FAST AND SLOW VELOCITY ALIGNMENTS IN A CUCKER-SMALE ENSEMBLE WITH ADAPTIVE COUPLINGS

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Abstract. We study the emergent dynamics of the Cucker-Smale (C-S for brevity) ensemble under adaptive couplings. For the adaptive couplings, we basically consider two types of couplings: Hebbian vs. anti-Hebbian. When the Hebbian rule is employed, we present sufficient conditions leading to the mono-cluster flocking using the Lyapunov functional approach. On the other hand, for the anti-Hebbian rule, the possibility of mono-cluster flocking mainly depends on the integrability of the communication weight function and the regularity of the adaptive law at the origin. In addition, we perform numerical experiments and compare them with our analytic results.

1. Introduction. Collective phenomena are ubiquitous in biological and physical systems, e.g., flashing of fireflies [4, 18], flocking of fish [15], neuronal synchronization in a brain [14], arrays of Josephson junctions [34] and application to the control of unmanned aerial vehicles (UAVs) such as drones or satellites [10, 41], etc. See [1, 16, 42] for brief surveys. Among many phenomenological models, our main interests lie in the Cucker-Smale model [13] which is a continuous dynamical system for position and velocity variables:

$$\dot{x}_i = v_i, \quad t > 0, \quad i = 1, \cdots, N,$$

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\[ \dot{v}_i = \frac{\kappa}{N} \sum_{j=1}^{N} \psi(\|x_i - x_j\|)(v_j - v_i), \quad (1.1) \]

where \(\| \cdot \|\) denotes the \(\ell^2\)-norm in \(\mathbb{R}^d\), \(\kappa\) and \(\psi = \psi(s)\) measure the coupling strength (gain) and communication weight between particles, respectively, and the latter satisfies the following structural properties:

\[ \psi \in \text{Lip}(\mathbb{R}^+; \mathbb{R}^+), \quad 0 \leq \psi(\cdot) \leq 1 \quad \text{and} \quad (\psi(r_1) - \psi(r_2))(r_1 - r_2) \leq 0, \quad \forall r_1, r_2 \geq 0. \]

Note that the C-S model (1.1) has been extensively studied from various perspectives, to name a few, stochastic C-S models [2, 6, 17, 21], interplay of time-delayed interaction [8, 11], rigorous mean-field limit and its kinetic model [5, 29], thermomechanical extension [28], etc. We refer the reader to [7] for a review paper. In the original C-S model (1.1), the coupling strength between particles is assumed to be mean-field and uniform over all interacting pairs. However, this mean-field and uniform assumption for the coupling strength is too restrictive to describe complex real-world phenomena. In order to relax these simplifying assumptions, we introduce plastic adaptiveness in the coupling strength depending on velocity configurations. Here, the plasticity is the terminology that has been used in neuroscience to illustrate the fact that interaction coupling between neurons is affected to change with respect to the state of the neurons. Of course, neuron plasticity has not been fully understood; however, it is known from [30] that neurons tend to follow a Hebbian rule in which the dynamics of couplings between neuron cells increase when they fire simultaneously. See also [36, 46] for an application to spike-timing-dependent plasticity as a fundamental mechanism for learning and memory in neuronal systems. Such neuronal plasticity is realized through the classical Kuramoto model [32, 33] together with the Hebbian dynamics. We refer the reader to [3, 19, 35, 40] for the Kuramoto model with various Hebbian dynamics which describes the behavior of a model neuron (or phase oscillator).

In this paper as a smooth extension, we adopt the scenario above into our system in order to apply the plasticity (or adaptiveness) of velocity configuration and analyze the role of the plasticity in view of asymptotic behavior. In [20], the authors considered nonlinear velocity couplings and studied how a velocity coupling can affect in the flocking analysis. However, it seems that there would be a lack of the justification for the reason why such velocity couplings have to be considered, whereas our approach can explain the relation by a Hebbian learning (see Remark 2.5 for the difference between two models). For the modeling point of view, depending on the behavior of the coupling strength for small relative velocity regime, there are two types of adaptive rules “Hebbian and anti-Hebbian rules” (see Definition 1.1). In this manner, the adaptive dynamics for coupling strength has been studied for first-order models for collective dynamics [3, 12, 22, 27, 31, 39] (see Section 2.4 for a brief literature review). Surprisingly, adaptive dynamics for time-varying coupling strengths has not been addressed for second-order models such as the C-S model.

Our purpose of this paper is to study asymptotic dynamics of the C-S model under adaptive coupling laws. To fix the idea, we set \(\kappa_{ij} = \kappa_{ij}(t)\) to be the coupling strength between \(i\)-th and \(j\)-th particles, and assume that \(\kappa_{ij}\) changes over time via a feedback mechanism due to the relative velocities. In this setting, the temporal dynamics of position-velocity variables is governed by the Cauchy problem to the
C-S model with adaptive couplings:

\[
\dot{x}_i = v_i, \quad t > 0, \quad 1 \leq i, j \leq N, \\
\dot{v}_i = \frac{1}{N} \sum_{j=1}^{N} \kappa_{ij} \psi(\|x_i - x_j\|)(v_j - v_i), \\
\dot{\kappa}_{ij} = \mu \Gamma(\|v_i - v_j\|) - \gamma \kappa_{ij}, \quad (1.2)
\]

\[
(x_i(0), v_i(0), \kappa_{ij}(0)) = (x_i^0, v_i^0, \kappa_{ij}^0).
\]

Here, \( \Gamma = \Gamma(s) \) is a continuous function of relative velocities, and it is called an adaptive rule or feedback law. The positive constants \( \mu \) and \( \gamma \) represent a learning enhancement rate and a friction constant, respectively. For a global well-posedness of (1.2), we refer the reader to the second assertion in Remark 2.3.

In this work, we consider two adaptive rules (Hebbian-type vs. anti-Hebbian-type) depending on the behavior at small relative velocities regime \( \|v_i - v_j\| \ll 1 \) (see Section 2.2 for details). The Hebbian-type rule implies stronger interactions as their velocity differences become smaller. In contrast, for the anti-Hebbian-type, the interaction becomes weaker when the velocity differences are smaller. Note that the Hebbian rule in neuroscience [30] was proposed in order to model adaptation of neurons in our brain, while they undergo learning process, and this rule essentially describes that the states of particles can influence their mutual interactions. This is why we call the interaction is adaptive (or state-dependent). In fact, several types of learning process have been studied in neuroscience, for instance, \( \Gamma(s) = |\sin s| \) in [43, 45] and \( \Gamma(s) = \cos s \) in [37, 44].

For an adaptive law \( \Gamma = \Gamma(s) \), let \( \omega_* \) be the smallest positive zero of \( \Gamma(s) \) if it exists:

\[
\omega_* := \inf \{ s \in [0, \infty) : \Gamma(s) = 0 \}.
\]

Then, we introduce two types of adaptive laws in the following definition.

**Definition 1.1.** (i) An adaptive law \( \Gamma(s) \) is called “Hebbian” if the following conditions hold:

\[
\Gamma(0) > 0 \quad \text{and} \quad \Gamma(\cdot) \text{ is decreasing on } [0, \omega_*]. \quad (1.3)
\]

(ii) An adaptive law \( \Gamma(s) \) is called “anti-Hebbian” if the following conditions hold:

\[
\Gamma(0) = 0, \quad \Gamma(s) > 0 \quad \text{if } s > 0, \quad \text{and} \quad \Gamma(\cdot) \text{ is increasing.} \quad (1.4)
\]

**Remark 1.2.** The condition \( \Gamma(0) = 0 \) in (1.4) is not necessary to make the coupling weaker when the states are close. However, the asymptotic dynamics essentially depends of the value and the slope of the function at origin. If \( \Gamma(0) > 0 \), the asymptotic behavior becomes similar to the Hebbian case and we can use similar estimates for it. Thus, we assume \( \Gamma(0) = 0 \) to contrast two cases.

Now, we consider a quasi-static regime for the coupling strengths in which \( \dot{\kappa}_{ij} \approx 0 \). In this regime, the coupling strength \( \kappa_{ij} \) can be approximated by

\[
\kappa_{ij} \approx \frac{\mu \Gamma(\|v_i - v_j\|)}{\gamma}. \quad (1.5)
\]

Then, we substitute the ansatz (1.5) into (1.2) to derive a C-S model with nonlinear velocity couplings:

\[
\dot{x}_i = v_i, \quad t > 0, \quad i = 1, \cdots, N,
\]
\[ \dot{v}_i = \frac{\mu}{\gamma N} \sum_{j=1}^{N} \psi(\|x_i - x_j\|) \Gamma(\|v_i - v_j\|)(v_j - v_i). \] (1.6)

This asymptotic model (1.6) exactly coincides with the C-S model with nonlinear velocity couplings [20] (see Remark 3.9 for details on the comparison between our model and the model in [20]). Of course, this asymptotic equivalence between (1.2) and (1.6) are only heuristic so that the full adaptive dynamics (1.2) might exhibit more complex and diverse asymptotic dynamics compared to (1.6). This is the main issue to be discussed in this work.

The main results of this work are three-fold. First, we apply the adaptiveness to the second-order C-S model for position configurations. In literature, the adaptive dynamics has been employed only to the first-order system such as the Kuramoto model and a swarm sphere model. Second, we provide diverse profiles toward a flocking state according the choices of \( \psi \) and \( \Gamma \) (see Table 1 in Section 3 for details). Third, we obtain the detailed convergence rates which would depend on the system functions. For instance, we attain fast decay to the flocking regime for an integrable \( \psi \) and Hebbian \( \Gamma \); in contrast, slow decay of the velocity difference can also arise for a non-integrable \( \psi \) and \( \Gamma \) whose form is given to be the relation (1.7) with \( \eta \geq 1 \).

Below, we briefly discuss our main results. First, we consider the Hebbian rule \( \Gamma \) with the property:

\[ \Gamma(0) > 0. \]

In this case, we will have fast velocity alignment of (1.2) for some class of initial data and positive initial coupling strengths (see Theorem 3.1). This can be easily understood using the asymptotic model (1.6). In a close-to-velocity alignment regime where \( \max_{i,j} \|v_i - v_j\| \ll 1 \), the velocity dynamics in (1.6) behaves like

\[ \dot{v}_i \approx \frac{\mu \Gamma(0)}{\gamma N} \sum_{j=1}^{N} \psi(\|x_i - x_j\|) \|v_i - v_j\|^\eta (v_j - v_i), \] (1.7)

which is exactly the same as the velocity dynamics in the C-S model (1.1) with \( \kappa = \frac{\mu \Gamma(0)}{\gamma} \). Hence, it is reasonable to guess the fast exponential velocity alignment. In fact, this heuristic argument really works for the original system (1.2). Second, we employ the anti-Hebbian rule with \( \Gamma(0) = 0 \). To fix the idea, we suppose that \( \Gamma \) is an algebraic function. Since the asymptotic dynamics, especially the convergence and decay rate, crucially depends on the slope of the function at the origin, the lowest-order term of the function matters. Thus, by the comparison principle, it suffices to choose the monomial ansatz for \( \Gamma \):

\[ \Gamma(s) = s^\eta, \quad \eta > 0. \] (1.8)

To see the plausible asymptotic dynamics of (1.2), we consider the asymptotic model (1.6) with (1.7):

\[ \dot{x}_i = v_i, \quad t > 0, \quad i = 1, \ldots, N, \]

\[ \dot{v}_i = \frac{\mu}{\gamma N} \sum_{j=1}^{N} \psi(\|x_i - x_j\|) \|v_i - v_j\|^\eta (v_j - v_i). \] (1.9)

Thus, our flocking estimates will depend on the far-field behavior of \( \psi \) at \( s = \infty \) and size of exponent \( \eta \):

(short-ranged interaction) : \( \int_0^\infty \psi(s)ds < \infty \) or
(long-ranged interaction): \[ \int_0^\infty \psi(s)ds = \infty; \quad \eta \geq 1 \text{ or } 0 < \eta < 1. \quad (1.9) \]

For a short-ranged \( \psi \) with \( \| \psi \|_{L^1(\mathbb{R}^+)} < \infty \), velocity alignment may not occur. This can be easily seen for a two-particle system (see Theorem 3.3), whereas the velocity alignment always occurs for a long-ranged communication weight. However, even for the two-particle system, group formation does not occur for \( \eta \geq 1 \) (see Theorem 3.5), but it does occur for \( 0 < \eta < 1 \) (see Theorem 3.7). Finally, we show that for a many-particle system, the mono-cluster flocking emerges, when \( \psi \) is long-ranged and \( 0 < \eta < 1 \) (see Theorem 3.8). See Table 1 for a summary of our main results at the beginning of Section 3.

The rest of this paper is organized as follows. In Section 2, we present basic lemmas for later use, and then we briefly review the emergent dynamics of the asymptotic model (1.6) and the previous results on the first-order collective models with adaptive dynamics. In Section 3, we summarize our main results on the velocity alignments. In Section 4, we present the proof of Theorem 3.1 on the emergence of mono-cluster flocking for the Hebbian rule. In Section 5, we provide the proofs of Theorems 3.3, 3.5, 3.7 and 3.8 in several subsections. In Section 6, we conduct several numerical experiments and compare them with our theoretical results. Finally, Section 7 is devoted to a brief summary of our main results and remaining issues for future work.

**Notation:** We set \( X := (x_1, \ldots, x_N) \in \mathbb{R}^{dN}, \ V := (v_1, \ldots, v_N) \in \mathbb{R}^{dN}, \ K := (\kappa_{ij})_{1 \leq i, j \leq N} \in \mathbb{R}^{N \times N}. \)

For the vector \( z = (z^1, \ldots, z^d) \in \mathbb{R}^d \), we use the \( \ell^p \)-norm as \( \| \cdot \|_p: \)

\[ \|z\|_p := \left( \sum_{\ell=1}^d (z^\ell)^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad \| \cdot \| := \| \cdot \|_2 \]

For position and velocity configurations \( X \) and \( V \), we introduce position and velocity diameters as follows:

\[ \mathcal{D}(X) := \max_{1 \leq i, j \leq N} \| x_i - x_j \|, \quad \mathcal{D}(V) := \max_{1 \leq i, j \leq N} \| v_i - v_j \|. \]

2. **Preliminaries.** In this section, we present basic lemmas and elementary estimates to be used later, and review previous results on the collective models with adaptive dynamics.

2.1. **Basic estimates.** We present basic a priori estimates and lemmas to be used later. First, we recall a definition of the mono-cluster flocking as follows.

**Definition 2.1.** Let \( (X, V, K) \) be a global solution to (1.2). Then, we say that system (1.2) exhibits an (asymptotic) mono-cluster flocking if the following relations hold.

1. (Group formation): the position diameter is uniformly bounded in time:

\[ \sup_{0 \leq t < \infty} \mathcal{D}(X(t)) < \infty. \]

2. (Velocity alignment): the velocity diameter converges to zero asymptotically:

\[ \lim_{t \to \infty} \mathcal{D}(V(t)) = 0. \]
Remark 2.2. (1). Note that the mono-cluster flocking roughly means that the whole spatial configuration moves like one big group asymptotically. In contrast, if several clusters are formed asymptotically, we call it the “multi-cluster flocking”. (2). In general, emergence of the mono-cluster flocking to (1.2) may depend on the initial data. If the mono-cluster flocking occurs for any initial configuration, we call it as the “global (or unconditional) flocking”. In contrast, if the mono-cluster flocking occurs for some restricted initial configuration, we call it as the “local (or conditional) flocking”.

For a velocity configuration $V = (v_1, \cdots, v_N)$, we set velocity moments as follows:

$$M_1(V) := \frac{1}{N} \sum_{k=1}^{N} v_k \quad \text{and} \quad M_2(V) := \frac{1}{N} \sum_{k=1}^{N} |v_k|^2.$$ 

Next, we study the basic properties of system (1.2).

**Proposition 1.** Let $(X,V,K)$ be a global solution to (1.2). Then, the following assertions hold.

1. Suppose the initial coupling strengths of $\kappa_{ij}$ are non-negative and symmetric:

$$\kappa_{ij}^0 = \kappa_{ji}^0 > 0, \quad 1 \leq i,j \leq N.$$

Then, the positivity and symmetric property are preserved:

$$\kappa_{ij}(t) = \kappa_{ji}(t) > 0, \quad t \geq 0, \quad 1 \leq i,j \leq N.$$

2. The velocity moments $M_1$ and $M_2$ satisfy

$$\frac{d}{dt} M_1(V) = 0 \quad \text{and} \quad \frac{d}{dt} M_2(V) = -\frac{1}{N^2} \sum_{i,j=1}^{N} \kappa_{ij} \psi(\|x_i - x_j\|)\|v_i - v_j\|^2, \quad t > 0.$$

**Proof.** (1) We directly integrate (1.2) with respect to time $t$ to find

$$\kappa_{ij}(t) = \kappa_{ij}^0 e^{-\gamma t} + \mu \int_{0}^{t} e^{\gamma(s-t)} \Gamma(\|v_i(s) - v_j(s)\|) ds. \quad (2.1)$$

Then, the positivity and symmetric property directly follow from the relation (2.1).

(2) For the first assertion, we use the symmetric property of $\kappa_{ij}$ and a standard symmetry trick $i \leftrightarrow j$ to see

$$\frac{d}{dt} M_1(V) = \frac{1}{N^2} \sum_{i,j=1}^{N} \kappa_{ij} \psi(\|x_i - x_j\|)(v_j - v_i)$$

$$= -\frac{1}{N^2} \sum_{i,j=1}^{N} \kappa_{ij} \psi(\|x_i - x_j\|)(v_j - v_i) = 0.$$

For the second assertion, we take the inner product (1.2) with $2v_i$, sum the resulting relation with respect to all $i$ and divide the final relation by $N$ to obtain

$$\frac{d}{dt} M_2(V) = \frac{2}{N^2} \sum_{i,j=1}^{N} \kappa_{ij} \psi(\|x_i - x_j\|)v_i \cdot (v_j - v_i)$$

$$= -\frac{1}{N^2} \sum_{i,j=1}^{N} \kappa_{ij} \psi(\|x_i - x_j\|)\|v_i - v_j\|^2. \quad \Box$$
Remark 2.3. Below, we discuss two implications of Proposition 1.

(i) Mean-zero position and velocity: Note that
\[
\sum_{i=1}^{N} v_i(t) = \sum_{i=1}^{N} v_i^0 \quad \text{and} \quad \sum_{i=1}^{N} x_i(t) = \sum_{i=1}^{N} x_i^0 + t \left( \sum_{i=1}^{N} v_i(t) \right), \quad t \geq 0.
\]
Thus, if we set the initial averages of position and velocity to be zero:
\[
\frac{1}{N} \sum_{k=1}^{N} x_k^0 = 0 \quad \text{and} \quad \frac{1}{N} \sum_{k=1}^{N} v_k^0 = 0,
\]
then one has
\[
\sum_{k=1}^{N} x_k(t) = 0 \quad \text{and} \quad \sum_{k=1}^{N} v_k(t) = 0, \quad t \geq 0. \tag{2.2}
\]
From this point, throughout the paper, we will assume zero sum conditions (2.2) without loss of generality.

(ii) Unique global solvability of (1.2): Below, we discuss a global unique solvability for system (1.2) with the anti-Hebbian rule \( \Gamma \):
\[
\Gamma(s) = \eta s, \quad \eta > 0,
\]
which is continuous in \([0, \infty)\), but it is not Lipschitz continuous at the origin for \(0 < \eta < 1\).

Global existence: Note that the terms in the right-hand side of (1.2) are continuous and sublinear in state variables \((X, V, K)\). This can be seen as follows. Thanks to the monotonicity of the second velocity moment, the particle velocities are uniformly bounded:
\[
\|v_i(t)\| < \sqrt{\sum_{k=1}^{N} \|v_k(t)\|^2} \leq \sqrt{NM_2(V^0)}, \quad t > 0, \quad i = 1, \ldots, N. \tag{2.3}
\]
Then, (1.2) yields
\[
\kappa_{ij} \leq -\gamma \kappa_{ij} + \mu (2 NM_2(V^0))^\frac{1}{2}, \quad t > 0.
\]
We integrate the differential inequality above to find a uniform positive upper bound for \(\kappa_{ij}\):
\[
\max_{1 \leq i, j \leq N} \kappa_{ij}(t) \leq \max \left\{ \max_{1 \leq i, j \leq N} \kappa_{ij}^0, \frac{\mu (2 NM_2(V^0))^\frac{1}{2}}{\gamma} \right\} =: \Lambda, \quad t > 0. \tag{2.4}
\]
Now, we use (2.3) and (2.4) to see that the nonlinear terms in the right-hand sides of (1.2) are uniformly bounded:
\[
\sup_{0 \leq t < \infty} \|v_i(t)\| = \sup_{0 \leq t < \infty} \left| \frac{1}{N} \sum_{j=1}^{N} \kappa_{ij} \psi(||x_i - x_j||)(v_j - v_i) \right| \leq 2\Lambda \sqrt{NM_2(V^0)},
\]
\[
\sup_{0 \leq t < \infty} |\Gamma(||v_i(t) - v_j(t)||)| \leq \left( 2\sqrt{NM_2(V^0)} \right)^\eta. \tag{2.5}
\]
This implies the vector field generated by (1.2) is continuous and it has at most linear growth in state variables \((X, V, K)\). Thus the standard Cauchy-Lipschitz theory yield a global existence of \(C^1\)-solution on the whole time interval \([0, \infty)\).
Uniqueness: First, it follows from (2.5) that \( \dot{v}_i \) is uniformly bounded, hence \( v_i \) is uniformly continuous on \( \mathbb{R}_+ \). Recall (1.2)_3:

\[
\dot{\kappa}_{ij} = \mu |v_i - v_j|^\eta - \gamma \kappa_{ij} := F(\kappa_{ij}, t), \quad t > 0. \tag{2.6}
\]

Then, the right-hand side of (2.6) is not Lipschitz continuous in \( v_i \) for \( \eta \in (0, 1) \). Hence, we cannot use the Cauchy-Lipschitz theory for the whole system (1.2) to guarantee the uniqueness of the solution. Moreover, it does not satisfy Osgood’s criterion [38] so that the global unique solvability is not that trivial as it is. However, we can still show that system (1.2) allows a unique \( C^1 \) solution. Since \( v_i \) is a uniformly continuous function in \( t \), the function \( F \) is uniformly Lipschitz continuous in \( \kappa_{ij} \) with the (uniform) Lipschitz constant \( L \) and continuous in time \( t \). Thus, we can apply the Cauchy-Lipschitz theorem to show that there exists a unique \( \kappa_{ij} \) to (2.6). More precisely, let \( \hat{\kappa}_{ij} \) and \( \tilde{\kappa}_{ij} \) be two solutions issued from the same initial data \( \kappa_{ij}^0 \). Then, by integrating the governing equation for \( \kappa_{ij} \), one has

\[
|\kappa_{ij}(t) - \hat{\kappa}_{ij}(t)| = \left| \int_0^t (F(\kappa_{ij}, s) - F(\hat{\kappa}_{ij}, s))ds \right| \leq L \int_0^t |\kappa_{ij}(s) - \hat{\kappa}_{ij}(s)|ds.
\]

Hence, the uniqueness for \( \kappa_{ij} \) follows from the Grönwall’s lemma. For such a solution \( \kappa_{ij} \), there exists a unique \( C^1 \)-solution \( v_i \) to (1.2)_2. Thus, we conclude that system (1.2) admits a unique global \( C^1 \)-solution.

Next, we derive the dynamics of the diameters \( D(X) \) and \( D(V) \).

**Proposition 2.** Suppose that the initial data and communication weight function satisfy

\[
\Lambda \psi(0) \leq 1, \tag{2.7}
\]

where \( \Lambda \) is defined in (2.4), and let \((X, V, K)\) be a solution to (1.2). Then, one has

\[
\left| \frac{d}{dt} D(X) \right| \leq D(V) \quad \text{and} \quad \frac{d}{dt} D(V) \leq -\dot{\kappa}(D(X)) D(V), \quad \text{a.e.} \quad t > 0, \tag{2.8}
\]

where \( \dot{\kappa}(t) := \kappa_{i_t,j_t}(t) \) and the index \((i_t, j_t)\) is chosen to satisfy the following relation:

\[
D(V(t)) := \|v_{i_t}(t) - v_{j_t}(t)\|, \quad t \geq 0.
\]

**Proof.** We follow the idea of the proof for Lemma 2.2 in [26]. We first set

\[
\phi_{ij}(t) := \kappa_{ij}(t) \psi(\|x_i(t) - x_j(t)\|), \quad t \geq 0
\]

and define the function \( \Phi_{ij} \) as

\[
\Phi_{ij}(t) := \frac{\phi_{ij}(t)}{N} + \left(1 - \frac{1}{N} \sum_{k=1}^{N} \phi_{ik}(t)\right) \delta_{ij}, \quad t \geq 0,
\]

where \( \delta_{ij} \) is the Kronecker delta, and we used (2.7) to see

\[
\phi_{ij}(t) \leq \left(\max_{1 \leq i, j \leq N} \kappa_{ij}(t)\right) \psi(0) \leq \Lambda \psi(0) \leq 1 \quad \text{so that} \quad 1 - \frac{1}{N} \sum_{k=1}^{N} \phi_{ik}(t) \geq 0.
\]

Then, \( \Phi_{ij} \) enjoys the following properties:

\[
\Phi_{ij} \geq \frac{\phi_{ij}}{N}, \quad \sum_{j=1}^{N} \Phi_{ij} = 1, \quad \sum_{j=1}^{N} \Phi_{ij} (v_j - v_i) = \sum_{j=1}^{N} \frac{\phi_{ij}}{N} (v_j - v_i).
\]
For each time $t \geq 0$, let $(i_t, j_t)$ be a pair of indices so that it gives the extremal value as

$$\mathcal{D}(V(t)) =: \|v_{i_t} - v_{j_t}\|, \quad t \geq 0.$$  

For notational simplicity, we denote the pair $(i_t, j_t)$ as $(i, j)$ by suppressing $t$-dependence, when it is obvious. By mimicking the proof of Lemma 2.2 in [26], we attain

$$\frac{1}{p} \frac{d}{dt} \|v_i - v_j\|^p \leq -\hat{\kappa}\psi(\mathcal{D}(X(t)))\|v_i - v_j\|^p, \quad t \geq 0.$$  

In particular, for $p = 2$, we obtain the desired estimate (2.8)

$$\frac{d}{dt} \mathcal{D}(V) \leq -\hat{\kappa}\psi(\mathcal{D}(X))\mathcal{D}(V), \quad t > 0.$$  

The first estimate (2.8) directly follows from the definition of $\mathcal{D}(X)$ and $\mathcal{D}(V)$.

**Remark 2.4.** (i) The proof of Proposition 2 does not depend on the type of adaptive law so that it can be applied for both Hebbian and anti-Hebbian cases.

(ii) The function $\hat{\kappa}$ is well-defined and almost everywhere differentiable. Furthermore, $\hat{\kappa}$ satisfies the following dynamics:

$$\dot{\hat{\kappa}} = -\gamma\hat{\kappa} + \mu \Gamma(\mathcal{D}(V)), \quad t > 0.$$

### 2.2. The C-S model with a nonlinear velocity coupling

In this part, we review the asymptotic behaviors of a C-S model with nonlinear velocity coupling proposed in [20] which appears as an asymptotic model for (1.2) in a quasi-static regime for coupling strengths.

Recall a C-S model with nonlinear velocity coupling $\mathcal{N}$:

$$\dot{x}_i = v_i, \quad t > 0, \quad i = 1, \ldots, N,$$

$$\dot{v}_i = \frac{\kappa}{N} \sum_{j=1}^{N} \psi(\|x_i - x_j\|)\mathcal{N}(v_j - v_i),$$

(2.9)

where $\mathcal{N}: \mathbb{R}^d \to \mathbb{R}^d$ is continuous and it satisfies the following two conditions: for $u \in \mathbb{R}^d$,

(i) $\mathcal{N}(-u) = -\mathcal{N}(u)$, $\mathcal{N}(u) \cdot u \geq 0$.

(ii) There exits a positive constant $C_0 > 0$ and $\delta \in (1, 3)$ such that

$$\mathcal{N}(u) \cdot u \geq C_0\|u\|^\delta. \quad (2.10)$$

In [20], the authors derived a system of differential inequalities for standard deviations, More precisely, we set

$$x_c := \frac{1}{N} \sum_{i=1}^{N} x_i, \quad v_c := \frac{1}{N} \sum_{i=1}^{N} v_i,$$

$$\sigma_x := \left( \frac{1}{N} \sum_{i=1}^{N} \|x_i - x_c\|^2 \right)^{\frac{1}{2}}, \quad \sigma_v := \left( \frac{1}{N} \sum_{i=1}^{N} \|v_i - v_c\|^2 \right)^{\frac{1}{2}},$$

and derived flocking estimates in three regimes: finite-in-time flocking $\delta \in (1, 2)$, fast flocking $\delta = 2$ and slow flocking $\delta \in (2, 3)$.

Next, we state the flocking estimates in [20] without proofs.
Case A: Finite-time flocking for \( \delta \in (1,2) \). There exists positive constants \( C_1 \) and \( T_s \) depending on system parameters such that
\[
\sup_{t \in [0,T_s]} \sigma_x(t) < \infty, \quad \sigma_v(t) \leq C_1(T_s - t)^{\frac{1}{2}}, \quad \forall \ t \in [0,T_s).
\]

Case B: Fast flocking for \( \delta = 2 \). There exists a positive constant \( C_2 \) such that
\[
\sup_{t \in [0,\infty)} \sigma_x(t) < \infty, \quad \sigma_v(t) \leq C_2 e^{-C_2 t}, \quad \forall \ t \in [0,\infty).
\]

Case C: Slow flocking for \( \delta \in (2,3) \). There exists a positive constant \( C_3 \) such that
\[
\sup_{t \in [0,\infty)} \sigma_x(t) < \infty, \quad \sigma_v(t) \leq C_3(1 + t)^{-\frac{1}{2}}, \quad \forall \ t \in [0,\infty). \tag{2.11}
\]

**Remark 2.5.** In (1.8) which can be obtained from a quasi-static limit of (1.2), if we associate the relation
\[
N(u) = \|u\|^\eta u, \quad u \in \mathbb{R}^d,
\]
one would recover the results in Cases B and C from the C-S model (1.2) with monomial anti-Hebbian rules for \( \eta = 0 \) and \( \eta \in (0,1) \) (see Table 1 in Section 3).

Case A corresponds to the case of \( \eta \in (-1,0) \) which might cause a problem for existence of a global solution due to the singularity at the origin. On the other hand, our approach also can provide a clue for the results of \( \delta \in [3,\infty) \). Since it correspond to the regime \( \eta \in [1,\infty) \), our result (in particular Theorem 3.5) implies that slow velocity alignment and failure of group formation will also occur in (2.9) with \( \delta \in [3,\infty) \). Thus, if one applies the similar technique presented in this paper and relevant literature, one would extend the results in [20] for \( \delta \in [3,\infty) \).

### 2.3. A Lyapunov functional approach.

In this part, we review a key tool in [23] for the flocking estimate using a Lyapunov functional. Let \((X,V)\) be a solution to the following dynamical system on \(\mathbb{R}^{Nd} \times \mathbb{R}^{Nd}\):
\[
\begin{cases}
\frac{dX}{dt} = V, & \frac{dV}{dt} = -L(X)V, \quad t > 0, \\
(X(0),V(0)) = (X^0,V^0),
\end{cases}
\tag{2.12}
\]
where \(L : \mathbb{R}^{Nd} \to \mathbb{R}^{Nd}\) is a linear operator satisfying the following coercivity condition: there exists a positive, non-increasing and measurable function \(\psi = \psi(s)\) such that
\[
\langle L(X)V, V \rangle \geq \psi(\|X\|)\|V\|^2.
\]
Here, \(\|\cdot\|\) is a Euclidean norm in \(\mathbb{R}^{Nd}\). Then, the norms \((\|X\|,\|V\|)\) satisfy the system of dissipative inequalities:
\[
\frac{d\|X\|}{dt} \leq \|V\| \quad \text{and} \quad \frac{d\|V\|}{dt} \leq -\psi(\|X\|)\|V\|, \quad \text{a.e.} \quad t > 0.
\]

We set Lyapunov functionals \(E_{\pm}\) as follows:
\[
E_{\pm}(\|X\|,\|V\|) := \|V\| \pm \Psi(\|X\|), \quad \Psi'(s) = \psi(s), \quad s > 0.
\]

Then, it is easy to check that \(E_{\pm}'(t) \leq 0\) and this implies the stability estimate:
\[
\|V(t)\| + \int_{\|X^0\|}^{\|X(t)\|} \psi(s)ds \leq \|V^0\|, \quad t \geq 0. \tag{2.13}
\]
The stability relation (2.13) yields the following theorem.
Theorem 2.6 ([23]). Let \((X, V)\) be a global solution to (2.12). Then, the following two assertions hold.

1. If the initial data satisfy
\[
\|V^0\| < \int_0^\|X^0\| \psi(s)ds,
\]
then there exists a positive constant \(x_m > 0\) such that
\[
\|X(t)\| \geq x_m, \quad \|X^0\| < \int_{x_m}^\|X^0\| \psi(s)ds, \quad t \geq 0.
\]

2. If the initial data satisfy
\[
\|V^0\| < \int_\|X^0\|^\infty \psi(s)ds,
\]
then there exists a positive constant \(x_M > 0\) such that
\[
\|X(t)\| \leq x_M, \quad \|V(t)\| \leq \|V^0\| e^{-\psi(x_M)t}, \quad t \geq 0,
\]
where the upper bound \(x_M\) is defined by the following implicit relation:
\[
\|V^0\| = \int_{x_M}^\|X^0\| \psi(s)ds.
\]

2.4. Particle models with adaptive couplings. In this part, we briefly review two first-order models (the Kuramoto model and a swarm sphere model) with adaptive couplings in [24, 27].

2.4.1. The Kuramoto model. Consider the Kuramoto model with adaptive couplings [27]:
\[
\dot{\theta}_i = \omega_i + \sum_{j=1}^N k_{ij} \sin(\theta_j - \theta_i), \quad t > 0,
\]
\[
\dot{k}_{ij} = \mu \Gamma(\theta_j - \theta_i) - \gamma k_{ij}, \quad 1 \leq i, j \leq N. \quad (2.14)
\]

In [27], two explicit adaptive laws for the plasticity function \(\Gamma(s)\) were considered:
\[
\Gamma(s) = \cos(s) \quad \text{and} \quad \Gamma(s) = |\sin(s)|.
\]

For the former case, they can obtain the exponential decay of the diameter. More precisely, there exist a finite entrance time \(t_1 > 0\) toward the exponential decay regime and two positive constants \(c_1, c_2 > 0\) such that
\[
e^{-c_1(t-t_1)} \leq D(\Theta(t)) := \max_{1 \leq i, j \leq N} |\theta_i(t) - \theta_j(t)| \leq e^{-c_2(t-t_1)}, \quad t \geq t_1.
\]

Here, the estimates crucially depend on the existence of a positive lower bound for \(\Gamma(s) = \cos s\). In contrast, for the latter case \(\Gamma(s) = |\sin s|\), (2.14) also exhibits the complete synchronization, but we cannot expect the exponential decay of the phase diameter. In particular, for the two-oscillator system, they can obtain a slow decay \(O(1/(t+1))\). To be more precise, there exist a finite entrance time \(t_2 > 0\) toward slow decay regime and positive constants \(d_1, d_2 > 0\) such that
\[
\frac{d_1}{t+1} \leq |\theta_1(t) - \theta_2(t)| \leq \frac{d_2}{t+1}, \quad t \geq t_2.
\]

In other words, the complete (phase) synchronization occurs algebraically slow.
2.4.2. A swarm sphere model. In [24], the authors studied a swarm sphere model with adaptive couplings:

\[
\dot{x}_i = \Omega_i \frac{x_i}{|x_i|^2} + \frac{1}{N} \sum_{j=1}^{N} \kappa_{ij} \left( x_j - \frac{(x_i, x_j)}{(x_i, x_i)} x_i \right), \quad t > 0,
\]

\[
\dot{\kappa}_{ij} = \mu \Gamma(x_i - x_j) - \gamma \kappa_{ij}, \quad 1 \leq i, j \leq N.
\]

Similarly, both Hebbian and anti-Hebbian rules were employed to establish the complete aggregation estimate using a two-point correlation function and a Lyapunov functional approaches. First, when the Hebbian rule admits a positive lower bound as the cosine function, they showed that the complete aggregation can occur. Furthermore, when the adaptive law allows negative values so that it can attain both positive and negative values, the bi-polar aggregation can emerge from some well-prepared initial data. On the other hand, for the anti-Hebbian rules, they constructed a Lyapunov functional

\[
L_{ij}(t) := L_{ij}(X(t), K(t)) = \frac{1}{2} \|x_i - x_j\|^2 + \frac{1}{4\mu N} \sum_{k=1}^{N} |\kappa_{ik} - \kappa_{jk}|^2, \quad i, j = 1, \ldots, N,
\]

to establish the complete aggregation. In this case, they cannot obtain the explicit decay rate, as in the anti-Hebbian rule for the Kuramoto model.

3. Presentation of main results. In this section, we present our main results without detailed proofs, and their proofs will be discussed in the following two sections.

Below, we discuss the detailed results in the following table.

### Table 1. Main results

| \(\Gamma\) | \(\psi\) | Asymptotic behavior | Corresponding result |
|----------|---------|----------------------|---------------------|
| **Hebbian** | \(\Gamma(0) > 0\) | Short-ranged | Conditional flocking | Theorem 3.1 |
| | | Long-ranged | Unconditional flocking | Figure 1 |

| **Anti-Hebbian** | \(\Gamma(s) = s^\eta\) | Short-ranged | No alignment | Theorem 3.3 |
| | | | | Figure 2 |
| | | Long-ranged | \(\eta \geq 1\) | Slow velocity alignment | Theorem 3.5 |
| | | | | Figure 3 |
| | | | \(0 < \eta < 1\) | Unconditional flocking | Theorems 3.7 and 3.8 |
| | | | | Figure 4 |

3.1. The Hebbian rule. In this subsection, we consider the Hebbian rule and present sufficient conditions leading to the mono-cluster flocking. As mentioned in Introduction, for Hebbian coupling rule, the asymptotic dynamics of system (1.2) can be given by the Cucker-Smale model heuristically:

\[
\dot{x}_i = v_i, \quad t > 0, \quad i = 1, \ldots, N,
\]

\[
\dot{v}_i = \frac{\mu \Gamma(0)}{\gamma N} \sum_{j=1}^{N} \psi(||x_i - x_j||)(v_j - v_i).
\]
Then, we can apply the Lyapunov functional approach in Section 2.3 to derive the following mono-cluster flocking estimate. Of course, this heuristic argument works for the original system (1.2). For a given initial data \((X^0, V^0, K^0)\), we set
\[
\kappa_m := \min \left\{ \min_{1 \leq i, j \leq N} \kappa_{ij}^0, \frac{\mu \Gamma(\sqrt{2N M_2(V^0)})}{\gamma} \right\},
\]
\[
\Lambda := \max \left\{ \max_{1 \leq i, j \leq N} \kappa_{ij}^0, \frac{\mu (2N M_2(V^0))^{\frac{1}{2}}}{\gamma} \right\}.
\]
Then, we are ready to state our first main result.

**Theorem 3.1.** Suppose that the initial data and initial coupling strengths satisfy
\[
\Lambda \psi(0) < 1, \quad D(V^0) < \kappa_m \int_0^\infty \psi(s) ds,
\]
and let \((X, V, K)\) be a global solution to (1.2). Then, there exists a positive constant \(x_M > 0\) such that
\[
\sup_{0 \leq t < \infty} D(X(t)) \leq x_M \quad \text{and} \quad D(V(t)) < D(V^0) e^{-\kappa_m \psi(x_M)t}, \quad \forall \ t \geq 0.
\]

**Proof.** The proof will follow from the following three steps.

Step 1: We use the uniform bound of total energy to derive a positive lower bound for \(\kappa_{ij}\):
\[
\exists \ \kappa_m > 0 \quad \text{such that} \quad \inf_{0 \leq t < \infty} \kappa_{ij}(t) \geq \kappa_m > 0.
\]

Step 2: With the positive lower bound \(\kappa_m\), we derive the system of dissipative differential inequalities (SDDI) for diameters:
\[
\left| \frac{dD(X)}{dt} \right| \leq D(V) \quad \text{and} \quad \frac{dD(V)}{dt} \leq -\kappa_m \psi(D(X)) D(V), \quad \text{a.e.} \ t > 0.
\]

Step 3: We perform a Lyapunov functional approach in Theorem 2.6 to derive our flocking estimate. The detailed justification of the steps above will be given in Section 4.

**Remark 3.2.** (i) If the communication weight function is long-ranged (or it is non-integrable), then one has
\[
\int_c^\infty \psi(s) ds = \infty
\]
for any finite \(c \geq 0\). Thus, any bounded initial data satisfy the second condition in (3.1). Hence, we have the unconditional flocking with a long-ranged interaction.

(ii) In general, it is a quite challenging question to find a necessary and sufficient condition of the emergence of flocking for a many-particle system. In Theorem 3.1, we provide a sufficient conditions leading to flocking. It is worthwhile to mention that the necessary and sufficient condition is presented in [9] for a two-particle system of the classical Cucker-Smale model. In [25], the necessary and sufficient condition for a many-particle Cucker-Smale model has been studied in one spatial dimension.
3.2. The anti-Hebbian rule. We now turn to the anti-Hebbian case, and divide this subsection into two parts: short-ranged and long-ranged interactions. Recall that the adaptive law $\Gamma$ has the following form:

$$\Gamma(s) = s^\eta, \quad \eta > 0.$$  \hspace{1cm} (3.3)

3.2.1. Short-ranged communications. Consider a short-ranged communication weight function $\psi = \psi(s)$:

$$\|\psi\|_{L^1} := \int_0^\infty \psi(s)ds < \infty.$$  \hspace{1cm} (3.4)

Then, we will see that if the communication weight $\psi$ is short-ranged, then velocity alignment fails, no matter what initial data are. This can be easily seen for a two-particle system. To be more specific, we set

$$x := x_1 - x_2, \quad v := v_1 - v_2 \quad \text{and} \quad \kappa := \kappa_{12} = \kappa_{21}.$$  

Then, (1.2) reduces to

$$\begin{cases}
\dot{x} = v, \\
\dot{v} = -\kappa \psi(\|x\|)v, \\
\dot{\kappa} = \mu \Gamma(\|v\|) - \gamma \kappa, \quad t > 0,
\end{cases} \hspace{1cm} (x, v, \kappa)(0) = (x^0, v^0, \kappa^0).$$  

For system (3.5), we show that the velocity difference $\|v\|$ converges to a positive constant.

**Theorem 3.3** (Non-existence of velocity alignment). Suppose that the initial difference of velocities are different, that is, $v^0 \neq 0$, and let $(x, v, \kappa)$ be a global solution to (3.5) with a short-ranged communication weight $\psi$. Then, velocity alignment does not occur, i.e., there exists a positive constant $v_\infty > 0$ such that

$$\lim_{t \to \infty} \|v(t)\| = v_\infty.$$  

**Proof.** Below, we briefly sketch why we cannot expect velocity alignment ($v_\infty = 0$) in three steps.

Step 1: We first show the following equivalence relation:

$$\lim_{t \to \infty} \|v(t)\| = 0 \iff \lim_{t \to \infty} \kappa(t) = 0. \hspace{1cm} (3.6)$$

Step 2: Under the assumption (3.6), we have

$$\lim_{t \to \infty} \frac{\kappa(t)}{\|v(t)\|^\eta} = \frac{\mu}{\gamma}. \hspace{1cm} (3.7)$$

Step 3: Suppose that

$$\lim_{t \to \infty} \|v(t)\| = 0. \hspace{1cm} (3.8)$$

Then, we derive

$$\frac{d}{dt} \|v\| = -\kappa \psi(\|x\|)\|v\| \geq -\frac{2\mu}{\gamma} \psi(\|x\|)\|v\|^{1+\eta}, \quad t \gg 1,$$  

or equivalently,

$$\frac{\|v\|^\prime}{\|v\|^{1+\eta}} \geq -\frac{2\mu}{\gamma} \psi(\|x\|), \quad t \geq \exists T_*.$$
We integrate the relation above to get
\[
\frac{1}{\|v(t)\|^\eta} - \frac{1}{\|v(T_*)\|^\eta} \leq \frac{2\mu}{\gamma} \int_{T_*}^t \psi(\|x(s)\|)ds.
\]
Letting \( t \to \infty \), the right-hand side tends to finite value, however, the left-hand side goes to infinity by (3.8). This gives a contradiction. Hence, we have
\[
\lim_{t \to \infty} \|v(t)\| > 0.
\]
Detailed arguments can be found in Section 5.1.

**Remark 3.4.** Theorem 3.3 is concerned with a two-particle system, and one might expect that this results can be extended to a many-particle system. It is directly linked to derivation of a sharp lower bound estimate of \( D(V) \), since the lower bound estimate of \( \|v\| \) is employed in (3.9). Note that only an upper bound estimate of \( D(V) \) is presented in Proposition 2. However, to the best of the authors’ knowledge, it is in general hard to derive a sharp lower estimate of \( D(V) \). Of course, one might heuristically guess that velocity alignment for a many-body system will fail, since it fails even for a two-body system. This plausible scenario is numerically justified in Figure 2(c) in which \( D(V) \) converges to some positive definite value.

### 3.2.2. Long-ranged communications

Now, we consider a long-ranged communication \( \psi = \psi(s) \):
\[
\|\psi\|_{L^1} = \int_0^\infty \psi(s)ds = \infty. \tag{3.10}
\]

The following two theorems for the two-particle system states that emergence of mono-cluster flocking depends on the regularity of the plasticity function \( \Gamma \) at the origin, i.e., \( \eta \geq 1 \) or \( 0 < \eta < 1 \).

**Theorem 3.5** (Slow velocity alignment). *Suppose that the initial difference of velocities are different, that is,
\[
v^0 \neq 0.
\]
Moreover, we suppose that the plasticity function \( \Gamma \) has the form of
\[
\Gamma(s) = s^\eta, \quad \eta \geq 1,
\]
and let \( (x, v, \kappa) \) be a global solution to (3.5) with a long-ranged communication weight \( \psi \). Then, the following assertions hold:

(i) Velocity alignment holds, but the group formation does not hold:
\[
\lim_{t \to \infty} \|v(t)\| = 0 \quad \text{and} \quad \sup_{0 \leq t < \infty} \|x(t)\| = \infty.
\]

More precisely, there exists a positive constant \( C_1 \) such that \( v \) does not decay faster than the following algebraic rate:
\[
\|v(t)\| \geq \frac{1}{C_1} \frac{1}{(1 + t)^\frac{1}{\eta}}, \quad ast \to \infty. \tag{3.11}
\]

(ii) If the long-ranged communication weight \( \psi \) satisfies the following lower bound condition: there exist positive constants \( d_* \) and \( r_0 \) such that
\[
\psi(r) \geq \frac{d_*}{r}, \quad r > r_0, \tag{3.12}
\]
then, we have
\[ \frac{1}{C_1} \frac{1}{(1 + t)^\frac{1}{d}} \leq \|v(t)\| \leq \frac{C_1}{(1 + \log t)^\frac{1}{d}}, \quad \text{as } t \to \infty. \]

**Proof.** We briefly introduce our strategy of the proof.

**Step 1:** We use the non-integrability of the communication weight \( \psi \) to show that velocity alignment always occur:
\[
\lim_{t \to \infty} \|v(t)\| = 0 \quad \text{and} \quad \lim_{t \to \infty} \kappa(t) = 0.
\]

**Step 2:** It follows from (3.7) and \( \psi(\cdot) \leq 1 \) that there exists a finite time \( T_1 \) such that
\[
-\frac{\mu}{2\gamma} \psi(\|x\|) \|v\|^{1+\eta} \geq \frac{d}{dt} \|v\| = -\kappa \psi(\|x\|) \|v\| \geq -\frac{2\mu}{\gamma} \|v\|^{1+\eta}, \quad \text{a.e. } t > T_1. \quad (3.13)
\]
Then, we integrate the second inequality in (3.13) to obtain
\[
\|v(t)\| \geq \frac{\|v(T_1)\|}{\left(1 + \frac{2\mu \|v(T_1)\|^{1+\eta}}{\gamma} (t - T_1)\right)^\frac{1}{\eta}}, \quad t > T_1.
\]

**Step 3:** For the upper decay rate of \( \|v\| \), we use the ansatz (3.12) for the communication weight \( \psi \) and the non-increasing property of \( \|v\| \) to find
\[
\psi(\|x(T_1)\| + \|v(T_1)\| t) \geq \frac{d^*_\gamma}{\|x(T_1)\| + \|v(T_1)\| t}, \quad t > T_1. \quad (3.14)
\]
We substitute (3.14) into the first inequality in (3.13) and then integrate the resulting relation to obtain the desired estimate. The detailed proof can be found in Section 5.2.1.

**Remark 3.6.** Note that the slow velocity alignment (3.11) implies that the relative position tends to infinity as time goes on. Hence, the mono-cluster flocking does not occur.

**Theorem 3.7** (Emergence of flocking). *Suppose that the initial difference of velocities are different, that is,
\[ v^0 \neq 0. \]
In addition, we suppose that the plasticity function \( \Gamma \) has the form of
\[ \Gamma(s) = s^\eta, \quad 0 < \eta < 1, \]
and let \( (x, v, \kappa) \) be a global solution to (3.5) with a long-ranged communication weight \( \psi \). Then, the following two assertions hold.

(i) The mono-cluster flocking emerges, i.e., there exists a positive constant \( x_M > 0 \) defined in the implicit relation (5.20) such that
\[
\lim_{t \to \infty} \|v(t)\| = 0 \quad \text{and} \quad \sup_{0 \leq t < \infty} \|x(t)\| \leq x_M.
\]

(ii) There exists a positive constant \( C_2 > 1 \) such that \( v \) decays to zero like
\[
\frac{1}{C_2} \frac{1}{(1 + t)^\frac{1}{d}} \leq \|v(t)\| \leq \frac{C_2}{(1 + t)^\frac{1}{d}}, \quad \text{as } t \to \infty. \quad (3.15)
\]
Proof. The proof will take the following three steps.
Step 1: We first show that there exists a positive constant $L_m$ such that
\begin{equation}
\kappa(t) \geq L_m \|v(t)\|^\eta, \quad t > 0.
\end{equation}
(3.16)
Step 2: In (3.9), we use (3.16) to find
\begin{equation}
\frac{d}{dt} \|v\| \leq -L_m \psi(\|x\|) \|v\|^{1+\eta}, \quad t > 0.
\end{equation}
Step 3: We utilize the Lyapunov functional approach in Theorem 2.6 to derive the flocking estimate. We postpone the justification of its steps in Section 5.2.1. □

Theorem 3.7 yields that for the mono-cluster flocking, it suffices to consider the plasticity function $\Gamma$ with $0 < \eta < 1$. Finally, we extend Theorem 3.7 to the many-particle system.

**Theorem 3.8 (Emergence of mono-cluster flocking).** Suppose that $\eta$, the initial data and initial coupling strengths satisfy
\begin{equation}
0 < \eta < 1, \quad \Lambda \psi(0) < 1,
\end{equation}
and let $(X, V, K)$ be a solution to (1.2). Then, the mono-cluster flocking emerges. More precisely, there exist positive constants $x_M$ and $C_3$ such that
\begin{equation}
\sup_{0 \leq t < \infty} D(X(t)) \leq x_M \quad \text{and} \quad D(V(t)) \leq \frac{C_3}{(1 + t)^{\frac{\eta}{2}}}, \quad \text{as} \quad t \to \infty.
\end{equation}

Proof. Its proof follows the same argument in Theorem 3.7.

Step 1: We find a positive constant $\alpha$ satisfying
\begin{equation}
\hat{\kappa}(t) \geq \alpha D(V(t))^\eta, \quad t > 0,
\end{equation}
where $\hat{\kappa}$ is defined in Proposition 2.

Step 2: In (4.7)\textsubscript{2}, we derive the system of dissipative differential inequalities for diameters:
\begin{equation}
\left| \frac{dD(X)}{dt} \right| \leq D(V) \quad \text{and} \quad \frac{dD(V)}{dt} \leq -\alpha \psi(D(X)) D(V)^{1+\eta}, \quad \text{a.e.} \quad t > 0.
\end{equation}

Step 3: We use the Lyapunov functional approach in Theorem 2.6 to show that the mono-cluster flocking emerges. We will verify these steps in Section 5.2.2 □

**Remark 3.9.** (i) For the two-particle system, we show that the decay rate of $\|v\|$ toward zero is exactly $(1 + t)^{-\frac{\eta}{2}}$ in Theorem 3.7. However, for the many-particle system, we can only show that the decay rate of $D(V)$ is at most $(1 + t)^{-\frac{\eta}{2}}$. This is basically due to the lack of the sharp lower estimate of $D(V)$ in Proposition 2. Our numeric results in Section 6 imply that the decay rate of $D(V)$ is close to $(1 + t)^{-\frac{\eta}{2}}$ (see Section 6.4 for details).

(ii) Here, we compare the results of our model and the model in [20]. Recall the approximated system (1.6) with anti-Hebbian ansatz (1.7):
\begin{equation}
\dot{x}_i = v_i, \quad t > 0,
\end{equation}
\begin{equation}
\dot{v}_i = \frac{\mu}{\gamma N} \sum_{j=1}^{N} \psi(\|x_i - x_j\|) |v_j - v_i|^\eta (v_j - v_i) = \frac{\mu}{\gamma N} \sum_{j=1}^{N} \psi(\|x_i - x_j\|) G(v_j - v_i).
\end{equation}
Then, the coupling function $G$ satisfies
\begin{equation}
G(u) = |u|^\eta u \quad \text{so that} \quad G(u) \cdot u = |u|^{2+\eta}, \quad u \in \mathbb{R}^d.
\end{equation}
Then, the exponent $\delta$ appearing in (2.10) and $\eta$ can be associated from the following relation:

$$\eta = \delta - 2.$$  

Thus, this relation yields

$$0 < \eta < 1, \quad 2 < \delta < 3$$

which corresponds to Case C (2.11) in Section 2.2 which says the formation of slow flocking. Hence our result in Theorem 3.8 is consistent with that of [20].

In the following two sections, we will provide the proofs of our main results.

4. **Hebbian interaction rule.** In this section, we study the flocking estimate of the C-S dynamics (1.2) with the Hebbian rule described in (1.3), and present the proof of Theorem 3.1. As a simple motivation, we begin with a two-particle system

$$\dot{x} = v, \quad \dot{v} = -\kappa \psi(\|x\|) v, \quad \dot{\kappa} = \mu \Gamma(\|v\|) - \gamma \kappa, \quad t > 0, \quad (x, v, \kappa)(0) = (x^0, v^0, \kappa^0). \quad (4.1)$$

4.1. **A two-particle system.** We first show that the condition $\Gamma(0) > 0$ yields the existence of a positive lower bound for $\kappa$.

**Lemma 4.1.** Suppose that $\Gamma(s)$ has the form of (1.3), and the initial velocity difference $v^0 = v_1^0 - v_2^0$ is bounded:

$$\|v^0\| < \infty,$$

and let $(x, v, \kappa)$ be a solution to (4.1). Then, there exists a positive constant $\tilde{\kappa}_m > 0$ such that

$$\kappa(t) \geq \tilde{\kappa}_m, \quad t > 0. \quad (4.2)$$

**Proof.** As long as $\kappa(t) \geq 0$, the velocity difference $v$ satisfies

$$\frac{d}{dt} \|v\| = -\kappa \psi(\|x\|) \|v\| \leq 0, \quad t > 0. \quad (4.3)$$

This yields $\|v(t)\| \leq \|v^0\|, \quad t > 0$. Since the plasticity function $\Gamma(s)$ is a decreasing function, we have

$$\Gamma(\|v(t)\|) \geq \Gamma(\|v^0\|), \quad t > 0.$$  

Thus, we obtain an inequality of $\kappa$ from (4.1):

$$\dot{\kappa} \geq -\gamma \kappa + \mu \Gamma(\|v^0\|), \quad t > 0.$$  

We use the method of integration factor to get

$$\kappa(t) \geq \left(\kappa^0 - \frac{\mu}{\gamma} \Gamma(\|v^0\|)\right) e^{-\gamma t} + \frac{\mu}{\gamma} \Gamma(\|v^0\|), \quad t > 0. \quad (4.4)$$

Then, we use (4.4) and the calculus fact

$$\inf_{0 \leq t < \infty} (ae^{-\gamma t} + b) \geq \min\{a + b, b\} \quad \text{for } \gamma > 0$$

to derive

$$\kappa(t) \geq \min \left\{\kappa_0, \frac{\mu}{\gamma} \Gamma(\|v^0\|)\right\} =: \tilde{\kappa}_m > 0. \quad (4.5)$$

Then, the relation (4.5) yields the desired estimate (4.2). \hfill \Box

From the existence of a positive lower bound $\tilde{\kappa}_m$, we can obtain the flocking estimate for the two-particle system as follows.
Theorem 4.2. Suppose that $\Gamma(s)$ has the form of (1.3), and the initial data $(x^0, v^0, \kappa^0)$ satisfy
\[ \|v^0\| < \infty \quad \text{and} \quad \|v^0\| < \tilde{\kappa}_m \int_0^\infty \psi(s)\|x\|ds, \]
where $\tilde{\kappa}_m$ is defined in (4.5), and let $(x, v, \kappa)$ be a solution to (4.1). Then, the two-particle system exhibits the (asymptotic) mono-cluster flocking: there exists a positive number $x_M > 0$ such that
\[ \sup_{0 \leq t < \infty} \|x(t)\| \leq x_M \quad \text{and} \quad \lim_{t \to \infty} \|v(t)\| = 0. \]

Proof. We combine (4.3) and (4.5) to obtain the system of dissipative differential inequalities:
\[ \left| \frac{d}{dt} \|x\| \right| \leq \|v\| \quad \text{and} \quad \frac{d}{dt} \|v\| \leq -\tilde{\kappa}_m \psi(\|x\|) \|v\|, \quad t \geq 0. \]
(4.6)
Once we have differential inequalities (4.6), we can apply Theorem 2.6 to establish the flocking estimate.

As a corollary of Theorem 4.2, we can derive the asymptotic value of $\kappa$.

Corollary 1. Under the same condition in Theorem 4.2, the coupling strength difference $\kappa = \kappa(t)$ converges to some positive definite value:
\[ \lim_{t \to \infty} \kappa(t) = \frac{\mu}{\gamma} \Gamma(0). \]

Proof. Since $\|v\|$ converges to zero from Theorem 3.1 and $\Gamma$ is continuous, we have
\[ \lim_{t \to \infty} \Gamma(\|v(t)\|) = \Gamma(0). \]
Then, we recall the dynamics (4.1) of $\kappa$:
\[ \dot{\kappa} = -\gamma \kappa + \mu \Gamma(\|v\|), \quad t \geq 0 \]
to obtain the desired convergence.

4.2. A many-particle system. In this subsection, we consider the many-particle system and use a similar argument in Section 4.1 to present the proof of Theorem 3.1. First, we show that the coupling strength $\kappa_{ij}$ has a positive lower bound.

Lemma 4.3. Suppose that the initial velocities are bounded
\[ \max_{1 \leq i \leq N} \|v_i^0\| < \infty, \]
and let $(X, V, K)$ be a global solution to (1.2). Then, there exists a positive constant $\kappa_m$ such that
\[ \inf_{0 \leq t < \infty} \kappa_{ij}(t) \geq \kappa_m. \]
(4.7)

Proof. It follows from Proposition 1(ii) that
\[ \sum_{i=1}^N \|v_i(t)\|^2 \leq NM_2(V^0), \quad t > 0. \]
Then, we use a rough estimate to find
\[ \|v_i(t) - v_j(t)\|^2 \leq 2(\|v_i(t)\|^2 + \|v_j(t)\|^2) \leq 2 \sum_{k=1}^N \|v_k(t)\|^2 \leq 2NM_2(V^0), \quad t > 0. \]
Then, since $\Gamma$ is non-increasing, we see that
\[
\Gamma(\|v_i - v_j\|) > \Gamma \left( \sqrt{2NM_2(V_0)} \right).
\] (4.8)
We combine (4.8) and (4.1) to obtain
\[
\dot{\kappa}_{ij} \geq -\gamma \kappa_{ij} + \mu \Gamma \left( \sqrt{2NM_2(V_0)} \right),
\] t > 0.
Then, we use the same argument as in the proof of Lemma 4.1 to see
\[
\kappa_{ij}(t) \geq \min \left\{ \min_{1 \leq i,j \leq N} \kappa_{ij}^0, \frac{\mu \Gamma \left( \sqrt{2NM_2(V_0)} \right)}{\gamma} \right\} =: \kappa_m, \quad \forall \ t \geq 0. \]
Proof of Theorem 3.1. Recall that for $t \geq 0$, $
\dot{\kappa}(t) = \kappa_{i_k,j_k}(t), \n$ where $(i_k,j_k)$ satisfies $\mathcal{D}(V(t)) = \|v_{i_k} - v_{j_k}\|$. Then, it follows from Lemma 4.3 that
\[
\dot{\kappa}(t) \geq \min_{1 \leq i,j \leq N} \kappa_{ij}(t) > \kappa_m, \quad t > 0.
\] This relation and Proposition 2 yield
\[
\mathcal{D}(X(t)) \leq \mathcal{D}(V), \quad \text{a.e.} \ t > 0,
\]
\[
\frac{d}{dt} \mathcal{D}(V) \leq -\dot{\kappa}_\psi(\mathcal{D}(X)) \mathcal{D}(V) \leq -\kappa_m \psi(\mathcal{D}(X)) \mathcal{D}(V).
\] (4.9)
Then, we use Theorem 2.6 to find a positive constant $x_M > 0$ such that
\[
\mathcal{D}(X(t)) \leq x_M, \quad t \geq 0 \quad \text{and} \quad \int_{\mathcal{D}(V)}^{x_M} \psi(s) \, ds = \frac{\mathcal{D}(V_0)}{\kappa_m}.
\] (4.10)
Now, we use (4.9) and (4.10) to yield the exponential decay of $\mathcal{D}(V)$:
\[
\mathcal{D}(V(t)) \leq \mathcal{D}(V_0) e^{-\kappa_m \psi(x_M) t}, \quad t \geq 0.
\] This completes the proof.

As a direct corollary of Theorem 3.1, we obtain the uniform asymptotic value of the adaptive coupling function $\kappa_{ij}$.

Corollary 2. Under the same condition in Theorem 3.1, $\kappa_{ij}$ converges to some positive definite value:
\[
\lim_{t \to \infty} \kappa_{ij}(t) = \frac{\mu}{\gamma} \Gamma(0), \quad 1 \leq i,j \leq N.
\] Proof. We use the same argument in Corollary 1 using (1.2) and the fact that
\[
\lim_{t \to \infty} \mathcal{D}(V(t)) = 0,
\] to derive the desired convergence.

5. Anti-Hebbian interaction rule. In this section, we study the flocking behavior for the C-S model equipped with the anti-Hebbian rule (1.4) depending on the asymptotic behaviors of $\psi = \psi(s)$, and present the proof of Theorems 3.3–3.8.
5.1. **Short-ranged communication weight.** We consider the two-particle system and show that the velocity difference converges to some positive definite value. This means that the mono-cluster flocking cannot occur. Before we present the proof of Theorem 3.3, we observe the equivalence relation on the zero convergences of \( v \) and \( \kappa \) which make the flocking estimate more subtle.

**Lemma 5.1.** Suppose that the plasticity function \( \Gamma \) is given by the anti-Hebbian rule (1.4), and the initial velocity difference is bounded \( \|v^0\| < \infty \), and let \((x,v,\kappa)\) be a global solution to a two-particle system (4.1). Then, the following two assertions hold.

(i) there exist \( \kappa_\infty \) and \( v_\infty \) such that

\[
\kappa_\infty := \lim_{t \to \infty} \kappa(t) \quad \text{and} \quad v_\infty := \lim_{t \to \infty} \|v(t)\|.
\]

(ii) The following equivalence holds:

\[
\lim_{t \to \infty} \|v(t)\| = 0 \iff \lim_{t \to \infty} \kappa(t) = 0.
\]

**Proof.** (i) Let \( \varepsilon > 0 \) be an arbitrary constant. Then, due to the uniform continuity of \( \Gamma(s) \), there exists \( \delta > 0 \) such that

\[
|s_1 - s_2| < \delta \implies |\Gamma(s_1) - \Gamma(s_2)| < \varepsilon.
\]

Then, it follows from (4.3) that \( \|v\| \) decreases in time and bounded below. Hence, there exists a non-negative constant \( v_\infty \) such that

\[
v_\infty := \lim_{t \to \infty} \|v(t)\| \geq 0.
\]

For a given \( \delta > 0 \), there exists a time \( T_\delta > 0 \) such that

\[
\|v(t)\| - v_\infty < \delta, \quad \forall \ t > T_\delta.
\]

By the continuity of \( \Gamma \), we have

\[
|\Gamma(\|v(t)\|) - \Gamma(v_\infty)| < \varepsilon, \quad \forall \ t > T_\delta.
\]  

We combine (5.1) and (4.1) to find

\[
-\gamma \kappa + \mu \Gamma(v_\infty) - \mu \varepsilon \leq \dot{\kappa}(t) \leq -\gamma \kappa + \mu \Gamma(v_\infty) + \mu \varepsilon, \quad t > T_\delta.
\]

This yields

\[
\frac{\mu}{\gamma} (\Gamma(v_\infty) - \varepsilon) \leq \liminf_{t \to \infty} \kappa(t) \leq \limsup_{t \to \infty} \kappa(t) \leq \frac{\mu}{\gamma} (\Gamma(v_\infty) + \varepsilon).
\]

Since \( \varepsilon \) was arbitrary, we obtain

\[
\kappa_\infty := \lim_{t \to \infty} \kappa(t) = \frac{\mu}{\gamma} \Gamma(v_\infty).
\]  

(5.2)

(ii) It follows from (1.4) and (5.2) that \( v_\infty = 0 \) implies

\[
\Gamma(v_\infty) = 0 \quad \text{and} \quad \kappa_\infty = 0.
\]

Conversely, if \( v_\infty > 0 \), then \( \kappa_\infty > 0 \). □

Note that the results of Lemma 5.1 say that if the velocity alignment occurs, then the coupling strength tends to zero asymptotically. Hence, the asymptotic size of the coupling strength will be highly dependent on some powers of the relative velocity \( \|v\| \). Next, we quantify the asymptotic relation between \( \kappa \) and \( \Gamma(\|v\|) \).
Lemma 5.2. Let \((x, v, \kappa)\) be a global solution to (4.1) with (3.3) satisfying a priori conditions:
\[
\lim_{t \to \infty} \|v(t)\| = 0 \quad \text{and} \quad \lim_{t \to \infty} \kappa(t) = 0.
\]
Then, the following assertions hold:

(i) The ratio between \(\kappa\) and \(\|v\|^\eta\) converges to a definite value:
\[
\lim_{t \to \infty} \frac{\kappa(t)}{\|v(t)\|^\eta} = \frac{\mu}{\gamma}.
\]

(ii) There exists a finite time \(T_1 > 0\) such that
\[
-\frac{2\mu}{\gamma} \psi(||x||)||v(t)||^{1+\eta} \leq \frac{d}{dt}||v(t)|| \leq -\frac{\mu}{2\gamma} \psi(||x||)||v(t)||^{1+\eta}, \quad \text{a.e.} \quad t > T_1. \tag{5.3}
\]

Proof. (i) We set an auxiliary function
\[
L(t) := \frac{\kappa(t)}{\Gamma(||v(t)||)} = \frac{\kappa(t)}{\|v(t)\|^\eta}, \quad t \geq 0.
\]
On the other hand, recall that
\[
\frac{d}{dt}||v|| = -\kappa \psi(||x||)||v|| \quad \text{and} \quad \dot{\kappa} = \mu||v||^\eta - \gamma \kappa, \quad \text{a.e.} \quad t > 0. \tag{5.4}
\]
These relations yield
\[
\dot{L} = -\left(\gamma - \eta \kappa \psi(||x||)\right)L + \mu, \quad \text{a.e.} \quad t > 0. \tag{5.5}
\]
Since we assume that \(\kappa\) converges to zero, there exists a finite time \(T_0 > 0\) such that
\[
\kappa(t) \leq \frac{\varepsilon}{\psi(0)}, \quad t > T_0. \tag{5.6}
\]
Then, (5.4)–(5.6) and non-increasing property of \(\psi(\cdot)\) yield
\[
-\gamma L + \mu \leq \dot{L} \leq (-\gamma + \eta \varepsilon)L + \mu, \quad \text{a.e.} \quad t > T_0.
\]
Hence, it follows from the same argument in Lemma 5.1 that
\[
\lim_{t \to \infty} \frac{\kappa(t)}{\|v(t)\|^\eta} = \frac{\mu}{\gamma}.
\]

(ii) Thus, there exists a finite time \(T_1 > 0\) such that
\[
\frac{\mu}{2\gamma} \|v(t)\|^\eta < \kappa(t) < \frac{2\mu}{\gamma} \|v(t)\|^\eta, \quad t > T_1. \tag{5.7}
\]
Finally, we combine (5.4) and (5.7) to get the desired estimate (5.3).

Remark 5.3. (i) Note that it follows from (5.5) that
\[
\dot{L} \geq -\gamma L + \mu, \quad \text{a.e.} \quad t > 0.
\]
Hence, there exists a positive lower bound \(L_m\) such that
\[
L(t) \geq L_m := \min \left\{L(0), \frac{\mu}{\gamma}\right\}, \quad \text{or equivalently,} \quad \kappa(t) \geq L_m \|v(t)\|^\eta, \quad t \geq 0. \tag{5.8}
\]

(ii) Since \(\|v\|\) is non-increasing in time and \(\kappa\) is uniformly bounded, we have
\[
\dot{\kappa} = -\gamma \kappa + \mu \|v^0\|^\eta, \quad \text{which implies} \quad \kappa(t) \leq \max \left\{\kappa^0, \frac{\mu \|v^0\|^\eta}{\gamma}\right\} =: \kappa_M. \tag{5.9}
\]
In (4.3), the relation (5.9) and $\psi(\cdot) \leq 1$ yields
\[
\frac{d}{dt} \|v\| \geq -\kappa_M \|v\|, \quad \text{or equivalently} \quad \|v(t)\| \geq \|v^0\| e^{-\kappa_M t}, \quad t > 0.
\]
Hence, $\|v\|$ cannot converge to zero in finite-time.

**Proof of Theorem 3.3.** Due to Lemma 5.1, the limit $v_\infty$ exists. We claim that $v_\infty > 0$. Suppose to the contrary that $v_\infty = 0$, i.e.,
\[
\lim_{t \to \infty} \|v(t)\| = 0. \tag{5.10}
\]
Hence, it follows from Lemma 5.2 that there exists a finite time $T_1 > 0$ such that
\[
\kappa(t) < \frac{2\mu}{\gamma} \|v(t)\|^{\eta}, \quad t > T_1.
\]
Then, (4.3) yields
\[
\frac{d}{dt} \|v(t)\| \geq -\frac{2\mu}{\gamma} \psi(\|x\|) \|v\|^{1+\eta}, \quad t > T_1. \tag{5.11}
\]
We integrate the relation (5.11) to obtain
\[
\frac{1}{\|v(t)\|^\eta} - \frac{1}{\|v(T_1)\|^\eta} \leq \frac{2\mu}{\gamma} \int_{T_1}^t \psi(\|x(s)\|) ds, \quad t > T_1. \tag{5.12}
\]
In (5.12), if we let $t \to \infty$, we use (5.10) to find that the left-hand of the resulting relation diverges to infinity, as $\|v\|$ does not touch zero in finite time, i.e., $\|v(T_1)\| \neq 0$. However, since the communication weight $\psi = \psi(s)$ satisfies the integrability condition (3.4), the right-hand side of the resulting relation is always finite. Hence, this contradicts (5.10) and completes the proof.

5.2. **Long-ranged communication weight.** In this subsection, we provide the proof of Theorem 3.5. To be more precise, we consider the two-particle system and show that the velocity difference always converges to zero, and group formation mainly depends on the exponent $\eta$ of the plasticity function (3.3).

5.2.1. **A two-particle system.** Here, we consider the asymptotic dynamics of the two-particle system with a long-ranged communication weight. Before we present the proof of Theorem 3.5, we first show that both $\|v\|$ and $\kappa$ converges to zero.

**Lemma 5.4.** Suppose that the plasticity function $\Gamma$ is anti-Hebbian, and the initial velocity difference is bounded:
\[
\Gamma(s) = s^\eta, \quad \eta > 0 \quad \text{and} \quad \|v^0\| < \infty,
\]
and the communication weight $\psi$ satisfies non-integrability condition (3.10), and let $(x, v, k)$ be a global solution to system (4.1). Then, we have
\[
\lim_{t \to \infty} \|v(t)\| = 0 \quad \text{and} \quad \lim_{t \to \infty} \kappa(t) = 0.
\]

**Proof.** It follows from Lemma 5.1 that the limits of $\|v\|$ and $\kappa$ exist, and we denote their limits $v_\infty$ and $\kappa_\infty$, respectively. Now, we claim:
\[
v_\infty = 0 \quad \text{and} \quad \kappa_\infty = 0.
\]
Suppose to the contrary that $v_\infty > 0$. Then, Lemma 5.1 yields $\kappa_\infty > 0$. It follows from the convergence of $\kappa$ that there exists a finite time $T_* > 0$ such that
\[
\kappa(t) > \frac{\kappa_\infty}{2}, \quad t > T_*.
\]
In (5.13), we let $t \to \infty$ to find
$$\frac{d}{dt} \|v\| \leq -\frac{\kappa_{\infty}}{2} \psi(||x||)\|v\|, \quad t > T_\ast.$$ 
We integrate the above relation to obtain
$$\|v(t)\| \leq \|v(T_\ast)\| \exp\left(-\frac{\kappa_{\infty}}{2} \int_{T_\ast}^{t} \psi(||x(s)||)ds\right), \quad t > T_\ast. \quad (5.13)$$

On the other hand, since $\frac{d}{dt} \|v\| \leq 0$ for $t > T_\ast$, we have
$$\|x(t)\| \leq \|x(T_\ast)\| + \|v(T_\ast)\| t, \quad t > T_\ast. \quad (5.14)$$

In (5.13), we let $t \to \infty$ to find
$$v_\infty \leq \|v(T_\ast)\| \exp\left(-\frac{\kappa_{\infty}}{2} \int_{T_\ast}^{\infty} \psi(||x(s)||)ds\right)$$
$$\leq \|v(T_\ast)\| \exp\left(-\frac{\kappa_{\infty}}{2} \int_{T_\ast}^{\infty} \psi(||x(T_\ast)|| + ||v(T_\ast)||s)ds\right)$$
$$= \|v(T_\ast)\| \exp\left(-\frac{\kappa_{\infty}}{2} \int_{T_\ast}^{\infty} \psi(r)r dr\right), \quad (5.15)$$

where $S_\ast := \|x(T_\ast)|| + \|v(T_\ast)||T_\ast$. However, since our communication weight is long-ranged, i.e., $\psi = \psi(r)$ is not integrable. The right-hand side of (5.15) is zero, which contradicts our assumption $v_\infty > 0$. Thus, we have $v_\infty = 0$, and by Lemma 5.1, we also have $\kappa_{\infty} = 0$. \qed

**Remark 5.5.** Note that Lemma 5.1 asserts that zero convergences of $\|v\|$ and $\kappa$ are equivalent. In contrast, Lemma 5.4 shows that the zero convergences actually occur.

**Proof of Theorem 3.5.** (i) We first consider the first assertion. By Lemma 5.4, velocity alignment holds. Thus, it suffices to check the uniform boundedness of $\|x\|$.

We multiply $\frac{1}{\|v\|^{1+\gamma}}$ on the both sides of (5.3) to get
$$-\frac{2\mu}{\gamma} \psi(||x(t)||) \leq \frac{\|v(t)||}{\|v(t)||^{1+\gamma}} = \frac{d}{dt} \left( -\frac{1}{\|v(t)||^\gamma} \right) \leq -\frac{\mu}{2\gamma} \psi(||x(t)||), \quad t > T_1.$$

We integrate the above relation to attain
$$-\frac{2\mu}{\gamma} \int_{T_1}^{t} \psi(||x(t)||) dt \leq \frac{1}{\|v(T_1)||^\gamma} - \frac{1}{\|v(t)||^\gamma} \leq -\frac{\mu}{2\gamma} \int_{T_1}^{t} \psi(||x(t)||) dt, \quad t > T_1. \quad (5.16)$$

We use the fact $\psi(\cdot) \leq 1$ and the first inequality of (5.16) to obtain
$$-\frac{2\mu}{\gamma} (t - T_1) \leq \frac{1}{\|v(T_1)||^\gamma} - \frac{1}{\|v(t)||^\gamma}, \quad t > T_1.$$ 

This yields the desired estimate (3.11): 
$$\|v(t)|| \geq \frac{\|v(T_1)||}{\left(1 + \frac{2\mu\|v(T_1)||^\gamma}{\gamma}(t - T_1)\right)^{\frac{1}{\gamma}}}, \quad t > T_1. \quad (5.17)$$

Moreover, it is obvious that $\|x\|$ is not bounded, i.e.,
$$\sup_{0 \leq t < \infty} \|x(t)|| = \infty,$$
This implies (5.17) is not integrable for \( \eta \geq 1 \). Hence, the mono-cluster flocking does not occur.

(ii) For the second claim, we use (3.12) and (5.14) to show that there exists \( T_2 > T_1 \) such that

\[
\psi(\|x(T_1)\| + \|v(t)\|) \geq \frac{\|x(T_1)\| + \|v(T_1)\|}{t} \quad \text{for} \quad t > T_2.
\]

By the second inequality of (5.16), we have

\[
\|v(t)\| \leq \frac{\|v(T_1)\|}{\left(1 + \frac{\|v(T_1)\|}{2\gamma} \int_{T_1}^t \psi(\|x(s)\|) \, ds\right)^{\frac{1}{\gamma}}}
\]

\[
\leq \frac{\|v(T_1)\|}{\left(1 + \frac{\|v(T_1)\|}{2\gamma} \int_{T_1}^t \psi(\|x(T_1)\| + \|v(T_1)\| s) \, ds\right)^{\frac{1}{\gamma}}}
\]

\[
\leq \frac{\|v(T_1)\|}{\left(C + \log(\|x(T_1)\| + \|v(T_1)\| t)\right)^{\frac{1}{\gamma}}}. \tag{5.18}
\]

From (5.17) and (5.18), we find the desired estimates.

Proof of Theorem 3.7. (i) For the first assertion, we use the relation (5.8) to find

\[
\frac{d}{dt} \|v(t)\| \leq -L_m \psi(\|x\|)\|v(t)\|^{1+\eta}, \quad \text{a.e.} \quad t > 0. \tag{5.19}
\]

Then, the relation (5.19) naturally induces the energy functional \( E_\pm \):

\[
E_\pm(\|x\|, \|v\|) := \frac{1}{1 - \eta} \|v\|^{1-\eta} - \int_{\pm x_0}^{\pm x(t)} \psi(\|x\|) \, ds,
\]

where \( \Psi = \Psi(r) \) is defined to be a primitive functional of \( \psi(r) \):

\[
\Psi(r)' = \psi(r), \quad r > 0.
\]

Then, we differentiate \( E_\pm \) to see that \( E_\pm \) is non-increasing in time \( t \): for a.e. \( t > 0 \),

\[
\frac{d}{dt} E_\pm = \|v\|^{\eta} \frac{d\|v\|}{dt} \pm L_m \psi(\|x\|) \frac{d\|x\|}{dt} \leq -L_m \psi(\|x\|)\|v\| \pm L_m \psi(\|x\|)\|v\| \leq 0.
\]

Hence, one has

\[
\frac{1}{1 - \eta} \left(\|v(t)\|^{1-\eta} - \|v_0\|^{1-\eta}\right) \leq -L_m \int_{\|x_0\|}^{\|x(t)\|} \psi(s) \, ds, \quad t > 0.
\]

This implies

\[
L_m \int_{\|x_0\|}^{\|x(t)\|} \psi(s) \, ds \leq \frac{1}{1 - \eta} \|v_0\|^{1-\eta}.
\]

Since the initial data always satisfy the following relation, as \( \psi \) is non-integrable,

\[
\|v_0\|^{1-\eta} < L_m (1 - \eta) \int_{\|x_0\|}^{\infty} \psi(s) \, ds,
\]
we can choose the largest value $x_M > 0$ such that
\[
\|v^0\|^{1-\eta} = L_m (1 - \eta) \left| \int_{\|x^0\|}^{x_M} \psi(s) ds \right|.
\] (5.20)
Then, we use Theorem 2.6 to find a positive constant $x_M > 0$ such that
\[
\|x(t)\| \leq x_M, \quad t \geq 0.
\] (5.21)
For the right inequality in (3.15), we put (5.21) into (5.19) to see
\[
\frac{d}{dt} \|v\| \leq -L_m \psi(x_M) \|v\|^{1+\eta}, \quad \text{a.e.} \; t > 0.
\] (5.22)
We integrate (5.22) to find
\[
\|v(t)\| \leq \left( \frac{1}{\|v^0\|^{1-\eta} + \eta L_m \psi(x_M) t} \right)^{\frac{1}{\eta}}.
\] (5.23)
This shows the first assertion and the right inequality in the second assertion.

(ii) Now, we show the left inequality of (3.15). In the left inequality of (5.3), we use $\psi(\cdot) \leq 1$ to obtain
\[
\frac{d}{dt} \|v\| \geq -\frac{2\mu}{\gamma} \|v\|^{1+\eta}, \quad \text{a.e.} \; t > T_1.
\] (5.24)
We integrate (5.24) to find
\[
\|v(t)\| \geq \left( \frac{1}{\|v(T_1)\|^{1-\eta} + \frac{2\mu_0}{\gamma} t} \right)^{\frac{1}{\eta}}, \quad t \geq T_1.
\]
Finally, we combine (5.23) and (5.24) to complete the proof of Theorem 3.7. \qed

5.2.2. A many-particle system. The argument for the many-particle system is similar to the one for the two-particle case as in the proof of Theorem 3.7. We first show that the velocity alignment always occurs for generic initial data, as in the two-particle system. For this, we recall the notation $\hat{\kappa}(t) := \kappa_{i,t,j,t}(t)$ in Proposition 2.3:
\[
\kappa_{i,t,j,t}(t), \quad t > 0,
\]
where $(i_t, j_t)$ are extremal indices such that
\[
D(V(t)) = \|v_{i_t}(t) - v_{j_t}(t)\|, \quad t > 0.
\]

Lemma 5.6. Let $(X, V, K)$ be a solution to (1.2) with a long-ranged communication weight $\psi = \psi(s)$. Then, the diameter $D_V$ and $\hat{\kappa}$ converge to zero:
\[
\lim_{t \to \infty} D(V(t)) = 0 \quad \text{and} \quad \lim_{t \to \infty} \hat{\kappa}(t) = 0.
\]
Proof. We split its proof into two steps.

Step A (equivalence of zero convergences): we first claim that
\[
\lim_{t \to \infty} D(V(t)) = 0 \iff \lim_{t \to \infty} \hat{\kappa}(t) = 0.
\]
We recall Proposition 2:
\[
\frac{d}{dt} D(V(t)) \leq -\hat{\kappa}(t) \psi(D(X)) D(V), \quad \frac{d}{dt} \hat{\kappa}(t) = -\gamma \hat{\kappa}(t) + \mu \Gamma(D(V)), \quad \text{a.e.} \; t > 0.
\]
Hence, the limit of $D(V(t))$ exists, i.e., there exists a nonnegative constant $V_\infty \geq 0$ such that
\[
\lim_{t \to \infty} D(V(t)) = V_\infty.
\]
Then, we follow the same argument in Lemma 5.1 to establish the desired equivalence.

Step B (zero convergence of $D_V$): suppose to the contrary that $V_\infty > 0$. Then, it follows from Step A that there exists a positive constant $\kappa_\infty$ such that

$$\lim_{t \to \infty} \hat{\kappa}(t) = \kappa_\infty.$$ Using the same argument in Lemma 5.4, we draw the desired contradiction. \qed

**Lemma 5.7.** Let $(X, V, K)$ be a solution to (1.2) with a long-ranged or short-ranged communication weight $\psi = \psi(s)$, and the anti-Hebbian rule $\Gamma$. Then, there exists a positive constant $\alpha > 0$ such that

$$\hat{\kappa}(t) > \alpha D(V(t))^\eta, \quad t \geq 0.$$ 

**Proof.** As in Lemma 5.2, we set

$$Z(t) := \frac{\hat{\kappa}(t)^{\frac{1}{\eta}}}{D(V(t))}, \quad t \geq 0. \quad (5.25)$$

For notational simplicity, we temporarily write $\nu := \frac{1}{\eta}$. Then, we differentiate $Z$ with respect to $t$ to find

$$\dot{Z} = \frac{\nu \hat{\kappa} V}{D(V)^2} \geq -\nu \gamma Z + \nu \mu \eta Z^{\frac{1}{\eta}} + \hat{\kappa} \psi(D(X)) D(V) \geq -\nu \gamma Z + \nu \mu \eta Z^{\frac{1}{\eta}}, \quad \text{a.e. } t > 0. \quad (5.26)$$

To solve (5.26), we set $Z(t) := Y(t)^\nu$. Then, it follows from (5.26) that $Y$ satisfies

$$\dot{Y} \geq -\gamma Y + \mu, \quad t > 0.$$ This yields

$$Y(t) \geq \left( Y(0) - \frac{\mu}{\gamma} \right) e^{-\gamma t} + \frac{\nu}{\gamma}, \quad t \geq 0.$$ Hence, one has

$$Y(t) \geq \min \left\{ Y(0), \frac{\mu}{\gamma} \right\}, \quad \text{or equivalently, } \quad Z(t) \geq \min \left\{ Z(0), \left( \frac{\mu}{\gamma} \right)^\nu \right\}, \quad t \geq 0.$$ In (5.25), we can find a positive constant $\alpha > 0$ such that

$$\hat{\kappa}(t) > \alpha D(V(t))^\eta = \alpha D(V(t))^\eta, \quad t \geq 0.$$ \qed

**Proof of Theorem 3.8.** For the proof of Theorem 3.8, since the velocity diameter $D(V)$ always converge to zero, as time goes to infinity, it suffices to show that the position diameter $D(X)$ is uniformly bounded. First, we recall Proposition 2:

$$\left| \frac{d}{dt} D(X) \right| \leq D(V) \quad \text{and} \quad \frac{d}{dt} D(V) \leq -\hat{\kappa} \psi(D(X)) D(V), \quad \text{a.e. } t > 0. \quad (5.27)$$

In (5.27), we use Lemma 5.7 to derive the dissipative inequality for $D(V) = D(V(t))$:

$$\frac{d}{dt} D(V) \leq -\alpha \psi(D(X)) D(V)^{1+\eta}, \quad \text{a.e. } t > 0. \quad (5.28)$$

Then, the inequality (5.28) induces a natural Lyapunov functional $\mathcal{E}_\pm$:

$$\mathcal{E}_\pm(D(X), D(V)) := \frac{1}{1-\eta} D(V)^{1-\eta} \pm \alpha \Psi(D(X)) \quad \text{where} \quad 
\Psi'(s) = \psi(s), \quad s \geq 0.$$
It follows from Theorem 2.6 that there exists a positive constant $x_M > 0$ such that

$$(D(V^0))^{1-\eta} = \alpha(1-\eta) \int_{D(V)}^x \psi(s)ds, \quad D(X(t)) \leq x_M, \quad t \geq 0, \quad (5.29)$$

where $(5.29)_1$ can be always achieved as $\psi$ is non-integrable. For the second assertion, we use $(5.29)_2$ in $(5.28)$ to find

$$\frac{d}{dt}D(V) \leq -\alpha\psi(x_M)D(V)^{1+\eta}, \quad \text{a.e. } t > 0. \quad (5.30)$$

Then, we integrate $(5.30)$ to find the desired decay rate:

$$D(V(t)) \leq \left(\frac{1}{(D(V^0))^{-\eta} + \alpha\eta\psi(x_M)t}\right)^{\frac{1}{\eta}}, \quad t \geq 0.$$

Hence, this completes the proof.

6. **Numerical simulations.** In this section, we demonstrate several numerical simulations for the C-S model (1.2) with adaptive couplings (1.7). For numerical implementations, we use the fourth-order Runge-Kutta method, and choose dimension $d$, number of particles $N$, learning enhancement rate $\mu$, positive friction constant $\gamma$ and the communication weight function $\psi = \psi(r)$ as follows:

$$d = 2, \quad N = 200, \quad \mu = 1, \quad \gamma = 2, \quad \psi(r) = \frac{1}{(1 + r^2)^{0.25}} \quad \text{for } r \geq 0.$$ 

Furthermore, the initial data $(x_i^0, v_i^0) \in [-1, 1]^4$ are chosen randomly, and $\kappa_{ij}^0$ is given to be symmetric and chosen from $[0, 1]$.

This section consists of four subsections which correspond to our four main results in Table 1 of Section 3, respectively.

6.1. **Hebbian rule.** We set the Hebbian adaptive rule (1.3) and the system functions

$$\Gamma(s) = \frac{1}{1 + s}, \quad \psi(r) = \frac{1}{(1 + r^2)^{0.25}}, \quad s, r \geq 0.$$

In Figure 1, we observe that $D(X)$ is uniformly bounded in time and $D(V)$ converges to zero, as time goes to infinity. Moreover, by the linear decline of $\log D(V(t))$, we can verify the exponential decay of $D(V(t))$. This yields the convergence of $\kappa_{ij}(t)$ as we can check in Figure 1(b), which coincides with Corollary 2:

$$\lim_{t \to \infty} \kappa_{ij}(t) = \frac{\mu}{\gamma} \Gamma(0) = \frac{1}{2}.$$ 

6.2. **Anti-Hebbian rule and short-ranged interaction.** We consider the anti-Hebbian case (1.4) with the system functions

$$\Gamma(s) = s^2, \quad \psi(r) = \frac{1}{1 + r^2}, \quad s, r \geq 0.$$ 

In Figure 2, we observe that $D(V)$ converges to some positive value which implies that $D(X)$ diverges to infinity with a linear growth. This coincides with what we proved in Theorem 3.3 for two particle case. In addition, we see that $\max_{1 \leq i,j \leq N} \kappa_{ij}$ does not converge to zero.
6.3. Anti-Hebbian rule with $\eta \geq 1$ and long-ranged interaction. We consider the anti-Hebbian rule with $\eta = 2$ and the system functions are given to be

$$\Gamma(s) = s^2, \quad \psi(r) = \frac{1}{(1 + r^2)^{0.1}}, \quad s, r \geq 0.$$  

We try to check the convergence of $D(V)$ toward zero. For a two-particle system, Theorem 3.5 shows that the velocity difference $\|v\|$ satisfies the following relation:

$$\frac{1}{C_1} \frac{1}{\sqrt{(1 + t)}} \leq \|v(t)\| \leq \frac{C_1}{(1 + \log t)^{0.5}}, \quad \text{as } t \to \infty.$$  

Here, we implement the dynamics of the many-particle system and look for the change of $D(V)$. Our natural guess for the many-particle case is that $D(V)$ has the similar decay rate of $\|v\|$: 

$$\frac{1}{C_1} \frac{1}{\sqrt{(1 + t)}} \leq D(V(t)) \leq \frac{C_1}{(1 + \log t)^{0.5}}, \quad \text{as } t \to \infty. \quad (6.1)$$

To this end, we compute and plot the change of $\sqrt{D}(V(t))$ and $\sqrt{\log(2 + t)}D(V(t))$. In Figure 3, we can observe increase of $\sqrt{D}(V(t))$ and decrease of $\sqrt{\log(2 + t)}D(V(t))$ as time goes on. Thus, after sufficiently large time, the numerical simulation gives a positive lower bound for $\sqrt{D}(V(t))$ and an upper bound for $\sqrt{\log(2 + t)}D(V(t))$, which justifies our hypothesis (6.1). Similarly, it follows from Lemma 5.7 that
the ratio between $\hat{\kappa}$ and $\mathcal{D}(V)$ satisfies

$$\lim_{t \to \infty} \frac{\hat{\kappa}(t)}{(\mathcal{D}(V(t)))^2} = \frac{\mu}{\gamma}. \tag{6.1}$$

Thus, we see that $t \max_{1 \leq i,j \leq N} \kappa_{ij}(t)$ is bounded below and $\log(2+t) \max_{1 \leq i,j \leq N} \kappa_{ij}(t)$ is bounded above as described in Figure 3(b) and 3(d). Note that when we plot Figure 3(c) and 3(d), we consider $\log(2+t)$ in order to avoid possible blow-up near $t = 0$.

6.4. Anti-Hebbian rule with $0 < \eta < 1$ and long-ranged interaction. We consider the anti-Hebbian rule with $\eta = 1/2$ under the given the system functions

$$\Gamma(s) = \sqrt{s}, \quad \psi(r) = \frac{1}{(1+r^2)^{0.1}}, \quad s, r \geq 0.$$

We are looking for the asymptotic behavior of $\mathcal{D}(V(t))$. Based on Theorem 3.7 for the two-particle system and Theorem 3.8 for the many-particle system, we anticipate the decay rate of $\mathcal{D}(V)$ as follows:

$$\frac{1}{C_3} \frac{1}{(1+t)^2} \leq \mathcal{D}(V(t)) \leq \frac{C_3}{(1+t)^2}, \quad \text{as } t \to \infty.$$

To specify the explicit decay rate of $\mathcal{D}(V(t))$, we compute $t^{\delta_1} \mathcal{D}(V(t))$ and $t^{\delta_2} \max_{1 \leq i,j \leq N} \kappa_{ij}(t)$ for some positive constants $\delta_1$ and $\delta_2$. For $\mathcal{D}(V(t))$, it seems that $t \mathcal{D}(V(t))$ decreases toward zero like order of $1/t$ and $t^2 \mathcal{D}(V(t))$ converges to
a certain value. Moreover, we can clearly observe the linear growth of $t^3 D(V(t))$ after sufficiently large time, which coincides with our expect

$$D(V(t)) = O\left(\frac{1}{t^2}\right) \quad \text{as} \quad t \to \infty.$$ 

We note that, based only on the plot of $t^2 D(V(t))$, it is not so clear whether $t^2 D(V(t))$ indeed converges or increases, due to the slow convergence. With a similar argument from the linear growth of $t^2 \max_{1 \leq i,j \leq N} \kappa_{ij}(t)$, we can attain

$$\max_{1 \leq i,j \leq N} \kappa_{ij}(t) = O\left(\frac{1}{t}\right) \quad \text{as} \quad t \to \infty.$$ 

7. Conclusion. In this paper, we have studied the asymptotic emergent dynamics of the Cucker-Smale (C-S) flocking model equipped with the adaptive dynamics, in which the coupling strength is influenced by the relative velocities. A coupling strength in the original C-S model is assumed to be a mean-field and uniform constant so that the interaction between particles becomes all-to-all. To describe the real-world phenomena, we employ two adaptive rules (Hebbian vs. anti-Hebbian). For the Hebbian rule, we present a mono-cluster flocking estimate based on the Lyapunov functional approach. In this case, we can obtain the fast decay toward the mono-cluster flocking regime. On the other hand, for the anti-Hebbian case, we determine the specific type of an adaptive law using monomial ansatz $\Gamma(s) = s^\eta$. 

Figure 3. Anti-Hebbian rule with $\eta = 2$ and long-ranged interaction
Then, our results mainly depend on the integrability of the communication function $\psi$ and the regularity of $\Gamma$ at the origin ($\eta \geq 1$ or $0 < \eta < 1$). More precisely, if the communication function is integrable (or short-ranged), then the velocity alignment does not hold, even for the two-particle system. For the non-integrable (or long-ranged) communication function, velocity alignment always occurs with polynomial decay. However, group formation relies on the exponent $\eta$. To be more specific, if $\eta \geq 1$, the group formation fails for the two-particle system, due to the slow decay rate of the velocity. In contrast, if $0 < \eta < 1$, the group formation occurs so that the mono-cluster flocking emerges. Of course, there are many remaining issues, e.g., the emergence of multi-cluster flocking when $\Gamma$ allows negative values, introducing
the fast learning algorithm which induces a degenerate system, etc. We leave these interesting issues for future work.

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