Quantum/Classical Interface: Fermion Spin

W. E. Baylis, R. Cabrera, and D. Keselica
Department of Physics, University of Windsor

Although intrinsic spin is usually viewed as a purely quantum property with no classical analog, we present evidence here that fermion spin has a classical origin rooted in the geometry of three-dimensional physical space. Our approach to the quantum/classical interface is based on a formulation of relativistic classical mechanics that uses spinors. Spinors and projectors arise naturally in the Clifford’s geometric algebra of physical space and not only provide powerful tools for solving problems in classical electrodynamics, but also reproduce a number of quantum results. In particular, many properties of elementary fermions, as spin-1/2 particles, are obtained classically and relate spin, the associated $g$-factor, its coupling to an external magnetic field, and its magnetic moment to Zitterbewegung and de Broglie waves. The relationship is further strengthened by the fact that physical space and its geometric algebra can be derived from fermion annihilation and creation operators. The approach resolves Pauli’s argument against treating time as an operator by recognizing phase factors as projected rotation operators.

I. INTRODUCTION

The intrinsic spin of elementary fermions like the electron is traditionally viewed as an essentially quantum property with no classical counterpart.[1, 2] Its two-valued nature, the fact that any measurement of the spin in an arbitrary direction gives a statistical distribution of either “spin up” or “spin down” and nothing in between, together with the commutation relation between orthogonal components, is responsible for many of its “nonclassical” properties. Nevertheless, a classical spinorial approach to the relativistic dynamics of charged particles displays these and many other quantum-like properties and suggests that intrinsic spin itself arises from geometric properties of three-dimensional physical space. In particular, the two-state property, the change in sign of the spinor under a $2\pi$ rotation, the $g$-factor of 2, and equations for spin distributions all arise classically. The spin itself is represented by a classical intrinsic rotation that arises as a rotational gauge freedom in the spinorial form of the Lorentz-force equation. The spin rotation is referred to as “intrinsic” because, in this approach, a free charge is not modeled by a classical structure, but rather it is defined as elementary only if it has no discernible structure other than a rest-frame direction; that direction is interpreted as the spin axis of the particle, but only the particle rotates, not its distribution. There is no known classical method of obtaining the rotation rate, but the rotation does give de Broglie waves, and measurements of the de Broglie wavelength imply rotation rates at the Zitterbewegung frequency. The main purpose of this paper is to present evidence of the classical origin of spin-1/2 fermions and their relation to the geometry of physical space that arises from the classical spinor approach to the quantum/classical (Q/C) interface.

Studies of the Q/C interface have long been of interest, both for shedding light on quantum processes, and also for providing insight into the unification of quantum theory with classical relativity. Spin-1/2 systems are basic qubits, and recent work in quantum computation and communication[3], together with the emerging field of spintronics[4], has focused attention on the dynamical control of both individual spins and spin currents. These fields stand to benefit from an improved understanding of spin at the Q/C interface. Traditional studies of the interface have largely concentrated on quantum systems in states of large quantum numbers and their relation to classical chaos.[5] to quantum states in decohering interactions with the environment.[6] or in continuum states, where semiclassical approximations are useful.[7] Our approach[8-11] is fundamentally different. We start with a description of classical relativistic dynamics using Clifford’s geometric algebra of physical space (APS). An important tool in the algebra is the amplitude of the Lorentz transformation that boosts the system from its rest frame to the lab. (In this paper by “rest frame”, we mean the inertial frame that is instantaneously commoving with the particle.) This amplitude enters as a spinor in a classical context, one that satisfies linear equations of evolution admitting superposition and interference. We explore its close relation to the quantum wave function.

Although APS is the Clifford algebra generated by a three-dimensional Euclidean space, it contains a four-dimensional linear space with a Minkowski spacetime metric that allows a covariant description of relativistic phenomena. The relativistic, spinorial treatment of APS is crucial in relating the classical and quantum formalisms. A number of fermionic, spin-1/2, properties follow from the spinor description of spatial rotations, and the extension to Lorentz transformations yields immediately the momentum-space form of the Dirac equation.[12] The formulation can be extended to the differential form of the Dirac equation and the Schrödinger equation in its low-velocity limit by considering a superposition of de Broglie waves. Because of its relativistic formulation, the classical spinor description of APS promises useful insights to the Q/C interface that may be useful not only in quantum information theory, quantum computation, and spintronics, but also in the foundations of quantum theory and some formulations of
quantum gravity.

The association of spin-1/2 systems, and more generally of two-level systems, to three-dimensional space is not new. States of a spin-1/2 system are commonly represented by points on the surface of the Bloch sphere if they are pure and inside the sphere if they are mixed.\textsuperscript{\[13, 14\]} Analogously, light polarization is represented by points on or inside the Poincaré sphere.\textsuperscript{\[15\]} The three-dimensional space of these spheres is often viewed as abstract with only indirect connections to physical space, but in spin-1/2 systems, the direction of the Bloch vector is the same as the polarization vector in physical space. It is also well known\textsuperscript{\[16\]} that the group \(SU(2)\) of unimodular, unitary, \(2 \times 2\) matrices, whose representation is carried by Pauli spinors, is the universal two-fold cover of the rotation group \(SO(3)\) in three dimensions and is isomorphic to \(Spin(3)\).\textsuperscript{\[17\]} We argue here that these associations are more than mathematical coincidence. The evidence suggests that fermionic spin-1/2 properties are inherent in the geometry of three-dimensional space. The paper resurrects the old idea of Kronig, Uhlenbeck, and Goudsmit\textsuperscript{\[2\]} that electron represents a spinning charge, but now in a geometrical approach that avoids the problems with superluminal velocities inherent in naïve physical models and that automatically includes Thomas-precession effects.\textsuperscript{\[8\]}

Much of the evidence presented here in APS can also be formulated other geometric algebras. Indeed, related work has been reported formulating Dirac theory in complex quaternions\textsuperscript{\[18\]} and the spacetime algebra (STA) of Hestenes\textsuperscript{\[19, 20, 21, 22, 23, 24\]} and coworkers.\textsuperscript{\[25\]} While each has its particular advantages and drawbacks,\textsuperscript{\[26\]} they share a common (isomorphic) spinorial basis for a geometric algebra describing Lorentz transformations in spacetime. Of the three, APS is most intimately tied to the spatial vectors and their associated geometry. It is half the size of STA on the one hand, but unlike complex quaternions, is readily extended by complexification. Hestenes and Gurtler\textsuperscript{\[27\]} have formulated the Pauli theory of the electron in APS and studied spin and other local variables in the nonrelativistic limit, in which their spinor is a scaled spatial rotation with no boost factor. They found that Planck’s constant enters the theory only in connection with the magnitude of the spin and that the unit imaginary \(i\) in the theory generally represents the spin plane.

This paper thus collects and extends evidence that fermionic spin-1/2 properties arise from the geometry of physical space. It is important to pull together various strands of evidence in order to minimize the danger of ascribing physical significance to what might be mere mathematical coincidences. While some of this evidence has been reported previously, we believe that many of the results presented here are new, in particular (1) a classical calculation of the magnetic moment of an electron or other elementary fermion and its relation to Zitterbewegung, which results from (2) a new approach to the magnetic interaction of the spin and an external magnetic field, a calculation that supports the interpretation of the quantum phase as an intrinsic rotation angle, that connects such rotation to de Broglie waves, and that reveals mass as the source of energy for a spin accelerating in a magnetic-field gradient, (3) the use of the magnetic-moment calculation to resolve the old issue of a time operator and especially Pauli’s argument\textsuperscript{\[28\]} against such, (4) the derivation of quantum formulas for spin distributions from classical spinors, (5) the introduction of a conserved covariant Pauli-Lubański spin current, and (6) the generation of APS as well as other Clifford algebras from fermion annihilation and creation operators.

In the next section, we see how APS arises as the natural algebra of vectors in three-dimensional physical space, but how it also includes a four-dimensional linear space that models spacetime of special relativity. In Section III, the eigenspinor is introduced and shown to be a valuable tool for finding the relativistic dynamics of particles. Although we consider it a classical object, it is an amplitude for particle dynamics and obeys linear equations of evolution as in quantum mechanics. Its transformation properties show that it changes sign under a rotation of \(2\pi\). Section IV introduces the concept of an “elementary” particle in a classical context and shows free elementary particles must be structureless although they generally have an orientation, and if charged, they have \(g\)-factors of \(2\). Spin as a rotational gauge freedom is also discussed together with its relation to de Broglie waves and Zitterbewegung. The classical Dirac gauge is derived and shown to be identical to the momentum-space Dirac equation of quantum theory, including representations of Dirac’s gamma matrices. In Section V, classical spin distributions are shown to be identical to quantum expressions. A simple mathematical description of the Stern-Gerlach experiment in terms of projectors as filters is given to show how the measurement process for the spin gives only spin-up and spin-down states, and how the uncertainty relation for spin components arise even if the spin itself has a well-defined direction. In the concluding section, consequences of the eigenspinor approach to spin are summarized and several possible extensions are briefly mentioned, including comparisons of classical solutions of the Lorentz-force for a charge with spin to quantum solutions of the Dirac equation, and the treatment of multiple-particle systems.

II. THE ALGEBRA OF PHYSICAL SPACE (APS)

One of the simplest ways to motivate the use of Clifford’s geometric algebra is to think of vectors as square matrices. It is more common, of course, to represent vectors as column matrices. In 3-dimensional Euclidean space, the fixed
orthonormal basis vectors are commonly written,

\[
e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},
\]

(1)

and the general vector is a linear combination of these. However, the vector-space properties are equally well served by a representation of vectors as square matrices, using for example the Pauli spin matrices

\[
e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

(2)

In a square-matrix representation, vectors can not only be added and scaled, they can also be multiplied together in an associative product. The vectors together with all their products form a vector algebra. It becomes Clifford’s geometric algebra of physical space (or equivalently, generated by the orthonormal basis \(\{e_1, e_2, e_3\}\)) if we constrain the product by one axiom, namely that the product of a vector with itself is its square length:

\[
v v = v^2 = v \cdot v.
\]

(3)

In particular, \(e_2^2 = 1\). Putting \(v = u + w\), we see that \(uw + wu = 2u \cdot w\). Thus, the familiar dot product of any two vectors is the symmetric part of the algebraic product. Furthermore, perpendicular vectors anticommute.

This approach also works to generate geometric algebras for pseudo-Euclidean spaces of higher dimensions, although in some cases, such as 5-dimensional Euclidean space, we need the additional assumption that the algebra is not equivalent to one generated by a smaller number a basis vectors. It should be emphasized that there are many possible matrix representations for the algebra in addition to the standard one. What they all share is their equivalence to one generated by a smaller number a basis vectors.

For example, \(e_1 e_2 = -e_2 e_1\) is a bivector representing the plane containing \(e_1\) and \(e_2\). More importantly, it is an operator that rotates vectors in the plane by \(\pi/2\). Thus, if \(v = v_1 e_1 + v_2 e_2\), then \(v e_1 e_2 = v_2 e_2 - v_1 e_1\) is \(v\) rotated counterclockwise in the plane by a right angle. To rotate in the plane by an angle \(\theta\), we use \(v \exp (e_1 e_2 \hat{\theta}) = v (\cos \theta + e_1 e_2 \sin \theta)\), whose Euler-like expansion follows from the relation \((e_1 e_2)^2 = -1\). We say that \(e_1 e_2\) is a unit bivector that generates rotations in the \(e_1 e_2\) plane.

### A. Rotations and Duals

More general rotations of a vector \(v\) that may have components perpendicular to the plane of rotation are realized by

\[
v \rightarrow R v R^\dagger,
\]

(4)

where the rotor \(R = \exp \Theta = \cos \Theta + \hat{\Theta} \sin \Theta\) is the exponential of a bivector \(\Theta\) that gives the plane of rotation and whose magnitude \(\Theta\) equals the area swept out by a unit vector in the plane by the rotation. That area is twice the rotation angle. The dagger (\(\dagger\)) on the second \(R\) indicates a conjugation called reversion: it reverses the order of all vectors in the term For example, given any two vectors \(u, w\), \((uw)^\dagger = wu\). More generally, reversion is equivalent to Hermitian conjugation in standard representations of APS, where the basis vectors \(e_k\) are represented by Hermitian matrices. Bivectors change sign under reversion,and it follows that rotors are unitary: \(R^\dagger = R^{-1}\).

The product \(e_1 e_2 e_3\) is the volume element of physical space and is called the pseudoscalar of APS. It commutes with all elements and squares to \((e_1 e_2 e_3)^2 = -1\). It is a trivector, a multivector of grade 3, whereas bivectors have grade 2, vectors grade 1, and scalars grade 0. We can identify \(e_1 e_2 e_3\) as the unit imaginary: \(e_1 e_2 e_3 = i\). Bivectors are then imaginary vectors (pseudovectors) normal to the plane, for example \(e_1 e_2 = e_1 e_2 e_3 e_3 = i e_3\). This establishes a duality between vectors and bivectors in APS, which we can use to express rotors in terms of the axes of rotation (but only in three dimensions): \(R = \exp (e_2 e_1 \phi/2) = \exp (-i e_3 \phi/2)\). The even elements (grades 0 and 2) of APS form a subalgebra isomorphic to the quaternions.

### B. Spinors, Projectors, and Spin-1/2

Rotors give a spinor representation of rotations in a classical context. To see what this implies, note that to combine the effect of several rotations, we simply multiply the rotors. Thus, \(R = R_2 R_1\) is the rotor for a rotation given by \(R_1\)
followed by one given by $R_2$. The rotors are elements of a group called $\text{Spin}(3)$, which is isomorphic to $SU(2)$ and the universal double covering group of the orthogonal rotation group $SO(3)$: $R \in \text{Spin}(3) \simeq SU(2) \simeq SO(3) \times Z_2$.\[17\]

The orientation of a system with respect to a reference frame is given by a spinor, say the rotor $R_1$. A further rotation can be viewed as a transformation of the spinor:

$$R_1 \rightarrow R = R_2R_1.$$ This is a spinorial transformation, which has a simpler form than the general one $\[3\]$ for vectors. Note that the rotation of a spinor by $2\pi$ about any axis $m$ introduces a factor of $\exp(-i\pi m) = -1$. This is also a distinguishing property of fermions. The rotor $R$ is a reducible rotational spinor. Irreducible spinors are formed by applying $R$ to a projector (a real idempotent), such as

$$P_3 = \frac{1}{2}(1 + e_3) = P_3^2 = P_3^\dagger.$$ The irreducible spinor is $RP_3$, and twice its even-grade part is $R$. Note, however, that projectors and irreducible spinors are not invertible. The existence of noninvertible elements demonstrates that APS is not a division algebra. Instead, it embodies a rich mathematical framework that admits such proven powerful tools as projectors and spinors, even in classical physics.

Elements that can be written in the form $xP_3$, where $x$ is an arbitrary element of APS, are said to lie within a minimal left ideal of APS that we denote $(\text{APS})P_3$. The elements of a minimal left ideal can all be expressed as even elements of the algebra times the projector of the ideal. Even elements of APS are quaternions, which may be considered spatial rotors times scalar dilation factors, and they do form a division algebra. The proof that any even element of $(\text{APS})P_3$ is equivalent to an even element of APS times $P_3$ is a trivial result of the “pacwoman” property $\[8\]$ that $e_3P_3 = P_3$. Elements of $(\text{APS})P_3$ can be specified by two complex-valued functions and are thus appropriate for a Hilbert-space treatment of quantum mechanics. In particular, the rotor for a rotation expressed in Euler angles can be written

$$R = \exp(e_{21}\phi/2)\exp(e_{13}\theta/2)\exp(e_{21}\chi/2),$$ and its projected ideal form is

$$RP_3 = \left( e^{-i\phi/2}\cos\frac{\theta}{2} + e_1 e^{i\phi/2}\sin\frac{\theta}{2} \right) e^{-i\chi/2}P_3.$$ The standard matrix representation is the usual two-component rotational spinor traditionally associated with a spin-1/2 system plus a second column of zeros:

$$RP_3 = e^{-i\chi/2} \begin{pmatrix} e^{-i\phi/2}\cos\theta/2 & 0 \\ e^{i\phi/2}\sin\theta/2 & 0 \end{pmatrix}.$$

All elements of the minimal left ideal $(\text{APS})P_3$ have a vanishing second column in the standard representation.

C. Physical Space from Fermion Annihilation/Creation Operators

Another way to relate physical space to fermions is to generate APS from a pair $a, a^\dagger$ of annihilation and creation operators for a given state. These are defined to be nilpotent elements $a^2 = 0 = (a^\dagger)^2$ that satisfy the anticommutation relation

$$aa^\dagger + a^\dagger a = 1.$$ In addition to $\[9\]$, the only other real elements that can be generated from $a, a^\dagger$ are

$$e_1 = a + a^\dagger$$
$$e_2 = i(a - a^\dagger)$$
$$e_3 = a^\dagger a - aa^\dagger$$
and these satisfy the Clifford relations $\mathbf{e}_j \mathbf{e}_k + \mathbf{e}_k \mathbf{e}_j = 2\delta_{jk}$ from which APS is derived. The relations (10) can be solved for $a, a^\dagger$ to give

$$a = \frac{1}{2} (\mathbf{e}_1 - i \mathbf{e}_2) = e_1 P_3$$

$$a^\dagger = P_3 e_1 = e_1 P_3$$

which identifies the annihilation and creation operators as null flags of APS.\[8, 31] Taken together with $P_3 = a a^\dagger$ and $P_3 = a^\dagger a$, they form a null basis for spacetime. Note that one may also view $a^\dagger, a$ as raising and lowering operators for a spin-1/2 system.

Fermion annihilation/creation operators for two or more states can be used to generate geometric algebras for high-dimensional spaces. For example, the orthonormal vectors of 4-dimensional Euclidean space can be expressed in terms of annihilation/creation operators for two fermion states:

$$\mathbf{e}_1 = a^\dagger_1 + a_1$$

$$\mathbf{e}_2 = i (a_1 - a_1^\dagger)$$

$$\mathbf{e}_3 = a^\dagger_2 + a_2$$

$$\mathbf{e}_4 = i (a_2 - a_2^\dagger)$$

The full geometric algebra can be generated from these basis vectors. Rotors in the four-dimensional space are elements of $\text{Spin}(4) \simeq SU(2) \otimes SU(2)$ and can be described by two independent spin-1/2 systems or qubits. However, rotors in four dimensions, generated by bivectors, do not span the whole Hilbert space, as shown in \[32\]. In higher dimensions more fermion states are required.

### III. PARA VECTORS AND SPACETIME

Every element of APS is some real linear combination of scalars (grade 0), vectors (grade 1), bivectors (grade 2), and trivectors (grade 3). However, as seen above, trivectors are pseudoscalars, which are expressed as imaginary scalars, and bivectors are pseudovectors, expressed as imaginary vectors. Thus, every element of APS is a linear combination of a complex scalar and a complex vector. The algebra of real vectors in three dimensions thereby forms a complex linear space of four dimensions. This possibility was actually evident from the $2 \times 2$ matrix representation of our original vectors. The vector elements of the four-dimensional space are called paravectors \[8, 17\] to distinguish them from vectors of the original real three-dimensional space. A paravector is the sum of a scalar and a vector, for example $p = p^0 + \mathbf{p}$, where $p^0$ is the scalar part. To reinforce the four-dimensional property of paravector space, we write $p = p^\mu e_\mu$, where our paravector basis elements are $e_0 = 1$ and $e_k = \mathbf{e}_k$, $k = 1, 2, 3$, and we adopt the Einstein summation convention of summing over indices that appear once as an upper index and once as a lower one. Note that APS also contains several linear subspaces of interest: the center of the algebra comprises scalars plus pseudoscalars: the complex numbers; the elements of even grade (scalars plus bivectors, the even subalgebra of APS) are quaternions; and elements of grades 0 and 1 are real paravectors.

The metric of paravector space suggests the physical significance of the fourth dimension. The original three-dimensional space has a Euclidean metric, and the paravector metric is determined by an appropriate quadratic form, that is a scalar expression representing the square length of a paravector. In the original vector space, the quadratic form was identified as the square of the vector, but the square of a paravector is not a scalar. As with complex numbers, we must multiply the paravector by a conjugate to be sure of getting a scalar. The appropriate conjugate is the Clifford conjugate $p = p^0 - \mathbf{p}$ because

$$p \bar{p} = \bar{p} p = (p^0)^2 - \mathbf{p}^2$$

is always a scalar. It can be adopted as the quadratic form. As long as the quadratic form $x \bar{x}$ of an element $x$ of APS does not vanish, the inverse of $x$ is

$$x^{-1} = \bar{x} (x \bar{x})^{-1} = (x \bar{x})^{-1} \bar{x}.$$  

The quadratic form $x \bar{x}$ of any element $x$ equals the determinant of its matrix representation. For consistency, the product of any two elements $x, y$, has the Clifford conjugate $x \bar{y} = \bar{y} \bar{x}$. 

\[5\]
TABLE I: Relations of paravector grade (pv-grade) to vector grade (v-grade). There exists a linear space for each vector and paravector grade. The number (no.) of independent elements is the dimension of the corresponding linear space.

| pv-grade | pv-type  | no. v-grades | basis elements |
|----------|----------|--------------|----------------|
| 0        | scalar   | 1            | e_0 = \langle e_0 \rangle_S |
| 1        | paravector | 4 + 1        | e_\mu = \langle e_\mu \rangle_R |
| 2        | biparavector | 6 + 2       | \langle e_\mu e_\nu \rangle_V or i e_\rho |
| 3        | triparavector | 4 + 3       | \langle e_\lambda e_\mu e_\nu e_\rho \rangle_3 |
| 4        | pseudoscalar | 1 + 3       | \langle e_\lambda e_\mu e_\nu e_\rho \rangle_3 |

We can use the Clifford conjugate to isolate the scalar-like (S) and vector-like (V) parts of any element p, and in the same way reversion (†) can be used to separate the “real” (R, or hermitian) and “imaginary” (3, or antihermitian) parts:

\[ \langle p \rangle_S = \frac{1}{2} (p + \bar{p}), \quad \langle p \rangle_V = \frac{1}{2} (p - \bar{p}) \]
\[ \langle p \rangle_R = \frac{1}{2} (p + p^\dagger), \quad \langle p \rangle_3 = \frac{1}{2} (p - p^\dagger). \] (17)

The scalar-like part of any element is half the trace of its matrix representation, and for any two elements p, x, \langle px \rangle_S = \langle xp \rangle_S . One can easily verify that \bar{p}p is its own Clifford conjugate. If we replace p by the sum p + q of two paravectors, we can determine the scalar product of p with q:

\[ \langle pq \rangle_S = \frac{1}{2} (pq + qp) = p^\mu q^\nu \langle e_\mu e_\nu \rangle_S = p^\mu q^\nu \eta_{\mu\nu}. \] (18)

The tensor (\eta_{\mu\nu}) = \langle e_\mu e_\nu \rangle_S = \text{diag}(1, -1, -1, -1) is the metric tensor of Minkowski spacetime. Had we chosen the quadratic form to be \(-p\bar{p}\), we would still have arrived at a Minkowski spacetime metric, but one with the opposite signature. In either case, we see that real paravectors can represent vectors in flat four-dimensional spacetime. Two paravectors are orthogonal if their scalar product vanishes.

We can now extend rotors to “rotations” in spacetime. The biparavector basis element \langle e_\mu e_\nu \rangle_V generates rotations in the spacetime plane containing the paravectors e_\mu and e_\nu. Lorentz rotors \( L = \pm \exp (W/2) \) with W a biparavector induce restricted Lorentz transformations of any spacetime vector p:

\[ p \rightarrow LpL^\dagger. \] (19)

The rotor \( L \in \text{Spin}_+ (1, 3) \approx SL(2, \mathbb{C}) \approx SO_+ (1, 3) \times \mathbb{Z}_2 \) is an amplitude for the Lorentz transformation. As with spatial rotors \( R \), the Lorentz rotor has unit spacetime length (is unimodular): \( LL = 1 \) and as with any invertible linear operator, there is a polar decomposition of \( L \) into the product of unitary and hermitian factors. Since \( L \) is unimodular, so are its factors, and we can write \( L = BR \), where \( R \) is a unitary spatial rotor and \( B = B^\dagger \) is a boost (velocity transformation). Since the Clifford conjugate of the transformation (19) is \( \bar{p} \rightarrow \bar{L}^\dagger \bar{pL} \), the Lorentz transformation of pq, where p and q are paravectors, takes the form \( pq \rightarrow LpqL^\dagger \), and it follows that the scalar product (18) is invariant.

Just as we identified multivectors of vector grades 0 to 3, we can identify other elements as multiparavectors of paravector grades 0 to 4. The relation is given in Table I. Note that whereas paravector grades 1, 2, and 3 have contributions from two neighboring vector grades (see Table I), the spacetime scalars are the same as vector scalars, and that the pseudoscalar element in spacetime is \( \iota \), the same as the vector pseudoscalar. This permits a simple calculation of Hodge-type duals of elements: if \( x \) is any element of APS, even a multigrade one, its Clifford-Hodge dual is defined to be \( \ast x = -ix \).

### A. Classical Eigenspinors for Relativistic Dynamics

The Lorentz rotor that transforms between the particle rest frame and the lab is useful for describing particle dynamics. With its help, the velocity and orientation of the particle can be calculated and, indeed, any property known in the rest frame can be transformed to the lab. Because of its special status, we give this Lorentz rotor a special designation: it is the eigenspinor \( \Lambda \) of the particle. For an accelerating particle, \( \Lambda \) is a function of the proper time \( \tau \) of the particle, representing at each instant the Lorentz rotor from the inertial frame commoving with the
particle (the “rest frame”) to the lab. For example, the proper velocity $u_0$ of the particle in the lab is (in units with $c = 1$) just the transformed time axis:

$$u_0 = \Lambda e_0 \Lambda^\dagger.$$  

(20)

The proper velocity is a spacetime vector and can be further transformed by a Lorentz rotor $L : u_0 \rightarrow Lu_0L^\dagger = L\Lambda e_0 \Lambda^\dagger L^\dagger$. This is equivalent to the Lorentz rotation $\Lambda \rightarrow LA$ of the eigenspinor. This form of a Lorentz transformation is distinct from that (19) for a spacetime vector or any product of vectors, and that is part of the justification for calling $\Lambda$ a spinor.

The other basis paravectors of an elementary system can be similarly transformed to the lab frame. The system tetrad $\{u_\mu\}$ is the set of transformed basis elements

$$u_\mu = \Lambda e_\mu \Lambda^\dagger.$$  

(21)

In addition to the special role played by the proper velocity $u_0$, the spacetime vector $u_3 = \Lambda e_3 \Lambda^\dagger$ may be identified with the Pauli-Lubanski (PL) spin. Whereas $u_0$ is a timelike unit paravector because $u_0 \bar{u}_0 = 1$, the PL spin is spacelike: $u_3 \bar{u}_3 = -1$ and is orthogonal to $u_0$:

$$\langle u_3 \bar{u}_0 \rangle_S = \langle e_3 \bar{e}_0 \rangle_S = 0.$$  

(22)

The spacetime dual of $u_3$ is

$$-iu_3 = \Lambda e_1 \bar{e}_2 e_0 \Lambda^\dagger = Su_0,$$  

(23)

where $S \equiv \Lambda e_1 \bar{e}_2 \Lambda = u_1 \bar{u}_2$ is recognized as the spacetime bivector (“biparavector”) for the plane orthogonal to both $u_0$ and $u_3$. If $\Lambda$ is a pure spatial rotation, $u_3$ is simply the unit spatial vector $s = Re_3 R^\dagger = iS$. The association of $e_3$ and $u_3$ with spin will be made more definite below.

A system of several parts will generally require several eigenspinors to describe its motion. A system is said to be elementary if all of its motion is described by a single $\Lambda(\tau)$. A free elementary system (“particle”) is necessarily unstructured, but it may have an orientation and is not necessarily point-like.

**B. Equation of Motion**

Since the eigenspinor at any instant $\tau$ is a Lorentz rotor and every rotor $L$ has an inverse $\bar{L}$, the eigenspinor at different times is related by

$$\Lambda(\tau_2) = L_2 \bar{L}_1 \Lambda(\tau_1) \equiv L(\tau_2, \tau_1) \Lambda(\tau_1),$$  

(24)

where $L_1 \equiv \Lambda(\tau_1), \ L_2 \equiv \Lambda(\tau_2)$, and we noted that by their group property, the product of Lorentz rotors is another Lorentz rotor. The proper-time derivative of the eigenspinor can be expressed

$$\dot{\Lambda} = (\dot{\Lambda} \dot{\Lambda}) \Lambda = \frac{1}{2} \Omega \Lambda = \frac{1}{2} \Lambda \Omega_{\text{rest}}$$  

(25)

with $\Omega = 2\dot{\Lambda} \dot{\Lambda}$, and it follows from the unimodularity of $\Lambda$ that $\Omega$ is a biparavector. From the infinitesimal time development

$$\Lambda(\tau + d\tau) = \left(1 + \frac{1}{2} \Omega \; d\tau\right) \Lambda(\tau) = \exp \left(\frac{1}{2} \Omega \; d\tau\right) \Lambda(\tau),$$  

(26)

one can interpret $\Omega$ as the spacetime rotation rate of the particle frame. Similarly, $\Omega_{\text{rest}} = \bar{\Lambda} \Omega \Lambda$ is the rotation rate in the rest frame of the particle. If $\Omega$ is known, we can find the proper time-rate of change of any property known in the rest frame. For example, the proper acceleration of the particle is given by

$$\ddot{u}_0 = \dot{\Lambda} e_0 \Lambda^\dagger + \Lambda e_0 \dot{\Lambda}^\dagger = \langle \Omega u_0 \rangle_R.$$  

(27)
C. Maxwell-Lorentz Theory

Comparison of \( \dot{u}_0 \) to the Lorentz-force equation \( \dot{p} = m\dot{u}_0 = e \langle F u_0 \rangle / \hbar \) suggests a covariant definition of the electromagnetic field as the spacetime rotation rate per unit charge-to-mass ratio

\[
F = \langle \partial \bar{A} \rangle_V = E + iB = m\Omega/e. \tag{26}
\]

It also gives a spinor form of Lorentz-force equation:

\[
\dot{\Lambda} = \frac{e}{2m}F\Lambda \tag{28}
\]

that simplifies many problems in electrodynamics. For example, if \( F \) is constant, we can integrate (28) immediately to get the eigenspinor \( \Lambda(\tau) = \exp \left( \frac{e}{2m} F \tau \right) \Lambda(0) \), which determines both the proper velocity (20) of the particle and the orientation of its reference frame. The spinor equation (28) also reveals surprising symmetries, for example, the fact that the field \( F_{\text{rest}} \) seen in the instantaneous rest frame of the particle is constant, even if the particle is accelerating:

\[
F_{\text{rest}}(\tau) = \bar{\Lambda}(\tau) F \Lambda(\tau) = F_{\text{rest}}(0). \tag{29}
\]

To complete the formulation of Maxwell-Lorentz theory, we need Maxwell’s equations relating \( F \) to the charge-current density \( j = eJ = e\rho + j \). These, in SI units, are just the scalar, vector, pseudovector, and pseudoscalar components of \( \partial F = \mu j \), where \( \mu \) is the permeability of space. Simple expansions of these algebraic equations in basis elements yield the traditional corresponding tensor equations. The eigenspinor approach has proved to be a powerful tool for finding exact solutions in classical relativistic dynamics.

Note that the spinor equation (28) is invariant under a change in the orientation of the rest frame:

\[
\Lambda \rightarrow \Lambda R, \tag{30}
\]

where \( R \) is any fixed spatial rotor. The transformation (30) may be considered a global gauge transformation of \( \Lambda \). The invariance can be extended to a local gauge transformation \( \Lambda \rightarrow \Lambda_{\omega_0} = \Lambda \exp (-i\epsilon_3\omega_0\tau) \) by adding a rotational gauge term to (28) that represents a rest-frame rotation or spin [see Eq. (23)]:

\[
\dot{\Lambda}_{\omega_0} = \frac{e}{2m}F\Lambda_{\omega_0} - i\omega_0\Lambda_{\omega_0}\epsilon_3 = \frac{1}{2} \left( \frac{e}{m}F + 2\omega_0 S \right) \Lambda_{\omega_0}. \tag{31}
\]

IV. Dirac Equation in APS

The spacetime momentum of a particle is given by \( p = \Lambda m\Lambda^\dagger \), but since the eigenspinor \( \Lambda \) is an invertible Lorentz rotor, we can equally well write the relation \( p\Lambda^\dagger = m\Lambda \) which may be called a real-linear form since real superpositions of solutions are also solutions. Note that we have not assumed that the particle whose dynamics are described by \( \Lambda \) has a point-like distribution. It may indeed be distributed in space with some density \( \rho \) in its rest frame. The current density \( J \) in the lab is then \( J = \Lambda\rho\Lambda^\dagger = \Psi e_0\Psi^\dagger \), where we have put \( \Psi = \rho^{1/2}\Lambda \). Since \( \rho \) is a real scalar, equation \( p\Lambda^\dagger = m\Lambda \) is also satisfied by the current amplitude \( \Psi \):

\[
p\Psi^\dagger = m\Psi. \tag{32}
\]

This is the classical Dirac equation.

To cast the equation in complex linear form required for a Hilbert-space formulation, we project it into minimal left ideals of APS

\[
p\Psi^\dagger P_3 = m\Psi P_3 \tag{33}
, \quad p\Psi^\dagger \bar{P}_3 = m\Psi \bar{P}_3 .
\]

We can now flip the ideal of the second equation with bar-dagger conjugation to get \( p\Psi P_3 = m\Psi^\dagger P_3 \) so that both equations lie in the same minimal left ideal of APS. As noted above, all elements of the ideal have only two independent complex components. If we stack them, using the Pauli-matrix representation of APS, we get a four-component column matrix identical to the Dirac spinor in the Weyl representation

\[
\psi^{(W)} = \frac{1}{\sqrt{2}} \left( \begin{array}{c}
\Psi P_{+3} \\
\Psi^\dagger \bar{P}_{+3}
\end{array} \right), \tag{34}
\]

\]
and the equation for it is exactly the Dirac equation in momentum form, complete with gamma matrices in the Weyl representation. The projection of the algebraic $\Psi$ by $P_3$ and $P_3 = P_{-3}$ picks out the upper and lower component pairs of $\psi^{(W)}$ and is seen to be equivalent to multiplication of $\psi^{(W)}$ by the traditional chirality projectors $\frac{1}{2} (1 \pm \gamma_5)$ with $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$:

$$\frac{1}{2} (1 \pm \gamma_5) \psi^{(W)} \leftrightarrow \Psi P_{\pm 3}.$$  (35)

We might therefore refer to the minimal left ideals $\mathcal{O}_3 P_{\pm 3}$ as the left and right chiral ideals of APS.

A. De Broglie Waves and Spin Interaction

The spin rotation of a free eigenspinor projects into a phase oscillation $\exp (-i e \omega_0 \tau) P_3 = e^{-i \omega_0 \tau} P_3$. A spatial distribution with a synchronized phase oscillation becomes a de Broglie wave in the lab frame after a boost by the eigenspinor $\Lambda$. The boost desynchronizes the phase oscillations across the distribution, giving a wave of wavelength $\lambda = 2\pi / (\gamma |v| \omega_0)$, where $v$ is the boost velocity and $\gamma = (1 - v^2)^{-1/2}$ is the Lorentz dilation factor. In terms of the Lorentz invariant $\tau = \langle x\bar{u} \rangle_S$, the phase factor in lab coordinates is

$$\exp (-i \omega_0 \tau) = \exp (-i \omega_0 \langle x\bar{u} \rangle_S) = \exp [-i \gamma \omega_0 (t - v \cdot x)].$$  (36)

The wavelength has the measured de Broglie value $\lambda = 2\pi h / (\gamma m |v|)$ if and only if the oscillations occur at the Zitterbewegung frequency $\omega_0 = E_0 / h = m / h$. We note that for a compound system, the rest energy $m$ includes internal motion and interaction. The frequency involved is very rapid, even for an electron, for which $\omega_0 \approx 0.776 \times 10^{21} \text{s}^{-1}$ and it is therefore clear that it can only refer to an intrinsic rotation, not to a rotation of the distribution as a whole. Any observation of an elementary system with such an intrinsic spin can only see it as essentially point-like, with no discernible physical extent. This is consistent with the standard Born interpretation of the quantum wave function as a probability amplitude.

We noted above that a magnetic field can be defined by the spatial rotation rate it induces. For an elementary charge at rest, a pure magnetic field $B$ causes a shift in the total spatial rotation rate from $|\Omega| = 2\omega_0$ to $|\Omega| = 2\omega_0 - (e / m) \bar{s} \cdot B$ for fields small compared to $2\omega_0 m / e \approx 4.414 \times 10^9$ tesla for an electron, and this shift in rotation frequency corresponds to a mass change and hence an interaction energy $-\mu \cdot B$, where

$$\mu = \frac{e \hbar}{2m}.$$  (37)

should evidently be interpreted as the magnetic moment of the fermion. The $g$-factor, which gives $2m / e$ times the ratio of the magnetic moment to the angular momentum, is given by the definition of an elementary particle: since its motion is described by a single eigenspinor field $\Lambda$, its cyclotron and Larmor-precession frequencies must be equal. This implies $g = 2$. Taken together, the $g$-factor and the magnetic moment imply a spin angular momentum of magnitude $\hbar / 2$ in the direction $\bar{s}$. The analysis not only derives the interaction of a spin in a magnetic field, it also supports the classical picture of the spin as a physical (but intrinsic) rotation at the rate $2\omega_0$ in a right-handed sense about the direction $\bar{s} = R e_3 R^\dagger$. The calculation also reveals the mass as a source of the energy when the magnetic moment of a spin is accelerated in an inhomogeneous but static magnetic field.

Our picture differs considerably from that of Hestenes, who models the electron as a point charge moving at the speed of light on a helical path that circles at the Zitterbewegung frequency.

B. Large and Small Components

For bound states and at low velocities, it is convenient to use $\langle \Psi \rangle_{\pm} = \frac{1}{2} (\Psi \pm \bar{\Psi}^\dagger)$, which are the even and odd parts of $\Psi$. They are even and odd not only in the Clifford algebra sense of containing only even-grade or only odd-grade elements of APS, but also in the sense of being even and odd under parity inversion. The even and odd parts of $\Psi$ correspond to the large and small components of positive-energy solutions at low velocities:

$$\langle \Psi \rangle_+ = \rho^{1/2} \langle B \rangle_+ R = \rho^{1/2} \sqrt{\frac{m + E}{2m}} R \approx \rho^{1/2} R$$  (38)

$$\langle \Psi \rangle_- = \rho^{1/2} \langle B \rangle_- R = \frac{P}{m + E} \langle \Psi \rangle_+.$$  (39)
where we noted that the scalar function $\rho^{1/2}$ and rotor $R$ are both even elements and that $B = (p/m)^{1/2} = (p + m)/\sqrt{2(m + E)}$. The third term on the RHS of (38) is for free particles of energy $E$, since for these

$$B = \sqrt{\frac{p}{m}} = \frac{p + m}{\sqrt{2m(E + m)}},$$

$$\langle B \rangle_+ = \sqrt{\frac{E + m}{2m}},$$

$$\langle B \rangle_- = \frac{p}{\sqrt{2m(E + m)}} = \frac{p}{E + m} \langle B \rangle_+ .$$

The last expression for $\langle \Psi \rangle_\pm$ on the RHS is the low-velocity approximation. In the rest frame, the small component $\langle \Psi \rangle_- \pm$ disappears and the eigenfunction is even. We say the particle has even intrinsic parity. Note that the spinors $\langle \Psi \rangle_\pm$ are easily extracted from the corresponding ideal spinors $\langle \Psi \rangle_\pm P_{+3}$:

$$\langle \Psi \rangle_+ = 2 \langle \langle \Psi \rangle_+ P_{+3} \rangle_+ , \quad \langle \Psi \rangle_- = 2 \langle \langle \Psi \rangle_- P_{+3} \rangle_- .$$

The Dirac bispinor in the Dirac-Pauli (or standard) representation is related by

$$\psi^{(DP)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \psi^{(W)} = \left( \frac{\langle \Psi \rangle_+ P_{+3}}{\langle \Psi \rangle_- P_{+3}} \right) ,$$

Generally, the solutions $\psi^{(W)}$ and $\psi^{(DP)}$ are represented by $4 \times 2$ matrices whose second columns are zero and whose first columns give the usual Dirac bispinors of quantum theory. If $R$ is replaced by the de Broglie spin rotor $R = \exp[-i\mathbf{e}_3 \langle \mathbf{p} \rangle S/\hbar]$, the solutions in the nonvanishing columns of $\psi^{(W)}$ and $\psi^{(DP)}$ are the usual momentum eigenstates of the Dirac equation.

To get the Dirac equation in the Dirac-Pauli representation, we split $p\bar{\Psi} = m\Psi$ into even and odd parts:

$$-\mathbf{p} \langle \Psi \rangle_- = (m - p^0) \langle \Psi \rangle_+ , \quad \mathbf{p} \langle \Psi \rangle_+ = (m + p^0) \langle \Psi \rangle_- .$$

The odd part

$$\langle \Psi \rangle_- = (m + p^0)^{-1} \mathbf{p} \langle \Psi \rangle_+$$

can be eliminated in the first equation to give a second-order form of the Dirac equation:

$$\mathbf{p} (m + p^0)^{-1} \mathbf{p} \langle \Psi \rangle_+ = (p^0 - m) \langle \Psi \rangle_+ .$$

C. Differential Operators and Commutation Relations

The differential form of momentum results when we (1) assume that the general $\Psi$ can be expressed as a linear superposition of de Broglie waves $R \exp(-i\langle \mathbf{p} \rangle S/\hbar \mathbf{e}_3)$ and (2) make the usual local gauge invariance argument for the spacetime vector potential $A$. In APS we can write for each component

$$p^\mu \Psi = i\hbar \partial^\mu \Psi \mathbf{e}_3 - eA^\mu \Psi .$$

When used with the classical Dirac equation, the result is fully equivalent to the usual Dirac equation in its differential form. When the $\mu = 0$ equation is applied to the even part of $\Psi$ and projected onto the minimal left ideal $(APS)P_3$, we find

$$p^0 \langle \Psi \rangle_+ P_3 = (i\hbar \partial_t - V) \langle \Psi \rangle_+ P_3 = (H - V) \langle \Psi \rangle_+ P_3 ,$$

where the Hamiltonian operator $H = i\hbar \partial_t$, operating on $\langle \Psi \rangle_+ P_3$ includes the rest energy $m$. The projected form of the second-order Dirac equation thus becomes

$$\mathbf{p} (m + p^0)^{-1} \mathbf{p} \langle \Psi \rangle_+ P_3 = (H - V - m) \langle \Psi \rangle_+ P_3 .$$
In the low-energy limit where the factor \((m + p^0)^{-1}\) on the LHS is approximated by \((2m)^{-1}\), Eq. (53) becomes the Pauli-Schrödinger equation. This demonstrates that the Pauli-Schrödinger wave function (with a two-component spinor of the form (3)) corresponds to the even ideal spinor \(\langle \Psi \rangle_+ P_3\) in the low-energy limit (58):

\[
\psi^{(Sch)} = \rho^{1/2} R P_3 .
\]

The differential-operator form of \(p^\mu\) (47) implies the commutation relations

\[
[p^\mu, x_\nu] \Psi = i\hbar \delta^\mu_\nu \Psi e_3
\]

In terms of the spin biparavector \(S = \Lambda e_1 \vec{e}_2 \vec{A} = -i\Lambda e_3 \vec{A}\) (23), we note

\[
i\Psi e_3 = -S\Psi
\]

so that relation (51) is equivalent to \([p^\mu, x_\nu] \Psi = -\hbar S \Psi\), and since this is true for any current amplitude \(\Psi\), we can simply write the operator relation

\[
[p^\mu, x_\nu] = -\hbar \delta^\mu_\nu S.
\]

In the minimal left ideal (APS)\(P_3\), the relation (53) reduces to the usual form, in which \(S\) is replaced by \(-i\).

We have treated the momentum components \(p^\mu\) as operators and the coordinates \(x^\mu\) as variables, but the inherent symmetry of momentum and position variables as apparent in phase-space treatments and in Eq. (55) allows us to reverse the roles and consider \(p^\mu\) the coordinates in momentum space and to write the \(x^\mu\) as differential operators on this space. However, there is a well-known objection by Pauli\(^{28}\) to considering time as an operator satisfying \([H, t] = i\hbar\), where we identify \(p^0 = H, x_0 = t\). He pointed out that given any energy eigenstate \(\psi_E\) with eigenvalue \(E\), \(H \psi_E = E \psi_E\), one could then form another eigenstate \(\psi_{E-\varepsilon} = \exp(\imath \varepsilon t/\hbar) \psi_E\) of eigenenergy \(E - \varepsilon\), since \(H \exp(\imath \varepsilon t/\hbar) = \exp(\imath \varepsilon t/\hbar) \exp(\imath \varepsilon t/\hbar) = \exp(\imath \varepsilon t/\hbar) (H + \imath \varepsilon [H, t]/\hbar) = \exp(\imath \varepsilon t/\hbar) (H - \varepsilon)\). The spectrum of \(H\) must then be continuous and unbounded. In the calculation (57) of the magnetic dipole moment of a fermion, we have shown that the “Pauli problem” is avoided in APS. The phase factor \(\exp(\imath \varepsilon t/\hbar)\) arises only from the projection of an actual rotation such as caused by a magnetic field, and such a constant rotation rate does indeed change the energy. It is that change in energy that gave us the correct magnetic moment and that can supply the energy when a spin is accelerated in a magnetic-field gradient.

Note that our approach also gives a natural, relativistic formulation of the Bohm/de Broglie theory\(^{37, 38, 39}\) of causal quantum mechanics with spin included. Flow lines are given by the current density \(J = \Psi e_0 \Psi^\dagger\), and the Pauli-Lubański spin distribution is found from \(S = \Psi e_3 \Psi^\dagger\). Of course, the existence of a simple formalism does not imply that Bohm’s causal interpretation is required.

The combined currents \(J_\pm = J \pm S\) are null elements, for which continuity equations are readily established:

\[
\langle \partial J_\pm \rangle_S = \langle \partial \left[ \Psi \left( e_0 \pm e_3 \right) \Psi^\dagger \right] \rangle_S = 2 \langle \left( \partial \Psi \right) \left( e_0 \pm e_3 \right) \Psi^\dagger \rangle_{\text{RS}}
\]

since from the Dirac equation (32) with (47), the relation

\[
\partial \Psi = -i\hbar^{-1} \left( \bar{\Psi} + e \vec{A} \right) \Psi e_3 = -i\hbar^{-1} \left( m \bar{\Psi}^\dagger + e \vec{A} \Psi \right) e_3
\]

gives

\[
\langle \partial J_\pm \rangle_S = -2\hbar^{-1} \langle i \left( (m \bar{\Psi}^\dagger + e \vec{A} \Psi) e_3 \right) (e_0 \pm e_3) \Psi^\dagger \rangle_{\text{RS}}
\]

\[
= -2\hbar^{-1} \langle i \left( (m \bar{\Psi}^\dagger + e \vec{A} \Psi) \right) (e_0 \pm e_3) \Psi^\dagger \rangle_{\text{RS}} = 0.
\]

This means that both \(J_\pm\) are conserved currents, and therefore, so are \(J\) and \(S\). The other Fierz identities for bilinear covariants also follow.\(^{40}\)

## V. SPIN DISTRIBUTIONS

Once the momentum has been replaced by a differential operator and the spatial form of the Dirac equation has been derived, we have crossed into the quantum side of the quantum/classical interface. While we needed to establish the correspondence between the Dirac spinor and Pauli-Schrödinger wave function to our classical eigenspinor, we consider here a more classical calculation of spin distributions. We want to show that the calculation is exactly equivalent to the corresponding quantum-mechanical one. To study spin distributions, the low-velocity limit

\[
\Psi \simeq \langle \Psi \rangle_+ \simeq \rho^{1/2} R
\]

(57)
is sufficient, with $R$ given in terms of Euler angles by \[ \theta \]. As we saw above \[ \theta \], the Schrödinger wave function $\psi^{(Sch)}$ with a two-component spinor, is the projection of this $\Psi$ \[ \{57\} \] onto the minimal left ideal $(APS)P_3$. To get expressions at relativistic speeds, we can always apply a subsequent boost to $\Psi$. The classical spin direction in a static system is $s = Re_3 R^\dagger A$ Pauli-Lubanski spin distribution of the state is

$$\mathcal{S} = \rho \ s = \rho Re_3 R^\dagger = \Psi e_3 \Psi^\dagger,$$

(58)

where the positive scalar $\rho = \rho(r)$ is the density of spins in the reference frame. As seen below, simple measurements of the spin direction give only one component at a time. The distribution of the component of the spin in the direction of an arbitrary unit vector $m$ is

$$\rho \cdot m = \langle \mathcal{S} m \rangle_S = \langle \Psi e_3 \Psi^\dagger m \rangle_S.$$

(59)

In terms of the projector $P_3 = P_3^2$, since $e_3 = P_3 - \bar{P}_3$, $\Psi$ is even, and for any elements $p, q$, $\langle pq \rangle_S = \langle qp \rangle_S = \langle \bar{pq} \rangle_S$, the distribution is

$$\langle \Psi P_3 \Psi^\dagger m \rangle_S - \langle \Psi \bar{P}_3 \Psi^\dagger m \rangle_S = \langle \Psi P_3 \Psi^\dagger m \rangle_S + \langle m \Psi^\dagger P_3 \Psi \rangle_S$$

$$= 2 \langle \Psi P_3 \Psi^\dagger m \rangle_S = 2 \rho \langle RP_3 R^\dagger m \rangle_S$$

$$= 2 \langle P_3 \Psi^\dagger m \Psi P_3 \rangle_S = \text{tr}\left\{ \psi^{(Sch)} \Psi \right\},$$

(60)

(61)

(62)

where $\psi^{(Sch)}$ has the standard matrix representation

$$\psi^{(Sch)} = \rho^{1/2} R P_{+3} e^{-i\phi/2} \rho^{1/2} \begin{pmatrix} e^{-i\phi/2} & 0 \\ e^{i\phi/2} \sin \theta/2 & 0 \end{pmatrix}$$

(63)

which, ignoring the inconsequential column of zeros, is the two-component spinor familiar from the usual nonrelativistic Pauli theory. The term $\psi^{(Sch)} \Psi \bar{\psi}^{(Sch)}$ is then a scalar and tr can be omitted from \[ \{62\} \]. If $\rho$ is normalized to unity,

$$\int d^3x \rho = 2 \int d^3x \left\{ \psi^{(Sch)} \bar{\psi}^{(Sch)} \right\}_S \equiv \left\{ \psi^{(Sch)} | \psi^{(Sch)} \right\} = 1,$$

(64)

then the average component of the spin in the direction $m$ is

$$2 \int d^3x \left\{ \psi^{(Sch)} \bar{\psi}^{(Sch)} m \right\}_S \equiv \left\{ \psi^{(Sch)} | m | \psi^{(Sch)} \right\}.$$

(65)

Although we derived the spin distribution as a classical expression, it has precisely the quantum form if we recognize that the matrix representation of the unit vector $m$, namely $m = m^i e_j \rightarrow m^1 \sigma_x + m^2 \sigma_y + m^3 \sigma_z$, is traditionally written $m \sigma$ (but this is misleading, since it represents a vector, not a scalar) and traditionally, $\sigma$ is thought of as the spin operator.

From expression \[ \{61\} \] we see that the real paravector $P_s = RP_{+3} R^\dagger = \frac{1}{2} (1 + s)$ embodies information about the classical spin state. Then $P_s = \psi^{(Sch)} \bar{\psi}^{(Sch)}$ is the spin density operator $\rho$ for the pure state $\psi^{(Sch)}$ of spin $s$. It is also a projector that acts as a state filter. To see whether a system with spin density $\rho$ is in a given state of spin $n$ we can apply the state filter to the spin density operator $\rho$ and see what remains:

$$P_n \rho P_n = \langle P_n \rho + \bar{P}_n \rho \rangle P_n = 2 \langle P_n \rangle_S P_n.$$

(66)

The scalar coefficient $2 \langle P_n \rangle_S = \langle (1 + n) \phi \rangle_S$ is the probability of finding the system described by $\rho$ in the state $n$. For a system in the pure state $\rho = P_s = \frac{1}{2} (1 + s)$, the probability is

$$2 \langle P_n P_s \rangle_S = \frac{1}{2} \langle (1 + n) (1 + s) \rangle_S = \frac{1}{2} (1 + n \cdot s).$$

(67)

This is unity if the system is definitely in the state $n$, whereas it vanishes if the system is in a state orthogonal to $n$. Thus, $s = n$ is required for the states to be the same and $s = -n$ for the states to be orthogonal. Note that the mathematics is the same as used to describe light polarization.\[ \{11\} \ [13\]
A. Spin $\frac{1}{2}$ and State Expansions

The value of $\frac{1}{2}$ for the spin of elementary spinors in physical space can be associated with the group-theoretical label for the irreducible spinor representation of the rotation group $SU(2)$ carried by ideal spinors, but it is also required by the fact that any rotation can be expressed as a linear superposition of two orthogonal rotations defined for any direction in space. The Euler-angle form of any rotor $R$ can be rewritten

$$ R = \exp \left( -i \frac{n \theta}{2} \right) \exp \left[ -i e_3 (\phi + \chi)/2 \right], $$

where $n = \exp ( -i e_3 \phi/2 ) e_2 \exp ( i e_3 \phi/2)$ is a unit vector in the $e_1 e_2$ plane. Therefore, any rotor $R$ is a real linear combination

$$ R = \cos \frac{\theta}{2} R_\uparrow + \sin \frac{\theta}{2} R_\downarrow $$

such that

$$ \langle R_\uparrow | \bar{R}_\downarrow \rangle = \langle -i n \rangle_S = 0. $$

The rotation $R_\uparrow$ maintains the $e_3$ component whereas $R_\downarrow$ flips it.

By projecting the rotors with $P_3$ onto the corresponding minimal left ideal, we obtain the equivalent relation of ideal spinors, which represent states with a given spin orientation:

$$\psi = R P_3 = \left( \cos \frac{\theta}{2} R_\uparrow + \sin \frac{\theta}{2} R_\downarrow \right) P_3 $$

By projecting the rotors with $P_3$ onto the corresponding minimal left ideal, we obtain the equivalent relation of ideal spinors, which represent states with a given spin orientation:

$$\psi = \cos \frac{\theta}{2} \psi_\uparrow + \sin \frac{\theta}{2} \psi_\downarrow $$

### Projection Operators

The projection operators $P_3$ and $\bar{P}_3$ operating from the left isolate the spin-up and spin-down parts:

$$ P_3 \psi = \cos \frac{\theta}{2} \psi_\uparrow $$

$$ \bar{P}_3 \psi = \sin \frac{\theta}{2} \psi_\downarrow $$

both of which are eigenstates of $e_3$:

$$ e_3 P_3 \psi = + P_3 \psi $$

$$ e_3 \bar{P}_3 \psi = - \bar{P}_3 \psi. $$

### Orthonormality Conditions

Traditional orthonormality conditions hold:

$$ 2 \langle \psi_\uparrow | \psi_\uparrow \rangle_S = 2 \langle \psi_\downarrow | \psi_\downarrow \rangle_S = 2 \rho \langle P_3 i n \rangle_S = 0 $$

$$ 2 \langle \psi_\downarrow | \psi_\uparrow \rangle_S = 2 \langle \psi_\uparrow | \psi_\downarrow \rangle_S = 2 \rho \langle P_3 \rangle_S = \rho . $$

It follows that the amplitudes are

$$ \langle \psi_\uparrow | \psi \rangle = 2 \langle \psi_\uparrow | \psi_\uparrow \rangle_S = \cos \frac{\theta}{2} $$

$$ \langle \psi_\downarrow | \psi \rangle = 2 \langle \psi_\downarrow | \psi_\downarrow \rangle_S = \sin \frac{\theta}{2} $$

giving probabilities as found above in Eq. (67).

$$ |\langle \psi_\uparrow | \psi \rangle|^2 = \cos^2 \frac{\theta}{2} = \frac{1}{2} (1 + s \cdot e_3) $$

$$ |\langle \psi_\downarrow | \psi \rangle|^2 = \sin^2 \frac{\theta}{2} = \frac{1}{2} (1 - s \cdot e_3). $$
B. Stern-Gerlach Experiment

The basic measurement of spin is that of the Stern-Gerlach experiment\textsuperscript{[41]}, in which a beam of ground-state silver atoms is split by a magnetic-field gradient into distinct beams of opposite spin polarization. It is a building block of real and thought experiments in quantum measurement\textsuperscript{[42]}. A description succeeds in the classical eigenspinor framework because of the linear form of the equations, the consequent possibility of superposition, and explicitly the ability to write rotors as superpositions of “spin-up” and “spin-down” rotors referenced to any direction.

Consider a nonrelativistic beam of ground-state atoms that travels with velocity $v = ve_1$ through a static magnetic field $B$ that vanishes everywhere except in the vicinity of the Stern-Gerlach magnet, where it has a gradient aligned with the $e_3$ ($z$) axis. The net effect of the magnetic gradient on atoms in the beam is to apply an impulse or boost in the $e_3$ direction proportional to the $z$ component $\mu_z$ of the magnetic dipole moment.

The generic form of the Stern-Gerlach experiment can be put into a form analogous to the action of a birefringent crystal on a beam of polarized light: the ideal state spinor \textsuperscript{[32]} is split into two parts

$$\psi = (P_3 + \bar{P}_3) \psi$$

which become separated spatially. In the Stern-Gerlach case, each part is associated with a distinct boost, so that the full state spinor $\Psi$ becomes

$$\Psi = 2B_+ \langle P_3 \psi \rangle_+ + 2B_- \langle \bar{P}_3 \psi \rangle_+,$$

where the boosts [see Eq. \textsuperscript{[10]}] combine the velocity $v = ve_1$ of the beam before the magnetic-field gradient with increments $\pm \Delta ve_3$ induced by the field gradient. In the nonrelativistic limit

$$B_\pm \simeq 1 + \frac{1}{2} (ve_1 \pm \Delta ve_3) \equiv 1 + \frac{1}{2} V_\pm.$$

With $\phi = 0$, $\chi = \langle p\bar{e} \rangle_S$, and $n = e_2$, we have $\psi_1 = \rho^{1/2} e^{-i(\phi + \chi)/2} P_3$, $\psi_\perp = -i n \psi_1$

$$P_3 \psi = \cos \frac{\theta}{2} \psi_1 = \rho^{1/2} e^{-i\chi/2} \cos \frac{\theta}{2} P_3$$

$$\langle P_3 \psi \rangle_+ = \rho^{1/2} e^{-i\chi e_3/2} \cos \frac{\theta}{2}$$

$$\bar{P}_3 \psi = \sin \frac{\theta}{2} \psi_1 = \rho^{1/2} e_1 e^{-i\chi e_3/2} \sin \frac{\theta}{2} P_3$$

$$\langle \bar{P}_3 \psi \rangle_+ = \rho^{1/2} e_1 e_3 e^{-i\chi e_3/2} \sin \frac{\theta}{2}.$$

Thus, $\Psi$ \textsuperscript{[31]} becomes

$$\Psi = \rho^{1/2} \left( B_+ \cos \frac{\theta}{2} + B_- \sin \frac{\theta}{2} e_1 e_3 \right) e^{-i\chi e_3/2}.$$

If the initial beam has finite profile $\rho (x)$, the action of the Stern-Gerlach magnet will eventually split the beam into two distinct beams moving with velocities $V_\pm$ and with opposite spin directions and distinct profiles $\rho_\pm \simeq \rho (x - V_\pm t)$. The proper-velocity profile is given by the current density

$$J = \Psi e_0 \Psi^\dagger = \left( B_+ \cos \frac{\theta}{2} + B_- \sin \frac{\theta}{2} e_1 e_3 \right) \rho \left( B_+ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} e_3 e_1 B_- \right)$$

$$= B^2_+ \rho_+ \cos^2 \frac{\theta}{2} + B^2_- \rho_- \sin^2 \frac{\theta}{2} + \rho \sin \theta \langle B_+ e_3 B_- \rangle_\mathbb{R}$$

since the cross terms cancel. At some distance down the beam past the magnet, the 2 sub-beams become non-overlapping and there are two distinct beams. The corresponding spin-density profile is

$$\mathcal{S} = \Psi e_3 \Psi^\dagger = \left( B_+ \cos \frac{\theta}{2} + B_- \sin \frac{\theta}{2} e_1 e_3 \right) e_3 \rho \left( B_+ \cos \frac{\theta}{2} + \sin \frac{\theta}{2} e_3 e_1 B_- \right)$$

$$= \rho_+ B_+ e_3 B_+ \cos^2 \frac{\theta}{2} - \rho_- B_- e_3 B_- \sin^2 \frac{\theta}{2} + \rho \sin \theta \langle B_+ e_1 B_- \rangle_\mathbb{R}.$$
Algebraically, the functions $\rho_{\pm}$ are the same as $\rho$ but the argument is appropriately transformed from the rest-frame coordinates. For consistency, we can take $B\rho^2 = (B\rho B)^{1/2}$ so that the $\rho$ factors in the $\sin\theta$ terms in the above expressions for $J$ and $\mathcal{G}$ are to be calculated as the geometric mean $(\rho_{+}\rho_{-})^{1/2}$. Using $B_{\pm} = 1 + \frac{1}{2}\mathbf{V}_{\pm}$ and discarding terms quadratic in $\mathbf{V}_{\pm}$, we get

\begin{align*}
B_{\pm}^2 &\simeq 1 + \mathbf{V}_{\pm}, \\
B_{+}\mathbf{e}_3 B_{+} &= \mathbf{e}_3 + 2\mathbf{V}_{+} \cdot \mathbf{e}_3 = \mathbf{e}_3 + \Delta v \\
B_{-}\mathbf{e}_3 B_{-} &= \mathbf{e}_3 - \Delta v \\
\langle B_{+}\mathbf{e}_3 \mathbf{e}_1 B_{-}\rangle_{\mathcal{G}} &= \frac{1}{2}(\mathbf{V}_{+} - \mathbf{V}_{-}) \mathbf{e}_3 \mathbf{e}_1 = \Delta v \mathbf{e}_1 \\
\langle B_{+}\mathbf{e}_1 B_{-}\rangle_{\mathcal{G}} &= \mathbf{e}_1 + v
\end{align*}

This gives

\begin{align*}
J &= \frac{1}{2}\rho_{+}(1 + \mathbf{V}_{+})(1 + \cos\theta) + \frac{1}{2}\rho_{-}(1 + \mathbf{V}_{-})(1 - \cos\theta) + \sqrt{\rho_{+}\rho_{-}} \Delta v \mathbf{e}_1 \sin\theta \\
\mathcal{G} &= \frac{1}{2}\rho_{+}(\mathbf{e}_3 + \Delta v)(1 + \cos\theta) - \frac{1}{2}\rho_{-}(\mathbf{e}_3 - \Delta v)(1 - \cos\theta) + \sqrt{\rho_{+}\rho_{-}} \sin\theta(\mathbf{e}_1 + v).
\end{align*}

The last term in each expression is an interference contribution that dies away as the distributions $\rho_{\pm}$ separate and cease to overlap. The scalar terms in the spin distribution arise because of relativity: it is really a distribution of the Pauli-Lubanski spin, and the boost of the rest-frame spin has a scalar contribution, but this is small because the velocities involved are much less than 1 (the speed of light).

### C. Uncertainty in Spin Measurements

We identified above the quantum spin operators usually denoted $\mathbf{\sigma} \cdot \mathbf{n}$, with unit vectors $\mathbf{n}$ in APS. Since the commutation relations of the operators are the same as those for the vectors, it is not too surprising that “uncertainty relations” exist among the measured classical spin components. Nevertheless, it is worthwhile to display the relation explicitly as it bears on our interpretation of spin. From the commutation relation $[\sigma_x, \sigma_y] = 2i\sigma_z$, one can derive the uncertainty relation

\[\Delta \sigma_x \Delta \sigma_y \geq |\langle \sigma_z \rangle|\]  \hspace{1cm} (91)

Analogous relations hold for cyclic permutations $x \rightarrow y \rightarrow z \rightarrow x$ of the indices. The relation holds classically as may be seen by a couple of examples.

For a system with classical spin $\mathbf{s} = \mathbf{e}_3$, we can write the rotor $R = 1$ as the superposition of rotors

\[R = \frac{1}{\sqrt{2}}[\exp(\mathbf{e}_1 \mathbf{e}_3 \pi/4) + \exp(-\mathbf{e}_1 \mathbf{e}_3 \pi/4)]\]  \hspace{1cm} (92)

which expands the state

\[RP_3 = \left[\frac{1}{2}(1 + \mathbf{e}_1 \mathbf{e}_3) + \frac{1}{2}(1 - \mathbf{e}_1 \mathbf{e}_3)\right]P_3 = P_1 P_3 + \bar{P}_1 P_3\]  \hspace{1cm} (93)

into eigenstates of $\mathbf{e}_1$ with equal portions filtered in the $+\mathbf{e}_1$ and $-\mathbf{e}_1$ directions. It follows for measurements of $\mathbf{s}$ along $\mathbf{e}_1$ that the root-mean-square deviation $\Delta \sigma_x$ is positive 1. The argument is similar for $\Delta \sigma_y$ except that we split $R$ into equal parts of $\exp(\pm \mathbf{e}_2 \mathbf{e}_3 \pi/4)$. We thus find $\Delta \sigma_x \Delta \sigma_y = 1 = |\langle \sigma_z \rangle|$. On the other hand, if we start in an eigenstate of $\mathbf{e}_1$, such as $RP_3 = \exp(\mathbf{e}_1 \mathbf{e}_3 \pi/4)P_3 = \sqrt{2}P_1 P_3$ we find $\Delta \sigma_x = 0 = |\langle \sigma_z \rangle|$, so that once again the uncertainty relation is satisfied.

More generally, any state $\psi = RP_3$ has the probability $2\langle \psi \psi|P_{\mathbf{n}}\rangle_S = 2\langle P_{\mathbf{s}}P_{\mathbf{n}}\rangle_S = \frac{1}{2}(1 + \mathbf{s} \cdot \mathbf{n})$ of being measured with spin $\mathbf{n}$, where $P_{\mathbf{s}} = RP_3 R^\dagger$. The average spin after filtering with $P_{\pm} = P_{\pm} P_3$ is therefore $\frac{1}{2}(1 + \mathbf{s} \cdot \mathbf{n}) - \frac{1}{2}(1 - \mathbf{s} \cdot \mathbf{n}) = \mathbf{s} \cdot \mathbf{n}$, and the mean square deviation of the measurement is

\[\langle (\Delta \mathbf{s} \cdot \mathbf{n})^2 \rangle = \frac{1}{2}(1 + \mathbf{s} \cdot \mathbf{n})(1 - \mathbf{s} \cdot \mathbf{n})^2 + \frac{1}{2}(1 - \mathbf{s} \cdot \mathbf{n})(1 + \mathbf{s} \cdot \mathbf{n})^2 = 1 - (\mathbf{s} \cdot \mathbf{n})^2\]  \hspace{1cm} (94)
The product $\Delta s \cdot e_1 \Delta s \cdot e_2$ of root-mean-square deviations is thus

$$\Delta s \cdot e_1 \Delta s \cdot e_2 = \sqrt{\left(1 - (s \cdot e_1)^2\right) \left(1 - (s \cdot e_2)^2\right)}$$

$$\geq \sqrt{1 - (s \cdot e_1)^2 - (s \cdot e_2)^2} = |s \cdot e_3|$$

which is equivalent to the uncertainty relation \[91\]

VI. CONCLUSIONS

Classical origins of fermionic spin 1/2 in the geometry of physical space are suggested by the classical formulation of elementary-particle dynamics in APS. APS itself, the geometric algebra of physical space, can be generated from a pair of fermion annihilation and creation operators, or equivalently from spin-1/2 raising and lowering operators. Spatial rotors, when projected onto a minimal left ideal of APS, are identified as two-component Pauli spinors, and experience the same change of sign under 360-degree rotations. This association is extended in a relativistic treatment to a correspondence between the classical eigenspinor $\Lambda$, the Lorentz amplitude for transformations between the rest and lab frames, and the four-component quantum Dirac spinor $\Psi$. The eigenspinor gives the orientation and motion of the particle frame as seen in the lab, and from it, an amplitude of the current density is formed, which satisfies linear equations and allows for superposition and quantum-like interference. A simple derivation of the classical Dirac equation, and its close relation to quantum formalism, illuminates the Q/C interface and demonstrates that many quantum phenomena have classical roots. Relativity is an essential part of this approach to Q/C interface. Although generated by the basis vectors of three-dimensional Euclidean space, APS includes a four-dimensional vector space with Minkowski metric of signatures $(1, 3)$ or $(3, 1)$. The classical eigenspinor and projectors in APS are powerful tools for solving problems in relativistic dynamics, but their demonstrated close relations to the quantum Dirac solutions and the implications of these relations for our understanding of quantum phenomena may be more significant.

A number of classical calculations of components, spin distributions, and measurements are seen to be fully equivalent to their quantum counterparts. An eigenspinor analysis of the Stern-Gerlach experiment shows how orthogonal components of any rotor lead to measurement results given by the eigenvalues of the measurement operator. The list of Q/C congruences grows when a classical spin rotation is included as allowed by rotational gauge freedom. The definition of a classical elementary particle as one whose motion is described by a single eigenspinor field means that the $g$-factor of such particles is 2. The boosted spin rotation gives de Broglie waves, and a measurement of their wavelength determines the rotation rate to be at the Zitterbewegung frequency. The calculated shift in the spin rotation rate in the presence of a magnetic field gives the magnetic dipole moment, and when this is combined with the $g$-factor, the magnitude of the spin as $\hbar/2$ is determined. The calculation of the magnetic moment provides new insight into the magnetic interaction of the spin with an external magnetic field and shows how the Pauli objection to putting time on the same footing as spatial coordinates is resolved in APS.

In spite of these many classical associations, there exists an essential core of quantum behavior that does not seem to have a classical correlate (at least none has yet been identified). Central to this core is the existence of quanta themselves and associated Born interpretation of the particle current $J$ as a probability current. These are key to the measurement problem in quantum mechanics and the attendant question of wave-function collapse. The classical eigenspinor apparently has little to say about these core problems except to emphasize the central role of amplitudes even in classical descriptions, and as argued above, the presence of a rapid intrinsic rotation would require that measurements of elemental quanta see them as point-like objects. One may also speculate that the generation of APS from annihilation/creation operators may yield a formulation of measurements that give quanta in analogy to the measurement of quantized states in a Stern-Gerlach experiment. In this case, however, the expansion in “filters” $1 = P_3 + P_3 = a a^+ + a^+ a$ would be imposed not on the left of the spinor, where we saw they would act as spin-polarization filters [Eq. 70] nor on the right, where we saw they would select chirality [Eq. 35], but in some other way yet to be imposed on the formalism.

Some might argue that because our spinor approach emphasizes the role of amplitudes that can interfere, it is essentially quantum rather than classical in nature. Indeed, the dividing line at the Q/C interface has become indistinct in places, but recall that the eigenspinor arises naturally in the APS treatment of classical dynamics. Furthermore, the analysis presented here is based on the classical eigenspinor approach rather than on a specific classical model. We have thus avoided assumptions, for example about the mechanical structure or charge distribution, that are not required by the geometry of space. While the lack of a concrete model will be frustrating to some, it appears necessary in order to ensure a secure mathematical and experiential footing for our approach. In the geometrical picture that emerges of spin-1/2 fermions, there are some differences from conventional pictures of quantum properties. In APS, the spin of any fermion in a pure state is a vector of length $\hbar/2$, equal to the magnitude of measured components.
Furthermore, the spin in a pure state has a definite direction with exact components. This differs from the conventional picture of fermion spin as a vector of length $\sqrt{3}\hbar/2$ with uncertain components except in the measured direction. The interpretation suggested by APS is instead that of the a traditional spin-density operator formulation or the description by Levitt. The uncertainty relations for the spin arise here from the measurement process, and the supposed length of $\sqrt{3}\hbar/2$ arises from the square root of $e_1^2 + e_2^2 + e_3^2$ and the (mis)-identification of the basis vectors with spin component operators.

We have concentrated here on single-particle systems and have therefore not discussed the important statistical properties of fermions or the classical view of entanglement. Much more work is needed. Nevertheless, significant progress in understanding single-fermion spin in classical terms has been reported here, and multiple qubit systems have been studied elsewhere with tensor products of APS and comparisons of explicit quantum and classical-eigenspinor solutions with spin have been undertaken.

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