A Uniform Spherical Goat (Problem): Explicit Solution for Homologous Collapse’s Radial Evolution in Time

Zachary Slepian1,2* and Oliver H. E. Philcox3,4†

1Department of Astronomy, University of Florida, 211 Bryant Space Science Center, Gainesville, FL 32611, USA
2Physics Division, Lawrence Berkeley National Laboratory, 1 Cyclotron Road, Berkeley, CA 94709, USA
3Department of Astrophysical Sciences, Princeton University, Princeton, NJ 08540, USA
4School of Natural Sciences, Institute for Advanced Study, 1 Einstein Drive, Princeton, NJ 08540, USA

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ABSTRACT

The homologous collapse from rest of a uniform density sphere under its self gravity is a well-known toy model for the formation dynamics of astronomical objects ranging from stars to galaxies. Equally well-known is that the evolution of the radius with time cannot be explicitly obtained because of the transcendental nature of the differential equation solution. Rather, both radius and time are written parametrically in terms of the development angle \( \theta \). We here present an explicit integral solution for radius as a function of time, exploiting methods from complex analysis recently applied to the mathematically-similar “geometric goat problem.” Our solution can be efficiently evaluated using a Fast Fourier Transform and allows for arbitrary sampling in time, with a simple Python implementation that is ~100\( \times \) faster than using numerical root-finding to achieve arbitrary sampling. Our explicit solution is advantageous relative to the usual approach of first generating a uniform grid in \( \theta \), since this latter results in a non-uniform radial or time sampling, less useful for applications such as generation of sub-grid physics models.

Key words: cosmology: large-scale structure of Universe, theory

1 INTRODUCTION

Spherical collapse is ubiquitous in astronomy and has been used to model the formation of stars up to the formation of galaxy halos. Going back many years (e.g. Lin et al. 1965; Tomita 1969; Gunn & Gott 1972), the model of a uniform density sphere collapsing homologously (no shell crosses another shell) from rest under its own gravity has been the simplest instantiation of this scenario. The governing equations (see e.g. Lin et al. 1965) are

\[
d^2r \over dt^2 = -GM_r \over r^2 = -\frac{4\pi G \rho_0}{3} \frac{r^3}{r^2},
\]

\[M_r \equiv \frac{4\pi}{3} \frac{r^3}{\rho_0},\]

where \( r \) is the radius of the sphere, \( G \) is Newton’s constant, \( M_r \) is the mass internal to radius \( r \), and \( \rho_0 \) is the (uniform) initial density when the sphere has its initial radius, \( r_0 \). The second equality in the first line above comes from inserting the form for \( M_r \) given in the second line. These have the cycloidal parametric solution

\[r(\theta) = r_0 \cos^2 \theta,\]

\[\theta + \frac{1}{2} \sin 2\theta = \frac{\pi}{2} t_{ff},\]

\[t_{ff} \equiv \sqrt{\frac{3\pi}{32G\rho_0}},\]

where \( t_{ff} \) is the free-fall time. \( \theta = 0 \) corresponds to the initial conditions of radius \( r_0 \) and zero velocity, and at \( \theta = \pi/2 \), the sphere has collapsed to zero radius. Since the equation for \( \theta \) is transcendental, one cannot explicitly obtain \( \theta \) as a function of \( t \) and thence \( r(t) \). Here, we show how using techniques from complex analysis recently developed to solve the “geometric goat problem” (which we will momentarily describe), an explicit integral solution for \( r(t) \) can be found.

The geometric goat problem is as follows. Suppose a goat is placed inside a circular (2-D) enclosure of radius \( R \), tethered to a fixed point on the circumference by a rope of length \( r \). How long must the rope be to permit the goat to graze on exactly half the area of the enclosure?

Writing down the appropriate integral expressions for the enclosed area as a function of \( r \) and \( R \), one obtains a transcendental equation. Following a number of (non-trivial) manipulations, this equation can be written as

\[\sin \beta - \beta \cos \beta = \frac{\pi}{2}\]

(Ullisch 2020). We observe that this equation is somewhat similar to our equation (2) for \( \theta \) if one treats \( t \) as a constant and \( \theta \) as analogous to \( \beta \). If one is able to solve an equation of the type above, it is worth considering whether the same method may be used to solve equation (2) for \( \theta(t) \). This indeed turns out to be so.

2 SOLUTION

We follow the approach described in Ullisch (2020) to obtain our solution.

First, we write our \( \theta \) equation (2) in terms of an entire function
f(z) defined on the complex plane,
\[ f(z) = \frac{1}{2} \sin 2z - \frac{\pi}{2} t_{\text{rf}}. \tag{4} \]

Here, we require f(z₀) = 0, where z₀ will give our desired solution \( \theta(t) \). By symmetry, this has \( \text{Im}(z) = 0 \). For real \( z \), and at fixed \( t > 0 \), \( f(z) \) is monotonically increasing on this interval and has exactly one zero (i.e., one solution for \( \theta(t) \)). This zero may be shown to be simple:
\[ \lim_{z \to z_0} \frac{z + (1/2) \sin z - (\pi/2)(t/t_{\text{rf}})}{z - z_0} = 1 + \cos 2z_0 \neq 0 \tag{5} \]
given the bounds on \( z_0 \).

**Theorem 1** of Ullisch (2020) (see also Jackson 1916, 1917; Luck et al. 2015) states that, on a simply-connected open subset \( U \) of the complex plane, for every simple zero \( z_0 \in U \) of a non-zero analytic function \( f(z) \), there exists a curve \( C \) such that
\[ z_0 = \frac{\oint_C z \, df(z)}{\oint_C f(z) \, dz}. \tag{6} \]

Indeed, this is true for any Jordan curve \( C \) (i.e., one which is continuous and does not self-intersect), enclosing \( z_0 \) such that \( z_0 \) is the only zero of \( f(z) \) on \( C \) and its interior. To apply this method to the spherical collapse scenario, we must first find a valid curve \( C \) by which to evaluate the result (6).

Motivated by the boundary conditions on \( \theta \) and the discussion in Ullisch (2020), we first consider the (simply-connected) rectangular region \( R = (0, \pi/2) \times (-M, M) \) in the complex plane for arbitrary \( M > 0 \). Via the argument principle, the number of zeros minus the number of poles contained within \( R \) is given by
\[ \frac{1}{2\pi i} \oint_{\partial R} \frac{f'(z)}{f(z)} \, dz = \frac{1}{2\pi} \Delta_{\partial R} \text{arg} f(z) \tag{7} \]
where \( \partial R \) is the (non-self-intersecting) boundary of \( R \) (traversed counter-clockwise), and \( \Delta_{\partial R} \text{arg} f(z) \) represents the total change in the argument of \( f(z) \) as one traverses \( \partial R \).

Given that \( f(z) \) contains no poles in \( R \), this simply counts the number of zeros within \( R \).

Denoting \( z = x + iy \), \( f(z) \) has the limiting forms
\[ f(0 + iy) = -\frac{\pi}{2} t_{\text{rf}} + i \left( y + \frac{1}{2} \sinh 2y \right) \]
\[ f\left( \frac{\pi}{2} + iy \right) = -\frac{\pi}{2} \left( 1 - \frac{t}{t_{\text{rf}}} \right) + i \left( y - \frac{1}{2} \sinh 2y \right) \]
\[ f(x + iM) \approx x - \frac{\pi}{2} \frac{t}{t_{\text{rf}}} + i e^{-2ix} e^{2M} \]
\[ f(x - iM) \approx x - \frac{\pi}{2} \frac{t}{t_{\text{rf}}} - i e^{2ix} e^{2M}, \tag{8} \]
where the third and fourth equations are exact in the limit \( M \to \infty \). Let us consider the change in \( \text{arg} f(z) \) along each of the four sides of \( \partial R \) in turn (assuming \( M \gg 0 \)).

(i) \( (0 + iM) \to (0 - iM) \). \( \text{Re} \{ f(z) \} \) takes the constant (negative) value \(-\pi(2)(t/t_{\text{rf}})\), whilst \( \text{Im} \{ f(z) \} \) decreases monotonically from \((1/4)\exp 2M\) to \(-(1/4)\exp 2M\). Thus \( \Delta \text{arg} f(z) \) = \(+\pi\).

(ii) \( (0 - iM) \to (\pi/2 - iM) \). For large \( M \), \( f(z) \approx (1/4)\exp [2M + 2ix + 3\pi/2] \), thus \( \Delta \text{arg} f(z) \) = \(+\pi\) as \( x \) increases from \( 0 \) to \( \pi/2 \).

(iii) \( (\pi/2 - iM) \to (\pi/2 + iM) \). \( \text{Re} \{ f(z) \} \) takes the constant (positive) value \((\pi/2)(1-t/t_{\text{rf}})\), whilst \( \text{Im} \{ f(z) \} \) increases monotonically from \((1/4)\exp e^{2M}\) to \((1/4)\exp -2M\). Thus \( \Delta \text{arg} f(z) \) = \(-\pi\).

(iv) \( (\pi/2 + iM) \to (0 + iM) \). For large \( M \), \( f(z) \approx (1/4)\exp [2M - 2ix + i\pi/2] \), thus \( \Delta \text{arg} f(z) \) = \(+\pi\) as \( x \) decreases from \( \pi/2 \) to \( 0 \).

Summing the regimes, we find \( \Delta \text{arg} f(z) = 2\pi \), indicating that \( R \) contains exactly one zero. Since the point \( z_0 \in R \), this must be the only zero in the region. Since \( M \) is arbitrary, we can thus write \( f(z) \neq 0 \) for all \( z \in U \setminus \{z_0\} \), where \( U = \{z : \text{Re} (z) \in (0, \pi/2)\} \).

Coupled with the theorem of Ullisch (2020), we see that any Jordan curve \( C \subset \overline{U} \) enclosing \( z_0 \) can be used to evaluate equation (6). Here, we set \( C \) equal to a circle with radius \( \pi/4 - \epsilon \) at center \((\pi/4, 0)\) where \( \epsilon > 0 \) is small. This is contained within \( U \) and, for sufficiently small \( \epsilon \), encloses \( z_0 \), thus the above conditions apply. A representative plot of \( f(z) \), alongside the region \( U \) and the contour \( C \) is shown in Figure 1.

We hence obtain the integral solution
\[ z_0(t) = \frac{\oint_{C} z \, df(z)}{\oint_{C} df(z)} \left[ z + (1/2) \sin 2z - (\pi/2)(t/t_{\text{rf}}) \right], \]
\[ r(t) = r_0 \cos \left( \frac{1}{2} e^{2\pi t} \right) \]
\[ C = \{ z : |z - \pi/4| = \pi/4 - \epsilon \}. \tag{9} \]

3 EVALUATION USING FAST FOURIER TRANSFORMS

Following Ullisch (2020), we consider how to evaluate equation (9) using Fast Fourier Transforms (FFTs). First, we parametrize the
and we define the Fourier coefficients $c_k$ by $c_k = \int_0^1 dx g(x; t) e^{-2\pi i k x}$ for integer $k$. The solution for $r(t)$ is thus

$$r(t) = r_0 \cos^2 \left( \frac{\pi}{4} + \left( \frac{\pi}{4} - \epsilon \right) \frac{c_{-2}(t)}{c_{-1}(t)} \right)$$

which can be computed to arbitrary precision for a given $t$ by estimating $c_k(t)$ using FFTs. We recall that $\epsilon$ enters the radius of the contour $C$ (see Figure 1) used for the integration; this radius is $\pi/4 - \epsilon$.

## 4 DISCUSSION

The above discussion has shown how one can obtain an efficient numerical approximation for the cycloid’s evolution using contour integration coupled with FFTs, just as for the geometric goal problem.

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3 Alternatively, we may numerically integrate (11) directly to compute only the $c_{-2}$ and $c_{-1}$ coefficients. In practice, this is slightly more efficient than using FFTs.

4 This is publicly available on GitHub.
Figure 4. Computation time for the FFT-based spherical collapse solver considered in this work (solid curves) versus the naïve root-finding approach discussed in Figure 3 (dashed lines). We show results for a range of FFT grid-sizes and three lengths of the input \( t \)-array, from \( 10^4 \) elements (top line/top curve, green) to \( 10^2 \) elements (bottom line/bottom curve, red). Both methods are simply implemented in \texttt{python}, and the FFT-based approach may be efficiently vectorized.

DATA AVAILABILITY

The data underlying this article will be shared on reasonable request to the corresponding author. A \texttt{python} implementation of our code is available on GitHub.

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