BETWEEN ULTRAPOWERS AND ASYMPTOTIC SEQUENCE ALGEBRAS

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Abstract. We relate the asymptotic sequence algebra of a separable C*-algebra to its ultrapower and use this to derive a general existence result for functors whose domain is the category of C*-algebras.

The method of classifying separable operator algebras by studying their position inside a massive quotient algebra dates back to the seminal [15] and [6]. In the case of tracial von Neumann algebras, ‘massive quotient algebras’ are invariably ultrapowers. Ultrapowers are a standard tool in the classification programme of C*-algebras in the situations when the simplicity of massive algebras is desirable (as in the Kirchberg–Phillips classification of Kirchberg algebras, [17]) and in the stably finite case, when one takes direct advantage of the tracial ultrapower whose fibres are ultrapowers of II_1 factors (see e.g., [16], [14], [4], and [20]). In some other arguments, the asymptotic sequence algebra \( \ell_\infty(B)/c_0(B) \) is more suitable because of the "reindexing technique" (see e.g., [17, Proposition 1.37] or [11, Theorem 4.3]). With the increase of sophistication in the classification programme it has become desirable to mix the ultrapowers and tracial von Neumann algebras techniques with the asymptotic sequence algebras (see the upcoming [5]).

Functorial classification results for C*-algebras have two components, existence and uniqueness. Given a functor \( F \) whose domain is the category of C*-algebras, the existence asserts that for separable C*-algebras \( A \) and \( B \) and a morphism \( \alpha: F(A) \rightarrow F(B) \), there exists a *-homomorphism \( \Phi: A \rightarrow B \) such that \( F(\Phi) = \alpha \). In this situation one says that \( \alpha \) is realized by \( \Phi \). An intermediate step in proving the existence is often to realize \( \alpha \) by a *-homomorphism from \( A \) into \( B^U \) or into \( B^\infty \) (see [11] for an explanation of the terminology). The following makes transfer between ultrapowers and asymptotic sequence algebras straightforward.

**Theorem A.** Suppose \( F \) is a functor whose domain is the category of C*-algebras. For unital and separable C*-algebras \( A \) and \( B \) and a morphism \( \alpha: F(A) \rightarrow F(B) \) the following are equivalent.

1. The morphism \( \alpha \) is realized by a *-homomorphism \( \Phi: A \rightarrow B^U \) for some (any) nonprincipal ultrafilter \( \mathcal{U} \) on \( \mathbb{N} \).
2. The morphism \( \alpha \) is realized by a *-homomorphism \( \Phi: A \rightarrow B^\infty \).

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The implication from (1) to (2) is more interesting because the former is often easier to achieve, while a standard reindexing argument shows that if $F$ satisfies additional conditions (preservation of injective inductive limits and invariance under the approximate unitary equivalence) then the latter can be combined with the reindexing technique to prove that $\Phi$ can be chosen so that its range is included in $B$.

Theorem $A$ is a consequence of our main result, Theorem 4.2. Although additional set-theoretic axioms are not used in its proof, the latter can be stated more elegantly with the aid of the Continuum Hypothesis.

**Theorem B.** Suppose that the Continuum Hypothesis holds, $U$ is a non-principal ultrafilter on $\mathbb{N}$, and $B$ is a separable and unital $C^*$-algebra. Then there are $^*$-homomorphisms $\Phi : B^\infty \to B^U$ and $\Theta : B^U \to B^\infty$ with the following properties.

1. The restriction to $B$ of each one of $\Theta$ and $\Phi$ is equal to the identity,
2. $\Phi \circ \Theta = \text{id}_{B^U}$, and
3. $\Theta \circ \Phi$ is a conditional expectation of $B^\infty$ onto the range of $\Phi$.

The Continuum Hypothesis is necessary for the conclusion of Theorem $B$ as stated, since if it fails then not all ultrapowers of $B$ associated with nonprincipal ultrafilters on $\mathbb{N}$ are isomorphic ([12], see also [9]). I do not know whether it can be proved in ZFC that the conclusion of Theorem $A$ always holds for some $U$.

The instances of Theorem $A$ when $F$ is any of the standard $K$-theoretic functors (in addition to $K_0$ and $K_1$, this includes the algebraic $K$-theory, $K$-theory with coefficients, KK, and KL; see e.g., [19], [3]) follow from Theorem $B$ by the standard metamathematical absoluteness arguments (similar to e.g., [1, Appendix 2]). We will take a more direct route and prove the ‘ZFC-variant’ of Theorem $B$ Theorem 4.2 and show that it implies Theorem $A$ for any functor $F$.

The proofs of Theorem 4.2 and Theorem $B$ use logic of metric structures ([2], [8]). In addition to the standard tools (Loś’s Theorem and countable saturation of ultrapowers), they use the metric analog of the Feferman–Vaught theorem ([13, Theorem 3.3]) and countable saturation of metric reduced products associated with the Fréchet ideal ([10, Theorem 1.1]). For the reader’s convenience, short proofs of these results are given in Appendix $A$ and Appendix $B$ respectively.

One more thing. The question which $C^*$-algebras have the property that tensoring with them preserves elementary equivalence has been raised in [8, Question 3.10.5]. In [5] we apply Theorem 4.2 to give a positive answer for $C(2^N)$ ($2^N$ denotes the Cantor space) and a negative answer for $Z$ (the Jiang–Su algebra) and all UHF algebras.

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1. Terminology

If $F$ is a functor whose domain is the category of $C^*$-algebras, a morphism $\alpha : F(A) \to F(B)$ is implemented by a $C^*$-homomorphism $\Phi : A \to B$ if $F(\Phi) = \alpha$. If $B$ is a $C^*$-subalgebra of $C$ and a $C^*$-homomorphism $\Phi : A \to C$ is such that (writing $\iota_B$ for the identity map on $B$) $F(\iota_B \circ \Phi) = \alpha$, then $\alpha$ is implemented by $\Phi$.

Our terminology and notation follow [8]. The ultrapower of a metric structure $M$ associated to an ultrafilter $\mathcal{U}$ is denoted $M^\mathcal{U}$. In particular by $B^\mathcal{U}$ we denote the $C^*$-algebra (norm) ultrapower. An element $b$ of $B^\mathcal{U}$ is determined by a representing sequence $(b_i)$, and the analogous remark applies to the elements of $B^\infty$. We identify $B$ with its diagonal image in $B^\mathcal{U}$ and with its diagonal image in $B^\infty$. By $\bar{a}$ we denote a tuple $(a_0, \ldots, a_{n-1})$ of an unspecified length (but ‘of the appropriate sort’). The arity of $\bar{a}$ will be routinely suppressed, and we will write $\bar{a} \in B$ for $\bar{a} \in B^n$ where $n$ is the arity of $\bar{a}$. For the sake of brevity, variables are sometimes omitted and a formula $\varphi(\bar{x})$ is written as $\varphi$. When dealing with tuples of representing sequences, in order to avoid confusion the entries of an $\bar{a}$ will be denoted $a(0), \ldots, a(n-1)$. If $\bar{a}$ is an $n$-tuple of elements of $B^\infty$ or $B^\mathcal{U}$, then $a_j(i)$, for $j \in \mathbb{N}$, is a representing sequence of the $i$-th entry of $\bar{a}$.

Given a substructure $\mathcal{C}$ of a metric structure $\mathcal{D}$, $\mathcal{C} \prec \mathcal{D}$ stands for ‘$\mathcal{C}$ is an elementary submodel of $\mathcal{D}$’. The standard language of $C^*$-algebras, as described in [8], §2.1, will be denoted $\mathcal{L}_{C^*}$. A pair $(A, B)$ where $A$ is a $C^*$-algebra and $B$ is a $C^*$-subalgebra of $A$ is construed as a metric structure in which separate sorts are used for $A$ and $B$ (and, as in $\mathcal{L}_{C^*}$, each of these sorts is divided into sorts corresponding to $n$-balls for $n \geq 1$). In addition, the language contains the function for the embedding from $B$ into $A$. Like the embeddings between $n$-balls for different $n$ in the standard axiomatization of $C^*$-algebras, this function is suppressed. The corresponding two-sorted language will be denoted $\mathcal{L}_{C^*,2}$. Similarly, the language corresponding to triples $(A, B, C)$ where $A$ is a $C^*$-algebra and $B$ and $C$ are $C^*$-subalgebras of $A$ will be denoted $\mathcal{L}_{C^*,3}$. In [12] and [13] we use further expansions of $\mathcal{L}_{C^*,3}$.

Given a metric language $\mathcal{L}$, a infinite indexed family of $\mathcal{L}$-structures $\mathcal{C}_j$, for $j \in J$, and an ideal $J$ on $J$, the reduced product is the quotient of $\prod_j \mathcal{C}_j$ corresponding to the pseudometric $d(\bar{a}, \bar{b}) = \inf_{x \in J} \sup_{j \in J} \chi d(a_j, b_j)$, denoted $\prod_j \mathcal{C}_j/J$. Both $B^\mathcal{U}$ and $B^\infty$ are special cases of the reduced product construction.

2. Reduced products

Fix a unital, separable, $C^*$-algebra $B$. By $\mathcal{L}_{C^*,3,B}$ we denote the expansion of $\mathcal{L}_{C^*,3}$ by constants for the elements of $B$. The $\mathcal{L}_{C^*,3,B}$-structures are triples $(A, C, D)$ where $B$ is a unital $C^*$-subalgebra of both $A$ and $C$ (but not of $D$). By making natural identifications, both $(B \otimes C(2^\mathcal{N}), B, C(2^\mathcal{N}))$ and $(B^\infty, B, \ell_\infty/c_0)$ are construed as metric $\mathcal{L}_{C^*,3,B}$-structures. We will
prove that \((B \otimes C(2^N), B, C(2^N))\) is isomorphic to an elementary submodel of \((B^{\infty}, B, \ell_\infty/c_0)\). One ingredient of the proof is the following general fact about an arbitrary metric language \(\mathcal{L}\).

**Proposition 2.1.** For every \(\mathcal{L}\)-formula \(\varphi(x)\) and every \(\varepsilon > 0\) there exist \(m \geq 1\) and \(\mathcal{L}\)-formulas \(\zeta_i(x)\) (with the free variables included among the free variables of \(\varphi\), for \(i < m\), such that for any two reduced products of \(\mathcal{L}\)-structures \(A = \prod_j A_j/\mathcal{J}\) and \(B = \prod_j B_j/\mathcal{J}\) and all \(\bar{a} \in A\) and \(\bar{b} \in B\) the following holds.

\[
\inf_{x \in \mathcal{J}} \sup_{j \in \mathbb{N}} \max_{0 < k < m} |\zeta_j(\bar{a}) - \zeta_j(\bar{b})| < \varepsilon/2
\]

then \(|\varphi(\bar{a}) - \varphi(\bar{b})| < \varepsilon\).

**Proof.** By replacing \(\varphi\) with \(s(\varphi - r)\) for appropriate real numbers \(r\) and \(s\), we may assume that the range of \(\varphi\) is included in the unit interval. By [13, Theorem 3.3] (or by Theorem [A.3] (2)) there are \(m(\varphi) \geq 1\) and formulas \(\zeta^\varphi_i(x)\) for \(i < m(\varphi)\) whose free variables are included among the free variables of \(\varphi\) such that the isomorphism type of the Boolean subalgebra of \(\mathcal{P}(\mathbb{N})/\mathcal{J}\) generated by \(Z^{\varphi}_{l/k}[\bar{b}] = \{j : \zeta^\varphi_i(b_j) > l/k\}\) for \(0 \leq l \leq k\) determines the value \(\varphi(\bar{b})^\mathcal{M}\) up to \(1/k\). Therefore \(\zeta^\varphi_i\), for \(i < m(\varphi)\), are as required. \(\square\)

**Lemma 2.2.**

1. For every nonzero projection \(p \in \ell_\infty/c_0\), the \(C^*\)-algebra \(B^{\infty}\) is isomorphic to the corner \(pB^{\infty}\).

2. Suppose that \(p_0, \ldots, p_{n-1}\) and \(q_0, \ldots, q_{n-1}\) are nonzero projections in \(\ell_\infty/c_0\) such that \(\sum_j p_j = \sum_j q_j = 1\). Then there exists an automorphism \(\Phi\) of \(B^{\infty}\) such that \(\Phi(p_j) = q_j\) for all \(j < n\).

**Proof.** \(\square\) The isomorphism is implemented by a reenumeration argument.

1. All \(p_j\) and \(q_j\) are central projections of \(B^{\infty}\), hence the isomorphisms guaranteed by (1) can be glued to obtain the required automorphism. \(\square\)

**Lemma 2.3.** A unital separable \(C^*\)-subalgebra of \(\ell_\infty/c_0\) is an elementary submodel of \(\ell_\infty/c_0\) if and only if it is isomorphic to \(C(2^N)\).

**Proof.** The \(C^*\)-algebras \(C(2^N)\) and \(\ell_\infty/c_0\) (the latter being isomorphic to \(C(\beta\mathbb{N} \setminus \mathbb{N})\)) are elementarily equivalent by [7, Corollary 5.10]. Since the theory of these algebras admits quantifier elimination in the language of \(C^*\)-algebras equipped with a constant for the unit ([7, Theorem 5.26]), any unital copy of \(C(2^N)\) inside \(\ell_\infty/c_0\) is an elementary submodel. Conversely, every separable model of the theory of \(C(2^N)\) is of the form \(C(X)\) where \(X\) is a compact, metricizable, totally disconnected space without isolated points; therefore \(X\) is homeomorphic to the Cantor space. \(\square\)

**Theorem 2.4.** Suppose \(B\) is a unital and separable \(C^*\)-algebra and \(A\) is a unital separable \(C^*\)-subalgebra of \(\ell_\infty/c_0\) (the latter is considered as a subalgebra of \(\ell_\infty(B)/c_0(B)\)) isomorphic to \(C(2^N)\). Then

1. \(C^*(B, A) \cong B \otimes A\), and
(2) The $L^{\epsilon}_{\omega,3,B}$-structure $\mathcal{C} = (B \otimes A, B, A)$ is an elementary submodel of the $L^{\epsilon}_{\omega,3,B}$-structure $\mathcal{D} = (B^\infty, B, \ell_\infty/c_0)$.

Proof. (1) Since $A$ is included in the centre of $B^\infty$, $A$ and $B$ commute. Since $A$ is nuclear, $C^\ast(B, A)$ is isomorphic to a quotient of the spatial tensor product $B \otimes A$.

In order to prove that the quotient map is injective, fix a nonzero $b \in B$ and a nonzero $a \in A$. The constant sequence $(b)$ is a representing sequence of $b$. For $a$ we can choose a representing sequence $(\lambda_j)$ such that each $\lambda_j$ is a scalar. If $\lambda_j \neq 0$ then $\lambda_j a \neq 0$. Since $\lim \sup_j |\lambda_j| = \|a\|$, we conclude that $\|ab\| = \|a\||b|$. Therefore the kernel $J$ of the quotient map does not contain any nontrivial elementary tensors. Suppose $J$ is nontrivial. By Kirchberg’s Slice Lemma ([19, Lemma 4.19]), there exists nonzero $c \in B \otimes A$ such that $cc^\ast \in J$ and $c^\ast c = b \otimes a$ for some $b \in B$ and $a \in A$. Therefore $c \in J$ and $b \otimes a \in J$; contradiction. We conclude that $J = \{0\}$ and the quotient map from $B \otimes A$ onto $C^\ast(B, A)$ is injective.

(2) By the Tarski–Vaught test ([8, Theorem 2.6.1]) it suffices to prove that if $\varphi(\bar{x}, \bar{y})$ is a formula and $\bar{a}$ in $\mathcal{C}$ is a tuple of the appropriate sort, then

$$\inf_{\bar{y} \in B \otimes A} \varphi(\bar{a}, \bar{y})^D = \inf_{\bar{y} \in B^\infty} \varphi(\bar{a}, \bar{y})^D.$$ 

This is equivalent to asserting that for every $\varepsilon > 0$ and every $d \in B^\infty$ there exists $c \in B \otimes A$ such that $\varphi(\bar{a}, c)^D < \varphi(\bar{a}, d)^D + \varepsilon$. Since the formulas in the prenex normal form are uniformly dense in the space of all formulas, it suffices to prove this under the additional assumption that $\varphi$ is in the prenex normal form. In addition, we may assume (by adding dummy quantifiers if necessary) that there are $k \geq 1$, continuous $f: \mathbb{R}^k \to \mathbb{R}$, and atomic formulas $\alpha_j$, for $j < k$, such that $\varphi(\bar{x}, \bar{z})$ is equal to

$$\sup \inf \ldots \inf_{y:\{0\}} f(\alpha_0(\bar{x}, \bar{z}, \bar{y}), \ldots, \alpha_{k-1}(\bar{x}, \bar{z}, \bar{y})).$$

Fix $\bar{a}$ in $\mathcal{C}$, $d$ in $\mathcal{D}$, and $\varepsilon > 0$. It is no loss of generality to assume that $\bar{a}$ belongs to $B \otimes A$ and $d$ belongs to $B^\infty$, since these are the largest sorts in $\mathcal{C}$ and $\mathcal{D}$ respectively.

Since the evaluation of $\varphi$ is uniformly continuous with a fixed modulus of uniform continuity, there is $\delta > 0$ such that any $\delta$-perturbation of the values of $\bar{x}$, $\bar{z}$, and $\bar{y}$ results in a perturbation of $f(\alpha_0(\bar{x}, \bar{z}, \bar{y}), \ldots, \alpha_{k-1}(\bar{x}, \bar{z}, \bar{y}))$ by less than $\varepsilon/2$. Since $B \otimes A$ is the completion of the algebraic tensor product of $A$ and $B$, we may assume that every entry of $\bar{a}$ is a finite sum of elementary tensors $a(i) = \sum_{j<m(i)} b(i, j) \otimes q(i, j)$ for $i < n$, where $n$ is the arity of $\bar{a}$.

The elements of $C(2^\mathbb{N})$ can be uniformly approximated by linear combinations of projections, and we may furthermore assume each $q(i, j)$, for

\footnote{It is not a loss of generality to assume that the free variables of $\varphi$ range over the largest sort, corresponding to $B \otimes A$ and $B^\infty$ in $\mathcal{C}$ and $\mathcal{D}$, respectively.}
Let $e_j$, for $j \in \mathbb{N}$, be orthogonal rank-one projections in $\ell_\infty$ naturally identified with a central $C^*$-subalgebra of $\ell_\infty(B)$ such that the partial sums $\sum_{j<n} e_j$ strictly converge to 1 as $n \to \infty$. (The strict topology is evaluated by considering $\ell_\infty(B)$ as the multiplier algebra of $c_0(B)$.) For $X \subseteq \mathbb{N}$ let $r_X$ denote the projection in $\ell_\infty/c_0$ that is the image of $\sum_{i \in X} e_i$ under the quotient map. For $l < m$ choose $X(l) \subseteq \mathbb{N}$ such that $p_l = r_{X(l)}$. Since $\sum_{l<m} p_l = 1$, we may choose $X(l)$ so that $\bigcup_{l<m} X(l) = \mathbb{N}$, by making finite changes to $X(l)$ if necessary.

By Corollary 2.1 there are $m \geq 1$ and $\mathcal{L}$-formulas $\zeta_i(x)$, for $i < m$ such that $\limsup_j \max_{i<m} |\zeta_i(a_j)|^B - \zeta_i(b_j)|^B| < \varepsilon$ implies $|\varphi(a)^B - \varphi(b)^B| < \varepsilon$.

For $j \in \mathbb{N}$ define an element of $\mathbb{R}^m$ by

$$\bar{s}(j) = ((\zeta_0(a_j, d_j), \ldots, \zeta_{m-1}(a_j, d_j))).$$

Since every formula has a bounded range, all $\bar{s}(j)$ are contained in a sufficiently large compact neighbourhood of 0 in $\mathbb{R}^m$. We can therefore find a partition $\bigcup_{m} Y(i)$ refining $\bigcup_{m} X(i)$ so that for every $i < m'$ and all $j$ and $j'$ in $Y(i)$ we have $\max_{i<m} |\bar{s}(j_i) - \bar{s}(j'_i)| < \varepsilon$. For $j$ and $j'$ that belong to the same $Y(i)$ Claim 2.5 implies $\bar{a}_j = \bar{a}_{j'}$. By applying Lemma 2.2 there exists an automorphism $\Phi$ of $B^\infty$ that is equal to the identity on $B$ and sends $1 \otimes r_{Y(i)}$ into $A$ for all $i < m$. By replacing $\bar{a}$ with its image under $\Phi$ we may assume that all $1 \otimes r_{Y(i)}$ belong to $A$.

For each $i < m'$ choose $j(i) \in Y(i)$ and define $\bar{c}_i \in B^\infty$ by its representing sequence, $\bar{c}_j = \bar{d}_{j(i)}$, if $j \in Y(i)$. Then $c(n) = \sum_{i < m'} d_{j(i)}(n) \otimes r_{Y(i)}$ for all $n$, and $\bar{c}$ is a tuple in $B \otimes A$. By the choice of $Y(i)$ and Corollary 2.1 we have $|\varphi(\bar{a}, \bar{c})^B - \varphi(\bar{a}, \bar{d})^B| < \varepsilon$, as required. \hfill $\Box$

**Lemma 2.6.** For a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, $(B \otimes C(2^\mathbb{N}), C(2^\mathbb{N}))^\mathcal{U}$ and $(B^\infty, \ell_\infty/c_0)$ are countably saturated, elementarily equivalent, $\mathcal{L}_{c', 2, B}$-structures.

**Proof.** The ultrapower $(B \otimes C(2^\mathbb{N}), C(2^\mathbb{N}))^\mathcal{U}$ is countably saturated (this is a folklore result, see [8, §4.3]). By Łoś’s Theorem, it is elementarily equivalent to $(B \otimes C(2^\mathbb{N}), C(2^\mathbb{N}))$. By Theorem 2.4 and taking $\mathcal{L}_{c', 2, B}$-reducts, $(B \otimes C(2^\mathbb{N}), C(2^\mathbb{N}))$ is isomorphic to an elementary submodel of
(B^∞, ℓ_∞/c_0). The latter structure is naturally identified with the reduced product of (B, ℂ) (C is identified with the unital subalgebra C1_B of B) associated with the Frechét ideal and therefore countably saturated by \[10\] Theorem 1.1 (Theorem B.2).

3. Ultrapowers of B ⊗ C(2^N)

Suppose B is a separable and unital C*-algebra. Consider the metric structure (B ⊗ C(2^N), B, C(2^N), Θ, Φ) where

(1) B and C(2^N) are identified with the C*-subalgebras B ⊗ 1_{C(2^N)} and 1_B ⊗ C(2^N), respectively, of B ⊗ C(2^N).

(2) Θ: B ⊗ C(2^N) → B is the conditional expectation Θ = id_B ⊗ ev_0 where ev_0: C(2^N) → ℂ is the evaluation character at a distinguished point 0 of 2^N (the actual choice of this point is inconsequential).

(3) Φ: B → B⊗C(2^N) is the *-homomorphism defined as Φ = id_B ⊗ 1_{C(2^N)}.

This is a structure in the expansion of \(L_{C^*,3,B}\) by the functions \(\Phi\) and \(\Theta\), denoted \(L_+\). The uniform continuity modulus of each one of these functions is the identity function. (In other words, both \(\Phi\) and \(\Theta\) are contractions and this is hardwired into the language.)

A short list of elementary properties of (B ⊗ C(2^N), B, C(2^N), Θ, Φ) follows.

(4) Both \(\Theta\) and \(\Phi\) are *-homomorphisms, and \(\Phi\) is injective.

(5) The range of \(\Theta\) is equal to the domain \(\Phi\), B. This is expressed as sup_{x∈B} inf_{y∈B⊗C(2^N)} \|\Theta(y) - x\| = 0\[3\]

(6) The *-homomorphism \(\Theta\) is a conditional expectation onto its range; this is expressed as sup_{x∈B⊗C(2^N)} \|\Theta(\Theta(x)) - \Theta(x)\| = 0\[4\]

(7) The composition \(\Theta \circ \Phi\) is equal to the identity on the range of \(\Theta\).

Using \[3\], this is expressed as sup_{x∈B⊗C(2^N)} \|\Theta\circ\Phi\circ\Theta(x) - \Theta(x)\| = 0.

(8) The composition \(\Phi \circ \Theta\) is equal to the identity on the range of \(\Theta\).

This is expressed as sup_{x∈B⊗C(2^N)} \|\Phi(\Theta(x)) - \Theta(x)\| = 0.

(9) The restriction of \(\Phi\) to C(2^N) is a character. Since \(\Phi\) is a character if and only if it sends every unitary to a scalar, this is expressed as sup_{x∈C(2^N), x unitary} inf_{|λ| = 1} \|\lambda \cdot 1_{B⊗C(2^N)} - \Phi(1_B ⊗ x)\| = 0\[5\]

\[2\]In the model-theoretic sense; the pun is accidental but appropriate.

\[3\]Here, and elsewhere, sup_{x∈D} is short for sup_{x∈D, ∥x∥ ≤ 1}, where D is any C*-algebra and z is any variable.

\[4\]This formula implicitly involves the suppressed injection from B into B ⊗ C(2^N).

\[5\]Quantifications over the set of unitaries of C(2^N) and over the unit circle in ℂ are justified as follows. Since the set of unitaries is definable in a unital C*-algebra (\[8\] Definition 3.2.1 and Example 3.2.7 (7)) and the unit circle \(T\), identified with \(T_1_B\), belongs to \(B^eq\) for any unital C*-algebra B (\[8\] §3), the left-hand side of the formula in \[3\] is a definable predicate.
Lemma 3.1. If $B$ is a unital and separable $C^*$-algebra and $U$ is a non-principal ultrafilter then the ultrapower $(B \otimes C(2^N), B, C(2^N), \Theta, \Phi)^U$ is the structure $((B \otimes C(2^N))^U, B^U, C(2^N))^U, \Theta^U, \Phi^U)$ which satisfies the following.

(10) Both $\Theta^U: (B \otimes C(2^N))^U \rightarrow B^U$ and $\Phi^U: B^U \rightarrow (B \otimes C(2^N))^U$ are *-homomorphisms, and $\Phi^U$ is injective.
(11) The range of $\Theta^U$ is equal to the domain of $\Phi^U$, $B^U$.
(12) The *-homomorphism $\Theta^U$ is a conditional expectation onto its range.
(13) The composition $\Theta^U \circ \Phi^U$ is equal to the identity on the range of $\Theta^U$.
(14) Both $\Phi^U$ and $\Theta^U$ are equal to the identity on the range of $\Theta^U$, and the restriction of $\Phi^U$ to $C(2^N)^U$ is a character.

Proof. The properties (10)–(14) are immediate consequences of (1)–(9) and Łoś’s Theorem ([8, Theorem 2.3.1]). □

Proof of Theorem 4.2. Elementarily equivalent saturated models of the same density character are isomorphic ([9, Corollary 4.14]). By applying this to the expansions of $(B \otimes C(2^N))^U, C(2^N)^U)$ and $(B^\infty, \ell_\infty/c_0)$ by constants for the elements of $B$, the Continuum Hypothesis implies that that there exists an isomorphism $\Psi: (B \otimes C(2^N))^U, C(2^N)^U) \rightarrow (B^\infty, \ell_\infty/c_0)$ that sends the diagonal copy of $B$ in $B^U$ to the diagonal copy of $B^\infty$. With $\Theta^U: B^U \rightarrow B^\infty$ and $\Phi^U: B^\infty \rightarrow B^U$ as in Lemma 3.1, $\Theta = \Theta^U \circ \Psi^{-1}$ and $\Phi = \Psi \circ \Phi^U$ are as required. □

4. $\sigma$-COMPLETE BACK-AND-FORTH SYSTEMS

The proof of Theorem 4.2 uses the same standard results about countably saturated models used in the proof of [9, Corollary 4.14]. Their combinatorial essence is given in the following definition (not used explicitly in the statement of Theorem 4.2).

Definition 4.1. Suppose $\mathcal{C}$ and $\mathcal{D}$ are nonseparable metric structures in the same language. A $\sigma$-complete back-and-forth system between $\mathcal{C}$ and $\mathcal{D}$ is a poset $\mathcal{F}$ with the following properties.

(1) The elements of $\mathcal{F}$ are triples $p = (C^p, D^p, g^p)$, where $C^p$ is a separable substructure of $\mathcal{C}$, $D^p$ is a separable substructure of $\mathcal{D}$, and $g^p: C^p \rightarrow D^p$ is an isometric isomorphism.
(2) We let $p \leq q$ if $C^p \subseteq C^q$, $D^p \subseteq D^q$, and $g^q | C^p = g^p$.
(3) ($\sigma$-completeness) Every increasing sequence $p(n)$, for $n \in \mathbb{N}$, in $\mathcal{F}$ has an upper bound in $\mathcal{F}$.
(4) (Density) For every $p \in \mathcal{F}$ and all $c \in \mathcal{C}$ and $d \in \mathcal{D}$ there exists $q \geq p$ in $\mathcal{F}$ such that $c \in C^q$ and $d \in D^q$.

The following is the ZFC analog of Theorem 4.2.

Theorem 4.2. Suppose that $U$ is a nonprincipal ultrafilter on $\mathbb{N}$ and $B$ is a separable and unital $C^*$-algebra. Then there is a partially ordered set $\mathcal{F}$ whose elements are quadruples $p = (C^p, D^p, \Theta^p, \Phi^p)$ satisfying the following:

(1) $C^p$ is a separable subalgebra of $B^U$ that includes $B$. 

Lemma 4.3. For any two elementarily equivalent, countably saturated, \(\mathcal{L}\)-structures \(\mathcal{C}\) and \(\mathcal{D}\) there is a \(\sigma\)-complete back-and-forth system \(\mathcal{F}\) between them. Moreover, every \((\mathcal{C}^p, \mathcal{D}^p, g^p)\) \(\in \mathcal{F}\) satisfies \(\mathcal{C}^p \preceq \mathcal{C}\) and \(\mathcal{D}^p \preceq \mathcal{D}\).

Proof. Let \(\mathcal{F}\) be the family of all triples \((\mathcal{C}^p, \mathcal{D}^p, g^p)\) such that \(\mathcal{C}^p\) is a separable elementary submodel of \(\mathcal{C}\), \(\mathcal{D}^p\) is a separable elementary submodel of \(\mathcal{D}\), and \(g^p: \mathcal{C}^p \rightarrow \mathcal{D}^p\) is an isomorphism. We claim that \(\mathcal{F}\) is a \(\sigma\)-complete back-and-forth system between \(\mathcal{C}\) and \(\mathcal{D}\). The \(\sigma\)-completeness is automatic since the union of a chain of elementary submodels is elementary. The density is proved using countable saturation, as follows.

Suppose that \(p = (\mathcal{C}^p, \mathcal{D}^p, g^p)\) \(\in \mathcal{F}\), \(c \in \mathcal{C}\), and \(d \in \mathcal{D}\). Let \(\mathcal{C}_1\) be a separable elementary submodel of \(\mathcal{C}\) that includes \(\mathcal{C}^p\) and contains \(c\). Enumerate a dense subset of \(\mathcal{C}_1\) as \(c_j\), for \(j \in \mathbb{N}\). For each \(j\) let \(t_j(x_0, x_1, \ldots, x_j)\) denote the type of \((c_0, \ldots, c_j)\) over \(\mathcal{C}^p\) (see \([8, \S 4.1]\)).

Let \(s_j(x_0, \ldots, x_j)\) be the type obtained by replacing \(a \in \mathcal{C}^p\) with \(g^p(a)\) in all conditions of \(t_j(x)\). We will recursively define \(d_j\), for \(j \in \mathbb{N}\), so that \((d_0, \ldots, d_k)\) realizes \(s_k\) for all \(k\). Since \(\mathcal{C}^p\) is an elementary submodel of \(\mathcal{C}\) and \(\mathcal{D}^p\) is an elementary submodel of \(\mathcal{D}\), the type \(s_0(x)\) is consistent. By the countable saturation of \(\mathcal{D}\) it is realized by some \(d_0\).

Suppose that \(d_j\), for \(j < k\), have been defined so that \((d_0, \ldots, d_{k-1})\) satisfies \(s_{k-1}\). Then the type \(s(x)\) obtained from \(s_k\) by replacing the variable \(x_j\)
with the constant $d_j$ for $j < k$ is consistent. Again using the countable saturation of $\mathcal{D}$, there exists $d_k$ that realizes $s$. Therefore $(d_0, \ldots, d_k)$ realizes $s_k$. This describes the recursive construction.

Extend $g^p$ to $\mathcal{E}^p \cup \{c_j : j \in \mathbb{N}\}$ by sending $c_j$ to $d_j$ for all $j$. This is an isometry, and the continuous extension of this function, denoted $g_1$, is an elementary embedding from $\mathcal{E}_1$ into $\mathcal{D}$. Then $\mathcal{D}_1 = g_1(\mathcal{E}_1)$ is an elementary submodel of $\mathcal{D}$ and the triple $p_1 = (\mathcal{E}_1, \mathcal{D}_1, g_1)$ belongs to $\mathcal{F}$. If $d \notin D_1$, let $\mathcal{D}_2$ be a separable elementary submodel of $\mathcal{D}$ that includes $\mathcal{D}_1 \cup \{d\}$. A proof analogous to the above produces a condition $(\mathcal{C}_2, \mathcal{D}_2, g_2)$ that extends $p_1$ and satisfies $c \in \mathcal{C}_2$ and $d \in \mathcal{D}_2$. □

**Proof of Theorem 4.2.** The $\mathcal{L}_{C^*, \mathcal{B}}$ structures $\mathfrak{A} = (B \otimes C(2^N), C(2^N))^H$ and $\mathfrak{B} = (B^\infty, \ell^\infty/c_0)$ are countably saturated and elementarily equivalent by Lemma 2.6. Lemma 2.3 implies there is a $\sigma$-complete back-and-forth system $\mathcal{F}_0$ of triples $(\mathfrak{A}^p, \mathfrak{B}^p, g^p)$ such that $\mathfrak{A}^p \prec \mathfrak{A}$, $\mathfrak{B}^p \prec \mathfrak{B}$, and $g^p : \mathfrak{A}^p \rightarrow \mathfrak{B}^p$ is an isomorphism.

The metric structure $\mathfrak{A}_1 = (B \otimes C(2^N), B, C(2^N), \Theta, \Phi)^H$ (as defined in [3]) is an expansion of $\mathfrak{A}$ to $\mathcal{L}_+$. For $\mathfrak{A}^p \prec \mathfrak{A}$ let $\mathfrak{A}^p_1$ denote the expansion of $\mathfrak{A}^p$ to $\mathcal{L}_+$. The set $\mathcal{F}_1 = \{\mathfrak{A}^p_1 : (\mathfrak{A}^p, \mathfrak{B}^p, g^p) \in \mathcal{F}_0 \text{ and } \mathfrak{A}^p \prec \mathfrak{A}^p_1\}$ is, by the standard Löwenheim–Skolem/Blackadar closing-up argument, cofinal in the poset of all separable substructures of $(B \otimes C(2^N), B, C(2^N), \Theta, \Phi)^H$ and $\sigma$-complete.

For $(\mathfrak{A}^p, \mathfrak{B}^p, g^p) \in \mathcal{F}_1$ let $A^p$ and $B^p$ denote the domains of $\mathfrak{A}^p$ and $\mathfrak{B}^p$, respectively, and let $E^p = \Theta^H[\mathcal{A}^p]$. Let $\mathcal{F}$ be the family of all quadruples $(B^p, E^p, \Theta^H \circ g^p, g^{-1} \circ \Phi^H \upharpoonright E^p)$ obtained in this manner (see the diagram). It remains to verify that $\mathcal{F}$ satisfies the requirements. The $\sigma$-completeness and density of $\mathcal{F}_1$ imply the corresponding conditions for $\mathcal{F}$. Both [1] and [2] are evidently true. Since Lemma 3.1 implies that $\mathfrak{A}_1$ satisfies [3]–[6], each $\mathfrak{A}^p_1$ is an elementary submodel of $\mathfrak{A}_1$, and [7]–[10] are axiomatizable (see [3]), the conclusion follows. □

5. **Preservation of elementarity by tensor products**

The question whether tensoring with any infinite-dimensional $C^*$-algebra preserves elementary equivalence has been raised in [8, Question 3.10.5]. By [8, Proposition 3.10.3], tensoring with $C([0,1])$ does not necessarily preserve elementary equivalence. Theorem 4.2 has the following consequence.
Corollary 5.1. If $C$ and $D$ are elementarily equivalent $C^*$-algebras, then so are $C \otimes C(2^\mathbb{N})$ and $D \otimes C(2^\mathbb{N})$.

Proof. By an application of Löwenheim–Skolem theorem it suffices to prove the statement in the case when both $C$ and $D$ are separable. By Theorem 2.4, $C \otimes C(2^\mathbb{N})$ is isomorphic to an elementary submodel of $C^\infty$ and $D \otimes C(2^\mathbb{N})$ is isomorphic to an elementary submodel of $D^\infty$. By [13, Proposition 3.6] (or by Theorem A.3), reduced products preserve elementary equivalence, and therefore $C^\infty$ and $D^\infty$ are elementarily equivalent. □

Corollary 5.1 implies that tensoring with $C(2^\mathbb{N}) \otimes F$ preserves elementary equivalence for any finite-dimensional $C^*$-algebra $F$. It is possible that these are the only infinite-dimensional $C^*$-algebras such that tensoring with them preserves elementarity. The following result, all but proven in [8, §3.5], transpired during a conversation with Chris Schafhauser, and it is included with his kind permission ([Z] denotes the Jiang–Su algebra).

Proposition 5.2. There are elementarily equivalent $C^*$-algebras $A$ and $B$ such that $A \otimes D$ and $B \otimes D$ are not elementarily equivalent if $D$ is $\mathcal{Z}$ or any UHF algebra $D$.

Proof. Let $A$ be the unital, monotracial, $C^*$-algebra constructed in [18, Theorem 1.4] such that $A^U$ does not have a unique trace and $B = A^U$. Then $A$ and $B$ are elementarily equivalent, $A \otimes \mathcal{Z}$ is monotracial, and $B \otimes \mathcal{Z}$ is not.

As the Cuntz–Pedersen nullset in $A \otimes \mathcal{Z}$ is definable (this has been essentially proved in [16], see [8, Theorem 3.5.5 (3)]), for every $\varepsilon > 0$ there exists $m(\varepsilon)$ such that every positive contraction $a$ in $A \otimes \mathcal{Z}$ can be $\varepsilon$-approximated by a sum $m(\varepsilon)$-commutators of elements of norm $\leq 1$. For a fixed $\varepsilon > 0$ this property can be expressed as a statement in the theory of $A \otimes \mathcal{Z}$. Since $B \otimes \mathcal{Z}$ does not have a unique trace, for a small enough $\varepsilon$ the corresponding assertion fails in $B \otimes \mathcal{Z}$ and therefore $A \otimes \mathcal{Z}$ and $B \otimes \mathcal{Z}$ are not elementarily equivalent.

If $D$ is a UHF algebra then $A \otimes D$ is monotracial and since $D$ absorbs $\mathcal{Z}$ tensorially so does $A \otimes D$. Therefore the Cuntz–Pedersen nullset of $A \otimes D$ is definable, and the proof proceeds as in the case of $\mathcal{Z}$. □

Appendix A. A proof of the metric Feferman–Vaught Theorem

In this appendix we provide a proof of [13, Theorem 3.3]. Given a language of the metric structures $\mathcal{L}$, an $\mathcal{L}$-formula is in \textit{prenex normal form} (PNF) if it is of the form $\sup_{x_1} \inf_{x_2} \sup_{x_3} \ldots \inf_{x_{2n}} g(||p_1(\bar{x})||, \ldots, ||p_k(\bar{x})||)$ for some $k \geq 1$, $*$-polynomials $p_j(\bar{x})$ for $j \leq k$ in non-commuting variables, and a continuous function $g$. Every formula can be uniformly (with respect to the seminorm defined in [8, §2.6]) approximated by PNF formulas (this is proved by combining [2, Theorem 6.3, Theorem 6.6, and Theorem 6.9]).

Let $\mathcal{L}^+_{\omega_1\omega}$ be the language of Boolean algebras equipped with the constants $Z_{i,s}^\varphi$, for all $\mathcal{L}$-formulas $\varphi$, $i \in \mathbb{N}$, and $s \in \mathbb{Q}$. By $\mathcal{L}^+_{\omega_1\omega}$ we denote
the extension of $\mathcal{L}^+$ that allows conjunctions and disjunctions of countable sets of formulas\footnote{In $\mathcal{L}_{\omega_1\omega}$ formulas $\bigvee_{\alpha \in \omega} \alpha$ and $\bigwedge_{\alpha \in \omega} \alpha$ are allowed only if the formulas $\alpha$ have at most finitely many distinct free variables. Since all subformulas of all $\vartheta^\zeta$ will contain only constant symbols, this will not be an issue for us.}. For an $\mathcal{L}$-formula $\varphi(\bar{x})$ let $R(\varphi)$ denote the convex closure of the range of $\varphi$. This is a compact interval in $\mathbb{R}$.

**Definition A.1.** An $\mathcal{L}$-formula $\varphi(\bar{x})$ is determined if objects with the following properties exist.

1. An $m(\varphi) \geq 1$ and $\mathcal{L}$-formulas $\zeta^\zeta_i(\bar{x})$, for $i < m(\varphi)$, whose free variables are included among the free variables of $\varphi(\bar{x})$.
2. Quantifier-free $\mathcal{L}_{\omega_1\omega}$-formulas $\vartheta^\zeta_t$, for $t \in R(\varphi)$\footnote{The theory of atomless Boolean algebras admits quantifier elimination, but this is not relevant to our discussion.}.

Given an indexed family of $\mathcal{L}$-structures $(M_j)_{j \in \mathbb{I}}$ and an ideal $\mathcal{J}$ on $\mathbb{I}$, for $i < m(\varphi)$, we write $\vartheta^\zeta_i[\bar{a}]$ for the value of the formula $\vartheta^\zeta_i$ in $\mathcal{P}(\mathbb{I})/\mathcal{J}$ when the constants are interpreted as $([x]_\mathcal{J}$ denotes the equivalence class of a set $X$ modulo $\mathcal{J}$) $\mathcal{Z}^\zeta_{i,t}(\bar{a}) = \{j : (\zeta^\zeta_i(\bar{a}_j))^{M_j} > t\}_{\mathcal{J}}$. Then the following holds.

3. If $s \leq t$ then $\vartheta^\zeta_t[\bar{a}]$ implies $\vartheta^\zeta_s[\bar{a}]$ for all $\bar{a}$ of the appropriate sort.
4. For every $s$, in the reduced product $M = \prod_j M_j/\mathcal{J}$, every tuple $\bar{a}$ in $M$ of the appropriate sort satisfies $\varphi(\bar{a})^M > s$ if and only if $\vartheta^\zeta_s[\bar{a}]$.

**Definition A.2.** For $k \geq 2$, an $\mathcal{L}$-formula $\varphi(\bar{x})$ is $1/k$-determined if in addition to $m(\varphi)$ and $\zeta^\zeta_i$, for $i < m(\varphi)$, as in Definition A.1 there are quantifier-free $\mathcal{L}^+$-formulas $\vartheta^\zeta_{t,k}$ for $t \in \mathbb{Q} \cap R(\varphi)$, such that the following holds. For every $k \geq 2$ and every $s$, in every reduced product $M = \prod_j M_j/\mathcal{J}$ every tuple $\bar{a}$ in $M$ of the appropriate sort with $\vartheta^\zeta_{s,k}[\bar{a}]$ as in Definition A.1 the following holds.

1. $\varphi(\bar{a})^M \geq t + (1/k)$ implies $\vartheta^\zeta_{s,k}[\bar{a}]$ and
2. $\vartheta^\zeta_{t,k}[\bar{a}]$ implies $\varphi(\bar{a})^M \geq t - (1/k)$.

In particular, the value of $\varphi(\bar{a})$ is determined up to $1/k$ by the (finite) set of all formulas $\vartheta^\zeta_{s,j/k}$ for $j \in \mathbb{Z}$ such that $j/k \in R(\varphi)$.

The following is essentially \cite[Theorem 3.3]{1}.

**Theorem A.3.**

1. Every $\mathcal{L}$-formula is determined.
2. Every $\mathcal{L}$-formula is $1/k$-determined for all $k \geq 2$.

**Proof.** By replacing $\varphi(\bar{x})$ with $r(\varphi(\bar{x}) - t)$ for appropriately chosen real numbers $r$ and $s$, we may assume that $R(\varphi) = [0, 1]$. A reduction on the complexity of $\varphi$ is given as follows. A formula $\varphi$ is called $\mathcal{F}_0$-restricted if it is obtained from the atomic formulas by recursively applying the functions $t \mapsto t/2$ and $(s, t) \mapsto s - t$ (where $s - t = \max(s - t, 0)$), the constant functions, and the quantifiers, and the quantifiers $\sup_x$ and $\inf_x$ (see \cite[Definition 6.5–Proposition 6.9]{2}).
By the Stone–Weierstrass theorem for lattices and \cite{2} Theorem 6.3, every \( \mathcal{L} \)-formula can be uniformly approximated by \( \mathcal{F}_0 \)-restricted formulas.

By induction on the complexity of a formula (following \cite{3} Definition 2.1.1), it suffices to prove that the set of all determined formulas satisfies the following closure properties:

1. All atomic formulas are determined.
2. If \( \varphi \) is determined, so is \( \frac{1}{2} \varphi \).
3. If \( \varphi \) and \( \psi \) are determined, so is \( \varphi \land \psi \).
4. If \( \varphi \) is determined, so are \( \sup_{x} \varphi \) and \( \inf_{x} \varphi \) for every variable \( x \).

For the readability of the ongoing proof, we will describe the Boolean formulas \( \theta_i^\varphi \) as in Definition \( \ref{def:formula} \) informally. We will also combine the recursive construction of the objects with a proof that they have the desired properties for arbitrary \( (M_j)_{j \in \mathbb{Z}} \) and \( \mathcal{J} \), with \( M = \prod_{j \in \mathbb{Z}} M_j / \mathcal{J} \) and \( \bar{a} \in M \) of the appropriate type. (Needless to say, the constructed objects will not depend on the choices of \( (M_j), \mathcal{J}, \) and \( \bar{a} \).) This will make the proof proceed smoother.

Let \( \theta_i^\varphi = \text{False} \) if \( t > \max(R(\varphi)) \) and \( \theta_i^\varphi = \text{True} \) if \( t \leq \min(R(\varphi)) \) for all \( \varphi \). This convention will be used tacitly.

If \( \varphi(\bar{x}) \) is atomic, then let \( m(\varphi) = 1 \), let \( \zeta_i^\varphi(\bar{x}) = \varphi(\bar{x}) \) and let \( \theta_i^\varphi \) be \( [Z_{0,t}^\varphi]_{\mathcal{J}} \neq 0 \). Since \( \varphi \) is atomic, \( \varphi(\bar{a})^M = \inf_{X \in \mathcal{J}} \sup_{j \in \mathbb{Z}} X \varphi(\bar{a})^{M_j} \). Then for any \( s \in \mathbb{R} \) we have \( \varphi(\bar{a})^M > s \) if and only if \( [Z_{0,s}^\varphi(\bar{a})]_{\mathcal{J}} \neq 0 \), if and only if \( \theta_i^{\varphi(\bar{a})} \) holds. Therefore \( \varphi \) is determined.

Assume \( \psi \) is determined and \( \varphi = \frac{1}{2} \psi \). Let \( m(\varphi) = m(\psi) \), \( \zeta_i^\psi = \frac{1}{2} \zeta_i^\psi \) for \( i < m(\varphi) \), and \( \theta_i^\psi = \theta_i^\psi \) for all \( t \). Then for an \( \mathcal{L} \)-structure \( N \) and \( \bar{a} \in N \) we have \( \varphi(\bar{a})^N = \frac{1}{2} \psi(\bar{a})^N \) and \( Z_{0,t}^\psi = Z_{0,2t}^\psi \) for all \( i < m(\varphi) \). Therefore with \( (M_j), \mathcal{J}, M, \) and \( \bar{a} \) as before we have \( \varphi(\bar{a})^M > t \) if and only if \( \psi(\bar{a})^M > 2t \) if and only if \( \theta_i^\psi(\bar{a}) \) if and only if \( \theta_i^\psi(\bar{a}) \). Hence \( \varphi \) is determined.

Assume \( \psi \) and \( \eta \) are determined and \( \varphi = \psi - \eta \). Let \( m(\varphi) = m(\psi) + m(\eta) \), \( \zeta_i^\psi = \zeta_i^\psi + 1 \) for \( i < m(\psi) \), and \( \zeta_i^\eta = \zeta_i^\eta \) for \( i < m(\eta) \). After replacing \( Z_{0,t+s}^\psi \) with \( Z_{0,t+s}^\psi \) in \( \theta_i^\psi \), and \( Z_{0,s}^\psi \) with \( Z_{0,s}^\psi \) in \( \theta_i^\psi \), let \( \theta_i^\varphi = \bigvee_{s \in \mathbb{Q}} (\theta_i^\psi \wedge \theta_i^\eta) \). With \( (M_j), \mathcal{J}, M, \) and \( \bar{a} \) as before we have \( \varphi(\bar{a})^M > t \) if and only if there exists \( s \in \mathbb{Q} \) such that \( \psi^M(\bar{a}) > s + t \) and \( \eta^M(\bar{a}) < s \). This is true if and only if there exists \( s \in \mathbb{Q} \) such that \( \theta_i^\psi(\bar{a}) \wedge \theta_i^\eta(\bar{a}) \). Therefore \( \psi \) is determined.

Assume \( \psi \) is determined, \( x \) is a variable, and \( \varphi = \sup_x \psi \). Let \( m(\varphi) = m(\psi), \zeta_i^\psi = \sup_x \zeta_i^\psi, \) and \( \theta_i^\psi \) for all \( t \). With \( (M_j), \mathcal{J}, M, \) and \( \bar{a} \) as before we have \( \varphi(\bar{a})^M > t \) if and only if there exists \( b \in M \) with representing sequence \( (b_j) \) such that \( \psi(\bar{a}, b)^M > t \). This is equivalent to \( \theta_i^\psi(\bar{a}, b) \), which is equivalent to \( \theta_i^{\psi|\bar{a}, b} \). Therefore \( \sup_x \psi \) is determined.

If \( \psi \) is determined and \( \varphi = \inf_x \psi \), the proof is analogous to the case when \( \varphi = \sup_x \psi \). This completes the proof of \( \ref{thm:determined} \).

\( \square \) The proof that every \( \mathcal{L} \)-formula \( \varphi(\bar{x}) \) is \( 1/k \)-determined for all \( k \geq 1 \) is analogous to the proof of \( \ref{thm:determined} \). The only difference is in the following.
Claim A.4. If $\psi$ and $\eta$ are $1/(2k)$-determined then $\psi - \eta$ is $1/k$-determined.

Proof. Writing $\varphi = \psi - \eta$, let $m(\varphi) = m(\psi) + m(\eta)$, $\zeta^\varphi_i = \zeta^\psi_i$ for $i < m(\psi)$, $\zeta^\varphi_{m(\psi) + i} = \zeta^\psi_i$ for $i < m(\eta)$, and $\theta^\varphi_{i,k} = \bigvee_{t=0}^{2k}(\theta^\psi_{t+i/2k} \land -\theta^\eta_{t+i/2k})$. With $(M_j), \mathcal{J}, M$, and $\bar{a}$ as before, if $\varphi^M(\bar{a}) > t - (1/k)$ then there is $j < 2k$ such that $\psi^M(\bar{a}) > s - j/(2k)$ and $\eta^M(\bar{a}) < j/(2k)$. Therefore $\theta^\psi_{s+j/(2k)} \land -\theta^\eta_{j/(2k)}$, which implies $\theta^\varphi_{i,k}$. By an analogous proof, $\theta^\varphi_{i,k}$ implies $\varphi^M(\bar{a}) < t+(1/k)$. ⬜

The claim shows that if all subformulas of $\varphi$ are $1/m$-determined for all $k$ then $\varphi$ is $1/k$-determined for all $k$. The proof now proceeds by recursion on the complexity of $\varphi$, for all $k$ simultaneously. ⬜

Appendix B. Countable saturation

In this appendix we give a short proof of a generalization [10, Theorem 1.1], using Theorem A.3. The following definition of layered ideals on an arbitrary infinite set $\mathbb{J}$ generalizes the definition of layered ideals on $\mathbb{N}$ given in [10, §2.5].

Definition B.1. An ideal $\mathcal{I}$ on an infinite set $\mathbb{J}$ is layered if $\mathbb{J} = \bigcup_{i \in \mathbb{N}} X_i$ for $X_i \in \mathcal{J}$, for $i \in \mathbb{N}$, and there is a function $\mu : \mathcal{P}(\mathbb{J}) \rightarrow [0, \infty]$ such that

1. $\mu$ is monotonic: If $X \subseteq Y$ then $\mu(X) \leq \mu(Y)$,
2. $\mathcal{I} = \{X : \mu(X) < \infty\}$, and
3. if $\mu(X) = \infty$ and $Y_n \in \mathcal{J}$, for $n \in \mathbb{N}$, satisfy $Y_n \subseteq Y_{n+1}$ for all $n$ and $X = \bigcup_n Y_n$, then $\sup_n \mu(Y_n) = \infty$.

The Fréchet ideal on $\mathbb{N}$ is layered: Take any partition of $\mathbb{N}$ into finite sets, and let $\mu(X)$ be the cardinality of $X$.

Theorem B.2. If $\mathcal{L}$ is a language of metric structures, $(M_j)_{j \in \mathbb{J}}$ is a family of $\mathcal{L}$-structures, and $\mathcal{J}$ is a layered ideal on $\mathbb{J}$ that contains the ideal of finite sets, then the quotient $M = \prod_j M_j / \bigoplus_{\mathcal{J}} M_j$ is countably saturated.

Proof. Fix a countable consistent type $t(\bar{x})$ over $M$, and enumerate it as a sequence of conditions $\varphi_i(\bar{x}) = s_i$, for $i \in \mathbb{N}$. We may assume that the range of each $\varphi_i$ is included in $[0, 1]$. Each $\varphi_i$ may have parameters in $M$, but we can expand the language by constants for these parameters. Applying Theorem A.3 to each $\varphi_i(\bar{x})$ and $k \geq i$, we obtain $m(i) \in \mathbb{N}$, formulas $\zeta^k_j(\bar{x})$, for $j < m(i)$, and Boolean formulas $\theta^i_{j,k}$, for $0 \leq l \leq k$, such that for every $\bar{a}$ in $M$ of the appropriate sort, the value of $\varphi^M_i(\bar{a})$ is $1/k$-determined by $\theta^i_{j,k}[\bar{a}]$, for $0 \leq l \leq k$. The value of this Boolean formula is determined by the Boolean algebra generated by the sets $Z^i_{j,l/k}(\bar{a}) = \{n : \zeta^k_j(\bar{a}_n)^M \geq l/k\}_{\mathcal{J}}$.

Since these sets generate a finite Boolean algebra, the value of any quantifier-free Boolean formula, including $\theta^i_{j,k}[\bar{a}]$, is uniquely determined by the following finite set (let $m(k) = \max_{i \leq k} m(i)$ and declare hitherto undefined.
sets to be empty; we are using the convention $n = \{0, \ldots, n-1\}$:

$$
\Upsilon_k(\bar{a}) = \{ S \subseteq k^2 \times m(k) : [\bigcap_{(i,j) \in S} Z^i_{j,k}(\bar{a})]_J = 0 \}.
$$

Since the type $t(\bar{x})$ is consistent, for every $k$ there exists $\bar{b}(k)$ in $M$ such that $\max_{i < k} |\varphi_i(\bar{b}(k)) - s_i| < 1/k$. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. For a fixed $k$ there is $\Upsilon^*_k \subseteq \mathcal{P}(k^2 \times m(k))$ such that $\{ n \in \mathbb{N} : \Upsilon_k(\bar{b}(n)) = \Upsilon^*_k \} \in \mathcal{U}$. For every $k$ choose $n(k)$ such that $\Upsilon_k(\bar{b}(n(k))) = \Upsilon^*_k$. Then

1. $\Upsilon^*_k = \{ S \cap (k^2 \times m(k)) : S \in \Upsilon^*_k \}$ for all $k < k'$,

because each one of these sets is equal to $\Upsilon^*_k(\bar{b}(n(k')))$. To produce $\bar{a} \in M$ which satisfies $t(\bar{x})$, it suffices to assure $\Upsilon_k(\bar{a}) = \Upsilon^*_k$ for all $k$. Writing

$$
\Phi_{S,k}(\bar{a}) = \bigcap_{(i,j) \in S} Z^i_{j,k}(\bar{a})
$$

this condition is equivalent to requiring $[\Phi_{S,k}(\bar{a})]_J = 0$ if and only if $S \in \Upsilon^*_k$, for every $S \subseteq k^2 \times m(k)$.

Fix a partition $\mathcal{J} = \bigsqcup_{i \in \mathcal{N}} X_i$ and $\mu : \mathcal{P}(\mathcal{J}) \to [0,\infty]$ as in Definition B.1. We will choose sets $\Upsilon_k \in \mathcal{J}$, for $2 \leq k$, which satisfy the following for all $k$:

2. $\bigcup_{j \leq k} \Upsilon_j \supseteq \bigcup_{j \leq k} X_j$.

3. $\mu(\Phi_{S,k}(\bar{b}(n(k))) \cap \Upsilon_k) \geq n$ for all $S \subseteq k^2 \times m(k)$ such that $S \in \Upsilon^*_k$, and

4. $\Phi_{S,k}(\bar{b}(n(k+1))) \subseteq \bigcup_{j \leq k} \Upsilon_j$ for all $S \subseteq k^2 \times m(k)$ such that $S \notin \Upsilon^*_k$.

We proceed to describe the recursive construction of the sequence $(\Upsilon_k)$. For $i \leq 2$ and $S \in \Upsilon^*_2$ we have $\Phi_{S,k}(\bar{b}(n(2))) \notin \mathcal{J}$, and by (3) of Definition B.1 we have $\mu(\Phi_{S,k}(\bar{b}(n(2)) \cap \bigcup_{j \leq m(S)} X_j) \geq 2$ for a large enough $m(S)$. With $m = \max\{m(S) : S \in \Upsilon^*_2 \}$ let

$$
\Upsilon_2 = \bigcup_{j \leq m} X_j \cup \{ \Phi_{S,3}(\bar{b}(n(3))) : S \subseteq 3^2 \times m(3), S \notin \Upsilon^*_3 \}.
$$

Then $\Upsilon_2$ satisfies (2), (3), and (4).

Suppose that the sets $\Upsilon_2, \ldots, \Upsilon_k$ have been chosen to satisfy (2), (3), and (4). We have $\Phi_{S,k+1}(\bar{b}(n(k+1))) \notin \mathcal{J}$ for every $S \in \Upsilon^*_k$, and therefore a large enough $m$ satisfies $\mu(\Phi_{S,k+1}(\bar{b}(n(k+1))) \cap (\bigcup_{j \leq m} X_j \setminus \bigcup_{j \leq k} \Upsilon_j)) \geq k+1$ for every $S \in \Upsilon^*_k$. Let

$$
\Upsilon_{k+1} = \bigcup_{j \leq m} X_j \setminus \bigcup_{j \leq k} \Upsilon_j
$$

$$
\cup \{ \Phi_{S,k+2}(\bar{b}(n(k+2))) : S \subseteq (k+2)^2 \times m(k+2), S \notin \Upsilon^*_{k+2} \}.
$$

Then the sets $\Upsilon_2, \ldots, \Upsilon_{k+1}$ satisfy (2), (3), and (4) with $k$ replaced by $k+1$.

This describes the recursive construction. The sets $\Upsilon_k$, for $k \in \mathbb{N}$, partition $\mathcal{J}$ into sets that belong to $\mathcal{J}$.

Define a representing sequence of $\bar{a} \in M$ by $\bar{a}_j = \bar{b}(n(k))$, if $j \in \Upsilon_k$. To prove that $\bar{a}$ realizes $t(\bar{x})$, it suffices to prove $\Upsilon_k(\bar{a}) = \Upsilon^*_k$ for all $k$. Fix $k$ and $S \subseteq k^2 \times m(k)$. If $S \notin \Upsilon^*_k$, then (3) implies that $\mu(\Phi_{S,k}(\bar{a})) \geq j$ for all $j \geq k$. Therefore $\mu(\Phi_{S,k}(\bar{a})) = \infty$, $S \notin \mathcal{J}$, and $S \in \Upsilon(\bar{a})$. Now suppose $S \notin \Upsilon^*_k$. Then $S \notin \Upsilon_j$ for all $j$. Fix any $l \geq k$. Then (4) and (1) together
imply $\Phi_{S,l}(\bar{a}) \cap Y_j = \emptyset$ for all $j \geq l + 1$. Hence $\Phi_{S,l}(\bar{a})$ is included in a set in $\mathcal{J}$, and since $l \geq k$ was arbitrary, $S \notin \mathcal{Y}_k(\bar{a})$. □

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