Cosine-of-phase sensitivity of a Mach–Zehnder interferometer for the Fock state inputs

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Abstract. In the case of the Mach-Zehnder interferometer fed with Fock states we speak of estimation of the cosine of phase instead of estimation of the phase. In case the numbers of input photons are equal, we pay attention to a proposal by Kim et al. (1998). Then we restrict ourselves to the estimation of the cosine of double phase. Although such restrictions can be lifted, they harbinger a unified approach both to the phase sensitivity and to the Cramér–Rao lower bound of the estimator variance.

1. Introduction
It is well-known that the interferometer is a fundamental apparatus in optical physics. Here we will deal with the Mach–Zehnder interferometer in the quantum regime. In classic work [1] this interferometer was studied together with other kinds of interferometers using group-theoretic formalism and the use of concepts related to the representation of the group SU(2) was typical of this interferometer, which consists of two lossless beamsplitters. These notions enable us to express the effect of the first beamsplitter, the effect of the phase shifts in the arms of the interferometer and the effect of the second beamsplitter on the light driving the interferometer. The apparatus is sensitive to the relative phase shift. The analysis has mainly been connected with the assumption that the input state of the interferometer is an eigenstate of the photon-number sum operator. As a consequence, the sum of quantum phases has the greatest possible uncertainty [2]. The analysis which would be content with the assumption of the input coherent states would “satisfy” also the assumption of input uncorrelated states. On selecting a photon-number sum $N$ in this situation, we generate a nonlocal correlation of the interferometric modes and this way the ordinary coherent states become the SU(2) generalized coherent states. In place of the Hilbert space of the two modes, a diminished Hilbert space can be considered.

Yurke, McCall, and Klauder have initiated an analysis showing that the states optimized respective to fluctuations of the photon-number difference and those of the phase difference are of greater importance than the SU(2) generalized coherent states [1]. On using illustrations of quasidistributions of suitable quadratic operators in annihilation and creation operators of the two modes, the role of such states has been shown at the input of the interferometer, i.e., at the input of the first beamsplitter, at the output of the first beamsplitter, or in the arms of the interferometer, and at the output of the second beamsplitter. From these illustrations as well as from a very simple explicit formula it is obvious that the reduction of fluctuations of the photon-number difference is realized in the input state, not the reduction of the phase difference noise. But for the distribution of the output photon-number difference to have good behaviour,
the state for interferometric estimation of the relative phase shift is to have an at least a little preferred value of difference phases at the input ports. Then the peak of the distribution of the photon-number difference at the output ports displaces in both directions according to the relative phase shift. In [1] an input state has been proposed which has the phase sensitivity of the order of $1/N$.

At the output of the first beamsplitter, the fluctuations of the photon-number difference are large and those of the phase difference are small. The use of the quasidistributions can be combined with the knowledge of a classical beamsplitter, where the zero value of the output phase difference follows from zero value of the input amplitude difference and zero of the output amplitude difference ensues from a zero of the input phase difference. A number of papers have focused on the search of the optimal states for this stage and have approached the solution independent of details [3, 4, 5, 6]. The study [7] has followed where it is shown that quantum estimators of the relative phase shift are investigated and that one must distinguish between a single measurement and a multiple measurement. In the situations of the multiple measurement, great performance differences derived for a single measurement level out. The Cramér–Rao lower bound and the Fisher information [8] have been used for the estimation analysis.

Although the argument in [1] is convincing, it can be formulated in terms of minimum-uncertainty states with respect to the suitable quadratic operators in annihilation and creation operators [9]. The phase sensitivity scaling as $1/N$ has been shown for the minimum-uncertainty states (intelligent states), which are very close to the Fock states with equal photon numbers. In [9] the input Fock states $|n_1, n_2\rangle$ have been studied, in which the difference of the quantum phases is quite uncertain (no preferred phases) and their good properties have been shown in the case $n_1 = n_2$. In this case, the phase sensitivity or the minimum detectable phase shift equals infinity, quite a pessimistic characteristic. Even if that paper does not comprise a quasidistribution, it presents the phase-difference distribution between the outputs of the beamsplitter both for the input state proposed and for the input coherent state $|\sqrt{n_1}, \sqrt{n_2}\rangle$. This comparison demonstrates that the distribution of the phase difference is at the Heisenberg limit in the input Fock states $|n_1, n_2\rangle$, $n_1 = n_2$. In [9] the distribution of the photon-number difference between the output arms of the interferometer is also presented. This distribution is concentrated in zero difference for zero relative phase shift and otherwise it spreads in both directions. The symmetric distribution expresses only the modulus of a (small) relative phase shift, not its sign. In [9] the phase sensitivity has not been expressed as the minimum detectable phase shift. The reason why it has not been realized will be obvious below. A Bayesian estimate of the phase shift has been provided. It serves for combining information from several (1, 2, 5 and 10) spatially independent modes in the interferometer. The restriction to a half of the symmetric distribution is quite natural and it is commented on neither for the applications where the sign of the phase shift does not matter nor for the cases where, unfortunately, this sign does matter. In a fashion appropriate for the Bayesian inference, the dependence of the quality of estimating the phase shift on the photon number of one of the input Fock states is shown. A “realization” (“trajectory” of “random process”) is visualized consisting in the mean and the standard error of the posterior distribution of the phase shift. On the same procedure in the interferometer driven by a coherent light, the Heisenberg and the classical limit can be discerned in the respective illustrations.

Paper [10] presents a thorough study on nonclassical interferometry with intelligent light. Paper [11] concerns the neutron (more generally fermion) interferometry. It is useful even for the light (more generally boson) interferometry, because the theory that depends only on the counted statistics is valid for both fermions and bosons. A repetition of measurements can be utilized not only for lowering the variance of the estimate of the cosine of the phase shift (the cosine of the double phase shift), but with an auxiliary shift $\pi/2$ (or $\pi/4$) in, let us say, a half of repeated measurements for the estimation of the sine of the phase shift (the sine of the double
phase shift). In [12] it has been indicated that, if the input Fock states have different photon numbers, the cosine of the phase shift has been estimated. If the input photon numbers are equal, only the cosine of the double phase shift is estimated. But the phase sensitivity of this estimate is of the order of $1/N$. With these observations in mind, we have found paper [12] interesting enough to base this one on it. A brief review [13] shows that the Fisher information provides meaningful error bound even in highly nonclassical regimes, namely for input Fock states.

In this paper we assume that the Mach–Zehnder interferometer is fed with a Fock state. We associate, with a measurement of the output photon-number difference, an unbiased estimator of the cosine of the phase shift. As is frequent with unbiased estimators, also this one may take on undesirable values, namely those outside the interval $[-1, 1]$. We investigate this and further estimators as the Hilbert-space operators. We associate, with a measurement of the square of the photon-number difference, an unbiased estimator of the cosine of the double phase shift. We express the standard deviation of the estimator of the cosine of the simple phase shift with the usual phase sensitivity and the standard deviation of the estimator of the cosine of the double phase shift with the phase sensitivity recently introduced in the literature. In fact, it is a reverse of the possible derivations of the two phase sensitivities. The Fisher information on the modulus of the phase-shift is the same for both kinds of measurement. Only the limit of zero-phase shift is well-known. We compare their reciprocal values with the square of the usual phase sensitivity and, remarkably, with the zero-phase-shift limit of the square of the novel phase sensitivity.

2. Mach–Zehnder interferometer and the modulus-of-phase sensitivity of measurement

We perform an analysis of a Mach–Zehnder interferometer consisting of two 50:50 beamsplitters $BS_1$ and $BS_2$ as shown in figure 1.

![Schematic of a Mach-Zehnder interferometer.](image)

The photon annihilation operators of the input modes of $BS_1$, $\hat{a}_1$, $\hat{a}_2$, satisfy the commutation relations

$$[\hat{a}_i, \hat{a}_j] = 0, \quad [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \hat{1}$$

for $i, j = 1, 2$. It is convenient to define four Hermitian operators

$$\hat{N} = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2, \quad \hat{J}_x = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1),$$
\[ \hat{J}_y = -\frac{i}{2}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1), \quad \hat{J}_z = \frac{1}{2}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2). \] (2)

The photon annihilation operators of the output modes of BS$_1$ are defined by the relations (here we differ from [12])

\[ \hat{a}_3 = \frac{1}{\sqrt{2}}\hat{a}_2 - i\frac{1}{\sqrt{2}}\hat{a}_1 = \hat{U}_1^\dagger \hat{a}_2 \hat{U}_1, \]
\[ \hat{a}_4 = -i\frac{1}{\sqrt{2}}\hat{a}_2 + \frac{1}{\sqrt{2}}\hat{a}_1 = \hat{U}_1^\dagger \hat{a}_1 \hat{U}_1, \] (3)

where

\[ \hat{U}_1 = \exp \left( -\frac{i\pi}{2} \hat{J}_x \right), \] (4)

and they satisfy commutation relations (1) for \( i, j = 3, 4 \). The photon-number sum is preserved,

\[ \hat{N} = \hat{a}_3^\dagger \hat{a}_3 + \hat{a}_4^\dagger \hat{a}_4. \] (5)

If we assume for simplicity that the phase shifts of the two arms of the interferometer are \( \theta_3 = 0, \theta_4 = -\theta \) (in fact we have \( \theta = \theta_3 - \theta_4 \)), the annihilation operators of the outputs, \( \hat{a}_5 \) and \( \hat{a}_6 \), are given by

\[ \hat{a}_5 = i\frac{1}{\sqrt{2}}\hat{a}_3 + \frac{1}{\sqrt{2}}\hat{a}_4 e^{-i\theta} \]
\[ = \hat{U}_2^\dagger_{1,2\rightarrow 4,3} \hat{U}_1^\dagger(\theta) \hat{a}_4 \hat{U}(\theta) \hat{U}_2^\dagger_{1,2\rightarrow 4,3}, \]
\[ \hat{a}_6 = \frac{1}{\sqrt{2}}\hat{a}_3 + i\frac{1}{\sqrt{2}}\hat{a}_4 e^{-i\theta} \]
\[ = \hat{U}_2^\dagger_{1,2\rightarrow 4,3} \hat{U}_1^\dagger(\theta) \hat{a}_3 \hat{U}(\theta) \hat{U}_2^\dagger_{1,2\rightarrow 4,3}, \] (6)

where

\[ \hat{U}_2 = \hat{U}_1^\dagger, \]
\[ \hat{U}(\theta) = \exp(-i\theta \hat{a}_3^\dagger \hat{a}_1), \] (7)

and the arrow indicates the replacements of subscripts, in fact, those of the annihilation operators. From the relations (6), (3) and (2) it follows that

\[ \hat{n}_5 = \hat{a}_3^\dagger \hat{a}_5 = \frac{1}{2} \hat{N} + \hat{J}_z \cos \theta - \hat{J}_x \sin \theta, \]
\[ \hat{n}_6 = \hat{a}_4^\dagger \hat{a}_6 = \frac{1}{2} \hat{N} - \hat{J}_z \cos \theta + \hat{J}_x \sin \theta. \] (8)

The photon-number sum is still conserved,

\[ \hat{N} = \hat{n}_5 + \hat{n}_6, \] (9)

but the other operators (2) can transform. For instance,

\[ \hat{J}_{z,\text{out}} = \frac{1}{2}(\hat{a}_3^\dagger \hat{a}_5 - \hat{a}_6^\dagger \hat{a}_6) = \hat{J}_z \cos \theta - \hat{J}_x \sin \theta, \] (10)

where we deviate in the middle of the formula.
We suppose that the state $|\psi\rangle$ with respect to which the distribution of the eigenvalues of any observable and a joint distribution of any set of compatible observables are computed is the Fock state $|n_1, n_2\rangle$. Such a state is an input state as to the property

$$\hat{n}_i|n_1, n_2\rangle = n_i|n_1, n_2\rangle, \quad i = 1, 2,$$

(11)

where

$$\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i, \quad i = 1, 2.$$

(12)

When the distribution of the eigenvalue of some operator is concentrated in some value, we say that the operator takes this value in the state $|\psi\rangle$. In the case under investigation, the operator $\hat{J}_z$ has the value $J_z = (n_1 - n_2)/2$, the operator $\hat{J}_z^2$ has the value $J_z^2 = (n_1 - n_2)^2/4$, and so on.

The modulus of the phase shift, $|\theta|$, can be inferred from the results of some measurements performed on the outputs. Let $M$ be a Hermitian operator corresponding to the measurement $M$ of a certain scheme. If the expectation value $\langle \hat{M}\rangle$ is a monotonous function of $|\theta|$, then from its inverse the modulus of the phase shift can be determined. We will see that the assumption is fulfilled for $\theta$ small. The variance of the operator $\hat{M}$ given by

$$\langle (\Delta \hat{M})^2 \rangle = \langle \hat{M}^2 \rangle - \langle \hat{M} \rangle^2$$

(13)

determines the modulus-of-phase sensitivity, or the uncertainty of the phase-shift measurement $\sqrt{\langle (\Delta |\theta|_M)^2 \rangle}$ through the relation

$$\langle (\Delta |\theta|_M)^2 \rangle = \left(\frac{\langle (\Delta \hat{M})^2 \rangle}{\frac{\partial \langle \hat{M} \rangle}{\partial |\theta|}}\right)^2.$$

(14)

### 3. Cosine-of-phase sensitivity of measurement

Let $M_1$ be a measurement of the photon number $\hat{n}_5$ at one output port and $M_2$ be a measurement of the operator $\hat{J}_z,_{out} = (\hat{n}_6 - \hat{n}_5)/2$, where $\hat{n}_6$ is the photon number at the other output port [12]. We search a quantum estimator or quantum estimators, to which these measurements lead. To find them, we will use the moment method. We consider several first moments, sometimes a single moment, of a random variable to be estimated with the sample moments, i.e., in the case of a sample of size one with respective powers of the random variable, the mean to be estimated with “the” random variable.

So let us observe that

$$\langle \hat{n}_5 \rangle = \frac{1}{2} N + J_z \cos \theta$$

(15)

and let us consider the moment on the right-hand side to be estimated with the measured number $n_5$. We will also speak of the quantum estimator $\hat{n}_5$. Since the quantities $N$ and $J_z$ are known, by introducing the quantum estimator $\hat{n}_5$ we also introduce the quantum estimator

$$\hat{\cos M_1} \theta = \frac{2\hat{n}_5 - (n_1 + n_2)}{n_1 - n_2}. $$

(16)

There is a simpler relation than (15),

$$\langle \hat{J}_z,_{out} \rangle = J_z \cos \theta,$$

(17)

and hence we arrive at the estimator

$$\hat{\cos M_2} \theta = \frac{2\hat{J}_z,_{out}}{n_1 - n_2}. $$

(18)
This operator differs from the operator (16) on the Hilbert space of the two modes, but, on the diminished Hilbert space, the two quantum estimators are the same. That is why we will restrict ourselves to the latter.

It appears that the moment method does not “operate” for \( n_1 = n_2 \) and it also appears that otherwise it does not provide a sensible estimator (the estimator (18) is not “sensible”), the eigenvalues of such a “quantum cosine” not being within the interval \([-1, 1]\). But it holds that an unbiased estimator of the parametric function \( \cos \theta \) has been derived

\[
\langle \cos_{M2}(\theta) \rangle = \cos \theta,
\]

and its variance is

\[
\langle (\Delta \cos_{M2}\theta)^2 \rangle = \langle (\Delta \hat{\theta}_{M2})^2 \rangle (\sin \theta)^2,
\]

where \( \sqrt{\langle (\Delta \hat{\theta}_{M2})^2 \rangle} \) means the usual phase sensitivity [12],

\[
\langle (\Delta \hat{\theta}_{M2})^2 \rangle = \frac{2n_1n_2 + n_1 + n_2}{(n_1 - n_2)^2}.
\]

An advantage of the phase sensitivity seems to be its not being dependent on any unknown parameter, while the cosine-of-phase sensitivity does depend on some. It seems that this characteristic offers an unrealistically perfect, extreme, sensitivity for \( \theta = 0 \). But with the dependence on an unknown parameter we deal either so that in a sense of a beforehand lost “game against the Nature” we select \( \theta \) least advantageous for us and most advantageous for the rival, or – because such an approach can be hardly accomplished – so that we adopt the Bayesian approach to the estimation [11].

In [13] the Fisher information associated with the modulus-of-phase estimation \( F(\theta)|_{\theta \neq 0} \) has been derived approximately. Introducing

\[
F(0) := \lim_{\theta \to 0} F(\theta)|_{\theta \neq 0} = 2n_1n_2 + n_1 + n_2,
\]

we see that \( F(\theta)|_{\theta \neq 0} \approx F(0) \) for \( \theta \to 0 \). Noting that the unitary matrix which can be used for transforming \( (\hat{a}_1, \hat{a}_2) \) onto \( (\hat{a}_5, \hat{a}_6) \) in the Fock-state basis has real elements, we derive easily that

\[
F(\theta)|_{\theta \neq 0} = 4 \left\langle \hat{J}_y^2 \right\rangle \\
= F(0) \quad \text{independent of } 0.
\]

Every unbiased estimator \( \hat{M} \) of the parametric function \( \cos \theta \), \( \langle \hat{M} \rangle = \cos \theta \), obeys the Cramér–Rao inequality,

\[
\langle (\Delta \hat{M})^2 \rangle \geq \frac{(\sin \theta)^2}{F(\theta)},
\]

where \( F(\theta) = F(0) \). To the second order in \( \theta \),

\[
\langle (\Delta \hat{M})^2 \rangle \geq \frac{(\sin \theta)^2}{2n_1n_2 + n_1 + n_2}.
\]
We may apply this result, take \( \hat{M} = \hat{\cos M^2} \) and it holds that
\[
\langle [\Delta \hat{\cos M^2}(\theta)]^2 \rangle \geq \frac{(\sin \theta)^2}{2n_1n_2 + n_1 + n_2}.
\] (27)

Let \( M_3 \) be the measurement of the operator \( \hat{J}^2_{z,\text{out}} \) and \( M_4 \) be the coincidence detection of the numbers, i.e., of the operator \( \hat{N}_c = : \hat{n}_5 \hat{n}_6 : \) [12], : : stands for the normal ordering. We search a quantum estimator, or quantum estimators to which these measurements lead. To this aim we will again use the moment method.

So let us observe that
\[
\langle \hat{\cos M^2}(\theta) \rangle = \frac{1}{8} A - \frac{1}{8} B \cos(2\theta),
\] (28)
where
\[
A = 4(\langle \hat{j}_{z}^2 \rangle + \langle \hat{j}_{x}^2 \rangle) = n_2(1 + n_1) + n_1(1 + n_2) + (n_1 - n_2)^2
\] 
\[= n_1(n_1 + 1) + n_2(n_2 + 1),
\] (29)
\[
B = 4(\langle \hat{j}_{z}^2 \rangle - \langle \hat{j}_{x}^2 \rangle) = n_2(1 + n_1) + n_1(1 + n_2) - (n_1 - n_2)^2
\] 
\[= 2n_1n_2 + n_1 + n_2 - (n_1 - n_2)^2,
\] (30)
and we consider the moment on the right-hand side of (28) to be estimated with the measured “squared component of angular momentum” \( \hat{J}^2_{z,\text{out}} \). We will also speak of the quantum estimator \( \hat{\cos M^3}(2\theta) = A \hat{1} - \frac{1}{8} B \cos(2\theta) \).

Similarly,
\[
\langle : \hat{n}_5 \hat{n}_6 : \rangle = \frac{1}{4}(n_1 + n_2)^2 - \langle \hat{j}_{z}^2 \rangle(\sin \theta)^2 - \langle \hat{j}_{x}^2 \rangle(\cos \theta)^2
\] 
\[= \frac{1}{4}(n_1 + n_2)^2 - \frac{1}{8} A + \frac{1}{8} B \cos(2\theta),
\] (32)
and hence we arrive at the estimator
\[
\hat{\cos M^4}(2\theta) = \frac{[A - 2(n_1 + n_2)^2] \hat{1} + : \hat{n}_5 \hat{n}_6 : }{B}.
\] (33)

This operator differs from the operator (31) on the Hilbert space of the two modes, but, on the diminished Hilbert space, the two quantum estimators are the same. That is why we will restrict ourselves to the former.

The estimator exists for \( n_1 = n_2 \). Independently of this, it is not sensible, the eigenvalues of such a quantum cosine may not be within the interval \([-1, 1]\). An unbiased estimator of the parametric function \( \cos(2\theta) \) has been derived
\[
\langle \hat{\cos M^3}(2\theta) \rangle = \cos(2\theta),
\] (34)
and its variance is
\[
\langle [\Delta \hat{\cos M^3}(2\theta)]^2 \rangle = \langle (\Delta \hat{\theta_{M^3}})^2 \rangle (2 \sin(2\theta))^2,
\] (35)
where $\sqrt{\langle (\Delta \hat{\theta}_{M3})^2 \rangle}$ means the phase sensitivity [12],

$$\langle (\Delta \hat{\theta}_{M3})^2 \rangle = \frac{4[C(\tan \theta)^2 + D]}{B^2},$$

(36)

with

$$C = \frac{1}{8} \left[ n_1^2 n_2^2 + n_1 n_2(n_1 + n_2 - 1) - (n_1 + n_2)(n_1 - n_2)^2 \right],$$

(37)

$$D = \frac{1}{4} \left[ ((n_1 - n_2)^2 + 1)(2n_1 n_2 + n_1 + n_2) - 2(n_1 - n_2)^2 \right].$$

(38)

We note that

$$\langle (\Delta \hat{\theta}_{M3})^2 \rangle \geq \frac{1}{F(\theta)},$$

(39)

but for $n_1 > 1$ or $n_2 > 1$, and $\theta \neq 0$ the equality sign is not attained. The phase sensitivity introduced in (36) seems to be useful for optics, although no estimator of the phase shift exists, on which – on whose variance and bias – the phase sensitivity would be based. Here in contrast we introduce the phase-of-cosine sensitivity and indicate that the phase sensitivity is derived from the cosine-of-phase sensitivity.

The phase sensitivity (36) depends on an unknown parameter. For $n_1 = n_2 = n$ the attainment of the Heisenberg limit can be derived in the case of a single measurement [12]. The meaning may also be doubted, since the derivation issues from the assumption $\theta = O\left(\frac{1}{n}\right)$, although in that paper only small values of $\theta$ are treated. For $\theta = 0$, the phase sensitivity is expressed as follows

$$\langle (\Delta \hat{\theta}_{M3})^2 \rangle_{\theta=0} = \frac{1}{2n(n+1)} = \frac{1}{F(0)} .$$

(40)

It seems that the cosine-of-phase sensitivity (35) offers an unrealistically perfect, extreme, sensitivity for $\theta = 0$. The interferometer can be adjusted so that $\theta = 0$. The output state is equal to the input state and the photon numbers are constant (being either different or the same), and the photon-number difference is constant as well. The variance of a constant is equal to zero (from the probability theory). Since $A = B + 2(n_1 - n_2)^2$, from relation (31) it follows that the measurement of the operator $\cos_{M3}(2\theta)$ results in unity, a constant.

Every unbiased estimator $\hat{M}$ of the parametric function $\cos(2\theta)$, $\langle \hat{M} \rangle = \cos \theta$, obeys the Cramér–Rao inequality,

$$\langle (\Delta \hat{M})^2 \rangle \geq \frac{[2\sin(2\theta)]^2}{F(\theta)} .$$

(41)

To the second order in $\theta$,

$$\langle (\Delta \hat{M})^2 \rangle \geq \frac{[2\sin(2\theta)]^2}{2n_1 n_2 + n_1 + n_2} .$$

(42)

For $n_1 = n_2 = n$ we may apply this result, take $\hat{M} = \cos_{M3}(2\theta)$ and it holds that

$$\langle (\Delta \cos_{M3}(2\theta))^2 \rangle \geq \frac{[2\sin(2\theta)]^2}{2n(n+1)} .$$

(43)

We see that the equality is attained – at least to the second order in $\theta$.

In brief, we mention the interesting topic of multiple phase-shift measurements. This is appropriate because the Heisenberg limit has not been attained for $\theta \neq 0$. We let $M$ denote the number of parallel or serial interferometric measurements. In [13], it has been derived, using a very convincing approximation, that the uncertainty of the estimation assessed as the variance of the posterior probability distribution for the phase-shift scales as $O(1/\ln N)$ for $M = 1$. The
estimation improves for $M = 2$ and the uncertainty scales as $O(1/N)$. For $M \geq 4$, it scales as $O(1/N^2)$, i.e., the Heisenberg limit is achieved. The posterior probability distribution has been considered only in the case where the result $n_1 = n_2 = n$ occurs $M$ times.

4. Conclusions
The usual phase sensitivity of a Mach–Zehnder interferometer in the quantum regime can be derived for an input Fock state with different photon numbers. Dealing with an input Fock state with equal photon numbers, one obtains an infinite phase (in)sensitivity, because the output photon number difference has the expectation value that does not depend on the phase shift. We have remarked that in both the cases rather a modulus-of-phase sensitivity is looked for.

From the heuristic viewpoint the expectation value of the observable measured at the output ports defines the function of the phase shift whose unbiased estimate is of interest. Usually we assume that the output photon-number difference is measured. This assumption is not quite evident, especially when we also assume that the Mach–Zehnder interferometer is fed with a Fock state with equal photon numbers. In this case then the expectation value of the photon-number difference does not depend on the phase shift and does not define an interesting function of the phase shift. It gets better, when the square of the photon-number difference is declared to be the observable under measurement as realized by Kim et al. in [12].

The Fisher information expresses the sensitivity of the distribution of the observable under measurement in its totality to a change of the parameter, not only a sensitivity of the expectation value of this observable, i.e., of the reduced distribution. The Fisher information is invariant. Passing over to another measured quantity, which is a one-to-one function of the original measured observable, the Fisher information expresses the sensitivity of the distribution whose dependence on the parameter has been formally changed, with the same value. This is valid also for two-to-one functions of the original measured observable, if the distribution features an appropriate symmetry. In the case under consideration the photon-number differences $n_1 - n_2 = 2m$ and $-2m$ are measured with equal probabilities, which is a symmetry appropriate to the squaring of the number difference.

The estimator of the cosine of the phase shift in terms of the output photon-number difference attains the Cramér–Rao lower bound only for $n_2 = 0$ and $n_1 = 0$. The estimator of the cosine of the double phase shift in terms of the square of the output photon-number difference achieves the Cramér–Rao lower bound only for $n_2 = 0$, $n_1 = n_2$, and $n_1 = 0$. For $n_1 = n_2 = n$ the standard deviation of the estimator of the cosine of the double phase shift scales as $1/N$, where $N = n_1 + n_2 = 2n$, but, we have found that when $n_1 > 1$ or $n_2 > 1$, the actual phase shift should scale as $1/N$ too.

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