1 Introduction

The purpose of this work is to prove Jiang’s conjecture [5] based on the analysis under the assumption of the functorial liftings and the endoscopic liftings so that we can see the existence of a $L$-function of a cuspidal representation of $GSp(4, \mathbb{A}) \times GSp(4, \mathbb{A})$ which has a pole of order 2 at $s = 1$, even for globally generic representations.

In [5], Dihua Jiang studies the degree 16 Rankin product $L$-function for $GSp(4) \times GSp(4)$, where $GSp(4)$ is the reductive group of symplectic similitudes of rank 2. More precisely, this $L$-function is defined as follows: Let $\pi_1$ and $\pi_2$ be irreducible automorphic cuspidal representations of $GSp(4, \mathbb{A})$ with trivial central characters and let $\rho$ be the standard representation of $GSp(4, \mathbb{C})$, the complex dual group of $GSp(4)$ [2]. The degree 16 standard $L$-function is $L^S(s, \pi_1 \otimes \pi_2, \rho \otimes \rho)$.

As explained by Jiang [5] the following commutative diagram with $L$-homomorphisms: $\sigma_2 = \sigma \circ \sigma_1$ and

$$\begin{array}{ccc}
\text{Sp}(4, \mathbb{C}) & \xrightarrow{\sigma_1} & \text{SL}(4, \mathbb{C}) \\
\text{SL}(2, \mathbb{C}) & \xrightarrow{\sigma_2} & \text{SL}(2, \mathbb{C})
\end{array}$$

will allow us to predict analytic properties of the $L$-function $L^S(s, \pi_1 \otimes \pi_2, \rho \otimes \rho)$ for $Re(s) > 0$ as follows:

1. If neither $\pi_1$ nor $\pi_2$ is an endoscopic lifting via $\sigma_1$, then

$$L^S(s, \pi_1 \otimes \pi_2, (\rho \otimes \rho) \circ \sigma) = L^S(s, \sigma(\pi_1) \otimes \sigma(\pi_2), (\rho \otimes \rho))$$

is holomorphic for all $s$ except at $s = 1$ where the L-function $L^S(s, \pi_1 \otimes \pi_2, \rho \otimes \rho)$ has a simple pole if and only if $\sigma(\pi_2) = \sigma(\pi_1)^\vee$, the contragredient representation of $\sigma(\pi_1)$.

2. If only one of $\pi_1$ and $\pi_2$ is an endoscopic lifting via $\sigma_1$, then the $L$-function $L^S(s, \pi_1 \otimes \pi_2, \rho \otimes \rho)$ is holomorphic for all $s$.

In fact, if, say $\pi_1 = \sigma_1(\pi_1^{(1)} \otimes \pi_2^{(2)})$, an endoscopic lifting via $\sigma_1$, then one has

$$\sigma(\pi_1) = \sigma_2(\pi_1^{(1)} \otimes \pi_2^{(1)}) = \pi_1^{(1)} \otimes \pi_2^{(1)} \text{ (automorphic induction)}.$$
2 Classification Theory

In this section, we will see the classification theorem to show that the transferred representation from GSp(4) to GL(4) is the isobaric sum of representations in GL(2)’s.

Let $F$ be a global field and $\mathbb{A}$ be the ring of adeles.

We can obtain a classification theorem for automorphic forms on GL(r) which is a precise analogue for this group of the known results for local groups by [6].

Accordingly let $P$ be a standard parabolic subgroup of GL(r) of type $(r_1, r_2, \cdots, r_a)$. The quotient of $P$ with its unipotent radical $U_P$ is isomorphic to the group

$$M = \text{GL}(r_1) \times \text{GL}(r_2) \times \cdots \times \text{GL}(r_a).$$
For each \( j, 1 \leq j \leq u \), let \( \sigma_j \) be an automorphic cuspidal representation of \( \text{GL}(r_j, \mathbb{A}) \). For each place \( v \) the representation \( \sigma_v = \otimes \sigma_{j,v} \) of the group \( \text{M}(F_v) \) can be regarded as a representation of \( \text{P}(F_v) \) trivial on \( \text{U}(F_v) \), it induces an admissible representation of \( \text{GL}(r, F_v) \) which we will denote by

\[
\xi_v = \text{Ind}(\text{GL}(r, F_v), \text{P}(F_v); \sigma_v).
\]

One obtains a family of irreducible admissible representations of \( \text{GL}(r, \mathbb{A}) \) by taking for each irreducible component \( \pi_v \) of the representation \( \xi_v \) and forming the tensor product \( \pi = \otimes \pi_v \). On the other hand, with \( \sigma = \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_u \), one can define globally an induced representation

\[
\xi = \text{Ind}(\text{GL}(r, \mathbb{A}), \text{P}(\mathbb{A}); \sigma).
\]

Of course \( \xi = \otimes_v \xi_v \).

Let \( Q \) be another standard parabolic say of type \( (s_1, s_2, \cdots, s_u) \) and \( \tau_j \) an automorphic cuspidal representation of \( \text{GL}(s_j, \mathbb{A}) \). As before let \( \tau_v = \otimes_j \tau_{j,v} \),

\[
\eta_v = \text{Ind}(\text{GL}(r, F_v), \text{Q}(F_v); \tau_v),
\]

and

\[
\eta = \text{Ind}(\text{GL}(r, \mathbb{A}), \text{Q}(\mathbb{A}); \tau),
\]

where \( \tau = \tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_u \). We may ask whether \( \xi \) and \( \eta \) have a common constituent. Suppose \( P \) and \( Q \) are associate and there is a permutation \( \phi \) of \( \{1, 2, \cdots, u\} \) such that \( s_j = r_{\phi(j)} \). Suppose moreover that \( \tau_j = \sigma_{\phi(j)} \). We will say in this situation that the pairs \( (\sigma, P) \) and \( (\tau, Q) \) are associate. When this is so the representations \( \xi_v \) and \( \eta_v \) have the same character, and therefore the same components. In particular if both \( \xi_v \) and \( \eta_v \) are unramified then their unique unramified components are the same. In other words the irreducible components of \( \xi \) and \( \eta \) are the same.

And the converse is also true by theorem 4.4 in [6].

**Proposition 1** Let \( P, Q, \sigma_j \) and \( \tau_1 \) be as above. Let \( S \) be a finite set of places containing all the places at infinity. Suppose that for \( v \notin S \) the representations \( \sigma_{j,v} \) and \( \tau_{v} \) are unramified and that the representations \( \xi_v \) and \( \eta_v \) of \( \text{GL}(r, F_v) \) they induce have the same unramified component. Then the pairs \( (\sigma, P) \) and \( (\tau, Q) \) are associate.

## 3 Transfer from GSO(4) to GL(4)

In this section, we will see the relation between GSO(4) and GL(4) which we will need later to show the existence of a representation transferred from GSO(4) to GL(4).

Let \( k \) be a number field with algebraic closure \( \bar{k} \). Let \( V \) be a finite dimensional vector space over \( k \) equipped with a non-degenerate symmetric bilinear form \( B : V \times V \to F \). Then the orthogonal similitude group of \( V \) with respect to the form \( B \) is the group \( \text{GO}(V, B) \) of all \( g \in \text{GL}(V) \) such that \( B(gv, gw) = \lambda(g)B(v, w) \) for any \( v, w \in V \) with \( \lambda(g) \in k^* \). The multiplicative character \( \lambda : \text{GO}(V, B) \to k^* \) is called the similitude character. Note that the orthogonal subgroup \( \text{O}(V, B) \) is equal to \( K(e) \).

Suppose \( V \) is a two dimensional vector space over \( k \) with a symplectic form \( \text{Sp} \) defined by the determinant. That is to say \( \text{Sp}(v, w) = \text{det}(v, w) \) for any \( v, w \in V \) which are expressed as column vectors with respect to a fixed base and \( (v, w) \) is written as a \( 2 \times 2 \) matrix. Then we can define a bilinear form \( B \) on \( V \otimes V \) by \( B(v_1 \otimes w_1, v_2 \otimes w_2) = \text{Sp}(v_1, v_2) \text{Sp}(w_1, w_2) \). It is easy to check that \( B \) is a non-degenerate and symmetric bilinear form on \( V \otimes V \) and the image of the tensor product from \( \text{GL}(2, k) \times \text{GL}(2, k) \) to \( \text{GL}(4, k) \) is a subgroup of \( \text{GO}(k^2, B_0) \) if we fix an isometry between \( (V \otimes V, B) \) and \( (k^2, B_0) \) where \( B_0 \) is the standard bilinear form of \( k^2 \) defined by \( B_0(v, w) = v^t w \) for any \( v, w \in k^2 \). Therefore we have the following exact sequence,

\[
1 \to k^* \to \text{GL}(2, k) \times \text{GL}(2, k) \to \text{GO}(k^2, B_0)
\]
since GL(2, k) is the symplectic similitude group of V with similitude character \( \lambda(g) = \text{det}(g) \). In particular, we have the following exact sequence:

\[
1 \rightarrow \{\pm(I_2, I_2)\} \rightarrow \text{SL}(2, k) \times \text{SL}(2, k) \rightarrow \text{SO}(k^4, B_0)
\]

From the discussion on page 57 of [3], the abelianization of \( \text{SO}(k^4, B_0) \) is isomorphic to \( k^* / k^* \), which is trivial if \( k = \bar{k} \). Therefore if we assume \( k = \bar{k} \), then \( \text{SO}(k^4, B_0) \) is equal to its commutator subgroup. By the discussion on page 59 of [3] \( \text{SO}(k^4, B_0) / \{\pm I_4\} \) is isomorphic to the group product \( \text{PSL}(2, k) \times \text{PSL}(2, k) \). Thus the map from \( \text{SL}(2, k) \times \text{SL}(2, k) \) to \( \text{SO}(k^4, B_0) \) is onto and the map from \( \text{GL}(2, k) \times \text{GL}(2, k) \) to \( \text{GO}(k^4, B_0) \) is onto \( \text{GO}(k^4, B_0) \).

If we use this:

\[
1 \rightarrow \mathbf{G}_m \rightarrow \text{GL}(2) \times \text{GL}(2) \rightarrow \text{GO}(B_0) \rightarrow 1
\]

and apply \( H^1 \), then we can get:

\[
1 \rightarrow k^* \rightarrow \text{GL}(2, k) \times \text{GL}(2, k) \rightarrow \text{GO}(k^4, B_0) \rightarrow 1
\]

since \( H^1(\text{Gal}(\bar{k}/k), \bar{k}^*) = 1 \).

**Lemma 1** Let \( k \) be a number field with algebraic closure \( \bar{k} \). Then

\[
\text{GSO}(4, k) = \frac{\text{GL}(2, k) \times \text{GL}(2, k)}{\{cI_2, c^{-1}I_2\}},
\]

where \( c \in k^* \)

*Proof.* Assume \( k = \bar{k} \).

We start with some notations. We let \( B(v, w) = vw \) be a non-degenerate symmetric bilinear form,

\[
\text{GO}(n, k) = \{g \in \text{GL}(n, k) | B(gv, gw) = \lambda(g)B(v, w), \lambda(g) \in k^*, v, w \in k^n\},
\]

where the multiplicative character \( \lambda : \text{GO}(n, k) \rightarrow k^* \) is called the similitude character,

\[
\text{O}(n, k) = \{g \in \text{GL}(n, k) | B(gv, gw) = B(v, w)\},
\]

\[
\text{SO}(n, k) = \{g \in \text{O}(n, k) | \text{det}g = 1\}
\]

and

\[
\text{Z}(n, k) = \text{center of GO}(n, k).
\]

For all \( g \in \text{GO}(4, k) \), \( (\text{det}g)^2 = \lambda(g)^4 \) and

\[
\text{GSO}(4, k) = \{g \in \text{GO}(4, k) | \text{det}g = \lambda(g)^2\}
\]

is generated by \( \text{SO}(4, k), \text{Z}(4, k) \) and \( \text{SO}(4, k) \cap \text{Z}(4, k) = \{\pm I_4\} \).

First, let \( W \) be \( k^2 \) with the standard symplectic form given by determinant. Then the induced bilinear form \( B_1 \) on \( W \otimes W \) is non-degenerated and symmetric, and \( B_1 \) is given by \( B_1(v_1 \otimes w_1, v_2 \otimes w_2) = \text{det}(v_1, v_2)\text{det}(w_1, w_2) \).

There is an isometry between \( (W \otimes W, B_1) \) and \( (k^4, B) \).

Since \( \text{GL}(2, k) \) is the symplectic similitude group of \( (W, \text{det}) \), we can get a sequence,

\[
1 \rightarrow k^* \rightarrow \text{GL}(2, k) \times \text{GL}(2, k) \xrightarrow{\beta} \text{GO}(4, k)
\]

in which the map \( \iota \) is given by \( \iota(c) = (cI_2, c^{-1}I_2) \).

Let \( \beta : \text{GL}(2, k) \times \text{GL}(2, k) \rightarrow \text{GO}(4, k) \) be defined as follows. The quadratic space \( (k^4, B) \) is isometric to \( (M_2(k), B_2) \) where \( B_2 \) is the symmetric bilinear map \( (X, Y) \rightarrow tr\langle XY \rangle \). Under this identification, \( \beta(g_1, g_2) \) is the automorphism of \( k^4 \) given by \( X \rightarrow g_1Xg_2 \) for all \( g_1, g_2 \in \text{GL}(2, k) \). And \( \ker \beta = \{(tI_2, r^{-1}I_2) | t \in k^* \} \).
We can calculate \( \det(\beta, g_1, g_2) = [\det(g_1) \det(g_2)]^2, \lambda(\beta(g_1, g_2)) = \det(g_1) \det(g_2) \). So, \( \det(\beta, g_1, g_2) = \lambda(\beta(g_1, g_2))^2 \). Therefore, image of \( \beta \subset \text{GSO}(4) \).

Since GSO(4, k) is generated by SO(4, k) and \( \mathbb{Z}(4, k) \) and SO(4, k) ∩ \( \mathbb{Z}(4, k) = \{ \pm I_4 \} \), it is enough to show that \( \mathbb{Z}(4, k) \) is contained in the image of \( \beta \) and SO(4, k) is contained in the image of \( \beta \). First part is clear in case \( k \) is an algebraically closed field and for the second part, we know

\[
\text{SO}(4, k) = \beta(\text{SL}(2, k) \times \text{SL}(2, k)) \subset \beta(\text{GL}(2, k) \times \text{GL}(2, k)).
\]

So \( \frac{\text{GL}(2, k) \times \text{GL}(2, k)}{\ker \beta} = \frac{\text{GL}(2, k) \times \text{GL}(2, k)}{\{(d_1, t^{-1}I_2)\}} \) is the image of \( \beta \) which is now GSO(4, k).

We can also deduce the following exact sequences when \( k \) is not an algebraically closed field:

\[
1 \to k^* \to \text{GL}(2, k) \times \text{GL}(2, k) \to \text{GSO}(4, k) \to H^1(\text{Gal}(\bar{k}/k), \bar{k}^*) = 1.
\]

Therefore, GSO(4, k) = \( \frac{\text{GL}(2, k) \times \text{GL}(2, k)}{\{(cI_2, c^{-1}I_2)\}} \). \( \square \)

\section{Langlands Parameter of GSp(4)}

Let \( k \) be a number field.

We have a commutative diagram:

\[
\begin{array}{cccccc}
0 & \to & k^* & \to & \text{GL}(2, k) \times \text{GL}(2, k) & \overset{\beta}{\to} & \text{GSO}(4, k) & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & k^* & \to & k^* \times k^* & \overset{\alpha}{\to} & k^* & \to & 0 \\
\end{array}
\]

where \( \beta \) is defined in the previous chapter and \( \alpha = \beta_{k^* \times k^*} \).

\textbf{Lemma 2} \( \square \) There is a bijection between cuspidal automorphic representations \( \pi \) of GSO(4, \( \mathbb{A} \)) and pairs \( (\pi, \tilde{\chi}) \) of a cuspidal automorphic representation \( \pi \) of GL(2, \( \mathbb{A} \)) × GL(2, \( \mathbb{A} \)) and a grössecharacter \( \tilde{\chi} : k^* \backslash \mathbb{A}^* \to \mathbb{C}^* \) such that \( \tilde{\chi} \circ \alpha \) is the central character of \( \pi \).

\textbf{Proof.} Since the bijection sends \( \pi \) to \( \{(f \circ \beta | f \in \pi), \chi_{\pi} \} \), where \( \chi_{\pi} \) denote the central character of \( \pi \) and \( \beta \) is the natural map from GL(2) × GL(2) to GSO(4) as above. In the other direction it sends the pair \( (\pi, \tilde{\chi}) \) to the set of functions from GSO(4, k) \( \backslash \) GSO(4, \( \mathbb{A} \)) to \( \mathbb{C} \) such that \( f \circ \beta \in \pi \) and \( f(zg) = \tilde{\chi}(z)f(g) \) for all \( z \in \mathbb{A}^* \) and all \( g \in \text{GSO}(4, \mathbb{A}). \square \)

Note that the second set in the lemma maps 2-1 to the set of cuspidal automorphic representations of GL(2, \( \mathbb{A} \)) × GL(2, \( \mathbb{A} \)) whose central character factors through the map \( \alpha \). Moreover, we can apply the same considerations for the local case. We consider the non-archimedean place \( v \). There is a bijection between irreducible admissible representations \( \pi_v \) of GSO(4, \( k_v \)) and pairs \( (\pi_v, \chi_{\pi_v}) \) of an irreducible representation \( \pi_v \) of GL(2, \( k_v \)) × GL(2, \( k_v \)) and a character \( \chi_{\pi_v} : k_v^* \to \mathbb{C}^* \) such that \( \chi_{\pi_v} \circ \alpha \) is the central character of \( \pi_v \).

Let \( G \) be GL(2) × GL(2). For the rest of this section induction will mean unitary induction. Let \( B_G \) denote the Borel subgroup of upper triangular matrices in \( G \). Four characters \( \chi_{11}, \chi_{21}, \chi_{12}, \chi_{22} \) of \( k_v^* \) give rise to a character \( (\chi_{11}, \chi_{21}, \chi_{12}, \chi_{22}) \) of \( B_G(k_v) \) by:

\[
(\chi_{11}, \chi_{21}, \chi_{12}, \chi_{22}) \begin{pmatrix}
d_1 & * & & \\
0 & d_2 & & \\
d_3 & * & & \\
0 & 0 & & d_4
\end{pmatrix} = \chi_{11}(d_1)\chi_{21}(d_2)\chi_{12}(d_3)\chi_{22}(d_4).
\]

We let \( T_G \) denote the torus of diagonal matrices.
Let $B_{GO(4)}$ denote the Borel subgroup of $GO(4)$

$$B_{GO(4)} = \left\{ \begin{pmatrix} a & * \\ b & c \\ c & * \\ d \end{pmatrix} \in GO(4) \right\}.$$ 

Let $T_{GO(4)}$ denote the Levi component

$$T_{GO(4)} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GO(4) \right\}.$$

Let $(\pi, \chi)$ be a pair as Lemma 2 corresponding to $\tilde{\pi}$. Suppose that $\pi_{c}$ is the principal series corresponding to a character $(\chi_{11}, \chi_{21}, \chi_{12}, \chi_{22})$ of $B_{G}(k_{v})$. Then $\chi_{11}\chi_{21} = \chi_{12}\chi_{22} = \chi_{v}$, by page 384[4].

Let $\mu$ and $\nu$ denote the multiplier characters of $GSp(4)$ and $GO(4)$ and let $Sp(4)$ and $O(4)$ (resp.) denote their kernels. Let $R = \{(g, h) \in GSp(4) \times GO(4) : \mu(g)\nu(h) = 1\}$.

The group $Q$ is a minimal parabolic subgroup of $GO(4)$. Let $R_{Q} = R \cup (GSp(4) \times Q)$.

The group

$$P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \in GSp(4) \right\}.$$ 

is a minimal parabolic subgroup of $GSp(4)$. Let $R_{P,Q} = R \cap (P \times Q)$.

From a standard calculation, we can get following result on the Langlands parameters.

**Lemma 3** The $L-$group of $GSp(4)$ is $GSp(4, \mathbb{C})$. If $\Pi$ is the unramified sub-quotient of the representation of $GSp(4, k_{v})$ unitarily induced from the character of $P(k_{v})$ which is trivial on the unipotent radical and sends:

$$\text{diag}(a, b, \mu a^{-1}, \mu b^{-1}) \rightarrow \chi_{1}(a)\chi_{2}(b)\chi_{3}(\mu),$$

then $\Pi$ has Langlands parameter $(\chi_{1}(v), \chi_{3}\chi_{1}(v), \chi_{2}\chi_{1}(v), \chi_{3}\chi_{2}(v)) \in GSp(4, \mathbb{C})$.

The following is from Rodier’s classification which we need for the proof of proposition 2.

**Lemma 4** (Rodier’s classification[8]) Suppose $\Pi$ is an irreducible pre-unitary representation of $GSp(4, k_{v})$ which is a subquotient of an unramified principal series representation with Langlands parameter $\text{diag}(\alpha, \beta, \gamma, \delta) \in GSp(4, \mathbb{C})$, then either $\Pi$ is the full induced representation or absolute value of $\alpha$, $\beta$, $\gamma$, $\delta$ are, up to the action of the Weyl group, $v$ to the power $(-\frac{1}{2}, -r, \frac{1}{2})$ with $0 \leq r \leq \frac{1}{4}$, or $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, or $(-\frac{3}{2}, -\frac{1}{2}, \frac{3}{2})$.

The main proposition to get the Langlands parameter for $GSp(4)$ when the representation is associated to the representation of $GL(2) \times GL(2)$ is the following.

**Proposition 2** [4] Suppose that $\pi = \pi_{1} \otimes \pi_{2}$ is an unramified irreducible pre-unitary principal series representation of $GL(2) \times GL(2)$ with Langlands parameters $\text{diag}(\alpha_{1}, \beta_{1})$ and $\text{diag}(\alpha_{2}, \beta_{2})$. Suppose that $\Pi$ is a pre-unitary irreducible admissible representation of $GSp(4)$ which is associated to the representation $(\pi, \tilde{\chi})$ obtained by theta lifting. Then $\Pi$ is an unramified irreducible principal series representation of $GSp(4)$ with Langlands parameter $\text{diag}(\alpha_{1}, \beta_{1}, \beta_{2}) \in GSp(4, \mathbb{C})$.
Proof. The representation π is induced from two pairs of characters (χ₁₁, χ₂₁) and (χ₁₂, χ₂₂) with \( \bar{x} = x₁₁x₂₁ = x₁₂x₂₂ \). Here, characters of the torus of GSO(4) are defined as

\[
χ₁(t₁, t₂, t₃) = \begin{pmatrix}
t₁ & t₂ \\
t₃t₁⁻¹ & t₃t₂⁻¹
\end{pmatrix}
\mapsto \left(\frac{X₁₁}{X₁₂}\right)(t₁)\left(\frac{X₁₂}{X₂₁}\right)(t₂)\left|\frac{t₃}{t₃}\right|^\frac{1}{2}χ₂₁(t₃)
\]

or one of its conjugates under the group \( W \) of order 8 which is generated by \( σ₁ \), which switches \( χ₁₁ \) and \( χ₂₁ \), and \( τ \) which switches \( χ₁j \) and \( χ₂j \) for \( j = 1, 2 \). Because \( π \) is unitary and irreducible principle series, \( χ₁j \neq χ₂j \). Let \( R = \ker μv \), where \( μ \) is the similitude character of GSp(4) and \( v \) is the similitude character of GO(4) and \( P \) is the minimal parabolic subgroup of GSp(4), \( Q \) is the minimal parabolic subgroup of GO(4).

Therefore, for one of the characters \( χ₁ \) above, \( Π \otimes χ₁ \) must be a quotient of the un-normalized induction from \( R \cap (P \times Q) \) to \( R \cap (GSp(4) \times Q) \) of the character which is trivial on the unipotent radical and sends

\[
\left(\begin{array}{c}
a \\
b \\
μa⁻¹ \\
μb⁻¹
\end{array}\right) \mapsto |μ|⁻²|ab|^2χ₁(a⁻¹t₁μ, b⁻¹t₂μ, 1)
\]

Therefore \( Π \otimes χ₁(1, 1, μ⁻¹) \) must be a quotient of the un-normalized induction from \( P \) to GSp(4) of a character which is trivial on unipotents and sends:

\[
\left(\begin{array}{c}
a \\
b \\
μa⁻¹ \\
μb⁻¹
\end{array}\right) \mapsto |μ|⁻²|b|^2χ₁(a⁻¹μ, b⁻¹μ, 1)
\]

for one of the characters \( χ₁ \).

Since

\[
χ₁(1, 1, μ⁻¹) = \left(\frac{X₁₁}{X₁₂}\right)(1)||\frac{X₁₂}{X₂₁}(1)|μ⁻¹|^\frac{1}{2}χ₂₁(μ⁻¹) = |μ|\frac{1}{2}χ₂₁(μ⁻¹),
\]

\[
|μ|\frac{1}{2}|b|^2χ₁(a⁻¹μ, b⁻¹μ, 1) \cdot |μ|\frac{1}{2}χ₂₁(μ)
\]

\[
= |μ|\frac{1}{2}|b|^2\left(\frac{X₁₁}{X₁₂}(a⁻¹μ)b⁻¹μ\left(\frac{X₁₁}{X₁₂}(b⁻¹μ)\right)^{-1}\right)^{-\frac{1}{2}}χ₂₁(1) \cdot |μ|\frac{1}{2}χ₂₁(μ)
\]

\[
= \left(\frac{X₁₂}{X₁₁}(a)\right)\left(\frac{X₂₁}{X₁₂}(b)\right)χ₁₁(μ).
\]

Therefore, \( Π \) is a quotient of the un-normalized induction from \( P \) of the character which sends

\[
\left(\begin{array}{c}
a \\
b \\
μa⁻¹ \\
μb⁻¹
\end{array}\right) \mapsto \left(\frac{X₁₂}{X₁₁}(a)\right)\left(\frac{X₂₁}{X₁₂}(b)\right)χ₁₁(μ)
\]

or one of its conjugates by \( W \).

The un-normalized induction of all these characters have unramified subquotients with Langlands parameters

\[
\begin{pmatrix}
χ₁₁(v) \\
χ₁₂(v) \\
χ₂₁(v) \\
χ₂₂(v)
\end{pmatrix} \in GSp(4)
\]
If Π is not the full induced representation, by Rodier’s classification, \(|\chi_{\Pi}(\nu)| = |\nu|^{\nu_0}\) with \((a_{11}, a_{12}, a_{21}, a_{22}) = (-\frac{1}{2}, -r, \frac{1}{2})\) with \(0 \leq r \leq \frac{1}{4}\); or \((-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\), or \((-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2})\). But since \(|\chi_{\Pi}(\nu)| < |\nu|^{\frac{1}{2}}\), Π is full induced representation. Therefore, the result follows. □

5 Transfer from GSp(4) to GL(4)

This section is from [1].

Let \(\mathbb{A} = \mathbb{A}_k\) denote the ring of adeles of a number field \(k\). Let \(\pi\) be a unitary cuspidal representation of GSp(4, \(\mathbb{A}_k\)), which we assume to be globally generic. Then \(\pi\) has a unique transfer to an automorphic representation \(\Pi\) of GL(4, \(\mathbb{A}_k\)). The transfer is generic (globally and locally) and satisfies \(\omega_{\Pi} = \omega_\pi^2\) and \(\Pi = \Pi \otimes \omega_\nu\). Here, \(\omega_\nu\) and \(\omega_\Pi\) denote the central characters of \(\pi\) and \(\Pi\), respectively. Moreover [1] gives a cuspidality criterion for \(\Pi\) and proves, when \(\Pi\) is not cuspidal, it is an isobaric sum of two unitary cuspidal representations of GL(2, \(\mathbb{A}_k\)). We define the similitude symplectic group of degree four via

\[
\text{GSp}(4) = \{g \in \text{GL}(4) : {}^t\!gJg = \mu(g)J\},
\]

where

\[
J = \begin{pmatrix}
1 & -1 \\
-1 & 1
\end{pmatrix}
\]

and \(\mu\) is the similitude character. We fix the following parametrization of the elements of the maximal torus \(T\) in GSp(4):

\[
T = \\{ t = (a_0, a_1, a_2) = \begin{pmatrix}
a_0a_1a_2 & a_0a_1 \\
0 & a_0
\end{pmatrix} \}.
\]

Let \(\pi = \otimes_v \pi_v\) be a globally \(\psi\)-generic unitary cuspidal automorphic representation of GSp(4, \(\mathbb{A}_k\)). Here, \(\psi = \otimes_v \psi_v\) is a non-trivial additive character of \(k \setminus \mathbb{A}\) defining a character of the unipotent radical of the standard upper-triangular Borel subgroup in the usual way. We fix \(\psi\) throughout this paper. Let \(S\) be any non-empty finite set of non-archimedean places \(v\), which includes those \(v\) with \(\pi_v\) or \(\psi_v\), ramified. Asgari and Shahidi prove that there exists an automorphic representation \(\Pi = \otimes_v \Pi_v\) of GL(4, \(\mathbb{A}_k\)) such that \(\Pi_v\) is a local transfer of \(\pi_v\) for outside of \(S\).

To be more explicit, assume that \(v \notin S\). If \(v\) is archimedean, then \(\pi_v\) is given by a parameter \(\phi_v : W_v \rightarrow \text{GSp}(4, \mathbb{C})\), where \(W_v\) is the Weil group of \(k_v\). Let \(\Phi_v : W_v \rightarrow \text{GL}(4, \mathbb{C})\) be given by \(\Phi_v = \iota \circ \phi_v\), where \(\iota : \text{GSp}(4, \mathbb{C}) \rightarrow \text{GL}(4, \mathbb{C})\) is the natural embedding. Then \(\Phi_v\) is the parameter of \(\Pi_v\).

If \(v \notin S\) is non-archimedean, then \(\pi_v\) is the unique unramified quotient of the representation induced from an unramified character \(\chi\) of \(T(k_v)\) to GSp(4, \(k_v\)). Writing \(\chi((a_0, a_1, a_2)) = \chi_0(a_0)\chi_1(a_1)\chi_2(a_2)\), where \(\chi_i\) are unramified characters of \(k_v^\times\) and \(a_i \in k_v^\times\), the representation \(\Pi_v\) is then the unique irreducible unramified subquotient of the representation \(\text{Ind}_{|\mathbb{A}_v|}^{\mathbb{A}_k} \otimes |\det|^{\iota} \sigma_i\) parabolically induced from the character

\[
\chi_1 \otimes \chi_2 \otimes \chi_2^{-1} \otimes \chi_1^{-1} \otimes \chi_0
\]

of \(T(k_v)\).

Moreover, they proved that \(\omega_{\Pi} = \omega_\pi^2\), where \(\omega = \omega_{\pi}\) and \(\omega_{\Pi}\) denote the central characters of \(\pi\) and \(\Pi\), respectively, and for \(v \notin S\) they have \(\Pi_v \sim \Pi_v \otimes \omega_{\Pi_v}\), i.e. \(\Pi\) is nearly equivalent to \(\Pi \otimes \omega\).

The representation \(\Pi\) equivalent to a subquotient of some representation

\[
\text{Ind}_{|\mathbb{A}_v|}^{\mathbb{A}_k} \otimes \cdots \otimes |\det|^{\iota} \sigma_i,
\]

where induction is from \(\text{GL}(n_1) \times \cdots \times \text{GL}(n_i)\) with \(n_1 + \cdots + n_i = 4\) to \(\text{GL}(4)\) and \(\sigma_i\) are the unitary cuspidal automorphic representation of \(\text{GL}(n_i, \mathbb{A}_k)\) and \(n_i \in \mathbb{R}\).
6 L-functions

Without loss of generality we may assume that \( r_1 \geq r_2 \geq \cdots \geq r_i \). Moreover, as \( \Pi \) is unitary we have \( n_1 r_1 + \cdots + n_i r_i = 0 \), which implies that \( r_i \leq 0 \). Let \( T = S \cup \{ v : v \mid \infty \} \) and consider

\[
L^T(s, \pi \times \sigma_i) = L^T(s, \Pi \times \sigma_i) = \prod_{i=1}^{r} L^T(s + r_i, \sigma_i \times \sigma_i).
\]

Here, \( L^T \) denotes the product over \( v \notin T \) of the local L-functions.

If \( n_i = 1 \), then the left-hand side is entire by a result of Piatetski-Shapiro [7]. Now consider the right-hand side at \( s_0 = 1 - r_i \geq 1 \). The last term in the product has a pole at \( s_0 \), whereas all of the others are non-zero there as \( R(s_0 + r_i) = 1 + r_i - r_i \geq 1 \). This is a contradiction.

Now assume that \( n_i = 2 \), i.e. \( t = 2 \) with \( n_1 = 1 \) and \( n_2 = 3 \). Replacing \( \pi \) and \( \Pi \) by their contragredients will change \( r_i \) to \(-r_i\) and takes us back to the above situation, which gives a contradiction again.

Therefore, \( n_i = 2 \). In this case, \( L^T(s, \pi \times \sigma_i) \) have a pole at \( s = 1 \) and if so, arguing as above, we conclude that \( r_i = 0 \). This means that we either have \( t = 2 \) with \( n_1 = n_2 = 2 \) or \( t = 3 \) with \( n_1 = n_2 = 1 \) and \( n_3 = 2 \). However, we can rule out the latter as follows.

Assume that \( t = 3 \) with \( n_1 = n_2 = 1 \) and \( n_3 = 2 \). Then, it follows from the fact that \( r_3 = 0 \) and contradictions \( r_1 \geq r_2 \geq r_3 \) and \( r_1 + r_2 + 2r_3 = 0 \) that all of the \( r_i \) would be zero in this case. This implies that if we consider the L-function of \( \pi \) twisted by \( \overline{\sigma}_i \), we have

\[
L^T(s, \pi \times \overline{\sigma}_i) = L^T(s, \sigma_1 \times \overline{\sigma}_1)L^T(s, \sigma_2 \times \overline{\sigma}_1)L^T(s, \sigma_3 \times \overline{\sigma}_1).
\]

Now the left-hand side is again entire by Piatetski-Shapiro’s result [7] mentioned above and the right-hand side has a pole at \( s = 1 \), which is a contradiction.

Therefore, the only possibilities are \( t = 1 \) i.e. \( \Pi \) unitary cuspidal or \( t = 2 \) and \( n_1 = n_2 = 2 \) with \( r_2 = 0 \). In the latter case, we also get \( r_1 = 0 \), as \( r_1 + r_2 = 0 \) by unitarity of the central character. Moreover, in this case we have \( \sigma_1 \neq \sigma_2 \) as, otherwise,

\[
L^T(s, \pi \times \overline{\sigma}_i) = L^T(s, \sigma_1 \times \overline{\sigma}_1)L^T(s, \sigma_2 \times \overline{\sigma}_1)
\]

must have a double pole at \( s = 1 \) while any possible pole of the left-hand side at \( s = 1 \) is simple.

Therefore, we can see the following.

**Proposition 3** [7] Let \( \pi \) be a globally generic unitary cuspidal automorphic representation of \( \text{GSp}(4, \mathbb{A}) \) and let \( \Pi \) be any transfer of \( \pi \) to \( \text{GL}(4, \mathbb{A}) \). Then \( \Pi \) is a subquotient of an automorphic representation as \( \text{Ind} = (|\text{det}|^t \sigma_1 \otimes \cdots \otimes |\text{det}|^t \sigma_i) \) with either \( t = 1 \), \( n_1 = 4 \) and \( r_1 = 0 \) (i.e. \( \Pi \) is unitary cuspidal) or \( t = 2 \), \( n_1 = n_2 = 2 \) and \( r_1 = r_2 = 0 \). In the latter case, we have \( \sigma_1 \neq \sigma_2 \).

In fact, we can get more precise information.

**Proposition 4** [7] Let \( \pi \) be a globally generic unitary cuspidal automorphic representation of \( \text{GSp}(4, \mathbb{A}) \) with \( \omega = \omega_\pi \) its central character and let \( \Pi \) be any transfer as above. Then, \( \Pi \simeq \Pi \simeq \omega \) (not just nearly equivalent). Moreover:

(a) the representation \( \Pi \) is cuspidal if and only if \( \pi \) is not obtained as a Weil lifting from \( \text{GSO}(4, \mathbb{A}) \)

(b) if \( \Pi \) is not cuspidal, then it is the isobaric sum of two representations \( \Pi = \Pi_1 \oplus \Pi_2 \), where each \( \Pi_i \) is a unitary cuspidal automorphic representation of \( \text{GL}(2, \mathbb{A}) \) satisfying \( \Pi_i \simeq \Pi_i \simeq \omega \) and \( \Pi_1 \neq \Pi_2 \).

6 L-functions

By the natural embedding from \( \text{GSp}(4, \mathbb{C}) \) to \( \text{GL}(4, \mathbb{C}) \), we can see a representation \( \Pi \) of \( \text{GL}(4) \) which is transferred from \( \text{GSp}(4) \) is not cuspidal when it is obtained as a Weil lifting from \( \text{GSO}(4, \mathbb{A}) \), and in this case it is the isobaric sum of two representations \( \Pi_i \)’s, where each \( \Pi_i \) is a unitary cuspidal automorphic representation of \( \text{GL}(2, \mathbb{A}) \) satisfying \( \Pi_i \simeq \Pi_i \simeq \omega \) and \( \Pi_1 \neq \Pi_2 \).
Theorem 5 Let $\pi_i, \ i = 1, 2,$ be cuspidal generic representations of $GSp(4, A)$ and $\Pi_i, \ i = 1, 2,$ be their transfers.

1. If neither of $\pi_i, \ i = 1, 2$ come from $GSO(4, A)$, then $L^S(s, \pi_1 \times \pi_2)$ has a pole at $s = 1$ if and only if $\pi_2 = \overline{\pi}_1$.

2. If only one of $\pi_i, \ i = 1, 2$ comes from $GSO(4, A)$, then $L^S(s, \pi_1 \times \pi_2)$ has no poles.

3. Suppose the representations $\pi_1, \pi_2$ of $GSp(4, A)$ are obtained as a Weil lifting from $GSO(4, A)$. Then $\Pi_1 = \Pi_1 \boxtimes \Pi_{12}$ and $\Pi_2 = \Pi_{21} \boxtimes \Pi_{22}$, and

$$L^S(\Pi_1 \times \Pi_2) = L^S((\Pi_1 \boxtimes \Pi_{12}) \times (\Pi_{21} \boxtimes \Pi_{22}))$$

$$= L^S(\Pi_1 \times \Pi_{12}) L^S(\Pi_{11} \times \Pi_{22}) L^S(\Pi_{12} \times \Pi_{21}) L^S(\Pi_1 \times \Pi_{22})$$

Consequently,

(a) if $\Pi_{11} \neq \overline{\Pi}_{21}$ and $\Pi_{12} \neq \overline{\Pi}_{22}$, then $L^S(\Pi_1 \times \Pi_2)$ has no poles, since $\Pi_{11} \neq \Pi_{12}, \Pi_{21} \neq \Pi_{22}$.

(b) if $\Pi_{11} \cong \overline{\Pi}_{21}$ and $\Pi_{12} \neq \overline{\Pi}_{22}$, then $L^S(\Pi_1 \times \Pi_2)$ has a simple pole at $s = 1$.

(c) if $\Pi_{11} \cong \overline{\Pi}_{21}$ and $\Pi_{12} \cong \overline{\Pi}_{22}$, then $L^S(\Pi_1 \times \Pi_2)$ has a double pole at $s = 1$.

Proof. We know that $L^S(\Pi \times \overline{\Pi})$ has a simple pole at $s = 1$ when $\Pi$ is the representation of $GL(2, A)$ and from the last section we know that $\Pi_{11} \neq \Pi_{12}, \Pi_{21} \neq \Pi_{22}$.

If the representation $\Pi_1$ is not obtained as a Weil lifting from $GSO(4, A)$ and $\Pi_2$ is obtained as a Weil lifting from $GSO(4, A)$, then $\Pi_2 = \Pi_{21} \boxtimes \Pi_{22}$. And

$$L^S(\Pi_1 \times \Pi_2) = L^S(\Pi_1 \times (\Pi_{21} \boxtimes \Pi_{22})) = L^S(\Pi_1 \times \Pi_{12}) L^S(\Pi_{11} \times \Pi_{22})$$

and we can see this $L$-function has no poles (c.f. [6]).

If representations $\Pi_1, \Pi_2$ of $GL(4, A)$ are obtained as Weil liftings from $GSO(4, A)$, then $\Pi_1 = \Pi_{11} \boxtimes \Pi_{12}$ and $\Pi_2 = \Pi_{21} \boxtimes \Pi_{22}$, and

$$L^S(\Pi_1 \times \Pi_2) = L^S((\Pi_{11} \boxtimes \Pi_{12}) \times (\Pi_{21} \boxtimes \Pi_{22}))$$

$$= L^S(\Pi_{11} \times \Pi_{21}) L^S(\Pi_{11} \times \Pi_{22}) L^S(\Pi_{12} \times \Pi_{21}) L^S(\Pi_{12} \times \Pi_{22})$$

Therefore, if $\Pi_{11} \neq \overline{\Pi}_{21}$ and $\Pi_{12} \neq \overline{\Pi}_{22}$, then $L^S(\Pi_1 \times \Pi_2)$ has no poles, since $\Pi_{11} \neq \Pi_{12}, \Pi_{31} \neq \Pi_{32}$. If $\Pi_{11} \cong \overline{\Pi}_{21}$ and $\Pi_{12} \neq \overline{\Pi}_{22}$, then $L^S(\Pi_1 \times \Pi_2)$ has a simple pole at $s = 1$ because $L^S(\Pi_{11} \times \Pi_{21})$ has a simple pole and $L^S(\Pi_{11} \times \Pi_{22}), L^S(\Pi_{12} \times \Pi_{21})$ and $L^S(\Pi_{12} \times \Pi_{22})$ have no poles.

If $\Pi_{11} \cong \overline{\Pi}_{21}$ and $\Pi_{12} \cong \overline{\Pi}_{22}$, then $L^S(\Pi_1 \times \Pi_2)$ has a double pole at $s = 1$ because $L^S(\Pi_{11} \times \Pi_{21}), L^S(\Pi_{12} \times \Pi_{22})$ each have a simple pole and $L^S(\Pi_{11} \times \Pi_{22}), L^S(\Pi_{12} \times \Pi_{21})$ have no poles and are non-zero at $s = 1$ [9]. Since $\Pi_1 \neq \Pi_2$ by Proposition 4, above cases are all for this theorem.

In fact, $L^S(\Pi \times \overline{\Pi})$ has a double pole at $s=1$ if $\Pi$ is a Weil lifting from $GSO(4)$. □

Thus part (c) shows the existence of a $L$-function of a cuspidal representation of $GSp(4, A) \times GSp(4, A)$ which has a pole of order 2 at $s = 1$, even for globally generic representations.

Theorem 6 If $\pi$ comes from $GSO(4, A)$, then $\pi$ is the Weil transfer of $\Pi_1 \otimes \Pi_2$ realized as a representation of $GSO(4, A)$. This agrees with Langlands Functoriality principle as $GSO(4)$ is an endoscopic group for $GSp(4)$. Moreover, it shows the data $\Pi_1 \otimes \Pi_2$ on $GSp(4)$ transfers to $\Pi_1 \boxtimes \Pi_2$ through the composite of the endoscopic transfer from $GSO(4)$ to $GSp(4)$ and the twisted endoscopic transfer from $GSp(4)$ to $GL(4)$.

Proof. By Lemma 2, we know that there is a bijection between cuspidal automorphic representations $\overline{\Pi}$ of $GSO(4, A)$ and pairs $(\Pi, \tilde{\chi})$ of a cuspidal automorphic representation $\Pi$ of $GL(2, A) \times GL(2, A)$ and a grössencharacter $\tilde{\chi}: k^* \setminus A^* \to \mathbb{C}^*$ such that $\tilde{\chi} \circ \alpha$ is the central character of $\Pi$. By Lemma 1, we see $GSO(4)$ is basically $GL(2) \times GL(2)$. Let $\Pi = \Pi_1 \boxtimes \Pi_2$ be an unramified irreducible pre-unitary principal series representation of $GL(2) \times GL(2)$ with Langlands parameters $\text{diag}(\alpha_1, \beta_1)$ and $\text{diag}(\alpha_2, \beta_2)$. Then by Proposition 2,
if we say $\pi$ is a pre-unitary irreducible admissible representation of GSp(4) which is associated to the representation $(\Pi, \tilde{\chi})$ obtained by the theta lifting, then $\pi$ is an unramified irreducible principal series representation of GSp(4) with Langlands parameter $\text{diag}(\alpha_1, \alpha_2, \beta_1, \beta_2) \in \text{GSp}(4, \mathbb{C})$.

Since $\pi$ is obtained as a Weil lifting from GSO(4, $A$), by Proposition 4, we know any transfer $\Pi'$ from $\pi$ is cuspidal and the isobaric sum of two representations $\Pi' = \Pi'_1 \boxplus \Pi'_2$, where each $\Pi'_i$ is a unitary cuspidal automorphic representation of GL(2, $A$).

From the classification theorem, we can say $\Pi_i = \Pi'_i$ for $i = 1, 2$ after reordering if it is necessary. Therefore $\Pi = \Pi' = \Pi_1 \boxplus \Pi_2$ and the result follows. □

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