Estimation of mean vector in elliptical models

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Abstract: In this paper, we are basically discussing on a class of Baranchik type shrinkage estimators of the vector parameter in a location model, with errors belonging to a sub-class of elliptically contoured distributions. We derive conditions under Schwartz space in which the underlying class of shrinkage estimators outperforms the sample mean. Sufficient conditions on dominant class to outperform the usual James-Stein estimator are also established. It is nicely presented that the dominant properties of the class of estimators are robust truly respect to departures from normality.

Key Words and Phrases: Elliptically contoured distribution, James-Stein estimator, Jeffreys’ prior, Minimax, Robustness, Shrinkage estimator, Schwartz space.

1 Introduction

The assumption of normality restricts the range of possible applications, especially in flatter densities. The elliptically contoured distributions (ECDs) are the parametric forms of the spherical symmetric distributions, which are invariant under orthogonal transformations and have equal density on sphere if densities exist. ECD’s primary purpose is to provide a highly impressive list of heavier/lighter tail alternatives to the multivariate Gaussian models. Materials involving vector-variate distributional properties and inferential problems will be found entirely in the work of a couple of statisticians like Das Gupta et al. (1972), Cambanis et al. (1981), Muirhead (1982), Anderson et al. (1986), Cellier et al. (1989), Anderson and Fang (1990), Fang and Zhang (1990), Fang et al. (1990) and Kibria and Haq (1999). Among others, the book of Gupta and Varga (1993) illustrates some significant results dealing with matrix-variate ECD. In addition, some of the well-known elliptical distributions are the Gaussian, Pearson Type II/VII, Student’s t, logistics, Kotz type, Laplace, Bessel and power exponential multivariate distributions.

In this paper, we consider the location model in a more general setup involving dependent errors. Initially let $S(p)$ denotes the set of all $p \times p$ positive definite matrices.
The precise set-up of the problem is as follows: Let $Y_i$ be an $p \times 1$ response vector with model

$$ Y_i = \theta + \epsilon_i, \quad 1 \leq i \leq N. \quad (1.1) $$

Here $\theta$ is a $p \times 1$ vector of location parameters and $\epsilon_i$ is a $p \times 1$ error vector such that

$$ E(\epsilon_i) = 0, \quad Cov(\epsilon_i, \epsilon_j) = \Sigma \in S(p), \quad i, j = 1, \cdots, N, \quad N > p. \quad (1.2) $$

It is assumed, in general, $\epsilon = (\epsilon_1, \cdots, \epsilon_N)'$ have a joint elliptically contoured distribution. Typically if it possess a density, it is followed by

$$ f(\epsilon|\Sigma) \propto |\Sigma|^\frac{-N}{2} g(\text{tr} \Sigma \sum_{i=1}^{N} \epsilon_i' \epsilon_i), \quad (1.3) $$

where $g(.)$ is a non-negative function over $\mathbb{R}_+$ such that $f(\cdot|\Sigma)$ is a density function with respect to (w.r.t.) a $\sigma$-finite measure $\mu$ on $\mathbb{R}^p$. In this case, notation $\epsilon_i \sim E_p(0, \Sigma, g)$ would probably be used.

Due to Chu (1973), each component of the aforementioned model being proposed in (1.3), possibly can be presented as the following form.

$$ f_{\epsilon_i}(x) = \int_0^\infty \mathcal{W}(t)\phi_{N_p(0, t^{-1}\Sigma)}(x) dt, \quad (1.4) $$

where $\phi_{N_p(0, t^{-1}\Sigma)}(.)$ is the pdf of $N_p(0, t^{-1}\Sigma)$.

$$ \mathcal{W}(t) = (2\pi)^{\frac{p}{2}} |\Sigma|^{\frac{p}{2}} t^{-\frac{p}{2}} \mathcal{L}^{-1}[f(s)], \quad (1.5) $$

$L^{-1}[f(s)]$ denotes the inverse Laplace transform of $f(s)$ with $s = t[x'\Sigma^{-1}x/2]$.

The inverse Laplace transform of $f(.)$ exists provided that the following conditions are satisfied.

(i) $f(t)$ is differentiable when $t$ is sufficiently large.

(ii) $f(t) = o(t^{-m})$ as $t \to \infty, m > 1$.

Although, it is rather difficult to derive the inverse Laplace transform of some functions, we are able to handle it for many density generators of elliptical densities. Refer to Debnath and Bhatta (2007) for more specific details.

The mean of $\epsilon_i$ is the zero-vector and the covariance-matrix of $\epsilon_i$ is

$$ Cov(\epsilon_i) = \int_0^\infty Cov(\epsilon_i|t)\mathcal{W}(t) dt $$

$$ = \int_0^\infty \mathcal{W}(t)Cov\{N_p(0, t^{-1}\Sigma)\} dt $$

$$ = \left( \int_0^\infty t^{-1}\mathcal{W}(t) dt \right) \Sigma, \quad (1.6) $$

provided the above integral exists.
Another sub-class of ECDs which includes the above class may be generated by a signed measure $W$ on the measurable space $(\mathbb{R}^+, \mathcal{B})$ such that the pdf $f(.)$ can be expressed through the following procedures:

\begin{enumerate}[(i)]
  \item $f(x) = \int_0^\infty \phi_{N_p(0, t^{-1}\Sigma)}(x)W(dt)$, \hspace{1cm} (1.7)
  \item $\int_0^\infty t^{-1}W^+(dt) < \infty$,
  \item $\int_0^\infty t^{-1}W^-(dt) < \infty$,
\end{enumerate}

where $W^+ - W^-$ is the Jordan decomposition of $W$ in positive and negative parts (see e.g. Srivastava and Bilodeau, 1989). Note that from (ii) – (iii) of (1.7),

\[ \int_0^\infty t^{-1}W(dt) < \infty \] \hspace{1cm} (1.8)

and thus, $\text{Cov}(\epsilon_i)$ exists under the sub-class defined above.

Now, under Bayesian framework, it is properly assumed that in distribution, little is known of course, a priori, about the parameters, the elements of $\theta$ and the $p(p + 1)/2$ distinct elements of $\Sigma \in \mathcal{S}(p)$. We shall first of all suppose that the elements of $\theta$ and those of $\Sigma$ are approximately independent (see Box and Tiao, 1992, page 425), i.e.

\[ \pi(\theta, \Sigma) \propto \pi(\theta)\pi(\Sigma). \] \hspace{1cm} (1.9)

Using the invariant theory due to Jeffreys (1961), we take

\[ \pi(\theta) \propto \text{constant}, \]
\[ \pi(\Sigma) \propto |\Sigma|^{-p+1}/2, \]

as the prior knowledge about the parameter space.

Next step being taken, is giving results for the marginal posterior distribution of the location parameter given responses.

**Lemma 1.1.** Assume in the location model (1.1), $\epsilon_i \sim \mathcal{E}_p(0, \Sigma, g)$, where $\Sigma \in \mathcal{S}(p)$. Then, w.r.t. the prior distribution given by (1.9), the posterior distribution of $\theta$ is multivariate Student’s $t$ distribution, denoted by $\theta|y \sim t_p(y, S, N-p)$, with the following pdf

\[ f(\theta|y) = \frac{N^{p+1/2}\Gamma(N/2)}{\pi^{p/2}(N-p)^{N/2}\Gamma(N/2)} \left[ 1 + N(\theta - \bar{Y})'S^{-1}(\theta - \bar{Y}) \right]^{-N/2} \]

where $\mathbf{Y} = (Y_1, \ldots, Y_N)$, and

\[ \bar{Y} = \frac{1}{N} \sum_{i=1}^N Y_i, \quad S = \sum_{i=1}^N (Y_i - \bar{Y})(Y_i - \bar{Y})'. \] \hspace{1cm} (1.10)
Proof: Using Proposition 1 of Ng (2002), one can directly obtain

\[ f(\theta|Y) \propto \left| \sum_{i=1}^{N} (Y_i - \theta)(Y_i - \theta)' \right|^{-\frac{N}{2}}, \tag{1.11} \]

which is the same as we take the errors to be normally distributed (Zellner, 1971, P.243).

At this level, through making conclusion based on the following equality

\begin{align*}
(Y_i - \theta)(Y_i - \theta)' &= (Y_i - \bar{Y})(Y_i - \bar{Y})' + (\bar{Y} - \theta)(\bar{Y} - \theta)' \\
&\quad + 2(Y_i - \bar{Y})(\bar{Y} - \theta)',
\end{align*}

we observe

\[ \left| \sum_{i=1}^{N} (Y_i - \theta)(Y_i - \theta)' \right| = |S + NA|, \tag{1.12} \]

where \( A = (\theta - \bar{Y})(\theta - \bar{Y})' \).

However, by taking advantage from Corollary A.3.1 of Anderson (2003) we reach the point that

\[ |S + NA| = \begin{vmatrix} S & -\sqrt{N}(\theta - \bar{Y})' \\ \sqrt{N}(\theta - \bar{Y}) & 1 \end{vmatrix}, \]

\[ = \begin{vmatrix} 1 & \sqrt{N}(\theta - \bar{Y})' \\ -\sqrt{N}(\theta - \bar{Y}) & S \end{vmatrix}, \]

\[ = |S| \{1 + N(\theta - \bar{Y})' S^{-1}(\theta - \bar{Y})\} \tag{1.13} \]

Therefore, by making use of equations (1.11)-(1.13) we come to realize the following formula

\[ f(\theta|Y) = c(N,p)|S|^{-\frac{N}{2}} \{1 + N(\theta - \bar{Y})' S^{-1}(\theta - \bar{Y})\}^{-\frac{N}{2}}, \]

where

\[ c(N,p) = \left\{ \int_{\theta \in \mathbb{R}^p} |S|^{-\frac{N}{2}} \{1 + N(\theta - \bar{Y})' S^{-1}(\theta - \bar{Y})\}^{-\frac{N}{2}} d\theta \right\}^{-1} \]

\[ = |S|^{-\frac{N}{2}} \begin{vmatrix} \pi(N-p)\frac{N}{2} \Gamma \left( \frac{N-p}{2} \right) |S|^{\frac{N}{2}} & \frac{\Gamma \left( \frac{N}{2} \right) |N(N-p)|^{\frac{N}{2}}}{\pi^{\frac{N}{2}} (N-p)^\frac{N-p}{2}} \\
\end{vmatrix}^{-1} \]

\[ = \frac{N^\frac{N}{2} \Gamma \left( \frac{N}{2} \right) |S|^{\frac{N}{2}}}{\pi^{\frac{N}{2}} (N-p)^{\frac{N-p}{2}} \Gamma \left( \frac{N-p}{2} \right)}. \]

This would prove our claim. \( \blacksquare \)

Throughout this paper, we shall also assume that the loss function is given by

\[ L(\hat{\theta}; \theta) = N(\hat{\theta} - \theta)' \Sigma^{-1}(\hat{\theta} - \theta) \tag{1.14} \]
for any estimator \( \hat{\theta} \) of \( \theta \).

It has been fully known that the Bayes estimator of \( \theta \) with respect to the loss (1.14) is the posterior mean (Proposition 2.5.1, Robert, 2001) given by

\[
\hat{\theta} = \bar{Y}.
\]  

(1.15)

As it can be realized from the estimator given by (1.15), the Bayes estimator, reduces to the sample mean, under the setup presented above. So there is no need to deal with the Bayesian aspects of \( \hat{\theta} \), and along the paper, we in fact concern sample mean rather than the Bayes estimator.

Then, from ECD properties (see Fang et al., 1990) we have

\[
\hat{\theta} \sim \mathcal{E}_p(\theta, N^{-1}\Sigma, g).
\]  

(1.16)

Under classical viewpoint, we devote a general class of Stein-type shrinkage estimators to the estimator \( \hat{\theta} \), given by

\[
\delta_r(\hat{\theta}) = \left[ 1 - \frac{r(\hat{\theta}) S^{-1}\hat{\theta}}{\hat{\theta} S^{-1}\hat{\theta}} \right] \hat{\theta},
\]  

(1.17)

where \( r : [0, \infty) \rightarrow [0, \infty) \) is an absolutely continuous function.

Furthermore, \( r \in \mathcal{S}(\mathbb{R}_+, \mu) \), (the Schwartz space or space of rapidly decreasing functions on \( \mathbb{R}_+ \) with the measure \( \mu \)) where

\[
\mathcal{S}(\mathbb{R}_+, \mu) = \{ r \in C^\infty(\mathbb{R}_+, \mu) : \| r \|_{\alpha, \beta} < \infty \; \forall \alpha, \beta \},
\]

\( \alpha \) and \( \beta \) are indices, \( C^\infty(\mathbb{R}_+, \mu) \) is the set of all smooth functions from \( \mathbb{R}_+ \) to \( \mathbb{C} \) (the set of all complex numbers) and

\[
\| r \|_{\alpha, \beta} = \| x^\alpha D^\beta r \|_\infty = \sup \{ | x^\alpha D^\beta r(x) | : x \in \text{domain of } r \}.
\]

Here \( D^\beta \) stands for the \( \beta \)th derivative of \( r \). See Folland (1999) for more details.

The latter condition plays strategic position in gaining main result. Note that for every function such as \( r(\cdot) \) belongs to \( \mathcal{S}(\mathbb{R}_+, \mu) \), we have

\[
\int_0^\infty r'(x) d\mu(x) < \infty, \quad \text{(1.18)}
\]

\[
\int_0^\infty r^2(x) d\mu(x) < \infty, \quad \text{(1.19)}
\]

More interesting that the Schwartz space is dense in the space of all functions satisfy the above conditions in (1.18) and (1.19).

The objective of this study is to construct conditions on \( r(\cdot) \) under which \( \delta_r(\hat{\theta}) \) performs better than \( \hat{\theta} \) in the sense of having smaller risk w.r.t. the loss function given by (1.14).

This study is highly motivated by the work of Srivastava and Bilodeau (1989). They chewed over a similar class of estimators to (1.17), substituting the function \( r(\cdot) \) with a constant under classical decision theory. Although, as noted above, considering Bayesian point of view does not offer substantial generality, taking vague prior, over the work of Srivastava and Bilodeau (1989), because of (1.16), the class specified in (1.17) contains the class which was previously stated as a special case.
2 Risk Derivations

In this section, we give some lemmas to evaluate the risk function of \( \delta_r(\hat{\theta}) \). Provided that if all expectations exist, we deserve the following Lemma.

**Lemma 2.1.** If \( x \sim \mathcal{N}_p(\theta, \alpha \Sigma) \), \( \alpha > 0 \), \( \Sigma \in S(p) \) is independent of \( S \sim W_p(\beta \Sigma, n) \), \( \beta > 0 \), \( n = N - 1 \), then

\[
E \left[ \frac{x' \Sigma^{-1}(x - \theta) r (x' S^{-1} x)}{x' S^{-1} x} \right] = \beta \alpha (n - p + 1) \left\{ (p - 2) E \left[ \frac{r(x' S^{-1} x)}{x' \Sigma^{-1} x} \right] + 2E \left[ r'(x' S^{-1} x) \right] \right\}
\]

and

\[
E \left[ \frac{x' \Sigma^{-1} x r^2 (x' S^{-1} x)}{(x' S^{-1} x)^2} \right] = \beta^2 (n - p + 1)(n - p + 3) E \left[ \frac{r^2(x' S^{-1} x)}{x' \Sigma^{-1} x} \right]
\]

**Proof:** Let \( y = \Sigma^{-\frac{1}{2}} x \), \( \theta^* = \Sigma^{-\frac{1}{2}} \theta \), and \( A = \beta^{-1} \Sigma^{-\frac{1}{2}} (\Sigma^{-\frac{1}{2}}) \), where \( \Sigma^{-\frac{1}{2}} \) is a symmetric square root of \( \Sigma^{-1} \). From independency of \( x \) and \( S \), \( y \sim \mathcal{N}_p(\theta^*, \alpha I_p) \) is independent of both \( A \sim W_p(I_p, n) \) and \( \text{diag} \left[ \frac{y' y}{y' A^{-1} y} \right] \sim \chi^2_{n-p+1} \). Also note that \( F = x' S^{-1} x = \beta^{-1} y' A^{-1} y \). Therefore using Stein’s (1981) identity we get

\[
E \left[ \frac{x' \Sigma^{-1}(x - \theta) r (x' S^{-1} x)}{x' S^{-1} x} \right] = E \left[ \frac{y'(y - \theta^*) r(F)}{y' y} \right] = E \left[ \frac{y'(y - \theta^*) r(F)}{y' y} \right] = \beta(n - p + 1) E \left[ \frac{y'(y - \theta^*) r(F)}{y' y} \right] = \alpha \beta(n - p + 1) \left\{ (p - 2) E \left[ \frac{r(F)}{y' y} \right] + 2E \left[ r'(F) \right] \right\}.
\]

Similarly

\[
E \left[ \frac{x' \Sigma^{-1} x r^2 (x' S^{-1} x)}{(x' S^{-1} x)^2} \right] = E \left[ \frac{y' y r^2(F)}{F^2} \right] = \beta^2 E \left[ \frac{r^2(F)}{y' y} \right] E \left[ \left( \frac{y' y}{y' A^{-1} y} \right)^2 \right] = \beta^2 (n - p + 1)(n - p + 3) E \left[ \frac{r^2(F)}{y' y} \right].
\]

**Lemma 2.2.** The risk function of the estimator \( \delta_r(\hat{\theta}) \) w.r.t. the loss function \( I_1 \) is
Theorem 3.1. 

dominates a James-Stein type shrinkage estimator. In this section, we demonstrate the minimaxity of the estimator.

3 Main Results

Proof: As far as the representation (1.4) is concerned, it is possible to continue this way

\[ R(\delta_r(\hat{\theta}); \theta) = N \left[ (\delta_r(\hat{\theta}) - \theta)' \Sigma^{-1}(\delta_r(\hat{\theta}) - \theta) \right] \]

\[ - \frac{2}{t^2} \int_0^\infty E \left[ \left( \frac{\theta' \Sigma^{-1}(\theta - \hat{\theta}) r \left( \theta' S^{-1} \hat{\theta} \right)}{\theta' S^{-1} \hat{\theta}} \right) | t \right] \mathcal{W}(dt) \]

\[ + N \int_0^\infty E \left[ \left( \frac{\theta' \Sigma^{-1} r \left( \theta' S^{-1} \hat{\theta} \right)}{(\theta' S^{-1} \hat{\theta})^2} \right) | t \right] \mathcal{W}(dt). \]  

(2.1)

But from $\epsilon_i | t \sim \mathcal{N}_p(0, t^{-1} \Sigma)$, it is concluded that using (1.10), $\hat{\theta} | t \sim \mathcal{N}_p(\theta, t^{-1} N^{-1} \Sigma)$ is independent of $S|t \sim W_p(t^{-1} \Sigma, n)$. Consequently, by making use of Lemma 2.1 for $\alpha = (tN)^{-1}$ and $\beta = t^{-1}$ we get

\[ E \left[ \frac{\theta' \Sigma^{-1}(\theta - \hat{\theta}) r \left( \theta' S^{-1} \hat{\theta} \right)}{\theta' S^{-1} \hat{\theta}} \right] | t \]  

\[ = \frac{N-p}{nt^2} \left\{ (p-2) E \left[ \left( \frac{r \left( \theta' S^{-1} \hat{\theta} \right)}{\theta' \Sigma^{-1} \hat{\theta}} \right) | t \right] \right\}, \]

\[ + 2E \left[ \left( \frac{r \left( \theta' S^{-1} \hat{\theta} \right)}{\theta' \Sigma^{-1} \hat{\theta}} \right) | t \right]\), \]

\[ E \left[ \frac{\theta' \Sigma^{-1} r \left( \theta' S^{-1} \hat{\theta} \right)}{(\theta' S^{-1} \hat{\theta})^2} \right] | t \]  

\[ = \frac{(N-p)(N-p+2)}{t^2} E \left[ \left( \frac{r^2 \left( \theta' S^{-1} \hat{\theta} \right)}{\theta' \Sigma^{-1} \hat{\theta}} \right) | t \right]. \]

After all, substituting the above expressions in (2.1), completes the proof.

3 Main Results

In this section, we demonstrate the minimaxity of the estimator $\delta_r(\hat{\theta})$, under some mild conditions made on the function $r(\cdot)$. Also we give conditions under which $\delta_r(\hat{\theta})$ dominates a James-Stein type shrinkage estimator.

Theorem 3.1. Assume in the model (1.1), $\epsilon_i \sim \mathcal{E}_p(0, \Sigma, q)$. Then w.r.t. the loss function (1.14), the estimator $\delta_r(\hat{\theta})$ is minimax in the sub-class (1.7), providing
Therefore $\delta d.f.$ and non-centrality parameter

\begin{equation}
   \text{Proof:} \text{ The estimator } \hat{\theta} \text{ given by (1.11) is minimax. Therefore, in order to show that } \delta_r(\hat{\theta}) \text{ is minimax it is enough to show that } R(\hat{\theta}; \theta) - R(\hat{\delta}_r; \theta) \geq 0. \text{ But from Lemma 2.2 we have } R(\delta_r; \theta) - R(\hat{\theta}; \theta) = A + B, \text{ where }
\end{equation}

\[ A = -4(N-p) \int_{0}^{\infty} E \left[ r' \left( \hat{\theta}' S^{-1} \hat{\theta} \right) \bigg| t \right] t^{-2} \mathcal{W}(dt) \]

\[ B = + \int_{0}^{\infty} E \left[ \frac{(N-p)r(\hat{\theta}' S^{-1} \hat{\theta})}{t^{2} \left( t^{-1} \Sigma \right)^{-1} \hat{\theta}} \right] 
\times \left[ N(N-p+2)r(\hat{\theta}' S^{-1} \hat{\theta}) - 2(p-2) \right] \bigg| t \right] \mathcal{W}(dt) \]

Whereof $r(.)$ is non-decreasing, $r' \left( \hat{\theta}' S^{-1} \hat{\theta} \right) \geq 0$. Also following Srivastava and Bilodeau (1989), we have

\[ \int_{0}^{\infty} E \left[ r' \left( \hat{\theta}' S^{-1} \hat{\theta} \right) \bigg| t \right] t^{-2} \mathcal{W}(dt) \]

\[ = \int_{0}^{\infty} E \left[ r' \left( \hat{\theta}' S^{-1} \hat{\theta} \right) \bigg| t \right] t^{-2} \mathcal{W}^+(dt) - \int_{0}^{\infty} E \left[ r' \left( \hat{\theta}' S^{-1} \hat{\theta} \right) \bigg| t \right] t^{-2} \mathcal{W}^-(dt) \geq 0. \]

Therefore $A \leq 0$ and by making use of (1.18), $A < \infty$. Taking $r \leq \frac{2(p-2)}{N(N-p+2)}$, $B \leq 0$ is achieved for finite $B$, which completes the proof. But for demonstrating that $B < \infty$, it is sufficient to show that

\[ (i) \int_{0}^{\infty} t^{-1} E \left[ \frac{1}{N\hat{\theta} (t^{-1} \Sigma)^{-1} \hat{\theta}} \bigg| t \right] \mathcal{W}(dt) < \infty, \quad (3.1) \]

\[ (ii) \int_{0}^{\infty} t^{-1} E \left[ \frac{r^{2} \left( \hat{\theta}' S^{-1} \hat{\theta} \right)}{N\hat{\theta} (t^{-1} \Sigma)^{-1} \hat{\theta}} \bigg| t \right] \mathcal{W}(dt) < \infty. \]

Note that for a fixed $t$, $N\hat{\theta} (t^{-1} \Sigma)^{-1} \hat{\theta}$ has non-central chi-square distribution with $p$ d.f. and non-centrality parameter $Nt\theta' \Sigma^{-1} \theta$. In conclusion, the

\[ E \left[ \frac{1}{N\hat{\theta} (t^{-1} \Sigma)^{-1} \hat{\theta}} \right] \leq E \left[ \frac{1}{t^{p-2}} \right] = \frac{1}{p-2} \]

is observed and (3.1) (i) is followed by (1.17) (ii)-(iii).

On the other hand, using the covariance inequality (see Lemma 6.6 page 370 of Lehmann and Casella, 1998) and equation (1.19)

\[ E \left\{ \frac{r^{2} \left( \hat{\theta}' S^{-1} \hat{\theta} \right)}{N\hat{\theta} (t^{-1} \Sigma)^{-1} \hat{\theta}} \bigg| t \right\} \leq E \left( r^{2} \left( \hat{\theta}' S^{-1} \hat{\theta} \right) \bigg| t \right) E \left( \frac{1}{N\hat{\theta} (t^{-1} \Sigma)^{-1} \hat{\theta}} \bigg| t \right) < \infty, \]

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Therefore (3.1) (ii) is satisfied by (1.7) (ii)-(iii).

In the following, we develop necessary conditions for the shrinkage estimator \( \delta_r(\hat{\theta}) \) to dominate the James-Stein type estimator given by

\[
\delta_{JS}(\hat{\theta}) = \left[ 1 - \frac{p-2}{\hat{\theta}'S^{-1}\hat{\theta}} \right] \hat{\theta}.
\] (3.2)

The performance of this estimator is discussed in Srivastava and Bilodeau (1989) extensively. The way we derive the necessary conditions honesty is due to Maruyama and Strawderman (2005). But this approach has the following superiorities comparing to their analysis. (1) We study correlated errors with unknown covariance matrix while they considered uncorrelated case. (2) They derived the dominating result for multivariate normal, and we extend it for ECDs. Although the item (1) is completely different from that of uncorrelated, it is worthwhile to note that their conditions are robust under departures from normality assumptions. The following result is the same as Corollary 2.1. of Maruyama and Strawderman (2005). They could also find the class of admissible estimators under normal theory with identity covariance matrix.

**Theorem 3.2.** Assume that the function \( r(.) \) is bounded and absolutely continuous. Necessary conditions for an estimator \( \delta_r(\hat{\theta}) \) to dominate \( \delta_{JS}(\hat{\theta}) \) are that

(i) for every \( \omega \), there exists \( \omega_0(>\omega) \) such that \( r'(\omega_0) \geq 0 \),

(ii) if \( \omega r'(\omega) \) has a limiting value as \( \omega \) approaches infinity, it must be 0,

(iii) if \( r(\omega) \) has a limiting value as \( \omega \) approaches infinity and \( \omega r'(\omega) \) converges to 0 as \( \omega \) approaches infinity, the limit value for \( r(\omega) \) must be \( \frac{p-2}{N(N-p+2)} \).

**Proof:** Proof of (i) directly follows from the proof of Corollary 2.1. of Maruyama and
For the proofs of (ii) of Corollary 2.1. of Maruyama and Strawderman (2005), it is enough to show that if \( \delta_r(\hat{\theta}) \) dominates \( \delta_{JS}(\hat{\theta}) \), then for every \( \omega \), there exists \( \omega_0(\omega) \) such that \( G_r(\omega_0) \geq 0 \). In this case we follow the proof of Theorem 2.1. of Maruyama and Strawderman (2005).

Suppose to the contrary that there exists \( \omega_0 \) such that \( G_r(\omega) < 0 \) for any \( \omega \geq \omega_0 \). Under the boundedness of \( G_r(.) \), there exists an \( M(> 0) \) such that \( G_r(\omega) \leq M \) for any \( \omega \). Under the assumption of absolute continuity of \( G_r(.) \) there exists two points \( (\omega_0, \omega_1, \omega_2) \) and \( \epsilon(> 0) \) such that \( G_r(\omega) < -\epsilon \) on \( \omega \in [\omega_1, \omega_2] \). Using \( M \) and \( \epsilon \), we have: 

\[
\Delta = R(\delta_{JS}(\hat{\theta}); \theta) - R(\delta_r(\hat{\theta}); \theta) \\
= 4(N - p) \int_0^\infty E \left[ r' \left( \dot{\theta}' S^{-1} \dot{\theta} \right) \left| t \right|^2 W(dt) \right] \\
+ \int_0^\infty E \left\{ \frac{(N - p)(p - 2)}{t \dot{\theta}' (t^{-1} \Sigma)^{-1} \dot{\theta}} \times [N(N - p + 2)(p - 2) - 2(p - 2)] \left| t \right|^2 W(dt) \right\} \\
- \int_0^\infty E \left\{ \frac{(N - p)r \left( \dot{\theta}' S^{-1} \dot{\theta} \right)}{t \dot{\theta}' (t^{-1} \Sigma)^{-1} \dot{\theta}} \times \left[ N(N - p + 2)r \left( \dot{\theta}' S^{-1} \dot{\theta} \right) - 2(p - 2) \right] \left| t \right|^2 W(dt) \right\} \\
= 4(N - p) \int_0^\infty E \left[ r' \left( \dot{\theta}' S^{-1} \dot{\theta} \right) \left| t \right|^2 W(dt) \right] \\
- (N - p) \int_0^\infty E \left[ \left\{ r \left( \dot{\theta}' S^{-1} \dot{\theta} \right) - (p - 2) \right\}^2 \left( \dot{\theta}' \Sigma^{-1} \dot{\theta} \right) \left| t \right|^2 W(dt) \right] \\
= 4(N - p) \int_0^\infty E \left[ r' \left( \dot{\theta}' S^{-1} \dot{\theta} \right) \left| t \right|^2 W(dt) \right] \\
- (N - p) \int_0^\infty E \left[ \left\{ r \left( \dot{\theta}' S^{-1} \dot{\theta} \right) - (p - 2) \right\}^2 \left( \dot{\theta}' \Sigma^{-1} \dot{\theta} \right) \left| t \right|^2 W(dt) \right] \\
= (N - p) \int_0^\infty E \left[ G_r(\dot{\theta}' S^{-1} \dot{\theta}) \left| t \right|^2 W(dt) \right],
\] 

where \( \dot{\theta}' S^{-1} \dot{\theta} \) and 

\[
G_r(\omega) = \frac{r(\omega) - (p - 2)}{(n - p - 1)\omega} + 4r'(\omega).
\]
define $\mathcal{G}_{r,\epsilon}(\omega)$ as

$$\mathcal{G}_{r,\epsilon}(\omega) = \begin{cases} M & \omega \leq \omega_0 \\ 0 & \omega_0 < \omega < \omega_1 \\ -\epsilon & \omega_1 \leq \omega \leq \omega_2 \\ 0 & \omega > \omega_2 \end{cases} \quad (3.5)$$

The inequality $\mathcal{G}_{r,\epsilon}(\omega) \geq \mathcal{G}_r(\omega)$ for any $\omega$ and using equation (3.3) imply

$$\Delta = (N - p) \int_0^\infty E \left[ \mathcal{G}_r(z'B^{-1}z) \right] t^{-2}W(dt) \leq M(N - p) \int_0^\infty P_\theta \left( W \leq \omega_0 \right) t^{-2}W(dt) - \epsilon(N - p) \int_0^\infty P_\theta \left( \omega_1 \leq W \leq \omega_2 \right) t^{-2}W(dt), \quad (3.6)$$

where $W = \|X\|^2$ for $X = (t^{-1}\Sigma)^{-\frac{1}{2}}\hat{\theta}$.

Based on the properties of the model under study, it can be realized that

$$\int_0^\infty P_\theta \left( W \leq \omega_0 \right) t^{-2}W(dt) \leq \int_0^\infty P_\theta \left( \omega_1 \leq W \leq \omega_2 \right) t^{-2}W(dt) \geq 0.$$

This phenomenon is also valid for $\int_0^\infty P_\theta \left( \omega_1 \leq W \leq \omega_2 \right) t^{-2}W(dt)$.

Now let $a$ be a fixed $p$-dimensional unit vector (see Fig. 1). Then the half plane $\{x : a'x \leq \sqrt{\omega_0}\}$ includes the $p$-dimensional hyper-ellipsoid $\{x : \|x\|^2 \leq \omega_0\}$. For $\theta = (\sqrt{\omega_0} + \lambda)(t^{-1}\Sigma)^{-\frac{1}{2}}a$, we have

$$P_\theta \left( W \leq \omega_0 \right) < \int_{a'x \leq \sqrt{\omega_0}} \frac{|t^{-1}\Sigma|^{-\frac{1}{2}}}{(2\pi)^{\frac{p}{2}}} \exp \left( -\frac{\|x - \theta\|^2}{2} \right) dx \leq \exp(\lambda\sqrt{\omega_0}) \exp \left( -\frac{\|\theta\|^2}{2} \right) \times \int_{a'x \leq \sqrt{\omega_0}} \frac{|t^{-1}\Sigma|^{-\frac{1}{2}}}{(2\pi)^{\frac{p}{2}}} \exp \left( -\frac{\|x\|^2}{2} + \sqrt{\omega_0}a'x \right) dx \leq \exp(\lambda\sqrt{\omega_0}) \exp \left( -\frac{\|\theta\|^2}{2} + \frac{\omega_0}{2} \right). \quad (3.7)$$

For $N = \{x : \omega_1 \leq \|x\|^2 \leq \omega_2, \sqrt{\omega_1} \leq a'x \leq \sqrt{\omega_2}\}$ and $\theta = (\sqrt{\omega_0} + \lambda)(t^{-1}\Sigma)^{-\frac{1}{2}}a$, we
get

\[
P_\theta \left( \omega_1 \leq W \leq \omega_2 \mid t \right) > \int N \frac{\left| t^{-1} \Sigma \right|^{-\frac{3}{2}}}{(2\pi)^{\frac{N}{2}}} \exp \left( - \frac{\| x - \theta \|^2}{2} \right) dx \\
\quad \geq \exp(\lambda \sqrt{\omega_1}) \exp \left( - \frac{\| \theta \|^2}{2} \right) \\
\quad \times \int N \frac{\left| t^{-1} \Sigma \right|^{-\frac{3}{2}}}{(2\pi)^{\frac{N}{2}}} \exp \left( - \frac{\| x \|^2}{2} + \sqrt{\omega_0} a' x \right) dx.
\]

By making use of the equations (3.6)-(3.8), we can obtain

\[
\Delta \leq (N - p) \int_0^\infty c_1 \exp \left( \sqrt{\omega_0} \lambda - \frac{\| \theta \|^2}{2} \right) \left( 1 - c_2 \exp \left[ (\sqrt{\omega_1} - \sqrt{\omega_0}) \lambda \right] \right) t^{-2} W(dt),
\]

where \( c_1 = M \exp \left( \frac{\omega_0}{2} \right) \) and

\[
c_2 = \frac{\epsilon}{M} \exp \left( \frac{\omega_0}{2} \right) \int N \frac{\left| t^{-1} \Sigma \right|^{-\frac{3}{2}}}{(2\pi)^{\frac{N}{2}}} \exp \left( - \frac{\| x \|^2}{2} + \sqrt{\omega_0} a' x \right) dx.
\]

Since \( c_1 \) and \( c_2 \) do not depend on \( \lambda \), \( \Delta \) is negative for sufficiently large \( \lambda \). This completes the proof.

Subsequently, we continue on giving an example of the function \( r(.) \).

Let

\[
r_*(x) = \frac{(p - 2)b}{1 + cx^{-1}}, \quad b = \frac{1}{N(N - p + 2)} \quad \text{and} \quad c \in \mathbb{R}_.
\]
Then we have
\[ 0 < r_*(x) \leq \frac{2(p - 2)}{N(N - p + 2)}, \quad Dr_*(x) = \frac{b(p - 2)^2}{(x + c)^2} > 0, \]
\[ \lim_{x \to \infty} r_*(x) = b(p - 2), \quad \lim_{x \to \infty} xDr_*(x) = 0, \]
which satisfy the conditions of Theorems 3.1 and 3.2.

The resulting shrinkage estimator using the function \( r_*(x) \) in (3.9), is the generalized type of Alam and Thompson (1969) estimator given by
\[
\delta_{r_*}(\hat{\theta}) = \left\{ 1 - \frac{r_*(\hat{\theta}' S^{-1} \hat{\theta})}{\hat{\theta}' S^{-1} \hat{\theta}} \right\} \hat{\theta} = \left\{ 1 - \frac{(p - 2)b}{\hat{\theta}' S^{-1} \hat{\theta} + c} \right\} \hat{\theta}.
\]

Also note that based on (1.18) and (1.19), the required conditions of the Schwartz space, for this example, are \( b(p - 2)^2 E(X + c)^{-2} < \infty \) and \( b^2(p - 2)^2 E \left( \frac{X}{X+c} \right)^2 < \infty \), which summarises to the sole condition \( E \left( \frac{X}{X+c} \right)^2 < \infty \).

4 Conclusions

In this paper, we utilized a broad class of Stein-type estimators which outperformed the consistent estimator of the mean of an elliptically contoured model. It is worthwhile to note that the minimaxity conditions are identical to that obtain under normal assumptions. Hence, those are robust with respect to departures from normality. Moreover, Bayesian perspective does not offer systematic generality over classical approaches taking flat prior information. The class of estimators considered in Srivastava and Bilodeau (1989) is broaden into a more general shrinkage estimators; and as a result, this work dominates series of Brandwein’s and Berger’s papers. To the best of my knowledge, it is not simple to prove the admissibility of the class of Bayes shrinkage estimator \( \delta_{r_*}(\theta) \) under elliptical symmetry and there exists no study in ECDs when the covariance matrix in unknown. But one may demonstrate it through taking the harmonic prior \( \|\theta\|^{2-p} \) for \( \pi(\theta) \) in (1.9) which leaves for further research. In this case, one may follow the work of Maruyama (2004) under the integral representation of elliptical models in (1.4). In this case, the work of Fourdrinier et al. (2003) has some interesting features.

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