Abstract

A version of the situation calculus in which situations are represented as first-order terms is presented. Fluents can be computed from the term structure, and actions on the situations correspond to rewrite rules on the terms. Actions that only depend on or influence a subset of the fluents can be described as rewrite rules that operate on subterms of the terms in some cases. If actions are bidirectional then efficient completion methods can be used to solve planning problems. This representation for situations and actions is most similar to the fluent calculus of Thielscher [10], except that this representation is more flexible and more use is made of the subterm structure. Some examples are given, and a few general methods for constructing such sets of rewrite rules are presented.

1 Introduction

The situation calculus permits reasoning about properties of situations that result from a given situation by sequences of actions [5]. In the situation calculus, situations (states) are represented explicitly by variables, and actions $a$ map states $s$ to states $do(a, s)$. Predicates and functions on a situation or state are called fluents. In some formalisms, a situation denotes a state of the world, specifying the values of fluents, so that two situations are equal if the values of all their fluents are the same. Other formalisms reserve the term situation for a sequence of states. A problem with the situation calculus or any formalism for reasoning about actions is the necessity to include a large number of frame axioms that express the fact that actions do not influence many properties (fluents) of a state. Since the early days of artificial intelligence research the frame problem has been studied, beginning with McCarthy and Hayes [5]. Lin [4] has written a recent survey of the situation calculus.

Reiter [8] proposed an approach to the frame problem in first-order logic that avoids the need to specify all of the frame axioms. The method of Reiter, foreshadowed by Haas [3], Pednault [6], Schubert [9] and Davis [2], essentially solves the frame problem by specifying that a change in the truth value of a fluent, caused by an action, is equivalent to a certain condition on the action. In this formalism, it is only necessary to list the actions that change each fluent, and it is not necessary to specify the frame axioms directly. If an action does not satisfy the condition, the fluent is not affected. In the following discussion the term “Reiter’s formalism” will be used for simplicity even though others have also contributed to its development. The fluent calculus [10] is another interesting approach to the frame problem. In this approach, a state is a conjunction of known facts.

Petrick [7] has adapted Reiter’s formalism to knowledge and belief and has also introduced the notion of a Cartesian situation that can decompose a situation into parts, in a way that appears to be similar to the aspect calculus. However, his formalism also considers a situation to include a sequence of states.

There are some problems with Reiter’s formalism, especially in its suitability for first-order theorem provers. In Reiter’s formalism, the successor state axiom for a fluent essentially says
that the fluent is true on a situation do(a, s) for fluent a and situation s if a is an action that makes the fluent true, or if the fluent was already true and a is not one of the actions that makes the fluent false. This requires one to know under what conditions an action changes the value of the fluent to "true" or "false." If for example the action is nondeterministic this may be difficult to know. Also, to formulate the successor state axiom, one needs a theory of equality between actions. If there are only a small number of actions that can make a fluent false, then Reiter’s formalism is concise because one need not list all of the actions that do not influence the fluent (the frame axioms for the fluent). However, if there are many actions (possibly thousands or millions) that influence the fluent, then this successor state axiom can become very long. Further, when converting Reiter’s approach to clause form, one needs an axiom of the form 'For all actions a, a = a_1 \lor a = a_2 \lor \cdots \lor a = a_n' where a_i are all the possible actions, as well as the axioms a_i \neq a_j for all i \neq j. If there are many actions, the first axiom will be huge. It is also difficult for many theorem provers to handle axioms of this form.

Even the successor state axiom itself, when translated into clause form, produces clauses having a disjunction of an equation and another literal. Using Φ(p, s) to denote the value of fluent p on situation s, a simple form of the successor state axiom would be

Φ(p, do(x, s)) \equiv [(Φ(p, s) \land (x \neq a_1) \land (x \neq a_2)) \lor (x = b_1 \lor x = b_2)]

where a_1 and a_2 are the only actions that can make p false and b_1 and b_2 are the only actions that make p true. Consider an even simpler form:

Φ(p, do(x, s)) \equiv [(Φ(p, s) \land (x \neq a_1)) \lor (x = b_1)]

The clause form of the latter is ¬Φ(p, do(x, s)) \lor Φ(p, s) \lor x = b_1, ¬Φ(p, do(x, s)) \lor x \neq a_1 \lor x = b_1, x \neq b_1 \lor Φ(p, do(x, s)), ¬Φ(p, s) \lor x = a_1 \lor Φ(p, do(x, s)). Such conjunctions of equations and inequations can be difficult for theorem provers to handle, especially if there are more actions in which case there would be more equations and inequations in the clauses.

2 Underlying Theory

We assume that there is some underlying set U of axioms in first-order logic concerning states, fluents, and actions. The semantics of this axiomatization will have domains for states and actions, with fluents mapping from states to various domains.

Actions in U are typically indicated by the letter a, possibly with subscripts, and fluents are typically indicated by the letters p and q, possibly with subscripts. F is the set of all fluents and A is the set of actions. States are denoted by s, t, and u, possibly with subscripts. The set of states is S.

If a is an action and s' is a state then do(a, s') is the result of applying action a in state s'. If p is a fluent then Φ(p, s') is the value of p on state s'. Thus fluents are essentially functions from states to various domains. If the value of a fluent is true or false, and it is not parameterized, then Φ(p, s') may be written as p(s') instead. The semantics (interpretation) of the underlying theory T is assumed to have sorts for fluents, states, and actions, in addition to possibly others.

3 Syntax

Term rewriting systems[1] have a simple syntax and semantics. In this paper a situation calculus based on first-order term rewriting is presented. Situations are represented by terms
and actions are represented by rewrite rules that operate on terms. This representation permits completion procedures \( \parallel \) to be used for planning if actions are bidirectional, and also makes use of the subterm structure of terms to separate actions whose effects are independent. This can improve the efficiency of the planning process. This representation is probably most closely related to the fluent calculus of Thieltscher \( \parallel \) among the approaches that have been proposed to date.

The basics of term rewriting systems are as follows.

### 3.1 Terms

The symbols \( f, g, h \) are function symbols, \( x, y, z \) are variables, \( r, s, t, u, v, w \) are terms, and \( a, b, c \) are individual constants. Also, \( i, j, k \) are variables that are intended to denote integers. \( F \) is the set of function symbols and \( X \) is the set of variables.

The arity of a function symbol is the number of arguments it takes. We assume there is a bound on the maximum arity of any function symbol. Terms are defined as follows: A variable or an individual constant is a term. Also, if \( f \) has arity \( n \) and \( t_1, \ldots, t_n \) are terms then \( f(t_1, \ldots, t_n) \) is a term. The set of terms over a set \( F \) of function symbols and a set \( X \) of variables is denoted \( T[F, X] \). A term is a ground term if it contains no variables. The notation \( s \equiv t \) for terms \( s \) and \( t \) indicates that the terms are syntactically identical. A term \( s \) is a subterm of \( f(t_1, \ldots, t_n) \) if \( s \equiv f(t_1, \ldots, t_n) \) or \( s \) is a subterm of \( t_i \) for some \( i \). Also, \( s \) is a proper subterm of \( f(t_1, \ldots, t_n) \) if \( s \) is a subterm of \( t_i \) for some \( i \). A maximal term in a set \( T \) of terms is a term in \( T \) that is not a proper subterm of any other term in \( T \). The size \( |s| \) of a term \( s \) is defined as follows: \( |x| = |c| = 1 \) for variables \( x \) and individual constants \( c \). Also, \( |f(t_1, \ldots, t_n)| = 1 + |t_1| + \cdots + |t_n| \).

A context is a term with one occurrence of \( \Box \) in it, such as \( f(a, \Box, b) \). This is written as \( t[\ ] \) and the result of substituting some term \( u \) for \( \Box \) is written as \( t[u] \). This notation can be extended to \( t[u_1, \ldots, u_n] \) indicating specific occurrences of the subterms \( u_1, \ldots, u_n \).

A substitution is a replacement of variables \( x_i \) by terms \( t_i \); this can be written as \( \{x_1 \leftarrow t_1, \ldots, x_n \leftarrow t_n \} \) or as \( \{x_1/t_1, \ldots, x_n/t_n\} \). Also, one can write \( t\{x_i \leftarrow t_i\} \) as \( t(x_i/t_i) \). Greek symbols such as \( \theta \) are commonly used for substitutions. The result of applying a substitution \( \theta \) to a term \( t \) is written as \( t\theta \). The term \( t\theta \) is called an instance of \( t \). Two terms \( r \) and \( s \) are unifiable if they have a common instance.

### 3.2 Term Rewriting Systems

A rewrite rule is of the form \( r \rightarrow s \) where \( r \) and \( s \) are terms and all variables in \( s \) also occur in \( r \). A term rewriting system is a finite or infinite set of rewrite rules. The relation \( \Rightarrow \) is defined by \( t_1 \Rightarrow_R t_2 \) iff there is a context \( t[\ ] \) such that \( t_1 \equiv t[r\theta] \) and \( t_2 \equiv t[s\theta] \) for some rewrite rule \( r \rightarrow s \) in \( R \) and some substitution \( \theta \). For example, if the rule \( f(g(x)) \rightarrow f(h(x, x)) \) is in \( R \), then \( f(f(h(a, b))) \Rightarrow_R f(f(h(a, h(a, b)))) \). The subterm occurrence \( r\Theta \) is called a redex. Also, \( \Rightarrow^* \) is the (reflexive) transitive closure of \( \Rightarrow \). A sequence \( t_1 \Rightarrow_R t_2 \Rightarrow_R t_3 \ldots \) is called a rewrite sequence. The system \( R \) is terminating if it has no infinite rewrite sequences. A term \( s \) is reducible for \( R \) if there is a term \( t \) such that \( s \Rightarrow_R t \); otherwise \( s \) is irreducible. If \( s \Rightarrow_R t \) and \( t \) is irreducible then one writes \( s \Rightarrow_R^* t \) and \( t \) is called a normal form of \( s \). A term rewriting system \( R \) is confluent if for all terms \( s, t_1, \) and \( t_2, \) if \( s \Rightarrow_R^* t_1 \) and \( s \Rightarrow_R^* t_2 \) then there is a term \( u \) such that \( t_1 \Rightarrow_R^* u \) and \( t_2 \Rightarrow_R^* u \). \( R \) is bidirectional or invertible if \( s \Rightarrow_R t \) implies \( t \Rightarrow_R s \) for all terms \( s, t \) in \( T[F, X] \).
4 Terminology

Instead of situations, we shall actually refer to states. Recall from above that the symbols \( s', t', u' \) refer to states and the set of states is \( S \). The set of actions is \( \mathcal{A} \) and \( a, b \) refer to actions. If \( s' \) is a state and \( a \) is an action then \( do(a, s') \) is a state obtained by performing action \( a \) on state \( s' \). There are also fluents, which map states onto various domains; \( \mathcal{F} \) is the set of fluents and the symbols \( p, q \) refer to fluents. If \( s' \) is a state and \( p \) is a fluent then \( \Phi(p, s') \) is the value of fluent \( p \) on state \( s' \). We assume that \( \mathcal{U} \) satisfies the fluent dependence condition:

\[ \chi(\{(p_1, v_1), \ldots, (p_n, v_n)\}) \text{ is true if there is a state } s' \in S \text{ such that } \Phi(p_i, s') = v_i \text{ for } 1 \leq i \leq n; \text{ otherwise it is false.} \]

Based on \( \mathcal{U} \) we construct a term-rewriting system \( R_\mathcal{U} \) to simulate the actions of \( \mathcal{U} \) using rewrite rules.

\[ T[F, X] \] be the set of first-order terms over a finite set \( F \) of function symbols and a set \( X \) of variables. The symbols \( r, s, t, u, v \) will represent terms.

Let \( T \subseteq T[F, X] \) be a set of first-order terms that represent states in \( \mathcal{U} \) and suppose \( \sigma \) is a function mapping such terms to states: \( \sigma : T \rightarrow S \). Thus for all \( t \in T \), \( \sigma(t) \) satisfies the fluent constraint in \( \mathcal{U} \). For now we assume all terms in \( T \) are ground terms so that for all \( t \in T \), for all fluents \( p \), \( \Phi(p, \sigma(t)) \) is defined.

We assume the following:

\[ \Phi(p, s') = \Phi(p, t') \text{ for all fluents } p \text{ then } do(a, s') = do(a, t') \text{ so that the effect of an action depends only on the fluents of the state. In fact, it may be best to assume that if } \Phi(p, s') = \Phi(p, t') \text{ for all fluents } p \text{ then } s' = t'. \]

There may be some combinations of fluents that do not correspond to states in the underlying theory \( \mathcal{U} \). Thus there may be a fluent constraint that specifies which combinations of fluents correspond to states in \( S \). In our examples the fluent constraint is typically always satisfied so that all combinations of fluents correspond to states in \( \mathcal{U} \).

4.1 Rewrite Rules for Situation Calculus

There are two kinds of rewrite rules in \( R_\mathcal{U} \): rearrangement rules that don’t affect the state and action rules that simulate actions on states. \( E_\sigma \) is the set of rearrangement rules and \( A_\sigma \) is the set of action rules. It is assumed that rules in \( E_\sigma \) are invertible. The rearrangement rules reformat the term without affecting the state; for example, they may permit a list of terms to be sorted in an arbitrary order.
Although $T$ is a set of ground terms, the rewrite rules in $R_U$ need not be ground rules, which will be clear from the examples. These rules are assumed to satisfy the following axioms:

If $\alpha \in E_\sigma \cup A_\sigma$ and $s \in T$ and $t$ is a term such that $s \Rightarrow \alpha t$
then $t \in T$ (so that $\sigma(t)$ also satisfies the fluent constraint for $U$).\hspace{1cm} (2)

For terms $s, t \in T$, (there exists $\alpha \in E_\sigma$ such that $s \Rightarrow \alpha t$) iff $\sigma(s) = \sigma(t)$.\hspace{1cm} (3)

Rules in $A_\sigma$ may correspond to more than one action in $A$. Thus we have the following axiom:

If $\alpha \in A_\sigma$ then for all $s, t \in T$, if $s \Rightarrow \alpha t$
then there exists $a \in A$ such that $\sigma(t) = do(a, \sigma(s))$.\hspace{1cm} (4)

In fact we assume that the action $a$ is computable, given $s, t$, and $\alpha \in A_\sigma$.

We also allow the possibility that actions in $A$ correspond to more than one rule in $A_\sigma$.

For all $s' \in S$ and all $a \in A$ and for all $s, t \in T$, if $\sigma(s) = s'$ and $\sigma(t) = do(a, \sigma(s'))$
then there are terms $u, v$ such that $s \Rightarrow \star E_\sigma u \Rightarrow A_\sigma v \Rightarrow \star E_\sigma t$.\hspace{1cm} (5)

Definition 7. A term-rewriting system $R_U$ represents the theory $U$ if there is a set $T$ of terms satisfying equation 1 and if $R_U = E_\sigma \cup A_\sigma$ where $E_\sigma$ and $A_\sigma$ are sets and there are functions $\sigma$ and $\hat{\Phi}$ such that $E_\sigma$ and $A_\sigma$ and the functions $\sigma$ and $\hat{\Phi}$ satisfy Definitions 4, 5, and 6, and Equations 2, 3, 4, and 5.

4.2 Planning Using Term Rewriting

Definition 8. For $s, t \in T$, $s \Rightarrow t$ if there are terms $u, v \in T$ such that $s \Rightarrow^* E_\sigma u \Rightarrow A_\sigma v \Rightarrow^* E_\sigma t$.

Theorem 9 (Planning Theorem). Suppose $s_1$ and $s_n$ are terms and there are states $s'_1, s'_2, \ldots, s'_n$ and actions $a_1, a_2, \ldots, a_{n-1}$ such that $s'_1 = \sigma(s_1), s'_n = \sigma(s_n)$, and for all $i$, $1 \leq i \leq n-1$, $s'_{i+1} = do(a_i, s_i)$. Then $s_1 \Rightarrow^* s_n$. The converse is also true.

Proof. By Definition 8 and Equation 5. The converse follows from Equations 3 and 4. \hspace{1cm} $\triangleright$

Thus to see if it is possible to reach $s'_n$ from $s'_1$ by a sequence of actions in $U$, one can construct terms $s_1$ and $s_n$ as in the theorem and test if $s_1 \Rightarrow s_n$.

Corollary 10. With conditions as in the planning theorem, if $A_\sigma$ is invertible, then $s_1 \Rightarrow^* s_n$.

In this case, rewrite strategies such as completion and unfailing completion \textsuperscript{11} can be used to test if $s_1 \Rightarrow^* s_n$. If it is shown that $s_1 \Rightarrow^* s_n$ then a plan (sequence of actions) can be extracted from the proof. This plan may not be optimal, but it may be possible to optimize it after it is found.
To extract a plan from a proof that \( s_1 \overset{*}{\rightarrow} s_n \), the notation \( sR^*_U[a_1 \cdots a_k \Rightarrow t] \) can be used indicating a sequence of actions leading from \( \sigma(s) \) to \( \sigma(t) \). These actions can be carried along in the completion procedure. We have the following rules:

\[
\text{If } sR^*_U[a_1 \cdots a_m \Rightarrow s] \text{ and } sR^*_U[a_{m+1} \cdots a_n \Rightarrow t] \text{ then } sR^*_U[a_1 \cdots a_n \Rightarrow t]. \tag{6}
\]

\[
\text{If } sR^*_U[a_1 \cdots a_k \Rightarrow t] \text{ then } tR^*_U[a_k \cdots a_2 a_1 \Rightarrow s] \text{ assuming } A_\sigma \text{ is invertible.} \tag{7}
\]

\[
sE^*_U[\epsilon \Rightarrow t] \text{ for all } s, t \text{ in } T \text{ where } \epsilon \text{ is the empty sequence.} \tag{8}
\]

If \( \sigma(t) = do(a, \sigma(s)) \) and \( s \Rightarrow_{A_\sigma} t \) then \( sR^*_U[a \Rightarrow t]. \tag{9} \)

However, there are some problems with this approach. One problem is that the same rewrite relation between \( s \) and \( t \) may be derived with more than one action sequence, complicating the search. Also, there may be more than one action corresponding to a given rewrite rule in \( A_\sigma \).

In fact, it’s not necessary to carry along sequences of actions. From a proof that \( s_1 \overset{*}{\rightarrow} s_n \) it is possible to construct a sequence \( s_1, s_2, \ldots, s_n \) of ground terms such that \( s_i \Rightarrow_{Rt} s_{i+1}, \ 1 \leq i \leq n-1 \). Then by repeated use of the decidability result following Equation \([3]\) one can find a sequence \( a_1, a_2, \ldots, a_{n-1} \) of actions such that \( \sigma(s_{i+1}) = do(a_i, \sigma(s_i)) \) for all \( i, 1 \leq i \leq n-1 \).

**Definition 11.** The constraint predicate \( \hat{\chi} \) is defined so that \( \hat{\chi}(t) \) is true for \( t \in T[F, X] \) if \( \chi(\{(p, \Phi(p, \sigma(t))) : p \in F\}) \) is true, otherwise \( \hat{\chi}(t) \) is false.

**Definition 12.** Let \( \text{Find} \) be a function which, given a set \( \{(p_1, v_1), \ldots, (p_n, v_n)\} \) of fluents and their values, finds a term \( t \) such that \( \Phi(p_i, \sigma(t)) = v_i \) for all \( i \) if such a term exists, else returns ‘none.’ We assume that the function \( \text{Find} \) is computable. There may be constraints on the fluents so that some combinations of fluents do not correspond to any state in \( S \) because of the fluent constraint for \( U \).

The procedure \( \text{Find} \) can be used to find the terms \( s_1 \) and \( s_n \) in the planning theorem if the values of all fluents on \( s'_1 \) and \( s'_n \) are known. The constraint predicate \( \hat{\chi} \) needs to be incorporated into planning if \( \chi \) is not always true. However, because of Equation \([2]\) if one finds a sequence \( s_1, s_2, \ldots, s_n \) of ground terms such that \( s_i \Rightarrow_{Rt} s_{i+1}, 1 \leq i \leq n-1 \), then all the terms \( s_i, 1 \leq i \leq n \) satisfy the constraint \( \hat{\chi} \).

So far the effect of rewrite rules operating on proper subterms of a term \( t \) in \( T \) has not been discussed, but this can be significant.

### 5 Examples

Some examples will illustrate the properties of this approach to the situation calculus. In these examples, the underlying theory \( U \) is described informally. The approach used for
rewriting is unfailing completion [H], which in the limit produces a term rewriting system that is confluent by ordered rewriting, we assume that \( R_u \) is bidirectional. Given a term \( s \) representing a starting state and a term \( t \) representing a goal state, both \( s \) and \( t \) are rewritten to a common term \( u \) using the limiting rewriting system and ordered rewriting. Then the plan to get from \( s \) to \( t \) is obtained from the rewrite sequence \( s \Rightarrow^* u \Leftarrow^* t \). This approach typically produces plans quickly but they are not always optimal. Frequently the plan produced can be made shorter by local optimizations.

### 5.1 Switches Example

In this example, there are \( n \) switches which can be on or off. The state of the switches is represented by a term in \( T \) of the form \( f(x_1, x_2, \ldots, x_n) \) where all \( x_i \) can be “on” or “off”.

Each switch can be turned on or off. So there is a rewrite rule in \( R_u \)

\[
f(x_1, \cdots, x_{i-1}, \text{off}, x_{i+1}, \cdots, x_n) \rightarrow f(x_1, \cdots, x_{i-1}, \text{on}, x_{i+1}, \cdots, x_n)
\]

to turn the \( i^{th} \) switch on and a rule

\[
f(x_1, \cdots, x_{i-1}, \text{on}, x_{i+1}, \cdots, x_n) \rightarrow f(x_1, \cdots, x_{i-1}, \text{off}, x_{i+1}, \cdots, x_n)
\]

to turn it off.

This set of rules is already bidirectional. Now, suppose the problem is to get from \( f(x_1, \cdots, x_n) \) to \( f(y_1, \cdots, y_n) \). How will this be solved using completion and term rewriting?

The two rewrite rules will be oriented in one direction, depending on the termination ordering. Suppose that “off” is larger than “on” in this ordering. Then both \( f(x_1, \cdots, x_n) \) and \( f(y_1, \cdots, y_n) \) will rewrite to \( f(\text{on}, \cdots, \text{on}) \), leading to a rewriting sequence \( f(x_1, \cdots, x_n) \Rightarrow^* f(\text{on}, \cdots, \text{on}) \Leftarrow^* f(y_1, \cdots, y_n) \), so that the plan will be to turn all the switches \( x_i \) off that are on, and then turn on all the switches \( y_i \) that are off. This could result in a switch that is on in both starting and ending states to be turned off and then on again. This plan can be optimized by removing such pairs of actions, resulting in a reasonable plan.

In this example, all actions can be encompassed in the single rules \( \text{off} \rightarrow \text{on} \) and \( \text{on} \rightarrow \text{off} \). If these rules are used, then the same plan will be derived as before. Notice here that a single rewrite rule corresponds to multiple actions.

Consider the effect of permitting rules in \( R_u \) to be non-ground and to rewrite on proper subterms of terms in \( T \). If the rules had to be ground rules and rewrite the whole term, then they would be of the form \( f(a_1, \cdots, a_n) \rightarrow f(b_1, \cdots, b_n) \) where \( a_i, b_i \in \{\text{off, on}\} \) and one of the \( b_i \) differs from \( a_i \). There would be \( n \times 2^n \) rules. By allowing the rules to be non-ground there are \( 2n \) rules of the form \( f(x_1, \cdots, a_i, \cdots, x_n) \rightarrow f(x_1, \cdots, b_i, \cdots, x_n) \) where \( a_i, b_i \in \{\text{off, on}\} \) and \( a_i \neq b_i \). Each rule is of length \( n \) leading to an overall complexity that is quadratic. If we now allow rewriting proper subterms, just the rules \( \text{off} \rightarrow \text{on} \) and \( \text{on} \rightarrow \text{off} \) suffice. Similar comments about non-ground rules and rewriting subterms apply to the following examples.

Now consider a slightly different example in which the state is represented as \( f(x_1, \cdots, x_n, z) \) where \( z \) is \textit{true} if all switches are on and \textit{false} otherwise. This introduces a constraint, and makes it more difficult to express the actions. One possibility is to have \( 2^n \) ground rewrite rules that specify the settings of all \( n \) switches and the effect of turning one switch on or off; this may or may not affect \( z \). Of course it is better to avoid so many rules.

Another possibility is to have the terms in \( T \) only represent the state of the switches and not the value of \( z \). This corresponds to a term in \( T \) only encoding a subset of the fluenets, if the other fluenets are determined by this subset.
Still another possibility is to have rules of the form
\[ f(x_1, \cdots, x_{i-1}, \text{on}, x_{i+1}, \cdots, x_n, z) \rightarrow f(x_1, \cdots, x_{i-1}, \text{off}, x_{i+1}, \cdots, x_n, \text{false}) \]
to turn the \( i^{th} \) switch off. To turn the \( i^{th} \) switch on one needs rules of the form
\[ f(x_1, \cdots, x_{i-1}, \text{off}, x_{i+1}, \cdots, \text{off}, \cdots, x_n, \text{false}) \rightarrow f(x_1, \cdots, x_{i-1}, \text{on}, x_{i+1}, \cdots, \text{off}, \cdots, x_n, \text{false}) \]
if another switch is off, and rules of the form
\[ f(\text{on}, \cdots, \text{on}, \text{off}, \cdots, \text{on}, \text{false}) \rightarrow f(\text{on}, \cdots, \text{on}, \text{on}, \cdots, \text{on}, \text{true}) \]
if all other switches are on.

Finally, this theory could be represented by constrained rewrite rules of the form
\[ f(x_1, \cdots, \text{off}, \cdots, x_n, z) \rightarrow f(x_1, \cdots, \text{on}, \cdots, x_n, z') \]
with the constraint that \( z' = \text{true} \) iff all \( x_i \) are on.

Another representation (for the problem without \( z \)) is to use a term \( f(g_1(x_1), g_2(x_2), \cdots, g_n(x_n)) \) to represent the state of the switches, where the \( g_i \) identify which switch is referred to and \( x_i \) gives its state, on or off. This permits the rewrite rules to be of the form \( g_i(\text{off}) \rightarrow g_i(\text{on}) \) and \( g_i(\text{on}) \rightarrow g_i(\text{off}) \), so that each rewrite rule refers to a different action.

### 5.2 Tower of Hanoi

For this example, there are \( n \) disks of different sizes on three pegs. On each peg the disks have to be in order of size, with the largest disk on the bottom. Only the top disk on a peg can be moved from one peg to another, and it has to be the smallest disk on the peg it is moved to. The problem is to rearrange the disks on the pegs, typically moving all of them from one peg to another.

The optimal sequence to move \( n \) disks from peg \( i \) to peg \( j \), for \( i \neq j \), consists of moving the \( n-1 \) smallest disks from peg \( i \) to peg \( k \), \( k \neq i \) and \( k \neq j \), then moving the largest disk from peg \( i \) to peg \( j \), then moving the \( n-1 \) smallest disks from peg \( k \) to peg \( j \). If \( n = 1 \) this consists of one move, so the general sequence has \( 2^n - 1 \) moves.

For this problem, terms in \( T \) can be of the form \( f(x_1, x_2, \cdots, x_n) \) where each \( x_i \) is either 1, 2, or 3 depending on which peg the disks are on, and \( x_1 \) refers to the largest peg, \( x_2 \) to the next largest, and so on.

The rules in \( R_T \) are of the form
\[ f(i, j, j, \cdots, j) \rightarrow f(k, j, j, \cdots, j) \]
where \( i, j, \) and \( k \) are distinct elements of the set \( \{1, 2, 3\} \) and also to move other disks, rules of the form
\[ f(x_1, x_2, \cdots, i, j, \cdots, j) \rightarrow f(x_1, x_2, \cdots, k, j, j, \cdots, j) \]
again where \( i, j, \) and \( k \) are distinct elements of \( \{1, 2, 3\} \). The \( i^{th} \) largest disk can only move from peg \( i \) to peg \( k \) if all the smaller disks are on peg \( j \), because otherwise, a smaller disk will be on peg \( i \) or \( k \), preventing the move. For example, consider the largest disk. To move it from peg 1 to peg 2, there can’t be any smaller disks on peg 1 because they would be on top of it, and only the top disk can be moved. There also can’t be any smaller disks on peg 2, because then a larger disk would be put on top of a smaller one, which is not permitted.

Now consider the smallest disk. It is always on top of one of the piles, and at any time it
can move to any other pile. In general, a disk is only constrained in moving by disks that are smaller than it is.

Completing this set of rules generates rules of the form

\[
f(x_1, x_2, \cdots, x_i, x_{i+1}, \cdots, x_n) \rightarrow f(x_1, x_2, \cdots, x_i, 1, 1, \cdots, 1)
\]

for all \( j \in \{1, 2, 3\} \) assuming a lexicographic ordering with \( 3 > 2 > 1 \). Then to get from one arrangement of disks to another, a plan will be generated to move all disks to peg 1, then move them back to the pegs in the goal situation. This is not optimal, but again, it may be possible to optimize this plan.

Another representation is to use the term \( f(x_1, f(x_2, \cdots, f(x_n, \perp \cdots)) \) instead of \( f(x_1, x_2, \cdots, x_n) \). This permits rewrite rules to be applied to proper subterms of the terms in \( T \).

For this problem, all actions are invertible. Consider the system with three disks; then there are the following rules:

\[
\begin{align*}
f(1, \perp) & \leftrightarrow f(2, \perp) \\
f(1, \perp) & \leftrightarrow f(3, \perp) \\
f(2, \perp) & \leftrightarrow f(3, \perp) \\
f(1, f(3, \perp)) & \leftrightarrow f(2, f(3, \perp)) \\
f(1, f(2, \perp)) & \leftrightarrow f(3, f(2, \perp)) \\
f(2, f(1, \perp)) & \leftrightarrow f(3, f(1, \perp)) \\
f(1, f(3, f(3, \perp))) & \leftrightarrow f(2, f(3, f(3, \perp))) \\
f(1, f(2, f(2, \perp))) & \leftrightarrow f(3, f(2, f(2, \perp))) \\
f(2, f(1, f(1, \perp))) & \leftrightarrow f(3, f(1, f(1, \perp)))
\end{align*}
\]

The rules will be oriented \( f(3, \perp) \rightarrow f(2, \perp) \rightarrow f(1, \perp) \). Then using these rules, the next three rules will become

\[
\begin{align*}
f(1, f(1, \perp)) & \leftrightarrow f(2, f(1, \perp)) \\
f(1, f(1, \perp)) & \leftrightarrow f(3, f(1, \perp)) \\
f(2, f(1, \perp)) & \leftrightarrow f(3, f(1, \perp))
\end{align*}
\]

which requires two rewrites for each rule, one on each side. The rule \( f(1, f(1, \perp)) \leftrightarrow f(3, f(1, \perp)) \) corresponds to the action sequence “Move the small disk to peg 2, then move the next smallest disk to peg 3, then move the small disk to peg 1.’ Similarly, rewriting using these six rules, the last three rules become

\[
\begin{align*}
f(1, f(1, f(1, \perp))) & \leftrightarrow f(2, f(1, f(1, \perp))) \\
f(1, f(1, f(1, \perp))) & \leftrightarrow f(3, f(1, f(1, \perp))) \\
f(2, f(1, f(1, \perp))) & \leftrightarrow f(3, f(1, f(1, \perp)))
\end{align*}
\]

which requires four rewrites on each rule, two on each side. In general completing the system in this way requires a quadratic number of rewrite operations even though the optimal action sequence requires an exponential number of actions. Finally, rewriting the starting term and the goal term to a common term requires a number of rewrites that is linear in \( n \).

If the problem is to show \( f(2, f(2, f(2, \perp))) \leftrightarrow f(3, f(3, f(3, \perp))) \) then both sides simplify to \( f(1, f(1, f(1, \perp))) \). The plan will then move all the disks from peg 2 to peg 1, and then move them all from peg 1 to peg 3. This is not optimal, but perhaps it can be optimized by local transformations. However, with a different ordering, the plan can be improved. If the ordering has \( 1 > 2 > 3 \) then the term \( f(2, f(2, f(2, \perp))) \) will be rewritten to \( f(3, f(3, f(3, \perp))) \) so the plan will move the disks from peg 2 to peg 3.
A closely related problem is to have a number of independent Towers of Hanoi; these can be represented by a term $g(t_1, t_2, \cdots, t_m)$ where $t_i$ is a term representing the state of the $i^{th}$ Tower of Hanoi problem.

5.3 Crossing a river

Here the problem is to arrange a group of people in some specified way on both banks of a river. There is a bridge across the river with three intermediate locations. If two people meet going opposite directions, they cannot cross each other, so one has to back up. This can be represented by a term $f(t_1, t_2, t_3, t_4, t_5)$ where the $t_i$ are lists of people that are at the given location. $t_1$ is the left bank, $t_5$ is the right bank, and $t_2, t_3,$ and $t_4$ are the intermediate locations on the bridge. $t_2, t_3,$ and $t_4$ can have at most one person at a time.

The actions are to sort the lists in $t_1$ and $t_5$ by exchanging adjacent elements (which is an action in $E_\sigma$, and to move an element at the head of the list from $t_i$ to $t_{i+1}$, $1 \leq i \leq 4$ and from $t_{i+1}$ to $t_i$, $1 \leq i \leq 4$. A person can be moved to $t_2, t_3$, or $t_4$ only if these locations are empty, but a person can be moved to $t_1$ or $t_5$ at any time.

A sample configuration would be $f(g(2, g(3, \bot)), \bot, g(4, \bot), g(5, \bot), g(1, \bot))$ indicating that persons 2 and 3 are on the left bank, person 1 is on the right bank, and persons 4 and 5 are crossing.

With a lexicographic ordering by the lengths of the lists $t_i$, given a problem of transforming $f(s_1, \cdots, s_5)$ to $f(t_1, \cdots, t_5)$, completion would result in a plan to move everyone to the right bank and then move them back to where they should be in the lists $t_i$. Again, it might be possible to apply local optimizations to this plan to make it more efficient.

5.4 Blocks world

In this domain, there are a fixed number of positions, each having a tower of blocks. The blocks can be piled in any order, and at any time the top block in any tower can be moved to top of any other tower. Then one wants a plan to transform some specified starting state to a goal state.

This can be represented by a list $f((s_1, x_1), f((s_2, x_2), \cdots, f((s_n, x_n), \bot) \cdots))$ where the $s_i$ are lists of blocks and the $x_i$ are their locations. The top block on a list $s_i$ appears first, then the blocks underneath it. The list of blocks is represented by $g(b_1, g(b_2, \cdots))$ where $b_1$ is the top block and $b_2$ is next under it, and so on. The actions include permuting the lists of blocks: $f(t_1, f(t_2, x)) \rightarrow f(t_2, f(t_1, x))$ to exchange adjacent towers of blocks (an action in $E_\sigma$) and an action $f((g(b_1, s_1), x_1), f((s_2, x_2), z)) \rightarrow f((s_1, x_1), f((g(b_1, s_2), x_2), z))$ to move the top block $b_1$ from the tower $g(b_1, s_1)$ at $x_1$ to the tower $s_2$ at $x_2$.

For this example, it’s not clear what kind of a plan the completion approach would generate, or what would be a suitable ordering. However, if the ordering is lexicographic by the sizes of the towers, then the plan will rewrite both starting and goal state terms to terms $t_1$ and $t_2$ with all blocks in one large tower. Then the blocks in these towers will be permuted to go from $t_1$ to $t_2$; this will be done using rules generated during completion.

6 Rewriting on Subterms

We show formally that sets $R_U$ of rewrite rules satisfying the conditions specified do exist for many theories $U$, and give an idea how they can be constructed. First, without using rewriting on proper subterms and using only ground terms, it is always possible to construct $R_U$. The following definition of $R_U$ only includes rules that rewrite an entire term in $T$. 


Definition 13. Let $U$ be any situation calculus satisfying the fluent dependence condition of Definition 7 let $T$ be a set of ground terms, and let $\sigma$ be a function satisfying Equation 7. Let $A_\sigma^0$ be a set of rules $\alpha_L \rightarrow \alpha_R$ where $\alpha_L, \alpha_R \in T$ are terms such that there are states $s, t$ in $U$ and an action $a \in A$ such that $\sigma(\alpha_L) = s$ and $\sigma(\alpha_R) = t$ and $t = do(a, s)$. One such rule is chosen for each $\alpha_L$ and each action $a$. Let $E_\sigma$ be some set of rules satisfying Equations 2 and 3. Let $R_\sigma^0$ be $A_\sigma^0 \cup E_\sigma$.

Theorem 14. $R_\sigma^0$ represents the theory $U$.

Proof. This is straightforward from the definitions. Equation 4 is a direct consequence of the construction. Equation 5 follows because for each $\alpha_L$ and each action $a$, one $\alpha_R$ is chosen such that $\sigma(\alpha_R) = do(a, \sigma(\alpha_L))$ but all such $\alpha_R$ are $E_\sigma$ equivalent by Equation 3 and the fluent dependence condition of Definition 1.

This construction however generally produces a huge term-rewriting system; the number of rules is at least as large as the number of states in $S$. It is helpful to understand in general how this number can be reduced.

The construction of $R_\sigma^0$ can be made more effective if one permits rewriting on proper subterms of terms in $T$, as shown in the examples. This corresponds to encoding frame axioms of $U$ because the fluents that depend on the term structure outside of the rewritten subterm will not change. It also may help to allow $R_\sigma^0$ to contain non-ground rules. These possibilities are considered in the following results. The first two definitions are interesting but are not used in the succeeding results.

Definition 15. Given a context $u$ over $T[F, X]$, $T|_u$ is the set of terms $t$ such that $u[t] \in T$.

Definition 16. A term $t$ in $T[F, X]$ is uniform for $T$ if for all contexts $u$ and $u'$ over $T[F, X]$, if $u[t] \in T$ and $u'[t] \in T$ then $T|_u = T|_{u'}$.

The following definition essentially specifies which rewrite rules can be used to represent actions in $U$.

Definition 17. Suppose $U$ is a situation calculus satisfying the fluent dependence condition of Definition 7 suppose $T$ is a set of ground terms, and let $\sigma$ be a function satisfying Equation 7. Then a term $t$ in $T[F, X]$ is an action support and the rule $t \rightarrow t'$ is an action rule if for all contexts $u$ over $T[F, X]$, if $u[t] \in T$ then there is a term $t'$ with $u[t'] \in T$ and an action $a \in A$ such that $\sigma(u[t']) = do(a, \sigma(u[t])).$

The next definition attaches actions and terms to action supports and action rules. More than one action and term can be attached to the same action rule.

Definition 18. Suppose $U$ is a situation calculus satisfying the fluent dependence condition of Definition 7 suppose $T$ is a set of ground terms, and let $\sigma$ be a function satisfying Equation 7. Suppose $u[t] \in T$ and $t$ is an action support. Suppose $t'$ is a term in $T[F, X]$ such that $t \rightarrow t'$ is an action rule and $\sigma(u[t']) = do(a, \sigma(u[t])).$ Then $t$ is an action support for action $a$ and term $u[t]$ and $t \rightarrow t'$ is an action rule for action $a$ and term $u[t]$.

The following definition specifies a way of choosing rewrite rules for $R_\sigma^0$ that will tend to choose rules that rewrite small subterms of terms in $T$.

Definition 19. Suppose $U$ is a situation calculus satisfying the fluent dependence condition of Definition 7 suppose $T$ is a set of ground terms, and let $\sigma$ be a function satisfying Equation 7. Let $A_\sigma^1$ be a set $\alpha_L \rightarrow \alpha_R$ of action rules for actions $a \in A$ and terms $u \in T$ such that...
one such rule is chosen for each action a and each term u. The rule that is chosen is one such that α_L is a minimal size action support for the action a and the term u. Let $E_\sigma$ be some set of rules satisfying Equations 2 and 3. Let $R^1_\mathcal{U}$ be $A^1_\alpha \cup E_\sigma$.

▶ **Theorem 20.** $R^1_\mathcal{U}$ represents the theory $\mathcal{U}$.

**Proof.** Again, this is straightforward from the definitions. This set of rules may be much smaller than $R^0_\mathcal{U}$ because of the use of minimal action support terms. The choices of which action support term to use and which term $\alpha_R$ to use do not matter, because of the fluent dependence condition of Definition 1.

In this respect, the switches example is interesting because a given rewrite rule can express more than one action. The rule $f \to g$, for example, can express turning on any of the switches, depending on where it is used in a term in $\mathcal{T}$.

The next definition lifts $R^1_\mathcal{U}$ to a possibly non-ground term rewriting system $R^2_\mathcal{U}$ that may be much more compact.

▶ **Definition 21.** Suppose $\mathcal{U}$ is a situation calculus satisfying the fluent dependence condition of Definition 1. Let $\mathcal{T}$ be a set of ground terms, and let $\sigma$ be a function satisfying Equation 5. Let $A^2_\beta$ be a set of possibly non-ground rules $\alpha_L \to \alpha_R$ such that for all ground instances $\alpha_L\beta$ of $\alpha_L$ with $\alpha_L\beta \in \mathcal{T}$, $\alpha_R\beta \in \mathcal{T}$ also and $\alpha_L\beta \to \alpha_R\beta \in A^1_\alpha$. Let $E_\sigma$ be some set of rules satisfying Equations 2 and 3. Let $R^2_\mathcal{U}$ be $A^2_\beta \cup E_\sigma$.

▶ **Theorem 22.** $R^2_\mathcal{U}$ represents the theory $\mathcal{U}$.

**Proof.** Again, this is straightforward from the definitions. The rewrite relation for $R^2_\mathcal{U}$ on $\mathcal{T}$ is the same as that for $R^1_\mathcal{U}$. The set $R^2_\mathcal{U}$ of rules may be much smaller even than $R^1_\mathcal{U}$ because of the use of non-ground rules.

For the first two examples, $E_\sigma$ is empty. For the river crossing example, $E_\sigma$ consists of rules that exchange adjacent elements of the lists in $t_1$ and $t_2$. For the blocks world example, $E_\sigma$ consists of the rules exchanging adjacent elements of the list of towers of blocks. The rules $R^1_\mathcal{U}$ given for the Tower of Hanoi problem are $R^1_\mathcal{U}$. The rules given for the switches problem are also $R^1_\mathcal{U}$. For both of these examples, nothing can be gained by using non-ground rules. For the crossing the river example, using subterms helps to select the lists in $t_1$ and $t_2$ and the use of non-ground terms also helps in general so this is an example where $R^2_\mathcal{U}$ is better than $R^1_\mathcal{U}$. For the blocks world example, both the use of subterms and the use of non-ground terms contribute to making $R^2_\mathcal{U}$ more concise than $R^1_\mathcal{U}$ and $R^0_\mathcal{U}$.

Now we examine which properties a term needs in order to be an action support term, in order to gain more understanding of the construction of $R_\mathcal{U}$.

▶ **Definition 23.** Suppose $\mathcal{F}'$ is a subset of $\mathcal{F}$. A subset $\mathcal{T}'$ of $\mathcal{T}$ is $\mathcal{F}'$-limited if for all pairs $t_1, t_2 \in \mathcal{T}'$, and all contexts $u$, if $u[t_1]$ and $u[t_2]$ are two terms in $\mathcal{T}$, then $\Phi(p, u[t_1]) = \Phi(p, u[t_2])$ for all $p \in \mathcal{F} \setminus \mathcal{F}'$. (The choice of $t_1$ or $t_2$ can only influence fluents in $\mathcal{F}'$.) The subset $\mathcal{T}'$ of $\mathcal{T}$ is $\mathcal{F}'$-expressive if for all states $s \in \mathcal{S}$ such that $\Phi(p, u[t_1]) = \Phi(p, s)$ for all $p \in \mathcal{F} \setminus \mathcal{F}'$, there is a term $t_3 \in \mathcal{T}'$ such that $\sigma(u[t_2]) = s$. (The choice of a term in the set can produce a full combination of fluents for that position in $u$, in some sense.)

▶ **Definition 24.** An action $a \in A$ is $\mathcal{F}$-limited if for all states $s \in \mathcal{S}$, for all fluents

$p \in \mathcal{F} \setminus \mathcal{F}'$, $\Phi(p, s) = \Phi(p, do(a, s))$ (a does not change any fluents outside of $\mathcal{F}'$) and if for all states $s_1, s_2 \in \mathcal{S}$, if $\Phi(p, s_1) = \Phi(p, s_2)$ for all $p \in \mathcal{F}'$ then $\Phi(p, do(a, s_1)) = \Phi(p, do(a, s_2))$ for all $p \in \mathcal{F}'$. (The action $a$ does not depend on any fluents outside of $\mathcal{F}'$).
The following result gives sufficient conditions for a rewrite rule $t \rightarrow t'$ to exist that represents an action, at least when it is applied to the term $t$.

**Theorem 25.** If $u[t]$ is a term in $T$, the set of $t'$ such that $u[t'] \in T$ is weakly $F'$ expressive for some $F' \subseteq F$, then for every $F'$ limited action $a \in A$ and every term $t$ such that $u[t] \in T$, there is a term $t'$ such that $u[t'] \in T$ and $\sigma(u[t']) = do(a, \sigma(u[t]))$.

**Proof.** The $F'$ expressive condition guarantees that such a term $t'$ exists. If the action $a$ is not $F'$ limited then it would have to change some of the term structure outside the occurrence of $t$.\hfill\square

The following definition and theorem give a weak necessary condition for a rewrite rule on a subterm to exist that expresses an action.

**Definition 26.** A set of terms is weakly $F'$ expressive for $F' \subseteq F$ if there is a context $u$ with $u[t] \in T$ for some term $t$ and there are at least two terms $t_1$ and $t_2$ such that $u[t_1]$ and $u[t_2]$ are both in $T$, $\hat{\Phi}(p, u[t_1]) = \hat{\Phi}(p, u[t_2])$ for all $p \in F \setminus F'$, but $\hat{\Phi}(p, u[t_1]) \neq \hat{\Phi}(p, u[t_2])$ for some $p \in F'$.

**Theorem 27.** Suppose $a \in A$, $a$ is $F'$ limited for some $F' \subseteq F$, $u[t] \in T$ for some term $u[t]$, there is a term $t'$ such that $\sigma(u[t']) = do(a, \sigma(u[t]))$, and for some $p \in F'$, $\hat{\Phi}(p, u[t']) \neq \hat{\Phi}(p, u[t])$. Then the set of terms $t$ such that $u[t] \in T$ is weakly $F'$ expressive.

**Proof.** For $t_1$ and $t_2$ one takes the terms $t$ and $t'$ of the theorem. These terms do not agree on all fluents because this is stated in the theorem, and it seems reasonable for an action to change at least one fluent in many cases. The terms $t$ and $t'$ agree on all fluents not in $F'$ because the action $a$ is $F'$ limited.\hfill\square

## 7 Conclusion

After a brief survey of the situation calculus, term-rewriting systems are introduced and situation calculus concepts are presented. Next an approach to encoding the situation calculus by term rewriting is presented. A general result for planning using this approach is given. Four examples illustrate the properties of this approach. General methods for constructing term rewriting systems embodying this approach are given, and special attention is given to rewriting on subterms, which corresponds to encoding frame axioms in the underlying theory, and lifting the rewrite rules to non-ground rules. Finally, some results are given about sufficient conditions and a necessary condition for a rewrite rule to exist that represents an action on a specific term.

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