On some generalizations of normality

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Abstract

Interrelation among various existing variants of normality is discussed. The situation in which the class of nearly normal spaces contains the class of (weak) $\theta$-normality is shown. A factorization of normality in presence of Hausdorff space is provided. Further, subspaces and preservation under mappings are also studied.

Keywords: normal, almost regular, almost normal, $\theta$-normal, $f\theta$-normal, $w\theta$-normal, $wf\theta$-normal, $\Delta$-normal, $w\Delta$-normal, $wf\Delta$-normal, nearly normal, $\pi$-normal, $\beta$-normal, $w\theta$-regular.

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1. Introduction and preliminaries

Several generalized forms of normality exists in the literature by using different types of closed sets. Singal and Arya [14] introduced almost normal spaces by using regularly closed sets. Veličko [16] in 1968 introduced the notion of $\theta$-closed and $\delta$-closed sets which was subsequently utilized by others to study different topological properties. Four variants of normality namely (weakly)(functionally)$\theta$-normal spaces [6] has been introduced by using $\theta$-closed sets. Similarly three more generalizations of normality namely(weakly)(functionally)$\Delta$-normal spaces [2] was introduced by using $\delta$-closed sets. Nearly normal spaces was introduced in 1998 by using $\delta$-closed and regularly closed sets. In this paper, interrelation among these variants of normality, their subspaces and preservation under mappings of some of these variants has been studied.

Let $X$ be a topological space and $A \subset X$. Throughout the present paper closure of $A$ is denoted by $\overline{A}$ and interior is denoted by $intA$. A point $x$ is said to be $\theta$-limit point [16] of $A$ if closure of every neighborhood containing $x$ intersects $A$. A set $A_\theta$ is the $\theta$-closure of $A$ which contains all $\theta$-limit points of $A$. A set $A$ is $\theta$-closed if $A = A_\theta$. Compliment of a $\theta$-closed set is known as $\theta$-open set. Similarly a point $x$ is called $\delta$-limit point [16] of $A$ if every regularly open neighborhood of $x$ intersects $A$. A set $A_\delta$ is said to be $\delta$-closed if $A_\delta$ containing all $\delta$-limit points of $A$ is same as $A$. Compliment of a $\delta$-closed set is known as $\delta$-open set. A set $A$ is said to be regularly closed [9] if $U = int\overline{U}$. Compliment of a regularly closed set is known as regularly closed.

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open. A finite union of regular open sets is called $\pi$-open set and a finite intersection of regular closed sets is called $\pi$-closed set.

The interrelation that exist among the above discussed types of closed sets is as follows:

\[
\text{regularly closed} \rightarrow \pi\text{-closed} \rightarrow \delta\text{-closed} \rightarrow \text{closed.}
\]

The above implications are not reversible which is evident from the following examples.

**Example 1.1.** Let $X$ be the set of positive integers. Define a topology on $X$ by taking every odd integer to be open and a set $U \subset X$ is open if for every even integer $p \in U$, the predecessor and successor of $p$ also belongs to $U$. Here $A = \{2, 3, 4\}$ is regularly closed but not $\theta$-closed. $B = \{6\}$ is intersection of two regularly closed sets $C = \{4, 5, 6\}$ and $D = \{6, 7, 8\}$, hence it is $\pi$-closed but not regularly closed. As every regularly closed set is $\delta$-closed, $A = \{2, 3, 4\}$ is $\delta$-closed but not $\theta$-closed.

**Example 1.2.** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{b, c\}, \{b\}\}$. Here $A = \{a, c\}$ is closed but not $\delta$-closed.

**Example 1.3.** Let $X$ denote the interior of the unit square $S$ in the plane together with the points $(0, 0)$ and $(1, 0)$, i.e., $X = \text{int}S \cup \{(0, 0), (1, 0)\}$. Every point in $\text{int}S$ has the usual Euclidean neighbourhoods. The points $(0, 0)$ and $(1, 0)$ have neighbourhoods of the form $U_n$ and $V_n$ respectively, where

\[
U_n = \{(0, 0)\} \cup \{(x, y) : 0 < x < \frac{1}{2}, 0 < y < \frac{1}{n}\}
\]

and

\[
V_n = \{(1, 0)\} \cup \{(x, y) : \frac{1}{2} < x < 1, 0 < y < \frac{1}{n}\}.
\]

Here points $\{(0,0)\}$ and $\{(1, 0)\}$ are $\delta$-closed but not $\pi$-closed.

### 2. Variants of normality

**Definition 2.1.** A topological space $X$ is said to be:

(i) $\theta$-normal [6] if every pair of disjoint closed sets one of which is $\theta$-closed are contained in disjoint open sets.

(ii) weakly $\theta$-normal (w$\theta$-normal) [6] if every pair of disjoint $\theta$-closed sets are contained in disjoint open sets.

(iii) functionally $\theta$-normal (f$\theta$-normal) [6] if for every pair of disjoint closed sets $A$ and $B$ one of which is $\theta$-closed there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

(iv) weakly functionally $\theta$-normal (w$\theta$-normal) [6] if for every pair of disjoint $\theta$-closed sets $A$ and $B$ there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

(v) $\Delta$-normal [2] if every pair of disjoint closed sets one of which is $\delta$-closed are contained in disjoint open sets.

(vi) weakly $\Delta$-normal (w$\Delta$-normal) [2] if every pair of disjoint $\delta$-closed sets are contained in disjoint open sets.

(vii) weakly functionally $\Delta$-normal (w$\Delta$-normal) [2] if for every pair of disjoint $\delta$-closed sets $A$ and $B$ there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$. 


(viii) almost normal [14] if every pair of disjoint closed sets one of which is regularly closed are contained in disjoint open sets.

(ix) nearly normal [10] if every pair of nonempty disjoint sets one of which is $\delta$-closed and the other is regularly closed are contained in disjoint open sets.

(x) $\pi$-normal [5] if every two disjoint closed subsets one of which is $\pi$-closed are contained in disjoint open sets.

(xi) $\beta$-normal [1] if any two disjoint closed sets $A$ and $B$ of $X$ there exist disjoint open subsets $U$ and $V$ of $X$ such that $(A \cap U) = A$, $(B \cap V) = B$ and $U \cap V = \emptyset$.

The following implications are obvious from the above definitions.

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normal $\downarrow$ $\downarrow$ $\pi$-normal $\rightarrow$ almost normal $\downarrow$ $\downarrow$
\Delta-normal $\rightarrow$ wf$\Delta$-normal $\downarrow$ $\downarrow$ nearly normal
\downarrow $\downarrow$
\downarrow $\downarrow
\downarrow $\downarrow
f_{\theta}$-normal $\rightarrow$ wf$\theta$-normal $\rightarrow$ w$\Delta$-normal
\downarrow $\downarrow$ $\downarrow$
\downarrow $\downarrow$
\downarrow $\downarrow$
\theta$-normal $\rightarrow$ w$\theta$-normal
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None of these implication is reversible (see [2, 3, 7, 8] and Example 2.2 below).

**Example 2.2.** Example of a space which is nearly normal but not w$\Delta$-normal. Let $X$ be the union of any infinite set $Y$ and two distinct one point sets $p$ and $q$. In Modified Fort space [15] any subset of $Y$ is open and any set containing $p$ or $q$ is open iff it contains all but a finite number of points in $Y$. Since every two disjoint regularly closed and $\delta$-closed sets can be separated, so $X$ is nearly normal but not w$\Delta$-normal because disjoint $\delta$-closed sets $p$ and $q$ can’t be separated by disjoint open sets.

The following example establishes that neither w$\theta$-normality nor $\theta$-normality imply near normality.

**Example 2.3.** Example of a space which is w$\theta$-normal but not nearly normal. Let $X$ be the set of positive integers. Define a topology on $X$ by taking every odd integer to be open and a set $U \subset X$ is open if for every even integer $p \in U$, the predecessor and successor of $p$ also belongs to $U$. The space $X$ is vacuously $\theta$-normal and so w$\theta$-normal because the only non-empty $\theta$-closed set in $X$ is the whole space. But the space is not nearly normal because for disjoint regularly closed set $A = \{2, 3, 4\}$ and $\delta$ closed set $B = \{6, 7, 8\}$ there does not exist disjoint open sets separating them.

From the above examples it is natural to ask “Which $\theta$-normal or w$\theta$-normal spaces imply near normality?” The following results provide a partial answer to this question. Recall that a space $X$ is almost regular [13] if every regularly closed set $A$ and a point outside it can be separated by disjoint open sets.

**Theorem 2.4.** In an almost regular space, every $\theta$-normal space is nearly normal.

**Proof.** Let $X$ be an almost regular, $\theta$-normal space and let $A$, $B$ be two disjoint closed sets out of which $A$ is regularly closed and $B$ is $\delta$-closed. Since every $\delta$-closed set is closed, $B$ is closed. As in an almost regular space every regularly closed set is $\theta$-closed [8], $A$ is $\theta$-closed. By $\theta$-normality of $X$, there exist two disjoint open sets separating $A$ and $B$. Thus $X$ is nearly normal.

The notion of w$\theta$-regularity [4] which is a simultaneous generalization of regularity and normality is useful for answering the question raised above.

**Definition 2.5.** [4] A space is w$\theta$-regular if for each $\theta$-closed set $F$ and each open set $U$ containing $F$, there exists a $\theta$-open set $V$ such that $F \subset V \subset U$. 


Theorem 2.6. In an almost regular wθ-regular space, every wθ-normal space is nearly normal.

Proof. Let $X$ be an almost regular, wθ-regular and wθ-normal space. Let $A$ and $B$ be two disjoint closed sets out of which $A$ is regularly closed and $B$ is δ-closed. Since $X$ is almost regular, by [8, Theorem 2.5], $A$ is θ-closed. As $U = X - B$ is an open set containing the θ-closed set $A$, by wθ-regularity of $X$, there exists a θ-open set $V$ such that $A \subset V \subset U$. Here $X - V$ is a θ-closed set containing $B$ which is disjoint from the θ-closed set $A$. Thus by wθ-normality of $X$, $A$ and $B$ can be separated by two disjoint open sets. Hence the space is nearly normal.

Lemma 2.7. [10] A Hausdorff nearly normal space is almost regular.

Theorem 2.8. A Hausdorff space $X$ is normal if it is nearly normal, β-normal and wθ-normal.

Proof. Normality implies β-normality, near normality and wθ-normality. Conversely, let $X$ be a Hausdorff nearly normal, wθ-normal and β-normal space. Let $A$ and $B$ be two disjoint closed sets in $X$. By β-normality of $X$, there exist open sets $U$ and $V$ such that $A \cap U = A$ and $B \cap V = B$ and $U \cap V = \emptyset$. Thus $U$ and $V$ contains $A$ and $B$ respectively. Since $X$ is a Hausdorff nearly normal space by Lemma 2.7, $X$ is almost regular. By almost regularity, the regularly closed sets $\overline{U}$ and $\overline{V}$ are two disjoint θ-closed sets [8] containing $A$ and $B$ respectively. Hence by wθ-normality, there exist two disjoint open sets separating $A$ and $B$ respectively. Therefore, the space is normal.

Theorem 2.9. A Hausdorff wθ-normal space is wθ-normal if it is nearly normal.

Proof. Let $X$ be a Hausdorff wθ-normal space which is nearly normal. Since every Hausdorff nearly normal space is almost regular [10], $X$ is an almost regular wθ-normal space. Thus by Theorem 5.18 of [7], $X$ is wθ-normal.

Corollary 2.10. A Hausdorff almost compact nearly normal space is wθ-normal.

Proof. Since every almost compact space is wθ-normal [6], the result is obvious.

Corollary 2.11. A Hausdorff Lindelöf nearly normal space is wθ-normal.

Proof. The prove directly follows from the result that every Lindelöf space is wθ-normal [6].

3. Subspaces and preservation under mappings

Definition 3.1. A subset $A$ of a topological space $X$ is said to be δ-embedded in $X$ if every δ-closed set in the subspace topology of $A$ is the intersection of $A$ with a δ-closed set in $X$.

Theorem 3.2. A closed δ-embedded subspace of a Δ-normal space is Δ-normal.

Proof. Let $X$ be a Δ-normal space and $Y$ be a closed δ-embedded subspace of $X$. Let $A$ and $B$ be two disjoint closed subsets of $Y$ out of which $A$ is δ-closed. Since $Y$ is δ-embedded in $X$ there exists a δ-closed set $C$ in $X$ such that $A = C \cap Y$. Since intersection of two δ-closed sets is δ-closed, $A$ is δ-closed in $X$ which is disjoint from the closed set $B$ of $X$. By Δ-normality of $X$, there exist disjoint open sets $U$ and $V$ in $X$ containing $A$ and $B$ respectively. Since $U \cap Y$ and $V \cap Y$ are disjoint open sets in $Y$ containing $A$ and $B$ respectively, $Y$ is Δ-normal.

Corollary 3.3. Every clopen subspace of a Δ-normal space is Δ-normal.

Theorem 3.4. Every clopen subspace of a nearly normal space is nearly normal.
Proof. Let $X$ be a nearly normal space and $Y$ be a clopen subspace of $X$. Every $\delta$-closed and regularly closed subsets of $Y$ are $\delta$-closed and regularly closed subsets in $X$ respectively. Thus the proof is obvious.

Definition 3.5. [11] A function $f : X \rightarrow Y$ is said to be $\delta$-continuous if for each $x \in X$ and each open neighborhood $V$ of $f(x)$, there exists an open neighborhood $U$ of $x$ such that $f(intU) \subseteq int(V)$.

Definition 3.6. [11] A function $f : X \rightarrow Y$ is said to be strongly $\theta$-continuous if for each $x \in X$ and each open neighborhood $V$ of $f(x)$, there exists an open neighborhood $U$ of $x$ such that $f(U) \subseteq V$.

Definition 3.7. [12] A function $f : X \rightarrow Y$ is said to be almost-continuous if for each $x \in X$ and each open neighborhood $V$ of $f(x)$, there exists an open neighborhood $U$ of $x$ such that $f(U) \subseteq int(V)$.

The following implications are obvious from definitions.

\[
\text{Strongly } \theta\text{-continuous } \implies \delta\text{-continuous } \implies \text{almost continuous.}
\]

Theorem 3.8. [11] For a function $f : X \rightarrow Y$, the following are equivalent

1. $f$ is $\delta$-continuous.
2. For every $\delta$-closed set $F$ of $Y$, $f^{-1}(F)$ is $\delta$-closed in $X$.

Theorem 3.9. A closed strongly $\theta$-continuous image of a $\Delta$-normal space is $\Delta$-normal.

Proof. Let $f : X \rightarrow Y$ be a strongly $\theta$-continuous closed function from a $\Delta$-normal space $X$ onto $Y$. Let $A$ and $B$ be two disjoint closed subsets of $Y$ out of which $A$ is $\delta$-closed. Then $f^{-1}(A)$ and $f^{-1}(B)$ are $\delta$-closed and closed sets in $X$ respectively. Since $X$ is $\Delta$-normal, there exist disjoint open sets $U$ and $V$ in $X$ containing $f^{-1}(A)$ and $f^{-1}(B)$ respectively. Since $Y - f(X - U)$ and $Y - f(X - V)$ are disjoint open sets in $Y$ containing $A$ and $B$ respectively, $Y$ is $\Delta$-normal.

Theorem 3.10. A closed $\delta$-continuous image of a weakly $\Delta$-normal space is weakly $\Delta$-normal.

Proof. Let $f : X \rightarrow Y$ be a $\delta$-continuous closed function from a weakly $\Delta$-normal space $X$ onto $Y$. Let $A$ and $B$ be two disjoint $\delta$-closed subsets of $Y$. Then by Theorem 3.8, $f^{-1}(A)$ and $f^{-1}(B)$ are $\delta$-closed sets in $X$ respectively. Since $X$ is weakly $\Delta$-normal, there exist disjoint open sets $U$ and $V$ in $X$ containing $f^{-1}(A)$ and $f^{-1}(B)$ respectively. Thus $Y - f(X - U)$ and $Y - f(X - V)$ are disjoint open sets in $Y$ containing $A$ and $B$. So $Y$ is weakly $\Delta$-normal.

Theorem 3.11. A closed strongly $\theta$-continuous image of a nearly normal space is nearly normal.

Proof. Let $f : X \rightarrow Y$ be a strongly $\theta$-continuous closed function from a nearly normal space $X$ onto $Y$. Let $A$ and $B$ be two disjoint closed subsets of $Y$ out of which $A$ is $\delta$-closed and $B$ is regularly closed. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint $\delta$-closed and regularly closed sets in $X$ respectively. Since $X$ is nearly normal, there exist disjoint open sets $U$ and $V$ in $X$ containing $f^{-1}(A)$ and $f^{-1}(B)$ respectively. Since $Y - f(X - U)$ and $Y - f(X - V)$ are disjoint open sets in $Y$ containing $A$ and $B$. Hence $Y$ is nearly normal.
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