AUGMENTED UPWIND NUMERICAL SCHEMES FOR A FRACTIONAL ADVECTION-DISPERSION EQUATION IN FRACTURED GROUNDWATER SYSTEMS

Amy Allwright and Abdon Atangana
Institute for Groundwater Studies, Faculty of Agricultural and Natural Sciences
University of the Free State, 9301
Bloemfontein, Free State, South Africa

Abstract. The anomalous transport of particles within non-linear systems cannot be captured accurately with the classical advection-dispersion equation, due to its inability to incorporate non-linearity of geological formations in the mathematical formulation. Fortunately, fractional differential operators have been recognised as appropriate mathematical tools to describe such natural phenomena. The classical advection-dispersion equation is adapted to a fractional model by replacing the time differential operator by a time fractional derivative to include the power-law waiting time distribution. The advection component is adapted by replacing the local differential by a fractional space derivative to account for mean-square displacement from normal to super-advection. Due to the complexity of this new model, new numerical schemes are suggested, including an upwind Crank-Nicholson and weighted upwind-downwind scheme. Both numerical schemes are used to solve the modified fractional advection-dispersion model and the conditions of their stability established.

1. Introduction. It has been well established that the fractional derivative has an important application in modelling anomalous diffusion [35, 28, 22, 23, 20]. Following on from this research, a case was made for the application of fractional derivatives to the advection-dispersion equation for modelling transport specifically in a fractured groundwater network [5, 4]. In this study, it is proposed to incorporate a fractional derivative for the advective term in space to capture anomalous behaviour in advection-driven transport within fractured systems; and in time to account for the waiting time distribution properties of the fractional derivatives. This approach differs from the usual approach, which incorporates the fractional derivative either in time [15, 26, 27], in space (but only for the diffusion/dispersion term) [5, 4, 32, 34], or in time and space (diffusion-dispersion term) [9, 25, 12].

For the resulting fractional advection-dispersion equation to be of use, the equation requires a solution. The complex nature of the fractional advection-dispersion equation makes an analytical solution challenging and limits its practical application, and thus numerical solutions are usually explored [5, 30, 21, 11, 6]. [6] found a substantial demand for readily usable tools to solve fractional derivative problems...
numerically when considering the growing body of research showing the successful description of complex natural phenomena by fractional derivatives. However, at that time only a limited collection of algorithms had been developed, which were described by [6] for the Riemann-Liouville fractional integral and Caputo-type derivatives.

There are a myriad of methods available to solve fractional differentials, including Laplace transform methods, Fourier transform methods, Adomian’s decomposition method, Homotopy analysis method, Finite difference methods, Finite element methods and many more [21, 10, 11, 24, 14, 3]. For this study, the focus will be on finite difference methods for numerically solving fractional partial differential equations, specifically the advection-dispersion equation. In 2005, several numerical methods to solve fractional differential equations had been developed, but limited numerical methods to solve fractional partial differential equations were available [19]. To add to the numerical solution methods for fractional partial differential equations, [19] proposed a simple explicit method and a semi-implicit method for the space fractional diffusion equation, which were analysed for efficiency and accuracy. The semi-implicit method was found to be more effective than the explicit method. A finite difference numerical method for the space fractional advection-dispersion equation, with the fractional derivative applied to the dispersion term, was presented by [21]. Making use of the standard Grünwald estimates, [21] describe unconditionally stable explicit and implicit Euler methods, and a Crank-Nicolson method. [16] considered the space-time fractional advection-dispersion equation, describing an explicit and implicit difference numerical solution method. The implicit difference method was found to be unconditionally stable and convergent, while the explicit difference method was conditionally stable and convergent.

The use of fractional derivatives to model anomalous dispersion or diffusion, and the related increase in research to find accurate and stable numerical methods that can be simply implemented was highlighted by [30]. Yet, investigation into the stability and convergence of these numerical schemes remained topical. In response, [30] generalised explicit finite difference approximations for a fractional advection-diffusion problem, where the fractional derivative was used to replace the second order derivative in the problem. One of these generalisations makes use of the finite difference upwind scheme for the advection term but with a classical derivative. A novel numerical approximation of the space fractional advection-dispersion equation was presented by [29], where a fractional centred difference scheme was applied to the Riesz fractional derivatives. The scheme was found to have second-order accuracy, unconditionally stable, consistent and convergent.

The development and analysis of numerical solutions for the fractional advection-dispersion equation has remained relevant, to continually pursue improved numerical solution methods [32, 34, 31, 17, 2, 3, 7, 13, 18]. Upwind finite difference numerical methods have been applied to fractional partial differential equations [38, 36, 37], and recently [1] presented augmented upwind schemes for the advection-dispersion equation with local operators, where an upwind Crank-Nicolson and weighted upwind-downwind finite difference schemes were developed. The numerical schemes were found to be an improvement on the traditional upwind approach, and are thus selected for application to the fractional advection-dispersion equation. In this paper, the upwind Crank-Nicolson and weighted upwind-downwind scheme are applied to the fractional advection-dispersion equation to investigate their suitability to anomalous advection in fractured systems.
2. Space-time fractional advection-dispersion equation (fADE) for fractured systems with non-local operators. Fractional derivatives are applied to the advection-dispersion equation to model not only anomalous dispersion, but also potentially anomalous advection in the form of preferential pathways in fractures within the groundwater system. It has been conceptualised that within an aquifer water is moving within the porous media at the expected rate, but additionally faster than expected within unknown fractures or faults, and slower than expected in other areas, such as closed faults/fractures.

The discrepancy in the manner in which groundwater flows is usually correlated to diffusion (anomalous diffusion), referred to as super-diffusion (faster than traditional methods predict) and sub-diffusion (slower than traditional methods predict). An analogy is drawn with the conceptual groundwater flow within a fractured aquifer, where super-advection is defined as flow faster than traditional methods predict, and sub-advection as flow slower than traditional methods predict. For this reason, the fractional derivative will be applied in space for the advection term of the advection-dispersion equation.

The fractional derivative will also be applied to time because of the properties of the fractional derivatives, where the waiting time distribution is defined. Incorporating the fractional derivative for time, thus allows these features to be activated in the advection-dispersion equation solution, i.e. for the Caputo derivative a power law distribution [33].

The traditional advection-dispersion equation,

\[ \frac{\partial}{\partial t} c(x,t) = -v \frac{\partial}{\partial x} c(x,t) + D_L \frac{\partial^2}{\partial x^2} c(x,t) \]  

Incorporating the fractional derivative for time and advection components,

\[ D_t^\alpha (c(x,t)) = -v D_x^\alpha (c(x,t)) + D_L \frac{\partial^2}{\partial x^2} c(x,t) \]  

where, \( D_t^\alpha \) indicates the use of a fractional derivative with a fractional order \( \alpha \).

3. Upwind numerical approximation schemes for the Caputo defined fADE. The first-order upwind scheme is applied to the one-dimensional, non-reactive fractional advection-dispersion equation for numerical approximation. Similar to the classical advection-dispersion equation, the upwind scheme for the fractional advection-dispersion equation finite difference approximation influences the advection term, where backward or forward differences are considered depending on the direction of the transporting velocity [1].

Applying the Caputo definition to the fractional advection-dispersion equation,

\[ C_0 D_t^\alpha (c(x,t)) + v C_0 D_x^\alpha (c(x,t)) - D_L \frac{\partial^2}{\partial x^2} c(x,t) = 0 \]  

where,

\[ C_0 D_t^\alpha f(x) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{d}{d\tau} f(\tau)(t - \tau)^{-\alpha} d\tau \]  

The forward finite difference in time is applied to the Caputo fractional derivative to illustrate the numerical approximation method. The Caputo fractional derivative is considered for a specific time \( (t_n) \),

\[ C_0 D_t^\alpha f(t_n) = \frac{1}{\Gamma(1 - \alpha)} \int_0^{t_n} \frac{d}{d\tau} f(\tau)(t_n - \tau)^{-\alpha} d\tau \]  

The time integer-order derivative \( \tau \) is replaced with the forward differences approximation at smaller increments in time \((k)\), and a summation is used to express the integral performed for each time step:

\[
C_a D_t^\alpha f(t_n) = \frac{1}{\Gamma(1 - \alpha)} \left[ \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{f_{i}^{k+1} - f_i^k}{\Delta t} (t_n - \tau)^{-\alpha} d\tau \right] \tag{5}
\]

The approximation of the continuous \( \tau \) function, results in two specific points of the function with respect to time \((t)\), which allows the approximated derivative to be taken out of the integral:

\[
C_a D_t^\alpha f(t_n) = \frac{1}{\Gamma(1 - \alpha)} \left[ \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n - \tau)^{-\alpha} d\tau \right] \tag{6}
\]

The fractional integral still requires approximation, let function \( y \) represent \((t_n - \tau), dy = -d\tau \); and the integral is reversed to consider a single time step:

\[
C_a D_t^\alpha f(t_n) = \frac{1}{\Gamma(1 - \alpha)} \left[ \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \left( - \int_{t_n - t_k}^{t_{n-k+1}} y^{-\alpha} dy \right) \right] \tag{7}
\]

Integrating,

\[
C_a D_t^\alpha f(t_n) = \frac{1}{\Gamma(1 - \alpha)} \left[ \sum_{k=0}^{n-1} \frac{f_{i}^{k+1} - f_i^k}{\alpha} \left( - \int_{t_n - t_k}^{t_{n-k+1}} \frac{1}{1 - \alpha} dy \right) \right] \tag{8}
\]

Considering the specific time can be represented as the number of time steps required to reach that time \((t_n = \Delta t \cdot n)\) and similarly, \(t_k = \Delta t \cdot k\). Thus, the same can be applied to achieve \(t_n - t_k = \Delta t (n - k)\) and \(t_n - t_{k+1} = \Delta t (n - k - 1)\):

\[
C_a D_t^\alpha f(t_n) = \frac{1}{\Gamma(1 - \alpha)} \left[ \sum_{k=0}^{n-1} \frac{f_{i}^{k+1} - f_i^k}{\Delta t} \left( - \frac{(\Delta t (n - k))^{1 - \alpha} - (\Delta t (n - k - 1))^{1 - \alpha}}{1 - \alpha} \right) \right] \tag{9}
\]

Simplifying by moving the constant \(\Delta t\), \(\Delta t^{1 - \alpha}\) and \(1 - \alpha\) out from the summation, and considering the properties of the gamma function, where \(\Gamma(2 - \alpha) = (1 - \alpha)\Gamma(1 - \alpha)\) when using integration by parts and L'Hôpital's rule:

\[
C_a D_t^\alpha f(t_n) = \frac{(\Delta t)^{1 - \alpha}}{\Gamma(2 - \alpha)} \left[ \sum_{k=0}^{n-1} f_{i}^{k+1} - f_i^k \left( (n - k)^{1 - \alpha} - (n - k - 1)^{1 - \alpha} \right) \right] \tag{10}
\]

Equation 10 is the numerically approximated Caputo fractional derivative, where the function is considered at two discrete points in time, yet additionally the fractional components are included to account for changes in between those two discrete points in time.

3.1 First-order upwind explicit. The numerical approximation of the Caputo fractional derivative with respect to time has been described in Equation 3 to 10, and is applied now for the Caputo fractional advection-dispersion equation as follows:

\[
C_0 D_t^\alpha (c(x_m, t_n)) = \frac{(\Delta t)^{1 - \alpha}}{\Gamma(2 - \alpha)} \left[ \sum_{k=0}^{n-1} (c_{i}^{k+1} - c_i^k) \left( (n - k)^{1 - \alpha} - (n - k - 1)^{1 - \alpha} \right) \right] \tag{11}
\]
where, a function \( \delta_{n,k}^{\alpha} \) is applied to simplify,

\[
\delta_{n,k}^{\alpha} = (n - k)^{1-\alpha} - (n - k - 1)^{1-\alpha}
\]

Thus,

\[
\frac{\partial}{\partial t} \bigg|_{t=0} D_{t}^{\alpha} (c(x_m, t_n)) = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \left[ \sum_{k=0}^{n-1} (c_{m}^{k+1} - c_{m}^{k}) \delta_{n,k}^{\alpha} \right] \tag{12}
\]

The first-order upwind scheme for the Caputo fractional advection-dispersion equation uses a one-sided finite difference in the upstream direction to approximate the advection term in the advection-dispersion equation (assuming \( v > 0 \)). Thus, the Caputo fractional derivative with respect to space (explicit) approximated by an explicit upwind scheme is,

\[
\frac{\partial}{\partial x} \bigg|_{x=0} D_{x}^{\alpha} (c(x_m, t_n)) = \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \left[ \sum_{i=1}^{m} (c_{i}^{n-1} - c_{i-1}^{n-1}) \left((m - i)^{1-\alpha} - (m - i - 1)^{1-\alpha}\right) \right] \tag{13}
\]

where, a function \( \delta_{m,i}^{\alpha} \) is applied to simplify,

\[
\delta_{m,i}^{\alpha} = (m - i)^{1-\alpha} - (m - i - 1)^{1-\alpha}
\]

Thus,

\[
\frac{\partial}{\partial x} \bigg|_{x=0} D_{x}^{\alpha} (c(x_m, t_n)) = \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \left[ \sum_{i=1}^{m} (c_{i}^{n-1} - c_{i-1}^{n-1}) \delta_{m,i}^{\alpha} \right] \tag{14}
\]

Substituting this back into Equation 3, and applying the traditional finite difference approach to the local second order derivative

\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \left[ \sum_{k=0}^{n-1} (c_{m}^{k+1} - c_{m}^{k}) \delta_{n,k}^{\alpha} \right] + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \left[ \sum_{i=1}^{m} (c_{i}^{n-1} - c_{i-1}^{n-1}) \delta_{m,i}^{\alpha} \right]
\]

\[
- D_{L} \left( \frac{c_{m+1}^{n-1} - 2c_{m}^{n-1} + c_{m-1}^{n-1}}{(\Delta x)^{2}} \right) = 0 \tag{15}
\]

Reformulating the following can be obtained,

\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \left( c_{m}^{n} - c_{m}^{n-1} \right) \delta_{n,n-1}^{\alpha} + \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \left( \sum_{k=0}^{n-2} (c_{m}^{k+1} - c_{m}^{k}) \delta_{n,k}^{\alpha} \right)
\]

\[
+ v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \left( c_{m}^{n-1} - c_{m-1}^{n-1} \right) \delta_{n,m}^{\alpha} + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \left[ \sum_{i=1}^{m-1} (c_{i}^{n-1} - c_{i-1}^{n-1}) \delta_{m,i}^{\alpha} \right]
\]

\[
- D_{L} \left( \frac{c_{m+1}^{n-1} - 2c_{m}^{n-1} + c_{m-1}^{n-1}}{(\Delta x)^{2}} \right) = 0 \tag{16}
\]

Rearranging,

\[
\left( \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n,n-1}^{\alpha} \right) c_{m}^{n} = \left( \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n,n-1}^{\alpha} + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n,m}^{\alpha} - \frac{2D_{L}}{(\Delta x)^{2}} \right) c_{m}^{n-1}
\]

\[
+ \left( v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n,m}^{\alpha} + \frac{D_{L}}{(\Delta x)^{2}} \right) c_{m-1}^{n-1} + \left( \frac{D_{L}}{(\Delta x)^{2}} \right) c_{m+1}^{n-1}
\]
where

\[ a = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n,n-1}; \ b = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n,n-1} + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n,m} - \frac{2D_L}{(\Delta x)^2}; \]
\[ d = v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n,m} + \frac{D_L}{(\Delta x)^2}; \ f = \frac{D_L}{(\Delta x)^2}; \ h = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n,n-1}; \]
\[ l = v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \]

3.2. First-order upwind implicit. Following the same approach as the explicit upwind numerical approximation Section 3.1, the following is obtained for the implicit upwind scheme,

\[ \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} (c_{m}^{k+1} - c_{m}^{k}) \delta_{n,k}^{\alpha} + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=1}^{m-1} (c_{i}^{n} - c_{i}^{n-1}) \delta_{m,i}^{\alpha} - D_L \left( \frac{c_{m+1}^{n} - 2c_{m}^{n} + c_{m-1}^{n}}{(\Delta x)^2} \right) = 0 \] (19)

Reformulating the following can be obtained,

\[ \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} (c_{m}^{n} - c_{m}^{n-1}) \delta_{n,n-1}^{\alpha} + \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \left( \sum_{k=0}^{n-2} (c_{m}^{k+1} - c_{m}^{k}) \delta_{n,k}^{\alpha} \right) + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} (c_{m}^{n} - c_{m}^{n-1}) \delta_{n,n}^{\alpha} + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=1}^{m-1} (c_{i}^{n} - c_{i}^{n-1}) \delta_{m,i}^{\alpha} - D_L \left( \frac{c_{m+1}^{n} - 2c_{m}^{n} + c_{m-1}^{n}}{(\Delta x)^2} \right) = 0 \] (20)

Rearranging,

\[ \left( \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n,n-1}^{\alpha} + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n,n}^{\alpha} - \frac{2D_L}{(\Delta x)^2} \right) c_{m}^{n} = \left( \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n,n-1}^{\alpha} \right) c_{m}^{n-1} + \left( v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n,n}^{\alpha} + \frac{D_L}{(\Delta x)^2} \right) c_{m-1}^{n-1} + \left( \frac{D_L}{(\Delta x)^2} \right) c_{m+1}^{n-1} - \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \left( \sum_{k=0}^{n-2} (c_{m}^{k+1} - c_{m}^{k}) \delta_{n,k}^{\alpha} \right) - v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=1}^{m-1} (c_{i}^{n} - c_{i}^{n-1}) \delta_{m,i}^{\alpha} \] (21)
The implicit upwind numerical scheme can be simplified by substituting placekeeper functions as follows,

\[
\begin{align*}
bc_m^n &= ac_m^{n-1} + dc_m^{n-1} + fc_m^{n+1} - h \left( \sum_{k=0}^{n-2} (c_{m+k}^n - c_m^n) \delta_{n,k}^\alpha \right) \\
&\quad - l \left( \sum_{i=1}^{m-1} (c_i^n - c_i^{n-1}) \delta_{m,i}^\alpha \right) 
\end{align*}
\] (22)

3.3. First-order upwind advection Crank-Nicolson scheme. The upwind Crank-Nicolson scheme for the Caputo fractional advection-dispersion equation is considered here \cite{1}. The time component remains the same as with the implicit and explicit schemes, but the space components change to,

\[
\begin{align*}
\partial_t^\alpha c(x_m, t_n) &= \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \\
&\quad \left[ \sum_{i=1}^{m} \left[ 0.5 \left( c_i^{n-1} - c_{i-1}^{n-1} \right) + 0.5 \left( c_i^n - c_{i-1}^n \right) \right] \left( (m - i)^1 - (m - i - 1)^1 - \alpha \right) \right] 
\end{align*}
\] (23)

Simplifying using the function \( \delta_{m,i}^\alpha \),

\[
\begin{align*}
\partial_t^\alpha c(x_m, t_n) &= \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \left[ \sum_{i=1}^{m} \left( 0.5 \left( c_i^{n-1} - c_{i-1}^{n-1} \right) + 0.5 \left( c_i^n - c_{i-1}^n \right) \right) \delta_{m,i}^\alpha \right] 
\end{align*}
\] (24)

Substituting this back into Equation 3,

\[
\begin{align*}
\frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \left[ \sum_{k=0}^{n-1} (c_{m+k}^n - c_m^n) \delta_{n,k}^\alpha \right] \\
&\quad + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \left[ \sum_{i=1}^{m} \left( 0.5 \left( c_i^{n-1} - c_{i-1}^{n-1} \right) + 0.5 \left( c_i^n - c_{i-1}^n \right) \right) \delta_{n,i}^\alpha \right] \\
&\quad - D_L \left( \frac{c_{m+1}^{n-1} - 2c_m^{n-1} + c_{m-1}^{n-1}}{(\Delta x)^2} \right) = 0 
\end{align*}
\] (25)

Reformulating the following can be obtained,

\[
\begin{align*}
\frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \left( c_m^n - c_m^{n-1} \right) \delta_{n,n-1}^{\alpha} + \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \left( \sum_{k=0}^{n-2} (c_{m+k}^n - c_m^n) \delta_{n,k}^\alpha \right) \\
&\quad + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \left( 0.5 \left( c_m^{n-1} - c_{m-1}^{n-1} \right) + 0.5 \left( c_m^n - c_{m-1}^n \right) \right) \delta_{n,m}^\alpha \\
&\quad + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \left[ \sum_{i=1}^{m-1} \left( 0.5 \left( c_i^{n-1} - c_{i-1}^{n-1} \right) + 0.5 \left( c_i^n - c_{i-1}^n \right) \right) \delta_{m,i}^\alpha \right]
\end{align*}
\]
where, 

\[
\theta \leq \frac{c_{m+1} - 2c_m + c_{m-1}}{(\Delta x)^2} = 0
\]  

(26)

Rearranging,

\[
\left( \frac{(\Delta t)^{-\alpha}}{\Gamma (2 - \alpha)} \right) \delta_{n,n-1}^{\alpha} + v \left( \frac{(\Delta x)^{-\alpha}}{\Gamma (2 - \alpha)} \right) 0.5 \delta_{n,m}^{\alpha} c_m^n =
\]

\[
\left( \frac{(\Delta t)^{-\alpha}}{\Gamma (2 - \alpha)} \delta_{n,n-1}^{\alpha} - v \left( \frac{(\Delta x)^{-\alpha}}{\Gamma (2 - \alpha)} \right) 0.5 \delta_{n,m}^{\alpha} - \frac{2D_L}{(\Delta x)^2} \right) c_{m-1}^{n-1}
\]

\[
+ v \left( \frac{(\Delta x)^{-\alpha}}{\Gamma (2 - \alpha)} 0.5 \delta_{n,m}^{\alpha} \right) c_{m-1}^{n-1} + \left( \frac{D_L}{(\Delta x)^2} \right) c_{m+1}^{n-1}
\]

\[
- v \left( \frac{(\Delta x)^{-\alpha}}{\Gamma (2 - \alpha)} \sum_{k=0}^{n-2} \left( c_{m+1}^{k+1} - c_m^{k} \right) \delta_{n,k}^{\alpha} \right)
\]

\[
- v \left( \frac{(\Delta x)^{-\alpha}}{\Gamma (2 - \alpha)} \sum_{i=1}^{m-1} \left( 0.5 \left( c_{i,n-1}^{n-1} - c_{i-1,n-1}^{n-1} \right) + 0.5 \left( c_i^n - c_{i-1}^n \right) \right) \delta_{m,i}^{\alpha} \right)
\]  

(27)

The following functions are used to simplify the upwind Crank-Nicolson numerical scheme as follows,

\[
\omega c_m^n = p c_m^{n-1} + q c_{m-1}^{n-1} + r c_{m-1}^{n-1} + f c_{m+1}^{n-1} - h \left( \sum_{k=0}^{n-2} \left( c_{m+1}^{k+1} - c_m^{k} \right) \delta_{n,k}^{\alpha} \right)
\]

\[
- l \left( \sum_{i=1}^{m-1} \left( 0.5 \left( c_{i,n-1}^{n-1} - c_{i-1,n-1}^{n-1} \right) + 0.5 \left( c_i^n - c_{i-1}^n \right) \right) \delta_{m,i}^{\alpha} \right)
\]  

(28)

where,

\[
\omega = \frac{(\Delta t)^{-\alpha}}{\Gamma (2 - \alpha)} \delta_{n,n-1}^{\alpha} + v \left( \frac{(\Delta x)^{-\alpha}}{\Gamma (2 - \alpha)} \right) 0.5 \delta_{n,m}^{\alpha}
\]

\[
p = \frac{(\Delta t)^{-\alpha}}{\Gamma (2 - \alpha)} \delta_{n,n-1}^{\alpha} - v \left( \frac{(\Delta x)^{-\alpha}}{\Gamma (2 - \alpha)} \right) 0.5 \delta_{n,m}^{\alpha} - \frac{2D_L}{(\Delta x)^2}
\]

\[
q = v \left( \frac{(\Delta x)^{-\alpha}}{\Gamma (2 - \alpha)} \delta_{n,m}^{\alpha} \right) + \frac{D_L}{(\Delta x)^2} ; r = v \frac{(\Delta x)^{-\alpha}}{\Gamma (2 - \alpha)} \delta_{n,m}^{\alpha}
\]

3.4. First-order upwind-downwind weighted scheme (explicit). Considering both the upwind and downwind direction for the advection term, a ratio of upwind to downwind is controlled by \( \theta \), where \( 0 \leq \theta \leq 1 \). The advection component can thus be represented as follows for the explicit upwind-downwind weighted scheme [1]:

\[
\hat{D}_x f \left( x_m, t_n \right) = \left( \frac{(\Delta x)^{-\alpha}}{\Gamma (2 - \alpha)} \right) \sum_{i=1}^{m} \left[ \theta \left( c_{i}^{n-1} - c_{i-1}^{n-1} \right) + (1 - \theta) \left( c_{i+1}^{n-1} - c_{i-1}^{n-1} \right) \right] \delta_{m,i}^{\alpha}
\]  

(29)
Substituting this back into Equation 3,

\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \left[ \sum_{k=0}^{n-1} (c_{m}^{k+1} - c_{m}^{k}) \delta_{n,k}^{\alpha} \right] + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \left[ \sum_{i=1}^{m} [\theta (c_{i}^{n-1} - c_{i-1}^{n-1}) + (1 - \theta) (c_{i+1}^{n-1} - c_{i}^{n-1})] \delta_{m,i}^{\alpha} \right] - D_{L} \left( \frac{c_{m+1}^{n-1} - 2c_{m}^{n-1} + c_{m-1}^{n-1}}{(\Delta x)^{2}} \right) = 0 \tag{30}
\]

Reformulating the following can be obtained,

\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} (c_{m}^{n} - c_{m}^{n-1}) \delta_{n,n-1}^{\alpha} + \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \left( \sum_{k=0}^{n-2} (c_{m}^{k+1} - c_{m}^{k}) \delta_{n,k}^{\alpha} \right) + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \langle \theta (c_{m}^{n-1} - c_{m-1}^{n-1}) + (1 - \theta) (c_{m+1}^{n-1} - c_{m}^{n-1}) \rangle \delta_{n,m}^{\alpha}
\]

\[
+ v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \left[ \sum_{i=1}^{m-1} [\theta (c_{i}^{n-1} - c_{i-1}^{n-1}) + (1 - \theta) (c_{i+1}^{n-1} - c_{i}^{n-1})] \delta_{m,i}^{\alpha} \right] - D_{L} \left( \frac{c_{m+1}^{n-1} - 2c_{m}^{n-1} + c_{m-1}^{n-1}}{(\Delta x)^{2}} \right) = 0 \tag{31}
\]

Rearranging,

\[
\frac{((\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,n-1}^{\alpha} \right) c_{m}^{n} = \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,n-1}^{\alpha} + v (2\theta - 1) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m}^{\alpha} - \frac{2D_{L}}{(\Delta x)^{2}} c_{m}^{n-1}
\]

\[
+ v \left[ \theta \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m}^{\alpha} + \frac{D_{L}}{(\Delta x)^{2}} \right] c_{m-1}^{n-1}
\]

\[
+ v (1 - \theta) \left[ \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m}^{\alpha} + \frac{D_{L}}{(\Delta x)^{2}} \right] c_{m+1}^{n-1}
\]

\[
- \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \sum_{k=0}^{n-2} (c_{m}^{k+1} - c_{m}^{k}) \delta_{n,k}^{\alpha}
\]

\[
- v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \sum_{i=1}^{m-1} [\theta (c_{i}^{n-1} - c_{i-1}^{n-1}) + (1 - \theta) (c_{i+1}^{n-1} - c_{i}^{n-1})] \delta_{m,i}^{\alpha} \tag{32}
\]

Similarly, place-keeper functions are implemented to simplify the explicit upwind-downwind weighted scheme.
\[ a c^n_m = s c^{n-1}_m + u c^{n-1}_{m-1} + w c^{n-1}_{m+1} - h \left( \sum_{k=0}^{n-2} (c^{k+1}_m - c^k_m) \delta^\alpha_{n,k} \right) \]
\[ - I \left[ \sum_{i=1}^{m-1} \left( \theta (c^n_i - c^{n-1}_i) + (1 - \theta) (c^n_{i+1} - c^n_i) \right) \delta^\alpha_{m,i} \right] \] (33)

where,
\[ s = \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \delta^\alpha_{n,n-1} + v (2\theta - 1) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta^\alpha_{n,m} - \frac{2D_L}{(\Delta x)^2} \]
\[ u = v \theta \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta^\alpha_{n,m} + \frac{D_L}{(\Delta x)^2} \]
\[ w = v (1 - \theta) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta^\alpha_{n,m} + \frac{D_L}{(\Delta x)^2} \]

3.5. First-order upwind-downwind weighted scheme (implicit). As with the explicit formulations, both the upwind and downwind direction for the advection term is considered for the implicit upwind-downwind weighted approximation. The advection component is approximated as follows using the Caputo fractional derivative [1]:
\[ c^n_0 \frac{D_x^\alpha}{D^n_x} c (x_m, t_n) = \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \left[ \sum_{i=1}^{m} \left[ \theta (c^n_i - c^{n-1}_i) + (1 - \theta) (c^n_{i+1} - c^n_i) \right] \delta^\alpha_{m,i} \right] \] (34)

Substituting this back into Equation 3,
\[ \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \left[ \sum_{k=0}^{n-1} (c^{k+1}_m - c^k_m) \delta^\alpha_{n,k} \right] + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \left[ \sum_{i=1}^{m} \left[ \theta (c^n_i - c^{n-1}_i) + (1 - \theta) (c^n_{i+1} - c^n_i) \right] \delta^\alpha \right] - D_L \left( \frac{c^n_{m+1} - 2c^n_m + c^n_{m-1}}{(\Delta x)^2} \right) = 0 \] (35)

Reformulating the following can be obtained,
\[ \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \left( c^n_m - c^{n-1}_m \right) \delta^\alpha_{n,n-1} + \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \left( \sum_{k=0}^{n-2} (c^{k+1}_m - c^k_m) \delta^\alpha_{n,k} \right) \]
\[ + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \left( \theta (c^n_m - c^{n-1}_m) + (1 - \theta) (c^n_{m+1} - c^n_m) \right) \delta^\alpha_{n,m} \]
\[ + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \left[ \sum_{i=1}^{m-1} \left[ \theta (c^n_i - c^{n-1}_i) + (1 - \theta) (c^n_{i+1} - c^n_i) \right] \delta^\alpha_{m,i} \right] \]
\[ - D_L \left( \frac{c^n_{m+1} - 2c^n_m + c^n_{m-1}}{(\Delta x)^2} \right) = 0 \] (36)
Rearranging,
\[
\left(\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n,n-1}^{\alpha} + v (2\theta - 1) \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n,m}^{\alpha} - \frac{2D_L}{(\Delta x)^2} \right) e^n_m
\]
\[
= \left(\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n,n-1}^{\alpha} \right) e^{n-1}_m + \left(\frac{v\theta (\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n,m}^{\alpha} + \frac{D_L}{(\Delta x)^2} \right) e^{n-1}_m
\]
\[
+ \left( v (1-\theta) \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n,m}^{\alpha} + \frac{D_L}{(\Delta x)^2} \right) e^n_{m+1} - \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \left( \sum_{k=0}^{n-2} (c_{m+1} - c_{m}) \delta_{n,k}^{\alpha} \right)
\]
\[
- v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \left[ \sum_{i=1}^{m-1} \left( \theta (c^n_i - c^n_{i-1}) + (1-\theta) (c^n_{i+1} - c^n_i) \right) \delta_{m,i}^{\alpha} \right]
\]

For simplification, place-keeper functions are also substituted into the implicit upwind-downwind weighted approximation as follows,
\[
s^n e^n_m = ac^n_{m-1} + uc^n_{m-1} + wc^n_{m+1} - h \left( \sum_{k=0}^{n-2} (c_{m+1} - c_{m}) \delta_{n,k}^{\alpha} \right)
\]
\[
- l \left[ \sum_{i=1}^{m-1} \left( \theta (c^n_i - c^n_{i-1}) + (1-\theta) (c^n_{i+1} - c^n_i) \right) \delta_{m,i}^{\alpha} \right]
\]

This concludes the formulation of the numerical approximations schemes to be investigated for the Caputo fractional advection-dispersion equation. In the following section, the numerical stability of each scheme will be analysed.

4. Numerical stability analysis. The recursive numerical stability method, as described by [2, 8, 1] is applied to the developed numerical approximation schemes for the Caputo fractional advection-dispersion equation. The numerical stability for the upwind schemes are presented to validate their use in solving the developed equation.

4.1. First-order upwind implicit. Considering the developed finite difference first-order upwind (implicit) numerical scheme discussed in Section 3.2, substituting induction method terms gives:
\[
bc = ac_{n-1} + dc_{n} e^{jk_i x} + fc_{n} e^{jk_i (x - \Delta x)} + g_{n} e^{jk_i (x + \Delta x)}
\]
\[
- h \left( \sum_{k=0}^{n-2} (c_{k+1} - c_k) \delta_{n,k}^{\alpha} \right) - l \left( \sum_{i=1}^{m-1} (c_{i} e^{jk_i x} - c_{i-1} e^{jk_i (x - \Delta x)}) \delta_{m,i}^{\alpha} \right)
\]

Multiple out and divide by $e^{jk_i x}$,
\[
bc = ac_{n-1} + dc_{n} e^{-jk_i \Delta x} + fc_{n} e^{jk_i \Delta x -}
\]
\[
- h \left( \sum_{k=0}^{n-2} (c_{k+1} - c_k) \delta_{n,k}^{\alpha} \right) - l \left( \sum_{i=1}^{m-1} (c_{i} e^{-jk_i \Delta x} - c_{i-1} e^{jk_i (x - \Delta x)}) \delta_{m,i}^{\alpha} \right)
\]

The induction numerical stability analysis is performed in two parts; firstly it is proved for a set $\forall n > 1$, $|c_n| < |c_o|$.
If $n = 1$, then

$$bc_1 = ac_0 + dc_1 e^{-jk_i \Delta x} + fc_1 e^{jk_i \Delta x} - l \left( \sum_{i=1}^{m-1} (c_1 - c_1 e^{-jk_i \Delta x}) \delta_{m,i} \right)$$  \hfill (41)

A subset for $m$ is now considered, where $m = 1$,

$$bc_1 = ac_0 + dc_1 e^{-jk_i \Delta x} + fc_1 e^{jk_i \Delta x}$$  \hfill (42)

Rearranging and applying the norm on both sides,

$$\frac{|c_1|}{|c_0|} = \frac{|a|}{(|b| + |d| e^{-jk_i \Delta x} + |f|)}$$  \hfill (43)

The condition required $|c_n| < |c_0|$, translates to:

$$\frac{|c_1|}{|c_0|} < 1$$

Remembering $|e^n| = 1$, the condition becomes

$$\frac{|a|}{(|b| + |d| + |f|)} < 1$$  \hfill (44)

The term is expanded using the simplification terms associated with Equation 22,

$$\frac{|(\Delta t)^{-\alpha}\delta_{n,n-1}^\alpha - 2D \Gamma(2-\alpha) \delta_{n,m}^\alpha + v^2 \Gamma(2-\alpha) \delta_{n,m}^\alpha + D \Gamma(2-\alpha) \delta_{n,n}^\alpha |}{\Gamma(2-\alpha) \delta_{n,n-1}^\alpha} < 1$$  \hfill (45)

The assumption is made where,

$$\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha) \delta_{n,n-1}^\alpha} + \frac{v(\Delta x)^{-\alpha}}{\Gamma(2-\alpha) \delta_{n,m}^\alpha} > \frac{2D \Gamma(2-\alpha)}{(\Delta x)^2}$$  \hfill (46)

Simplifying,

$$2v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha) \delta_{n,m}^\alpha} > 0$$  \hfill (47)

Under this assumption Equation 46, the implicit upwind numerical scheme for the Caputo fractional advection-dispersion equation is unconditionally stable.

When the complementary assumption is made where,

$$\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha) \delta_{n,n-1}^\alpha} + \frac{v(\Delta x)^{-\alpha}}{\Gamma(2-\alpha) \delta_{n,m}^\alpha} < \frac{2D \Gamma(2-\alpha)}{(\Delta x)^2}$$  \hfill (48)

Simplifying,

$$\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha) \delta_{n,n-1}^\alpha} < \frac{2D \Gamma(2-\alpha)}{(\Delta x)^2}$$  \hfill (49)

For the complementary assumption Equation 48, the implicit upwind numerical scheme for the Caputo fractional advection-dispersion equation is conditionally stable.

A subset for $m$ is now considered for all $m > 1$,

$$bc_1 = ac_0 + dc_1 e^{-jk_i \Delta x} + fc_1 e^{jk_i \Delta x} - l \left( \sum_{i=1}^{m-1} (c_1 - c_1 e^{-jk_i \Delta x}) \delta_{m,i} \right)$$  \hfill (50)
Simplifying and expanding the summation,
\[
(b - de^{-jk_i\Delta x} - fe^{jk_i\Delta x} + l (1 - e^{-jk_i\Delta x}) \left(m^{1-\alpha} - (-1)^{1-\alpha}\right) ) c_1 = ac_0 \tag{51}
\]
Let a new function simplify to,
\[
\phi = k_i \Delta x
\]
where,
\[
e^{-j\phi} = e^{-jk_i\Delta x}
\]
Remembering Euler’s formula for complex numbers,
\[
(b - de^{-jk_i\Delta x} - fe^{jk_i\Delta x} + l (1 - \cos\phi + is\sin\phi) \left(m^{1-\alpha} - (-1)^{1-\alpha}\right) ) c_1 = ac_0 \tag{52}
\]
\[
(|b| + |d| + |f| + |l| (2 - 2\cos\phi) |\beta_m|) |c_1| = |a||c_0| \tag{53}
\]
where,
\[
\beta_m = m^{1-\alpha} - (-1)^{1-\alpha}
\]
Rearranging,
\[
\frac{|c_1|}{|c_0|} = \frac{|a|}{(|b| + |d| + |f| + |l| (2 - 2\cos\phi) |\beta_m|)} \tag{54}
\]
The condition required \(|c_n| < |c_o|\), translates to:
\[
\frac{|a|}{(|b| + |d| + |f| + |l| (2 - 2\cos\phi) |\beta_m|)} < 1 \tag{55}
\]
The term is expanded using the simplification terms associated with Equation 22,
\[
\left|\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\delta_{n,n-1}^\alpha + v (\Delta x)^{-\alpha}\delta_{n,m}^\alpha - \frac{2D_L}{(\Delta x)^2} | + |v (\Delta x)^{-\alpha}\delta_{n,m}^\alpha + \frac{D_L}{(\Delta x)^2} \right| < 1 \tag{56}
\]
The assumption is made where,
\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\delta_{n,n-1}^\alpha + v (\Delta x)^{-\alpha}\delta_{n,m}^\alpha > 2D_L \tag{57}
\]
Simplifying the resulting condition gives,
\[
v (\Delta x)^{-\alpha} (2\delta_{n,m}^\alpha + (2 - 2\cos\phi) \beta_m) > 0 \tag{58}
\]
According to the assumption made in Equation 57, the implicit upwind scheme is unconditionally stable.

The apposed assumption is then made where,
\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)}\delta_{n,n-1}^\alpha + v (\Delta x)^{-\alpha}\delta_{n,m}^\alpha < \frac{2D_L}{(\Delta x)^2} \tag{59}
\]
The condition thus becomes,

\[
\frac{2 (\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,n-1}^\alpha < \frac{4D_L}{(\Delta x)^2} + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} (2 - 2\cos\phi) (\beta_m)
\]

(60)

Under the apposed assumption Equation 59, the implicit upwind numerical scheme is conditionally stable.

Secondly, making the assumption that \(|c_{n-1}| < |c_o|\) is true for all time steps, the second part of the numerical stability analysis is to demonstrate for a set \(\forall n \geq 1\), that

\[
|c_n| < |c_o|
\]

Rearranging Equation 40 for \(c_n\),

\[
\left( b - de^{-jk_i \Delta x} - f e^{jk_i \Delta x} + l (1 - e^{-jk_i \Delta x}) \left( \sum_{i=1}^{m-1} \delta_{m,i}^\alpha \right) \right) c_n = ac_{n-1} - h \left( \sum_{k=0}^{n-2} (c_{k+1} - c_k) \delta_{n,k}^\alpha \right)
\]

(61)

Following a similar process as previously,

\[
\left( b - de^{-jk_i \Delta x} - f e^{jk_i \Delta x} + l (1 - \cos\phi + isin\phi) \left( m^{1-\alpha} - (-1)^{1-\alpha} \right) \right) c_n = ac_{n-1} - h \left( \sum_{k=0}^{n-2} (c_{k+1} - c_k) \delta_{n,k}^\alpha \right)
\]

(62)

Applying a norm on both sides,

\[
|\left( b - de^{-jk_i \Delta x} - f e^{jk_i \Delta x} + l (1 - \cos\phi + isin\phi) \left( m^{1-\alpha} - (-1)^{1-\alpha} \right) \right) c_n| = |ac_{n-1} - h \left( \sum_{k=0}^{n-2} (c_{k+1} - c_k) \delta_{n,k}^\alpha \right)|
\]

(63)

Thus,

\[
|b - de^{-jk_i \Delta x} - f e^{jk_i \Delta x} + l (1 - \cos\phi + isin\phi) \left( m^{1-\alpha} - (-1)^{1-\alpha} \right) ||c_n| < |a||c_{n-1}|
\]

\[
+ |h|| \sum_{k=0}^{n-2} (c_{k+1} - c_k) \delta_{n,k}^\alpha |
\]

(64)

Remembering that it has been proved that for a set \(\forall n \geq 1\),

\[
|c_{n-1}| < |c_o|
\]

Then,

\[
|b - de^{-jk_i \Delta x} - f e^{jk_i \Delta x} + l (1 - \cos\phi + isin\phi) \left( m^{1-\alpha} - (-1)^{1-\alpha} \right) ||c_n| < |a||c_{n-1}|
\]

\[
+ |h|| \sum_{k=0}^{n-2} (c_{k+1} - c_k) \delta_{n,k}^\alpha |
\]
and, it can be inferred that,

\[
|b - de^{-jk_i \Delta x} - fe^{jk_i \Delta x} + l \left(1 - \cos \phi + isin \phi \right) (m^{1-\alpha} - (-1)^{1-\alpha}) ||c_n|| < |a||c_0| + |h||c_0|\beta_n
\]  

where

\[
\beta_n = n^{1-\alpha} + 2^{1-\alpha} - 1^{1-\alpha}
\]

and, the solution will be stable when,

\[
\beta_m = m^{1-\alpha} - (-1)^{1-\alpha}
\]

Simplifying and rearranging,

\[
\frac{|c_n|}{|c_0|} < \frac{|a| + |h|\beta_n}{|b| + |d| + |f| + |l| \left(2 - 2\cos \phi \right) \beta_m}
\]  

Thus, the solution will be stable when,

\[
|a| + |h|\beta_n < 1
\]  

The term is expanded using the simplification terms associated with Equation 22,

\[
\frac{\left(\frac{\Delta t}{\Gamma(2-\alpha)} \delta_{n-1}^\alpha + \frac{\Delta t}{\Gamma(2-\alpha)} \delta_{n,m}^\alpha - \frac{2D_k}{\Delta x^2} \right) + \left| v \left(\frac{\Delta x}{2-\alpha} \right) \delta_{n,m}^\alpha - \frac{D_k}{\Delta x^2} \right|}{\left(\frac{\Delta t}{\Gamma(2-\alpha)} \delta_{n-1}^\alpha + \frac{\Delta t}{\Gamma(2-\alpha)} \delta_{n,m}^\alpha - \frac{2D_k}{\Delta x^2} \right) + \left| v \left(\frac{\Delta x}{2-\alpha} \right) \delta_{n,m}^\alpha - \frac{D_k}{\Delta x^2} \right|} < 1
\]  

\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n-1}^\alpha + v \left(\frac{\Delta x}{\Gamma(2-\alpha)} \right)^{-\alpha} \delta_{n,m}^\alpha > \frac{2D_k}{\Delta x^2}
\]  

An assumption is made as follows,

\[
\beta_n < v \left(\frac{\Delta x}{\Gamma(2-\alpha)} \right)^{-\alpha} \left(2\delta_{n,m}^\alpha + (2 - 2\cos \phi) \beta_m \right)
\]  

The assumption made in Equation 73, results in the implicit upwind numerical scheme being conditionally stable.
When the complementary assumption is made, as follows

\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)\delta_{n,n-1}^\alpha + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)\delta_{n,m}^\alpha}} < \frac{2D_L}{(\Delta x)^2} \quad (75)
\]

Simplifying the resulting condition gives,

\[
(2\delta_{n,n-1}^\alpha + \beta_n) \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)\beta_n} < \frac{4D_L}{(\Delta x)^2} + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} (2-2\cos\phi) \beta_m \quad (76)
\]

where,

\[
\delta_{n,n-1}^\alpha = (n-(n-1))^{1-\alpha} - (n-(n-1)-1)^{1-\alpha} = 1
\]

The complementary assumption Equation 75, results in a conditionally stable implicit upwind scheme.

It has been proved by means of the induction method that the implicit upwind finite difference scheme for the Caputo fractional advection-dispersion equation has the following stability criterion,

\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)\beta_n} < v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} (2\delta_{n,m}^\alpha + (2-2\cos\phi) \beta_m)
\]

and

\[
(2 + \beta_n) \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} < \frac{4D_L}{(\Delta x)^2} + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} (2-2\cos\phi) \beta_m
\]

Because under these conditions, the error of the approximation is not propagated throughout the solution, but rather decreases with each time step, as according to the induction method, where for all values of \( n \), \( |c_{n+1}| < |c_n| \).

### 4.2. First-order upwind explicit.

Considering the developed explicit upwind numerical scheme discussed in Section 3 Equation 18, substituting induction method terms gives

\[
ac_n = bc_{n-1} + dc_{n-1}e^{-jk_i\Delta x} + fc_{n-1}e^{jk_i\Delta x} - h \left( \sum_{k=0}^{n-2} (c_{k+1} - c_k) \delta_{n,k}^\alpha \right) - l \left( \sum_{i=1}^{m-1} (c_{n-1} - c_{n-1}e^{-jk_i\Delta x}) \delta_{m,i}^\alpha \right) \quad (77)
\]

Following the same recursive method as shown in Section 4.4, when \( n = 1 \), and a subset for \( m \) is considered, where \( m = 1 \), the upwind (explicit) numerical scheme for the Caputo fractional advection-dispersion equation is unstable, under the assumption,

\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)\beta_n} > \frac{2D_L}{(\Delta x)^2} \quad (78)
\]

When the opposite assumption is made, the explicit upwind numerical scheme is conditionally stable according to,

\[
\frac{2D_L}{(\Delta x)^2} < \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \quad (79)
\]
For the subset where \( m \) is considered for all \( m > 1 \), the finite difference first-order upwind (explicit) numerical scheme for the Caputo fractional advection-dispersion equation is unstable, under the assumption,

\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m} > \frac{2D_L}{(\Delta x)^2}
\]  \( (80) \)

Making the alternative assumption, the explicit upwind scheme is also conditionally stable according to the condition as follows,

\[
\frac{4D_L}{(\Delta x)^2} + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} (2 - 2\cos \phi) \beta_m < \frac{2(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)}
\]  \( (81) \)

For the second step of the numerical stability analysis, the explicit upwind numerical scheme for the Caputo fractional advection-dispersion equation is unstable under the assumption,

\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m} > \frac{2D_L}{(\Delta x)^2}
\]  \( (82) \)

When the complementary assumption is made where, the explicit upwind scheme is conditionally stable according to the condition,

\[
\frac{4D_L}{(\Delta x)^2} + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} (2 - 2\cos \phi) \beta_m + \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \beta_n < \frac{2(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)}
\]  \( (83) \)

It has been proved by means of the induction method that the explicit upwind finite difference scheme for the Caputo fractional advection-dispersion equation has the following stability criterion,

\[
\frac{4D_L}{(\Delta x)^2} + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} (2 - 2\cos \phi) \beta_m + \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \beta_n < \frac{2(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)}
\]  \( (84) \)

However, for the explicit upwind formulation there are assumptions that result in numerical instabilities.

**4.3. First-order upwind Crank-Nicolson scheme.** Substituting the induction method terms into the developed upwind Crank-Nicolson numerical scheme as discussed in Section 3.3 Equation 28, gives

\[
oc_n = p c_{n-1} + q c_{n-1} e^{-j k_1 \Delta x} - r c_n e^{-j k_1 \Delta x} + f c_{n-1} e^{j k_1 \Delta x} - h \left( \sum_{k=0}^{n-2} (c_{k+1} - c_k) \delta_{n,k} \right) - l \left( \sum_{i=1}^{m-1} \left( 0.5 (c_{n-1} - c_{n-1} e^{-j k_1 \Delta x}) + 0.5 (c_n - c_{n-1} e^{-j k_1 \Delta x}) \right) \delta_{m,i} \right)
\]  \( (85) \)

As presented in Section 4.1, a similar recursive stability analysis is performed. When \( n = 1 \), and a subset for \( m \) is considered, where \( m = 1 \), the upwind Crank-Nicolson numerical scheme for the Caputo fractional advection-dispersion equation is unconditionally stable, under the assumption,

\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} > v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m} + \frac{2D_L}{(\Delta x)^2}
\]  \( (86) \)
When the alternative assumption is made, the upwind Crank-Nicolson numerical scheme is found to be conditionally stable according to the condition
\[
\frac{2D_L}{(\Delta x)^2} < \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \tag{87}
\]

For the subset where \( m \) is considered for all \( m > 1 \), the finite difference first-order upwind Crank-Nicolson numerical scheme is unconditionally stable, under the assumption,
\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} v \frac{\Delta x}{\Gamma(2 - \alpha)} \delta_{n,m} + 2D_L \frac{(\Delta x)^2}{(\Delta x)^2} \tag{88}
\]

Considering the apposed assumption, the upwind Crank-Nicolson scheme is determined to be conditionally stable as defined by the stability condition,
\[
\frac{2D_L}{(\Delta x)^2} < \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \tag{89}
\]

For the subsequent step in the induction method, to demonstrate that for a set \( \forall n \geq 1 \), that \( |c_n| < |c_0| \), the upwind Crank-Nicolson scheme was determined to be conditionally stable when the following assumption is made,
\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,n-1} > v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m} \tag{90}
\]

Under the condition,
\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \beta_n < v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} (\delta_{n,m} + \beta_m (1 - \cos\phi)) \tag{91}
\]

The complementary assumption is made and the upwind Crank-Nicolson scheme is then conditionally stable according to
\[
\frac{4D_L}{(\Delta x)^2} + \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \beta_n < 2 \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} (1 - \cos\phi) \beta_m \tag{92}
\]

It has been demonstrated by means of the induction method that the upwind Crank-Nicolson scheme for the Caputo fractional advection-dispersion equation has the following stability criterion,
\[
\frac{2D_L}{(\Delta x)^2} < \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} ;
\]
\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \beta_n < v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} (\delta_{n,m} + \beta_m (1 - \cos\phi)) ;
\]
and,
\[
\frac{4D_L}{(\Delta x)^2} + \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \beta_n < 2 \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} + v \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} (1 - \cos\phi) \beta_m
\]

Only under these conditions, is the approximation error not transported throughout the solution, but rather decreases with each time step.
4.4. **First-order upwind-downwind weighted scheme (implicit).** Considering the developed implicit upwind-downwind weighted scheme discussed in Section 3.5 Equation 38, substituting induction method terms gives,

\[
sc_n = ac_{n-1} + uc_ne^{-jk_i \Delta x} + wc_ne^{jk_i \Delta x} - \frac{h}{\Delta x} \left( \sum_{k=0}^{n-2} (c_{k+1} - c_k) \delta_{n,k}^{\alpha} \right) - l \left( \sum_{i=1}^{m-1} \left( \theta (c_n - c_ne^{-jk_i \Delta x}) + (1 - \theta) (c_ne^{jk_i \Delta x} - c_n) \right) \delta_{n,i}^{\alpha} \right)
\]

The recursive method as presented in Section 4.4 is applied for the implicit upwind-downwind weighted scheme discussed in Section 4.1, and the implicit upwind-downwind weighted scheme is found to be unconditionally stable, under the assumption,

\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} + v (2\theta - 1) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m}^{\alpha} > \frac{2D_L}{(\Delta x)^2}
\]

(94)

The opposite assumption results in the implicit upwind-downwind weighted scheme to be conditionally stable, according to the following condition

\[
v (2 - \theta) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m}^{\alpha} + \frac{4D_L}{(\Delta x)^2} > \frac{2 \beta}{\Gamma(2 - \alpha)} (\Delta t)^{-\alpha}
\]

(95)

For the subset where \( m \) is considered for all \( m > 1 \), the implicit upwind-downwind weighted scheme is unconditionally stable, under the assumption,

\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} + v (2\theta - 1) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m}^{\alpha} > \frac{2D_L}{(\Delta x)^2}
\]

(96)

When the complementary assumption is made, the implicit upwind-downwind weighted scheme is found to be conditionally stable according to the following condition,

\[
v (1 - \theta) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m}^{\alpha} + v (1 - 3\theta (\cos \phi)) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \beta_m + \frac{2D_L}{(\Delta x)^2} > \frac{\beta}{\Gamma(2 - \alpha)} (\Delta t)^{-\alpha}
\]

(97)

The second step of the numerical stability analysis is performed as described in Section 4.1, and the implicit upwind-downwind weighted scheme is found to be conditionally stable when the following assumption is made,

\[
\frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} + v (2\theta - 1) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m}^{\alpha} > \frac{2D_L}{(\Delta x)^2}
\]

(98)

With the stability condition,

\[
v (\theta + 1) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m}^{\alpha} + 2v (1 - \cos \phi) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \beta_m > \frac{\beta}{\Gamma(2 - \alpha)} (\Delta t)^{-\alpha}
\]

(99)

The alternative assumption results in the implicit upwind-downwind weighted scheme to be conditionally stable, according to the following condition

\[
2v (1 - \theta) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m}^{\alpha} + 2v (1 - \cos \phi) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \beta_m + \frac{4D_L}{(\Delta x)^2} > \frac{\beta}{\Gamma(2 - \alpha)} (\Delta t)^{-\alpha}
\]

(100)
It has thus been proved, by means of the induction method, that the implicit upwind-downwind weighted finite difference scheme for the Caputo fractional advection-dispersion equation (Caputo) has the following stability criterion, 

\[ v(2 - \theta) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m}^\alpha + \frac{4D_L}{(\Delta x)^2} > \frac{2(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \]

\[ v(1 - \theta) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m}^\alpha + v(1 - 3\theta \cos\phi) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \beta_m + \frac{2D_L}{(\Delta x)^2} > \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \]

\[ v(\theta + 1) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m}^\alpha + 2v(1 - \cos\phi) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \beta_m > \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \beta_n \]

and,

\[ 2v(1 - \theta) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m}^\alpha + 2v(1 - \cos\phi) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \beta_m + \frac{4D_L}{(\Delta x)^2} > (2 + \beta_n) \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \]

Under these conditions, the error of the approximation is not propagated throughout the solution, but rather decreases with each time step, as according to the induction method, where for all values of \( n \), \( |c_{n+1}| < |c_n| \).

4.5. First-order upwind-downwind weighted scheme (explicit). The induction method terms associated with the numerical stability analysis are substituted into the developed explicit upwind-downwind weighted scheme as presented in Section 3.4 Equation 33,

\[ ac_n = sc_{n-1} + uc_{n-1}e^{-jk_i\Delta x} + wc_{n-1}e^{jk_i\Delta x} - h \left( \sum_{k=0}^{n-2} (c_{k+1} - c_k) \delta_{n,k}^\alpha \right) \]

\[-l \left( \sum_{i=1}^{m-1} \left[ \theta \left( c_{n-1} - c_{n-1}e^{-jk_i\Delta x} \right) + (1 - \theta) \left( c_{n-1}e^{jk_i\Delta x} - c_{n-1} \right) \right] \delta_{n,i}^\alpha \right) \]

(101)

A similar method as followed in Section 4.1 is used, and when \( n = 1 \), and a subset for \( m \) is considered, where \( m = 1 \), the explicit upwind-downwind weighted numerical scheme is unstable, under the assumption,

\[ \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,n-1}^\alpha + v(2\theta - 1) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m}^\alpha > \frac{2D_L}{(\Delta x)^2} \]

(102)

Making the complementary assumption, the explicit upwind-downwind weighted numerical scheme is conditionally stable, according to the following condition

\[ v(1 - 2\theta) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m}^\alpha + \frac{2D_L}{(\Delta x)^2} < \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} \]

(103)

For the subset where \( m \) is considered for all \( m > 1 \), the explicit upwind-downwind weighted numerical scheme is found to be unstable, under the assumption,

\[ \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} + v(2\theta - 1) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \delta_{n,m}^\alpha > \frac{2D_L}{(\Delta x)^2} \]

(104)

When the opposite assumption is made, the scheme is conditionally stable, according to

\[ \frac{2D_L}{(\Delta x)^2} < \frac{(\Delta t)^{-\alpha}}{\Gamma(2 - \alpha)} + v(\delta_{n,m}^\alpha (\theta + 1) + \beta_m (1 - \cos\phi)) \frac{(\Delta x)^{-\alpha}}{\Gamma(2 - \alpha)} \]

(105)
Lastly, making the assumption that $|c_{n-1}| < |c_n|$ is true for all time steps and following the same process, the explicit upwind-downwind weighted scheme is also found to be unstable when the assumption is made,

$$\frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n,n-1} + v (2\theta - 1) \frac{(\Delta x)^{-\alpha}}{\Gamma(2-\alpha)} \delta_{n,m} > \frac{2D_L}{(\Delta x)^2}$$  \hspace{1cm} (106)

Making the alternative assumption, the explicit upwind-downwind weighted scheme is also unstable. Thus, the second assumption, $|c_n| < |c_o|$, failed to produce a stable solution.

This concludes the numerical stability analysis for the proposed numerical solution schemes, and in the following section, the found stabilities are further assessed.

5. **Comparison of numerical schemes stability conditions.** Upwind numerical schemes were developed for the Caputo fractional advection-dispersion equation, including the traditional upwind as well as the newly proposed upwind Crank-Nicolson and a weighted upwind-downwind scheme. The numerical schemes were subjected to numerical stability analysis using the recursive method, with a summary of these results in Table 1.

The traditional upwind (explicit) numerical scheme for the space-time fractional advection-dispersion equation (Caputo) is found to be unstable under certain assumptions, while the traditional upwind (implicit) numerical scheme is conditionally stable under all assumptions. The upwind Crank-Nicolson scheme, which has a half-weighted approach to implicit and explicit for the advection term, is found to be conditionally stable under all assumptions. The weighted upwind-downwind (explicit) scheme is found to be unstable under certain assumptions, similarly to the traditional upwind (explicit) numerical scheme. The weighted upwind-downwind (implicit) scheme is found to be conditionally stable under all assumptions, again similar to the traditional upwind (implicit) scheme. This trend of implicit numerical scheme formulations being more stable than the corresponding explicit formulation has been found by other researchers [19, 21, 16].

6. **Conclusions.** The new weighted upwind-downwind (explicit) scheme is not more applicable than the traditional upwind (explicit) scheme, because the new weighted (explicit) scheme tends to be unstable under more assumptions. On the other hand, the weighted (implicit) scheme does provide an improvement on the traditional upwind (implicit) scheme in terms of stability, where the inclusion of the weighting factor ($\theta$) provides a means to improve the likelihood of upholding the stability condition. Thus, the upwind Crank-Nicolson and weighted upwind-downwind (implicit) schemes are applicable for solution of the space-time Caputo fractional advection-dispersion equation, if the stability criterion are upheld.

**REFERENCES**

[1] A. Allwright and A. Atangana, Augmented upwind numerical schemes for the groundwater transport advection-dispersion equation with local operators, *International Journal for Numerical Methods in Fluids*, 87 (2018), 437–462.

[2] A. Atangana, On the stability and convergence of the time-fractional variable order telegraph equation, *Journal of Computational Physics*, 293 (2015), 104–114.

[3] A. Atangana and K. M. Owolabi, New numerical approach for fractional differential equations, *Mathematical Modelling of Natural Phenomena*, 13 (2018), Art. 3, 21 pp.

[4] D. A. Benson, S. W. Wheatcraft and M. M. Meerschaert, Application of a fractional advection-dispersion equation, *Water Resources Research*, 36 (2000), 1403–1412.
| Scheme          | Assumptions                                                                 | Stability condition                                                                 |
|-----------------|-----------------------------------------------------------------------------|-------------------------------------------------------------------------------------|
| Upwind (explicit) | \( \frac{(\Delta t)^{-\alpha} \delta_{n,n-1} + \alpha (\Delta x)^{-\alpha} \delta_{n,m}}{(1 - \alpha)} > \frac{2D_x}{(\Delta t)^2} \) | Unstable                                                                            |
|                 | \( \frac{(\Delta t)^{-\alpha} \delta_{n,n-1} + v (\Delta x)^{-\alpha} \delta_{n,m}}{(1 - \alpha)} < \frac{2D_x}{(\Delta t)^2} \) | Conditionally stable / Conditionally stable \( \beta_n < \frac{(\Delta t)^{-\alpha}}{(1 - \alpha)} (2 + \beta_n) + \frac{v (\Delta x)^{-\alpha}}{(1 - \alpha)} (2 - 2 \cos \phi) \beta_m \) |
| Upwind (implicit) | \( \frac{(\Delta t)^{-\alpha} \delta_{n,n-1} + v (\Delta x)^{-\alpha} \delta_{n,m}}{(1 - \alpha)} > \frac{2D_x}{(\Delta t)^2} \) | Conditionally stable \( \frac{2D_x}{(\Delta t)^2} < \frac{(\Delta t)^{-\alpha}}{(1 - \alpha)} \beta_n < \frac{(\Delta t)^{-\alpha}}{(1 - \alpha)} (\delta_{n,m} + \beta_m (1 - \cos \phi)) \) |
|                 | \( \frac{(\Delta t)^{-\alpha} \delta_{n,n-1} + v (\Delta x)^{-\alpha} \delta_{n,m}}{(1 - \alpha)} < \frac{2D_x}{(\Delta t)^2} \) | Conditionally stable \( \frac{2D_x}{(\Delta t)^2} + \frac{(\Delta t)^{-\alpha}}{(1 - \alpha)} \beta_n < \frac{2(\Delta t)^{-\alpha}}{(1 - \alpha)} + \frac{v (\Delta x)^{-\alpha}}{(1 - \alpha)} (1 - \cos \phi) \beta_m \) |
| Upwind Crank-Nicolson | \( \frac{(\Delta t)^{-\alpha} \delta_{n,n-1}}{(1 - \alpha)} > \frac{0.5(\Delta x)^{-\alpha} \delta_{n,m}}{(1 - \alpha)} + \frac{2D_x}{(\Delta t)^2} \) | Unconditionally stable / Conditionally stable \( \frac{(\Delta t)^{-\alpha} \delta_{n,m}}{(1 - \alpha)} + \frac{2D_x}{(\Delta t)^2} \) |
|                 | \( \frac{(\Delta t)^{-\alpha} \delta_{n,n-1}}{(1 - \alpha)} < \frac{0.5(\Delta x)^{-\alpha} \delta_{n,m}}{(1 - \alpha)} + \frac{2D_x}{(\Delta t)^2} \) | Conditionally stable / Unstable / conditionally stable \( \beta_n < \frac{(\Delta t)^{-\alpha}}{(1 - \alpha)} (\delta_{n,m} + \beta_m (1 - \cos \phi)) \) |

**Table 1.** Summary of the established stability condition, and corresponding assumption, for each numerical approximation scheme.
[5] D. A. Benson, *The Fractional Advection-Dispersion Equation: Development and Application*, PhD thesis, University of Nevada, Reno, 1998.

[6] K. Diethelm, N. J. Ford, A. D. Freed and Y. Luchko, Algorithms for the fractional calculus: A selection of numerical methods, *Computer methods in applied mechanics and engineering*, 194 (2005), 743–773.

[7] R. Fazio and A. Jannelli, A finite difference method on quasi-uniform mesh for time-fractional advection-diffusion equations with source term, *arXiv preprint arXiv1801.07160*.

[8] R. Gnitchogna and A. Atangana, New two step laplace adam-bashforth method for integer a noninteger order partial differential equations, *Numerical Methods for Partial Differential Equations*, 34 (2018), 1739–1758.

[9] F. Huang and F. Liu, The fundamental solution of the space-time fractional advection-dispersion equation, *Journal of Applied Mathematics and Computing*, 18 (2005), 339–350.

[10] Q. Huang, G. Huang and H. Zhan, A finite element solution for the fractional advection-dispersion equation, *Advances in Water Resources*, 31 (2008), 1578–1589.

[11] H. Jafari and H. Tajadodi, Numerical solutions of the fractional advection-dispersion equation, *Prog. Fract. Differ. Appl.*, 1 (2015), 37–45.

[12] S. Javadi, M. Jani and E. Babolian, A numerical scheme for space-time fractional advection-dispersion equation, *arXiv preprint arXiv:1512.06629*.

[13] X. Li and H. Rui, A high-order fully conservative block-centered finite difference method for the time-fractional advection-dispersion equation, *Applied Numerical Mathematics*, 124 (2018), 89–109.

[14] Z. Li, Z. Liang and Y. Yan, High-order numerical methods for solving time fractional partial differential equations, *Journal of Scientific Computing*, 71 (2017), 785–803.

[15] F. Liu, V. V. Anh, I. Turner and P. Zhuang, Time fractional advection-dispersion equation, *Journal of Applied Mathematics and Computing*, 13 (2003), 233–245.

[16] F. Liu, P. Zhuang, V. Anh, I. Turner and K. Burrage, Stability and convergence of the difference methods for the space–time fractional advection–diffusion equation, *Applied Mathematics and Computation*, 191 (2007), 12–20.

[17] T. Liu and M. Hou, A fast implicit finite difference method for fractional advection-dispersion equations with fractional derivative boundary conditions, *Advances in Mathematical Physics*, 2017 (2017), Art. ID 8716752, 8 pp.

[18] Z. Liu and X. Li, A crank–nicolson difference scheme for the time variable fractional mobile–immobile advection–dispersion equation, *Journal of Applied Mathematics and Computing*, 56 (2018), 391–410.

[19] V. E. Lynch, B. A. Carreras, D. del Castillo-Negrete, K. Ferreira-Mejias and H. Hicks, Numerical methods for the solution of partial differential equations of fractional order, *Journal of Computational Physics*, 192 (2003), 406–421.

[20] M. M. Meerschaert, Fractional calculus, anomalous diffusion, and probability, in *Fractional Dynamics: Recent Advances*, World Scientific, 2012, 265–284.

[21] M. M. Meerschaert and C. Tadjeran, Finite difference approximations for fractional advection-dispersion flow equations, *Journal of Computational and Applied Mathematics*, 172 (2004), 65–77.

[22] R. Metzler, W. G. Glöckle and T. F. Nonnenmacher, Fractional model equation for anomalous diffusion, *Physics A: Statistical Mechanics and its Applications*, 211 (1994), 13–24.

[23] R. Metzler and J. Klafter, The random walk’s guide to anomalous diffusion: A fractional dynamics approach, *Physics reports*, 339 (2000), 1–77.

[24] G. Pang, W. Chen and Z. Fu, Space-fractional advection–dispersion equations by the kansa method, *Journal of Computational Physics*, 293 (2015), 280–296.

[25] Y. Povstenko, Space-time-fractional advection diffusion equation in a plane, in *Advances in Modelling and Control of Non-Integer-Order Systems*, Springer, 320 (2015), 275–284.

[26] Y. Povstenko, Fundamental solutions to time-fractional advection diffusion equation in a case of two space variables, *Mathematical Problems in Engineering*, 2014 (2014), Art. ID 705364, 7 pp.

[27] Q. Rubbab, I. A. Mirza and M. Z. A. Qureshi, Analytical solutions to the fractional advection-diffusion equation with time-dependent pulses on the boundary, *AIP Advances*, 6 (2016), 075318.

[28] W. Schneider and W. Wyss, Fractional diffusion and wave equations, *Journal of Mathematical Physics*, 30 (1989), 134–144.
[29] S. Shen, F. Liu, V. Anh, I. Turner and J. Chen, A novel numerical approximation for the space fractional advection–dispersion equation, *IMA journal of Applied Mathematics*, 79 (2014), 431–444.

[30] E. Sousa, Finite difference approximations for a fractional advection diffusion problem, *Journal of Computational Physics*, 228 (2009), 4038–4054.

[31] E. Sousa and C. Li, A weighted finite difference method for the fractional diffusion equation based on the riemann–liouville derivative, *Applied Numerical Mathematics*, 90 (2015), 22–37.

[32] L. Su, W. Wang and Q. Xu, Finite difference methods for fractional dispersion equations, *Applied Mathematics and Computation*, 216 (2010), 3329–3334.

[33] A. A. Tateishi, H. V. Ribeiro and E. K. Lenzi, The role of fractional time-derivative operators on anomalous diffusion, *Frontiers in Physics*, 5 (2017), 52.

[34] K. Wang and H. Wang, A fast characteristic finite difference method for fractional advection–diffusion equations, *Advances in Water Resources*, 34 (2011), 810–816.

[35] W. Wyss, The fractional diffusion equation, *Journal of Mathematical Physics*, 27 (1986), 2782–2785.

[36] Y. Yirang, L. Changfeng and S. Tongjun, The second-order upwind finite difference fractional steps method for moving boundary value problem of oil-water percolation, *Numerical Methods for Partial Differential Equations*, 30 (2014), 1103–1129.

[37] Y. Yirang, Y. Qing, L. Changfeng and S. Tongjun, Numerical method of mixed finite volume-modified upwind fractional step difference for three-dimensional semiconductor device transient behavior problems, *Acta Mathematica Scientia*, 37 (2017), 259–279.

[38] Y. Yuan, The upwind finite difference fractional steps methods for two-phase compressible flow in porous media, *Numerical Methods for Partial Differential Equations: An International Journal*, 19 (2003), 67–88.

Received June 2018; revised July 2018.

E-mail address: AllwrightAJ@ufs.ac.za
E-mail address: AtanganaA@ufs.ac.za