Random Paraunitary Projections

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Abstract—Transforms using random matrices have been found to have many applications. We are concerned with the projection of a signal onto Gaussian-distributed random orthogonal bases. We also would like to easily invert the process through transposes in order to facilitate iterative reconstruction. We derive an efficient method to implement random unitary matrices of larger sizes through a set of Givens rotations. Random angles are hierarchically generated on-the-fly and the inverse merely requires traversing the angles in reverse order. Hierarchical randomization of angles also enables reduced storage. Using the random unitary matrices as building blocks we introduce random paraunitary systems (filter banks). We also highlight an efficient implementation of the paraunitary system and of its inverse. We also derive an adaptive under-decimated system, wherein one can control and adapt the amount of projections the signal undergoes, in effect, varying the sampling compression ratio as we go along controlling and adapting the amount of projections the signal undergoes, in effect, varying the sampling compression ratio as we go along. We can completely avoid storing or calculating the matrix $M$, so that the resulting matrix $U$ is perfectly unitary. Nevertheless, the statistical correlation and probability distribution function (PDF) of the samples of $U$ are not trivially found from the PDF of $\theta_{ij}$ [8],[9]. If $\theta$ is uniformly distributed, the PDFs of $\cos(\theta)$ and $\sin(\theta)$ are biased and the many operations derived from the successive plane rotations tend to further concentrate the samples. Under certain restrictions it can be shown that [9] each angle is randomly chosen according to the following distribution:

$$p(\theta_{ij}) = \frac{\Gamma \left( \frac{L-1}{2} \right)}{\pi^{\frac{L}{2}}} \Gamma \left( \frac{L-1}{2} \right) \cos^{j-1} \theta_{ij},$$

(2)

then the resulting PDF would tend to be Gaussian. Also, of great importance to us is the autocorrelation of the resulting matrix, which ideally should be an impulse. The trivial approach is to generate random matrix entries in order to ensure decorrelation. Even though we use an indirect method, generating random rotation angles, our simulations show that even though we use an indirect method, the statistical correlation and probability distribution function (PDF) of the samples of $U$ are not trivially found from the PDF of $\theta_{ij}$ [8],[9]. If $\theta$ is uniformly distributed, the PDFs of $\cos(\theta)$ and $\sin(\theta)$ are biased and the many operations derived from the successive plane rotations tend to further concentrate the samples. Under certain restrictions it can be shown that [9] each angle is randomly chosen according to the following distribution:

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then the resulting PDF would tend to be Gaussian. Also, of great importance to us is the autocorrelation of the resulting matrix, which ideally should be an impulse. The trivial approach is to generate random matrix entries in order to ensure decorrelation. Even though we use an indirect method, generating random rotation angles, our simulations show that the above distribution of angles leads to decorrelated near-perfect Gaussian-distributed entries.

It is a case of interest to project an $M$-tuple $x$ into $N << M$ bases, i.e. an orthogonal $N \times M$ Gaussian random transform $A$, so that $y = Ax$. This can be easily accomplished through pruning the unnecessary rotations. One can obtain $A$ through generating $U$ and discarding $M - N$ rows. For that, it can be shown that we may also discard the last $M - N - 1$ stages in [1]. Its transpose (rather than its inverse transform) $x = A^T y$, requires traversing the rotations backwards from $N$ samples to an approximation of the original $M$-tuple.

Low-memory implementation

If we implement the transform through in-place rotations, we can completely avoid storing or calculating the matrix entries. Angles are generated as they are used in sequence following (1). Each stage (rotation) demands calculating $\cos(\theta_{ij})$ and $\sin(\theta_{ij})$, 4 multiplications and 2 additions. There is no need for matrix inversion and we only store the input vector. The inverse transform is accomplished by reversing the order of the planes and by applying negative rotations. However, the angles would have to be generated in reverse order which normally would require to buffer all the random angles. Let the set of $\ell = M(M-1)/2$ angles be $\Theta = \{\theta_0, \ldots, \theta_{\ell-1}\}$. We
divide the set of angles into \( N_s \) subsets \( \Theta \), of \( \ell/N_s \) angles each as \( \Theta = \{ \Theta_1, \Theta_2, \ldots, \Theta_{N_s} \} \). We first generate a seed \( S_0 \), from which we generate \( N_s \) seeds \( S_k \), \( 1 \leq k \leq N_s \). Each of the subsets \( \Theta_k \) is then randomly generated using seed \( S_k \). For the inverse transform angles and seeds, one would need to convey \( S_0 \) to both forward and inverse transform stages and to buffer only the set of seeds \( \{ S_k \} \). At a time, only \( \ell/N_s \) angles for one subset need to be generated and visited in reverse order. Thus, the only storage required would be necessary for \( 3 \) vectors: \( \Theta \), \( \frac{\ell}{N} \), and \( s \). Input/output, seeds, and random subgroup, of sizes, \( \ell, N_s \), respectively. If \( N_s = M \), then total storage is roughly \( 2.5M \). This makes it possible to make random orthogonal projections (and their inverses) of very large vectors.

As a note, we disregarded the computation and storage to produce the random angles with PDF given by (2).

### III. Random Paraunitary Systems

An \( M \)-input, \( M \)-output \((M \times M)\) paraunitary system \( H(z) \) of order \( K \) can be constructed using a cascade of \( K \) simpler order-1 paraunitary systems:

\[
H(z) = U_{0} \prod_{i=1}^{K} A(z) U_{i},
\]

where \( A(z) = \text{diag}\{1, \ldots, 1, z^{-1}, \ldots, z^{-1}\} \) and \( U_{i} \) are unitary matrices.

We can make a random, Gaussian, paraunitary system if we cascade random unitary matrices, i.e. if we make \( U_{i} \) random unitary matrices as explained in the previous Section. To see that we begin by showing that \( H(z) \) has random independent entries \( \{h_{i,j}(z)\} \). Let \( H(z) = H'(z) U_{i} \) such that \( h_{i,j}(z) = \sum_{k} h_{i,k}(z) u_{n,k,j} \) and that the \( \{ u_{n,i,j} \} \) have zero mean. Then, \( E(h_{i,j}(z) h_{k,l}(z)) = \sum_{uv} E(h_{i,v}(z) h_{k,u}(z)) E(u_{i,v} u_{n,k,u}) = 0 \) if \( i \neq j, k \neq \ell \) and if the entries \( \{ h_{i,j}' \} \) are independent. Since \( U_{0} \) has independent entries, so does \( U_{0} A(z) \) and, hence, the same applies to \( H(z) \) for any order.

Let \( F(z) = \sum_{i=0}^{n} F_{i} z^{-i} \) be a system of order \( n \), obtained through appending an order-1 stage to \( E(z) = \sum_{i=0}^{n} E_{i} z^{-i} \), i.e. \( F(z) = E(z) A(z) U_{n} \). Let \( A(z) U_{n} = U_{n} + z^{-1} U_{n}' \). Then, we get

\[
F_{k} = E_{k} U_{n}' + E_{k-1} U_{n}'' \quad 0 < k < n
\]

\[
F_{0} = E_{0} U_{n}' \quad F_{n} = E_{n-1} U_{n}'.
\]

The entries \( \{ f_{k,j} \} \) of \( F(z) \) are sums of products of random variables. For large \( M \), it is expected that the distribution of \( \{ f_{k,j} \} \) approaches a Gaussian PDF. As we can make all \( U_{i} \) to have Gaussian-distributed entries as in Section II, we expect all \( \{ h_{k,j} \} \) entries in \( H(z) = \sum_{i=0}^{n} H_{i} z^{-i} \) to be approximately Gaussian distributed, for large \( M \).

Let \( A^{H} \) denote the Hermitian of matrix \( A \), i.e. its transposed conjugate. Also, let the subscript \( A^{(s)}(z) \) denote to conjugate only the coefficients of its polynomials, i.e. if \( A(z) \) has entries \( \{ \sum_{n} a_{n} z^{-n} \} \), then \( A^{(s)}(z) \) has entries \( \{ \sum_{n} a_{n}^{*} z^{-n} \} \). Then, the inverse of a paraunitary system \( H(z) \) is simply its para-conjugated version \( H(z) \).

\(1\)We mean the order of the system as the highest polynomial degree of its entries, and not the McMillan degree of the system \( [10] \).

\[
H^{-1}(z) = H^{T(*)}(1/z) = \sum_{i=0}^{K} H_{i}^{H} z^{i}.
\]

A paraunitary \( H(z) \) is, thus, unitary on the unit circle.

#### Low-memory implementation

The system can be easily implemented using \( A \), wherein each random unitary stage \( U_{i} \) can be implemented using the techniques described in the previous section. Each \( M \)-sample vector of the input signal undergoes a chain of random transforms, for each \( U_{i} \) interspersed with shuffles \( A(z) \). One has to budget memory to store the \( KM/2 \) internal states, in between the random unitary transforms. The inverse transform can be accomplished by traversing the system backwards. Let \( A(z) = z^{-1} A(1/z) \), such that

\[
\tilde{A}(z) A(z) = z^{-1} I.
\]

Let

\[
G(z) = U_{0}^{H} \prod_{i=1}^{K} \tilde{A}(z) U_{i}^{H},
\]

Then, \( G(z) H(z) = z^{K} I \) and the above inverse has the same structure as \( A \). Hence, both the forward and inverse transforms have the same implementation, with the proper adaptation of the shuffles in between stages of unitary transforms. The transpose (inverse) of a unitary transform can be accomplished by going through the plane rotations in reverse order, with inverse angles.

As a note, the order can be changed on-the-fly by adding or removing stages to the cascade of unitary transforms.

### IV. Adaptive Sampling Matrices

The \( M \times M \) system \( H(z) \) can be seen as an \( M \)-channel filter bank, where each filter is down-sampled by a factor of \( M \). The filters have length \( L = (K+1)/M \). In such, \( M \) samples enter the system and \( M \) samples leave it at a time, in what is referred as a critically decimated system. If we increase the decimation, the systems becomes over-decimated, and not invertible. Let us denote as \( H_{ijn}(z) \) the representation of the paraunitary system, but with down-sampling by a factor of \( m \). Then,

\[
H_{ijm}(z) = H(z) = \sum_{i=0}^{K} H_{i} z^{-i}
\]

\[
H_{i2j}(z) = \sum_{i=0}^{[K/2]} [H_{2i}, H_{2i+1}] z^{-i}
\]

where \( [x] \) is the “floor” rounding operation. For a more general down-sampling factor,

\[
H_{iqm}(z) = \sum_{i=0}^{[K/q]} [H_{q1}, H_{q1+1}, \ldots, H_{q1+q-1}] z^{-i}
\]

where \( H_{0} = 0 \) for \( n > K \). In the extreme case, if \( q = K + 1 \), i.e. \( \downarrow 1 \), then
Fig. 1. A $K = 4$ paraunitary system lattice and diagram illustrating the blocks ignored in the ↓ $L$ case (white filled). Each branch carries $M/2$ samples.

$$H_{1:L}(z) = [H_0, H_1, \ldots, H_K], \quad (8)$$

i.e. the under-sampled matrix becomes a (scalar) sampling matrix.

There are $qM$ input samples per $M$ output ones, i.e. $q$ is the sampling compression parameter. The down-sampling and, thus, $q$ can be changed on the fly. The system can go from $M$ input to $M$ output samples per clock, to an $L$-to-$M$ one, and all stages in between. In all cases, the bases are random in nature and are approximately Gaussian distributed as discussed in the previous Section. Hence, $q$ can be made adaptive, perhaps adapting how compressive the sensing is, in order to track local sparseness of non-stationary signals.

**Low-memory implementation**

Starting from a ↓ $M$ system in (3), a ↓ $qM$ system can be implemented by picking one out of every set of $q$ output blocks of $M$ samples. If we view (3) as a flow graph, the system can then be more efficiently implemented by simply pruning out the stages which are not necessary and will not be used to produce output. Figure 1 illustrates a case $K = 4$ and indicates unitary stages to be pruned in the ↓ $L$ (sampling matrix) case. It can be shown that maximum pruning yields computational savings in the order of $K - (2K)^{-1}$. The reverse (transpose) system can be accomplished through reversing the system, i.e. using (4) and setting to zero the samples which are not produced using the over-decimated system.

**V. Conclusions**

In this letter, we present a method to construct Gaussian-distributed random paraunitary systems. The system can be dynamically applied in order to create an adaptive compressive sensing framework wherein sampling is compressed by a factor of $q$ which can be changed on-the-fly. The bases are unitary in nature and the projection can be easily implemented and reversed. We discussed the efficient implementation of random unitary matrices, which are used as building blocks to construct the random paraunitary filter banks. An efficient implementation of such filter banks is also discussed. Applications of such an adaptive system are being investigated.

**References**

[1] M. L. Mehta, *Random Matrices*, Academic Press: San Diego, CA, USA, 1990.
[2] R. Baraniuk, “Compressive sensing,” *IEEE Signal Processing Magazine*, Vol. 24, no. 4, pp. 118–121, July 2007.
[3] D. Needell and J. A. Tropp, *COSAMP: Interactive Signal Recovery from Incomplete and Inaccurate Samples*, Technical Report, California Institute of Technology, 2008.
[4] F. E. Hohn, *Elementary Matrix Algebra*, Second Edition, New York, NY: MacMillan, 1964.
[5] J. Wishart, “The generalised product moment distribution in samples from a normal multivariate population,” *Biometrika*, Vol. 20A, pp. 32–52, 1928.
[6] F. Mezzadri, “How to generate random matrices from the classical compact groups,” *Notices of the American Mathematical Society*, Vol. 54, No. 5, pp. 592–604, May 2007.
[7] G. W. Stewart, “The efficient generation of random orthogonal matrices with an application to condition estimators,” *SIAM Journal on Numerical Analysis*, Vol. 17, No. 3, pp. 403–409, June 1980.
[8] T. W. Anderson, I. Olkin, and L. G. Underhill, “Generation of random orthogonal matrices,” *SIAM Journal on Scientific and Statistical Computing*, Vol. 8, No. 4, July 1987.
[9] W. D. Heiss, “Distributions of angles of a random unit vector and random orthogonal matrices,” *Zeitschrift Fur Physik A*, 349, pp. 9–12, 1994.
[10] P.P. Vaidyanathan, *Multirate Systems and Filter Banks*. Englewood Cliffs, NJ: Prentice-Hall, 1993.