The Constraints of Unitary on $\pi\pi$ Scattering Dispersion Relations

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Abstract

A new dispersion relation for the partial wave $\pi\pi$ scattering $S$ matrix is set up. Using the dispersion relation we generalize the single channel unitarity condition, $SS^+=1$, to the entire complex $s$ plane, which is equivalent to the generalized unitarity condition in quantum mechanics. The pole positions of the $\sigma$ resonance and the $f_0(980)$ resonance are estimated based on the theoretical relations we obtained. The central value of the $\sigma$ pole position is $M_\sigma \simeq 410\text{MeV}$, $\Gamma_\sigma \simeq 550\text{MeV}$, obtained after including the the constraint of the Adler zero condition.

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In a series of recent publications [1, 2, 3], the present authors presented a dispersive approach to discuss the single channel and coupled channel $\pi\pi$ interaction physics. The essence of the method is to use dispersion relations for physical quantities containing poles and also cuts except those endowed by unitarity. Different from more traditional methods, like the $K$ matrix method, Padé approximation, etc., in the present scheme the role of the dynamical cuts can be traced explicitly. In Ref. [1, 2] we have established a dispersion relation for $\sin(2\delta_{\pi})$, where $\delta_{\pi}$ is the $\pi\pi$ scattering phase shift defined in the single channel unitarity region,

$$\sin(2\delta_{\pi}) = \rho F,$$

$$F(s) = \alpha + \sum_i \frac{\beta_i}{2i\rho(s_i)(s-s_i)} - \sum_j \frac{1}{2i\rho(z^{II}_j)S'(z^{II}_j)(s-z^{II}_j)} + \frac{1}{\pi} \int_L \frac{\text{Im}L F(s')}{s'-s} ds' + \frac{1}{\pi} \int_R \frac{\text{Im}R F(s')}{s'-s} ds', \quad (1)$$

where $F \equiv \frac{1}{2\rho} (S-1/S)$ is the analytic continuation to the real part of the scattering $T$ matrix defined in the physical region (times a factor of 2). In Eq. (1), $s_i$ denote the possible bound state pole positions and $\beta_i$ are the corresponding residues of $S$; $z^{II}_j$ denote the possible resonance pole positions on the second sheet. The integrals denote the cut contributions to $\sin(2\delta_{\pi})$, $L = (-\infty, 0]$ is the left hand cut (LHC) and $R$ starts from the $\bar{K}K$ threshold once the $4\pi$ cut are neglected, $\alpha$ denotes the subtraction constant. The sum of contributions from the cut integrals and from the subtraction constant is sometimes called the background contribution. By evaluating
the left hand integral in the above dispersion relation using the $O(p^4)$ amplitude (and its Padé unitarization) of chiral perturbation theory result, it is found that the LHC contribution to $\sin(2\delta_\pi)$ is negative and concave in the $I=J=0$ channel. Therefore the $\sigma$ resonance must be introduced to explain the experimental data.

The present note is a supplementary and an extension to our previous studies. It will be shown that the discontinuities of the partial wave $S$ matrix and all other physical quantities across the unitarity cut can be expressed as an explicit dependence on the kinematic factor $\rho = \sqrt{1 - 4m^2_\pi/s}$. In other words, the presence of the discontinuity on the right is solely due to the presence of the kinematic factor. Furthermore we are able to re-express the unitarity constraint in a non-trivial and analytic expression which holds on the entire complex $s$ plane. For the reason of simplicity we will confine our discussion in the single channel unitarity region. As an exercise we will estimate the pole positions of the $\sigma$ and the $f_0(980)$ mesons using the formalism discussed in this note. Some comments related to the coupled channel physics will also be made.

We start from the single channel unitarity region, or more precisely, the center of mass energy squared, $s$, is greater than $4m^2_\pi$ and less than $16m^2_\pi$. The relation between the partial wave unitary $S$ matrix and $T$ matrix is defined as

$$S(s) = 1 + 2i\rho(s)T(s), \quad (2)$$

where $\rho = \sqrt{1 - 4m^2_\pi/s}$. With this definition the single channel unitary relation,

$$\text{Im} T(s) = T(s)\rho(s)T^*(s), \quad (3)$$

is being used together with the property of real analyticity,

$$T^*(s + i\epsilon) = T(s - i\epsilon), \quad (4)$$

to analytically continue the $S$ matrix and the $T$ matrix, which are analytic functions on the physical cut plane, to the second sheet of the Riemann surface:

$$T^{II}(s + i\epsilon) = T^I(s - i\epsilon) = \frac{T^I(s)}{S^I(s)}, \quad \text{and} \quad S^{II} = \frac{1}{S^I}. \quad (5)$$

One can then verify that the function $\tilde{F}$ defined as

$$\tilde{F} \equiv \frac{1}{2}(S + \frac{1}{S}) \quad (6)$$

has no discontinuity across the real axis when $0 < s < 16m^2_\pi$, since

$$\tilde{F}(s - i\epsilon) = \frac{1}{2}(S(s - i\epsilon) + \frac{1}{S(s - i\epsilon)}) = \frac{1}{2}(S^{II}(s + i\epsilon) + \frac{1}{S^{II}(s + i\epsilon)})$$

$$= \frac{1}{2}(S^I(s + i\epsilon) + S^I(s + i\epsilon)) = \tilde{F}(s + i\epsilon), \quad (7)$$

and the left hand cut it contains starts from $-\infty$ to 0. The cut structure of $\tilde{F}$ is very similar to the cut structure of the function $F$ studied previously. The function $\tilde{F}$ is the analytic continuation of $\cos(2\delta_\pi)$ defined in the single channel unitarity region.
According to the analytic structure of $\tilde{F}$, as discussed above we can set up the following dispersion relation,

$$
\cos(2\delta) = \tilde{F} = \tilde{\alpha} + \sum_i \frac{\beta_i}{2(s - s_i)} + \sum_j \frac{1}{2S'(z^I_j)(s - z^I_j)} + \frac{1}{\pi} \int_L \frac{\text{Im}L\tilde{F}(s')ds'}{s' - s} + \frac{1}{\pi} \int_R \frac{\text{Im}R\tilde{F}(s')ds'}{s' - s},
$$

(8)

where $\tilde{\alpha}$ is the subtraction constant and one subtraction to the cut integrals in the above expression is understood. The right hand cut $R$ starts from $16m^2_\pi$ in principle but becomes important only when $s$ approaches the $\bar{K}K$ threshold. Using Eqs. (8) and (1), we get an analytic expression of $S$ on the complex $s$ plane in terms of poles, dynamical cuts, and the kinematic factor:

$$
S(z) = \cos(2\delta) + i\sin(2\delta) = \tilde{\alpha} + i\alpha\rho(z) + \sum_i \frac{\beta_i}{2(z - s_i)} + \sum_i \frac{\rho(z)\beta_i}{2\rho(s_i)(z - s_i)} + \sum_j \frac{\rho(z^I_j) - \rho(z)}{2\rho(z^I_j)S'(z^I_j)(z - z^I_j)} + \frac{1}{\pi} \int_L \frac{\text{Im}L\tilde{F}ds'}{s' - z} + \frac{i\rho(z)}{\pi} \int_L \frac{\text{Im}L\tilde{F}ds'}{s' - z} + \frac{1}{\pi} \int_R \frac{\text{Im}R\tilde{F}ds'}{s' - z} + \frac{i\rho(z)}{\pi} \int_R \frac{\text{Im}R\tilde{F}ds'}{s' - z}.
$$

(9)

One may use the definition $S(4m^2_\pi) = 1$ to re-express $\tilde{\alpha}$ in Eqs. (9) and (8) in terms of other parameters. The above expression respects the well known properties of $S$ matrix theory. For example, the physical sheet $S(z)$ does not contain resonance poles though the phase motion of $S$ is affected by resonance poles on the second sheet. The Eq. (9), though simple to derive, is an exact relation. The Eqs. (8) and (1) must satisfy a relation on the whole complex $s$ plane:

$$\sin^2 2\delta + \cos^2 2\delta \equiv 1,$$

(10)

which is the analytic continuation of the single channel unitarity relation, $S^+S = 1$, on the complex $s$ plane. The Eq. (10) is equivalent to the generalized unitarity condition $S(k)S(k^*)^* = 1$ in quantum mechanics and it contains all information about single channel unitarity and analyticity. For example, Eq. (8) must obey another relation,

$$S(z^I_j) = 0.$$

(11)

This equation is derived by the analytic structure of $S$ matrix, that is, the physical $S$ matrix has zero at the same energy, $z^I_j$, as the pole energy on the second sheet. The Eq. (11) is actually equivalent to the requirement of the vanishing of the first order pole terms on the l.h.s. of Eq. (10) (the second order poles disappear automatically). The Eq. (11) demands correlations between various parameters: $s_i$, $\beta_i$, $z^I_j$, $S'(z^I_j)$,

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1We neglect the $4\pi$ cut from now on in the text. According to the conventional wisdom the $4\pi$ cut becomes important only above, say, 1.2GeV.

2Since all cuts at higher energies are actually included in those cut integrals.

3We are in debt to the referee who points out this fact to us.
\[\alpha,\text{ and cut integrals. However, these relations are in general very complicated to use directly.}\]

We find it helpful, for pedagogical reasons, to analyze Eq. (9) together with Eq. (10) in some very simple situations. For example, we neglect all the cut integrals in Eq. (9), and assume only one pole at \(s = s_0\) exist. Then we found two solutions satisfying Eqs. (9) and (10):

1. A bound state:
   \[\tilde{\alpha} = 1 - \frac{s_0}{2}, ~ \alpha = -\frac{1}{2} \sqrt{s_0(4 - s_0)}, ~ \beta = s_0(4 - s_0). \tag{12}\]
   The scattering length is \(a = -\sqrt{\frac{s_0}{4-s_0}}\) (taking the mass of the scattering particles to be 1).

2. A virtual state:
   \[\tilde{\alpha} = 1 - \frac{s_0}{2}, ~ \alpha = \frac{1}{2} \sqrt{s_0(4 - s_0)}, ~ \beta = \frac{1}{S'(s_0)} = s_0(4 - s_0). \tag{13}\]
   The scattering length is \(a = \sqrt{\frac{s_0}{4-s_0}}\).

We can learn some lessons from these two simple solutions. Comparing with the nonrelativistic version of the toy model \[4\],

\[S = \frac{1 + ika}{1 - ika}, \tag{14}\]

where \(k = \sqrt{s/4 - 1}\). The nonrelativistic version contains a bound state pole when \(a < 0\), and a virtual state pole when \(a > 0\). This agrees with the qualitative behavior of the relativistic case. But the pole locates at \(s_0 = 4(1 - \frac{1}{a^2})\) whereas in our case the pole locates at \(s_0 = 4/(1 + \frac{1}{a^2})\). In the nonrelativistic case the pole can locate at anywhere between \(-\infty\) and 4, but in the present case \(s_0 \in (0, 4)\). The latter of course makes sense by eliminating the possible existence of tachyons. Furthermore, in the nonrelativistic case the phase shift \(\delta(\infty)\) goes to \(\pm \pi/2\) as dictated by the “weak” Levinson’s theorem. That is not the case in the present situation, even there is no dynamical cut. All these differences come from the use of relativistic kinematics (to use \(\rho\) instead of \(k\)), which really makes physical sense, as shown above. The relativistic kinematic factor introduces an additional cut from the square root of \(s\) which is conveniently placed at from 0 to \(-\infty\). This additional (kinematical) cut, though carefully excluded from the dynamical cuts \[1, 2\], does function in its own way. For solutions with more than one pole, one can prove that a two–pole (a pair of resonances) solution does not exist (in the absence of dynamical cuts), which is different from the non-relativistic case. A three–pole solution however exists. As an existence proof one can construct the \(S\) matrix in the following form:

\[S = \frac{s - M^2 - i\rho g}{s - M^2 + i\rho g}; ~ g > 0, \tag{15}\]

\[4\text{We call the dynamical cuts as those appear as left hand integrals in Eqs. (8) and (1).}\]
which, for $M^2 > 4$ and sufficiently small $g$, contains a pair of resonances and a virtual state pole. In general, however, there is no simple correspondance between $S$ matrix poles and the physical resonances [4].

In the phenomenological discussion on the realistic $\pi\pi$ scatterings, we follow the method of Refs. [1, 2] to study the properties of resonance poles after the cut integrals are estimated. In the following we focus on the IJ=00 channel, using the phase shift data from Refs. [6, 7, 8, 9]. The difference between the fit made in Refs. [1] and here is that in Refs. [1] we only fit $\sin(2\delta_\pi)$, or the imaginary part of the $S$ matrix. In here we fit the $S$ matrix itself using Eq. (9),

$$
\delta_\pi(s) = \frac{1}{2i} \ln[S(s)] = \text{Re} \left[ \frac{1}{2i} \ln[S(s)] \right] + i \text{Im} \left[ \frac{1}{2i} \ln[S(s)] \right].
$$

(16)

In the single channel unitarity region, $\delta_\pi$ is real. For the given set of the experimental value of $\delta_\pi$: $\{s_j, \delta_j, \Delta \delta_j\}$, one may construct the expression of total $\chi^2$ containing two terms, $\chi^2 = \chi^2_1 + \chi^2_2$, in which,

$$
\chi^2_1 = \sum_j \frac{|\delta_j - \text{Re} \left[ \frac{1}{2i} \ln[S(s_j)] \right]|^2}{|\Delta \delta_j|^2}; \quad \chi^2_2 = \sum_j \frac{|\text{Im} \left[ \frac{1}{2i} \ln[S(s_j)] \right]|^2}{|\Delta \delta_j|^2}.
$$

(17)

Notice that single channel unitarity in here, unlike most conventional approaches, is not guaranteed automatically. If we use the above expression of $\chi^2$ to make the fit it may happen that the $\chi^2$ minimization program prefers a solution with non-vanishing $\chi^2_2$, i.e., violating the single channel unitarity. In order to circumvent such a problem the unitarity constraint Eq. (10) has to be taken into account to confine the violation of unitarity in a numerically acceptable range, which substantially complicates the fit. What we gain with such a price paid is that we can, at least in principle, clearly keep track of all kinds of dynamical singularities in their right places. This property is not easy to maintain in other approaches which automatically guarantee unitarity.

The circumvent is possible, noticing that the term $\chi^2_2$ in Eq. (17) is in fact quite arbitrary, since there is no experimental error bar for the ‘imaginary part of $\delta_\pi$’. Therefore we can freely chose, for example, another expression of $\chi^2_2$,

$$
\chi^2_2 = \frac{1}{\epsilon^2} \sum_j |\text{Im} \left[ \frac{1}{2i} \ln[S(s_j)] \right]|^2,
$$

(18)

with sufficiently small $\epsilon$ parameter which will guarantee Eq. (10) in a numerically satisfiable range. Actually what we do here is an example of the so called ‘penalty function method’ in the theory of probability and statistics [12]. In here the term $\chi^2_2$ defined in Eq. (18) is called the penalty term and $1/\epsilon^2$ is called the penalty factor.

In order to make use of Eq. (9) to study the properties of resonance poles, it is necessary at first to estimate various cut integrals. The discontinuities of function $F$ and $\tilde{F}$ on the left can be rewritten as,

$$
\text{Im}_L F = 2\text{Im}_L \text{Re}_R T(s),
$$

(19)

$$
\text{Im}_L \tilde{F} = -2\rho(s)\text{Im}_L \text{Im}_R T(s),
$$

(20)
since $F = 2 \text{Re}_R T$ and $\tilde{F} = 1 - 2 \rho \text{Im}_R T$. The $r.h.s.$ of Eq. (13) has been estimated in Ref. [1], that is one expands $\text{Im}_L \text{Re}_R T(s)$ to $O(p^4)$ in chiral perturbation theory ($\chi$PT). But it is easy to see that $\text{Im}_L \text{Im}_R T(s)$ vanishes up to $O(p^4)$ since $\text{Im}_R T^{(4)}(s) = \rho \left( \frac{2s-m^2_K}{32\pi F^2} \right)^2$. Therefore $\text{Im}_L \text{Im}_R T(s)$ must be expanded to $O(p^6)$,

$$\text{Im}_L \text{Im}_R T(s) = 2 \rho T^{(2)} \text{Im}_L \text{Re}_R T^{(4)}(s),$$

(21)

to get a non-vanishing result. Hence the bad high energy behavior of the chiral amplitude gets even worse when estimating Eq. (21), which means when estimating the cut-off version of the dispersion integral [1] the numerical result will be very sensitive to the cut off parameter. In our understanding the vanishing of $\text{Im}_L \text{Im}_R T$ at $O(p^4)$ implies that the quantity and its integral are indeed very small, at least at moderately low energies. This suggestion is confirmed by the prediction of the [1,1] Padé amplitude which is very small in magnitude. We therefore in the following fix the left hand integral of $\tilde{F}$ by using the result from the Padé amplitude. We use the same strategy as in Ref. [1] to estimate $\text{Im}_L F$ and its integral. That is we use both the $O(p^4)$ $\chi$PT and the Padé approximant to estimate $\text{Im}_L F$, for the former we truncate the left hand integral at certain scale $-\Lambda^2$ which varies within a reasonable range.

One of the lessons one may draw from Ref. [2] is that it is not absolutely necessary to go to coupled channel situation when discussing, at qualitative level, the property of the narrow $f_0(980)$ resonance on the second sheet. Therefore we include the $f_0$ pole in our discussion within the current formalism which only makes use of the data in the single channel unitarity region. In some sense, introducing the $f_0$ pole in the fit improves the determination on the pole location of the $\sigma$ resonance as done in Ref. [1], since in here we no longer need to truncate the data (at around $\sqrt{s} \simeq 900\text{MeV}$) which is somewhat arbitrary. The right hand cut integrals induced by the $\bar{K}K$ threshold have to be taken into account in here since they will develop a cusp structure below the $\bar{K}K$ threshold. Here we follow the same strategy as in Ref. [2] to estimate the right hand integrals by using the $T$ matrix parameterization above the $\bar{K}K$ threshold given in Refs. [10] and [11], and cut the integral at $\sqrt{s} \simeq 1.5\text{GeV}$. For the integrand we have,

$$\text{Im}_R \tilde{F} = \frac{1}{2} (\eta - \frac{1}{\eta}) \sin(2\delta_\pi),$$

$$\text{Im}_R F = \frac{1}{2\rho} (\frac{1}{\eta} - \eta) \cos(2\delta_\pi).$$

(22)

In fig. [1], we can see the two estimates of the right hand integral in the IJ=00 channel. Even though they are not coincide with each other, both of them give the same trend when approaching $4m^2_K$. When we only fit $\sin(2\delta_\pi)$, the $r.h.c.$ barely

\[\text{Unlike the situation in the IJ=20 case where the spurious physical sheet resonance (SPSR) contribution to cos}(2\delta_\pi) \text{ is large}[3], \text{ in the IJ=00 case the SPSR contribution to cos}(2\delta_\pi) \text{ is rather small. The smallness of the SPSR contribution may be considered a necessary condition for the predictions of Padé amplitudes to be numerically reasonable.} \]
have any effect to the pole position of $f_0(980)$. But in here, we see that the effects of the right hand integrals are no longer negligible.

As already stated earlier the unitarity constraint, Eq. (10), has to be taken into account in our fit. Instead of trying to solve the constraints among parameters provided by Eq. (10) explicitly we make use of the so called ‘penalty function’ method in data fit with constraints among parameters. In principle, increasing the penalty factor will drive the fit result moving towards a solution respecting unitarity exactly. But since there are uncertainties in the input, i.e., the cut contributions and the number of pole terms, increasing the penalty factor does not always lead to reasonable results. For example, in the present case, for a too large penalty factor corresponding to $\epsilon \sim 0.01$ the quality of the fit near the $\bar{K}K$ threshold becomes very bad. The masses and widths of the $\sigma$ and the $f_0(980)$ poles can be estimated from the fit by varying the left and right cut contributions. The variation range of the cutoff parameter $\Lambda$ in evaluating the left hand integral for $\sin(2\delta_{\pi})$ is taken from $600\text{MeV}$ to $800\text{MeV}$ here, as we find that larger values of $\Lambda$ also lead the fit quality below the $\bar{K}K$ threshold to be rather bad. The $\epsilon$ parameter is therefore taken to be around 0.02. The results are listed in the following:

$$M_\sigma \simeq 440 - 530\text{MeV} \ , \ \Gamma_\sigma \simeq 540 - 590\text{MeV} \ ;$$

$$M_{f_0} \simeq 976 - 987\text{MeV} \ , \ \Gamma_{f_0} \simeq 22 - 44\text{MeV} \ ;$$

$$a_0^0 \simeq 0.230 - 0.276 \ . \quad (23)$$

The above results are compatible with the results of Ref. [1] (the table 1 there), and especially Ref. [2] (the Eq. (44) there) though the methods are somewhat different. The uncertainty for the width of $f_0$ is larger here when comparing with that of Ref. [2], which may be partly due to the fact that in here we only work in the single channel unitarity region. In Ref. [2], the unitarity constraint is not considered since only the imaginary part of the $S$ matrix (or $\sin(2\delta_{\pi})$ ) is fitted there. When $\sin(2\delta_{\pi})$ approaches 1, its error behaves as $2\cos(2\delta_{\pi}) \Delta \delta_{\pi}$ and hence approaches 0. Therefore the violation of unitarity is automatically confined in an acceptable range in Ref. [2].

\footnote{See fig. 2, if one takes $\epsilon = 0.01$ the fit curve of $a_0^0$ would simply miss the data point which is just below the $\bar{K}K$ threshold.}
From the results we find that the global fit favors a larger value of $a_0^0$ comparing with the results of Refs. [9, 13]. A typical fit result is plotted in fig. 2a. The problem of having a larger scattering length is due to that we have not put the constraint of the Adler zero condition in our data fit. In Ref. [1] this problem is solved by putting the constraint of the scattering length parameter by hand. Here we improve the fit by including the constraint of the Adler zero condition. If we fix the Adler zero position at $s = m_\pi^2/2$ the results are the following:

\begin{align*}
M_\sigma &\approx 380 - 440 \text{MeV}, \quad \Gamma_\sigma \approx 510 - 580 \text{MeV} ; \\
M_{f_0} &\approx 976 - 983 \text{MeV}, \quad \Gamma_{f_0} \approx 43 - 64 \text{MeV} ; \\
a_0^0 &\approx 0.190 - 0.212 .
\end{align*}

The above results are obtained using the same range of parameters as in obtaining Eq. (23). The most important change of Eq. (24) comparing with Eq. (23) is that now the scattering length parameter decreases and is in better agreement with the results of Ref. [9, 13]. We observe again that the decreasing of the scattering length parameter drives the $\sigma$ pole moving towards left on the complex $s$ plane, in agreement with the observation made in Ref. [1]. Another major difference between Eq. (23) and (24) is that the latter gives a larger $f_0$ width. But Fig. 2b reveals that the $f_0$ pole is not fitted very well in the latter case. A more reliable determination of the $f_0(980)$ resonance requires a coupled channel analysis.

In above discussions one of the major uncertainty in obtaining our results comes from the estimates on the left hand cuts which are very difficult to determine accurately from pure theoretical calculations. Our estimates on the $l.h.c.$ effects are based on $O(p^4)$ chiral perturbation theory. It means that these estimates are no longer trustworthy at high energies, or more precisely, at $s \approx m_\rho^2$ and above. Indeed, in our calculation to estimate $\text{Im} F$ using $O(p^4)$ chiral perturbation theory, the resonance effects are not taken into account because the resonance only contributes at $O(p^4)$ the real part of the amplitudes. Taking the $\rho$ resonance as an example, the $t$ channel $\rho$ exchange will enhance the $t$ channel $\pi\pi$ cut at around $t = m_\rho^2$ and will
influence the l.h.c. of s channel partial wave amplitude through Eq. (53) of Ref. [1]. This enhancement will result in a contribution to the l.h.c. starting from around $4m^2_\pi - m^2_\rho$ to further left. If we approximate the effect by a tree level $\rho$ exchange then the effect is approximated by an effective l.h.c. from $4m^2_\pi - m^2_\rho$ to $-\infty$.\footnote{Notice that, exactly speaking, there is only an enhancement to the $\pi\pi$ l.h.c., but no new cut being generated, since the $\rho$ resonance is not stable.} No systematic method is known to exist to estimate these resonance contributions to the dynamical cuts, in the non-perturbative scheme.\footnote{A non-perturbative treatment is necessary because otherwise the derivative coupling of $\rho$ to $\pi$ will cause the once-subtracted dispersive integral divergent.} However, it is reasonable to expect that the power counting rule for resonances in perturbation theory\footnote{A thorough investigation to this problem needs a complete new study and goes beyond the scope of the present note.} also works in the non-perturbation scheme, at low energies. Therefore it is a reasonable speculation that resonance contribution to the left hand integral is $O(p^6)$ at low energies and therefore does not distort any of our qualitative conclusion on the cut integral at low energy, though their effects will become important at around $s = m^2_\rho$ and above. Our speculation seems to be supported by a phenomenological analysis of Ref. [13] (see fig. 2 in that paper). A thorough investigation to this problem needs a complete new study and goes beyond the scope of the present note.

To conclude we in this note further extend the previous method we proposed to study the partial wave scattering problem by establishing a dispersion relation for $\cos(2\delta_\pi)$. In our procedure the effects of the unitarity cut are fully exposed by the explicit dependence of physical quantities on the kinematic factor. The constraint of single channel unitarity, Eq. (3), is re-expressed as an analytic relation which holds on the whole $s$ plane, i.e., Eq. (10). The Eq. (11) is equivalent to the generalized unitarity condition in quantum mechanics. Applications of our approach are made to determine the pole positions of the $\sigma$ and $f_0(980)$ resonances, after estimating various cut contributions from both chiral perturbation theory and experiments. We find that the central value of the $\sigma$ pole position is about $M_\sigma \simeq 410\text{MeV}$, $\Gamma_\sigma \simeq 550\text{MeV}$, according to our fit with the constraint of the Adler zero condition.

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