Hyperkähler structures and
infinite-dimensional Grassmannians

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Abstract

In this paper, we describe an example of a hyperkähler quotient of a Banach manifold by a Banach Lie group. Although the initial manifold is not diffeomorphic to a Hilbert manifold (not even to a manifold modelled on a reflexive Banach space), the quotient space obtained is a Hilbert manifold, which can be furthermore identified either with the cotangent space of a connected component \( Gr^j_{res} \), \( j \in \mathbb{Z} \), of the restricted Grassmannian or with a natural complexification of this connected component, thus proving that these two manifolds are isomorphic hyperkähler manifolds. Moreover, Kähler potentials associated with the natural complex structure of the cotangent space of \( Gr^j_{res} \) and with the natural complex structure of the complexification of \( Gr^j_{res} \) are computed using Kostant-Souriau’s theory of prequantization.

Résumé

Dans cet article, nous présentons un exemple de quotient hyperkählérien d’une variété banachique par un groupe de Lie banachique. Bien que la variété initiale ne soit pas difféomorphe à une variété hilbertienne (ni même à une variété modélée sur un espace de Banach réflexif), l’espace quotient obtenu est une variété hilbertienne, qui peut être identifiée, selon la structure complexe distinguée, soit à l’espace cotangent d’une composante connexe \( Gr^j_{res} \) \( j \in \mathbb{Z} \) de la grassmannienne restreinte, soit à une complexification naturelle de cette même composante connexe. De plus, les potentiels kählériens associés respectivement à la structure complexe naturelle de l’espace cotangent de \( Gr^j_{res} \) et à la structure complexe naturelle de la complexification de \( Gr^j_{res} \) sont calculés à l’aide de la théorie de préquantisation de Kostant-Souriau.

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1 Introduction

The restricted Grassmannian in a Hilbert manifold closely related to Loop groups (see [38], [41]) and equations of the KdV type (see [39], [37], [40], [43]). It is also a Hermitian symmetric space. Therefore it is natural to ask (see below) whether its complexification or its cotangent space admit hyperkähler structures.

Recall that a Kähler manifold of finite dimension is a Riemannian manifold endowed with a complex structure that is parallel for the Levi-Civita connection, or equivalently, a manifold equipped with a closed symplectic real form, called the Kähler form, and a compatible integrable complex structure. A hyperkähler manifold of finite dimension is a manifold endowed with a Riemannian metric $g$ and three complex structures $I, J, K$ such that $IJK = -1$, and $g$ is Kähler with respect to each complex structure. Hence a hyperkähler manifold admits three Kähler forms $\omega_1, \omega_2, \omega_3$ and the choice of one complex structure, for instance $I$, allows to build a holomorphic symplectic form, namely $\Omega = \omega_2 + i\omega_3$.

D. Kaledin and B. Feix have found independently in [16] and [12] that, given a finite-dimensional manifold $N$ endowed with a Kähler metric $g_N$, there exists a hyperkähler metric $g$ defined on a neighborhood of the zero section of the cotangent space $T^*N$ of $N$, compatible with the natural holomorphic symplectic structure of the cotangent space, and such that the restriction of $g$ to $N$ is $g_N$. In addition, $g$ is unique if one requires $S^1$-invariance. D. Kaledin uses for his proof the theory of Hodge manifolds, whereas B. Feix uses twistor spaces. As far as we know, this result has not been extended to the infinite-dimensional Banach setting. Moreover this existence result does not lead to an explicit expression of the metric and examples of explicit hyperkähler metrics are rare.

O. Biquard and P. Gauduchon provided in [4] a construction of hyperkähler metrics on cotangent bundles of Hermitian symmetric spaces, and in [5] hyperkähler metrics on coadjoint orbits of symmetric type of a complex semi-simple Lie group. Furthermore they established formulas for Kähler potentials that allow explicit expressions of these metrics. In [5] the same authors identified these hyperkähler structures, showing that the cotangent space and the complexified coadjoint orbit are, in the case of Hermitian symmetric spaces of finite dimension, two aspects of the same hyperkähler object, which appear in accordance with the chosen complex structure within the 2-sphere of complex structures.

In the case of the cotangent space of the Grassmannian $Gr(p,n)$ of subspaces of dimension $p$ in $\mathbb{C}^n$, the aforementioned hyperkähler structure can be obtained by a hyperkähler quotient, and the corresponding Kähler potential can be computed via the theory of Kostant-Souriau’s prequantization. It is the study of this particular example of Hermitian symmetric space that leads to the theory of O. Biquard and P. Gauduchon as developed in [4]. In the present work, we show that each connected component $Gr_{res}^j$ ($j \in \mathbb{Z}$) of the restricted Grassmannian $Gr_{res}$ introduced by Pressley and Segal in [38], gives an example of an infinite-dimensional Hermitian symmetric space whose cotangent space can be obtained by an infinite-dimensional hyperkähler quotient of a Banach manifold by a Banach Lie group. Moreover, we show that the resulting quotient space can also be identified with a natural complexification $O_C$ of $Gr_{res}^j$, also called complexified orbit, consisting of pairs $(P, Q)$ of elements of $Gr_{res}^j$ such that $P \cap Q = \{0\}$. In this way, the study of this particular example provides a first step towards the generalization of the aforementioned results of O. Biquard and P. Gauduchon to the infinite-dimensional setting. The full generalization of these results has been carried out in [39] by the construction of hyperkähler metrics on complexifications of Hermitian-symmetric affine coadjoint orbits of semi-simple $L^*$-groups of compact type, and by the identification of these complexifications
with the cotangent spaces of the orbits under consideration. This generalization is based on Mostow’s Decomposition Theorem (see [13]) and on the notion of strongly orthogonal roots of a $L^*$-algebra (see the Appendix in [19]).

The theory of symplectic quotients was initiated by J. E. Marsden and A. Weinstein in [25]. In finite dimension it was used in particular to construct new examples of symplectic manifolds. In [27], J. E. Marsden and T. Ratiu applied this theory to infinite-dimensional manifolds and obtained new developments of V. I. Arnold’s idea that fluid motion equations are the equations of geodesics on a suitable infinite-dimensional Lie group. Another example of infinite-dimensional symplectic reduction is given by J. E. Marsden and A. Weinstein in [30] in relation with the Maxwell-Vlasov equation. (For an overview of applications of the symplectic reduction see [27].) Kähler and hyperkähler reductions are refinements of this theory. An infinite-dimensional version based on the study of Nahm’s equations was used by P. B. Kronheimer in [22] and [23] in order to construct hyperkähler structures on maximal semi-simple and nilpotent coadjoint orbits of semi-simple complex (finite-dimensional) Lie groups. These results were generalized to all orbits by O. Biquard in [3] and A. G. Kovalev in [21]. In [24], the study of the same Nahm’s equations but with different boundary conditions leads P. B. Kronheimer to prove that the cotangent space of any complex Lie group carries a hyperkähler structure.

In [15], V. G. Kac classifies infinite-dimensional Lie groups and algebras into four (overlapping) categories:

1. Groups of diffeomorphisms of manifolds and the corresponding Lie algebras of vector fields;
2. Lie groups (resp. Lie algebras) of maps from a finite-dimensional manifold to a finite-dimensional Lie group (resp. Lie algebra);
3. Classical Lie groups and algebras of operators on Hilbert and Banach spaces;
4. Kac-Moody algebras.

The examples of infinite-dimensional reduction mentioned above concern only the first two classes of groups. In this work we construct an example of hyperkähler reduction involving the third class of groups.

The structure of the paper is as follows. In Section 2 we introduce the necessary background on Kähler and hyperkähler quotients of Banach manifolds as well as the theory of Kähler potential on a quotient induced by Kostant-Souriau’s theory of prequantization. In the infinite-dimensional setting, the definition of a hyperkähler manifold has to be specified so as to avoid problems such as the possible non-existence of a Levi-Civita connection for weak Riemannian metrics. The conditions needed to get a smooth Kähler structure on a Kähler quotient of a Banach manifold by a Banach Lie group have to be strengthened in comparison to the finite-dimensional case. In this Section, a basic definition of the notion of stable manifold associated with a level set and an holomorphic action of a complex Banach Lie group on a Kähler manifold is given. It is the most adapted to our purpose, but can be related to more sophisticated definitions as the one appearing in the Hilbert-Mumford Geometric Invariant Theory (see [33]) or the one appearing in Donaldson’s work (see [11] for an exposition of the circle of ideas around this notion). In Subsection 2.2, the existence of a smooth projection of the stable manifold to the level set is proved and used in the identification of a smooth Kähler quotient with the complex quotient of the associated stable manifold by the complexification of the group. In Subsection 2.3, we give a survey of the theory of Kähler potential on a Kähler quotient which includes a natural generalization to the Banach setting of the formula for the Kähler potential proved by O. Biquard and P. Gauduchon in Theorem 3.1 in [11].

In Section 3, we construct a smooth hyperkähler quotient of the tangent bundle $T\mathcal{M}_k$ of a flat non-reflexive Banach space $\mathcal{M}_k$ (indexed by $k \in \mathbb{R}^*$) by a Banach Lie group $G$. The key point in the proof of this result is the existence of a $G$-equivariant slice of the tangent space to the level set, which is orthogonal to the $G$-orbits and allows one to define a structure of smooth Riemannian manifold on the quotient. As far as we know, the general procedure for finding closed complements to closed subspaces of a Banach space recently developed by D. Beltiţă and B. Prunaru in [2] does not apply in our cases, so that the existence of closed complements has to be worked out by hand. For this purpose, the properties of Schatten ideals (see [44]) are extensively used.

In Section 4 we show that the quotient space obtained in Section 3 can be identified with the cotangent bundle of a connected component $Gr^\circ_{res}$ of the restricted Grassmannian, which is therefore endowed with a (1-parameter family of strong) hyperkähler structure(s). To prove this identification, we use the stable manifold associated with one of the complex structures of the quotient space and the general results of Section 2. At the end of Section 4 we compute the Kähler potential associated with this complex structure using the theory explained in Subsection 2.3 and we give an expression of this potential using the curvature of $Gr_{res}$. The formulas obtained by this method are analogous to the ones proved by O. Biquard and P. Gauduchon in the finite-dimensional setting. By restriction to the zero section, the theorems proved in Section 4 realize each connected component of the restricted
Grassmannian as a Kähler quotient and provide the expression of the Kähler potential of the restricted Grassmannian induced by Plücker’s embedding (for a description of Plücker embedding, see [38] and [45]). The realization of the restricted Grassmannian as a symplectic quotient was independently obtained by T. Wurzbacher in unpublished work and explained in numerous talks (see [51]).

In Section 2, we consider a Hamiltonian action of $\mathbf{G}$ on a weakly symplectic Banach manifold $(M,\omega)$, where $\mathbf{G}$ denotes the Lie group and $\omega$ is a smooth 2-form on $M$. The realization of the restricted Grassmannian as a symplectic quotient was independently obtained by T. Wurzbacher in unpublished work and explained in numerous talks (see [51]).

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In Section 2.4, we consider a Hamiltonian action of $\mathbf{G}$ on a weakly symplectic Banach manifold $(M,\omega)$, where $\mathbf{G}$ denotes the Lie group and $\omega$ is a smooth 2-form on $M$. The realization of the restricted Grassmannian as a symplectic quotient was independently obtained by T. Wurzbacher in unpublished work and explained in numerous talks (see [51]).
Suppose that the manifold for all $x$ is of the form $\tilde{\omega}$ and $(\tilde{\omega})$ is isomorphic to a Hilbert space. This implies in particular that for a non-reflexive Banach space $A$ even if $A$ is endowed with a weak Riemannian metric, equality (1) is certainly not fulfilled by every closed subspace $A$. 

**Remark 2.8** Let $b$ be a continuous bilinear form on a Banach vector space $B$. Suppose that $b$ realizes an injection of $B$ into its topological dual $B'$ by $b(X) := b(X, \cdot)$, for $X \in B$. For any linear subspace $A$ of $B$, we have the inclusion

$$A \subset (A^{\perp_b})^{\perp_b}$$

but not necessarily the equality. The equality means that any continuous linear form vanishing on $A^{\perp_b}$ is of the form $b(X)$ for some $X \in A$, which is a particular property of the subspace $A$. Along the same lines, if $b$ is a positive definite symmetric bilinear form on $B$, we have:

$$A \cap A^{\perp_b} = \{0\},$$

but in general we do not have:

$$B = A \oplus A^{\perp_b}$$

even if $A$ is closed, since the right hand side may not be closed. Furthermore J. Lindenstrauss and L. Tzafriri have proved in [26] that a Banach space in which every closed subspace is complemented is isomorphic to a Hilbert space. This implies in particular that for a non-reflexive Banach space $B$ endowed with a weak Riemannian metric, equality (1) is certainly not fulfilled by every closed subspace $A$.

**Proposition 2.9** If $\mu^{-1}(\xi)/G$ has a Banach manifold structure such that the quotient map is a submersion, and if $G$ acts by symplectomorphisms, the condition

$$T_x G \cdot x = \left((T_x G \cdot x)^{\perp_{\omega}}\right)^{\perp_{\omega}}$$

for all $x \in \mu^{-1}(\xi)$ implies that $\mu^{-1}(\xi)/G$ is a weakly symplectic manifold.

**□ Proof of Proposition 2.9**

Denote by $\pi$ the quotient map $\pi : \mu^{-1}(\xi) \to \mu^{-1}(\xi)/G$. Let us show that the following expression

$$\omega^{\text{red}}_{[x]}(X, Y) := \omega_{\pi}(\tilde{X}, \tilde{Y}),$$

(2)

where $X, Y$ are in $T_{[x]}(\mu^{-1}(\xi)/G)$ and where $\pi_*\tilde{X} = X$ and $\pi_*\tilde{Y} = Y$, defines a weak symplectic structure on the quotient. Note that for all $x \in \mu^{-1}(\xi)$, the tangent space $T_x(\mu^{-1}(\xi))$ is precisely the kernel of the differential $d\mu_x$, so that for all $a \in g$, the 1-form $i_a \omega$ vanishes on $\mu^{-1}(\xi)$. This implies that the right-hand side of (2) does not depend on the choice of $\tilde{X}$ and $\tilde{Y}$. Since $G$ acts by symplectomorphisms, it does not depend on the choice of the element $x$ in the class $[x]$ either. It follows that $\pi^*\omega^{\text{red}} = \omega_{\mu^{-1}(\xi)}$. Since $\omega$ is closed, so is $\omega^{\text{red}}$. The kernel of $\omega^{\text{red}}$ at a point $[x]$ in the quotient space is:

$$\pi_* \left(T_x \left((\mu^{-1}(\xi))^{\perp_{\omega}}\right)^{\perp_{\omega}}\right).$$

Note that the tangent space $T_x G \cdot x$ to the $G$-orbit of $x$ is spanned by $\{X^\alpha(a), a \in g\}$, hence we have

$$T_x(\mu^{-1}(\xi)) = (T_x G \cdot x)^{\perp_{\omega}},$$

and

$$\left((T_x (\mu^{-1}(\xi)))^{\perp_{\omega}}\right)^{\perp_{\omega}} = \left((T_x G \cdot x)^{\perp_{\omega}}\right)^{\perp_{\omega}}.$$

For $\omega^{\text{red}}$ to be symplectic, one needs $T_x(\mu^{-1}(\xi))^{\perp_{\omega}} = T_x G \cdot x$, which is precisely the hypothesis. □

Recall the following definition:

**Definition 2.10** A $G$-equivariant slice of the manifold $\mu^{-1}(\xi)$ is a subbundle $H$ of the tangent bundle $T(\mu^{-1}(\xi))$ such that, for every $x \in \mu^{-1}(\xi)$, $H_x$ is a closed complement to the tangent space $T_x G \cdot x$ of the $G$-orbit $G \cdot x$, and such that

$$H_g x = g_* H_x,$$

for all $x \in \mu^{-1}(\xi)$ and for all $g \in G$.

**Remark 2.11** Suppose that the manifold $\mathcal{M}$ is endowed with a weakly Riemannian $G$-invariant metric $g$. Then the existence of a $G$-invariant slice $H$ of $\mu^{-1}(\xi)$ allows one to define a weakly Riemannian metric $g^{\text{red}}$ on the quotient $\mu^{-1}(\xi)/G$, as follows. For every $x \in \mu^{-1}(\xi)$, we set

$$g^{\text{red}}_{[x]} : T_{[x]}(\mu^{-1}(\xi)/G) \times T_{[x]}(\mu^{-1}(\xi)/G) \to \mathbb{R}$$

$$T_{[x]}(\mu^{-1}(\xi)/G) \to g_x(\tilde{X}, \tilde{Y}),$$

where $\tilde{X}$ and $\tilde{Y}$ are the unique elements of $H_x$ such that $\pi_* (\tilde{X}) = X$ and $\pi_* (\tilde{Y}) = Y$. 

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**Definition 2.12** A weak Kähler manifold is a Banach manifold $\mathcal{M}$ endowed with a weak symplectic form $\omega$ and a weak Riemannian metric $g$ (i.e. such that at every $x \in \mathcal{M}$, $g$ defines an injection of $T_x \mathcal{M}$ into its dual), satisfying the following compatibility condition:

(C) the endomorphism $I$ of the tangent bundle of $\mathcal{M}$ defined by $g(IX, Y) = \omega(X, Y)$ satisfies $I^2 = -1$ and the Nijenhuis tensor $N$ of $I$ vanishes.

Recall that the Nijenhuis tensor has the following expression at $x \in \mathcal{M}$:

$$N_x(X, Y) := [X, Y] + I[X, IY] + I[IX, Y] - [IX, IY],$$

where $X$ and $Y$ belong to $T_x \mathcal{M}$.

**Theorem 2.13** Let $\mathcal{M}$ be a smooth Kähler Banach manifold endowed with a free and proper Hamiltonian action of a Banach Lie group $G$ preserving the Kähler structure. If, for every $x$ in the preimage $\mu^{-1}(\xi)$ of an $Ad^*(G)$-invariant regular value of the moment map $\mu$, the tangent space $T_xG \cdot x$ satisfies the direct sum condition

$$T_xG \cdot x \oplus (T_xG \cdot x)^{\perp_\omega} = T_x(\mu^{-1}(\xi)),$$

then the quotient space $\mathcal{M}/G := \mu^{-1}(\xi)/G$ is a smooth Kähler manifold.

**Proof of Theorem 2.13**

Let us denote by $(g, \omega, I)$ the Kähler structure of $\mathcal{M}$. For $x$ in $\mu^{-1}(\xi)$, the condition

$$T_xG \cdot x \oplus (T_xG \cdot x)^{\perp_\omega} = T_x\mu^{-1}(\xi),$$

implies that the tangent space of the orbit $G \cdot x$ satisfies the following property:

$$\left( (T_xG \cdot x)^{\perp_\omega} \right)^{\perp_\omega} = T_xG \cdot x,$$

(3)

(the converse may not be true). In particular, $T_xG \cdot x$ is closed and splits, so one is able to define on the quotient space a Banach manifold structure by use of Proposition 2.7. Now equality (3) is equivalent to

$$\left( (T_xG \cdot x)^{\perp_\omega} \right)^{\perp_\omega} = T_xG \cdot x,$$

since $I$ is orthogonal with respect to $g$. So the condition needed for the definition of the symplectic structure on the quotient in Proposition 2.7 is fulfilled. Moreover the orthogonal $H_x := (T_xG \cdot x)^{\perp_\omega}$ of $T_xG \cdot x$ in $T_x(\mu^{-1}(\xi))$ defines a $G$-equivariant slice for the manifold $\mu^{-1}(\xi)$, and, by Remark 2.10, allows one to define a Riemannian metric on the quotient. It remains to define a compatible complex structure $I^{\text{red}}$ on $\mu^{-1}(\xi)/G$. For this purpose, let us remark that

$$IH_x = I(T_xG \cdot x)^{\perp_\omega} \subset T_x\mu^{-1}(\xi) = (T_xG \cdot x)^{\perp_\omega},$$

since $\omega(X^a, IU) = g(IX^a, IU) = g(X^a, U) = 0$ for all $a$ in $\mathfrak{g}$ and $U$ in $H_x$. In addition, $IH_x$ is orthogonal to $T_xG \cdot x$ with respect to $g$ since $g(IU, X^a) = \omega(U, X^a) = 0$ for $U \in T_x(\mu^{-1}(\xi)) = (T_xG \cdot x)^{\perp_\omega}$. This implies that $H_x$ is stable under $I$ and one can define a complex structure $I^{\text{red}}$ on the quotient space by:

$$I^{\text{red}} : T_{[x]}(\mu^{-1}(\xi)/G) \rightarrow T_{[x]}(\mu^{-1}(\xi)/G) \quad \Rightarrow \quad \pi_*I_\mathcal{X},$$

where $x$ is in $[x]$ and where $\mathcal{X}$ is the unique element of $H_x$ whose projection on $T_{[x]}(\mu^{-1}(\xi)/G)$ is $X$. Hence one has:

$$I^{\text{red}}\pi_*\mathcal{X} = \pi_*I\mathcal{X}.$$

The application $\pi$ being a submersion, given two vector fields $X$ and $Y$ on $\mu^{-1}(\xi)/G$, one has:

$$[X, Y] = \pi_*\left( [\mathcal{X}, \mathcal{Y}] \right),$$

where again $\mathcal{X}$ satisfies $\mathcal{X}(x) \in H_x$ and $\pi_*(\mathcal{X}) = X$, and similar conditions for $\mathcal{Y}$. Therefore the formal integrability condition on $I$ implies the formal integrability condition on $I^{\text{red}}$ since the Nijenhuis tensor of $I^{\text{red}}$ has the following expression:

$$N(X, Y) := [X, Y] + I^{\text{red}}[X, I^{\text{red}}Y] + I^{\text{red}}[I^{\text{red}}X, Y] - [I^{\text{red}}X, I^{\text{red}}Y] = \pi_*[\mathcal{X}, \mathcal{Y}] + I^{\text{red}}\pi_*[\mathcal{X}, I\mathcal{Y}] + I^{\text{red}}\pi_*[I\mathcal{X}, \mathcal{Y}] = \pi_*N(\mathcal{X}, \mathcal{Y}).$$

where $X, Y$ are in $T_{[x]}(\mu^{-1}(\xi)/G)$ and $\mathcal{X}, \mathcal{Y}$ are as before.
Remark 2.14 In contrast to the finite-dimensional case illustrated by the Newlander-Nirenberg Theorem (see [34]), the formal integrability condition of an almost complex structure $I$ on a Banach manifold $\mathcal{M}$ given by a vanishing Nijenhuis tensor is not sufficient for $\mathcal{M}$ to admit a system of holomorphic charts. An example of a formally integrable complex structure on a real Banach manifold which does not admit any open subset biholomorphic to an open subset of a complex Banach manifold was recently constructed by I. Patiño in [35]. However, if $\mathcal{M}$ is a real analytic manifold and $I$ a formally integrable analytic complex structure, then $\mathcal{M}$ can be endowed with a holomorphic atlas (see [36], and [11] for the details of this result). Note also that, in the Fréchet context, L. Lempert showed in [25] that the complex structure defined in [31] by J. E. Marsden and A. Weinstein on the space of knots does not lead to the existence of holomorphic charts, although this structure was shown to be formally integrable by J. L. Brylinski in [11]. In the context of formally integrable complex structures, we will call a map $f$ between two complex manifolds $(M, I_M)$ and $(N, I_N)$ holomorphic if $df \circ I_M = I_N \circ df$.

### 2.2 Stable manifold

Let $(\mathcal{M}, \omega, g, I)$ be a smooth Kähler Banach manifold endowed with a smooth Hamiltonian action of a Banach Lie group $G$ preserving the Kähler structure. Let $\xi$ be an $\text{Ad}^*(G)$-invariant regular value of the moment map $\mu$. Assume that there exists a complex Lie group $G^C$ with Lie algebra $g^C := g \oplus ig$ which acts holomorphically and smoothly on $\mathcal{M}$ extending the action of $G$, and that the following assumption holds:

- $(H)$ for every $x$ in $\mu^{-1}(\xi)$, one has: $T_x \mathcal{M} = T_x (\mu^{-1}(\xi)) \oplus I T_x G \cdot x$ as topological sum.

(Note that, by definition of the moment map, one has $T_x (\mu^{-1}(\xi)) = (I T_x G \cdot x)^{\perp}$, so $(H)$ states that the direct sum of $I T_x G \cdot x$ and its orthogonal is closed in $T_x \mathcal{M}$, which is not always the case as mentioned in Remark 2.15, but nevertheless a natural assumption to make.) The action of $G^C$ on $\mathcal{M}$ allows one to define a notion of stable manifold associated with the level set $\mu^{-1}(\xi)$:

**Definition 2.15** The stable manifold $\mathcal{M}^s$ associated with the level set $\mu^{-1}(\xi)$ is defined by:

$$\mathcal{M}^s := \{ x \in \mathcal{M} \mid \exists g \in G^C, g \cdot x \in \mu^{-1}(\xi) \}.$$  

**Remark 2.16** The assumption $(H)$ implies that $\mathcal{M}^s$ is open in $\mathcal{M}$ since $T_x \mathcal{M}^s = T_x \mathcal{M}$ for every element $x$ in $\mu^{-1}(\xi)$, hence, by translation by an element of $G^C$, for every $x$ in $\mathcal{M}^s$.

**Proposition 2.17** If $G^C$ admits a polar decomposition $G^C = \exp igG$, then for every $x$ in $\mu^{-1}(\xi)$, one has:

$$G^C \cdot x \cap \mu^{-1}(\xi) = G \cdot x.$$  

**Proof of Proposition 2.17** The argument below has already been used by F. Kirwan (see [17], [18], [19]). It goes like this. Clearly $G \cdot x \subset G^C \cdot x \cap \mu^{-1}(\xi)$ since $\xi$ is $\text{Ad}^*(G)$-invariant. Let us show that $G^C \cdot x \cap \mu^{-1}(\xi) \subset G \cdot x$. Suppose that there exists $\gamma \in G^C$ such that $\gamma \cdot x \in \mu^{-1}(\xi)$. Since $\mu^{-1}(\xi)$ is $G$-invariant and since $G^C = \exp igG$, it is sufficient to consider the case when $\gamma = \exp ia$, $a \in g$.

Define the function $h : \mathbb{R} \to \mathbb{R}$ by $h(t) = \mu((\exp ita) \cdot x)(a)$. One has $h(0) = h(1) = \xi(a)$, hence there exists $t_0 \in (0, 1)$ such that:

$$0 = h'(t_0) = d_{x, y}((\exp ita) \cdot y)(a) = -\omega_{\mu}(ia \cdot y, a \cdot y) = \|a \cdot y\|^2,$$

where $y = \exp(it_0a) \cdot x$. Hence $a \cdot y = 0$ and $\exp(iaR)$ fixes $y$, thus also $x$. It follows that:

$$\exp(iaR) \cdot x \cap \mu^{-1}(\xi) = \{ x \}.$$  

From now on and till the end of Section 2 it will be assumed that $G^C$ admits a polar decomposition.

**Corollary 2.18** If $G$ acts freely on $\mu^{-1}(\xi)$, then $G^C$ acts freely on $\mathcal{M}^s$.

**Proof of Corollary 2.18** Let $x$ be an element of $\mu^{-1}(\xi)$ and let $\gamma \in G^C$ be such that $\gamma \cdot x = x$. Since $G^C = \exp igG$, there exists $u \in G$ and $a \in g$ such that $\gamma = \exp(ia)u$, and one has $\exp(ia)u \cdot x = x$. From the proof of the previous Proposition, one has:

$$\exp(iaR) \cdot (ux) \cap \mu^{-1}(\xi) = \{ ux \}.$$  

It follows that $ux = x$, thus $u = e$, since $G$ acts freely on $\mu^{-1}(\xi)$. Now, the condition $\exp(iaR) \cdot x = x$ implies that $a$ fixes $x$, hence $a = 0$. 

$\square$
Proposition 2.19 Assume that $G$ acts freely on $\mu^{-1}(\xi)$. Then, for every $y$ in $M^s$, there is a unique element $g(y)$ in exp $i g$ such that $g(y)$ maps $y$ to the level set. The resulting application

$$g : M^s \to \exp i g$$

$$y \mapsto g(y)$$

is smooth and the projection $q$

$$q : M^s \to \mu^{-1}(\xi)$$

$$y \mapsto g(y) \cdot y$$

is smooth and $G$-equivariant.

Proof of Proposition 2.19:

Let $y$ be in $M^s$ and suppose that there exist two elements $a$ and $b$ in $g$ such that both $\exp i a \cdot y$ and $\exp i b \cdot y$ belong to $\mu^{-1}(\xi)$. Since $\exp i a \cdot y$ and $\exp i b \cdot y$ are in the same $G^C$-orbit, by Proposition 2.17 there exists $u$ in $G$ such that $\exp i a \cdot y = u \cdot \exp i b \cdot y$. Since $G$ acts freely on the level set, Corollary 2.18 implies that $G^C$ acts freely on $M^s$. It follows that $\exp i a = u \cdot \exp i b$, hence, by uniqueness of the polar decomposition, $u$ is the unit element of $G$ and $\exp i a = \exp i b$. Therefore $g(y) = \exp i a$ is well defined, and so is the projection $q$.

Let us show that the application:

$$g : M^s \to \exp i g$$

$$y \mapsto g(y)$$

is smooth. Since $G^C$ acts smoothly on $M^s$, it will imply that the projection $q$ is smooth also. Consider the following map:

$$\phi : \exp i g \times M^s \to M$$

$$(\exp i a, y) \mapsto \exp i a \cdot y,$$

which maps $\exp i g \times M^s$ onto $M^s$. (Recall that $\exp i g$ inherits a Banach manifold structure from its identification with the homogeneous space $G^C/G$.) We will prove that $\phi$ is transversal to $\mu^{-1}(\xi)$ (see §5.11.6 and §5.11.7 of [3] for a definition of this notion), so that the subset

$$\phi^{-1}(\mu^{-1}(\xi)) = \{ (g(y), y), y \in M^s \}$$

is a smooth submanifold of $\exp i g \times M^s$. The smoothness of the application $g$ will therefore follow from the smoothness of the projection $p_1 : \exp i g \times M^s \to \exp i g$ on the first factor. We will denote by $R_y$ the right translation by $y$ on $G^C$. The differential of $\phi$ at a point $(\exp i a, y)$ in $\exp i g \times M^s$ reads:

$$(d\phi)_{(\exp i a, y)} \left( ((R_{\exp i a})_*(i b), Z) \right) := i b \cdot (\exp i a \cdot y) \oplus (\exp i a)_*(Z).$$

Note that, for every element $(\exp i a, y)$ in $\phi^{-1}(\mu^{-1}(\xi))$, one has:

$$(d\phi)_{(\exp i a, y)} \left( \{0\} \times T_y M^s \right) = T_x M^s$$

where $x := \exp i a \cdot y$, so that $(d\phi)_{(\exp i a, y)}$ is surjective. It remains to show that the subspace

$$(d\phi)^{-1}_{(\exp i a, y)} \left( T_x (\mu^{-1}(\xi)) \right)$$

is complemented. For this purpose recall that, by assumption (H), the tangent space $T_y M^s$ is isomorphic to

$$(\exp(-i a))_*(T_x (\mu^{-1}(\xi))) \oplus (\exp(-i a))_*(\exp i y) \cdot x,$$

so that the tangent space

$$T_{(\exp i a, y)} (\exp i g \times M^s) = T_{\exp i a} (\exp i g) \times T_y M^s$$

is isomorphic to $g \times T_x (\mu^{-1}(\xi)) \times g$ by the following isomorphism:

$$j : g \times T_x (\mu^{-1}(\xi)) \times g \to T_{\exp i a} (\exp i g) \times T_y M^s$$

$$((R_{\exp i a})_*(i b), (\exp(-i a))_*(\exp i y) \cdot (i c \cdot x)) \mapsto ((R_{\exp i a})_*(i b), (\exp(-i a))_*(W) + (\exp(-i a))_*(i c \cdot x)).$$

The element $j(b, W, c)$ belongs to $(d\phi)^{-1}_{(\exp i a, y)} (T_x (\mu^{-1}(\xi)))$ whenever $i b \cdot x + W + i c \cdot x \in T_x (\mu^{-1}(\xi))$. Since $G$ acts freely on $M$, it follows that the subspace:

$$(d\phi)^{-1}_{(\exp i a, y)} (T_x (\mu^{-1}(\xi)))$$

is isomorphic to $g \times T_x (\mu^{-1}(\xi)) \times g$.
equals 
\[ \{ j(b, W - b) , \ b \in g, W \in T_x (\mu^{-1}(\xi)) \}, \]
and 
\[ \{ j(b, 0, b) , \ b \in g \} \]
is a closed complement to it.

\[
\text{Let us check the } G\text{-equivariance of } q. \text{ Since } \mu \text{ is } \mu^{-1}(\xi) \text{-invariant, one has : } \\
\mu(u \cdot g(y) \cdot y) = (Ad)^*(u)(\mu(g(y) \cdot y)) = (Ad)^*(u)(\xi) = \xi
\]
for all \( u \in G \) and \( y \in M^s \). We can write \( g(y) = \exp i a \) for some \( a \in g \). Now the equality \( u \cdot \exp i a = \exp (Ad(u)(i a)) \cdot u \) and the uniqueness of the element \( g(u \cdot y) \) satisfying \( g(u \cdot y) \cdot (u \cdot y) \in \mu^{-1}(\xi) \) proved above, imply that:
\[
g(u \cdot y) = \exp (Ad(u)(i a))
\]
Hence \( q \) satisfies the \( G\)-equivariant condition:
\[
q(u \cdot y) = u \cdot q(y)
\]
for all \( u \in G \) and \( y \in M^s \).

\[ \square \]

**Proposition 2.20** If \( G \) acts freely and properly on \( \mu^{-1}(\xi) \), then \( G^C \) acts (freely and) properly on \( M^s \).

\[ \square \ \text{Proof of Proposition 2.20} \]

By Proposition 2.18 \( G^C \) acts freely on \( M^s \). By Proposition 2.6, \( G^C \) acts properly on \( M^s \) if and only if the graph \( \tilde{C} \) of the equivalence relation defined by \( G^C \) is closed in \( M^s \times M^s \) and the canonical map from \( \tilde{C} \) to \( G^C \) is continuous.

Let us show that \( \tilde{C} \) is closed in \( M^s \times M^s \). Denote by \( C \) the graph of the equivalence relation defined by \( G^C \) on \( \mu^{-1}(\xi) \). Let \( \{(y_n, v_n \cdot y_n)\}_{n \in \mathbb{N}} \) be a sequence in \( \tilde{C} \), where \( y_n \in M^s \) and \( v_n \in G^C \), which converges to an element \( (y_\infty, z_\infty) \) in \( M^s \times M^s \). From Proposition 2.19 and from the continuity of the projection \( q \), the sequence \( \{(q(y_n), q(v_n \cdot y_n))\}_{n \in \mathbb{N}} \) belongs to \( C \) and converges to \( (q(y_\infty), q(z_\infty)) \). Since \( C \) is closed in \( \mu^{-1}(\xi) \times \mu^{-1}(\xi) \), it follows that \( q(z_\infty) = u_\infty \cdot q(y_\infty) \) for some \( u_\infty \in G \). Hence \( z_\infty = g(z_\infty)^{-1} u_\infty g(y_\infty) \cdot y_\infty \). Thus \( \tilde{C} \) is closed in \( M^s \times M^s \).

Let us show that the canonical map \( \tilde{i} \) from \( \tilde{C} \) to \( G^C \) is continuous. Denote by \( i \) the canonical map from \( C \) to \( G \). One has:
\[
\tilde{i}(y, z) \mapsto g(z)^{-1} \circ i(q(y), q(z)) \circ g(y),
\]
and the continuity of \( \tilde{i} \) follows from the continuity of the applications \( i, q \) and \( g \).

\[ \square \]

We conclude this Subsection with

**Theorem 2.21** Let \( M \) be a Banach Kähler manifold endowed with a smooth, free and proper Hamiltonian action of a Banach Lie group \( G \), which preserves the Kähler structure and extends to a smooth holomorphic action of a complex Lie group \( G^C = \text{exp} i G \). Let \( \xi \) be an \( Ad(G)\)-invariant regular value of the moment map \( \mu \). If for every \( x \) in \( \mu^{-1}(\xi) \), the orthogonal \((T_x G \cdot x)^{\perp} \) of \( T_x G \cdot x \) in \( T_x (\mu^{-1}(\xi)) \) satisfies the following direct sum condition
\[
(D + H) \quad T_x M = T_x G \cdot x \oplus (T_x G \cdot x)^{\perp} \oplus I(T_x G \cdot x),
\]
then the quotient space \( M^s/G^C \) is a smooth complex manifold isomorphic to the smooth Kähler quotient \( M//G := \mu^{-1}(\xi)/G \) as complex smooth manifold. Moreover the full integrability of the complex structure on \( M \) implies the full integrability of the complex structure on \( M^s/G^C \), hence on \( M//G \).

\[ \square \ \text{Proof of Theorem 2.21} \]

By Proposition 2.18 and Proposition 2.20 \( G^C \) acts freely and properly on \( M^s \). The smoothness of the application \( g \) and the \( G\)-equivariant slice \( H \) of \( \mu^{-1}(\xi) \) given by \( H_x := (T_x G \cdot x)^{\perp} \), allows one to define a \( G\)-equivariant slice on \( M^s \), also denoted by \( H \), by the following formula:
\[
H_y := g(y)^{-1}(H_{q(y)}) \subset T_y M^s,
\]
for all \( y \) in \( M^s \). The following decomposition of the tangent space \( T_y M^s \) holds:
\[
T_y M^s = H_y \oplus T_y (G^C \cdot y).
\]
By Proposition 2.7 it follows that \( M^s/G^C \) has a unique (real) Banach manifold structure such that the quotient map is a submersion. Since for every element \( y \) in \( M^s \), \( H_{q(y)} \) is invariant under the complex
structure $I$ of $\cal M$, and since the $G^C$-action on $\cal M$ is holomorphic, for every $y$ in $\cal M^*$ the subspace $H_y$ of $T_y\cal M^*$ is $I$-invariant and, further, $\cal M^*/G^C$ inherits a natural complex structure. Moreover, since the complex structure of the quotient $\cal M^*/G^C$ comes from the complex structure of $\cal M$, the natural injection $\mu^{-1}(\xi) \to \cal M^*$ induces a complex isomorphism between $\mu^{-1}(\xi)/G$ and $\cal M^*/G^C$. Finally, $\cal M^*$ being an open subset of $\cal M$ because of (H), the existence of holomorphic charts on $\cal M$ allow one to apply Proposition 2.24 to the holomorphic quotient $\cal M^*/G^C$, thus implies the existence of holomorphic charts on $\cal M^*/G^C$.

\section{Hyperkähler quotient}

Let $\cal M$ be a Banach manifold endowed with a weak hyperkähler metric $g$, with Kähler forms $\omega_1$, $\omega_2$ and $\omega_3$, and corresponding complex structures $I_1$, $I_2$, and $I_3$. Let $G$ be a connected Banach Lie group with Lie algebra $\mathfrak{g}$, acting freely on $\cal M$ by hyperkähler diffeomorphisms. We assume in this Subsection that there exists a $G$-equivariant moment map $\mu$ for each symplectic structure. Define $\mu : \cal M \to \mathfrak{g}^* \otimes \mathbb{R}^3$ by $\mu = \mu_1 \oplus \mu_2 \oplus \mu_3$. Let $\xi = (\xi_1, \xi_2, \xi_3)$ be a regular $\text{Ad}^*(G)$-invariant value of the moment map $\mu$.

\textbf{Theorem 2.22} If $G$ acts freely and properly on $\cal M$ and if, for every $x$ in $\mu^{-1}(\xi)$, the orthogonal $(T_xG : x)^{\perp}$ satisfies the direct sum condition

\[(D) \quad T_xG : x \oplus (T_xG : x)^{\perp} = T_x \mu^{-1}(\xi),\]

then the quotient space $\mu^{-1}(\xi)/G$ carries a structure of smooth Banach hyperkähler manifold.

\textbf{Proof of Theorem 2.22}

This is a direct application of Theorem 2.13 with respect to each complex structure $I_1$, $I_2$, and $I_3$ of $\cal M$. By the very definition of the complex structures and symplectic forms on the reduced space, $\mu^{-1}(\xi)/G$ inherits a hyperkähler structure from the hyperkähler structure of $\cal M$.

Let us now assume that the orthogonal $(T_xG : x)^{\perp}$ of $T_xG : x$ in $T_x \mu^{-1}(\xi)$ satisfies the direct sum condition

\[(S) \quad \text{for every } x \text{ in } \mu^{-1}(\xi), \text{ one has: } T_x\cal M = T_xG : x \oplus (T_xG : x)^{\perp} \oplus I_1T_xG : x \oplus I_2T_xG : x \oplus I_3T_xG : x \text{ as a topological direct sum.}\]

For each complex structure $I_\ell = I_{\ell 1} I_{\ell 2} I_{\ell 3}$ in the 2-sphere of complex structures on $\cal M$, indexed by $\ell = (k_1, k_2, k_3)$ in $S^2$, let us define an action $\iota_\ell$ of $\mathfrak{g}$ on $\cal M$ by:

$$ia \iota_\ell x = I_\ell(a \cdot x),$$

for all $a$ in $\mathfrak{g}$ and for all $x$ in $\cal M$. Assume that for a given $\ell$, the action $\iota_\ell$ integrates into an $I_{\ell}$-holomorphic action of $G^C$ on $\cal M$. Let us choose an orthogonal complex structure to $I_{\ell 1}$ denoted by $I_{\ell 1}$, and define a third complex structure by $I_\ell := I_{\ell 1} I_{\ell 2} I_{\ell 3}$ so that $(I_{\ell 1}, I_{\ell 2}, I_{\ell 3})$ satisfies the quaternionic identities. Denote by $\mu_\ell$ the combination $\mu_{\ell 1} = k_1 \mu_1 + k_2 \mu_2 + k_3 \mu_3$ and similarly $\mu_2$ and $\mu_3$. Consider $\xi_{\ell 1} := l_1 \xi_1 + l_2 \xi_2 + l_3 \xi_3$ and $\xi_{\ell 3} := m_1 \xi_1 + m_2 \xi_2 + m_3 \xi_3$, and similarly $\omega_{\ell 1} := l_1 \omega_1 + l_2 \omega_2 + l_3 \omega_3$ and $\omega_{\ell 3} := m_1 \omega_1 + m_2 \omega_2 + m_3 \omega_3$.

\textbf{Proposition 2.23} The map $\mu^C := \mu_{\ell 1} + i\mu_{\ell 3}$ is an $G^C$-equivariant holomorphic moment map for the $I_{\ell}$-complex symplectic structure $\omega_{\ell 1} + i\omega_{\ell 3}$.

\textbf{Proof of Proposition 2.23}

By the same algebraic arguments as the ones given in [14], section 3 (D), the map $\mu^C$ is holomorphic and a moment map for $\omega_{\ell 1} + i\omega_{\ell 3}$. Since $G$ is connected, the $G^C$-equivariance of $\mu^C$ will be done by following from

$$\langle d\mu^C(ia \cdot x), b \rangle = \langle \mu^C(x), [ia, b] \rangle,$$

for all $x$ in $\cal M$, and all $a, b \in \mathfrak{g}$, where the bracket denotes the duality pairing. But this is an easy consequence of the $G$-equivariance of $\mu_{\ell 1}$ and $\mu_{\ell 3}$ and the following identities

$$d\mu_{\ell 1} := d\mu_{\ell 1}(I_{\ell 2}X^a) = i\mu_{\ell 1}(X^a) \quad \text{and} \quad \mu_{\ell 3}(ia \cdot b) = i\mu_{\ell 3}(x)([ia, b]),$$

where $X^a$ denotes the vector field generated by $a$ in $\mathfrak{g}$.

\textbf{Lemma 2.24} The stable manifold $\cal M^*_\xi$ with respect to the complex structure $I_{\ell 1}$, associated with the level set $\mu^{-1}(\xi)$, is a submanifold of the Banach manifold $\cal M$, contained in the preimage by $\mu^C$ of the $G^C$-coadjoint orbit of $\xi_{\ell 1} + i\xi_{\ell 3}$. In particular, if $\xi_{\ell 1} + i\xi_{\ell 3}$ is in the center of $\mathfrak{g}^C$, then $\cal M^*_\xi \subset \mu_{\ell 1}^{-1}(\xi_{\ell 1}) \cap \mu_{\ell 3}^{-1}(\xi_{\ell 3})$. 

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The fact that $\mathcal{M}^\ast \ast$ is included in $(\mu^\ast)^{-1} (\text{Ad}^* (G^\ast) (\xi + i\xi_m))$ is a direct consequence of the $G^\ast$-equivariance of $\mu^\ast$ (Proposition 2.28). Let $x$ be an element in the level set $\mu^{-1}(\xi)$. Since $\xi$ is a regular value of $\mu$, the level set $\mu^{-1}(\xi)$ is a Banach submanifold of $\mathcal{M}$. Consider an adapted chart $(\mathcal{V}, E \times F, \varphi)$ of the submanifold $\mu^{-1}(\xi)$ at $x$, where $\mathcal{V}$ is a neighborhood of $x$ in $\mathcal{M}$, $E$ and $F$ are two Banach spaces, and $\varphi$ is an homeomorphism from $\mathcal{V}$ onto a neighborhood of $0$ in $E \times F$ such that $\mathcal{U}_1 := \varphi (\mu^{-1}(\xi) \cap \mathcal{V}) \subset E$. By assumption (S), one has:

$$T_x \mathcal{M} = T_x G \cdot x \oplus (T_x G \cdot x)^{\perp} \oplus I_{\overline{\xi}} (T_x G \cdot x) \oplus I_{\overline{\xi}} (T_x G \cdot x) \oplus I_{\overline{\xi}} (T_x G \cdot x),$$

hence $F$ is isomorphic (as Banach space) to $I_{\overline{\xi}} (T_x G \cdot x) \oplus I_{\overline{\xi}} (T_x G \cdot x) \oplus I_{\overline{\xi}} (T_x G \cdot x).$ For $a$ in the Lie algebra $\mathfrak{g}$ of $G$, denote by $X^a$ the vector field on $\mathcal{M}$ generated by $a$. Since $G$ acts freely on $\mathcal{M}$, the map $\mathfrak{g} \to I_{\overline{\xi}} (T_x G \cdot x)$ which assigns to $a \in \mathfrak{g}$ the vector $I_{\overline{\xi}} (T_x G \cdot x)$ is a continuous bijection of Banach spaces, hence an isomorphism, and similarly for the indexes $\overline{\eta}$ and $\overline{\mu}$. Since by hypothesis the $G$-action on $\mathcal{M}$ extend to a $I_{\overline{\xi}}$-holomorphic action of $G^\ast$ on $\mathcal{M}$, the vector fields $I_{\overline{\xi}} X^a$, for $a \in \mathfrak{g}$, are complete, whereas the flows $t \to f_{I_{\overline{\xi}} X^a + I_{\overline{\mu}} X^ b}$ of the vector fields $(I_{\overline{\xi}} X^a + I_{\overline{\mu}} X^ b)$, with $b$ and $c$ in $\mathfrak{g}$, may not be globally defined. Nevertheless, by the smooth dependance of the solutions of a differential equation with respect to a parameter, there exists a small neighborhood $U_2$ of $0$ in $I_{\overline{\xi}} (T_x G \cdot x) \oplus I_{\overline{\mu}} (T_x G \cdot x) \simeq \mathfrak{g} \oplus \mathfrak{g}$ for which the flows $f_{I_{\overline{\xi}} X^a + I_{\overline{\mu}} X^ b}$ are defined for $t \in [0,1]$. Now consider the following map:

$$\Psi : \mathcal{U}_1 \ominus \mathfrak{g} \ominus \mathcal{U}_2 \to \mathcal{M}, \quad (u, a, (b, c)) \mapsto \exp \frac{i \alpha}{\overline{\xi}} \int_{I_{\overline{\xi}} X^a + I_{\overline{\mu}} X^ b} (\varphi^{-1}(u)) \cdot$$

Then $\Psi$ provides an adapted chart in the neighborhood of any $y$ in the fiber $q_{\overline{\xi}}^{-1}(x)$ where $q_{\overline{\xi}} : \mathcal{M}^\ast \to \mu^{-1}(\xi)$ is the projection defined in Proposition 2.28.

With the notation above, Theorem 2.24 reads:

**Theorem 2.25** If $G$ acts freely and properly on $\mathcal{M}$, and if, for every $x$ in $\mu^{-1}(\xi)$, the orthogonal of $T_x G \cdot x$ in $T_x \mu^{-1}(\xi)$ satisfies the direct sum conditions (D) and (S), then the quotient space $\mathcal{M}^\ast \ast / G^\ast$ is a smooth $I_{\overline{\xi}}$-complex manifold which is diffeomorphic to $\mu^{-1}(\xi)/G$ as a $I_{\overline{\xi}}$-complex smooth manifold.■

### 2.4 Kähler potential on a Kähler quotient

This Subsection is a generalization to the Banach setting of results obtained in the finite-dimensional case by O. Biquard and P. Gauduchon in [4] (see Theorem 3.1 there), and based on an idea in [13]. In this Subsection, we will again make use of the setting of Subsection 2.2 and we will suppose that the complex structure $I$ of $\mathcal{M}$ is formally integrable.

Recall that the complex structure $I$ acts on $n$-differential forms by $\eta \mapsto I \eta$ where

$$(I \eta)_x (X_1, \ldots, X_n) := (-1)^n \eta (IX_1, \ldots, IX_n).$$

and where $x$ belongs to $\mathcal{M}$, and $X_1, \ldots, X_n$ are elements in $T_x \mathcal{M}$. This action allows one to define the corresponding differential operator $d^c$ by $d^c := 1dI^{-1}$. Note that $dd^c = 2i\partial \bar{\partial}$. Recall also the following definitions:

**Definition 2.26** A Kähler potential on $\mathcal{M}$ is a function $K$ on $\mathcal{M}$ such that $\omega = dd^c K$.

In the following we will make the assumption that $\mathcal{M}$ admits a $G$-invariant globally defined Kähler potential, which will be the case in the next Section. Under this assumption, the action of $G$ is Hamiltonian with respect to the moment map $\mu$ defined by:

$$\mu(x)(a) := dK_x (IX^a),$$

for all $x$ in $\mathcal{M}$, and for all $a$ in $\mathfrak{g}$.

**Lemma 2.27** If $K$ is a globally defined Kähler potential on $(\mathcal{M}, \omega, I)$, then the trivial bundle $L = \mathcal{M} \times \mathbb{C}$ endowed with the Chern connection $\nabla$ associated with the Hermitian product $h$ on $L$ given by:

$$h(\sigma(x), \sigma(x)) := e^{-2K(x)},$$

where $\sigma$ is the canonical section $\sigma(x) = (x, 1)$, prequantifies $\mathcal{M}$ in the sense that $R^\nabla = i\omega$.\[\square\]
\[ R^\nabla = \frac{1}{2i} dd^c \log h(\sigma, \sigma) \]

since the \( \partial \) operator on \( L = M \times \mathbb{C} \) satisfies the formal integrability condition \( d^\partial \circ d^\partial = 0 \). Hence the Hermitian product \( h \) satisfies : \( dd^c \log h(\sigma, \sigma) = -2\omega = -2dd^c K \), i.e. \( \log h = -2K + \eta \), where \( \eta \) is in the kernel of \( dd^c \) and can be chosen to be 0.

The aim of this Subsection is to compute a Kähler potential on a smooth Kähler quotient \( \mathcal{M}/G := \mu^{-1}(\xi)/G \), given a globally defined Kähler potential on \( \mathcal{M} \). For this purpose we will construct a holomorphic line bundle \( (L, \hat{h}) \) over \( \mu^{-1}(\xi)/G \) which prequantifies \( \mu^{-1}(\xi)/G \). The previous Lemma establishes the link between the Hermitian scalar product on a trivial line bundle and a globally defined potential. If the pull-back of \( (\hat{L}, \hat{h}) \) to the stable manifold \( \mathcal{M}^s \) (which is an open subset of \( \mathcal{M} \) since we assume hypothesis (H) satisfied) is a trivial Hermitian line bundle, then the pull-back of the Hermitian scalar product \( \hat{h} \) induces a globally defined Kähler potential on \( \mathcal{M}^s \).

Consider the following action of the Lie algebra \( g \) on the vector space \( \Gamma(L) \) of sections of the trivial line bundle \( L : \)

\[ a.\sigma = -\nabla_{X^a}\sigma - i\mu^a\sigma + i\xi(a)\sigma, \]

for all \( \sigma \) in \( \Gamma(L) \) and \( a \) in \( g \). This action corresponds to the action of \( g \) on the total space of \( L \) which assigns to an element \( a \in g \) the vector field \( X^a \) over \( L \) whose value at a point \( \zeta \in L \) over \( x \in M \) is :

\[ \tilde{X}^a(\zeta) = \tilde{X}^a(\zeta) + i\mu^a(x).T(\zeta) - i\xi(a).T(\zeta), \]

where \( \tilde{X}^a(\zeta) \) denotes the horizontal lift at \( \zeta \) of \( X^a \) for the trivial connection, and where \( T \) denotes the vertical vector field given by \( T(\zeta) = \zeta \). This action integrates into an action of the group \( G \) if and only if the following integrability condition is satisfied :

(I) \( a \mapsto i\xi(a) \) is the differential of a group homomorphism from \( G \) to \( S^1 \) at the unit element \( e \) of \( G \).

Suppose it is the case and denote by \( \chi \) the homomorphism such that \((d\chi)_e = -i\xi \). The homomorphism \( \chi \) extends to a homomorphism of \( G^C \) to \( C^* \), which will be also denoted by \( \chi \). The corresponding action of \( G^C \) on \( L \) is given by :

\[ g \cdot (x, \zeta) = (g \cdot x, \chi(g)^{-1} \cdot \zeta), \]

where \( g \) is an element in \( G^C \). This induces an action of \( G^C \) on \( \Gamma(L) \) by :

\[ (g \cdot \sigma)(x) := g(\sigma(g^{-1} \cdot x)). \]

where \( \sigma \) belongs to \( \Gamma(L) \), \( g \) is an element in \( G^C \) and \( x \) is in \( \mathcal{M} \). In the remainder of this Subsection, we will assume that the integrability condition (I) is satisfied.

**Definition 2.28** Let \( \hat{L} \) be the complex line bundle over \( \mu^{-1}(\xi)/G \) obtained as the \( G \)-orbit space of the restriction of the trivial line bundle \( L \) to the level set \( \mu^{-1}(\xi) \). The fiber of \( \hat{L} \) over an element \([x] \) in \( \mu^{-1}(\xi)/G \) is :

\[ \hat{L}([x]) = [[x], \zeta]], \]

where \( (x, \zeta) \sim (g \cdot x, \chi(g)^{-1} \cdot \zeta) \) for all \( g \in G \).

**Definition 2.29** Define an Hermitian scalar product \( \hat{h} \) on \( \hat{L} \) by :

\[ \hat{h}(\hat{\sigma}_1, \hat{\sigma}_2) = h(\sigma_1, \sigma_2), \]

where, for \( i \in \{1, 2\} \), \( \sigma_i \) is the \( G \)-invariant section of \( L|_{\mu^{-1}(\xi)} \) whose projection to \( \Gamma(\hat{L}) \) is \( \hat{\sigma}_i \).

**Proposition 2.30** Let \( \sigma \) be a \( G \)-invariant section of \( L|_{\mu^{-1}(\xi)} \) whose projection to \( \Gamma(\hat{L}) \) will be denoted by \( \hat{\sigma} \), and let \( X \) be a vector field on \( \mu^{-1}(\xi)/G \) whose horizontal lift with respect to an arbitrary \( G \)-invariant connection \( \nabla \) on the bundle \( \mu^{-1}(\xi) \rightarrow \mu^{-1}(\xi)/G \) will be denoted by \( \hat{X} \). Then the identity

\[ \nabla_X \hat{\sigma} := \nabla_{\hat{X}} \sigma \]

where \( \nabla \) denotes the Chern connection on \( L \), defines a connection \( \nabla \) on \( \hat{L} \), which is independent of \( \nabla \) and for which \( \hat{h} \) is parallel.
Proposition 2.31: The Hermitian line bundle \((\hat{L}, \hat{h}, \hat{\nabla})\) prequantifies the quotient \(\mu^{-1}(\xi)/G\) in the sense that \(R^\nabla = i\omega^{\text{red}}\).

Proposition 2.33: The 2-form \(p^*\omega^{\text{red}}\) is the curvature of the line bundle \((L_{|\mathcal{M}^*}, \hat{h})\), where \(\hat{h}\) is defined by:
\[
\hat{h}(\zeta, \zeta) = h(g(x) \cdot \zeta, g(x) \cdot \zeta),
\]
for \(x \in \mathcal{M}^*\) and \(\zeta \in L_x\).

Corollary 2.32: The complex structure of the Hermitian line bundle \((L, h, \hat{\nabla})\) is formally integrable and the connection \(\hat{\nabla}\) is the Chern connection.
Since the element $q(x)$ in the class $[q(x)]$ is distinguished, the fiber $(x, [L_{q(x)}])$ can be identified with the fiber $(x, L_{q(x)})$. One can therefore define an isomorphism $\Phi$ of complex bundles by:

$$
\Phi : L_{|M^*} \rightarrow p^* \hat{L},
$$

$$
\zeta \in L_x \mapsto g(x) \cdot \zeta \in \left(p^* \hat{L}\right)_x.
$$

Clearly $\Phi^* \hat{h} = \hat{h}$. Now let $\hat{\nabla}$ be the Chern connection of the trivial Hermitian bundle $(L_{|M^*}, \hat{h})$. Its curvature is:

$$
R^{\hat{\nabla}} = \Phi^{-1} \circ R^{p^* \hat{\nabla}} \circ \Phi.
$$

Since the bundle is a complex line bundle, one has:

$$
R^{\hat{\nabla}} = R^{p^* \hat{\nabla}} = ip^* \omega^\text{red}.
$$

**Theorem 2.34** The 2-form $p^* \omega^\text{red}$ on $M^*$ satisfies $ip^* \omega^\text{red} = dd^c \hat{K}$, where for every $x \in M^*$,

$$
\hat{K}(x) := K(g(x) \cdot x) + \frac{1}{2} \log |\chi(g(x))|^2.
$$

**Proof of Theorem 2.34**

The curvature of the Chern connection of the Hermitian line bundle $(L_{|M^*}, \hat{h})$ is given by:

$$
R = \frac{1}{2i} dd^c \log \hat{h}(\sigma, \sigma),
$$

where $\sigma$ is the canonical section. Moreover:

$$
\hat{h}(\sigma, \sigma) = h(g(x) \cdot \sigma, g(x) \cdot \sigma) = |\chi(g(x))|^{-2} h(\sigma, \sigma).
$$

Thus:

$$
p^* \omega^\text{red}(x) = -\frac{1}{2} dd^c \log |\chi(g(x))|^{-2} h(\sigma, \sigma) = dd^c K(g(x) \cdot x) + \frac{1}{2} dd^c \log |\chi(g(x))|^2.
$$

**3 An Example of hyperkähler quotient of a Banach manifold by a Banach Lie group**

**3.1 Notation**

Let us summarize the notation used in the remainder of this paper.

$H$ will stand for a separable Hilbert space endowed with an orthogonal decomposition $H = H_+ \oplus H_-$ into two closed infinite-dimensional subspaces. The orthogonal projection from $H$ onto $H_+$ (resp. $H_-$) will be denoted by $p_+$ (resp. $p_-$). The restriction of $p_\pm$ to a subspace $P$ will generally be denoted by $pr_\pm$ and the space $P$ be specified.

Given two Banach spaces $E$ and $F$, the set of Fredholm operators from $E$ to $F$ will be denoted by $\text{Fred}(E, F)$, the Hilbert space of Hilbert-Schmidt operators from $E$ to $F$ will be denoted by $L^2(E, F)$, the Banach space of trace class operators from $E$ to $F$ by $L^1(E, F)$, and the Banach space of bounded operators from $E$ to $F$ by $B(E, F)$. The argument $F$ in the previous operator spaces will be omitted when $F = E$.

The set of self-adjoint trace class operators on a complex Hilbert space $E$ will be denoted by $\mathcal{S}^1(E)$, and the set of skew-Hermitian trace class operators on $E$ by $\mathcal{A}^1(E)$. Similarly, $\mathcal{S}^2(E)$ will stand for self-adjoint Hilbert-Schmidt operators on $E$, and $\mathcal{A}^2(E)$ for skew-Hermitian Hilbert-Schmidt operators on $E$.

The unitary group of $H$ will be denoted by $U(H)$ and the identity map of a Hilbert space $E$ by $\text{Id}_E$. $\text{Ran}(A)$ will stand for the range of an operator $A$ and $\text{Ker}(A)$ for its kernel.
3.2 Introduction

The restricted Grassmannian $Gr_{res}$ of $H$, studied for instance in [38] and [50], is defined as follows:

$$Gr_{res}(H) = \{ P \text{ closed subspace of } H \text{ such that } pr_+ : P \to H_+ \in \text{Fred}(P, H_+) \},$$

where $pr_\pm$ denotes the orthogonal projection from $P$ to $H_\pm$. The space $Gr_{res}$ is a Hilbert manifold and a homogeneous space under the restricted unitary group:

$$U_{res} = \left\{ \begin{pmatrix} U_+ & U_{+-} \\ U_{-+} & U_- \end{pmatrix} \in U(H) \mid U_{+-} \in L^2(H_-, H_+), \ U_{-+} \in L^2(H_+, H_-) \right\}.$$  

Note that the stabilizer of $H_+$ is $U(H_+) \times U(H_-)$. The space $Gr_{res}$ is a strong Kähler manifold whose Kähler structure is invariant under $U_{res}$. The expressions of the metric $g_{Gr}$, the complex structure $I_{Gr}$ and the symplectic form $\omega_{\text{res}}$ at the tangent space of $Gr_{res}$ at $H_+$ are:

$$g_{Gr}(X, Y) = \Re \text{ Tr } X^* Y,$$

$$I_{Gr} Y = i Y,$$

$$\omega_{\text{res}}(X, Y) = g_{Gr}(iX, Y) = \Im \text{ Tr } X^* Y,$$

where $X$ and $Y$ belong to the tangent space $T_H Gr_{res}$ which can be identified with $L^2(H_+, H_-)$. Two elements $P_1$ and $P_2$ of $Gr_{res}$ are in the same connected component $Gr^0_{res}$ if and only if the projections $pr_1^+ : P_1 \to H_+$ and $pr_2^+ : P_2 \to H_+$ have the same index $j$. In particular $Gr^0_{res}$ denotes the connected component of $Gr_{res}$ containing $H_+$.

The aim of this Section is to construct a hyperkähler quotient whose quotient space will be identified with the cotangent space of $Gr_{res}$ in Section 3. This will make $T'Gr_{res}$ into a strong hyperkähler manifold. In Section 4 the same quotient space will be identified with a complexified orbit of $Gr^0_{res}$, which will therefore carry a hyperkähler structure in its own right. Since all the constructions that follow can be carried out substituting an arbitrary element of another connected component $Gr^1_{res}$, it will follow in particular that the cotangent space of the whole restricted Grassmannian is strongly hyperkähler and that it can be identified with the union of the complexifications of all connected components $Gr^j_{res}$. The latter union is nothing but the orbit of $p_+$ under the action by conjugation of the (non-connected) group $GL_{res}$ which is the complexification of $U_{res}$. In other words, $T'Gr_{res} \simeq GL_{res} \cdot p_+$.

3.3 A weak hyperkähler affine space $T\mathcal{M}_k$

For $k \in \mathbb{R}^*$, let $\mathcal{M}_k$ be the following affine Banach space:

$$\mathcal{M}_k := \left\{ x = \begin{pmatrix} x_+ \\ x_- \end{pmatrix} \in B(H_+, H) \mid x_+ - \text{Id}_{H_+} \in L^1(H_+), x_- \in L^2(H_+, H_-) \right\}.$$  

Then $\mathcal{M}_k$ is modeled over the Banach space $L^1(H_+, H_+) \times L^2(H_+, H_-)$. The tangent space of $\mathcal{M}_k$

$$T\mathcal{M}_k = \left\{ (x, X) \in \mathcal{M}_k \times B(H_+, H) \mid p_+ \circ X \in L^1(H_+), p_- \circ X \in L^2(H_+, H_-) \right\}.$$  

injects into the continuous cotangent space $T'\mathcal{M}_k$ of $\mathcal{M}_k$ via the application:

$$(x, X) \mapsto (x, (Y \mapsto \text{Tr } X^* Y)).$$

Thus $T\mathcal{M}_k$ inherits a structure of weak complex symplectic manifold with symplectic form $\Omega$ given by

$$\Omega((Z_1, T_1); (Z_2, T_2)) = \text{Tr } (T_1^* Z_2) - \text{Tr } (T_2^* Z_1),$$

where $(Z_1, T_1)$ and $(Z_2, T_2)$ belong to $T_{(x,X)}(T\mathcal{M}_k)$. We will denote by $\omega_2$ and $\omega_3$ the real symplectic forms given respectively by the real and imaginary parts of $\Omega$. Besides, from the natural inclusion of $L^1(H_+)$ into $L^2(H_+)$, it follows that $T\mathcal{M}_k$ admits a natural weak Riemannian metric whose expression is

$$g_{(x,X)}((Z_1, T_1); (Z_2, T_2)) = \Re \text{ Tr } Z_1^* Z_2 + \Re \text{ Tr } T_1^* T_2$$

where $(Z_1, T_1)$ and $(Z_2, T_2)$ belong to $T_{(x,X)}(T\mathcal{M}_k)$. The complex symplectic form $\Omega$ and the Riemannian metric $g$ give rise to a hyperkähler structure on $T\mathcal{M}_k$ with complex structures:

$$I_1(Z, T) = (iZ, -iT)$$

$$I_2(Z, T) = (T, -Z)$$

$$I_3(Z, T) = (iT, iZ).$$
The real symplectic form associated with $I_1$ is given by
\[ \omega_1 ((Z_1, T_1) : (Z_2, T_2)) = g(x, X) (I_1(Z_1, T_1) : (Z_2, T_2)) = 3 \Tr (Z_1 T_2) - 3 \Tr (T_1^* T_2). \]

### 3.4 Tri-Hamiltonian action of a unitary group $G$

Let $G$ be the following Banach Lie group of unitary operators:
\[ G := \mathcal{U}(H_+) \cap \{ \Id + L^1(H_+) \}, \]

The Lie algebra $g$ of $G$ is the Lie algebra of skew-Hermitian operators of trace class. We will denote by $G^\mathbb{C}$ the complexification of $G$:
\[ G^\mathbb{C} := GL(H_+) \cap \{ \Id + L^1(H_+) \}, \]

and by $g^\mathbb{C}$ its complex Lie algebra $g \oplus ig$. The group $G$ acts on $TM_k$ by
\[ u \cdot (x, X) := (x \circ u^{-1}, X \circ u^{-1}), \]

for all $u$ in $G$ and for all $(x, X)$ in $TM_k$. This action extends to an $I_1$-holomorphic action of $G^\mathbb{C}$ by
\[ g \cdot (x, X) := (x \circ g^{-1}, X \circ g^*), \]

for all $g$ in $G^\mathbb{C}$ and for all $(x, X)$ in $TM_k$.

**Proposition 3.1** The action of $G^\mathbb{C}$ on $TM_k$ is Hamiltonian with respect to the complex symplectic form $\Omega$, with moment map $\mu^\mathbb{C}$:
\[ \mu^\mathbb{C} : TM_k \rightarrow (g^\mathbb{C})' \]
\[ (x, X) \mapsto (a \mapsto \Tr (X^* xa)). \]

**Proof of Proposition 3.1**

Let us check that $\mu^\mathbb{C}$ satisfies:
\[ \langle d\mu^\mathbb{C}_{(x,X)} ((Z,T)) , a \rangle = i_a (a(X) \Omega (Z,T)), \]

for all $(Z,T)$ in $T_{(x,X)}M_k$ and for all $a$ in $g^\mathbb{C}$, where $\langle , , \rangle$ denotes the duality pairing and where $a \cdot (x, X) = (-x \circ a, X \circ a^*)$ is the vector induced by the infinitesimal action of $a$ on $(x, X)$. One has:
\[ \langle d\mu^\mathbb{C}_{(x,X)} ((Z,T)) , a \rangle = \Tr ((X^* Z + T^* x)a) = \Tr (X^* Za + T^* xa) \]
\[ = \Tr (aX^* Z) - \Tr (T^* (-x \circ a)) \text{ since } a \in L^1(H_+) \]
\[ = \Tr ((X \circ a^*)^* Z) - \Tr (T^* (-x \circ a)) \]
\[ = i_{(-x_{ao},-Xao^*)}\Omega ((Z,T)). \]

\[ \square \]

It follows from Proposition 3.1 that the real symplectic forms $\omega_2$ and $\omega_3$ are Hamiltonian with respect to the real moment maps:
\[ \mu_2 = \Re(\mu^\mathbb{C}) : TM_k \rightarrow g' \]
\[ (x, X) \mapsto (a \mapsto \frac{1}{2} \Tr (X^* x - x^* X)a) \]
\[ \mu_3 = \Im(\mu^\mathbb{C}) : TM_k \rightarrow g' \]
\[ (x, X) \mapsto (a \mapsto -\frac{1}{2} \Tr (X^* x + x^* X)a) \]

**Proposition 3.2** The action of $G$ on $TM_k$ is Hamiltonian with respect to the real symplectic form $\omega_1$ with moment map:
\[ \mu_1 : TM_k \rightarrow g' \]
\[ (x, X) \mapsto (a \mapsto -\frac{i}{2} \Tr (x^* x - X^* X)a). \]

**Proof of Proposition 3.2**

One has to check that $\mu_1$ satisfies:
\[ \langle (d\mu_1)_{(x,X)} ((Z,T)) , a \rangle = i_{(-x_{ao},-Xao)} \omega_1 ((Z,T)), \]

\[ \square \]
for all \((Z,T)\) in \(T_{(x,X)}M_k\) and all \(a\) in \(g\). One has:

\[
\langle d\mu_1(x,X) ((Z,T)), a \rangle = -\frac{1}{2} \text{Tr} \left( Z^* xa + x^* Z a \right) + \frac{1}{2} \text{Tr} \left( T^* X a + X^* T a \right)
\]

\[
= \frac{1}{2} \text{Tr} \left( Z^* (-x \circ a) - ax^* Z \right) - \frac{1}{2} \text{Tr} \left( T^* (-X \circ a) - aX^* T \right)
\]

\[
= \frac{1}{2} \text{Tr} \left( Z^* (-x \circ a) - (x \circ a)^* Z \right) - \frac{1}{2} \text{Tr} \left( T^* (-X \circ a) - (X \circ a^*)^* T \right)
\]

\[
= \frac{3}{2} \text{Tr} \left( -x \circ a \right) Z - \frac{3}{2} \text{Tr} \left( -X \circ a \right) T
\]

\[
i_{(-x \circ a, -X \circ a)} \omega_1 ((Z,T)).
\]

In the following, we will denote by \(\mu\) the \(g' \otimes \mathbb{R}^3\)-valued moment map defined by:

\[
\mu : T_{\mathcal{M}_k} \rightarrow g' \otimes \mathbb{R}^3
\]

\[
(x,X) \mapsto (\mu_1(x,X), \mu_2(x,X), \mu_3(x,X)).
\]

In the next Subsection we will consider the Ad\(^+-\)(G)-invariant value \(\xi_k := (-\frac{1}{2}k^2 \text{Tr} \ 0, 0, 0)\) of the moment map \(\mu\) and the level set:

\[
W_k := \mu^{-1}(\xi_k).
\]

### 3.5 Smooth Banach manifold structure on the level set \(W_k\)

This Subsection is devoted to the proof of the following Theorem:

**Theorem 3.3** \(W_k := \mu^{-1} \left( (-\frac{1}{2}k^2 \text{Tr} \ 0, 0, 0) \right)\) is a smooth Riemannian submanifold of \(T\mathcal{M}_k\).

We will prove that \(\xi_k = (-\frac{1}{2}k^2 \text{Tr} \ 0, 0, 0)\) is a regular value of the moment map \(\mu\). For this purpose we will need the following fact, which will be useful in other parts of the paper, so that we single it out here:

**Lemma 3.4** Let \(B\) be a Banach space which injects continuously into a Hilbert space \(H\). Let \(F\) be a closed subspace of \(B\) and \(\bar{F}\) its closure in \(H\). If the orthogonal projection of \(H\) onto \(\bar{F}\) maps \(B\) onto \(F\), then \(F\) admits a closed complement in \(B\) which is \(F^\perp \cap B\).

**\(\triangle\) Proof of Lemma 3.4** Since the orthogonal projection \(p\) of \(H\) onto \(\bar{F}\) maps \(B\) onto \(F\), \(B\) is the algebraic sum of \(F\) and \(F^\perp \cap B\). Since \(B\) injects continuously into \(H\) and \(p\) is continuous, the projection from \(B\) onto \(F\) with respect to \(F^\perp \cap B\) is continuous. Hence

\[
B = F \oplus (F^\perp \cap B)
\]

as a topological sum. \(\triangle\)

**\(\square\) Proof of Theorem 3.3** Consider the following smooth map of Banach manifolds

\[
\mathcal{F} : T\mathcal{M}_k \rightarrow L^1(H_+) \times S^1(H_+)
\]

\[
(x,X) \mapsto (X^*x, x^*x - X^*X),
\]

We have:

\[
d_{(x,X)}\mathcal{F} : T_{(x,X)}T\mathcal{M}_k \rightarrow L^1(H_+) \times S^1(H_+)
\]

\[
(Z,T) \mapsto (X^*Z + T^*x, x^*Z + Z^*x - X^*T - T^*X).
\]

The level set \(W_k\) is the preimage of \((0, k^2)\) under \(\mathcal{F}\). To prove that \(W_k\) is a smooth Banach submanifold of \(T\mathcal{M}_k\) it is sufficient to prove that the differential of \(\mathcal{F}\) at a point \((x,X)\) of \(W_k\) is surjective and that its kernel splits.

For this purpose consider the following decomposition of \(H_+;\)

\[
H_+ = \text{Ker} \ X \oplus (\text{Ker} \ X)^\perp.
\]

The operator \(x\) is a Fredholm operator hence it has closed range, and \(X\) is a compact operator. The equality \(X^*x = 0\) implies that the range \(\text{Ran} \ x\) of \(x\) is orthogonal to the range \(\text{Ran} \ X\) of \(X\). From the continuity of the orthogonal projection of \(H\) onto \(\text{Ran} \ x\) it follows that: \(\text{Ran} \ x \perp \text{Ran} \ X\). Let us introduce the following decomposition of \(H;\)

\[
H = \text{Ran} \ X \oplus (\text{Ran} \ X)^\perp \cap (\text{Ran} \ x)^\perp \oplus \text{Ran} \ x|_{\text{Ker} \ X} \oplus \text{Ran} \ x|_{\text{Ker} \ X}.
\]

\[
(5)
\]
With respect to the decompositions 1 and 2 of $H_+$ and $H$ into closed subspaces, $x$ and $X$ have the following expressions:

$$x = \begin{pmatrix} 0 & 0 & 0 \\ x_{31} & 0 & 0 \\ 0 & x_{42} & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & X_{12} \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where $x_{31}$ and $x_{42}$ are continuous bijections, thus isomorphisms, and where $X_{12}$ is 1-1 but not onto. Let $(Z, T)$ be a tangent vector to $TM_k$ at $(x, X)$, and denote by $(Z_{ij})_{1 \leq i,j \leq 2}$ and $(T_{ij})_{1 \leq i,j \leq 2}$ the block decompositions of $Z$ and $T$ with respect to the direct sums 1 and 2. Note that:

$$X^*Z + T^*x = \begin{pmatrix} T_{31}^* x_{31} & T_{32}^* x_{42} \\ X_{12}^* Z_{11} + T_{32}^* x_{31} & X_{12}^* Z_{12} + T_{32}^* x_{42} \end{pmatrix},$$

and that $x^*Z + Z^*x - X^*T - T^*X$ equals:

$$\begin{pmatrix} x_{31}^* Z_{31} + Z_{31}^* x_{31} & x_{32}^* Z_{32} + Z_{32}^* x_{42} - T_{11}^* x_{12} \\ x_{41}^* Z_{41} + Z_{41}^* x_{41} - T_{11}^* x_{11} & x_{42}^* Z_{42} + Z_{42}^* x_{42} - T_{12}^* x_{12} - T_{12}^* x_{12} \end{pmatrix}.$$

To show that the differential of $F$ is onto, consider an element $(U, V)$ of $L^1(H_+) \times S^1(H_+)$ and denote be $(U_{ij})_{1 \leq i,j \leq 2}$ and $(V_{ij})_{1 \leq i,j \leq 2}$ the block decompositions of $U$ and $V$ with respect to the direct sum $H_+ = \text{Ker } X \oplus (\text{Ker } X)^\perp$. A preimage of $(U, V)$ by $d_{(x, X)}F$ is given by the following ordered pair $(Z, T)$:

$$Z = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} x_{31}^{-1} v_{11} & \frac{1}{2} x_{31}^{-1} v_{12} \\ \frac{1}{2} x_{42}^{-1} v_{21} & \frac{1}{2} x_{42}^{-1} v_{22} \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 0 \\ -x_{31}^{-1} u_{11} & -x_{31}^{-1} u_{21} \\ x_{42}^{-1} u_{12} & x_{42}^{-1} u_{22} \end{pmatrix}.$$

Let us now show that for every $(x, X)$ in $\mathcal{W}_k$, the kernel of the differential $d_{(x, X)}F$ splits. It is given by the following subspace of $T_{(x, X)}\mathcal{M}_k$:

$$T_{(x, X)}\mathcal{W}_k := \{ (Z, T) \in T_{(x, X)}\mathcal{M}_k \mid X^*Z + T^*x = 0, \ x^*Z + Z^*x = X^*T + T^*X \}.$$

Consider the space $L^2(H_+, H) \times L^2(H_+, H)$ endowed with the complex structure $I(Z, T) = (iZ, -iT)$ and the strong Riemannian metric $g$ given by the real part of the natural Hermitian product. Let $E$ be the closure of $T_{(x, X)}\mathcal{W}_k$ in $L^2(H_+, H) \times L^2(H_+, H)$ and $E^{1\downarrow}$ its orthogonal in $L^2(H_+, H) \times L^2(H_+, H)$. By Lemma 3.1 to show that $T_{(x, X)}\mathcal{W}_k$ is complemented in $T_{(x, X)}(T, \mathcal{M}_k)$, it is sufficient to show that the orthogonal projection of $L^2(H_+, H) \times L^2(H_+, H)$ onto $E$ maps $T_{(x, X)}(T, \mathcal{M}_k)$ onto $T_{(x, X)}\mathcal{W}_k$. An ordered pair $(Z, T) \in L^2(H_+, H) \times L^2(H_+, H)$ is in $E$ if and only if $Z$ and $T$ are of the following form:

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \\ x_{31}^{-1} a_1 & x_{31}^{-1} a_1 \\ x_{42}^{-1} (X_{12} T_{12} - Z_{32} x_{31}) & x_{42}^{-1} \left( \frac{1}{2} (X_{12} T_{12} + T_{12} X_{12}) + a_2 \right) \end{pmatrix}, \quad T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \\ 0 & -x_{31}^{-1} Z_{11} X_{12} \\ 0 & -x_{42}^{-1} Z_{12} X_{12} \end{pmatrix},$$

where $a_1$ (resp. $a_2$) is an element of the space $A^2((\text{Ker } X)^\perp)$ of skew-Hermitian Hilbert-Schmidt operators on $\text{Ker } X$ (resp. on $(\text{Ker } X)^\perp$). An ordered pair $(Z, T) \in L^2(H_+, H) \times L^2(H_+, H)$ is in $E^{1\downarrow}$ if and only if $Z$ and $T$ have the following form:

$$Z = \begin{pmatrix} X_{12} T_{22}^{-1} x_{31}^{-1} & X_{12} T_{42}^{-1} x_{42}^{-1} \\ 0 & x_{31}^{-1} s_1 \\ x_{31}^{-1} s_1 & x_{31}^{-1} s_1 \\ Z_{41} & Z_{42} \end{pmatrix}, \quad T = \begin{pmatrix} -X_{12} x_{42}^{-1} Z_{41} & -X_{12} x_{42}^{-1} \end{pmatrix},$$

where $s_1$ (resp. $s_2$) is an element of the space $S^2((\text{Ker } X)^\perp)$ of self-adjoint Hilbert-Schmidt operators on $\text{Ker } X$ (resp. $(\text{Ker } X)^\perp$). We will denote by $p_E(Z)$, $p_E(T)$ the orthogonal projection of an element $(Z, T) \in L^2(H_+, H) \times L^2(H_+, H)$ to $E$, and $(p_E(Z_{ij}))_{1 \leq i \leq 4, 1 \leq j \leq 2}$ (resp.
that the element is continuous. Thus establishes an isomorphism between equivalence relation defined by Theorem 3.5. In this Subsection we will prove that the assumptions (D) and (S) of Subsection 2.3 are satisfied, as since and the Theorem follows.

\[ T_{(x,y)}(TM_k) = T_{(x,y)}W_k \oplus \left( E^{\pm x} \cap T_{(x,y)}TM_k \right). \]

and the Theorem follows.

3.6 Hyperkähler quotient of $TM_k$ by $G$

In this Subsection we will prove that the assumptions (D) and (S) of Subsection 2.3 are satisfied, as well as the following Theorem.

**Theorem 3.5.** The hyperkähler quotient of the weak hyperkähler space $TM_k$ with respect to the tri-symplectic action of $G$ provides $W_k/G$ with a structure of strong hyperkähler smooth Hilbert manifold.

For the remainder of the paper, we will denote by \( (g^{\text{red}}, I_1^{\text{red}}, I_2^{\text{red}}, I_3^{\text{red}}, \omega_1^{\text{red}}, \omega_2^{\text{red}}, \omega_3^{\text{red}}) \) the hyperkähler structure induced by $TM_k$ on the reduced space $W_k/G$.

**Proof of Theorem 3.5.**

Let us first show that the quotient space $W_k/G$ endowed with the quotient topology is Hausdorff. Since $G$ acts freely on $W_k$, by Propositions 2.3 and 2.4 it sufficient to prove that the graph $C$ of the equivalence relation defined by $G$ is closed in $W_k \times W_k$ and that the canonical application from $C$ to $G$ is continuous.

Note that, for all $(y, Y)$ in $W_k$, the operator $y$ is an injective Fredholm operator from $H_+^\perp$ to $H$, thus establishes an isomorphism between $H_+^\perp$ and its (closed) range. We will denote by $y^{-1}$ the inverse operator of $y$, and extend it to an operator from $H$ to $H$ by demanding that the restriction of $y^{-1}$ to the orthogonal complement of $\text{Ran} y$ vanishes.

Let us remark that the canonical map from the graph $C$ to $G$ assigns to an ordered pair $((x, X), (y, Y))$ the element $g = y^{-1}x$. It is continuous with respect to the topology of $C$ inherited from the topology of $W_k \times W_k$, and the topology of $G$ induced by the $L^1$-norm on $\text{Id} + L^1(H_+)$. Let $\{ (x_n, X_n) ; (x_n \circ g_n^{-1}, X_n \circ g_n^{-1}) \}$ be a sequence in $C$ which converges to an element $((x, X); (y, Y))$ in $W_k \times W_k$. The sequence $\{ g_n^{-1} = x_n^{-1} \circ (x_n \circ g_n^{-1}) \}$ is a sequence of elements of $G$ which converges to the element $x^{-1}y$ in $\text{Id} + L^1(H_+)$. Since $G$ is closed in $\text{Id} + L^1(H_+)$, it follows that $x^{-1}y$ is in $G$ and that $C$ is closed in $W_k \times W_k$. 

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By Theorem 2.22 for \( W_k/G \) to be hyperkähler, it is sufficient to have the following topological direct sum, referred to as (D):

\[
(D) \quad T(x, X)G \cdot (x, X) \oplus (T(x, X)G \cdot (x, X))^\perp \times = T(x, X)W_k.
\]

For this purpose, we will use Lemma 3.4. Let us again consider the space \( L^2(H_+, H) \times L^2(H_+, H) \) endowed with the complex structure \( I(Z, T) = (iZ, -iT) \) and the strong Riemannian metric \( \bar{g} \) given by the real part of the natural Hermitian product. Let \( E \) (resp. \( F \)) be the closure of \( T(x, X)W_k \) (resp. \( T(x, X)G \cdot (x, X) \)) in \( L^2(H_+, H) \times L^2(H_+, H) \). We will show that the \( g \)-orthogonal projection of \( E \) onto \( F \) maps \( T(x, X)W_k \) onto \( T(x, X)G \cdot (x, X) \). The orthogonal of \( F \) in \( E \) is the set of ordered pairs \((Z, T) \in L^2(H_+, H) \times L^2(H_+, H)\) of the form:

\[
Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}, \quad T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}.
\]

The orthogonal projection of \( E \) onto \( F \) maps an ordered pair \((Z, T) \in E\) having the block decompositions with respect to the direct sums \( a \) and \( b \):

\[
Z = \begin{pmatrix} Z_{11} & 0 \\ Z_{21} & Z_{22} \\ x_{31}^{-1}a_1 & x_{32}^{-1} \end{pmatrix}, \quad T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \\ 0 & 0 \end{pmatrix}.
\]

To an ordered pair \((p_G(Z), p_G(T))\) with decompositions:

\[
p_G(Z) = \begin{pmatrix} 0 & 0 \\ x_{31}^{-1}a_1 & x_{31}^{-1}(T_{11}X_{12} - x_{31}Z_{32})x_{42}^{-1}x_{42}^{-1} \\ x_{42}^{-1}(X_{12}T_{11} - Z_{32}x_{31}) & x_{42}^{-1} \end{pmatrix}
\]

\[
p_G(T) = \begin{pmatrix} 0 & 0 \\ x_{31}^{-1}(T_{11}X_{12} - x_{31}Z_{32})x_{42}^{-1}x_{42}^{-1} & 0 \\ 0 & 0 \end{pmatrix}.
\]

where \( a \) satisfies:

\[
a_2 + \frac{1}{2}(X_{12}^*T_{12} - T_{12}^*X_{12}) = k^2a - (x_{42}x_{42}a + ax_{42}x_{42}).
\]

Let us show that the projection \( p_G \) from \( E \) to \( F \) maps an element of \( T(x, X)W_k \) into \( T(x, X)G \cdot (x, X) \). Let

\[
p_+ = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \end{pmatrix}
\]

be the expression of the orthogonal projection onto \( H_+ \) with respect to the direct sums \( a \) and \( b \). For \((x, X) \in W_k\), the operators \( p_{11}, p_{12}, p_{14}, p_{21}, p_{22}, p_{23}\) are of trace class as well as \( p_{13} - x_{31}^{-1} \) and \( p_{24} - x_{42}^{-1} \). The condition \( p_+(Z) \in L^1(H_+) \) implies in particular that \( a_1, Z_{32}, \) and \( a_2 \) are of trace class. To conclude that \( p_G \) maps \( T(x, X)W_k \) into \( T(x, X)G \cdot (x, X) \), it remains to show that \( a \) defined by Equation (6) is of trace class. This will follow by Lemma 3.6 below. As a consequence, \( p_+ \circ p_G(Z) \) and \( p_+ \circ p_G(T) \) are of trace class. Hence, \( F^\perp \cap T(x, X)W_k \) is a closed complement to \( T(x, X)G \cdot (x, X) \). Since \( g \) is the restriction of \( \bar{g} \) to \( T(x, X)W_k \), it follows that:

\[
T(x, X)G \cdot (x, X) \oplus (T(x, X)G \cdot (x, X))^\perp = T(x, X)W_k.
\]
Lemma 3.6 The map $\mathcal{E}_{x,42}$ defined by

$$
\mathcal{E}_{x,42} : L^2 \left( (\text{Ker } X)^\perp \right) \xrightarrow{\text{a}} L^2 \left( (\text{Ker } X)^\perp \right)
$$

is an isomorphism which restricts to an isomorphism of $L^1 \left( (\text{Ker } X)^\perp \right)$.

Proof of Lemma 3.6:
Indeed, $x_{42}^* x_{42}$ is a self-adjoint positive definite operator on the space $(\text{Ker } X)^\perp$ satisfying

$$
x_{42}^* x_{42} = k^2 \text{Id} + X_{12}^* X_{12},
$$

where $X_{12}^* X_{12}$ is a compact operator. It follows that there exists a diagonal operator $D$ with respect to an orthogonal basis $\{ f_j \}_{j \in J}$ of $(\text{Ker } X)^\perp$, and a unitary operator $u \in U \left( (\text{Ker } X)^\perp \right)$ such that $x_{42}^* x_{42} = u D u^*$. Denote by $D_i$ the eigenvalues of $D$. Remark that $D_i > k^2$. It follows that the equation

$$
k^2 a = x_{42}^* x_{42} a + a x_{42}^* x_{42}
$$

is equivalent to

$$
k^2 u^* a u = u^* a u D + D u^* a u,
$$

and implies in particular that, for every $i$ and $j$ in $J$, one has:

$$
k^2 \langle (u^* a u)(f_j), f_i \rangle = (D_j + D_i) \langle (u^* a u)(f_j), f_i \rangle.
$$

From $(D_j + D_i) > k^2$, we get that $\langle (u^* a u)(f_j), f_i \rangle = 0$ for every $i, j \in J$. Thus $\text{Ker } x_{42}$ is $0$. To see that the map $\mathcal{E}_{x,42}$ is surjective, consider an element $V$ in $L^2 \left( (\text{Ker } X)^\perp \right)$, and define an operator $\tilde{a} \in L^2 \left( (\text{Ker } X)^\perp \right)$ by:

$$
\langle \tilde{a}(f_j), f_i \rangle = \frac{1}{(k^2 - (D_j + D_i))} \langle (u^* V) u(f_j), f_i \rangle.
$$

The operator $u \tilde{a} u^*$ is a preimage of $V$ by $\mathcal{E}_{x,42}$. Moreover if $V \in L^1 \left( (\text{Ker } X)^\perp \right)$, then $u \tilde{a} u^* \in L^1 \left( (\text{Ker } X)^\perp \right)$. Since $\mathcal{E}_{x,42}$ is clearly continuous, it follows that $\mathcal{E}_{x,42}$ is an isomorphism of $L^2 \left( (\text{Ker } X)^\perp \right)$ that restricts to an isomorphism of $L^1 \left( (\text{Ker } X)^\perp \right)$. △

Proposition 3.7 For every $(x, X)$ in $W_k$ one has:

$$
T_{(x,X)} M_k = T_{(x,X)} G^* (x, X) \oplus H_{(x,X)} \oplus I_1 \left( T_{(x,X)} G^* (x, X) \right) \oplus I_2 \left( T_{(x,X)} G^* (x, X) \right) \oplus I_3 \left( T_{(x,X)} G^* (x, X) \right),
$$

where $H_{(x,X)}$ is the orthogonal of $T_{(x,X)} G^* (x, X)$ in $T_{(x,X)} W_k$.

Proof of Proposition 3.7:
With the previous notation, it follows from the proof of Theorem 3.3 that the orthogonal projection from $L^2(H_+, H) \times L^2(H_+, H)$ to $T_{(x,X)} W_k$ with respect to the strong Riemannian metric $\tilde{g}$ takes $T_{(x,X)} M_k$ to $T_{(x,X)} W_k$. It also follows from the proof of Theorem 3.3 that the orthogonal projection from $T_{(x,X)} W_k$ to $T_{(x,X)} G^* (x, X)$ with respect to $\tilde{g}$ takes $T_{(x,X)} W_k$ onto $T_{(x,X)} G^* (x, X)$. Now let us remark that the complex structures $I_j$, $j = 1, 2, 3$, extend to complex structures of $L^2(H_+, H) \times L^2(H_+, H)$ by the same formulas, making $L^2(H_+, H) \times L^2(H_+, H)$ into a hyperkähler space. Since, for $j = 1, 2, 3$, the complex structure $I_j$ fixes $T_{(x,X)} M_k$ and is orthogonal with respect to $\tilde{g}$, it follows that for $j = 1, 2, 3$, the orthogonal projection from $L^2(H_+, H) \times L^2(H_+, H)$ onto $I_j \left( T_{(x,X)} G^* (x, X) \right)$ takes $T_{(x,X)} M_k$ onto $I_j \left( T_{(x,X)} G^* (x, X) \right)$. Hence, from the orthogonal sum

$$
L^2(H_+, H) \times L^2(H_+, H) = \frac{T_{(x,X)} G^* (x, X) \oplus H_{(x,X)} \oplus I_1 \left( T_{(x,X)} G^* (x, X) \right)}{I_2 \left( T_{(x,X)} G^* (x, X) \right) \oplus I_3 \left( T_{(x,X)} G^* (x, X) \right)},
$$

it follows that

$$
T_{(x,X)} M_k = T_{(x,X)} G^* (x, X) \oplus H_{(x,X)} \oplus I_1 \left( T_{(x,X)} G^* (x, X) \right) \oplus I_2 \left( T_{(x,X)} G^* (x, X) \right) \oplus I_3 \left( T_{(x,X)} G^* (x, X) \right).
$$

□
4 A 1-parameter family of hyperkähler structures on the cotangent bundle of the restricted Grassmannian

4.1 The stable manifold $W_k^{s_1}$ associated with the complex structure $I_1$

Recall that the complex Banach Lie group $G^C := GL(H_+) \cap \{ \Id + L^1(H_+) \}$ acts $I_1$-holomorphically on $TM_k$ by

$$ g \cdot ((x, X)) = (x \circ g^{-1}, X \circ g^*) $$

for all $g$ in $G^C$, and for all $(x, X)$ in $TM_k$. Let $W_k^{s_1}$ be the stable manifold associated with $W_k$ with respect to the complex structure $I_1$, i.e. the union of $G^C$-orbits (for the above action) intersecting $W_k$. Since by the proof of Theorem 3.5 and Proposition 3.4, assumptions (D) and (S) are satisfied, one has the following Corollary of Lemma 2.24 and Theorem 2.25.

**Corollary 4.1** The space $W_k^{s_1}$ is a $I_1$-complex submanifold of $TM_k$ and the quotient space $W_k^{s_1}/G^C$ is a smooth complex manifold. The map from $W_k/G$ to $W_k^{s_1}/G^C$ induced by the natural injection of $W_k$ into $W_k^{s_1}$ is an $I_1$-holomorphic diffeomorphism. □

In the following Proposition we give an explicit characterization of the stable manifold $W_k^{s_1}$ and we compute the projection:

$$ q_1 : W_k^{s_1} \to W_k $$

$$ (x, X) \mapsto q_1((x, X)) = g(x, X) \cdot (x, X). $$

defined by Proposition 2.19.

**Proposition 4.2** The stable manifold $W_k^{s_1}$ is the set

$$ \{(x, X) \in TM_k \text{ such that } X^*x = 0 \text{ and } x \text{ is one-to-one}, \} $$

and, for all $(x, X)$ in $W_k^{s_1}$, the unique element $g(x, X)$ of $\exp g$ such that $g(x, X) \cdot (x, X)$ belongs to $W_k$ is defined by

$$ g(x, X) = \left( \frac{k^2}{2} (x^*x)^{-1} + \frac{k^2}{2} (x^*x)^{-\frac{1}{2}} \left( \Id_{H_+} + \frac{4}{k^2} (x^*x)^{\frac{1}{2}} X^*X(x^*x)^{-\frac{1}{2}} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}. $$

□ **Proof of Proposition 4.2**

Let $A$ be the set of elements $(x, X)$ in $TM_k$ such that $X^*x = 0$ and $x$ is one-to-one. Let us show that $W_k^{s_1} \subset A$. Consider $(x, X) \in W_k^{s_1}$ and $g \in G^C$ such that $g \cdot (x, X) \in W_k$. We have:

$$ (X \circ g^*)^*(x \circ g^{-1}) = gX^*xg^{-1} = 0. $$

If $g^{-1} = |g|^{-1}u^{-1}$ denotes the polar decomposition of $g^{-1}$, the equality:

$$ (x \circ g^{-1})^*(x \circ g^{-1}) = (X \circ g^*)^*(X \circ g^*) = k^2 \Id_{H_+}, $$

reads:

$$ |g|^{-1}x^*x|g|^{-1} = |g|X^*X|g| = k^2 \Id_{H_+}. $$

Thus $|g|^{-1}x^*x|g|^{-1} = k^2 \Id_{H_+} + |g|X^*X|g|$ is a positive definite self-adjoint operator. Since $|g|^{-1}$ is an isomorphism, the same holds true for $x^*x$. It follows that $x$ is one-to-one.

To see that $A \subset W_k^{s_1}$, consider an element $(x, X)$ of $A$. We are looking for a positive definite self-adjoint operator $g(x, X)$ such that:

$$ g(x, X)x^*xg(x, X) - g(x, X)X^*Xg(x, X) = k^2 \Id_{H_+}. $$

The operator $x$ being one-to-one, $x^*x$ is positive definite and its square root $(x^*x)^{\frac{1}{2}}$ is an invertible operator on $H_+$. Hence it is sufficient to find a positive definite operator $\gamma := (x^*x)^{\frac{1}{2}}g^{-1}(x, X)$ such that:

$$ \gamma^*\gamma - \gamma^{-1}(x^*x)^{\frac{1}{2}}X^*X(x^*x)^{\frac{1}{2}}\gamma^{-1} = k^2 \Id_{H_+} $$

$$ \iff (\gamma^*\gamma)^2 - k^2(\gamma^*\gamma) - (x^*x)^{\frac{1}{2}}X^*X(x^*x)^{\frac{1}{2}} = 0 $$
The unique positive definite solution of the latter equation is:

\[
\gamma \gamma^* = \frac{k^2}{2} \left( \text{Id}_{H_+} + \left( \text{Id}_{H_+} + \frac{4}{k^2}(x^*x) \right)^* X(x^*x)^{\frac{3}{2}} \right).
\]

Therefore

\[g_{(x,X)} := (x^*x)^{-\frac{1}{2}} \gamma \gamma^*(x^*x)^{-\frac{1}{2}} \]  

(7)
is positive definite and self-adjoint, and its square root satisfies the required condition. □

4.2 Identification of \(W^\psi_k/G^C\) with the cotangent space \(T'Gr_{res}^0\) of the restricted Grassmannian

In this Subsection, we will use the following Theorem to identify the complex quotient space \(W^\psi_k/G^C\) with the cotangent bundle \(T'Gr_{res}^0\) of the connected component \(Gr_{res}^0\) of the restricted Grassmannian. Recall that \(\text{Ran} \, x\) denotes the range of an operator \(x\).

**Theorem 4.3** The map \(\Psi\) defined by

\[
\Psi : \quad (x,X) \quad \mapsto \quad \text{Ran} \, x, \; x \circ X^*
\]
is an \(I_1\)-holomorphic submersion whose fibers are the orbits under the \(I_1\)-holomorphic action of the complex group \(G^C\) on \(W^\psi_k \subset T \mathcal{M}_k\).

**Proof of Theorem 4.3**

- For \((x,X)\) in \(W^\psi_k\), the range \(P\) of \(x\) is an element of \(Gr_{res}^0\) since \(p_+ \circ x\) belongs to \(\{ \text{Id}_{H_+} + L^1(H_+) \}\), thus this is a Fredholm operator with index 0, and \(p_- \circ x\) is a Hilbert-Schmidt operator. Furthermore, the condition \(X^*x = 0\) implies that the restriction of \(\eta := \frac{1}{2} x \circ X^*\) to \(P\) vanishes. Thus \(\eta\) can be identified with an element of \(L^2(P^-,P)\) which is the cotangent space of \(Gr_{res}^0\) at \(P\).

- Let us check that \(\Psi\) is onto. For \(P\) in \(Gr_{res}^0\), denote by \(x_P\) the operator from \(H_+\) to \(H\) whose columns are the vectors of the canonical basis of \(P\) as defined in [38]. Then \(k \circ x_P\) is in \(\mathcal{M}_k\) (see [47] for the details of this affirmation). On the other hand, for every \(V \in L^2(P^-,P)\), the operator \(X\) defined by \(X := k^2 V^* \circ x_p^{-1}\) (where \(x_P\) is viewed as an isomorphism between \(H_+\) and \(P\)) satisfies \(\frac{1}{2} x_P \circ X^* = V\) and is an element of \(L^2(H_+,P^\perp)\). Moreover, since \(\varphi_- : P \rightarrow H_-\) is Hilbert-Schmidt, \(\varphi_+ : P^\perp \rightarrow H_+\) is also Hilbert-Schmidt and it follows that \(p_+ \circ x \in L^1(H_+)\). Thus \(\Psi ((k \circ x_P, k^2 V^* \circ x_p^{-1})) = (P,V)\).

- Let us show that two elements \((x_1,X_1)\) and \((x_2,X_2)\) in \(W^\psi_k\) have the same image by \(\Psi\) if and only if they are in the same orbit under \(G^C\). We have:

\[
\text{Ran} \, x_1 = \text{Ran} \, x_2 \iff 0 = x_2 \circ x_1 = x_2 \circ g^{-1} \text{ for some } g^{-1} \in G^C,
\]

thus:

\[
x_2 \circ X_2^* = x_1 \circ X_1^* = x_2 \circ g \circ X_1^*
\]

which is equivalent to: \(X_2 = X_1 \circ g^*\) since \(x_2\) is one-to-one.

- Let us explicit the differential of \(\Psi\) at \((x,X)\). Denote by \(P\) the range of \(x\), by \(U_P \subset Gr_{res}^0\) the open subset of elements \(P' \in Gr_{res}^0\) such that the orthogonal projection of \(P'\) onto \(P\) is an isomorphism and by \(\varphi_P\) the chart from \(U_P\) onto \(L^2(P,P^\perp)\) which maps \(P'\) to the unique element \(U\) in \(L^2(P,P^\perp)\) whose graph is \(P'\). Let

\[
(Z,T) \in T(x,X)W^\psi_k
\]

and

\[
(x(t),X(t)) \in C^1((-\epsilon,\epsilon),W^\psi_k)
\]

be such that:

\[
\dot{x}(0) = Z \text{ and } \dot{X}(0) = T.
\]

Denote by \((U(t),V(t))\) the curve \(\varphi \circ \Psi (x(t),X(t))\). Since

\[
\text{Ran} \, x(t) = \text{Ran} \, (\text{Id}_P + U(t))
\]

and since \(U(0) = 0\), there exists \(g(t) \in C^1((-\epsilon,\epsilon),G^C)\) such that

\[
x(t) \circ g(t)^{-1} = \text{Id}_P + U(t) \text{ and } x(0) \circ g(0)^{-1} = \text{Id}_P.
\]
Considering the decomposition of $H$ into $P \oplus P^\perp$, one has: $g(t) = pr_P \circ x(t)$, where $pr_P$ denotes the orthogonal projection onto $P$. Thus $U(t) = pr_{P^\perp} \circ x(t) \circ (pr_P \circ x(t))^{-1}$ and

$$
\frac{d}{dt}_{t=0} U(t) = pr_{P^\perp} \circ Z \circ x(0)^{-1}.
$$

Moreover one has $V(t) = \frac{1}{k^2} pr_P \circ x(t) \circ X(t) \big|_{P^\perp}$ and

$$
\frac{d}{dt}_{t=0} V(t) = \frac{1}{k^2} (pr_P(Z) \circ X^* + x \circ pr_{P^\perp}(T)^*)
$$

Therefore

$$
d\varphi \circ d\Psi((Z,T)) = \left( pr_{P^\perp} \circ Z \circ x^{-1}, \frac{1}{k^2} (pr_P(Z) \circ X^* + x \circ pr_{P^\perp}(T)^*) \right).
$$

It follows that:

$$
d\varphi \circ d\Psi((I_1(Z,T)) = i d\varphi \circ d\Psi((Z,T))
$$

thus $\Psi$ is $I_1$-holomorphic. Furthermore $d\varphi \circ d\Psi((x,X))$ is surjective, a preimage of $(U,V) \in L^2(P, P^\perp) \times L^2(P^\perp, P)$ being given by $(U \circ x, k^2 V^* x^{-1})$. At last, from the above considerations it follows that the kernel of $d\Psi((x,X))$ is the tangent space of the $G^C$-orbit $G^C((x, X))$, which splits by Proposition 3.7. □

**Corollary 4.4** The quotient space $W^*_k / G^C$ is isomorphic as a smooth complex manifold to the cotangent space $T^* Gr^0_{res}$ endowed with its natural complex structure via the following isomorphism

$$
\Psi : \frac{W^*_k}{G^C} \rightarrow T^* Gr^0_{res}
$$

$$
[(x,X)] \rightarrow (\text{Ran}_x \frac{1}{k^2} x \circ X^*).
$$

Hence $T^* Gr^0_{res}$ carries a 1-parameter family of hyperkähler structures indexed by $k \in \mathbb{R}^*$. 

**Remark 4.5** By exchanging $H_+ \subset \ker L$ with a subspace of another connected component of $Gr^0_{res}$, we obtain the cotangent space of every connected component of $Gr^0_{res}$ as a hyperkähler quotient.

By restriction to the zero section of the tangent space $T M^*_k$ one deduces from the previous Theorem the following result, which has been, as already mentioned in the Introduction, partially obtained by T. Wurzbacher (cf. [51]):

**Corollary 4.6** For every $k \in \mathbb{R}^*$, the connected component $Gr^0_{res}$ of the restricted Grassmannian is diffeomorphic to the Kähler quotient of the space $M^*_k$ by the Hamiltonian action of the unitary group $G$, with level set

$$
\{ x \in M^*_k, x^* x = k^2 \text{Id} \}.
$$

### 4.3 The Kähler potential $K_1$ of $T^* Gr^0_{res}$

The hyperkähler manifold $T M^*_k$ admits a globally defined hyperkähler potential, i.e. a Kähler potential with respect to all complex structures, which has the following expression:

$$
K : T M^*_k \rightarrow \mathbb{R}
$$

$$(x,X) \rightarrow \frac{1}{4} \text{Tr} (x^* x + X^* X - k^2 \text{Id}).
$$

The theory of Subsection 2.4 applied to the particular case of $T^* Gr^0_{res}$ yields the following Theorem:

**Theorem 4.7** For all $k^2 \in \mathbb{N}^*$, the 2-form $\Psi^* \omega^\text{red}_1$ on $W^*_k$ satisfies $i \Psi^* \omega^\text{red}_1 = dd^c K_1$, where for all $(x,X)$ in $W^*_k$,

$$
K_1 ((x,X)) = \frac{k^2}{4} \log \det \left( \frac{x^* x}{k^2} \right) + \frac{k^2}{2} \text{Tr} \left( \frac{X X^*}{k^2} - \text{Id} \right) - \frac{k^2}{4} \text{Tr} \left( \log \frac{X X^*}{k^2} \right),
$$

with $\gamma \gamma^* := \frac{k^2}{4} \left( \text{Id}_{H_+} + \left( \text{Id}_{H_+} + \frac{1}{2k} (x^* x)^{1/2} X X^* (x^* x)^{1/2} \right) \right)$. 



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Thus, after conjugation by $g_{(x,X)}^{-1}$:

$$K (g_{(x,X)}^{-1}(x,X)) := K \left( g_{(x,X)}^{-1}(x,X) \right) + \frac{1}{2} \log \left| \chi_{\overline{k}} (g_{(x,X)}) \right|^2.$$ 

Since $g_{(x,X)}X^*Xg_{(x,X)} = g_{(x,X)}^{-1}x^*xg_{(x,X)}^{-1} - k^2 \text{Id}$, one has:

$$K \left( g_{(x,X)}^{-1}(x,X) \right) := \frac{1}{2} \text{Tr} \left( g_{(x,X)}^{-1}x^*xg_{(x,X)}^{-1} + g_{(x,X)}X^*Xg_{(x,X)} - k^2 \text{Id} \right)$$

Thus, after conjugation by $g_{(x,X)}^{-1}$:

$$K \left( g_{(x,X)}^{-1}(x,X) \right) = \frac{k^2}{2} \text{Tr} \left( \frac{x^*x}{k^2} - \text{Id} \right),$$

and by (7)

$$g_{(x,X)}^{-2}x^*x = (x^*x)^{-\frac{k}{2}} \gamma \gamma^*(x^*x)^{\frac{k}{2}}.$$ 

After conjugation by $(x^*x)^{-\frac{k}{2}}$, we have:

$$K \left( g_{(x,X)}^{-1}(x,X) \right) = \frac{k^2}{2} \text{Tr} \left( \frac{\gamma \gamma^*}{k^2} - \text{Id} \right).$$

On the other hand,

$$\frac{1}{2} \log \left| \chi_{\overline{k}} (g_{(x,X)}) \right|^2 = - \frac{1}{2} \log \det \left( g_{(x,X)}^{-2} \right) \frac{k^2}{2}$$

Furthermore, the operator

$$A := \frac{\gamma \gamma^*}{k^2} - \text{Id} = \frac{1}{2} \left( \left( \text{Id} + \frac{4}{k^4} (x^*x)^{\frac{k}{2}}X^*X(x^*x)^{\frac{k}{2}} \right)^{\frac{k}{2}} - \text{Id} \right)$$

is self-adjoint positive and of trace class. Thus:

$$\log \det \left( \frac{\gamma \gamma^*}{k^2} \right) = \text{Tr} \left( \frac{\gamma \gamma^*}{k^2} \right).$$

**Proposition 4.8** For every $\overline{k}^2 \in \mathbb{N}^*$, the 2-form $\Psi^* e_\overline{k}$ on $W_{\overline{k}}^*$ satisfies $i \Psi^* e_\overline{k} = dx \circ K_1$, with:

$$K_1 ((x,X)) = \frac{k^2}{4} \text{det} \left( \frac{4}{k^4} x^* x \right) + \frac{k^2}{4} \text{Tr} \left( \left( \text{Id} + 4V^*X \right)^{\frac{k}{2}} - \text{Id} \right)$$

where $V^* = \frac{1}{4} x \circ X^*$ is the image of the class $[x,X]$ under the identification $\hat{\Psi} : W_{\overline{k}}^* / G^C \to T^*Gr_{\text{res}}$ given by Corollary 4.4.

**Proof of Proposition 4.8**

Since:

$$\frac{\gamma \gamma^*}{k^2} = \frac{1}{2} \left( \text{Id} + \left( \text{Id} + \frac{4}{k^4} x^* X^* x \right)^{\frac{k}{2}} \right),$$

the operator $\frac{\gamma \gamma^*}{k^2}$ is conjugate to:

$$\frac{1}{2} \left( \text{Id} + \left( \text{Id} + \frac{4}{k^4} x^* X^* x \right)^{\frac{k}{2}} \right).$$

Hence one has:

$$\text{Tr} \left( \frac{\gamma \gamma^*}{k^2} - \text{Id} \right) - \frac{k}{2} \text{Tr} \left( \log \frac{\gamma \gamma^*}{k^2} \right) = \text{Tr} \left( \left( \text{Id} + 4V^*X \right)^{\frac{k}{2}} - \text{Id} \right)$$

$$- \frac{k}{2} \text{Tr} \left( \text{Id} + \left( \text{Id} + 4V^*X \right)^{\frac{k}{2}} \right).$$
4.4 The Kähler potential $K_1$ as a function of the curvature of $Gr^0_{res}$

Theorem 4.9 The potential $K_1$ has the following expression in terms of the curvature:

$$K_1((x, X)) = \frac{k^2}{4} \log \det \left( \frac{x^* x}{k^2} \right) + k^2 g_{Gr} (f(I_1 R_{I_1, V, V}) V, V),$$

with $V = \frac{1}{k^2} X \circ x^*$ and $f(u) = \frac{1}{u} \left( \sqrt{1 + u} - 1 - \log \frac{1 + \sqrt{1 + u}}{2} \right)$.

Proof of Theorem 4.9

The Grassmannian $Gr^0_{res}$ is a Hermitian-symmetric orbit of the connected component $U^0_{res}$ of the restricted unitary group. Its curvature is therefore given by (see [46]):

$$R_{X, Y, Z} = Y X^* Z - Z Y^* X + Z X^* Y - X Y^* Z,$$

for all $X, Y, Z \in T_p Gr^0_{res}$. The operator $R_{I_1, V, V}$ acts on $T_p Gr^0_{res}$ by:

$$R_{I_1, V, V} Y = -2i(VV^* Y + YV^* V).$$

It follows that:

$$g_{Gr}(I_1 R_{I_1, V, V} V, V) = 2\Re \operatorname{Tr} (VV^* V + V^* V V) = 4\Re \operatorname{Tr} ((V^* V)^2)$$

and:

$$g_{Gr}((I_1 R_{I_1, V, V})^2 V, V) = \Re \operatorname{Tr} (4(V^* V)^{j+1})$$

$$= \frac{1}{4} \operatorname{Tr} ((4V^* V)^{j+1}).$$

Therefore one has:

$$\frac{1}{4} \operatorname{Tr} \left( \left( \operatorname{Id} + 4V^* V \right)^{\frac{1}{2}} - \operatorname{Id} \right) - \frac{1}{2} \operatorname{Tr} \log \frac{1}{2} \left( \operatorname{Id} + (\operatorname{Id} + 4V^* V)^{\frac{1}{2}} \right)$$

$$= g_{Gr}(f(I_1 R_{I_1, V, V}) V, V),$$

with

$$f(u) = \frac{1}{u} \left( \sqrt{1 + u} - 1 - \log \frac{1 + \sqrt{1 + u}}{2} \right).$$
Proof of Proposition 5.1

Let us denote by $\varepsilon$ the operator $ik^2p_+$. The stabilizer of $\varepsilon$ under the action of $GL_2(H)$ by conjugation is $GL_2(H_+) \times GL_2(H_-)$. The tangent space at $\varepsilon$ of the homogeneous space $GL_2(H) \cdot \varepsilon$ is isomorphic to $gl_2(H)/(gl_2(H_+) \times gl_2(H_-))$ which can be identified with $L^2(H_+, H_-) \times L^2(H_-, H_+)$. For $g \in GL_2(H)$, $g \varepsilon g^{-1} = ik^2pr_{g,H_+}$, where $pr_{g,H_+}$ denotes the projection on $g.H_+$ parallel to $g.H_-$. Since $g$ belongs to $GL_2(H)$, the orthogonal projection of $g.H_+$ to $H_-$ is an Hilbert-Schmidt operator, and the orthogonal projection of $g.H_+$ to $H_+$ is a Fredholm operator of index 0. Similarly, the orthogonal projection of $g.H_-$ to $H_+$ is Hilbert-Schmidt, and the orthogonal projection of $g.H_-$ to $H_-$ is a Fredholm operator of index 0. Thus $g.H_+$ belongs to $G_{res}^0$ and $g.H_-$ to $G_{res}^0$. Moreover $g.H_+ \cap g.H_- = \{0\}$.

5.2 The stable manifold $W_{k}^{3s}$ associated with the complex structure $I_3$

Recall that $I_3(Z,T) = (iT, iZ)$ for $(Z,T) \in T(x,X)TM_k$, and that the action of the Lie algebra $g$ on $(x,X) \in TM_k$ is given by

$$a.(x,X) = (-x \circ a, -X \circ a),$$

for all $a$ in $g$. The action $\cdot_3$ of the complexification $g^C$ of the Lie algebra $g$ on $TM_k$ compatible with $I_3$ is defined by

$$i a \cdot_3 (x,X) := I_3(a)(x,X) = (iX \circ a, -ix \circ a) = (x,X) \begin{pmatrix} 0 & -ia \\ -ia & 0 \end{pmatrix},$$

for all $a$ in $g$. This action integrates into an $I_3$-holomorphic action of $G^C = \exp i g$. $G$ on $TM_k$, also denoted by $\cdot_3$, and given by

$$\exp(iu)u \cdot_3 (x,X) := (x \circ u^{-1}, X \circ u^{-1}) \begin{pmatrix} \cosh ia & -\sinh ia \\ -\sinh ia & \cosh ia \end{pmatrix},$$

for all $a$ in $g$ and for all $u$ in $G$. By Lemma 2.23 the stable manifold associated with $I_3$ is the $I_3$-complex submanifold of $TM_k$, contained in $\mu_1^{-1}( -\frac{1}{2} k^2 \text{Tr} ) \cap \mu_2^{-1}(0)$, defined as

$$W_{k}^{3s} := \{ (x,X) \in TM_k, \exists a \in g, \exp i a \cdot_3 (x,X) \in W_k \}.$$

5.3 Identification of $W_{k}^{3s}/G^C$ with $O^C$

Theorem 5.2 The map $\psi$ defined by :

$$\psi : W_{k}^{3s} \rightarrow O^C \quad (x,X) \mapsto z = i(x + X)(x^* - X^*)$$

is an $I_3$-holomorphic submersion whose fibers are the orbits of the $I_3$-holomorphic action of $G^C$ on $TM$.

Proof of Theorem 5.2

Let us show that $\psi$ takes all its values in $O^C$. Recall that, for $(x,X) \in W_{k}^{3s}$, one has :

$$x^*x - X^*X = k^2 \text{Id} \quad \text{and} \quad X^*x = x^*X.$$

Thus :

$$(x^* - X^*)(x + X) = k^2 \text{Id},$$

$$(x^* + X^*)(x - X) = k^2 \text{Id}. \quad (8)$$

It follows that $\text{Ker} (x + X) = \{0\}$ and $\text{Ker} (x - X) = \{0\}$. The kernel of $z$ is therefore :

$$\text{Ker} z = \text{Ker} (x^* - X^*) = (\text{Ran} (x - X))^\perp.$$

Moreover, since for all $v \in H$ :

$$z ((x + X)v) = i(x + X)(x^* - X^*)(x + X)v = ik^2(x + X)v,$$

the subspace $\text{Ran} (x + X)$ is contained in the eigenspace of $z$ corresponding to the eigenvalue $ik^2$. Hence :

$$\text{Ran} (x + X) \cap \text{Ker} (x^* - X^*) = \{0\}.$$

Further, the projection of $H$ onto $\text{Ran} (x + X)$ is given by

$$p_1 : H \rightarrow \text{Ran}(x + X) \quad v \mapsto \frac{1}{ik}(x + X)(x^* - X^*)v.$$
and is continuous. Since $\text{Id}_H - p_1$ takes its values in $\text{Ker} \,(x^* - X^*)$, one has
\[
\text{Ran}(x + X) \oplus \text{Ker} \,(x^* - X^*) = H
\]
as a direct topological sum. Moreover, for $(x, X) \in T\mathcal{M}_k$, $\text{Ran}(x + X)$ and $\text{Ran}(x - X)$ are elements of $\mathcal{G}^0_{\text{res}}$, thus $\text{Ker} \,(x^* - X^*) = \{ \text{Ran}(x - X) \}^\perp$ is an element of $\mathcal{G}^0_{\text{res}}$. It follows that $\psi$ takes values in $\mathcal{O}^C$.

Let us show that the fibers of $\psi$ are the orbits under the $I_3$-holomorphic action of $G^C$. Suppose that $\psi((x_1, X_1)) = \psi((x_2, X_2))$ where $(x_1, X_1)$ and $(x_2, X_2)$ are in $W^k_{\mathcal{G}}$. It follows that :
\[
\text{Ran}(x_1 + X_1) = \text{Ran}(x_2 + X_2) \quad \text{and} \quad \text{Ran}(x_1 - X_1) = \text{Ran}(x_2 - X_2).
\]
Therefore there exists $g \in \text{GL}(H_+)$ such that $(x_2 + X_2) = (x_1 + X_1) \circ g$ and $g' \in \text{GL}(H_+)$ such that $(x_2 - X_2) = (x_1 - X_1) \circ g'$. This implies that:
\[
\begin{align*}
2x_2 & = x_1(g + g') + X_1(g - g') \\
2X_2 & = x_1(g - g') + X_1(g + g')
\end{align*}
\]
Recall that for $i = 1, 2, x_i^*x_i - X_i^*X_i = k^2 \text{Id}$ and $X_i^*x_i = x_i^*X_i$. We have:
\[
4(x_i^*x_2 - X_i^*X_2) = (g^* + g^*)(x_i^*x_1 - X_i^*X_1)(g + g') + (g^* - g^*)(x_i^*X_1 - x_i^*x_1)(g - g') + (g^* - g^*)(X_i^*x_1 - x_i^*X_1)(g - g') + (g^* + g^*)(X_i^*x_1 - X_i^*x_1)(g + g'),
\]
i.e.
\[
g^*g' + g^*g = 2 \text{Id},
\]
and
\[
4(X_i^*x_2 - x_i^*X_2) = (g^* - g^*)(x_i^*x_1 - X_i^*X_1)(g + g') + (g^* + g^*)(x_i^*X_1 - x_i^*x_1)(g - g') + (g^* - g^*)(X_i^*x_1 - x_i^*X_1)(g - g') + (g^* + g^*)(X_i^*x_1 - X_i^*x_1)(g + g'),
\]
that is :
\[
g^*g' = g^*g.
\]
Thus $g' = g^{-1}$. Denoting by $\exp(-i\alpha)u^{-1} = g^{-1}$ the polar decomposition of $g^{-1}$, with $u \in U(H_+)$ and $\alpha \in u(H_+)$, it follows that:
\[
\begin{align*}
x_2 & = x_1 \cosh(\alpha)u + X_1 \sinh(\alpha)u \\
X_2 & = -x_1 \sinh(\alpha)u + X_1 \cosh(\alpha)u.
\end{align*}
\]
Consequently : $(x_2, X_2) = \exp(-i\alpha)u^{-1} \cdot (x_1, X_1)$, in other words $(x_1, X_1)$ and $(x_2, X_2)$ belong to the same $G^C$-orbit.

Let us show that $\psi$ is onto. Let $P \in \mathcal{G}_{\text{res}}$ and $Q \in \mathcal{G}_{\text{res}}$ be such that $P \cap Q = \{ 0 \}$. $Q^\perp$ is the graph of a Hilbert-Schmidt operator $A : P \rightarrow P^\perp$ and $Q$ is the graph of $-A^* : P^\perp \rightarrow P$. Let $f$ be the map that takes an orthonormal basis $\{ e_i \} \in H_+$ to the associated canonical basis of $P$ and that takes an orthonormal basis $\{ e_{i+1} \} \in H_- \cap H_-$ to the associated canonical basis of $P^\perp$. Denote by $g$ the unitary element $f \circ |f|^{-1}$. Let us remark that $g$ belongs to $U_2(H)$ and satisfies : $p_+ \circ g|_{H_+} \in \text{Id}_{H_+} + L^1(H_+)$ as well as $p_- \circ g|_{H_-} \in \text{Id}_{H_-} + L^1(H_-)$. Define :
\[
\left\{ \begin{array}{c}
x &= k(\text{Id}_P + \frac{1}{2}A) \circ g|_{H_+} \\
X &= -\frac{1}{2}A \circ g|_{H_-}
\end{array} \right.
\]
One has: $\text{Ran} \,(x + X) = \text{Ran} \,(g|_{H_+}) = P$ and $\text{Ran} \,(x - X) = \text{Ran} \,(\text{Id}_P + A) \circ g|_{H_+} = Q^\perp$. Let us check that $(x, X)$ is an element of $TM$. Denote by :
\[
\text{Id}_H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
the block decomposition of the identity operator with respect to the direct sums $H = P \oplus P^\perp$ and $H = H_+ \oplus H_-$, where $a$ (resp. $d$) belongs to $\text{Fred}(P, H_+)$ (resp. $\text{Fred}(P^\perp, H_-)$) and where $b$ (resp. $c$) belongs to $L^2(P^\perp, H_+)$ (resp. $L^2(P, H_-)$). Further, denote by :
\[
g = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}
\]
the block decomposition of $g$ with respect to the direct sums $H = H_+ \oplus H_-$ and $H = P \oplus P^\perp$. The block decomposition of $g$ with respect to $H = H_+ \oplus H_-$ is:

$$g = \begin{pmatrix} a_{11} & b_{12} \\ c_{11} & d_{12} \end{pmatrix}.$$  

As mentioned above $p_+ \circ g|_{H_+} = a_{11}$ belongs to $\text{Id}_{H_+} + L^1(H_+)$, and $p_- \circ g|_{H_-} = d_{12}$ belongs to $\text{Id}_{H_-} + L^1(H_-)$. It follows that with respect to the direct sum $H = H_+ \oplus H_-$, the operator $x$ has the following expression:

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} k \text{Id}_P \\ \frac{k}{2} A \end{pmatrix} \circ u_1 = \begin{pmatrix} k a_1 + \frac{k}{2} b A_1 \\ k c_1 + \frac{k}{2} d A_1 \end{pmatrix}.$$  

It follows that:

$$p_+ \circ x = k a_1 + \frac{k}{2} b A_1 \in k \text{Id}_{H_+} + L^1(H_+)$$

and

$$p_- \circ x = k c_1 + \frac{k}{2} d A_1 \in L^2(H_+, H_-).$$

Similarly, 

$$p_+ \circ X = -\frac{k}{2} b A_1 \in L^1(H_+)$$

and

$$p_- \circ X = -\frac{k}{2} d A_1 \in L^2(H_+, H_-).$$

Hence the ordered pair $(x, X)$ is in $TM_4$. Besides, $x^* x - X^* X = k^2 \text{Id}_{H_+}$ and $X^* x = x^* X = 0$. It remains to prove that $(x, X) \in W_k^{4^3}$. For this purpose, observe that:

$$x^* x + X^* X = k^2 \text{Id}_{H_+} + \frac{k^2}{2} u_1^* A^* A u_1$$

$$X^* x + x^* X = -\frac{k^2}{2} u_1^* A^* A u_1.$$

The condition $\exp ia \cdot (x, X) \in W_k$ is equivalent to the following equation:

$$\cosh ia \circ (\text{Id}_{H_+} + \frac{1}{2} u_1^* A^* A u_1) \circ \sinh ia + \sinh ia \circ (\text{Id}_{H_+} + \frac{1}{2} u_1^* A^* A u_1) \circ \cosh ia$$

$$+ \cosh ia \circ (\frac{1}{2} u_1^* A^* A u_1) \circ \sinh ia + \sinh ia \circ (\frac{1}{2} u_1^* A^* A u_1) \circ \sinh ia = 0,$$

whose solution is:

$$a = \frac{i}{4} \log (\text{Id}_{H_+} + u_1^* A^* A u_1),$$

which belongs to $A^1(H_+)$. 

The differential of $\psi$ at an element $(x, X) \in W_k^{4^3}$ maps the ordered pair $(Z, T) \in T_{(x, X)} W_k^{4^3}$ to:

$$d\psi_{(x, X)} ((Z, T)) = i(Z + T)(x^* - X^*) + i(x + X)(Z^* - T^*).$$

One has $d\psi_{(z, X)} (I_3 ((Z, T))) = id\psi_{(z, X)} ((Z, T))$, thus $\psi$ is holomorphic. Let $z$ be in $\mathcal{O}^C$, and let $P$ (resp. $Q$) be the eigenspace of $z$ with respect to the eigenvalue $i k^2$ (resp. 0). Let $(U, V)$ be an element of $L^2(P, P^\perp) \times L^2(Q, Q^\perp)$. A preimage of $(U, V)$ by $d\psi_{(z, X)}$ is given by the ordered pair $(Z, T) \in L^2(H_+, H) \times L^2(H_+, H)$ defined by:

$$U = i(Z + T)(x^* - X^*)$$

$$V = i(x + X)(Z^* - T^*).$$

Using equations \[\square\] page 27, one gets:

$$(Z + T) = -\frac{i}{k^2} U(x + X)$$

$$(Z - T) = \frac{i}{k^2} V^*(x - X).$$

Hence:

$$Z = -\frac{i}{2k^2} (U(x + X) - V^*(x - X))$$

$$T = -\frac{i}{2k^2} (U(x + X) + V^*(x - X)).$$

Moreover, for $(x, X) \in W_k^{4^3}$, $p_+ \circ Z$ and $p_+ \circ T$ are trace class operators. It follows that the differential $d\psi_{(x, X)}$ is onto.

\[\square\]

**Corollary 5.3** The quotient space $W_k^{4^3} / G^C$ is diffeomorphic to the complexified orbit $\mathcal{O}^C$ via the isomorphism:

$$\tilde{\psi} : \quad W_k^{4^3} / G^C \quad \mapsto \quad \mathcal{O}^C$$

$$[(x, X)] \quad \mapsto \quad z = i(x + X)(x^* - X^*)$$

\[\square\]
5.4 The Kähler potential $\hat{K}_3$ of $T^*Gr_{res}^0$

From the general theory it follows that:

$$\psi^*\omega_3^{red}((x, X)) = \dd c\hat{K}(q_3(x, X)),$$

where: $K((x, X)) = \frac{1}{2} \text{Tr} (x^*x + X^*X - k^2 \text{Id})$, and where $q_3$ is the projection from $W^*_k$ to $W_k$. This Subsection is devoted to the computation of the Kähler potential $K_3 := K \circ q_3$ associated with the complex structure $I_3$ at a point $(x, X)$ of the stable manifold $W^*_k$ by the use of an invariant of the $G^C$-orbits. We will use the following notation. The injection of $g$ into $g'$ given by the trace allows to identify the moment map $\mu_3$ with the map (still denoted by $\mu_3$) defined by

$$\mu_3 ((x, X)) = \frac{i}{2}(X^*x + x^*X).$$

Define a function $\mu_4$ by:

$$\mu_4 ((x, X)) := \frac{i}{2}(x^*x + X^*X).$$

We have the following Lemma:

**Lemma 5.4** For every $(x, X)$ in $W^*_k$, one has

$$K_3 ((x, X)) = \frac{1}{4} \text{Tr} \left( \left( \mu_4^2 ((x, X)) - \mu_3^2 ((x, X)) \right) \right)^\frac{1}{2} - k^2 \text{Id}.$$  

**Proof of Lemma 5.4**

One has:

$$\mu_3 (\exp i\alpha \cdot 3 (x, X)) = \cosh i\alpha \circ \mu_4 ((x, X)) \circ \sinh i\alpha + \sinh i\alpha \circ \mu_4 ((x, X)) \circ \cosh i\alpha,$$

$$\mu_4 (\exp i\alpha \cdot 3 (x, X)) = \cosh i\alpha \circ \mu_4 ((x, X)) \circ \cosh i\alpha + \sinh i\alpha \circ \mu_4 ((x, X)) \circ \sinh i\alpha,$$

$$+ \cosh i\alpha \circ \mu_3 ((x, X)) \circ \sinh i\alpha + \sinh i\alpha \circ \mu_3 ((x, X)) \circ \cosh i\alpha.$$

Therefore:

$$(\mu_3 + \mu_4) (\exp i\alpha \cdot 3 (x, X)) = \exp i\alpha \circ (\mu_3 + \mu_4) \circ \exp i\alpha,$$

$$+ \mu_4 (\exp i\alpha \cdot 3 (x, X)) = \exp(-i\alpha) \circ (\mu_3 - \mu_4) \circ \exp(-i\alpha),$$

and

$$(\mu_4^2 - \mu_3^2) (\exp i\alpha \cdot 3 (x, X)) = \exp(i\alpha) \circ \left( \mu_4^2 ((x, X)) - \mu_3^2 ((x, X)) \right) \circ \exp(-i\alpha).$$

For $\alpha \in g$ such that $\exp i\alpha \cdot 3 ((x, X)) = q_3 ((x, X))$, it follows that

$$\mu_4 (q_3 ((x, X))) = \exp(i\alpha) \circ \left( \mu_4^2 ((x, X)) - \mu_3^2 ((x, X)) \right) \circ \exp(-i\alpha),$$

and

$$K_3 ((x, X)) = \frac{1}{4} \text{Tr} \left( \left( \mu_4^2 ((x, X)) - \mu_3^2 ((x, X)) \right) \right)^\frac{1}{2} - k^2 \text{Id}.$$  

**Proposition 5.5**

For every $(x, X)$ in $W^*_k$, one has

$$K_3 ((x, X)) = \frac{k^2}{4} \text{Tr} \left( \text{Id} + \frac{4}{k^2} (xX^*X^*x - X^*x X^*x) \right)^\frac{1}{2} - \text{Id}.$$  

**Proof of Proposition 5.5**

Since $W^*_k \subset \mu_3^{-1} \left( -\frac{4}{k^2} \text{Tr} \right) \cap \mu_2^{-1}(0)$, for all $(x, X)$ in $W^*_k$, one has $x^*x - X^*X = k^2 \text{Id}$ and $X^*x = x^*X$. Hence:

$$(\mu_4^2 ((x, X)) - \mu_3^2 ((x, X))) = x^*x X^*x + x^*X x^*X + X^*X x^*x + X^*X X^*x X$$

$$= x^*x (X^*X + k^2) + x^*x X^*x + X^*X x^*x + x^*x = k^4 + 4x^*x X^*x - 4x^*x X^*x.$$  

The result then follows after conjugation by $x^{s-1}$ viewed as an operator of $H_+$ onto $\text{Ran} x.$  

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Theorem 5.6:
The symplectic form \( \omega^{red}_3 \) on the cotangent space \( T^*Gr^0_{res} \) admits a globally defined Kähler potential \( \hat{K}_3 \) on \( T^*Gr_{res} \), whose expression at \( (P, V^*) \in T^*Gr_{res} \) is given by

\[
\hat{K}_3((P, V^*)) = \frac{k^2}{4} \text{Tr} \left( (I + 4V^*V)^{\frac{1}{2}} - I \right)
\]

where:

\[
h(u) := \frac{1}{u}(\sqrt{1 + u} - 1).
\]

**Proof of Theorem 5.6**

When \((x, X)\) belongs to the level set, \(x^*X = 0\) and

\[
K_3((x, X)) = \frac{k^2}{4} \text{Tr} \left( (I + \frac{1}{4}(xX^*Xx^*))^{\frac{1}{2}} - I \right)
\]

where \(V = \frac{1}{4^2}X \circ x^*\). The theorem then follows from the identities:

\[
g_{Gr}(I_xR_{I_y}V, V) = \frac{1}{4} \text{Tr} ((4V^*V)^2)
\]

and

\[
g_{Gr}((I_xR_{I_y}V)^2V, V) = \frac{1}{4} \text{Tr} ((4V^*V)^{2+1})
\]

and from the fact that \(p^*_3\omega^{red}_3 = dd^c\hat{K}_3 = dd^c\hat{K}_3\) since \(p_3\) is holomorphic with respect to the complex structure \(I_3\).

### 5.5 The Kähler potential \( \hat{K}_3 \) of \( \mathcal{O}^C \) as a function of the characteristic angles

In this Subsection, we show that the general formulas of [6] giving \( K_3 \) in terms of the characteristic angles have an analogue in the infinite-dimensional setting. For this purpose, we use a section of the application \( \psi \) (defined in Theorem 5.2), which has been already used in the proof of Theorem 5.2.

Theorem 5.2 states that every ordered pair \((P, Q)\) belonging to \(Gr^0_{res} \times Gr^0_{res}\) with \(P \cap Q = \{0\}\) represents an element of the complexified orbit. A preimage \((x, X)\) of \((P, Q)\) by the application \(\psi: \mathcal{W}_k^{31} \rightarrow \mathcal{O}^C\) is given by:

\[
\begin{cases}
x = k(Id_P + \frac{1}{2}A)g_{H^+} \\
x = -\frac{1}{2}Ag_{H^+}
\end{cases}
\]

(9)

where \(A\) is a Hilbert-Schmidt operator from \(P\) to \(P^\perp\) whose graph is \(Q^\perp\) (determined modulo the right action of \(GL(P)\)), and where \(g\) is a unitary operator uniquely defined if \(P\) and \(Q\) are endowed with their canonical bases. Note that the eigenvalues \(\{a_i^2\}_{i \in \mathbb{N}}\) of \(A^*A\) are independent of the operator \(A\) chosen to represent the ordered pair \((P, Q)\). If \(A\) is generic, i.e. if all the eigenvalues \(a_i^2\) are distinct, it is possible to define pairs of characteristic lines \(\{l_i, l'_i\}\), \(i \in \mathbb{N}\), as follows. The complex line \(l_i\) is the eigenspace in \(P\) of the operator \(A^*A\) with respect to the eigenvalue \(a_i^2\), and \(l'_i\) is the complex line in \(\mathcal{Q}^\perp\) which is the image of \(l_i\) under the operator \(Id_P + A\). The angle \(\theta_i\) between the two complex line \(l_i\) and \(l'_i\) is defined by

\[
\cos \theta_i = \frac{|\langle e_i, e'_i \rangle|}{|\langle e_i, e'_i \rangle|},
\]

where \(e_i\) is a unitary generator of \(l_i\) and where:

\[
e'_i := \frac{e_i + A(e_i)}{|e_i + A(e_i)|}.
\]

The angle \(\theta_i\) is related to the eigenvalue \(a_i^2\) by the following formula:

\[
\cos \theta_i = \frac{1}{\sqrt{1 + a_i^2}}.
\]

The latter expression makes sense even in the non-generic case, and allows one to uniquely define the set of characteristic angles \(\theta_i \in (-\frac{\pi}{2}, +\frac{\pi}{2})\), \(i \in \mathbb{N}\).
Remark 5.7 The orbit of an ordered pair \((P, Q)\) in \(Gr^0_{res} \times Gr^{\omega}_{res}\) under the natural action of \(GL_2(H)\) is characterized by the dimension of \(P \cap Q\). The orbit of \((P, Q)\) under the action of \(U_2(H)\) on \(Gr^0_{res} \times Gr^{\omega}_{res}\) is characterized by the set of characteristic angles \(\theta_i\).

Proposition 5.5 allows to express the Kähler potential \(K_3\) on the complexified orbit either in terms of the eigenvalues \(a_i^2\) of \(A^*A\) or in terms of the characteristic angles \(\theta_i\):

**Theorem 5.8** The form \(\omega^{red}_3\) defined on the complexified orbit \(\mathcal{O}^C\) and associated with the natural complex structure of \(\mathcal{O}^C\) satisfies \(\omega^{red}_3 = d\tilde{\omega} K_3\) with:

\[
\tilde{K}_3((P, Q)) = k^2 \text{Tr} \left( (\text{Id}_P + A^*A)^{\frac{1}{2}} - \text{Id}_P \right),
\]

for \((P, Q)\) in \(\mathcal{O}^C\), where \(A\) is such that \(\text{Ran}(\text{Id}_P + A) = Q^\perp\). Denoting by \(a_i^2\) the eigenvalues of the operator \(A^*A\), and by \(\theta_i\) the characteristic angles of the pair \((P, Q)\), one has:

\[
\tilde{K}_3((P, Q)) = k^2 \sum_{i \in \mathbb{N}} \left( \sqrt{1 + a_i^2} - 1 \right) = k^2 \sum_{i \in \mathbb{N}} \left( \frac{1}{\cos \theta_i} - 1 \right).
\]

**Proof of Theorem 5.8**
From the proof of Theorem 5.5 it follows that the potential \(K_3\) is given at an element \((x, X)\) of the stable manifold \(W^r_k\) by:

\[
K_3((x, X)) = \text{Tr} \left( (k^4 + 4x^*xX^*X - 4x^*Xx^*X)^{\frac{1}{2}} - k^2 \text{Id}_P \right).
\]

To proceed, let us recall the element \((x, X)\) of \(W^r_k\) defined in the proof of Theorem 5.5 by:

\[
x = k(\text{Id}_P + \frac{1}{2}A) \circ u_1
\]
\[
X = -\frac{1}{2}A \circ u_1,
\]

where \(u_1\) is a unitary operator from \(H^+\) to \(P\). One has \(\psi((x, X)) = (P, Q)\) and

\[
\tilde{K}_3((P, Q)) = K_3((x, X)) = k^2 \text{Tr} \left( (\text{Id}_P + u_1^*A^*Au_1)^{\frac{1}{2}} - \text{Id}_P \right),
\]

which, after conjugation by \(u_1\), gives the result. \(\blacksquare\)

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