Constituent quarks as solitons

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ABSTRACT

We exhibit soliton solutions of QCD in two dimensions that have the quantum numbers of quarks. They exist only for quarks heavier than the dimensional gauge coupling, and have infinite energy, corresponding to the presence of a string carrying the non-singlet color flux off to spatial infinity. The quark solitons also disappear at finite temperature, as the temperature-dependent effective quark mass is reduced in the approach to the quark/hadron phase transition.

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1. Introduction

One of the key problems in non-perturbative QCD is the understanding of “constituent” quarks from first principles. Historically, the first indications for the physical reality of quarks came from the successes of the constituent quark model in light hadron spectroscopy and matrix elements, assuming $m_u,d \simeq 300$ MeV and $m_s \simeq 500$ MeV.\cite{1} Subsequently, current algebra was abstracted from the lagrangian for the light quarks ($u,d,s$) and applied successfully to calculate the properties and interactions of pions and kaons.\cite{2} However, the success of quark current algebra could be understood only in the context of approximate chiral symmetry, according to which the $u,d$ and $s$ masses in the lagrangian were much smaller that the original “constituent” masses.\cite{3} The so-called “current quark” masses are a few MeV for $u$ and $d$ and $O$(100 to 200) MeV for the $s$ quark.\cite{4} Thus the question arises: what is the relationship between “current” and “constituent” quarks in QCD? In the cases of the “heavy” quarks ($c, b, t$), the difference between “constituent” and “current” quarks is presumably not so great, although even the successful quarkonium potential model has not been derived completely from QCD. Even more of a challenge\cite{5−9} is to derive the constituent versions of the light $u, d$ and $s$ quarks from QCD.

Refs. [6],[8] proposed that the constituent quarks are soliton solutions of a hypothetical effective lagrangian of QCD, with effective bosonic fields carrying both color and flavor. It is not known whether such an effective lagrangian exists in a well defined sense, nor is it known whether there are stable, non-trivial classical solutions. Sidestepping these questions, ref. [8] makes the interesting observation that the conjectured group-theoretical structure favors solitons with quantum numbers of quarks. In the absence of a well-justified effective dynamics of QCD, however, further progress requires a theoretical laboratory where the relevant phenomena are explicitly calculable.

A useful laboratory for studying this problem is QCD in 2 dimensions.$^{[10−20]}$ This theory can be written in bosonized form$^{[14]}$ for arbitrary numbers of colors
\(N_c\) and flavors \(N_f\).\(^{[16]}\) It reflects accurately the phenomena of quark confinement and condensation in the vacuum that we expect to occur in QCD in 4 dimensions. Even though Goldstone bosons cannot exist in 2 dimensions, QCD\(_2\) yields a relation between the pion mass, light quark mass and quark condensate similar to that expected on the basis of approximate chiral symmetry in QCD\(_4\).\(^{[11]}\)

Moreover, QCD\(_2\) has finite-energy soliton solutions for arbitrary values of \(N_c\) and \(N_f\) that can be interpreted as baryons,\(^{[16−19]}\) in close analogy with the Skyrmion interpretation of baryons as solitons in QCD\(_4\).\(^{[21−23]}\)

In this paper we start an investigation of constituent quarks in QCD\(_2\). Specifically, we examine QCD\(_2\) with one heavy quark flavor, and look for soliton solutions corresponding to a single quark, i.e. baryon number 1 (the baryon number is normalized to be \(N_c\) for the nucleon). They exist, for a sufficiently heavy quark \(Q\), but have infinite energy, corresponding to a string carrying the non-singlet color flux off to spatial infinity. These quark soliton solutions disappear when the meson mass parameter \(M\) is reduced to become comparable to the gauge coupling strength \(e_c\) (which, we recall, has the dimension of mass in QCD\(_2\)). They disappear when it is energetically favoured to create a new \(\bar{Q}Q\) pair, rather than to create a string with tension \((\alpha')^{-1} \sim e_c^2\) (for the analogous result in QED\(_2\), see ref. [24]). The quark solitons also disappear when the temperature \(T\) is increased to \(\mathcal{O}(M)\), because of a reduction in the \(T\)-dependent effective meson mass. The quark condensate and the baryonic solitons disappear at a somewhat higher temperature, signalling deconfinement at the quark/hadron phase transition.
2. Quarks as Solitons

We start with the bosonized effective action for QCD

\[
\begin{align*}
A(N_F=1) &= S[h] + \frac{1}{2} \int d^2x (\partial_\mu \phi)^2 \\
&+ M^2 \int d^2x \mathrm{Tr} \left[ he^{i\sqrt{\frac{4\pi}{N_c}}\phi} + h^+ e^{-i\sqrt{\frac{4\pi}{N_c}}\phi} \right] \\
&- \frac{e_c^2}{2\pi} \int d^2x \mathrm{Tr} H^2
\end{align*}
\]

where we took one flavor \( N_f=1 \) for simplicity, \( N_c \) is the number of colors, \( h \) is the SU\((N_c)\) color matrix of field operators, and

\[
S[h] = \frac{1}{8\pi} \int d^2x \mathrm{Tr} (\partial_\mu h \partial^\mu h^+) + \frac{1}{12\pi} \int_B d^3y e^{ijk} \mathrm{Tr} (h^+ \partial_i h) (h^+ \partial_j h) (h^+ \partial_k h)
\]

where the last term is the Wess-Zumino term. In eq. (1) \( e_c \) is the dimensional gauge coupling and \( \phi \) is the baryon number field. The mass scale \( M \) is related to the current quark mass \( m_Q \) and to the gauge coupling, in the limit where \( e_c \gg m_Q \), by

\[
M = [CN_c m_Q \left( \frac{e_c}{\sqrt{\pi}} \right)^p]^{\frac{1}{1+p}}; \quad p = 1 - \frac{1}{N_c}
\]

where \( C = \frac{1}{2} e^\gamma \approx 0.89 \), and \( \gamma \) is Euler’s constant. We will be interested in the opposite limit \( m_Q \gg e_c \), hence we choose \( M=m_Q \). The field \( H \) in (1) is related to \( h \) through

\[
\partial_- H = i h \partial_- h^+
\]

We define hermitian fields \( \varphi \) by

\[
h = e^{i\sqrt{4\pi} \varphi}
\]

and take \( \mathrm{Tr} \varphi = 0 \), absorbing a possible trace term \( \mathrm{Tr} \varphi = \frac{2\pi n}{\sqrt{4\pi}} \) in \( \phi \).
To search for the lowest-energy classical solutions, we take $\varphi$ diagonal with entries $\varphi_k$, ($k = 1, 2, \ldots, N_c$).

\[
A = \int \left[ \frac{1}{2} \sum_{k=1}^{N_c} \left( \partial_\mu \varphi_k \right)^2 + \frac{1}{2} \left( \partial_\mu \phi \right)^2 \right] + 2M^2 \sum_{k=1}^{N_c} \int \cos \left( \sqrt{4\pi} \varphi_k + \sqrt{\frac{4\pi}{N_c}} \phi \right) - 2e_c^2 \sum_{k=1}^{N_c} \int \varphi_k^2
\]

(5)

Defining the shifted variables

\[
\chi_k = \varphi_k + \frac{1}{\sqrt{N_c}} \phi \quad (k = 1, 2, \ldots, N_c)
\]

(6)

we find that for static $\chi_k$ the Hamiltonian takes the form

\[
H(\chi_k) = \frac{1}{2} \sum_{k=1}^{N_c} (\partial_1 \chi_k)^2 + V
\]

\[
V = 2M^2 \sum_{k=1}^{N_c} \left[ 1 - \cos \sqrt{4\pi} \chi_k \right] + 2e_c^2 \sum_{k=1}^{N_c} \chi_k^2 - \frac{2e_c^2}{N_c} \left( \sum_{l=1}^{N_c} \chi_l \right)^2
\]

(7)

and the baryon number is\(^{16, 17}\)

\[
B = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{N_c} [\chi_k(\infty) - \chi_k(-\infty)]
\]

(8)

The field equations for the static case are

\[
\chi''_k - 4M^2 \sqrt{\pi} \sin \sqrt{4\pi} \chi_k - 4e_c^2 \left[ \chi_k - \frac{1}{N_c} \sum_{l=1}^{N_c} \chi_l \right] = 0 \quad (k = 1, 2, \ldots, N_c)
\]

(9)

Let us look for solutions where $\chi_k(-\infty) = 0$ for all $k$. Then, at $x = \infty$, we must have

\[
M^2 \sqrt{\pi} \sin \sqrt{4\pi} \chi_k(\infty) + e_c^2 \left[ \chi_k(\infty) - \frac{1}{N_c} \sum_{l=1}^{N_c} \chi_l(\infty) \right] = 0
\]

(10)

If we seek a solution with $\chi_k(\infty) \equiv \chi(\infty)$ for all $k$, we find that $\chi_k(\infty) = \frac{1}{2} \sqrt{\pi} n$, and $B = \frac{1}{2} n N_c$. But to have the eigenvalues of the squared mass-matrix...
\[ \frac{\partial^2 V}{\partial \chi_k \partial \chi_l} \] all positive, we must have even \( n \), and thus integer baryon number \( B = N_c \), corresponding to a nucleon state.

In our search for quark solitons, let us first consider the simplest case \( N_c = 2 \). Then, the equations (10) become

\[
\sin \sqrt{4\pi} \chi_1(\infty) = -\epsilon \sqrt{\pi} (\chi_1(\infty) - \chi_2(\infty)) \quad (11a)
\]

\[
\sin \sqrt{4\pi} \chi_2(\infty) = -\epsilon \sqrt{\pi} (\chi_2(\infty) - \chi_1(\infty)) \quad (11b)
\]

where \( \epsilon \equiv e_c^2 / 2\pi M^2 \). Looking at the equations (11a),(11b), we see that we can have

\[
\chi_2(\infty) = -\chi_1(\infty) + n \sqrt{\pi} \quad (12a)
\]

or

\[
\chi_2(\infty) = \chi_1(\infty) + (n + \frac{1}{2}) \sqrt{\pi} \quad (12b)
\]

In case (a) we have \( B = n \), and in case (b) we have \( B = \frac{2}{\sqrt{\pi}} \chi_1(\infty) + (n + \frac{1}{2}) \). However, looking again at the eigenvalues of the matrix of second derivatives of the potential, and requiring positivity, we exclude case (b) (which otherwise would have non-integer baryon number).

For case (a), where \( B = n \), we may have quarks for \( n = 1 \), and other non-baryonic solitons for odd values of \( n \). As seen in the figure, there is no such solution when \( \epsilon = e_c^2 / 2\pi M^2 > 1 \) (for the analogous result in QED_2, see ref. [24]). This can also be seen directly from the fact that \( |\sin(x)/x| \leq 1 \) for any \( x \). For \( \epsilon \ll 1 \), however, we find a series of solutions with positive second derivative matrix.* Defining

\[
\xi = \sqrt{4\pi} \left[ \chi_1(\infty) - \frac{1}{2} n \sqrt{\pi} \right] \quad (13)
\]

* See ref. [26] for comparison.
these are
\[ \xi_l = \left[ \pi - \frac{e_c^2}{2M^2} \right] \left\{ \begin{array}{ll} (2l) & \text{for } n \text{ even} \\ (2l + 1) & \text{for } n \text{ odd} \end{array} \right\} \] (14)
in the limit where \( \frac{e_c^2}{M^2} \ll \frac{\pi}{2} \). The number of solutions increases as \( \epsilon = \frac{e_c^2}{2\pi M^2} \) decreases, as seen in the figure, and one can find solutions numerically for the general case \( \epsilon < 1 \). The solutions (14) above have infinite energy, with classical string tension\( (\alpha')^{-1} \approx \pi e_c^2 \left\{ \begin{array}{ll} (2l)^2 & \text{for } n \text{ even} \\ (2l + 1)^2 & \text{for } n \text{ odd} \end{array} \right\} \) (15)
They correspond to excitations of “colored” states. The case \( n=1, (2l+1)=1 \) is the single constituent quark soliton.

Note that the string tension (15) grows like \( l^2 \) for even \( n \), and like \( (l + \frac{1}{2})^2 \) for odd \( n \). A possible interpretation is that the classical string tension is growing like \( I^2 \), where \( I \) is the color isospin of the state: quantum mechanically it obviously has to become \( I(I + 1) \).

3. Generalization to \( N_c > 2 \)

It is a simple matter to generalize the above asymptotic solution (12a) to an arbitrary number of colors. The equations (10) clearly require \( \sqrt{4\pi \chi_k}(\infty) = n_k \pi + \mathcal{O}(e_c^2/\sqrt{\pi M^2}) \) for some integers \( \{n_k\} \). The positivity of the eigenvalues of \( [\partial^2 V/\partial \chi_k \partial \chi_l] \) in fact requires that the \( \{n_k\} \) be even: \( n_k = 2p_k \), where the \( \{p_k\} \) are integers.

Hence we look for asymptotic solutions, for \( \frac{e_c^2}{2\pi M^2} \ll 1 \), of the form
\[
\sqrt{4\pi \chi_k}(\infty) = 2p_k \pi + c_k \frac{e_c^2}{2M^2} + \mathcal{O}\left(\frac{e_c^4}{M^4}\right) \quad (k = 1, \ldots, N_c) \] (16)
The equations (10) in fact tell us that for integer \( \{p_k\} \) the corrections \( \{c_k\} \) are
given by

\[ c_k = 2 \left( -p_k + \frac{1}{N_c} \left( \sum_{j=1}^{N_c} p_j \right) \right) \quad (k = 1, \ldots, N_c) \quad (17) \]

Equation (8) then tells us that the baryon number

\[ B = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{N_c} \chi_k(\infty) = \sum_{k=1}^{N_c} p_k \quad (18) \]

as \( \sum_{k=1}^{N_c} c_k = 0 \). Hence \( B \) is an integer. The variable \( \xi \) (13) for the case \( N_c=2 \) may be generalized to the variables

\[ \xi_k \equiv \sqrt{4\pi} \chi_k(\infty) - \left( \sum_{l=1}^{N_c} p_l \right) \pi \quad (k = 1, \ldots, N_c) \quad (19) \]

which take the values

\[ \xi_k = (2p_k - B)\pi - \frac{e^2}{M^2} \left( p_k - \frac{1}{N_c} \sum_{l=1}^{N_c} p_l \right) + \mathcal{O} \left( \frac{e^2}{M^2} \right)^2 \quad (k = 1, \ldots, N_c) \quad (20) \]

generalizing the expression (14) for \( N_c=2 \). Correspondingly, the string tension (15) becomes

\[ (\alpha')^{-1} = 2\pi e_c^2 \sum_{k=1}^{N_c} (B/N_c - p_k)^2 \quad (21) \]

We see that when \( B \) is some integer multiple of \( N_c \), corresponding to a multiple-baryon state, \(^{18}\) the string tension vanishes, \( (\alpha')^{-1} = 0 \), if all the \( \{p_k\} \) are taken equal to the multiple-baryon number.
4. Finite temperature

Further insight into the interpretation of these QCD$_2$ solitons can be gained by considering the bosonized form of the action (1),(5) at finite temperature. The temperature-dependent one-loop corrections to a generic 2-dimensional action can be written in the form\[^{[25]}\]

$$\delta V(T) = \frac{T}{2\pi} \int_0^\infty dk \ Tr \ \ln \left( 1 - e^{-\beta \sqrt{k^2 + M^2}} \right)$$  \hspace{1cm} (22)

where $T = 1/\beta$ is the temperature and $M$ is the mass matrix. The leading field-dependent term in (22) has the form

$$\delta V_1(T) = c T \ Tr \sqrt{M^2}$$ \hspace{1cm} (23)

where $c$ is a coefficient independent of the specific theory. In the case $N_c = 2$, the finite-temperature correction (23) to the effective potential (7) takes the form

$$\delta V_1(T) = c T \left( \sqrt{2} M \left( \sqrt{\cos \sqrt{4\pi \chi_1}} + \sqrt{\cos \sqrt{4\pi \chi_2}} \right) + \frac{\epsilon_c^2}{8\sqrt{2\pi}} \left( \frac{1}{M \sqrt{\cos \sqrt{4\pi \chi_1}}} + \frac{1}{M \sqrt{\cos \sqrt{4\pi \chi_2}}} \right) \right)$$ \hspace{1cm} (24)

We now discuss some likely implications of this correction whilst recognizing\[^{[25]}\] that a complete discussion of the phase structure of QCD$_2$ would require evaluating the effective potential to all orders in the loop expansion.

As discussed in Section 2, “baryonic” soliton solutions to the zero-temperature $N_c = 2$ field equations (even $B$) exist for arbitrary values of $\epsilon = \epsilon_c^2/2\pi M^2$, whilst “quark” (odd $B$) solitons exist only for $\epsilon < 1$. As follows from equations (13),(14),
they correspond to

\[ \chi_1(\infty) \simeq m\sqrt{\pi} + \epsilon\sqrt{\pi} \left( \frac{B}{2} - m \right) \quad (m \in \mathbb{Z}) \]  

(25)
in the limit \( \epsilon \ll 1 \), and \( \chi_2(\infty) = -\chi_1(\infty) + n\sqrt{\pi} \). In the case of even \( B \), there are baryonic solitons with \( m = B/2 \) for which the \( \epsilon \)-dependent correction in (25) vanishes, whereas it is always present for quark solitons with odd \( B \), and indeed becomes large when \( \epsilon \to 1 \), corresponding to the previously noted disappearance of the soliton. When \( \epsilon \ll 1 \), the leading effect of the finite-temperature correction (24) is to replace the zero-temperature solution (25) by

\[ \chi_1^T(\infty) \simeq m\sqrt{\pi} + \epsilon(T)\sqrt{4\pi} \left( \frac{B}{2} - m \right) \]  

(26)

where

\[ \epsilon(T) \simeq \epsilon/ \left( 1 - \frac{cT}{M\sqrt{2}} \right) \]  

(27)
The effect of the finite-temperature modification (27) is in the direction of destabilizing the quark solitons (and also baryonic solitons with \( m \neq B/2 \)), which presumably disappear at critical temperatures \( T_{cB,m} = \mathcal{O}(M) \), with solitons whose values of \( |B/2 - m| \) are largest disappearing first. In particular, the last to disappear would be the quark solitons with \( |B/2 - m| = \frac{1}{2} \), and finally the baryons with \( B = m \). The latter actually disappear only when the coefficient of the periodic contribution to the zero-temperature potential (7) is finally overwhelmed by the finite-temperature corrections (24), which occurs at \( T_{cB,\frac{B}{2}} = T_{q/h} = \mathcal{O}(M) \).

We now discuss briefly some possible interpretations of these results. We interpret \( T_{q/h} \simeq \mathcal{O}(M) \) as the quark/hadron phase transition temperature, where the fermion condensates dissolve,\(^{[27,28]}\), the baryonic solitons disappear, and correspondingly quarks are deconfined, as discussed in refs. \([29],[30]\). We are not surprised by the fact that \( T_{q/h} \) seems to be quark mass- (and hence flavor-) dependent in QCD\(_2\), whereas it is expected to be \( \mathcal{O}(\Lambda_{QCD}) \) and flavor-independent.
in QCD. In QCD, \( T_{q/h} \) can be determined by entropy considerations (the flux tube fluctuates more and more until flux lines eventually fill all space), whereas in QCD\(_2\) the flux line cannot fluctuate out of the single space dimension, and \( T_{q/h} \) is determined by the energetics of quark pair-creation, which becomes favoured when \( T = \mathcal{O}(M) \). The presence of fermion condensation is necessary for solitons to form, but not sufficient. Even when \( T = 0 \), for any fixed value of \( \epsilon \) solitons with large values of \( |B - m| \) do not exist and the number of such solitons diminishes as \( T \) increases, with the “quark” solitons disappearing before the \( B = 2m \) “baryons”.

We plan to return to these and other issues at finite temperature and chemical potential in a future publication\(^{[31]} \)

5. Conclusions

We have shown in this paper that QCD\(_2\) has quark soliton solutions if the quark mass is sufficiently large. We have also shown how these quark solitons disappear when the quark mass \( m_Q \) is reduced until the meson mass \( M \) (3a) becomes comparable to the dimensional gauge coupling strength \( e_c \), or when the temperature is increased to \( \mathcal{O}(M) \) in the approach to the quark/hadron phase transition. The next step in this approach to the derivation of constituent quarks from QCD is to look for meson solutions to the QCD\(_2\) field equations which contain one heavy and one light quark. The light meson cloud in the presence of a heavy quark would correspond to the concept of a light constituent quark in QCD.\(^{[32]} \) Once these solutions are obtained, we hope to be able to abstract the relevant features of the field equations in QCD\(_4\) and then solve them to construct constituent quark solitons also in 4 dimensions.
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**FIGURE CAPTION**

Comparison of the left- and right-hand sides of the soliton eq. (11a), applying the relation (12a) corresponding to a quark soliton with $n = 1$, for $\epsilon = e_c^2/2\pi M^2 = 0.1$ (solid line) and $\epsilon = 0.5$ (dotted line). There are no solutions for $\epsilon > 1$ (dot-dashed line, drawn for $\epsilon = 2$).