Kinetic theory of collisionless relaxation for systems with long-range interactions

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We develop the kinetic theory of collisionless relaxation for systems with long-range interactions in relation to the statistical theory of Lynden-Bell. We treat the multi-level case. We make the connection between the kinetic equation obtained from the quasilinear theory of the Vlasov equation and the relaxation equation obtained from a maximum entropy production principle. We propose a method to close the infinite hierarchy of kinetic equations for the phase level moments and obtain a kinetic equation for the coarse-grained distribution function in the form of a generalized Landau, Lenard-Balescu or Kramers equation associated with a generalized form of entropy [P.H. Chavanis, Physica A 332, 89 (2004)]. This allows us to go beyond the two-level case associated with a Fermi-Dirac-type entropy. We discuss the numerous analogies with two-dimensional turbulence. We also mention possible applications of the present formalism to fermionic and bosonic dark matter halos.

I. INTRODUCTION

Recently, the dynamics and thermodynamics of systems with long-range interactions has been a subject of considerable interest in statistical mechanics [1-3]. For systems with long-range interactions, the relaxation towards statistical equilibrium (Boltzmann distribution) is governed by the homogeneous [4, 7] or inhomogeneous [8, 9] Lenard-Balescu equation, which is a generalization of the homogeneous [10] or inhomogeneous [11] Landau equation taking into account collective effects. The collisional relaxation time generically scales with the number of particles $N$ as $t_{\text{relax}} \sim N t_D$, where $t_D$ is the dynamical time.\(^1\) When $N \to +\infty$, the relaxation time diverges and the Boltzmann statistical equilibrium state is never reached. The system is then governed by the Vlasov equation [10, 11] which describes a collisionless evolution driven only by mean field effects. The Vlasov equation is reversible and conserves the Boltzmann entropy among an infinite number of Casimir invariants. This seems to preclude the relaxation towards an equilibrium state. However, systems governed by the Vlasov equation can experience a process of violent relaxation on the coarse-grained scale towards a metaequilibrium state. This process of collisionless relaxation was first evidenced by King [18] and Hénon [19] in the case of stellar systems described by the Vlasov-Poisson equations. It can account for the structure and regularity of galaxies whose collisional relaxation time exceeds the age of the Universe by many orders of magnitude [12]. Similarly, incompressible and inviscid flows in 2D hydrodynamics are described by the Euler-Poisson equations. Systems governed by these equations also experience a process of violent relaxation leading to the formation of large-scale vortices like Jupiter’s Great Red spot [20, 21].

A statistical theory of violent relaxation has been developed by Lynden-Bell [22] for stellar systems. The metaequilibrium state is obtained by maximizing a mixing entropy while accounting for all the constraints of the dynamics. This leads to the Lynden-Bell distribution function (DF) which can be viewed as a Fermi-Dirac-type DF in the two-level case, or a superposition of Fermi-Dirac-type DFs in the multi-level case. A similar statistical theory has been developed by Miller [23] and Robert and Sommeria [24] for 2D incompressible and inviscid flows. The numerous analogies between stellar systems and 2D vortices are described by Chavanis [20, 25-28]. The Lynden-Bell statistical theory has also been applied to other systems with long-range interactions such as the HMF model [29-36]. However, the power of prediction of the Lynden-Bell statistical theory is limited by the problem of incomplete relaxation. The Lynden-Bell statistical theory relies on an assumption of ergodicity which is not always fulfilled in practice. It is difficult to know a priori if the system will mix efficiently, as required by the ergodicity assumption.

The Lynden-Bell theory is an equilibrium theory. It is then important to develop a kinetic theory of (violent or quiescent) collisionless relaxation and obtain an evolution equation for the coarse-grained DF.\(^2\) This project was

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\(^1\) The “collisional” evolution of systems with long-range interactions described by the Lenard-Balescu equation is induced by two-body correlations among the particles. The Lenard-Balescu equation is valid at the order $1/N$ in a proper thermodynamic limit where $N \to +\infty$ with $m \sim 1/N$. Accordingly, the collisional relaxation time generically scales as $t_{\text{relax}} \sim N t_D$ [12]. For self-gravitating systems, the Chandrasekhar relaxation time scales as $t_{\text{relax}} \sim (N/\ln N) t_D$ because of logarithmic corrections [12]. A similar scaling is obtained in plasma physics where the particle number $N$ is replaced by the plasma parameter $\Lambda$ giving the number of electrons in the Debye sphere [11]. For spatially homogeneous 1D systems with long-range interactions, the Lenard-Balescu collision term vanishes [12] and one has to account for higher order correlations among the particles. A kinetic equation, taking into account three-body correlations, has recently been derived in Refs. [13, 14]. This equation is valid at the order $1/N^2$, leading to a collisional relaxation time scaling as $t_{\text{relax}} \sim N^2 t_D$.

\(^2\) Lynden-Bell [22] heuristically described the evolution of the coarse-grained DF by a simple Fokker-Planck equation.
first considered by Kadomtsev and Pogutse (KP) \cite{40} for the Vlasov equation of plasma physics. They developed a quasilinear theory of collisionless relaxation and, in the two-level case, obtained a fermionic-like Landau or Lenard-Balescu equation which relaxes towards the Lynden-Bell DF. Because of their assumptions, their kinetic theory can only describe the late quiescent stages of the relaxation process (gentle relaxation). Their approach was extended by Severne and Luwel (SL) \cite{41} in the multi-level case in an astrophysical context. The quasilinear theory of collisionless relaxation was further discussed by Chavanis \cite{42,44}. He used it to derive a truncated model (a sort of fermionic King model) taking into account the evaporation of high energy stars. This DF has a finite mass in contrast to the original Lynden-Bell DF which is not normalizable when coupled to gravity. The quasilinear theory of the Vlasov-Poisson equations has been exported to the case of 2D incompressible and inviscid flows described by the Euler-Poisson equations \cite{43,50}.

A completely different approach has been developed by Chavanis, Sommeria and Robert (CSR) \cite{26} by using a maximum entropy production principle (MEPP) previously introduced in 2D hydrodynamics \cite{47,49}. A relaxation equation is constructed heuristically by maximizing the rate of production of Lynden-Bell’s entropy while accounting for all the constraints of the dynamics. This leads to a generalized Fokker-Planck equation with a time-dependent temperature that evolves so as to conserve the energy. In the two-level case, this relaxation equation reduces to a fermionic-like Kramers equation which generalizes the Fokker-Planck equation for the coarse-grained DF introduced heuristically by Lynden-Bell \cite{22}.

It is possible to obtain the CSR equation from the KP and SL equations by making a sort of thermal bath approximation. This procedure provides the explicit expression of the diffusion coefficient in the CSR equation which is not given by the MEPP \cite{26}. The connection between the KP and the CSR equations was made in Refs. \cite{42,44} in the two-level case and is generalized in the present paper to the multi-level case. Then, there remains a complicated closure problem: The SL and CSR equations yield an infinite hierarchy of equations for the moments of the distribution. Following the suggestion of Ref. \cite{44}, we introduce a simple method to close this hierarchy of equations. We first remark that the equilibrium coarse-grained DF extremizes a “generalized entropy” at fixed mass and energy. We also show that the variance of the distribution at equilibrium is related to the coarse-grained DF through this generalized entropy. We then propose to close the hierarchy of kinetic equations at the level of the coarse-grained DF by extending this relation out-of-equilibrium (this can be justified by a local thermodynamical equilibrium assumption). This leads to a form of generalized Landau or Lenard-Balescu equation associated with a generalized entropy. This equation has been studied in detail in \cite{44}. It conserves mass and energy and satisfies an $H$-theorem for the generalized entropy. In the thermal bath approximation, it reduces to a generalized Kramers equation and the diffusion coefficient can be calculated explicitly.

We show how the kinetic theory is able to account for the problem of incomplete relaxation through a space and time-dependent diffusion coefficient.

We also discuss the nonlinear dynamical stability of stationary solutions of the Vlasov equation. We introduce an energy principle which provides the most refined condition of dynamical stability. A stationary solution of the Vlasov equation is dynamically stable if and only if it is a minimum of energy with respect to symplectic perturbations (i.e. perturbations that conserve all the Casimirs). We then propose a relaxation equation that minimizes the energy while conserving all the Casimirs. This relaxation equation can serve as a numerical algorithm to construct stable steady states of the Vlasov equation. We also show that the maximization of a pseudo-entropy at fixed mass and energy (thermodynamic-looking microcanonical principle) provides a sufficient condition of dynamical stability. As a result, the generalized Landau, Lenard-Balescu and Kramers equations of Refs. \cite{44,51,52} can be used as numerical algorithms to construct stable steady states of the Vlasov equation.

Finally, we discuss the notion of $H$-functions and “selective decay” following the work of Tremaine et al. \cite{52}. We propose an evolution equation that monotonically increases all the $H$-functions while conserving the mass and the energy. It differs from the generalized Landau, Lenard-Balescu and Kramers equations of Refs. \cite{44,51,52} which monotonically increase only one particular $H$-function (the generalized entropy) at fixed mass and energy.

II. LYNDEN-BELL’S STATISTICAL THEORY

A. Vlasov equation

We consider a Hamiltonian system of $N$ particles interacting via a long-range binary potential $u(|r-r'|)$ which decays at large distances as $r^{-\gamma}$ with $\gamma \leq d$, where $d$ is the dimension of space. Let $f(r,v,t)$ denotes the DF defined such that $f \, dr\,dv$ gives the total mass of particles with position $r$ and velocity $v$ at time $t$. In the collisionless (or mean field) regime, the evolution of the DF is governed by the Vlasov equation

$$ \frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \nabla \Phi \cdot \frac{\partial f}{\partial v} = 0, \quad (1) $$
where

\[ \Phi(r, t) = \int u(|r - r'|)f(r', v', t) \, dr' \, dv' \]  

is the mean potential produced self-consistently by the particles. The Vlasov equation can be obtained from the N-body Liouville equation by making a mean-field approximation, i.e., by writing the N-body DF as a product of N one-body DFs. The Vlasov equation is valid when \( t \ll t_{\text{relax}} \), where \( t_{\text{relax}} \) is the collisional relaxation time. Since \( t_{\text{relax}} \) grows algebraically with \( N \), the Vlasov equation becomes exact when \( N \rightarrow +\infty \) in a proper thermodynamic limit where the mass of the particles scales as \( m \sim 1/N \) [5]. We note that the individual mass of the particles does not appear in the Vlasov equation. This implies that the collisionless dynamics does not lead to a segregation by mass.

The Vlasov (or collisionless Boltzmann) equation simply states that, in the absence of encounters, the DF \( f \) is conserved by the flow in phase space. This can be written as \( Df/Dt = 0 \) where \( D/Dt = \partial/\partial t + v \cdot \nabla \). The conservation of the DF together with the incompressibility of the flow in phase space imply that the total mass (or hypervolume) of all phase elements with phase density between \( f \) and \( f + \delta f \) is conserved. This is equivalent to the conservation of an infinite number of invariants called the Casimir integrals \( I_h = \int h(f) \, dr \, dv \) for any continuous function \( h(f) \). The conservation of the Casimirs is equivalent to the conservation of all the moments of the DF, denoted \( M_n = \int f^n \, dr \, dv \), which include the total mass \( M = \int f \, dr \, dv \) as a special case. The Vlasov equation also conserves the total energy \( E = 1/2 \int f v^2 \, dr \, dv + 1/2 \int f \Phi \, dr \, dv \) (kinetic + potential), the total impulse \( P = \int f v \, dr \, dv \), and the total angular momentum \( L = \int f r \times v \, dr \, dv \) (see Appendices A and B). The Vlasov equation admits an infinite number of stationary solutions whose general form is given by the Jeans theorem [53]. For example, any DF of the form \( f = f(\epsilon) \), where \( \epsilon = v^2/2 + \Phi(r) \) is the individual energy of the particles by unit of mass, is a steady state of the Vlasov equation. In astrophysics, it describes spherically symmetric and isotropic stellar systems [54].

### B. Metaequilibrium state

The Vlasov-Poisson equations develop very complex filaments as a result of a mixing process in phase space (collisionless mixing). In this sense, the fine-grained DF \( f(r, v, t) \) will never reach a stationary state but will rather produce intermingled filaments at smaller and smaller scales. However, if we introduce a coarse-graining procedure, the coarse-grained DF \( \bar{f}(r, v, t) \) is expected to reach a metaequilibrium state \( \bar{f}(r, v) \) on a very short timescale, of the order of the dynamical time \( \tau_D \). This is because the evolution continues at scales smaller than the scale of observation (coarse-grained). This process is known as “phase mixing” and “violent relaxation” (or collisionless relaxation) [54]. Lynden-Bell [22] has tried to describe this metaequilibrium state in terms of statistical mechanics. In the following, we summarize his theory and provide some complements (see also [25, 39, 55]).

Let \( f_0(r, v) \) denote the initial (fine-grained) DF. We discretize \( f_0(r, v) \) in a series of levels \( \eta \) over which \( f_0(r, v) \sim \eta \) is approximately constant. Thus, the levels \( \{ \eta \} \) represent all the values taken by the fine-grained DF. If the initial condition is unsteady or unstable, the DF \( f(r, v, t) \) will be stirred in phase space (phase mixing) but will conserve its values \( \eta \) and the corresponding hypervolumes \( \gamma(\eta) = \int \delta(f(r, v, t) - \eta) \, dr \, dv \) as a property of the Vlasov equation (this is equivalent to the conservation of all the Casimirs).

Let us introduce the probability density \( \rho(r, v, \eta) \) of finding the level of phase density \( \eta \) in a small neighborhood of the position \( (r, v) \) in phase space. This probability density satisfies at each point the normalization condition

\[ \int \rho(r, v, \eta) \, d\eta = 1. \]  

The locally averaged (coarse-grained) DF is then expressed in terms of the probability density as

\[ \bar{f}(r, v) = \int \rho(r, v, \eta) \eta \, d\eta, \]  

and the associated potential satisfies \( \bar{\Phi}(r, t) = \int u(|r - r'|)\bar{f}(r', v', t) \, dr' \, dv' \). The conserved quantities of the Vlasov equation can be decomposed into two groups.\(^3\) The mass and energy are called robust integrals because they are (approximately) conserved by the coarse-grained DF: \( \bar{M}[\bar{f}] = M[f] \) and \( \bar{E}[\bar{f}] \simeq E[f] \). Hence

\[ M = \int \bar{f} \, dr \, dv, \]  

\(^3\) This distinction was first made in [37, 14].
The potential is smooth since it is expressed as an integral of the DF, so we can express the energy in terms of the coarse-grained fields \( \overline{\Phi} \) and \( \overline{\Phi} \) neglecting the internal energy of the fluctuations \( \overline{\Phi} \). Therefore, the mass and the energy can be calculated at any time of the evolution from the coarse-grained field \( \overline{\Phi} \). By contrast, the moments \( M_n \) with \( n \geq 2 \) are called *fragile integrals* because they are altered on the coarse-grained scale since \( \overline{\Phi}^\eta \neq \overline{\Phi} \), where \( \overline{\Phi}^\eta = \int \rho(\vec{r}, \vec{v}; \eta) d\eta \). Therefore, only the moments of the fine-grained DF, \( M_n^{f.g.} = M_n[\overline{\Phi}] = \int \overline{\Phi} d\vec{v} \), are conserved, i.e.

\[
M_n^{f.g.} = \int \rho(\vec{r}, \vec{v}, \eta) \eta^n \, d\vec{v} \, d\eta.
\]  

The moments of the coarse-grained DF, \( M_n^{c.g.}[\overline{\Phi}] = \int \overline{\Phi}^\eta \, d\vec{v} \), are not conserved along the evolution since \( M_n[\overline{\Phi}] \neq M_n[\overline{\Phi}] \). Instead of conserving the fine-grained moments, we can equivalently conserve the total hypervolume

\[
\gamma(\eta) = \int \rho(\vec{r}, \vec{v}, \eta) \, d\vec{v}
\]

of each phase level \( \eta \). We note that \( M_n^{f.g.} = \int \gamma(\eta) \eta^n \, d\eta \).

After a complex evolution, we may expect the system to be in the most probable, i.e. most mixed state, consistent with all the constraints imposed by the dynamics. Surviving the metaequilibrium state is obtained by maximizing the Lynden-Bell mixing entropy\(^5\)

\[
S_{LB}[\rho] = -\int \rho(\vec{r}, \vec{v}, \eta) \ln \rho(\vec{r}, \vec{v}, \eta) \, d\vec{v} \, d\eta,
\]

while conserving the mass \( M \), the energy \( E \) and all the Casimirs (or the fine-grained moments \( M_n^{f.g.} \)). We also need to account for the local normalization condition \( \overline{\Phi} \). The Lynden-Bell entropy \( S_{LB} \) can be obtained from a standard combinatorial analysis taking into account the specificities of the Vlasov equation (see Refs. \(22, 29, 53\) for details).

Introducing Lagrange multipliers, the first variations satisfy

\[
\delta S_{LB} - \beta \delta E - \alpha \delta M - \sum_{n>1} \alpha_n \delta M_n^{f.g.} - \int \zeta(\vec{r}, \vec{v}) \delta \left( \int \rho(\vec{r}, \vec{v}, \eta) d\eta \right) d\vec{v} = 0,
\]

where \( \beta = 1/T \) is the inverse temperature associated with the conservation of energy and \( \alpha_n \) are the “chemical potentials” associated with the conservation of the fine-grained moments \( M_n^{f.g.} \) (the function \( \zeta(\vec{r}, \vec{v}) \) accounts for the local normalization condition).\(^6\) This variational principle leads to the Gibbs state

\[
\rho(\vec{r}, \vec{v}, \eta) = \frac{1}{Z(\epsilon)} \chi(\eta) e^{-\eta(\beta \epsilon + \alpha)},
\]

where \( \epsilon = \frac{v^2}{2} + \overline{\Phi}(\vec{r}) \) is the energy of a particle by unit of mass.\(^7\) In writing Eq. \(11\), we have distinguished the Lagrange multipliers \( \alpha \) and \( \beta \) associated with the robust integrals \( M \) and \( E \) from the Lagrange multipliers \( \alpha_n, n>1 \), associated with the conservation of the fragile moments \( M_n^{f.g.} \), which have been regrouped in the function

\[
\chi(\eta) \equiv \exp(- \sum_{n>1} \alpha_n \eta^n).
\]

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\(^4\) This statement relies on an assumption of ergodicity which may not always be realized in practice (see Refs. \(22, 53, 36\) for a discussion of the concept of incomplete relaxation).

\(^5\) We take \( k_B = 1 \) throughout the paper.

\(^6\) For brevity, we shall not consider here the conservation of linear impulse \( \vec{P} \) and angular momentum \( \vec{L} \). Actually, if we work in the barycentric frame of reference, the Lagrange multiplier \( \Omega \) associated with the conservation of \( \vec{P} \) vanishes. On the other hand, the conservation of angular momentum implies that the statistical equilibrium state has a solid rotation. When \( \vec{L} = 0 \), it can be shown that the Lagrange multiplier \( \Omega \) associated with the conservation of \( \vec{L} \) also vanishes \(22, 29\). We will assume that we are in this situation, even though the general case can be treated straightforwardly.

\(^7\) We note that the Lynden-Bell distribution \(11\) does not lead to a segregation by mass since the individual mass of the particles does not appear in the Vlasov equation on which the whole theory is based. However, it leads to a segregation by phase levels \( \eta \).
This distinction will make sense in the following (see also Appendix C). Under this form, we see that the equilibrium distribution of phase levels is a product of a universal Boltzmann factor $e^{-\eta(\beta\epsilon+\alpha)}$ by a non-universal function $\chi(\eta)$ which depends on the initial condition. The partition function $Z$ is determined by the local normalization condition yielding

$$Z(\epsilon) = \int \chi(\eta) e^{-\eta(\beta\epsilon+\alpha)} \, d\eta.$$  

(13)

The partition function $Z(\epsilon)$ can be used as a generating function for constructing the moments of the fine-grained distribution (see Appendix D). We note that the Lynden-Bell statistics (11) has a form similar to a superstatistics (see Ref. [53] for the development of this analogy). The equilibrium coarse-grained DF defined by Eq. (4) can be written as

$$\mathcal{F} = \frac{1}{Z(\epsilon)} \int \chi(\eta) \eta e^{-\eta(\beta\epsilon+\alpha)} \, d\eta = \frac{\int \chi(\eta) \eta e^{-\eta(\beta\epsilon+\alpha)} \, d\eta}{\int \chi(\eta) e^{-\eta(\beta\epsilon+\alpha)} \, d\eta}. \quad (14)$$

One can easily check that

$$\mathcal{F} = -\frac{1}{\beta} \frac{\partial \ln Z}{\partial \epsilon} = F(\beta\epsilon+\alpha) = \mathcal{F}(\epsilon). \quad (15)$$

We note that the coarse-grained DF predicted by Lynden-Bell depends only on the individual energy $\epsilon$ of the particles. As such, it is a particular stationary solution of the Vlasov equation. We also note that $\mathcal{F}(\epsilon)$ is a monotonically decreasing function of energy. Indeed, from Eqs. (11) and (15), it is easy to establish that (see Appendix D)

$$\mathcal{F}(\epsilon) = -\beta f_2, \quad (16)$$

where

$$f_2 = \int \rho(\eta - \mathcal{F})^2 d\eta = \mathcal{F}^2 - \mathcal{F}^2 \geq 0 \quad (17)$$

is the local centered variance of the distribution $\rho(\mathbf{r}, \mathbf{v}, \eta)$. Equation (16) is a form of fluctuation-dissipation theorem. We note that $\mathcal{F}(\epsilon) \leq 0$ since $\beta \geq 0$ is required to make the velocity profile normalizable. We can also easily show [53] that $\mathcal{F}(\mathbf{r}, \mathbf{v}) \leq f_0^{\max}$, where $f_0^{\max}$ is the maximum value of the initial (fine-grained) DF. The inequality $0 \leq \mathcal{F} \leq f_0^{\max}$ is clear from physical considerations since the coarse-grained DF can only decrease by mixing. Finally, one can show that the coarse-grained DF predicted by Lynden-Bell is nonlinearly dynamically Vlasov stable (see Sec. VIII).

For a given initial condition, the statistical theory of Lynden-Bell selects a particular stationary solution of the Vlasov equation (the most probable – most mixed – one) among an infinity of stationary solutions. The Lynden-Bell equilibrium state is obtained by solving the integral equation

$$\chi(\eta) = \int u(|\mathbf{r} - \mathbf{r}'|) \mathcal{F}_{\alpha_n, \beta} \left[ \frac{\eta^2}{2} + \chi(\eta') \right] \, d\mathbf{r'} \, d\mathbf{v'} \quad (18)$$

and relating the Lagrange multipliers ($\alpha_n, \beta$) to the constraints ($M_n^{c, g}$, $E$). We also have to make sure that the equilibrium state is an entropy maximum not a minimum or a saddle point (see Appendix C). We note that the coarse-grained DF $\mathcal{F}(\epsilon)$ can take different forms depending on the function $\chi(\eta)$ determined by the fragile moments. In the present context, the function $\chi(\eta)$ is determined from the constraints a posteriori. Indeed, we have to solve the full problem in order to get the $\alpha_n$’s and obtain the expression of $\chi(\eta)$.[8] We emphasize that the Lynden-Bell statistical equilibrium state $\mathcal{F}(\epsilon)$ resulting from a violent collisionless relaxation depends on the details of the initial condition. This is different from the Boltzmann statistical equilibrium state resulting from a collisional relaxation which depends only on the value of the mass $M$ and the energy $E$. In the present case, we need to know the value of the fine-grained moments $M_n^{c, g}$ which are accessible only in the initial condition (or from the fine-grained field) since the observed moments $M_n^{c, g}$ are altered for $t > 0$ by the coarse-graining as the system undergoes a mixing process ($M_n^{c, g} \neq M_n^{c, g}$).

Remark: Similar results have been derived by Miller [23] and Robert and Sommeria [24] in 2D turbulence. The analogy between the Lynden-Bell theory and the Miller-Robert-Sommeria theory is discussed in [20] [26].

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[8] In this sense, the constraints associated with the conservation of the fine-grained moments are treated microcanonically. Following the approach of [53] in 2D turbulence, we have suggested in [53] that, when the system is forced by an external medium, the fine-grained constraints may be treated canonically. In that case, the function $\chi(\eta)$ should be considered as given a priori (it is determined by the external forcing). Treating the Casimirs canonically also allows us to derive a sufficient condition of thermodynamical stability in the sense of Lynden-Bell (see Appendix C).
C. Two-level case

If the initial DF takes only two values \( f_0 = \eta_0 \) and \( f_0 = 0 \) (vacuum), the Lynden-Bell entropy reduces to

\[
S = -\int \left\{ \frac{T}{\eta_0} \ln \frac{T}{\eta_0} + \left( 1 - \frac{T}{\eta_0} \right) \ln \left( 1 - \frac{T}{\eta_0} \right) \right\} \, dr \, dv,
\]

which is similar to the Fermi-Dirac entropy. Furthermore, the constraints reduce to the conservation of mass \( M \) and energy \( E \) since \( M_{\eta_0} = \int f_0 \, dr \, dv = \eta_0 \) and \( E_{\eta_0} = \int f_0 \, dr \, dv = \eta_0 \). The metaequilibrium state is then given by

\[
\mathcal{T} = \frac{\eta_0}{1 + e^{(\beta \epsilon + \alpha)}},
\]

which is similar to the Fermi-Dirac DF \[22, 57\]. Morphologically, the Lynden-Bell statistics corresponds to a 4th type of statistics since the particles are distinguishable but subject to an exclusion principle \( f \leq \eta_0 \) due to the incompressibility of the flow in phase space \[22\]. This constraint plays a role similar to the Pauli exclusion principle in quantum mechanics. In the dilute (nondegenerate) limit of the Lynden-Bell theory \( \mathcal{T} \ll \eta_0 \), the entropy \(19\) and the DF \(20\) reduce to

\[
S = -\int \frac{\mathcal{T}}{\eta_0} \left( \ln \frac{\mathcal{T}}{\eta_0} - 1 \right) \, dr \, dv,
\]

\[
\mathcal{T} = \eta_0 e^{-\eta_0 (\beta \epsilon + \alpha)}.
\]

which are similar to the Boltzmann entropy and to the Boltzmann distribution.

Remark: We note that the effective temperature \( T = 1/\beta \) in the DF \[21\] has not the dimension of a temperature. Indeed, the mass \( m \) of the particles does not appear in the Lynden-Bell theory since it is based on the Vlasov equation for collisionless systems which is independent of the mass of the particles. However, \( T/\eta_0 \) can be interpreted as a velocity dispersion (in the nondegenerate limit). In this sense, one can say that the temperature in Lynden-Bell’s theory is proportional to the mass of the particles (or more precisely to the ratio \( m/\eta_0 \)) \[22\].

D. Generalized entropy

Since the coarse-grained DF \( \mathcal{F}(\epsilon) \) predicted by the statistical theory of Lynden-Bell depends only on the individual energy and is monotonically decreasing, it extremizes a “generalized entropy” of the form \[37, 39, 44, 55\]

\[
S[\mathcal{F}] = -\int C(\mathcal{F}) \, dr \, dv
\]

at fixed mass \( M \) and energy \( E \), where \( C(\mathcal{F}) \) is a convex function (i.e. \( C'' > 0 \)). Indeed, introducing Lagrange multipliers \( \alpha \) and \( \beta \), and writing the variational principle under the form

\[
\delta S - \beta \delta E - \alpha \delta M = 0,
\]

we find that

\[
C'(\mathcal{F}) = -\beta \epsilon - \alpha.
\]

Since \( C'(\mathcal{F}) \) is a monotonically increasing function of \( \mathcal{F} \), we can inverse this relation to obtain

\[
\mathcal{F} = F(\beta \epsilon + \alpha) = \mathcal{F}(\epsilon),
\]

where the function \( F(x) = (C')^{-1}(-x) \) is determined by the generalized entropy \( C(\mathcal{F}) \). Inversely, for a given \( F(x) \), the generalized entropy is given by

\[
C(\mathcal{F}) = -\int \mathcal{F} \, F^{-1}(x) \, dx.
\]

From the identity

\[
\mathcal{F}'(\epsilon) = -\frac{\beta}{C''[\mathcal{F}]}.
\]

obtained from Eq. (24), we find that $\mathcal{F}(\epsilon)$ is a monotonically decreasing function, i.e., $\mathcal{F}(\epsilon) < 0$ (since $\beta > 0$ as explained in [58]).

Therefore, for any Gibbs state of the form (11), there exists a generalized entropy of the form (22) that the coarse-grained DF $\mathcal{F}$, given by Eq. (15), extremizes at fixed mass $M$ and energy $E$. From the statistical theory of Lynden-Bell, we have $F(x) = -(\ln Z)'(x)$ [see Eq. (13)], where $Z(x)$ depends only on $\chi(\eta)$ [see Eq. (13)]. Substituting this relation into Eq. (20), we find that the generalized entropy is given by

$$C(\mathcal{F}) = - \int (\ln Z)'^{-1}(-x) \, dx. \quad (28)$$

We expect that in many cases the coarse-grained distribution (15) maximizes the generalized entropy $S$ at fixed mass $M$ and energy $E$ (robust constraints) although this is not necessarily the case (see Appendix C). We emphasize that the generalized entropy (28) is a non-universal function which depends on the initial condition. Indeed, it is determined by the function $\chi(\eta)$ which depends indirectly on the initial condition through the complicated procedure discussed at the end of Sec. II B. In general, the generalized entropy (22) with (28) is not the ordinary Boltzmann entropy $S_B = -\int \mathcal{F} \ln \mathcal{F} \, dx \, dv$ because of the existence of fine-grained constraints (Casimirs) that modify the form of the entropy that we would naively expect. These constraints are sometimes referred to as hidden constraints since they are not accessible from the coarse-grained dynamics [38, 39, 44, 55].

**Remark:** Similar results have been derived by Chavanis [59–62] in 2D turbulence.

### E. $H$-functions and selective decay principle

In order to quantify the importance of mixing during the process of violent relaxation, Tremaine et al. [52] have introduced the notion of $H$-functions. They are defined by

$$H(\mathcal{F}) = - \int C(\mathcal{F}) \, dr \, dv, \quad (29)$$

where $C$ is any convex function (i.e. $C'' > 0$). It can be shown that the $H$-functions $H(\mathcal{F})$ calculated with the coarse-grained DF increase during violent relaxation in the sense that $H(\mathcal{F}(r,v,t)) \geq H(\mathcal{F}(r,v,0))$ for $t > 0$ where it is assumed that, initially, the system is not mixed so that $\mathcal{F}(r,v,0) = f(r,v,0)$. This is similar to the $H$-theorem in kinetic theory. However, contrary to the Boltzmann equation, the Vlasov equation does not single out a unique functional (the above inequality is true for all $H$-functions) and the time evolution of the $H$-functions is not necessarily monotonic (nothing is implied concerning the relative values of $H(t)$ and $H(t')$ for $t,t' > 0$). Yet, this observation suggests a notion of generalized selective decay principle (for $-H$). Among all the invariants of the collisionless dynamics, the $H$-functions (fragile constraints) tend to increase ($-H$ tend to decrease) on the coarse-grained scale while the mass and the energy (robust constraints) are approximately conserved. According to this phenomenological principle, we may expect that the metaequilibrium state reached by the system as a result of violent relaxation will maximize a certain $H$-function (non-universal), denoted $H^*(\mathcal{F})$, at fixed mass and energy. This would guarantee that the metaequilibrium state is nonlinearly dynamically stable (see Sec. V). If the evolution is ergodic, the above-mentioned statement is presumably correct. The $H$-function $H^*(\mathcal{F})$ that is effectively maximized at metaequilibrium

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9 In the case where the system experiences an external forcing (footnote 8), the function $\chi(\eta)$ and, consequently, the generalized entropy $C(\mathcal{F})$ should be considered to be given a priori, being determined by the forcing.

10 A similar selective decay principle has been advocated in 2D turbulence and magnetohydrodynamics (see, e.g., Refs. [61, 63–66] and references therein). It is either due to a small dissipation or to a coarse-graining (for dissipationless systems). In 2D turbulence, it has been argued that the enstrophy $\Gamma_2 = \int \omega^2 \, dr$ decreases while the circulation $\Gamma = \int \omega \, dr$ and the energy $E = \frac{1}{2} \int \omega \psi \, dr$ are conserved so that the system reaches a minimum enstrophy state [66, 67]. The minimization of enstrophy at fixed circulation and energy (which is mathematically equivalent to the minimization of energy at fixed circulation and enstrophy) leads to a linear relationship $\omega = \lambda \psi + \mu$ between vorticity and stream function. The minimization of “generalized enstrophies” has also been considered. In magnetohydrodynamics [70, 71], it has been argued that the magnetic energy $E = \int B^2 \, dr$ decreases while the helicity $H = \int A \cdot B \, dr$ is conserved so that the system reaches a minimum energy state. The minimization of magnetic energy at fixed helicity (which is mathematically equivalent to the maximization or minimization of helicity at fixed energy) leads to a linear relationship $\nabla \times B = \lambda B$ which characterizes a force-free configuration. These variational principles ensure the nonlinear dynamical stability of the system with respect to a dissipationless evolution (we note that all the above functionals are conserved for a purely inviscid and fine-grained evolution).

11 This is, however, not necessary: All the $H$-functions could increase at fixed mass and energy without necessarily implying that the coarse-grained DF reaches a steady state that maximizes one of them (see Sec. IX).
is the generalized entropy $S[f]$ defined by Eqs. (22) and (23), as obtained from Lynden-Bell’s theory (according to the comment that follows Eq. (25) this is expected to be true in many cases but not in all cases). Furthermore, under the assumption of Appendix C it can be shown that the generalized entropy monotonically increases during violent relaxation (see Sec. IV D). In case of incomplete relaxation, it is possible (but not necessary) that the metaequilibrium state maximizes a certain $H$-function (sometimes also called a generalized entropy) which is different from the one associated with the Lynden-Bell theory ($H[f]^* \neq S[f]$). This discussion shows that the notion of selective decay for collisionless systems with long-range interactions is quite subtle.

F. Incomplete relaxation

The statistical theory of Lynden-Bell relies on the assumption that the evolution is ergodic so that the equilibrium state maximizes the mixing entropy $\mathcal{H}$ under the constraints of the dynamics. In reality, this is not always the case. It has been understood since the beginning [22] that violent relaxation may be incomplete so that the mixing entropy $\mathcal{H}$ is not maximized in the whole available phase space. This is obvious in the case of 3D self-gravitating systems since there is no maximum entropy state, even in theory (the Lynden-Bell DF has an infinite mass). However, even for simpler systems for which a maximum entropy state (in the sense of Lynden-Bell) exists, there are cases where this maximum entropy state is not reached (see the discussion in [38] and in Sec. 6 of [39]). For example, in the context of the HMF model [28, 30, 72, 73] and in the context of 2D turbulence [74, 80], situations have been reported where the Lynden-Bell prediction works well and situations have been reported where the Lynden-Bell does not work well (!). Some authors have proposed to account for incomplete relaxation by changing the form of entropy and by using for example the Tsallis entropy [81]. Sometimes, the Tsallis distribution provides a good fit of the metaequilibrium state (see the above-mentioned references). However, this is not general. Furthermore, this type of approach leads to some arbitrariness since the generalized entropy depends on unknown parameters (like, e.g., Tsallis’ $q$ parameter) that are not predicted by the theory. In practice, these parameters have to be fitted to the observed distribution. In Sec. V we shall discuss an alternative approach to take into account incomplete relaxation, based on kinetic theory, where there is no such indetermination.

III. KINETIC THEORY OF QUIESCENT COLLISIONLESS RELAXATION

In this section, we recall the kinetic theory of quiescent collisionless relaxation, based on a quasilinear theory of the Vlasov equation, initially developed by Kadomtsev and Pogutse [40], Severne and Luwel [41], and Chavanis [42–44]. We give some details of derivation and complements.

A. Quasilinear theory

Basically, a collisionless system with long-range interactions is described in a self-consistent mean field approximation by the Vlasov equation [see Eqs. (1) and (2)]. In principle, this equation completely determines the evolution of the DF $f(r, v, t)$. However, as discussed in Sec. III, we are not interested in practice by the finely striated structure of the flow in phase space but only by its macroscopic, i.e. smoothed-out, structure. Indeed, the observations and the numerical simulations are always realized with a finite resolution. Moreover, the coarse-grained DF $\overline{f}(r, v)$ is likely to converge towards a steady state $\overline{f}(r, v)$ (metaequilibrium state) contrary to the exact distribution $f(r, v, t)$ which develops smaller and smaller scales for all times.

If we decompose the DF and the potential in a mean and fluctuating part ($f = \overline{f} + \delta f$, $\Phi = \overline{\Phi} + \delta \Phi$) and take the local average of the Vlasov equation (1), we readily obtain an equation for the coarse-grained DF of the form

$$\frac{\partial \overline{f}}{\partial t} + v \cdot \frac{\partial \overline{f}}{\partial r} - \nabla \overline{\Phi} \cdot \frac{\partial \overline{f}}{\partial v} = - \frac{\partial}{\partial v} \cdot \mathbf{J}_f$$

(30)

with a diffusion current $\mathbf{J}_f = -\delta \overline{f} \nabla \delta \Phi$ related to the correlations of the fine-grained fluctuations. The right hand side of Eq. (30) can be interpreted as an effective “collision” term. Any systematic calculation of the diffusion current starting from the Vlasov equation must necessarily introduce an evolution equation for the fluctuation $\delta f$. This equation is simply obtained by subtracting Eq. (30) from Eq. (1). This yields

$$\frac{\partial \delta f}{\partial t} + v \cdot \frac{\partial \delta f}{\partial r} - \nabla \delta \Phi \cdot \frac{\partial \delta f}{\partial v} = \nabla \delta \Phi \cdot \frac{\partial \overline{f}}{\partial v} + \nabla \delta \Phi \cdot \frac{\partial \delta f}{\partial v} - \nabla \delta \Phi \cdot \frac{\partial \delta f}{\partial v}.$$

(31)
Equations (30) and (31) are exact since no approximation has been made for the moment. To go further, we need to implement some approximations. In the sequel, we shall develop a quasilinear theory which was introduced by Kadomtsev and Pogutse [40] by analogy with the quasilinear theory of collisional relaxation based on the Klimontovich equation. This will provide a precise theoretical framework to analyze the process of collisionless relaxation in systems with long-range interactions. The essence of the quasilinear theory is to assume that the fluctuations are weak and neglect the nonlinear terms in Eq. (31) altogether. In that case, Eqs. (30) and (31) reduce to the coupled system

\[ \frac{\partial \tilde{f}}{\partial t} + \mathbf{v} \cdot \frac{\partial \tilde{f}}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial \tilde{f}}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \cdot \delta \tilde{f} \nabla \delta \Phi, \]  

(32)

\[ \frac{\partial \delta f}{\partial t} + \mathbf{v} \cdot \frac{\partial \delta f}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial \delta f}{\partial \mathbf{v}} - \nabla \Phi \cdot \frac{\partial \tilde{f}}{\partial \mathbf{v}} = 0. \]  

(33)

Physically, these equations describe the coupling between a subdynamics (played here by the small scale fluctuations \( \delta f \)) and a macrodynamics (played by the coarse-grained DF \( f \)). Due to the strong simplifications implied by the neglect of nonlinear terms in Eq. (31), the quasilinear theory only describes the late quiescent stages of the violent relaxation process, when the fluctuations have weakened (gentle relaxation). Although this is essentially an asymptotic theory, it is of importance to develop this theory in detail since it provides an explicit expression of the effective “collision operator” which appears on the coarse-grained scale.

If we restrict ourselves to spatially homogeneous distributions, the field \( -\nabla \Phi \) vanishes. In that case, Eqs. (32) and (33) reduce to the coupled equations

\[ \frac{\partial \tilde{f}}{\partial t} = \frac{\partial}{\partial \mathbf{v}} \cdot \delta \tilde{f} \nabla \delta \Phi, \]  

(34)

\[ \frac{\partial \delta f}{\partial t} + \mathbf{v} \cdot \frac{\partial \delta f}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial \delta f}{\partial \mathbf{v}} = 0. \]  

(35)

We shall assume that the fluctuations evolve rapidly compared to the evolution of the coarse-grained fields, so that the time variation of \( \tilde{f} \) and \( \Phi \) can be neglected in the calculation of the collision term. This is similar to the Bogoliubov ansatz in the collisional theory. Therefore, for the purpose of solving Eq. (35) and obtaining the correlation function \( \delta f \delta \Phi \), we shall regard \( \tilde{f}(\mathbf{v}) \) as constant in time. With this approximation, Eqs. (34) and (35) can be solved with the aid of Fourier-Laplace transforms and the collision term can be explicitly calculated. The derivation proceeds similarly to the derivation of the Lenard-Balescu equation from the Klimontovich equation in the collisional theory (see, e.g., [12]). The collisional and collisionless kinetic theories are analogous because the Klimontovich equation is formally similar to the Vlasov equation. However, the Klimontovich equation involves a DF which is a sum of \( \delta \)-functions while the Vlasov equation involves a continuous DF.

B. Dielectric function

The Fourier-Laplace transform of the fluctuations of the DF \( \delta f \) is defined by

\[ \delta \hat{f}(\mathbf{k}, \mathbf{v}, \omega) = \int \frac{d\mathbf{r}}{(2\pi)^d} \int_0^{+\infty} dt e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \delta f(\mathbf{r}, \mathbf{v}, t). \]  

(36)

The inverse transform is

\[ \delta f(\mathbf{r}, \mathbf{v}, t) = \int d\mathbf{k} \int_{C} \frac{d\omega}{2\pi} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \delta \hat{f}(\mathbf{k}, \mathbf{v}, \omega), \]  

(37)

where the Laplace contour \( C \) in the complex \( \omega \) plane must pass above all poles of the integrand. Similar expressions hold for the fluctuations of the potential \( \delta \Phi \). We note that, for periodic potentials, the integral over \( \mathbf{k} \) is replaced by a discrete summation over the different modes. If we take the Fourier-Laplace transform of Eq. (35), we find that

\[ -\delta \hat{f}(\mathbf{k}, 0) - i\omega \delta \hat{f}(\mathbf{k}, \omega) + i\mathbf{k} \cdot \mathbf{v} \delta \hat{f}(\mathbf{k}, \omega) - i\mathbf{k} \cdot \frac{\partial \tilde{f}}{\partial \mathbf{v}} \delta \Phi(\mathbf{k}, \omega) = 0, \]  

(38)
where the first term is the spatial Fourier transform of the initial value of the fluctuations

$$\delta \hat{f}(k, v, 0) = \int \frac{d r}{(2\pi)^d} e^{-i k \cdot r} \delta f(r, v, 0).$$  \hspace{1cm} (39)$$

Equation (38) can be rewritten as

$$\delta \tilde{f}(k, v, \omega) = \frac{k \cdot \frac{\delta T}{\delta v}}{k \cdot v - \omega} \delta \Phi(k, \omega) + \frac{\delta \hat{f}(k, v, 0)}{i(k \cdot v - \omega)},$$  \hspace{1cm} (40)$$

where the first term takes into account “collective effects”. The fluctuations of the potential are related to the fluctuations of the DF by a convolution

$$\delta \Phi(r, t) = \int u(|r - r'|) \delta f(r', v', t) \, dr' \, dv'.$$  \hspace{1cm} (41)$$

Taking the Fourier-Laplace transform of this equation, we obtain

$$\delta \tilde{\Phi}(k, \omega) = (2\pi)^d \hat{u}(k) \int \delta \tilde{f}(k, v, \omega) \, dv.$$  \hspace{1cm} (42)$$

Substituting Eq. (40) into Eq. (42), we find that the Fourier-Laplace transform of the fluctuations of the potential is given by

$$\delta \tilde{\Phi}(k, \omega) = (2\pi)^d \hat{u}(k) \int \frac{k \cdot \frac{\delta T}{\delta v}}{k \cdot v - \omega} \, dv \delta \tilde{\Phi}(k', \omega') \hspace{1cm} \text{where}$$

$$\epsilon(k, \omega) = 1 - (2\pi)^d \hat{u}(k) \int \frac{k \cdot \frac{\delta T}{\delta v}}{k \cdot v - \omega} \, dv,$$  \hspace{1cm} (43)$$

is the dielectric function. The Fourier-Laplace transform of the fluctuations of the DF is then given by Eq. (40) with Eq. (43). The dispersion relation associated with the linearized Vlasov equation corresponds to \(\epsilon(k, \omega) = 0\). It determines the proper pulsations of the system. If the DF is Vlasov stable,\(^{12}\) then \(|\omega| < 0\) for all modes \(\omega\). In particular, \(\epsilon(k, \omega)\) does not vanish when \(\omega\) is real so that Eq. (43) is well-defined. If collective effects were neglected in Eq. (40), we would obtain Eq. (40) with \(\epsilon(k, \omega) = 1\). This shows that, because of collective effects, the bare potential of interaction \(\hat{u}(k)\) is replaced by a “dressed” potential \(\hat{u}(k)/\epsilon(k, \omega)\) taking into account the polarization of the medium.

We can use the foregoing equations to compute the effective collision term appearing on the right hand side of Eq. (34). One has

$$\hat{f}(k, v, \omega) = \int dk \int C \frac{d \omega}{2\pi} \int C \frac{d \omega'}{2\pi} i k' e^{i(k \cdot r - \omega t)} e^{i(k' \cdot r - \omega' t)} \delta f(k, v, \omega) \delta \tilde{\Phi}(k', \omega').$$  \hspace{1cm} (45)$$

Using Eq. (40), we find that

$$\frac{\delta f(k, v, \omega) \delta \tilde{\Phi}(k', \omega')}{k \cdot v - \omega} = \frac{k \cdot \frac{\delta T}{\delta v}}{k \cdot v - \omega} \delta \tilde{\Phi}(k, \omega) \delta \tilde{\Phi}(k', \omega') + \frac{\delta \hat{f}(k, v, 0) \delta \tilde{\Phi}(k', \omega')}{i(k \cdot v - \omega)}.$$  \hspace{1cm} (46)$$

As we shall see, the first term accounts for a diffusion and the second term accounts for a friction. Let us consider these two terms separately.

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\(^{12}\) The initial DF may be dynamically unstable or even unsteady but we focus on a regime of quiescent relaxation where the DF is dynamically stable.
C. Diffusion

From Eq. 45, we obtain
\[
\delta \Phi(k, \omega) \delta \Phi(k', \omega') = -(2\pi)^d \frac{\hat{u}(k) \hat{u}(k')}{\epsilon(k, \omega) \epsilon(k', \omega')} \int d\nu d\nu' \frac{\delta \hat{f}(\nu, \omega) \delta \hat{f}(\nu', \omega')}{(k \cdot \nu - \omega)(k' \cdot \nu' - \omega')}.
\] (47)

To proceed further, we have to evaluate the correlation function \( \frac{\delta \hat{f}(r, \nu, t) \delta \hat{f}(r', \nu', t)}{\epsilon} \). Following Kadomtsev and Pogutse [40] we shall assume that the mixing in phase space is sufficiently efficient that the scale of the kinematic correlations is small with respect to the coarse-graining mesh size. In that case, we can write
\[
\frac{\delta \hat{f}(r, \nu, t) \delta \hat{f}(r', \nu', t)}{\epsilon} = \epsilon_r \epsilon_v \delta(r - r') \delta(\nu - \nu') f_2(\nu),
\] (48)

where \( \epsilon_r \) and \( \epsilon_v \) are the resolution scales in position and velocity respectively and
\[
f_2 \equiv (\delta \hat{f})^2 = (f - f)^2 = \overline{f}^2 - \overline{f}^2
\] (49)
is the local variance of the fine-grained fluctuations. Note that \( \epsilon_r \epsilon_v \) can be interpreted as the hypervolume of a macroweb in Lynden-Bell’s statistical theory.\(^{13}\)

Taking the Fourier transform of Eq. 48, we get
\[
\frac{\delta \Phi(k, \nu, 0) \delta \Phi(k', \nu', 0)}{\epsilon} = \frac{1}{(2\pi)^d} \epsilon_r \epsilon_v (k + k') \delta(\nu - \nu') f_2(\nu).
\] (50)

Substituting Eq. 50 into Eq. 47, we find that
\[
\frac{\delta \Phi(k, \omega) \delta \Phi(k', \omega')}{\epsilon} = (2\pi)^d \epsilon_r \epsilon_v \frac{\hat{u}(k)^2}{\epsilon(k, \omega) \epsilon(-k, \omega')} \delta(k + k') \int d\nu \frac{f_2(\nu)}{(k \cdot \nu - \omega)(k' \cdot \nu' + \omega')}.
\] (51)

Considering only the contributions that do not decay in time, it can be shown [82] that \( |(k \cdot \nu - \omega)(k \cdot \nu' + \omega')|^{-1} \) can be substituted by \( (2\pi)^d \delta(\omega + \omega') \delta(k \cdot \nu - \nu') \).

Then, using the property \( \epsilon(-k, \omega) = \epsilon(k, \omega)^* \), one finds that the correlations of the fluctuations of the potential are given by
\[
\frac{\delta \Phi(k, \omega) \delta \Phi(k', \omega')}{\epsilon} = (2\pi)^d \epsilon_r \epsilon_v \frac{\hat{u}(k)^2}{\epsilon(k, \omega)^2} \delta(k + k') \delta(\omega + \omega') \int d\nu \frac{f_2(\nu)}{(k \cdot \nu - \omega)f_2(\nu')} \delta(k \cdot \nu' - \omega).
\] (52)

From Eq. 52, we get the contribution to Eq. 45 of the first term of Eq. 46. This yields the diffusion term
\[
(\delta f \nabla \delta \Phi)_{11}^{\text{Diff}} = -i(2\pi)^d \epsilon_r \epsilon_v \frac{\hat{u}(k)^2}{\epsilon(k, \omega)^2} \int dk \int \frac{d\omega}{2\pi} \int d\nu \frac{k \cdot \frac{\partial}{\partial \nu} \hat{u}(k)^2}{(k \cdot \nu - \omega)^2} f_2(\nu') \delta(k \cdot \nu' - \omega).
\] (53)

Using the Landau prescription \( \omega \rightarrow \omega + i0^+ \) and the Plemelj formula,
\[
\frac{1}{x \pm i0^+} = \mathcal{P} \left( \frac{1}{x} \right) \mp i\pi \delta(x),
\] (54)

where \( \mathcal{P} \) denotes the principal value, we can replace \( 1/(k \cdot \nu - \omega - i0^+) \) by \( +i\pi \delta(k \cdot \nu - \omega) \). Then, integrating over \( \omega \), we obtain
\[
(\delta f \nabla \delta \Phi)_{11}^{\text{Diff}} = \pi(2\pi)^d \epsilon_r \epsilon_v \frac{\hat{u}(k)^2}{\epsilon(k, \omega)^2} \int dk \int d\nu' \frac{k \cdot \frac{\partial}{\partial \nu'} \hat{u}(k)^2}{(k \cdot \nu' - \omega)^2} f_2(\nu') \delta(k \cdot \nu' - \omega).
\] (55)

\(^{13}\) The Lynden-Bell entropy can be obtained from a combinatorial analysis by dividing the phase space into macrocells and microcells and by counting the number of microstates associated with a given macrostate. The mixing entropy is equal to the logarithm of this number.

\(^{14}\) See also Appendix A of [83] for a more precise justification of this procedure through a detailed calculation of the integral (Laplace transform) obtained by substituting Eq. 46 with Eq. 51 into Eq. 45.
D. Friction

Proceeding similarly, we obtain

$$\frac{\delta f(k, v, 0) \delta \Phi(k', \omega')}{i(k \cdot v - \omega)} = (2\pi)^d \frac{\hat{u}(k')}{\epsilon(k', \omega')} \frac{1}{i(k \cdot v - \omega)} \int dk' v' \frac{\delta \bar{f}(k, v, 0) \delta f(k', v', 0)}{i(k' \cdot v' - \omega')}$$ (56)

and

$$\frac{\delta f(k, v, 0) \delta \Phi(k', \omega')}{i(k \cdot v - \omega)} = \epsilon_\nu \frac{\hat{u}(k')}{\epsilon(k', \omega')} \delta(k + k') \frac{1}{i(k \cdot v - \omega)} f_2(v).$$ (57)

Considering only the contributions that do not decay in time, it can be shown \cite{82} (see footnote 14 with Eq. \ref{51}) that Eq. \ref{57} can be substituted by

$$\frac{\delta f(k, v, 0) \delta \Phi(k', \omega')}{i(k \cdot v - \omega)} = (2\pi)^d \epsilon_\nu \frac{\hat{u}(k')}{\epsilon(k', \omega')} \delta(k + k') \delta(\omega + \omega') \delta(k \cdot v - \omega) f_2(v).$$ (58)

From Eq. \ref{58}, we get the contribution to Eq. \ref{45} of the second term of Eq. \ref{46}. This yields the friction term

$$\langle \delta f \nabla \delta \Phi \rangle^{\text{Fric}} = \epsilon_\nu \epsilon_\nu \int dk k_i \frac{\hat{u}(k)}{\epsilon(k, \mathbf{k} \cdot \mathbf{v})} \Im \epsilon(k, \mathbf{k} \cdot \mathbf{v}) f_2(v).$$ (59)

Using the Landau prescription \( \omega \to \omega + i0^+ \) and the Plemelj formula (54), the imaginary part of the dielectric function \( \epsilon \) reads

$$\Im \epsilon(k, \omega) = -\pi (2\pi)^d \hat{u}(k) \int k \cdot \frac{\partial T}{\partial v} \delta(k \cdot v - \omega) dv.$$ (60)

Substituting this expression into Eq. \ref{59}, we obtain

$$\langle \delta f \nabla \delta \Phi \rangle^{\text{Fric}} = -\pi (2\pi)^d \epsilon_\nu \epsilon_\nu \int dk dv' k_i k_j \frac{\hat{u}(k)^2}{\epsilon(k, \mathbf{k} \cdot \mathbf{v})^2} \delta[k \cdot (v - v')] f_2(v) \frac{\partial T}{\partial v'}(v').$$ (61)

E. Collision term: Kadomtsev-Pogutse (1970) equation

Regrouping Eqs. \ref{34}, \ref{55} and \ref{61}, we end up with the kinetic equation

$$\frac{\partial \bar{f}}{\partial t} = -\pi (2\pi)^d \epsilon_\nu \epsilon_\nu \frac{\partial}{\partial v_i} \int dk dv' k_i k_j \frac{\hat{u}(k)^2}{\epsilon(k, \mathbf{k} \cdot \mathbf{v})^2} \delta[k \cdot (v - v')] \left( f_2 \frac{\partial \bar{T}}{\partial v_j} - f_2 \frac{\partial \bar{T}}{\partial v'_j} \right),$$ (62)

where \( \bar{T} = T(v, t), \bar{T}' = T(v', t), f_2 = f_2(v, t), \) and \( f_2' = f_2(v', t). \) This equation, which takes collective effects into account, was first derived by Kadomtsev and Pogutse \cite{40}. It is formally similar to the Lenard-Balescu equation of the collisional theory (see, e.g., \cite{12}), except that \( m f \) in the Lenard-Balescu equation is replaced by \( \epsilon_\nu \epsilon_\nu f_2 \) in the KP equation. We also note that, contrary to the Lenard-Balescu equation, the KP equation is not closed in the general case since it involves the variance \( f_2 \) of the fine-grained distribution. It can be closed exactly only in the two-level case (see Sec. IV C).

If we neglect collective effects and take \( |\epsilon(k, \mathbf{k} \cdot \mathbf{v})| = 1, \) Eq. \ref{62} reduces to

$$\frac{\partial \bar{f}}{\partial t} = -\pi (2\pi)^d \epsilon_\nu \epsilon_\nu \frac{\partial}{\partial v_i} \int dk dv' k_i k_j \hat{u}(k)^2 \delta[k \cdot (v - v')] \left( f_2 \frac{\partial \bar{T}}{\partial v_j} - f_2 \frac{\partial \bar{T}}{\partial v'_j} \right).$$ (63)

\[15\] It is closely related to an equation previously derived by Dupree \cite{84}.
The integral over $k$ can be performed explicitly (see, e.g., [28]) and we obtain

$$\frac{\partial \mathbf{F}}{\partial t} = K_d \frac{\partial}{\partial v_i} \int d\mathbf{v}' \frac{w^2 \delta_{ij} - w_i w_j}{w^3} \left( f_2 \frac{\partial \mathbf{F}}{\partial v_j} - f_2 \frac{\partial \mathbf{F}}{\partial v_{j'}} \right),$$

(64)

where $\mathbf{w} = \mathbf{v} - \mathbf{v}'$ is the relative velocity and $K_d$ is a constant with value $K_3 = 8\pi^2 \varepsilon^2 \varepsilon_i^3 \int_0^{+\infty} k^3 \hat{u}(k)^2 dk$ in $d = 3$ and $K_2 = 8\pi^2 \varepsilon^2 \varepsilon_i^3 \int_0^{+\infty} k^2 \hat{u}(k)^2 dk$ in $d = 2$. This kinetic equation, that neglects collective effects, is formally similar to the Landau equation [14] in the collisional theory (see, e.g., [12]) with the substitution $mf \to \varepsilon_i^d \varepsilon_i^d f_2$. For a 3D plasma, using $(2\pi)^3 \hat{u}(k) = 4\pi e^4 / m^2 k^2$, we get $K_3 = (2\pi e^4 / m^3) \ln \Lambda$ where $\ln \Lambda = \int_0^{+\infty} dk / k$ is the Coulomb logarithm that has to be regularized with appropriate cut-offs. The large-scale cut-off is the Debye length $\lambda_D$ (the Debye length appears naturally when we take into account collective effects) and the small-scale cut-off is the spatial resolution scale $\varepsilon_i$ which replaces the Landau length $\lambda_L$ in the collisional theory. This yields $\ln \Lambda = \ln(\lambda_D / \varepsilon_i)$. For a 2D plasma, using $(2\pi)^2 \hat{u}(k) = 2\pi e^2 / m^2 k^2$ and introducing a large-scale cut-off at the Debye length, we obtain $K_2 = 2\pi e^4 / m^3 k_D$.

There is no need to introduce a small-scale cut-off in that case but the integration should be stopped at $\varepsilon_i$ in principle. Returning to Eqs. (62) and (63), we note that collective effects can be taken into account simply by replacing the bare potential $\hat{u}(k)$ in the Landau equation by a “dressed” potential $\hat{u}_d(k) = \hat{u}(k) / \varepsilon_i(k, k, \mathbf{v})$, including the dielectric function, without changing the overall structure of the kinetic equation. Physically, this means that the particles are “dressed” by their polarization cloud. In plasma physics, collective effects are important because they account for screening effects and regularize, at the scale of the Debye length, the logarithmic divergence that occurs in the Landau equation. This avoids the introduction of ad hoc cut-offs at large scales.

We can write the kinetic equation (62) in the compact form

$$\frac{\partial \mathbf{F}}{\partial t} = \frac{\partial}{\partial v_i} \int d\mathbf{v}' K_{ij} \left( f_2 \frac{\partial \mathbf{F}}{\partial v_j} - f_2 \frac{\partial \mathbf{F}}{\partial v_{j'}} \right),$$

(65)

by introducing the tensor

$$K_{ij} = \pi (2\pi)^d \varepsilon_i^d \varepsilon_i^d \int dk k_i k_j \frac{\hat{u}(k)^2}{\varepsilon(k, k, \mathbf{v})^2} \delta(k - (\mathbf{v} - \mathbf{v}')).$$

(66)

We note that it satisfies the identity $K_{ij} w_j = 0$. If we neglect collective effects, the tensor $K_{ij}$ is explicitly given by (see, e.g., [28])

$$K_{ij}^{\text{bare}} = K_d \frac{1}{w} \left( \delta_{ij} - \frac{w_i w_j}{w^2} \right).$$

(67)

We also note that the kinetic equation (63) has the structure of a generalized Fokker-Planck equation involving a diffusion term and a friction term.

Remark: An equation similar to the KP equation has been introduced in 2D turbulence [15, 16] (see also Eq. (148) of [41] and Eq. (88) of [48]). It reads

$$\frac{\partial \mathbf{F}}{\partial t} = 2\pi^2 \varepsilon_i^2 \varepsilon_i \frac{1}{r} \frac{\partial}{\partial r} \sum_n \int_0^{+\infty} r' dr' |n| |G(n, r, r', n\Omega)|^2 \delta(\Omega - \Omega') \left( \omega_2 \frac{1}{r} \frac{\partial \mathbf{F}}{\partial r} - \omega_2 \frac{1}{r'} \frac{\partial \mathbf{F}}{\partial r'} \right).$$

(68)

F. 3D self-gravitating systems

Systems with attractive long-range interactions are generically spatially inhomogeneous. It is important to develop a kinetic theory of collisionless relaxation for such systems. If we implement a quasilinear approximation and neglect collective effects, we can derive a generalized kinetic equation of the form [12, 44]

$$\frac{\partial \mathbf{F}}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathbf{F}}{\partial r} - \nabla \Phi \cdot \frac{\partial \mathbf{F}}{\partial \mathbf{v}} = \varepsilon_i^d \varepsilon_i^d \frac{\partial}{\partial v_i} \int_0^t ds \int d\mathbf{v}' F_i(\mathbf{r}' \to \mathbf{r}) F_j(\mathbf{r}' \to \mathbf{r}) v_{i-s} \left( f_2 \frac{\partial \mathbf{F}}{\partial v_j} - f_2 \frac{\partial \mathbf{F}}{\partial v_{j'}} \right) t_{-s},$$

(69)

that is valid for systems that are not necessarily spatially homogeneous and not necessarily Markovian. Here, $\mathbf{F} = \mathbf{F}(\mathbf{r}, \mathbf{v}, t)$, $\mathbf{F} = \mathbf{F}(\mathbf{r}', \mathbf{v}', t)$, $f_2 = f_2(\mathbf{r}, \mathbf{v}, t)$ and $f_2' = f_2(\mathbf{r}', \mathbf{v}', t)$. On the other hand, $\mathbf{F}(\mathbf{r}' \to \mathbf{r})_t$ denotes the force by
unit of mass exerted by a particle located in \( \mathbf{r}' \) on a particle located in \( \mathbf{r} \) at time \( t \). In certain cases, e.g. for 3D self-gravitating systems, we can make a local approximation and proceed as if the system were spatially homogeneous. If we also implement a Markovian approximation, the kinetic equation (69) is replaced by

\[
\frac{\partial \mathcal{F}}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathcal{F}}{\partial \mathbf{r}} - \nabla \mathcal{F} \cdot \frac{\partial \mathcal{F}}{\partial \mathbf{v}} = \epsilon_i^r \epsilon_i^\gamma \frac{\partial}{\partial \mathbf{v}_i} \int_0^{+\infty} ds \int d\mathbf{v}' d\mathbf{v}' \int_{\mathbf{F}_i(\mathbf{r}' \to \mathbf{r})} \int_{\mathbf{F}_j(\mathbf{r}' \to \mathbf{r})} \left( f_2 \frac{\partial \mathcal{F}}{\partial \mathbf{v}_j} - f_2 \frac{\partial \mathcal{F}}{\partial \mathbf{v}'_j} \right),
\]

where now \( \mathcal{F} = \mathcal{F}(\mathbf{r}, \mathbf{v}, t), \mathcal{F}' = \mathcal{F}(\mathbf{r}', \mathbf{v}', t), f_2 = f_2(\mathbf{r}, \mathbf{v}, t) \) and \( f_2' = f_2(\mathbf{r}, \mathbf{v}', t) \). Passing in Fourier space, we obtain

\[
\frac{\partial \mathcal{F}}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathcal{F}}{\partial \mathbf{r}} - \nabla \mathcal{F} \cdot \frac{\partial \mathcal{F}}{\partial \mathbf{v}} = \pi (2\pi)^3 \epsilon_i^3 \epsilon_i^\gamma \frac{\partial}{\partial \mathbf{v}_i} \int d\mathbf{k} d\mathbf{v}' k_i k_j \delta[k \cdot (\mathbf{v} - \mathbf{v}')] \left( f_2 \frac{\partial \mathcal{F}}{\partial \mathbf{v}_j} - f_2' \frac{\partial \mathcal{F}}{\partial \mathbf{v}'_j} \right).
\]

In this equation, the effects of spatial inhomogeneity are kept only in the advection (Vlasov) term, while the collision term is calculated as if the system were spatially homogeneous. The integral over \( \mathbf{k} \) can be performed explicitly \(^28\) and the foregoing equation can be rewritten as

\[
\frac{\partial \mathcal{F}}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathcal{F}}{\partial \mathbf{r}} - \nabla \mathcal{F} \cdot \frac{\partial \mathcal{F}}{\partial \mathbf{v}} = K_3 \frac{\partial}{\partial \mathbf{v}_i} \int d\mathbf{v} w'^2 \delta_{ij} - w_i w_j w^3 \left( f_2 \frac{\partial \mathcal{F}}{\partial \mathbf{v}_j} - f_2' \frac{\partial \mathcal{F}}{\partial \mathbf{v}'_j} \right).
\]

For a 3D self-gravitating system, using \((2\pi)^3 \delta(k) = -4\pi G/k^2\), we get \( K_3 = 2\pi c_i^3 c_i^\gamma G^2 \ln \Lambda \) where \( \ln \Lambda = \int_0^{+\infty} dk/k \) is the Coulomb factor that has to be regularized with appropriate cut-offs. The large-scale cut-off is the Jeans length \( \lambda_J \) (which is of the order of the system size \( R \)) and the small-scale cut-off is the spatial resolution scale \( \epsilon_r \) which replaces the gravitational Landau length \( \lambda_L \) in the collisional theory. This yields \( \ln \Lambda = \ln(\Lambda_J/\epsilon_r) \).

**Remark:** We can also extend the KP equation to spatially inhomogeneous systems by making a local approximation. In this manner, we can heuristically take collective effects into account. In Sec. \( \text{[VII]} \) we introduce more general kinetic equations written with angle-action variables that take into account spatial inhomogeneity and collective effects without relying on a local approximation.

### IV. MULTI-LEVEL CASE

In the general case, the kinetic equation (69) for the coarse-grained DF \( \mathcal{F}(\mathbf{v}, t) \) is not closed because it depends on the local centered variance \( f_2(\mathbf{v}, t) \) of the distribution \( \rho(\mathbf{v}, \eta, t) \). It is therefore necessary to extend the kinetic theory to the multi-level case and work in term of the DF \( \rho(\mathbf{v}, \eta, t) \) for each level.

#### A. Severne-Luwel (1980) equation

A kinetic equation for \( \rho(\mathbf{v}, \eta, t) \) has been derived by Severne and Luwel \(^{11} \) by generalizing the quasilinear theory of Kadomtsev and Pogutse \(^{40} \) to the multi-level case.\(^{16} \) They obtained an equation of the form\(^{17} \)

\[
\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial \mathbf{v}_i} \int d\mathbf{v}' K_{ij} \left( f_2' \frac{\partial \rho}{\partial \mathbf{v}'_j} - \rho(\eta - \mathcal{F}) \frac{\partial \mathcal{F}}{\partial \mathbf{v}'_j} \right).
\]

We can check that the normalization condition \( \int \rho \, d\eta = 1 \) is preserved with time and that the equation for the coarse-grained DF \( \mathcal{F} = \int \rho_t \, d\eta \) returns Eq. (65). We can also show that the SL equation conserves the energy and all

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\(^{16}\) The derivation is similar to the one detailed in Sec. \( \text{[III]} \). In the multi-level case, one has to work in terms of \( \rho_\lambda(\mathbf{r}, \mathbf{v}, \eta, t) = \delta(f(\mathbf{r}, \mathbf{v}, t) - \eta) \) and \( \rho(\mathbf{v}, \eta, t) = \delta(f(\mathbf{r}, \mathbf{v}, t) - \eta) \) which generalize \( f(\mathbf{r}, \mathbf{v}, t) \) and \( \mathcal{F}(\mathbf{v}, t) \), respectively. Equation (65) is generalized into

\[
\frac{\delta \rho(\mathbf{r}, \mathbf{v}, \eta, t)}{\delta \rho(\mathbf{r}', \mathbf{v}', \eta', t')} = \epsilon_i^r \epsilon_i^\gamma \rho(\mathbf{r}, \mathbf{v}, \eta, t) \delta(\eta - \eta') - \rho(\mathbf{r}, \mathbf{v}, \eta, t) \delta(\mathbf{r} - \mathbf{r}') \delta(\mathbf{v} - \mathbf{v}'),
\]

corresponding to Eq. (3.13) of \( \text{[11]} \).

\(^{17}\) Severne and Luwel \(^{11} \) developed their theory for spatially inhomogeneous stellar systems. However, at the end of their calculations, in order to obtain an explicit kinetic equation, they made a local approximation which amounts to proceeding as if the system were infinite and homogeneous (see Sec. \( \text{[III]} \)).
the Casimirs, that it satisfies and $H$-theorem for the Lynden-Bell entropy \[ S = \int \rho \eta^n \, d\eta \] and that it relaxes towards the Gibbs state \[ \rho = \frac{1}{Z} \exp(-\beta H) \] (see Appendix E).

From the SL equation, we can derive a hierarchy of equations for the moments \[ \bar{f}^n = \int \rho \eta^n \, d\eta \] of the distribution. The general term of this hierarchy is

\[
\frac{\partial \bar{f}^n}{\partial t} = \frac{\partial}{\partial v_i} \int d\nu' K_{ij} \left[ f'_j \frac{\partial \bar{f}^n}{\partial v_j} - (\bar{f}^{n+1} - \bar{f}^n) \frac{\partial \bar{f}^n}{\partial \nu'} \right].
\]

For $n = 1$ we recover the KP equation \[ Eq. (65) \].

Remark: An equation similar to the SL equation can be derived in 2D turbulence \[ 46 \]. It reads

\[
\frac{\partial \rho}{\partial t} = 2\pi \epsilon^2 \int_0^{+\infty} r' dr' \sum_n \int_{+\infty}^{1} G(n, r, r', n\Omega)^2 \delta(\Omega - \Omega') \left[ \omega_1^2 \frac{1}{\epsilon} \frac{\partial \rho}{\partial r} + \rho(\sigma - \Omega') \frac{1}{\epsilon} \frac{\partial \omega'}{\partial \nu'} \right].
\]

B. Chavanis-Sommeria-Robert (1996) equation

By using a different approach based on a Maximum Entropy Production Principle (MEPP), Chavanis, Sommeria and Robert \[ 26 \] have proposed an equation for \[ \rho(v, \eta, t) \] that relaxes towards the Lynden-Bell distribution. \[ 18 \] This phenomenological equation is expected to describe the whole process of violent relaxation, including the very nonlinear early regime. In the late regime of quiescent relaxation, it is possible to connect the CSR equation to the KP and SL equations as follows. \[ 19 \]

The kinetic equation \[ (74) \] obtained from the quasilinear theory of the Vlasov equation is an integrodifferential equation. The current in \( v \) depends on the value of \( f \) and \( f^2 \) in \( v' \) through an integral over \( v' \). We can transform the integrodifferential equation \[ (74) \] into a differential equation by replacing \( f' \) and \( f^2' \) by their equilibrium values obtained from the Gibbs state \[ (11) \]. This amounts to making a thermal bath approximation. Using the identity from Eq. \[ (16) \], we find that

\[
\frac{\partial f'}{\partial \nu'} = -\beta f_2' v'.
\]

Substituting Eq. \[ (77) \] into Eq. \[ (74) \] we get

\[
\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial v_i} \int d\nu' K_{ij} \left[ f'_j \frac{\partial \rho}{\partial v_j} + \beta \rho(\eta - \bar{f}) v_j \right].
\]

In principle, \( f^2 \) should be calculated at equilibrium. However, in order to be more general, we shall evaluate \( f^2 \) in Eq. \[ (78) \] at time \( t \), not at equilibrium. Using the identity \( K_{ij}(v_j - v'_j) = 0 \), we can replace \( v'_j \) by \( v_j \) in the last term of Eq. \[ (78) \]. In this manner, we obtain

\[
\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial v_i} \left\{ D_{ij} \left[ \frac{\partial \rho}{\partial v_j} + \beta \rho(\eta - \bar{f}) v_j \right] \right\}
\]

with

\[
D_{ij} = \int d\nu' K_{ij} f^2.
\]

This equation can be viewed as a generalized Fokker-Planck equation of the Kramers type involving a diffusion term and a friction term. The friction term is the counterpart of Chandrasekhar’s dynamical friction \[ 87 \] in the kinetic theory of collisional stellar systems. The friction coefficient is given by a form of Einstein relation \( \xi_{ij} = D_{ij} \beta \eta \).

By making the thermal bath approximation from Eq. \[ (77) \], we have lost the conservation of energy. Indeed, we have passed from a microcanonical description (conservation of energy \( E \) and \( H \)-theorem for the entropy \( S \)) to a canonical

---

\[ 18 \] The MEPP can be formulated in the inhomogeneous case, as discussed in Appendix E, but in this section we consider spatially homogeneous systems.

\[ 19 \] This connection was made in \[ 42, 43 \] in the two-level case and is extended here to the multi-level case.
description (fixed temperature $T$ and $H$-theorem for the free energy $F = E - TS$). However, the conservation of energy can be artificially restored by letting $\beta(t)$ depend on time in a suitable manner. To that purpose, using Eq. (79), we first write the equation for the coarse-grained DF $\mathcal{F} = \int \rho \eta \, d\eta$ which reads

$$\frac{\partial \mathcal{F}}{\partial t} = \frac{\partial}{\partial v_i} \left\{ D_{ij} \left[ \frac{\partial \mathcal{F}}{\partial v_j} + \beta f_2 v_j \right] \right\}. \quad (81)$$

This equation can also be obtained by applying the thermal bath approximation to Eq. (62). Then, we compute

$$\dot{E} = \int \frac{\partial \mathcal{F}}{\partial t} v^2 \, dv = - \int \frac{\partial \mathcal{F}}{\partial v} \mathcal{F} \, dv = - \int D_{ij} \left[ \frac{\partial \mathcal{F}}{\partial v_j} + \beta f_2 v_j \right] v_i \, dv. \quad (82)$$

We can enforce the conservation of energy ($\dot{E} = 0$) by taking

$$\beta(t) = - \frac{\int D_{ij} v_i \frac{\partial \mathcal{F}}{\partial v_j} \, dv}{\int D_{ij} f_2 v_i v_j \, dv}. \quad (83)$$

This approach returns the CSR equations formed by Eqs. (79) and (83) above. It also provides the explicit expression of the diffusion coefficient – actually a tensor $D_{ij}$ – which was not given by the MEPP (26). Equation (79) with Eqs. (80) and (83) is an integrodifferential equation since $\beta$ and $D_{ij}$ are expressed as integrals of $\mathcal{F}$ and $f_2$, but it is simpler than the KP and SL equations. We can show (see Ref. 26 and Appendix C) that the CSR equations conserve the energy and all the Casimirs, and that they monotonically increase the Lynden-Bell entropy ($H$-theorem). They usually relax towards the Lynden-Bell DF except in the cases reported in Sec. VI where the diffusion tensor $D_{ij}$ vanishes.

From the CSR equation (79), we can derive a hierarchy of equations for the moments $\mathcal{F}^n = \int \rho \eta^n \, d\eta$ of the distribution. The general term of this hierarchy is

$$\frac{\partial \mathcal{F}^n}{\partial t} = \frac{\partial}{\partial v_i} \left\{ D_{ij} \left[ \frac{\partial \mathcal{F}^n}{\partial v_j} + \beta(\mathcal{F}^{n+1} - \mathcal{F}^{n}) v_j \right] \right\}. \quad (84)$$

For $n = 1$ we recover Eq. (81). This hierarchy of equations can also be obtained by applying the thermal bath approximation to Eq. (79).

**Remark:** In the context of 2D turbulence, the MEPP has been introduced by Robert and Sommeria (17). It can be viewed as a variational formulation of the linear thermodynamics of Onsager (88–90) (see 51, 91). The relaxation equations for the coarse-grained vorticity involves a diffusion term and a drift term (20, 44, 49). The drift is the counterpart of Chandrasekhar’s dynamical friction (20, 88, 92). The moments of the relaxation equation have been derived in 48, 61, 62, 93. The connection between the quasilinear theory of the 2D Euler equation and the MEPP has been discussed by Chavanis (45). This connection allowed us to compute the diffusion coefficient of 2D vortices (not given by the MEPP) and recover the heuristic expression given in Ref. 48. A more detailed discussion of the quasilinear theory of the 2D Euler equation is given in 46.

**C. Two-level case: Fermionic-like equations**

If the initial condition in phase space consists of patches of uniform DF $f = \eta_0$ surrounded by vacuum $f = 0$ (two-level approximation), we can write $\mathcal{F}^2 = \eta_0 \times \mathcal{F} = \eta_0 \mathcal{F}$ yielding

$$f_2 = \mathcal{F}(\eta_0 - \mathcal{F}). \quad (85)$$

In that case, the local centered variance $f_2$ of the distribution can be trivially related to the coarse-grained DF $\mathcal{F}$. We see how the self-correlations and the incompressibility of the flow in phase space give rise to an effective “exclusion principle”. Substituting Eq. (85) into Eq. (65), we obtain a kinetic equation of the form

$$\frac{\partial \mathcal{F}}{\partial t} = \beta \frac{\partial}{\partial v_i} \int d\mathbf{v'} K_{ij} \left[ \mathcal{F}(\eta_0 - \mathcal{F}) \frac{\partial \mathcal{F}}{\partial v_j} - \mathcal{F}(\eta_0 - \mathcal{F}) \frac{\partial \mathcal{F}}{\partial v'_j} \right]. \quad (86)$$

This equation is similar to the fermionic Landau (or Lenard-Balescu) equation (26, 44, 94). It conserves the mass and the energy and monotonically increases the Lynden-Bell (or Fermi-Dirac-like) entropy (94). In the nondegenerate
limit $\bar{f} \ll \eta_0$, it becomes similar to the classical Landau (or Lenard-Balescu) equation associated with the Boltzmann entropy.

On the other hand, the CSR equations (79), (80) and (83) reduce to

$$\frac{\partial \bar{f}}{\partial t} = \frac{\partial}{\partial v_i} \left\{ D_{ij} \left[ \frac{\partial \bar{f}}{\partial v_j} + \beta(t) (\eta_0 - \bar{f}) v_j \right] \right\}$$

with

$$D_{ij} = \int d\nu' K_{ij} (\eta_0 - \bar{f})$$

and

$$\beta(t) = - \frac{\int D_{ij} v_i \frac{\partial \bar{f}}{\partial v_j} d\nu}{\int D_{ij} (\eta_0 - \bar{f}) v_i v_j d\nu}.$$ (89)

Equation (87) is similar to the fermionic Kramers equation. It can be obtained from Eq. (86) by using the thermal bath approximation (77). This equation has been derived and studied in [42–44, 94]. It conserves the mass and the energy and monotonically increases the Lynden-Bell (or Fermi-Dirac-like) entropy. In the nondegenerate limit $\bar{f} \ll \eta_0$, it becomes similar to the classical Kramers equation [22].

D. Chavanis (2004) equation

For more complicated initial conditions (multi-level case), we have to solve the equation for $\rho(r, v, \eta, t)$, the probability density of finding the phase level $\eta$ in $(r, v)$ at time $t$ [see Eq. (74) or Eq. (79)]. The strategy is to discretize the initial condition into $N$ levels $\eta$. This approach then leads us to a closed system of $N$ coupled equations (one for each level $\eta$). However, for generic initial conditions, we have to deal with a great number of levels and these equations are not convenient to solve when $N \gg 1$. We can alternatively try to solve the hierarchy of equations for the moments $\bar{f}^n$ [see Eq. (75) or Eq. (84)] but we then encounter a difficult closure problem. In practice, we are mainly interested in the evolution of the first moment, namely the coarse-grained DF $\bar{f}$. As we have seen, the equation for $\bar{f}$ [see Eq. (86) or Eq. (89)] depends on the variance $f_2$. In order to obtain a self-consistent kinetic equation for $\bar{f}$, we need to relate the variance $f_2$ to the coarse-grained DF $\bar{f}$. In Ref. [44], we have proposed a closure approximation that leads to a simple kinetic equation. While not being exact, this equation preserves the robust features of the process of violent relaxation and is amenable to an easier numerical implementation. Its main interest is to go beyond the two-level approximation while leaving the problem tractable. The idea is to observe that Eqs. (65) and (81) lead to the important relation

$$f_2 = \frac{1}{C''(\bar{f})}.$$

(90)

This relation is valid at equilibrium but we propose to use it as a closure approximation in Eqs. (75) and (84). This is expected to be a reasonable approximation if we are close to equilibrium, which is in fact dictated by the quasilinear approximation. Of course, this procedure assumes that we know the function $C(\bar{f})$ in advance. This is the case if we have already determined the equilibrium state by the procedure discussed in Sec. II D and we want to describe the dynamics close to equilibrium. A justification of the closure relation (90) is given in Appendix C following an argument first given in Ref. [50].

If we close the hierarchy of equations (75) with Eq. (90), we obtain a self-consistent kinetic equation of the form

$$\frac{\partial \bar{f}}{\partial t} = \frac{\partial}{\partial v_i} \int d\nu' K_{ij} \left[ \frac{1}{C''(\bar{f})} \frac{\partial \bar{f}}{\partial v_j} - \frac{1}{C''(\bar{f})} \frac{\partial \bar{f}}{\partial v'_j} \right].$$

(91)

20 This is also the case when the generalized entropy $C(\bar{f})$ is determined by the external forcing as discussed in footnotes 8 and 9.
It can be viewed as a generalized Landau (or Lenard-Balescu) equation. It conserves the mass and the energy and monotonically increases the generalized entropy \[ (H \text{-theorem}) \] \[ (22) \] in the two-level case, using \[ f_2 = \frac{1}{C''(f)} \] \[ f_2 = f(\eta) - f \]. The generalized Landau equation \[ (91) \] has been studied in detail in Ref. \[ 44 \].

If we close the hierarchy of equations \[ (84) \] with Eq. \[ (90) \], we obtain a self-consistent kinetic equation of the form \[ (92) \]

\[
\frac{\partial \bar{f}}{\partial t} = \frac{\partial}{\partial v_i} \left\{ D_{ij} \left[ \frac{\partial \bar{f}}{\partial v_j} + \beta(t) \frac{C''(f)}{C''(f)} v_j \right] \right\}
\]

where

\[
D_{ij} = \int dv' K_{ij} \frac{1}{C''(f)}
\]

and

\[
\beta(t) = -\frac{1}{\int D_{ij} v_i \frac{\partial \bar{f}}{\partial v_j} dv} \int D_{ij} \frac{1}{C''(f)} v_i v_j dv
\]

Equation \[ (92) \] can also be derived from Eq. \[ (91) \] by making the thermal bath approximation \[ (77) \]. It can be viewed as a generalized Kramers equation. It conserves the mass and the energy and monotonically increases the generalized entropy \[ (22) \] \[ (H \text{-theorem}) \] \[ (50) \]. In two-level case, using \[ f_2 = \frac{1}{C''(f)} = f(\eta) - f \], we recover Eqs. \[ (87)-(89) \]. The generalized Kramers equations \[ (92)-(94) \] have been studied in detail in Ref. \[ 50 \].

**Remark:** Similar equations have been introduced in the context of 2D turbulence in Refs. \[ 50, 55, 59–62 \].

**E. One-dimensional systems**

For one-dimensional (1D) systems, recalling the expression of \[ K_{ij} \] from Eq. \[ (66) \], the SL equation \[ (74) \] becomes

\[
\frac{\partial \rho}{\partial t} = 2\pi^2 \epsilon_v \epsilon_v \frac{\partial}{\partial v} \int dk' k' \hat{u}(k')^2 \delta(k(v - v')) \left[ f_2 \frac{\partial \rho}{\partial v} - \rho(\eta - \bar{f}) \frac{\partial \bar{f}}{\partial v} \right].
\]

Using the identity \[ \delta(k(v - v')) = \frac{1}{|k|} \delta(v - v') \], it can be rewritten as

\[
\frac{\partial \rho}{\partial t} = 2\pi^2 \epsilon_v \epsilon_v \frac{\partial}{\partial v} \int dk |k| \frac{\hat{u}(k)^2}{|\epsilon(k, kv)|^2} \left[ f_2 \frac{\partial \rho}{\partial v} - \rho(\eta - \bar{f}) \frac{\partial \bar{f}}{\partial v} \right].
\]

The corresponding hierarchy of moment equations takes the form

\[
\frac{\partial \bar{f}^n}{\partial t} = 2\pi^2 \epsilon_v \epsilon_v \frac{\partial}{\partial v} \int dk |k| \frac{\hat{u}(k)^2}{|\epsilon(k, kv)|^2} \left[ f_2 \frac{\partial \bar{f}^n}{\partial v} - \bar{f}^{n+1} - \bar{f}^n - \bar{f} \frac{\partial \bar{f}}{\partial v} \right].
\]

For \( n = 1 \) we get

\[
\frac{\partial \bar{f}}{\partial t} = 0.
\]

This result can be directly obtained from the KP equation \[ (62) \] applied to 1D systems. It implies that the coarse-grained DF does not change with time. However, the higher moments evolve in time.

**Remark:** We note that the “kinetic blocking” of the coarse-grained DF in 1D occurs only for spatially homogeneous systems. For 1D inhomogeneous systems, the coarse-grained DF evolves in time according to the inhomogeneous KP equation \[ (108) \]. On the other hand, for 1D homogeneous systems, the CSR equation \[ (81) \] does not show such a “kinetic blocking”. This may be related to the fact that this equation is more justified in the phase of violent relaxation than in the phase of quiescent relaxation.
V. INCOMPLETE VIOLENT RELAXATION

We have seen that the kinetic equation derived from the quasilinear theory relaxes towards the Lynden-Bell distribution. In this sense, it provides a justification of the maximum entropy principle and implies that the evolution is ergodic. However, direct numerical simulations of the Vlasov equation (or direct simulations of the N-body problem performed in the collisionless regime) show that violent relaxation is in general incomplete \cite{22, 37, 39}. The fluctuations of the potential that are the engine of the collisionless relaxation can die out before the system has reached the statistical equilibrium equilibrium state. How can we reconcile these apparently contradictory results?

First, we have to recall that the quasilinear theory, which is based on the assumption that the nonlinear terms in the equation for the fluctuations can be neglected, describes only a regime of late quiescent relaxation. Therefore, the relaxation toward the Lynden-Bell DF may be limited to this “gentle” situation. In addition, we have assumed that the correlation function is given by Eq. (48) in which the resolution scales $\epsilon_r$ and $\epsilon_v$ in position and velocity are constant in time. This is also a strong assumption. At the end of their paper, Kadomtsev and Pogutse \cite{40} argue that the scale of correlations may decrease in time as the variations of the potential $\Phi$ decay. In that case, the kinetic equation becomes

$$\frac{\partial \rho}{\partial t} = \pi(2\pi)^d \epsilon_r(t)^d \epsilon_v(t)^d \frac{\partial}{\partial \eta} \int d\nu \, d\mathbf{k} \, k_i k_j \frac{\hat{u}(k)^2}{\epsilon(k, \mathbf{k} \cdot \mathbf{v})^2} \delta[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')] \left[ f_2 \frac{\partial \rho}{\partial \nu_j} - \rho(\eta - \bar{f}) \frac{\partial f}{\partial \nu_j} \right],$$

(99)

where $A(t) \equiv \epsilon_r(t)^d \epsilon_v(t)^d$ tends to zero for $t \to +\infty$. If the scale of correlations decreases rapidly in time, the relaxation towards the Lynden-Bell distribution may be inhibited. This effect may account for incomplete relaxation.

The heuristic CSR approach \cite{26} aims at describing the very nonlinear regime of violent relaxation. It leads to a relaxation equation of the form

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \frac{\partial \rho}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial \rho}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}} \left\{ D(\mathbf{r}, \mathbf{v}, t) \left[ \frac{\partial \rho}{\partial \mathbf{v}} + \beta(t)(\eta - \bar{f})\mathbf{v} \right] \right\},$$

(100)

with a diffusion coefficient $D(\mathbf{r}, \mathbf{v}, t)$ given by Eq. \eqref{99}. The diffusion coefficient depends on the local centered variance $f_2$ of the distribution. Therefore, it vanishes in the regions of phase space where there are no fluctuations. Said differently, the fluctuations $\Phi$ of the potential must be strong enough to provide an efficient mixing. The vanishing of the diffusion coefficient can “freeze” the system in a subdomain of phase space and account for incomplete relaxation and non-ergodicity \cite{20, 22}. For the same reason, the KP equation may also experience a process of incomplete relaxation when $f_2 \to 0$.

This kinetic justification of incomplete relaxation is interesting because it is not based on a generalized entropy such as the Tsallis entropy (see Sec. \ref{1F}), so it does not involve any free parameter like $q$ \cite{51}. However, it demands to solve a dynamical equation [Eq. \eqref{99} or Eq. \eqref{100}] with Eqs. \eqref{80} and \eqref{83} in order to predict the incompletely mixed equilibrium state reached by the system. The idea is that, in case of incomplete relaxation (non-ergodicity), the prediction of the equilibrium state is impossible without considering the dynamics \cite{38}.

Finally, the kinetic theory allows us to take into account the evaporation of high energy particles, which also prevents the relaxation of the system towards a true (Lynden-Bell) statistical equilibrium state. \cite{23} Truncated models such as the fermionic King model have been derived in Refs. \cite{12, 44}. These truncated DFs differ from the Lynden-Bell DF and have a finite mass contrary to the Lynden-Bell DF for 3D self-gravitating systems.

Remark: Similar arguments have been developed in 2D turbulence to explain the process of incomplete violent relaxation \cite{22, 48, 86}.

VI. COLLISIONLESS VERSUS COLLISIONAL RELAXATION

In this section, we discuss the collisionless relaxation time associated with the KP equation \cite{12} and compare it with the collisional relaxation time associated with the ordinary Lenard-Balescu equation \cite{12}. Our discussion follows

\begin{itemize}
  \item[21] Kadomtsev and Pogutse \cite{40} argue that $A(t)$ decreases like $t^{-1/2}$ or even more rapidly.
  \item[22] This “freezing” has been observed numerically in 2D turbulence \cite{48} and led to the concept of maximum entropy “bubbles” \cite{62}. It is not clear if the vanishing of the diffusion coefficient in certain regions of phase space completely stops the relaxation or simply slows it down. However, if the relaxation is strongly slowed down (as observed in \cite{48}) the result is essentially the same on a practical point of view.
  \item[23] Evaporation is particularly important for 3D self-gravitating systems since there is no statistical equilibrium state in a strict sense. The Lynden-Bell DF coupled to the Poisson equation has an infinite mass.
\end{itemize}
and completes the discussion given in Refs. \cite{40, 42}.

The Lenard-Balescu equation describing the evolution of the system sourced by finite $N$ effects (collisions) is (see, e.g., \cite{12})

$$\frac{\partial f}{\partial t} = \pi(2\pi)^d m \frac{\partial f}{\partial v_i} \int dk' f k' j \frac{\hat{u}(k)^2}{\left|\mathbf{k} - \mathbf{k}'\right|^2} \delta[\mathbf{k} - \mathbf{v} - \mathbf{v}'] \left( f' \frac{\partial f}{\partial v_j} - f \frac{\partial f'}{\partial v_j'} \right).$$

As discussed previously, we can obtain the KP equation \cite{62} describing the collisionless relaxation of the system by making the substitution $mf \rightarrow \epsilon^d N f_2$ in the right hand side of Eq. \cite{101}. In the two-level case, this substitution becomes $mf \rightarrow \epsilon^d \epsilon_0^d \eta \mathbf{f}$. In the nondegenerate limit $\mathbf{f} \ll \eta_0$, it reduces to $mf \rightarrow \epsilon^d \epsilon_0^d \eta_0$; i.e., $m \rightarrow \epsilon^d \epsilon_0^d \eta_0$. In other words, we have to replace the mass $m$ of the particles by the effective mass

$$m_{\text{eff}} = \epsilon^d \epsilon_0^d \eta_0,$$

which is the mass of a completely filled macrocell. In the terminology of Dupree \cite{84} and Kadomtsev and Pogutse \cite{40}, this can be viewed as the effective mass of “macroparticles” or “clumps”, i.e., correlated regions. Using the fact which is the mass of a completely filled macrocell. In the terminology of Dupree \cite{84} and Kadomtsev and Pogutse (see Sec. 4.2 of Ref. \cite{86}).

Detailed estimates of the relaxation time are given in \cite{12, 94, 96, 97}. Similar results are obtained in 2D turbulence (see Sec. 4.2 of Ref. \cite{84}).

Remark: In the case of 3D plasmas and 3D stellar systems we have to account for logarithmic corrections. In that case, we get

$$m_{\text{eff}} = \epsilon^d \epsilon_0^d \eta_0 \ln \left( \frac{\lambda_{D,J}}{\lambda_r} \right).$$

Detailed estimates of the relaxation time are given in \cite{12, 94, 96, 97}. Similar results are obtained in 2D turbulence (see Sec. 4.2 of Ref. \cite{84}).

\footnote{In the case of 3D plasmas and 3D stellar systems we have to account for logarithmic corrections yielding $t_R \sim \left[ N \ln(\lambda_{D,J}/\lambda_r) \right] t_D$. In $d = 1$, the KP and Lenard-Balescu operators vanish for spatially homogeneous systems, implying that the relaxation is longer (see Sec. \ref{4.2}). In that case, the collisional relaxation time scales as $N^2 t_D$. \cite{12, 14, 16}.

\footnote{Similarly, the collisional relaxation time of self-gravitating fermions is larger than the collisional relaxation time of self-gravitating classical particles. \cite{84}.}
VII. KINETIC THEORY OF QUIESCENT COLLISIONLESS RELAXATION FOR SPATIALLY INHOMOGENEOUS SYSTEMS

We can easily extend the quasilinear theory of quiescent collisionless relaxation to the case of spatially inhomogeneous systems by introducing angle-action variables and using the formalism developed in Ref. [8]. The inhomogeneous Lenard-Balescu equation describing the evolution of the system sourced by finite $N$ effects (collisions) is

$$
\frac{\partial f}{\partial t} = \pi(2\pi)^d m \frac{\partial}{\partial J} \sum_{k,k'} \int dJ' k |A^d_{k,k'}(J, J', k \cdot \Omega)|^2 \delta(k \cdot \Omega - k' \cdot \Omega') \left( f'k \cdot \frac{\partial f}{\partial J} - f k' \cdot \frac{\partial f'}{\partial J'} \right),
$$

(107)

where $A^d_{k,k'}(J, J', \omega)$ is the dressed potential of interaction (it is written as $-1/D_{k,k'}(J, J', \omega)$ in Refs. [8, 9]) and $\Omega(J)$ is the pulsation of the orbit of a particle with action $J$. This equation can be obtained from a quasilinear theory based on the Klimontovich equation [9]. As we have previously explained, the quasilinear theory based on the Vlasov equation is similar to the quasilinear theory based on the Klimontovich equation provided that we make the substitution $m f \rightarrow e^d_{ij} e^d_{ij} f_2$. As a result, the inhomogeneous KP equation describing the quiescent collisionless relaxation of the system is

$$
\frac{\partial \mathcal{T}}{\partial t} = \pi(2\pi)^d e^d_{ij} e^d_{ij} \frac{\partial}{\partial J} \sum_{k,k'} \int dJ' k |A^d_{k,k'}(J, J', k \cdot \Omega)|^2 \delta(k \cdot \Omega - k' \cdot \Omega') \left( f'k \cdot \frac{\partial \mathcal{T}}{\partial J} - f k' \cdot \frac{\partial \mathcal{T}'}{\partial J'} \right).
$$

(108)

In the multilevel case, we obtain the inhomogeneous SL equation

$$
\frac{\partial \rho}{\partial t} = \pi(2\pi)^d e^d_{ij} e^d_{ij} \frac{\partial}{\partial J} \sum_{k,k'} \int dJ' k |A^d_{k,k'}(J, J', k \cdot \Omega)|^2 \delta(k \cdot \Omega - k' \cdot \Omega') \left( f'k \cdot \frac{\partial \rho}{\partial J} - \rho(\eta - \bar{\mathcal{T}})k' \cdot \frac{\partial \mathcal{T}}{\partial J} \right).
$$

(109)

If we close the hierarchy of moment equations with the ansatz from Eq. (109), we obtain the inhomogeneous Chavanis equation

$$
\frac{\partial \mathcal{T}}{\partial t} = \pi(2\pi)^d e^d_{ij} e^d_{ij} \frac{\partial}{\partial J} \sum_{k,k'} \int dJ' k |A^d_{k,k'}(J, J', k \cdot \Omega)|^2 \delta(k \cdot \Omega - k' \cdot \Omega') \left[ \frac{1}{C''(\mathcal{T})} k \cdot \frac{\partial \mathcal{T}}{\partial J} - \frac{1}{C''(\mathcal{T})} k' \cdot \frac{\partial \mathcal{T}'}{\partial J'} \right].
$$

(110)

In the two-level case, the foregoing equations reduce to

$$
\frac{\partial \mathcal{T}}{\partial t} = \pi(2\pi)^d e^d_{ij} e^d_{ij} \frac{\partial}{\partial J} \sum_{k,k'} \int dJ' k |A^d_{k,k'}(J, J', k \cdot \Omega)|^2 \delta(k \cdot \Omega - k' \cdot \Omega') \left[ \mathcal{T}(\eta_0 - \mathcal{T})k \cdot \frac{\partial \mathcal{T}}{\partial J} - \mathcal{T}(\eta_0 - \mathcal{T})k' \cdot \frac{\partial \mathcal{T}'}{\partial J'} \right],
$$

(111)

which can be viewed as a form of inhomogeneous fermionic Lenard-Balescu equation.

If we make a thermal bath approximation, using the identity from Eq. (109) and the relation $\Omega(J) = \partial \epsilon/\partial J$, we obtain

$$
\frac{\partial \mathcal{T}}{\partial J'} = -\beta f_2 \Omega(J').
$$

(112)

Substituting Eq. (112) into Eq. (109) we get

$$
\frac{\partial \rho}{\partial t} = \pi(2\pi)^d e^d_{ij} e^d_{ij} \frac{\partial}{\partial J} \sum_{k,k'} \int dJ' k |A^d_{k,k'}(J, J', k \cdot \Omega)|^2 \delta(k \cdot \Omega - k' \cdot \Omega') \left[ f'k \cdot \frac{\partial \rho}{\partial J} + \beta f_2 \rho(\eta - \mathcal{T})k' \cdot \Omega' \right].
$$

(113)

Using the properties of the $\delta$-function (resonance condition), we can replace $k' \cdot \Omega'$ by $k \cdot \Omega$ in the last term in brackets. We can then rewrite the foregoing equation as

$$
\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial J_i} \left\{ D_{ij} \left[ \frac{\partial \rho}{\partial J_j} + \beta \rho(\eta - \mathcal{T})\Omega \right] \right\}
$$

(114)

with

$$
D_{ij} = \pi(2\pi)^d e^d_{ij} e^d_{ij} \sum_{k,k'} \int dJ' k_i k_j |A^d_{k,k'}(J, J', k \cdot \Omega)|^2 \delta(k \cdot \Omega - k' \cdot \Omega') f_2(J').
$$

(115)
The equation for the coarse-grained DF reads

\[
\frac{\partial \mathcal{f}}{\partial t} = \frac{\partial}{\partial J_i} \left\{ D_{ij} \left[ \frac{\partial \mathcal{f}}{\partial J_j} + \beta f_2 \Omega_j \right] \right\}.
\] (116)

As in Sec. IV B we can enforce the conservation of energy \( E = \int f(\mathbf{J}) \epsilon(\mathbf{J}) \, d\mathbf{J} \) by letting the inverse temperature evolve in time according to

\[
\beta(t) = -\frac{\int D_{ij} \Omega_i \frac{\partial \mathcal{f}}{\partial J_j} \, d\mathbf{J}}{\int D_{ij} f_2 \Omega_i \Omega_j \, d\mathbf{J}}.
\] (117)

In this manner, we obtain another type of CSR equations for inhomogeneous systems written with angle-action variables. Note that the diffusion coefficient from Eq. (115) does not display a logarithmic divergence at large scales for self-gravitating systems, contrary to the case where a local approximation is made (see Sec. III F), since spatial inhomogeneity has been properly accounted for.

The above kinetic equations conserve the energy and all the Casimirs and monotonically increase the Lynden-Bell entropy (H-theorem).

**VIII. NONLINEAR DYNAMICAL STABILITY AND NUMERICAL ALGORITHMS**

In this section, we consider the nonlinear dynamical stability of steady states of the Vlasov equation based on variational principles (see [39, 58, 98] and references therein for additional discussions). We also introduce relaxation equations that can serve as numerical algorithms to compute stable steady states of the Vlasov equation. Similar results obtained for the 2D Euler equation are given in [61, 62].

**A. Energy principle**

The Vlasov equation conserves the energy and an infinite class of Casimirs. It can be shown that a DF which is an extremum of energy \( \delta E = 0 \) with respect to symplectic perturbations (i.e. perturbations that conserve all the Casimirs) is a stationary solution of the Vlasov equation. Furthermore, this DF is dynamically stable if and only if it is a minimum of energy \( \delta^2 E > 0 \) with respect to symplectic perturbations (see, e.g., [58] for a brief presentation of these results). This energy principle is the most refined stability criterion because it takes into account all the constraints of the Vlasov equation (an infinity of Casimirs). This stability criterion has been introduced in astrophysics by Bartholomew [99] and Kandrup [100] for the Vlasov-Poisson equations. It is similar to the Kelvin-Arnold energy principle for 2D inviscid incompressible hydrodynamical flows governed by the Euler-Poisson equations [61]. We are led therefore to considering the minimization problem

\[
\min_f \{ E[f] \mid \text{symplectic perturbations} \} \tag{118}
\]

or, equivalently,

\[
\min_f \{ E[f] \mid M_{n\geq 1}[f] = M_{n\geq 1} \} \tag{119}
\]

Here, the perturbations must conserve all the Casimirs, which is equivalent to the conservation of all the moments of the DF.\(^{26}\) If we restrict ourselves to DFs of the form \( f = f(\epsilon) \) with \( f'(\epsilon) < 0 \), it can be shown [58] that \( f \) is a local minimum of \( E \) for isovortical perturbations if and only if

\[
\delta^2 \mathcal{E}[\delta f] \equiv -\frac{1}{2} \int \frac{(\delta f)^2}{f'(\epsilon)} \, d\mathbf{v} + \frac{1}{2} \int \delta f \delta \Phi \, d\mathbf{v} > 0, \forall \delta f \mid \delta E = \delta M_{n\geq 1} = 0. \tag{120}
\]

\(^{26}\) For the Newtonian gravitational interaction, it can be shown that all the DFs of the form \( f = f(\epsilon) \) with \( f'(\epsilon) < 0 \) are minima of energy with respect to symplectic perturbations so, according to the stability criterion (118), they are dynamically Vlasov stable (see Refs. [100, 102] for linear stability and Ref. [104] for nonlinear stability). This is, however, no more true in general relativity (see the discussion in [107]).
For the Coulombian potential of interaction in plasma physics, the second term in Eq. (120) is positive implying that all the DFs of the form \( f = f(\epsilon) \) with \( f'(\epsilon) < 0 \) are stable (\( \epsilon = v^2/2 \) for homogeneous plasmas). For the Newtonian potential of interaction in astrophysics, the second term in Eq. (120) is negative. Still, it can be shown that all the DFs of the form \( f = f(\epsilon) \) with \( f'(\epsilon) < 0 \) are stable (see footnote 26).

**Numerical algorithm:** We can easily construct a modified dynamics for the DF that conserves all the Casimirs and that monotonically dissipates the energy. Let us consider the equation

\[
\frac{\partial f}{\partial t} + \{ f, \epsilon \} = \alpha \{ f, \{ f, \epsilon \} \},
\]

(121)

where \( \{ f, g \} \) is the Poisson bracket defined by Eq. (A1). When \( \alpha = 0 \), we recover the Vlasov equation which conserves the energy and all the Casimirs (see Appendix B). When \( \alpha > 0 \), we show below that Eq. (121) conserves all the Casimirs while the energy decreases monotonically. Therefore, it relaxes towards a minimum of energy with respect to symplectic perturbations. By construction, this is a dynamically stable steady state of the Vlasov equation. Therefore, Eq. (121) can be used as a numerical algorithm to construct stable steady states of the Vlasov equation. This is interesting because it is generally difficult to construct steady states of the Vlasov equation and be sure that they are dynamically stable.

**Proof:** We first show that Eq. (121) conserves all the Casimirs. We have

\[
\dot{I}_h = \int h'(f) \frac{\partial f}{\partial t} \, dr \, dv = \alpha \int h'(f) \{ f, \{ f, \epsilon \} \} \, dr \, dv.
\]

(122)

Using the identity from Eq. (A2), we get

\[
\dot{I}_h = \alpha \int \{ f, \epsilon \} \{ h'(f), f \} \, dr \, dv.
\]

(123)

Then, using the identity from Eq. (A3) and the fact that \( \{ f, f \} = 0 \), we obtain

\[
\dot{I}_h = \alpha \int h''(f) \{ f, \epsilon \} \{ f, f \} \, dr \, dv = 0.
\]

(124)

We now show that the energy decreases monotonically. We have

\[
\dot{E} = \int \epsilon \frac{\partial f}{\partial t} \, dr \, dv = \alpha \int \epsilon \{ f, \{ f, \epsilon \} \} \, dr \, dv.
\]

(125)

Using the identity from Eq. (A2), we get

\[
\dot{E} = \alpha \int \{ f, \epsilon \} \{ \epsilon, f \} \, dr \, dv.
\]

(126)

Then, using the identity from Eq. (A4), we obtain

\[
\dot{E} = -\alpha \int \{ f, \epsilon \}^2 \, dr \, dv \leq 0.
\]

(127)

Therefore, the energy is non increasing. At equilibrium (\( \dot{E} = 0 \)), we have \( \{ f, \epsilon \} = 0 \) implying that \( f \) is a stationary solution of the Vlasov equation.

**B. Sufficient conditions of dynamical stability**

We have seen that a DF is a dynamically stable steady state of the Vlasov equation if and only if it is a minimum of energy for perturbations that conserve all the Casimirs. Therefore, a sufficient condition of dynamical stability is that \( f \) is a minimum of energy for perturbations that conserve the mass \( M \) and one Casimir of the form

\[
S[f] = -\int C(f) \, dr \, dv,
\]

(128)

\[\text{footnote text}\]

\[\text{footnote text}\]
where $C(f)$ is a convex function, i.e. $C'' > 0$. In that case, it is a fortiori a minimum of energy for perturbations that conserve all the Casimirs (i.e. for symplectic perturbations). We are therefore led to considering the two-constraint minimization problem

$$
\min_f \{ E[f] \mid M[f] = M, \ S[f] = S \}. \tag{129}
$$

It is shown in [58] that this minimization problem is equivalent to the maximization problem\footnote{If we view the coarse-grained DF $\overline{f}$, this maximization problem can be related to the selective decay principle (for $-S$) of Sec. 11B.}

$$
\max_f \{ S[f] \mid M[f] = M, \ E[f] = E \}. \tag{130}
$$

The first variations can be treated like in Sec. II D leading to the DF from Eq. (25). It can be shown [58] that $f$ is a local minimum of $E$ at fixed $M$ and $S$ or a local maximum of $S$ at fixed $E$ and $M$ if and only if

$$
\delta^2 \mathcal{E}[\delta f] = -\frac{1}{2} \int \frac{(\delta f)^2}{f'(c)} \, dv + \frac{1}{2} \int \delta f \delta \Phi \, dv > 0,
$$

$$
\forall \delta f \mid \delta E = \delta M = 0. \tag{131}
$$

Clearly, Eq. (131) implies Eq. (129). Indeed if $\delta^2 \mathcal{E}$ is positive for all perturbations that conserve mass and energy at first order, it is a fortiori positive for all perturbations that conserve mass, energy and all the Casimirs at first order. If we view the functional $E$ as a “pseudo (or effective) entropy” [31, 110, 111] the maximization problem (129) is similar to a condition of microcanonical stability in thermodynamics, i.e., to the maximization of the entropy at fixed mass and energy.\footnote{We stress that we are just making a “thermodynamical analogy” [31, 110, 111]. There is no thermodynamics involved in the dynamical stability problem of the Vlasov equation. This thermodynamical analogy (or effective thermodynamics) may provide an interpretation of the Tsallis entropy $S_{\alpha} = -\frac{1}{q-1} \int f^{1/q}(f^{1/q}-f) \, dv$, leading to the Tsallis distribution $f = (1/q)^{1/(q-1)}[1 - (q-1)(\beta \epsilon + \alpha)]^{1/(q-1)}$, in terms of a “pseudo entropy” [31, 110, 111] in the sense given above. This “Tsallis pseudo entropy” may be useful for dynamical (not thermodynamical) stability problems. The maximization of the Tsallis (pseudo) entropy at fixed mass and energy ensures the dynamical stability of a particular class of stationary solutions of the Vlasov equation known as polytropic DFs (see [74, 104, 111] for a more detailed discussion).}

Therefore, a maximum of pseudo entropy at fixed mass and energy is a dynamically stable steady state of the Vlasov equation. In particular, considering the Boltzmann entropy $S = -\int f \ln f \, dv$ of statistical mechanics leading to the Boltzmann distribution $f = e^{-\beta \epsilon - \alpha}$, we conclude that microcanonical stability implies (Vlasov) dynamical stability. However, the reciprocal is wrong: a dynamically stable steady state of the Vlasov equation is not necessarily a maximum of pseudo entropy at fixed mass and energy. For example, we have indicated in footnote 26 that, in the case of Newtonian self-gravitating systems, all the DFs of the form $f = f(c)$ with $f'(c) < 0$ are dynamically (Vlasov) stable, even those that do not maximize a pseudo entropy at fixed mass and energy. The stability criteria (129) and (130) are less refined than the stability criterion (118) because they do not take into account all the constraints of the Vlasov equation. This is similar to a notion of ensemble inequivalence in thermodynamics (see below).

An even less refined condition of dynamical stability is that $f$ maximizes $J = S - \beta E$ at fixed mass or, equivalently, minimizes $F = E - TS$ at fixed mass, where $J$ or $F$ is the Legendre transform of the pseudo entropy with respect to the energy. We are therefore led to considering the one-constraint minimization problem

$$
\min_f \{ F[f] = E[f] - TS[f] \mid M[f] = M \}. \tag{132}
$$

The first variations return the results of Sec. II E so that (130) and (132) have the same critical points. It can be shown [58] that $f$ is a local minimum of $F$ at fixed $M$ if and only if

$$
\delta^2 \mathcal{E}[\delta f] = -\frac{1}{2} \int \frac{(\delta f)^2}{f'(c)} \, dv + \frac{1}{2} \int \delta f \delta \Phi \, dv > 0,
$$

$$
\forall \delta f \mid \delta M = 0. \tag{133}
$$

Clearly, Eq. (133) implies Eq. (131). Indeed if $\delta^2 \mathcal{E}$ is positive for all perturbations that conserve mass, it is a fortiori positive for all perturbations that conserve mass and energy at first order. If we view the functional $F$ as a “pseudo (or effective) free energy” [31, 110, 111] the minimization problem (132) is similar to a condition of canonical stability
in thermodynamics, i.e., to the minimization of the free energy at fixed mass. Therefore, a minimum of pseudo free energy at fixed mass is a dynamically stable steady state of the Vlasov equation. The fact that (132) implies (130) means that a minimum of free energy at fixed mass is necessarily a maximum of entropy at fixed mass and energy. However, the reciprocal is wrong: A maximum of entropy at fixed mass and energy is not necessarily a minimum of free energy at fixed mass. Therefore, canonical stability implies microcanonical stability but not the converse [58]. This corresponds the notion of ensemble inequivalence in thermodynamics for systems with long-range interactions [5, 58, 112–114]. Transposed to the present (dynamical) context, the minimization of pseudo free energy at fixed mass. Therefore, canonical stability implies microcanonical stability but not the converse [58]. This corresponds to a microcanonical description. By construction, this equilibrium state is a stable steady state of the Vlasov equation. If we fix $\beta$ in Eq. (135) or Eqs. (136) and (137) can be used as numerical algorithms to construct stable steady states of the Vlasov equation. If we fix $\beta$, Eq. (136) can be viewed as a generalized Kramers equation. It relaxes towards a minimum of free energy at fixed mass. This corresponds to a canonical description. By construction, the equilibrium state of the generalized Kramers equation is a stable steady state of the Vlasov equation. Therefore, Eq. (136) with fixed $\beta$ can be used as a numerical algorithm to construct stable steady states of the Vlasov equation. We can also obtain simpler numerical algorithms by taking the hydrodynamic moments of the generalized Landau and Kramers equations [50, 51]. Similar numerical algorithms have been introduced in 2D turbulence [50, 61, 62].

### C. Dynamical and thermodynamical stability

It can be shown that a thermodynamical equilibrium state in the sense of Lynden-Bell is nonlinearly dynamically stable. Indeed, the coarse-grained DF $\bar{f}$ obtained from the Gibbs state [11] which maximizes the Lynden-Bell entropy

\[ \Delta H = -k_B \int f \ln f \, dv \]

is a time-dependent inverse temperature. \[31\] These equations conserve the mass $M$ and the energy $E$ and monotonically increase the pseudo entropy $S$ ($H$-theorem). They relax towards a maximum entropy state at fixed mass and energy. This corresponds to a microcanonical description. By construction, this equilibrium state is a stable steady state of the Vlasov equation. Therefore, Eq. (136) or Eqs. (136) and (137) can be used as numerical algorithms to construct stable steady states of the Vlasov equation. If we fix $\beta$, Eq. (136) can be viewed as a generalized Kramers equation. It conserves the mass $M$ and monotonically decreases the pseudo free energy $F = E - TS$ [50, 51]. It relaxes towards a minimum of free energy at fixed mass. This corresponds to a canonical description. By construction, the equilibrium state of the generalized Kramers equation is a stable steady state of the Vlasov equation. Therefore, Eq. (136) with fixed $\beta$ can be used as a numerical algorithm to construct stable steady states of the Vlasov equation. We can also obtain simpler numerical algorithms by taking the hydrodynamic moments of the generalized Landau and Kramers equations [50, 51], and closing the hierarchy of equations with a local thermodynamic equilibrium assumption, leading to generalized Navier-Stokes, Euler and Smoluchowski equations [50, 51]. Similar numerical algorithms have been introduced in 2D turbulence [50, 61, 62].

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30 We can also introduce a no-constraint problem by considering the maximization of the grand potential $G = S - \beta E - \alpha M$. This is the least refined stability criterion (see Ref. [61]).

31 Here, we use the kinetic equations (136) and (139) as numerical algorithms to compute stable steady states of the Vlasov equation, not as parametrizations of the coarse-grained dynamics. As a result, we can make the following simplifications: (i) we can write these equations for spatially inhomogeneous systems and take $f' = f(r, v', t)$ in Eq. (139) even if the local approximation is not justified for the true evolution of the system; (ii) we can ignore collective effects in Eq. (139) and define $K_{ij}$ by Eq. (67) instead of Eq. (60); (iii) we can replace $D_{ij}$ by $D_{ij} = D\delta_{ij}$ in Eq. (139), where $D$ is a strictly positive constant, in order to make the equation simpler and make sure that it relaxes towards a maximum entropy state at fixed mass and energy without experiencing a situation of kinetic blocking (see Sec. V).
at fixed mass, energy and Casimir constraints is a minimum of energy $E[f]$ with respect to perturbations that conserve the coarse-grained moments $M^{f,g}_{n \geq 1} \{ \cdot \}$ (see Sec. 7.8 of [61]). Therefore, according to Eq. (119), this is a nonlinearly dynamically stable steady state of the Vlasov equation. By contrast, the initial condition $f_0$, even though it has the same energy as the metaequilibrium state, is generically not a minimum of energy $E[f_0]$ with respect to perturbations that conserve the fine-grained moments $M^{f,g}_{n \geq 1} \{ \cdot \}$, so it is dynamically unstable and relaxes towards the metaequilibrium state.

It can be shown that a DF which maximizes the generalized entropy defined by Eqs. (22) and (28) at fixed mass and energy is (i) nonlinearly dynamically stable (see VIII B) and (ii) thermodynamically stable in the sense of Lynden-Bell (see Appendix C). We stress, however, that this is just a sufficient condition of dynamical and thermodynamical stability. In particular, the DF $f$ associated with a thermodynamical equilibrium state in the sense of Lynden-Bell does not necessarily maximizes the generalized entropy defined by Eqs. (22) and (28) at fixed mass and energy (see Appendix C).

We would be tempted to believe that Vlasov nonlinear dynamical stability implies Lynden-Bell’s thermodynamical stability. More precisely, we would be tempted to believe that a DF which is a monotonically decreasing function of $\epsilon$ and which is a minimum of energy with respect to symplectic perturbations is a thermodynamically equilibrium state in the sense of Lynden-Bell. However, this is not true as shown by the following counter-example. For collisionless self-gravitating systems, all the DFs of the form $f = f(\epsilon)$ with $f'(\epsilon) < 0$ are dynamically stable (see footnote 26) even those that are not thermodynamically stable in the sense of Lynden-Bell. In particular, all the extrema – including saddle points – of Lynden-Bell’s entropy at fixed mass, energy and Casimir constraints are dynamically stable (since they are of the form $\mathcal{F} = \mathcal{F}(\epsilon)$ with $\mathcal{F}'(\epsilon) < 0$) even if they are not maxima of Lynden-Bell’s entropy at fixed mass, energy and Casimir constraints. This is because the dynamical stability criterion involves the coarse-grained moments $M^{f,g}_{n \geq 1} \{ \mathcal{F} \}$ while the Lynden-Bell thermodynamical criterion involves the fine-grained moments $M^{f,g}_{n \geq 1} \{ \rho \}$. This can be easily understood in the two-level case. In that case, the Lynden-Bell statistical equilibrium state is obtained by maximizing the Fermi-Dirac-like entropy (19) at fixed mass and energy (the fine-grained moments $M^{f,g}_{n \geq 1} \{ \rho \}$ are all proportional to the mass) or, equivalently, by minimizing the energy at fixed mass and Fermi-Dirac-like entropy. However, we have seen that this optimization problem is just a sufficient condition of dynamical stability. A more refined condition of dynamical stability is that $\mathcal{F}$ is a minimum of energy $E[\mathcal{F}]$ with respect to perturbations that conserve all the coarse-grained moments $M^{f,g}_{n \geq 1} \{ \mathcal{F} \}$ (not just the Fermi-Dirac-like entropy). For self-gravitating systems this is the case for all DFs of the form $f = f(\epsilon)$ with $f(\epsilon) < 0$. Thus, there exist DFs which are dynamically Vlasov stable while they do not maximize the Fermi-Dirac-like entropy at fixed mass and energy. Such DFs are dynamically stable but not thermodynamically stable in the sense of Lynden-Bell.

IX. AN EQUATION THAT CONSERVES THE MASS AND THE ENERGY AND THAT MONOTONICALLY INCREASES ALL THE $H$-FUNCTIONS

Using the same method as the one developed in Sec. VIII A we can easily construct a modified dynamics for the DF that conserves the mass and energy and that monotonically increases all the $H$-functions (see also Appendix H). Let us consider the equation

$$\frac{\partial \mathcal{F}}{\partial t} + \{ \mathcal{F}, \epsilon \} = \alpha \{ \epsilon, \} \{ \mathcal{F}, \epsilon \}. \tag{138}$$

When $\alpha = 0$, we recover the Vlasov equation which conserves the energy and all the Casimirs (see Appendix B). When $\alpha > 0$, we show below that Eq. (138) conserves the energy while it increases all the $H$-functions monotonically. We note that all the stationary solutions of the Vlasov equation (satisfying $\{ \mathcal{F}, \epsilon \} = 0$) are stationary solutions of Eq. (138). Equation (138) may admit other stationary solutions (satisfying $\{ \mathcal{F}, \epsilon \} = \alpha \{ \epsilon, \} \{ \mathcal{F}, \epsilon \}$) but, according to the result derived below Eq. (143), they are necessarily unstable.

Proof: We first show that Eq. (138) conserves the energy. We have

$$\dot{E} = \int (\frac{\partial \mathcal{F}}{\partial t}) d\mathbf{r} d\mathbf{v} = \alpha \int \{ \mathcal{F}, \epsilon \} \{ \mathcal{F}, \epsilon \} d\mathbf{r} d\mathbf{v}. \tag{139}$$

Using the identity from Eq. (A2) and the fact that $\{ \epsilon, \} = 0$, we get

$$\dot{E} = \alpha \int \{ \mathcal{F}, \epsilon \} \{ \epsilon, \} d\mathbf{r} d\mathbf{v} = 0. \tag{140}$$
We now show that Eq. (138) monotonically increases all the generalized $H$-functions. We have
\[
\dot{H} = \int C'(\mathcal{J}) \frac{\partial \mathcal{J}}{\partial t} \, d\mathbf{v} = \alpha \int C'(\mathcal{J}) \{\epsilon, \{\mathcal{J}, \epsilon\}\} \, d\mathbf{v}.
\]
(141)
Using the identity from Eq. (A2), we get
\[
\dot{H} = \alpha \int \{\mathcal{J}, \epsilon\} \{C'(\mathcal{J}), \epsilon\} \, d\mathbf{v}.
\]
(142)
Then, using the identity from Eq. (A3), we obtain
\[
\dot{H} = \alpha \int C''(\mathcal{J}) (\mathcal{J}^2 - \epsilon^2) \, d\mathbf{v} \geq 0.
\]
(143)
Therefore, the $H$-functions are non decreasing. At equilibrium ($\dot{H} = 0$), Eq. (143) implies $\{\mathcal{J}, \epsilon\} = 0$. Therefore, Eq. (138) relaxes towards a stationary solution of the Vlasov equation. Note that this stationary solution does not necessarily maximize a particular $H$-function at fixed mass and energy. It cannot be predicted \textit{a priori}. One has to solve the kinetic equation (138) numerically to determine its equilibrium state.

X. CONCLUSION

In this paper, we have discussed the kinetic theory of collisionless relaxation for systems with long-range interactions. We have recalled the basics of the quasilinear theory of the Vlasov equation developed by Kadomtsev and Pogutse [40], Severne and Luwel [41], and Chavanis [42-44]. We have established a connection between the kinetic equations derived from the quasilinear theory and the CSR relaxation equations obtained from a phenomenological MEPP [26]. We have proposed a method to close the hierarchy of moment equations leading to a self-consistent kinetic equation for the coarse-grained DF which is valid beyond the two-level case [14]. This equation [see Eq. (91)] depends on a generalized entropy $C(\mathcal{J})$ which can be obtained from the equilibrium state and then used out-of-equilibrium, or which can be obtained at any time of the dynamics by using the procedure explained in Appendix C. We have also discussed the nonlinear dynamical stability of steady states of the Vlasov equation and proposed numerical algorithms in the form of kinetic (relaxation) equations that can be used to construct nonlinearly stable steady states. Similar results can be obtained in 2D turbulence and vortex dynamics by exploiting the analogy between the Vlasov and the 2D Euler equations. This will be discussed in a specific paper [46].

The statistical mechanics of violent relaxation was initiated by Lynden-Bell [22] in the context of collisionless stellar systems. However, the present paper was motivated by the possibility to apply these ideas to the context of fermionic or bosonic dark matter [115-119]. Indeed, these systems also exhibit a process of violent relaxation (known as gravitational cooling [120] in the case of boson stars). For these systems, we have to take into account the quantum nature of the particles. In the case of fermionic dark matter, the quantum potential arising from the Heisenberg uncertainty principle is negligible and we can use the classical Vlasov equation (Thomas-Fermi approximation). The Lynden-Bell theory of violent relaxation can justify the establishment of a Fermi-Dirac-like DF on a timescale shorter than the age of the universe [115-117, 119]. This leads to dark matter halos with a “core-halo” structure. The quantum core (fermion ball) solves the core-cusp problem of classical cold dark matter and the isothermal halo leads to flat rotation curves in agreement with the observations [113]. In the case of bosonic dark matter the quantum potential is important and we must replace the Vlasov equation by the Wigner equation. A generalization of the Lynden-Bell theory of violent relaxation taking into account the specificities of the Wigner equation has been recently proposed in [118]. This theory also leads to dark matter halos with a “core-halo” structure where the quantum core is a self-gravitating Bose-Einstein condensate (soliton) surrounded by a halo made of quantum interferences. The collisional kinetic theory of fermions and bosons has been studied in [94, 121, 122]. Fermions and bosons behave antisymmetrically regarding their collisional relaxation. The Pauli blocking $f(\eta_0 - f)$ for fermions has the tendency to slow down the relaxation and the Bose enhancement $f(\eta_0 + f)$ for bosons, leading to the formation of “granules” or “quasiparticles”, has the tendency to accelerate the relaxation. Gravitational encounters (“collisions”) are completely negligible in fermionic dark matter halos. In bosonic dark matter halos, they manifest themselves on a (secular) timescale of the order of the age of the universe (see [94] and references therein).

32 For self-gravitating fermions, gravitational encounters are completely negligible and cannot establish a statistical equilibrium state on a relevant timescale. However, a collisional relaxation may be relevant if the fermions are self-interacting [119].
Appendix A: Basic properties of the Poisson bracket

The Poisson brackets are defined by

\[ \{f, g\} = \nabla_r f \cdot \nabla_r g - \nabla_v f \cdot \nabla_v g. \tag{A1} \]

We recall below some basic properties of the Poisson brackets that can be established straightforwardly:

\[ \int f\{g, h\} \, drd\mathbf{v} = \int h\{f, g\} \, drd\mathbf{v}, \tag{A2} \]

\[ \int \{h(f), g\} \, drd\mathbf{v} = \int h'(f)\{f, g\} \, drd\mathbf{v}, \tag{A3} \]

\[ \{f, g\} = -\{g, f\}. \tag{A4} \]

Appendix B: Basic properties of the Vlasov equation

In this appendix, we establish some basic properties of the Vlasov equation [1].

(i) The conservation of the Casimirs can be established as follows:

\[ \dot{I}_h = \int h'(f) \frac{\partial f}{\partial t} \, drd\mathbf{v} = -\int h'(f) \left( \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} - \nabla \Phi \cdot \frac{\partial h}{\partial \mathbf{r}} \right) \, drd\mathbf{v} = -\int \left\{ \frac{\partial}{\partial \mathbf{r}} \cdot [h(f)v] - \frac{\partial}{\partial \mathbf{v}} \cdot [h(f)\nabla \Phi] \right\} \, drd\mathbf{v} = 0. \tag{B1} \]

(ii) The conservation of the energy can be established as follows:

\[ \dot{E} = \int \epsilon \frac{\partial f}{\partial t} \, drd\mathbf{v} = -\int \epsilon \left( \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} - \nabla \Phi \cdot \frac{\partial h'}{\partial \mathbf{r}} \right) \, drd\mathbf{v} = \int \frac{\partial}{\partial \mathbf{r}} \cdot (f\mathbf{v}) - \frac{\partial}{\partial \mathbf{v}} \cdot (f\nabla \Phi) \, drd\mathbf{v} = 0. \tag{B2} \]

(iii) The conservation of the impulse can be established as follows:

\[ \dot{\mathbf{P}} = \int \mathbf{v} \frac{\partial f}{\partial t} \, drd\mathbf{v} = -\int \left( \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} \right) \, drd\mathbf{v} = -\int \left( \frac{\partial}{\partial \mathbf{r}} \cdot (f\mathbf{v}) - \frac{\partial}{\partial \mathbf{v}} \cdot (f\nabla \Phi) \right) \, drd\mathbf{v} = \int \frac{\partial}{\partial \mathbf{v}_j} \left( f\partial_j\Phi \right) \, drd\mathbf{v} = -\int \delta_{ij} \partial_i f \partial_j \Phi \, drd\mathbf{v} = -\int \partial \nabla \Phi \, drd\mathbf{v} = -\int \rho \nabla \Phi \, dr = 0. \tag{B3} \]

The last equality results from the fact that the sum of the forces acting on the system vanishes. Indeed, using Eq. [2], we get

\[ -\int \rho \nabla \Phi \, dr = -\int drd\mathbf{r}' \rho(\mathbf{r})\rho(\mathbf{r}')\nabla u(|\mathbf{r} - \mathbf{r}'|) = \int drd\mathbf{r}' \rho(\mathbf{r})\rho(\mathbf{r}')\nabla u(|\mathbf{r} - \mathbf{r}'|) = 0. \tag{B4} \]

To get the second equality, we have interchanged the dummy variables \( \mathbf{r} \) and \( \mathbf{r}' \), and to get the last equality we have added the half sum of the two preceding expressions.

(iv) The conservation of the angular momentum can be established as follows:

\[ \dot{\mathbf{L}} = \int (\mathbf{r} \times \mathbf{v}) \frac{\partial f}{\partial t} \, drd\mathbf{v} = -\int (\mathbf{r} \times \mathbf{v}) \left( \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} \right) \, drd\mathbf{v} = -\int (\mathbf{r} \times \mathbf{v}) \left( \frac{\partial}{\partial \mathbf{r}} \cdot (f\mathbf{v}) - \frac{\partial}{\partial \mathbf{v}} \cdot (f\nabla \Phi) \right) \, drd\mathbf{v} = -\int \epsilon_{ijk} x_j v_k \left( \frac{\partial}{\partial \mathbf{v}_l} \cdot (f\mathbf{v}_l) - \frac{\partial}{\partial \mathbf{v}_l} \cdot (f\partial_l \Phi) \right) \, drd\mathbf{v} = -\int \epsilon_{ijk} x_j f \partial_k \Phi \, drd\mathbf{v} = \int f \times \mathbf{v} \, drd\mathbf{v} - \int f \mathbf{r} \cdot \nabla \Phi \, drd\mathbf{v} = -\int \rho \mathbf{r} \times \nabla \Phi \, dr = 0. \tag{B5} \]
The last equality results from the fact that the sum of torques acting on the system vanishes. Indeed, using Eq. (2), we get

\[- \int \rho \times \nabla \Phi \, dx = - \int dr dr' \rho(r) \rho(r') \times \nabla u(|r - r'|) = \int dr dr' \rho(r) \rho(r') \times \nabla u(|r - r'|)\]

\[= - \frac{1}{2} \int dr dr' \rho(r) \rho(r')(r - r') \times \nabla u(|r - r'|) = - \frac{1}{2} \int dr dr' \rho(r) \rho(r') u'(|r - r'|)(r - r') \times \frac{r - r'}{|r - r'|} = 0. \] (B6)

To get the second equality, we have interchanged the dummy variables \(r\) and \(r'\), and to get the third equality we have added the half sum of the two preceding expressions.

We can establish these results in a slightly different manner, by using the properties of the Poisson brackets (see Appendix A). The Vlasov equation can be written as

\[\frac{\partial f}{\partial t} + \{f, \epsilon\} = 0.\] (B7)

The steady states of the Vlasov equation satisfy \(\{f, \epsilon\} = 0\). The conservation of the Casimirs can be be established as follows:

\[\dot{I}_h = \int h'(f) \frac{\partial f}{\partial t} \, dr \, dv = - \int h'(f) \{f, \epsilon\} \, dr \, dv = - \int \epsilon \{h'(f), f\} \, dr \, dv = - \int \epsilon h''(f) \{f, f\} \, dr \, dv = 0.\] (B8)

The conservation of the energy can be established as follows:

\[\dot{E} = \int \epsilon \frac{\partial f}{\partial t} \, dr \, dv = - \int \epsilon \{f, \epsilon\} \, dr \, dv = - \int \{f, \epsilon, \epsilon\} \, dr \, dv = 0.\] (B9)

**Appendix C: Canonical treatment of the Casimir constraints**

In the statistical theory of Lynden-Bell [22], the Casimir constraints are treated microcanonically. This is the correct approach of the problem for an isolated system since these quantities are conserved by the Vlasov equation. However, it makes the problem quite complicated to solve because we have to relate a large number of Lagrange multipliers \(\alpha_n\) (chemical potentials) to the moments \(M^{f, g}_n\) of the fine-grained DF. For that reason, we may consider a simpler problem where the Casimir constraints are treated canonically (note that the energy and the mass are still treated microcanonically). In that case, we assume that the Lagrange multipliers \(\alpha_n\) for \(n > 1\) are prescribed instead of the moments \(M^{f, g}_n\).

There are several justifications for treating the Casimir constraints canonically:

1. If the system is not isolated, we may assume that forcing and dissipation will destroy the conservation of the fine-grained moments \(M^{f, g}_n\) and fix the Lagrange multipliers \(\alpha_n\) (chemical potentials) instead. While this is an interesting and convenient suggestion, it does not rest on a firm solid basis.

2. Treating the Casimir constraints canonically provides a simpler maximization problem which determines a sufficient condition of thermodynamical stability in the sense of Lynden-Bell (see Appendices C1 C3).

3. In the kinetic theory of collisionless relaxation, a canonical description of the Casimir constraints is justified to close the hierarchy of moments equations if the Lagrange multipliers \(\alpha_n\) do not differ too much from their equilibrium value (see Appendix C4).

**Remark:** Similar results have been obtained for the Euler equation in 2D hydrodynamics [56, 59–62, 123] and their adaptation to the Vlasov equation has been discussed in [39].

### 1. Sufficient condition of Lynden-Bell’s thermodynamical stability

In the Lynden-Bell theory, the statistical equilibrium state is obtained by maximizing the mixing entropy \(S_{LB}[\rho]\) at fixed mass \(M\), energy \(E\), Casimirs \(M^{f, g}_{n \geq 1}\), and normalization condition (see Sec. III B). This is a necessary and sufficient condition of thermodynamical stability in the sense of Lynden-Bell. It determines the most probable state of the system. We thus have to solve the maximization problem

\[\max_{\rho} \left\{ S_{LB}[\rho] \mid M[f] = M, \quad E[f] = E, \quad M^{f, g}_{n \geq 1}[\rho] = M^{f, g}_{n \geq 1}, \quad \int \rho \, d\eta = 1 \right\}. \] (C1)
The variational problem determining the extrema of $S_{\text{LB}}$ at fixed $M$, $E$, $M_{n>1}^{f.g.}$ and normalization condition is given by Eq. (10), leading to the Gibbs state (11). This equilibrium state is a local maximum of $S_{\text{LB}}$ at fixed $M$, $E$, $M_{n>1}^{f.g.}$ and normalization condition if and only if

$$\delta^2 J[\delta \rho] \equiv -\frac{1}{2} \int \frac{(\delta \rho)^2}{\rho} \, d\mathbf{r} d\mathbf{v} d\eta - \frac{\beta}{2} \int \delta^T \delta \mathbf{F} \, d\mathbf{r} d\mathbf{v} < 0,$$

$$\forall \, \delta \rho \mid \delta E = \delta M = \delta M_{n>1}^{f.g.} = \int \delta \rho \, d\eta = 0. \tag{C2}$$

Let us now consider the maximization of the relative entropy

$$S_\chi = S_{\text{LB}} - \sum_{n>1} \alpha_n M_{n}^{f.g.} \tag{C3}$$

at fixed mass $M$, energy $E$ and normalization condition. $S_\chi$ is the Legendre transform of $S_{\text{LB}}$ with respect to the fine-grained moments. As compared to the original maximization problem, this amounts to treating the Casimir constraints canonically instead of microcanonically. We thus have to solve the maximization problem

$$\max_{\rho} \left\{ S_\chi[\rho] \mid M[\mathcal{F}] = M, \ E[\mathcal{F}] = E, \ \int \rho \, d\eta = 1 \right\}. \tag{C4}$$

The variational problem determining the extrema of $S_\chi$ at fixed $M$, $E$ and normalization is again given by Eq. (10), leading to the same Gibbs state (11) as in the original problem. This equilibrium state is a local maximum of $S_\chi$ at fixed $M$, $E$ and normalization if and only if

$$\delta^2 J[\delta \rho] \equiv -\frac{1}{2} \int \frac{(\delta \rho)^2}{\rho} \, d\mathbf{r} d\mathbf{v} d\eta - \frac{\beta}{2} \int \delta^T \delta \mathbf{F} \, d\mathbf{r} d\mathbf{v} < 0,$$

$$\forall \, \delta \rho \mid \delta E = \delta M = \int \delta \rho \, d\eta = 0. \tag{C5}$$

The critical points (first variations) of (C4) and (C1) are the same but the condition of stability (second variations) is different. A maximum of $S_\chi$ at fixed $M$, $E$ and normalization condition is always a maximum of $S$ at fixed $M$, $E$, $M_{n>1}^{f.g.}$ and normalization condition, but the converse is wrong. Indeed if inequality (C5) is satisfied for all variations that satisfy the conservation at first order of mass, energy and normalization condition, it is a fortiori satisfied for all variations that satisfy the conservation at first order of mass, energy, normalization condition and Casimirs. Therefore (C5) implies (C2) but this is not reciprocal. As a result, (C4) provides just a sufficient condition of thermodynamical stability (in the sense of Lynden-Bell). Making the relative entropy explicit, we get

$$S_\chi = -\int \rho(\mathbf{r}, \mathbf{v}, \eta) \ln \rho(\mathbf{r}, \mathbf{v}, \eta) \, d\mathbf{r} d\mathbf{v} d\eta - \sum_{n>1} \alpha_n \int \rho(\mathbf{r}, \mathbf{v}, \eta) \eta^n \, d\mathbf{r} d\mathbf{v} d\eta$$

$$= -\int \rho(\mathbf{r}, \mathbf{v}, \eta) \left[ \ln \rho(\mathbf{r}, \mathbf{v}, \eta) + \sum_{n>1} \alpha_n \eta^n \right] \, d\mathbf{r} d\mathbf{v} d\eta$$

$$= -\int \rho(\mathbf{r}, \mathbf{v}, \eta) \ln \left[ \frac{\rho(\mathbf{r}, \mathbf{v}, \eta)}{\chi(\eta)} \right] \, d\mathbf{r} d\mathbf{v} d\eta, \tag{C6}$$

where we have used Eq. (12) to get the last equality.

Let us finally consider the maximization of the generalized entropy $S[\mathcal{F}]$ at fixed mass $M$ and energy $E$ (see Sec. 11.2). We have to solve the maximization problem

$$\max_{\mathcal{F}} \{ S[\mathcal{F}] \mid M[\mathcal{F}] = M, \ E[\mathcal{F}] = E \}. \tag{C7}$$

The variational problem determining the extrema of $S$ at fixed $M$ and $E$ is given by Eq. (23), leading to the equilibrium state (25) corresponding to the Lynden-Bell coarse-grained DF. This equilibrium state is a local maximum of $S$ at
fixed \( M \) and \( E \) if and only if\(^{33}\)

\[
\delta^2 J[\delta \mathcal{F}] = -\frac{1}{2} \int C''(\mathcal{F})(\delta \mathcal{F})^2 \, dr \, dv - \frac{\beta}{2} \int \delta \mathcal{F} \delta F \, dr \, dv < 0,
\]

\( \forall \, \delta \mathcal{F} \mid \delta E = \delta M = 0. \) \hspace{1cm} (C8)

Below we show that the maximization of the relative entropy \( S_\chi[\rho] \) at fixed mass \( M \), energy \( E \) and normalization condition is equivalent to the maximization of the generalized entropy \( S[\mathcal{F}] \) defined by \(^{25}\) at fixed mass \( M \) and energy \( E \). As a result, \((C7)\) provides a sufficient condition of thermodynamical stability (in the sense of Lynden-Bell).

In summary

\[ (C7) \leftrightarrow (C4) \Rightarrow (C4) \] \hspace{1cm} (C9)

Remark: We may miss important solutions by maximizing the relative entropy \( S_\chi \) at fixed mass and energy instead of maximizing the Lynden-Bell entropy \( S_{LB} \) at fixed mass, energy and Casimirs. This is similar to the notion of ensemble inequivalence for systems with long-range interactions \(^3 \, 58 \, 112 \, 114\). For example, for systems with long-range interactions, equilibrium states with negative specific heats are forbidden in the canonical ensemble (fixed \( T \)) while they are allowed in the microcanonical ensemble (fixed \( E \)). Similarly, we may miss important solutions by treating the Casimirs canonically instead of microcanonically.

### 2. Equivalence for global maximization

We first show the equivalence of \((C4)\) and \((C7)\) for global maximization. To maximize \( S_\chi[\rho] \) at fixed mass \( M[\mathcal{F}] \), energy \( E[\mathcal{F}] \) and normalization condition \( \int \rho(r,v,\eta) \, d\eta = 1 \) we can proceed in two steps:

(i) In a first step, we maximize \( S_\chi[\rho] \) at fixed mass \( M[\mathcal{F}] \), energy \( E[\mathcal{F}] \) and normalization condition \( \int \rho(r,v,\eta) \, d\eta = 1 \) for a given DF \( \mathcal{F}(r,v) \). Since the specification of \( \mathcal{F}(r,v) \) determines \( M[\mathcal{F}] \) and \( E[\mathcal{F}] \), this is equivalent to maximizing \( S_\chi[\rho] \) at fixed normalization condition \( \int \rho(r,v,\eta) \, d\eta = 1 \) and with the constraint \( \int \rho(r,v,\eta) \eta \, d\eta = \mathcal{F}(r,v) \). Writing the variational problem as

\[
\delta S_\chi - \int \zeta(r,v) \delta \left( \int \rho(r,v,\eta) \, d\eta \right) \, dr \, dv - \int \Psi(r,v) \delta \left( \int \rho(r,v,\eta) \eta \, d\eta \right) \, dr \, dv = 0,
\]

where \( \zeta(r,v) \) and \( \Psi(r,v) \) are Lagrange multipliers, we get

\[
\rho_\times(r,v,\eta) = \frac{1}{Z[\Psi(r,v)]} \chi(\eta) e^{-\Psi(r,v)}. \] \hspace{1cm} (C11)

This is the global maximum of entropy with the previous constraints since \( \delta^2 S_\chi = -\frac{1}{4} \int [(\delta \rho)^2 / \rho_\times] \, dr \, dv \, d\eta < 0 \) (the constraints are linear in \( \rho \) so their second variations vanish). The functions \( Z[\Psi] \) and \( \Psi \) are determined by

\[
Z[\Psi(r,v)] = \int \chi(\eta) e^{-\Psi(r,v)} \, d\eta, \quad \mathcal{F}(r,v) = \frac{1}{Z[\Psi(r,v)]} \int \chi(\eta) \eta e^{-\Psi(r,v)} \, d\eta \] \hspace{1cm} (C12)

expressing the normalization condition and the specification of the DF \( \mathcal{F}(r,v) \). These results are similar to those of Sec.\(^{113}\) provided that we replace \( \beta \epsilon + \alpha \) by \( \Psi \). Then, we have

\[
\mathcal{F} = F(\Psi) = - (\ln Z)'(\Psi), \quad \mathcal{F}'(\Psi) = - f_2(\Psi), \] \hspace{1cm} (C13)

where \( F \) and \( f_2 \) are defined in Sec.\(^{113}\).

We can then determine \( S[\mathcal{F}] = S_\chi[\rho_\times] \). Substituting Eq. \((C11)\) into Eq. \((C6)\) we get

\[
S[\mathcal{F}] = \int \mathcal{F} \Psi \, dr \, dv + \int \ln Z \, dr \, dv. \] \hspace{1cm} (C14)

\(^{33}\) Using the identity from Eq. \((27)\), we can check that Eq. \((C8)\) is equivalent to Eq. \((131)\).
This is of the form of Eq. (22) with
\[ C(\mathcal{J}) = -\mathcal{J} \Psi - \ln Z. \] (C15)

Using Eq. (C18) we find that
\[ C'(\mathcal{J}) = -\Psi - \mathcal{J} \frac{\partial \Psi}{\partial \mathcal{J}} - \frac{\partial \ln Z}{\partial \mathcal{J}} \frac{\partial \Psi}{\partial \mathcal{J}} = -\Psi - \mathcal{J} \frac{\partial \Psi}{\partial \mathcal{J}} + \mathcal{J} \frac{\partial \Psi}{\partial \mathcal{J}} = -[\ln Z]^{-1}(-\mathcal{J}). \] (C16)

Therefore,
\[ C(\mathcal{J}) = -\int [\ln Z]^{-1}(-x) \, dx. \] (C17)

This returns the result from Eq. (28) establishing the fact that \( S[\mathcal{J}] \) is the generalized entropy from Sec. II D. Therefore, the generalized entropy \( S[\mathcal{J}] \) is equal to the relative entropy \( S_{\chi}[\rho] \) calculated at \( \rho_* \) when the Casimir constraints are treated canonically (this is also true for the Lynden-Bell entropy \( S_{LB}[\rho] \) calculated at \( \rho_* \) when the Casimir constraints are treated microcanonically).

(ii) In a second step, we maximize \( S[\mathcal{J}] \equiv S_{\chi}[\rho_*] \) at fixed mass \( M[\mathcal{J}] \) and energy \( E[\mathcal{J}] \). Proceeding as in Sec. II D, the cancellation of the first variations yields
\[ C'(\mathcal{J}) = -\beta \epsilon(r, v) - \alpha. \] (C18)

Comparing Eqs. (C16) and (C18) we find (at equilibrium) that
\[ \Psi(r, v) = \beta \epsilon(r, v) + \alpha. \] (C19)

Substituting this relation into Eq. (C11) we recover the Gibbs state (11). However, we have proven more than that. The present approach shows that \( \rho(r, v, \eta) \) is the global maximum of \( S_{\chi}[\rho] \) at fixed \( M, E \) and normalization condition if and only if \( \mathcal{J}(r, v) \) is the global maximum of \( S[\mathcal{J}] \) at fixed \( M \) and \( E \) (this is where we need to treat the Casimir constraints canonically in order to have a fixed shape of the generalized entropy).

Remark: Equation (C13) implies
\[ C''(\mathcal{J}) = -\frac{1}{\mathcal{J} \Psi'}. \] (C20)

Comparing this relation with Eq. (C13) we obtain the important relation
\[ f_2 = \frac{1}{C''(\mathcal{J})}. \] (C21)

We stress that this relation is valid even before maximizing \( S[\mathcal{J}] \equiv S_{\chi}[\rho_*] \) at fixed mass \( M[\mathcal{J}] \) and energy \( E[\mathcal{J}] \). In this sense, it is expected to remain valid (or approximately valid) when the coarse-grained DF \( \mathcal{J}(r, v) \) is out-of-equilibrium (see Appendix C).

3. Equivalence for local maximization

We now show the equivalence of (C4) and (C7) for local maximization, i.e. \( \rho(r, v, \eta) \) is a (local) maximum of \( S_{\chi}[\rho] \) at fixed \( E, M \) and normalization condition if and only if the corresponding coarse-grained DF \( \mathcal{J}(r, v) \) is a (local) maximum of \( S[\mathcal{J}] \) at fixed \( E \) and \( M \). To that purpose, we show the equivalence between the stability criteria (C6) and (C7).

Let us determine the perturbation \( \delta \rho_*(r, v, \eta) \) that maximizes \( \delta^2 J[\delta \rho] \) given by (C5) with the constraints \( \delta \mathcal{J} = \int \delta \rho \, d\eta \) and \( \int \delta \rho \, d\eta = 0 \), where \( \delta \mathcal{J}(r, v) \) is prescribed (assumed to conserve energy and mass at first order). Since the specification of \( \delta \mathcal{J} \) determines \( \delta \Psi \), hence the second integral in Eq. (C5), we can write the variational problem under the form
\[ \delta \left( -\frac{1}{2} \int \frac{(\delta \rho)^2}{\rho} \, dv \, d\eta \right) + \int \lambda(r, v) \delta \left( \int \delta \rho \, d\eta \right) \, dv = \int \zeta(r, v) \delta \left( \int \delta \rho \, d\eta \right) \, dv = 0, \] (C22)
where \( \lambda(\mathbf{r}, \mathbf{v}) \) and \( \zeta(\mathbf{r}, \mathbf{v}) \) are Lagrange multipliers. This gives
\[
\delta \rho_\ast(\mathbf{r}, \mathbf{v}, \eta) = -\rho(\mathbf{r}, \mathbf{v}, \eta)[\lambda(\mathbf{r}, \mathbf{v})\eta + \zeta(\mathbf{r}, \mathbf{v})],
\]  
(C23)
which is the global maximum of \( \delta^2 J[\delta \rho] \) with the previous constraints since \( \delta^2 J[\delta \rho] = -\int \{[\delta(\delta \rho)]^2/2\rho \} \, d\mathbf{r} d\mathbf{v} d\eta < 0 \) (the constraints are linear in \( \delta \rho \) so their second variations vanish). The Lagrange multipliers are determined from the constraints \( \delta \mathbf{f} = \int \delta \rho \eta \, d\mathbf{r} \) and \( \int \delta \rho \, d\mathbf{r} = 0 \) yielding \( \delta \mathbf{f} = -\lambda \mathbf{f}^2 - \zeta \mathbf{f} \) and \( 0 = -\lambda \mathbf{f} - \zeta \). Therefore, the optimal perturbation \( \delta \rho_\ast \) can finally be written
\[
\delta \rho_\ast = \frac{\delta \mathbf{f}}{f_2} \rho(\eta - \mathbf{f}).
\]  
(C24)
Since it maximizes \( \delta^2 J[\delta \rho] \), we have \( \delta^2 J[\delta \rho] \leq \delta^2 J[\delta \rho_\ast] \). Explicating \( \delta^2 J[\delta \rho_\ast] \) using Eqs. (C5) and (C24), we obtain
\[
\delta^2 J[\delta \rho] \leq -\frac{1}{2} \int \frac{(\delta \mathbf{f})^2}{f_2} \, d\mathbf{r} d\mathbf{v} - \frac{1}{2} \beta \int \delta \mathbf{f} \delta \mathbf{f} \, d\mathbf{r} d\mathbf{v}.
\]  
(C25)
Finally, using Eq. (C10), which is rigorously valid at equilibrium, the foregoing inequality can be rewritten as
\[
\delta^2 J[\delta \rho] \leq -\frac{1}{2} \int C''(\mathbf{f})(\delta \mathbf{f})^2 \, d\mathbf{r} d\mathbf{v} - \frac{1}{2} \beta \int \delta \mathbf{f} \delta \mathbf{f} \, d\mathbf{r} d\mathbf{v} \equiv \delta^2 J[\delta \mathbf{f}],
\]  
(C26)
where the r.h.s. is precisely the functional appearing in Eq. (C8). Furthermore, there is equality in Eq. (C26) if and only if \( \delta \rho = \delta \rho_\ast \). This proves that the stability criteria (C5) and (C8) are equivalent. Indeed: (i) if inequality (C8) is fulfilled for all perturbations \( \delta \mathbf{f} \) that conserve mass and energy at first order, then according to Eq. (C26), we know that inequality (C5) is fulfilled for all perturbations \( \delta \rho \) that conserve mass, energy, and normalization condition at first order; (ii) if there exists a perturbation \( \delta \mathbf{f}_c \) that makes \( \delta^2 J[\delta \mathbf{f}_c] > 0 \), then the perturbation \( \delta \rho_c \) given by Eq. (C24) with \( \delta \mathbf{f} = \delta \mathbf{f}_c \) makes \( \delta^2 J[\delta \rho_c] = \delta^2 J[\delta \mathbf{f}_c] > 0 \) (this is where we need to treat the Casimir constraints canonically otherwise this perturbation might not be allowed by the Casimir constraints). In conclusion, the stability criteria (C5) and (C8) are equivalent.

Remark: We can also derive this result by using the method of orthogonal perturbations [124] developed in the Appendix of [39].

4. Out-of-equilibrium distribution and justification of the closure relation from Eq. (90)

We can use the strategy developed above to propose a closure of the hierarchy of moment equations [75] describing the collisionless relaxation of systems with long-range interactions.\(^{34}\) The idea is to maximize, out-of-equilibrium, the relative entropy \( S_\chi[\rho] \) at fixed normalization \( \int \rho(\mathbf{r}, \mathbf{v}, \eta, t) \, d\mathbf{r} d\mathbf{v} d\eta = 1 \) and coarse-grained DF \( \mathbf{f}(\mathbf{r}, \mathbf{v}, t) = \int \rho(\mathbf{r}, \mathbf{v}, \eta, t) \eta \, d\mathbf{r} d\mathbf{v} d\eta \). This amounts to constructing a thermodynamical equilibrium distribution \( \rho_\ast(\mathbf{r}, \mathbf{v}, \eta, t) \) corresponding to an out-of-equilibrium coarse-grained DF \( \mathbf{f}(\mathbf{r}, \mathbf{v}, t) \), just like in the first step of Appendix C. This returns, at each time \( t \), the equations of the first step of Appendix C. In particular, one has
\[
\rho_\ast(\mathbf{r}, \mathbf{v}, \eta, t) = \frac{1}{Z[\Psi(\mathbf{r}, \mathbf{v}, t)]} \chi(\eta)e^{-\eta \Psi(\mathbf{r}, \mathbf{v}, t)},
\]  
(C27)
where \( Z[\Psi(\mathbf{r}, \mathbf{v}, t)] \) and \( \Psi(\mathbf{r}, \mathbf{v}, t) \) are determined in terms of \( \chi(\eta) \) and \( \mathbf{f}(\mathbf{r}, \mathbf{v}, t) \) by Eq. (C12). As a result, Eq. (C21) is valid at any time (under the previous assumption) yielding
\[
f_2(\mathbf{r}, \mathbf{v}, t) = \frac{1}{C''(\mathbf{f}(\mathbf{r}, \mathbf{v}, t))}.
\]  
(C28)
As discussed in Sec. (14), this important relation allows us to close the hierarchy of kinetic equations. This leads to Eqs. (91) and (92–94). We have already indicated in Sec. (14) that these equations conserve mass and energy and satisfy an \( H \)-theorem for the generalized entropy \( S[\mathbf{f}] \). Since \( S_\chi[\rho_\ast] = S[\mathbf{f}] \), we conclude that the entropy \( S_\chi(t) \) increases monotonically with time until the Gibbs state is reached.

\(^{34}\) This method was first introduced in Appendix C of [55] and in [62] in the context of 2D turbulence.
Remark: If we treat the Casimir constraints microcanonically, we find the same results as above except that, at each time \( t \), we have to relate \( \chi(\eta) \) to the Casimirs \( M_{\alpha\beta}^f \) and to the coarse-grained DF \( \overline{f}(r, v, t) \). As a result, \( \chi(\eta) \) and \( C_i(\overline{f}) \) become functions of time. Therefore, the shape of the generalized entropy changes with time. The kinetic equation (91) remains valid except that we have to replace \( C(\overline{f}) \) by \( C_i(\overline{f}) \). A manner to justify treating the Casimir constraints canonically is to assume that the function \( \chi(\eta) \) is always close to its equilibrium value so that it does not change substantially. Actually, maximizing out-of-equilibrium the relative entropy \( S_\chi[\rho] \) at fixed normalization and coarse-grained DF to get Eq. (C27) is only valid close to equilibrium so the two assumptions are conditioned to each other. In the canonical closure approach, we just have to solve the equilibrium problem to get \( \chi(\eta) \) and \( C(\overline{f}) \) once for all. Then, Eq. (91) determines the dynamical evolution of the system for all times \( t \) provided that we are sufficiently close to equilibrium for the above assumptions to be valid. Alternatively, in the microcanonical closure approach, we have to determine \( \chi(\eta) \) and \( C_i(\overline{f}) \) at each time in order to obtain Eq. (91). Since \( S_{LB}[\overline{f}] = S_i(\overline{f}) \), we conclude that the Lynden-Bell entropy \( S_{LB}(t) \) increases monotonically with time until the Gibbs state is reached. This microcanonical closure approach is more precise, but it is also much more complicated.

5. The equation for the distribution of phase levels

In the approach developed in the previous section, the coarse-grained DF \( \overline{f}(v, t) \) evolves according to Eq. (91) or Eqs. (92)-(94). The distribution \( \rho_*(v, \eta, t) \) is then given by Eq. (C27). It may be of interest to determine the relaxation equation satisfied by \( \rho_*(v, \eta, t) \) explicitly. According to Eq. (C27), we have

\[
\ln \rho_* = -\eta \Psi + \ln \chi(\eta) - \ln Z(\Psi),
\]

where \( \Psi(v, t) \) is related to \( \overline{f}(v, t) \) according to Eq. (C12). Differentiating Eq. (C29) with respect to \( t \) and using Eqs. (C13) and (C20), we obtain

\[
\frac{\partial \rho_*}{\partial t} = -\rho_*(\eta - \overline{f}) \frac{\partial \Psi}{\partial t} = \rho_*(\eta - \overline{f}) C''(\overline{f}) \frac{\partial \overline{f}}{\partial t},
\]

(C30)

Similarly, we have

\[
\frac{\partial \rho_*}{\partial \mathbf{v}} = -\rho_*(\eta - \overline{f}) \frac{\partial \Psi}{\partial \mathbf{v}} = \rho_*(\eta - \overline{f}) C''(\overline{f}) \frac{\partial \overline{f}}{\partial \mathbf{v}}.
\]

(C31)

Combining Eq. (C30) with Eqs. (91) and (92), we get

\[
\frac{\partial \rho_*}{\partial t} = \rho_*(\eta - \overline{f}) C''(\overline{f}) \frac{\partial}{\partial v_i} \int d \mathbf{v'} K_{ij} \left[ \frac{1}{C''(\overline{f})} \frac{\partial \overline{f}}{\partial v_j} - \frac{1}{C''(\overline{f})} \frac{\partial \overline{f}}{\partial v_j} \right]
\]

(C32)

and

\[
\frac{\partial \rho_*}{\partial \mathbf{v}} = \rho_*(\eta - \overline{f}) C''(\overline{f}) \frac{\partial}{\partial v_i} \left\{ D_{ij} \left[ \frac{\partial \overline{f}}{\partial v_j} + \frac{C''(\overline{f})}{\partial \mathbf{v}} v_j \right] \right\}.
\]

(C33)

Using Eqs. (C21) and (C31), the foregoing equations can be rewritten as

\[
\frac{\partial \rho_*}{\partial t} = \rho_*(\eta - \overline{f}) \frac{\partial}{\partial v_i} \int d \mathbf{v'} K_{ij} \frac{f_2}{\rho_*(\eta - \overline{f})} \left[ f_2 \frac{\partial \rho_*}{\partial v_j} - \rho_*(\eta - \overline{f}) \frac{\partial \overline{f}}{\partial v_j} \right]
\]

(C34)

and

\[
\frac{\partial \rho_*}{\partial \mathbf{v}} = \rho_*(\eta - \overline{f}) \frac{\partial}{\partial v_i} \left\{ D_{ij} \frac{f_2}{\rho_*(\eta - \overline{f})} \left[ \frac{\partial \rho_*}{\partial v_j} + \beta(t) \rho_*(\eta - \overline{f}) v_j \right] \right\}.
\]

(C35)

Under that form, we see some analogies (but also crucial differences) with the SL and CSR equations (74) and (79). Remark: Similar equations have been obtained in the context of 2D turbulence [62]. By proceeding similarly to Sec. 4.2 of [62], it is also possible to derive a relaxation for \( \rho(\mathbf{r}, \mathbf{v}, t) \) associated with the maximization problem (C4) where the Casimir constraints are treated canonically. This equation can be used as a numerical algorithm to solve the maximization problem (C4).
6. Log-entropy

In the previous sections, we have treated the fine-grained moments $M_{n>1}^{f,g}$ canonically. If we do not take into account at all the contribution of the fine-grained moments $M_{n>1}^{f,g}$ in the variational principle, the Gibbs state reduces to

$$\rho_*(r, v, \eta) = \frac{1}{Z[\Psi(r, v)]} e^{-\eta \Psi(r, v)}. \quad (C36)$$

This amounts to writing $\chi(\eta) = 1$ in Eq. (C11). Using Eqs. (C12) and (C13) it is easy to establish that

$$Z = \frac{1}{\Psi}, \quad \mathcal{T} = \frac{1}{\Psi}, \quad f_2 = \frac{1}{\Psi^2} = \mathcal{T}^2. \quad (C37)$$

We can then rewrite Eq. (C36) as

$$\rho_*(r, v, \eta) = \frac{1}{\mathcal{T}(r, v)} e^{-\eta/\mathcal{T}(r, v)}. \quad (C38)$$

The generalized entropy associated with this distribution can be obtained from the relation [see Eq. (C28)]

$$\frac{1}{C''(f)} = f_2 = \mathcal{T}^2, \quad (C39)$$

leading to the functional

$$S = \int \ln \mathcal{T} \, dr \, dv. \quad (C40)$$

This is what we have called the log-entropy in Ref. [55]. The kinetic equation (91) associated with the log-entropy has been studied in [94, 125]. Using Eq. (C19), the equilibrium DF is given by

$$\mathcal{T} = \frac{1}{\beta \epsilon + \alpha}. \quad (C41)$$

This is the Lorentzian DF. Note that this DF is not normalizable in $d = 3$, so there is no equilibrium state in that case.

Appendix D: Cumulant generating function

In the multi-level case, the equilibrium distribution of the statistical theory of Lynden-Bell is the Gibbs state

$$\rho(r, v, \eta) = \frac{1}{Z(\epsilon)} \chi(\eta) e^{-\eta(\beta \epsilon + \alpha)}, \quad (D1)$$

where

$$Z(\epsilon) = \int \chi(\eta) e^{-\eta(\beta \epsilon + \alpha)} \, d\eta \quad (D2)$$

is the partition function. The coarse-grained DF $\mathcal{T} = \int \rho \eta \, d\eta$ (first moment) is given by

$$\mathcal{T} = \frac{1}{Z(\epsilon)} \int \eta \chi(\eta) e^{-\eta(\beta \epsilon + \alpha)} \, d\eta$$

$$= -\frac{1}{Z(\epsilon)} \frac{\partial}{\partial \epsilon} \int \chi(\eta) e^{-\eta(\beta \epsilon + \alpha)} \, d\eta$$

$$= -\frac{1}{Z(\epsilon)} \frac{\partial Z}{\partial \epsilon}$$

$$= -\frac{1}{\beta \epsilon} (\ln Z)'(\epsilon). \quad (D3)$$
To the probability density (D1) we associate the cumulant generating function
\[
\kappa(\lambda, \epsilon) = \ln e^{-\lambda \beta \eta}.
\] (D4)

It satisfies
\[
e^{-\lambda \beta \eta} = \frac{1}{Z(\epsilon)} \int e^{-\lambda \beta \eta} \chi(\eta) e^{-\eta (\beta \epsilon + \alpha)} d\eta
= \frac{1}{Z(\epsilon)} \int \chi(\eta) e^{-\eta [\beta (\epsilon + \lambda) + \alpha]} d\eta
= \frac{Z(\epsilon + \lambda)}{Z(\epsilon)}.
\] (D5)

Therefore,
\[
\kappa(\lambda, \epsilon) = \ln Z(\epsilon + \lambda) - \ln Z(\epsilon).
\] (D6)

Taking the partial derivative of this expression with respect to \(\lambda\), we get
\[
\frac{\partial \kappa(\lambda, \epsilon)}{\partial \lambda} = (\ln Z)^{\prime}(\epsilon + \lambda).
\] (D7)

Combined with Eq. (D3), we obtain
\[
\frac{\partial \kappa(\lambda, \epsilon)}{\partial \lambda} = -\beta f(\epsilon + \lambda).
\] (D8)

Expanding both sides of Eq. (D8) in powers of \(\lambda\), we find that the cumulants \(\kappa_n(\epsilon) = \kappa(n)(0, \epsilon)\) are related to the derivatives of \(f(\epsilon)\) by
\[
\kappa_{n+1}(\epsilon) = -\beta \frac{d^n f}{d \epsilon^n}.
\] (D9)

For example,
\[
\beta^2 (\overline{f^2} - \overline{f}^2) = -\beta \frac{df}{d\epsilon},
\] (D10)
\[
\beta^3 (\overline{f^3} - 3 \overline{f} \overline{f^2} + 2 \overline{f}^3) = \beta \frac{d^2 f}{d\epsilon^2},
\] (D11)
\[
\beta^4 (\overline{f^4} - 3 \overline{f^2} \overline{f^2} - 4 \overline{f} \overline{f^3} + 12 \overline{f}^2 \overline{f}^2 - \overline{f}^4) = -\beta \frac{d^3 f}{d\epsilon^3}.
\] (D12)

Appendix E: Properties of the KP and SL equations

1. Conservation of energy, linear impulse and Casimirs

It is easy to show that the KP equation (62) conserves the energy and the linear impulse. Indeed,
\[
\dot{E} = \int dv \frac{v^2}{2} \frac{\partial \overline{f}}{\partial t} = -\int d\nu \nu' \nu K_{ij} \left( f'_2 \frac{\partial \overline{f}}{\partial \nu_j} - f_2 \frac{\partial \overline{f}}{\partial \nu'_j} \right) = \int d\nu \nu' \nu' K_{ij} \left( f'_2 \frac{\partial \overline{f}}{\partial \nu_j} - f_2 \frac{\partial \overline{f}}{\partial \nu'_j} \right)
= -\frac{1}{2} \int d\nu \nu' \nu' K_{ij} \left( f'_2 \frac{\partial \overline{f}}{\partial \nu_j} - f_2 \frac{\partial \overline{f}}{\partial \nu'_j} \right) = 0,
\] (E1)

where we have interchanged the dummy variables \(\nu\) and \(\nu'\) to obtain the third equality and used the identity \(K_{ij} \nu_j = 0\) to obtain the last equality. Similarly,
\[
\dot{P}_i = \int dv v_i \frac{\partial \overline{f}}{\partial \nu} = -\int d\nu \nu' K_{ij} \left( f'_2 \frac{\partial \overline{f}}{\partial \nu_j} - f_2 \frac{\partial \overline{f}}{\partial \nu'_j} \right) = \int d\nu \nu' K_{ij} \left( f'_2 \frac{\partial \overline{f}}{\partial \nu_j} - f_2 \frac{\partial \overline{f}}{\partial \nu'_j} \right) = 0.
\] (E2)

Since the KP equation (62) is the first moment of the SL equation (74), the SL equation conserves the energy and the linear impulse. The SL equation also (trivially) conserves the hypersurface \(\gamma(\eta) = \int \rho d\nu\) of each level. This is equivalent to the conservation of all the Casimirs.
2. $H$-theorem

We can also show that the SL equation (74) satisfies an $H$-theorem for the Lynden-Bell entropy (6). The SL equation (74) can be rewritten in a more symmetric form as

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial v_i} \int d\nu' d\eta' K_{ij} \left[ \rho'(\eta' - T) \frac{\partial \rho}{\partial v_j} - \rho(\eta - T) \frac{\partial \rho'}{\partial v_j} \right].$$

(E3)

The rate of change of the Lynden-Bell entropy (9) is

$$\dot{S}_{LB} = -\int d\nu d\eta (1 + \ln \rho) \frac{\partial \rho}{\partial t}.$$  

(E4)

Substituting Eq. (E3) into Eq. (E4), we get

$$\dot{S}_{LB} = \int d\nu d\nu' d\eta d\eta' \frac{1}{\rho \rho'} K_{ij} \left[ \frac{\partial}{\partial v_i} \rho'(\eta' - T) \frac{\partial \rho}{\partial v_j} - \rho(\eta - T) \frac{\partial \rho'}{\partial v_j} \right] \left[ \frac{\partial}{\partial v_i} \rho'(\eta' - T) \frac{\partial \rho}{\partial v_j} - \rho(\eta - T) \frac{\partial \rho'}{\partial v_j} \right].$$

(E5)

To obtain the first line we have integrated by parts, to obtain the second line we have interchanged the primed and unprimed variables, and to obtain the third line we have taken the half-sum of the first and second lines. Equation (E5) can be rewritten as

$$\dot{S}_{LB} = \frac{1}{2} \int d\nu d\nu' d\eta d\eta' \frac{1}{\rho \rho'} \left[ \rho'(\eta' - T) \frac{\partial \rho}{\partial v_i} - \rho(\eta - T) \frac{\partial \rho'}{\partial v_i} \right] \left[ \frac{\partial}{\partial v_i} \rho'(\eta' - T) \frac{\partial \rho}{\partial v_j} - \rho(\eta - T) \frac{\partial \rho'}{\partial v_j} \right] + I,$$

(E6)

where $I$ is the integral

$$I = \frac{1}{2} \int d\nu d\nu' d\eta d\eta' \frac{1}{\rho \rho'} \left[ \rho'(\eta' - T) \frac{\partial \rho}{\partial v_i} - \rho(\eta - T) \frac{\partial \rho'}{\partial v_i} \right] \left[ \frac{\partial}{\partial v_i} \rho'(\eta' - T) \frac{\partial \rho}{\partial v_j} - \rho(\eta - T) \frac{\partial \rho'}{\partial v_j} \right].$$

(E7)

Expanding the terms in brackets, it can written as the sum of four integrals. The first integral

$$I_1 = \frac{1}{2} \int d\nu d\nu' d\eta d\eta' \frac{1}{\rho} \left( \eta' \frac{\partial \rho}{\partial v_i} \right) K_{ij} \left[ \rho'(\eta' - T) \frac{\partial \rho}{\partial v_j} \right]$$

(E8)

vanishes because $\int d\eta' \rho'(\eta' - T) = T - \bar{T} = 0$. The second integral

$$I_2 = \frac{1}{2} \int d\nu d\nu' d\eta d\eta' \left( \eta' \frac{\partial \rho}{\partial v_i} \right) K_{ij} \left[ \eta - \bar{T} \frac{\partial \rho}{\partial v_j} \right] = 0$$

(E9)

vanishes because $\int d\eta' (\partial \rho'/\partial v_j) = 0$ (recall that $\int d\eta' \rho' = 1$). The two other integrals $I_3$ and $I_4$ vanish for the same reasons. As a result, we find that $I = 0$. The rate of change of the Lynden-Bell entropy (E6) can therefore be written as

$$\dot{S}_{LB} = \frac{1}{2} \int d\nu d\nu' d\eta d\eta' \frac{1}{\rho \rho'} X_i K_{ij} X_j$$

(E10)

with

$$X = \rho'(\eta' - T) \frac{\partial \rho}{\partial \nu} - \rho(\eta - T) \frac{\partial \rho'}{\partial \nu'}.$$  

(E11)

$^{35}$ This $H$-theorem was not derived in [1].
Since
\[ X_i K_{ij} X_j = \pi (2\pi)^d \epsilon^d \epsilon^d \int d\mathbf{k} (\mathbf{k} \cdot \mathbf{X})^2 \frac{\hat{u}(k)^2}{|\epsilon(k, \mathbf{k} \cdot \mathbf{v})|^2} \delta[\mathbf{k} \cdot (\mathbf{v} - \mathbf{v}')] \],
we conclude that \( \dot{S}_{\text{LB}} \geq 0 \) with equality if and only if \( \mathbf{X} \) is parallel to \( \mathbf{v}' - \mathbf{v} \). Therefore, the Lynden-Bell entropy increases monotonically (H-theorem).

3. Gibbs state

Let us check that the Gibbs state \((11)\) is a stationary solution of the SL equation \((74)\). From Eq. \((11)\) we have
\[ \ln \rho = -\eta(\beta \epsilon + \alpha) + \ln \chi(\eta) - \ln Z. \] Taking the derivative of Eq. \((E13)\) and using Eq. \((13)\) we get
\[ \frac{\partial \rho}{\partial \mathbf{v}} = -\beta \rho(\eta - \bar{f}) \mathbf{v}. \] On the other hand, according to Eq. \((10)\), we have
\[ \frac{\partial f}{\partial \mathbf{v}} = f'(\epsilon) \mathbf{v} = -\beta f_2 \mathbf{v}. \] Therefore, at statistical equilibrium,
\[ f'_2 \frac{\partial \rho}{\partial \mathbf{v}} - \rho(\eta - \bar{f}) \frac{\partial f}{\partial \mathbf{v}} = -f'_2 \beta \rho(\eta - \bar{f}) \mathbf{w}. \] Since \( K_{ij} w_j = 0 \), we find that the current in Eq. \((74)\) vanishes implying that \( \partial \rho / \partial t = 0 \).

Inversely, the condition that \( \mathbf{X} \) must be parallel to \( \mathbf{v}' - \mathbf{v} \) at equilibrium (this condition results from the H-theorem as shown above) can be written as
\[ \frac{1}{\eta - \bar{f}} \frac{\partial \ln \rho}{\partial \mathbf{v}} - \frac{1}{\eta' - \bar{f}} \frac{\partial \ln \rho'}{\partial \mathbf{v}'} = -A(\eta, \eta', \mathbf{v}, \mathbf{v}') (\mathbf{v} - \mathbf{v}'). \] From the symmetry of the left hand side of Eq. \((E17)\) it can be shown \([8]\) that \( A(\eta, \eta', \mathbf{v}, \mathbf{v}') \) is a constant that we shall denote \( \beta \). This then implies that
\[ \frac{\partial \ln \rho}{\partial \mathbf{v}} + \beta(\eta - \bar{f})(\mathbf{v} - \mathbf{u}) = 0, \] where \( \mathbf{u} \) is another constant. At that stage, we can repeat the argument of \([26]\) (see also Appendix \([3]\) to show that Eq. \((E18)\) leads to the Gibbs state \((11)\). In conclusion, the SL equation relaxes towards the Lynden-Bell distribution.

Appendix F: Interpretation of the global temperature in the CSR equations

In this Appendix, we provide a physical interpretation of the inverse temperature \( \beta(t) \) in the CSR equation which was introduced in \([26]\) as a Lagrange multiplier associated with the conservation of energy.

1. Spatially inhomogeneous systems

For spatially inhomogeneous systems, the CSR equations can be written as
\[ \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \frac{\partial \rho}{\partial \mathbf{r}} - \nabla \bar{\Phi} \cdot \frac{\partial \rho}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}_i} \left\{ D_{ij} \left[ \frac{\partial \rho}{\partial \mathbf{v}_j} + \beta(t) \rho(\eta - \bar{f}) \mathbf{v}_j \right] \right\}. \]
with

\[ D_{ij} = \int d\mathbf{v}' K_{ij} f_2' \]  \hspace{1cm} (F2) 

and

\[ \beta(t) = -\frac{\int D_{ij} v_i \frac{\partial f'}{\partial v_j} \, dr \, dv}{\int D_{ij} f_2 v_i v_j \, dr \, dv}, \]  \hspace{1cm} (F3) 

where we have made a local approximation \( f'_2 = f_2(\mathbf{r}, \mathbf{v}', t) \) in Eq. (F2). If we consider a simplified model where \( D_{ij} = D\delta_{ij} \) with \( D \) constant, we obtain after an integration by parts

\[ \beta(t) = \frac{d}{dt} \int f \, dr \, dv \]  \hspace{1cm} (F4) 

In the two-level case, and in the nondegenerate limit, the CSR equations reduce to

\[ \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = \sum_{ij} D_{ij} \left[ \frac{\partial f'}{\partial v_j} + \beta(t) \eta_0 f v_j \right] \]  \hspace{1cm} (F5) 

with

\[ D_{ij} = \int d\mathbf{v}' K_{ij} \eta_0 f' \]  \hspace{1cm} (F6) 

and

\[ \beta(t) = -\frac{\int D_{ij} v_i \frac{\partial f'}{\partial v_j} \, dr \, dv}{\int D_{ij} \eta_0 f v_i v_j \, dr \, dv}. \]  \hspace{1cm} (F7) 

Equation (F5) is similar to the classical Kramers equation, except that it involves a time-dependent temperature. If we consider a simplified model where \( D_{ij} = D\delta_{ij} \) with \( D \) constant, we obtain after an integration by parts

\[ \beta(t) = \frac{d}{dt} \int f \, dr \, dv = \frac{dM}{\eta_0 K(t)}, \]  \hspace{1cm} (F8) 

where \( M = \int f \, dr \, dv \) is the total mass and \( K(t) = \frac{1}{2} \int f v^2 \, dr \, dv \) is the total kinetic energy. Writing \( \beta = 1/T \), we get

\[ T(t) = \frac{2\eta_0 K(t)}{dM} \] \hspace{1cm} \( \Leftrightarrow \) \hspace{1cm} \[ K(t) = \frac{dM}{2\eta_0} T(t). \]  \hspace{1cm} (F9) 

This relation shows that \( T(t) \) can be interpreted as a global kinetic temperature. It is, however, different from the spatial average value of the local kinetic temperature. The local kinetic temperature is defined by

\[ T_{\text{kin}}(\mathbf{r}, t) = \frac{\eta_0 \int [\mathbf{v} - \mathbf{u}(\mathbf{r}, t)]^2 \, dv}{d \int f \, dv}, \]  \hspace{1cm} (F10) 

where \( \mathbf{u}(\mathbf{r}, t) = \frac{1}{\rho} \int f \mathbf{v} \, dv \) is the local velocity. The spatial average of the kinetic temperature is

\[ \langle T_{\text{kin}} \rangle(t) = \frac{\rho T_{\text{kin}}(\mathbf{r}, t) \, d\mathbf{r}}{\rho \, d\mathbf{r}} = \frac{\eta_0}{dM} \left( \int f v^2 \, dr \, dv - \int \rho \mathbf{u}^2 \, d\mathbf{r} \right). \]  \hspace{1cm} (F11) 

We have the following relation

\[ T(t) = \langle T_{\text{kin}} \rangle(t) + \frac{\eta_0}{dM} \int \rho \mathbf{u}^2 \, d\mathbf{r} \]  \hspace{1cm} (F12) 

between the global kinetic temperature \( T(t) \) and the spatial average value of the local kinetic temperature \( \langle T_{\text{kin}} \rangle(t) \). 

---

36 See another possible expression of \( D_{ij} \) in Appendix B of [26]. More generally, we can leave \( D_{ij} \) unspecified provided that the quadratic form \( D_{ij} X_i X_j \geq 0 \) for any \( X \) is definite positive.
2. Spatially homogeneous systems

For spatially homogeneous systems, the energy reduces to the kinetic energy \( (K = E) \) implying that the inverse temperature defined by Eq. (F6) is constant

\[
\beta = \frac{1}{T} = \frac{dM}{2\eta_0 E}.
\]  

(F13)

In that case, the CSR equation (F5) becomes

\[
\frac{\partial \overline{f}}{\partial t} = D \frac{\partial}{\partial \mathbf{v}} \left( \frac{\partial \overline{f}}{\partial \mathbf{v}} + \beta \eta_0 \overline{f} \mathbf{v} \right),
\]  

(F14)

which is similar to the usual Kramers (or Klein-Kramers-Chandrasekhar) equation \[127\,128\,127\]. For the initial condition \( \overline{f}_0(\mathbf{v}) = M\delta(\mathbf{v} - \mathbf{v}_0) \) it has the analytical solution

\[
\overline{f}(\mathbf{v}, t) = M \left[ \frac{\beta \eta_0}{2\pi(1 - e^{-2D\beta\eta_0 t})} \right]^{d/2} e^{-\frac{\beta \eta_0 (\mathbf{v} - \mathbf{v}_0)^2}{2(1 - e^{-2D\beta\eta_0 t})}}.
\]  

(F15)

We can check that this solution relaxes towards the Boltzmann DF \( \overline{f}(\mathbf{v}) = M(\beta\eta_0/2\pi)^{d/2}e^{-\beta\eta_0\mathbf{v}^2/2} \). The solution (F15) was first found by Lord Rayleigh \[128\] long before the seminal paper of Einstein \[129\] on Brownian motion (see \[130\] for more details). Taking the time derivative of \( E = \frac{1}{2} \int \overline{f} \mathbf{v}^2 d\mathbf{v} \) and using Eq. (F14), we get\[37\]

\[
\dot{E} + 2D\beta\eta_0 E = dDm.
\]  

(F16)

This equation can be integrated into

\[
E(t) = \left( E_0 - \frac{dM}{2\beta\eta_0} \right) e^{-2D\beta\eta_0 t} + \frac{dM}{2\beta\eta_0}.
\]  

(F17)

This result can also be directly obtained from Eq. (F15). The Kramers equation (F14) satisfies an \( H \)-theorem for the free energy \( F = E - TS \) where \( S = -\int (\overline{f}/\eta_0) \ln(\overline{f}/\eta_0) d\mathbf{v} \) is the Boltzmann entropy. Indeed,

\[
\dot{F} = -\int \frac{D\mathbf{v}}{\eta_0} \left( \frac{\partial \overline{f}}{\partial \mathbf{v}} + \beta \eta_0 \overline{f} \mathbf{v} \right)^2 \leq 0.
\]  

(F18)

Equations (F15)-(F18) are valid for arbitrary values of \( \beta \). In general, the energy is not conserved since the Kramers equation is associated with the canonical ensemble (thermal bath). However, when \( \beta \) is exactly given by Eq. (F13) it turns out that \( E(t) = E_0 \) is constant. In that case, Eq. (F14) satisfies an \( H \)-theorem for the Boltzmann entropy \( S \).

Appendix G: Generalized CSR equations

In the CSR equations \[20\] the energy, the linear impulse and the angular momentum are conserved globally thanks to uniform time-dependent Lagrange multipliers (inverse temperature \( \beta(t) \), linear velocity \( \mathbf{U}(t) \) and angular velocity \( \mathbf{R}(t) \)). It is possible to introduce more general relaxation equations that conserve the energy, the linear impulse and the angular momentum locally. The equation for \( \rho(\mathbf{r}, \mathbf{v}, \eta, t) \) reads\[38\]

\[
\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \frac{\partial \rho}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial \rho}{\partial \mathbf{v}} = \frac{\partial}{\partial v_i} \left\{ D_{ij} \left[ \frac{\partial \rho}{\partial v_j} + \beta(\mathbf{r}) \rho(\eta - \overline{f})(\mathbf{v} - \mathbf{u}(\mathbf{r}, t)) \right] \right\}.
\]  

(G2)

37 If we make the correspondence \( \mathbf{v} \leftrightarrow \mathbf{r} \), the Kramers equation (F4) is equivalent to the Smoluchowski equation \[131\] for a Brownian particle in a harmonic potential. In that case, the kinetic energy is equivalent to the moment of inertia and Eq. (F10) can be interpreted as a form of virial theorem.

38 This equation can be obtained from the SL equation (77) by first extending it to spatially inhomogeneous systems, making a local approximation (see Sec. 1110), then by computing the term \( \partial \overline{f}/\partial \mathbf{v}^i \) with the distribution

\[
\rho(\mathbf{r}, \mathbf{v}', \eta, t) = \frac{1}{Z(\mathbf{r}, \mathbf{v}', t)} \chi(\eta) e^{-\frac{\beta(\mathbf{r}, t)(\mathbf{v} - \mathbf{u}(\mathbf{r}, t))^2}{2} + \alpha(\mathbf{r}, t)},
\]  

(G1)

which relies on a local thermodynamic equilibrium approximation. In that case, the diffusion tensor in Eq. (G2) is given by Eq. (80) with \( f_2^2 = f_2^2(\mathbf{r}, \mathbf{v}', t) \). The usual CSR equations (26) are recovered for \( \beta(\mathbf{r}, t) = \beta(t) \) and \( \mathbf{u}(\mathbf{r}, t) = \mathbf{U}(t) - \mathbf{R}(t) \times \mathbf{r} \).
Multiplying Eq. (G2) by $\eta$ and integrating over the phase levels, we get
\[
\frac{\partial \tilde{T}}{\partial t} + \mathbf{v} \cdot \frac{\partial \tilde{T}}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial \tilde{T}}{\partial \mathbf{v}} = \frac{\partial}{\partial v_i} \left\{ D_{ij} \left[ \frac{\partial \tilde{T}}{\partial v_j} + \beta(\mathbf{r}, t)f_2(v - \mathbf{u}(\mathbf{r}, t)) \right] \right\}.
\] (G3)

These relaxation equation can be written as
\[
\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \frac{\partial \rho}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial \rho}{\partial \mathbf{v}} = -\frac{\partial}{\partial v_i} \cdot \mathbf{J} \quad \text{and} \quad \frac{\partial \tilde{T}}{\partial t} + \mathbf{v} \cdot \frac{\partial \tilde{T}}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial \tilde{T}}{\partial \mathbf{v}} = -\frac{\partial}{\partial v_i} \cdot \mathbf{J}_f,
\] (G4)
where $\mathbf{J}$ is the current of the phase levels $\eta$ and $\mathbf{J}_f$ is the current of the coarse-grained DF given by
\[
\mathbf{J}_i = -D_{ij} \left[ \frac{\partial \rho}{\partial v_j} + \beta(\mathbf{r}, t)\rho(\eta - \tilde{T})(\mathbf{v} - \mathbf{u}(\mathbf{r}, t)) \right] \quad \text{and} \quad \mathbf{J}_j = -D_{ij} \left[ \frac{\partial \tilde{T}}{\partial v_j} + \beta(\mathbf{r}, t)f_2(v - \mathbf{u}(\mathbf{r}, t)) \right].
\] (G5)

We note that $\int \mathbf{J} \, d\eta = 0$ (according to the normalization condition) and $\mathbf{J}_f = \int \mathbf{J}_f \, d\eta$. The local conservation of linear impulse and energy imposes that
\[
\int \mathbf{J}_f \, d\mathbf{v} = 0 \quad \text{and} \quad \int \mathbf{J}_f \cdot \mathbf{v} \, d\mathbf{v} = 0.
\] (G6)

Substituting the current $\mathbf{J}_f$ from Eq. (G5) into the constraints from Eq. (G4), we obtain a set of two linear equations
\[
\int D_{ij} \frac{\partial \tilde{T}}{\partial v_j} \, d\mathbf{v} + \beta(\mathbf{r}, t) \int D_{ij} f_2(v - \mathbf{u}(\mathbf{r}, t)) \, d\mathbf{v} = 0,
\] (G7)
\[
\int D_{ij} v_i \frac{\partial \tilde{T}}{\partial v_j} \, d\mathbf{v} + \beta(\mathbf{r}, t) \int D_{ij} v_i f_2(v - \mathbf{u}(\mathbf{r}, t)) \, d\mathbf{v} = 0,
\] (G8)
which determine $\beta(\mathbf{r}, t)$ and $\mathbf{u}(\mathbf{r}, t)$.

The $H$-theorem can be derived as follows. First we note that the Lynden-Bell entropy, and more generally all the functionals of $\rho$, are conserved by the advection term of Eq. (G2). The proof is similar to the one given in Appendix B for the Vlasov equation:
\[
\dot{I}_h = \int h'(\rho) \frac{\partial \rho}{\partial t} \, d\mathbf{r} d\mathbf{v} d\eta = -\int h'(\rho) \left( \mathbf{v} \cdot \frac{\partial \rho}{\partial t} - \nabla \Phi : \frac{\partial \rho}{\partial \mathbf{v}} \right) \, d\mathbf{r} d\mathbf{v} d\eta = -\int \left[ \mathbf{v} \cdot \frac{\partial h(\rho)}{\partial t} - \nabla \Phi \cdot \frac{\partial h(\rho)}{\partial \mathbf{v}} \right] \, d\mathbf{r} d\mathbf{v} d\eta
\] = \int \left\{ \frac{\partial}{\partial \mathbf{r}} \cdot \left[ h(\rho) \mathbf{v} - \frac{\partial}{\partial \mathbf{v}} \cdot \left[ h(\rho) \nabla \Phi \right] \right) \right\} \, d\mathbf{r} d\mathbf{v} d\eta = 0.
\] (G9)

Therefore, the change of entropy is only due to the current $\mathbf{J}$. It is given by
\[
\dot{S}_{\text{LB}} = -\int (\ln \rho + 1) \frac{\partial \rho}{\partial t} \, d\mathbf{r} d\mathbf{v} d\eta = -\int (\ln \rho + 1) \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{J} \, d\mathbf{r} d\mathbf{v} d\eta = -\int \frac{\partial}{\partial \rho} \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{J} \, d\mathbf{r} d\mathbf{v} d\eta = -\int \frac{\mathbf{J}}{\rho} \cdot \frac{\partial}{\partial \mathbf{v}} \, d\mathbf{r} d\mathbf{v} d\eta.
\] (G10)

The last term of this equation can be rewritten as
\[
\dot{S}_{\text{LB}} = -\int \frac{\mathbf{J}}{\rho} \cdot \left[ \frac{\partial}{\partial \mathbf{v}} + \beta(\eta - \tilde{T})(\mathbf{v} - \mathbf{u}) \right] \, d\mathbf{r} d\mathbf{v} d\eta + \int \beta(\eta - \tilde{T}) \frac{\mathbf{J}}{\rho} \cdot (\mathbf{v} - \mathbf{u}) \, d\mathbf{r} d\mathbf{v} d\eta.
\] (G11)

Integrating over $\eta$ and using the normalization condition and the local conservation of impulse and energy from Eq. (G6) we see that the second term in Eq. (G11) vanishes:
\[
\int \beta(\eta - \tilde{T}) \mathbf{J} \cdot (\mathbf{v} - \mathbf{u}) \, d\mathbf{r} d\mathbf{v} d\eta = \int \beta \mathbf{J}_f \cdot (\mathbf{v} - \mathbf{u}) \, d\mathbf{r} d\mathbf{v} = 0.
\] (G12)

As a result, there remains
\[
\dot{S}_{\text{LB}} = -\int \frac{\mathbf{J}}{\rho} \cdot \left[ \frac{\partial}{\partial \mathbf{v}} + \beta(\eta - \tilde{T})(\mathbf{v} - \mathbf{u}) \right] \, d\mathbf{r} d\mathbf{v} d\eta,
\] (G13)
which, using Eq. (G15), can be written as
\[ \dot{S}_{LB} = \int \frac{1}{\rho} \left[ \frac{\partial \rho}{\partial \mathbf{v}} + \beta \rho \mathbf{v} - \mathbf{f} \right] \cdot \mathbf{D}_{ij} \left[ \frac{\partial \rho}{\partial \mathbf{v}} + \beta \rho \mathbf{v} - \mathbf{f} \right] \cdot d\mathbf{r} d\mathbf{v} d\eta. \]  

(G14)

Assuming that the quadratic form \( X_i D_{ij} X_j \geq 0 \) for any \( X \) is positive definite (we can check that this is the case with the expression of \( D_{ij} \) from Eqs. (G13), (G17) and (G20)) we conclude that \( \dot{S}_{LB} \geq 0 \). At equilibrium, the current \( \mathbf{J} \) vanishes leading to the Gibbs state (11). This can be proven as follows. The condition \( \mathbf{J} = 0 \) can be written as
\[ \frac{\partial \ln \rho}{\partial \mathbf{v}} + \beta (\mathbf{v} - \mathbf{f}) = 0. \]  

(G15)

Applying this relation to a reference level \( \eta_0 \), we get
\[ \frac{\partial \ln \rho_0}{\partial \mathbf{v}} + \beta (\eta_0 - \mathbf{f}) (\mathbf{v} - \mathbf{u}) = 0, \]  

(G16)

where \( \rho_0 = \rho(\mathbf{r}, \mathbf{v}, \eta_0) \). Subtracting Eqs. (G15) and (G16), we obtain
\[ \frac{\partial}{\partial \mathbf{v}} \ln \left( \frac{\rho}{\rho_0} \right) + \beta (\eta - \eta_0) (\mathbf{v} - \mathbf{u}) = 0. \]  

(G17)

This equation can be integrated into
\[ \ln \left( \frac{\rho}{\rho_0} \right) + \frac{1}{2} \beta (\eta - \eta_0)(\mathbf{v} - \mathbf{u})^2 = A(\mathbf{r}, \eta), \]  

(G18)

where \( A(\mathbf{r}, \eta) \) is a constant of integration. At equilibrium, the advection term in Eq. (G2) must also vanish yielding
\[ \mathbf{v} \cdot \frac{\partial \rho}{\partial \mathbf{v}} - \nabla \Phi \cdot \frac{\partial \rho}{\partial \mathbf{v}} = 0. \]  

(G19)

Repeating the same procedure as above, we get
\[ \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} \ln \left( \frac{\rho}{\rho_0} \right) - \nabla A(\mathbf{r}, \eta) \cdot \frac{\partial \rho}{\partial \mathbf{v}} = 0. \]  

(G20)

One can show from the combination of Eqs. (G14) and (G19) that, at equilibrium, \( \beta \) must be uniform and \( \mathbf{u} \) must vanish (this can be viewed as a consequence of the Jeans theorem [53]). Therefore, \( \lim_{t \to +\infty} \beta(\mathbf{r}, t) = \beta \) and \( \lim_{t \to +\infty} \mathbf{u}(\mathbf{r}, t) = \mathbf{0} \). Then, Eq. (G18) reduces to
\[ \ln \left( \frac{\rho}{\rho_0} \right) + \frac{1}{2} \beta (\eta - \eta_0) \mathbf{v}^2 = A(\mathbf{r}, \eta). \]  

(G21)

Taking its gradient with respect to \( \mathbf{r} \), we get
\[ \frac{\partial}{\partial \mathbf{r}} \ln \left( \frac{\rho}{\rho_0} \right) = \nabla A(\mathbf{r}, \eta). \]  

(G22)

Substituting Eq. (G17) and Eq. (G22) into Eq. (G20) we get \( \mathbf{v} \cdot \nabla A + \beta (\eta - \eta_0) \nabla \Phi = 0 \). This equality must be true for all \( \mathbf{v} \), implying that \( \nabla A + \beta (\eta - \eta_0) \nabla \Phi = \mathbf{0} \), which can be integrated into \( A(\mathbf{r}, \eta) = -\beta (\eta - \eta_0) \Phi(\mathbf{r}) - B(\eta) \), where \( B(\eta) \) is a constant of integration. Finally, Eq. (G21) can be rewritten as
\[ \ln \left( \frac{\rho}{\rho_0} \right) = -\beta (\eta - \eta_0) \epsilon - B(\eta), \]  

(G23)

which is equivalent to the Gibbs state (11) with \( 1/Z(\mathbf{r}, \mathbf{v}) = \rho(\mathbf{r}, \mathbf{v}, \eta_0) e^{\beta \eta_0 \epsilon(\mathbf{r}, \mathbf{v})} \) and \( \chi(\eta) e^{-\eta \alpha} = e^{-B(\eta)} \). Inversely, starting from the Gibbs state (11) and using Eqs. (E13) and (E14), we get \( \mathbf{J} = 0 \).

---

39 Note that in the CSR approach [24], at each time \( t \), the inverse temperature \( \beta(t) \) is uniform and \( \mathbf{u} = \mathbf{0} \). One then have \( \lim_{t \to +\infty} \beta(t) = \beta \).
Remark: If we assume that \( D_{ij} = D \delta_{ij} \) with \( D \) constant, the linear equations \((G7)\) and \((G8)\) reduce to
\[
\begin{align*}
\mathbf{u}(\mathbf{r}, t) &= \frac{\int f_2 \mathbf{v} \, d\mathbf{v}}{\int f_2 \, d\mathbf{v}}, & T(\mathbf{r}, t) &= \frac{\int f_2 (\mathbf{v} - \mathbf{u}(\mathbf{r}, t))^2 \, d\mathbf{v}}{d \int f \, d\mathbf{v}}. \\
\end{align*}
\]
(G24)

In the two-level case and in the nondegenerate limit, we obtain
\[
\frac{\partial \mathcal{F}}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathcal{F}}{\partial \mathbf{r}} - \nabla \Phi \cdot \frac{\partial \mathcal{F}}{\partial \mathbf{v}} = D \frac{\partial}{\partial \mathbf{v}} \left[ \mathcal{F} + \beta(t) f_0 (\mathbf{v} - \mathbf{u}(\mathbf{r}, t)) \right]
\]
(G25)

with
\[
\mathbf{u}(\mathbf{r}, t) = \frac{\int \mathcal{F} \mathbf{v} \, d\mathbf{v}}{\int \mathcal{F} \, d\mathbf{v}}, \quad T(\mathbf{r}, t) = \frac{\int (\mathbf{v} - \mathbf{u}(\mathbf{r}, t))^2 \, d\mathbf{v}}{d \int \mathcal{F} \, d\mathbf{v}}.
\]
(G26)

In that case, we recover the usual expressions of the local velocity and local kinetic temperature. Equations \((G25)\) and \((G26)\) are similar to the kinetic equations introduced by Dougherty [132] for collisional systems.

Appendix H: Another equation that conserves the mass and the energy and that monotonically increases all the \(H\)-functions

In Sec. [IX] we have introduced an equation that conserves the mass and the energy and that monotonically increases all the \(H\)-functions. In this Appendix, we introduce another equation that satisfies the same properties.

1. Anisotropic diffusion equation

The CSR equation is given by Eq. \((79)\) with Eq. \((83)\). If we assume for simplicity that the diffusion tensor is isotropic and constant, so that \( D_{ij} = D \delta_{ij} \), this equation reduces to
\[
\frac{\partial \rho}{\partial t} = D \frac{\partial}{\partial \mathbf{v}} \cdot \left[ \frac{\partial \rho}{\partial \mathbf{v}} + \beta(t) \rho (\eta - \mathcal{F}) \mathbf{v} \right]
\]
(H1)

with
\[
\beta(t) = - \frac{\int \mathbf{v} \cdot \frac{\partial \mathcal{F}}{\partial \mathbf{v}} \, d\mathbf{v}}{\int f_2 \mathbf{v}^2 \, d\mathbf{v}}.
\]
(H2)

If we get rid of the integrals in Eq. \((H2)\), we get
\[
\beta(\mathbf{v}, t) = - \frac{\mathbf{v} \cdot \frac{\partial \mathcal{F}}{\partial \mathbf{v}}}{f_2 \mathbf{v}^2}.
\]
(H3)

Substituting this relation into Eq. \((H1)\), we obtain
\[
\frac{\partial \rho}{\partial t} = D \frac{\partial}{\partial \mathbf{v}} \cdot \left[ \frac{\partial \rho}{\partial \mathbf{v}} - \frac{\mathbf{v} \cdot \frac{\partial \mathcal{F}}{\partial \mathbf{v}}}{f_2 \mathbf{v}^2} \rho (\eta - \mathcal{F}) \mathbf{v} \right].
\]
(H4)

This equation can also be obtained by applying the MEPP with a local conservation of energy \( \mathbf{J}_f \cdot \mathbf{v} = 0 \) in velocity space.\(^{40}\) It conserves the normalization condition, the Casimirs (or the total hypervolume of each phase level \( \eta \)) and the energy. It also increases the mixing entropy (9) monotonically (\(H\)-theorem). The proof is essentially the same as for the CSR equations (see Ref. [26] and Appendix [G]). However, it does not relax towards the Gibbs state (11).\(^{41}\)

\(^{40}\) An equation similar to Eq. \((H4)\), but acting in position space instead of velocity space, has been obtained in the context of 2D turbulence in Ref. [27].

\(^{41}\) It is not clear if this property is a drawback of this equation or if it can account for situations of incomplete relaxation where the quasistationary state is different from the Lynden-Bell statistical equilibrium state.
To see that, let us consider the equation for the coarse-grained DF \( \langle S \rangle \) which, for an isotropic and constant diffusion tensor, can be written as

\[
\frac{\partial \tilde{f}}{\partial t} = D \frac{\partial}{\partial v} \cdot \left[ \frac{\partial \tilde{f}}{\partial v} + \beta(t)f_2v \right].
\]  

(H5)

Replacing \( \beta(t) \) by Eq. (H3) we obtain

\[
\frac{\partial \tilde{f}}{\partial t} = D \frac{\partial}{\partial v} \cdot \left( \frac{\partial \tilde{f}}{\partial v} - \frac{v \cdot \partial \tilde{f}}{v^2}v \right).
\]  

(H6)

This equation can also be obtained by multiplying Eq. (H4) by \( \eta \) and integrating over \( \eta \). We note that, unlike Eq. (H5), this equation is closed since the second moment \( f_2 \) has cancelled out. Equation (H6) can be rewritten as

\[
\frac{\partial \tilde{f}}{\partial t} = D \frac{\partial}{\partial v} \left[ \left( \delta_{ij} - \frac{v_iv_j}{v^2} \right) \frac{\partial \tilde{f}}{\partial v_j} \right].
\]  

(H7)

This is an anisotropic diffusion equation of the form

\[
\frac{\partial \tilde{f}}{\partial t} = \frac{\partial}{\partial v} \left( D_{ij} \frac{\partial \tilde{f}}{\partial v_j} \right)
\]  

(H8)

with a diffusion tensor

\[
D_{ij} = D \frac{v^2\delta_{ij} - v_iv_j}{v^2}.
\]  

(H9)

The diffusion tensor \( D_{ij} \) has the property that \( D_{ij}v_j = 0 \). As a result, all isotropic DFs are stationary solutions of Eq. (H8). Indeed, for a DF of the form \( \tilde{f} = \tilde{f}(v) \) with \( v = |v| \), we have \( \frac{\partial \tilde{f}}{\partial v_j} = \tilde{f}(v)v_j/v \). Since \( D_{ij}v_j = 0 \), we obtain \( D_{ij}\frac{\partial \tilde{f}}{\partial v_j} = 0 \), hence \( \frac{\partial \tilde{f}}{\partial t} = 0 \). When the initial DF \( \tilde{f}_0(v) \) is anisotropic, the system evolves until \( \tilde{f}(v, t) \) becomes isotropic. Therefore, the effect of the diffusion equation (H6) is to “isotropize” an initially anisotropic DF.\(^{42}\)

Remark: For 1D systems, like the HMF model, Eq. (H6) reduces to

\[
\frac{\partial \tilde{f}}{\partial t} = 0
\]  

(H10)

so there is no evolution in that case.

2. Properties of Eq. (H6)

Let us write Eq. (H6) under the conservative form

\[
\frac{\partial \tilde{f}}{\partial t} = -\frac{\partial}{\partial v} \cdot \mathbf{J}_f
\]  

(H11)

with the diffusion current

\[
\mathbf{J}_f = -D_{ij} \frac{\partial \tilde{f}}{\partial v_j} = -D \left[ \frac{\partial \tilde{f}}{\partial v} - \left( v \cdot \frac{\partial \tilde{f}}{\partial v} \right) \frac{v}{v^2} \right] = -D \left( \frac{\partial \tilde{f}}{\partial v} - \frac{\partial \tilde{f}}{\partial v} v \frac{v}{v^2} \right).
\]  

(H12)

We note that the diffusion current is normal to the velocity:

\[
\mathbf{J}_f \cdot v = 0.
\]  

(H13)

\(^{42}\) For simplicity, we have considered spatially homogeneous systems. However, Eq. (H6) remains valid for spatially inhomogeneous systems provided that we introduce an advection term in the left hand side. In that case, it relaxes towards an isotropic DF of the form \( f(\epsilon) \) where \( \epsilon = v^2/2 + \Phi(r) \) which cancels both the advection term and the “collision” term.
As a result, Eq. (H6) trivially conserves the energy (6). Indeed
\[
\dot{E} = \int \frac{v^2}{2} \frac{\partial \overline{f}}{\partial t} dv = - \int \frac{v^2}{2} \frac{\partial}{\partial v} \cdot J_f dv = \int J_f \cdot v dv = 0. \tag{H14}
\]

We can also show that Eq. (H6) monotonically increases all the $H$-functions (29). We have
\[
\dot{H} = - \int C'(\overline{f}) \frac{\partial \overline{f}}{\partial t} dv = \int C'(\overline{f}) \frac{\partial}{\partial v} \cdot J_f dv = - \int C''(\overline{f}) J_f \cdot \frac{\partial \overline{f}}{\partial v} dv. \tag{H15}
\]

Using Eq. (H12), the last equality of this equation can be rewritten as
\[
\dot{H} = \int C''(\overline{f}) \frac{1}{D} J_f \cdot \left[ J_f - D \left( v \cdot \frac{\partial \overline{f}}{\partial v} \right) \frac{v}{v^2} \right] dv. \tag{H16}
\]

Using Eq. (H13), we get
\[
\dot{H} = \int C''(\overline{f}) \frac{J_f^2}{D} dv \geq 0. \tag{H17}
\]

Therefore, all the $H$-functions increase monotonically. At equilibrium, we have $J_f = 0$. This determines an isotropic DF of the form $\overline{f} = f(v)$.

Remark: We note that Eqs. (138) and (H6) share similar general properties (conservation of energy and monotonic increase of all the $H$-functions). However, these two equations are very different. In particular, Eq. (138) reduces to $\partial_t \overline{f} = 0$ for spatially homogeneous systems contrary to Eq. (H6), and Eq. (H6) reduces to $\partial_t f = 0$ for 1D systems contrary to Eq. (138).

3. Analytical solution of Eq. (H6)

It turns out that Eq. (H6) can be solved analytically. Taking the divergence of the current from Eq. (H12), we can rewrite Eq. (H6) as
\[
\frac{\partial \overline{f}}{\partial t} = D \left( \Delta_v \overline{f} - \Delta_\theta \overline{f} \right) = \frac{D}{v^2} \Delta_S \overline{f}, \tag{H18}
\]

where $\Delta_v$ is the Laplacian operator in velocity space, $\Delta_\theta = \frac{\partial^2}{\partial \theta^2} + \frac{d-1}{v} \frac{\partial}{\partial v}$, is the part of the Laplacian operator which involves derivatives with respect to the modulus of $v$, and $\Delta_S$ is the part of the Laplacian operator which involves derivatives with respect to the orientation of the vector $v$ (on the unit sphere).

In $d = 2$, introducing a polar system of coordinates, we have
\[
\Delta_S = \frac{\partial^2}{\partial \theta^2}. \tag{H19}
\]

The solution of Eq. (H18) is then
\[
\overline{f}(v, \theta, t) = \sum_{n=-\infty}^{+\infty} c_n(v) e^{in\theta} e^{-Dn^2 t/v^2} \tag{H20}
\]

with
\[
c_n(v) = \frac{1}{2\pi} \int_0^{2\pi} \overline{f}_0(v, \theta) e^{-in\theta} d\theta. \tag{H21}
\]

For $t \to +\infty$, we get
\[
\overline{f}(v, \theta, t) \to c_0(v) = \frac{1}{2\pi} \int_0^{2\pi} \overline{f}_0(v, \theta) d\theta = (\overline{f}_0(v, \theta))_\theta. \tag{H22}
\]

Therefore, $\overline{f}(v, t)$ tends to an isotropic DF which is equal to the average over the angle $\theta$ of the initial DF $\overline{f}_0(v) = f_0(v, \theta)$.
In $d = 3$, introducing a spherical system of coordinates, we have
\[
\Delta s = \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right).
\] (H23)

In that case, the solution of Eq. (H18) is
\[
\mathcal{T}(v, \theta, \phi, t) = \sum_{l=0}^{+\infty} \sum_{|m|=l} c_{lm}(v) Y_{lm}(\theta, \phi) e^{-D(l+1)t/v^2}
\] (H24)
with
\[
c_{lm}(v) = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \frac{\partial}{\partial \theta} \mathcal{T}_0(v, \theta, \phi) Y_{lm}^*(\theta, \phi),
\] (H25)
where $Y_{lm}(\theta, \phi)$ are the spherical harmonics. For $t \to +\infty$, we get
\[
\mathcal{T}(v, \theta, \phi, t) \to c_{00}(v) Y_{00}(\theta, \phi) = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \mathcal{T}_0(v, \theta, \phi) = \langle \mathcal{T}_0(v, \theta, \phi) \rangle_{\theta, \phi}.
\] (H26)

Therefore, $\mathcal{T}(v, t)$ tends to an isotropic DF which is equal to the average over the angles $\theta$ and $\phi$ of the initial DF $\mathcal{T}_0(v) = f_0(\theta, \phi, v)$.

4. Another type of equation

If we integrate Eq. (H2) by parts, we find that
\[
\beta(t) = \frac{d \int \mathcal{T} d\nu}{\int f_2 v^2 d\nu}.
\] (H27)

If we get rid of the integrals, we get
\[
\beta(v, t) = \frac{d \mathcal{T}}{f_2 v^2}.
\] (H28)

Substituting this relation into Eq. (H5) we obtain
\[
\frac{\partial \mathcal{T}}{\partial t} = D \frac{\partial}{\partial v} \cdot \left( \frac{\partial \mathcal{T}}{\partial v} + d\mathcal{T} \frac{v}{v^2} \right).
\] (H29)

Again, this is a closed equation. Equation (H29) conserves the energy $\langle \mathcal{E} \rangle$. Indeed:
\[
\dot{\mathcal{E}} = \int \frac{v^2}{2} \frac{\partial \mathcal{T}}{\partial t} d\nu = \int \frac{v^2}{2} D \frac{\partial}{\partial v} \cdot \left( \frac{\partial \mathcal{T}}{\partial v} + d\mathcal{T} \frac{v}{v^2} \right) d\nu = -\int D \left( \frac{\partial \mathcal{T}}{\partial v} + d\mathcal{T} \frac{v}{v^2} \right) \cdot v d\nu = -\int D \left( \frac{\partial \mathcal{T}}{\partial v} \cdot v + d\mathcal{T} \right) d\nu = 0,
\] (H30)
where the last equality is obtained after performing an integration by parts. By contrast, nothing general can be said about the sign of $H$.

Remark: Equation (H30) can be interpreted as a Smoluchowski equation describing the evolution of a Brownian particle coupled to a thermal bath of unit temperature $T_c = 1$ and submitted to an attractive logarithmic potential $U(v) = d \ln |v|$, where $v$ plays the role of the position $r$. The stationary solution $\mathcal{T}_\infty = A e^{-U(v)} = A/v^d$ is not normalizable (the normalization factor diverges logarithmically at both small and large velocities). From the general theory of Fokker-Planck equations, we know that Eq. (H30) satisfies an $H$-theorem ($\dot{\mathcal{E}} \leq 0$) for the free energy $F = \int \mathcal{T} U(v) d\nu + T_c \int \mathcal{T} \ln \mathcal{T} d\nu$. At the critical temperature $T_c = 1$, Eq. (H29) has the particularity to conserve the energy $E = (1/2) \int \mathcal{F} v^2 d\nu$ which is analogous to the moment of inertia if we make the correspondence $r \leftrightarrow v$ (see above). In this respect, Eq. (H30) can be interpreted as a form of virial theorem. The study of Eq. (H29) at $T \leq T_c$ is subtle because it displays a form of “collapse” or a form of Bose-Einstein condensation in the state $v = 0$ leading to a Dirac peak $\delta(v)$. This is an example of Bessel process that has been studied in, e.g., Ref. [133].
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