On the infinitude of Prime $k$-tuples

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Abstract

Starting with Zhang’s theorem on the infinitude of prime doubles [1], we give an inductive argument that there exists an infinite number of prime $k$-tuples for at least one admissible set $\mathcal{H}_k = \{h_1, \ldots, h_k\}$ for each $k$.

1 Introduction

The eventual lesson learned from counting single primes was “If you want the asymptotics of primes, you have to look at $\zeta(s)|_{\Re(s)=1}$”. Indeed, in the single prime case it required a thorough understanding of $\zeta(s)$ to finally nail down the prime number theorem (PNT). Even the Selberg-Erdős ‘elementary proof’ of the PNT conceals $\zeta(s)$ lurking in the background [2]. Consequently, it seems unlikely that the Hardy-Littlewood $k$-tuple conjecture [3] can be settled without possessing the $k$-tuple analog of zeta.

Nevertheless, number theorists have made impressive gains in the quest to count prime $k$-tuples — especially recently. As important as these recent advances are, the situation for counting prime $k$-tuples is rather like that for single primes prior to Riemann, Hadamard, and de la Vallée Poussin: Without a $k$-tuple zeta function to exploit, the focus has been on showing the infinitude of prime $k$-tuples.

Perhaps the most germane are Zhang’s theorem [1] and the Maynard-Tao theorem [4],[5] paraphrased by Granville [6]:

Theorem 1.1 There exists an integer $k$ such that the following is true: If $x+a_1, \ldots, x+a_k$ is an admissible set of forms then there are infinitely many integers $n$ such that at least two of $n+a_1, \ldots, n+a_k$ are prime numbers.

Theorem 1.2 For any given integer $m \geq 2$, there exists an integer $k$ such that the following is true: If $x+a_1, \ldots, x+a_k$ is an admissible set of forms then there are infinitely many integers $n$ such that at least $m$ of $n+a_1, \ldots, n+a_k$ are prime numbers.

Both of these theorems imply a corollary, again given by Granville [6]:

Corollary 1.1 There is an integer $h$; $0 < h \leq B$ such that there are infinitely many pairs of primes $p, p+h$. (for some finite bound $B$)

To justify this corollary, choose a sufficiently large finite interval $(0, B]$ and string it together to cover the positive integers in the obvious way. Each interval contains at least two primes out of a finite number of combinations that could occur. Since there are an infinitude of intervals, at least one of those combinations must be represented an infinite
number of times. Moreover, such a combination is necessarily admissible because \( h \) will be even.

Unfortunately, the same reasoning doesn’t work for \( k \)-tuples with \( k > 2 \). The problem is, one can deduce an infinite number of at least one particular combination but there is no guarantee it will be an *admissible* \( k \)-tuple. This is disappointing because belief in the \( k \)-tuple conjecture is strong, so one strongly expects an infinitude of admissible prime \( k \)-tuples.

Of course Euclid (and later several others) figured out a way to get the total number of primes without using \( \zeta(s) \). And Zhang first did it for (certain) prime doubles. On the other hand, Euler found a way to utilize \( \zeta(s) \) to deduce the infinitude of single primes. Can one generalize Euler to the prime double case and thereby get a handle on a \( k \)-tuple zeta function? Unfortunately, it is well-known the sum over prime-double reciprocals does not diverge. So a straightforward generalization is thwarted from the start.

But maybe a straightforward generalization is not the best approach. Let’s briefly re-interpret Euler’s method for guidance. First recall that, if \( \mathcal{P} \) is the set of all primes, then

\[
\log(\zeta(s)) \approx \sum_{p \in \mathcal{P}} p^{-s}.
\]

We aim to show that \( \lim_{s \to 1^+} \log(\zeta(s)) = \infty \), but suppose we do not know Euler’s product representation of \( \zeta(s) \). We can still conclude the result by a simple argument.

Assume the contrary. Then \( \sum \Lambda(n)/\log(n)n^s \) would converge uniformly at \( s = 1 \), and we would have

\[
\lim_{s \to 1^+} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\log(n)n^s} = \lim_{N \to \infty} \sum_{n \leq N} \frac{\Lambda(n)}{\log(n)n}.
\]

Hence, partial summation would yield

\[
\lim_{s \to 1^+} \log(\zeta(s)) = \log(\log(N_{\text{max}})) - \epsilon + O\left(\frac{1}{\log(N_{\text{max}})}\right)
\]

where \( \epsilon \) is an end-point contribution and

\[
N_{\text{max}} := \lim_{N \to \infty} \sum_{n=1}^{N} \Lambda(n) = \lim_{N \to \infty} \sum_{p \leq N} \log(p).
\]

But from Euclid we know the right-most sum must diverge, and so we have a contradiction. This \( \log(\log(N)) \) behavior for \( \sum_{p \leq N} 1/p \) is well-known, and it shows the reciprocal of primes ‘just barely’ diverges with \( N \).

The point of reviewing Euler’s method is to give a preview of our plan for prime doubles. We don’t yet have a representation of the prime double zeta function \( \zeta(2)(s) \). So we will adapt the above argument to the prime-double case and utilize Zhang’s result to infer the divergence of a certain pertinent sum for at least one admissible \( \mathcal{H}_2 = \{0, h\} \).

It turns out that the pertinent sum to consider is

\[
\log'(\zeta(2)(s)) := \sum_{n=1}^{\infty} \frac{\Lambda(n)\Lambda(n+h) \log(n(n+h)^{1/2})}{\log(n)\log(n+h)(n(n+h))^{s/2}} \asymp \sum_{p \in \mathcal{P}} \frac{\log(p(p+h))}{(p(p+h))^{s/2}}.
\]
This sum happens to exhibit the $\log(\log(N))$ behavior.

Of course one could just guess this sum. But it is more satisfying and reassuring to see that it comes from the definition

$$\log(\zeta(2)(s)) := \sum_{n=1}^{\infty} \frac{\Lambda(n)\Lambda(n+h)}{\log(n)\log(n+h)(n(n+h))^{s/2}},$$

which in turn comes from explicit formulae relating exact and average summatory functions for prime doubles [7, 8].

The prime-double case is then extended to $k$-tuples by induction. The reasoning relies crucially on Zhang’s theorem: Given an admissible prime $k$-tuple $(p + h_1, \ldots, p + h_k)$, Zhang’s theorem implies there exists at least one $h$ such that the amended $(k+1)$-tuple $(p + h_1, \ldots, p + h_k, p + h_k + h)$ is also prime (but not necessarily admissible). Of course this is consistent with the Maynard-Tao theorem.

The final step is to show there must be at least one admissible prime $(k+1)$-tuple by this construction. Here we rely on a lemma establishing the fortunate circumstance that $\log'(\zeta(2)(s)) \approx \log'(\zeta_{2i}(s))$ where the left side is associated with admissible $h$ and the right side represents an equivalence class of prime doubles $[2i]$ determined by $(2i)^l \sim 2i$ for all $l' \in \mathbb{N}_+$ with $2i$ such that $(2i)^l = h$ for some $i, l \in \mathbb{N}_+$.

The end game and our main result is the following theorem:

**Theorem** Assume $\mathcal{H}_k = \{0, h_2, \ldots, h_k\}$ is admissible. Then

$$\lim_{s \to 1^+} \frac{\sum_{p \in \mathcal{P}_k} \log^{k-1}(p_{(k)})}{p_{(k)}} (-1)^{k-1} \log^{(k-1)l'}(\zeta_{(k)}(s)) = 1$$

(1.1)

and

$$\lim_{s \to 1^+} (-1)^{k-1} \log^{(k-1)l'}(\zeta_{(k)}(s)) = \infty$$

(1.2)

for at least one admissible $\mathcal{H}_k$.

Here $\mathcal{P}_k$ is the set of admissible prime $k$-tuples and $p_{(k)} := (p(p + h_2) \cdots (p + h_k))^{1/k}$.

To briefly recap, possessing exact and average summatory functions for admissible prime $k$-tuples allows us to infer certain relevant objects $\log(\zeta_{(k)}(s))$. Although $\zeta_{(k)}(s)$ remains elusive, $\log(\zeta_{(k)}(s))$ together with Euler’s method and Zhang’s theorem allow us to deduce an infinitude of at least one admissible prime $k$-tuple for all $k$. History suggests we will need to understand $\zeta_{(k)}(s)$ to go further.

## 2 Definitions and lemmas

This section will introduce some notation/definitions and establish some useful lemmas.

- $\mathcal{P}_k$ is the set of admissible prime $k$-tuples.
- $\mathcal{P}_k \supset p_k := (p, p + h_2, \ldots, p + h_k)$ where $p$ is prime and $\mathcal{H}_k = \{0, h_2, \ldots, h_k\}$ is admissible.
\[ n_{(k)} := (n(n + h_2)(n + h_k))^{1/k} \text{ for integer } n. \]

\[ \mu_{(k)}(n) := (-1)^k \mu(n) \cdots \mu(n + h_k). \]

\[ \lambda_{(k)}(n) := \Lambda(n) \cdots \Lambda(n + h_k)/\log(n) \cdots \log(n + h_k). \]

\[ \Lambda_{(k)}(n) := \lambda_{(k)}(n) \log^k(n_{(k)}). \]

\[ \log \left( \zeta_{(k)}(s) \right) := \sum_{n=1}^{\infty} \lambda_{(k)}(n)/n_{(k)}^s, \quad \Re(s) > 1. \]

Now some lemmas.

**Lemma 2.1**

\[ \log \left( \zeta_{(2)}(s) \right) \asymp \sum_{p_2 \in \mathbb{P}_2} \frac{1}{p_{(2)}^s} =: \log \left( \zeta_{(2)}(s) \right). \tag{2.1} \]

**proof:** By definition,

\[ \log \left( \zeta_{(2)}(s) \right) = \sum_{p_2 \in \mathbb{P}_2} \frac{1}{\omega p^{\mu s/2} \omega' (p^{\omega} + 2i)^{\omega' s/2}}, \quad \Re(s) > 1 \]

\[ = \sum_{p_2 \in \mathbb{P}_2} \frac{1}{p^{s/2} (p + 2i)^{s/2}} + \sum_{p_2 \in \mathbb{P}_2} \sum_{\omega = 2}^{\infty} \sum_{\omega' = 2}^{\infty} \frac{1}{\omega p^{\mu s/2} \omega' (p^{\omega} + 2i)^{\omega' s/2}} \]

\[ =: \log \left( \zeta_{(2)}(s) \right) + S(s) \tag{2.2} \]

where \( p_{(2)}^s := (p^{\omega}, (p^{\omega} + 2i)^{\omega'}) \) is a prime-power double. But, for \( s = \sigma + it \) with \( \sigma, t \in \mathbb{R} \) and \( \sigma > 1 \),

\[ |S(s)| \leq \sum_{p_2 \in \mathbb{P}_2} \sum_{\omega = 2}^{\infty} \sum_{\omega' = 2}^{\infty} \left| \frac{1}{\omega p^{\mu s/2} \omega' (p^{\omega} + 2i)^{\omega' s/2}} \right| \]

\[ < \sum_{p_2 \in \mathbb{P}_2} \sum_{\omega = 2}^{\infty} \sum_{\omega' = 2}^{\infty} \left| \frac{1}{p^{s/2} (p^{\omega} + 2i)^{\omega' s/2}} \right| \]

\[ < \sum_{p_2 \in \mathbb{P}_2} \sum_{\omega = 2}^{\infty} \sum_{\omega' = 2}^{\infty} \left| \frac{1}{p^{s/2} (p + 2i)^{\omega' s/2}} \right| \]

\[ \leq \sum_{p_2 \in \mathbb{P}_2} \frac{1}{p^{s/2} (p^{\omega} - p^{\omega/2}) ((p + 2i)^{\sigma} - (p + 2i)^{\sigma/2})} \]

\[ < \sum_{n=2}^{\infty} \frac{1}{(n^{\sigma} - n^{\sigma/2}) ((n + 2i)^{\sigma} - (n + 2i)^{\sigma/2})} \]

\[ < \sum_{n=2}^{\infty} \frac{1}{n^{1/2} (n^{1/2} - 1)} \frac{1}{(n + 2i)^{1/2} ((n + 2i)^{1/2} - 1)} \]

\[ < \sum_{n=2}^{\infty} \frac{1}{(n - 1)^{3/2}} = \zeta(3/2). \tag{2.3} \]

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1The subscript \((k)\) is supposed to indicate the level \(k\) and implicitly the admissible \(\mathcal{H}_k\). Sometimes we will make the dependence on \(\mathcal{H}_k\) explicit by writing for example \(p_{(h_2)}\) for a particular prime double.
The following two lemmas utilize a particularly useful interpretation of sums of the form \( \sum \mu(2)(n) \lambda(2)(n) f(n) \). Since the pre-factor localizes onto prime doubles \( \mathbb{Z} \), it is advantageous to view the sum as a double sum over the coprime 2-lattice.

**Lemma 2.2** Denote the coprime 2-lattice by \( \{(n_1, n_2) \in \mathbb{N}_+^2 \mid \gcd(n_1, n_2) = 1\} \), and let \( 2i \leq 2j = (2i)^l \) with fixed \( l \in \mathbb{N}_+ \). Then

\[
\sum_{n=1}^{\infty} \frac{\mu(2)(n)\lambda(2)(n)}{n^{s/2}(n+2i)^{s/2}} \log(n(2i)) \geq \sum_{n=1}^{\infty} \frac{\mu(2)(n)\lambda(2)(n)}{n^{s/2}(n+2i)^{s/2}} \log(n(2j)) - R(2j, 2i; \infty)
\]

where \( R(2j, 2i; \infty) := \lim_{N \to \infty} \sum_{n=N}^{N+1} \frac{\mu(2)(n)\lambda(2)(n)}{n^{s/2}(n+2i)^{s/2}} \log(n(2j)) \).

**proof:** For \( N > 1 + (2j - 2i) \),

\[
\sum_{n=1}^{n=N} \frac{\mu(2)(n)\lambda(2)(n)}{n^{s/2}(n+2i)^{s/2}} \log(n(2i)) = \sum_{n_1 \leq N} \sum_{n_2 \leq N^2 + 2i} \frac{\mu(n_1)\lambda(1)(n_1)\mu(n_2)\lambda(1)(n_2)}{n_1^{s/2}n_2^{s/2}} \log(n_1n_2) \delta(n_2, n_1 + 2i)
\]

\[
= \sum_{n_1 = 1 + (2j - 2i)}^{N} \frac{\mu(n_1)\lambda(n_1)}{n_1^{s/2}} \left( \sum_{n_2 \leq N^2 + 2i} \frac{\mu(n_2)\lambda(n_2)}{n_2^{s/2}} \log(n_1n_2) \delta(n_2, n_1 + 2i) \right)
\]

\[
\geq \sum_{n_1 = 1 + (2j - 2i)}^{N} \frac{\mu(n_1)\lambda(n_1)}{n_1^{s/2}} \left( \sum_{n_2 \leq N^2 + 2i} \frac{\mu(n_2)\lambda(n_2)}{n_2^{s/2}} \log(n_1n_2) \delta(n_2, n_1 + 2i) \right)
\]

\[
= \sum_{n_1 = 1 + (2j - 2i)}^{N-2} \frac{\mu(n_1)\lambda(n_1)}{n_1^{s/2}} \left( \sum_{n_2 \leq N^2 + 2i} \frac{\mu(n_2)\lambda(n_2)}{n_2^{s/2}} \log(n_1n_2) \delta(n_2, n_1 + 2i) \right)
\]

\[
= \sum_{n_1 = 1 + (2j - 2i)}^{N-2} \frac{\mu(n_1)\lambda(n_1)}{n_1^{s/2}} \left( \sum_{n_2 \leq N^2 + 2i} \frac{\mu(n_2)\lambda(n_2)}{n_2^{s/2}} \log(n_1n_2) \delta(n_2, n_1 + 2i) \right)
\]

\[
\geq \sum_{n_1 = 1 + (2j - 2i)}^{N-2} \frac{\mu(n_1)\lambda(n_1)}{n_1^{s/2}} \left( \sum_{n_2 \leq N^2 + 2i} \frac{\mu(n_2)\lambda(n_2)}{n_2^{s/2}} \log(n_1n_2) \delta(n_2, n_1 + 2i) \right)
\]

The delta function in the second line restricts the double sum to the appropriate ray \( \mathfrak{r}(2i) \) in the coprime 2-lattice. The fourth line is a simple truncation of the outer sum. The fifth line holds since \( n_1 \mapsto n_1' = n_1 + 2i - 2j \) bijectively maps lattice intersections of \( \mathfrak{r}(2i) \) to lattice intersections of \( \mathfrak{r}(2j) \) precisely because \( 2j = (2i)^l \) which ensures that \( \gcd(n_1', n_1 + 2j) = 1 \) and the inner sum gets evaluated on a congruent set of coprimes. The result follows as \( N \to \infty \) since both series converge for \( \Re(s) > 1 \). ✷

By the same token,

**Lemma 2.3** Let \( 2j \leq 2m = (2j)^l \) with fixed \( l \in \mathbb{N}_+ \), then

\[
\sum_{n=1}^{\infty} \frac{\mu(2m)(n)\lambda(2m)(n)}{n^{s/2}(n+2m)^{s/2}} \log(n(2m)) \geq \sum_{n=1}^{\infty} \frac{\mu(2)(n)\lambda(2)(n)}{n^{s/2}(n+2j)^{s/2}} \log(n(2j)) - R(2m, 2j; 1)
\]

(2.6)
where \( R(2m, 2j; 1) := \sum_{n=1}^{(2m-2j)} \frac{\mu(2j(n))\lambda(2j)(n)}{n^{s/2}(n+2j)^{s/2}} \log(n+2j) \).

**proof:** The proof follows the same argument as the preceding lemma.

\[
\sum_{n=1}^{N} \frac{\mu(2m)(n)\lambda(2m)(n)}{n^{s/2}(n+2m)^{s/2}} \log(n+2m)
\]

\[
\geq \sum_{n'=1}^{N+(2m-2j)} \frac{\mu(n'_1)\lambda(n'_1)}{n'_1^{s/2}} \left( \sum_{n_2 \leq N+2m} \frac{\mu(n_2)\lambda(n_2)}{n_2^{s/2}} \log(n'_1, n_2, n'_1 + 2j) \right) \]

\[
\geq \sum_{n'=1}^{N} \frac{\mu(2j)(n)\lambda(2j)(n)}{n^{s/2}(n+2j)^{s/2}} \log(n+2j) - R(2m, 2j; 1). \tag{2.7}
\]

Again, the crucial fact is that \( \gcd(n_1, n_1 + 2m) = 1 \) and the map \( n_1 \mapsto n'_1 = n_1 + 2m - 2j \) ensures that \( \gcd(n'_1, n'_1 + 2j) = 1 \). \( \square \)

Therefore choosing \( 2i \leq 2j = (2i)^l \) or \( (2j)^l = 2i \geq 2j \),

\[
\log' \left( \zeta_{(2i)}(s) \right) = \sum_{n=1}^{\infty} \frac{\mu(2i)(n)\lambda(2i)(n)}{n^{s/2}(n+2i)^{s/2}} \log(n_{(2i)})
\]

\[
\geq \sum_{n=1}^{\infty} \frac{\mu(2j)(n)\lambda(2j)(n)}{n^{s/2}(n+2j)^{s/2}} \log(n_{(2j)}) \pm |2j-2i|
\]

\[
= \log' \left( \zeta_{(2j)}(s) \right) \pm |2j-2i| \tag{2.8}
\]

follows from Lemma 2.2 and Lemma 2.3 because the remainder in both cases is certainly bounded by \( |2j-2i| \). So if \( 2i \) is square-free and \( 2j = (2i)^l \), then

\[
\log' \left( \zeta_{(2i)}(s) \right) \asymp \log' \left( \zeta_{(2j)}(s) \right), \quad \Re(s) > 1 \tag{2.9}
\]

where the equivalence class includes all integer powers of \( 2i \).

Together with Lemma 2.1 conclude

**Corollary 2.1**

\[
\log' \left( \zeta_{(2i)}(s) \right) \asymp \log' \left( \zeta_{(2j)}(s) \right), \quad \Re(s) > 1. \tag{2.10}
\]

### 3 Euler’s lead

With these preliminaries, we can follow Euler’s method for prime doubles. Recall the definition

**Definition 3.1**

\[
\log \left( \zeta_{(2)}(s) \right) := \sum_{n=1}^{\infty} \frac{\lambda_{(2)}(n)}{n^{s_{(2)}}}, \quad \Re(s) > 1. \tag{3.1}
\]
We have

**Proposition 3.1** Let \( 2j = h \) where \( h \) comes from Zhang’s theorem. Then

\[
\lim_{s \to 1^+} \log' \left( \zeta(2j)(s) \right) = \infty. \tag{3.2}
\]

**Proof:** Assume the contrary. Then \( \lim_{s \to 1^+} \log' \left( \zeta(2j)(s) \right) \) is bounded and so converges. Consequently, for every \( \varepsilon > 0 \) there exists an \( M \) such that \( m > M \) implies

\[
\left| \sum_{n=m+1}^{m+l} \frac{\lambda(2j)(n)}{(n+2j)} \log(n(2j)) \right| < \varepsilon \tag{3.3}
\]

for each \( l \in \mathbb{N}_+ \). It follows that \( \log' \left( \zeta(2j)(s) \right) \) converges uniformly for \( \Re(s) \geq 1 \).

Now, the PNT implies \( \lambda(1)(n+2j) = \lambda(n+2j) = O(1/\log(n+2j)) \). Moreover, \( \log(n(2j))/\log(n) = O(1) \). So by uniform convergence, the PNT, and partial summation we get

\[
\lim_{s \to 1^+} \log' \left( \zeta(2j)(s) \right) > -\lim_{N \to \infty} \sum_{n=1}^{N} \frac{\lambda(2j)(n)}{(n+2j)} \log(n(2j)) \approx -\lim_{N \to \infty} \sum_{n=1}^{N} \frac{w(2j)(n)\Lambda(n)}{(n+2j)\log(n+2j)} = \log(\log(N_{\text{max}}^{(j)})) - \epsilon_{2j} + O \left( \frac{1}{\log(N_{\text{max}}^{(j)})} \right) \tag{3.4}
\]

where the weight \( w(2j)(n) = 1 \) if \( \Lambda(n) \neq 0 \land \Lambda(n+2j) \neq 0 \) and \( w(2j)(n) = 0 \) otherwise, \( N_{\text{max}}^{(j)} := \lim_{N \to \infty} \sum_{n=1}^{N} w(2j)(n)\Lambda(n) \), and the constant \( \epsilon_{2j} \) is an inconsequential end-point contribution. But the PNT and Zhang’s theorem imply \( N_{\text{max}}^{(j)} = \infty \), and we arrive at a contradiction. \( \Box \)

Therefore, for at least one \( 2j \), Proposition 3.1 and Corollary 2.1 imply

**Corollary 3.1**

\[
\lim_{s \to 1^+} \log' \left( j_{[2j]}(s) \right) = \sum_{p_2 \in \mathcal{P}_2} \frac{\log(p_{[2j]})}{p_{[2j]}} = \infty \tag{3.5}
\]

where \( [2i] \) is the equivalence class determined by \( (2i)^{l'} \sim 2i \) for all \( l' \in \mathbb{N}_+ \) with \( 2i \) such that \( (2i)^{l'} = 2j \) for some \( i, l \in \mathbb{N}_+ \).

The sum must include an infinite number of terms, and so there are infinitely many \( p_2 \in \mathcal{P}_2 \) for each admissible \( H_2 = \{0, [2i]\} \).
4 Proof of Theorem

We restate the theorem for easy reference.

**Theorem 4.1** Assume \( \mathcal{H}_k \) is admissible. Then

\[
\lim_{s \to 1^+} \frac{\sum_{p_k \in \mathcal{P}_k} \frac{\log^{k-1}(p_k)}{p_k^s}}{(-1)^{k-1} \log^{(k-1)'}(\zeta_k(s))} = 1 \tag{4.1}
\]

and

\[
\lim_{s \to 1^+} (-1)^{k-1} \log^{(k-1)'}(\zeta_k(s)) = \infty \tag{4.2}
\]

for at least one admissible \( \mathcal{H}_k \).

**Proof:** It is straightforward to show that (2.1) generalizes to

\[
\log^{(k-1)'}(\zeta_k(s)) \asymp \sum_{p_k \in \mathcal{P}_k} p_k^{-s}(k) \quad \text{with} \quad |S(s)| < \zeta(k+1). \]

So

\[
\sum_{p_k \in \mathcal{P}_k} \frac{\log^{k-1}(p_k)}{p_k^s} \asymp (-1)^{k-1} \log^{(k-1)'}(\zeta_k(s)). \tag{4.3}
\]

Now let \( w_{(k+1)}(n) = 1 \) encode the condition that \( \Lambda_{(k)}(n) \neq 0 \land \Lambda(n + h_{k+1}) \neq 0 \) and \( w'_{(k+1)}(n) = 0 \) otherwise. Evidently \( w_{(k+1)}(n) = w_{(k)}(n)w'(1)(n + h_{k+1}) \). In particular

\[
w_{(k+1)}(n)\Lambda(n + h_{k+1}) = w_{(k)}(n)\Lambda(n + h_{k+1}). \tag{4.4}
\]

Also note that

\[
\log^{(k-1)'}(\zeta_k(s)) = (-1)^{k-1} \sum_{n=1}^{\infty} \frac{\lambda_{(k)}(n)}{n_{(k)}^s} \log^{k-1}(n_{(k)})
\]

\[
> (-1)^{k-1} \sum_{n=1}^{\infty} \frac{\lambda_{(k)}(n)}{(n + h_{k})^s} \log^{k-1}(n_{(k)})
\]

\[
= (-1)^{k-1} \sum_{n=1}^{\infty} \frac{w_{(k)}(n)\lambda_{(k)}(n)}{\log^{1-k}(n_{(k)})} \frac{1}{(n + h_{k})^s}. \tag{4.5}
\]

So assuming \( w_{(k)}(n)\lambda_{(k)}(n) = O(\lambda(n + h_{k}) \log^{1-k}(n_{(k)})) \) with \( \mathcal{H}_k \) admissible is tantamount to assuming (4.2) which in turn implies infinitely many \( p_k \in \mathcal{P}_k \).
Consequently, adopting the assumption that \( w_{(k)}(n) \lambda_{(k)}(n) = O(\lambda(n+h_k) \log^{1-k}(n(k))) \) for at least one admissible \( \mathcal{H}_k \) — which is tautological for \( k = 1 \), get
\[
(-1)^k \log^{(k)'}(\zeta(k+1)(s)) = \sum_{n=1}^{\infty} \frac{\lambda_{(k+1)}(n)}{n^{s_{(k+1)}}} \log^k(n_{(k+1)}) \\
> \sum_{n=1}^{\infty} \frac{\lambda_{(k+1)}(n)}{(n + h_{k+1})^s} \log^k(n_{(k+1)}) \\
> \sum_{n=1}^{\infty} \frac{\lambda_{(k+1)}(n)}{(n + h_{k+1})^s} \left( \frac{k}{k+1} \right)^k \log^k(n_{(k)}) \\
\sim \left( \frac{k}{k+1} \right)^k \sum_{n=1}^{\infty} \frac{w(2)(n + h_k) \Lambda(n + h_k)}{\log(n + h_{k+1})^s(n + h_{k+1})^s} \tag{4.6}
\]
where the binomial expansion was used in the third line (keeping only the first term), and the last line uses the assumption as well as \( \lambda(n+h_{k+1}) = O(1/\log(n+h_{k+1})) \) implied by the PNT and \( \log(n(k))/\log(n + h_k) = O(1) \).

As in the prime double case, assume \((-1)^k \log^{(k)'}(\zeta(k+1)(s))\) is bounded at \( s = 1 \). Partial summation yields
\[
\lim_{s \to 1^+} (-1)^k \log^{(k)'}(\zeta(k+1)(s)) \sim \log(\log(\max x)) - \epsilon_{h_{k+1}}. \tag{4.7}
\]
But again we have a contradiction, because \( \max x = \lim_{N \to \infty} \sum_{n=1}^{N+h_k} w(2)(n) \Lambda(n) = \infty \) follows from Zhang’s work for at least one \( 2j_{k+1} \) with \( 0 < 2j_{k+1} \leq h_{k+1} - h_k \) provided \( h_{k+1} - h_k \) is chosen large enough. Hence, \( \lim_{s \to 1^+} (-1)^k \log^{(k)'}(\zeta(k+1)(s)) = \infty \) for the associated equivalence class \([2i_{k+1}]\) by previous arguments.

If it happens that \( 2i_{k+1} \equiv 0 \pmod{k+1} \), then \( h_k + [2i_{k+1}] \) and \( h_k \) belong to the same residue class \((\pmod{k+1})\) so \( \mathcal{H}_{k+1} = \{0, \ldots, h_k, h_k + [2i_{k+1}]\} \) is automatically admissible. Conversely, if \( 2i_{k+1} \not\equiv 0 \pmod{k+1} \), then \( \mathcal{H}_{k+1} \) can be rendered admissible for a suitable choice of representative in \([2i_{k+1}]\). For example, if \( 2i_{k+1} \not\equiv 0 \pmod{k+1} \), then \( h_{k+1} = h_k + (2i_{k+1}) \) and \( h_{k+1} = h_k + (2i_{k+1})^2 \) belong to different residue classes \((\pmod{k+1})\). Consequently at least one of them will yield an admissible \( \mathcal{H}_{k+1} \), because at least one case will not occupy the complete set of residue classes modulo primes.

Finally, Zhang’s result guarantees the induction assumption is true at \( k = 2 \) for at least one \( \mathcal{H}_2 = \{0, 2j_2\} \). It follows that \( \lim_{s \to 1^+} (-1)^k \log^{(k-1)'}(\zeta(k)(s)) = \infty \) for at least one admissible \( \mathcal{H}_k = \{0, [2i_2], [2i_2] + [2i_3], \ldots, \sum_{i \leq k} [2i_i]\} \) for all \( k \) by induction. \( \square \)

**Corollary 4.1** Given an admissible \( \mathcal{H}_k \) satisfying Theorem 4.1, if some \( 2j_k = 2^a \) for integer \( a \), then there are infinitely many twin primes.

**Proof:** The \( (0, \ldots, k, \ldots) \) components of \( \mathbf{p}_k \) determine an admissible \( \mathcal{H}_2 = \{0, 2^a\} \) and its associated prime doubles \( \mathcal{P}_2 \) of which there are necessarily infinitely many. Then previous arguments imply the claim. Of course similar statements for other types of
prime doubles can be made for any component of $p_k$ with $2j_k = (2i_k)^a$. □

One would like to extend this theorem to all admissible $\mathcal{H}_k$. But such an extension would require that $\max x \to \infty$ for all prime doubles. In this case, the assumption used for the induction argument could be made for all admissible $\mathcal{H}_k$ and verified at $k = 2$.

**Corollary 4.2** If $\zeta(2)(s)$ is meromorphic on $\mathbb{C}$ with a single simple pole at $s = 1$, then Theorem 4.1 holds for all admissible $\mathcal{H}_k$.

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