SCHWARZIAN DERIVATIVE FOR CONVEX MAPPINGS OF ORDER ALPHA

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Abstract. The main purpose of this paper is to obtain sharp bounds of the norm of Schwarzian derivative for convex mappings of order \( \alpha \) in terms of the value of \( f''(0) \), in particular, when this quantity is equal to zero. In addition, we obtain sharp bounds for distortion and growth for this mappings and we generalized the results obtained by Suita \([9]\) and Yamashita \([11]\) for this particular case.

1. Introduction

Let \( \mathcal{C} \) be the class of conformal functions such that \( f(\mathbb{D}) \) is a convex region in the complex plane, normalized by the conditions \( f(0) = 0 \) and \( f'(0) = 1 \), where \( \mathbb{D} \) is the unit disk in \( \mathbb{C} \). A very important characterization of these functions is that \( f \in \mathcal{C} \) if and only if

\[
\text{Re}\left\{ 1 + z \frac{f''}{f'}(z) \right\} > 0, \quad z \in \mathbb{D}.
\]

Nehari in \([6]\) shows that if \( f \in \mathcal{C} \) then

\[
(1 - |z|^2)^2 |Sf(z)| \leq 2, \quad z \in \mathbb{D},
\]

where the differential operator \( Sf \) is known as the Schwarzian derivative of \( f \) and it is defined as:

\[
Sf(z) = \left( \frac{f''}{f'}(z) \right)' - \frac{1}{2} \left( \frac{f''}{f'}(z) \right)^2,
\]

considering that \( f \) is locally univalent in \( \mathbb{D} \). Inequality \([10]\) was obtained by Nehari applying geometric arguments via the Schwarz-Christoffel formula for convex polygons. In the same time, Trimble in \([10]\) showed that if \( f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \) maps \( \mathbb{D} \) into a convex domain, then \( |a_3 - a_2^2| \leq (1 - |a_2|^2)/3 \). Since \( \mathcal{C} \) is a linear invariant family (see Remark \([4]\) this inequality implies also \([11]\). Later, Chuaqui et al. in \([1]\) proved the same result by applying the Schwarz-Pick lemma and the the fact that the expression \( 1 + z(f''/f')(z) \) is subordinate to the half plane mapping \( \ell(z) = (1 + z)/(1 - z) \), which is

\[
1 + z \frac{f''}{f'}(z) = \ell(w(z)) = \frac{1 + w(z)}{1 - w(z)},
\]

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for some function \( w : \mathbb{D} \to \mathbb{D} \) holomorphic and such that \( w(0) = 0 \). The expression defined in (2) allowed the authors to obtain other characterizations for convex functions:

\[
\text{(3)} \quad f \in \mathcal{C} \text{ if and only if, } \Re \left\{ 1 + z \frac{f''(z)}{f'(z)} \right\} \geq \frac{1}{4} (1 - |z|^2) \left| \frac{f''(z)}{f'(z)} \right|^2 ,
\]

and

\[
\text{(4)} \quad f \in \mathcal{C} \text{ if and only if, } \left| (1 - |z|^2) \frac{f''(z)}{f'(z)} - 2z \right| \leq 2 ,
\]

for all \( z \in \mathbb{D} \).

One of the many extensions of the class of convex functions was given by Robertson in [8]: \( f \) is a convex function of order \( \alpha \), with \( 0 \leq \alpha < 1 \), if

\[
\Re \left\{ 1 + z \frac{f''(z)}{f'(z)} \right\} > \alpha .
\]

We denote the class of convex function of order \( \alpha \) as \( \mathcal{C}_\alpha \). Observe that \( \mathcal{C}_0 = \mathcal{C} \) and \( \mathcal{C}_\alpha \subset \mathcal{C} \) for all \( 0 \leq \alpha < 1 \). Robertson [8] observed that functions \( f \) in \( \mathcal{C}_\alpha \), have the following geometric property: the ratio of the angle between adjacent tangents of the unit circle over the angle between the corresponding tangents in the image of \( f \) is less than \( 1/\alpha \). Hence the closer \( \alpha \) is to 1 the “rounder” is the image. For instance, segments in the boundary of \( f(\mathbb{D}) \) are prohibited as soon as \( \alpha > 0 \). An analogous bound associated to (1) was obtained by Suita in [9] for this class. We present this result as Theorem A in section 3. The author obtains his result characterizing the expression \( 1 + z(f''/f')(z) \) in terms of their Herglotz representation associated to the Poisson Kernel (see [3, p. 15]). In [4, p. 54] there is a review of the most important properties of the class \( \mathcal{C}_\alpha \).

In this work, we apply some of the ideas presented in [1], in particular the Schwarz Lemma and its generalizations, to obtain analogous inequalities associated to (3) and (4) for the convex of order \( \alpha \) case, and a sharp bound for the norm of the Schwarzian derivative when \( f \in \mathcal{C}_\alpha \) with the additional condition that \( f''(0) = 0 \). In the same path, Kanas and Sugawa in [5] proved similar results, but for the case of strongly convex mappings, which can not be improved with our additional assumption since the extremal mapping satisfies that \( f''(0) = 0 \). Yamashita in [11] showed that the norm of the pre-Schwarzian derivative for convex mappings with order alpha is less than or to \( 4(1 - \alpha) \), which is improved in the Proposition 3 assuming that \( f''(0) = 0 \). In fact, we shall show that \( (1 - |z|^2) |f''(z)/f'(z)| \leq 2(1 - \alpha) \). Also, for restricted values of \( |f''(0)| \neq 0 \), we obtain a better bound than the one presented by Suita for \( \|Sf\| \) in the general case.

2. Bounds on convex mappings with order \( \alpha \).

The Möbius transformation:

\[
\ell_\alpha(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} ,
\]
sends $\mathbb{D}$ to the half plane $H_\alpha = \{ z \in \mathbb{C} : \text{Re}(z) > \alpha \}$. If $f \in C_\alpha$ then there exists $w : \mathbb{D} \to \mathbb{D}$ such that:

$$1 + z \frac{f''}{f'}(z) = \ell_\alpha(w(z)) = \frac{1 + (1 - 2\alpha)w(z)}{1 - w(z)},$$

with $w(0) = 0$, analogously as in [2]. We can assume that $w(z) = z\varphi(z)$ for some holomorphic mapping $\varphi$ that satisfies $\varphi(\mathbb{D}) \subseteq \mathbb{D}$, from where:

$$\frac{f''}{f'}(z) = 2(1 - \alpha) \frac{\varphi(z)}{1 - z\varphi(z)},$$

The next theorem generalizes the characterizations (3) and (4) to convex functions of order $\alpha$.

**Theorem 1.** If $0 \leq \alpha < 1$, then the following are equivalent:

i. $f \in C_\alpha$.

ii. $Re \left\{ 1 + z \frac{f''}{f'}(z) \right\} \geq \alpha + \frac{1}{4(1 - \alpha)}(1 - |z|^2) \left| \frac{f''}{f'}(z) \right|^2$.

iii. $|z| \left| \frac{f''}{f'}(z) - 2(1 - \alpha)\overline{z} \right| \leq 2(1 - \alpha)$.

**Proof.** We prove that i. is equivalent to ii. and ii. is equivalent to iii.. From (5) we obtain that:

$$\varphi(z) = \frac{(f''/f')(z)}{2(1 - \alpha) + z(f''/f')(z)}.$$

From the fact that $|\varphi(z)|^2 \leq 1$ we have:

$$4(1 - \alpha)^2 + 2 \text{Re} \left\{ 2(1 - \alpha)z \frac{f''}{f'}(z) \right\} + |z|^2 \left| \frac{f''}{f'}(z) \right|^2 \geq \left| \frac{f''}{f'}(z) \right|^2.$$

Factorizing we obtain,

$$4(1 - \alpha) \left[ 1 - \alpha + \text{Re} \left\{ z \frac{f''}{f'}(z) \right\} \right] \geq (1 - |z|^2) \left| \frac{f''}{f'}(z) \right|^2.$$

Since $\alpha \neq 1$ we conclude that

$$\text{Re} \left\{ 1 + z \frac{f''}{f'}(z) \right\} \geq \alpha + \frac{1}{4(1 - \alpha)}(1 - |z|^2) \left| \frac{f''}{f'}(z) \right|^2.$$

If we multiply both sides of inequality (7) by $(1 - |z|^2)$, we obtain:

$$(1 - |z|^2) \left| \frac{f''}{f'}(z) \right|^2 \leq 4(1 - \alpha)^2(1 - |z|^2) + 4(1 - \alpha)(1 - |z|^2)\text{Re} \left\{ \frac{f''}{f'}(z) \right\}.$$

This is equivalent to:

$$\left( (1 - |z|^2) \left| \frac{f''}{f'}(z) \right| \right)^2 - 2 \text{Re} \left\{ 2(1 - \alpha)z(1 - |z|^2) \frac{f''}{f'}(z) \right\} + (2(1 - \alpha)|z|)^2 \leq 4(1 - \alpha)^2,$$
from which we have that:
\[
\left| (1 - |z|^2) \frac{f''}{f'}(z) - 2(1 - \alpha) \bar{z} \right| \leq 2(1 - \alpha).
\]

It is a well-known fact that if \( f \in \mathcal{C} \), then for any \( a \in \mathbb{D} \) the corresponding *Koebe Transform* \( f_a \) given by
\[
(8) 
\]
\[
f_a(z) = \frac{f \left( \frac{z + a}{1 + \bar{a}z} \right) - f(a)}{(1 - |a|^2)f'(a)},
\]
belongs to \( \mathcal{C} \). However, if \( f \in \mathcal{C}_\alpha \) then it is not necessarily that \( f_a \in \mathcal{C}_\alpha \). For this case we have the following proposition:

**Proposition 2.** If \( f \in \mathcal{C}_\alpha \) then for any \( a \in \mathbb{D} \), \( f_a \in \mathcal{C}_\beta \) with \( \beta = \alpha \left( \frac{1 - |a|}{1 + |a|} \right) \).

**Proof.** By equation (8) we have
\[
\frac{f''}{f'}(z) = \frac{f''}{f'} \left( \frac{z + a}{1 + \bar{a}z} \right) \frac{1 - |a|^2}{(1 + \bar{a}z)^2} - \frac{2\bar{a}}{1 + \bar{a}z}.
\]
If we assume that \( |z| = 1 \) (because of the maximum modulus principle) we have that:
\[
1 + z \frac{f''}{f'}(z) = 1 + \left( \frac{z + a}{1 + \bar{a}z} \right) \frac{f''}{f'} \left( \frac{z + a}{1 + \bar{a}z} \right) \frac{z(1 - |a|^2)(1 + \bar{a}z) - 2\bar{a}z}{(1 + \bar{a}z)^2(z + a)} - \frac{2\bar{a}}{1 + \bar{a}z}
\]
\[
= 1 + \left( \frac{z + a}{1 + \bar{a}z} \right) \frac{1 - |a|^2}{|1 + \bar{a}z|^2} - \frac{2\bar{a}}{1 + \bar{a}z}.
\]

Let \( \zeta = \left( \frac{z + a}{1 + \bar{a}z} \right) \in \mathbb{D} \), from which \( \text{Re} \left\{ \zeta \frac{f''}{f'}(\zeta) \right\} > \alpha - 1 \). Then, for \( |z| = 1 \) we have that
\[
\text{Re} \left\{ 1 + z \frac{f''}{f'}(z) \right\} = 1 + \left( \frac{1 - |a|^2}{|1 + \bar{a}z|^2} \right) \text{Re} \left\{ \zeta \frac{f''}{f'}(\zeta) \right\} - 2 \text{Re} \left\{ \frac{\bar{a}z}{1 + \bar{a}z} \right\}
\]
\[
\geq 1 + \left( \frac{1 - |a|^2}{|1 + \bar{a}z|^2} \right) (\alpha - 1) - 2 \text{Re} \left\{ \frac{\bar{a}z}{1 + \bar{a}z} \right\}
\]
\[
= 1 + \alpha \left( \frac{1 - |a|^2}{|1 + \bar{a}z|^2} \right) - \frac{1 - |a|^2 + 2 \text{Re}\{\bar{a}z(1 + \bar{a}z)\}}{|1 + \bar{a}z|^2}
\]
\[
= 1 + \alpha \left( \frac{1 - |a|^2}{|1 + \bar{a}z|^2} \right) - \frac{1 + 2 \text{Re}\{\bar{a}z\} + |a|^2}{|1 + \bar{a}z|^2}
\]
\[
= \alpha \left( \frac{1 - |a|^2}{|1 + \bar{a}z|^2} \right) \geq \alpha \left( \frac{1 - |a|^2}{1 + |a|} \right) = \alpha \left( \frac{1 - |a|}{1 + |a|} \right).
\]
From the above we can conclude that if \( f \in \mathcal{C}_\alpha \) then \( f_a \in \mathcal{C}_\beta \). \( \square \)
Now, with the additional condition that \( f''(0) = 0 \), equation (5) shows that \( \varphi \) satisfies that
\[
\varphi(z) = z \psi(z),
\]
for some holomorphic mapping \( \psi \) with \( |\psi| < 1 \). We say that \( f \in C^0_\alpha \) if \( f \in C_\alpha \) and \( f''(0) = 0 \). Thus, a straightforward calculation shows that:

**Proposition 3.** If \( f \in C^0_\alpha \) then
\[
(1 - |z|^2) \left| \frac{f''}{f'}(z) \right| \leq 2|z|(1 - \alpha), \quad \forall z \in \mathbb{D}.
\]

**Proof.** Since \( \varphi(z) = z \psi(z) \), with \( |\psi(z)| < 1 \), then in equation (5) we have:
\[
\left| \frac{f''}{f'}(z) \right| \leq 2(1 - \alpha) \frac{|z| |\psi(z)|}{1 - |z|^2 |\psi(z)|} \leq 2(1 - \alpha) \frac{|z|}{1 - |z|^2}.
\]

\( \square \)

Robertson in [8] shows that if \( f \in C_\alpha \) then
\[
1 \leq |f'(z)| \leq \frac{1}{1 - (1 - |z|^2)^{2\alpha - 1}},
\]
and, if \( \alpha \neq 1/2 \)
\[
\frac{(1 + |z|)^{2\alpha - 1} - 1}{2\alpha - 1} \leq |f(z)| \leq \frac{1 - (1 - |z|)^{2\alpha - 1}}{2\alpha - 1},
\]
for \( \alpha = 1/2 \),
\[
\log(1 + |z|) \leq |f(z)| \leq -\log(1 - |z|).
\]

With the additional condition \( f''(0) = 0 \), we improve these results which are summarized in the following theorem.

**Theorem 4.** If \( f \in C^0_\alpha \) then for all \( z \in \mathbb{D} \), we have that
\[
\frac{1}{(1 + |z|^2)^{1-\alpha}} \leq |f'(z)| \leq \frac{1}{(1 - |z|^2)^{1-\alpha}},
\]
and
\[
\int_0^{\bar{z}} \frac{1}{(1 + \zeta)^{1-\alpha}} d\zeta \leq |f(z)| \leq \int_0^{\bar{z}} \frac{1}{(1 - \zeta)^{1-\alpha}} d\zeta.
\]

We will see in Example 11 that this bounds are sharp within \( C^0_\alpha \). In particular, when \( \alpha = 0 \), which corresponds to the subclass \( \mathcal{C} \) with \( f''(0) = 0 \) denoted by \( \mathcal{C}^0 \), we have that
\[
\tan^{-1}(|z|) \leq |f(z)| \leq \frac{1}{2} \log \left( \frac{1 + |z|}{1 - |z|} \right).
\]

Clearly, these bounds are better than the well-known bounds for the entire convex class \( \mathcal{C} \) (see [4, p. 43]). We will see in Theorem 5 that \( \|Sf\| \leq 2(1 - \alpha^2) \) for any \( f \in C^0_\alpha \), therefore, a comparison is in order between these bounds and those obtained by Chuaqui and Osgood in [2]. These authors proved certain sharp bounds for distortion and growth for mappings with the condition that \( (1 - |z|^2)^2 |Sf| \leq 2t \).
when \( t \in [0,1] \). They showed that if \( f \) is holomorphic in \( \mathbb{D} \) such that \( f(0) = 0 \), \( f'(0) = 1 \) and \( f''(0) = 0 \), then:

\[
A(|z|, -t) \leq |f(z)| \leq A(|z|, t),
\]

\[
A'(|z|, -t) \leq |f'(z)| \leq A'(|z|, t),
\]

where

\[
A(z, t) = \frac{1}{\sqrt{1 - t}} \frac{(1 + z)^{\sqrt{1-t}} - (1 - z)^{\sqrt{1-t}}}{(1 + z)^{\sqrt{1-t}} + (1 - z)^{\sqrt{1-t}}}.
\]

Our case (\( \alpha = 0 \)) corresponds to their case when \( t = 1 \), where the upper bound for growth coincides with our upper bound because the mapping for which equality holds belongs to both classes. However, for the lower bound the situation is completely different since their lower bound is less than or equal to our lower bound. Moreover, when \( \alpha = 1/2 \), the growth inequality in Theorem 3 is \( \sinh^{-1}(|z|) \leq |f(z)| \leq \sin^{-1}(|z|) \), but the corresponding parameter is \( t = 3/4 \), hence our result is more precise than the one obtained by Chuaqui and Osgood in [2], which is not surprising since in their case the family is larger than \( C^0_\alpha \).

**Proof of Theorem 4.** From equation (6) and considering that \( \varphi(0) = 0 \) (because \( f''(0) = 0 \)), then by the Schwarz Lemma we have that

\[
|\frac{(f''/f')(z)}{2(1 - \alpha) + z(f''/f')(z)}|^2 \leq |z|^2,
\]

which is equivalent to

\[
(1 - |z|^4) \left| \frac{f''}{f'}(z) \right|^2 \leq 4|z|^2(1 - \alpha)^2 + 4|z|^2(1 - \alpha)\text{Re} \left\{ \frac{zf''}{f'}(z) \right\}.
\]

If we first multiply by \( (1 - |z|^4) \) and then we add \( 4(|z|^2(1 - \alpha)|z|)^2 \) on both sides, we obtain

\[
(1 - |z|^4)^2 \left| \frac{f''}{f'}(z) \right|^2 - 4|z|^2(1 - |z|^4)(1 - \alpha)\text{Re} \left\{ \frac{zf''}{f'}(z) \right\} + 4(|z|^2(1 - \alpha)|z|)^2
\]

\[
\leq 4(1 - |z|^4)|z|^2(1 - \alpha)^2 + 4|z|^4(1 - \alpha)^2|z|^2 = 4|z|^2(1 - \alpha)^2.
\]

Then multiplying on both sides by \( |z| \) we have

\[
(1 - |z|^4)|z|\frac{f''}{f'}(z) - 2|z|^4(1 - \alpha) \leq 2|z|^2(1 - \alpha).
\]

This implies

\[
\frac{-2|z|^2(1 - \alpha) + 2|z|^4(1 - \alpha)}{1 + |z|^4} \leq \text{Re} \left\{ \frac{zf''}{f'}(z) \right\} \leq \frac{2|z|^2(1 - \alpha) + 2|z|^4(1 - \alpha)}{1 - |z|^4}.
\]

Which it is equivalent to
\[-2(1-\alpha)|z|^2 \leq \text{Re} \left\{ z \frac{f''}{f'(z)} \right\} \leq \frac{2(1-\alpha)|z|^2}{1-|z|^2}.\]

Let \( z = re^{i\theta} \), then we have
\[-2(1-\alpha)r \leq \frac{\partial}{\partial r} (\log |f'(re^{i\theta})|) \leq \frac{2(1-\alpha)r}{1-r^2}.\]

If we integrate respect to \( r \), we obtain
\[\log(1+r^2)^{-(1-\alpha)} \leq \log|f'(re^{i\theta})| \leq \log(1-r^2)^{-(1-\alpha)}.\]

Exponentiating we can conclude that
\[\frac{1}{(1+|z|^2)^{1-\alpha}} \leq |f'(z)| \leq \frac{1}{(1-|z|^2)^{1-\alpha}}.\]

Now, for the growth part of the theorem, from the upper bound it follows that
\[|f(re^{i\theta})| = \left| \int_0^r f'(te^{i\theta})e^{i\theta} \, dt \right| \leq \int_0^r |f'(te^{i\theta})| \, dt \leq \int_0^r \frac{1}{(1-t^2)^{1-\alpha}} \, dt.\]

Which implies that
\[|f(z)| \leq \int_0^{\frac{1}{1-|z|^2}} \frac{1}{(1-\zeta^2)^{1-\alpha}} \, d\zeta,\]
for all \( z \in \mathbb{D} \). It is well known that if \( f(z_0) \) is a point of minimum modulus on the image of the circle \( |z| = r \) and \( \gamma = f^{-1}(\Gamma) \), where \( \Gamma \) is the line segment from 0 to \( f(z_0) \), then
\[|f(z)| \geq |f(z_0)| = \int_\gamma |dw| = \int_\gamma |f'(\zeta)| \, |d\zeta| \geq \int_0^r \frac{1}{(1+|\zeta|^2)^{1-\alpha}} \, d|\zeta|.\]

Thus, the proof is completed. \( \square \)

3. On the norm of the Schwarzian derivative when \( f''(0) = 0 \).

This section is devoted to finding the sharp bound of the norm of the Schwarzian Derivative in terms of the parameter \( \alpha \) under the assumption that \( f''(0) = 0 \). We define the \( C^0_\alpha \) class as those \( f \in C_\alpha \) such that \( f''(0) = 0 \). Firstly, form equation (5) it follows that
\[Sf(z) = 2(1-\alpha) \left( \frac{\varphi'(z) + \alpha \varphi^2(z)}{(1-z \varphi(z))^2} \right).\]

Suita in [9] proved the sharp bounds of \( \|Sf\| \) for the hole class \( C_\alpha \). We present this results as the following Theorem:

**Theorem A.** If \( f \in C_\alpha \) then for all \( z \in \mathbb{D} \) we have that if \( 0 \leq \alpha \leq 1/2 \):
\[(1-|z|^2)^2 |Sf(z)| \leq 2,\]
and for \( 1/2 < \alpha < 1 \):
\[(1-|z|^2)^2 |Sf(z)| \leq 8\alpha(1-\alpha).\]
The next theorem gives us a sharp bound for the expression $(1 - |z|^2)^2 |Sf(z)|$ when $f \in C^0_\alpha$ by a direct application of the Schwarz Lemma.

**Theorem 5.** If $f \in C^0_\alpha$ then

$$(1 - |z|^2)^2 |Sf(z)| \leq 2(1 - \alpha^2), \quad \forall z \in \mathbb{D}.$$ 

The inequality is sharp.

**Proof.** From (9) we obtain, by applying the triangle inequality and the Schwarz-Pick lemma, that:

$$(10) \quad (1 - |z|^2)^2 |Sf(z)| \leq 2(1 - \alpha) \frac{(1 - |z|^2)^2}{|1 - z\varphi(z)|^2} \left[ \frac{|1 - \varphi(z)|^2}{1 - |z|^2} + \alpha|\varphi(z)|^2 \right].$$

We define the function $\Phi : \mathbb{D} \to \mathbb{C}$ such that:

$$\Phi(z) = \frac{\bar{z} - \varphi(z)}{1 - z\varphi(z)}.$$

Since $\varphi(\mathbb{D}) \subset \mathbb{D}$, then $(1 - |z|^2)(1 - |\varphi(z)|^2) > 0$, which implies that

$$|\bar{z} - \varphi(z)|^2 < |1 - z\varphi(z)|^2,$$

so, we can conclude that $|\Phi(z)|^2 < 1$. Therefore,

$$1 - |\Phi(z)|^2 = \frac{(1 - |\varphi(z)|^2)(1 - |z|^2)}{|1 - z\varphi(z)|^2},$$

and,

$$\frac{(1 - |z|^2)^2}{|1 - z\varphi(z)|^2} = \frac{(1 - |\Phi(z)|^2)(1 - |z|^2)}{(1 - |\varphi(z)|^2)}.$$ 

If we replace the latter expression in (10) we have:

$$(11) \quad (1 - |z|^2)^2 |Sf(z)| \leq 2(1 - \alpha)(1 - |\Phi(z)|^2) \left[ 1 + \alpha \left( \frac{|\varphi(z)|^2(1 - |z|^2)}{1 - |z|^2} \right) \right].$$

From the normalization and using the equation (5) we conclude that $\varphi(0) = 0$. Then, by Schwarz lemma, we obtain that $|\varphi(z)| \leq |z|$ for all $z \in \mathbb{D}$ and from this we can conclude that:

$$\frac{|\varphi(z)|^2}{1 - |\varphi(z)|^2} \leq \frac{|z|^2}{1 - |z|^2}.$$ 

Applying the above in (10) we obtain:

$$(1 - |z|^2)^2 |Sf(z)| \leq 2(1 - \alpha)(1 - |\Phi(z)|^2)(1 + \alpha|z|^2).$$

We know that $1 - |\Phi(z)|^2 \leq 1$, then:

$$(1 - |z|^2)^2 |Sf(z)| \leq 2(1 - \alpha)(1 + \alpha) = 2(1 - \alpha^2).$$

The sharpness follows from the next example.  \[\square\]
Example 1. The family of parameterized functions defined as:

\[ f_\alpha(z) = \int_0^z \frac{1}{(1 - \zeta^2)^{1-\alpha}} \, d\zeta, \quad z \in \mathbb{D}, \]

maximizes the Schwarzian norm defined as:

\[ \|Sf\| = \sup_{z \in \mathbb{D}} (1 - |z|^2)^2 |Sf(z)|, \]

and from this, the sharpness of the inequality given in Theorem 2 holds for any \(0 \leq \alpha < 1\). Notice that:

\[ \frac{f''_\alpha}{f'_\alpha}(z) = 2(1 - \alpha) \frac{z}{1 - z^2}, \quad \text{and} \quad Sf_\alpha(z) = 2(1 - \alpha) \cdot \frac{1 + \alpha z^2}{(1 - z^2)^2}. \]

from which,

\[ (1 - |z|^2)^2 |Sf_\alpha(z)| = 2(1 - \alpha) \left(1 - |z|^2\right)^2 |1 + \alpha z^2| \leq 2(1 - \alpha) |1 + \alpha z^2|. \]

We can conclude that:

\[ \|Sf_\alpha\| = \sup_{z \in \mathbb{D}} (1 - |z|^2)^2 |Sf_\alpha(z)| = 2(1 - \alpha)(1 + \alpha) = 2(1 - \alpha^2). \]

In general, the integral formula for \(f_\alpha\) given in (12) does not give primitives in terms of elementary functions, however, when \(\alpha = 0\) we obtain:

\[ f_0(z) = \int_0^z \frac{1}{1 - \zeta^2} \, d\zeta = \frac{1}{2} \log \left( \frac{1 + z}{1 - z} \right), \]

where \(\|Sf_0\| = 2\). Also, for \(\alpha = 1/2\) we have that:

\[ f_{1/2}(z) = \int_0^z \frac{1}{(1 - \zeta^2)^{1/2}} \, d\zeta = \sin^{-1}(z), \]

where \(\|Sf_{1/2}\| = 3/2\).

Remark 1. In [1] is presented the next inequality for \(f \in C\):

\[ (1 - |z|^2)^2 |Sf(z)| + 2 \left| \frac{\varphi(z) - \bar{z}}{1 - z \varphi(z)} \right|^2 \leq 2, \]

which is equivalent to:

\[ (1 - |z|^2)^2 |Sf(z)| + 2 \left| \bar{z} - \frac{1}{2}(1 - |z|^2) \frac{f''}{f'}(z) \right|^2 \leq 2. \]

However, for the class \(C_\alpha\) we can obtain from (11) an analogous inequality associated to (13) and (14). In fact, we have that

\[ (1 - |z|^2)^2 |Sf(z)| \leq 2(1 - \alpha) \left[ (1 + \alpha) - \left| \varphi(z) - \frac{\bar{z}}{1 - z \varphi(z)} \right|^2 - \alpha A \right], \]
where, using the equation (15),

\[
A = 1 - \left( \frac{(1 - |z|^2)|\varphi(z)|}{|1 - z\varphi(z)|} \right)^2 = 1 - \left( \frac{1 - |z|^2}{2(1 - \alpha)} \left| f''(z) \right| \right)^2.
\]

Now, from (15) we have that:

\[
(1 - |z|^2)^2 |Sf(z)| + 2(1 - \alpha) \left| \frac{\varphi(z) - \bar{z}}{1 - z\varphi(z)} \right|^2 + 2(1 - \alpha)\alpha A \leq 2(1 - \alpha^2),
\]

which is equivalent to

\[
(1 - |z|^2)^2 |Sf(z)| + 2(1 - \alpha) \left| \bar{z} - \frac{1 - |z|^2}{2(1 - \alpha)} f''(z) \right|^2 + 2(1 - \alpha)\alpha A \leq 2(1 - \alpha^2).
\]

Thus, when \( A \geq 0 \) we have that \( |Sf| \leq 2(1 - \alpha^2) \), which occurs when \( \varphi(0) = 0 \), because of \( f''(0) = 0 \). In this case, using the Schwarz Lemma we can show that \( A \geq 1 - |\varphi(z)| \). Therefore we can assert the following results that improve Theorem 1 in [11].

The next theorem gives us a bound for the expression \((1 - |z|^2)^2 |Sf(z)|\) for \( f \in C_\alpha \) in terms of value \( |f''(0)| \), which is not necessarily zero. To do this, we need the next lemma (its proof can be found in [7], p. 167) which gives us a generalization of the Schwarz Lemma for bounded holomorphic functions:

**Lemma 6.** If \( \varphi : \mathbb{D} \to \mathbb{D} \) be a holomorphic function then

\[
|\varphi(z)| \leq \frac{|\varphi(0)| + |z|}{1 + |\varphi(0)||z|}.
\]

**Theorem 7.** Let \( 0 \leq \alpha < 1 \), \( f \in C_\alpha \) and \( p = \frac{|f''(0)|}{2(1 - \alpha)} \). Then:

\[
(1 - |z|^2)^2 |Sf(z)| \leq 2(1 - \alpha) \left( 1 + \alpha \frac{1 + p}{1 - p} \right).
\]

**Proof.** Let \( p = |\varphi(0)| \). Applying Lemma 6, we have that:

\[
\frac{|\varphi(z)|^2}{1 - |\varphi(z)|^2} \leq \frac{(p + |z|)^2}{(1 + p|z|)^2 - (p + |z|)^2} = \frac{(p + |z|)^2}{(1 - p^2)(1 - |z|^2)}.
\]

If we replace the above expression in (11), at the proof of Theorem 5 we obtain that:

\[
(1 - |z|^2)^2 |Sf(z)| \leq 2(1 - \alpha)(1 - |\Phi(z)|^2) \left( 1 + \alpha \frac{(p + |z|)^2}{1 - p^2} \right).
\]

From the fact that \( |z| \ll 1 \) and \( 1 - |\Phi(z)|^2 < 1 \) we can conclude that:

\[
(1 - |z|^2)^2 |Sf(z)| \leq 2(1 - \alpha) \left( 1 + \alpha \frac{(p + 1)^2}{1 - p^2} \right) = 2(1 - \alpha) \left( 1 + \alpha \frac{1 + p}{1 - p} \right).
\]

\( \square \)
Remark 2. Although the bound obtained is not sharp in general, for the $C_\alpha$ class, we have that in certain cases the bound in Theorem 7 is better than the one given in Theorem A, (see [9]). In fact, for $0 \leq \alpha \leq 1/2$ and $f \in C_\alpha$ such that $p \leq \alpha(2 - \alpha)^{-1}$ then $(1 + p)(1 - p)^{-1} \leq (1 - \alpha)^{-1}$, which implies

$$2(1 - \alpha) \left( 1 + \frac{\alpha (1+p)}{1-p} \right) \leq 2.$$ 

On the other side, if $1/2 < \alpha < 1$ and $f$ is such that $p \leq (3\alpha - 1)(5\alpha - 1)^{-1}$ then $(1 + p)(1 - p)^{-1} \leq (8\alpha - 2)(2\alpha)^{-1}$, from which

$$2(1 - \alpha) \left( 1 + \frac{\alpha (1+p)}{1-p} \right) \leq 8\alpha(1 - \alpha).$$

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References

[1] Chuaqui, M., Duren, P., Osgood, B., Schwarzian derivatives of convex mappings, Ann. Acad. Sci. Fenn. Math. 36(2), 449–460 (2011).

[2] Chuaqui, M., Osgood, B., Sharp distortion theorems associated with the Schwarzian derivative, J. London Math. Soc. 48(2), 289-298 (1993).

[3] Duren P. L., Univalent functions, Springer-Verlag, New York, (1983).

[4] Graham, I., Kohr, G., Geometric Function Theory in One and Higher Dimensions, Marcel Dekker Inc., New York, (2003).

[5] Kanas, S., Sugawa, T., Sharp norm estimate of Schwarzian derivative for a class of convex functions, Ann. Polon. Math. 101(1), 75-86 (2011).

[6] Nehari, Z., A property of convex conformal maps. J. Analyse Math. 30, 390–393 (1976).

[7] Nehari, Z., Conformal Mapping, Dover Publications, New York (1972).

[8] Robertson, M.I.S., On the theory of univalent functions. Annals of Mathematics 37(2), 374–408 (1936).

[9] Suita, N., Schwarzian derivatives of convex functions. J. Hokkaido Univ.Ed. Sect. II A 46(2), 113–117 (1996).

[10] Trimble, S., A coefficient inequality for convex univalent functions. Proc. Am. Math. Soc, 48, 266–267 (1975).

[11] Yamashita, S., Norm estimates for function starlike or convex of order alpha, Hokkaido Math. J. 28, 217-230 (1999).
