Strichartz estimates
for the Wave and Schrödinger Equations
with Potentials of Critical Decay

Nicolas Burq       Fabrice Planchon       John G. Stalker
A. Shadi Tahvildar-Zadeh

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Abstract

We prove weighted $L^2$ estimates for the solutions of linear Schrödinger and wave equation with potentials that decay like $|x|^{-2}$ for large $x$, by deducing them from estimates on the resolvent of the associated elliptic operator. We then deduce Strichartz estimates for these equations.

1 Introduction and Main Results

Consider the following linear equations

\begin{align}
  i\partial_t u + \Delta u - V(x)u &= 0 \quad u(0, x) = f(x) \\
  -\partial_t^2 u + \Delta u - V(x)u &= 0 \quad u(0, x) = f(x), \quad \partial_t u(0, x) = g(x)
\end{align}

where $\Delta$ is the $n$ dimensional Laplacian. Throughout this paper we will assume $n \geq 3$.

In [1] we showed that in the case where $V(x) = \frac{a}{|x|^2}$ the solution to the above equations satisfies generalized spacetime Strichartz estimates as long as $a > -(n - 2)^2 / 4$. We intend to extend this result to potentials which, in a sense to be made precise below, behave like the inverse square potential.

Let $\lambda(n)$ be defined as follows

$$
  \lambda := \frac{n - 2}{2}
$$

We also define multiplication operators $\Omega_s$ by

$$(\Omega^s \phi)(x) = |x|^s \phi(x).$$

Let $\Delta$ denote the spherical Laplacian and $\nabla$ the spherical gradient on the unit sphere. Let $r(x) := |x|$ denote the polar radius. For a given function $V \in C^1(\mathbb{R}^n \setminus \{0\})$ let $\tilde{V}$ be defined by

$$
  \tilde{V}(x) := -\partial_r (rV(x)),
$$

We denote the positive and negative parts of a function $V$ by $V_+ := \max\{V, 0\}$ and $V_- := \max\{-V, 0\}$ respectively. Thus $V = V_+ - V_-$. 

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In this paper we consider time-independent potentials \( V(x) \in C^1(\mathbb{R}^n \setminus \{0\}) \) satisfying the following assumptions.

(A1) \( \gamma_\pm^2 := \sup_{x \in \mathbb{R}^n} |x|^2 V_\pm(x) < \infty \)

(A2) The operator \(-\Delta + \Omega^2 V + \lambda^2\) is positive on every sphere, i.e., there exists a \( \delta > 0 \) such that for every \( r > 0 \),
\[
\int_{|x|=r} |\nabla u(x)|^2 + (\lambda^2 + |x|^2 V(x))|u(x)|^2 \, d\sigma(x) \geq \delta^2 \int_{|x|=r} |u(x)|^2 \, d\sigma(x) \quad (4)
\]

(A3) The operator \(-\Delta + \Omega^2 \tilde{V} + \lambda^2\) is positive on every sphere, i.e. (4) holds with \( \tilde{V} \) in place of \( V \).

**Remark 1** The potential is thus allowed to have one point singularity, which without loss of generality we take to be at the origin of coordinates. Note that no sign condition is assumed on \( V \), and that only differentiability in the radial direction is actually used.

Note also that for an inverse-square potential \( V = a|x|^{-2} \) assumptions (A2) and (A3) are the same, and require that \( a > -\lambda^2 \). More generally, for potentials that are homogeneous functions of degree \(-2\), i.e. \( V(x) = |x|^{-2} a(x/|x|) \) with \( a \) a function defined on the unit sphere, these two assumptions are again the same, namely that \(-\Delta + a + \lambda^2\) be a positive operator on the unit sphere. In section 4 we will consider an application where such a potential appears.

**Remark 2** While the approach we present recovers the results of [1] as a special case, it should be noted that it turns out to be much simpler, and hence more flexible. In particular, it should be possible to include time-dependent potentials as well, a strategy which will be pursued elsewhere.

Before stating the main results of the paper, let us examine the above assumptions more closely. Let \( Q(u) \) denote the quadratic form naturally associated with the operator \(-\Delta + V\), i.e.
\[
Q(u) := \int_{\mathbb{R}^n} |\nabla u(x)|^2 + V(x)|u(x)|^2 \, dx \quad (5)
\]

We then have

**PROPOSITION 1** Under the assumptions (A1-A2) for \( V \), there are constants \( 0 < c_1 \leq 1 \leq c_2 \) such that
\[
c_1 \|\nabla u\|^2_{L^2} \leq Q(u) \leq c_2 \|\nabla u\|^2_{L^2}
\]

**Proof:** Recall the celebrated Hardy’s inequality:
\[
\|\Omega^{-1} u\|_{L^2(\mathbb{R}^n)} \leq \frac{2}{n-2} \|\frac{\partial u}{\partial r}\|_{L^2(\mathbb{R}^n)}
\]
for \( n \geq 3 \) (see [6] for a proof). By (A1) we thus have
\[
Q(u) \leq \int |\nabla u|^2 + \gamma_\pm^2 |u|^2 \, dx \leq (1 + \gamma_\pm^2) \|\nabla u\|^2_{L^2}
\]
so that $c_2 = 1 + \frac{\gamma^2}{\lambda^2}$. On the other hand, by (A2),

$$Q(u) = \int_0^\infty \int_{|x|=r} \left( \frac{\partial u}{\partial x} \right)^2 + \frac{1}{|x|^2} |\nabla u|^2 + V(x)|u(x)|^2 \, d\sigma(x) dr,$$

$$\geq \int_0^\infty \frac{1}{r^2} \int_{|x|=r} |\nabla u|^2 + (r^2V(x) + \lambda^2)|u(x)|^2 d\sigma(x) dr,$$

$$\geq \int_0^\infty \frac{\delta^2}{r^2} \int_{|x|=r} |u(x)|^2 d\sigma(x) dr = \delta^2 \|\Omega^{-1}u\|_{L^2}^2,$$

(6)

and thus, if we set $c_1 := \frac{\delta^2}{\delta^2 + \gamma^2}$, then using (A1) we have

$$Q(u) - c_1 \|\nabla u\|_{L^2}^2 \geq \int_{\mathbb{R}^n} \left( -c_1 |x|^2 V_-(x) + (1 - c_1)\delta^2 \right) \frac{|u|^2}{|x|^2} dx \geq 0.$$

An important consequence of the above proposition is the following equivalence result:

**COROLLARY 1** Let $\dot{H}^s(\mathbb{R}^n)$ denote the scale of homogeneous Sobolev spaces based on the powers of the operator $P = -\Delta + V$. I.e., the completion of $C^\infty_c(\mathbb{R}^n \setminus \{0\})$ with respect to the seminorm

$$\|u\|_{\dot{H}^s(\mathbb{R}^n)} := \|P^{s/2}u\|_{L^2(\mathbb{R}^n)}.$$ 

If $V$ satisfies (A1-A2) then the spaces $\dot{H}^s$ are equivalent to the standard Sobolev spaces $\dot{H}^s$ (based on the powers of $-\Delta$) for $|s| \leq 1$.

For $s = 1$ this follows immediately from the above Proposition, noting that $Q(u) = \|P^{1/2}u\|_{L^2}^2$. The case $s = -1$ then follows by duality, and by interpolation we get the $s$ in between.

Our main result for the Schrödinger equation (1) is

**THEOREM 1** Let $f \in L^2$, $p \geq 2, q$ such that

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad n \geq 3.$$ (7)

Let $u$ be the unique solution of (1). Then there exists a constant $C > 0$ such that

$$\|u\|_{L^p_t(L^q_x)} \leq C\|f\|_{L^2}.$$ (8)

For the wave equation (2) we have

**THEOREM 2** Let $u$ be the solution to (2) with Cauchy data $(f,g) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}$. Let $p > 2$, and $q$ be such that $\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}$. Then

$$\|(-\Delta)^{\sigma/2}u\|_{L^p_t(L^q_x)} \leq C(\|f\|_{\dot{H}^{1/2}} + \|g\|_{\dot{H}^{-1/2}}),$$ (9)

where $\sigma = \frac{1}{p} + \frac{2}{q} - \frac{n-1}{2}$ (gap condition).
Remark 3 Notice we include the end-point for the Schrödinger equation while we exclude it for the wave equation. This is strictly intended to make the argument shorter, and one could adapt the argument from [1] to recover the endpoint case for the wave equation as well.

The strategy for proving the above Strichartz estimates is to deduce them from the corresponding estimates for the free case $V \equiv 0$, using Duhamel’s principle. This was the approach taken in [18] where Strichartz estimates for 3D Schrödinger where proved for (1) under the assumption that $V(x)$ decays like $|x|^{-2-\epsilon}$ for large $x$. For the 3D wave equation, dispersive estimates were recently proven in [4], under the following assumptions:

$$V(x) \geq 0, \quad V(x) \in C^0(\mathbb{R} \setminus \{0\}), \quad V(x) \lessapprox \inf(|x|^{-2+0}, |x|^{-2-0})$$

and the usual spectral assumption that zero is neither an eigenvalue nor a resonance. The method involves rather delicate resolvent estimates, and once the dispersive estimate is proven, Strichartz estimates follows by standard considerations. Note that the $1/|x|^2$ is barely missed and therefore appears like a borderline case. Indeed, when the potential is admitted to have a singularity at $x = 0$ slightly stronger, space-time estimates may fail, as the example provided in [3] shows.

Here we follow the strategy from [18] and bypass dispersive estimates to obtain directly Strichartz estimates. The key ingredient in this argument is the availability of a weighted spatiotemporal $L^2$ estimate for the solutions of the above equations. More precisely, for (1) one needs the estimate

$$\|\Omega^{-1}u\|_{L^2_tL^2_x} \leq C\|f\|_{L^2} \quad (10)$$

which for the free case is a particular instance of the Kato-Yajima smoothing estimate [8], while for (2) the corresponding estimate needed turns out to be

$$\|\Omega^{-1}u\|_{L^2_tL^2_x} \leq C\left(\|f\|_{\dot{H}^{1/2}} + g\|_{\dot{H}^{-1/2}}\right), \quad (11)$$

which can be thought of as a generalization of the Morawetz estimate [11] (See [5] for a proof of (11) in the free case).

Using an abstract machinery largely due to Kato [7] (see [17, §XIII.7]), the above weighted-$L^2$ estimates are deduced from a weighted resolvent estimate for the elliptic operator $P$ which is a particular self-adjoint extension of $-\Delta + V$ (see Theorem 4). We use the method of multipliers to prove this resolvent estimate (see Theorem 3). (see [14] where multipliers are used to prove a similar estimate).

The outline of this paper is as follows: In Section 2 we prove the resolvent estimate (12). Weighted-$L^2$ and Strichartz estimates are proved in Section 3. We consider an application in Section 4.

## 2 Resolvent estimates

We prove weighted $L^2$ estimates for the resolvent of $-\Delta + V$. Note that for the potentials that we are considering here, $-\Delta + V$ is not a compact perturbation of $-\Delta$. In order to define the resolvent however, we first need to consider self-adjoint extensions of $-\Delta + V(x)$, which is a symmetric operator but a priori only defined on $C^2(\mathbb{R}^n \setminus \{0\})$. We refer to [15] for a similar discussion in the case of inverse-square potentials $V = a|x|^{-2}$. In that case it is well-known that self-adjoint extensions are not unique when $-\lambda^2 < a < 1 - \lambda^2$. In
particular, there are two extensions that are both rotation and dilation invariant. One of the two corresponding domains contains functions with infinite energy (i.e. infinite $\dot{H}^1$ norm).

It will be clear in what follows that having finite energy is crucial to the arguments that we present, and that is why we are going to consider the Friedrichs extension of the operator $-\Delta + V$, i.e. using the corresponding quadratic form to define the extension. It was shown in [6] (Theorem 3) that for the class of potentials we are considering, this extension has the property that the domain of the extended operator is contained in $\dot{H}^1$. We denote by $P$ the Friedrichs extension of $-\Delta + V$.

It was shown in [6] (Theorem 3) that for the class of potentials we are considering, this extension has the property that the domain of the extended operator is contained in $\dot{H}^1$. We denote by $P$ the Friedrichs extension of $-\Delta + V$. $P$ is thus self-adjoint, and an application of Hardy’s inequality, together with assumption (A2) implies that $P$ is a positive operator, and

$$\sigma(P) = \sigma_{ac}(P) = [0, +\infty).$$

It follows that the resolvent $R(\mu) := (P - \mu)^{-1}$ is a well-defined bounded operator on $L^2$ for $\mu \notin \mathbb{R}^+$. The goal of this section is to prove

**THEOREM 3** Let $V(x)$ satisfy (A1-A3). Then there is a constant $C > 0$ such that

$$\sup_{\mu \notin \mathbb{R}^+} \|\Omega^{-1} R(\mu) \Omega^{-1} f\|_{L^2} \leq C \|f\|_{L^2}$$

(12)

The proof uses the method of multipliers, and is based on Morawetz’s radial identity [12]. Given $f \in L^2(\mathbb{R}^n)$ and $\mu \in \mathbb{C} \setminus \mathbb{R}^+$, let $u \in D(P) \subset H^1_0(\mathbb{R}^n)$ be the unique solution of the inhomogeneous Helmholtz equation

$$Pu + z^2 u = f$$

(13)

where $z = \sqrt{-\mu}$, with the branch chosen such that Re $z = \sigma > 0$. Thus $u = R(\mu)f$. In order to carry out the integration by parts argument below, one needs to know something about the behavior at the origin and at infinity of $u$, to check that the contributions of these points have “the good sign”. In the case of the potential $a|x|^{-2}$, this can be done by using the explicit asymptotic behavior of Hankel functions near 0 and infinity. In the general case, it is actually enough to know that $u \in H^1(\mathbb{R}^n)$ (but the argument requires some care, see below).

To prove (12) we first note that by density, we can take $f \in C^\infty_0(\mathbb{R}^n \setminus \{0\})$. Then $u$ is clearly a classical solution of (13). Let $w : \mathbb{R}^+ \times S^{n-1} \to \mathbb{C}$ be defined by

$$w(r, \theta) := r^{\lambda+1/2} e^{rz} u(r\theta).$$

Then $w$ satisfies

$$-\partial_r^2 w - \frac{1}{r^2} \Delta w + (\lambda^2 - \frac{1}{4} + r^2 V(r\theta)) \frac{w}{r^2} + 2z \partial_r w = e^{rz} r^{\lambda+1/2} f$$

(14)

For $R > \epsilon > 0$ fixed, let $\phi = \phi_{\epsilon,R}(r)$ be a smooth cut-off function, $0 \leq \phi \leq 1$, that is zero outside $[0, R + 1]$ and is equal to one on $[\epsilon, R]$. Multiplying (14) by

$$re^{-2r\sigma} \phi(r) \partial_r \bar{w}$$
and taking the real part, we obtain
\[ -\frac{1}{2}r e^{-2\sigma} \phi \partial_r |\partial_r w|^2 + \frac{1}{2r} e^{-2\sigma} \phi \partial_r |\nabla w|^2 + 2\sigma r e^{-2\sigma} \phi |\partial_r w|^2 + \frac{1}{2r} e^{-\sigma} \phi (\lambda^2 - \frac{1}{4} + r^2 V) |\partial_r w|^2 = \text{Re}(r^{\lambda+3/2} e^{r(z-2\sigma)} \partial_r \bar{w} f) \]

Integrating the above equality on \( \mathbb{R}^+ \times S^{n-1} \) and performing the integration by parts we obtain
\[ \frac{1}{2} \int_0^\infty \int_{|\theta|=1} \phi e^{-2\sigma} \left[ (1 + 2\sigma r)(|\partial_r w|^2 - \frac{1}{4r^2} |w|^2) + \frac{1}{r^2} \left( |\nabla w|^2 + (r^2 \tilde{V}(r\theta) + \lambda^2) |w|^2 \right) + \frac{2r\sigma}{r^2} \left( |\nabla w|^2 + (r^2 V(r\theta) + \lambda^2) |w|^2 \right) \right] d\theta dr + \frac{1}{2} \int_0^\infty \int_{|\theta|=1} r e^{-2\sigma} \phi'(r) \left[ |\partial_r w|^2 + \frac{1}{4r^2} |w|^2 - \frac{1}{r^2} \left( |\nabla w|^2 + (r^2 V(r\theta) + \lambda^2) |w|^2 \right) \right] d\theta dr = \int_0^\infty \int_{|\theta|=1} \text{Re}(r^{\lambda+3/2} e^{r(z-2\sigma)} \partial_r \bar{w} f) d\theta dr \] (15)

By Cauchy’s inequality, for any \( a > 0 \) the right hand side of the above is less than or equal to
\[ \frac{1}{4a^2} \|f\|_{L^2}^2 + a^2 \int \phi e^{-2\sigma} |\partial_r w|^2 d\theta dr. \] (16)

The difficulty in the analysis of (15) is twofold: first we have to check that the first integral in the left-hand side controls \( w \) in some suitable space. Second we have to show that the contributions of the second integral in this left-hand side are non negative as \( \varepsilon \to 0 \) and \( R \to +\infty \). We start by considering the second problem. We note that \( \text{supp}\phi' \subset I_\varepsilon \cup I_R \), where \( I_\varepsilon := [0,\varepsilon] \) and \( I_R := [R,R+1] \). On \( I_\varepsilon \) we have \( 0 \leq \phi' \leq C/\varepsilon \) and on \( I_R \) we know that \( -C \leq \phi' \leq 0 \). Since the left hand side of (15) is to be estimated from below, we only need to estimate the negative terms in this integral. In particular, it is enough to show
\[ \lim_{R \to \infty} \int_{|\theta|=1}^{R+1} r e^{-2\sigma} (|\partial_r w|^2 + \frac{1}{4r^2} |w|^2) d\theta dr = 0 \] (17)
\[ \lim_{\varepsilon \to 0} \int_{0}^{\varepsilon} \int_{|\theta|=1} \frac{1}{r^2} (|\nabla w|^2 + |w|^2) d\theta dr = 0 \] (18)

Let us first consider (17). It is in fact enough to show that there exists a sequence \( R_n \to \infty \) along which it holds. We note that
\[ \partial_r w = r^{\lambda+1/2} e^{r/z} ((\partial_r + z)u + \frac{\lambda + 1/2}{r} u). \]
We thus have, using Hardy’s inequality, that
\[
\int_R^{R+1} \int_{|\theta|=1} \frac{r e^{-2r\sigma}(|\partial_r w|^2 + \frac{1}{4r^2}|w|^2)}{d\theta dr} + 1 \\
\leq C(n) \int_R^{R+1} \int_{|\theta|=1} r(|\partial_r u|^2 + |z|^2 |u|^2) d\theta r^{n-1} dr \\
\leq C(n, |z|) \int_R^{R+1} rh(r) dr
\]
where
\[
h(r) := r^{n-1} \int_{|\theta|=1} |\partial_r u(r\theta)|^2 + |u(r\theta)|^2 d\theta
\]
so that by virtue of \( u \) being in \( H^1(\mathbb{R}^n) \), we know
\[
\int_0^\infty h(r) dr < \infty.
\]
It thus follows that given \( \mu_m > 0 \) there exists a sequence \( R_n^{(m)} \to \infty \) such that
\[
\int_{R_n^{(m)}}^{R_n^{(m)}+1} h(r) dr < \frac{\mu_m}{R_n^{(m)}},
\]
because otherwise the integral \( \int_0^\infty h dr \) would diverge. Using a diagonal argument it thus follows that there exists a sequence \( R_n \to \infty \) such that
\[
\int_{R_n}^{R_n+1} rh(r) dr \to 0 \quad \text{as } n \to \infty,
\]
which establishes (17) along a sequence.

Similarly, using that the \( H^1 \) norm of \( u \) on a ball is finite, we have
\[
\int_\epsilon^\infty \int_{|\theta|=1} \frac{1}{r^2} (|\nabla w|^2 + |w|^2) d\theta dr \leq C \int_\epsilon^\infty \int_{|\theta|=1} |\nabla u|^2 r^{n-1} d\theta dr \to 0
\]
as \( \epsilon \to 0 \), establishing (18).

We can thus focus our attention on the first integral on the left in (15). Using the assumptions (A2), (A3) on the potential, it can be estimated from below by
\[
\frac{1}{2} \int_0^\infty \int_{|\theta|=1} \phi e^{-2r\sigma} (1 + 2r\sigma) \left[ |\partial_r w|^2 + \left( \delta^2 - \frac{1}{4} \right) \frac{|w|^2}{r^2} \right] d\theta dr.
\]
We need the following weighted version of Hardy’s inequality:

**LEMMA 2.1** Let \( \psi \in C^2(\mathbb{R}^+; \mathbb{R}) \) be such that
\[
\psi(r) \geq 0, \quad \psi'(r) \leq 0, \quad r(\psi'(r)^2 + 2\psi(r)\psi''(r)) - 2\psi(r)\psi'(r) \geq 0
\]
for all \( r \geq 0 \). Let \( f : \mathbb{R}^+ \to \mathbb{C} \) be such that \( f(0) = 0 \). Then
\[
\int_0^\infty \psi^2 \frac{|f|^2}{r^2} dr \leq 4 \int_0^\infty \psi^2 |f'|^2 dr
\]
Proof: (inspired by [19]) Let $G$ be the following densely-defined symmetric operator on $L^2(\mathbb{R}^+)$

$$G := \frac{1}{i}(\psi \partial_r + \frac{1}{2} \psi')$$

We have $[G, m] = -i \psi m'$ where $m$ is the operator of multiplication by the function $m(r)$. We thus have

$$0 \leq \|G - im)f\|^2 = \langle (G + im)(G - im)f, f \rangle = \|Gf\|^2 - \langle (\psi m' - m^2)f, f \rangle$$

Using the definition of $G$,

$$\|Gf\|^2 = \int_0^\infty \psi^2 |f'|^2 + \frac{1}{4}(\psi')^2 |f|^2 + \frac{1}{2} \psi \psi' \partial_r |f|^2 \, dr$$

$$= \int_0^\infty \psi^2 |f'|^2 + \frac{1}{4}(\psi')^2 |f|^2 - \frac{1}{2} \partial_r (\psi \psi') |f|^2 \, dr$$

$$+ \frac{1}{2} \psi(R) \psi'(R) |f(R)|^2_{R=\infty} - \frac{1}{2} \psi(R) \psi'(R) |f(R)|^2_{R=0}$$

$$\leq \int_0^\infty \psi^2 |f'|^2 - \left(\frac{1}{4}(\psi')^2 + \frac{1}{2} \psi \psi'' + \psi m' - m^2\right) |f|^2 \, dr$$

Thus

$$0 \leq \int_0^\infty \psi^2 |f'|^2 - \left(\frac{1}{4}(\psi')^2 + \frac{1}{2} \psi \psi'' + \psi m' - m^2\right) |f|^2 \, dr$$

To establish the lemma we thus need to choose $m$ such that the coefficient of $|f|^2$ in the above is greater than $\psi^2/(4r^2)$. We now check that this is satisfied if we set $m = -\frac{\psi}{2r}$, provided

$$\frac{1}{4}(\psi')^2 + \frac{1}{2} \psi \psi'' - \frac{\psi \psi'}{2r} \geq 0 \tag{19}$$

which is equivalent to the condition of the Lemma.

To apply the above Lemma to $w$, we set

$$f(r) := \left(\int_{|\theta|=1} |w(r, \theta)|^2 \, d\theta \right)^{1/2}$$

and

$$\psi(r) := e^{-\sigma r}(1 + 2\sigma r)^{1/2}.$$  

We check that

$$\frac{1}{4}(\psi')^2 + \frac{1}{2} \psi \psi'' - \frac{\psi \psi'}{2r} = 3\sigma^4 r e^{-2\sigma r} (1 + 2\sigma r)^{-1} \geq 0$$

Moreover,

$$\int_0^1 \frac{f(r)^2}{r^2} \, dr = \int_0^1 \int_{|\theta|=1} e^{2r \Re z} \left|\frac{u(r\theta)}{r^2}\right|^2 r^{n-1} \, dr \, d\theta$$

$$\leq C(z)\|\Omega^{-1} u\|_{L^2(\mathbb{R}^n)}$$
which is finite by Hardy’s inequality. Similarly, \( \int_0^1 |f'(r)|^2 dr < \infty \) since \( u \in H^1 \), and this implies that \( f \in C^{1/2}((0, 1)) \) and thus \( f(0) = 0 \). By the above Lemma then
\[
\int_0^\infty \int_{|\theta|=1} e^{-2r\sigma}(1+2r\sigma)|w|^2 \frac{d\theta dr}{4r^2} \leq \int_0^\infty \int_{|\theta|=1} e^{-2r\sigma}(1+2r\sigma)|\partial_r w|^2 d\theta dr \tag{20}
\]
Using (16) and assumption (A3), and taking the limits \( R \to \infty, \epsilon \to 0 \), we deduce from (15) that
\[
\frac{1}{2} \int_0^\infty \int_{|\theta|=1} e^{-2r\sigma}(1+2r\sigma) \left\{ (1-a^2)|\partial_r w|^2 + \left( \delta^2 - \frac{1}{4} \right) \frac{|w|^2}{r^2} \right\} d\theta dr \leq \frac{1}{4a^2} \| \Omega f \|_{L^2}^2
\]
and thus using (20) and optimizing on \( a \) obtain
\[
\| e^{-\sigma r} \frac{w}{r} \|_{L^2(\mathbb{R}^+ \times \mathbb{S}^{n-1})} \leq \frac{1}{2\delta^2} \| \Omega f \|_{L^2(\mathbb{R}^n)}
\]
which by the definition of \( w \) gives
\[
\| \Omega^{-1} u \|_{L^2(\mathbb{R}^n)} \leq \frac{1}{2\delta^2} \| \Omega f \|_{L^2(\mathbb{R}^n)}
\]
establishing (12).

3 Morawetz and Strichartz estimates

3.1 From resolvent to Morawetz

We recall the result stating that one can deduce weighted-\( L^2 \) spacetime estimates for a Hamiltonian evolution from a weighted resolvent estimate for the associated elliptic operator (see Corollary to Theorem XIII.25, [17, p. 146].)

**Theorem 4** (Kato [7]) Let \( H \) be a self-adjoint operator on the Hilbert space \( X \), and for \( \mu \not\in \mathbb{R} \) let \( R(\mu) := (H - \mu)^{-1} \) denote the resolvent. Suppose that \( A \) is a closed, densely defined operator, possibly unbounded, from \( X \) into a Hilbert space \( Y \). Suppose that
\[
\Gamma := \sup_{\mu \not\in \mathbb{R}} \| AR(\mu) A^* \chi \|_Y < \infty
\]
for \( \chi \in D(A^*) \), \( \| \chi \| = 1 \)

Then \( A \) is \( H \)-smooth and
\[
\| A \|_H := \sup_{\phi \in X, \| \phi \| = 1} \frac{1}{2\pi} \int_{-\infty}^{\infty} \| Ae^{-itH} \phi \|^2_Y dt \leq \Gamma^2/\pi^2
\]

We use this result to prove (10) and (11).
3.1.1 Schrödinger’s equation

Consider first the case of equation (1). Set $H = P$, $X = Y = L^2(\mathbb{R}^n)$ and let $A = \Omega^{-1}$, i.e. multiplication by $\frac{1}{|x|}$. Thus $A^* = A$, $\mathcal{R} = R$, and if we let $z = \sqrt{-\mu}$, with the square root branch chosen such that Re $z > 0$, by Theorem 3 we then have

$$\|A \mathcal{R}(\mu) A^* \chi\|_Y = \|\Omega^{-1}(P + z^2)^{-1}\Omega^{-1} \chi\|_{L^2} \leq \frac{1}{2\delta^4} \|\chi\|_{L^2}$$

Taking the supremum over $\mu$ we see that the hypothesis of Theorem 4 is satisfied, and $\Gamma \leq 1/(2\delta^2)$. Thus for $u$ the solution to (1) we have the desired estimate

$$\|\Omega^{-1} u\|_{L^2} \leq \frac{1}{\delta^2 \sqrt{2\pi}} \|f\|_{L^2}$$

3.1.2 The wave equation

For the wave equation (2), we instead make the following identifications: $X = \mathcal{H}^{1/2} \times \mathcal{H}^{-1/2}$, $Y = L^2$, $A = (\Omega^{-1}, 0)$

Recall that $\mathcal{H}^s$, defined in Corollary 1 are homogeneous Sobolev spaces based on the powers of $P$, and thus $A^* = (P^{-1/2}\Omega^{-1}, 0)$.

We also let

$$H = \begin{pmatrix} 0 & -i \\ iP & 0 \end{pmatrix}$$

so that the solution to the wave equation (2) is $u = e^{itH} \begin{pmatrix} f \\ g \end{pmatrix}$. The resolvent of $H$ is

$$\mathcal{R}(z) = (H - z)^{-1} = \begin{pmatrix} (P - z^2)^{-1} & -i(P - z^2)^{-1} \\ i(P - z^2)^{-1}P & z(P - z^2)^{-1} \end{pmatrix}$$

so that

$$B := A\mathcal{R}(z)A^* = \Omega^{-1} z(P - z^2)^{-1}P^{-1/2}\Omega^{-1}$$

Let

$$D := \Omega^{-1} P^{-1} \Omega^{-1}$$

(21)

**Lemma 3.1** The operator $D$ is bounded on $L^2$.

*Proof:* Let

$$E := \Omega^{-1} P^{-1/2}$$

Then $D = EE^*$ and it’s thus enough to prove that $E$ is bounded. This amounts to proving the Hardy inequality for $P$, i.e.

$$\|\Omega^{-1} u\|_{L^2} \leq c\|P^{1/2} u\|_{L^2}$$

which has already been shown (6), with $c = 1/\delta$.

We are going to use complex interpolation to prove boundedness of $B$. This will require the following fact from operator theory, to be proved in the Appendix.
THEOREM 5 Suppose $\Lambda$ and $\Omega$ are self-adjoint operators on a Hilbert space $X$ and that $\Lambda$ is non-negative with zero nullspace. If there is a constant $c$ such that

$$\|[\Omega, \Lambda^2]f\| \leq c\|\Lambda f\|$$

(22)

for all $f \in X$ then $[\Omega, \Lambda]$ is a bounded operator on $X$.

LEMMA 3.2 The operator $B$ is bounded on $L^2$, uniformly in $z$.

Proof:

We define

$$P_0 := -\Delta + V_0, \quad \Lambda := P^{1/2}, \quad \Lambda_0 := P_0^{1/2}, \quad R := \Lambda_0\Lambda^{-1}$$

where

$$V_0(x) := a|x|^{-2}, \quad a \geq 0.$$ 

Let $r = |x|$. On $\Sigma_l$, the $l$’th spherical harmonic subspace of $L^2(\mathbb{R}^n)$, we have $P_0 = A_{\nu}$ where

$$A_{\nu} := \partial_r^2 + (n-1)r^{-1}\partial_r + \nu^2r^{-2}$$

and

$$\nu^2 := (\lambda + l)^2 + a, \quad \lambda := \frac{n-2}{2}.$$ 

If $a$ is chosen sufficiently large then $\nu^2 > 1$. We will assume this from now on. In this case $A_{\nu}$ agrees with the operator of the same name in [15] and [1], the positive branch having been chosen for the square root. For $n > 4$ we can choose $a = 0$, but this does not result in any real simplification of the argument below.

Let us define the following operators:

$$E_0 := \Omega^{-1}\Lambda_0^{-1}, \quad M_0 := [\Lambda_0^2, \Omega]\Lambda_0^{-1}, \quad C_1 := \Omega\Lambda_0^{-1}\Omega^{-1}\Lambda_0, \quad C_2 := \Omega\Lambda_0\Omega^{-1}\Lambda_0^{-1}.$$ 

Note that $E_0$ is the same as the operator $E = \Omega^{-1}\Lambda^{-1}$, but with the potential $V$ replaced by $V_0$, hence the $L^2$-boundedness of $E_0$ follows from Hardy’s inequality by the same argument as in the proof of Lemma 3.1. The boundedness of $M_0$ is reduced to that of $E_0$ by a direct computation. Meanwhile,

$$C_2 = [\Omega, \Lambda_0]E_0 + I.$$ 

Thus the boundedness of $C_2$ follows from that of $M_0$, using Theorem 5. Finally we have the following lemma, the proof of which will be given in the Appendix.

LEMMA 3.3 The operator $C_1$ is bounded on $L^2$.

To proceed with complex interpolation, we define the following family of operators: For $s \in \mathbb{C}$, $0 \leq \Re s \leq 1$, let

$$T_s := z^{2s}e^{s^2-1/4\Omega^{-1}(P - z^2)^{-1}\Lambda_0^{-1}\Omega^{-1}\Lambda_0^{-1}2s}.$$ 

Up to a constant,

$$T_0 = \Omega^{-1}(P - z^2)^{-1}\Lambda_0^{-1}\Omega^{-1}\Lambda_0 = \Omega^{-1}(P - z^2)^{-1}\Omega^{-1}C_1.$$ 

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Meanwhile

\[ T_1 = \Omega^{-1} z^2 (P - z^2)^{-1} \Lambda_0^{-1} C_0 = \Omega^{-1} (P (P - z^2)^{-1} - I) \Lambda_0^{-1} C_0 = \Omega^{-1} (P - z^2)^{-1} \Omega^{-1} F - C_0^2, \]

where

\[ F := \Omega P \Lambda_0^{-1} C_0 = C_2 + (\Omega^2 V - a) C_0^2. \]

By (A1), \( \Omega^2 V \) is bounded in \( L^\infty \). We have proved the uniform \( L^2 \) boundedness of \( \Omega^{-1} (P - z^2)^{-1} \Omega^{-1} \) for \( z^2 \not\in \mathbb{R}^+ \) in Theorem 3. So \( T_0 \) and \( T_1 \) are bounded. The contribution of the imaginary part of \( s \) to the power of \( \Lambda_0 \) only puts a unitary operator at the tail end, which doesn’t affect boundedness, and its contribution to the \( z \) power is taken care of by the exponential term. Therefore \( T_0 + i t \) and \( T_1 + it \) are bounded uniformly in \( t \) and in \( z \). By complex interpolation \( T_{1/2} \) is bounded uniformly in \( z \):

\[ T_{1/2} = z \Omega^{-1} (P - z^2)^{-1} \Lambda_0^{-1} \Omega^{-1} \Omega^{-1}. \]

We have

\[ B = T_{1/2} \Omega \Lambda_0 \Lambda^{-1} \Omega^{-1} =: T_{1/2} G \]

and

\[ G = \Lambda_0 \Omega \Lambda^{-1} \Omega^{-1} + [\Omega, \Lambda_0] \Lambda^{-1} \Omega^{-1} =: J + [\Omega, \Lambda_0] E^* \]

Finally,

\[ J = \Lambda_0 \Lambda^{-1} + \Lambda_0 [\Omega, \Lambda^{-1}] \Omega^{-1} = R + R[\Lambda, \Omega] E^*. \]

Moreover,

\[ M := [\Lambda^2, \Omega] \Lambda^{-1} = [\Lambda_0^2, \Omega] \Lambda^{-1} = M_0 R, \]

Note that \( R \) is bounded by the equivalence of Sobolev norms (Corollary 1). Therefore \( M \) is bounded too. The boundedness of \([\Lambda, \Omega]\) and \([\Lambda_0, \Omega_0]\) now follows from Theorem 5. These in turn imply the boundedness of \( J, G \), and hence that of \( B \). This concludes the proof of Lemma 3.2.

We can therefore again apply Theorem 4 to deduce the Morawetz estimate (11), except that the norms on the right will be \( P \)-based norms. By the equivalence result of Corollary 1, however, we can replace those with standard Sobolev norms.

### 3.2 From Morawetz to Strichartz

#### 3.2.1 Schrödinger’s equation

We consider the potential term as a source term,

\[ i \partial_t u + \Delta u = V(x) u, \quad u(0) = f \quad (23) \]

and integrate using \( S_0(t) = e^{-it\Delta} \), the free evolution, to get

\[ u(t) = S_0(t) f + \int_0^t S_0(t-s) V u(s) ds \quad (24) \]

The first term can be ignored since it satisfies the estimate we want to prove, and we can focus on the Duhamel term. Given that \( n \geq 3 \), one has Strichartz estimates up to the
end-point for the free evolution, i.e. for the pair \((p, q) = (2, \frac{2n}{n-2})\). We recall that these Strichartz estimates hold in a slightly relaxed setting,

\[
\left\| \int_0^t S_0(t-s)F(x,s)ds \right\|_{L_t^2(L_x^{\frac{2n}{n-2}})} \leq C \|F\|_{L_t^2(L_x^{\frac{2n}{n-2}})},
\]

where \(L^{\alpha,\beta}\) are Lorentz spaces. Hence to prove our estimate, all we need to check is \(F = Vu \in L_t^2(L_x^{\frac{2n}{n-2}})\). However, from (11) we have \(\Omega^{-1}u \in L_t^2 L_x^2\), while assumption (A1) implies \(\Omega V \in L^n,\infty\). Thus, using O’Neil’s inequality (Hölder inequality for Lorentz spaces) we have

\[
\left\| \int_0^t S_0(t-s)Vu(s)ds \right\|_{L_t^2(L_x^{\frac{2n}{n-2}})} \leq C \|Vu\|_{L_t^2(L_x^{\frac{2n}{n-2}})} \\
\leq C \|\Omega V\|_{L^n,\infty} \|\Omega^{-1}u\|_{L_t^2 L_x^2} \\
\leq C \|f\|_{L^2}
\]

which proves (8) at the end-point \((p, q) = (2, \frac{2n}{n-2})\). Interpolating between this and the conservation of the \(L^2\) norm for (1), which corresponds to \((p, q) = (\infty, 2)\) in (8), one obtains the full range of Strichartz estimates.

### 3.2.2 Wave equation

We write the solution to (2) as the sum of the solution to the free wave equation plus a Duhamel term

\[
u(t) = \hat{W}(t)f + W(t)g - \int_0^t W(t-s)V(x)u(s)ds,
\]

where \(W(t) = \frac{\sin(t\sqrt{\lambda_n})}{\sqrt{\lambda_n}}\), and \(\hat{W} = \partial_t W\). We again ignore the first two terms in the above and focus on the Duhamel term. Since \(W(t-s) = -\hat{W}(t)W(s) + W(t)\hat{W}(s)\), this splits into two terms. We will deal with the first one, the treatment of the second term being similar. We are going to use the following lemma,

**Lemma 3.4** Let \(X, Y\) be two Banach spaces and let \(T\) be a bounded linear operator from \(L^\beta(\mathbb{R}^+; X)\) to \(L^\gamma(\mathbb{R}^+; Y)\), \(Tf(t) = \int_0^\infty K(t,s)f(s)ds\). Then the operator \(\hat{T}f(t) = \int_0^t K(t,s)f(s)ds\) is bounded from \(L^\beta(\mathbb{R}^+; X)\) to \(L^\gamma(\mathbb{R}^+; Y)\) when \(\beta < \gamma\), and \(\|\hat{T}\| \leq c_{\beta,\gamma}\|T\|\) with \(c_{\beta,\gamma} = (1 - 2^{1/\gamma - 1/\beta})^{-1}\).

We set

\[
Th(t) := \hat{W}(t) \int W(s)\Omega^{-1}h(s)ds.
\]

Using the following Strichartz estimate for the free wave equation

\[
\|\hat{W}(t)F\|_{L_t^6 H_x^2} \leq C \|F\|_{\dot{H}^{1/2}(\mathbb{R}^n)},
\]

combined with the *dual* to the Morawetz estimate (11) for the free wave equation, namely

\[
\| \int W(s)G(s)ds \|_{\dot{H}^{1/2}(\mathbb{R}^n)} \leq C \|\Omega G\|_{L^2(\mathbb{R}^{n+1})}
\]
we obtain
\[
\|Th\|_{L^pH^s_q} \leq C\int W(s)\Omega^{-1}h(s)ds\|H^{1/2}(\mathbb{R}^n) \leq C\|h\|_{L^2(\mathbb{R}^{n+1})}
\]
with \(p\), \(q\) and \(\sigma\) as in the statement of the Theorem. By Lemma 3.4, the corresponding operator \(\tilde{T}\) satisfies the same estimate as \(T\) (with a different constant). On the other hand, the solution to (2) is
\[
u(t) = \tilde{W}(t)f + W(t)g + \tilde{T}(\Omega V u)
\]
By assumption (A1) and (11) we conclude
\[
\|\Omega Vu\|_{L^2(\mathbb{R}^{n+1})} \leq \max\{\gamma_2^2, \gamma_2^2\}\|\Omega^{-1}u\|_{L^2(\mathbb{R}^{n+1})} \leq C(\|f\|_{H^{1/2}} + \|g\|_{H^{-1/2}})
\]
which establishes (9).

4 The point-dipole potential

An example of a physical potential satisfying our assumptions (A1-A3) is that of an electrical point-dipole. The Schrödinger equation with this potential arises for example in the study of electron capture by polar molecules [10]. Let \(\psi\) be the wave function of an electron in the electric field of a dipole that is supposed to be point-like and fixed at the origin. The equation then reads
\[
i\partial_t \psi = -\Delta \psi + \frac{p \cdot x}{|x|^3} \psi
\]
(27)
where \(p := \frac{2meD}{\hbar^2}\) is dimensionless. Here \(m\), \(e\) are the mass and charge of the electron and \(D\) is the electric dipole moment of the molecule. Choosing coordinates such that \(p = (0,0,p)\), the potential \(V(x) = px_3/|x|^3\) is homogeneous of degree -2, so that assumptions (A2) and (A3) coincide, and for the weighted-\(L^2\) (10) and Strichartz (8) estimates to hold for \(\psi\), all we need is that the lowest eigenvalue of the operator \(-\Delta + px_3\) on \(S^2\) be larger than \(-1/4\). This is clearly the case if \(p < 1/4\), and if we let \(p_0\) denote the largest value of \(p\) for which this continues to hold, it is known that \(p_0 \approx 1.28\) (see [10] for the calculation of this “critical value” of the dipole moment).

5 Appendix

5.1 Proof of Theorem 5

From this point on \(c\) will denote a constant which depends only on the dimension, but whose value may differ from equation to equation.

We define rescaled versions of \(\Omega\) and \(\Lambda\),
\[
\Omega_s := s^{-1/2}\Omega, \quad \Lambda_s := s^{1/2}\Lambda.
\]
We have
**Lemma 5.1** For $\alpha \geq 0$, let $Q_\alpha(\sigma) := \Lambda_\alpha^\alpha \exp(-\Lambda_\alpha^2)$, then

1. the operator $Q_\alpha(\sigma)$ is bounded on $L^2$, uniformly in $s$, for all $\alpha \geq 0$.

2. For $\alpha, \gamma > 0$,
   \[
   \Lambda^{\alpha-2\gamma} = \Gamma(\gamma)^{-1} \int_0^\infty s^{\gamma-\alpha/2} Q_\alpha(s) \frac{ds}{s}. \tag{28}
   \]

3. For $\alpha > 0$,
   \[
   \int_0^\infty \|Q_\alpha(s)g\|^2 \frac{ds}{s} = 2^{-\alpha} \Gamma(\alpha) \|g\|^2. \tag{29}
   \]

By spectral theory we may assume without loss of generality that $X = L^2(\mathcal{X})$ where $\mathcal{X}$ is a measure space and that $\Lambda_\alpha$ is multiplication by a non-negative real measurable function on $\mathcal{X}$,

\[(\Lambda g)(\xi) = m(\xi)g(\xi).\]

All of the results above then follow immediately. For example,

\[
\left(\int_0^\infty s^{\gamma-\alpha/2} Q_\alpha(s) \frac{ds}{s} g \right)(\xi) = \int_0^\infty s^{\gamma} m^\alpha(\xi) \exp(-sm^2(\xi)) \frac{ds}{s} g(\xi) = \Gamma(\gamma) m^{\alpha-2\gamma}(\xi) g(\xi) = \Gamma(\gamma) \left(\Lambda^{\alpha-2\gamma} g\right)(\xi),
\]

so that

\[
\int_0^\infty s^{\gamma-\alpha/2} Q_\alpha(s) \frac{ds}{s} = \Gamma(\gamma) \Lambda^{\alpha-2\gamma}.
\]

Similarly,

\[
\int_0^\infty \|Q_\alpha(s)g\|^2 \frac{ds}{s} = \int_0^\infty \int_{\xi \in \mathcal{X}} s^{m^2(\xi)} \exp(-2sm^2(\xi)) |g(\xi)|^2 d\xi \frac{ds}{s} = \int_{\xi \in \mathcal{X}} \int_0^\infty s^{m^2(\xi)} \exp(-2sm^2(\xi)) \frac{ds}{s} |g(\xi)|^2 d\xi = \int_{\xi \in \mathcal{X}} 2^{-\alpha} \Gamma(\alpha) |g(\xi)|^2 d\xi = 2^{-\alpha} \Gamma(\alpha) \|g\|^2.
\]

For our present purposes, the main use of the operators $Q_\alpha(s)$ is the following boundedness criterion, which may be seen as a simple form of the Cotlar-Stein lemma.

**Lemma 5.2** Let $\epsilon > 0$, and suppose that $T$ is an operator such that

\[
\|Q_\alpha(s) T Q_\alpha(t)\| \leq c \exp(-\epsilon |\log s - \log t|), \tag{30}
\]

then

\[
\|T\| \leq 2^{3/2} \epsilon^{-3/2} \Gamma(\alpha)^{-1} c.
\]
By Lemma 5.1 we may write
\[
Q_\alpha(s) T = \Gamma(\alpha)^{-1} \int_0^\infty Q_\alpha(s) T Q_\alpha(2t) \frac{dt}{t},
\]
and, by the triangle inequality,
\[
\|Q_\alpha(s) T f\| \leq \Gamma(\alpha)^{-1} \int_0^\infty \|Q_\alpha(s) T Q_\alpha(t)\| \|Q_\alpha(t) f\| \frac{dt}{t}.
\]
Let
\[
d(s, t) = e^{\log s - \log t}.
\]
Using (30) and then Cauchy-Schwarz,
\[
\|Q_\alpha(s) T f\| \leq c \Gamma(\alpha)^{-1} \int_0^\infty d(s, t)^{-\epsilon} \|Q_\alpha(t) f\| \frac{dt}{t}
\]
\[
\leq c \Gamma(\alpha)^{-1} \left[ \int_0^\infty d(s, t)^{-\epsilon} \frac{dt}{t} \right]^{1/2} \left[ \int_0^\infty d(s, t)^{-\epsilon} \|Q_\alpha(t) f\|^2 \frac{dt}{t} \right]^{1/2}
\]
\[
= 2^{\epsilon - 1} c \Gamma(\alpha)^{-1} \left[ \int_0^\infty d(s, t)^{-\epsilon} \|Q_\alpha(t) f\|^2 \frac{dt}{t} \right]^{1/2}.
\]
Then, squaring this last inequality and integrating over \(s\),
\[
\int_0^\infty \|Q_\alpha(s) T f\|^2 \frac{ds}{s} \leq 4 \epsilon^{-2} c^2 \Gamma(\alpha)^{-2} \int_0^\infty \int_0^\infty d(s, t)^{-\epsilon} \|Q_\alpha(t) f\|^2 \frac{dt}{t} \frac{ds}{s}
\]
or, switching the order of integration on the right and using (29),
\[
\|T f\|^2 \leq 8 \epsilon^{-3} c^2 \Gamma(\alpha)^{-2} \|f\|^2
\]
which is the claim.

**Lemma 5.3** Define \(K_\alpha(\sigma) := [\Omega, Q_\alpha(\sigma)]\) and \(L(s) := [\Lambda^2, \Omega] Q_0(s)\).

1. Both operators \(K_0(s)\) and \(L(s)\) are bounded on \(L^2\), uniformly in \(s\).
2. The operator \(Q_2(r) K_2(s)\) is bounded on \(L^2\), and
\[
\|Q_2(r) K_2(s)\| \leq cd(r, s)^{-1/2}.
\]

(31)

We start with \(K_0\):
\[
\partial_s (s^{1/2} K_0(s) f) = [\Omega, \partial_s Q_0(s)] f = [\Omega, -\Lambda^2 Q_0(s)] f = -\Lambda^2 (s^{1/2} K_0(s)) f + [\Lambda^2, \Omega] Q_0(s) f.
\]
Thus, taking the scalar product with \(s^{1/2} K_0(s) f\),
\[
\partial_s \|s^{1/2} K_0(s) f\|_2 \leq 2 \|[\Lambda^2, \Omega] Q_0(s) f\|_2 \leq c s^{-1/2} \|Q_1(s) f\|_2 \leq c s^{-1/2} \|f\|_2,
\]
where we have used that \(\Lambda\) satisfies (22). Integrating both sides of the above on \([0, s]\) we obtain the boundedness of \(K_0\).
Next, using again (22),

$$\|L(s)f\| = \|\Lambda_2^2 \Omega_2|Q_0(s)f\| = \|s^{-1/2}[\Lambda^2, \Omega]|Q_0(s)f\| \leq c|s^{-1/2}Q_0(s)f| = c\|Q_1(s)f\| \leq c\|f\|.$$ 

Finally, we establish the bound on $Q_2(r)K_2(s)$. We can write it in either of the two forms

$$Q_2(r)K_2(s) = \sqrt{\frac{s}{r}}Q_2(r)|\Omega_r, \Lambda_r|Q_0(s) + Q_2(r)\Lambda^2_2\Omega_2Q_0(s) - Q_2(r)Q_0(\Omega s)$$

$$= \sqrt{\frac{s}{r}}Q_2(r)|\Omega_r, \Lambda_r|Q_0(s) + \frac{s}{r}Q_4(r)K_0(s)$$

$$= 2^{3/2}\sqrt{\frac{s}{r}}Q_2(r/2)(-L^*(r/2))Q_0(s) + \frac{s}{r}Q_4(r)K_0(s).$$

or

$$Q_2(r)K_2(s) = \frac{r}{s}Q_0(r) ([\Lambda^2_r, \Omega]Q_2(s) + [\Omega_r, Q_4(s)])$$

$$= 2^{3/2}\frac{r}{s}Q_0(r)(L(s/2)Q_2(s/2) + K_2(s/2)Q_2(s/2) + Q_2(s/2)K_2(s/2))$$

$$= 2^{3/2}\frac{r}{s}Q_0(r)(L(s/2)Q_2(s/2) - L(s/2)^*Q_2(s/2) + K_0(s/2)Q_4(s/2)$$

$$- Q_2(s/2)L(s/2) + Q_4(s/2)K_0(s/2)).$$

From the uniform boundedness of $Q_\alpha$, $K_0$ and $L$ we see that

$$\|Q_2(r)K_2(s)\| \leq c\sqrt{\frac{s}{r}}$$

and

$$\|Q_2(r)K_2(s)\| \leq c\sqrt{\frac{r}{s}}.$$ 

which yields (31).

Theorem 5 will follow from the following identity:

$$[\Lambda, \Omega] = \int_0^\infty Q_1(s)\Lambda \frac{ds}{s} - \int_0^\infty \Omega \Lambda Q_1(s) \frac{ds}{s} = \int_0^\infty Q_2(s)\Omega - \Omega s Q_2(s) \frac{ds}{s} = \int_0^\infty K_2(s) \frac{ds}{s}.$$ 

The forthcoming Lemma 5.4 then provides a bound on $Q_2(r)[\Lambda, \Omega]Q_2(s)$ which we use to apply Lemma 5.2 and obtain boundedness of $[\Lambda, \Omega]$.

**LEMMA 5.4** Define

$$E(r, t) := \int_0^\infty Q_2(r)K_2(s)|Q_2(t)\frac{ds}{s}.$$ 

Then

$$||E(r, t)|| \leq cd(r, t)^{-1/4}.$$ 

Since $E(r, t)^* = -E(t, r)$ we may assume without loss of generality that $r \leq t$. We write

$$E(r, t) = E_<(r, t) + E_>(r, t)$$

where

$$E_<(r, t) := \int_{0}^{\sqrt{rt}} Q_2(r)K_2(s)|Q_2(t)\frac{ds}{s}, \quad E_>(r, t) := \int_{\sqrt{rt}}^{\infty} Q_2(r)K_2(s)|Q_2(t)\frac{ds}{s}. $$
Remark that $K_2(s)Q_2(t) = -(Q_2(t)K_2(s))^*$. Then, we use (31) and for $s \leq \sqrt{rt}$,
\[
\|Q_2(r)K_2(s)Q_2(t)\| \leq \|Q_2(r)\|\|K_2(s)Q_2(t)\| \leq cd(s, t)^{-1/2} = c\sqrt{s/t}.
\]
while for $s \geq \sqrt{rt}$,
\[
\|Q_2(r)K_2(s)Q_2(t)\| \leq \|Q_2(r)K_2(s)\|\|Q_2(t)\| \leq cd(r, s)^{-1/2} = c\sqrt{r/s}.
\]
Integrating,
\[
\|E_{<}(r, t)\| \leq c(r/t)^{1/4} = cd(r, t)^{-1/4}
\]
and
\[
\|E_{>}(r, t)\| \leq c(r/t)^{1/4} = cd(r, t)^{-1/4},
\]
which completes the proof.

5.2 Proof of Lemma 3.3

Recall the following from [15]: The Mellin transform, its inverse
\[
(\mathcal{M}\phi)(z) = \int_0^\infty r^{-n}\phi(r)r^{n-1}dr, \quad (\mathcal{M}^{-1}f)(r) = -\frac{1}{2\pi i} \int_C r^{-z}f(z)dz,
\]
and its action on multiplication by powers of $r$,
\[
(\mathcal{M}\Omega^\sigma\phi)(z) = (\mathcal{M}\phi)(z + \sigma).
\]
We also recall the Hankel transform,
\[
(\mathcal{H}_\nu\phi)(r) = \int_0^\infty (rs)^{-\lambda}J_\nu(rs)\phi(s)s^{n-1}ds,
\]
where $J_\nu$ is the Bessel function of the first kind of order $\nu$, its composition with the Mellin transform,
\[
(\mathcal{M}\mathcal{H}_\nu\phi)(z) = 2z^{-\lambda-1} \frac{\Gamma(\frac{z+\lambda+\nu}{2})}{\Gamma(1 - \frac{z-\lambda-\nu}{2})}(\mathcal{M}\phi)(2\lambda + 2 - z)
\]
and the representation of $\Lambda_0^\sigma$ on $\Sigma_l$ via the Hankel transform
\[
(A_\nu)^{\sigma/2} = \mathcal{H}_\nu\Omega^\sigma\mathcal{H}_\nu.
\]
Now Plancherel’s formula can be written in the form
\[
\langle \varphi, \psi \rangle = \frac{\text{Vol}(S^{n-1})}{2\pi} \int_{-\infty}^\infty (\mathcal{M}\varphi)(\lambda + 1 + iy)(\overline{\mathcal{M}\psi})(\lambda + 1 + iy)dy,
\]
from which
\[
\|\varphi\|^2 = \frac{\text{Vol}(S^{n-1})}{2\pi} \int_{-\infty}^\infty |(\mathcal{M}\varphi)(\lambda + 1 + iy)|^2dy.
\]
Hence, if $\mathcal{O}$ is an operator whose action is given in terms of the Mellin transform by
\[
(\mathcal{M}\mathcal{O}\varphi)(z) = \mathcal{O}(z)(\mathcal{M}\varphi)(z)
\]
then its $L^2$ to $L^2$ norm is

$$\|O\| = \sup |O(\lambda + 1 + iy)|.$$  

Applying this with $O = C_1|_{\Sigma_l}$ we obtain

$$O = \Omega A^{-1/2}_\nu \Omega^{-1} A^{1/2}_\nu = \Omega \mathcal{H}_\nu \Omega^{-1} \mathcal{H}_\nu \Omega \mathcal{H}_\nu,$$

whose action on the Mellin transform side is multiplication by

$$O(z) = \frac{\Gamma\left(\frac{\nu + z - \lambda + 1}{2}\right)^2 \Gamma\left(\frac{\nu - z + \lambda}{2}\right)}{\Gamma\left(\frac{\nu - z + \lambda + 1}{2}\right)^2 \Gamma\left(\frac{\nu + z - \lambda + 2}{2}\right)}.$$  

We thus get

$$\|O\| = \frac{\nu^2}{\nu^2 - 1}$$

for $\nu > 1$, because

$$|O(\lambda + 1 + iy)| = \left| \frac{\Gamma\left(\frac{\nu + iy + 2}{2}\right)^2 \Gamma\left(\frac{\nu - iy - 1}{2}\right)}{\Gamma\left(\frac{\nu + iy + 1}{2}\right)^2 \Gamma\left(\frac{\nu - iy + 1}{2}\right)} \right| = \left| \frac{(\nu + iy)^2}{(\nu + iy)^2 - 1} \right|.$$  

For the $P_0$ on the $l$'th spherical harmonic subspace we have

$$\nu = \sqrt{(\lambda + l)^2 + a} \geq \sqrt{\lambda^2 + a} > 1,$$

for $a > 1 - \lambda^2$. This show the boundedness of $C_1|_{\Sigma_l}$. The $L^2$ boundedness of $C_1$ then follows from the orthogonality of the subspaces $\Sigma_l$ with respect to the $L^2$ innerproduct.

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