AN UPPER BOUND ON THE DEGREE OF SINGULAR VECTORS FOR
E(1, 6)

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Abstract. The aim of this work is to prove a technical result, that had been stated by Boyallian, Kac and Liberati [3], on the degree of singular vectors of finite Verma modules over the exceptional Lie superalgebra $E(1, 6)$ that is isomorphic to the annihilation superalgebra associated with the conformal superalgebra $CK_6$.

1. INTRODUCTION

Finite simple conformal superalgebras were completely classified in [12] and consist in the list: Cur $\mathfrak{g}$, where $\mathfrak{g}$ is a simple finite–dimensional Lie superalgebra, $W_n(n \geq 0)$, $S_{n,b}$, $\tilde{S}_n(n \geq 2, b \in \mathbb{C})$, $K_n(n \geq 0, n \neq 4)$, $K'_4$, $CK_6$. The finite irreducible modules over the conformal superalgebras Cur $\mathfrak{g}$, $K_0$, $K_1$ were studied in [7]. The classification of all finite irreducible modules over the conformal superalgebras Cur $\mathfrak{g}$, $K_N$, for $N = 2, 3, 4$ was obtained in [10]. Boyallian, Kac, Liberati and Rudakov classified all finite irreducible modules over the conformal superalgebras of type $W$ and $S$ in [4]; Boyallian, Kac and Liberati classified all finite irreducible modules over the conformal superalgebras of type $K_N$ for $N \geq 4$ in [2]. All finite irreducible modules over the conformal superalgebra $K'_4$ were classified in [1]. Finally a classification of all finite irreducible modules over the conformal superalgebra $CK_6$ was obtained in [3] and [16] with different approaches.

In [3] the classification of all finite irreducible modules over the conformal superalgebra $CK_6$ is obtained by their correspondence with irreducible finite conformal modules over the annihilation superalgebra $\mathcal{A}(CK_6)$ associated with $CK_6$. The annihilation superalgebra $\mathcal{A}(CK_6)$ is isomorphic to the exceptional Lie superalgebra $E(1, 6)$ (see [8], [9], [17], [14]). In [3], in order to obtain this classification, the authors classify all highest weight singular vectors of finite Verma modules, i.e. induced modules $\text{Ind}(F) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\geq 0})} F$, where $F$ is a finite–dimensional irreducible $\mathfrak{g}_{\geq 0}$–module [15][10]. In [3] the classification of highest weight singular vectors is based on a technical lemma, whose proof is missing (Lemma 4.4 in [3]), that provides an upper bound on the degree of singular vectors for $E(1, 6)$.

The aim of this paper is to prove that technical lemma stated in [3]. The proof of this lemma completes the classification of singular vectors for $E(1, 6)$ given in [3].

The paper is organized as follows. In section 2 we recall some notions on conformal superalgebras. In section 3 we recall the definition of the conformal superalgebra $CK_6$ and some of its properties. Finally, in section 4 we prove the bound on the degree of singular vectors for $E(1, 6)$.

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2. Preliminaries on Conformal Superalgebras

We recall some notions on conformal superalgebras. For further details see [13, Chapter 2], [11], [4], [2].

Let \( \mathfrak{g} \) be a Lie superalgebra; a formal distribution with coefficients in \( \mathfrak{g} \), or equivalently a \( \mathfrak{g} \)-valued formal distribution, in the indeterminate \( z \) is an expression of the following form:

\[
a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1},
\]

with \( a_n \in \mathfrak{g} \) for every \( n \in \mathbb{Z} \). We denote the vector space of formal distributions with coefficients in \( \mathfrak{g} \) in the indeterminate \( z \) by \( \mathfrak{g}[[z, z^{-1}]] \). We denote by \( \text{Res}(a(z)) = a_0 \) the coefficient of \( z^{-1} \) of \( a(z) \). The vector space \( \mathfrak{g}[[z, z^{-1}]] \) has a natural structure of \( \mathbb{C}[\partial_z] \)-module.

We define for all \( a(z) \in \mathfrak{g}[[z, z^{-1}]] \) its derivative:

\[
\partial_z a(z) = \sum_{n \in \mathbb{Z}} (-n - 1) a_n z^{-n-2}.
\]

A formal distribution with coefficients in \( \mathfrak{g} \) in the indeterminates \( z \) and \( w \) is an expression of the following form:

\[
a(z, w) = \sum_{m,n \in \mathbb{Z}} a_{m,n} z^{-m-1} w^{-n-1},
\]

with \( a_{m,n} \in \mathfrak{g} \) for every \( m, n \in \mathbb{Z} \). We denote the vector space of formal distributions with coefficients in \( \mathfrak{g} \) in the indeterminates \( z \) and \( w \) by \( \mathfrak{g}[[z, z^{-1}], w, w^{-1}] \). Given two formal distributions \( a(z) \in \mathfrak{g}[[z, z^{-1}]] \) and \( b(w) \in \mathfrak{g}[[w, w^{-1}]] \), we define the commutator \([a(z), b(w)]\):

\[
[a(z), b(w)] = \left[ \sum_{n \in \mathbb{Z}} a_n z^{-n-1}, \sum_{m \in \mathbb{Z}} b_m w^{-m-1} \right] = \sum_{m,n \in \mathbb{Z}} [a_n, b_m] z^{-n-1} w^{-m-1}.
\]

**Definition 2.1.** Two formal distributions \( a(z), b(z) \in \mathfrak{g}[[z, z^{-1}]] \) are called local if:

\[
(z - w)^N [a(z), b(w)] = 0 \quad \text{for some } N \gg 0.
\]

We call \( \delta \)-function the following formal distribution in the indeterminates \( z \) and \( w \):

\[
\delta(z - w) = z^{-1} \sum_{n \in \mathbb{Z}} \left( \frac{w}{z} \right)^n.
\]

See Corollary 2.2 in [13] for the following equivalent condition of locality.

**Proposition 2.2.** Two formal distributions \( a(z), b(z) \in \mathfrak{g}[[z, z^{-1}]] \) are local if and only if \([a(z), b(w)]\) can be expressed as a finite sum of the form:

\[
a(z, w) = \sum_j (a(w)(j)b(w)) \frac{\partial^j}{j!} \delta(z - w),
\]

where the coefficients \( (a(w)(j)b(w)) := \text{Res}_z (z - w)^j [a(z), b(w)] \) are formal distributions in the indeterminate \( w \).

**Definition 2.3** (Formal Distribution Superalgebra). Let \( \mathfrak{g} \) be a Lie superalgebra and \( \mathcal{F} \) a family of mutually local \( \mathfrak{g} \)-valued formal distributions in the indeterminate \( z \). The pair \((\mathfrak{g}, \mathcal{F})\) is called a formal distribution superalgebra if the coefficients of all formal distributions in \( \mathcal{F} \) span \( \mathfrak{g} \).
We define the $\lambda$–bracket between two formal distributions $a(z), b(z) \in \mathfrak{g}[[z, z^{-1}]]$ as the generating series of the $(a(z)_{(j)} b(z))$'s:

\[
[a(z)_{\lambda} b(z)] = \sum_{j \geq 0} \frac{\lambda^j}{j!} (a(z)_{(j)} b(z)).
\]

**Definition 2.4** (Conformal superalgebra). A **conformal superalgebra** $R$ is a left $\mathbb{Z}_2$–graded $\mathbb{C}[\partial]$–module endowed with a $\mathbb{C}$–linear map, called $\lambda$–bracket, $R \otimes R \to \mathbb{C}[\lambda] \otimes R$, $a \otimes b \mapsto [a_{\lambda} b]$, that satisfies the following properties for all $a, b, c \in R$:

\begin{enumerate}[(i)]
  \item **conformal sesquilinearity**: $[\partial a_{\lambda} b] = -\lambda[a_{\lambda} b]$, \quad $[a_{\lambda} \partial b] = (\lambda + \partial)[a_{\lambda} b]$;
  \item **skew–symmetry** : $[a_{\lambda} b] = -(-1)^{p(a)p(b)} [b_{-\lambda-\partial a}]$;
  \item **Jacobi identity** : $[a_{\lambda} [b_{\mu} c]] = [[a_{\lambda} b]_{\lambda+\mu} c] + (-1)^{p(a)p(b)} [b_{\mu} [a_{\lambda} c]]$;
\end{enumerate}

where $p(a)$ denotes the parity of the element $a \in R$ and $p(\partial a) = p(a)$ for all $a \in R$.

We call $n$–products the coefficients $(a_{(n)} b)$ that appear in $[a_{\lambda} b] = \sum_{n \geq 0} \frac{\lambda^n}{n!} (a_{(n)} b)$ and give an equivalent definition of conformal superalgebra.

**Definition 2.5** (Conformal superalgebra). A **conformal superalgebra** $R$ is a left $\mathbb{Z}_2$–graded $\mathbb{C}[\partial]$–module endowed with a $\mathbb{C}$–bilinear product $(a_{(n)} b) : R \otimes R \to R$, defined for every $n \geq 0$, that satisfies the following properties for all $a, b, c \in R, m, n \geq 0$:

\begin{enumerate}[(i)]
  \item $p(\partial a) = p(a)$;
  \item $\ (a_{(n)} b) = 0$, for $n \gg 0$;
  \item $(\partial a_{(0)} b) = 0$ and $(\partial a_{(n+1)} b) = -(n+1)(a_{(n)} b)$;
  \item $\ (a_{(n)} b) = -(-1)^{p(a)p(b)} \sum_{j \geq 0} (-1)^{j+n} \frac{j!}{m!} (b_{(n+j)} a)$;
  \item $\ (a_{(m)} (b_{(n)} c)) = \sum_{j=0}^{m} \sum_{j=0}^{n-j} (a_{(j)} b_{(m+n-j)} c) + (-1)^{p(a)p(b)} (b_{(n)} (a_{(m)} c))$.
\end{enumerate}

Using (iii) and (iv) in Definition 2.5 it is easy to show that for all $a, b \in R, n \geq 0$:

\[
(a_{(n)} \partial b) = \partial (a_{(n)} b) + n(a_{(n-1)} b).
\]

Due to this relation and (iii) in Definition 2.5 the map $\partial : R \to R$, $a \mapsto \partial a$ is a derivation with respect to the $n$–products.

**Remark 2.6.** Let $(\mathfrak{g}, \mathcal{F})$ be a formal distribution superalgebra, endowed with $\lambda$–bracket $[\ ]$. The elements of $\mathcal{F}$ satisfy sesquilinearity, skew-symmetry and Jacobi identity with $\partial = \partial_{\zeta}$; for a proof see Proposition 2.3 in [13].

We say that a conformal superalgebra $R$ is **finite** if it is finitely generated as a $\mathbb{C}[\partial]$–module. An **ideal** $I$ of $R$ is a $\mathbb{C}[\partial]$–submodule of $R$ such that $(a_{(n)} b) \in I$ for every $a \in R, b \in I, n \geq 0$. A conformal superalgebra $R$ is **simple** if it has no non-trivial ideals and the $\lambda$–bracket is not identically zero. We denote by $R'$ the **derived subalgebra** of $R$, i.e. the $\mathbb{C}$–span of all $n$–products.

**Definition 2.7.** A module $M$ over a conformal superalgebra $R$ is a left $\mathbb{Z}_2$–graded $\mathbb{C}[\partial]$–module endowed with $\mathbb{C}$–linear maps $R \to \text{End}_\mathbb{C} M$, $a \mapsto a_{(n)}$, defined for every $n \geq 0$, that satisfy the following properties for all $a, b \in R, v \in M, m, n \geq 0$:

\begin{enumerate}[(i)]
  \item $a_{(n)} v = 0$ for $n \gg 0$;
  \item $(\partial a)_{(n)} v = [\partial, a_{(n)}] v = -na_{(n-1)} v$;
\end{enumerate}
(iii) \([a_{(m)}, b_{(n)}]v = \sum_{j=0}^{m} \binom{m}{j} (a_{(j)}b)_{(m-n+j)}v.\]

Given a module \(M\) over a conformal superalgebra \(R\), we define for all \(a \in R\) and \(v \in M\):

\[
a_{\lambda}v = \sum_{n \geq 0} \frac{\lambda^n}{n!}a_{(n)}v.
\]

A module \(M\) is called \textit{finite} if it is a finitely generated \(\mathbb{C}[\partial]\)-module.

We can construct a conformal superalgebra starting from a formal distribution superalgebra \((g, \mathcal{F})\). Let \(\mathcal{F}\) be the closure of \(\mathcal{F}\) under all the \(n\)-products, \(\partial_z\) and linear combinations. By Dong’s Lemma, \(\mathcal{F}\) is still a family of mutually local distributions (see [13]). It turns out that \(\mathcal{F}\) is a conformal superalgebra. We will refer to it as the conformal superalgebra associated with \((g, \mathcal{F})\).

Let us recall the construction of the annihilation superalgebra associated with a conformal superalgebra \(R\). Let \(\tilde{R} = R[y, y^{-1}]\), set \(p(y) = 0\) and \(\tilde{\partial} = \partial + \partial_y\). We define the following \(n\)-products on \(\tilde{R}\), for all \(a, b \in R, f, g \in \mathbb{C}[y, y^{-1}], n \geq 0:\)

\[
(af_{(n)}bg) = \sum_{j \in \mathbb{Z}_+} (a_{(n+j)}b)\left(\frac{\partial_j}{j!}f\right)g.
\]

In particular if \(f = y^m\) and \(g = y^k\) we have for all \(n \geq 0:\)

\[
(ay_{m_{(n)}}by_{k}) = \sum_{j \in \mathbb{Z}_+} \binom{m}{j} (a_{(n+j)}b)y^{m+k-j}.
\]

We observe that \(\tilde{\partial}\tilde{R}\) is a two sided ideal of \(\tilde{R}\) with respect to the \(0\)-product. The quotient \(\text{Lie } R := \tilde{R}/\tilde{\partial}\tilde{R}\) has a structure of Lie superalgebra with the bracket induced by the \(0\)-product, i.e. for all \(a, b \in R, f, g \in \mathbb{C}[y, y^{-1}]:\)

\[
[af, bg] = \sum_{j \in \mathbb{Z}_+} (a_{(j)}b)\left(\frac{\partial_j}{j!}f\right)g.
\]

**Definition 2.8.** The annihilation superalgebra \(\mathcal{A}(R)\) of a conformal superalgebra \(R\) is the subalgebra of \(\text{Lie } R\) spanned by all elements \(ay^n\) with \(n \geq 0\) and \(a \in R\).

The extended annihilation superalgebra \(\mathcal{A}(R)^e\) of a conformal superalgebra \(R\) is the Lie superalgebra \(\mathbb{C}\partial \ltimes \mathcal{A}(R)\). The semidirect sum \(\mathbb{C}\partial \ltimes \mathcal{A}(R)\) is the vector space \(\mathbb{C}\partial \oplus \mathcal{A}(R)\) endowed with the structure of Lie superalgebra determined by the bracket:

\[
[\partial, ay^m] = -\partial_y(ay^m) = -may^{m-1},
\]

for all \(a \in R\) and the fact that \(\mathbb{C}\partial, \mathcal{A}(R)\) are Lie subalgebras.

For all \(a \in R\) we consider the following formal power series in \(\mathcal{A}(R)[[\lambda]]:\)

\[
a_{\lambda} = \sum_{n \geq 0} \frac{\lambda^n}{n!}ay^n.
\]

For all \(a, b \in R\), we have: \([a_{\lambda}, b_{\mu}] = [a_{\lambda}b]_{\lambda+\mu}\) and \((\partial a)_{\lambda} = -\lambda a_{\lambda}\) (for a proof see [5]).
Proposition 2.9 ([7]). Let $R$ be a conformal superalgebra. If $M$ is an $R$-module then $M$ has a natural structure of $\mathcal{A}(R)^c$-module, where the action of $ay^n$ on $M$ is uniquely determined by $a_k v = \sum_{n \geq 0} \frac{\lambda^n}{n!} ay^n.v$ for all $v \in V$. Viceversa if $M$ is a $\mathcal{A}(R)^c$-module such that for all $a \in R$, $v \in M$ we have $ay^n.v = 0$ for $n \gg 0$, then $M$ is also an $R$-module by letting $a_k v = \sum_n \frac{\lambda^n}{n!} ay^n.v$.

Proposition 2.9 reduces the study of modules over a conformal superalgebra $R$ to the study of a class of modules over its (extended) annihilation superalgebra. The following proposition states that, under certain hypotheses, it is sufficient to consider the annihilation superalgebra. We recall that, given a $\mathbb{Z}$-graded Lie superalgebra $g = \oplus_{l \in \mathbb{Z}} g_l$, we say that $g$ has finite depth $d \geq 0$ if $g_{-d} \neq 0$ and $g_i = 0$ for all $i < -d$.

Proposition 2.10 ([2, 10]). Let $g$ be the annihilation superalgebra of a conformal superalgebra $R$. Assume that $g$ satisfies the following conditions:

- **L1:** $g$ is $\mathbb{Z}$-graded with finite depth $d$;
- **L2:** There exists an element whose centralizer in $g$ is contained in $g_0$;
- **L3:** There exists an element $\Theta \in g_{-d}$ such that $g_{-d} = [\Theta, g]$, for all $i \geq 0$.

Finite modules over $R$ are the same as modules $V$ over $g$, called finite conformal, that satisfy the following properties:

1. For every $v \in V$, there exists $j_0 \in \mathbb{Z}$, $j_0 \geq -d$, such that $g_{j}.v = 0$ when $j \geq j_0$;
2. $V$ is finitely generated as a $\mathbb{C}[[\Theta]]$-module.

Remark 2.11. We point out that condition **L2** is automatically satisfied when $g$ contains a grading element, i.e. an element $t \in g$ such that $[t, b] = \deg(b)b$ for all $b \in g$.

Let $g = \oplus_{i \in \mathbb{Z}} g_i$ be a $\mathbb{Z}$-graded Lie superalgebra. We will use the notation $g_{>0} = \oplus_{i \geq 0} g_i$, $g_{<0} = \oplus_{i < 0} g_i$, and $g_{\geq 0} = \oplus_{i \geq 0} g_i$. We denote by $U(g)$ the universal enveloping algebra of $g$.

Definition 2.12. Let $F$ be a $g_{>0}$-module. The generalized Verma module associated with $F$ is the $g$-module $\text{Ind}(F)$ defined by:

$$\text{Ind}(F) := \text{Ind}^g_{>0}(F) = U(g) \otimes_{U(g_{>0})} F.$$

If $F$ is a finite-dimensional irreducible $g_{>0}$-module we will say that $\text{Ind}(F)$ is a finite Verma module. We will identify $\text{Ind}(F)$ with $U(g_{<0}) \otimes F$ as vector spaces via the Poincaré–Birkhoff–Witt Theorem. The $\mathbb{Z}$-grading of $g$ induces a $\mathbb{Z}$-grading on $U(g_{<0})$ and $\text{Ind}(F)$. We will invert the sign of the degree, so that we have a $\mathbb{Z}_{\geq 0}$-grading on $U(g_{<0})$ and $\text{Ind}(F)$. We will say that an element $v \in U(g_{<0})k$ is homogeneous of degree $k$. Analogously an element $m \in U(g_{<0})k \otimes F$ is homogeneous of degree $k$. For a proof of the following proposition see [1].

Proposition 2.13. Let $g = \oplus_{i \in \mathbb{Z}} g_i$ be a $\mathbb{Z}$-graded Lie superalgebra. If $F$ is an irreducible finite-dimensional $g_{>0}$-module, then $\text{Ind}(F)$ has a unique maximal submodule. We denote by $\text{I}(F)$ the quotient of $\text{Ind}(F)$ by the unique maximal submodule.

Definition 2.14. Given a $g$-module $V$, we call singular vectors the elements of:

$$\text{Sing}(V) = \{v \in V \mid g_{>0}.v = 0\}.$$

Homogeneous components of singular vectors are still singular vectors so we often assume that singular vectors are homogeneous without loss of generality. If $V = \text{Ind}(F)$, for a
\(g_{\geq 0}\)-module \(F\), we will call \textit{trivial singular vectors} the elements of \(\text{Sing}(V)\) of degree 0 and \textit{nontrivial singular vectors} the nonzero elements of \(\text{Sing}(V)\) of positive degree.

**Theorem 2.15** ([15], [10]). Let \(g\) be a Lie superalgebra that satisfies \(L1, L2, L3\), then:

(i) if \(F\) is an irreducible finite-dimensional \(g_{\geq 0}\)-module, then \(g_{\geq 0}\) acts trivially on it;

(ii) the map \(F \mapsto I(F)\) is a bijective map between irreducible finite-dimensional \(g_{0}\)-modules and irreducible finite conformal \(g\)-modules;

(iii) the \(g\)-module \(\text{Ind}(F)\) is irreducible if and only if the \(g_{0}\)-module \(F\) is irreducible and \(\text{Ind}(F)\) has no nontrivial singular vectors.

3. The Conformal Superalgebra \(CK_6\)

In this section we recall the definition and some properties of the conformal superalgebra \(CK_6\) from [3]. Let \(\Lambda(N)\) be the Grassmann superalgebra in the \(N\) odd indeterminates \(\xi_1, \ldots, \xi_N\). Let \(t\) be an even indeterminate and \(\Lambda(1,N) = \mathbb{C}[t, t^{-1}] \otimes \Lambda(N)\). We consider the Lie superalgebra of derivations of \(\Lambda(1,N)\):

\[
W(1,N) = \left\{ D = a\partial_t + \sum_{i=1}^N a_i\partial_i \mid a, a_i \in \Lambda(1,N) \right\},
\]

where \(\partial_t = \frac{\partial}{t}\) and \(\partial_i = \frac{\partial}{\xi_i}\) for every \(i \in \{1, \ldots, N\}\).

Let us consider the contact form \(\omega = dt - \sum_{i=1}^N \xi_i d\xi_i\). The contact Lie superalgebra \(K(1,N)\) is defined by:

\[
K(1,N) = \left\{ D \in W(1,N) \mid D\omega = f_D\omega \text{ for some } f_D \in \Lambda(1,N) \right\}.
\]

Analogously, let \(\Lambda(1,N)_+ = \mathbb{C}[t] \otimes \Lambda(N)\). We define the Lie superalgebra \(W(1,N)_+\) (resp. \(K(1,N)_+\)) similarly to \(W(1,N)\) (resp. \(K(1,N)\)) using \(\Lambda(1,N)_+\) instead of \(\Lambda(1,N)\). We can define on \(\Lambda(1,N)\) a Lie superalgebra structure as follows. For all \(f, g \in \Lambda(1,N)\) we let:

\[
[f, g] = \left(2f - \sum_{i=1}^N \xi_i \partial_i f\right) (\partial_t g) - (\partial_t f) \left(2g - \sum_{i=1}^N \xi_i \partial_i g\right) + (-1)^{p(f)} \left(\sum_{i=1}^N \partial_i f \partial_i g\right).
\]

We recall that \(K(1,N) \cong \Lambda(1,N)\) as Lie superalgebras via the following map (see [9]):

\[
\Lambda(1,N) \longrightarrow K(1,N)
\]

\[
f \mapsto 2f \partial_t + (-1)^{p(f)} \sum_{i=1}^N (\xi_i \partial_i f + \partial_i f)(\xi_i \partial_t + \partial_t).
\]

We will always identify elements of \(K(1,N)\) with elements of \(\Lambda(1,N)\) and we will omit the symbol \(\wedge\) between the \(\xi_i\)'s. We consider on \(K(1,N)\) the standard grading, i.e. for every \(t^m \xi_i \cdots \xi_s \in K(1,N)\) we have \(\deg(t^m \xi_i \cdots \xi_s) = 2m + s - 2\).

We consider the following family of formal distributions:

\[
\mathcal{F} = \left\{ A(z) := \sum_{m \in \mathbb{Z}} (At^m) z^{-m-1} = A\delta(t - z), \forall A \in \Lambda(N) \right\}.
\]

The pair \((K(1,N), \mathcal{F})\) is a formal distribution superalgebra and the conformal superalgebra \(\overline{\mathcal{F}}\) can be identified with \(K_N := \mathbb{C}[\partial] \otimes \Lambda(N)\) (for a proof see [1]). We will refer to it as the conformal superalgebra of type \(K\).
On \( K \) the \( \lambda \)-bracket for \( f, g \in \Lambda(N) \), \( f = \xi_{i_1} \cdots \xi_{i_r} \) and \( g = \xi_{j_1} \cdots \xi_{j_s} \), is given by (see \[2, 12\]):

\[
[f, g] = (r - 2) \partial (fg) + (-1)^r \sum_{i=1}^{N} (\partial_i f)(\partial_i g) + \lambda (r + s - 4) fg.
\]

The associated annihilation superalgebra is (see \[2, 12\]):

\[
\text{A}(K) = K(1, N)_+.
\]

We adopt the following notation: we denote by \( \mathcal{I} \) the set of finite sequences of elements in \( \{1, \ldots, N\} \); we will write \( I = i_1 \cdots i_r \) instead of \( I = (i_1, \ldots, i_r) \). Given \( I = i_1 \cdots i_r \) and \( J = j_1 \cdots j_s \), we will denote \( i_1 \cdots i_r j_1 \cdots j_s \) by \( IJ \); if \( I = i_1 \cdots i_r \in \mathcal{I} \) we let \( \xi_I = \xi_{i_1} \cdots \xi_{i_r} \) and \( |I| = |I| = r \). We denote by \( \mathcal{I}_\neq \) the subset of \( \mathcal{I} \) of sequences with distinct entries and by \( \mathcal{I}_\leq \) the subset of \( \mathcal{I}_\neq \) of increasingly ordered sequences. We focus on \( N = 6 \). We let \( \xi_* = \xi_{123456} \).

Following [5], for \( \xi_I \in \Lambda(6) \) we define the modified Hodge dual \( \xi_I^* \) to be the unique monomial such that \( \xi_I \xi_I^* = \xi_* \). We extend the definition of modified Hodge dual to elements \( \sum k,l \alpha_{k,l} t^k \xi_I \in \Lambda(1, 6)_+ \) letting \( (\sum k,l \alpha_{k,l} t^k \xi_I)^* = \sum k,l \alpha_{k,l} t^k \xi_I^* \).

The conformal superalgebra \( CK_6 \) is the subalgebra of \( K_6 \) defined by (see construction in [5]):

\[
CK_6 = \mathbb{C}[\partial] - \text{span} \left\{ \xi_L - i(-1)^{|L|/2} (|\partial|)^{3-|L|} \xi_L^* : L \in \mathcal{I}_\neq, 0 \leq |L| \leq 3 \right\}.
\]

We introduce the linear operator \( A : K(1, 6)_+ \to K(1, 6)_+ \):

\[
A(t^k \xi_L) = (-1)^{|L|/2} \left( \frac{d}{dt} \right)^{3-|L|} (t^k \xi_L)^*,
\]

where \( \left( \frac{d}{dt} \right)^{-1} \) indicates integration with respect to \( t \) (i.e. it sends \( t^k \) to \( t^{k+1}/(k+1) \)) and \( A \) is extended by linearity (cf. Remark 5.3.2 in [5]). The annihilation superalgebra associated with \( CK_6 \), that we will denote by \( \mathfrak{g} \), is the subalgebra of \( K(1, 6)_+ \) given by the image of \( Id - iA \); it is isomorphic to the exceptional Lie superalgebra \( E(1, 6) \) (see [5, 8, 17, 14]). The bracket on \( \mathfrak{g} \) is given by \([\cdot, \cdot]\).

**Remark 3.1.** We point out that, \( \mathfrak{g} \) is in bijective correspondence with the span of elements \( (Id - iA)(t^k \xi_L) \) with \( L \in \mathcal{I}_\neq, |L| \leq 3, k \geq 0 \). Indeed for \( L \in \mathcal{I}_\neq \), with \(|L| > 3\):

\[
(Id - iA)(t^k \xi_L) = (Id - iA) \left( -i(-1)^{|L|/2} \frac{t^{k+|L|-3}}{k(k+1) \cdots (k+|L|-3)} \xi_L^* \right).
\]

The map \( A \) preserves the \( \mathbb{Z} \)-grading, then \( \mathfrak{g} \) inherits the \( \mathbb{Z} \)-grading. The homogeneous components of non-positive degree of \( \mathfrak{g} \) and \( K(1, 6)_+ \) coincide and are:

\[
\mathfrak{g}_{-2} = \langle 1 \rangle, \\
\mathfrak{g}_{-1} = \langle \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6 \rangle, \\
\mathfrak{g}_0 = \langle t, \xi_{ij} : 1 \leq i, j \leq 6 \rangle.
\]

The annihilation superalgebra \( \mathfrak{g} \) satisfies \( \mathbf{L1}, \mathbf{L2}, \mathbf{L3} \): \( \mathbf{L1} \) is straightforward; \( \mathbf{L2} \) follows by Remark \[2, 11\] since \( t \) is a grading element for \( \mathfrak{g} \); \( \mathbf{L3} \) follows from the choice \( \Theta := -1/2 \in \mathfrak{g}_{-2} \). Let us focus on \( \mathfrak{g}_0 = \langle t, \xi_{ij} : 1 \leq i < j \leq 6 \rangle \cong \mathfrak{Cl} \oplus \mathfrak{so}(6), \) where \( \mathfrak{so}(6) \) is the Lie algebra of \( 6 \times 6 \) skew-symmetric matrices and \( \xi_{ij} \in \mathfrak{g}_0 \) corresponds to \( E_{ji} - E_{ij} \in \mathfrak{so}(6) \). We recall
the following notation from [3]. We choose as basis of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{so}(6)$ the elements:

$$H_1 = -i\xi_{12}, 
H_2 = -i\xi_{34}, 
H_3 = -i\xi_{56}.$$ 

Let $\varepsilon_j \in \mathfrak{h}^*$ such that $\varepsilon_j(H_k) = \delta_{jk}$. The roots are $\Delta = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq l < j \leq 3\}$, the positive roots are $\Delta^+ = \{\varepsilon_l \pm \varepsilon_j : 1 \leq l < j \leq 3\}$ and the simple roots are $\Pi = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_2 + \varepsilon_3\}$. The root decomposition is $\mathfrak{so}(6) = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$, where $\mathfrak{g}_\alpha = \mathbb{C}\mathfrak{e}_\alpha$ and the $\mathfrak{e}_\alpha$’s are, for $1 \leq l < j \leq 3$:

$$E_{\varepsilon_l - \varepsilon_j} = -\xi_{2l-1,2j-1} - \xi_{2l,2j} - i\xi_{2l-1,2j} + i\xi_{2l,2j-1},$$

$$E_{\varepsilon_l + \varepsilon_j} = -\xi_{2l-1,2j-1} + \xi_{2l,2j} + i\xi_{2l-1,2j} + i\xi_{2l,2j-1},$$

$$E_{-(\varepsilon_l - \varepsilon_j)} = -\xi_{2l-1,2j-1} - \xi_{2l,2j} + i\xi_{2l-1,2j} - i\xi_{2l,2j-1},$$

$$E_{-(\varepsilon_l + \varepsilon_j)} = -\xi_{2l-1,2j-1} + \xi_{2l,2j} - i\xi_{2l-1,2j} - i\xi_{2l,2j-1}.$$ 

We denote by $N_{\mathfrak{g}_0}$ the nilpotent subalgebra $\oplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. 

We introduce the following notation. Given a proposition $P$, we let

$$\chi_P = \begin{cases} 
1 & \text{if } P \text{ is true,} \\
0 & \text{if } P \text{ is false.}
\end{cases}$$

From now on $F$ will be a finite-dimensional irreducible $\mathfrak{g}_0$-module, such that $\mathfrak{g}_{>0}$ acts trivially on it. We point out that $\text{Ind}(F) \cong \mathbb{C}[\Theta] \otimes \Lambda(6) \otimes F$. Indeed, let us denote by $\eta_i$ the image in $U(\mathfrak{g})$ of $\xi_i \in \Lambda(6)$, for all $i \in \{1,2,3,4,5,6\}$. In $U(\mathfrak{g})$ we have that $\eta_i^2 = \Theta$, for all $i \in \{1,2,3,4,5,6\}$: since $[\xi_i, \xi_i] = -1$ in $\mathfrak{g}$, we have $\eta_i\eta_i = -\eta_i\eta_i - 1$ in $U(\mathfrak{g})$. We will make the following abuse of notation: if $I, J \in \mathcal{I}_\neq$ we will denote by $I \cap J$ the increasingly ordered sequence whose elements are the elements of the intersection of the underlying sets of $I$ and $J$. Given $I = i_1, \ldots, i_k \in \mathcal{I}_\neq$, we will use the notation $\eta_I$ to denote the element $\eta_{i_1} \cdots \eta_{i_k} \in U(\mathfrak{g}_{<0})$ and we will denote $|\eta_I| = |I| = k$. We will denote $\eta_\star = \eta_{123456}$. Given $I, J \in \mathcal{I}_\neq$, we define:

$$\xi_i \star \eta_I = \chi_{I \cap J = \emptyset} \eta_I \eta_I,$$

$$\eta_I \star \xi_i = \chi_{I \cap J = \emptyset} \eta_I \eta_1.$$ 

We will also use the following notation: if $i_1, \ldots, i_k \in \mathcal{I}_\neq$ and $i \in \{1,2,3,4,5,6\}$ we let

$$\partial_{i_1, \ldots, i_k} = \begin{cases} 
(-1)^{j+1}\eta_{i_1, \ldots, i_j, \ldots, i_k} & \text{if } i = i_j \text{ for some } j \\
0 & \text{otherwise.}
\end{cases}$$

and for $a \in \mathbb{C}, I = (i_1, i_2, \ldots, i_k), J \in \mathcal{I}_\neq$:

$$\partial_{i_1, i_2, \ldots, i_k} \eta_I J = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} \eta_I J,$$

$$\partial_{i_1, i_2, \ldots, i_k} \eta_I J = a \partial_{i_1} \eta_I J,$$

$$\partial_{i_1, i_2, \ldots, i_k} \eta_\star = \eta_\star,$$

$$\partial_{i_1, i_2, \ldots, i_k} \xi_I J = \partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} \xi_I;$$

$$\partial_{i_1, i_2, \ldots, i_k} \xi_I J = a \partial_{i_1} \xi_I J,$$

$$\partial_{i_1, i_2, \ldots, i_k} \xi_\star = \xi_\star.$$ 

We extend the definition of modified Hodge dual to the elements of $U(\mathfrak{g}_{<0})$ in the following way: for $\eta_I \in U(\mathfrak{g}_{<0})$, we let $\eta_I^* \eta_I^* \eta_I^* \eta_I^* \eta_I^* \eta_I^*$ to be the unique monomial such that $\xi_I \star \eta_I^* = \eta_I$. 

Moreover we define the Hodge dual of elements of $\Lambda(6)$ (resp. $U(\mathfrak{g}_{<0})$) in the following way: for $\xi_I \in \Lambda(6)$ (resp. $\eta_I \in U(\mathfrak{g}_{<0})$), we let $\xi_I$ (resp. $\eta_I$) to be the unique monomial such that
\[ \overline{\xi}_I \xi_I = \xi_s \] (resp. \( \overline{\eta} \ast \xi_I = \eta_* \)). Then we extend by linearity the definition of Hodge dual to elements \( \sum \alpha_I \xi_I \) (resp. \( \sum \alpha_I \eta_I \)) and we set \( t^k \overline{\xi}_I = t^k \overline{\xi}_I \) (resp. \( \Theta^k \eta_I = \Theta^k \eta_I \)). We point out that for \( \eta_I \in U(g_{<0}) \), \( \overline{\eta} \) = \((-1)^{|I|} \eta_I \).

In order to study singular vectors, it is important to find an explicit form for the action of \( g \) on \( \text{Ind}(F) \) using the \( \lambda \)-action notation \([3]\). Due to the fact that the homogeneous components of non-positive degree of \( E(1,6) \) are the same as those of \( K(1,6)_+ \), the \( \lambda \)-action is given by restricting the \( \lambda \)-action for \( K(1,6)_+ \):

\[ \xi_{L \lambda}(g \otimes v) = \sum_{j \geq 0}^{\lambda^j} t^j \xi_{L \lambda}(g \otimes v), \]

for \( L \in \mathcal{I} \), \( g \otimes v \in \text{Ind}(F) \), described explicitly in Theorem 4.1 in \([2]\). We recall the following result proved in \([2]\), Theorem 4.3 for the \( \lambda \)-action in the case of \( K(1,6)_+ \).

**Proposition 3.2** (\([2]\)). Let \( T \) be the vector spaces isomorphism \( T : \text{Ind}(F) \rightarrow \text{Ind}(F) \), \( g \otimes v \mapsto \overline{g} \otimes v \), for all \( g \otimes v \in \text{Ind}(F) \cong \mathbb{C}[\Theta] \otimes \Lambda(6) \otimes F \). Let \( L, I \in \mathcal{I}_\# \). Then

\[
T \circ \xi_{L \lambda} \circ T^{-1}(\eta_I \otimes v) = (-1)^{\frac{|L|(|L|+1)}{2} + |L||I|} \left\{ \left( |L| - 2 \right) \Theta(\xi_L \ast \eta_I) \otimes v - (-1)^{|L|} \sum_{i=1}^{6} (\partial_i \xi_L \ast \partial_i \eta_I) \otimes v - \sum_{r<s} (\partial_{rs} \xi_L \ast \eta_I) \otimes \xi_{sr}.v \\
+ \lambda \left( (\xi_L \ast \eta_I) \otimes t.v - (-1)^{|L|} \sum_{i=1}^{6} \partial_i (\xi_{L_i} \ast \eta_I) \otimes v + (-1)^{|L|} \sum_{i \neq j} (\partial_i \xi_{L_i} \ast \eta_I) \otimes \xi_{ji}.v \right) \\
- \lambda^2 \sum_{i<j} (\xi_{L_{ij}} \ast \eta_I) \otimes \xi_{ji}.v \right\}.
\]

The following lemma allows to compute \( \xi_{L \lambda}(\Theta^k \xi_I \otimes v) \).

**Lemma 3.3.** Let \( L, I \in \mathcal{I}_\# \) and \( k \geq 0 \). The following holds:

\[ \xi_{L \lambda}(\Theta^k \xi_I \otimes v) = (\Theta + \lambda)^k(\xi_{L \lambda} \xi_I \otimes v). \]

**Proof.** The proof is analogous to Lemma 5.11 in \([1]\). \( \square \)

Let \( \vec{m} \) be a vector of the \( E(1,6) \)-module \( \text{Ind}(F) \). From \([3]\) we know that \( \vec{m} \) is a highest weight singular vector if and only if:

**S0:** \( N_{\vec{m}, \vec{m}} = 0 \).

**S1:** For all \( L \in \mathcal{I}_\# \), with \( 0 \leq |L| \leq 3 \):

\[ \frac{d^2}{d\lambda^2} \left( \xi_{L \lambda} \vec{m} - i( -1 )^{|L|(|L|+1)} \lambda^{3-|L|} (\xi^*_{L \lambda} \vec{m}) \right) = 0. \]

**S2:** For all \( L \in \mathcal{I}_\# \), with \( 1 \leq |L| \leq 3 \):

\[ \frac{d}{d\lambda} \left( \xi_{L \lambda} \vec{m} - i(-1)^{|L|(|L|+1)} \lambda^{3-|L|} (\xi^*_{L \lambda} \vec{m}) \right) \big|_{\lambda=0} = 0. \]

**S3:** For all \( L \in \mathcal{I}_\# \), with \( |L| = 3 \):

\[ \left( \xi_{L \lambda} \vec{m} - i(-1)^{|L|(|L|+1)} \lambda^{3-|L|} (\xi^*_{L \lambda} \vec{m}) \right) \big|_{\lambda=0} = 0. \]
In particular condition **S0** is equivalent to impose that \( \vec{m} \) is a highest weight vector; conditions **S1-S3** are equivalent to impose that \( \vec{m} \) is a singular vector. Indeed condition **S1** is equivalent to

\[
\sum_{j \geq 2} j(j-1)\frac{\lambda_j}{j!}(t^j \xi_L)\vec{m} - i(-1)^{(\frac{i}{2}(|L|+1))} \sum_{j \geq 2} (3 - |L| + j)(2 - |L| + j)\frac{\lambda_1^{1-|L|+j}}{j!}(t^j \xi_L^*)\vec{m} = 0,
\]

which implies \((t^j \xi_L - i(-1)^{(\frac{i}{2}(|L|+1))} t^j \xi_L^*)\vec{m} = 0\) for all \( L \in \mathcal{I}_k \), with \( 0 \leq |L| \leq 3 \) and \( j \geq 2 \).

Condition **S2** is equivalent to \((t^j \xi_L - i(-1)^{(\frac{i}{2}(|L|+1))} t^j \xi_L^*)\vec{m} = 0\) for all \( L \in \mathcal{I}_k \), with \( 1 \leq |L| \leq 3 \).

Condition **S3** is equivalent to \((t^j \xi_L - i\xi_L^*)\vec{m} = 0\) for all \( L \in \mathcal{I}_k \) such that \( |L| = 3 \). Therefore, by Remark **3.4**, **S1-S3** are equivalent to impose that \( \vec{m} \) is a singular vector.

**Remark 3.4.** We point out that, by the previous conditions, a vector \( \vec{m} \in \text{Ind}(F) \) is a highest weight singular vector if and only if it satisfies **S0-S3**. Since \( T \), defined as in Proposition **3.2**, is an isomorphism, the fact that \( \vec{m} \in \text{Ind}(F) \) satisfies **S0-S3** for \((T \circ (\xi_L - i(-1)^{\frac{i}{2}(|L|+1)} \lambda^{3-|L|}\xi_L^*) \circ T^{-1})T(\vec{m})\), using the expression given by Proposition **3.2**

Therefore in the following results we will consider a vector \( T(\vec{m}) \in \text{Ind}(F) \) and we will impose that the expression for \((T \circ (\xi_L - i(-1)^{\frac{i}{2}(|L|+1)} \lambda^{3-|L|}\xi_L^*) \circ T^{-1})T(\vec{m}) = (T \circ (\xi_L - i(-1)^{\frac{i}{2}(|L|+1)} \lambda^{3-|L|}\xi_L^*))\vec{m}\) given by Proposition **3.2** satisfies conditions **S0-S3**. We will have that \( \vec{m} \) is a highest weight singular vector.

Motivated by Remark **3.4** we consider a singular vector \( \vec{m} \in \text{Ind}(F) \) such that:

\[
T(\vec{m}) = \sum_{k=0}^{N} \Theta^k \sum_{I \in \mathcal{I}_<} \eta_I \otimes v_{I,k}.
\]

We will denote \( v_{123456,k} = v_{*,k} \) for all \( k \).

4. **Main result**

In [3] the following Lemma is stated without proof (Lemma 4.4 in [3]). In particular, this Lemma is used in [4] to completely classify the highest weight singular vectors of finite Verma modules over \( g \).

**Lemma 4.1.** Let \( \vec{m} \in \text{Ind}(F) \) be a singular vector, such that \( T(\vec{m}) \) is written as in [3]. Then the degree of \( \vec{m} \) with respect to \( \Theta \) is at most 2. Moreover, \( T(\vec{m}) \) has the following form:

\[
T(\vec{m}) = \Theta^2 \sum_{|I| \geq 5} \eta_I \otimes v_{I,2} + \Theta \sum_{|I| \geq 3} \eta_I \otimes v_{I,1} + \sum_{|I| \geq 1} \eta_I \otimes v_{I,0}.
\]

The rest of this section is the dedicated to the proof of Lemma **4.1**

**Lemma 4.2.** A singular vector \( \vec{m} \in \text{Ind}(F) \), such that \( T(\vec{m}) \) is written as in [3], has degree at most 4 with respect to \( \Theta \).
Proof. By Remark 3.4, condition S1 for \( \xi_1 \) reduces to:

\[
\frac{d^2}{d\lambda^2} \left( T(\xi_1, \tilde{m} + i\lambda^2(\xi_{23456}, \tilde{m})) \right) = 0.
\]

Using Proposition 3.2 and Lemma 3.3 the previous equation reduces to:

\[
0 = \frac{d^2}{d\lambda^2} \sum_{k=0}^{N} \sum_{l} (\lambda + \Theta)^{k} (-1)^{1+|l|} \left\{ -\Theta(\xi_1 \star \eta_l) \otimes v_{l,k} + \partial_1 \eta_l \otimes v_{l,k} + \lambda \left( (\xi_1 \star \eta_l) \otimes t.v_{l,k} - \sum_{l<j} (\xi_{1l} \star \eta_l) \otimes \xi_{jl} \right) \right\}
\]

\[
= \sum_{k=0}^{N} \sum_{l} (\lambda + \Theta)^{k} (-1)^{1+|l|} \left\{ -2 \sum_{l<j} (\xi_{1l} \star \eta_l) \otimes \xi_{jl} \right\}
\]

\[
+ 2 \sum_{k=2}^{N} \sum_{l} k(k-1)(\lambda + \Theta)^{k-2} (-1)^{1+|l|} \left\{ -\Theta(\xi_1 \star \eta_l) \otimes v_{l,k} + \partial_1 \eta_l \otimes v_{l,k}
\]

\[
- 2\lambda \sum_{l<j} (\xi_{1l} \star \eta_l) \otimes \xi_{jl} \right\}
\]

\[
+ \sum_{k=0}^{N} \sum_{l} (\lambda + \Theta)^{k} (-1)^{1+|l|} \left\{ 3\Theta(\xi_{23456} \star \eta_l) \otimes v_{l,k} + \sum_{l=1}^{6} \partial_l (\xi_{23456} \star \partial_l \eta_l) \otimes v_{l,k} - \sum_{r<s} (\partial_{rs} \xi_{23456} \star \eta_l) \otimes \xi_{sr} \right\}
\]

\[
+ \lambda \left( (\xi_1 \star \eta_l) \otimes t.v_{l,k} - \sum_{j \neq l} (\xi_{1l} \star \eta_l) \otimes \xi_{jl} \right) \right\}
\]

\[
+ \sum_{k=0}^{N} \sum_{l} (-1)^{1+|l|} \left( 4i\lambda(\lambda + \Theta)^{k} + 4i\lambda k(\lambda + \Theta)^{k-1} + i\lambda^2 k(k-1)(\lambda + \Theta)^{k-2} \right).
\]
We consider the previous expression as a polynomial in $\lambda$ and $\lambda + \Theta$, by writing $\Theta$ as $(\lambda + \Theta) - \lambda$. We look at the coefficient of $\lambda^3(\lambda + \Theta)^s$, for a fixed $s \geq 0$, in (7) and we obtain that:

\[
\sum_{I} (-1)^{|I|} \left[ \right. \\
-3(\xi_{23456} \ast \eta_I) \otimes v_{I,s+2} + (\xi_{23456} \ast \eta_I) \otimes t.v_{I,s+2} \\
+ \sum_{l=1}^6 \partial_l(\xi_{23456l} \ast \eta_I) \otimes v_{I,s+2} - \sum_{l \neq j} (\partial_l \xi_{23456j} \ast \eta_I) \otimes \xi_{jl}.v_{I,s+2} \left. \right] = 0.
\]

We consider the coefficient of $\lambda^2(\lambda + \Theta)^s$, for a fixed $s \geq 1$, in (8) and we obtain that:

\[
\sum_{I} (-1)^{|I|} (s+1) \left\{ - (s+2) \sum_{l<j} (\xi_{1lj} \ast \eta_I) \otimes \xi_{jl}.v_{I,s+2} \\
+ 4i \left[ -3(\xi_{23456} \ast \eta_I) \otimes v_{I,s+1} + (\xi_{23456} \ast \eta_I) \otimes t.v_{I,s+1} \\
+ \sum_{l=1}^6 \partial_l(\xi_{23456l} \ast \eta_I) \otimes v_{I,s+1} - \sum_{l \neq j} (\partial_l \xi_{23456j} \ast \eta_I) \otimes \xi_{jl}.v_{I,s+1} \right] \\
+ is3(\xi_{23456} \ast \eta_I) \otimes v_{I,s+1} + i(s+2) \left\{ \sum_{l=1}^6 (\partial_1 \xi_{23456l} \ast \partial_l \eta_I) \otimes v_{I,s+2} - \sum_{r<p} (\partial_{rp} \xi_{23456} \ast \eta_I) \otimes \xi_{pr}.v_{I,s+2} \right\} \\
+ 2i \left\{ (\xi_{23456} \ast \eta_I) \otimes t.v_{I,s+1} + \sum_{l=1}^6 \partial_l(\xi_{23456l} \ast \eta_I) \otimes v_{I,s+1} - \sum_{l \neq j} (\partial_l \xi_{23456j} \ast \eta_I) \otimes \xi_{jl}.v_{I,s+1} \right\} \right\} = 0.
\]

Using (7), we obtain that the sum over $I$ of the terms in the second and third rows is zero, and the sum over $I$ of the last row is equal to $\sum_I (-1)^{|I|} 6i(\xi_{23456} \ast \eta_I) \otimes v_{I,s+1}$. Hence for $s \geq 1$:

\[
\sum_{I} (-1)^{|I|} \left\{ - \sum_{l<j} (\xi_{1lj} \ast \eta_I) \otimes \xi_{jl}.v_{I,s+2} + 3i(\xi_{23456} \ast \eta_I) \otimes v_{I,s+1} \\
+ i \left\{ \sum_{l=1}^6 (\partial_1 \xi_{23456l} \ast \partial_l \eta_I) \otimes v_{I,s+2} - \sum_{r<p} (\partial_{rp} \xi_{23456} \ast \eta_I) \otimes \xi_{pr}.v_{I,s+2} \right\} \right\} = 0.
\]

We consider the coefficient of $\lambda(\lambda + \Theta)^s$, for a fixed $s \geq 2$, in (9) and we obtain that:

\[
\sum_{I} (-1)^{|I|} \left\{ - 4(s+1) \sum_{l<j} (\xi_{1lj} \ast \eta_I) \otimes \xi_{jl}.v_{I,s+1} \\
+ (s+1)(s+2) \left[ (\xi_1 \ast \eta_I) \otimes v_{I,s+2} + (\xi_1 \ast \eta_I) \otimes t.v_{I,s+2} + \sum_{l=1}^6 \partial_l(\xi_{1l} \ast \eta_I) \otimes v_{I,s+2} \\
- \sum_{j \neq 1} (\xi_j \ast \eta_I) \otimes \xi_{j1}.v_{I,s+2} \right] \\
+ 2i \left[ -3(\xi_{23456} \ast \eta_I) \otimes v_{I,s} + (\xi_{23456} \ast \eta_I) \otimes t.v_{I,s} \right] \right\} = 0.
\]
\[ + \sum_{l=1}^{6} \partial_l (\xi_{23456l} \ast \eta_l) \otimes v_{l,s} - \sum_{l \neq j} (\partial_l \xi_{23456j} \ast \eta_l) \otimes \xi_{jl} v_{l,s} \]

\[ + 12i s(\xi_{23456} \ast \eta_l) \otimes v_{l,s} + 4i(s + 1) \left[ \sum_{l=1}^{6} (\partial_l \xi_{23456} \ast \partial_l \eta_l) \otimes v_{l,s+1} - \sum_{r < p} \partial_{rp}(\xi_{23456} \ast \eta_l) \otimes \xi_{pr} v_{l,s+1} \right] \]

\[ + 4i \left[ (\xi_{23456} \ast \eta_l) \otimes t v_{l,s} + \sum_{l=1}^{6} \partial_l (\xi_{23456l} \ast \eta_l) \otimes v_{l,s} - \sum_{l \neq j} (\partial_l \xi_{23456l} \ast \eta_l) \otimes \xi_{jl} v_{l,s} \right] \right) = 0. \]

We use (7) to point out that the sum over \( I \) of the terms in the fourth and fifth rows is zero. Moreover, due to (7), the sum over \( I \) of the terms in the last row is equal to \( \sum_{I}(-1)^{1+|I|}12i(\xi_{23456} \ast \eta_l) \otimes v_{l,s} \). Finally, the sum of \( \sum_{I}(-1)^{1+|I|}12i(\xi_{23456} \ast \eta_l) \otimes v_{l,s} \) plus the sum over \( I \) of the terms from the first and sixth rows is zero due to (8).

Therefore for \( s \geq 2 \):

\[ \sum_{I}(-1)^{1+|I|} \left[ (\xi_1 \ast \eta_l) \otimes v_{l,s+2} + (\xi_1 \ast \eta_l) \otimes t v_{l,s+2} + \sum_{l=1}^{6} \partial_l (\xi_1 \ast \eta_l) \otimes v_{l,s+2} \right. \]

\[ - \sum_{j \neq 1} (\xi_j \ast \eta_l) \otimes \xi_{j1} v_{l,s+2} \right] = 0. \]

Finally, we consider the coefficient of \( (\lambda + \Theta)^s \), for a fixed \( s \geq 3 \), in (6) and we obtain that:

\[ \sum_{I}(-1)^{1+|I|} \left\{ - 2 \sum_{l<j} (\xi_{1lj} \ast \eta_l) \otimes \xi_{j1} v_{l,s} \right. \]

\[ + 2(s + 1) \left[ (\xi_1 \ast \eta_l) \otimes t v_{l,s+1} + \sum_{l=1}^{6} \partial_l (\xi_1 \ast \eta_l) \otimes v_{l,s+1} - \sum_{j \neq 1} (\xi_j \ast \eta_l) \otimes \xi_{j1} v_{l,s+1} \right] \]

\[ - s(s + 1)(\xi_1 \ast \eta_l) \otimes v_{l,s+1} + (s + 1)(s + 2)\partial_1 \eta_l \otimes v_{l,s+2} \]

\[ + 6i(\xi_{23456} \ast \eta_l) \otimes v_{l,s+1} + 2i \left[ \sum_{l=1}^{6} (\partial_l \xi_{23456} \ast \partial_l \eta_l) \otimes v_{l,s} - \sum_{r < p} (\partial_{rp} \xi_{23456} \ast \eta_l) \otimes \xi_{pr} v_{l,s} \right] \right) = 0. \]

Using (8), we observe that the sum over \( I \) of the terms from the first and the last row is zero. Using (9) we obtain that the sum of the terms from the second row is equal to \( -2 \sum_{I} (s + 1)(-1)^{1+|I|}(\xi_1 \ast \eta_l) \otimes v_{l,s+1} \). Thus for \( s \geq 3 \):

\[ \sum_{I} (-1)^{1+|I|}(\xi_1 \ast \eta_l) \otimes v_{l,s+1} - \partial_1 \eta_l \otimes v_{l,s+2} = 0. \]

By linear independence, we obtain:

\[ \sum_{I} (-1)^{1+|I|}(\xi_1 \ast \eta_l) \otimes v_{l,s+1} = 0. \]

Therefore \( v_{I,k} = 0 \) for \( |I| \leq 5 \), \( 1 \notin I \) and \( k \geq 4 \). We point out that \( 1 \notin I \) is not necessary, since we could have chosen at the beginning any \( \xi_i \) instead of \( \xi_1 \). Finally, the coefficient of \( \eta_{l}^* \) in (10) is \( v_{s,s+2} \). Hence \( v_{s,k} = 0 \) if \( k \geq 5 \).
By Lemma \[\text{4.2}\] for a singular vector \(\vec{m}\), \(T(\vec{m})\) has the following form:
\[
T(\vec{m}) = \Theta^4 \sum_{l \in I_<} \eta_l \otimes v_{l,4} + \Theta^3 \sum_{l \in I_<} \eta_l \otimes v_{l,3} + \Theta^2 \sum_{l \in I_<} \eta_l \otimes v_{l,2} + \Theta \sum_{l \in I_<} \eta_l \otimes v_{l,1} + \sum_{l \in I_<} \eta_l \otimes v_{l,0}.
\]

Following \[\text{3}\], we write the \(\lambda\)-action in the following way, using Proposition \[\text{3.2}\] and Lemma \[\text{3.3}\].
\[
\begin{align*}
T(\xi_L, \lambda \vec{m}) &= b_0(\xi_L) + \lambda(B_0(\xi_L) - a_0(\xi_L)) + \lambda^2 C_0(\xi_L) \\
&\quad + (\lambda + \Theta)[a_0(\xi_L) + b_1(\xi_L)] + (\lambda + \Theta)\lambda(B_1(\xi_L) - a_1(\xi_L)) + (\lambda + \Theta)\lambda^2 C_1(\xi_L) \\
&\quad + (\lambda + \Theta)^2[a_1(\xi_L) + b_2(\xi_L)] + (\lambda + \Theta)^2\lambda(B_2(\xi_L) - a_2(\xi_L)) + (\lambda + \Theta)^2\lambda^2 C_2(\xi_L) \\
&\quad + (\lambda + \Theta)^3[a_2(\xi_L) + b_3(\xi_L)] + (\lambda + \Theta)^3\lambda(B_3(\xi_L) - a_3(\xi_L)) + (\lambda + \Theta)^3\lambda^2 C_3(\xi_L) \\
&\quad + (\lambda + \Theta)^4[a_3(\xi_L) + b_4(\xi_L)] + (\lambda + \Theta)^4\lambda(B_4(\xi_L) - a_4(\xi_L)) + (\lambda + \Theta)^4\lambda^2 C_4(\xi_L) \\
&\quad + (\lambda + \Theta)^5a_4(\xi_L),
\end{align*}
\]
where the coefficients \(a_p(\xi_L), b_p(\xi_L), B_p(\xi_L), C_p(\xi_L)\) depend on \(\xi_L\) for all \(0 \leq p \leq 4\) and are explicitly defined as follows. For all \(0 \leq p \leq 4\) we let:
\[
\begin{align*}
a_p(\xi_L) &= \sum_i (-1)^{(iL(|L|+1)/2)+|L||I|} \left([|L| - 2](\xi_L \ast \eta_i) \otimes v_{l,p}\right); \\
b_p(\xi_L) &= \sum_i (-1)^{(iL(|L|+1)/2)+|L||I|} \left(- (|L| - 1)^6 \sum_{i=1}^6 (\partial_i \xi_L \ast \partial_i \eta_i) \otimes v_{l,p} - \sum_{r<s} (\partial_{rs} \xi_L \ast \eta_i) \otimes \xi_{sr,vI,p}\right); \\
B_p(\xi_L) &= \sum_i (-1)^{(iL(|L|+1)/2)+|L||I|} \left((\xi_L \ast \eta_i) \otimes t.v_{l,p} - (|L| - 1)^6 \sum_{i=1}^6 \partial_i (\xi_Li \ast \eta_i) \otimes v_{l,p}\right) + (-1)^{|L|} \sum_{i \neq j} (\partial_i \xi_Lj \ast \eta_i) \otimes \xi_{ji,vI,p}; \\
C_p(\xi_L) &= \sum_i (-1)^{(iL(|L|+1)/2)+|L||I|} \left(- \sum_{i<j} (\xi_Lij \ast \eta_i) \otimes \xi_{ji,vI,p}\right).
\end{align*}
\]

We will write \(a_p\) instead of \(a_p(\xi_L)\) if there is no risk of confusion, and similarly for the others. Analogously:
\[
\begin{align*}
T(\xi_L^*, \lambda \vec{m}) &= bd_0(\xi_L) + \lambda(Bd_0(\xi_L) - ad_0(\xi_L)) + \lambda^2 Cd_0(\xi_L) \\
&\quad + (\lambda + \Theta)[ad_0(\xi_L) + bd_1(\xi_L)] + (\lambda + \Theta)\lambda(Bd_1(\xi_L) - ad_1(\xi_L)) + (\lambda + \Theta)\lambda^2 Cd_1(\xi_L) \\
&\quad + (\lambda + \Theta)^2[ad_1(\xi_L) + bd_2(\xi_L)] + (\lambda + \Theta)^2\lambda(Bd_2(\xi_L) - ad_2(\xi_L)) + (\lambda + \Theta)^2\lambda^2 Cd_2(\xi_L) \\
&\quad + (\lambda + \Theta)^3[ad_2(\xi_L) + bd_3(\xi_L)] + (\lambda + \Theta)^3\lambda(Bd_3(\xi_L) - ad_3(\xi_L)) + (\lambda + \Theta)^3\lambda^2 Cd_3(\xi_L) \\
&\quad + (\lambda + \Theta)^4[ad_3(\xi_L) + bd_4(\xi_L)] + (\lambda + \Theta)^4\lambda(Bd_4(\xi_L) - ad_4(\xi_L)) + (\lambda + \Theta)^4\lambda^2 Cd_4(\xi_L) \\
&\quad + (\lambda + \Theta)^5a_4(\xi_L),
\end{align*}
\]
Lemma 4.3. Let $\vec{m}$ be a singular vector, such that $T(\vec{m})$ is written as in (11).

(i) Condition $S2$ for $L = j$ implies:
\[
4a_4 + B_4 = 3a_3 + B_3 + 4b_4 = 2a_2 + B_2 + 3b_3 = B_1 + a_1 + 2b_2 = B_0 + b_1 = 0.
\]

(ii) Condition $S2$ for $L = ijk$ implies:
\[
4a_4 + B_4 - i(4ad_4 + Bd_4) = 3a_3 + B_3 + 4b_4 - i(3ad_3 + Bd_3 + 4bd_4)
\]
\[
= 2a_2 + B_2 + 3b_3 - i(2ad_2 + Bd_2 + 3bd_3) = B_1 + a_1 + 2b_2 - i(Bd_1 + ad_1 + 2bd_1)
\]
\[
= B_0 + b_1 - i(Bd_0 + bd_1) = 0.
\]

(iii) Condition $S3$ for $L = ijk$ implies:
\[
a_4 - iad_4 = a_3 + b_4 - i(ad_3 + bd_4) = a_2 + b_3 - i(ad_2 + bd_3) = a_1 + b_2 - i(ad_1 + bd_2)
\]
\[
= a_0 + b_1 - i(ad_0 + bd_1) = b_0 - ibd_0 = 0.
\]

(iv) Condition $S1$ for $|L| = 0$ implies:
\[
\begin{align*}
(14) & \quad C_3 + 4B_4 + 6a_4 = 0, \\
(15) & \quad C_2 + 3a_3 + 3B_3 + 6b_4 = 0, \\
(16) & \quad 2C_3 + 2(B_4 - a_4) - iad_1 - ibd_2 = 0, \\
(17) & \quad 4C_2 + 3B_3 - 3a_3 - 3iad_0 - 3ibd_1 = 0, \\
(18) & \quad C_3 - 2iBd_1 - 2ibd_2 = 0, \\
(19) & \quad C_2 - 6iBd_0 + 3iad_0 - 3ibd_1 = 0, \\
(20) & \quad 10Cd_0 + 4Bd_1 - 3iad_1 + bd_2 = 0.
\end{align*}
\]

Proof. It follows by direct computations using notation (12) and (13). □

Lemma 4.4. Let $\vec{m}$ be a singular vector, such that $T(\vec{m})$ is written as in (11).
Condition $S2$ for $L = j$ implies:
\[
0 = \sum_I (-1)^{1+|I|} \left[ (\xi_j \star \eta_I) \otimes t.v_{I,1} + \sum_{i=1}^6 \partial_i(\xi_{ji} \star \eta_I) \otimes v_{I,1} - \sum_{i \neq l} (\partial_i \xi_{jl} \star \eta_I) \otimes \xi_{li} v_{I,1} \\
- (\xi_j \star \eta_I) \otimes v_{I,1} + 2\partial_j \eta_I \otimes v_{I,2} \right];
\]
\[
0 = \sum_I (-1)^{1+|I|} \left[ (\xi_j \star \eta_I) \otimes t.v_{I,0} + \sum_{i=1}^6 \partial_i(\xi_{ji} \star \eta_I) \otimes v_{I,0} - \sum_{i \neq l} (\partial_i \xi_{jl} \star \eta_I) \otimes \xi_{li} v_{I,0} + \partial_j \eta_I \otimes v_{I,1} \right];
\]
\[
0 = \sum_I (-1)^{1+|I|} \left[ -2(\xi_j \star \eta_I) \otimes v_{I,2} + (\xi_j \star \eta_I) \otimes t.v_{I,2} \right]
\]
By Lemma 4.3, relations \( B \) is

\[
0 = \sum_{l=1}^{6} (-1)^{l+1} \left[ (\xi_{ijk} \ast \eta_l) \otimes v_{l,1} + \sum_{i \neq l} \partial_i (\xi_{ijkl} \ast \eta_l) \otimes v_{l,1} \right] .
\]

Conditions \( S2 \) and \( S3 \) for \( L = ijk \) imply:

\[
0 = \sum_{l=1}^{6} (-1)^{l+1} \left[ (\xi_{ijk} \ast \eta_l) \otimes v_{l,0} + \sum_{i \neq l} \partial_i (\xi_{ijkl} \ast \eta_l) \otimes v_{l,0} \right] .
\]

Proof. These are the explicit expression of some of equations of Lemma 4.3. Equation (21) is \( B_1(j) + a_1(j) + 2b_2(j) = 0 \), \( (22) \) is \( B_0(j) + b_1(j) = 0 \), \( (23) \) is \( 2a_2(j) + B_2(j) + 3b_3(j) = 0 \). By Lemma 4.3, relations \( S2 \) and \( S3 \) for \( L = ijk \) imply, taking linear combinations:

\[
B_1 - a_1 - i(Bd_1 - ad_1) = a_0 - B_0 - i(ad_0 - Bd_0) = -a_2 + B_2 - i(-ad_2 + Bd_2) = 0 .
\]

Equation (24) is \( B_1(ijk) - a_1(ijk) - i(Bd_1(ijk) - ad_1(ijk)) = 0 \), equation (25) is \( a_0(ijk) - B_0(ijk) - i(ad_0(ijk) - Bd_0(ijk)) = 0 \), equation (26) is \( -a_2(ijk) + B_2(ijk) - i(-ad_2(ijk) + Bd_2(ijk)) = 0 \).

Proof of Lemma 4.7. By Lemma 4.2 for a singular vector \( \bar{m} \), \( T(\bar{m}) \) is written as in (11). Let us consider (21) for \( L = j \); the coefficient of \( \eta_j \) is:

\[
t.v_{j,1} - 6v_{j,1} = 0 .
\]
Let us consider (22) for $L = j$; the coefficient of $\eta_j$ is:

$$t.v_{\emptyset,0} - 5v_{\emptyset,0} = 0.$$  

(28)

Let us consider (23) for $L = j$; the coefficient of $\eta_j$ is:

$$t.v_{\emptyset,2} - 7v_{\emptyset,2} = 0.$$  

(29)

The coefficient of 1 in (22) for $L = j$ is $v_{j,1} = 0$. The coefficient of 1 in (21) for $L = j$ is $v_{j,2} = 0$. Now let us consider (24) for $L = ijk$; the coefficient of $\eta_{ijk}$ is $t.v_{\emptyset,1} - 4v_{\emptyset,1} = 0$. Hence, by (27) we deduce $v_{\emptyset,1} = 0$.

Moreover, let us consider (25) for $L = ijk$; the coefficient of $\eta_{ijk}$ is $-t.v_{\emptyset,0} + 4v_{\emptyset,0} = 0$. Hence by (28) we deduce $v_{\emptyset,0} = 0$.

Finally, let us consider (26) for $L = ijk$; the coefficient of $\eta_{ijk}$ is $t.v_{\emptyset,2} - 4v_{\emptyset,2} = 0$. Hence by (29) we deduce $v_{\emptyset,2} = 0$.

So far we have shown that, for all $i \in \{1, 2, 3, 4, 5, 6\}$, $v_{\emptyset,0} = v_{\emptyset,1} = v_{\emptyset,2} = v_{i,1} = v_{i,2} = 0$.

Let us now show that $v_{j,l,1} = 1$ for all $ji \in I_c$. The coefficient of $\eta_l$ in (22) for $L = j$ is $-\eta_l \otimes v_{j,l,1} + \eta_l \otimes \xi_{li}v_{\emptyset,0} = 0$. Therefore $v_{j,l,1} = 0$.

We know by (20) that $bd_2 = -4Bd_1 + 3ad_1$, since $Cd_0(\emptyset) = 0$. Using this relation we have that Equations (14), (16), (18) reduce to:

$$C_3 + 4B_4 + 6a_4 = 2C_3 + 2(B_4 - a_4) - 4iad_1 + 4iBd_1 = C_3 - 6iad_1 + 6iBd_1 = 0.$$  

We consider the following linear combinations of the previous equations:

$$3B_4 + 7a_4 + 2iad_1 - 2iBd_1 = B_4 - a_4 + 4iad_1 - 4iBd_1 = 0.$$

Since $ad_1(\emptyset)$ and $Bd_1(\emptyset)$ involve only terms in $\eta$ with $v_{\emptyset,1}$ that is 0, we obtain $a_4(\emptyset) = 0$. Therefore

$$\sum \eta_l \otimes v_{l,1} = 0.$$

Using linear independence of distinct $\eta_l$’s, we get $v_{l,4} = 0$ for all $l \in I_c$. Now Equations (15), (17), (19) reduce to:

$$C_2 + 3a_3 + 3B_3 = 4C_2 + 3B_3 - 3a_3 - 3iad_0 - 3iBd_1 = C_2 - 6Bd_0 + 3iad_0 - 3iBd_1 = 0.$$  

We observe that $ad_0(\emptyset)$ and $Bd_0(\emptyset)$ involve only terms with $v_{\emptyset,0}$ that is 0, $bd_1(\emptyset)$ involves only terms with $v_{\emptyset,1}, v_{l,1}$ where $|I| = 1, 2$, that are zero. Then these equations reduce to:

$$C_2 + 3a_3 + 3B_3 = 4C_2 + 3B_3 - 3a_3 = C_2 = 0.$$  

Therefore $a_3(\emptyset) = 0$. As before we deduce $v_{l,3} = 0$ for all $l \in I_c$.

Thus we have shown that, for a singular vector $\vec{m}$, $T(\vec{m})$ has the following form:

$$T(\vec{m}) = \Theta^2 \sum_{|I|\geq 2} \eta_l \otimes v_{l,2} + \Theta \sum_{|I|\geq 3} \eta_l \otimes v_{l,1} + \sum_{|I|\geq 1} \eta_l \otimes v_{l,0}.$$  

This means that there are singular vectors $\vec{m}$ of at most degree 8 and, in particular, $T(\vec{m})$ has the following form:

$$T(\vec{m}) = \Theta^2 \sum_{|I|=2} \eta_l \otimes v_{l,2} \text{ degree 8},$$
\[ T(\vec{m}) = \Theta^2 \sum_{|I|=3} \eta_I \otimes v_{I,2} \quad \text{degree 7}, \]
\[ T(\vec{m}) = \Theta^2 \sum_{|I|=4} \eta_I \otimes v_{I,2} \quad \text{degree 6}, \]
\[ T(\vec{m}) = \Theta^2 \sum_{|I|=5} \eta_I \otimes v_{I,2} + \Theta \sum_{|I|=3} \eta_I \otimes v_{I,1} + \sum_{|I|=1} \eta_I \otimes v_{I,0} \quad \text{degree 5}, \]
\[ T(\vec{m}) = \Theta^2 \sum_{|I|=6} \eta_I \otimes v_{I,2} + \Theta \sum_{|I|=4} \eta_I \otimes v_{I,1} + \sum_{|I|=2} \eta_I \otimes v_{I,0} \quad \text{degree 4}, \]
\[ T(\vec{m}) = \Theta \sum_{|I|=5} \eta_I \otimes v_{I,1} + \sum_{|I|=3} \eta_I \otimes v_{I,0} \quad \text{degree 3}, \]
\[ T(\vec{m}) = \Theta \sum_{|I|=6} \eta_I \otimes v_{I,1} + \sum_{|I|=4} \eta_I \otimes v_{I,0} \quad \text{degree 2}, \]
\[ T(\vec{m}) = \sum_{|I|=5} \eta_I \otimes v_{I,0} \quad \text{degree 1}. \]

If we look respectively at vectors of degree 8, 7 and 6, we can use relation \( B_1(j) + a_1(j) + 2b_2(j) = 0 \) from condition S2 for \( L = j \). In both these three cases it reduces to \( b_2(j) = 0 \) since there are no \( v_{I,1} \)'s involved. We get that:
\[ b_2(j) = \sum_I \text{sgn}_I \partial_j \eta_I \otimes v_{I,2} \]
where \( \text{sgn}_I = \pm 1 \) and is not needed explicitly here, for \( |I| = 2, 3, 4 \) respectively. By linear independence we get \( v_{I,2} = 0 \) for \( |I| = 2, 3, 4, I \in \mathcal{I}_< \).

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