Passivity Analysis of Singular Neural Systems with Variable Delays

Ya-tao LAI¹,∗ and Xu-Y. LOU¹

¹Key Laboratory of Advanced Process Control for Light Industry (Ministry of Education)
Jiangnan University, Wuxi 214122, China
Laiyt_jn@163.com
*Corresponding author

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Abstract. In this paper, some sufficient conditions for singular Hopfield neural networks (SHNNs) with variable delays to be passive are presented. The passivity of delayed singular Hopfield neural networks without uncertainties is studied, and then the result is extended to the case with time-varying parametric uncertainties. The results are based on a Lyapunov functional construction. Numerical examples are also given to demonstrate the effectiveness of the theoretical results. In addition, these criteria possess important leading significance in design and applications of passivity analysis of SHNNs with variable delays.

Introduction

In many applications such as optimization, control and image processing, it is of prime importance to ensure that the designed neural network be stable. Stability of neural networks with time delay are problems of recurring interest and has received a lot of attention (see, for example [1-10]).

The passivity theory intimately related to the circuit analysis methods [11-12] has received a lot of attention from the control community since the 70s (see [13-16], to cite only a few). The passivity theory intimately related to the circuit analysis methods [11] has received a lot of attention in control theory. The passivity theory provides a nice tool for analyzing the stability of systems, and has found applications in diverse areas such as stability, complexity, signal processing, chaos control and synchronization, and fuzzy control. In [17], the authors studied the passivity properties for delayed neural networks and derived the passivity condition for delayed neural networks without uncertainties, and then extend the result to the case with time-varying parametric uncertainties.

In this paper we shall consider the passivity problem based stabilization of delayed singular Hopfield neural networks (DSHNNs) with or without uncertainties. The passivity conditions are presented in terms of linear matrix inequalities (LMIs), which can be easily solved by using the effective interior-point algorithm. The layout of this paper is as follows. Problem formulation and preliminaries are given in Section 2. In Section 3, our results are given to ascertain the passivity of DSHNNs based on Lyapunov method and we give concluding remarks of results. In Section 4, we present an illustrative example. Finally, conclusions are drawn in Section 5.

System Description and Preliminaries

We consider the following DSHNNs model

\[ E\dot{x}(t) = -Ax(t) + Bg(x(t)) + Cg(x(t - \tau(t))) + u(t), \quad (1) \]

where \( x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T \in \mathbb{R}^n \) is the neuron state vector, \( E \in \mathbb{R}^{n \times n} \) may be singular, that is, \( \text{rank}(E) = r \leq n \), \( A = \text{diag}(a_1, a_2, \cdots, a_n) \) is a positive diagonal matrix, \( B^{n \times m} \) and \( C^{n \times m} \) are interconnection weight matrices, \( 0 \leq \tau(t) \leq \tau_0 \) is the time delay, and it is assumed that \( 0 \leq \tau \leq \tau^* < 1 \). \( u(t) \) is the input vector, \( g(x) = [g(x_1), g(x_2), \cdots, g(x_n)]^T \) denotes the neuron activation function, and
we let \( y(t) = g(x(t)) \) be the output of the neural networks. As in many papers, we assume that each activation function in (1) satisfies the following sector condition:

\[
  g_j(x_j)(g_j(x_j) - k x_j) \leq 0, \quad j = 1, 2, \ldots, n
\]  

(2)

where \( k > 0 \) is a real constant.

Definition 1 [17]. The system (1) is called passive if there exists a scalar \( \gamma \) such that

\[
  2\int_0^t y^T(s)u(s)ds \geq -\gamma\int_0^t u^T(s)u(s)ds
\]  

(3)

for all \( t_p \geq 0 \) and for all solution of (1) with \( x_0 = 0 \).

**Passivity of DSHNNs without Uncertainties**

In this section, we analyze the passivity of delayed singular neural network (1) without uncertainties and give a sufficient condition.

**Theorem 1.** If there exist symmetric positive definite matrices \( P, Q > 0 \), a positive diagonal matrix \( D = \text{diag}(d_1, d_2, \ldots, d_n) \) and a scalar \( \gamma > 0 \) such that the following LMI holds

\[
  M_1 = \begin{bmatrix}
  -E^T PA - A^T PE & E^T PB & E^T PC & E^T P \\
  B^T PE & -2kDEA + DEB + B^T E^T D + Q & DEC & D - I \\
  C^T PE & C^T E^T D & (1 - \tau^*)Q & 0 \\
  PE & D - I & 0 & -\gamma I \\
  \end{bmatrix} < 0,
\]  

(4)

where \( I \) is the identity matrix of appropriate dimension. Then, the DSHNNs (1) is passive in the sense of Definition 1.

**Proof:** Consider a Lyapunov functional

\[
  V(t) = (Ex(t))^T PEx(t) + 2\sum_{i=1}^n d_i \sum_{j=1}^n e_{ij} \int_0^{\tau(t)} g_i(s)ds + \int_{\tau(t)}^t g^T(x(s))Qg(x(s))ds. 
\]  

(5)

Calculating the derivative of the Lyapunov functional \( V \) along the solution of (1), we obtain that
\[ \dot{V}(t) - 2y^T(t)u(t) - \gamma u^T(t)u(t) \]

\[ = (Ex(t))^TPEx(t) + (Ex(t))^TPEx(t) \]

\[ + 2g^T(x(t))DE\dot{x}(t) + g^T(x(t))Qg(x(t)) \]

\[ - (1 - \dot{\tau}(t))g^T(x(t - \tau(t)))Qg(x(t - \tau(t))) \]

\[ - 2y^T(t)u(t) - \gamma u^T(t)u(t) \]

\[ \leq -x^T(t)(E^TPA + A^TPE)x(t) + 2x^T(t)E^TPB(x(t)) \]

\[ + 2x^T(t)E^TPCg(x(t - \tau(t))) + x^T(t)E^TPu(t) + u^TPEx(t) \]

\[ - 2g^T(x(t))DEAx(t) + 2g^T(x(t))DEB(x(t)) \]

\[ + 2g^T(x(t))DECG(x(t - \tau(t))) + 2g^T(x(t))DEu(t) \]

\[ + g^T(x(t))Qg(x(t)) - (1 - \tau^*)g^T(x(t - \tau(t)))Qg(x(t - \tau(t))) \]

\[ - 2g^T(x(t))u(t) - \gamma u^T(t)u(t). \]  \hspace{1cm} (6)

Let \( z = [x^T(t), g^T(x(t)), g^T(x(t - \tau(t))), u^T(t)]^T \), we get

\[ \dot{V}(t) - 2y^T(t)u(t) - \gamma u^T(t)u(t) \leq z^TM_1z. \]  \hspace{1cm} (7)

Using the LMI (4), it follows

\[ \dot{V}(t) - 2y^T(t)u(t) - \gamma u^T(t)u(t) < 0. \]  \hspace{1cm} (8)

By integrating (8) with respect to \( t \) over the time period \( 0 \rightarrow t_p \), we have

\[ 2\int_0^{t_p} y^T(s)u(s)ds \geq V(x(t_p)) - V(x(0)) - \gamma \int_0^{t_p} u^T(s)u(s)ds \]  \hspace{1cm} (9)

for \( x(0) = 0 \), we have \( V(x(0)) = 0 \), so (3) holds, and hence the neural network (1) is passive in the sense of Definition 1.
Passivity of DSHNNs with Parametric Uncertainties

In this section, we extend the result of the above section to the case with time-varying parametric uncertainties, that is, we consider the following DSHNNs

\[
\dot{x}(t) = -(A + \Delta A(t))x(t) + (B + \Delta B(t))g(x(t)) + (C + \Delta C(t))g(x(t - \tau(t))) + u(t),
\]

where \( \Delta A(t), \Delta B(t), \Delta C(t) \) are time-varying parametric uncertainties and are defined by

\[
\Delta A(t) = L_0 F_0(t) H_0, \quad \Delta B(t) = L_1 F_1(t) H_1, \quad \Delta C(t) = L_2 F_2(t) H_2,
\]

where \( L_0, L_1, L_2, H_0, H_1, H_2 \) are known constant matrices of appropriate dimensions, and \( F_0(t), F_1(t), F_2(t) \) are unknown time-varying matrices with Lebesgue measurable elements bounded by

\[
F_0^T(t) F_0(t) \leq I, \quad F_1^T(t) F_1(t) \leq I, \quad F_2^T(t) F_2(t) \leq I.
\]

Theorem 2. If there exist symmetric positive definite matrices \( P, Q > 0 \), a positive diagonal matrix \( D = \text{diag}(d_1, d_2, \ldots, d_n) \), and a scalar \( \gamma > 0 \) such that the following LMI holds

\[
M_2 = \begin{bmatrix}
-E^T PA - A^T PE & E^T PB & E^T PC & -E^T PL_0 & E^T PL_1 & E^T PL_2 & E^T P \\
B^T PE - DA & D E^T C & (1 - \tau^*) Q & 0 & 0 & 0 & 0 \\
C^T PE & D^T E^T C & (1 - \tau^*) Q & 0 & 0 & 0 & 0 \\
-L_0^T PE & -L_0^T E^T D & 0 & 0 & 0 & 0 & 0 \\
L_1^T PE & L_1^T E^T D & 0 & 0 & 0 & 0 & 0 \\
L_2^T PE & L_2^T E^T D & 0 & 0 & 0 & 0 & 0 \\
PE & D - I & 0 & 0 & 0 & 0 & -\gamma I \\
\end{bmatrix} < 0,
\]

then the DSHNNs (1) is passive in the sense of Definition 1.

Proof: Consider a Lyapunov functional as

\[
V(t) = (Ex(t))^T PEx(t) + 2 \sum_{i=1}^{n} d_i \sum_{j=1}^{n} e_{ij} \int_{0}^{x(t)} g_i(s)ds + \int_{t-\tau(t)}^{t} g^T(x(s))Qg(x(s))ds.
\]

Then, we have that
\[ V(t) - 2y^T(t)u(t) - \gamma u^T(t)u(t) \]
\[ = (Ex(t))^T P\dot{x}(t) + (Ex(t))^T PEx(t) \]
\[ + 2g^T(x(t))DE\dot{x}(t) + g^T(x(t))Qg(x(t)) \]
\[ -(1-\tau(t))g^T(x(t-\tau(t)))Qg(x(t-\tau(t))) \]
\[ -2y^T(t)u(t) - \gamma u^T(t)u(t) \]
\[ \leq -x^T(t)(E^T PA + A^T PE)x(t) + 2(Ex(t))^T PBg(x(t)) \]
\[ + 2(Ex(t))^T PCg(x(t-\tau(t))) + (Ex(t))^T Pu(t) + u^T P(Ex(t)) \]
\[ -x^T(t)(E^T PL_0F_0^T(t)H_0 + (L_0F_0(t)H_0)^T PEx(t) \]
\[ + (Ex(t))^T P(L_1F_1^T(t)H_1)g(x(t)) + g^T(x(t))(L_1F_1^T(t)H_1)^T P(Ex(t)) \]
\[ + (Ex(t))^T P(L_2F_2^T(t)H_2)g(x(t-\tau(t))) \]
\[ + g^T(x(t-\tau(t)))(L_2F_2^T(t)H_2)^T P(Ex(t)) \]
\[ + (Ex(t))^T Pu(t) + u^T P(Ex(t)) \]
\[ -2g^T(x(t))DEAx(t) - 2g^T(x(t))DE(L_0F_0(t)H_0)x(t) \]
\[ + 2g^T(x(t))DEB(x(t)) + 2g^T(x(t))DE(L_1F_1^T(t)H_1)g(x(t)) \]
\[ + 2g^T(x(t))DE(L_2F_2^T(t)H_2)g(x(t-\tau(t))) \]
\[ + 2g^T(x(t))DECg(x(t-\tau(t))) + 2g^T(x(t))Du(t) \]
\[ + g^T(x(t))Qg(x(t)) - (1-\tau^*)g^T(x(t-\tau(t)))Qg(x(t-\tau(t))) \]
\[ -2y^T(t)u(t) - \gamma u^T(t)u(t). \quad (15) \]

Let
\[
z = [x^T(t), g^T(x(t)), g^T(x(t-\tau(t))), (F_0^T(t)H_0x(t))^T, \]
\[
(F_1^T(t)H_1g(x(t)))^T, (F_2^T(t)H_2g(x(t-\tau(t))))^T, u^T(t)]^T.
\]

One can have
\[ V(t) - 2y^T(t)u(t) - \gamma u^T(t)u(t) \leq z^TMz. \quad (16) \]

Using the LMI (13), it follows that
\[ V(t) - 2y^T(t)u(t) - \gamma u^T(t)u(t) < 0. \quad (17) \]
By integrating (17) with respect to $t$ over the time period $0 \sim t_p$, we have

$$2\int_0^{t_p} y^T(s)u(s)ds \geq V(x(t_p)) - V(x(0)) - \gamma \int_0^{t_p} u^T(s)u(s)ds$$

(18)

for $x(0) = 0$, we have $V(x(0)) = 0$, so (3) holds, and hence the neural network (10) is passive in the sense of Definition 1.

**Illustrative Example**

Example 1. Consider a delayed singular Hopfield neural network (1) without uncertainties, whose activation function is $y_j = g(x_j(t)) = \tanh(x_j(t))$, $j = 1, 2, 3$. Obviously, this activation function satisfies the sector condition (2) with $k \equiv 1$. We let $u_1 = u_2 = u_3 = 1$, $v_1 = v_2 = v_3 = 2$,

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0.1 & 0.5 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}.$$

and let the time delay be $\tau(t) = 0.5 + 0.1\cos(t)$. So, we have $\tau^* = 0.1 < 1$. Applying Theorem 1 to this example and using the MATLAB LMI toolbox, we can obtain the following feasible solutions for the LMI (4)

$$P = \begin{bmatrix} 0.3796 & -0.0599 \\ -0.0599 & 0.3796 \end{bmatrix} > 0, Q = \begin{bmatrix} 1.2258 & -0.2007 \\ -0.2007 & 1.1715 \end{bmatrix} > 0, D = \begin{bmatrix} 0.7837 & 0 \\ 0 & 1.0991 \end{bmatrix} > 0,$$

and $\gamma = 1.2151 > 0$,

which means that the above neural network is passive in the sense of Definition 1.

**Conclusions**

In this paper, the problem of passivity analysis for singular Hopfield neural networks with variable delays is investigated. Based on Lyapunov stability theory and some analysis techniques, the passivity conditions are given in terms of linear matrix inequalities. The proposed approach is more flexible in computation. A numerical example is also given to illustrate the effectiveness of the theoretical results. In addition, these criteria possess important leading significance in design and applications of passivity analysis of SHNNs with variable delays.

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