AN ALGORITHM FOR DETECTING "LINEAR" SOLUTIONS OF NONLINEAR POLYNOMIAL DIFFERENTIAL EQUATIONS.

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Abstract

A symbolic computational algorithm which detects "linear" solutions of nonlinear polynomial differential equations of single functions, is developed throughout this paper.

1 Introduction

The problem of obtaining general solutions of differential equations via symbolic algorithms has been studied in the past by many authors, [2], [4], [5], [6], [7], [8]. These algorithms allowed new calculation techniques to be accomplished, much more efficiently, faster and without approximation errors.

In this paper we treat with differential equations of the form:

\[ p(x, y(x), y'(x), \ldots, y^{(n)}(x)) = 0 \quad (1) \]

where \( p \) is a polynomial function and \( y(x) \) a real function of a single variable. Our aim is to discover possible "linear" solutions of (1). By the term "linear" we mean solutions which can be obtained by solving linear differential equations. Our approach is focused on the construction of an algorithm which faces the problem symbolically. What this algorithm is essentially doing is that helps us to rewrite \( p \) as follows:

\[
p = c_1(x, y', \ldots, y^{(k-1)}, W_{i,\sigma,\varphi})[W_{0,k} + W_{1,k}y + W_{2,k}y' + \cdots + y^{(k)}]^{j_0,1} \\
+ \cdots + c_n(x, y', \ldots, y^{(k-1)}, W_{i,\sigma,\varphi})[W_{0,k} + W_{1,k}y + W_{2,k}y' + \cdots + y^{(k)}]^{j_0,1}
\]

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\[
\begin{align*}
[W_{0,k+1,\nu} + W_{1,k+1,\nu}y + W_{2,k+1,\nu}y' + \cdots + y^{(k)}]^{j_{1,\nu}} \\
\cdots [W_{0,n,\nu} + W_{1,n,\nu}y + W_{2,n,\nu}y' + \cdots + y^{(k)}]^{j_{n-k,\nu}} + R
\end{align*}
\]

Where \( k \) is given, \( n \) is the order of \( p \), and \( j_{a,b} \) are specific positive whole numbers. The quantities \( W_{i,\sigma,\varphi} \) are undetermined parameters which can take certain values, \( c_j \) are the coefficients, depending from the parameters \( W_{i,\sigma,\varphi} \) and \( R \) a polynomial of the variables \( x, y, y', \ldots, y^{(k-1)} \) and the parameters \( W_{1,\sigma,\varphi} \) called the remainder. Afterwards, we seek for those values of the parameters which eliminate the remainder. If this is possible, then the linear differential polynomial \( W_{0,k} + W_{1,k}y + \cdots + y^{(k-1)} \), with \( W_{0,k}, W_{1,k}, \ldots \) evaluated over those values which vanish the remainder, is a factor of \( p \), (where the operation of differentiation has been taken under consideration). This means that any solution of \( W_{0,k} + W_{1,k}y + \cdots + y^{(k-1)} = 0 \) is a solution of the equation (1), too. Since we do it for every \( k = 0, \ldots, n \), we collect likewise, all the ”linear” solutions of the equation \( p = 0 \).

Our method is an extension of a similar procedure, introduced by the author and applied in the study of difference equations and feedback design [1], [9]. Its main merit is its computational orientation. It turns to be a useful tool, implemented on a computer machine and gives useful results. Moreover, despite our method resembles with the approach of differential algebra, [5], there are some differences. Concretely, (i) We are working with a single differential polynomial whilst Ritt’s algorithm deals with sets of differential polynomials. (ii) The existence of the parameters \( W_{i,\sigma,\varphi} \) permits us to find classes of linear solutions. We can then select among them, these particular solutions which satisfy additional conditions. For instance, we can search for those values of \( W_{i,\sigma,\varphi} \), if any, which do not only eliminate the remainder but also yield stable linear factors. (iii) In the classical Ritt’s approach we find the minimum number of differential polynomials which generate a differential ideal. In our method we check if a given polynomial belongs to a differential ideal, produced by linear differential polynomials. Throughout the text, \( \mathbb{R} \) and \( \mathbb{Z}^+ \) will denote the sets of real numbers and positive integers, correspondingly.

2 The Algebraic Framework

Let \( \mathbb{R}[x] \) be the ring of polynomials of a single variable with real coefficients. This polynomial ring is a differential ring too, with the usual derivation [3], [2]. Let \( y(x) \) be a real function and \( y^{(i)}(x) \), \( i = 0, 1, 2, \ldots \) its derivatives. A differential polynomial \( p \), in \( y(x) \) or shortly in \( y \), is a polynomial in \( y \) and its derivatives with coefficients in \( \mathbb{R}[x] \). \( p \) can be written as follows:

\[
p = \sum_{\lambda=1}^{\varphi} s_{\lambda} x^{a_{\lambda}} \prod_{i=0}^{n} [y^{(i)}(x)]^{\theta_{i,\lambda}}
\]

where \( s_{\lambda} \in \mathbb{R} \) and some of the exponents \( a_{\lambda}, \theta_{i,\lambda} \in \mathbb{Z}^+ \) are not equal to zero. The number \( n \), which represents the highest order derivative of \( y(x) \), is called the order of \( p \). An equation of the form \( p = 0 \), with \( y(x) \) as unknown function, is called a polynomial differential equation. Any function which satisfies it, is called a solution or
a general solution. An expression of the form: \( L = \sum_{i=0}^{n} a_i y^{(i)}(x), a_i \in \mathbb{R} \), is called a linear differential polynomial and the equation \( L = 0 \) a linear differential equation. Its solutions are called “linear” solutions of order \( n \).

The highest derivative of \( y(x) \), appeared in the polynomial \( p \), is called the leader. Let \( p_1 = s_1 x^{s_1} \prod_{i=0}^{n} [y^{(i)}(x)]^{\theta_{i,1}}, p_2 = s_2 x^{s_2} \prod_{i=0}^{n} [y^{(i)}(x)]^{\theta_{i,2}} \) be two, not identical, terms of \( p \). This means that there is at least one index \( k, 1 \leq k \leq n \), such that \( \theta_{k,1} \neq \theta_{k,2} \) or \( \theta_{i,1} = \theta_{i,2}, i = 0, 1, \ldots, n \) and \( a_1 \neq a_2 \). We say that the term \( p_2 \) is ordered higher than \( p_1 \) with respect to lexicographical order and we write \( p_1 \prec p_2 \), if either there is an index \( s \) such that \( \theta_{s,1} < \theta_{s,2} \) and \( \theta_{j,1} = \theta_{j,2} = s, j = s + 1, \ldots, n \) or \( \theta_{i,1} = \theta_{i,2}, i = 0, 1, \ldots, n \) and \( a_1 < a_2 \). By means of this rank we can order all the terms of \( p \) in an ascending way. The term which is ordered higher, is called the maximum term of \( p \).

The differential ideal, generated by a finite set of differential polynomials: \( \Phi = \{ \Phi_1, \Phi_2, \ldots, \Phi_m \} \) and denoted by \( [\Phi] \) is a set which consists of all differential polynomials that can be formed of elements in \( \Phi \) by multiplication with arbitrary polynomials, addition and differentiation.

Let \( \mathcal{V} = \{ W_{i,\sigma,\varphi} \} \) be a set of undetermined parameters, taking values in \( \mathbb{R} \). A Formal- \( k \)-Factorization of \( p \), denoted by \( Formal(p,k) \), is an expression of \( p \) of the form:

\[
Formal(p,k) = \sum_{\mu=1}^{\nu} c_\mu(x, y'(x), \ldots, y^{(k-1)}(x), W_{i,\sigma,\varphi}) \cdot 
\prod_{\lambda=0}^{n-k} ((W_{0,k+i,\mu} + W_{1,k+i,\mu} y(x) + W_{2,k+i,\mu} y'(x) + \cdots + y^{(k)}(x))^{j_\lambda} + R
\]

where the coefficients \( c_\mu(x, y'(x), \ldots, y^{(k-1)}(x), W_{i,\sigma,\varphi}) \) and the remainder \( R \) are polynomials of the terms \( x, y'(x), \ldots, y^{(k-1)}(x), W_{i,\sigma,\varphi} \) only. Some of the exponents \( j_\lambda \in \mathbb{Z}^+ \) may be equal to zero. Sometimes, (2) is written briefly as

\[
Formal(p,k) = \sum_{\mu=1}^{\nu} c_\mu \prod_{\lambda=0}^{n-k} [L_{i,\mu}]^{j_\lambda} + R
\]

where we used the notation \( L_{i,k} \) for the linear differential polynomial \( W_{0,k+i,\mu} + W_{1,k+i,\mu} y(x) + W_{2,k+i,\mu} y'(x) + \cdots + y^{(k)}(x) \).

We can take different expressions of the \( Formal(p,k) \) of a concrete differential polynomial \( p \), by giving to the parameters \( W_{i,\sigma,\varphi} \) certain values. Such procedures are called evaluations of the \( Formal(p,k) \). A most rigorous approach is the following: Let \( \mathcal{V} = \{ W_{i,\sigma,\varphi} \} \) be the set of the variables, appeared in the Formal- \( k \)-Factorization of a given polynomial \( p \). By arranging the parameters in an increasing order we form the vector \( \mathcal{V} = (W_{i_k,\sigma_k,\varphi_k})_{h=1,2,\ldots,n} \). Let \( r = (a_h)_{h=1,2,\ldots,n} \) be a vector of real numbers, which has the same length with the vector \( \mathcal{V} \). We say that the parameters \( \mathcal{V} \) follow the rules \( r \) and we write \( \mathcal{V} \to r \) if the following substitution is valid: \( W_{i_h,\sigma_h,\varphi_h} = a_h, h = 1,2,\ldots,n \). Let \( M \) a set of rules, \( M = \{ r_1, r_2, \ldots, r_\lambda \} \) then

\[
Formal(p,k) \mid_M = \bigcup_{\nu=1}^{\lambda} \left( \sum_{\mu=1}^{\nu} c_\mu \prod_{\lambda=0}^{n-k} ([W_{0,k+i,\mu} + W_{1,k+i,\mu} y(x) + \cdots + y^{(k)}(x)]^{j_\lambda} + R
\]

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The set of substitutions $M$, may be finite or infinite. When we evaluate $\text{Formal}(p, k)$ over $M$, the linear differential polynomials $L_{i, \mu}$ and the remainder $R$ take specific values, these are denoted by $L_{i, \mu} \mid_M$ and $R \mid_M$. A case of particular interest is when we can find values of the parameters which eliminate the remainder $R$. Whenever this happens, $p$ is a "combination" of linear differential polynomials or, in a more formal language, $p$ is a member of the differential ideal produced by these linear differential polynomials. Relevant is the following theorem:

**THEOREM 1.** Let $p$ be a differential polynomial of order $n$, let $k$ be given and $\text{Formal}(p, k) = \sum_{\mu=1}^{\nu} c_\mu \prod_{i=1}^{n-k} L_{i, \mu}^{|y_i|} + R$ its Formal-k-Factorization. Let us suppose that there is a set of rules, denoted by $\mathcal{R}$, which eliminates the remainder $R$, i.e. $R \mid_{\mathcal{R}} = 0$, then $p \in \{ L_{i, \mu} \mid_{\mathcal{R}} ; i = 0, \ldots, n - k, \mu = 1, \ldots, \nu \}$.

**PROOF.** The proof comes straightforward from the definition of the Formal-k-Factorization. \qed

It is obvious that if the linear differential equations $L_{i, \mu} \mid_{\mathcal{R}} = 0$ have a common solution, this is a solution of the nonlinear equation $p = 0$, too. This is the cornerstone of our approach.

**EXAMPLE 1.** Let us consider the differential polynomial

$$p = 4y'' - 4(y'')^2 + y'y'' - \frac{1}{16}(y')^2 - 1$$

then,

$$\text{Formal}(p, 2) = 4(W_{0,3,1} + W_{1,3,1}y + W_{2,3,1}y' + y'')' - 4(W_{0,2,2} + W_{1,2,2}y + W_{2,2,2}y' + y'')^2 + 8W_{0,2,2} - 4W_{2,3,1} + 8W_{1,2,2} + (8W_{2,2,2} + 1)y' - (W_{0,2,3} + W_{1,2,3}y + W_{2,2,3}y' + y'') - R$$

where the remainder $R$ is $R = 4W_{0,2,2}^2 - 8W_{0,2,3}W_{0,2,2}$

$$+ 4W_{2,3,1}W_{0,2,2} - 1 - (8W_{1,2,2}W_{0,2,2} - 8W_{1,2,3}W_{0,2,2} - 8W_{0,2,3}W_{1,2,2} + 4W_{1,2,3}W_{2,3,1})y + (8W_{2,2,2} - 8W_{1,2,2}W_{1,2,3})y^2 + 8W_{2,2,2}W_{0,2,2} - 8W_{0,2,3}W_{2,2,2} - 8W_{2,2,3}W_{0,2,2} - 8W_{1,2,3}W_{2,2,2} - 8W_{2,2,2}W_{2,2,2} - 8W_{1,2,2}W_{2,2,2}y^y.$$ The following rules eliminate the remainder: $r_1 = \{W_{0,3,1} = s, W_{0,2,2} = \omega, W_{2,2,2} = \varphi, W_{0,2,3} = k, W_{1,2,1} = \frac{1}{2}\omega - \frac{(\omega^2 - 1)\varphi}{4k}, W_{1,2,3} = 0, W_{2,2,2} = -\frac{1}{8}, W_{1,2,2} = 0, W_{2,3,1} = 2\omega + \frac{4\omega^2 - 1}{4k} \}$ and $r_2 = \{W_{0,3,1} = s, W_{0,2,2} = \omega, W_{2,2,2} = \varphi, W_{0,2,3} = k, W_{1,2,1} = \frac{1}{2}[\omega + (\omega - 2k)(k - 4(\omega - 2k)\varphi)], W_{1,2,3} = 0, W_{2,2,2} = 2\varphi + \frac{1}{8}, W_{1,2,2} = 0, W_{2,3,1} = 2\omega + \frac{4\omega^2 - 1}{4k} \}$ with $\omega, \varphi, k, s \in \mathbb{R}$. Indeed, for instance

$$\text{Formal}(p, 2) \mid_{r_1} = 4(s - \frac{1}{4} \left(\omega + \frac{(4\omega^2 - 1)\varphi}{k}\right) y + \left(2\omega + \frac{4\omega^2 - 1}{4k}\right) y' + y''')$$

$$- 4(\omega - \frac{y'}{8} + y'')^2 + \left(\frac{4\omega^2 - 1}{k}\right) (\omega + \varphi y' + y'')$$

and the differential ideal which contains is $\{s - \frac{1}{4} \left(\omega + \frac{(4\omega^2 - 1)\varphi}{k}\right) y, + \left(2\omega + \frac{1}{4k}\right) y' + y'', \omega - \frac{y'}{8} + y'', \omega + \varphi y' + y''\} \omega, \varphi, s, k \in \mathbb{R}$. By setting $\omega = \varphi = s = 0, k = 1$ we take the simplified ideal $\{y'', -\frac{y'}{8} + y''\}$. We can take analogous expression for $\text{Formal}(p, 2) \mid_{r_2}$.
3 Detection of the Linear Solutions

The scope of this section is to present the algorithm which constructs for a given $k$, a special $Formal(p,k)$ with a linear differential polynomial as a common factor to each term. We denote this factor by $L_{c,k}$. Afterwards, by finding proper sets of values for the parameters $W_{i,\sigma,\varphi}$, we eliminate the remainder. It is then clear that any solution of the linear equation $L_{c,k} = 0$, where the polynomial $L_{c,k}$ has been evaluated over this set, is a solution of the original system, too. By repeating the whole procedure for every $k = 0, \ldots, n$ we discover all the linear solutions. As we pointed out, the crucial issue is how can we eliminate the remainder. This is carried out by solving a system of algebraic equations. Finally, we have to elucidate that in this paper we do not take into account initial conditions. We are only focused on how we obtain general solutions, that is solutions which ” contain ” constants. We present now the algorithm upon discussion.

Let us suppose that an algorithm which solves an algebraic system of polynomial equations, is available. These algorithms are classical in computational algebra and there are many of them in the literature [3]. We name a such algorithm as $SysAlgEqs$.

THE DIF-FORMAL ALGORITHM

**Input:**
- A differential polynomial $p$ of order $n$.
- A set of undetermined parameters $W = \{W_{i,\sigma,\varphi}\}$, taking values in $R$.

**Output:** The quantities $S_k, k = 0, \ldots, n$

**FOR** $k = 0$ TO $n$

**Step 1:** $R = p, \mu = 0$.

**Step 2:** REPEAT the following steps UNTIL $R$ does not contain terms of order $\geq k$.

**Step 2a:** Set $\mu = \mu + 1$.

**Step 2b:** Find the maximum term of $R$, with respect to the lexicographical order. We denote it by

$$ p_\mu = s_\mu \cdot x^{a_\mu} \cdot \prod_{i=0}^{n} [y^{(i)}(x)]^{\lambda_{i,\mu}} $$

where $a_\mu, \lambda_{i,\mu}$ are positive integers and $y^{(i)}(x)$ the derivatives of $y(x)$ of order $i$. At the first iteration $s_\mu$ is a constant, then it becomes a function of the free parameters $W_{i,\sigma,\varphi}$ as well.

**Step 2c:** Construct the linear formal differential polynomials:

$$ L_{c,k} = W_{0,k} + W_{1,k}y(x) + W_{2,k}y'(x) + \cdots + W_{k,k}y^{(k-1)}(x) + y^{(k)}(x) = $$

$$ = W_{0,k} + \sum_{j=0}^{k-1} W_{j+1,k}y^{(j)}(x) + y^{(k)}(x) $$

$$ L_{i,k} = W_{0,k+i,\mu} + \sum_{j=0}^{k-1} W_{j+1,k+i,\mu}y^{(j)}(x) + y^{(k)}(x), \quad i = 1, \ldots, n - k $$

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Step 2d: Execute the operation:

\[ R = R - s_{\mu} x^{a_{\mu}} \prod_{i=0}^{k-1} [y^{(i)}(x)]^{\lambda_{i,\mu}} \cdot [L_{c,k}^{(0)}]^{\lambda_{c,\mu}} \prod_{i=1}^{n-k} [L_{i,k}^{(i)}]^{\lambda_{i,\mu}} \]

END of REPEAT

Step 3: By means of the SysAlgEqs-Algorithm we find the set \( S_k \) of those values of the parameters, \( W_{i,\sigma,\phi} \), which eliminate the remainder. In other words: \( R \mid_{S_k} = 0 \).

END of FOR

It is obvious that the DIF-FORMAL algorithm terminates after a finite number of iterations.

THEOREM 2. Let \( S_k, k = 0, \ldots, n \) be the outputs of the DIF-FORMAL Algorithm. If \( S_k \neq \emptyset \) for some values of \( k \), then the solutions of the linear differential equations \( L_{c,k} \mid_{S_k} = 0 \) are solutions of the nonlinear polynomial differential equation \( p = 0 \), too.

PROOF. Let \( p \) a differential polynomial and \( k \) fixed. By substituting backwards the successive results of the step 2d we find that

\[ p = \sum_{\mu=1}^{m} s_{\mu} x^{a_{\mu}} \prod_{i=0}^{k-1} [y^{(i)}]^{\lambda_{i,\mu}} \cdot [L_{c,k}^{(0)}]^{\lambda_{c,\mu}} \prod_{i=1}^{n-k} [L_{i,k}^{(i)}]^{\lambda_{i,\mu}} \]

This is the Formal-k-Factorization of \( p \) with \( c_{\mu} = s_{\mu} x^{a_{\mu}} \prod_{i=0}^{k-1} [y^{(i)}]^{\lambda_{i,\mu}} \) and \( L_{c,k} \) as a common factor in all the terms but the remainder. We denote it by \( C_{\text{Formal}}(p, k) \). Let us now suppose that \( S_k \neq \emptyset \), this means that \( R \mid_{S_k} = 0 \) and thus, the linear differential polynomial \( L_{c,k} \mid_{S_k} = 0 \) is a factor of every term of \( C_{\text{Formal}}(p, k) \mid_{S_k} = 0 \). This implies that any solution of the linear differential equation \( L_{c,k} \mid_{S_k} = 0 \) is also a solution of the nonlinear equation \( p = 0 \). Since this argument is true for any \( k \), the theorem has been proved. □

The above result can be restated, using ideals, in the following way.

COROLLARY 1. Let \( S_k, k = 0, \ldots, n \) be the outputs of the DIF-FORMAL Algorithm. If \( S_k \neq \emptyset \) for some values of \( k \), then \( p \in \{ L_{c,k} \mid_{S_k} \}, k = 0, \ldots, n \).

In order to eliminate the remainder we have to solve a system of algebraic polynomial equations. This can be done via several methods. Groebner basis, \[3\], is a popular powerful tool, with satisfactory results.

We can extend the algorithm posted above toward different directions. (a) Instead of the constant term in the common linear differential polynomial we can use a single polynomial of \( x \) of a given degree with parametrical coefficients. These coefficients can be determined through the remainder elimination, too. (b) Instead of linear differential polynomial, with constant coefficients, we can use linear differential polynomials with variable coefficients which correspond to solvable differential equations (Euler equations, for instance). The last two cases are currently under studying.

EXAMPLE 2. Let us consider the differential equation

\[ (y'')^3 - 2y'(y'')^2 - 4(y'')^2 + (y')^2y''' + 4y'y''' + 4y''' = 0 \]
or $p = 0$, where $y = y(x)$ is the unknown function and the order of $p$ is $n = 3$. We want to find all the "linear" solutions which are included into this equation. The application of the DIF-FORMAL algorithm gave the following results: $y(x) = c_1c_2$, $y(x) = c_1e^x - 2x + c_2$, $c_1, c_2 \in \mathbb{R}$. To clarify our ideas and to indicate how the algorithm works in practice, we shall present the case $k = 2$ in details. Since $k = 2$ we are going to detect linear polynomials of second order included into the original equation. At the first iteration the maximum term is $[(y')^2]'$. We construct the linear polynomial:

$$L_{c,2} = W_{0,2} + W_{1,2}y + W_{2,2}y' + y''$$

and we execute the subtraction: $p_1 = p - [(y')^2] \cdot L_{c,2}$. This operation will eliminate the $y'''$ term. In the next iteration $4y'y'''$ is the maximum term. Since we are looking for a common second order linear equation included into $p$, we use the same $L_{c,2}$ and we calculate $p_2 = p_1 - 4y' \cdot L_{c,2}$. Working this way we finally get:

$$\text{Formal}(p, 2) = [(y')^2 + 4y' + 4] \cdot L_{c,2} + L_{c,2}^3 + [-3W_{0,2} - 3W_{1,2}y -$$

$$-3W_2,2y' - 2y' - 4]L_{c,2}^2 + [3W_{0,2}^2 + 6W_{1,2}W_{0,2}y + 6W_{2,2}W_{0,2}y' + 4W_{0,2}y' +$$

$$+8W_{0,2} + 3W_{1,2}y^2 + 3W_{2,2}y^2 + 3W_{2,2}(y')^2 - 4W_{2,2} + 8W_{1,2}y + 4W_{2,2}y' +$$

$$+4W_{2,2}yy' + 6W_{1,2}W_{2,2}yy']L_{c,2} + R$$

where $R = (-3W_{1,2}W_{0,2}^2 + 8W_{1,2}W_{0,2} + 4W_{1,2}W_{2,2})y + (-3W_{0,2}W_{1,2} - 4W_{2,2})y^2 - W_{1,2}y' +$$

$$+(-3W_{2,2}W_{0,2}^2 - 2W_{0,2} - 4W_{2,2}W_{0,2} + 4W_{2,2}^2 - 4W_{1,2})y' + (-3W_{0,2}W_{1,2} +$$

$$-6W_{0,2}W_{2,2}W_{1,2} - 4W_{2,2}W_{1,2}yy' + (-3W_{2,2}W_{1,2}^2 - 2W_{1,2}^2)y' + (-3W_{0,2}W_{2,2}^2 - 3W_{0,2}W_{2,2} -$$

$$-4W_{1,2})(y')^2 + (-3W_{1,2}W_{2,2}^2 - 3W_{1,2}W_{2,2}y(y')^2 + (-W_{2,2}W_{1,2} - W_{1,2}^2)y(y')^3 + 4W_{2,2}W_{0,2} -$$

$$-W_{2,2}W_{1,2})y) + (W_{0,2} = -2, W_{1,2} = 0, W_{2,2} = -1) and (W_{0,2} = 0, W_{1,2} = 0, W_{2,2} = 0), eliminate the remainder. By substituting them to $L_{c,2}$ we get the linear differential equations $y'' - y' - 2 = 0$, $y'' = 0$. The general solutions of those equations will be solutions of the original nonlinear differential equation too. Therefore $y(x) = c_1e^x + c_2 - 2x$, $y(x) = c_2x + c_1$, are solutions of the equation $p = 0$, too. Working similarly, we obtain for $k = 0$ and $k = 1$ analogous results. What we have actually proved is that $p$ is a member of the differential ideal $[y'' - y' - 2, y'''].$

**EXAMPLE 3.** Let us illustrate now the case where, instead of constant terms, we have polynomials of $x$. We consider the differential equation:

$$(x^2 - x)y' + xy'' - x^2y''' + (-2x^3 - 3x^2 + 3x) = 0$$

In this case we work with the extended algorithm which uses common linear factors of the form: $y' + W_{1,1}y + W_{-2,0,1}x^2 + W_{-1,0,1}x + W_{0,0,1}$. This modified DIF-FORMAL Algorithm will give that $W_{1,1} = -1$, $W_{-2,0,1} = 1$, $W_{-1,0,1} = 3$ and $W_{0,0,1}$ free or $W_{1,1} = 0$, $W_{-2,0,1} = 0$, $W_{-1,0,1} = -2$ and $W_{0,0,1} = -5$. The corresponding general linear solution is $y(x) = c_1e^x + x^2 + 5x + c_2$. 


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