On a generalized Kirchhoff equation with sublinear nonlinearities

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In this paper, we consider a generalized Kirchhoff equation in a bounded domain under the effect of a sublinear nonlinearity. Under suitable assumptions on the data of the problem, we show that, with a simple change of variable, the equation can be reduced to a classical semilinear equation and then studied with standard tools. Copyright © 2016 John Wiley & Sons, Ltd.

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1. Introduction

In this article, we study the existence of solutions $u : \Omega \to \mathbb{R}$ for the following nonlocal problem in divergence form:

$$
\begin{aligned}
-\text{div} \left( m(u, |\nabla u|^2) \nabla u \right) &= f(x, u) \quad \text{in} \ \Omega, \\
u &= 0 \quad \text{on} \ \partial\Omega,
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary,

$$m : \mathbb{R} \times [0, \infty) \to \mathbb{R}, \quad f : \Omega \times \mathbb{R} \to \mathbb{R}$$

are given functions satisfying suitable conditions, which will be given later. Hereafter, we denote with $| \cdot |_p$ the usual $L^p(\Omega)$—norm. By a solution of the aforementioned problem, we mean a function $u_\ast \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that

$$\forall \psi \in H_0^1(\Omega) : \int_{\Omega} m(u_\ast(x), |\nabla u_\ast|^2) \nabla u_\ast \cdot \nabla \psi \, dx = \int_{\Omega} f(x, u_\ast) \psi \, dx,$$

whenever the integrals make sense.

When the function $m$ does not depend on $u$, we have the classical problem

$$
\begin{aligned}
-\Delta u &= f(x, u) \quad \text{in} \ \Omega, \\
u &= 0 \quad \text{on} \ \partial\Omega,
\end{aligned}
$$

which is the $N$-dimensional version, in the stationary case, of the Kirchhoff equation introduced in [1]. We do not list here the huge amount of papers concerning this equation. On the other hand, in many physical problems, rather then on $|\nabla u|^2$, the function $m$ depends on the unknown $u$, or even on quantities related to $u$, as its $L^1$—norm (e.g., [2]).

Problem (P) studied in this paper can be considered as a slight generalization of the Kirchhoff equation. It can modelize the following physical situation. Consider an elastic membrane with shape $\Omega$ and fixed edge $\partial\Omega$, initially at rest. Let $f$ be a given external force acting on $\Omega$ and $u(x)$ the transverse displacement at a point $x \in \Omega$, with respect to initial position, of the equilibrium solution. When the displacement is small, the number $(1/2) \int_{\Omega} |\nabla u|^2 \, dx$ gives us a good approximation of the variation of the superficial area of the
membrane $\Omega$. What we are assuming here is that the velocity of the displacement of the membrane is proportional to the gradient of the displacement with a factor $m$ depending not only on the variation of the superficial area of $\Omega$ but also on the same displacement:

$$
\mathbf{v} = -m\left(u, |\nabla u|^2\right) \nabla u.
$$

Such a model is quite reasonable, and to the best of our knowledge, it has not been considered before.

Beside the physical interest, the equation is also challenging from a mathematical point of view. In this paper, we will treat the situations in which $f = f(x)$ and $f = f(x, u)$ with sublinear growth in $u$. However, our results will be stated and proved in the next sections; indeed, to state our theorems (especially Theorem 4.2), some preliminaries are needed. Here, we say that the function $m$ will satisfy quite general assumptions. In particular, in contrast to the case in which $m$ depend only on $|\nabla u|^2$, here, there is no restriction in the growth at infinity of $m$ with respect to $|\nabla u|^2$.

The main novelty of our approach is that the proofs are based on a simple ‘change of variable’ device, which seems not to have been used for these kind of nonlocal and nonlinear equations. With the use of the change of variable, Eq. (P) is reduced to a ‘local’ semilinear equation, for which various tools are available to solve it.

We point out that a change of variable of the same type to that used here is already present in the book of Lions [3, chap.2] where he treated a parabolic problem of type $\partial_t u - \text{div}(k(u)\nabla u) = f(x)$.

For other type of change of variables in this type of problems, see also [4, 5].

**Remark 1**
Here, based on a result in [6], we will assume a sublinear assumption on $f$, and we will use topological tools. However, depending on other type of assumptions on the nonlinearity $f$, other methods, for instance variational, can be employed to solve the local equation in which the problem is transformed after the change of variable (and then recover a solution of the original equation).

**Remark 2**
We believe that change of variable of this type can be used also to deal with other nonlocal equations.

The paper is organized as follows.

In Section 2, we present the general approach to solve problem (P), we state the main assumptions on $m$, we introduce its primitive $M$, and we give some of its properties (Lemma 2.1). In Section 3, we state and prove our main result in the case $f = f(x)$, that is, Theorem 3.1. In Section 4, we consider the general case $f = f(x, u)$; actually, we give a criterion (Theorem 4.1) that ensures the existence of a solution to problem (P). Then, two applications are given in Section 5.

**2. Assumptions on $m$ and a useful Lemma**

Our approach to treat problem (P), in both cases $f = f(x)$ and $f = f(x, u)$, consists in the following steps. Firstly, for every fixed $r \geq 0$, we consider the auxiliary problem

$$
\begin{cases}
-\text{div} \left(m(u, r)\nabla u\right) = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
$$

associated with (P) for which the existence of a unique solution $u_r$ will be guaranteed. Secondly, we show that the map

$$
S : r \mapsto \int_{\Omega} |\nabla u_r|^2 \, dx
$$

has a fixed point, which is of course a solution of the original problem (P).

Before to state our basic assumptions, let us introduce the following convention: for every $r \geq 0$, let us denote with $m$, the map

$$
m_r : t \in \mathbb{R} \mapsto m(t, r) \in \mathbb{R}.
$$

We suppose that $m : \mathbb{R} \times [0, \infty) \to (0, \infty)$ is a function satisfying the following conditions:

- (m0) $m : \mathbb{R} \times [0, +\infty) \to (0, +\infty)$ is a continuous function;
- (m1) there is $m > 0$ such that $m(t, r) \geq m$ for all $t \in \mathbb{R}$ and $r \in (0, \infty)$; and
- (m2) for each $r \in [0, +\infty)$, the map $m_r : \mathbb{R} \to (0, +\infty)$ is strictly decreasing in $(-\infty, 0)$ and strictly increasing in $(0, +\infty)$.

The class of functions satisfying the aforementioned assumptions is very large; the aforementioned conditions are satisfied, for example, by $m(t, r) = t^2(r^p + 1) + 1$, with $p > 0$, or $m(t, r) = t^2(e^{rt^2} + 1) + 1$ which, en passant, achieve their minimum $m = 1$ in points of type $(0, r)$.

**Remark 3**
It would be interesting to see what happens in case $m$ might vanish.

Let us define the map $M(t, r) := \int_0^t m(s, r) \, ds$; we set also

$$
M_r : = M(\cdot, r) : \mathbb{R} \to \mathbb{R}.
$$
As we have mentioned before, the result of this section provides some properties of $M_r$, which will play an important role in the study of problem (P).

**Lemma 2.1**
Assume (m0)-(m1). Then,

(a) for each $r \in [0, \infty)$, the map $M_r : \mathbb{R} \to \mathbb{R}$ is a strictly increasing diffeomorphism; and

(b) for each $r \in [0, \infty)$, the inverse map $M_r^{-1}$ is Lipschitz continuous with Lipschitz constant $m^{-1}$. In particular, $|M_r^{-1}(s)| \leq m^{-1}|s|$ for all $s \in \mathbb{R}$.

Assume now also (m2). Then,

(c) if $s_n \to s_0$ and $r_n \to r_0$ then $M_{r_n}^{-1}(s_n) \to M_{r_0}^{-1}(s_0)$; and

(d) for each $r \in [0, +\infty)$, the map $s \mapsto M_r^{-1}(s)/s$ is (continuous and) strictly decreasing in $(0, +\infty)$ and strictly increasing in $(-\infty, 0)$.

**Proof**

(a) It is obvious.

(b) Fixed $r \geq 0$, $s_1, s_2 \in \mathbb{R}$ and setting $t_i = M_r^{-1}(s_i), i = 1, 2$, we are reduced to prove

$$m|t_1 - t_2| \leq |M_r(t_1) - M_r(t_2)|.$$

Assume $t_1 > t_2$. Then,

$$|M_r(t_1) - M_r(t_2)| = M_r(t_1) - M_r(t_2) = \int_{t_2}^{t_1} m(\sigma, r)d\sigma \geq m(t_1 - t_2) = m|t_1 - t_2|.$$

(c) Let us see first a simple fact. Let $s > 0$ be fixed. By the mean value theorem and (m2), for each $r \in [0, \infty)$, there is a unique $t_r, s$ between 0 and $M_r^{-1}(s)$ such that

$$M_r^{-1}(s) = M_r^{-1}(s) - M_r^{-1}(0) = (M_r^{-1})'(M_r(t_r))(s - 0) = \frac{s}{m(t_r, r)}.$$  \quad (2.1)

The unicity follows as $m$ is strictly increasing in $(0, +\infty)$. Equivalently, $t_r, s$ is the unique positive number satisfying

$$s = M_r \left( \frac{s}{m(t_r, r)} \right).$$  \quad (2.2)

Moreover, by (m1) and (2.1), it follows that

$$t_{r, s} < M_r^{-1}(s) \leq m^{-1}s.$$  \quad (2.3)

If $s < 0$, using that $m_r$ is strictly decreasing in $(-\infty, 0)$, we conclude again the existence of a unique $t_{r, s}$ between $M_r^{-1}(s)$ and 0, which satisfy (2.2) and the inequalities in (2.3) hold with absolute values.

Now, let $\{s_n\}, \{r_n\}$ be sequences such that $r_n \to r_0 \geq 0, s_n \to s_0$.

If $s_0 \neq 0$, from (2.3) (or (2.3) with absolute values in case $s_0 < 0$), up to a subsequence, there is $t_* \in \mathbb{R}$ such that $t_{r_n, s_n} \to t_*$. Passing to the limit in $n$ in the identity

$$s_n = M_r \left( \frac{s_n}{m(t_{r_n, r_n}, r_n)} \right)$$

(recall (2.2)) and using the continuity of $m$ and $M_r$, it follows that

$$s_0 = M_r \left( \frac{s_0}{m(t_*, r_0)} \right).$$

By the unicity, we infer that $t_*= t_{r_0, s_0}$. Consequently,

$$M_{r_0}^{-1}(s_0) = \frac{s_0}{m(t_{r_0, s_0}, r_0)},$$

showing that

$$M_{r_0}^{-1}(s_n) = \frac{s_n}{m(t_{r_n, s_n}, r_n)} \to \frac{s_0}{m(t_{r_0, s_0}, r_0)} = M_{r_0}^{-1}(s_0).$$

If $s_0 = 0$, then from (b), we have $M_{r_0}^{-1}(s_n) \to 0 = M_{r_0}^{-1}(0)$. 


(d) Let \( r \geq 0 \) be fixed. Consider the case \( s_1 > s_2 > 0 \). Setting
\[
t_i := M_i^{-1}(s_i) > 0, \quad i = 1, 2,
\]
we are reduced to show that
\[
\frac{M_i(t_1)}{t_1} > \frac{M_i(t_2)}{t_2}.
\]
(2.4)

From (m2) and
\[
M_i(t) = \int_0^t m(\sigma, r) d\sigma < m(t, r) \int_0^t d\sigma = m(t, r) t,
\]
we deduce
\[
\left( \frac{M_i(t)}{t} \right)' = \frac{m(t, r)t - M_i(t)}{t^2} > 0 \quad \forall t > 0,
\]
which implies (2.4).

The case \( s_1 < s_2 < 0 \) is treated similarly.

\[\Box\]

3. The case \( f(x, u) = f(x) \)

In this section, we address the problem

\[
\begin{cases}
-\text{div} \left( m(u, |\nabla u|_2^2) \nabla u \right) = f(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( f \in L^q(\Omega), q > N/2 \) and \( N \geq 2 \).

We prove the following.

Theorem 3.1

If (m0)–(m2) hold, \( 0 \neq f \in L^q(\Omega) \) and \( q > N/2 \), then problem (3.1) has a nontrivial weak solution \( u_+ \).

Let us consider for every \( r \geq 0 \) the problem

\[
\begin{cases}
-\text{div} \left( m_i(u) \nabla u \right) = f(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(P_r)

whose weak solution is, by definition, a function \( u_r \in H^1_0(\Omega) \cap L^\infty(\Omega) \) satisfying

\[
\forall \varphi \in H^1_0(\Omega) : \int_\Omega m_i(u_r) \nabla u_r \nabla \varphi dx = \int_\Omega f(x) \varphi dx.
\]

By our assumptions, the aforementioned equality makes sense.

We observe that \( u \in H^1_0(\Omega) \cap L^\infty(\Omega) \) solves (P_r) if and only if \( v := M_i(u) \in H^1_0(\Omega) \cap L^\infty(\Omega) \) satisfies

\[
\begin{cases}
-\Delta v = f(x) & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega,
\end{cases}
\]

in a weak sense. Note also that \( v \) does not depend on \( r \), because of the unicity of the solution of (3.2). Because \( f \in L^q(\Omega) \) with \( q > N/2 \), from Sobolev embeddings, the weak solution \( v \) of (3.2) belongs to \( C(\overline{\Omega}) \). Then, \( u_r := M_i^{-1}(v) \) is a weak solution of (P_r) and belongs to \( C(\overline{\Omega}) \), being composition of continuous functions.

The next step in the study of (3.1) is given by the next proposition.

Proposition 3.2

If (m0)–(m2) hold and \( 0 \neq f \in L^q(\Omega), q > N/2 \), then for each \( r \geq 0 \), the auxiliary problem (P_r) has a unique nontrivial weak solution \( u_r \). Moreover, the map

\[
S : r \in [0, +\infty) \mapsto \int_\Omega |\nabla u_r|^2 dx \in [0, +\infty)
\]

is continuous.
Proof
As we have seen, the unicity of the solution of \((P_r)\) is a consequence of the unicity of the solution of (3.2); we just need to show the second statement in the proposition. Let \(\{r_n\}\) be a sequence of nonnegative numbers such that \(r_n \to r_0 \geq 0\). Setting \(u_n := u_{r_n}\), it holds
\[
\forall n \in \mathbb{N} : \int_\Omega |\nabla u_n|^2 \, dx = \int_\Omega \frac{1}{[m_n(u_n)]^2} |\nabla v|^2 \, dx,
\]
where \(v\) is the solution of (3.2). From Lemma 2.1(c) and the continuity of \(m\), we have pointwise in \(\Omega\)
\[
u_n(x) = M_0^{-1}(v(x)) \to M_0^{-1}(v(x)) = u_n(x)
\]
\[
\frac{1}{[m_n(u_n)]^2} |\nabla v|^2 \to \frac{1}{[m_n(u_n)]^2} |\nabla v|^2 \quad \text{a.e. in } \Omega.
\]
On the other hand, by \((m1)\)
\[
\frac{1}{[m_n(u_n)]^2} |\nabla v|^2 \leq \frac{1}{m^2} |\nabla v|^2 \quad \text{a.e. in } \Omega.
\]
We deduce by the dominated convergence theorem that
\[
\int_\Omega |\nabla u_n|^2 \, dx = \int_\Omega \frac{1}{[m_n(u_n)]^2} |\nabla v|^2 \, dx \to \int_\Omega \frac{1}{[m_n(u_n)]^2} |\nabla v|^2 \, dx = \int_\Omega |\nabla u_0|^2 \, dx,
\]
concluding the proof. \(\square\)

**Proof of Theorem 3.1.**
The idea is to show that the continuous map \(S\) defined in Proposition 3.2 has a fixed point.
Clearly, being \(u_0\), a nontrivial solution of \((P_r)\) with \(r = 0\), it holds
\[
S(0) = \int_\Omega |\nabla u_0|^2 \, dx > 0.
\]
Introducing the map \(T : [0, \infty) \to [0, \infty)\) given by
\[
T(r) := \int_\Omega m(u, r) |\nabla u|^2 \, dx = \int_\Omega \frac{1}{m(u, r)} |\nabla v|^2 \, dx,
\]
it is
\[
\forall r \geq 0 : \quad S(r) \leq \frac{1}{m} T(r) \leq \frac{1}{m^2} \int_\Omega |\nabla v|^2 \, dx.
\]
In particular, there is \(R > 0\) such that
\[
\forall r \geq R : \quad S(r) < r.
\]
The aforementioned considerations allow us to conclude that \(S\) has a fixed point \(r_* > 0\), and then for \(u_* := u_{r_*}\), we have
\[
r_* = S(r_*) = \int_\Omega |\nabla u_*|^2 \, dx.
\]
In other words, \(u_*\) is a solution of (3.1). \(\square\)

**Remark 4**
Observe that we have showed that the map \(S\) is bounded.

### 4. The general case \(f = f(x, u)\)

Throughout this section, \(f : \Omega \times \mathbb{R} \to \mathbb{R}\) is a function satisfying the following condition:

\((C)\) is a Carathéodory function and
\[
\exists c > 0 : \quad |f(x, t)| \leq c(1 + |t|^p), \quad (x, t) \in \Omega \times \mathbb{R},
\]
where \(1 < p < 2^* - 1\) and \(2^* = \frac{N}{N-2}\).

As in the previous section, we begin by introducing, for fixed \(r \geq 0\), the auxiliary problem
\[
\begin{align*}
-\text{div} \left( m_r(u) \nabla u \right) &= f(x, u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
associated to \((P)\).
Consequently, passing to the limit in
\[\begin{align*}
-\Delta v &= h_r(x, v) \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega,
\end{align*}\]
where
\[h_r(x, t) := f(x, M_r^{-1}(t)).\]
Of course, a weak solution of \((SP_r)\) is a function \(v \in H^1_0(\Omega) \cap L^\infty(\Omega)\) such that
\[\forall \psi \in H^1_0(\Omega) : \int_\Omega \nabla v \nabla \psi \, dx = \int_\Omega h_r(x, v) \psi \, dx.\]
In view of Lemma 2.1 (b) and (C), the aforementioned right-hand side makes sense.

Now, we state the criterium anticipated in Section 1, which gives us a sufficient condition for the existence of solutions for problem (P).

**Theorem 4.1**
Let \(m\) and \(f\) satisfy \((m_0) - (m_2)\) and (C), respectively. Suppose, additionally, that for each \(r \geq 0\), problem \((SP_r)\) has a unique nontrivial weak solution \(v_r \in H^1_0(\Omega) \cap L^\infty(\Omega)\) and the map
\[V : r \in [0, +\infty) \mapsto \int_\Omega |\nabla v_r|^2 \, dx \in [0, +\infty)\]
is in \(L^\infty([0, +\infty))\). Then, (P) possesses a nontrivial weak solution.

**Proof**
By assumptions, for each \(r \geq 0\), the auxiliary problem \((P'_r)\) has a unique nontrivial solution \(u_r = M_r^{-1}(v_r) \in H^1_0(\Omega) \cap L^\infty(\Omega)\). The idea is to show that the map
\[T : r \in [0, +\infty) \mapsto \int_\Omega |\nabla u_r|^2 \, dx \in [0, +\infty)\]
has a fixed point \(u_\infty\), which will be a solution of (P). Let us begin by proving that \(T\) is continuous.

Let \(\{r_n\}\) be a sequence of nonnegative numbers such that \(r_n \to r_0 \geq 0\). Then, setting for brevity \(u_n := u_{r_n}\) and \(v_n := v_{r_n}\), the boundedness of \(V\) implies that \(\{v_n\}\) is bounded in \(H^1_0(\Omega)\). Then, there exists \(v \in H^1_0(\Omega)\), such that, up to subsequences,
\[v_n \rightharpoonup v \text{ in } H^1_0(\Omega) \quad \text{and} \quad v_n \to v \text{ a.e. in } \Omega\]
and, for some \(g \in L^2(\Omega)\),
\[|v_n(x)| \leq g(x) \quad \text{a.e. in } \Omega.\]
Consequently, passing to the limit in \(n\) in the identity
\[\int_\Omega \nabla v_n \nabla \psi \, dx = \int_\Omega h_n(x, v_n) \psi \, dx = \int_\Omega f(x, M_{r_n}^{-1}(v_n)) \psi \, dx,\]
where \(h_n := h_{r_n}\), and using (C) and Lemma 2.1 (b), (c), by the dominated convergence theorem, we obtain
\[\int_\Omega |\nabla v|^2 \, dx = \int_\Omega f(x, M_{r_0}^{-1}(v)) \psi \, dx.\]
(4.1)
Similarly, passing to the limit in \(n\) in
\[\int_\Omega |\nabla v_n|^2 \, dx = \int_\Omega h_n(x, v_n) \psi \, dx = \int_\Omega f(x, M_{r_n}^{-1}(v_n)) \psi \, dx,\]
it follows that
\[\int_\Omega |\nabla v|^2 \, dx \to \int_\Omega f(x, M_{r_0}^{-1}(v)) \psi \, dx.\]
(4.2)
From (4.1) and (4.2), we conclude that
\[\int_\Omega |\nabla v_n|^2 \, dx \to \int_\Omega |\nabla v|^2 \, dx.\]
Then, \(v_n \to v\) in \(H^1_0(\Omega)\). Then, up to a subsequence,
\[|\nabla v_n(x)|^2 \to |\nabla v(x)|^2 \quad \text{a.e. in } \Omega\]
and for some \( \tilde{g} \in L^1(\Omega) \),

\[
|\nabla v_n(x)|^2 \leq \tilde{g}(x) \quad \text{a.e. in } \Omega,
\]

so that,

\[
\frac{1}{|m(u_n, r_n)|^2} |\nabla v_n|^2 \leq \frac{1}{m^2} \tilde{g}(x) \quad \text{a.e. in } \Omega.
\]  \tag{4.3}

Finally, from Lemma 2.1 (c), we have

\[
u_n(x) = M_{r_n}^{-1}(v_n(x)) \rightarrow M_{r_0}^{-1}(v(x)) \quad \text{a.e. in } \Omega.
\]

As before, passing to the limit in \( n \) in the identity

\[
\int \nabla v_n \nabla \varphi \, dx = \int h_n(x, v_n) \varphi \, dx = \int f(x, M_{r_n}^{-1}(v_n)) \varphi \, dx,
\]

we have

\[
\int \nabla \varphi \, dx = \int f(x, M_{r_0}^{-1}(v)) \varphi \, dx,
\]

which implies, by uniqueness of the solution, that \( M_{r_0}^{-1}(v(x)) = u_n(x) \) and so \( u_n \rightarrow u_0 \) a.e. in \( \Omega \). Then

\[
\frac{1}{|m(u_n, r_n)|^2} |\nabla v_n|^2 \rightarrow \frac{1}{|m(u_0, r_0)|^2} |\nabla v|^2 \quad \text{a.e. in } \Omega
\]  \tag{4.4}

and from (4.4), (4.3), and the dominated convergence theorem, we obtain

\[
\int |\nabla u_n|^2 \, dx = \int \frac{1}{|m(u_n, r_n)|^2} |\nabla v_n|^2 \, dx \rightarrow \int \frac{1}{|m(u_0, r_0)|^2} |\nabla v|^2 \, dx = \int \frac{1}{|m(u_0, r_0)|^2} |\nabla v_n|^2 \, dx,
\]

proving the continuity of \( T \).

To prove that \( T \) has a fixed point, we observe that \( T(0) = \int |\nabla u_0|^2 \, dx > 0 \), and being \( r \mapsto \int |\nabla v_r|^2 \, dx \) bounded, there exists \( R > 0 \) such that

\[
\forall r \geq R : \quad T(r) = \int \frac{1}{|m(u, r)|^2} |\nabla v_r|^2 \, dx \leq \frac{1}{m^2} \int |\nabla v_r|^2 \, dx \leq r.
\]

Then, the existence of a fixed point is guaranteed.

\[ \Box \]

Remark 5

Of course, the main ingredients (and difficulties) in Theorem 4.1 are the existence of a unique solution to the auxiliary problem \((SP_r)\) as well as the \textit{a priori} bound of the solutions with respect to \( r \).

However, the hypothesis on the unicity of the solution to problem \((SP)\) is not strictly necessary and actually not satisfied in many semilinear elliptic problems. All that we need is the existence of a fixed point for the map \( T \), which can be achieved, for example, in the following case.

Assume that, even though problem \((SP)\) has not a unique solution, it is possible to choose continuously in \( r \) the solution \( v_r \) of \((SP)\), so that we can define a continuous branch of solutions and the map \( V : r \mapsto \int |\nabla v_r|^2 \, dx \) is continuous. In this case, being the map \( r \mapsto v_r \in H_0^1(\Omega) \) continuous, it is easy to see, using Lemma 2.1 (c) that also \( r \mapsto u_r = M_{r_n}^{-1}(v_r) \in H_0^1(\Omega) \), and hence \( T \), is continuous. The existence of a fixed point for \( T \) is then guaranteed if one can prove that \( \lim_{r \to +\infty} V(r)/r < m^2 \).

5. Two particular cases

As we mentioned before the goal of this section is to present some particular examples of problems covered by Theorem 4.1. Certainly, Theorem 4.1 covers a range of different situations; however, we limit ourselves to consider two special cases.

5.1. First case

Under assumptions (m0)–(m2) on \( m \), let \( f : \Omega \times [0, +\infty) \to \mathbb{R} \) be a function that satisfies the following conditions:

(f1) there exists \( c > 0 \) such that

\[
0 \leq f(x, t) \leq c(1 + t) \text{ a.e. in } \Omega \text{ and } \forall t \in [0, +\infty);
\]

(f2) \( \text{for a.e. } x \in \Omega, \text{ the function } t \mapsto f(x, t) \text{ is continuous on } [0, +\infty); \)
(f3) for each \( t \geq 0 \), the function \( x \mapsto f(x, t) \) belongs to \( L^\infty(\Omega) \); and

(f4) for a.e. \( x \in \Omega \), the function \( t \mapsto f(x, t)/t \) is decreasing on \( (0, +\infty) \).

In what follows, we extend \( f \) on the negative numbers by giving the constant value \( f(x, 0) \). This extension, denoted again with \( f \), satisfies obviously (f1)–(f4).

Recall that \( h_r(x, t) = f(x, M^{-1}_r(t)) \). We prove the following.

**Claim:** \( h_r \) satisfies (f1)–(f4).

Indeed, it follows from (f2), (f3), and (m1) (see also Lemma 2.1 (b)) that \( h_r \) satisfies also (f2) and (f3) for all \( r \geq 0 \) and \( t \in \mathbb{R} \). Moreover, from (f1) and (m1) (see Lemma 2.1 (b) again), the function \( h_r \) verifies, for every \( r \geq 0 \), the growth condition

\[
\exists c > 0 : \quad |h_r(x, t)| \leq c(1 + |t|) \quad \text{a.e. in } \Omega, \forall t \in \mathbb{R},
\]

that is, a (f1)-like condition. Moreover, from (f4) and Lemma 2.1 (a) and (d), we have, for \( t_1 > t_2 > 0 \), a.e. in \( \Omega \):

\[
\begin{align*}
\frac{h_r(x, t_1)}{t_1} &= \frac{f(x, M^{-1}_r(t_1)) M^{-1}_r(t_1)}{M^{-1}_r(t_1)} \\
&\leq \frac{f(x, M^{-1}_r(t_2)) M^{-1}_r(t_1)}{M^{-1}_r(t_1)} \\
&< \frac{f(x, M^{-1}_r(t_2)) M^{-1}_r(t_2)}{M^{-1}_r(t_2)} \\
&= \frac{h_r(x, t_2)}{t_2},
\end{align*}
\]

showing that \( h_r \) satisfies also condition (f4), proving the claim.

Note that the unique point in this case where the positivity of \( f \) is used is in the above computation.

Then, according to [6, Theorem 1, first part], there is at most one solution to \( (SP) \). To prove the existence, define the functions (following [6])

\[
\alpha_0(x) := \lim_{t \to 0} \frac{f(x, t)}{t} \left( \begin{array}{cc} \geq f(x, 1) \\ \leq f(x, 1) \end{array} \right)
\]

\[
\alpha_\infty(x) := \lim_{t \to \infty} \frac{f(x, t)}{t} \left( \begin{array}{cc} \geq f(x, 1) \\ \leq f(x, 1) \end{array} \right);
\]

then, as a consequence of (f3) and (f4), there is a constant \( C \geq 0 \) such that

\[
\alpha_0(x) \in [0, +\infty) \quad \text{and} \quad \alpha_\infty(x) \in [0, C] \quad \text{for a.e. } x \in \Omega.
\]

At the same way, it is natural to introduce, for every \( r \geq 0 \), the functions

\[
\alpha'_0(x) := \lim_{t \to 0} \frac{h_r(x, t)}{t} \quad \text{and} \quad \alpha'_\infty(x) := \lim_{t \to \infty} \frac{h_r(x, t)}{t}.
\]

Note that the limits exist, as \( h_r \) satisfies (f4), with

\[
\alpha'_0(x) \in [h_r(x, 1), +\infty] \quad \text{and} \quad \alpha'_\infty(x) \in [0, h_r(x, 1)], \quad \text{for a.e. } x \in \Omega.
\]

Because \( h_r \) satisfies also (f3), for \( C_r := |h_r(., 1)|_\infty \), it is actually

\[
\alpha'_0(x) \in [0, +\infty] \quad \text{and} \quad \alpha'_\infty(x) \in [0, C_r], \quad \text{for a.e. } x \in \Omega.
\]

Of course it may happens \( \alpha'_0 = +\infty \).

**Remark 6**

We recall ([6, section 3]) that given a measurable function \( \alpha \) that is bounded above or below, we denote with

\[
\lambda_1(-\Delta - \alpha(x)) = \inf_{u \in C^2_0(\Omega)} \frac{\int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} \alpha(x) u^2 \, dx}{\int_{\Omega} u^2 \, dx}
\]

the first eigenvalue of the operator \(-\Delta - \alpha(x)\) with Dirichlet boundary condition. Note that it can be also \(+\infty\) or \(-\infty\). However, in our case, due to (5.2), it is

\[
\lambda_1(-\Delta - \alpha'_0(x)) \in [-\infty, +\infty) \quad \text{and} \quad \lambda_1(-\Delta - \alpha'_\infty(x)) \in (-\infty, +\infty).
\]
Then, if we impose the further condition

\[(f5) \text{ for every } r \geq 0, \]

\[\lambda_1 (-\Delta - \alpha'_0(x)) < 0 < \lambda_1 (-\Delta - \alpha'_\infty(x)),\]

we can apply [6, Theorem 1, second part] to deduce the existence of a unique nontrivial and nonnegative weak solution \(v_r\) to problem \((SP_r)\), with \(v_r \in W^{2,q}(\Omega) \cap L^\infty(\Omega), \) for all \(q \in (1, +\infty).\)

The main result of this section is the following.

**Theorem 4.2**

Under the conditions \((m0)-(m2)\) and \((f1)-(f5)\), problem \((P)\) admits a nontrivial and nonnegative solution \(u \in W^{2,q}(\Omega) \cap L^\infty(\Omega), \) for all \(q \in (1, +\infty).\)

**Proof**

To prove the result, we will use Theorem 4.1. We have already shown that under our hypothesis, problem \((SP_r)\) has a unique solution \(v_r\) for every \(r \geq 0.\) It remains to show the boundedness of the map \(V : r \mapsto \int_\Omega |\nabla v_r|^2 dx \) for every \(r \geq 0.\) Then, all the assumptions of Theorem 4.1 are satisfied, and we deduce the existence of a solution for \((P).\)

\[0 \leq v_r \leq C, \quad \forall r \geq 0.\]

From this, the boundedness of the map \(V : r \mapsto \int_\Omega |\nabla v_r|^2 dx\) easily follows; indeed,

\[
\begin{align*}
\int_\Omega |\nabla v_r|^2 dx &= \int_\Omega h_r(x,v_r)v_r dx \\
&\leq \int_\Omega c(1 + v_r)v_r dx \\
&\leq c(1 + C) \int_\Omega v_r dx \\
&\leq C \left( \int_\Omega |\nabla v_r|^2 dx \right)^{1/2}.
\end{align*}
\]

5.1.1. Some comments on the verification of \((f5).\)

Observe that assumption \((f5)\) involves the function \(f\) via the change of variable, and also the parameter \(r\); then, it can be not easy to verify. However, by assuming simpler conditions (see examples in the succeeding discussion), the verification of \((f5)\) can be simplified.

First of all, it is possible to give the dependence of \(\alpha'_0\) and \(\alpha'_\infty\) with respect to \(r.\)

Fixed \(x,\) by the definition of \(h_r,\)

\[\alpha'_0(x) = \lim_{t \to 0} \frac{h_r(x,t)}{t} = \lim_{t \to 0} \frac{f(x,M_r^{-1}(t))}{M_r^{-1}(t)} \frac{M_r^{-1}(t)}{t}.\]

Denoting \(t = M_r(s),\) by the L'Hôpital rule, we obtain

\[\lim_{t \to 0} \frac{M_r^{-1}(t)}{t} = \lim_{t \to 0} \frac{1}{m(t)} = \frac{1}{m(0)}\]

with \(0 < 1/m(0, r) \leq 1/m\) for all \(r \geq 0.\) Then

\[\alpha'_0(x) = \frac{\alpha_0(x)}{m(0,r)} \in [0, +\infty]. \tag{5.3}\]

In an analogous way, if we try to estimate

\[\alpha'_\infty(x) = \lim_{t \to +\infty} \frac{f(x,M_r^{-1}(t))}{M_r^{-1}(t)} \frac{M_r^{-1}(t)}{t},\]

because \(M_r^{-1}(t) \to +\infty\) as \(t \to +\infty\) (Lemma 2.1 (a)), we derive again by the L'Hôpital rule that

\[\alpha'_\infty(x) = \frac{\alpha_\infty(x)}{m(\infty,r)} \quad \text{where } m(\infty,r) := \lim_{s \to +\infty} m_r(s).\]
Then, from Lemma 2.1 (b),

\[ \alpha_{\infty}^r(x) = \lim_{t \to \infty} \frac{f(x, M^{-1}_r(t))}{t} \leq m^{-1} \lim_{t \to \infty} \frac{f(x, m^{-1} t)}{m^{-1} t} = m^{-1} \alpha_{\infty}(x), \quad \forall r \geq 0. \]

Consequently,

\[ \lambda_1 \left( -\Delta - m^{-1} \alpha_{\infty}(x) \right) \leq \lambda_1 \left( -\Delta - \alpha_{\infty}^r(x) \right), \quad \forall r \geq 0. \quad (5.4) \]

Without loss of generality let \( m = \inf_{t \in J} m(t, r) \) and assume the further condition

\( m(0, r) = m, \quad \forall r \geq 0. \) (m3)

Then, by (5.3), \( \alpha_{\infty}^r(x) = m^{-1} \alpha_0(x) \) and therefore

\[ \lambda_1 \left( -\Delta - \alpha_0(x) \right) = \lambda_1 \left( -\Delta - m^{-1} \alpha_0(x) \right), \quad \forall r \geq 0. \quad (5.5) \]

In this case, from (5.4) and (5.5), we see that condition

\[ \lambda_1 \left( -\Delta - m^{-1} \alpha_0(x) \right) < 0 < \lambda_1 \left( -\Delta - m^{-1} \alpha_{\infty}(x) \right) \quad (5.6) \]

(which does not involve \( r \)) implies condition (f5). We are then reduced to verify condition (5.6), as in [6], which just involve the pure \( f \).

Example 5.2

If \( m(\infty, r) = +\infty \), for all \( r \geq 0 \) then \( \lambda_1 \left( -\Delta - \alpha_{\infty}(x) \right) = \lambda_1 \left( -\Delta \right) > 0 \); then, the second inequality in the assumption (f5) is automatically satisfied.

Example 5.3

When \( m(0, r) < 1 \) for all \( r \geq 0 \) then \( \lambda_1 \left( -\Delta - \alpha_0(x) \right) < \lambda_1 \left( -\Delta - \alpha_0(x) \right) \) and the first inequality in the assumption (f5) reduces to prove an inequality, which just involve the original nonlinearity \( f \), as in [6].

As we have seen, Theorem 4.2 is based on the fact that problem (SP1) has a unique solution, thanks to a result of [6]. However, other simple cases in which there is the unicity of the solution at (SP2) in the sublinear case are easily found in the literature (e.g., [7]) then other assumptions on \( f \) can be given in order to obtain a solution of (P).

5.2. Second case

Under the same assumptions (m0)–(m2) on \( m \), let \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) be Carathéodory function satisfying

(f7) \( f(x, 0) \neq 0 \),

(f8) there exists \( \mu \in L^2(\Omega), \delta \in (0, 1), \nu > 0 \), such that

\[ |f(x, t)| \leq \mu(x) + \nu |t|^\delta \text{ a.e. in } \Omega \text{ and } \forall t \in \mathbb{R}, \]

(f9) thereis \( \theta \in (0, m\lambda_1) \)such

\[ |f(x, t_1) - f(x, t_2)| \leq \theta |t_1 - t_2| \text{ a.e. in } \Omega, \]

for all \( t_1, t_2 \in \mathbb{R} \). Hereafter, \( \lambda_1 \) is the first eigenvalue of \( -\Delta \) in \( H^1_0(\Omega) \).

Because \( f \) satisfies (f8), it is clear that the same holds for \( h_r \), for any \( r \geq 0 \). On the other hand, from (f9) and Lemma 2.1 (b), we conclude that

\[ |h_r(x, s_1) - h_r(x, s_2)| \leq \frac{\theta}{m} |s_1 - s_2| \text{ a.e. in } \Omega, \quad (5.7) \]

for all \( r \geq 0 \) and \( s_1, s_2 \in \mathbb{R} \). This redly implies that problem (SP1) has a unique nontrivial solution for each \( r \geq 0 \). The argument is known; however, we revise it here for completeness. Defining the solution operator \( S_r : H^1_0(\Omega) \to H^1_0(\Omega) \), which associates to each \( \nu \in H^1_0(\Omega) \) the unique solution \( \nu \) of problem...
it follows from (5.7) and the Sobolev embedding that (hereafter \( \| \cdot \| \) is the \( H^1_0 \)-norm)

\[
\| S_r(w_1) - S_r(w_2) \|^2 \leq \int_\Omega |h_r(x, w_1) - h_r(x, w_2)| |S_r(w_1) - S_r(w_2)| \, dx
\]

\[
\leq \frac{\theta}{m} \int_\Omega |w_1 - w_2| |S_r(w_1) - S_r(w_2)| \, dx
\]

\[
\leq \frac{\theta}{m \lambda_1} \| w_1 - w_2 \| |S_r(w_1) - S_r(w_2)|
\]

showing that \( S_r \) is a contraction for each \( r \geq 0 \).

It follows by the Banach fixed point theorem that, for each \( r \geq 0 \), there is a unique nontrivial solution \( v_r \in H^1_0(\Omega) \cap L^\infty(\Omega) \) to problem (\( SP_r \)).

Finally, from

\[
\| v_r \|^2 = \int_\Omega h_r(x, v_r) v_r \, dx \leq \lambda_1^{-1/2} \| v_r \| |\mu + v| v_r|^{\delta} \|_2
\]

\[
\leq \lambda_1^{-1/2} \| v_r \| \left( |\mu|_2 + v| v_r|_2^{\delta} \right)
\]

\[
\leq \lambda_1^{-1/2} \| v_r \| \left( |\mu|_2 + v \lambda_1^{-\delta/2} | v_r|_2^{\delta} \Omega^{(1-\delta)/2} \right)
\]

\[
\leq \lambda_1^{-1/2} \| v_r \| \left( |\mu|_2 + v \lambda_1^{-\delta/2} | v_r|^{\delta} \Omega^{(1-\delta)/2} \right)^{\delta+1/2}
\]

we deduce the boundedness of the map \( V \).

Then, by Theorem 4.1, we obtain

**Theorem 4.3**

Under the conditions (m0)–(m2) and (f7)–(f9), problem (P) admits a nontrivial solution \( u \in W^{2,q}(\Omega) \cap L^\infty(\Omega) \), for all \( q \in (1, \infty) \).

Of course the theorem also holds if in (f8) we allow \( \delta = 1 \) with \( v \in (0, \lambda_1) \).

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