Topical Review

Magnetoelectric effects in Josephson junctions

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Abstract
The review is devoted to the fundamental aspects and characteristic features of the magnetoelectric effects, reported in the literature on Josephson junctions (JJs). The main focus of the review is on the manifestations of the direct and inverse magnetoelectric effects in various types of Josephson systems. They provide a coupling of the magnetization in superconductor/ferromagnet/superconductor JJs to the Josephson current. The direct magnetoelectric effect is a driving force of spin torques acting on the ferromagnet inside the JJ. Therefore it is of key importance for the electrical control of the magnetization. The inverse magnetoelectric effect accounts for the back action of the magnetization dynamics on the Josephson subsystem, in particular, making the JJ to be in the resistive state in the presence of the magnetization dynamics of any origin. The perspectives of the coupling of the magnetization in JJs with ferromagnetic interlayers to the Josephson current via the magnetoelectric effects are discussed.

Keywords: magnetoelectric effect, Josephson junction, spin–orbit coupling, magnetization dynamics, anomalous Josephson effect

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1. Introduction

Since the discovery of the Josephson effect in 1962 [1–3], there has been a growing interest in the fundamental physics [4] and applications of this effect. The achievements in Josephson-junction technology enabled the development of sensors for detecting ultralow magnetic fields and weak electromagnetic radiation, ultrafast digital rapid single flux quantum circuits, the design of large-scale integrated circuits for signal processing and general-purpose computing as well as adiabatic superconducting cells operating as an artificial neuron and synapse [5–7].

Theoretical investigations of hybrid structures involving superconductors and ferromagnets and subsequent experimental realization of superconductor/ferromagnet/superconductor Josephson junctions (JJs) [4, 8, 9] have led to the discovery of spin-triplet Cooper pairs, thus giving rise to a synergy between superconductivity and spintronics. The emergent new field was called superconducting spintronics and is being actively developed now [10, 11]. One of the key effects in spintronics is the so-called magnetoelectric effects. In the most general sense the field embrace all the effects related to the coupling and interconversion of the charge and spin degrees of freedom. The field already went beyond the framework of the fundamental physics only and, in particular, a scalable energy-efficient magnetoelectric spin–orbit logic has been proposed [12] thus potentially opening new technology paradigm for improving energy efficiency in beyond-CMOS computing devices.

Here our goal is to review of current understanding of fundamental aspects of magnetoelectric effects in JJs, which potentially open new perspectives in superconducting spintronics. As an introduction, we discuss the fundamental aspects of the related magnetoelectric effects in nonsuperconducting systems briefly and then their analogues in superconducting materials and structures. The main part of the review is devoted to the magnetoelectric effects in a particular type of superconducting hybrids—JJs. In section 2 we discuss the manifestations of the direct and inverse magnetoelectric effects in different types of JJs, section 3 is devoted to the role of the magnetoelectric effects in the magnetization dynamics and electrical control of the ferromagnet magnetization in the JJs via ferromagnets. A specific for superconductivity magnetoelectric effect—generation of triplet superconductivity by a moving condensate is discussed in section 4. Section 5 provides a short summary of the current situation in the field. Studies of magnetoelectric effects have a long history [13–15]. The most common view is that the magnetoelectric media are characterized by unconventional equilibrium responses to an electric field \( \mathbf{E} \) and a magnetic field \( \mathbf{B} \). While \( \mathbf{E} \) induces only an electric polarization in ordinary materials, it also creates a magnetization in magnetoelectric materials. Similarly, the magnetic field \( \mathbf{B} \) in magnetoelectric materials generates an electric polarization in addition to a magnetization. Further, the advent of multiferroic materials [16, 17] with their large magnetoelectric couplings has greatly boosted current interest in magnetoelectricity. But we do not touch on this physics in the review. Here we focus on the related phenomena of current-induced spin polarization and the inverse effects. In this field besides the well-established spin Hall effect and inverse spin Hall effect [18–30] the direct and inverse magnetoelectric effects are also known. The essence of the direct magnetoelectric effect is creating a stationary spin density \( S_x \) along the \( a \) direction in spin space in response to an electric field \( E_k \) applied in the \( k \) direction in the real space:

\[
S^n = \sigma^b_k E_k. \tag{1}
\]

The effect is known for a wide class of systems. It is theoretically investigated and measured for spin–orbit coupled materials [23, 31–33], where it is also called the Edelstein effect. In this case the Edelstein conductivity \( \sigma^a_k \) is proportional to the spin-orbit coupling (SOC) constant of the material. The mutual orientation of the applied electric field and the induced electron spin polarization is determined by the particular form of the SOC. Let’s consider the examples of Rashba and Dresselhaus SOC. These types of SOC were originally discussed for noncentrosymmetric zinc-blende or wurtzite semiconductors by Dresselhaus [34], and Rashba [35]. The Rashba-type SOC also arises due to the structural inversion asymmetry (SIA). SIA typically occurs at the surfaces or interfaces. An important realization of a system with Rashba-type spin–orbit coupling is a 2D electron gas in doped semiconductor heterostructures [36, 37], that support an electron gas at the interface between two materials. Another possibility to study the Rashba-effect in 2DEG are surfaces that support a surface state, e.g. in Au(111) [38]: the electrons of the surface state move in a potential gradient that is provided by the surface itself. The Hamiltonian term accounting for the Rashba SOC takes the form \( H_R = \alpha \hat{z} (\sigma \times \mathbf{p}) \), where \( \alpha \) is the Rashba constant and \( \hat{z} \) is the unit vector along the \( z \)-axis, chosen along the polar vector of the material, which determines the direction of the broken inversion symmetry, \( \sigma = (\sigma_z, \sigma_y, \sigma_x) \) is the vector of Pauli matrices in spin space and \( \mathbf{p} \) is the electron momentum. For Rashba SOC \( \sigma^z_k = -\sigma^a_k \), while the other components of \( \sigma^a_k \) are zero. Therefore, for this case the induced spin polarization lies in the plane perpendicular to the polar vector and is perpendicular to the applied electric field. The simplest form of the Dresselhaus SOC Hamiltonian, realized in the presence of strain along the \( (001) \) direction is \( H_D = \beta_D (p_x \sigma_z - p_z \sigma_y) \), where \( \beta \) is the Dresselhaus SOC constant. If the current direction coincides with \( x \) or \( y \) axes, the induced spin polarization is directed along the current.

The reason for the electrically induced spin polarization and qualitative understanding of its direction with respect to the current in normal quasi-2D systems is clearly seen from figure 1, where the helical Fermi surfaces of the Rashba and Dresselhaus materials are demonstrated. The electric current results in the total shift of all the Fermi surfaces by \( q \) along the current direction. This leads to the nonzero average spin polarization of the corresponding electronic states in each of the helical bands. Due to the different spin structure of the helical Fermi surfaces the directions of the resulting electron polarizations in the Rashba and Dresselhaus cases are different. The average polarization is perpendicular to the applied current for the Rashba case and can have different mutual
orientations with the current for the Dresselhaus SOC depending on the orientation of the current with respect to the crystal axes. The split helical Fermi surfaces contribute to the polarization in opposite directions, as it is seen from figure 1. This leads to a great reduction of the total polarization, the resulting effect is nonzero only due to the difference between the Fermi momenta for the helical subbands. Therefore, the current induced spin polarization is always proportional to the ratio $\Delta_{so}/\varepsilon_F$, where $\Delta_{so}$ is the energy splitting of the helical subbands, $\Delta_{so} \sim \alpha(\beta D)p_F$ for the Rashba (Dresselhaus) case.

The direct magnetoelectric effect has also been predicted and measured in topological insulators (TIs) [39–43], where the mutual orientation of the spin polarization and the current is the same as for the case of Rashba materials. In addition, the direct magnetoelectric effect also exists in spin-textured ferromagnets, where the induced spin polarization takes the form:

$$S_\perp = -\frac{b_{ij}}{J_{sd}M^2}M \times \partial_t M + \frac{c_{ij}}{J_{sd}M} \partial_t M,$$

(2)

which results in the well-known spin transfer torque [44] acting on the magnetization according to:

$$T = J_{sd}M \times S_\perp = b_{ij}\partial_t M - \frac{c_{ij}}{M} M \times \partial_t M,$$

(3)

where $M$ is the saturation magnetization in the ferromagnet, $J_{sd}$ is the coupling constant of the exchange interaction between the s-band conduction electrons and d-band localized electrons responsible for the magnetism. $S_\perp$ means the component of the current-induced spin polarization, perpendicular to the magnetization direction, because it is this component that leads to a torque on the magnetization.

There is also an inverse magnetoelectric effect (it is also called by the spin-galvanic effect), which consists of generating a charge current $j_k$ by a steady spin imbalance, which can be induced, for example, by a time-dependent magnetic field via the paramagnetic effect [45]:

$$j_k = \sigma_k^a(\mu_B \dot{B}^a),$$

(4)

where $g$ is the Lande factor, $\mu_B$ is the Bohr magneton, and $\dot{B}^a$ is the time derivative of the magnetic field component along the $a$ axis. The inverse magnetoelectric effect has been observed in experiments with spin–orbit coupled materials [46, 47] and TIs [48, 49].

The inverse magnetoelectric effect also takes place in spin-textured metallic ferromagnets. In this case, it manifests itself as the so-called electromotive force (emf) induced by the magnetization dynamics [50–60]:

$$F_i = \frac{\hbar}{2}[m(\partial_t m \times \nabla m) + \beta(\partial_t m \nabla m)],$$

(5)

where $m(r,t)$ is the unit vector in the direction of the magnetization and $\beta$ is a phenomenological parameter. Due to the existence of the electromotive force the magnetization dynamics leads to appearance of an additional voltage drop. This voltage can vary in the range from nV to $\mu$V [59] and in special situations can be used for electrical detection of the presence of magnetization dynamics [61].

Conceptually the same magnetoelectric effects also take place in superconducting systems. However, here the physical situation is somewhat different because of the presence of the superconducting condensate. In contrast to the normal case, in a superconductor an equilibrium electric supercurrent can flow in the absence of an external electric field and is directly related to the gauge invariant superconducting condensate superfluid velocity, $v_F$, and $\varphi$ is the phase of the superconducting order parameter and $A$ is the vector potential. That leads to two consequences: (a) a supercurrent can generate an equilibrium spin polarization in the presence of intrinsic SOC [62–65], extrinsic impurity-induced SOC [66, 67], in TI-based superconducting heterostructures [68, 69] and in superconductor/ferromagnet hybrids with spin-textured ferromagnets [70, 71] and (b) in contrast to the normal case, in superconductors a static Zeeman field $B$ can induce a supercurrent $j_k$:

$$j_k = \chi^a h^a,$$

(6)

where $h^a = (1/2)\mu_B B^a$. This effect has been obtained for a case of a 2D superconductor with Rashba SOC [72]. It was also discussed for heterostructures consisting of the superconducting and ferromagnetic layers with SOC [73–77] or for a ferromagnet/superconducting TI hybrid structures [78], when the exchange field is not induced by the externally applied magnetic field, but is generated by the proximity to the ferromagnet. In the case of heterostructures with thick enough superconducting layer (the thickness should be much larger

Figure 1. Spin-split helical Fermi surfaces for (a) Rashba and (c) Dresselhaus spin–orbit coupled materials. (b), (d) Current-induced shift of the Fermi surface. The original Fermi surface in the absence of the applied current is shown by the dashed lines. The direction of the accumulated spin in each of the subbands is shown by arrows.
than the superconducting coherence length) [73, 76] such a state with a spontaneous supercurrent flowing along the S/F interface and decaying into the depth of the superconductor can be a true ground state of the system. The same is valid if the exchange field is spatially inhomogeneous [74, 75] and even a topologically nontrivial vortex states can appear under the appropriate conditions [77, 78].

But for the case of the superconductors with an intrinsic SOC in the homogeneous Zeeman field the state carrying homogeneous nonzero supercurrent is not the true ground state. In the true ground state the superconducting phase gradient is developed in order to compensate the supercurrent. The resulting state is characterized by the zero supercurrent and nonzero superconducting phase gradient. It is called by the helical state and is a specific for superconducting systems manifestation of the inverse magnetoelectric effect. There is an important difference between the helical state and the well-known inhomogeneous FFLO state [79–82]. While in the phase-modulated FFLO state the direction of the superconducting phase gradient does not depend on the direction of the exchange field, in the helical phase they are directly related. This fact results in strong coupling between the magnetization and the condensate phase. Therefore, it leads to the possibilities of the electrical control of the magnetization dynamics, which look perspective from the point of view of spintronics applications. The helical state has been predicted for superconductors with intrinsic Rashba SOC under the applied Zeeman field [83–88] and superconducting hybrids with spin-textured ferromagnets [70, 71], where the magnetic inhomogeneity plays a role of the effective SOC.

The helical state is a kind of inverse magnetoelectric effect, which is realized in the simply connected superconducting systems. Similar effects occur also in an S–X–S JJ, between two superconductors and a normal or ferromagnetic interlayer X with an intrinsic SOC or if the interlayer is a spin-textured ferromagnet. In a JJ the supercurrent depends on the phase difference $\phi$ between the superconducting electrodes. Similar to the simply connected superconductors a Zeeman field may induce a supercurrent through the junction at zero phase difference between the superconductors according to equation (6). In the ground state of the junction this ‘anomalous supercurrent’, generated by the inverse magnetoelectric effect, is compensated by the phase shift $\phi_0 \neq 0, \pi$. It is called by the anomalous ground state phase shift and the JJs manifesting this effect are called $\varphi_0$-junctions.

The $\varphi_0$-junctions have been predicted in a wide class of systems including S/F/S junctions with intrinsic SOC, S/N/S junctions with intrinsic SOC under applied Zeeman field [89–100], S/topological insulator/S junctions under the applied Zeeman field or if the Zeeman field in the TI surface states is induced by the proximity to a ferromagnet (S/TI–F/S junctions) [101–105] and also in S/F/S junctions with spin-textured interlayers [71, 106–118]. Below we will discuss all the mentioned classes of systems in more details. The anomalous phase shift has been observed experimentally for Al/InAs/Al JJs [119], in JJs via nanowire quantum dots [120], in Bi$_2$Se$_3$ JJs [121] and in JJs via bismuth nanowires [122] under the applied magnetic field. The magnetoelectric nature of the anomalous phase shift has been unveiled in [98]. It has also been reported very recently that the anomalous phase shift is a key ingredient of the mechanism providing an extremely long-range interaction of magnetic moments in a coupled system of JJs with magnetic interlayers [123]. There is also a recent review [124], specially devoted to the physics of the anomalous phase shift.

Naturally, the equilibrium direct magnetoelectric effect discussed above can also occur in JJs. Indeed, it has been predicted for JJs via normal interlayers with Rashba SOC in diffusive [125] and ballistic [126] systems and via TI interlayers [68]. The effect plays a key role in the electrical control of magnetization in S/F/S JJs, which is also discussed in detail below.

For completeness, we briefly mention other reported effects in superconducting hybrids, which can also be viewed as magnetoelectric ones but are not discussed in detail in this review. One group of effects is related to different types of quasiparticle nonequilibrium in the system. Among them are spin-charge conversion effects in superconducting hybrids. They involve a nonequilibrium spin polarization and spin current pumped into a superconducting system by some external source. It has been shown that for the Rashba SOC and SOC caused by spin–orbit impurities such a nonequilibrium spin distribution can generate the electric current and electric potential in superconductors [127–129]. This nonequilibrium situation resembles much the analogous inverse magnetoelectric and spin Hall effects in normal systems.

2. Direct and inverse magnetoelectric effects in Josephson junctions: main properties and physical systems

2.1. Anomalous phase shift—a realization of inverse magnetoelectric effect in Josephson junctions with spin–orbit coupling and Zeeman field

The minimal form of the current-phase relation (CPR) characterizing the dc Josephson effect is given by $j(\phi) = j_c \sin \phi$. Here $j_c$ is the total superconducting current flowing across the junction, $|j_c|$ is the critical current and $\phi$ is the phase difference between superconducting electrodes [3, 4]. The ordinary JJs have $j_c > 0$ yielding the zero phase difference ground state $\phi = 0$. In certain cases $j_c < 0$ leading to the ground state $\phi = \pi$. Such $\pi$-junctions are realized in S/F/S JJs [8, 130–132], non-equilibrium S/N/S JJs [133], non-equilibrium S/F/S systems [134], $d$-wave superconductors [135, 136], semiconductor nanowires [137], gated carbon nanotubes [138] or multi-terminal Josephson systems [139]. The $\pi$-junctions can be used in scalable superconducting logic and quantum computers [140–143].

Even more exotic situation occurs in systems with magnetoelectric effects where the $\varphi_0$-junctions are realized. They are described by the CPR [92]:

$$j(\phi) = j_c \sin(\phi + \varphi_0),$$

(7)
with **anomalous (spontaneous) phase shift** $\phi_0 \neq 0, \pi$. In this case there is a finite supercurrent at zero phase difference $I_{an} = j_r \sin \phi_0$ called the **anomalous (spontaneous) current**.

The Josephson energy $E_J = j_r [1 - \cos(\phi + \phi_0)]$ yields the ground state with non-trivial phase difference $\phi = -\phi_0$ and zero current $I(\phi_0) = 0$. Such a phase-shifted ground state is analogous to the helical state in the homogeneous superconductor discussed above. The general symmetry requirements for obtaining the $\phi_0$-Josephson junctions are the same as for having magnetoelastic coupling in the homogeneous superconductor. That is, we need to combine the two symmetry breaking mechanism. First, the time-reversal symmetry is to be broken by the Zeeman field $h$ inside the JJ. Second, the orbital and spin degrees of freedom should be coupled so that it is impossible to invert the magnetic moment by spin rotation independently from the orbital coordinates. By analogy with spontaneous current (6) one can construct the phenomenological expression for the anomalous phase shift $\phi_0 \propto \chi^2 h^a$, where $a$ is the axis across JJ. This type of symmetry breaking is enabled e.g. by Rashba-type SOC $\chi^2 \propto \varepsilon \alpha h^b$ with anisotropy vector $n$. In this case the spontaneous current is $j \propto n \times h$ and the anomalous phase shift is $\phi_0 \propto J \cdot (n \times h)$.

Theoretical description of the $\phi_0$-junctions is significantly more challenging than that of the 0 and $\pi$-junctions. To obtain the anomalous phase shift one has to include the magnetoelastic coupling which is beyond the standard quasiclassical approximation [64, 66, 68, 71, 96, 98, 116, 118, 126, 144–152]. For Rashba-type SOC described by the hamiltonian $H_{R} = \alpha [p \times n] \sigma$ in the ballistic regime and for large Rashba constant $\alpha$, the anomalous phase shift is given by [92]:

$$\phi_{0b} = \frac{4\hbar c d}{\hbar v_F} \frac{1}{(\hbar v_F)^2},$$

(8)

where $d$ is the length of the JJ interlayer and $v_F$ is the Fermi velocity of the electrons in the interlayer. In the diffusive regime for weak $\alpha$, highly transparent interfaces and neglecting spin-relaxation, the predicted result for the anomalous phase shift is:

$$\phi_{0d} = \frac{\pi m^* \hbar (\alpha d)^3}{3m D},$$

(9)

where $\tau$ is the elastic scattering time, $m^*$ is the effective electron mass and $D$ is the diffusion constant [96].

The anomalous Josephson effect based on the systems with SOC has been found in recent experiments. Anomalous phase junctions were demonstrated in Bi-nanowires [122], InSb nanowires in a quantum dot geometry [120], in JJ using Bi$_2$Se$_3$ [121], in heterostructures formed by InAs and epitaxial superconducting Al [119] and also in JJs via InAs nanowires [153]. In the quantum dot realization [120] and in [119] the gate-tunable phase has been achieved. The JJs of [120] support a few modes and consequently exhibit small critical currents, and the structures investigated in [119] have high interface transparency and large critical currents. In Bi$_2$Se$_3$, which is a TI, large planar $\phi_0$-junction are possible [121], however, they are not gate-tunable. In all the experimental works, the exchange field inside the interlayer of the JJ, which is required for the generation of the anomalous phase shift, has been created by the externally applied in-plane magnetic field via the Zeeman effect $h = (1/2) g \mu_B B_z$, where $g$ is the electron $g$-factor for the corresponding material and $B_z$ is the magnetic field component, perpendicular to the anisotropy vector $n$ (along $z$, see figure 2) and to the Josephson current direction.

The anomalous phase shift has been observed directly through measurements of the current-phase relationship in a Josephson interferometer. A typical current-biased measurement of a single JJ shows no measurable signature. Under the applied current, the phase difference across the JJ changes to maximize the critical current. This means that any phase shift applied to such a system will be invisible. Therefore, the experimental works use SQUID geometry, whose primary property is phase sensitivity.

One scheme of measurements, which was realized in [120–122], is based on the asymmetric SQUID configuration. The SQUID consists of two junctions in parallel with very different critical currents $I_{1} > I_{2}$, where $I_{1}$ is the critical current of the reference JJ and $I_{2}$ is the critical current of the investigated JJ. The sketch of the asymmetric SQUID is presented in figure 2(a). The phase differences $\varphi_{1}$ and $\varphi_{2}$ for the two junctions are linked by the relation $\varphi_{1} - \varphi_{2} = 2 \pi \Phi_0 / \Phi_0$, where $\Phi = B_S$ is the magnetic flux enclosed in the SQUID of surface $S$, $B_z$ is a magnetic field component perpendicular to the sample, i.e. along $e_c$ and $\Phi_0$ is the flux quantum. As the critical current $I_{1}$ is much higher than $I_{2}$, then $\varphi_{1} = \pi / 2$ and $I_{1} = I_{1c} + I_{1c} \cos[2 \pi \Phi / \Phi_0 - \varphi_{0}]$. Thus, a measurement of the critical current $I_{1c}$ as function of $B_z$ provides a measure of the current $I_{2}$ as function of $\varphi_{2}$, i.e. the CPR. In the experiments the Zeeman field $h$ has been induced by the externally applied in-plane magnetic field $h = (1/2) g \mu_B B_z$, where $B_z = B \cos \theta$ is the in-plane component of the applied magnetic field, see figure 3(b). Then the anomalous phase results in the increased oscillation frequency of the critical current as a function of the applied magnetic field [121] $2 \pi \Phi / \Phi_0 - \varphi_{0} = \omega B_z$, where $\omega = 2 \pi \sin \theta / \Phi_0 - \varphi_{0} / B$. The corresponding experimental results adopted from [121] are presented in figure 3(a).

The other measurement scheme is to consider the symmetric SQUID, where the JJs has the same critical currents $I_{1} = I_{2} = I_{c}$ [153]. In this case the total supercurrent through the interferometer is:
concluded that their JJs agree with the theory developed in [119]. The authors compare their results to the theoretical predictions expressed by equations (8) and (9) and concluded that both results return values of $\phi_0$ which are much smaller than the observed ones. A possible explanation is that the investigated JJs were in the short junction limit with a fully developed proximity effect inside the interlayer, which is not described by those theoretical results.

As it was mentioned earlier, in all the discussed experiments, the exchange field generating $\phi_0$ has been created by the externally applied field. Assouline et al [121] and Mayer et al [119] report the linear dependence of the anomalous phase shift on the in-plane magnetic field, as it is predicted by equations (8) and (9). The experimental results reported in [153] are more complicated. The nonmonotonic dependence of the anomalous phase shift on the applied in-plane field with a maximum shift at $B_y \approx \pm 5 \text{ mT}$ and saturation for $|B_y| > 30 \text{ mT}$ has been observed, see figure 4(b). The authors suggest that this behavior is due to magnetic Kondo impurities in the nanowire. Due to the antiferromagnetic nature of the Kondo interaction, the effective exchange field created by these unpaired spins is opposite to the Zeeman field generated by $B_y$, so that the two contributions are competing in the anomalous phase with a partial cancellation. The total anomalous phase shift is divided into two contributions $\phi_0(B_y) = \phi_{\text{int}}(B_y) + \phi_{\text{ext}}(B_y)$, where $\phi_{\text{int}}(B_y)$ is an intrinsic phase shift, which is present even in the absence of the in-plane magnetic field if a finite $B_y$ has been previously applied. Since it stems from a ferromagnetic ordering, $\phi_{\text{int}}$ depends only on the history of $B_y$ and shows a hysteresis in the back and forth sweep direction. The extrinsic contribution to the phase shift, $\phi_{\text{ext}}$ stems directly from the external magnetic field. The dependence $\phi_{\text{ext}}(B_y)$ is characterized by a linear increase at low magnetic fields up to a maximum phase shift of $\pm \pi/2$. Based on the diffusive theory [96], Strambini et al [153] developed a model explaining the observed behavior of $\phi_{\text{ext}}$. It leads to the relation $\phi_{\text{ext}} \approx C \phi_0 B_y + O(B^2)$. This model provides a reasonable explanation of the obtained data and predicts a nonuniversal

$$I_s = 2I_c \sin \delta_0 \cos \left[ \frac{1}{2} \frac{\Phi}{\Phi_0} + \frac{\phi_{\text{tot}}}{2} \right],$$

where $\delta_0 = (\phi_0^{(1)} - \phi_0^{(2)}) / 2 - (\phi_1 + \phi_2) / 2$ and $\phi_{\text{tot}} = \phi_0^{(1)} - \phi_0^{(2)}$ is the total anomalous phase shift built in the interferometer. With the geometry realized in [153] and shown in figure 2(b), the two junctions experience the same in-plane magnetic field orientation but the supercurrent flow in opposite directions resulting in $\phi_0^{(1)} = -\phi_0^{(2)}$ and $\phi_{\text{tot}} = 2\phi_0$. The stable state configuration of the SQUID is achieved by minimizing the total Josephson free energy obtained at $\delta_0 = \pi/2$. Therefore, the maximum supercurrent, which can be sustained by the system, is:

$$I_s(\Phi) = 2I_c \cos \left[ \frac{\Phi}{\Phi_0} + \frac{1}{2} \phi_{\text{tot}} \right].$$

Therefore, the dependence $I_s(\Phi)$ contains the anomalous phase shift. Notably, there is a replica of the $I_s(\Phi)$ oscillations in the voltage drop $\Delta V(\Phi)$ when $I > I_s$, and the SQUID operates in the dissipative regime, as conventionally realized with strongly overdamped JJs. This replica is used in [153] for measurements of the anomalous phase shift. The results, adopted from [153], are represented in figure 4. Figure 4(a) demonstrates evolution of $\Delta V(\Phi)$ at constant current bias as a function of the applied in-plane magnetic field. Figure 4(b) shows the resulting anomalous phase shift extracted from the data in figure 4(a) according to the resistively shunted junction (RSJ) model relation $\Delta V = (R/2) \sqrt{T^2 - 4I_c^2 \cos(\pi \Phi/\Phi_0 + \phi_{\text{tot}}/2)^2}$. Experimental data provide rather large values of the anomalous phase shift of the order of $\pi$. For example, in [121] $\phi_0 \approx 0.9\pi$ was measured at $B_y \approx 100 \text{ mT}$. It was compared to the theoretical predictions for the ballistic equation (8) $\phi_{\text{bal}} \approx 0.01\pi$ and diffusive equation (9) $\phi_{\text{diff}} \approx 0.94\pi$. The authors concluded that their JJs agree with the theory developed in [96]. In [153] it was also reported that their data could be fitted by the diffusive theory [96]. Rather high values of $\phi_0 \approx \pi/2$ at $B_y \approx 400 \text{ mT}$ (the particular value of $\phi_0$ depends on the gate voltage) have also been observed in [119].
strong dependence of the CPR on the magnetization direction and the conductive order parameter \( \Delta \) of the TI due to the proximity effect with a high-

effective exchange field was induced in the surface states of the 3D TI. For a given momentum direction the spinless retarded Green’s function of the surface states of TI by the chemical potential, \( \mu \), is assumed to be of the form \( \tilde{A}_r^{\tau} = \tilde{A}^{\tau} + (i/v_{F}) [h, \tilde{r}, \tilde{A}]_{\tau} \). Equation (14) should be supplemented by the normalization condition \( \tilde{g} \otimes \tilde{g} = 1 \) and the boundary conditions at \( x = \mp d/2 \). It is assumed that the JJ is formed at the interfaces and, therefore, these interfaces are considered as fully transparent. In this case, the boundary conditions are reduced to continuity of the spinless retarded Green’s function \( \tilde{g} \) obeys the following transport equations in the ballistic limit:

\[
-iv_{F} n_{F} \nabla \tilde{g} = \left[ \epsilon_{s} - \Delta, \tilde{g} \right], \tag{14}
\]

where \( [A, B]_{\otimes} = A \otimes B - B \otimes A \) and \( A \otimes B = \exp[(i/2)\partial_{\epsilon_{s}}\partial_{\epsilon_{t}} - \partial_{\epsilon_{s}}\partial_{\epsilon_{t}}]A(\epsilon_{s}, \epsilon_{t})B(\epsilon_{s}, \epsilon_{t}) \) are Pauli matrices in particle-hole space with \( \tau_{\pm} = (\tau_{x} \pm i\tau_{y})/2 \). \( \Delta \) is the superconducting order parameter \( \Delta(x) \) in the particle-hole space. We assume \( \Delta(x) = \Delta \sqrt{\epsilon^{2} - d^{2}/2} \) in the particle-hole space. The spin-momentum locking allows for including \( h \) into the gauge-covariant gradient \( \nabla \tilde{A} = \nabla \tilde{A} + (i/v_{F})[h, \tilde{r}, \tilde{A}]_{\tau} \).

The 3D TI surface states host Dirac quasiparticles, exhibiting full spin-momentum locking: an electron spin always makes a right angle with its momentum. The sketch of the system under consideration and the Fermi surface of the conducting electrons in the 3D TI surface states are demonstrated in figure 5. The spin-momentum locking gives rise to a very strong dependence of the CPR on the magnetization direction. The opposite spin direction is not allowed.

2.2. Anomalous phase shift in topological insulator-based S/F/S junctions

Here we consider the JJ through TIs where the SOC is so strong that only one helical band is present. In such systems, one can expect the strongest possible spontaneous phase shift effect. At the same time, having only one helical band allows for the significant simplification of the theoretical description using the generalized quasiclassical theory. More particularly, we consider the anomalous phase shift in superconductor/ferromagnet/superconductor (S/F/S) JJs designed on a three dimensional topological insulator (3D TI) surface. At present, great progress has been made in the experimental implementation of F(TI) hybrid structures. In particular, a structure was successfully implemented experimentally, in which a sufficiently strong exchange field was induced in the surface states of the TI due to the proximity effect with a high-\( T_{c} \) ferromagnetic insulator (FI) [154–157].

The 3D TI surface states host Dirac quasiparticles, exhibiting full spin-momentum locking: an electron spin always makes a right angle with its momentum. The sketch of the system under consideration and the Fermi surface of the conducting electrons in the 3D TI surface states are demonstrated in figure 5. The spin-momentum locking gives rise to a very strong dependence of the CPR on the magnetization direction [101, 102, 104, 148]. In particular, the anomalous ground state phase shift proportional to the in-plane magnetization component perpendicular to the supercurrent direction was reported.

It is assumed that the magnetization \( M(r) \) of the ferromagnet induces an effective exchange field \( h(r) \sim M(r) \) in the underlying conductive surface layer. The discussed theory is also applicable if the Zeeman term in the Hamiltonian of the TI surface states is induced by the applied magnetic field. The hamiltonian that describes the TI surface states in the presence of an in-plane exchange field \( h(r) \) reads:

\[
\hat{H} = \int d^{2}r \psi^{\dagger}(r') \hat{h}(r') \psi(r'), \tag{12}
\]

\[
\hat{H}(r) = -iv_{F}(\nabla \times \hat{e}_{z}) \hat{\sigma} + h(r) \hat{\sigma} - \mu, \tag{13}
\]

where \( \psi = (\psi_{x}, \psi_{y}) \), \( v_{F} \) is the Fermi velocity, \( \hat{e}_{z} \) is a unit vector normal to the surface of TI, \( \mu \) is the chemical potential, and \( \hat{\sigma} = (\sigma_{x}, \sigma_{y}, \sigma_{z}) \) is a vector of Pauli matrices in the spin space. It was shown [68, 104] that in the quasiclassical approximation \( (h, \epsilon, \Delta) \ll \mu \) the Green’s function has the following spin structure: \( \hat{g}(n_{F}, r, \epsilon) = \hat{g}(n_{F}, r, \epsilon) + (1 + n_{z} \sigma_{z})/2 \), where \( n_{z} = (n_{F}, -n_{F}, 0, 0) \) is the unit vector perpendicular to the direction of the quasiparticle trajectory \( n_{F} = p_{F}/|p_{F}| \) and \( \hat{g} \) is the spinless \( 4 \times 4 \) matrix in the particle-hole and Keldysh spaces containing normal and anomalous quasiclassical Green’s functions. The spin structure above reflects the fact that the spin and momentum of a quasiparticle at the surface of the 3D TI are strictly locked and make a right angle. It was demonstrated [68, 104, 158] that the spinless retarded Green’s function \( \hat{g}(n_{F}, r, \epsilon) \) obeys the following transport equations in the ballistic limit:

\[
-iv_{F} n_{F} \nabla \hat{g} = \left[ \epsilon_{s} - \Delta, \hat{g} \right], \tag{14}
\]
satisfying asymptotic conditions $f^R \to (\Delta/\epsilon)e^{\pm \imath \sqrt{2}/\epsilon}$ at $x \to \pm \infty$ and continuity conditions at $x = \mp d/2$ takes the form:

$$f^R_\pm = \frac{\Delta e^{\mp \imath \sqrt{2}/\epsilon}}{\epsilon} \exp \left[ \mp 2\imath (\hbar n_{\perp} - \epsilon)(d/2 \pm x)/v_x \right],$$

where the subscript $\pm$ corresponds to the trajectories $\text{sgn} v_x = \pm 1$.

The density of electric current along the $x$-axis is:

$$j_x = -\frac{eN_F v_F}{4} \int_{\epsilon_c}^{\infty} d\epsilon \int_{-\pi/2}^{\pi/2} d\phi \cos \phi \frac{\Delta^2}{\epsilon^2} \times \left[ -\frac{2\epsilon d}{v_F \cos \phi} \right] \cos \left[ \frac{2h d \tan \phi}{v_F} \right],$$

where $\phi$ is the angle, which the quasiparticle trajectory makes with the $x$-axis, $\varphi_\perp$ is the distribution function corresponding to the trajectories $\text{sgn} v_x = \pm 1$. In equilibrium $\varphi_\perp = \tanh [\epsilon/2T]$. Exploiting the normalization condition one can obtain $g^R_\pm \approx 1 - R_\pm f^R_\pm/2$. Taking into account that $g^R_\pm = -g^R_\perp$ the following final expression for the Josephson current has been obtained:

$$j_x = j_s \sin (\chi - \chi_0),$$

$$j_e = e\nu_F n_T \sum_{\epsilon_c > 0} \int_{-\pi/2}^{\pi/2} d\phi \cos \phi \frac{\Delta^2}{\epsilon^2} \times \left[ -\frac{2\epsilon d}{v_F \cos \phi} \right] \cos \left[ \frac{2h d \tan \phi}{v_F} \right],$$

$$\chi_0 = 2h d/v_F,$$

where $\epsilon_c = \pi T (2n + 1)$. It is seen that the CPR equation (17) contains the anomalous phase shift $\chi_0$. At high temperatures $T \approx T_c, \Delta$ the main contribution to the current comes from the lowest Matsubara frequency and equation (18) can be simplified further:

$$j_e = \frac{e\nu_F \pi T}{\sqrt{2}} \int_{-\pi/2}^{\pi/2} d\phi \cos \phi \times \left[ -\frac{2\pi T d}{v_F \cos \phi} \right] \cos \left[ \frac{2h d \tan \phi}{v_F} \right].$$

Equations (19) and (20) demonstrate that the dependence of the Josephson current on the exchange field and opposite spin directions. At first, we assume that the exchange field is along the $y$-axis, that is $h = h e_y$. Because the spin of an electron is strictly perpendicular to its momentum, each of the electrons composing the pair ‘sees’ its own effective exchange field, which is equal to the projection of $h$ onto its spin direction. If the electron momentum makes an angle $\phi$ with the $x$-axis, the effective field seen by the electron is $h \cos \phi$, see figure 6(a). After passing the distance $x$ along the $x$-axis the electron acquires a phase $\Phi = p_F x + (h \cos \phi / v_{Fx}) x$. The second electron from the pair with the opposite values of the momentum and spin sees the opposite effective exchange field and, therefore, acquires the same phase. The total phase gained by the pair is $(2h \cos \phi / v_{Fx}) x$ in agreement with equation (15). Taking into account that $v_{Fx} = v_F \cos \phi$ we see that for the given exchange field direction the phase acquired by the pair does not depend on the trajectory direction. Therefore, the contributions from the different trajectories to the Josephson current do not cancel each other due to the momentum averaging. As a result, the critical current is not suppressed by the exchange field, and the influence of the field only appears via the overall phase $\pm 2h d/v_F$ acquired by all the pairs traveling through the interlayer to the right (left). Because of the independence of the phase gain on the momentum direction between two scattering events, the consideration is also applicable to the diffusive case resulting in the same answer. Interestingly that the same phase
factors can result in the electric current oscillations through the Andreev interferometer [159].

Now let us compare the previous consideration to the case of the exchange field along the x-axis, that is $h = h_0 e_x$. In this case the effective exchange field which is seen by the electron is $-h \sin \phi$ and the same reasoning results in the phase acquired by the pair $\Phi = -(2k \pi \tan \phi / \sqrt{v_F}) x$. The phase depends on the trajectory direction and, therefore, the Josephson current carried by all the pairs is greatly reduced due the averaging of these acquired phases. It has also been reported that due to the finite width of the junction along the y-direction, the decay of the critical current upon the increase of the x-component of the exchange field is accompanied by the Fraunhofer-like oscillations [160].

In summary, we can see that the Josephson current via the 3D TI-based S/F/S junction depends strongly on the exchange field direction. The physical reason for this dependence is the spin-momentum locking. In particular, the Josephson current exhibits the anomalous phase shift $\chi_0 = 2h_0 d / V_F$. It is determined by the exchange field component perpendicular to the current direction analogously to the case of Rashba materials, described above. However, the anomalous phase shift in TI-based JJs is much larger than in the materials with Rashba SOC because it does not contain the reducing factor $\Delta_0 / \epsilon_F \sim \alpha_0 / \epsilon_F$. It is directly connected to the fact that the 3D TI has only one helical Fermi surface (see figure 5(b)) in contrast to the Fermi surface of a Rashba material, which consists of two helical bands with opposite helicity (see figure 1(a)).

2.3. Anomalous phase shift in S/F/S junctions via inhomogeneous ferromagnets

The interaction of conduction electron spin with a magnetic texture can be described using an artificial spin-orbital coupling potential. In general the local transformation $\tilde{U}(r) = e^{i\sigma \theta(r) / 2}$ rotates spin axes to the local frame where $h \parallel z$. It is parametrized by the spin vector $\theta = \theta n$ defined by the spatial texture of the exchange field distribution $h(r) = R(\theta(r)) h_0$, where $R$ is the spatially-dependent rotation matrix and we choose $h = h_0 z$. This transformation generates the spin-dependent potential $\tilde{\sigma}_a [M^a_r p_i] / 2$ with pure gauge SU(2) field $M^a_r = -i \Gamma^a \left( \partial_\tau \hat{U} \partial_\tau \hat{U} \right) / 2m$. This artificial SOC leads to the spontaneous current [146] in the form (6) with $\chi^a \sim M^a_k$ so that $J \sim h_0 \chi^a_k$. This spontaneous current is nonzero provided that $M^a_k \neq 0$ which physically means that the magnetic texture is non-coplanar. On the qualitative level, the non-coplanarity is needed to obtain the different orbital states of conduction electrons in magnetic textures $\pm h(r)$ connected by the time-reversal transformation $\tilde{U} \rightarrow -\tilde{U}$. In the non-coplanar case it is not possible to compensate this sign change by the global, that is coordinate-independent spin rotation. The minimal non-coplanar texture consists of three magnetic moments $m_{1,2,3}$ with nonzero scalar spin chirality [161] $\chi = m_i (m_2 \times m_3)$. As demonstrated in [118] already the arrangement of three pointwise non-coplanar magnetic impurities leads to the generation of the spontaneous supercurrent and superconducting phase gradients $j, \nabla \phi \propto \chi$.

The anomalous phase shifts have been studied in JJs through various non-coplanar magnetic textures, including the three-layer systems [99, 106, 108, 110, 111, 113, 116, 162, 163], magnetic helices [118, 146] and skyrmions [118]. In most cases the magnetic systems are assumed to be metallic consisting either of strong ferromagnets or the half-metals. However, in several papers it has been shown that magnetoelectric effects can be engineered even with the help of FI [116, 163] which are currently considered as the promising platform for coupling of the superconductivity and ferromagnetism [164–168]. Below we discuss the particular example of such system [163].

Let us consider the system shown in figure 7 consisting of two superconducting electrodes $S_{1,2}$ separated by a single FI interlayer acting as a spin-filtering barrier with polarization $P$. The outer FI layers generate Zeeman fields $h_{1,2}$ in $S_{1,2}$ due to the magnetic proximity effect [169, 170]. To calculate the currents across spin-filtering barriers we use generalized Kuprianov-Lukichev boundary conditions [171], that include spin-polarized tunneling at the SF interfaces [11, 114, 115, 172]. The matrix tunneling current from $S_1$ to $S_2$ is given by:

$$I_{12} = \left[ \tilde{\Gamma}_1 \tilde{g}_1, \tilde{g}_2 \right],$$

where $\tilde{g}_k$ for $k = 1, 2$ are the matrix Green’s functions in the superconducting electrodes $S_k$. The spin-polarized tunneling matrix has the form $\tilde{\Gamma} = t_s \tilde{g}_0 \tau_0 + t_- (m \tilde{\sigma}) \Gamma_3$, where $m$ is the direction of barrier magnetization, $t_s = \sqrt{(1 \pm \sqrt{1 - P^2}) / 2}$ and $P$ being the spin-filter efficiency of the barrier that ranges from 0 (no polarization) to 1 (100% filtering.

![Figure 7.](image-url)
tering time. Due to the normalization condition and for the JJ's and can be expressed by the general equation:

\[ \frac{\Delta}{\tau_{so}} \frac{1}{2} \] is impurity-induced. The spin–orbit coupling can be responsible for the spin relaxation described by \( \tau_{so} = (S \cdot gS)/8\tau_{so} \), where \( \tau_{so} \) is the SO scattering time. Due to the normalization condition \( g^2 = 1 \).

The solution of equation (22) can be found in the form:

\[ g = \tau_3 [g_{03} + g_{33}(\sigma h)] + \tau_1 [g_{01} + g_{11}(\sigma h)] \]  

The terms diagonal in Nambu space (\( \tau_3 \)) correspond to the normal correlations which determine the density of states. The off-diagonal components (\( \tau_1 \)) describe spin-singlet \( g_{01} \) and spin-triplet \( g_{11} \) superconducting correlations which appear as a result of the exchange splitting.

The CPR (7) can be calculated for the spin-valve shown in figure 7 using the general matrix current (21). The usual \( j_0 = j_0 \cos \varphi_0 \) and anomalous \( j_{an} = j_0 \sin \varphi_0 \) Josephson currents through tunnel barrier are given by:

\[ \frac{R_{Nh0}}{\pi eF} = \sum_{\omega_n} \left[ r \left( \frac{\theta_{+}}{\theta_{-}} h_+ h_\perp \right) \right] \quad (24) \]

\[ \frac{R_{Nh0}}{\pi eF} = \chi P \sum_{\omega_n} g_{31} \]  

where \( \chi = P(h_1 \times h_2) \) is the spin chirality, \( h_{\perp} \) are the projections of \( h_0 \) on the plane perpendicular to \( P \).

Expressions (24) and (25) show that the anomalous current \( j_{an} \) is mediated by spin-triplet component \( g_{31} \). Physically the phase-shifting term \( j_{an} \) appears as a result of the additional phase picked up by the spin-triplet Cooper pairs when tunneling between two superconductors with non-collinear exchange fields through the spin-polarizing barrier. Therefore \( \varphi_0 \) Josephson effect is the directly observable signature of the spin-triplet superconducting current across the junction. For the ideal spin filter \( P \) is equations (24) and (25) yield a temperature-independent phase shift of CPR \( \varphi_0 = \theta_h \), where \( \theta_h \) is the geometric angle between the vectors \( h_{\perp 1} \) and \( h_{\perp 2} \). In the general case \( \varphi_0 \) can be quite different, as shown in figure 7.

2.4. Direct magnetoelectric effect in Josephson junctions via spin–orbit materials

The direct magnetoelectric effect was predicted for superconducting systems in the presence of spin–orbit coupling, where it represents the generation of an equilibrium spin polarization in response to supercurrent. The spin–orbit coupling can be both of intrinsic type, that is arising due to the inversion symmetry breaking [62, 63, 125, 126] and of extrinsic type, that is impurity-induced [66, 67]. Physically, the effect is the same for superconductors in the presence of the spin–orbit coupling and for the JJ's and can be expressed by the general equation:

\[ S^a = \frac{1}{ev_F} \kappa \frac{j_{a,k}}{2} \]  

where \( j_{a,k} \) is the \( k \)-component of the supercurrent density. However, in this review we describe the effect and the used theoretical approaches focusing on the JJ's. The direct magnetoelectric effect was considered both for ballistic [126] and diffusive JJ's [125] via spin–orbit coupled materials.

We will focus on the case of ballistic S/NSO junction with Rashba-type spin–orbit coupling in the interlayer and discuss the value of the electron spin polarization induced by the applied supercurrent and the theoretical approach used for solving the problem in [126]. The sketch of the system under consideration is shown in figure 8. The S/NSO interfaces are at \( x = \pm d/2 \). The Hamiltonian of a singlet superconductor in the presence of an arbitrary linear in momentum spin–orbit (SO) coupling [174, 175]:

\[ \hat{H} = \int d^2r \hat{\psi}_r \left( \hat{H}_0 (r', r) \hat{\psi}^\dagger_r (r') + \Delta (r) \hat{\psi}^\dagger_{-} (r) \hat{\psi}^\dagger_{+} (r) \right) \]

\[ + \Delta^* (r) \hat{\psi}^\dagger_{+} (r) \hat{\psi}^\dagger_{-} (r), \]  

\[ \hat{H}_0 (r) = \frac{p^2}{2m} - \frac{1}{2} \hat{\mathcal{A}} \hat{p} - \mu, \]  

where \( \Delta (r) \) is the superconducting parameter, which is nonzero only in the superconducting leads. \( \hat{H}_0 \) is the Hamiltonian of the normal metal in the presence of the spin–orbit coupling (NSO). The general linear in momentum SO is expressed by the term \( \hat{\mathcal{A}} \hat{p} = \hat{\mathcal{A}}^\dagger \hat{p} \hat{\mathcal{A}}^\alpha \), where \( \hat{\mathcal{A}}^\alpha \) are Pauli matrices in spin space. \( \psi = (\psi^\dagger_{-} \psi^\dagger_{+}) \) is the chemical potential. For particular case of the Rashba SOC \( \hat{\mathcal{A}}^\dagger = -\hat{\mathcal{A}} = \alpha \) and for the Dresselhaus SOC \( \hat{\mathcal{A}}^\dagger = -\hat{\mathcal{A}} = \beta \).

In the superconducting systems under the applied supercurrent the Cooper pairs acquire nonzero total momentum, which leads to the analogous shift of the Fermi surfaces
We focus on the approach developed in Fermi momenta for both helical subbands, that is, approximation disregards the difference between the values of the optical reason for this can be immediately seen from figure of singlet correlations to the triplet ones absence of an exchange field do not provide a transformation netoelectric effects, which are of the first order with respect to less than the Fermi energy which makes it possible to effectively solve spatially inhomogeneous systems: generation of triplet pairs under the applied electric current. Now our goal is to describe this effect in JJs.

Superconducting hybrid mesoscopic systems are often considered within the framework of the quasiclassical theory, which makes it possible to effectively solve spatially inhomogeneous problems. The SOC can be treated in the quasiclassical approximation when its characteristic energy $\Delta_\alpha$ is much less than the Fermi energy $\varepsilon_F$. This situation is typical. However, the quasiclassical consideration does not capture the magnetoelectric effects, which are of the first order with respect to the parameter $\Delta_\alpha/\varepsilon_F$. In particular, it is known that the standard quasiclassical equations in the presence of SOC and in the absence of an exchange field do not provide a transformation of singlet correlations to the triplet ones [174, 175]. The physical reason for this can be immediately seen from figure 9. The opposite-momenta pairing occurs between the electrons of the same helical band. It is obvious that for a given momentum direction one can compose a pair $\Psi_{p_+, \uparrow \downarrow}$ at one of the subbands and a pair $\Psi_{p_-, \uparrow \downarrow}$ for the other subband, where $p_\pm$ are Fermi momenta of the both helical subbands. Each of the pairs is a singlet–triplet mixture. Therefore, taking into account the fact that wave functions of the pairs differ at different Fermi surfaces $\Psi_{p_+, \uparrow \downarrow} \neq \Psi_{p_-, \uparrow \downarrow}$, we obtain a triplet admixture of the pair wave function [176]. However, the quasiclassical approximation disregards the difference between the values of the Fermi momenta for the both helical subbands, that is $p_+ = p_-$ with the quasiclassical accuracy. In this case the singlet pairs do not accompanied by such a triplet admixture. Therefore, in order to describe the triplet correlations in SOC materials and the resulting magnetoelectric effects, a generalization of the quasiclassical theory to include the first order corrections with respect to the parameter $\Delta_\alpha/\varepsilon_F$ is required. Several approaches have been reported in the literature [98, 125, 126]. We focus on the approach developed in [126].
Making use of the above equations it is possible to evaluate the average spin polarization:

\[ S = \frac{1}{2} \langle \hat{\psi}(r, t) \sigma \hat{\psi}(r, t) \rangle. \]  

(34)

The supercurrent-induced spin polarization is directly related to the triplet pairing. It can be written as [126]:

\[ S \propto \int d^2r \tanh \frac{\varepsilon}{2T} \left[ \frac{1}{R_{++}} (\hat{\rho}_+^R \hat{\rho}_-^R + \hat{\rho}_-^R \hat{\rho}_+^R) + \frac{1}{g_{s,s}^R} (\hat{\rho}_-^{R,\tau} \hat{\rho}_+^{R,\tau} + \hat{\rho}_+^{R,\tau} \hat{\rho}_-^{R,\tau}) \right] \left( f_s^R + f_f^R \sigma \right) \left( f_s^R + f_f^R \sigma \right), \]  

(35)

where \( g_s(f_s) \) and \( g_f(f_f) \) are the singlet and triplet components of the normal (anomalous) Green’s function, which can be found from the matrix Green’s function structure:

\[ g = \begin{pmatrix} g_s & g_s + g_f \sigma \\ g_f \sigma & g_s + g_f \sigma \end{pmatrix}, \]  

(36)

where the structure in the particle-hole space is shown explicitly and the matrix structure in the spin space is encoded in the Pauli matrices \( \sigma \). The anomalous Green’s function \( \tilde{f} \) can be expressed as \( \tilde{f}_s^R = -f_s^R (\Delta_s - \chi, -v_F, \tau) \). The only component of the triplet anomalous Green’s function \( f \), which is not an odd function of \( p_x \), and, therefore, survives after integration over \( p_x \), is \( f_s^R \), expressed by equation (32). As a result, only the \( \gamma \)-component of the spin polarization is nonzero in the JJJ via the Rashba SOC material, according to the qualitative consideration above.

In a vector form the induced spin polarization can be represented as:

\[ S = \kappa \left[ c \times \frac{j_s}{eV_F} \right]. \]  

(37)

For the considered case \( \alpha V_F / 2\pi T = \Delta_m / 2\pi T \gg 1 \):

\[ \kappa = \frac{\alpha V_F}{8eF}. \]  

(38)

It is the same value as for homogeneous superconductors [62]. Whether this universal behavior holds for S/NSO/S tunnel junctions has not yet been investigated. It is worth to note here that the direct magneto-electric effect should also take place in S/NSO/S junctions with very strong SO coupling in the interlayer (\( \Delta_m \sim \varepsilon_F \)), but the described theory is not able to consider this case quantitatively. This problem can be solved on the basis of the different quasichanical formalism, where the SO interaction is so strong that the coupling between the two helical subbands is disregarded [88, 179].

Another technique accounting for the magnetoelectric effects is a gauge-covariant approach to establish the transport equations [98]. In this approach, both the electromagnetic and spin interactions are described in terms of U(1) Maxwell and SU(2) Yang–Mills equations, respectively. The additional terms to the Eilenberger equation are expressed via the gauge field \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \). In the framework of this approach the well-known Edelstein’s results for supercurrent induced spin polarization in homogeneous superconductors, which were obtained in the diffusive [63] and ballistic limits [62], have been generalized for the case of arbitrary disorder strength [65].

The supercurrent-induced spin polarization in the S/NSO/S JJs has also been calculated in the diffusive limit [125]. The first-order corrections in \( \Delta_m/\varepsilon_F \) were added to the Usadel equation for the Green’s functions. The triplet anomalous Green’s function has been obtained in the form:

\[ f_s = -i \frac{\alpha}{\sqrt{2}} \frac{\partial}{\partial x} \]  

(39)

The supercurrent-induced electron spin polarization is:

\[ S = \frac{eN_F\alpha}{\varepsilon_F} [c \times j_s]. \]  

(40)

where \( \sigma \) is the conductivity of the NSO interlayer, \( N_F \) is the density of states at the Fermi level and \( \tau \) is the elastic-scattering time. The ratio \( \varepsilon_F j_s(v) \sim \alpha p_j / \varepsilon_F \) in equation (40) is also of the first order in \( \Delta_m/\varepsilon_F \). In the subsequent paper [180] of the same group, the S/NSO/S junction under the applied voltage has also been considered. It was predicted that the spin-Hall current does not turn to zero in contrast to the stationary Josephson effect. The physical reason is that besides a direct proximity effect caused by a Cooper pair’s transition into a triplet state, the spin current and polarization are also driven by a periodic electric field associated with the charge imbalance.

2.5. Direct magnetoelectric effect in S/TI/S junctions

As it is discussed in the previous section, the value of the supercurrent-induced electron spin polarization in JJs via SO-materials is rather small and \( \propto \Delta_m / \varepsilon_F \). The physical reason for this is the presence of two helical Fermi surfaces in the material with intrinsic SOC, which contribute to the direct magnetoelectric effect in opposite directions. In contrast, the 2D Fermi surface of the conductive surface states of the 3D topological insulators (TIs) consists of the only helical Fermi surface, see figure 5(b). It leads to the absence of the partial compensation of the current-induced electron spin polarization produced by the helical Fermi surfaces with the opposite helicities. As a result, the current-induced spin polarization does not contain the reducing factor \( \Delta_m / \varepsilon_F \) and takes the form:

\[ \langle s \rangle = -\frac{1}{2eV_F} [c_x \times j_s]. \]  

(41)

Equation (41) has been obtained, making use of the quasi-classical theory for TI-based superconducting hybrid structures (assuming no Zeeman field in the system) [68], which has already been described in section 2.2. Equation (41) coincides with the result for the current-induced spin polarization in normal state TI surface states [181].

While the supercurrent-induced spin polarization in TIs has not been measured yet, the current-induced spin polarization in normal state TIs has been directly measured as a voltage
3. Magnetoelectric effects and magnetization dynamics

3.1. Direct coupling between magnetization and superconducting condensate: basic physics and equations

The discussed above direct and inverse magnetoelectric effects are of great interest for JJs with ferromagnetic interlayers because they result in the coupling between the ferromagnet magnetization and the Josephson current. This leads to several interesting ways of electrical control of magnetization and electrical detection in S/F/S junctions. The present section aims to a discussion of those effects.

First of all we formulate the basic equations and describe qualitatively the physics underlying the direct coupling between the supercurrent and the magnetization. In the considered S/F/S junction the coupled dynamics of magnetization $M$ and Josephson phase difference $\chi$ is determined by the following closed set of equations:

$$\chi(t) = j_{c}(M) \sin \left( \chi(t) - \chi_{0}(M) \right) + \frac{\chi(t) - \chi_{0}(M)}{2eRS}$$

Equation (42) represents the non-equilibrium CPR generalizing RSJ model. The capacitive term is neglected here. This relation is written [149] in a gauge-invariant form amended to include the anomalous phase shift $\chi_{0}(M)$ defined by SOC and magnetic texture. In contrast to the previously used gauge non-invariant formulations [182, 183] equation (42) describes the normal spin-galvanic effects when $j_{c} = 0$ such as the electromotive force and charge current generated in the ferromagnet due to the time derivative of the Berry phase [52, 54–57, 184, 185]. The analogous equation is also valid for a more general non-sinusoidal CPR.

The dynamic is described by the Landau–Lifshitz–Gilbert (LLG) equation (43), where $\alpha$ is the Gilbert damping constant. The polarization of the conductivity electrons, which occurs in the interlayer of the JJ due to the direct magnetoelectric effect discussed above, interacts with the ferromagnet magnetization by the exchange mechanism. Suppose this interaction is internal for the ferromagnet. In that case, it can be accounted for in the framework of the s–d model [44], which describes the exchange interaction between the conductivity electrons and localized electrons responsible for the magnetization according to $H_{ex} = -J_{s} \sum \langle S_{i} \rangle \langle \breve{s} \rangle$, where $S_{i}$ is the spin operator of the localized spin at site $i$ and $\breve{s}$ is the spin operator of the conductivity electron. The examples are interlayers made of spin-textured ferromagnets or ferromagnets with intrinsic SOC. In hybrid interlayers consisting of a ferromagnet and a nonmagnetic material (NM) with SOC or a TI, see examples in figure 11, the spin polarization induced by the current in the NM due to the direct magnetoelectric effect, see sections 2.4.

Figure 10. Sketch of the experimental setup for the electrical measurement of the direct magnetoelectric effect at the surface of the 3D TI. If the magnetization ($M$) of the ferromagnetic detector ($F$, blue) has a component along the conductivity electron polarization (red arrows), the voltage $V$ is nonzero. If the magnetization of the detector is along the current axis, $V = 0$.

Figure 11. Examples of Josephson junctions, where the current-induced electron polarization is possible. The direction of the spin polarization relative to the current is shown by red arrows. (a) a JJ via a spin-textured ferromagnet ($F$); (b) a JJ via a combined ferromagnet ($F$) + SOC material interlayer ($N$); (c) a JJ via a ferromagnet on top of a 3D topological insulator (TI). For all the cases it is assumed that the interlayer length should not exceed a few normal state coherence lengths of the interlayer material to ensure a sizeable Josephson current through the JJ.
and 2.5, interacts with the ferromagnet magnetization via the exchange interaction at the F/NM interface:

\[ H_{int} = -\int d^2 \mathbf{r} \hat{\Psi}^\dagger(r) J_e \mathbf{S} \sigma \hat{\Psi}(r), \]  

(44)

where \( \hat{\Psi} = (\Psi_+^T, \Psi_-^T) \), \( \mathbf{S} \) is the localized spin operator in the ferromagnet, \( J_e \) is the interface exchange constant and the integration is performed over the 2D interface.

In both cases the torque can be represented in the form:

\[ \mathbf{T} = J \mathbf{M} \times \langle \mathbf{s} \rangle, \]

(45)

where \( \langle \mathbf{s} \rangle \) is the averaged polarization of conductivity electrons. \( J \) is the exchange constant of the s–d model or \( J = J_e/d_F \) for the hybrid F/HM interlayers with \( d_F \) standing for the ferromagnet thickness.

Below, the particular examples of the electrically induced magnetization dynamics in JJ via ferromagnets and the backward influence of the dynamics on the JJ are discussed.

3.2. Supercurrent-induced magnetization dynamics in S/F/S junctions via spin-textured ferromagnets

In the context of inhomogeneous ferromagnetic interlayers, the supercurrent-induced spin torques have been calculated theoretically in JJs through single spins [186–188] two [189–191], three [111] FM layers and bulk inhomogeneous ferromagnets [147]. The supercurrent-induced dynamics has also been studied. In particular, the supercurrent-induced magnetization switching in S/F/N/F/S JJs between parallel and antiparallel configurations has been predicted in [190]. The supercurrent-induced DW motion has been studied in [147].

The spin torque in these systems is commonly viewed as a spin transfer from the spin-polarized triplet supercurrent to the magnetization. However, it can also be understood in terms of the supercurrent-induced conductivity electron spin polarization. From this point of view, it can be considered in a unified manner with the supercurrent-induced torque in JJs with SOC in the interlayer. Below we illustrate this concept by the example of the supercurrent-induced DW motion in S/F/S JJs.

Let us consider an S/F/S JJ via a strong ferromagnet with the exchange energy \( h \) of the order of the Fermi energy. This model is relevant for most classical ferromagnets, including Fe, Ni, Co and permalloy. It is well-known that in this case, the opposite spin pairs decay very rapidly into the depth of the ferromagnet (on the length scale of a few nanometers), and the Josephson current is presumably carried by equal spin pairs [8, 9]. These pairs can be generated via different mechanisms, including rotation of the pair spin at magnetic inhomogeneities and spin-flip scattering at S/F interfaces [11]. The theory of equal-spin Josephson current via homogeneous strong ferromagnets has been developed in [113, 192]. In order to describe the Josephson current in strong inhomogeneous ferromagnets and its influence on the magnetic texture, this theory has been generalized for the case of inhomogeneous magnetization in [117, 147].

The Hamiltonian of the ferromagnetic interlayer is:

\[ \hat{H}(r,t) = -\frac{\mathbf{\Pi}_e^2}{2m_F} + (\partial \mathbf{h}(r,t)) - i(\hat{\sigma} \hat{\mathbf{B}} \mathbf{\Pi}_e), \]

(46)

where \( \mathbf{\Pi}_e = \nabla - i(e/c) \mathbf{A}(r) \) and \( \mathbf{A} \) is the vector potential of the electromagnetic field. The last term in equation (46) is the general form of a linear in momentum spin–orbit coupling (SOC) determined by the constant tensor coefficient \( \hat{B} \).

The quasiclassical theory is formulated in terms of the spinless quasiclassical Keldysh–Green’s function \( \hat{g}_\sigma(t_1, t_2) \) \( (\sigma = \pm 1) \), which is defined separately at each of the Zeeman split Fermi-surfaces for spin-up and spin-down electrons and in general depends on the spatial coordinates \( \mathbf{r} \) and the two time variables \( t_{1,2} \). This theory only accounts for the equal-spin Cooper pairs residing at the same Fermi surface. In contrast, the opposite spin correlations residing at different Fermi surfaces are neglected due to their strong suppression resulting from the large Zeeman splitting of the Fermi surfaces. The Usadel equation takes the form:

\[ \{ \hat{\tau}_3 \hat{\partial}_r \hat{g}_\sigma \}, - D_\sigma \hat{\partial}_r (\hat{g}_\sigma \circ \hat{\partial}_r \hat{g}_\sigma) = 0, \]

(47)

where \( D_\sigma \) are the spin-dependent diffusion coefficients, in the isotropic case given by \( D_\sigma = \tau_\sigma v_F^2 / 3 \). The spin-dependent Fermi velocities \( v_\pm = \sqrt{2(\mu \pm h)/m_F} \) are determined on each of the spin-split Fermi surfaces with \( m_F \) being the electron mass. The time dependence appears due to the dynamical character of the problem under consideration and \( \circ \)-product is defined as \( (A \circ B)(t_1, t_2) = \int_{-\infty}^{\infty} dt_A(t_1) dt_B(t_2) \). The commutator of the Green’s function with an arbitrary operator \( C \) is defined as \( \{C, \hat{g}_\sigma\} = \hat{C}(t_1, r_1) \hat{g}_\sigma - \hat{g}_\sigma \hat{C}(t_2, r_2) \). The anticommutator \( \{C, \hat{g}_\sigma\} \), is defined analogously with the plus sign. \( \hat{\tau}_3 \) and \( \hat{\tau}_1 \) are Pauli matrices in spin and Nambu spaces, respectively, and:

\[ \hat{\partial}_r = \nabla - i e [\mathbf{A}, \hat{\mathbf{\tau}}_3, \cdot] + i e [\mathbf{Z}, \hat{\mathbf{\tau}}_3, \cdot], \]

(48)

The spin-dependent gauge field is given by the superposition of two terms \( \mathbf{Z} = \mathbf{Z}^m + \mathbf{Z}^w \), where \( \mathbf{Z}^m = -iTr(\hat{\sigma}_z \hat{U}^\dagger \partial \hat{U} / 2 \) is the texture-induced part. \( \hat{U} = \hat{U}(r,t) \) is in general the time- and space-dependent unitary 2 \( \times \) 2 matrix that rotates the spin quantization axis \( \mathbf{z} \) to the local frame determined by the exchange field, so that \( \mathbf{h} \parallel \mathbf{z} \). The term \( \mathbf{Z}^w = m_F \mathbf{m}_B \) (where \( m = M / M \) appears due to the SOC.

Equation (47) is a spin-scalar equation but do not describe conventional spin-singlet superconducting correlations, unlike the standard spin-scalar form of the non-stationary Usadel equation. It is only applicable for strong ferromagnets and describes equal-spin triplet correlations.

In the framework of this theory the torque equation (45) consists of two terms:

\[ \mathbf{T} = \mathbf{T}_{st} + \mathbf{T}_{so}, \]

(49)

\[ \mathbf{T}_{st} = 2\mu_B (\hat{\mathbf{J}} \nabla) \mathbf{m}, \]

(50)

\[ \mathbf{T}_{so} = 4\mu_B m_F (\mathbf{m} \times \mathbf{B}) \hat{J}_z, \]

(51)
where $\mathbf{B}_j = (B_{jx}, B_{jy}, B_{jz})$ is a vector, which is determined by $j$-th coordinate component of the tensor $\hat{B}$ and $\mathbf{j}'$ is the spin current in the local frame determined by the exchange field, which is represented by difference between the Josephson current carried by spin-up and spin-down pairs. In equations (49)–(51) $T_{\alpha}$ is the supercurrent spin transfer torque [193–195]. In the considered approximation it takes the form of the adiabatic torque and does not contain a non-adiabatic torque term. $T_{\alpha}$ is the spin–orbit torque [196–198]. Its particular structure strongly depends on the type of the spin–orbit coupling, realized in the system.

Equations (49)–(51) can be viewed in terms of the supercurrent induced spin polarization. If the Josephson current flows along the $x$ direction, the spin polarization takes the form:

$$\langle s_m^z \rangle = -\frac{2\mu_B J_j}{J_z^M} [m \times \partial_x m + 2 m \times m \times (m \times B_j)],$$

where only perpendicular to $m$ component of the polarization is considered. The first term is the current-induced spin polarization originated from the spin texture and is analogous to the first term in equation (2). The second term of equation (2) associated with the nonadiabatic torque does not appear here because locally spin-up and spin-down pairs are not coupled in the framework of this theory and, therefore, the pair analogue of spin-flip processes, which account for this term, is not allowed. The second term represents the current-induced polarization due to the SOC.

It was shown [147] that the current-induced torques, equations (50) and (51) allow for the DW motion in the S/F/S junction in full analogy with the case of a nonsuperconducting ferromagnets. In the absence of the spin–orbit torque, the DW motion driven by the adiabatic torque only occurs if the Josephson current exceeds a threshold value $J_j^c > J_j^{c, \text{crit}}$, where $J_j^{c, \text{crit}}$ corresponds to the electric current density $eK_d j_{DW}/\hbar \sim eK_d j_{DW}/\hbar$, where $K_d$ and $K_{\alpha}$ are the easy- and hard-axis anisotropy constants of the considered ferromagnet, respectively, and $d_{DW}$ is the DW width. This estimate is in full agreement with the result obtained in [199] for nonsuperconducting ferromagnetic strips under the action of the adiabatic torque, where it has been concluded that $j_{DW}^{\alpha} \sim K_d d_{DW}$, while in the numerical analysis of [147] only the case $K_{\alpha} \sim K$ has been investigated. Taking for estimations the material parameters of $\text{CrO}_2$ nanowires [200] one can obtain that $j_{DW}^{\alpha} \sim 10^{10} \sim 10^{11}$ A m$^{-2}$, what is one-two orders of magnitude larger than the Josephson critical current density $j_c \sim 10^9$ A m$^{-2}$, measured in such nanowires [201]. Other S/F/S devices with large critical current density use usual Co and Ni ferromagnets [202–204], rare earth ferromagnet Ho [205] and half-metal manganite [206]. Recently the Josephson S/F/S junction thorough the pinned domain wall in Ni was realized [204]. In principle, it should be possible to realize the supercurrent-controlled domain wall motion in such systems.

The DW motion of an unpinned DW under the action of the spin–orbit torque is possible for arbitrary small values of the applied supercurrent. For example, if we consider a Neel DW in the $(x, y)$-plane, see figure 12, and a Rashba-type spin–orbit coupling with $B_z = (0, B_R, 0)$ the spin–orbit torque equation (51) at small applied supercurrents leads to the stationary motion of the DW with the velocity:

$$v_{DW} = -\frac{\mu_B B_R}{\alpha} j_c,$$

where $u = 2\mu_B J_j/M$ is the characteristic velocity associated with the value of spin current $J_j = (1/2e)(J_\uparrow - J_\downarrow)$ flowing through the ferromagnetic interlayer and $\beta = 2\mu eB_R d_{DW}$ is the dimensionless SO coupling parameter.

3.3. Resistive state of the Josephson junctions in the presence of magnetization dynamics

The reciprocal effect to the torque is the appearance of a gauge spin-dependent vector potential $\mathbf{Z}$ in the local spin basis [174, 207–213]. The gauge spin-dependent vector potential generates an anomalous phase shift. In case the magnetization depends on time, it also produces an electromotive force [214–216]. It has been shown [149] that due to the presence of this electromotive force, the magnetization dynamics deprives the JJs of the nondissipative regime, i.e. they cannot support a supercurrent because of the appearance of a nonzero voltage between the leads. The voltage appears to compensate the electromotive force. This voltage that maintains the DW motion in spin-textured interlayers or precession of a homogeneous magnetization in JJs, compensating the dissipation power occurring due to Gilbert damping by the work done by a power source.
Suppose that we apply a constant electric current $I = jS$ (here $S$ is the junction area) to the JJ and consider a steady motion of the DW across the junction with a constant velocity defined by equation (53) under the action of the spin–orbit torque. In this case the time-averaged voltage induced at the junction can be calculated from equation (42) and takes the form:

$$V(t) = RS\sqrt{\beta^2 - j_c^2} + \frac{\pi \beta^2 u}{\cos \theta} \frac{t}{\sin \theta},$$

where the first term is well-known and represents the conventional Josephson voltage appearing at $j > j_c$. The second term $V_M$ is nonzero both at $j > j_c$ and at $j < j_c$ and leads to the fact that the JJ is in the resistive state if the DW is driven by current. The corresponding IV- characteristics of the junction are shown in figure 13. For numerical estimates of $V_M$ one can take $\alpha = 0.01$, $d_y = 60$ nm, $u = 1$ m s$^{-1}$, what corresponds to the maximal Josephson current density [201] through the CrO$_2$ nanowire $j_c \sim 10^9$ A m$^{-2}$. The dimensionless SOC constant $\beta$ can vary in wide limits. Having in mind that experimentally the predictions can be realized, for example, for hybrid interlayers consisting of a ferromagnet/heavy metal bilayers, $\beta = 1$–10 considering that the SOC $\sigma_{\text{sk}}$ ranges from $10^{-11}$ to $10^{-10}$ eV m at interfaces of heavy-metal systems [217]. Then one can obtain $V_M|_{j = j_c}$ up to $10^{-5}$–$10^{-3}$ V.

The resistance of the junction at $j < j_c$ caused by the DW motion is given by:

$$R_{DW} = \left(\frac{\partial V}{\partial t}\right)_{j < j_c} = \frac{\pi \gamma \beta^2 h}{2e^2 S \alpha_{DW} M}$$

(55)

It is important that according to equation (55) $R_{DW}$ per unit area does not depend on the JJ parameters, such as $j_c$ and $R$, and is determined only by the characteristics of the magnetic subsystem. In this case, the work done by a power source is exactly equal to the energy losses in the magnetic subsystem due to the Gilbert damping and is not spent on compensating the Joule losses in the interlayer. Indeed, at $j < j_c$ the normal current through the JJ is zero despite nonzero voltage generated at the junction. It is seen directly from equation (42) because for $j < j_c$ it has the solution $\chi(t) = \chi_0(t)$. The equivalent circuit scheme of the junction is presented in the insert to figure 13. The voltage is compensated by the electromotive force induced in the junction by the emergent electric field $(h/e)\mathcal{Z}^{\omega}$.

In real setups the time of the DW motion through the junction is limited by the finite junction length: $t_{DW} \approx d/\nu_{DW} = (\alpha/\beta) (d/u)$. Therefore, the voltage should be averaged over $t < t_{DW}$. Although experiments on the DW motion in JJs have not yet been carried out, the estimates $t_{DW} \geq 0.5(\alpha/\beta) \times 10^{-6}$ s has been obtained [149] for $j_c \sim 10^9$ A m$^{-2}$. For other setups, which report the Josephson current carried by equal-spin triplet correlations [202, 205], this time can be several orders of magnitude higher due to much less values of the critical current density.

From the practical point of view, it is convenient to induce DW motion by large current pulses. For short pulses $j(t) = j(t)\theta(T - t)$ with $T < t_{DW}$, the DW does not leave the junction during the impulse time. The resulting voltage signal consists of two parts of different physical origins. The first part is the conventional Josephson response with the characteristic time $t_J = 1/2eR_l$. The other part is of purely magnetic origin and vanishes if there is no motion of the domain wall in the junction. The resulting voltage signals for $j < j_c$ are shown in figure 14. In this regime, the typical $V(t)$ curve consists of an initial sharp Josephson voltage impulse, a final sharp impulse of the same nature and a gradual voltage increase and decrease of purely magnetic origin, which takes the form $V(t) = -\pi \beta v(t)/ed_y$. This gradual voltage increase does not occur if the DW does not move.

![Figure 13](https://example.com/fig13.png)

**Figure 13.** IV-characteristics of the SFS junction with a DW at rest (blue) and a moving DW (red). $\beta = 1$, $\alpha = 0.1$, $eK_{DW}/(\pi j_c) = 5$. Insert: the equivalent circuit scheme of the junction. Reprinted figure with permission from [149], Copyright (2019) by the American Physical Society.

![Figure 14](https://example.com/fig14.png)

**Figure 14.** $V(t)$ for rectangular current impulses. Different curves correspond to different impulse periods $T$; the solid lines correspond to $\beta = 1$ (the anomalous phase due to the DW motion is nonzero) and the dashed lines are for $\beta = 0$ (the anomalous phase shift is zero). Different colors correspond to different values of a characteristic time scale $t_J$, where the DW velocity reaches its stationary value: (a) $j = 0.5j_c$, $T = 3t_J$ (blue), $T = 2t_J$ (yellow), $T = t_J$ (red). $\alpha = 0.1$, $eK_{DW}/(\pi j_c) = 5$, $t_J = 40\pi$. Reprinted figure with permission from [149], Copyright (2019) by the American Physical Society.
3.4. Electrical control of magnetization in S/F/S junctions

3.4.1. Magnetization dynamics under the applied voltage. In general, a Josephson current can induce magnetization dynamics. The coupling is described by equation (45) and realized via the generation of the electron spin polarization in the interlayer region, carried by triplet pairs. The pair spin should be misaligned with the ferromagnet magnetization to exert a torque on the magnetization. It can be achieved in JJs with misaligned or spin-textured ferromagnets or in the presence of the SOC.

The torque can be obtained from a direct calculation of the average electron polarization s [191]. However, there is another widely used approach for calculating the torque. Its strategy is to determine the Josephson energy of the system $E_J$ and then find the additional contribution to the effective field $\delta H_{\text{eff}}$ according to the relation $\delta H_{\text{eff}} = -(1/V_F)\delta E_J/\delta M$, where $V_F$ is the ferromagnet volume [111, 148, 182, 183, 189, 190, 218]. One should understand that this approach only takes into account the torque caused by the supercurrent. At the same time, under the applied voltage, the supercurrent via the JJ is accompanied by the normal current described by the second term in equation (42). The normal current also contributes to the current-induced electron polarization and, therefore, to the torque. The same thing should also be considered for problems under the applied current if the situation cannot be considered stationary, for example in the case of finite current pulses. However, for complex interlayers composed of ferromagnetic and non-ferromagnetic materials, the supercurrent and the normal current can presumably flow via the different layers. Therefore the approach based on the Josephson energy is applicable. The example is an S/F/S JJ via a metallic ferromagnet on top of a 3D TI [148], where the supercurrent flows via the TI conductive surface states and the normal current flows via the ferromagnet and does not contribute to the torque.

The magnetization dynamics has been analyzed in voltage-biased JJs via single spins [187, 188], where Josephson mutations were predicted. In addition, a possibility of obtaining supercurrent-induced magnetization switching in voltage-biased JJ, where the interlayer is composed of two misaligned ferromagnets separated by a normal spacer, has been reported [190]. The effect is based on the generation of triplet superconducting correlations in the misaligned configuration of the ferromagnet magnetizations. The pair spin polarization of these triplet correlations is not aligned with the magnetizations of the layers exerting a torque on the free layer. The magnetization dynamics has also been considered for the interlayer with three misaligned ferromagnets, two of which are fixed in misaligned configurations [219]. It has been demonstrated that the magnetic system exhibits a range of different behaviors, from simple harmonic oscillations to fractional-frequency periodic behavior and chaotic motion depending on the ratio between the Josephson frequency $\omega_J = 2eV$ and the characteristic frequency of the magnetic system.

Magnetization dynamics has also been studied in S/F/S JJs in the presence of the Rashba SOC inside the F interlayer [182]. The sketch of the considered setup is presented in figure 15. The ferromagnet has been assumed to be an easy-axis type with the easy axis along the z-direction. It was obtained that in the low-frequency regime $\hbar \omega_J \ll T_c$ the Josephson current can cause rich dynamics of the magnetization. The influence of the Josephson subsystem on the magnetic subsystem is controlled by the parameter $\Gamma = rE_J/E_M$, where $E_J$ is the Josephson energy, $E_M$ is the energy of the easy-axis magnetic anisotropy and $r \propto (\Delta_m/e_p) d$—is the parameter characterizing the SOC strength and the magnitude of the anomalous phase shift, see section 2.1. In the ‘weak coupling regime’ $\Gamma \ll 1$, the magnetic moment precesses around the $z$-axis with the Josephson frequency $\omega_J$. In the limit of the strong coupling $\Gamma \gg 1$, the solution of the LLG equation yields:

$$m_y(t) = \sin \left( \frac{\Gamma}{\omega} (1 - \cos \omega_J t) \right),$$
$$m_z(t) = \cos \left( \frac{\Gamma}{\omega} (1 - \cos \omega_J t) \right),$$

where $\omega = \omega_J/\omega_F$ is the ratio of the Josephson frequency to the ferromagnetic resonance one. It is seen that in this regime, the magnetic moment processes around the $y$-axis, which is the direction of the pseudo magnetic Rashba field induced by the current. If $\Gamma/\omega > \pi/2$, the full magnetization reversal occurs in the system. For general coupling regimes, the magnetic dynamics was found to be complicated and strongly nonharmonic.

3.4.2. Electrical control of the magnetization easy axis. It has been demonstrated that in S/F/S JJs with SOC in the interlayer, the stable position of the ferromagnet easy-axis can be dynamically reoriented under the applied voltage. The system in which such a feature was first studied is called the Kapitza pendulum. Particularly, in a pendulum with a vibrating point of suspension, the external sinusoidal force can invert the stability position of the pendulum [220]. In [221] it was predicted that the S/F/S JJ with SOC exhibits the analogous behavior: the unstable fixed point, which does not coincide with the equilibrium ferromagnet easy-axis, can become dynamically...
stable under the applied voltage. It was found that if the equilibrium easy axis of the magnet is the z-axis (the sketch of the considered system is shown in figure 15), then under the applied voltage, the new stability point is in the (x, y)-plane. The angle $\Theta$ between the stability direction and the z-axis is determined by:

$$
\sin \Theta = -\frac{1 + \sqrt{1 + 4\beta^2}}{2\beta},
$$

(57)

where $\beta = \Gamma^2r\alpha / 2\omega(1 + \alpha^2)$. Equation (57) determines two dynamical stability points, as it is shown in figure 16 instead of the equilibrium stability points $m_s = \pm 1$. The physical origin of the phenomenon is clear: in the presence of the Josephson current, the magnetic moment is influenced not only by the magnetic anisotropy field, but also by the pseudo magnetic Rashba field aligned with the y-axis. In the limit $\Gamma \gg 1$, the dynamical easy axis lies along the y-direction. The full time evolution of the y-component of the magnetization is shown in figure 17 for different values of the coupling between the magnetic and Josephson subsystems.

Another interesting effect related to the electrical control of the ferromagnet easy axis is the easy axis splitting in S/F/S JJs on top of the 3D TI [148]. The qualitative difference of this system from the case of the S/F/S JJ with the SOC in the interlayer is in that in 3D TI-based JJs the critical current demonstrates strong dependence on the x-component of magnetization. At the same time in S/F/S JJ with the SOC it has been considered as independent on the magnetization direction. The dependence and its physical origin were discussed in section 2.2. The suppression of the critical current as a function of $m_s = M_s/M_s$ has been discussed in [150] and is presented in figure 18. For estimates we take $d = 50$ nm, $v_F = 10^6$ m s$^{-1}$ and $T_e = 10$ K, what corresponds to the parameters of NbBi$_2$Te$_3$/Nb JJs [222]. In this case $\xi_N = v_F/2\pi T_e \approx 12$ nm. $j_c(m_s)$ for $T_e = 1.8$ K has also been plotted, what corresponds to the JJs with Al leads.

Now let us assume that the ferromagnetic interlayer of the JJ has an easy axis along the y-direction. The dependence of the anomalous phase shift on $m_s$ equation (19) results in the additional contribution to the y-component of the effective magnetic field $\delta H_{eff,y} = -(1/V_F)\delta E_j / \delta M_s$, where $E_j = \Phi_0(j_c/2\pi)[1 - \cos(\chi - \chi_0)]$ with $j_c$ and $\chi_0$ determined by equations (18) and (19). The effective magnetic field is added to the magnetic anisotropy field and does not cause the magnetization dynamics if the magnetization is along the easy axis. At the same time the dependence of the critical current on $m_s$ leads to nonzero $H_{eff,y} = \Delta m_s$ at small $m_s$. This means that the easy y-axis can become unstable in a voltage-driven

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**Figure 16.** Dynamical stability points (red circles) of the ferromagnet easy axis for the S/F/S JJ with the Rashba SOC in the interlayer. Reproduced from [221]. Copyright (2018) EPLA. All rights reserved.

**Figure 17.** Dynamical evolution of $m_s$ for the voltage-biased S/F/S JJ with the Rashba SOC in the interlayer. Different colors represent the data corresponding to different values of the parameter $\Gamma / r$. The numerical data and picture are provided by I Rahamonov.

**Figure 18.** Results of the numerical calculation of the critical current of the S/F/S JJ on top of the 3D TI as a function of $m_s$. The parameters are chosen to be relevant for a JJ with Nb superconducting leads (solid lines) and Al leads (dashed lines). The value of the effective exchange field is $h_{eff} = 20$ K for red lines. In this case, the suppression of the critical by $m_s$ is not very strong. For comparison the blue lines represent the critical current at $h_{eff} = 100$ K. It is seen that in this case it is strongly suppressed $j_c$ is normalized to $j_c(0)$, where $j_c(0)$ is the critical current at $h_{eff} = 100$ K. Reprinted figure with permission from [150], Copyright (2020) by the American Physical Society.
or current-driven junction, while this axis is always stable if the critical current does not depend on magnetization direction. Moreover, there is no difference for the system between $\pm m_x$-components of the magnetization. This leads to the fact that in the driven system an easy axis does not reorient, keeping two stable magnetizations directions, as it was already obtained earlier, but splits. As a result, four stable directions of magnetization appear. This splitting effect can be realized in a range of the parameter $A$ values, which can be achieved experimentally according to the estimates of [148].

Figure 19 demonstrate the corresponding four stability points (red arrows). The dynamically stable easy axes are in the $(x, y)$-plane, therefore if the magnetization direction is parametrized as $m = (\sin \Theta \cos \Phi, \cos \Theta, \sin \Theta \sin \Phi)$, the stable points correspond to $\Phi = 0, \pi$. Red circles shown them at the vector fields, which demonstrate the time evolution of the magnetization starting from an arbitrary initial magnetization position. Panel (a) corresponds to the small value of the parameter $A$ when the easy-axis is not split yet. Panels (b) and (c) are in the parameter range where the splitting occurs, and for panel (d), the $A$ value already exceeds the upper boundary of the range. In this case, the easy axis is reoriented to the $x$-direction.

3.4.3. Magnetization reversal by electric current pulses. Cryogenic memory elements. Another intriguing effect related to the electrical control of the magnetization in S/F/S JJs is the ferromagnet magnetization reversal by the Josephson current pulses. One of the key challenges towards developing ultra-low-power computers is the fabrication of a reliable and scalable cryogenic memory architecture. Superconductor–ferromagnet–superconductor (S–F–S) junctions are promising structures suggested for such memories [142, 143, 223–233].

The magnetization reversal by the electric current pulses discussed here has also been suggested as one of the possible realizations of the JJ-based cryogenic memory elements.

The magnetization reversal by current pulses has been originally discussed in [183] for the S/F/S JJ with the Rashba SOC in the interlayer. The magnetization dynamics in the system shown in figure 15 has been considered in the regime of applied electric current pulses. The torque acting on the magnetization has been calculated via contribution to the effective magnetic field, produced by the supercurrent $\delta H_{eff} = -(1/V_F)\delta E_J/\delta M$. The possibility of the moment reversal has been demonstrated. The possibility depends strongly on the pulse amplitude and duration, as well as on the value of the SOC parameter $r$ and the coupling strength between the magnetic and Josephson subsystems $\Gamma$. The ideas suggested in [183] have been developed in several subsequent papers. In particular, a periodicity in the appearance of intervals of the reversal of the magnetic moment under the variation of the spin–orbit coupling $r$, Gilbert damping parameter, and the coupling parameter $\Gamma$ [234] has been predicted. An example of the corresponding periodic patterns is represented in figure 20. Furthermore, an analytical criterion of the most efficient reversal for the easy-axis magnet has been formulated [235]. The criterion allows for optimization of the pulse parameters.
In [218] the idea to exploit the S/F/S JJ with the Rashba SOC as a cryogenic memory element has been investigated. The two memory states are encoded in the direction of the out-of-plane magnetization and the current pulses switch between them. The robustness of the current-induced magnetization reversal against thermal fluctuations has been explored [218, 236]. It has also been suggested [218] that the readout of the memory state can be nondestructively performed by direct measurement of the magnetization state through a dc SQUID inductively coupled to the junction.

The magnetization reversal by the electric current pulses has also been investigated for the 3D TI-based S/F/S JJs [151]. The advantage of this system is the very strong value of the SOC parameter $r$ provided by the spin-momentum locking in the TI surface states, what allows to use low-current pulses for the reversal of the magnetization. The full spin–orbit torque containing the contributions both from the supercurrent and the normal current can be found in terms of the current-induced polarization of the surface states conductivity electrons equation (41) and takes the form:

$$N = J_{\alpha} M \times (s) = -\frac{\gamma_{\text{TI}}}{eMv_{\text{F}}} [m \times e_{y}].$$

Whether there will be a reversal of the magnetic moment under the action of a given current pulse—depends strongly on the amplitude and duration of the pulse. It is illustrated by numerical data represented in figure 21. This diagram shows regions where the reversal occurs/does not occur in the $(dt, A_{s})$-plane, where $dt$ is the current pulse duration, and $A_{s}$ is its amplitude. It is seen that the regions where the reversal occurs (colored) and does not occur (white) are separated by striped regions, where the behavior of the system is very difficult to predict. Therefore, the result of the operation (yes/no reversal) is very sensitive to the pulse parameters. It was reported that the widths of the uncertainty regions depend on the magnetic anisotropy field in the ferromagnet. In [151] the magnetic anisotropy field in the ferromagnet was chosen as follows:

$$H_{\text{eff}} = -\frac{K}{M} m_{x} e_{z} + \frac{K_{y}}{M} m_{y} e_{x},$$

where $K$ and $K_{y}$ are the hard axis and the easy axis anisotropy constants, respectively. Therefore, an easy-plane anisotropy was considered in addition to the easy axis anisotropy, investigated earlier. This situation corresponds to the experimental data reported for YIG thin films [237]. The parameter $k = K/K_{y}$ can describe the ratio of the hard and easy-axis anisotropy parameters. The nonzero value of this parameter results in the appearance of the uncertainty regions in the reversal diagrams, figure 21. These regions grow with an increase of $k$ and disappear at $k \to 0$.

In addition, it has been suggested in [151] to exploit the voltage induced at the junction due to the magnetization dynamics for electrical detection of the magnetization reversal. To detect the reversal $m = e_{x} \to -e_{x}$ it is efficient to measure the transverse voltage generated between the additional leads, as it is shown in figure 22. This voltage is measured in the open circuit geometry when the electric current between the additional transverse electrodes is zero. In this case the solution of equation (42) takes the form $\chi = \chi_{0}$. Then the voltage generated between the additional electrodes due to magnetization dynamics is determined by the dynamics of $m_{x}$ and can be written as follows [150]:

$$V_{t} = \frac{h_{\text{TI}}}{e v_{\text{F}}}.\tag{60}$$

This voltage is the same both for superconducting additional electrodes and for nonsuperconducting electrodes and is only determined by the electromotive force. If the magnetization dynamics is caused by the pulse of electric current applied in the $x$-direction, then:
Suggested that this mechanism should generate equal-spin interest for several decades. Triplet pairing states provides effective manipulation of the odd-frequency spin. More specifically, it was predicted that the condensate motion can ferromagnet hybrids with interfacial SOC controllable con-

densate motion can be achieved, for example, by the magnetic field through the Meissner effect. The interplay of condensate momentum $p_z$, SOC and exchange field $h$ leads to the generation of long-range $s$-wave spin-triplet component. Taken from [239].

$$\int V(t)dt = \frac{\hbar}{e} \frac{\Delta m_z}{2},$$

where $\Delta m_z$ is the full change of $m_z$ caused by the pulse. If the magnetization reversal $m = e_x \rightarrow -e_x$ occurred, then $\Delta m_z = -2$, otherwise it is zero. Therefore, the integrated over time value of the voltage between the additional electrodes can be used as a criterion of the magnetization reversal.

4. Triplet correlations generated by the moving condensate

Another type of magnetoelectric effect, specific for superconducting systems, is the generation of triplet $S = 1$ Cooper pairs by the condensate motion. It is well-established that the nonzero momentum of moving superconducting condensate is a pair-breaking factor. The reason is that it makes the momenta of two paired electrons to be not exactly opposite, and such a finite-momentum pairing has less binding energy. This fundamental mechanism called the orbital depairing effect exists in any superconducting system and leads to the suppression of superconductivity by the magnetic field or by the supercurrent [238].

4.1. Triplets induced by the static Meissner currents

In [239] it has been demonstrated that in superconductor/ferromagnet hybrids with interfacial SOC controllable condensate motion can induce superconducting correlations. More specifically, it was predicted that the condensate motion provides effective manipulation of the odd-frequency spin-triplet pairing states [240] which have attracted continual interest for several decades [9, 107, 241–262]. It has been suggested that this mechanism should generate equal-spin triplet Cooper pairs in currently available experimental setups with SOC [263–270]. In the context of superconductor/ferromagnet hybrid structures these correlations are known as long-range triplets (LRT) because they can penetrate at large distances into the ferromagnetic material [9, 106, 110, 112, 173, 192, 201–203, 205, 206, 271–276]. Consequently, in S/F/S JJs the magnetic field can in fact stimulate Josephson current by generating long-range equal-spin odd-frequency triplet correlations.

The sketch of the basic structure is shown in figure 23. Without the supercurrent, LRT are absent in the generic S/F structures such as shown in figure 23(a). Here the exchange field $h \parallel z$ produces only short-range triplets (SRT) with $S_z = 0$, shown schematically by the blue arrows, which decay at a short length of the order $\xi_z \sim 1$ nm in usual ferromagnets. It has been demonstrated [174, 175] that, in principle, the SOC in combination with the exchange field can induce LRTs in diffusive systems. However, pure Rashba or Dresselhaus SO coupling does not induce the LRTs in a transversal geometry with an in-plane magnetization [174]. It is this situation that is the most common experimental setup and is depicted in figure 23. The generation of LRT with $S_z = \pm 1$ shown schematically by red arrows in figure 23(b) can be achieved by inducing the superconducting condensate momentum $p_z$ satisfying the condition $h \times (n \times p_z) \neq 0$. The required condensate motion can be achieved, for example, by applying the external magnetic field along the $y$-direction, which causes the Meissner currents along the $z$-direction.

The qualitative physics of the effect can be described as follows. The general structure of anomalous function describing the pairing in a spin-triplet channel can be parameterized as $(\vec{\sigma} \cdot \vec{d})$, where $\vec{\sigma}$ is the vector of Pauli matrices. The role of $h \parallel z$ is to generate $S_z = 0$ spin triplet correlations with $d_{SRT} \propto h$. The role of SOC $H_{soc} = \alpha (n \times p_z)$ is to convert them to $S_z = \pm 1$ correlations due to the momentum-dependent spin rotation by the SOC. The resulting $p$-wave spin

![Figure 22](image-url)

**Figure 22.** (a) Sketch of the superconductor/ferromagnetic insulator/superconductor (S/F/S) JJ on top of the 3D topological insulator (TI) with additional normal (N) electrodes, which are used for the electrical detection of $m = e_x \rightarrow -e_x$, reversal in comparison to (b) the basic S/F/S JJ. Reprinted figure with permission from [151]. Copyright (2019) by the American Physical Society.

![Figure 23](image-url)

**Figure 23.** Schematic picture of the simplest system, where the LRTs are generated by the moving condensate: a diffusive superconductor/ferromagnet junction with Rashba SOC at the S/F interface induced by the thin heavy metal Pt layer. (a) In the absence of the moving condensate, only short-range superconducting correlations are present. (b) The condensate motion along the exchange field direction is induced, for example, by the magnetic field through the Meissner effect. The interplay of condensate momentum $p_z$, SOC and exchange field $h$ leads to the generation of long-range $s$-wave spin-triplet component. Taken from [239].
vector is \( d_{\text{pw}} = F_{\text{pw}}(\omega) \mathbf{h} \times (n \times p) \) with the amplitude \( F_{\text{pw}}(\omega) \) which is an even function of the Matsubara frequency \( \omega \). Such momentum-odd correlations are greatly suppressed in diffusive systems due to the efficient impurity-induced momentum averaging.

The externally induced superflow \( p_s \neq 0 \) induces Doppler shift of the quasiparticle energy levels \([277, 278]\) \( v_F \cdot p_s \). It results in the suppression of pairing on one part of Fermi surface, namely for electrons with momentum \( p \parallel p_s \). In the simplest case of homogeneous system this leads to the shift of imaginary frequencies so that the amplitude of triplet correlations is given by \( F_{\text{pw}}(\omega - i v_F \cdot p_s) \approx F_{\text{pw}}(\omega) - i(v_F \cdot p_s) \partial_\omega F_{\text{pw}} \).

This modification of the pairing amplitude results in the additional component of the spin vector \( \delta \mathbf{d} = -i\partial_\omega F_{\text{pw}}(v_F \cdot p_s) \mathbf{h} \times (n \times p) \). The s-wave component \( d_{\text{sw}} = \langle \delta \mathbf{d} \rangle_p \) is given by \( d_{\text{sw}} = (2/3i)E_F (\partial_\omega F_{\text{pw}}) \mathbf{h} \times (n \times p_s) \).

By the order of magnitude \( |d_{\text{sw}}| \sim \hbar v_F c / \Delta^2 \gg v_F c / E_F \). Then the typical amplitude of the s-wave correlations is \( \langle |d_{\text{sw}}| \rangle \sim (p_s \hbar) v_F c / \Delta^2 \) where \( \xi \) is the coherence length. It means that the magnitude of this magnetoelectric effect can be considerably larger than that of the effects discussed before because the magnitude of the electron spin polarization and current-induced triplet correlations due to the Rashba SOC is governed by the parameter \( v_F c / E_F \), see section 2.4.

Technically the triplet correlations can be calculated on the basis of the quasiclassical Usadel equation \([239]\). If the Rashba SOC is only present at the S/F interfaces, the (SOC + supercurrent)-induced SRT-LRT conversion can be described by the effective boundary condition at the S/F interface, which in the framework of the linearized with respect to the anomalous Green’s function approach takes the form \([239]\):

\[
\eta_n \nabla \cdot \mathbf{f}_{\text{LRT}} = 4i\tilde{\alpha} \mathbf{f}_{\text{SRT}} \times (p_s \times n),
\]

where the surface SOC strength \( \tilde{\alpha} = \int d\alpha(x) \). The solution of the linearized Usadel equations for the LRT anomalous Green’s function:

\[
\frac{D}{2} \nabla^2 \mathbf{f}_{\text{LRT}} = |\omega| \mathbf{f}_{\text{LRT}},
\]

supplemented by the boundary condition \((62)\) gives the LRT anomalous Green’s function, which in the \( \xi_F \ll d_F \ll \xi_N \) takes the form:

\[
\mathbf{f}_{\text{LRT}} \propto \frac{\gamma^2 2 \tilde{\alpha}}{d_F \omega} \frac{\Delta}{\sqrt{\Delta^2 + \omega^2}} \mathbf{h} \times (n \times p_s),
\]

where \( \xi_F = \sqrt{D/\hbar} \) is the coherence length of the SRTs in the ferromagnet and \( \gamma \) is the S/F interface transparency \([9, 171]\). The amplitude of long-range spin-triplets is proportional to the condensate momentum \( p_s \). If it is generated by the external magnetic field through the Meissner effect, then \( p_s \propto B \).

If now two S/F interfaces with Rashba SOC are combined into the S/F/S JJ, the amplitude of critical current grows as \( I_c \propto B^2 \) for a small external magnetic field, when the total flux through the junction area \( \Phi = 2\lambda_p LB \) is small \( \Phi \ll \Phi_0 \). Here \( L \) is the length of the junction. For larger fields, one needs to take into account phase variation along the junction which leads to the usual factor \( (L \sin \phi) / \phi \) in the critical current, where \( \phi = 2\pi \Phi / \Phi_0 \). It results in \( I_c \propto B \) envelope dependence of the critical current shown in figure 24. This growth is bounded from above by the depairing effects. The \( I_c(B) \) pattern in figure 24 drastically differs from the ones observed previously in non-ferromagnetic JJs with SOC \([121, 279]\) and ferromagnetic ones without SOC \([280]\). This behaviour can be considered as the fingerprint of the LRT produced by the moving condensate in the presence of SOC.

### 4.2. Dynamic triplets induced by alternating electric fields

The moving condensate can be also induced by the alternating electric field \( E(t) \), which is described by the time-dependent vector potential \( E = - (1/c) \partial_\mathbf{A} \). It produces an oscillating condensate motion with the momentum \( p_s = -(2c/e)A(t) \). It has been demonstrated in \([281]\) that in the S/F/S JJ sketched in figure 25 with Rashba SOC at the S/F interfaces this oscillating condensate motion produces triplet correlations with the energy and time-dependent spin vector constructed as follows:

\[
d(\varphi, t) = \int dt' K_d(\varphi, t - t') (E(t') \times n) \times h.
\]

The scalar kernel \( K_d(\varphi, t - t') \) is determined in the framework of a particular microscopic model.

The triplets are long-range and result in the controllable appearance of the Josephson effect in the setups shown in figure 25 under irradiation or by applying an ac current source. This mechanism can help to achieve switching rates in the terahertz and even the visible light frequency domains. If the applied electric field has zero time-average value, the time-average value of the dynamic triplets also vanishes. In spite of this fact, they result in nonzero dc component of the Josephson current via the JJ. For the case of a harmonic electromagnetic wave, the following CPR has been obtained:
radiation there are no currents in the loop. Radiation switches on (b) which is well within the measurable limits. (c) Generation of LRT correlations due to the irradiation of the setup with electromagnetic wave (b) and the ac current source (c) both producing the electric field $E(t) = E_{0} e^{i\omega t}$ in the ferromagnetic interlayer. The LRT are shown schematically by the red spheres with co-directed arrows corresponding to the spin states aligned with the exchange field $h$. They sustain the Josephson current. Reprinted figure with permission from [281]. Copyright (2021) by the American Physical Society.

$$I(\chi, t) = [F_{dc} + F_{2\Omega} \cos(2\Omega t)] \sin \chi.$$  \hspace{1cm} (66)

Both the dc and double-frequency critical current amplitudes are determined by the alternating electric field $F_{dc} \propto E_{0} E_{\pm \Omega}$ and $F_{2\Omega} \propto E_{2\Omega}^2$. By the order of magnitude $F_{dc}, F_{2\Omega} \sim I_{0}$, where

$$I_{0} = \sigma_{e} S (\Delta / ed_{f}) (2 \tilde{\alpha} \gamma_{2} / \pi)^{2} (\Delta / T)^{2} P / P_{c},$$  \hspace{1cm} (67)

where $S$ is the junction area, $P = |e|E_{0}^{2}$ is the radiation power, $P_{c} = (\hbar c / e^{2}) \hbar \Omega^{2} / \xi_{s}^{2}$ is the radiation power needed to speed up the Cooper pairs to the depairing velocity. In [281] it has been estimated that $I_{0} / (P / P_{c})^{2} \sim 10^{-1} - 10^{-3}$ A for typical parameters of JJs with ferromagnetic interlayers and taking $\tilde{\alpha} \approx 0.1$–1 [255, 262, 282–284]. Assuming $\xi_{s} \approx 30$ one can estimate $P_{c} \approx 10^{3} (\Omega \text{ GHz}^{-1})^{2} W$ m$^{-2}$. Therefore such a JJ is quite sensitive to the radio-frequency and microwave irradiation. For example, a cell phone at one meter distance generates microwave radiation with $\Omega \approx 3$–4 GHz and $P \sim P_{c}$, which induces rather large currents $I_{0} \sim 10^{-1} - 10^{-3}$ A. At the same time the frequency rise strongly suppresses the power sensitivity. For the frequency of the cosmic background radiation $P_{c} \approx 10^{6}$ W m$^{-2}$ so that the power density $P \approx 10^{-5}$ W m$^{-2}$ enters rather small critical current $I_{0} \sim 10^{-12} - 10^{-15}$ A. However, even THz and visible light radiation sources can induce large critical current. For example, a THz radiation with power $1$ mW mm$^{-2}$ yields $I_{0} \sim 10^{-5} - 10^{-7}$ A. Laser beam of the frequency about $\omega_{ph} = \sqrt{2 \pi P_{c} / C_{ph} 0}$ is the plasma frequency corresponding to the $\pi$-JJ. Equation (68) has been obtained under the condition $\Omega \ll \omega_{ph}$, where $\omega_{ph} = 1 / \sqrt{LC}$ is the eigen frequency of the superconducting loop, in order to avoid parametric effects due to the time-dependent current amplitude of the photo-assisted JJ. In case of the typical values $\omega_{ph} \sim 10$ GHz the threshold value in the r.h.s. of equation (68) about $10^{-6}$ A.

Once the condition (68) is satisfied the SQUID switches to the state with spontaneous dc current $I_{dc}$ and constant magnetic field $B_{dc}$. The photo-induced magnetic flux magnitude was estimated as $\sim 10^{-5} \Phi_{0}$.

One can also obtain the photo-magnetic response without any threshold for the incoming power, provided the second branch of the SQUID contains the Josephson anomalous phase junction. Such photo-magnetic element generates dc current $I_{dc} \approx I_{dc} \cos \varphi_{0}$ and the corresponding magnetic field $B_{dc}$ being exposed to any arbitrary small radiation power. It has been found that the zero-current state becomes unstable under the following condition:

$$I_{dc} > \frac{\Phi_{0}}{2\pi} \frac{\omega_{ph} \omega_{p}^{2}}{\omega_{ph}^{2} + \omega_{p}^{2}},$$  \hspace{1cm} (68)

where $\omega_{p} = \sqrt{2 \pi P_{c} / C_{ph} 0}$.

**Figure 25.** S/S/S junction with Rashba SOC at the interfaces induced by the thin layer of heavy metal Pt. (a) Only short-range superconducting correlations are present shown by the blue and red spheres with opposite arrows. Therefore, if the interlayer length is of the order of several normal state coherence lengths, the Josephson current through the junction is strongly suppressed. (b), (c) Generation of LRT correlations due to the irradiation of the setup with electromagnetic wave (b) and the ac current source (c) both producing the electric field $E(t) = E_{0} e^{i\omega t}$ in the ferromagnetic interlayer. The LRT are shown schematically by the red spheres with co-directed arrows corresponding to the spin states aligned with the exchange field $h$. They sustain the Josephson current. Reprinted figure with permission from [281]. Copyright (2021) by the American Physical Society.

5. **Conclusions**

In this review, we have discussed the fundamental aspects and characteristic features of the magnetoelectric effects, which were reported in the literature on JJs. The main focus of the review is on the manifestations of the direct and inverse magnetoelectric effects in various types of Josephson systems. The coupling of the magnetization in JJs with ferromagnetic interlayers to the Josephson current via the magnetoelectric effects and perspectives of this coupling are also discussed.

To summarize, the direct magnetoelectric effect, that is, the current-induced spin polarization of the conductivity power, it has been found that the zero-current state becomes unstable under the following condition:

$$I_{dc} > \frac{\Phi_{0}}{2\pi} \frac{\omega_{ph} \omega_{p}^{2}}{\omega_{ph}^{2} + \omega_{p}^{2}},$$  \hspace{1cm} (68)

where $\omega_{p} = \sqrt{2 \pi P_{c} / C_{ph} 0}$.
The effect is a driving force of the spin torques acting on the ferromagnet inside the JJ and, therefore, is of key importance for the electrical control of the magnetization. The inverse magnetoelectric effect in JJs takes the form of the anomalous ground state phase shift and has been reported for JJs via spin-textured ferromagnets, multilayered ferromagnetic systems, homogeneous ferromagnets with SOC and combined interlayers consisting of TIs or materials with SOC and ferromagnets. The effect accounts for the back action of the magnetization dynamics on the Josephson subsystem, making the JJ be in the resistive state in the presence of the magnetization dynamics of any origin. Another manifestation of the magnetoelectric effects in JJs is the generation of LRT pairs in S/F/S JJs by the moving condensate, which allows for controllable and low-dissipative manipulation by the critical current of the JJ.

Although by now progress has been most pronounced on the theoretical understanding of the magnetoelectric effects in JJs, the experimental activity has in the past few years started to catch up. In particular, a number of experiments confirmed the anomalous ground state phase shift in JJs via SOC materials and TIs under the applied Zeeman field. There is also a growing activity in the field of spin pumping experiments in superconducting hybrids and, in particular, JJs [268, 285, 286], where some interesting results concerning the influence of the superconducting subsystem on the ferromagnetic resonance are obtained. Nevertheless, there remains a plethora of interesting physics to investigate, and we hope that the most valuable experiments in the near future will directly verify the role of magnetoelectric effects in S/F/S JJs thus opening a way to applications in low-dissipative spintronics.

**Data availability statement**

No new data were created or analyzed in this study.

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