inside the core will be constant only in the equatorial plane $\cos \theta = 0$. Therefore, the Kerr singularity is regularized and the curvature is constant in string-like region $r < r_0$ and $\theta = \pi/2$ near the former Kerr singular ring. At the same time, when considering the energy density in central regions of the disc corresponding to $\theta = 0$, and taking into account the values of parameters for spinning particle $\alpha$, one finds that its ratio to the energy density at $\theta = \pi/2$ will be negligible $\frac{\rho_{\theta=\pi/2}}{\rho_{\theta=0}} \approx r^2/a^4 < r_e^2/a^4 \sim 10^{-48}$. Similarly, estimating the ratio $2G/D$ near the axis of the disc one finds that energy density is also negligible in respect to the stress of domain wall in this region $|2G/D| \sim |2(f'r - f)/f''a^2| < (r_e/a)^2 < 10^{-4}$ and the ratio $\frac{\text{stress}|_{\theta=\pi/2}}{\text{stress}|_{\theta=0}} < (r_e/a)^4 < 10^{-8}$ shows a strong increase of the stress near the string-like boundary of the disc.
3 Field model for the bag-like Kerr source

The known models of the bags [30] and cosmic bubbles [31, 35] with smooth domain wall boundaries are based on the Higgs scalar field $\phi$ with a Lagrange density of the form $L = -\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{\lambda}{8} (\phi^2 - \eta^2)^2$ leading to the kink planar solution (the wall is placed in $xy$-plane at $z = 0$)

$$\phi(z) = \eta \tanh(z/\delta),$$

where $\delta = \frac{2}{\lambda \eta}$ is the wall thickness. The kink solution describes two topologically distinct vacua $< \phi > = \pm \eta$ separated by the domain wall.

The stress–energy tensor of the domain wall is

$$T^\nu_{\mu} = \frac{\lambda^2 \eta^4}{4} \cosh^{-4}(z/\delta) \text{diag}(1, 1, 1, 0),$$

indicating a surface stress within the plane of the wall which is equal to the energy density. When applied to the spherical bags or cosmic bubbles [30, 31, 35], the thin wall approximation is usually assumed $\delta \ll r_0$, and a spherical domain wall separates a false vacuum inside the ball ($r < r_0$) $< \phi >_{in} = -\eta$ from a true outer vacuum $< \phi >_{out} = \eta$.

In the gauge string models [10], the Abelian Higgs field provides confinement of the magnetic vortex lines in superconductor. Similarly, in the models of superconducting bags [31, 32], the gauge Yang-Mills or quark fields are confined in a bubble (or cavity) in superconducting QCD-vacuum.

A direct application of the Higgs model for modelling superconducting properties of the Kerr source is impossible since the Kerr source has to contain the external long range Kerr-Newman electromagnetic field, while in the models of strings and bags the situation is quite opposite: vacuum is superconducting in external region and electromagnetic field acquires a mass there from Higgs field turning into a short range field. An exclusion represents the $U(I) \times \tilde{U}(I)$ cosmic string model given by Vilenkin-Shellard and Witten [11, 12] which represents a doubling of the usual Abelian Higgs model. The model contains two sectors, say $A$ and $B$, with two Higgs fields $\phi_A$ and $\phi_B$, and two gauge fields $A_\mu$ and $B_\mu$ yielding two sorts of superconductivity $A$ and $B$. It can be adapted to the bag-like source in such a manner that the gauge field $A_\mu$ of the $A$ sector has to describe a long-range electromagnetic field in outer region of the bag while the chiral scalar field of this sector $\phi_A$ has to
form a superconducting core inside the bag which must be unpenetrable for $A_\mu$ field.

The sector $B$ of the model has to describe the opposite situation. The chiral field $\phi_B$ must lead to a $B$-superconductivity in outer region confining the gauge field $B_\mu$ inside the bag.

The corresponding Lagrangian of the Witten $U(I) \times \tilde{U}(I)$ field model is given by \[ L = -(D^\mu \phi_A)(\bar{D}_\mu \phi_A) - (\bar{D}^\mu \phi_B)(\bar{D}_\mu \phi_B) - \frac{1}{4} F^\mu_\nu F_A^\nu_\mu - \frac{1}{4} \tilde{F}^\mu_\nu \tilde{F}_B^\nu_\mu - V, \] (17)

where $F_A^\mu_\nu = \partial^\mu A_\nu - \partial^\nu A_\mu$ and $\tilde{F}_B^\mu_\nu = \partial^\mu B_\nu - \partial^\nu B_\mu$ are field stress tensors, and the potential has the form

\[ V = \lambda(\bar{\phi}_B \phi_B - \eta^2)^2 + f(\bar{\phi}_B \phi_B - \eta^2)\bar{\phi}_A \phi_A + m^2 \bar{\phi}_A \phi_A + \mu(\bar{\phi}_A \phi_A)^2. \] (18)

Two Abelian gauge fields $A_\mu$ and $B_\mu$ interact separately with two complex scalar fields $\phi_B$ and $\phi_A$ so that the covariant derivative $\bar{D}_\mu \phi_A = (\partial + i e A_\mu) \phi_A$ is associated with $A$ sector, and covariant derivative $\bar{D}_\mu \phi_B = (\partial + i g B_\mu) \phi_B$ is associated with $B$ sector. Field $\phi_B$ carries a $\tilde{U}(1)$ charge $\bar{q} \neq 0$ and a $U(1)$ charge $q = 0$, and field $\phi_A$ carries $\tilde{U}(1)$ and $U(1)$ charges of $\bar{q} = 0$ and $q \neq 0$, respectively.

Outside the string core, the group $U(I)$ is unbroken and $A_\mu$ describes a long range electromagnetic field. At the same time group $\tilde{U}(I)$ is broken outside the string. Within the string core group $U(I)$ gets broken giving rise to superconducting condensate and nonvanishing current. Parameters $\lambda$, $\mu$, $f$, and $m$ are to be positive. The vacuum state outside the core is characterized by $\bar{\phi}_B \phi_B = \eta^2$ and $\phi_A = 0$. In the string core $\phi_B = 0$, and in the range of parameters $(f \eta^2 - m^2) > 0$, there appears a superconducting $\phi_A$-condensate $|\phi_A| = \phi_0 > 0$ breaking the gauge group $U(I)$ and generating the mass $m_A = e^2 \phi_0^2$ for the electromagnetic field.

Therefore, the model fully retains the properties of the usual bag models which are described by $B$ sector providing confinement of $B_\mu$ gauge field inside bag, and it acquires the long range electromagnetic field $A_\mu$ in the outer-to-the-bag region described by sector $A$. The A and B sectors are almost independent interacting only through the potential term for scalar fields. This interaction has to provide synchronized phase transitions from

\[\text{We use signature } (- + + +).\]
superconducting B-phase inside the bag to superconducting A-phase in the outer region. The synchronization of this transition occurs explicitly in a supersymmetric version of this model given by Morris [19].

4 Supersymmetric Morris model

In Morris model, the main part of Lagrangian of the bosonic sector is similar to the Witten field model. However, model has to contain an extra scalar field $Z$ providing synchronization of the phase transitions in $A$ and $B$ sectors.

The effective Lagrangian of the Morris model has the form

$$L = -2(D^\mu \phi)(\bar{D_\mu} \phi) - 2(\bar{D^\mu} \sigma)(\bar{D_\mu} \sigma) - \partial^\mu Z \partial_\mu \bar{Z}$$
$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}F_B^{\mu\nu}F_{B\mu\nu} - V(\sigma, \phi, Z),$$

where the potential $V$ is determined through the superpotential $W$ as

$$V = \sum_{i=1}^5 |W_i|^2 = 2|\partial W/\partial \phi|^2 + 2|\partial W/\partial \sigma|^2 + |\partial W/\partial Z|^2.$$  

The following superpotential, yielding the gauge invariance and renormalizability of the model, was suggested [6]

$$W = \lambda Z(\sigma \bar{\sigma} - \eta^2) + (cZ + m)\phi \bar{\phi},$$

where the parameters $\lambda$, $c$, $m$, and $\eta$ are real positive quantities.

The resulting scalar potential $V$ is then given by

$$V = \lambda^2 (\sigma \bar{\sigma} - \eta^2)^2 + 2\lambda c(\sigma \bar{\sigma} - \eta^2)\phi \bar{\phi} + c^2 (\bar{\phi} \phi)^2 + 2\lambda^2 \bar{Z}Z \sigma \bar{\sigma} + 2(c \bar{Z} + m)(cZ + m)\phi \bar{\phi}.$$  

6In fact the Morris model contains five complex chiral fields $\phi_i = \{Z, \phi_-, \phi_+, \sigma_-, \sigma_+\}$. However, the following identification of the fields is assumed $\phi = \phi_+; \bar{\phi} = \phi_- and \sigma = \sigma_+; \bar{\sigma} = \sigma_-$. In previous notations $\phi \sim \phi_A$ and $\sigma \sim \phi_B$.

7Superpotential is holomorphic function of $\{Z, \phi, \bar{\phi}, \sigma, \bar{\sigma}\}$. 

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4.1 Supersymmetric vacua

From (20) one sees that the supersymmetric vacuum states, corresponding to the lowest value of the potential, are determined by the conditions

\[ F_\sigma = -\partial \bar{W} / \partial \bar{\sigma} = 0; \tag{23} \]
\[ F_\phi = -\partial \bar{W} / \partial \bar{\phi} = 0; \tag{24} \]
\[ F_Z = -\partial \bar{W} / \partial \bar{Z} = 0, \tag{25} \]

and yield \( V = 0. \) These equations lead to two supersymmetric vacuum states:

\[ I) \quad Z = 0; \quad \phi = 0; \quad |\sigma| = \eta; \quad W = 0; \tag{26} \]

and

\[ II) \quad Z = -m/c; \quad \sigma = 0; \quad |\phi| = \eta \sqrt{\lambda/c}; \quad W = \lambda m \eta^2 / c. \tag{27} \]

We shall take the state \( I \) as a true vacuum state for external region of the bag, and the state \( II \) as a false vacuum state inside the bag.

The field equations following from this Lagrangian have the form

\[ \bar{D}_\mu \bar{D}^\mu \sigma - \sigma \left[ \lambda^2 (\bar{\sigma} \sigma - \eta^2) + \lambda c \bar{\phi} \phi + \lambda^2 \bar{Z} \bar{Z} \right] = 0, \tag{28} \]
\[ D_\mu D^\mu \phi - \phi \left[ \lambda c (\bar{\sigma} \sigma - \eta^2) + c^2 \bar{\phi} \phi + (c \bar{Z} + m) (c \bar{Z} + m) \right] = 0, \tag{29} \]
\[ \nabla_\mu \nabla^\mu \bar{Z} - 2\lambda^2 Z \sigma \sigma - 2c (c \bar{Z} + m) \bar{\phi} \phi = 0, \tag{30} \]
\[ - \nabla_\nu \nabla^\nu B_\mu = I_{B\mu} = i2g \left[ \bar{\sigma} (\bar{D}_\mu \sigma) - \sigma (\bar{D}_\mu \bar{\sigma}) \right], \tag{31} \]
\[ - \nabla_\nu \nabla^\nu A_\mu = I_{A\mu} = i2e \left[ \bar{\phi} (\bar{D}_\mu \phi) - \phi (\bar{D}_\mu \bar{\phi}) \right], \tag{32} \]

where \( \nabla_\mu \) is a covariant derivative.\(^8\)

\(^8\)Factors 2 by charges are connected with factors 2 in effective Lagrangian and can be removed by a rescaling of the fields \( \sigma \) and \( \phi. \)
4.2 Gauge fields and superconducting sources

It is convenient to select amplitudes and phases of the chiral fields

\[ \phi = \Phi e^{i \chi_A}, \sigma = \Sigma e^{i \chi_B}. \]  

Field \( Z \) is considered as real. By this ansatz the real part of the equation (29) yields

\[ \frac{D^\mu D_\mu \Phi}{\Phi} - \left[ \lambda c (\Sigma^2 - \eta^2) + c^2 \Phi^2 + (cZ + m)^2 + U_A \right] = 0, \]  

where \( U_A = (\chi_{A,\mu} + eA_\mu)^2 \), and its imaginary part is

\[ [\nabla^\mu + 2(ln \Phi)_{,\mu}] (\chi_{A,\mu} + eA_\mu) = 0. \]  

Similarly, one obtains the real part of (28)

\[ \frac{\tilde{D}^\mu \tilde{D}_\mu \Sigma}{\Sigma} - \left[ \lambda c (\Sigma^2 - \eta^2) + c^2 \Phi^2 + (cZ + m)^2 + U_B \right] = 0, \]  

where \( U_B = (\chi_{B,\mu} + gB_\mu)^2 \), and imaginary part

\[ [\nabla^\mu + 2(ln \Sigma)_{,\mu}] (\chi_{B,\mu} + eB_\mu) = 0. \]  

The equations (30), (31) and (32) take the form

\[ \nabla_\mu \nabla^\mu Z - 2 \lambda^2 \Sigma^2 - 2c(cZ + m)\Phi^2 = 0, \]  

\[ \nabla_\nu \nabla^\nu B_\mu = -I_{B\mu} = 4g\Sigma^2 (\chi_{B,\mu} + gB_\mu), \]  

\[ \nabla_\nu \nabla^\nu A_\mu = -I_{A\mu} = 4e\Phi^2 (\chi_{A,\mu} + eA_\mu). \]  

This is a very complicated system of coupled equations. However, one can see that the behavior of the gauge fields can be studied independently in the nearly independent sectors \( A \) and \( B \) when taking into account the known vacuum states (26) and (27) for internal and external regions of the bag. At the same time, as the first approximation, the interaction of these sectors can be studied setting aside the gauge fields. In our analysis we will follow this line.
4.2.1 Charged superconducting sphere

When considering sector $A$ equations (34),(35) can be simplified by assuming that the amplitudes are functions of radial coordinate $r$. Setting $\Phi = \Phi(r)$ and $\Sigma = \Sigma(r)$, one obtains

$$\Phi'' + \frac{2}{r} \Phi' - \Phi \left[ \lambda c(\Sigma^2 - \eta^2) + e^2 \Phi^2 + (cZ + m)^2 + U_A \right] = 0,$$

(41)

where

$$U_A = (\chi_{A,\mu} + e A_\mu)^2.$$  

(42)

$$(\partial^\mu + 2 \Phi' \vec{n}^\mu)(\chi_{A,\mu} + e A_\mu) = 0,$$

(43)

$$\nabla_\mu \nabla^\nu A_\mu = -I_{A\mu} = 4e \Phi^2 (\chi_{A,\mu} + e A_\mu),$$

(44)

where $\vec{n}^\mu = (0, x, y, z)/r$ is the unit radial vector. Equation (44) shows that field $A_\mu$ is massless if $\Phi = 0$ and acquires mass in the region of the non-zero field $\Phi$. As it follows from (26),(27) for true vacuum outside the core we have $\Phi = 0$, while inside the core $\Phi = \eta \sqrt{\lambda/c}$, and field $A_\mu$ acquires the mass $m_A = e\eta \sqrt{2\lambda/c}$ inside the core as the consequence of equation (44). The $U(I)$ gauge symmetry is broken showing that the core is superconducting with respect to the field $A_\mu$. Indeed, it follows from (44) and (43) that for $\Phi \neq 0$

$$\partial^\mu (\chi_{A,\mu} + e A_\mu) = -\Phi' (\vec{n}^\mu I_{A\mu})/(e \Phi^2) = 0,$$

(45)

so far as $\vec{n}^\mu I_{A\mu} = I_{Ar}$ must vanish for the stationary system as a radial component of e.m. current. Rewriting (43) in the form

$$\partial^\mu I_{A\mu} = 0,$$

(46)

one sees the conservation of the e.m. $A$-current corresponding to superconductivity.

For a stationary charged ball-like source field $A_\mu$ can be described as $A = A_\mu dx^\mu = \frac{e}{r} A(r)(dt + dr)$ where the factor $A$ describes exponential fall-off inside the superconductor. The symmetry of the problem shows that $I_\theta = I_r = I_\phi = 0$, and the stationary condition is $I_\theta = 0$, that yields $(\dot{\chi}_A + e A_0) = -I_{A0}/(2e \Phi^2)$. Therefore, the term $U_A$ in the equation (41) acquires the form $U_A = - (\omega_A + e A_0)^2$, where $\omega_A = \dot{\chi}_A$.

The treatment of the gauge field $B_\mu$ in $B$ sector is similar in many respects because of the symmetry between $A$ and $B$ sectors allowing one to consider
the state $\Sigma = \eta$ in outer region as superconducting one in respect to the
gauge field $B_\mu$. Field $B_\mu$ acquires the mass $m_B = g\eta$ in outer region, and
the $U(1)$ gauge symmetry is broken, which provides confinement of the $B_\mu$
field inside the bag. However, in the bag models the QCD-vacuum is “dual”
to usual superconductivity forming a magnetic superconductor [31, 32]. The
bag can also be filled by quantum excitations of fermionic, or non Abelian
fields. The interior space of the Kerr bag is regularized in this model since
the Kerr singularity and twofoldedness are suppressed by function $f = f_0(r)$.
However, a strong increase of the fields near the former Kerr singularity can
be retained leading to the appearance of traveling waves along the boundary
of the disc.

4.2.2 Superconducting disc-like source of the Kerr-Newman e.m. field

A similar treatment can be performed for the disc-like Kerr source if we con-
sider the $r$-coordinate as ellipsoidal coordinate of the Kerr geometry [14].
The exact treatment on the Kerr-Newman background is rather complicated
and will be given elsewhere [56]. Because of that, we consider here the case
corresponding to small values of the mass and charge parameters taking into
account the $e/a$ and $m/a$ ratios [4]. The Kerr coordinates $r, \theta, \phi$ are con-
ected with Cartesian coordinates $(x, y, z)$ by relations

$$
x + iy = (r + ia)e^{i\phi}\sin \theta, \quad (47)
$$

$$
z = r \cos \theta. \quad (48)
$$

In the considered limit we have in the Kerr coordinates

$$
\nabla^\mu \nabla_\mu = \frac{1}{r^2 + a^2 \cos^2 \theta} \left[ \partial_r (r^2 + a^2) \partial_r + \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta \right.
$$

$$
+ \frac{1}{\sin^2 \theta} \partial_\phi^2 - \partial_\phi^2 + 2a \partial_r \partial_\phi \right]. \quad (49)
$$

The surfaces $r = \text{const.}$ represent oblated spheroids and $\theta = \text{const.}$ are
confocal hyperboloids. By assuming that the scalar amplitudes are only
functions of $r$, it is reduced to

$$
\nabla^\mu \nabla_\mu = \frac{1}{r^2 + a^2 \cos^2 \theta} \partial_r (r^2 + a^2) \partial_r. \quad (50)
$$
In this case the equations (34),(36) and (38) take the form
\[ \frac{1}{r^2 + a^2 \cos^2 \theta} \left( \Phi'' + 2r \Phi' \right) - \Phi \left[ \lambda c(\Sigma^2 - \eta^2) + c^2 \Phi^2 + (cZ + m)^2 + U_A \right] = 0, \]
where \( U_A = (\chi_{A,\mu} + eA_\mu)^2; \)
\[ \frac{1}{r^2 + a^2 \cos^2 \theta} \left( \Sigma'' + 2r \Sigma' \right) - \Sigma \left[ \lambda c(\Sigma^2 - \eta^2) + c^2 \Phi^2 + (cZ + m)^2 \right] = 0, \]
where \( U_B = g^{\mu\nu}(\chi_{B,\mu} + gB_\mu)(\chi_{B,\nu} + gB_\nu); \)
\[ \frac{1}{r^2 + a^2 \cos^2 \theta} \left( Z'' + 2r Z' \right) - \lambda^2 Z \Sigma^2 - c(cZ + m)\Phi^2 = 0, \]
and by \( r \ll a \cos \theta \) the equations tend to ones with a planar symmetry depending only on coordinate \( z = r \cos \theta \). Such a region takes the place near the axis of the disc-like source where the solution tends to a planar domain wall. Another limiting case corresponds to the region near the Kerr singularity when \( r \ll a \) and \( \cos \theta = 0 \). Delambertian in this region takes the form
\[ \frac{1}{r^2} (a \partial_r^2 + 2 \partial_r). \]
Finally, in the region \( r \gg a \) Delambertian takes the usual form corresponding to the spherical symmetry of the problem. Let us now consider specific behavior of the Kerr-Newman e.m. field surrounding the disc-like superconducting core.

In our approximation contravariant form of metric in Kerr coordinates is
\[ g^{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \frac{r^2 + a^2}{\Sigma} & 0 & \frac{a}{\Sigma} \\
0 & 0 & 1 & 0 \\
0 & \frac{a}{\Sigma} & 0 & \frac{1}{\Sigma \sin^2 \theta}
\end{pmatrix}, \]
and
\[ \sqrt{-g} = \Sigma \sin \theta. \]
From physical reason and from the symmetry of the problem we set the constraints that current must vanish in radial \( \vec{n} = (0,1,0,0) \) and angular \( \vec{e}_\theta = (0,0,1,0) \) directions
\[ I^r = (I\vec{n}) = 0; \quad I^\theta = (I\vec{e}_\theta) = 0. \]
By using (56) one obtains
\[ \chi,\theta = -eA, \tag{59} \]
and
\[ I_r = -\frac{a}{r^2 + a^2} I_\phi. \tag{60} \]
Taking into account (57), (56) and (58) and stationarity \( \partial_0 I^\mu = 0 \), the equation of current conservation (45) is reduced to the form
\[ D_\mu I^\mu = \partial_\phi I^\phi = 0, \tag{61} \]
which yields
\[ \chi,\phi\phi = 0. \tag{62} \]
Vector potential of the Kerr-Newman e.m. field has the form
\[ A_{KN} = \frac{e r}{\Sigma} (dt + dr - a \sin^2 \theta d\phi). \tag{63} \]
For the Kerr-Newman field connected with superconducting source we retain this form adding an extra factor \( A(r) \) that describes fall-off of the field inside the source
\[ A = A(r) \frac{e r}{\Sigma} (dt + dr - a \sin^2 \theta d\phi). \tag{64} \]
Since \( A_\theta = 0 \) we have from (59)
\[ \chi,\theta = 0, \tag{65} \]
and consequently
\[ \chi,\theta = \chi,\phi\phi = 0. \tag{66} \]
Together with (62) and condition of stationarity \( \partial_0 I^0 \) it gives the integral
\[ \chi = n\phi + \omega t + \chi_1(r), \tag{67} \]
where \( n \) and \( \omega \) are constants, and \( n \) has to be integer to provide single-valuedness of phase \( \chi \). Since \( \chi,\theta = \chi_1'(r) \) one obtains from (59)
\[ \chi_1' = -\frac{an + eA(r)}{r^2 + a^2}, \tag{68} \]
that can be integrated leading to the expression
\[ \chi(r, t, \phi) = n\phi + \omega t - \int \frac{a_n + eA(r)}{r^2 + a^2}dr. \] (69)

Now one can calculate the extra contribution to scalar potential \( U_A \) caused by e.m. field
\[ U_A = \frac{1}{(r^2 + a^2)}\left(\frac{n}{\sin^2 \theta} - aA(r)\right)^2 - \left(\omega + eA(r)\right)^2. \] (70)

One sees that by \( n \neq 0 \) potential depends from \( \theta \), and moreover, it is singular at \( \theta = 0 \) that corresponds to the appearance of the Dirac monopole string.

### 4.3 Supersymmetric domain wall and phase transition

Now we would like to analyze a supersymmetric domain wall based on the Morris field model. It is described by the system of equations (34), (36), (38).

To simplify the problem we consider the planar approximation and neglect the contributions to potential \( U_A \) and \( U_B \) resulting from the gauge fields. The supersymmetric Lagrangian for the chiral matter fields (bosonic sector) has the form
\[ (-g)^{-1/2}L_{\text{mat}} = -K_{ij}(D_{\mu}\Phi^i)(\overline{D}^\mu\Phi^j) - V; \] (71)

where
\[ V = K_{ij}(D_i W)(\overline{D}_j W), \] (72)

\( K_{ij} \) is Kähler metric, \( D_i = \partial/\partial\Phi + \overline{\Phi}^i \), and \( W \) is a superpotential. Stress-energy tensor is given by
\[ T_{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\partial L_{\text{mat}}}{\partial g^{\mu\nu}} = K_{ij}(D_{\mu}\Phi^i)(\overline{D}_\nu\Phi^j) - \frac{1}{2}g_{\mu\nu}[K_{ij}(D_\lambda\Phi^i)(\overline{D}_j\Phi^j) + V]. \] (73)

In our case the set of fields \( \Phi^i = (Z, \Sigma_+, \Sigma_-, \Phi_+, \Phi_-) \), the superpotential \( W \) is
\[ W = \lambda Z(\Sigma_+\Sigma_- - \eta^2) + (cZ + m)\Phi_+\Phi_-; \] (74)

and \( K_{i,j} = \delta_{ij} \).

For the planar domain wall the fields depend on the coordinate \( z \) only, and, neglecting the contributions from gauge fields, one can consider the
fields as real. The non-zero components of the stress-energy tensor take the form

\[ T_{00} = -T_{xx} = -T_{yy} = \frac{1}{2} \left[ \delta_{ij}(\Phi^i_{,z})(\Phi^j_{,z}) + V \right]; \]  

\[ T_{zz} = \frac{1}{2} \left[ \delta_{ij}(\Phi^i_{,z})(\Phi^j_{,z}) - V \right]. \]

(75)

(76)

Since in our case the potential is positive \( V \geq 0 \), the problem permits a mathematical analogy with the problem of motion of a particle of unit mass propagating in the potential \( U_{\text{part}} = -V(\Phi^i) \) in a space-time having the space coordinates \( \Phi^i \) and time coordinate \( z \), see [38]. The field equations, following from (71) in this case

\[ 2\Phi^i_{,zz} - V,_{\Phi_i} = 0, \]  

(77)

lead to the equation

\[ 2\Phi^i_{,zz} \Phi^i_{,zz} - V,_{\Phi_i} \Phi^i_{,zz} = 0, \]  

(78)

that can be represented in the form

\[ \partial_z[(\Phi^i_{,z})^2 - V] = 0. \]  

(79)

Integral of this equation is

\[ \frac{1}{2}(\Phi^i_{,z})^2 + U_{\text{part}} = \text{const.}, \]  

(80)

where \( U_{\text{part}} = -\frac{1}{2}V \) an effective potential for the dynamics of the fictitious particle, and \( (80) \) describes conservation of its kinetic and potential energy. Since the wall interpolates between two disconnected vacuum regions, we have asymptotic behavior

\[ U_{\text{part}}|_{z=\pm\infty} = -\frac{1}{2}(\Phi^i_{,z})^2|_{z=\pm\infty} = 0, \]  

(81)

that yields \( \text{const.} = 0 \) and

\[ V = (\Phi^i_{,z})^2. \]  

(82)

As a result of this equation we obtain

\[ T_{zz} = 0; \]  

\[ T_{00} = -T_{xx} = -T_{yy} = (\Phi^i_{,z})^2 = V. \]  

(83)

(84)
Therefore, similarly to the simplest kink solution for domain wall (15), (16) the wall has a positive tangential stress which is equal to energy density and zero pressure in the transverse to wall direction. It should be noted that this property of the wall has been derived without obtaining an explicit solution of the field equations and without specifying the form of potential $V$ on the basis of only its positivity and boundary conditions (81).

Let us now assume that the wall has a spherical form with a small thickness in respect to the radius. One can calculate a gravitational mass produced by the spherical wall in outer region. Using the Tolman relation
\[ M = \int dx^3 \sqrt{-g} (-T_0^0 + T_1^1 + T_2^2 + T_3^3), \]
replacing coordinate $z$ on radial coordinate $r$, and integrating over sphere one obtains
\[ M_{\text{bubble}} = -4\pi \int V(r) r^2 dr = -4\pi \int (\Phi^i, r)^2 r^2 dr. \] (85)
The resulting effective mass is negative, which is caused by gravitational contribution of the tangential stress. The repulsive gravitational field was obtained in many singular and smooth models of domain walls [42, 28, 39, 41]. In particle models based on singular charged bubbles, this negative gravitational energy of domain wall cancels the gravitational part of the external electromagnetic field energy [28]. One should note, that similar gravitational contribution to the mass caused by interior of the bag will be $M_{\text{gr.int}} = \int Dr^2 dr = -\frac{2}{3} \Lambda r_0^3$, and it depends on the sign of curvature inside the bag.

An additional information can be extracted by the choice of a reasonable approximation to optimal trajectory of the phase transition from vacuum state I ($z = -\infty$) to vacuum state II ($z = +\infty$) in the space of fields $\Phi^i(z) = \{Z(z), \Phi(z), \Sigma(z)\}$ with a possible minimization of the potential [38]. The potential (23) for real fields takes the form
\[ V = \lambda^2[(\Sigma^2 + \Phi^2 - \eta^2)^2 + Z^2 \Sigma^2 + (Z + m/\lambda)^2 \Phi^2], \] (86)
where for simplicity we set $c = \lambda$ and the position of domain wall $z_0 = 0$. To cancel the first term of this expression one can set the following parametrization for $\Phi$ and $\Sigma$:
\[ \Sigma = \eta \sin \beta, \Phi = \eta \cos \beta. \] (87)
The subsequent minimization of the next two terms yields the parametrization for $Z$
\[ Z(z) = -(m/\lambda) \cos^2 \beta. \] (88)
The phase transition occurs for $\beta \subset [0, \pi/2]$. The resulting dependence of the potential on $z$ will be

$$V_{\text{opt}}(\beta) = \eta^2 m^2 \sin^2 2\beta/4. \quad (89)$$

Substituting (87), (88) and (89) in (82) one obtains for dependence $\beta(z)$ the equation

$$d\beta/dz = m \sin 2\beta/\sqrt{1 + (m^2/\lambda^2 \eta^2) \sin^2 2\beta}. \quad (90)$$

### 4.3.1 Bogomol’nyi transformation, BPS bound and total mass of the bag

Starting from the expressions (76) and (72) by $K_{i,j} = \delta_{i,j}$ one can use the Bogomol’nyi transformation [43] and represent the energy density as follows

$$\rho = T_{00} = \frac{1}{2} \delta_{ij}[(\Phi^i_{,z})(\Phi^j_{,z}) + (\partial W/\partial \Phi^i)(\partial W/\partial \Phi^j)] \quad (91)$$

$$= \frac{1}{2} \delta_{ij} [\Phi^i_{,z} + \partial W/\partial \Phi^i] [\Phi^j_{,z} + \partial W/\partial \Phi^j] - \partial W/\partial \Phi^i \Phi^i_{,z}, \quad (92)$$

where the last term is full derivative. Then, integrating over the wall depth $z$ one obtains for the surface density of the wall energy

$$\epsilon = \int_0^\infty \rho dz = \frac{1}{2} \int \Sigma_i (\Phi^i_{,z} + \partial W/\partial \Phi^i)^2 dz + W(0) - W(\infty). \quad (93)$$

The minimum of energy is achieved by solving the first-order Bogomol’nyi equations $\Phi^i_{,z} + \partial W/\partial \Phi^i = 0$, or in terms of $Z, \Phi, \Sigma$

$$Z' = -\lambda (\Sigma^2 - \eta^2) - c\Phi^2, \quad (94)$$

$$\Sigma' = -\lambda Z\Sigma, \quad (95)$$

$$\Phi' = -(cZ + m)\Phi. \quad (96)$$

Its value is given by $\epsilon = \epsilon_{\text{min}} = W(0) - W(\infty) = \lambda m \eta^2 / c$. Therefore, this domain wall is BPS-saturated solution. One can see that energy is proportional to parameter $m$ which determines amplitude of variation of the field $Z$. Therefore, the additional field $Z$, which appears only in the supersymmetric version of the model, plays an essential role for the formation of this domain wall. Substituting approximate solution (87), (88), (90) in Bogomol’nyi equations one sees that this approximation neglects the term $(Z')^2$. 21
By using the relations (84),(85) and (83) one obtains that the total energy of a uncharged bubble forming from the supersymmetric BPS saturated domain wall is

\[ E_{\text{bubble}} = E_{\text{wall}} = 4\pi \int_0^\infty \rho r^2 dr \approx 4\pi r_0^2 \epsilon_{\text{min}}, \]  

where \( r_0 \) is radius of the bubble. Corresponding total mass following from the Tolman relation will be negative

\[ M_{\text{bubble}} = -E_{\text{wall}} \approx -4\pi r_0^2 \epsilon_{\text{min}}. \]  

It is the known fact showing that the uncharged bubbles are unstable and form the time-dependent states \([39, 41]\).

For charged bubbles there are extra positive terms: contribution caused by the energy and mass of the external electromagnetic field

\[ E_{\text{e.m.}} = M_{\text{e.m.}} = \frac{e^2}{2r_0}, \]  

and contribution to mass caused by gravitational field of the external electromagnetic field (determined by Tolman relation for the external e.m. field)

\[ M_{\text{grav.e.m.}} = E_{\text{e.m.}} = \frac{e^2}{2r_0}. \]  

As a result the total energy for charged bubble is

\[ E_{\text{tot.bubble}} = E_{\text{wall}} + E_{\text{e.m.}} = 4\pi r_0^2 \epsilon_{\text{min}} + \frac{e^2}{2r_0}, \]  

and the total mass will be

\[ M_{\text{tot.bubble}} = M_{\text{0.bubble}} + M_{\text{e.m.}} + M_{\text{grav.e.m.}} = -E_{\text{wall}} + 2E_{\text{e.m.}} = -4\pi r_0^2 \epsilon_{\text{min}} + \frac{e^2}{r_0}. \]  

\[ \text{This analysis represents only a qualitative estimation in the spirit of the typical treatments of the bag models. There can also be other contributions to the total mass and energy, in particular, the energy of the interior of the bag and the energy of the tail of electromagnetic field penetrating in superconducting core, which can be comparable with } E_{\text{e.m.}}. \]
Minimum of the total energy is achieved by

\[ r_0 = \left( \frac{e^2}{16\pi\epsilon_{\text{min}}} \right)^{1/3}, \]  

(103)

which yields the following expressions for total mass and energy of the stationary state

\[ M^*_\text{tot} = E^*_\text{tot} = \frac{3e^2}{4r_0}. \]  

(104)

One sees that the resulting total mass of charged bubble is positive, however, due to negative contribution of \( M^\text{bubble} \) it can be lower than BPS energy bound of the domain wall forming this bubble. This remarkable property of the bubble models (called as ‘ultra-extreme’ states for the Type I domain walls in [39]) allows one to overcome BPS bound [44] and opens the way to get the ratio \( m^2 \ll e^2 \) which is necessary for particle-like models.

### 4.3.2 Supergravity domain wall

In supergravity the scalar potential has a more complicated form [40, 38, 41, 37]

\[ V_{sg} = e^{k^2K} (K^{ij} D_i W \bar{D}_j \bar{W} - 3k^2 W \bar{W}), \]  

(105)

where \( K \) is Kähler potential \( K^{ij} = \frac{\partial^2 K}{\partial \bar{\phi}_i \partial \phi_j} \), and \( k^2 = 8\pi G_N \), \( G_N \) is the Newton constant. For small values of \( kW \), this expression turns into potential of global susy (72) and the above treatment of the charged domain wall bubble is valid in supergravity in this approximation. However, in the first order in \( k^2 \) the internal vacuum state II) does not preserve supersymmetry since \( W = \lambda m \eta^2/c \) inside the bag, and \( D_i W \approx k^2 K_i W \neq 0 \) there. Besides, there appears an extra contribution to stress-energy tensor having the leading term

\[ T_{\mu\nu} = 3(k^2/8\pi)e^{k^2K}|W|^2 g_{\mu\nu}, \]  

(106)

and yielding the negative cosmological constant \( \Lambda = -3k^4 e^{k^2K}|W|^2 \) and to anti-de Sitter space-time for the bag interior. General expression for cosmological constant inside the bag has the form

\[ \Lambda = k^4 e^{k^2K} \sum_i \{k^2|K_i W|^2 - 3|W|^2 \}, \]  

(107)

and leads to AdS vacuum if \( k^2|K_i W|^2 - 3|W|^2 < 0 \).
In the same time the vacuum state I) in external region has $W = 0$, it preserves supersymmetry for strong values of the chiral fields and does not give contribution to stress-energy tensor.

4.4 Conclusion

The analysis of the stress-energy tensor of the Kerr source in the Kerr-Schild class of metrics shows that the simple and natural demands of the regularity of the energy density of the source lead to the conclusion concerning a bag-like structure of the core of the Kerr-Newman solution. The boundary of the bag represents a smooth (thick) domain wall with a tangential stress playing the role of a Poincarè stress of the core. For the thin domain wall this domain wall can be considered as being in a rigid relativistic rotation.

Among the known field models suggested for the description of extended objects such as strings, domain walls and bubbles, only the Witten $U(I) \times \tilde{U}(I)$ field model based on two gauge and two Higgs fields can provide the long range electromagnetic external field which is necessary for modelling the Kerr bag-like superconducting source.

The application of Morris supersymmetric version of this model to the bag-like source was considered, and it was shown that this model provides a phase transition between two disconnected supersymmetric vacua. The model possesses two superconducting sectors and two vacuum states: the vacuum state inside the bag is superconducting in respect to the external Kerr-Newman electromagnetic field, while the vacuum of the external region is a standard QCD-vacuum of the bag models which can be considered as a magnetic superconductor in respect to the inhabiting the bag quantum gauge fields. These superconducting sectors are almost independent and interact only via a special scalar field $Z$ providing a synchronization of phase transition. The Bogomol’nyi equations are obtained showing that the domain wall is BPS saturated solution. Due to negative gravitational contribution of the domain wall to the mass of the charged bag the total mass of the bag can be lower than BPS energy bound of this domain wall, which represents a remarkable property of the bubble-like core opening the way for particle-like models with a close to real parameter ratio.

The case N=1 supergravity is considered, and it is shown that strong chiral fields can lead to a negative cosmological constant and AdS-geometry inside the bag.
The current analysis can be considered only as a preliminary one. The complexity of the model does not allow one to hope to obtain an analytical solution of the full system of equations for this model, and numerical calculations must be used. However, one can expect essential further progress in obtaining analytical solutions for some sectors of this model. In particular, exact solutions for the Kerr-Newman electromagnetic field generated by the superconducting disc-like core on the Kerr-Newman background apparently can be obtained, as well as the existence of the exact analytical solutions for quantum excitations of the bosonic and fermionic fields inside the Kerr disc-like bag on the regularized Kerr-Schild background can be assumed.

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