1. Introduction

In this paper, we consider the equivariant classification of locally standard 2-torus manifolds. A 2-torus manifold is a closed smooth manifold of dimension $n$ with an effective action of a 2-torus group $(\mathbb{Z}_2)^n$ of rank $n$, and it is said to be locally standard if it is locally isomorphic to a faithful representation of $(\mathbb{Z}_2)^n$ on $\mathbb{R}^n$. The orbit space $Q$ of a locally standard 2-torus $M$ by the action is a nice manifold with corners. When $Q$ is a simple convex polytope, $M$ is called a small cover and studied in [4]. A typical example of a small cover is a real projective space $\mathbb{R}P^n$ with a standard action of $(\mathbb{Z}_2)^n$. Its orbit space is an $n$-simplex. On the other hand, a typical example of a compact non-singular toric variety is a complex projective space $\mathbb{C}P^n$ with a standard action of $(\mathbb{C}^*)^n$ where $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. $\mathbb{C}P^n$ has complex conjugation and its fixed point set is $\mathbb{R}P^n$. More generally, any compact non-singular toric variety admits complex conjugation and its fixed point set often provides an example of a small cover. Similarly to the theory of toric varieties, an interesting connection among topology, geometry and combinatorics is discussed for small covers in [4], [5] and [7]. Although locally standard 2-torus manifolds form a much wider class than small covers, one can still expect such a connection. See [9] for the study of 2-torus manifolds from the viewpoint of cobordism.

The orbit space $Q$ of a locally standard 2-torus manifold $M$ contains a lot of topological information on $M$. For instance, when $Q$ is a simple convex polytope (in other words, when $M$ is a small cover), the betti numbers of $M$ (with $\mathbb{Z}_2$ coefficient) are described in terms of face numbers of $Q$ ([4]). This is not the case for a general $Q$, but the euler characteristic of $M$ can be described in terms of $Q$ (Theorem 4.1). Although $Q$ contains a lot of topological information on $M$, $Q$ is not sufficient to reproduce $M$, i.e., there are many locally standard 2-torus manifolds with the same orbit space in general. We need two data to reproduce $M$ from $Q$. One is a characteristic function on $Q$ introduced in [4]. It is a map from the set of codimension-one faces of $Q$ to $(\mathbb{Z}_2)^n$ satisfying a certain linearly independence condition. Roughly speaking, a characteristic function provides information on the set of non-free orbits in $M$. The other data is a principal $(\mathbb{Z}_2)^n$-bundle over $Q$ which provides information on the set of free orbits in $M$. It turns out that the orbit space $Q$ together with these two data uniquely determines a locally standard 2-torus manifold up to equivariant homeomorphism (Lemma 3.1).

Keywords. 2-torus manifold, equivariant classification.

2000AMS Classification: 57S10, 14M25, 52B70.

The first author is supported by grants from NSFC (No. 10371020 and No. 10671034) and JSPS (No. P02299).
When $Q$ is a simple convex polytope, any principal $(\mathbb{Z}_2)^n$-bundle over it is trivial; so only a characteristic function matters in this case \((\mathcal{H})\).

The set of isomorphism classes in all principal $(\mathbb{Z}_2)^n$-bundles over $Q$ can be identified with $H^1(Q; (\mathbb{Z}_2)^n)$. Let $\Lambda(Q)$ be the set of all characteristic functions on $Q$. Then each element in $H^1(Q; (\mathbb{Z}_2)^n) \times \Lambda(Q)$ determines a locally standard 2-torus manifold with orbit space $Q$. However, different elements in the product may produce equivariantly homeomorphic locally standard 2-torus manifolds. Let $\text{Aut}(Q)$ be the group of self-homeomorphisms of $Q$ as a manifold with corners. It naturally acts on $H^1(Q; (\mathbb{Z}_2)^n) \times \Lambda(Q)$ and one can see that equivariant homeomorphism classes in locally standard 2-torus manifolds with orbit space $Q$ can be identified with the coset \((H^1(Q; (\mathbb{Z}_2)^n) \times \Lambda(Q))/\text{Aut}(Q)\), see Proposition \(5.5\).

It is not easy in general to count elements in the coset above, but we can manage when $Q$ is a compact surface with only one boundary. In this case, codimension-one faces sit in the boundary circle, so a characteristic function on $Q$ is nothing but a coloring on a circle (with vertices) with three colors.

The paper is organized as follows. In section 2, we introduce the notion of locally standard 2-torus manifold and give several examples. Following Davis and Januszkiewicz \(\mathcal{H}\), we define a characteristic function and construct a locally standard 2-torus manifold from a characteristic function and a principal bundle in section 3. In section 4 we describe the euler characteristic of a locally standard 2-torus manifold in terms of its orbit space. Section 5 discusses three equivalence relations among locally standard 2-torus manifolds and identify them with some cosets. We count the number of colorings on a circle in section 6. Applying this result, we find in section 7 the number of equivariant homeomorphism classes in locally standard 2-torus manifolds when the orbit space is a compact surface with only one boundary.

2. 2-TORUS MANIFOLDS

We denote the quotient additive group $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}_2$ throughout this paper. The natural action of a 2-torus $(\mathbb{Z}_2)^n$ of rank $n$ on $\mathbb{R}^n$ defined by

\[(x_1, \ldots, x_n) \mapsto ((-1)^{g_1}x_1, \ldots, (-1)^{g_n}x_n), \quad (g_1, \ldots, g_n) \in (\mathbb{Z}_2)^n\]

is called the standard representation of $(\mathbb{Z}_2)^n$. The orbit space is a positive cone $\mathbb{R}^n_{\geq 0}$. Any real $n$-dimensional faithful representation of $(\mathbb{Z}_2)^n$ is obtained from the standard representation by composing a group automorphism of $(\mathbb{Z}_2)^n$, up to isomorphism. Therefore the orbit space of the faithful representation space can also be identified with $\mathbb{R}^n_{\geq 0}$.

A 2-torus manifold $M$ is a closed smooth manifold of dimension $n$ with an effective smooth action of $(\mathbb{Z}_2)^n$. We say that $M$ is locally standard if for each point $x$ in $M$, there is a $(\mathbb{Z}_2)^n$-invariant neighborhood $V_x$ of $x$ such that $V_x$ is equivariantly homeomorphic to an invariant open subset of a real $n$-dimensional faithful representation space of $(\mathbb{Z}_2)^n$.

Remark. The notion of a torus manifold is introduced in \(\mathcal{S}\). It is a closed smooth manifold of dimension $2n$ with an effective smooth action of a compact $n$-dimensional
torus \((S^1)^n\) having a fixed point. (More precisely speaking, an orientation data on \(M\) called an omniorientation in \([2]\) is incorporated in the definition.) There is also a notion of local standardness in this setting (\([4]\)). Although many interesting examples of torus manifolds are locally standard (e.g. this is the case for compact non-singular toric varieties with restricted action of the compact torus, more generally for torus manifolds with vanishing odd degree cohomology, \([11]\)), the local standardness is not assumed in the study of \([8]\) and \([10]\) because a combinatorial object called a multi-fan can be defined without assuming it. However, the existence of a fixed point is not assumed for a 2-torus manifold unlike a torus manifold.

For a locally standard 2-torus manifold \(M\), the orbit space \(Q\) of \(M\) naturally becomes a manifold with corners (see \([3]\) for the details of a manifold with corners). Therefore the notion of a face can be defined for \(Q\). In this paper we assume that a face is connected. We call a face of dimension 0 a vertex, a face of dimension one an edge and a codimension-one face a facet.

An \(n\)-dimensional convex polytope \(P\) is said to be simple if exactly \(n\) facets meet at each of its vertices. Each point of a simple convex polytope \(P\) has a neighborhood which is affine isomorphic to an open subset of the positive cone \(\mathbb{R}_+^n\), so \(P\) is an \(n\)-dimensional manifold with corners. A locally standard 2-torus manifold \(M\) is said to be a small cover when its orbit space is a simple convex polytope, see \([4]\).

We call a closed, connected, codimension-one submanifold of \(M\) characteristic if it is a connected component of the set fixed pointwise by some \(\mathbb{Z}_2\) subgroup. Since \(M\) is compact, \(M\) has only finitely many characteristic submanifolds. The action of \((\mathbb{Z}_2)^n\) is free outside the union of all characteristic submanifolds, in other words, a point of \(M\) with non-trivial isotropy subgroup is contained in some characteristic submanifold of \(M\).

Through the quotient map \(M \to Q\), a fixed point in \(M\) corresponds to a vertex of \(Q\) and a characteristic submanifold of \(M\) corresponds to a facet of \(Q\). A connected component of the intersection of \(k\) characteristic submanifolds of \(M\) corresponds to a codimension-\(k\) face of \(Q\), so a codimension-\(k\) face of \(Q\) is a connected component of the intersection of \(k\) facets. In particular, any codimension-two face of \(Q\) is a connected component of the intersection of two facets of \(Q\), which means that \(Q\) is nice, see \([3]\).

We shall give examples of locally standard 2-torus manifolds.

**Example 2.1.** A real projective space \(\mathbb{R}P^n\) with the standard \((\mathbb{Z}_2)^n\)-action defined by

\[
[x_0, x_1, \ldots, x_n] \mapsto [x_0, (-1)^{g_1}x_1, \ldots, (-1)^{g_n}x_n], \quad (g_1, \ldots, g_n) \in (\mathbb{Z}_2)^n
\]

is a locally standard 2-torus manifold. It has \(n+1\) isolated points and \(n+1\) characteristic submanifolds. The orbit space of \(\mathbb{R}P^n\) by this action is an \(n\)-simplex, so this locally standard 2-torus manifold is actually a small cover.

**Example 2.2.** Let \(S^1\) denote the unit circle in the complex plane \(\mathbb{C}\) and consider two involutions on \(S^1 \times S^1\) defined by

\[
t_1: (z, w) \mapsto (-z, w), \quad t_2: (z, w) \mapsto (z, \bar{w}).
\]
Since $t_1$ and $t_2$ are commutative, they define a $(\mathbb{Z}_2)^2$-action on $S^1 \times S^1$, and it is easy to see that $S^1 \times S^1$ with this action is a locally standard 2-torus manifold. It has no fixed point and the orbit space is $\mathbb{R}P^1 \times I = S^1 \times I$ where $I$ is a closed interval.

**Example 2.3.** If $M_1$ and $M_2$ are both locally standard 2-torus manifolds of the same dimension, then the equivariant connected sum of them along their free orbits produces a new locally standard 2-torus manifold. For example, we take $\mathbb{R}P^2$ in Example 2.1 and $S^1 \times S^1$ in Example 2.2 and do the equivariant connected sum of them along their free orbits. The orbit space of the resulting locally standard 2-torus manifold $M$ is the connected sum of a 2-simplex with $S^1 \times I$ at their interior points. $M$ has five characteristic submanifolds and three of them have a fixed point but the other two have no fixed point.

If $M$ is a locally standard 2-torus manifold of dimension $n$ and a subgroup of $(\mathbb{Z}_2)^n$ has an isolated fixed point, then the isolated point must be fixed by the entire group $(\mathbb{Z}_2)^n$. This follows from the local standardness of $M$. The following is an example of a closed $n$-manifold with an effective $(\mathbb{Z}_2)^n$-action which is not a locally standard 2-torus manifold.

**Example 2.4.** Consider two involutions on the unit sphere $S^2$ of $\mathbb{R} \times \mathbb{C}$ defined by

\[ t_1 : (x, z) \mapsto (-x, -z), \quad t_2 : (x, z) \mapsto (x, \bar{z}). \]

Since $t_1$ and $t_2$ are commutative, they define a $(\mathbb{Z}_2)^2$-action on $S^2$. But $S^2$ with this action is not a locally standard 2-torus manifold because the fixed point set of $t_1t_2$ consists of two isolated points $(0, \pm \sqrt{-1})$ but they are not fixed by the entire group $(\mathbb{Z}_2)^2$.

### 3. Characteristic functions and principal bundles

Let $Q$ be an $n$-dimensional nice manifold with corners. We denote by $\mathcal{F}(Q)$ the set of facets of $Q$. A codimension-$k$ face of $Q$ is a connected component of the intersection of $k$ facets. We call a map

\[ \lambda : \mathcal{F}(Q) \longrightarrow (\mathbb{Z}_2)^n \]

a characteristic function on $Q$ if it satisfies the following linearly independent condition:

if a codimension-$k$ face $F$ of $Q$ is a connected component of the intersection of $k$ facets $F_1, \ldots, F_k$, then $\lambda(F_1), \ldots, \lambda(F_k)$ are linearly independent when viewed as vectors of the vector space $(\mathbb{Z}_2)^n$ over the field $\mathbb{Z}_2$.

We denote by $G_F$ the subgroup of $(\mathbb{Z}_2)^n$ generated by $\lambda(F_1), \ldots, \lambda(F_k)$.

**Remark.** When $n \leq 2$, it is easy to see that any $Q$ admits a characteristic function. When $n = 3$, $Q$ admits a characteristic function if the boundary of $Q$ is a union of 2-spheres, which follows from the Four Color Theorem, but $Q$ may not admit a characteristic function otherwise. When $n \geq 4$, there is a simple convex polytope which admits no characteristic function, see [1, Nonexamples 1.22].

A characteristic function arises naturally from a locally standard 2-torus manifold $M$ of dimension $n$ with orbit space $Q$. A facet of $Q$ is the image of a characteristic...
submanifold of $M$ by the quotient map $\pi: M \to Q$. To each element $F \in \mathcal{F}(Q)$ we assign the nonzero element of $(\mathbb{Z}_2)^n$ which fixes pointwise the characteristic submanifold $\pi^{-1}(F)$. The local standardness of $M$ implies that this assignment satisfies the linearly independent condition above required for a characteristic function.

Besides the characteristic function, a principal $(\mathbb{Z}_2)^n$-bundle over $Q$ will be associated with $M$ as follows. We take a small invariant open tubular neighborhood for each characteristic submanifold of $M$ and remove their union from $M$. Then the $(\mathbb{Z}_2)^n$-action on the resulting space is free and its orbit space can naturally be identified with $Q$, so it gives a principal $(\mathbb{Z}_2)^n$-bundle over $Q$.

We have associated a characteristic function and a principal $(\mathbb{Z}_2)^n$-bundle with a locally standard 2-torus manifold. Conversely, one can reproduce the locally standard 2-torus manifold from these two data. This is done by Davis-Januszkiewicz [4] when $Q$ is a simple convex polytope, but their construction still works in our setting. Let $\xi = (E, \kappa, Q)$, where $\kappa: E \to Q$, be a principal $(\mathbb{Z}_2)^n$-bundle over $Q$ and let $\lambda: \mathcal{F}(Q) \to (\mathbb{Z}_2)^n$ be a characteristic function on $Q$. We define an equivalence relation $\sim$ on $E$ as follows: for $u_1, u_2 \in E$

$$u_1 \sim u_2 \iff \kappa(u_1) = \kappa(u_2) \text{ and } u_1 = u_2g$$

for some $g \in G_F$ where $F$ is the face of $Q$ containing $\kappa(u_1) = \kappa(u_2)$ in its relative interior and $G_F$ is the subgroup of $(\mathbb{Z}_2)^n$ defined at the beginning of this section. Then the quotient space $E/\sim$, denoted by $M(\xi, \lambda)$, naturally inherits the $(\mathbb{Z}_2)^n$-action from $E$.

The following is proved in [4] when $Q$ is a simple convex polytope, but the same proof works in our setting.

**Lemma 3.1.** If a locally standard 2-torus manifold $M$ over $Q$ has $\xi$ as the associated principal $(\mathbb{Z}_2)^n$-principal bundle and $\lambda$ as the characteristic function, then there is an equivariant homeomorphism from $M(\xi, \lambda)$ to $M$ which covers the identity on $Q$.

### 4. Euler characteristic of a locally standard 2-torus manifold

The following formula describes the euler characteristic $\chi(M)$ of a locally standard 2-torus manifold $M$ in terms of its orbit space.

**Theorem 4.1.** If $M$ is a locally standard 2-torus manifold over $Q$, then

$$\chi(M) = \sum_F 2^{\dim F} \chi(F, \partial F) = \sum_F 2^{\dim F} (\chi(F) - \chi(\partial F))$$

where $F$ runs over all faces of $Q$.

**Proof.** As observed in Section 3, $M$ is the disjoint union of $2^{\dim F}$ copies of $F \setminus \partial F$ over all faces $F$ of $Q$. This implies the former identity in the theorem. The latter identity is well-known. In fact, it follows from the homology exact sequence for a pair $(F, \partial F)$. \(\square\)

When $\dim M = 2$, $Q$ is a surface with boundary and each boundary component is a circle with at least two vertices if it has a vertex.

**Corollary 4.2.** If $\dim M = 2$ and $Q$ has $m$ vertices, then $\chi(M) = 4\chi(Q) - m$. 
Proof. Since $\partial Q$ is a union of circles, $\chi(Q, \partial Q) = \chi(Q)$. If a boundary circle has no vertex, then it is an edge without boundary and its euler characteristic is zero. So we may neglect it. If $F$ is an edge with a vertex, then it has two endpoints and $\chi(F, \partial F) = \chi(F) - \chi(\partial F) = -1$, and if $F$ is a vertex, then $\chi(F, \partial F) = \chi(F) = 1$. Since the number of edges with a vertex and the number of vertices are both $m$, it follows from Theorem 4.1 that

$$\chi(M) = 2^2\chi(Q) - 2m + m = 4\chi(Q) - m.$$ 

□

Remark. When $\dim M = 2$, it is not difficult to see that $M$ is orientable if and only if $Q$ is orientable and the characteristic function $\lambda: F(Q) \to (\mathbb{Z}_2)^2$ associated with $M$ assigns exactly two elements to each boundary component of $Q$ with a vertex, cf. [12]. Therefore one can find the homeomorphism type of $M$ from the corollary above and the characteristic function $\lambda$.

5. Classification of locally standard 2-torus manifolds

In this section we introduce three notions of equivalence in locally standard 2-torus manifolds over $Q$ and identify each set of equivalence classes with a coset of $H^1(Q; (\mathbb{Z}_2)^n) \times \Lambda(Q)$ by some action.

Following Davis and Januszkiewicz [4] we say that two locally standard 2-torus manifolds $M$ and $M'$ over $Q$ are equivalent if there is a homeomorphism $f: M \to M'$ together with an element $\sigma \in \text{GL}(n, \mathbb{Z}_2)$ such that

1. $f(gx) = \sigma(g)f(x)$ for all $g \in (\mathbb{Z}_2)^n$ and $x \in M$, and
2. $f$ induces the identity on the orbit space $Q$.

When we classify locally standard 2-torus manifolds up to the above equivalence, it suffices to consider locally standard 2-torus manifolds of the form $M(\xi, \lambda)$ by Lemma 3.1. We denote by $\xi'$ the principal $(\mathbb{Z}_2)^n$-bundle $\xi$ with $(\mathbb{Z}_2)^n$-action through $\sigma \in \text{GL}(n, \mathbb{Z}_2)$. Then it would be obvious that $M(\xi', \lambda')$ is equivalent to $M(\xi, \lambda)$ if and only if there exists $\sigma \in \text{GL}(n, \mathbb{Z}_2)$ such that $\xi' = \xi^\sigma$ and $\lambda' = \sigma \circ \lambda$.

We denote by $\mathcal{P}(Q)$ the set of all principal $(\mathbb{Z}_2)^n$-bundles over $Q$. Since the classifying space of $(\mathbb{Z}_2)^n$ is an Eilenberg-Maclane space $K((\mathbb{Z}_2)^n, 1)$, $\mathcal{P}(Q)$ can naturally be identified with $H^1(Q; (\mathbb{Z}_2)^n)$ and the action of $\sigma$ sending $\xi$ to $\xi^\sigma$ is nothing but the action on $H^1(Q; (\mathbb{Z}_2)^n)$ induced from the automorphism $\sigma$ on the coefficient $(\mathbb{Z}_2)^n$. With this understood, the above fact implies the following.

**Proposition 5.1.** The set of equivalence classes in locally standard 2-torus manifolds over $Q$ bijectively corresponds to the coset

$$\text{GL}(n, \mathbb{Z}_2) \backslash \left( H^1(Q; (\mathbb{Z}_2)^n) \times \Lambda(Q) \right)$$

by the diagonal action.

The action of $\text{GL}(n, \mathbb{Z}_2)$ on $H^1(Q; (\mathbb{Z}_2)^n) \times \Lambda(Q)$ is free when $Q$ has a vertex by the following lemma.
Lemma 5.2. If $Q$ has a vertex, then the action of $\text{GL}(n, \mathbb{Z}_2)$ on $\Lambda(Q)$ is free and $|\Lambda(Q)| = |\text{GL}(n, \mathbb{Z}_2) \setminus \Lambda(Q)| \prod_{k=1}^{n}(2^{n} - 2^{k-1})$.

Proof. Suppose that $\lambda = \sigma \circ \lambda$ for some $\lambda \in \Lambda(Q)$ and $\sigma \in \text{GL}(n, \mathbb{Z}_2)$. Take a vertex of $Q$ and let $F_1, \ldots, F_n$ be the facets of $Q$ meeting at the vertex. Then

$$(\lambda(F_1), \ldots, \lambda(F_n)) = \sigma(\lambda(F_1), \ldots, \lambda(F_n)).$$

Since the matrix $(\lambda(F_1), \ldots, \lambda(F_n))$ is non-singular, $\sigma$ is the identity matrix. This proves the former statement in the lemma. Then the latter statement follows from the well-known fact that $|\text{GL}(n, \mathbb{Z}_2)| = \prod_{k=1}^{n}(2^n - 2^{k-1})$, see [1]. □

Lemma 5.2 is also helpful to count the number of elements in $\Lambda(Q)$. Here is an example.

Example 5.3. (The number of characteristic functions on a prism.) There exist seven combinatorially inequivalent 3-polytopes with six vertices (see [6] Theorem 6.7) and only one of them is simple, which is a prism $P^3$.

Let us count the number of characteristic functions on $P^3$. $P^3$ has five facets, consisting of three square facets and two triangular facets. We denote the three square facets by $F_1, F_2, F_4$, and the two triangular facets by $F_3, F_5$. The facets $F_1, F_2, F_3$ intersect at a vertex and we may assume that a characteristic function $\lambda$ on $P^3$ takes the standard basis $e_1, e_2, e_3$ of $(\mathbb{Z}_2)^3$ on $F_1, F_2, F_3$ respectively through the action of $\text{GL}(3, \mathbb{Z}_2)$ on $(\mathbb{Z}_2)^3$. The characteristic function $\lambda$ must satisfy the linearly independent condition at each vertex of $P^3$. This requires that the values of $\lambda$ on the remaining facets $F_4, F_5$ must be as follows:

$$(\lambda(F_1), \lambda(F_5)) = (e_1 + e_2 + e_3, e_3) \quad \text{or} \quad (e_1 + e_2, ae_1 + be_2 + e_3)$$

where $a, b \in \mathbb{Z}_2$. Therefore

$$|\text{GL}(3, \mathbb{Z}_2) \setminus \Lambda(P^3)| = 5 \quad \text{and} \quad |\Lambda(P^3)| = 5|\text{GL}(3, \mathbb{Z}_2)| = 840$$

by Lemma 5.2.

Another natural equivalence relation among locally standard 2-torus manifolds is equivariant homeomorphism. An automorphism of $Q$ is a self-homeomorphism of $Q$ as a manifold with corners, and we denote the group of automorphisms of $Q$ by $\text{Aut}(Q)$. Similarly, an automorphism of $\mathcal{F}(Q)$ is a bijection from $\mathcal{F}(Q)$ to itself which preserves the poset structure of $\mathcal{F}(Q)$ defined by inclusions of faces, and we denote the group of automorphisms of $\mathcal{F}(Q)$ by $\text{Aut}(\mathcal{F}(Q))$. An automorphism of $Q$ induces an automorphism of $\mathcal{F}(Q)$, so we have a natural homomorphism

$$\Phi: \text{Aut}(Q) \to \text{Aut}(\mathcal{F}(Q)).$$

We note that $\text{Aut}(\mathcal{F}(Q))$ acts on $\Lambda(Q)$ by sending $\lambda \in \Lambda(Q)$ to $\lambda \circ h$ for $h \in \text{Aut}(\mathcal{F}(Q))$.

Lemma 5.4. $M(\xi, \lambda)$ is equivariantly homeomorphic to $M(\xi', \lambda')$ if and only if there is an $h \in \text{Aut}(Q)$ such that $\lambda' = \lambda \circ \Phi(h)$ and $h^*(\xi') = \xi$ in $\mathcal{P}(Q)$, where $h^*(\xi')$ denotes the bundle induced from $\xi'$ by $h$. 

Proof. If \( M(\xi, \lambda) \) is equivariantly homeomorphic to \( M(\xi', \lambda') \), then there is an equivariant homeomorphism \( H: M(\xi', \lambda') \to M(\xi, \lambda) \) and it is easy to see that the automorphism of \( Q \) induced from \( H \) is the desired \( h \) in the theorem.

Conversely, suppose that there is an \( h \in \Lambda(Q) \) such that \( \lambda' = \lambda \circ \Phi(h) \) and \( \xi' = h^*(\xi) \) in \( \mathcal{P}(Q) \). Then there is a bundle map \( \hat{h}: \xi' \to \xi \) which covers \( h \), and \( \hat{h} \) descends to a map \( H \) from \( M(\xi', \lambda') \) to \( M(\xi, \lambda) \) because \( \lambda' = \lambda \circ \Phi(h) \). It is not difficult to see that \( H \) is an equivariant homeomorphism. \( \square \)

\( \text{Aut}(Q) \) naturally acts on \( H^1(Q; (\mathbb{Z}_2)^n) \) and the canonical bijection between \( \mathcal{P}(Q) \) and \( H^1(Q; (\mathbb{Z}_2)^n) \) is equivariant with respect to the actions of \( \text{Aut}(Q) \).

Proposition 5.5. The set of equivariant homeomorphism classes in all locally standard 2-torus manifolds over \( Q \) bijectively corresponds to the coset

\[
(H^1(Q; (\mathbb{Z}_2)^n) \times \Lambda(Q))/\text{Aut}(Q)
\]

by the diagonal action of \( \text{Aut}(Q) \). If \( Q \) is a simple convex polytope, then the set of equivariant homeomorphism classes in all small covers over \( Q \) bijectively corresponds to the coset \( \Lambda(Q)/\text{Aut}(\mathcal{F}(Q)) \).

Proof. The former statement in the proposition follows from Lemma 5.4. If \( Q \) is a simple polytope, then \( H^1(Q; (\mathbb{Z}_2)^n) = 0 \). Therefore, the latter statement in the proposition follows if we prove that the map \( \Phi \) in (5.1) is surjective when \( Q \) is a simple convex polytope.

A simple polytope \( Q \) has a simplicial polytope \( Q^* \) as its dual and the face poset \( \mathcal{F}(Q) \) is same as \( \mathcal{F}(Q^*) \) with reversed inclusion relation. Therefore \( \text{Aut}(\mathcal{F}(Q)) = \text{Aut}(\mathcal{F}(Q^*)) \). Since \( Q^* \) is simplicial, an element \( \varphi \) of \( \text{Aut}(\mathcal{F}(Q^*)) \) is realized by a simplicial automorphism on the boundary of \( Q^* \), so it extends to an automorphism of \( Q^* \). Since \( Q \) is dual to \( Q^* \), the automorphism of \( Q^* \) determines a bijection on the vertex set of \( Q \) and hence an automorphism of \( Q \) which induces the chosen \( \varphi \). \( \square \)

Our last equivalence relation is a combination of the previous two relations. We say that two locally standard 2-torus manifolds \( M \) and \( M' \) over \( Q \) are weakly equivariantly homeomorphic if there is a homeomorphism \( f: M \to M' \) together with \( \sigma \in \text{GL}(n, \mathbb{Z}_2) \) such that \( f(gx) = \sigma(g)f(x) \) for any \( g \in (\mathbb{Z}_2)^n \) and \( x \in M \). We note that \( f \) induces an automorphism of \( Q \) but it may not be the identity on \( Q \). The observation above shows that \( M(\xi, \lambda) \) and \( M(\xi', \lambda') \) are weakly equivariantly homeomorphic if and only if there are \( h \in \text{Aut}(Q) \) and \( \sigma \in \text{GL}(n, \mathbb{Z}_2) \) such that \( \xi' = h^*(\xi\sigma) \) and \( \lambda' = \sigma \circ \lambda \circ h \). It follows that

Proposition 5.6. The set of weakly equivariant homeomorphism classes in locally standard 2-torus manifolds over \( Q \) bijectively corresponds to the double coset

\[
\text{GL}(n, \mathbb{Z}_2)\backslash(H^1(Q; (\mathbb{Z}_2)^n) \times \Lambda(Q))/\text{Aut}(Q)
\]

by the diagonal actions of \( \text{Aut}(Q) \) and \( \text{GL}(n, \mathbb{Z}_2) \). If \( Q \) is a simple convex polytope, then the set of weakly equivariant homeomorphism classes in small covers over \( Q \) bijectively corresponds to the double coset

\[
\text{GL}(n, \mathbb{Z}_2)\backslash\Lambda(Q)/\text{Aut}(\mathcal{F}(Q)).\tag{5.2}
\]
Remark. When \( Q \) is a right-angled regular hyperbolic polytope (such \( Q \) is the dodecahedron, the 120-cell or an \( m \)-gon with \( m \geq 5 \)), it is shown in [7, Theorem 3.3] that the double coset (5.2) agrees with the set of hyperbolic structures in small covers over \( Q \). This together with Mostow rigidity implies that when \( \dim Q \geq 3 \), that is, when \( Q \) is the dodecahedron or the 120-cell, the double coset (5.2) agrees with the set of homeomorphism classes in small covers over \( Q \) ([7, Corollary 3.4]), i.e., the natural surjective map from the double coset to the set of homeomorphism classes in small covers over \( Q \) is bijective for such \( Q \). However, this last statement does not hold for an \( m \)-gon \( Q \) with \( m \geq 6 \) although it holds for \( m = 3, 4, 5 \), see the remark following Example 6.5 in the next section.

6. Enumeration of colorings on a circle

When \( \dim Q = 2 \), each boundary component is a circle with at least two vertices if it has a vertex, and any two non-zero elements in \((\mathbb{Z}_2)^2\) form a basis of \((\mathbb{Z}_2)^2\); so a characteristic function on \( Q \) is equivalent to coloring arcs on the boundary circles with three colors in such a way that any two adjacent arcs have different colors.

Let \( S(m) \) be a circle with \( m \) (\( \geq 2 \)) vertices. A coloring on \( S(m) \) (with three colors) means to color arcs of \( S(m) \) in such a way that any adjacent arcs have different colors. We denote by \( \Lambda(m) \) the set of all colorings on \( S(m) \) and set

\[
A(m) := |\Lambda(m)|.
\]

Lemma 6.1. \( A(m) = 2^m + (-1)^m 2 \).

Proof. Let \( L(m) \) be a segment with \( m + 1 \) vertices including the endpoints, so \( L(m) \) has \( m \) segments. The number of coloring segments of \( L(m) \) with three colors in such a way that any adjacent segments have different colors is \( 3 \cdot 2^{m-1} \). If the two end segments have different colors, then it produces a coloring on \( S(m) \) by gluing the end points of \( L(m) \). If the two end segments have the same color, then it produces a coloring on \( S(m-1) \) by gluing the end segments of \( L(m) \). Thus, we have that

\[
A(m) + A(m-1) = 3 \cdot 2^{m-1}.
\]

It follows that

\[
A(m) - 2A(m-1) = -(A(m-1) - 2A(m-2)) = \cdots = (-1)^{m-3}(A(3) - 2A(2))
\]

and a simple observation shows that \( A(3) = A(2) = 6 \), so

\[
A(m) - 2A(m-1) = (-1)^{m-6}.
\]

The lemma then follows from (6.1) and (6.2). \( \square \)

We think of \( S(m) \) as the unit circle of \( \mathbb{C} \) with \( m \) vertices \( e^{2\pi k/m} \) (\( k = 0, 1, \ldots, m-1 \)). Let \( \mathcal{D}_m \) be the dihedral group of order \( 2m \) consisting of \( m \) rotations of \( \mathbb{C} \) by angles \( 2\pi k/m \) (\( k = 0, 1, \ldots, m-1 \)) and \( m \) reflections with respect to lines in \( \mathbb{C} \) obtained by rotating the real axis by angles \( \pi k/m \) (\( k = 0, 1, \ldots, m-1 \)). Then the action of \( \mathcal{D}_m \) on \( S(m) \) preserves the vertices so that \( \mathcal{D}_m \) acts on the set \( \Lambda(m) \). With this understood we have
Theorem 6.2. Let $\varphi$ denote the Euler’s totient function, that is, $\varphi(1) = 1$ and $\varphi(N)$ for a positive integer $N (\geq 2)$ is the number of positive integers both less than $N$ and coprime to $N$. Then

$$|\Lambda(m)/\mathfrak{D}_m| = \frac{1}{2m} \left( \sum_{2 \leq d | m} \varphi(m/d)A(d) + \frac{1 + (-1)^m}{2} \cdot 3 \cdot 2^{m/2} \cdot \frac{m}{2} \right).$$

Proof. The famous Burnside Lemma or Cauchy-Frobenius Lemma (see [1]) says that if $G$ is a finite group and $X$ is a finite $G$-set, then

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

where $X^g$ denotes the set of $g$-fixed points in $X$. We apply this formula to our $\mathfrak{D}_m$-set $\Lambda(m)$. Let $a \in \mathfrak{D}_m$ be the rotation by angle $2\pi/m$ and $b \in \mathfrak{D}_m$ be the reflection with respect to the real axis. Then we have

$$(6.3) \quad |\Lambda(m)/\mathfrak{D}_m| = \frac{1}{2m} \sum_{k=0}^{m-1} \left(|\Lambda(m)^a|^k + |\Lambda(m)^{ab}|^k\right).$$

Here, if $d$ is the greatest common divisor of $k$ and $m$, then $\Lambda(m)^a = \Lambda(m)^{ad}$ because the subgroup generated by $a^k$ is same as that by $a^d$. Since $\Lambda(m)^{ad} = \Lambda(d)$ and $\Lambda(1)$ is empty, we have

$$(6.4) \quad \sum_{k=0}^{m-1} |\Lambda(m)^a|^k = \sum_{2 \leq d | m} \varphi(m/d)A(d).$$

On the other hand, since $a^{kb}$ is a reflection with respect to the line in $\mathbb{C}$ obtained by rotating the real axis by angle $\pi k/m$, we have

$$(6.5) \quad |\Lambda(m)^{a^k}| = \begin{cases} 3 \cdot 2^{m/2} & \text{when } m \text{ is even and } k \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Putting (6.4) and (6.5) into (6.3), we obtain the formula in the theorem. \hfill \Box

Example 6.3. As is well known, $\varphi(p^n) = p^{n-1}(p-1)$ for any prime number $p$ and positive integer $n$, and $\varphi(ab) = \varphi(a)\varphi(b)$ for relatively prime positive integers $a$ and $b$. We set

$$B(m) := |\Lambda(m)/\mathfrak{D}_m|.$$
Using the formula in Theorem 6.2 together with Lemma 6.1 one finds that
\[ B(2) = 3, \quad B(3) = 1, \quad B(4) = 6, \quad B(5) = 3, \quad B(6) = 13, \]
\[ B(7) = 9, \quad B(8) = 30, \quad B(9) = 29, \quad B(10) = 78, \]
\[ B(2^k) = 2^{2k-k-1} + 3 \cdot 2^{2k-1-2} + \sum_{i=1}^{k} 2^{2^{i-1}-1} \]
\[ B(p^k) = \sum_{i=1}^{k} \frac{1}{2p^i}(2^{p^i} - 2^{p^i-1}) \]
\[ B(2p) = \frac{1}{4p}(4p + (3p + 1)2^p + 6p - 6) \]
\[ B(pq) = \frac{1}{2pq}(2^{pq} - 2^p - 2^q + 2) + \frac{1}{2p}(2^p - 2) + \frac{1}{2q}(2^q - 2) \]
where \( p \) is an odd prime number and \( q \) is another odd prime number.

Remark. The same argument as above works for coloring \( S(m) \) with \( s \) colors. In this case the identity in Lemma 6.1 turns into
\[ A_s(m) = (s - 1)^m + (-1)^m(s - 1) \]
and if we denote by \( \Lambda_s(m) \) the set of all coloring on \( S(m) \) with \( s \) colors, then the formula in Theorem 6.2 turns into
\[ |\Lambda_s(m)/\mathcal{D}_m| = \frac{1}{2m} \left( \sum_{2 \leq d|m} \varphi(m/d)A_s(d) + \frac{1 + (-1)^m}{2} \cdot s \cdot (s - 1)^{m/2} \cdot \frac{m}{2} \right). \]

The computation of \( |GL(2, \mathbb{Z}_2)\backslash\Lambda(m)/\mathcal{D}_m| \) can be done in a similar fashion to the above but is rather complicated. We note that the action of \( GL(2, \mathbb{Z}_2) \) on \( \Lambda(m) \) is permutation of the 3 colors used to color \( S(m) \). \( GL(2, \mathbb{Z}_2) \) consists of 6 elements and three of them are of order 2 and two of them is of order 3.

Theorem 6.4. Let \( \alpha \) and \( \beta \) be the functions defined as follows:
\[ \alpha(1) = 1, \quad \alpha(2) = 3, \quad \alpha(3) = 2, \quad \alpha(6) = 4, \]
\[ \beta(1) = 0, \quad \beta(2) = 2, \quad \beta(3) = 2, \quad \beta(6) = 4. \]
Then \( |GL(2, \mathbb{Z}_2)\backslash\Lambda(m)/\mathcal{D}_m| \) is given by
\[ \frac{1}{2m} \left[ \sum_{d|m} \varphi(m/d) \cdot \frac{1}{6} \left( \alpha((m/d, 6))A(d) + \beta((m/d, 6))A(d - 1) \right) \right] + E(m) \]
where \((m/d, 6)\) denotes the greatest common divisor of \( m/d \) and 6, \( A(q) = 2^q + (-1)^{q/2} \) as before, and
\[ E(m) = \begin{cases} \frac{m}{6}A\left(\frac{m+1}{2}\right) & \text{if } m \text{ is odd}, \\ m \cdot 2^{m/2-1} & \text{if } m \text{ is even}. \end{cases} \]
Proof. Applying the Burnside Lemma to our $\mathcal{D}_m$-set $\Gamma(m) := \text{GL}(2, \mathbb{Z}_2) \setminus \Lambda(m)$, we have
\[
|\text{GL}(2, \mathbb{Z}_2) \setminus \Lambda(m)/\mathcal{D}_m| = \frac{1}{2m} \sum_{g \in \mathcal{D}_m} |\Gamma(m)^g|
\]
(6.6)
\[
= \frac{1}{2m} \sum_{k=0}^{m-1} (|\Gamma(m)^a| + |\Gamma(m)^a^b|) = \frac{1}{2m} \left[ \sum_{d|m} \varphi(m/d)|\Gamma(m)^a^d| + \sum_{k=0}^{m-1} |\Gamma(m)^a^b| \right].
\]
We need to analyze $|\Gamma(m)^a^d|$ with $d|m$ and $|\Gamma(m)^a^b|$.

First we shall treat $|\Gamma(m)^a^d|$ with $d|m$. Note that $\lambda \in \Lambda(m)$ is a representative of $\Gamma(m)^a^d$ if and only if there is $\sigma \in \text{GL}(2, \mathbb{Z}_2)$ such that
\[
\sigma \circ \lambda = \lambda \circ a^d.
\]
Since $a^d$ is of order $m/d$, the repeated use of (6.7) shows that
\[
\sigma^{m/d} = 1.
\]
The identity (6.7) implies that the $\lambda$ satisfying (6.7) can be determined by the coloring restricted to the union of a consecutive $d$ arcs, say $T$, and it also tells us how to recover $\lambda$ from the coloring on $T$.

Let $\mu$ be a coloring on $T$. We shall count colorings $\lambda$ on $S(m)$ which are extensions of $\mu$ and satisfy (6.7) for some $\sigma \in \text{GL}(2, \mathbb{Z}_2)$. To each $\sigma$ satisfying (6.8), there is a unique extension to $S(m)$ which satisfies (6.7). However, the extended one may not be a coloring, i.e., two arcs meeting at a junction of $T$ and its translations by rotations $(a^d)^r$ ($r = 1, \ldots, m/d - 1$) may have the same color. Let $t$ and $t'$ be the end arcs of $T$ such that the rotation of $t$ by $a^{d-1}$ is $t'$. (Note: When $d = 1$, we understand $t = t'$ and then the subsequent argument works.) The extended one is a coloring if and only if
\[
\sigma(\mu(t)) \neq \mu(t').
\]
As is easily checked, the number of $\sigma$ satisfying conditions (6.8) and (6.9) is $\alpha((m/d, 6))$ if $\mu(t) \neq \mu(t')$ and is $\beta((m/d, 6))$ if $\mu(t) = \mu(t')$. On the other hand, the number of $\mu$ with $\mu(t) \neq \mu(t')$ is $A(d)$ and that with $\mu(t) = \mu(t')$ is $A(d-1)$. It follows that the number of $\lambda$ satisfying (6.7) for some $\sigma$ is $\alpha((m/d, 6))A(d) + \beta((m/d, 6))A(d-1)$. This proves that
\[
|\Gamma(m)^a^d| = \frac{1}{6} \left( \alpha((m/d, 6))A(d) + \beta((m/d, 6))A(d-1) \right)
\]
(6.10)
since the action of $\text{GL}(2, \mathbb{Z}_2)$ on $\Lambda(m)$ is free by Lemma 5.2 and the order of $\text{GL}(2, \mathbb{Z}_2)$ is 6.

Next we shall treat $|\Gamma(m)^a^b|$. The argument is similar to the above. As before, $\lambda \in \Lambda(m)$ is a representative of $\Gamma(m)^a^b$ if and only if there is $\sigma \in \text{GL}(2, \mathbb{Z}_2)$ such that
\[
\sigma \circ \lambda = \lambda \circ a^b.
\]
Since $a^b$ is of order two, the repeated use of (6.11) shows that
\[
\sigma^2 = 1.
\]
Suppose that $m$ is odd. Then the line fixed by $a^b$ goes through a vertex, say $v$, of $S(m)$ and the midpoint of the arc, say $e'$, of $S(m)$ opposite to the vertex $v$. Let $H$ be
the union of \((m + 1)/2\) consecutive arcs starting from \(v\) and ending at \(e'\). Let \(e\) be the other end arc of \(H\) different from \(e'\). The arc \(e\) has \(v\) as a vertex. Let \(\nu\) be a coloring on \(H\) and let \(\sigma \in \text{GL}(2, \mathbb{Z}_2)\) satisfy (6.12). Then \(\nu\) has an extension to a coloring of \(S(m)\) satisfying (6.11) if and only if

\[
\sigma(\nu(e)) \neq \nu(e) \quad \text{and} \quad \sigma(\nu(e')) = \nu(e').
\]

It follows that \(\nu(e)\) must be different from \(\nu(e')\) and there is only one \(\sigma\) satisfying the two identities above for each such \(\nu\). Since the number of \(\nu\) with \(\nu(e) \neq \nu(e')\) is \(A((m + 1)/2)\), so is the number of \(\lambda \in \Lambda(m)\) satisfying (6.11) for some \(\sigma\). It follows that

\[
\sum_{k=0}^{m-1} |\Gamma(m)^a_{b'}| = \frac{m}{6} A((m + 1)/2).
\]

(6.13)

Suppose that \(m\) is even and \(k\) is odd. Then the line fixed by \(a_{b'}\) goes through the midpoints of two opposite arcs, say \(e\) and \(e'\), of \(S(m)\). Let \(H\) be the union of consecutive \(m/2 + 1\) arcs starting from \(e\) and ending at \(e'\). Let \(\nu\) be a coloring on \(H\) and let \(\sigma \in \text{GL}(2, \mathbb{Z}_2)\) satisfy (6.12). Then \(\nu\) has an extension to a coloring of \(S(m)\) satisfying (6.11) if and only if

\[
\sigma(\nu(e)) = \nu(e) \quad \text{and} \quad \sigma(\nu(e')) = \nu(e').
\]

If \(\nu(e) \neq \nu(e')\) then such \(\sigma\) must be the identity, and if \(\nu(e) = \nu(e')\) then there are two such \(\sigma\) one of which is the identity. Since the number of \(\nu\) with \(\nu(e) \neq \nu(e')\) is \(A(m/2 + 1)\) and that with \(\nu(e) = \nu(e')\) is \(A(m/2)\), the number of \(\lambda \in \Lambda(m)\) satisfying (6.11) for some \(\sigma\) is \(A(m/2 + 1) + 2A(m/2)\). It follows that

\[
\sum_{k=0, k: \text{odd}}^{m-1} |\Gamma(m)^a_{b'}| = \frac{m}{12} \left( A(m/2 + 1) + 2A(m/2) \right).
\]

(6.14)

Suppose that \(m\) is even and \(k\) is even. Then the line fixed by \(a_{b'}\) goes through two opposite vertices, say \(v\) and \(v'\), of \(S(m)\). Let \(H\) be the union of consecutive \(m/2\) arcs starting from \(v\) and ending at \(v'\). Let \(e\) and \(e'\) be the end arcs of \(H\) which respectively have \(v\) and \(v'\) as a vertex. Let \(\nu\) be a coloring on \(H\) and let \(\sigma \in \text{GL}(2, \mathbb{Z}_2)\) satisfy (6.12). Then \(\nu\) has an extension to a coloring of \(S(m)\) satisfying (6.11) if and only if

\[
\sigma(\nu(e)) \neq \nu(e) \quad \text{and} \quad \sigma(\nu(e')) \neq \nu(e').
\]

If \(\nu(e) \neq \nu(e')\) then there is only one such \(\sigma\), and if \(\nu(e) = \nu(e')\) then there are two such \(\sigma\). Since the number of \(\nu\) with \(\nu(e) \neq \nu(e')\) is \(A(m/2)\) and that with \(\nu(e) = \nu(e')\) is \(A(m/2 - 1)\), the number of \(\lambda \in \Lambda(m)\) satisfying (6.11) for some \(\sigma\) is \(A(m/2) + 2A(m/2 - 1)\). It follows that

\[
\sum_{k=0, k: \text{even}}^{m-1} |\Gamma(m)^a_{b'}| = \frac{m}{12} \left( A(m/2) + 2A(m/2 - 1) \right).
\]

(6.15)
Thus, when $m$ is even, it follows from (6.14) and (6.15) that

\[ (6.16) \quad \sum_{k=0}^{m-1} |\Gamma(m)^{\alpha_k}| = \frac{m}{12} \left( A(m/2 + 1) + 3A(m/2) + 2A(m/2 - 1) \right) = m \cdot 2^{m/2-1} \]

where we used $A(q) = 2^q + (-1)^q2$ at the latter identity.

The theorem now follows from (6.6), (6.10), (6.13) and (6.16). \qed

Remark. When $m$ is even, $\Lambda(m)$ contains exactly three colorings with two colors and it defines the unique element in the double coset $\text{GL}(2, \mathbb{Z}_2) \backslash \Lambda(m)/\mathcal{D}_m$.

Example 6.5. We set

\[ C(m) := |\text{GL}(2, \mathbb{Z}_2) \backslash \Lambda(m)/\mathcal{D}_m|. \]

Using the formula in Theorem 6.4, one finds that
\[
C(2) = 1, \quad C(3) = 1, \quad C(4) = 2, \quad C(5) = 1, \quad C(6) = 4, \quad C(7) = 3 \\
C(8) = 8, \quad C(9) = 8, \quad C(10) = 18, \quad C(11) = 21, \quad C(12) = 48.
\]

We conclude this section with a remark. When $Q$ is an $m$-gon ($m \geq 3$), a small cover over $Q$ is a closed surface with euler characteristic $4 - m$ and the cardinality of the set of homeomorphism classes in small covers over $Q$ is one (resp. two) when $m$ is odd (resp. even). On the other hand, the double coset (5.2) agrees with $\text{GL}(2, \mathbb{Z}_2) \backslash \Lambda(m)/\mathcal{D}_m$ and we see from Theorem 6.4 that its cardinality is strictly larger than 2 when $m \geq 6$. So, the natural surjective map from the double coset (5.2) to the set of homeomorphism classes in small covers over $Q$ is not injective when $Q$ is an $m$-gon with $m \geq 6$. However, it is bijective when $m = 3, 4, 5$, see Example 6.5.

7. Locally standard 2-torus manifolds of dimension two

We shall enumerate the number of equivariant homeomorphism classes in locally standard 2-torus manifolds with orbit space $Q$ when $Q$ is a compact surface with only one boundary.

Theorem 7.1. Suppose that $Q$ is a compact surface with only one boundary component with $m$ ($\geq 2$) vertices and set

\[ h(Q) := |H^1(Q; (\mathbb{Z}_2)^2)/\text{Aut}(Q)|. \]

Then the number of equivariant homeomorphism classes in locally standard 2-torus manifolds over $Q$ is $h(Q)B(m)$, where $B(m) = |\Lambda(m)/\mathcal{D}_m|$ is the number discussed in the previous section.

Proof. By Corollary 5.5 it suffices to count the number of orbits in $H^1(Q; (\mathbb{Z}_2)^2) \times \Lambda(Q)$ under the diagonal action of $\text{Aut}(Q)$. Since $Q$ has only one boundary component and $m$ vertices, $\Lambda(Q)$ can be identified with $\Lambda(m)$ in Section 6 and $\text{Aut}(\mathcal{F}(Q))$ is isomorphic to the dihedral group $\mathcal{D}_m$.

Let $H$ be the normal subgroup of $\text{Aut}(Q)$ which acts on $H^1(Q; (\mathbb{Z}_2)^2)$ trivially. We claim that the restriction of the natural homomorphism

\[ (7.1) \quad \text{Aut}(Q) \to \text{Aut}(\mathcal{F}(Q)) \cong \mathcal{D}_m \]
to $H$ is still surjective. An automorphism of $Q$ (as a manifold with corners) which rotates the boundary circle and fixes the exterior of its neighborhood is an element of $H$. Therefore $H$ contains all rotations in $Q_m$. It is not difficult to see that any closed surface admits an involution which has one-dimensional fixed point component and acts trivially on the cohomology with $\mathbb{Z}_2$ coefficient. Since $Q$ is obtained from a closed surface by removing an invariant open disk centered at a point in the one-dimensional fixed point set, $Q$ admits an involution which reflects the boundary circle and lies in $H$. This implies the claim.

Let $K$ be the kernel of the homomorphism $\text{Aut}(Q) \to \text{Aut}(\mathcal{F}(Q))$. Then
\begin{equation}
\text{(7.2)} \quad |(H^1(Q; (\mathbb{Z}_2)^2) \times \Lambda(Q))/\text{Aut}(Q)| = |(H^1(Q; (\mathbb{Z}_2)^2)/K \times \Lambda(Q))/\text{Aut}(Q)|.
\end{equation}

For any element $g$ in $\text{Aut}(Q)$, there is an element $h$ in $H$ such that $gh$ lies in $K$ because the map (7.1) restricted to $H$ is surjective. Since $H$ acts trivially on $H^1(Q; (\mathbb{Z}_2)^2)$, this shows that an $\text{Aut}(Q)$-orbit in $H^1(Q; (\mathbb{Z}_2)^2)$ is same as an $K$-orbit. This means that the induced action of $\text{Aut}(Q)$ on $H^1(Q; (\mathbb{Z}_2)^2)/K$ is trivial. Therefore the right hand side at (7.2) reduces to
\[ |H^1(Q; (\mathbb{Z}_2)^2)/\text{Aut}(Q)| \cdot |\Lambda(Q)/\text{Aut}(Q)|. \]

Here the first factor is $h(Q)$ by definition and the second one agrees with $|\Lambda(m)/Q_m| = B(m)$ because of the surjectivity of the map (7.1), proving the theorem. \hfill $\square$

**Example 7.2.** $H^1(Q; (\mathbb{Z}_2)^2)$ is isomorphic to $H^1(Q; \mathbb{Z}_2) \oplus H^1(Q; \mathbb{Z}_2)$ and the action of $\text{Aut}(Q)$ on the direct sum is diagonal. When $Q$ is a disk, $h(Q) = 1$. When $Q$ is a real projective plane with an open disk removed, $H^1(Q; \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2$ and the action of $\text{Aut}(Q)$ on it is trivial. Therefore, $h(Q) = 4$ in this case. When $Q$ is a torus with an open disk removed, $H^1(Q; \mathbb{Z}_2)$ is isomorphic to $(\mathbb{Z}_2)^2$. The action of $\text{Aut}(Q)$ on it is non-trivial and it is not difficult to see that $h(Q) = 5$ in this case.

**References**

[1] J. L. Alperin and R. B. Bell, *Groups and representations*, Graduate Texts in Mathematics 162 (1995), Springer-Verlag.

[2] V. M. Buchstaber and T.E. Panov, *Torus actions and their applications in topology and combinatorics*, University Lecture Series, 24. American Mathematical Society, Providence, RI, 2002.

[3] M. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. 117 (1983), 293-324.

[4] M. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. 61 (1991), 417-451.

[5] M. Davis, T. Januszkiewicz and R. Scott, *Nonpositive curvature of blow-ups*, Sel. math. New ser. 4 (1998), 491-547.

[6] G. Ewald, *Combinatorial Convexity and Algebraic Geometry*, Graduate Texts in Math., Springer, 1996.

[7] A. Garrison and R. Scott, *Small covers of the dodecahedron and the 120-cell*, Proc. Amer. Math. Soc. 131 (2002), 963-971.

[8] A. Hattori and M. Masuda, *Theory of multi-fans*, Osaka J. Math., 40 (2003), 1-68.

[9] Z. Lii, *2-torus manifolds, cobordism and small covers*, arXiv:math/0701928.

[10] M. Masuda, *Unitary toric manifolds, multi-fans and equivariant index*, Tohoku Math. J. 51 (1999), 237-265.
[11] M. Masuda and T. Panov, *On the cohomology of torus manifolds*, Osaka J. Math. **43** (2006), 711-746.

[12] H. Nakayama and Y. Nishimura, *The orientability of small covers and coloring simple polytopes*, Osaka J. Math. **42** (2005), 243-256.

**Institute of Mathematics, School of Mathematical Science, Fudan University, Shanghai, 200433, P.R. China.**

*E-mail address: zlu@fudan.edu.cn*

**Department of Mathematics, Osaka City University, Sumiyoshi-ku, Osaka 558-8585, Japan.**

*E-mail address: masuda@sci.osaka-cu.ac.jp*