Kinematical Hilbert Spaces for Fermionic and Higgs Quantum Field Theories

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Abstract

We extend the recently developed kinematical framework for diffeomorphism invariant theories of connections for compact gauge groups to the case of a diffeomorphism invariant quantum field theory which includes besides connections also fermions and Higgs fields. This framework is appropriate for coupling matter to quantum gravity.

The presence of diffeomorphism invariance forces us to choose a representation which is a rather non-Fock-like one: the elementary excitations of the connection are along open or closed strings while those of the fermions or Higgs fields are at the end points of the string.

Nevertheless we are able to promote the classical reality conditions to quantum adjointness relations which in turn uniquely fixes the gauge and diffeomorphism invariant probability measure that underlies the Hilbert space.

Most of the fermionic part of this work is independent of the recent preprint by Baez and Krasnov and earlier work by Rovelli and Morales-Tecótl because we use new canonical fermionic variables, so-called Grassman-valued half-densities, which enable us to solve the difficult fermionic adjointness relations.

1 Introduction

A fair amount of intuition about the kinematical structure of quantum field theories comes from free, scalar quantum field theories in a Minkowski background. In this paper we will see that this intuition coming from free scalar field theories is misleading once we give up one or both of the following two essential ingredients of this field theory:
1) The quantum configuration space is a vector space,
2) There is available, a fixed (Minkowski) background metric.

More concretely, we have the following:

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The Osterwalder-Scrader axioms of constructive quantum field theory \[1\] are first of all only appropriate for kinematically linear theories, that is, the space of quantum fields is a vector space (before imposing the quantum dynamics). This follows from the fact that right from the beginning the measure on the space of quantum fields is required to be supported on the space of tempered distributions $\mathcal{S}'$ on $\mathbb{R}^{d+1}$ where $d$ is the dimension of a spacelike hypersurface in Minkowski space. This is, of course, very natural in case that the space of classical fields is a vector space as well (before imposing the equations of motion), that is, for the case of scalar fields. That the quantum configuration space is required to be $\mathcal{S}'$ is furthermore justified 1) by the fact that the Gaussian measure underlying, say, the Klein-Gordon field is actually supported on $\mathcal{S}'$ and 2) by the Bochner-Minlos theory \[2\] which tells us that under natural assumptions about the characteristic functional of any measure (satisfied by free scalar field theories) the support is guaranteed to be contained in $\mathcal{S}'$.

Secondly, the Euclidean group of spacetime plays an important role in that the measure is required to be invariant under the motions of the Euclidean group and that the time translation subgroup acts ergodically on the measure space. In particular, the discrete time reflection subgroup enables us to state the condition of “reflection positivity” \[1\] by means of which one makes contact with physics, that is, one constructs a Hamiltonian and a Hilbert space. Again, all those properties are satisfied by the free scalar field theories and even some interacting scalar field theories such as for instance the $P(\phi)^2$ theories.

However, what if the kinematical structure stated above is absent? We are particularly interested in a diffeomorphism invariant field theory on a manifold with a fixed topology and whose classical configuration space is possibly non-linear. Then there is no Euclidean group but there is the diffeomorphism group and the measure of such a field theory should therefore be diffeomorphism invariant, rather than Euclidean invariant. In general the Euclidean group will not even be a subgroup of the diffeomorphism group in question.

Diffeomorphism invariant field theories are not at all of academic interest only, rather, they are the only physically interesting field theories! This is because the physical theory of nature is not the standard model but the standard model (or beyond) coupled to the fully dynamical gravitational field which is the theory of quantum gravity. This field theory is indeed four-diffeomorphism invariant and contains kinematically non-linear fields such as gauge fields, Higgs fields and the gravitational field.

Since a diffeomorphism invariant quantum field theory lacks the whole arsenal of techniques that come with the presence of a fixed, non-dynamical Minkowski metric (the Poincaré group as symmetry group fixes vacuum and Hamiltonian operator, determines the irreducible representations under which the fields transform, the whole Wightman axiom system, a notion of time and so forth) we must expect that its underlying kinematical structure is radically different from what one is used to from Euclidean field theory (whose Wick rotation, another process that depends on the presence of a Minkowski background metric, results in the Wightman field theory). In particular, we do not expect the measure to be supported on $\mathcal{S}'$ any longer.

An already well-established example for such a departure from $\mathcal{S}'$ is provided by diffeomorphism invariant theories of connections for a compact gauge group in $d+1$ dimensions. In \[3\] such theories were canonically quantized and the representation of the canonical commutation relations that was chosen is generated by the Abelian $C^*$ algebra $W$ of Wilson loop observables via the GNS construction \[4\]. The quantum configuration space of distributional gauge fields $\mathcal{A}$ (or $\mathcal{A}/\mathcal{G}$ after moding by the gauge freedom) is naturally
identified with the compact Hausdorff space that one gets from \( W \) by the Gel’fand isomorphism (recall that an Abelian \( C^\star \) algebra is isometric isomorphic with the algebra of continuous functions on a compact Hausdorff space). The Hilbert space of this class of theories, in other words the measure on \( \mathfrak{A} \), is uniquely determined by promoting the reality structure of the classical phase space into adjointness relations with respect to the inner product.

What is interesting from the point of view of constructive field theory whose Hilbert space is determined by a measure on \( \mathcal{S}'(\mathbb{R}^d) \) is that the space \( \mathfrak{A} \) is 1) not a vector space, 2) not a subset of of \( \mathcal{S}'(\mathbb{R}^d) \) and 3) \( \mathcal{S}'(\mathbb{R}^d) \) is not contained in \( \mathfrak{A} \). Thus, as argued above, taking the non-linear structure of the quantum configuration space seriously we obtain a fairly non-standard Hilbert space in any spatial dimension \( d \) which still represents the canonical commutation relations faithfully. Of course, from the point of view of, say, the free Maxwell field (scattering processes) this Hilbert space is worthless (except in \( d = 2 \)) because it does not coincide with Fock space. However, it turns out as shown in \[9, 10]\] that for a diffeomorphism invariant theory the Hilbert space constructed is very useful: all the solutions of the diffeomorphism constraint are bona fide distributions on a dense space of test functions and therefore standard procedures known as “group averaging techniques” \[11, 12\] lead to a well-defined inner product on this space of distributional solutions \[3, 13\]. The fact that for \( d = 2 \) also Yang-Mills theory is represented on this Hilbert space \[10\] can be traced back to the fact that this theory is almost diffeomorphism invariant, it is invariant under area-preserving diffeomorphisms.

In this paper we extend the canonical framework of \[3\], which applies to any kind of gauge field theory, to the rest of the matter fields, specifically, we are looking at fermionic and Higgs field theories (including standard scalar fields which do not transform under any kind of gauge group besides the diffeomorphism group). Notice that \[3\] also applies to quantum gravity when formulated as a dynamical theory of connections \[13\]. We need it in its manifestly real formulation as advocated, for the Lorentzian signature, first in \[14\] because \[3\] only applies when the gauge group is compact (in this case \( SU(2) \)).

We develop the framework for the Higgs field, in the diffeomorphism invariant context, by pushing the analogy between a usual path holonomy and a “point holonomy” constructed from a Higgs Field (see below) and can therefore translate step by step the framework \[4, 5, 6, 7, 8\] developed for gauge fields to arrive at the Higgs field Hilbert space.

For the Fermion Field two works are to be mentioned: in \[15\] the authors introduce an algebra of observables for complex Ashtekar gravity and the Maxwell field coupled to fermions. The fermions couple by making insertions at the endpoints of Wilson lines. The authors compute the Poisson and commutator algebra of this (overcomplete) set of gauge invariant observables. The framework was incomplete because no inner product for complex Ashtekar gravity was known at that time. Recently, the viewpoint for the kinematical framework of quantum gravity has shifted towards a manifestly real formulation as advocated in \[14\]: This is because 1) the rigorous kinematical framework of \[1, 3, 4, 8\] can be employed and 2) the Wheeler-DeWitt constraint operator can be completely rigorously formulated while with complex variables \[16, 17, 18\] this turned out to be too difficult. Moreover, the same techniques can be employed to quantize a) Euclidean 2+1 gravity in the real-valued Ashtekar- (rather than Witten-) formulation \[13\] giving the expected results and b) the ADM energy \[20\].
The authors of [21] have followed this trend and formulated a kinematical framework for
diffeomorphism invariant theories of fermions coupling to real-valued Ashtekar gravity and
other gauge fields. That is, they define a natural inner product by means of the Berezin
integral, show that a certain class of the operators defined in [13] are well-defined on
it and indicate how to define diffeomorphism invariant distributions involving fermions
along the lines of [3]. However, certain difficulties having to do with the awkward reality
structure of the fermion field make their framework incomplete in the sense that most
of the interesting operators defined in [15] do not have an adjoint on the Hilbert space
defined. In particular, this prevented the authors from proving that their inner product
is selected by the classical reality conditions. Since the adjointness relations on an inner
product are the only conditions that we can impose in lack of a background structure (in
constructive field theory it is an axiom that the inner product be Poincaré invariant which
usually fixes it uniquely) it is unclear from their framework whether their inner product
is the appropriate one.

In this paper we define new fermionic canonical variables by casting the fermion field
into a half-density. Not only does this trivialize the classical reality conditions on the
fermion field, it is also forced on us:

If one does not work with half densities then it turns out that
a) The gravitational connection becomes complex valued. Thus, the gravitational gauge
   group would become become $SL(2, \mathbb{C})$ which is non-compact and would make the arsenal
   of techniques developed in [4, 5, 6, 7, 8, 3] inapplicable.

b) The faithful implementation of the reality conditions on scalar fermion fields at the
   quantum level is inconsistent with the implementation of the canonical Anti-Poisson
   brackets as observed in [21].

With the fermionic half-densities on the other hand we can then basically use the natural
Berezin integral also employed in [21] to define the inner product. However, now we
can rigorously compute adjoints for any, not even gauge-invariant operator constructed
from fermion variables and then prove that the inner product is uniquely selected by the
reality conditions. We do not need to deal with the complicated path observables defined
in [13, 21] in order to establish this result because we can prove everything at the
non-gauge-invariant level.

The kinematical framework developed in this paper is applied in [22] which extends
the dynamical framework of [16, 17, 18, 13, 20] to the standard model coupled to gravity
and other kind of matter.

The plan of the paper is as follows:

In section 2 we introduce the notation and review [4, 5, 6, 7, 8, 3]. Since we are using
the canonical formalism, we assume that the four-dimensional manifold is locally of the
form $M = \mathbb{R} \times \Sigma$ where $\Sigma$ can be any three-dimensional manifold. The classical
configuration subspace of the phase space can then be coordinatized by functions on $\Sigma$
which take values, respectively, in

a) the space of $SU(2) \times G$ connections $\omega_a$ for the gravitational and gauge field where $G$
is a compact gauge group,
b) the space of $Lie(G)$-valued scalars $\phi$ for the Higgs field and
c) the space of Grassman-valued fields $\eta$ transforming according to the fundamental rep-
   resentation of $SU(2) \times G$ for the Fermion Field.
We recall the refined algebraic quantization programme [3] whose major input is the choice of a star representation of the classical Poisson algebra. In sections 3 and 4 we will then make the following choice of a star representation:

We require that the following classical configuration observables (together with certain conjugate momentum observables) become densely defined operators on the Hilbert space:

1) Gravitational and Gauge Fields = Path Holonomies:
Given an open path e in Σ compute the holonomy (or path ordered exponential) $H_e(\omega)$, along e.

2) Higgs Field = Point Holonomy:
Given a point p in Σ and we may exponentiate the element of Lie(G) corresponding to the Higgs field $\phi$ at p to obtain a G-valued “point-holonomy” $U(p)$ (called this way due to its transformation properties under gauge transformations).

3) Fermion Field = Grassman-valued Half-Density:
Given a point p in Σ we may evaluate the fermion field $\eta$ at this point to obtain a Grassman number $\eta(p)$ which transforms as a scalar under diffeomorphisms. From this we construct the half-density $\xi(p) := \sqrt[4]{\det(q)}(p)\psi(p)$ where $q_{ab}$ is the dynamical metric on Σ.

This choice is radical because the fields are not smeared with test functions in three space directions. In fact, the phase of the theory that we obtain will be very different from the Fock phase, a phase appropriate for a manifestly diffeomorphism invariant description of the theory!

We show that the faithful implementation of the star relations fixes the Hilbert space completely.

In section 5 we will finally combine 1), 2), 3) and obtain the gauge and diffeomorphism invariant Hilbert space by means of standard techniques [11, 12].

2 Preliminaries

We begin by describing the field content of the theory.

The topology of the four-dimensional manifold is chosen, as always in the canonical approach, to be $M = \mathbb{R} \times \Sigma$ where Σ is a smooth 3-manifold which admits smooth Riemannian metrics.

The gravitational sector can be described [14] in terms of “real-valued Ashtekar variables”, that is, by a canonical pair $(A_i^a, E_i^a/\kappa)$ where $a, b, c$ etc. are tensorial indices for Σ, $i, j, k$ etc. are $su(2)$ indices, $A_i^a$ is an $SU(2)$ connection and $E_i^a$ is its conjugate momentum, an $ad_{SU(2)}$ transforming vector density of weight one. The relation with the usual geometrodynamical variables is established by $E_i^a = e^{abc}\epsilon_{ijk}e^j_c e^k_i/2$ where $e_i^a$ is the co-triad and $q_{ab} = e_i^a e_i^b$ is the intrinsic metric of Σ while $A_i^a - \Gamma_i^a = \text{sgn}(\det((e_i^a)))K_{abc}e^b_i$ where $K_{ab}$ is the extrinsic curvature of Σ, $\Gamma_i^a$ the spin-connection of $e_i^a$ and $e_i^a$ its inverse. $\kappa$ is the gravitational constant.

Now we come to the matter sector. Let $G$ be an arbitrary compact gauge group, for instance the gauge group of the standard model. Denote by $I, J, K,$... Lie(G) indices. We
introduce classical Grassman-valued spinor fields \( \eta = (\eta_{A,\mu}) \) where \( A, B, C, .. \) denote indices associated with the gravitational \( SU(2) \) and \( \mu, \nu, \rho, .. \) with the group \( G \). Clearly, the fermion species \( \eta \) transforms like a scalar and according to an irreducible representation of \( SU(2) \times G \). It turns out that in its manifestly real form (the associated conjugation is just complex conjugation for non-spinorial variables and for spinorial fields it involves a cyclic reversal of order in products) the most convenient description of the constraints is in terms of half-densities \( \xi := \sqrt{\det(q)} \eta \). The momentum conjugate to \( \xi_{A,\mu} \) is then just given by \( \pi_{A,\mu} = i\xi_{A,\mu} \).

Namely, as we will show in the next section, the gravitational connection of the Einstein-Dirac theory is real valued (that is, it is an \( su(2) \) rather than \( sl(2, \mathbb{C}) \) connection) only if one uses half-densities. This is important because for complex valued connections the techniques developed in [5, 6, 7, 8, 3] are inaccessible.

Notice that it is no lack of generality to restrict ourselves to just one helicity as we can always perform the canonical transformation \((i\bar{\xi}, \xi) \rightarrow (i\epsilon \xi, \epsilon \bar{\xi})\) where \( \epsilon \) is the spinor-metric, the totally skew symbol in two dimensions. Notice that there is no minus sign missing in this canonical transformation because we take the fermion fields to be anti-commuting, the action is invariant under this transformation [23, 22].

In the gauge sector we have canonical pairs \((A^I_a, E^a_I/Q^2)\) where the first entry is a \( G \) connection and the second entry is the associated electric field, \( Q \) is the Yang-Mills coupling constant. Finally, we may have scalar Higgs fields described by a canonical pair \((\phi^I, p^I)\) transforming according to the adjoint representation of \( G \). Without loss of generality we can take these as real valued by suitably raising the number of Higgs families. Here and in what follows we assume that indices \( I, J, K, .. \) are raised and lowered with the Cartan-Killing metric \( \delta_{IJ} \) of \( G \) which we take to be semi-simple up to factors of \( U(1) \).

Rarity-Schwinger fields can be treated by similar methods as here but we refrain from doing so because it is a simple exercise to apply the methods developed in this paper to supersymmetric field theories.

We will denote by \( \tau_i \) the generators of the Lie algebra of \( su(2) \) with the convention \([\tau_i, \tau_j] = i\epsilon_{ijk} \tau_k\), by \( \omega_a := A_a + A^A_a \) the \( SU(2) \times G \) connection. We have written only one family member of the possibly arbitrary large family of field species, in particular, we can have an arbitrary number of gauge fields all associated with different gauge groups and associated “quarks” and “Higgs” fields transforming under different irreducible representations. Also, one could easily deal with a more complicated “unified gauge group” which is not of the product type \( SU(2) \times G \) but contains it as a subgroup. However, for simplicity we will not deal with these straightforward generalizations and consider only one field species transforming under the fundamental representation of both \( SU(2) \) and \( G \) (fermions) or \( G \) (Higgs field) respectively. This furnishes the classical description of the field content.

We now come to the quantum theory. We can immediately apply the techniques of [4, 3, 2, 1] to write down a kinematical inner product for the gravitational and Yang-Mills sector that faithfully incorporates all the reality conditions. We get a Hilbert space \( L_2(\mathcal{A}_{SU(2)} \times \mathcal{A}_G, d\mu_{AL,SU(2)} \otimes d\mu_{AL,G}) \) where the index “AL” stands for Ashtekar-Lewandowski measure and the group index indicates to which gauge group the Ashtekar-Lewandowski measure is assigned. The reader interested in the constructions and techniques around the space of generalized connections modulo gauge transformations is urged
to consult the papers listed. In particular, the probability measure \( \mu_{\text{AL}} \) is very natural and gauge and diffeomorphism invariant.

The extension of the framework to Higgs and fermionic fields is the subject of this paper to which we turn in the next two sections. The quantization programme consists, roughly, of the following steps (see [3]):
1) Choose a representation of the canonical commutation relations, that is, choose a vector space of functions \( \Phi \) and an (over)complete set of basic operators corresponding to classical observables which leave \( \Phi \) invariant such that the Poisson bracket relations among those functions become canonical (anti)commutation relations between the corresponding operators.
2) Choose an inner product \( \langle \ldots, \ldots \rangle \) on \( \Phi \) such that the classical reality conditions among the basic functions become adjointness relations among the corresponding operators. Show that \( \langle \ldots, \ldots \rangle \) is unique, given the representation chosen, when requiring that the Poisson star algebra is faithfully represented.
3) Complete \( \Phi \) with respect to \( \langle \ldots, \ldots \rangle \) to obtain a kinematical Hilbert space \( \mathcal{H} \). The basic operators thus become densely defined. Identify the gauge invariant subspace of \( \mathcal{H} \).
4) Equip \( \Phi \) with a topology of its own to make it a topological vector space. Apply the group averaging procedure of [3] to construct solutions to the diffeomorphism constraint which lie in the topological dual \( \Phi' \) of \( \Phi \) and a scalar product \( \langle \ldots, \ldots \rangle_{\text{Diff}} \) on these diffeomorphism invariant states.
5) So far all steps could be applied to all fields (gauge, fermion, Higgs) separately. Now put everything together and construct the gauge and diffeomorphism invariant states depending on all three kinds of fields.

3 Grassman-valued half-densities

In this section we develop the integration theory for Grassman-valued half-densities since to our knowledge this is a situation usually not considered in the literature.

The reason for dealing with half-density-valued rather than scalar-valued Grassman variables are two-fold:

The first reason comes from the fact that in the latter case the momentum conjugate to \( \xi \) would be the density \( \pi = i \sqrt{\det(q)} \xi \) [24]. The density weight itself is not troublesome, what is troublesome is that the density weight comes from the function \( \sqrt{\det(q)} \) which in quantum gravity becomes itself an operator. This immediately leads to inconsistencies as already stated in [21] if one wants to construct an inner product on functions of the fermion fields which incorporates the reality condition \( \pi = -i \sqrt{\det(q)} \xi \).

Let us repeat the argument of [21] for the sake of clarity. Namely, assume, for instance, that we have managed to construct an inner product such that \( \pi^\dagger = -i \sqrt{\det(q)} \xi \) and if \( f = f(A) \) is any non-trivial real-valued function of the gravitational connection \( A \) then \( 0 = 0^\dagger = [\pi, f(A)]^\dagger = i[\sqrt{\det(q)}, f(A)] \eta \neq 0 \) which is a contradiction unless the commutator vanishes which would be unnatural because the classical Poisson bracket is non-zero. Thus, a different approach is needed. In [21] the authors partly solve the problem by restricting the adjoint operation to a certain sub-algebra of “real elements” of the (gauge-invariant) observable algebra. However, as one can show, the action of the Hamil-
tonian constraint derived in section 3 does not preserve any cyclic subspace generated by that subalgebra. Moreover, we will show now that if one does not work with half-densities, then the gravitational connection becomes actually complex-valued so that one could not apply the techniques associated with $A/G$ which hold only for compact gauge groups.

The second reason becomes evident when recalling that for the Einstein-Dirac theory, formulated in terms of the variables $(A_i^a, E_i^a)$ \[22\] which have proved to be successful for quantizing the Wheeler-DeWitt constraint \[16, 17\], the symplectic structure for each of the fermionic degrees of freedom is given by $\Theta = \frac{i}{2} \int_\Sigma d^3x \sqrt{\det(q)} \left[ \dot{\eta} - \dot{\bar{\eta}} \right]$ with scalar-valued $\eta$. Assume now that we would define $\pi := i \sqrt{\det(q)} \eta$ to be the momentum conjugate to $\eta$. Upon an integration by parts we see that the symplectic structure becomes up to a total time derivative equal to $\int_\Sigma d^3x \left[ \pi \dot{\eta} + \frac{1}{4} \dot{\eta} E^i_a \dot{E}_i^a \right]$ where $E^i_a$ denotes the inverse of $E_a^i$. But since $E^a_i$ is the momentum conjugate to $A^i_a$ this implies that the real-valued gravitational connection $A^i_a$ acquires a purely imaginary correction term $-\frac{1}{4} \pi \psi = -i \frac{\bar{\eta} \eta}{4} \xi E^a_i$ which leads to new mathematical difficulties because the techniques developed in \[4, 5, 6, 7, 8, 3\] break down for complex valued connections. In fact, both problems are related because if one uses $\psi$ as a canonical variable then $A$ is not real-valued as we just proved and then $0 = [\pi, f(A)] = i [\sqrt{\det(q)} \eta, f(A)]$ is not a contradiction but very messy and it will be hard to write $f(A)$ for general $f$. In other words, in \[21\] the contradiction arises from dealing with a non-real-valued (even when neglecting a total time drivative) symplectic structure which is physically inacceptable (“the world is real-valued”). We therefore follow the approach outlined below.

The key observation is that if we introduce the Grassman-valued half-density

$$\xi := \sqrt{\det(q)} \eta$$

then we have the identity $\Theta = \frac{i}{2} \int_\Sigma d^3x \left[ \dot{\xi} - \dot{\bar{\xi}} \right]$ because the time derivative of the half-density factor drops out in the antisymmetrized sum. The last equation, upon dropping a total time derivative, can also be written as $\Theta = \int_\Sigma d^3x \left[ i \bar{\xi} \dot{\xi} \right]$ thus displaying

$$\pi := \bar{\xi}$$

as the momentum conjugate to $\xi$ without that $A^i_a$ gets a correction thus leaving the compactness of the gravitational gauge group intact. Moreover, the reality condition is now very simple namely

$$\pi = -i \xi$$

(3.3)

which does not lead to the inconsistencies displayed above because the $\sqrt{\det(q)}$ does not show up in (3.3).

We begin with only one Grassman variable and recall some facts from Berezin integration \[24\]. Later we will generalize to the field theory case. All what we say here can be reformulated in terms of supermanifold theory \[25\] but since we only need a tiny fraction of the apparatus we refrain from introducing the relevant notions and keep the vocabulary to a minimum.

Let $\theta$ be a Grassman variable and $\bar{\theta}$ its adjoint, enjoying the familiar anti-commutation relations $\theta^2 = \bar{\theta}^2 = \theta \bar{\theta} + \bar{\theta} \theta = 0$ which state that $\theta, \bar{\theta}$ are Grassman-valued. On the other
hand, \( \theta, \bar{\theta} \) obey an anti-Poisson algebra \( \{ \theta, \theta \}_+ = \{ \bar{\theta}, \bar{\theta} \}_+ = 0 \), \( \{ \theta, \bar{\theta} \}_+ = -i \). Here \( \{ f, g \}_+ = (-1)^{n_f n_g + 1} \{ f, g \} \) where \( n_f \in \{0,1\} \) denotes the Grassman parity of \( f \).

Any “holomorphic” function of \( \theta \) is of the form \( f(\theta) = a + b\theta \) where for our purposes it will be sufficient to take \( a, b \) to be complex numbers. A general function of \( \theta, \bar{\theta} \) is of the form \( F(\bar{\theta}, \theta) = a + b\theta + c\bar{\theta} + d\bar{\theta}\theta \). Then the “measure” \( d\bar{\theta}d\theta \) on the Grassman space (or superspace) \( \mathcal{S} \) coordinatized by \( \theta, \bar{\theta} \) is defined by \( \int_{\mathcal{S}} d\bar{\theta}d\theta F = \delta \). Superspace becomes a measurable space in the sense of measure theory \( \mathbb{P} \) upon equipping it with the trivial \( \sigma \) algebra consisting only of \( \mathcal{S} \) itself and the empty set. This Berezin “measure” is insufficient for our purposes because we want to construct an \( L_2 \) space of holomorphic functions which requires a positive definite inner product. We therefore define a positive measure on \( \mathcal{S} \) by \( dm(\bar{\theta}, \theta) := e^{\theta \bar{\theta}} d\bar{\theta}d\theta = (1 + \theta \bar{\theta})d\bar{\theta}d\theta \) and readily verify that \( \langle f, f \rangle := \int d\mu \overline{f}f = |a|^2 + |b|^2 \).

Thus \( ||f|| = 0 \) iff \( f = 0 \) which would not be the case without the additional “density” \( dm/(d\bar{\theta}d\theta) \). Here we say that a function is positive if it is a linear combination of functions of the form \( \bar{f}f \) where \( f \) is holomorphic (i.e. depends only on \( \theta \) but not on \( \bar{\theta} \)). Then the integral of positive functions is positive and therefore \( dm \) is a positive measure.

As usual we now quantize the Grassman algebra by \( \hat{\theta}f := \theta f, \quad \hat{\bar{\theta}}f := df/d\theta \) which is easily checked to verify \( [\hat{\theta}, \hat{\bar{\theta}}]_+ = 1 \) which shows that the anti Poisson algebra is faithfully represented provided that \( \pi = i\theta \) is the momentum conjugate to \( \theta \) so that \( \hat{\pi} = id/d\theta \).

We must show that with respect to \( \mathcal{H} := L_2(dm, \mathcal{S}) \) it holds that \( \hat{\pi}^\dagger = \hat{\bar{\theta}} \). We compute

\[
\langle f, \hat{\bar{\theta}}g \rangle = \int d\bar{\theta}d\theta(1 + \theta \bar{\theta})\overline{f}g = \int d\bar{\theta}d\theta \overline{f}g = \int d\bar{\theta}d\theta \frac{df}{d\theta} \overline{g} = \int d\bar{\theta}d\theta \overline{\theta} \frac{df}{d\theta} \overline{g} = \int d\bar{\theta}d\theta(1 + \theta \bar{\theta}) \frac{df}{d\theta} \overline{g} = \langle \hat{\bar{\theta}}f, g \rangle. \tag{3.4}
\]

In the second equality we used that \( \theta^2 = 0 \) in order to get rid of the \( \bar{\theta}\theta \), in the third we used that \( d\bar{\theta}d\theta \) picks the term proportional to \( \bar{\theta}\theta \) which comes from the term of \( f \) linear in \( \theta \) and the factor of \( \theta \) already present, in the fourth and fifth we used the definition of the involution of the Grassman variables and that \( \bar{\theta}\theta \) is bosonic, in the sixth we recover the density of the measure because the additional integral is identically zero since \( df/d\theta \overline{g} \) cannot have a term proportional to \( \bar{\theta}\theta \). Thus, we verified that the reality conditions are faithfully implemented.

We wish to show now that, given the representation on holomorphic functions of the anti-Poisson bracket algebra defined by \( \hat{\theta}f = \theta f, \quad \hat{\pi}f = idf/d\theta \), the measure \( dm \) is the unique probability measure selected by asking that \( \hat{\pi}^\dagger = \hat{\bar{\theta}} \). This follows from considering the most general Ansatz for a measure given by \( dm(\bar{\theta}, \theta) = F(\bar{\theta}, \theta)d\bar{\theta}d\theta \) with \( F = a + b\theta + c\bar{\theta} + d\bar{\theta}\theta \) as above and studying the implications of asking that \( \langle 1, 1 \rangle = 1, \quad \langle f, \theta g \rangle = \langle \frac{df}{d\theta}, g \rangle, \quad \langle f, \frac{dg}{d\theta} \rangle = \langle \theta f, g \rangle \). We leave it as an exercise to check that indeed the first condition
leads to $d = 1$, the second to $a = 1, c = 0$ and the third to $b = 0$. Thus $\mathcal{H} := L_2(\mathcal{S}, dm)$ is the unique Hilbert space selected by the * relations.

These results are readily generalized to the case of a finite number, say $n$, of Grassman variables with the anti-Poisson algebra $\{\theta^i, \theta^j\} = \{\bar{\theta}^i, \bar{\theta}^j\} = 0, \{\bar{\theta}^i, \theta^k\} = -i\delta^{ij}$. Now a holomorphic function has the form

$$ f = \sum_{k=0}^{n} \frac{1}{k!} a_{i_1..i_k} \theta^i_1..\theta^i_k $$

with totally skew complex valued coefficients $a_{i_1..i_k}$. We introduce the product measure $dm(\{\bar{\theta}^i\}, \{\theta^i\}) = \prod_{i=1}^{n} dm(\bar{\theta}^i, \theta^i)$ and find that

$$ <f, f> = \sum_{k=0}^{n} \sum_{1 \leq i_1 < .. < i_k \leq n} |a_{i_1..i_k}|^2 \tag{3.5} $$

which shows that $L_2(\mathcal{S}, dm)$ is a $2^n$ dimensional Hilbert space with orthonormal basis $\theta^i, \bar{\theta}^i, \bar{\theta}^i < .. < i_k \leq n$. Moreover, by defining $\hat{\theta}^i f = \theta^i f$, $\hat{\bar{\theta}}^i f = \partial f/\partial \theta^i$ the anti-Poisson algebra is faithfully represented together with the * relations.

This furnishes the quantum mechanical case.

Let us now address the field theoretic case. Recall that we have the classical anti-Poisson brackets $\{\xi_{A\mu}(x), \pi_{B\nu}(y)\} = \delta_{AB} \delta_{\mu\nu} \delta(x, y)$. Notice that on both sides of this equation we have a density of weight one. Moreover, we have the classical * relations $\pi_{A\mu} = \pi^*_{A\mu} - i\xi_{A\mu}$. Let now $f$ be a general holomorphic functional of the $\xi_{A\mu}(x)$. We define operators $\hat{\xi}_{A\mu}(x) f := \xi_{A\mu}(x) f$, $\hat{\pi}_{A\mu}(x) f := i \hbar \frac{\delta f}{\delta \xi_{A\mu}(x)}$ or, equivalently, $\hat{\bar{\xi}}_{A\mu}(x) f := \hbar \frac{\delta f}{\delta \xi_{A\mu}(x)}$. It is easy to see that this satisfies the required anti-commutator relation $[\hat{\xi}_{A\mu}(x), \hat{\pi}_{B\nu}(y)] = i \hbar \delta_{AB} \delta_{\mu\nu} \delta(x, y)$ by using that the functional derivative is also Grassman-valued.

Now, however, we must take seriously into account that $\xi(x)$ is a half-density (we will suppress the label $A\mu$ in this paragraph). The first observation is that it poses no problem to extend the above framework to the case of countably infinite number of variables. Let us therefore introduce a triangulation $\Sigma = \cup_{n} B_n$ of $\Sigma$, that is, a splitting into countably many, mutually disjoint (up to common faces, edges and vertices) closed solid boxes $B_n$ such that $\Sigma = \cup_{n} B_n$. Each $B_n$ has a Lebesgue measure $\epsilon_n$ and a centre $v_n = v(B_n)$. Let $\chi_n(x) := \chi_{B_n}(x)$ denote the characteristic function of $B_n$. Let us define new Grassman-valued variables

$$ \theta_n := \int_{\Sigma} \frac{\chi_n(x)}{\epsilon_n^{3}} \xi(x) \tag{3.6} $$

Then it is easy to see that

$$ \sum_{n} \hat{\theta}_n \theta_n = \sum_{n} \int d^3x \chi_n(x) \int d^3y \frac{\chi_n(y)}{\epsilon_n^{3}} \xi(x) \xi(x) \rightarrow \int d^3x \xi(x) \xi(x) \text{ as } \epsilon_n \rightarrow 0 \forall n. \tag{3.7} $$

Furthermore, the corresponding operators

$$ \hat{\theta}_n := \int_{\Sigma} \frac{\chi_n(x)}{\epsilon_n^{3}} \xi(x), \quad \hat{\bar{\theta}}_n := \int_{\Sigma} \frac{\chi_n(x)}{\epsilon_n^{3}} \bar{\xi}(x) \tag{3.8} $$
enjoy discrete canonical anti-commutator relations (even at finite $\epsilon_n$)

\[
[\hat{\theta}_m, \hat{\theta}_n]_+ = [\hat{\bar{\theta}}_m, \hat{\bar{\theta}}_n]_+ = 0 \\
[\hat{\bar{\theta}}_m, \hat{\bar{\theta}}_n]_+ = \int d^3x \int d^3y \frac{\chi_m(x)\chi_n(y)}{\sqrt{\epsilon_m^3 \epsilon_n^3}} \delta(x, y) \\
= \int d^3x \frac{\chi_m(x)}{\sqrt{\epsilon_m^3 \epsilon_n^3}} = \delta_m,n \int d^3x \frac{\chi_m(x)}{\epsilon_m^3} = \delta_m,n \quad (3.9)
\]

because $\chi_m(x)\chi_n(x) = \delta_m,n \chi_n(x)$. A similar calculation using the anti-bracket relation $
\{\xi(x), \xi(y)\}_+ = -i\delta(x, y)$ reveals that $\{\theta_m, \bar{\theta}_n\} = -i\delta_m,n$.

The quantities $\theta_n$ have a formal limit as $\epsilon_n \to 0$ (i.e. the triangulation becomes the continuum) and $B_n \to v(B_n) =: x$ : Namely, because $\chi_n(x) = \sqrt{\chi_n(x)}$ it follows that

\[
\theta(x) := \lim_{\epsilon_n \to 0} \theta_n = \lim_{\epsilon_n \to 0} \int d^3y \frac{\chi_n(y)}{\epsilon_n^3} \xi(y)^i = \int \Sigma d^3y \sqrt{\delta(x, y)} \xi(y) \quad (3.10)
\]

where $\delta(x, y)$ is the three-dimensional $\delta$-distribution. Notice that the $\delta$ distribution is a density of weight one so that due to the square root and the half-density $\xi$ the quantities $\theta(x)$ transform as scalars! This is important for the construction of diffeomorphism invariant states as we will see below.

We are now ready to construct an inner product. At finite $\epsilon_n$ we define a regulated Fermion measure by

\[
d\mu_F(\theta, \bar{\theta}) := \prod_n dm(\theta_n, \bar{\theta}_n) \quad (3.11)
\]

where $\epsilon := \inf_n \{\epsilon_n\} > 0$ and $dm$ is the measure constructed above. It is easy to see that the * relations $\hat{\bar{\theta}}_n^\dagger = \hat{\bar{\theta}}_n$ hold for each $n$. The formal limit, as $\epsilon \to 0$, of this measure is given by

\[
d\mu_F(\xi, \bar{\xi}) := (\prod_{x \in \Sigma}[d\bar{\theta}(x)d\theta(x)]) e^{\sum_{x \in \Sigma} \bar{\theta}(x)\theta(x)} \\
= (\prod_{x \in \Sigma}[d\bar{\theta}(x)d\theta(x)]) e^{\int_{\Sigma} d^3x \bar{\xi}(x)\xi(x)} \quad (3.12)
\]

In the first step we have used the fact that the quantities $\bar{\theta}_n \theta_n$ mutually commute to write the product of the exponentials as the exponential of the sum of exponents and then took the limit. In the second step we exploited (3.7) in addition. By inspection, the measure $d\mu_F$ is diffeomorphism invariant and obviously one formulates the theory better in terms of the $\theta(x)$ than in terms of the half-densities $\xi(x)$.

All identities that we will state in the sequel can be rigorously justified by checking them at finite $\epsilon$ and then take the limit. For example, we already verified that $\hat{\bar{\theta}}_n^\dagger = \hat{\bar{\theta}}_n$ for all $n$ with respect to $\mu_F'$. Taking $\epsilon \to 0$ we conclude that

\[
\hat{\bar{\theta}}_n^\dagger (x) = \hat{\bar{\theta}}_n (x) \quad (3.13)
\]
for all $x$ with respect to $\mu_F$. This is obviously equivalent to the statement $\hat{\xi}(x) = \hat{\xi}(x)$ for all $x$ with respect to $\mu_F$ in view of (3.10). Thus we have implemented the reality conditions faithfully and the measure $\mu_F$ is the unique probability measure that does it (the uniqueness statement follows 1) from the uniqueness for each $m_x = m(\theta(x), \theta(y))$ separately as seen above and 2) from the fact that all points of $\Sigma$ are uncorrelated).

Another example is the anti-commutator $[\hat{\theta}_m, \hat{\theta}_n]_+ = \hbar \delta_{m,n}$ whose limit is given by

$$[\hat{\theta}(x), \hat{\theta}(y)]_+ = \hbar \delta_{x,y}. \tag{3.14}$$

Notice that $\delta_{x,y}$ is a Kronecker $\delta$ not a $\delta$-distribution which is precisely in accordance with the classical symplectic structure $\{\theta(x), \theta(y)\} = -i\delta_{x,y}$. In fact, we have by the same reasoning as in (3.7)

$$\sum_n \theta_n\dot{\theta}_n \to \sum_x \theta_x\dot{\theta}_x = \int d^3 x \check{\xi}(x)\xi(x)$$

as $\epsilon \to 0$.

This motivates the following.

We will choose a representation for which the scalar Grassman-valued field $\hat{\theta}(x)$ is the basic quantum configuration variable, in particular, it is densely defined.

As a last example we quote the following orthogonality relations $<\theta_m, \theta_n> = \delta_{m,n}$ which extend to $<\theta(x), \theta(y)> = \delta_{x,y}$.

Summarizing and re-introducing the labels $A\mu$, the Fermion measure is given by

$$d\mu_F([\theta], \{\theta\}) := \prod_{x \in \Sigma} dm([\theta_{A\mu}(x)], \{\theta_{A\mu}(x)\}) =: \prod_{x \in \Sigma} dm_x \tag{3.15}$$

where $dm_x$ is the measure on the Grassman space $S_x$ at $x$ which is defined by the $2 \times 2d$ Grassman variables at $x$ introduced above, $d$ the dimension of the fundamental representation of $G$. This measure is more precisely defined as follows: we say that a function $f$ is cylindrical whenever it depends only on the fermion fields at a finite number of points $v_1, \ldots, v_n$ in our case turn out to be the vertices of a piecewise analytic graph $\gamma$. We then have the simple definition $\mu_F(f) = (m_{v_1} \otimes \ldots \otimes m_{v_n})(f) =: m_{v_1,\ldots,v_n}(f)$. These cylindrical product measures are consistently defined, that is, we have $m_{v_1,\ldots,v_n}(f) = m_{v_1,\ldots,v_n}(f)$ whenever $f$ depends only on the fermion fields on the fermion fields at the first $m \leq n$ points. Since then we have an uncountable product of probability measures the Kolmogorov theorem for the uncountable direct product (3.15) tells us that the consistent family $\{m_{v_1,\ldots,v_n}\}$ has the unique $\sigma$-additive extension (3.11) to the infinite product measurable space $\overline{S} := \times_x S_x$. The Hilbert space is just given by

$$\mathcal{H}_F = L_2(\overline{S}, d\mu_F) = \otimes_x L_2(S_x, dm_x).$$

Thus, $\mathcal{H}_F := L_2(\overline{S}, d\mu_F)$ is the appropriate and unique kinematical Hilbert space on holomorphic Grassman functionals (in terms of $\theta(x)$) satisfying the appropriate adjointness relations. Notice that the measure $\mu_F$ is diffeomorphism invariant in the sense that it assigns the same integral to any cylindrical function before and after mapping the points $\vec{v}$ with a diffeomorphism. It is also gauge invariant since a gauge transformation at $x$ is in particular a unitary transformation on the on the $2d$ dimensional vector space with respect to the inner product (3.5).
An orthonormal basis for $\mu_F$ can be constructed as follows:
Order the labels $A\mu$ in some way from $i = 1$ to $i = 2d$. Let $\vec{v}$ be an ordered, finite set of mutually different points as above. For each $v \in \vec{v}$ denote by $I_v$ an array $(i_1(v) < .. < i_k(v))$ where $0 \leq k \leq 2d$ and $1 \leq i_j(v) \leq 2d$ for each $1 \leq j \leq k$. Also, let $|I_v| = k$ in this case.
Finally let $\vec{I} := \{I_v\}_{v \in \vec{v}}$.

**Definition 3.1** A fermionic vertex function $F_{\vec{v},\vec{I}}(\theta)$ is defined by

$$F_{\vec{v},\vec{I}} := \prod_{v \in \vec{v}} F_{v,I_v}$$

where

$$F_{v,I_v} := \prod_{k,j=1}^{2d} \theta^{i_j(v)}$$

It follows that a cylindrical function $f_{\vec{v}}$ can be written as

$$f_{\vec{v}} = \sum_{\vec{I}} a_{\vec{v},\vec{I}} F_{\vec{v},\vec{I}}$$

for some complex valued coefficients $a_{\vec{v},\vec{I}}$ and the sum runs for each $v \in \vec{v}$ over the $2^{2d}$ values of $I_v$.

**Lemma 3.1** The collection of fermionic vertex functions provides an orthonormal basis for $H_F$.

**Proof**:
By construction the vertex functions provide a dense set of vectors in $H$. To see the orthonormality one just needs to recall (3.5).

We now proceed to the solution of the diffeomorphism constraint.
First of all, let us denote by $\Phi_F$ the space of cylindrical functions constructed only from fermionic fields. Thus, any element is a finite linear combination of products of the functions $F_{v,I_v}$ of Definition 3.1. We equip $\Phi_F$ with a topology by assigning to the element $f_{\vec{v}}$ the “Fourier-norm”

$$||f_{\vec{v}}||_1 := \sum_{\vec{I}} |< F_{\vec{v},\vec{I}}, f_{\vec{v}}>|$$

the name being motivated by the fact that this is like an $L_1$ norm on the “Fourier coefficients” $a_{\vec{v},\vec{I}}$ in (3.13). The space $\Phi'_F$ is the topological dual of $\Phi_F$ with respect to $||.||_1$. We then have the inclusion $\Phi_F \subset H_F \subset \Phi'_F$ since $||f||_2^2 \leq ||f||_1^2$ so that the assertion follows from the Schwarz inequality.

We wish to construct distributions in $\Phi'_F$ which are diffeomorphism invariant with respect to smooth diffeomorphisms (in case that the points $\vec{v}$ of a vertex function are actually vertices of a piecewise analytic $\gamma$ graph with vertex set $V(\gamma)$ defined by the gauge fields, consider only analyticity preserving smooth diffeomorphisms [13]). This is straightforward given the framework of [3] : Notice that a unitary representation of the diffeomorphism group on $H_F$ is defined by $\hat{U}(\varphi)f_{\vec{v}} := f_{\varphi(\vec{v})}$. This transformation law under diffeomorphisms is compatible with the fact that the $\theta_i(x)$ are scalars rather than half-desities. This is rather important because as we will see later, there is no obvious way to group average the $\xi_i(x)$ such that the result is a distribution.

Consider the orbit of the basis vector $F_{\vec{v},\vec{I}}$, that is, consider the set of states $\{F_{\vec{v},\vec{I}}\}$ := $\{\hat{U}(\varphi)F_{\vec{v},\vec{I}}, \varphi \in \text{Diff}(\Sigma)\}$. We define the group average of the basis vectors $F_{\vec{v},\vec{I}}$ by

$$[F_{\vec{v},\vec{I}}] := \sum_{F \in \{F_{\vec{v},\vec{I}}\}} F.$$
The group average of a general element (3.16) is defined by requiring linearity of the group averaging procedure, that is,

\[ [f_{\vec{v}}] := \sum_I a_{\vec{v},I}[F_{\vec{v},I}] \, . \]

Why the group averaging has to be done in terms of a chosen basis is explained in [18]. Since we are eventually interested in averaging gauge invariant states the problem of the “graph symmetry factors” outlined in [18] is resolved in the process of averaging the gauge fields. Finally, the diffeomorphism invariant inner product is given by [3, 18]

\[ <[f], [g]>_{\text{diff}} := [f](g) \]

where the latter means evaluation of the distribution \([f] \in \Phi'_F\) on the cylindrical function \(g \in \Phi_F\).

We conclude this section with showing why the shift from \(\xi(x)\) to \(\theta(x)\) becomes mandatory in the diffeomorphism invariant context:

Consider a cylindrical function \(f_{v_1, \ldots, v_n}\) with \(n_v \leq 2d\) fermionic half-densities \(\xi\) at the point \(v \in \{v_1, \ldots, v_n\}\). Consider the transformation law of that function under a diffeomorphism \(\varphi \in \text{Diff}(\Sigma)\). Due to the half-density weight we get \(f_{v_1, \ldots, v_n} \rightarrow \prod_{k=1}^n \frac{1}{J_\varphi(v_k)} f_{\varphi(v_1), \ldots, \varphi(v_n)}\) where \(J_\varphi(v) = |\det(\partial \varphi^a/\partial x^b)(v)|^{1/2}\). Now let \(\varphi\) be an (analyticity preserving) diffeomorphism which leaves the points \(v_1, \ldots, v_n\) invariant (and the underlying graph) but such that the \(J_\varphi(v) \neq 1\). Such diffeomorphisms certainly exist as the reader can convince himself by considering in two dimensions the diffeomorphism \(\varphi(v, y) = (\alpha^2 x, y)\) which leaves, say, the interval \([0, 1]\) along the \(y\)-axis invariant but \(J_\varphi(0, 0) = |\alpha|\). We see that all the states \(\lambda f_{v_1, \ldots, v_n}, \lambda > 0\) are related by a diffeomorphism and therefore the group average of this state along the lines of [3] would include the object \([\sum_{\lambda > 0} \lambda] f_{v_1, \ldots, v_n}\) which is meaningless even when considered as a distribution on \(\Phi_F\) because it has non-vanishing inner product with \(f_{v_1, \ldots, v_n} \in \Phi_F\). Notice that for scalar valued fermions \(\theta\) instead of \(\xi\) on the other hand the diffeomorphisms just discussed would not alter the orbit of the state and therefore would not contribute to the group average.

4 Point Holonomies

We have made for the gravitational and gauge fields the (rather radical) assumption that the holonomies along edges can be promoted to a densely defined operator. This assumption has lead to the Ashtekar-Isham quantum configuration space \(\mathcal{A}/\mathcal{G}\) of distributional connections modulo gauge transformations and to the Ashtekar-Lewandowski measure \(\mu_{\text{AL}}\) which is the unique measure (up to a positive constant) that promotes the reality conditions on the classical phase space into adjointness relations on the corresponding Hilbert space \(L_2(\mathcal{A}/\mathcal{G}, d\mu_{\text{AL}})\).

In this section we are going to make an even more radical assumption for the Higgs field: Namely, given a classical Higgs configuration field \(\phi_I(v)\) we may exponentiate it to obtain the so-called holonomy at the point \(v\), \(U(v) := \exp(\phi_I(v)\tau_I)\), where \(\tau_I\) are the generators of \(\text{Lie}(G)\). The \(U(v)\) are then \(G\)-valued classical objects which transform under the adjoint representation of \(G\) since \(\phi_I \tau_I \rightarrow g[\phi_I(v)\tau_I]g^{-1}\). Here and in what follows we consider only unitary and unimodular compact groups (so that the Cartan Killing metric is given by \(\delta_{IJ}\)). The case of a scalar field would correspond to \(U(v) = e^{i\phi(v)} \in U(1)\) and
can be handled similarly. The key assumption is now

The point holonomies $U(v)$ can be promoted to densely defined operators on a Hilbert space that implements the appropriate adjointness relations!

The adjointness relations that we would like to implement are, of course, that (in a sense to be made precise below) $[\phi_I(v)]^\dagger = \phi_I(v)$, $[p_I^f(v)]^\dagger = p_I^f(v)$.

Let us first pause and give a rational why it is more appropriate, in the diffeomorphism invariant context, to base the quantization on the point holonomies $U(v)$ rather than on the Higgs (or scalar) fields $\phi_I(v)$ itself:

Assume that we choose as our basic configuration field variables the $\phi_I(x)$. Having made this assumption, then a natural choice for the inner product for scalar fields is a Gaussian measure: some sort of Gaussian joint distribution is needed because $\phi_I$ is real valued rather than valued in a compact set. We will now argue that this sort of a kinematical measure is in conflict with diffeomorphism invariance when one implements the canonical commutation relations and the adjointness relations (we consider the case of scalar field $\phi$, even more problems associated with gauge invariance result with Higgs fields). In an attempt to resolve the problems that occur we will see that we are naturally lead to the consideration of point holonomies.

Recall [1] that a measure for a field whose quantum configuration space is a vector space is rigorously defined in terms of its characteristic functional. The characteristic functional is given by the integral of the functional $e^{i\phi(f)}$, that is, $\chi(f) := \mu(e^{i\phi(f)})$ where $\phi(f) = \int_{\Sigma} d^3x f(x)\phi(x)$ and $f$ is some sort of real valued test function on $\Sigma$ (usually of rapid decrease). The point is now that the label $f(x)$ is a function of the coordinates $x^a$ whose only meaningful transformation law under diffeomorphisms amounts to saying that they are scalars and therefore $f$ is a scalar. Therefore the pairing $\phi(f)$ only has a covariant meaning (observing that $\Sigma$ is invariant under diffeomorphisms) provided that $\phi$ is a scalar density! This can be achieved by performing the classical canonical transformation $(\phi, p) \rightarrow (p, -\phi)$ and so does not pose any immediate problem. In this case we have a representation of the diffeomorphism group given by $\hat{U}(\varphi)\phi(f) = \phi([\varphi^{-1}]^* f)$. Therefore, the measure $\mu$ is diffeomorphism invariant if and only if $\chi(\varphi^* f) = \chi(f)$ for any $\varphi$. Now a Gaussian measure is defined through its covariance $C$ which is the kernel of an operator in the sense that

$$\chi(f) = e^{-\frac{i}{2}C(f,f)}$$

where $C(f,g) := \int d^3x \int d^3y f(x)C(x,y)g(y)$.

Obviously diffeomorphism invariance imposes that

$$C(x,y) = |\det(\partial \varphi(x)/\partial x)\det(\partial \varphi(y)/\partial y)|C(\varphi(x),\varphi(y))$$

which is the transformation law of a double density. The only quantity merely built from the coordinates, which transforms like a double density and is diffeomorphism invariant is of the form $C(x,y) = \sum_{z \in \Sigma} \delta(z,x)\delta(z,y)$ (notice that this involves a sum rather than an integral over $\Sigma$). But $\sum_{z \in \Sigma} |f(z)|^2$ does not diverge only if the test function $f$ is not smooth but vanishes at all but a finite number of points. But then $\phi(f) = 0$ for almost all tempered distributions $\phi$ except for those which are of the form $\phi(x) = \sum_{z \in \Sigma} \delta(x,z)\Phi(z)$ where $\Phi(z)$ is a scalar. We see that $\exp(i\phi(f))$ equals the point holonomy $\exp(i\sum_{x \in \text{supp}(f)} f(x)\Phi(x))$ which therefore comes out of the analysis naturally. But let us further stick with the assumption that the $\phi(x)$ can be made well-defined operators. Then the diffeomorphism invariant measure

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automatically is supported on those $\phi$ only which are of the form $\sum_{z \in \Sigma} \delta(x, z) \Phi(z)$. The corresponding Gaussian measure is formally given by $\left( \frac{dx}{2\pi} \right) \exp \left( -1/2 \sum_{z \in \Sigma} \Phi(z)^2 \right)$, a “discrete white noise measure” which is not even expressible in terms of $\phi(x)$. We see that the measure is an infinite product of Gaussian measures, one for each point in $\Sigma$.

We see that the assumption that $\phi$ was a density has lead to a more natural description in terms of scalar $\Phi$ so let us start with scalar $\phi := \Phi$ right from the beginning and that the Gaussian measure is the discrete white noise measure in terms of $\phi(x)$. Then multiplication by $\phi(x)$ and the operator $-i\hbar [\partial/\partial \phi(x) - 1/2 \sum_{z \in \Sigma} \delta(x, z)] = \hat{\phi}(x)$ smeared with a three-dimensional test function become formally self-adjoint operators on the associated $L_2$ space which has an orthonormal basis given by finite linear combinations of Hermite monomials: $\phi(x_1)^m : \ldots : \phi(x_m)^m$ (Wick ordering with respect to the Gaussian measure above). The trouble with this approach is that multiplication by $\phi(x)$ but not $\hat{\phi}(x)$ are densely defined, not even when smeared with a test function: to see this consider the simplest state, the “vacuum state” $f(\phi) = 1$. Let $B$ be a compact region in $\Sigma$, then $f' := f_B d^3y \hat{\phi}(y) f = i\hbar/2 \sum_{z \in B} \phi(z)$ which is a state of infinite norm because $<\phi(x), \phi(y)> = \delta_{x,y}$, for any compact set $B$. Thus, we conclude that the Gaussian measure defined above and which comes from the assumption that the $\phi$ are well-defined operators leads to rather difficult mathematical problems given our adjointness relations.

The rest of this section is devoted to constructing an appropriate quantum configuration space $\mathcal{U}$ of Higgs fields where $\mathcal{U}$ denotes the space of classical, that is, smooth Higgs fields and to determine a unique $\sigma$–additive measure $\mu_\mathcal{U}$ on $\mathcal{U}$ selected by the adjointness relations. As we will see, we can roughly copy the procedure of [4, 5].

**Definition 4.1** A Higgs vertex function is a function $H_{\vec{v}, \vec{\pi}, \vec{i}, \vec{j}}$ on $\mathcal{U}$ labelled by a quadruple $(\vec{v}, \vec{\pi}, \vec{i}, \vec{j})$. Here $\vec{v} := \{v_1, \ldots, v_n\}$ is a finite set of points in $\Sigma$ and $\vec{\pi} := \{\pi_1, \ldots, \pi_n\}$ is a corresponding set of irreducible representations of $G$ (choose once and for all a representation from the class of independent equivalent representations). Finally, $\vec{i} := \{i_1, \ldots, i_n\}$, $\vec{j} := \{j_1, \ldots, j_n\}$ are corresponding labels for the matrix elements of $\pi_1, \ldots, \pi_n$. We say that the point $v_i$ is dressed by $\pi_i$ and obtain a complex valued function

$$H_{\vec{v}, \vec{\pi}, \vec{i}, \vec{j}}(\phi) := [\otimes_{k=1}^n [\sqrt{d_{\pi_k}} \pi_k(U(v_k))]_{i_k,j_k}] := \prod_{k=1}^n [\sqrt{d_{\pi_k}} \pi_k(U(v_k))]_{i_k,j_k}$$

where $d_{\pi}$ denotes the dimension of the irreducible representation $\pi$.

Let us now consider the * algebra $\mathcal{V}$ (with unit 1, the constant function with value one) of functions consisting of the finite linear combinations of vertex functions where the * operation is just complex conjugation. We turn it into an Abelian $C^*$ algebra $V$ by completing it with respect to the supremum norm, $||f|| := \sup_{\phi \in \mathcal{U}} |f(\phi)| |vf \in \mathcal{V}|$. Notice that the supremum norm makes sense because $|\pi_{ij}(U(v))| \leq 1$.

**Definition 4.2** The space $\mathcal{U}$ of distributational Higgs fields is the Gel’fand spectrum of $V$ equipped with the Gel’fand topology.

Recall that the Gel’fand spectrum of an (Abelian) $C^*$ algebra is the set of its characters, that is, the set of all * homomorphisms from $V$ into the complex numbers. That is, given an element $\phi \in \mathcal{U}$ and an element $f \in V$ we obtain a complex number $\phi(f)$, moreover, $\phi(f + g) = \phi(f) + \phi(g)$, $\phi(fg) = \phi(f)\phi(g)$, $\phi(f^*) = \overline{\phi(f)}$, $\phi(1) = 1$. This shows that
\( \mathcal{U} \subset \overline{\mathcal{U}} \). The Gel’fand topology on the spectrum is the weak * topology, that is, a net \( \phi_\alpha \) in \( \overline{\mathcal{U}} \) converges if the net of complex numbers \( \phi_\alpha(f) \) converges for each element \( f \in V \) pointwise.

The Gel’fand isomorphism \( V \to C(\mathcal{U}); \ f \to \hat{f}; \ \hat{f}(\phi) := \phi(f) \) shows that \( V \) is isomorphic with the set of continuous functions on its spectrum. That \( \hat{f} \) is indeed continuous follows from \( \hat{f}(\phi_\alpha) = \phi_\alpha(f) \to \phi(f) = \hat{f}(\phi) \) by definition of convergence in \( \overline{\mathcal{U}} \). The isomorphism is in fact an isometry since on the one hand

\[
||\hat{f}|| := \sup_{\phi \in \overline{\mathcal{U}}} |\hat{f}(\phi)| = \sup_{\phi \in \mathcal{U}} |\phi(f)| \geq \sup_{\phi \in \mathcal{U}} |f(\phi)| = ||f||
\]

and on the other hand \( |\phi(f)| \leq ||f|| \sup_{g \in V, ||g||=1} |\phi(g)| = ||f|| ||\phi|| \). But \( \phi \) is a character, so \( ||\phi|| = 1 \) (to see this, consider any \( f \) with \( ||f|| = 1 \). Then \( \phi(|f|^2) = |\phi(f)|^2 \) so taking the supremum over all \( f \) gives \( ||\phi|| = ||\phi||^2 \). Therefore \( ||\hat{f}|| \leq ||f|| \). Finally, \( \mathcal{U} \) is densely embedded into \( \overline{\mathcal{U}} \) by standard Gel’fand theory.

The spectrum is a rather abstract construction. However, just as in the case of \( \overline{\mathcal{A}}/\mathcal{G} \) we can describe it in more intuitive terms.

**Theorem 4.1 (Characterization of the spectrum)** \( \overline{\mathcal{U}} \) is in one to one correspondence with the set \( \text{Fun}(\Sigma, G) \) of \( G \)-valued functions on \( \Sigma \), the correspondence being given by

\[
\phi \leftrightarrow U_\phi; \ (U_\phi(v))_{\mu\nu} = \hat{\phi}(H_{v,\pi_\text{fun},\mu,\nu})
\]

where \( \pi_\text{fun} \) denotes the fundamental representation of \( G \).

**Proof**

The proof is pretty trivial:

Given \( \phi \in \overline{\mathcal{U}} \) we need to check that \( \phi(H_{v,\pi_\text{fun},\mu,\nu}) \) are the matrix elements of an element of \( G \). But this follows from the fact that \( \phi \) is a homomorphism from the algebra of vertex functions into the complex numbers. Therefore, all the relations satisfied between elements of that algebra continue to hold after mapping with \( \phi \). The assertion now follows from the fact that \( H_{v,\pi_\text{fun},\mu,\nu} \) satisfies all the properties of the matrix elements of an element of \( G \).

Conversely, given an element \( U \in \text{Fun}(\Sigma, G) \) we define \( \phi_U \in \overline{\mathcal{U}} \) by \( \phi_U(H_{v,\pi_\text{fun},\mu,\nu}) := U_{\mu\nu}(v), \ \phi_U(1) := 1 \) and extend by linearity and multiplicativity. Then all we have to show in order to establish that \( \phi_U \) qualifies as an element of the spectrum is that \( \overline{\phi(f)} = \phi(\overline{f}) \).

This follows from the fact that \( G \) is unimodular and unitary, therefore \( U(v)_{\mu\nu} \) is actually a polynomial in \( U(v)_{\mu\nu} \).
\[ \square \]

Since the spectrum is a compact Hausdorff space, positive linear functionals \( \Gamma \) on \( C(\overline{\mathcal{U}}) \) are in one to one correspondence with regular Borel probability measures on \( \overline{\mathcal{U}} \), the correspondence being given by \( \Gamma(f) = \int_{\overline{\mathcal{U}}} f d\mu f \). We will define the measure \( \mu_U \) through its corresponding characteristic functional \( \Gamma_U \).

**Definition 4.3**

\[
\Gamma_U(H_{\v,\pi,\mu,\nu}) = \begin{cases} 
1 : H_{\v,\pi,\mu,\nu} = 1 \\
0 : \text{otherwise} 
\end{cases} \quad (4.2)
\]

That this completely defines \( \Gamma_U \) follows from the fact that finite linear combinations of vertex functions are dense in \( V \) and thus by the Gel’fand isomorphism dense in \( C(\overline{\mathcal{U}}) \). It
is immediate from this definition that the measure $\mu_U$ is diffeomorphism invariant and gauge invariant. It is easy to check that Definition (4.3) is equivalent to the following cylindrical definition of $\mu_U$:

A function $f$ of $\phi$ is said to be cylindrical with respect to a vertex set $\vec{v}$ if it depends only on the finite number of variables $U_{\phi}(v_1), \ldots, U_{\phi}(v_n)$, that is, $f(\phi) = f_\vec{v}(U_{\phi}(v_1), \ldots, U_{\phi}(v_n)) =: f_\vec{v}(p_\vec{v}(\phi))$ where $f_\vec{v}$ is a complex-valued function on $G^n$. We define the integral of $f$ by

$$\int_{U} d\mu_U(\phi)f(\phi) := \int_{\mathcal{U}} d\mu_\vec{v}(\phi)f(\phi) := \int_{G^n} d\mu_H(g_1) \ldots d\mu_H(g_n) f_\vec{v}(g_1, \ldots, g_n)$$

(4.3)

where $\mu_H$ is the Haar measure on $G$. The so defined family of measures $\{\mu_\vec{v}\}$ is of course consistently defined because the Haar measure is a probability measure. That (4.3) coincides with (4.2) is now immediate because of the Peter&Weyl theorem which says that the functions $\sqrt{d_\pi} \pi_{ij}(g)$ form an orthonormal basis for $L_2(G, d\mu_H)$: Equation (4.2) is nothing else than the statement that the vertex function 1 is orthogonal to every other one with respect to $\mu_U$. But this is also the case by (4.3) using the Peter&Weyl theorem.

**Corollary 4.1** The vertex functions provide an orthonormal basis for $L_2(\mathcal{U}, d\mu_U)$.

This follows again from the Peter&Weyl theorem.

Another definition of the measure $\mu_U$ is as follows:

$$d\mu_U(\phi) := \prod_{v \in \Sigma} d\mu_H(U_{\phi}(v)).$$

(4.4)

Notice that this definition is rigorous: As is shown in [2], a measure which on cylindrical subspaces is a product of probability measures corresponds uniquely to a $\sigma$-additive measure on the uncountable product measurable space.

We also have the analogue of the Marolf-Mourão result [8]: while $\mathcal{U}$ is topologically dense in $U$ it is contained in a measurable set of $\mu_U$ measure zero. To see this we show a stronger result: the set of $\phi$'s which are continuous at any point and in any direction is of measure zero. We will give a new proof of this result below.

**Theorem 4.2** Let $p \in \Sigma$ be a point and $c$ a curve in $\Sigma$ starting in $p$. Let $F_{p,c} := \{\phi \in U; \lim_{t \to 0} \sum_{\mu} |\phi(H_{\pi_{\mu}} - H_{\pi_\phi})|^2 = \lim_{t \to 0} 2d(1 - \frac{1}{\pi} \text{tr}(U_{\phi}(c(t))U_{\phi}(p)^{-1}) = 0\}$ be the set of characters which are continuous at $p$ along the curve $c$ ($d$ is the dimension of the fundamental representation $\pi_{\text{fund}}$ of $G$).

Then $\mu_U(F_{p,c}) = 0$.

**Proof**

Notice that $F_{p,c}$ is measurable since it can be characterized in terms of measurable functions. Consider the measurable function $f_{\epsilon,t}(\phi) := \exp(-\frac{1}{\epsilon^2}[1 - \frac{1}{\pi} \text{tr}(U_{\phi}(c(t))U_{\phi}(p)^{-1})]$.

We have $\lim_{\epsilon \to 0} \lim_{t \to 0} f_{\epsilon,t}(\phi) = \chi_{F_{p,c}}(\phi)$ where $\chi$ denotes the characteristic function. The two-parameter set of functions $f_{\epsilon,t}$ are bounded by the measurable function 1 and the limit $\lim_{\epsilon \to 0, t \to 0} f_{\epsilon,t} = \chi_{F_{p,c}}$ exists pointwise. Thus, since $\mu_U$ is a Borel measure, we may apply the Lebesgue dominated convergence theorem to exchange limits and integration so that

$$\mu_U(F_{p,c}) = \lim_{\epsilon \to 0, t \to 0} \int_{\mathcal{U}} d\mu_U(\phi)f_{\epsilon,t}(\phi) = \lim_{\epsilon \to 0} \int_{G^2} d\mu_H(g) d\mu_H(h) e^{-\frac{1}{\epsilon^2}[1 - \frac{1}{\pi} \text{tr}(gh^{-1})]}$$

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by the definition of the measure $\mu_U$. We can now see where the measure zero property has its origin: the measure is diffeomorphism invariant and therefore the $t$-dependence has completely dropped out of the integral, two points are “arbitrarily far away” as soon as they are different. To complete the proof use the translation invariance and normalization of the Haar measure to see that the last integral becomes

$$\int_G d\mu_H(g) e^{-\frac{1}{2t} \text{tr}(g)}$$

Now as $\epsilon \to 0$ we may expand $g = \exp(t^i \tau_i)$ around 1 which is where the main contribution comes from. Then $1 - \frac{1}{2t} \text{tr}(g) = 1/2t^i \delta_{ij} + o(t^i)^2$ and we conclude with a similar expansion for the Jacobian that the resulting (essentially Gaussian) integral is of order $\epsilon^{\dim(G)}$.

\[ \square \]

**Corollary 4.2** Since $U \subset F_{p,c}$ for any $p,c$ we have $\mu_U(U) = 0$.

Another result that we can copy is the following: the Gel'fand topology can be characterized by the fact that the (Gel'fand transforms of the) vertex functions are continuous. Now we can see that the spectrum is homeomorphic with the uncountable direct product $X := \prod_{v \in \Sigma} X_v$ of copies of $G$ (that is, $X_v = G$) equipped with the Tychonov topology. To see this, recall that the Tychonov topology is the weakest topology such that all the projections $p_v : X \to X_v$ are continuous. Therefore, every continuous function on $X$ is of the form of sums of products of functions of the form $p_v f_v$ where $f_v$ is a continuous function on $X_v = G$. But these are precisely the vertex functions. Therefore the topologies on $U$ and $X$ are generated by the same functions. That both sets are bijective and inverse bijective with each other follows from the fact that the infinite array of group elements $(g_v)_{v \in \Sigma}$ which is an element of $X$ defines obviously a $G$-valued function $U(v) := g_v$ and now the assertion follows from Theorem 1.1. This is the precise analogue of the result [8] that $\overline{A/G}$ is the projective limit (rather than direct product) of spaces $(\overline{A/G})_\gamma$, one for each graph $\gamma$, which is isomorphic with $G^n$, $n$ the number of edges of $\gamma$.

We now wish to show that $\mu_U$ is the unique probability measure on $\overline{U}$ which incorporates the reality condition that $\phi_I, p_J$ are real-valued and that realizes the canonical commutation relations $[\hat{\phi}_I(x), \hat{p}^J(y)] = i\hbar \delta^I_J \delta(x,y)$. Notice that for $\phi \in U$ we have that $U_{\phi}(v)$ is a unitary matrix. The operator $(\hat{U}_{\mu\nu}(v)f)(\phi) := (U_{\phi}(v))_{\mu\nu} f(\phi)$ is densely defined on $L_2(\overline{U}, d\mu_U)$, thus the appropriate adjointness relation to be implemented is $[\hat{U}_\phi(v)]^\dagger = [(U_\phi(v))^{-1}]_{\mu\nu}$. The operator $\hat{p}^I(x) := -i\hbar \delta/\delta \phi_I(x)$ on the other hand, which formally implements the canonical commutation relations $[\hat{\phi}_I(x), \hat{p}^J(y)] = i\hbar \delta^I_J \delta(x,y)$, is certainly not densely defined on the $L_2$ space. This is to be expected because our states depend on $\phi$ at a given point, that is, $\phi$ is not at all smeared with test functions. The situation is somewhat similar to the case of $\overline{A/G}$ where the connection is only smeared in one direction along the path of the associated holonomy: the electric field operators are not densely defined. What is densely defined though is the integral of the electric field over a two-dimensional surface. The surface and the path form together three smearing directions which are sufficient to absorb the delta-distribution that appears in the canonical commutation relations. Likewise we are lead here to consider the operators $\hat{p}^I(B) := \int_B d^3x \hat{p}^I(x)$ which gives the required three-dimensional smearing.
over a three-dimensional compact subset \( B \subset \Sigma \). We ask that \([\hat{p}^I(B)]^\dagger = \hat{p}^I(B)\) for each \( B \). Notice that the integral is covariant under diffeomorphisms because \( p_I \) is a density of weight one.

Let us check that \( \hat{p}^I(B) \) is indeed densely defined. Let \( f_{\vec{v}} \) be a cylindrical function. We are going to evaluate \( \hat{p}^I(B) \) on \( f_{\vec{v}} \) first of all for a smooth field \( \phi \) and then are able to generalize to \( \mathcal{U} \). We have formally

\[
\hat{p}^I(B) f_{\vec{v}}(\phi) = -i\hbar \int_B d^3 x \sum_{v \in \vec{v}} \frac{\delta U_{\nu,\mu\nu}(\phi)}{\delta \phi_I(x)} \frac{\partial f_{\vec{v}}(\phi)}{\partial U_{\nu,\mu\nu}(\phi)}
\]

and everything boils down to the appropriate definition of the functional derivative involved in (4.3). As it stands, it is ill-defined. We will proceed as follows: let \( h_\epsilon(x, y) \) be a one-parameter family of smooth functions such that \( \lim_{\epsilon \to 0} \int_\Sigma d^3 y f(y) h_\epsilon(x, y) = f(x) \) for every smooth function, say, of rapid decrease. Let \( \alpha_{r,s} \) be a two parameter family of smooth loops which embed an abstract solid cylinder into \( \Sigma \) with \( r, s \in [0, 1] \) and such that \( \nu \) lies inside the image of \([0, 1]^2 \times S^1\). It follows that the equation \( \alpha_{r,s}(t) = \nu \) has only one solution in \([0, 1]^3\). Consider the regulated quantity

\[
U^\epsilon_{\nu}(\phi) := \mathcal{P} \exp\left( \int_0^1 dt \phi(\alpha, \epsilon, \nu, t) \right) \quad \text{with} \quad \phi(\alpha, \epsilon, \nu, t) := \int_0^1 dr \int_0^1 d\nu e_{\nu}(\alpha_{r,s}(t), v) \det\left( \frac{\partial \alpha_{r,s}(t)}{\partial (r, s, t)} \right) |\phi_I(\alpha_{r,s}(t))| \tau_I.
\]

where \( \mathcal{P} \) means path ordering. Notice that \( \lim_{\epsilon \to 0} U^\epsilon_{\nu}(\phi) = U_{\nu}(\phi) \). There are two reasons for why we chose the particular smearing \((4.5)\) of \( U_{\nu}(\phi) \) which involves the path ordering:

The first motivation is that the point holonomies \( U_{\nu}(\phi) \) are in a certain sense limits of usual loop holonomies as the the loop shrinks to a point, evaluated on a distributional connection (for a smooth connection the limit would equal \( 1_G \)). Indeed, for each loop \( \alpha_{r,s}(t) \) we may associate a G-“connection”

\[
\hat{A}_0(\alpha_{r,s}(t)) := h_\epsilon(\alpha_{r,s}(t), v) \det\left( \frac{\partial \alpha_{r,s}(t)}{\partial (r, s, t)} \right) |\phi_I(\alpha_{r,s}(t))| \tau_I \quad \text{with} \quad \hat{A}_0(\alpha_{r,s}(t))^\dagger = \hat{A}_0(\alpha_{r,s}(t)).
\]

where we have used a Euclidean norm \(||.||\) to raise indices. The meaning of this quantity is that \( \int dr \int ds \int d\nu A_\alpha, A = \int dt \phi(\alpha, \epsilon, \nu, t) \).

The second motivation is that the naive definition \( \delta U_{\nu}(\phi)/\delta \phi_I(x) := \delta(x, v) \partial U_{\nu}(\phi)/\partial \phi_I(\nu) \), when integrated over \( B \), does not result in a function which can be expressed as a polynomial in \( U_{\nu,\mu\nu} \) again so that \( p^I(B) \) would not be densely defined. The particular choice \((4.6)\) will lead to a densely defined operator.

The functional derivative of the quantity \((4.6)\) is well-defined: We get just zero unless there is a solution \( \alpha_{r,s}(t_x) = x \) with \((r_x, s_x, t_x) \in (0, 1)^2 \times S^1\) in which case we find

\[
\frac{\delta U^\epsilon_{\nu}(\phi)}{\delta \phi_I(x)} = \int_{[0, 1]^3} dr ds dt \delta(x, \alpha_{r,s}(t)) h_\epsilon(\alpha_{r,s}(t), v) \det\left( \frac{\partial \alpha_{r,s}(t)}{\partial (r, s, t)} \right) |U^\epsilon_{\nu}(0, t_x)\tau_I U^\epsilon_{\nu}(t_x, 1)|
\]
\[ h_\epsilon(x, v) \int_{[0,1]^3} dr ds dt \delta(r, r_x) \delta(s, s_x) \frac{1}{2} \left[ \delta(t, t_x) + \delta(t, t_x^+) \right] \times \]
\[ \times \left| \det \left( \frac{\partial \alpha_{r,s,t}(t)}{\partial(r,s,t)} \right) \right| U_v^\epsilon(0, t_x) \tau_I \left( U_v^\epsilon(t_x, 1) \right) \]
\[ = h_\epsilon(x, v) \frac{1}{2} \left[ U_v^\epsilon(0, t_x^+) \tau_I U_v^\epsilon(t_x^+, 1) + U_v^\epsilon(0, t_x^-) \tau_I U_v^\epsilon(t_x^-, 1) \right]. \quad (4.7) \]

Here we mean by \( U_v^\epsilon(a, b) \) the path ordered exponential \((4.6)\) for \( t \in [a, b] \) where we identify values modulo 1. Now we easily see that \( \lim_{\epsilon \to 0} U_v^\epsilon(t_x, t_x + 1) \) gives \( U(v)(\phi) \) if \( t_x = 0 \) and \( 1_G \) otherwise. However, since \( h_\epsilon \) gives zero as \( \epsilon \to 0 \) unless \( x = v \) for which indeed \( t_x = 0 \equiv 1 \) (mod 1) we find the functional derivative of \( U_v(\phi) \) to be
\[ \frac{\delta U_{\text{gau}}(\phi)}{\delta \phi_I(x)} := \lim_{\epsilon \to 0} \frac{\delta U_{\text{gau}}(\phi)}{\delta \phi_I(x)} = \delta (x, v) \frac{1}{2} \left[ U_v(\phi) \tau_I + \tau_I U_v(\phi) \right]_{\mu\nu}. \quad (4.8) \]

The end result \((4.8)\) is independent of the particular function \( h_\epsilon \) or family of loops \( \alpha_{r,s} \) of loops used and is therefore satisfactory.

Putting everything together we finally find the remarkably simple formula
\[ p^I(B)f_{\tilde{v}} = -i h \sum_{\nu \in B \cap \tilde{v}} X^I_{\nu} f_{\tilde{v}} \quad (4.9) \]
where
\[ X^I_{\nu} := X^I(U_v(\phi)), \ X^I(g) := \frac{1}{2}[X^I_R(g) + X^I_L(g)], \]
\[ X^I_{\nu}(g) := \text{tr}(\tau_I g) \frac{\partial}{\partial g}, \]
\[ X^I_L(g) := \text{tr}(g \tau_I) \frac{\partial}{\partial g}. \quad (4.10) \]

The vector fields \( X^I_R \) and \( X^I_L \) on \( G \) are respectively right and left invariant in the sense \( X^I_R(hg) = X^I_R(g), \ X^I_L(hg) = X^I_L(g) \) for each \( h \in G \) as one can easily check. However, \( X^I_R(hg) = \text{Ad}(h^{-1})^{IJ} X^J_R(g), \ X^I_L(hg) = \text{Ad}(h)^{IJ} X^J_L(g) \) where \( h^{-1} = \text{Ad}(h)^{IJ} \tau^J \) denotes the matrix elements of the adjoint representation of \( G \) on \( \text{Lie}(G) \). This is an important consistency check because it implies that \( X^I(hgh^{-1}) = \text{Ad}(h)^{IJ} X^J(g) \) transforms according to adjoint representation as well and so \((4.3)\) is covariant under gauge transformations.

Another intuitively appealing feature of \((4.3)\) is the following: recall that the functional derivative of a usual path holonomy \( h_\epsilon \) at \( x \) is proportional to a right invariant vector field \( X^I_R(h_\epsilon) \) if \( x \) is the starting point of \( e \) and proportional, with the same coefficient, to a left invariant vector field \( X^I_L(h_\epsilon) \) if \( x \) is the end point of \( e \). Therefore, formally, we expect that if \( e \) shrinks to a point (or if \( e \) is a loop) then we must get the average \( X(h_\epsilon) = (X_R(h_\epsilon) + X_L(h_\epsilon))/2 \) which is exactly the result \((4.9)\), thus reassuring that our interpretation of \( U(v) \) as a point holonomy is correct.

As one can easily check, if \([\tau_I, \tau_J] = f_{IJK} \tau_K\) define the structure constants of \( \text{Lie}(G) \) then \([X^I_R, X^J_R] = -f_{IJK} X^{JK}_R, [X^I_L, X^J_L] = +f_{IJK} X^{JK}_L, [X^I_R, X^J_L] = 0 \) which implies that \([X^I, X^J] = f_{IJK}(X^K_R + X^K_L)/4 \) so that the algebra of the \( X^I \) is not closed while the algebra of the \((-X_R + X_L)/2 \) is closed. We note that if \( X^I_{\pm} = \frac{1}{2}[X^I_L \pm X^I_R] \) then
\[ [X_\mu^I, X_\nu^J] = [X_\mu^I, X_\nu^J] = \frac{\hbar \mu \kappa}{4} X^K, \quad [X_\mu^I, X_\nu^J] = \frac{\hbar \mu \kappa}{4} X^K. \]

That we do not find the right hand side of \([p^I(B), p^J(B')]\) to be expressible in terms of a \(p^I(B')\) again is not surprising as the Poisson bracket between the corresponding functions should actually vanish. The reason for this “anomaly” is due to the fact that we did not properly smear the \(\phi\)'s as explained in \[26\] : we can expect the correspondence as they should. We can consider the quantity \(\{\ldots\}\) to be a \(\{\ldots\}\) function of \(p\) not commute which suggests to build gauge invariant quantities. In fact, classically only \(U\) commute as \(U\) is not the case here. We conclude that nothing is inconsistent. See \[26\] for a more detailed explanation for this phenomenon.

Interestingly, we do not encounter any such “anomaly” if we restrict attention to gauge invariant functions of \(p\) which are the only ones in which we are interested in anyway: indeed, as we saw the “anomaly” arose only due to the fact that the \(X^I\) do not commute which suggests to build gauge invariant quantities. In fact, classically only the modulus of the vector valued density \(p\) is gauge invariant so that we are lead to consider the quantity \(\hat{K}(B) := \hat{\rho}^I(B)\hat{\rho}^J(B)\delta_{IJ}\) which on vertex functions gives \(\hat{K}(B) = -\hbar^2 \sum_{v \in B \cap \delta} \delta_{IJ} X_v^I X_v^J f_v.\) Choosing \(B = B_v\) small enough a neighbourhood of a point \(v\) we find that \(\hat{K}(B_v) =: \hat{K}_v^2 = -\hbar^2 X_v^I X_v^J \delta_{IJ}\) which is gauge invariant and all the \(\hat{K}_v\) actually commute as they should.

In any case we are now in the position to show that the adjointness relations \(\hat{U}_{v, \mu \nu}(\phi)^\dagger = \hat{U}_{v, -\nu \mu}(\phi)\), \((\rho^I(B))^\dagger = \rho^I(B)\forall v, B\) hold on \(H = L_2(|\mathcal{U}|, d\mu_U)\). The first relation follows trivially from the fact that \(\hat{U}_{v, \mu \nu}(\phi)\) is a multiplication operator on \(H\) and so its hermitean conjugate is its complex conjugate. The second follows from the fact that \(X_v^I\) is a linear combination of left and right invariant vector fields on \(G\) both of which annihilate the volume form corresponding to the (left and right invariant) Haar measure. Moreover, by results following from \[1\] one can show that \(\hat{\rho}^I(B)\) is essentially self-adjoint and that the adjointness relation on \(\hat{U}_{\nu, \mu \nu}\) is identically satisfied.

This suffices to show that \(\mu_U\) implements the \(*\) relations. Now, since \(\rho^I\) is linear in left and right invariant vector fields and \(\hat{U}_{\nu, \mu \nu}\) is a multiplication operator, the measure \(\mu_U\) is uniquely selected (if we insist that it be a probability measure) by the above reality conditions as was shown for a general theory in \[3\] simply because the Haar measure on a compact gauge group is by definition annihilated by precisely the left and right invariant vector fields.

We now address the diffeomorphism invariance. The space \(\Phi_U\) is the space of finite linear combinations of vertex functions equipped with the “Fourier topology” (similar as in the case of the fermion field)

\[ ||f||_1 := \sum_{\bar{\varepsilon}, \bar{\mu}, \bar{\nu}} |< H_{\bar{\varepsilon}, \bar{\mu}, \bar{\nu}}, f >| \]

and \(\Phi_U^\prime\) its topological dual (alternatively, as in \[3\], we could equip \(\Phi_U\) with a standard nuclear topology on \(G^n\) where \(n\) is the number of vertices of the cylindrical function in question). Again we have the inclusion \(\Phi_U \subset \mathcal{H}_U \subset \Phi_U^\prime\) and we are looking for solutions of the diffeomorphism constraint in \(\Phi_U^\prime\) by a straightforward application of the group averaging method from \[3\]. The unitary representation of the diffeomorphism group is defined by \(\hat{U}(\varphi)\hat{f}_{\bar{\varepsilon}} := \hat{f}_{\varphi(\bar{\varepsilon})}\) for each smooth diffeomorphism \(\varphi\) of \(\Sigma\) (in case that \(\bar{\varepsilon}\)}

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are vertices of a graph $\gamma$ defined by gauge fields as below consider only those smooth
diffeomorphisms that leave $\gamma$ piecewise analytic). Let us use a compound label $I$ for
($\vec{p}, \vec{\mu}, \vec{v}$). Consider the orbit \( \{H_{\vec{v},I} := \{H_{\varphi(v),I}, \varphi \in \text{Diff}(\Sigma)\} \) . Then we define
\[
[H_{\vec{v},I}] := \sum_{H \in \{H_{\vec{v},I}\}} H
\]
and extend by linearity, that is, decompose every $f \in \Phi_U$ first in terms of $H_{\vec{v},I}$'s and then average each $T_{\vec{v},I}$ separately. We quickly verify that this method always results
in elements of $\Phi'_U$. The diffeomorphism invariant Hilbert space is then defined through
\[
\langle [f], [g] \rangle_{\text{Diff}} = [f(g)] \text{ for any } f, g \in \Phi_U.
\]

5 Gauge Invariant States

Let us summarize : our Hilbert space of diffeomorphism non-invariant (not necessarily
gauge invariant) functions of gravitational, gauge, spinor and Higgs fields is given by
\[
\mathcal{H} := L_2(\mathcal{A}_{SU(2)}, d\mu_{AL}(SU(2))) \otimes L_2(\mathcal{A}_G, d\mu_{AL}(G)) \otimes L_2(\mathfrak{S}, d\mu_F) \otimes L_2(\mathcal{U}, d\mu_U).
\]

The Hilbert space of gauge invariant functions will be just the restriction of $\mathcal{H}$ to
gauge invariant functions. It is easy to see that, because our total measure is a probability
measure, gauge invariant functions will be still integrable with respect to it.

A natural gauge invariant object associated with spinor fields, Higgs fields and gauge
fields are “spin-colour-network states”. By this we mean the following : Let $\gamma$ be a
piecewise analytic graph with edges $e$ and vertices $v$ which is not necessarily connected
or closed. By suitably subdividing edges into two halves we can assume that all edges are
outgoing at a vertex. Given a (generalized) connection $\omega_a = A_a + A_a$ we can compute the
holonomies $h_e(A), h_e(A), H_e(\omega) = h_e(A)h_e(\mathbf{1})$. With each edge $e$ we associate a spin $j_e$
and a colour $c_e$ corresponding to irreducible representations of $SU(2)$ and of $G$ respectively
(for instance for $G = SU(N)$, $c_e$ is an array of $N - 1$ not increasing integers corresponding
to the frame of a Young diagramme). Furthermore, with each vertex $v$ we associate an
integer $0 \leq n_v \leq 2d$, yet another colour $C_v$ and two projectors $p_v, q_v$. The integer
$n_v$ corresponds to the subvector space of $Q_v$, the vector space spanned by holomorphic
functions of $\theta_i(v)$, spanned by those vectors $F_I, v$ such that $|I| = n(v)$. Likewise, the colour
$C_v$ stands for an irreducible representation of $G$, evaluated at the point holonomy $U(v)$.
The projector $p_v$ is a certain $SU(2)$ invariant matrix which projects onto one of the linearly
independent trivial representations contained in the decomposition into irreducibles of the
tensor product consisting of
a) the $n_v$-fold tensor product of fundamental representations of $SU(2)$ associated with
the subvector space of $Q_v$ spanned by the $F_I, v$, $|I| = n(v)$ and
b) the tensor product of the irreducible representations $j_e$ of $SU(2)$ of spin $j_e$ where $e$
runs through the subset of edges of $\gamma$ which start at $v$.

Likewise, the projector $q_v$, repeats the same procedure just that $SU(2)$ is being replaced
by $G$ and that we need to consider in addition the adjoint representation associated with
$C_v$ coming from the Higgs field at $v$. Now we simply contract all the indices of the tensor
product of
1) the irreducible representations evaluated at the holonomy of the given connection,
2) the fundamental representations evaluated at the given spinor field and
3) the adjoint representations evaluated at the given scalar field,
all associated with the same vertex \( v \), with the projectors \( p_v, q_v \) in the obvious way and
for all \( v \in V(\gamma) \). The result is a gauge invariant state

\[ T_{\gamma,[j,\vec{a},\vec{p},e,\vec{c},\vec{C},\vec{q}]} \]

which we will call a spin-colour-network state because they extend the definition of the
pure spin-network states which arise in the source-free case (e.g. [3]).
These spin-colour-networks turn out to be a basis for the subspace of gauge invariant func-
tions. They are not orthonormal, but almost : we just need to decompose the fermionic
dependence into an orthonormal basis of the \( Q_v \).
This means that we get a gauge invariant Hilbert space

\[ \mathcal{H}_{\text{inv}} := L_2(\mathcal{A}_{SU(2)} \times \mathcal{A}_G \times \mathcal{S} \times \mathcal{U}/\mathcal{G}, d\mu_{AL}(SU(2)) \otimes d\mu_{AL}(G) \otimes d\mu_F \otimes d\mu_U) \]

where \( \mathcal{G} \) denotes the action of the gauge group \( SU(2) \times G \) on all the fields at each point
of \( \Sigma \).

Next, in order to get the full space of diffeomorphism invariant distributions consider
in addition to the spaces \( \Phi_F, \Phi_U \) the spaces \( \Phi_{SU(2)}, \Phi_G \) of gravitational and gauge field
finite linear combinations of not necessarily gauge invariant (generalized) spin-network
and colour-network functions [3] equipped with a nuclear topology from \( SU(2)^n \) and \( G^n \) respectively, \( n \) the number of edges of the graph of the cylindrical function in question.
Let \( \Phi'_{SU(2)}, \Phi'_G \) be their topological duals. We now consider the product topological vector
space and its dual

\[ \Phi = \Phi_{SU(2)} \times \Phi_G \times \Phi_F \times \Phi_U \quad \text{and} \quad \Phi' = \Phi'_{SU(2)} \times \Phi'_G \times \Phi'_F \times \Phi'_U \]

take the gauge invariant subspaces thereof and construct diffeomorphism invariant elements
of \( \Phi' \) as follows : take a spin-colour-network function \( T \), consider its orbit under
analyticity preserving diffeomorphisms \( \{T\} \) and define \( [T] := \sum_{T' \in \{T\}} T' \) (slightly modified as in [18] through the presence of the graph symmetries). Take any \( f \in \Phi \), decom-
pose it in terms of \( T' \)’s, average each \( T \) separately and define the result to be \( [f] \). Then
\( <[f],[g]>_{\text{Diff}} := [f](g) \) for each \( f, g \in \Phi \) defines the Hilbert space \( \mathcal{H}_{\text{Diff}} \) upon comple-
tion.

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