NON-PERTURBATIVE ANALYSIS, GRIBOV HORIZONS AND THE BOUNDARY OF THE FUNDAMENTAL DOMAIN †

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ABSTRACT

In this contribution to the proceedings we will describe some of the details for constructing the Gribov horizon and the boundary of the fundamental modular domain, when restricting to some low energy modes of pure SU(2) gauge theory in a spherical spatial geometry. The fundamental domain is a one-to-one representation of the set of gauge invariant degrees of freedom, in terms of transverse gauge fields. Boundary identifications are the only remnants of the Gribov copies.

1. Introduction

At the conference one of us gave an overview of the finite volume analysis on a torus, relevant for comparison with lattice Monte Carlo results [1]. Recently, for technical reasons, this analysis was extended to a spherical spatial geometry [2], results of which were only briefly touched upon during the talk. The other author has, from a different perspective, been interested in this geometry for quite some years now [3]. Thus we found some common interest, whose hitherto unpublished fruits we will discuss in these proceedings. It involves computing the Gribov horizon and the boundary of the fundamental domain in a spherical spatial geometry. The remaining part of the talk is summarized in the contribution to the Shanxi conference [4]. For the applications and further references see [1-4].

In gauge theories one can fix to the transverse or Coulomb gauge, \( \partial_i A_i = 0 \), using a functional method, which allows one to pick from the different Gribov copies [5] (transverse gauge fields that are nevertheless gauge equivalent) an (almost) unique representative. The collection of these configurations should thus form a fundamental modular domain, in other words it should form a one-to-one mapping with the Yang-

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Mills configuration space. The relevant functional is the $L^2$-norm of the gauge field:
$$\|A_i\|^2 = \int_M \text{Tr}(A_i^\dagger A_i).$$
For each gauge invariant field configuration this gives a Morse functional on the gauge orbit. One easily verifies that stationary points of this Morse functional satisfy the Coulomb gauge condition and that the Hessian (second order derivative) at the stationary point is precisely given by the Faddeev-Popov operator, whose determinant measures the volume of the gauge orbit. It is natural to choose among the various Gribov copies the one with lowest norm. The collection of transverse fields thus obtained has a boundary. Points on this boundary are usually degenerate in norm and gauge equivalent to at least one other point on the boundary [6]. This gauge equivalence induces boundary identifications that make the set into a fundamental modular domain $\Lambda$ (no Gribov copies occur at the interior). In general $\Lambda$ is well contained in the Gribov region $\Omega$, which is by definition the collection of transverse potentials for which the norm functional (when considered as a function of the gauge orbit) has a local minimum, i.e. the Hessian or Faddeev-Popov operator $FP(A) = -\partial_iD_i(A)$ is positive semi-definite ($D_i(A)$ is the usual adjoint covariant derivative). There can [6], however, be points at the boundary of $\Lambda$ that coincide with the boundary of $\Omega$. Since the latter is the Gribov horizon, which is where the lowest eigenvalue of the Faddeev-Popov operator (and hence the Faddeev-Popov determinant) vanishes, these points will still require some extra care. They have, so far, not been considered in the subsector of the theory we will be studying here.

2. Defining the subspace

The subsector we will study is given by 18 modes that are degenerate in energy, to lowest (quadratic) order. There are only twelve modes with a lower energy, also all degenerate. These 18 modes contain degrees of freedom relevant for the tunnelling from the $A = 0$ vacuum to the two vacua that have Chern-Simons number $Q(A) = \frac{1}{8\pi^2} \int_{S^3} \text{Tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A)$ one or minus one. Here $A = A_i dx_i = iA_i^a \tau_a dx_i/2$ is the connection one-form for a SU(2) vector potential $A_i^a$ on the three-sphere. In other words the two vacua with $Q(A) = \pm 1$, are transverse vector potentials, that are pure gauge ($A = gdg^{-1}$) with a gauge function $g$ that has a winding number $n(g) = \pm 1$, where $n(g) = \frac{1}{24\pi^2} \int_{S^3} \text{Tr}((g^{-1}dg)^2)$. The tunnelling between the degenerate vacua is of course described by instantons, easily obtained from the well-known instanton solutions on $\mathbb{R}^4$ by the conformal transformation that relates $S^3 \times \mathbb{R}$ to $\mathbb{R}^4$. To be precise, the 18 modes contain the tunnelling paths that describe the transition over the lowest barrier separating the nearest-neighbour vacua. This particular saddle point of the energy functional is also known as a sphaleron.

One way of describing these modes is by using [2] the 't Hooft $\eta$ symbols (useful in giving an explicit expression for the instanton vector potentials), to define a framing of the three-sphere. It is easy to specify the $\eta$ symbol in terms of a basis (and its dual) of the unit quaternions $\sigma_\mu$ ($\bar{\sigma}_\mu$) with $\sigma_4 = \bar{\sigma}_4 = 1$ and $\sigma_\alpha = -\bar{\sigma}_\alpha = i\tau_\alpha$. One has $2i\eta_{\mu\nu}^a \tau_\alpha = \sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu$ and if we parametrize the three-sphere by the unit vectors $n_\mu$ in four dimensions, one can define a dreibein by $e^i_\mu = n^i_\mu n_\nu$. The 18 modes now split in two categories: there are 9 modes described by constant
components \( A_i^a(n) = e_i^a \) (it is essential that the index \( i \) refers to a flat index with respect to the above defined framing), whereas the other 9 modes are constant up to a coordinate-dependent rotation, \( V_i^b(n) = \frac{1}{2} \text{Tr}(n^\mu \sigma_\mu n^\nu \tilde{\sigma}_\nu \sigma_b) \), of the dreibein, i.e. \( A_i^a(n) = -V_i^b(n) d_b^a \). One easily verifies that both vector potentials are transverse. It is furthermore not too difficult to show that (minus) the rotated frame is precisely the one obtained by replacing \( \eta \) by \( \tilde{\eta} \) (defined by \( 2i \tilde{\eta}^a_{\mu \nu} \tau_a = \tilde{\sigma}_\mu \sigma_\nu - \tilde{\sigma}_\nu \sigma_\mu \)), which is used to express the anti-instantons. The instanton is represented by \( c_i^a = -2 \delta_i^a / (1 + e^{-2t}) \) and \( A_0 = 0 \), with an identical expression for the anti-instanton in terms of \( d_i^a \).

In these proceedings we will restrict ourselves to the two-dimensional cross-section of the field space corresponding to the direction of these particular instanton and anti-instanton configurations which, since they describe tunnelling through the sphaleron (the lowest barrier separating two nearest-neighbour vacua), were called sphaleron modes [2]. Thus we will consider \( A_i^a(n) = v V_i^b(n) - u \delta_i^a \).

The classical vacua in the \((u, v)\) plane are located at \((0, 0)\), \((0, 2)\) and \((2, 0)\), whereas the sphalerons can be found at \((0, 1)\) and \((1, 0)\). Furthermore, applying the gauge transformation \( g(n) = n^\mu \tilde{\sigma}_\mu \) (which has winding number \( n(g) = -1 \)) to the configuration \((u, 0)\) can be shown [2] to yield \((0, 2 - u)\). As it should, this maps the vacua \((2, 0)\) and \((0, 0)\) (which have respectively \( Q(A) = 1 \) and \( Q(A) = 0 \)) to the vacua \((0, 0)\) and \((0, 2)\) (where \((0, 2)\) has \( Q(A) = -1 \)). Furthermore it maps the sphaleron at \((1, 0)\) (with \( Q(A) = \frac{1}{2} \)) to the sphaleron at \((0, 1)\) (with \( Q(A) = -\frac{1}{2} \)).

3. The Gribov horizon

To diagonalize the Faddeev-Popov operator \( FP(A) \) in the subspace \((u, v)\) it is convenient to introduce angular momentum operators [2] \( L_1^a = \frac{1}{2} e_\mu^a \partial_\mu = -\frac{1}{2} \eta_{\mu \nu} n^\mu \partial_\nu \), \( L_2^a = -i \frac{1}{2} \tilde{\eta}^a_{\mu \nu} n^\mu \partial_\nu \), \( T^a = \frac{1}{2} \text{ad}(\tau_a) \), and \( \vec{J} = \vec{L}_1 + \vec{L}_2 + \vec{T} \). It is not too difficult to check that \( FP(u, v) \) commutes both with \( \vec{L}_1^2 = \vec{L}_2^2 \) and with \( \vec{J} \). For this it is convenient to write \( FP(u, v) = 4 \vec{L}_1^2 + 2 u \vec{T} \cdot \vec{L}_1 - v \text{ad}(\tilde{\sigma}_\mu n^\mu \tilde{\sigma}_\nu n^\nu \cdot \vec{L}_1) \); only the last term in this expression requires some care in computing the commutators with \( \vec{J} \) and \( \vec{L}_1^2 \).

Thus \( FP(u, v) \) can be diagonalized in the subspace defined by \( \ell \neq 0, j \) and \( j_z \). There is an obvious degeneracy in \( j_z \) and the eigenvalues will be denoted by \( \lambda_{2 \ell + j}(u, v) \). Note that the Coulomb gauge does not fix the constant gauge transformations, which means that the constant modes (\( \ell = 0 \)) are eliminated from the spectrum of \( FP(A) \).

The remaining invariance under constant gauge transformations is easily taken into account without further gauge fixing. As the Gribov horizon is defined as the set of configurations where the lowest eigenvalue of \( FP(A) \) vanishes, it suffices to diagonalize \( FP(u, v) \) for \( \ell = \frac{1}{2} \). There are 12 eigenfunctions in this sector that split in one \( j = 0 \) singlet \((n_0 \tau_a)\), two \( j = 1 \) triplets \((n_0 \tau_a \text{ and } \varepsilon_{abc} n_b \tau_c)\) and one \( j = 2 \) quintet \((n_0 \tau_a + n_b \tau_a - \frac{2}{3} \varepsilon_{abc} n_c \tau_c \delta_{ab})\). For the singlet and quintet the problem of diagonalizing \( FP(u, v) \) becomes one-dimensional, whereas for the two triplets one has to diagonalize a \( 2 \times 2 \) matrix. One easily checks that \( \lambda_{1,0} = 3 - 2s \), \( \prod \lambda_{1,1} = (9 - 3s - 2p^2)^3 \) and \( \lambda_{1,2} = 3 + s \) (for convenience we introduced the scalar and pseudoscalar (even and odd) helicity combinations \( s \equiv u + v \) and \( p \equiv u - v \)). In fig. 1 the full curves give the solutions of \( \lambda_{1,j}(u, v) = 0 \), whereas the Gribov horizon is indicated by fat
sections. The result is in accordance with the convexity of $\Omega$. This convexity is a simple consequence of the linear dependence of $FP(A)$ on the vector potential. Note that the sphaleron configurations are well within the Gribov region, but that the vacua nearest to $A = 0$ are outside (but not on) the Gribov horizon. Nevertheless, it is easily proved [6] that at these vacua the Faddeev-Popov determinant has to vanish. It turns out that the relevant eigenvalues (there are actually 3 of them) that vanish at the vacua nearest to $A = 0$ have $\ell = 1$. So it is instructive to diagonalize $FP(u,v)$ in this sector too. One finds one $j = 0$ singlet, three $j = 1$ triplets, two $j = 2$ quintets and one $j = 3$ septet. We thus have to diagonalize at most a $3 \times 3$ matrix to obtain the eigenvalues for $FP(u,v)$, with the following result

$$\lambda_{2,0} = 8 - 2s, \quad \Pi \lambda_{2,1} = 512((8 - 2s)^2 - s^2 + (2s - 7)p^2)^3, \quad \Pi \lambda_{2,2} = (64 - s^2 - 3p^2)^5$$

and $\lambda_{2,3} = 8 + 2s$. This allows us to determine the location of the zero’s for the Faddeev-Popov determinant in this sector, indicated in fig. 1 by the dashed curves.

Figure 1: Location of the classical vacua (large dots), sphalerons (smaller dots), zeros of the Faddeev-Popov determinant (full curves at $\ell = \frac{1}{2}$, dashed curves at $\ell = 1$), the Gribov horizon (fat sections) and part of the boundary of the fundamental domain (dotted curves) in the plane specified by $A_i = vV_i(n) - u\delta_i$. 

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4. The boundary of the fundamental modular domain

To construct the boundary of the fundamental modular domain requires one to find transverse gauge copies that are degenerate in norm. This is unfortunately a rather non-local problem. We can, however, make use of the fact that the two sphalerons are transverse, have the same norm and are gauge copies of each other, as we discussed above. We can therefore start from \((0, 1)\) and construct \((u, r(u))\), expanding in powers of \(u\), such that the gauge transformation of \(A(u, r(u))\) (coinciding with the sphaleron configuration at \((1, 0)\)) is again transverse and has the same norm (note that for \(u \neq 0\), the gauge transformed potential is not part of the \((u, v)\) plane).

Obviously, also \((r(v), v)\) is part of the boundary of the fundamental modular domain. These two branches are indicated by the dotted curves in figure 1. To find the desired gauge transformation one writes 
\[ g(n) = n^\mu \sigma_\mu \exp(X(n)) \]
and one minimizes 
\[ M(u, v) \equiv |g(n)A(u, v)g(n)^{-1} + g(n)\tilde{g}(n)^{-1}|^2 - ||\tilde{A}(u, v)||^2 \]
with respect to \(X\). It can be verified a posteriori (by checking transversality of the gauge transformed vector potential) that the solution is of the form 
\[ X(n) = if(n)\vec{n} \cdot \vec{r}. \]
Using this as an ansatz, reduces the problem of finding the stationary solution to solving an ordinary 2nd order differential equation. Its solution can be written as
\[ f(x) = x \sum_{j=1}^{\infty} a_j,k(x)u^j x^k. \]

One finds:
\[ a_{1,0}(v) = 2(2 + v)^{-1}, \quad a_{2,0} = -2(v^2 + 6v - 16)(2 + v)^{-3}(10 + v)^{-1}, \]
\[ a_{2,1}(v) = 4(6 + v)(v + 2)^{-2}(v + 10)^{-1}, \] etc. By solving \(M(u, r(u)) = 0\) we find
\[ r(u) = 1 - \frac{1}{9}u^2 - \frac{2}{81}u^3 - \frac{25}{2673}u^4 - \frac{1238}{264627}u^5 - \frac{172442}{66950631}u^6 - \frac{687429956}{457339760361}u^7 + O(u^8). \]

The result is exhibited in figure 1 by the dotted curves. Similar to the Gribov region, the fundamental modular domain is convex and its boundary has “corners”, where different branches of copies that are degenerate in norm intersect. Usually this will keep \(\partial \Lambda\) from touching the Gribov horizon [6].

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