Categorical Perspective on Quantization of Poisson Algebra

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Abstract

We propose a generalization of quantization using a categorical approach. For a fixed Poisson algebra quantization, categories are defined as subcategories of the $\mathbb{R}$-module category equipped with the structure of classical limits. We then construct the generalized quantization categories including matrix regularization, strict deformation quantization, prequantization, and Poisson enveloping algebra. It is shown that the categories of strict deformation quantization, prequantization, and matrix regularization with certain conditions are equivalent categories. On the other hand, the categories of Poisson enveloping algebra are not equivalent to the other categories.

1 Introduction

Noncommutative geometry is regarded as one of the key concepts for formulating the quantum gravity theory or non-perturbative string theory. There are many ways to construct noncommutative geometry, including deformation quantization, geometric quantization, $C^*$-algebra, matrix regularization, and so on. To find the best approach for quantum gravity or other physics, a unified perspective and a more general formulation containing the existing quantization models would be useful.

In this article, we define a generalized quantization of Poisson manifolds as a subcategory of the category of modules over a ring. It is shown that matrix regularization, strict deformation quantization, and prequantization are included in the generalization, and each pair of them are equivalent categories under some conditions described later. In addition, universal enveloping algebra derived from Poisson manifolds is also formulated as the generalized quantization method for Poisson manifolds.

In preparation for the following sections, we review several definitions of noncommutative geometries or quantizations.

Dirac introduces the quantization rule as replacing the Poisson brackets by commutators. When we regard quantization as a map $\hat{\cdot}$ from functions to operators acting linearly on a Hilbert space, it is generally considered that the axioms for quantization maps $\hat{\cdot}$ satisfy the following conditions: (1) $(\hat{H_1} + \hat{H_2}) = \hat{H_1} + \hat{H_2}$. (2) $(\lambda \hat{H}) = \lambda \hat{H}$, $\lambda \in \mathbb{R}$. (3) $[\hat{H_1}, \hat{H_2}] = i [H_1, H_2]$ where $[H_1, H_2] = H_1H_2 - H_2H_1$. (4) $\hat{1} = \text{Id}$ (1 is a constant 1 and $\text{Id}$ is an identity operator.) (5) $\hat{q_i}$ is the multiplication operator of the function $q_i$, and $\hat{p_i} = \frac{1}{i} \frac{\partial}{\partial q_i}$. However, because no theory satisfies all these conditions, each quantization is defined under weaker conditions or with some restrictions.

Let us consider the definition of matrix regularization of a symplectic manifold $(M, \omega)$. Matrix regularization [17] has evolved from the ideas of Berezin-Toeplitz quantization [10, 34], Fuzzy space [25], and so on. We employ the definition by [3].

Definition 1.1. Let $N_1, N_2, \ldots$ be a strictly increasing sequence of positive integers and $h$ be a real-valued strictly positive decreasing function such that $\lim_{N \to \infty} Nh(N)$ converges. Let $T_k$ be a linear map from $C^\infty(M)$ to $N_k \times N_k$ Hermitian matrices for $k = 1, 2, \ldots$. If the following conditions are satisfied, then we call the pair $(T_k, h)$ a $C^\ast$-convergent matrix regularization of $(M, \omega)$.
1. $\lim_{k \to \infty} \|T_k(f)\| < \infty$,
2. $\lim_{k \to \infty} \|T_k(fg) - T_k(f)T_k(g)\| = 0$,
3. $\lim_{k \to \infty} \|\frac{1}{i\hbar(N_k)} [T_k(f), T_k(g)] - T_k(\{f, g\})\| = 0$,
4. $\lim_{k \to \infty} 2\pi \hbar(N_k) \text{Tr} T_k(f) = \int_M f \omega$,

where $\|\|$ is the operator norm, $\omega$ is a symplectic form on $M$ and $\{ , \}$ is the Poisson bracket induced by $\omega$.

Formal deformation quantization is defined as follows \cite{[12, 13, 19, 31]}. The formal deformation quantization has been widely adopted. However, it is difficult to regard the deformation quantization as a theory of physics when the theory remains formal. We thus employ a strict deformation quantization introduced by Rieffel \cite{[32, 33]}. There are various definitions of this quantization. We use a definition similar to that of \cite{[20]} in this article. Before defining the strict deformation quantization, therefore, we will consider the definition of strict quantization in \cite{[20]}.

**Definition 1.2.** Let $\mathcal{F}$ be a set of formal power series in $\hbar$ with coefficients of $C^\infty$ functions on $M$

$$\mathcal{F} := \left\{ f \mid f = \sum_{k} \hbar^k f_k, \; f_k \in C^\infty(M) \right\}, \quad (1.1)$$

where $\hbar$ is a noncommutative parameter. A star product is defined on $\mathcal{F}$ by

$$f \ast g = \sum_{k} \hbar^k C_k(f, g), \quad (1.2)$$

such that the product satisfies the following conditions.

1. $\ast$ is an associative product.
2. $C_k$ is a bidifferential operator.
3. $C_0$ and $C_1$ are defined as
   $$C_0(f, g) = fg, \quad C_1(f, g) - C_1(g, f) = i\{f, g\}, \quad (1.3, 1.4)$$
   where $\{f, g\}$ is the Poisson bracket.
4. $f \ast 1 = 1 \ast f = f$.

Several variations of the deformation quantization with some minor changes from this definition exist. In this definition, algebra is treated as a set of formal power series of smooth functions. For an arbitrary Poisson manifold, there exists a deformation quantization \cite{[19]}. The formal deformation quantization has been widely adopted. However, it is difficult to regard the deformation quantization as a theory of physics when the theory remains formal. We thus employ a strict deformation quantization introduced by Rieffel \cite{[32, 33]}. There are various definitions of this quantization. We use a definition similar to that of \cite{[20]} in this article. Before defining the strict deformation quantization, therefore, we will consider the definition of strict quantization in \cite{[20]}.

**Definition 1.3.** Let $A_0$ be a Poisson algebra which is densely contained in the self-adjoint part $C^\infty_{\mathbb{R}}$ of an abelian $C^\ast$-algebra $C^\infty$. Let $I$ be a subset of real numbers which contains 0. Strict quantization of the Poisson algebra $A$ is a family of maps $(Q^\hbar : A_0 \to C^\infty_{\mathbb{R}})$, where

1. $C^\infty$ is a $C^\ast$-algebra with an associative product $\times_\hbar$ and a $C^\ast$-norm $\|\|_\hbar$. For $a, b \in C^\infty_{\mathbb{R}}$ which is the self-adjoint part of $C^\infty$,
   $$a \ast_\hbar b := \frac{1}{2}(a \times_\hbar b + b \times_\hbar a),$$
   $$[a, b]_\hbar := (a \times_\hbar b - b \times_\hbar a).$$

2. $\forall \hbar \in I$, $Q^\hbar : A_0 \to C^\infty_{\mathbb{R}}$ is $\mathbb{R}$-linear and $Q^0$ is just the inclusion map such that
   (a) For $f \in A_0$, the map $\hbar \to \|Q^\hbar(f)\|_\hbar$ is continuous.
   (b) For $f, g \in A_0$, $\|Q^\hbar(f) \ast_\hbar Q^\hbar(g) - Q^\hbar(fg)\|_\hbar \to 0$ as $\hbar \to 0$.
   (c) For $f, g \in A_0$, $\|Q^\hbar(f), Q^\hbar(g)\|_\hbar - i\hbar Q^\hbar(\{f, g\})\|_\hbar \to 0$ as $\hbar \to 0$.
   (d) $\forall \hbar \in I$, $Q^\hbar(A_0)$ is dense in $C^\infty_{\mathbb{R}}$.

We define a strict deformation quantization based on this strict quantization.
Definition 1.4. If \( \forall h \in I, \mathcal{Q}^h(A_0) \) is a subalgebra of \( C^0_k(M) \) and \( \mathcal{Q}^h \) is injective, a strict quantization \( \mathcal{Q}^h \) is called a strict deformation quantization.

Especially, a strict deformation quantization of a Poisson manifold \( M \) is defined by a Poisson subalgebra \( A_0(M) \) of \( C^\infty(M) \) composed of bounded functions. In this article, we use not formal but strict deformation quantization \((A_0(M), \mathcal{Q}^h)\).

The prequantization is defined as follows. (See for example \([9, 36]\). For the general Poisson manifolds case, Vaisman established the geometric quantization \([23]\).)

**Definition 1.5.** Let \( M \) be a Poisson manifold. The prequantization is a correspondence between a real smooth function \( H_i \in C^\infty(M) \) and a Hermitian operator \( Q(H) = \hat{H} \) acting on a Hilbert space \( \mathcal{H} \), such that

1. \((\hat{H_1} + \hat{H_2}) = \hat{H_1} + \hat{H_2}\)
2. \((\lambda \hat{H}) = \lambda \hat{H}, \quad \lambda \in \mathbb{R}\)
3. \([\hat{H_1}, \hat{H_2}] = i\{\hat{H_1}, \hat{H_2}\} \text{ where } [\hat{H_1}, \hat{H_2}] = \hat{H_1}\hat{H_2} - \hat{H_2}\hat{H_1}\).
4. \(\hat{1} = \text{Id}, \quad (1 \text{ is a constant } 1 \text{ and } \text{Id is an identity operator.)}\)

The following theorems are important to understand the prequantization through the concrete construction of the quantization map \(\hat{f}\).

**Theorem 1.6.** Let \((M, \omega)\) be a prequantizable symplectic manifold, \(L\) be a line bundle called prequantum line bundle, and \(S\) be the subset of smooth square integrable sections of \(L\) with compact support. There exist a quantization map \(C^\infty(M) \rightarrow \text{Op}(S) : f \rightarrow \hat{f}\) satisfying the above 1-4. Here, \(\text{Op}(S)\) is a set of linear operators acting on \(S\). The map is constructed concretely:

\[
\hat{f} : s \mapsto \frac{1}{\hbar} (\nabla_{X_f})s + fs, \tag{1.5}
\]

where \(X_f\) is a Hamilton vector field of \(f\) and \(\nabla_{X_f}s\) is a covariant derivative of the section \(s\) along \(X_f\).

In this article, we put forward a new framework of quantization including the above quantization theories by using categorical methods. In Section 2 we introduce categories including matrix regularization, strict deformation quantization, and prequantization. We show that they are the categories of quantization of the Poisson algebras. In Section 3 we study the universal enveloping algebra derived from Poisson algebras. It is found that the categories including matrix regularization, strict deformation quantization and prequantization are equivalent categories under some conditions in Section 4. In Section 5 we summarize all results and make some remarks. In addition, we discuss applications of the quantization category to physics.

## 2 Quantization Category

In this section, we give a generalization of quantization as a subcategory of the category of modules. We call this subcategory “quantization category” and show that some quantization theories are included in this category in the following sections. A limit in the category theory corresponds to a classical limit by considering a sequence of morphisms as quantization maps. Before introducing the quantization category, we define a category \(\mathcal{P}(A(M))\).

**Definition 2.1.** Let \(M\) be a fixed Poisson manifold. Let \(R\text{-Mod}\) be a category of \(R\)-module for a commutative ring \(R\) over \(\mathbb{C}\). For a Poisson algebra \(A(M)\) on the Poisson manifold \(M\), a subcategory \(\mathcal{P}(A(M))\) of \(R\text{-Mod}\) is defined as follows.

1. \(A(M) \in ob(\mathcal{P})\)
2. \(\forall M_i \in ob(\mathcal{P}), \) at least a morphism \(T_i \in \mathcal{P}(A(M), M_i)\) exists. We call \(T_i\) a quantization map.
3. \(\forall M_i \in ob(\mathcal{P}), M_i\) is an associative algebra with a Lie bracket \([\cdot, \cdot]\), where the Lie bracket is the commutator for the associative product of algebra, i.e. \([A, B] = A \cdot B - B \cdot A\) for \(A, B \in M_i\).
4. The Lie bracket \([\cdot, \cdot]\) satisfies

\[
[T_k(f), T_k(g)]_k = \sqrt{-1}h(T_k)T_k([f, g]) + O(h^{1+\epsilon}(T_k)) \quad (\epsilon > 0) \tag{2.1}
\]

by noncommutative complex parameter \(h(T_k)\) for all quantization maps \(T_k\).
(5) \( \forall A, B \in \text{ob}(\mathcal{P}) \), if \( T_{AB} \in \mathcal{P}(A, B) \) is an algebraic isomorphism, then there exist \( T_{AB}^{-1} \in \mathcal{P}(B, A) \) which satisfies
\[
T_{AB}T_{AB}^{-1} = \text{id}_B, \quad T_{AB}^{-1}T_{AB} = \text{id}_A.
\]

We call the subcategory \( \mathcal{P}(A(M)) \) the pre-\( \mathcal{P} \) category.

Note that the Lie bracket \([f, g]_{A(M)}\) is not the Poisson bracket but \( fg - gf = 0 \). For \( T = \text{id}_{A(M)} \in \mathcal{P}(A(M), A(M)) \), \( h(\text{id}_{A(M)}) = 0 \) from (2).

Below we denote \( \mathcal{P}(A(M)) \) by \( \mathcal{P} \) for simplicity.

**Definition 2.2.** A map \( \chi : \text{ob}(\mathcal{P}) \to \mathbb{R} \) is defined by the absolute maximum of \( h \):
\[
\chi(M_i) := \max_{T_i \in \mathcal{P}(A(M), M_i)} |h(T_i)|.
\]

We call the map \( \chi \) a noncommutative character.

Let us define an index category of \( \mathcal{P} \).

**Definition 2.3.** \( J^* \) is an index category as disjoint union \( J^* := \bigsqcup_{\alpha} J^\alpha \), where \( J^\alpha \) is a connected component of \( J^* \). \( J^\alpha \) is defined as follows.

(i) \( \forall M_i \in \text{ob}(\mathcal{P}) \setminus \{A(M)\}, \exists ! J^\alpha, i \in \text{ob}(J^\alpha) \),
(ii) \( \forall T^i_j \in \mathcal{P}(M_i, M_j), \chi(M_i) \leq \chi(M_j) \Rightarrow \exists J^\alpha, f^k \in J^\alpha(i, j) \),
(iii) \( \forall f^k \in J^\alpha(i, j), \exists T^i_j \in \mathcal{P}(M_i, M_j), \chi(M_i) \leq \chi(M_j) \).

Note that \( J^* \) is uniquely determined by \( \mathcal{P} \) and \( \chi \). Each connected component \( J^\alpha \) is also an index category.

**Definition 2.4.** \( F^* \) is a set of diagrams of \( J^* \) as \( F^* := \{F^1, F^2, \cdots\} \), where \( F^\alpha \) is a functor called a diagram of \( J^\alpha \). \( F^\alpha : J^\alpha \to \mathcal{P} \) is defined by
\[
i : \text{ob}(J^\alpha) \to M_i \in \text{ob}(\mathcal{P}), \quad f^k \in J^\alpha(i, j) \Rightarrow T^i_j \in \mathcal{P}(M_i, M_j)
\]
for all \( i, j \in \text{ob}(J^\alpha) \) and \( f^k \in J^\alpha(i, j) \).

Note that, since \( J^\alpha \) is a connected category \( F(J^\alpha) \) is also connected. However, objects which are connected in \( \mathcal{P} \) are not always connected from the condition (ii) of Definition 2.3.

**Proposition 2.5.** There exists \( \lim M^\alpha_{\infty} \) in \( \mathcal{P} \) for each \((J^\alpha, F^\alpha)\).

**Proof.** From the condition (1) of Definition 2.1 \( A(M) \) is a candidate for the limit of all \((J^\alpha, F^\alpha)\). \( \square \)

The definition of the limit for \((J, F)\) used in this article can be found, for example, in Chapter 5 in [24] and [22]. The limit \( M^\alpha_{\infty} \) is usually written as a combination with projections \( \pi \) like \((M^\alpha_{\infty}, \pi)\). We often denote \((M^\alpha_{\infty}, \pi)\) as \( M^\alpha_{\infty} \) for short.

We can now define the quantization category.

**Definition 2.6.** A quantization category \( \mathcal{Q}(\mathcal{P}(A(M))), J^*, F^*, \chi \) of a Poisson algebra \( A(M) \) is a pre-\( \mathcal{P} \) category \( \mathcal{P}(A(M)) \) satisfying the following quantization conditions at each limit \( M^\alpha_{\infty} \) of \((J^\alpha, F^\alpha)\):
\begin{align*}
&\text{(Q1)} \quad T(fg) - T(f)T(g) = 0, \\
&\text{(Q2)} \quad [T(f), T(g)]_{\infty} - i\hbar T(\{f, g\}) = 0.
\end{align*}

We denote \( \mathcal{Q}(\mathcal{P}(J^*, F^*, \chi)) \) by \( \mathcal{Q} \) for short. Note that all morphisms are linear maps since \( \mathcal{Q} \subset \text{RMod} \).

In general, therefore, a quantization map \( T_i \) is not a ring homomorphism.

We show that this definition includes matrix regularization, deformation quantization, quantization, and Poisson enveloping algebra in the following sections.
3 Matrix Regularization

In this section, we construct a quantization category to include matrix regularization of Definition \[1.1\]

**Definition 3.1.** Let \(\{N_i\}\) be a strictly increasing sequence of positive integers, and let \(h\) be a real-valued strictly positive decreasing function such that \(\lim_{N \to \infty} Nh(N)\) converges. For a Poisson manifold \(M\), a subcategory \(\mathcal{P}_{MR}\) of \(\text{RMd}\) is defined as follows:

\[
\text{ob}(\mathcal{P}_{MR}) := \{A(M), \text{Mat}_{N_i}, \text{Mat}_\infty \mid k = 1, 2, \ldots\}
\]

where \(A(M)\) is \(C^\infty(M)\) on a Poisson manifold \(M\), \(\text{Mat}_{N_k}\) is a \(N_k \times N_k\) matrix algebra and \(\text{Mat}_\infty\) is a matrix algebra which contains matrices of the limit of matrix regularization in Definition \[1.1\]. The operator norm is defined for all \(\text{Mat}_{N_i}\) and \(\text{Mat}_\infty\). A morphism

\[
T_i : A(M) \to \text{Mat}_{N_i}
\]

and a morphism

\[
T : A(M) \to \text{Mat}_\infty
\]

exist and they are quantization maps of Definition \[1.1\] for all \(\text{Mat}_{N_i} \in \text{ob}(\mathcal{P}_{MR})\). In addition, if and only if \(N_i \leq N_j\) (\(N_i, N_j \in \mathbb{Z}\)), morphisms \(T_{ij} \in \mathcal{P}_{MR}(\text{Mat}_{N_j}, \text{Mat}_{N_i})\) satisfying

\[
T_i = T_{ij} \circ T_j,
\]

exist. For all morphisms \(T_{ik} \in \mathcal{P}_{MR}(A(M), \text{Mat}_{N_k})\) \(k = 1, 2, \ldots, \infty\), this codomain is equipped with the Lie bracket \([\cdot, \cdot]_k\) as the commutator such that

\[
[T_k(f), T_k(g)]_k = ih(N_k)T_k(\{f, g\}) + O(h(N_k)^2) \quad (f, g \in A(M)).
\]

If there exists some algebraic isomorphism \(T_{iso}\) in the morphism of \(\mathcal{P}_{MR}\), then \(T_{iso}^{-1}\) also exists.

The consistency of this definition is confirmed in the proofs of the following lemmas.

**Lemma 3.2.** When \(\text{Mat}_{N_i}\) and \(\text{Mat}_{N_j}\) are decomposed into \(\text{Im} \ T_i \oplus \text{Coker} \ T_i\) and \(\text{Im} \ T_j \oplus \text{Coker} \ T_j\), respectively, for all \(T_{ij} \in \mathcal{P}_{MR}(\text{Mat}_{N_j}, \text{Mat}_{N_i})\), \(T_{ij}\) is block diagonal, i.e.,

\[
T_{ij} = \begin{pmatrix}
T_{i1}^{11} & 0 \\
0 & T_{ij}^{22}
\end{pmatrix} : \begin{pmatrix}
\text{Im} \ T_j \\
\text{Coker} \ T_j
\end{pmatrix} \to \begin{pmatrix}
\text{Im} \ T_i \\
\text{Coker} \ T_i
\end{pmatrix}.
\]

**Proof.** Let us denote \(T_{ij}\) as

\[
T_{ij} = \begin{pmatrix}
T_{i1}^{11} & T_{i1}^{12} \\
T_{i2}^{1j} & T_{ij}^{22}
\end{pmatrix} \begin{pmatrix}
\text{Im} \ T_j \\
\text{Im} \ T_j
\end{pmatrix} \to \begin{pmatrix}
\text{Im} \ T_i \\
\text{Im} \ T_i
\end{pmatrix}.
\]

Since \(T_i = T_{ij} \circ T_j\), \(T_{ij}^{21} = T_{ij}^{22} = 0\) is derived. \(\Box\)

**Lemma 3.3.** \(\mathcal{P}_{MR}\) exist as a pre-$\mathcal{D}$ category equipped with \(T_{ij} \in \mathcal{P}_{MR}(\text{Mat}_{N_j}, \text{Mat}_{N_i})\) for all \(i, j\) \((i \leq j)\).

**Proof.** From the definition of \(\mathcal{P}_{MR}\), the conditions (1), (2) and (3) of Definition \[2.1\] are satisfied, where \(h(T_k) = h(N_k)\). For \(\mathcal{P}_{MR}\) to be a category, the consistency \(T_{ij} \circ T_{jk} = T_{ik}\) should be satisfied for all \(i, j\) and \(k\) such that \(N_i \leq N_j \leq N_k\). We show that there is no contradiction even if \(T_{ik}\) is simply defined as a composition of \(T_{ij} \circ T_{jk}\). For adjacent objects \(\text{Mat}_{N_i}\) and \(\text{Mat}_{N_{i+1}}\), let us define \(T_{i(i+1)}\) so that \(T_{i(i+1)}T_{i+1} = T_i\) for every \(i\). Since \(N_i < N_{i+1}\), \(T_{i(i+1)}\) in Lemma \[3.2\] exists. So the existence of this \(T_{i(i+1)}\) for all \(i\) is guaranteed. We define \(T_{ij}\) by the ordered product as follows.

\[
T_{ij} := T_{i(i+1)}T_{(i+1)(i+2)} \cdots T_{(j-1)j}.
\]

These \(T_{ij}\) trivially satisfy \(T_{ij} \circ T_{jk} = T_{ik}\) for every \(N_i \leq N_j \leq N_k\) at least if we put every \(T_{ij}^{22} = 0\). Now, we check that they do not contradict the condition \(T_i = T_{ij} \circ T_j\). For the block matrix

\[
T_{ij} = \begin{pmatrix}
T_{i1}^{11} & 0 \\
0 & T_{ij}^{22}
\end{pmatrix},
\]
\[ T_{ij}T_j = T_{(i+1)(i+2)} \cdots T_{(j-2)(j-1)}T_{(j-1)(j)} \]
\[ = T_{(i+1)(i+2)} \cdots T_{(j-2)(j-1)}T_{(j-1)(j)} \]
\[ \vdots \]
\[ = T_i. \]

Thus, (3.2) is no contradiction.

From this proof, if \( T_{ij}T_{jk} = T_{ik} \) is satisfied, then \( \mathcal{P}_{MR} \) is a consistently pre-\( \mathcal{Q} \) category. These morphisms \( T_i, T_{ij} \) for \( i, j = 1, 2, \ldots \) can be specifically configured in the case of Berezin-Toeplitz quantization as follows.

**Example 3.4 ([21]).** Let \( (M, \omega) \) be a compact, connected, Kähler manifold with a line bundle \( L \to M \). We denote \( L \otimes^k L \) by \( L^k \) for all \( k \geq 1 \), and an algebra of smooth sections of \( L^k \) by \( C^\infty(M, L^k) \). The quantum space at \( k \) given as

\[ \mathcal{H}_k = H^0(M, L^k) \]

of global holomorphic sections of \( L^k \to M \). Here, \( H^0(M, L^k) \) is Dolbeault cohomology. Let \( L^2(M, L^k) \) be the completion of \( C^\infty(M, L^k) \) with respect to the inner product \( \langle \cdot, \cdot \rangle_k \), and \( \Pi_k \) be the orthogonal projector from \( L^2(M, L^k) \) to \( \mathcal{H}_k \). The Berezin-Toeplitz operator is defined as

\[ T_k = \Pi_k : C^\infty(M) \to \text{End}(\mathcal{H}_k). \]

That is, \( T_k \) is a linear map from the commutative algebra \( C^\infty(M) \) to a matrix algebra.

**Theorem 3.5 ([34]).** The dimension of the \( \mathcal{H}_k \) satisfies

\[ \dim \mathcal{H}_k = \left( \frac{k}{2\pi} \right)^n \text{vol}(M) + O(k^{n-1}) \]

when \( k \) goes to infinity, where \( 2n = \text{dim } M \).

From this theorem, \( T_{ij} \) is constructed as follows. Let \( |n \rangle \) be an orthonormal basis that diagonalizes \( T_k \), i.e., \( \forall T_k = \Pi_k = \sum_n \dim \mathcal{H}_k |n \rangle \langle n | \). For \( \dim \mathcal{H}_j \geq \dim \mathcal{H}_i \),

\[ T_{ij}T_j = U_{ij}^* \Pi_j U_{ij} \]
\[ = T_i, \]

where \( U_{ij} = \sum_n \dim \mathcal{H}_i \langle n \rangle_1 \langle n | \).

For the Berezin-Toeplitz quantization map \( T_k \), the following theorem is known.

**Theorem 3.6 ([34]).** Let \( (M, \omega) \) be a compact, connected, Kähler manifold and \( T_k \) be the Berezin-Toeplitz operator:

(a) For any \( f \in C^\infty(M) \),

\[ \| T_k(f) \| \leq \| f \|_\infty, \]

where \( \| T \| \) stands for the operator norm and \( \| T \|_\infty \) stands for the uniform norm.

(b) For any \( f, g \in C^\infty(M) \),

\[ T_k(f)T_k(g) = T_k(fg) + O(k^{-1}). \]

(c) For any \( f, g \in C^\infty(M) \),

\[ [T_k(f), T_k(g)] = \frac{1}{ik} T_k(\{f, g\}) + O(k^{-2}). \]
Thus, the Berezin-Toeplitz quantization map $T_k$ is the homomorphism of algebra between $C^\infty(M)$ and $\text{End}(H_k)$ when $k \to \infty$.

Note that $\mathcal{P}_{MR}(A(M), \text{Mat}_N)$ has only one quantization map $T_i$ by definition. In such a case, $\chi = h$. Therefore the character $\chi$ is obtained by $\chi(\text{Mat}_N) = h(N)$. This $\chi$ gives the index category $J^1_{MR}$ and the set of diagrams $F^1_{MR}$ as follows. In the case of $J^1_{MR}$, $J^1_{MR}$ has only one connected component $J^1_{MR}$, i.e., $J^1_{MR} = \{J^1_{MR}\}$. So, $J^1_{MR}$ is given as a directed set such that there exists a morphism $i \to j$ if and only if $i \leq j$ from Lemma 3.3. For $F^1_{MR} = \{F^1_{MR}\}$, a diagram $F^1_{MR}$ of $J^1_{MR}$ is given by

$$F^1_{MR} : \begin{array}{l}
\text{ob}(J^1_{MR}) \to \text{ob}(\mathcal{P}_{MR}) \\
i \to \text{Mat}_N_i
\end{array}$$

$$F^1_{MR} : \begin{array}{l}
\mathcal{P}_{MR}(i,j) \to \mathcal{P}_{MR}(M_i, M_j) \\
j^i_j \to T^i_j
\end{array}$$

Since $\{N_i\}$ is a strictly increasing sequence, $F^1_{MR}$ is a diagram which satisfies Definition 2.4 from a sequence in $J^1_{MR}$

$$\cdots \to k \overset{T_k}{\to} j \overset{T_j}{\to} i \overset{T_i}{\to} \cdots$$

in $\mathcal{P}_{MR}$, where $k \leq j \leq i$.

**Theorem 3.7.** For $(\mathcal{P}_{MR}, J^1_{MR}, F^1_{MR}, \chi)$ given as above, $\mathcal{D}_{MR} := \mathcal{Q}(\mathcal{P}_{MR}, J^1_{MR}, F^1_{MR}, \chi)$ is a quantization category of the matrix regularization with the only limit $\text{Mat}_\infty$ of $(J^1_{MR}, F^1_{MR})$.

**Proof.** From the Lemma 3.3 $\mathcal{P}_{MR}$ exists as a pre-$\mathcal{D}_{MR}$ category. A limit $(M_\infty, \pi)$ of $F_{MR}$ is given as the following commutative diagram for any $i, j$.

$$\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\text{Mat}_N_i & \xrightarrow{\pi_i} & M_\infty & \xrightarrow{\pi_j} & \text{Mat}_N_j \\
\text{Mat}_N_k & \xrightarrow{T_{ik}} & \text{Mat}_N_l & \xrightarrow{T_{lj}} & \text{Mat}_N_j
\end{array}$$

From the condition (3.1) of morphisms, $M_\infty$ is $A(M)$ or $\text{Mat}_\infty$. Lemma 3.3 shows that $M_\infty = \text{Mat}_\infty$, i.e., $T = T_\infty$, $\pi_i = T_\infty$ and $\pi_j = T_\infty$, and then $\pi_i \circ T = T_j$ and $\pi_i \circ T_m = \pi_m$ for all $l, m$ with $X = A(M)$. From the definition of $\text{Mat}_\infty$, the quantization conditions (Q1), (Q2) and (Q3) in Definition 2.6 are satisfied on $M_\infty$. Thus $\mathcal{D}_{MR}$ is a quantization category of the matrix regularization.

This quantization category $\mathcal{D}_{MR}$ can be further minimized.

**Corollary 3.8.** For the quantization category $\mathcal{D}_{MR}$, a restriction $\mathcal{D}_{MR}|_{\text{Mat}_N_i}$ is given as follows:

$$\text{ob}(\mathcal{D}_{MR}|_{\text{Mat}_N_i}) := \{A(M), \text{Mat}_N_i, \text{Mat}_\infty\}$$

where $i$ is fixed. Morphisms are restricted to $T_i, T_\infty, T_\infty$ in $\mathcal{D}_{MR}$. Then $\mathcal{D}_{MR}|_{\text{Mat}_N_i}$ is a quantization category of the matrix regularization with the limit $\text{Mat}_\infty$ of $(J^1_{MR}, F^1_{MR}|_{i})$. Here, $J^1_{MR}|_{i}$ is given by $\text{ob}(J) = \{i, \infty\}$ and the only morphism of $J^1_{MR}$ is $i \to \infty$, and $F^1_{MR}|_{i}$ is given by $F(i) = \text{Mat}_N_i$, $F(\infty) = \text{Mat}_\infty$, $F(i \rightarrow \infty) = T_\infty$.

**Proof.** Since the diagram of $\mathcal{D}_{MR}|_{\text{Mat}_N_i}$ is given by

$$\begin{array}{ccc}
A(M) & \xrightarrow{T} & \text{Mat}_N_i \\
\downarrow & & \downarrow \text{id}_{\text{Mat}_N_i} \\
\text{Mat}_N_i & \xrightarrow{\pi_i} & M_\infty & \xrightarrow{\pi_j} & \text{Mat}_N_j
\end{array}$$

(3.3)
4 Deformation Quantization

In the following, we consider the strict deformation quantization. In other words, we consider not \((F, \ast)\) but \((A_0(M), C^0)\). (These examples are shown in \([32, 33]\).)

**Definition 4.1.** Let \((A_0(M), C^0)\) be the strict deformation quantization of a Poisson manifold \(M\) (see Definitions 1.3 and 1.4). A subcategory \(P_{DQ}\) of RMod is defined as follows:

\[
\text{ob}(P_{DQ}) := \{A_0(M), C^0 \mid \forall \hbar \in I\},
\]

where \(I\) is a subset of real numbers which contains 0. The morphisms of \(P_{DQ}\) are defined by quantization maps \(Q^h : A_0(M) \to C^h_\hbar \subset C^h\) for all \(\hbar \in I\) of Definition 1.3. In addition, if and only if \(h \geq h'\), \(T_{h'} \in P_{DQ}(C^h, C^h)\) satisfying

\[
Q^h = T_{hh'} \circ Q^{h'}
\]

exist. For all morphisms \(Q^h \in P_{DQ}(A_0(M), C^0)\), the codomain is equipped with the Lie bracket \([\cdot, \cdot]_h\) as the commutator such that

\[
[Q^h(f), Q^h(g)]_h = \sqrt{-\hbar}Q^h(\{f, g\}) + O(h^2) \quad (f, g \in A_0(M)).
\]

If there exists some algebraic isomorphism \(T_{iso}\) in the morphism of \(P_{DQ}\), then there also exists inverse \(T_{iso}^{-1}\) of \(T_{iso}\).

**Lemma 4.2.** \(P_{DQ}\) is a pre-\(\mathcal{D}_{DQ}\) category.

**Proof.** The diagram of \(P_{DQ}\) is given by

\[
\begin{array}{ccc}
C^h & \xrightarrow{T_{h0}} & C^0 \\
\downarrow{Q^h} & & \uparrow{Q^0} \\
A_0(M) & \xrightarrow{T_{h0}} & C^h \\
\uparrow{\chi} & & \downarrow{\chi} \\
C^0 & \xrightarrow{T_{h0}} & C^h \\
\end{array}
\]

Since \(T_{h'} = T_{hh'} \circ T_{h''}\) can be confirmed in the same manner as Lemma 3.3 \(P_{DQ}\) is a category, and the Definition 2.1 is also satisfied by the definition of \(P_{DQ}\).

The character \(\chi\) is obtained by \(\chi(C^0) = |h|\). This character \(\chi\) gives the index category \(J^*_{DQ}\) and the set of diagrams \(F_{DQ}\) as follows. For \(J^*_{DQ} := \{J_{DQ}\}\), \(J_{DQ}\) is given as a directed set. For \(F_{DQ} := \{F_{DQ}\}\), a diagram \(F_{DQ}\) of \(J_{DQ}\) is surjective from \(\text{ob}(J_{DQ}) \to \text{ob}(P_{DQ})\)\(\{A_0(M)\}\). Since \(\chi(C^0) = |h|\) for all \(h\), if and only if \(|h| \geq |h'|\) there exist morphisms \(h' \in \text{ob}(J_{DQ}) \to h \in \text{ob}(J_{DQ})\) such that

\[
F_{DQ} : \begin{array}{ccc}
\text{ob}(J_{DQ}) & \xrightarrow{\cup} & \text{ob}(P_{DQ}) \\
\hbar & \mapsto & Q^0 \\
F_{DQ} : J_{DQ}(h, h') & \xrightarrow{\cup} & P_{DQ}(C^0, C^h) \\
f_{ij}^k & \mapsto & T_{h'k}.
\end{array}
\]

**Theorem 4.3.** For \((P_{DQ}, J^*_{DQ}, F_{DQ}, \chi)\) given as above, \(\mathcal{D}_{DQ} := \mathcal{L}(P_{DQ}, J^*_{DQ}, F_{DQ}, \chi)\) is a quantization category \(\mathcal{D}_{DQ}\) of the strict deformation quantization with a limit \(C^0\) of \(J_{DQ}, F_{DQ}\).

**Proof.** The proof is given in a completely parallel manner as the proof of Theorem 3.7. From Lemma 4.2 \(P_{DQ}\) is a pre-\(\mathcal{D}\) category. We consider a limit \(M_{\infty}\) of \(F_{DQ}\) as the following commutative diagram.

\[
\begin{array}{ccc}
A_0(M) & \xrightarrow{\chi} & C^0 \\
\downarrow{Q^h} & & \uparrow{Q^0} \\
C^h & \xrightarrow{T_{h0}} & C^0 \\
\downarrow{M_{\infty}} & & \uparrow{T_{h0}} \\
C^h & \xrightarrow{T_{h0}} & C^h \\
\end{array}
\]

Thus, the limit of \(F_{DQ}\) is \(C^0\). Trivially, the quantization conditions are satisfied on \(C^0\), so \(\mathcal{D}_{DQ}\) is a quantization category of the strict deformation quantization. \(\square\)
The quantization category $\mathcal{D}_{DQ}$ can be further minimized in a manner similar to $\mathcal{D}_{MR}$.

**Corollary 4.4.** For the quantization category $\mathcal{D}_{DQ}$, a restriction $\mathcal{D}_{DQ}|_{ch}$ is given as follows:

$$\text{ob}(\mathcal{D}_{DQ}|_{ch}) := \{C^0, C^\hbar, A_0(M)\},$$

where $\hbar$ is fixed. Similarly, the morphisms are restricted to $Q^0, Q^\hbar, T_{\hbar}$.

![Diagram](image)

$$A_0(M) \quad \xrightarrow{\hat{\cdot}} \quad \text{id}_{C^\hbar}$$

Then $\mathcal{D}_{DQ}|_{ch}$ is a quantization category of the strict deformation quantization with the limit $C^0$.

## 5 Prequantization

It is possible to construct a quantization category including the prequantization whose definition is given in Definition 1.

**Definition 5.1.** Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold with a prequantum line bundle $L \to M$ and $\Gamma_{hol}(M, L)$ be a holomorphic global section on $L$. A subcategory $\mathcal{P}_{PQ}$ of $RMod$ is defined as follows.

The objects are $A(M) = C^\infty(M)$ and $\text{End}(\Gamma_{hol}(M, L))$:

$$\text{ob}(\mathcal{P}_{PQ}) := \{A(M), \text{End}(\Gamma_{hol}(M, L))\}.$$  

Its morphisms of $\mathcal{P}_{PQ}$ are given by identities and a quantization map $\hat{\cdot}$:

$$\hat{\cdot}: \quad A(M) \quad \xrightarrow{\psi} \quad \text{End}(\Gamma_{hol}(M, L))$$

where $\hat{f}$ is $\text{Def} 1.5$. For $T := \hat{\cdot} \in \mathcal{P}_{PQ}(A(M), \text{End}(\Gamma_{hol}(M, L)))$, the codomain is equipped with the Lie bracket $[\cdot, \cdot]$ as the commutator such that

$$[T(f), T(g)]_{\hbar} = i\hbar T(\{f, g\}).$$

If $\hat{\cdot}$ is a algebraic isomorphism, then the inverse of $\hat{\cdot}$ exists as a morphism of $\mathcal{P}_{PQ}$.

**Lemma 5.2.** $\mathcal{P}_{PQ}$ is a pre-$\mathcal{P}_{PQ}$ category.

**Proof.** The diagram of $\mathcal{P}_{DQ}$ is given by

![Diagram](image)

where $\pi = \hat{\cdot}$. Trivially, $\mathcal{P}_{PQ}$ is a category, and the Definition 2.1 is also satisfied by the definition of $\mathcal{P}_{PQ}$.

There is only one quantization map $\hat{\cdot}$, so we obtain $\chi(\text{End}(\Gamma_{hol}(M, L))) = |\hbar|$. This $\chi$ gives the index category $J_{PQ}^*$ and a set of diagrams $F_{PQ}^*$ as follows. For $J_{PQ}^* := \{J_{PQ}\}$, $J_{PQ}$ is given as a category of only one object, and we denote it by $\mathcal{A}$. For $F_{PQ}^* := \{F_{PQ}\}$, a diagram $F_{PQ}$ of $J_{PQ}$ is given by

$$F(\mathcal{A}) = \text{End}(\Gamma_{hol}(M, L)).$$

$F_{PQ}$ is a diagram which satisfies the Definition 2.4

**Theorem 5.3.** For $(\mathcal{P}_{PQ}, J_{PQ}^*, F_{PQ}^*, \chi)$ given as above, $\mathcal{P}_{PQ} := (\mathcal{P}_{PQ}, J_{PQ}^*, F_{PQ}^*, \chi)$ is a quantization category $\mathcal{D}_{PQ}$ of the prequantization with the limit $A(M)$ of $(J_{PQ}, F_{PQ})$. 


Proof. From Lemma \[5.2\], $\mathcal{D}_{PQ}$ is a pre-$\mathcal{D}$ category. We consider a limit $(M_\infty, \pi)$ of $F_{PQ}$ as the following commutative diagram.

$$\begin{array}{ccc}
\mathcal{A}(M) & \xrightarrow{id_{\mathcal{A}(M)}} & \mathcal{A}(M) \\
\downarrow & & \downarrow \\
\text{End}(\Gamma_{hol}(M, L)) & \xrightarrow{\pi} & \text{End}(\Gamma_{hol}(M, L))
\end{array}$$

Thus, the limit of $F_{PQ}$ is $(\mathcal{A}(M), \pi)$. It is trivial that the quantization conditions are satisfied on $\mathcal{A}(M)$, so $\mathcal{D}_{PQ}$ is a quantization category of the deformation quantization. \hfill \Box

6 Poisson Enveloping Algebra

In this section, we describe the relationship between the quantization category and Poisson enveloping algebra.

First, we will review the definition of the Poisson enveloping algebra.

**Definition 6.1** \((28, 29)\). Let $\mathcal{A} = (A, \cdot, \{ \ , \ })$ be a Poisson algebra. For an algebra $U$, let $\alpha$ be an algebra homomorphism from $(A, \cdot)$ to $U$ and $\beta$ be a Lie homomorphism from $(A, \{ \ , \ })$ to $U$. These morphisms satisfy the following conditions for all $a, b \in A$.

$$\begin{align*}
\alpha(\{a, b\}) &= \beta(a)\alpha(b) - \alpha(b)\beta(a), \\
\beta(ab) &= \alpha(a)\beta(b) + \alpha(b)\beta(a).
\end{align*}$$

(6.1)\hspace{1cm} (6.2)

Let $X$ be an arbitrary algebra such that there exist an algebra homomorphism $\alpha'$ from $(A, \cdot)$ to $X$ and a Lie homomorphism $\beta'$ from $(A, \{ \ , \ })$ to $X$ such that

$$\begin{align*}
\alpha'('\{a, b\}) &= \beta'(a)\alpha'(b) - \alpha'(b)\beta'(a), \\
\beta'(ab) &= \alpha'(a)\beta'(b) + \alpha'(b)\beta'(a).
\end{align*}$$

(6.3)\hspace{1cm} (6.4)

If there exists a unique algebra homomorphism $h$ from $U$ to $X$ that makes the following diagram (6.5) commutative, then $(U, \alpha, \beta)$ is called a Poisson enveloping algebra of $A$.

$$\begin{array}{ccc}
X & \xrightarrow{\alpha'} & U \\
\downarrow & \searrow & \downarrow \\
(A, \cdot) & \xrightarrow{\alpha} & (A, \{ \ , \ })
\end{array}$$

(6.5)

**Example 6.2** \((35)\). A Poisson symplectic algebra $P_n$ is given by the polynomial algebra $R[x_1, \ldots, x_n, y_1, \ldots, y_n]$ that is equipped with the Poisson bracket defined by

$\{x_i, y_j\} = \delta_{ij}$, $\{x_i, x_j\} = 0$, $\{y_i, y_j\} = 0$, where $1 \leq i, j \leq n \in \mathbb{Z}$. Let $A_{2n}$ be the Weyl algebra such that $A_{2n}$ is an associative algebra given by generators $x_1, \ldots, x_{2n}, y_1, \ldots, y_{2n}$ and defined relations

$$[x_i, y_j] = \delta_{ij}, \quad [x_i, x_j] = 0, \quad [y_i, y_j] = 0,$$

where $1 \leq i, j \leq 2n \in \mathbb{Z}$. Then, the enveloping algebra $P_n^\ast$ of $P_n$ and the Weyl algebra $A_{2n}$ are isomorphic.

From Example 6.2 and so on, we are convinced that the Poisson enveloping algebra is a kind of quantization.

**Definition 6.3.** Let $U := (U, \alpha, \beta)$ be the Poisson enveloping algebra with a Poisson algebra $\mathcal{A} = (A, \cdot, \{ \ , \ })$. A subcategory $\mathcal{P}_{env}$ of $\text{RMod}$ is defined as follows:

$\text{ob}(\mathcal{P}_{env}) := \{ \mathcal{A}, (A, \cdot), (A, \{ \ , \ }), U, M_1, M_2, \ldots \}$,

\(^1\)To limit ourselves primarily to the case of noncommutative geometry, we chose $\mathcal{A}(M)$ as a Poisson algebra in Definition \[27\]. However, we have only used the fact that $\mathcal{A}(M)$ is a Poisson algebra until now. Therefore, it is possible to replace $\mathcal{A}(M)$ with a Poisson algebra $A$. In this definition, we employ the Poisson algebra $A$ instead of $\mathcal{A}(M)$ to get closer to the definition of Poisson enveloping algebra.
where $M_i$ for $i=1,2,\cdots$, are Lie algebras such that the following diagram commutes.

$$
\begin{array}{c}
(A, \cdot) \\
\downarrow_{\varphi_0} \\
U
\end{array} \xleftarrow{\Pi_1} \xrightarrow{\Pi_2} \begin{array}{c}
(A, \{,\})
\end{array}
$$

Morphisms of $\mathcal{P}_{\text{env}}$ are given by

$$\Pi_1: A \rightarrow (A, \cdot) \quad \Pi_2: A \rightarrow (A, \{,\}),$$

where $\Pi_1$ is the identity forgetting the $\{,\}$ structure and $\Pi_2$ is the identity, and the homomorphisms and Lie homomorphisms are as follows. These homomorphisms and Lie homomorphisms satisfy \( (6.1) - (6.5) \), respectively. The other morphisms are simply defined by a composition such that $\mathcal{P}_{\text{env}}$ is a category. For all quantization maps $T_k$, whose domain is $A$, this codomain is equipped with the Lie bracket $[\cdot,\cdot]_k$ as the commutator such that

$$[T_k(f), T_k(g)]_k = \sqrt{-1}h(T_k)((f,g)).$$

If there exists some algebraic isomorphism $T_{\text{iso}}[-1]$ of $T_{\text{iso}}$, then there also exists an inverse $T_{\text{iso}}^{-1}$ of $T_{\text{iso}}$.

**Lemma 6.4.** $\mathcal{P}_{\text{env}}$ is a pre-$\mathcal{P}_{\text{env}}$ category.

**Proof.** This lemma follows immediately from the Definition 6.3. \qed

A character $\chi$ gives the index category $J_{\text{env}}^\bullet$ and $F_{\text{env}}^\bullet$ as follows. For $J_{\text{env}}^\bullet := \{J_{\text{env}}\}$, $J_{\text{env}}$ is given as a directed set. For $F_{\text{env}}^\bullet := \{F_{\text{env}}\}$, a diagram $F_{\text{env}}$ of $J_{\text{env}}$ is surjective from $ob(J_{\text{env}}) \rightarrow ob(\mathcal{P}_{\text{env}})\setminus\{A\}$. For the object $U$, $h(\alpha_0 \circ \Pi_1) = 0$ and $h(\beta_0 \circ \Pi_2) = 1/\sqrt{-1}$ because $\alpha$ is an algebra homomorphism and $\beta$ is a Lie homomorphism, i.e.,

$$0 = [\alpha(a), \alpha(b)] = \sqrt{-1}h(\alpha_0 \circ \Pi_1)\alpha(a,b))$$

and

$$\beta_0(\{a,b\}) = [\beta_0(a), \beta_0(b)] = \sqrt{-1}h(\beta_0 \circ \Pi_2)\beta(\{a,b\}).$$

Therefore $\chi(U) = |1/\sqrt{-1}| = 1$. Since $\chi(A) = \chi((A,\cdot)) = \chi((A,\{,\})) = 0$ and $\chi(U) = 1$, there at least exist morphisms between them in $F_{\text{env}}(J_{\text{env}})$. If $\chi(M_i) \geq 1$, $h_i \in \mathcal{P}_{\text{env}}(U,M_i)$ exists in $F_{\text{env}}(J_{\text{env}})$.

**Theorem 6.5.** For $(\mathcal{P}_{\text{env}}, J_{\text{env}}^\bullet, F_{\text{env}}^\bullet, \chi)$ given as above, $\mathcal{Q}_{\text{env}} := \mathcal{Q}(\mathcal{P}_{\text{env}}, J_{\text{env}}^\bullet, F_{\text{env}}^\bullet, \chi)$ is a quantization category $\mathcal{Q}_{\text{env}}$ with a limit $A$ of $(J_{\text{env}}, F_{\text{env}})$.

**Proof.** From the Lemma 6.4, $\mathcal{P}_{\text{env}}$ is a pre-$\mathcal{Q}$ category. We consider a limit $(M_\infty, \pi)$ of $F_{\text{env}}$ as the following commutative diagram.

$$
\begin{array}{c}
(C^\infty, \cdot) \\
\downarrow_{\varphi_0} \\
U
\end{array} \xleftarrow{\Pi_1} \xrightarrow{\Pi_2} \begin{array}{c}
(C^\infty, \{,\})
\end{array}
$$
Note that there exist only morphisms that satisfy the condition (ii) of the Definition 2.3. The only candidate for a domain of \( \text{Hom}(\cdot, F_{\text{env}}(i)) \) for all \( i \in \text{ob}(J_{\text{env}}) \) is \( A \). Thus, a limit of \( F_{\text{env}} \) is \( (A, \text{Hom}(A, F_{\text{env}}(i))) \). Because the quantization map to \( A \) is the identity, the quantization conditions are satisfied on \( A \) trivially, so \( Q_{\text{env}} \) is a quantization category of the Poisson enveloping algebra.

7 Categorical Equivalence

In this section, we discuss the equivalence of the quantization categories appearing in Section 3-5.

Let \( Q_{\text{MR}}|_{\text{Mat}_{N_i}} \) and \( Q_{\text{DQ}}|_{C^\bullet} \) be those appearing in Corollary 3.8 and Corollary 4.3. The following proposition is trivially obtained.

**Proposition 7.1.** If there is no algebraic isomorphism, or if there are isomorphisms on the same edges in the commutative diagrams \( [3.3] \) and \( [4.1] \), then the quantization categories \( Q_{\text{MR}}|_{\text{Mat}_{N_i}} \) and \( Q_{\text{DQ}}|_{C^\bullet} \) are equivalent categories. More strictly, they are isomorphic.

This proposition can be immediately extended to the following proposition.

**Proposition 7.2.** Let \( \sharp N \) be a cardinality of the sequence of \( \{N_i\} \) in \( Q_{\text{MR}} \) and \( \sharp \hbar \) be a cardinality of \( \hbar \) in \( Q_{\text{DQ}} \). When \( \sharp N = \sharp \hbar \), if there is no algebraic isomorphism, or if there are isomorphisms on the same edges in each of the commutative diagrams, then \( Q_{\text{MR}} \) and \( Q_{\text{DQ}} \) are equivalent categories. More strictly, they are isomorphic.

**Proposition 7.3.** If \( A(M) \) and \( \text{Mat}_{\infty} \) are isomorphic in \( Q_{\text{MR}}|_{\text{Mat}_{N_i}} \), then \( Q_{\text{MR}}|_{\text{Mat}_{N_i}} \) and \( Q_{\text{PQ}} \) are equivalent categories.

Proof. We denote \( Q_{\text{MR}}|_{\text{Mat}_{N_i}} \) and \( \text{End}(T_{\text{hol}}(M, L)) \) by \( (Q_{\text{MR}})' \) and \( H \), respectively, for simplicity. Let us introduce a functor \( F : (Q_{\text{MR}})' \to Q_{\text{PQ}} \) defined as

\[
F(A(M)) = F(\text{Mat}_{\infty}) = A(M),
F(\text{Mat}_{N_i}) = H,
F(T_i) = F(id_{A(M)}) = F(id_{\text{Mat}_{\infty}}) = F(T_i^{-1}) = id_{A(M)},
F(\pi_i) = F(T_i) = T_i,
F(id_{\text{Mat}_{N_i}}) = id_H,
\]

and a functor \( G : Q_{\text{DQ}} \to (Q_{\text{MR}})' \) defined as

\[
G(A(M)) = A(M),
G(H) = \text{Mat}_{N_i},
G(id_{A(M)}) = id_{A(M)},
G(id_H) = id_{\text{Mat}_{N_i}},
G(T_i) = T_i.
\]

For the \( F \) and \( G \), let us show that \( FG \simeq id_{Q_{\text{PQ}}} \) and \( GF \simeq id_{(Q_{\text{MR}})'} \). The natural transformation \( \theta' : FG \to id_{Q_{\text{PQ}}} \) is trivial. The diagrams of \( \theta : GF \to id_{(Q_{\text{MR}})'} \) is given by

\[
\begin{array}{ccc}
GF(\text{Mat}_{N_i}) & \xrightarrow{\theta_{\text{Mat}_{N_i}}} & \text{Mat}_{N_i} \\
GF(id_{\text{Mat}_{N_i}}) & \downarrow & \\
GF(\text{Mat}_{N_i}) & \xrightarrow{\theta_{\text{Mat}_{N_i}}} & \text{Mat}_{N_i}
\end{array}
\quad
\begin{array}{ccc}
GF(A(M)) & \xrightarrow{\theta_A} & A(M) \\
GF(id_{A(M)}) & \downarrow & \\
GF(A(M)) & \xrightarrow{\theta_A} & A(M)
\end{array}
\quad
\begin{array}{ccc}
GF(\text{Mat}_{\infty}) & \xrightarrow{\theta_{\text{Mat}_{\infty}}} & \text{Mat}_{\infty} \\
GF(id_{\text{Mat}_{\infty}}) & \downarrow & \\
GF(\text{Mat}_{\infty}) & \xrightarrow{\theta_{\text{Mat}_{\infty}}} & \text{Mat}_{\infty}
\end{array}
\quad
\begin{array}{ccc}
GF(A(M)) & \xrightarrow{\theta_A} & A(M) \\
GF(T_i) & \downarrow & \\
GF(T_i) & \xrightarrow{\theta_A} & A(M)
\end{array}
\quad
\begin{array}{ccc}
GF(\text{Mat}_{N_i}) & \xrightarrow{\theta_{\text{Mat}_{N_i}}} & \text{Mat}_{N_i} \\
GF(id_{\text{Mat}_{N_i}}) & \downarrow & \\
GF(\text{Mat}_{N_i}) & \xrightarrow{\theta_{\text{Mat}_{N_i}}} & \text{Mat}_{N_i}
\end{array}
\quad
\begin{array}{ccc}
GF(A(M)) & \xrightarrow{\theta_A} & A(M) \\
GF(T_i) & \downarrow & \\
GF(T_i) & \xrightarrow{\theta_A} & A(M)
\end{array}
\]
Thus, if $\theta_{\text{Mat}_\infty} = T$, $\theta_{\mathcal{A}(M)} = \text{id}_{\mathcal{A}(M)}$ and $\theta_{\text{Mat}_{N_i}} = \text{id}_{\text{Mat}_{N_i}}$, then $GF \simeq \text{id}_{\mathcal{D}_{MR}'}$. \hfill \Box

Example 7.4. Let $(M, \omega)$ be a quantizable compact Kähler manifold, $L$ some very ample line bundle over $M$, and $\mathcal{A}(M)$ a Poisson algebra of smooth functions on $M$. Let $H^{(m)} = \Gamma_{\text{hol}}(M, L^m)$ be the Hilbert space of holomorphic sections in $L^m$, where $L^m := L \otimes^m$. For $f \in \mathcal{A}(M)$, The Toeplitz operator is defined as

$$T^{(m)}(f) := \Pi^{(m)} \circ M_f \circ \Pi^{(m)},$$

where $\Pi^{(m)} : L^2(M, L^m) \to \Gamma_{\text{hol}}(M, L^m)$ is a projection and $M_f$ is the operator of multiplication by $f$. If $m \to \infty$ then $\text{End}(H^{(m)})$ and $\mathcal{A}(M)$ are isomorphic (See [34]). Therefore, this is the case of the above Proposition 7.3.

Proposition 7.5. If $\mathcal{A}_0(M) \text{ and } \mathcal{C}_0^\circ$ isomorphic in $\mathcal{D}_{MQ}|_{\mathcal{C}_K}$, then $\mathcal{D}_{MQ}|_{\mathcal{C}_K}$ and $\mathcal{D}_{PQ}$ are equivalent categories.

Proof. From Proposition 7.1 and 7.3

$$\mathcal{D}_{MQ}|_{\mathcal{C}_K} \simeq \mathcal{D}_{MR}|_{\text{Mat}_{N_i}} \simeq \mathcal{D}_{PQ}.$$ \hfill \Box

Proposition 7.6. Suppose that $\mathcal{D}_{MQ}$ and $\mathcal{D}_{MR}$ have more than three objects each pair of which are not isomorphic, respectively. Then, the quantization category $\mathcal{D}_{PQ}$ is not categorically equivalent to either $\mathcal{D}_{MQ}$ or $\mathcal{D}_{MR}$.

Proof. Suppose that $\mathcal{D}_{PQ}$ and $\mathcal{D}_{MQ}$ are equivalent categories. In other words, functors $F : \mathcal{D}_{PQ} \to \mathcal{D}_{MQ}$ and $G : \mathcal{D}_{MQ} \to \mathcal{D}_{PQ}$ exist, such that $FG \simeq \text{id}_{\mathcal{D}_{MQ}}$ and $GF \simeq \text{id}_{\mathcal{D}_{PQ}}$. Then there are functors $X, Y \in \text{ob}(\mathcal{D}_{MQ})$ such that

$$F(\mathcal{A}(M)) = X,$$

$$F(\text{End}(\Gamma_{\text{hol}}(M, L))) = Y.$$ At least, if $GF \simeq \text{id}_{\mathcal{D}_{PQ}}$ then $G$ must be

$$G(X) = \mathcal{A}(M),$$

$$G(Y) = \text{End}(\Gamma_{\text{hol}}(M, L)), \text{ or } G(Y) = \mathcal{A}(M).$$

We show some contradictions: namely, cases (i) $\mathcal{A}(M)$ and $\text{End}(\Gamma_{\text{hol}}(M, L))$ are not linear isomorphic, and cases (ii) $\mathcal{A}(M)$ and $\text{End}(\Gamma_{\text{hol}}(M, L))$ are linear isomorphic.

(i) Let us consider the case that $\mathcal{A}(M)$ and $\text{End}(\Gamma_{\text{hol}}(M, L))$ are not linear isomorphic. From $GF \simeq \text{id}_{\mathcal{D}_{MQ}}$, $G(Y) = \text{End}(\Gamma_{\text{hol}}(M, L))$. Let $Z$ be a object of $\mathcal{D}_{MQ}$ which is not isomorphic to $X$ or $Y$. Note that there is a morphism from $Z$ to $X$ or from $X$ to $Z$. First we will consider the case that there is a morphism from $X$ to $Z$. The functor $G$ must be $G(Z) = \mathcal{A}(M)$ or $G(Z) = \text{End}(\Gamma_{\text{hol}}(M, L))$. Since $FG(X) = X$ and $FG(Z) = X$ or $Y$, if $FG \simeq \text{id}_{\mathcal{D}_{MQ}}$ then

$$X \xrightarrow{\theta_X} X \xrightarrow{\theta_X} X$$

is naturally isomorphic. However, arbitrary object $Z$ is not isomorphic to $X$ and $Y$. This is a contradiction. Similarly, the case that there is a morphism from $Z$ to $X$ also has a contradiction.

(ii) Let us consider the case that $\mathcal{A}(M)$ and $\text{End}(\Gamma_{\text{hol}}(M, L))$ are linear isomorphic.
Then $X$ and $Y$ must be isomorphic. There exists $Z \in \text{ob}(\mathcal{D}_{DQ})$ such that $Z$ and $X$ are not isomorphic. If $FG \simeq \text{id}_{\mathcal{D}_{DQ}}$ then at least a natural transformation of one of the following four diagrams

$$
\begin{align*}
X & \xrightarrow{\theta_2} Z \\
G(F(f)) & \xrightarrow{G(f)} \text{id}_Z \\
\quad & \\
Y & \xrightarrow{\theta_2} Z \\
G(F(f)) & \xrightarrow{G(f)} \text{id}_Z \\
\end{align*}
$$

is isomorphic. This is also a contradiction.

Thus, $\mathcal{D}_{PQ}$ is not categorically equivalent to $\mathcal{D}_{DQ}$. The case of $\mathcal{D}_{MR}$ is proved in the same way. □

**Proposition 7.7.** $\mathcal{D}_{\text{env}}$ is categorically equivalent to neither $\mathcal{D}_{PQ}$, nor $\mathcal{D}_{MR}$ nor $\mathcal{D}_{DQ}$.

**Proof.** We consider the following diagram.

$$
\begin{array}{ccc}
(A, \cdot) & \xrightarrow{\Pi_1} & A & \xrightarrow{\Pi_2} & (A, \{ , , \}) \\
\downarrow{\gamma_1} & & \downarrow{\gamma_2} & & \downarrow{\gamma_3} \\
\gamma_1 & \xrightarrow{\alpha \circ \Pi_1} & \gamma_2 & \xrightarrow{\beta \circ \Pi_2} & \gamma_3 \\
\downarrow{h_1} & & \downarrow{h_2} & & \downarrow{h_3} \\
M_i & & & & M_f \\
\end{array}
$$

Suppose that $\mathcal{D}_{\text{env}}$ and $\mathcal{D}_{DQ}$ are equivalent categories, i.e., for functors $F : \mathcal{D}_{\text{env}} \rightarrow \mathcal{D}_{DQ}$ and $G : \mathcal{D}_{DQ} \rightarrow \mathcal{D}_{\text{env}}$, $FG \simeq \text{id}_{\mathcal{D}_{DQ}}$ and $GF \simeq \text{id}_{\mathcal{D}_{\text{env}}}$. Then there exists $X, Y \in \text{ob}(\mathcal{D}_{DQ})$ such that

$$
F(A(M)) = X, \\
F(U) = Y.
$$

$\alpha \circ \Pi_1$ and $\beta \circ \Pi_2$ must be mapped to $\mathcal{D}_{DQ}(F(A(M)), F(U))$ by $F$:

$$
F(\alpha \circ \Pi_1), \ F(\beta \circ \Pi_2) \in \mathcal{D}_{DQ}(F(A(M)), F(U)).
$$

Since the morphisms of $\mathcal{D}_{DQ}$ or $\mathcal{D}_{PQ}$ are unique,

$$
GF(\alpha \circ \Pi_1) = GF(\beta \circ \Pi_2) \in \mathcal{D}_{\text{env}}(GF(A(M)), GF(U)),
$$

for arbitrary functor $G : \mathcal{D}_{DQ} \rightarrow \mathcal{D}_{\text{env}}$. However, $\alpha \circ \Pi_1 \neq \beta \circ \Pi_2$ in $\mathcal{D}_{\text{env}}$. This is a contradiction. Thus, $\mathcal{D}_{\text{env}}$ is not categorically equivalent to $\mathcal{D}_{DQ}$. The other cases for both $\mathcal{D}_{DQ}$ and $\mathcal{D}_{PQ}$ are proved in the same way. □

### 8 Conclusions and Discussions

In this article, we discuss a category of quantization of Poisson manifolds or Poisson algebras as a subcategory of $R\text{Mod}$, but its objects are commutative and noncommutative algebras. We define the quantization category as a generalization of quantizations of the Poisson algebra, and show that this category contains categories of some known quantizations of the Poisson algebra. We also discuss relationships between the categories of various types of quantizations.

The pre-$\mathcal{D}$ category $\mathcal{P}$ is defined by choosing a fixed Poisson algebra $A(M)$ and algebras $M_i$ of $A(M)$’s quantized algebras as objects, and choosing quantization linear maps $T_i : A(M) \rightarrow M_i$ and linear maps between each $M_i$ as morphisms. If a morphism is an algebraic isomorphism, then its inverse is also a morphism. Each quantization map $T_i$ has a noncommutative parameter $h(T_i)$. The character $\chi(M_i)$ is
introduced as the maximum absolute value of $h(T_i)$. The index category $J^*$ and a set of functors $F^*$ are determined by the noncommutative character $\chi$ to consider the classical limit. In addition to these structures, we defined the quantization category as being equipped with Lie homomorphisms of the algebra between Poisson brackets in $\mathcal{A}(M)$ and Lie brackets in the limit determined by $F^*$ of $J^*$. As concrete examples, the quantization category of matrix regularization (including Berezin-Toeplitz quantization), strict deformation quantization, prequantization and Poisson enveloping algebra are constructed. The equivalence or non-equivalence of these categories is also discussed. In particular, we show the equivalence of the quantization category of matrix regularization and the quantization category of strict deformation quantization when $\sharp N = \sharp h$. In addition, we show that equivalence between $\mathcal{D}_{DQ}|_{C^\infty}$, $\mathcal{D}_{MR}|_{Mat_N}$ and $\mathcal{D}_{PQ}$ under a condition that the quantization map $T$ from $\mathcal{A}(M)$ to the limit $M_\infty$ and $Q^\alpha$ from $\mathcal{A}_0(M)$ to the limit $M_\infty$ are an isomorphism, respectively. For example, this condition is satisfied for compact Kähler manifolds in the case of Berezin-Toeplitz quantization. On the other hand, it is shown that the quantization category of Poisson enveloping algebra is not equivalent to the other quantization categories.

We have focused on the quantization category which contains one quantization procedure. However, we define the quantization category such that it is possible to include multiple types of quantization theories. For example, if the union of index categories $J^*_1$ and $J^*_2$ of two quantization categories $\mathcal{D}_1$ and $\mathcal{D}_2$ with the same Poisson algebra $\mathcal{A}(M)$ is empty, a category consisting of the sum of $\mathcal{D}_1$ and $\mathcal{D}_2$ is also a quantization category. Here, the category made up by this summation is a category whose object set is the union of the object sets of $\mathcal{D}_1$ and $\mathcal{D}_2$, and its set of morphisms is created from the union of morphisms of $\mathcal{D}_1$ and $\mathcal{D}_2$ and adding composite maps to the union so that the whole becomes a category. For example, a category created by the sum of $\mathcal{D}_{MR}$ and $\mathcal{D}_{DQ}$ is a quantization category. One of the future tasks will be to examine the concrete construction and to study properties of such a category made up of the sum of quantization categories. It is also necessary to consider the sums of more complex categories whose $J^*_1$ and $J^*_2$ are not disjoint. Such researches should be done as a next step.

In this paper, we formulated a quantization category by adopting (1), (2) and (3) among the conditions by Dirac enumerated at the beginning of this article. However, we can choose other combinations. Therefore, the quantization category studied in this paper might be an example of a series of quantization categories that have a variety of quantization conditions. The task of investigating such a large area of quantization categories remains for future work.

Finally, we will consider potential applications of the quantization category to physics. The universe we live is classically described as a vector bundle. The base manifold is a Riemannian manifold. The fibers are for the electromagnetic field, non-Abelian gauge fields, matter fields and so on. In the case of the particle physics given by a Hamiltonian formulation of mechanics, a cotangent bundle over a Riemannian manifold is its geometry. The cotangent bundle is the Poisson manifold. When we consider $M$ a cotangent bundle over a Riemannian manifold, then the $\mathcal{A}(M)$ in $\mathcal{D}$ corresponds with a classical physics. In that case, the character $\chi$ or $h$ should be chosen as the Planck constant or energy scale. To make concrete predictions or to clarify physical properties, we have to import further structures into the quantization category. The findings of several previous studies might be useful when introducing physical structures. For example, Ojima and Takeori [30] studied the correspondence between classical and quantum physics, which is called Micro-macro duality by categorical approach. Alternatively, categorical approaches to quantum mechanics have shown how to describe the fine structure of physics as functors [1, 8].

As an example, let us consider the IKKT matrix model or noncommutative gauge theory in the context of $\mathcal{D}_{MR}$ [3, 11, 18]. In Section 3, we considered matrix regularization. The classical IKKT matrix model is regarded as a matrix regularization of the type IIB string theory with the Schild gauge. The bosonic part of the Lagrangian is given as

$$\{X^\mu, X^\nu\}\{X^\mu, X^\nu\},$$

where $X^\mu$ is a map from a parameter space to a world sheet and $\{X^\mu, X^\nu\}$ is the Poisson bracket on the world sheet. Using a determinant of the world sheet metric $g$ and Levi-Civita symbol $\epsilon^{ab}$, the Poisson bracket is defined as

$$\{X^\mu, X^\nu\} := \frac{1}{\sqrt{g}} \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu.$$
The quantization map $T_i$ of the quantization category of matrix regularization in Section 3 maps this Poisson bracket to a commutator

$$[X^\mu, X^\nu] := X^\mu X^\nu - X^\nu X^\mu,$$

up to higher order terms of $\hbar$. Then the Lagrangian is also obtained as

$$[X^\mu, X^\nu][X^\mu, X^\nu],$$

at the limit $\text{Mat}_\infty$. This matrix model is also regarded as a noncommutative $U(1)$ gauge theory on noncommutative Euclidean space at the limit of the quantization category of matrix regularization. In this context, type IIB string theory with Schild gauge is defined on the object $A(M)$ and the IKKT matrix model is defined on the object $\text{Mat}_\infty$.

In this article, only the ordinary Poisson structure has been considered for quantization. However, there are many other types of classical mechanics, such as Nambu mechanics [27]. To attack the quantization problem of the membrane theory, quantization of the Nambu bracket has been shown to be an effective approach. For this purpose, the Nambu bracket should be replaced with the Lie 3 bracket by the quantization morphism [2, 5, 6, 7, 14, 15]. The category of quantization defined in this article is naturally generalized to such a quantization type. This would be a suitable problem to address in the next stage of investigation.

The quantization category proposed in this article involves many basic or applied problems, including pure mathematical and physical problems, as mentioned above. All of these should be solved in the future.

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