An Improved Deterministic Parameterized Algorithm for Cactus Vertex Deletion

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Abstract

A cactus is a connected graph that does not contain $K_4 - e$ as a minor. Given a graph $G = (V,E)$ and integer $k \geq 0$, Cactus Vertex Deletion (also known as Diamond Hitting Set) is the problem of deciding whether $G$ has a vertex set of size at most $k$ whose removal leaves a forest of cacti. The current best deterministic parameterized algorithm for this problem was due to Bonnet et al. [WG 2016], which runs in time $26^k n^{O(1)}$, where $n$ is the number of vertices of $G$. In this paper, we design a deterministic algorithm for Cactus Vertex Deletion, which runs in time $17.64^k n^{O(1)}$. As a straightforward application of our algorithm, we give a $17.64^k n^{O(1)}$-time algorithm for Even Cycle Transversal. The idea behind this improvement is to apply the measure and conquer analysis with a slightly elaborate measure of instances.

1 Introduction

A connected graph is a cactus if every edge belongs to at most one cycle. A cactus forest is a graph such that every connected component is a cactus. In this paper, we consider the following problem.

Definition 1 (Cactus Vertex Deletion). Given a graph $G = (V,E)$ and an integer $k \geq 0$, the problem asks whether $G$ has a vertex set $X \subseteq V$ with $|X| \leq k$ whose removal leaves a cactus forest.

The problem is one of vertex deletion problems for hereditary properties, which have been both intensively and extensively studied in parameterized algorithms and complexity. The best known problem in this context is Vertex Cover. The problem asks whether an input graph has a vertex cover of size at most $k$. A naive algorithm solves Vertex Cover in $O^*(2^k)$ time and after a series of improvement, the fastest known algorithm is due to [4], which runs in time $O^*(1.2738^k)$. Another example of this kind of problems is Feedback Vertex Set. The problem asks whether an input graph $G = (V,E)$ has a vertex set of size at most $k$ that hits all the cycles in the graph. In other words, the goal of this problem is to compute $X \subseteq V$ with $|X| \leq k$ such that the graph obtained from $G$ by deleting $X$ is a forest. The problem is also intensively studied, and several deterministic and randomized algorithms have been proposed so far. The current best running time is due to Iwata and Kobayashi [14] for deterministic algorithms and Li and Nederlof [16] for randomized algorithms, which run in time $O^*(3.460^k)$ and $O^*(2.7^k)$, respectively.

The gap between the running time of deterministic and randomized algorithms sometimes emerges for vertex deletion problems to “sparse” hereditary classes of graphs, such as Feedback Vertex Set. For instance, Pseudo Forest Vertex Deletion can be solved deterministically in time $O^*(3^k)$ [12] and randomized in time $O^*(2.85^k)$ [12] and Bounded Degree-2 Vertex Deletion can be solved deterministically in time $O^*(2.85^k)$ [12].
$O^*(3.0645^k)$ [19] and randomly in time $O^*(3^k)$ [8]. Among others, the known gap on CACTUS VERTEX DELETION is remarkable: Bonnet et al. [3] presented a deterministic $O^*(26^k)$-time algorithm, while Koley et al. [15] presented a randomized $O^*(12^k)$-time algorithm.

In this paper, we narrow the gap between the running time of deterministic and randomized algorithms by giving an improved deterministic algorithm for CACTUS VERTEX DELETION.

**Theorem 1.** CACTUS VERTEX DELETION can be solved deterministically in time $O^*(17.64^k)$.

As a variant of CACTUS VERTEX DELETION, we consider EVEN CYCLE TRANSVERSAL defined as follows. A cactus is called an odd cactus if every cycle in it has an odd number of vertices.

**Definition 2 (EVEN CYCLE TRANSVERSAL).** Given a graph $G = (V, E)$ and an integer $k \geq 0$, the problem asks whether $G$ has a vertex set $X \subseteq V$ with $|X| \leq k$ whose removal leaves a forest of odd cacti.

Note that a graph has no cycles of even length if and only if it is a forest of odd cacti [15]. Kolet et al. [15] gave an $O^*(12^k)$-time randomized algorithm and Misra et al. [17] gave an $O^*(50^k)$-time deterministic algorithm for EVEN CYCLE TRANSVERSAL. In this paper, we improve the running time of the deterministic algorithm for EVEN CYCLE TRANSVERSAL.

**Theorem 2.** EVEN CYCLE TRANSVERSAL can be solved deterministically in time $O^*(17.64^k)$.

The idea of our algorithms follows the one used in [3]. We solve the disjoint version of CACTUS VERTEX DELETION with a branching algorithm. To solve this problem, we use the measure and conquer analysis [10] with an elaborate measure compared with the one used in [3]. We believe that although our measure is slightly involved, the algorithm itself and its analysis would be simpler than theirs.

## 2 Preliminaries

### Graphs.
Throughout the paper, graphs have no self-loops but may have multiedges. Let $G = (V, E)$ be a graph. We write $V(G)$ and $E(G)$ to denote the sets of vertices and edges of $G$, respectively. For two distinct vertices $u, v$ in $G$, we denote by $m(u, v)$ the number of edges between $u$ and $v$. Let $v \in V$. The degree of $v$ is the number of edges incident to it. We denote by $N_G(v)$ the set of neighbors of $v$ in $G$. Note that as $G$ may have multiedges, $|N_G(v)|$ may not be equal to its degree. For $X \subseteq V$, the subgraph of $G$ induced by $X$ is denoted as $G[X]$.

For a graph $H$, we denote by $cc(H)$ the number of connected components in $H$ and by $b(H)$ the number of bridges in $H$.

**Lemma 1.** Let $H$ be a multigraph with $h$ vertices. Then, it holds that $cc(H) + b(H) \leq h$.

**Proof.** We prove this lemma by induction on $h$. Suppose that $H$ has two or more connected components $H_1, \ldots, H_l$. Then, by the induction hypothesis, we have $\sum_{1 \leq i \leq l}(cc(H_i) + b(H_i)) \leq h$. Since $cc(H_i) = 1$ and $\sum_{1 \leq i \leq l}b(H_i) = b(H)$, we have $cc(H) + b(H) = t + \sum_{1 \leq i \leq l}b(H_i) \leq h$.

Suppose that $H$ is connected. If $H$ has no bridges, the lemma trivially holds. Then, let $b$ be a bridge of $H$. By the induction hypothesis, the two subgraphs $H_1$ and $H_2$ obtained from $H$ by removing $b$ satisfies $1 + b(H_1) \leq h_1$ and $1 + b(H_2) \leq h_2$, where $h_1$ and $h_2$ are the numbers of vertices in $H_1$ and $H_2$, respectively. Then, $cc(H) + b(H) = 1 + b(H_1) + b(H_2) + 1 = h$, and hence lemma follows.

A block $B$ of a graph $G$ is either a biconnected component or an isolated vertex in $G$. Note that a graph consisting of two vertices with at least one edge is a block. It is easy to see that every block in a cactus forest is either a cycle, an edge, or an isolated vertex. In particular, we call $B$ a leaf block if it has at most one cut vertex. We say that vertices $v_1, \ldots, v_t \in V(B)$ are consecutive in $B$ if for each $1 \leq i < t$, $v_i$ is adjacent to $v_{i+1}$ in $B$.

### Iterative compression.
Our algorithm employs the well-known iterative compression technique invented by Reed, Smith, and Vetta [18]. They gave an algorithm for ODD CYCLE TRANSVERSAL based on this technique. The essential idea can be generalized as follows. Let $\mathcal{C}$ be a hereditary class of graphs, that is, for $G \in \mathcal{C}$, every induced subgraph of $G$ also belongs to $\mathcal{C}$. The technique is widely used for designing algorithms of vertex deletion problems to hereditary classes of graphs. The crux of the technique can be described as the following lemma.
Lemma 2 (IL). Let \( C \) be a hereditary class of graphs. Given a graph \( G = (V, E) \) and an integer \( k \), the problem of computing \( X \subseteq V \) with \(|X| \leq k\) such that \( G[V \setminus X] \in C \) can be solved in time \( O^*(k+1)^k \). If one can solve the following problem in time \( O^*(c^k) \): Given a subset \( S \subseteq V \) of cardinality at most \( k+1 \) with \( G[V \setminus S] \in C \), the problem asks to find \( X \subseteq V \setminus S \) with \(|X| \leq k\) such that \( G[V \setminus X] \in C \).

For Cactus Vertex Deletion, the latter problem is defined as follows.

Definition 3 (Disjoint Cactus Vertex Deletion). Given a graph \( G = (V, E) \), an integer \( k \geq 0 \), and \( S \subseteq V \) such that \(|S| \leq k+1\) and \( G[V \setminus S] \) is a cactus forest, the problem asks whether \( G \) has a vertex set \( X \subseteq V \setminus S \) with \(|X| \leq k\) whose removal leaves a cactus forest.

Let us note that we can assume that both \( G[S] \) and \( G[V \setminus S] \) are cactus forests as otherwise the problem is trivially infeasible.

Measure and conquer analysis. Our algorithm for Disjoint Cactus Vertex Deletion is based on a standard branching algorithm with the measure and conquer analysis [10]. Given an instance \( I \) of the problem, we define a measure \( \mu(I) \) that is non-negative real and design a branching algorithm that generates subinstances \( I_1, \ldots, I_t \) with \( \mu(I) > \mu(I_i) \) for \( 1 \leq i \leq t \). To measure the running time of the algorithm, we use a branching factor \((b_1, \ldots, b_t)\), where \( \mu(I) - \mu(I_i) \leq b_i \) for each \( i \). It is known that the total running time of this branching algorithm is upper bounded by \( O^*(c^{\mu(I)}) \), where \( c \) is the unique positive real root of equation

\[
x^{-b_1} + x^{-b_2} + \cdots + x^{-b_t} = 1,
\]

assuming that from any instance, its subinstances can be generated in polynomial time. We refer the reader to the book [11] for a detailed exposition for the measure and conquer analysis.

3 Improved algorithm for Disjoint Cactus Vertex Deletion

This section is devoted to developing an algorithm for Disjoint Cactus Vertex Deletion that runs in time \( O^*(16.64^k) \), proving, by Lemma 2 Theorem 1

Lemma 3. Disjoint Cactus Vertex Deletion can be solved in time \( O^*(16.64^k) \).

Let \( I = (G, S, k) \) be an instance of Disjoint Cactus Vertex Deletion, where \( G = (V, E) \) is a multigraph, \( S \subseteq V \), and \(|S| \leq k+1\). Note that \( G[V \setminus S] \) and \( G[S] \) are both cactus forests.

Let \( \mu(I) = \alpha \cdot k + \beta \cdot cc(G[S]) + \gamma \cdot b(G[S]) \), where \( \alpha, \beta, \gamma \) are chosen later. In the following, we assume that \( \beta \geq \gamma \). For the sake of simplicity, we write, for \( X \subseteq V \), \( cc(X) \) and \( b(X) \) to denote \( cc(G[X]) \) and \( b(G[X]) \), respectively.

As \( G \) may have multiedges, every cactus forest can be characterized as the following form.

Proposition 1 (IL). Let \( D \) be the graph of two vertices and three parallel edges between them. A graph is a cactus forest if and only if it does not contain a subgraph isomorphic to any subdivision of \( D \).

We call a subdivision of \( D \) an obstruction. In particular, \( D \) itself is also an obstruction.

The algorithm consists of several branching rules and reduction rules. We say that a reduction rule is safe if the original instance has a feasible solution if and only if so does the instance obtained by applying the rule. We also say that a branching rule is safe if the original instance is a Yes-instance if and only if at least one of the instances obtained by applying the rule is a Yes-instance. We apply these rules in the order of their appearance. The algorithm terminates if \( V(G) = S \) or \( k = 0 \), and it answers “YES” if and only if \( k \geq 0 \) and \( G \) is a cactus forest.

The following reduction and branching rules are trivially safe.

Reduction Rule 1. If \( G[V \setminus S] \) contains a component \( C \) that has no neighbors in \( S \), then delete all the vertices in \( C \).

Reduction Rule 2. If \( G[V \setminus S] \) contains a vertex of degree one in \( G \), then delete it.

Reduction Rule 3. If \( G[V \setminus S] \) contains \( v \) such that \( G[S \cup \{v\}] \) is not a cactus forest, then delete \( v \) and decrease \( k \) by one.
Branching Rule 1. If $G[V \setminus S]$ contains a vertex $u$ such that there is a vertex $v \in V \setminus S$ with $m(u, v) \geq 3$, branch into two cases: (1) delete $u$ and decrease $k$ by one; (2) delete $v$ and decrease $k$ by one.

The branching factor of Branching rule is $(\alpha, \alpha)$. By applying these rules, we make the following assumption on each vertex in $V \setminus S$.

Assumption 1. Every vertex $v \in V \setminus S$ has degree at least two in $G$ and there are at most two edges between two vertices.

As $G$ is a multigraph, some vertex may have only one neighbor even if its degree is greater than one. If $G[V \setminus S]$ contains a vertex $v$ with $|N_G(v)| = 1$, this vertex also can be removed since it is not a part of an obstruction, assuming that $m(u, v) \leq 2$ with $u \in N_G(v)$. This implies the following reduction rule.

Reduction Rule 4. If $G[V \setminus S]$ contains a vertex $v$ with $|N_G(v)| = 1$, then delete it.

Thus, we further make the following assumption on each vertex in $V \setminus S$.

Assumption 2. Every vertex $v \in V \setminus S$ has at least two neighbors in $G$.

Suppose that there is a vertex $v \in V \setminus S$ that has at least two neighbors in $S$. By Reduction rule there is no component in $G[S]$ that contains at least three vertices of $N_G(v) \cap S$. Let $W = N_G(v) \cap S$. We denote by $t_1$ (resp. by $t_2$) the number of components in $G[S]$ that contain exactly one vertex (resp. two vertices) of $W$. Let $C$ be a component in $G[S]$ that has at least one vertex of $W$. If $|W \cap C| = 2$, say $w, w' \in W \cap C$, there is an unique path between $w$ and $w'$ in $G[C]$ as otherwise $S$ contains an obstruction, which implies that $v$ is removed by Reduction rule. Then, there is at least one bridge on the path between $w$ and $w'$ in $G[C]$. Thus, $b(C \cup \{v\}) \leq b(C) - 1$. If $|W \cap C| = 1$, $G[C \cup \{v\}]$ has $b(C) + 1$ bridges. Hence, we have

$$\beta \cdot cc(S \cup \{v\}) \leq \beta \cdot cc(S) - \beta(t_1 + t_2 - 1),$$
$$\gamma \cdot b(S \cup \{v\}) \leq \gamma \cdot b(S) + \gamma(t_1 - t_2).$$

Consider the value $t_1(\beta - \gamma) + t_2(\beta + \gamma) - \beta$, that is, a lower bound of $\mu((G, S, k)) - \mu((G, S \cup \{v\}, k))$. If $t_1 + t_2 \geq 2$, the value is at least $\beta - 2\gamma$. If $t_1 + t_2 = 1$, $t_1$ must be zero since $|W| \geq 2$. In this case, the value is at least $\gamma$. This implies the following branching rule, which is clearly safe, has branching factor $(\alpha, \min(\beta - 2\gamma, \gamma))$.

Branching Rule 2. Suppose $G[V \setminus S]$ contains a vertex $v$ that has at least two neighbors $u, w$ in $S$. Then, branch into two cases: (1) delete $v$ and decrease $k$ by one; (2) put $v$ into $S$.

Thus, we make the following assumption on each vertex in $V \setminus S$.

Assumption 3. Every vertex $v \in V \setminus S$ has at least two neighbors in $G$ and at most one of them belongs to $S$.

We can remove a vertex having exactly two neighbors by adding an edge between its neighbors. The following lemma justifies this reduction.

Lemma 4. Let $v \in V \setminus S$ have exactly two neighbors $u, w$ in $G$. Let $G'$ be the graph obtained from $G$ by deleting $v$ and adding $p$ parallel edges between $u$ and $w$, where $p = \max(m(u, v), m(v, w))$. Suppose that $p \leq 2$. Then, $G$ has a cactus deletion set of size at most $k$ if and only if $G'$ has a cactus deletion set of size at most $k$.

Proof. Since every obstruction in $G$ containing $v$ also has both $u$ and $w$, there is a smallest cactus deletion set $X$ that does not contain $v$. Such a set is also a cactus deletion set of $G'$ and vise versa. 

By Assumption at least one of two neighbors $u$ and $w$ of $v$ is contained in $V \setminus S$. By Lemma the following reduction rule is safe.

Reduction Rule 5. Suppose that $G[V \setminus S]$ contains a vertex $v$ with $N_G(v) = \{u, w\}$. Then delete $v$ and add $\max(m(u, v), m(v, w))$ parallel edges between $u$ and $w$.

This implies that the following assumption is made.

Assumption 4. Every vertex $v \in V \setminus S$ has at most one neighbors in $S$ and at least two neighbors in $V \setminus S$. 

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Since $G[V \setminus S]$ is a cactus forest, there is a leaf block $B$. By Assumption 1, $B$ contains at least three vertices. Moreover, there is a vertex $v \in V(B)$ that is not a cut vertex of $G[V \setminus S]$. Suppose that $B$ has exactly three vertices $u, v, w$. As $B$ is a leaf block, we can assume that $u \in V(B)$ is not a cut vertex of $G[V \setminus S]$. By Assumptions 3 and 4, both $u$ and $v$ have exactly one neighbor in $S$, which can be an identical vertex. If there is a component $C$ in $G[S]$ that contains both a neighbor of $u$ and a neighbor of $v$, then $G[C \cup \{u, v, w\}]$ has an obstruction, which yields the following branching rule with branching factor $(\alpha, \alpha, \alpha, \beta)$.

**Branching Rule 3.** Suppose that there is a leaf block $B$ with $V(B) = \{u, v, w\}$ in $G[V \setminus S]$. Suppose moreover that each of $u$ and $v$ has exactly one neighbor in $S$ and that these neighbors belong to a single component in $G[S]$. Then, branch into three cases: (1) delete $u$; (2) delete $v$; (3) delete $w$. For each case, decrease $k$ by one.

Otherwise, the neighbors of $u$ and $v$ belong to distinct components in $G[S]$. Let $C_u$ and $C_v$ be the components of $G[S]$ that have neighbors of $u$ and $v$, respectively. If $w$ has a neighbor in $C_u$ or $C_v$, then $G[S \cup \{u, v, w\}]$ contains an obstruction. In this case, we apply Branching rule 3 as well. Thus, either $w$ has no neighbor in $S$ or $w$ has exactly one neighbor in a component $C_w$ in $G[S]$ with $C_w \neq C_u$ and $C_w \neq C_v$. Suppose $w$ has no neighbor in $S$. Then, $G[S \cup \{u, v, w\}]$ contains $cc(S) - 1$ components and $b(S) + 2$ bridges. Thus, the following rule has branching factor $(\alpha, \alpha, \alpha, \beta - 2\gamma)$.

**Branching Rule 4.** Suppose that there is a leaf block $B$ with $V(B) = \{u, v, w\}$ in $G[V \setminus S]$. Suppose moreover that each of $u$ and $v$ has exactly one neighbor in $S$ and that these neighbors belong to distinct components in $G[S]$. Then, branch into four cases: (1) delete $u$; (2) delete $v$; (3) delete $w$; (4) put $u, v,$ and $w$ into $S$. For (1), (2), and (3), decrease $k$ by one.

Suppose otherwise that $w$ has an exactly one neighbor in a component $C_w$ in $G[S]$ with $C_w \neq C_u$ and $C_w \neq C_v$. Then, $G[S \cup \{u, v, w\}]$ contains $cc(S) - 2$ components and $b(S) + 3$ bridges. Thus, Branching rule 4 has branching factor $(\alpha, \alpha, \alpha, 2\beta - 3\gamma)$.

By Branching rules 3 and 4, the following assumption is made.

**Assumption 5.** There is a leaf block $B$ in $G[V \setminus S]$ that contains three consecutive vertices, each of which is not a cut vertex in $G[V \setminus S]$ and has exactly one neighbor in $S$.

Let $u, v, w$ be three consecutive vertices in $B$, each of which is not a cut vertex in $G[V \setminus S]$ and has exactly one neighbor in $S$. Let $u', v', w'$ be the neighbors of $u, v, w$ in $S$, respectively. There are four cases (Figure 1).

Suppose that there is a component $C$ in $G[S]$ that contains these neighbors ((a) in Figure 1). Then, $G[C \cup \{u, v, w\}]$ has an obstruction, yielding a similar rule to Branching rule 3 that has branching factor $(\alpha, \alpha, \alpha)$.

Suppose next that exactly two of $u', v', w'$ are contained in a single component $C$ in $G[S]$. There are essentially two cases: (1) $u'$ and $v'$ are contained in $C$ ((b) in Figure 1) or (2) $u'$ and $w'$ are contained in $C$ ((c) in Figure 1). In case (1), $G[S \cup \{u, v, w\}]$ contains $cc(S) - 1$ components and $b(S) + 2$ bridges. In case (2), $G[S \cup \{u, v, w\}]$ contains $cc(S) - 1$ components and $b(S) + 1$ bridges. For these cases, we apply Branching rule 4. Let us note that, for case (1), there may be multiple edges between $u$ and $v$. In this case, we omit branch (4) in Branching rule 4. Hence, Branching rule 4 has branching factors $(\alpha, \alpha, \alpha, \beta - 2\gamma)$ and $(\alpha, \alpha, \alpha, \beta - \gamma)$ for these cases.

Finally, suppose any two of $u', v', w'$ are not contained in a single component in $G[S]$. Again, we apply Branching rule 4 to this case. Since $G[S \cup \{u, v, w\}]$ contains $cc(S) - 2$ components and $b(S) + 5$ bridges,
Algorithm 1 A pseudocode of the algorithm for Disjoint Cactus Vertex Deletion

1: procedure DCVD($G = (V, E), S, k$)
2:   if $k \geq 0$ and $V = S$ then
3:     return true
4:   if $k < 0$ then
5:     return false
6:   if $G[V \setminus S]$ contains a component $C$ that has no neighbors in $S$ then
7:     return DCVD($G[V \setminus C], S, k$)
8:   if $G[V \setminus S]$ contains a vertex $v$ of degree one in $G$ then
9:     return DCVD($G[V \setminus \{v\}], S, k$)
10:  if $G[V \setminus S]$ contains $v$ such that $G[V \cup \{v\}]$ is not a cactus forest then
11:     return DCVD($G[V \setminus \{v\}], S, k - 1$)
12:  if $G[V \setminus S]$ contains vertices $u$ and $v$ with $m(u, v) \geq 3$ then
13:     return DCVD($G[V \setminus \{u\}], S, k - 1 \lor DCVD(G[V \setminus \{v\}], S, k - 1$)
14:  if $G[V \setminus S]$ contains a vertex $v$ with $|N_G(v)| = 1$ then
15:     return DCVD($G[V \setminus \{v\}], S, k$)
16:  if $G[V \setminus S]$ contains a vertex $v$ that has at least two neighbors in $S$ then
17:     return DCVD($G[V \setminus \{v\}], S, k - 1 \lor DCVD(G[V], S \cup \{v\}, k$)
18:  if $G[V \setminus S]$ contains a vertex $v$ with $N_G(v) = \{u, v\}$ then
19:     Let $G' = G[V \setminus \{v\}]$ and add max{$m(u, v), m(v, w)$} parallel edges between $u$ and $w$.
20:     return DCVD($G', S, k$)
21:  if $G[V \setminus S]$ has a leaf block $B$ with $V(B) = \{u, v, w\}$ then
22:     for $x \in V(B)$ do
23:       if $DCVD(G[V \setminus \{x\}], S, k - 1$ then
24:         return true
25:     if $G[S \cup V(B)]$ is a cactus forest then
26:       return DCVD($G[V], S \cup V(B), k$)
27:     return false
28:  if $G[V \setminus S]$ has a leaf block $B$ with $|V(B)| \geq 4$ then
29:     Let $B' = \{u, v, w\}$ be consecutive vertices in $B$ that are not cut vertices in $G[V \setminus S]$.
30:     for $x \in V(B')$ do
31:       if $DCVD(G[V \setminus \{x\}], S, k - 1$ then
32:         return true
33:     if $G[S \cup V(B')]$ is a cactus forest then
34:       return DCVD($G[V], S \cup V(B'), k$)
35:     return false

Branching rule 4 has branching factor $(\alpha, \alpha, \alpha, 2\beta - 5\gamma)$. The entire algorithm for Disjoint Cactus Vertex Deletion is given in Algorithm 1.

The reduction and branching rules cover all cases for the instance $I$ and all the rules are safe. Thus, the algorithm correctly computes a cactus deletion set $X \subseteq V \setminus S$ with $|X| \leq k$ if it exists. By choosing $\alpha = 1$, $\beta = 0.4052$, $\gamma = 0.0726$, the running time is dominated by the branching factor $(\alpha, \alpha, \alpha, 2\gamma) = (1, 1, 1, 0.26)$. By Lemma 1 we have $\beta \cdot cc(S) + \gamma \cdot b(S) \leq \beta \cdot k$. Therefore, the running time of the algorithm is

$$O^*(c^{\beta(I)}) \subseteq O^*(c^{\alpha - k + \beta \cdot cc(S) + \gamma \cdot b(S)}) \subseteq O^*(c^{1.4052k}),$$

where $c < 7.3961$ is the unique positive real root of equation $3x^{-1} + x^{-0.26} = 1$. This yields the running time bound $O^*(16.64^k)$ for Disjoint Cactus Vertex Deletion.
4 An improved algorithm for Even Cycle Transversal

Recall that Even Cycle Transversal asks whether, given a graph $G = (V, E)$ and an integer $k$, $G$ has a vertex set $X$ of size at most $k$ such that $G[V \setminus X]$ is a forest of odd cacti. As in the previous section, we solve the disjoint versions of Even Cycle Transversal.

**Definition 4 (Disjoint Even Cycle Transversal).** Given a graph $G = (V, E)$, an integer $k \geq 0$, and $S \subseteq V$ such that $|S| \leq k + 1$ and $G[V \setminus S]$ is a forest of odd cacti, the problem asks whether $G$ has a vertex set $X \subseteq V \setminus S$ with $|X| \leq k$ whose removal leaves a forest of odd cacti.

A key difference from Disjoint Cactus Vertex Deletion is that we need to take the length of cycles into account. However, in Reduction rule 3, we replace (a chain of) cycles with two multiple edges between two extreme vertices, which does not preserve the length of cycles in the original graph. Given this, we consider a slightly general problem. In addition to the input of Disjoint Even Cycle Transversal, we are given a binary weight function $w : E \to \{0, 1\}$ on edges, and the length of a cycle is defined to be the total weight of edges in it. Indeed, when $w(e) = 1$ for all $e \in E$, the problem corresponds to Disjoint Even Cycle Transversal.

Let $S \subseteq V$ with $|S| \leq k + 1$ such that $G[V \setminus S]$ is a forest of odd cacti. We first apply Reduction rules 1 to 4 and Branching rules 1 and 2, which are trivially safe for Disjoint Even Cycle Transversal. Moreover, we add the following reduction rule, which is also trivially safe.

**Reduction Rule 6.** If there is a vertex $v \in V \setminus S$ such that $G[S \cup \{v\}]$ has a cycle of even length, then delete it and decrease $k$ by one.

It is not obvious to find a vertex in $V \setminus S$ that satisfies the condition in Reduction rule 6. An important observation to do this is that $G[S]$ has treewidth at most two. We refer to [5] for the detailed definition of treewidth and its algorithmic usage. More specifically, we have the following proposition, which allows us to check the condition in linear time.

**Proposition 2.** Let $S \subseteq V$ that induces a cactus forest and let $X \subseteq V \setminus S$. Suppose that $X$ has a constant number of vertices. Then, we can check in linear time whether $G[S \cup X]$ has a cycle of even length.

Up to this point, Assumption 3 is made. By Reduction rule 3 and Branching rule 4 we also assume that $m(u,v) \leq 2$ for every pair of vertices in $G$. Suppose that $m(u,v) = 2$. If the parities of two edges between $u$ and $v$ are the same, the length cycle consisting of these edges is even. Thus, we apply the following branching rule in this case.

**Branching Rule 5.** Suppose that $u, v \in V \setminus S$ and $m(u,v) = 2$ for some $u$. Let $f, f'$ be the edges between $u$ and $v$. If $w(f) = w(f')$, branch into two cases: (1) delete $u$ and decrease $k$ by one; (2) delete $v$ and decrease $k$ by one.

By Reduction rule 6 and Branching rule 3, the following assumption is made.

**Assumption 6.** For every pair of vertices $u, v$ with $m(u,v) = 2$, the parities of the weights of edges between them are opposite.

Now, let us consider a vertex $v \in V \setminus S$ that has exactly two neighbors in $G$. Let $u$ and $w$ be the neighbors of $v$. By Assumption 6, at least one of $u$ and $w$ belongs to $V \setminus S$. Similarly to Lemma 4, we define a graph $G'$ by deleting $v$ from $G$ and adding $p$ parallel edges between $u$ and $v$, where $p = \max(m(u,v), m(v,u))$. We define the weight function $w'$ for $G'$ as follows. If $p = 1$, we set the weight of the introduced edge $e = \{u, w\}$ as $w'(e) = w(f) + w(f')$, where $f$ (resp. $f'$) is the edge between $u$ (resp. between $v$ and $w$) and the sum is taken under addition modulo two. If $p = 2$, at least one of the pairs $\{u, v\}$ or $\{v, w\}$ has multiple edges. By Assumption 6 these two edges have different parities. A crucial observation is that if there is a cycle passing through exactly one of these edges, there is another cycle passing through the other edges, which has the different parity. By setting $w'(e) = 0$ and $w'(e') = 1$, such cycles are preserved in $G'$.

**Lemma 5.** The instance $(G, w, S, k)$ is feasible if and only if so is $(G', w', S, k)$.

*Proof.* Consider a cycle $C$ of even length that passes through $v$ in $G$. By Assumption 6, $C$ must pass through both $u$ and $w$. Thus, there is an optimal solution $X \subseteq V \setminus S$ for $(G, w, S, k)$ not containing $v$. By the construction
of \(G'\), there is a cycle obtained from \(C\) by omitting \(v\) is a cycle of \(G'\) that has even length. Hence, \(X\) is an optimal solution for \((G',w',S,k)\). It is not hard to see that this correspondence is reversible and hence the lemma follows.

This lemma ensures that the weighted version of Reduction rule is safe for **Even Cycle Transversal**, and then Assumption is made as well. The rest of branching rules are the same with **Disjoint Cactus Vertex Deletion**, which yields an \(O^*(16.64^k)\)-time algorithm that solves **Disjoint Even Cycle Transversal** as well.

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