Asymptotic and Non-asymptotic Results in the Approximation by Bernstein Polynomials

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Abstract. This paper deals with the approximation of functions by the classical Bernstein polynomials in terms of the Ditzian–Totik modulus of smoothness. Asymptotic and non-asymptotic results are respectively stated for continuous and twice continuously differentiable functions. By using a probabilistic approach, known results are either completed or strengthened.

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1. Introduction and Statements of the Main Results

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. As usual, $C[0, 1]$ denotes the space of all real continuous functions defined on $[0, 1]$, and $C^m[0, 1]$, $m \in \mathbb{N}_0$, denotes the subspace of all $m$-times continuously differentiable functions, with the obvious understanding that $C^0[0, 1] = C[0, 1]$. For $m \in \mathbb{N}$, we denote by $C^m[0, 1] \supset C^m[0, 1]$ the set of functions $f \in C^{m-1}[0, 1]$ such that $f^{(m-1)}$ is absolutely continuous, i. e.,

$$f^{(m-1)}(y) - f^{(m-1)}(x) = \int_x^y g(u)du, \quad x, y \in [0, 1],$$

for some bounded measurable function $g$, which can be denoted by $g = f^{(m)}$.

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The indicator function of a set $A$ is denoted by $1_A$, and $E$ stands for mathematical expectation.

Let $f \in C[0, 1]$. The sup-norm of $f$ is simply denoted by $\|f\|$, although, more generally, we use the notation $\|f\|_A = \sup\{|f(x)| : x \in A\}$, $A \subseteq [0, 1]$.

The second order central difference of $f$ is defined by
$$
\Delta^2_h f(x) = f(x + h) - 2f(x) + f(x - h), \quad h \geq 0,
$$
whenever $x \pm h \in [0, 1]$. The Ditzian–Totik modulus of smoothness of $f$ with weight function $\varphi(x) = \sqrt{x(1 - x)}$ is defined by
$$
\omega^\varphi_2(f; \delta) = \sup \left\{ \frac{1}{h^2} |\Delta^2_h \varphi(x)f(x)| : 0 \leq h \leq \delta, \ x \pm h\varphi(x) \in [0, 1] \right\}, \quad \delta \geq 0.
$$
The classical first order modulus of continuity is simply denoted by $\omega(f; \delta)$.

In this paper, we will make use of the following important inequality proved by Bustamante [2]:
$$
\omega^\varphi_2(f; \lambda \delta) \leq (2 + 3\lambda^2)\omega^\varphi_2(f; \delta), \quad \lambda, \delta \geq 0, \quad \lambda \delta \in [0, 1).
$$

Finally, the $n$th Bernstein polynomial of $f$ is defined as
$$
B_n f(x) = \sum_{k=0}^{n} f \left( \frac{k}{n} \right) p_{n,k}(x), \quad p_{n,k}(x) = \binom{n}{k} x^k (1 - x)^{n-k}, \quad k = 0, 1, \ldots, n.
$$
We have the probabilistic representation
$$
B_n f(x) = Ef \left( \frac{S_n(x)}{n} \right),
$$
where $S_n(x)$ is a random variable having the binomial law with parameters $n$ and $x$, that is to say,
$$
P(S_n(x) = k) = p_{n,k}(x), \quad k = 0, 1, \ldots, n.
$$

Throughout this paper, whenever we write $f$, $n$, $x$, and $y$, we are assuming that $f \in C[0, 1]$, $n \in \mathbb{N}$, and $x, y \in [0, 1]$.

Following the works by Ditzian and Ivanov [4] and Totik [9], the rates of uniform convergence for the Bernstein polynomials are characterized by
$$
K_1 \omega^\varphi_2 \left( f; \frac{1}{\sqrt{n}} \right) \leq \|B_n f - f\| \leq K_2 \omega^\varphi_2 \left( f; \frac{1}{\sqrt{n}} \right),
$$
for some absolute constants $K_1$ and $K_2$. Whereas no specific values for $K_1$ have been provided yet, different authors completed statement (4) by showing specific values for the constant $K_2$. In this regard, Adell and Sangüesa [1] gave $K_2 = 4$, Gavrea et al. [5] and Bustamante [2] provided $K_2 = 3$, and finally, Păltănea [7] proved the validity of $K_2 = 2.5$, this being the best result up to date and up to our knowledge.

This notwithstanding, if additional smoothness conditions on $f$ are added, then the second inequality in (4) may be valid for values of $K_2$
smaller than 2.5. In this respect, Bustamante and Quesada [3] and Păltânea [8] obtained the following asymptotic result

$$\lim_{n \to \infty} \frac{\| B_n f - f \|}{\omega^\varphi_2 (f; 1/\sqrt{n})} = \frac{1}{2}, \quad f \in C^2[0, 1],$$

provided that $f$ is not an affine function.

The contribution of this paper is twofold. In first place, we strengthen statement (5) by giving a non-asymptotic version of it. In fact, we prove the following result.

**Theorem 1.** If $f \in C^2[0, 1]$, then

$$\left| \frac{1}{2} \omega^\varphi_2 \left( f; \frac{1}{\sqrt{n}} \right) - \frac{1}{2} \omega^\varphi_2 \left( f'; \frac{1}{3\sqrt{n}} \right) + \frac{1}{4} \omega^\varphi_2 \left( f''; \frac{1}{\sqrt{n}} \right) \right| \leq \frac{1}{4n} \left( \omega \left( f''; \frac{1}{3\sqrt{n}} \right) + \frac{1}{4} \omega^\varphi_2 \left( f''; \frac{1}{\sqrt{n}} \right) \right).$$

(6)

As a consequence, statement (5) holds true.

In second place, we complete statement (4) in the following asymptotic form.

**Theorem 2.** Let $(\tau_n)_{n \geq 1}$ be a sequence of positive real numbers such that

$$\tau_n \to \infty, \quad \frac{\tau_n}{n} \to 0, \quad n \to \infty.$$

If $f \in C[0, 1]$ is not an affine function, then

$$\limsup_{n \to \infty} \frac{1}{\omega^\varphi_2 \left( f; \frac{1}{\sqrt{n}} \right)} \| B_n f - f \|_{[\tau_n/n, 1-\tau_n/n]} \leq \frac{3}{2}.$$  

(7)

Moreover, we have in (4),

$$K_2 \geq 1.$$  

(8)

This result is based upon Theorem 3 in Sect. 3, which gives estimates of the form

$$| B_n f(x) - f(x) | \leq K_2(n, x) \omega^\varphi_2 \left( f; \frac{1}{\sqrt{n}} \right),$$

for some explicit constants $K_2(n, x)$ depending on $n$ and $x$.

The paper is organized as follows. The proof of Theorem 1 is given in Sect. 2. We show Theorem 2 in Sect. 3 with the aid of two kinds of auxiliary results. On the one hand, we define certain smooth approximants $Q_n^h f$ of the function $f \in C[0, 1]$, by antisymmetrizing in an appropriate way the classical Steklov means of $f$. On the other hand, we estimate the tail probabilities and the truncated variance of the random variable $S_n(x)$ appearing in the probabilistic representation of $B_n f$ given in (2).
2. Proof of Theorem 1

2.1. Preliminaries
The Taylor’s formula of order $m \in \mathbb{N}$ for $f \in C^m[0,1]$, with remainder in integral form can be written as

\[
  f(y) - \sum_{j=0}^{m-1} \frac{f^{(j)}(x)}{j!}(y-x)^j = \frac{(y-x)^m}{(m-1)!} \int_0^1 (1-\theta)^{m-1} f^{(m)}(x+(y-x)\theta) d\theta
\]

\[
  = \frac{(y-x)^m}{m!} \mathbb{E}f^{(m)}(x+(y-x)\beta_m), \tag{9}
\]

where $\beta_m$ is a random variable with the beta density $\rho_m(\theta) = \frac{m(1-\theta)^{m-1}}{m!}$, $0 \leq \theta \leq 1$.

**Lemma 1.** If $f \in C^2[0,1]$ and $\delta \geq 0$, then

\[
  \left| \omega_2^\varphi(f;\delta) - \delta^2 \| \varphi^2 f'' \| \right| \leq \frac{\delta^2}{8} \omega_2^\varphi(f'';\delta).
\]

**Proof.** Let $h \geq 0$ with $x \pm h \in [0,1]$. Using (9) with $m = 2$, we get

\[
  f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2
\]

\[
  + \frac{h^2}{2} \mathbb{E}(f''(x-h\beta_2) - f''(x)),
\]

as well as

\[
  f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2
\]

\[
  + \frac{h^2}{2} \mathbb{E}(f''(x+h\beta_2) - f''(x)).
\]

Adding these two identities, we obtain

\[
  \Delta_h^2 f(x) = f''(x)h^2
\]

\[
  + \frac{h^2}{2} \mathbb{E}(f''(x+h\beta_2) - 2f''(x) + f''(x-h\beta_2)). \tag{10}
\]

Replacing in (10) $h$ by $h\varphi(x)$ and applying the reverse triangular inequality, we have

\[
  \left| \omega_2^\varphi(f;\delta) - \delta^2 \| \varphi^2 f'' \| \right| \leq \frac{\delta^2}{2} \| \varphi^2 \| \omega_2^\varphi(f'';\delta)
\]

\[
  = \frac{\delta^2}{8} \omega_2^\varphi(f'';\delta),
\]

thus completing the proof. \qed
Gonska et al. [6] showed that
\[
\left\| B_n f - f - \frac{1}{2n} \varphi^2 f'' \right\| \leq \frac{1}{4n} \omega \left( f''; \frac{1}{3\sqrt{n}} \right).
\] (11)

### 2.2. Proof of Theorem 1

Statement (6) is an immediate consequence of (11), Lemma 1 with \( \delta = 1/\sqrt{n} \), and the reverse and direct triangular inequalities. On the other hand, we have from Lemma 1
\[
\omega_2^\varphi \left( f; \frac{1}{\sqrt{n}} \right) = \frac{1}{n} \left\| \varphi^2 f'' \right\| + o \left( \frac{1}{n} \right),
\]
since \( f \in C^2[0,1] \). Thus, statement (5) readily follows from (6), and completes the proof.

### 3. Proof of Theorem 2

#### 3.1. Auxiliary Results

Let \( 0 < h \leq 1/3 \). We consider the Steklov means of \( f \) defined as
\[
P_h f(y) = \int_{-1}^{1} \int_{-1}^{1} f \left( y + \frac{h}{2}(v_1 + v_2) \right) dv_1 dv_2
\]
\[
= \int_{-1}^{1} f(y + hv)\rho(v)dv, \quad h \leq y \leq 1 - h,
\]
where
\[
\rho(v) = (1 + v)1_{[-1,0]} + (1 - v)1_{[0,1]}, \quad -1 \leq v \leq 1.
\]

In probabilistic terms, the Steklov means of \( f \) can be written as follows. Let \( V_1 \) and \( V_2 \) be independent identically distributed random variables having the uniform distribution on \([-1,1]\) and set \( V = (V_1 + V_2)/2 \). Since \( \rho(v) \) is the probability density of \( V \), we can write
\[
P_h f(y) = \mathbb{E} f(y + hV), \quad h \leq y \leq 1 - h.
\] (12)

**Lemma 2.** Let \( 0 < h \leq 1/3 \) and let \( P_h f(y) \) be as in (12). Then,

(a) \[
|P_h f(y) - f(y)| \leq \frac{1}{2} \omega_2^\varphi \left( f; \frac{h}{\varphi(y)} \right).
\]

(b) \[
|(P_h f)''(y)| \leq \frac{1}{h^2} \omega_2^\varphi \left( f; \frac{h}{\varphi(y)} \right).
\]
Proof. Since $V$ takes values in $[-1, 1]$ and is symmetric (i.e., $V$ and $-V$ have the same law), we see that

$$|P_h f(y) - f(y)| = \frac{1}{2} |\mathbb{E}(f(y + hV) + f(y - hV) - 2f(y))| \leq \frac{1}{2} \omega^\varphi_n \left( f; \frac{h}{\varphi(y)} \right),$$

thus showing (a). On the other hand, it can be checked that

$$P_h f(y) = \frac{1}{h^2} \left( f(2)(y + h) + f(2)(y - h) - 2f(2)(y) \right),$$

where $f(2)$ is a second antiderivative of $f$. This readily implies part (b) and completes the proof. □

We will make use of the approximant $P_h f$, whose domain is the interval $[h, 1 - h]$, to define a further one whose domain is the whole interval $[0, 1]$, keeping at the same time analogous properties to those given in Lemma 2. To this end, we assume that

$$n \geq 3, \quad 0 < a < \frac{\varphi(a/2)}{\sqrt{n}} + a \leq 1. \quad (13)$$

and take

$$h = \frac{\varphi(ax)}{\sqrt{n}}, \quad \frac{1}{a(n + 1)} \leq x \leq \frac{1}{2}. \quad (14)$$

It turns out that

$$h \leq \min(ax, 1/3). \quad (15)$$

Now, we define the approximant $Q^a_h f(y)$ by antisymmetrizing $P_h f(y)$ around the axes $y = ax$ and $y = 1 - ax$ as follows

$$Q^a_h f(y) = \begin{cases} 2P_h f(ax) - P_h f(2ax - y), & y \in [0, ax); \\ P_h f(y), & y \in [ax, 1 - ax]; \\ 2P_h f(1 - ax) - P_h f(2(1 - ax) - y), & y \in (1 - ax, 1]. \end{cases} \quad (16)$$

The fact that $Q^a_h f$ is well defined readily follows from (13) and (14). Also, note that $Q^a_h f$ is twice differentiable except at the points $ax$ and $1 - ax$. In these two points, $Q^a_h f$ only has sided second derivatives. This implies that $Q^a_h f \in \mathcal{C}^2[0, 1]$.

Lemma 3. Let $R_a = [ax, 1 - ax]$. Under assumptions (13) and (14), we have

(a) If $y \in R_a$, then

$$|Q^a_h f(y) - f(y)| \leq \frac{1}{2} \omega^\varphi_n \left( f; \frac{1}{\sqrt{n}} \right), \quad |(Q^a_h f)'(y)| \leq \frac{1}{h^2} \omega^\varphi_n \left( f; \frac{1}{\sqrt{n}} \right).$$

(b) If $y \notin R_a$, then

$$|Q^a_h f(y) - f(y)| \leq \left( \frac{7}{2} + \frac{3\sqrt{anx}}{(1 - a)^{3/2}} \right) \omega^\varphi_n \left( f; \frac{h}{\varphi(h)} \right).$$
and
\[ |(Q_h^a f)''(y)| \leq \frac{1}{h^2} \omega_2^\varphi \left( f; \frac{h}{\varphi(h)} \right). \]

**Proof.** (a) If \( y \in R_a \), then
\[ \frac{h}{\varphi(y)} = \frac{\varphi(ax)}{\varphi(y)} \leq \frac{1}{\sqrt{n}}. \] (17)
Thus, the first inequality in part (a) follows from Lemma 2(a) and definition (16), whereas the second one follows from Lemma 2(b).

(b) Suppose that \( y \in [0, ax) \). By (16), we can write
\[
Q_h^a f(y) - f(y) = 2(P_h f(ax) - f(ax)) - (P_h f(2ax - y) - f(2ax - y)) - (f(2ax - y) + f(y) - 2f(ax)). \] (18)
Since \( h \leq ax \leq 2ax - y \leq 1 - h \), we see that
\[ \varphi(ax) \geq \varphi(h), \quad \varphi(2ax - y) \geq \varphi(h). \] (19)
We therefore have from Lemma 2(a)
\[
|Q_h^a f(y) - f(y)| \leq \frac{3}{2} \omega_2^\varphi \left( f; \frac{h}{\varphi(h)} \right) + \omega_2^\varphi \left( f; \frac{ax}{\varphi(ax)} \right). \] (20)
Applying (1) with \( \lambda = ax \varphi(h)/(h \varphi(ax)) \) and \( \delta = h/\varphi(h) \), we obtain
\[
\omega_2^\varphi \left( f; \frac{ax}{\varphi(ax)} \right) \leq \left( 2 + \frac{3(ax)^2 \varphi^2(h)}{\varphi^2(ax) h^2} \right) \omega_2^\varphi \left( f; \frac{h}{\varphi(h)} \right) \leq \left( 2 + \frac{3\sqrt{a}nx}{(1-a)^{3/2}} \right) \omega_2^\varphi \left( f; \frac{h}{\varphi(h)} \right), \] (21)
as follows from (14) and some simple computations. Hence, the first inequality follows from (20) and (21).

On the other hand, we have from (16), (19), and Lemma 2(b)
\[ |(Q_h^a f)''(y)| = |(P_h f)''(2ax - y)| \leq \frac{1}{h^2} \omega_2^\varphi \left( f; \frac{h}{\varphi(2ax - y)} \right) \]
\[ \leq \frac{1}{h^2} \omega_2^\varphi \left( f; \frac{h}{\varphi(h)} \right). \]
If \( y \in (1 - ax, 1] \), the proof es similar. \( \square \)

The following estimates concerning the random variable \( S_n(x)/n \) will be needed.

**Lemma 4.** In the setting of Lemma 3, denote by \( r = 1 - a \). Then,
(a) 
\[ P\left( \frac{S_n(x)}{n} \notin R_a \right) \leq e^{-nxr^2/2} + 3e^{-nxr^2/(2e)} =: \epsilon_n(x). \]

(b) 
\[ \frac{1}{h^2} \mathbb{E} \left( \frac{S_n(x)}{n} - x \right)^2 1\{S_n(x)/n \notin R_a\} \leq \frac{nx}{a(1-ax)} \left( e^{-nxr^2/2} + 6e^{-(n-2)xr^2/(2e)} \right) =: \delta_n(x). \]

Proof. (a) As follows from (3), we have
\[ \mathbb{E} e^{\theta S_n(x)} = \left( 1 + x(e^\theta - 1) \right)^n, \quad \theta \in \mathbb{R}. \] (22)
Let \( \theta \geq 0 \). By (22) and Chebyshev’s inequality, we have
\[
P(S_n(x) < anx) = P\left( e^{-\theta S_n(x)} > e^{-\theta anx} \right) \leq \mathbb{E}e^{-\theta S_n(x)} + \theta anx
\]
\[= e^{-n(-\log(1-x(1-e^{-\theta}))-\theta ax)} \leq e^{-nx((1-e^{-\theta})-a\theta)} \leq e^{-nx(r\theta - \theta^2/2)}, \]
where we have used the inequalities
\[-\log(1-u) \geq u, \quad u \geq 0, \quad 1 - e^{-\theta} \geq \theta - \theta^2/2, \quad \theta \geq 0.\]
Choosing \( \theta = r \) in (23) (the value minimizing the exponent), we get
\[ P(S_n(x) < anx) \leq e^{-nxr^2/2}. \] (24)

On the other hand, we claim that
\[ P(S_n(x) > n(1-ax)) \leq P(S_n(x) > n(1-ax) - 1) \leq 3e^{-nxr^2/(2e)}. \] (25)
Indeed, let \( 0 \leq \theta \leq 1 \). Using the inequalities
\[ \log(1+u) \leq u, \quad u \geq 0, \quad e^\theta - 1 \leq \theta + \frac{e\theta^2}{2}, \quad 0 \leq \theta \leq 1, \]
we have, as in the proof of (24),
\[
P(S_n(x) > n(1-ax) - 1) = P\left( e^{\theta S_n(x)} > e^{\theta n(1-ax) - \theta} \right)
\]
\[\leq \mathbb{E}e^{\theta S_n(x) - n\theta(1-ax) + \theta} \leq 3\mathbb{E}e^{\theta S_n(x) - n\theta(1-ax)}
\]
\[= 3e^{n\log(1+x(e^{\theta} - 1)) - \theta(1-ax)} \leq 3e^{n(x\theta + e\theta^2/2 - \theta(1-ax))}
\]
\[= 3e^{n\theta(2x-1)}e^{nx(e^{\theta^2}/2 - r\theta)} \leq 3e^{nx(e^{\theta^2}/2 - r\theta)}, \]
(26)
since \( x \leq 1/2 \). Thus, claim (25) follows by choosing \( \theta = r/e \) in (26). Hence, part (a) follows from (24) and (25).
(b) From (24), we see that
\[
\mathbb{E} \left( \frac{S_n(x)}{n} - x \right)^2 1\{S_n(x) < anx\} \\
\leq x^2 P(S_n(x) < anx) \leq x^2 e^{-nxr^2/2}.
\]

(27)

On the other hand, since \(1 - ax \geq 1/2\), we have
\[
\frac{k}{k - 1} \leq \frac{n/2 - 1}{n - 2} = \frac{n}{n - 2}, \quad k > n(1 - ax).
\]

We therefore have
\[
\mathbb{E} \left( \frac{S_n(x)}{n} - x \right)^2 1\{S_n(x) > n(1 - ax)\} \leq \frac{1}{n^2} \mathbb{E} S_n(x)^2 1\{S_n(x) > n(1 - ax)\}
\]
\[
= \frac{1}{n^2} \sum_{k > n(1 - ax)} \binom{n}{k} k^2 x^k (1 - x)^{n-k}
\]
\[
= \frac{n - 1}{n} x^2 \sum_{k > n(1 - ax)} \binom{n - 2}{k - 2} \frac{k}{k - 1} x^{k-2} (1 - x)^{n-k}
\]
\[
\leq \frac{n - 1}{n} x^2 P(S_{n-2}(x) > n(1 - ax) - 2)
\]
\[
\leq 2x^2 P(S_{n-2}(x) > n(1 - ax) - 2),
\]

(28)
since \(n \geq 3\). Observe that
\[
n(1 - ax) - 2 = (n - 2)(1 - ax) - 2ax \geq (n - 2)(1 - ax) - 1,
\]
as follows from assumptions (13) and (14). By (25), the right-hand side in (28) can be bounded above by
\[
2x^2 P(S_{n-2}(x) > n(1 - ax) - 1) \leq 6x^2 e^{-(n-2)xr^2/(2e)}.
\]

This, together with (27) and (28), shows part (b) and completes the proof.

\[\square\]

We are in a position to give the following local estimate.

**Theorem 3.** In the setting of Lemma 4, we have
\[
|B_n f(x) - f(x)| \leq \left( 1 + \frac{1}{2} \varphi^2(ax) \right) \omega_2^\varphi \left( f; \frac{1}{\sqrt{n}} \right) + \nu_n(x) \omega_2^\varphi \left( f; \frac{h}{\varphi(h)} \right),
\]

where
\[
\nu_n(x) = \left( \frac{7}{2} + \frac{3\sqrt{a_n x}}{(1 - a)^{3/2}} \right) \epsilon_n(x) + \frac{1}{2} \delta_n(x).
\]

(29)
Proof. We use the notation $Qf(y) = Q_h f(y)$ and write

$$B_n f(x) - f(x) = (Qf(x) - f(x)) + (B_n f(x) - B_n (Qf)(x))$$

$$+ (B_n (Qf)(x) - Qf(x)) =: I + II + III. \quad (30)$$

By Lemma 3(a), we have

$$|I| \leq \frac{1}{2} \omega^f_2 \left( f; \frac{1}{\sqrt{n}} \right). \quad (31)$$

By (2) and Lemma 3(a) and (b), we see that

$$|II| = \left| \mathbb{E} Qf \left( \frac{S_n(x)}{n} \right) - \mathbb{E} f \left( \frac{S_n(x)}{n} \right) \right|$$

$$\leq \frac{1}{2} \omega^f_2 \left( f; \frac{1}{\sqrt{n}} \right)$$

$$+ \left( \frac{7}{2} + \frac{3 \sqrt{anx}}{(1-a)^{3/2}} \right) P \left( \frac{S_n(x)}{n} \notin R_a \right) \omega^f_2 \left( f; \frac{h}{\varphi(h)} \right). \quad (32)$$

Finally, denote by $\xi_n(x) = x + (S_n(x)/n - x)\beta_2$. Applying (9) with $m = 2$ and Lemma 3, we get

$$|III| = \frac{1}{2} \mathbb{E} Qf''(\xi_n(x)) \left( \frac{S_n(x)}{n} - x \right)^2$$

$$\leq \frac{1}{2 h^2} \omega^f_2 \left( f; \frac{1}{\sqrt{n}} \right) \mathbb{E} \left( \frac{S_n(x)}{n} - x \right)^2 1\{S_n(x)/n \in R_a\}$$

$$+ \frac{1}{2 h^2} \omega^f_2 \left( f; \frac{h}{\varphi(h)} \right) \mathbb{E} \left( \frac{S_n(x)}{n} - x \right)^2 1\{S_n(x)/n \notin R_a\}$$

$$\leq \frac{1}{2} \frac{\varphi^2(x)}{\varphi^2(ax)} \omega^f_2 \left( f; \frac{1}{\sqrt{n}} \right)$$

$$+ \frac{1}{2 h^2} \omega^f_2 \left( f; \frac{h}{\varphi(h)} \right) \mathbb{E} \left( \frac{S_n(x)}{n} - x \right)^2 1\{S_n(x)/n \notin R_a\}, \quad (33)$$

where we have used (14), the inequality $1/\sqrt{n} \leq h/\varphi(h)$, and the well known fact that

$$\mathbb{E} \left( \frac{S_n(x)}{n} - x \right)^2 = \frac{\varphi^2(x)}{n}. \quad (34)$$

The result follows from (30)–(33) and Lemma 4. \hfill \square

3.2. Proof of Theorem 2

Since the random variables $S_n(x)$ and $n - S_n(1 - x)$ have the same law, we have

$$B_n f(1 - x) - f(1 - x) = \mathbb{E} f \left( 1 - \frac{S_n(x)}{n} \right) - f(1 - x).$$
On the other hand, if \( g(y) = f(1 - y) \), we obviously have
\[
\omega^\varphi_2(g; \delta) = \omega^\varphi_2(f; \delta), \quad \delta \geq 0.
\]
Thus, without loss of generality, we can assume that \( 0 < x \leq 1/2 \).

In the setting of Lemma 4, we claim that
\[
\omega^\varphi_2\left(f; \frac{h}{\varphi(h)}\right)
\leq \left(2 + 3\sqrt{\frac{anx}{1-a}}\right) \omega^\varphi_2\left(f; \frac{1}{\sqrt{n}}\right).
\] (34)

Actually, choose \( \lambda = h\sqrt{n}/\varphi(h) \) and \( \delta = 1/\sqrt{n} \). By definition (14) and the fact that \( h \leq ax \), we see that
\[
\lambda^2 = \frac{h^2n}{\varphi^2(h)} = \frac{\varphi^2(ax)}{\varphi^2(h)} = \frac{ax(1-ax)}{h(1-h)} \leq \frac{ax\sqrt{n}}{h} = \frac{ax\sqrt{n}}{\varphi(ax)} = \frac{\sqrt{anx}}{1-ax} \leq \sqrt{\frac{anx}{1-a}}.
\]

This, in conjunction with (1), shows claim (34).

From Theorem 3 and (34), we have
\[
|B_n f(x) - f(x)| \leq 1 + \frac{1}{2a} \varphi^2(ax) + \left(2 + 3\sqrt{\frac{anx}{1-a}}\right) \nu_n(x),
\] (35)

where \( \nu_n(x) \) is defined in (29). Recalling the definitions of \( \epsilon_n(x) \) and \( \delta_n(x) \) given in Lemma 4, we see that
\[
\left(2 + 3\sqrt{\frac{anx}{1-a}}\right) \nu_n(x) \leq P_3(\sqrt{nx})e^{-cx},
\] (36)

where \( P_3(\cdot) \) is a polynomial of degree three and \( c \) is a positive constant not depending on \( n \) and \( x \). Observe that, whenever \( t_n \to \infty \), as \( n \to \infty \), we have
\[
\limsup_{n \to \infty} P_3(\sqrt{u}) e^{-cu} = 0.
\] (37)

Let \( \tau_n \) be as in Theorem 2. From (35) and (36), we get
\[
\frac{1}{\omega^\varphi_2(f; 1/\sqrt{n})} \|B_n f - f\|_{[\tau_n/n, 1/2]} \leq 1 + \frac{1}{2a} + \sup_{u \geq \tau_n} P_3(\sqrt{u}) e^{-cu}.
\]

By (37) and the fact that \( \tau_n \to \infty \), as \( n \to \infty \), this implies that
\[
\limsup_{n \to \infty} \frac{1}{\omega^\varphi_2(f; 1/\sqrt{n})} \|B_n f - f\|_{[\tau_n/n, 1/2]} \leq 1 + \frac{1}{2a},
\]

which shows (7), since \( 0 < a < 1 \) is arbitrary.

On the other hand, let \( x \in (0, 1/n) \). Consider the function
\[
f_x(y) = \left(1 - \frac{y}{x}\right) 1_{[0,x]}(y).
\]

Observe that \( \omega^\varphi_2(f_x; 1/\sqrt{n}) = 1 \), as well as
\[
B_n f_x(x) - f_x(x) = \mathbb{E} f_x \left( \frac{S_n(x)}{n} \right) = P(S_n(x) = 0) = (1 - x)^n,
\]
thus implying that \( K_2 \geq (1 - x)^n \). Therefore, letting \( x \to 0 \), we see that \( K_2 \geq 1 \). This shows (8) and completes the proof.

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**Declarations**

**Competing interests** The authors declare that they have no competing interests.

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