INTRINSICALLY SPHERICAL 3-LINKED GRAPHS

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Abstract: We exhibit several families of planar graphs that are minor-minimal intrinsically spherical 3-linked. A graph is intrinsically spherical 3-linked if it is planar graph that has, in every spherical embedding, a non-split 3-link consisting of two disjoint cycles ($S^1$s) and two disjoint vertices ($S^0$), or a cycle and two pairs of disjoint vertices. We conjecture that $K_4 \cup K_4, K_{3,2} \cup K_{3,2}$, and $K_4 \cup K_{3,2}$ form the complete set of minor-minimal intrinsically type I spherical 3-linked graphs (that is, in every spherical embedding, have a non-split link of two cycles and one $S^0$).

1. Introduction

Dekhordi and Farr [4] showed that the complete set of minor-minimal intrinsically spherical linked graphs consists of $K_4 \cup K_1, K_{3,2} \cup K_1$, and $K_{3,1,1}$, where $K_n$ stands for the complete graph on $n$ vertices, $\cup$ stands for the disjoint union, and $K_{n_1,n_2,...,n_k}$ stands for the complete multipartite graph on $n_1 + ... + n_k$ vertices, with $k$ partition sets. Intrinsically spherical linked graphs (also known as separating planar graphs) are planar graphs that have every spherical embedding containing a non-split link consisting of a cycle ($S^1$) and two disjoint vertices ($S^0$). Here, we exhibit several families of planar graphs that are minor-minimal intrinsically spherical 3-linked, that is planar graphs that have, in every spherical embedding, a non-split 3-link consisting of two disjoint cycles ($S^3$s) and two disjoint vertices ($S^0$), or a cycle and two pairs of disjoint vertices.

During the 1980’s, Conway and Gordon [3], and Sachs [12], [13], independently proved that $K_6$ is intrinsically linked, that is, every spatial embedding of $K_6$ contains a pair of cycles that form a non-split link. Robertson, Seymour, and Thomas [11] proved Sachs’ conjecture, that the Petersen family of graphs (the seven graphs including $K_6$, obtained from $K_6$ by $\Delta - Y$ and $Y - \Delta$ exchanges) form the complete minor-minimal set of graphs that are intrinsically linked.

Robertson and Seymour’s Minors Theorem [10] states that if $P$ is a minor-closed graph property, then the minor-minimal forbidden graphs for $P$ form a finite set. This implies that the complete set of minor-minimal intrinsically 3-linked graphs must be finite. Recall that a graph is intrinsically 3-linked if it contains, in every spatial embedding, cycles that form a non-split link of 3 components. It is still open to classify the complete set of minor-minimal intrinsically 3-linked graphs, and it appears that the list of such graphs will be significantly larger than 7 (see, for example, [6], [1], [9]). The difficulty of this problem has been the inspiration and motivation for this paper; we aimed to find the complete set of minor-minimal intrinsically spherical 3-linked graphs. The hope is that this question will be easier for the spherical case. Though the question remains open, we conjecture that

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$K_4 \cup K_4$, $K_{3,2} \cup K_{3,2}$, and $K_4 \cup K_{3,2}$ form the complete set of minor-minimal intrinsically type I spherical 3-linked graphs (that is, in every spherical embedding, have a nonsplit link of two cycles and one $S^0$). We also exhibit several graphs that are minor-minimal intrinsically type II spherical 3-linked. In general, it remains open to find families of planar graphs that are minor-minimal intrinsically (spherical) $n$--linked for $n \geq 3$.

2. Terminology and Background

First, let us define $S^k$ to be the $k$-sphere. We call a particular way to place a graph, $G$, into $S^k$ an embedding of $G$. A link is a collection of disjoint spheres of various dimensions, embedded within a sphere of greater dimension. We say that a link, $\ell$, embedded in $S^k$ is split if there exists an $S^{k-1}$ embedded in $S^k - \ell$ that bounds only part of $\ell$. We say that a graph, $G$, is intrinsically linked if every embedding of $G$ into $S^3$ contains a non-split link.

A planar embedding of a graph, $G$, is spherical linked if it contains a non-split link of one embedded $S^1$ and one embedded $S^0$, as depicted below:

A planar embedding of a graph, $G$, is intrinsically spherical linked if every embedding of $G$ into $S^2$ contains a non-split spherical link.

A spherical embedding of a graph, $G$, is type I spherical 3-linked if it contains a non-split link of two embedded $S^1$'s and one embedded $S^0$, as depicted below:

A planar embedding of a graph, $G$, is type II spherical 3-linked if $G$ contains a non-split link of one embedded $S^1$ and two embedded $S^0$'s, as depicted below:
When speaking of planar links, it doesn’t make topological sense to call 0-spheres “components,” since an $S^0$ is not connected. Henceforth we will refer to an $S^3$ or an $S^0$ as a piece of an $n$-link. Note that the following definition only makes sense after we have chosen which points form $S^0$s.

**Theorem 2.1.** (Dekhordi and Farr [4]). A graph, $G$, is intrinsically spherical linked if and only if $G$ contains $K_4 \cup K_1$, $K_{3,2} \cup K_1$, or $K_{3,1,1}$ as a minor.

A spherical $n$-link is a disjoint collection of $n - m$ 1-spheres and $m$ 0-spheres, embedded into $S^2$. We say a spherical $n$-link, $\ell$, is split if there exists an $S^1$ embedded in $S^2 - \ell$ that bounds only part of $\ell$. We say that a spherical embedding of a graph, $G$, is linked if it contains a non-split link of one $S^1$ and one $S^0$.

A graph, $H$, is a minor of another graph, $G$, if and only if $H$ can be obtained from $G$ by a sequence of the following: vertex deletion(s), edge deletion(s), (and/or) edge contraction(s). We say that a graph, $G$, is minor-minimal (with respect to a property) if $G$ has that property and no minor of $G$ has that property.

A graph, $G$, is planar if and only if $G$ can be embedded into $S^2$. Kuratowski’s famous theorem [8] asserts that a graph, $G$, is planar if and only if $G$ does not contain $K_5$ nor $K_{3,3}$ as a minor.

A graph, $G$, is outerplanar if $G$ can be embedded in $S^2$ with all vertices on a common face.

**Theorem 2.2.** [7], [2] A graph, $G$, is outerplanar if and only if $G$ does not contain $K_4$ nor $K_{3,2}$ as a minor.

A planar graph, $G$, is $n$-connected if $G$ is connected, and removing $n - 1$ or fewer vertices from $G$ always results in a connected planar graph.

3. Intrinsically Spherical 3-Linked Graphs

### Type I Spherical 3-Linked Graphs

A spherical embedding of a graph, $G$, is type I spherical 3-linked if it contains a non-split link of two embedded $S^1$’s and one embedded $S^0$, as depicted in Figure 3.1.

![Figure 2. A type I non-split 3-link.](image)
Figure 3. $K_4 \hat{\cup} K_4$ has a unique embedding in $S^2$, up to equivalence.

Figure 4. $K_{3,2} \hat{\cup} K_{3,2}$ has a unique embedding in $S^2$, up to equivalence.

Figure 5. [Left] $K_4 \hat{\cup} K_4$ minus an edge. [Right] $K_4 \hat{\cup} K_4$ with a contracted edge.
A graph, $G$, is \textit{intrinsically type I spherical 3-linked} if every spherical embedding of $G$ is type I spherical 3-linked.

**Proposition 3.1.** The graphs $K_4 \cup K_4$, $K_3,2 \cup K_3,2$, and $K_4 \cup K_3,2$ are minor-minimal with respect to being intrinsically type I spherical 3-linked.

**Proof.** First, we embed $K_4$ into $S^2$. This embedding determines four faces, each of which are equivalent. Now, we embed another $K_4$ into our $S^2$. As each face was equivalent, we have that $K_4 \cup K_4$ has a unique planar embedding, up to equivalence. By examining Figure 3, we see that $K_4 \cup K_4$ is intrinsically type I spherical 3-linked.

Now, we will verify that $K_4 \cup K_4$ is minor-minimal with respect to being intrinsically type I spherical 3-linked. Before we begin, we note that each edge in $K_4 \cup K_4$ is equivalent to any other edge. Now, by examining Figure 5 [Left], we see that $K_4 \cup K_4$ minus an edge fails to be intrinsically type I spherical 3-linked. Thus, if we delete any vertex from $K_4 \cup K_4$ the resulting graph will not be intrinsically type I spherical 3-linked, since each vertex is connected to an edge in $K_4 \cup K_4$. Finally, by examining Figure 5 [Right], we see that $K_4 \cup K_4$ with a contracted edge fails to intrinsically type I spherical 3-linked. Hence, no minor of $K_4 \cup K_4$ is intrinsically type I spherical 3-linked. Therefore, $K_4 \cup K_4$ is minor-minimal with respect to being intrinsically type I spherical 3-linked.

Next, we embed $K_3,2$ into $S^2$. This embedding determines three faces, each of which are equivalent. Now, we embed another $K_3,2$ into our $S^2$. As each face was equivalent, we have that $K_3,2 \cup K_3,2$ has a unique planar embedding, up to equivalence. By examining Figure 4, we see that $K_3,2 \cup K_3,2$ is intrinsically type I spherical 3-linked.

Now, we will verify that $K_3,2 \cup K_3,2$ is minor-minimal with respect to being intrinsically type I spherical 3-linked. Before we begin, we note that each edge in $K_3,2 \cup K_3,2$ is equivalent to any other edge. Now, by examining Figure 6 [Left], we see that $K_3,2 \cup K_3,2$ minus an edge fails to be intrinsically type I spherical 3-linked. Thus, if we delete any vertex from $K_3,2 \cup K_3,2$ the resulting graph will not be intrinsically type I spherical 3-linked, since each vertex is connected to an edge in $K_3,2 \cup K_3,2$. Finally, by examining Figure 6 [Right], we see that $K_3,2 \cup K_3,2$ with a contracted edge fails to intrinsically type I spherical 3-linked. Hence, no minor
of $K_{3,2} \cup K_{3,2}$ is intrinsically type I spherical 3-linked. Therefore, $K_{3,2} \cup K_{3,2}$ is minor-minimal with respect to being intrinsically type I spherical 3-linked.

Lastly, we note that the proof that $K_{4} \cup K_{4}$ is minor-minimal with respect to being intrinsically type I spherical 3-linked is analogous to the above argument.

This completes our proof. □

**Conjecture 3.2.** The graphs $K_{4} \cup K_{4}$, $K_{3,2} \cup K_{3,2}$, and $K_{4} \cup K_{3,2}$ form the complete minor-minimal set of intrinsically type I spherical 3-linked graphs.

### 3.2. Type II Spherical 3-Linked

A planar embedding of a graph, $G$, is **type II spherical 3-linked** if $G$ contains a non-split link of one embedded $S^1$ and two embedded $S^0$'s, as depicted in Figure 7.

![Figure 7. A type II non-split spherical 3-link.](image)

We say that a graph, $G$, is **intrinsically type II spherical 3-linked** if every spherical embedding of $G$ is type II spherical 3-linked.

#### 3.2.1. Vertices-Bar Exchange

In this sub-section, we will define the Vertex-Bar exchange, and then show how it preserves the property of being intrinsically minor-minimal intrinsically type II spherical linked.

![Figure 8. An illustration of the Vertices-Bar exchange.](image)

**Proposition 3.3.** Suppose that a graph, $G$, satisfies the following:

i. $G$ is minor-minimal with respect to being intrinsically type II spherical 3-linked,

ii. $G = G_0 \cup \{v_1, v_2, ..., v_n\}$, for some $n \in \mathbb{N}$ with $n \geq 3$ (where $G_0$ is a connected planar graph),

iii. $G_0 \cup \{v_1, v_2\}$ contains a type II spherical 3-link whenever $G_0$ is embedded into $S^2$ and the two vertices $v_1$ and $v_2$ are embedded into any one face of $G_0$,

iv. In each embedding $\varphi$ of $G_0 \setminus e$ into $S^2$, there is at least one face, $F_\varphi$, such that $G_0 \cup \{v_1, v_2, ..., v_n\}$ is not type II spherical 3-linked when $\{v_1, v_2, ..., v_n\}$ is embedded into $F_\varphi$,

v. In each embedding $\varphi$ of $G_0 \setminus e$ into $S^2$, there is at least one face, $F_\varphi$, such that $G_0 \cup \{v_1, v_2, ..., v_n\}$ is not type II spherical 3-linked when $\{v_1, v_2, ..., v_n\}$ is embedded into $F_\varphi$. 

Then, $G_0 \cup K_2$ is minor-minimal with respect to being intrinsically type II spherical 3-linked.

Proof. Suppose that $G$ is a graph that satisfies the properties above.

Notice, $K_2$ contains exactly two vertices and can only possibly be embedded into any one face of each embedding of $G_0$ into $S^2$. Thus, by hypothesis iii, $G_0 \cup K_2$ is intrinsically type II spherical 3-linked.

Now, we will verify that $G_0 \cup K_2$ is minor-minimal with respect to being intrinsically type II spherical 3-linked. Note that each edge in $G$ is exclusively contained in $G_0$ or $K_2$. Consider $G_0 \cup K_2$ minus an edge. By hypothesis iv, we know that there exists an embedding $\varphi$ of $G_0 \cup K_2$ minus an edge that is not type II spherical 3-linked whenever we delete an edge contained in $G_0$; namely, when we embed $K_2$ into $F_\varphi$. Furthermore, since $G_0 \cup K_2$ minus an edge equals $G_0 \cup \{v_1, v_2\}$ when we delete the edge contained in $K_2$, we know that each embedding of $G_0 \cup K_2$ minus an edge fails to be intrinsically type II spherical 3-linked, by hypotheses i and ii.

Hence, as $G_0$ is connected, we have that each embedding of $G_0 \cup K_2$ minus a vertex is not intrinsically type II spherical 3-linked, as well, since we cannot delete a vertex from $G_0 \cup K_2$ without also deleting an edge.

Lastly, consider $G_0 \cup K_2$ with a contracted edge. By hypothesis v, we know that there exists an embedding $\phi$ of $G_0 \cup K_2$ with a contracted edge that is not type II spherical 3-linked whenever we contract an edge contained in $G_0$; namely, when we embed $K_2$ into $F_\phi$. Furthermore, since $G_0 \cup K_2$ with a contracted edge equals $G_0 \cup \{v_1\}$ when we contract the edge contained by $K_2$, we know that each embedding of $G_0 \cup K_2$ with a contracted edge fails to be intrinsically type II spherical 3-linked, by hypotheses i and ii. Thus, no minor of $G_0 \cup K_2$ can be intrinsically type II spherical 3-linked, so $G_0 \cup K_2$ is therefore minor-minimal with respect to being intrinsically type II spherical 3-linked.

□

3.2.2. Subdivisions-Dangle Move. In this sub-section, we will define the Subdivisions-Dangle move, and then show how it preserves the property of being intrinsically minor-minimal intrinsically type II spherical linked.

We define a dangle in a graph, $G$, to be a $K_2$ with exactly one vertex identified with a subdivision on some edge of $G$ (see Figure 4.6).

We say that $G$ is dangle-able if $G$ meets the criterion in Proposition 3.4.

We call $G$ basic, with respect to being dangle-able, if $G$ is dangle-able and does not contain any dangles.

![Figure 9. An illustration of the Subdivisions-Dangle move.](image)
Proposition 3.4. Suppose that a graph, $G$, satisfies the following:

1. $G$ is minor-minimal with respect to being intrinsically type II spherical 3-linked.
2. $G$ is a connected planar graph.
3. $G$ contains an edge, $e$, with two subdivisions $s_1$ and $s_2$ on $e$.
4. $G \setminus \{v\}$ with a contracted edge between $s_2$ and the adjacent endpoint of $e$ is not intrinsically type II spherical 3-linked.
5. $G$ with a contracted edge between $s_2$ and the adjacent endpoint of $e$ with a vertex of $K_2$ identified with an endpoint of $e$ is not intrinsically type II spherical 3-linked.

Then, the graph $G'$ obtained by contracting the edge between $s_2$ and the adjacent endpoint of $e$ and then identifying a vertex of $K_2$ with $s_1$ is minor-minimal with respect to being intrinsically type II spherical 3-linked; we call this particular $K_2$, $K_2'$, and we call this new edge including $K_2'$, $e'$.

Proof. Suppose that $G$ is a graph that satisfies the properties above. Consider $G'$. Notice that, for any spherical embedding, $G'$ and $G$ share the same number of faces and that each face of $G'$ is equivalent to that of $G$. Further, notice that $e'$ can only be embedded into the equivalent faces of $G'$ that $e$ can be embedded into of $G$. Thus, $G'$ is intrinsically type II spherical 3-linked, as $G$ is.

Now, we will verify that $G'$ is minor-minimal with respect to being intrinsically type II spherical 3-linked. Consider $G'$ minus and edge. As $e'$ can only be embedded into the equivalent faces that $e$ can, we see that $G'$ minus an edge fails to be intrinsically type II planar 3-linked whenever we delete an edge of $G'$ that is not contained in $e'$, by hypothesis i. Furthermore, we notice that deleting an edge contained in $e'$ that is not also contained in $K_2'$ is equivalent to deleting an edge between an endpoint of $e$ and an adjacent subdivision of $e$ in $G$, and so $G'$ minus an edge fails to be intrinsically type II planar 3-linked whenever we delete one of these two edges, also by hypothesis i. Finally, $G'$ minus the edge contained in $K_2'$ is the same graph as described in hypothesis iii, so in general, $G'$ minus an edge fails to be intrinsically type II spherical 3-linked. Thus, deleting any vertex of $G'$ that is connected to an edge fails to be intrinsically type II spherical 3-linked. Also, since $e'$ can only be embedded into the equivalent faces that $e$ can, $G'$ minus a disjoint vertex (if $G'$ has one) fails to be intrinsically type II planar 3-linked in the same way that $G$ would fail, by hypothesis i. So, in general, $G'$ minus a vertex fails to be intrinsically type II spherical 3-linked. Lastly, consider $G'$ with a contracted edge. Similar to above, as $e'$ can only be embedded into the equivalent faces that $e$ can, we see that $G'$ with a contracted edge fails to be intrinsically type II planar 3-linked whenever we contract an edge of $G'$ that is not contained in $e'$, by hypothesis i. Furthermore, we notice that contracting an edge contained in $e'$ that is not also contained in $K_2'$ is the same graph as described in hypothesis iv, so $G'$ with a contracted edge fails to be intrinsically type II planar 3-linked whenever we contract one of these two edges. Lastly, $G'$ with the edge contained in $K_2'$ contracted is a minor of $G$, and so in general, $G'$ with a contracted edge fails to be intrinsically type II spherical 3-linked, by hypothesis i. Thus, no minor of $G'$ can be intrinsically type II spherical 3-linked, so $G'$ is therefore minor-minimal with respect to being intrinsically type II spherical 3-linked. \qed
### Figure 10

[Top Row] Embeddings of our basic minor-minimal intrinsically type II spherical 3-linked graphs (with the exceptions of $\mathcal{D}_1$ and $\mathcal{D}_2$). [Lower Rows] Embeddings of our basic graphs with Sub-Dangle moves applied to them.

### Figure 11

[Top Rows] Embeddings of our alpha versions of our minor-minimal intrinsically type II spherical 3-linked graphs (with the exceptions of $\mathcal{D}_{6}$, $\mathcal{D}_{12}$, and $\mathcal{D}_{13}$). [Bottom Rows] Embeddings of our beta versions of our minor-minimal intrinsically type II spherical 3-linked graphs.

| $n$ | 1   | 2   | 3   | 4   | 5   |
|-----|-----|-----|-----|-----|-----|
| $\mathcal{D}_n$ | ![Image](image1.png) | ![Image](image2.png) | ![Image](image3.png) | ![Image](image4.png) | ![Image](image5.png) |
| $\mathcal{D}_n'$ | ![Image](image6.png) | ![Image](image7.png) | ![Image](image8.png) | ![Image](image9.png) | ![Image](image10.png) |
| $\mathcal{D}_n''$ | ![Image](image11.png) | ![Image](image12.png) | ![Image](image13.png) | ![Image](image14.png) | ![Image](image15.png) |
| $\mathcal{D}_n'''$ | ![Image](image16.png) | ![Image](image17.png) | ![Image](image18.png) | ![Image](image19.png) | ![Image](image20.png) |

| $n$ | 6   | 7   | 8   | 9   |
|-----|-----|-----|-----|-----|
| $\mathcal{D}_{n,1}$ | ![Image](image21.png) | ![Image](image22.png) | ![Image](image23.png) | ![Image](image24.png) |
| $\mathcal{D}_{n,2}$ | ![Image](image25.png) | ![Image](image26.png) | ![Image](image27.png) | ![Image](image28.png) |

| $n$ | 10  | 11  | 12  | 13  |
|-----|-----|-----|-----|-----|
| $\mathcal{D}_{n,1}$ | ![Image](image29.png) | ![Image](image30.png) | ![Image](image31.png) | ![Image](image32.png) |
| $\mathcal{D}_{n,2}$ | ![Image](image33.png) | ![Image](image34.png) | ![Image](image35.png) | ![Image](image36.png) |
Proposition 3.5. The graphs $D_1$, $D_2$, $D_3$, $D_4$, $D_5$, $D_6$, $D_7$, $D_8$, $D_9$, $D_{10}$, $D_{11}$, $D_{12}$, and $D_{13}$ are minor-minimal with respect to being intrinsically type II spherical 3-linked.

Proof. First, notice that the graphs $D_i$ for $i \in \mathbb{N}$ with $1 \leq i \leq 5$ are connected, so it suffices to show that deleting or contracting any edge from these graphs will result in a loss of our graph being intrinsically type II spherical 3-linked, in order to show minor-minimality (since we will not be able to delete a vertex from these graphs without also deleting an edge). Similarly, notice that we can characterize the vertices in the graphs $D_j$ for $j \in \mathbb{N}$ with $6 \leq j \leq 13$ as being incident to an edge or not. With this characterization, it suffices to show that deleting or contracting any edge from these graphs, or deleting vertex component from these graphs, will result in a loss of our graph being intrinsically type II spherical 3-linked, in order to show minor-minimality.

First, we embed $K_4$ into $S^2$. This embedding is unique, up to equivalence, as $K_4$ is 3-connected. Additionally, this embedding determines four faces, each of which are equivalent, and we also note that each edge and vertex in $K_4$ are pairwise equivalent. In regards to Figure 14, we see that by identifying just one vertex of a copy of $K_2$ to each vertex of $K_4$, we are forced to embed the other vertex of each copy of $K_2$ into a face of $K_4$, although at most three remaining vertices can be embedded into one face of $K_4$. Since the faces of $K_4$ are all equivalent, we embed one vertex of a copy of $K_2$ into any face of $K_4$; call this face $F$. As each face is adjacent to this face, we are left with only four possible ways to embed the last three vertices:

1. Embed exactly two of the remaining vertices into $F$,
2. Embed exactly one of the remaining vertices into $F$, and embed the last two remaining vertices into one distinct face,
3. Embed exactly one of the remaining vertices into $F$, and embed the last two remaining vertices into two distinct faces.

or

4. Embed exactly one of the remaining vertices into each of the remaining faces.

Thus, the graph $D_2$ has only four distinct embeddings into $S^2$, up to equivalence. We note that in a similar way, $K_4$ with one subdivision on two nonadjacent edges and $K_4$ with one subdivision on three pairwise adjacent edges have a unique embedding into $S^2$, which determines four equivalent faces.

In regards to the graph $D_7$, as in Figure 24 we see that after embedding one vertex into a face $F_1$ of $K_4$ with one subdivision on two nonadjacent edges, we are only left with the options (1), (3) (minus a vertex), and (4).

In regards to the graph $D_8$, as in Figure 25 we see that after embedding one vertex into a face $F_2$ of $K_4$ with one subdivision on three pairwise adjacent edges, we have all of the four options above to place our remaining vertices, as well as a fifth option to place all three of the remaining vertices into $F_2$.

Now, regarding the graph $D_3$, as in Figure 16 we see that subdividing an edge of $K_4$ yields two equivalence classes of faces; those that are bounded by an edge with a subdivision, and those that are not. So, when we identify the two endpoints of a copy of $K_2$ with two subdivisions to the two vertices of $K_4$ with one subdivision that are not incident to an edge with a subdivision, we see that we are forced to embed this copy of $K_2$ with two subdivisions into one of two equivalent faces of $K_4$. 
with a subdivision; a face that is not bounded by an edge with a subdivision. Thus, \( \mathcal{D}_3 \) has a unique embedding into \( S^2 \), up to equivalence.

Hence, as can be seen in Figures 14, 16, 24, and 26, \( \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_7, \) and \( \mathcal{D}_8, \) are intrinsically type II planar 3-linked.

Now, consider the graph \( \mathcal{D}_6 \). First, we embed \( K_4,2 \) into \( S^2 \). Because of the symmetry of \( K_4,2 \), this embedding is unique, up to equivalence. Also, this embedding determines four faces, each of which are equivalent. In regards to Figure 22, we see that there are only three ways to embed two disjoint vertices into faces of \( K_4,2 \) in \( S^2 \); embedding both vertices into one face, embedding one vertex into two adjacent faces, or embedding one vertex into two nonadjacent face. Thus, there are three distinct ways to embed \( \mathcal{D}_6 \) into \( S^2 \). Hence, as seen in Figure 22, \( \mathcal{D}_6 \) is intrinsically type II planar 3-linked.

Consider the graph \( \mathcal{D}_{9,\alpha} \). We embed \( K_5 - e \) into \( S^2 \). This embedding is unique, up to equivalence, as \( K_5 - e \) is 3-connected, and determines six faces which are equivalent. This leaves us with three options; embed all three vertex components into one face, embed exactly two vertex components into one face, or embed exactly one vertex component into any face. Moreover, when we embed exactly two vertex components into one face of \( K_5 - e \), we have sub-options to embed the last vertex component into one of two adjacent faces (these two faces are distinct since two vertices have been embedded into one particular face), or into a nonadjacent face. Furthermore, when we embed exactly one vertex into any face, we have sub-options to have one vertex in three pairwise adjacent faces, one vertex in three non-pairwise adjacent faces, or one vertex in two adjacent faces and one vertex into a pairwise nonadjacent face. Additionally, when we embed one vertex into three non-pairwise adjacent faces, we have two sub-sub-options when we choose how to place the three vertices. Thus, \( \mathcal{D}_{9,\alpha} \) has a total of eight distinct embeddings into \( S^2 \). Hence, as seen in Figure 28, \( \mathcal{D}_{9,\alpha} \) is intrinsically type II planar 3-linked.

Next, consider the graph \( \mathcal{D}_{10,\alpha} \). We embed \( K_3,2 \) with one subdivision on three pairwise adjacent edges into \( S^2 \). Because of the symmetry of \( K_3,2 \) with one subdivision on three pairwise adjacent edges, this embedding is unique, up to equivalence, and determines three equivalent faces. In regards to Figure 30, we see that there are only four ways to embed our remaining four disjoint vertices; embed exactly four vertices into one face, embed exactly three vertices into one face, embed exactly two vertices into any face, or embed exactly two vertices into exactly one face. Thus, \( \mathcal{D}_{10,\alpha} \) has four distinct embeddings into \( S^2 \). Hence, by Figure 30, \( \mathcal{D}_{10,\alpha} \) is intrinsically type II planar 3-linked.

Now, consider the graph \( \mathcal{D}_{11,\alpha} \). In regards to Figure 32, we see that \( \mathcal{D}_{11,\alpha} \) minus three disjoint vertices has a unique embedding into \( S^2 \), as the only vertex left that is not contained by \( K_3,2 \) can only be embedded into one of two equivalent faces of \( \mathcal{D}_{11,\alpha} \) minus three disjoint vertices. So, we embed this particular vertex into any one of these two possible faces; call this face \( H \). Now, we notice that whenever we embed another vertex into \( H \), the resultant embedding is type II spherical 3-linked. Similarly, whenever we embed any two vertices into one face, we have that the resultant embedding is type II spherical 3-linked. Hence, as there are only three faces in \( \mathcal{D}_{11,\alpha} \) minus three disjoint vertices, we have that \( \mathcal{D}_{11,\alpha} \) is intrinsically type II planar 3-linked.
Continuing, consider the graph $D_{12}$. In regards to Figure 34, we see that $D_{12}$ minus two disjoint vertices has two distinct embeddings into $S^2$, as the only two vertices left that are not contained by $K_{3,2}$ can be embedded into a common face, or not. Notice, when $D_{12}$ minus two disjoint vertices is embedded in these two vertices in a common face, the resultant embedding is type II spherical 3-linked no matter where we embed the two disjoint vertices. Furthermore, when we embed $D_{12}$ minus two disjoint vertices with these two vertices in adjacent faces, we have that whenever we embed one vertex into one of these two faces, or whenever we embed both disjoint vertices into a common face, the resultant embedding will be type II spherical 3-linked. Thus, we have that $D_{12}$ is intrinsically type II planar 3-linked.

Lastly, consider the graph $D_{13}$. In regards to Figure 36, we see that $D_{13}$ minus one disjoint vertex has only two distinct embeddings into $S^2$, up to equivalence, one of which is type II spherical 3-linked no matter where another vertex is added. Moreover, observing Figure 36, we see that no matter where we place a vertex in any embedding of $D_{13}$ minus one disjoint vertex results in an embedding that is type II spherical 3-linked. Thus, $D_{13}$ is intrinsically type II planar 3-linked.

Finally, for each $1 \leq n \leq 6$ and each $7 \leq m \leq 13$, we note that the graphs $D_n$ and $D_m$ are minor-minimal with respect to being intrinsically type II spherical 3-linked, as can be seen in Figures $nB$ and $mB$, respectively.

Our proof of Proposition 3.5 is now complete.

\[\square\]

Corollary 3.6. The graphs $D_3'$, $D_4'$, $D_4''$, $D_5''$, $D_5'''$, $D_7\alpha$, $D_8\alpha$, $D_9\alpha$, $D_{10}\alpha$, and $D_{11}\alpha$ are minor-minimal with respect to being intrinsically type II spherical 3-linked.

Proof. Recall, the Sub-Dangle and Vert-Bar moves preserve minor-minimality of intrinsic type II spherical 3-linkings when certain criteria are met.

By Proposition 3.5, $D_3$, $D_4$, and $D_5$ fulfill requirement $i$ of the Sub-Dangle move. By examining Figures 16, 18, and 20, we see that each of these graphs also fulfills requirement $ii$. Lastly, by examining Figures 39, 41, and 45 we see that each of these graphs fulfill requirements $iii$ and $iv$, as well. Thus, $D_3'$, $D_4'$, and $D_5'$ are minor-minimal with respect to being intrinsically type II spherical 3-linked.

Now, we see that $D_4'$ and $D_5'$ fulfill requirements $i$ and $ii$ of the Sub-Dangle move. By examining Figures 43 and 47 we see that these two graphs fulfill requirements $iii$ and $iv$, as well. Thus, $D_4'''$ and $D_5'''$ are minor-minimal with respect to being intrinsically type II spherical 3-linked. Similarly, by examining Figure 49 we see that $D_5'''$ is minor-minimal with respect to being intrinsically type II spherical 3-linked.

Continuing, by Proposition 3.5 we have that $D_{7\alpha}$, $D_{8\alpha}$, $D_{9\alpha}$, $D_{10\alpha}$, and $D_{11\alpha}$ fulfill requirement $i$ of the Vert-Bar move. By examining Figures 24, 26, 28, 30, and 32 we see that these graphs also fulfill requirements $ii$ and $iii$. Finally, by examining Figures 25, 27, 29, 31, and 33 we see that these graphs fulfill requirements $iv$ and $v$, as well. Thus, $D_{7\beta}$, $D_{8\beta}$, $D_{9\beta}$, $D_{10\beta}$, and $D_{11\beta}$ are minor-minimal with respect to being intrinsically type II spherical 3-linked.

This completes our proof.

\[\square\]
Figure 12. $D_1$ has a unique embedding into $S^2$, up to equivalence.

Figure 13. $D_1$ has only two equivalence classes of edges; those that are incident to a vertex of degree 2, and those that are not.

Figure 14. $D_2$ has four distinct embeddings into $S^2$, up to equivalence.

Figure 15. $D_2$ has only two equivalence classes of edges; those that are contained by $K_4$, and those that are not.
Figure 16. $D_3$ has a unique embedding into $S^2$, up to equivalence.

Figure 17. $D_3$ has seven equivalence classes of edges, as highlighted in Figure 16.

Figure 18. $D_4$ has a unique embedding into $S^2$, up to equivalence, since $D_4$ is 3-connected (disregarding subdivisions).

Figure 19. $D_4$ has three equivalence classes of edges; those that are incident to two vertices of degree 2, those that are incident to two vertices of degree 3, and those that are neither.
Figure 20. $\mathcal{D}_5$ has a unique embedding into $S^2$, up to equivalence, by the symmetry of $K_{3,1,1}$ with one subdivision on three pairwise adjacent edges.

Figure 21. $\mathcal{D}_5$ has three equivalence classes of edges; that are incident to two vertices of degree 2, those that are incident to two vertices of degree 4, and those that are neither.

Figure 22. $\mathcal{D}_6$ has three distinct embeddings in $S^2$, up to equivalence.

Figure 23. Every edge in $\mathcal{D}_6$ is equivalent, based on the symmetry of $K_{4,2}$. 
Figure 24. $D_7\alpha$ has three distinct embeddings into $S^2$, up to equivalence.

Figure 25. $D_7\alpha$ has only two equivalence classes of edges; those that are incident to two vertices of degree 3, and those that are not.

Figure 26. $D_8\alpha$ has five distinct embeddings into $S^2$, up to equivalence.

Figure 27. $D_8\alpha$ has only two equivalence classes of edges; those that are incident to two vertices of degree 3, and those that are not.
Figure 28. $\mathcal{D}_{9\alpha}$ has eight distinct embeddings into $S^2$, up to equivalence.

Figure 29. $\mathcal{D}_{9\alpha}$ has only two equivalence classes of edges; those that are incident to two vertices of degree 4, and those that are not.

Figure 30. $\mathcal{D}_{10\alpha}$ has four distinct embeddings into $S^2$, up to equivalence.
Figure 31. $D_{10\alpha}$ has only two equivalence classes of edges; those that are incident to two vertices of degree 2, and those that are not.

Figure 32. $D_{11\alpha}$ has ten distinct embeddings into $S^2$, up to equivalence.

Figure 33. $D_{11\alpha}$ has four equivalence classes of edges; those that are incident to a vertex of degree 1, those that are incident to two vertices of degree 2, those that are incident to two vertices of degree 3, and those that are neither.
Figure 34. \( \mathcal{D}_{12} \) has four equivalence classes of edges.

Figure 35. \( \mathcal{D}_{12} \) has four equivalence classes of edges; those that are incident to a vertex of degree 1, those that are incident to two vertices of degree 2, those that are incident to two vertices of degree 3, and those that are neither.

Figure 36. \( \mathcal{D}_{13} \) has four distinct embeddings into \( S^2 \), up to equivalence.

Figure 37. \( \mathcal{D}_{13} \) has only two equivalence classes of edges; those that are incident to a vertex of degree 1, and those that are not.
Figure 38. $D'_3$ is the graph obtained by applying a sub-dangle move to $D_3$.

Figure 39. $D_3$ satisfies the requirements to apply the sub-dangle move.

Figure 40. $D'_4$ is the graph obtained by applying a sub-dangle move to $D_4$.

Figure 41. $D_4$ satisfies the requirements to apply the sub-dangle move.

Figure 42. $D''_4$ is the graph obtained by applying a sub-dangle move to $D'_4$. 
Figure 43. $D_4'$ satisfies the requirements to apply the sub-dangle move.

Figure 44. $D_5'$ is the graph obtained by applying a sub-dangle move to $D_5$.

Figure 45. $D_5$ satisfies the requirements to apply the sub-dangle move.

Figure 46. $D_5''$ is the graph obtained by applying a sub-dangle move to $D_5'$.

Figure 47. $D_5'$ satisfies the requirements to apply the sub-dangle move.
Figure 48. $D_5'''$ is the graph obtained by applying a sub-dangle move to $D_5''$.

Figure 49. $D_5''$ satisfies the requirements to apply the sub-dangle move.

Figure 50. $D_7\beta$ is the graph obtained by applying a vert-bar move to $D_7\alpha$.

Figure 51. $D_8\beta$ is the graph obtained by applying a vert-bar move to $D_8\alpha$.

Figure 52. $D_9\beta$ is the graph obtained by applying a vert-bar move to $D_9\alpha$. 
Figure 53. $\mathcal{D}_{10,\beta}$ is the graph obtained by applying a vert-bar move to $\mathcal{D}_{10,\alpha}$.

Figure 54. $\mathcal{D}_{11,\beta}$ is the graph obtained by applying a vert-bar move to $\mathcal{D}_{11,\alpha}$.

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