Level-dependent interpolatory Hermite subdivision schemes and wavelets

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Abstract

We study many properties of level-dependent Hermite subdivision, focusing on schemes preserving polynomial and exponential data. We specifically consider interpolatory schemes, which give rise to level-dependent multiresolution analyses through a prediction-correction approach. A result on the decay of the associated multiwavelet coefficients, corresponding to an uniformly continuous and differentiable function, is derived. It makes use of the approximation of any such function with a generalized Taylor formula expressed in terms of polynomials and exponentials.

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1 Introduction

Hermite subdivision schemes are iterative procedures that allow, through a refinement process, to generate curves from a given set of discrete data points, consisting of function and derivative values. Such data naturally occurs in applications of motion control

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where position, velocity, acceleration and even higher derivatives of the motion are computed on a discrete grid, cf. [20], and have then to be interpolated by the Numerical Control when the motion is actually performed.

The polynomial reproduction property of such schemes has been thoroughly investigated in the last years, cf. [3, 8, 9, 14, 19]. As in standard (non-Hermite) schemes, such a property is crucial for assuring not only the convergence of the scheme, but also the smoothness and the approximation order of its limit function [6, 12, 15].

Recently, some research has focused both on standard [4, 16] and Hermite [1, 2, 5, 25] schemes preserving not only polynomial, but also exponential data, that is, sequences of the form \( (e^{\lambda k} : k \in \mathbb{Z}) \). This generalization allows the generation of curves which also exhibit transcendental features. Such schemes necessarily have a level-dependent nature, which means that the subdivision operator varies at each step of the subdivision process.

The reproduction property of level-dependent schemes has been explored from the point of view of wavelet analysis [11, 27], where it translates into a vanishing moment property for both exponentials and polynomials. Using the strong connection between wavelets and subdivision schemes, [2] proposes a construction of Hermite multiwavelets and corresponding multiresolution analysis (MRA) with polynomial and exponential vanishing moments. This construction is based on the interpolatory Hermite schemes possessing the polynomial and exponential preservation property introduced in [1].

We remark that the vanishing moment property is very desirable in wavelet analysis, as it ensures compression capabilities of the wavelet system as long as the processed data contains many “small” negligible details and is of a certain smoothness otherwise. How small such details are depends on the decay of the wavelet coefficients and indeed, the decay rate can serve as a measure of smoothness of the underlying function.

The aim of this paper is threefold. We start by providing some basic results on level-dependent Hermite schemes in Sections 2 and 3. The main focus is on the properties of the limit functions of such schemes, which are the building blocks of the corresponding level-dependent MRA.

If such schemes possess the property of reproducing polynomial and exponential data, then the wavelet coefficients satisfy a generalized vanishing moment condition which is important for assuring sparse representations of any function \( f \in C^d(\mathbb{R}) \). This property is due to a certain decay rate of the wavelet coefficients as the scale increases. In order to prove such a decay, usually a Taylor expansion is used. We thus introduce and analyze in the second part of this paper, Section 4 a generalized Taylor formula, which expands a given function using polynomials and exponentials. We also compare this generalization to the classical Taylor formula and derive an error bound between the two.

With this result at hand, we are able to determine the decay of the wavelet coefficients connected to the MRA generated by level-dependent interpolatory Hermite
subdivision procedure. This constitutes the third and final part of this paper, namely Section 5.

Our analysis is tailored for interpolatory Hermite schemes from which it is very easy and natural to construct wavelet systems that depend only on point evaluations by means of prediction-correction methods. Moreover, from the viewpoint of extending our results also to manifold-valued data, this is a reasonable approach: In [18] it is shown that in general manifold-valued analogues of even scalar, non-Hermite wavelets are not possible any more, while perfect reconstruction, stability and wavelet coefficient decay results can still be achieved for many interpolatory examples, see [17, 18]. We do expect that our result on the decay of Hermite multiwavelet coefficients (Theorem 11) can be transferred to the manifold setting by combining it with results from [17] and [23, 24].

2 Preliminaries and first results

We start by fixing the notation and introducing some basics concepts. Vectors and matrices in \( \mathbb{R}^{d+1} \) and in \( \mathbb{R}^{(d+1)\times(d+1)} \) are denoted by boldface lower and upper case letters, respectively. If the particular coordinates are of interest, a column vector \( v \) is also written as \( v = [v_j]_{j=0}^d \). For the canonical basis in \( \mathbb{R}^{d+1} \) we write \( e_j, j = 0, \ldots, d \). On \( \mathbb{R}^{d+1} \) we use both the infinity-norm \( \| \cdot \|_\infty \) and the Euclidean norm \( \| \cdot \|_2 \), while for matrices in \( \mathbb{R}^{(d+1)\times(d+1)} \) we use the operator norms, induced by the respective norms on \( \mathbb{R}^{d+1} \) and again denote them by \( \| \cdot \|_\infty \) and \( \| \cdot \|_2 \).

The space of all vector-valued sequences \( c = (c_j : j \in \mathbb{Z}) \) is written as \( \ell(\mathbb{Z})^{d+1} \), and \( \ell(\mathbb{Z})^{(d+1)\times(d+1)} \) stands for the space of all matrix-valued sequences \( A = (A_j : j \in \mathbb{Z}) \). By \( \ell_\infty(\mathbb{Z})^{d+1} \) we denote all bounded vector-valued sequences, i.e., sequences \( c \in \ell(\mathbb{Z})^{d+1} \) with

\[
\|c\|_\infty := \sup_{j \in \mathbb{Z}} |c_j|_\infty < \infty.
\]

Similarly, \( \ell_\infty(\mathbb{Z})^{(d+1)\times(d+1)} \) is the space of matrix-valued sequences where

\[
\|A\|_\infty := \sup_{j \in \mathbb{Z}} |A_j|_\infty < \infty.
\]

The symbol of a finitely supported matrix sequence, which we write as \( A \in \ell_0(\mathbb{Z})^{(d+1)\times(d+1)} \), is the matrix valued Laurent polynomial

\[
A^*(z) = \sum_{j \in \mathbb{Z}} A_j z^j, \quad z \in \mathbb{C} \setminus \{0\}.
\]

The scalar peak sequence \( \delta \in \ell_0(\mathbb{Z}) \) with \( \delta_j = \delta_{j_0}, j \in \mathbb{Z} \), can also be used to build matrix valued sequences \( \delta C \) of any given dimension, where \( (\delta C)_0 = C \) and all other values equal to the zero matrix.
The space of all $d$-times continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ is denoted by $C^d(\mathbb{R})$. Similarly, $C_u(\mathbb{R})$ is the space of uniformly continuous and bounded functions, whereas $C^d_u(\mathbb{R})$ contains all $d$-times continuously differentiable functions with derivatives $f^{(j)} \in C_u(\mathbb{R})$, $j = 0, \ldots, d$. On $C_u(\mathbb{R})$ we use the infinity-norm
\[ \|f\|_{\infty} = \sup_{x \in \mathbb{R}} |f(x)|, \]
while for a vector-valued function $g \in C_u(\mathbb{R})^{d+1}$ we employ the norm
\[ \|g\|_{\infty} = \sup_{x \in \mathbb{R}} |g(x)|_{\infty}. \]
In the case that $g = [g^{(j)}]_{j=0}^d$, $g \in C^d_u(\mathbb{R})$, we so obtain the Sobolev norm
\[ \|g\|_{\infty} = \|g\|_{d,\infty} = \max_{j=0, \ldots, d} \|g^{(j)}\|_{\infty}. \]
For matrix-valued functions $G \in C_u(\mathbb{R})^{(d+1)\times(d+1)}$, the norm is given by the matrix infinity-norm
\[ \|G\|_{\infty} = \sup_{x \in \mathbb{R}} |G(x)|_{\infty}. \]

### 2.1 Hermite subdivision schemes

For $n \in \mathbb{N}$, we define the level-$n$ subdivision operator $S_{A[n]} : \ell(\mathbb{Z}) \to \ell(\mathbb{Z})$ with finitely supported matrix mask $A[n] \in \ell_0(\mathbb{Z})^{(d+1)\times(d+1)}$ as
\[ (S_{A[n]} c)_j = \sum_{k \in \mathbb{Z}} A[n]_{j-2k} c_k, \quad j \in \mathbb{Z}, \ c \in \ell(\mathbb{Z}). \quad (1) \]

One immediately notices the well-known fact that the subdivision operator is a composition of the upsampling operator $\uparrow$, defined as $(\uparrow B)_{2j} = B_j$ and $(\uparrow B)_{2j+1} = 0$, $j \in \mathbb{Z}$, $B \in \ell(\mathbb{Z})^{(d+1)\times(d+1)}$ and the convolution $\ast$. In view of this, we denote by $\ast_2$ the following operation between two finitely supported masks $A, B \in \ell_0(\mathbb{Z})^{(d+1)\times(d+1)}$:
\[ A \ast_2 B = S_A B = A \ast (\uparrow B), \]
that is
\[ (A \ast_2 B)_j = \sum_{k \in \mathbb{Z}} A_{j-2k} B_k, \quad j \in \mathbb{Z}. \]

Note that $\ast_2$ is neither commutative nor associative, due to which we define iterated products as
\[ A^{[n]} \ast_2 \cdots \ast_2 A^{[1]} = A^{[n]} \ast_2 (A^{[n-1]} \ast_2 \cdots \ast_2 A^{[1]}), \]
with \( A^{[j]} \in \ell((\mathbb{Z})^{(d+1)\times(d+1)}), j = 1, \ldots, n \).

Starting with \( A *_{2} B *_{2} C = (A *_{2} B) *_{4} C \), one can then easily prove by induction that
\[
A^{[n]} *_{2} \cdots *_{2} A^{[1]} *_{2} C = (A^{[n]} *_{2} \cdots *_{2} A^{[1]}) *_{2n} C,
\]
so that the application of \( m + 1 \) subdivision steps (1) to an initial sequence \( c \) can be written, for \( n, m \in \mathbb{N} \), as
\[
S_{A^{[n+m]}} \cdots S_{A^{[n]}} c = A^{[n+m]} *_{2} \cdots *_{2} A^{[n]} *_{2} c. \tag{2}
\]

Let \( (A^{[n]} : n \geq 0) \), be a sequence of finitely supported masks. A level-dependent Hermite subdivision scheme \( S(A^{[n]} : n \geq 0) \) is the procedure of iteratively constructing vector sequences by the rule
\[
D^{n+1} c^{[n+1]} = S_{A^{[n]}} D^n c^{[n]}, \quad n \in \mathbb{N}, \tag{3}
\]
starting from an initial sequence \( c^{[0]} \) of vector-valued data. The \( k \)-th component of \( c^{[n]} \) is interpreted as the \( k \)-th derivative of a function evaluated at the grid \( 2^{-n} \mathbb{Z} \). In (3), \( D \) denotes the diagonal matrix \( D = \text{diag} \{ 1, 1/2, \ldots, 1/2^n \} \) and the sequence \( c^{[n+1]} \) is related to the evaluation of function values and consecutive derivatives on the dyadic grid \( 2^{-(n+1)} \mathbb{Z} \), where the powers of \( D \) in the iteration of (3) correspond to a chain rule for the derivatives.

If the same mask \( A \) is used at all levels of the subdivision process, i.e., \( A^{[n]} = A \), for \( n \in \mathbb{N} \), the associated Hermite subdivision scheme is also called “stationary” sometimes. In the following, unless explicitly specified, we always refer to the level-dependent case.

A Hermite subdivision scheme as in (3) is called interpolatory if \( c^{[n+1]}_{2j} = c^{[n]}_j \), \( j \in \mathbb{Z} \), for any \( n \in \mathbb{N} \). In this case, all the masks satisfy \( A^{[n]}_{2j} = D \delta_j \), \( j \in \mathbb{Z} \).

The scheme is said to be \( C^d \)-convergent for some \( d \geq 1 \), if for any input data \( c^{[0]} \in \ell_{\infty}(\mathbb{Z})^{d+1} \) there exists a function \( \Phi = [\phi_j]_{j=0}^d : \mathbb{R} \rightarrow \mathbb{R}^{d+1} \), such that the sequence \( c^{[n]} \) of refinements satisfies
\[
\lim_{n \to \infty} \sup_{j \in \mathbb{Z}} |c^{[n]}_j - \Phi(2^{-n}j)|_\infty = 0,
\]
and where \( \phi_0 \in C^d_c(\mathbb{R}) \) as well as \( \frac{d^j \phi_0}{dx^j} = \phi_j \), \( j = 0, \ldots, d \), see [9, 22]. Moreover, convergence requests that the scheme is nontrivial, i.e., that there exists at least one \( c^{[0]} \in \ell_{\infty}(\mathbb{Z})^{d+1} \) such that the resulting limit function satisfies \( \Phi \neq 0 \).

### 2.2 Polynomial and exponential reproduction

We are interested in Hermite subdivision schemes that reproduce polynomials and exponentials, i.e., elements of the \((d+1)\)-dimensional space
\[
V_{p,A} = \text{span} \{ 1, x, \ldots, x^p, e^{\lambda x}, \ldots, e^{\lambda x} \}
\]
schemes have already been studied in [1, 2], however restricted to pairs of exponential frequencies ±λ_k in Λ, due to technical reasons. This restriction is not needed here, the only requirement is that λ_j ≠ λ_k, j ≠ k. In the sequel we will always assume that the frequencies are all distinct.

Following [1], this reproduction property is formulated in terms of the \( V_{p,\Lambda} \)-spectral condition. In order to give a definition, we need the following notation: For \( f \in C^d(\mathbb{R}) \) we denote by \( v_f : \mathbb{R} \to \mathbb{R}^{d+1} \) the vector valued function

\[
v_f(x) = \left[ f^{(j)}(x) \right]_{j=0}^{d}, \quad x \in \mathbb{R},
\]

and by \( v_f^{[n]} := v_f|_{2^{-n}\mathbb{Z}} \in \ell(\mathbb{Z})^{d+1} \) the vector-valued sequences with components

\[
(v_f^{[n]})_j = v_f(2^{-n}j) = \left[ f^{(k)}(2^{-n}j) \right]_{k=0}^{d}, \quad j \in \mathbb{Z}.
\]

**Definition 1.** A Hermite subdivision scheme \( S(A^{[n]} : n \geq 0) \) is said to satisfy the \( V_{p,\Lambda} \)-spectral condition if

\[
S_{A^{[n]}}D^n v_f^{[n]} = D^{n+1} v_f^{[n+1]}, \quad f \in V_{p,\Lambda}, n \in \mathbb{N}.
\]

**Remark 1.** The \( V_{p,\Lambda} \)-spectral condition is a generalization of the polynomial reproduction property of classical Hermite schemes introduced in [10], see also [22]. In [10], the reproduction of polynomials is called the spectral condition. Since \( V_{d,0} = \Pi_d \), i.e., the space of polynomials of order up to \( d \), the \( V_{d,0} \)-spectral condition is simply called polynomial reproduction or spectral condition.

For \( n, m \in \mathbb{N} \), the \( V_{p,\Lambda} \)-spectral condition implies

\[
S_{A^{[n+m]}} \cdots S_{A^{[n]}} D^n v_f^{[n]} = D^{n+d+1} v_f^{[n+m+1]}, \quad f \in V_{p,\Lambda},
\]

which reduces to

\[
S_{A^{[n+m]}} D^n v_f^{[n]} = D^{n+d+1} v_f^{[n+m+1]}, \quad f \in \Pi_d,
\]

whenever \( A^{[n]} = A, n \in \mathbb{N} \).

We end the section by some remarks related to the possibility of factorizing the subdivision operator once it satisfies the exponential and polynomial preservation property. It is shown in [1] that such a factorization can be given in terms of the so-called cancellation operator \( H^{[n]} : \ell(\mathbb{Z})^{d+1} \to \ell(\mathbb{Z})^{d+1} \), which is a convolution operator \( H^{[n]} c = H^{[n]} * c \) for some \( H^{[n]} \in \ell_0(\mathbb{Z})^{(d+1) \times (d+1)} \) and \( c \in \ell(\mathbb{Z})^{d+1} \) whose action is described by

\[
0 = (H^{[n]} v_f^{[n]})_j = \sum_{k \in \mathbb{Z}} H^{[n]}_{j-k}(v_f^{[n]})_k, \quad j \in \mathbb{Z}, \quad f \in V_{p,\Lambda}.
\]
Recall from [2] that a cancellation operator $H$ for $V_{p,\Lambda}$ is called minimal if any other convolution operator $H'$ with $H' V_{p,\Lambda} = 0$ has a factorizable impulse response in the convolution algebra, i.e., $H' = C \ast H$ for some finitely supported matrix-valued sequence $C$. More specifically, the following theorem holds.

**Theorem 1.** If, for $n \geq 0$, the subdivision operator $S_{A^{[n]}}$ satisfies the $V_{p,\Lambda}$-spectral condition, then there exist a minimal cancellation operator $H^{[n]}$ and a finitely supported mask $R^{[n]} \in \ell_0(\mathbb{Z})^{(d+1) \times (d+1)}$ such that the factorization property

$$H^{[n+1]} S_{A^{[n]}} = S_{R^{[n]}} H^{[n]}$$

holds true.

The structure of $H^{[n]}$ is given in [1] for the case where the exponentials in $V_{p,\Lambda}$ are associated to pairs of frequencies $\pm \lambda_k$. In our more general setting, a similar structure can be derived, as shown in the following lemma. For its formulation, we use the notation

$$D(e^\Lambda) = \text{diag}(e^{\lambda_1}, \ldots, e^{\lambda_r}), \quad W_n(\Lambda) = \begin{bmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \ldots & \lambda_r^{n-1} \end{bmatrix}$$

and keep in mind that both matrices are nonsingular since the $\lambda_1, \ldots, \lambda_r$ are nonzero and disjoint; the latter guarantees that the Vandermonde matrix $W_n(\Lambda)$ is invertible.

**Lemma 2.** The unique minimal cancellation operator for $V_{p,\Lambda}$ on level $n$ is given as

$$H^{[n]} = H_{2^{-n} \Lambda},$$

where $H_\Lambda$ is the convolution operator with associated symbol

$$H^*(z) := H_\Lambda^*(z) = \begin{bmatrix} z^{-1} I + T_0 & Q \\ 0 & z^{-1} I + R_0 \end{bmatrix},$$

defined by the scalar matrices

$$T_0 = \begin{bmatrix} -1 & -1 & \cdots & -1 \\ 0 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & -1 \end{bmatrix} \in \mathbb{R}^{(p+1) \times (p+1)},$$

$$Q = - (W_{p+1}(\Lambda) D(e^\Lambda) + T_0 W_{p+1}(\Lambda)) D(e^\Lambda)^{-p-1} W_r(\Lambda)^{-1} \in \mathbb{R}^{(p+1) \times r},$$

$$R_0 = - W_r(\Lambda) D(e^\Lambda) W_r(\Lambda)^{-1} \in \mathbb{R}^{r \times r}.$$
Proof. Proceeding like in [1], we denote by $T^* p(z)$ the symbol of the complete Taylor operator [22], and can obtain $H[n]$ from the operator $H_\Lambda$ with symbol

$$H^* (z) = \begin{bmatrix} T^* p(z) & Q \\ 0 & R^* (z) \end{bmatrix} = \begin{bmatrix} z^{-1} I + T_0 & Q \\ 0 & z^{-1} I + R_0 \end{bmatrix}$$

for some $Q$ and $R_0$, which must satisfy

$$H^* (e^{-\lambda_j}) D^n [\lambda_j^k]_{k=0}^d = 0, \quad j = 1, \ldots, r.$$

The conditions

$$\begin{bmatrix} e^{\lambda_j} I + T_0 & Q \\ 0 & e^{\lambda_j} I + R_0 \end{bmatrix} \begin{bmatrix} [\lambda_j^k]_{k=0}^p \\ [\lambda_j^k]_{k=p+1}^d \end{bmatrix} = 0, \quad j = 1, \ldots, r,$$

can be written as

$$W_{p+1}(\Lambda) D(e^{\Lambda}) + T_0 W_{p+1}(\Lambda) + Q W_r(\Lambda) D(e^{\Lambda})^{p+1} = 0$$

and

$$W_r(\Lambda) D(e^{\Lambda}) + R_0 W_r(\Lambda) = 0,$$

from which it follows that

$$Q = -(W_{p+1}(\Lambda) D(e^{\Lambda}) + T_0 W_{p+1}(\Lambda)) D(e^{\Lambda})^{-p-1} W_r(\Lambda)^{-1},$$

$$R_0 = -W_r(\Lambda) D(e^{\Lambda}) W_r(\Lambda)^{-1}.$$ 

The remaining arguments, in particular minimality, are as in [2]. □

2.3 Basic limit functions and refinability

Applying a $C^d$-convergent Hermite subdivision scheme $S(A[n] : n \geq 0)$ to the input data $\delta e_j$, we obtain, for each $j = 0, \ldots, d$, a vector consisting of a limit function $\phi_j$ and all its derivatives. Together, all such $\phi_j$, $j = 0, \ldots, d$, give rise to the basic limit function

$$F = \begin{bmatrix} \phi_0 & \phi_1 & \ldots & \phi_d \\ \phi'_0 & \phi'_1 & \ldots & \phi'_d \\ \vdots & \vdots & \ddots & \vdots \\ \phi^{(d)}_0 & \phi^{(d)}_1 & \ldots & \phi^{(d)}_d \end{bmatrix}.$$ 

In addition to the scheme $S(A[n] : n \geq 0)$ it is also useful to consider the subdivision schemes $S(A^{n+\ell} : n \geq 0)$, $\ell \geq 0$, with the iteration

$$(D^{n+1} c^{n+1})_j = \sum_{k \in \mathbb{Z}} A_{j-2k}^{n+\ell} D^n c_k^n, \quad j \in \mathbb{Z}. \quad (6)$$
Remark 2. The assumption that $S(A^{[n+\ell]} : n \geq 0), \ell \geq 0$, is convergent is a standard one in level-dependent subdivision as it ensures the existence of a refinement equation which we will give in the following Lemma 3. As shown in [13], this property follows from the convergence of $S(A^{[n]} : n \geq 0)$, provided that the scheme is asymptotically equivalent to some $C^d$-convergent classical Hermite subdivision scheme based on a mask $A \in \ell_0(\mathbb{Z})^{(d+1) \times (d+1)}$. This property is defined as

$$\sum_{n=0}^{\infty} \| S_{A^{[n]}} - S_A \|_{\infty} < \infty,$$

cf. [13]. Also recall from [1, 2] the fact that the $V_{p,A}$-preserving schemes built by (5) with $R^{[n]} := R, n \geq 0, \ell \geq 0$, for some $R$ are asymptotically equivalent to the Hermite scheme based on a Taylor factorization [22] with factor $R$. Their limit mask is the one related to the Hermite subdivision scheme preserving $V_{p+r,0} = \Pi_d$. Note that this is the "Hermite analogy" of the exponential B–spline introduced in [26].

If we suppose that $S(A^{[n+\ell]} : n \geq 0), \ell \geq 0$, are all convergent and apply all such schemes to the same initial data $c^{[0]} = \delta I_{d+1}$, we obtain a sequence of basic limit functions $(F^{[\ell]} : \ell \geq 0)$, where $F^{[0]} = F$. Those basic limit functions are connected by a level-dependent refinement equation which we prove in the next lemma for the sake of completeness. We mention, however, that this refinement equation is already stated in [7], and well-known and popular in the level-dependent non-Hermite [15, Section 2.3] as well as in the stationary Hermite [9, Theorem 19] setting. The extension here is an adaption of technique given there.

Lemma 3. If $S(A^{[n+\ell]} : n \geq 0), \ell \geq 0$, are $C^d$-convergent Hermite subdivision schemes, then the associated sequence of basic limit functions $(F^{[\ell]} : \ell \geq 0)$ satisfies

$$F^{[\ell]} = \sum_{k \in \mathbb{Z}} D^{-1} F^{[\ell+1]} (2 \cdot -k) A^{[\ell]}_k, \quad \ell \in \mathbb{N}. \quad (7)$$

Proof. We iterate the subdivision scheme on matrix valued data. For $C^{[0]} = \delta I_{d+1}$, the first iteration of (3) yields for $\ell \in \mathbb{N}$, that $DC^{[1,\ell]} = A^{[\ell]}$, hence

$$(D^{n+1} C^{[n+1,\ell]})_j = (A^{[\ell+n]} \ast_2 \cdots \ast_2 A^{[\ell+1]} \ast_{2n} DC^{[1,\ell]}$$

or, equivalently

$$D^{n+1} C^{[n+1,\ell]} = A^{[\ell+n]} \ast_2 \cdots \ast_2 A^{[\ell+1]} \ast_{2n} A^{[\ell]} \quad (8)$$

Interpreting (8) as

$$(A^{[\ell+n]} \ast_2 \cdots \ast_2 A^{[\ell+1]} \ast_{2n} A^{[\ell]} = (D^n C^{[n,\ell+1]} \ast_{2n} A^{[\ell]}),$$

Proof.
it follows that
\[ (DC_{j}^{n+1,\ell})_j = \sum_{k \in \mathbb{Z}} C_{j-2^n k}^{n,\ell} A_k^{[\ell]}, \quad j \in \mathbb{Z}. \]

Let \( x \in \mathbb{R} \) and let \( (j_n : n \in \mathbb{N}) \) be a sequence of integers such that \( j_n/2^n \to x \) as \( n \to \infty \). Let \( F^{[n,\ell]} \) denote the piecewise linear matrix valued functions such that
\[ C_{j_n}^{n,\ell} = F^{[n,\ell]} \left( \frac{j_n}{2^n} \right), \quad n, \ell \geq 0. \]

We have that
\[ DF_{j_n}^{n+1,\ell} \left( \frac{j_n}{2^n+1} \right) = \sum_{k \in \mathbb{Z}} F_{j_n}^{[n,\ell+1]} \left( \frac{j_n - 2^n k}{2^n} \right) A_k^{[\ell]}, \]
and the uniform convergence of \( F^{[n,\ell]} \) to \( F^{[\ell]} \) for \( n \to \infty \) yields (7).

Note that in the case \( A^{[n]} = A, n \geq 0 \), there is only one basic limit function \( F \) which satisfies the refinement equation
\[ F = \sum_{k \in \mathbb{Z}} D^{-1} F(2 \cdot -k) A_k. \tag{9} \]

Therefore Lemma 3 is a generalization of [9, Theorem 19], where (9) can already be found.

For interpolatory subdivision schemes, the values of the basic limit functions \( F^{[\ell]} \) at the dyadics \( 2^{-n} \mathbb{Z} \) are exactly the coefficients of the corresponding schemes at level \( n \) and \( F^{[\ell]} \) is cardinal, that is
\[ F^{[\ell]}(k) = \delta_k I_{d+1}, \quad k \in \mathbb{Z}, \quad \ell \geq 0. \tag{10} \]

By iteration of (7), it is easy to see that the sequence of basic limit functions of a \( C^d \)-convergent Hermite subdivision scheme satisfies
\[ F^{[\ell]} = \sum_{k \in \mathbb{Z}} D^{-m} F^{[\ell+m]}(2^m \cdot -k)(A^{[\ell+m-1]} \ast_2 \cdots \ast_2 A^{[\ell]})_k, \quad \ell \geq 0, m \geq 1, \tag{11} \]
which reduces to
\[ F = \sum_{k \in \mathbb{Z}} D^{-m} F^{[2^m \cdot -k]} A^m_k, \quad m \in \mathbb{N}, \]
if all masks coincide. From (10) and (11) we get explicit representations for the basic limit functions of an interpolatory \( C^d \)-convergent Hermite subdivision scheme at the integers, namely,
\[ F^{[\ell]}(2^{-m} k) = D^{-m} (A^{[\ell+m-1]} \ast_2 \cdots \ast_2 A^{[\ell]})_k, \quad k \in \mathbb{Z}, \quad \ell \geq 0, m \geq 1, \tag{12} \]
and
\[ F(2^{-m} k) = D^{-m} A^m_k, \quad k \in \mathbb{Z}, \quad m \geq 1, \]
respectively.
3 Wavelets defined by interpolatory Hermite subdivision

In this section we briefly review the construction of a level-dependent MRA based on interpolatory Hermite subdivision which was suggested in [7].

To that end, we start with a $C^d$-convergent interpolatory Hermite subdivision scheme $S(A^{[n]} : n \geq 0)$. The sequence of basic limit functions $\{F^{[n]} : n \geq 0\}$, more precisely their first rows $\Phi^{[n]} = \{\phi_j^{[n]}\}_{j=0}^d$, $n \geq 0$, span a level-dependent MRA $(V_n : n \geq 0)$ for the space $C_u^d(\mathbb{R})$. This means that the spaces $V_n$ are still nested but the refinement property between $V_n$ and $V_{n+1}$ depends on $n$ as it is now based on (7). Since the subdivision scheme is interpolatory, the projection of $f \in C_u^d(\mathbb{R})$ onto $V_n$ is given by the Hermite interpolant

$$P_n f = \sum_{k \in \mathbb{Z}} (\Phi^{[n]})^T (2^n \cdot -k)c_k^{[n]},$$

(13)

with

$$c_k^{[n]} = D^n v_f^{[n]}.$$ 

The associated wavelet space $W_n$ is the complement of $V_n$ in $V_{n+1}$. Taking into account that

$$P_{n+1} f = P_n f + (P_{n+1} - P_n) f = P_n f + Q_n f,$$

the projection $Q_n f$ onto the wavelet space is given by

$$Q_n f = P_{n+1} f - P_n f,$$

(14)

and we set $W_n = Q_n V_{n+1}$. It is shown in [7] that

$$Q_n f = \sum_{k \in \mathbb{Z}} (\Phi^{[n+1]})^T (2^{n+1} \cdot -k)d_k^{[n]},$$

(15)

where the wavelet coefficients are given by the prediction-correction scheme

$$d_k^{[n]} = c_k^{[n+1]} - S A^n c_k^{[n]}.$$

(16)

The interpolatory Hermite wavelet transform associates to any $f \in C_u^d(\mathbb{R})$ a representation in terms of the vector-valued decomposition sequences:

$$c^{[0]}, d^{[0]}, d^{[1]}, d^{[2]}, \ldots$$

Conversely, the coefficient sequence connected to the projection (13) can be reconstructed as

$$c_k^{[n]} = d_k^{[n-1]} + S A^{[n-1]} c_k^{[n-1]}$$

$$= d_k^{[n-1]} + S A^{[n-1]} (d_k^{[n-2]} + S A^{[n-2]} c_k^{[n-2]})$$

$$= d_k^{[n-1]} + S A^{[n-1]} d_k^{[n-2]} + \cdots + S A^{[n-1]} \cdots S A^{[0]} c_k^{[0]}.$$
Incorporating also the derivatives of $f$ into (13) and (15), we find that

$$v_{P_n f} = \sum_{k \in \mathbb{Z}} D^{-n} F^{[n]} (2^n - k) \left( D^n v_f^{[n]} \right)_k$$

(17)

and

$$v_{Q_n f} = \sum_{k \in \mathbb{Z}} D^{-n-1} F^{[n+1]} (2^{n+1} - k) d_k^{[n]}.$$

(18)

From (14) it also follows that

$$v_{Q_n f} = v_{P_{n+1} f} - v_{P_n f}. \quad (19)$$

We recall that in the classical (non-Hermite) situation, the reproduction of polynomials up to the degree $d$ by the subdivision scheme implies polynomial vanishing moments for the wavelets. In our setting, polynomial reproduction is replaced by the $V_{p, \Lambda}$-spectral condition from Definition 1, which is a condition on the function $f \in V_{d, \Lambda}$ and all its derivatives up to order $d$. This has also consequences for the corresponding MRA, resulting in a $V_{p, \Lambda}$ vanishing moment property, that is, the wavelet coefficients (16) connected to any $f \in V_{p, \Lambda}$ are all zero. A first property of the projections, which is useful for proving the result on the decay of the wavelet coefficients in Section 5, is stated and proved in the following lemma.

**Lemma 4.** Let $S(A^n : n \geq 0)$ be an interpolatory $C^d$-convergent Hermite subdivision scheme satisfying the $V_{p, \Lambda}$-spectral condition. Then we have

$$v_{P_n f} = v_f, \quad f \in V_{p, \Lambda}, \quad n \in \mathbb{N}. \quad (20)$$

**Proof.** Since $f$ and all its derivatives are continuous by assumption, and since the dyadic points are dense in $\mathbb{R}$, it is sufficient to verify (20) for dyadic points of the form $2^{-m}k$, $k \in \mathbb{Z}$, $m \in \mathbb{N}$. We fix $n \in \mathbb{N}$ and distinguish between the cases $m = n, m < n$ and $m > n$.

If $m = n$, then $v_{P_n f} (2^{-n}k) = v_f (2^{-n}k)$ follows directly from (10) and (17), since
the scheme is interpolatory. In the case that \( r := n - m \) is positive, get
\[
D^n v_{p,f}(2^{-m}k) = \sum_{\ell \in \mathbb{Z}} F[n]((2^n2^{-m}k - \ell)D^n v_f(2^{-n}\ell))
\]
\[
= \sum_{\ell \in \mathbb{Z}} F[n](2^r(2^{2r}k - 2^r\ell))\left(D^n v_f[n]\right)_\ell
\]
\[
= D^{-r} \left( (A^{[n+r-1]} *_2 \cdots *_2 A^{[n]} *_{2^r} D^n v_f[n] \right)_{2^r k}
\]
\[
= D^{-r} D^{r-1+n+1} \left( A^{[r-1+n+1]} \right)_{2^r k} = \left( D^n v_f[n+r] \right)_{2^r k}
\]
\[
= D^n v_f(2^{-n-r}2^r k) = D^n v_f(2^{-m}k),
\]
and in the final case \( r < 0 \), we set \( s = -r > 0 \) and compute likewise
\[
D^n v_{p,f}(2^{-m}k) = \sum_{\ell \in \mathbb{Z}} F[n](2^{-s}k - \ell)\left(D^n v_f[n]\right)_\ell
\]
\[
= D^{-s} \left( (A^{[n+s-1]} *_2 \cdots *_2 A^{[n]} *_{2^s} D^n v_f[n] \right)_k
\]
\[
= D^{-s} D^{s-1+n+1} \left( A^{[s-1+n+1]} \right)_{k} = \left( D^n v_f[n] \right)_k = D^n v_f(2^{-m}k),
\]
which completes the proof. \( \square \)

4 Taylor formula with exponentials

As already mentioned, estimates for the decay rate of the wavelet coefficients are usually based on a local Taylor polynomial approximation and the polynomial reproduction property of the operator. When dealing not only with polynomial but also exponential vanishing moments, a more general tool is needed to fully explore the approximation power of the space \( V_{p,\Lambda} \).

In this section we derive such a generalized Taylor formula, by means of elements in the space \( V_{p,\Lambda} \), namely an approximation of the form
\[
f(x + h) \approx T_{p,\Lambda} f(x, h) := \sum_{j=0}^{p} \frac{f^{(j)}(x)}{j!}h^j + \sum_{k=1}^{r} \mu_k(f) e^{\lambda_k h}.
\] (21)

We show that for proper functionals \( \mu_k \), such an expression can obtain an error of the order \( h^{d+1} \) for a function \( f \in C^d(R) \) where \( d = p + r \). In Lemma we give the
appropriate choice for $\mu_1, \ldots, \mu_k$, such that the same approximation rate as that of the usual Taylor operator

$$T_d f(x, h) = \sum_{j=0}^{d} \frac{f^{(j)}(x)}{j!} h^j$$

is obtained.

In our construction, we use the nonsingular Vandermonde matrix

$$V_d := \left[ \begin{array}{ccc} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p! & \lambda_1^p & \cdots & \lambda_r^p \\ \lambda_1^{p+1} & \cdots & \lambda_r^{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^d & \cdots & \lambda_r^d \end{array} \right] =: \left[ \begin{array}{ccc} X & Y \\ Z \end{array} \right] \in \mathbb{R}^{(d+1) \times (d+1)},$$
corresponding to the poised, i.e., uniquely solvable, Hermite interpolation problem at the $(d+1)$-fold point $0$ and at $\Lambda$. Note that the square matrix $Z \in \mathbb{R}^{r \times r}$ satisfies

$$Z = \left[ \begin{array}{ccc} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ \lambda_{1}^{p+1} & \cdots & \lambda_{r}^{p+1} \end{array} \right] \left[ \begin{array}{ccc} \lambda_1^{p+1} & \cdots & \vdots \\ \vdots & \ddots & \vdots \\ \lambda_1^d & \cdots & \lambda_r^d \end{array} \right]$$

and, therefore, is the product of a Vandermonde matrix and a nonzero diagonal matrix, hence invertible. The inverse of $V_d$ is then

$$V_d^{-1} = \left[ \begin{array}{ccc} X^{-1} & -X^{-1}YZ^{-1} \\ Z^{-1} \end{array} \right].$$

**Lemma 5.** If $f \in C^d(\mathbb{R})$ then

$$T_{p,\Lambda} f(x, h) := [1, \ldots, h^p, e^{\lambda_1 h}, \ldots e^{\lambda_r h}] V_d^{-1} \psi_f(x)$$

satisfies, for any $R < 1$,

$$|T_{p,\Lambda} f(x, h) - T_d f(x, h)| \leq C_{\Lambda,R} |\psi_f(x)|_2 |h|^{d+1}, \quad |h| \leq R,$$

where the constant $C_{\Lambda,R}$ depends on $\Lambda$ and $R$ only.

**Proof.** Recall from [2] that

$$\left[ \begin{array}{c} [x^j]_{j=0}^p \\ [e^{\lambda_j x}]_{j=1}^r \end{array} \right] = [V_d^T | M_\Lambda V_d^T | M_\Lambda^2 V_d^T | \cdots ] [x^j]_{j \in \mathbb{N}}$$
with

\[ M_\Lambda = \begin{bmatrix} 0_{(p+1) \times (p+1)} & \lambda_1^{d+1} & \cdots & \lambda_r^{d+1} \end{bmatrix} \].

Hence,

\[
\left( V_d^{-1} \left[ f^{(j)}(x) \right]_{j=0}^d \right)^T \begin{bmatrix} [h_j]_{j=0}^p \\ \left[ e^{\lambda_j h} \right]_{j=1}^r \end{bmatrix} =
\]

\[
\left( f^{(j)}(x) \right)_{j=0}^d \begin{bmatrix} I & V_d^{-T} M_\Lambda V_d^T & V_d^{-T} M_\Lambda^2 V_d^T & \ldots \end{bmatrix} \begin{bmatrix} [h_j]_{j=1}^r \end{bmatrix} \in \mathbb{N}
\]

\[
= \sum_{j=0}^d \frac{f^{(j)}(x)}{j!} h^j + \left( f^{(j)}(x) \right)_{j=0}^d \begin{bmatrix} I & V_d^{-T} M_\Lambda V_d^T & V_d^{-T} M_\Lambda^2 V_d^T & \ldots \end{bmatrix} \begin{bmatrix} [h_j]_{j=1}^r \end{bmatrix} \in \mathbb{N}
\]

\[
= T_d f(x, h) + v_f(x)^T \sum_{k=1}^\infty \sum_{j=0}^d V_d^{-T} M_\Lambda^k V_d^T e_j \frac{h_j^{k(d+1)+j}}{(k(d+1) + j)!}.
\]

The spectral radius in the sum satisfies

\[
\rho \left( V_d^{-T} M_\Lambda V_d^T \right) = \rho(M_\Lambda) = \max_{j=1, \ldots, r} |\lambda_j| =: \rho,
\]

and, since \( V_d^{-T} M_\Lambda^k V_d^T = \left( V_d^{-T} M_\Lambda V_d^T \right)^k \), there exists for any \( \varepsilon > 0 \) a constant \( C > 0 \), depending on \( \Lambda \) and \( \varepsilon \) such that

\[
| V_d^{-T} M_\Lambda^k V_d^T | \leq C (\rho + \varepsilon)^k, \quad k \geq 0.
\] (26)

Hence,

\[
| T_{p, \Lambda} f(x, h) - T_d f(x, h) |
\]

\[
\leq |v_f(x)|_2 \sum_{k=1}^\infty \sum_{j=0}^d |V_d^{-T} M_\Lambda^k V_d^T|_2 \frac{|h_j|^{k(d+1)+j}}{(k(d+1) + j)!}
\]

\[
\leq C |v_f(x)|_2 \sum_{k=1}^\infty (\rho + \varepsilon)^k |h_j|^{k(d+1)} \sum_{j=0}^d \frac{|h_j|^j}{(k(d+1) + j)!}
\]

\[
= C |v_f(x)|_2 (\rho + \varepsilon) |h|^{d+1} \sum_{k=0}^\infty \left( (\rho + \varepsilon)|h|^d+1 \right) \sum_{j=0}^d \frac{|h_j|^j}{((k+1)(d+1) + j)!}.
\]
Since for $|h| \leq R < 1$,

$$\sum_{j=0}^{d} \frac{|h|^j}{((k+1)(d+1)+j)!} \leq \frac{1}{((k+1)(d+1))!} \sum_{j=0}^{d} |h|^j \leq \frac{1}{k!} \frac{1-R^{d+1}}{1-R},$$

we can conclude that

$$\sum_{k=0}^{\infty} \frac{((\rho+\varepsilon)|h|^{d+1})^k}{k!} \sum_{j=0}^{d} \frac{|h|^j}{((k+1)(d+1)+j)!} \leq \frac{1-R^{d+1}}{1-R} e^{(\rho+\varepsilon)|h|^{d+1}} \leq \frac{1}{1-R} e^{\rho+\varepsilon}$$

which is a constant that depends only on $\Lambda$ and $R$. Hence,

$$|T_{p,\Lambda}f(x,h) - T_df(x,h)| \leq C \left| \psi_f(x) \right|_2 \frac{1}{1-R} e^{\rho+\varepsilon} (\rho + \varepsilon) |h|^{d+1}$$

uniformly in $|h| \leq R < 1$ and if we combine all these numbers into a single constant, we get (24).

**Remark 3.** We find it worthwhile to note that even when $f$ is only $d$-times continuously differentiable, the deviation between $T_{p,\Lambda}f(x,h)$ and $T_df(x,h)$, which both use derivatives up to order $d$, is of order $h^{d+1}$ for small $h$ and not only of order $h^d$. In other words, the difference between the operators is smaller than their approximation to $f$ due to which they can indeed be considered equivalent.

Since

$$V_{d^{-1}} = \begin{bmatrix}
1 & * & \ldots & * \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 1/p! & * & \ldots & * \\
* & \ldots & * & \ldots & \ddots & \vdots \\
* & \ldots & * & \ldots & \ldots & * \\
\end{bmatrix},$$

the initial polynomial part of $T_{d,\Lambda}f$ is indeed the usual Taylor polynomial $T_pf$ which means that the operator defined in (23) is indeed of the form (21), the $\mu_j$ being defined by the inverse of the Vandermonde matrix.

**Corollary 6.** With the setting of Lemma 5 we have that

$$|T_{p,2^{-n}\Lambda}f(x,h) - T_df(x,h)| \leq 2^{-n(d+1)}C_{\Lambda,R} \left| \psi_f(x) \right|_2 |h|^{d+1}, \quad |h| \leq R.$$
Proof. We only have to notice that \( M_{2-n\Lambda} = 2^{-n(d+1)} M_{\Lambda} \), hence, like in the preceding proof,

\[
|T_d,2-n\Lambda f(x,h) - T_d f(x,h)| 
\leq |v_f(x)|_2 \sum_{k=1}^{\infty} 2^{-n(d+1)k} \sum_{j=0}^{d} |V_d^T M_j^k V_d^T|_2 \frac{|h|^{k(d+1)+j}}{(k(d+1)+j)!}
\]

and the rest follows as above with an even smaller constant. \( \square \)

Like in the classical Taylor formula we also have results for the derivatives of the approximant from \( V_p,\Lambda \).

**Lemma 7.** For \( f \in C^d(\mathbb{R}) \) and \( x \in \mathbb{R} \) one has

\[
(T_{p,\Lambda} f)^{(j)}(x, \cdot) = T_{p-j,\Lambda} f^{(j)}(x, \cdot), \quad j = 0, \ldots, p,
\]

and

\[
\frac{d^j}{dh^j} (T_{p,\Lambda} f)(x,h) = \left[ e^{\lambda_j h}, \ldots, e^{\lambda_r h} \right] \begin{bmatrix}
\lambda_1^{p+1-j} & \cdots & \lambda_r^{p+1-j} \\
\vdots & \ddots & \vdots \\
\lambda_1^{-j} & \cdots & \lambda_r^{-j} \\
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
\lambda_1^{d-j} & \cdots & \lambda_r^{d-j}
\end{bmatrix}^{-1} \begin{bmatrix}
[f^{(k)}(x)]_{k=p+1}^{j-1} \\
[f^{(k)}(x)]_{k=j}^{d}
\end{bmatrix},
\]

for \( j = p + 1, \ldots, d \).

Proof. For \( j = 0, \ldots, p \) we have

\[
(T_{p,\Lambda} f)^{(j)}(x,h) = \left[ 0, \ldots, 0, j! \frac{p!}{(p-j)!} h^{p-j}, \lambda_1^j e^{\lambda_1 h}, \ldots, \lambda_r^j e^{\lambda_r h} \right] V_d^{-1} v_f(x)
\]

\[
= \left[ 0, \ldots, 0, 1, \ldots, h^{p-j}, e^{\lambda_1 h}, \ldots, e^{\lambda_r h} \right] \text{diag} \left( 1, \ldots, \frac{p!}{(p-j)!}, \lambda_1^j, \ldots, \lambda_r^j \right) V_d^{-1} v_f(x).
\]
Furthermore,
\[
\text{diag} \left( 1, \ldots, \frac{p!}{(p-j)!}, \lambda_1^j, \ldots, \lambda_r^j \right) \cdot V_d^{-1}
\]
\[
= (V_d \cdot \text{diag} \left( 1, \ldots, \frac{p!}{p!}, \lambda_1^{-j}, \ldots, \lambda_r^{-j} \right))^{-1}
\]
\[
= \begin{bmatrix}
1 & \cdots & \lambda_1^{-j} & \cdots & \lambda_r^{-j} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\lambda_1^{-1} & \cdots & 1 & \cdots & 1 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
(p-j)! & \cdots & \lambda_1^{p-j} & \cdots & \lambda_r^{p-j} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\lambda_1^{d-j} & \cdots & \lambda_1^{d-j} & \cdots & \lambda_r^{d-j}
\end{bmatrix}^{-1}
\]
\[
= \left[ I \left| W \right| V_{p-j}^{-1} \right]^{-1}
\]

By (22) we obtain
\[
(T_{p,A}f)^{(j)}(x, h) = \left[ 0, \ldots, 0, 1, \ldots, h^{p-j}, e^{\lambda_1 h}, \ldots, e^{\lambda_r h} \right] \left[ I \left| -\frac{W V_{p-j}^{-1}}{V_{p-j}} \right| \right] \nu_f(x)
\]
\[
= \left[ 1, \ldots, h^{p-j}, e^{\lambda_1 h}, \ldots, e^{\lambda_r h} \right] V_{p-j}^{-1} \left[ f^{(k)}(x) \right]_{k=j}^d
\]
\[
= T_{p-j,A}f^{(j)}(x, h).
\]

For \( j = p+1, \ldots, d \), the polynomial part vanishes completely. Equation (28) then follows from similar computations as the ones just carried out.

Using the preceding result, we obtain, in analogy with Lemma 5, similar error bounds between the derivatives of the Taylor polynomial and (23).

**Corollary 8.** Let \( f \in C^d(\mathbb{R}) \). For any \( R < 1 \) and \( j = 0, \ldots, d \) we have
\[
| (T_{p,A}f)^{(j)}(x, h) - T_{d-j}f^{(j)}(x, h) | \leq C_{\lambda, R} \left\| \nu_{f^{(j)}}(x) \right\|_2 | h |^{d+1-j}, \quad | h | \leq R,
\]
where \( k_j = \min\{j, p+1\} \).

**Proof.** For \( j = 0, \ldots, p \) the statement follows by combining (24) and (27). The proof of the case \( j = p+1, \ldots, d \), does not follow directly from (24) and (28), because of the special form of (28). Still the proof is very similar to that of Lemma 5. Let
\[ j = p + 1, \ldots, d \] and define
\[
W = \begin{bmatrix}
\lambda_1^{p+1-j} & \cdots & \lambda_r^{p+1-j} \\
\vdots & \ddots & \vdots \\
\lambda_1^{d-j} & \cdots & \lambda_r^{d-j}
\end{bmatrix} \in \mathbb{R}^{r \times r},
\]
where the upper part of \( W \) is of dimension \((j - p - 1) \times r\), and the lower part a \((d - j + 1) \times r\) matrix. Furthermore, we introduce the \( r \times r \) matrix
\[
N_\Lambda = \begin{bmatrix}
\lambda_1^{d-j+1} & \cdots \\
\vdots \\
\lambda_r^{d-j+1}
\end{bmatrix}.
\]
Similar to (25) we have
\[
[e^{\lambda_k x}]_k^r = [W^T | N_\Lambda W^T | N_\Lambda^2 W^T | \ldots] \begin{bmatrix}
0_{1-p-1} \\
\frac{x^k}{k!} \end{bmatrix}_{k \in \mathbb{N}}.
\]
Therefore, by (28)
\[
(T_{p,A} f)^{(j)}(x, h) = \left( [f^{(k)}(x)]_{k=p+1}^d \right)^T W^{-T} [e^{\lambda_h x}]_k^r
\]
\[
= \left( [f^{(k)}(x)]_{k=p+1}^d \right)^T \begin{bmatrix}
I_r | W^{-T} N_\Lambda W^T | W^{-T} N_\Lambda^2 W^T | \ldots
\end{bmatrix} \begin{bmatrix}
0_{1-p-1} \\
\frac{x^k}{k!} \end{bmatrix}_{k \in \mathbb{N}}.
\]
Using the same arguments as in (26) we have for \(|h| \leq R < 1\) that
\[
|(T_{p,A} f)^{(j)}(x, h) - T_{d-j} f^{(j)}(x, h)|
\]
\[
\leq |v_{f^{(p+1)}}(x)|_2 \sum_{k=1}^{\infty} \sum_{j=1}^{r} \left| W^{-T} N_\Lambda^j W^T \right|_2 |h|^{d-j+(\ell-1)r+j} \frac{|h|^{d-j+(\ell-1)r+j}}{(d - j + (\ell - 1)r + j)!}
\]
\[
\leq |v_{f^{(p+1)}}(x)|_2 |h|^{d-j+1} \sum_{\ell=0}^{\infty} \sum_{j=0}^{r-1} (\rho + \epsilon)رئيس+1 \frac{|h|^{\ell r+j}}{(d - j + \ell r + j + 1)!}
\]
\[
\leq |v_{f^{(p+1)}}(x)|_2 |h|^{d-j+1} (\rho + \epsilon)e^{Rr(\rho+\epsilon)} \frac{1 - R^r}{1 - R}.
\]
This gives
\[ |(T_{p,A} f)^{(j)}(x, h) - T_{d-j} f^{(j)}(x, h)| \leq C_{A,R} |v_{f^{(j+1)}}(x)|_2 |h|^{d-j+1} \]
as claimed. \(\square\)

In the case of functions in \(C^d_a(\mathbb{R})\), Corollary 8 immediately gives a bound independent of \(x\).

**Corollary 9.** For \(f \in C^d_a(\mathbb{R})\), \(j = 0, \ldots, d\) and any \(R < 1\)
\[ |(T_{p,A} f)^{(j)}(x, h) - T_{d-j} f^{(j)}(x, h)| \leq C_{A,R,f} |h|^{d-j+1}, \quad |h| < R. \]

**Proof.** Since \(f \in C^d_a(\mathbb{R})\) we have for \(x \in \mathbb{R}\) and \(j = 0, \ldots, d\) that
\[ |v_{f^{(j)}}(x)|_2 \leq |v_{f^{(j)}}(x)|_\infty = \max_{k=0,\ldots,d} |f^{(k)}(x)| \leq \max_{k=0,\ldots,d} \sup_{y \in \mathbb{R}} |f^{(k)}(y)| = \|f\|_{d,\infty}. \]

Therefore \(|v_{f^{(j)}}(x)|_2\) is bounded by a finite constant independent of \(x \in \mathbb{R}\) and for \(j = 0, \ldots, d\). \(\square\)

Finally we are able to determine the asymptotic behavior of the remainder term for any function (and derivatives) approximated with the generalized Taylor formula.

**Lemma 10.** Let \(f \in C^d_a(\mathbb{R})\). Then for any \(R < 1\) we have
\[ \left| h^j \left( f^{(j)}(x + h) - (T_{d} f^{(j)})(x, h) \right) \right|_\infty \leq C_{A,R,f} h^{d+1}, \quad |h| < R. \]

**Proof.** It is well-known that the derivatives in the usual Taylor formula of \(f\) are given by
\[ (T_d f)^{(j)}(x, h) = (T_{d-j} f^{(j)}) (x, h), \quad j = 0, \ldots, d, \]
with the remainder terms satisfying
\[ |h^j \left( f^{(j)}(x + h) - (T_d f^{(j)})(x, h) \right) | \leq C_{A,R,f} h^{d+1}, \quad |h| < R, \quad j = 0, \ldots, d. \quad (29) \]

We then have
\[ \left| h^j \left( f^{(j)}(x + h) - (T_{p,A} f)^{(j)}(x, h) \right) \right|_\infty = \max_{j=0,\ldots,d} \left| h^j \left( f^{(j)}(x + h) - (T_{p,A} f)^{(j)}(x, h) \right) \right| \]
\[ \leq \max_{j=0,\ldots,d} \left( |h^j (f^{(j)}(x + h) - (T_{d} f)^{(j)}(x, h))| \right) \]
\[ + \sum_{j=0}^d \left( |h^j ((T_d f)^{(j)}(x, h) - (T_{p,A} f)^{(j)}(x, h)))| \right). \]

Therefore, in virtue of (29) and Corollary 8, it follows that:
\[ \left| h^j \left( f^{(j)}(x + h) - (T_{p,A} f)^{(j)}(x, h) \right) \right|_\infty \leq C_{A,R,f} h^{d+1}, \quad |h| < R. \quad \square \]
5 Decay of wavelet coefficients

In this section we give our main result, namely we prove that the wavelet coefficients associated to an interpolatory MRA generated by a level-dependent Hermite subdivision scheme satisfying the $V_{p,\Lambda}$-spectral condition decrease of a certain well-defined order as the scale increases and we give estimates of such a decay. To our knowledge this has never been investigated in the Hermite setting, even in the case of only polynomial reproduction. Our proof exploits the generalized Taylor formula associated to a function in $C^d_u(\mathbb{R})$ given in the previous section.

We start with some remarks concerning the support of the basic limit functions associated to a Hermite subdivision scheme. Let us consider a sequence of masks $(A^{[n]}: n \geq 0)$ whose support is contained in a finite interval $[-N, N]$ for all $n \in \mathbb{N}$. Moreover, the associated Hermite subdivision scheme $S(A^{[n]}: n \geq 0)$ is assumed to be $C^d$-convergent. Denote by $(F^{[n]}: n \geq 0)$ its sequence of basic limit functions. Using similar arguments as in [15, Section 2.3], it is easy to see that also

$$\text{supp}(F^{[n]}) \subseteq [-N, N], \quad n \in \mathbb{N}. \quad (30)$$

This fact is essential in the proof of the main theorem:

**Theorem 11.** Let $S(A^{[n]}: n \geq 0)$ be a $C^d$-convergent interpolatory Hermite subdivision scheme satisfying the $V_{p,\Lambda}$-spectral condition. Moreover assume that there exists $N \in \mathbb{N}$ such that $\text{supp}(A^{[n]}) \subseteq [-N, N]$ for all $n \in \mathbb{N}$, and that $\sup_{n \in \mathbb{N}} \|F^{[n]}\|_\infty < \infty$. For $f \in C^d_u(\mathbb{R})$ the associated wavelet coefficients $d^{[n]}$ defined in (16) satisfy the following property: For $R < 1$, there exist $m \in \mathbb{N}$ and a constant $C > 0$, depending on $\Lambda, R, f, N$ and the Hermite subdivision scheme, such that

$$\|d^{[n]}\|_\infty \leq C 2^{-n(d+1)}, \quad n \geq m.$$  

**Proof.** Due to (10) and (17), we have that

$$v_{p_n f}(2^{-n} \ell) = v_f(2^{-n} \ell) \quad \ell \in \mathbb{Z}, n \in \mathbb{N}, \quad (31)$$

holds true whenever $f \in C^d_u(\mathbb{R})$; note that Lemma 4 cannot be applied here as it is only valid for functions in $V_{p,\Lambda}$. The representations (18) and (19) allow us to express the wavelet coefficients in the following way

$$d^{[n]}_\ell = D^{n+1} v_{Q_n f}(2^{-n-1} \ell) = D^{n+1} \left( v_{p_{n+1} f}(2^{-n-1} \ell) - v_{p_n f}(2^{-n-1} \ell) \right),$$

and, by (31),

$$d^{[n]}_\ell = D^{n+1} v_f(2^{-n-1} \ell) - D^{n+1} v_{p_n f}(2^{-n-1} \ell), \quad \ell \in \mathbb{Z}.$$
Define \( g \in V_{p,A} \) by \( g(x) := T_{p,A} f (x - 2^{-n-1}, 2^{-n-1}), x \in \mathbb{R} \). Then we have

\[
\left| d_n^{[p]} \right|_\infty = \left| D^{n+1} v_f (2^{-n-1} \ell) - D^{n+1} v_{p_n f} (2^{-n-1} \ell) \right|_\infty \\
\leq \left| D^{n+1} v_f (2^{-n-1} \ell) - D^{n+1} v_g (2^{-n-1} \ell) \right|_\infty \\
+ \left| D^{n+1} v_g (2^{-n-1} \ell) - D^{n+1} v_{p_n f} (2^{-n-1} \ell) \right|_\infty \\
=: I_1 + I_2.
\]

We start by estimating \( I_1 \). With \( m \geq 1 - \log_2 R \) we have \( 2^{-m} < R \) for \( n \geq m \) and Lemma 10 with \( h = 2^{-m} \) and \( x = 2^{-m-1} (\ell - 1) \) gives

\[
I_1 = \left| D^{n+1} v_f (2^{-n-1} \ell) - D^{n+1} v_g (2^{-n-1} \ell) \right|_\infty \\
= \left| \left[ 2^{-(n+1)j} (f^{(j)} (2^{-n-1} \ell) - (T_{p,A} f)^{(j)} (2^{-n-1} (\ell - 1), 2^{-n-1})) \right] \right|_\infty \\
\leq C_{A,R,f} 2^{-(n+1)j} (2^{-m} - 2^{-m-1}) \quad \ell \in \mathbb{Z},
\]

for \( n \geq m \). The bound (32) is even independent of \( \ell \in \mathbb{Z} \) since \( f \) is uniformly continuous.

It remains to estimate \( I_2 \). Due to Lemma 4, \( v_g (2^{-n-1} \ell) = v_{p_n f} (2^{-n-1} \ell) \) holds for \( n \in \mathbb{N}, \ell \in \mathbb{Z} \). By (30) there exists \( N \in \mathbb{N} \) such that \( \text{sup} (F^{[n]}) \subseteq [-N, N], n \in \mathbb{N} \). Therefore, \( F^{[n]} (2^{-1} \ell - k) \neq 0 \) if and only if \( k \in J_\ell := (\ell/2 + [-N, N]) \cap \mathbb{Z} \). Using (17), we get

\[
I_2 = \left| D \sum_{k \in \mathbb{Z}} F^{[n]} (2^{-1} \ell - k) D^n (v_g (2^{-n} k) - v_f (2^{-n} k)) \right|_\infty \\
= \left| D \sum_{k \in J_\ell} F^{[n]} (2^{-1} \ell - k) D^n (v_g (2^{-n} k) - v_f (2^{-n} k)) \right|_\infty \\
\leq \left| D \right|_\infty \sum_{k \in J_\ell} \left| F^{[n]} (2^{-1} \ell - k) \right|_\infty \left| D^n (v_g (2^{-n} k) - v_f (2^{-n} k)) \right|_\infty \\
\leq \left| D \right|_\infty \left( \sum_{k \in J_\ell} \left| F^{[n]} (2^{-1} \ell - k) \right|_\infty \right) \left( \sup_{k \in J_\ell} \left| D^n (v_g (2^{-n} k) - v_f (2^{-n} k)) \right|_\infty \right) \\
\leq \left| D \right|_\infty \#J_\ell \left\| F^{[n]} \right\|_\infty \left( \sup_{k \in J_\ell} \left| D^n (v_g (2^{-n} k) - v_f (2^{-n} k)) \right|_\infty \right). \quad (33)
\]

The estimate is now completely similarly to \( I_1 \), using the generalized Taylor expansion: For \( R < 1 \) there exists \( m \in \mathbb{N} \) such that for \( n \geq m \) we have

\[
\sup_{k \in J_\ell} \left| D^n (v_g (2^{-n} k) - v_f (2^{-n} k)) \right|_\infty \leq C_{A,R,f} 2^{-n(d+1)}.
\]

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Therefore, by continuing from (33), we obtain:

\[ I_2 \leq |D|_{\infty}(2N+1)\|F[n]\|_{\infty}C_{\Lambda,R,f}2^{-n(d+1)}, \quad n \in \mathbb{N}, \quad (34) \]

and since the norm of \( F[n] \) is bounded uniformly in \( n \) we combine (32) and (34) to obtain that for \( R < 1 \), there exist \( m \in \mathbb{N} \) and a constant \( C > 0 \), depending on \( \Lambda, R, f, N \) and the Hermite subdivision scheme, such that

\[ \|d^[[n]]_{\ell}\|_{\infty} \leq I_1 + I_2 \leq C2^{-n(d+1)}, \quad \ell \in \mathbb{Z}, \quad n \geq 0. \]

Since \( C \) is independent of \( \ell \in \mathbb{Z} \), this concludes the proof. \( \square \)

**Remark 4.** The assumption \( \sup_n \|F[n]\|_{\infty} < \infty \) of uniform boundedness of the limit functions is crucial and not easy to verify in general. However, like in Remark 2 it is valid again for asymptotically equivalent schemes, see once more [13]. In particular it holds true for \( V_{p,\Lambda} \)-preserving interpolatory schemes mentioned in Remark 2. Also note that uniform boundedness of the supports of the \( A^[n] \) is needed for describing convergence like in [2].

By a careful inspection of the proof of Theorem 11, we can derive the following improved version of this result which shows that the decay rate of the wavelet coefficients measures the local regularity of the function as it is typical for compactly supported wavelets.

**Corollary 12.** With the assumptions as in Theorem 11 the wavelet coefficients \( d^[[n]]_{\lfloor 2n+1x \rfloor} \), \( x \in \mathbb{R} \), depend on \( v_f(y), y \in x+2^{-n}[-N - \frac{1}{2}, N + \frac{1}{2}] \).

**Proof.** Let \( \ell := \lfloor 2n+1x \rfloor \), hence \( \ell \in [2n+1x, 2n+1x+1] \). The estimate of \( I_1 \) in (32) depends on the values of \( f \) and its derivatives at \( 2^{-n-1}\ell \in [x, x+2^{-n-1}] \) and \( 2^{-n-1}(\ell-1) \in [x-2^{-n-1}, x] \), hence, in total on the behavior of \( f \) on \( x+2^{-n-1}[-1, 1] \). On the other hand, the set \( J_\ell \) in the estimate of (34) of \( I_2 \) satisfies \( J_\ell \subset \ell/2 + [-N, N] \), hence

\[ 2^{-n}J_\ell \subset 2^{-n-1}\lfloor 2n+1x \rfloor + 2^{-n}[-N, N] \subset x+2^{-n}[-N, N+1/2] \).

Thus, the term \( v_f(2^{-n}k), k \in J_\ell \), involves only values from that interval while \( v_g(2^{-n}k), k \in J_\ell \), uses \( x+2^{-n}[-N - \frac{1}{2}, N] \).

Corollary 12 is the justification to use multiwavelets for edge detection. If the sampled vector data leads to wavelet coefficients that do not decay like \( 2^{-n(d+1)} \), then the underlying function cannot be \( C^d \) at a position specified by the location of the slowly decaying wavelet coefficients. The higher the level of the wavelet coefficients is, the better the localization of the singularity, reproducing a well-known wavelet effect, cf. [21].
Conclusion

In this paper we presented results on level-dependent Hermite subdivision schemes preserving polynomial and exponential data, focusing on the interpolatory case, which allows to naturally obtain multiwavelet systems via the prediction-correction approach. Such wavelets possess a generalized vanishing moment property with respect to elements in the space spanned by exponentials and polynomials. Vanishing moments can be crucial for data compression purposes, in particular when such systems are applied to data exhibiting transcendental features. In addition, a result concerning the decay of the wavelet coefficients corresponding to any \( f \in C^d_{\alpha}(\mathbb{R}) \) is proved, yielding an analogous extension of the classical result in the standard wavelet theory. To the best of our knowledge, this result has never been presented even in the case of standard (non level-dependent) Hermite multiwavelets. In order to prove it, a generalized Taylor formula in the space \( V_{d,A} \) is introduced, which may be of independent interest, and error bounds on the deviation from the classical Taylor polynomial approximation are given. Future research includes the extension of our results to the case of manifold-valued data.

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