The general QCD parametrization and the large $N_c$ description: Some remarks

G. Morpurgo
Università di Genova and Istituto Nazionale di Fisica Nucleare, Genova, Italy.

Abstract. Stimulated by a recent paper of Buchmann and Lebed, a comparison is presented of the two methods mentioned in the title for treating hadron properties in QCD. Doubts arise on the equivalence of the large $N_c$ description to real QCD. (PACS: 12.38.Aw; 11.15.Pg; 13.40.Dk)

1. Introduction.

A recent paper by Buchmann and Lebed (Large $N_c$, Constituents quarks and $N$, $\Delta$ charge radii) \cite{1} compares, in a specific case, the $1/N_c$ description and the general parametrization (GP) method of QCD \cite{2,3,4}. Ref.\cite{1} seems to imply that, for the case at hand, the general parametrization should be looked as a very good (though not fully exact) approximation to the $1/N_c$ method, which is regarded as more fundamental.

I recall that both the $1/N_c$ and the GP methods are parametrizations to describe hadronic properties, but that the GP method, although down to earth, is founded on QCD, while the same is not so clear for $1/N_c$. One reason for my doubts on $1/N_c$ is that expressed concisely in \cite{5}: “The basis for the large $N_c$ approach is the assumption that $N_c = 3$ QCD is similar to QCD in the limit $N_c = \infty$. In particular it is assumed that there are no phase transitions as we go from $N_c = 3$
to $N_c \to \infty$. Currently the status of these assumptions is not clear, because not much is known about QCD($N_c = \infty$)”.

Note, incidentally, that the factor $g^2/3$ producing the hierarchy in the $1/N_c$ method is, approximately, the same factor that empirically emerges in depressing the diagrams with one added gluon in the GP method; so that the two approaches are characterized, in practice, by a similar hierarchy.

I will exemplify the general parametrization in a few cases, to clarify the situation. But, before doing this, I note two points:

1. The GP method is an exact consequence of QCD, based only on few general properties of the QCD Lagrangian. For many physical quantities of the lowest multiplet of hadrons (e.g. masses, magnetic moments, electromagnetic and semileptonic matrix elements, e.m. form factors etc.) it leads to an exact spin-flavor parametrization, independent of the choice of the renormalization point of the quark masses in the QCD Lagrangian. It turns out that, for a given quantity, the number of terms in this exact QCD parametrization is rather small, indeed smaller than one might have anticipated. The GP method -which, even if not covariant, is fully relativistic- was developed originally to explain the unexpected semiquantitative success of the non relativistic quark model (NRQM) [6]; it did this [2a] long before the $1/N_c$ treatment, and much more directly. It emerged that the structure of the terms in the GP is similar to that of the NRQM. Because terms of increasing complexity in the GP have decreasing coefficients, few terms usually suffice to reproduce the data reasonably well, explaining why the NRQM works already in its most naive form.

2. Although $SU_6$ was important in suggesting the NRQM [3], it does not play a role after that. For baryons the essential point in the construction of the NRQM was that the space part of the octet and decuplet wave function has an overall zero orbital angular momentum: $L = 0$. This implies [4] the factorizability of the
baryon (octet or decuplet) NRQM model state as:

\[ \phi_B = X_{L=0}(r_1, r_2, r_3) \cdot W_B(s, f) \]  

(1)

where \( X \) is the space part and \( W_B(s, f) \) is the spin-flavor factor. (Color is understood.) The \( W_B \)'s are symmetric in the three quark variables and have necessarily \( J = 1/2 \) and \( J = 3/2 \) for the octet and decuplet, so that, automatically, the \( W_B \) spin-flavor part of \( \phi_B \) has the form prescribed by \( SU_6 \), without the need of invoking \( SU_6 \) at all. The factorizability of \( \phi_B \) (1) into a space and spin-flavor factor is essential to derive the simple structure of the general parametrization. In the GP there is no need to relate the states to \( SU_6 \) representations as in the \( 1/N_c \) method; nor to rename [1] constituent quarks as “representation quarks”.

Although I will not derive here the GP method -this was done repeatedly [2a,3a,4]- I recall some notation, in order to compare GP and \( 1/N_c \) in a few cases. The symbol \(|\phi_B\rangle\) indicates, in the quark-gluon Fock space, the state corresponding to no gluons and three quarks with wave function \( \phi_B \). The exact eigenstate of the QCD hamiltonian \( H_{QCD} \) for the baryon \( B \) (with mass \( M_B \) at rest is written \(|\psi_B\rangle\). It is \( H_{QCD}|\psi_B\rangle = M_B|\psi_B\rangle \). A unitary transformation \( V \) defined in [2a], acting on the auxiliary state \(|\phi_B\rangle\), transforms it into the exact eigenstate \(|\psi_B\rangle\) of \( H_B \), so that:

\[ |\psi_B\rangle = |qqq\rangle + |qqq\bar{q}\rangle + |qqq, \text{Gluons}\rangle + \cdots \]  

(2)

where the last form of (2) recalls that \( V|\phi_B\rangle\) is a superposition of all possible quark-antiquark-gluon states with the correct quantum numbers. In particular, configuration mixing is automatically included in \( V|\phi_B\rangle \). The mass of a baryon is:

\[ M_B = \langle \psi_B|H_{QCD}|\psi_B\rangle = \langle \phi_B|V^\dagger H_{QCD}V|\phi_B\rangle = \langle W_B|“\text{parametrized mass}”|W_B\rangle \]  

(3)

The last step (eliminating the space variables) is due to the factorizability of \( \phi_B \) (eq.(1)). In the next section I discuss the “parametrized mass” in (3).
2. The parametrization of the baryon masses in the GP method.

The “parametrized mass” in (3) following from the GP method is \([2e,3a]\):

\[
\text{"parametrized mass"} = M_0 + B \sum_i P_s^i + C \sum_{i>k} (\sigma_i \cdot \sigma_k) + \\
D \sum_{i>k} (\sigma_i \cdot \sigma_k) (P_s^i + P_s^k) + E \sum_{i \neq k \neq j, (i > k)} (\sigma_i \cdot \sigma_k) P_s^j + a \sum_{i>k} P_s^i P_s^k + \\
b \sum_{i>k} (\sigma_i \cdot \sigma_k) P_s^i P_s^k + c \sum_{i \neq k \neq j, (i > k)} (\sigma_i \cdot \sigma_k) (P_s^i + P_s^k) P_s^j + d P_1^s P_2^s P_3^s
\]

where the notation is defined in [2e]; \(P_s^i\)s are the projectors on the strange quarks; \(M_0, B, C, \ldots, D\) are parameters. Of the two parameters \(a\) and \(b\) only the combination \((a + b)\) intervenes.

A comment on (4): Because the different masses of the lowest octet and decuplet baryons are 8 (barring e.m. and isospin corrections), Eq.(4), with 8 parameters \((M_0, B, C, D, E, a + b, c, d)\), is certainly true, no matter what is the underlying theory. Yet the general parametrization (4) is not trivial: The values of the above 8 parameters are seen to decrease strongly on moving to terms with increasing number of indices (Eq.(5)). In deriving (4) from QCD, the term \(\Delta m \bar{\psi} P^s \psi\) in the QCD Lagrangian is treated exactly; Eq. (4) is correct to all orders in flavor breaking and the derivation takes into account all possible closed loops. In (4) the parameters (in MeV) are -Ref.[3a]:

\[
M_0 = 1076 \quad , \quad B = 192 \quad , \quad C = 45.6 \quad , \quad D = -13.8 \pm 0.3 \\
(a + b) = -16 \pm 1.4 \quad , \quad E = 5.1 \pm 0.3 \quad , \quad c = -1.1 \pm 0.7 \quad , \quad d = 4 \pm 3
\]

The hierarchy of these numbers is evident and, as shown in [3a], it corresponds to a reduction factor \(\approx 1/3\) for an additional pair of indices and \(\approx 1/3\) for each flavor breaking factor \(P_s^i\). The values (4) decrease strongly with increasing complexity of the accompanying spin-flavor structure. Barring \(c\) and \(d\), the following mass formula results [2e], a generalization of the Gell-Mann Okubo formula that includes
octet and decuplet:

\[ \frac{1}{2}(p + \Xi^0) + T = \frac{1}{4}(3\Lambda + 2\Sigma^+ - \Sigma^0) \]  

(6)

The symbols stay for the masses and \( T \) is the following combination of decuplet masses:

\[ T = \Xi^* - \frac{1}{2}(\Omega + \Sigma^*) \]  

(7)

Because of the level of accuracy reached in comparing Eq. (6) with the data, we wrote (6) so as to be free of electromagnetic effects. (It can be easily checked the combinations in (6) are independent of electromagnetic and isospin effects, to zero order in flavor breaking.) The data satisfy (6) as follows:

\[ \text{l.h.s.} = 1133.1 \pm 1.0 \quad \text{r.h.s.} = 1133.3 \pm 0.04 \]  

(8)

an impressive agreement confirming the smallness of the terms neglected in (4).

One more remark [3a]: A QCD calculation, if feasible, would express each \((M_0, B, C, D, E, a, b, c, d)\) in (4) in terms of the quantities in the QCD Lagrangian, the running quark masses -normalized at any \( q \) that we like to select- and the dimensional (mass) parameter \( \Lambda \equiv \Lambda_{QCD} \); for instance, setting for simplicity \( m_u = m_d = m \):

\[ M_0 \equiv \Lambda \hat{M}_0 (m(q)/\Lambda, m_s(q)/\Lambda) \]  

(9)

where \( \hat{M}_0 \) is some function. Similarly for \( B, C, D, E, a, b, c, d \). The numerical value of the coefficients should be seen as the result of a QCD exact calculation performed with an arbitrary choice of the renormalization point of the running quark masses.

3. A comparison with the large \( N_c \) method.

We now compare the parametrized baryon mass (4), with that obtained in the \( 1/N_c \) method. There (compare ref. [8], Eq.3.4) the parametrization of the baryon
masses is also expressed in terms of 8 parameters (from \( c_{10} \) to \( c_{64,0}^{(3)} \)), but, note, the quantities they multiply are collective rather than individual quark variables. It is again true that, setting to zero the smaller coefficients, one finds a relation (Eq.(4.6) in [8]) between octet and decuplet baryon masses, which is equivalent to Eq. (6).

Neither in Ref.[8] nor in other papers [9] it was stated [10] that this relation coincides -except for the notation- with (6), published long before. I note only, here, the following: The general QCD parametrization can reproduce the good results of \( 1/N_c \) simply using, as we did, the conjecture that the empirical hierarchy apparent in Eqs. (4,5) for the baryon masses, applies to many or all properties, at least for the lowest baryons with a factorizable \( \phi_B \). It is, I repeat, what we always did in [2, 3] (see [2g], fig.1). We explained [2, 3] in this way a variety of facts about the magnetic moments, \( \Delta^+ \rightarrow p\gamma \), semileptonic decays and many other quantities. Note that the GP method in principle includes all diagrams, not only the planar ones; the closed loops, related in the GP to the Trace terms (see in [3a], the ref.14) are also taken into account; their contribution is depressed or not depending on the number of additional gluons that are necessary, due to the Furry theorem (see [3c], in particular fig.1). For instance the Trace terms that were written in Ref.[3f] are depressed by the Furry theorem and can be neglected as we did. The phrasing of Buchmann and Lebed [1] did not clearly express this.

By the way, on reading Ref.[1] it looks as if by neglecting, as we did, from the start, terms proportional to \( m_u - m_d \) (which are of the order \( |m_u - m_d|/(3\Lambda_{QCD}) \approx 5 \times 10^{-3} \)) we had imposed a “mild physical constraint”. This is not so. But, except for these points of language, Buchmann and Lebed seem to state that the relationship between the radii of \( N \) and \( \Delta \) implied by the GP and \( 1/N_c \) methods is the same. One may ask: Is it then really necessary to start from \( N_c = \infty \)?

Finally I comment on the Coleman-Glashow (CG) relation considered in a
ref.[3h](see also [11]). In ref.[3h], we recalled the GP result of ref.[2f] (only three index flavor breaking terms violate the CG relation) and showed that neither the $u-d$ mass difference, nor the Trace terms modify this conclusion. This explains the “miraculous” precision of the CG relation, which neglects entirely flavor breaking in its original derivation; such a precision is much better tested after a recent measurement of the $\Xi^0$ mass [12].

After the appearance of [3h](as hep-ph/004198,20 apr 2000) a preprint by Jenkins and Lebed [13] implied by its title that in the large $N_c$ description it is quite natural (not ”miraculous”) that the CG relation is so beautifully verified. It is asserted in [13] that the neglected terms are naturally expected to be of an order in $1/N_c$ sufficiently high to guarantee their smallness.

This confidence, however, is not supported by their theory. E.g. the Trace terms present in the general parametrization (corresponding to closed loops) are many [3h]. Their negligible or vanishing global contribution cannot be established using only the order in $1/N_c$ of a typical term. This is another reason, in addition to the doubts raised in [3], confirming that it is not established that the $1/N_c$ expansion can make predictions having a real QCD foundation.

Acknowledgement. I am very indebted to G.Dillon for frequent discussions.
References

[1] Buchmann A. and Lebed R.F., arXiv:hep-ph/0003167, 16 march 2000 (LANL preprint): Large $N_c$, Constituents quarks and $N, \Delta$ charge radii.

[2] Morpurgo G., a) Phys. Rev. D 40 (1989) 2997 ; b) Phys. Rev. D 40 (1989) 3111 ; c) Phys. Rev. D 41 (1990) 2865 ; d) Phys. Rev. D 42 (1990) 1497 ; e) Phys. Rev. Lett. 68 (1992) 139 ; f) Phys. Rev. D 45 (1992) 1686 ; g) Phys. Rev. D 46 (1992) 4068 ; h) Morpurgo G., in “The rise of the standard model; particle physics in the 1960 and 1970” - ed. by L.Hoddeson et al., Cambridge U.P., Cambridge U.K. 1997, Chapt.31, p.561.

[3] Dillon G. and Morpurgo G., a) Phys. Rev. D 53 (1996) 3754 ; b) Z. Phys. C 62 (1994) 31 ; c) Z. Phys. C 64 (1994) 467 ; d) Z. Phys. C 7 (1997) 547 ; e) Proc. of the “Conf. on perspectives in hadronic physics”, Trieste 1997 - ed. by S.Boffi, C.Ciofi degli Atti and M.M.Giannini (World Sci., Singapore 1997) p.516; f) Phys. Lett. B 448 (1999) 107; g) Phys. Lett. B 459 (1999) 321; h) Phys. Lett. B 481 (2000) 239 and Erratum (to appear).

[4] Morpurgo G., La Rivista del Nuovo Cimento. vol.22 (1999) nr.2, p.1.

[5] Schäfer T. and Shuryak E., arXiv:hep-lat/0005025, 30 may 2000 (LANL preprint).

[6] Morpurgo G., Physics 2 (1965) 95 [also reprinted in Kokkedee J.J.J., The quark model, Benjamin, New York, 1969-p.132].

[7] An additional pair of indices in a term of the GP implies the exchange of an additional gluon; this carries a reduction factor $\approx 0.30$. This number is estimated using Eq.(4) and taking the average between the values for the standard and pole masses (compare refs.[2e,3a]).

[8] Jenkins E. and Lebed R.F., Phys. Rev. D 52 (1995) 282.

[9] see the references in [3] and Jenkins E., Ann.Rev. Nucl.Part.Science 48 (1998) 81.

[10] R.F.Lebed cited (Nucl. Phys. B430 (1994) 295) the GP method [2a,e,f] as “a very general quark model”. Because I noted this misinterpretation, Lebed replied (18/8/1995) that “the above qualification was unfortunate and misleading” and...
that “future reference to your work will state that the results follow from QCD in an interacting quark picture”.

[11] The title of [3h] is: “On the miracle of the Coleman-Glashow and other baryon mass formulas”. The Erratum (to appear) corrects the following misprints: a) At the very end of Sect.3, $\approx 0.02$ MeV should be read $0.02 - 0.1$ MeV; b) Eq. (13) should be: 
\[
\frac{1}{2}(n + \Xi^0) + T = \frac{1}{4}(3\Lambda + \hat{\Sigma}^+) \quad \text{where} \quad \hat{\Sigma}^+ = 2\Sigma^+ - \Sigma^0 + 2(n - p).
\]

[12] NA 48 Collab., Fanti V. et al., Eur. Phys. J. C12 (2000) 69.

[13] Jenkins E. and Lebed R.F., “Naturalness of the Coleman-Glashow Mass Relation in the $1/N_c$ Expansion: an Update” (preprint, arXiv:hep-ph/0005038, 4 May 2000).