Remarkable relations between the central binomial series, Eulerian polynomials, and poly-Bernoulli numbers

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Abstract. The central binomial series at negative integers are expressed as a linear combination of values of certain two polynomials. We show that one of the polynomials is a special value of the bivariate Eulerian polynomial and the other polynomial is related to the antidiagonal sum of poly-Bernoulli numbers. As an application, we prove Stephan’s observation from 2004.

1. Introduction

The central binomial series is a Dirichlet series defined by

\[ \zeta_{CB}(s) = \sum_{n=0}^{\infty} \frac{1}{n^{s} \binom{2n}{n}} \quad (s \in \mathbb{C}). \]

Borwein, Broadhurst, and Kamnitzer [6] studied special values \( \zeta_{CB}(k) \) at positive integers and recovered some remarkable connections. A classical evaluation is \( \zeta_{CB}(4) = \frac{1728}{945} = \frac{4}{3} \zeta(4) \). In particular, for \( k \geq 2 \) Borwein–Broadhurst–Kamnitzer showed that \( \zeta_{CB}(k) \) can be written as a \( \mathbb{Q} \)-linear combination of multiple \( \zeta \) values and multiple Clausen and Glaisher values.

On the other hand, Lehmer [15] proved that for \( k \leq 1 \), \( \zeta_{CB}(k) \) is a \( \mathbb{Q} \)-linear combination of \( 1 \) and \( \pi/\sqrt{3} \). For example, we have

\[ \zeta_{CB}(1) = \frac{1}{3} \frac{\pi}{\sqrt{3}}, \quad \zeta_{CB}(0) = \frac{1}{3} + \frac{2}{9} \frac{\pi}{\sqrt{3}}, \quad \zeta_{CB}(-1) = \frac{2}{3} + \frac{2}{9} \frac{\pi}{\sqrt{3}}, \quad \zeta_{CB}(-2) = \frac{4}{3} + \frac{10}{27} \frac{\pi}{\sqrt{3}}. \]

He considered the general sum

\[ \sum_{n=1}^{\infty} \frac{(2n)^{2n}}{n^{2n} \binom{2n}{n}} = 2 \frac{x \arcsin(x)}{\sqrt{1-x^2}} \quad (|x| < 1) \]

and its derivatives to derive interesting series evaluations. More precisely, Lehmer provided the following explicit formula for the special values \( \zeta_{CB}(k) \) at negative integers. Define two sequences of polynomials \( (p_k(x))_{k \geq -1} \) and \( (q_k(x))_{k \geq -1} \) by the initial values \( p_{-1}(x) = 0, q_{-1}(x) = 1 \) and the recursion

\[ p_{k+1}(x) = 2(kx + 1)p_k(x) + 2x(1-x)p_k'(x) + q_k(x), \]

\[ q_{k+1}(x) = (2(k+1)x + 1)q_k(x) + 2x(1-x)q_k'(x). \]

Then for \( k \geq -1 \), we have

\[ \sum_{n=1}^{\infty} \frac{(2n)^k (2x)^{2n}}{n^{2n} \binom{2n}{n}} = \frac{x}{(1-x^2)^{k+1}} \left( x \sqrt{1-x^2} p_k(x^2) + \arcsin(x) q_k(x^2) \right). \]

Consequently,

\[ \zeta_{CB}(-k) = \frac{1}{3} \left( \frac{1}{3} \right)^k \left( \frac{2}{3} \right) p_k \left( \frac{1}{4} \right) + \frac{1}{3} \left( \frac{2}{3} \right)^{k+1} \left( \frac{1}{4} \right) \frac{\pi}{\sqrt{3}} \in \mathbb{Q} + \mathbb{Q} \frac{\pi}{\sqrt{3}}. \]

The first few polynomials are: \( p_0(x) = 1, p_1(x) = 3, p_2(x) = 8x + 7; \) and \( q_0(x) = 1, q_1(x) = 2x + 1, q_2(x) = 4x^2 + 10x + 1. \)
In 2004, Stephan [23, A098830] observed that the rational part of (1.4) is nothing but (a third of) a sum of poly-Bernoulli numbers of negative indices. Poly-Bernoulli numbers $B_n^{(k)}$ are a generalization of classical Bernoulli numbers using polylogarithm functions and were introduced by Kaneko [12]. We will give a precise definition in Section 3.

**Conjecture 1.1.** [13, stated by Kaneko, Stephan’s conjecture] For any $n \geq 0$,

$$
\left( \frac{2}{3} \right)^n p_n \left( \frac{1}{4} \right) = \sum_{k=0}^{n} p_{n-k}.
$$

In this article, we connect both polynomials $p_n(x)$ and $q_n(x)$ to known numbers and polynomials. More precisely, we prove Stephan’s conjecture (relating this way $p_n(x)$ to the poly-Bernoulli numbers) using the fact that the polynomial sequence $q_n(x)$ is a generalization of the classical Eulerian polynomials.

### 2. The polynomials $q_n(x)$ and bivariate Eulerian polynomials

Eulerian polynomials have been studied by Euler himself. Since then they have been studied and became classical. Several extensions, generalizations and applications are known today.

Let $S_n$ denote the set of permutations $\pi = \pi_1 \pi_2 \ldots \pi_n$ of $[n] = \{1, 2, \ldots, n\}$. For each $\pi \in S_n$, the excedance set is defined as $\text{Exc}(\pi) = \{ i \in [n] \mid \pi_i > i \}$. We set $\text{exc}(\pi) = |\text{Exc}(\pi)|$. It is well-known that the Eulerian number $A(n, k)$ counts the number of permutations $\pi \in S_n$ with $\text{exc}(\pi) = k$. For instance, $A(3, 1) = 4$ because there are 4 permutations of $\{1, 2, 3\}$ with $\text{exc}(\pi) = 1$, namely, $132, 213, 312, 321$. A map $f : S_n \to \mathbb{Z}_{\geq 0}$ satisfying $|\{ \pi \in S_n \mid f(\pi) = k \}| = A(n, k)$ is often called an Eulerian statistic. The map exc is an example of Eulerian statistics. By Foata’s fundamental transformation, it is also known that the number of permutations with $k$ excedances is the same as the number of permutations with $k$ descents, or equivalently formulated, with $k + 1$ ascending runs, (see Bóna’s book [5]).

The Eulerian polynomial is defined by

$$
A_n(x) = \sum_{\pi \in S_n} x^{\text{exc}(\pi)} = \sum_{k=0}^{n} A(n, k) x^k.
$$

The generating function of the Eulerian polynomials is given as

$$
\sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!} = \frac{1 - x}{e^{t(x-1)} - x}.
$$

For a more detailed history and properties on the Eulerian numbers (polynomials) and Eulerian statistics, the articles [5, 9, 19] are good references.

We recall now a generalization of the Eulerian polynomial introduced by Foata–Schützenberger [9, Chapter IV-3]. Here we define a shifted version. Let $\text{cyc}(\pi)$ denote the number of cycles in the disjoint cycle representation of $\pi \in S_n$.

**Definition 2.1** (Bivariate Eulerian polynomial). For any integer $n \geq 0$, let $F_0(x, y) = 1$ and define

$$
F_n(x, y) = \sum_{\pi \in S_n} x^{\text{exc}(\pi)} y^{\text{cyc}(\pi)}, \quad (n > 0).
$$

**Example 2.2.** We have $F_3(x, y) = y^3 + 3xy^2 + x^2y + xy$.

| $S_3$ | 123 = (1)(2)(3) | 132 = (1)(23) | 213 = (12)(3) | 231 = (123) | 312 = (132) | 321 = (13)(2) |
|-------|-----------------|---------------|-----------------|-----------------|-----------------|-----------------|
| exc(\pi) | 0 | 1 | 1 | 2 | 1 | 1 |
| cyc(\pi) | 3 | 2 | 2 | 1 | 1 | 2 |

Table 1. Permutations in $S_3$ with their weights.

The generating function of the bivariate Eulerian polynomials is given by

$$
\mathcal{F}(x, y; t) := \sum_{n=0}^{\infty} F_n(x, y) \frac{t^n}{n!} = \left( \frac{1 - x}{e^{t(x-1)} - x} \right)^y.
$$

Savage–Viswanathan [22] derived several identities for the polynomials. Here we recall their recursion formula.

\[(2.1)\]
Proposition 2.3. For $n \geq 0$,

$$F_{n+1}(x, y) = \left( x(1-x) \frac{d}{dx} + nx + y \right) F_n(x, y),$$

with the initial value $F_0(x, y) = 1$.

Note that by the definition, we have $F_n(x, 1) = A_n(x)$. Moreover, for $y = r \in \mathbb{Z}_{\geq 2}$, the polynomials $F_n(x, r)$ are the $r$-Eulerian polynomials originally studied by Riordan [20]. In addition, we have $F_{n+1}(x, -1) = -(x-1)^n$ and $F_{n+1}(1, y) = y(y+1) \cdots (y+n)$ for any $n \geq 0$.

The surprising fact is however that the values at $y = 1/k$, for any positive integer $k$, have also nice combinatorial interpretations. Namely, for a sequence $s = (s_i)_{i \geq 1}$ of positive integers, let the $s$-inversion sequence of length $n$ be defined as

$$I_n^{(s)} = \{(e_1, \ldots, e_n) \in \mathbb{Z}^n | 0 \leq e_i < s_i \text{ for } 1 \leq i \leq n\}.$$

The ascent statistic on $e \in I_n^{(s)}$ is

$$\text{asc}(e) = \left\{ 0 \leq i < n : \frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}} \right\},$$

with the convention that $e_0/s_0 = 0$. Then the $s$-Eulerian polynomials are defined by

$$E_n^{(s)}(x) = \sum_{e \in I_n^{(s)}} x^{\text{asc}(e)}.$$

For more properties of the $s$-Eulerian polynomials, see also Savage–Visontai [21]. Note that for $s = (i)_{i \geq 1} = (1, 2, 3, \ldots)$, the $s$-Eulerian polynomials are the classical Eulerian polynomials, $E_n^{(s)} = A_n(x)$. Savage–Viswanathan [22] showed that for $s = ((i-1)k+1)_{i \geq 1} = (1, k+1, 2k+1, 3k+1, \ldots)$ where $k$ is a positive integer, it holds

$$E_n^{(s)}(x) = k^n F_n \left( x, \frac{1}{k} \right).$$

They called the coefficients in this special case the $1/k$-Eulerian numbers. The $1/k$-Eulerian numbers play role in the theory of $k$-lecture hall polytopes [22] and enumerate certain statistics in $k$-Stirling permutations [17]. We now show that for $k = 2$, the $E_n^{(1, 3, 5, \ldots)}(x) = 2^n F_n(x, 1/2)$ is the same as the $q_n(x)$ polynomial sequence in Lehmer’s identity.

| $n$ | $2^n F_n(x, 1/2)$ | $q_n(x)$ |
|-----|-----------------|----------|
| -1  | -               | 1        |
| 0   | 1               | 1        |
| 1   | 1               | 2x + 1   |
| 2   | $2x + 1$        | $4x^2 + 10x + 1$ |
| 3   | $4x^2 + 10x + 1$| $8x^3 + 60x^2 + 36x + 1$ |
| 4   | $8x^3 + 60x^2 + 36x + 1$| $16x^4 + 296x^3 + 516x^2 + 116x + 1$ |
| 5   | $16x^4 + 296x^3 + 516x^2 + 116x + 1$| $\ldots$ |

Table 2. The polynomials $2^n F_n(x, 1/2)$ and $q_n(x)$.

Theorem 2.4. The generating function

$$Q(x, t) := \sum_{n=0}^{\infty} q_{n-1}(x) \frac{t^n}{n!}$$

equals $\mathcal{F}(x, 1/2; 2t)$, that is, $q_n(x) = 2^{n+1} F_{n+1}(x, 1/2)$ for any $n \geq -1$.

PROOF. By translating the recursion in (1.2), the generating function $Q(x, t)$ is characterized by the differential equation

$$\left( (2xt - 1) \frac{d}{dt} + 2x(1-x) \frac{d}{dx} + 1 \right) Q(x, t) = 0$$

and the initial condition $Q(x, 0) = 1$. We can check that the function

$$\left( \frac{1-x}{e^{2t(x-1)} - x} \right)^{1/2} = \mathcal{F}(x, 1/2; 2t) = \sum_{n=0}^{\infty} 2^n F_n(x, 1/2) \frac{t^n}{n!}$$

satisfies these conditions by a direct calculation.

The generating function \( Q(x, t) = \mathcal{F}(x, 1/2; 2t) \) tells us that the coefficients of \( q_n(x) \) count perfect matchings with the restriction on the number of matching pairs have odd smaller entries (see [16] and [23, A185411]) and \( q_n(1) = (2n + 1)!! \).

The relation between the polynomials \( F_n(x, 1/2) \) and \( q_n(x) \) shed light on a proof of Stephan’s conjecture which follows in the next section.

3. The polynomials \( p_n(x) \) and a proof of Stephan’s conjecture

In this section, we focus on the polynomial sequence \( p_n(x) \) in the expression of Lehmer (1.3). We prove the observation of Stephan who noticed a relation of the sequence with the poly-Bernoulli numbers. Poly-Bernoulli numbers were introduced by Kaneko [12] by the polylogarithm function (\( \text{Li}_k(z) = \sum_{m=1}^{\infty} z^m / m^k \) for any integer \( k \)) as a generalization of the classical Bernoulli numbers. The poly-Bernoulli numbers \( B_n^{(k)} \in \mathbb{Q} \) are defined by

\[
\sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!} = \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}}.
\]

Poly-Bernoulli numbers have attractive properties. In particular, the values with negative indices \( k \) enumerate several combinatorial objects, (see for instance, [3, 4, 8, 11] and the references therein).

As one of the most basic properties, Arakawa and Kaneko [1] showed that

\[
\sum_{k=0}^{n} (-1)^k B_{n-k}^{(-k)} = 0
\]

holds for any positive integer \( n \). Since then, several authors have generalized the formula for the alternating anti-diagonal sum in [14, 18], but not much is known about the anti-diagonal sum in Conjecture 1.1.

In most of the combinatorial interpretations, the roles of \( n \) and \( k \) are separately significant, hence it is not natural to consider the anti-diagonal sum. However, one of the interpretations, where this is natural, is the set of permutations with ascending-to-max property [10]. A permutation \( \pi \in \mathfrak{S}_n \) is called ascending-to-max, if for any integer \( i, 1 \leq i \leq n - 2 \)

a. if \( \pi^{-1}(i) < \pi^{-1}(n) \) and \( \pi^{-1}(i + 1) < \pi^{-1}(n) \) then \( \pi^{-1}(i) < \pi^{-1}(i + 1) \), and
b. if \( \pi^{-1}(i) > \pi^{-1}(n) \) and \( \pi^{-1}(i + 1) > \pi^{-1}(n) \), then \( \pi^{-1}(i) > \pi^{-1}(i + 1) \).

In other words: record a permutation in one-line notation and draw an arrow from value \( i \) to \( i + 1 \) for each \( i \). Then, the permutation has the ascending-to-max property if all the arrows starting from the left of \( n \) point forward and all the arrows starting from an element to the right of \( n \) point backward. For instance, 47518362 has the property, but 41385762 has not. It follows from the results of Bényi and Hajnal [2] that the number of permutations \( \pi \in \mathfrak{S}_{n+1} \) with the ascending-to-max property is given by the anti-diagonal sum \( b_n = \sum_{k=0}^{n} B_{n-k}^{(-k)} \). However, no explicit formula or recursion was known about the sequence \( b_n \).

Our first result is a recursion for the sequence \( b_n \).

**Proposition 3.1.** The sequence \( (b_n)_{n \geq 0} \) satisfies the recurrence relation \( b_0 = 1 \) and

\[
3b_{n+1} = 2b_n + \sum_{k=0}^{n} \binom{n+1}{k} b_k + 3.
\]

In order to prove this theorem, we need some preparations. Recall that by [1, p.163], we have

\[
\sum_{n=0}^{\infty} b_n x^n = \sum_{j=0}^{\infty} \frac{(j!)^2 x^{2j}}{(1-x)^2(1-2x)^2 \cdots (1-(j+1)x)^2} = \frac{1}{(1-x)^2} \sum_{j=0}^{\infty} f_j \left( 2 - \frac{1}{x} - 2 - \frac{1}{x} \right),
\]

where \( (x)_j = x(x+1)(x+2) \cdots (x+j-1) \) is the Pochhammer symbol and we put

\[
f_j(x, y) = \frac{(j!)^2}{(x)_j(y)_j}.
\]
By a direct calculation, we have
\[
\sum_{n=k=0}^{\infty} \binom{n+k}{k} b_k x^n = \sum_{k=0}^{\infty} b_k x^k \sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \sum_{k=0}^{\infty} b_k x^{k-1} \left( \frac{1}{(1-x)^{k+1}} - 1 \right).
\]

\[
= \frac{1-x-x^2}{x(1-2x)^2} \sum_{j=0}^{\infty} f_j \left( 3 - \frac{1}{x}, 3 \frac{1}{x} \right) - \frac{1}{x(1-x)^2} \sum_{j=0}^{\infty} f_j \left( 2 - \frac{1}{x}, 2 - \frac{1}{x} \right).
\]

Thus, the desired recursion in (3.1) is equivalent to
\[
\frac{2(2-x)}{(1-x)^2} \sum_{j=0}^{\infty} f_j \left( 2 - \frac{1}{x}, 2 - \frac{1}{x} \right) = \frac{3}{1-x} + \frac{1-x}{(1-2x)^2} \sum_{j=0}^{\infty} f_j \left( 3 - \frac{1}{x}, 3 \frac{1}{x} \right).
\]

To prove (3.3), we derive a useful equation.

Lemma 3.2. For any \(j \in \mathbb{Z}_{\geq 0}\), we have
\[
(x-1)(x-2) (f_j(x-2,y) - f_{j-1}(x-2,y)) + (x-1)(2x-5)f_{j-1}(x-1,y)
\]
\[- (x-1)(x-y-1)f_j(x-1,y) - (x-2)^2 f_{j-1}(x,y)
\]
\[
= \begin{cases} 
(x-1)(y-1) & \text{if } j = 0, \\
0 & \text{if } j > 0,
\end{cases}
\]
where we put \(f_{-1}(x,y) = 0\).

Proof. By direct calculation, one can verify it.

Proof of Proposition 3.1. We prove (3.3). Setting \(x \to 3 - 1/x\) and \(y \to 2 - 1/x\) and applying Lemma 3.2, we obtain
\[
f_j \left( 2 - \frac{1}{x}, 2 - \frac{1}{x} \right) = \frac{(1-x)^2}{(1-2x)(2-x)} f_j \left( 3 - \frac{1}{x}, 2 - \frac{1}{x} \right)
\]
\[- \frac{1-x}{2-x} \left( f_{j+1} \left( 1 - \frac{1}{x}, 2 - \frac{1}{x} \right) - f_j \left( 1 - \frac{1}{x}, 2 - \frac{1}{x} \right) \right).
\]

Summing up both sides over \(j = 0, 1, 2, \ldots\), we have
\[
\sum_{j=0}^{\infty} f_j \left( 2 - \frac{1}{x}, 2 - \frac{1}{x} \right) = \frac{1-x}{2-x} + \frac{(1-x)^2}{(1-2x)(2-x)} \sum_{j=0}^{\infty} f_j \left( 3 - \frac{1}{x}, 2 - \frac{1}{x} \right).
\]

From Lemma 3.2 again for \(x \to 3 - 1/x\) and \(y \to 3 - 1/x\), we conclude (3.3).

Remark 3.3. Unfortunately, we could not provide a combinatorial proof for this recurrence, though it would be very interesting to find one using for instance the permutations with the ascending-to-max property.

To relate the sequence \((b_n)_{n \geq 0}\) to the polynomial sequence \((p_n(x))_{n \geq -1}\), we next derive the generating function for \(p_n(x)\).

Proposition 3.4. We have
\[
P(x, t) := \sum_{n=0}^{\infty} p_{n-1}(x) \frac{t^n}{n!} = \frac{e^{(1-x)t} (\arcsin(x^{1/2} e^{(1-x)t}) - \arcsin(x^{1/2}))}{x^{1/2} (1 - xe^{2(1-x)t})^{1/2}}.
\]

Proof. From (1.3), we have
\[
\sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \binom{2n}{n} k^{k-1} (2x)^{2n} \frac{k}{k!} = \frac{x}{(1-x^2)^{1/2}} \left( x \sqrt{1-x^2} P \left( x^2, \frac{t}{1-x} \right) + \arcsin(x) Q \left( x^2, \frac{t}{1-x^2} \right) \right).
\]

By applying (1.3) with \(k = -1\) again, the left-hand side equals
\[
\sum_{n=1}^{\infty} \frac{(2n)^{-1} (2xe^{t})^{2n} (2n)!}{(n^n)!} = \frac{x e^t \arcsin(x e^t)}{(1-x e^{2t})^{1/2}}.
\]
Thus, Combining with Theorem 2.4, we have
\[
P(x^2, t) = \frac{e^{\arcsin(xe^t)}}{x(1 - x^2e^{2t})^{1/2}} \frac{\arcsin(x)}{x\sqrt{1 - x^2}} F\left(x^2, \frac{1}{2}, 2t\right)
\]
which implies the claim. \(\square\)

We define the sequence \(a_n\) as special values of \(p_n(x)\),
\[
a_n = \left(\frac{2}{3}\right)^n p_n\left(\frac{1}{4}\right).
\]

Using the generating function of \(p_n(x)\), we obtain the recurrence formula that the sequence \((a_n)_{n \geq 0}\) satisfies.

**Proposition 3.5.** The sequence \((a_n)_{n \geq 0}\) defined in (3.4) satisfies \(a_0 = 1\) and
\[
3a_{n+1} = 2a_n + \sum_{k=0}^{n} \binom{n+1}{k} a_k + 3.
\]

**Proof.** By Proposition 3.4, the generating function for \(a_n\) is given by
\[
\sum_{n=0}^{\infty} a_n \frac{t^{n+1}}{(n+1)!} = \frac{3}{2} P\left(\frac{1}{4}, \frac{2}{3}\right) = \frac{6e^{t/2}(\arcsin(e^{t/2}/2) - \arcsin(1/2))}{(4 - e^t)^{1/2}}.
\]
Since this function satisfies the differential equation
\[
(4 - e^t) \frac{d}{dt} - 2 \frac{3}{2} P\left(\frac{1}{4}, \frac{2}{3}\right) = 3e^t,
\]
the coefficients \(a_n\) satisfy the desired recurrence formula. \(\square\)

In conclusion, we have the main theorem.

**Theorem 3.6.** Conjecture 1.1 is true, i.e. for any \(n \geq 0\),
\[
\left(\frac{2}{3}\right)^n p_n\left(\frac{1}{4}\right) = \sum_{k=0}^{n} P_{n-k}^{(-k)}.
\]

**Proof.** Proposition 3.1 and Proposition 3.5 imply the theorem. \(\square\)

In the course of our proof, we obtain two types of generating functions in (3.2) and (3.5) for the sequences \((a_n)_{n \geq 0}\) = \((b_n)_{n \geq 0}\). As a corollary, we have an explicit formula for the anti-diagonal sum, (see [3, p.24]).

**Corollary 3.7.**
\[
b_n = \sum_{k=0}^{n} B_{n-k}^{(-k)} = \frac{(-1)^{n+1}}{2} \sum_{j=1}^{n+1} (-1)^{j} j! \binom{n+1}{j} \left(\frac{2j}{3}\right) \sum_{i=0}^{j-1} \frac{3^i}{(2i+1)(2i)!}.
\]

**Proof.** The result follows from the explicit formula by Borwein–Girgensohn [7] and Theorem 3.6. \(\square\)

As a final remark, we show that the polynomial \(p_n(x)\) can also be expressed in terms of bivariate Eulerian polynomials.

**Theorem 3.8.** For any \(n \geq 0\), we have
\[
p_n(x) = 2^n \sum_{k=0}^{n} \binom{n+1}{k} F_{n-k}(x, 1/2) F_k(x, 1/2).
\]

**Proof.** Consider
\[
P(x, t) = \frac{e^{(1-x)t}(\arcsin(x^{1/2}e^{(1-x)t}) - \arcsin(x^{1/2}))}{x^{1/2}(1 - xe^{2(1-x)t})^{1/2}}
\]
\[
= \mathcal{F}\left(x^2, \frac{1}{2}, 2t\right) \frac{1}{x^{1/2} (1-x)^{1/2}}(\arcsin(x^{1/2}e^{(1-x)t}) - \arcsin(x^{1/2})).
\]
Since
\[
\frac{d}{dt} x^{1/2}(1 - x)^{1/2} \left( \arcsin(x^{1/2}e^{(1-x)t}) - \arcsin(x^{1/2}) \right) = \mathcal{F} \left( x, \frac{1}{2}, 2t \right),
\]
it holds that
\[
\frac{1}{x^{1/2}(1 - x)^{1/2}} \left( \arcsin(x^{1/2}e^{(1-x)t}) - \arcsin(x^{1/2}) \right) = \sum_{n=0}^{\infty} \frac{2^n F_n(x, 1/2)}{(n + 1)!} t^{n+1}.
\]
Thus, we have
\[
P(x, t) = \sum_{n=0}^{\infty} \left( 2^{n-1} \sum_{k=0}^{n-1} \binom{n}{k} F_{n-k-1}(x, 1/2) F_k(x, 1/2) \right) \frac{t^n}{n!},
\]
which concludes the proof. \(\square\)

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