The Yangian Symmetry of the Hubbard Models with Variable Range Hopping

Frank Göhmann† and Vladimir Inozemtsev‡

†Department of Physics, Faculty of Science, University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113, Japan
‡ISSP, University of Tokyo, 7-2-21 Roppongi, Minato-ku, Tokyo 106, Japan

Abstract

We present two pairs of \( Y(sl_2) \) Yangian symmetries for the trigonometric and hyperbolic versions of the Hubbard model with non-nearest-neighbour hopping. In both cases the Yangians are mutually commuting, hence can be combined into a \( Y(sl_2) \oplus Y(sl_2) \) Yangian. Their mutual commutativity is of dynamical origin. The known Yangians of the Haldane-Shastry spin chain and the nearest neighbour Hubbard model are contained as limiting cases of our new representations.

Yangian quantum groups were introduced by Drinfel’d more than ten years ago. His original intention was to put the algebraic structure underlying the Yang-Baxter exchange relations with rational \( R \)-matrix, responsible for the integrability of the most prominent integrable systems, into the mathematically more conventional context of Hopf algebras. Yangian quantum groups and their representation theory are thus intimately connected with the classification of integrable quantum systems.

* e-mail: frank@monet.phys.s.u-tokyo.ac.jp
However, during the past few years it became apparent that Yangians can also play a physically very interesting role as additional symmetries of integrable systems, and moreover, that Yangians are part of the symmetry algebra of such well studied integrable systems as the nearest neighbour Heisenberg model [2], or the nearest neighbour Hubbard model [3], if considered on infinite lattices. These symmetries have been overlooked for many years, since it was unusual to deal directly with infinite systems. Instead all conventional approaches to integrable systems like the various Bethe Ansätze start from finite systems, usually under periodic boundary conditions, and the thermodynamic limit is only performed at a later stage of calculation. The Yangian symmetries of the nearest neighbour Hubbard and Heisenberg models are incompatible with periodic boundary conditions. For this reason they do not combine with Bethe Ansatz methods.

Now there are two recent developments that make a utilisation of Yangians for integrable systems feasible. First, methods have been developed to deal directly with infinite systems. Within this so-called symmetry based approach it became possible to calculate for instance higher order spin correlators for the XXZ-chain [4]. Quantum groups play an essential role here. Second, interesting integrable systems with a finite number of degrees of freedom have been discovered which exhibit Yangian symmetry compatible with periodic boundary conditions, most prominent among these the Haldane-Shastry spin chain [5, 6]. Until the discovery of its Yangian symmetry [7] the high degeneracy of its spectrum remained a puzzle, which is now resolved by Yangian representation theory [8].

Below we present a new pair of mutually commuting representations of the $Y(sl_2)$ Yangian in terms of Fermi operators, which form a Yangian symmetry of the Hubbard model with non-nearest-neighbour hopping of Gebhard and Ruckenstein [9, 10]. The model itself contains the usual Hubbard model as well as the Haldane-Shastry spin chain as certain limiting cases, and so do the generators of its Yangian symmetry.

The model describes itinerant electrons of spin $\sigma$ created by $c^+_j \sigma$ at site $j$ of a one dimensional lattice. The probability amplitude for hopping between sites $j, k$ will be denoted by $t_{jk}$. Two electrons of different spin encountering each other on the same
lattice site feel an on-site repulsion $U > 0$.

$$H = \sum_{j,k} t_{jk} c^+_j \sigma c_k \sigma + 2U \sum_j (c^+_j c_j - \frac{1}{2}) (c^+_j c_{j+\sigma} - \frac{1}{2}) .$$  

(1)

The constants here have been chosen for later convenience, and $t_{jj} := 0$ by definition. Throughout this letter we are using sum convention with respect to Greek indices. We will consider two different choices of translational invariant hopping amplitudes, $t_{jk} = t_{j-k}$. For $n \neq 0$ let

$$t_n := -i \text{sh} \kappa \text{sh}^{-1} \kappa n , \quad n \neq 0 .$$  

(2)

Then our choices are given by $\kappa = i \pi/N$ for a finite lattice of $N$ sites, and $\kappa > 0$ for an infinite lattice. The energy scale has been chosen such as to give hopping amplitudes of absolute value 1 between neighbouring sites. The summation indices run from 0 to $N - 1$ in the trigonometric case, and over all integers in the hyperbolic case. The thermodynamic limit of the trigonometric model and the limit $\kappa \to 0$ of the hyperbolic model coincide. In both cases $t_{jk}$ turns into $-i/(j-k)$. This is the true $1/r$ Hubbard model.

In the limit $\kappa \to \infty$, the hyperbolic model turns, up to a canonical transformation, which is described below, into the nearest neighbour Hubbard model.

In order to understand the physical meaning of the above kind of hopping amplitudes, one has to consider the dispersion relation of the free model ($U = 0$) \[3, 10\]. In the trigonometric case we obtain

$$\varepsilon(p) = \sum_{n=1}^{N-1} t_n e^{ipn} = \frac{N}{\pi} \sin \left( \frac{p}{N} \right) (\pi - p) ,$$  

(3)

where $p = 2\pi(m + 1/2)/N, \ m = 0, \ldots, N - 1$. In the thermodynamic limit this yields $\varepsilon(p) \to \pi - p$. The dispersion relation \[3\] is linear in the first Brillouin zone, the model is chiral. It contains only left moving particles. The physically most interesting point about this chiral model is the appearance of a Mott transition at finite $U > 0$ \[3, 11\]. In

---

*It may be interesting to notice, that the trigonometric and hyperbolic hopping amplitudes above can be interpreted as $q$-deformed $1/r$-hopping. The notion of $q$-deformation is defined by $r_q := (q^r - q^{-r})/(q - q^{-1})$. Setting $q$ equal to $e^\kappa$ the hopping amplitudes become $t_{jk} = -i/(j-k)$. The trigonometric case corresponds to $q$ being the $N$-th root of unity, the hyperbolic case to $q > 1$.\]
the hyperbolic case the dispersion relation is

\[ \epsilon(p) = \sum_{n=1}^{\infty} t_n e^{i\rho n} = 2 \text{sh}(\kappa) \sum_{n=1}^{\infty} \frac{\sin(pn)}{\text{sh}(\kappa n)}. \]

(4)

The last expression is easily recognised as being, up to a redefinition of scales, the logarithmic derivative of the Jacobi theta function \( \vartheta_4 \). As a function of \( \kappa \) it interpolates between the sinusoidal dispersion relation of the of the nearest neighbour model and the saw tooth shaped dispersion relation of the \( 1/r \) model.

The local U(1) transformation \( c_{j\sigma} \rightarrow e^{i\phi_j} c_{j\sigma}, \phi_j \) real, does not alter the canonical anticommutation relations between the Fermi operators. The local electron densities \( c_{j\sigma}^\dagger c_{j\sigma} \) are invariant under this transformation, hence the interaction part of the Hamiltonian (1) is as well. This means that we can always use a U(1) transformation to modify the hopping term to our convenience. The modified model will be completely equivalent to the original one. Consider the case \( \phi_j = j\pi \). This transformation introduces a factor of \((-1)^{j+k}\) into the expression for the hopping amplitudes and shifts the dispersion relations by a half period. Using this transformation our conventions meet the conventions of Gebhard and Ruckenstein [9]. To recover the nearest neighbour Hubbard model in its familiar form, we do not only have to consider \( \kappa \rightarrow \infty \), but in addition the above transformation with \( \phi_j = j\pi/2 \). This transformation removes the factor of “i” in front of the hopping amplitude, changes the hopping amplitude to an even function, and shifts the dispersion relation by a quarter period. Hence the quadratic bottom of the sinusoidal band is shifted to \( p = 0 \).

There is yet another important canonical transformation, namely

\[ c_{j\downarrow} \rightarrow c_{j\downarrow}, \quad c_{j\uparrow} \rightarrow c_{j\uparrow}^\dagger, \quad U \rightarrow -U. \]

(5)

This transformation leaves every Hamiltonian of the form (1) with antisymmetric hopping matrix invariant, but the global spin operators and, in our case, the Yangian generators (see below) are not. It is responsible for the doubling of the Yangian.

The natural language for writing down the \( sl_2 \) generators of the rotational symmetry of the Hamiltonian (1) is, of course, in terms of spin operators, which are linear combinations of products of one creation and one annihilation operator at the same site. For
the formulation of our Yangian generators below it turns out to be useful to extend this concept to spin-like operators with indices corresponding to different sites. We arrange the pair of operators $c_{j\sigma}^+ c_{k\tau}$ in a $2 \times 2$-matrix labeled by spin indices $\sigma$, $\tau$ in the usual tensor product convention, $(S_{jk})^\sigma := c_{j\sigma}^+ c_{k\tau}$, and then set

$$
S^\alpha_{jk} := \text{tr}(\bar{\sigma}^\alpha S_{jk}), \quad S^0_{jk} := \text{tr}(S_{jk}), \quad S^\alpha_j := S^\alpha_{jj}, \quad S^0_j := S^0_{jj},
$$

(6)

where the $\sigma^\alpha$ are the Pauli matrices, and the bar denotes complex conjugation. Our definition implies $(S^\alpha_{jk})^+ = S^\alpha_{kj}$, $(S^0_{jk})^+ = S^0_{kj}$. $\frac{1}{2} S^0_j$ and $S^0_j$ are the spin density and electron density operators, respectively. The algebra of the operators $S^\alpha_{jk}$ is rather rich.

We obtain the commutators

$$
[S^0_{jk}, S^0_{lm}] = \delta_{kl} S^0_{jm} - \delta_{mj} S^0_{lk},
$$

(7)

$$
[S^0_{jk}, S^\alpha_{lm}] = \delta_{kl} S^\alpha_{jm} - \delta_{mj} S^\alpha_{lk},
$$

(8)

$$
[S^\alpha_{jk}, S^\beta_{lm}] = \delta^{\alpha\beta} \left( \delta_{kl} S^0_{jm} - \delta_{mj} S^0_{lk} \right) + i \epsilon^{\alpha\beta\gamma} \left( \delta_{kl} S^\gamma_{jm} + \delta_{mj} S^\gamma_{lk} \right).
$$

(9)

However, there are other relations. For the construction of our Yangian generator and the verification of the Yangian Serre relations below, we further need the following,

$$
S^\alpha_{jk} S^\alpha_{lm} + S^\alpha_{jk} S^\gamma_{lm} + 2 S^0_{jm} S^0_{lk} = 4 \delta_{kl} S^0_{jm} + 2 \delta_{lm} S^0_{jk},
$$

(10)

$$
S^0_{jk} S^\alpha_{lm} + S^0_{lm} S^\alpha_{jk} + S^\alpha_{jm} S^\alpha_{lk} + S^\alpha_{jm} S^\gamma_{lk} = \delta_{jk} S^\alpha_{lm} + \delta_{lm} S^\alpha_{jk} + \delta_{lk} S^\alpha_{jm} + \delta_{jm} S^\alpha_{lk},
$$

(11)

$$
S^\alpha_{jk} S^\beta_{lm} + S^\beta_{jk} S^\alpha_{lm} + S^\alpha_{jm} S^\beta_{lk} + S^\beta_{jm} S^\alpha_{lk} = \delta^{\alpha\beta} \left( S^0_{jm} (2 \delta_{lk} - S^0_{lk}) + S^\gamma_{jm} S^\gamma_{lk} \right),
$$

(12)

$$
- i \epsilon^{\alpha\beta\gamma} S^\beta_{jk} S^\gamma_{lm} - S^0_{jm} S^\alpha_{lk} + S^0_{jm} S^\alpha_{lk} = 2 \delta_{lk} S^\alpha_{jm} + \delta_{jk} S^\alpha_{lm} - \delta_{lm} S^\alpha_{jk}.
$$

(13)

These relations generate a long list of successively less general relations by systematically equating all possible combinations of site indices. Setting $j = k$ and $l = m$ in eq. (13), for example, implies that the operators $\frac{1}{2} S^\alpha_j$ are spin density operators,

$$
[S^\alpha_j, S^\beta_k] = \delta_{jk} 2 i \epsilon^{\alpha\beta\gamma} S^\gamma_j.
$$

(14)

The Hamiltonian (11) now assumes the following form

$$
H = \sum_{j,k} t_{jk} S^0_{jk} + U \sum_j \left( (S^0_j - 1)^2 - \frac{1}{2} \right).
$$

(15)
Since the particle number $I^0 = \sum_j S_j^0$ is conserved, only the term $(S_j^0)^2$ is relevant in the interaction part of the Hamiltonian. The other terms can be removed by a shift of the chemical potential. We retained them here to make obvious the invariance of $H$ under the transformation (5). The operators of the total spin are $I^\alpha := \frac{1}{2} \sum_j S_j^\alpha$. It follows from (7) that they commute with the Hamiltonian. Their $sl_2$ commutation relations are obtained by summing (14) over $j$ and $k$,

$$[I^\alpha, I^\beta] = i \varepsilon^{\alpha\beta\gamma} I^\gamma .$$  (16)

Now everything is prepared to formulate our main result. Consider the Hamiltonian (15) with yet unspecified antisymmetric hopping matrix, $t_{jk} = -t_{kj}$. Let

$$J^\alpha := \frac{1}{2} \sum_{j,k} \left( (f_{jk} + h_{jk}(S_j^0 + S_k^0 - 2))S_j^{\alpha} + g_{jk} \varepsilon^{\beta\gamma\alpha} S_j^\beta S_j^\gamma \right) ,$$  (17)

where $g_{jk}$ and $h_{jk}$ are odd functions, and $f_{jj} = g_{jj} = h_{jj} = 0$ by convention. Then $H$ commutes with $J^\alpha$ if and only if the following functional equations between the coefficients are satisfied,

$$ (g_{jl} - g_{kl})h_{jk} = \frac{1}{2} h_{jl} h_{kl} , \quad j \neq k \neq l \neq j ,$$  (18)

$$i U f_{jk} / 2 h_0 + g_{jk} h_{jk} = -\frac{1}{4} \sum_l h_{jl} h_{kl} , \quad j \neq k ,$$  (19)

$$\sum_l (f_{jl} h_{kl} - f_{kl} h_{jl}) = 0 ,$$  (20)

$$t_{jk} = h_0 h_{jk} .$$  (21)

Here $h_0$ is a free parameter which fixes the scale for $J^\alpha$. The only solutions to these equations correspond to the cases of trigonometric and hyperbolic hopping amplitudes (4) under consideration. In the trigonometric case we find

$$f_{jk} = 0 , \quad g_{jk} = \frac{1}{2} \cotg(\pi(j - k)/N) , \quad h_{jk} = i \sin^{-1}(\pi(j - k)/N) ,$$  (22)

whereas in the hyperbolic case

$$f_{jk} = \frac{\sinh(\kappa(j - k))}{U \sinh(\kappa(j - k))} , \quad g_{jk} = \frac{1}{2} \coth(\kappa(j - k)) , \quad h_{jk} = i \sinh^{-1}(\kappa(j - k)) .$$  (23)
h_0 has to be real in order for J^\alpha to be selfadjoint. We choose h_0 = -\sin(\pi/N) in the trigonometric case and h_0 = -\text{sh}(\kappa) in the hyperbolic case. It is an unexpected fact that J^\alpha does not depend on U in the trigonometric case. Hopping part and interaction part of the Hamiltonian commute separately with J^\alpha.

One easily checks that our conserved operator J^\alpha turns into known generators of Yangians in various limiting cases. In the nearest neighbour Hubbard limit of the hyperbolic model (\kappa \to \infty) we obtain f_{jk} \to \delta_{j-k}\{-1/U, g_{jk} \to \text{sign}(j-k)/2, h_{jk} \to 0. After a canonical transformation c_{j\sigma} \to i^\lambda c_{j\sigma} we recover the Yangian generator of Uglov and Korepin [3]. In the limit U \to \infty at less than half filling the model reduces to the “t-0” chain [12] with all states with double occupancies projected out from the Hilbert space. Because at half filling hopping is not allowed anymore in this limit, one can set S^0_j = 1, and recover the Yangian generator of the Haldane-Shastry chain [7] or its hyperbolic counterpart to leading order in (t/U)^2.

Indeed J^\alpha itself generates a representation of a Y(sl_2) Yangian. This is our second result. The spin operators I^\alpha and the conserved quantities J^\alpha satisfy the relations

\[
[I^\lambda, J^\mu] = c_{\lambda\mu\nu}J^\nu,
\]

\[
[[J^\lambda, J^\mu], [I^\rho, J^\sigma]] + [[J^\rho, J^\sigma], [I^\lambda, J^\mu]] = -4\delta(a_{\lambda\mu\nu\alpha\beta\gamma}c_{\rho\sigma\nu} + a_{\rho\sigma\nu\alpha\beta\gamma}c_{\lambda\mu\nu})\{I^\alpha, I^\beta, J^\gamma\},
\]

where \delta = -1 in the trigonometric case, \delta = 1 in the hyperbolic case, and the further abbreviations

\[
c_{\lambda\mu\nu} := i\varepsilon^{\lambda\mu\nu}
\]

\[
a_{\lambda\mu\nu\alpha\beta\gamma} := c_{\lambda\alpha\rho}c_{\mu\beta\sigma}c_{\nu\gamma\tau}c_{\rho\sigma\tau},
\]

\[
\{x_1, x_2, x_3\} := \frac{1}{6} \sum_{i \neq j \neq k \neq i} x_ix_jx_k
\]

have been used. Eqs. (24) and (25) together with the defining relation (16) of the sl_2 Lie algebra are Drinfel’d’s definition of the Y(sl_2) Yangian [1]. Equation (24) says that the J^\alpha transform like a vector representation of sl_2, and is easily confirmed for our J^\alpha. Since both equations (24) and (25) are homogeneous, we could have introduced a deformation parameter h^2 on the right hand side of the Yangian Serre relation (25).
Since this parameter merely fixes the scale of \( J^\alpha \) and has no deeper physical meaning, we suppressed it here. We have confirmed (24) by direct calculation. The calculation is lengthy. Before we comment on it we formulate our third result.

Under the transformation (5) the generators \( I^\alpha, J^\beta \) transform into an independent set of generators \( I'^\alpha, J'^\beta \) of another representation of the \( Y(sl_2) \) Yangian. The two representations commute, hence can be combined to a \( Y(sl_2) \oplus Y(sl_2) \) double Yangian. Their commutativity is non trivial and is of dynamical origin, i.e. it relies on the functional equations (18) - (21) between the coefficients that define \( J^\alpha, J'^\beta \).

To check the Yangian Serre relation, the original formulation (25) is rather inappropriate. We used the following simplification in the \( sl_2 \) case instead. Let

\[
K^\alpha := -i\varepsilon^{\alpha\beta\gamma}[J^\beta, J^\gamma] - 4\delta(I^\beta)^2 I^\alpha . \tag{29}
\]

Then a short but slightly tricky calculation shows that (25) is equivalent to the equation

\[
[J^\alpha, K^\beta] + [J^\beta, K^\alpha] = 0 . \tag{30}
\]

The left-hand side of (30) has a property that turns out to be very useful in practical calculations. It is traceless. Assume we are given an operator \( J^\alpha \), and we do already know that it transforms as a vector representation of \( sl_2 \). Then this knowledge assures the identity \([J^\alpha, K^\alpha] = 0 \). It is therefore sufficient to show that the left-hand side of equation (30) is proportional to \( \delta^{\alpha\beta} \). This is a severe simplification, since the symmetrisation of the commutator produces a lot of terms proportional to \( \delta^{\alpha\beta} \), which can be neglected according to the above argument. The explicit expression for \( K^\alpha \) in our case is

\[
K^\alpha = \frac{1}{2}\sum_{j,k,l}\left\{(8g_{jk}g_{jl} - \delta)S_j^\alpha S_k^\beta S_l^\gamma + 2A_{jl}A_{lk}S_j^\alpha - 4iA_{jk}(g_{jl} - g_{kl})S_j^\alpha S_{jkl}^\gamma
\right.
\]

\[
+2A_{jk}(g_{jl} + g_{kl})\varepsilon^{\alpha\beta\gamma}S_j^\beta S_l^\gamma + ih_{jl}\varepsilon^{\alpha\beta\gamma}(A_{jk}S_j^\beta - A_{kj}S_k^\beta)(S_j^\gamma - S_k^\gamma)\right\} , \tag{31}
\]

where \( A_{jk} := f_{jk} + h_{jk}(S_j^0 + S_k^0 - 2) \). To verify (31), we used the following relations among the coefficients \( f_{jk}, g_{jk}, h_{jk} \) in addition to their defining functional equations above.

\[
f_{jk}(g_{jl} - g_{kl}) = \frac{1}{2}(f_{jl}h_{kl} - f_{kl}h_{jl}) , \quad j \neq k \neq l \neq j \tag{32},
\]

\[
g_{jk}g_{jl} + g_{kl}g_{kj} + g_{ij}g_{ik} = \delta/4 \quad , \quad j \neq k \neq l \neq j \quad , \tag{33}
\]

\[
4g_{jk}^2 + h_{jk}^2 = \delta \quad , \quad j \neq k \quad . \tag{34}
\]
The homogeneity of the lattices has not been used in the verification of the Yangian Serre relation in the bulk. However, it is necessary to guarantee the commutativity of $J^\alpha$ with the Hamiltonian. This situation is similar to the case of the Yangian symmetric spin chains. Therefore we conjecture the existence of a Yangian symmetric long range Hubbard Hamiltonian on an inhomogeneous lattice. In analogy to the spin chain case the generator of its Yangian symmetry might be constructed by adding “potential terms” to the second order Yangian generator $K^\alpha$, eq. (29).

At this point we would like to emphasize that Yangian symmetry does not imply integrability. Nevertheless, we strongly believe that the models considered here are integrable, and are special cases of a more general integrable non Yangian symmetric model with elliptic hopping amplitudes. The proof of integrability would provide the basis for an understanding of the Haldane-Shastry chain and the nearest neighbour Hubbard model on a common ground. At the present state of knowledge these models appear rather unlike. The integrability of the Haldane-Shastry chain has been shown by exploiting a mapping to a related dynamical model, whereas the integrability of the nearest neighbour Hubbard model follows from its connection to an integrable system of two coupled six-vertex models. We expect that a proof of the integrability of the non-nearest-neighbour Hubbard models will reveal a more generic structure.

A first application of our new $Y(sl_2)$ Yangian will be the classification of the Jastrow-like eigenfunctions of the “$t$-0” model. For the system at finite on-site energy the situation is more complicated. Not even the ground state wave function is known. There is evidence that the wave functions are neither of Jastrow-type as for the Haldane-Shastry chain nor of Bethe Ansatz form as in case of the nearest neighbour Hubbard model.

Acknowledgments. This work has been supported by the Japan Society for the Promotion of Science and the Ministry of Science, Culture and Education of Japan. One of the authors (V. I.) would like to express his sincere gratitude to Professor Minoru Takahashi for useful discussions and kind hospitality extended to him at the ISSP. The other one (F. G.) likes to thank Professor Miki Wadati for his hospitality and the warm atmosphere in his group at Tokyo University.
References

[1] V. G. Drinfel’d, Soviet Math. Dokl. 32, 254 (1985).

[2] D. Bernard, Int. J. Mod. Phys. B 7, 3517 (1993).

[3] D. B. Uglov and V. E. Korepin, Phys. Lett. A 190, 238 (1994).

[4] M. Jimbo, K. Miki, T. Miwa, and A. Nakayashiki, Phys. Lett. A 168, 256 (1992).

[5] F. D. M. Haldane, Phys. Rev. Lett. 60, 635 (1988).

[6] B. S. Shastry, Phys. Rev. Lett. 60, 639 (1988).

[7] F. D. M. Haldane, Z. N. C. Ha, J. C. Talstra, D. Bernard, and V. Pasquier, Phys. Rev. Lett. 69, 2021 (1992).

[8] D. Bernard, M. Gaudin, F. D. M. Haldane, and V. Pasquier, J. Phys. A 26, 5219 (1993).

[9] F. Gebhard and A. E. Ruckenstein, Phys. Rev. Lett. 68, 244 (1992).

[10] P.-A. Bares and F. Gebhard, J. Low. Temp. Phys. 99, 565 (1995).

[11] F. Gebhard, A. Girndt, and A. E. Ruckenstein, Phys. Rev. B 49, 10926 (1994).

[12] D. F. Wang, Q. F. Zhong, and P. Coleman, Phys. Rev. B 48, 8476 (1993).

[13] K. Hikami, Nucl. Phys. B 441, 530 (1995).

[14] J. C. Talstra and F. D. M. Haldane. J. Phys. A 28, 2369 (1995).

[15] B. S. Shastry, J. Stat. Phys. 50, 57 (1988).