FEYNMAN-KAC FORMULA FOR LÉVY PROCESSES WITH DISCONTINUOUS KILLING RATE

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ABSTRACT. The challenge to fruitfully merge state-of-the-art techniques from mathematical finance and numerical analysis has inspired researchers to develop fast deterministic option pricing methods. As a result, highly efficient algorithms to compute option prices in Lévy models by solving partial integro differential equations have been developed. In order to provide a solid mathematical foundation for these methods, we derive a Feynman-Kac representation of variational solutions to partial integro differential equations that characterize conditional expectations of functionals of killed time-inhomogeneous Lévy processes. We allow for a wide range of underlying stochastic processes, comprising processes with Brownian part, and a broad class of pure jump processes such as generalized hyperbolic, multivariate normal inverse Gaussian, tempered stable, and α-semi stable Lévy processes. By virtue of our mild regularity assumptions as to the killing rate and the initial condition of the partial differential equation, our results provide a rigorous basis for numerous applications, not only in financial mathematics but also in probability theory and relativistic quantum mechanics.

Time-inhomogeneous Lévy process, killing rate, Feynman-Kac representation, weak solution, variational solution, parabolic evolution equation, partial integro differential equation, pseudo differential equation, nonlocal operator, fractional Laplace operator, Sobolev-Slobodeckii spaces, option pricing, Laplace transform of occupation time, relativistic Schrödinger equation

1. Introduction

Feynman-Kac formulas play a distinguished role in probability theory and functional analysis. Ever since their birth in 1949, Feynman-Kac type formulas have been a constant source of fascinating insights in a wide range of disciplines. They originate in the description of particle diffusion by connecting Schrödinger’s equation and the heat equation to the Brownian motion, see Kac [1943]. A type of Feynman-Kac formula also figures at the beginning of modern mathematical finance: In their seminal article of 1973, Black and Scholes derived their Nobel Prize-winning option pricing formula by expressing the price as a solution to a partial differential equation, thereby rediscovering Feynman and Kac’s deep link between the heat equation and the Brownian motion.

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The fundamental contribution of Feynman-Kac formulas is to link stochastic processes to solutions of deterministic partial differential equations. Thus they establish a connection between different disciplines that have evolved separately. Although both enjoy great success, transfer between them has remained only incidental. This may very well be the reason for applications of Feynman-Kac still appearing so surprisingly fresh. In computational finance, they enable the development of option pricing methods by solving deterministic evolution equations. These have proven to be highly efficient, particularly when compared to Monte Carlo simulation. Thus, like other deterministic methods, they come into play whenever efficiency is essential and the complexity of the pricing problem is not too high. This is the case for recurring tasks such as calibration and real-time pricing and has given rise to extensive research in computing option prices by solving partial differential equations over the last decades. The challenge to extend these methods to price options in advanced jump models has inspired researchers to develop highly efficient and widely applicable algorithms in recent years, see for instance Cont and Voltchkova (2005b), Hilber, Reich, Schwab and Winter (2009), Hilber, Reichmann, Schwab and Winter (2013), Salmi, Toivanen and Sydow (2014) and Itkin (2015).

In this article we derive a Feynman-Kac formula so as to provide a solid mathematical basis for fast option pricing in time-inhomogeneous Lévy models using partial integro differential equations. While large parts of the literature focus on numerical aspects of these pricing methods, only little is known about the precise link between the related deterministic equations and the corresponding conditional expectations representing option prices. Our main question therefore is: Under which conditions is there a Feynman-Kac formula linking option prices given by conditional expectations with solutions to evolution problems?

In order to further specify the problem, we focus on time-inhomogeneous Lévy models and options whose path dependency may be expressed by a possibly discontinuous killing rate. In this setting with $A = (A_t)_{t \in [0,T]}$ the Kolmogorov operator of a time-inhomogeneous Lévy process, killing rate (or potential) $\kappa : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, source $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and initial condition $g : \mathbb{R}^d \to \mathbb{R}$, the Kolmogorov equation is of the form

$$\partial_t u + A_{T-t} u + \kappa_{T-t} u = f, \quad u(0) = g.$$  \hspace{1cm} (1)

Proceeding in an unsophisticated manner, one would typically assume that equation (1) has a classical solution. If the solution $u$ is sufficiently regular to allow for an application of Itô’s formula and moreover satisfies an appropriate integrability condition, it is straightforward to derive the following Feynman-Kac representation

$$u(T-t, L_t) = E \left( g(L_T) e^{-\int_{T-t}^T \kappa_s(L_s) \, dh} + \int_{T-t}^T f(T-s, L_s) e^{-\int_s^{T-t} \kappa_r(L_r) \, dh} \, ds \bigg| \mathcal{F}_t \right). \hspace{1cm} (2)$$

Then, the conditional expectation can be obtained by solving the equation by means of a deterministic numerical scheme. Such an argumentation hinges on a strong regularity assumption on the solution $u$ and thus implicitly on the data of the equation, $g, f, A$ and $\kappa$. This constitutes a serious restriction on the applicability of such an approach.

We, however, pay special attention to identifying conditions for the validity of (2) that are appropriate for financial applications. Here, the choice of killing rates as indicator functions turns out to be the key to a variety of applications, not only in mathematical finance, but also in probability theory as we will show in section 2. The fundamental role of killing rates of indicator type is killing the process outside...
a specified domain which relates them to occupation times. Moreover, exit times of stochastic processes may conveniently be approximated by means of killing rates of the indicator type. Thus discontinuous killing rates form a common root of exit probabilities, the distribution of the supremum process and prices of path-dependent options such as those of barrier, lookback, and American type. For a characterization of prices of barrier options and the distribution function of the past supremum of time-inhomogeneous Lévy processes obtained with the help of killing rates see Glau (2010) and Glau (2015). For these reasons we will allow for non-smooth and even discontinuous killing rates in Kolmogorov equation (1).

Discontinuities in the killing rate $\kappa$ result in non-smoothness of the solution $u$ of Kolmogorov equation (1). In particular, one cannot expect $u \in C^{1,2}$. Assume $u(0) \neq 0$ and $\kappa = 1_{(-\infty,0)}d$ in (1), then $x \mapsto u(t,x) \in C^2$ implies $x \mapsto \kappa(x)u(t,x) \in C$, which obviously is a contradiction. Hence, for our purposes, the assumption Itő’s formula can be applied to the solution $u$ is fruitless. Neither is it reasonable to presume that equation (1) has a classical solution. Let us not only emphasize that such irregularity is inherent in equation (1) if the killing rate is discontinuous, but also that it is a typical feature of Kolmogorov equations for path-dependent option prices. Prominent examples are boundary value problems related to barrier options in Lévy models as well as free boundary value problems for American option prices. In all of these cases, the use of a generalized solution concept is necessary and does equation (1) allow for a weak, also called variational, formulation for a wide class of stochastic processes.

We prefer weak formulations to other general solution concepts such as viscosity solutions for the following reasons. First, weak formulations are sufficiently general to apply to pricing equations of most of the relevant option and model types. Second, given existence and uniqueness of a weak solution, it only depends on its regularity properties whether it is also a viscosity solution or a classical solution. The main reason for our choice, however, is that weak formulations are the theoretical foundation of Galerkin methods, a rich class of versatile numerical methods to solve partial differential equations. Galerkin methods rely on an elegant problem formulation in Hilbert spaces that by its very construction leads to convergent schemes as well as to a lucid error analysis. They furthermore distinguish themselves by their enormous flexibility towards problem types as well as compression techniques. Both theory and implementation of Galerkin methods have experienced a tremendous advancement over the past fifty years. They have become indispensable for today’s technological developments in such diverse areas as aeronautical, biomechanical, and automotive engineering.

In mathematical finance, Galerkin pricing algorithms have already been developed even for basket options in jump models. Furthermore, numerical experiments and error estimates have confirmed their efficiency both in theory as well as in practice. See Hilber et al (2013), and e.g. Matache, von Petersdorff and Schwab (2004), Matache, Schwab and Wihler (2005), von Petersdorff and Schwab (2004). Beyond space discretizations, the freedom in the choice of the approximating finite dimensional function space is exploited in Galerkin based model reduction techniques that have a great potential in financial applications, see Cont, Lantos and Pironneau (2011), Pironneau (2011), and Sachs and Schui (2013), Haasdonk, Salomon and Wohlmuth (2012) and Haasdonk et al (2012). Hence, our specific question is: Under which conditions on the time-inhomogeneous Lévy process $L$, the killing rate $\kappa$, the source $f$ and initial condition $g$ is there a unique weak solution of Kolmogorov equation (1) that allows for a stochastic representation of form (2)? Feynman-Kac representations for viscosity solutions with application to option pricing in Lévy models have been derived in Cont and Voltchkova (2005a) and
Cont and Voltchkova (2005b). Results linking jump processes with Brownian part to variational solutions had already been proven earlier in Bensoussan and Lions (1982). However, in order to cover some of the most relevant financial models, we have to consider pure jump processes, i.e. processes without a Brownian component, as well. Pure jump Lévy models are able to fit market data with high accuracy, see e.g. Eberlein (2001) and, as a consequence, see Schoutens (2003), Cont and Tankov (2004). Moreover, statistical analysis of high-frequency data supports the choice of pure jump models, and Aït-Sahalia and Jacod (2014).

Pure jump processes differ significantly from processes with a Brownian part. For example, almost surely every path of a Lévy process is of finite quadratic variation only in the presence of a Brownian component. The Brownian component translates to a second order derivative in the Kolmogorov operator, while the pure jump part corresponds to an integro differential operator of lower order of differentiation. Accordingly, the second order derivative is only present in Kolmogorov operators of processes with a Brownian component. As a consequence, the solution to the Kolmogorov equation of a pure jump Lévy process does not lie in the Sobolev space $H^1$ that is the space of quadratic integrable functions with a square integrable weak derivative. Therefore a more general solution space needs to be chosen. In order to make an appropriate choice, recall that Lévy processes are nicely characterized through the Lévy-Khinchine formula by the Fourier transform of their distribution functions, we use exponentially weighted Sobolev-Slobodeckii spaces as to allow for a wide range of initial conditions, such as the payoff function of a call option in logarithmic variables and the Heaviside step function that relates to pure jump models, and Aït-Sahalia and Jacod (2014).

In order to present the main result of the present article, we introduce the underlying stochastic processes, present the Kolmogorov equation with killing rate, its weak formulation as well as the solution spaces of our choice. We denote by $C^\infty_0(\mathbb{R}^d)$ the set of smooth real-valued functions with compact support in $\mathbb{R}^d$ and let

\[ F(\varphi) := \int_{\mathbb{R}^d} e^{i\langle \xi, x \rangle} \varphi(x) \, dx \]  

the Fourier transform of $\varphi \in C^\infty_0(\mathbb{R}^d)$ and let $F^{-1}$ its inverse.

Since Lévy models typically lead to a considerably better fit to the time-evolution of financial data when time-dependent parameters are chosen, we base our analysis on the class of time-inhomogeneous Lévy processes. Let us be given a stochastic basis $(\Omega, F, (F_t)_{0 \leq t \leq T}, P)$ and let $L$ be an $\mathbb{R}^d$-valued time-inhomogeneous Lévy process with characteristics $(b_t, \sigma_t, F_t; h)_{t \geq 0}$. I.e. $L$ has independent increments and for fixed $t \geq 0$ its characteristic function is

\[ E e^{i\langle \xi, L_t \rangle} = e^{-\int_0^t A_s(-i\xi) \, ds} \]  

where the symbol of the process $A_t$ for any fixed $t \geq 0$ equals

\[ A_t(\xi) := \frac{1}{2} \langle \xi, \sigma_t \xi \rangle + i \langle \xi, b_t \rangle - \int_{\mathbb{R}^d} \left( e^{-i\langle \xi, y \rangle} - 1 + i \langle \xi, h(y) \rangle \right) F_t(dy). \]  

Here, for every $s > 0$, $\sigma_s$ is a symmetric, positive semi-definite $d \times d$-matrix, $b_s \in \mathbb{R}^d$, and $F_s$ is a Lévy measure i.e. a positive Borel measure on $\mathbb{R}^d$ with $F_s(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (|x|^2 \wedge 1) F_s(dx) < \infty$, and $h$ is a truncation function i.e. $h : \mathbb{R}^d \to \mathbb{R}$. Such
that \( \int_{|x|>1} h(x)F_t(dx) < \infty \) with \( h(x) = x \) in a neighbourhood of 0. The maps 
\[ s \mapsto \sigma_s, s \mapsto b_s \text{ and } s \mapsto \int (|x|^2 \wedge 1) F_s(dx) \]
are Borel-measurable with 
\[ \int_0^T (|b_s| + \| \sigma_s \|_{M(d \times d)} + \int (|x|^2 \wedge 1) F_s(dx)) \, ds < \infty \]
for every \( T > 0 \), where \( \| \cdot \|_{M(d \times d)} \) is a norm on the vector space formed by the 
\( d \times d \)-matrices.

The **Kolmogorov operator of the process** \( L \) is given by 
\[ A_t \varphi(x) := -\frac{1}{2} \sum_{j,k=1}^d \sigma_{i,j}^k \frac{\partial^2 \varphi}{\partial x_j \partial x_k}(x) - \sum_{j=1}^d b_i^j \frac{\partial \varphi}{\partial x_j}(x) 
- \int_{\mathbb{R}^d} \left( \varphi(x + y) - \varphi(x) - \sum_{j=1}^d \frac{\partial \varphi}{\partial x_j}(x)y^j \right) F_t(dy) \]
for every \( \varphi \in C^\infty_0(\mathbb{R}^d) \), where \( y^j \) denotes the \( j \)-th component of the truncation function \( h \). An elementary calculation shows 
\[ A_t \varphi = \mathcal{F}^{-1}(A_t \mathcal{F}(\varphi)) \quad \text{for all} \quad \varphi \in C^\infty_0(\mathbb{R}^d), \]
(8)
hence Kolmogorov operator \( A \) is pseudo differential operator with symbol \( A \).

In order to give the weak formulation of Kolmogorov equation (1) let \( L^2(0, T; H) \) denote the space of weakly measurable functions \( u : [0, T] \rightarrow H \) with \( \int_0^T \| u(t) \|_H^2 \, dt < \infty \). For a **Gelfand triplet** \((V, H, V^*)\), i.e. separable Hilbert spaces \( V \) and \( H \) such that there exist continuous embeddings from \( V \) into \( H \) and from \( H \) into the dual space \( V^* \) of \( V \); let 
\[ W^1(0, T; V, H) := \left\{ u \in L^2(0, T; V) \left| \partial_t u \in L^2(0, T; V^*) \right. \right\} \]
(9)
where \( \partial_t u \) denotes the derivative with respect to time of \( u \) in distributional sense. For the definition of weak measurability and for a detailed introduction of the space \( W^1(0, T; V, H) \) that relies on the Bochner integral, we refer to section 24.2 in [Wloka 1987]. Details on Gelfand triplets can be found e.g. in section 17.1 in [Wloka 1987] for details.

For operator \( A : L^2(0, T; V) \rightarrow L^2(0, T; V^*) \) given by 
\[ \kappa(u)(v) = \langle ku, v \rangle_H, \]
\( f \in L^2(0, T; V^*) \) and \( g \in H \), \( u \in W^1(0, T; V, H) \) is called **weak solution** of Kolmogorov equation (1), if 
\[ \partial_t u + A_{T-t}u + \kappa_{T-t}u = f \]
(10)
is satisfied in \( L^2(0, T; V^*) \) and \( H - \lim_{t \downarrow 0} u = g \).

As solution spaces we consider **exponentially weighted Sobolev-Slobodeckii spaces** \( H^\alpha_\eta(\mathbb{R}^d) \) with index \( \alpha \geq 0 \) and weight \( \eta \in \mathbb{R}^d \) the completion of \( C^\infty_0(\mathbb{R}^d) \) with respect to the norm \( \| \cdot \|_{H^\alpha_\eta} \) given by 
\[ \| \varphi \|_{H^\alpha_\eta}^2 := \int_{\mathbb{R}^d} (1 + |\xi|)^{2\alpha} \mathcal{F}(\varphi)(\xi - i\eta) \, d\xi, \]
(11)
which is a separable Hilbert space. For \( \eta = 0 \) the space \( H^\alpha_\eta(\mathbb{R}^d) \) coincides with the Sobolev-Slobodeckii space \( H^\alpha(\mathbb{R}^d) \) as it is defined e.g. in [Wloka 1987]. For \( \alpha = 0 \) the space \( H^\alpha_\eta(\mathbb{R}^d) \) coincides with the weighted space of square integrable functions 
\[ L^2_\eta(\mathbb{R}^d) := \{ u \in L^1_{\text{loc}}(\mathbb{R}^d) \mid x \mapsto u(x) e^{(\eta, x)} \in L^2(\mathbb{R}^d) \}. \]
Furthermore, we denote the dual space of \( H^\alpha_\eta(\mathbb{R}^d) \) by \( (H^\alpha_\eta(\mathbb{R}^d))^* \).
Let $a : [0, T] \times H^\alpha_0(\mathbb{R}^d) \times H^\alpha_0(\mathbb{R}^d) \to \mathbb{R}$ a family $(a_t)_{t \in [0, T]}$ of bilinear forms, measurable in $t$, with associated linear operators $A_t : H^\alpha_0(\mathbb{R}^d) \to (H^\alpha_0(\mathbb{R}^d))^*$ via

$$A_t(u)(v) = a_t(u, v) \quad \text{for all } u, v \in H^\alpha_0(\mathbb{R}^d) \quad (12)$$

and related symbol $A : \mathbb{R}^d \to \mathbb{C}$ via

$$A_t(\varphi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle \xi, \varphi \rangle} A_t(\xi) F(\varphi)(\xi) \, d\xi \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d). \quad (13)$$

One classical way to proof existence of weak solutions of parabolic equations is to verify continuity and a Gårding inequality of the bilinear form and we will rely on the following

**Definition 1.1.** We call operator $A$ respectively the bilinear form $a$ parabolic in $H^\alpha_0(\mathbb{R}^d)$, $L^2(\mathbb{R}^d)$ uniformly in time and weight $[0, T] \times \mathbb{R}$, if $t \mapsto a_t(u, v)$ is càdlàg for all $u, v \in H^\alpha_0(\mathbb{R}^d)$ and there exist constants, $C, G > 0$, $G' \geq 0$ such that uniformly for all $t \in [0, T]$, all $\eta \in \mathbb{R} \subset \mathbb{R}^d$, and all $u, v \in H^\alpha_0(\mathbb{R}^d)$,

$$|a_t(u, v)| \leq C \|u\|_{H^\alpha_0(\mathbb{R}^d)} \|v\|_{H^\alpha_0(\mathbb{R}^d)} \quad \text{(Continuity (Cont-a))}$$

$$a_t(u, u) \geq G \|u\|^2_{H^\alpha(\mathbb{R}^d)} - G' \|u\|^2_{L^2(\mathbb{R}^d)} \quad \text{(Gårding inequality (Gård-a))}$$

As highlighted in equation in [8], the Kolmogorov operator of a time-inhomogeneous Lévy process is a pseudo differential operator. Its symbol is explicitly known for various classes and in general is characterized by the exponent of the Lévy-Khintchine representation. Thus, it is interesting express parabolicity of the bilinear form in terms of the symbol. This characterization will allow us in section [3] to show parabolicity of the related operator for number of classes of processes. The characterization, moreover is one of the key steps in our proof of Feynman-Kac representation [10].

An extension of the bilinear form to weighted Sobolev-Slobodeckii spaces corresponds to a shift of the symbol in the complex plane and we introduce the appropriate notion for the symbol.

**Definition 1.2.** We say the symbol $A = (A_t)_{t \in [0, T]}$ has Sobolev index $\alpha$ uniformly in time and weight from $[0, T] \times \mathbb{R}$, if $t \mapsto A_t(\xi + i\eta)$ is càdlàg for each $\eta \in \mathbb{R}$ and there exist constants, $0 \leq \beta < \alpha$, $C, G_1 > 0$, $G_2 \geq 0$ such that uniformly for all $t \in [0, T]$, all $\eta \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^d$,

$$|A_t(\xi + i\eta)| \leq C(1 + |\xi|)^\alpha \quad \text{(Continuity condition (Cont-A))}$$

$$\Re(A_t(\xi + i\eta)) \geq G_1(1 + |\xi|)^\alpha - G_2(1 + |\xi|)^\beta \quad \text{(Gårding condition (Gård-A))}$$

If $A$ is the symbol of a time-inhomogeneous Lévy process, we also say $L$ has Sobolev index $\alpha$ uniformly in time and weight $[0, T] \times \mathbb{R}$.

For weight $\eta = (\eta^1, \ldots, \eta^d)$ let

$$U_\eta := \{ z \in \mathbb{C}^d \mid \exists (z^j) \in \{ 0 \} \cup \text{sgn}(\eta^j)[0, |\eta^j|) \text{ for } j = 1, \ldots, d \}, \quad (14)$$

$$R_\eta := \text{sgn}(\eta^1)[0, |\eta^1|] \times \cdots \times \text{sgn}(\eta^d)[0, |\eta^d|]. \quad (15)$$

The assertion as well as its proof of the next theorem are a straightforward generalization of Theorem 3.2 in [Glauc 2014]. In order to provide a self-contained presentation the proof is given in Appendix [3].

**Theorem 1.3.** Let the symbol $A = (A_t)_{t \in [0, T]}$ of pseudo differential operator $A = (A_t)_{t \in [0, T]}$ have for each $t \in [0, T]$ a continuous extension on $U_{-\eta}$ that is analytic
in the interior $\bar{U}_{-\eta}$ and satisfies for each $t \in [0, T]$

$$|A_t(z)| \leq C(t)(1 + |z|)^{m(t)} \quad \text{for all } z \in U_{-\eta}$$

(16)

for some constant $C(t), m(t) > 0$. Then the following assertions are equivalent.

(i) The operator $A_t$ is parabolic in $H^\alpha_{\eta}(\mathbb{R}^d), L^2_{\eta}(\mathbb{R}^d)$ uniformly in time and weight $[0, T] \times R_{-\eta}$.

(ii) The symbol $A_t$ has Sobolev index $2\alpha$ uniformly in time and weight $[0, T] \times R_{-\eta}$.

For a time-inhomogeneous Lévy process $L$ and $\eta \in \mathbb{R}^d$, Theorem 25.17 in Sato (1999) implies that $\int_{|x|>1} e^{(\eta, x)} F_t(dx) < \infty$ for every $0 \leq t \leq T$ is equivalent to the Exponential Moment condition

$$E[e^{(\eta, L_t)}] < \infty \quad \text{for every } 0 \leq t \leq T \quad (EM(\eta))$$

and for such $\eta$ we have

$$E[e^{(\eta, L_t)}] = e^{-\int_0^t A_t(-\xi + i\eta) ds} \quad \text{for all } \xi \in \mathbb{R}^d \text{ and all } t \geq 0. \quad (17)$$

Moreover, Lemma 2.1 (c) in Eberlein and Glau (2014) shows that if

$$EM(\eta') \quad \text{holds for every } \eta' \in R_{-\eta}, \quad (EM(R_{-\eta}))$$

then for every $0 \leq t \leq T$ the map $z \mapsto A_t(z)$ has a continuous extension to the domain $\bar{U}_{-\eta}$ which is analytic in the interior $\bar{U}_{-\eta}$. Moreover, Theorem 25.17 in Sato (1999) and Lemma [A.1] show that inequality (16) is satisfied with $m(t) = 2$ for some constant $C(t) > 0$ for each $t \in [0, T]$. We obtain the following

**Corollary 1.4.** Let $L$ a time-inhomogeneous Lévy process with $EM(R_{-\eta})$. Then the following assertions are equivalent.

(i) The Kolmogorov operator of $L$ is parabolic in $H^\alpha_{\eta}(\mathbb{R}^d), L^2_{\eta}(\mathbb{R}^d)$ uniformly in time and weight $[0, T] \times R_{-\eta}$.

(ii) $L$ has Sobolev index $2\alpha$ uniformly in time and weight $[0, T] \times R_{-\eta}$.

For an integrable or nonnegative random variable $X$ we denote

$$E_{t,x}(X) := E(X|L_t = x), \quad \text{for } t > 0, \quad E_{0,x}(X) := E_x(X) \quad (18)$$

where $x \mapsto E(X|L_t = x)$ is the factorization of the conditional expectation $E(X|L_t)$ and denoting by $E_x$ the expectation w.r.t. $P_x$ which is a probability measure such that $P_x(L_0 = x) = 1$.

**Theorem 1.5.** Let $L$ an $\mathbb{R}^d$-valued time-inhomogeneous Lévy process with $EM(R_{-\eta})$ and with Sobolev index $2\alpha$ uniformly in time and weight $[0, T] \times R_{-\eta}$. Then

(i) for $\kappa: [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ measurable and bounded, $f \in L^2(0, T; (H^\alpha_{\eta}(\mathbb{R}^d))^*)$ and $g \in L^2_{\eta}(\mathbb{R}^d)$ Kolmogorov equation (1) has a unique weak solution $u \in W^1(0, T; (H^\alpha_{\eta}(\mathbb{R}^d), L^2_{\eta}(\mathbb{R}^d)))$;

(ii) if additionally, $f \in L^2(0, T; H^1_{\eta}(\mathbb{R}^d))$ for some $l \geq 0$ with $l > (d - 2\alpha)/2$ then for every $t \in [0, T]$ and a.e. $x \in \mathbb{R}^d$

$$u(T-t,x) = E_{t,x}(g(L_T) e^{-\int_t^T \kappa_u(L_s) ds} + \int_t^T f(T-s,L_s) e^{-\int_t^s \kappa_u(L_r) dr} ds). \quad (19)$$
The assertion of part (i) of the theorem directly follows from Corollary 1.4 and the classical result on existence and uniqueness of weak solutions to parabolic equations, see e.g. Theorem 23.A in Zeidler (1990). Section 5 is dedicated to the proof of part (ii) of Theorem 1.5.

As an immediate consequence of Theorem 1.5 and Sobolev embedding result Theorem 8.2 in Nezza, Palatucci and Valdinoci (2011) we obtain Corollary 1.6.

Under the assumptions and notations of Theorem 1.5 in the univariate case, i.e. for $d = 1$, for $\alpha \in (1/2, 1]$ and any fixed $t \in (0, T)$, the function $x \mapsto u(t, x)$ is $\lambda$-Hölder continuous with $\lambda = \frac{2\alpha - 1}{2}$, i.e.

$$\sup_{x, y \in \mathbb{R}, x \neq y} \frac{|u(t, x) - u(t, y)|}{|x - y|^\lambda} < \infty.$$ 

In particular, $x \mapsto u(t, x)$ is continuous and equality (19) in Theorem 1.5 holds for every $x \in \mathbb{R}$.

The rest article is organized as follows: In the next section we outline various applications of Feynman-Kac Theorem 1.5 and in the third section we present examples of stochastic processes that satisfy the assumptions of the theorem. We dedicate section 4 to a robustness result for weak solutions which is required in our proof of Theorem 1.5 that we present in section 5. The section also contains regularity result Lemma 5.1 for the solutions to the Kolmogorov equation. A provides essential properties of the symbol and the operator, and B concludes with the proof of Theorem 1.3.

2. Applications

We choose examples from different fields such as physics, probability theory and finance to illustrate the interdisciplinary benefit of Theorem 1.5. We present each of the examples in a self-consistent way to facilitate its usage on the one side and the readability on the other.

Remark 2.1. In several applications there does not exist an $\eta \in \mathbb{R}^d$ such that the initial condition $g \in L^2_{\eta}(\mathbb{R}^d)$. In these cases, $g$ may be split up in $2^d$ summands that are supported in the $2^d$ orthants. By the linearity of the expectation, respectively of the PIDE, the problem can be split additively in $2^d$ separate problems. Then typically for each of the summands $g^j$ for $j = 1, \ldots, d$, an exponential damping factor $\eta^j \in \mathbb{R}^d$ exists such that $g^j \in L^2_{\eta^j}(\mathbb{R}^d)$ and the results of Theorem 1.5 can be applied for each initial condition $g^j$ separately. Consider e.g. the initial condition $g \equiv 1$. In the one-dimensional setting, write e.g. $g = 1_{(-\infty, 0)} + 1_{(0, \infty)}$, where $1_{(-\infty, 0)} \in L^2_{\eta^-}(\mathbb{R})$ for every $\eta^- > 0$ and $1_{(0, \infty)} \in L^2_{\eta^+}(\mathbb{R})$ for every $\eta^+ < 0$. For numerical purposes, a splitting in smooth functions instead of Heaviside step functions is preferable.

2.1. Employee options. Consider rewarding the management board of a corporation according to the performance of the corporation’s stock price. Financial instruments used in this context are called employee stock options and typically are based on European call options, where at a certain maturity, say after 5 years, the owner of the option has the right—but not the obligation— to purchase the stock value for the strike price that was fixed in advance. Thus, he will be rewarded if the stock exceeds the strike at a certain time in the future. Shareholders though typically are interested in the performance of the stock during the hole period and not only at fixed time points. They wish to support management decisions that push the stock price constantly to a high level and even more, they wish to
reward according to the level. Moreover, it is arguably fairer to reward the management board according to the performance of the stock value relative to the market evolution. Therefore we introduce additional reference assets. Denote by $S$ the $d$-dimensional stochastic process, modelling the stock of the company and $d - 1$ reference assets. Let $G : \mathbb{R}^d \to \mathbb{R}$ be a payoff profile and let $\kappa : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ be a reward rate function. Moreover, we incorporate a continuously paid salary by the salary function $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}$. We suggest a class of employee options for flexible rewarding of the management board of the following type: At maturity $T$ the employee obtains the payoff

$$G(S_T) e^{\int_0^T \kappa_h(S_s) \, dh},$$

(20)

additionally at each instant $t \in [0, T]$, the salary

$$f(t, S_t) e^{\int_0^t \kappa_h(S_s) \, dh} \, dt$$

(21)

is paid. Thus, the payoff profile $G$ may depend on the level of the stock and the reference assets, the rewarding rate and the salary function may additionally be time-dependent.

We use the following notation. For $x = (x^1, \ldots, x^d) \in \mathbb{R}^d$ let $e^x := (e^{x^1}, \ldots, e^{x^d})$ and $\tilde{G}(x) := G(e^x)$, $\tilde{\kappa}(\cdot, x) := \kappa(\cdot, e^x)$ and $\tilde{f}(\cdot, x) := f(T - \cdot, e^x)$.

Let the interest rate $(r_t)_{t \geq 0}$ be deterministic, measurable and bounded and let $S = (S_0^1 e^{L_1}, \ldots, S_0^d e^{L_d})$ with a time-inhomogeneous Lévy process $L$ with local characteristics $(b, c, F)$ such that the so-called drifted condition is satisfied,

$$b^j_t = r_t - \frac{1}{2} \sigma^2_j - \int \left( e^{\xi_j} - 1 - h^j(x) \right) F_1(dx),$$

(22)

where $h^j$ is the $j$-th component of the truncation function $h$. Then, under assumption (A1) for some $\bar{y} \in \mathbb{R}^d$ with $\bar{y}_j > 1$ for each component $j = 1, \ldots, d$, the discounted asset price process $\tilde{S} := S e^{-\int_0^T r_s \, ds}$ is a martingale, see e.g. the proof of Proposition 4.4. in Eberlein, Jacob and Raible (2005), and $S, r$ is a no-arbitrage asset price model driven by a time-inhomogeneous Lévy process.

The following assertion shows that the model price of the employee option (20), (21) can be computed solving the related Kolmogorov PIDE.

**Corollary 2.2.** Let $\eta \in \mathbb{R}^d$ such that $\tilde{G} \in L^2_0(\mathbb{R}^d)$ and let the time-inhomogeneous Lévy process $L$ satisfy assumptions (A1)-(A4). Assume additionally (22) and that (A1) is satisfied for some $\eta \in \mathbb{R}^d$ with $\bar{y}_j > 1$ for each component $j = 1, \ldots, d$. Then the time-$t$ value of the employee option with payout profile (20), (21),

$$u(T, x) := E_x \left( \tilde{G}(L_T) e^{\int_0^T (\hat{\kappa}_h(L_s) - r_s) \, dh} + \int_0^T \tilde{f}(T - s, L_s) e^{\int_0^s (\hat{\kappa}_h(L_u) - r_u) \, dh} \, ds \right),$$

is given by the unique weak solution $u \in W^1(0, T; H^{\alpha/2}_0(\mathbb{R}^d), L^2_0(\mathbb{R}^d))$ of

$$u + A_{T-} u + \tilde{\kappa}_{T-} u = -\tilde{f}, \quad u(0) = \tilde{G}.$$  

(23)

**Proof.** The assertion of the proposition is an immediate consequence of Theorem 1.56. \hfill \Box

2.2. **Lévy-driven short rate models.** As another application in finance, we consider bond prices in a Lévy driven short rate model. In Eberlein and Raible (1999) Lévy driven term structure models have been introduced first. Consider a short rate of the form

$$r_t := r(t, L_t)$$

(24)
with an \(\mathbb{R}^d\)-valued time-inhomogeneous Lévy process \(L\) and a measurable interest rate function \(r: [0, T] \times \mathbb{R}^d \to \mathbb{R}\) such that the discount factor \((e^{-\int_0^T r_h \, dh})_{0 \leq t \leq T}\) is a martingale. Basic interest rate derivatives are the so-called zero coupon bonds. At maturity, the holder of the bond receives 1 unity of currency. In accordance with the no-arbitrage principle, the time-\(t\) value of the zero-coupon bond with maturity \(0 \leq t < T\) is modelled by
\[
P(t, T) := E\left( e^{-\int_t^T r_h \, dh} \mid \mathcal{F}_t \right).
\]

We reuse the notation Corollary 2.6 and with analogous arguments as for its proof we obtain

**Corollary 2.3.** If \(L\) satisfies the assumptions of Corollary 2.6 with \(\epsilon, \alpha, \eta\) and \(\eta^j, O^j\) for \(j = 1, \ldots, 2^d\) as in Corollary 2.6 and the interest rate function \(r\) from (24) is bounded, then for every \(0 \leq t < T\), the price of the zero coupon bond is given as
\[
P(t, T) = \sum_{j=1}^{2^d} u^j(T - t, L_t) \quad \text{a.s.}
\]
where \(u^j\) is the unique weak solution \(u^j \in W^1(0, T; H^{\alpha/2}_\eta(d) \circ \tilde{L}_t^j(\mathbb{R}^d))\) of
\[
\dot{w}^j + \mathcal{A}_{T-t}w^j + rw^j = 0, \quad w^j(0) = \mathbb{1}_{O^j}.
\]

Let us furthermore consider an option on a zero-coupon bond with payoff given by \(G(P(T_1, T))\) at the options maturity \(T_1 \in (0, T)\) for some measurable function \(G\). Since the bond prices are bounded \(0 \leq P(t, T') \leq \bar{P}\) for every \(0 \leq t < T'\), if the interest rate function is bounded from below, it is enough to consider bounded payoff functions \(G : [0, \bar{P}] \to [0, \bar{P}]\). Then, according to Proposition 2.3, \(G(P(T_1, T_2)) = G \circ u(T_1, L_{T_1})\) and by the no-arbitrage principle, the time-\(t\) price of the option is
\[
\Pi_t = E\left( G(P(T_1, T)) e^{-\int_t^T r(h, L_h) \, dh} \mid \mathcal{F}_t \right).
\]

Noting that \(u^j\) in Corollary 2.6 is bounded for each \(j = 1, \ldots, 2^d\), we obtain, again analogue to the proof of Corollary 2.6, the following

**Corollary 2.4.** Under the assumptions of Proposition 2.6,
\[
\Pi_t = \sum_{j=1}^{2^d} v^j(T_1 - t, L_t) \quad \text{a.s.,}
\]
where for each \(j = 1, \ldots, 2^d\), the function \(v^j\) is the unique weak solution \(v^j \in W^1(0, T; H^{\alpha/2}_\eta(d) \circ \tilde{L}_t^j(\mathbb{R}^d))\) of
\[
\dot{v}^j + \mathcal{A}_{T_1-t}v^j + rv^j = 0, \quad v^j(0) = u(T_1, \cdot)\mathbb{1}_{O^j},
\]
where \(u\) is given in Proposition 2.6.

As further applications we mention bankruptcy probabilities in the model of Albrecher et al. (2011), the value of barrier strategies in the bankruptcy model of Albrecher and Lautscham (2013) as well as reduced form modelling of credit risk, see Jeanblanc and Le Cam (2007).

### 2.3. Penalization of the domain

Feynman-Kac representation (19) is essential for the method of penalization of domain in order to derive a Feynman-Kac correspondence for boundary value problems. Under the presence of a dominating diffusion part, the method is outlined in Bensoussan and Lions (1982). In a forthcoming article, Glau (2015), the method is used for PHACs, compare also Glau (2014). The argument is based on the following...
Corollary 2.5. Let $L$ be an $\mathbb{R}^d$-valued time-inhomogeneous Lévy process satisfying assumptions (A1)–(A5) for some $\alpha \in (0,2]$ and some $\eta \in \mathbb{R}^d$. Let $f \in L^2(0,T;H^\alpha_{\eta}(\mathbb{R}^d))$ for some $l \geq 0$ with $l > (d - \alpha)/2$, let $\kappa : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be measurable and bounded, and $g \in L^2(\mathbb{R}^d)$. Then, for each $\lambda > 0$ and $D \subset \mathbb{R}^d$ open, the unique weak solution $u^\lambda \in W^1(0,T;H^\alpha_{\eta}(\mathbb{R}^d),L^2_D(\mathbb{R}^d))$ of
\begin{equation}
\partial_t u^\lambda + A_{T-t} u^\lambda + \kappa_{T-t} u^\lambda + \lambda \mathcal{M}_T u^\lambda = f, \quad u^\lambda(0) = g,
\end{equation}
has for every $t \in (0,T]$ almost surely the stochastic representation
\begin{align}
&u^\lambda(T-t, L_t) = E^\lambda \left( g(L_T) e^{-\int_0^T \kappa_h(L_h-) dh} e^{-\lambda \int_0^T \mathcal{M}_T (L_h-) dh} \right. \\
&\left. + \int_t^T (T-s, L_s) e^{-\int_s^T \kappa_h(L_h-) dh} e^{-\lambda \int_s^T \mathcal{M}_T (L_h-) dh} ds \Big| \mathcal{F}_t \right)
\end{align}
Proof. The assertion follows directly from Theorem 1.3. □

2.4. Laplace transform of occupation times of Lévy processes. Setting $\kappa \equiv 0$, $f \equiv 0$ and initial condition $g \equiv 1$, $f \equiv 0$, and inserting $L_0 = x$, Corollary 2.5 equation (30) becomes
\begin{equation}
u(T,x) = E_x \left( e^{-\alpha \int_0^T 1_D(L_h) dh} \right),
\end{equation}
which is the Laplace transform at $\alpha$ of the occupation time $\int_0^T 1_D(L_h) dh$ the process $L$ spent in the domain $D$ until time $T$. In Landriault, Renaud and Zhou (2011), the Laplace transforms of occupation times of spectrally negative Lévy processes are discussed based on fluctuation identities. In the next corollary we show that, for a wide class of time-inhomogeneous Lévy process, these transforms are characterized by parabolic PIDEs. Here we use the given right-hand-side and killing rate to further generalize the assumptions using Remark 2.4. We split the initial conditions in the following way, $1 \equiv g(x) = \sum_{j=1}^{2^d} 1_{O_j}(x)$ a.e. with the distinct orthants $O_j$ of $\mathbb{R}^d$. More precisely, for $j = 1,\ldots, 2^d$ let $p^j := (p^j_1,\ldots, p^j_d)$ with $p^j_i \in \{-1,1\}$ for the $2^d$ different possible configurations and let $O_j := \{(x_1,\ldots, x_d) \in \mathbb{R}^d | p^j_i x_i \geq 0 \text{ for all } i = 1,\ldots, d\}$.
For each $j = 1,\ldots, 2^d$ we choose $\eta_j \in \mathbb{R}^d$ such that $1_{O_j} e^{\eta_j \cdot \cdot} \in L^2(\mathbb{R}^d)$. If the distribution $P^{\mathbb{R}^d}$ of $L_T$ has a Lebesgue density we may rewrite equation (31) as
\begin{equation}
u(T,x) = \sum_{j=1}^{2^d} \nu^j(T,x), \quad \text{with } \nu^j(T,x) := E_x \left( 1_{O_j}(L_T) e^{-\alpha \int_0^T 1_D(L_h) dh} \right).
\end{equation}
Corollary 2.6. Let $L$ be a time-inhomogeneous Lévy process with local characteristics $(b,\sigma,F)$ and $\epsilon > 0$ with $\int_0^T \int_{|x| > \epsilon} a^j |x|^2 F(dx) dt < \infty$. If there exist $\alpha > 0$ such that the symbol $A$ of $L$ satisfies assumption (A4) for $\eta \in \mathbb{R}^d$ with $|\eta| < \epsilon$, and to assure (A2) and (A3) we assume there exist constants $C_i$ for $i = 1,2,3$ such that for each $\eta \in \mathbb{R}^d$ with $|\eta| < \epsilon$ uniformly for every $t \in [0,T]$
\begin{equation}|A_\xi (\eta + i \xi)| \leq C_1 (1 + |\xi|)^\alpha,
\end{equation}
\begin{equation}\Re (A_\xi (\eta + i \xi)) \geq C_2 (1 + |\xi|)^\alpha - C_3 (1 + |\xi|)^\beta
\end{equation}
for all $\xi \in \mathbb{R}^d$. Let $\eta^j := -\epsilon d^{-1/2} p^j$. Then, $\nu^j$ from equation (32) is the unique weak solution $u^j \in W^1(0,T;H^\alpha_{\eta^j} (\mathbb{R}^d),L^2_D(\mathbb{R}^d))$ of
\begin{equation}\dot{u}^j + A_{T-t} u^j + 1D u^j = 0, \quad u(0) = 1_{O_j}.
\end{equation}
Moreover equation (32) is true, and if \( \alpha > 1 \), then for each \( t \in [0, T] \), the mapping \( x \mapsto u(t, x) := E_x \left( e^{-\alpha \int_0^T \frac{\tau}{2} \psi(t)} \right) \) is \( \lambda \)-Hölder continuous with \( \lambda = \frac{\alpha - 1}{2} \), in particular it is continuous.

Proof. For each \( j = 1, \ldots, 2^d \), the assumptions (A1)–(A4) are satisfied for \( \alpha \) and \( \eta^j \). According to part (ii) of Remark 3.3, \( P^\tau_\eta \) has a Lebesgue density which shows equation (32). Hence the assertion follows from Theorem 1.5 and Corollary 1.6. □

2.5. Relativistic Schrödinger equation. The relation between the nonrelativistic Schrödinger operator and the Brownian motion is usually referred to under the names Feynman and Kac. Carmona, Masters and Simon (1990) present without proof an analogous link between relativistic Schrödinger operators and Normal Inverse Gaussian Lévy (NIG) processes. Baeumer, Meerschaert and Naber (2010) use this relation to model the relativistic diffusion of a particle as NIG process. We briefly present their derivation of the relativistic Schrödinger equation and the connection to NIG processes.

The nonrelativistic Schrödinger equation for a single particle in a quantum system described by the potential energy \( V : \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{R} \) is the following partial differential equation for the wave-function \( \psi : \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{C} \),

\[
\frac{\hbar}{i} \frac{\partial \psi}{\partial t}(x,t) = \left( -\frac{\hbar^2}{2m} \Delta + V(x,t) \right) \psi(x,t), \tag{36}
\]

where \( i \) is the imaginary unit, \( \frac{\partial}{\partial t} \) denotes the time derivative of \( \psi \), \( 2\pi\hbar \) is Planck’s constant, \( m \) is the particle’s mass, and the Laplace operator \( \Delta \) is given by \( \Delta \psi(x,t) := \sum_{j=1}^{d} \frac{\partial^2 \psi}{\partial x_j^2}(x,t) \).

For a free particle i.e. \( V \equiv 0 \), a formal connection to the Kolmogorov backward equation of the Brownian motion is obtained by the analytic continuation of the Schrödinger equation (36) in time to \( \tau = it \). For \( V \not\equiv 0 \), this relates equation (36) to the Kolmogorov backward equation of the Brownian motion killed with rate \( V \).

Let us pass to the relativistic Schrödinger equation. According to Baeumer, Meerschaert and Naber (2010), the relativistic kinetic energy of a particle with rest mass \( m \) and momentum \( p \) is given by

\[
E(p) = \sqrt{||p||^2c^2 + m^2c^4} - mc^2 \tag{37}
\]

where \( c \) denotes the speed of light. The relativistic energy (37) serves as pseudo differential operator to define the relativistic Schrödinger operator

\[
\mathcal{H}_0(\psi)(\cdot, t) := \mathcal{F}^{-1}(\mathcal{F}(\psi(\cdot, t))) \tag{38}
\]

for the free particle. Thus, the relativistic Schrödinger equation for a single particle in a quantum system described by the potential energy \( V \) is given by

\[
\frac{\hbar}{i} \frac{\partial \psi}{\partial t}(x,t) = \left( \mathcal{H}_0 + V(x,t) \right) \psi(x,t). \tag{39}
\]

Analogous to the nonrelativistic case, formally inserting \( \tau = it \) in equation (39) and setting \( V(x, it) := V(x, t) \) for every \( x \) and \( t \), yields

\[
\frac{\partial \psi}{\partial t}(x,t) + \left( \mathcal{H}_0 + V(x,t) \right) \psi(x,t) = 0. \tag{40}
\]

Note that \( E(p) \) is the symbol of the NIG process \( L \) with parameters \( \tilde{\alpha} = mc^2 \), \( \beta = 0 \), \( \delta = 1 \), \( \mu = 0 \) and \( \Delta = c^2 \text{Id}_d \) where we use the notation of Example 3.3 and \( \text{Id}_d \) denotes the identity matrix in \( \mathbb{R}^d \times \mathbb{R}^d \).

The following proposition formally justifies the Feynman-Kac for equation (40) in terms of weak solutions.
Corollary 2.7. Let the potential energy $V$ be measurable and bounded. Let $g \in L^2_{\text{loc}}(\mathbb{R}^d)$ for some $\eta \in \mathbb{R}^d$ such that $\|\eta\|^2 \leq m^2 c^2$. Then the unique weak solution $u \in W^1(0, T; H^1 \cap L^2, L^2_{\text{loc}}(\mathbb{R}^d))$ of
\[
\dot{u} + \mathcal{H}_0 u + Vu = 0, \quad u(0) = g,
\]
has for every $t \in (0, T]$ the stochastic representation
\[
u(T - t, L_t) = E \left( g(L_T) e^{-\int^T_t V_{T-u} \, dh} \bigg| \mathcal{F}_t \right) \quad \text{a.s.} \tag{42}
\]
Corollary 2.7 is a direct consequence of Theorem 1.5 and Example 3.8.

3. Examples of classes of time-inhomogeneous Lévy processes

First, we provide some general assertions concerning the conditions of Feynman-Kac Theorem 1.5. Then we consider Lévy processes with a Brownian component

\[
\text{for every } t \in (0, T]
\]
and the same

(2014) for the case of Lévy processes. The symbol of a Lévy process is constant in

remark 3.2.

Remark 3.1. Let $A$ be the symbol of a time-inhomogeneous Lévy processes satisfying $(EM(R_{-\eta})$. By virtue of Lemma 3.1 and the continuity of Lévy symbols validity of (A2) for $A$ is equivalent to the following asymptotic condition: For every $N > 0$ there exist a constant $G > 0$ such that
\[
\mathbb{R}(A_x(\xi + \iota \eta)) \geq G|\xi|^\alpha - A(i\eta) \quad \text{for every } \xi \in \mathbb{R}^d \text{ such that } |\xi| > N.
\]
In Eberlein and Glau (2014), examples of PIHACs satisfying assumptions (A1)–(A3) are discussed. For $\eta = 0$, assumptions (A2) and (A3) are studied in Glau (2014) for the case of Lévy processes. The symbol of a Lévy process is constant in time, hence assumption (A4) is trivially satisfied for their symbols.

We start with some general assertions that can be read as construction principles for Lévy processes with symbols satisfying (A1)–(A4).

Remark 3.2. For $j = 1, 2$ let $L^j$ be two stochastically independent time-inhomogeneous Lévy processes with symbol $A^j$ such that $(A1)–(A4)$ are satisfied for some $\alpha^j$ and the same $\eta \in \mathbb{R}^d$. Then the sum $L := L^1 + L^2$ is a (time-inhomogeneous) Lévy process with symbol $A := A^1 + A^2$, and $(A1)–(A4)$ are satisfied for $\alpha := \max(\alpha^1, \alpha^2)$, Compare Remark 4.5. in Glau (2014) for the case $\eta = 0$ and conditions (A1)–(A3).

It is obvious that conditions (A2) and (A3) are not satisfied for every time-inhomogeneous Lévy process and not for every Lévy process. On the one hand, the nature of the class of processes satisfying such a continuity and Gårding condition can be characterized by its distributional properties:

Remark 3.3. Let $L$ be a time-inhomogeneous Lévy process with symbol $A = (A_t)_{t \geq 0}$ and let $\eta \in \mathbb{R}^d$. If condition (Gård-A) is satisfied for some $\alpha \in (0, 2]$, we have for $0 \leq s \leq t \leq T$,
\[
|e^{-\int_s^t A_u(\cdot - i\eta) \, du}| \leq C_1 e^{-(t-s)C_2|\xi|^\alpha}.
\]
In particular, (Gård-A) implies for every $t \in (0, T]$ that the distribution of $L_t$ has a smooth Lebesgue density.
On the other hand, continuity and Gårding condition (A2) and (A3) relate to the path behaviour of the process:

**Remark 3.4.** A Lévy process with symbol $A$ satisfying (A2) and (A3) for some $\alpha \in (0, 2)$ has Blumenthal-Getoor index $\alpha$, see Glau (2014). Hence, every pure jump Lévy process satisfying assumptions (A2) and (A3) has infinite jump activity. In particular, compound Poisson processes do not satisfy (A3). Variance Gamma processes have Blumenthal-Getoor index 0 and thus also do not satisfy both of the conditions, (A2) and (A3), compare Example 4.4 in Glau (2014).

For the Variance Gamma process, the small jumps may be approximated by a scaled Brownian motion as introduced in Asmussen and Rosiński (2001). Thus, the pure jump process is approximated by a series of jump diffusion Lévy processes, a class for which assumptions (A2) and (A3) are satisfied:

**Example 3.5.** [Multivariate Lévy processes with Brownian part] $\mathbb{R}^d$-valued Lévy processes $L$ with characteristics $(b, \sigma, F)$ with a positive definite matrix $\sigma$ and a Lévy measure satisfying (A1). Then the symbol of the process satisfies (A2) and (A3) with $\alpha = 2$, compare Example 4.6 in Glau (2014). For the time-inhomogeneous extension, see section 3.2.

**3.1. Pure jump Lévy processes and operators of fractional order.** Typically, either the Lévy measure or the characteristic function of a Lévy process is known explicitly. Our first example provides sufficient conditions on the Lévy density under which the main assumptions are satisfied.

**Example 3.6.** [Lévy processes with Lévy density] Let $L$ be a real-valued Lévy process and a special semimartingale with characteristic triplet $(b, \sigma, F)$ with respect to the truncation function $h(x) = x$ and some $\eta \in \mathbb{R}$. Let the Lévy measure $\lambda$ with $\int_{|x|>1} e^{\eta |x|} \lambda(dx) < \infty$ have a Lebesgue density $F(dx) = f(x)\,dx$. For its symmetric part $f_s(x) := \frac{1}{2} (f(x) + f(-x))$ assumes $f_s(x) = \frac{1}{|x|^{1+\eta}} + g(x)$ with $g(x) = O(|x|^{-1-Y+\delta})$ for $x \to 0$

with $0 < \delta$. In the following cases, the symbol of $L$ satisfies (A2) and (A3) with Sobolev index $\alpha = Y$ and weight $\eta$.

a) Let $1 < Y < 2$.

b) The antisymmetric part $f_{as}(x) := f(x) - f_s(x)$ of the Lévy density $f$ satisfies $f_{as}(x) = O(|x|^{-1-\alpha})$ for $x \to 0$

1) with $\alpha < Y = 1$, or

2) let $0 < \alpha \leq Y < 1$, $\int |x| f(x) \,dx < \infty$, and moreover $b = \int x F(dx)$.

For $\eta = 0$ the assertion is proved in Proposition 4.14 in Glau (2014). The case $\eta \neq 0$ thus follows by Lemma 3.2.

**Example 3.7.** [Univariate generalized tempered stable Lévy process] A generalized tempered stable Lévy process $L$ with parameters $C_-, C_+ \geq 0$ such that $C_- + C_+ > 0$ and $G$, $M > 0$ and $Y_-, Y_+ < 2$, is a pure jump Lévy process $L$ whose Lévy measure $F_{temp}$ is given by its Lebesgue density

\[ f_{temp}(x) = \begin{cases} \frac{C_-}{|x|^{1+\alpha}} e^{G x} & \text{for } x < 0 \\ \frac{C_+}{|x|^{1+\alpha}} e^{-M x} & \text{for } x \geq 0, \end{cases} \]

compare Poirot and Tankov (2003). For $C_{\pm} = 0$ we set $Y_{\pm} := 0$ and denote by $(b, 0, F)$ the characteristic triplet of $L$ with respect to some truncation function $h$. Example 4.15 in Glau (2014) shows that in each of the following cases, assumptions (A1)−(A3) are satisfied for $\alpha := \max\{Y_+, Y_-\}$ and $\eta \in (-G, M)$.

(i) $\alpha = \max\{Y_+, Y_-\} > 1$. 


Example 3.8. [Multivariate Normal Inverse Gaussian (NIG) processes] Let \( L \) be an \( \mathbb{R}^d \)-valued NIG-process i.e. a Lévy process with \( L_1 = (L_1^1, \ldots, L_1^d) \sim \text{NIG}_d(\tilde{\alpha}, \delta, \mu, \Delta) \) with parameters \( \tilde{\alpha}, \delta > 0, \) \( \mu \in \mathbb{R}^d \) and symmetric positive definite matrix \( \Delta \in \mathbb{R}^{d \times d} \) with \( \tilde{\alpha}^2 > \langle \beta, \Delta \beta \rangle \). Then the symbol of \( L \) is given by

\[
A(u) = i\langle u, \mu \rangle - \delta \left( \sqrt{\tilde{\alpha}^2 - \langle \beta, \Delta \beta \rangle} - \sqrt{\tilde{\alpha}^2 - \langle \beta + iu, \Delta (\beta - iu) \rangle} \right),
\]

where by \( \langle \cdot, \cdot \rangle \) we denote the product \( (z, z') = \sum_{j=1}^d z_j z_j' \) for \( z \in \mathbb{C}^d \), compare e.g. equation (2.3) in Hammerstein (2014).

Assumptions (A1*), (A2) and (A3) are satisfied for the index \( \alpha = 1 \) for any \( \eta \in \mathbb{R}^d \) such that \( \tilde{\alpha}^2 > \langle \beta - \eta', \Delta (\beta - \eta') \rangle \) for all \( \eta' \in \mathbb{R}^d \), see Example 7.3 in Eberlein and Glau (2014). This is in particular the case, if \( \|\beta\|^2 + \|\eta\|^2 \leq \tilde{\alpha}^2/|\Delta| \).

Further examples are discussed in Examples 4.8–4.10 in Glau (2014): For \( \eta = 0 \), generalized student-t processes and Cauchy processes satisfy assumptions (A2) and (A3) with \( \alpha = 1 \). And for multivariate \( \alpha \)-semi stable with \( \alpha \in [1,2] \) and univariate strictly stable Lévy processes with \( \alpha = 1 \), (A2) and (A3) are satisfied with index \( \alpha \) and \( \eta = 0 \). Also for univariate generalized hyperbolic processes, validity of assumptions (A2) and (A3) can be derived with index \( \alpha = 1 \), see Example 7.2 in Eberlein and Glau (2014).

3.2. Time-inhomogeneous processes. Time-inhomogeneous Lévy processes arise naturally in financial applications because they provide a considerably better fit to the time-evolution of data than Lévy processes. We present two construction principles and conditions under which the resulting time-inhomogeneous ones satisfy (A1)–(A4). A natural family of time-inhomogeneous Lévy processes is obtained by inserting time-dependent parameters into a given parametric class of Lévy processes. It is straightforward to show the following

**Lemma 3.9.** Let \((A(p, \cdot))_{p \in P}\) a parametrized family of symbols. Fix some \( \eta \in \mathbb{R}^d \) and some \( \alpha \in (0,2) \). Let (A2) and (A3) be satisfied for \( A \), uniformly for all \( p \in P \). Then for \( t \mapsto p(t) \) measurable, (A2) and (A3) are satisfied for

\[
A_t(\xi) := A(p(t), \xi) \quad \text{for} \ t \in [0, T] \ \text{and} \ \xi \in U_\eta.
\]

If moreover, \((p, \xi) \mapsto A(p, \xi)\) and \( t \mapsto p(t) \) are continuous, \((A_t)_{t \geq 0}\) is the symbol of a time-inhomogeneous Lévy process \( L' \) and satisfies also (A4). If additionally (A1) is satisfied for \( L \), then it is also satisfied for \( L' \).

As another natural construction we consider stochastic integrals of deterministic functions with respect to Lévy processes. Let \( L \) be an \( \mathbb{R}^d \)-valued Lévy process. Then

\[
X_t := f \cdot L_t := \int_0^t f(s) \, dL_s := \left( \sum_{k=1}^d \int_0^t f^{jk}(s) \, dL^k_s \right)_{j \leq d}
\]

with a deterministic and \( L \)-integrable \( \mathbb{R}^{n \times d} \)-valued function \( f \) is an \( \mathbb{R}^n \)-valued semi-martingale with deterministic characteristics. Denote by \((b, c, F)\) the characteristics of \( L \) w.r.t. the truncation function \( h \). Then, by elementary arguments based on
their definition, the characteristics \((b^X_t, c^X_t, F^X_t)_{t \geq 0}\) of \(X\) w.r.t. \(\tilde{h}\) are shown to be given by
\[
\begin{align*}
    b^X_t &= f(t)b + \int_{\mathbb{R}^d} (\tilde{h}(f(t)x) - f(t)h(x)) F(dx), \\
    c^X_t &= f(t)c \langle f(t) \rangle, \\
    F^X_t(B) &= \int_{\mathbb{R}^d} 1_B(f(t)x) F(dx) \quad \text{for every } B \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).
\end{align*}
\]
(44)

In particular, \(X\) is a time-inhomogeneous Lévy process in the sense of our definition if integrability condition (10) is satisfied for its characteristics. Moreover, if \(A\) denotes the symbol of \(L\), the symbol \(A^X\) of \(X\) is given by
\[
A^X_t(\xi) = A(f(t)^{tr} \xi) + i \langle \xi, b(\tilde{h}, h, f) \rangle \quad \text{for every } \xi \in \mathbb{R}^d.
\]
(45)

where \(b(\tilde{h}, h, f) := \int_{\mathbb{R}^d} (\tilde{h}(f(t)x) - f(t)h(x)) F(dx)\). This generalizes Example 7.6 in Eberlein and Glau (2014) for the case \(f : [0, \infty) \to \mathbb{R}_+\).

**Lemma 3.10.** Let \(L\) a Lévy process and a special semimartingale with symbol \(A\). Let \(f : [0, \infty) \to \mathbb{R}^{n \times d}\) measurable and such that there exist constants \(0 < f_*, f^*\) with
\[
\sup_{0 \leq t \leq T} \|f(t)f(t)^{tr}\|^{1/2} \leq f_* \quad \text{and} \quad \sup_{0 \leq t \leq T} \|f(t)f(t)^{tr}\|^{1/2} \leq f^*,
\]
(46)

where \(\| \cdot \|\) denotes the spectral norm. Then \(X := f \cdot L\) is a time-inhomogeneous Lévy process and a special semimartingale with symbol
\[
A^X_t(\xi) = A(f(t)^{tr} \xi) \quad \text{for all } \xi \in \mathbb{R}^n.
\]

Fix some \(\rho > 0\), \(\eta^X \in \mathbb{R}^d\) with \(|\eta^X| \leq \frac{\rho}{f_*}\) and some \(\alpha > 0\). If \(E e^{\rho|L_t|} < \infty\) for some, respectively for all \(t > 0\), then \(X\) satisfies (EM(\(R_{-\rho}\))). If additionally \(A\) satisfies (A2) and (A3) for every weight \(\eta \in \mathbb{R}^d\) with \(|\eta| \leq \rho\) and index \(\alpha > 0\), then (A2) and (A3) hold for \(A^X\) with the same index \(\alpha\) and weight \(\eta^X\). Moreover, if (A4) holds for \(A\) it also satisfied for \(A^X\).

**Proof.** From the assumptions it is immediate that \(f\) is integrable with respect to \(L\) and hence \(X\) is a semimartingale with characteristics of form (44). Integrability condition (10) also follows directly, so \(X\) is a time-inhomogeneous Lévy process. Since \(L\) is a special semimartingale, where \(\int_{|x| > 1} \frac{x}{|x|} F(dx) < \infty\) where \(F\) denotes the Lévy measure of \(L\) and (46) implies
\[
\int_0^T \int_{|x| > 1} \frac{x}{|x|} F(dx) \leq T f^* \int_{|x| > 1/f_*} |x| F(dx) < \infty
\]
(47)

which shows that also \(X\) is a special semimartingale. Therefore we may choose \(\tilde{h}\) and \(h\) as the identity so that \(b(\tilde{h}, h, f) = 0\) and from (45) we obtain the equality \(A^X_t(\xi) = A(f(t)^{tr} \xi)\). The assertion on the exponential moment condition (A1) follows analogously to (47), and the assertions on (A2)–(A4) are immediate consequences of (46), the continuity Lévy symbols and Lemma A.1. \(\square\)

Combining the examples of Lévy processes with Lemma 3.10 one obtains various classes of PHAC satisfying our main assumptions. Here we consider the following
Example 3.11. \textit{Time-inhomogeneous Lévy jump diffusion} \ Let $L$ a PIAC whose pure jump part’s symbol and Lévy measure satisfy (A1), (A2), (A4) for some $\eta \in \mathbb{R}^d$ and some $\alpha \in [0, 2]$ as well as assumption (A3) for $\eta$ and $\alpha = 2$ and has a time-dependent covariance matrix $\sigma(t)$ such that the mapping $t \mapsto \sigma(t)$ from $[0, T]$ to the positive definite $d \times d$-matrices is continuous, then the symbol of the process satisfies (A1)–(A4) for weight $\eta \in \mathbb{R}^d$ and Sobolev index $\alpha = 2$.

4. Robustness of the weak solutions

We provide the following general robustness result, a special case of which is crucial for our derivation of the Feynman-Kac Theorem.

(An1) There exists constants $C_i > 0$ for $i = 1, 2, 3$ such that uniformly for all $n \in \mathbb{N}$ and all $t \in [0, T],$

$$|a_t(u, v)| \leq C_1 \|u\|_X \|v\|_X \quad \text{for all } u, v \in X, \quad (48)$$

$$\min\{a_t^n(u, u), a_t(u, u)\} \geq C_1 \|u\|_X^2 - C_2 \|u\|_H^2 \quad \text{for all } u \in X. \quad (49)$$

(An2) For each $n \in \mathbb{N}$ there exists a constant $C_i(n) > 0$ for $i = 3, 4, 5$ such that uniformly for all $t \in [0, T],$

$$|a_t^n(u, v)| \leq C_3(n) \|u\|_Y \|v\|_Y \quad \text{for all } u, v \in Y, \quad (50)$$

$$a_t^n(u, u) \geq C_4(n) \|u\|_Y^2 - C_5(n) \|u\|_H^2 \quad \text{for all } u \in Y. \quad (51)$$

(An3) There exists a sequence $C_0(n) \downarrow 0$ for $n \to \infty$ such that uniformly for all $t \in [0, T],$

$$\left|(a_t^n - a_t)(u, v)\right| \leq C_0(n) \|u\|_X \|v\|_X \quad \text{for all } u, v \in X. \quad (52)$$

Proposition 4.1. \textit{Let the operators $A$ and $A^n$ for $n \in \mathbb{N}$ satisfy (An1), (An2). Let $f^n, g \in L^2(0, T; H)$ with $f^n \to f$ in $L^2(0, T; H)$ and $g^n, g \in H$ with $g^n \to g$ in $H$. Then the sequence of unique weak solutions $u^n \in W^1(0, T; Y, H)$ of}

$$\dot{u}^n + A^n u^n = f^n, \quad u^n(0) = g^n \quad (53)$$

\textit{converges strongly in } $L^2(0, T; X) \cap C(0, T; H)$. \textit{Moreover, the limit $u$ belongs to } $W^1(0, T; X, H)$ \textit{and is the unique weak solution of}

$$\dot{u} + A u = f, \quad u(0) = g. \quad (54)$$

\textit{Proof.} We insert the weak solution $u^n$ of equation (53) as test function in (53). Using $\int_0^t (\dot{u}^n(s), u(s)) \, ds = \frac{1}{2} (\|u^n(t)\|_H^2 - \|u^n(0)\|_H^2)$, compare Wloka (1987), equation (2) on p. 394, inserting Gårding inequality (19) and the inequality of Young, integrating over time and applying the lemma of Gronwall yields the existence of constants $C_1, C_2 > 0$ with

$$\sup_{t \in [0, T]} \|u^n(t)\|_H^2 + C_1 \|u^n\|^2_{L^2(0, T; X)} \leq C_2 \left(\|f\|^2_{L^2(0, T; X)} + \|g\|_H^2\right). \quad (55)$$
Subtracting equation (53) for $n$ and $m$ and inserting $w^{nm} := u^n - u^m$ with the solutions $u^n$ and $u^m$ of equation (53) for $n$ and $m$ as test function yields
\[
\int_0^T \left[ (\dot{w}^{nm}(t), w^{nm}(t)) + a_i^n(w^{nm}(t), w^{nm}(t)) \right] dt = \int_0^T \left[ (f^n(t) - f^m(t), w^{nm}(t)) \right]_H - (a_i^n - a_i^m)(u^m(t), w^{nm}(t)) dt.
\]
(56)

Thanks to inequality (52) and (55) and Young’s inequality we obtain
\[
\int_0^T \left| (a_i^n - a_i^m)(u^m(t), w^{nm}(t)) \right| dt \leq \epsilon^{nm} + \epsilon \|u^{nm}\|^2_{L^2(0,T;X)}
\]
with $0 < \epsilon, 0 < \epsilon^{nm} \to 0$ for $n,m \to \infty$. Inserting the last inequality in equation (52), applying again Wloka (1987), equation (2) on p. 394, Gårding inequality (49), the inequality of Young and the lemma of Gronwall yield the existence of constants $C_3, C_4 > 0$ with
\[
\sup_{t \in [0,T]} \|u^{nm}(t)\|^2_H + C_4 \|u^{nm}\|^2_{L^2(0,T;X)} 
\leq C_4 (\epsilon^{nm} + \|f^n - f^m\|^2_{L^2(0,T;X^*)} + \|g^n - g^m\|^2_H),
\]
from where the strong convergence of the sequence $u^n$ in the space $L^2(0,T;X) \cap L^\infty(0,T;H)$ follows. We have
\[
- \int_0^T \langle u(t), \dot{v}(t) \rangle_H dt + \int_0^T a_t(u(t), v(t)) dt 
= - \int_0^T \langle u^n(t), \dot{v}(t) \rangle_H dt + \int_0^T a_t^n(u^n(t), v(t)) dt - R^n(u^n - u, v)
\]
with
\[
R^n(\varphi, v) := \int_0^T \langle \dot{\varphi}(t), \dot{v}(t) \rangle_H dt + \int_0^T (a_t - a_t^n)(\varphi(t), v(t)) dt 
+ \int_0^T a_t(\varphi(t), v(t)) dt.
\]
Due to the convergence $u^n \to u$ in $L^\infty(0,T;H) \cap L^2(0,T;X)$, from equation (52) and from the continuity of $a$ we get $R^n(u^n - u, v) \to 0$ for $n \uparrow \infty$ and hence inserting equation (53), we have
\[
- \int_0^T \langle u(t), \dot{v}(t) \rangle_H dt + \int_0^T a_t(u(t), v(t)) dt 
= \int_0^T \langle f(t), v(t) \rangle dt + \langle g, v(0) \rangle_H
\]
(57)
for all $v \in C^1([0,T];Y)$ with $v(T) = 0$. From the density of $Y$ in $X$ and from Proposition 23.20 in Zeidler (1990) we deduce that $u \in W^1(0,T,X,H)$ and $u$
In particular, \( \tilde{u} \) and hence the unique weak solution of (58)–(59) and \( \tilde{u} \) solves (54). Moreover, due to inequality (48) and (49), equation (54) has a unique solution. Then the following assertions are valid.

(i) Let \( m \geq 1 \). If \( g \in H^1_0(m^{-1})\alpha/2(\mathbb{R}^d) \), \( f \in L^2_\alpha(0,T; H^1_0(m^{-1})\alpha/2(\mathbb{R}^d)) \) and for every \( 1 \leq k \leq m \) we have \( \partial^k u \in L^2(0,T; H^1_0(m^{-1})\alpha/2(\mathbb{R}^d)) \) for all \( h \in L^2(0,T; H^1_0(\mathbb{R}^d)) \), then \( u \in L^2(0,T; H^1_0(\mathbb{R}^d)) \) and \( \tilde{u} \in L^2(0,T; H^1_0(\mathbb{R}^d)) \).

(ii) If \( g \in H^\beta_0(\mathbb{R}^d) \) for \( \beta = m + d/2 + \max(\alpha, 1/2) \), \( f \in L^2(0,T; H_\alpha(\mathbb{R}^d)) \) for \( \gamma = m + (d + 1)/2 \) and \( \kappa \in C_0^\infty([0,T] \times \mathbb{R}^d) \), then for every multi-index \( k = (k_1, \ldots, k_d) \) with \( |k| \leq m \) the derivative \((1 + \partial^\beta)D^k u\) is in \( C([0,T] \times \mathbb{R}^d) \).

If moreover \( \mathcal{A} \) is the infinitesimal generator of a Lévy process and \( f \) is continuous, then equality (58) holds pointwise for every \((t,x) \in (0,T) \times \mathbb{R}^d\).

Proof. We derive the regularity assertion by explicit operations on the Fourier transform of the solution of (55)–(59). Let \( u \in W^1(0,T; H^1_0(\mathbb{R}^d), L^2_\alpha(\mathbb{R}^d)) \) be the unique weak solution of (55)–(59) and \( \tilde{u} = u^1 + u^2 + u^3 \), with

\[
F_\eta(u^1(t)) := F_\eta(g)e^{-\int_{-t}^t A_s(-i\eta)du},
\]

\[
F_\eta(u^2(t)) := \int_0^t F_\eta(f(s))e^{-\int_{-t}^s A_u(-i\eta)du}ds,
\]

\[
F_\eta(u^3(t)) := -\int_0^t F_\eta(\kappa u(s))e^{-\int_{-t}^s A_u(-i\eta)du}ds
\]

and hence

\[
\partial_t F_\eta(u^1(t)) = -\mathcal{A}T_{-t}(-i\eta)F_\eta(u^1(t)),
\]

\[
\partial_t F_\eta(u^2(t)) = -\mathcal{A}T_{-t}(-i\eta)F_\eta(u^2(t)) + F_\eta(f(t)),
\]

\[
\partial_t F_\eta(u^3(t)) = -\mathcal{A}T_{-t}(-i\eta)F_\eta(u^3(t)) - F_\eta(\kappa u(t)).
\]

In particular, \( \tilde{u} \) satisfies (58). From inequality (48) with constants \( C_1, C_2 > 0 \) and the inequality of Cauchy-Schwarz follows the existence of a constant \( c_1, c_2 > 0 \) s.t.
for every \((t, \xi) \in [0, T] \times \mathbb{R}^d\),

\[
|F_\eta(u^1(t))(\xi)| \leq C_1 |F_\eta(g)(\xi)| e^{-tC_2|\xi|^\alpha}
\]

\[
|F_\eta(u^j(t))(\xi)| \leq C_1 \left( \int_0^t |F_\eta(f(s))(\xi)|^2 \, ds \right)^{1/2} \left( \int_0^t e^{-(t-s)2C_2|\xi|^\alpha} \, ds \right)^{1/2}
\]

\[
\leq C_2 \left( \int_0^T |F_\eta(f^j(s))(\xi)|^2 (1 + |\xi|)^{-\alpha} \, ds \right)^{1/2} ,
\]

as well as

\[
|F_\eta(\partial_t u^1(t))(\xi)| \leq C_1 |F_\eta(g)(\xi)| (1 + |\xi|)^\alpha e^{-tC_2|\xi|^\alpha},
\]

\[
|F_\eta(\partial_t u^j(t))(\xi)| \leq C_2 \left( \int_0^T |F_\eta(f^j(s))(\xi)|^2 (1 + |\xi|)^\alpha \, ds \right)^{1/2} + |F_\eta(f^j(s))(\xi)| ,
\]

for \(j = 1, 2\) with \(f^1 = f\) and \(f^2 = -\kappa u\). Hence there exist constants \(c_4, c_5 > 0\) with

\[
\|\bar{u}\|_{L^1(0,T;H^{m-1}/2(\mathbb{R}^d))} \leq c_4 (\|g\|_{H^{m-1}/2(\mathbb{R}^d)}) + \|f\|_{L^2(0,T;H^{m-1}/2(\mathbb{R}^d))} + \|\kappa u\|_{L^2(0,T;H^{-m+1}/2(\mathbb{R}^d))}
\]

as well as

\[
\|\partial_t \bar{u}\|_{L^2(0,T;H^{m-2}/2(\mathbb{R}^d))} \leq c_4 (\|g\|_{H^{m-1}/2(\mathbb{R}^d)}) + \|f\|_{L^2(0,T;H^{m-1}/2(\mathbb{R}^d))} + \|\kappa u\|_{L^2(0,T;H^{-m+1}/2(\mathbb{R}^d))}.
\]

For \(m = 1\), inserting \(u \in L^2(0,T;H^{\alpha/2}(\mathbb{R}^d))\) as well as \(\kappa u \in L^2(0,T;H^{\alpha/2}(\mathbb{R}^d))\) we obtain \(\bar{u} \in L^2(0,T;H^{\alpha/2}(\mathbb{R}^d))\) and \(\partial_t \bar{u} \in L^2(0,T;\mathbb{H}_0^{\alpha/2}(\mathbb{R}^d))\). In particular, \(\bar{u} \in W^1(0,T;H^{\alpha/2}(\mathbb{R}^d),L^2(\mathbb{R}^d))\) is the unique weak solution \(\bar{u} = u\) of equation (58) – (59).

For \(m = 2\) it is thus sufficient to notice that \(\kappa u \in L^2(0,T;H^{\alpha/2}(\mathbb{R}^d))\) yields \(u \in L^2(0,T;H^{-1/2}(\mathbb{R}^d))\) as well as \(\partial_t \bar{u} \in L^2(0,T;L^2(\mathbb{R}^d))\). Iteration shows part (i) of the Lemma.

\(iii\) By the inequality of Cauchy-Schwarz and \(\int_{\mathbb{R}^d} (1 + |\xi|)^{-d+\epsilon} \, d\xi < \infty\) for \(\epsilon > 0\), we obtain for \(\beta = m + d/2 + \max(\alpha,1/2)\) and \(\gamma = m + (d + 1)/2\) the existence of a constant \(c_3 > 0\) s.t.

\[
\int_{\mathbb{R}^d} \left| (1 + \partial_t)F_\eta((u^1 + u^2)(t))(\xi) \right| (1 + |\xi|)^\beta \, d\xi \leq c_3 (\|g\|_{H^{\beta+1}(\mathbb{R}^d)} + \|f\|_{L^2(0,T;H^{\beta+1}(\mathbb{R}^d))}) < \infty.
\]

Moreover, the mappings \(t \mapsto F_\eta(\bar{u}(t))(\xi)\) and \(t \mapsto \partial_t F_\eta(\bar{u}(t))(\xi)\) are continuous for each \(\xi \in \mathbb{R}^d\). Dominated convergence yields \(D^k_\eta(1 + \partial_t)(u^1 + u^2) \in C([0,T] \times \mathbb{R}^d)\) for every multiindex \(k = (k_1, \ldots, k_d)\) with \(|k| \geq 0\).

Moreover, there exists a constant \(c_4 > 0\) such that

\[
\int_{\mathbb{R}^d} \left| (1 + \partial_t)F_\eta(u^j(t))(\xi) \right| (1 + |\xi|)^\gamma \, d\xi \leq c_4 (\kappa u\|_{L^2(0,T;H^{\gamma+1}(\mathbb{R}^d))} < \infty\)
\]

and dominated convergence yields \(D^k_\eta(1 + \partial_t)u^3 \in C([0,T] \times \mathbb{R}^d)\) for every multiindex \(k = (k_1, \ldots, k_d)\) with \(|k| \geq 0\).

In order to derive (65) pointwise, fix a \(t \in T\) for which the equation holds as operator equation over \(L^2(\mathbb{R}^d)\) and choose a sequence \(u_n \in C_0^\infty ([0,T) \times \mathbb{R}^d)\) such that
Theorem of Fubini and Parseval’s identity yield inserting inequality (43) and the inequality of Cauchy-Schwarz yields assertion (i).\[ u_n(t) \to u(t) \] in the norm of \( H^{n/2}_{\sigma}(\mathbb{R}^d) \) and \( \dot{u}_n(t) \to \dot{u} \) in the norm of \( L^2_{\eta}(\mathbb{R}^d) \). Moreover, let \( \varphi \in C_0^\infty(\mathbb{R}^d) \). Since \( u(t) \in C^1(\mathbb{R}^d) \), \( A\dot{u} \) is defined pointwise and an elementary manipulation shows
\[
\int_{\mathbb{R}^d} A u(x) \varphi(x) e^{-2t|\eta|^2} \, dx = \langle u, A^{-\eta^*} \varphi \rangle_{L^2_\eta} = \lim_{n \to \infty} \langle u_n, A^{-\eta^*} \varphi \rangle_{L^2_\eta}
\]
with the adjoint operator \( A^{-\eta^*} \) defined in Lemma A.2. By equation \[ (\ref{eq:adjoint}) \] in Rudin (1966), we obtain a Lebesgue density. Applying Parseval’s identity, see e.g. equality (10) on p. 187 of \[ (\ref{eq:parseval}) \], continuity of the scalar product and of the bilinear form, we obtain
\[
\lim_{n \to \infty} \langle u_n, A^{-\eta^*} \varphi \rangle_{L^2_\eta} = \lim_{n \to \infty} a(u_n, \varphi) = a(u, \varphi)
\]
and hence
\[
\langle \dot{u}, \varphi \rangle_{L^2_\eta} + \langle A \dot{u}, \varphi \rangle_{L^2_\eta} = \langle f, \varphi \rangle_{L^2_\eta} \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^d).
\]
Thanks to the fundamental lemma of variational calculus the equality follows for \( t \) and a.e. \( x \in \mathbb{R}^d \). Since we can choose \( t \) arbitrary from a dense subset in \((0, T)\) the assertion follows by continuity of \( \dot{u} + A \dot{u} - f \).

\[ \square \]

**Lemma 5.2.** Let \( L \) be a PIIAC with symbol \( A = (A_t)_{t \in [0, T]} \) such that satisfies \( EM(R_{-\eta}) \) for some \( \eta \in \mathbb{R}^d \) and \( (\text{G"{a}rd-A}) \) for some \( \alpha > 0 \) and every \( \eta \in R_{-\eta} \). Then

(i) for \( t > 0 \) there exists a constant \( C(t) > 0 \) with \( E[|\varphi(L_t)|] \leq C(t) \| \varphi \|_{L^2_{\sigma}(\mathbb{R}^d)} \)
uniformly for all \( \varphi \in L^2_{\eta}(\mathbb{R}^d) \) and \( s \in [t, T] \),

(ii) for \( l > (d-\alpha)/2 \) and every \( 0 \leq t < T \) there exists a constant \( C_1 > 0 \) with \( |E(\int_t^T \varphi(s, L_s) \, ds \, | F_t) | \leq C_1 \| \varphi \|_{L^2(0, T; H^l_{\sigma}(\mathbb{R}^d))} \)
uniformly for all \( \varphi \in L^2(0, T; H^l_{\eta}(\mathbb{R}^d)) \).

**Proof.** (i) By Remark 3.3 assumption (A3) yields that the distribution of \( L_t \) has a Lebesgue density. Applying Parseval’s identity, see e.g. equality (10) on p. 187 in Rudin (1966), we obtain
\[
E[|\varphi(L_t)|] = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} F(|\varphi|)(\xi - i\eta) e^{-\int_0^t A_s(\xi - i\eta) \, ds} \, d\xi,
\]
inserting inequality (43) and the inequality of Cauchy-Schwarz yields assertion (i).

(ii) W.l.o.g. \( \varphi \geq 0 \). We have
\[
E(\int_t^T \varphi(s, L_s) \, ds \, | F_t) = G(L_t)
\]
with
\[
G(y) = E\left( \int_0^{T-t} \varphi(s + t, L_{t+s} - L_t + y) \, ds \right).
\]
The theorem of Fubini and Parseval’s identity yield
\[
G(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_0^{T-t} \mathcal{F}(\tau_y \varphi(s + t)) (\xi - i\eta) e^{-\int_0^t A_s(\xi - i\eta) \, ds} \, ds \, d\xi,
\]
where \( \tau_y f(x) := f(x + y) \). Notice that \( \mathcal{F}_y(\tau_y f)(\xi) = e^{-\langle \xi, y \rangle} \mathcal{F}_y(\xi) \) inserting the inequality of Cauchy-Schwarz and equation (43) with constants \( C_1, C_2 > 0 \), we
obtain for $l > d - \alpha$ some constants $c_1, c_2 > 0$ with

$$|G(y)| \leq c_1 \int_{\mathbb{R}^d} \left( \int_0^{T-t} \left| \mathcal{F}(\tau \varphi(s+t))(\xi - i\eta) \right|^2 \, ds \right)^{1/2} \, d\xi$$

$$\leq c_1 \int_0^T \int_{\mathbb{R}^d} \left| \mathcal{F}(\varphi(s+t))(\xi - i\eta) \right| \left(1 + |\xi|\right)^\alpha \, d\xi \, ds$$

$$\leq c_2 \|\varphi\|_{L^2(0,T;H^\alpha/2(\mathbb{R}^d))}.$$}

We are now in a position to prove part (ii) of Theorem 1.5.

**Proof.** First, let $t \mapsto A_t(\xi - i\eta)$ continuous for every $\xi \in \mathbb{R}^d$. By density arguments, we choose sequences $g^n \in C_0^\infty(\mathbb{R}^d)$, $f^n \in C_0^\infty([0,T] \times \mathbb{R}^d)$ as well as $\kappa^n \in C_0^\infty([0,T] \times \mathbb{R}^d)$ s.t. each partial derivative of $\kappa$ is bounded and such that for $n \uparrow \infty$

$$g^n \to g \quad \text{in } L_\eta^2(\mathbb{R}^d),$$

$$f^n \to f \quad \text{in } L_\eta^2(0,T;H_\eta^1(\mathbb{R}^d)),$$

$$\kappa^n \to \kappa \quad \text{in } L^\infty([0,T] \times \mathbb{R}^d).$$

Let $u^n \in W^1(0,T;H_\eta^{\alpha/2}(\mathbb{R}^d),L_\eta^2(\mathbb{R}^d))$ be the unique weak solution of

$$\dot{u}^n + A_{t+} u^n + \kappa u^n = f^n, \quad u^n(0) = g^n. \quad (61)$$

Proposition 4.1 yields the convergence $u^n \rightharpoonup u$ in $L^2(0,T;H_\eta^{\alpha/2}(\mathbb{R}^d)) \cap C(0,T;L_\eta^2(\mathbb{R}^d))$ to the weak solution $u \in W^1(0,T;H_\eta^{\alpha/2}(\mathbb{R}^d),L_\eta^2(\mathbb{R}^d))$ of

$$\dot{u} + A_{t+} u + \kappa u = f, \quad u(0) = g. \quad (62)$$

Lemma 5.1 shows that the equality holds pointwise and that $u$ is regular enough to apply Itô’s formula. Therefore let $w^n(t,x) := u^n(T - t, x)$ and $(h_t, \sigma_t, F_t)_{t \in [0,T]}$ denote the local characteristics of $L$, then Itô’s formula for semimartingales, see e.g. Theorem I.4.57 in Jacod and Shiryayev (2003) yields

$$w^n(T,L_T) e^{-\int_0^T \kappa^n(L_s) \, d\lambda} - w^n(s,L_s) e^{-\int_0^s \kappa^n(L_u) \, d\lambda}$$

$$= \int_s^T \left[ w^n - A_{h_s} w^n - \kappa w^n \right](h,L) e^{-\int_0^h \kappa^n(L_u) \, d\lambda} \, dh$$

$$+ \int_s^T \left( \sigma_h^{1/2} \nabla w^n(h,L) \right) e^{-\int_0^h \kappa^n(L_u) \, d\lambda} \, dW_h$$

$$+ \left( e^{-\int_0^h \kappa^n(L_u) \, d\lambda} (w^n(\cdot,L_u + x) - w^n(\cdot,L_u)) \mathbb{1}_{(s,\infty)}(\cdot) \right) \ast (\mu - \nu)_T.$$

Thanks to our assumptions on $g^n$, $f^n$ and $\kappa^n$, we may decompose $u^n$ in three summands along the lines of (60) and application of part (ii) of Lemma 5.1 it is elementary to conclude that $w^n$ and $\nabla w^n$ belong to $L^2(\mathbb{R}^d)$. Hence, the integrals with respect to $W$ and $\mu - \nu$ are martingales, compare Theorem II.1.33 a) in Jacod and Shiryayev (2003). Inserting the identity $w^n - A_{h_s} w^n - \kappa w^n = \overline{f}^n$ with $\overline{f}(t,x) := f^n(T - t, x)$, multiplication of the equation with the term $e^{\int_0^T \kappa^n(L_s) \, d\lambda}$
and taking the conditional expectation yields for $0 \leq s \leq T$,

$$
E \left( w^n(T, L_T) e^{-\int_T^T \kappa^n_\lambda(L_\lambda) d\lambda} \bigg| F_s \right) - w^n(s, L_s)
= E \left( \int_s^T (h, L_h) e^{-\int_h^T \kappa^n_\lambda(L_\lambda) d\lambda} d\lambda \bigg| F_s \right).
$$

(63)

Let w.l.g. $0 < s \leq T$. We derive the stochastic representation by letting $n \to \infty$ for each term in (63):

Denote $w(t, x) := u(T - t, x)$. From the convergence $w^n(s, \cdot) \to w(s, \cdot)$ in $L^2_\eta(\mathbb{R}^d)$ and Lemma 5.2 (i) for $s > 0$, we get the convergence

$$
w^n(s, L_s) \to w(s, L_s) \text{ in } L^1(P) \text{ and a.s for a subsequence.}
$$

Since $\kappa^n$ converges to $\kappa$ in $L^\infty$, dominated convergence yields $\int_a^b \kappa^n_\lambda(L_\lambda) d\lambda \to \int_a^b \kappa_\lambda(L_\lambda) d\lambda$ and uniform boundedness of the sequence for $0 \leq a \leq b \leq T$.

Together with $w^n(s, L_s) \to w(s, L_s)$ in $L^1(P)$ the convergence

$$
E \left( \left| w^n(t, L_t) e^{-\int_t^T \kappa^n_\lambda(L_\lambda) d\lambda} - w(t, L_t) e^{-\int_t^T \kappa_\lambda(L_\lambda) d\lambda} \right| \bigg| F_s \right) \to 0
$$

for $n \to \infty$ follows elementary using the triangle inequality.

Next, denote $\bar{T}(t, x) := f(T - t, x)$. From part (ii) of Lemma 5.2 there exists a constant $c_2 > 0$ for $t > (d - \alpha)/2$ with

$$
E \left( \int_s^T \left| \bar{T}(h, L_h) \right| d\lambda \bigg| F_s \right) \leq c_1 \|f\|_{L^2(t, T; H^d_0(\mathbb{R}^d))} \to 0
$$

due to $f^n \to f \in L^2(t, T; H^d_0(\mathbb{R}^d))$. Again from elementary application of the triangle inequality we obtain the convergence of the second line in equation (63) and thus the assertion of Theorem 1.3 under the additional assumption that the mapping $t \mapsto A_t(\xi - i\eta)$ is continuous for every $\xi \in \mathbb{R}^d$. Thanks to the tower rule of conditional expectations and that existence and uniqueness of the weak solution does not require continuity of the bilinear form, the claim follows by induction over the continuity periods also under the more general assumption that $t \mapsto A_t(\xi - i\eta)$ is càdlàg for every $\xi \in \mathbb{R}^d$. $\Box$

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New proofs

APPENDIX A. ADJOINT OPERATOR

For a Lévy process $L$ with characteristics $(b, \sigma, F; h)$ we denote by $A^{(b,\sigma,F)}$ and $\hat{A}(b,\sigma,F)$ its Kolmogorov operator and its symbol.

Lemma A.1. Let Lévy process with characteristics $(b, \sigma, F; h)$ satisfy $EM(\eta)$ and let $A$, $\hat{A}$ its Kolmogorov operator and symbol. Then

$$A^{\eta}(\xi) := A(\xi + i\eta) = A^{(b^{\eta},\sigma,F^{\eta})}(\xi) + A(i\eta) \quad \text{for all } \xi \in \mathbb{R}^d$$

with

$$b^{\eta} = b + \sigma \cdot \eta + \int_{\mathbb{R}^d} (1 - e^{i\langle \eta, y \rangle}) h(y) F(dy),$$

$$F^{\eta}(dy) = e^{i\langle \eta, y \rangle} F(dy)$$

i.e. $A^{\eta}$ is the symbol of a Lévy process with killing rate $A(i\eta)$. Moreover, its Kolmogorov operator $A^{\eta}$ satisfies

$$A^{\eta} \phi = e^{-i\langle \eta, \cdot \rangle} A(e^{i\langle \eta, \cdot \rangle} \phi) = A^{(b^{\eta},\sigma,F^{\eta})} \phi + A(i\eta) \phi \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^d).$$

Proof. It is elementary to verify the assertion on the symbol which can be nicely used to verify the assertion for the operator: Let $\phi \in C_0^\infty(\mathbb{R}^d)$ then $F(e^{i\langle \eta, \cdot \rangle} \phi)(\xi) =
$F(\varphi)(\xi - i\eta)$ and

\[
A \left( e^{i\eta \cdot \varphi} \right)(x) = F^{-1}(A F(e^{i\eta \cdot \varphi})) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i(\xi, x)} A(\xi) F(\varphi)(\xi - i\eta) \, d\xi
\]

\[
= \frac{e^{i(\eta, x)}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i(\xi, x)} A(\xi + i\eta) F(\varphi)(\xi) \, d\xi.
\]

For all $\varphi \in C_0^\infty(\mathbb{R}^d)$ let

\[
F_\eta(\varphi) := e^{-i(\eta, \cdot)} F(\varphi e^{i\eta \cdot \cdot}) \quad \text{and} \quad F^{-1}_\eta(\varphi) := e^{-i(\eta, \cdot)} F^{-1}(\varphi e^{i\eta \cdot \cdot}).
\]

Theorem 4.1 in Eberlein and Glau (2014) shows that for pseudo differential operator $A$ whose symbol $\lambda$ has a continuous extension to $U_{-\eta}$ that is analytic in the interior of $U_{-\eta}$ and satisfies the continuity condition (CA),

\[
A \varphi = F^{-1}(A \varphi) = F^{-1}_\eta(A^{-\eta} F_\eta(\varphi)) \quad \text{for all} \quad \varphi \in C_0^\infty(\mathbb{R}^d)
\]

and Parseval’s equality yields for all $\varphi, \psi \in C_0^\infty(\mathbb{R}^d)$,

\[
a(\varphi, \psi) = \langle A \varphi, \psi \rangle_{L_2^\eta} = \frac{1}{(2\pi)^d} \langle A^{-\eta} F_\eta(\varphi), F_\eta(\psi) \rangle_{L_2^\eta}.
\]

We denote by $A^{-\eta \ast}$ and $A^{-\eta \ast}$ the $L_2^\eta$-adjoint of the pseudo differential operator $A$ and its symbol, given for all $\varphi, \psi \in C_0^\infty(\mathbb{R}^d)$ by

\[
\langle A, \psi \rangle_{L_2^\eta} = \langle \varphi, A^{-\eta \ast} \psi \rangle_{L_2^\eta}, \quad \langle A^{-\eta} F_\eta(\varphi), F_\eta(\psi) \rangle_{L_2^\eta} = \langle \varphi, A^{-\eta \ast} F_\eta(\psi) \rangle_{L_2^\eta}.
\]

**Lemma A.2.** Let $L$ a Lévy process with characteristics $(b, \sigma; F; h)$ satisfies that $EM(\eta)$ and let $A$, $A$ its Kolmogorov operator and symbol. Then

\[
A^{-\eta \ast} = A^{b^{-\eta \ast}, \sigma, F^{-\eta \ast}} + A(-i\eta), \quad A^{-\eta \ast} \varphi = e^{-i(\eta, \cdot)} A\left(e^{i\eta \cdot \cdot} \varphi \right) = A^{b^{-\eta \ast}, \sigma, F^{-\eta \ast}} \varphi + A(-i\eta)\varphi,
\]

where

\[
b^{-\eta \ast} = -b^{-\eta}, \quad F^{-\eta \ast}(B) = F^{-\eta}(B) - F^{-\eta}_{\text{asym}}(B), \quad \text{for all Borel sets } B \neq \{0\}.
\]

Moreover $F^{-\eta \ast}$ is a Lévy measure.

**Proof.** For every $\varphi \in C_0^\infty(\mathbb{R}^d)$ we have

\[
\langle A^{-\eta} F_\eta(\varphi), F_\eta(\psi) \rangle_{L_2^\eta} = \langle A^{-\eta} F\left(e^{i\eta \cdot \cdot} \varphi \right), F\left(e^{i\eta \cdot \cdot} \varphi \right) \rangle_{L_2^\eta}
\]

\[
= \langle F\left(e^{i\eta \cdot \cdot} \varphi \right), A^{-\eta} F\left(e^{i\eta \cdot \cdot} \varphi \right) \rangle_{L_2^\eta},
\]

and by Lemma A.1 and since $A(\zeta) \in \mathbb{R}$ for $\Im(\zeta) \in \mathbb{R}^d$,

\[
A^{-\eta} = A^{b^{-\eta \ast}, \sigma, F^{-\eta}} + A(-i\eta).
\]

Since $A^{b^{-\eta \ast}, \sigma, F^{-\eta}}$ is the symbol of a Lévy process,

\[
A^{b^{-\eta \ast}, \sigma, F^{-\eta}}(\xi) = A^{b^{-\eta \ast}, \sigma, F^{-\eta}}(-\xi) \quad \text{for all} \quad \xi \in \mathbb{R}^d,
\]

from where the assertion of the lemma follows directly.

\[\square\]
Appendix B. Proof of Theorem 1.3

By the assumption on the analyticity of $A$ and $\alpha$ in Eberlein and Glau (2014) for all $t \in [0, T]$, all $\eta' \in R_\eta$ and every $\varphi, \psi \in C_0^\infty(\mathbb{R}^d)$,

$$a_t(\varphi, \psi) = \frac{1}{(2\pi)^d} (A_t(\varphi), \varphi)_L^z = \frac{1}{(2\pi)^d} (A_t(1 - i\eta')\varphi, \psi_\eta(\psi))_L^z,$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} A_t(1 - i\eta')\varphi(\xi - i\eta')\psi(\xi - i\eta') \, d\xi$$

(68)

This equality directly entails that (Cont-A) implies (Cont-a). Additionally, together with the following elementary inequalities it yields the implication of (Gård-a) by (Gård-A): For $C_1 > 0$, $C_2 > 0$, $0 \leq \beta < \alpha$ and $0 < C_3 < C_4$ there exists a constant $C_4 > 0$ such that $C_1 x^\alpha - C_2 x^{\beta} \geq C_3 x^\alpha - C_4$ for every $x \geq 0$ and

$$C_2 |\xi|^{2\alpha} - C_3 (1 + |\xi|^2)^{\beta} \geq C_2 |\xi|^{2\alpha} - C_3 (1 + |\xi|^{2\beta}) \geq C_3 (1 + |\xi|) \geq C_3$$

(69)

with a strictly positive positive constant $c_2$ and $C_3, c_3 \geq 0$.

Moreover, piecewise continuity of $t \mapsto a_t(u, v)$ for every $u, v \in H_\eta^0(\mathbb{R}^d)$ follows from the piecewise continuity of $t \mapsto A_t(z)$ for every $z \in U - \eta$ and dominated convergence thanks to (Cont-A).

In order to derive implication (i) $\Rightarrow$ (ii), let us first note that following closely the derivation of the fundamental lemma of variational calculus yields for any continuous function $\gamma$, that if for all $u \in H_\eta^0(\mathbb{R}^d)$ such that $\mathcal{F}_\eta(u)$ is compactly supported

$$\int_{\mathbb{R}^d} \gamma(\xi)|\mathcal{F}_\eta(u)(\xi)|^2 e^{-2(\eta' \cdot \xi)} \, d\xi \geq 0$$

(70)

holds, then $\gamma(\xi) \geq 0$ for all $\xi \in \mathbb{R}^d$. To this end, let us for a moment assume $\gamma(\xi) < 0$ for some $\xi \in \mathbb{R}^d$. Thanks to the continuity of the function, the integrand would be negative on a nonempty open subset of $U \subset \mathbb{R}^d$. We may choose a function $u$ such that its Fourier transform $\mathcal{F}_\eta(u)$ is smooth, not constant and such that its compact support is contained in $U$. Noting that those functions lie in $H_\eta^0(\mathbb{R}^d)$, we would obtain a contradiction to inequality (70).

Since (Cont-a) implies inequality (70) for the continuous mappings $\xi \mapsto C(1 + |\xi|)^{2\alpha} + \Re(A(\xi - i\eta'))$ and $\xi \mapsto C(1 + |\xi|)^{2\alpha} + \Im(A(\xi - i\eta'))$ for all $t \in [0, T]$ and all $\eta' \in R_\eta$, (Cont-A) follows. Similarly, using once again inequality (69), we obtain that (Gård-a) implies (Gård-A).

Finally, we observe that $\lim_{s \to t} a_s(u, u) = a_t(u, u)$ implies

$$\lim_{s \to t} \int_{\mathbb{R}^d} A_s(\xi - i\eta')|\mathcal{F}_{\eta'}(u)(\xi)|^2 \, d\xi = \int_{\mathbb{R}^d} A_t(\xi - i\eta')|\mathcal{F}_{\eta'}(u)(\xi)|^2 \, d\xi$$

while on the other hand dominated convergence shows

$$\lim_{s \to t} \int_{\mathbb{R}^d} A_s(\xi - i\eta')|\mathcal{F}_{\eta'}(u)(\xi)|^2 \, d\xi = \int_{\mathbb{R}^d} \lim_{s \to t} A_s(\xi - i\eta')|\mathcal{F}_{\eta'}(u)(\xi)|^2 \, d\xi.$$

Hence, (70) yields $\lim_{s \to t} A_s(\xi - i\eta') = A_t(\xi - i\eta')$ for all $\xi \in \mathbb{R}^d$ and all $\eta' \in R_\eta$, so piecewise continuity of the bilinear form entails piecewise continuity of the symbol.