Diameter of Io-Decomposable Riordan Graphs of the Bell Type

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Abstract
Recently, in the paper (Cheon et al. in Linear Algebra Appl 579:89–135, 2019) we suggested the two conjectures about the diameter of io-decomposable Riordan graphs of the Bell type. In this paper, we give a counterexample for the first conjecture. Then we prove that the first conjecture is true for the graphs of some particular size and propose a new conjecture. Finally, we show that the second conjecture is true for some special io-decomposable Riordan graphs.

Keywords Riordan graph · Io-decomposable Riordan graph · Diameter · Catalan graph

Mathematics Subject Classification 05C75 · 05A15

1 Introduction

Let $\kappa[[z]]$ be the ring of formal power series in the variable $z$ over an integral domain $\kappa$. A Riordan matrix [12] $L = (\ell_{ij})_{i,j \geq 0}$ is defined by a pair of formal power series $(g,f) \in \kappa[[z]] \times \kappa[[z]]$ with $f(0) = 0$ such that $[z^i]g^j = \ell_{ij}$ for integers $i, j \geq 0$ where $[z^i]$ is the coefficient extraction operator. Usually, the Riordan matrix is denoted by $L = (g,f)$ and its leading principal submatrix of order $n$ is denoted by $(g,f)_n$, i.e. $(g,f)_n = (\ell_{ij})_{0 \leq i,j \leq n-1}$. Since $f(0) = 0$, every Riordan matrix $(g,f)$ is an infinite lower triangular matrix. Most studies on the Riordan matrices were related to combinatorics [1, 8, 14, 15, etc.] or algebraic structures [2, 5, 6, 13, etc.].
Throughout this paper, we write \( a \equiv b \pmod{2} \) for \( a \equiv b \) (mod 2) and denote the \( n \)-set \( \{1, 2, \ldots, n\} \) by \([n]\).

Recently, we in [3, 4] introduced a Riordan graph by using the notion of the Riordan matrix modulo 2 as follows.

**Definition 1.1** A simple labelled graph \( G \) with \( n \) vertices is a Riordan graph of order \( n \) if the adjacency matrix \( A(G) = [r_{i,j}]_{i,j \in [n]} \) of \( G \) is an \( n \times n \) symmetric \((0, 1)\)-matrix given by

\[
A(G) \equiv (zg, f)_n + (zg, f)^T_n, \quad \text{i.e.} \; r_{i,j} = r_{j,i} = \begin{cases} [z^{i-2}]gf^{j-1} & \text{if } i > j \\ 0 & \text{if } i=j \end{cases}
\]

for some Riordan matrix \((g, f)\) over \( \mathbb{Z} \). We denote such \( G \) by \( G_n(g, f) \). In particular, the Riordan graph \( G_n(g, f) \) is called proper if \([z^0]g = [z^1]f = 1\).

For example, consider the Catalan graph \( C_{G_n} = G_n(C(z), zC(z)) \) where \( C(z) \) is the generating function for the Catalan numbers, i.e.

\[
C(z) = \frac{1 - \sqrt{1 - 4z}}{2z} = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} z^n = 1 + z + 2z^2 + 5z^3 + 14z^4 + \cdots. \tag{1}
\]

Then the Catalan matrix is given by

\[
(C(z), zC(z)) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 0 \\
5 & 5 & 3 & 1 & 0 \\
14 & 14 & 9 & 4 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}.
\]

By Definition 1.1, we have the Catalan graph \( C_{G_n} := G_n(C(z), zC(z)) \) of order \( n \) whose adjacency matrix is

\[
A(C_{G_n}) \equiv (zC(z), zC(z))_n + (zC(z), zC(z))^T_n.
\]

We note that \((zC(z), zC(z))_n = L_n(C(z), zC(z))_n \) if \( L_n := [\delta_{i,j+1}]_{0 \leq i,j \leq n-1} \) is a row shift matrix where \( \delta_{i,j} \) is the Kronecker delta symbol. In particular, when \( n = 6 \) we have Fig. 1.

An induced subgraph of a graph is another graph, formed from a subset of the vertices of the graph and all of the edges connecting pairs of vertices in that subset. For a subset \( S \) of vertices \( V \), we let \( \langle S \rangle \) denote the graph induced by the vertices in \( S \subseteq V \).

In [3], we studied the structural properties of families of Riordan graphs obtained from infinite Riordan graphs, which include a fundamental decomposition theorem and certain conditions on Riordan graphs to have an Eulerian trail/cycle or a Hamiltonian cycle. A Riordan graph \( G_n(g, f) \) is called Bell type if \( f = zg \). Moreover, we studied the following Riordan graphs of special Bell type.
Definition 1.2 [3] Let \( G_n = G_n(g,f) \) be a proper Riordan graph with the odd and even vertex sets \( V_o = \{ i \in V(G_n) \mid i \equiv 1 \} \) and \( V_e = \{ i \in V(G_n) \mid i \equiv 0 \} \), respectively. The graph \( G_n \) is said to be io-decomposable if \( \langle V_o \rangle \cong G_{[n/2]}(g,f) \) and \( \langle V_e \rangle \) is a null graph. We note that “io” stands for “isomorphically odd”.

It is known [3, Example 4.8] that the Catalan graph \( CG_n \) is io-decomposable. For instance, let us consider \( CG_6 \). Then we see that \( \langle \{1,3,5\} \rangle \cong CG_3 \) and \( \langle \{2,4,6\} \rangle \) is a null graph from Fig. 1.

A vertex in a graph \( G \) is universal if it is adjacent to all other vertices in \( G \). A \( k \)-partite graph is a graph whose vertices can be partitioned into \( k \) different independent sets. The chromatic number of a graph \( G \) is the smallest number of colors needed to color the vertices of \( G \) so that no two adjacent vertices share the same color. A clique is a subset of vertices of a graph \( G \) such that its induced subgraph is a complete graph. The clique number of \( G \) is the number of vertices in a maximum clique in \( G \). The distance between two vertices \( u \) and \( v \) in a graph \( G \) is the number of edges in a shortest path between \( u \) and \( v \). The diameter of \( G \) is the maximum distance between all pairs of vertices, and it is denoted by \( \text{diam}(G) \).

We found several properties of an io-decomposable Riordan graph of the Bell type as follows.

Lemma 1.3 [3] Let \( G_n = G_n(g,zg) \) be an io-decomposable Riordan graph of the Bell type. Then we have the following.

(i) If \( n = 2^k + 1 \) for \( k \geq 0 \), then \( G_n \) and \( G_{n+1} \) have at least one universal vertex, namely \( 2^k + 1 \).

(ii) \( G_n \) is a \( \lceil \log_2 n \rceil + 1 \)-partite graph.

(iii) The chromatic number and the clique number of \( G_n \) are \( \lceil \log_2 n \rceil + 1 \).

(iv) The diameter of \( G_n \) is bounded by \( \text{diam}(G_n) \leq \lceil \log_2 n \rceil \). In particular, if \( n = 2^k + 2 \) or \( 2^{k+1} + 1 \), for \( k \geq 1 \), then \( \text{diam}(G_n) = 2 \).

(v) If \( 2^k + 1 < n < 2^{k+1} \), then \( \text{diam}(G_n) \leq \lceil \log_2 (n - 2^k) \rceil + 1 \).

A graph is called weakly perfect if its chromatic number equals its clique number. By Lemma 1.3(iii), every io-decomposable Riordan graph of the Bell type is weakly perfect.
It is known [9] that almost all $K_k$-free graphs are $(k - 1)$-partite for $k \geq 3$. By (ii) and (iii) of Lemma 1.3, every io-decomposable Riordan graph $G_n(g, zg)$ of the Bell type is $K_{\lceil \log_2 n \rceil + 2}$-free and $(\lceil \log_2 n \rceil + 1)$-partite for $n \geq 2$. Thus the io-decomposable Riordan graph of the Bell type is a very interesting object in Riordan graph theory.

It is known [3] that the Pascal graph $PG_n = G_n(1/(1 - z), z/(1 - z))$ and the Catalan graph $CG_n = G_n(C(z), zC(z))$ are the io-decomposable Riordan graphs of the Bell type. The following two conjectures introduced in [3] show significance of the Pascal graph $PG_n$ and the Catalan graph $CG_n$.

**Conjecture 1** [3] Let $G_n$ be an io-decomposable Riordan graph of the Bell type. Then

$$2 = \text{diam}(PG_n) \leq \text{diam}(G_n) \leq \text{diam}(CG_n)$$

for $n \geq 4$. Moreover, $PG_n$ is the only graph in the class of io-decomposable graphs of the Bell type whose diameter is 2 for all $n \geq 4$.

**Conjecture 2** [3] We have that $\text{diam}(CG_{2^k}) = k$ and there are no io-decomposable Riordan graphs $G_{2^k} \not\cong CG_{2^k}$ of the Bell type satisfying $\text{diam}(G_{2^k}) = k$ for all $k \geq 1$.

We note that $\text{diam}(PG_n) = 1$ if $n = 2, 3$ and $\text{diam}(PG_n) = 2$ if $n \geq 4$ since the vertex 1 is adjacent to all other vertices, $PG_n \cong K_n$ if $n = 2, 3$, and $PG_n \not\cong K_n$ if $n \geq 4$.

In this paper, we first give a counterexample of the upper bound in Conjecture 1. Then we prove that the upper bound in Conjecture 1 is true for the graph of some particular size and we propose a new conjecture for an upper bound of the diameter of an io-decomposable Riordan graph of the Bell type. Finally, we show that Conjecture 2 is true for some special io-decomposable Riordan graphs.

### 2 Upper Bound of Conjecture 1

It is known [10] that an infinite lower triangular matrix $L = [\ell_{i,j}]_{i,j \geq 0}$ with $\ell_{0,0} \neq 0$ is a proper Riordan matrix if and only if there is a unique sequence $(a_0, a_1, \ldots)$ with $a_0 \neq 0$ such that, for $i \geq j \geq 0$,

$$\ell_{i+1,j+1} = a_0 \ell_{i,j} + a_1 \ell_{i,j+1} + \cdots + a_{i-j} \ell_{i,i}.$$  

The sequence $(a_i)_{i \geq 0}$ is called the $A$-sequence of the Riordan array. Also, if $L = (g,f)$ then

$$f = zA(f), \quad \text{or equivalently } \quad A = z/f$$

where $A = \sum_{i \geq 0} a_i z^i$ is the generating function for the $A$-sequence of $(g,f)$ and $f$ is the compositional inverse of $f$. The $(0, 1)$-matrix $M$, an integer matrix in which each element is a 0 or 1, is called binary Riordan matrix if there is a pair of generating functions $g$ and $f$ such that $M \equiv (g,f)$. If $B$ is a binary Riordan matrix $B = [b_{i,j}]_{i,j \in \mathbb{N}_0} \equiv (g,f)$ with $f'(0) = 1$ then the sequence is called the binary A-sequence.
(1, a₁, a₂, ...) where ak ∈ {0, 1}. In particular, if B is the Bell type, i.e. f = zg, then for i ≥ j ≥ 0

\[ b_{0,0} = 1, \quad b_{i+1,0} = a_1 b_{i,0} + a_2 b_{i,1} + \cdots + a_i b_{i,j}; \]
\[ b_{i+1,j+1} = b_{i,j} + a_1 b_{i,j+1} + \cdots + a_i b_{i,i}. \]  

(3)

Hence each binary Riordan matrix B ≡ (g, zg) with g(0) = 1 is uniquely determined by its binary A-sequence.

We in [3] characterized the io-decomposable Riordan graph Gn(g, zg) of the Bell type, see the following lemma.

**Lemma 2.1** [3] Let Gn = Gn(g, zg) be a Riordan graph of the Bell type. Then Gn is io-decomposable if and only if the binary A-sequence of (g, zg) is (1, 1, a₂, a₃, a₄, a₅, ...) where a₂j ∈ {0, 1} for all j ≥ 1.

Let Gₙ = Gₙ(g, zg) be the io-decomposable Riordan graph of the Bell type with its binary A-sequence generating function A(z) = \[ \sum_{i=0}^{15} z^i = \frac{1-z^{16}}{1-z}. \] We note that the function g satisfies the functional equation 1 – g + zg² - z¹⁶g¹⁶ = 0 by the first equation of (2). By Lemma 2.1, the Riordan graph Gₙ is io-decomposable. By using the Sagemath [11] which is a free open-source mathematics software system licensed under the GPL, we compare the diameters between CGₙ and Gₙ up to degree n = 100. Then we find the following 13 counterexamples for the upper bound of Conjecture 1.

| n   | 44  | 45  | 46  | 47  | 48  | 78  | 79  | 80  | 87  | 88  | 89  | 90  | 91  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| diam(CGₙ) | 3   | 3   | 3   | 3   | 3   | 3   | 3   | 3   | 3   | 3   | 3   | 3   | 3   |
| diam(Gₙ)  | 4   | 4   | 4   | 4   | 4   | 4   | 4   | 4   | 4   | 4   | 4   | 4   | 4   |

Since the binary A-sequence generating function of Gₙ is A(z) = \[ \sum_{i=0}^{15} z^i \], we can obtain the adjacency matrix Fig. 2 of G₄₄ by (3). From the 16th row of G₄₄ in Fig. 2, we see that the vertex 16 is adjacent to vertices 15, 17 and 33. From the 44th row of G₄₄ in Fig. 2, we also see that the vertex 44 is adjacent to vertices 39 and 43. By checking the ith row in Fig. 2 for each i ∈ {15, 17, 33, 39, 43}, 16 → 15 → 13 → 39 → 44 is the shortest path between vertices 16 and 44, i.e. the distance between 16 and 44 is four. Thus the diameter of G₄₄ is four by Lemma 1.3(v). It is known [3] that the binary A-sequence generating function of CGₙ is A(z) = (1 – z)⁻¹. Then we can obtain the adjacency matrix Fig. 3 of CG₄₄ by (3). From the 44th row of CG₄₄ in Fig. 3, the vertex 33 is adjacent to all other vertices except seven vertices in X := {36, 38, 39, 40, 42, 43, 44}. Therefore a distance between any pair of vertices in V(CG₄₄) \ X is at most 2. Let u ∈ V(CG₄₄) and v ∈ X. Then we now remain to show that a distance between two vertices u and v is at most 3. We checked that this is true however this is complicated part.

If for a Riordan graph Gₙ(g, f) with [z¹]f = 1, the relabelling is done by reversing the vertices in [n], that is, by replacing a label i by n + 1 – i for each i ∈ [n], then
the resulting graph will always be a Riordan graph given by the following lemma. We denote the reverse relabelling of $G_n$ by $G_n^r$.

Lemma 2.2 [3] The reverse relabelling of a Riordan graph $G_n(g,f)$ with $f'(0) = 1$ is the Riordan graph.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{G_44}
\caption{The adjacency matrix of $G_{44}$}
\end{figure}
\[ G'_n(g, f) = G_n(g(\tilde{f}) \cdot (\tilde{f})' \cdot (z/\tilde{f})^{n-1}, \tilde{f}) \]

where \( \tilde{f} \) is the compositional inverse of \( f \).

Now, we prove that the upper bound in Conjecture 1 is true if \( n = 2^k \), \( n = 2^k - 1 \) or \( n = 1 + 2^m + 2^k \) where \( k \geq 2 \) and \( 1 \leq m < k \).
Lemma 2.3 Let $G_n = G_n(g, zg)$ be a proper Riordan graph and $A(z)$ be the generating function for its binary $A$-sequence. Then the reverse relabelling of $G_n$ is the Riordan graph given by

$$G'_n = G_n((zA(z))', A^{n-2}(z), z/A(z)).$$

In particular, if $G_n$ is an io-decomposable Riordan graph of the Bell type then the reverse relabelling of $G_n$ is the Riordan graph given by

$$G'_n = G_n(A'(z) \cdot A^{n-2}(z), z/A(z)).$$

Proof Let $f = zg$ and $\bar{f}$ be the compositional inverse of $f$. Since (2) leads to

$$g(\bar{f}) = z/\bar{f} = A(z) \quad \text{and} \quad (\bar{f})' = \left(\frac{z}{A(z)}\right)' = \frac{A(z) + zA'(z)}{A^2(z)} = (zA(z))'A^{-2}(z),$$

we obtain

$$g(\bar{f}) \cdot (\bar{f})' \cdot (z/\bar{f})^{n-1} \equiv (zA(z))' \cdot A^{n-2}(z) \quad \text{and} \quad \bar{f} = z/A(z).$$

Thus, by Lemma 2.2, we obtain the desired result. In particular, if $G_n$ is an io-decomposable Riordan graph of the Bell type then it follows from Lemma 2.1 that $(zA(z))' \equiv A'(z)$. Hence the proof follows.

If the base $p$ (a prime) expansion of $n$ and $m$ is $n = n_0 + n_1p + n_2p^2 + \cdots$ and $m = m_0 + m_1p + m_2p^2 + \cdots$, respectively, then

$$\binom{n}{m} \equiv \prod_i \binom{n_i}{m_i} \pmod{p}.$$

This is called the Lucas’s theorem.

Let $G_n$ be an io-decomposable Riordan graph of the Bell type. Since $\text{diam}(G_n) \leq k$ if $n = 2^k$ with $k \geq 0$ by Lemma 1.3(iv), the following theorem shows that the upper bound of Conjecture 1 is true if $n = 2^k$ for $k \geq 1$. We denote the distance between two vertices $u$ and $v$ in a graph $G$ by $d_G(u, v)$.

Theorem 2.4 For an integer $k \geq 1$, we obtain

$$\text{diam}(CG_{2^k}) = k.$$

Proof First we show that $CG_{2^k} = G_{2^k}(1, z + z^2)$ for $k \geq 1$. It follows from (1) and (2) that the generating function for $A$-sequence of $(C, zC)$ is $1/(1-z)$. By Lemma 2.3, we obtain
\[
A'(z) \cdot A^{2^k-2}(z) = \left(\frac{1}{1-z}\right)^2 \cdot \left(\frac{1}{1-z}\right)^{2^k-2} = \left(\frac{1}{1-z}\right)^{2^k}
\]
so that by Lemma 2.2 the reverse relabelling of the Catalan graph is
\[
CG_{2^k}^\ast = G_{2^k}((1-z)^{-2^k}, z + z^2).
\] (4)
Since \((1-z)^{-2^k} = \sum_{j \geq 0} \binom{2^k+j-1}{j} w^j\), by Lucas’s theorem we obtain
\[
\binom{2^k+j-1}{2^k-1} \equiv 1 \quad \text{for } j = 0 \quad \text{and} \quad \binom{2^k+j-1}{2^k-1} \equiv 0 \quad \text{for } 1 \leq j \leq 2^k - 1
\]
which imply \(G_{2^k}((1-z)^{-2^k}, z + z^2) = G_{2^k}(1, z + z^2)\). Thus, by (4), \(CG_{2^k} = G_{2^k}(1, z + z^2)\). Let \(m_i = \max\{j \in V(CG_{2^k}) \mid ij \in E(CG_{2^k})\}\). For each \(i \in \{1, 2, \ldots, 2^{k-1}\}\), we obtain
\[
m_i = \max\{j \in V(CG_{2^k}) \mid [z^{j-2}](z + z^2)^{i-1} = 1\} = 2i.
\] (5)
Since \(m_1 < m_2 < \cdots < m_{2^k-1}\), a unique shortest path from 1 to \(2^k\) in \(CG_{2^k}\) is \(1 \rightarrow 2^1 \rightarrow \cdots \rightarrow 2^{k-1} \rightarrow 2^k\) and so \(d_{CG_{2^k}}(1, 2^k) = k\). Hence, by (iv) of Lemma 1.3, we obtain the desired result.

By Lemma 2.1, the following lemma is obtained from [3, Theorem 3.11] when \(\ell = 0\).

**Lemma 2.5** Let \(G = G_{n}(g,f)\) be an io-decomposable Riordan graph of the Bell type. Then, for each \(i, j \in V(G)\) with \(i < j\), we obtain
\[
<\{i, i+1, \ldots, j\} > \cong <\{i + m(j-i), j + m(j-i) + 1, \ldots, i + m(j-i)\}>
\]
for a positive integer \(m\) satisfying
\[
\frac{1-i}{j-i} \leq m \leq \frac{n-j}{j-i}.
\]

We can ask that how many vertex pairs \((u, v)\) can have the maximal distance \(k\) in \(CG_{2^k}\). By using Lemma 2.5, the answer is given by the following theorem.

**Theorem 2.6** Let \(k \geq 1\) be an integer. There exist exactly \(2^{k-1}\) vertex pairs \((i, 2^k)\) with \(i \in \{1, \ldots, 2^{k-1}\}\) such that \(d_{CG_{2^k}}(i, 2^k) = k\) is the maximal distance in \(CG_{2^k}\).

**Proof** Since \(CG_{2^k}^\ast\) is the reverse relabelling of \(CG_{2^k}\), this theorem is equivalent to the following:
- \(d_{CG_{2^k}}(i,j) = k\) if \(i = 1\) and \(j \in \{2^{k-1}+1, \ldots, 2^k\}\);
- \(d_{CG_{2^k}}(i,j) \leq k - 1\) otherwise.
Since by (i) of Lemma 1.3 the vertex $2^{k-1} + 1$ is adjacent to all vertices $1, \ldots, 2^{k-1}$ in $CG_{2^k}$, the vertex $2^{k-1}$ is adjacent to all vertices $2^{k-1} + 1, \ldots, 2^k$ in $CG'_{2^k}$. Therefore, by (5), the shortest path from 1 to $j$ in $CG'_{2^k}$ is

$$2^0 \to 2^1 \to \cdots \to 2^{k-1} \to j$$

where $j \in \{2^{k-1} + 1, \ldots, 2^k\}$

and so $d_{CG'_{2^k}}(i,j) = k$ if $i = 1$ and $j \in \{2^{k-1} + 1, \ldots, 2^k\}$.

Let $V_1 = \{u \in V(CG'_{2^k}) \mid 1 \leq u \leq 2^{k-1}\}$ and $V_2 = \{v \in V(CG'_{2^k}) \mid 2^{k-1} < v \leq 2^k\}$. Then, by applying Lemma 2.5 to $m = 2$, we obtain $<V_1> \cong <V_2> \cong CG'_{2^{k-1}}$. Since $CG'_{2^{k-1}} \cong CG_{2^{k-1}}$, by Theorem 2.4, we obtain

$$d_{CG'_{2^{k-1}}}(i,j) \leq k - 1$$

if $i,j \in V_1$ or $i,j \in V_2$.

Now it is enough to show that $d_{CG'_{2^k}}(i,j) < k$ if $i \in V_1 \setminus \{1\}$ and $j \in V_2$ for $k \geq 2$. We prove this by induction on $k \geq 2$. Let $k = 2$. Since the adjacency matrix of $CG'_2$ is given by

$$A(CG'_2) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix},$$

we see that $d_{CG'_2}(2,3) = d_{CG'_2}(2,4) = 1 < 2$. Thus it holds for $k = 2$. Let $k \geq 3$.

Since $<V_1> \cong CG'_{2^{k-1}}$ and the vertex $2^{k-1}$ is adjacent to all vertices $j \in V_2$ in $CG'_{2^k}$, we obtain

$$d_{CG'_{2^k}}(i,j) \leq d_{CG'_{2^k}}(i,2^{k-1}) + d_{CG'_{2^k}}(2^{k-1},j) \leq d_{CG'_{2^{k-1}}}(i,2^{k-1}) + 1$$

$$\leq k - 1$$ (by induction)

where $i \in V_1 \setminus \{1\}$ and $j \in V_2$. Hence the proof follows.

**Example 2.7** Let us consider the Catalan graph $CG_8 = G_8(C(z), zC(z))$ of order 8.

Since its reverse relabeling is $CG'_8 = G_8(1, z + z^2)$, we obtain Fig. 4 from the adjacency matrix

![Graph](image-url)
Thus we can see that the four vertex pairs (1, 5), (1, 6), (1, 7) and (1, 8) in $CGr_8$ have maximal distance 3 i.e., the four vertex pairs (8, 4), (8, 3), (8, 2) and (8, 1) in $CG_8$ have the maximal distance 3.

Let $G_n$ be an io-decomposable Riordan graph of the Bell type. Since it follows from (iv) of Lemma 1.3 that $\text{diam}(G_n) \leq k - 1$ if $n = 2^k - 1$, the following corollary shows that the upper bound of Conjecture 1 is true if $n = 2^k - 1$ for $k \geq 1$.

**Corollary 2.8** For an integer $k \geq 1$, we obtain

$$\text{diam}(CG_{2^k-1}) = k - 1.$$  

**Proof** By Theorem 2.6, we obtain $\text{diam}(CG_{2^k-1}) \leq k - 1$. It follows from Lemma 2.3 that one can show $CG_{2^k-1} = G_n(1 + z, z + z^2)$. By using the similar proof in Theorem 2.4, we can show that $2^1 - 1 \rightarrow 2^2 - 1 \rightarrow \cdots \rightarrow 2^k - 1$ is the shortest path from 1 to $2^k - 1$ in $CG_{2^k-1}$, i.e. $d_{CG_{2^k-1}}(1, 2^k - 1) = k - 1$. Since it follows from Theorem 2.6 that $\text{diam}(CG_{2^k-1}) \leq k - 1$, we obtain $\text{diam}(CG_{2^k-1}) = k - 1$. Hence the proof follows. 

The following lemma is useful to obtain Theorem 2.10 and Conjecture 3.

**Lemma 2.9** Let $n = 1 + 2^m + \sum_{j=0}^s 2^{k+j}$ be an integer with $k > m \geq 1$. If $G_n$ be the io-decomposable Riordan graph of the Bell type, then we obtain

$$\text{diam}(G_n) \leq \begin{cases} s + 2 & \text{if } m = 1; \\ s + 3 & \text{otherwise.} \end{cases}$$  

**Proof** We prove this by induction on $s \geq 0$. Let $s = 0$, i.e. $n = 1 + 2^m + 2^k$. If $m = 1$ then it follows from (v) of Lemma 1.3 that $\text{diam}(G_{2^k+3}) = 2$. For $k > m \geq 2$, let $V_1 = \{i \in V(G_n) \mid 1 \leq i \leq 2^k + 1\}$ and $V_2 = \{j \in V(G_n) \mid 2^k + 1 \leq j \leq n\}$. Since $\langle V_1 \rangle \cong G_{2^k+1}$ and $\langle V_2 \rangle \cong G_{2^m+1}$ by Lemma 2.5, it follows from (iv) of Lemma 1.3 that $\text{diam}(\langle V_1 \rangle) = \text{diam}(\langle V_2 \rangle) = 2$. Let $i \in V_1 \setminus \{2^k + 1\}$ and $j \in V_2 \setminus \{2^k + 1\}$. Now it is enough to show that $d(i, j) \leq 3$. Since the vertices $2^k + 1$
and $2^k + 2^m + 1$ are the universal vertices in $<V_1>$ and $<V_2>$ respectively, we obtain
\[
\begin{align*}
d_{G_n}(i,j) &\leq d_{G_n}(i,2^k+1) + d_{G_n}(2^k+1,2^k+2^m+1) + d_{G_n}(2^k+2^m+1,j) \\
&\leq d_{<V_1>}((i,2^k+1) + d_{<V_2>}((2^k+1,2^k+2^m+1) + d_{<V_2>}((2^k+2^m+1,j) \\
&\leq 3.
\end{align*}
\]

Thus the theorem holds for $s = 0$.

Let $s \geq 1$, i.e. $n = 1 + 2^m + \sum_{j=0}^{s-1} 2^{k+j}$. For $k > m \geq 1$, let $W_1 = \{i \in V(G_n) \mid 1 \leq i \leq 2^{k+s} + 1\}$ and $W_2 = \{j \in V(G_n) \mid 2^{k+s} + 1 \leq j \leq n\}$. Since by Lemma 2.5 we obtain $<W_1> \cong G_{2^{k+s}+1}$ and $<W_2> \cong G_{n-2^{k+s}}$, by (iv) of Lemma 1.3 we obtain $\text{diam}(<W_1>) = 2$ and by induction we obtain $\text{diam}(<W_2>) \leq s + 1$ if $m = 1$ or $\text{diam}(<W_2>) \leq s + 2$ if $k > m > 1$. Let $i \in W_1 \setminus \{2^{k+s} + 1\}$ and $j \in W_2 \setminus \{2^{k+s} + 1\}$. Now it is enough to show that $d_{G_n}(i,j) \leq s + 2$ if $m = 1$ or $d_{G_n}(i,j) \leq s + 3$ if $k > m > 1$. Since the vertices $2^{k+1} + 1$ are the universal vertices in $<W_1>$, we obtain
\[
d_{G_n}(i,j) \leq d_{G_n}(i,2^{k+s} + 1) + d_{G_n}(2^{k+s} + 1,j) \leq 1 + d_{G_{n-2^s}}(2^{k+s} + 1,j).
\]

Hence, by induction, we obtain the desired result. \hfill \square

From Lemma 2.9, the following theorem shows that the upper bound of Conjecture 1 is true if $n = 1 + 2^m + 2^k$ for $k > m \geq 1$.

**Theorem 2.10** Let $k$ and $m$ be integers with $k > m \geq 1$. Then
\[
\text{diam}(CG_{1+2^m+2^k}) = \begin{cases} 2 & \text{if } m = 1; \\ 3 & \text{otherwise.} \end{cases}
\]

**Proof** Since by Lemma 2.9 we obtain $\text{diam}(CG_{2^k+3}) = 2$, it is enough to show that $\text{diam}(CG_{1+2^m+2^k}) = 3$ for $k > m > 1$. Now let $k$ and $m$ be integers with $k > m > 1$. By Lemma 2.2, the reverse relabelling of the Catalan graph $CG_{2^k+2^m+1}$ is
\[
CG_{1+2^m+2^k}^r = G_{1+2^m+2^k}(\frac{1}{1-z} - \frac{2^m - 2^k}{z + z^2}). \tag{6}
\]

Let $A(CG_{1+2^m+2^k}) = [c_{i,j}]$ and $A(CG_{1+2^m+2^k}^r) = [r_{i,j}]$. By (6), we obtain
\[
c_{2^m+2^k,j} = r_{2^m+2^k+2^j,j} = \begin{cases} 1 & \text{if } j = 2^k + 2^m + 1; \\ 0 & \text{if } j = 2^k + 2^m; \\ [z^{2^k+2^j-2^m}]z(1-z)^{2^m-2^j - 2^m} & \text{otherwise.} \end{cases} \tag{7}
\]

Since
\[
\left[ z^{2^k+2^m-j} \right] z(1-z)^{-2^{k+m}} = \left( \frac{2^{k+1} + 2^{m+1} - j - 2}{2^{k} + 2^{m} - 1} \right),
\]

by Lucas’s theorem we obtain for \( j = 1, \ldots, 2^k + 2^m - 1 \)

\[
c_{2^k+2^m,j} = \begin{cases} 
1 & \text{if } j \in \{2^{m+1} + r2^{m} - 1 \mid t = 0, \ldots, 2^{k-m} - 1\}; \\
0 & \text{otherwise}.
\end{cases}
\]  

(8)

By (7) and (8), the set \( N(2^k + 2^m) \) of neighbors of the vertex \( 2^k + 2^m \) in \( CG_{2^k+2^m+1} \) is

\[
N(2^k + 2^m) = \{2^{m+1} + t2^m - 1 \mid t = 0, \ldots, 2^{k-m} - 1\} \cup \{2^k + 2^m + 1\}.
\]

It is known [7] that \( [z^n] C(z) \equiv 1 \) if and only if \( n = 2^k - 1 \) for \( k \geq 1 \). It implies

\[
c_{i,1} = \begin{cases} 
1 & \text{if } j \in \{2^{s} + 1 \mid s = 0, 1, \ldots, k\}; \\
0 & \text{otherwise}.
\end{cases}
\]

Thus the set \( N(1) \) of neighbors of the vertex 1 in \( CG_{2^k+2^m+1} \) is

\[
N(1) = \{2^s + 1 \mid s = 0, \ldots, k\}.
\]

Since

\[
2^k + 2^m \not\in N(1), \quad 1 \not\in N(2^k + 2^m) \quad \text{and} \quad N(1) \cap N(2^k + 2^m) = \emptyset,
\]

the distance between vertices 1 and \( 2^k + 2^m \) in \( CG_{1+2^m+2^k} \) is at least 3 so that by Lemma 2.9 we obtain \( \text{diam}(CG_{1+2^m+2^k}) = 3 \). Hence the proof follows.

We end this section with the following conjecture.

**Conjecture 3** Let \( n = 1 + 2^m + \sum_{j=0}^{s} 2^{k+j} \) be an integer with \( k \geq m \geq 1 \) and \( s \geq 1 \). Then

\[
\text{diam}(CG_n) = \begin{cases} 
s + 2 & \text{if } m = 1; \\
s + 3 & \text{otherwise}.
\end{cases}
\]

**Remark 2.11** If Conjecture 3 is true, then by Lemma 2.9 the upper bound of Conjecture 1 is true if \( n = 1 + 2^m + \sum_{j=0}^{s} 2^{k+j} \) for \( k \geq m \geq 1 \) and \( s \geq 1 \). By using the Sagemath, we have checked that Conjecture 3 is true for \( n \leq 255 = 1 + 2 + \sum_{j=0}^{5} 2^{2+j} \). We note that the next value of 255 is 387 = 1 + 2 + \sum_{j=0}^{1} 2^{2+j} \) to check Conjecture 3 but the Sagemath cannot evaluate this value.
Lemma 3.1 Let $G_2 = G_2(g, zg)$ be an io-decomposable Riordan graph. If there exists $k \geq 2$ such that $\text{diam}(G_{2^k}) = s$ then $\text{diam}(G_{2^k+m}) \leq s + m$ for all $m \geq 1$.

Proof Let $V_1 = \{i \in V(G_n) \mid 1 \leq i \leq 2^k+1 \}$ and $V_2 = \{j \in V(G_n) \mid 2^k+1 \leq j \leq 2^{k+m} \}$ be the vertex subsets of $V(G_{2^k+m})$. Then, by Lemma 2.5, we obtain $<V_2> \cong G_{2^k+m-1}$. Since $<V_1> \cong G_{2^k+m-1+1}$ has a universal vertex $2^k+m-1$, we obtain

$$\text{diam}(G_{2^k+m}) \leq \text{diam}(<V_2>)+1 = \text{diam}(G_{2^k+m-1})+1.$$  

(9)

Let $\text{diam}(G_2) = s$. Applying for $m = 1$ in (9), we obtain $\text{diam}(G_{2^{k+1}}) \leq s + 1$. Applying again for $m = 2$ in (9), we obtain $\text{diam}(G_{2^{k+2}}) \leq s + 2$. By repeating this process, we obtain the desired result. □

Let $B(g,f)$ denote a binary Riordan matrix, i.e. $B(g,f) = (g,f)$. We note that a Riordan matrix $[b_{i,j}]_{i,j \geq 0}$ is of the Bell type given by $B(g,zg)$ with $g(0) = 1$ if and only if, for $i \geq j \geq 0$,

$$b_{i+1,0} = a_1 b_{i,0} + a_2 b_{i,1} + \cdots + a_{i+1} b_{i,i},$$ 

$$b_{i+1,j+1} = b_{i,j} + a_1 b_{i,j+1} + \cdots + a_{i-j} b_{i,i}$$

(10)

where $(1, a_1, \ldots)$ is the binary A-sequence of $B(g,zg)$. Let $G_n = G_n(g,zg)$ and $\mathcal{A}(G_n) = [r_{ij}]_{1 \leq i,j \leq n}$ where $g(0) = 1$. Since $r_{ij} = b_{i-2,j-1}$ for $i \geq j \geq 1$, by (10) we need the finite term $(1, a_1, \ldots, a_{n-2})$ of the binary A-sequence to determine $\mathcal{A}(G_n)$.

Theorem 3.2 Let $G_{2^k} = G_{2^k}(g,zg)$ be an io-decomposable Riordan graph. If the binary A-sequence of $(g, zg)$ is of the following form

$$\underbrace{(1, 1, \ldots, 1, 0, 0}_2 a_2^{m-2}, \underbrace{a_2^{m}, a_2^{m+2}, a_2^{m+4}, \ldots, a_2^{m+2}}_2 a_j \in \{0, 1\}, m \geq 4 \),$$

(11)

then for $k \geq 4$ we obtain

$$\text{diam}(G_{2^k}) < \text{diam}(CG_{2^k}) = k.$$
Table 2  Diameters of io-decomposable Riordan graphs of the Bell type with degree 16 such that the first 6 entries of its \( A \)-sequence are all 1s

| \( A \)-seq. of \( G_{16} \)  | \( \text{diam}(G_{16}) \) | \( A \)-seq. of \( G_{16} \)  | \( \text{diam}(G_{16}) \) |
|-----------------|---------|-----------------|---------|
| (1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) | 3       | (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0) | 3       |
| (1, 1, 1, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0) | 3       | (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0) | 3       |
| (1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0) | 3       | (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0) | 3       |
| (1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) | 3       | (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0) | 3       |
| (1, 1, 1, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0) | 3       | (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0) | 3       |
| (1, 1, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0) | 3       | (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0) | 3       |
| (1, 1, 1, 1, 1, 0, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0) | 3       | (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0) | 3       |
| (1, 1, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1) | 3       | (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0) | 3       |
| (1, 1, 1, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1) | 3       | (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0) | 3       |
| (1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1) | 3       | (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0) | 3       |
| (1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) | 3       | (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0) | 3       |
Proof. First we show that \(\text{diam}(G_{2m}) = m - 1\). Since the induced subgraph \(H\) of \(\{1, 2, \ldots, 2^m - 1\}\) in \(G_{2m}\) is \(H = CG_{2^{m-1}}\) and \(CG_{2^{m-1}} = G_{2^{m-1}}(1 + z, z + z^2)\), the \((2^m - 1)\)th row of \(A(G_{2m}) = [r_{i,j}]\) is given by

\[
(0, \ldots, 0, 1, 1, 0, 1) = (r_{2^m-1,i})_{i=1}^{2^m}.
\]

By (10), (11) and (12), the \(2^m\)th row in \(A(G_{2m}) = [r_{i,j}]\) is given by

\[
(1, 0, \ldots, 0, 1, 0) = (r_{2^m,i})_{i=1}^{2^m}
\]

which means the only two vertices 1 and \(2^m - 1\) are adjacent to the vertex \(2^m\) in \(G_{2m}\). Let \(V_1 = \{1, \ldots, 2^{m-1} + 1\}\) and \(V_2 = \{2^{m-1} + 1, \ldots, 2^m - 1\}\) be the vertex subsets of \(V(G_{2m})\). Since \(<V_1>\) has the universal vertex \(2^{m-1} + 1\) and \(<V_2> \cong CG_{2^{m-1}-1}\), if \(v_1 \in V_1\) and \(v_2 \in V_2\) then we respectively obtain \(d_{G_{2m}}(v_1, 2^m) \leq 3\) and

\[
d_{G_{2m}}(v_2, 2^m) \leq \text{diam}(CG_{2^{m-1}-1}) + 1 \leq 2^m - 1 \quad \text{(by Corollary 2.8)}
\]

which implies \(\text{diam}(G_{2m}) = m - 1\). Hence, by Lemma 3.1, we obtain the desired result.

By Lemma 3.1, using the results in Tables 1 and 2 we obtain the following theorem.

**Theorem 3.3** For \(k \geq 4\), let \(G_{2^k} = G_{2^k}(g, zg)\) be an io-decomposable Riordan graph and \(G_{2^k} \neq CG_{2^k}\). If the first 16 entries in the binary A-sequence of \((g, zg)\) are not all 1s then

\[
\text{diam}(G_{2^k}) < \text{diam}(CG_{2^k}) = k.
\]

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