Quantum electrodynamics in two dimensions at a finite-temperature thermofield bosonization approach

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Abstract
The Schwinger model at finite temperature is analyzed using the thermofield dynamics formalism. The operator solution due to Lowenstein and Swieca is generalized to the case of finite temperature within the thermofield bosonization approach. The general properties of the statistical–mechanical ensemble averages of observables in the Hilbert subspace of gauge invariant thermal states are discussed. The bare charge and chirality of the Fermi thermofields are screened, giving rise to an infinite number of mutually orthogonal thermal ground states. One consequence of the bare charge and chirality selection rule at finite temperature is that there are innumerably many thermal vacuum states with the same total charge and chirality of the doubled system. The fermion charge and chirality selection rules at finite temperature turn out to imply the existence of a family of thermal theta-vacua states parametrized with the same number of parameters as in the zero temperature case.

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1. Introduction

Quantum electrodynamics of massless fermions in two dimensions (Schwinger model) was exactly solved by Schwinger using functional methods [1]. The original motivation was to show that local gauge invariance does not necessarily require the existence of a massless physical particle. However, it was only later that the subtleties which make this model particularly interesting were displayed by the work of Lowenstein and Swieca [2]. Paralleling Klaiber’s paper on the Thirring model [3], a complete operator solution in terms of Bose fields (bosonization) was presented, and ‘only then does the striking simplicity of the physical
content of the model become obvious’ [2]. For four decades, the Schwinger model has been extensively discussed within different approaches, applications and extensions. (For a general survey and a complete list of references, see [4, 5].)

Using the functional integral approach within the imaginary time formalism, the extension of the Schwinger model to the case of finite temperature has been the subject of several publications [5–11]. The main conclusions are as follows: (i) the temperature independence of the anomaly, which in turn is responsible for the temperature independence of the Schwinger mass; (ii) a lengthy computation of the temperature dependence of the chiral condensate shows that the spontaneous chiral symmetry breaking persists at finite temperature; and (iii) the correlator of the Polyakov loop-operator in the ‘zero instanton’ sector leads to clustering violation for any finite temperature. This suggests the existence of a degeneracy of the ground state, although Lorentz invariance is not manifest at finite temperature. Further approaches to finite-temperature field theory models in 1+1 and 2+1 dimensions are found in [12] and [13], respectively.

However, despite a number of publications on the Schwinger model at finite temperature, some questions related to the basic structural properties of the model have not been fully appreciated and clarified in a convincing manner in the literature, such as (i) the role played by the bare fermionic charge and chirality selection rules in the construction of the Hilbert space of thermal physical states; (ii) the charge screening in the Higgs–Schwinger mechanism at finite temperature, giving rise to an infinite number of thermal vacuum states; and (iii) the formulation of the gauge invariant Hilbert subspace in terms of a thermal theta-vacuum parametrization, displaying the spontaneous symmetry breaking at finite temperature. These properties are more easily discussed in the operator formulation of thermofield dynamics. It is the purpose of this paper to discuss and clarify these aspects of the model, as well as to acquire a more detailed understanding of the physical grounds of gauge theories at finite temperature within the thermofield dynamics approach.

As is well known, some properties of a two-dimensional quantum field model are more transparent in the operator formulation, which resumes in a compact way the solution for a general correlation function, and others are better seen in the functional integral formulation. In this paper, we formulate the operator solution of the Schwinger model with massless fermions at finite temperature within the thermofield dynamics approach, using the bosonization of Fermi fields at finite temperature (thermofield bosonization) introduced in [14, 15]. We hereby provide the complement to the ‘real time formalism’ of the functional approach as a natural extension of the Lowenstein–Swieca zero-temperature solution. From the present analysis, the intrinsic mathematical structural aspects of the model are displayed and the physical content of the model becomes clear. As a byproduct, some known results, formerly obtained within the functional integral formulation [5–7, 9–11], are easily obtained within the thermofield bosonization approach.

The introduction of finite temperature first requires a doubling of the Hilbert space at zero temperature in order to allow for a later extension to $T \neq 0$. In this case this requires a doubling of the Lowenstein–Swieca operator solution [2, 16], which, as we shall explain, involves some subtleties related to the thermofield bosonization of the Fermi field [14, 15]. This is done in section 2. In order to have a clear understanding of the role played by the bare charge and chirality selection rules, a detailed discussion of the superselection rules and vacuum structure in the doubled Hilbert space at zero temperature is presented.

In section 3 we discuss the operator solution of the Schwinger model at finite temperature. In thermofield dynamics the statistical–mechanical ensemble averages are expressed in the form of expectation values of $T = 0$ operators in a temperature-dependent vacuum. In the case of a gauge theory, such statistical averages can for instance be performed in the gauge invariant
subspace defined by the Gupta–Bleuler subsidiary condition [2, 17]. We show that at finite temperature the free fermionic charge and chirality of the total combined system are screened, giving rise to an infinite number of degenerate thermal vacua states. The thermal selection rules imply that a theta-vacuum representation in the Hilbert space of the gauge invariant thermal states can be given in terms of a family of thermal theta vacua parametrized by the same number of parameters as in the zero-temperature case, contrary to naïve expectation.

In section 4 we compute the thermal theta-vacuum expectation value of the mass operator and obtain the analytic expression for the chiral condensate for any temperature in terms of the mean number of massive particles in the ensemble. The high-temperature behavior of the chiral condensate is then easily obtained within the thermofield bosonization approach. In section 5 we conclude and present a brief discussion of other physical properties of the model.

In the appendix we consider the two-dimensional free massive scalar thermofield theory. The diagonal two-point function of the massive boson is computed displaying the manifest invariance under the KMS [18] condition. This describes the physical dynamics of the model since, analogously to the zero-temperature case, in T ≠ 0 the algebra of observables is isomorphic to the algebra of the massive scalar boson thermofield. We also show how the off-diagonal thermal two-point function in the doubled Hilbert space is obtained from the diagonal one by analytic continuation in time to t − i β, in agreement with the view of Umezawa [19]. This argument is general in the sense that it holds for Bose or Fermi thermofields (using the corresponding statistical weights), massive or massless fields, and is independent of the spacetime dimension. This streamlines the presentation of [14, 15].

2. QED2 with doubled Hilbert space at T = 0

In thermofield dynamics, a ‘tilde’ operator is introduced for each of the operators describing the system under consideration [6, 17, 19–21]. This entails a doubling of the Hilbert space \( \mathcal{H} \rightarrow \mathcal{H} \otimes \tilde{\mathcal{H}} \). In this section we shall reconsider the operator solution of the Schwinger model, as formulated by Lowenstein and Swieca [2], taking account of this doubling of the Hilbert space and the ‘tilde’ conjugation rule within the thermofield bosonization scheme. Although at T = 0, \( \mathcal{H} \) merely plays the role of a spectator, this will set the stage for making the transition to finite temperature in section 3. The temperature dependence of the expectation values of operators living in this doubled Hilbert space will then be entirely contained in the thermal vacuum state, which in turn will lead to a coupling of \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \).

The doubled Schwinger model at zero temperature is defined by the classical total Lagrangian of the combined system,

\[
\hat{\mathcal{L}}(x) = \mathcal{L}(x) - \tilde{\mathcal{L}}(x),
\]

where \( \mathcal{L} \) is the Lagrangian of the original system\(^4\),

\[
\mathcal{L}(x) = -\frac{1}{4} \mathcal{F}_{\mu\nu}(x) \mathcal{F}^{\mu\nu}(x) + \overline{\psi}(x) \gamma^{\mu}(i\partial_{\mu} + eA_{\mu}(x)) \psi(x), \tag{2.1}
\]

and \( \tilde{\mathcal{L}} \) is the Lagrangian of the ‘tilde’ system obtained from (2.1) by the ‘tilde’ conjugation rule [17],

\[
\tilde{\mathcal{L}}(x) = -\frac{1}{4} \tilde{\mathcal{F}}_{\mu\nu}(x) \tilde{\mathcal{F}}^{\mu\nu}(x) + \overline{\tilde{\psi}}(x) \gamma^{\mu}(i\partial_{\mu} + e\tilde{A}_{\mu}(x)) \tilde{\psi}(x),
\]

\(^4\) The conventions used are as follows:

\[
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^5 = \gamma^0 \gamma^1, \quad \epsilon^{01} = 1, \quad g^{00} = 1,
\]

\[
\gamma^{\mu}\gamma^{\nu} = \epsilon^{\mu\nu} \gamma^{\rho}, \quad x^{\pm} = x^{0} \pm x^{1}, \quad \partial_{\pm} = \partial_{0} \pm \partial_{1}.
\]
with \( F(x) \) and \( \tilde{F}(x) \) the field-strength tensors for the vector fields \( A_\mu(x) \) and \( \tilde{A}_\mu(x) \), respectively. At \( T = 0 \) the contribution of the tilde and untilde fields decouples in the correlation functions. The operator solution of the model will be given in terms of Wick-ordered exponentials of a set of free Bose fields and their corresponding ‘tilde conjugated’ partners\(^5\) acting on the Fock-vacuum state \( |\tilde{0}, 0\rangle = |\tilde{0}\rangle \otimes |0\rangle \). At zero temperature, the corresponding quantum field theory is defined by the sets of field operators \{\( \psi, A_\mu \)\} and \{\( \tilde{\psi}, \tilde{A}_\mu \)\}, which define two independent field algebras \( A \) and \( \tilde{A} \), respectively, and generate the Hilbert space as the direct product 

\[
\tilde{H} \otimes H = \tilde{A}|\tilde{0}\rangle \otimes A|0\rangle.
\]

In the local covariant operator formulation, the operator solutions for the Dirac equations

\[
\text{(2.2)} \quad i\gamma^\mu \partial_\mu \psi(x) + \frac{e}{2} \gamma^\mu \lim_{\varepsilon \to 0} \{ A_\mu(x + \varepsilon) \psi(x) + \psi(x) A_\mu(x - \varepsilon) \} = 0,
\]

\[
\text{(2.3)} \quad -i\gamma^\mu \partial_\mu \tilde{\psi}(x) + \frac{e}{2} \gamma^\mu \lim_{\varepsilon \to 0} \{ \tilde{A}_\mu(x + \varepsilon) \tilde{\psi}(x) + \tilde{\psi}(x) \tilde{A}_\mu(x - \varepsilon) \} = 0
\]

are essentially obtained by the doubling of the Lowenstein–Swieca solution [2], taking proper care of signs as required by the thermofield bosonization scheme [15]. By essentially we mean that in the representation of the fermion fields as Wick-ordered exponentials of bosonic field, the corresponding representation of the tilde fields is not obtained by the usual tilde conjugation operation. We can nevertheless implement in general this operation by introducing two parameters \( s \) and \( \tilde{s} \) which take the values \( s = 1 \) and \( \tilde{s} = -1 \). The free zero-mass fermion field can then be written as the following Wick-ordered exponential of free massless scalar fields [15]\(^6\),

\[
\psi^{(0)}(x) = \left( \frac{\mu}{2\pi} \right)^{\frac{1}{2}} : e^{i\sqrt{\pi} \gamma^5 \Sigma_1(x) + \phi(x)} :,
\]

where \( \mu \) is an infrared regulator of the massless scalar field \( \phi \), and \( \phi_D \) is the dual of \( \phi \),

\[
\partial_\mu \phi_D(x) = \varepsilon_{\mu\nu} \partial^\nu \phi(x).
\]

The corresponding equations for the tilde fields are now straightforwardly obtained by applying the tilde conjugation operation to both sides of the equations, keeping in mind that the parameter \( s = 1 \) is also taken into \( \tilde{s} = -1 \) by this operation\(^7\):

\[
\tilde{\psi}^{(0)}(x) = \left( \frac{\mu}{2\pi} \right)^{\frac{1}{2}} : e^{-i\sqrt{\pi} \gamma^5 \tilde{\Sigma}_1(x) + \tilde{\phi}(x)} :,
\]

The solution to the Dirac equations (2.2) and (2.3) can then be written in the form

\[
\psi(x) = e^{i\sqrt{\pi} \gamma^5 [\Sigma(x) + \eta(x)]} : \psi^{(0)}(x),
\]

\[
A_\mu(x) = -s \frac{\sqrt{\pi}}{e} \varepsilon_{\mu\nu} \partial^\nu (\Sigma(x) + \eta(x)),
\]

together with their ‘tilde’ counterparts.

The fields \( \Sigma \) and \( \tilde{\Sigma} \) are free pseudoscalar fields of mass \( m = e/\sqrt{\pi} \), and \( \eta, \tilde{\eta} \) are free massless fields reflecting the gauge degrees of freedom, quantized with indefinite metric [2]. Moreover, the tilde fields are quantized in general with metric opposite to that of the untilde fields [20].

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\(^5\) The ‘tilde conjugation’ of a c-number \( c \) times an operator \( O \), \( c\tilde{O} \), is defined by \( \tilde{c}\tilde{O} = c^* \tilde{O} \).

\(^6\) We have suppressed the Klein factors needed to ensure the correct anticommutation relations [14].

\(^7\) Thus, for example, \( e^{i\tilde{\phi}(x)} = e^{-i\tilde{\phi}(x)} = e^{\tilde{\phi}(x)} \).
Let us remark that the sign associated with both the $\tilde{\Sigma}$ and $\tilde{\eta}$ fields in equation (2.5) is dictated by the bosonized form of the free Fermi field. This bosonization prescription for $\tilde{\psi}^{(0)}$ proves necessary in order to obtain agreement with the real-time formalism of Umezawa et al [19] for the off-diagonal two-point function $(\tilde{\psi}^{(0)} \tilde{\psi}^{(0)})$ at finite temperature, as well as agreement with the revised thermofield approach for fermions due to Ojima [17]. For a more detailed discussion of the bosonization of the free Fermi thermofield, we refer the reader to [14, 15].

In what follows, all equations for the tilde fields are obtained by simply applying the ‘tilde conjugation operation’ as described above. Aside from the parameter $s$, which only plays a role in the tilde conjugation operation, they represent nothing but a brief summary of the Lowenstein–Swieca work.

The two-point functions of the left- and right-moving components of the free massless scalar fields $\phi$ are given by

\begin{equation}
\langle 0 | \phi(x^+) \phi(0) | 0 \rangle = -\frac{1}{2\pi} \ln \{ i\mu (x^+) - i\epsilon \},
\end{equation}

and for the massless free fields $\eta(x)$, quantized with indefinite metric, we have [2]

\begin{equation}
\langle 0 | \eta(x^+) \eta(0) | 0 \rangle = \frac{1}{2\pi} \ln \{ i\tilde{\mu} (x^+) - i\epsilon \}.
\end{equation}

Here $\mu$ and $\tilde{\mu}$ are independent infrared cutoffs. The fermion charge and chirality selection rules are carried by the Wick-ordered exponentials of the fields $\phi$ and $\tilde{\phi}$. The indefinite metric fields $\eta$ and $\tilde{\eta}$ do not carry these selection rules and the cutoff $\tilde{\mu}$ will be maintained at a fixed finite, though small, value.

The bosonization of free massless fermions in two dimensions is based on the fact that the correlation functions of Wick-ordered exponentials of the free massless scalar field satisfy Wightman positivity, provided we associate with the exponential a conserved charge $\lambda$ [4, 5, 22, 23]. This superselection rule, contained in the exponentials of the bosonized fermion theory, implements the fermion charge and pseudo charge conservation. The infrared regulator $\mu$, that appears in the bosonized expressions (2.4) and (2.5), ensures that in the limit $\mu \rightarrow 0$ the only non-vanishing Wightman functions are those for which the bare charge and chirality are conserved. Following the spirit of [22], this will be explained in detail later for both the $T = 0$ and $T \neq 0$ case.

The vector current, computed as the short distance limit of the standard gauge invariant point-splitting prescription [2, 4, 5], is given by

\begin{equation}
J_\mu(x) = -\frac{s}{\sqrt{\pi}} \epsilon_{\mu\nu} \partial^\nu \Sigma(x) + \ell_\mu(x),
\end{equation}

where $\ell_\mu(x)$ are longitudinal zero mass contributions to the currents

\begin{equation}
\ell_\mu(x) = -\frac{s}{\sqrt{\pi}} \epsilon_{\mu\nu} \partial^\nu \psi(x) = -\frac{s}{\sqrt{\pi}} \partial_\mu \phi_D(x),
\end{equation}

with

\begin{equation}
\psi(x) = \eta(x) + \phi(x).
\end{equation}

The relative sign between the currents $J_\mu(x)$ and $\tilde{J}_\mu(x)$ ($\delta = -s$) is a consequence of the commutation relations of tilde versus untilde fields already mentioned.

Due to the opposite metric quantization of the fields $\eta$ and $\tilde{\eta}$ with respect to $\phi$ and $\tilde{\phi}$, the longitudinal currents $\ell_\mu$ and $\tilde{\ell}_\mu$ both generate zero norm states on the vacuum $| 0 \rangle \otimes \tilde{| 0 \rangle}$:

\begin{equation}
(0, 0) | \ell_\mu(x) \ell_\nu(y) | 0, 0 \rangle = (0, 0) | \tilde{\ell}_\mu(x) \tilde{\ell}_\nu(y) | 0, 0 \rangle = 0, \quad \forall (x, y).
\end{equation}

8 For notational simplicity, we have dropped the $l$ and $r$ subscripts in the decompositions $\phi(x) = \phi_l(x^+) - \phi_l(x^-)$ and $\phi_D(x) = \phi_r(x^+) + \phi_r(x^-)$.
Note that, since at $T = 0$ the fields $\phi$ ($\eta$) and $\tilde{\phi}$ ($\tilde{\eta}$) are independent
\begin{equation}
\langle 0, \bar{0} | \eta(x) \tilde{\eta}(y) | \bar{0}, 0 \rangle = 0, \quad \langle 0, \bar{0} | \phi(x) \tilde{\phi}(y) | \bar{0}, 0 \rangle = 0,
\end{equation}
we also have
\begin{equation}
\langle 0, \bar{0} | \ell_{\mu}(x) \tilde{\ell}_{\nu}(y) | \bar{0}, 0 \rangle = 0, \quad \forall (x, y),
\end{equation}
regardless of the sign of the metric quantization of the fields.

The free charge and pseudo charge operators are formally defined by
\begin{equation}
\tilde{\ell}_{\mu}(x) = \int j^\mu(x) \tilde{\phi}(x), \quad \tilde{Q}_{\mu}^5 = \int j_{\mu}^5(z) \tilde{\phi}(z),
\end{equation}
and similarly for $\tilde{\eta}_{\mu}(x)$ and $\tilde{Q}_{\mu}^5$. Using the equal-time commutation relations [15]
\begin{equation}
[\phi_D(x), \partial_0 \phi_D(y)] = i\delta(x^1 - y^1) = -[\tilde{\phi}_D(x), \partial_0 \tilde{\phi}_D(y)],
\end{equation}
one finds that $\tilde{\psi}^{(0)}$ and $\psi^{(0)}$ carry the same charge and chirality quantum numbers
\begin{equation}
[\tilde{Q}_{\mu}, \psi^{(0)}(x)] = -\psi^{(0)}(x), \quad [\tilde{Q}_{\mu}^5, \psi^{(0)}(x)] = -\gamma^5 \psi^{(0)}(x),
\end{equation}
with similar relations for $\tilde{Q}_f$ and $\tilde{Q}_f^5$. In thermofield dynamics, the above algebraic relations will be retained at finite temperature.

As in the standard model [2], the physical (gauge invariant) states $|\Phi\rangle$, $|\tilde{\Phi}\rangle$ are defined by the subsidiary conditions
\begin{equation}
\langle \Phi | \ell_{\mu}(x) | \Phi \rangle = 0, \quad \langle \tilde{\Phi} | \tilde{\ell}_{\mu}(x) | \tilde{\Phi} \rangle = 0.
\end{equation}
On the physical states, Maxwell’s equations are satisfied in the weak sense:
\begin{equation}
\langle \Phi | (\partial_\mu F^{\mu\nu}(x) + e J^{\nu}(x)) | \Phi \rangle = 0,
\end{equation}
with a similar equation for the tilde fields.

The physical Hilbert subspace $\mathcal{H}_\text{phys}$ is obtained by applying on the Fock vacuum $|0, \bar{0}\rangle = |0\rangle \otimes |0\rangle$ Wightman polynomials of operators $\mathcal{O}$ that are strictly gauge invariant [24], i.e.
\begin{equation}
[\mathcal{O}, \ell_{\mu}(x)] = 0, \quad [\mathcal{O}, \tilde{\ell}_{\mu}(x)] = 0.
\end{equation}
At $T = 0$ the set of operators $\mathcal{O}$ and $\tilde{\mathcal{O}}$ are independent, and a general gauge invariant state is given by
\begin{equation}
|\Phi, \Psi\rangle = |\tilde{\Phi}\rangle \otimes |\Psi\rangle = \tilde{\Phi}(\bar{0}) \otimes \Psi(0).
\end{equation}
Using the gauge freedom of the theory, one may consider the set of gauge invariant field operators $\{\Psi, \tilde{\Psi}, A_\mu, \tilde{A}_\mu\}$, related to the set of fields $\{\psi, \tilde{\psi}, A_\mu, \tilde{A}_\mu\}$, by the operator-valued gauge transformations [2, 24] (the gauge $\sqrt{\gamma}$ of [2])
\begin{equation}
\Psi(x) = :\psi(x)e^{i\sqrt{\gamma}\eta_0(x)}: \left( \frac{\mu}{2\pi} \right)^{\frac{1}{2}} :e^{i\sqrt{\gamma}y^5\Sigma(x)} : \sigma(x),
\end{equation}
\begin{equation}
A_\mu(x) = \tilde{A}_\mu(x) + \frac{s}{e} \partial_\mu \eta_0(x) = -s\frac{\sqrt{\gamma}}{e} \epsilon_{\mu\nu} \partial_\nu \Sigma(x),
\end{equation}
where we have defined the dimensionless operator [2]
\begin{equation}
\sigma_\mu(x) = e^{2i\sqrt{\gamma} \psi(x^5)} = \left( \frac{\mu}{\bar{\mu}} \right)^{\frac{1}{2}} e^{i\sqrt{\gamma}(\gamma_\mu \psi(x^5) + \gamma_{\bar{\mu}}(x))} = \left( \frac{\mu}{\bar{\mu}} \right)^{\frac{1}{2}} :e^{2i\sqrt{\gamma} \psi(x^5)},
\end{equation}
with an analogous expression for \( \hat{\sigma}_a \). Here \( x^\pm \) refers to the distinct chirality labels of the \( \sigma_a \) fields, \( \sigma_1 = \sigma(x^+) \) and \( \sigma_2 = \sigma(x^-) \). Note that for \( T = 0 \) all tilde-operators just play the role of 'spectators'. Analogous to the standard Schwinger model, the operators \( \sigma \) and \( \tilde{\sigma} \) are constant operators that commute among themselves and with all gauge invariant operators [2]. The only reason for \( \sigma \) and \( \tilde{\sigma} \) not being the identity operator on the Hilbert space is that they carry a selection rule corresponding to the charge and the chirality associated with each free fermion field \( \psi(0) \) and \( \tilde{\psi}(0) \),

\[
[Q_f, \sigma] = -\sigma, \quad [Q_5, \sigma] = -\gamma^5 \sigma,
\]

with similar relations for the tilde operators.

2.1. Superselection rule and vacuum structure

In order to display in the zero-temperature case the role played by the superselection rules carried by the Wick-ordered exponentials of the free Bose fields \( (\phi, \tilde{\phi}) \) in (2.4) and (2.5), let us compute the general Wightman functions of the operators \( \sigma \) and \( \tilde{\sigma} \). To this end we make use of the two-point functions (2.7) and (2.8), implying for the fields \( \psi \) and \( \tilde{\psi} \) in (2.10) the constant correlation functions

\[
\langle 0, \tilde{0} | \psi(x) \bar{\psi}(y) | 0, \tilde{0} \rangle = -\frac{1}{4\pi} \ln \frac{\mu}{\mu}, \quad (2.16)
\]

\[
\langle 0, \tilde{0} | \tilde{\psi}(x) \bar{\psi}(y) | 0, \tilde{0} \rangle = 0. \quad (2.17)
\]

Hence the operators \( \sigma \) and \( \tilde{\sigma} \) generate constant Wightman functions. Since at \( T = 0 \) the fields \( \psi \) and \( \tilde{\psi} \) are independent, as expressed by (2.17), there are no correlations between the fields \( s \) and \( \tilde{s} \), and a general Wightman function factorizes into a direct product. Using (2.16), as a consequence of the bare charge and chirality selection rules carried by the Wick-ordered exponentials of the free massless fields \( \phi(x) \) and \( \tilde{\phi}(x) \), the Wightman functions are given by [2]

\[
\langle 0 | \sigma_1(x_1) \cdots \sigma_1(x_m) \sigma_2(y_1) \cdots \sigma_2(y_{m_2}) | 0 \rangle = \lim_{\mu \to 0} \left( \frac{\mu}{\mu} \right)^{\frac{1}{2}(m_1m_2 + m_2m_1)} \delta_{m_1-m_2,0}, \quad (2.18)
\]

with similar relations for the tilde operators.

For each spinor field component \( \sigma_a \), the total bare charge \( q_a \) and chirality \( q_a^5 \) quantum numbers carried by the Wightman functions above are given by

\[
q_a = m_a - n_a, \quad q_a^5 = \gamma^5 (m_a - n_a). \quad (2.19)
\]

In the zero-temperature case, the selection rules require two independent conservation laws, implying that the only non-zero, cutoff-independent Wightman functions are those with \( q_a = 0 \) and \( \tilde{q}_a = 0 \). The cluster property is violated independently for each of the Hilbert spaces \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \). The Hilbert spaces \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \) generated cyclically from the Fock vacua \( |0\rangle \) and \( |\tilde{0}\rangle \) will contain an infinite number of vacuum states [2]. As in the case of the usual Schwinger model [2], the \( s \) and \( \tilde{s} \) form an Abelian algebra of constant unitary operators, \( \sigma_a(x) = \sigma_a(0) = \sigma_a \), \( \tilde{\sigma}_a(x) = \tilde{\sigma}_a(0) = \tilde{\sigma}_a \), which merely carry the selection rules (2.18) and its tilde conjugate, and imply the existence of an infinite number of mutually orthogonal vacua generated by repeated application of both \( \sigma_a \) and \( \tilde{\sigma}_a \) to the Fock vacuum \( |0, \tilde{0}\rangle \equiv |0\rangle \otimes |\tilde{0}\rangle \equiv |0_1, \tilde{0}_2\rangle \otimes |0_2, \tilde{0}_1\rangle \) (1 and 2, stand for the left- and right-chiral components, respectively),

\[
|\vec{n}_1, \vec{n}_2; n_1, n_2\rangle = \sigma_1^{n_1} \sigma_2^{n_2} |0, \tilde{0}\rangle \otimes \sigma_1^{n_1} \sigma_2^{n_2} |0, 0\rangle = |\vec{n}_1, \vec{n}_2\rangle \otimes |n_2, n_2\rangle. \quad (2.20)
\]
satisfying $H |\tilde{n}_1, \tilde{n}_2; n_1, n_2 \rangle = \tilde{H} |\tilde{n}_1, \tilde{n}_2; n_1, n_2 \rangle = 0$. Since in the zero-temperature case the Hilbert space factorizes, one has

$$
|0, 0\rangle \sigma^{n, n^\prime} \sigma^n \sigma^{n^\prime} |0, 0\rangle = \langle 0| \sigma^{n, n^\prime} \sigma^n \sigma^{n^\prime} |0\rangle = \lim_{\mu \to 0^+} \left( \frac{\mu}{\mu} \right)^{1/2} |(\theta - \tilde{\theta})^2 + |n - n^\prime|^2 \rangle \tag{2.21}
$$

for each chirality. In the limit $\mu \to 0$, this result will be cutoff independent provided the selection rules $n - n^\prime = 0$ and $\tilde{n} - \tilde{n}^\prime = 0$ are satisfied. In particular, one has $|0, 0\rangle \sigma \tilde{\sigma} |0, 0\rangle = 0$.

As in the standard treatment of the $T = 0$ model, the observables commute with the fields $\sigma$ and $\tilde{\sigma}$ for all spacetime separations. Following [2], we can introduce a set of irreducible vacuum states with respect to which the observables are diagonal, as follows:

$$
|\theta_1, \theta_2; \tilde{\theta}_1, \tilde{\theta}_2\rangle = \frac{1}{2\pi^2} \sum_{n, n^\prime = -\infty}^{+\infty} e^{-in\theta_1} e^{-in^\prime \theta_1} e^{in\theta_2} e^{in^\prime \theta_2} |n, n^\prime; \theta_1, \theta_2, \tilde{\theta}_1, \tilde{\theta}_2\rangle, \tag{2.22}
$$

such that the $\sigma$ and $\tilde{\sigma}$ fields turn out to be ‘spurionized’ [4]

$$
\sigma_n |\theta_1, \theta_2; \tilde{\theta}_1, \tilde{\theta}_2\rangle = e^{in\theta_1} |\theta_1, \theta_2; \tilde{\theta}_1, \tilde{\theta}_2\rangle, \tag{2.23}
$$

In order to simplify the notation we shall write $\theta = \{\theta_1, \theta_2\}$, $\tilde{\theta} = \{\tilde{\theta}_1, \tilde{\theta}_2\}$, $|n, n^\prime; \theta_1, \theta_2, \tilde{\theta}_1, \tilde{\theta}_2\rangle = |n\rangle$, etc. As a consequence of the two independent selection rules (2.19) and its tilde-conjugate, the decoupling between distinct $\theta$-sectors is complete ($\delta \langle \theta | \delta (\theta^\dagger)$):

$$
|\theta', \tilde{\theta}'\rangle |\theta, \tilde{\theta}\rangle = \frac{1}{(2\pi)^4} \sum_{n, n^\prime} \sum_{n, n^\prime} e^{-i\theta_n^\prime - \theta_{n^\prime}} e^{i\tilde{\theta}_{n^\prime} - \tilde{\theta}_n} \delta_n n_n^\prime = \delta (\theta' - \theta) \delta (\tilde{\theta}' - \tilde{\theta}),
$$

such that we have two independent theta vacua sectors, $|\theta; \tilde{\theta}\rangle = |\theta\rangle \otimes |\tilde{\theta}\rangle$.

The irreducible representations of the field algebras $A$ and $\tilde{A}$ are obtained by defining averaged vacuum expectation values of arbitrary Wightman polynomials $\mathcal{P} = \mathcal{O} \otimes \tilde{\mathcal{O}}$ of gauge invariant operators [2] by

$$
\langle \mathcal{P} \rangle_{\theta, \tilde{\theta}} \equiv \int_0^{2\pi} d^2 \theta' \int_0^{2\pi} d^2 \tilde{\theta} \langle \theta' | \mathcal{P} | \tilde{\theta} \rangle, \tag{2.24}
$$

In this $\theta$-vacuum representation, one has

$$
\langle \sigma_n \tilde{\sigma}^\dagger \rangle_{\theta, \tilde{\theta}} = \langle \sigma_n \rangle_{\theta} \otimes \langle \tilde{\sigma}^\dagger \rangle_{\tilde{\theta}} = e^{in\theta_n},
$$

displaying the spontaneous symmetry breaking in the physical Hilbert space $\mathcal{H}_{\text{phys}} \otimes \tilde{\mathcal{H}}_{\text{phys}}$.

### 3. QED$_2$ at finite temperature

The above considerations have set the stage for introducing the temperature dependence. To this end we depart from the operator solution for the combined doubled system at $T = 0$, as given in terms of the independent free Bose fields $\Sigma, \eta, \phi$, and their corresponding tilde fields. For $T \neq 0$, the statistical–mechanical ensemble averages are then given by the expectation values of observables in a temperature-dependent vacuum, $|0(\beta)\rangle$ [17, 19, 20], obtained via a Bogoliubov transformation from the $T = 0$ Fock vacuum $|0, 0\rangle$. These thermal averages factorize into products of independent thermal averages corresponding to the distinct free
Bose field sectors. Let us thus introduce the creation and annihilation operators, acting on the Fock vacuum, \((a, a^\dagger, \tilde{a}, \tilde{a}^\dagger, b, b^\dagger, \tilde{b}, \tilde{b}^\dagger, c, c^\dagger, \tilde{c})\) for the fields \(\Sigma, \tilde{\Sigma}, \phi, \tilde{\phi}, \eta, \tilde{\eta}\) and \(\tilde{\eta}\), respectively. The thermal vacuum is then defined in terms of independent Bogoliubov transformations as follows:

\[
|0(\beta)\rangle = U(\beta)|\tilde{0}, 0\rangle = U_\Sigma[\vartheta(\beta)]U_\phi[\vartheta(\beta)]U_\eta[\vartheta(\beta)]|\tilde{0}, 0\rangle,
\]

where the unitary operators above are given in terms of the creation and annihilation operators of the fields \((\Sigma, \tilde{\Sigma})\), \((\phi, \tilde{\phi})\) and \((\eta, \tilde{\eta})\),

\[
U_\Sigma[\vartheta(\beta)] = e^{\int_{-\infty}^{\infty} dp \left( \alpha(p)^\dagger \tilde{\alpha}(p) - \alpha(p) \tilde{\alpha}^\dagger(p) \right) \Delta_\Sigma(p^0; \beta)},
\]

with \(p^0 = \sqrt{p_1^2 + m^2}\), and

\[
U_\phi[\vartheta(\beta)] = e^{\int_{-\infty}^{\infty} dp \left( \beta(p)^\dagger \tilde{\beta}(p) - \beta(p) \tilde{\beta}^\dagger(p) \right) \Delta_\phi(p^1; \beta)},
\]

\[
U_\eta[\vartheta(\beta)] = e^{\int_{-\infty}^{\infty} dp \left( \eta(p)^\dagger \tilde{\eta}(p) - \eta(p) \tilde{\eta}^\dagger(p) \right) \Delta_\eta(p^1; \beta)}.
\]

The minus sign in the exponential in (3.1) is due to indefinite metric quantization for the fields \(\eta\) and \(\tilde{\eta}\) [17],

\[c(p^1), c^\dagger(k^1) = -\delta(p^1 - k^1).\]

The Bogoliubov parameters are implicitly defined by [6, 19, 20]

\[
\sinh \vartheta_\Sigma(p^0; \beta) = \frac{e^{-\frac{\vartheta_\Sigma}{2} p^0} - e^{\frac{\vartheta_\Sigma}{2} p^0}}{\sqrt{1 - e^{-\vartheta_\Sigma p^0}}}, \quad \cosh \vartheta_\Sigma(p^0; \beta) = e^{\frac{\vartheta_\Sigma}{2} p^0} \sinh \vartheta_\Sigma(p^0; \beta),
\]

\[
\sinh \vartheta(p^1; \beta) = \frac{e^{-\frac{\vartheta(p^1)}{2} |p^1|} - e^{\frac{\vartheta(p^1)}{2} |p^1|}}{\sqrt{1 - e^{-\vartheta(p^1) |p^1|}}}, \quad \cosh \vartheta(p^1; \beta) = e^{\frac{\vartheta(p^1)}{2} |p^1|} \sinh \vartheta(p^1; \beta),
\]

and the corresponding Bose–Einstein statistical weights are given by

\[
N_\Sigma(p^0; \beta) = \sinh^2 \vartheta_\Sigma(p^0; \beta) = \frac{1}{e^{\vartheta_\Sigma p^0} - 1},
\]

\[
N(p^1; \beta) = \sinh^2 \vartheta(p^1; \beta) = \frac{1}{e^{\vartheta(p^1) |p^1|} - 1}.
\]

For any free bosonic field, the corresponding Bogoliubov transformed annihilation operators \(e(p^1) = \{a, b, c\}(p^1)\) are given by [6, 19, 20]

\[
e(p^1; \beta) = U(-\vartheta(\beta))e(p^1)U(\vartheta(\beta)) = e(p^1) \cosh \vartheta(p^0; \beta) - \tilde{e}(p^1) \sinh \vartheta(p^0; \beta),
\]

(3.2)

\[
\tilde{e}(p^1; \beta) = U(-\vartheta(\beta))\tilde{e}(p^1)U(\vartheta(\beta)) = \tilde{e}(p^1) \cosh \vartheta(p^0; \beta) - e^\dagger(p^1) \sinh \vartheta(p^0; \beta).
\]

(3.3)

3.1. Gauge invariant thermal averages

As in the zero-temperature case, in the extension of thermofield dynamics formalism to gauge theories [17], the physical content of the model at finite temperature lies in the gauge invariant subspace of states. The statistical–mechanical ensemble averages of physical relevance are those expressed as expectation values with respect to physical thermal states.

Let \(\Phi\) be the set of gauge invariant operators at \(T = 0\), as defined by (2.12). At finite temperature a physical state is defined in terms of the temperature-dependent vacuum state \(|0(\beta)\rangle\) as

\[|\Phi(\beta)\rangle \equiv \Phi|0(\beta)\rangle = U(\beta)\Phi(\beta)|\tilde{0}, 0\rangle,\]
The physical (gauge invariant) Hilbert space $H_{\text{phys}}(\beta) = \{|\Phi(\beta)\rangle\}$ at finite temperature is now defined by the subsidiary conditions

\[
\langle \Phi'(\beta)| \ell_\mu(x; \beta) \Phi(\beta) \rangle = \langle 0, \tilde{\ell}_\mu(x) \Phi(\beta)|\Phi(\beta)\rangle = 0, \\
\langle \Phi'(\beta)| \tilde{\ell}_\mu(x) \Phi(\beta) \rangle = \langle 0, \tilde{\ell}_\mu(x)|\Phi(\beta)\rangle = 0, 
\]

where $\ell_\mu(x; \beta)$ and $\tilde{\ell}_\mu(x; \beta)$ are the zero norm thermal longitudinal currents

\[
\ell_\mu(x; \beta) = -\frac{1}{\sqrt{\pi}} \epsilon_{\mu\nu} \partial^\nu \varphi(x; \beta) = -\frac{1}{\sqrt{\pi}} \partial_\mu \varphi_{D}(x; \beta), \\
\tilde{\ell}(x; \beta) = \frac{1}{\sqrt{\pi}} \epsilon_{\mu\nu} \partial^\nu \varphi(x; \beta) = \frac{1}{\sqrt{\pi}} \partial_\mu \varphi_{D}(x; \beta),
\]

with

\[
\varphi(x; \beta) = \phi(x; \beta) + \eta(x; \beta), 
\]

and similarly for $\varphi_{D}$, $\tilde{\varphi}$ and $\tilde{\varphi}_{D}$. The thermal vector currents are

\[
\mathbf{J}_\mu(x; \beta) = -\frac{1}{\sqrt{\pi}} \epsilon_{\mu\nu} \partial^\nu \varphi(x; \beta) + \ell_\mu(x; \beta), \\
\tilde{\mathbf{J}}_\mu(x; \beta) = \frac{1}{\sqrt{\pi}} \epsilon_{\mu\nu} \partial^\nu \varphi(x; \beta) + \tilde{\ell}_\mu(x; \beta),
\]

and the Maxwell equations are satisfied in the weak form

\[
\langle \Phi'(\beta)|(\tilde{\partial}_\mu \mathcal{F}^{\mu\nu}(x) + \epsilon^{\mu\nu}\mathcal{J}^\nu(x))|\Phi(\beta)\rangle = 0, \\
\langle \Phi'(\beta)|(\tilde{\partial}_\mu \mathcal{F}^{\mu\nu}(x) + \epsilon^{\mu\nu}\tilde{\mathcal{J}}^\nu(x))|\Phi(\beta)\rangle = 0,
\]

with

\[
\left(\Box + \frac{\pi^2}{\pi}\right) \Sigma(x; \beta) = \left(\Box + \frac{\pi^2}{\pi}\right) \tilde{\Sigma}(x; \beta) = 0.
\]

The two-point functions of the free massive thermofields are discussed in detail in the appendix.

The field combination (3.4) generates constant correlation functions. The two-point functions of the free massless thermofields are given by [14, 15]

\[
\langle 0, \tilde{\ell}_\mu(x^\pm; \beta)|\phi(0; \beta)\rangle = D^{(+)}(x^\pm; \beta) = D^{(+)}(x^\pm; \beta) \\
= -\frac{1}{4\pi} \ln \left\{ \frac{i\mu}{\beta} \sinh \frac{\pi}{\beta} (x^\pm - i\epsilon) \right\} + \frac{1}{2\pi} z(\beta, \mu'), \\
\langle 0, \tilde{\ell}_\mu(x^\pm; \beta)|\tilde{\varphi}(0; \beta)\rangle = 0 = -\frac{1}{4\pi} \ln \left\{ \frac{i\mu}{\beta} \sinh \frac{\pi}{\beta} (x^\pm + i\epsilon) \right\} + \frac{i}{4} + \frac{1}{2\pi} z(\beta, \mu'), \\
\langle 0, \tilde{\varphi}(x^\pm; \beta)|\tilde{\eta}(0; \beta)\rangle = 0 = \frac{1}{4\pi} \ln \left\{ \frac{i\bar{\mu}}{\beta} \sinh \frac{\pi}{\beta} (x^\pm - i\epsilon) \right\} - \frac{1}{2\pi} z(\beta, \bar{\mu'}), \\
\langle 0, \tilde{\varphi}(x^\pm; \beta)|\tilde{\eta}(0; \beta)\rangle = 0 = \frac{1}{4\pi} \ln \left\{ \frac{i\bar{\mu}}{\beta} \sinh \frac{\pi}{\beta} (x^\pm + i\epsilon) \right\} + \frac{i}{4} - \frac{1}{2\pi} z(\beta, \bar{\mu'}). 
\]

For the off-diagonal two-point functions, one has

\[
\langle 0, \tilde{\varphi}(x^\pm; \beta)|\phi(0; \beta)\rangle = -D^{(+)}(x^\pm - i\frac{\beta}{2}; \beta), \\
\langle 0, \tilde{\eta}(x^\pm; \beta)|\eta(0; \beta)\rangle = 0 = -\langle 0, \tilde{\varphi}(x^\pm - i\frac{\beta}{2}; \beta)\rangle \eta(0; \beta)(0, 0) 
\]

(3.5)
where $\mu'$ and $\bar{\mu}'$ are infrared cutoffs, and $z(\beta \mu')$ is the mean number of massless particles having momenta in the range $[\mu', \infty]$ (the same for $z(\beta \bar{\mu}')$)

\[
z(\beta \mu') = \int_{\mu'}^{\infty} \frac{dp}{p} N(p, \beta).
\]

(3.6)

In this way one has for the field combinations $\phi$ and $\bar{\phi}$, acting as potentials for the longitudinal currents, the following space- and time-independent thermal two-point functions:

\[
\langle 0(\beta) | \psi(x) \psi(y) | 0(\beta) \rangle = \langle 0(\beta) | \tilde{\psi}(x) \bar{\psi}(y) | 0(\beta) \rangle = -\frac{1}{4\pi} \ln \left( \frac{\mu}{\bar{\mu}} e^{-2(z(\beta \mu') - z(\beta \bar{\mu}'))} \right),
\]

(3.7)

\[
\langle 0(\beta) | \tilde{\psi}(x) \-bar{\psi}(y) | 0(\beta) \rangle = +\frac{1}{4\pi} \ln \left( \frac{\mu}{\bar{\mu}} e^{-2(z(\beta \mu') - z(\beta \bar{\mu}'))} \right).
\]

(3.8)

In contrast to the zero-temperature case (see equation (2.17)) at $T \neq 0$ the fields $\phi$ and $\bar{\phi}$ exhibit a non-zero, constant, crossed correlation function, as expressed in (3.8). As we shall see, this will play a fundamental role in the thermal selection rule, and is a consequence of property (3.5) also displayed by the ‘real-time’ formalism of Umezawa et al. We have again

\[
\langle 0, \bar{0} | \ell_\mu(x; \beta) \ell_\nu(y; \beta) | 0, \bar{0} \rangle = \langle 0, \bar{0} | \tilde{\ell}_\mu(x; \beta) \tilde{\ell}_\nu(y; \beta) | 0, \bar{0} \rangle = 0, \quad \forall (x, y),
\]

\[
\langle 0, \bar{0} | \ell_\mu(x; \beta) \bar{\ell}_\nu(y; \beta) | 0, \bar{0} \rangle = 0, \quad \forall (x, y).
\]

3.2. Thermal selection rule

In order to discuss the thermal selection rule, we shall consider the general thermal correlation functions of the gauge invariant field operators $\Psi$ and $\bar{\Psi}$ defined in section 2. Since these fields are given in terms of Wick-ordered exponentials of massive and massless free scalar fields, one should review some aspects of the thermodfield bosonization introduced in [14, 15]. The Wick-ordered thermofield exponentials of the fields $\phi$, $\eta$ and $\Sigma$ are obtained from the corresponding exponentials at zero temperature as follows (the same for the corresponding tilde fields):

\[
\mathcal{U}[-\vartheta(\beta)] e^{i\phi(x)} ; \mathcal{U}[\vartheta(\beta)] = e^{-\frac{i}{4\pi}z(\beta \mu') z(\beta \bar{\mu}')},
\]

\[
\mathcal{U}[-\vartheta(\beta)] e^{i\eta(x)} ; \mathcal{U}[\vartheta(\beta)] = e^{i\frac{z(\beta \mu') z(\beta \bar{\mu}')}{}},
\]

\[
\mathcal{U}[-\vartheta \Sigma(\beta)] e^{i\Sigma(x)} ; \mathcal{U}[\vartheta(\beta)] = e^{-\frac{i}{\pi} z(\beta m)} e^{i\frac{1}{\pi} \Sigma(x; \beta)},
\]

(3.9)

where $z(\beta \mu')$ ($z(\beta \bar{\mu}')$) is defined in (3.6), $z_\gamma(\beta m)$ is the mean number of particles with mass $m$ having momenta $p^1 \equiv p$ in the range $[0, \infty]$, and the Wick-ordered exponentials of the free thermodfields are defined by [14, 15]

\[
\mathcal{U}^\psi(x; \beta) = e^{i\bar{\Sigma}(x; \beta)} e^{i\psi(x; \beta)},
\]

where the ordering is with respect to the zero-temperature creation and annihilation operators. Taking this into account, one finds, from (2.13) for the gauge invariant thermodfields and their tilde conjugate,

\[
\Psi_\alpha(x; \beta) = \mathcal{U}[-\vartheta(\beta)] \Psi(x) \mathcal{U}[\vartheta(\beta)] = \left( \frac{N(Z_\Sigma(\beta m))}{2\pi} \right)^{\frac{1}{4}} e^{i\vartheta \Sigma(x; \beta)} \sigma_\alpha(x; \beta),
\]

\[
\bar{\Psi}_\alpha(x; \beta) = \mathcal{U}[-\vartheta(\beta)] \bar{\Psi}(x) \mathcal{U}[\vartheta(\beta)] = \left( \frac{N(Z_\Sigma(\beta m))}{2\pi} \right)^{\frac{1}{4}} e^{i\vartheta \Sigma(x; \beta)} \bar{\sigma}(x; \beta),
\]

(3.10)
where we have defined
\[ Z_\sigma(\beta m) = e^{-\pi \mu (\beta m)}. \]

The thermal operators \( \sigma(x; \beta) \) and \( \tilde{\sigma}(x; \beta) \), generalizing to finite temperature those introduced in [2], are
\[
\begin{align*}
\sigma_\alpha(x; \beta) &= e^{2i\sqrt{\mu} \phi(x; \beta)} = \left( \frac{\mu}{\beta} e^{-2i(\beta \mu' - \beta (\beta \mu'))} \right)^{\frac{1}{2}} e^{2i\sqrt{\mu} \phi(x; \beta)}, \\
\tilde{\sigma}_\alpha(x; \beta) &= e^{2i\sqrt{\mu} \phi(x; \beta)} = \left( \frac{\mu}{\beta} e^{-2i(\beta \mu' - \beta (\beta \mu'))} \right)^{\frac{1}{2}} e^{2i\sqrt{\mu} \phi(x; \beta)},
\end{align*}
\]

where \( x^+(x^-) \) goes with \( \sigma_\alpha(\sigma_\alpha) \). The general thermal correlation function of the gauge invariant field operators \( \Psi \) and \( \tilde{\Psi} \) is then
\[
\langle 0(\beta) | \prod_{i=1}^n \Psi(x_i) \prod_{j=1}^n \tilde{\Psi}(\tilde{x}_j) \prod_{k=1}^{n'} \Psi^\dagger(y_k) \prod_{\ell=1}^{n'} \tilde{\Psi}^\dagger(\tilde{y}_\ell) | 0(\beta) \rangle = \left( \frac{\pi}{2\pi} Z_\sigma(\beta m) \right)^{\frac{1}{2} (n' + n \delta(n' + n))} \times e^{-\pi F_\Sigma(x, y; \beta)} e^{-\pi H_\Sigma(x, \tilde{x}, y, \tilde{y}; \beta)} \langle 0| \prod_{i=1}^n \sigma_\alpha_i(x_i; \beta) \prod_{j=1}^{n'} \tilde{\sigma}_\alpha_j(y_j; \beta) \tilde{\sigma}_\alpha_j(\tilde{y}_j; \beta) | 0, 0 \rangle.
\]

The functions \( F_\Sigma \) and \( H_\Sigma \) collect the spacetime-dependent contributions of the two-point functions of the massive thermofields, \( \Sigma \) and \( \tilde{\Sigma} \). These functions, \( F_\Sigma \) and \( H_\Sigma \), generalize to finite temperature the gauge invariant subspace of the fermion Wightman function at zero temperature obtained by Schwinger [1, 2] and are given by
\[
F_\Sigma(x, y; \beta) = \sum_{i < i'} \gamma_\xi_i^5 \gamma_\gamma_i^5 \Delta^{(s)}(x_i - x_i'; m, \beta) + \sum_{k < k'} \gamma_\xi_k^5 \gamma_\gamma_k^5 \Delta^{(s)}(y_k - y_k'; m, \beta)
\]
\[
+ \sum_{i=1}^n \sum_{k=1}^{n'} \gamma_\xi_i^5 \gamma_\gamma_i^5 \Delta^{(s)}(x_i - y_k; m, \beta)
\]
and
\[
H_\Sigma(x, \tilde{x}, y, \tilde{y}; \beta) = \sum_{i < i'} \gamma_\tilde{\xi}_i^5 \gamma_\tilde{\gamma}_i^5 \Delta^{(s)}(\tilde{x}_i - \tilde{x}_i'; m, \beta) + \sum_{k < k'} \gamma_\tilde{\xi}_k^5 \gamma_\tilde{\gamma}_k^5 \Delta^{(s)}(\tilde{y}_k - \tilde{y}_k'; m, \beta)
\]
\[
+ \sum_{i=1}^n \sum_{k=1}^{n'} \gamma_\tilde{\xi}_i^5 \gamma_\tilde{\gamma}_i^5 \Delta^{(s)}(\tilde{x}_i - y_k; m, \beta) - \sum_{i=1}^n \sum_{k=1}^{n'} \gamma_\tilde{\xi}_i^5 \gamma_\tilde{\gamma}_i^5 \Delta^{(s)}(\tilde{x}_i - \tilde{y}_k; m, \beta)
\]
\[
- \sum_{j=1}^n \sum_{k=1}^{n'} \gamma_\xi_j^5 \gamma_\gamma_j^5 \Delta^{(s)}(x_j - y_k; m, \beta)
\]
\[
+ \sum_{j=1}^n \sum_{k=1}^{n'} \gamma_\tilde{\xi}_j^5 \gamma_\tilde{\gamma}_j^5 \Delta^{(s)}(\tilde{x}_j - \tilde{y}_k; m, \beta) - \sum_{j=1}^n \sum_{k=1}^{n'} \gamma_\xi_j^5 \gamma_\gamma_j^5 \Delta^{(s)}(x_j - \tilde{y}_k; m, \beta),
\]

where (see the appendix)
\[
\begin{align*}
\langle 0, 0 | \Sigma(x; \beta) \Sigma(y; \beta) | 0, 0 \rangle &= 0, \\
\langle 0, 0 | \tilde{\Sigma}(x; \beta) \tilde{\Sigma}(y; \beta) | 0, 0 \rangle &= \Delta^{(s)}(x - y; m, \beta), \\
\langle 0, 0 | \tilde{\Sigma}(x; \beta) \Sigma(\tilde{y}; \beta) | 0, 0 \rangle &= -\Delta^{(s)}(\tilde{x} - y; m, \beta),
\end{align*}
\]

(3.13)
with
\[ x = \left( x^0 - \frac{i\beta}{2} x^1 \right). \]

Note that equation (3.13) is in agreement with Umezawa’s view of the thermal ‘off-diagonal’ functions as an analytic continuation of the ‘diagonal’ ones \[ [19, 20] \] and, as we show in the appendix, is a general structural property of the thermofield dynamics formalism.

The thermal operators \( \sigma(x; \beta) \) and \( \tilde{\sigma}(x; \beta) \) generate constant correlation functions,
\[
\langle 0, \tilde{0} | \prod_{i=1}^{\ell'} \sigma_{\alpha_i}^\dagger(\mathbf{x}_i; \beta) \prod_{k=1}^{n'} \sigma_{\alpha_k}(\mathbf{y}_k; \beta) \tilde{\sigma}_{\alpha_j}(\mathbf{z}_j; \beta) \tilde{0}, 0 \rangle \\
= \langle 0, \tilde{0} | \sigma_{\alpha_1}^\dagger(\beta) \tilde{\sigma}_{\alpha_1}(\beta) \sigma_{\alpha_2}^\dagger(\beta) \tilde{\sigma}_{\alpha_2}(\beta) \ldots \sigma_{\alpha_n}(\beta) \tilde{\sigma}_{\alpha_n}(\beta) \tilde{0}, 0 \rangle \\
\times \langle 0, \tilde{0} | \sigma_{\alpha_1}^\dagger(\beta) \tilde{\sigma}_{\alpha_1}(\beta) \sigma_{\alpha_{n+1}}^\dagger(\beta) \tilde{\sigma}_{\alpha_{n+1}}(\beta) \ldots \tilde{\sigma}_{\alpha_{n'}}(\beta) \sigma_{\alpha_{n'}}(\beta) \tilde{0}, 0 \rangle,
\]
where the expectation value has been factorized into its positive and negative chirality parts, and \( \sigma_{\alpha}(\beta) \equiv \sigma_{\alpha}(0, \beta), \tilde{\sigma}_{\alpha}(\beta) \equiv \tilde{\sigma}_{\alpha}(0, \beta). \) As we now show, this expectation value implements the charge and chirality thermal selection rules. Using (3.11) and (3.12) we obtain for each chirality the thermal correlation function
\[
\langle 0, \tilde{0} | \sigma_{\alpha}^\dagger(\beta) \tilde{\sigma}_{\alpha}^\dagger(\beta) \sigma_{\alpha_{n'}}(\beta) \tilde{\sigma}_{\alpha_{n'}}(\beta) \tilde{0}, 0 \rangle = \left( \frac{\mu e^{-2z(\beta\mu')}}{\beta e^{-2z(\beta\mu')}} \right)^{\frac{1}{2}} |\langle n-n'\rangle-(\bar{n}-\bar{n}')|^2.
\]

Compare this expression with the zero-temperature result (2.21). The difference is a consequence of the ‘entanglement’ of the \( \sigma^\dagger(\beta) \) and \( \tilde{\sigma}^\dagger(\beta) \) field components (the corresponding two-point function does not vanish) resulting from the Bogoliubov transformation (3.2) and (3.3) which mixes untilded and tilded fields. In the limit \( \mu \to 0 \) the result will be different from zero and independent of the cutoffs \( \bar{\mu} \) and \( \bar{\mu}' \), if we require that
\[
n - n' = \bar{n} - \bar{n}'.
\]

By considering the free massless scalar thermofield theory as the zero mass limit of the free massive scalar thermofield theory, the regulator \( \mu \) of the zero-temperature two-point function should be identified with the infrared cutoff \( \mu' = \frac{\mu}{\bar{\mu}} [14, 15] \). Under this assumption, for a fixed temperature, the infrared singular asymptotic behavior of the integral (3.6) is given by
\[
\zeta(\beta\mu)_{n=0} = \frac{1}{\beta \mu} + \frac{1}{2} \ln(\beta \mu),
\]
and one has for the two-point function (3.7),
\[
\langle 0(\beta)|\varphi(x)\varphi(y)|0(\beta)\rangle = -\frac{1}{4\pi} \ln \left( \frac{\mu}{\beta \mu} e^{2z(\beta\mu')} \right).
\]
In this case the selection rule arises from the limit
\[
\langle 0, \tilde{0} | \sigma_{\alpha}^\dagger(\beta) \tilde{\sigma}_{\alpha}^\dagger(\beta) \sigma_{\alpha_{n'}}(\beta) \tilde{\sigma}_{\alpha_{n'}}(\beta) \tilde{0}, 0 \rangle = \lim_{\mu \to 0} \left( \frac{\mu}{\beta \mu} e^{2z(\beta\mu')} \right)^{\frac{1}{2}} |\langle n-n'\rangle-(\bar{n}-\bar{n}')|^2 = \delta_{\alpha_{n'}}, \delta_{\alpha_{n'}}.
\]
Without loss of generality, and for economy of arbitrary constants, one may choose \( \mu' = \bar{\mu}' \), such that the temperature-dependent infrared contributions cancel and one finds for (3.7) the constant correlation functions
\[
\langle 0(\beta)|\varphi(x)\varphi(y)|0(\beta)\rangle = -\langle 0(\beta)|\varphi(x)\varphi(y)|0(\beta)\rangle = -\frac{1}{4\pi} \ln \left( \frac{\mu}{\bar{\mu}} \right).
\]
In this case, one has
\[
\langle 0, \tilde{0} | \sigma_{\alpha}^\dagger(\beta) \tilde{\sigma}_{\alpha}^\dagger(\beta) \sigma_{\alpha_{n'}}(\beta) \tilde{\sigma}_{\alpha_{n'}}(\beta) \tilde{0}, 0 \rangle = \lim_{\mu \to 0} \left( \frac{\mu}{\bar{\mu}} \right)^{\frac{1}{2}} |\langle n-n'\rangle-(\bar{n}-\bar{n}')|^2 = \delta_{\alpha_{n'}}, \delta_{\alpha_{n'}}.
\]
Hence, for each chirality we obtain the thermal selection rule
\[ \langle 0, 0 | \sigma^\nu (\beta) \tilde{\sigma}^{\nu'} (\beta) \sigma^n (\beta) \tilde{\sigma}^h (\beta) | 0, 0 \rangle = \delta_{n, -n', \beta - \beta'}. \]  
(3.16)

We further have
\[ \sigma^h (\beta) = 1 = \tilde{\sigma}^h (\beta). \]

The thermal selection rule (3.16) has a natural interpretation. Since the algebraic relations at \( T = 0 \) are retained at finite temperature, the commutation relations (2.15) hold at finite temperature. The operators \( \sigma^h (\beta) \) and \( \tilde{\sigma}^h (\beta) \) carry the charge and chirality of the free fermion thermofield. However, in contrast to the zero-temperature case, the thermal vacuum state \( | 0(\beta) \rangle \) is neither an eigenstate of the free charge operator \( \hat{Q}_f \) nor \( \tilde{\hat{Q}}_f \). Nevertheless, for the total free charge of the combined system defined by
\[ \hat{Q}_f = Q_f - \tilde{Q}_f, \]
one finds
\[ [\hat{Q}_f, \sigma^h (\beta) \tilde{\sigma}^h (\beta)] = -(n_u - \bar{n}_u) \sigma^h (\beta) \tilde{\sigma}^h (\beta). \]

The bosonized expressions for the zero-temperature Hamiltonians \( H \) and \( \tilde{H} \) are given in terms of the free field Hamiltonians of the Bose fields \( (\Sigma, \bar{\Sigma}, \eta, \bar{\eta}, \phi, \bar{\phi}) \) as follows:
\[ H = H^{(0)}_{\Sigma} + H^{(0)}_{\eta} + H^{(0)}_{\phi}, \]
\[ \tilde{H} = \tilde{H}^{(0)}_{\Sigma} + H^{(0)}_{\eta} + H^{(0)}_{\phi}. \]

The thermal vacuum state \( | 0(\beta) \rangle \) is not an eigenstate of either \( H \) or \( \tilde{H} \) [6], and thus the thermal states,
\[ | n_u, \bar{n}_u; \beta \rangle = \sigma^h | 0(\beta) \rangle, \]
are not vacuum eigenstates of \( H \) or \( \tilde{H} \). This follows from the fact that, since \( H \) and \( \tilde{H} \) commute with \( \sigma \) and \( \tilde{\sigma} \), we get
\[ \hat{H} | n, \bar{n}; \beta \rangle = \sigma^h \tilde{\sigma}^h \hat{H} | 0(\beta) \rangle = \sigma^h \tilde{\sigma}^h \hat{U} (\beta) | 0(\beta) \rangle = \sigma^h \tilde{\sigma}^h \hat{H} | 0(\beta) \rangle = \sigma^h \tilde{\sigma}^h \hat{U} (\beta) | 0(\beta) \rangle = 0, \]
\[ \tilde{H} | n, \bar{n}; \beta \rangle = \sigma^h \tilde{\sigma}^h \tilde{H} | 0(\beta) \rangle = \sigma^h \tilde{\sigma}^h \hat{U} (\beta) \tilde{H} | 0(\beta) \rangle = 0, \]

However, \( | n, \bar{n}; \beta \rangle \) are eigenstates of the total Hamiltonian \( \tilde{H} = H - \tilde{H} \) of the combined system \( \tilde{H} (\beta) = \tilde{H} \).

\[ \hat{H} | n, \bar{n}; \beta \rangle = \sigma^h \tilde{\sigma}^h \hat{H} | 0(\beta) \rangle = \sigma^h \tilde{\sigma}^h \hat{U} (\beta) \hat{H} | 0(\beta) \rangle = 0, \]

since \( \tilde{H} (\beta) = \tilde{H} \). For the total bare charge of the combined system \( \hat{Q}_f \), one finds
\[ \hat{Q}_f | n, \bar{n}; \beta \rangle = \sigma^h n (\beta) | n, \bar{n}; \beta \rangle. \]

At finite temperature the free fermionic charge and chirality of the total combined system are condensed into the vacua states.

\[ \hat{U} (\beta) \left[ a^\dagger (p) a (p) - \bar{a}^\dagger (p) \bar{a} (p) \right] \hat{U}^{-1} (\beta) = a^\dagger (p) a (p) - \bar{a}^\dagger (p) \bar{a} (p). \]

\[ 10 \text{ This follows from the fact that} \]

\[ \hat{U} (\beta) \left[ a^\dagger (p) a (p) - \bar{a}^\dagger (p) \bar{a} (p) \right] \hat{U}^{-1} (\beta) = a^\dagger (p) a (p) - \bar{a}^\dagger (p) \bar{a} (p). \]
3.3. Thermal theta vacuum states

For \( T = 0 \) we have no correlation between the fields \( \sigma \) and \( \tilde{\sigma} \), and hence two independent selection rules for each chirality, corresponding to the conservation of the independent quantum numbers \( q_\alpha \) and \( \tilde{q}_\alpha \) carried by the vacua \( |n_\alpha\rangle \otimes |\tilde{n}_\alpha\rangle \). However, for \( T \neq 0 \) we have correlations between the fields \( \sigma(\beta) \) and \( \tilde{\sigma}(\beta) \), such that the total Hilbert space of thermal states does not factorize as a direct product. As we have seen only one selection rule emerges in this case for each chirality, corresponding to the conservation of the total quantum number \( \tilde{q}_\alpha = q_\alpha - \tilde{q}_\alpha \) of the combined system. Thus, one expects the existence of only one set of theta vacua, say \( |\theta_\beta\rangle \), in accordance with the tilde conjugation rule, and

\[
\langle 0|\beta_0\rangle|\sigma_\alpha(x)\tilde{\sigma}_\alpha(y)|0(\beta_0)\rangle = 1,
\]

independent of the vacuum representation.

Let us now perform the statistical ensemble average with respect to the thermal \( \theta \) vacuum. This is achieved by replacing the zero-temperature Fock vacuum \( |0,0\rangle \) in (22.22) by the thermal vacuum state \( |0(\beta)\rangle \). For notational simplicity, we shall use the notation introduced in section 2. One has

\[
|\theta, \tilde{\theta}; \beta\rangle \equiv \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{\infty} e^{-im\theta} \sum_{\tilde{n}=-\infty}^{\infty} e^{i\tilde{m}\tilde{\theta}} |n, \tilde{n}; \beta\rangle = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{\infty} e^{-im\theta} \sum_{\tilde{n}=-\infty}^{\infty} e^{i\tilde{m}\tilde{\theta}} |n, \tilde{n}; \beta\rangle.
\]

We still have the ‘spurionization’, equation (2.23),

\[
\sigma_\alpha|\theta, \tilde{\theta}; \beta\rangle = e^{i\tilde{m}_\alpha|\theta, \tilde{\theta}; \beta\rangle}, \quad \tilde{\sigma}_\alpha|\theta, \tilde{\theta}; \beta\rangle = e^{-im_\alpha|\theta, \tilde{\theta}; \beta\rangle}.
\]

Using the thermal selection rule (3.16), the orthonormalization condition among the thermal theta vacua becomes

\[
\langle \beta; \tilde{\beta}', \theta'|\theta, \tilde{\theta}; \beta\rangle = \frac{1}{(2\pi)^2} \sum_{n,n'} e^{-i(n\theta-n'\theta')} \sum_{\tilde{n},\tilde{n}'} e^{i(\tilde{n}\tilde{\theta}-\tilde{n}'\tilde{\theta}')} \delta_{n,-n'} \delta_{\tilde{n},-\tilde{n}'}.
\]

Defining \( n_\pm = n + n' \), \( \tilde{n}_\pm = \tilde{n} + \tilde{n}' \), the selection rule (3.16) now reads \( \delta_{n_- n_+} \), and we get

\[
\langle \beta; \tilde{\beta}', \theta'|\theta, \tilde{\theta}; \beta\rangle = \frac{1}{(2\pi)^2} \sum_{n_+} e^{-i(n_+\theta-n_-\theta')} \sum_{\tilde{n}_+} e^{i(\tilde{n}_+\tilde{\theta}-\tilde{n}_-\tilde{\theta}')} \sum_{n_-} e^{i(n_\theta-n_-\theta')} \sum_{\tilde{n}_-} e^{i(\tilde{n}_\tilde{\theta}-\tilde{n}_-\tilde{\theta}')} \delta_{n_+ n_-} \delta_{\tilde{n}_+ \tilde{n}_-}.
\]

(3.17)

Analogous to the \( T = 0 \) computation, the summation over \( n_+ \) and \( \tilde{n}_+ \) implements Dirac delta function constraints within each theta vacuum. The last summation in (3.17), obeying the thermal selection rule, owes its origin to the infinite number of states with the same eigenvalues of the total charge \( q_\alpha = n_\alpha - \tilde{n}_\alpha \) of the combined system, which have the same orthogonality relations upon thermalization and ensure the symmetry of the total combined system under the tilde conjugation operation. One finds

\[
\langle \beta; \tilde{\beta}', \theta'|\theta, \tilde{\theta}; \beta\rangle = (2\pi)^2 \delta^2(\theta - \tilde{\theta}) \delta^2(\tilde{\theta} - \tilde{\theta}') \delta^2(\theta - \theta').
\]

In this way we naturally lead to define the averaged thermal expectation values of observables with respect to the thermal theta-vacua states as follows:

\[
\langle \mathcal{O} \rangle_\beta \equiv \frac{1}{(2\pi)^2} \int_0^{2\pi} d\theta' \int_0^{2\pi} d\tilde{\theta}' \int_0^{2\pi} d\theta \int_0^{2\pi} d\tilde{\theta} \langle \beta; \tilde{\beta}', \theta'|\mathcal{O}|\theta, \tilde{\theta}; \beta\rangle.
\]

This result can be interpreted as saying that, within the thermofield dynamics, the thermalization selects just one theta vacuum parameter for each chirality out of the two,
\( \theta \) and \( \tilde{\theta} \), that label the zero-temperature theta vacua of the doubled system. This comes about as an intrinsic property of the model for the thermal vacuum states, which enable the existence of correlations only within the same thermal theta world. We have

\[
\langle \sigma_\alpha \rangle_\theta = e^{i\theta_\alpha}, \quad \langle \tilde{\sigma}_\alpha \rangle_\theta = e^{-i\theta_\alpha}.
\]

For a given theta-vacuum sector, and \( T \neq 0 \), \( \sigma_\alpha \) and \( \tilde{\sigma}_\alpha \) are the beta-independent \( c \)-numbers:

\[
\sigma_\alpha(\beta) = e^{i\theta_\alpha}, \quad \tilde{\sigma}_\alpha(\beta) = e^{-i\theta_\alpha},
\]

such that

\[
\sigma_\alpha(\beta)\tilde{\sigma}_\alpha(\beta) = 1.
\]

4. Chiral condensate at finite temperature and symmetry breaking

As is well known, some properties of a two-dimensional quantum field model are more transparent in the operator formulation, and others are better seen in the functional integral formulation. In order to emphasize the usefulness of the thermofield bosonization, we shall compute the expression for the chiral condensate and the corresponding high-temperature behavior. This enables one to obtain a statistical–mechanical interpretation for the high-temperature behavior of the chiral condensate, which is easily obtained within the thermofield bosonization approach.

In [5, 9, 10], using the functional integral formulation in Euclidean space, the high-temperature dependence of the chiral condensate in the Schwinger model has been shown to be given by [10]

\[
\langle \bar{\psi}\psi \rangle_\beta \sim \frac{1}{\beta} e^{-\frac{\pi}{\beta} \cos \theta}.
\]

The dependence on the \( \theta \)-vacuum parameter has been considered in [9] as an ad hoc assumption.

Let us consider the computation of the theta-vacuum expectation value of the chiral density \( \langle \langle \bar{\Psi}(x)\Psi(x) \rangle \rangle_\theta \). To this end we shall consider the mass operator defined by the point-split operator product at zero temperature. From (2.13) we have

\[
\langle \bar{\Psi}(x)\Psi(x) \rangle = \lim_{\epsilon \to 0} f^{-1}(\epsilon) \bar{\Psi}(x + \epsilon)\Psi(x) = \frac{\pi}{2\pi} \left( \sigma_1^\dagger \sigma_2 e^{2i\sqrt{\pi} \Sigma(x)} + \sigma_2^\dagger \sigma_1 e^{-2i\sqrt{\pi} \Sigma(x)} \right),
\]

where the renormalization constant \( f(\epsilon) \) is defined by

\[
f(\epsilon) = e^{-\pi \langle \langle 0|\Sigma(\epsilon)|0 \rangle \rangle_0}.
\]

Now, using (3.9) and (3.10) [14, 15], we get

\[
\langle \bar{\Psi}(x, \beta)\Psi(x, \beta) \rangle = U(\mu(\beta); \bar{\Psi}(x)\Psi(x)) U^{-1}(\mu(\beta))
\]

\[
= \frac{\pi}{2\pi} e^{-2\pi \zeta_{c}(\beta)m} \left( \sigma_1^\dagger(\beta)\sigma_2(\beta) : e^{2i\sqrt{\pi} \Sigma(x; \beta)} + \sigma_2^\dagger(\beta)\sigma_1(\beta) : e^{-2i\sqrt{\pi} \Sigma(x; \beta)} \right).
\]

With respect to the thermal theta vacuum, a straightforward computation leads to the following gauge-invariant expression for the chiral condensate:

\[
\langle \langle \bar{\Psi}(x)\Psi(x) \rangle \rangle_\theta = \frac{\pi}{\pi} e^{-2\pi \zeta_{c}(\beta)m} \cos(\theta_2 - \theta_1),
\]

(4.1)

exhibiting spontaneous chiral symmetry breaking.

At zero temperature, the chiral condensate behaves as

\[
\langle \langle \bar{\Psi}(x)\Psi(x) \rangle \rangle_\theta = \frac{\pi}{\pi} \cos(\theta_2 - \theta_1).
\]
From the statistical–mechanical point of view, one sees that the thermal chiral condensate is given in terms of the mean number of massive particles in the ensemble (see the appendix).

The chiral symmetry breaking at finite temperature emerges in accordance with the violation of the cluster decomposition property for the mass operator, which imply the existence of degenerate thermal vacua states. One finds

\[ \langle 0(\beta) | (\bar{\psi}(x + \lambda) \psi(x + \lambda) : ) (\bar{\psi}(x) \psi(x) : ) | 0(\beta) \rangle |_{\lambda \to \infty} \approx \left( \frac{\mu}{\pi} \right)^2 e^{-4\pi z_\chi(\beta m)}. \]  

(4.2)

In order to show that the chiral symmetry remains broken at all finite temperature, let us consider the high-temperature behavior of the chiral condensate. To this end let us consider the mean number of massive particles at temperature \( \beta \) (see the appendix):

\[ z_\chi(\beta m) = \frac{1}{\pi} \int_0^{\infty} \frac{dp}{\sqrt{p^2 + m^2}} \frac{1}{e^\beta \sqrt{p^2 + m^2} - 1} = \frac{1}{\pi} \sum_{k=1}^{\infty} K_0(k \beta m) \]

\[ = \frac{1}{2\pi} \ln \left( \frac{m \epsilon^\beta}{4\pi} \right) + \frac{1}{2\beta m} + \sum_{\ell=1}^{\infty} \left\{ \frac{1}{\sqrt{(\beta m)^2 + (2\ell \pi)^2}} - \frac{1}{2\ell \pi} \right\}. \]  

(4.3)

For high temperatures, the leading terms are

\[ z_\chi(\beta m) \approx \frac{1}{2\pi} \ln \left( \frac{m \epsilon^\beta}{4\pi} \right) + \frac{1}{2\beta m}. \]  

(4.4)

From (4.4) and (4.2) one has for high temperatures

\[ \langle 0(\beta) | (\bar{\psi}(x + \lambda) \psi(x + \lambda) : ) (\bar{\psi}(x) \psi(x) : ) | 0(\beta) \rangle |_{\lambda \to \infty} \approx \left( \frac{4\mu}{m \epsilon^\beta} \right)^2 (kT)^2 e^{-\frac{\mu}{m \epsilon^\beta} kT}, \]

which shows that, as expected, there is no critical temperature for which the cluster decomposition property is restored, the chiral symmetry remaining broken for all finite temperature. With respect to the \( \theta \) vacuum, the high-temperature behavior of the chiral condensate (4.1) is then given by

\[ \langle \bar{\psi}(x) \psi(x) : |^{\beta}_{\theta} \approx \left( \frac{4\mu}{m \epsilon^\beta} \right) (kT) e^{-\frac{\mu}{m \epsilon^\beta} kT} \cos (\theta_2 - \theta_1). \]  

(4.5)

5. Concluding remarks

We have considered the operator solution for quantum electrodynamics in two dimensions at finite temperature within the thermofield dynamics approach. Using the bosonization scheme developed in \cite{14, 15} we have obtained the generalization of the operator solution due to Lowenstein and Swieca \cite{2} at zero temperature to finite temperature. Our thermofield bosonization approach for the free fermion field is in agreement with the revised version of thermofield dynamics formalism for fermions due to Ojima \cite{17}, which plays a crucial role in obtaining consistency with the free fermion thermofield selection rule.

With respect to the physical content of the model a remark is in order: we have defined as physical states the set of gauge invariant states, regardless of whether they are tilde or untilde states. From the observable point of view, the fields that describe the physical system under consideration are the untilded gauge invariant operators obtained from \( \{ \bar{\psi}, \psi, A_\mu \} \). The statistical–mechanical thermal averages of these operators are those of physical relevance. However, the gauge invariant crossed thermal correlation functions (off-diagonal) involving both tilde and non-tilde fields participate as intermediary (virtual) processes in a perturbative
computation (as for instance, in the perturbative expansion in the fermion mass for the case of the massive Thirring model [15]).

Within the operator approach one easily shows that the chiral symmetry breaking at \( T = 0 \) persists for any finite temperature; the demonstration of this proved much simpler in this formalism as compared to the functional approach [5]. The existence of only one set of theta vacua characterizing the chiral symmetry breakdown emerges here as a byproduct of the thermal selection rule which ensures that the bare fermionic charge and chirality of the total combined system are screened.

Analogously to the \( T = 0 \) case [2], at \( T \neq 0 \) the algebra of observables is isomorphic to the algebra of the massive thermofield \( \Sigma(x; \beta) \). Since it is a free field, one is able to compute the corresponding two-point function (see the appendix) and the model turns out to be exactly solvable. Since the equal-time properties of commutators remain unaffected by the unitary transformation taking one from \( T = 0 \) to \( T \neq 0 \), it is furthermore easy to see that the usual arguments demonstrating the screening of the charge and confinement remain valid.

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Appendix. Two-dimensional free massive scalar thermofield theory

The free massive scalar thermofields in two dimensions are given by [14] \( p^0 = \sqrt{(p^1)^2 + m^2} \)

\[
\Sigma(x; \beta) = U^{-1}[\vartheta(\beta)]\Sigma(x)U[\vartheta(\beta)] = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{dp_1}{\sqrt{p^0}} \times [e^{-ip_\mu x_\mu} [a(p^1) \cosh \vartheta(p^0, \beta) - \tilde{a}(p^1) \sinh \vartheta(p^0, \beta)] + e^{ip_\mu x_\mu} [\tilde{a}(p^1) \cosh \vartheta(p^0, \beta) - a(p^1) \sinh \vartheta(p^0, \beta)]],
\]

\[
\bar{\Sigma}(x; \beta) = U^{-1}[\vartheta(\beta)]\bar{\Sigma}(x)U[\vartheta(\beta)] = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{dp_1}{\sqrt{p^0}} \times [e^{ip_\mu x_\mu} [\tilde{a}(p^1) \cosh \vartheta(p^0, \beta) - a(p^1) \sinh \vartheta(p^0, \beta)] + e^{-ip_\mu x_\mu} [a(p^1) \cosh \vartheta(p^0, \beta) - \tilde{a}(p^1) \sinh \vartheta(p^0, \beta)]].
\]

The diagonal and off-diagonal two-point functions are

\[
\langle 0, \bar{0} | \Sigma(x; \beta) \Sigma(y; \beta) | 0, \bar{0} \rangle = \Delta^{(s)}(x - y; \beta, m)
\]

\[
= \frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{dp_1}{\sqrt{p^2 + m^2}} [e^{-ip_\mu(\bar{x} - y)_\mu} \cosh^2 \vartheta(p^0, \beta) + e^{ip_\mu(\bar{x} - y)_\mu} \sinh^2 \vartheta(p^0, \beta)] \tag{A.1}
\]

\[
\langle 0, \bar{0} | \bar{\Sigma}(x; \beta) \Sigma(y; \beta) | 0, \bar{0} \rangle = \nabla^{(s)}(x - y; \beta, m)
\]

\[
= -\frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{dp_1}{\sqrt{p^2 + m^2}} [e^{-ip_\mu(\bar{x} - y)_\mu} \cosh \vartheta(p^0, \beta) \sinh \vartheta(p^0, \beta) + e^{ip_\mu(\bar{x} - y)_\mu} \sinh \vartheta(p^0, \beta) \cosh \vartheta(p^0, \beta)]. \tag{A.2}
\]
In order to show that the off-diagonal two-point function \((A_2)\) corresponds to an analytic continuation of the diagonal ones \((A_1)\), we shall use that\(^{12}\)

\[
\sinh \vartheta(p^0, \beta) = e^{-\frac{1}{2}p^0} \cosh \vartheta(p^0, \beta).
\]

The off-diagonal two-point function \((A_2)\) can be written as

\[
\nabla^{(+)}(x-y; \beta, m) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{dp^1}{\sqrt{p^2 + m^2}} \times \left\{ e^{-i(p^\mu(x-y)_\mu) - i^2 p^0} \cosh^2 \vartheta(p^0, \beta) + e^{i(p^\mu(x-y)_\mu - i^2 p^0)} \sinh^2 \vartheta(p^0, \beta) \right\},
\]

that is,

\[
\nabla^{(+)}(x-y; \beta, m) = -\frac{1}{\Delta_1^{(+)}(x-y; \beta, m)}.
\]

Let us now compute the two-point function \((A_1)\), which can be decomposed as

\[
\Delta^{(+)}(x; m, \beta) = \Delta_0^{(+)}(x; m) + I(x; m, \beta),
\]

where \(\Delta_0^{(+)}(x; m)\) is the zero-temperature contribution

\[
\Delta_0^{(+)}(x; m) = \frac{1}{2\pi} \frac{1}{\sinh \left[ \frac{\beta}{2} \sqrt{p^2 + m^2} \right]},
\]

and the finite-temperature contribution is given by \((p \equiv |p^1|)\)

\[
I(x; m, \beta) = \frac{1}{\pi} \int_{0}^{\infty} \frac{dp}{\sqrt{p^2 + m^2}} \cos(x^0 \sqrt{p^2 + m^2}) \cos[p^1 x^1] \sinh^2 \vartheta(p, \beta).
\]

The mean number of particles with mass \(m\) corresponds to

\[
z_m(\beta m) = I(0; \beta m).
\]

The Bose–Einstein statistical weight can be written as

\[
\sinh^2 \theta = \frac{1}{e^{\frac{1}{2} \sqrt{p^2 + m^2}} - 1} = \frac{1}{2} e^{\frac{1}{2} \sqrt{p^2 + m^2}} \cosh \left[ \frac{\beta}{2} \sqrt{p^2 + m^2} \right] = \sum_{n=1}^{\infty} e^{-n\beta \sqrt{p^2 + m^2}}.
\]

The temperature-dependent contribution \((A.5)\) is then

\[
I(x; \beta, m) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{dp}{\sqrt{p^2 + m^2}} \cos(p^1 x^1)(e^{-[\beta - i\xi] \sqrt{p^2 + m^2}} + e^{-[\beta + i\xi] \sqrt{p^2 + m^2}}).
\]

Introducing \(x^\pm = x^0 \pm x^1\), one can write \((A.6)\) in terms of a Bessel function series as \([25]\)

\[
I(x; \beta, m) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \left[ K_0(m \sqrt{(x^+ + in\beta)(x^- + in\beta)}) + K_0(m \sqrt{(x^- + in\beta)(x^+ - in\beta)}) \right].
\]

\(^{12}\)This argument is general, independent of the spacetime dimensions and also holds for the two-point function of free Fermi thermofields, upon replacing the hyperbolic functions by the corresponding trigonometric functions \([6, 17, 19]\),

\[
\sin \vartheta_F(p^0, \beta) = e^{-\frac{1}{2}p^0} \cos \vartheta_F(p^0, \beta),
\]

with the Fermi–Dirac statistical weight given by

\[
\sin^2 \vartheta_F(p^0, \beta) = \frac{1}{e^{\beta p^0} + 1}.
\]
Taking into account (A.4) and (A.7), one can write the thermal two-point function (A.3) as

\[
\Delta^{(+)}(x; m, \beta) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} K_0(m \sqrt{-(x^+ + in\beta)(x^- + in\beta)}).
\]  

(A.8)

(i) **Zero mass limit.**

For \( m \approx 0 \), one has (\( \mu = m e^\gamma / 2 \) and \( \gamma \) is a de Euler constant)

\[
\Delta^{(+)}_0(x, m) \approx -\frac{1}{4\pi} \ln(\mu^2 x^+ x^-),
\]

and for the temperature-dependent contribution, we have

\[
I(x; \beta, m \approx 0) \approx -\frac{1}{4\pi} \sum_{n=1}^{\infty} \{ \ln((i\mu)^2(x^+ + in\beta)(x^+ - in\beta)) + \ln((i\mu)^2(x^- + in\beta)(x^- - in\beta)) \}. 
\]

Using that [25] \( (x \equiv \frac{ix^+}{16\pi}) \)

\[
\sum_{n=1}^{\infty} \ln \left( 1 - \frac{x^2}{n^2 \pi^2} \right) = \ln(\sin x) - \ln x, \quad 0 < x < \pi,
\]

the zero mass limit of the massive thermofield two-point function is given by [14]

\[
\Delta^{(+)}(x; \beta, m \approx 0) \approx -\frac{1}{4\pi} \ln \left[ \left( \frac{i\mu\beta}{\pi} \right)^2 \sinh \left( \frac{\pi}{\beta} x^+ \right) \sinh \left( \frac{\pi}{\beta} x^- \right) \right] - \frac{1}{\pi} \sum_{n=1}^{\infty} \ln(i\mu n\beta),
\]

where the last term corresponds to the mean number of massless particles with momentum in the range \([\mu, \infty]\).

(ii) **Mean number of massive particles.**

For the mean number of massive particles in the ensemble, one has

\[
z_{\tau}(\beta m) = I(0; \beta m) = \frac{1}{\pi} \sum_{n=1}^{\infty} K_0(n\beta m).
\]

Using that [25]

\[
\sum_{k=1}^{\infty} K_0(kx) \cos(kxt) = \frac{1}{2} \left( \nu + \ln \frac{x}{4\pi} \right) + \frac{\pi}{2x\sqrt{1+t^2}} \sum_{\ell=-\infty}^{\infty} \left\{ \frac{1}{\sqrt{x^2 + (2\ell\pi - t)x}^2} - \frac{1}{2\ell\pi} \right\}
\]

one obtains expression (4.3).
References

[1] Schwinger J 1962 Phys. Rev. 128 2425
Schwinger J 1963 Theoretical Physics (Trieste Lectures 1962) (Vienna: IAEA) p 89

[2] Lowenstein J H and Swieca J A 1971 Ann. Phys. 68 172

[3] Klaiber B 1968 Lectures in Theoretical Physics (Boulder Lectures 1967) (New York: Gordon and Breach) p 141

[4] Abdalla E, Abdalla M C B and Rothe K D 1991 2 Dimensional Quantum Field Theory (Singapore: World Scientific)

[5] Abdalla E, Abdalla M C B and Rothe K D 2000 2 Dimensional Quantum Field Theory revised 2nd edn (Singapore: World Scientific)

[6] Das A 1997 Finite Temperature Field Theory (Singapore: World Scientific)

[7] Dolan L and Jackiw R 1974 Phys. Rev. D 9 3320

[8] Love S T 1981 Phys. Rev. D 23 420

[9] Kao Y C 1992 Mod. Phys. Lett. A 7 1411

[10] Sachs I and Wipf A 1992 Helv. Phys. Acta 65 1411

[11] Kao Y C and Lee Y W 1994 Phys. Rev. D 50 1165

[12] Schön V and Thies M 2001 At the Frontier of Particle Physics: Handbook of QCD (Boris Ioffe Festschrift vol 3) ed M Shifman (Singapore: World Scientific) chapter 33, p 1945

[13] Vshivtsev A S, Magnitsky B V and Zhukovsky V Ch 1998 Phys. Part. Nucl. 29 523

[14] Amaral R L P G, Belvedere L V and Rothe K D 2005 Ann. Phys. 320 399

[15] Amaral R L P G, Belvedere L V and Rothe K D 2008 Ann. Phys. 323 2662

[16] Rothe K D and Swieca J A 1977 Phys. Rev. D 15 541
    Rothe K D and Swieca J A 1977 Phys. Rev. D 15 1675

[17] Ojima I 1981 Ann. Phys. 137 1

[18] Kubo R 1957 J. Phys. Soc. Japan 12 570
    Martin P C and Schwinger J 1959 Phys. Rev. 115 1342

[19] Umezawa H, Matsumoto H and Tachiki M 1982 Thermo Field Dynamics and Condensed States (Amsterdam: North-Holland)

[20] Leplae L, Mancini F and Umezawa H 1974 Phys. Rep. 10 C 151
    Takahashi Y and Umezawa H 1975 Collect. Phenom. 2 55
    Matsumoto H 1977 Fortschr. Phys. 25 1

[21] Matsumoto H, Ojima I and Umezawa H 1984 Ann. Phys. 152 348

[22] Swieca J A 1977 Fortschr. Phys. 25 303

[23] Wightman A S 1964 Introduction to Some Aspects of Quantized Fields (Lectures Notes, Cargèse Summer School) (New York: Gordon and Breach)

[24] Symanzik K 1968 Lectures on Lagrangian Quantum Field Theory DESY report T-71/1

[25] Gradshteyn I S and Ryzhik I M 2007 Table of Integrals, Series and Products 7th edn ed A Jeffrey and D Zwillinger (New York: Academic)