Non–Commutative Field Theories beyond Perturbation Theory

W. Bietenholz 1, F. Hofheinz 1,2 and J. Nishimura 3

1 Humboldt Universität zu Berlin, Invalidenstr. 110, D–10115 Berlin, Germany
2 Freie Universität Berlin, Arnimallee 14, D–14195 Berlin, Germany
3 Dept. Physics, Nagoya University, Nagoya 464–8602, Japan

Abstract: We investigate two models in non-commutative (NC) field theory by means of Monte Carlo simulations. Even if we start from the Euclidean lattice formulation, such simulations are only feasible after mapping the systems onto dimensionally reduced matrix models. Using this technique, we measure Wilson loops in 2d NC gauge theory of rank 1. It turns out that they are non-perturbatively renormalizable, and the phase follows an Aharonov-Bohm effect if we identify $\theta = 1/B$. Next we study the 3d $\lambda\phi^4$ model with two NC coordinates, where we present new results for the correlators and the dispersion relation. We further reveal the explicit phase diagram. The ordered regime splits into a uniform and a striped phase, as it was qualitatively conjectured before. We also confirm the recent observation by Ambjørn and Catterall that such stripes occur even in $d = 2$, although they imply the spontaneous breaking of translation symmetry. However, in $d = 3$ and $d = 2$ we observe only patterns of two stripes to be stable in the range of parameters investigated.

1 NC field theories as matrix models

NC field theory is highly fashionable, in particular due to its potential rôle as a low energy effective description of string and M theory. Here we study Euclidean NC field theory as such; for a discussion of the Wick rotation and the related issue of unitarity, see e.g. Ref. [1]. We assume two coordinates to obey the standard NC relation

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta \epsilon_{\mu\nu}. \quad (1)$$

This implies an uncertainty relation $\Delta x_\mu \Delta x_\nu = O(\theta)$, and therefore UV and IR divergences are mixed. That property complicates perturbative renormalization drastically [2]. We can also observe non-perturbative manifestations of UV/IR mixing, see below.

NC gauge theory can be formulated by a consistent use of the star product. Even for $U(1)$ a YM term appears in the gauge action,

$$S[A] = \frac{1}{4} \int d^2x \text{Tr}(F_{\mu\nu} \star F_{\mu\nu}), \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig(A_\mu \star A_\nu - A_\nu \star A_\mu). \quad (2)$$

A lattice formulation can be written down, but its use in simulations is hardly possible: already the generation of star gauge invariant link variables appears as a fatal obstacle.

§ corresponding author: hofheinz@physik.hu-berlin.de
However, $U(n)$ NC gauge theories can be mapped onto certain forms of the twisted Eguchi-Kawai model (TEK) (which is defined on a single site),

$$S_{\text{TEK}}[U] = -N\beta \sum_{\mu \neq \nu} Z_{\mu \nu} \text{Tr}(U_{\mu} U_{\nu} U_{\mu}^\dagger U_{\nu}^\dagger), \quad Z_{\mu \nu} = Z_{\nu \mu} = e^{2\pi ik/L} \quad (\mu < \nu, \; k \in \mathbb{N}, \; L = N^{2/d}),$$

where $U_{\mu}$ are unitary $N \times N$ matrices and $Z_{\mu \nu}$ is the twist. This mapping is based on “Morita equivalence”, which is exact at $N \to \infty$. At finite $N$ there is an exact mapping onto a NC gauge theory on a lattice with spacing $a$ in a finite volume, if we choose $k = (N + 1)/2$, which implies $\theta = Na^2/\pi$. After this mapping the system can indeed be simulated. Thus we by-pass the problems with a direct lattice formulation. The limit $N \to \infty$ corresponds to the simultaneous continuum and infinite volume limit (these limits are entangled due to UV/IR mixing).

The action of the NC $\lambda \phi^4$ model reads

$$S[\phi] = \int d^d x \left[ \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4} \phi \star \phi \star \phi \star \phi \right].$$

(3)

Here the star product does not affect the bilinear terms, so the strength of the self-coupling $\lambda$ also determines the extent of NC effects.

We consider $d = 3$ with a commutative time direction and two space coordinates obeying relation (1). Again we start from a lattice formulation — $\phi(t, \vec{x})$ is defined on a $T \times N^2$ lattice — and map it on a matrix model. Its action takes the form

$$S[\hat{\phi}] = \text{Tr} \sum_{t=1}^{T} \left[ \frac{1}{2} \sum_{\mu} \left( \Gamma_\mu \hat{\phi}(t) \Gamma_\mu^\dagger - \hat{\phi}(t) \right)^2 + \frac{1}{2} \left( \hat{\phi}(t + 1) - \hat{\phi}(t) \right)^2 + \frac{m^2}{2} \hat{\phi}^2(t) + \frac{\lambda}{4} \hat{\phi}^4(t) \right],$$

where $\hat{\phi}(t)$ are Hermitian $N \times N$ matrices. The “twist eaters” $\Gamma_\mu$ provide the spatial shift, since they obey the Weyl-’t Hooft commutation relation $\Gamma_\mu \Gamma_\nu = Z_{\nu \mu} \Gamma_\nu \Gamma_\mu$ (where $Z$ is the same twist as in the TEK describing NC gauge theory). The final formulation has a remarkable similarity to the corresponding model on a “fuzzy sphere”.

2 Numeric results

2.1 Gauge theory on a NC plane

We simulated pure 2d NC $U(1)$ gauge theory — for analytical work on this model, see Ref. [1]. The large $N$ limit at fixed $\beta$ coincides with the planar limit of the commutative theory, which was solved by Gross and Witten. It exhibits an exact area law for the Wilson loop. From this connection we identify the “physical area” of one lattice plaquette as $a^2 \approx 1/\beta$. However, in contrast to the planar limit we take the double scaling limit where $N \to \infty$, $\beta \to \infty$, so that $\theta \propto N/\beta$ is kept constant. Thus we can extrapolate to a NC continuum theory in an infinite volume.

The (analogue of the) square shaped Wilson loop in the TEK takes the form

$$W_{12}(I \times I) = Z_{12}^{I^2} \text{Tr}(U_{\mu}^I U_{\nu}^I U_{\mu}^\dagger U_{\nu}^\dagger) = W_{21}^*(I \times I),$$

(4)

\footnote{Including also the time in the NC relation seems more consistent from the gravity inspired motivation for NC geometry but it causes especially severe problems related to causality.}

\footnote{A corresponding matrix model formulation for 3d fermions was suggested in Ref. [12].}
and it corresponds to the Wilson loop in NC gauge theory. Note that the expectation value \( W(I) := \frac{1}{N} \langle W_{12}(I \times I) \rangle \) is complex. Fig. 1 shows its behavior in polar coordinates (for \( N/\beta = 32 \)). The large \( N \) double scaling reveals that the Wilson loop is indeed non-perturbatively renormalizable. At small area (relative to \( \theta \)) the result follows the Gross-Witten area law, hence in this regime the double scaling limit coincides with the planar limit of both, the commutative and the NC model. At larger areas, however, \( |W(I)| \) does not decay any further, but the phase \( \varphi \) increases linearly in the area. Comparison of different values for \( N/\beta \) shows in that regime the simple relation

\[
\varphi = \frac{A}{\theta} = AB \quad (A = (aI)^2 = \text{physical area})
\]

(5)
to hold to a very high precision. Here \( B \) is a magnetic field across the plane, which we introduce formally as \( B = 1/\theta \), in agreement with the Seiberg-Witten map \[14\] and with solid state applications \[15\]. This relation is illustrated in Fig. 2 (on the left).

We also found large \( N \) double scaling in some regime for the Wilson 2-point function and for the 2-point function of the Polyakov line (see Fig. 2 on the right), if a suitable (universal) wave function renormalization is applied \[2\].

Figure 1: The polar coordinates of the complex Wilson loop \( W(I) \) plotted against the physical area \( A = a^2I^2 \). At small areas it follows the Gross-Witten area law (solid line). At larger areas the absolute value does not decay any more, but the phase grows linearly.

Figure 2: On the left: The phase of the Wilson loop \( W(I) \) at \( N = 125 \) for various values of \( N/\beta \), corresponding to different values of \( \theta \). At large areas \( A > O(\theta) \) the phase \( \varphi \) agrees with the formula \( \varphi = A/\theta \), which is shown by straight lines. On the right: The Polyakov 2-point function at \( N = 195 \), after performing a universal wave function renormalization, which provides a double scaling regime.
2.2 The NC $\lambda \phi^4$ model in three dimensions

The NC $\lambda \phi^4$ model has been studied analytically with various techniques. As a qualitatively new feature (compared to the commutative world), it was conjectured that in part of the ordered regime (at strongly negative $m^2$, which roughly corresponds to low temperature) stripe patterns may dominate over the Ising-type uniform ordering. Gubser and Sondhi performed a self-consistent Hartree-Fock type one-loop calculation \cite{GubserSondhi}, which would be exact in the $O(N)$ model at large $N$. They conjectured that it could also apply to the $\lambda \phi^4$ model, which suggests a striped phase in $d = 4$ and $d = 3$, but not in $d = 2$ (in the dimensions $d < 4$ they considered an effective action as suggested by Brazovskii \cite{Brazovskii}). However, stripes did occur in 2d simulation results by Ambjørn and Catterall \cite{AmbjornCatterall}; we add some remarks on that case in the Appendix. In $d = 3$ and 4, Gubser and Sondhi predicted a first order phase transition from the uniform to a striped phase if $\theta$ increases in the ordered regime. At very strong $\theta$ they expect more complicated patterns (like rectangles or rhombi) to become stable.

Chen and Wu carried out a renormalization group analysis \cite{ChenWu} in $d = 4 - \varepsilon$. They predicted a striped phase for $\theta > 12/\sqrt{\varepsilon}$, which should therefore not occur in $d = 4$.

We considered this model in $d = 3$ with two NC coordinates, and simulated the corresponding matrix model (described in Section 1) at $T = N = 15$, 25, 35 and 45. Thus we obtained the explicit phase diagram shown in Fig. 3. It is based on the order parameter

$$M(k) = \frac{1}{NT} \max_{|\vec{n}|=k} \left| \sum_{t=1}^{T} \tilde{\phi}(t, \vec{p}) \right|, \quad \vec{p} = \frac{2\pi}{N} \vec{n}, \quad n_1, n_2 \in \{0, 1, \ldots, N-1\}.$$  

$(M(k))$ detects a uniform order at $k = 0$ and a pattern of two stripes at $k = 1$. Typical snapshots of ordered and disordered configurations at small and at large $\lambda$ are shown in Fig. 4. We did not find more complicated patterns to be stable, but we consider it as an open question if this may set in at larger $N$ or $\lambda$ (we recall that this corresponds to larger $\theta$). The order-disorder transition seems to be of second order, whereas the uniform-striped transition appears to be first order — it takes the factor $N^2$ on the axes to stabilize it. In order to localize the transitions accurately we also looked for peaks in the connected 2-point functions of $M(k)$; some examples were given in Ref. \cite{ChenWu}. 

![Figure 3: The phase diagram of the 3d NC $\phi^4$ theory. The connected points show the separation line between the disordered and the ordered regime, and the vertical lines mark the transition region between the uniformly ordered and the striped phase.](image)
Here we illustrate the different phases further by showing the spatial correlator

\[ C(\vec{x}) = \frac{1}{TN^2} \sum_{t=1}^{T} \langle \phi(t, \vec{0}) \phi(t, \vec{x}) \rangle \]  

(7)

in Fig. 5. It does not decay exponentially in the disordered phase, see Fig. 6 on the left. Hence NC effects are visible also in the disordered phase.

On the other hand, the temporal correlator in momentum space

\[ G(\tau, \vec{p}) = \frac{1}{TN^2} \sum_{t=1}^{T} \text{Re} \langle \tilde{\phi}(t, \vec{p})^* \tilde{\phi}(t + \tau, \vec{p}) \rangle \]  

(8)

does follow an exponential decay (resp. a \( \cosh \) behavior at finite \( T \)). Fig. 6 on the right is an example at \( \vec{p} = \vec{0} \). Varying \( \vec{p} \) we can identify the dispersion relation \( E(p) \), which we show in Fig. 7 still in the disordered phase. It follows closely the linear behavior \( E^2 = \vec{p}^2 + M_{\text{eff}}^2 \) (solid line, where \( M_{\text{eff}} \) is a free parameter). The deviation at large momenta is a trivial lattice artifact. However, we also observe a systematic jump at \( \vec{p} = \vec{0} \), only in the case where we are close to the striped phase. This confirms the energy minimum at \( k = N|\vec{p}|/(2\pi) = 1 \), i.e. the trend towards stripes rather than uniform ordering. Such irregularities in the dispersion at significant \( \theta \) may be of importance; note that a \( \theta \) deformed dispersion relation has also been postulated for photons [20].

Figure 5: The spatial correlator \( C(\vec{x}) \) defined in equation (7) at \( N^2 \lambda = 90 \) (a,b) and \( N^2 \lambda = 900 \) (c,d) at \( N = 45 \), against \( x_1 \). The circles and squares correspond to two orthogonal directions; in particular in plot (d) the circles (squares) show the correlator parallel (vertical) to the stripes.

\[ (a) \ N^2 m^2 = -45 \quad (b) \ N^2 m^2 = -225 \quad (c) \ N^2 m^2 = -360 \quad (d) \ N^2 m^2 = -945 \]
Figure 6: Correlation functions at $N = 35$ in the disordered phase (near the phase transition) at $N^2m^2 = -70, N^2\lambda = 490$. We show the spatial correlator \(^\square\) on the left, and the temporal correlator in momentum space \(^\blacksquare\) at $\vec{p} = 0$ on the right.

Figure 7: The dispersion relation $E^2(p^2)$ in the disordered phase at $N = 45$. We are close to the uniform phase $N^2m^2 = -22.5, N^2\lambda = 90$ (left) resp. close to the striped phase at $N^2m^2 = -280, N^2\lambda = 900$ (right). The solid line represents the continuum behavior in a Lorentz invariant theory.

3 Conclusions

We presented non-perturbative results for two types of NC field theories. Such theories are non-local in the range of $O(\sqrt{\theta})$. We observe basic deviations from the commutative world rather at large scales though, which can be understood from the UV/IR mixing.

Simulating the TEK we obtained results for 2d NC $U(1)$ gauge theory. The scaling behavior that we observed is the first evidence for the non-perturbative renormalizability of a NC field theory. The Wilson loop follows an area law at small area. At large area it can be described by the Aharonov-Bohm effect if we identify $B = 1/\theta$, which appears here as a dynamical relation (on tree level it was used before in the literature).

In the 3d $\lambda\phi^4$ model on the lattice we explored the phase diagram in the $m^2 - \lambda$ plane. In the ordered regime (at strongly negative $m^2$) we found at small $\lambda$ a uniform order of the Ising type, as in the commutative case. At larger $\lambda$ — which amplifies the NC effects — we observed a pattern of two stripes to be stable for $N = 15 \ldots 45$. It is an open question if multiple stripes etc. stabilize at larger values of $N$ or $\lambda$, as conjectured by Gubser and Sondhi. We observed the same behavior in $d = 2$, where we also verified up to $N = 45$ that only two stripes are ultimately stable (see Appendix). Still this implies the spontaneous breaking of translation symmetry, which is possible due to the non-locality \[^{18}\].

In the ordered regime, the spatial correlations are dictated by the dominant pattern: uniform as in the commutative case, or striped with strong correlation in the direction of the stripes and anti–correlation vertical to them. \[^3\] This agrees with the predictions in Ref. \[^{17}\]. In the disordered phase the spatial correlators deviate from the exponential

\[^{3}\text{The stripes that we observed were always parallel to one of the axes.}\]
decay. However, the correlators in momentum space do decay exponentially in time for all momenta.

That property allowed us to study the dispersion relation in the disordered phase: at small $\lambda$ also the dispersion behaves qualitatively as in the commutative model, but at large $\lambda$ there appears a jump at $|\vec{p}| = 0$ as a NC effect. Hence the energy minimum is at $|\vec{p}| = 2\pi/N$, in agreement with the trend towards two stripes. It remains to be clarified how this irregular dispersion behaves at large $N$, and if it can be related to the expected non-linear photon dispersion at $\theta > 0$; the latter makes experimentalists hope for a precise measurement of $\theta$ (for a discussion, see for instance Ref. [21]).

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A The NC $\lambda\phi^4$ model in $d = 2$

The occurrence of stripes in the ground state implies the spontaneous breaking of translation invariance. The case $d = 2$ is particularly interesting in this respect. Gubser and Sondhi argued based on the Mermin-Wagner Theorem that stripes cannot be stable in $d = 2$. However, Ambjørn and Catterall pointed out that this Theorem is in general not applicable to non-local theories, and in fact they did observe non-uniform patterns in their numeric results for the NC $\lambda\phi^4$ model in $d = 2$ [18] (where the two coordinates obey the NC relation (1)).

We also performed simulations in $d = 2$, and we fully agree that also there stripe vacua exist. Starting from hot configurations and proceeding in Metropolis steps which propose a full Hermitian matrix to be added to $\hat{\phi}(t)$ (with a coefficient tuned for a good acceptance rate), we find after about 500 steps a rich structure of patterns, where in general $p_1$ and $p_2$ are both non-zero, and often larger than $2\pi/N$. However, if we continue to thermalize much longer we see that only the two-stripe pattern is ultimately stable, just as in $d = 3$. This we tested again up to $N = 45$. The same can be seen if we switch to a more efficient algorithm, which updates the (conjugate pairs of) matrix elements one by one (this reduces the thermalization time by a factor $\propto 1/N$). Still the variety of meta-stable patterns may be of importance; we present some examples for them in Fig. 8. The Goldstone Theorem still holds for NC field theories [22], hence it is an interesting question to investigate the massless fluctuations related to translation symmetry.

Figure 8: Examples for meta-stable patterns in two dimensions at $N = 35$, $\lambda = 0.43$, $m^2 = -2$, which is deeply in the striped phase. We show the same sort of maps as in Fig. 7.

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4One might expect that the Theorem extends to theories, which are non-local in only a small range, which is invisible at very large scales. However, this argument is not adequate for NC models due to the UV/IR mixing.

5However, we verified carefully, both in $d = 2$ and in $d = 3$, that those two stripes are really stable; one consistently runs into them starting from all sort of hot configurations, and they show absolutely no trend to disappear in very long histories.
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