Quantum Field Theories on a Noncommutative Euclidean Space: Overview of New Physics

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ABSTRACT
In this talk I briefly review recent developments in quantum field theories on a noncommutative Euclidean space, with Heisenberg-like commutation relations between coordinates. I will be concentrated on new physics learned from this simplest class of non-local field theories, which has applications to both string theory and condensed matter systems, and possibly to particle phenomenology.

1. Noncommutative Field Theories

In this talk I will give a brief overview of the new physics recently learned from quantum field theories on a noncommutative Euclidean space. Below I will call them simply as noncommutative field theories (NCFT). (Because of the time limitation, it is regretful that many significant topics and contributions in this field have to be left out.)

By definition, a noncommutative Euclidean space is a space with noncommuting spatial coordinates, satisfying the Heisenberg-like commutation relations:

\[ [x^i, x^j] = i\theta^{ij}, \]

where \( \theta^{ij} = -\theta^{ji} \) are real constants. (There are other types of noncommutative spaces, with coordinates satisfying Lie-algebraic or Yang-Baxter relations, or with space-time noncommutativity. In this talk I am restricted only to the case of Eq. (1).) There are two ways of interpreting these commutation relations.

The first way is to interpret \( x^i \) as operators in a Hilbert space satisfying Eq. (1). In this interpretation, the noncommutative space can be mathematically viewed as generalization of phase space in usual quantum mechanics. The second is to interpret coordinates \( x^i \) as functions that generate a noncommutative algebra of functions (or fields) on the space. One may develop a classical field theory from this point of view, by realizing the algebra of fields in the set of ordinary functions of commuting variables \( \{x^\mu\} \) with the Moyal star product [1] (i.e. deformation quantization)
\[(f \ast g)(x) = \exp\left\{ \frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \right\} f(x) g(y) \big|_{x=y}. \tag{2} \]

(Here \(\mu = 0\) labels the time component, and we set \(\theta^{0i} = 0\). It is easy to check that \(x^\mu \ast x^\nu - x^\mu \ast x^\nu = i \theta^{\mu\nu}\).) Then we can transplant the usual differentiation and integration to the algebra of fields. So the (classical) action principle and the Euler-Lagrange equations of motion make sense. The key difference from the usual field theory is that everywhere the usual multiplication of two fields is replaced by the non-local star product (2). For example, for a scalar \(\phi^4\) theory, we have

\[ S = \int \frac{1}{2} \partial_\mu \phi \ast \partial^\mu \phi + \frac{1}{2} m^2 \phi \ast \phi + \frac{g}{4!} \phi \ast \phi \ast \phi \ast \phi. \tag{3} \]

For a gauge theory, the noncommutative Yang-Mills (NCYM) action is

\[ S_{YM} = -\int \frac{1}{4} F_{\mu\nu} \ast F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \ast A_\nu - A_\nu \ast A_\mu. \tag{4} \]

Note that even \(U(1)\) theory becomes non-abelian, since \(F_{\mu\nu}\) always contain the star-commutator term. In the following I will give a summary of new physic in this class of NCFT, with coordinate noncommutativity (1). Possible applications to the real world will be addressed in the last section.

### 2. New Interaction Vertices

In the deformation quantization approach, noncommutative quantum field theory (NCQFT) is developed using path integral formalism. The propagator of a field is the same as usual, since ignoring boundary terms we have

\[ \int \phi \ast \psi = \int \phi \psi. \tag{5} \]

So changing the ordinary product to the star product only affects the interaction terms, giving rise to new vertices in NCQFT.

In the NCFT (3), say, the \(\phi^4\) interaction term contains an infinite number of higher-order derivatives. So it gives rise to non-local interactions characterized by a new momentum scale \(\Lambda_{NC} = 1/\sqrt{\theta_{nc}}\). (Here we assume all \(\theta^{ij}\) are of the same magnitude, denoted as \(\theta_{nc}\).) In path integral, the star product in this term generates a momentum dependent phase factor in the four-\(\phi\) vertex:

\[ \frac{g}{6} \exp\left\{ \frac{1}{2} \sum_{i<j} k_i \wedge k_j \right\}, \tag{6} \]
where $k_i \wedge k_j = \theta^{ij} k_i k_j$. Note that this vertex is invariant only under cyclic permutation of particle legs. This leads to the distinction between planar and non-planar diagrams in perturbation.

New interaction vertices appear inNCYM too. According to the action (4), noncommutative quantum electrodynamics (NCQED) contains $3$–photon and $4$–photon vertices, both having a momentum dependent phase factor arising from the star product. Moreover, in NCQCD the star commutators in Eq. (4) does not close for $SU(3)$ group, so the latter has to be extended to $U(3)$. Also the $U(1)$ and $SU(3)$ subgroups do not decouple, and there are new type of vertices like $U(1) – SU(3) – SU(3)$ couplings etc.

The effects of these new interactions are the origin of new physics in noncommutative space, and they may be experimentally explored to probe possible coordinate noncommutativity in the real world.

3. UV-IR Entanglement

In perturbation theory, the non-locality gives rise to a novel entanglement of the UV and IR behavior. One manifestation of this effect is the one-loop non-planar correction to the 1PI two-point function [2]:

$$\Gamma_{np}^{(2)} = \frac{g^2}{96}[\Lambda_{eff}^2 - m^2 \ln \frac{\Lambda_{eff}^2}{m^2} + O(1)],$$

with $\Lambda_{eff}$ the effective cut-off which is related to the true cut-off $\Lambda$ by

$$\Lambda_{eff}^2 = \frac{1}{\Lambda^{-2} + p \circ p}, \quad p \circ q = -p_i (\theta^2)^{ij} q_j.$$

It is obvious that the UV limit $\Lambda \rightarrow \infty$ and the IR limit $p \rightarrow 0$ do not commute, so that the UV and IR behavior are entangled!

The UV-IR entanglement is a novel, essential and far-reaching feature of any NCFT. This is because of the non-local nature of interactions associated with coordinate noncommutativity: One has the space-space uncertainty relations like

$$\Delta x^1 \Delta x^2 \sim \theta^{12}.$$ 

Assume $\theta^{12} \neq 0$. If one squeezes $\Delta x^1 \rightarrow 0$, then $\Delta x^2$ must tend to $\infty$ and vice versa.

Some researchers feel very uncomfortable with the UV-IR entanglement and are worried that it may make the renormalization group (RG) not work. Indeed some features of RG in local field theories should not work in NCFT, which are known to be non-local. But this does not imply that RG should not work (see Sec. 8 below). Of course, how RG survives in NCFT should be studied carefully case by case.
4. Noncommutative Solitons

Another piece of significant new physics in NCFT is the existence of new static solitons. For example, in ordinary scalar theories, there is a famous Derrick’s theorem: Static solitons exist only in space with dimension \( d = 1 \). This can be understood by a simple scaling argument: If one shrinks the size of the soliton by changing \( x \to \lambda^{-1}x \), then the kinetic energy scales as \( K \to \lambda^{2-d}K \), while the scalar potential scales like \( V \to \lambda^{-d}V \). In \( d \geq 2 \), a soliton configuration can reduce its energy by shrinking to zero size.

Obviously such an argument is inapplicable to noncommutative space, in which a point does not make sense and there is a natural minimum length scale, given by \( \sqrt{\theta_{nc}} \). Mathematically this is because \( V(\phi) \), as a star polynomial, contains derivatives of \( \phi \), so shrinking the size may lead to an increase, rather than decrease, in interaction energy. The existence of static scalar solitons in \( d \geq 2 \) is shown explicitly in ref. [3]. For example, in the large \( \theta \)-limit, the scalar potential dominates and a static soliton can be generated from a projector function satisfying

\[
\phi_0(x) \star \phi_0(x) = \phi_0(x),
\]  

which can be easily solved in \( d = 2 \). The correspondence between noncommutative fields and operators in Hilbert space is extremely useful in generating multi-soliton solutions. Physically the new scalar solitons represent bubbles of false vacuum with size set by \( \sqrt{\theta_{nc}} \). They are stable when the potential \( V(\phi) \) has at least two minima.

New solitons also appear in noncommutative gauge theories. In particular, there are nonsingular \( U(1) \) monopoles [4] and \( U(1) \) instantons [3,5]. Moreover, the small-instanton singularity in the moduli space of ordinary instantons gets resolved inNCYM, since there exists a minimum size for instantons, set by \( \sqrt{\theta_{nc}} \) [5].

The existence of new solitons dramatically changes the spectrum, and greatly enriches nonperturbative physics of the theory.

5. Spacetime Symmetry Breaking

The third significant new feature of an NCFT is the natural breaking of spacetime symmetry by coordinate noncommutativity. This is because the noncommutativity parameters \( \theta^{ij} \) behave like a background in space. In \( 3 + 1 \) dimensional NCQED, \( \theta^{ij} \) is unchanged under \( x \to -x \), parity \( P \) is invariant, while \( C \) and \( T \) are non-invariant because they lead to \( \theta \to -\theta \). However, \( CPT \) is still a symmetry[6], \( CP \) is broken too.

Physically these could be understood in the following way: an electron in NCQED has a tree-level momentum dependent electric dipole moment, given by
\[ \mu_e = -\frac{e}{4\hbar}(\theta \times p). \]  

(10)

It is easy to verify the above statements on discrete symmetries for this expression.

Moreover, the \( \theta \)-background makes space anisotropic and breaks Lorentz boost symmetry in the active sense, namely if one rotates or Lorentz-boosts a physical system, its behavior will become different. Of course, if one makes passive rotational and boost transformations of coordinates, the physics would be unchanged. Therefore, the effects of space-time symmetry breaking effects can be used to probe the coordinate noncommutativity parameters \( \theta^{ij} \) or, at least, to set observational upper limits for them. (See Sec. 7 below.)

6. New Physics from Radiative Corrections

Since the amplitude of planar diagrams in NCQFT differs from the ordinary cases by an overall phase factor that depends merely on external momenta, introducing coordinate noncommutativity does not make a UV divergent ordinary field theory finite. However, in NCFT non-planar diagrams are suppressed due to loop-momentum dependent phase factors, and there are new interaction vertices due to star product, so the UV behavior and radiative corrections are affected.

The renormalizability, i.e. the counterterms being of the same form and with the same \( \theta \) parameter(s) in the star product as in the bare action, has been verified explicitly at least at one loop for many NCFT’s. The most dramatic change of the UV behavior occurs in \( U(1) \) NCYM theory in 3 + 1 dimensions, which is asymptotically free. The one-loop \( \beta \)-function of \( U(N) \) NCYM is given by [7]

\[ \beta(g) = -11 \frac{g^3N}{3 (4\pi)^2}, \]  

(11)

which is valid even for \( N = 1! \)

Similarly \( U(1) \) noncommutative Chern-Simons (NCCS) theory in 2 + 1 dimensions has a non-vanishing one-loop level shift, which shows up in ordinary Chern-Simons theory only for \( SU(N) \) with \( N \geq 2 \). The one-loop level shift in \( U(N) \) NCCS is given by [8]

\[ k \rightarrow k + N \text{sign}(k), \]  

(12)

which is valid also for \( N = 1! \) In ordinary \( U(1) \) Chern-Simons theory, even with coupling to matter, there is no such shift at least at one loop [9]. The result (12) indicates that the topology of the gauge group on noncommutative space should be different from that in ordinary space.
7. Observational Limits

Whether our real world is a noncommutative space or not can be determined only by observations. We need to probe new physics brought up by NCQFT. Here we cite two methods as examples; the second one gives the best observational upper limit for the noncommutativity parameter(s) $\theta_{nc}$.

One method is to explore the Lamb shift of the hydrogen atom. In NCQED, there is a tree-level electric dipole moment for the electron. Both the electric and magnetic dipole moment receive one-loop radiative corrections. Their contributions to the $2S_{1/2} - 2P_{3/2}$ hyperfine splitting are estimated in ref. [10]. Comparing with the experimental error bar for the most recent precise measurement gives the following observational limit:

$$|\theta_{nc}| \leq (0.1 \text{TeV})^{-2}. \quad (13)$$

Another method is to probe possible effects on space anisotropy and Lorentz violation due to $\theta^{ij}$, $(i, j = x, y, z)$, which look like a background. More concretely, there should be change in clock rate due to rotation of the Earth. An atomic clock comparison experiment was carried out, quite a bit time ago [11], to monitor closely the difference between two atomic clocks and to search for variations as the Earth rotates. New analysis of the old data has been done from the point of view of NCQED [12], which sets the upper limit

$$|\theta^{YZ}, \theta^{ZX}| \leq (10 \text{TeV})^{-2}. \quad (14)$$

Here $X, Y, Z$ are refereed to the non-rotating celestial equatorial coordinates.

8. Renormalization Group and Critical Exponents

In condensed matter physics, renormalization group equations (RGE) provide a systematic and powerful tool to study the low energy or large distance behavior of a many-body system, in particular the critical behavior near a second order (or continuous) phase transition point. A prototype of such phase transition occurs in the Landau-Ginzburg model, namely a real $\phi^4$ theory with a “wrong-sign” mass term, which leads to spontaneous symmetry breaking. In ordinary space, depending on the sign of $m^2$ there are only two phases possible: With $m^2 > 0$, the system is in a disordered phase with $<\phi> = 0$, while for $m^2 < 0$ it is in an ordered phase with $<\phi> \neq 0$, with a phase transition at $m^2 = 0$. For the critical behavior near the phase transition, it is well-known that $D = 4$ is the critical dimension: Above it (for $D > 4$), mean field theory is valid, and below it ($D < 4$) one needs to go beyond mean field theory and exploit the RGE to get correct critical exponents.
With my postdoc, G. H. Chen, we have studied [13] the phase diagram and phase transitions in the noncommutative Landau-Ginzburg model (NCLGM), described by the action (3). This model is known to be one-loop renormalizable, and the counterterm of the $\phi^4$ interaction has a star product with the same $\theta$-parameters. Therefore, the $\theta$-parameters, that characterize noncommutative geometry of the space, are not renormalized. Though this is natural from the geometric point of view, it is contrary to the intuition from local quantum field theory, which considers coordinate noncommutativity as short-distance effects, which should be washed out at large distances. In our opinion, this intuition is not justified in noncommutative space, because of the non-local $UV-IR$ entanglement: The large-distance behavior of the system does carry fingerprints of the noncommutative geometry at short distances.

We exploited the so-called functional RGE approach [14] popular in condensed matter theory, to overcome the difficulty due to the $UV-IR$ entanglement. The idea is that to derive RGE, we only need to perform the integration over a thin shell in momentum space at one loop, which gives us the effects of changing the cutoff from $\Lambda$ to $\Lambda - d\Lambda$. This has the advantage of avoiding possible IR singularity, best suited to our purpose. Using this approach we obtained the RGE for the dimensionless variables $r \equiv m^2/\Lambda^2$ and the coupling constant $u \equiv g\Lambda^{4-D}$ as follows:

$$\frac{dr}{dt} = 2r + \frac{u}{2}K_D(1 - r), \quad \frac{du}{dy} = (4 - D)u - \frac{3}{2}K_Du^2,$$

which are the same as in ordinary Landau-Ginzburg model (LGM), and there is no need to consider the RGE for $\theta_{nc}$. Here $t = s - 1$ is the RG flow parameter defined by $\Lambda \to \Lambda/s$. What is novel in the noncommutative case is there is a non-vanishing wave function renormalization from the non-planar part of the one-loop tadpole diagram for the propagator, which leads to a negative $\theta$-dependent anomalous dimension for the order parameter field $\phi$:

$$\gamma = -\frac{u}{48}K_D\tilde{\theta}^2,$$

with $\tilde{\theta} \equiv \theta\Lambda^2$ being dimensionless, and $K_D$ the area of unit sphere in $D$ dimensions.

For small values of $\tilde{\theta}_{nc}$ in dimension $D = 4 - \epsilon$, we still have the same Wilson-Fisher fixed point as before: $r^* = -\epsilon/6, u^* = 16\pi^2\epsilon/3$. However, the critical behavior near phase transition is altered in one aspect: The critical exponent $\eta$ becomes negative and $\tilde{\theta}$-dependent: $\eta = -\epsilon\tilde{\theta}^2/72$. (In ordinary LGM, $\eta$ vanishes.)

9. Non-uniform Ordered Phase and the Lifshitz Point

The negative anomalous dimension (16) leads to an instability for sufficiently large $\tilde{\theta}$ parameters, when it makes the total dimension of $\phi$ zero or negative. In other
words, when $\tilde{\theta} \geq \tilde{\theta}_c$, e.g. at the WF fixed point $\tilde{\theta}_c = 12/\sqrt{\epsilon}$, the sign of the quadratic kinetic term will become negative, signaling a momentum-space instability.

In such a case, one has to include a positive fourth-order derivative term to maintain the stability of the system. (We have verified that if at tree level there is no such term, it will be induced at one loop.) Then the low-$r$ (i.e. low temperature) ordered phase becomes non-uniform because of the instability: The order parameter now must be modulating in space:

$$\phi(x) = \phi_0 \cos(k \cdot x), \quad (17)$$

with a wave vector $k$ at the minimum of the combined kinetic energy.

Therefore, the phase diagram of the NCLGM in $r - \tilde{\theta}$ space is complicated, as shown in Fig. 1. There are three possible phases: For positive $r$, the system is always in disordered phase. However, for large and negative $r$, the ordered phase can be uniform or non-uniform, a sort of striped phase, depending on the value of $\tilde{\theta}$.

On the ordered phase side, the transition from a uniform to a non-uniform phase is of first order. On the other hand, the transition from a disordered to an ordered phase (by tuning $r$) is always of second order. Therefore there is a Lifshitz critical point [15] at the intersection of the first and second phase transition lines. (A similar scenario has been proposed earlier with a different argument in ref. [16].) We have
used a mean field theory to determine the critical behavior near the Lifshitz point. For details, we refer the interested reader to the original paper [13]. Our feeling is that the appearance of a non-uniform, ordered phase at large values of $\theta$ should be a general feature of noncommutative space.

10. Applications or Realizations

To conclude my talk I discuss applications or realizations of NCFT in physics.

One realization is known to be the lowest Landau level (LLL) in the quantum Hall systems: In some semi-conductor devices, electrons may be constrained to moving in a plane. In a strong transverse magnetic field and at very low temperature, electrons may be further restricted to the LLL, in which the cyclotron motion has a fixed minimal radius. It is well-known that the guiding center coordinates of the cyclotron orbit in the LLL do not commute [17]:

$$[X, Y] = \frac{i \hbar c}{eB}. \tag{18}$$

This is obviously a realization of Eq. (1). So it is generally believed that NCFT should be useful to the theory of quantum Hall systems. However, nobody has been able to discover new physics directly with the help of NCFT.

More applications can be found in string theory. For example, the coordinates of an open string endpoint on a D-brane, in an anti-symmetric tensor $B$-background, are shown to be noncommuting [18], with a form similar to Eq. (18). If our observed world happens to be on such a D3-brane, then the NCQFT on space (1) would become relevant to the real world, particularly to particle physics phenomenology! Another application is to Witten’s second quantized string field theory [19], in which the product between two string fields is noncommutative. Recently it has been shown [20] that the string-field product can be rewritten as an infinite tensor product of Moyal’s star products in single string Hilbert space. Thus, the techniques and physics of NCFT are expected to be very useful in string field theory.

For physics readers interested in NCFT, there is an excellent review paper in Review of Modern Physics [21]. Also a reprint volume of original papers in NCFT has been published by the Rinton Press [22]. Most papers quoted here can be found in that volume.

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