One-way Resynchronizability of Word Transducers

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Abstract. The origin semantics for transducers was proposed in 2014, and it led to various characterizations and decidability results that are in contrast with the classical semantics. In this paper we add a further decidability result for characterizing transducers that are close to one-way transducers in the origin semantics. We show that it is decidable whether a non-deterministic two-way word transducer can be resynchronized by a bounded, regular resynchronizer into an origin-equivalent one-way transducer. The result is in contrast with the usual semantics, where it is undecidable to know if a non-deterministic two-way transducer is equivalent to some one-way transducer.

Keywords: String transducers · Resynchronizers · One-way transducers

1 Introduction

Regular word-to-word functions form a robust and expressive class of transformations, as they correspond to deterministic two-way transducers, to deterministic streaming string transducers [1], and to monadic second-order logical transductions [11]. However, the transition from word languages to functions over words is often quite tricky. One of the challenges is to come up with effective characterizations of restricted transformations. A first example is the characterization of functions computed by one-way transducers (known as rational functions). It turns out that it is decidable whether a regular function is rational [14], but the algorithm is quite involved [3]. In addition, non-determinism makes the problem intractable: it is undecidable whether the relation computed by a non-deterministic two-way transducer can be also computed by a one-way transducer, [2]. A second example is the problem of knowing whether a regular word function can be described by a first-order logical transduction. This question is still open in general [16], and it is only known how to decide if a rational function is definable in first-order logic [13].

Word transducers with origin semantics were introduced by Bojańczyk [4] and shown to provide a machine-independent characterization of regular word-to-word functions. The origin semantics, as the name suggests, means tagging the output by the positions of the input that generated that output.
A nice phenomenon is that origins can restore decidability for some interesting problems. For example, the equivalence of word relations computed by one-way transducers, which is undecidable in the classical semantics [18,19], is PSPACE-complete for two-way non-deterministic transducers in the origin semantics [7]. Another, deeper, observation is that the origin semantics provides an algebraic approach that can be used to decide fragments. For example, [4] provides an effective characterization of first-order definable word functions under the origin semantics. As for the problem of knowing whether a regular word function is rational, it becomes almost trivial in the origin semantics.

A possible objection against the origin semantics is that the comparison of two transducers in the origin semantics is too strict. Resynchronizations were proposed in order to overcome this issue. A resynchronization is a binary relation between input-output pairs with origins, that preserves the input and the output, changing only the origins. Resynchronizations were introduced for one-way transducers [15], and later for two-way transducers [7]. For one-way transducers rational resynchronizations are transducers acting on the synchronization languages, whereas for two-way transducers, regular resynchronizations are described by regular properties over the input that restrict the change of origins. The class of bounded\(^4\) regular resynchronizations was shown to behave very nicely, preserving the class of transductions defined by non-deterministic, two-way transducers: for any bounded regular resynchronization \(\mathcal{R}\) and any two-way transducer \(T\), the resynchronized relation \(\mathcal{R}(T)\) can be computed by another two-way transducer [7]. In particular, non-deterministic, two-way transducers can be effectively compared modulo bounded regular resynchronizations.

As mentioned above, it is easy to know if a two-way transducer is equivalent under the origin semantics to some one-way transducer [4], since this is equivalent to being order-preserving. But what happens if this is not the case? Still, the given transducer \(T\) can be “close” to some order-preserving transducer. What we mean here by “close” is that there exists some bounded regular resynchronizer \(\mathcal{R}\) such that \(\mathcal{R}(T)\) is order-preserving and all input-output pairs with

\(^4\) “Bounded” refers here to the number of source positions that are mapped to the same target position. It rules out resynchronizations such as the universal one.
origins produced by $T$ are in the domain of $\mathcal{R}$. We call such transducers one-way resynchronizable. Figure 1 gives an example.

In this paper we show that it is decidable if a two-way transducer is one-way resynchronizable. We first solve the problem for bounded-visit two-way transducers. A bounded-visit transducer is one for which there is a uniform bound for the number of visits of any input position. Then, we use the previous result to show that one-way resynchronizability is decidable for arbitrary two-way transducers, so without the bounded-visit restriction. This is done by constructing, if possible, a bounded, regular resynchronization from the given transducer to a bounded-visit transducer with regular language outputs. Finally, we show that bounded regular resynchronizations are closed under composition, and this allows to combine the previous construction with our decidability result for bounded-visit transducers.

**Related work and paper overview.** The synthesis problem for resynchronizers asks to compute a resynchronizer from one transducer to another one, when the two transducers are given as input. The problem was studied in [6] and shown to be decidable for unambiguous two-way transducers (it is open for unrestricted transducers). The paper [21] shows that the containment version of the above problem is undecidable for unrestricted one-way transducers.

The origin semantics for streaming string transducers (SST) [1] has been studied in [5], providing a machine-independent characterization of the sets of origin graphs generated by SSTs. An open problem here is to characterize origin graphs generated by aperiodic streaming string transducers [10,16]. Going beyond words, [17] investigates decision problems of tree transducers with origin, and regains the decidability of the equivalence problem for non-deterministic top-down and MSO transducers by considering the origin semantics. An open problem for tree transducers with origin is that of synthesizing resynchronizers as in the word case.

We will recall regular resynchronizations in Section 3. Section 4 provides the proof ingredients for the bounded-visit case, and the proof of decidability of one-way resynchronizability in the bounded-visit case can be found in Section 5. Finally, in Section 6 we sketch the proof in the general case. Missing proofs can be found in the appendix.

## 2 Preliminaries

Let $\Sigma$ be a finite input alphabet. Given a word $w \in \Sigma^*$ of length $|w| = n$, a position is an element of its domain $\text{dom}(w) = \{1, \ldots, n\}$. For every position $i$, $w(i)$ denotes the letter at that position. A cut of $w$ is any number from 1 to $|w| + 1$, so a cut identifies a position between two consecutive letters of the input. The cut $i = 1$ represents the position just before the first input letter, and $i = |w| + 1$ the position just after the last letter of $w$.

**Two-way transducers.** We use two-way transducers as defined in [3,6], with a slightly different presentation than in classical papers such as [22]. As usual for
two-way machines, for any input \( w \in \Sigma^* \), \( w(0) = \uparrow \) and \( w(|w| + 1) = \downarrow \), where \( \uparrow, \downarrow \notin \Sigma \) are special markers used as delimiters.

A two-way transducer (or just transducer from now on) is a tuple \( T = (Q, \Sigma, \Gamma, \Delta, I, F) \), where \( \Sigma, \Gamma \) are respectively the input and output alphabets, \( Q = Q_- \sqcup Q_+ \) is the set of states, partitioned into left-reading states from \( Q_- \) and right-reading states from \( Q_+ \), \( I \subseteq Q_- \) is the set of initial states, \( F \subseteq Q_+ \) is the set of final states, and \( \Delta \subseteq Q \times (\Sigma \sqcup \{ \uparrow, \downarrow \}) \times \Gamma^* \times Q \) is the finite transition relation. Left-reading states read the letter to the left, whereas right-reading states read the letter to the right. This partitioning will also determine the head movement during a transition, as explained below.

As usual, to define runs of transducers we first define configurations. Given a transducer \( T \) and a word \( w \in \Sigma^* \), a configuration \( \rho \) of \( T \) on \( w \) is a state-cut pair \((q, i)\), with \( q \in Q \) and \( 1 \leq i \leq |w| + 1 \). A configuration \((q, i)\), \( 1 \leq i \leq |w| + 1 \) means that the automaton is in state \( q \) and its head is between the \((i-1)\)-th and the \(i\)-th letter of \( w \). The transitions that depart from a configuration \((q, i)\) and read \( a \) are denoted \( (q, i) \xrightarrow{a} (q', i') \), and must satisfy one of the following:

1. \( q \in Q_- \), \( q' \in Q_+ \), \( a = w(i) \), \( (q, a, v, q') \in \Delta \), and \( i' = i + 1 \),
2. \( q \in Q_+ \), \( q' \in Q_- \), \( a = w(i) \), \( (q, a, v, q') \in \Delta \), and \( i' = i \),
3. \( q \in Q_- \), \( q' \in Q_- \), \( a = w(i-1) \), \( (q, a, v, q') \in \Delta \), and \( i' = i - 1 \).

In this case we call \( T \) a one-way transducer.

A run of \( T \) on \( w \) is a sequence \( \rho = (q_1, i_1) \xrightarrow{a_{j_1}v_{k_1}} (q_2, i_2) \xrightarrow{a_{j_2}v_{k_2}} \cdots \xrightarrow{a_{j_m}v_{k_m}} (q_{m+1}, i_{m+1}) \) of configurations connected by transitions. Note that the positions \( j_1, j_2, \ldots, j_m \) of letters do not need to be ordered from smaller to bigger, and can differ slightly (by \( +1 \) or \( -1 \)) from the cuts \( i_1, i_2, \ldots, i_{m+1} \), since cuts take values in between consecutive letters.

A configuration \((q, i)\) on \( w \) is initial (resp. final) if \( q \in I \) and \( i = 1 \) (resp. \( q \in F \) and \( i = |w| + 1 \)). A run is successful if it starts with an initial configuration and ends with a final configuration. The output associated with a successful run \( \rho \) as above is the word \( v_1 v_2 \cdots v_m \in \Gamma^* \). A transducer \( T \) defines a relation \([T] \subseteq \Sigma^* \times \Gamma^* \) consisting of all the pairs \((u, v)\) such that \( v \) is the output of some successful run \( \rho \) of \( T \) on \( u \).

Origin semantics. In the origin semantics for transducers [4] the output is tagged with information about the position of the input where it was produced. If reading the \( i \)-th letter of the input we output \( v \), then all letters of \( v \) are tagged with \( i \), and we say they have origin \( i \). We use the notation \((v; i)\) for \( v \in \Gamma^* \) to denote that all positions in the output word \( v \) have origin \( i \), and we view \((v, i)\) as word over the alphabet \( \Gamma \times \mathbb{N} \). The outputs associated with a successful run \( \rho = (q_1, i_1) \xrightarrow{b_{k_1}v_{j_1}} (q_2, i_2) \xrightarrow{b_{k_2}v_{j_2}} (q_3, i_3) \cdots \xrightarrow{b_{k_m}v_{j_m}} (q_{m+1}, i_{m+1}) \) in the origin semantics are the words of the form \( v = (v_1, j_1)(v_2, j_2) \cdots (v_m, j_m) \) over \( \Gamma \times \mathbb{N} \) where, for all \( 1 \leq k \leq m \), \( j_k = i_k \) if \( q_k \in Q_- \), and \( j_k = i_k - 1 \) if \( q_k \in Q_+ \). Under the origin semantics, the relation defined by \( T \), denoted \([T]_{\text{o}} \), is the set of pairs
σ = (u, ν) — called synchronized pairs — such that u ∈ Σ* and ν ∈ (Γ × ℕ)* is the output of some successful run on u.

Equivalently, a synchronized pair (u, ν) can be described as a triple (u, ν, orig), where ν is the projection of u on Γ, and orig : dom(ν) → dom(u) associates with each position of ν its origin in u. So for ν = (ν1, j1)(ν2, j2)⋯(vm, jm) as above, ν = v1⋯vm and, for all positions i s.t. |v1⋯vk−1| < i ≤ |v1⋯vk|, we have orig(i) = jk. Given two transducers T1, T2, we say they are origin-equivalent if [T1]o = [T2]o. Note that two transducers T1, T2 can be equivalent in the classical semantics, [T1] = [T2], while they can have different origin semantics, so [T1]o ≠ [T2]o.

Bounded-visit transducers. Let k > 0 be some integer, and ρ some run of a two-way transducer T. We say that ρ is k-visit if for every i ≥ 0, it has at most k occurrences of configurations from Q × {i}. We call a transducer T k-visit if for every σ ∈ [T]o there is some successful, k-visit run ρ of T with output σ (actually we should call the transducer k-visit in the origin semantics, but for simplicity we omit this). For example, the relation {{w, wσ} | w ∈ Σ*}, where w* denotes the reverse of w, can be computed by a 3-visit transducer. A transducer is called bounded-visit if it is k-visit for some k.

Common guess. It is often useful to work with a variant of two-way transducers that can guess beforehand some annotation on the input and inspect it consistently when visiting portions of the input multiple times. This feature is called common guess [5], and strictly increases the expressive power of two-way transducers, including bounded-visit ones.

3 One-way resynchronizability

3.1 Regular resynchronizers

Resynchronizations are used to compare transductions in the origin semantics. A resynchronization is a binary relation R ⊆ (Σ* × (Γ × ℕ)*)^2 over synchronized pairs such that (σ, σ′) ∈ R implies that σ = (u, ν, orig) and σ′ = (u, ν, orig′) for some origin mappings orig, orig′ : dom(ν) → dom(u). In other words, a resynchronization will only change the origin mapping, but neither the input, nor the output. Given a relation S ⊆ Σ* × (Γ × ℕ)* with origins, the resynchronized relation R(S) is defined as R(S) = {σ′ | (σ, σ′) ∈ R, σ ∈ S}. For a transducer T we abbreviate R([T]o) by R(T). The typical use of a resynchronization R is to ask, given two transducers T, T′, whether R(T) and T′ are origin-equivalent.

Regular resynchronizers (originally called MSO resynchronizers) were introduced in [7] as a resynchronization mechanism that preserves definability by two-way transducers. They were inspired by MSO (monadic second-order) transductions [9,12] and are formally defined as follows. A regular resynchronizer is a tuple R = (T, Γ, opar, ipar, move_r, next_r, τ, τ′) consisting of

- some monadic parameters (colors) T = (I1, ..., Im) and Γ = (O1, ..., On),
MSO sentences $\mathit{ipar}$, $\mathit{opar}$, defining languages over expanded input and output alphabets, i.e. over $\Sigma' = \Gamma \times 2^{\{1,\ldots,m\}}$ and $\Gamma' = \Gamma \times 2^{\{1,\ldots,n\}}$, respectively,

- MSO formulas $\mathit{move}_\tau(y, z)$, $\mathit{next}_{\tau,\tau'}(z, z')$ with two free first-order variables and parametrized by expanded output letters $\tau, \tau'$ (called types, see below).

To apply a regular resynchronizer as above, one first guesses the valuation of all the predicates $I_1, O_k$, and uses it to interpret the parameters $\bar{I}$ and $\bar{O}$. Based on the chosen valuation of the parameters $\bar{O}$, each position $x$ of the output $v$ gets an associated type $\tau_x = (v(x), b_1, \ldots, b_n) \in \Gamma \times \{0,1\}^n$, where $b_j$ is 1 or 0 depending on whether $x \in O_j$ or not. We refer to the output word together with the valuation of the output parameters as annotated output, so a word over $\Gamma \times \{0,1\}^n$. Similarly, the annotated input is a word over $\Sigma \times \{0,1\}^m$. The annotated input and output word must satisfy the formulas $\mathit{ipar}$ and $\mathit{opar}$, respectively.

The origins of output positions are constrained using the formulas $\mathit{move}_\tau$ and $\mathit{next}_{\tau,\tau'}$, which are parametrized by output types and evaluated over the annotated output. Intuitively, the formula $\mathit{move}_\tau(y, z)$ states how the origin of every output position of type $\tau$ changes from $y$ to $z$. We refer to $y$ and $z$ as source and target origin, respectively. The formula $\mathit{next}_{\tau,\tau'}(z, z')$ instead constrains the target origins $z, z'$ of any two consecutive output positions with types $\tau$ and $\tau'$, respectively.

Formally, $\mathcal{R} = (\bar{I}, \bar{O}, \mathit{ipar}, \mathit{opar}, (\mathit{move}_\tau), (\mathit{next}_{\tau,\tau'}))$ defines the resynchronization consisting of all pairs $(\sigma, \sigma')$, with $\sigma = (u, v, \mathit{orig})$, $\sigma' = (u, v, \mathit{orig}')$, $u \in \Sigma^*$, and $v \in \Gamma^*$, for which there exist $u' \in \Sigma'^*$ and $v' \in \Gamma'^*$ such that

- $\pi_\Sigma(u') = u$ and $\pi_\Gamma(v') = v$
- $u'$ satisfies $\mathit{ipar}$ and $v'$ satisfies $\mathit{opar}$,
- $(u', \mathit{orig}(x), \mathit{orig}'(x))$ satisfies $\mathit{move}_\tau$ for all $\tau$-labeled output positions $x \in \mathit{dom}(v')$, and
- $(u', \mathit{orig}'(x), \mathit{orig}'(x+1))$ satisfies $\mathit{next}_{\tau,\tau'}$ for all $x, x+1 \in \mathit{dom}(v')$ such that $x$ and $x+1$ have label $\tau$ and $\tau'$, respectively.

**Example 1.** Consider the following resynchronization $\mathcal{R}$. A pair $(\sigma, \sigma')$ belongs to $\mathcal{R}$ if $\sigma = (uv, uvv, \mathit{orig})$, $\sigma' = (uv, uvv, \mathit{orig}')$, with $u, v, w \in \Sigma^+$. The origins $\mathit{orig}$ and $\mathit{orig}'$ are both the identity over $u$ and $v$. The origin of every position of $w$ in $\sigma$ (hence a source origin) is either the first or the last position of $v$. The origin of every position of $w$ in $\sigma'$ (a target origin) is the first position of $v$.

This resynchronization is described by a regular resynchronizer that uses two input parameters $I_1$, $I_2$ to mark the last and the first positions of $v$ in the input, and one output parameter $O$ to mark the factor $w$ in the output. The formula $\mathit{move}_\tau(y, z)$ is either $(I_1(y) \lor I_2(y)) \land I_2(z)$ or $(y = z)$, depending on whether the type $\tau$ describes a position inside $w$ or a position outside $w$.

We now turn to describing some important restrictions on (regular) resynchronizers. Let $\mathcal{R} = (\bar{I}, \bar{O}, \mathit{ipar}, \mathit{opar}, (\mathit{move}_\tau), (\mathit{next}_{\tau,\tau'}))$ be a resynchronizer.

- $\mathcal{R}$ is $k$-bounded (or just bounded) if for every annotated input $u' \in \Sigma'^*$, every output type $\tau \in \Gamma'$, and every position $z$, there are at most $k$ positions $y$ such that $(u', y, z)$ satisfies $\mathit{move}_\tau$. Recall that $y, z$ are input positions.
– $\mathcal{R}$ is $T$-preserving for a given transducer $T$, if every $\sigma \in [[T]]_o$ belongs to the domain of $\mathcal{R}$.

– $\mathcal{R}$ is partially bijective if each move$_T$ formula defines a partial, bijective function from source origins to target origins. Observe that this property implies that $\mathcal{R}$ is 1-bounded.

The boundedness restriction rules out resynchronizations such as the universal one, that imposes no restriction on the change of origins. It is a decidable restriction [7], and it guarantees that definability by two-way transducers is effectively preserved under regular resynchronizations, modulo common guess. More precisely, Theorem 16 in [7] shows that, given a bounded regular resynchronizer $\mathcal{R}$ and a transducer $T$, one can construct a transducer $T'$ with common guess that is origin-equivalent to $\mathcal{R}(T)$.

Example 1 (continued). Consider again the regular resynchronizer $\mathcal{R}$ described in the previous example. Note that $\mathcal{R}$ is 2-bounded, since at most two source origins are redirected to the same target origin. If we used an additional output parameter to distinguish, among the positions of $w$, those that have source origin in the first position of $v$ and those that have source origin in the last position of $v$, we would get a 1-bounded, regular resynchronizer.

We state below two crucial properties of regular resynchronizers (the second lemma is reminiscent of Lemma 11 from [21], which proves closure of bounded resynchronizers with vacuous next$_{\tau,\tau'}$ relations).

**Lemma 1.** Every bounded, regular resynchronizer is effectively equivalent to some 1-bounded, regular resynchronizer.

**Lemma 2.** The class of bounded, regular resynchronizers is effectively closed under composition.

### 3.2 Main result

Given a two-way transducer $T$ one can ask if it is origin-equivalent to some one-way transducer. It was observed in [4] that this property holds if and only if all synchronized pairs defined by $T$ are order-preserving, namely, for all $\sigma = (u, v, \text{orig}) \in [[T]]_o$ and all $y, y' \in \text{dom}(v)$, with $y < y'$, we have orig$(y) \leq$ orig$(y')$.

The decidability of the above question should be contrasted to the analogous question in the classical semantics: “is a given two-way transducer classically equivalent to some one-way transducer?” The latter problem turns out to be decidable for functional transducers [14,3], but is undecidable for arbitrary two-way transducers [2].

Here we are interested in a different, more relaxed notion:

**Definition 1.** A transducer $T$ is called one-way resynchronizable if there exists a bounded, regular resynchronizer $\mathcal{R}$ that is $T$-preserving and such that $\mathcal{R}(T)$ is order-preserving.
Note that if $T'$ is an order-preserving transducer, then one can construct rather easily a one-way transducer $T''$ such that $T' = o T''$, by eliminating non-productive U-turns from accepting runs.

Moreover, note that without the condition of being $T$-preserving every transducer $T$ would be one-way resynchronizable, using the empty resynchronization.

**Example 2.** Consider the transducer $T_1$ that moves the last letter of the input $wa$ to the front by a first left-to-right pass that outputs the last letter $a$, followed by a right-to-left pass without output, and finally by a left-to-right pass that produces the remaining $w$. Let $R$ be the bounded regular resynchronizer that redirects the origin of the last $a$ to the first position. Assuming an output parameter $O$ with an interpretation constrained by $\text{opar}$ that marks the last position of the output, the formula $\text{move}_{(a,1)}(y,z)$ says that target origin $z$ (source origin $y$, resp.) of the last $a$ is the first (last, resp.) position of the input. It is easy to see that $R(T_1)$ is origin-equivalent to the one-way transducer that on input $wa$, guesses $a$ and outputs $aw$. Thus, $T_1$ is one-way resynchronizable. See also Figure 1.

**Example 3.** Consider the transducer $T_2$ that reads inputs of the form $u\#v$ and outputs $vu$ in the obvious way, by a first left-to-right pass that outputs $v$, followed by a right-to-left pass, and a finally a left-to-right pass that outputs $u$. Using the characterization with the notion of cross-width that we introduce below, it can be shown that $T_2$ is not one-way resynchronizable.

In order to give a flavor of our results, we anticipate here the two main theorems, before introducing the key technical concepts of cross-width and inversion (these will be defined further below).

**Theorem 1.** For every bounded-visit transducer $T$, the following are equivalent:

1. $T$ is one-way resynchronizable,
2. the cross-width of $T$ is finite,
3. no successful run of $T$ has inversions,
4. there is a partially bijective, regular resynchronizer $R$ that is $T$-preserving and such that $R(T)$ is order-preserving.

Moreover, condition (3) is decidable.

We will use Theorem 1 to show that one-way resynchronizability is decidable for arbitrary two-way transducers (not just bounded-visit ones).

**Theorem 2.** It is decidable whether a given two-way transducer $T$ is one-way resynchronizable.

Let us now introduce the first key concept, that of cross-width:

**Definition 2 (cross-width).** Let $\sigma = (u, v, \text{orig})$ be a synchronized pair and let $X_1, X_2 \subseteq \text{dom}(v)$ be sets of output positions such that, for all $x_1 \in X_1$ and $x_2 \in X_2$, $x_1 < x_2$ and $\text{orig}(x_1) > \text{orig}(x_2)$. We call such a pair $(X_1, X_2)$ a cross
and define its width as \( \min(|\text{orig}(X_1)|, |\text{orig}(X_2)|) \), where \( \text{orig}(X) = \{ \text{orig}(x) \mid x \in X \} \) is the set of origins corresponding to a set \( X \) of output positions.

The cross-width of a synchronized pair \( \sigma \) is the maximal width of the crosses in \( \sigma \). A transducer has bounded cross-width if for some integer \( k \), all synchronized pairs associated with successful runs of \( T \) have cross-width at most \( k \).

The other key notion of inversion will be introduced formally in the next section (page 12), as it requires a few technical definitions. The notion however is very similar in spirit to that of cross, with the difference that a single inversion is sufficient for witnessing a family of crosses with arbitrarily large cross-width.

4 Proof overview for Theorem 1

This section provides an overview of the proof of Theorem 1, and introduces the main ingredients.

We will use flows (a concept inspired from crossing sequences [22,3] and revised in Section 4.1) in order to derive the key notion of inversion. Roughly speaking, an inversion in a run involves two loops that produce outputs in an order that is reversed compared to the order on origins. Inversions were also used in the characterization of one-way definability of two-way transducers under the classical semantics [3]. There, they were used for deriving some combinatorial properties of outputs. Here we are only interested in detecting inversions, and this is a simple task.

Flows will also be used to associate factorization trees with runs (the existence of factorization trees of bounded height was established by the celebrated Simon’s factorization theorem [23]). We will use a structural induction on these factorization trees and the assumption that there is no inversion in every run to construct a regular resynchronization witnessing one-way resynchronizability of the transducer at hand.

Another important ingredient underlying the main characterization is given by the notion of dominant output interval (Section 4.2), which is used to formalize the invariant of our inductive construction.

4.1 Flows and inversions

Intervals. An interval of a word is a set of consecutive positions in it. An interval is often denoted by \( I = [i, i'] \), with \( i = \min(I) \) and \( i' = \max(I) + 1 \). Given two
intervals $I = [i, i')$ and $J = [j, j')$, we write $I < J$ if $i' \leq j$, and we say that $I, J$ are adjacent if $i' = j$. The union of two adjacent intervals $I = [i, i')$, $J = [j, j')$, denoted $I \cdot J$, is the interval $[i, j')$ (if $I, J$ are not adjacent, then $I \cdot J$ is undefined).

Subruns. Given a run $\rho$ of a transducer, a subrun is a factor of $\rho$. Note that a subrun of a two-way transducer may visit a position of the input several times. For an input interval $I = [i, j)$ and a run $\rho$, we say that a subrun $\rho'$ of $\rho$ spans over $I$ if $i$ (resp. $j$) is the smallest (resp. greatest) input position labeling some transition of $\rho'$. The left-hand-side of the figure at page 11 gives an example of an interval $I$ of an input word together with the subruns $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1$ that span over it. Subruns spanning over an interval can be left-to-right, left-to-left, right-to-left, or right-to-right depending on where the starting and ending positions are w.r.t. the endpoints of the interval.

Flows. Flows are used to summarize subruns of a two-way transducer that span over a given interval. The definition below is essentially taken from [3], except for replacing “functional” by “K-visit”. Formally, a flow of a transducer $T$ is a graph with vertices divided into two groups, L-vertices and R-vertices, labeled by states of $T$, and with directed edges also divided into two groups, productive and non-productive edges. The graph satisfies the following requirements. Edge sources are either an L-vertex labeled by a right-reading state, or an R-vertex labeled by a left-reading state, and symmetrically for edge destinations; moreover, edges are of one of the following types: LL, LR, RL, RR. Second, each node is the endpoint of exactly one edge. Finally, L (R, resp.) vertices are totally ordered, in such a way that for every LL (RR, resp.) edge $(v, v')$, we have $v < v'$. We will only consider flows of $K$-visiting transducers, so flows with at most $2K$ vertices. For example, the flow in the left-hand-side of the figure at page 11 has six L-vertices on the left, and six R-vertices on the right. The edges $\alpha_1, \alpha_2, \alpha_3$ are LL, LR, and RR, respectively.

Given a run $\rho$ of $T$ and an interval $I = [i, i')$ on the input, the flow of $\rho$ on $I$, denoted $flow_{\rho}(I)$, is obtained by identifying every configuration at position $i$ (resp. $i'$) with an L (resp. R) vertex, labeled by the state of the configuration, and every subrun spanning over $I$ with an edge connecting the appropriate vertices (this subrun is called the witnessing subrun of the edge of the flow). An edge is said to be productive if its witnessing subrun produces non-empty output.

Flow monoid. The composition of two flows $F$ and $G$ is defined when the R-vertices of $F$ induce the same sequence of labels as the L-vertices of $G$. In this case, the composition results in the flow $F \cdot G$ that has as vertices the L-vertices of $F$ and the R-vertices of $G$, and for edges the directed paths in the graph obtained by glueing the R-vertices of $F$ with the L-vertices of $G$ so that states are matched. Productiveness of edges is inherited by paths, implying that an edge of $F \cdot G$ is productive if and only if the corresponding path contains at least one edge (from $F$ or $G$) that is productive. When the composition is undefined, we simply write $F \cdot G = \bot$. The above definitions naturally give rise to a flow monoid associated with the transducer $T$, where elements are the flows of $T$, extended
with a dummy element \( \perp \), and the product operation is given by the composition of flows, with the convention that \( \perp \) is absorbing. It is easy to verify that for any two adjacent intervals \( I < J \) of a run \( \rho \), \( \text{flow}_\rho(I) \cdot \text{flow}_\rho(J) = \text{flow}_\rho(I \cdot J) \). We denote by \( M_T \) the flow monoid of a \( K \)-visiting transducer \( T \).

Let us estimate the size of \( M_T \). If \( Q \) is the set of states of \( T \), there are at most \( |Q|^{2K} \) possible sequences of \( L \) and \( R \)-vertices; and the number of edges (marked as productive or not) is bounded by \( \binom{2K}{K} \cdot (2K)^K \cdot 2^K \leq (2K+1)^{2K} \). Including the dummy element \( \perp \) in the flow monoid, we get \( |M_T| \leq (|Q| \cdot (2K+1))^{2K} + 1 =: \mathbf{M} \).

**Loops.** A loop of a run \( \rho \) over input \( w \) is an interval \( I = [i, j] \) with a flow \( F = \text{flow}_\rho(I) \) such that \( F \cdot F = F \) (call \( F \) idempotent). The run \( \rho \) can be pumped on a loop \( I = [i, j] \) as expected: given \( n > 0 \), we let \( \text{pump}_n^\rho(\rho) \) be the run obtained from \( \rho \) by glueing the subruns that span over the intervals \([1, i]\) and \([j, |w| + 1]\) with \( n \) copies of the subruns spanning over \( I \) (see figure to the right).

The lemma below shows that the occurrence order relative to subruns witnessing LR or RL edges of a loop (called straight edges, for short) is preserved when pumping the loop. This seemingly straightforward lemma is needed for detecting inversions and its proof is surprisingly non-trivial. For example, the external edge connecting the two \( L \)-vertices 1, 2 in the figure above appears before edge \( \alpha_2 \), and also before every copy of \( \alpha_2 \) in the run where loop \( I \) is pumped.

**Lemma 3.** Let \( \rho \) be a run of \( T \) on \( u \), let \( J < I < K \) be a partition of the domain of \( u \) into intervals, with \( I \) loop of \( \rho \), and let \( F = \text{flow}_\rho(J) \), \( E = \text{flow}_\rho(I) \), and \( G = \text{flow}_\rho(K) \) be the corresponding flows. Consider an arbitrary edge \( f \) of either \( F \) or \( G \), and a straight edge \( e \) of the idempotent flow \( E \). Let \( \rho_f \) and \( \rho_e \) be the witnessing subruns of \( f \) and \( e \), respectively. Then the occurrence order of \( \rho_f \) and \( \rho_e \) in \( \rho \) is the same as the occurrence order of \( \rho_f \) and any copy of \( \rho_e \) in \( \text{pump}^n_\rho(\rho) \).

We can now formalize the key notion of inversion:
Definition 3 (inversion). An inversion of $\rho$ is a tuple $(I, e, I', e')$ such that
- $I, I'$ are loops of $\rho$ and $I < I'$,
- $e, e'$ are productive straight edges in $\text{flow}_\rho(I)$ and $\text{flow}_\rho(I')$ respectively,
- the subrun witnessing $e'$ precedes the subrun witnessing $e$ in the run order.

(see the figure to the right).

4.2 Dominant output intervals

In this section we identify some particular intervals of the output that play an important role in the inductive construction of the resynchronizer for a one-way resynchronizable transducer.

Given $n \in \mathbb{N}$, we say that a set $B$ of output positions is $n$-large if $|\text{orig}(B)| > n$; otherwise, we say that $B$ is $n$-small. Recall that here we work with a $K$-visiting transducer $T$, for some constant $K$, and that $M = (|Q| \cdot (2K + 1))^{2K} + 1$ is an upper bound to the size of the flow monoid $M_T$. We will extensively use the derived constant $C = M^{2K}$ to distinguish between large and small sets of output positions. The intuition behind this constant is that any set of output positions that is $C$-large must traverse a loop of $\rho$. This is formalized in the lemma below. The proof uses algebraic properties of the flow monoid $M_T$ [20] (see also Theorem 7.2 in [3], which proves a similar result, but with a larger constant derived from Simon’s factorization theorem):

Lemma 4. Let $I$ be an input interval and $B$ a set of output positions with origins inside $I$. If $B$ is $C$-large, then there is a loop $J \subseteq I$ of $\rho$ such that $\text{flow}_\rho(J)$ contains a productive straight edge witnessed by a subrun that intersects $B$ (in particular, $\text{out}(J) \cap B \neq \emptyset$).

We need some more notations for outputs. Given an input interval $I$ we denote by $\text{out}_\rho(I)$ the set of output positions whose origins belong to $I$ (note that this might not be an output interval). An output block of $I$ is a maximal interval contained in $\text{out}_\rho(I)$.

The dominant output interval of $I$, denoted $\text{bigout}_\rho(I)$, is the smallest output interval that contains all $C$-large output blocks of $I$. In particular, $\text{bigout}_\rho(I)$ either is empty or begins with the first $C$-large output block of $I$ and ends with the last $C$-large output block of $I$. We will often omit the subscript $\rho$ from the notations $\text{flow}_\rho(I)$, $\text{out}_\rho(I)$, $\text{bigout}_\rho(I)$, etc., when no confusion arises.

We now fix a successful run $\rho$ of the $K$-visiting transducer $T$. The rest of the section presents some technical lemmas that will be used in the inductive constructions for the proof of the main theorem. In the lemmas below, we assume that all successful runs of $T$ (in particular, $\rho$) avoid inversions.
Lemma 5. Let $I_1 < I_2$ be two input intervals and $B_1, B_2$ output blocks of $I_1, I_2$, respectively. If both $B_1, B_2$ are C-large, then $B_1 < B_2$.

Proof (sketch). If the claim would not hold, then Lemma 4 would provide some loops $J_1 \subseteq I_1$ and $J_2 \subseteq I_2$, together with some productive edges in them, witnessing an inversion.

Lemma 6. Let $I = I_1 \cdot I_2$, $B = \text{bigout}(I)$, and $B_i = \text{bigout}(I_i)$ for $i = 1, 2$. Then $B \setminus (B_1 \cup B_2)$ is $4K\text{C}$-small.

Proof (sketch). By Lemma 5, $B_1 < B_2$. Moreover, all C-large output blocks of $I_1$ or $I_2$ are also C-large output blocks of $I$, so $B$ contains both $B_1$ and $B_2$. Suppose, by way of contradiction, that $B \setminus (B_1 \cup B_2)$ is $4K\text{C}$-large. This means that there is a $2K\text{C}$-large set $S \subseteq B \setminus (B_1 \cup B_2)$ with origins entirely to the left of $I_2$, or entirely to the right of $I_1$. Suppose, w.l.o.g., that the former case holds, and decompose $S$ as a union of maximal output blocks $B'_1, B'_2, \ldots, B'_n$ with origins either entirely inside $I_1$, or entirely outside. Since $S \cap B_1 = \emptyset$, every block $B'_i$ with origins inside $I_1$ is C-small. Similarly, by Lemma C.1 in Appendix C, every block $B'_i$ with origins outside $I_1$ is C-small too. Moreover, since $\rho$ is K-visiting, we get $n \leq 2K$. Altogether, this contradicts the assumption that $S$ is $2K\text{C}$-large.

Lemma 7. Let $I = I_1 \cdot I_2 \cdots I_n$, such that $I$ is a loop and $\text{flow}(I) = \text{flow}(I_k)$ for all $k$. Then $\text{bigout}(I)$ can be decomposed as $B_1 \cdot J_1 \cdot B_2 \cdot J_2 \cdots J_{n-1} \cdot B_n$, where

1. for all $1 \leq k \leq n$, $B_k = \text{bigout}(I_k)$ (with $B_k$ possibly empty);
2. for all $1 \leq k < n$, the positions in $J_k$ have origins inside $I_k \cup I_{k+1}$ and $J_k$ is $2K\text{C}$-small.

Proof (sketch). The proof idea is similar to the previous lemma. First, using properties of idempotent flows, one shows that all output positions strictly between $B_k$ and $B_{k+1}$, for any $k = 1, \ldots, n - 1$, have origin in $I_k \cup I_{k+1}$. Then, one observes that every output block of $I_k$ disjoint from $B_k$ is C-small, and since $T$ is K-visiting there are at most $K$ such blocks. This shows that every output interval $J_k$ between $B_k$ and $B_{k+1}$ is $2K\text{C}$-small. For an illustration see the figure to the right. The C-large blocks in $I_1$ are shown in red; in blue those for $I_2$, in purple those for $I_3$. So $\text{bigout}(I_1)$ is the entire output between the two red dots, $\text{bigout}(I_2)$ between the two blue dots, and $\text{bigout}(I_3)$ between the purple dots. All three blocks are non-empty, and $\text{bigout}(I_1 \cdot I_2 \cdot I_3)$ goes from the first red to the second purple dot. Black non-dashed arrows stand for C-small blocks. □
5 Proof of Theorem 1

This section is devoted to proving the characterization of one-way resynchronization in the bounded-visit case. We will use the notion of bounded-traversal from [21], that was shown to characterize the class of bounded regular resynchronizers, in as much as bounded-delay characterizes rational resynchronizers [15].

Definition 4 (traversal [21]). Let \((u,v,\text{orig})\) and \((u,v,\text{orig}')\) be two synchronized pairs with the same input and output words.

Given two input positions \(y, y' \in \text{dom}(u)\), we say that \(y\) traverses \(y'\) if there is a pair \((y,z)\) of source and target origins associated with the same output position such that \(y'\) is between \(y\) and \(z\), with \(y' \neq z\) and possibly \(y' = y\). More precisely:

- \((y,y')\) is a left-to-right traversal if \(y \leq y'\) and for some output position \(x\), \(\text{orig}(x) = y\) and \(z = \text{orig}'(x) > y'\);
- \((y,y')\) is a right-to-left traversal if \(y \geq y'\) and for some output position \(x\), \(\text{orig}(x) = y\) and \(z = \text{orig}'(x) < y'\).

A pair \((\sigma,\sigma')\) of synchronized pairs with input \(u\) and output \(v\) is said to have \(k\)-bounded traversal, with \(k \in \mathbb{N}\), if every \(y' \in \text{dom}(u)\) is traversed by at most \(k\) distinct positions of \(\text{dom}(u)\).

A resynchronizer \(R\) has bounded traversal if there is some \(k \in \mathbb{N}\) such that every \((\sigma,\sigma') \in R\) has \(k\)-bounded traversal.

Lemma 8 ([21]). A regular resynchronizer is bounded if and only if it has bounded traversal.

Proof (of Theorem 1). First of all, observe that the implication \(4 \rightarrow 1\) is straightforward. To prove the implication \(1 \rightarrow 2\), assume that there is a \(k\)-bounded, regular resynchronizer \(R\) that is \(T\)-preserving and such that \(R(T)\) is order-preserving. Lemma 8 implies that \(R\) has \(t\)-bounded traversal, for some constant \(t\). We head towards proving that \(T\) has cross-width bounded by \(t+k\). Consider two synchronized pairs \(\sigma = (u,v,\text{orig})\) and \(\sigma' = (u,v,\text{orig}')\) such that \(\sigma \in [T]_o\) and \((\sigma,\sigma') \in R\), and consider a cross \((X_1,X_2)\) of \(\sigma\). We claim that \(\text{orig}(X_1)\) or \(\text{orig}(X_2)\) is at most \(t+k\). Let \(x_1 = \min(\text{orig}(X_1))\), \(x'_1 = \max(\text{orig}'(X_1))\), \(x_2 = \max(\text{orig}(X_1))\), and \(x'_2 = \min(\text{orig}'(X_2))\). Since \((X_1,X_2)\) is a cross, we have \(x_1 > x_2\), and since \(\sigma'\) is order-preserving, we have \(x'_1 \leq x'_2\). Now, if \(x'_1 > x_2\), then at least \(|\text{orig}(X_2)| - k\) input positions from \(X_2\) traverse \(x'_1\) to the right (the \(-k\) term is due to the fact that at most \(k\) input positions can be resynchronized to \(x'_1\)). Symmetrically, if \(x'_1 \leq x_2\), then at least \(|\text{orig}(X_1)| - k\) input positions from \(X_1\) traverse \(x_2\) to the left (the \(-k\) term accounts for the case where some positions are resynchronized to \(x'_1\) and \(x'_1 = x_2\)). This implies \(\min(|\text{orig}(X_1)|,|\text{orig}(X_2)|) \leq t + k\), as claimed.

The remaining implications rely on the assumption that \(T\) is bounded-visit.

The implication \(2 \rightarrow 3\) is shown by contraposition: one considers a successful run \(\rho\) with an inversion, and shows that crosses of arbitrary width emerge after pumping the loops of the inversion (here Lemma 3 is crucial).
The proof of $3 \rightarrow 4$ is more involved, we only sketch it here. Assuming that no successful run of $T$ has inversions we build a partially bijective, regular resynchronizer $R$ that is $T$-preserving and $R(T)$ is order-preserving. The resynchronizer $R$ uses some parameters to guess a successful run $\rho$ of $T$ on $u$ and a factorization tree of bounded height for $\rho$. Formally, a factorization tree for a sequence $\alpha$ of monoid elements (e.g. the flows $\text{flow}_{\rho}([y,y])$ for all input positions $y$) is an ordered, unranked tree whose yield is the sequence $\alpha$. The leaves of the factorization tree are labeled with the elements of $\alpha$. All other nodes have at least two children and are labeled by the monoid product of the child labels (in our case by the flows of $\rho$ induced by the covered factors in the input). In addition, if a node has more than two children, then all its children must have the same label, representing an idempotent element of the monoid. By Simon’s factorization theorem [23], every sequence of monoid elements has some factorization tree of height at most linear in the size of the monoid (in our case, at most $3|M_T|$, see e.g. [8]).

**Parameters.** We use input parameters to encode the successful run $\rho$ and a factorization tree for $\rho$ of height at most $H = 3|M_T|$. These parameters specify, for each input interval corresponding to a subtree, the start and end positions of the interval and the label of the root of the subtree. Correctness of these annotations can be enforced by an MSO sentence $ipar$. The run and the factorization tree also need to be encoded over the output, using output parameters. More precisely, given a level in the tree and an output position, we need to be able to determine the flow and the productive edge that generated that position. The technical details for checking correctness of the output annotation using the formulas $opar$, $move_\tau$ and $next_{\tau,\tau'}$ can be found in Appendix D.

**Moving origins.** For each level $\ell$ of the factorization tree, a partial resynchronization relation $R_\ell$ is defined. The relation is partial in the sense that some output positions may not have a source-target origin pair defined at a given level. But once a source-target pair is defined for some output position at a given level, it remains defined for all higher levels.

In the following we write $\text{bigout}(p)$ for the dominant output interval associated with the input interval $I(p)$ corresponding to a node $p$ in the tree. For every level $\ell$ of the factorization tree, the resynchronizer $R_\ell$ will be a partial function from source origins to target origins, and will satisfy the following:

- the set of output positions for which $R_\ell$ defines target origins is the union of the intervals $\text{bigout}(p)$ for all nodes $p$ at level $\ell$;
- $R_\ell$ only moves origins within the same interval at level $\ell$, that is, $R_\ell$ defines only pairs $(y,z)$ of source-target origins such that $y,z \in I(p)$ for some node $p$ at level $\ell$;
- the target origins defined by $R_\ell$ are order-preserving within every interval at level $\ell$, that is, for all output positions $x < x'$, if $R_\ell$ defines the target origins of $x,x'$ to be $z,z'$, respectively, and if $z,z' \in I(p)$ for some node $p$ at level $\ell$, then $z \leq z'$;
- $R_\ell$ is $\ell \cdot 4KC$-bounded, namely, there are at most $\ell \cdot 4KC$ distinct source origins that are moved by $R_\ell$ to the same target origin.
The construction of $\mathcal{R}_\ell$ is by induction on $\ell$. For a binary node $p$ at level $\ell$ with children $p_1, p_2$, the resynchronizer $\mathcal{R}_\ell$ inherits the source-origin pairs from level $\ell - 1$ for output positions that belong to $\bigout(p_1) \cup \bigout(p_2)$. Note that $\bigout(p_1) < \bigout(p_2)$ by Lemma 5, so $\mathcal{R}_\ell$ is order-preserving inside $\bigout(p_1) \cup \bigout(p_2)$. Output positions inside $\bigout(p) \setminus (\bigout(p_1) \cup \bigout(p_2))$ are moved in an order-preserving manner to one of the extremities of $I(p)$, or to the last position of $I(p_1)$. Boundedness of $\mathcal{R}_\ell$ is guaranteed by Lemma 6.

The case where $p$ is an idempotent node at level $\ell$ with children $p_1, p_2, \ldots, p_n$ follows a similar approach. For brevity, let $I_i = I(p_i)$ and $B_i = \bigout(p_i)$, and observe that, by Lemma 5, $B_1 < B_2 < \cdots < B_n$. Lemma 7 provides a decomposition of $\bigout(p)$ as $B_1, J_1, B_2, J_2, \ldots, J_{n-1}, B_n$, for some $2K$-small output intervals $J_k$ with origins inside $I_k \cup I_{k+1}$, for $k = 1, \ldots, n - 1$. As before, the resynchronizer $\mathcal{R}_\ell$ behaves exactly as $\mathcal{R}_{\ell-1}$ for the output positions inside the $B_k$’s. For any other output position, say $x \in J_k$, the resynchronizer $\mathcal{R}_\ell$ will move the origin either to the last position of $I_k$ or to the first position of $I_{k+1}$, depending on whether the source origin of $x$ belongs to $I_k$ or $I_{k+1}$.

\section{Proof overview of Theorem 2}

The main obstacle towards dropping the bounded-visit restriction from Theorem 1, while maintaining the effectiveness of the characterization, is the lack of a bound on the number of flows. Indeed, for a transducer $T$ that is not necessarily bounded-visit, there is no bound on the number of flows that encode successful runs of $T$, and thus the proofs of the implications $2 \rightarrow 3 \rightarrow 4$ are not applicable anymore. However, the proofs of the implications $1 \rightarrow 2$ and $4 \rightarrow 1$ remain valid, even for a transducer $T$ that is not bounded-visit.

The idea for proving Theorem 2 is to transform $T$ into an equivalent bounded-visit transducer $\low(T)$, so that the property of one-way resynchronizability is preserved. More precisely, given a two-way transducer $T$, we construct:

1. a bounded-visit transducer $\low(T)$ that is classically equivalent to $T$,
2. a $1$-bounded, regular resynchronizer $\mathcal{R}$ that is $T$-preserving and such that $\mathcal{R}(T) =_o \low(T)$.

We can apply our characterization of one-way resynchronizability in the bounded-visit case to the transducer $\low(T)$. If $\low(T)$ is one-way resynchronizable, then by Theorem 1 we obtain another partially bijective, regular resynchronizer $\mathcal{R}'$ that is $\low(T)$-preserving and such that $\mathcal{R}'(\low(T))$ is order-preserving. Thanks to Lemma 2, the resynchronizers $\mathcal{R}$ and $\mathcal{R}'$ can be composed, so we conclude that the original transducer $T$ is one-way resynchronizable. Otherwise, if $\low(T)$ is not one-way resynchronizable, we show that neither is $T$. This is precisely shown in the lemma below.

\textbf{Lemma 9.} For all transducers $T, T'$, with $T'$ bounded-visit, and for every partially bijective, regular resynchronizer $\mathcal{R}$ that is $T$-preserving and such that
\( R(T) =_o T', \) \( T \) is one-way resynchronizable if and only if \( T' \) is one-way resynchronizable.

There are however some challenges in the approach described above. First, as \( T \) may output arbitrarily many symbols with origin in the same input position, and \( \text{low}(T) \) is bounded-visit, we need \( \text{low}(T) \) to be able to produce arbitrarily long outputs within a single transition. For this reason, we allow \( \text{low}(T) \) to be a transducer with regular outputs. The transition relation of such a transducer consists of finitely many tuples of the form \((q, a, L, q')\), with \( q, q' \in Q, \ a \in \Sigma, \) and \( L \subseteq \Gamma^* \) a regular language over the output alphabet. The semantics of a transition rule \((q, a, L, q')\) is that, upon reading \( a \), the transducer can switch from state \( q \) to state \( q' \), and move its head accordingly, while outputting any word from \( L \). We also need to use transducers with common guess. Both extensions, regular outputs and common guess, already appeared in prior works (cf. \([5,7]\)), and the proof of Theorem 1 in the bounded-visit case can be easily adapted to these features.

There is still another problem: we cannot always expect that there exists a bounded-visit transducer \( \text{low}(T) \) classically equivalent to \( T \). Consider, for instance, the transducer that performs several passes on the input, and on each left-to-right pass, at an arbitrary input position, it copies as output the letter under its head. It is easy to see that the Parikh image of the output is an exact multiple of the Parikh image of the input, and standard pumping arguments show that no bounded-visit transducer can realize such a relation.

A solution to this second problem is as follows. Before trying to construct \( \text{low}(T) \), we test whether \( T \) satisfies the following condition on vertical loops (these are runs starting and ending at the same position and at the same state). There should exist some \( K \) such that \( T \) is \( K \)-sparse, meaning that the number of different origins of outputs generated inside some vertical loop is at most \( K \). If this condition is not met, then we show that \( T \) has unbounded cross-width, and hence, by the implication 1 \( \rightarrow \) 2 of Theorem 1, \( T \) is not one-way resynchronizable. Otherwise, if the condition holds, then we show that a bounded-visit transducer \( \text{low}(T) \) equivalent to \( T \) can indeed be constructed.

### 7 Complexity

We discuss the effectiveness and complexity of our characterization. For a \( k \)-visit transducer \( T \), the effectiveness of the characterization relies on detecting inversions in successful runs of \( T \). It is not difficult to see that this can be decided in space that is polynomial in the size of \( T \) and the bound \( k \). We can also show that one-way resynchronizability is \( \text{Pspace} \)-hard. For this we recall that the emptiness problem for two-way finite automata is \( \text{Pspace} \)-complete. Let \( A \) be a two-way automaton accepting some language \( L \), and let \( \Sigma \) be a binary alphabet disjoint from that of \( L \). The function \( \{(w \cdot a_1 \ldots a_n, a_n \ldots a_1) \mid w \in L, a_1 \ldots a_n \in \Sigma^*, n \geq 0\} \) can be realized by a two-way transducer \( T \) of size polynomial in \(|A|\), and \( T \) is one-way resynchronizable if and only if \( L \) is empty.
In the unrestricted case, we showed that one-way resynchronizability is decidable (Theorem 2). We briefly outline the complexity of the decision procedure:

1. First one checks that $T$ is $K$-sparse for some $K$. To do this, we construct from $T$ the regular language $L$ of all inputs with some positions marked that correspond to origins produced within the same vertical loop. Bounded sparsity is equivalent to having a uniform bound on the number of marked positions in every input from $L$. Standard techniques for two-way automata allow to decide this in space that is polynomial in the size of $T$. Moreover, this also gives us a computable exponential bound to the largest constant $K$ for which $T$ can be $K$-sparse.

2. Next, we construct from the $K$-sparse transducer $T$ a bounded-visit transducer $T'$ that is classically equivalent to $T$ and has exponential size.

3. Finally, we decide one-way resynchronizability of $T'$ by detecting inversions in successful runs of $T'$ (Theorem 1).

Summing up, one can decide one-way resynchronizability of unrestricted two-way transducers in exponential space. It is open if this bound is optimal. We also do not have any interesting bound on the size of the resynchronizer that witnesses one-way resynchronizability, both in the bounded-visit case and in the unrestricted case. Similarly, we lack upper and lower bounds on the size of the resynchronized one-way transducers, when these exist.

8 Conclusions

As the main contribution of this paper, we provided a characterization for the subclass of two-way transducers that are one-way resynchronizable, namely, that can be transformed by some bounded, regular resynchronizer, into an origin-equivalent one-way transducer.

There are similar definability problems that emerge in the origin semantics. For instance, one could ask whether a given two-way transducer can be resynchronized, through some bounded, regular resynchronization, to a relation that is origin-equivalent to a first-order transduction. This can be seen as a relaxation of the first-order definability problem in the origin semantics, namely, the problem of telling whether a two-way transducer is origin-equivalent to some first-order transduction, shown decidable in [4]. It is worth contrasting the latter problem with the challenging open problem whether a given transduction is equivalent to a first-order transduction in the classical setting.

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Appendix

A Proofs from Section 3.1

Lemma 1. Every bounded, regular resynchronizer is effectively equivalent to some 1-bounded, regular resynchronizer.

Proof. Let \( R = (\mathcal{T}, \mathcal{O}, \text{ipar}, \text{opar}, (\text{move}_\tau), (\text{next}_{\tau,\tau'})) \) be a k-bounded, regular resynchronizer. Let \( \hat{u} \) and \( \hat{v} \) be a pair of annotated input and output satisfying \( \text{ipar} \) and \( \text{opar} \) respectively. To construct an equivalent 1-bounded regular resynchronizer \( R' \) we introduce additional output parameters. Specifically, each output position will be annotated with an output type \( \tau \) from \( R \) and an additional index in \( \{1, \ldots, k\} \). The intended meaning of the index is as follows: if \( (y, z) \) is the source/target origin pair associated with an output position labeled by \( (\tau, i), i \in \{1, \ldots, k\} \), then there are exactly \((i-1)\) positions \( y' < y \) such that \( (\hat{u}, y', z) \models \text{move}_\tau \).

Note that this indexing depends on the choice of the target origin \( z \). Therefore, different indexing are possible for different choice of the target origin \( z \).

Based on the resynchronizer \( R \), we define the new resynchronizer as \( R' = (\mathcal{T}, \mathcal{O}', \text{ipar}, \text{opar}', (\text{move}'_{(\tau, i)}), (\text{next}'_{(\tau, i), (\tau', i')})_{\tau,\tau',i}) \), where

- \( \mathcal{O}' = \mathcal{O} \cup \{O'_1, \ldots, O'_k\} \) consists of the old output parameters \( \mathcal{O} \) of \( R \) plus some new parameters \( O'_1, \ldots, O'_k \) for representing indices in \( \{1, \ldots, k\} \);
- \( \text{opar}' \) defines language of all output annotations whose projections over \( \mathcal{I}' \) (the output alphabet extended with the parameters of \( R \)) satisfy \( \text{opar} \) and each position is marked by exactly one index;
- given a type \( \tau' \) that encodes a type \( \tau \) of \( R \) and an index \( i \in \{1, \ldots, k\} \), \( \text{move}'_{(\tau, i)}(y, z) \) states that \( y \) is the \( i \)-th position \( y' \) satisfying \( \text{move}_{\tau}(y', z) \); This property can be expressed by the MSO-formula

\[
\exists y_1 < \cdots < y_k = y \bigwedge_j \text{move}_{\tau}(y_j, z) \\
\land \forall y' \leq y \ (\text{move}_{\tau}(y', z) \rightarrow \bigvee_j y' = y_j);
\]

- \( \text{next}'_{(\tau, i), (\tau', i')}(z, z') \) enforces the same property as \( \text{next}_{\tau,\tau'}(z, z') \).

The resynchronizer \( R' \) is 1-bounded by definition of \( \text{move}'_{(\tau, i)} \). If for positions \( y < y' \), \( (\hat{u}, y, z) \models \text{move}'_{(\tau, i)} \) and \( (\hat{u}, y', z) \models \text{move}'_{(\tau, i)} \), then \( y \) and \( y' \) are both the \( i \)-th source position in \( \hat{u} \) satisfying \( \text{move}_{\tau} \) with target \( z \), which is a contradiction.

We now prove that \( R \) and \( R' \) define the same relation between synchronized pairs. First we show \( R' \subseteq R \). Consider \( ((u, v), (u', v')) \in R' \). Therefore, there exists \( \hat{u} \models \text{ipar} \) and \( \hat{v} \models \text{opar}' \) such that \( \text{move}' \) applied to positions of \( \hat{v} \) give the \( v' \) witnessing \( ((u, v), (u', v')) \in R' \). By definition of \( \text{opar}' \), \( \hat{v} \models \text{opar} \). Suppose, a position \( x \) of output type \( (\tau, i) \) is moved from origin \( y \) in \( v \) to \( z \) in \( v' \). This means \( (\hat{u}, y, z) \models \text{move}'_{(\tau, i)} \). Then, by definition of \( \text{move}'_{(\tau, i)} \), \( (\hat{u}, y, z) \models \text{move}_{\tau} \). This shows \( R' \subseteq R \).
For the containment $R \subseteq R'$, consider $((u, v), (u, v')) \in R$. Therefore, there exists $\hat{u} \models \text{ipar}$ and $\hat{v} \models \text{opar}$ such that $\text{move}$ applied to each position in $\hat{v}$ witnesses $((u, v), (u, v')) \in R$. This means for every position $x \in \text{dom}(\hat{v})$ with output-type $\tau$, there exist $y, z$, such that $(\hat{u}, y, z) \models \text{move}_\tau$, $y = \text{orig}(v(x))$ and $z = \text{orig}(v'(x))$. For such a position $x \in \text{dom}(\hat{v})$ of output type $\tau$, let $i \in \{1, \ldots, k\}$ be such that there are exactly $i-1$ positions $y_1 < y_2 < \ldots < y_{i-1} < y$ such that $(\hat{u}, y_j, z) \models \text{move}_\tau$. Let $\hat{v}'$ be the annotation of $\hat{v}$ where every position $x$ is annotated with the index $i$ as above. Clearly $\hat{v}' \models \text{opar}$ and therefore, $((u, v), (u, v')) \in R'$. We conclude $R = R'$.

Lemma 2. The class of bounded, regular resynchronizers is effectively closed under composition.

Proof. Let $R = (\mathcal{T}, \mathcal{O}, \text{ipar}, \text{opar}, (\text{move}_\tau)_\tau, (\text{next}_{\tau, \tau'})_{\tau, \tau'})$ and $R' = (\mathcal{T}', \mathcal{O}', \text{ipar}', \text{opar}', (\text{move}_{\tau, \lambda})_\lambda, (\text{next}_{\lambda, \lambda'})_{\lambda, \lambda'})$ be two bounded, regular resynchronizers. In view of Lemma 1, we can assume that both resynchronizers are 1-bounded. The composition $R \circ R'$ can be defined by combining the effects of $R$ and $R'$ almost component-wise. Some care should be taken, however, in combining the formulas next and next'. Formally, we define the composed resynchronizer $R'' = (\mathcal{T}'', \mathcal{O}'', \text{ipar}'', \text{opar}'', (\text{move}_{\tau, \lambda}')_{\tau, \lambda}, (\text{next}_{\tau, \tau'}')_{\tau, \tau'})$, where

- $\mathcal{T}''$ is the union of the parameters $\mathcal{T}$ and $\mathcal{T}'$,
- $\mathcal{O}''$ is the union of the parameters $\mathcal{O}$ and $\mathcal{O}'$,
- $\text{ipar}''$ is the conjunction of the formulas $\text{ipar}$ and $\text{ipar}'$;
- $\text{opar}''$ is the conjunction of the formulas $\text{opar}$ and $\text{opar}'$;
- $\text{move}_{\tau, \lambda}'(y, z)$ states the existence of some position $t$ satisfying both formulas $\text{move}_\tau(t, z)$ and $\text{move}'_\lambda(y, t)$;
- $\text{next}_{\tau, \lambda}'(z, z')$ requires that $\text{next}_{\tau, \tau'}(z, z')$ holds and, moreover, that there exist some positions $t, t'$ satisfying $\text{move}_\tau(t, z)$, $\text{move}_{\tau'}(t', z')$, and $\text{next}_{\lambda, \lambda'}(t, t')$; note that these positions $t, t'$ are uniquely determined from $z, z'$ since $R$ is 1-bounded, and they act, at the same time, as source origins for $R$ and as target origins for $R'$.

By definition, $\text{move}_{\tau, \lambda}'$ is 1-bounded, thus $z$ and $\tau$ determine a unique $t$, which together with $\lambda$ determines a unique $y$. It is also easy to see that $R''$ is equivalent to $R \circ R'$ as the positions corresponding to $t$ in formulas $\text{move}_{\tau, \lambda}'$ and $\text{next}_{\tau, \tau'}'$ correspond to the source origin of $R$ and target origin of $R'$.

B Proofs from Section 4.1

Lemma 3. Let $\rho$ be a run of $T$ on $u$, let $J < I < K$ be a partition of the domain of $u$ into intervals, with $I$ loop of $\rho$, and let $F = \text{flow}_\rho(J)$, $E = \text{flow}_\rho(I)$, and $G = \text{flow}_\rho(K)$ be the corresponding flows. Consider an arbitrary edge $f$ of either $F$ or $G$, and a straight edge $e$ of the idempotent flow $E$. Let $\rho_f$ and $\rho_e$ be the witnessing subruns of $f$ and $e$, respectively. Then the occurrence order of $\rho_f$ and $\rho_e$ in $\rho$ is the same as the occurrence order of $\rho_f$ and any copy of $\rho_e$ in pump$T^\rho(\rho)$. 

Proof. It is convenient to rephrase the claim of the lemma in terms of a juxtaposition operation on flows and in terms of an induced accessibility relation on edges. Formally, given two flows $F,G$, we define the juxtaposition $FG$ in a way similar to concatenation, with the only exception that in the result we maintain as an additional group of vertices the $R$ vertices of $F$, glued with the state-matching $L$ vertices of $G$ (strictly speaking, the result of a juxtaposition of two flows is not a flow, since it has three distinguished groups of vertices). We denote by $E \ldots E$ the $n$-fold juxtaposition of the flow $E$ with itself (this must not be confused with the $n$-fold concatenation $E \cdot \ldots \cdot E$).

Let $F,E,G, f,e, \rho_f, \rho_e$ be as stated in the lemma, and let $\preceq$ denote the accessibility order between edges in a juxtaposition of flows, e.g. $FEG$ (note that, due to the type of flows considered here, $\preceq$ turns out to always be a total order on $FEG$). Observe that the relative occurrence order of $\rho_f$ and $\rho_e$ inside $\rho$ is the same as the accessibility order $\preceq$ of the edges $f$ and $e$ on the graph $FEG$. A similar claim holds for the occurrence order of $\rho_f$ and any copy of $\rho_e$ inside the pumped run $\text{pump}^n_\gamma(\rho)$, which corresponds to the accessibility order of $f$ and any copy of $e$ in the graph $FE \ldots EG$. Thanks to these correspondences, to prove the lemma it suffices to consider any copy $e'$ of $e$ in $FE \ldots EG$, and show that

$$f \preceq e \text{ in } FEG \iff f \preceq e' \text{ in } FE \ldots EG.$$  

We thus prove the above claim. Consider the maximal path $\pi$ inside $E \ldots E$ that contains the edge $e'$. Note that this path starts and ends at some extremal vertices of $E \ldots E$ (otherwise the path could be extended while remaining inside $E \ldots E$). Also recall that concatenation can be defined from juxtaposition by removing the intermediate groups of vertices, leaving only the extremal ones, and by shortcutting paths into edges. We call this operation flattening, for short. In particular, since $E$ is idempotent, we have that $E = E \cdot \ldots \cdot E$ can be obtained from the flattening of $E \ldots E$, and this operation transforms the path $\pi$ into an edge $e''$. By construction, we have that $f \preceq e$ in $FE \ldots EG$ if and only if $f \preceq e''$ in $FEG$. So it remains to prove that

$$f \preceq e \text{ in } FEG \iff f \preceq e'' \text{ in } FEG.$$  

Clearly, this latter claim holds if the edges $e$ and $e''$ coincide. This is indeed the case when $e$ is a straight edge and $E$ is idempotent. The formal proof that this holds is rather tedious, but follows quite easily from a series of results we have already proven in [3]. Roughly speaking, one proves that:

- the edges of an idempotent flow $E$ can be grouped into components (cf. Definition 6.4 from [3]), so that each component contains exactly one straight edge (cf. Lemma 6.6 from [3], see also the figure at page 11, where components are represented by colors);
- every path inside the juxtaposition $EE$, with $E$ idempotent, consists of edges from the same component, say $C$; moreover, after the flattening from $EE$ to $E$, this path becomes an edge of $E$ that belongs again to the component $C$ (cf. Claims 7.3 and 7.4 in the proof of Theorem 7.2 from [3]);
– every maximal path in $E \ldots E$ that contains a straight edge starts and end at opposite sides of $E \ldots E$ (simple observation based on the definition of concatenation and Lemma 6.6 from [3]).

To conclude, recall that $\pi$ is a path inside $E \ldots E$ that contains a copy $e'$ of the straight edge $e$, and that becomes the edge $e''$ after the flattening into $E$. The previous properties immediately imply that $e = e''$.

\section{Proofs from Section 4.2}

As explained in the body, the technical lemmas involving output blocks are only applicable to transducers that avoid inversions. Hereafter we assume that $T$ is a transducer that avoids inversions, and we denote by $\rho$ an arbitrary successful run of $T$.

\begin{lemma}
Let $I_1 < I_2$ be two input intervals and $B_1, B_2$ output blocks of $I_1, I_2$, respectively. If both $B_1, B_2$ are $C$-large, then $B_1 < B_2$.
\end{lemma}

\begin{proof}
$B_1$ and $B_2$ are clearly disjoint. By way of contradiction, assume that $B_1$ and $B_2$ are $C$-large, but $B_1 \nless B_2$. By Lemma 4, we can find for both $i = 1$ and $i = 2$ a loop $J_i \subseteq I_i$ and a productive straight edge $e_i \in \text{flow}(J_i)$ that is witnessed by a subrun intersecting $B_i$. Clearly, we have $J_1 < J_2$, and since $B_1 > B_2$, the subrun witnessing $e_1$ follows the subrun witnessing $e_2$. Thus, $(J_1, e_1, J_2, e_2)$ is an inversion of $\rho$, which contradicts the assumption that $T$ avoids inversions.
\end{proof}

We now turn to the proofs of Lemmas 6 and 7, which both require auxiliary lemmas relying again on the assumption that $T$ avoids inversion.

\begin{lemma}
Let $I$ be an input interval, $B_1 < B_2$ two output blocks of $I$, and $S$ the set of output positions strictly between $B_1$ and $B_2$ and with origins outside $I$. If $B_1, B_2$ are $C$-large, then $S$ is $2C$-small.
\end{lemma}

\begin{proof}
By way of contradiction, suppose that $S$ is $2C$-large. This means that $|\text{orig}(S) \cap I'| > C$ for some interval $I'$ disjoint from $I$, say $I' < I$ (the case of $I' > I$ is treated similarly). By Lemma 4, we can find two loops $J \subseteq I$ and $J' \subseteq I'$ and some productive straight edges $e \in \text{flow}(J)$ and $e' \in \text{flow}(J')$ that are witnessed by subruns intersecting $B_1$ and $S$, respectively. Since $S > B_1$, we know that the subrun witnessing $e$ follows the subrun witnessing $e'$. As in the previous proof, this shows the inversion $(J, e, J', e')$, which contradicts the assumption that $T$ avoids inversions.
\end{proof}

\begin{lemma}
Let $I = I_1 \cdot I_2$, $B = \text{bigout}(I)$, and $B_i = \text{bigout}(I_i)$ for $i = 1, 2$. Then $B \setminus (B_1 \cup B_2)$ is $4KC$-small.
\end{lemma}

\begin{proof}
By Lemma 5, we have $B_1 < B_2$. Moreover, all $C$-large output blocks of $I_1$ or $I_2$ are also $C$-large output blocks of $I$, so $B$ contains both $B_1$ and $B_2$. Let $I_0$
be the maximal interval to the left of $I_1$, and thus adjacent to it, and, similarly, let $I_3$ be the maximal interval to the right of $I_2$, and thus adjacent to it.

Suppose, by way of contradiction, that $B \setminus (B_1 \cup B_2)$ is $4K_C$-large. This means that there is a $2K_C$-large set $S \subseteq B \setminus (B_1 \cup B_2)$ with origins entirely inside $I_0 \cdot I_1$ or entirely inside $I_2 \cdot I_3$. Suppose, w.l.o.g., that the former case holds, and decompose $S$ as a union of maximal output blocks $B_{1}', B_{2}', \ldots, B_{n}'$ of either $I_0$ or $I_1$. Since $S \cap B_1 = \emptyset$, we have that every block $B_{i}'$ has origins inside $I_1$ is $C$-small. Similarly, by Lemma C.1, every block $B_{i}'$ with origins inside $I_0$ is $C$-small too. Moreover, since $\rho$ is $K$-visiting, we have that the number $n$ of maximal output blocks of either $I_0$ or $I_1$ that are contained in $S$ is at most $2K$.

All together, this contradicts the assumption that $S$ is $2K_C$-large.

**Lemma C.2.** Let $I$ be a loop of $\rho$. Then $\text{flow}(I)$ has at most one productive straight edge, and this edge must be LR.

**Proof.** Suppose, by way of contradiction, that there are two productive straight edges in $\text{flow}(I)$, say $e$ and $f$, with $e$ before $f$ in $\rho$ (the reader may refer again to the figure at page 11, and think of $e$ and $f$, for instance, as the edges labeled by $\alpha_2$ and $\gamma_1$, respectively). Suppose that we pump $I$ twice, and let $I_1 < I_2$ be the copies of $I$ in the pumped run $\rho'$. Let also $e_1, e_2$ (resp. $f_1, f_2$) be the corresponding copies of $e$ (resp. $f$), so that $e_j, f_j$ belong to the flow $\text{flow}_{\rho'}(I_j)$.

It is easy to check the following properties:

- if $e$ is an LR edge, then the subrun witnessed by $e_1$ occurs in $\rho'$ before the subrun witnessed by $e_2$ (and the other way around if $e$ is RL);
- the subruns witnessed by $e_1$ and $e_2$ occur in $\rho'$ before the subruns witnessed by $f_1, f_2$ (this property follows easily from the observation that when building the product $\text{flow}(I) \cdot \text{flow}(I)$, the edges $e_1, e_2$ will be “part” of the edge $e$ in the product, whereas $f_1, f_2$ will be “part” of the edge $f$).

Let us assume first that $e$ is an RL edge. Then observe that $(I_1, e_1, I_2, e_2)$ is an inversion in $\rho'$. But this contradicts $T$ being inversion-free. Therefore, both $e, f$ are LR edges. But then, $(I_1, f_1, I_2, e_2)$ is an inversion in $\rho'$, and we have again a contradiction.

**Remark C.1.** The statement of Lemma C.2 can be strengthened by observing the following property of productive edges in an idempotent flow. Assume that $I$ is a loop and $e$ is the unique productive straight edge in $\text{flow}(I)$. Let $f$ be some productive (non-straight) edge of $\text{flow}(I)$ with $f \neq e$. When $I$ is pumped then the subruns witnessing the copies of $f$ are part of the subrun witnessing $e$ in the product flow. This means for example, that in the figure on page 11 the productive edges are either all among the blue edges, or all among the gray edges (none of the red edges can be productive, because the straight edge is RL, and would result in a productive RL edge on pumping).

**Lemma 7.** Let $I = I_1 \cdot I_2 \cdots I_n$, such that $I$ is a loop and $\text{flow}(I) = \text{flow}(I_k)$ for all $k$. Then $\text{bigout}(I)$ can be decomposed as $B_1 \cdot J_1 \cdot B_2 \cdot J_2 \cdots J_{n-1} \cdot B_n$, where
1. for all $1 \leq k \leq n$, $B_k = \text{bigout}(I_k)$ (with $B_k$ possibly empty);
2. for all $1 \leq k < n$, the positions in $J_k$ have origins inside $I_k \cup I_{k+1}$ and $J_k$ is $2K\mathbf{C}$-small.

Proof. By Lemma C.2, we can assume that $\text{flow}(I) = \text{flow}(I_k)$ has a unique productive straight edge $e$, which is an LR edge. Let $B_k'$ be the output block corresponding to $e$ in $\text{flow}(I_k)$. Since $\text{flow}(I)$ is idempotent, any output block of $I$ has one of the following shapes (see also Remark C.1):

(a) A block $B = B'_1 \cdot J'_1 \cdot \ldots \cdot J'_{n-1} \cdot B'_n$, for some intervals $J'_1, \ldots, J'_{n-1}$ such that

\[ \text{out}(I_k) \text{ is included in } J'_{k-1} \cdot B'_k \cdot J'_k \text{ for all } 1 < k < n, \]

(b) At most $2K$ output blocks $L_1, \ldots, L_p, R_1, \ldots, R_s$, where each $L_i$ and $R_j$ corresponds to an edge of $\text{flow}(I_1)$ and $\text{flow}(I_n)$, respectively: the blocks $L_i, R_j$ appear before, respectively after the straight edge.

Moreover, the order of the output blocks of $I$ is $L_1, \ldots, L_p, B, R_1, \ldots, R_s$. To illustrate the statement (a) above, the reader can take as example $p = s = 2$, $L_1 = \alpha$, $L_2 = \beta$, $R_1 = \kappa$, $R_2 = \zeta$, $J'_1 = \cdots = J'_{n-1} = \alpha\kappa\beta\zeta$ in Figure 2. For statement (b), notice that in $I_1 \cdot I_2 \cdot I_3$, we have the output blocks $L_1 = \alpha, L_2 = \beta$ of $I_3$, the straight edge $(\gamma\alpha\kappa\beta\zeta)^2\gamma$ (the purple zigzag) followed by $R_1 = \kappa, R_2 = \zeta$ of $I_3$.

Note that $B_k = \text{bigout}(I_k)$ is contained in $J'_{k-1} \cdot B'_k \cdot J'_k$ for all $1 < k < n$. Moreover, $B_1 = \text{bigout}(I_1)$ is contained in $L_1 \cdot L_p \cdot B'_1 \cdot J'_1$, and $B_n = \text{bigout}(I_n)$ is contained in $J'_{n-1} \cdot B'_n \cdot R_1 \cdots R_s$. Also by Lemma 5, $B_j$ precedes $B_{j+1}$ for all $j$.

If one of the $L_k$ is $\mathbf{C}$-large, then $B_1$ is non-empty, hence $\text{bigout}(I)$ is non-empty and starts at the first position of $B_1$. Similarly, if one of the $R_k$ is $\mathbf{C}$-large then $B_n$ is non-empty, hence $\text{bigout}(I)$ is non-empty and ends with the last position of $B_n$. Otherwise, if all $L_j, R_j$ are $\mathbf{C}$-small then $\text{bigout}(I)$ is either empty or equal to $B$. In all cases we can write $\text{bigout}(I) = B_1 \cdot J_1 \cdot B_2 \cdot J_2 \cdot \ldots \cdot J_{n-1} \cdot B_n$, with each $J_k$ consisting of at most $K \mathbf{C}$-small blocks of $I_k$ and $K \mathbf{C}$-small blocks of $I_{k+1}$, namely those left over after gathering the $\mathbf{C}$-large blocks into $\text{bigout}(I_k)$ and $\text{bigout}(I_{k+1})$, respectively. Therefore, each $J_k$ is $2K\mathbf{C}$-small.
D Proof of Theorem 1.

Recall that the implication 4 $\rightarrow$ 1 is straightforward, and the implication 1 $\rightarrow$ 2 was already proven in full detail in the main body. Below, we provide detailed proofs of the implications 2 $\rightarrow$ 3 $\rightarrow$ 4.

The implication 2 $\rightarrow$ 3 is shown by contradiction. Consider a successful run $\rho$ of $T$ on some input $u$ and suppose there is an inversion: $\rho$ has disjoint loops $I < I'$, whose flows contain productive straight edges, say $e$ in $\text{flow}_\rho(I)$ and $e'$ in $\text{flow}_\rho(I')$, such that $e'$ precedes $e$ in the run order. Let $u = u_1 w u_2 w' u_3$ so that $w$ and $w'$ are the factors of the input delimited by the loops $I$ and $I'$, respectively. Further let $v$ and $v'$ be the outputs produced along the edges $e$ and $e'$, respectively. Consider now the run $\rho_k$ obtained from $\rho$ by pumping the input an arbitrary number $k$ of times on the loops $I$ and $I'$. This run is over the input $u_1 (w)^k u_2 (w')^k u_3$, and in the output produced by $\rho_k$ there are $k$ (possibly non-consecutive) occurrences of $v$ and $v'$. By Lemma 3 all occurrences of $v'$ precede all occurrences of $v$. In particular, if $X_1$ (resp. $X_2$) is the set of positions corresponding to all the occurrences of $v$ (resp. $v'$) in the output produced by $\rho_k$, then $(X_1, X_2)$ is a cross of width at least $k$.

Now we prove the implication 3 $\rightarrow$ 4. We assume that no run of $T$ has any inversion. We want to build a partially bijective, regular resynchronizer $R$ that is $T$-preserving and such that $R(T)$ is order-preserving. The resynchronizer $R$ will use input and output parameters to guess a successful run $\rho$ of $T$ on the input $u$ and a corresponding factorization tree for $\rho$ of height at most $H = 3|MT|$ (see page 5 for the formal definition and the existence of a factorization tree).

The resynchronizer $R$ that we will define is functional, which means here that every source origin is mapped by each move$_r$ formula to at most one target position.

Notations. For a node $p$ of a factorization tree we write $I(p)$ for the input interval which is the yield of the subtree of $p$. Recall that the leaves of the factorization tree correspond to singleton intervals on the input. The set of output positions with origins in $I(p)$ is denoted by $\text{out}(p)$ (note that this might not be an interval).

Recall that an output block $B$ of $\text{out}(p)$ is a maximal interval of output positions with origins in $I(p)$, and hence the position just before and the position just after $B$ have origins outside $I(p)$. We also write $\text{bigout}(p)$, instead of $\text{bigout}(I(p))$, for the dominant output interval of $I(p)$ (see page 12 for the definition). Finally, given a position $x$ in the output and a level $\ell$ of the factorization tree of $\rho$, we denote by $p_{x,\ell}$ the unique node at level $\ell$ such that $I(p_{x,\ell})$ contains the source origin of $x$.

Input Parameters. The successful run $\rho$ together with its factorization tree of height at most $H = 3|MT|$ can be easily encoded over the input using input parameters $ipar$. The parameters describe each input interval $I(p)$ and the label $\text{flow}(I(p))$ of each node $p$ in the factorization tree. Formally, an input interval
$I(p)$ is described by marking the begin and end with two distinguished parameters for the specific level. The label $flow(I(p))$ annotates every position inside $I(p)$. This accounts for $H(2 + |M_T|)$ input parameters. Correctness of the annotations with the above input parameters can be expressed by a formula $ipar$. In particular, on the leaves, $ipar$ checks that every interval is a singleton of the form $\{y\}$ and its flow is the one induced by the letter $u(y)$. On the internal nodes, $ipar$ checks that the label of a node coincides with the monoid product of the labels of its children, which is a composition of flows. It also checks that for every node with more than two children, the node and the children are labelled by the same idempotent flow.

**Output Parameters.** We also need to encode the run $\rho$ on the output, because the resynchronizer will determine the target origin of an output position, not only on the basis of the flow at the source origin, but also on the basis of the productive transition that generated that particular position. The annotation that encodes the run $\rho$ on the output is done using output parameters (one for each transition in $\Delta$), and its correctness will be enforced by a suitable combination of the formulas $opar$, $move_{\tau}$, and $next_{\tau,\tau'}$. This will take a significant amount of technical details and will rely on specific properties of formulas $move_{\tau}$, so we prefer to temporarily postpone those details.

Below, we explain how the origins are transformed by a series of partial resynchronizers $R_\ell$ that “converge” in finitely many steps to a desired resynchronization, under the assumption that the output annotation correctly encodes the same run $\rho$ that is represented in the input annotation.

**Moving origins.** Here we will work with a fixed successful run $\rho$ and a factorization tree for it, that we assume are correctly encoded by the input and output annotations. For every level $\ell$ of the factorization tree, we will define a functional, bounded, regular resynchronizer $R_\ell$. Each resynchronizer $R_\ell$ will be partial, in the sense that for some output positions it will not define source-target origin pairs. However, the set of output positions with associated source-target origin pairs increases with the level $\ell$, and the top level resynchronizer $R_\ast$ will specify source-target origin pairs for all output positions. The latter resynchronizer will almost define the resynchronization that is needed to prove item (4) of the theorem; we will only need to modify it slightly in order to make it 1-bounded and to check that the output annotation is correct.

To enable the inductive construction, we need the resynchronizer $R_\ell$ to satisfy the following properties, for every level $\ell$ of the factorization tree:

- the set of output positions for which $R_\ell$ defines target origins is the union of the dominant output intervals $bigout(p)$ of all nodes $p$ at level $\ell$;
- $R_\ell$ only moves origins within the same interval at level $\ell$, that is, $R_\ell$ defines only pairs $(y, z)$ of source-target origins such that $y, z \in I(p)$ for some node $p$ at level $\ell$;
- the target origins defined by $R_\ell$ are order-preserving within the same interval at level $\ell$, that is, for all output positions $x < x'$, if $R_\ell$ defines the target
origins of \(x, x'\) to be \(z, z'\), respectively, and if \(z, z' \in I(p)\) for some node \(p\) at level \(\ell\), then \(z \leq z'\).

- \(\mathcal{R}_\ell\) is \(\ell \cdot 4K\mathcal{C}\)-bounded, namely, there are at most \(\ell \cdot 4K\mathcal{C}\) distinct source origins that are moved by \(\mathcal{R}_\ell\) to the same target origin.

The inductive construction of \(\mathcal{R}_\ell\) will basically amount to defining appropriate formulas \(\text{move}_\ell(y, z)\).

**Base Case.** The base case is \(\ell = 0\), namely, when the resynchronization is acting at the leaves of the factorization tree. In this case, the regular resynchronizer \(\mathcal{R}_\ell\) is vacuous, as the input intervals \(I(p)\) associated with the leaves \(p\) are singletons, and hence all dominant output intervals \(\text{bigout}(p)\) are empty. Formally, for this resynchronizer \(\mathcal{R}_\ell\), we simply let \(\text{move}_\ell(y, z)\) be false, independently of the underlying output type \(\tau\) and of the source and target origins. This resynchronization is clearly functional, \(0\)-bounded, and order-preserving.

**Inductive Step.** For the inductive step, we explain how the origins of an output position \(x \in \text{bigout}(p)\) are moved within the interval \(I(p)\), where \(p = p_{x, \ell}\) is the node at level \(\ell\) that “generates” \(x\). Even though we explain this by mentioning the node \(p_{x, \ell}\), the definition of the resynchronization will not depend on it, but only on the level \(\ell\) and the underlying input and output parameters. In particular, to describe how the origin of a \(\tau\)-labeled output position \(x\) is moved, the formula \(\text{move}_\ell(y, z)\) has to determine the productive edge that generated \(x\) in the flow that labels the node \(p_{x, \ell}\). This can be done by first determining from the output type \(\tau\) the productive transition \(t_x\) that generated \(x\), and then inspecting the annotation at the source origin \(y\) to “track” \(t_x\) inside the productive edges of the flow \(\text{flow}(I_p)\), for each node \(p'\) along the unique path from the leaf \(p_{x, 0}\) to node \(p_{x, \ell}\). In the case distinction below, we implicitly rely on this type of computation, which can be easily implemented in MSO.

1. **\(p_{x, \ell}\) is a binary node.** We first consider the case where \(p = p_{x, \ell}\) is a binary node (the annotation on the source origin \(y\) will tell us whether this is the case). Let \(p_1, p_2\) be the left and right children of \(p\). If \(x\) belongs to one of the dominant output blocks \(\text{bigout}(p_1)\) and \(\text{bigout}(p_2)\) (again, this information is available at the input annotation), then the resynchronizer \(\mathcal{R}_\ell\) will inherit the source-target origin pairs associated with \(x\) from the lower level resynchronization \(\mathcal{R}_{\ell - 1}\). Note that \(\text{bigout}(p_1) < \text{bigout}(p_2)\) by Lemma 5, so \(\mathcal{R}_\ell\) is order-preserving at least for the output positions inside \(\text{bigout}(p_1) \cup \text{bigout}(p_2)\).

   We now describe the source-target origin pairs when \(x \in \text{bigout}(p) \setminus (\text{bigout}(p_1) \cup \text{bigout}(p_2))\). The idea is to move the origin of \(x\) to one of the following three input positions, depending on the relative order between \(x\) and the positions in \(\text{bigout}(p_1)\) and in \(\text{bigout}(p_2)\):
   - the first position of \(I(p_1)\), if \(x < \text{bigout}(p_1)\);
   - the last position of \(I(p_1)\), if \(\text{bigout}(p_1) < x < \text{bigout}(p_2)\);
   - the last position of \(I(p_2)\), if \(x > \text{bigout}(p_2)\).

   Which of the above cases holds can be determined, again, by inspecting the output type \(\tau\) and the annotation of the source origin \(y\), in a way similar
to the computation of the productive edge that generated $x$ at level $\ell$. So
the described resynchronization can be implemented by an MSO formula $\text{move}_\ell(y, z)$.

The resulting resynchronization $R_\ell$ is functional and order-preserving inside
every interval at level $\ell$. It remains to argue that $R_\ell$ is $\ell \cdot 4KC$-bounded.

To see why this holds, assume, by the inductive hypothesis, that $R_{\ell-1}$ is
$(\ell - 1) \cdot 4KC$-bounded. Recall that the new source-target origin pairs that
are added to $R_\ell$ are those associated with output positions in $\text{bigout}(p) \setminus
(\text{bigout}(p_1) \cup \text{bigout}(p_2))$. Lemma 6 tells us that there are at most $4KC$
distinct positions that are source origins of such positions. So, in the worst
case, at most $(\ell - 1) \cdot 4KC$ source origins from $R_{\ell-1}$ and at most $4KC$
new source origins from $R_\ell$ are moved to the same target origin. This shows that
$R_\ell$ is $\ell \cdot 4KC$-bounded.

2. $p_{x, \ell}$ is an idempotent node. The case where $p = p_{x, \ell}$ is an idempotent
node with children $p_1, p_2, \ldots, p_n$ follows a similar approach. For brevity, let
$I_i = I(p_i)$ and $B_i = \text{bigout}(p_i)$. By Lemma 5, we have $B_1 < B_2 < \cdots < B_n$.

Lemma 7 then provides a decomposition of $\text{bigout}(p)$ as
$B_1 \cdot B_2 \cdot B_3 \cdot \cdots \cdot B_{n-1} \cdot B_n$, for some $2KC$-small output intervals $J_k$, for $k = 1, \ldots, n-1$, that
have origins inside $I_k \cup I_{k+1}$.

As before, the resynchronizer $R_\ell$ behaves exactly as $R_{\ell-1}$ for the output
positions inside the $B_k$’s. For any other output position, say $x \in J_k$ for some
$k = 1, 2, \ldots, n-1$, we first recall that the source origin $y$ of $x$ is either inside
$I_k$ or inside $I_{k+1}$. Depending on which of the two intervals contains $y$, the
resynchronizer $R_\ell$ will define the target origin $z$ to be either the last position
of $I_k$ or the first position of $I_{k+1}$. However, since we cannot determine using
MSO the index $k$ of the interval $J_k$ that contains $x$, we proceed as follows.

First observe that any block $B_i$ can be identified by some flow edge at level
$\ell - 1$, and the latter edge can represented in MSO by suitable monadic predicates over the input. Let $B, B'$ be the two consecutive blocks among
$B_1, \ldots, B_n$ such that $B < x < B'$. Note that these blocks can be determined
in MSO once the productive edge that generated $x$ is identified. Further let
$I$ be the interval among $I_1, \ldots, I_n$ that contains the origin $y$ of $x$. By the
previous arguments, we have that the interval $I$ contains either all the origins
of $B$ or all the origins $B'$. Moreover, which of the two sub-cases holds can again
be determined in MSO by inspecting the annotations. The formula $\text{move}_\ell(y, z)$ can thus define the target origin $z$ to be

- the last position of $I$, if $I$ contains the origins of $B$;
- the first position of $I$, if $I$ contains the origins of $B'$.

The above construction yields a functional regular resynchronization $R_\ell$ that
associates with any two output positions $x < x'$ with source origins in the
same interval $I(p)$, some target origins $z \leq z'$. In other words, the resynchronization $R_\ell$ is order-preserving in each interval at level $\ell$.

It remains to show that $R_\ell$ is $\ell \cdot 4KC$-bounded, under the inductive hy-
pothesis that $R_{\ell-1}$ is $(\ell - 1) \cdot 4KC$-bounded. This is done using a similar
argument as before, that is, by observing that the output positions in
$\text{bigout}(p) \setminus (\bigcup_{1 \leq k \leq n} \text{bigout}(p_k))$ belong to some $J_k$, and in the worst case
all source origins \( y \) of positions from \( J_k \) are moved to the last position of \( I_k \).

By Lemma 7, there are at most \( 2KC \) such positions \( y \).

**Top level resynchronizer.** Let \( R_* \) be the resynchronizer \( R_\ell \) obtained at the top level \( \ell \) of the factorization tree. Based on the above constructions, \( R_* \) defines target origins for all output positions, unless the dominant output interval \( \text{bigout}(p) \) associated with the root \( p \) is empty (this can indeed happen when the number of different origins in the output is at most \( C \), so not sufficient for having at least one \( C \)-large output factor). In particular, if \( \text{bigout}(p) \neq \emptyset \), then \( \text{bigout}(p) \) is the whole output, and \( R_\ell \) is basically the desired resynchronization, assuming that the output annotations are correct.

Let us now discuss briefly the degenerate case where \( \text{bigout}(p) = \emptyset \), which of course can be detected in MSO. In this case, the appropriate resynchronizer \( R_* \) should be redefined so that it moves all source origins to the same target origin, say the first input position. Clearly, this gives a functional, regular resynchronizer that is order-preserving and \( C \)-bounded.

**Correctness of output annotation.** Recall that the properties of the top level resynchronizer \( R_* \), in particular, the claim that \( R_* \) is bounded, were crucially relying on the assumption that every output position \( x \) is correctly annotated with the productive transition that generated it. This assumption cannot be guaranteed by the MSO sentence \( \text{opar} \) alone (the property intrinsically talks about a relation between input and output annotations). Below, we explain how to check correctness of the output annotation with the additional help of the formulas \( \text{move}_\tau(y,z) \) (that will be modified for this purpose) and \( \text{next}_\tau,\tau'(z,z') \).

Let \( \rho \) be the successful run as encoded by the input annotation. The idea is to check that the sequence of productive transitions \( t_x \) that annotate the positions \( x \) in the output is the maximal sub-sequence of \( \rho \) consisting only of productive transitions. Besides the straightforward conditions (concerning, for instance, the first and last productive transitions of \( \rho \), or the possible multiple symbols that could be produced within a single transition), the important condition to be verified is the following:

\[ \text{For every pair of consecutive output positions } x, x+1 \text{ with source origins } y, y', \text{ respectively, then on the run } \rho \text{ that is annotated on the input, one can move from transition } t_x \text{ at position } y \text{ to transition } t_{x+1} \text{ at position } y' \text{ by using at the intermediate steps only non-productive transitions.} \] (†)

The above property is easily expressible by an MSO formula \( \varphi^\dagger_{\tau,\tau'}(y,y') \), assuming that \( \tau, \tau' \) are the output types of \( x, x+1 \) and the free variables \( y \) and \( y' \) are interpreted by the source origins of \( x \) and \( x+1 \), with \( x \) ranging over all output positions. This is very close to the type of constraints that can be enforced by the formula \( \text{next}_{\tau,\tau'} \) of a regular resynchronizer, with the only difference that the latter formula can only access the target origins \( z, z' \) of \( x, x+1 \).

We thus need a way to uniquely determine from the target origins \( z, z' \) of \( x \) the source origins \( y, y' \) of \( x \). For this, we could rely on the formulas \( \text{move}_\tau(y,z), \)
if only they were defining partial bijections between \( y \) and \( z \). Those formulas are in fact close to define partial bijections, as they are functional and \( k \)-bounded, for \( k = H \cdot 4K_C \). The latter boundedness property, however, depends again on the assumption that the output annotation is correct. We overcome this problem by gradually modifying the resynchronizer \( R_* \) so as to make it functional and 1-bounded (i.e., partially bijective), independently of the output annotations.

We start by modifying the formulas \( \text{move}_\tau(y, z) \) to make them “syntactically” \( k \)-bounded. Formally, we construct from \( \text{move}_\tau(y, z) \) the formula

\[
\text{move}^{\prime}_\tau(y, z) = \text{move}_\tau(y, z) \land \forall y_1, \ldots, y_k, y_{k+1} \left( \bigwedge_i \text{move}_\tau(y_i, z) \rightarrow \left( \bigvee_{i \neq j} y_i = y_j \right) \right).
\]

Intuitively, the above formula is semantically equivalent to \( \text{move}_\tau(y, z) \) when there are at most \( k \) input positions \( y' \) that can be paired with \( z \) via the same formula \( \text{move}_\tau \), and it is false otherwise.

Let \( R'_* \) be the regular resynchronizer obtained from \( R_* \) by replacing the formulas \( \text{move}_\tau \) by \( \text{move}^{\prime}_\tau \), for every output type \( \tau \). By construction, \( R'_* \) is functional and \( k \)-bounded, independently of any assumption on the output annotations. We can then apply Lemma 1 and obtain from \( R'_* \) an equivalent regular resynchronizer \( R''_* = (T'', \mathcal{O}'', \text{ipar}'', \text{opar}'', (\text{move}''_\tau)_{\tau}, (\text{next}''_{\tau, \tau'})_{\tau, \tau'}) \) that is 1-bounded. So each \( \text{move}''_\tau \) is a partial bijection.

We are now ready to verify the correctness of the output annotation. Recall that the idea is to enforce the property (†) by exploiting the previously defined formula \( \varphi^{\dagger}_{\tau, \tau'}(y, y') \) and the partial bijection between the source origins \( y, y' \) and the target origins \( z, z' \), as defined by \( \text{move}''_{\tau}(y, z) \) and \( \text{move}''_{\tau'}(y', z') \). Formally, we define

\[
\text{next}''_{\tau, \tau'}(z, z') = \text{next}''_{\tau, \tau'}(z, z') \land \exists y, y' \; \text{move}''_{\tau}(y, z) \land \text{move}''_{\tau'}(y', z') \land \varphi^{\dagger}_{\tau, \tau'}(y, y').
\]

To conclude, by replacing in \( R'' \) the formulas \( \text{next}''_{\tau, \tau'} \), with \( \text{next}''_{\tau, \tau'} \), we obtain a regular resynchronizer \( R \) that is partially bijective, \( T \)-preserving and such that \( R(T) \) is order-preserving. This completes the proof of the implication 3 \( \rightarrow \) 4 of our main theorem.

### E Proof of Theorem 2

We provide here the missing details of the proof of Theorem 2, as sketched in Section 6. We recall that the goal is to construct, from a given arbitrary two-way transducer \( T \):

1. a bounded-visit transducer \( \text{low}(T) \) that is classically equivalent to \( T \),
2. partially bijective, regular resynchronizer \( R \) that is \( T \)-preserving and such that \( R(T) =_o \text{low}(T) \).

We will reason with a fixed input \( u \) at hand and with an induced accessibility relation on productive transitions of \( T \), tagged with origins. Formally, a tagged
transition is any pair \((t, y)\) consisting of a transition \(t \in \Delta\) and a position \(y\) on the input \(u\), such that \(t\) occurs at position \(y\) in some successful run on \(u\).

The accessibility preorder on tagged transitions is such that \((t, y) \preceq_u (t', y')\) whenever \(T\) has a run on \(u\) starting with transition \(t\) at position \(y\) and ending with transition \(t'\) at position \(y'\). This preorder induces an equivalence relation, denoted \(\sim_u\). Intuitively, \((t, y) \sim_u (t', y')\) means that \(T\) can cycle an arbitrary number of times between these two tagged transitions (possibly \((t, y) = (t', y')\)).

A \(\sim_u\)-equivalence class \(C\) is called realizable on \(u\) if there is a successful run on \(u\) that uses at least once a tagged transition from the class \(C\).

We say that \(T\) is \(K\)-sparse if for every input \(u\) and every realizable \(\sim_u\)-equivalence class \(C\), there are at most \(K\) productive tagged transitions in \(C\) (recall that a productive transition is one that produces non-empty output).

Intuitively, bounded sparsity means that the number of origins of outputs produced by vertical loops in successful runs of \(T\) is uniformly bounded. If \(T\) is not \(K\)-sparse for any \(K\), then we say that \(T\) has unbounded sparsity.

When \(T\) is \(K\)-sparse, the productive tagged transitions from the same realizable \(\sim_u\)-equivalence class can be lexicographically ordered and distinguished by means of numbers from a fixed finite range, say \(\{1, \ldots, K\}\). An important observation is that the equivalence \(\sim_u\) is a regular property, in the sense that one can construct, for instance, an MSO formula \(\varphi_{\sim_u}(y, y')\) that holds on input \(u\) if and only if \((t, y) \sim_u (t', y')\). In particular, this implies that unbounded sparsity can be effectively tested: it suffices to construct the regular language consisting of every possible input \(u\) with a distinguished realizable \(\sim_u\)-equivalence class marked on it, and check whether this language contains words with arbitrarily many marked positions that correspond to productive tagged transitions (this boils down to detecting special loops in a classical finite-state automaton).

**Lemma E.1.** If \(T\) has unbounded sparsity, then \(T\) is not one-way definable.

**Proof.** The assumption that \(T\) has unbounded sparsity and the definition of \(\sim_u\) imply that, for every \(n \in \mathbb{N}\), there exist an input \(u\), a successful run \(\rho\) on \(u\), and \(2n\) tagged transitions \((t_1, y_1), \ldots, (t_n, y_n), (t'_1, y'_1), \ldots, (t'_n, y'_n)\) such that the \(t_i\)'s occur before the \(t'_i\)'s in \(\rho\) and the \(y'_i\) are to the right of the \(y_i\). Since \(n\) can grow arbitrarily, this witnesses precisely the fact that \(T\) has unbounded cross-width. Thus, by the implication 1 \(\rightarrow\) 2 of Theorem 1, which is independent of \(T\) being bounded-visit, we know that \(T\) is not one-way resynchronizable. \(\Box\)

Let us now show how to construct a bounded-visit transducer \(\text{low}(T)\) with regular outputs and common guess that is equivalent to \(T\), under the assumption that \(T\) is \(K\)-sparse for some constant \(K\). Intuitively, \(\text{low}(T)\) simulates successful runs of \(T\) on input \(u\) by shortcutting maximal vertical loops. Formally, for an input \(u\) and a tagged transition \((t, y)\), a vertical loop at \((t, y)\) is any run on \(u\) that starts and ends with transition \(t\) at position \(y\). We will tacitly focus on vertical loops that are realizable on \(u\), exactly as we did for \(\sim_u\)-equivalence classes. The output of a vertical loop is the word spelled out by the productive transitions in it.
Of course, all tagged transitions in a vertical loop at \((t, y)\) are \(\sim_u\)-equivalent to \((t, y)\). In particular, as \(T\) is \(K\)-sparse, there are at most \(K\) productive tagged transitions in a (realizable) vertical loop, and hence the language \(L_{t, y}\) of outputs of vertical loops at \((t, y)\) is regular. In addition, there are only finitely many languages \(L_{t, y}\) for varying \((t, y)\). This can be seen as follows: we can assume an order on the elements of the \(\sim_u\)-class \(C\) of \((t, y)\), and a strongly connected graph with nodes corresponding to \(C\) and edges reflecting the accessibility preorder. The correctness of the graph can be checked with regular annotations on the input, and the graph itself can be turned into an automaton accepting \(L_{t, y}\).

Therefore, using common guess in \(\text{low}(T)\), we can assume that every position \(y\) carries as annotation the language \(L_{t, y}\) for each transition \(t\). By definition, \(L_{t, y}\) is non-empty if and only if there is some productive vertical loop at \((t, y)\).

Consider an arbitrary successful run \(\rho\) of \(T\) on \(u\). Let \(\text{low}(\rho)\) be the run obtained by replacing, from left to right, every maximal vertical loop at \((t, y)\) by the single transition \((t, y)\). For short, we call \(\text{low}(\rho)\) the normalization of \(\rho\) and we observe that this is a successful, \(|\Delta|\)-visit run. This means that (i) \(\text{low}(\rho)\) can be finitely encoded on the input as a sequence of flows of height at most \(|\Delta|\), and (ii) the language consisting of inputs annotated with such encodings is regular.

The transducer \(\text{low}(T)\) guesses the encoding of a normalization \(\text{low}(\rho)\) and uses it to simulate a possible run \(\rho\) of \(T\). In particular, every time \(\text{low}(T)\) traverses a transition \(t\) from the flow of \(\text{low}(\rho)\) at position \(y\), it outputs a word from the language \(L_{t, y}\). However, in order to simplify later the construction of a resynchronizer \(\mathcal{R}\) such that \(\mathcal{R}(T) =_o \text{low}(T)\), it is convenient that \(\text{low}(T)\) outputs the word from \(L_{t, y}\) in a possibly different origin, which is uniquely determined by the \(\sim_u\)-equivalence class of \((t, y)\). Formally, we define the anchor of a \(\sim_u\)-equivalence class \(C\), denoted \(\text{an}(C)\), to be the leftmost input position \(z\) such that \((t', z) \in C\) for some transition \(t'\). After traversing a transition \(t\) from the flow at position \(y\), and before outputting a word from \(L_{t, y}\), the transducer \(\text{low}(T)\) moves to the anchor \(\text{an}([(t, y)]_{\sim_u})\). Then it outputs the appropriate word and moves back to position \(y\), where it can resume the simulation of the normalized run \(\text{low}(\rho)\). Note that the position \(y\) can be recovered from the anchor \(\text{an}([(t, y)]_{\sim_u})\) since the elements inside the equivalence class \([(t, y)]_{\sim_u}\) can be identified by numbers from \(\{1, \ldots, K\}\) (recall that \(T\) is \(K\)-sparse), and since the relationship between any two such elements is a regular property. It is routine to verify that the described transducer \(\text{low}(T)\) is equivalent to \(T\) and bounded-visit.

Let us now explain how to construct a partially bijective, regular resynchronizer \(\mathcal{R}\) that is \(T\)-preserving and such that \(\mathcal{R}(T) =_o \text{low}(T)\). We proceed as in the construction of \(\text{low}(T)\) by annotating the input word \(u\) with flows that encode the normalization \(\text{low}(\rho)\) of a successful run \(\rho\) of \(T\) on \(u\). As for the output word \(v\), we annotate every position \(x\) of \(v\) with the productive transition \(t = (q, a, v, q')\) of \(\rho\) that generated \(x\). For short, we call \(t\) the transition of \(x\). In addition, we fix an MSO-definable total ordering on tagged transitions (e.g. the lexicographic ordering). Then, we determine from each output position \(x\) the \(\sim_u\)-equivalence class \(C = [(t, y)]_{\sim_u}\), where \(u\) is the underlying input, \(t\) is the
productive transition that generated $x$, and $y$ is its origin, and we extend the annotation of $x$ with the index of the element $(t, y)$ inside the equivalence class $C$, according to the fixed total ordering on tagged transitions. This number $i$ is called the index of $x$.

The resynchronizer $R$ needs to redirect the source origin $y$ of any output position generated by a transition $t$ to a target origin $z$ that is the anchor of the $\sim_u$-equivalence class of $(t, y)$. To simplify the explanation, we temporarily assume that the input and output are correctly annotated as described above. By inspecting the type $\tau$ of an output position $x$, the formula $\text{move}_\tau(y, z)$ of $R$ can determine the transition $t$ of $x$, and enforce that $(t, y) \sim_u (t', z)$, for some transition $t'$, and that $(t, y) \not\sim_u (t'', z')$, for all $z' < z$ and all transitions $t''$. Under the assumption that the input and output annotations are correct, this would result in a bounded resynchronizer $R$. Indeed, for every position $z$, there exist at most $K \cdot |\Delta|$ positions $y$ that, paired with some productive transition, turn out to be $\sim_u$-equivalent to $(t', z)$ for some transition $t'$. Once again, we need to further constrain the relation $\text{move}_\tau(y, z)$ so that it describes a partial bijection between source and target origins (this will be useful later). For this, it suffices to additionally enforce that $(t, y)$ is the $i$-th element in its $\sim_u$-equivalence class, accordingly to the fixed total ordering on tagged transitions, where $i$ is the index specified in the output type $\tau$ of $x$. This latter modification also guarantees that $i$ is the correct index of $x$.

Unless we further refine our constructions, we cannot claim that they always result in a 1-bounded resynchronizer $R$, since the above arguments crucially rely on the assumption that the input and output annotations are correct. However, we can apply the same trick that we used in the proof of Theorem 1, to make the resynchronizer $R$ “syntactically” 1-bounded, even in the presence of badly-formed annotations. Formally, let $\text{move}_\tau(y, z)$ be the formula that transforms the origins in the way described above, and define

$$\text{move}'_\tau(y, z) = \text{move}_\tau(y, z) \land \forall y' \ (\text{move}_\tau(y', z) \rightarrow y' = y).$$

By construction, the above formula defines a partial bijection entailing the old relation $\text{move}_\tau$ (in the worst case, when the annotations are not correct, the above formula may not hold for some pairs of source and target origins). In addition, if the annotations are correct, then $\text{move}'_\tau(y, z)$ is semantically equivalent to $\text{move}_\tau(y, z)$, as desired. In this way, we obtain a regular resynchronizer $R = (T, \mathcal{O}, \text{ipar}, \text{opar}, \text{move}'_\tau, \text{next})$ that is always 1-bounded, no matter how we define $\text{ipar, opar}$, and $\text{next}$.

We now explain how to check that the annotations are correct. The input annotation does not pose any particular problem, since the language of inputs annotated with normalized runs is regular, and can be checked using the first formula $\text{ipar}$ of the resynchronizer. As for the output annotation, correctness of the indices was already enforced by the $\text{move}'_\tau$ relation. It remains to enforce correctness of the transitions. Once again, this boils down to verifying the following
property (†):

For every pair of consecutive output positions \(x, x+1\) with source origins \(y, y'\), respectively, if \(t, t'\) are the productive transitions specified in the output types of \(x, x+1\), then on the flows that annotate the input, one can move from transition \(t\) at position \(y\) to transition \(t'\) at position \(y'\) by using as intermediate edges only non-productive transitions. (†)

From here we proceed exactly as in the proof of Theorem 1. We observe that Property (†) is expressible by an MSO formula \(\phi^\tau_{\tau', \tau}(y, y')\), assuming that \(\tau, \tau'\) are the output types of \(x, x+1\), that \(y, y'\) are interpreted by the source origins of \(x, x+1\), and that \(x\) ranges over all output positions. We then recall that \(\text{move}_\tau(y, z)\) and \(\text{move}_{\tau'}(y', z')\) describe partial bijections between source and target origins, and exploit this enforce (†) by means of the last formula of \(R\):

\[
\text{next}_{\tau, \tau'}(z, z') = \exists y, y' \text{ move}_\tau(y, z) \land \text{move}_{\tau'}(y', z') \land \varphi^\tau_{\tau', \tau}(y, y').
\]

This guarantees that all annotations are correct, and proves that \(R\) is a partially bijective, regular resynchronizer satisfying \(R(T) =_\alpha \text{low}(T)\). It is also immediate to see that \(R\) is \(T\)-preserving.

We finally prove that one-way resynchronizability of \(T\) reduces to one-way resynchronizability of \(\text{low}(T)\), which can be effectively tested using Theorem 1 since \(\text{low}(T)\) is bounded-visit:

**Lemma 9.** For all transducers \(T, T'\), with \(T'\) bounded-visit, and for every partially bijective, regular resynchronizer \(R\) that is \(T\)-preserving and such that \(R(T) =_\alpha T'\), \(T\) is one-way resynchronizable if and only if \(T'\) is one-way resynchronizable.

**Proof.** For the right-to-left implication, suppose that \(T' =_\alpha R(T)\) is bounded-visit and one-way resynchronizable. Since \(T'\) is bounded-visit, we can use the implications \(1 \rightarrow 2 \rightarrow 3 \rightarrow 4\) in Theorem 1 to get the existence of a partially bijective, regular resynchronizer \(R'\) that is \(T'\)-preserving and such that \(R'(T')\) is order-preserving. By Lemma 2, there is a bounded, regular resynchronizer \(R''\) that is equivalent to \(R' \circ R\). In particular, \(R''(T)\) is order-preserving. It remains to verify that \(R''\) is also \(T\)-preserving. Consider any synchronized pair \(\sigma \in [T]_o\). Since \(R\) is \(T\)-preserving, \(\sigma\) belongs to the domain of \(R\), and hence \((\sigma, \sigma') \in R\) for some synchronized pair \(\sigma' \in [T']_o\). Since \(R\) is \(T'\)-preserving, \(\sigma'\) belongs to the domain of \(R\), and hence there is \((\sigma, \sigma'') \in (R' \circ R) = R''\). This shows that \(R''\) is \(T\)-preserving, and hence \(T\) is one-way resynchronizable.

For the converse direction, suppose that \(T''\) is bounded-visit, but not one-way resynchronizable. We apply again Theorem 1, but now we use the contrapositives of the implications \(2 \rightarrow 3 \rightarrow 4 \rightarrow 1\), and obtain that \(T''\) has unbounded cross-width (see Definition 2).

We also recall that \(R = (T, \Omega, \text{ipar}, (\text{move}_\tau)_{\tau}, (\text{next}_{\tau, \tau'})_{\tau, \tau})\) is partially bijective. This means that every formula \(\text{move}_\tau(y, z)\) defines a partial bijection from
source to target positions. A useful property of every MSO-definable partial bijection is that, for every position \( t \), it can only define boundedly many pairs \((y, z)\) with either \( y \leq t < z \) or \( z \leq t < y \) — for short, we say call such a pair \((y, z)\) \( t \)-separated. This follows from compositional properties of regular languages. Indeed, let \( A \) be a deterministic automaton equivalent to the formula that defines the partial bijection. For every pair \((y, z)\) in the partial bijection, let \( q_{y,z} \) be the state visited at position \( t \) by the successful run of \( A \) on the input annotated with the pair \((y, z)\). If \( A \) accepted more than \( |Q| \) pairs that are \( t \)-separated, where \( Q \) is the state space of \( A \), then at least two of them, say \((y, z)\) and \((y', z')\), would satisfy \( q_{y,z} = q_{y',z'} \). But this would imply that the pair \((y, z')\) is also accepted by \( A \), which contradicts the assumption that \( A \) defines a partial bijection.

We now exploit the above result to prove that the property of having unbounded cross-width transfers from \( T' \) to \( T \). Consider a cross \((X_1, X_2)\) of arbitrarily large width \( h \) in some synchronized pair \( \sigma = (u, v, orig) \) of \( T' \). Without loss of generality, assume that all positions in \( X_1 \cup X_2 \) have the same type \( \tau \). Let \( Z_i = orig(X_i) \), for \( i = 1, 2 \), and \( t = \max(Z_2) \). By definition of cross, we have \( X_1 < X_2 \) and \( Z_2 \leq t < Z_1 \). Recall that \text{move}_\tau \) defines a partial bijection, and that this implies that there are only boundedly many pairs of source-target origins that are \( t \)-separated, say \((y_1, z_1), \ldots, (y_k, z_k)\) for a constant \( k \) that only depends on \( R \). Moreover, since \( R(T) =_\tau T' \), the positions in \( Z \) can be seen as target origins of the formula \text{move}_\tau \) of \( R \). Now, let \( X'_1 = X_1 \setminus orig^{-1}({z_1, \ldots, z_k}) \) and \( Y'_i = orig'(X'_i) \), for any synchronized pair \( \sigma' = (u, v, orig') \) such that \((\sigma, \sigma') \in R \). By construction, we have \( X'_1 < X'_2 \) and \( Y'_2 \leq t < Y'_1 \) (the latter condition follows from the fact that the source origins from \( Y'_2 \) can only be moved to target origins on the same side w.r.t. \( t \)). This means that \((X'_1, X'_2)\) is a cross of width \( h - k \). As \( h \) can be taken arbitrarily large and \( k \) is constant, this proves that \( T \) has unbounded cross-width as well.

Finally, by the contrapositive of the implication \( 1 \rightarrow 2 \) of Theorem 1 (which does not need the assumption that \( T \) is bounded-visit), we conclude that \( T \) is not one-way resynchronizable.

\[ \square \]

Summing up, the algorithm that decides whether a given two-way transducer \( T \) is one-way resynchronizable first verifies that \( T \) is \( K \)-sparse for some \( K \) (if not, it claims that \( T \) is not one-way resynchronizable), then it constructs a bounded-visit transducer \( low(T) \) equivalent to \( T \), and finally decides whether \( low(T) \) is one-way resynchronizable (which happens if and only if \( T \) is one-way resynchronizable). This concludes the proof of Theorem 2.

\[ \square \]