A BILINEAR T(B) THEOREM FOR SINGULAR INTEGRAL OPERATORS

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ABSTRACT. In this work, we present a bilinear T(b) theorem for singular integral operators of Calderón-Zygmund type. We prove some new accretive type Littlewood-Paley theory and bilinear paraproduct for a para-accretive function setting. We also introduce a criterion for extending certain $L^p$ Calderón reproducing formulas to convergence in $H^1$.

1. INTRODUCTION

In the development of Calderón-Zygmund singular integral operator theory, measuring cancellation of operators via testing conditions has become a central theme through T1 and Tb theorems. In the 1980’s, David-Journé [8] proved the original T1 theorem, which gave a characterization of $L^2$ boundedness for Calderón-Zygmund operators. Driven by the Cauchy integral operator, in the late 1980’s David-Journé-Semmes [9] and McIntosh-Meyer [25] proved Tb theorems, which are also characterizations of $L^2$ bounds for Calderón-Zygmund operators based on perturbed testing conditions. We state the version from [9] to compare to the bilinear version we present in this work.

**Tb Theorem.** Let $b_0, b_1$ be para-accretive functions. Assume that $T$ is a singular integral operator of Calderón-Zygmund type associated to $b_0, b_1$. Then $T$ can be extended to a bounded operator on $L^2$ if and only if $M_{b_0}T M_{b_1}$ satisfies the weak boundedness property and $M_{b_0} T b_1, M_{b_1} T^* b_0 \in BMO$.

From the late 1980’s to the early 2000’s, multilinear Calderón-Zygmund theory was developed and a multilinear T1 theorem was proved by Christ-Journé [7], Kenig-Stein [21], and Grafakos-Torres [14], but to date there has been no multilinear Tb theorem. In this work we prove a bilinear Tb theorem, which can be naturally extended to a multilinear Tb theorem. The proof presented in this work does not rely on the linear Tb theorem of David-Journé-Semmes [9] or McIntosh-Meyer [25]. Furthermore a new proof of the linear Tb theorem can be easily extracted from the work in this paper. We now state the main result of this work.

**Theorem 1.1.** Let $b_0, b_1, b_2$ be para-accretive functions. Assume that $T$ is a bilinear singular integral operator of Calderón-Zygmund type associated to $b_0, b_1, b_2$. Then $T$ can be extended to a bounded operator from $L^{p_1} \times L^{p_2}$ into $L^p$ for all $1 < p_1, p_2 < \infty$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ if and only if $M_{b_0} T (M_{b_1} \cdot, M_{b_2} \cdot)$ satisfies the weak boundedness property and $M_{b_0} T (b_1, b_2), M_{b_1} T^{+1} (b_0, b_2), M_{b_2} T^{+2} (b_1, b_0) \in BMO$. 

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The meaning of $M_{b_t}T(b_1, b_2) \in BMO$ is not necessarily clear here, but we define this notations in Section 2. We also define other terminology use here, para-accrctive, singular integral operators of Calderón-Zygmund type, weak boundedness property, etc.

Calderón [4] proved some convergence results for a reproducing formula of the form

$$\int_0^\infty \phi_t * \phi_t * f \frac{dt}{t} = f,$$

for appropriate functions $\phi_t$, which came to be known as Calderón’s reproducing formula. The convergence of Calderón’s reproducing formula holds in many topologies: In certain $H^p$ spaces due to Calderón [5], in distribution due to Janson-Taibleson [19], and in $L^2$ by Frazier-Jawerth-Weiss [11], many others among others. This formula has since been generalized and reformulated in many ways. For example, Han [15] proved a perturbed Calderón reproducing formula for $L^p$ in a perturbed, para-accrctive setting. In this work, we consider discrete versions of Calderón’s formula where we replace convolution with $\phi_t$ with certain non-convolution integral operators indexed by a discrete parameter $k \in \mathbb{Z}$ instead of the continuous parameter $t > 0$. We prove criterion for extending the convergence of perturbed discrete Calderón reproducing formulas from $L^p$ spaces to $H^1$. More precisely, we will prove:

**Theorem 1.2.** Let $b \in L^\infty$ be para-accrctive functions and $\Theta_k$ be a collection of Littlewood-Paley square function kernels such that $\Theta_k b = \Theta_k^* b = 0$ for all $k \in \mathbb{Z}$. Also assume that

$$\sum_{k \in \mathbb{Z}} M_{b} \Theta_k M_{b} f = b f$$

for any $f \in C^\infty_0$ such that $b f$ has mean zero, where the convergence holds in $L^p$ for some $1 < p < \infty$. If $\phi \in C^\infty_0$ for some $1 < \delta \leq 1$ such that $b \phi$ has mean zero, then $b \phi \in H^1$ and

$$\sum_{k \in \mathbb{Z}} M_{b} \Theta_k M_{b} \phi = b \phi,$$

where the convergence holds in $H^1$.

Here, we take the typical definition of $H^1$ with norm $||f||_{H^1} = ||f||_{L^1} + \sum_{\ell=1}^n ||R_\ell f||_{L^1}$, where $R_\ell$ is the $\ell^{th}$ Reisz transform in $\mathbb{R}^n$ for $\ell = 1, \ldots, n$, $R_\ell f = c_n \left( p.v. \frac{1}{|y|^n} * f \right)$ and $c_n$ is a dimensional constant. Theorem 1.2 tells use that anytime we have convergence of Calderón’s reproducing formula in $L^p$ for some $p$, then it also converges in $H^1$, for appropriate operators and functions.

This article is organized the following way: In Section 2, we set notation and give a few pertinent definitions. In Section 3, we prove a few almost orthogonality estimates for bilinear Littlewood-Paley square function kernels and operators. In Section 4, we prove a number of convergence results in various spaces, including the $H^1$ convergence stated in Theorem 1.2. In Section 5, we prove an estimate closely related to bilinear Littlewood-Paley square function theory, which will serve as an estimate for truncated Calderón-Zygmund operators. In Section 6, we complete the proof of Theorem 1.1 by proving a reduced Tb theorem and constructing a bilinear paraproduct for para-accrctive perturbed setting.

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2. Definitions and Preliminaries

We first define para-accretive functions as one of several equivalent definitions provided by David-Journé-Semmes [9].

**Definition 2.1.** A function \( b \in L^\infty \) is para-accretive if \( b^{-1} \in L^\infty \) and there is a \( c_0 > 0 \) such that for every cube \( Q \), there exists a sub-cube \( R \subset Q \) such that

\[
\frac{1}{|Q|} \left| \int_R b(x)dx \right| \geq c_0.
\]

Many results involving para-accretive functions were proved by David-Journé-Semmes [9], McIntosh-Meyer [25], and by Han in [15]. We will use a number of the results from [9] and [15] in this work.

2.1. Bilinear Singular Integrals Associated to Para-Accretive Functions. Next we introduce the Hölder continuous spaces and para-accretive perturbed Hölder spaces. These are the functions spaces that we use to form our initial weak continuity assumption for \( T \) in Theorem 1.1, similar to the linear \( T_b \) theorem in [9].

**Definition 2.2.** Define for \( 0 < \delta \leq 1 \) and \( f : \mathbb{R}^n \to \mathbb{C} \)

\[
\|f\|_\delta = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{1+\delta}},
\]

and the space \( C^\delta = C^\delta(\mathbb{R}^n) \) to be the collection of all functions \( f : \mathbb{R}^n \to \mathbb{C} \) such that \( \|f\|_\delta < \infty \). Also define \( C^\delta_0 = C^\delta_0(\mathbb{R}^n) \) to be the subspace of all compactly supported functions in \( C^\delta \). It follows that \( \| \cdot \|_\delta \) is a norm on \( C^\delta_0 \). Despite conventional notation, we will take \( C^1 \) and \( C^1_0 \) to be the spaces of Lipschitz continuous functions to keep our notation consistent. Let \( b \) be a para-accretive function and define \( bC^\delta_0 \) to be the collection of functions \( bf \) such that \( f \in C^\delta_0 \) with norm \( \|bf\|_\delta = \|f\|_\delta \). Also let \( (bC^\delta_0)' \) be the collection of all sequentially continuous linear functionals on \( bC^\delta_0 \), i.e. a linear functional \( W : bC^\delta_0 \to \mathbb{C} \) is in \( (bC^\delta_0)' \) if and only if

\[
\lim_{k \to \infty} \|f_k - f\|_\delta = 0 \text{ where } f_k, f \in C^\delta_0 \implies \lim_{k \to \infty} \langle W, bf_k \rangle = \langle W, bf \rangle,
\]

where these are both limits of complex numbers. Given a topological space \( X \), we say that an operator \( T : X \to (bC^\delta_0)' \) is continuous if

\[
\lim_{k \to \infty} x_k = x \text{ in } X \implies \lim_{k \to \infty} \langle T(x_k), bf \rangle = \langle T(x), bf \rangle \text{ for all } f \in C^\delta_0.
\]

Given a bilinear operator \( T : b_1C^\delta_0 \times b_2C^\delta_0 \to (b_0C^\delta_0)' \) for some \( \delta > 0 \), define the adjoints of \( T \) for \( f_1, f_2, b \in C^\delta_0 \)

\[
\langle T^{1*}(b_0f_0, b_2f_2), b_1f_1 \rangle = \langle T^{2*}(b_1f_1, b_0f_0), b_1f_1 \rangle = \langle T(b_1f_1, b_2f_2), b_0f_0 \rangle.
\]

Then the adjoints of \( T \) are bilinear operators acting on the following spaces: \( T^{1*} : b_0C^\delta_0 \times b_2C^\delta_0 \to (b_1C^\delta_0)' \) and \( T^{2*} : b_1C^\delta_0 \times b_0C^\delta_0 \to (b_2C^\delta_0)' \). One could more generally define the adjoint \( T^{1*} \) on \( (b_1C^\delta_0)'' \times b_1C^\delta_0 \), but this is not necessary for this work. So we restrict the first spot of \( T^{1*} \) to \( b_1C^\delta_0 \) instead of \( (b_1C^\delta_0)'' \). Likewise for \( T^{2*} \).
Definition 2.3. A function $K : \mathbb{R}^{3n} \setminus \{(x,x,x) : x \in \mathbb{R}^n\} \to \mathbb{C}$ is a standard bilinear Calderón-Zygmund kernel if

$$|K(x,y_1,y_2)| \lesssim \frac{1}{(|x-y_1| + |x-y_2|)^{2n}} \text{ when } |x-y_1| + |x-y_2| \neq 0$$

$$|K(x,y_1,y_2) - K(x',y_1,y_2)| \lesssim \frac{|x-x'|}{(|x-y_1| + |x-y_2|)^{2n+1}}$$

when $|x-x'| < \max(|x-y_1|,|x-y_2|)/2$

$$|K(x,y_1,y_2) - K(x,y_1',y_2)| \lesssim \frac{|y_1 - y_1'|}{(|x-y_1| + |x-y_2|)^{2n+1}}$$

when $|y_1 - y_1'| < \max(|x-y_1|,|x-y_2|)/2$

$$|K(x,y_1,y_2) - K(x,y_1,y_2')| \lesssim \frac{|y_2 - y_2'|}{(|x-y_1| + |x-y_2|)^{2n+1}}$$

when $|y_2 - y_2'| < \max(|x-y_1|,|x-y_2|)/2$.

Let $b_0, b_1, b_2 \in L^\infty(\mathbb{R}^n)$ be para-accretive functions. We say a bilinear operator $T : b_1 C_0^\delta \times b_2 C_0^\delta \to (b_0 C_0^\delta)'$ is a bilinear singular integral operator of Calderón-Zygmund type associated to $b_0, b_1, b_2$, or for short a bilinear C-Z operator associated to $b_0, b_1, b_2$, if $T$ is continuous from $b_1 C_0^\delta \times b_2 C_0^\delta$ into $(b_0 C_0^\delta)'$ for some $\delta > 0$ and there exists a standard Calderón-Zygmund kernel $K$ such that for all $f_1, f_2, f_3 \in C_0^\delta$ with disjoint support

$$\langle T(M_{b_1} f_1, M_{b_2} f_2), M_{b_0} f_0 \rangle = \int_{\mathbb{R}^{3n}} K(x, x_1, x_2) \prod_{i=1}^{2} f_i(y_i) b_1(y_i) dy_i.$$

Note that this continuity assumption for $T$ from $b_1 C_0^\delta \times b_2 C_0^\delta$ into $(b_0 C_0^\delta)'$ is equivalent to the following: For any $f_0, f_1, f_2, g, g_k \in C_0^\delta$ such that $g_k \to g$ in $C_0^\delta$, we have

$$\lim_{k \to \infty} \langle T(M_{b_1} g_k, M_{b_2} f_2), M_{b_0} f_0 \rangle = \langle T(M_{b_1} g, M_{b_2} f_2), M_{b_0} f_0 \rangle,$$

$$\lim_{k \to \infty} \langle T(M_{b_1} f_1, M_{b_2} g_k), M_{b_0} f_0 \rangle = \langle T(M_{b_1} f_1, M_{b_2} g), M_{b_0} f_0 \rangle,$$

$$\lim_{k \to \infty} \langle T(M_{b_1} f_1, M_{b_2} f_2), M_{b_0} g_k \rangle = \langle T(M_{b_1} f_1, M_{b_2} f_2), M_{b_0} g \rangle.$$

It follows that the continuity assumptions for a bilinear singular integral operator $T$ associated to para-accretive functions $b_0, b_1, b_2$ is symmetric under adjoints. That is, $T$ is a bilinear C-Z operator associated to $b_0, b_1, b_2$ if and only if $T^\ast$ is a bilinear C-Z operator associated to $b_1, b_2, b_0$ if and only if $T^\ast$ is a bilinear C-Z operator associated to $b_2, b_1, b_0$.

Definition 2.4. A function $\phi \in C_0^\infty$ is a normalized bump of order $m \in \mathbb{N}$ if $\text{supp}(\phi) \subset B(0,1)$ and

$$\sup_{|\alpha| \leq m} ||\partial^\alpha \phi||_{L^\infty} \leq 1.$$

Let $b_0, b_1, b_2 \in L^\infty$ be para-accretive functions, and $T$ be an bilinear C-Z operator associated to $b_0, b_1, b_2$. We say that $M_{b_0} T(M_{b_1}, M_{b_2})$ satisfies the weak boundedness property (written $M_{b_0} T(M_{b_1}, M_{b_2}) \in WBP$) if there exists an $m \in \mathbb{N}$ such that for all normalized bumps $\phi_0, \phi_1, \phi_2 \in C_0^\infty$ of order $m, x \in \mathbb{R}^n$, and $R > 0$

$$\left| \langle T(M_{b_1} \phi_{1}^{x,R}, M_{b_2} \phi_{2}^{x,R}), M_{b_0} \phi_{0}^{x,R} \rangle \right| \lesssim R^m$$

where $\phi_{i}^{x,R}(u) = \phi(u/R^x)$. 

It follows by the symmetry of this definition that $M_{b_0} T(M_{b_1} \cdot M_{b_2} \cdot) \in \text{WBP}$ if and only if $M_{b_0} T^{1*} (M_{b_1} \cdot M_{b_2} \cdot) \in \text{WBP}$ if and only if $M_{b_0} T^{2*} (M_{b_1} \cdot M_{b_2} \cdot) \in \text{WBP}$. Next we define $T$ on $(b_1 C_0^\delta \cap \text{L}^\infty) \times (b_2 C_0^\delta \cap \text{L}^\infty)$, so that we can make sense of the testing condition $M_{b_0} T(b_1, b_2) \in \text{BMO}$ as well as adjoint conditions. The definition we give is essentially the same as the one given by Torres [35] in the linear setting and by Grafakos-Torres [14] in the multilinear setting. Here we define this for the accretive functions $b_0, b_1, b_2$ with the necessary modifications. A benefit of this definition versus the ones in some other works (see e.g. [8] or [9]) is that we define $T(b_1, b_2)$ paired with any element of $b_0 C_0^\delta$, not just the ones with mean zero. Although one must still take care to note that the definition of $T(b_1, b_2)$ agrees with the given definition of $T$ when paired with elements of $b_0 C_0^\delta$ with mean zero. This is all precise in the next definition and the remarks that follow it.

**Definition 2.5.** Let $b_0, b_1, b_2$ be para-accretive function, $T$ be a bilinear singular integral operator associated to $b_0, b_1, b_2$, and $f_1, f_2 \in C_0^\delta \cap \text{L}^\infty$. Also fix functions $\eta_R^i \in C_0^\infty$ for $R > 0$, $i = 1, 2$ such that $\eta_R^i \equiv 1$ on $B(0, R)$ and $\text{supp}(\eta_R^i) \subset B(0, 2R)$. Then we define

$$T(b_1, b_2) = \lim_{R \to \infty} T(b_1, b_2)$$

(2.1)

where this limit is taken in the weak* topology of $(b_0 C_0^\delta)'$. For $f_0 \in C_0^\delta$, there exists $R_0 > 1$ such that $\text{supp}(f_0) \subset B(0, R_0/2)$. When $R > 2R_0$, we have

$$\langle T(\eta_R^1 b_1, \eta_R^2 b_2), 0 \rangle = \langle T(\eta_R^1 b_1, \eta_R^2 b_2), 0 \rangle$$

(2.1)

The first term $I$ is well defined since $\eta_R^i b_1, b_2 \in b_0 C_0^\delta$ for a fixed $R_0$ (depending on $f_0$). We check that the first integral term $II$ is absolutely convergent: The integrand of $II$ is bounded by $\|b_1\|_{L^2} \sum_{i=1}^2 \|b_i\|_{L^2} |f_i|_{L^2}$ times

$$|K(y_0, y_1, y_2) \eta_R^1(y_1)(\eta_R^2(y_2) - \eta_R^2(y_2)) f_0(y_0)| \lesssim \frac{|\eta_R^1(y_1)(\eta_R^2(y_2) - \eta_R^2(y_2)) f_0(y_0)|}{(|y_0 - y_1| + |y_0 - y_2|)^{2n}} \leq \frac{|\eta_R^1(y_1)(\eta_R^2(y_2) - \eta_R^2(y_2)) f_0(y_0)|}{(|y_0 - y_1| + |y_0 - y_2|/2 + (R_0 - R/2)^2)^{2n}} \leq \frac{|\eta_R^1(y_1) f_0(y_0)|}{(R_0 + |y_0 - y_2|)^{2n}}.$$
This is an $L^1(\mathbb{R}^n)$ function that is independent of $R$ (as long as $R > 4R_0$),
\[
\int_{\mathbb{R}^n} \frac{|\eta_R^l(y_1)f_0(y_0)|}{(R_0 + |y_0 - y_2|)2^n} dy_0 dy_1 dy_2 \lesssim \int_{\mathbb{R}^n} \frac{|\eta_R^l(y_1)f_0(y_0)|}{R_0^3} dy_0 dy_1 \lesssim ||f_0||_{L^\infty}.
\]

Since $\eta_R \to 1$ pointwise, by dominated convergence the following limit exists:
\[
\lim_{R \to \infty} \int_{\mathbb{R}^n} K(y_0, y_1, y_2)\eta_R^l(y_1)(\eta_R^2(y_2) - \eta_R^2(y_2)) \prod_{i=0}^2 b_i(y_i)f_i(y_i)dy_0 dy_1 dy_2
\]
\[
= \int_{\mathbb{R}^n} K(x, y_1, y_2)\eta_R^l(y_1)(1 - \eta_R^2(y_2)) \prod_{i=0}^2 b_i(y_i)f_i(y_i)dy_0 dy_1 dy_2.
\]

So $\lim_{R \to \infty} II$ exists. A symmetric argument holds for $\lim_{R \to \infty} III$. Finally, we consider IV minus the integral term from (2.1)
\[
IV - \left( \int_{|y_1|,|y_2| > 1} K(0, y_1, y_2) \prod_{i=1}^2 f_i(y_i)\eta_R^l(y_i)b_i(y_i)dy_1 dy_2, b_0 f_0 \right)
\]
\[
= \int_{\mathbb{R}^n} (K(y_0, y_1, y_2) - K(0, y_1, y_2))b_0(y_0) f_0(y_0)
\]
\[
\times \prod_{i=1}^2 (\eta_R^l(y_i) - \eta_R^2(y_i)) f_i(y_i) b_i(y_i) dy_1 dy_2 dy_0.
\]

Again we bound the integrand by $||b_0||_{L^\infty} \prod_{i=1}^2 ||b_i||_{L^\infty} ||f_i||_{L^\infty}$ times
\[
|K(y_0, y_1, y_2) - K(0, y_1, y_2)| ||f_0(y_0)|| (\eta_R^l(y_1) - \eta_R^2(y_1)) \lesssim \frac{|y_0|^2|\eta_R^l(y_1) - \eta_R^2(y_1)|}{(|y_0 - y_1| + |y_0 - y_2|)^{2n}} ||f_0(y_0)||
\]
\[
\lesssim \frac{|y_0|^2|\eta_R^l(y_1) - \eta_R^2(y_1)|}{(R_0 + |y_0 - y_1| + |y_0 - y_2|)^{2n}} ||f_0(y_0)||
\]
\[
\lesssim \frac{R_0^2 ||f_0(y_0)||}{(R_0 + |y_0 - y_1| + |y_0 - y_2|)^{2n}}.
\]

which is an $L^1(\mathbb{R}^n)$ function:
\[
\int_{\mathbb{R}^n} \frac{R_0^2 ||f_0(y_0)||}{(R_0 + |y_0 - y_1| + |y_0 - y_2|)^{2n}} dy_0 dy_1 dy_2 \lesssim \int_{\mathbb{R}^n} \frac{R_0^2 ||f_0(y_0)||}{(R_0 + |y_0 - y_1|)^{2n}} dy_0 dy_1
\]
\[
\lesssim \int_{\mathbb{R}^n} ||f_0(y_0)|| dy_0 \lesssim ||f_0||_{L^\infty}.
\]

Then it follows again by dominated convergence that
\[
\lim_{R \to \infty} \left< T((\eta_R^l - \eta_R^2)b_1 f_1, (\eta_R^2 - \eta_R^2)b_2 f_2), b_0 f_0 \right>
\]
\[
= \int_{\mathbb{R}^n} (K(x, y_1, y_2) - K(0, y_1, y_2))b_0(x)f_0(x) \prod_{i=1}^2 (1 - \eta_R^l(y_i)) f_i(y_i) b_i(y_i) dy_1 dy_2 dx,
\]
which is an absolutely convergent integral. Therefore $T(b_1 f_1, b_2 f_2)$ is well defined as an element of $(b_0 C_0^2)'$ for $f_1, f_2 \in C_0^2 \cap L^n$. Furthermore if $f_0, f_1, f_2 \in C_0^2$ and $b_0 f_0$ has mean
zero, then this definition of $T$ is consistent with the a priori definition of $T$ since

$$
\lim_{K \to \infty} \left( \int_{|y_1|,|y_2|>1} K(0,y_1,y_2) \prod_{i=1}^{2} \eta_{E_{k_i}}(y_i) b_i(y_i) f_i(y_i) dy_1 dy_2, b_0 f_0 \right)
= \left( \int_{|y_1|,|y_2|>1} K(0,y_1,y_2) \prod_{i=1}^{2} b_i(y_i) f_i(y_i) dy_1 dy_2 \left( \int_{\mathbb{R}^n} b_0(y_0) f_0(y_0) dy_0 \right) = 0, \right.
$$

since both of these integrals are absolutely convergent. Also, when $b_0 f_0$ has mean zero in this way, the definition of $(T(b_1,b_2),b_0 f_0)$ is independent of the choice of $\eta_{E_{k_i}}$ and $\eta_{E_{k_i}}$. We will also use the notation $M_{b_0}T(b_1,b_2) \in BMO$ or $M_{b_0}T(b_1,b_2) = \beta$ for $\beta \in BMO$ to mean that for all $f_0 \in C_0^\infty$ such that $b_0 f_0$ has mean zero, the following holds:

$$
\langle T(b_1,b_2), b_0 f_0 \rangle = \langle \beta, b_0 f_0 \rangle.
$$

Here the left hand side makes sense since $T(b_1,b_2)$ is defined in $(b_0 C_0^\infty)'$. The right hand side also makes sense since $b_0 f_0 \in H^1$ for $f_0 \in C_0^\infty$ where $b_0 f_0$ has mean zero. The condition $M_{b_0}T(b_1,b_2) \in BMO$ defined here is weaker than (possibly equivalent to) $T(b_1,b_2) \in BMO$ when we can make sense of $T(b_1,b_2)$ as a locally integrable function. This is because our definition of $M_{b_0}T(b_1,b_2) \in BMO$ only requires this equality to hold when paired with a subset of the predual space of $BMO$, namely we require this to hold for $\{b_0 f : f \in C_0^\infty \text{ and } b_0 f \text{ has mean zero} \} \subseteq H^1$. It is possible that this is equivalent through some sort of density argument, but that is not of consequence here. So we do not pursue it any further, and use the definition of $M_{b_0}T(b_1,b_2) \in BMO$ that we have provided. Furthermore, if $T$ is bounded from $L^p_1 \times L^p_2$ into $L^p$ for some $1 \leq p_1, p_2, p < \infty$, then $T$ can be defined on $L^\infty \times L^\infty$, and is bounded from $L^\infty \times L^\infty$ into $BMO$. This is result is due to Peetre [26], Spanne [29], and Stein [31] in the linear setting and Grafakos-Torres [14] in the bilinear setting. Hence, if $T$ is bounded, then $M_{b_0}T(b_1,b_2), M_{b_1}T^{-1}(b_0,b_2), M_{b_2}T^{-2}(b_1,b_0) \in BMO$.

2.2. Function, Operator, and General Notations. Define for $N > 0, k \in \mathbb{Z}$, and $x \in \mathbb{R}^n$

$$
\Phi_k^N(x) = \frac{2^{kn}}{(1+2^k|x|)^N}.
$$

For $f : \mathbb{R}^n \to \mathbb{C}$, we use the notation $f_k(x) = 2^{kn} f(2^k x)$. We will say indices $0 < p_1, p_2 < \infty$ satisfy a Hölder relationship if

$$
\frac{1}{p_1} = \frac{1}{p_2} = \frac{1}{p_2}.
$$

**Definition 2.6.** Let $\theta_k$ be a functions from $\mathbb{R}^{2n}$ into $\mathbb{C}$ for each $k \in \mathbb{Z}$. We call $\{\theta_k\}_{k \in \mathbb{Z}}$ a collection of Littlewood-Paley square function kernels of type $LPK(A,N,\gamma)$ for $A > 0$, $N > n$, and $0 < \gamma \leq 1$ if for all $x, y, y' \in \mathbb{R}^n$ and $k \in \mathbb{Z}$

$$
|\theta_k(x,y)| \leq A \Phi_k^{N+\gamma}(x-y)
$$

(2.3)

$$
|\theta_k(x,y) - \theta_k(x,y')| \leq A (2^k|y-y'|)^\gamma \left( \Phi_k^{N+\gamma}(x-y) + \Phi_k^{N+\gamma}(x-y') \right).
$$

(2.4)

We say that $\{\theta_k\}_{k \in \mathbb{Z}}$ is a collection of smooth Littlewood-Paley square function kernels of type $SLPK(A,N,\gamma)$ for $A > 0$, $N > n$, and $0 < \gamma \leq 1$ if it satisfies (2.3), (2.4), and for all $x, x', y \in \mathbb{R}^n$ and $k \in \mathbb{Z}$

$$
|\theta_k(x,y) - \theta_k(x',y')| \leq A (2^k|x-x'|)^\gamma \prod_{j=1}^{2} \left( \Phi_k^{N+\gamma}(x-y) - \Phi_k^{N+\gamma}(x'-y) \right).
$$

(2.5)
If \( \{ \theta_k \} \) is a collection of Littlewood-Paley square function kernels of type \( LPK(A, N, \gamma) \) (respectively \( SLPK(A, N, \gamma) \)) for some \( A > 0, N > n, \) and \( 0 < \gamma \leq 1 \), then write \( \{ \theta_k \} \in LPK \) (respectively \( \{ \theta_k \} \in SLPK \)). We also define for \( k \in \mathbb{Z} \), \( x \in \mathbb{R}^n \), and \( f \in L^1 + L^\infty \)

\[
\Theta_k f(x) = \int_{\mathbb{R}^n} \theta_k(x, y) f(y) dy.
\]

**Definition 2.7.** Let \( \theta_k \) be a functions from \( \mathbb{R}^{3n} \) into \( \mathbb{C} \) for each \( k \in \mathbb{Z} \). We call \( \{ \theta_k \}_{k \in \mathbb{Z}} \) a collection of bilinear Littlewood-Paley square function kernels of type \( BLPK(A, N, \gamma) \) for \( A > 0, N > n, \) and \( 0 < \gamma \leq 1 \) if for all \( x, y_1, y_2, y_1', y_2' \in \mathbb{R}^n \) and \( k \in \mathbb{Z} \)

\[
\begin{align*}
(2.6) \quad |	heta_k(x, y_1, y_2)| & \leq A \Phi_k^{N+\gamma}(x - y_1) \Phi_k^{N+\gamma}(x - y_2) \\
(2.7) \quad |	heta_k(x, y_1, y_2) - \theta_k(x, y_1', y_2)| & \leq A (2^k |y_1 - y_1'|)^2 \Phi_k^{N+\gamma}(x - y_2) \\
(2.8) \quad |	heta_k(x, y_1, y_2) - \theta_k(x, y_1, y_2')| & \leq A (2^k |y_2 - y_2'|)^2 \Phi_k^{N+\gamma}(x - y_1)
\end{align*}
\]

We say that \( \{ \theta_k \}_{k \in \mathbb{Z}} \) is a collection of smooth Littlewood-Paley square function kernels of type \( SBLPK(A, N, \gamma) \) for \( A > 0, N > n, \) and \( 0 < \gamma \leq 1 \) if it satisfies (2.3)-(2.5) and for all \( x, x', y_1, y_2 \in \mathbb{R}^n \) and \( k \in \mathbb{Z} \)

\[
(2.9) \quad |	heta_k(x, y_1, y_2) - \theta_k(x', y_1, y_2)| \leq A (2^k |x - x'|)^2 \prod_{i=1}^2 \left( \Phi_k^{N+\gamma}(x - y_i) - \Phi_k^{N+\gamma}(x' - y_i) \right).
\]

If \( \{ \theta_k \} \) is a collection of bilinear Littlewood-Paley square function kernels of type \( BLPK(A, N, \gamma) \) (respectively of type \( SBLPK(A, N, \gamma) \)) for some \( A > 0, N > n, \) and \( 0 < \gamma \leq 1 \), then we write \( \{ \theta_k \} \in BLPK \) (respectively \( \{ \theta_k \} \in SBLPK \)). We also define for \( k \in \mathbb{Z} \), \( x \in \mathbb{R}^n \), and \( f_1, f_2 \in L^1 + L^\infty \)

\[
\Theta_k(f_1, f_2)(x) = \int_{\mathbb{R}^{2n}} \theta_k(x, y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2.
\]

**Remark 2.8.** Let \( \theta_k \) be a function from \( \mathbb{R}^{3n} \) to \( \mathbb{C} \) for each \( k \in \mathbb{Z} \). There exists \( A_1 > 0, N_1 > n, \) and \( 0 < \gamma_1 \leq 1 \) such that \( \{ \theta_k \} \) is a collection of Littlewood-Paley square function kernels of type \( SBLPK(A_1, N_1, \gamma_1) \) if and only if there exist \( A_2 > 0, N_2 > n, \) and \( 0 < \gamma_2 \leq 1 \) such that for all \( x, y_1, y_2, y_1', y_2' \in \mathbb{R}^n \) and \( k \in \mathbb{Z} \)

\[
(2.10) \quad |	heta_k(x, y_1, y_2)| \leq A_2 \Phi_k^{N_2}(x - y_1) \Phi_k^{N_2}(x - y_2)
\]

\[
(2.11) \quad |	heta_k(x, y_1, y_2) - \theta_k(x, y_1', y_2)| \leq A_2 2^{2nk} (2^k |y_1 - y_1'|)^2
\]

\[
(2.12) \quad |	heta_k(x, y_1, y_2) - \theta_k(x, y_1, y_2')| \leq A_2 2^{2nk} (2^k |y_2 - y_2'|)^2
\]

\[
(2.13) \quad |	heta_k(x, y_1, y_2) - \theta_k(x', y_1, y_2)| \leq A_2 2^{2nk} (2^k |x - x'|)^2.
\]

A similar equivalence holds for smooth square function kernels of type \( BLPK(A, N, \gamma) \), \( LPK(A, N, \gamma) \), and \( SLPK(A, N, \gamma) \) with the obvious modifications.
Proof. Assume that \( \{ \theta_k \} \in SBLPK(A_1, N_1, \gamma_1) \), and define \( A_2 = 2A_1, N_2 = N_1 + \gamma_2 \), and \( \gamma_2 = \gamma_1 \). It follows easily that (2.10) holds. Also
\[
|\theta_k(x, y_1, y_2) - \theta_k(x, y'_1, y_2)| \leq A_1 (2^k |y_1 - y'_1|)^{N_1 + \gamma_1} (x - y_2) \\
\times \left( \Phi_k^{N_1 + \gamma_1}(x - y_1) + \Phi_k^{N_1 + \gamma_1}(x - y'_1) \right) \\
\leq 2A_1 2^{nk} (2^k |y_1 - y'_1|)^{\gamma_2}.
\]
A similar argument holds for regularity in the \( y_2 \) and \( x \) spots. Then \( \theta_k \) satisfies (2.10)-(2.13).

Conversely we assume that (2.10)-(2.13) hold. Define \( \eta = \frac{N_2 - n}{2(N_2 + \gamma_2)} \). \( A_1 = A_2, N_1 = N_2(1 - \eta) - \eta \gamma_2 \), and \( \gamma_1 = \eta \gamma_2 \). Estimate (2.6) easily follows since \( N_1 + \gamma_1 < N_2 \). Estimate (2.7) also follows since
\[
|\theta_k(x, y_1, y_2) - \theta_k(x, y'_1, y_2)| \leq A_2 (2^k |y_1 - y'_1|)^{N_2(1 - \eta)} (x - y_2) \\
\times \left( \Phi_k^{N_2(1 - \eta)}(x - y_1) + \Phi_k^{N_2(1 - \eta)}(x - y'_1) \right) \\
\leq A_1 (2^k |y_1 - y'_1|)^{\eta} \Phi_k^{N_1 + \gamma_1}(x - y_2) \\
\times \left( \Phi_k^{N_1 + \gamma_1}(x - y_1) + \Phi_k^{N_1 + \gamma_1}(x - y'_1) \right).
\]

Note that this selection satisfies
\[
N_1 = N_2 - \eta(N_2 + \gamma_2) = N_2 - \frac{N_2 - n}{2} = \frac{N_2 + n}{2} > n.
\]
Then (2.7) holds for this choice of \( A_1, N_1 \), and \( \gamma_1 \) as well. Estimates (2.8) and (2.9) follow with a similar argument, and hence \( \{ \theta_k \} \) is a collection of Littlewood-Paley square function kernel of type \( BLPK(A_1, N_1, \gamma_1) \). The proofs of the other equivalences are contained in the proof of this one.

Remark 2.9. If \( \{ \lambda^{i}_k \} \in LPK \) (respectively \( \{ \lambda^{i}_k \} \in SLPK \)) for \( i = 1, 2 \), then \( \{ \theta_k \} \in BLPK \) (respectively \( \{ \theta_k \} \in SBLPK \)) where \( \theta_k \) is defined, \( \theta_k(x, y_1, y_2) = \lambda^{1}_k(x, y_1) \lambda^{2}_k(x, y_2) \).

3. Almost Orthogonality Estimates

In this section, we prove some almost orthogonality estimates for kernel functions and for operators. These type of estimate have been well-developed over the years. In the linear setting, they go back to Besov [1, 2], Taibleson [32, 33, 34], Peetre [26, 27, 28], Triebel [36, 37], and Lizorkin [22], among others. In the bilinear setting, some of these estimates are proved by Maldonado [23], Maldonado-Naibo [24], the author [16] with addendum [17], and Grafakos-Liu-Maldonado-Yang [13]. Here we prove all estimates even though some of the results were proved in the works [23, 24, 16, 17, 13].

3.1. Kernel Almost Orthogonality. We first mention a well known almost orthogonality estimate for non-negative functions: If \( M, N > n \), then for all \( j, k \in \mathbb{Z} \)
\[
\int_{\mathbb{R}^n} \Phi_j^M(x-u) \Phi_k^N(u-y) du \lesssim \Phi_j^M(x-y) + \Phi_k^N(x-y).
\]
Then next result is also a result for integrals with non-negative integrands, but this one involves regularity estimates on the functions.
Proposition 3.1. If \( \{ \theta_k \}_{k \in \mathbb{Z}} \in BL PK \), then for all \( j,k \in \mathbb{Z}, x,y_1,y_2 \in \mathbb{R}^n \)

\[
\int_{\mathbb{R}^n} |\theta_j(x,y_1,y_2) - \theta_j(x,u,y_2)| \Phi_k^{N+\gamma}(u-y_1) \, du \\
\lesssim 2^{(j-k)} \left( \Phi_j^N(x-y_1) + \Phi_k^N(x-y_1) \right) \Phi_j^N(x-y_2),
\]

\[
\int_{\mathbb{R}^n} |\theta_j(x,y_1,y_2) - \theta_j(x,y_1,u)| \Phi_k^{N+\gamma}(u-y_2) \, du \\
\lesssim 2^{(j-k)} \Phi_j^N(x-y_1) \left( \Phi_j^N(x-y_2) + \Phi_k^N(x-y_2) \right),
\]

and

\[
\int_{\mathbb{R}^{2n}} |\theta_j(x,y_1,y_2) - \theta_j(x,u_1,u_2)| \Phi_k^{N+\gamma}(u_1-y_1) \Phi_k^{N+\gamma}(u_2-y_2) \, du_1 \, du_2 \\
\lesssim 2^{(j-k)} \prod_{i=1}^2 (\Phi_j^N(x-y_i) + \Phi_k^N(x-y_i)).
\]

Proof. Since \( \{ \theta_k \}_{k \in \mathbb{Z}} \) is of type \( BL PK(A,N,\gamma) \), it follows that

\[
\int_{\mathbb{R}^n} |\theta_j(x,y_1,y_2) - \theta_j(x,u,y_2)| \Phi_k^{N+\gamma}(u-y_1) \, du \\
\lesssim \Phi_k^N(x-y_2) \int_{\mathbb{R}^n} (2^j |u-y_1|)^\gamma \left( \Phi_j^{N+\gamma}(x-y_1) + \Phi_j^{N+\gamma}(x-u) \right) \Phi_k^{N+\gamma}(u-y_1) \, du \\
\leq 2^{(j-k)} \Phi_j^N(x-y_2) \int_{\mathbb{R}^n} \left( \Phi_j^{N+\gamma}(x-y_1) + \Phi_j^{N+\gamma}(x-u) \right) \Phi_k^N(u-y_1) \, du \\
\lesssim 2^{(j-k)} \left( \Phi_j^N(x-y_1) + \Phi_k^N(x-y_1) \right) \Phi_j^N(x-y_2).
\]

By symmetry the second estimate holds as well. For the third estimate, we make a similar argument,

\[
\int_{\mathbb{R}^{2n}} |\theta_j(x,y_1,y_2) - \theta_j(x,u_1,u_2)| \Phi_k^{N+\gamma}(u_1-y_1) \Phi_k^{N+\gamma}(u_2-y_2) \, du_1 \, du_2 \\
\leq \int_{\mathbb{R}^{2n}} |\theta_j(x,y_1,y_2) - \theta_j(x,y_1,u_2)| \Phi_k^{N+\gamma}(u_1-y_1) \Phi_k^{N+\gamma}(u_2-y_2) \, du_1 \, du_2 \\
+ \int_{\mathbb{R}^{2n}} |\theta_j(x,y_1,u_2) - \theta_j(x,u_1,u_2)| \Phi_k^{N+\gamma}(u_1-y_1) \Phi_k^{N+\gamma}(u_2-y_2) \, du_1 \, du_2 \\
\lesssim 2^{(j-k)} \int_{\mathbb{R}^{2n}} \Phi_j^N(x-y_1) \left( \Phi_j^N(x-y_2) + \Phi_j^N(x-u_2) \right) \Phi_k^{N+\gamma}(u_1-y_1) \Phi_k^N(u_2-y_2) \, du_1 \, du_2 \\
+ 2^{(j-k)} \int_{\mathbb{R}^{2n}} \left( \Phi_j^N(x-y_1) + \Phi_j^N(x-u_1) \right) \Phi_j^N(x-u_2) \Phi_k^{N+\gamma}(u_1-y_1) \Phi_k^N(u_2-y_2) \, du_1 \, du_2 \\
\lesssim 2^{(j-k)} \left( \Phi_j^N(x-y_1) + \Phi_k^N(x-y_1) \right) \left( \Phi_j^N(x-y_2) + \Phi_k^N(x-y_2) \right).
\]

This completes the proof of the proposition. \( \square \)

3.2. Operator Almost Orthogonality Estimates. It is well-known that if \( N > n \) and \( f \in L^1 + L^\infty \), then \( \Phi_k \ast |f|(x) \lesssim M f(x) \) for all \( k \in \mathbb{Z} \), where \( M \) is the Hardy-Littlewood maximal function

\[
M f(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| \, dy,
\]

and here the supremum is taken over all balls \( B \) containing \( x \). Next we use the kernel function almost orthogonality estimates to prove pointwise estimates for some operators.
Proposition 3.2. If \( \{ \lambda_k \}, \{ \theta_k \} \in L^p_k \) and there exists a para-accretive function \( b \) such that \( \Lambda_k(b) = \Theta_k(b) = 0 \) for all \( k \in \mathbb{Z} \), then for all \( f \in L^1 + L^\infty \) and \( j, k \in \mathbb{Z} \)

\[
(3.1) \quad |\Theta_j M_b \Lambda_k^i f(x)| \lesssim 2^{-\gamma(j-k)} M f(x).
\]

If \( \{ \lambda_k \} \in L^p K, \{ \theta_k \} \in B L^p K \) and there exists a para-accretive functions \( b \) such that \( \Lambda_k(b) = 0 \) and

\[
\int_{\mathbb{R}^n} \Theta_k(x, y_1, y_2) b(x) dx = 0
\]

for all \( k \in \mathbb{Z} \) and \( y_1, y_2 \in \mathbb{R}^n \), then for all \( f_1, f_2 \in L^1 + L^\infty \) and \( j, k \in \mathbb{Z} \)

\[
(3.2) \quad |\Lambda_k M_b \Theta_j(f_1, f_2)(x)| \lesssim 2^{-\gamma(j-k)} M (M f_1 \cdot M f_2)(x)
\]

Finally, if \( \{ \lambda^1_k \}, \{ \lambda^2_k \} \in L^p K, \{ \theta^1_k \} \in S B L^p K \) and there exist para-accretive functions \( b_1, b_2 \) and \( i \in \{ 1, 2 \} \) such that \( \Lambda^i_k(b_1) \cdot \Lambda^i_k(b_2) = \Theta_k(b_1, b_2) = 0 \) for all \( k \in \mathbb{Z} \), then for all \( f_1, f_2 \in L^1 + L^\infty \) and \( j, k \in \mathbb{Z} \)

\[
(3.3) \quad |\Theta_j(M_{b_1} \Lambda^{i*}_k f_1, M_{b_2} \Lambda^{j*}_k f_2)(x)| \lesssim 2^{-\gamma(j-k)} M f_1(x) M f_2(x).
\]

Here we use capital \( \Lambda_k \) to be the operator defined by integration against the kernel lower case \( \lambda_k \), just like \( \Theta_k \) and \( \theta_k \).

Proof. We first prove (3.1). Using that \( \Lambda^i_k(b) = 0 \) and Proposition 3.1

\[
|\Theta_j M_b \Lambda^i_k f(x)| \leq \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} (\theta_j(x, u) - \theta_j(x, y)) b(u) \lambda_k(y, u) du \right| f(y) |dy|
\]

\[
\lesssim \int_{\mathbb{R}^n} |\theta_j(x, u) - \theta_j(x, y)| \Phi_k(y - u) f(y) |dy|
\]

\[
\lesssim 2^{\gamma(j-k)} (\Phi_j * |f|(x) + \Phi_k * |f|(x))
\]

\[
\lesssim 2^{\gamma(j-k)} M f(x).
\]

With a symmetric argument, the same estimate holds replacing \( 2^{\gamma(j-k)} \) with \( 2^{\gamma(k-j)} \). Therefore (3.1) holds. Now we prove (3.2). We first use that \( \Lambda_k(b) = 0 \) to estimate

\[
|\Lambda_k M_b \Theta_j(f_1, f_2)(x)|
\]

\[
\leq \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \lambda_k(x, u) b(u) (\theta_j(u, y_1, y_2) - \theta_j(x, y_1, y_2)) du \right| f_1(y_1) f_2(y_2) |dy_1 dy_2
\]

\[
\lesssim \int_{\mathbb{R}^n} \Phi_k^{N+\gamma}(x-u)(2^j |x-u|^\gamma (\Phi_j^{N+\gamma}(u-y_1) + \Phi_j^{N+\gamma}(u-y_2))
\]

\[
\times |f_1(y_1) f_2(y_2)| dy_1 dy_2
\]

\[
\lesssim 2^{\gamma(j-k)} \int_{\mathbb{R}^n} \Phi_k^{N}(x-u)(\Phi_j^{N+\gamma}(u-y_1) + \Phi_j^{N+\gamma}(u-y_2))
\]

\[
\times |f_1(y_1) f_2(y_2)| dy_1 dy_2
\]

\[
= 2^{\gamma(j-k)} \int_{\mathbb{R}^n} \Phi_k^{N}(x-u) \prod_{i=1}^2 (\Phi_i^{N} * |f_i|(u) + \Phi_i^{N} * |f_i|(x)) du
\]

\[
\lesssim 2^{\gamma(j-k)} M (M f_1 \cdot M f_2)(x).
\]
We also have

\[
|\Lambda_k M_k \Theta_j (f_1, f_2)(x)|
\]

\[
\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\lambda_k(x,u) - \lambda_k(x,y_1)) b(u) \Theta_j(u,y_1,y_2) du \left| f_1(y_1)f_2(y_2) dy_1 dy_2 \right|
\]

\[
\lesssim 2^{(k-j)\gamma} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\Phi_k^N(x-u) + \Phi_k^N(x-y_1)) \prod_{i=1}^2 \Phi_j^N(u-y_i) |f_i(y_i)| dy_i du
\]

\[
\lesssim 2^{(k-j)\gamma} \Phi_k^N(x-u) \prod_{i=1}^2 \Phi_j^N(u-y_i) |f_i(y_i)| dy_i du
\]

\[
+ 2^{(k-j)\gamma} \int_{|x-y_1| \leq |x-u|/2} \Phi_k^N(x-y_1) \prod_{i=1}^2 \Phi_j^N(u-y_i) |f_i(y_i)| dy_i du
\]

\[
+ 2^{(k-j)\gamma} \int_{|x-y_1| > |x-u|/2} \Phi_k^N(x-y_1) \prod_{i=1}^2 \Phi_j^N(u-y_i) |f_i(y_i)| dy_i du
\]

\[
= 2^{(k-j)\gamma} (I + II + III).
\]

Note that \(2^{(k-j)\gamma} I \lesssim \mathcal{M}(\mathcal{M} f_1 \cdot \mathcal{M} f_2)(x)\) just like in the proof of the first case, which is on the right hand side of (3.2). In II, replace \(\Phi_k^N(x-y_1)\) with \(\Phi_k^N((x-u)/2)\) and it follows that \(II \lesssim I\). So II is bounded appropriately as well. The final term, III is bounded by

\[
\int_{|x-y_1| < |x-u|/2} \Phi_k^N(x-y_1) \frac{2^{jm} |f_1(y_1)|}{(1 + 2^j(|x-u| - |x-y_1|))^{2N}} \Phi_j^N \ast |f_2(u)| dy_1 du
\]

\[
\lesssim \int_{|x-y_1| < |x-u|/2} \Phi_k^N(x-y_1) \Phi_j^N(x-u) |f_1(y_1)| \Phi_j^N \ast |f_2(u)| dy_1 du
\]

\[
\lesssim \left( \int_{\mathbb{R}^n} \Phi_k^N(x-y_1) |f_1(y_1)| dy_1 \right) \left( \int_{\mathbb{R}^n} \Phi_j^N(x-u) \Phi_j^N \ast |f_2(u)| du \right)
\]

\[
\lesssim \Phi_k^N \ast |f_1|(x) \Phi_j^N \ast |f_2|(x)
\]

\[
\lesssim \mathcal{M}(\mathcal{M} f_1 \cdot \mathcal{M} f_2)(x).
\]

This verifies that (3.2) holds. For estimate (3.3) when \(j < k\), we use that \(\Lambda_k^1 (b_1) \cdot \Lambda_k^2 (b_2) = 0\) and Proposition 3.1

\[
|\Theta_j (M_{b_1} \Lambda_k^1 \ast M_{b_2} \Lambda_k^2 \ast f_2)(x)|
\]

\[
\leq \int_{\mathbb{R}^n} |\Theta_j(x,u_1,u_2) - \Theta_j(x,y_1,y_2)| \prod_{i=1}^2 |b_i(u)\lambda_k^i(y_i,u_i)f_i(y_i)| dy_1 du_i
\]

\[
\lesssim 2^{\gamma (j-k)} \int_{\mathbb{R}^n} \prod_{i=1}^2 (\Phi_j^N(x-u_i) + \Phi_j^N(x-y_i)) \Phi_k^N(u_i-y_i) |f_i(y_i)| du_i dy_i
\]

\[
\lesssim 2^{\gamma (j-k)} \prod_{i=1}^2 \int_{\mathbb{R}^n} (\Phi_j^N(x-y_i) + \Phi_j^N(x-y_i)) |f_i(y_i)| dy_i
\]

\[
\lesssim 2^{\gamma (j-k)} \mathcal{M} f_1(x) \mathcal{M} f_2(x).
\]
Finally using that $\Theta_j(b_1, b_2) = 0$, it follows that
\[
|\Theta_j(M_{b_1} \lambda_k^1 f_1, M_{b_2} \lambda_k^2 f_2)(x)|
\leq \int_{\mathbb{R}^n} |\theta_j(x,u_1,u_2)| \left[ \prod_{i=1}^2 \lambda_i^j(y_i,u_i) - \prod_{i=1}^2 \lambda_i^j(y_i,x) \right] \prod_{i=1}^2 |b_i(u_i) f_i(y_i)| dy_i du_i
\lesssim \int_{\mathbb{R}^n} \left( \prod_{i=1}^2 \lambda_i^j(y_i,u_i) - \prod_{i=1}^2 \lambda_i^j(y_i,x) \right) \prod_{i=1}^2 \Phi_N^{x+y}(u_i - y_i) du_i \prod_{i=1}^2 |f_i(y_i)| dy_i
\lesssim 2^{n(k-j)} (\Phi_N^x f_1(x) + \Phi_N^y f_1(x)) (\Phi_N^x f_2(x) + \Phi_N^y f_2(x))
\lesssim 2^{n(k-j)} M f_1(x) M f_2(x).
\]
Note that we use Remark 2.9 to see that $\lambda_k^1(x,y_1)\lambda_k^2(x,y_2)$ form a collection of kernels of type BLPK. Then (3.3) holds as well.

4. CONVERGENCE RESULTS

In this section, we prove convergence results for various function spaces. Most of these results are well known, see e.g. the work of Davie-Journé-Semmes [9] or Han [15], but for convenience we include them here. In this section, we also introduce a criterion for extending the convergence of some reproducing formulas in $L^p$ for to convergence in $H^1$.

4.1. APPROXIMATION TO IDENTITIES.

Proposition 4.1. Suppose $p_k : \mathbb{R}^n \to \mathbb{C}$ for $k \in \mathbb{Z}$ satisfy $|p_k(x,y)| \lesssim \Phi_N^x(x-y)$ and $N > n$, and define $P_k$

\[
P_k f(x) = \int_{\mathbb{R}^n} p_k(x,y) f(y) dy
\]

for $f \in L^1 + L^\infty$. If

\[
\int_{\mathbb{R}^n} p_k(x,y) dy = 1
\]

for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, then $P_k f \to f$ in $L^p$ as $k \to \infty$ for all $f \in L^p$ when $1 \leq p < \infty$ and $P_k f \to 0$ in $L^p$ as $k \to -\infty$ for all $f \in L^p \cap L^q$ for $1 \leq q < \infty$.

Proof. For $f \in L^p$ with $1 \leq p < \infty$

\[
||P_k f - f||_{L^p} = \left( \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} p_k(x,y) f(y) dy - \int_{\mathbb{R}^n} p_k(x,y) f(x) dy \right|^p dx \right)^{\frac{1}{p}}
\]

\[
= \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} p_k(x,x-2^{-k}y) (f(x-2^{-k}y) - f(x)) 2^{-kn} dy dx \right)^{\frac{1}{p}}
\]

\[
\lesssim \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \Phi_N^x(y) |f(x-2^{-k}y) - f(x)|^p dx \right)^{\frac{1}{p}} dy
\]

\[
\lesssim \int_{\mathbb{R}^n} \Phi_N^x(y) ||f(\cdot - 2^{-k}y) - f||_{L^p} dy.
\]

Note that $\Phi_N^x(y) ||f(\cdot - 2^{-k}y) - f||_{L^p} \leq 2 ||f||_{L^p} \Phi_N^0(y)$ which is an $L^1(\mathbb{R}^n)$ function independent of $k$. So by Lebesgue dominated convergence and the continuity of translation in $|| \cdot ||_{L^p}$,

\[
\lim_{k \to \infty} ||P_k f - f||_{L^p} \leq \int_{\mathbb{R}^n} \Phi_N^0(y) \lim_{k \to \infty} ||f(\cdot - 2^{-k}y) - f||_{L^p} dy = 0.
\]
Next we compute
\[ |P_k f(x)| \lesssim |\Phi_k^N|_{L^p} |f|_{L^q} = 2^{kn/q} |\Phi_k^N|_{L^p} |f|_{L^q}. \]
So \( P_k f \to 0 \) almost everywhere as \( k \to -\infty \). We also have
\[ |P_k f(x)| \lesssim \Phi_k^N * |f|(x) \lesssim M f(x), \]
and since \( f \in L^p(\mathbb{R}^n) \), it follows that \( M f \in L^p(\mathbb{R}^n) \) as well when \( 1 < p < \infty \). So by dominated convergence
\[ \lim_{k \to -\infty} \|P_k f\|_{L^p} = \int_{\mathbb{R}^n} \lim_{k \to -\infty} |P_k f(x)|^p \, dx = 0. \]
This proves the proposition. \( \square \)

**Corollary 4.2.** Let \( b \) be a para-accretive function. Suppose \( s_k : \mathbb{R}^{2n} \to \mathbb{C} \) for \( k \in \mathbb{Z} \) satisfy \( |s_k(x,y)| \lesssim \Phi_k^N(x-y) \) for some \( N > n \), and define \( S_k \)
\[ S_k f(x) = \int_{\mathbb{R}^n} s_k(x,y) f(y) \, dy \]
for \( f \in L^1 + L^\infty \). If
\[ \int_{\mathbb{R}^n} s_k(x,y) b(y) \, dy = 1 \]
for all \( k \in \mathbb{Z} \) and \( x \in \mathbb{R}^n \), then \( S_k M_k f \to f \) and \( M_k S_k f \to f \) in \( L^p \) as \( k \to -\infty \) for all \( f \in L^p(\mathbb{R}^n) \) when \( 1 \leq p < \infty \). Also \( S_k M_k f \to 0 \) and \( M_k S_k f \to 0 \) in \( L^p \) as \( k \to -\infty \) for all \( f \in L^p \cap L^q \) for \( 1 \leq q < p < \infty \).

**Proof.** Define \( P_k f = S_k M_k f \) with kernel \( p_k \). It is obvious that \( |p_k(x,y)| \lesssim \Phi_k^N(x-y) \), and \( P_k(1) = S_k(b) = 1 \). So by Proposition 4.1, since \( f \in L^p \) it follows that \( S_k M_k f = P_k f \to f \) in \( L^p \) when \( f \in L^p \) and \( 1 \leq p < \infty \). Also when \( f \in L^p \cap L^q \) for \( 1 \leq q < p < \infty \), it follows that \( S_k M_k f = P_k f \to 0 \) as \( k \to -\infty \). Also \( M_k S_k f = M_k P_k(b^{-1} f) \), so the same convergence properties hold for \( M_k S_k \). \( \square \)

These approximations to identities perturbed by para-accretive functions are important to this work. They have been studied in depth by David-Journé-Semmes [9] and Han [15], among others.

**Definition 4.3.** Let \( b \in L^\infty \) be a para-accretive function. A collection of operators \( \{S_k\}_{k \in \mathbb{Z}} \)
defined by
\[ S_k f(x) = \int_{\mathbb{R}^n} s_k(x,y) f(y) \, dy \]
for kernel functions \( s_k : \mathbb{R}^{2n} \to \mathbb{C} \) is an approximation to identity with respect to \( b \) if \( \{s_k\} \in SLPK \), and
\[ |s_k(x,y) - s_k(x',y) - s_k(x,y') + s_k(x',y')| \leq A2^{kn}(2^k|x-x'|)^\gamma(2^k|y-y'|)^\gamma \times \left( \Phi_k^{N+\gamma}(x-y) + \Phi_k^{N+\gamma}(x'-y) + \Phi_k^{N+\gamma}(x-y') + \Phi_k^{N+\gamma}(x'-y') \right) \]
\[ \int_{\mathbb{R}^n} s_k(x,y) b(y) \, dy = \int_{\mathbb{R}^n} s_k(x,y) b(x) \, dx = 1. \]
We say that an approximation to identity with respect to \( b \) has compactly supported kernel if \( s_k(x,y) = 0 \) whenever \( |x-y| > 2^{-k} \).
Remark 4.4. Given a para-accretive function \( b \), we define a particular approximation to the identity with respect to \( b \). Let \( \varphi \in C_0^\infty \) be radial with integral 1 and \( \text{supp}(\varphi) \subset B(0, 1/8) \). Define \( S_k^b = P_k M_{(p_k b)^{-1}} P_k \). It follows that \( S_k^b \) is an approximation to identity with respect to \( b \). Furthermore, \( S_k^b \) is self-adjoint and has compactly supported kernel. It is not trivial to see that \( M_{(p_k b)^{-1}} \) is well a defined operator, but it was proved in [9] that whenever \( b \) is a para-accretive function there exists \( \varepsilon > 0 \) such that \( |P_k b| \geq \varepsilon > 0 \) uniformly in \( k \). With this fact, the proof of this remark easily follows.

Proposition 4.5. Let \( b \) be a para-accretive function, \( \{S_k\} \) be the approximation to identity with respect to \( b \) that has compactly supported kernel, and \( \delta_0 > 0 \). Then \( M_b S_N f \to b f \) and \( M_b S_{-N} f \to 0 \) in \( BC_0^b \) as \( N \to \infty \) for all \( f \in C_0^\infty \) and \( 0 < \delta < \min(\delta_0, \gamma) \), where \( \gamma \) is the smoothness parameter for \( \{s_k\} \in \text{SLPK} \). In particular these convergence results hold for the operators defined in Remark 4.4.

Proof. Let \( f \in C_0^\infty \) and \( 0 < \delta < \delta_0 \). Without loss of generality assume that \( \gamma = \delta \), where \( \gamma \) is the smoothness parameter of \( s_k \). We must check that \( ||S_N M_b f - f||_\delta \to 0 \) as \( N \to \infty \). So we start by estimating

\[
||S_N M_b f(x) - f(x) - (S_N M_b f(y) - f(y))|| \\
= \left| \int_{\mathbb{R}^n} (s_N(x,u)(f(u) - f(x))b(u)du - \int_{\mathbb{R}^n} (s_N(y,u)(f(u) - f(y))b(u)du \right| \\
\leq ||b||_\delta \int_{\mathbb{R}^n} |F_N^b(u) - F_N^y(u)| du
\]

where \( F_N^b(u) = s_N(x,u)(f(u) - f(x)) \). Consider \( u \in B(y, 2^{-N}) \), and it follows that

\[
|F_N^b(u) - F_N^y(u)| = |s_N(x,u)(f(u) - f(x)) - s_N(y,u)(f(u) - f(y))| \\
\leq |s_N(x,u)||f(y) - f(x)| + |s_N(y,u) - s_N(x,u)||f(u) - f(y)|| \\
\lesssim ||f||_\delta 2^nN|x-y|^\delta + ||f||_\delta 2^nN(2^N|x-y|^\delta_0 + |y - u|\delta_0) \\
\lesssim ||f||_\delta 2^nN|x-y|^\delta_0
\]

With a similar argument, it follows that for \( u \in B(x, 2^{-N}) \), \( |F_N^x(u) - F_N^y(u)| \lesssim ||f||_\delta 2^nN|x-y|^\delta_0 \). Now we may also estimate \( |F_N^x(u)| \) in the following way for \( u \in B(x, 2^{-N}) \),

\[
|F_N^x(u)| \leq 2^nN||f(u) - f(x)|| \leq ||f||_\delta_0 2^nN|u - x|\delta_0 \leq ||f||_\delta 0 2^nN 2^{-\delta_0} N.
\]

Using the support properties of \( s_k \), we have that \( \text{supp}(F_N^x - F_N^y) \subset B(x, 2^{-N}) \cup B(y, 2^{-N}) \). Then it follows from (4.1), (4.2), and \( \delta_0 \in (0, 1) \) that

\[
|F_N^x(u) - F_N^y(u)| \lesssim \left( ||f||_\delta_0 2^nN|x-y|^\delta_0 \right)^{\frac{\delta}{\delta_0}} \left( ||f||_\delta 0 2^nN 2^{-\delta_0} N \right)^{1-\frac{\delta}{\delta_0}} \\
\lesssim ||f||_\delta 2^nN|x-y|^\delta 2^{-(\delta_0-\delta)N}.
\]

Therefore \( S_N M_b f \to f \) in \( || \cdot ||_\delta \) since

\[
\frac{||S_N M_b f(x) - f(x) - (S_N M_b f(y) - f(y))||}{|x-y|^\delta} \leq \frac{1}{|x-y|^\delta} \int_{\mathbb{R}^n} |F_N^b(u) - F_N^y(u)| du \\
\lesssim ||f||_\delta 2^{-\delta_0} N \int_{B(x, 2^{-N}) \cup B(y, 2^{-N})} 2^nN du \\
\lesssim ||f||_\delta 2^{-\delta_0} N.
\]
This proves that $S_N M_b f \to f$ in $C_0^\delta$ as $N \to \infty$. Now we consider $S_{-N} M_b f$ as $N \to \infty$. We also have

$$\frac{|S_{-N} M_b f(x) - S_{-N} M_b f(y)|}{|x-y|^{1+\delta}} \leq \frac{1}{|x-y|^{1+\delta}} \int_{\mathbb{R}^n} |s_{-N}(x,u) - s_{-N}(y,u)| |b(u)f(u)| du$$

$$\lesssim \frac{||f||_{L^1}}{|x-y|^{1+\delta}} \left( \int_{|u|<|x-y|^2} + \int_{|u|<|y|^2} \right) 2^{-\delta N} 2^{N|y-x|} |u|^{1+\delta} du$$

$$\lesssim ||f||_{L^1} 2^{-\delta N}.$$

Note that $||f||_{L^1} < \infty$ since $f$ is continuous and compactly supported. Therefore $S_N M_b f \to f$ and $S_{-N} M_b f \to 0$ as $N \to \infty$ in the topology of $C_0^\delta$.

4.2. Reproducing Formulas. We state a Calderón type reproducing formula for the para-accretive setting, which was constructed by Han in [15].

Theorem 4.6. Let $b \in L^\infty$ be a para-accretive function and $S_k^b$ for $k \in \mathbb{Z}$ be approximation to the identity operators with respect to $b$. Define $D_k^b = S_k^b - S_{k+1}^b$. There exist operators $\tilde{D}_k^b$ such that

$$(4.3) \quad \sum_{k \in \mathbb{Z}} \tilde{D}_k^b M_b D_k^b f = bf$$

in $L^p$ for all $1 < p < \infty$ and $f \in C_0^\delta$ such that $bf$ has mean zero. Furthermore, $\tilde{D}_k^b(b) = \tilde{D}_k^{b*}(b) = 0$ and $\tilde{D}_k^b$ is defined by

$$\tilde{D}_k^b f(x) = \int_{\mathbb{R}^n} \tilde{d}_k^b(x,y) f(y) dy$$

where $(\tilde{d}_k^{b*}) \in LPK$, where $\tilde{d}_k^b(y,x) = \tilde{d}_k^b(y,x)$ are the kernels associated with $\tilde{D}_k^{b*}$.

We will use this formula extensively, and in fact, we need this formula in $H^1$ as well to construct the accretive type para-product in Section 6. We will prove that this reproducing formula holds in $H^1$ in Theorem 1.2 and its Corollary 4.8. First we prove a lemma.

Lemma 4.7. If $f : \mathbb{R}^n \to \mathbb{C}$ has mean zero and

$$|f(x)| \lesssim \Phi_j^N(x) + \Phi_k^N(x)$$

for some $N > n$ and $j, k \in \mathbb{Z}$, then $f \in H^1$ and $||f||_{H^1} \lesssim 1 + |j-k|$, where the suppressed constant is independent of $j$ and $k$.

This is an extension of a result of Uchiyama [38], which is Lemma 4.7 when $j = k$. Initially, we were only able to obtain a quadratic bound, $|j-k|^2$, for Lemma 4.7 using an argument involving atomic decompositions in $H^1$, but thanks to suggestions from Atanas Stefanov we are able to obtain the linear bound stated here. Both are sufficient for the purposes in this work, but the proof due to Stefanov, which we present here, is more natural and obtains a better linear bound for Lemma 4.7.

Proof. The conclusion of Lemma 4.7 is well known for $j = k$, see e.g. the work of Uchiyama [38] or Wilson [39]. So without loss of generality we take $j \neq k$, and furthermore we suppose that $j < k$. It is easy to see that

$$||f||_{L^1} \lesssim ||\Phi_j^N||_{L^1} + ||\Phi_k^N||_{L^1} \lesssim 1,$$

so we may reduce the problem to proving that $||R_{\ell} f||_{L^1} \lesssim k - j$ for $\ell = 1, \ldots, n$. The strategy here is to split the norm $||R_{\ell} f||_{L^1}$ into two sets, where $|x| \leq 2^{-j}$ and where $|x| > 2^{-j}$. We
will control the first by \( k - j \) and the second by 1. Define \( p = 1 + \frac{1}{k - j} > 1 \), and use that \( \| R_\ell \|_{L^q \to L^q} \lesssim q' \) to estimate

\[
\| \chi_{|x| \leq 2^{-j} R_\ell f} \|_{L^1} \leq \| \chi_{|x| \leq 2^{-j}} \|_{L^{q'}} \| R_\ell f \|_{L^p} \\
\lesssim 2^{-nj/p'} \| f \|_{L^1} \\
(4.4)
\]

Note that here we use that \( p' = k - j + 1 \) and hence \( 2^{n(k-j)/p'} \leq 2^n \). Now it remains to control

\[
\| \chi_{|x| > 2^{-j} R_\ell f} \|_{L^1} \leq \sum_{m = -j}^{\infty} \| \chi_{2^m < |x| \leq 2^{m+1}} R_\ell f \|_{L^1} \\
\leq \sum_{m = -j}^{\infty} \| \chi_{2^m < |x| \leq 2^{m+1}} R_\ell f \chi_{|y| \leq 2^{m+1}} \|_{L^1} \\
(4.5)
\]

In order to estimate \( I \) from (4.5), we bound the terms of the sum by first breaking them into two pieces using the mean zero hypothesis on \( f \):

\[
\| \chi_{2^m < |x| \leq 2^{m+1}} R_\ell (f \chi_{|y| \leq 2^{m+1}}) \|_{L^1} = \int_{|x| > 2^m} \int_{|y| \leq 2^{m+1}} \left| R_\ell (f \chi_{|y| \leq 2^{m+1}})(x) - \int_R \frac{x_\ell}{|x-y|^{n+1}} f(y) dy \right| dx \\
\leq \int_{|x| > 2^m} \int_{|y| \leq 2^{m+1}} \frac{|x_\ell - y_\ell|}{|x-y|^{n+1}} \left| f(y) \right| dy dx \\
+ \int_{2^m < |x| \leq 2^{m+1}} \int_{|y| > 2^{m+1}} \frac{|f(y)|}{|x|} dy dx = I_a + I_b. \\
(4.6)
\]

Let \( \delta = \min(1, (N-n)/2) \) and \( N' = N - \delta > n \). Then the first term of (4.6) is bounded by

\[
I_a \leq \int_{|x| > 2^m} \int_{|y| \leq 2^{m+1}} \frac{|y|}{|x|^{n+\delta}} |f(y)| dy dx \leq \int_{|x| > 2^m} \int_{|y| \leq 2^{m+1}} \frac{|y|^{\delta}}{|x|^{n+\delta}} |f(y)| dy dx \\
\lesssim 2^{-m\delta} \int_{\mathbb{R}} |y|^{\delta} \left( \Phi_1^N(y) + \Phi_k^N(y) \right) dy \\
\leq 2^{-m\delta} \int_{\mathbb{R}} \left( 2^{-j\delta} \Phi_1^{N'}(y) + 2^{-j\delta} \Phi_k^{N'}(y) \right) dy \\
\lesssim 2^{-(j+m)\delta}.
\]

Note that we absorb the \( 2^{-k\delta} \) term into the \( 2^{-j\delta} \) term since \( k > j \). The second term of (4.6) is bounded by

\[
I_b \leq \int_{2^m < |x| \leq 2^{m+1}} \int_{|y| > 2^{m+1}} \frac{1}{|x|^{n}} |f(y)| dy dx \leq 2^{-mn} \int_{2^m < |x| \leq 2^{m+1}} \int_{|y| > 2^{m+1}} |f(y)| dy dx \\
\leq \int_{|y| > 2^{m+1}} \left( \frac{2^{-j(N-n)}}{|y|^N} + \frac{2^{-k(N-n)}}{|y|^N} \right) dy \\
\lesssim 2^{-(j+m)(N-n)} + 2^{-(k+m)(N-n)} \\
\lesssim 2^{-(j+m)(N-n)}.
\]
Again we use that $2^{-k(N-n)} \leq 2^{-j(N-n)}$ since $k > j$ and $N > n$. Now in order to estimate $II$ from (4.5), we bound the terms of the sum using an $L^2$ bound for $R_\ell$

$$\|\chi_{2m<x|}x|\leq 2^{m+1}R_\ell(f\chi_{|y|>2^m})\|_{L^1} \leq \|\chi_{2m<x|}x|\leq 2^{m+1}\|R_\ell(f\chi_{|y|>2^m})\|_{L^2} \leq 2^{m/2}\left(\int_{|y|>2^m} (\Phi_0^N(y) + \Phi_k^N(y))^2 \, dy \right)^{1/2} \leq 2^{m/2} \left( \frac{2^{2j(n-N)}}{|y|^{2N}} + \frac{2^{2k(n-N)}}{|y|^{2N}} \right) \leq 2^{m/2} \left( 2^{-j(N-n)} + 2^{-k(N-n)} \right) \left( \int_{|y|>2^m} \frac{1}{|y|^{2N}} \, dy \right)^{1/2} \leq 2^{-(j+m)(N-n)}.$$

Using these estimates, it follows that (4.5) is bounded in the following way:

$$I + II \leq \sum_{m=-j}^{\infty} 2^{-(j+m)\delta} + \sum_{m=-j}^{\infty} 2^{-(j+m)(N-n)} \leq 1.$$

Therefore using (4.4) and (4.5), it follows that $\|R_\ell f\|_{L^1} \lesssim k - j$ for $\ell = 1, \ldots, n$ and hence $\|f\|_{H^1} \lesssim k - j$. \hfill \Box

Now we prove Theorem 1.2.

\textbf{Proof.} Define for $k \in \mathbb{Z}, f_k(x) = M_b \Theta_k M_b \phi$. It easily follows that

$$\int_{\mathbb{R}^n} f_k(x) \, dx = \int_{\mathbb{R}^n} M_b \phi(x) \Theta_k \phi(x) \, dx = 0.$$

Let $R$ be large enough so that $\text{supp}(\phi) \subset B(0, R)$. We estimate

$$|f_k(x)| \leq \|b\|_{L^\infty} \left| \int_{\mathbb{R}^n} (\Theta_k(x,y) - \Theta_k(x,0)) b(y) \phi(y) \, dy \right| \leq \int_{\mathbb{R}^n} 2^k |y|^\gamma (\Phi^N_k(x-y) + \Phi^N_k(x)) |\phi(y)| \, dy \leq 2^k R^\gamma (\Phi^N_k + \Phi^N_k(x)) \lesssim 2^k (\Phi^N_k(x) + \Phi^N_k(x)) \, dy \leq 2^k \Phi^N_k(x) + \Phi^N_k(x)).$$

We also estimate

$$|f_k(x)| \leq \|b\|_{L^\infty} \left| \int_{\mathbb{R}^n} \Theta_k(x,y) b(y) (\phi(y) - \phi(x)) \, dy \right| \leq \int_{\mathbb{R}^n} \Phi^N_k (x-y) |x-y|^\gamma (\Phi^N_k(y) + \Phi^N_k(x)) \, dy \leq 2^{-\gamma k} \int_{\mathbb{R}^n} \Phi^N_k (x-y) (\Phi^N_k(y) + \Phi^N_k(x)) \, dy \leq 2^{-\gamma k} \Phi^N_k(x) + \Phi^N_k(x)).$$

So we have proved that $|f(x)| \lesssim 2^{-\gamma k} (\Phi^N_k(x) + \Phi^N_k(x))$. It follows from Lemma 4.7 applied with $j = 0$ that

$$\|f_k\|_{H^1} \lesssim (1 + |k|) 2^{-|k|\gamma}.$$
Therefore
\[ \left\| \sum_{|k| < M} f_k \right\|_{L^1} \leq \sum_{|k| < M} \| f_k \|_{L^1} \lesssim \sum_{k \in \mathbb{Z}} (1 + |k|)^{-1/2} < \infty. \]

Hence \( \sum_{|k| < M} f_k \) is a Cauchy sequence in \( L^1 \), and there exists \( \tilde{\phi} \in H^1 \) such that
\[ \tilde{\phi} = \sum_{k \in \mathbb{Z}} f_k = \sum_{k \in \mathbb{Z}} M_k \Theta_k M_\phi. \]

But since the reproducing formula holds for \( b\phi \) in \( L^p \) for some \( 1 < p < \infty \), it follows that \( \tilde{\phi} = b\phi \) and the reproducing formula holds for \( b\phi \) in \( H^1 \), which completes the proof. \( \square \)

**Corollary 4.8.** Let \( b \in L^\infty \) be a para-accretive function, \( S^\phi_k, D^\phi_k, \) and \( \tilde{D}^\phi_k \) be approximation to identity and reproducing formula operator with respect to \( b \) as in Theorem 4.6. Then for all \( \delta > 0 \) and \( \phi \in C_c^\delta \) such that \( b\phi \) has mean zero,
\[ \sum_{k \in \mathbb{Z}} M_k \tilde{D}_k M_k b\phi = \sum_{k \in \mathbb{Z}} M_k D_k M_k \phi = b\phi \]
in \( H^1 \).

**Proof.** By Theorem 4.6, it follows that the kernels of \( \tilde{D}_k M_k D_k \) and \( D_k \) are Littlewood-Paley square function kernels of type \( LPK \), that
\[ \tilde{D}_k M_k D_k(b) = (\tilde{D}_k M_k D_k)^*(b) = D_k(b) = D_\phi(b) = 0, \]
and finally that
\[ \sum_{k \in \mathbb{Z}} M_k \tilde{D}_k M_k b\phi = \sum_{k \in \mathbb{Z}} M_k D_k M_k \phi = b\phi \]
in \( L^p \) for all \( 1 < p < \infty \) when \( \phi \in C_c^\delta \) when \( b\phi \) has mean zero. Therefore it follows that the formula holds in \( H^1 \) as well. \( \square \)

**5. A Square Function-Like Estimate**

In this section, we work with Littlewood-Paley type square function kernel adapted to para-accretive functions, but we do not actually prove any square function bounds. Instead we prove an estimate for a sort of “dual pairing” that will be useful to approximate Lebesgue space norms for the singular integral operators in the next section.

**Theorem 5.1.** If \( \{\Theta_k\} \in SLpk \) and there exist para-accretive functions \( b_0, b_1, b_2 \) such that
\[ \int_{\mathbb{R}^n} \Theta_k(x, y_1 + y_2) b_0(x) \, dx = \int_{\mathbb{R}^{2n}} \Theta_k(x, y_1, y_2) b_1(y_1) b_2(y_2) \, dy_1 \, dy_2 = 0 \]
for all \( x, y_1, y_2 \in \mathbb{R}^n \) and \( k \in \mathbb{Z} \), then for all \( 1 < p, p_1, p_2 < \infty \) satisfying (2.2), \( f_i \in L^{p_i} \) for \( i = 0, 1, 2 \) where \( p_0 = p' \)
\[ \sum_{k \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \Theta_k(f_1, f_2)(x) f_0(x) \, dx \right| \lesssim \prod_{i=0}^2 \| f_i \|_{L^{p_i}} \]

**Proof.** Fix \( 1 < p, p_1, p_2 < \infty \) satisfying (2.2), \( f_i \in L^{p_i} \cap C_c^\delta \) for \( i = 0, 1, 2 \) and some \( \delta \) where \( b_i f_i \) has mean zero. Define
\[
\Pi_i^j(f_1, f_2)(x) = M_{b_1} D_{k_1}^j M_{b_2} f_1(x) M_{b_2} S_{k_1}^{D_{k_1}} M_{b_1} f_2(x)
\]
\[
\Pi_i^j(f_1, f_2)(x) = M_{b_1} S_{k_1}^j M_{b_2} f_1(x) M_{b_2} D_{k_1}^j M_{b_1} f_2(x).
\]
where $S_k^{b_1}$ and $D_k^j$ are defined as in Theorem 4.6. Then it follows that

$$
\Theta_k(f_1, f_2) = \lim_{N \to \infty} \Theta_k(M_{b_1} S_k^{b_1} M_{b_1} f_1, M_{b_2} S_k^{b_2} M_{b_2} f_2) - \Theta_k(M_{b_1} S_{k+1}^{b_1} M_{b_1} f_1, M_{b_2} S_{k+1}^{b_2} M_{b_2} f_2) \\
= \lim_{N \to \infty} \sum_{j=-N}^{N-1} \Theta_k(M_{b_1} S_k^{b_1} M_{b_1} f_1, M_{b_2} S_k^{b_2} M_{b_2} f_2) - \Theta_k(M_{b_1} S_{k+1}^{b_1} M_{b_1} f_1, M_{b_2} S_{k+1}^{b_2} M_{b_2} f_2) \\
= \lim_{N \to \infty} \sum_{j=-N}^{N-1} \Theta_k \Pi_j^1(f_1, f_2) + \Theta_k \Pi_j^2(f_1, f_2)
$$

where the convergence holds in $L^p$. Then we approximate the above dual pairing in the following way

$$
\left| \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} \Theta_k(f_1, f_2)(x) f_0(x) dx \right| \leq \sum_{j,k \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \Theta_k \Pi_j^1(f_1, f_2)(x) f_0(x) dx \right| + \sum_{j,k \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \Theta_k \Pi_j^2(f_1, f_2)(x) f_0(x) dx \right|.
$$

These two terms are symmetric, so we only bound the first one. The bound for the other term follows with a similar argument. By the convergence in Theorem 4.6, we have that

$$
\left| \sum_{j,k \in \mathbb{Z}} \int_{\mathbb{R}^n} \Theta_k \Pi_j^1(f_1, f_2)(x) f_0(x) dx \right| \leq \sum_{j,k,l,m \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \tilde{D}_m^{b_1} \tilde{D}_n^{b_1} M_{b_0} \Theta_k \Pi_j^1(M_{b_1} \tilde{D}_l^{b_1} M_{b_1} \tilde{D}_l^{b_1} f_1, f_2)(x) f_0(x) dx \right|
$$

By Proposition 3.2 we also have the following three estimates

$$
|D_m^{b_1} M_{b_0} \Theta_k \Pi_j^1(M_{b_1} \tilde{D}_l^{b_1} M_{b_1} \tilde{D}_l^{b_1} f_1, f_2)(x)| \lesssim 2^{-\gamma|m-k|} M^2 \left( \Pi_j^1(M_{b_1} \tilde{D}_l^{b_1} M_{b_1} \tilde{D}_l^{b_1} f_1, f_2) \right)(x)
$$

$$
|D_m^{b_1} M_{b_0} \Theta_k \Pi_j^1(M_{b_1} \tilde{D}_l^{b_1} M_{b_1} \tilde{D}_l^{b_1} f_1, f_2)(x)| \lesssim 2^{-\gamma|m-k|} M^2 \left( \Theta_k \Pi_j^1(M_{b_1} \tilde{D}_l^{b_1} M_{b_1} \tilde{D}_l^{b_1} f_1, f_2) \right)(x)
$$

$$
|D_m^{b_1} M_{b_0} \Theta_k \Pi_j^1(M_{b_1} \tilde{D}_l^{b_1} M_{b_1} \tilde{D}_l^{b_1} f_1, f_2)(x)| \lesssim 2^{-\gamma|k-j|} M^2 \left( \tilde{D}_l^{b_1} f_1 \cdot \tilde{M}_f \right)(x)
$$

Taking the geometric mean of these three estimates, we have the following pointwise bound

$$
|D_m^{b_1} M_{b_0} \Theta_k \Pi_j^1(M_{b_1} \tilde{D}_l^{b_1} M_{b_1} \tilde{D}_l^{b_1} f_1, f_2)(x)| \lesssim 2^{-\gamma(m-n) + \gamma|m-k| + \gamma|k-j|} M^2 \left( \tilde{D}_l^{b_1} f_1 \cdot \tilde{M}_f \right)(x).
$$
Therefore
\[
\sum_{j,k,l,m\in\mathbb{Z}} \int_{\mathbb{R}^n} \left| f_m b_{\Theta} \Pi_1 (M_{b_0} \tilde{b}_m, b_1, b_2) f_0 (x) \right| dx
\]
\[
\lesssim \left\langle \sum_{j,k,l,m\in\mathbb{Z}} 2^{-\tau \left[ \frac{|m|}{2} + \frac{|k|}{2} + \frac{|l|}{2} \right]} M^2 \left( \mathcal{M} (D_{m}^{b_1} f_1) \cdot \mathcal{M} (D_m^{b_2} f_2) \right) \right\rangle_{L^p}
\]
\[
\times \left\langle \left( \sum_{j,k,l,m\in\mathbb{Z}} 2^{-\tau \left[ \frac{|m|}{2} + \frac{|k|}{2} + \frac{|l|}{2} \right]} \tilde{D}_m^{b_3} f_0 \right)^2 \right\rangle_{L^{p'}}
\]
\[
\lesssim \left\langle \left( \sum_{\ell \in \mathbb{Z}} M^2 (\mathcal{M} (D_{\ell}^{b_1} f_1) \cdot \mathcal{M} (D_{\ell}^{b_2} f_2))^2 \right) \right\rangle_{L^p}
\]
\[
\|f_0\|_{L^{p'}} \|f_0\|_{L^{p'}} \lesssim \prod_{i=0}^2 \|f_i\|_{L^{p_i}}.
\]

In the last three lines, we apply the Fefferman-Stein vector valued maximal inequality [10], Hölder’s inequality, and the square function bounds for \(D_{\ell}^{b_1}\) and \(\tilde{D}_m^{b_3}\) proved by David-Journé-Semmes in [9]. By symmetry and density, this completes the proof. \(\square\)

6. SINGULAR INTEGRAL OPERATORS

In the last section of this work, we prove a reduced \(Tb\) theorem, construct a para-accretive paraproduct, and prove a full \(Tb\) theorem all in the bilinear setting. First, we prove a few technical lemmas that relate the work in the preceding sections to singular integral operators.

6.1. Two Technical Lemmas.

**Lemma 6.1.** Let \(b_0, b_1, b_2 \in L^\infty\) be para-accretive functions, and assume that \(T\) is a bilinear \(C-Z\) operator associated to \(b_0, b_1, b_2\) such that \(M_{b_0} T (M_{b_1} \cdot, M_{b_2} \cdot) \in WBP\) for normalized bumps of order \(m\). Then for all normalized bumps \(\phi_0, \phi_1, \phi_2, R > 0\) of order \(m\), and \(y_0, y_1, y_2 \in \mathbb{R}^n\) such that \(|y_0 - y_1| \leq tR\)
\[
\left| \left\langle T (M_{b_0} \phi_1^{y_1,R}, M_{b_2} \phi_2^{y_2,R}), M_{b_0} \phi_0^{y_0,R} \right\rangle \right| \lesssim (1 + t)^{n + 3m} R^n.
\]

**Proof.** Let \(y_0, y_1, y_2 \in \mathbb{R}^n, R > 0\), and define \(D = 1 + 2t\). Then it follows that
\[
\left| \left\langle T (M_{b_0} \phi_1^{y_1,R}, M_{b_2} \phi_2^{y_2,R}), M_{b_0} \phi_0^{y_0,R} \right\rangle \right| = \left| \left\langle T (M_{b_0} \phi_1^{y_1,D R}, M_{b_2} \phi_2^{y_2,D R}), M_{b_0} \phi_0^{y_0,D R} \right\rangle \right|.
\]
where \(\phi_0 (u) = \phi_0 (D u)\) and \(\phi_i (u) = \phi_i (D u + \frac{y_0 - y_i}{R})\) for \(i = 1, 2\). If \(|u| > 1\), then clearly \(D |u| > 1\), and
\[
\left| D u + \frac{y_0 - y_1}{R} \right| \geq D |u| - \frac{|y_0 - y_1|}{R} \geq (1 + 2t) |u| - t \geq 1.
\]
So we have that \( \text{supp}(\widetilde{\phi}_i) \subset B(0,1) \). It follows that \( D^{-m}\phi_i \in C_0^\infty \) are normalized bumps of order \( m \), and it follows that
\[
\left\| \left( T(M_{b_1}\widetilde{\phi}_1^{\mu,\nu}, M_{b_2}\widetilde{\phi}_2^{\mu,\nu}), M_{b_0}\widetilde{\phi}_0^{\mu,\nu} \right) \right\| \lesssim \|D^m(DR)\|^n \lesssim (1 + t)^{n + 3m}R^n.
\]
This completes the proof. \( \Box \)

**Lemma 6.2.** Let \( b_0, b_1, b_2 \in L^m \) be para-accretive functions. Suppose \( T \) is an bilinear \( C\)-\( Z \) operator associated to \( b_0, b_1, b_2 \) with standard kernel \( K \), and that \( M_{b_0}T(M_{b_1}, M_{b_2}) \in WBP \). Also let \( S_k^0 \) be approximations to the identity with respect to \( b_1 \) and \( D_k^{b_0} = S_{k+1}^{b_0} - S_k^{b_0} \) with compactly supported kernels \( s_k^{b_0} \) and \( d_k^{b_0} \) for \( k \in \mathbb{Z} \). Then
\[
\theta_k(x,y_1, y_2) = \left\langle T \left( b_1 s_k^{b_1}(\cdot, y_1), b_2 s_k^{b_2}(\cdot, y_2) \right), b_0 d_k^{b_0}(x, \cdot) \right\rangle
\]
is a collection of Littlewood-Paley square function kernels of type \( \text{SBLPK} \). Furthermore \( \theta_k \) satisfies
\[
\int_{\mathbb{R}^n} \theta_k(x,y_1, y_2) b_0(x)dx = 0
\]
for all \( y_1, y_2 \in \mathbb{R}^n \).

**Proof.** Fix \( x, y_1, y_2 \in \mathbb{R}^n \) and \( k \in \mathbb{Z} \). We split estimate (2.6) into two cases: \( |x - y_1| + |x - y_2| \leq 2^{3-k} \) and \( |x - y_1| + |x - y_2| > 2^{3-k} \). Note that
\[
\phi_1(u) = s_k^{b_1}(u + 2^k y_1, 2^k y_1)
\]
is a normalized bump up to a constant multiple and \( s_k^{b_1}(u, y_1) = 2^{-kn} \phi_1^{y_1, 2^{-k}}(u) \). Likewise \( s_k^{b_2}(u, y_2) = 2^{-kn} \phi_2^{y_2, 2^{-k}}(u) \) and \( d_k^{b_0}(x,u) = 2^{-kn} \phi_0^{y_0, 2^{-k}}(u) \) where \( \phi_0 \) and \( \phi_2 \) are normalize bumps up to a constant multiple. Then
\[
|\theta_k(x,y_1, y_2)| = \left| \left\langle T \left( b_1 s_k^{b_1}(\cdot, y_1), b_2 s_k^{b_2}(\cdot, y_2) \right), b_0 d_k^{b_0}(x, \cdot) \right\rangle \right| = \frac{2^{kn}}{2} \left| \left\langle T \left( b_1 \phi_1^{y_1, 2^{-k}}, b_2 \phi_2^{y_2, 2^{-k}} \right), b_0 \phi_0^{y_0, 2^{-k}} \right\rangle \right| \lesssim 2^{2kn}
\]
Now if we assume that \( |x - y_1| + |x - y_2| > 2^{3-k} \), then it follows that \( |x - y_0| > 2^{2-k} \) for at least one \( i_0 \in \{1,2\} \) and hence
\[
\text{supp}(d_k^{b_0}(x, \cdot)) \cap \text{supp}(s_k^{b_1}(\cdot, y_1)) \cap \text{supp}(s_k^{b_2}(\cdot, y_2)) \subset B(x, 2^{-k}) \cap B(y_{i_0}, 2^{-k}) = \emptyset.
\]
Therefore, we can estimate \( \theta_k \) the kernel representation of \( T \) in the following way
\[
|\theta_k(x,y_1, y_2)|
\]
\[
\lesssim \frac{|u_0 - x|^\gamma 2^{3nk}du_0 du_1 du_2}{2^{-k} 2^{3nk}du_0 du_1 du_2}
\]
\[
\lesssim \frac{2^{-k} 2^{3nk}du_0 du_1 du_2}{2^{-k}}
\]
\[
\lesssim \Phi_k^{s+\gamma/2}(x - y_1) \Phi_k^{s+\gamma/2}(x - y_2).
\]
For (2.7), note that by the continuity from $b_1C_0^5 \times b_2C_0^5$ into $(b_0C_0^5)'$ and that $S_k^b = P_iM_{[b,b]}^{-1}P_k$ has a $C_0^0$ kernel, we have for $\alpha \in \mathbb{N}_0^m$ with $|\alpha| = 1$

$$|\partial_\alpha \theta_k(x,y,z) - 0| = \left| \left\langle T \left( b_1s_{b_1}^{j_1}(\cdot,y_1), b_2s_{b_2}^{j_2}(\cdot,y_2) \right), b_0\partial_\alpha (d_k(\cdot,\cdot)) \right\rangle \right| \lesssim 2^{k+2kn}.$$ 

Estimate (2.7) easily follows in light of Remark 2.8. By symmetry, it follows that $\{\theta_k\}$ is a collection of smooth bilinear Littlewood-Paley square function kernels. Now we verify that $\theta_k$ has integral 0 in the $x$ spot: By the continuity of $T$ from $b_1C_0^5 \times b_2C_0^5$ into $(b_0C_0^5)'$

$$\int_{\mathbb{R}^n} \theta_k(x,y_1,y_2)b_0(x)dx = \lim_{R \to \infty} \left\langle T(b_1s_k^{b_1}(-,y_1), b_2s_k^{b_2}(-,y_2)), b_0 \int_{|x|<R} d_k^0(x,\cdot)b_0(x)dx \right\rangle$$

$$= \lim_{R \to \infty} \left\langle T(b_1s_k^{b_1}(-,y_1), b_2s_k^{b_2}(-,y_2)), \lambda_R \right\rangle$$

where we take this to be the definition of $\lambda_R$. Now if we take $R > 2^{1-k}$, then for $|u| < R - 2^{-k}$ it follows that

$$\supp(d_k^0(\cdot,\cdot)) \subset B(u,2^{-k}) \subset B(0,|u|+2^{-k}) \subset B(0,R),$$

and hence for $|u| < R - 2^{-k}$ we have that

$$\lambda_R(u) = b_0(u) \int_{|x|<R} d_k^0(x,u)b_0(x)dx = b_0(u)D_k^0 \lambda_R(u) = 0.$$ 

Also when $|u| > R + 2^{-k}$, it follows that $\supp(d_k^0(\cdot,\cdot)) \cap B(0,R) = \emptyset$, and hence that $\lambda_R(u) = 0$. So we have $\lambda_R(x) = 0$ for $|x| < R - 2^{-k}$ and for $|x| > R + 2^{-k}$. Finally $||\lambda_R||_{L^1} \leq \sup_u ||d_k^0(\cdot,\cdot)||_{L^1} \lesssim 1$. Since $\supp(d_k^0(x,\cdot)) \subset B(0,R+2^{-k}) \cap B(0,R-2^{-k})$, it follows that for $R > 4(2^{-k}+|y_1|)$, we may use the integral representation

$$\left| \left\langle T(b_1s_k^{b_1}(-,y_1), b_2s_k^{b_2}(-,y_2)), \lambda_R \right\rangle \right|$$

$$\leq \int_{\mathbb{R}^n} |K(u,v_1,v_2)b_1(v_1)s_k^{j_1}(v_1,y_1)b_2(v_2)s_k^{j_2}(v_2,y_2)\lambda_R(u)| dudv_1dv_2$$

$$\lesssim \int_{|v_2-v_1|<2^{-k}} \int_{|v_1-y_1|<2^{-k}} \int_{|u-v_1|+|u-v_2|}^{2kn} \left[ (|u-v_1|+|u-v_2|)2^{-n} dudv_1dv_2 \right.$$ 

$$\leq \int_{|v_2-v_1|<2^{-k}} \int_{|v_1-y_1|<2^{-k}} \int_{|u-v_1|+|u-v_2|}^{2kn} \left[ (|u|+|v_1-v_1|-|y_1|)2^{-n} dudv_1dv_2 \right.$$ 

$$\leq \int_{|v_2-v_1|<2^{-k}} \int_{|v_1-y_1|<2^{-k}} \int_{|u-v_1|+|u-v_2|}^{2kn} \left[ (R-2^{-k}+|v_1-y_1|-|y_1|)2^{-n} dudv_1dv_2 \right.$$ 

$$\lesssim |\supp(\lambda_R)| \lesssim 2^{-k} R^{-2n}$$

This tends to zero as $R \to \infty$. Hence $\theta_k(x,y_1,y_2)$ has integral zero in the $x$ variable. □

6.2. Reduced Multilinear $T(b)$ Theorem. It has become a standard argument in $T1$ and $Tb$ theorems to first prove a reduced version, see e.g. [8], [9], and [18]. The general idea of the argument is to first assume a stronger $Tb = 0$ type cancellation condition, and then prove that an operator satisfying the weaker $Tb \in BMO$ type cancellation condition is a perturbation of an operator satisfying the stronger cancellation condition. More precisely
this is done through a paraproduct operator, which we will construct later in this section. First we state and prove our reduced Tb theorem.

**Theorem 6.3.** Let and T be an bilinear C-Z operator associated to para-accretive functions $b_0, b_1, b_2$. If $M_{b_0}T(M_{b_1}, M_{b_2}) \in WBP$ and

$$M_{b_0}T(b_1, b_2) = M_{b_0}T^{*1}(b_0, b_2) = M_{b_2}T^{*2}(b_1, b_0) = 0,$$

then T can be extended to a bounded linear operator from $L^{p_1} \times L^{p_2}$ into $L^p$ for all $1 < p_1, p_2 < \infty$ satisfying (2.2).

Note that in the hypothesis of Theorem 6.3, we take $M_{b_0}T(b_1, b_2) = 0$ in the sense of Definition 2.5: For appropriate $\eta^1_R, \eta^2_R$ and all $\phi \in C^0$ such that $b_0 \phi$ has mean zero

$$\lim_{R \to 0} \langle T(\eta^1_R b_1, \eta^2_R b_2), b_0 \phi \rangle = 0.$$ 

The meaning of $M_{b_1}T^{*1}(b_0, b_2) = M_{b_2}T^{*2}(b_1, b_0) = 0$ are expressed in a similar way.

**Proof.** Let T be as in the hypothesis, $1 < p, p_1, p_2 < \infty$ satisfy (2.2), and $f_0, f_1, f_2 \in C^0$ such that $b_i f_i$ have mean zero. Then by Proposition 4.5 and the continuity of T from $b_1 C^0 \times b_2 C^0$ into $(b_0 C^0)^\prime$, it follows that

$$|\langle T(b_1 f_1, b_2 f_2), b_0 f_0 \rangle| = \lim_{N \to \infty} \left| \left\langle T(M_{b_1} S_{N}^0 b_1 f_1, M_{b_2} S_{N}^0 b_2 f_2), M_{b_0} S_{N}^0 b_0 f_0 \right\rangle ight. + \left. \left\langle T(M_{b_1} S_{N}^0 b_1 f_1, M_{b_2} S_{N}^0 b_2 f_2), M_{b_0} S_{N}^0 b_0 f_0 \right\rangle \right|$$

$$\leq \sum_{k \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \Theta_k^0(b_1 f_1, b_2 f_2) b_0 f_0(x) dx \right|$$

$$+ \left| \int_{\mathbb{R}^n} \Theta_k^1(b_0 f_0, b_2 f_2) b_1 f_1(x) dx \right|$$

$$+ \left| \int_{\mathbb{R}^n} \Theta_k^2(b_1 f_1, b_0 f_0) b_2 f_2(x) dx \right| .$$

where

$$\Theta_k^0(f_1, f_2) = \mathcal{D}_k^0 M_{b_0} T(M_{b_1} S_{k+1} b_1 f_1, M_{b_2} S_{k+1}^0 b_2 f_2),$$

$$\Theta_k^1(f_1, f_2) = \mathcal{D}_k^1 M_{b_1} T^{*1}(M_{b_0} S_{k}^0 b_0 f_1, M_{b_2} S_{k}^0 b_2 f_2),$$

$$\Theta_k^2(f_1, f_2) = \mathcal{D}_k^2 M_{b_2} T^{*2}(M_{b_1} S_{k+1}^0 f_1, M_{b_0} S_{k+1}^0 f_2).$$

We focus on $\Theta_k^0 = \Theta_k$ to simplify notation; the other terms are handled in the same way. Since $M_{b_0}T(M_{b_1}, M_{b_2}, \cdot) \in WBP$ and T has a standard kernel, it follows from Lemma 6.2 that $\{ \theta_k \} \in SBLPK$ and $\theta_k(x,y_1,y_2) b_0(x)$ has mean zero in the $x$ variable for all $y_1, y_2 \in \mathbb{R}^n$.

Now we show that $\Theta_k(b_1, b_2) = 0$, which follows from the assumption that $M_{b_0}T(b_1, b_2) = 0.$
where

\[ T \{ b_1 \eta_k^b \eta_k^b \} \cdot b_0 d_k \eta_k^b (x, \cdot) \right) = 0, \]

We’ve used that \( M_{b_0} T (b_1, b_2) = 0 \), and that \( \eta_k^b \in C^\infty \), \( \eta_k^b \equiv 1 \) on \( B(0, R) \), and \( \text{supp}(\eta_k^b) \subset B(0, 2R) \) for \( R \) sufficiently large. Then by Theorem 5.1, it follows that

\[
\sum_{k \in \mathbb{Z}} |\langle \Theta_k^b (M_{b_1}, f_1, M_{b_2}, f_2), M_{b_0} f_0 \rangle| \lesssim ||f_0||_{L^p_0} ||f_1||_{L^{p_1}} ||f_2||_{L^{p_2}}.
\]

A similar argument holds for \( \Theta_k^i \) with \( i = 1, 2 \) again taking advantage of the facts \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_1} \) and \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_1} \). Therefore \( T \) can be extended to a bounded operator from \( L^{p_1} \times L^{p_2} \) into \( L^p \) for all \( 1 < p, p_1, p_2 < \infty \) satisfying (2.2).

6.3. A Para-Product Construction. In the original proof of the T1 theorem, David-Journé [8] used the Bony paraproduct [3] to pass from their reduced T1 theorem to the full T1 theorem. Following the same idea, David-Journé-Semmes [9] proved the Tb theorem by constructing a para-accretive version of the Bony paraproduct. In [18], we constructed a bilinear Bony-type paraproduct, which allowed us to transition from a reduce bilinear T1 theorem to a full T1 theorem. Here we construct a bilinear paraproduct in a para-accretive function setting. First we prove a quick lemma, which is a bilinear version of an observation made by Coifman-Meyer [2].

Lemma 6.4. Suppose \( \{\theta_k\} \in SBLPK \) with decay parameter \( N > 2n \), and define \( K : \mathbb{R}^{3n} \setminus \{(x, x, x) : x \in \mathbb{R}^n \} \rightarrow \mathbb{C} \)

\[ K(x, y_1, y_2) = \sum_{k \in \mathbb{Z}} \theta_k(x, y_1, y_2). \]

Then \( K \) is a bilinear standard Calderón-Zygmund kernel.

Proof. To prove the size estimate, we take \( d = |x - y_1| + |x - y_2| \neq 0 \) and compute

\[
|K(x, y_1, y_2)| \leq \sum_{k \in \mathbb{Z}} \frac{2^{kn}}{(1 + 2^k |x - y_1|)^{N+\gamma}(1 + 2^k |x - y_2|)^{N+\gamma}}
\]

\[
\lesssim \sum_{2^k \leq d^{-1}} 2^{kn} \sum_{2^k > d^{-1}} \frac{2^{2kn}}{(2^k d)^{N+\gamma}} \lesssim d^{-2n}. \]

For the regularity in \( x \), we take \( x, x', y_1, y_2 \in \mathbb{R}^n \) with \( |x - x'| < \max(|x - y_1|, |x - y_2|) / 2 \) and define \( d' = |x' - y_1| + |x' - y_2| \). Then

\[
|K(x, y_1, y_2) - K(x', y_1, y_2)| \lesssim \sum_{k \in \mathbb{Z}} \frac{(2^k |x - x'|)^{2kn}}{(1 + 2^k |x - y_1|)^{N+\gamma}(1 + 2^k |x - y_2|)^{N+\gamma}}
\]

\[
+ \sum_{k \in \mathbb{Z}} \frac{(2^k |x - x'|)^{2kn}}{(1 + 2^k |x' - y_1|)^{N+\gamma}(1 + 2^k |x' - y_2|)^{N+\gamma}}
\]

\[
= I + II.
\]
We first bound $I$ by $|x-x'|^2$ times

$$\sum_{2^k \leq d^{-1}} 2^{k(2n+\gamma)} + \sum_{2^k > d^{-1}} \frac{2^{k(2n+\gamma)}}{2^k d^N} \lesssim d^{-(2n+\gamma)} + d^{-(N+\gamma)} \sum_{2^k > d^{-1}} 2^{k(2n-N)} \lesssim d^{-(2n+\gamma)},$$

By symmetry, it follows that $II \lesssim |x-x'| d^{-(2n+\gamma)}$, but since $|x-x'| < \max(|x-y_1|, |x-y_2|)/2$, without loss of generality say $|x-y_1| \geq |x-y_2|$ it follows that

$$II \lesssim \frac{|x-x'|^2}{(|x-y_1| + |x-y_2|)^{2n+\gamma}} \lesssim \frac{|x-x'|^2}{|x-y_1|^{2n+\gamma}} \lesssim \frac{|x-x'|^2}{d^{2n+\gamma}}.$$

With a similar computation for $y_1, y_2$, it follows that $K$ is a standard kernel. \hfill \Box

**Theorem 6.5.** Given para-accretive functions $b_0, b_1, b_2 \in L^\infty$ and $\beta \in BMO$, there exists a bilinear Calderón-Zygmund operator $L$ that is bounded from $L^{p_1} \times L^{p_2}$ into $L^{p}$ for all $1 < p_1, p_2 < \infty$ satisfying (2.2) with $p = 2$ such that $M_{b_0} T(b_1, b_2) = \beta$, $M_{b_1} T^{*1}(b_0, b_2) = M_{b_2} T^{*2}(b_1, b_0) = 0$.

**Proof.** Let $b_0, b_1, b_2$ be para-accretive functions, and $S_k^b, D_k^b,$ and $\bar{D}_k^b$ be the approximation to identity and reproducing formula operators with respect to $b_i$ for $i = 0, 1, 2$ that have compactly supported kernels as defined in Remark 4.4 and Theorem 4.6. Define

$$L(f_1, f_2) = \sum_{k \in \mathbb{Z}} L_k(f_1, f_2) = \sum_{k \in \mathbb{Z}} D_k^b M_{b_0} \left( (\bar{D}_k^b, M_{b_0} \beta)(S_k^b f_1)(S_k^b f_2) \right),$$

$$\ell(x, y) = \sum_{k \in \mathbb{Z}} \ell_k(x, y) = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} d_k^b (x, u) b_0(u) \bar{D}_k^b M_{b_0} \beta(u) s_k^b(u, y_1) b_2(u, y_2) du.$$

It follows that $L$ is bounded from $L^{p_1} \times L^{p_2}$ into $L^2$ for all $1 < p_1, p_2 < \infty$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$:

$$\left| \int_{\mathbb{R}^n} L(f_1, f_2)(x) f_0(x) dx \right| \leq \sum_{k \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} \bar{D}_k^b M_{b_0} \beta(x) s_k^b f_1 (x) s_k^b f_2 (x) D_k^b f_0(x) b_0(x) dx \right|$$

$$\lesssim \left\| \sum_{k \in \mathbb{Z}} |M_{\bar{D}_k^b, M_{b_0} \beta} s_k^b f_1 s_k^b f_2|^2 \right\|^{\frac{1}{2}} \left\| \sum_{k \in \mathbb{Z}} |D_k^b f_0|^2 \right\|^{\frac{1}{2}}$$

$$\lesssim \left( \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} \left[ \Phi_k^N * |f_1| (x) \Phi_k^N * |f_2| (x) \right]^2 |\bar{D}_k^b M_{b_0} \beta(x)|^2 dx \right)^{\frac{1}{p_1}}$$

$$\times \left( \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} \left[ \Phi_k^N * |f_2| (x) \right]^2 |\bar{D}_k^b M_{b_0} \beta(x)|^2 dx \right)^{\frac{1}{p_2}}$$

$$\lesssim \| f_0 \|_{L^2} \| f_1 \|_{L^{p_1}} \| f_2 \|_{L^{p_2}}.$$ 

Note that in the last line we apply the discrete version of a well-known result relating Carleson measure and square functions due to Carleson [6] and Jones [20] for the Carleson measure

$$d\mu(x, t) = \sum_{k \in \mathbb{Z}} |\bar{D}_k^b b_0 \beta(x)|^2 \delta_{a=2^{-k}}.$$
For details of the discrete version of this result, see for example the book by Grafakos [12], Theorems 7.3.7 and 7.3.8(c). This proves that $L$ is bounded from $L^{p_1} \times L^{p_2}$ into $L^2$ for all $1 < p_1, p_2 < \infty$ satisfying (2.2) with $p = 2$. It is easy to check that $\{\ell_k\} \in SBLPK$ with size index $N > 2n$: Since $d_k^{b_0}$ and $s_k^{b_1}$ are compactly supported kernels, for $i = 1, 2$ it follows that

$$|\ell_k(x, y_1, y_2)| \leq \|b_0 D_k^{b_0} M_{b_0}\|_{L^\infty} \int_{\mathbb{R}^n} |d_k^{b_0}(x - u)| s_k^{b_i}(u - y_1) s_k^{b_j}(u - y_2) du$$

$$\leq 2^{2n} \int_{\mathbb{R}^n} \Phi_k^{4(n+1)}(x - u) \Phi_k^{4(n+1)}(u - y_1) du \leq 2^{2n} \Phi_k^{4(n+1)}(x - y_i).$$

Hence the size condition (2.6) with size index $N = 2n + 1$ and $\gamma = 1$ follows

$$|\ell_k(x, y_1, y_2)| \leq \prod_{i=1}^2 \left( 2^{2n} \Phi_k^{4(n+1)}(x - y_i) \right)^{\frac{1}{2}} = \Phi_k^{2n+2}(x - y_i) \Phi_k^{2n+2}(x - y_2).$$

The regularity estimates (2.7)-(2.9) follow easily from the regularity of $d_k^{b_0}$, $s_k^{b_1}$, and $s_k^{b_2}$. Then by Lemma 6.4, $L$ has a standard Calderón-Zygmund kernel $\ell$. It follows from a result of Grafakos-Torres [14] that $L$ is bounded from $L^{p_1} \times L^{p_2}$ into $L^p$ where $1 < p_1, p_2 < \infty$ satisfy (2.2). Next we compare $M_{b_0} L(b_1, b_2)$: Let $\delta > 0$, $\phi \in C_0^\infty$ such that $\text{supp}(\phi) \subset B(0, N)$ and $b_0 \phi$ has mean zero. Let $\eta_R(x) = \eta(x/R)$ where $\eta \in C_0^\infty$ satisfies $\eta \equiv 1$ on $B(0, 1)$, and $\text{supp}(\eta) \subset B(0, 2)$. Then

$$\langle L(b_1, b_2), b_0 \phi \rangle$$

$$= \lim_{R \to \infty} \sum_{2^{k} < R/4} \int_{\mathbb{R}^n} \tilde{D}_k^{b_0} M_{b_0} \beta(x) s_k^{b_1} M_{b_1} \eta_R(x) s_k^{b_2} M_{b_2} \eta(x) M_{b_0} D_k^{b_0}(b_0 \phi)(x) dx$$

$$+ \lim_{R \to \infty} \sum_{2^{-k} \geq R/4} \int_{\mathbb{R}^n} \tilde{D}_k^{b_0} M_{b_0} \beta(x) s_k^{b_1} M_{b_1} \eta_R(x) s_k^{b_2} M_{b_2} \eta(x) M_{b_0} D_k^{b_0}(b_0 \phi)(x) dx,$$

where we may write this only if the two limits on the right hand side of the equation exist. As we are taking $R \to \infty$ and $N$ is a fixed quantity determined by $\phi$, without loss of generality assume that $R > 2N$. Note that for $2^{-k} \leq R/4$ and $|x| < N + 2^{-k}$,

$$\text{supp}(s_k^{b_i}(x, \cdot)) \subset B(x, 2^{-k}) \subset B(0, N + 2^{1-k}) \subset B(0, R).$$

Since $\eta_R \equiv 1$ on $B(0, R)$, it follows that $s_k^{b_i} M_{b_i} \eta_R(x) = 1$ for all $|x| < N + 2^{-k}$ when $2^{-k} \leq R/4$. Therefore

$$\lim_{R \to \infty} \sum_{2^{-k} < R/4} \int_{\mathbb{R}^n} \tilde{D}_k^{b_0} M_{b_0} \beta(x) M_{b_0} D_k^{b_0}(b_0 \phi)(x) dx = \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} M_{b_0} \tilde{D}_k^{b_0} M_{b_0} D_k^{b_0}(b_0 \phi)(x) \beta(x) dx = \langle \beta, b_0 \phi \rangle.$$

Here we use the convergence of the accretive type reproducing formula in $H^1$ from Corollary 4.8. For any $k \in \mathbb{Z}$, we have the estimates

$$(6.1) \quad \|s_k^{b_i} M_{b_i} \eta_R\|_{L^1} \lesssim \|\eta_R\|_{L^1} \lesssim R^n,$$

$$(6.2) \quad \|s_k^{b_i} M_{b_i} \eta_R\|_{L^\infty} \lesssim \|\eta_R\|_{L^\infty} = 1,$$
and for any \( x \in \mathbb{R}^n \)

\[
|D_k^{b_0} M_{b_0} \phi(x)| \leq \int_{\mathbb{R}^n} |d_k^{b_0}(x,y) - d_k^{b_0}(x,0)| \, |b_0(y)\phi(y)| \, dy
\]

\[
\lesssim \int_{\mathbb{R}^n} (2^k|y|)^\gamma |\phi(y)| \, dy \lesssim N^\gamma \|\phi\|_{L^1} 2^{k(n+\gamma)}.
\]

Here we know that \( \{d_k^{b_0}\} \in \mathrm{LPK} \), so without loss of generality we take the corresponding smoothness parameter \( \gamma \leq \delta \). Later we will use that \( \gamma \leq \delta \) implies \( \phi \in \mathcal{C}_0^\delta \subseteq \mathcal{C}_0^\gamma \), so we have that \( |\phi(x) - \phi(y)| \lesssim |x-y|^\gamma \). Therefore

\[
\sum_{2^{-k} > R/4} \int_{\mathbb{R}^n} |\tilde{D}_k^{b_0} M_{b_0} \beta(x) S_k^{b_1} M_{b_1} \eta_R(x) S_k^{b_2} M_{b_2} \eta_R(x) M_{b_0} D_k^{b_0}(b_0\phi)(x)| \, dx
\]

\[
\leq \sum_{2^{-k} > R/4} ||\tilde{D}_k^{b_0} M_{b_0} \beta||_{L^\infty} ||S_k^{b_1} M_{b_1} \eta_R||_{L^1} ||S_k^{b_2} M_{b_2} \eta_R||_{L^\infty} ||M_{b_0} D_k^{b_0}(b_0\phi)||_{L^\infty}
\]

(6.3)

\[
\lesssim R^n \sum_{2^{-k} > R/4} 2^{k(n+\gamma)} \lesssim R^{-\gamma}.
\]

Hence the second limit above exists and tends to 0 as \( R \to \infty \). Then \( \langle L(b_1, b_2), b_0\phi \rangle = \langle \beta, b_0\phi \rangle \) for all \( \phi \in \mathcal{C}_0^\delta \) such that \( b_0\phi \) has mean zero and hence \( M_{b_0} L(b_1, b_0) = \beta \) as defined in Section 2. Again for any \( \phi \in \mathcal{C}_0^\delta \) such that \( b_1 \phi \) has mean zero and \( \text{supp}(\phi) \subseteq B(0,N) \), we have

\[
\|L_{L^1}(b_1, b_2, b_1 \phi)\|
\]

\[
= \lim_{R \to \infty} \left| \sum_{k \in \mathbb{Z}} \sum_{2^{-k} > R/4} \int_{\mathbb{R}^n} |\tilde{D}_k^{b_0} M_{b_0} \beta(x) S_k^{b_1} M_{b_1} \phi(x) S_k^{b_2} M_{b_2} \eta_R(x) D_k^{b_0}(b_0\phi)(x)| \, dx \right|
\]

\[
\lesssim \lim_{R \to \infty} \sum_{k \in \mathbb{Z}} ||\tilde{D}_k^{b_0} M_{b_0} \beta||_{L^\infty} ||S_k^{b_1} M_{b_1} \phi||_{L^1} ||S_k^{b_2} M_{b_2} \eta_R||_{L^\infty} ||D_k^{b_0}(b_0 \phi)||_{L^\infty}
\]

\[
\lesssim \lim_{R \to \infty} \sum_{k \in \mathbb{Z}} ||S_k^{b_1} M_{b_1} \phi||_{L^1} ||D_k^{b_0} M_{b_0} \eta_R||_{L^\infty}.
\]

We will now show that \( ||S_k^{b_1} M_{b_1} \phi||_{L^1} ||D_k^{b_0} M_{b_0} \eta_R||_{L^\infty} \) bounded by a in integrable function in \( k \) (i.e., summable) independent of \( R \), so that we can bring the limit in \( R \) inside the sum. To do this we start by estimating

\[
|S_k^{b_1} M_{b_1} \phi(x)| \leq \int_{\mathbb{R}^n} |s_k^{b_1}(x,y) - s_k^{b_1}(x,0)| \, |\phi(y)b_1(y)| \, dy
\]

\[
\leq N^\gamma ||\phi||_{L^1} ||b_1||_{L^\infty} 2^k (F_0^N(x) + F_0^N(x))
\]

and so \( ||S_k^{b_1} M_{b_1} \phi||_{L^1} \lesssim 2^k \). We also have that \( ||s_k^{b_1} M_{b_1} \phi||_{L^1} \lesssim ||\phi||_{L^1} \lesssim 1 \), so \( ||S_k^{b_1} M_{b_1} \phi||_{L^1} \lesssim \min(1, 2^k) \). Also

\[
|D_k^{b_0} M_{b_0} \eta_R(x)| \leq \int_{\mathbb{R}^n} |d_k^{b_0}(x,y)| \, |\eta_R(y) - \eta_R(x)| \, |b_0(y)| \, dy
\]

\[
\lesssim 2^{-\gamma} R^{-\gamma} \int_{\mathbb{R}^n} \Phi^{N+\gamma}(x-y)(2^k|x-y|)^\gamma dy \lesssim 2^{-\gamma} R^{-\gamma}.
\]

It follows that \( ||D_k^{b_0} M_{b_0} \eta_R||_{L^\infty} \lesssim ||\eta_R||_{L^\infty} \lesssim 1 \), and hence \( ||D_k^{b_0} M_{b_0} \eta_R||_{L^\infty} \lesssim \min(1, 2^{-k}) \). So when \( R > 1 \), we have

\[
||D_k^{b_0} M_{b_0} \eta_R||_{L^\infty} ||S_k^{b_1} M_{b_1} \phi||_{L^1} \lesssim \min(2^{-\gamma} R^{-\gamma}, 2^{-k}) \lesssim 2^{-\gamma k}.
\]
and hence by dominated convergence
\[ \left| \langle L^{1} \phi, b_{1} \phi \rangle \right| \leq \sum_{k \in \mathbb{Z}} \lim_{R \to \infty} \| D^{k}_{b_{0}} M_{b_{1}} \phi \|_{L^{1}} \| D^{k}_{b_{0}} \eta_{R} \|_{L^{\infty}} \leq \sum_{k \in \mathbb{Z}} \lim_{R \to \infty} 2^{-k \gamma} R^{-\gamma} = 0. \]

Then \( M_{b_{0}} L^{1}(b_{0}, b_{2}) = 0 \) and a similar argument shows that \( M_{b_{2}} L^{2}(b_{1}, b_{0}) = 0 \), which concludes the proof. □

Now to complete the proof of Theorem 1.1 is a standard argument using the reduced \( T_{b} \) Theorem 6.3 and paraproducts construction in Theorem 6.5.

**Proof.** Assume that \( M_{b_{0}} T(M_{b_{1}}, M_{b_{2}}) \) satisfies the weak boundedness property and
\[ M_{b_{0}} T(b_{1}, b_{2}), M_{b_{1}} T^{1}(b_{0}, b_{2}), M_{b_{2}} T^{2}(b_{1}, b_{0}) \in BMO \]
for \( i = 0, 1, 2 \). By Theorem 6.5, there exist bounded bilinear Calderón-Zygmund operators \( L_{s} \) such that
\[
\begin{align*}
M_{b_{0}} L_{0}(b_{1}, b_{2}) &= M_{b_{0}} T(b_{1}, b_{2}) & M_{b_{1}} L_{0}^{1}(b_{0}, b_{2}) &= M_{b_{1}} T^{1}(b_{0}, b_{2}) = M_{b_{2}} L_{0}^{2}(b_{1}, b_{0}) = 0 \\
M_{b_{1}} L_{1}^{1}(b_{0}, b_{2}) &= M_{b_{1}} T^{1}(b_{0}, b_{2}) & M_{b_{0}} L_{1}(b_{1}, b_{2}) &= M_{b_{2}} L_{1}^{2}(b_{1}, b_{0}) = 0 \\
M_{b_{2}} L_{2}^{2}(b_{1}, b_{0}) &= M_{b_{2}} T^{2}(b_{1}, b_{0}) & M_{b_{0}} L_{2}^{1}(b_{0}, b_{2}) &= M_{b_{0}} L_{2}(b_{1}, b_{2}) = 0
\end{align*}
\]

Now define the operator
\[ S = T - \sum_{i=0}^{2} L_{i}, \]
which is continuous from \( b_{1}C^{0}_{0} \times b_{2}C^{0}_{0} \) into \( (b_{0}C^{0}_{0})' \). Also \( M_{b_{0}} S(M_{b_{1}}, M_{b_{2}}) \) satisfies the weak boundedness property since \( M_{b_{0}} T(M_{b_{1}}, M_{b_{2}}) \) and \( M_{b_{0}} L_{i}(M_{b_{1}}, M_{b_{2}}) \) for \( i = 0, 1, 2 \) do. Finally we have
\[
\begin{align*}
M_{b_{0}} S(b_{1}, b_{2}) &= M_{b_{0}} T(b_{1}, b_{2}) - \sum_{i=0}^{2} M_{b_{0}} L_{i}(b_{1}, b_{2}) = 0 \\
M_{b_{1}} S^{1}(b_{0}, b_{2}) &= M_{b_{1}} T^{1}(b_{0}, b_{2}) - \sum_{i=0}^{2} M_{b_{1}} L_{i}^{1}(b_{0}, b_{2}) = 0 \\
M_{b_{2}} S^{2}(b_{1}, b_{0}) &= M_{b_{2}} T^{2}(b_{1}, b_{0}) - \sum_{i=0}^{2} M_{b_{2}} L_{i}^{2}(b_{1}, b_{0}) = 0
\end{align*}
\]
Then by Theorem 6.3, \( S \) can be extended to a bounded linear operator from \( L^{p_{1}} \times L^{p_{2}} \) into \( L^{p} \) for all \( 1 < p, p_{1}, p_{2} < \infty \) satisfying (2.2). Therefore \( T \) is bounded on the same indices, and by results from [14] \( T \) is also bounded without restriction on \( p \). The converse is also a well-known result from [14]. □

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