Relational electromagnetism and the
electromagnetic force

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Abstract

The force exerted by an electromagnetic body on another body in relative motion, and its minimal expression, the force on moving charges or Lorentz’ force constitute the link between electromagnetism and mechanics. Expressions for the force were produced first by Maxwell and later by H. A. Lorentz, but their expressions disagree. The construction process was the result, in both cases, of analogies rooted in the idea of the ether. Yet, the expression of the force has remained despite its production context. We present a path to the electromagnetic force that starts from Ludwig Lorenz’ relational electromagnetism which was available at the time of Lorentz’ work. The present mathematical abduction does not rest on analogies. Following this path we show that relational electromagnetism, as pursued by the Göttingen school, is consistent with Maxwell’s transformation laws and compatible with the idea that the “speed of light” takes the same value in all (inertial) frames of reference, while it cannot be conceived on the basis of analogies with material motion.

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1 Introduction and Historical Background

By the mid XIX-th century two cultures, or rather two forms of conceiving science, came into conflict around the issues posed by electromagnetic phenomena. We will label them the Göttingen and Berlin schools. Franz Neumann as well as his son Carl, Weber, Riemann and Gauss belong to the Göttingen school and support action-at-a-distance. Hertz, Heaviside, Clausius and others represent the Berlin school, who denies action-at-a-distance and relies on the ether as a form of electromagnetic medium sustaining the electromagnetic interaction.
Maxwell has an intermediate position. He acknowledges the achievements of the Göttgen school, while he cannot conceive electromagnetic interactions without a medium supporting the interaction while “travelling” from source to detector. Indeed, he ends the Treatise by commenting on the Göttgen school, finally stating ([866], Maxwell [1873]) (our highlight):

But in all of these theories the question naturally occurs: If something is transmitted from one particle to another at a distance, what is its condition after it has left the one particle and before it has reached the other? If this something is the potential energy of the two particles, as in Neumann’s theory, how are we to conceive this energy as existing in a point of space, coinciding neither with the one particle nor with the other? In fact, whenever energy is transmitted from one body to another in time, there must be a medium or substance in which the energy exists after it leaves one body and before it reaches the other, for energy, as Torricelli remarked, ’is a quintessence of so subtile a nature that it cannot be contained in any vessel except the inmost substance of material things.’ **Hence all these theories lead to the conception of a medium in which the propagation takes place**, and if we admit this medium as an hypothesis, I think it ought to occupy a prominent place in our investigations, and that we ought to endeavour to construct a mental representation of all the details of its action, and this has been my constant aim in this treatise.

The paragraph shows Maxwell’s urge to represent interactions in terms of analogies with matter. Since bodies have a place in (subjective) space, the paragraph assumes, without saying it, that we should consider interactions (an abstract concept) in terms of the more accessible (intuitive) idea of bodies. Because bodies, not interactions, have a place in space.

On the other hand, the Berlin school resents of Maxwell’s construction as being “impure” noticing that his formulae are not always the result of elaborations in terms of the ether. The Berlin school linked understanding with images, mental representations of the observed, the *bild* theory (D’Agostino, 2004) after Helmholz and Hertz, whereas the Göttgen school aimed to provide a mathematical organisation to electromagnetic knowledge where equations and their abduction/conceptualisation/construction constitute an unity. Another remarkable characteristic of the ether culture was the rejection of the mathematical structures of mechanics (in particular action-at-a-distance). In his posthumous work, Hertz attempted to reformulate Mechanics ([Hertz and Walley, 1899]). He begins the preface by fostering doubts on Newton’s mechanics in relation to the

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1It is worth to keep in mind that non-scientific, irrational, factors may have played a role as well since by 1868 Göttgen (Hannover) was annexed to Prussia (hosting the school of Berlin). Clausius, a Prussian nationalist and scientist, promoted at the same time an attack on the scientific position on electromagnetism held by the Göttgen school, from the point of view of the ether and of Berlin’s epistemic approach (See [Clausius, 1869] and [Archibald, 1986; Natiello and Solarz, 2019]).
ether:

All physicists agree that the problem of physics consists in tracing the phenomena of nature back to the simple laws of mechanics. But there is not the same agreement as to what these simple laws are [...] But we have here no certainty as to what is simple and permissible, and what is not: it is just here that we no longer find any general agreement [...] So, for example, it is premature to attempt to base the equations of motion of the ether upon the laws of mechanics until we have obtained a perfect agreement as to what is understood by this name.

Lorentz’ place in the development of electromagnetism is at a turning point of it. Lorentz was fluent in both epistemic traditions and his re-elaboration of Maxwell tries to keep a balance on the different views, while committed to the ether interpretation. However, these two views cannot coexist and Lorentz’ approach turned to be incompatible with both.

The history of what we know today as Lorentz’ force has been presented in (Appendix A, Assis, 1994). It appears in Maxwell’s treatise as:

\[ F = qE + J \times B \]

where \( J = j + \epsilon_0 \frac{\partial E}{\partial t} \) (written in modern notation). The term associated to \( j \) was further decomposed in contributions of galvanic currents and that of charges in relative motion and it was inferred after considerations of energy associate to the interaction of two circuits. The term \( \epsilon_0 \frac{\partial E}{\partial t} \times B \) was added by analogy under the idea that the displacement current was a current associated to the ether. Thompson (Thomson, 1881) obtained the force following Maxwell’s method and some special considerations, Heaviside (Heaviside, 1889) later obtained the expression introducing some changes in the argumentation but basically along the same lines.

Lorentz deduction of the electromagnetic force on a charged particle from Maxwell’s background is presented in (§74-§80, Lorentz, 1892). Lorentz attempts to reintroduce mathematical structures but his argumentation is marred by the use of steps founded only in the idea of the ether. Lorentz assumed a simultaneous displacement of a body with respect to the ether and to the complement of electromagnetic bodies. As soon as the argumentation rests on the ether, it falls apart when the ether is suppressed, unless there is an alternative for producing the result resting on the complement of electromagnetic bodies.

The positive side of both Maxwell’s and Lorentz’ attempts is that they derive the mechanical force due to electromagnetic phenomena as a basic part of the electromagnetic theory. The theory in this way is connected (unified) with mechanics from within itself. Indeed, Lorentz writes down the five “fundamental equations” of electrodynamics (Maxwell’s four and the mechanical force) (Lorentz, 1899). The downside is that both Maxwell (through the displacement current and the so-called Ampère-Maxwell law) and Lorentz include the ether in
their derivations. The ether has subsequently been banned from electrodynamics, leaving us with the uncomfortable situation of having a theory rejecting the ether while its fundamental pillars at the same time rest on the ether.

In the classical textbooks of the XXth. century (Panofsky and Phillips, 1955; Jackson, 1962; Feynman et al, 1965; Purcell, 1965), the Lorentz force is presented as complementing Maxwell’s equations and its ultimate support is left to its success –after using it intuitively– as well as to its consistence with Lorentz transformations. There is no derivation of the force from observations or first principles in the cited books. Indeed, some of the mentioned books present the Lorentz force as an independent posit, others refer to the original work of Lorentz, and finally other books (Soper, 1975; Rohrlich, 2007; Zangwill, 2012) “reverse-engineer” a lagrangian, formally deriving the force from that lagrangian. It must be observed that the ad-hoc lagrangian is not the one in Lorentz’ while the mathematical operations assume independence of currents and fields, only to arrive to the conclusion that they are not independent, this is, showing that the approach is inconsistent. From a historical point of view, Faraday writes, in the manuscript entitled “Thoughts on ray vibrations” (Faraday, 1855, p. 447):

The point intended to be set forth for consideration of the hearers was, whether it was not possible that the vibrations in certain theory are assumed to account for radiation and radiant phenomena cannot occur in the lines of force which connect particles, and consequently masses of matter together; a motion that, as far as it is admitted, will dispense with the æther, which in another view is supposed to be the medium in which this vibrations take place.

You are aware of the speculation which I sometimes since uttered respecting that view of the nature of matter that considers its ultimate atoms as centres of force, and not as so many little bodies surrounded by forces, the bodies being considered in the abstract as independent of the forces and capable of existing without them. In the later view, this little atoms have a definite form and a certain limited size; in the former view such is not the case, for that which represents size may be considered as extending to any distance to which the lines of force of the particle extends: the particle indeed is supposed to exist only by these forces, and where they are it is.

This influential work is cited by Maxwell (Maxwell, 1865) in his ground-breaking work. The proper conception of Maxwell’s electromagnetism rests upon the idea that force-matter or field-matter constitute dualities, hence it is not possible to conceive a change in matter (or its state of motion) and a change of its action (real or potential) upon other bodies independently. Maxwell’s equations have this duality in their foundation.

The main goal of the present work is to obtain the expression of an electromagnetic force without any reference to the ether, using the theoretical tools available by 1873 (Maxwell’s Treatise). Our claim is that at that historical point it was possible to formulate the electromagnetic force on charged particles (and
Maxwell equations) completely ether-free, resorting to research that was known around 1892.

We will show how the line of thought that Lorentz followed can be implemented taking inspiration in Maxwell’s approach without resorting to the ether, thus moving away from absolute space and back to a relational space. In so doing, we need to begin by reconsidering Maxwell’s electromagnetism from the standpoint of Ludwig Lorenz, who was the first in deducing from experiments the wave equations for light as a transversal wave phenomena [Lorenz, 1861]. After removing the ether, Lorenz argument (Lorenz, 1867) must be acknowledged as the first 2 correct conception of electromagnetism.

After his experimental observations ([Lorenz, 1861]), Ludwig Lorenz proposed that light is a transverse vibration phenomenon responding to the wave equation. The idea of electromagnetic waves was circulating at least since 1857 after work by Kirchhoff (Kirchhoff, 1857) and Weber (Poggendorff, 1857; Weber, 1864). Their waves, however, were present inside material media (e.g., in wires). Lorenz’ work (Lorenz, 1867) combines elements from both Weber and Kirchhoff, regarding light as the perception of an electromagnetic oscillation outside matter. Lorenz considered that Weber’s equations had experimental support only in experiments involving slow motion and could be regarded as a first order approximation to wave propagation 3.

With a philosophical position well aligned with that of Faraday who wrote

But it is always safe and philosophic to distinguish, as much as is in our power, fact from theory; the experience of past ages is sufficient to show us the wisdom of such a course; and considering the constant tendency of the mind to rest on an assumption, and, when it answers every present purpose, to forget that it is an assumption, we ought to remember that it, in such cases, becomes a prejudice, and inevitably interferes, more or less, with a clear-sighted judgement. (Faraday, 1844, p.285)

Lorenz self-restrained from formulating physical hypothesis and considered the matter (perhaps in more practical terms)

Hence it would probably be best to admit that in the present state of science we can form no conception of physical reason of forces and of their working in the interior of bodies; and therefore (at present, at all events) we must choose another way, free from all physical hypothesis, in order, if possible, to develop theory step by step in such a manner that further progress of a future time will not nullify the results obtained.

The duality between matter and action, central to the thought of Faraday in his vibrating rays theory (Faraday, 1855, p. 447), was put in more mathematical

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2 In his Treatise, Maxwell reclaims the primacy in time of his work over Lorenz’ while he does not contend Lorenz’ rational argument ([805], Maxwell, 1873).
3 Notice that by those times Weber’s theory was able to explain all electromagnetic phenomena experimentally known, as Maxwell recognised in his 1865 foundational work.
terms by Lorenz: light corresponded with the electrical activity in matter and its propagation at distance was subject to matching conditions with the electrical activity inside matter. This is the meaning of the “identity of the vibrations of light with electrical currents” announced in the title of his contribution.

We will show in the next Section that Lorenz’ approach provides an alternative to the standard argument introducing Maxwell’s displacement current in the Ampère-Maxwell law (see eq. 4 below).

We will attempt a reconstruction of electromagnetism starting from Lorenz standpoint, recovering the energy formula by Maxwell and further discussing Lorentz Lagrangian as an alternative starting point to Lorenz. Next, we will address the Lorentz force in this setting showing how it belongs to it. We will further show that, consistent with this deduction, there is an alternative form for the force. The transformation from one form to the other was envisioned by Maxwell ([600-601], Maxwell, 1873). We give further evidence that the current view of electromagnetism is more the consequence of the epistemological shift towards pragmatism than to experimental facts. The actual use of electromagnetism and its confrontation with experiments corresponds to the relational view, while the constructive inconsistencies of the Lorentz force correspond to its relation to the ether as an absolute reference.

2 The Lorentz Force

What came to be known as the Lorentz force is the outcome of the derivation in (§74-§80, Lorentz, 1892) for the force exerted on a distinguished body (the probe) by the ether. The action of the ether is assumed to be conveyed by the fields \( E \) and \( B \), which satisfy Maxwell equations according to Heaviside, i.e., regarded from a reference system at rest with the ether. Lorentz takes the expressions for kinetic and potential electromagnetic energies from Maxwell, combining them in Lagrangian form to obtain (in §77) \( \nabla \times E = -\partial B/\partial t \), the “fourth” Maxwell equation (not explicitly stated by Maxwell). Subsequently, in §80 the Lorentz force is extracted from the Lagrangian, for a probe taken to be a rigid solid, consisting of particles with some charge density (nonzero only at the location of each particle) that adds up to a smooth charge density \( \rho \). The procedure in both §77 and §80 is not purely mathematical but it responds to the assumption on the ether, to which fields and velocities refer. Three main differences with Maxwell are that (a) Maxwell never wrote down a Lagrangian but worked with the energy contributions instead, (b) Maxwell considers virtual displacements of the secondary circuit while Lorentz considers that a virtual displacement corresponds at the same time (and the same amount) to a displacement relative to the electromagnetic bodies producing the fields and with respect to the ether (in §76). One can hardly imagine something different than the ether being the reference space associated to the sources of the field. Finally, (c), the velocity of the charge in motion is for Lorentz the velocity with respect to the ether, while Maxwell’s velocity in [598] could be interpreted either as relative to the primary circuit (the source of electromagnetic fields) –as
in Faraday’s original concept— or relative to the ether. The current use of the force adheres to Faraday’s view, thus becoming inconsistent with the derivation, inasmuch the latter rests on the ether.

Before we proceed to a derivation of the force, it is worth to consider the difference between Lorentz’ and Maxwell’s expressions for the force, since Lorentz appears to be following Maxwell steps up to some point. Maxwell restrained from making an early decision on the nature of electricity. He considered a piece of matter carrying a current and included the contribution of the “displacement current” just by analogy with galvanic currents. The resulting force was determined up to a contribution consisting of the gradient of a potential. Lorentz, in turn, adopted the idea of material corpuscles (Weber’s hypothesis \cite{Weber1846}). He considered that no galvanic current was present in the moving body and introduced the Lorentz current which is no more than the description of a moving (charged) body with respect to the reference frame of the ether, made in classical terms. For both of them, Maxwell and Lorentz, the body was a rigid solid and the force was to be determined by a variational method. Lorentz acknowledges the influence of the Göttingen school, when he claims: “The influence that was suffered by a particle B due to the vicinity of a second one A, indeed depends on the motion of the latter, but not on its instantaneous motion. Much more relevant is the motion of A some time earlier, and the adopted law corresponds to the requirement for the theory of electrodynamics, that was presented by Gauss in 1845 in his known letter to Weber (bd.5 p. 627-629, \cite{Gauss1870}).” Nevertheless, the extent of this influence is difficult to gauge, since his derivation of the electromagnetic force rests on different premises. Maxwell, on the other hand, acknowledges that Ludwig Lorenz’ approach leads to the same electromagnetic equations but he does not explore contact points and differences any further.

2.1 A word about the ether and absolute space.

Absolute space had earned a bad reputation by the beginning of the XIX century, only the relational space appeared to matter for scientists. The difficulties of conceiving the wave propagation of electromagnetic phenomena as well as the philosophical belief that “matter acts where it is”\footnote{Reverted brilliantly by Faraday into \textit{matter is where it acts}, indicating that matter can only be inferred but not sensed, thus matter is a belief while action is real.} emerged as the concept of an electromagnetic ether (having different and sometimes contradictory properties for different authors). Lorentz shortly addressed this problem:

That we cannot speak about an absolute rest of the aether, is self-evident; this expression would not even make sense ... When I say for the sake of brevity, that the aether would be at rest, then this only means that one part of this medium does not move against the other one and that all perceptible motions are relative motions of the celestial bodies in relation to the aether. (p. 1, \cite{Lorentz1895})
For Lorentz, the ether had to be material but transparent to the (ponderable) bodies. He did not fully convince other authors such as Einstein (1907), Ritz (1908) for whom Lorentz’ ether was just absolute space.

2.2 Ether-free electrodynamics

2.2.1 Maxwell’s equations

The starting point of our task are Maxwell equations [Maxwell, 1873] that in modern notation, with \( C^2 = (\mu_0 \varepsilon_0)^{-1} \), can be stated as

\[
\begin{align*}
\mathbf{B} & = \nabla \times \mathbf{A} \\
\mathbf{E} & = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V \\
\varepsilon_0 \nabla \cdot \mathbf{E} & = \rho \\
\mu_0 \mathbf{j} + \frac{1}{C^2} \frac{\partial \mathbf{E}}{\partial t} & = \nabla \times \mathbf{B}
\end{align*}
\]

All quantities are evaluated at a reference point, \( x \), relative to the reference frame in which the distribution of charge is given by \( \rho(x,t) \) and satisfies the continuity equation \( \frac{\partial \rho(x,t)}{\partial t} + \nabla \cdot \mathbf{j}(x,t) = 0 \) ([295], Maxwell, 1873). In other words, it is the frame where the fields \( \mathbf{E}, \mathbf{B} \) have been determined through \( \rho \) and \( \mathbf{j} \). The equations correspond to the electromagnetic momentum, \( \mathbf{A} \) (today called vector potential), the magnetic induction, \( \mathbf{B} \) ([619] eq. A, Maxwell, 1873), the electric field intensity \( \mathbf{E} \) ([619], eq. B, Maxwell, 1873) (being the sum of an induction contribution ([598], Maxwell, 1873) and a gradient generalising the electrostatic potential) and the galvanic current, \( \mathbf{j} \). The third equation, taken from ([612], Maxwell, 1873) and inspired in Coulomb and Faraday, generalises here Poisson’s equation from electrostatics, while the fourth equation ([619], eq. E, Maxwell, 1873) refers to ([610], eq. H*, Maxwell, 1873).

Maxwell obtained eq. 4 through considerations about the ether. He argued that the propagation of electromagnetic waves asks for a propagation medium, by analogy with “the flight of a material substance through space” ([866] Maxwell, 1873). However, Ludwig Lorentz (1867) expresses discomfort with the ether hypothesis, which had only been useful to “furnish a basis for our imagination” (p. 287). Consequently, he offered an ether-free derivation of the field equations that took into account Gauss’ suggestion of delayed action at a distance (bd.5 p. 627-629, Gauss, 1870). For static situations, the electromagnetic potentials at a point \( x \) (and time \( t \)) originating in a charge and current distribution over a domain in space described through the coordinate \( y \)

\[5\]

The same equations can be found in (Maxwell, 1865) Section III under the equation labels: B, 35, G, C (taking into account A) and H.

\[6\]

Other authors arrived to similar equations, also without involving the ether (Riemann, 1867; Betti, 1867; Neumann, 1868). These authors (but not Lorentz) were in turn criticised for not involving the ether in their conceptualisation (Clauss, 1863).
are defined through Poissons’s equation starting from charges and currents, i.e.,
\[ (A, V_C)(x, t) = \frac{\mu_0}{4\pi} \int \left( \frac{(j, \rho C)(y, t)}{|x - y|} \right) d^3y, \] (5)

For the vector potential \( A \) this idea was originally advanced by Franz Neumann (Neumann, 1846).

Inspired in Gauss’ proposal, Lorenz considered that the charges and currents contributing to the potentials at time \( t \) act with delay, after being originated in a previous time \( s = t - \frac{|x - y|}{C} \):
\[ (A, V_C)(x, t) = \frac{\mu_0}{4\pi} \int \left( \frac{(j, \rho C)(y, t - \frac{1}{C}|x - y|)}{|x - y|} \right) d^3y, \] (6)

where only galvanic currents and actual charges are involved. After standard operations of vector calculus (see Lemma 5, Appendix A) it is found that
\[ \square A = -\mu_0 j \] (7)
\[ \square V = -\frac{1}{\epsilon_0} \rho \]

where \( \square = \Delta - \frac{\partial^2}{\partial t^2} \) is D’Alembert’s wave operator. In Lemma 7, Appendix A we derive eq. 4 from Lorenz’ setup, taking advantage of the Lorenz gauge. We note on passing that eq. 5 is not the only possible definition of \( A \) and \( V \) that is compatible with the wave equation.

2.2.2 Charge and current distribution of electrified bodies in motion in a relational framework

Following Lorentz, charges are assumed to be rigid bodies (§ 75c, Lorentz, 1892) or “corpuscles” and all charges are considered to be accounted for directly, so there is no need to introduce polarisation/induction fields. With this setup, the force on a probe charge will be identical to the electromotive force.

While the idea that a charge in movement corresponds to a current goes back (at least) to Weber (Weber, 1846), the form of constructing the current had been the subject of controversy. For Maxwell the velocities that matter were relative velocities between primary and secondary circuits in full agreement with Faraday-Maxwell ([568-583] 1873). For the sake of the argument we consider both circuits as rigid solids. Lorentz in turn considered that the velocities had to be considered relative to the ether, an idea that has to be disregarded since no evidence of the ether can be found. Hence, only the Faraday-Weber-Maxwell (and others) idea of relative velocity appears to be tenable. We let \( \dot{x}(t) \) be the relative velocity between two reference points in the primary and secondary bodies. We hereby define current according to
\( (x,t) = (y + \bar{x}(t), t) \)
\( \bar{\rho}_2(x,t) = \rho_2(x - \bar{x}(t), t) \)
\( \bar{j}_2(x,t) = j_2(x - \bar{x}(t), t) + \dot{x} \rho_2(x - \bar{x}(t), t) \) \hspace{1cm} (8)

where \( y \) is a “local” coordinate of the secondary body (labelled with the index 2, the body where we aim to compute the acting force) and \( x \) is the coordinate relative to the reference frame above in which the potentials are \( A \) and \( V \). Hence, neither ether nor any inertial frame is considered, but only the relative coordinate between the electromagnetic bodies undergoing mutual interaction. The charge density and internal current \( (\rho_2(y,t), j_2(y,t)) \) are described in a frame at rest with respect to the rigid body 2, while \( (\bar{\rho}_2(x,t), \bar{j}_2(x,t)) \) are their expressions with reference to the primary circuit. Also, it is assumed that the body does not rotate. In either reference frame, charge and current for body 2 also satisfy the continuity equation.

This definition of current (eq. 8) reduces to Lorentz’ current for the particular case \( j_2 = 0 \), only that the velocity is now relational, while the first term in eq. 8 allows for other sources of current, as it was entertained by Faraday and Maxwell.

2.2.3 Maxwell’s energy and Lorentz’ Lagrangian revisited

Maxwell’s energy is introduced through a process in which matter acquires its electromagnetic state \([\text{Maxwell, } 1873]\). The electrostatic energy is obtained in \([630-631]\) bringing charges from infinity. Next, the magnetostatic energy is obtained in similar form in \([632-633]\), based upon magnetostatic results previously obtained in \([389]\). Maxwell proceeds to add an electrokinetic energy due to the currents \([634-635]\). In the even numbered articles he presents the physical idea and in the odd numbered articles he transforms the expression using integration by parts.

Thus, the energy required to create a given electromagnetic state (a distribution of charges and currents) can be regarded as the time-integral of the power, first bringing charges from a condition of zero energy (from “infinity”) working against \(-\nabla V\) and bringing also the current distribution, now working against the electromagnetic momentum \( A \):

\[
\mathcal{P} = \left( \nabla V + \frac{\partial A}{\partial t} \right) \cdot j = -E \cdot j
\]

the time-integral from a situation in which \( E(0) = 0 \), leads to

\[
\mathcal{E} = -\int_0^t dt \int d^3x (E \cdot j) = \int_0^t dt \int d^3x \left( \left( \nabla V + \frac{\partial A}{\partial t} \right) \cdot j \right).
\] \hspace{1cm} (9)

Assuming that all of \(|B|^2, |E|^2, A \cdot j, V \rho\) decrease faster than \(\frac{1}{r^2}\) at infinity (a hypothesis needed for most manipulations performed by Maxwell and Lorentz)
and applying Gauss theorem to convert volume integrals of a divergence into surface integrals (vanishing at infinity by the assumption), a straightforward computation (Lemma 8, Appendix A) shows that the energy provided by the electrification of the body is

$$E = \frac{1}{2} \int d^3x \left( \frac{1}{\mu_0} |B|^2 + \epsilon_0 |E|^2 \right).$$

(10)

From the electrostatic and magnetostatic situations it is clear that the individual terms correspond to the electrokinetic, 

$$T = \frac{1}{2} \int d^3x \left( \frac{1}{\mu_0} |B|^2 \right),$$

and potential energies, 

$$U = \frac{1}{2} \int d^3x \left( \epsilon_0 |E|^2 \right),$$

thus suggesting that the electromagnetic Lagrangian reads 

$$\mathcal{L} = T - U$$

and the action

$$A = \int dt \mathcal{L}.$$  

Another straightforward computation and application of Gauss theorem (Lemma 9, Appendix A) leads to

$$A = \frac{1}{2} \int dt \int \left( \frac{1}{\mu_0} |B|^2 - \epsilon_0 |E|^2 - \kappa \cdot (A - A) \chi + \lambda (V - V) \chi \right) d^3x + f(t) - f(t_0)$$

(11)

The dynamic equations are independent of $f$, so we can proceed from here by taking $f \equiv 0$. While the intuition of terms can be taken as a suggestion, we must ask: What kind of dynamical situation is reflected by this action?

**Theorem 1.** Let $(A, V)$ be the known values of the electromagnetic potentials in a piece of matter supported on a region of space with characteristic function $\chi$. Then, under the assumptions of Lemma 6, Hamilton’s principle of least action (Ch 3, 13 Ap. 59, Arnold, 1989), $\delta A = 0$, subject to the constraints given by $(A, V)$ implies that the manifestation of the potentials outside matter obeys the wave equation.

**Proof.** The result follows from the computation of the minimal action under

$$(V - V) \chi = 0$$

$$(A - A) \chi = 0$$

Multiplying the constraints by the Lagrange multipliers $\lambda$ and $\kappa$ (the latter a vector), while we use the shorthand notations $B = \nabla \times A$ and $E = (-\frac{\partial A}{\partial t} - \nabla V)$, we need to variate

$$A = \frac{1}{2} \int dt \int \left( \frac{1}{\mu_0} |B|^2 - \epsilon_0 |E|^2 - \kappa \cdot (A - A) \chi + \lambda (V - V) \chi \right) d^3x.$$ 

After variation and applying Gauss theorem as usual leads to

$$\Box A = -\chi \kappa$$

$$\Box V = -\chi \lambda$$

$^7$The continuity equation, eq. (6) and $\nabla \cdot A + \frac{1}{C^2} \frac{\partial V}{\partial t} = 0$, see Appendix.
which allows us to identify \( \mu_0 j = \chi \kappa \) (the density of current inside the material responsible for \( A \)) and \( \frac{\rho}{\varepsilon_0} = \chi \lambda \) (the density of charge responsible for \( V \)).

This theorem deserves to be called Lorenz’ theorem since he wrote about these relations (p.300, Lorenz [1867]): “This result is a new proof of the identity of the vibrations of light with electrical currents; for it is clear now that not only the laws of light can be deduced from those of electrical currents, but that the converse way may be pursued, provided the same limiting conditions are added which the theory of light requires.”

Thus, electromagnetism has been summarised in terms of the potential \( V \) and vector potential (or electrokinetic momentum, according to Maxwell-Faraday) \( A \) (defined via eq. 6), the principle of least action and the continuity equation. All other relations consist in either naming intermediate quantities, such as \( B = \nabla \times A \) and \( E = -\left( \frac{\partial A}{\partial t} + \nabla V \right) \), or stating consequences of these definitions, such as \( \frac{\partial}{\partial t} \nabla \times A = \nabla \times \frac{\partial A}{\partial t} \) (which leads immediately to Faraday’s induction law), or \( \nabla \cdot E = \rho \) (that follows after the use of the continuity equation).

The connection of this theoretical structure with observations in nature is given by the phenomenological map [Solari and Natiello, 2021] providing observational content to charges and currents in matter. We are now ready to show how Lorentz’ force belongs to this context and is already inscribed in the former equations.

### 2.2.4 Deduction of Lorentz’ force revisited

When we consider two pieces of electrified matter in interaction we can envisage a different form of constructing the system. In the first step, the bodies are far apart, so that we can assume that they do not interact, and are electrified to reach their actual state. Next, they will be brought together to their corresponding mechanical positions in terms of a thought process called a virtual displacement. The formalisation of this idea already present in Maxwell is called a virtual variation (Ch 4, 21 B p. 92, Arnold [1989]). The force associated to this virtual displacement will then result from the variation of the interaction terms in the Lagrangian. Given the electromagnetic contribution to the action, determining the contribution to the force amounts to applying Hamilton’s principle using a virtual displacement of the probe (which we indicate with subindex 2) with respect to the primary circuit producing the fields (subindex 1).

In formulae, we must request

\[
\delta \bar{x}(t) \delta A = 0
\]

with \( \delta \bar{x}(t_0) = 0 \) and \( \delta \bar{x}(t) = 0 \), where \( \bar{x}(t) \) denotes the relative distance between probe and primary circuit. In so doing, we must take into account that there is a corresponding variation of the velocity \( \delta \dot{x}(t) \). Recall that according to 7 and \# \( \bar{x} \) occurs in \( (\rho_2, j_2) \) and \( \dot{x} \) occurs only in \( j_2 \). We state the result as a theorem:
Theorem 2. Assuming that all of $|B|^2$, $|E|^2$, $A$, $j$, $V$, $\rho$ decrease faster than $\frac{1}{r^2}$ at infinity and under the conditions of Lemmas [5–10] (in particular the action of eq. [11]) the electromagnetic force

$$F_{em} = \vec{j}_2 \times B_1 + \vec{\rho}_2 E_1$$

on the probe can be deduced from Hamilton’s principle of minimal action ($\delta \bar{x}(t) A = 0$) using a virtual displacement $\delta z$ of the probe (which we indicate with subindex 2) with respect to the primary circuit producing the fields (subindex 1).

For the proof, see Subsection A.0.1, Appendix A. Lorentz considered only the case $j_2 = 0$, hence $\bar{j}_2 = \bar{\rho}_2 \ddot{x}$. Maxwell considered the general case in Maxwell ([602] 1873) but using his “total current” instead, consistent with his belief in the ether-based displacement current as a material current. However, his elaboration is based on galvanic currents, finally modifying the final result by analogy. [619 eq. C]maxw73. The present approach differs with Lorentz’ in the broader concept of current and in that the participating quantities are fully relational and describe the interaction between probe and primary circuit.

2.3 Subjective perspective

Since the Lagrangian is expressed by an integral, we are free to change the integration variable by a fixed translation leaving the integral unchanged. Actually, we can use a different translation for each time in (11). We propose to change integration variable from $x$ to $z$, with $x = z + \bar{x}(t)$. Instead of performing the change in eq.(13) we will save effort and perform it in eq.(14), prior to the partial integration in time, namely

$$\delta A = \int \! dt \int \! d^3 z \left[ j_2 (z, t) + \dot{\bar{x}} \rho_2 (z, t) \right] \cdot (\delta \bar{x} \cdot \nabla) \bar{A} - \bar{x} \nabla \cdot (\rho_2 \bar{A})$$

(with $\bar{j}_2(x, t)$, $\bar{\rho}_2(x, t)$ given by eq[8]). We introduce the following notation

$$z = x - \bar{x}(t)$$

$$V(z, t) = V(z + \bar{x}(t), t)$$

$$A(z, t) = A(z + \bar{x}(t), t)$$

Hence, the variation reads now

$$\delta \int \mathcal{L} dt = \int \! dt \int \! d^3 z \left[ (j_2(z, t) + \dot{\bar{x}} \rho_2(z, t)) \cdot (\delta \bar{x} \cdot \nabla) \bar{A} - \rho_2(z, t) \cdot (\delta \bar{x} \cdot \nabla) \bar{V} \right]$$

$$+ \int \! dt \int \! d^3 z [\delta \bar{x} \cdot (\bar{A} \rho_2(z, t))]$$

integrating by parts in time the last term and using the relations.
\[
\int dt \left[ \delta \ddot{x} \cdot (A \rho_2) \right] = \int dt \left[ -\rho_2 \delta \ddot{x} \cdot \left[ \frac{\partial A}{\partial t} \right] + \delta \ddot{x} \cdot \left[ A \frac{\partial \rho_2}{\partial t} \right] \right]
\]

\[
\left[ \frac{\partial A}{\partial t} \right] = \left. \frac{\partial}{\partial t} A(z + \ddot{x}(t), t) \right|_{x = z + \ddot{x}(t)} - (\ddot{x} \cdot \nabla) A(z + \ddot{x}(t), t)
\]

\[
-\nabla V - \left[ \frac{\partial A}{\partial t} \right] = -\nabla V - \left. \frac{\partial}{\partial t} A(x, t) \right|_{x = z + \ddot{x}(t)} - (\ddot{x} \cdot \nabla) A
\]

(along with the continuity equation) we arrive after some algebra and further use of Gauss’ theorem to:

\[
\delta \int L \, dt = \int dt \int d^3z \left[ j_2 \cdot (\delta \ddot{x} \cdot \nabla) A - j_2 \cdot \nabla (\delta \ddot{x} \cdot A) \right]
\]

\[
\quad + \int dt \int d^3z \left[ (\rho_2 \ddot{x}) \cdot (\delta \ddot{x} \cdot \nabla) A + \rho_2 \delta \ddot{x} \cdot (-\nabla V - \frac{\partial A}{\partial t}) \right]
\]

Finally, the following relations (the second one valid for any sufficiently differentiable scalar function \( \Phi \))

\[
\ddot{x} \cdot (\delta \ddot{x} \cdot \nabla) A = -\delta \ddot{x} \cdot \nabla (\ddot{x} \cdot A)
\]

\[
(\delta \ddot{x} \cdot \nabla) \Phi(x, t) = -\delta \ddot{x} \times (\nabla \times \Phi) + (\nabla \delta \ddot{x} \cdot \Phi)
\]

\[
j_2 \cdot (\delta \ddot{x} \cdot \nabla) A - j_2 \cdot \nabla (\delta \ddot{x} \cdot A) = j_2 \cdot (-\delta \ddot{x} \times (\nabla \times A))
\]

lead us to the next result:

\[
\delta \int L \, dt = \int dt \int d^3z \left[ j_2 \cdot (\delta \ddot{x} \times (\nabla \times A)) \right]
\]

\[
\quad + \int dt \int d^3z \left[ \delta \ddot{x} \cdot \rho_2 \left( - \left[ \frac{\partial A}{\partial t} \right] - \nabla (V - \ddot{x} \cdot A) \right) \right]
\]

\[
\quad = \int dt \int d^3z \delta \ddot{x} \cdot \left[ j_2 \times B + \rho_2 \left( - \left[ \frac{\partial A}{\partial t} \right] - \nabla (V - \ddot{x} \cdot A) \right) \right]
\]

Hence we have two expressions for the mechanical contribution of the electromagnetic force: The one obtained from eq.(14) above and the present one, i.e.,

\[
F_{em} = \int d^3x \left[ j_2 \times B + \rho_2 E \right] = \int d^3z \left[ j_2 \times B + \rho_2 \left( - \left[ \frac{\partial A}{\partial t} \right] - \nabla (V - \ddot{x} \cdot A) \right) \right]
\]

(recall the relation among functions defined in eqs. \([8]\) and \([12]\)).

We call this relation “Maxwell’s transformation theorem”, since Maxwell showed the following result in Maxwell ([601], 1873):

\[8\] Maxwell’s result covers also rotating reference systems, but we skip this case to keep the argumentation simple.
**Theorem 3. (Maxwell’s invariance theorem)**: Let \( x = x' + \vec{x}(t) \), and correspondingly \( v = v' + \dot{x} \). Define \( A'(x', t) \equiv A(x, t) \), then the value of the electromotive force at a point \( x \) does not depend on the choice of reference system if and only if \( \psi(x, t) \) transforms as \( \psi'(x', t) \equiv \psi(x' - \vec{x}, t) - \dot{x} \cdot A'(x', t) \). In formulae, \( \mathcal{E}'(x', t) = \mathcal{E}(x, t) \), where \( \psi \) is an undetermined electrodynamic potential (introduced for the sake of generality):

\[
\mathcal{E}'(x', t) = v' \times (\nabla \times A'(x', t)) - \frac{\partial A'(x', t)}{\partial t} - \nabla \psi'(x', t).
\]

**Proof.** First, according to the definition, we have \( A'(x', t) = A(x' + \vec{x}, t) \).

Next, we note that by straightforward vector calculus identities, Maxwell’s electromotive force (eq. B in [Maxwell, 1873]) can be restated as

\[
\mathcal{E}(x, t) = - \frac{\partial A(x, t)}{\partial t} - (v \cdot \nabla) A(x, t) - \nabla (\psi(x, t) - v \cdot A(x, t))
\]

In the new coordinate system we compute:

\[
\mathcal{E}'(x', t) = v' \times (\nabla \times A'(x', t)) - \frac{\partial A'(x', t)}{\partial t} - \nabla \psi'(x', t)
\]

\[
= - \frac{\partial A'(x', t)}{\partial t} (x', t) - (v' \cdot \nabla) A' - \nabla (\psi'(x', t) - v' \cdot A'(x', t)).
\]

Subsequently, under the present assumption \( A'(x', t) \equiv A(x, t) \) we may rewrite

\[
\frac{\partial A'(x', t)}{\partial t} = \left. \frac{\partial A(x, t)}{\partial t} \right|_{x' + \vec{x}} + (\dot{x}, \nabla) A
\]

leading to

\[
\mathcal{E}'(x', t) = - \frac{\partial A(x, t)}{\partial t} - (\dot{x} \cdot \nabla) A - (v' \cdot \nabla) A' - \nabla (\psi'(x', t) - v' \cdot A'(x', t))
\]

\[
= - \frac{\partial A(x, t)}{\partial t} - (v \cdot \nabla) A - \nabla (\psi(x, t) - \dot{x} \cdot A(x, t) - v' \cdot A'(x', t))
\]

\[
= - \frac{\partial A(x, t)}{\partial t} - (v \cdot \nabla) A - \nabla (\psi(x, t) - \dot{x} \cdot A(x, t) - v' \cdot A'(x', t))
\]

\[
= - \frac{\partial A(x, t)}{\partial t} - (v \cdot \nabla) A - \nabla (\psi(x, t) - v \cdot A(x, t)) = \mathcal{E}(x, t)
\]

where to proceed from the second line to the third, we used the condition \( \psi(x', t) \equiv \psi(x, t) - \dot{x} \cdot A(x, t) \). Note that \( \psi \) is Maxwell’s undetermined potential that once determined in one system of reference transforms according to the theorem to other systems. 

---

Maxwell refers to this expression as: “the theory of the motion of a body of invariable form”. For any property of matter, this relation is immediate.
Maxwell’s Art. [601] of the Treatise states “It appears from this that the electromotive intensity is expressed by a formula of the same type, whether the motions of the conductors be referred to fixed axes or to axes moving in space, the only difference between the formulae being that in the case of moving axes the electric potential $\psi$ must be changed into $\psi + \psi'$.\textsuperscript{a}

2.3.1 No Arbitrariness Principle

The invariance of the force as determined from the point of view of the source (primary circuit) or of the target (probe, secondary circuit) in the previous subsection follows from the general invariance of integrals in front of coordinate changes, supported in Maxwell’s invariance theorem. The result is a special case of the No Arbitrariness Principle (NAP) \cite{Solari and Natieldo, 2018}, stating that the description of natural processes cannot depend on arbitrary choices (in this case the choice of subjective reference frame). Indeed, we can attain a fully relational description of electromagnetic phenomena.

The consideration of three electrified bodies will allow us to inspect this problem. Let $x_{ij}(t)$ the relative position between the $i$ and the $j$ body, with $i,j \in \{1,2,3\}$, clearly, $x_{ii} = 0$. The relative positions satisfy $x_{12} + x_{23} + x_{31} = 0$ and correspondingly, the relative velocities satisfy $v_{12} + v_{23} + v_{31} = 0$. Let $\zeta_i$ be each of the components of the four dimensional vector $(j_i, \rho_i)$ describing the density of currents and the density of charges in body $i$, as perceived from a frame fixed to itself, and $\zeta_j$ the same components as described from the frame of body $j$. We will denote by $\mathcal{W}$ the operator that produces the delayed propagation of the EM situation in the body and, finally, $T(x)$ the operator that applies a time-dependent translation to current+charge vector as in \S

Clearly $T(0) = \text{Id}$. Then, given the potentials of body $i$ in its own frame, we get the potentials in the body $j$ frame as

$$(A_i^j, V_i^j) = (\mathcal{W}T(x_{ji})\Box) (A_i, V_i)$$

The operators $T(x_{ij}(t))$ form a group of transformations. It is easy to verify that the product law is:

$$T(x(t))T(y(t)) = T((x + y)(t))$$

Letting

$$\tilde{T}(x_{ij}) = WT(x_{ji})\Box$$

we notice that the operators $\tilde{T}(x_{ij})$ are conjugated to $T(x_{ij})$ since the relation $\Box \mathcal{W} = \text{Id}$ and $\mathcal{W} \Box = \text{Id}$ produce the conjugation relation

$$\Box \tilde{T}(x_{ij}) = T(x_{ij})\Box$$
$$\tilde{T}(x_{ij})\mathcal{W} = \mathcal{W}T(x_{ij})$$

Thus, the rigid translation of charge densities and currents $T(x_{ij})$ acts with a conjugate representation on the wave representatives. This is the abstract content of Maxwell’s theorem.
We can finally write all the fields in terms of one reference body, say, the body with \( i = 1 \) as

\[
(A_1, V_1) = W (\zeta_1^1 + T(x_{12})\zeta_2^2 + T(x_{13})\zeta_3^3)
\]

We next apply \( \tilde{T}(x_{21}) \) to this expression, we get

\[
\tilde{T}(x_{21})(A_1, V_1) = \tilde{T}(x_{21})W (\zeta_1^1 + T(x_{12})\zeta_2^2 + T(x_{13})\zeta_3^3)
\]

\[
= (WT(x_{21})\nabla) W (\zeta_1^1 + T(x_{12})\zeta_2^2 + T(x_{13})\zeta_3^3)
\]

\[
= (WT(x_{21})) (\zeta_1^1 + T(x_{12})\zeta_2^2 + T(x_{13})\zeta_3^3)
\]

\[
= W (T(x_{21})\zeta_1^1 + T(x_{21})T(x_{12})\zeta_2^2 + T(x_{21})T(x_{13})\zeta_3^3)
\]

\[
= W (T(x_{21})\zeta_1^1 + \zeta_2^2 + T(x_{23})\zeta_3^3)
\]

\[
= (A_2, V_2)
\]

Which shows how the subjective representation of fields transforms consistently with the time-dependent-translations group. When the admitted reference is restricted to inertial bodies (Natiello and Solari, 2019), the group of transformations is the Galilean group.

Let us specify the above construction for the electromagnetic force that bodies 2 and 3 exert on a moving charge \( q_1 \). Expressing the fields as computed by bodies 2 and 3 respectively, this force reads,

\[
F_1 = q_1 (v_{12} \times B_2^2 + E_2^2) + q_1 (v_{13} \times B_3^2 + E_3^2)
\]

From Maxwell’s invariance theorem, the relation \((A_3^2, V_3^2) = T_{23}(A_3^1, V_3^1)\) is just a recasting of the vector potential \( A_3^3 \) in the coordinates of body 2. Hence, both potentials take the same value on any given point of space. Consequently \( B_3^2 = B_3^3 \). The transformation of the electric field and scalar potential reads (also by Theorem 3),

\[
E_3^3 = -\nabla V_3^3 - \frac{\partial}{\partial t} A_3^3 = -\nabla (v_{32} \cdot A_3^3) - \frac{\partial}{\partial t} A_3^3 - (v_{32} \cdot \nabla) A_3^3
\]

\[
= v_{32} \times (\nabla \times A_3^3) - \nabla V_3^3 - \frac{\partial}{\partial t} A_3^3
\]

\[
= v_{32} \times B_3^2 + E_3^2
\]

Hence,

\[
F_1 = q_1 (v_{12} \times B_2^2 + E_2^2) + q_1 (v_{13} \times B_3^2 + (v_{32} \times B_3^2 + E_3^2))
\]

\[
= q_1 (v_{12} \times (B_2^2 + B_3^2) + (E_2^2 + E_3^2))
\]

i.e., we have proven

**Corollary 4.** Under the assumptions of Theorem 3, the interaction force between electromagnetic bodies is invariant in front of arbitrary (subjective) Galilean translations of the reference system.
In the present framework the electromagnetic force is a Galilean invariant describing an interaction between primary and secondary circuits. Changing the reference system for the description while keeping the relative motion does not affect the force. This is the context in which translational invariance is proper (Solari and Natiello, 2018) and where the Galilean transformation emerges. The situation must be distinguished from that in which the secondary circuit is put in relative motion with respect to the primary circuit and, in addition, the reference frame is changed so that it follows the secondary circuit. Let us call the primary circuit the source and the secondary circuit the receiver. We know that if we put the receiver in relative motion with respect to the source, it will detect a different signal (Doppler’s effect). Thus, a valid (experimentally accessible) physical question is: which is the relation between what the receiver perceives and what it would have perceived had it not been put in relative motion? However, nothing changes in the relation between source and detector if we decide to describe it while jogging around the experiment. What confuses matters is that, in the case of instantaneous action at distance (not our case), the same Galilean transformation can be used to connect reference frames of description in (constant) relative motion and to relate the perceptions of a receiver at rest or in (constant) motion relative to the source. In short: proper Galilean invariance and Maxwell’s theorem are not in contradiction with experimental facts or with the theoretical use of Lorentz’ transformations for cases of relative motion.

3 Discussion

The attitude towards error defines our science. When our observations are incompatible with a hypothesis, i.e., they refute it, we can only think of suppressing and/or replacing the indicted hypothesis as well as all of its consequences, going back to the point where we had the wrong idea, and restarting our progress from that point. This has not been the path historically followed, hence we must ask: which forces made it impossible? To answer the question we suggest to search for historical, social and epistemological constrains.

Maxwell worked out his electromagnetism following the path of the Göttingen school, but he needed to persuade himself of analogies with matter (Maxwell, 1856), hence his introduction of the ether. Lorentz as well worked out from this mixed epistemological position. More than a hundred years later we can say that their success comes from the mathematisation while the problems with their approaches come from their analogical thoughts, as we have observed for example in Maxwell’s force and Lorentz deduction of his force. Modern textbooks have a mixed attitude towards Lorentz force, some of them pragmatically accepting it as an independent posit (see Introduction). Even the nature of the velocity in the expression of the force is not completely clear (Assis, 1994) after dropping the ether.

In this work we have shown how to construct the basic elements of electro-dynamics developing a relational electromagnetism that reaches a higher level of consistency and harmony than the accepted electromagnetism at the time.
of Lorentz. Within this approach, the electromagnetic force corresponding to Lorentz’ emerges from the same principles that produce Maxwell equations, this is, they form a consistent theory a-priori rather than a-posteriori. The resulting relational electromagnetism admits a subjective form that complies with the non-arbitrariness-principle [Solari and Natie11d 2018], a principle that is more general and demanding than Poincaré’s principle-of-relativity. The construction begins with one concept of space and ends within the same concept. We have further discussed how Galilean invariance takes two different meanings, one that is proper and derives from the non-arbitrariness-principle and one that is accidental and restricted to instantaneous-action-at-a-distance. When each matter is given its proper place, there is no contradiction between Galilean invariance and the use of Lorentz transformations or experimental results on Doppler’ shifts.

Not surprisingly, the elements of our construction can be found in Gauss, Maxwell, Lorenz, and the Göttingen school. The correspondence between Lorentz’ current and Maxwell’s invariance theorem is a key element for this state of harmony. Lorentz’ and Maxwell’s transformations, when restricted to inertial systems reduce to not-so-obvious presentations of the Galilean group of transformations.

In this relational approach, electromagnetic interactions manifest themselves in any reference system as waves of speed $C$, refuting the idea that this is incompatible with classical space-time. But then, what do we need to drop to accept this theory? The answer is clear: analogy must be trusted no more that Maxwell did:

“...It appears to me, however, that while we derive great advantage from the recognition of the many analogies between the electric current and a current of material fluid, we must carefully avoid making any assumption not warranted by experimental evidence, and that there is, as yet, no experimental evidence to shew whether the electric current is really a current of a material substance, or a double current, or whether its velocity is great or small as measured in feet per second.” ([574], Maxwell 1873)

The conflict between the Göttingen and Berlin schools illustrates a radical difference in the conception of science. When physics left the safe waters of mechanics to penetrate the phenomena that develop inside matter, the Göttingen school was prepared to resign the matter-space idealisation, as proposed by Faraday and Lorenz, maintaining the tradition initiated by Leibniz and Newton that demands science not to introduce physical hypotheses. The Berlin school struggled to preserve the matter-space idealisation, introducing metaphysical entities such as the ether or a body-like carrier of interactions, whose ultimate function is to facilitate analogical thinking. When we separate action from matter we need a form to propagate the action of a material entity at distance. In contrast, when action and matter form a duality as proposed by Faraday we do not need the ether or action carriers but our intuitive view of the material world is shaken. While a social decision on this matter has been taken, the voice of Faraday
haunts us from the past: “we ought to remember that it, in such cases, becomes a prejudice, and inevitably interferes, more or less, with a clear-sighted judgement” (Faraday, 1844, p.285).

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A Lemmas and Proofs

Lemma 5. \( A(x, t) = \frac{\mu_0}{4\pi} \int_U \left( j(y, t - \frac{1}{C} |x - y|) \right) d^3y \Rightarrow \square A = -\mu_0 j \), and similarly for \( \epsilon_0 \square V = -\rho \), where \( \square \equiv \Delta - \frac{1}{C^2} \frac{\partial^2}{\partial t^2} \).

Proof. We perform the calculation in detail only for \( A \), since the other one is similar. We use the shorthand \( r = |x - y| \).

\[
\begin{align*}
\nabla_x A_i &= \frac{\mu_0}{4\pi} \int d^3y \left( j_i \nabla_x \left( 1 - \frac{\partial}{\partial t} j_i \nabla_x \frac{r}{r} \right) \right) \\
\Delta A_i &= \nabla_x \cdot \nabla_x A_i \\
&= \frac{\mu_0}{4\pi} \int d^3y \left( j_i \Delta \frac{1}{r} - 2 \left( \nabla_x \frac{1}{r} \right) \cdot \left( \frac{\partial}{\partial t} j_i \nabla_x \frac{r}{C} \right) - \frac{\partial}{\partial t} j_i \Delta \frac{r}{r} + \frac{\partial^2}{\partial t^2} \left( \nabla_x \frac{r}{C} \right)^2 \right)
\end{align*}
\]

Moreover, standard vector calculus identities give

\[
\begin{align*}
\frac{\partial}{\partial t} j_i \left( 2 \nabla_x \frac{1}{r} \cdot \nabla_x \frac{r}{C} + \frac{\Delta \frac{r}{C}}{r} \right) &= 0 \\
|\nabla_x \frac{r}{C}|^2 &= \frac{1}{C^2}
\end{align*}
\]

and therefore

\[
\Delta A_i(x, t) = \frac{\mu_0}{4\pi} \int d^3y j_i(y, t - \frac{r}{C}) \Delta \left( \frac{1}{r} \right) + \left( \frac{1}{C^2} \right) \frac{\mu_0}{4\pi} \int d^3y \frac{\partial^2}{\partial t^2} j_i(y, t - \frac{r}{C})
\]

The time derivative in the last term can be extracted outside the integral, thus yielding,

\[
\begin{align*}
\square A_i(x, t) &= \Delta A_i(x, t) - \left( \frac{1}{C^2} \right) \frac{\mu_0}{4\pi} \int d^3y \frac{\partial^2}{\partial t^2} j_i(y, t - \frac{r}{C}) \\
&= \Delta A_i(x, t) - \left( \frac{1}{C^2} \right) \frac{\partial^2}{\partial t^2} A_i(x, t) \\
&= \frac{\mu_0}{4\pi} \int d^3y j_i(y, t - \frac{1}{C} |x - y|) \Delta \left( \frac{1}{r} \right) \\
&= -\mu_0 j_i(x, t)
\end{align*}
\]

Lemma 6. The continuity equation along with eq\(\square\) imply \( \nabla \cdot A + \frac{1}{C^2} \frac{\partial V}{\partial t} = 0 \) (the Lorenz gauge).
Proof. Still using the shorthand \( r = |x - y| \), and applying Gauss theorem over volume integrals of total divergences of functions vanishing sufficiently fast at infinity,

\[
0 = \nabla \cdot A + \frac{1}{C^2} \frac{\partial V}{\partial t} = \\
\int d^3y \left( \nabla \cdot \frac{j(y,t - \frac{r}{C})}{|x - y|} + \frac{\partial}{\partial t} \frac{\rho(y,t - \frac{r}{C})}{|x - y|} \right) \\
= \int d^3y \left( \nabla \cdot \frac{1}{|x - y|} \cdot j(y,t - \frac{r}{C}) - \frac{1}{|x - y|} \frac{\partial}{\partial s} j(y,s) \right)_{s = t - \frac{r}{C}} \cdot \nabla \frac{r}{C} + \frac{\partial}{\partial t} \frac{\rho(y,t - \frac{r}{C})}{|x - y|} \\
= \int d^3y \left( \frac{\nabla \cdot j(y,t - \frac{r}{C})}{|x - y|} - \frac{1}{|x - y|} \frac{\partial}{\partial s} j(y,s) \right)_{s = t - \frac{r}{C}} \cdot \nabla \frac{r}{C} + \frac{\partial}{\partial t} \frac{\rho(y,t - \frac{r}{C})}{|x - y|} \\
= \int d^3y \left( \frac{\nabla \cdot j(y,s)}{|x - y|} \right)_{s = t - \frac{r}{C}} - \frac{1}{|x - y|} \frac{\partial}{\partial s} j(y,s) \left( \nabla + \frac{\nabla}{|x - y|} \cdot \nabla \frac{r}{C} + \frac{\partial}{\partial t} \frac{\rho(y,t - \frac{r}{C})}{|x - y|} \right) \\
= \int d^3y \left( \frac{\partial}{\partial s} \rho(y,s) + \nabla \cdot j(y,s) \right)_{s = t - \frac{r}{C}}
\]

since \((\nabla + \nabla_x) |x - y| = 0\).

\[ \square \]

**Lemma 7.** Lemma 6, together with eqs. 1, 2 and 6 imply eq. 4.

Proof. The proof requires standard applications of vector calculus. From eq. 6 we derive

\[
\mu_0 j = - \left( \nabla (\nabla \cdot A) - \nabla \times (\nabla \times A) - \frac{1}{C^2} \frac{\partial^2 A}{\partial t^2} \right) \\
= \frac{1}{C^2} \left( \nabla \frac{\partial V}{\partial t} + \frac{\partial^2 A}{\partial t^2} \right) + \nabla \times B \\
= - \frac{1}{C^2} \frac{\partial E}{\partial t} + \nabla \times B
\]

\[ \square \]

**Lemma 8.** Eq. 9 is equivalent to eq. 10 (the total electromagnetic energy), up to the volume integral of the gradient of a function that vanishes at infinity.
Proof. Under the general assumption that \( \int \nabla \cdot F(x,t) \, d^3x \) vanishes at infinity, being \( F \) a vector function that decays sufficiently fast (i.e., faster than \( r^{-2} \)), we obtain

\[
\int_0^t dt \int d^3x \left( \left( \nabla V + \frac{\partial A}{\partial t} \right) \cdot j \right) = \\
\int_0^t dt \int d^3x \left( \nabla \cdot (Vj) - V \nabla \cdot j + \frac{\partial A}{\partial t} \cdot \left( -\epsilon_0 \frac{\partial E}{\partial t} + \frac{1}{\mu_0} \nabla \times B \right) \right) \\
\int_0^t dt \int d^3x \left( V \frac{\partial \rho}{\partial t} + \frac{\partial A}{\partial t} \cdot \left( -\epsilon_0 \frac{\partial E}{\partial t} + \frac{1}{\mu_0} \nabla \times B \right) \right) \\
\int_0^t dt \int d^3x \left( \epsilon_0 \nabla \cdot \left( \frac{\partial E}{\partial t} \right) - \epsilon_0 \nabla \cdot \frac{\partial E}{\partial t} - \epsilon_0 \frac{\partial A}{\partial t} \cdot \frac{\partial E}{\partial t} + \frac{1}{\mu_0} \frac{\partial B}{\partial t} \cdot B \right) \\
\int_0^t dt \int d^3x \left( \epsilon_0 E \frac{\partial E}{\partial t} + \frac{1}{\mu_0} \frac{\partial B}{\partial t} \cdot B \right) \\
\frac{1}{2} \int_0^t dt \frac{\partial}{\partial t} \int d^3x \left( \epsilon_0 |E|^2 + \frac{1}{\mu_0} |B|^2 \right) \\
\frac{1}{2} \int d^3x \left( \epsilon_0 |E|^2 + \frac{1}{\mu_0} |B|^2 \right)
\]

where all integrals involving total divergences have been set to zero by Gauss’ theorem. We have also used the continuity equation.

**Lemma 9.** Up to an overall function of time and the divergence of a function vanishing sufficiently fast at infinity, the electromagnetic action satisfies

\[
\mathcal{A} = \frac{1}{2} \int dt \int \left( \frac{1}{\mu_0} |B|^2 - \epsilon_0 |E|^2 \right) \, d^3x \\
= \frac{1}{2} \int dt \int (A \cdot j - \rho V) \, d^3x
\]

Proof. The proof requires standard vector calculus operations on the integrand, namely

\[
A \cdot j - \rho V = A \cdot \left( \frac{1}{\mu_0} \nabla \times B - \epsilon_0 \frac{\partial E}{\partial t} \right) - \epsilon_0 V \nabla \cdot E \\
= \frac{1}{\mu_0} \left( |B|^2 - \nabla \cdot (A \times B) \right) - \epsilon_0 A \cdot \frac{\partial E}{\partial t} - \epsilon_0 \nabla \cdot (VE) + \epsilon_0 E \cdot \nabla V \\
= \frac{1}{\mu_0} \left( |B|^2 - \nabla \cdot (A \times B) \right) - \epsilon_0 A \cdot \frac{\partial E}{\partial t} - \epsilon_0 V \nabla \cdot (VE) + \epsilon_0 E \cdot \left( -E - \frac{\partial A}{\partial t} \right) \\
= \frac{1}{\mu_0} |B|^2 - \epsilon_0 |E|^2 - \nabla \cdot \left( \frac{1}{\mu_0} A \times B + \epsilon_0 VE \right) - \epsilon_0 \frac{\partial}{\partial t} (A \cdot E)
\]
Lemma 10. The result of Lemma 9 is independent of the choice of gauge.

Proof. Modifying the potentials with a sufficiently smooth function \( \Lambda(x,t) \) also vanishing adequately at infinity and such that \( A' = A + \nabla \Lambda \) and \( V' = V - \frac{\partial \Lambda}{\partial t} \), we obtain

\[
A' \cdot j - \rho V' = A \cdot j + (\nabla \Lambda \cdot j) - \rho \left( V - \frac{\partial \Lambda}{\partial t} \right)
\]

\[
= A \cdot j - \rho V + \left( \nabla \Lambda \cdot j + \rho \frac{\partial \Lambda}{\partial t} \right)
\]

\[
= A \cdot j - \rho V + \nabla \cdot (\Lambda j) - \rho \frac{\partial \Lambda}{\partial t} \nabla \cdot (\rho \Lambda)
\]

A.0.1 Proof of Theorem 2

Proof. For the sake of convenience, we may express the action using Lemma 9 before proceeding:

\[
\delta \bar{x}(t) A = \delta \bar{x}(t) \int dt \int d^3 x \left[ A \cdot j - V \rho \right] = \int dt \int d^3 x \left[ \delta A \cdot j + A \delta j - \delta V \rho - V \delta \rho \right] = 0
\]

(13)

Potentials charge and current can be split in primary and secondary circuits, namely \( A = A_1 + A_2 \) (and similarly for \( V, j, \rho \)) and the variation can be formulated as moving the secondary circuit with respect to a fixed primary circuit. Only the interacting terms actually vary, so the integrand reads \( \left( \delta A_2 \cdot j_1 - \delta V_2 \rho_1 \right) + \left( A_1 \cdot \delta j_2 - V_1 \delta \rho_2 \right) \). We transform the first parenthesis using Maxwell equations, Lemma 5:

\[
\int dt \int d^3 x \left[ \delta A_2 \cdot j_1 - \delta V_2 \rho_1 \right] = \int dt \int d^3 x \left[ -\mu_0 \delta A_2 \cdot \square A_1 + \epsilon_0 \delta V_2 \square V_1 \right]
\]

The particular variation reflecting the problem (moving subsystem 2 with respect to a fixed subsystem 1) and partial integrations in space and time allow to transform

\[
\int dt \int d^3 x \left[ -\mu_0 \delta A_2 \cdot \square A_1 + \epsilon_0 \delta V_2 \square V_1 \right] = \int dt \int d^3 x \left[ -\mu_0 \delta A_1 \cdot \square A_2 + \epsilon_0 \delta V_1 \square V_2 \right]
\]

up to an overall divergence whose contribution vanishes by Gauss theorem and an overall function of time that do not contribute to the variation. Hence, the required variation reads (dropping the index \( \bar{x}(t) \)),

\[
\delta A = \int dt \int d^3 x \left[ A_1 \cdot \delta j_2 - V_1 \delta \rho_2 \right]
\]
The variation of the current and of the charge distribution due to the motion of the probe relative to the primary circuit are, after eq. 

\[
\delta \tilde{j}_2 = -(\delta \tilde{x} \cdot \nabla) \tilde{j}_2 + \tilde{\rho}_2 \delta \tilde{x} \\
\delta \tilde{\rho}_2 = -(\delta \tilde{x} \cdot \nabla) \tilde{\rho}_2
\]

We have then

\[
\delta A = \delta \int dt \mathcal{L} = \int dt \int d^3x \left[ A_1 \cdot \delta \tilde{j}_2 - V_1 \delta \tilde{\rho}_2 \right]
\]

\[
= \int dt \int d^3x \left[ A_1 \cdot (-(\delta \tilde{x} \cdot \nabla) \tilde{j}_2 + \tilde{\rho}_2 \delta \tilde{x}) - V_1 \left( -(\delta \tilde{x} \cdot \nabla) \tilde{\rho}_2 \right) \right]
\]

\[
= \int dt \int d^3x \left[ \tilde{j}_2 \cdot (\delta \tilde{x} \cdot \nabla) A_1 - \tilde{\rho}_2 (\delta \tilde{x} \cdot \nabla) V_1 - \delta \tilde{x} \cdot \frac{\partial}{\partial t} (A_1 \tilde{\rho}_2) \right]
\]

(14)

The last line is obtained after some integrations by parts and following the cancellation of a whole divergence via Gauss theorem and of an overall function of time via the variational constraints. In particular:

\[
((\delta \tilde{x} \cdot \nabla) \tilde{j}_2) \cdot A_1 = (\delta \tilde{x} \cdot \nabla) (\tilde{j}_2 \cdot A_1) - \tilde{j}_2 \cdot (\delta \tilde{x} \cdot \nabla) A_1
\]

\[
V_1 (\delta \tilde{x} \cdot \nabla) \tilde{\rho}_2 = \delta \tilde{x} \cdot \nabla (V_1 \tilde{\rho}_2) - \tilde{\rho}_2 (\delta \tilde{x} \cdot \nabla) V_1
\]

\[
A_1 \cdot (\tilde{\rho}_2 \delta \tilde{x}) = \frac{\partial}{\partial t} (\rho_2 A_1 \cdot \delta \tilde{x}) - \delta \tilde{x} \cdot \frac{\partial}{\partial t} (A_1 \tilde{\rho}_2)
\]

Further transformation with mathematical identities allows us to write

\[
\tilde{j}_2 \cdot (\delta \tilde{x} \cdot \nabla) A_1 - \delta \tilde{x} \cdot A_1 \frac{\partial}{\partial t} \tilde{\rho}_2 = \tilde{j}_2 \cdot (\delta \tilde{x} \cdot \nabla) A_1 + (\delta \tilde{x} \cdot A_1) (\nabla \cdot \tilde{j}_2)
\]

\[
= \tilde{j}_2 \cdot (\delta \tilde{x} \cdot \nabla) A_1 + \nabla \cdot (\tilde{j}_2 (\delta \tilde{x} \cdot A_1)) - \tilde{j}_2 \cdot \nabla \delta \tilde{x} \cdot A_1
\]

\[
= \nabla \cdot (\tilde{j}_2 (\delta \tilde{x} \cdot A_1)) - \tilde{j}_2 \cdot \delta \tilde{x} \times (\nabla \times A_1)
\]

\[
= \nabla \cdot (\tilde{j}_2 (\delta \tilde{x} \cdot A_1)) + \delta \tilde{x} \cdot \tilde{j}_2 \times (\nabla \times A_1)
\]

and finally, after applying Gauss theorem again,

\[
\int dt \int d^3x \delta \tilde{x} \cdot [\tilde{j}_2 \times B_1 + \tilde{\rho}_2 E_1]
\]

This is, following the standard use of Hamilton’s principle in mechanics we arrive to an electromagnetic contribution to the force on the probe

\[
F_{em} = \tilde{j}_2 \times B_1 + \tilde{\rho}_2 E_1
\]