Bohm’s realist interpretation of Quantum mechanics

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Abstract

A brief account of the world view of classical physics is given first. We then recapitulate as to why the Copenhagen interpretation of the quantum mechanics had to renounce most of the attractive features of the classical world view such as a causal description, locality, scientific realism and introduce a fundamental distinction between system and apparatus. The crucial role is played in this by the Bohr’s insistence on the wavefunction providing the most complete description possible for an even individual system. The alternative of introducing extra dynamical variables, called hidden variables, in addition to the wavefunction of the system so as to be able to retain at least some of the desirable features of classical physics, is then explored. The first such successful attempt was that of Bohm in 1952 who showed that a realistic interpretation of the quantum mechanics can be given which maintains a causal description as well as does not treat systems and measuring apparatus differently. We begin with the construction of the Bohm’s theory. He introduces particle positions as the hidden variables. The particle positions play a special role in Bohm theory. The particle trajectories are guided by the wavefunction. The Bohm theory is deterministic. The probability enters through a special assumption, “quantum equilibrium” hypothesis, for the initial conditions on the ensemble of particle trajectories. The “wave or particle” dilemma is resolved by a “wave and particle” resolution. The measurements in Bohm theory can be described without mysticism. Bohm’s theory is however nonlocal. It is however without nonlocal signalling. After Bell’s work, and the experimental work on testing Bell’s inequalities, it has however, become clear that quantum mechanics is basically nonlocal. We also describe briefly the “Bohmian mechanics” reformulation of the Bohm theory. In the end we discuss some discontents with the Bohm’s theory as well as it’s future prospects. The writeup is supplemented with mathematical and bibliographical notes.
1. The World view of Classical Physics:

The description of the physical universe, as given in classical physics, was in many ways a very attractive one. It firmly subscribed to scientific realism. It aimed for internal logical consistency and completeness of description. As it described the world as it is, it was very satisfying.

The basic ontological entities were point-particles, fields and space-time. They obeyed the well defined causal dynamical laws having the form of differential equations and were deterministic. Thus in Newtonian mechanics if the positions and momentums of all the particle were specified at any one time, their motion could be determined for all times. Similarly Maxwell’s equations for electromagnetic field and Einstein’s equations for gravitational (-metric) field were determined if they were specified for any one time together with appropriate boundary conditions. There was no fundamental role for randomness in it’s physical description. The role of the probability consideration was thus only present when either we were not interested in full details of the situation and we wanted to have a simplified description of a complex situation using only a few variables. Such examples are provided by classical statistical mechanics and the theory of Brownian motion. With the discovery of chaotic nonlinear systems in classical physics one has now to make a distinction between determinism and predictability. For these systems there is an extreme sensitive dependance of the dynamical motion on initial conditions.

In it’s mature form classical physics also shuns “action at a distance” theories of influence. This is achieved through the modalities of fields. Particles generate fields which then act on other particles elsewhere. As Faraday said “matter can not act where it is not”. The physical effects and signals can not propagate instantaneously. They can do so at most with the speed of light.

Another aspect of classical physics, which we have come to admire more in the post classical physics days is it’s unitary nature. Both the physical systems and the measuring apparatus used to study them obey laws of classical physics. The measurement does not constitute an epistomological problem. Of course every measuring apparatus used to probe a system will disturb it somewhat and we would be learning about the disturbed system. But in classical physics the disturbance can be reduced to be as small as we like by using gentler probes.

2. Coming of the Quantum:

As is well known this beautiful edifice of classical physics, after successfully serving for the description of macroscopic physical world since Newton till Einstein, ie from the seventeenth century to the nineteenth century, was found empirically inadequate in microscopic world of atoms and radiation. The first quarter of the twentieth century was the period of struggle for the new quantum ideas. The final mathematical formulation quantum mechanics, needed to describe new phenomenon in the microscopic domain, was finally achieved around 1925.
Soon thereafter the “Copengahen interpretation” of what the new mathematical quantum formalism means emerged. It became the ruling orthodoxy for a long time so that any other interpretation of the formalism was not encouraged to get a foothold. In the Copenhagen interpretation the scientific realism, the bedrock of the classical physics, was given up along with determinism, unitarity and many other appealing features of classical physical description. That a realist interpretation of quantum mechanics could be given by realised by David Bohm and published in 1952. A precursor was de Broglie’s attempt called “pilot wave” interpretation given in 1927. In view of the dominate of the Copenhagen interpretation, it was however not taken seriously until the important work of John Bell on foundations of quantum mechanics in late nineteen sixtees.

In order to put things in perspective and bring out the magnitude of Bohm’s achievement in proposing his realist interpretation, we will first understand as to why Copenhagen interpretation was forced to renounce so many of appealing aspects of classical physics. As a further preliminary we now present a brief account of the “rules of game” on which all interpretations of quantum mechanics agree in order to make use of the mathematical formalism.

3. Rules of the Game:

(i) **The States**: The state of a system at any time is described by a state vector. A state vector, when multiplied by a complex number, also describes the same state. The coordinate representative of the state vector, ie the wavefunction, is generally a complex quantity. If two different state vectors are appropriate to describe the state of a system, then a linear combination of these two also describes a possible state of the system. This principle of linear superpositions of state vectors is of the same nature as for classical electromagnetic waves. It is this feature of quantum mechanics which helps explain the wave nature of quantum particles like electrons in the electron interference experiments. All state vectors of a physical system belong to the Hilbert space of the system. The state vectors will normally be assumed to be normalised to unity.

(ii) **Physical observables**: All physical observables are represented by linear self-adjoint operators operating on the state vectors in the Hilbert space of the system. All eigenvalues of such operators are real numbers. Any measurement of an observable always results in getting one of it’s eigenvalues. The energy of the system is given by it’s Hamiltonian operator.

(iii) **Dynamics**: The time evolution of a state vector is described by a linear Schrödinger equation. The evolution is unitary with Hamiltonian acting as time-translation operator. If the system at any one time is prepared in any particular state it fixes the state vector at any other later time during it’s free evolution ie before it is measured.

All the postulates so far are quite consistent with a deterministic theory. The probability enters into the theory through the following rule.
(iv) **Statistical postulate:** As we noted earlier the measurement of any observables results in only one of its eigenvalues being observed. The probability of any particular eigenvalue being observed is given by Born’s rule i.e. it is equal to absolute square of the component of the eigenvector corresponding to the observed eigenvalue in the state-vector at the time of measurement.

4. **Renunciations of Copenhagen interpretation:**

The Copenhagen interpretation was hammered out by Niels Bohr and collaborators including Heisenberg, Pauli, Rosenfeld and others. They had the difficult job of making some sense of puzzling quantum phenomenon with which they had to struggle using the above rules of the game. There was also the problem of the nature of quantum entities. The light behaved as waves in some situation involving their interference and diffraction, while it behaved as particles, called ‘photons’, in situations involving interaction of light with matter, such as photoelectric effect and Compton effect. Electrons were regarded as charged point-particles when discovered by J.J. Thomson in 1897 but electron beams were later seen to exhibit diffraction from crystals in the experiments by Davisson and Germer in 1927 showing that they too had a wave behavior. Then there was the notorious problem, to which we will come back, of how measurements of a physical observables of a quantum systems produce definite answers.

Niels Bohr was quite ascetic in his attitude towards new concepts. He took it as bedrock the idea that the description of a quantum system provided by the state vector was complete in itself for even an individual system. No more completer description was possible. It was not a statistical description of an ensemble of similar systems as was advocated by Einstein. All the renunciations follow from this standpoint.

From the quantum rules it follows that when we measure an observable for a individual system which is in a superposition of two eigenvalues of this observable, we will obtain the result to be one of these two eigenvalues. We can not say which one it will be. Born rule only gives the probability for each of these two outcomes. This clearly leads to violation of causality and determinism if the wavefunction provides a complete description of the system.

We next look at the double slit experiment with electrons. Each electron in the beam after it has passed through the slits is detected at a single point on the screen in a detector. It exhibits a discreteness as expected from the point particles. However the vertical distributions of electron clicks, produced by the arrival of electrons at the screen exhibits an interference pattern indicating of the wavenature of the electron. In Copenhagen view, as quantum rules apply to each individual process, the interference pattern is due to each electron interfering with itself. It is not due to an average produced by many electrons in an ensemble. Classically electron must have gone either one or the other of the slits. It can
then however not produce the interference pattern on the screen. We could try to actually check as to through which slit a particular electron went by putting an electron detector at each slit and noting whether it clicks or not. We then would know it’s path, ie the “which way” informations about it. We then find that each electron goes either one or the other slit only. But now the interference pattern is no longer seen. What we see is a superposition of two distributions, each one corresponding to electrons coming from one of the slits. It is just as we would get for classical bullets. So we get a point-particle like pattern when we do know the “which way” information and a wave like pattern when we do not know the “which way” information about the electrons.

The moral Bohr would draw from the double slit experiment on electrons that the phenomenon we observe are not produced by a physical system, as it exists out there independently of us, but only through the combined setup of physical system plus the apparatus used for probing it. “No phenomenon is a phenomenon until it is observed” as Wheeler puts it. It thus forces us to renounce scientific realism for quantum phenomenon.

For Bohr the measuring apparatus is described by classical physics while the quantum system is to be described by the quantum rules. For him it is a logical necessity as the language of classical physics is the only means to communicate the results of an experiments to each other. Thus in view of Bohr the description the quantum physics is not unitary. The system and the apparatus are not described using the same framework as in classical physics.

John von-Neumann would rather have the measuring apparatus also described using quantum dynamics. He then has to introduce an additional type of dynamics, not given by Schrödinger equation, according to which every measurement, when completed, results in the wavefunction of the system suddenly changing to the system being in the eigenstate corresponding to the measured value of the observable. This is the postulate of “the collapse of the wavefunction” at the completion of the measurement. The quantum rules of the game here are also not enough since we have two kind of dynamics ie one applicable to measurement interaction apart from the normal dynamics giving Schrödinger equation. The “rules of game” had included only the Schrödinger dynamics in the dynamics.

Besides all the renunciations of scientific realism, causality, determinism, unitary nature of system and apparatus description, it seemed that even the “action at a distance” is required when Einstein-Podolsky and Rosen (EPR) discovered in 1935, certain nonlocal correlations in quantum phenomenon. They found that two systems, which are in an entangled state, even if separated as far as you like from each other, retain correlations, called EPR correlations, which do not decrease with increasing separation. Here by entangled state is meant those states of the two systems which can not be written as product of their individual systems in any basis whatsoever. An example is the two electrons in an spin singlet state. Bohr felt this discovery to be rather a bolt from the blue. His response was basically that it does not make sense to discuss parts of a combined system.
5. The Hidden variable program:

We thus have seen that the assumption of the “completeness of the description by only the wavefunction of the system” forces us to renounce scientific realism, determinism, causality, locality and loss of unitary description of both the system and the apparatus. It might appear natural then to give up this assumption and entertain the possibility of a more complete description of the state of the system than that provided by the wavefunction alone by introducing additional physical variables. Such additional variable are now called the “hidden variables” though the terminology is not always a happy one. To Bohr and Copenhagen school any such considerations were considered an anathema and were ruled out of court. They would have presumably lessened the mystique of new discoveries.

Apart from the role played by the reigning Copenhagen interpretation, it was a theorem, proved by the great mathematician John von-Neumann, in 1932, which proved most discouraging to anybody trying to follow a hidden variable program toward completion of quantum mechanics. According to this theorem any such completion through hidden variables would not be able to reproduce all the objective results of quantum mechanics.

David Bohm, nevertheless, published in 1954 a hidden variable theory, which was not supposed to be possible by von-Neumann, for the nonrelativistic quantum theory. He proposed that the wavefunction together with particle positions provide a valid completion of quantum theory. The particle positions are the “hidden variables” in this Bohm’s realistic causal interpretation of the quantum mechanics. A similar proposal had been made earlier by Louis de Broglie, who had earlier associated the concept of the waves for the particles, in 1927. But under criticism from Pauli and Einstein, he had given up this attempt. Bohm was able to deal with these early criticisms as well. This interpretation therefore is sometimes referred to as de-Broglie-Bohm causal interpretation. We will mostly refer to it as Bohm’s theory.

Under the spell of von-Neumann’s theorem and of Copenhagen interpretation, the work of Bohm also did not receive much attention. The spell was only broken after John Bell started doing his important work on foundations of quantum mechanics in 1966-67. Bell analysed the von-Neumann’s proof, and since he had an explicit hidden variable model of a spin one-half particle, he could pin-point an assumption in von-Neumann’s proof, which while looking mathematically nice, was not necessarily physically required for the possible hidden variable theories. He also advocated Bohm’s theory strongly. Bell also reformulated the theory.

6. Constructing Bohm’s theory:

Bohm begins with the nonrelativistic Schrödinger equation for the wavefunction for the case of a particle in a potential. As we have noted earlier the wavefunction is a complex function of space coordinates and time. Just as real numbers can be put in a one-to-one correspondence with a line, the complex numbers can be put in a one-to-one correspondence
to an two dimensional Euclidean plane. A complex number can be specified by giving the
distance of it’s corresponding point in the plane from the origin, ie “modulus”, and its
“phase” ie the angle which the line from the origin to the corresponding point make with
one of the two perpendicular lines (called x and y axis), say x axis. Both the “modulus”
and the “phase” of a complex number are real numbers.

We now rewrite the Schrödinger equation for time development. The complex wave
function \( \psi \) in terms of two equations, involving only real quantities, for the time development
of the square of its modulus \( R^2 \) and the phase \( \phi \). Note that \( \psi = Re^{i\phi} \), and \( R^2 = |\psi|^2 \). We
also define action \( S = \hbar \phi \) where \( \hbar \) is Planck’s constant \( \hbar \) divided by \( 2\pi \).

The equation for the time evolution of the action looks quite similar to Hamiltonian-
Jacobi equation for the time development of action in classical dynamics except for an
additional term, which we will call “quantum potential”, \( Q \). It is given by \( Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} \)
and formally vanishes as \( \hbar \to 0 \). It adds to the potential \( V \) in which the particle was placed.
Effectively the potential which the particle feels in Bohm’s theory is \( V + Q \). Thus \( S \) can be
taken as the “quantum action”.

The equation for the time evolution of the square modulus \( R^2 = |\psi|^2 \) has the form
of a equation of continuity for the density \( R^2 \) provided the momentum of the particle is
identified with the gradient of the action \( S \), as is natural in the Hamilton-Jacobi theory.
Madelung, in 1926, had tried to identify the \( R^2 \equiv |\psi|^2 \) as the fluid density of electron fluid
in his hydrodynamical interpretation of quantum mechanics. That however was untenable
as electrons were found to be localised objects when they were detected and not spread out
as in a fluid. It was however fully clear, after Born’s work of 1926, that the value of \( |\psi|^2 \)
at a given location has to be interpreted as the probability density of finding the quantum
particles at that location. Bohm also subscribed to it.

The identification of momentum with the gradient of the phase of the wave function also
leads to an expression for the velocity of the particle since momentum is equal to mass time
velocity. We will refer to it as the guidance equation for the particles.

The multiparticle generalisation of the above procedure is straightforward. We begin
with the \( N \)-particle Schrödinger equation and following the same procedure we find that
particle momenta are again given by the respective gradients of the action function. The
Quantum potential now is given by the ratio of a sum of \( N \)-particle Laplacians of \( R \), each
multiplied by a factor \( -\hbar^2/2m \) for the appropriate mass \( m \), and divided by the \( R \).

We now briefly recapitulate Bohm theory. In Bohm’s theory the basic ontological entities
are the wavefunction of the system and all the particle positions. Both the wavefunction and
particle positions obey time evolution equations, ie Schrödinger equation for wavefunction
and the guidance equation for particle problems, are of first order in time. As a result once
the wavefunction and particle positions are given at an initial time they are determined at
all later times. The trajectories are guided by the wavefunction. It is therefore sometimes referred to as a “pilot wave” theory. The wavefunction is however not affected by the particle motion.

7. Role of probability in Bohm theory:

Bohm’s theory is fully deterministic. So where does the randomness in quantum phenomenon come from? It is taken, in Bohm theory, that we are unable to control the particle positions precisely, so we are able to prepare only that ensemble of particles in which the particle positions are distributed, at a given time, say \( t = 0 \), a recording to the probability distribution given by \( |\psi(q, t = 0)|^2 \). We shall refer this hypothesis as Bohm’s “quantum equilibrium hypothesis”. Once this hypothesis is accepted Bohm’s theory and standard quantum mechanics lead to same observable consequence. Once this initial ensemble is prepared, then the laws of Bohmian dynamics make sure that the particle positions are distributed according the probability distribution given \( |\psi(q, t)|^2 \) at later times.

The probability considerations thus appears in Bohm’s theory in the same way as they do in the classical statistical mechanics ie through our ignorance of the precise initial conditions. They are however not intrinsic to the theory.

8. Special Role of Particle Positions:

It will be noticed that the particle positions play a rather special role in the Bohm’s theory. It is conceptually independent of the wavefunction and has it’s own dynamical motion. Since we unable to produce particle ensembles, as we can not control the particle position in it, other than those conforming to “quantum equilibrium hypothesis”, they are called hidden variables of the theory. Further in Bohm’s theory it is assumed that they are, as Bohm and Hiley put it, “intrinsic and not inherently dependent ..... on the overall context”. They can be measured without being changed. In Bell’s terminology they are ‘beables’ of the theory and not just ‘observables’. In view of this, all measurements are reducible, in Bohm’s theory, to the pointer readings of the measuring apparatus.

The particle momenta, given by the gradient of the action, depends on the wavefunction of the system as a whole. It is also a hidden variable of the theory. It is however not regarded as an intrinsic property and is not a beable of the system. A measurement does not reveal a momentum value given by the Bohmian expression.

9. Waves or (/and) Particles:

How does Bohm’s theory view the particle or wave conundrum for quantum objects like electron. Bohm’s theory associates both a position and velocity, and therefore a trajectory, as well as a wavefunction to the electrons. So the simple answer is that electrons are not either a particle or a wave but rather both a particle and a wave.

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In the double slit experiment the electron trajectory goes through only one of the slits but the electron wave, described by the wavefunction, of course goes through both the slits. This produces the observed interference pattern on the screen.

If we wish to obtain the “which way” information ie to know as to which particular slit any electron went through, we will have to put electron track detectors near the two slits. To discuss this new situation we have to also include the detectors, along with electrons and the slits, in our quantum description. This discussion requires a theory of system-detector interaction and will be dealt with later. We will then see that getting this “which way” information destroys the observed interference pattern.

10. Nonlocality:

The guidance equation for a particle velocity explicitly depends on the gradient of phase of the wavefunction evaluated at the positions of all the particles at that time and some of the other particles can be quite far away. This dependance of the particle velocities, on the far away positions of other particle, is clearly nonlocal. Same point can be made through a consideration of the quantum potential which also has a dependance on the position of other particles, some of which could far away. Bohm’s theory is thus manifestly nonlocal. What is the origin of this nonlocality in quantum mechanics. As Bell says, “that the guiding wave, in the general case, propagates not in ordinary three-space but in a multidimensional-configuration space is the origin of the notorious ‘nonlocality’ of quantum mechanics. It is a merit of the de Broglie-Bohm version to bring this out so explicitly that it can not be ignored”.

The nonlocal dependence on the far away particle positions, of the guidance equation for the motion of a particle, however disappears if the wavefunction is separable in coordinates. In general for all two-particle wavefunction, which are entangled, nonlocal Einstein-Podolsky-Rosen correlations will be there. They are easy to understand in a natural way through Bohm’s theory as it provides a causal mechanism to generate them.

These nonlocal correlations however donot produce nonlocal controllable effects. So they can not be used for signalling instantaneously and are thus from a physical point of view comparatively benign. This comes about since in quantum mechanics such a signalling is not possible, and in view of Bohm’s “quantum equilibrium” hypothesis, all the observable consequences of Bohm’s theory agree with the standard quantum mechanics.

11. Describing the “measurements”:

How does Bohm’s theory cope with the notorious “measurement” problem of the quantum mechanics? For Bohm the measuring appartus is also to be described by quantum mechanics.

Let initially the quantum system be in a definite state $i$, with wavefunction $\psi(i)$, of the physical observable to be measured, and let the measuring appparatus be in some fixed known
base state with its pointer reading at \( a_0 \), with wavefunction \( \phi(a_0) \) ie the initial state of the round system is given by \( \psi(i)\phi(a_0) \) interaction between the system and the apparatus causes the joint system-apparatus state to evolve into the apparatus state to get correlated with the system in the state \( i \), so that at the completion of the measurement, the Schrödinger unitary evolution of the joint system leads to its wavefunction becoming \( \psi(i)\phi(a_i) \). By reading the pointer reading of the measuring apparatus to \( a_i \), we will conclude that the system was in the state \( \psi(i) \). If the initial state of the physical system is a linear combination of different states given by the normalised
\[ \psi = c_1\psi_1 + c_2\psi_2 + \cdots, \]
that the initial joint state \( \psi\phi(a_0) \) would evolve to the joint state given by
\[ \psi = c_1\psi_1\phi(a_1) + c_2\psi_2\phi(a_2) + \cdots, \]
as the Schrödinger evolution is linear.

Now in view of the quantum equilibrium hypothesis, the configuration of the system plus apparatus will be distributed according the configuration probability density equal to \( |\psi|^2 \). Now
\[ |\psi|^2 = |c_1\psi_1\phi(a_1)|^2 + |c_2\psi_2\phi(a_2)|^2 + \cdots. \]
We have here taken into account the fact that the different pointer states, being macroscopically different, would have nonoverlapping support in the configuration space of the apparatus ie \( \phi(a_i)\phi(a_j) = 0 \) for \( i \neq j \). For the pointer reading to be equal to \( a_i \), the probability would be given by \( |c_i|^2 \). This agrees with Born’s probability rule. Further the system would be effectively in the state \( \psi(I) \). We thus reproduce the results obtained from the “collapse of the wavefunction” postulate of the standard quantum mechanics without requiring any collapse of the wavefunction since the wavefunction of the joint system \( \psi \) does not collapse.

Let us call each \( \psi(i) \) a channel for the system-particles. After the pointer reading is \( a_I \), the system particles would be in the channel \( \psi(I) \). Since their future particle motion in Bohm’s theory depends on their present positions, the only relevant part of the wavefunction for it would be \( \psi(I) \). The other channels \( \psi(i) \) for \( i \neq I \), are called empty waves. They will continue to evolve according to Schrödinger equation but are irrelevant for the future motion of the system particles.

This discussion can be applied in a straightforward way to situation of two slit experiments for the case when we position electron path detectors near the two slits. Bohm theory would reproduce the result that in this case the interference pattern disappears. In fact it was not even necessary for us to point this out explicitly in view of our demonstration above of the equivalence for all observable predictions between Bohm’s theory and standard quantum rules provided “quantum equilibrium” hypothesis holds.

12. Bohmian Mechanics:

In Bohm’s original formulation the modified Hamilton-Jacobi equation, and the continuity equation played an important role. The momentum of the particle was defined as in
Hamilton-Jacobi theory. Bohm regarded particles moving under the influence of the forces just as in Newtonian theory except that now they were subject an additional Quantum force due the new quantum potential $Q$. The Quantum potential was used extensively to understand various quantum phenomenon. It served as a measure of deviation of the quantum dynamics from the Newtonian one. It was useful in many other contexts. In fact the textbooks of Holland, and of Bohm and Hiley on Bohm’s theory follow this approach.

Bohm was, of course, aware that his theory can be reformulated as a first order theory by taking the Schrödinger equation for time evolution of the wavefunction $\psi$ and the guidance equation for the particle velocities specified in terms of wavefunction and it’s gradients. This formulation was preferred by John Bell in his presentation of theory. It has been used Dürr and his collaborators extensively and they have named it Bohmian Mechanics.

Bohmian Mechanics appears to be a clearer and deeper formulation of the theory. Within this formulation one has been able to probe the nature of quantum equilibrium hypothesis. As we noted earlier if the probability for the configuration $q$ is given $|\psi(q, t = 0)|^2$ at some time, say $t = 0$, then the distribution is given by $|\psi(q, t)|^2$ for $t > 0$ ie the form of the distribution in terms of the wavefunction does not change with time. Thus assumed “quantum equilibrium” has this attractive property which has been called “equivariance” by Dürr et. al. It has been shown later that it is the unique equivariant distribution which is a local functional of $\psi$ by Goldstein et. al. This concept of equivariance generalises the concept of equilibrium distribution we come across in classical statistical mechanics, e.g. Maxwellian distribution of particle velocities in a gas.

Dürr, Goldstein and Zhangi tried to argue that if deal with the wavefunction of the whole universe $\psi$, then $|\psi|^2$ is a natural measure of probability for initial configurations of the whole universe, which yields Born’s rule for all subsystems at a later time. They argue that this measure is necessary if there has to exist the notion of an effective wavefunction for the subsystem.

Now while we normally get the Maxwellian distribution in a gas, we can conceive of situations, admittedly nonequilibrium ones, where it may not be there e.g. by perturbing the thermal equilibrium of the gas. The deviation from Maxwellian distribution, however, rapidly tend to vanish. Could it be that “quantum equilibrium hypothesis” in Bohmian theory is of similar nature? Valentini and Westman have tried to argue that this is indeed quite plausible using an analogue of classical coarse graining $H$-theorem of Boltzmann. The $H$-function defined by them is the

$$H = \int dq \rho \ln(\rho/|\psi|^2)$$

where $\rho$ is the arbitrary initial probability density in configuration which tends to $|\psi|^2$ quickly. This develops an approach taken earlier by Bohm and Vigier.
13. Discontents with the Bohm theory:

The most common objection against entertaining Bohm’s realist interpretation is since it has identical prediction to standard quantum theory what is gained by introducing “hidden variables” referring to the positions of the particles in the theory. If a theory is nothing more than a set of calculational algorithms for predicting the result of the experiments then obviously nothing is gained. If the theory however is also supposed to provide an understanding of the physical phenomenon then Bohm’s theory definitely does so better than the bare “quantum rules”. It gets rid of the notorious “measurement problem” of standard quantum mechanics and provides a unitary description of system and apparatus within the same framework.

The position and momentum, are treated in a similar manner by the standard quantum kinematics. This feature is lost in the Bohmian theory which gives particle position a special role in contrast to momentum. Of course the dynamics treats the position and momentum asymmetrically e.g. the Hamiltonian is not symmetrical in the two. So it is not necessary for Bohm’s theory to do so even if the standard quantum kinematics does so. Besides there is an attempt to write down a version of Bohm’s theory which actually treats the position and momentum symmetrically.

Many people dread a return to the days of orderly classical physics after having tasted the revolutionary fervour of the Copenhagen interpretation. However Bohm’s theory by no means does that. The Bohm theory, though sharply formulated as opposed to the fuzzy formulation of Bohr and collaborators, does not return us to Newtonian mechanics. The trajectories of the particles are very different in behavior. Some times they are so far from Newtonian expectations that they have been called surreal. For example the electrons are at rest in the bound states of a Hydrogen atom. Bohm theory has also been criticised, from the opposite side, for leading to such non-newtonian trajectories. The explicit nonlocality of Bohm theory also did not endear it to people like Einstein. This has however now, that we know from experiments on Bell’s inequality, is to be regarded rather a virtue than as a defect.

Bell, however, found de Broglie-Bohm theory very instructive. As he advocated in 1982, “should it not be taught, not as the only way, but as an antidote to the prevailing complacency? To show that vagueness, subjectivity, and indeterminism, are not forced on us by experimental facts, but by deliberate choice?”

14. Future Prospects:

14.1 Spin and Relativity

Our discussion so far has been restricted spinless nonrelativistic quantum mechanics. Can it be extended to include spin and special relativistic considerations.
Let us first discuss electrons which have a spin of one-half in the units of $\hbar$. We can use nonrelativistic Pauli equations, instead of Schrödinger equation for spinless case, for this and it is relatively easy to give a Bohm theory for this case. This was done by Bohm, Schiller and Tiomno. It’s interpretation is that of a particle which is spinning rigidly. One here defines a spin vector for each electron. This faces some problems for the many electron problems. A more satisfactory approach is to regard nonrelativistic electrons as described by the nonrelativistic limit of relativistic Dirac equation for an electron. Now the spin is not regarded as an inherent property in addition to the particle velocity. The spin effects arise due to an extra term in the expression for the particle velocity itself.

Even though there are approaches to Dirac-Bohm theory using Dirac equation for the relativistic quantum mechanics, we do not discuss them here. We know that the formulation of relativistic quantum mechanics without introducing quantum fields has not been a great success theory irrespective of Bohm theory.

14.2 Field Theory

We shall now discuss whether Bohm like formulation can be given for quantum field theories. It was believed for a long time that it can not be done. This was so despite the fact that Bohm himself had applied his theory to electromagnetic fields in an appendix to his 1952 papers. Bohm and Hiley applied it scalar fields later. Thus application to Bosonic fields seems to present no undue difficulties. Here one introduces “field beables”. Bosons, e.g. photons, do not have a trajectory.

It is however true that an application of the Bohm theory for the fermionic fields waited till Bell’s attempt to introduce fermionic particle beables for them in 1984. The attempts using fermionic field beables do not seem to be very successful. Bell’s work was done for a “lattice” cutoff model of the field theory and was stochastic in nature. He however suspected that the “stochastic element introduced here goes away in some sense in the continuum limit” ie when lattice cut off’s are removed. In a continuum model later developed by Colin et. al., this seems to be the case.

Dürr et. al. have developed a Bohmian mechanics version of continuum field theory which they have called Bell-type quantum field theories. They associate particle ontology with both bosonic and fermionic fields. The interaction part of the Hamiltonian is associated with jump-like stochastic processes like the “particle-antiparticle pair” creation or annihilation.

A common feature of the work on Bohmian field theory is lack of manifest Lorentz covariance. From what one has said clearly much work remains to be done in this area.

14.3 New Problems

We had emphasised earlier that all the observable consequences of the standard quantum mechanics are reproduced by the Bohm theory provided quantum equilibrium hypothesis
is accepted. But there is a prospect that since Bohm theory is a sharper formulation of the quantum mechanics, it allows us to formulate new problems which can not even be formulated in the old language of the standard quantum mechanics. An example of such a problem is “How much time does a particle spends in the potential barrier?” Bohm theory, having trajectories, has a definite answer while the standard quantum mechanics does not even allow us to ask the question. If the feasible experiments can be devised for measuring these “dwell-times”, then clearly one has some thing to look for.

A remoter possibility is follows. Suppose a future technology allows us to prepare ensembles of particle, which are not having “quantum ensemble” distribution, then we should observe derivations from the predictions of ordinary quantum mechanics. Of course, not all possible deviations from “quantum equilibrium” hypothesis can occur as some of them would lead to instantaneous signalling which would violate special theory of relativity. A possibility has also been considered by Valentini that the quantum equilibrium was not established at the time of big bang of the universe, and if so, it would have observable consequences in that there would be corrections to the usual inflationary model predictions for cosmic microwave background and super-Hubble field correlations and relic nonequilibrium particles.

15. Mathematical Notes

Here we collect some mathematical material relevant to Bohm’s theory. This section can be skipped by nonphysicists. For physicists however this section would add to their deeper understanding and enjoyment.

15.1 Equations for de Broglie-Bohm’s causal theory

We first consider a single particle, with mass $m$, moving in a potential $V(\vec{r})$. The time $t$, development of the wavefunction $\psi(\vec{r})$ of the system, with Hamiltonian $H$, where

$$H = \frac{(\vec{p})^2}{2m} + V(\vec{r}), \quad (1)$$

is given by

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(\vec{r})\psi. \quad (2)$$

Let us rewrite the wavefunction in it’s polar decomposition given by

$$\psi = Re^{iS/\hbar} \quad (3)$$

where $R$ and $S$ are real functions of $\vec{r}$. Substituting the decomposition (3) in eqn.(2) and separating out the real and imaginary parts of the equation we obtain

$$\frac{\partial R}{\partial t} + \frac{1}{2m} \left[ R \nabla^2 S + 2\nabla R \cdot \nabla S \right] = 0 \quad (4)$$
and
\[ \frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V(x) - \frac{\hbar^2}{2m} \nabla^2 R = 0. \] (5)

We now define
\[ P(\vec{r},t) = |\psi(\vec{r},t)|^2 = |R(\vec{r},t)|^2, \] (6)
\[ m\vec{v}(\vec{r},t) = \nabla S(\vec{r},t), \] (7)
and
\[ Q(\vec{r},t) = -\frac{\hbar^2}{2m} \nabla^2 R(\vec{r},t), \] (8)

Using these definitions we can rewrite eqns. (4) and (5) in more suggestive forms
\[ \frac{\partial P}{\partial t} + \nabla(P\vec{v}) = 0 \] (9)
\[ \frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V + Q = 0. \] (10)

The first of these equations is the continuity of equation for density \( P \) with an associated current density \( P\vec{v} \). The second of these equations is of the form of Hamilton-Jacobi equation in Newtonian dynamics for a mass \( m \) particle moving in the potential \( V + Q \). The \( Q \) is thus an added potential of quantum origin and is referred to as Quantum potential.

The de Broglie-Bohm theory also introduces the particle positon \( \vec{q}(t) \) as the extra variable needed to describe the system fully in addition to the wavefunction \( \psi(\vec{r},t) \). We further make the identification of the particle velocity, \( dq(t)/dt \) with \( \vec{v}(\vec{r},t) \), defined above in equation (7) at \( (\vec{r},t) \equiv (\vec{q}(t),t) \) ie
\[ \frac{d\vec{q}}{dt} = \vec{v}(\vec{q},t) = \frac{1}{m} \nabla S(\vec{r},t) \bigg|_{(\vec{r},t)=(\vec{q},t)}, \] (11)
since it is natural in Hamilton-Jacobi theory to have particle momentum \( \vec{p}(t) \)
\[ \vec{p}(t) = m \frac{d\vec{q}}{dt} = \nabla S(\vec{r},t) \bigg|_{(\vec{r},t)=(\vec{q},t)}. \] (12)

With these definitions, it can be shown that
\[ \left[ \frac{\partial}{\partial t} + \frac{1}{m} \nabla S \cdot \nabla \right] \nabla S = -\nabla(V + Q). \] (13)

This can be rewritten as
\[ \frac{d}{dt} \vec{p}(t) = -\nabla(V + Q) \] (14)
where
\[ \frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla, \] (15)
which is the Newtonian equation of motion for a particle of mass $m$ in the potential $V + Q$. This justifies the name “Quantum Potential” for $Q$ for it plays the same role as potential $V$.

Following Born, Bohm also identifies the probability density $\rho(\vec{q}, t)$ for the particle positions $\vec{q}(t)$, as follows,

$$\rho(\vec{q}, t) = P(\vec{q}, t) = |\psi(\vec{q}, t)|^2. \quad (16)$$

The “quantum equilibrium” hypothesis is that

$$\rho(q, t = 0) = |\psi(\vec{q}, t = 0)|^2. \quad (17)$$

It then follows from de Broglie-Bohm equations of motion that

$$\rho(q, t) = |\psi(\vec{q}, t)|^2. \quad (18)$$

The $N$-particle generalisation is straightforward. One has now to have $N$ trajectory functions $\vec{q}_1(t), \vec{q}_2(t), \cdots, \vec{q}_N(t)$ in addition to the wavefunction $\psi(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_N; t)$ for a complete description of the system. We now have, apart from the usual Schrödinger equation for the $N$-particle system

$$m_i \frac{d\vec{q}_i}{dt} = \nabla_i S \bigg|_{\vec{r}_i=\vec{q}_i}, \quad (19)$$

where $m_i$ is the mass of the $i$-th particle. The Quantum potential $Q$ is given by

$$Q = -\hbar^2 \left( \frac{1}{2m_1} \nabla_1^2 R + \frac{1}{2m_2} \nabla_2^2 R + \cdots + \frac{1}{2m_N} \nabla_N^2 R \right) \rho \bigg|_{\vec{r}_i=\vec{q}_i}. \quad (20)$$

15.2 Equations for Bohmian Mechanics

Newtonian equations of motion, eqn. (14), are second order in time as they specify the particle accelerations $d\vec{p}/dt$ or $d^2\vec{q}/dt^2$. The equations of Bohmian mechanics are first order in time. They are

(i) Schrödinger equation for the wavefunction $\psi(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_N, t)$ for a $N$-particle system, with $i$-th particle having mass $m_i$, and moving in a potential $V(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_N)$ given by

$$i\hbar \frac{\partial \psi}{\partial t} = -\left( \frac{\hbar^2}{2m_1} \nabla_1^2 + \frac{\hbar^2}{2m_2} \nabla_2^2 + \cdots + \frac{\hbar^2}{2m_N} \nabla_N^2 \right) \psi + V(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_N) \psi, \quad (21)$$

and

(ii) particle guidance equations

$$m_i \frac{d\vec{q}_i(t)}{dt} = \hbar \left( \frac{\nabla \psi}{\psi} - \frac{\nabla \psi^*}{\psi^*} \right) \bigg|_{\vec{r}_i=\vec{q}_i}. \quad (22)$$

The quantum equilibrium hypothesis is same as before.
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