Boundedness in a quasilinear fully parabolic Keller–Segel system with logistic source

Qingshan Zhang and Yuxiang Li

Abstract. This paper deals with the Neumann boundary value problem for the system

\[
\begin{align*}
  u_t &= \nabla \cdot (D(u) \nabla u) - \nabla \cdot (S(u) \nabla v) + f(u), & x \in \Omega, & t > 0, \\
  v_t &= \Delta v - v + u, & x \in \Omega, & t > 0
\end{align*}
\]

in a smooth bounded domain \( \Omega \subset \mathbb{R}^n \) \((n \geq 1)\), where the functions \( D(u) \) and \( S(u) \) are supposed to be smooth satisfying 

\[
D(u) \geq Mu^{-\alpha} \quad \text{and} \quad S(u) \leq Mu^\beta
\]
with \( M > 0 \), \( \alpha \in \mathbb{R} \) and \( \beta \in \mathbb{R} \) for all \( u \geq 1 \), and the logistic source \( f(u) \) is smooth fulfilling \( f(0) \geq 0 \) as well as \( f(u) \leq a - \mu u^\gamma \) with \( a \geq 0 \), \( \mu > 0 \) and \( \gamma \geq 1 \) for all \( u \geq 0 \). It is shown that if

\[
\alpha + 2\beta < \begin{cases} 
  \gamma - 1 + \frac{2}{n}, & \text{for } 1 \leq \gamma < 2, \\
  \gamma - 1 + \frac{4}{n+2}, & \text{for } \gamma \geq 2, 
\end{cases}
\]

then for sufficiently smooth initial data, the problem possesses a unique global classical solution which is uniformly bounded.

Mathematics Subject Classification. 35K59 · 92C17 · 35K55.

Keywords. Quasilinear chemotaxis system · Logistic source · Global solution · Boundedness.

1. Introduction

In this paper, we consider the initial-boundary value problem for the parabolic–parabolic quasilinear chemotaxis system with logistic source

\[
\begin{align*}
  u_t &= \nabla \cdot (D(u) \nabla u) - \nabla \cdot (S(u) \nabla v) + f(u), & x \in \Omega, & t > 0, \\
  v_t &= \Delta v - v + u, & x \in \Omega, & t > 0, \\
  \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, & t > 0, \\
  u(x, 0) &= u_0(x), & v(x, 0) &= v_0(x), & x \in \Omega
\end{align*}
\]  

(1.1)

in a bounded domain \( \Omega \subset \mathbb{R}^n \) \((n \geq 1)\) with smooth boundary \( \partial \Omega \), where \( u = u(x, t) \) denotes the density of bacteria, and \( v = v(x, t) \) is the concentration of oxygen. \( \frac{\partial}{\partial \nu} \) represents differentiation with respect to the outward normal \( \nu \) on \( \partial \Omega \). The initial data \( u_0 \in C^0(\Omega) \) and \( v_0 \in W^{1,\theta}(\Omega) \) with \( \theta > \max\{2, n\} \) are nonnegative functions. The parameter functions \( D(u), S(u) \) with \( S(0) = 0 \) from \( C^2([0, \infty)) \) are supposed to satisfy

The study was supported in part by National Natural Science Foundation of China (No. 11171063). The first author is also supported by the Scientific Research Foundation of Graduate School of Southeast University (No. YBJJ1445) and the Key Technologies R&D Program of Henan Province (No. 132102110120).
\[ D(u) \geq M_1(u + 1)^{-\alpha} \text{ for all } u \geq 0 \]  
with \( M_1 > 0 \) and \( \alpha \in \mathbb{R} \),  
\[ S(u) \leq M_2(u + 1)^{\beta} \text{ for all } u \geq 0 \]  
with \( M_2 > 0 \) and \( \beta \in \mathbb{R} \), and \( f(u) \) is smooth satisfying \( f(0) \geq 0 \) and  
\[ f(u) \leq a - \mu u^\gamma \text{ for all } u \geq 0 \]  
with \( a \geq 0, \mu > 0, \) and \( \gamma \geq 1 \).

Keller and Segel [9] proposed the chemotaxis system (1.1) to describe the biased movement of biological cell in response to chemical gradients. Since then, the model has attracted significant attention in mathematical biology, and one of the main issues is under what conditions the solutions of (1.1) blow up or exist globally.

When \( D(u) = 1, S(u) = \chi u \) and \( f(u) \equiv 0 \), system (1.1) corresponds to the so-called minimal model, which has been extensively studied. It is proved that the solutions never blow up if \( n = 1 \) [17]. In the two-dimensional case, if \( \int_{\Omega} u_0 < 4\pi/\chi \), the solutions are global and bounded [13], whereas when \( \int_{\Omega} u_0 > 4\pi/\chi \), the solutions blow up in finite time [6,20]. In the case \( n \geq 3 \), there is finite-time blowup for arbitrarily small prescribed \( \int_{\Omega} u_0 [26] \). In many applications, the blow-up phenomenon is an extreme case, so a logistic growth restriction of type (1.4) in the model (1.1) is expected to rule out the possibility of blowup for solutions. When \( D(u) = 1, S(u) = \chi u \), and \( f(u) \leq a - \mu u^2 \) in the model (1.1), all solutions are global and bounded provided that \( n \leq 2 \) or \( \mu > \mu_0 \) with some \( \mu_0 > 0 \) in higher dimensions \( n \geq 3 \) [16,17,23]. In the special case \( f(u) = u - \mu u^2 \), whenever \( \frac{\chi}{\mu} \) is suitably large, the solution \((u,v)\) stabilizes to the spatially homogeneous steady-state \( \left( \frac{1}{\mu}, \frac{1}{\mu} \right) \) as \( t \to \infty [27] \). Moreover, when \( n \geq 3 \), there exists at least one global weak solution for any \( \mu > 0 [10] \). However, it is unclear whether in higher dimensions \( n \geq 3 \), the logistic source \( f(u) \) with \( \gamma = 2 \) in the problem (1.1) might be sufficient to rule out blowup for arbitrarily small \( \mu > 0 [23] \).

Superlinear logistic growth is not always rule out chemotactic collapse in the Keller–Segel model. The initial-boundary value problem for the related system

\[
\begin{cases}
  u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \lambda u - \mu u^\gamma, & x \in \Omega, \ t > 0, \\
  0 = \Delta v - m(t) + u, & x \in \Omega, \ t > 0
\end{cases}
\]  
(1.5)

with \( m(t) := \frac{1}{|\Omega|} \int_{\Omega} u(x,t) \) was considered in [25]. It was shown that if \( \lambda \in \left( 1, \frac{3}{2} + \frac{1}{2n-2} \right) \) with \( n \geq 5 \), then there exist initial data such that the solutions blow up in finite time.

The authors in [14,15] introduced the model with nonlinear production of a chemoattractant, namely the second equation in (1.5) is replaced by \( v_t = \Delta v - v + g(u) \) for \( g(u) = u(u + 1)^{\beta-1} \) with \( \beta > 0 \), and constructed the global bounded solutions of the system in two- and three-dimensional smooth bounded domains under certain relations between the degradation and production orders.

On the other hand, the volume-filling effect can also prevent blowup [5,18]. In the case \( D(u) = 1 \) and \( f(u) \equiv 0 \) in (1.1), Horstmann and Winkler [7] proved that if \( S(u) \leq K(u + 1)^{\beta} \) with \( \beta < \frac{2}{3n} \) and some \( K > 0 \), then the solutions are global and bounded, while if \( S(u) \geq K(u + 1)^{\beta} \) with \( \beta > \frac{2}{3n} \) and some \( K > 0 \), then the solutions blow up in finite or infinite time.

As to the Neumann boundary value problem for the associated parabolic–elliptic system

\[
\begin{cases}
  u_t = \nabla \cdot (D(u) \nabla u) - \nabla \cdot (S(u) \nabla v), & x \in \Omega, \ t > 0, \\
  0 = \Delta v - m(t) + u, & x \in \Omega, \ t > 0
\end{cases}
\]