Witten’s loop in the minimal flipped SU(5) unification revisited

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In the simplest potentially realistic renormalizable variants of the flipped SU(5) unified model the right-handed neutrino masses are conveniently generated by means of the Witten’s two-loop mechanism. As a consequence, the compactness of the underlying scalar sector provides strong correlations between the low-energy flavor observables such as neutrino masses and mixing and the flavor structure of the fermionic currents governing the baryon and lepton number violating nucleon decays. In this study, the associated two-loop Feynman integrals are fully evaluated and, subsequently, are used to draw quantitative conclusions about the central observables of interest such as the proton decay branching ratios and the absolute neutrino mass scale.

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I. INTRODUCTION

Though not a genuine grand unified theory (GUT), the flipped SU(5) gauge theory still attracts significant attention due to several rather unique features it exhibits. In particular, one-stage symmetry breaking down to the Standard Model (SM) can be achieved regardless of whether or not a TeV-scale supersymmetry is assumed. The corresponding Higgs sector can also be very small, as it is sufficient to employ just a single 10-dimensional representation to accomplish the necessary symmetry breaking. This is to be compared to the 24 of the Georgi-Glashow SU(5) GUTs (see, e.g., Refs. [9, 10] and references therein).

Flipped SU(5) models also share several other nice features with their truly unified cousins. From the point of view of phenomenology, two such features stand out as being particularly relevant due to their immediate experimental consequences. Firstly, as in the SO(10) GUTs, 3 right-handed (RH) neutrinos are enforced in the spectrum, allowing for the use of a type-I seesaw mechanism to generate the light neutrino masses. Additionally, as in SU(5) there is only one heavy gauge boson, which typically yields somewhat stronger correlations between the flavor structure of the baryon and lepton number violating (BLNV) currents and the low-energy flavor observables, and hence one can often say quite a bit about, e.g., the proton lifetime. However, upon closer inspection one finds a certain level of tension between the practical implications of these two points. For example, in order to implement the standard type-I seesaw with the RH neutrinos at hand, a 50-dimensional four-index scalar 50S of SU(5) is typically added together with a 3 × 3 complex symmetric Yukawa matrix Y50 in order to generate the desired RH Majorana mass term via a renormalizable coupling such as YR50 10−10.50.50 10.50.50. Besides enlarging the scalar sector enormously (and, hence, disposing of the uniquely small size of the “minimal” Higgs sector noted above as one of the most attractive structural features of the framework), the extra scalar and the associated Yukawa at play reduces the value of the low-energy neutrino masses and the lepton mixing data as constraints for the proton lifetime estimates as it essentially leaves the neutrino sector on its own.

Remarkably enough, this dichotomy may be overcome by noticing that the RH neutrino masses in flipped SU(5) models may be generated even without the unpleasant extra 50S at the two-loop level by means of a variant of the mechanism first identified by Witten in the SO(10) context [14]. The two main features of this scenario are, first, a simple relation among the seesaw and the GUT scales where the former is, roughly speaking, given by the latter times an extra two-loop suppression and, second, a rigid correlation between the flavor structures of the neutrino and charged sectors, which in most cases may be transformed into a set of strong constraints for, e.g., the proton decay partial widths and branching ratios.

To this end, the Witten’s-loop-equipped flipped SU(5) may even be viewed as the most economical renormalizable theory of the BLNV nucleon decays, much simpler than, e.g., the potentially realistic variants of the SO(10) and even the SU(5) GUTs.

From this perspective, it is interesting that in Ref. [13] most of the basic features of this framework may have been identified even without an explicit calculation of the graphs involved in Witten’s mechanism. In this work we intend to overcome this drawback by a careful inspection of the Feynman graphs and their evaluation which, as we shall see, will clarify several other points left unaddressed in the preceding studies. In particular, the calculation will make it very clear that the minimal potentially realistic and renormalizable incarnation of the scheme under consideration is the variant featuring a pair of 5-dimensional scalars in the Higgs sector (besides a single copy of the “obligatory” 10-dimensional 10H scalar).

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Second, it will be shown that, in this framework, the light neutrino spectrum is always forced to be on the heavy side (actually, within the reach of the KATRIN experiment)\footnote{It may be worth pointing out here that, due to the non-zero $U(1)_X$ charge of 10$_H$, inherent to the flipped SU(5) models, there is no way to build a non-renormalizable $d=5$ operator (presumably Planck-scale suppressed) that might, in the broken phase, affect the gauge-kinetic form and hence introduce significant theoretical uncertainties in the high-scale gauge-matching conditions and the determination of the GUT scale. As a result, one of the primary sources of irreducible uncertainties hindering the predictive power of the “standard” GUTs (such as the Georgi-Glashow SU(5) or the non-minimal SO(10) models with either 54 or 210 breaking the unified symmetry) is absent from this class of models.}, which, among other things, provides a clear smoking gun signal of the scheme.

In Section II we first provide a brief review of the flipped SU(5) gauge theory context, identify the Feynman graphs underpinning the radiative RH neutrino mass generation in the minimal and next-to-minimal models, and exploit the seesaw formula in order to get strong constraints on their parameter space. Section III is devoted to a detailed analysis of the relevant two-loop graphs in the scenario with one copy of the 5-dimensional scalar in the Higgs sector; this setting is simple enough to allow for a complete analytic understanding of the results. In Section IV these findings are used for the identification of the minimal potentially realistic model of this kind, which is subsequently shown to be strongly constrained and potentially highly predictive. Most of the technical details of the lengthy calculations are deferred to a set of appendices.

II. FLIPPED SU(5) À LA WITTEN

The defining feature of the flipped SU(5) unifications is the “non-standard” embedding of the SM hypercharge generator within its SU(5) $\otimes$ U(1)$_X$ gauge symmetry algebra, namely

$$Y = \frac{1}{5}(X - T_{24}),$$

(1)

where $T_{24}$ stands for the usual hypercharge-like generator of the standard SU(5) (normalized in such a way that the electric charge obeys $Q = T_{24}^2 + T_{24}$) and $X$ is the unique non-trivial anomaly-free generator of the additional U(1) normalized in such a way that it receives integer values over the three basic irreps accommodating each generation of the SM matter, \[ \bar{5}_M \equiv (\bar{5}, -3), \quad 10_M \equiv (10, +1), \quad 1_M \equiv (1, +5), \]

(2)

where the first number in brackets gives the SU(5) representation and the second the charge under U(1)$_X$. Compared to the standard SU(5) case, the SM matter fields $u^c_L$ and $d^c_L$ are swapped with respect to their usual assignments, i.e., the former is a member of $\bar{5}_M$ while the latter resides in 10$_M$. Similarly, $c^c_L$ is found in the SU(5) singlet and the compulsory RH neutrino $\nu^c_L$ replaces it in the 10-plet.

As for the gauge fields, the $(24,0) \oplus (1,0)$ adjoint of SU(5) $\otimes$ U(1)$_X$ in this context contains a multiplet $X_\nu$ transforming under $SU(3)_C \otimes SU(2)_L \otimes U(1)$_Y as $(3, \bar{2}, +\frac{1}{6})$, plus its hermitian conjugate, rather than the traditional hypercharge $\frac{5}{3}$ gauge bosons of the standard SU(5). The remaining degrees of freedom account for the 12 SM gauge fields and one additional heavy singlet.

The minimal Higgs sector sufficient for breaking the SU(5) $\otimes$ U(1)$_X$ symmetry down to the SM and, subsequently, to the SU(3) $\otimes$ U(1) of QCD+QED consists of 10$_H = (10, +1)^1$, in which the SM singlet occupies the same position as the RH neutrino does in 10$_M$, and 5$_H = (5, -2)$ containing the SM Higgs doublet. The breakdown of SU(5) $\otimes$ U(1)$_X$ to the SM gauge symmetry takes place after the SM singlet present in 10$_H$ develops a non-zero vacuum expectation value (VEV), $V_G$, generating masses

$$m^2_{X} = \frac{g^2 V^2_G}{2}$$

(3)

for the gauge bosons $X_\mu$, where $g_5$ is the SU(5) gauge coupling. The color triplet, SU(2)$_L$ singlet components of 10$_H$ and 5$_H$ also mix at this stage to form a pair of massive color triplets $\Delta_{1,2}$ transforming under the SM gauge symmetry as $(3, 1, -\frac{1}{3})$, with masses $m_{\Delta_{1,2}}$. Further details regarding the tree-level scalar spectrum in this minimal flipped SU(5) model are given in Appendix B.

For the above embedding of the SM matter content and minimal set of Higgs scalars, one can readily write the most general renormalizable\footnote{Note that in non-renormalizable settings the benefits of the scheme may be lost as the Witten’s loop contribution may be swamped by the effects of, e.g., the $d = 5$ non-renormalizable operators of the 10$_M$10$_M$10$_H$10$_H$ type.} Yukawa Lagrangian (suppressing all flavor indices)

$$L \supset Y_{10}10_M10_M5_H + Y_510_M\bar{5}_M5_H + Y_T\bar{5}_M1_M5_H + h.c.,$$

(4)

with $Y_{10}$, $Y_5$ and $Y_T$ denoting the relevant 3 x 3 complex Yukawa coupling matrices; note that the first of these, unlike the latter two, is required to be symmetric in its flavor indices, i.e., $Y_{10} = Y_{10}^T$. In the broken phase, the second term in Eq. (4) yields a strong correlation among the Dirac neutrino mass matrix $M_D^\nu$, and the up-type quark mass matrix $M_u$, namely,

$$M_D^\nu = M_u^T$$

(5)

at the GUT scale. The flavor symmetric nature of $Y_{10}$ also means that the down-type quark mass matrix satisfies $M_d = M_d^T$, while the couplings in Eq. (4) say nothing specific about the mass matrix $M_e$ of the charged leptons.
As we shall see, these correlations will turn out to be central for the high degree of predictivity of this framework\(^3\) entertained in the following sections.

### A. The RH neutrino masses and type-I seesaw

So far, we have left aside any discussion of the physical light neutrino masses in the current scenario. Obviously, Eq. (5) cannot be the whole story here and, thus, one has to employ a variant of the seesaw mechanism in some way; since the type-II and/or type-III options cannot be realized with the minimal scalar and fermionic sectors at hand one is left with the type-I seesaw implemented through the Majorana mass term for the RH neutrinos.

This may be most easily devised by employing a 50-dimensional scalar \(^{11}\) that can couple to the \(10^T_C - 10^T_M\) fermionic bilinear; the VEV of a singlet therein then gives rise to the desired mass term. As was noted in Section\(^4\) however, the associated single-purpose extra Yukawa matrix does not bring any additional insight into the flavor structure of the model, and limits the extent to which low-energy data can be used in constraining proton decay observables. Therefore we do not adopt this option here and, instead, consider the effects emerging at the quantum level in the minimal model.

#### 1. The Witten’s loop structure

The simultaneous presence of the diquark-type of interactions, mediated by the \(X_\mu\) and \(\Delta_{1,2}\) bosons, together with their leptoquark counterparts (involving the same set of fields) in the model implies that even \(\Delta L = 2\) Feynman diagrams corresponding to the Majorana type of neutrino masses may be constructed at some higher order level. This, indeed, is the central point behind every radiative (Majorana) neutrino mass generation mechanism; in the flipped \(SU(5)\) framework, it finds its incarnation in a pair of two-loop topologies depicted in FIG. 1 which can be viewed as “reduced” versions of Witten’s original \(SO(10)\) graph(s) \(^{14}\).

Note that in our analysis we shall work in the broken phase perturbation theory with masses in the free Hamiltonian\(^4\) and in the unitary gauge so that there are no Goldstone modes around. This reduces the number of relevant graphs considerably, albeit at the cost of making the Feynman integrals somewhat more complicated compared to other cases.

Based on the graphs in FIG. 1 that remain in this approach, it is immediately possible to make several comments on both the flavor structure and overall scale of the generated Majorana mass matrix \(M_{\nu}^M\). The flavor structure in particular plays a central role in what follows, and is governed by the Yukawa couplings appearing in each of the contributing graphs. In each of the two topologies there is only a single Yukawa coupling present, associated with the couplings of \(\Delta_1\) to the fermions. These couplings involve only the \(5_H\) components of \(\Delta_1\), since it is only these components that couple to the fermions through the Yukawa interactions in Eq. (4). Moreover, since all of the fermions appearing in the two graphs in FIG. 1 reside in \(10_M\), the single relevant Yukawa coupling matrix is the symmetric \(Y_{10}\). Hence, in the minimal model there is a tight correlation between the radiatively generated RH neutrino Majorana mass matrix and the mass matrix of the down-type quarks, making the scheme rather predictive.

The overall scale of \(M_{\nu}^M\), on the other hand, depends on both the Yukawa couplings in \(Y_{10}\) as well as the gauge couplings and the sizes of the mass parameters entering into each of the graphs. One can initially estimate it to be proportional to the dominant mass entry in the relevant graphs suppressed by the appropriate two-loop factor and the combination of gauge (entering raised to the fourth power) and Yukawa couplings.

Of the various mass parameters appearing in the evaluation of the graphs, the fermionic masses \(m_f\) should play no role in the integrals as the singlet Majorana mass generation does not rely on the electroweak symmetry relevant mixing matrix in the scalar sector, Eq. (15).
be such that the unification pattern is consistent with the low-energy data and compatible with the non-observation of proton decay with at least $10^{14}$ years of lifetime [16].

Hence, demanding consistency of Eq. (7) with the data one can derive constraints on $m_{\nu}^{\text{max}}$ and, in particular, on $U_{\nu}$, which is central to the BLNV phenomenology of the model. Indeed, $U_{\nu}$ drives all the proton decay branching ratios into neutral mesons including the “golden channel” $p \to p^{0}e^{+}$ final state:

$$\frac{\Gamma(p \to \pi^{0}e^{+})}{\Gamma(p \to \pi^{+}\pi^{-})} = \frac{1}{2}|(V_{\text{CKM}})_{11}|^2|\langle V_{\text{PMNS}}U_{\nu}\rangle_{\alpha 1}|^2;$$

$$\frac{\Gamma(p \to \eta e^{+})}{\Gamma(p \to \pi^{+}\pi^{-})} = \frac{C_2}{C_1}|(V_{\text{CKM}})_{11}|^2|\langle V_{\text{PMNS}}U_{\nu}\rangle_{\alpha 1}|^2,$$

$$\frac{\Gamma(p \to K^{0}e^{+})}{\Gamma(p \to \pi^{+}\pi^{-})} = \frac{C_3}{C_1}|(V_{\text{CKM}})_{12}|^2|\langle V_{\text{PMNS}}U_{\nu}\rangle_{\alpha 1}|^2,$$

where the $C_i$’s are various low-energy factors calculable using chiral Lagrangian techniques (see, e.g., Ref. [17] and references therein) and $V_{\text{CKM}}$ and $V_{\text{PMNS}}$ are the Cabibbo-Kobayashi-Maskawa and the Pontecorvo-Maki-Nakagawa-Sakata mixing matrices, respectively.

In this sense, the minimal flipped $SU(5)$ unification equipped with the Witten’s loop mechanism can be viewed as a particularly simple (if not the most minimal of all) theory of the absolute neutrino mass scale and, at the same time, the two-body BLNV nucleon decays.

### B. Consistency constraints and implications

Let us now work out the aforementioned consistency constraints in more detail and give some basic examples of their possible implications. Firstly, it should be noted that there is a lower limit on the largest entry of $W_{\nu}$ that depends on $m_{\nu}^{\text{max}}$ and the shape of $U_{\nu}$. Taking into account the typical 50% reduction of the running top quark Yukawa between $M_{Z}$ and the unification scale (at around $10^{16}$ GeV) and taking, for example, $m_{\nu}^{\text{max}} = 1$ eV and $U_{\nu} = 1$ one finds that the $(3,3)$ entry of $W_{\nu}$ is as large as about

$$|\langle W_{\nu}\rangle_{33}| \sim 6.4 \times 10^{12} \text{ GeV.}$$

The same magnitude, however, may not so easily be achieved for the $(3,3)$ entry of $M_{\nu}^{M}$ as required by Eq. (7) due to the generic $10^{-3}$ geometrical suppression in the relevant two-loop graphs and a possible further suppression associated with the Yukawa coupling $Y_{10}$; the latter may be especially problematic in the minimal scenario [14] because then $Y_{10}$ is fixed by the down-type quark masses and, thus, brings about another suppression of some $10^{-2}$ to $(M_{\nu}^{M})_{33}$.

However, this correlation is loosened if there is more than a single copy of $5_{H}$ in the scalar sector. As was already indicated in Ref. [13], the additional $Y_{10}$ associated to an extra $5_{H}$ can conspire with the original $Y_{10}$ to do two things at once: they may partially cancel in the

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5 Assuming, implicitly, that the renormalization scale dependence eventually disappears as a consequence of the assumed UV-finiteness of the full result.
down-type quark mass formula to account for the moderate suppression of $M_{d}/M_{u}$ yet their other combination governing $M^{M}_{\nu}$ (weighted by the appropriate scalar mixings) may still remain large, thus avoiding the problematic additional $10^{-2}$ suppression. In what follows, we shall model this situation by imposing a humble $|y| \lesssim 4\pi$ perturbativity criterion on all the $Y_{10}$ and $Y'_{10}$ entries.

However, even in such a case the $\sim 10^{13}$ GeV lower limit on the largest entry $(W'_{\nu})_{33}$, may still be problematic because, for $U_{\nu} \neq 1$, it may be further enhanced by the admixture of the yet larger $(2, 2)$ and, in particular, the $(1, 1)$ entry of $(m^{\text{diag}}_{\nu})^{-1}$; as a matter of fact the latter is not constrained at all given that the lightest neutrino mass eigenstate may still be extremely light. Thus, the lower bound on the magnitude of the largest element of $W_{\nu}$ gets further boosted over the naïve estimate of $10^{13}$ GeV whenever $U_{\nu}$ departs significantly from unity, which in turn constrains all of the partial widths, Eq.s (3).

Hence, a thorough evaluation of the graphs in FIG. 1 will decide several important questions, namely:

1. Can the elements of $M^{M}_{\nu}$ ever be big enough to be consistent (at least in the most optimistic scenario with $U_{\nu} \sim 1$) with $W_{\nu}$, as required by Eq. (7), in the case of the single $5_{H}$ scenario with its typical extra $10^{-2}$ suppression at play?

2. If not, can the two-$5_{H}$ scenario work? What would be then the corresponding lower limit for $m_{\nu}^{\text{max}}$ in this scenario?

3. In either case, what is the allowed domain for the entries of $U_{\nu}$ and, thus, for the corresponding BLNV nucleon decay rates?

This is what we turn our attention to in the remainder of this article.

III. WITTEN’S LOOP CALCULATION

The leading contribution to the radiatively generated RH neutrino mass in the current scheme may be computed by considering the graphs in FIG. 1 evaluated at zero external momentum, see Appendix C, with the relevant interaction terms given in Appendix A. In the minimal renormalizable model containing only a single $10_{H}$ and one or more $5_{H}$ representations, no one-loop contribution to the RH neutrino mass matrix can be generated, nor do there exist any one-loop counterterm graphs. The resulting expression for the RH Majorana neutrino mass matrix in the case of a single $5_{H}$ multiplet reads

$$(M_{\nu}^{M})^{IJ} = -\frac{3g^{4}_{s}}{(4\pi)^{4}} V_{G}^{2} \sum_{i=1}^{2} (-8Y_{10}^{IJ}(U_{\Delta})_{i1}(U_{\Delta}^{\ast})_{i2}I_{3}(s_{i}),$$

where the scalar mixing matrix elements $(U_{\Delta})_{ij}$ are given in Appendix C, and $I_{3}(s_{i})$ is the sum of the corresponding loop integrals evaluated at zero external incoming momentum,

$$I_{3}(s_{i}) = -(4\pi)^{4}(\Sigma^{P}_{1}(0) + 2\Sigma^{P}_{2}(0)),$$

regarded as a function of $s_{i} = m^{2}_{\Delta_{i}}/m^{2}_{X}$. Recall that there is an overall extra factor of 2 included in Eq. (10) related to the permutation of the two external neutral field lines (for $I = J$) or to the symmetry of $Y_{10}$ (for $I \neq J$). The integrals $\Sigma^{P}_{1}(0)$ and $\Sigma^{P}_{2}(0)$, corresponding to topology 1 and 2 respectively, are given by

$$i\Sigma^{P}_{1}(0) = i \int \frac{d^{4}p}{(2\pi)^{4}} \int \frac{d^{4}q}{(2\pi)^{4}} \gamma_{\nu} \frac{1}{q^{\mu}g^{\nu}} \frac{1}{(p + q)^{2} - m^{2}_{\Delta_{1}}} \frac{-g^{\mu\nu} + \frac{1}{m^{2}_{X}} p^{\mu}p^{\nu} - g^{\nu} + \frac{1}{m^{2}_{X}} g_{\nu} q^{\nu}}{q^{2} - m^{2}_{X}},$$

$$i\Sigma^{P}_{2}(0) = i \int \frac{d^{4}p}{(2\pi)^{4}} \int \frac{d^{4}q}{(2\pi)^{4}} \gamma_{\nu} \frac{1}{q^{\mu}g^{\nu}} \frac{1}{(p + q)^{2} - m^{2}_{\Delta_{1}}} \frac{-g^{\mu\nu} + \frac{1}{m^{2}_{X}} p^{\mu}p^{\nu} - g^{\nu} + \frac{1}{m^{2}_{X}} (p + q)_{\nu}(p + q)^{\nu}}{q^{2} - m^{2}_{X}}.$$
and

\[ -(4\pi)^4\Sigma_{\nu,\text{div}}(0) = -\frac{3}{4\epsilon} + \frac{m_{\Delta_i}^4}{4m_X^4} \left( \frac{1}{2\epsilon^2} + \frac{3}{2\epsilon} - \frac{1}{\epsilon \log \frac{m_{\Delta_7}^2}{Q^2}} \right). \]  

\[ (15) \]

It follows from Eq. (11) that the total contribution \( I_3(s) \) to the RH neutrino mass matrix is UV finite, as must be the case here due to the absence of the necessary counterterms.

\section{RESULTS}

The behavior of the result for the purely kinematic piece of the RH neutrino mass matrix, \( I_3(s) \), is shown in FIG. 2. Notably, the magnitude of \( I_3(s) \) is bounded for all \( s \geq 0 \). Indeed, from the analytic result given in Eq. (D31), one has that for \( s \to 0 \),

\[ I_3(s \to 0) = 3 + s \left( 3 \log s + \pi^2 - \frac{15}{2} \right) + O(s^2 \log^2 s), \]  

\[ (16) \]

while in the opposite limit with \( s \to \infty \),

\[ I_3(s \to \infty) = -3 + O(s^{-1} \log^2 s). \]  

\[ (17) \]

\section{RH neutrino masses in the minimal model}

With \( I_3(s) \) determined, we may proceed to evaluate the size of \( M_{\nu}^M \) in Eq. (10). Substituting in the explicit forms of the mixing matrix elements in Eq. (B8), one obtains

\[ M_{\nu}^M = \frac{3g_5^4}{(4\pi)^4}(-8Y_{10})V_G\bar{I}, \]  

\[ (18) \]

where

\[ \bar{I} = \sum_{i=1}^{2} \frac{2\nu_\ast (2\lambda_2 + g_5^2 s_i)}{4|\nu|^2 + (2\lambda_2 + g_5^2 s_i)^2}f_3(s_i), \]  

\[ (19) \]

and \( \nu = \mu/V_G \). We note that \( \bar{I} \to 0 \) as \( \mu \to 0 \), reflecting the fact that the graphs rely on the 10_H \(- 5_H \) mixing. It is also clear from Eq. (19) that, since \( I_3(s) \) is bounded, \( \bar{I} \) cannot be made arbitrarily large to compensate for the suppression factors noted in Section III. To develop some sense of the allowed size of \( \bar{I} \), it is useful to substitute for \( s_i \) from Eq. (B7) and inspect \( \bar{I} \) as a function of \( \nu, \lambda_2, \lambda_5 \), and \( g_5 \), neglecting all terms that are of the order of \( \nu^2/V_G^2 \), where \( \nu \) is the electroweak VEV, see Eq. (B2).

Requiring that the tree-level vacuum be locally stable implies \( \lambda_2, \lambda_5 < 0 \) and

\[ |\nu| \leq \sqrt{\lambda_2 \lambda_5}. \]  

\[ (20) \]

When this bound is saturated, i.e., when \( |\nu| = \sqrt{\lambda_2 \lambda_5} \), the mass \( m_{\Delta_7} \) vanishes for all values of \( \lambda_2, \lambda_5 \) while \( m_{\Delta_2}^2 = -(\lambda_2 + \lambda_5)V_G^2 \). The resulting value of \( \bar{I} \) for this special case is shown in the \( (\lambda_2, \lambda_5) \) plane in FIG. 3. In particular, it should be noted that the value of \( \bar{I} \) is unchanged under the interchange \( \lambda_2 \leftrightarrow \lambda_5 \), as can be easily verified from Eqs. (19) and (B7), and \( |\bar{I}| \leq 3 \) for all values of \( \lambda_2 \) and \( \lambda_5 \). The maximal value of \( |\bar{I}| \) is achieved for \( \lambda_2 = \lambda_5 \), with \( |\bar{I}| \to 3 \) as \( \lambda_2 = \lambda_5 \to -\infty \).

Qualitatively different behavior results in the more general case that \( \nu \) does not saturate the bound given in Eq. (20). This is demonstrated in FIG. 4, in which the value of \( \bar{I} \) is plotted as a function of \( \lambda_2 = \lambda_5 = \lambda \) with

\[ \nu = \alpha \sqrt{\lambda_2 \lambda_5}, \quad \alpha \in [0, 1], \]  

\[ (21) \]

FIG. 2: Plot of the function \( I_3(s) \) appearing in the RH neutrino mass matrix.

FIG. 3: Contour plot of \( \bar{I} \), as defined in Eq. (19), in the \( (\lambda_2, \lambda_5) \)-plane, with \( g_5 = 0.5 \) and \( \nu = \alpha \sqrt{\lambda_2 \lambda_5} \) for \( \alpha = 1 \), corresponding to the maximal value of \( |\nu| \) consistent with a locally stable SM vacuum.
for several values of $\alpha$. Although $\tilde{I}$ remains invariant under $\lambda_2 \leftrightarrow \lambda_3$, for values of $|\alpha| < 1$, $|\tilde{I}|$ now tends to zero for large values of the scalar couplings $\lambda_2$, $\lambda_3$. This is due to the fact that, for $|\alpha| \neq 1$, both $s_1$, $s_2$ grow with increasing $|\lambda|$ such that $I_3(s_1)$, $I_3(s_2) \rightarrow -3$, while the coefficients of each in Eq. (19) are equal in magnitude but of opposite sign, resulting in the two terms cancelling. Physically, this corresponds to the expected dynamical decoupling of the heavy scalar states in the $m_{\Delta_{1,2}} \rightarrow \infty$ limit. For $\alpha = 1$, at least one color triplet scalar is massless at tree-level for all values of $\lambda_2$ and $\lambda_3$. Consequently, this state never decouples and $\tilde{I}$ therefore does not vanish. Technically, this arises because $I_3(s_1) = 3$ while $I_3(s_2) \rightarrow -3$, with the two contributions still entering $\tilde{I}$ with coefficients of equal magnitude but opposite sign.

However, even in the most optimistic case with $|\tilde{I}| \rightarrow 3$, the above results make it clear that there is little hope for a viable prediction of the light neutrino spectrum in the minimal scenario under consideration. For acceptable values of $m_X \sim 10^{17}$ GeV, and taking $g_0 \approx 0.5$, the elements of $M^L_{5}$ are found to be $\lesssim 10^{12}$ GeV after taking into account the $\sim 10^{-2}$ suppression associated with presence of $Y_{10}$. This is to be compared with the (optimistic) lower bound of $\sim 10^{13}$ GeV for the elements of the left-hand side of Eq. (7). Evidently, in the case when only a single $5_H$ is present in the spectrum the answer to whether Eq. (7) can be satisfied is negative. In fact, in this minimal model the problem is exacerbated by the fact that $Y_{10} \propto M_d$, which implies a far too hierarchical pattern of light neutrino masses irrespective of their absolute size, as was previously noted in Ref. [12]. Thus we are immediately led to consider the remaining questions raised in Section III concerning the viability of the model with an additional $5_H$ representation instead.

### B. Minimal potentially realistic model

As noted above, the addition of a second $5_H$ multiplet in principle allows both the $Y_{10}$ suppression and the overly hierarchical flavor structure to be avoided. At the same time, the overall predictive power of the theory is not significantly harmed by this addition; in particular, doing so does not spoil the key Yukawa relations used in obtaining Eq. (7). With a second $5'_H$ multiplet, the Yukawa sector of the model reads

$$\mathcal{L} \supset Y_{10}10M10_{5H} + Y'_{10}10M10_{5'H}$$

holding at tree-level, where $\nu'$ is the VEV associated with the electrically neutral component of $5'_H$, see Appendix B2. By contrast, the analogous relationship between the down-type quark masses and the generated RH neutrino Majorana masses, $M_d$, $M^M_{\nu} \propto Y_{10}$, is no longer preserved. While $M_d \propto Y_{10} \nu$ + $Y'_{10} \nu'$, the appropriate generalization of Eq. (10) reads

$$M^D_{\nu} = M^T_u \propto Y_5 \nu + Y'_{5} \nu'$$

where $Y_{10}'$ is of course also flavor symmetric. In this scenario, the Dirac neutrino mass matrix still remains tightly correlated with the up-type quark masses, with the GUT scale relation

$$M^M_{\nu}^{ij} = \frac{3g_0^4}{(4\pi)^4} Y_G \sum_{i=1}^{3} \sum_{j=2}^{3} (-8Y_j Y_j)(U_\Delta)_{i1}(U^\Delta^\dagger)_{ij} I_3(s_i),$$

where $Y_j = Y_{10}$ when $j = 2$ and $Y_j = Y'_{10}$ when $j = 3$, with $U_\Delta$ now a $3 \times 3$ mixing matrix as defined in Eq. (B16). Thus, in general, $M_d$ and $M^M_{\nu}$ are determined by different linear combinations of the Yukawa couplings $Y_{10}$ and $Y'_{10}$. In turn, this means that the generic suppression of $M^M_{\nu}$ by a factor $\propto M_d$ may be avoided in the two-$5_H$ scenario. On the other hand, it is still the case that the elements of $M^M_{\nu}$ are bounded from above, at least so long as it is required that all couplings remain perturbative.

#### 1. Phenomenology of the minimal potentially realistic model

As the ignorance of yet higher-order effects makes any such perturbativity constraints somewhat arbitrary in general, in what follows we shall give two examples of the $M^M_{\nu}$ estimates corresponding to two different choices of the upper limits on the effective (running) SM down-quark Yukawa couplings. These, according to Eq. (A3), obey $Y_d \equiv 8Y_{10}$ and $Y'_d \equiv 8Y'_{10}$ at the matching scale. The two cases to be considered are i) $|Y_d|, |Y'_d| \lesssim 1$ and ii) $|Y_d|, |Y'_d| \lesssim 4\pi$. For the former case (motivated by the SM value of the top Yukawa coupling) one has the
following upper limit on $M^M_\nu$ calculated from Eq. (24)

\[ |M^M_\nu| \lesssim 6.4 \times 10^{12} \left( \frac{m_X}{10^{17} \text{GeV}} \right) \text{ GeV}, \quad (25) \]

while for the latter one obtains

\[ |M^M_\nu| \lesssim 8.0 \times 10^{13} \left( \frac{m_X}{10^{17} \text{GeV}} \right) \text{ GeV}. \quad (26) \]

Note that in both cases we have used the (numerical) upper limit

\[ \sum_{i=1}^{3} \sum_{j=2}^{3} |(U_\Delta)_{i1}(U_\Delta)_{ij}I_3(s_i)| \leq 3 \quad (27) \]

which is completely analogous to the limit discussed in Section IVA for the single-$5_H$ case.

Remarkably, for the typical flipped $SU(5)$ value of $m_X = 10^{17}$ GeV (see, e.g., Ref. [13]) the case i) limit, Eq. (25), is just on the borderline of compatibility with the optimistic lower limit in Eq. (9) on $|W^\nu|$, while the latter case ii) in principle admits lower\(^6\) values of $m_X$.

This, in turn, implies that there is generally not much room for any significant admixture of the second neutrino (inverse) mass within the element $(W_\nu)_{33}$, hence, the only allowed $U_{\nu}$’s in Eq. (7) are those for which $(U_\nu)_{13}$ and $(U_\nu)_{23}$ are small.

To this end, the model clearly calls for a dedicated numerical analysis including a detailed calculation of the heavy spectrum that conforms to, among other things, the requirement of a significant spread of the scalar triplets in order to maximize $|\tilde{I}|$. This, however, is beyond the scope of the current study and will be elaborated on elsewhere.

At this point, let us just illustrate the typical situation by evaluating the most significant proton-decay two-body branching ratios (neglecting the kinematically suppressed vector-meson channels for simplicity) in the $(U_\nu)_{13} = (U_\nu)_{23} = 0$ limit with the 1-2 mixing angle $\theta_{12}$ therein chosen in such a way that $\Gamma(p \to \pi^+ \mu^+)$ is maximized (see Ref. [13] for further details):

\[ \text{Br}(p \to \pi^+ \nu) \approx 80.0\%, \]

\[ \text{Br}(p \to \pi^0 e^+) \approx 14.2\%, \]

\[ \text{Br}(p \to \pi^0 \mu^+) \approx 5.5\%, \]

\[ \text{Br}(p \to K^0 e^+) \approx 0.1\%. \quad (28) \]

Needless to say, for non-extremal values of $\theta_{12}$ these branching ratios may vary; in particular, $\text{Br}(p \to \pi^0 e^+)/\text{Br}(p \to \pi^0 \mu^+)$ should increase.

Finally, let us say a few words about the lower limits on the mass of the heaviest SM neutrino in the two cases (25) and (26). As for the former, one obtains\(^{7}\)

\[ m_3 \gtrsim \left( \frac{10^{17} \text{GeV}}{m_X} \right) \text{ eV} \quad (29) \]

while for the latter one has

\[ m_3 \gtrsim 0.08 \left( \frac{10^{17} \text{GeV}}{m_X} \right) \text{ eV} \quad (30) \]

which, actually, turns out to be independent on the specific form of the $U_\nu$ matrix as long as the 1-3 and 2-3 mixings therein are small (see the discussion above). With this at hand, any specific experimental upper limit on the absolute neutrino mass scale may be readily translated into a lower limit on $m_X$ and, subsequently, the proton lifetime.

V. CONCLUSIONS AND OUTLOOK

The two-loop radiative RH neutrino mass generation mechanism originally identified by Witten in 1980s in the $SO(10)$ context finds a beautiful incarnation in the class of renormalizable flipped $SU(5)$ unified theories where, among other effects, it avoids the need for the 50-dimensional scalar representation. This, in turn, renders the simplest potentially realistic scenarios perhaps the most minimal (partially) unified gauge theories on the market, with strong implications for some of the key beyond-Standard-Model observables such as the absolute neutrino mass scale and proton decay.

In this work we have focused on a thorough evaluation of the relevant Feynman graphs in these scenarios paying particular attention to their analytic properties and the absolute size of the effect which turns out to be the key to the consistency of the scenario as a whole. It has been shown that there is no way to be consistent with the data with only one 5-dimensional scalar multiplet at play and, hence, the minimal potentially realistic setup must include two such irreps in the scalar sector (along with the 10-dimensional tensor).

As it turns out, such a minimal flipped $SU(5)$ model is subject to strong constraints on its allowed parameter space that lead to rather stringent limits on the absolute light neutrino mass scale as well as the BLMN two-body nucleon decays. A thorough numerical analysis of the corresponding correlations is deferred to a future study.

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\(^{6}\) These, however, may not be that simple to get within potentially realistic unification chains, see Appendix C of Ref. [13].

\(^{7}\) Given the structure of the seesaw formula in the current context together with the tight constraints on the structure of the $U_\nu$ matrix we generally assume the hierarchy of the light neutrino mass eigenstates to be normal.
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Appendix A: The interaction Lagrangian

The radiative generation of the RH neutrino masses involves only a small subset of the interactions associated with the full flipped SU(5) Lagrangian. Working in the SU(5) ⊗ U(1)X broken phase, we extract the required interactions from the kinetic terms and general Yukawa Lagrangian, Eq. (4), making use of FeynRules [24,25] and FeynArts [26,27] to verify that all terms and contributing diagrams are accounted for. As discussed in Section IV, when the model contains only a single 5H representation the relevant diagrams are found to be those in Fig. 1 arising from the interaction Lagrangian

\[ \mathcal{L}_{\text{int}} \supset \frac{g_5}{\sqrt{2}} \epsilon_{ijk} \epsilon^{\alpha \beta} V G X^{\mu \nu} X^j_{\mu \beta} \mathcal{T}^k + \frac{g_5}{\sqrt{2}} \epsilon_{ijk} X^{\mu \nu} X^j_{\mu \beta} \mathcal{T}^k + 8 Y_{10}^{IJ} \frac{\epsilon_{\gamma \delta}}{\sqrt{2}} \nu_{L_i}^\gamma \nu_{L_j}^\delta - 4 Y_{10}^{IJ} \epsilon_{\gamma \delta} (Q_{L_i}^{j \alpha})^T C^{-1} Q_{L_j}^{\alpha \nu} + h.c. \]

where \( i, j, k \) and \( \alpha, \beta \) denote the SU(3)_C and SU(2)_L indices, respectively, and \( \epsilon_{ijk} \) and \( \epsilon^{\alpha \beta} \) are the relevant fully antisymmetric tensors with \( \epsilon_{123} = -\epsilon_{132} = 1 \). In this expression, \( \mathcal{T} \) denotes the \((3,1,1/3)\) components of the scalar 10\(_H\), \( T \) the \((3,1,-1/3)\) components of 5\(_H\), \( Q_{L_i} \) the quark doublet \((3,2,1/3)\) in 10\(_M\), \( d_{L_i} \) the down-type quark singlet \((3,1,1/3)\) in 10\(_M\), and \( \nu_{L_{ij}} \) the \((1,1,0)\) components of 10\(_M\). The charged vector bosons \( X_\mu \) associated with the breaking of SU(5) ⊗ U(1)_X have SM quantum numbers \((3,2,1/3)\). Following the breakdown of the SU(5) ⊗ U(1)_X symmetry due to the non-zero VEV \( V_G \), the scalar states \( T \) and \( \mathcal{T} \) mix to form the SU(3)_C ⊗ SU(2)_L ⊗ U(1)_Y eigenstates \( \Delta_{1,2} \), as described in Appendix [3].

Let us note that in deriving the central formula Eq. (10), especially the overall factor of 3 therein, the color and isospin factors in Eq. (11) play a crucial role. It is also worth noting that the exact cancellation of the UV divergences discussed in Section III which relies on the extra factor of 2 in Eq. (11), emerges from the difference of the overall numerical factors in the last two terms in Eq. (11).

After including an additional 5\(_H^\prime\) to arrive at the minimal realistic model discussed in Section IVB, the interaction Lagrangian remains rather similar. The addition of Yukawa couplings involving 5\(_H^\prime\) leads to the set of interaction terms (with color indices suppressed for simplicity)

\[ \mathcal{L}_{\text{int}}^{\text{THSM}} = \mathcal{L}_{\text{int}} - \left[ 8 Y_{10}^{IJ} d_{L_i}^\nu T^{-1} \nu_{L_j}^\nu T + 4 Y_{10}^{IJ} \epsilon_{\gamma \delta} (Q_{L_i}^{j \alpha})^T C^{-1} Q_{L_j}^{\alpha \nu} + h.c. \right] \]

where \( T' \) denotes the additional \((3,1,-1/3)\) multiplet contained in 5\(_H^\prime\), which mixes with the states \( \mathcal{T} \) and \( T \) to yield a set of SU(3)_C ⊗ SU(2)_L ⊗ U(1)_Y eigenstates \( \Delta_{1,2,3} \).

For the sake of completeness and matching to the SM Yukawa couplings we also present the terms involving the doublet Higgs interactions here:

\[ -\mathcal{L}_{\text{int}} \supset 8 Y_{10}^{IJ} \epsilon_{\gamma \delta} H_0^\nu d_{L_i}^\nu T^{-1} C^{-1} Q_{L_j}^{\alpha \nu} + Y_{10}^{IJ} \epsilon_{\gamma \delta} c_{L_i}^\nu T^{-1} C^{-1} c_{L_j}^{\alpha \nu} + h.c. \]

where the SM Higgs doublet \( H \) consists of the components of 5\(_H\) transforming under the SM gauge group as \((1,2,1/2)\), \( u_{L_i}^\nu \) and \( c_{L_i}^\nu \) are the components of 5\(_M\) transforming as \((3,1,-1/3)\) and \((1,2,-1/2)\) respectively, and \( c_{L_i}^\nu \) denotes the single component of 1\(_M\), transforming as \((1,1,+1)\).

Appendix B: Triplet scalar spectrum and mixing

1. Model with a single 5\(_H\) representation

The tree-level scalar potential in the model with a single 5\(_H\) may be written

\[ V = \frac{1}{2} m_0^2 \text{Tr}(10_H^T 10_H) + m_5^2 5_H^T 5_H + \frac{1}{8} \left( \mu_{ijklm} 10_H^T 10_H^T 10_H^T 10_H^T + h.c. \right) + \frac{1}{4} \lambda_1 \left[ \text{Tr}(10_H^T 10_H) \right]^2 + \frac{1}{4} \lambda_2 \text{Tr}(10_H^T 10_H 10_H^T 10_H^T) + \frac{1}{2} \lambda_3 (5_H^T 5_H)^2 + \frac{1}{2} \lambda_4 \text{Tr}(10_H^T 10_H 5_H^T 5_H^T) + \frac{1}{4} \lambda_5 (5_H^T 10_H 10_H^T 5_H^T) \]

The scalar basis is chosen such that the spontaneous breaking of SU(5) ⊗ U(1)_X and the subsequent electroweak symmetry breaking takes place via the non-zero VEVs

\[ \langle 10_H \rangle^{45} = -\langle 5_H \rangle^{54} = V_G, \quad \langle 5_H \rangle^4 = v. \]

Requiring that this corresponds to a stationary point of the scalar potential yields the conditions

\[ V_G \left[ m_0^2 + V_G^2 (2 \lambda_1 + \lambda_2) + v^2 (\lambda_4 + \lambda_5) \right] = 0, \quad \lambda_3 v \left[ m_5^2 + 2 \lambda_3 v^2 + V_G^2 (\lambda_4 + \lambda_5) \right] = 0, \]

which permit the parameters \( m_0^2, m_5^2 \) to be eliminated in favor of the VEVs.

After the breakdown of SU(5) ⊗ U(1)_X to SU(3)_C ⊗ SU(2)_L ⊗ U(1)_Y, the charged vector bosons \( X_\mu \) associated with the broken generators acquire masses \( m_X \) given by Eq. (3). The scalar states \( T \) and \( \mathcal{T} \) of relevance to the generation of the RH neutrino masses mix, with the mass matrix (in the basis \( (\mathcal{T}, T) \))

\[ M_\Delta^2 = \left( \begin{array}{cc} -\lambda_2 V_G^2 & \mu^* V_G \\ \mu V_G & m_5^2 + \lambda_4 V_G^2 \end{array} \right), \]

where \( \mu = \sqrt{\lambda_2 V_G^2 + \mu^2} \).
where Eq. (B3) with \( v = 0 \) has been used to eliminate \( m^2_{\tilde{t}_0} \). This is diagonalized by a unitary matrix \( U_\Delta \) according to

\[
U_\Delta M^2_\Delta U_\Delta^\dagger = \begin{pmatrix} m^2_{\Delta_1} & 0 \\ 0 & m^2_{\Delta_2} \end{pmatrix},
\]

with

\[
m^2_{\Delta_{1,2}} = \frac{1}{2} \left( m^2_\mu + (\lambda_4 - \lambda_2) V^2_G \right) \pm \sqrt{\left( m^2_\mu + (\lambda_4 - \lambda_2) V^2_G \right)^2 + 4|\mu|^2 V^2_G},
\]

which, in the electroweak vacuum, simplifies into

\[
m^2_{\Delta_{1,2}} = \frac{V^2_G}{2} \left\{ - (\lambda_2 + \lambda_5) \pm \sqrt{(\lambda_2 - \lambda_5)^2 + 4|\mu|^2 V^2_G} \right\}.
\]

The elements of the mixing matrix \( U_\Delta \) read

\[
\begin{align*}
(U_\Delta)_{11} &= \frac{\mu^* V_G}{\sqrt{|\mu|^2 V^2_G + (m^2_{\Delta_1} + \lambda_2 V^2_G)^2}}, \\
(U_\Delta)_{12} &= \frac{m^2_{\Delta_2} + \lambda_2 V^2_G}{\sqrt{|\mu|^2 V^2_G + (m^2_{\Delta_1} + \lambda_2 V^2_G)^2}}, \\
(U_\Delta)_{21} &= \frac{\mu^* V_G}{\sqrt{|\mu|^2 V^2_G + (m^2_{\Delta_2} + \lambda_2 V^2_G)^2}}, \\
(U_\Delta)_{22} &= \frac{m^2_{\Delta_2} + \lambda_2 V^2_G}{\sqrt{|\mu|^2 V^2_G + (m^2_{\Delta_2} + \lambda_2 V^2_G)^2}}.
\end{align*}
\]

2. Model with two \( 5_H \) representations

In the minimal realistic model with two \( 5_H \) representations, we take the tree-level scalar potential to be given by

\[
V = \frac{1}{2} m^2_\tilde{t}_0 \text{Tr}(10_H^t 10_H) + m^2_{\tau_1} 5_H + m^2_{\tilde{\tau}_1} 5^*_H + m^2_{\tilde{\tau}_1} \lambda^2 \tilde{\tau}_1 5^*_H \\
+ \frac{1}{4} \lambda_1 \left[ \text{Tr}(10_H^t 10_H) \right]^2 + \frac{1}{4} \lambda_2 \text{Tr}(10_H^t 10_H 10_H^t 10_H) \\
+ \lambda_3 (5^H H)^2 + \lambda_3 (5^H H^t 5^*_H)^2 + \lambda_5 (5^H H^t 5^*_H)(5^H H^t 5^*_H) \\
+ \lambda_6 (5^H H^t 5^*_H)(5^H H^t 5^*_H) + \frac{1}{2} \lambda_4 5^H 5_H \text{Tr}(10_H^t 10_H) \\
+ \frac{1}{2} \lambda_4 5^H 5^*_H \text{Tr}(10_H^t 10_H) + \lambda_5 5^H 10_H^t 10_H^t 5^*_H \\
+ \lambda_5 5^H 10_H^t 10_H^t 5^*_H + \left[ m^2_{\tilde{\tau}_1} 5_H \right] \\
+ \frac{\mu}{8} \epsilon_{ijklm} 10^H 10^H 10^H 10^H = \frac{\mu'}{8} \epsilon_{ijklm} 10^H 10^H 10^H 10^H \\
+ \eta_1 (5^H 5_H)(5^H 5_H) + \eta_2 (5^H 5_H)^2 \\
+ \eta_3 (5^H 5_H)(5^H 5_H) + \frac{1}{2} \lambda_7 5^H 5^*_H \text{Tr}(10_H^t 10_H) \\
+ \lambda_8 5^H 10_H^t 10_H^t 5^*_H + h.c. \right].
\]

The field basis is again chosen such that the fields \( 10_H \) and \( 5_H \) acquire non-zero VEVs given by Eq. (B2), while

\[
\langle 5_H \rangle^4 = v'.
\]

The corresponding conditions that must hold for this to be a stationary point of the potential are

\[
f_i = 0, \quad i = 1, 2, 3,
\]

where

\[
\begin{align*}
f_1 &= v_1 m^2_\mu + v_2 m^2_{\tilde{\tau}_1} + 3 v_2^2 v_{\eta_1} + v_2^3 \eta_3 \\
&+ v_2 V^2_G (\lambda_7 + \lambda_8) + 2 v_2^4 \lambda_3 + v_1 V^2_G (\lambda_4 + \lambda_5) \\
&+ v_1 v_2^2 (\lambda_6 + \lambda_5 + \lambda_2) + 2 \eta_2, \\
f_2 &= v_2 m^2_\mu + v_2 m^2_{\tilde{\tau}_1} + v_2^4 \eta_1 + 3 v_2^3 \eta_3 \\
&+ v_1 V^2_G (\lambda_7 + \lambda_8) + 2 v_2^3 \lambda_3 + v_2 V^2_G (\lambda_4 + \lambda_5) \\
&+ v_1^2 v_2 (\lambda_6 + \lambda_5 + \lambda_2) + 2 \eta_2, \\
f_3 &= V_G m^2_{\tilde{\tau}_1} + V^2_G (2 \lambda_1 + \lambda_2) + v_1^2 V_G (\lambda_4 + \lambda_5) \\
&+ v_2^3 V_G (\lambda_4 + \lambda_5) + 2 v_1 v_2 V_G (\lambda_7 + \lambda_8).
\end{align*}
\]

In deriving the above, and in all expressions below, we restrict our attention to the case where all couplings are real.

In the \( SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \) symmetric phase, i.e., for \( V_G \neq 0, v = v' = 0 \), the set of scalar color triplets that mix is extended to include the color triplet \( T' \) associated with \( 5_H \). The \( 3 \times 3 \) mass matrix, in the basis \( (\tilde{D}_1^t, T, T') \), reads

\[
M^2_\Delta = \begin{pmatrix} -\lambda_2 V^2_G & \mu V_G & m^2_\mu + \lambda_4 V^2_G \\
\mu V_G & m^2_\mu + \lambda_4 V^2_G & \mu' V_G \\
m^2_\mu + \lambda_4 V^2_G & \mu' V_G & m^2_\mu + \lambda_4 V^2_G \end{pmatrix},
\]

where Eq. (B14) with \( v = v' = 0 \) has been used to eliminate the dependence on \( m^2_{\tilde{t}_0} \). The resulting mass eigenstates \( (\Delta_1, \Delta_2, \Delta_3) \) are obtained through the rotation

\[
(U_\Delta)_{11} = \frac{\mu^* V_G}{\sqrt{|\mu|^2 V^2_G + (m^2_{\Delta_1} + \lambda_2 V^2_G)^2}}, \\
(U_\Delta)_{12} = \frac{m^2_{\Delta_2} + \lambda_2 V^2_G}{\sqrt{|\mu|^2 V^2_G + (m^2_{\Delta_1} + \lambda_2 V^2_G)^2}}, \\
(U_\Delta)_{21} = \frac{\mu^* V_G}{\sqrt{|\mu|^2 V^2_G + (m^2_{\Delta_2} + \lambda_2 V^2_G)^2}}, \\
(U_\Delta)_{22} = \frac{m^2_{\Delta_2} + \lambda_2 V^2_G}{\sqrt{|\mu|^2 V^2_G + (m^2_{\Delta_2} + \lambda_2 V^2_G)^2}},
\]

\[
\begin{pmatrix} -\lambda_2 V^2_G & \mu V_G & m^2_\mu + \lambda_4 V^2_G \\
\mu V_G & m^2_\mu + \lambda_4 V^2_G & \mu' V_G \\
m^2_\mu + \lambda_4 V^2_G & \mu' V_G & m^2_\mu + \lambda_4 V^2_G \end{pmatrix},
\]

where the unitary matrix \( U_\Delta \) diagonalizes \( M^2_\Delta \) according to

\[
U_\Delta M^2_\Delta U_\Delta^\dagger = \text{diag}(m^2_{\Delta_1}, m^2_{\Delta_2}, m^2_{\Delta_3}).
\]

\[
\begin{pmatrix} \Delta_1 \\
\Delta_2 \\
\Delta_3 \end{pmatrix} = U_\Delta \begin{pmatrix} \tilde{D}_1^t \\
T \\
T' \end{pmatrix},
\]

Appendix C: Radiative fermion mass generation

In general, the physical mass of a single spin-1/2 fermion is obtained as the value of \( m \) for which

\[
(\bar{k} + m) \Gamma^{(2)}(k) = 0 \quad \forall k 
\]

where \( \Gamma^{(2)}(k) \) is the renormalized two-point 1PI Green’s function,

\[
\Gamma^{(2)}(k) = Z(k)\bar{k} - \Sigma(0).
\]
In this expression, \( Z(k) \) corresponds to the wavefunction renormalization and \( \Sigma(0) \) is the zero incoming momentum contribution to the appropriate sum of Feynman diagrams. Taken together, Eq. \((C1)\) and Eq. \((C2)\) imply that

\[
mZ(m^2) = \Sigma(0), \tag{C3}
\]

which generally amounts to a transcendental equation to be solved for the physical mass \( m \). An expression for \( m \) may be obtained perturbatively by writing \( Z(m^2) = 1 + \Delta Z(m^2) \), \( \Sigma(0) = m_0 + \Delta m_0 \), where the first and second term in each expression correspond to the tree-level and loop corrections to each quantity, respectively. One finds the result

\[
m = m_0 + [\Delta m_0 - m_0 \Delta Z(m_0^2)] + \ldots, \tag{C4}
\]

where we show only the leading part of the higher-order contribution. Therefore, in the general case with \( m_0 \neq 0 \), a calculation of the leading higher-order contribution to the physical mass would require the evaluation of the loop corrections to both \( \Sigma(0) \) and \( Z(k^2) \).

However, for the case studied in this article in which the RH neutrinos are massless at tree-level, Eq. \((C4)\) reads simply \( m = \Delta m_0 = \Sigma(0) \) at leading order.

**Appendix D: Evaluation of the two loop Feynman integrals**

1. Veltman-Van der Bij brackets

Remarkably enough, there is an entire industry concerning the evaluation methods for the zero-external-momentum two-point 1PI graphs, see, e.g., Ref. [18] or Ref. [23] and references therein.

The principal object in these methods are the so-called Veltman-Van der Bij brackets. As the original paper uses an Euclidean metric and a different choice of dimensional regularization parameter \( \epsilon \), we give here all of the relevant expressions in our particular convention, i.e., in Minkowski metric \( g = \text{diag}(1, -1, -1, -1) \) and with the number of spacetime dimensions equal to \( D = 4 - 2\epsilon \).

We introduce the brackets in the following way

\[
\{M_{11}, M_{12}, \ldots; M_{21}, \ldots; M_{31}, \ldots\} = \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \frac{1}{(p^2 - M_{11}^2)(p^2 - M_{12}^2) \ldots (q^2 - M_{21}^2) \ldots [(p + q)^2 - M_{31}^2] \ldots}, \tag{D1}
\]

\[
\{M_{11}, M_{12}, \ldots\} = \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - M_{11}^2)(p^2 - M_{12}^2) \ldots}, \tag{D2}
\]

\[
\{M_{11}, \ldots; M_{21}, \ldots; M_{31}, \ldots\} \{A(p, q)\} = \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \frac{1}{(p^2 - M_{11}^2) \ldots (q^2 - M_{21}^2) \ldots [(p + q)^2 - M_{31}^2] \ldots} A(p, q). \tag{D3}
\]

With the last expression we have introduced a shorthand notation that simplifies the form of this appendix.

Note that the brackets are invariant under the exchange of positions of the individual groups of components, which can be obtained by the change of variables \( p \leftrightarrow q \) and \( p + q \rightarrow p, -q \rightarrow q \).

By a partial cancellation of fractions we can derive various reduction formulae of the type

\[
\{M_A; m_a; M_B, m_b; M_C\} \{p^2\} = \{M_A; M_B, m_b; M_C\} + m_a^2 \{M_A, m_a; M_B, m_b; M_C\}. \tag{D4}
\]

A similar trick using \( p^2 - M_B^2 - (p^2 - M_A^2) = M_A^2 - M_B^2 \)

\[
\{M_A, m_a; M_B, m_b; M_C\} \{p^2\} = \{M_A; M_B, m_b; M_C\} + m_a^2 \{M_A, m_a; M_B, m_b; M_C\}. \tag{D5}
\]

Note that this simplification relates together the Passarino-Veltman integrals \( A_0 \) and \( B_0 \).

\[
B_0(0, 0, M_A^2) = \{M_A, 0\} = \frac{1}{M_A^2} \{M_A\} = \frac{1}{M_A^2} A_0(M_A^2). \tag{D5}
\]
\{M_A; M_B; M_C\}. It is of use to rewrite them further into double brackets
\begin{equation}
\{2M_A; M_B; M_C\} \equiv \{M_A, M_A; M_B; M_C\},
\end{equation}
which are dimensionless (cf. Ref. [18]). The operation transcribing simple brackets into double brackets is ’t Hooft’s p-operation [28]. In our notation it reads
\begin{equation}
\{M_A; M_B; M_C\} = \frac{1}{D-3} \left( M_A^2 \{2M_A; M_B; M_C\} 
+ M_B^2 \{2M_B; M_C; M_A\} 
+ M_C^2 \{2M_C; M_A; M_B\} \right).
\end{equation}

2. Topology 1

Topology 1 of FIG. 1 leads to the kinematic form (i.e., neglecting the specific form of the vertices) of the integral given in Eq. (12). By using D-dimensional gamma matrix gymnastics, it can be simplified into
\begin{equation}
\Sigma_1^\mu (0) = - \{m_X, 0; m_X, 0; m_{\Delta}\} \left[ (D-4) g^\mu + 4 p \cdot q 
- \frac{p^2 + q^2}{m_X^2} g^\mu + \frac{p^2 q^2}{m_X^2} p \cdot q \right].
\end{equation}
The slashed product can be rewritten into $p q = p \cdot q - i p^\mu \sigma_{\mu \nu} q^\nu$. After performing the p integration the second term would have to be of the form $i q^\mu \sigma_{\mu \nu} q^\nu$ and, due to the antisymmetry of $\sigma_{\mu \nu}$, such a term will not contribute. After the operations given above, we obtain
\begin{equation}
\Sigma_1^\mu (0) = - \frac{m_X^2}{2 m_X^2} \{0; 0; m_{\Delta}\} - (D-1) \left( \frac{1}{2 m_X^2} \} A_0(m_X^2)^2 
+ \frac{m_X^2}{2} \{m_X, 0; m_X, 0; m_{\Delta}\} 
- \{m_X, 0; m_X; m_{\Delta}\} \right).
\end{equation}
This may be rewritten in terms of the simple brackets using relations similar to those in Eq. (D6).

3. Topology 2

Neglecting the specific form of the vertices, Topology 2 of FIG. 1 leads to the second integral in Eq. (12). It can be simplified into (again making use of the antisymmetry of $\sigma_{\mu \nu}$)
\begin{equation}
\Sigma_2^\mu (0) = - \{m_X, 0; m_{\Delta}, 0; m_X\} \left[ (2 - D) p \cdot q 
- \frac{2 p^2 q^2}{m_X^2} - \frac{2 p^2 + q^2}{m_X^2} p \cdot q + \frac{p^4 q^2}{m_X^2} 
+ \frac{p^2 (q^2 + p^2)}{m_X^2} p \cdot q + \frac{p^2}{m_X^2} (p \cdot q)^2 \right].
\end{equation}
The result after simplification reads
\begin{equation}
\Sigma_2^\mu (0) = \frac{2 - D}{2} \{0; m_{\Delta}, 0; m_X\} 
+ \frac{3 - D}{2} \{m_X, 0; m_{\Delta}, m_X\} 
+ \frac{m_{\Delta}^2}{4 m_X^2} \left( 2 \{m_X, 0; m_{\Delta}\} - \{m_X; m_X; m_{\Delta}\} \right) 
+ \frac{D - 2}{2 m_{\Delta}^2 m_X^2} A_0(m_X^2)^2 A_0(m_{\Delta}^2) - \frac{1}{4 m_X^2} A_0(m_X^2)^2.
\end{equation}

4. Integrals

For the reader’s convenience, we list here the results of the integrals appearing in the expressions in our convention. As integrals $A_0(M^2)$ appear in the results in the second power, we need to evaluate also the term linear in $\epsilon$. This gives
\begin{equation}
A_0(M^2) = Q^{1-D} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2 - M^2} 
= -i \left[ \frac{M^2}{4\pi^2} \left( \frac{1 - \epsilon}{L_A - \frac{\epsilon}{2}} \left( L_A + 1 + \frac{\pi^2}{6} \right) \right) + O(\epsilon^2) \right],
\end{equation}
where
\begin{equation}
L_A = \log \frac{M^2}{Q^2} - \log 4\pi + \gamma - 1,
\end{equation}
with $Q$ being the renormalization scale and $\gamma$ the Euler-Mascheroni constant.

As was already stated, all of the simple brackets can be obtained from the double brackets using Eq. (D9). Therefore, we give here the result only for them. It reads
\begin{equation}
\{2M; M_a; M_b\} = \frac{1}{(4\pi)^4} \left( S(M) - f(a, b) \right) + O(\epsilon),
\end{equation}
where
\begin{equation}
S(M) = -\frac{1}{2\epsilon^2} + \frac{1}{\epsilon} \left( L + \frac{1}{2} \right) - \left( L^2 + L + \frac{1}{2} + \frac{\pi^2}{12} \right),
\end{equation}
\begin{equation}
a = \frac{M^2}{M^2}, \quad b = \frac{M^2}{M^2},
\end{equation}
and the function $f(a, b)$ is given by
\[
f(a, b) = \frac{1}{2} \log a \log b + \frac{1 - a - b}{2\sqrt{q}} \left[ \text{Li}_2 \left( -\frac{x_2}{y_1} \right) + \text{Li}_2 \left( -\frac{y_2}{x_1} \right) - \text{Li}_2 \left( -\frac{x_1}{y_1} \right) - \text{Li}_2 \left( -\frac{y_1}{x_2} \right) \right] \\
+ \text{Li}_2 \left( \frac{b - a}{x_2} \right) + \text{Li}_2 \left( \frac{a - b}{y_2} \right) - \text{Li}_2 \left( \frac{b - a}{x_1} \right) - \text{Li}_2 \left( \frac{a - b}{y_1} \right) \right],
\]

(D19)

\[
f(b, b) = -\frac{(2b - 1) \left( 2 \text{Li}_2 \left( \frac{x^2 - 4y}{\sqrt{1 - 4b}} \right) + \frac{\pi^2}{6} + \frac{1}{2} \log^2 \left( -\frac{x^2 - 4y}{\sqrt{1 - 4b}} \right) \right)}{\sqrt{1 - 4b}} - \frac{1}{2} \log^2(b).
\]

(D20)

In Eq. (D19) and Eq. (D20) the quantities \( q, x_{1,2}, \) and \( y_{1,2} \) are defined by

\[
q \equiv 1 - 2(a + b) + (a - b)^2,
\]

(D21)

\[
x_{1,2} \equiv \frac{1}{2} (1 + b - a \pm \sqrt{q}),
\]

(D22)

\[
y_{1,2} \equiv \frac{1}{2} (1 + a - b \pm \sqrt{q}).
\]

(D23)

In addition to Eq. (D20) giving the value of \( f(a, b) \) when \( a = b, \) it is helpful to note the additional special cases

\[
f(0, 0) = \frac{\pi^2}{6},
\]

(D24)

\[
f(0, b) = \text{Li}_2(1 - b),
\]

(D25)

\[
f(0, b^{-1}) = -\frac{1}{2} \log^2 b - f(0, b).
\]

(D26)

5. The kinematic structure of the self-energies

Rewriting Eq. (D11) and Eq. (D13) yields the expressions in terms of double brackets,

\[
\Sigma_1^f(0) = -\frac{1}{D - 3} \frac{m^4_{\Delta}}{2m^4_X} \{2m_{\Delta}; 0; 0\} - \frac{1}{2m^4_X} A_0(m^2_X)^2 + \frac{D - 1}{D - 3} \left( 2 \{2m_X; m_X; m_{\Delta}\} - \{2m_X; 0; m_{\Delta}\} \right)
\]

\[
+ \frac{D - 1}{D - 3} \frac{m^4_{\Delta}}{2m^4_X} \{2m_{\Delta}; m_X; 0\} - \{2m_{\Delta}; m_X; m_X\} - \{2m_{\Delta}; 0; 0\}
\]

\[
+ \frac{D - 1}{D - 3} \frac{m^4_{\Delta}}{2m^4_X} \{2m_X; 0; m_{\Delta}\} - \{2m_X; m_X; m_{\Delta}\} + \{2m_{\Delta}; m_X; m_X\} - \{2m_{\Delta}; 0; m_{\Delta}\},
\]

(D27)

\[
\Sigma_2^f(0) = \frac{D - 2}{2m^2_{\Delta} m^2_X} A_0(m^2_X) A_0(m^2_{\Delta}) - \frac{1}{4m^4_X} A_0(m^2_X)^2 + \frac{D - 2}{D - 3} \frac{m^2_{\Delta}}{2m^2_X} \{2m_X; 0; 0\} - \{2m_X; 0; m_{\Delta}\}
\]

\[
+ \frac{m^2_{\Delta}}{2m^4_X} \{2m_{\Delta}; m_X; 0\} - \{2m_{\Delta}; m_X; m_X\} + \frac{1}{D - 3} \frac{m^2_{\Delta}}{2m^2_X} \{2m_X; 0; m_{\Delta}\} - \{2m_X; m_X; m_{\Delta}\}
\]

\[
- \frac{D - 2}{2(D - 3)} \{2m_{\Delta}; m_X; 0\} - \frac{1}{2} \{2m_{\Delta}; m_X; m_{\Delta}\} - \{2m_{\Delta}; 0; m_{\Delta}\}
\]

\[
+ \frac{1}{D - 3} \frac{m^4_{\Delta}}{4m^4_X} \{2m_{\Delta}; m_X; 0\} - \{2m_{\Delta}; m_X; m_X\}.
\]

(D28)
Using the explicit expression for the double brackets, Eq. (D16), $\Sigma^P_1(0)$ and $\Sigma^P_2(0)$ are then finally found to be given by (where $s_i = \frac{m_\Delta}{m_X}$ as above)

\[
(4\pi)^4\Sigma(0)_1^P = -\frac{3}{2e} + 3L_X - 2 + \frac{s_i^2}{2} \left[ \frac{1}{2e^2} - \frac{1}{e} \left( L_{\Delta_i} - \frac{1}{2} \right) + \left( L_{\Delta_i} - L_{\Delta} + \frac{3}{2} + \frac{\pi^2}{12} \right) \right]
+ 3f(0, s_i) - 2f(1, s_i) + \frac{3}{2} s_i^2 [f(s_i^{-1}, s_i^{-1}) - 2f(0, s_i^{-1})]
+ 3s_i[f(1, s_i) - f(0, s_i) - f(s_i^{-1}, s_i^{-1}) + f(0, s_i^{-1})] + 2s_i^2 f(0, 0),
\]

\[
(4\pi)^4\Sigma(0)_2^P = \frac{3}{4e} - \frac{1}{2} \left[ L_X + 2L_{\Delta} - (L_{\Delta_i} - L_X) - 1 \right] - \frac{s_i^2}{4} \left[ \frac{1}{2e^2} - \frac{1}{e} \left( L_{\Delta_i} - \frac{1}{2} \right) + \left( L_{\Delta_i} - L_{\Delta} + \frac{3}{2} + \frac{\pi^2}{12} \right) \right]
- s_i^{-1}[f(0, 0) - f(0, s_i)] + f(0, s_i^{-1}) + f(1, s_i) - \frac{1}{2} f(0, s_i)
- \frac{s_i}{2} [f(s_i^{-1}, 0) - f(s_i^{-1}, s_i^{-1}) + f(0, s_i) - f(1, s_i)] - \frac{s_i^2}{4} [2f(0, s_i^{-1}) - f(s_i^{-1}, s_i^{-1})].
\]

(D30)

Note that the individual diagrams are UV divergent, with the divergent terms given by Eq. (14) and Eq. (15). However, as noted in Section III, their combination appearing in Eq. (11) yielding the total contribution to the RH neutrino mass matrix is finite and compact,

\[
I_3(s) = 1 + 2 \log s + s(1 - 2s) \log^2 s + 2(s^{-1} - 1) \left[ f(0, 0)(1 + s + s^2) + 2sf(1, s) + f(0, s)(1 + s)(1 + 2s) + s^2 f(s^{-1}, s^{-1}) \right].
\]

(D31)
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