CONTINUOUS TIME MIXED STATE BRANCHING PROCESSES AND STOCHASTIC EQUATIONS

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Abstract A continuous time and mixed state branching process is constructed by a scaling limit theorem of two-type Galton-Watson processes. The process can also be obtained by the pathwise unique solution to a stochastic equation system. From the stochastic equation system we derive the distribution of local jumps and give the exponential ergodicity in Wasserstein-type distances of the transition semigroup. Meanwhile, we study immigration structures associated with the process and prove the existence of the stationary distribution of the process with immigration.

Key words mixed state branching process; weak convergence; stochastic equation system; Wasserstein-type distance; stationary distribution.

2010 MR Subject Classification 60J80; 60H20; 60G51

1 Introduction

Branching processes were introduced as probabilistic models describing the evolution of populations. The study of branching processes was initiated by Bienaymé (1845) and Galton and Watson (1874), independently, and the processes were referred to as discrete time and discrete state branching processes (GW-processes). To increase specificity, several naturally generalized processes including continuous time discrete state branching processes (DB-processes) with or without immigration and continuous time continuous state branching processes (CB-processes) with or without immigration, were subsequently introduced and studied by researchers.

The DB-processes are continuous time discrete state Markov processes with lifetimes that are independent and with exponentially distributed random variables. There have been many works on DB-processes, including ones pertaining to their construction, to the properties of moments, to limit theorems and so on; we refer to [1] for the details regarding these. The application of stochastic equations to branching processes has been developed in recent decades. Let \( N = \{0, 1, 2, \cdots\} \) and let \( \sharp(\cdot) = \sum \delta_i(\cdot) \) be a counting measure on \( N \). Let \( X = \{X_t : t \geq 0\} \) be a DB-process with immigration with a branching rate \( c > 0 \), offspring distribution \( (p_i : i \in N) \),
an immigration rate \( \eta > 0 \) and immigration distribution \((q_i : i \in \mathbb{N})\). The two distributions satisfy \( \sum_{k=1}^{\infty} kp_k < \infty \) and \( \sum_{k=1}^{\infty} kq_k < \infty \). It is known that \( X \) can be obtained as a pathwise unique strong solution to the stochastic equation

\[
X_t = X_0 + \int_0^t \int_0^t \left( z - 1 \right) M(ds, dz, du) + \int_0^t \int N(ds, dz),
\]

where \( X_0 \) is a random variable taking values in \( \mathbb{N} \), \( M(ds, dz, du) \) is a Poisson random measure on \( (0, \infty) \times \mathbb{N} \times (0, \infty) \) with intensity measure \( cp_x ds dz du \), \( N(ds, dz) \) is a Poisson random measure on \( (0, \infty) \times \mathbb{N} \) with intensity measure \( \eta q_x ds dz dz \), and \( X_0, M(ds, dz, du) \) and \( N(ds, dz) \) are independent of each other. In particular, if \( \eta = 0, q_x = 0 \) for all \( x \in \mathbb{N} \), this reduces things to the DB-process. Moreover, here and in the sequel, we understand that for any \( b \geq a \geq 0 \),

\[
\int_a^b = \int_{(a, b]}, \quad \int_a^\infty = \int_{(a, \infty)}.
\]

CB-processes were first introduced in [16], to model the random evolution of large population dynamics. Denoting the law on \( \mathcal{D}([0, \infty) \times [0, \infty)) \) by \( \mathbb{P}_x \) for each initial value \( x \geq 0 \), the branching property of processes can be described by \( \mathbb{P}_x * \mathbb{P}_y \). The semigroup of CB-processes with immigration (CBI-processes) \((Q_t)_{t \geq 0}\) can be characterized uniquely by the Laplace transform

\[
\int_{[0, \infty)} e^{-\lambda y} Q_t(x, dy) = \exp \left\{ -x v_t(\lambda) - \int_0^t \psi(v_s(\lambda)) ds \right\}, \quad \lambda \geq 0, x \geq 0,
\]

where, for any \( \lambda \geq 0, t \mapsto v_t(\lambda) \) uniquely solves the equation

\[
v_t(\lambda) = \lambda - \int_0^t \phi(v_s(\lambda)) ds, \quad t \geq 0,
\]

and the branching mechanism \( \phi \) and immigration mechanism \( \psi \) defined on \([0, \infty)\) take the form of

\[
\phi(z) = az + \alpha z^2, \quad \psi(z) = bz + \int_0^\infty (1 - e^{-zu}) n(du),
\]

with Lévy measures \( m, n \) satisfying \( \int_0^\infty u \wedge u^2 m(du) + \int_0^\infty u n(du) < \infty \), and \( a \in \mathbb{R}, b, \alpha \geq 0 \). In particular, when \( \psi \equiv 0 \), this reduces matters to the CB-process. There are several ways to construct such processes. [11] proved that a diffusion process may arise in a limit theorem of GW-processes. In [21], the authors systematically studied the limit theorems of GW-processes with immigration and characterized the class of the limit process as a CBI-process; the conditions of the main theorem involved iterations of the probability generating functions. Some simpler conditions for the weak convergence were provided in [22], and [27] extended the result of [22] to a two-type CBI-process. Letting \( Y = \{Y_t : t \geq 0\} \) be a CBI-process, then similarly, \( Y \) can also be represented as a pathwise unique strong solution to the stochastic equation

\[
Y_t = Y_0 + \int_0^t (b - a Y_s) ds + \sqrt{2\alpha} \int_0^t \int_0^t Y_s W(ds, du) + \int_0^t \int_0^\infty z N(ds, dz) + \int_0^t \int_0^\infty z \tilde{M}(ds, dz, du),
\]

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where $Y$ is a random variable taking values in $\mathbb{R}_+$, $W(ds, du)$ is a time-space white noise with intensity measure $ds du$, $M(ds, dz, du)$ is a Poisson random measure on $(0, \infty)^3$ with intensity measure $ds m(ds, dz, du)$, $N(ds, dz)$ is a Poisson random measure on $(0, \infty)^2$ with intensity measure $ds m(ds)$ and $\tilde{M}(ds, dz, du) = M(ds, dz, du) - ds m(ds, dz, du)$ is the compensated measure of $M(ds, dz, du)$. Moreover, $Y_0, \tilde{M}, \tilde{M}$ and $N$ are independent of each other. We mention that the moment condition $\int_0^\infty z n(ds) < \infty$ was removed in [13]. The sample paths of $Y$ can also be obtained as a unique strong solution to a stochastic equation driven by Brownian motions and Poisson random measures. One finds that the formulation (1.3) is nicer for analysing the flows of CBI-processes and other applications; see [6] for the specific construction. We refer to [3, 5, 6, 13, 23, 25, 30] for more on this approach and further properties of the above stochastic equations. Based on the stochastic equations established above, [14] studied the explicit expression of the distribution of jumps. [17] gave the criteria for the existence of general moments for CB-processes with or without immigration under a more general branching mechanism, where the characterization of the processes in terms of stochastic equations plays an essential role, and [18] extended those results to the processes in Lévy random environments. Some applications for finance can be found in [19]. A two-type CBI-process obtained as a unique strong solution of a stochastic equation system was studied in [28, 29].

We can rewrite (1.3) without an immigration part by extending the $M$ to a Poisson random measure denoted again by $M$ on $(0, \infty)^3 \times N$ with intensity $ds m(ds, dz) du (\lambda z)^k e^{-\lambda z^k} dk$ for some $\lambda > 0$ as follows:

$$Y_t = Y_0 - a \int_0^t Y_s ds + \sqrt{2\alpha} \int_0^t \int_0^{Y_s} W(ds, du) + \int_0^t \int_0^\infty \int_0^{Y_s} \int_{N} z \tilde{M}(ds, dz, du, dk)$$

$$= Y_0 - \phi'(\lambda) \int_0^t Y_s ds + \sqrt{2\alpha} \int_0^t \int_0^{Y_s} W(ds, du)$$

$$+ \int_0^t \int_0^\infty \int_0^{Y_s} z \tilde{M}^0(ds, dz, du) + 2\alpha \lambda \int_0^t Y_s ds$$

$$+ \int_0^t \int_0^\infty \int_0^{Y_s} z M^1(ds, dz, du) + \int_0^t \int_0^\infty \int_0^{Y_s} z M^2(ds, dz, du).$$

(1.4)

Here, $M^0(ds, dz, du) = M(ds, dz, du, \{k = 0\})$ and

$$M^1(ds, dz, du) = M(ds, dz, du, \{k = 1\}), \quad M^2(ds, dz, du) = M(ds, dz, du, \{k \geq 2\}).$$

Recently, [9] gave another SDE-type description for one-dimensional CB-processes based on (1.4) with $a < 0, \lambda \geq \lambda^*$; here $\lambda^*$ is the unique root of $\phi$ on $(0, \infty)$. One of the results in [9] shows that the last three integrals on the right-hand side of (1.4) are identified with the mass that immigrates from the skeleton construction. More precisely, the following stochastic equation system has a unique strong solution:

$$Y_t = Y_0 + \int_0^t \left(2\alpha X_s - \phi'(\lambda) Y_s \right) ds + \sqrt{2\beta} \int_0^t \int_0^{Y_s} W(ds, du)$$

$$+ \int_0^t \int_0^\infty \int_0^{Y_s} z \tilde{M}^0(ds, dz, du) + \int_0^t \int_0^\infty \int_1^{X_s} z M^3(ds, dz, dk)$$

$$+ \int_0^t \int_{\mathbb{N}} \int_1 z M^4(ds, dz, dk),$$

(1.5)
The specific form of the stochastic equation system is as follows: the process is called the continuous time mixed state branching process (MSB-process). The authors prove that, for any \( \eta \) and \( \lambda \), \( \eta \) is Poisson distributed with parameter \( \lambda \eta \). Moreover, (1.5)–(1.6) includes the prolific skeleton decomposition when \( \lambda = \lambda^* \); see [2] for the properties of this special decomposition.

We refer to [10] for a similar construction of (1.5)–(1.6) in the setting of superprocesses.

Inspired by the formulations (1.5)–(1.6), the first objective of this paper is to construct a two-dimensional branching Markov process \( \{(Y_1(t), Y_2(t)) : t \geq 0\} \) taking values in \( \mathbb{M} \) obtained as a unique strong solution to a more generalized stochastic equation system than (1.5)–(1.6); the process is called the continuous time mixed state branching process (MSB-process). The specific form of the stochastic equation system is as follows:

\[
X_t = X_0 + \int_0^t \int_{\mathbb{N}} \int_{\mathbb{N}} Y_{s-} \, dN(s, ds, du, dk).
\]

\( M = \mathbb{R}_+ \times \mathbb{N} \), \( M^3(ds, dz, dk) \) is a Poisson random measure on \( (0, \infty)^2 \times \{N \setminus \{0\}\} \) with intensity measure \( ds \cdot e^{-\lambda z} \, m(dz) \), \( M^4(ds, dz, dk) \) is a Poisson random measure on \( (0, \infty) \times \mathbb{M} \times \{N \setminus \{0\}\} \) with intensity measure \( \phi(\lambda) ds \cdot m(dz) \), \( p_z \) and one can see the specific definitions of two distributions \( (\eta_k)_{k \in \mathbb{N}} \) and \( (p_k)_{k \in \mathbb{N}} \) in [9, p.1127], so we omit them here. The authors prove that, for any \( y \geq 0 \), \( \{Y_t : t \geq 0\} \) is a weak solution of (1.4) with initial value \( Y_0 = y \) if \( X_0 \) is Poisson distributed with parameter \( \lambda y \).

The existence of the stationary distribution and the ergodic rates are both important topics in the theory of Markov processes. Demonstration of a necessary and sufficient condition for the existence of the stationary distribution of one-dimensional CBI-processes was initiated by [31]; see also [23] for a proof. The sufficient condition for the multi-type case can be found in [20]. The strong Feller property and exponential ergodicity of such processes in the

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total variation distance were given in [24] by a coupling of CBI-processes constructed by the stochastic equation driven by time-space noises and Poisson random measures; see also [25]. In a recent work, [26] considered the ergodicities and exponential ergodicities in Wasserstein and total variation distances of Dawson-Watanabe superprocesses with or without immigration; this clearly includes the multi-type CBI-process case. After constructing the MSB-processes, we also want to study the ergodic theory of such processes and prove the exponential ergodicity in the $L^1$-Wasserstein distance by establishing upper bound estimates for the variations of the transition probabilities; this is inspired by similar results on measure-valued branching processes in [26]. Moreover, by adding the immigration structures, we give a sufficient and necessary condition for the existence of the stationary distribution of MSB-processes with immigration (MSBI-processes).

The remainder of this paper is organized as follows: in Section 2, we prove a weak convergence theorem from GW-processes to DB-processes. In Section 3, we obtain the MSB-process arising in a limit theorem of rescaled two-type GW-processes. In Section 4, we provide another construction of MSB-processes by stochastic equation systems. The analysis of distributions of jumps is given in Section 5. In Section 6, we study both the estimates for the variations and the exponential ergodicity in the $L^1$-Wasserstein distance of the transition semigroup of such processes. Finally, we prove the existence of the stationary distribution of such processes with immigration in Section 7.

2 The Construction of DB-processes

Let $\{p_j : j \in \mathbb{N}\}$ be a probability distribution on $\mathbb{N}$, and denote the generating function by $g(z) = \sum_{j=0}^{\infty} p_j z^j$ on $|z| \leq 1$. Let $u(z) = a (g(z) - z)$ for some $a > 0$. A Markov process $\{X_t : t \geq 0\}$ with state space $\mathbb{N}$ is called a DB-process with branching rate $a > 0$ and offspring distribution $\{p_j : j \in \mathbb{N}\}$ if its transition probabilities $Q_{ij}(t)$ satisfy

$$\sum_{j=0}^{\infty} Q_{ij}(t) z^j = \left[ \sum_{j=0}^{\infty} Q_{1j}(t) z^j \right]^i, \quad i \in \mathbb{N}, \quad t \geq 0, \quad z \in [0,1],$$

(2.1)

which implies the branching property of the process. Denote $F(z,t) = \sum_{k=0}^{\infty} Q_{1k}(t) z^k$. Clearly, $F = (F(\cdot,t) : t \geq 0)$ satisfies the semigroup property $F(\cdot,t+s) = F(F(\cdot,t),s)$ for $t,s \geq 0$ and is the unique solution of the following differential equation:

$$\frac{\partial}{\partial t} F(z,t) = u[F(z,t)], \quad F(z,0) = z.$$

(2.2)

We call $F$ the compound semigroup for the DB-process, and refer to [1, p.106-107] for more details.

We now provide a sufficient condition for the weak convergence of GW-processes to the DB-process. Assume that there exists a sequence of GW-processes $\{X_k(n) : n \geq 0\}_{k \geq 1}$ with parameters $\{g_k\}_{k \geq 1}$, and let $\{\gamma_k\}_{k \geq 1}$ be a sequence of positive numbers. Denote the $n$-step transition probability for $\{X_k(n) : n \geq 1\}$ by $Q^n_k$, and let $[x]$ be the integral part of $x$. One can
see that
\[
\sum_{j=0}^{\infty} Q_k^{|\gamma_k t|}(i,j)z^j = [g_k^{|\gamma_k t|}(z)]^i := (F_k(z,t))^i, \quad i \in \mathbb{N}, \quad z \in [0,1], \quad t \geq 0,
\]
and
\[
F_k(z,t) = z + \sum_{i=1}^{|\gamma_k t|} (g_k^o(z) - g_k^o(i-1)(z))
\]
\[
= z + \gamma_k^{-1} \sum_{i=1}^{|\gamma_k t|} U_k(F_k(z,\frac{i-1}{\gamma_k}))
\]
\[
= z + \int_0^{\gamma_k} U_k(F_k(z,r)) dr,
\]
where \(g^o(z)\) is defined by \(g^o(z) = g(g^o(\gamma_k^{-1})(z))\), successively with \(g^o(z) = z\) and \(U_k(z) = \gamma_k(g_k(z) - z)\), \(0 \leq z \leq 1\). For convenience, we formulate the following conditions:

(A) \(\gamma_k \to \infty\) as \(k \to \infty\).

(B) The sequence \(U_k(z)\) is uniformly Lipschitz on \([0,1]\), and converges to a continuous function \(u(z)\) as \(k \to \infty\).

**Proposition 2.1** (i) Suppose that (A, B) hold. Then the limit function of sequence \(\{U_k(z)\}_{k \geq 1}\) has representation \(u(z) = a(g(z) - z)\) as \(k \to \infty\) for all \(0 \leq z \leq 1\), where \(a\) is a strictly positive constant, \(g(z)\) is a generation function and \(g'(1-) < \infty\).

(ii) For any given \(u(z) = a(g(z) - z)\), there exists a sequence of \(\{U_k\}_{k \geq 1}\) such that (A, B) hold with \(U_k(z) \to u(z)\).

**Proof** (i) The desired result is a corollary of Proposition 3.1 (i), to be demonstrated later. Indeed, it suffices to consider the offspring distribution corresponding to two-type GW-processes cases satisfying \(v_k{(\{i,\cdot\})} \equiv 0\) for all \(i \geq 1\).

(ii) For a given \(g(z) - z = \sum_{i=0}^{\infty} p_i z^i - z\) and \(a > 0\), set \(\gamma_k = ak\) and
\[
p_{ki} = \begin{cases} \frac{p_i}{k}, & \text{if } i \neq 1; \\ (p_1 - 1)/k + 1, & \text{if } i = 1. \end{cases}
\]
Defining \(U_k(z) = ak(\sum_{i=0}^{\infty} p_{ki} z^i - z)\), it is not hard to see that \(U_k\) satisfies condition (B), and converges to \(u(z)\) for all \(z \in [0,1]\).

**Lemma 2.2** Suppose that (A, B) hold. Then there are constants \(\lambda, N \geq 0\) such that \(F_k(z,t) \in [e^{\lambda t}, 1]\) for every \(t \geq 0\), \(z \in [0,1]\) and \(k \geq N\).

**Proof** Let \(b_k := \gamma_k(g_k'(1) - 1)\). Under condition (B), there exists \(\lambda \geq 0\) such that \(2|b_k| \leq \lambda\) for all \(k \geq 1\). It is not hard to obtain that
\[
\sum_{j=1}^{\infty} jQ_k^{|\gamma_k t|}(i,j) = ig_k'(1-|\gamma_k t|) = i(1 + \frac{b_k}{\gamma_k} |\gamma_k t|).
\]
Since \(\gamma_k \to \infty\) as \(k \to \infty\), there is a \(N \geq 1\) such that, for all \(k \geq N\),
\[
0 \leq (1 + \frac{b_k}{\gamma_k})^\frac{1}{\gamma_k} \leq \frac{1}{2}\lambda \frac{1}{2\gamma_k} \leq \epsilon,
\]
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so for \( t \geq 0 \) and \( k \geq N \),
\[
\sum_{j=1}^{\infty} j Q_k^{(\gamma_k t)}(i, j) \leq i \exp \left\{ \frac{\lambda}{\gamma_k} \gamma_k t \right\} \leq ie^{\lambda t}.
\]
We get the desired result by Jensen’s inequality.

**Lemma 2.3** Suppose that \((A, B)\) hold. For any \( c > 0 \), we have \( F_k(z, t) \to F(z, t) \) uniformly on \([0, e^{-c}] \times [0, c]\) as \( k \to \infty \), and the limit function solves (2.2).

**Proof** We may rewrite
\[
F_k(z, t) = z + \int_0^t U_k(F_k(z, r)) dr - \left( t - \frac{\gamma_k t}{\gamma_k} \right) U_k(F_k(z, \frac{\gamma_k t}{\gamma_k})).
\]
By Proposition 2.1 and Lemma 2.2, for \( \varepsilon \in (0, 1] \), we can take \( N \geq 1 \) large enough such that
\[
|U_k(z) - u(z)| \leq \varepsilon, \quad e^{-e \varepsilon} \leq z \leq 1, \quad k \geq N.
\]
Denote the last term on the right hand of equation (2.3) by \( \varepsilon_k(t, z) \). Then
\[
|\varepsilon_k(t, z)| \leq \gamma_k^{-1} M, \quad 0 \leq t \leq c, \quad 0 \leq z \leq e^{-c},
\]
where \( M = 1 + \sup_{\Theta} |u(z)| \), \( \Theta = \{ z : e^{-e \varepsilon} \leq z \leq 1 \} \). On the other hand, for \( n \geq k \geq N \), we set
\[
K_{k,n}(t, z) = \sup_{0 \leq r \leq t} \left| F_n(z, r) - F_k(z, r) \right|,
\]
and it follows that
\[
K_{k,n}(t, z) \leq 2 \gamma_k^{-1} M + \varepsilon c + \rho \int_0^t K_{k,n}(r, z) dr, \quad 0 \leq t \leq c, \quad 0 \leq z \leq e^{-c},
\]
where \( \rho = \sup_{\Theta} |u'(z)| \). By applying Gronwall’s inequality, we obtain that
\[
K_{k,n}(t, z) \leq 2 \gamma_k^{-1} M + \varepsilon c e^{\rho t}, \quad 0 \leq t \leq c, \quad 0 \leq z \leq e^{-c},
\]
from which it follows that \( F_k(z, t) \to F(z, t) \), and the limit function satisfies (2.2).

By (2.1), we see that the transition probabilities \( Q = \{ Q_{ij}(t) : i, j \in \mathbb{N}, t \geq 0 \} \) of the DB-process \( \{ X_t : t \geq 0 \} \) can be determined by
\[
\sum_{j=0}^{\infty} Q_{ij}(t) z_j = (F(z, t))^i, \quad i \geq 1, t \geq 0.
\]
Based on Proposition 2.1 and Lemma 2.3, by similar arguments as to those of Theorem 2.9 in [25], it is not hard to see that the transition probability \( Q \) of \( \{ X_t : t \geq 0 \} \) is a limit of a sequence of transition probabilities \( \{ Q_k^{(\gamma_k t)}(i, j) : i, j \in \mathbb{N}, t \geq 0 \} \) associated with GW-processes in the sense of weak convergence under conditions \((A, B)\); this indeed implies another construction of DB-processes by a rescaling approach.

### 3 The Construction of MSB-processes

For a two-type GW-process \( \{ Y(n) = (Y_1(n), Y_2(n)) : n \in \mathbb{N} \} \) of \( n \in \mathbb{N} \), we define two corresponding generation functions for \( i = (i_1, i_2) \in \mathbb{N}^2 \) and \( s_1, s_2 \in [0, 1] \):
\[
g_j(s_1, s_2) = \sum_{i \in \mathbb{N}^2} p_j(i)s_1^{i_1}s_2^{i_2}, \quad p_j(i) = P(Y(1) = i|Y(0) = e_j), \quad j = 1, 2.
\]
Here $e_1 = (1, 0), e_2 = (0, 1)$. Note that $g_j^{n_1} = g_j(g_1^{n_1-1}) g_2^{n_1-1})$ for $j = 1, 2$ and $n \geq 1$. It is known that the one-step transition matrix $P(i, j)$ of $\{Y(n) : n \geq 0\}$ is uniquely determined by

$$
\sum_{j \in \mathbb{N}^2} P(i, j) s_1^j s_2^j = \left[ g_1(s_1, s_2) \right]^{i_1} \left[ g_2(s_1, s_2) \right]^{i_2}.
$$

Let $\{Y_k(n) = (Y_{k,1}(n), Y_{k,2}(n)) : n \in \mathbb{N}\}$ be a sequence of two-type GW-processes corresponding to $G_k = (g_{k,1}, g_{k,2})$, and let $\{\gamma_k\}$ be a sequence of positive numbers. Denoting the $n$-step transition probability for $\{(k^{-1}Y_{k,1}(n), Y_{k,2}(n)) : n \geq 0\}$ by $P^n_k$, it is not hard to see that, for $t \geq 0$ and $\lambda \in \mathbb{R}^+$,

$$
\int_{\mathbb{M}_k} e^{-(\lambda,y)} P^{|\gamma_k| t}_k(x, dy) = \left[ g_{k,1}^{\gamma_k t}(e^{-\lambda_1/k}, e^{-\lambda_2}) \right]^{k \lambda_2} \left[ g_{k,2}^{|\gamma_k| t}(e^{-\lambda_1/k}, e^{-\lambda_2}) \right]^{\gamma_k t}, \ x \in \mathbb{M}_k,
$$

where $\mathbb{M}_k := \{(i/k, j) : (i, j) \in \mathbb{N}^2\}$. For $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^+$, we define a vector function $V_k(t, \lambda) = (V_{k,1}(t, \lambda), V_{k,2}(t, \lambda))$ with

$$
V_{k,1}(t, \lambda) = -k \log g_{k,1}^{\gamma_k t}(e^{-\lambda_1/k}, e^{-\lambda_2}), \quad V_{k,2}(t, \lambda) = -\log g_{k,2}^{\gamma_k t}(e^{-\lambda_1/k}, e^{-\lambda_2}),
$$

so we can rewrite

$$
\int_{\mathbb{M}_k} e^{-(\lambda,y)} P^{|\gamma_k| t}_k(x, dy) = \exp\{-\langle x, V_k(t, \lambda) \rangle\}, \ x \in \mathbb{M}_k,
$$

and $V_k(t, \lambda)$ satisfies the equation

$$
V_{k,1}(t, \lambda) = \lambda_1 - \int_0^{|\gamma_k| t} \Phi_{k,1}(V_k(s, \lambda)) ds, \quad V_{k,2}(t, \lambda) = \lambda_2 - \int_0^{|\gamma_k| t} \Phi_{k,2}(V_k(s, \lambda)) ds,
$$

where two functions $\Phi_{k,1}$ and $\Phi_{k,2}$ are defined on $\mathbb{R}^+$, taking the form

$$
\Phi_{k,1}(\lambda_1, \lambda_2) = k \gamma_k \log(1 - (k\gamma_k)^{-1} \Phi_{k,1}(\lambda_1, \lambda_2) e^{\frac{\lambda_1}{k}}),
\Phi_{k,1}(\lambda_1, \lambda_2) = \gamma_k [e^{-\lambda_1/k} - g_{k,1}(e^{-\frac{\lambda_1}{k}}, e^{-\lambda_2})],
\Phi_{k,2}(\lambda_1, \lambda_2) = \gamma_k \log(1 - g_{k,2}(\lambda_1, \lambda_2) e^{\lambda_2}),
\Phi_{k,2}(\lambda_1, \lambda_2) = \gamma_k [e^{-\lambda_2} - g_{k,2}(e^{-\frac{\lambda_2}{k}}, e^{-\lambda_2})].
$$

We further define two functions $\Phi_1$ and $\Phi_2$ on $[0, \infty)^2$:

$$
\Phi_1(\lambda_1, \lambda_2) = -a_{11} \lambda_1 - a_1 \lambda_2^2 - \int_\mathbb{M} (e^{-(\lambda_1, z)} - 1 + \lambda_1 z_1) n_1(dz),
\Phi_2(\lambda_1, \lambda_2) = a_{21} \lambda_1 + \int_\mathbb{M}_{\geq 1} (1 - e^{-(\lambda_1, z)}) n_2(dz).
$$

Here $a_{11}$ is a constant, $a_{11}, a_1 \geq 0$, $n_1$ and $n_2$ are finite measures on $\mathbb{M}$ and $\mathbb{M}_{\geq 1}$, supported by $\mathbb{M} \setminus \{0\}$ and $\mathbb{M}_{\geq 1} \setminus \{0\}$, respectively, such that $n_2(\mathbb{R} \times \{-1\}) < \infty$ and

$$
\int_\mathbb{M} (z_1 + z_1^2 + z_2) n_1(dz) + \int_{\mathbb{M}_{\geq 1}} (z_1 + z_2) n_2(dz) < \infty.
$$

For convenience, let us consider the following conditions:

(A) $\gamma_k \to \infty$.

(C) The sequence $\{\Phi_{k,1}(\lambda_1, \lambda_2)\}_{k \geq 1}$ is uniformly Lipschitz in $(\lambda_1, \lambda_2)$ on each bounded rectangle, and converges to a continuous function as $k \to \infty$.

(D) The sequence $\{e^{1/k} \Phi_{k,2}(\lambda_1, \lambda_2)\}_{k \geq 1}$ is uniformly Lipschitz in $(\lambda_1, \lambda_2)$ on each bounded rectangle, and converges to a continuous function as $k \to \infty$. 

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Proposition 3.1  
(i) Assume that \((A, C, D)\) hold. Then the limit function \(\Phi_1(\lambda_1, \lambda_2)\) of \(\{\Phi_{k,1}(\lambda_1, \lambda_2)\}_{k \geq 1}\) has the representation (3.3), and the limit \(\Phi_2(\lambda_1, \lambda_2)\) of \(\{e^{\lambda_2}\Phi_{k,2}(\lambda_1, \lambda_2)\}_{k \geq 1}\) has the representation (3.4):

(ii) For any \(\Phi_1\) and \(\Phi_2\) given by (3.3) and (3.4), there are sequences \(\{\gamma_k\}\) and \(\{(g_k,1, g_k,2)\}\) as above such that \((A, C, D)\) hold with \(\Phi_{k,1}(\lambda_1, \lambda_2) \rightarrow \Phi_1(\lambda_1, \lambda_2)\) and \(e^{\lambda_2}\Phi_{k,2}(\lambda_1, \lambda_2) \rightarrow \Phi_2(\lambda_1, \lambda_2)\) for all \((\lambda_1, \lambda_2) \in \mathbb{R}_+^2\) as \(k \rightarrow \infty\).

Proof  
(i) We first prove the representation result for \(\Phi_2\), which is inspired by [27].

(1) Fix \(k \geq 1\). Let \(M_{-1,k} = \{(i/k, j-1) : \{i,j\} \in \mathbb{N}^2\}\), and let \(\rho_k\) be the measure defined by

\[
\rho_k(\cdot) = \gamma_k \sum_{i,j=0}^{\infty} \nu_k(\{i,j\}) \delta_{(i/k,j-1)}(\cdot),
\]

where \(\nu_k\) is the probability measure on \(\mathbb{N}^2\) corresponding to \(g_{k,2}\), so \(\rho_k\) is a finite measure on \(M_{-1,k}\). Let \(l(z) = (z_1 + |z_2|) \land 1, \nu_k = \int_{M_{-1,k}} l(z) \rho_k(dz)\). If \(\rho_k > 0\), we define \(P_k(dz) = (l(z)/\rho_k) \rho_k(dz)\), and if \(\rho_k = 0\), we let \(P_k(\cdot)\) be the Dirac measure at some point \(z_0 \in M_{-1,k}\setminus\{0\}\). In both cases, we have that \(P_k(\cdot)\) is a probability measure on \(M_{-1,k}\) and that

\[
e^{\lambda_2}\Phi_{k,2}(\lambda_1, \lambda_2) = \gamma_k \int_{M_{-1,k}\setminus\{0\}} (1 - e^{-\langle \lambda, z \rangle}) l(z)^{-1} P_k(dz).
\]

(2) Let \(M^\Delta_{-1} = M_{-1} \cup \{\Delta\}\) be the one-point compactification of \(M_{-1}\). Then \(\{P_k\}\) is the sequence of probability measures on \(M^\Delta_{-1}\), so it is relatively compact. Choose any subsequence denoted again by \(\{P_k\}\) which converges to a probability \(P\) on \(M^\Delta_{-1}\). Letting \(E = \{\varepsilon \mid P(\|z\| = \varepsilon) = 0\}\), for \(\varepsilon \in E\), we define a compact space of \(M^\Delta_{-1}\) by \(Q := \{z \in M^\Delta_{-1}, \|z\| \leq \varepsilon\}\). We have

\[
e^{\lambda_2}\Phi_{k,2}(\lambda_1, \lambda_2) = \gamma_k a_{k,1}^{(1)} \lambda_1 + \gamma_k a_{k,2}^{(2)} \lambda_2 - \gamma_k (J_{k,\varepsilon} + I_{k,\varepsilon}),
\]

where

\[
a_{k,1}^{(i)} = \int_{Q \setminus \{0\}} \chi(z_i) l(z)^{-1} P_k(dz), \quad i = 1, 2, \quad \chi(z_i) = (1 \land z_i) \lor (-1),
\]

\[
J_{k,\varepsilon} = \int_{M_{-1}\setminus Q} \left( e^{-\langle \lambda, z \rangle} - 1 \right) l(z)^{-1} P_k(dz),
\]

\[
I_{k,\varepsilon} = \int_{Q \setminus \{0\}} \left( e^{-\langle \lambda, z \rangle} - 1 + \chi(z_1) \lambda_1 + \chi(z_2) \lambda_2 \right) l(z)^{-1} P_k(dz).
\]

It is not hard to see that \(a_{k,\varepsilon}^{(2)} = 0\), \(\lim_{E\ni\varepsilon \uparrow 0} k \rightarrow \infty I_{k,\varepsilon} = 0\).

(3) Fix \(\varepsilon \in E, 0 < \varepsilon < 1\). If \(\lim \inf_{k \rightarrow \infty} \rho_k = 0\), then \(\Phi_2 = 0\). If \(\lim \inf_{k \rightarrow \infty} \rho_k > 0\), then there exists a subsequence denoted again by \(\{g_k\}\) which converges to \(g \in (0, \infty]\). We can prove that \(P(\{\Delta\}) = 0\). Actually,

\[
\int_{M_{-1}\setminus Q} e^{-\langle \lambda, z \rangle} l(z)^{-1} P_k(dz) \rightarrow \text{some } L(\lambda),
\]

where \(L(\lambda)\) is a continuous function. Since \(l(z)^{-1} \rightarrow 1\) as \(z \rightarrow \Delta\), and is the continuous function of \(z \in M^\Delta_{-1}\setminus Q\), it follows from the monotone convergence theorem that

\[
\int_{M_{-1}\setminus Q} l(z)^{-1} P(dz) = \lim_{k \rightarrow \infty} \int_{M_{-1}\setminus \{0\}} l(z)^{-1} P_k(dz) = L(0),
\]

\[
\int_{M^\Delta_{-1}\setminus Q} l(z)^{-1} P(dz) = \lim_{n \rightarrow \infty} \int_{M_{-1}\setminus Q} e^{-\langle \lambda, z \rangle} l(z)^{-1} P(dz) = \lim_{n \rightarrow \infty} L(\frac{1}{n}) = L(0).
\]
Defining \( v(\{0\}) = 0 \) and \( v(dz) = l(z)^{-1}P(dz) \) on \( \{ z \in \mathbb{M} : \| z \| > 0 \} \), we have
\[
\lim_{E \ni \xi, 0} \lim_{k \to \infty} J_{k,\epsilon} = \int_{\mathbb{M} - 1} (e^{-(\lambda, z)} - 1) v(dz), \quad \lim_{E \ni \xi, 0} \lim_{k \to \infty} a_{k,\epsilon}^{(1)} := a^{(1)} \geq 0.
\]

Based on the calculations above, we obtain that
\[
\frac{1}{\theta} \Phi_2(\lambda_1, \lambda_2) = a^{(1)}\lambda_1 + \int_{\mathbb{M} - 1} (1 - e^{-(\lambda, z)}) v(dz).
\]

(4) Now we need to verify that \( 1/\theta > 0 \). If not, the r.h.s of (3.7) is the zero function. Assuming that \( v(\mathbb{M} \setminus \{0\}) = 0 \), we have that \( P(\{0\}) = 1 \), and by the representations of \( a_{k,\epsilon}^{(1)} \), we can have \( a^{(1)} = \lim_{E \ni \xi, 0} P(\| z \| \leq \epsilon) = 1 \), but it follows from the representation of Lévy-Khintchine type functions that the parameters are unique, therefore \( 1/\theta = 0 \) is impossible.

Letting \( n_2(\cdot) = \rho v(\cdot), a_{21} = \rho a^{(1)} \), we can rewrite \( \Phi_2 \) as follows:
\[
\Phi_2(\lambda_1, \lambda_2) = a_{21}\lambda_1 + \int_{\mathbb{M} - 1} (1 - e^{-(\lambda, z)}) n_2(dz).
\]

By applying the monotone convergence theorem, we get
\[
\frac{\partial \Phi_2}{\partial \lambda_1}(0+, 0) = a_{21} + \int_{\mathbb{M} - 1} z_1 n_2(dz), \quad \frac{\partial \Phi_2}{\partial \lambda_2}(0, 0+) = \int_{\mathbb{M} - 1} z_2 n_2(dz).
\]

From the fact that \( \Phi_2 \) is locally Lipschitz, we have \( \int_{\mathbb{M} - 1} (z_1 + z_2) n_2(dz) < \infty \) and \( n_2(\mathbb{R}^+ \times \{-1\}) < \infty \).

Now we apply \( \Phi_{k,1} \). Fix \( k \geq 1 \), and let \( W = [-1, \infty) \times \mathbb{N}, W_k = \{(i - 1)/k, j) : (i, j) \in \mathbb{N}^2\} \).

We define \( \tilde{\rho}_k \) by
\[
\tilde{\rho}_k(\cdot) = k\gamma_k \sum_{i,j=0}^{\infty} \tilde{v}_k(\{i, j\}) \delta_{\left(\frac{i-1}{k}, j\right)}(\cdot),
\]
where \( \tilde{v}_k \) is the probability measure on \( \mathbb{N}^2 \) corresponding to \( g_{k,1} \). Letting \( \tilde{l}(z) = (z_1^2 + z_2) \wedge 1 \), we have
\[
e^{\lambda_1/k} \Phi_{k,1}(\lambda_1, \lambda_2) = \tilde{\beta}_{k,1}\lambda_1 + \tilde{\Delta}_{k,\epsilon} \tilde{\lambda}_2 - \tilde{\Delta}_{k,\epsilon} \tilde{J}_{k,\epsilon},
\]
where
\[
\tilde{\beta}_{k,1} = \int_W \chi(z_1)\tilde{\rho}_k(dz), \quad \tilde{\Delta}_{k,\epsilon} = \frac{1}{2} \int_{Q \setminus \{0\}} \chi(z_1)^2 \tilde{l}(z)^{-1} \tilde{P}_k(dz),
\]
\[
\tilde{\Delta}_{k,\epsilon} = \int_{Q \setminus \{0\}} \chi(u_2)\tilde{l}(u)^{-1} \tilde{P}_k(du), \quad h(u, \lambda) = e^{-(\lambda, z)} - 1 + \chi(z_1)\lambda_1,
\]
\[
\tilde{J}_{k,\epsilon} = \int_{W \setminus Q} h(z, \lambda)\tilde{l}(z)^{-1} \tilde{P}_k(dz),
\]
\[
\tilde{I}_{k,\epsilon} = \int_{Q \setminus \{0\}} \left(h(z, \lambda) + \chi(z_2)\lambda_2 - \frac{1}{2} \chi(z_1)^2 \tilde{l}(z)^{-1}\right) \tilde{P}_k(dz).
\]

Note that \( \tilde{\Delta}_{k,\epsilon}^{(2)} = 0 \). By a similar argument as that in [27], we have that
\[
\Phi_1(\lambda_1, \lambda_2) = -a_{11}\lambda_1 - \alpha\lambda_1^2 - \int_\mathbb{M} (e^{-(\lambda, z)} - 1 + \lambda_1 z_1) n_1(dz),
\]
and \( n_1 \) satisfies
\[
\int_\mathbb{M} (z_1 \wedge z_1^2 + z_2) n_1(dz) < \infty.
\]
(ii) Given the function from (3.4), we set $\tilde{D}_k = \{z \in \mathbb{M}_{-1} : z_1 > \frac{1}{\sqrt{k}}\}$, $\tilde{g}_{k,1} = a_{21} k, \tilde{g}_{k,2} = n_2(\tilde{D}_k)$, $g^k_{k,1}(x_1, x_2) = x_1 x_2$ and $g^k_{k,2}(x_1, x_2) = \tilde{g}_{k,2}^{-1} f_{\tilde{D}_k} x_1 \gamma_{k,1}^{-1} x_2 \gamma_{k,2}^{-1} n_2(\mathbb{R})$. Then, if we set sequences $\tilde{\gamma}_k = \gamma_{k,1} + \gamma_{k,2}$ and $\tilde{g}^{(2)}_k = \gamma_{k,2}^{-1}(\gamma_{k,1} \tilde{g}_{k,1} + \gamma_{k,2} \tilde{g}_{k,2})$, we find the sequences $\{\tilde{\gamma}_k\}$ and $\{\tilde{g}^{(2)}_k\}$ such that conditions (A, D) hold with $e^{\lambda_2} \Phi_{k,2}(\lambda_1, \lambda_2) \rightarrow \Phi_2(\lambda_1, \lambda_2)$ for all $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2$ as $k \to \infty$.

Given the function from (3.3), we set $\tilde{\gamma}_{k,1} = a_{11}, \tilde{g}_{k,1}(x_1, x_2) = 1, \tilde{g}_{k,2} = (2\alpha + 1) k, \tilde{g}_{k,2}(x_1, x_2) = x_1 + \frac{\alpha}{2\alpha+1}(1-x_1)^2$, and let $\tilde{D}_k = \{z \in \mathbb{M} : z_1 > \frac{1}{\sqrt{k}}, z_2 > \frac{1}{\sqrt{k}}\}$ and $\sigma_k = \int_{\tilde{D}_k} (z_1 - \frac{1}{k}) n_1(\mathbb{R})$. Then we set the sequences

$$\tilde{\gamma}_{k,3} = \sigma_k + \frac{1}{k} n_1(\tilde{D}_k) + 1,$$

$$\tilde{g}_{k,3}(x_1, x_2) = \frac{1}{\tilde{\gamma}_{k,3}^2} \int_{\tilde{D}_k} x_1 x_2 n_1(\mathbb{R}) + \frac{\sigma_k + \frac{1}{k} n_1(\mathbb{R})}{\gamma_{k,3}^2} \left( x_1 + \frac{\sigma_k}{\sigma_{k+1}} (1 - x_1) \right).$$

Then, if we let $\tilde{\gamma}_k = \tilde{\gamma}_{k,1} + \tilde{\gamma}_{k,2} + \tilde{\gamma}_{k,3}$ and $\tilde{g}^{(1)}_k = \gamma_{k,1}^{-1}(\gamma_{k,1} \tilde{g}_{k,1} + \gamma_{k,2} \tilde{g}_{k,2} + \gamma_{k,3} \tilde{g}_{k,3})$, we find that we have the sequences $\{\tilde{\gamma}_k\}$ and $\{\tilde{g}^{(1)}_k\}$ such that conditions (A, C) hold with $\Phi_{k,1}(\lambda_1, \lambda_2) \rightarrow \Phi_1(\lambda_1, \lambda_2)$ for all $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2$ as $k \to \infty$.

Now letting $\gamma_k = \tilde{\gamma}_{k} + \tilde{\gamma}_{k}$ and

$$g_{k,1}(x_1, x_2) = \gamma_{k,1}^{-1}(\gamma_{k,1} \tilde{g}_{k,1} + \gamma_{k,2} \tilde{g}_{k,2}),$$

$$g_{k,2}(x_1, x_2) = \gamma_{k,2}^{-1}(\gamma_{k,2} \tilde{g}_{k,2} + \gamma_{k,3} \tilde{g}_{k,3}),$$

we finally find the common sequences $\{\gamma_k\}$ and $\{\tilde{g}^{(1)}_k, \tilde{g}^{(2)}_k\}$ such that conditions (A, C, D) hold with $\Phi_{k,1}(\lambda_1, \lambda_2) \rightarrow \Phi_1(\lambda_1, \lambda_2)$ and $e^{\lambda_2} \Phi_{k,2}(\lambda_1, \lambda_2) \rightarrow \Phi_2(\lambda_1, \lambda_2)$ for all $(\lambda_1, \lambda_2) \in \mathbb{R}_+^2$ as $k \to \infty$. □

**Proposition 3.2** Assume that (A, C, D) hold. Then, for any $a \geq 0$, we have $V(t, \lambda) \rightarrow$ some $V(t, \lambda)$ uniformly on $[0, a]^3$ as $k \to \infty$, and the limit function solves the following integral equations:

$$V_1(t, \lambda) = \lambda_1 + \int_0^t \Phi_1(V(s, \lambda)) \, ds, \quad V_2(t, \lambda) = \lambda_2 + \int_0^t \Phi_2(V(s, \lambda)) \, ds.$$  \hfill (3.8)

Moreover, suppose that $(\Phi_1, \Phi_2)$ is given by (3.3)–(3.4). Then, for any $\lambda \in \mathbb{R}_+$, the solution $t \mapsto V(t, \lambda)$ of (3.8) is unique, and the solution satisfies the semigroup property

$$V(r + t, \lambda) = V(r, V(t, \lambda)), \quad r, t \geq 0.$$  \hfill (3.9)

**Proof** By a similar argument as to that of Lemma 2.2 in [25], it follows from Proposition 3.1 that $\tilde{\Phi}_{k,1} \rightarrow -\Phi_1$ and $\tilde{\Phi}_{k,2} \rightarrow -\Phi_2$ uniformly on each bounded rectangle as $k \to \infty$. We can rewrite

$$V_{k,1}(t, \lambda) = \lambda_1 - \int_0^t \tilde{\Phi}_{k,1}(V_k(s, \lambda)) \, ds + \varepsilon_{k,1}(t, \lambda),$$

$$V_{k,2}(t, \lambda) = \lambda_2 - \int_0^t \tilde{\Phi}_{k,2}(V_k(s, \lambda)) \, ds + \varepsilon_{k,2}(t, \lambda),$$

where

$$\varepsilon_{k,i}(t, \lambda) = -(t - \gamma_{k,1}^{-1}[\gamma_k t]) \tilde{\Phi}_{k,i}(V_k(\gamma_{k,1}^{-1}[\gamma_k t], \lambda)), \quad i = 1, 2.$$  

For $\theta \in \mathbb{R}_+$, it is not hard to obtain that

$$\int_{\mathbb{M}} (\theta_1 y_1 + \theta_2 y_2) \, P_k^{[\gamma_k t]}(x, dy).$$

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It follows from Jensen’s inequality that
\[
\frac{1}{k} \left[ \theta_1 (\gamma_k^{-1} \Phi_{k,1}(1) (0) + 1) + \theta_2 (\gamma_k^{-1} \Phi_{k,2}(0) (0) + 1) \right] A_k \left[ (\gamma_k t - 1) (kx_1, x_2) \right] \]

where
\[
A_k = \left( \begin{array}{cc} g_{k,1,1} (1) & g_{k,2,1} (1) \\ g_{k,1,2} (1) & g_{k,2,2} (1) \end{array} \right), \quad A_k^{(n)} = A_k \times A_k^{(n-1)},
\]

and \( g_{k,1,1}, g_{k,2,1}, \Phi_{k,1} \) and \( \Phi_{k,2} \) denote the derivative with respect to \( \lambda_i, i = 1, 2 \). By assumption (C, D), there exists \( B \geq 0 \) such that \( |\Phi_{k,i,j}(x)| \leq B \) for all \( i, j \in \{1, 2\} \) and \( k \geq 1 \).

It follows from Jensen’s inequality that
\[
\langle x, V_k (t, \theta) \rangle \leq (\gamma_k^{-1} B + 1) (\theta_1 + \theta_2) (k^{-1}, 1) A_k \left[ (\gamma_k t - 1) (kx_1, x_2) \right],
\]

so
\[
\|V_k (t, \theta)\| = \sup_{\|x\|=1} \langle x, V_k (t, \theta) \rangle \leq (\gamma_k^{-1} B + 1) (\theta_1 + \theta_2) \sup_{\|x\|=1} (k^{-1}, 1) B_k \left[ (\gamma_k t - 1) (kx_1, x_2) \right] \leq (2\gamma_k^{-1} B + 1)^{\frac{1}{\gamma_k t}} \sqrt{2} (\theta_1 + \theta_2),
\]

where
\[
B_k = \left( \begin{array}{cc} \gamma_k^{-1} B + 1 & \gamma_k^{-1} B \\ (k\gamma_k)^{-1} B & \gamma_k^{-1} B + 1 \end{array} \right).
\]

By a modification of the proof of Lemma 2.6 and Theorem 2.7 in [25], we get (3.8). For given \( (\Phi_1, \Phi_2) \) by (3.3)–(3.4), it follows from Proposition 3.1 (ii) that there is a sequence \( \{\Phi_{k,1}, \Phi_{k,2}\} \) satisfying (A, C, D), so if we let a sequence \( \{V_k\} \) be given by (3.1) and (3.2), the existence of the solution is immediate. The uniqueness of the solution follows by Gronwall’s inequality, and the semigroup property follows from the uniqueness of the solution.  

**Proposition 3.3** Suppose \( (\Phi_1, \Phi_2) \) are given by (3.3)–(3.4). For any \( \lambda \in \mathbb{R}_+, \) let \( t \mapsto V(t, \lambda) \) be the unique positive solution to (3.8). Then we can define a transition semigroup \( (P_t)_{t \geq 0} \) by
\[
\int_\mathbb{M} e^{-\langle \lambda, y \rangle} P_t (x, dy) = \exp \left\{ -\langle x, V(t, \lambda) \rangle \right\}, \quad x \in \mathbb{M}.
\]

**Proof** Given \( (\Phi_1, \Phi_2) \) by (3.3)–(3.4), by Proposition 3.1, there is a sequence \( (\Phi_{k,1}, \Phi_{k,2}) \) satisfying (A, C, D). By Proposition 3.2, for any \( a \geq 0 \) we have \( V_k (t, \lambda) \rightarrow V(t, \lambda) \) uniformly on \([0, a] \) as \( k \rightarrow \infty \). Take \( x_k \in \mathbb{M}_k \) satisfying \( x_k \rightarrow x \) as \( k \rightarrow \infty \). Then, by a continuity theorem (see, e.g., Theorem 1.18 in [23]), (3.10) defines a probability measure on \( \mathbb{M} \) and
\[
\lim_{k \rightarrow \infty} P_k (\gamma_k t, \cdot) = P_l (x, \cdot)
\]

weakly. The semigroup property of the family of \( (P_t)_{t \geq 0} \) follows from (3.9) and (3.10).

**Definition** A Markov process \( \{Y(t) = (Y_1(t), Y_2(t)) : t \geq 0\} \) is called a MSB-process with state space \( \mathbb{M} \) if it has the transition semigroup \( (P_t)_{t \geq 0} \) in (3.10).

**Proposition 3.4** Let \( (P_t)_{t \geq 0} \) be the transition semigroup defined by (3.10). Then we have
\[
\int_\mathbb{M} \langle \lambda, y \rangle P_t (x, dy) = \langle x, \pi(t, \lambda) \rangle, \quad \lambda \in \mathbb{R}_+, x \in \mathbb{M},
\]

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where \( t \mapsto \pi(t, \lambda) = (\pi_1(t, \lambda), \pi_2(t, \lambda)) \in \mathbb{R}_+^2 \) is the unique solution to the equations
\[
\begin{align*}
\frac{d\pi_1}{dt}(t, \lambda) &= -a_{11}\pi_1(t, \lambda) + \pi_2(t, \lambda) \int_M z_2 n_1(dz), \\
\frac{d\pi_2}{dt}(t, \lambda) &= a_{21}\pi_1(t, \lambda) + \int_{M^2} (z, \pi(t, \lambda)) n_2(dz)
\end{align*}
\]
with initial condition \( \pi(0, \lambda) = \lambda \).

**Proof** One can see that \( \phi(t, 0+) = 0 \) for \( t \geq 0 \). By differentiating both sides of (3.10) with respect to \( \lambda_1 \) and \( \lambda_2 \), we have
\[
\int_M y_1 P_t(x, dy) = x_1 \frac{\partial V_1(t, 0+)}{\partial \lambda_1} + x_2 \frac{\partial V_2(t, 0+)}{\partial \lambda_1},
\]
\[
\int_M y_2 P_t(x, dy) = x_1 \frac{\partial V_1(t, 0+)}{\partial \lambda_2} + x_2 \frac{\partial V_2(t, 0+)}{\partial \lambda_2}.
\]

It follows from (3.8) that
\[
\begin{align*}
\frac{\partial V_1(t, 0+)}{\partial \lambda_1} &= 1 + \int_0^t \left( -a_{11} \frac{\partial V_1(s, 0+)}{\partial \lambda_1} + \frac{\partial V_2(s, 0+)}{\partial \lambda_1} \int_M z_2 n_1(dz) \right) ds, \\
\frac{\partial V_1(t, 0+)}{\partial \lambda_2} &= \int_0^t \left( -a_{11} \frac{\partial V_1(s, 0+)}{\partial \lambda_2} + \frac{\partial V_2(s, 0+)}{\partial \lambda_2} \int_M z_2 n_1(dz) \right) ds,
\end{align*}
\]
and \( \phi_1 := a_{21} + \int_{M^2} z_1 n_2(dz) \)
\[
\begin{align*}
\frac{\partial V_2(t, 0+)}{\partial \lambda_1} &= \int_0^t \left( \frac{\partial V_1(s, 0+)}{\partial \lambda_1} \phi_1 + \frac{\partial V_2(s, 0+)}{\partial \lambda_1} \int_{M^2} z_2 n_2(dz) \right) ds, \\
\frac{\partial V_2(t, 0+)}{\partial \lambda_2} &= 1 + \int_0^t \left( \frac{\partial V_1(s, 0+)}{\partial \lambda_2} \phi_1 + \frac{\partial V_2(s, 0+)}{\partial \lambda_2} \int_{M^2} z_2 n_2(dz) \right) ds.
\end{align*}
\]

For \( \theta = (\theta_1, \theta_2) \in \mathbb{R}_+^2 \) and \( t \geq 0 \), we define \( \pi(t, \theta) = (\pi_1(t, \theta), \pi_2(t, \theta)) \) by
\[
\begin{align*}
\pi_1(t, \theta) &= \theta_1 \frac{\partial V_1(t, 0+)}{\partial \lambda_1} + \theta_2 \frac{\partial V_1(t, 0+)}{\partial \lambda_2}, \\
\pi_2(t, \theta) &= \theta_1 \frac{\partial V_2(t, 0+)}{\partial \lambda_1} + \theta_2 \frac{\partial V_2(t, 0+)}{\partial \lambda_2},
\end{align*}
\]
and we can conclude from the calculations above that
\[
\begin{align*}
\pi_1(t, \theta) &= \theta_1 + \int_0^t \left( -a_{11} \pi_1(s, \theta) + \int_M z_2 n_1(dz) \pi_2(s, \theta) \right) ds, \\
\pi_2(t, \theta) &= \theta_2 + \int_0^t \left( a_{21} \pi_1(s, \theta) + \int_{M^2} (z, \pi(s, \theta)) n_2(dz) \right) ds,
\end{align*}
\]
and the desired assertion follows.

By a modification of the proof of Theorem 2.11 in [25], one can see that the semigroup defined by (3.10) is a Feller semigroup. Then the MSB-process has a càdlàg realization. Moreover, the MSB-process can also be characterized in terms of the martingale problem described as follows (see Corollary 4.4 below for the proof): for \( f \in C^2(M) \), let \( L \) be an operator acting on \( C^2(M) \) defined by
\[
Lf(x) = x_1\left( \alpha f''_1(x) - a_{11} f'_1(x) + \int_M \left\{ f(x + z) - f(x) - z f'_1(x) \right\} n_1(dz) \right) \\
+ x_2\left( a_{21} f'_1(x) + \int_{M^2} \left\{ f(x + z) - f(x) \right\} n_2(dz) \right).
\]
Suppose that \( \{ (Y_1(t), Y_2(t)) : t \geq 0 \} \) is a non-negative càdlàg process with \( \mathbb{E}[Y_i(0)] < \infty, i = 1, 2 \). Then \( \{ (Y_1(t), Y_2(t)) : t \geq 0 \} \) is a MSB-process with transition semigroup \((P_t)_{t \geq 0}\) if and only if, for every \( f \in C^2(\mathbb{M})\),

\[
f(Y(t)) = f(Y(0)) + \int_0^t Lf(Y(s)) \, ds + \text{local mart}.
\]

**Theorem 3.5** Assume that \((A, C, D)\) hold, and that \((Y_{k,1}(0)/k, Y_{k,2}(0))\) converges to \((Y_1(0), Y_2(0))\) in distribution. Then

\[
\{(k^{-1}Y_{k,1}(\lfloor gt \rfloor), Y_{k,2}(\lfloor gt \rfloor)) : t \geq 0\} \Rightarrow \{(Y_1(t), Y_2(t)) : t \geq 0\}
\]
in distribution on \(D([0, \infty), \mathbb{M})\) as \( k \to \infty \).

**Proof** Let \( L \) be the generator of the MSB-process. For \( \lambda = (\lambda_1, \lambda_2) \gg 0, x \in \mathbb{M} \), set \( e_\lambda(x) = e^{-(\lambda \cdot x)} \). We have

\[
L e_\lambda(x) = -e^{-(\lambda \cdot x)} \{ x_1 \Phi_1(\lambda) + x_2 \Phi_2(\lambda) \}.
\]

Denote by \( D_1 \) the linear hull of \( \{ e_\lambda, \lambda \gg 0 \} \). Then \( D_1 \) is an algebra which strongly separates the points of \( \mathbb{M} \). Let \( C_0(\mathbb{M}) \) be the space of the continuous function on \( \mathbb{M} \) vanishing at infinity. By the Stone-Weierstrass theorem, \( D_1 \) is dense in \( C_0(\mathbb{M}) \) for the supremum norm. Noting that \( D_1 \) is invariant under \( P_t \) by (3.10), it follows from Proposition 3.3 in Chapter I of [8] that \( D_1 \) is the core of \( L \). Note that \( \{ Y_{k,1}(n)/k, Y_{k,2}(n) : n \geq 0 \} \) is a Markov chain with state space \( \mathbb{M}_k \), and the one-step transition probability is determined by

\[
\int_{\mathbb{M}_k} e^{-(\lambda \cdot y)} P_k(x, dy) = (g_{k,1}(e^{-\lambda_1/k}, e^{-\lambda_2}))^{kx_1} (g_{k,2}(e^{-\lambda_1/k}, e^{-\lambda_2}))^{kx_2}.
\]

The (discrete) generator \( L_k \) of \( \{(k^{-1}Y_{k,1}(\lfloor gt \rfloor), Y_{k,2}(\lfloor gt \rfloor)) : t \geq 0\} \) is given by

\[
L_k e_\lambda(x) = \gamma_k \left\{ (g_{k,1}(e^{-\lambda_1/k}, e^{-\lambda_2}))^{kx_1} \cdot (g_{k,2}(e^{-\lambda_1/k}, e^{-\lambda_2}))^{kx_2} - e^{-(\lambda \cdot x)} \right\}
\]

\[
= e^{-\lambda \cdot x} \gamma_k \left\{ \exp[kx_1 \log(1 - (k \gamma_k)^{-1} \Phi_{k,1}(\lambda_1, \lambda_2)e^{\lambda_1/k}) + x_2 \log(1 - (k \gamma_k)^{-1} \Phi_{k,2}(\lambda_1, \lambda_2)e^{\lambda_2})] - 1 \right\}
\]

\[
e^{-\lambda \cdot x} \left\{ x_1 \Phi_{k,1}(\lambda_1, \lambda_2) + x_2 \Phi_{k,2}(\lambda_1, \lambda_2) \right\} + o(1),
\]

where \( \Phi_{k,i}, \Phi_{k,i}, i = 1, 2 \) are defined as before. It follows from Proposition 3.1 that

\[
\lim_{k \to \infty} \sup_{x \in \mathbb{E}_k} |L_k e_\lambda(x) - L e_\lambda(x)| = 0.
\]

From Corollary 8.9 in Chapter 4 of [8], we prove the desired result. \( \square \)

**Theorem 3.6** Suppose that \( \{(Y_1(t), Y_2(t)) : t \geq 0\} \) is any MSB-process with \((\Phi_1, \Phi_2)\). Then, there exist a sequence of positive numbers \( \{ \gamma_k \} \) and a sequence of two-type GW-processes \( \{(Y_{k,1}(n), Y_{k,2}(n)) : n \in \mathbb{N}\} \) with generation functions \((g_{k,1}, g_{k,2})\) such that the sequence \( \{(k^{-1}Y_{k,1}(\lfloor gt \rfloor), Y_{k,2}(\lfloor gt \rfloor)) : t \geq 0\} \) converges in distribution on \(D([0, \infty), \mathbb{M})\) to the process \( \{(Y_1(t), Y_2(t)) : t \geq 0\} \) as \( k \to \infty \).

**Proof** By Proposition 3.1, there exist \( \{ \gamma_k \}, \{(g_{k,1}, g_{k,2})\} \) such that conditions \((A, C, D)\) hold. The desired result follows from Theorem 3.5. \( \square \)
4 The Construction of MSB-processes by Stochastic Equations

Let \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) be a complete filtered probability space satisfying the usual hypotheses, let \(\{B(t)\}\) be a standard Brownian motion, let \(\{N_1(ds,du,dz)\}\) be a Poisson random measure on \((0, \infty)^2 \times \mathbb{M}\) with intensity \(dsduN_1(dz)\), and let \(\{N_2(ds, du, dz)\}\) be a Poisson random measure on \((0, \infty)^2 \times \mathbb{M}\) with intensity \(dsduN_2(dz)\), \(z = (z_1, z_2)\). Suppose that \(B, N_1, N_2\) are independent of each other. Let us recall the stochastic integral equation system (1.7)–(1.8)

\[
Y_1(t) = Y_1(0) - \int_0^t a_{11}Y_1(s)\,ds + \int_0^t \sqrt{2\alpha Y_1(s)}\,dB(s) + \int_0^t \int_M z_1\tilde{N}_1(ds,du,dz) + \int_0^t \int_{M-1} z_1 N_1(ds,du,dz)
\]

\[
Y_2(t) = Y_2(0) + \int_0^t \int_0^t z_2\int_M N_1(ds,du,dz) + \int_0^t \int_{M-1} z_2 N_2(ds,du,dz),
\]

where \(\tilde{N}_1(ds,du,dz)\) is the compensated Poisson random measure of \(N_1\).

**Proposition 4.1** Suppose that \(\{Y(t)\}\) satisfies (1.7)–(1.8) and \(P\{Y(0) \geq 0\} = 1\). Then \(P\{Y(t) \geq 0, \forall t \geq 0\} = 1\).

**Proof** By equation (1.8), if \(Y_2(0) \geq 0\), it is not hard to see that, for all \(t \geq 0\), \(Y_2(t) \geq 0\). Now suppose that there exists \(\varepsilon > 0\) such that \(\tau := \inf\{t > 0, Y_1(t) \leq -\varepsilon\} < \infty\) with strictly positive probability. Then there exists \(t_0 > 0, Y_1(t_0) = 0\), and on the time interval \([t_0, \tau], t \mapsto Y_1(t)\) is a strictly negative continuous function. Hence there are some \(t_1 \in [t_0, \tau]\) and \(\delta > 0\) such that, for all \(s \in [t_0, t_1], -a_{11}Y_1(s) + a_{21}Y_2(s) \geq \delta\). Then

\[
Y_1(t_1) = Y_1(t_1) - Y_1(t_0) \geq \int_{t_0}^{t_1} (-a_{11}Y_1(s) + a_{21}Y_2(s))\,ds \geq \delta(t_1 - t_0) > 0,
\]

and since \(Y_1(t) < 0, \forall t \in (t_0, \tau]\), we get a contradiction.

To analyze the property of above equation system, we first construct a sequence of functions \(\{\phi_k\}\) on \(\mathbb{R}\) as follows: for each integer \(k \geq 0\), define \(a_k = \exp\{-k(k + 1)/2\}\). Then \(a_k \to 0\) decreasingly as \(k \to \infty\) and \(\int_{a_k}^{a_{k-1}} z^{-1}\,dz = k\) for \(k \geq 1\). Let \(x \mapsto \psi_k(x)\) be a non-negative continuous function on \(\mathbb{R}\) which has support in \((a_k, a_{k-1})\) and satisfies \(\int_{a_k}^{a_{k-1}} \psi_k(x)\,dx = 1\) and \(0 \leq \psi_k(x) \leq 2(kx)^{-1}\) for \(a_k < x < a_{k-1}\). For each \(k \geq 1\), let

\[
\phi_k(z) = \int_0^{|z|} dy \int_0^y \psi_k(x)\,dx, \quad z \in \mathbb{R}.
\]

Moreover, for function \(f\) on \(\mathbb{R}\), we denote

\[
\Delta_z f(x) = f(x + z) - f(x).
\]

**Theorem 4.2** The pathwise uniqueness for (1.7)–(1.8) holds.

**Proof** Suppose that \(\{Y(t)\}\) and \(\{Y'(t)\}\) are two solutions of (1.7)–(1.8). Let \(\zeta_i(t) = Y_i(t) - Y_i'(t), i = 1, 2\) for \(t \geq 0\). We have

\[
\zeta_1(t) = \zeta_1(0) - \int_0^t (a_{11}\zeta_1(s) + a_{21}\zeta_2(s))\,ds + \int_0^t \sqrt{2\alpha Y_1(s)}\,dB(s)
\]

\[
+ \int_0^t \int_{Y_1(s-)} z_1 \mathbf{1}_{\{\zeta_1(s-) > 0\}}\tilde{N}_1(ds,du,dz).
\]
- \int_0^t \int_{Y_1(s-)} \int_M z_1 \mathbf{1}_{\{\zeta_1(s-) \leq 0\}} \tilde{N}_1(ds, du, dz) \\
+ \int_0^t \int_{Y_2(s-)} \int_{M-1} z_1 \mathbf{1}_{\{\zeta_2(s-)> 0\}} N_2(ds, du, dz) \\
- \int_0^t \int_{Y_2(s-)} \int_{M-1} z_1 \mathbf{1}_{\{\zeta_2(s-) \leq 0\}} N_2(ds, du, dz).

Let \( \tau_m = \inf \{ t \geq 0 : Y_1(t) \lor Y_2(t) \lor Y_1'(t) \lor Y_2'(t) \geq m \} \) for \( m \geq 1 \). By similar calculations as to those in Theorem 3.4 in [29], there exists \( C_1 > 0 \) such that

\[
E[\zeta_1(t \land \tau_m)] \leq C_1 \int_0^t E\left( |\zeta_1(s \land \tau_m)| + |\zeta_2(s \land \tau_m)| \right) ds.
\]

On the other hand,

\[
\zeta_2(t) = \zeta_2(0) + \int_0^t \int_{Y_1(s-)} \int_M z_2 \mathbf{1}_{\{\zeta_1(s-) > 0\}} N_1(ds, du, dz) \\
- \int_0^t \int_{Y_1(s-)} \int_M z_2 \mathbf{1}_{\{\zeta_1(s-) \leq 0\}} N_1(ds, du, dz) \\
+ \int_0^t \int_{Y_2(s-)} \int_{M-1} z_2 \mathbf{1}_{\{\zeta_2(s-)> 0\}} N_2(ds, du, dz) \\
- \int_0^t \int_{Y_2(s-)} \int_{M-1} z_2 \mathbf{1}_{\{\zeta_2(s-) \leq 0\}} N_2(ds, du, dz).
\]

By Itô’s formula,

\[
\phi_k(\zeta_2(t \land \tau_m)) = \phi_k(\zeta_2(0)) + \int_0^{t \land \tau_m} \int_{Y_1(s-)} \int_M \Delta_{zz} \phi_k(\zeta_2(s-)) \mathbf{1}_{\{\zeta_1(s-) > 0\}} ds du n_1(dz) \\
+ \int_0^{t \land \tau_m} \int_{Y_1(s-)} \int_M \Delta_{zz} \phi_k(\zeta_2(s-)) \mathbf{1}_{\{\zeta_1(s-) \leq 0\}} ds du n_1(dz) \\
+ \int_0^{t \land \tau_m} \int_{Y_2(s-)} \int_{M-1} \Delta_{zz} \phi_k(\zeta_2(s-)) \mathbf{1}_{\{\zeta_2(s-) > 0\}} ds du n_2(dz) \\
+ \int_0^{t \land \tau_m} \int_{Y_2(s-)} \int_{M-1} \Delta_{zz} \phi_k(\zeta_2(s-)) \mathbf{1}_{\{\zeta_2(s-) \leq 0\}} ds du n_2(dz) + \text{mart.}
\]

Similarly, there exists \( C_2 > 0 \) such that

\[
E[\zeta_2(t \land \tau_m)] \leq C_2 \int_0^t E\left( |\zeta_1(s \land \tau_m)| + |\zeta_2(s \land \tau_m)| \right) ds.
\]

In conclusion,

\[
E\left( |\zeta_1(t \land \tau_m)| + |\zeta_2(t \land \tau_m)| \right) \leq (C_1 + C_2) \int_0^t E\left( |\zeta_1(s \land \tau_m)| + |\zeta_2(s \land \tau_m)| \right) ds.
\]

By Gronwall’s inequality, for all \( t \geq 0 \),

\[
E\left( |\zeta_1(t \land \tau_m)| + |\zeta_2(t \land \tau_m)| \right) = 0.
\]

Since \( \{Y(t)\} \) and \( \{Y'(t)\} \) have càdlàg sample paths, we conclude that \( P\{Y(t) = Y'(t), \forall t \geq 0\} = 1 \) as \( m \to \infty \).
**Theorem 4.3** There is a unique non-negative strong solution to (1.7)–(1.8).

**Proof** Since \( \nu_1 \) is supported on \( \mathbb{M} \setminus \{0\} \), we can rewrite (1.7)–(1.8) as

\[
Y_1(t) = Y_1(0) + \int_0^t \left( a_{21} Y_2(s) - a_{11} Y_1(s) \right) ds + \int_0^t \sqrt{2a Y_1(s)} dB(s) \\
+ \int_0^t \int_0^t \int_{\mathbb{M} - 1} z_1 \tilde{N}_1(ds, du, dz) + \int_0^t \int_0^t \int_{\mathbb{M} - 1} z_1 N_2(ds, du, dz), \\
Y_2(t) = Y_2(0) + \int_0^t \int_0^t \int_{\mathbb{M} - 1} z_2 N_1(ds, du, dz) + \int_0^t \int_0^t \int_{\mathbb{M} - 1} z_2 N_2(ds, du, dz).
\]

For any fixed \( n \geq 1 \), let \( V_n = \{ z \in \mathbb{M} : \| z \| \geq 1/n \} \), so \( n_1(V_n) + n_2(V_n) < \infty \). For \( m \geq 1 \) and \( x \in \mathbb{M} \), define

\[
b(x, m) = a_{21}(x_2 \land m) - a_{11}(x_1 \land m), \quad \theta(m, n) = \int_{V_n} (z_1 \land m) n_1(dz), \\
\beta_1(m) = \int_{\mathbb{M} - 1} (z_1 - z_1 \land m) n_1(dz), \quad \beta_2(m) = \int_{\mathbb{M} - 1} (z_2 - z_2 \land m) n_2(dz).
\]

By the results for continuous-type stochastic equations in [15, p.169], one can show that there is a non-negative weak solution to the following stochastic equation system:

\[
Y_1(t) = Y_1(0) + \int_0^t \left( b(Y(s), m) - (\beta_1(m) + \theta(m, n))(Y_1(s) \land m) \right) ds \\
+ \int_0^t \sqrt{2a(Y_1(s) \land m)} dB(s), \\
Y_2(t) = Y_2(0) - \int_0^t \beta_2(m)(Y_2(s) \land m) ds.
\]

The pathwise uniqueness holds for the above system of equations by similar arguments as to those in Theorem 4.2. Then it has a unique strong solution. By similar arguments as to those in the proof of Proposition 2.2 in [13], we can get a pathwise unique non-negative strong solution \( \{ Y_{m,n}(t) : t \geq 0 \} \) to (4.1)–(4.2) as follows:

\[
Y_1(t) = Y_1(0) + \int_0^t b(Y(s), m) - \beta_1(m)(Y_1(s) \land m) ds + \int_0^t \sqrt{2a(Y_1(s) \land m)} dB(s) \\
+ \int_0^t \int_{V_n} \int_{\mathbb{M} - 1} (z_1 \land m) \tilde{N}_1(ds, du, dz) \\
+ \int_0^t \int_{V_n} \int_{\mathbb{M} - 1} (z_1 \land m) N_2(ds, du, dz),
\]

\[
Y_2(t) = Y_2(0) - \int_0^t \beta_2(m)(Y_2(s) \land m) ds + \int_0^t \int_{V_n} \int_{\mathbb{M} - 1} (z_2 \land m) N_1(ds, du, dz) \\
+ \int_0^t \int_{V_n} \int_{\mathbb{M} - 1} (z_2 \land m) N_2(ds, du, dz).
\]

As in the proof of Lemma 4.3 in [13], one can see that the sequence \( \{ Y_{m,n}(t) : t \geq 0 \}, n = 1, 2, \cdots \) is tight in \( D([0, \infty), \mathbb{M}) \). Following the proof of Theorem 4.4 in [13], it is easy to show that any weak limit point \( \{ Y_m(t) : t \geq 0 \} \) of the sequence is a non-negative weak solution to

\[
Y_1(t) = Y_1(0) + \int_0^t b(Y(s), m) - \beta_1(m)(Y_1(s) \land m) ds + \int_0^t \sqrt{2a(Y_1(s) \land m)} dB(s).
\]
where \( G \) is an optimal random measure on \([0, 1)\). We have characterized it uniquely by the martingale problem. By a standard stopping time argument, \( \mathbb{P} \) is a transition semigroup defined by (3.8) and (3.10). Then the distributions of \( L \) are characterized uniquely by the martingale problem associated with the generator \( L \). Proposition 2.4 in [13].

By Theorem 4.2, the pathwise uniqueness holds for (4.3)–(4.4), so the system of equations has a unique strong solution. Finally, the desired result follows from a modification of the proof of Proposition 2.4 in [13].

**Corollary 4.4** A càdlàg non-negative process is a MSB-process with transition semigroup \((P_t)_{t \geq 0}\) defined by (3.8) and (3.10) if and only if it is a weak solution of (1.7)–(1.8).

**Proof** Suppose that \( \{(Y_1(t), Y_2(t))\}_{t \geq 0} \) is a weak solution of (1.7)–(1.8). By Itô’s formula, one can see that \( \{(Y_1(t), Y_2(t))\}_{t \geq 0} \) solves the martingale problem associated with the generator \( L \). By the arguments in Section 3, we infer that \( \{(Y_1(t), Y_2(t))\}_{t \geq 0} \) is a MSB-process with a transition semigroup \((P_t)_{t \geq 0}\) defined by (3.8) and (3.10). Conversely, suppose that \( \{(Y_1(t), Y_2(t))\}_{t \geq 0} \) is a càdlàg realization of the MSB-process with transition semigroup \((P_t)_{t \geq 0}\) defined by (3.8) and (3.10). Then the distributions of \( \{(Y_1(t), Y_2(t))\}_{t \geq 0} \) on \( D([0, \infty), \mathbb{M}) \) can be characterized uniquely by the martingale problem. By a standard stopping time argument, we have

\[
Y_1(t) = Y_1(0) - \int_0^t (a_{11} Y_1(s) - a_{21} Y_2(s)) \, ds + \int_0^t \int_{M_{-1}} Y_2(s) z_1 \, ds \, n_2(dz) + G_1(t),
\]

\[
Y_2(t) = Y_2(0) + \int_0^t \int_{M_1} Y_1(s) z_2 \, ds \, n_1(dz) + \int_0^t \int_{M_{-1}} Y_2(s) z_2 \, ds \, n_2(dz) + G_2(t),
\]

where \( G_1(t) \) and \( G_2(t) \) are two square-integrable local martingales. Let \( N_0(ds, dz) \) be the optimal random measure on \([0, \infty) \times M_{-1} \) defined by

\[
N_0(ds, dz) := \sum_{s > 0} \mathbf{1}_{\{(Y_1(s), Y_2(s)) \neq (Y_1(s-), Y_2(s-))\}} \delta_{(s, Y(s)-Y(s-))}(ds, dz).
\]

It follows from [7, p.376] that

\[
G_1(t) = G_1^t(0) + \int_0^t \int_{M_{-1}} z_1 \, \tilde{N}_0(ds, dz), \quad G_2(t) = G_2^t(0) + \int_0^t \int_{M_{-1}} z_2 \, \tilde{N}_0(ds, dz),
\]

where \( t \mapsto G_1^t(0) \) and \( t \mapsto G_2^t(0) \) are two continuous local martingales with quadratic variations \( t \mapsto C_1(t) \) and \( t \mapsto C_2(t) \), respectively. Applying Itô’s formula to (4.5)–(4.6), and using the uniqueness of canonical decompositions of semi-martingales, we find that \( N_0(ds, dz) \) has a predictable compensator

\[
\tilde{N}_0(ds, dz) = Y_1(s-) \, ds \, n_1(dz) + Y_2(s-) \, ds \, n_2(dz),
\]

d\( C_1(t) = 2\alpha Y_1(t) \, dt \) and \( dC_2(t) = 0 \). Then we obtain the equation (1.7)–(1.8) on an extension of the probability space by applying martingale representation theorems; see, e.g., [15, p.93, p.84]. This completes the proof.

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5 The Distribution of Local Jumps

For any initial time \( r \geq 0 \), let \( Y = (\Omega, \mathcal{F}, \mathcal{F}_r, Y(t), \mathbf{P}_{r,y}: t \geq r, y \geq 0) \) be a Hunt realization of the MSB-process with transition semigroup \((P_t)_{t \geq 0}\) defined by (3.8) and (3.10). Here, \( \{\mathbf{P}_{r,y}: y \geq 0\} \) is a family of probability measures on \( (\Omega, \mathcal{F}, \mathcal{F}_r) \) satisfying \( \mathbf{P}_{r,y}\{Y(r) = y\} = 1 \) for all \( y \geq 0 \). For any \( t \geq r \geq 0 \) and \( \lambda \in [0, \infty)^2 \), we have

\[
\mathbf{P}_{r,y} \exp \left\{ -\langle \lambda, Y(t) \rangle \right\} = \exp \left\{ -\langle y, \bar{V}(r, \lambda) \rangle \right\},
\]

where \( r \rightarrow \bar{V}(r, \lambda) := V(t - r, \lambda) \) satisfies

\[
\bar{V}_1(r, \lambda) = \int_r^t \Phi_1(\bar{V}(s, \lambda)) \, ds + \lambda_1, \quad \bar{V}_2(r, \lambda) = \int_r^t \Phi_2(\bar{V}(s, \lambda)) \, ds + \lambda_2, \quad 0 \leq r \leq t.
\]

By modifying the arguments of Proposition 4.1, Theorem 4.2 and Corollary 4.4 in [25], we have

**Proposition 5.1** For \( \{t_1 < \cdots < t_n\} \subset [0, \infty) \) and \( \{\lambda_1, \cdots, \lambda_n\} \subset [0, \infty)^2 \), we have

\[
\mathbf{P}_{r,y} \exp \left\{ -\sum_{j=1}^n \langle \lambda_j, Y(t_j) \rangle 1_{\{r \leq t_j\}} \right\} = \exp \left\{ -\langle y, \bar{V}(r) \rangle \right\}, \quad 0 \leq r \leq t_n,
\]

where \( \bar{V}(r) = \bar{V}(r, \lambda_1, \cdots, \lambda_n) \) on \( [0, t_n] \) satisfies

\[
\bar{V}_1(r) = \int_r^{t_n} \Phi_1(\bar{V}(s)) \, ds + \sum_{j=1}^n \lambda_j 1_{\{r \leq t_j\}}, \quad \bar{V}_2(r) = \int_r^{t_n} \Phi_2(\bar{V}(s)) \, ds + \sum_{j=1}^n \lambda_{2j} 1_{\{r \leq t_j\}}.
\]

**Proposition 5.2** Suppose that \( t \geq 0 \) and that \( \mu \) is a finite measure supported by \( [0, t] \). Letting \( s \mapsto \lambda(s) = (\lambda_1(s), \lambda_2(s)) \) be a bounded positive Borel function on \( [0, t] \), we have

\[
\mathbf{P}_{r,y} \exp \left\{ -\int_{[r,t]} \langle \lambda(s), Y(s) \rangle \mu(ds) \right\} = \exp \left\{ -\langle y, \bar{V}(r) \rangle \right\}, \quad 0 \leq r \leq t,
\]

where \( r \mapsto \bar{V}(r) = \bar{V}(r, \lambda_1, \lambda_2) \) is the positive solution on \( [0, t] \) of

\[
\bar{V}_1(r) = \int_r^t \Phi_1(\bar{V}(s)) \, ds + \int_r^t \lambda_1(s) \mu(ds), \quad \bar{V}_2(r) = \int_r^t \Phi_2(\bar{V}(s)) \, ds + \int_r^t \lambda_2(s) \mu(ds).
\]

**Corollary 5.3** Let \( Y = (\Omega, \mathcal{F}, \mathcal{F}_t, Y(t), \mathbf{P}_y) \) be a Hunt realization of the MSB-process started from time zero. Then, for \( t \geq 0, \lambda = (\lambda_1, \lambda_2) \in [0, \infty)^2 \), we have

\[
\mathbf{P}_y \exp \left\{ -\int_0^t \langle \lambda, Y(s) \rangle \, ds \right\} = \exp \left\{ -\langle y, \bar{V}(t) \rangle \right\},
\]

where \( t \mapsto \bar{V}(t) = \bar{V}(t, \lambda) \) is the positive solution of

\[
\bar{V}_1(t) = \int_0^t \Phi_1(\bar{V}(s)) \, ds + \lambda_1 t, \quad \bar{V}_2(t) = \int_0^t \Phi_2(\bar{V}(s)) \, ds + \lambda_2 t.
\]

We shall introduce some notations before presenting the main results in this section. Let \( r = (r_1, r_2) \in [0, \infty)^2 \), \( A_r = (r_1, \infty) \times (r_2, \infty) \) and \( n(A_r) = (n_1(A_r), n_2(A_r)) \). We define two functions on \( [0, \infty)^2 \):

\[
\Phi_1^r(\lambda_1, \lambda_2) = -a_{11}^r \lambda_1 + b_{11}^r \lambda_2 - \alpha \lambda_1^2 - \int_{\mathbb{R} \setminus A_r} (e^{-\langle \lambda, z \rangle} - 1 + \langle \lambda, z \rangle) n_1(dz),
\]

\[
\Phi_2^r(\lambda_1, \lambda_2) = a_{21}^r \lambda_1 + b_{21}^r \lambda_2 - \alpha \lambda_2^2 - \int_{\mathbb{R} \setminus A_r} (e^{-\langle \lambda, z \rangle} - 1 + \langle \lambda, z \rangle) n_2(dz).
\]
Here

\[ a_{11}' = a_{11} + \int_{A_r} z_1 n_1(dz), \quad a_{21}' = a_{21} + \int_{M - 1 \setminus A_r} z_1 n_2(dz), \]

\[ b_{11}' = \int_{M \setminus A_r} z_2 n_1(dz), \quad b_{21}' = \int_{M - 1 \setminus A_r} z_2 n_2(dz). \]

The following theorem gives a characterization of the distribution of the local maximal jump of the MSB-process:

**Theorem 5.4** Let \( \tau_r = \inf \{ s \geq 0 : \Delta Y_1(s) > r_1 \text{ or } \Delta Y_2(s) > r_2 \} \). Then we have

\[ P_y(\tau_r > t) = \exp \left\{ - \langle y, \bar{V}_r(t, n(A_r)) \rangle \right\}, \]

where \( \bar{V}_r(t) = (\bar{V}_1(t), \bar{V}_2(t)) \) is the solution of

\[ \bar{V}_1(t) = \int_0^t \Phi_1^r(\bar{V}_r(s)) ds + n_1(A_r) t, \quad \bar{V}_2(t) = \int_0^t \Phi_2^r(\bar{V}_r(s)) ds + n_2(A_r) t. \]

**Proof** We can rewrite equations (1.7)–(1.8) as

\[ Y_1(t) = Y_1(0) + \int_0^t \left( a_{11}' Y_2(s) - a_{11} Y_1(s) \right) ds + \int_0^t \sqrt{2a Y_1(s)} dB(s) \]

\[ + \int_0^t \int_0^s \int_{M \setminus A_r} z_1 \tilde{N}_1(ds, du, dz) + \int_0^t \int_0^s \int_{M - 1 \setminus A_r} z_1 \tilde{N}_2(ds, du, dz) \]

\[ + \int_0^t \int_0^s \int_{A_r} z_1 N_1(ds, du, dz) + \int_0^t \int_0^s \int_{A_r} z_1 N_2(ds, du, dz), \]

\[ Y_2(t) = Y_2(0) + \int_0^t \left( b_{11}' Y_1(s) + b_{21}' Y_2(s) \right) ds \]

\[ + \int_0^t \int_0^s \int_{M \setminus A_r} z_2 \tilde{N}_1(ds, du, dz) + \int_0^t \int_0^s \int_{M - 1 \setminus A_r} z_2 \tilde{N}_2(ds, du, dz) \]

\[ + \int_0^t \int_0^s \int_{A_r} z_2 N_1(ds, du, dz) + \int_0^t \int_0^s \int_{A_r} z_2 N_2(ds, du, dz). \]

Let \( (Y_1^{r_1}(t), Y_2^{r_2}(t)) : t \geq 0 \) be the strong solution to

\[ Y_1^{r_1}(t) = Y_1(0) + \int_0^t \left( a_{11}' Y_2^{r_1}(s) - a_{11} Y_1^{r_1}(s) \right) ds + \int_0^t \sqrt{2a Y_1^{r_1}(s)} dB(s) \]

\[ + \int_0^t \int_0^s \int_{M \setminus A_r} z_1 \tilde{N}_1(ds, du, dz) + \int_0^t \int_0^s \int_{M - 1 \setminus A_r} z_1 \tilde{N}_2(ds, du, dz), \]

\[ Y_2^{r_2}(t) = Y_2(0) + \int_0^t \left( b_{11}' Y_1^{r_1}(s) + b_{21}' Y_2^{r_2}(s) \right) ds \]

\[ + \int_0^t \int_0^s \int_{M \setminus A_r} z_2 \tilde{N}_1(ds, du, dz) + \int_0^t \int_0^s \int_{M - 1 \setminus A_r} z_2 \tilde{N}_2(ds, du, dz). \]

Then \( (Y_1^{r_1}(t), Y_2^{r_2}(t)) : t \geq 0 \) is a MSB-process with branching mechanism \( (\Phi_1^r, \Phi_2^r) \). It is easy to see that \( (Y_1^{r_1}(t), Y_2^{r_2}(t)) = (Y_1(s), Y_1(s)) \) for \( 0 \leq s < \tau_r \) and that

\[ \{ \tau_r > t \} = \left\{ \max_{0 \leq s \leq t} \Delta Y_1(s) \leq r_1, \max_{0 \leq s \leq t} \Delta Y_2(s) \leq r_2 \right\} \]

\[ = \left\{ \int_0^t \int_{A_r} N_1(ds, du, dz) = 0, \int_0^t \int_{A_r} N_2(ds, du, dz) = 0 \right\}. \]
\[= \left\{ \int_0^t \int_0^1 Y_{11}^{(s-)}(s) \int_{A_r} N_1(ds, du, dz) = 0, \int_0^t \int_0^1 Y_{22}^{(s-)}(s) \int_{A_r} N_2(ds, du, dz) = 0 \right\}.\]

Note that \(N_1\) and \(N_2\) restricted to \((0, \infty)^2 \times A_r\) are independent of \(\{(Y_{11}^{(t)}, Y_{22}^{(t)}(t)) : t \geq 0\}\).

It follows that
\[P_y\{\tau_r > t\} = P_y \exp \left\{ -n_1(A_r) \int_0^t Y_{11}^{(s)}(s) ds - n_2(A_r) \int_0^t Y_{22}^{(s)}(s) ds \right\}.\]

Finally, the desired result follows from (5.1)–(5.4).

**Corollary 5.5** Suppose that both \(n_1\) and \(n_2\) have unbounded supports. As \(r \to \infty\), we have
\[P_y\{\tau_r \leq t\} \sim \left( y_1, y_2 \right) \int_0^t e^{(t-s)H} ds \begin{pmatrix} n_1(A_r) \\ n_2(A_r) \end{pmatrix},\]
where
\[H = \begin{pmatrix} -a_{11} & \int_M z_2 n_1(dz) \\ a_{21} + \int_{M \setminus A_q} z_1 n_2(dz) & \int_{M \setminus A_q} z_2 n_2(dz) \end{pmatrix} \]
and \(e^{(t-s)H} = \sum_{k=0}^\infty \frac{(t-s)^k H^k}{k!}\).

**Proof** For \(r, q \in [0, \infty)^2, q_i \geq r_i, i = 1, 2\), we have, obviously, that \(\Phi_i \geq \Phi_i^r \geq \Phi_i^q, i = 1, 2\). Then, by Proposition 5.2, we see that \(\tilde{V}_i \geq \tilde{V}_i^r \geq \tilde{V}_i^q, i = 1, 2\). It follows that
\[1 - \exp\left\{ -\left( y, \tilde{V}_i^q(t, n(A_r)) \right) \right\} \leq P_y\{\tau_r \leq t\} \leq 1 - \exp\left\{ -\left( y, \tilde{V}(t, n(A_r)) \right) \right\}.\]

Note that \(\tilde{V}_i^q(t, 0+) = \tilde{V}_i^r(t, 0+) = \tilde{V}(t, 0+) = 0\). Moreover, we can calculate that
\[\frac{\partial}{\partial t} \frac{\partial}{\partial \lambda} \tilde{V}_i(t, 0) = e^{(1)} - a_{11} \frac{\partial}{\partial \lambda} \tilde{V}_i(t, 0) + \int_M z_2 n_1(dz) \cdot \frac{\partial}{\partial \lambda} \tilde{V}_2(t, 0),\]
\[\frac{\partial}{\partial t} \frac{\partial}{\partial \lambda} \tilde{V}_2(t, 0) = e^{(2)} + (a_{21} + \int_{M \setminus A_q} z_1 n_2(dz)) \frac{\partial}{\partial \lambda} \tilde{V}_1(t, 0) + \int_{M \setminus A_q} z_2 n_2(dz) \cdot \frac{\partial}{\partial \lambda} \tilde{V}_2(t, 0),\]
\[\frac{\partial}{\partial \lambda} \tilde{V}_1(0, 0) = \frac{\partial}{\partial \lambda} \tilde{V}_2(0, 0) = 0.\]

We can solve the above equations to get
\[\begin{pmatrix} \frac{\partial}{\partial \lambda} \tilde{V}_1(t, 0) \\ \frac{\partial}{\partial \lambda} \tilde{V}_2(t, 0) \end{pmatrix} = \int_0^t e^{(t-s)H} ds. \quad (5.5)\]

Similarly, we have
\[\begin{pmatrix} \frac{\partial}{\partial \lambda} \tilde{V}_1^q(t, 0) \\ \frac{\partial}{\partial \lambda} \tilde{V}_2^q(t, 0) \end{pmatrix} = \int_0^t e^{(t-s)H_q} ds, \quad (5.6)\]
where
\[H_q = \begin{pmatrix} -a_{11}^q & \int_{M \setminus A_q} z_2 n_1(dz) \\ a_{21} + \int_{M \setminus A_q} z_1 n_2(dz) & \int_{M \setminus A_q} z_2 n_2(dz) \end{pmatrix}.\]
By (5.5) and (5.6), as \( r \to \infty \),
\[
1 - \exp \left\{ - \langle y, \tilde{V}(t, n(A_r)) \rangle \right\} \sim (y, \tilde{V}(t, n(A_r)))
\]
\[
\sim (y_1, y_2) \int_0^t e^{(t-s)H} \, ds \begin{pmatrix} n_1(A_r) \\ n_2(A_r) \end{pmatrix}
\]
and
\[
1 - \exp \left\{ - \langle y, \tilde{V}^q(t, n(A_r)) \rangle \right\} \sim (y, \tilde{V}^q(t, n(A_r)))
\]
\[
\sim (y_1, y_2) \int_0^t e^{(t-s)H_q} \, ds \begin{pmatrix} n_1(A_r) \\ n_2(A_r) \end{pmatrix}.
\]
Then we complete the proof by noticing that \( \lim_{q \to \infty} H_q = H \). \( \square \)

6 Exponential Ergodicity in Wasserstein Distances

In order to present our results in this section, we first introduce some notations. Given two probability measures \( \mu \) and \( \nu \) on \( \mathbb{M} \), the standard \( L^p \)-Wasserstein distance \( W_p \) for all \( p \geq 1 \) is given by
\[
W_p(\mu, \nu) = \inf_{\Pi \in \mathcal{C}(\mu, \nu)} \left( \int_{\mathbb{M} \times \mathbb{M}} |x - y|^p \, \Pi(dx, dy) \right)^{1/p},
\]
where \( | \cdot | \) denotes the Euclidean norm and \( \mathcal{C}(\mu, \nu) \) stands for the set of all coupling measures of \( \mu \) and \( \nu \), i.e., \( \mathcal{C}(\mu, \nu) \) is the collection of measures on \( \mathbb{M} \times \mathbb{M} \) having \( \mu \) and \( \nu \) as marginals. Denoting \( \mathcal{P}_p(\mathbb{M}) \) as the set of probability measures having a finite moment of order \( p \), it is known that \( (\mathcal{P}_p(\mathbb{M}), W_p) \) becomes a Polish space.

The next theorem gives the upper and lower bounds for the variations in the \( L^1 \)-Wasserstein distance \( W_1 \) of the transition probabilities of the MSB-process started from two different initial states.

**Theorem 6.1** Let \( (P_i)_{i \geq 0} \) be the transition semigroup defined by (3.10). Then for all \( x, y \in \mathbb{M} \) and \( t \geq 0 \), we have
\[
|\langle x - y, \pi(t, 1) \rangle| \leq W_1(\delta_x P_t, \delta_y P_t) \leq \sum_{i=1}^2 |x_i - y_i| \pi_i(t, 1),
\]
where \( \delta_x P_t(\cdot) := P_t(\cdot, x) \) and \( \pi(t, 1) \) is defined as in Proposition 3.4 with \( \lambda = (1, 1) \).

**Proof** The proof is based on the same idea as that of Theorem 2.2 in [26]. By Proposition 3.4, we see that \( \int_{\mathbb{M}} (y_1 + y_2) P_t(x, dy) = \langle x, \pi(t, 1) \rangle \). It follows from Theorem 5.10 in [4] that
\[
W_1(\delta_x P_t, \delta_y P_t) \geq \int_{\mathbb{M}} (z_1 + z_2) \left( P_t(x, dz) - P_t(y, dz) \right) = \langle x - y, \pi(t, 1) \rangle.
\]
Similarly, \( W_1(\delta_x P_t, \delta_y P_t) \geq \langle y - x, \pi(t, 1) \rangle \). Then the first inequality follows. On the other hand, for \( x, y \in \mathbb{M} \), let \( (x - y)_\pm := ((x_1 - y_1)_\pm, (x_2 - y_2)_\pm) \), and \( x \wedge y := x - (x - y)_+ = y - (x - y)_- \). Let \( P_t(x, y, d\gamma_1, d\gamma_2) \) be the image of the product measure
\[
P_t(x \wedge y, d\gamma_1) P_t((x - y)_+, d\gamma_1) P_t((x - y)_-, d\gamma_2)
\]
\( \square \) Springer
under the mapping \((\gamma_0, \gamma_1, \gamma_2) \mapsto (\eta_1, \eta_2) := (\gamma_0 + \gamma_1, \gamma_0 + \gamma_2)\). It is not hard to see that \(P_t(x, y, d\eta_1, d\eta_2)\) is a coupling of \(P_t(x, d\eta_1)\) and \(P_t(y, d\eta_2)\). Then
\[
W_1(\delta_x P_t, \delta_y P_t) \leq \int_{\mathbb{M}^2} |\eta_1 - \eta_2| P_t(x, y, d\eta_1, d\eta_2) \\
\leq \int_{\mathbb{M}} P_t((x-y)_+, d\gamma_1) \int_{\mathbb{M}} (\gamma_{11} + \gamma_{12} + \gamma_{21} + \gamma_{22}) P_t((x-y)_-, d\gamma_2) \\
= \int_{\mathbb{M}} (\zeta_1 + \zeta_2) P_t((|x_1 - y_1|, |x_2 - y_2|), d\zeta) \\
= \sum_{i=1}^2 |x_i - y_i| \pi_i(t, 1),
\]
where we have used the branching property \(P_t(a, \cdot) \ast P_t(b, \cdot) = P_t(a + b, \cdot)\) for all \(a, b \in \mathbb{M}, t \geq 0\) in the third row. Therefore the proof is finished. \(\square\)

Based on Theorem 6.1, we can establish the exponential ergodicity with respect to \(W_1\). Recalling that a \(2 \times 2\) matrix \(H = [H_{ij}]_{2 \times 2}\) in Corollary 5.5 is defined as
\[
H = \begin{pmatrix}
-a_{11} & \int_{\mathbb{M}} z_2 n_1(dz) \\
\int_{\mathbb{M}_i} z_1 n_2(dz) & \int_{\mathbb{M}_i} z_2 n_2(dz)
\end{pmatrix},
\]
we have the following result:

**Theorem 6.2** Assume that \(H_{11}H_{22} - H_{12}H_{21} > 0\) and \(H_{11} + H_{22} < 0\). Then there exist \(\lambda, \vartheta > 0\) such that, for any \(t \geq 0\) and \(x, y \in \mathbb{M}\),
\[
W_1(\delta_x P_t, \delta_y P_t) \leq \vartheta |x - y| e^{-\lambda t}.
\]

**Proof** By assumption, it is easy to see that
\[
\lambda^2 - (H_{11} + H_{22})\lambda + H_{11}H_{22} - H_{12}H_{21} = 0
\]
has two different roots: \(\lambda_1 = 2^{-1}(H_{11} + H_{22} + \sqrt{\Delta})\), \(\lambda_2 = \lambda_1 - \sqrt{\Delta}\) and \(\lambda_2 < \lambda_1 < 0\). Here \(\Delta = (H_{11} - H_{22})^2 + 4H_{12}H_{21} > 0\). If \(H_{12} = H_{21} = 0\), then \(\lambda_1 = H_{11}\) and \(\lambda_2 = H_{22}\).

By Proposition 3.4, \(\pi_i(t, 1) = e^{\lambda_i t}\) for \(i = 1, 2\), and the desired result follows. Next we only consider \(H_{12} > 0\). We can calculate that
\[
\pi_1(t, 1) = \frac{H_{11} + H_{12} - \lambda_2}{\sqrt{\Delta}} e^{\lambda_1 t} + \frac{\lambda_1 - H_{11} - H_{12}}{\sqrt{\Delta}} e^{\lambda_2 t} \\
:= \theta_{11} e^{\lambda_1 t} + \theta_{12} e^{\lambda_2 t},
\]
\[
\pi_2(t, 1) = \frac{(H_{11} + H_{12} - \lambda_2)(\lambda_1 - H_{11})}{\sqrt{\Delta} H_{12}} e^{\lambda_1 t} + \frac{(\lambda_1 - H_{11} - H_{12})(\lambda_2 - H_{11})}{\sqrt{\Delta} H_{12}} e^{\lambda_2 t} \\
:= \theta_{21} e^{\lambda_1 t} + \theta_{22} e^{\lambda_2 t}.
\]

It is easy to see that \(\theta_{11}, \theta_{21} > 0\). It follows from Theorem 6.1 that
\[
W_1(\delta_x P_t, \delta_y P_t) \leq |x_1 - y_1| (|\theta_{11} e^{\lambda_1 t} + \theta_{12} e^{\lambda_2 t}| + |x_2 - y_2| (|\theta_{21} e^{\lambda_1 t} + \theta_{22} e^{\lambda_2 t}| \\
\leq |x - y|(|\theta_{11}| + |\theta_{21}|) e^{\lambda_1 t} + |x - y|(|\theta_{12}| + |\theta_{22}|) e^{\lambda_2 t} \\
\leq (\theta_{11} + \theta_{21} + |\theta_{12}| + |\theta_{22}|)|x - y| e^{\lambda_1 t},
\]
and we obtain the desired result by setting \(\vartheta = \theta_{11} + \theta_{21} + |\theta_{12}| + |\theta_{22}| > 0\) and \(\lambda = -\lambda_1 > 0\). \(\square\)
Corollary 6.3 Assume that the conditions of Theorem 6.2 hold. Then there exist a unique \( \pi \in \mathcal{P}_1(\mathbb{M}) \) and \( \vartheta, \lambda > 0 \) such that, for any \( x \in \mathbb{M} \) and \( t \geq 0 \),

\[
W_1(\delta_x P_t, \pi) \leq \vartheta W_1(\delta_x, \pi)e^{-\lambda t}.
\]

Proof By Theorem 7.5 below, there exists a unique invariant measure. Arguing similarly as to the proof of Theorem 3.2 in [12], one can see that \( \pi \in \mathcal{P}_1(\mathbb{M}) \), and the desired assertion is easily obtained by Theorem 6.2. \( \Box \)

7 MSBI-processes

Suppose that \( \Phi_1, \Phi_2 \) are two functions on \([0, \infty)^2\) defined as in (3.3)–(3.4), and that there exists function \( \Psi \) on \([0, \infty)^2\) defined by

\[
\Psi(\lambda_1, \lambda_2) = b\lambda_1 + \int_{\mathbb{M}} (1 - e^{-(\lambda_1, \lambda_2)}) m(dz), \quad \lambda \in \mathbb{R}_+^2,
\]

where \( b > 0 \) and \( m \) is a \( \sigma \)-finite measure on \( \mathbb{M} \) supported by \( \mathbb{M} \setminus \{0\} \) such that

\[
\int_{\mathbb{M}} (1 + z_1 + z_2) m(dz) < \infty.
\]

A Markov process \( \{Z(t) = (Z_1(t), Z_2(t)) : t \geq 0\} \) is called a MSBI-process on \( \mathbb{M} \) if it has a transition semigroup \( (P_t)_{t \geq 0} \) uniquely determined by

\[
\int_{\mathbb{M}} e^{-(\lambda, y)} P_t^\gamma(x, dy) = \exp \left\{ -\langle x, V(t, \lambda) \rangle - \int_0^t \Psi(V(s, \lambda)) ds \right\}, \quad x \in \mathbb{M}, \lambda \in \mathbb{R}_+^2,
\]

where \( V(t, \lambda) = (V_1(t, \lambda), V_2(t, \lambda)) \) takes values on \( \mathbb{R}_+^2 \) and satisfies (3.8). One can see that the semigroup defined by (7.2) is a Feller semigroup, so the MSBI-process has a càdlàg realization. We can also establish a similar result as to that of Theorem 6.1 for MSBI-processes; indeed, we have the following:

Theorem 7.1 Let \( (P_t^\gamma)_{t \geq 0} \) be the transition semigroup defined by (7.2). Assume that \( \int_{\mathbb{M}} (z_1 + z_2) m(dz) < \infty \). Then, for \( t \geq 0 \) and \( x, y \in \mathbb{M} \), we have

\[
|\langle x - y, \pi(t, 1) \rangle| \leq W_1(\delta_x P_t^\gamma, \delta_y P_t^\gamma) \leq \sum_{i=1}^2 |x_i - y_i| \pi_i(t, 1),
\]

where \( \pi(t, 1) \) is defined as in Proposition 3.4 with \( \lambda = (1, 1) \).

Proof The proof is based on the same idea as that of Theorem 4.1 in [26]. One can see that

\[
\int_{\mathbb{M}} \langle y_1 + y_2, P_t^\gamma(x, dy) \rangle = \langle x, \pi(t, 1) \rangle + (b + \int_{\mathbb{M}} z_1 m(dz)) \int_0^t \pi_1(s, 1) ds + \int_{\mathbb{M}} z_2 m(dz) \int_0^t \pi_2(s, 1) ds,
\]

which yields that

\[
W_1(\delta_x P_t^\gamma, \delta_y P_t^\gamma) \geq \int_{\mathbb{M}} (z_1 + z_2) |P_t^\gamma(x, dz) - P_t^\gamma(y, dz)| = \langle x - y, \pi(t, 1) \rangle.
\]

Similarly, \( W_1(\delta_x P_t^\gamma, \delta_y P_t^\gamma) \geq \langle y - x, \pi(t, 1) \rangle \), and the first inequality follows. Next, we want to construct a coupling measure of \( P_t^\gamma(x, \cdot) \) and \( P_t^\gamma(y, \cdot) \). It is known that there exists a family of
probability measures \((\gamma_t)_{t \geq 0}\) such that \(P_t^\gamma(x, \cdot) = P_t(x, \cdot) * \gamma_t(\cdot)\) for \(t \geq 0, x \in \mathbb{M}\), and
\[
\int_{\mathbb{M}} e^{-\langle \lambda, y \rangle} \gamma_t(dy) = \exp \left\{ -\int_0^t \Psi(V(s, \lambda)) \, ds \right\},
\]
and we call \((\gamma_t)_{t \geq 0}\) a skew convolution semigroup associated with \((P_t)_{t \geq 0}\); see, e.g., Chapter 9 in [23]. Let \(P_t(x, y, dm_1, dm_2)\) be the coupling measure of \(P_t(x, dm_1)\) and \(P_t(y, dm_2)\) constructed in the proof of Theorem 6.1 and let \(P_t^\gamma(x, y, d\sigma_1, d\sigma_2)\) be the image of \(\gamma_t(dm_0)P_t(x, y, dm_1, dm_2)\) under the mapping \((\eta_0, \eta_1, \eta_2) \mapsto (\sigma_1, \sigma_2) = (\eta_0 + \eta_1, \eta_0 + \eta_2)\). By the relation \(P_t^\gamma(x, \cdot) = P_t(x, \cdot) * \gamma_t(\cdot)\), we see that \(P_t^\gamma(x, y, d\sigma_1, d\sigma_2)\) is a coupling measure of \(P_t^\gamma(x, d\sigma_1)\) and \(P_t^\gamma(y, d\sigma_2)\). It follows that
\[
W_1(\delta_x P_t^\gamma, \delta_y P_t^\gamma) \leq \int_{\mathbb{M}^2} |\sigma_1 - \sigma_2| P_t^\gamma(x, y, d\sigma_1, d\sigma_2)
\]
\[
= \int_{\mathbb{M}} \gamma_t(dm_0) \int_{\mathbb{M}^2} |\eta_1 - \eta_2| P_t(x, y, dm_1, dm_2)
\]
\[
= \int_{\mathbb{M}^2} |\eta_1 - \eta_2| P_t(x, y, dm_1, dm_2) \leq \sum_{i=1}^2 |x_i - y_i| \pi_t(1, 1),
\]
where the last inequality follows from Theorem 6.1.

By a similar argument as to that of Theorem 6.2, we have

**Theorem 7.2** Assume that \(H_{11}H_{22} - H_{12}H_{21} > 0\) and \(H_{11} + H_{22} < 0\). Then, there exist \(\lambda, \vartheta > 0\) such that, for any \(t \geq 0\) and \(x, y \in \mathbb{M}\),
\[
W_1(\delta_x P_t^\gamma, \delta_y P_t^\gamma) \leq \vartheta |x - y| e^{-\lambda t}.
\]

### 7.1 The construction of MSBI-processes by stochastic equations

We now give a construction of MSBI-processes by stochastic equations. Let us consider the stochastic equation system
\[
Z_1(t) = Z_1(0) + \int_0^t \left( b - a_{11}Z_1(s) + a_{21}Z_2(s) \right) \, ds + \int_0^t \sqrt{2a Z_1(s)} \, dB(s)
\]
\[
+ \int_0^t \int_0^t \int_{\mathbb{M}} Z_1(s^{-}) \, dN_1(ds, du, dz) + \int_0^t \int_{\mathbb{M}} Z_1(s^{-}) \, dM(ds, dz)
\]
\[
+ \int_0^t \int_0^t \int_{\mathbb{M}^{-1}} Z_1(s^{-}) \, dN_2(ds, du, dz),
\]
\(7.3\)
\[
Z_2(t) = Z_2(0) + \int_0^t \int_0^t \int_{\mathbb{M}} Z_2(s^{-}) \, dN_1(ds, du, dz) + \int_0^t \int_{\mathbb{M}} Z_2(s^{-}) \, dM(ds, dz)
\]
\[
+ \int_0^t \int_0^t \int_{\mathbb{M}^{-1}} Z_2(s^{-}) \, dN_2(ds, du, dz),
\]
\(7.4\)
where \(b \geq 0, M(ds, dz)\) is a Poisson random measure on \([0, \infty) \times \mathbb{M}\) with intensity measure \(ds m(dz)\), and the other coefficients are the same as in Section 4. Furthermore, we assume that those random elements are independent of each other. By a modification of the proof of Section 4, as well as that in [28], we see that \((7.3)-(7.4)\) has a unique strong solution and is a MSBI-process with branching mechanism \((\Phi_1, \Phi_2)\) defined by \((3.3)-(3.4)\) and an immigration mechanism \(\Psi\) defined by \((7.1)\).
7.2 Stationary distribution

In order to characterize the stationary distribution of MSBI-processes, we need to estimate the upper and lower bounds of $|V(t, \lambda)|$ for $t > 0, \lambda \in \mathbb{R}^2_+$; this will play an important role in the sequel.

Lemma 7.3 Let $(Y_t)_{t \geq 0}$ be a MSB-process with semigroup $(P_t)_{t \geq 0}$ satisfying (3.10). Let $H = [H_{ij}]_{2 \times 2}$ be a $2 \times 2$ matrix defined as in Corollary 5.5. Suppose that all the eigenvalues of $H$ have strictly negative real parts. Then there exist some strictly positive constants $c_1(\lambda)$ and $c_2$ where $c_1$ depends on $\lambda$ such that

$$|V(t, \lambda)| \leq c_1(\lambda) \exp \{-c_2 t\}, \quad \lambda \in \mathbb{R}^2_+, \quad t \geq 0.$$ 

Proof We follow the same calculations as those in Proposition 3.4 to see that

$$\begin{pmatrix} \frac{\partial V_1(t, 0+)}{\partial \lambda_1} \\ \frac{\partial V_2(t, 0+)}{\partial \lambda_1} \end{pmatrix} = e^{tH} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and so

$$\int_M y_1 P_t(x, dy) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T e^{tH} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$ Similarly,

$$\int_M y_2 P_t(x, dy) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T e^{tH} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ By Jensen’s inequality, we deduce that, for all $x = (x_1, x_2) \in M,$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} V_1(t, \lambda) \\ V_2(t, \lambda) \end{pmatrix} \leq \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T e^{tH} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.$$ Since all the eigenvalues of $H$ have strictly negative real parts, there exist some strictly positive $c, c_2 > 0$ such that, for all $t > 0,$

$$\|e^{tH}\| := \sup_{|x|=1} |e^{tH}x| \leq ce^{-c_2 t};$$

see, e.g., equation (2.8) in [32], which implies that $|V(t, \lambda)| \leq |\lambda|ce^{-c_2 t}$. We finish the proof by setting $c_1(\lambda) = |\lambda|c.$

Lemma 7.4 Under the conditions of Lemma 7.3, for every $\lambda \in \mathbb{R}^2_+$, there exist two strictly positive constants $A(\lambda)$ and $B(\lambda)$ such that

$$V_1(t, \lambda) \geq A_1e^{-A(\lambda)t}, \quad V_2(t, \lambda) \geq A_2e^{-B(\lambda)t}, \quad t \geq 0.$$ 

Proof

$$V_1(t, \lambda) = \lambda_1 + \int_0^t \left\{H_{11}V_1(s, \lambda) + H_{12}V_2(s, \lambda) - \alpha V_1^2(s, \lambda) - \int_M (e^{-\langle V(s, \lambda), z \rangle} - 1 + \langle V(s, \lambda), z \rangle) n_1(dz) \right\} ds,$$ 

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\[ V_2(t, \lambda) = \lambda_2 + \int_0^t \left\{ a_{21} V_1(s, \lambda) + \int_{\mathbb{M}^{-1}} (1 - e^{-\langle V(s, \lambda), z \rangle}) n_2(dz) \right\} ds. \]

It follows from Lemma 7.3, the comparison theorem and the fact that
\[ e^{-\lambda x} - 1 + \lambda x \leq \left( \frac{\lambda^2}{2} + \lambda \right) (x \wedge x^2), \quad x, \lambda \geq 0, \]
that there exists \( A(\lambda) = |H_{11} - \kappa - (\alpha + \kappa/2)c_1(\lambda)|, \) where \( \kappa = \int_{\mathbb{R}^2} z_1 \wedge z_1^2 n_1(dz) \) such that
\[ V_1(t, \lambda) \geq \lambda_1 e^{-A(\lambda)t}, \quad t \geq 0, \quad \lambda \in \mathbb{R}^2_+. \]

On the other hand, letting \( \theta = n_2(\mathbb{R}^+ \times \{-1\}) < \infty, \)
\[ V_2(t, \lambda) \geq \lambda_2 - \theta \int_0^t (e^{V_2(s, \lambda)} - 1) ds \]
\[ \geq \lambda_2 - 2\theta \int_0^t \left( e^{c_1(\lambda)} e^{-r_2 s} V_2(s, \lambda) \right) ds \]
\[ \geq \lambda_2 - 2\theta e^{c_1(\lambda)} \int_0^t V_2(s, \lambda) ds, \]
and by the comparison theorem we deduce that \( V_2(t, \lambda) \geq \lambda_2 e^{-2\theta e^{c_1(\lambda)} t}, \) and we obtain the desired result by setting \( B(\lambda) = 2\theta e^{c_1(\lambda)}. \)

We now give our main result.

**Theorem 7.5** Let \((Z_t)_{t \geq 0}\) be a MSBI-process with semigroup \((P_t^\gamma)_{t \geq 0}\) satisfying (7.2). Suppose that all the eigenvalues of \( H \) have strictly negative real parts. Then \( P_t^\gamma(x, \cdot) \) converges to a probability measure \( \pi \) on \( \mathbb{M} \) as \( t \to \infty \) for all \( x \in \mathbb{M} \) if and only if
\[ \int_{\{|z| \geq 1\}} \log |z| m(dz) < \infty. \] (7.5)

**Proof** By Lemma 7.3 we have \(|V(t, \lambda)| \to 0\) as \( t \to \infty. \) Supposing that \((Z_t)_{t \geq 0}\) has a stationary distribution \( \pi, \) one can see that
\[ \int_{\mathbb{M}} e^{-\langle \lambda, y \rangle} \pi(dy) = \exp \left\{ -\int_0^\infty \Psi(V(s, \lambda)) ds \right\}, \quad \lambda \in \mathbb{R}^2_+, \]
which implies that \( \int_0^\infty \Psi(V(s, \lambda)) ds < \infty \) for all \( \lambda \in \mathbb{R}^2_+. \) Thus
\[ \int_0^\infty ds \int_{\{|z| \geq 1\}} \left( 1 - e^{-\langle \lambda_1 \wedge \lambda_2, y \rangle} \exp(-\langle A(\lambda) + B(\lambda), y \rangle(z_1 + z_2)) \right) m(dz) < \infty. \]
If we set \( C(\lambda) := |H_{11}| + \theta + (\alpha + \theta/2)c_1(\lambda) + 2\theta e^{c_1(\lambda)} > 0 \) for all \( \lambda \in \mathbb{R}^2_+, \) then \( C(\lambda) \geq A(\lambda) + B(\lambda). \) Choose a proper \( \tilde{\lambda} \) such that \( \tilde{\lambda}_1 \wedge \tilde{\lambda}_2 > 0 \) and let \( t := e^{-C(\tilde{\lambda})s} |z| \). We then have
\[ dt = -C(\tilde{\lambda}) t ds, \]
and
\[ \int_{\{|z| \geq 1\}} m(dz) \int_0^{\log |z|} \frac{1 - e^{-\langle \tilde{\lambda}_1 \wedge \tilde{\lambda}_2, y \rangle t}}{t} dt < \infty, \]
which yields that
\[ \int_{\{|z| \geq 1\}} \log |z| m(dz) < \infty, \]
since
\[ \int_0^{\log |z|} \frac{1 - e^{-\langle \tilde{\lambda}_1 \wedge \tilde{\lambda}_2, y \rangle t}}{t} dt \sim \log |z| \quad \text{as} \quad |z| \to \infty. \]
On the other hand, it suffices to prove \(\int_0^\infty \int_M (1 - e^{-V(s,\lambda, z)}) \, ds \, m(dz) < \infty\) for all \(\lambda \in \mathbb{R}_+^2\), provided that \(\int_{\{|z| \geq 1\}} |z| \, m(dz) < \infty\). From Lemma 7.3 and Fubini’s theorem,

\[
\int_0^\infty \, ds \int_M \left(1 - e^{-V(s,\lambda, z)}\right) m(dz) \leq \int_0^\infty \, ds \int_{\{|z| < 1\}} \left(1 - e^{-c_1(\lambda)e^{-c_2s}(z_1 + z_2)}\right) m(dz) \\
= \int_0^\infty \, ds \int_{\{|z| < 1\}} \left(1 - e^{-c_1(\lambda)e^{-c_2s}(z_1 + z_2)}\right) m(dz) \\
+ \int_0^\infty \, ds \int_{\{|z| \geq 1\}} \left(1 - e^{-c_1(\lambda)e^{-c_2s}(z_1 + z_2)}\right) m(dz)
\]

\[:= I_*(\lambda) + I^*(\lambda).\]

For \(I_*(\lambda)\), by a change of variables \(t := c_1(\lambda)e^{-c_2s}(z_1 + z_2)\), we get that

\[I_*(\lambda) = \frac{1}{c_2} \int_{\{|z| < 1\}} m(dz) \int_0^{c_1(\lambda)(z_1 + z_2)} \frac{1 - e^{-t}}{t} \, dt \leq \frac{c_1(\lambda)}{c_2} \int_{\{|z| < 1\}} (z_1 + z_2) \, m(dz) < \infty,
\]

where the last inequality follows from \(\int_M (1 \wedge z_1 + 1 \wedge z_2) \, m(dz) < \infty\). Moreover, by a change of variables \(t := c_1(\lambda)e^{-c_2s}|z|\), we have that

\[I^*(\lambda) = \frac{1}{c_2} \int_{\{|z| \geq 1\}} m(dz) \int_0^{c_1(\lambda)|z|} \frac{1 - e^{-2t}}{t} \, dt \leq \frac{1}{c_2} \int_{\{|z| \geq 1\}} m(dz) \int_0^{c_1(\lambda)|z|} \frac{1 - e^{-2t}}{t} \, dt,
\]

which implies that \(I^*(\lambda) < \infty\), by observing

\[\int_0^{c_1(\lambda)|z|} \frac{1 - e^{-2t}}{t} \, dt \sim \log |z|, \quad |z| \to \infty
\]

and \(\int_{\{|z| \geq 1\}} \log |z| \, m(dz) < \infty\). \hfill \Box

**Corollary 7.6** Assume that \(H_{11}H_{22} - H_{12}H_{21} > 0\) and \(H_{11} + H_{22} < 0\) hold. Moreover, suppose that Lévy measure \(m\) satisfies that \(\int_{\{|z| > 1\}} |z| \, m(dz) < \infty\). Then there exist \(\lambda, \vartheta > 0\) and a unique \(\pi \in \mathcal{P}_1(M)\) such that, for any \(t \geq 0\) and \(x \in M\),

\[W_1(\delta_x P_t^\gamma, \pi) \leq \vartheta W_1(\delta_x, \pi) e^{-\lambda t}.
\]

**Proof** It follows from Theorem 7.5 and the assumptions that there exists a unique stationary distribution \(\pi\). We can easily derive that \(E_x[|Z_t|] < \infty\) for all \(t \geq 0\) and \(x \in M\) by the assumption that \(\int_{\{|z| > 1\}} |z| \, m(dz) < \infty\). By a modification of the proof of Corollary 6.3, we have that \(\pi \in \mathcal{P}_1(M)\), and the desired result follows from Theorem 7.2. \hfill \Box

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