THE PHASE OF THE SCATTERING MATRIX

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Vacuum polarization in external fields is treated by way of calculating —exactly and then perturbatively— the phase of the quantum scattering matrix in the Shale–Stinespring approach to field theory. The link between the Shale–Stinespring method and the Epstein–Glaser renormalization procedure is highlighted.

1. Introduction

Renormalization theory is bound to suffer a reappraisal in the light of the reconstruction of Zimmermann’s forest formula in Hopf-algebraic terms, together with the interpretation of the dimensional regularization method given by Connes and Kreimer [1], and the birth of quantum field theory on noncommutative spaces [2, 3], together with the evidence that Yang–Mills theories on noncommutative manifolds are ultraviolet divergent —see [4] and the other references in that paper.

Now, part of the advantages and new popularity (see, for instance, [5–8]) of the Epstein–Glaser renormalization method [9,10] stems from the fact that it is locally defined, and so in principle applies to models on nonflat manifolds. There is however some contention on whether, as claimed by some practitioners [11], the Epstein–Glaser method is a fundamental one.

In cases like these, it sometimes helps to look at a simpler problem, for which an absolutely reliable method is known, to see how the marketed procedures fare in its respect.

The chosen problem is that of vacuum polarization in external fields and the chosen reliable method is the Shale–Stinespring approach to linear quantum field theories. This is an entirely rigorous algebraic method; in [3] its \textit{prima facie} applicability to “implementable” theories on noncommutative as well as commutative spaces was brought to the fore.

After reviewing the Shale-Stinespring theorem in Section 2, in the body of this article we show, by a refinement of its technique, that the phase of the scattering matrix is well defined and finite for implementable linear theories. We give explicit formulae for the phase.

We then exploit in QED a perturbative version of this approach, which leads in a direct way to the formulas reached by Scharf [11] in his account of vacuum polarization in QED by the Epstein–Glaser method. Some simplification of his calculations results from using gauge-invariant variables. The contention by Scharf and followers that the Epstein–Glaser renormalization procedure is a fundamental one is vindicated to some extent.

2. A reminder on Shale–Stinespring theory

Let $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ be a Hilbert space, graded by the projections $P_\pm$ on the positive and negative spectral subspaces for a free Dirac operator. Operators on $\mathcal{H}$...
are presented in block form:

\[ A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix}. \]

We have in mind particularly the classical (or “first-quantized”) scattering matrix

\[ S = \begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix}. \]

Introduce a nomenclature for its even and odd parts:

\[ p_S := S_{\text{even}} = \begin{pmatrix} S_{++} & 0 \\ 0 & S_{--} \end{pmatrix}, \quad q_S := S_{\text{odd}} = \begin{pmatrix} 0 & S_{+-} \\ S_{-+} & 0 \end{pmatrix}. \]

Similarly,

\[ S^{-1} = S^\dagger = \begin{pmatrix} S^\dagger_{++} & S^\dagger_{+-} \\ S^\dagger_{-+} & S^\dagger_{--} \end{pmatrix}, \quad p_S^\dagger = \begin{pmatrix} S^\dagger_{++} & 0 \\ 0 & S^\dagger_{--} \end{pmatrix}, \quad q_S^\dagger = \begin{pmatrix} 0 & S^\dagger_{+-} \\ S^\dagger_{-+} & 0 \end{pmatrix}. \]

Unitarity of \( S \) gives the identities

\[ S_{++}S^\dagger_{++} + S_{--}S^\dagger_{--} = S^\dagger_{++}S_{++} + S^\dagger_{--}S_{--} = P_+, \]
\[ S_{--}S^\dagger_{--} + S_{-+}S^\dagger_{-+} = S^\dagger_{--}S_{--} + S^\dagger_{-+}S_{-+} = P_-, \]
\[ S_{++}S^\dagger_{+-} + S_{+-}S^\dagger_{-+} = S^\dagger_{++}S_{+-} + S^\dagger_{+-}S_{-+} = 0, \]
\[ S_{--}S^\dagger_{+-} + S_{-+}S^\dagger_{-+} = S^\dagger_{--}S_{+-} + S^\dagger_{-+}S_{-+} = 0. \]

It is clear that \( p_S^{-1} \) exists if and only if \( S_{++} \) and \( S_{--} \) are invertible as operators on \( \mathcal{H}^+ \) and on \( \mathcal{H}^- \), respectively; this is the generic case, that always holds when \( S \) is close to the identity, and will be the only one considered in the sequel. We then define the skewadjoint operators

\[ T_S := q_S p_S^{-1} = \begin{pmatrix} 0 & S_{--}^{-1}S_{+-}^{-1} \\ S^{-1}_{++}S_{+-}^{-1} & 0 \end{pmatrix}, \quad \tilde{T}_S := T_S^\dagger = \begin{pmatrix} 0 & -S_{++}^{-1}S_{--}^{-1} \\ -S_{+-}^{-1}S_{-+}^{-1} & 0 \end{pmatrix}. \]

Consider the Fock space constructed on \( \mathcal{H} \) with the new scalar product

\[ \langle \eta \mid \varphi \rangle := \langle \eta_+ \mid \varphi_+ \rangle + \langle \varphi_- \mid \eta_- \rangle, \tag{1} \]

where \( \eta_\pm := P_\pm \eta \). Let \( \{ \phi_k \} \) and \( \{ \psi_k \} \) denote arbitrary orthonormal bases for \( \mathcal{H}^+ \) and \( \mathcal{H}^- \), respectively; we shall abbreviate \( b_k := b(\phi_k) \), \( d_k := d(\psi_k) \) in the notation of the “particle” and “antiparticle” annihilation operators, and similarly for the creation operators \( b_k^\dagger \), \( d_k^\dagger \). For any operator \( A \) on \( \mathcal{H} \) we have the quantum (or “second-quantized”) counterpart, acting on Fock space:

\[ d\Lambda(A) := b^\dagger A_{++}b + b^\dagger A_{+-}d^\dagger + dA_{-+}b + :dA_{--}d^\dagger:. \]
Here, for instance, \( :dA_{-}d^\dagger: \) denotes \(-\sum_{j,k} d^\dagger_k \langle \psi_j | A_{-} \psi_k \rangle d_j \), the double colon meaning, as usual, a normally ordered product, and \( b^\dagger A_{+}d^\dagger \) is \( \sum_{j,k} b^\dagger_k \langle \phi_k | A_{+} \psi_j \rangle d^\dagger_j \); the other cases should be clear.

This rule corresponds to the infinitesimal spin representation [12]; it is independent of the drafted orthonormal bases and makes sense only when \( A_{+}, A_{-} \) are Hilbert–Schmidt. The rule is mainly applied to selfadjoint operators, and yields (at least formally) selfadjoint operators in turn. For instance, in QED the free Dirac equation is written as

\[
i \frac{\partial}{\partial t} \psi = \beta m \psi - i \bar{\alpha} \frac{\partial}{\partial \vec{x}} \psi = :D_0 \psi,
\]

where the Dirac matrices, say in the chiral representation, are given by

\[
\bar{\alpha} := \gamma^0 \vec{\gamma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}, \quad \beta := \gamma^0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

To this classical selfadjoint operator \( D_0 \) corresponds the quantum free Hamiltonian

\[
H_0 := d\Lambda(D_0) = b^\dagger D_0 b + :dD_0d^\dagger:,
\]

which is a positive operator. The classical interaction Hamiltonian is of the form

\[
H(t) = e(A^0(t) - \bar{\alpha} \cdot \vec{A}(t)),
\]

where \( e \) denotes the electromagnetic coupling constant and \( (A^0, \vec{A}) = :A \) the electromagnetic vector potential, a real \( c \)-number function. Note the covariant form \( H(t) = e\gamma^0 A \), with \( A := \gamma^\mu A_\mu \). In the interaction picture one considers

\[
V(t) := e^{iD_0 t} H(t) e^{-iD_0 t}. \tag{2}
\]

The quantum interaction Hamiltonian is then

\[
V(t) := d\Lambda(V(t)) = b^\dagger V_{++}(t)b + b^\dagger V_{-+}(t)d^\dagger + dV_{++}(t)b + :dV_{-+}(t)d^\dagger:. \tag{3a}
\]

This can be rewritten in terms of the formal fermion field \( \Psi \), as

\[
V(t) = \int d^3 x \cdot \overline{\Psi}(x) A(x) \Psi(x), \tag{3b}
\]

with \( x = (t, \vec{x}) \) and the bar meaning the Dirac adjoint. For that, just write the fermion field in the form

\[
\Psi(x) = \sum_k (b_k \phi_k(x) + d^\dagger_k \psi_k(x)),
\]

where \( \phi_k(x) = e^{-iD_0 t} \phi_k(\vec{x}) \), and similarly for the \( \psi \)'s.
For the quantum scattering matrix $S$ we need instead the global spin representation \([12,13]\), which we call $\Lambda$. It is given by

$$S := e^{i\theta} \Lambda(S) = e^{i\theta} |\langle 0_{\text{in}} | 0_{\text{out}} \rangle| : \exp d\Lambda(I) : = |\langle 0_{\text{in}} | 0_{\text{out}} \rangle| : \exp d\Lambda(I) : .$$

Here $0_{\text{in}}$ denotes the incoming vacuum, $0_{\text{out}} := S0_{\text{in}}$, and

$$I := \begin{pmatrix} (S_{++}^\dagger)^{-1} - 1 & S_{+-}S_{-1}^{-1} \\ S_{-1}^{-1}S_{++} & 1 - S_{-1}^{-1} \end{pmatrix} .$$

Again, this makes sense if and only if $S_{+-}, S_{-+}$ are Hilbert–Schmidt, and then we say that $S$ is implementable. The absolute value of the vacuum persistence amplitude $|\langle 0_{\text{in}} | 0_{\text{out}} \rangle|$ is given by

$$|\langle 0_{\text{in}} | 0_{\text{out}} \rangle| = \det^{-1/4}(1 - T_2^2) = \det^{1/2}(S_{-+}S_{+-}^\dagger) = \det^{1/2}(S_{++}S_{++}^\dagger) = \det^{1/2}(1 - S_{+-}S_{-+}^\dagger) .$$

For our present purposes, this is the content of the Shale–Stinespring theorem \([12, 14]\). The phase $\theta$ is in principle undetermined and conventionally taken equal to zero; this is all that is needed to compute transition probabilities.

The phase of the vacuum persistence amplitude does matter physically, however: the current density is modified with respect to the free field situation by the vacuum polarization effect (that bears on the radiative correction to the photon propagator in the nonlinear theory), and the interacting current density is found by functional derivation of $S$ with respect to the gauge potential, in which the phase intervenes \([15]\).

The question is then to find an appropriate and computable definition for the phase of the quantum scattering matrix.

3. Computing the phase in the Shale–Stinespring framework

The difficulty comes from the fact that the global spin representation is projective. Let $U(s,t)$ be the classical unitary propagator in the interaction representation, which interpolates between the identity and $S$. Then $U(s,t)$ solves the equation

$$U(s,t) = 1 - i \int_t^s V(u)U(u,t) \, du ; \tag{4}$$

an explicit form being given by the Dyson expansion

$$U(s,t) = 1 + \sum_{n=1}^{\infty} (-i)^n \int_t^s V(t_1) \int_t^{t_1} V(t_2) \cdots \int_t^{t_{n-1}} V(t_n) \, dt_n \cdots dt_2 dt_1 ,$$

from which follow the propagator properties:

$$U(t,t) = 1, \quad U(t,s)U(s,r) = U(t,r).$$
Here $S = U(\infty, -\infty)$.

Now, for unitary implementable operators $U_1, U_2$, in general

$$\Lambda(U_1) \Lambda(U_2) = c(U_1, U_2) \Lambda(U_1 U_2),$$

where the cocycle $c$ (with the vanishing phase convention) is given [12, 13] by

$$c(U_1, U_2) = \exp \left( i \arg \det^{1/2} (1 - T_{U_2} \hat{T}_{U_1}) \right) = \exp \left( i \arg \det^{1/2} (p_{U_1}^{-1} p_{U_1} U_2 p_{U_2}^{-1}) \right). \quad (5)$$

This is not a trivial cocycle because the determinants of the $p_U$ operators are not individually defined in general. Assume that the interpolating family $U(s, t)$ is implementable and (strongly) differentiable with respect to its parameters — this will happen if the external field is sufficiently well behaved. Then

$$\Lambda(U(s, t)) \Lambda(U(t, r)) = c(s, t, r) \Lambda(U(s, r)), \quad (6)$$

with an obvious notation. Still, $c(t, t, r) = c(s, t, t) = 1$. On the other hand, it can be shown [12] that

$$i \frac{\partial}{\partial s} \bigg|_{s=t} \Lambda(U(s, t)) = V(t).$$

We now seek to redefine $\Lambda(U(s, t))$ by multiplying a phase factor $e^{i\theta(s, t)}$ so that the new quantum family $U(s, t) := e^{i\theta(s, t)} \Lambda(U(s, t))$ fulfils

$$U(t, t) = 1; \quad U(s, t) U(t, r) = U(s, r), \quad (7)$$

just like the classical propagator. If we manage that, then $\theta(+-\infty)$ will have every right to be called the phase of the quantum scattering operator $S$. Let $c(s, t, r) := \exp(i \xi(s, t, r))$.

Differentiating equation (6), one gets:

$$V(t) \Lambda(U(t, r)) = i \frac{\partial}{\partial s} \bigg|_{s=t} c(s, t, r) \Lambda(U(t, r)) + i \frac{\partial}{\partial t} \Lambda(U(t, r)).$$

Then, we redefine

$$U(s, t) := \exp \left( i \int_t^s \frac{\partial}{\partial \lambda} \bigg|_{\lambda=t} \xi(\lambda, t, \tau) \, d\tau \right) \Lambda(U(s, t)), \quad (8)$$

which clearly satisfies

$$U(s, t) = 1 - i \int_t^s V(u) U(u, s) \, du.$$

This equation is the quantized version of (4) and sports the same kind of iteration solution:

$$U(s, t) = 1 + \sum_{n=1}^\infty (-i)^n \int_t^s V(t_1) \int_t^{t_1} V(t_2) \cdots \int_t^{t_{n-1}} V(t_n) \, dt_n \cdots dt_2 \, dt_1. \quad (8)$$
In the present case, the quantum Dyson expansion is rigorous: although $V$ is an unbounded operator, it is a pretty tame one. Let $E_m$ denote the projector on states containing at most $m$ particles. Then, in view of (3), $VE_m$ is a bounded operator from $E_m$ into $E_{m+2}$, and the norms of

$$U(s, t)_n E_m := \int_s^t V(t_1) \int_s^{t_1} V(t_2) \cdots \int_s^{t_{n-1}} V(t_n) dt_n \cdots dt_2 dt_1 E_m$$

can be easily estimated. To see that, one introduces the norm

$$||V(t)|| := ||V_{\text{even}}(t)|| + ||V_{\text{odd}}(t)||_2,$$

where the latter is the Hilbert–Schmidt norm. By continuity and uniform boundedness, $||V(t)|| \leq a(s, r)$ holds for some finite function $a(s, r)$, when $r \leq t \leq s$, and is not difficult to check that there are constants $C_m$ such that

$$||V(t)E_m|| \leq C_m a(s, r).$$

Consult [16, 17] for precise analyses of these bounds (encompassing also the boson case). On the other hand, from the integral equation (4),

$$||U(s, r)|| \leq 1 + (s - r) a(s, r).$$

Putting both inequalities together, one gets the estimate

$$||U(s, t)_n E_m|| \leq C_m C_{m+2} \cdots C_{m+2n-2} \frac{(s - t)^n a(s, t)^n}{n!},$$

with the result that the series in (8) indeed converges to a unitary operator, for $s - t$ small enough. Equation (7) then follows from (8) and allow us to extend the validity of the last conclusion—and, in turn, its own domain of validity. More detail on this is found in the important paper [18].

We can give an exact formula for the phase now. Use the standard identities

$$\frac{d}{dt}(\arg z(t)) = \Im \frac{d}{dt}(\log z(t)), \quad \frac{d}{dt}(\log \det A(t)) = \text{Tr} \left( A(t)^{-1} \frac{dA(t)}{dt} \right),$$

which give, from (5),

$$\left. \frac{\partial}{\partial \lambda} \right|_{\lambda=\tau} \xi(\lambda, \tau, t) = -\frac{1}{2} \Im \text{Tr} \left( T_{U(\tau, t)} \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=\tau} \hat{T}_{U(\lambda, \tau)} \right).$$

Therefore,

$$\theta(s, t) = -\frac{1}{2} \Im \int_t^s \text{Tr} \left( T_{U(\tau, t)} \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=\tau} \hat{T}_{U(\lambda, \tau)} \right) d\tau = -\frac{i}{4} \int_t^s \text{Tr} \left[ \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=\tau} \hat{T}_{U(\lambda, \tau)}, T_{U(\tau, t)} \right] d\tau.$$
The last expression is more symmetrical; the trace of this commutator is not zero, because it is taken in Fock space, whereupon, in view of the form of the scalar product (1),

$$\text{Tr} \left( \begin{array}{cc} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{array} \right) = \text{Tr} A_{++} + \text{Tr} A_{--}.$$ 

The phase of the scattering matrix is then

$$\theta = -\frac{1}{2} \Im \int_{-\infty}^{\infty} \text{Tr} \left( T_{U(\tau, -\infty)} \frac{\partial}{\partial \lambda} \right|_{\lambda = \tau} \widehat{T}_{U(\lambda, \tau)} \right) d\tau. \quad (9)$$

The analogous formula, with the same notation, for the boson case was first given, to the best of our knowledge, by Várilly and the author [19]; it differs only by a sign. Then, equivalent formulae both for the boson and fermion cases were found by Langmann [18]. The latter apply to charged fields, which are the ones considered in this paper. However, under the form (9) and with a suitable interpretation, the phase formula is applicable to Majorana fields, which are more general than charged fields [20].

Also note, before continuing, that

$$\frac{\partial}{\partial s} \bigg|_{s=t} \theta(s, t) = \frac{\partial}{\partial \lambda} \bigg|_{\lambda=s} \xi(\lambda, s, s) = 0. \quad (10)$$

In other words, there is no contribution from the coincidence points of \( \widehat{T}_{U(\lambda, \tau)} \) and \( T_{U(\tau, t)} \).

This will prove to be the crucial remark.

One has simply

$$\frac{\partial}{\partial \lambda} \bigg|_{\lambda=\tau} \widehat{T}_{U(\lambda, \tau)} = iV_{\text{odd}}(\tau)$$

in our present framework. Therefore, on calling \( T(\tau, t) := T_{U(\tau, t)} \), finally:

$$\theta(s, t) = \frac{1}{2} \int_{t}^{s} \text{Tr} \left( V_{-+}(\tau)T_{-+}(\tau, t) - T_{++}(\tau, t)V_{++}(\tau) \right) d\tau,$$

a rather elegant expression. Note that it differs from zero only at second order in perturbation theory. At that order,

$$\theta_{:2}(s, t) = \frac{1}{2} \int_{t}^{s} \text{Tr} \left( V_{-+}(\tau)U_{-+;1}(\tau, t) - U_{-+;1}(\tau, t)V_{++}(\tau) \right) d\tau$$

with an obvious notation. Since \( U_{-;1}(\tau, t) = -i \int_{\tau}^{t} V(\tau) \, d\tau \), we set out to compute

$$\theta_{:2}(s, t) = -\frac{i}{2} \int_{t}^{s} \text{Tr} \left[ V_{-+}(\tau) \left( \int_{\tau}^{t} V(\lambda) \, d\lambda \right) - \left( \int_{\tau}^{t} V(\lambda) \, d\lambda \right) V_{++}(\tau) \right] d\tau.$$

Of course, (10) still applies at this approximation. The total phase \( \theta_{:2} := \theta_{:2}(\infty, -\infty) \) at this approximation is then

$$\theta_{:2} = -\frac{i}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \theta(t_{1} - t_{2})(V_{-+}(t_{1})V_{++}(t_{2}) - V_{-+}(t_{2})V_{++}(t_{1})) \, dt_{1} \, dt_{2}. \quad (11a)$$
It should be clear that, at the same order of approximation, this is precisely
\[ -\frac{1}{2} \Im \left\langle 0_{\text{in}} \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T[V(t_1) V(t_2)] \, dt_1 \, dt_2 \, 0_{\text{in}} \right. \right. \],

where \( T \) denotes the time-ordered product.

4. Perturbative calculation of the phase in quantum electrodynamics

In QED, from (2), or working like in the derivation of (3b), the integrals (11) are recast as
\[
\theta_{2} = \Im e^{2} \int \theta(t_1 - t_2) \, \text{tr} \left[ A(x_1) S^{-}(x_1 - x_2) A(x_2) S^{+}(x_2 - x_1) \right.
\]
\[
- A(x_1) S^{+}(x_1 - x_2) A(x_2) S^{-}(x_2 - x_1) \left. \right] \, d^4x_1 \, d^4x_2.
\]

Here \( S^{\pm} \) denote the Wightman “functions”.

The first thing to remark is that, since
\[
S^{-}(x) \gamma^{\nu} S^{+}(-x) - S^{+}(x) \gamma^{\nu} S^{-}(-x) = S_{JP}(x) \gamma^{\nu} S^{+}(-x) - S^{+}(x) \gamma^{\nu} S_{JP}(-x),
\]
and \( S_{JP} \) has support inside the lightcone, then the integrand has support inside the lightcone. That allows one to substitute for \( \theta(t_1 - t_2) \) the covariant expression \( \theta((v(x_1 - x_2))) =: \chi(x_1 - x_2) \), where \( v \) is an arbitrary timelike vector, which can thus be varied at will, and a parenthesis has been used to denote the Minkowski product \( g_{\mu\nu} y^\mu x^\nu =: (yx) \) of two four-vectors \( y, x \). Let then
\[
F^{\mu\nu}(x) := \chi(x) \tilde{F}^{\mu\nu}(x),
\]
\[
\tilde{F}^{\mu\nu}(x) := \text{tr} \left[ \gamma^{\mu} S^{-}(x) \gamma^{\nu} S^{+}(-x) - \gamma^{\mu} S^{+}(x) \gamma^{\nu} S^{-}(-x) \right].
\]

The second thing to remark is that the last expression —and hence (12a)— is only formal: \( \chi \tilde{F}^{\mu\nu} \) is actually undefined as a product of distributions in view of the singularities of \( \tilde{F}^{\mu\nu} \) on the lightcone. Because the apparent trouble occurs at the coincidence points and since \( \chi(x) \tilde{F}^{\mu\nu}(x) \) makes sense for \( x \neq 0 \), one can try to define \( \chi \tilde{F}^{\mu\nu} \) as a distributional extension —or “regularization” in the terminology of [21]— of the latter. The scaling degree [5] or singular order [11] of the integral of the product of the two Wightman functions is 2; therefore, distinct extensions of this quantity will differ by linear combinations of the delta function at the origin and its derivatives up to order two —i.e., by polynomials in \( k \) of degree at most two in momentum space. The procedure is undoubtedly sound in the present case, as we know a priori the phase to be finite. Moreover, our framework will allow to select the “good” extension.

It is indeed convenient to work in momentum space. There, \( \theta_{2} = \frac{e^2}{2} (2\pi)^2 \Im \int F^{\mu\nu}(k) A_{\mu}(k) \overline{A}_{\nu}(k) \, d^4k \),

(12b)
where we take into account that $A(-k) = \overline{A}(k)$, with the bar meaning here complex conjugation, because $A(x)$ is real; and, formally, $F^{\mu\nu}(k) = (2\pi)^{-1/2} \chi * \tilde{F}^{\mu\nu}(k)$ with * denoting ordinary convolution. We recall that, for timelike $k$ and when choosing a frame in which $k = (k^0, \vec{0})$,

$$\chi(k) = \frac{-i\delta(\vec{k})}{\sqrt{2\pi(k^0 - i\varepsilon)}} = \frac{-i\delta(\vec{k})}{\sqrt{2\pi}} \left( P \frac{1}{k^0} + i\pi\delta(k^0) \right). \quad (13)$$

To compute $\tilde{F}^{\mu\nu}(k)$, one looks at the Fourier transform of $\text{tr}[\gamma^\mu S^+(x)\gamma^\nu S^-(x)]$. By using the well known expressions of the Wightman functions in momentum space, this is expressed as

$$-\frac{1}{(2\pi)^4} \int \text{tr}[\gamma^\mu(\slashed{q} + \slashed{m})\gamma^\nu(\slashed{q} - \slashed{m})]\theta(p^0)\theta(q^0)\delta(p^2 - m^2) \times \delta(q^2 - m^2)\delta^4(k - p - q) \, d^4q \, d^4p =: -\frac{1}{(2\pi)^4} T^{\mu\nu}(k). \quad (14)$$

Moreover, $\text{tr}[(\slashed{q} + \slashed{m})\gamma^\mu(\slashed{q} - \slashed{m})\gamma^\nu] = 4(p^\mu q^\nu + q^\mu p^\nu - ((pq) - m^2)g^{\mu\nu})$. Then one of the integrations in (14) is immediately disposed of, with the help of the $\delta^4$-function. The other is easily performed, with the help of the remaining $\delta$-functions, again by choosing a frame in which $k = (k^0, \vec{0})$ so that $\tilde{F}^{\mu\nu}$ can be regarded as a function of only one variable; we obtain

$$T^{\mu\nu}(k) = \frac{2\pi}{3} \left( \frac{k^\mu k^\nu}{k^2} - g^{\mu\nu} \right) \left[ k^2(1 + \gamma(k^2))(1 - 2\gamma(k^2))^{1/2}\theta(1 - 2\gamma(k^2))\theta(k^0) \right],$$

where $\gamma(k^2) := 2m^2/k^2$. All this is found in many books [11, 22, 23].

Thus we have been led formally to compute $\chi * \tilde{F}^{\mu\nu}$, where $\tilde{F}^{\mu\nu}(k)$ equals

$$\frac{1}{3(2\pi)^3} \left( \frac{k^\mu k^\nu}{k^2} - g^{\mu\nu} \right) \left[ k^2(1 + \gamma(k^2))(1 - 2\gamma(k^2))^{1/2}\theta(1 - 2\gamma(k^2))\text{sign}(k^0) \right].$$

That indeed behaves as a polynomial of degree two at high momentum transfer, confirming that the singular degree of $F^{\mu\nu}$ is two.

The correct (unique) recipe to regularize the imaginary part of $\chi * \tilde{F}^{\mu\nu}$ is selected by prescribing that the result $F^{\mu\nu}$ vanishes, together with derivatives up to order two, at zero momentum. This kills the delta function at the origin and its derivatives in configuration space, which otherwise would give a nonzero contribution to $\partial/\partial s \big|_{s=t} \theta(s, t)$, contradicting (10). It is clear now, from the $\delta$-function in (13), that

$$\Re F^{\mu\nu}(k) = \frac{1}{2} \tilde{F}^{\mu\nu}(k),$$

whereas the relation between the real and imaginary parts of $F^{\mu\nu}$ is then given by a subtracted (at the origin) dispersion relation

$$\Im F^{\mu\nu}(k^0) = -\frac{(k^0)^3}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\Re F^{\mu\nu}(\zeta)}{\zeta^3(\zeta - k^0)} \, d\zeta. \quad (15)$$
The prescription that $F^{\mu\nu}$ possess a zero of the third (indeed, fourth) order at $k = 0$ can be independently justified by an argument that, although heuristic, we deem very strong in the present context. On invoking the Maxwell equations (in the Lorentz gauge) $A^\mu(k) = j^\mu(k)/k^2$ to conjure up the source $j$ of the classical field, and on introducing

$$G(k) := \frac{1}{k^2} (1 + \gamma(k^2))(1 - 2\gamma(k^2))^{1/2} \theta(1 - 2\gamma(k^2)) \text{sign}(k^0),$$

one gets

$$\frac{e^2}{2}(2\pi)^2 \Re \int F^{\mu\nu}(k) A_\mu(k) A_\nu(k) \, d^4k = -\frac{e^2}{24\pi} \int (j(k)\bar{j}(k)) G(k) \, d^4k.$$  \hfill (16)

The continuity equation $(j(k)k) = 0$ has been employed to simplify the result. This simple expression exhibits only gauge-invariant variables.

Now, we remark that (classical) gauge transformations are in general not implementable in $1 + 3$ dimensions. This is a very good indicator of the existence of ultraviolet divergences in the nonlinear theory, and indeed it was used by J. C. Várilly and the author in [3] to point out that QFT theories on noncommutative manifolds had to be ultraviolet divergent. On the other hand, in the linear theory selfinteraction is absent, so we would not expect ultraviolet divergences on physical grounds. That nonimplementability is the only source of spurious divergence difficulties. We therefore expect to be able to express the phase in terms of gauge-invariant variables, in a similar way to (16):

$$\theta_{\mu = 2} = -\frac{e^2}{24\pi} \int (j(k)\bar{j}(k)) H(k) \, d^4k,$$  \hfill (17a)

with $H(k)$ is regular at $k = 0$; this is equivalent to $F^{\mu\nu}$ having the aforementioned behaviour at the origin in momentum space.

Taking into account that $G$ is odd, equation (15) leads immediately to a simple form for $H$:

$$H(k) = \frac{1}{\pi} \text{P} \int_{4m^2}^{\infty} \frac{(1 + \gamma(\lambda))(1 - 2\gamma(\lambda))^{1/2}}{\lambda(\lambda - k^2)} \, d\lambda.$$  \hfill (17b)

The restriction to timelike $k$ can be removed by analytic continuation. Making the change of variable $\lambda =: 4m^2/(1 - v^2)$, we get

$$H(k) = \frac{3}{\pi k^2} \int_{0}^{1} \frac{v^2 - v^4/3}{v^2 - 1 + 4m^2/k^2} \, dv,$$

which is essentially the expression one finds in textbooks [24, pp. 249–252] for the vacuum polarization functional, after renormalization. The last integral can be easily carried out analytically, and it is then an instructive exercise to check that the function $H$ is perfectly smooth at $k = 0$ (at $k^2 = 4m^2$, the onset of the absorptive part, $H$ has a cusp). We shall not go into the details.

Before rushing to the conclusions, a comment is in order: we have more or less treated the $A(x)$ as test functions, guaranteeing implementability of the interpolating operators,
for the sake of the argument. However, it is clear that the final formula for the phase (9) is acceptable with only the milder requirement of the implementability of the scattering operator; consult [25] for a very efficient removal of technical conditions on the potentials, for this last purpose.

5. Conclusions

In this paper we have performed what amounts to an \textit{ab initio} finite calculation of the “bubble” diagrams in linear quantum field theory.

Now, at first significant order in QED this is essentially the same as the one-loop vacuum polarization or “photon self-energy” diagram (see in this respect [26, pp. 195–196]). The computation done here does not appear to have been pushed to that finish line before now, although the tools have been there since the seventies at least [27]. (Besides Várilly and the author [19,13], Langmann and Mickelsson [18,25] came close in the nineties.) The main point is that the “local causality condition” (10) selects the correct prescription among all the (finite) regularizations, with recourse to neither heuristic arguments [11] nor the extremely long and complicated “nonperturbative proof” in [28].

On the other hand, it will not have escaped the reader’s attention that, in order to avoid pitfalls, we reorganize the calculation in the same way as [11]. The whole procedure is thus in the spirit of the Epstein–Glaser renormalization procedure, where there are Feynman graphs, but the Feynman rules do not necessarily apply—we avoided rewriting (12a) in terms of Feynman propagators. It is remarkable that, in the boson case, the quantum scattering matrix was found long ago by the Epstein–Glaser method by Bellissard [29]—without the phase.

From formulae (16) and (17) one can easily verify \textit{a posteriori} Bogoliubov’s causality condition

$$\frac{\delta}{\delta A(x_1)} \langle 0_{\text{in}} | S^\dagger \frac{\delta S}{\delta A(x_2)} 0_{\text{in}} \rangle = 0 \quad \text{for} \quad x_1^0 > x_2^0,$$

which was the starting point of Epstein and Glaser.

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