Nucleon parton distributions in a light-front quark model

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I. INTRODUCTION

In Refs. 1, 2 we proposed phenomenological light-front wave functions (LFWFs) for the nucleon, which produce a description of electromagnetic form factors of nucleons consistent with data and with the correct power behavior at higher scales 3, 4. The difference in the two papers 1, 2 concerns the modeling of the x-dependence, which has an impact on the scaling behavior of nucleon parton distributions. In the first case 1 the nucleon parton distributions have the correct x behavior at large scales, while at the initial scale μ ~ 1 GeV they were different from the results of the world data analysis. In the second paper 2, we improved the x-dependence of the LFWFs in a such way that the modified LFWFs produced the correct helicity-independent parton distributions at the starting point for the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution 5–8. In the latter case 2 we also had some freedom in setting up the LFWFs, because we did not consider helicity-dependent parton distributions. A similar application of the pion LFWFs, resulting in good agreement with data and in the correct scaling of form factors and parton distributions, was extensively studied in the literature starting from a pioneer paper by Brodsky et al. 9 and continuing by progress done by many groups in derivation of the LFWFs and its applications to nucleon phenomenology (see e.g. Refs. 11–27).

In the present manuscript we derive the nucleon LFWFs where now the x-dependence is encoded by knowledge of the helicity-independent qv(x) and helicity-dependent δqv(x) valence parton distributions. The main advantage of our approach is that the derived LFWF does not depend on phenomenological parameters like masses of quark/diquark, which are not directly related to QCD. Restricting to zero current quark masses we obtain a reasonable description of data on nucleon form factors. The paper is organized as follows. In Sect. II we construct the nucleon LFWFs, which will be used for the calculation of parton distributions and form factors using the presentations of these quantities in terms of the LFWFs. In Sect. III we collect the well-known decompositions of the nucleon Dirac and Pauli form factors, parton distributions (including longitudinal, transverse, Wigner and Husimi distributions) in terms of the LFWFs. In Sect. IV we present our numerical results and discussion. Finally, Sect. V contains our summary and conclusions. We have collected some technical material on the Wigner and Husimi parton distributions in the appendix.

II. NUCLEON LIGHT-FRONT WAVE FUNCTIONS

For simplicity we consider the quark-scalar diquark model, where the generic ansatz for the massless LFWFs at the initial scale μ0 = 1 GeV reads 1, 2

\[
\begin{align*}
\psi^+_{1q}(x, k_\perp) &= \varphi^{(1)}_q(x, k_\perp), \\
\psi^-_{1q}(x, k_\perp) &= \frac{k_1 + ik_2}{M_N} \varphi^{(2)}_q(x, k_\perp), \\
\psi^+_{-q}(x, k_\perp) &= \frac{k_1 - ik_2}{M_N} \varphi^{(2)}_q(x, k_\perp), \\
\psi^-_{-q}(x, k_\perp) &= \varphi^{(1)}_q(x, k_\perp),
\end{align*}
\]
where \( \varphi_q^{(1)} \) and \( \varphi_q^{(2)} \) are the LFWFs:

\[
\varphi_q^{(1)}(x, k_\perp) = \frac{4\pi}{M_N} \sqrt{q_v(x) + \delta q_v(x)} \sqrt{D_q^{(1)}(x)} \\
\times \exp\left(-\frac{k_\perp^2}{2M_N} D_q^{(1)}(x)\right),
\]

\[
\varphi_q^{(2)}(x, k_\perp) = \eta_q \frac{4\pi}{M_N} \sqrt{q_v(x) - \delta q_v(x)} \sqrt{D_q^{(2)}(x)} \\
\times \exp\left(-\frac{k_\perp^2}{2M_N} D_q^{(2)}(x)\right).
\]

Here \( M_N \) is the nucleon mass, \( q_v(x) \) and \( \delta q_v(x) \) are the helicity-independent and helicity-dependent valence quark parton distributions (for these quantities the exact expressions from a world data analysis at the initial scale are understood), \( D_q^{(1)} \) and \( D_q^{(2)} \) are the longitudinal wave functions, connected to the electromagnetic form factors of the nucleon, \( \eta_u = 1 \) and \( \eta_d = -1 \).

For simplicity we therefore choose a scale coinciding with the nucleon mass \( \Lambda = M_N \). Our functions \( \varphi_q^{(1)} \) and \( \varphi_q^{(2)} \) are normalized as

\[
\int \frac{d^2k_\perp}{16\pi^3} \left[ \varphi_q^{(1)}(x, k_\perp) \right]^2 = \frac{q_v(x) + \delta q_v(x)}{2},
\]

\[
\int \frac{d^2k_\perp}{16\pi^3} \frac{k_\perp^2}{M_N^2} \left[ \varphi_q^{(2)}(x, k_\perp) \right]^2 = \frac{q_v(x) - \delta q_v(x)}{2}.
\]

Note that the derived LWF is not symmetric under the exchange \( x \to 1 - x \). This asymmetry results from the matching of matrix elements of the bare electromagnetic current between the dressed LWF in light-front QCD and of the dressed electromagnetic current between hadronic wave functions in AdS/QCD.

Concerning the \( k_\perp \) dependence of the \( \varphi_q^{(1,2)} \) functions we use a specific functional form for them — Gaussian ansatz. However, a generalized ansatz for \( \varphi_q^{(1)} \) reads

\[
\varphi_q^{(1)}(x, k_\perp) = \frac{4\pi}{M_N} \sqrt{q_v(x) + \delta q_v(x)} \sqrt{D_q^{(1)}(x)} \\
\times \exp\left(-\frac{k_\perp^2}{2M_N} D_q^{(1)}(x)\right),
\]

\[
\varphi_q^{(2)}(x, k_\perp) = \eta_q \frac{4\pi}{M_N} \sqrt{q_v(x) - \delta q_v(x)} \sqrt{D_q^{(2)}(x)} \\
\times \exp\left(-\frac{k_\perp^2}{2M_N} D_q^{(2)}(x)\right).
\]

where the functions \( \psi_1 \) and \( \psi_2 \) must satisfy the normalization conditions following from Eq. (5)

\[
\int \frac{d^2k_\perp}{16\pi^3} \left[ \psi_1(-k_\perp^2) \right]^2 = \frac{\pi}{2},
\]

\[
\int \frac{d^2k_\perp}{16\pi^3} \frac{k_\perp^2}{M_N^2} \left[ \psi_2(-k_\perp^2) \right]^2 = \frac{\pi}{4}.
\]

III. LIGHT-FRONT DECOMPOSITIONS FOR THE NUCLEON QUANTITIES

A. Form factors and parton distributions

In this section we collect the well-known decompositions of the nucleon form factors and parton distributions in terms of the nucleon LFWFs. First we quote the connection of the nucleon Dirac and Pauli form factors \( F_{1,2}^N \) (\( N = p, n \)) with the valence quark distributions \( F_{1,2} \) (\( q = u, d \)) in nucleons with

\[
F_{i}^{p(n)}(Q^2) = \frac{2}{3} F_{i}^{u(d)}(Q^2) - \frac{1}{3} F_{i}^{d(u)}(Q^2).
\]

The valence quark distributions are related to the nucleon nonforward parton densities (NPDs) \( \tilde{H}^i(x, Q^2) \).
and $\mathcal{E}^q(x, Q^2)$ evaluated at zero skewness $\xi = 0$ as

$$F_1^q(Q^2) = \int_0^1 dx \mathcal{H}^q(x, Q^2),$$
$$F_2^q(Q^2) = \int_0^1 dx \mathcal{E}^q(x, Q^2),$$

where $Q^2 = -q^2 > 0$ is the Euclidean momentum squared. At $Q^2 = 0$ the NPDs are related to the quark densities — valence $q_v(x)$ and magnetic $q_m(x)$ as

$$\mathcal{H}^q(x, 0) = q_v(x), \quad \mathcal{E}^q(x, 0) = \mathcal{E}^q(x),$$

which are normalized as

$$n_q = F_1^q(0) = \int_0^1 dx q_v(x),$$
$$\kappa_q = F_2^q(0) = \int_0^1 dx \mathcal{E}^q(x),$$

where $\kappa_q$ is the anomalous quark magnetic moment.

The nucleon Sachs form factors $G_{E/M}^N(Q^2)$ and the electromagnetic radii $\langle r^2_{E/M} \rangle^N$ are given in terms of the Dirac and Pauli form factors as

$$G_E^N(Q^2) = F_1^N(Q^2) - \frac{Q^2}{4M_N^2} F_2^N(Q^2),$$
$$G_M^N(Q^2) = F_1^N(Q^2) + F_2^N(Q^2),$$

$$\langle r_E^2 \rangle^N = -6 \left. \frac{dG_E^N(Q^2)}{dQ^2} \right|_{Q^2=0},$$
$$\langle r_M^2 \rangle^N = -6 \left. \frac{dG_M^N(Q^2)}{dQ^2} \right|_{Q^2=0},$$

where $G_M^N(0) \equiv \mu_N$ is the nucleon magnetic moment.

The light-front representation [11, 13, 40] for the Dirac and Pauli quark form factors is

$$F_1^q(Q^2) = \int_0^1 dx \int \frac{d^2k_\perp}{16\pi^3} \left[ \psi_{\uparrow q}^+(x, k'_\perp) \psi_{\uparrow q}^+(x, k_\perp) \right],$$
$$F_2^q(Q^2) = -\frac{2M_N}{q^4 - i q^2} \int_0^1 dx \int \frac{d^2k_\perp}{16\pi^3} \left[ \psi_{\downarrow q}^+(x, k'_\perp) \psi_{\downarrow q}^+(x, k_\perp) \right],$$

where $k'_\perp = k_\perp + q_\perp(1-x)$. Here $\psi_{\lambda q}^N(x, k_\perp)$ are the LFWFs at the initial scale $\mu_0$ with specific helicities for the nucleon $\lambda_N = \pm$ and for the struck quark $\lambda_q = \pm$, where plus and minus correspond to $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively. We work in the frame with $q = (0, 0, q_\perp)$, and where the Euclidean momentum squared is $Q^2 = q^2_\perp$. For the initial scale we choose the value $\mu_0 \sim 1$ GeV which is used in the most of the global fits.

The expressions for the quark helicity-independent NPDs $\mathcal{H}^q$ and $\mathcal{E}^q$ in the nucleon read

$$\mathcal{H}^q(x, Q^2) = \frac{q_v(x)}{2} e^{-\frac{t^{(11)}_q(x, Q^2)}{2}} + \frac{q_m(x)}{2} e^{\frac{t^{(22)}_q(x, Q^2)}{2}} \times \left[ 1 - t^{(22)}_q(x, Q^2) \right],$$
$$\mathcal{E}^q(x, Q^2) = \mathcal{E}^q(x) e^{-\frac{t^{(11)}_q(x, Q^2)}{2}},$$

where

$$t^{(ij)}_q(x, Q^2) = \frac{Q^2}{4M_N^2} \left( \frac{2D^{(i)}_q(x)}{D^{(i)}_q(x) + D^{(j)}_q(x)} \right) (1 - x)^2.$$
B. Transverse momentum-dependent parton distributions

In the quark-diquark model, the light-front decomposition for the transverse momentum-dependent parton distributions (TMDs) is discussed in detail in Ref. 17 (see also Ref. 26). For recent progress in the extraction of TMDs from data, see e.g. Refs. 11-43. The set of the valence quark $T$-even TMDs for the case of the quark-scalar diquark model is given by [17]:

$$ f_1^{q^v}(x, k_\perp) = \frac{1}{16\pi^3} \left| \psi_{+q}^+(x, k_\perp) \right|^2 + \left| \psi_{-q}^+(x, k_\perp) \right|^2 $$

$$ = \frac{1}{16\pi^3} \left[ \left( \varphi_q^{(1)}(x, k_\perp) \right)^2 + \frac{k_\perp^2}{M_N^2} \left( \varphi_q^{(2)}(x, k_\perp) \right)^2 \right], \quad (26) $$

$$ g_1^{q^v}(x, k_\perp) = \frac{1}{16\pi^3} \left[ \left| \psi_{+q}^+(x, k_\perp) \right|^2 - \left| \psi_{-q}^+(x, k_\perp) \right|^2 \right] $$

$$ = \frac{1}{16\pi^3} \left[ \left( \varphi_q^{(1)}(x, k_\perp) \right)^2 - \frac{k_\perp^2}{M_N^2} \left( \varphi_q^{(2)}(x, k_\perp) \right)^2 \right], \quad (27) $$

$$ h_1^{q^v}(x, k_\perp) = \frac{1}{16\pi^3} \left[ \psi_{+q}^+(x, k_\perp) \psi_{-q}^+(x, k_\perp) \frac{M_N}{k_\perp^2} \right] + \psi_{+q}^+(x, k_\perp) \psi_{+q}^+(x, k_\perp) \frac{M_N}{k_\perp^2} \right] $$

$$ = \frac{1}{16\pi^3} \varphi_q^{(1)}(x, k_\perp) \varphi_q^{(2)}(x, k_\perp), \quad (28) $$

Using our expressions for the LFWFs we can express the TMDs through the PDFs

$$ f_1^{q^v}(x, k_\perp) = \frac{1}{4\pi M_N} \frac{q_v(x) + \delta q_v(x)}{2} D_q^{(1)}(x) $$

$$ g_1^{q^v}(x, k_\perp) = \frac{1}{4\pi M_N} \frac{q_v(x) - \delta q_v(x)}{2} \frac{k_\perp^2}{M_N^2} \left( D_q^{(2)}(x) \right)^2 $$

$$ \frac{k_\perp^2}{2M_N^2} h_1^{q^v}(x, k_\perp) = -\frac{F_2(x, k_\perp)}{2}, \quad (31) $$

where

$$ F_1(x, k_\perp) = \frac{1}{\pi M_N} \frac{q_v(x) + \delta q_v(x)}{2} D_q^{(1)}(x) $$

$$ \times e^{\frac{k_\perp^2}{\pi M_N^2} D_q^{(1)}(x)} $$

$$ F_2(x, k_\perp) = \frac{1}{\pi M_N} \frac{q_v(x) - \delta q_v(x)}{2} \frac{k_\perp^2}{M_N^2} \left( D_q^{(2)}(x) \right)^2 $$

$$ \times e^{\frac{k_\perp^2}{\pi M_N^2} D_q^{(2)}(x)} $$

$$ F_3(x, k_\perp) = \eta_q \sqrt{\frac{4M_N^2}{k_\perp^2} \tilde{F}_1(x, k_\perp) F_2(x, k_\perp)} $$

$$ \times e^{-\frac{k_\perp^2}{\pi M_N^2} \left( D_q^{(1)}(x) + D_q^{(2)}(x) \right)} \ . \quad (32) $$

Performing the $k_\perp$-integration over the TMDs with

$$ \langle \text{TMD} \rangle = \int d^2k_\perp \text{TMD}(x, k_\perp), $$

$$ \langle \text{TMD} \rangle = \int d^2k_\perp \frac{k_\perp^2}{2M_N^2} \text{TMD}(x, k_\perp) \quad (33) $$

results in the identities

$$ f_1^{q^v}(x) = h_1^{q^v}(x) = q_v(x), $$

$$ g_1^{q^v}(x) = \delta q_v(x), $$

$$ g_1^{\bar{q}^v}(x) = -h_1^{\bar{q}^v}(x) = \bar{q}_v(x) \frac{1 + \sigma_q(x)}{2(1 - x)}, $$

$$ h_1^{q^v}(x) = q_v(x) + \delta q_v(x), $$

$$ h_1^{\bar{q}^v}(x) = -q_v(x) + \delta q_v(x) \ . \quad (34) $$
Finally, the integration over \( x \) leads to the normalization conditions
\[
\int_0^1 dx f_1^{q\nu}(x) = \int_0^1 dx h_1^{q\nu}(x) = n_q,
\]
\[
\int_0^1 dx g_1^{q\nu}(x) = g_A^q,
\]
\[
\int_0^1 dx h_1^{q\nu}(x) = g_T^q,
\]
where \( g_T^q \) is the tensor charge.

Our TMD, independently on the longitudinal functions \( D_1^{(i)} \), satisfy all relations and inequalities found before in theoretical approaches (see detailed discussion in Refs. [26, 46–49]. In particular, our TMDs in agreement with QCD and other models [48, 49] (see also Ref. [26]) satisfy the following inequality relations:
\[
f_1^{q\nu}(x, k_{\perp}) > 0,
\]
\[
|g_1^{q\nu}(x, k_{\perp})| \leq |f_1^{q\nu}(x, k_{\perp})|,
\]
\[
|h_1^{q\nu}(x, k_{\perp})| \leq |f_1^{q\nu}(x, k_{\perp})|,
\]
\[
|g_1^{q\nu}(x, k_{\perp})| \leq |F_1^{q\nu}(x, k_{\perp})|,
\]
which follow from the simple positivity condition for our functions \( F_1(x, k_{\perp}) \) and \( F_2(x, k_{\perp}) \)
\[
[F_1(x, k_{\perp}) - F_2(x, k_{\perp})]^2 \geq 0.
\]

Additionally, we confirm the inequality between the tensor and axial charges found in lattice QCD and different model (see discussion in Refs. [50] and [26]) and the general inequality
\[
|h_1^{q\nu}(x, k_{\perp})| \geq |g_1^{q\nu}(x, k_{\perp})|
\]
observed before in the framework of parton model [50] and derived recently in the quark-diquark model in Ref. [26]. Finally, our TMDs satisfy the non-linear relation found in Ref. [50] and recently confirmed in Ref. [26]:
\[
h_1^{q\nu}(x, k_{\perp})h_1^{q\nu}(x) = \frac{1}{2} \left[ h_{1L}^{q\nu}(x) \right]^2.
\]

We would like to stress that the last inequality condition in Eq. \((38)\) relating \( g_1^{q\nu} \) and \( f_1^{q\nu} \) after integration over \( k_{\perp} \) is also fulfilled in our approach. In particular, after integration over \( k_{\perp} \) we get
\[
g_1^{q\nu}(x) = E(x) \frac{1 + \sigma_q(x)}{2(1 - x)} \leq F_1^{q\nu}(x) = \sqrt{\pi} D_1^{(1)}(x)
\]
\[
\times \left[ q_v(x) + \delta_q(x) q_v(x) - \delta_q(x) \right].
\]

The inequality \((40)\) is fulfilled because it is reduced to more trivial inequality
\[
[1 + \sigma_q(x)]^2 \geq \frac{8}{\pi} \sigma_q(x),
\]
which occurs because of
\[
[1 + \sigma_q(x)]^2 \geq 8\sigma_q(x)
\]
and
\[
\sqrt{8} > \frac{8}{\pi}.
\]

In Sect. [5] we present a plot where we compare our predictions for the \( g_1^{p\nu}(x) \) TMDs with corresponding upper limits defined by right-hand side of Eq. \((40)\).

### C. Wigner distributions

In light-front QCD the Wigner distributions read [27, 51–54]
\[
\rho^{\gamma}[\gamma](x, b_{\perp}, k_{\perp}; S) = \int \frac{d^2 \Delta}{4\pi^2} e^{-i\Delta \cdot b_{\perp}} W_q^{\gamma}[\gamma](x, \Delta_{\perp}, k_{\perp}; S),
\]
where \( W_q^{\gamma}[\gamma](x, \Delta_{\perp}, k_{\perp}; S) \) is the matrix element of the Wigner operator for \( \Delta^+ = 0 \) and \( z^+ = 0 \). The light-front decomposition of the Wigner matrix elements \( W_q^{\gamma}[\gamma](x, \Delta_{\perp}, k_{\perp}; S) \) is given by [58]
\[
W_q^{\gamma\gamma^*}(x, \Delta_{\perp}, k_{\perp}) = \frac{1}{16\pi^3} \left[ \psi_{q^+}(x, k_{\perp})\psi_{q^+}(x, k_{\perp}) + \psi_{q^-}(x, k_{\perp})\psi_{q^-}(x, k_{\perp}) \right],
\]
\[
W_q^{\gamma\gamma^*}(x, \Delta_{\perp}, k_{\perp}) = \frac{1}{16\pi^3} \left[ \psi_{q^+}(x, k_{\perp})\psi_{q^+}(x, k_{\perp}) - \psi_{q^-}(x, k_{\perp})\psi_{q^-}(x, k_{\perp}) \right],
\]
where \( k_{\perp} = k_{\perp} \pm (1 - x)\Delta_{\perp}/2 \).

Next we use the standard definitions of the Wigner distributions, specified by the nucleon helicity \( \lambda_N \) and the quark helicity \( \lambda_q \)[53]
\[
\rho_{\lambda_N\lambda_q}(x, b_{\perp}, k_{\perp}, \lambda_N e_z) = \frac{1}{2} \left[ \rho^{\gamma\gamma^*}(x, b_{\perp}, k_{\perp}, \lambda_N e_z) \right.
\]
\[
+ \lambda_q \rho^{\gamma\gamma^*}(x, b_{\perp}, k_{\perp}, \lambda_N e_z),
\]
which can be further expressed in terms of distributions where the proton and the struck quark are unpolarized.
The Fourier transforms \( \omega_{AB} = \frac{1}{2} \left[ \rho^q_{UU} + \lambda_N \rho^q_{LU} + \lambda_q \rho^q_{LL} + \lambda_N \lambda_q \rho^q_{LL} \right] \), where the quark orbital angular momentum (OAM) \( L \) is given by the integral representation over \( x, b_\bot, k_\bot \):

\[
\omega_{AB} = \int d^2b_\bot e^{i\Delta_\perp b_\bot} \rho^q_{AB}(x, b_\bot, k_\bot).
\]

The expressions for the Wigner distributions \( \rho^q_{AB} \) and \( \omega^q_{AB} \) in the light-front quark-diquark approach are listed in Appendix [A].

### D. Quark orbital angular momentum

Following Ji [55], we define the quark contribution to the nucleon angular momentum:

\[
J^q_z = L^q_z + S^q_z
\]

where the quark orbital angular momentum (OAM) \( L^q_z \) and internal spin \( S^q_z \) contributions are defined as

\[
L^q_z = \frac{1}{2} \int_0^1 dx \left( x \left[ H^q(x, 0) + \mathcal{E}^q(x, 0) \right] - H^q(x, 0) \right)
\]

\[
= \frac{1}{2} \int_0^1 dx \left( q_v(x) + \mathcal{E}^q(x) - \delta q_v(x) \right)
\]

and

\[
S^q_z = \frac{1}{2} \int_0^1 dx H^q(x, 0) = \frac{1}{2} \int_0^1 \delta q_v(x).
\]

Integrating the TMD \(-\frac{k^2}{2M_N^2} h_{11T}^{qz}(x, k_\bot)\) over \( x \) and \( k_\bot \) one can derive the quantity \( \mathcal{L}^q_z \)

\[
\mathcal{L}^q_z = -\frac{1}{2} \int_0^1 dx \int d^2k_\bot \frac{k^2}{2M_N^2} h_{11T}^{qz}(x, k_\bot)
\]

which is some quark models [18, 49, 56, 57] is equal to the quark OAM, but in general, in a gauge theory, it is not the case and \( \mathcal{L}^q_z \neq L^q_z \) (see discussion in Refs. 58 and 21). In particular, in our approach the quantity \( \mathcal{L}^q_z \) is not related to the quark OAM \( L^q_z \)

\[
\mathcal{L}^q_z = \frac{1}{2} \int_0^1 dx \left( q_v(x) - \delta q_v(x) \right) = \frac{n_q - g_A^q}{2} \\
\neq L^q_z.
\]

Using \( n_u = 2 \) and \( n_d = 1 \) we get relation between quantities \( \mathcal{L}^q_z \) and \( S^q_z \):

\[
L^u_z = 1 - S^u_z, \quad L^d_z = 1 - S^d_z.
\]

The next interesting quantity is the averaged quark orbital angular momentum (OAM) in a nucleon which is polarized in the z-direction [21, 22, 53]:

\[
l^q_z = \langle L^q \rangle^{(+)}(\hat{e}_z)
\]

\[
= \int_0^1 dx \int d^2k_\bot d^2b_\bot (b_\bot \times k_\bot)_z
\]

\[
\times \rho^q_{UU}(b_\bot, k_\bot, x, \hat{e}_z)
\]

\[
= \int_0^1 dx \int d^2k_\bot d^2b_\bot (b_\bot \times k_\bot)_z
\]

\[
\times \rho^q_{LU}(b_\bot, k_\bot, x)
\]

\[
= \frac{1}{2} \int_0^1 dx (1 - x) \left( q_v(x) - \delta q_v(x) \right).
\]

One can see that the \( l^q_z \) is related with TMD \( h_{11T}^{qz}(x, k_\bot) \) by the integral representation over \( x \) and \( k_\bot \) as

\[
l^q_z = -\frac{1}{2} \int_0^1 dx \int d^2k_\bot \frac{k^2}{2M_N^2} (1 - x) h_{11T}^{qz}(x, k_\bot).
\]

Another relevant quantity is the correlation between the quark spin and the orbital angular momentum [21, 22, 53] with

\[
C^q_z = \int_0^1 dx \int d^2k_\bot d^2b_\bot (b_\bot \times k_\bot)_z
\]

\[
\times \rho^q_{UL}(b_\bot, k_\bot, x),
\]

which in quark-scalar diquark model [52] is opposite to the quantity \( l^q_z \) with

\[
C^q_z = -l^q_z
\]

because of \( \rho^q_{UL}(b_\bot, k_\bot, x) = -\rho^q_{LU}(b_\bot, k_\bot, x) \). It is also confirmed in our calculations.
E. Husimi distribution

Finally, we consider the Husimi distribution function for the nucleon, which was recently discussed in detail in Refs. [60, 61]. As was stressed in [60, 61] this distribution is better behaved and positive in comparison to the Wigner distribution. It also gives a probabilistic interpretation and can be used to define the entropy of the nucleon as a measure of the complexity of the partonic structure. It also could be connected to the color glass condensate approach at small $x$.

The Husimi distribution $h_{AB}^q(x, b_{\perp}, k_{\perp})$ is defined as the integral of the Wigner distribution $\rho_{AB}^q(x, b_{\perp}, k_{\perp})$ over the impact parameter $b_{\perp}$ and the transverse momentum $k_{\perp}$

$$h_{AB}^q(x, b_{\perp}, k_{\perp}) = \frac{1}{\pi^2} \int d^2 b'_{\perp} d^2 b_{\perp} e^{-\frac{(b_{\perp}-b'_{\perp})^2}{2}} \times e^{-\frac{r^2(k_{\perp}-k'_{\perp})^2}{2}} \rho_{AB}^q(x, b_{\perp}', k'_{\perp})$$

where $1/r^2 = \langle k_{\perp}^2 \rangle$ is the average transverse momentum squared.

Note that the double moment of the Husimi distribution $h_{UU}^q$ and $h_{LL}^q$ is the ordinary PDF:

$$f_{q}^U(x) = \int d^2 b_{\perp} d^2 k_{\perp} h_{UU}^q(x, b_{\perp}, k_{\perp})$$

$$= \int d^2 b_{\perp} d^2 k_{\perp} h_{UU}^q(x, b_{\perp}, k_{\perp}),$$

$$g_{LL}^q(x) = \int d^2 b_{\perp} d^2 k_{\perp} h_{LL}^q(x, b_{\perp}, k_{\perp})$$

$$= \int d^2 b_{\perp} d^2 k_{\perp} h_{LL}^q(x, b_{\perp}, k_{\perp}).$$

In case of $h_{UU}^q$ and $h_{LL}^q$, the double moments equal zero.

In quantum mechanics the Husimi distribution $h_{QM}$ is positive definite and one can define the so-called the Husimi-Wahrl entropy [62], which in our case can be extended to define the entropy of the nucleon [60]

$$S(x) = -\int d^2 b_{\perp} d^2 k_{\perp} h(x, b_{\perp}, k_{\perp}) \times \log \left[ h(x, b_{\perp}, k_{\perp}) \right].$$

In particular, it is convenient to define two quantities

$$S_1^q(x) = -\int d^2 b_{\perp} d^2 k_{\perp} h_{q}^1(x, b_{\perp}, k_{\perp})$$

$$\times \log \left[ h_{q}^1(x, b_{\perp}, k_{\perp}) \right],$$

where $h_{q}^1 = (h_{UU}^q \pm h_{LL}^q)/2$. Expressions for $S_1^q(x)$ are listed in the Appendix [A].

IV. RESULTS

In this paper we do not pretend on a precise analysis of the available nucleon data. Instead we first would like to illustrate how our method works. For this purpose we use the results for the NLO helicity-independent and helicity-dependent parton distributions at $\mu_{NLO}^2 = 0.40$ GeV$^2$ from Refs. [63] and [64] as input:

$$q(x) = q_u(x) + \bar{q}(x), \quad \delta q(x) = \delta q_u(x) + \delta \bar{q}(x),$$

$$x u_{u}(x) = 0.632 x^{0.43} (1 - x)^{3.09} (1 + 18.2 x),$$

$$d_{u}(x) = 0.394 (1 - x) u_{u}(x),$$

$$x (\bar{u} + d)(x) = 1.24 x^{0.20} (1 - x)^{8.5} (1 - 2.3 \sqrt{x} + 5.7 x),$$

$$x (d - \bar{u})(x) = 0.2 x^{0.43} (1 - x)^{12.4} (1 - 13.3 \sqrt{x} + 60 x),$$

$$x \delta u_{u}(x) = 2.04 x^{0.97} (1 - x)^{0.64} u(x),$$

$$x \delta d_{u}(x) = -2.709 x^{1.26} (1 - x)^{1.06} d(x),$$

$$x \delta \bar{u}(x) = 1.727 x^{0.73} (1 - x)^{2.00} \bar{u}(x),$$

$$\delta \bar{d}(x) = \delta \bar{u}(x) \frac{x \delta \bar{u}(x)}{\delta d(x)}.$$ (64)

The $D_1^q(x)$ are specified as

$$D_1^{(1)}(x) = A_q \log(1/x) (x + 0.001)^q (1 - x)^{3/2},$$

$$\sigma_q(x) = N_q e^{-\gamma q x} x^{\alpha q} (1 - x)^{3/2},$$ (65)

where

$$A_u = 6.3385, \quad A_d = 1.71396,$$

$$\alpha_u = 0.37, \quad \alpha_d = -0.31, \quad \beta_u = 0.09, \quad \beta_d = -0.50,$$

$$N_u = 12.6, \quad N_d = 2.8, \quad \gamma_u = 3.70, \quad \gamma_d = 0.45,$$

$$\alpha_u = 0.045, \quad \alpha_d = 0, \quad \beta_u = -0.60, \quad \beta_d = 0.$$ (66)

In Table I we present our results for the valence quark properties ($J_1^q$, $L_2^q$, $L_2^\perp$, $h_\perp^q$, $C_2^q$, $k_q$) and compare them to results of other calculations (light-cone constituent quark model (LCCQM) and chiral quark-soliton model (χQSM)) [21]. One can see that most of our results are different from the predictions of the LCCQM and χQSM approaches. This is caused by the difference in the magnetization PDFs $E_q(x)$ (anomalous quark magnetic moments $\kappa_q$), helicity-dependent PDFs $\delta q_{\perp}(x)$ (quark contributions to internal spin $S^\perp_q$). Note that our magnetization PDFs are consistent with data for nucleon electromagnetic form factors and helicity-dependent PDFs $\delta q_{\perp}(x)$ are taken from Refs. [63] [64]. Also we would like to stress that our results for the quantities $L_2^q$ are clearly understood because they are related to the quantities $S_2^q$ by the relations [64].

In Figs. [1] [22] we plot the results for the $x$-dependence of the unpolarized and polarized PDFs, TMDs, Wigner and Husimi distributions, and indicate selected results for the quark and nucleon electromagnetic form factors. The data on the quark decomposition of the nucleon form factors are taken from Refs. [63] [64]. In particular, in Fig. [3] we show our predictions for magnetization PDFs $E^q$ and compare them with results of Ref. [63]. In Fig. [5] we present a comparison of our predictions for $x g_{1T}^q(x)$ quark TMDs with the corresponding upper limits $x F_1^q(x)$. One can see that our results for $g_{1T}^q(x)$ are consistent with model-independent inequalities derived.
TABLE I: Valence quark properties.

| Quantity | LCCQM [21] | hQCM [21] | Our |
|----------|------------|------------|-----|
| $J_u^z$  | 0.569      | 0.566      | 0.358 |
| $J_d^z$  | $-0.069$   | $-0.066$   | $-0.010$ |
| $L_u^z$  | 0.071      | $-0.008$   | 0.055 |
| $L_d^z$  | 0.055      | 0.077      | $-0.001$ |
| $S_u^z$  | 0.498      | 0.574      | 0.303 |
| $S_d^z$  | $-0.124$   | $-0.143$   | $-0.009$ |
| $C_u^z$  | 0.169      | 0.093      | 0.697 |
| $C_d^z$  | $-0.042$   | $-0.023$   | 0.509 |
| $l_u^z$  | 0.131      | 0.073      | 0.598 |
| $l_d^z$  | $-0.005$   | $-0.004$   | 0.404 |
| $K_u^z$  | 0.227      | 0.130      | $-0.598$ |
| $K_d^z$  | 0.187      | 0.109      | $-0.404$ |

in Ref. [48]. Note that before in Sect. [11] we proved it analytically. Our Wigner distributions are negative for longitudinal-logitudinal polarized case of the $d$-quark and for unpolarized-longitudinal polarized case for both quark flavors. Note that negative Wigner distributions have been obtained in some approaches, e.g., after including the gluons (see discussion in Refs. [60, 61, 69]).

V. CONCLUSION

We want to summarize the main result of our paper. In the quark-scalar diquark picture we propose LFWFs for the nucleon which analytically reproduce the quark PDFs in the nucleon at the initial scale $\mu \sim 1$ GeV. Our LFWFs contain four longitudinal wave functions $D_q^{(i)}$, $q = u, d$ and $i = 1, 2$ depending on the $z$ variable, which are fixed from the analysis of nucleon form factors. Then we present a list of different types of nucleon parton distributions (TMDs, Wigner and Husimi distributions) in terms of the quark PDFs and the longitudinal functions $D_q^{(i)}$. Finally, we present the numerical analysis for the quark distributions in the nucleon, we also indicate selected results for the quark and nucleon form factors using a specific ansatz for the NLO helicity-independent and helicity-dependent parton distributions at $\mu_{NLO} = 0.40$ GeV$^2$ [63, 64]. The resulting valence quark densities in the nucleon (e.g., TMDs) can be evolved to higher scales and can be compared to results for these quantities extracted in a data analysis.

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Appendix A: Wigner and Husimi parton distributions in the light-front quark model

The Wigner distributions $\rho_{UU}^q(b_\perp,k_\perp,x)$ and $\omega_{UU}^q(\Delta_\perp,k_\perp,x)$ read

$$
\rho_{UU}^q = \frac{1}{\pi^2(1-x)^2} \left[ q_v(x) + \delta q_v(x) \right] e^{-\frac{k_t^2}{M_N^2} \alpha_q^{(1)}(x)} e^{-b_\perp^2 M_N^2 \beta_q^{(1)}(x)} + \frac{q_v(x) - \delta q_v(x)}{2} e^{-\frac{k_t^2}{M_N^2} \alpha_q^{(2)}(x)} e^{-b_\perp^2 M_N^2 \beta_q^{(2)}(x)} \left[ -1 + \frac{k_t^2}{M_N^2} \alpha_q^{(2)}(x) + b_\perp^2 M_N^2 \beta_q^{(2)}(x) \right], \tag{A1}
$$

$$
\rho_{LL}^q = \frac{1}{\pi^2(1-x)^2} \left[ q_v(x) + \delta q_v(x) \right] e^{-\frac{k_t^2}{M_N^2} \alpha_q^{(1)}(x)} e^{-b_\perp^2 M_N^2 \beta_q^{(1)}(x)} - \frac{q_v(x) - \delta q_v(x)}{2} e^{-\frac{k_t^2}{M_N^2} \alpha_q^{(2)}(x)} e^{-b_\perp^2 M_N^2 \beta_q^{(2)}(x)} \left[ -1 + \frac{k_t^2}{M_N^2} \alpha_q^{(2)}(x) + b_\perp^2 M_N^2 \beta_q^{(2)}(x) \right], \tag{A2}
$$

$$
\rho_{UL}^q = -\rho_{LU}^q = \frac{1}{\pi^2(1-x)^2} e^{i\gamma} k^i b^j \left[ q_v(x) - \delta q_v(x) \right] e^{-\frac{k_t^2}{M_N^2} \alpha_q^{(2)}(x)} e^{-b_\perp^2 M_N^2 \beta_q^{(2)}(x)} \tag{A3}
$$

and

$$
\omega_{UU}^q(\Delta_\perp,k_\perp,x) = \frac{1}{\pi M_N^2} \left[ q_v(x) + \delta q_v(x) \right] \frac{k_t^2 + \Delta_t^2 (1-x)^2}{M_N^2} \alpha_q^{(1)}(x) e^{-\frac{k_t^2}{M_N^2} \alpha_q^{(1)}(x)} + \frac{q_v(x) - \delta q_v(x)}{2} \left( \alpha_q^{(2)}(x) \right)^2 k_t^2 + \frac{\Delta_t^2 (1-x)^2}{M_N^2} e^{-\frac{k_t^2}{M_N^2} \alpha_q^{(2)}(x)} - \frac{q_v(x) - \delta q_v(x)}{2} \left( \alpha_q^{(2)}(x) \right)^2 k_t^2 + \frac{\Delta_t^2 (1-x)^2}{M_N^2} e^{-\frac{k_t^2}{M_N^2} \alpha_q^{(2)}(x)} \right], \tag{A4}
$$

$$
\omega_{LL}^q(\Delta_\perp,k_\perp,x) = \frac{1}{\pi M_N^2} \left[ q_v(x) + \delta q_v(x) \right] \frac{k_t^2 + \Delta_t^2 (1-x)^2}{M_N^2} \alpha_q^{(2)}(x) e^{-\frac{k_t^2}{M_N^2} \alpha_q^{(2)}(x)} - \frac{q_v(x) - \delta q_v(x)}{2} \left( \alpha_q^{(2)}(x) \right)^2 k_t^2 + \frac{\Delta_t^2 (1-x)^2}{M_N^2} e^{-\frac{k_t^2}{M_N^2} \alpha_q^{(2)}(x)} \right], \tag{A5}
$$

$$
\omega_{UL}^q(\Delta_\perp,k_\perp,x) = -\omega_{LU}^q(\Delta_\perp,k_\perp,x) = \frac{1}{\pi M_N^2} i e^{i\gamma} k^i \Delta_\perp^j \left[ q_v(x) - \delta q_v(x) \right] e^{-\frac{k_t^2}{M_N^2} \alpha_q^{(2)}(x)} e^{-b_\perp^2 M_N^2 \beta_q^{(2)}(x)} \tag{A6}
$$

where $\alpha_q^{(i)}(x) = D_q^{(i)}(x)$, $\beta_q^{(i)}(x) = 1/(1-x)^2 D_q^{(i)}(x)$, $e^{12} = -e^{21} = 1$.

The integrals over the Wigner distributions are related to the TMDs, NPDFs and PDFs by

$$
\int d^2b_\perp \rho_{UU}^q(b_\perp,k_\perp,x) = f^q_\perp(x,k_\perp), \tag{A7}
$$

$$
\int d^2b_\perp \rho_{LL}^q(b_\perp,k_\perp,x) = g_{LL}^q(x,k_\perp), \tag{A7}
$$

$$
\int d^2b_\perp \omega_{UU}^q(\Delta_\perp,k_\perp,x) = \mathcal{H}(x,0,\Delta_\perp^2), \tag{A8}
$$

$$
\int d^2b_\perp \omega_{LL}^q(\Delta_\perp,k_\perp,x) = \tilde{\mathcal{H}}(x,0,\Delta_\perp^2), \tag{A8}
$$

and

$$
\int d^2k_\perp d^2b_\perp \rho_{UU}^q(x,k_\perp,b_\perp) = q_v(x) = f_q^v(x), \tag{A9}
$$

$$
\int d^2k_\perp d^2b_\perp \rho_{LL}^q(x,k_\perp,b_\perp) = \delta q_v(x) = g_q^v(x) \tag{A9}
$$
where

\[ \int_0^1 dx \int d^2 k_\perp d^2 b_\perp \rho_{UU}^q(x, k_\perp, b_\perp) = n_q, \]

\[ \int_0^1 dx \int d^2 k_\perp d^2 b_\perp \rho_{LL}^q(x, k_\perp, b_\perp) = g_\perp^q, \]

\[ \int_0^1 dx \int d^2 k_\perp d^2 b_\perp \rho_{UL}^q(x, k_\perp, b_\perp) = \frac{1}{1} \int_0^1 dx \int d^2 k_\perp d^2 b_\perp \rho_{LU}^q(x, k_\perp, b_\perp) = 0. \]

The Husimi parton distributions are given by

\[ h_{UU}^q(x, b_\perp, k_\perp) + h_{LL}^q(x, b_\perp, k_\perp) = \frac{M_N^2 l^2}{\pi^2} (q_v(x) + \delta q_v(x)) \rho_q^{(1)}(x) \sigma_q^{(1)}(x) e^{-k_\perp^2 l^2} e^{-b_\perp^2 M_N^2 \sigma_q^{(1)}(x)}, \]

\[ h_{UU}^q(x, b_\perp, k_\perp) - h_{LL}^q(x, b_\perp, k_\perp) = \frac{M_N^2 l^2}{\pi^2} (q_v(x) - \delta q_v(x)) \rho_q^{(2)}(x) \sigma_q^{(2)}(x) e^{-k_\perp^2 l^2} e^{-b_\perp^2 M_N^2 \sigma_q^{(2)}(x)} \times \left[ 1 + \frac{M_N^2 l^2}{D_q^{(2)}(x)} \rho_q^{(2)}(x) \left( k_\perp^2 l^2 \rho_q^{(2)}(x) - 1 \right) \right] + D_q^{(2)}(x) \sigma_q^{(2)}(x)(1-x)^2 \left( b_\perp^2 M_N^2 \sigma_q^{(2)}(x) - 1 \right), \]

\[ h_{UL}^q(x, b_\perp, k_\perp) = -h_{LU}^q(b_\perp, k_\perp, x) \]

\[ = \frac{1}{\pi^2} e^{ij} k_\perp^i b_\perp^j \left[ q_v(x) - \delta q_v(x) \right] (1-x) M_N^4 l^4 \left( \rho_q^{(2)}(x) \sigma_q^{(2)}(x) \right)^2 \times e^{-k_\perp^2 l^2} e^{-b_\perp^2 M_N^2 \sigma_q^{(2)}(x)}, \]

where

\[ \rho_q^{(i)}(x) = \frac{D_q^{(i)}(x)}{M_N^2 l^2 + D_q^{(i)}(x)(1-x)^2}, \quad \sigma_q^{(i)}(x) = \frac{1}{M_N^2 l^2 + D_q^{(i)}(x)(1-x)^2}. \]

The expressions for the entropies of the nucleon \( S_q^P(x) \) are given by

\[ S_q^P(x) = (q_v(x) + \delta q_v(x)) \left[ 1 - \frac{1}{2} \log \left( \frac{q_v(x) + \delta q_v(x)}{2\pi^2} \right) \right], \]

\[ S_q^P(x) = (q_v(x) - \delta q_v(x)) \left[ 1 - \frac{1}{2} \log \left( \frac{q_v(x) - \delta q_v(x)}{2\pi^2} \right) \right] \]

\[ - \frac{1}{2} \log(B) - \frac{A_1 + A_2}{4} \left( A_1 \int_0^\infty \frac{dt e^{-t}}{A_1 t + B} + A_2 \int_0^\infty \frac{dt e^{-t}}{A_2 t + B} \right) \]

\[ - \frac{A_1^2 + A_2^2}{4} \int_0^\infty \int_0^\infty \frac{dt_1 dt_2 e^{-(t_1 + t_2)}}{A_1 t_1 + A_2 t_2 + B} \]

where

\[ A_1 = \frac{D_q^{(2)}(x)(1-x)^2}{M_N^2 l^2 + D_q^{(2)}(x)(1-x)^2}, \quad A_2 = \frac{M_N^2 l^2}{M_N^2 l^2 + D_q^{(2)}(x)}, \quad B = 1 - A_1 - A_2. \]

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FIG. 1: $u$ quark PDFs multiplied with $x$.

FIG. 2: $d$ quark PDFs multiplied with $x$.

FIG. 3: quark magnetization PDFs: $x\xi^q$ - our results, $x\xi^q_{GP\text{RV}}$ - results of Ref. 68.

FIG. 4: $u$ quark TMDs multiplied with $x$.

FIG. 5: $d$ quark TMDs multiplied with $x$.

FIG. 6: Comparison our predictions for $xg_{1T}^q(x)$ quark TMDs multiplied with corresponding upper limits $xF_{1T}^q(x)$.

FIG. 7: Dirac $u$ quark form factor multiplied by $Q^4$.

FIG. 8: Dirac $d$ quark form factor multiplied by $Q^4$. 
FIG. 9: Pauli u quark form factor multiplied by $Q^4$.

FIG. 10: Pauli d quark form factor multiplied by $Q^4$.

FIG. 11: Dirac proton form factor multiplied by $Q^4$.

FIG. 12: Dirac neutron form factor multiplied by $Q^4$.

FIG. 13: Wigner distribution $\rho^U_{UU}(x, b_{\perp}, k_{\perp})$ at $x = 0.5$, $k_x = k_y = 0.5$ GeV.

FIG. 14: Wigner distribution $\rho^U_{UU}(x, b_{\perp}, k_{\perp})$ at $x = 0.5$, $k_x = k_y = 0.5$ GeV.

FIG. 15: Wigner distribution $\rho^L_{LL}(x, b_{\perp}, k_{\perp})$ at $x = 0.5$, $k_x = k_y = 0.5$ GeV.

FIG. 16: Wigner distribution $\rho^L_{LL}(x, b_{\perp}, k_{\perp})$ at $x = 0.5$, $k_x = k_y = 0.5$ GeV.
FIG. 17: Wigner distribution $\rho_{UL}(x, b_\perp, k_\perp)$ at $x = 0.5$, $k_x = k_y = 0.5$ GeV.

FIG. 18: Wigner distribution $\rho_{UL}(x, b_\perp, k_\perp)$ at $x = 0.5$, $k_x = k_y = 0.5$ GeV.

FIG. 19: Husimi distribution $h^u_{UL}(x, b_\perp, k_\perp)$ at $x = 0.5$, $l = 1$ GeV$^{-1}$, $k_x = k_y = 0.5$ GeV.

FIG. 20: Husimi distribution $h^d_{UL}(x, b_\perp, k_\perp)$ at $x = 0.5$, $l = 1$ GeV$^{-1}$, $k_x = k_y = 0.5$ GeV.

FIG. 21: Husimi distribution $h^u_{UL}(x, b_\perp, k_\perp)$ at $x = 0.5$, $l = 1$ GeV$^{-1}$, $k_x = k_y = 0.5$ GeV.

FIG. 22: Husimi distribution $h^d_{UL}(x, b_\perp, k_\perp)$ at $x = 0.5$, $l = 1$ GeV$^{-1}$, $k_x = k_y = 0.5$ GeV.

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