ON DEFORMATIONS OF HAMILTONIAN ACTIONS

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Abstract. In this paper we generalize to coisotropic actions of compact Lie groups a theorem of Guillemin on deformations of Hamiltonian structures on compact symplectic manifolds. We show how one can reconstruct from the moment polytope the symplectic form on the manifold.

1. Introduction

Let $G$ be a compact connected Lie group acting in a Hamiltonian fashion on a $2n$-dimensional compact symplectic manifold $M$. We will denote by $\mu$ the corresponding moment map from $M$ to the dual of the Lie algebra of $G$. Throughout the following we will indicate Lie groups and their Lie algebras with capital and gothic letters respectively.

Fix a maximal torus $T$ in $G$ and denote by $t^*$ the dual of its Lie algebra.

A well known result of Kirwan [K2] states that the moment map image of $M$ meets the closure of a Weyl chamber $t^*_+$ in a convex polytope $\Delta(M)$. Delzant [D2] conjectured that $\Delta(M)$, together with some additional invariants of the action, determines the manifold up to $G$-equivariant symplectomorphisms (this has been solved in special cases, see e.g. [D1], [D2], [I], [W1] and [C]). We deal with a local version of this conjecture. We consider pairs $(\omega, \varphi)$ where $\omega$ denotes the symplectic form on $M$ and $\varphi$ the Hamiltonian differentiable action of a compact Lie group $G$ on $M$. We study smooth deformations of Hamiltonian pairs on a fixed manifold proving the following

Theorem 1. Let $(M, \omega)$ be a compact symplectic manifold acted on coisotropically and effectively by a compact Lie group $G$ in a Hamiltonian fashion. The moduli space of Hamiltonian $G$-structures $(\omega, \varphi)$ on $M$ is a smooth manifold whose dimension $k$ is equal to the second Betti number of $M$. Moreover there are $2k$ points, not necessarily distinct, $q_1, q_2, \cdots, q_k, q'_1, q'_2, \cdots, q'_k$ in the moment map image such that, as one varies the pair $(\omega, \varphi)$, the distances between $q_i$ and $q'_i$ are coordinates of this moduli space.

Guillemin in [G] solves the problem in the abelian case under the following additional assumptions

i) A torus $T$ acts locally freely on an open dense subset of $M$;

ii) the set of fixed points, $M^T$, is finite;

iii) the restriction of the moment map on the set $M^T$ is injective.

In the same paper the author remarks also that one can treat the non-abelian case replacing the images of $M^T$ with the set of vertices of the Kirwan’s polytope under some additional
hypotheses on the action. Actually, in our proof we show that the relevant points are exactly
the images of points fixed by a previously chosen maximal torus of \( G \).

The paper is organized as follows: we firstly prove a version of Guillemin’s theorem in case
the moment map restricted to \( M^T \) \textit{is not injective}. Then, with an elementary argument, we
show how to recover the image of the moment map for the action of the maximal torus from
the Kirwan’s polytope (Lemma 4). We finally show that if the \( G \)-action is coisotropic the
problem boils down to the abelian situation treated in [G] in the case the map \( \mu_{|MT} \) is not
injective.

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2. The abelian case

Throughout this section a Hamiltonian action of a torus \( T \) on a compact symplectic manifold
\((M, \omega)\) is fixed. Denote by \( \mu : M \to t^* \) the corresponding moment map.

For the reader’s convenience we here recall some notations from [G] adapted to our purposes.
Let \( p \) be a \( T \)-fixed point, and let \( \Sigma_p = \{ \alpha_1^p, \alpha_2^p, \ldots, \alpha_n^p \} \subset t^* \) be the set of weights of the isotropy representation on \( T_p M \). From now on we assume that \( M^T \) is finite. Choose a topological generator of \( T \), say \( \xi \in t \) such that \( \alpha_i(\xi) \neq 0 \) for all \( \alpha_i \in \cup \Sigma_p \) where \( p \) ranges through \( M^T \). Denote by \( \sigma_p \) the cardinality of the set \( \{ \alpha \in \Sigma_p : \alpha(\xi) < 0 \} \). Consider now the real valued function \( \mu_\xi : x \mapsto \mu(x)(\xi) \). The defining properties of \( \mu \) implies
that the critical points of \( \mu_\xi \) are the fixed points of \( T \). Moreover \( \mu_\xi \) is a Morse function (see e.g. [A] and [F]) and the index of a critical point \( p \in M^T \) equals \( 2\sigma_p \).

The fact that there are no critical points of odd index implies that \( \mu_\xi \) is perfect, hence the
even Betti numbers \( b_{2k} \) equals the number of points in \( M^T \) with \( \sigma_p = k \) and the odd ones
vanish.

The moment map image of \( M \) is convex [A], and the directions of the edges starting from a
vertex \( q \in \mu(M^T) \) are given by the weights of the representations at points \( p_i \) in the fiber
of \( q \). More precisely, let \( q \) be the image of a \( T \)-fixed point \( p \) in \( M \), via the moment map.
On every direction \( \alpha \in \Sigma_p \) starting from \( q \) there exists another point \( q' \in \mu(M^T) \) [C]. It is
natural to investigate whether one can or not “determine” the even Betti numbers of \( M \) from
the image of the moment map.

When the restriction of the moment map to \( M^T \) is not injective \textit{this is not always possible};
if two points \( p, p' \) have the same image \( q \) via the moment map they describe the same cones
\( C_p = cone(\alpha_1^p, \alpha_2^p, \ldots, \alpha_n^p) \) and \( C_{p'} = cone(\alpha_1^{p'}, \alpha_2^{p'}, \ldots, \alpha_n^{p'}) \) (this can be seen combining Example 4.21 and Theorem 6.5 in [S] since the fibers of the moment map are connected)
nevertheless they can have \( \sigma_p \neq \sigma_{p'} \), for any choice of the topological generator of \( T \), as
Example 4 shows.
**Example 1.** Let $T$ be a maximal torus in $\text{SO}(5)$. The usual choice for a basis for $t$ gives isomorphisms $t \cong \mathbb{R}^2$ and

$$t^* \cong \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq x\}.$$  

Let $\Theta_\gamma$ be the coadjoint orbit through $(\gamma,\gamma)$ and $\Theta_\delta$ be the coadjoint orbit through $(\delta,0)$ where $\gamma$ and $\delta$ are positive real numbers. Consider the diagonal action of $T$ on the product $M = \Theta_\gamma \times \Theta_\delta$. If $\gamma = \delta$ the image of the moment map is given in Fig. 1 and the dots are the image of the $T$-fixed points. The interior dots represent points that have two inverse images. One can determine the weights of the representations on the tangent spaces at two different points in the same fiber. As one can check from the picture, they differ for one weight, hence for any choice of $\xi$ there is always a pair of $T$-fixed points with the same image and different $\sigma$. In the figure the filled arrows denote the common weights, while the dashed ones come from different pre-images of the same point.

**Figure 1.** Moment polytope of a $T^2$-action on the product of two coadjoint orbits of $\text{SO}(5)$.

The previous example clearly shows that one cannot determine the Betti numbers of $M$ starting from $\mu_T(M)$ without coming back to the manifold and counting the contribution of every point in $M^T$. Nevertheless, with the same arguments used in [G] (pag. 230 Lemma 2), it is always possible to associate to each point in $p \in M^T$ of index $k$ an element $c_p \in H_{2k}(M)$ and show that the set of $\{c_p, \ \sigma_p = k\}$ yields a basis of $H_{2k}(M)$.

Now consider, in particular, the set of points $p_i \in M^T$ with $\sigma_{p_i} = 1$; the images of these points can be distinct or not. Note that for each $p_i$ in this set there exists only one $\alpha_{p_i}^{p_i} \in \Sigma_{p_i}$ with $\alpha_{p_i}^{p_i}(\xi) < 0$. Moreover on the ray

$$q + \alpha_{j}^{p_i} t \quad 0 < t < \infty$$

there is at least another point $q' \in \mu(M^T)$. More precisely, we have

**Lemma 2.** Let $t_0 = [\omega](c_p)$. Then the point $q' = q + t_0 \alpha_j^p$, where $p$ has $\sigma_p = 1$, belongs to $\mu(M^T)$.

This can be proven following the same ideas of [G]. Thus, in particular, if $p_1, p_2, \ldots, p_k$ are points of index 1, the distances between their (not necessarily distinct) images, via the
moment map, \(q_1, q_2, \ldots, q_k\) and \(q'_1, q'_2, \ldots, q'_k\) determine the cohomology class of \(\omega\) in \(H^2(M)\).

Now we recall some definitions and facts on deformations. By a differentiable deformation of a differentiable action \(\varphi\) of a group \(G\) on a manifold \(M\) we mean a one-parameter family \(\varphi_t (t \in I = [0, 1])\) of differentiable actions such that \(\varphi_0 = \varphi\) and the map \((g, p, t) \mapsto \varphi_t(g, p)\) of \(G \times M \times I\) into \(M\) is differentiable. Recall that a deformation of \(M\) is a one-parameter family \(\psi_t\) of diffeomorphisms of \(M\) such that \(\psi_0\) is the identity and the map \((p, t) \mapsto \psi_t(p)\) is differentiable. A deformation action \(\psi_t\) of a manifold \(M\) under the action \(\varphi\) of a group \(G\) induces a differentiable deformation \(\varphi_t\) defined by

\[
\varphi_t(g, p) := \psi_t(\varphi(g, \psi_t^{-1}(p))).
\]

Such a deformation \(\varphi\) is called trivial. Palais and Stewart showed that

**Proposition 3.** [PS] If the group \(G\) and the manifold \(M\) are assumed to be compact then any differentiable deformation of a given \(G\)-action \(\varphi\) on \(M\) is differentiably trivial.

Hence we can assume that the \(G\)-action is fixed. Moreover, if we “move” the symplectic form in its cohomology class, using a \(G\)-equivariant version of Moser’s Theorem, we get that the moment map image does not change. Henceforth the only way to deform the Hamiltonian pair \((\omega, \varphi)\) is to deform the class \([\omega]\).

After a deformation of the class of \(\omega\), the directions of the edges of the \(\mu_T(M)\) are preserved, while the distances of certain points give us a “measure” of the deformation. Indeed, if we put on the manifold \(M\) a \(T\)-invariant metric \(g\) and consider the map that associates to each class \(c\) in \(H^2(M, \mathbb{R})\) the 2-form \(\omega_c = \omega + h_{c-c_0}^g\), where \(c_0\) is the cohomology class of \(\omega\) and \(h_{c-c_0}^g\) the unique harmonic representative of \(c - c_0\), we get a \(T\)-invariant closed 2-form, \(\omega_c\), which is symplectic if \(c\) is close to \([\omega]\). Moreover, since \(H^1(M, \mathbb{R}) = 0\), the action \(\varphi\) is Hamiltonian on \((M, \omega_c)\). Therefore the set of pairs \((\omega_c, \varphi)\), with \(c\) close to \([\omega]\), is an open subset of the moduli space of \(G\)-Hamiltonian actions on \(M\) whose coordinates are thus given by the distances of \(q_1, q_2, \ldots, q_k\) and \(q'_1, q'_2, \ldots, q'_k\), according to Lemma 2 and the subsequent remarks.

3. The non abelian case: proof of Theorem 1

In this section we consider a compact, non necessarily abelian, Lie group \(G\) acting in a Hamiltonian fashion on a compact symplectic manifold \((M, \omega)\). Fix once and for all a maximal torus \(T\) in \(G\). We will denote by \(\mu\) and \(\mu_T\) the corresponding moment maps for the \(G\)- and the \(T\)-action respectively. The following lemma shows how to recover the \(\mu_T\) image from the Kirwan polytope \(\Delta(M)\), more precisely from \(\mu(M) \cap \mathfrak{t}^*\) (which is obtained from \(\Delta(M)\) by reflections through the walls of the Weyl chambers). We state and prove it since we could not find it in the literature, although it is probably well-known.

**Lemma 4.** The set \(\mu_T(M)\) coincides with the convex hull of \(\mu(M) \cap \mathfrak{t}^*\).

**Proof.** Firstly recall that the image of the moment map for the \(T\)-action is the convex hull of the finite set \(\mu_T(M^T)\) [A]. Note that \(\mu\) and \(\mu_T\) coincide on \(M^T\) indeed the restriction of the moment map \(\mu\) to the fixed point set of a closed subgroup \(H\) of \(G\), takes values in the dual of the Lie algebra of the centralizer of \(H\) in \(G\), \(\xi_{\mathfrak{h}}(h)^*\), hence for \(H = T\) the image is
contained in $t^*$. Therefore $\mu_T(M) = \text{conv}(\mu(M^T)) \subset \text{conv}(\mu(M) \cap t^*)$. The other inclusion follows from the fact that $\mu_T$ is the composition of $\mu$ with the projection $g^* \to t^*$ induced by the inclusion $t \hookrightarrow g$.

The previous lemma is still true if the moment map $\mu$ is assumed to be proper, and the manifold is not necessarily compact.

If we assume that the $T$-fixed point set is finite, and that $T$ acts on an open dense subset $M_0 \subseteq M$ locally freely, we can apply the results of section 2 to the non-abelian case. A large class of actions for which $M^T$ is finite is provided by coisotropic ones. Recall that a Hamiltonian $G$-action is called coisotropic if the principal $G$-orbits $G \cdot p$ are coisotropic with respect to $\omega$, i.e. $(T_p G \cdot p)\omega \subseteq T_p G \cdot p$.

Before proving our main theorem we here state a theorem that gives a characterization of coisotropic actions on non necessarily compact symplectic manifolds. We will use it only in the compact setting, however we give here the proof in full generality since it might have an autonomous interest (see also [W2] in the compact setting).

**Theorem 5.** The following conditions for a Hamiltonian $G$-action on a connected symplectic manifold $M$ with connected moment map fibers are equivalent:

(i) The $G$-action is coisotropic.

(ii) The cohomogeneity of the $G$-action is equal to the difference between the rank of $G$ and the rank of a principal isotropy subgroup of $G$.

(iii) The set of $G$-invariant smooth functions $C^\infty(M)^G$ is an abelian Poisson algebra.

(iv) The moment map $\mu$ separates $G$-orbits.

We here enclose condition (ii) as it can be used in proving that Example 5 in Section 4 is coisotropic.

**Proof.** We get our claim, combining a result of [HW] and a result of [ACG].

The first two conditions are equivalent thanks to Theorem 3.3 in [HW]. The equivalence of (i) and (iii) is straightforward. We want to show that if (iii) holds then all the reduced spaces $M_q = \mu^{-1}(q)/G_q = \mu^{-1}(G \cdot q)/G_q$ are points. In general $M_q$ is not necessarily smooth, however Theorem 2 in [ACG] says that, for arbitrary $q$ the algebra

$$C^\infty(M_q) := C^\infty(M)^G / T_q^G$$

where $T_q^G$ is the ideal of $G$-invariant functions vanishing on $\mu^{-1}(q)$, is a non-degenerate Poisson algebra. That is the Poisson bracket vanishes only on constant functions on $M_q$. If $C^\infty(M)^G$ is abelian, then $C^\infty(M)^G / T_q^G$ is abelian and hence $M_q$ is a point, whenever it is connected (this naturally holds when the fibers of $\mu$ are connected). Finally, (i) follows from (iv) immediately because (i) means that the moment map fibers are generically tangent to $G$-orbits.

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** We firstly show that if the action is coisotropic then the $T$-fixed points are isolated. Since $\mu(M^T)$ is finite and $\mu$ separates orbits (iv), the set $\mu^{-1}(\mu(M^T))$ is contained in the union of finitely many $G$-orbits. The claim follows from the fact that the number of
T-fixed points in a $G$-orbit $\mathcal{O}$ is given by its Euler characteristic $\chi(\mathcal{O})$.

Now we only need the $T$-action to be locally-free on an open dense subset of $M$. In fact this is the case when the $G$-action is effective. Indeed, if $M_o \subseteq M$ is the dense open subset of $G$-principal points, and $L$ is a principal isotropy subgroup of $T$ acting on $M_o$, we have that $L$ acts trivially both on the tangent space to the orbit and on the slice, therefore it is contained in the kernel of the action.

Finally we can apply the non-injective version of Guillemin’s theorem, in order to obtain the coordinates of the moduli space of $T$-invariant Hamiltonian structures on $M$. Note that, since we start from a $G$-invariant symplectic form $\omega$, we can choose a $G$-invariant metric $g$ on $M$, compatible with $\omega$, thus the $g$-harmonic forms $h_{c-c_0}^{g,c}$ will be $G$-invariant, and the claim follows. $\square$

In the following example in which the action is not coisotropic, and the $T$-fixed point set is not finite, Theorem 1 does not hold.

Example 2. Consider the manifold $M$ given by the product of a full flag $G/T$ and a manifold $N$ acted on transitively on the first factor and trivially on the second one by $G$. The moment polytope for this action is a point. The convex hull of the points obtained by reflections through the Weyl walls gives us $\mu_T(M)$. The “deformations parameters” are at most $l = \text{rank}(G)$, while the second Betti number can be larger, whenever $b_2(N) > l$.

4. Examples

We have emphasized that the relevant points in the moment polytope in order to study deformations of Hamiltonian structures are the images of the $T$-fixed points. It would be interesting to distinguish from the combinatoric of the moment image the points in $\mu(M^T)$.

We here collect some facts in this regard.

We say that a convex polytope $\Delta(M) \subset t^*_+$ is reflective at $q \in \Delta(M)$ if and only if

(i) The set of hyperplanes that intersect $\Delta(M)$ in codimension 1 faces that contain $q$ is invariant under the stabilizer $W_q$ of $q$, where $W$ is the Weyl group of $G$;

(ii) any open codimension 1 face of $\Delta(M)$ containing $q$ in its closure is contained in the open positive Weyl chamber in $t^*_+$. 

The first condition is equivalent to requiring that if $H$ is an hyperplane such that $H \cap \Delta(M)$ is a codimension 1 face containing $q$ and $w \in W_q$ then $H \cap w\Delta(M)$ is a codimension 1 face of $w\Delta(M)$. That is, the codimension 1 faces continue through the walls.

(1) Let $q \in \mu(M)$ be a non reflective point. Then $q$ is the image of a $T$-fixed point.

Let $q$ be a non reflective vertex on a wall. Let $\alpha \in t^*$, be a direction, starting from $q$, such that it “does not continue” beyond the wall. Let $\mathfrak{t} = \{ \eta \in t, \alpha(\eta) = 0 \}$. Since $\alpha$ belongs to $t, \mathfrak{t}$ is the lie algebra of a closed connected Lie subgroup, $K$, of $T$; the quotient group $T/K$ is a circle group with Lie algebra $t/\mathfrak{t}$. Let $X$ be the connected component of $M^K$ that is sent by $\mu$ onto the line $q + t\alpha$. The group $K$ acts trivially on $X$ and the $T/K$-action on it is Hamiltonian with moment map $\mu_\alpha : X \to \mathbb{R}$ defined by $\mu_\alpha(x) = \langle \mu(x), \alpha \rangle$. The critical points of $\mu_\alpha$ are those $p \in X$ where the fundamental field associated to $\alpha$ vanishes, i.e. points fixed
by the torus \( T^\alpha \) generated by \( \alpha \). The function \( \mu_\alpha \) achieves its maximum and its minimum on the extremal points of the segment \( \{ q + t\alpha : t \in [0,t_0] \} \). Therefore \( q \) is the image of a point \( p \in M^K \) critical for \( \mu_\alpha \), and it is fixed by \( T \).

(2) The interior vertices \( q \), i.e., vertices that belongs to the interior of a Weyl chamber, are reflective (\( W_q \) is trivial) and images of \( T \)-fixed points. Since in this case the stabilizer \( G_q \) is exactly the maximal torus \( T \), the last statement is a consequence of the necessary condition for a point \( p \) of \( M \) to be sent to a vertex (see pag.81 Theorem 6.7 [S])

\[ g_q = [g_q, g_q] + g_p, \]

from which one gets \( G_p = G_q = T \).

Furthermore observe that, if the action is coisotropic, the fiber of an interior vertex contains only one \( T \)-fixed point \( p \), indeed each \( T \)-fixed point in the orbit \( G \cdot p \) is sent to a different Weyl chamber. Note that there can be points \( p \) with \( G_p = T \) that are sent to points on the boundary of a Weyl chamber (see fig. 2). The maximal points in \( \mu(M) \), i.e., points whose distance from the origin is maximal, are non reflective henceforth belong to \( \mu(M^T) \). In the Kähler setting, this can be seen also as a consequence of the fact that if \( p \) is a point in which \( \| \mu \| \) reaches its maximum, the \( G \)-orbit through \( p \) is complex [GP]. We remark here that if \( p \) is a point in which \( \| \mu \| \) achieves its minimum the \( G \)-orbit through \( p \) should be symplectic but not complex (see Remark 6 for an example).

**Example 3.** Consider the natural action of SU(\( n+1 \)) on the complex \( n \)-dimensional projective space \( \mathbb{P}^n \)

\[ A \mapsto \{ [x] \mapsto [Ax] \}. \]

This action preserves the Fubini-Study Kähler form \( \omega_0 \) of \( \mathbb{P}^n \) and it is obviously Hamiltonian. If we identify the Lie algebra \( \mathfrak{su}(n+1) \) with its dual by means of the Killing form, the corresponding moment map (which is unique, due to the semisimplicity of the special unitary group SU(\( n+1 \))) can be written as follows

\[ \mu : \mathbb{P}^n \to \mathfrak{su}(n+1) \]

\[ [x] \mapsto \frac{1}{2\pi i \|x\|^2} \left( x x^* - \frac{\text{Tr}(xx^*)}{n+1} I_{n+1} \right) \]

we can obtain a Hamiltonian action of SU(\( n+1 \)) on the symplectic product of \( \mathbb{P}^n \) with itself in two ways: the natural one

\[ A \in \text{SU}(n+1) \mapsto \{ ([x], [y]) \mapsto ([Ax], [Ay]) \} \]

or the skew one

\[ A \in \text{SU}(n+1) \mapsto \{ ([x], [y]) \mapsto ([Ax], [\bar{Ay}]) \}. \]

(A) We consider, for simplicity, the case \( n = 2 \) and we deal first with the natural action. The case \( n > 2 \) can be treated with the same arguments.

The principal orbits have codimension 1 in \( M = \mathbb{P}^2 \times \mathbb{P}^2 \) (they are diffeomorphic to SU(3)/U(1)),
hence the action is naturally coisotropic. Note that the action of SU($n+1$) is not locally free, but since it is effective, the action of a maximal torus is locally free on an open dense subset of $M$. Since $b_2(\mathbb{P}^2 \times \mathbb{P}^2) = 2$ we can find a 2-parameter family of SU(3)-invariant symplectic forms on $M$. Denote by $\omega_{t,s}$ the 2-form $t\omega_0 \oplus s\omega_0$. For $(t, s)$ close to $(1, 1)$ this is symplectic on $\mathbb{P}^2 \times \mathbb{P}^2$. The corresponding moment map $\phi^{t,s} : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathfrak{su}(3)$ is given by

$$\phi^{t,s}(x, y) = t\mu(x) + s\mu(y).$$

To draw the picture of Kirwan’s polytope it suffices to find the image of two points belonging to the two singular orbits, and then consider the maximal torus of SU(3) such that the dual structure in such a way that $t\mu$ $\neq s\mu$.

Note that for $t = s$ the image of $p_2$ lies on a Weyl wall. When we deform the symplectic structure in such a way that $t \neq s$, $q_2 = \phi^{t,s}(p_2)$ lies in the interior of a Weyl chamber; therefore $p_2$ must be fixed by $T$. By reflection through the wall we can find another point $q_3 = \phi^{t,s}(p_3)$ where $p_3 \in M^T$. Note that $\phi^{t,t}(p_2) = \phi^{t,s}(p_3)$, hence when $t = s$ the moment map is not injective on $M^T$. For the $T$-generator $\xi$ opportune chosen $\sigma_{p_2} = \sigma_{p_3} = 1$ and the distances $d(\phi^{t,s}(p_2), \phi^{t,s}(p_1))$ and $d(\phi^{t,s}(p_3), \phi^{t,s}(p_1))$ are coordinates of the moduli space of Hamiltonian structures on $\mathbb{P}^2 \times \mathbb{P}^2$.

Note that the orbit through $p_2$ does not inherit the symplectic structure $\omega$ of $M$; indeed suppose by contradiction that $\omega$ is non degenerate on $\mathcal{O}_2 = \text{SU}(3) \cdot p_2$ now the restriction of $\phi^{t,t}$ on $\mathcal{O}_2$ should be a covering on $\text{SU}(3) \cdot \phi^{t,t}(p_2)$, hence $\text{SU}(3)_{p_2} = T$ should be equal to $\text{SU}(3)_{\phi^{t,t}(p_2)} = U(2)$ which is not the case.

**B** The picture changes drastically for the skew action of SU(3) on the same symplectic manifold. Once again the principal stabilizer is 1-dimensional, and the action is coisotropic. The moment map $\tilde{\phi}^{t,s} : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathfrak{su}(3)$ w.r.t. the symplectic form $\omega_{t,s}$ is given by

$$\tilde{\phi}^{t,s}(x, y) = t\mu(x) - s\mu(y).$$

In this case the two singular orbits are $\mathcal{O}_1 = \{[[x], [x]], x \in \mathbb{C}^3\}$ and $\mathcal{O}_2 = \{[[x], [y]], < x, y >= 0, x, y \in \mathbb{C}^3\}$. We can therefore determine $\Delta(M)$, choosing the same points $p_1$ and $p_2$, as before. We see that at least three points $p_2, p_3, p_4$ of $M^T$ are sent by $\tilde{\phi}^{t,t}$ to 0. When $t \neq s$ the images of these three points move on the walls away from the origin, note that for $t = s$ the preimage of 0 is totally real in $\mathbb{P}^2 \times \mathbb{P}^2$, while this is no more true for the preimages of the three points on the walls. Since the four weights of the representation of $T$ on $T_{p_i}M$ $i = 2, 3, 4$ occur in opposite pairs (both $\alpha$ and $-\alpha$ are weights), $\sigma_{p_i} = 2$ for any choice of the
topological generator \( \xi \). Hence the two relevant vertices (with index \( \sigma = 1 \)) come from two distinct points of the complex orbit \( O_2 \).

In the previous example, when considering the diagonal action of SU(3) on the Kähler manifold \((M, \omega) = (\mathbb{P}^2 \times \mathbb{P}^2, \omega_o \oplus \omega_o)\), we remarked that \( O_1 \) is a codimension 2 complex orbit (it corresponds to a maximum). Then, we can blow up \( M \) along \( O_1 \) in a SU(3)-equivariant manner, obtaining a Kähler manifold \( \tilde{M} \) whose second Betti number is 3. Note that the cohomogeneity of the SU(3)-action on \( \tilde{M} \) is unchanged and the action is therefore coisotropic.

Using the technique of symplectic cuts (see e.g. [W1], and [L]) it is not hard to draw the moment polytope as in figure 4. Note that now the “degrees of freedom” are 3 (the “blown up vertex ” has left the Weyl wall). The singular orbits in \( \tilde{M} \) both have codimension 1, then a further blow-up will not affect \( b_2 \).

**Remark 6.** The \( G \)-action on the \( G \)-equivariant blow-up \( \tilde{M} \) of a Kähler manifold \( M \) acted on coisotropically by a group of isometries \( G \), along a complex orbit \( O \), is still coisotropic.
Recall that, in this setting, when $G$ is a compact Lie subgroup of the full isometry group, if all Borel subgroups of $G^C$ act with an open orbit on $M$, then the $G^C$-open orbit $\Omega$ is called a spherical homogeneous space and $M$ is called a spherical embedding of $\Omega$. The $G$-action is coisotropic if and only if the Kähler manifold $M$ is projective algebraic, $G^C$-almost homogeneous and a spherical embedding of the open $G^C$-orbit $[\Omega]$. Recall that on the $G$-equivariant blow-up $\tilde{M}$ of a Kähler manifold $M$ there is a well defined $G$-invariant Kähler structure (see [E]). Denote by $\pi : \tilde{M} \rightarrow M$ the canonical projection, by $\tilde{O} = \pi^{-1}(O)$ the exceptional divisor and by $\Omega$ the open (hence dense) orbit of a Borel subgroup $B$ of $G^C$ in $M$. Since $O$ is closed in $M$ and the restriction $\pi|_{\tilde{M}\setminus\tilde{O}} : \tilde{M}\setminus\tilde{O} \rightarrow M\setminus O$ is a biholomorphism, the submanifold $\tilde{\Omega} = \pi^{-1}(\Omega)$ is a $B$-orbit open in $M\setminus\tilde{O}$ hence also in $\tilde{M}$.

We conclude this remark pointing out that a symplectic blow-up $\tilde{M}$ of a Kähler manifold $M$ along a non complex orbit, does not necessarily admit an invariant compatible Kähler structure as Theorem 7.3 [W1] shows. In particular, one can argue that the $G$-orbit through a minimal point of $\|\mu\|^2$ is not complex in this case.

**Example 4.** Now we go back to Example 1 and we consider the action of the semisimple group $SO(5,\mathbb{R})$ instead of its maximal torus. We refer to [W1], in drawing the picture on the left. With the same notations of Example 1 we consider the non transversal case ($\gamma = \delta$). Note that the image in Example 1 can be drawn taking the convex hull of this picture as Lemma 4 says. By Kunneth’s formula the second Betti number of $M$ is 2, thus, once again deformations have two degrees of freedom. Note that the vertex $q_{t,s} \notin \mu(M^T)$, hence it never leaves the wall (remember that interior vertices come from $M^T$ see fact (2)). The vertex $r_{t,t}$ is the image of a point in $M^T$ since is non-reflective, see fact (1).

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Figure 5. The moment map images for the action of SO(5) on $\frac{SO(5)}{U(2)} \times \frac{SO(5)}{U(2)}$ on the left when the symplectic form is not deformed, on the right after a deformation.

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