"Discrete" vacuum geometry as a tool for Dirac fundamental quantization of Minkowskian Higgs model.

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Abstract

We demonstrate that assuming the "discrete" vacuum geometry in the Minkowskian Higgs model with vacuum BPS monopole solutions can justify the Dirac fundamental quantization of that model.

The important constituting of this quantization is getting various rotary effects, including collective solid rotations inside the physical BPS monopole vacuum, and just assuming the "discrete" vacuum geometry seems to be that thing able to justify these rotary effects.

More precisely, assuming the "discrete" geometry for the appropriate vacuum manifold implies the presence of thread topological defects (side by side with point hedgehog topological defects and walls between different topological domains) inside this manifold in the shape of specific (rectilinear) threads: gauge fields located in the spatial region intimately near the axis \( z \) of the chosen (rest) reference frame.

Just around these (topologically nontrivial) threads, collective solid rotations proceed inside the BPS monopole vacuum suffered the Dirac fundamental quantization.

The presence of topologically nontrivial threads inside the discrete vacuum manifold involves the effect annihilating equal magnetic charges, say \( \mathbf{m}_1 = \mathbf{m}_2 = \mathbf{m} \), colliding crossing a topologically nontrivial thread.

On the other hand, in the Higgs sector of the discussed model, it is possible to construct solutions joining in a smooth wise Higgs vacuum BPS monopole solutions.

This submit the collective solid rotations inside the physical BPS monopole vacuum to the Gauss-shell reduction of the Minkowskian Higgs model with vacuum BPS monopoles.

It will be argued that indeed the first-order phase transition occurs in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac.
This comes to the coexistence of two thermodynamic phases inside the appropriate BPS monopole vacuum.

There are the thermodynamic phases of collective solid rotations and superfluid potential motions.

The important consequence of assuming the "discrete" geometry for the vacuum manifold in the Minkowskian Higgs model (with vacuum BPS monopoles) quantized by Dirac is supplanting the widespread notion about the confinement in Yang-Mills QCD as provoked (for instance, in the maximal Abelian gauge has been fixed) by center ($\mathbb{Z}_2$) vortices by that taking place against the background of the countable ($O(\mathbb{Z})$) number of topological domains inside the discrete vacuum manifold and the presence therein of topologically nontrivial threads.

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1 Introduction.

In the recent studies [1, 2, 3], the Gauss-shell reduction of the Minkowskian Higgs model with vacuum BPS monopole solutions was discussed.

Such Gauss-shell reduction is, indeed, the particular case of the Dirac fundamental quantization [4] of gauge theories, when such (Minkowskian) gauge model is written down in terms of topological Dirac variables $A^D_i$ ($i = 1, 2$): transverse and gauge invariant (i.e. automatically physical) functionals of gauge fields.

As a result of this Gauss-shell reduction, the picture of the BPS monopole vacuum as a medium possessing manifest superfluid properties and various rotary effects was given.

More exactly, superfluid properties of the BPS monopole vacuum suffered the Dirac fundamental quantization [4] come to its potentiality, described correctly by the Bogomol’nyi [1, 2, 6, 7, 8, 9, 10, 11],

$$B = \pm D\Phi,$$

(1.1)

and Gribov ambiguity [2, 7, 8, 12],

$$[D^2_k(\Phi^{(0)}_a)]^{ab}\Phi_{(0)b} = 0,$$

(1.2)

equations.

These Eqs. implicate the vacuum "magnetic" field $B$ and Higgs vacuum BPS monopole modes $\Phi$ (in particular, topologically trivial modes $\Phi_{(0)a}$).

$\Phi_k^{(0)}$ are Yang-Mills (YM) vacuum BPS monopole modes, and $D$ is the (covariant) derivative.

The connection between the Bogomol’nyi and Gribov ambiguity equations is realised via the Bianchi identity $D B = 0$ (the latter identity is equivalent to say that the vacuum "magnetic" field $B$ is "transverse").

In Ref. [6] it was pointed out to the transparent analogy between the BPS monopole vacuum and a liquid helium II specimen, possessing the manifest superfluidity.

This analogy comes, in particular, to the mathematical resemblance between the Bogomol’nyi equation (1.1) and the expression [6, 13]

$$v_0 = \frac{\hbar}{m}\nabla\Phi(t, r),$$

(1.3)

for the critical velocity $v_0$ of superfluid potential motions inside a liquid helium II specimen 1.

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1 $v_0 = \min (\epsilon/p)$ for the ratio of the energy $\epsilon$ and momentum $p$ for quantum excitations spectrum in the given liquid helium II specimen. Herewith at velocities of the liquid exceeding a critical velocity $v_0 = \min (\epsilon/p)$ for the ratio of the energy $\epsilon$ and momentum $p$ for quantum excitations spectrum in the liquid helium II, the dissipation of the liquid helium energy occurs via arising excitation quanta with momenta $p$ directed antiparallel to the velocity vector $v$. Such dissipation of the liquid helium energy becomes advantageous just at

$$\epsilon + pv < 0 \Rightarrow \epsilon - pv < 0.$$
Eq. (1.3) implicates the phase \( \Phi(t, r) \) of the complex-value helium Bose condensate wave function \( \Xi(t, r) \in \mathbb{C} \), given as [6, 13]
\[
\Xi(t, r) = \sqrt{n_0(t, r)} \ e^{i\Phi(t, r)},
\]  
(1.4)

with \( n_0(t, r) \) being the number of particles in the ground energy state \( \epsilon = 0 \).
m is the mass of a helium atom.
On the other hand, for any potential motion, the condition [6]
\[
\text{rot grad } \Phi = 0 
\]  
(1.5)
is always fulfilled (to within a constant) for a scalar field \( \Phi \).
In particular, the potentiality condition (1.5) is fulfilled for the crucial velocity \( \mathbf{v}_0 \), (1.3), of superfluid potential motions inside a liquid helium II, i.e.
\[
\text{rot } \mathbf{v}_0 = 0.
\]
This implies the absence of vortices and friction forces inside the superfluid component in a liquid helium II specimen.
The similar conclusion may be drawn also about the vacuum "magnetic" field \( \mathbf{B} \) in the Minkowskian Higgs model with vacuum BPS monopole solutions described correctly by the Bogomol’nyi equation [1, 2, 6, 7, 8, 9, 10, 11] (with the correction that in the latter case the operator grad would be replaced with the [covariant] derivative \( D \)).
Additionally, the Gribov ambiguity equation (Gribov equation) (1.2) describes the Minkowskian BPS monopole vacuum as an incompressible medium.
This fact was discussed in Ref. [2]; it comes to the presence of typical topological invariants in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [4].
There are the magnetic charge \( m \) and the degree of the map referring to the \( U(1) \subset SU(2) \) embedding.
At resolving the YM Gauss law constraint [1, 2, 3, 7, 8, 12]
\[
\frac{\delta W}{\delta A_0^c} = 0 \iff [D^2(A)]^{ac}A_0^c = D_i^{ac}(A)\partial_0 A_i^c 
\]  
(1.6)
in terms of topological Dirac variables \( A_i^D \), satisfying the gauge [5]
\[
D_i^{ab}(A^D)\partial_0 (A_i^D) = 0, 
\]  
(1.7)
the right-hand side of this constraint vanishes, and it becomes the homogeneous equation [8, 7, 8, 12]
\[
[D_i^2(\Phi^{(0)})]^{ac}A_0^c = 0, 
\]  
(1.8)
permitting, in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [4], the family of zero mode solutions [12, 16],
\[
A_0^c(t, \mathbf{x}) = \dot{N}(t)\Phi^{(0)}(\mathbf{x}) \equiv Z^c, 
\]  
(1.9)
implicating the topological variable $\dot{\bar{N}}(t)$ and Higgs (topologically trivial) vacuum Higgs BPS monopole modes $\Phi_0^a(\mathbf{x})$.

The topological variables $\bar{N}(t)$ (respectively, $N(t)$), may be specified [3, 7, 8, 12] via the relation

$$\nu[A_0, \Phi(0)] = \frac{g^2}{16\pi^2} \int_{t_{\text{in}}}^{t_{\text{out}}} dt \int d^3x F_{\mu\nu}^a \tilde{F}^{a\mu\nu} = \frac{\alpha_s}{2\pi} \int d^3x F_{i0}^a B_i^a(\Phi(0))[N(t_{\text{out}}) - N(t_{\text{in}})]$$

$$= N(t_{\text{out}}) - N(t_{\text{in}}) = \int_{t_{\text{in}}}^{t_{\text{out}}} dt \dot{\bar{N}}(t),$$

(1.10)

taking account of the natural duality between the tensors $F_{i0}^a$ and $F_{ij}^a$.

Herewith $\nu[A_0, \Phi(0)]$ is referred to as the vacuum Chern-Simons functional, implicating the asymptotical states "in" and "out" taking in the time instants $t_{\text{in}}$ and $t_{\text{out}}$, respectively.

As it was discussed in Ref. [3] (repeating the arguments [17]), it may be set

$$t_{\text{in}} \to -\infty; \quad t_{\text{out}} \to \infty.$$ 

The nontrivial topological dynamics inherent in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac comes to the specific Josephson effect [17] in the enumerated model, i.e. to the existence of collective solid rotations inside the physical BPS monopole vacuum.

These collective solid rotations are described correctly by the free rotator action functional

$$W_N = \int d^4x \frac{1}{2} (F_{0i}^c)^2 = \int dt \frac{\dot{N}^2 I}{2}$$

(1.11)

involving the rotary momentum [5]

$$I = \int_V d^3x (D_i^c(\Phi_0^k)\Phi_{0c})^2 = \frac{4\pi^2\epsilon}{\alpha_s} = \frac{4\pi^2}{\alpha_s^2} \frac{1}{V <B^2>}.$$ 

(1.12)

The YM coupling constant

$$\alpha_s = \frac{g^2}{4\pi(hc)^2}$$

enters this expression for $I$ together with the typical size $\epsilon$ [7, 8] of BPS monopoles and the vacuum expectation value $<B^2>$ of the "magnetic" field.

In Ref. [3] there was argued that $\dot{\bar{N}}(t)$ play the role of angular velocities of collective solid rotations inside the Minkowskian Higgs vacuum suffered the Dirac fundamental quantization [4].

Meanwhile, the action functional (1.11) contains vacuum "electric" monopoles (in the terminology [7, 8])

$$F_{i0}^a = \dot{\bar{N}}(t) D_i^c(\Phi_0^k)\Phi_{0c}(\mathbf{x})$$

(1.13)
and is associated with the real spectrum

\[ P_N = \dot{N} I = 2\pi k + \theta; \quad \theta \in [-\pi, \pi]; \quad k \in \mathbb{Z}; \] (1.14)

of the topological momentum \( P_N \).

The one of manifestations of the Josephson effect \([17]\) occurring inside the Minkowskian Higgs vacuum suffered the Dirac fundamental quantization are the existence of never vanishing (excluding the value \( \theta = 0 \) of the \( \theta \)-angle) vacuum “electric” fields (“electric” monopoles) \([13]\) in the shape \([5]\)

\[ F^a_{i0} \equiv E^a_i = \dot{N}(t) (D_i(\Phi_k(0)) \Phi(0))^a = P_N \frac{\alpha_s}{4\pi^2 \epsilon} B^a_i(\Phi(0)) = (2\pi k + \theta) \frac{\alpha_s}{4\pi^2 \epsilon} B^a_i(\Phi(0)). \] (1.15)

Any vacuum “electric” field \((1.15)\) achieves its minimum in the zero topological sector of the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac:

\[ (E^a_i)_{\min} = \theta \frac{\alpha_s}{4\pi^2 \epsilon} B^a_i; \quad -\pi \leq \theta \leq \pi. \] (1.16)

Such is, briefly, the picture of the (topologically nontrivial) dynamics inherent in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac \([4]\).

This picture seems to be correct at least at the absolute zero temperature \( T = 0 \), when collective solid rotations inside the BPS monopole vacuum proceed in the “non-stop” regime \([17]\) and “friction forces” between this BPS monopole vacuum and its surroundings are absent.

In Ref. \([3]\), there was asserted that ”geometrically”, collective solid rotations inside the BPS monopole vacuum suffered the Dirac fundamental quantization \([4]\) are, indeed, rotations around the infinitely narrow cylinder of the effective diameter \( \sim \epsilon \) (with \( \epsilon \) being the typical size of BPS monopoles) along the axis \( z \) of the chosen (rest) reference frame \([2]\).

In the present study we attempt to ground this assertion.

This will be associated immediately with assuming the ”discrete vacuum geometry” \([1, 3]\)

\[ R_{YM} \equiv SU(2)/U(1) \simeq \mathbb{Z} \otimes G_0/U_0, \]

with

\[ \pi_1(U_0) = \pi_1(G_0) = 0 \]

and

\[ SU(2) \equiv G; \quad U(1) \equiv U, \]

for the degeneration space (vacuum manifold) in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac, involving \([9]\) the existence of (rectilinear, topologically nontrivial) threads inside the mentioned vacuum manifold, just located intimately near the axis \( z \) of the chosen (rest) reference frame.

\(^2\)For the first time, the idea of ”cylinder topologies” in (non-Abelian) gauge theories was proposed, probably in the paper \([18]\) (where also the topological dynamical variable \( N(t) \) was introduced).
Vortices generated by such threads, similar to those [15] one can observe in a (rested) liquid helium II specimen, may be identified naturally with collective solid rotations inside the BPS monopole vacuum suffered the Dirac fundamental quantization [4].

Unlike the spatial region near the axis \( z \), far off this axis (including the region \( |x| \to \infty \)), superfluid potential motions inside the BPS monopole vacuum occur.

These motions are described correctly by the Bogomol’nyi, (1.1), and Gribov, (1.2), equations.

The coexistence of (delimited somewhat in the space) collective solid rotations and superfluid potential motions inside the BPS monopole vacuum suffered the Dirac fundamental quantization is thus on hand [3].

It is the sign of the first-order phase transition occurring in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac.

This phase transition is additional to the second-order one associated with the spontaneous breakdown of the initial \( SU(2) \) gauge symmetry down to the \( U(1) \) one.

In the present study we reveal the role of the discrete geometry assumed for the vacuum manifold \( R_{YM} \) in this first-order phase transition just as in the series of physical effects (including collective solid vacuum rotations) taking place in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac.

### 2 Statement of problem.

The ensuing exposition is organized as follows.

In Section 3 (repeating partially the theses of Ref. [1]) we, utilizing the general theory of topological defects (see e.g. §\( \Phi 1 \) in [2]), construct explicitly the degeneration space (vacuum manifold) \( R_{YM} \) for the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [4].

As we have seen already, it is possible to postulate the look [1] for this manifold, implicating the discrete multiplier \( Z \).

Such look of the YM vacuum manifold \( R_{YM} \) is destined by the discrete factorisation [1] of the initial, \( SU(2) \), and residual, \( U(1) \), gauge symmetries groups in the enumerated model:

\[
G \equiv SU(2) \simeq G_0 \otimes Z; \quad H \equiv U(1) \simeq U_0 \otimes Z.
\]

In Section 2, repeating the arguments [3], we discover three kinds of topological defects inside the vacuum manifold \( R_{YM} \). There are thread and point hedgehog (stable) defects and domain walls between different topological sectors of \( R_{YM} \).

Herewith thread topological defects are of a great importance for us as those determined physical rotary effects and the first-order phase transition taking place in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac. We shall make sure in this soon.

As it is well known [3], the sufficient condition for thread topological defects to exist in a gauge theory involving the spontaneous breakdown of the initial symmetry group is
the isomorphism
\[ \pi_1(R) = \pi_0(H) \neq 0 \] (2.1)
between the appropriate homotopical groups of the degeneration space \( R \) and residual symmetry group \( H \) in the considered gauge theory.

This isomorphism is correct, in particular, for the vacuum manifold \( R_{YM} \).

We ground this utilizing the holonomies group arguments (developed in Ref. [7]; see also [9]).

The existence of point hedgehog topological defects inside the same degeneration space \( R \) is controlled by the isomorphism [6, 9]
\[ \pi_2(R) = \pi_1(H) \neq 0. \] (2.2)

These topological defects are always associated, in (Minkowskian) Higgs non-Abelian models, with various monopole solutions (typical of which will be us discussed below). Geometrically, they are located at the origin of coordinates.

Besides two said types of topological defects, the YM vacuum manifold \( R_{YM} \) possesses the some more kind of topological defects. There are domain walls between different topological sectors of this manifold.

The cause of said domain walls is in the manifest "discrete" geometry of the Minkowskian YM vacuum manifold \( R_{YM} \) [9]:
\[ \pi_0(R_{YM}) = \mathbb{Z}. \]

Indeed, the width of a domain wall is roughly proportional to the inverse of the lowest mass of all the physical particles being present in the (gauge) model considered [19].

In the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac it is just the typical size \( \epsilon \sim (m/\sqrt{\lambda})^{-1} \) of BPS monopoles.

Thus in the Bogomol’nyi-Prasad-Sommerfeld (BPS) limit [6, 7, 8, 9, 10]
\[ m \to 0; \quad \lambda \to 0 \] (2.3)
for the Higgs mass \( m \) and selfinteraction constant \( \lambda \), widths of domain walls inside \( R_{YM} \) can take arbitrary values, including infinite ones.

Actually [5, 7, 8], \( \epsilon \) is the function of the spatial volume \( V \sim r^3 \):
\[ \epsilon(V) \sim V^{-1} \propto r^{-3}. \]

and the (gluonic) coupling constant \( g \). In the paper [3] it was argued that in the limit \( V \to \infty \) and simultaneously \( g \to 0 \), \( \epsilon(0) \) can be finite.

This implies that walls between topological domains are indeed of the typical width \( \epsilon(0) \neq 0 \) in the spatial region \( r \to 0 \) (including the infinite narrow cylinder of the effective diameter \( \epsilon(\infty) \to 0 \) with its axis symmetry \( z \)).

Vice verse, as \( |x| \to \infty \), domain walls become infinitely thin (of the typical width \( \epsilon(\infty) \to 0 \)).
At the conclusion of the paper [3], some additional important suggestion was made about the domain structure inside the vacuum manifold $R_{YM}$. Concretely, it was assumed the location of discrete topological domains inside $R_{YM}$ in the spheroidal region with an effective radius $r_1 \ll 1$ fm. The next, in turn, suggestion was that topological domains inside the vacuum manifold $R_{YM}$ are located parallel the axis $z$ at short distances $O(r_1)$. They possess finite sizes and are separated by domain walls also of finite sizes.

All these suggestions should be in agreement with the asymptotical freedom of quarks and gluonic fields at distances $r \to 0$ and the confinement behaviour of these fields at $r \sim 1$ fm. And this matching is one of the key tasks of the Dirac fundamental quantization of the Minkowskian YM model involving vacuum BPS modes. We will often return to this task in the present study.

In Ref. [3], it was also argued (implicating the general QFT arguments) that asymptotical (vacuum) states ”in” and ”out” (cf. (1.10)) in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac would be separated by the infinite time interval $T \to \infty$ [17].

Section 4 of the present study we devote to the discussion about (topologically non-trivial) rectilinear threads containing inevitably inside the vacuum manifold $R_{YM}$.

It will be shown, repeating the arguments [9], that there are YM fields

$$A_{\theta}(\rho, \theta, z) = A_{\mu} \partial x^{\mu} / \partial \theta$$

These fields may be always represented as [9]

$$A_{\theta}(\rho, \theta, z) = \exp(i M \theta) A_{\theta}(\rho) \exp(-i M \theta),$$

with $M$ being the generator of the group $G_1$ of global rotations compensating changes in the vacuum (Higgs-YM) configuration $(\Phi^a, A_{\mu}^a)$ at rotations around the axis $z$ of the chosen (rest) reference frame.

The elements of $G_1$ may be set as [9]

$$g_{\theta} = \exp(i M \theta).$$

YM fields $A_{\theta}$ are manifestly invariant with respect to shifts along the axis $z$.

It is important that rectilinear threads $A_{\theta}$ don’t coincide with vacuum YM BPS monopole solutions, and, on the contrary, there are, indeed, gaps between directions of ”magnetic” tension vectors: $B_1$,

$$|B_1| \sim \partial_{\rho} A_{\theta}(\rho, \theta, z),$$

and $B$, given by the Bogomol’nyi equation (1.1) and diverging as $r^{-2}$ at the origin of coordinates [10].

These gaps just testify in favour the above discussed first-order phase transition occurring in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [4].
In the Higgs sector of that model, there exist z-invariant (vacuum) Higgs solutions in a (small) neighbourhood of the origin of coordinates:

\[ \Phi^{(n)}(\rho, \theta, z) = \exp(M\theta) \phi(\rho) \quad (n \in \mathbb{Z}), \quad (2.7) \]

that can join (in a smooth wise) vacuum Higgs BPS monopoles, belonging to the same topology \( n \) and disappearing at the origin of coordinates.

In the monograph \[9\], the claim

\[ \nabla_\mu \phi(\rho) \leq \text{const} \rho^{-1-\delta} \quad \delta > 0; \quad (2.8) \]

\( \rho = \sqrt{x^2 + y^2} \), onto the field \( \phi(\rho) \) was imposed.

Herewith, speaking "in a smooth wise", we imply that the covariant derivative \( D\Phi \) of any vacuum Higgs field \( \Phi^{(n)} \) (this derivative must be now proportional to \( (2.8) \)) merges with the covariant derivative of a vacuum Higgs BPS monopole solution (in its topological class).

In the present study we shall give the explicit evaluation of \( \delta \).

This requirement for vacuum Higgs fields \( \Phi^{(n)} \) to be smooth is quite natural if the goal is pursued, in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac \[4\], to justify various rotary effects inherent in this model.

In particular, vacuum "electric" monopoles (1.15) \[5\] are directly proportional to \( D_i(\Phi^{(0)}_k) \Phi^{(0)} \).

These vacuum "electric" monopoles, in turn, enter explicitly the action functional (1.11), describing, in the Dirac fundamental quantization scheme \[4\], collective solid rotations inside the Minkowskian BPS monopole vacuum.

Such (smooth) sawing together appropriate vacuum Higgs modes \( \Phi^{(n)} \) and BPS monopoles is intended to remove the seeming contradiction between the manifest superfluid properties of the Minkowskian BPS monopole vacuum (suffered the Dirac fundamental quantization \[4\]), setting by the Bogomolnyi, (1.1), and Gribov ambiguity, (1.2), equations.

Moreover, one can assert \[16\] that

\[ D_B \sim D_E = 0 \quad (2.9) \]

for vacuum "magnetic" and "electric" tensions in the quested Minkowskian Higgs model, i.e. that these tensions are, indeed, "transverse" vectors collinear each other.

Note that Eq. (1.15) \[3\] just reflects this collinearity.

Going out from this contradiction seems to be just in locating (topologically nontrivial) threads in the infinitely narrow cylinder of the effective diameter \( \epsilon \) around the axis \( z \) and in joining (in a smooth wise) vacuum Higgs fields \( \Phi^{(n)}_a \) and BPS monopole solutions.

In this case collective solid rotations (vortices) inside the Minkowskian BPS monopole vacuum, occurring actually in that spatial region around the axis \( z \) and described correctly by the action functional (1.11), become quite "legitimate", and simultaneously, the Gauss law constraint (1.8) is satisfied outward this region with smooth vacuum "electric" monopoles \( E^a_i \) \[5\], (1.15).
The said is the one more argument in favour of the first-order phase transition occurring in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac and coming therein to the coexistence of two thermodynamic phases: the thermodynamic phases of collective solid rotations and superfluid potential motions, inside the appropriate Minkowskian BPS monopole vacuum.

Herewith the enough clear-cut picture can be observed how the enumerated thermodynamic phases are distributed inside the discrete vacuum manifold $R_{YM}$.

Thread topological defects (vortices), associated with rectilinear threads $A_\theta$, are located intimately near the axis $z$ of the chosen (rest) reference frame. Actually, they refer to the cylinder of the effective diameter $d \sim 2r_1 \neq 0$ with $z$ serving its symmetry axis.

Indeed, geometrically, it is the region of the vacuum manifold $R_{YM}$ possessing a non-trivial domain structure determined by the value $\epsilon(0) \neq 0$, the typical wide of domain walls in this region. Under these circumstances, one can suggest a highly complicated structure of vortices therein. In particular, these have finite lengths in each topological domain.

Simultaneously, superfluid potential motions refer to the spatial region out of this cylinder, including the spatial region $|x| \to \infty$ (corresponding to the infrared region of the momentum space).

The important consequence of the presence of rectilinear threads $A_\theta$ in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [4] and involving the "discrete" vacuum geometry for the vacuum manifold $R_{YM}$ is the effect [9] of annihilating two equal magnetic charges $m_1 = m_2 = m(n) \neq 0$ ($n \in \mathbb{Z}$) colliding at crossing a rectilinear topologically nontrivial thread $A_\theta(n)$.

The said can lead to the annihilation of all the topologically nontrivial YM vacuum BPS monopole modes and excitations over this BPS monopole vacuum (suffered the Dirac fundamental quantization [4]) during a definite time (mathematically this can be written as $<m> = 0$).

As a consequence of such possible annihilation, Higgs (BPS monopole) modes become free electric fields: their electric charges $e$ are dual (due to the Dirac quantization [21] of electric and magnetic charges) to zero magnetic charges can only survive upon the above described annihilation.

Such situation when Higgs modes possess arbitrary electric charges $e$, while magnetic charges $m \neq 0$ are confined is referred to as the Higgs phase in modern physical literature (see e.g. [22]).

If quarks are incorporated in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [4], disappearing topologically nontrivial YM modes via the "colliding" mechanism [9] can cause the possibility to observe free "coloured" quarks in the spatial region near the axis $z$.

The said can serve as a (perhaps, enough rough) representation for the asymptotical freedom of quarks in that model.

Vice verse, in the spatial region $|x| \to \infty$, where walls between different topological domains inside $R_{YM}$ become highly thin, the infrared topological and physical confinement
of quarks (in the spirit \[12, 23\]).

Such confinement may be provided, for instance, by surviving, upon colliding \[9\] at threads \(A_\theta\), only topologically trivial YM modes.

In Section 5 we discuss important consequences for QCD (based on the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac) of assuming the ”discrete” geometry.

The thing is that the Bogomol’nyi equation (1.1) is fulfilled to within the sign before the (covariant) derivative \(D\).

Herewith it may be shown (repeating the arguments \[9\]), and it will be done in Section 5, that this duality in specifying the sign of the vacuum ”magnetic” field \(B\) becomes unavoidable when there exists the (already mentioned) group \(G_1\) of global rotations, compensating changes in the vacuum (YM-Higgs) configuration \((\Phi^a, A^a_\mu)\) at rotations around the axis \(z\) of the chosen (rest) reference frame.

This causes inverting the sign before the generator \[9\] \(h \equiv h(\Phi) \equiv \Phi/a\) (with \(\Phi\) being a Higgs BPS monopole mode and \(a^2 \equiv m/\sqrt{\lambda}\)) of the residual \(U(1)\) gauge symmetry group in the quested Minkowskian Higgs model.

More exactly, to within an isomorphism, the elements \[9\]

\[\alpha(n) = (P \exp(- \frac{2\pi}{0} A_\theta d\theta))^{-1}\]  

(with \(P\) standing for parallel transports along appropriate curves [integration ways]) of the holonomies group \([7, 9]\) \(H \simeq U(1)\) may be constructed involving (topologically nontrivial) threads \(A_\theta\).

In this case there exist gauge transformations \[9\]

\[\check{h}(\Phi) = \alpha(2k)h(\Phi)\alpha(2k)^{-1} = -h(\Phi)\]  

mapping the topological domain with a topological number \(k\) in the discrete group space \(H\) (and by that the appropriate topological domain inside the degeneration space \(R_{YM}\)) into the topological domain with the topological number \(-k\).

Herewith the change in the sign of \(B\) in the Bogomol’nyi equation (1.1) occurs automatically at such mapping, accompanied also by changing the signs of magnetic charges \(m\).

Thus vacuum ”magnetic” fields \(B\) and \(-B\) may be identified by means of gauge transformations (2.11), and this implies, regarding the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac, the manifest invariance of the appropriate vacuum ”magnetic” energy, squared by \(|B|\), with respect to changes in the sign of the vacuum ”magnetic” field \(B\).

The same gauge transformations (2.11) influence also the ”electric” energy of the Minkowskian BPS monopole vacuum (suffered the Dirac fundamental quantization [4]), squared by the topological momentum \(P_N\) and given by the action functional (1.11).
As a result, the "electric" energy of this vacuum and the action functional corresponding to this "electric" energy prove to be also invariant with respect to changes in signs of topological charges $k$.

Just this reasoning about the vacuum "magnetic" and "electric" energies of the Minkowskian BPS monopole vacuum allows to draw the conclusion that the look of the initial and residual gauge symmetries groups in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac would be modified: respectively,

$$G_M = G_0 \otimes \mathbb{Z}/\mathbb{Z}_2; \quad U_1 = U_0 \otimes \mathbb{Z}/\mathbb{Z}_2.$$

This implies the modification

$$R'_{YM} = (\mathbb{Z}/\mathbb{Z}_2) \otimes G_0 / U_0$$

in the look of the appropriate degeneration space (vacuum manifold) $R_{YM}$.

On the other hand, the manifest account of the centre symmetry in the "modified" looks $G_M$ and $U_1$ of the initial and residual gauge symmetries groups in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac (inducing the look $R'_{YM}$ for the vacuum manifold in that model) via the quotient $\mathbb{Z}_2$ does not involves additional singular gauge fields.

Such singular gauge fields arise in the case of the continuous, by neglecting centre vortices, $SU(2)$ group geometry.

The latter kind of topological defects plays an important role in the modern sight about QCD (stated, in particular, in the review and in the papers [24, 25]); that is why the enough much place will be given centre vortices in the present study.

We pursue herewith the goal to demonstrate the uselessness of centre vortices in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac and involving the "discrete" vacuum geometry.

Center vortices in a non-Abelian model involving, for simplicity, initial $SU(2)$ gauge symmetry group, at the "continuous" group geometry being assumed, are associated with the nontrivial centre of $SU(2)$ consisting of two elements, $e$ and $-e$, with $e$ being the unit element of $SU(2)$.

Herewith when one requires the centre symmetry to be present upon gauge fixing, the isotropy group formed by the centre reflections must survive the "symmetry breakdown" induced by the elimination of redundant variables.

In this way, one can change effectively the gauge group:

$$SU(2) \rightarrow SU(2)/\mathbb{Z}_2.$$

So long as

$$\pi_1(SU(2)/\mathbb{Z}_2) \simeq \pi_1(SO(3)) = \mathbb{Z}_2,$$

3In the present study, also the case of $\mathbb{Z}_3$ vortices, associated with the (initial) $SU(3)_{col}$ gauge symmetry group in "realistic" QCD will be discussed.
the group space of $SU(2)/\mathbb{Z}_2$ proves to be containing the specific kind of topological defects, referring to as center vortices.

On the other hand, the real projective space

$$\mathbb{RP}^2 \simeq S^2/\mathbb{Z}_2$$

possesses the same topology that

$$SU(2)/\mathbb{Z}_2 \simeq SO(3)$$

More precisely,

$$\pi_1(SU(2)/\mathbb{Z}_2) = \pi_1(\mathbb{RP}^2) = \mathbb{Z}_2.$$ 

Furthermore, so long as $\mathbb{RP}^2 \subset \mathbb{R}^3$, this implies that the group space of $SU(2)/\mathbb{Z}_2$ contains nontrivial singularity lines in $\mathbb{R}^3$ similar to those one can discover in liquid nematic crystals possessing a one symmetry axis [9].

This similarity seems to be very didactic for understanding $\mathbb{Z}_2$ vortices in the YM model; that is why it will be the one of topics in Section 5.

In particular, it will be demonstrated (repeating the arguments [9]) that the degeneration space (vacuum manifold) in a liquid nematic crystal with a one symmetry axis is indeed

$$R_n = SO(3)/O(2) \simeq S^2;$$

it is the surface over which the free energy of the crystal achieves its minimum.

This degeneration space corresponds to the (global) $SO(3)$ symmetry of such a liquid nematic crystal that is violated thereupon down to its $O(2)$ subgroup.

On the other hand,

$$R_n \simeq S^2/\mathbb{Z}_2 \simeq \mathbb{RP}^2.$$ 

Just this induce topologically nontrivial singularity lines along which the thermodynamic equilibrium is violated in the liquid nematic degeneration space $R_n \simeq \mathbb{RP}^2$. They refer to as disclinations in the modern physical literature [9, 49].

Herewith disclinations are separated by domain walls (the one in the given liquid nematic crystal) from topologically trivial lines, that may be contracted into a point; thus the appropriate degeneration space $R_n$ is two-connected.

Disclinations in liquid nematic crystals possessing a one symmetry axis are patterns of topological defects belonging to the type of center vortices.

The sign of disclinations existing in a liquid nematic crystal is the relation [9, 49]

$$\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2,$$

which one encounters also in the case of the $SU(2)/\mathbb{Z}_2$ gauge symmetry group in the YM theory.

Just the latter topological relation allows to drawn the conclusion about the similarity of geometries in the case of liquid nematic crystals possessing a one symmetry axis and
the YM theory (if it implicating, at appropriate gauge fixing \cite{49}, the [initial] \(SU(2)/\mathbb{Z}_2\) gauge symmetry group).

Following \cite{49}, it will be shown that singularity lines in the YM model involving the initial \(SU(2)\) gauge group are always associated with purely gauge transformations

\[
A^\mu_{\mathbb{Z}_2}(x) = \frac{1}{ig} U_{\mathbb{Z}_2}(x) \partial^\mu U_{\mathbb{Z}_2}^\dagger(x).
\]

Herewith the gauge matrices \(U_{\mathbb{Z}_2}\) have the look \cite{49}

\[
U_{\mathbb{Z}_2}(\varphi) = \exp i \frac{\varphi}{2} \tau^3
\]

in the cylindrical coordinates \(\rho, \varphi, z, t\).

The gauge matrices \(U_{\mathbb{Z}_2}\) prove to be singular on the sheet \(\rho = 0\) (for all \(z, t\)), and thus these gauge matrices exhibit the essential properties of singular gauge transformations referring to center vortices.

It is correctly because

\[
U_{\mathbb{Z}_2}(2\pi) = -U_{\mathbb{Z}_2}(0),
\]

i.e. since any \(U_{\mathbb{Z}_2}\) gauge transformation is continuous in \(SU(2)/\mathbb{Z}_2\) but discontinuous as an element of \(SU(2)\).

The next evidence in favour of singularity of the gauge transformations \(U_{\mathbb{Z}_2}\) on the sheet \(\rho = 0\) is \cite{49} the definite behaviour of Wilson loops \(W_{c,\mathbb{Z}_2}\):

\[
W_{c,\mathbb{Z}_2} = \frac{1}{2} \text{tr} \{U_{\mathbb{Z}_2}(2\pi) U_{\mathbb{Z}_2}^\dagger(0)\}. \tag{2.12}
\]

There was demonstrated in \cite{49} that this expression equal to -1 for an arbitrary path \(C\) enclosing a center vortex.

This Eq. allows to calculate the components of gauge fields corresponding to matrices \(U_{\mathbb{Z}_2}\) and Wilson loops \(W_{c,\mathbb{Z}_2}\).

These turn out to be \cite{49}

\[
A^\varphi_{\mathbb{Z}_2}(x) = -\frac{1}{2g\rho} \tau^3,
\]

manifestly singular on the sheet \(\rho = 0\).

Herewith singular YM fields \(A^\varphi_{\mathbb{Z}_2}(x)\) represent correctly center vortices in the gauge model involving the \(SU(2)/\mathbb{Z}_2\) symmetry group.

In the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac \cite{4} and involving the ”discrete” vacuum geometry, the quotient \(\mathbb{Z}_2\) is also present in the explicit expression for \(R'_{\text{YM}}\).

But the situation now is rather different than in the YM theory involving the \(SU(2)/\mathbb{Z}_2\) gauge symmetry group and center vortices \cite{49}.

In the former case, gauge transformations (2.11), represented by holonomies elements \(\alpha(n)\) \cite{9}, ”overlap” completely gauge transformations \(U_{\mathbb{Z}_2}\) \cite{49} regarding changes in signs
of topological charges in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac.

However the holonomies elements $\alpha(n)$ are associated there actually with (topologically nontrivial) thread solutions $A_\theta$ [9], i.e. with thread topological defects.

Just this kind of topological defects replaces centre vortices in the mentioned Minkowskian Higgs model, leaving no room for the latter ones.

In spite of this, the discussion about center vortices seems to be very instructive: for instance, in order to understand that the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [4] gives the rather new confinement picture in QCD in comparison with the one getting at assuming (as it was done, for instance, in Ref. [49]) the initial $SU(n)_{\text{col}}/\mathbb{Z}_n$ ($n = 2, 3$) gauge group.

Indeed, it may be supposed additionally that this initial $SU(n)_{\text{col}}/\mathbb{Z}_n$ ($n = 2, 3$) gauge symmetry group is violated in the

$$SU(n)_{\text{col}}/\mathbb{Z}_n \rightarrow U(n-1)/\mathbb{Z}_n$$

wise [4].

But irrespective of either the initial $SU(n)_{\text{col}}/\mathbb{Z}_n$ ($n = 2, 3$) gauge symmetry group is exact or it is violated down to its Abelian subgroup $U(n-1)/\mathbb{Z}_n$, the presence of center vortices implies [49, 24] the correct quark confinement description.

More precisely, it may be shown [49, 24] that the area law, serving the confinement criterion in the QCD model involving the Mandelstam linearly increasing potential $E = K r$ between some quark $q$ and antiquark $\bar{q}$, is satisfied for Wilson loops $U_{C,\mathbb{Z}_2}$, (2.12), in the case of the initial $SU(2)_{\text{col}}/\mathbb{Z}_2$ gauge symmetry group:

$$\langle W \rangle = \sum_{n=1}^{N} (-1)^n p_n \rightarrow \exp(-2\nu A_W).$$

Such area law is got for the random distribution of intersection points of vortices with the enough large area $\mathcal{A}$ with those with a much smaller area $A_W$.

This random distribution has the look [49, 24]

$$p_n = \binom{N}{n} \left( \frac{A_W}{\mathcal{A}} \right)^n \left( 1 - \frac{A_W}{\mathcal{A}} \right)^{N-n}.$$

Herewith $\nu$ is the density of intersection points.

The above area law remains invariable at the spontaneous breakdown, in the (2.13) wise, of the $SU(2)_{\text{col}}/\mathbb{Z}_2$ gauge symmetry.

In the latter case, Wilson loops $U_{C,\mathbb{Z}_2}$, (2.12), contain gauge matrices $U_{\mathbb{Z}_2}$ belonging to $U(1)/\mathbb{Z}_2$ (the Abelian subgroup of $SU(2)_{\text{col}}/\mathbb{Z}_2$).

In particular,

$$SU(3)_{\text{col}}/\mathbb{Z}_3 \rightarrow U(1) \otimes U(1)/\mathbb{Z}_3 \simeq U(2)/\mathbb{Z}_3.$$
Herewith the residual $U(1)/\mathbb{Z}_2$ (and, generally, $U(n-1)/\mathbb{Z}_n$) gauge symmetry group may be chosen by fixing the so-called maximal Abelian gauge (MAG) for YM (gluonic) fields.

This gauge comes to the maximal diagonalization of the appropriate gauge matrices. This may be achieved (in the simpler case of the YM theory) by means of maximizing the integral
\[
\int d^4x \left[ (A^1_\mu)^2 + (A^2_\mu)^2 \right]; \quad A_\mu = A^a_\mu \tau_a.
\]
It is equivalent to the fact (see e.g. Ref. [47]) of maximization of the quantity
\[
\sum_x \sum_{\mu=1}^4 \text{Tr}[\tau_3 U_\mu(x) \tau_3 U_\mu^\dagger(x)],
\]
where $U_\mu(x)$ are $SU(2)$ gauge matrices.

MAG may be got [48] by imposing the "Lorentz" gauge
\[
D_{ab}^\mu A^b_\mu = 0.
\]
Indeed, the combination of MAG and the center vortices in the case when the initial $SU(2)_{\text{col}}/\mathbb{Z}_2$ gauge symmetry group is violated in the (2.13) wise, is a "mixed"

\section{How one can introduce "discrete" geometry for degeneration spaces in Minkowskian Higgs models.}
Recall again the recent paper [6]. In this paper some modern Minkowskian Higgs models with stationary vacuum monopole solutions were analysed.

Besides already mentioned BPS monopole model [7, 8, 9, 10, 11], one can mention the 't Hooft-Polyakov monopole model [26, 27].

As it was discussed in [6], such (vacuum) monopole solutions are compatible with the "continuous" vacuum geometry
\[
R \equiv SU(2)/U(1) \simeq S^2.
\]
Also it was argued that Minkowskian Higgs models with stationary vacuum monopole solutions, involving the "continuous" vacuum geometry (3.1), can be described quite correctly in the framework of the Faddeev-Popov (FP) "heuristic" quantization scheme [28], coming [1, 6] to actual fixing the Weyl gauge $A_0 = 0$ (for instance via the multiplier $\delta(A_0)$ in appropriate FP path integrals).

From the topological viewpoint, the continuous vacuum manifold $R$ (3.1) possesses the one kind of topological defects, namely the point hedgehog topological defects.

\footnote{Following [9], $R$ may be called the degeneration space.}

It is quite correct herewith to interpret the initial symmetry group $G$ in a Minkowskian Higgs model
These topological defects are associated with vacuum monopole solutions in appropriate Minkowskian Higgs models.

Point hedgehog topological defects inside $R$ are determined by the topological chain

$$\pi_2(R) \equiv \pi_2S^2 = \pi_1(H) \equiv \pi_1U(1) = \mathbb{Z}. \quad (3.2)$$

From the thermodynamic points of view, all the kinds of topological defects may be explained (repeating the arguments [9]) by violating the thermodynamic equilibrium over a region in the appropriate coordinate (in particular, Minkowski) space.

For example, point hedgehog topological defects inside $R$ are associated with violating the thermodynamic equilibrium in an infinitesimal neighbourhood of the origin of coordinates. Such neighbourhoods are topologically equivalent to the two-sphere $S^2$.

In the present study we attempt to demonstrate our readers that going out from the FP "heuristic" quantization scheme [28] to the Dirac "fundamental" one [4] in the Minkowskian Higgs model with vacuum BPS monopole solutions claims the in principle new geometrical approach to constructing the appropriate vacuum manifold (in comparison with assuming the "continuous" $\sim S^2$ vacuum geometry in the former case).

This "new geometrical approach" (whose outlines were contemplated in the recent paper [1]) comes to assuming the "discrete" geometry for the vacuum manifold $SU(2)/U(1)$.

As we shall make sure soon, this assuming explains enough good various vacuum rotary effects inherent in the physical BPS monopole vacuum suffered the Dirac fundamental quantization [4].

So, let us represent [1] the initial, $SU(2) \equiv G$, and residual, $U(1) \equiv H$, gauge symmetries groups in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac in the shape of discrete spaces

$$SU(2) \simeq G_0 \otimes \mathbb{Z}; \quad U(1) \simeq U_0 \otimes \mathbb{Z}, \quad (3.3)$$

respectively 6.

From the topological viewpoint, the discrete representation (3.3) for the gauge groups $G$ and $H$ extracts "small" (topologically trivial) and "large" (corresponding to topological

as that does not change the energy functional (Hamiltonian) of that model, while the residual symmetry group $H \subset G$ as that consisting of transformations that keep invariant a fixed equilibrium state.

All these states (at a fixed temperature $T < T_c$, with $T_c$ being the appropriate Curie point in which the initial symmetry $G$ is violated and the second-order phase transition occurs) just form the appropriate degeneration space $R = G/H$. The natural claim to this space is herewith to be topological.

The structure of a degeneration space may be investigated with the aid of the Landau theory of second-order phase transitions.

An equilibrium state is determined by the condition for the free energy of the given system to be minimal.

In the Landau theory of second-order phase transitions (the pattern of which is the Minkowskian Higgs model) one supposes that an equilibrium state may be found at minimizing the free energy of the given system by the set of states specified by a finite number of parameters (called order parameters), but not by the set of all the states.

6Such representation for gauge groups spaces was proposed for the first time in the paper [16].
numbers \( n \neq 0 \) gauge transformations in the complete set of appropriate gauge transformations (the idea of such subdividing for gauge transformations was suggested in Ref. [18]).

According to the terminology [18], the complete groups \( G_0 \) and \( H_0 \) just contain ”small” gauge transformations, that implies

\[ \pi_n G_0 = \pi_n H_0 = 0 \]  

for loops in the group spaces \( G_0 \) and \( H_0 \) in all the dimensions \( n \geq 1 \).

Simultaneously, in definition,

\[ \pi_0 G_0 = \pi_0 H_0 = 0, \]  

i.e. \( G_0 \) and \( H_0 \) are maximal connected components [9] in their gauge groups (respectively, \( G \) and \( H \)).

Later Eq. implies [9] that

\[ \pi_0 [G_0 \otimes \mathbb{Z}] = \pi_0 [G_0 \otimes \mathbb{Z}] = \pi_0 (\mathbb{Z}) = \mathbb{Z}. \]  

The latter relation indicates transparently the discrete nature of the appropriate group spaces [7].

To construct the degeneration space (vacuum manifold) \( R_{YM} \equiv G/H \) corresponding to the discrete representation (3.3) for the gauge groups \( G \) and \( H \), let us multiply Eq. [6, 29]

\[ G = H \oplus G/H \]  

left by \( \mathbb{Z} \), substituting simultaneously in (3.7) the ”discrete” representations (3.3) for the gauge symmetries groups:

\[ \mathbb{Z} \otimes G_0 = \mathbb{Z} \otimes U_0 + \mathbb{Z} \otimes G_0/U_0. \]  

From latter Eq. we learn that the degeneration space in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [11] is

\[ R_{YM} = \mathbb{Z} \otimes G_0/U_0. \]  

Again \( \pi_1(R_{YM}) = \mathbb{Z} \) since \( U_0 \otimes \mathbb{Z} \) is multi-connected.

It is correctly because the isomorphism (2.1) (the proof of this isomorphism was given in Ref. [9], in §T20, we recommend our readers for understanding the matter).

\[ \pi_1(R_{YM}) = \mathbb{Z} \]  

It is the particular case of the more general relation

\[ \pi_i(K) = \pi_i(L_1) + \cdots + \pi_i(L_r) \]

for a group \( K \) which is the product of the groups \( L_1 \ldots L_r \) at a fixed \( i \) (it is correctly for the Lie groups of the series \( SU, U \) and \( SO \), with which modern theoretical physics deals).
Further, it becomes transparent from Eq. (3.9) that the ”small” coset $G_0/U_0$ is one-connected:

$$\pi_1(G_0/U_0) = 0.$$ 

Really, the coset $G_0/U_0$ is treated as the space of $U_0$-orbits on $G_0$; the latter space is one-connected.

One can see also the topological equivalence between $G_0/U_0$ and the subset of one-dimensional ways on $SU(2)/U(1) \simeq S^2$ that may be contracted into a point, i.e. between the unit elements of the fundamental homotopical groups of these spaces.

The vacuum manifold $R_{YM}$ is transparently multi-connected:

$$\pi_0(R_{YM}) = \mathbb{Z}. \quad (3.10)$$

This Eq. may be explained in the framework of our above remark [9] about homotopical groups of products of groups.

Latter Eq. has very important physical sense.

It implies [9] that domain walls exist between different topological sectors in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac.

The origin of said domain walls is in the ”discrete” factorisation (3.3) of the residual gauge symmetry group $U(1)$.

As it is well known (see e.g. §7.2 in [30] or the paper [19]), the width of a domain (or Bloch, in the terminology [19]) wall is roughly proportional to the inverse of the lowest mass of all the physical particles in the (gauge) model considered.

In Minkowskian Higgs models the typical such scale is the (effective) Higgs mass $m/\sqrt{\lambda}$.

In particular, in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [4], $m/\sqrt{\lambda}$ is the only mass scale different from zero.

This fact was explained in Ref. [2]. This was argued by zero masses of YM fields represented by topological Dirac variables $A_i^D (i = 1, 2)$.

These variables become massless at the Dirac removal [2, 5]

$$U(t, x)(A_0^{(0)} + \partial_0)U^{-1}(t, x) = 0. \quad (3.11)$$

of temporal YM components $A_0$, that are, indeed, nondynamical fields possessing zero canonical momenta

$$E_0 \equiv \partial \mathcal{L}/\partial (\partial_0 A_0) = 0.$$ 

In this context [7, 8],

$$U(t, x) = v(x)T \exp\left\{ \int_{t_0}^{t} \frac{1}{D^2(\Phi_{BPS})} \partial_0 D_k(\Phi_{BPS}) \hat{A}^k \right\}, \quad (3.12)$$

with $v(x)$ being stationary Gribov topological multipliers (depending indeed on topological numbers $n$).
If, additionally, the transverse gauge \( A^D \) is fixed for topological Dirac variables \( A_i^D \), this just implies the masslessness of YM fields.

Here is a transparent analogy with the photon case in QED.

In that case one gets plane waves by fixing the Lorentz gauge

\[
\phi = 0; \quad \partial_i A^i = 0 \quad (i = 1, 2, 3) \Leftrightarrow \partial_\mu A^\mu = 0
\]

(with \( A_0 \equiv \phi \) being the scalar potential).

As it was argued in the monograph \( [31] \), such fixing the (relativistic invariant) Lorentz gauge implies the equation of motion (referred to as the D’alambert or wave equation)

\[
\Delta A - \frac{\partial^2 A}{\partial t^2} = 0
\]

(at setting \( c = 1 \) for the light velocity). Latter Eq. shows transparently the absence of the (rest) mass for a photon.

As it was stressed in Ref. \( [3] \) and in Section 2 of the present work, in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac \( [4] \), the typical value of the length dimension inversely proportional to \( m/\sqrt{\lambda} \) is the size \( \epsilon \) of BPS monopoles.

It may be given as \( [5, 7, 8] \)

\[
\frac{1}{\epsilon} = \frac{gm}{\sqrt{\lambda}} \sim \frac{g^2 < B^2 > V}{4\pi}.
\]

Thus \( \epsilon \) is, in turn, inversely proportional to the spatial volume \( V = \int d^3x \) occupied by the appropriate (YM-Higgs) field configuration. This volume is fixed.

The said allows to assert that \( \epsilon \) (and \( \sqrt{\lambda}/m \) in turn) disappears at the spatial infinity in the infinite spatial volume limit \( V \to \infty \). On the other hand, in the zone of asymptotical freedom of quarks at the origin of coordinates, when the YM (gluonic) constant \( g \) goes to zero, \( \epsilon \) can take any finite values (due to the \( (0 \times \infty) \)^\(-1\) uncertainty in the case of the fixed infinite volume).

This means, due to the above reasoning \( [19] \), that walls between topological domains inside \( R_{\text{YM}} \) can be of finite widens \( O(\epsilon(0)) \neq 0 \), at the origin of coordinates.

The fact \( \epsilon(\infty) \to 0 \) is also meaningful. This implies actual merging of topological domains inside the vacuum manifold \( R_{\text{YM}} \), \( (3.9) \), at the spatial infinity. This is just the confinement region.

This promotes the infrared topological confinement (destructive interference) of Gribov ”large” multipliers \( \nu^{(n)}(x) \) in gluonic and quark Green functions in all the orders of the perturbation theory.

The latter fact was demonstrated utilizing the strict mathematical language in Ref. \( [23] \) (partially these arguments \( [23] \) were reproduced in \( [1] \)).

The existence of domain walls between different topological sectors of the vacuum manifold \( R_{\text{YM}} \), \( (3.9) \), permits their treatment as quite separated sets.
The notion of quite separated sets means in this case that always there exists such a function $f : \mathbb{R}Y \rightarrow I$ ($I \equiv [0, 1]$) for two homotopical classes $A$ and $B$ inside $\mathbb{R}Y$ that $f(x) = 0$ as $x \in A$, while $f(x) = 1$ as $x \in B$. Here $x$ is a (one-dimensional) way.

One speaks in this case that $f$ separates the sets $A$ and $B$.

A domain walls structure, like that we discuss now for $\mathbb{R}Y$, must give its nontrivial contribution in the model Lagrangian (Hamiltonian).

In magnetism we have examples of such domain walls as transitions between different magnetic moments. More precisely, a domain wall is a gradual reorientation of individual moments across a finite distance. The energy of a domain wall herewith is simply the difference between the magnetic moments before and after the domain wall was created (with correct accounting of the physical units).

A simple exemple of such domain walls structure is a Bloch wall. It is a narrow transition region at the boundary between magnetic domains, over which the magnetization changes from its value in one domain to that in the next. The magnetization rotates through the plane of the domain wall. Bloch domain walls appear in bulk materials, i.e. when sizes of magnetic material are considerably larger than domain wall width.

A mechanism transition between magnetic domains was suggested by L. Landau and E. Lifshitz in which the magnetization vector $M$ turns in the domain wall plane changing its direction to the opposite.

Thus an important point here is just a nontrivial contribution in the Lagrangian (Hamiltonian) of the magnetization theory.

The next example of domain walls tells us the notion of step voltage. It is, actually, the difference of potentials $\phi_1(r) - \phi_2(r)$. If these potentials possess two ”neighboring” topologies: say, $\phi_1(n_1)$ and $\phi_2(n_1+1)$, then such difference is $\phi_2(n_1+1) - \phi_1(n_1)$. It is just the potential energy associated with the domain wall between two domains ”neighboring on the topology” if we identify surfaces swept by these two potentials with such topological domains. In this context, the potentials $\phi$ are appropriate (effective) YM potentials. Note that these are disconnected functions of the radius $r$ due to the first order phase transition taking place in the considered model (in the same way as disconnecting in the plot of the vacuum magnetic tension $B$).

On the other hand, as it is well known, a potential $\phi$ is defined as energy per unit charge, and due to the Dirac hypothesis about the quantization of electric and magnetic charges, such (effective) potential can be specified in terms of electric as well as magnetic charges. The most important thing in this context is the contributions to the energy of the YM-Higgs system (expressed via the appropriate Hamiltonian) of the concrete topologies. Such contributions for the Minkowskian YM model with vacuum BPS monopole solutions is set by the topologically degenerated vacuum Hamiltonian

$$H_{\text{cond}} = \frac{2\pi}{g^2} [P^2_N (\frac{g^2}{8\pi^2})^2 + 1]$$

(3.15)
with $P_N$ given in (1.14). The later equation was obtained at the direct summation of
the Hamiltonians corresponding to the free rotator action (1.11) and vacuum "magnetic"
ergy (1.50) [5, 8], we shall derive in Section 4.

This Hamiltonian can be rewritten more clearly as the sum of topological contributions

$$H_{\text{cond}} = \sum_k \frac{2\pi}{g^2 \epsilon} [P_k^2 \left( \frac{g^2}{8\pi^2} \right)^2 + 1] \equiv \sum_k H_k. \quad (3.16)$$

Now, in terms of the vacuum (condensation) Hamiltonian (3.15), one can introduce
the effective "magnetic" potential for the $k$th topological sector of the discussed model:

$$\phi_k = H_k/m_k, \quad (3.17)$$

with $m_k$ being the total magnetic charge referring to this $k$th topological sector. As A. S. Schwartz shows in his monograph [9] (in §Φ7) there can be shown that a magnetic charge $m$ is a linear function of the topological charge $\zeta$:

$$m(\Phi, A) = C \zeta(\Phi, A), \quad \zeta(\Phi, A) \in \mathbb{Z}. \quad (3.18)$$

Here $\zeta(\Phi, A)$ is the topological number corresponding to the chain

$$\pi_2 S^2 = \pi_3(SU(2)) = \pi_1(U(1)) = \pi_1 S^1 = \mathbb{Z}, \quad (3.19)$$
in a (vacuum) YMH configuration $(\Phi, A)$.

Moreover, the author [9] also shows that $C = \nu/4\pi$, where $\nu$ can be found from the conditions

$$\exp(\nu h) = 1; \quad \exp(\lambda h) \neq 1 \quad (3.20)$$

($h \equiv h(\Phi) \equiv \Phi a; \quad a = m/\sqrt{\lambda}$ as $0 \leq \lambda \leq \nu^2$). One also can speak that $\nu$ is the minimal positive number for which $\exp(\nu h) = 1$. From the geometrical point of view, $\nu$ is characterized as the length of the circle $U(1) \simeq S^1$ (of the unit radius).

All the said is correct also for any topological charge $m_k$ in the YMH theory Minkowskian Higgs model with vacuum BPS monopole solutions quantized byDirac.

Finally, we now can write down the formula for the "step voltage" between the neighboring potentials. This is

$$\Delta_m = \phi_{k+1} - \phi_k = H_{k+1}/m_{k+1} - H_k/m_k. \quad (3.21)$$

\[8\]On the other hand, the general formula for the magnetic charge

$$m = \frac{1}{4\pi} \int \mathbf{B} d\mathbf{S}$$
can be utilized. Then it can see that the above mentioned disconnecting the "magnetic" field $\mathbf{B}$ implies disconnecting potentials $\phi_k(r)$.

\[9\]In the original paper [9], it was set $h \equiv h(\Phi) \equiv \Phi/a$. The different setting for $h$ in the present study ensures the finite value of $1/a$, the typical width of a domain wall in Minkowskian YMH models with vacuum BPS monopole solutions. The crucial point in this change is that $\exp(\nu h)$ is dimensionless.

\[10\]Since $\Phi/a$ should be dimensionless, the (absolute) value of the Higgs field $\Phi$ should have the dimension of mass ($a$ has the dimension of mass since the Higgs self-interaction constant $\lambda$ is dimensionless in $D = 4$).
Now it will be useful to find the connection between the typical width $1/a = \sqrt{\lambda}/m$ of a domain wall and the value $m_k$ of the (total) magnetic charge of the $k^{th}$ topological domain. Expanding the exponent $\exp(\nu h)$ in (3.20) as
\[
\exp(\nu h) = 1 + \nu h + (\nu h)^2/2 + \ldots,
\]
we get from the condition (3.20) the homogeneous algebraic equation
\[
\nu h + (\nu h)^2/2 + \cdots + (\nu h)^i/i! + \cdots = 0.
\]
(3.22)
Actually, we are interested only in the terms up to the third order in (3.22). Thus we get the equation
\[
\nu h + (\nu h)^2/2 + (\nu h)^3/3 = 0 \iff \nu h((\nu h)^2 + \frac{3}{2}(\nu h) + 3) = 0.
\]
(3.23)
The first solution of this equation, $\nu h = 0$ can imply
1. $|\Phi| = 0$ (at $a \neq 0$ in the BPS limit $m \to 0$, $\lambda \to 0$ adopted in the present study()): in other words, a zero mode of the Higgs (vacuum) field; in the case discussed in the present issue it is a zero mode of the Higgs BPS monopole.
2. Alternatively, $\nu = 0$ involves $C = 0$, i.e. the absence of magnetic charges.

The second solution of Eq. (3.23) is complex. It can be treated as a constraint superimposed onto the (total) magnetic charge at the fixed topological number $n$. It is remarkable that the above constraint does not depend onto the concrete topology.

Let us attempt now to solve the second equation in (3.23). It is
\[
(\nu h)_{1,2} = -\frac{3}{4} \pm \frac{i}{4}\sqrt{15}.
\]
(3.24)
Since $h(\Phi) = \Phi/a$, it is possible now to estimate the typical length (scale) $1/a = \sqrt{\lambda}/m$ of domain walls inside the given vacuum manifold in a (Minkowskian) YM-Higgs model. Note that it is an universal calculation for such models. Setting $\nu = 4\pi C$, we have from the solution (3.24), taking herewith only the real part of this solution. Then
\[
|1/a| \sim \frac{16\pi C|\Phi|}{3}.
\]
(3.25)

It is a very interesting formula. It connects the (typical) width of a domain wall $1/a$ with the geometry of $U(1) \simeq S^1$ (via the constant $C$) and the absolute value of the Higgs field. At first sight, the latter equation implies infinitely wide domain walls between different topological sectors inside a vacuum manifold. But that’s not the case indeed for the vacuum manifold $R_{YM}$, (3.3) in the discussed YMH theory [7, 8, 9, 10, 11] with vacuum (Higgs and YM) BPS monopole solutions quantized by Dirac. It’s all about the Higgs BPS ansatz $f_0^{BPS}(r)$ [5, 7, 8, 20] (see Eq. (4.24) below) possesses the behavior $f_0^{BPS}(0) \to 0$, while $f_0^{BPS}(\infty) \to 1$. Thus the $1/r$ multiplied this ansatz in the ”Higgs” vacuum solution results therein the uncertainty $0/0$ at the origin of coordinates and zero at the spatial infinity. It is just what is required for the finite width of a domain wall $1/a$
at the origin of coordinates and it disappearing at the spatial infinity inside the vacuum manifold in the discussed YMH theory.

Indeed, we run into a problem with Eq. (3.17). The fact is that it is taken $\epsilon \to 0$ at all the calculations [7, 8, 12] about $H_{\text{cond}}$. This implies the actual divergence of the condensation Hamiltonian and, as a consequence, the divergence of effective potentials $\phi_k$ at any $k$.

The essence of the problem in the following. The actual behaviour of potentials $\phi$ should be anti-Coulomb to provide the asymptotical freedom in the YM-like gauge models: in particular, in QCD, in the spatial region near the origin of coordinates. This anti-Coulomb behaviour comes to the $r^\alpha$ ($\alpha > 0$) dependence of potentials as $r \to 0$. In the opposite spatial region $r \to \infty$ gluonic potentials must infinitely increase and this corresponds to the quarks (colours) confinement. The same behavior is, of course, is proper to the difference of two ("topologically neighboring") potentials, $\phi_2(n_1 + 1) - \phi_1(n_1)$.

As a solution of the problem we offer introducing a new tuning parameters $\Lambda_k(r)$ ($k \in \mathbb{Z}$) in such a wise that $\Lambda_k(0)/\epsilon \to 0$ for $k \neq 0$ while $\Lambda_0(0)/\epsilon \to 0$ and $\Lambda_0(0)/(\epsilon \mathfrak{m}_0) \to 0$.

Then the condensation Hamiltonian $H_{\text{cond}}$, (3.16) can be transformed into:

$$H'_{\text{cond}}(r) = \sum_k \Lambda_k(r) H_k. \quad (3.26)$$

As it is easy to see, all the effective potentials $\phi_k(0)$, (3.17), disappear at such fit; and this is just the proper anti-Coulomb behavior for effective potentials.

To get also the correct behavior of the model at the spatial infinity (and thus specify the behavior of the parameters $\Lambda_k(r)$ at the spatial infinity and at intermediate values of the distance $r$), i.e. in the infrared confinement region where $\epsilon(\infty) \to 0$ and the condensation Hamiltonian $H_{\text{cond}}$ diverges once again, one should postulate the finite behavior of $\Lambda_k(\infty)$ at $k \neq 0$. As to $\Lambda_0(\infty)$, we demand $\Lambda_0(\infty)/\mathfrak{m}_0$ to be fixed.

Thus accounting the domain structure inside the vacuum manifold $R_{YM}$, (3.9), requires the introduction of the additional (along with the cut-off parameter $\epsilon(\infty)$) tuning parameters $\Lambda_k$ ($k \in \mathbb{Z}$) in each topological sector of the Minkowskian YMH theory in the Dirac quantization scheme [4]. It is worth to remember again that these tuning parameters reflect the presence of thread topological defects inside the vacuum manifold $R_{YM}$. Eventually, this takes into account the (topologically nontrivial) dynamics (1.11), (1.12) (see [3] and the references therein). In turn, such (topologically nontrivial) dynamics inside the vacuum manifold $R_{YM}$ arises at resolving the YM Gauss law constraint (1.6) in terms of the family of zero mode solutions $Z^a$ (1.9) [12, 16].

In the case of BPS monopoles, as those which "inbuilt" in the vacuum manifold $R_{YM}$, (3.9), it is well known (see e.g. the paper [20]) that the BPS ansatz

$$f^{\text{BPS}}_1(r) = 1 - \frac{r}{\epsilon \sinh(r/\epsilon)}$$

behaves as $f^{\text{BPS}}_1(r) \to 0$ at $r \to 0$. This implies disappearing YM BPS monopole modes at the origin of coordinates, i.e. it is an example of the above mentioned anti-Coulomb
behaviour. In this the specific role of the YM BPS ansatz $f_1^{BPS}(r)$ is manifested (recall also the discussion in [33] about the role of the YM BPS ansatz $f_1^{BPS}(r)$ as some electric form-factor screening the [elementary] charge $e$).

Note that the Wu-Yang (vacuum) monopole solutions [34], merging with the BPS solutions [10] at the spatial infinity, possess, on the other hand, the Coulombic $1/r$ behaviour as $r \to 0$. Thus the Wu-Yang (vacuum) monopole background can not ensure the asymptotical freedom of gluons and quarks at the origin of coordinates, unlike that background consisting of YM BPS monopole modes possessing the anti-Coulomb behavior there. The said is very important for constructing realistic QCD on the base of vacuum BPS monopole solutions.

Of course, the "effective" magnetic potentials $\phi_k$, defined via (3.17), must be agreed with (vacuum) BPS YM monopole potentials, at least in their spatial asymptotic: at the origin of coordinates as well as at the spatial infinity.

The nontrivial isomorphism (2.1) [9] satisfying (as we have discussed above) for the vacuum manifold $R_{YM}$, (3.9), implies the presence of thread topological defects inside this manifold.

On the level of phenomenology this implies [9] violating the thermodynamic equilibrium along definite lines in the given vacuum manifold.

In the simple case when such lines, threads, are straight, the thermodynamic equilibrium is violated in an infinitely narrow tube around each of these threads. Thus regions of thread topological defects have the topology of a cylinder (topologically equivalent to the circle $S^1$). But it is well known that $\pi_1 S^1 = \mathbb{Z}$.

In our researches about the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [4] we assume that the region of thread topological defects inside the vacuum manifold $R_{YM}$, (3.9), is the infinitely narrow tube of the infinite length around the axis $z$ in the chosen rest reference frame. The effective "diameter" of this cylinder of the infinite length is just $O(\epsilon(\infty))$, that is, indeed, an infinitely small value.

In the next section we shall ground the above assumption investigating the properties [9] of rectilinear threads $A_9$ inside $R_{YM}$.

Furthermore, it may be shown that

$$\pi_2(R_{YM}) = \mathbb{Z} = \pi_1 H.$$  \hspace{1cm} (3.27)

The isomorphism $\pi_1(H) = \mathbb{Z}$ follows from the arguments of the holonomies group, stated in Refs. [7, 9].

Actually, these come to the existence of one-dimensional loops in the $H$ group space, forming holonomies elements

$$b_\gamma = P \exp(- \oint_\Sigma \mathbf{T} \cdot A_\mu dx^\mu)$$  \hspace{1cm} (3.28)

over one-dimensional cycles $\Sigma$.  

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The holonomies elements $b_\gamma$, belonging to the $U(1) \subset SU(2)$ embedding, make the complete holonomies group (we shall denote alls as $H$) isomorphic to $U(1)$ \cite{7}.

Eq. (3.28) makes the isomorphism $\pi_1(H) = \mathbb{Z}$ highly transparent (as that coming to $\pi_1 S^1 = \mathbb{Z}$).

The relation

$$\pi_2(R_{YM}) = \pi_1 H,$$

is the particular case of more general Eq. (2.2).

The good proof of the isomorphism \cite{9} was demonstrated in the monograph \cite{9}, in §7T20, and we recommend our readers this monograph to acquaint with it.

The nontrivial isomorphism (3.27) testifies in favour of point hedgehog topological defects inside the vacuum manifold $R_{YM}$ in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac \cite{4}.

These point hedgehog topological defects and vacuum BPS monopole solutions are indivisible.

As in the case of (vacuum) 't Hooft-Polyakov monopole solutions \cite{26, 27}, associated with the "continuous" vacuum $\sim S^2$ geometry (3.1) and the FP "heuristic" quantization scheme \cite{28}, point hedgehog topological defects inside the discrete vacuum manifold $R_{YM}$, (3.9), come to violating the thermodynamic equilibrium in an infinitesimal neighbourhood of the origin of coordinates.

In particular, this violating implies the singularity of the appropriate vacuum "magnetic" fields $B$, serving as the order parameters in the mentioned Higgs models, at the origin of coordinates.

In spite of definite distinctions between the 't Hooft-Polyakov and BPS Higgs models (these distinctions were pointed out, for instance, in the recent paper \cite{6}; the principal of them is the absence of superfluid properties in the 't Hooft-Polyakov model \cite{26, 27}) the vacuum "magnetic" field $B$ diverges as $r^{-2}$ at the origin of coordinates in both models\footnote{for the 't Hooft-Polyakov model \cite{26, 27} this was demonstrated in Ref. \cite{29}, while for the BPS Higgs model, in the original papers \cite{10}.}

$$B_k \sim \frac{r_k}{g r^3}. \quad (3.29)$$

Moreover, there is a likeness between Eqs. (3.27) and (3.2) as the particular cases of the general relation (2.2) \cite{9}, inducing point topological defects in (gauge) theories.

On the other hand, there are no thread topological defects inside the continuous vacuum manifold $R$, (3.1), since $\pi_1 S^2 = 0$ and moreover when the "continuous" geometry is assumed for the residual $U(1)$ gauge symmetry in the (Minkowskian) Higgs model, also $\pi_0 U(1) = 0$.

In the next section it will be argued that the first-order phase transition occurs indeed in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac \cite{4}.

To facilitate this job, now we should like discuss an useful analogy, us give the liquid helium II model.
Just in the latter model, possessing manifest superfluid properties [14], also the first-order phase transition happens.

It is associated with (spontaneous) violating the superfluidity [14] in a (resting) liquid helium II specimen along definite (rectilinear) lines, threads.

This implies [15] the presence of thread (rectilinear) vortices in that specimen.

The similar situation with arising (rectilinear) vortices takes place when this specimen turns together with the cylindrical vessel in which it is contained.

The phenomenology of the latter case was stated enough good in the monograph [13], in §29. We don’t intend, in the present study, to retail all the said in the monograph [13] about this subject; however we, for all that, would like to elucidate some theses of the helium turning model. This will be very helpful for us in the next section.

A good argument in favour of the first-order phase transition occurring in the liquid helium II turning model is the nontrivial contribution $\Delta E$ [13] in the total helium ($\text{He}^3$)
Hamiltonian from the global solid potential rotations of the helium specimen turning together with the cylindrical vessel where it is contained.

This contribution is given with Eq.

$$\Delta E \sim L \pi \rho_s \frac{\hbar^2}{m^2} \ln \frac{R}{a}$$

with $L$ being the length of the vessel, $\rho_s$ being the density of the superfluid component in the helium II specimen, $m$ being the mass of a helium atom; at last $R$ and $a$ are, respectively, the radius of the vessel and an arbitrary distance of atomic scales.

The shortcoming of this Eq. is its logarithmic divergence at $R/a \to \infty$.

The important peculiarity of the liquid helium II turning model is the appearance of rectilinear threads (accompanied by appropriate vortices), parallel to the rotation axis 12 This has the typical look

$$\hat{\mathcal{H}} = \int d^3r \hat{a}^+(r, t) \left( -\frac{\hbar^2}{2m} \nabla^2 \right) \hat{a}(r, t) + \frac{1}{2} \int d^3r \int d^3r' \hat{a}^+(r, t) \hat{a}^+(r', t) V(|r-r'|) \hat{a}(r', t) \hat{a}(r, t).$$

Boson creation and annihilation operators: $a^+(r, t)$ and $a+(r, t)$, respectively, may be expressed in terms of the phase $\Phi(t, r)$ of the Bose condensate wave function.

Note that such expressing boson creation and annihilation operators in terms of the one phase function $\Phi(t, r)$ is in a good agreement with the quantum-mechanical calculations about the liquid helium II proposed by Bogolubov and co-authors, in particular, with the Bogolubov transformations.

$$\hat{b_p} = u_p \hat{\xi}^+_p + v_p \hat{\xi}^-_p,$$

with

$$u_p^2 - v_p^2 = 1$$

and the creation (respectively, annihilation) operators

$$\hat{b}_p^+ = \frac{\hat{a}_p^+ \hat{a}_0^+}{\sqrt{n_0}}, \quad \hat{b}_p = \frac{\hat{a}_0^+ \hat{a}_p^+}{\sqrt{n_0}},$$

expressed through the "initial" creation (annihilation) operators entering actually the Bogolubov model Hamilton operator, may be represented as

$$\hat{H} = -\sum_{a=1}^{N} \frac{\hbar^2}{2m} \Delta_a + \frac{1}{2} V(|r_a - r_b|),$$

(with $V(|r_a - r_b|)$ being the interaction energy between particles $a$ and $b$ and $N$ being the complete number of particles in the considered system), brought in the diagonal form

$$\hat{H} = \hat{H}_0 + \sum_{p \neq 0} \epsilon(p) \hat{\xi}_p^+ \hat{\xi}_p$$

via the above Bogolubov transformations.

Herewith the creation (respectively, annihilation) operators $\hat{a}_0^+$ and $\hat{a}_0$ corresponds to the zero momenta $p = 0$ and the number $n_0$ of helium atoms possessing these zero momenta.
(altitude) of the cylindrical vessel (herewith chosen to coincide with the axis $z$ of the given rest reference frame).

The liquid helium II turning model \[13\] possesses initially the manifest $U(1) \otimes O(2)$ symmetry.

In this model $O(2)$ is the group of \textit{global} (rigid) potential rotations around the axis $z$ of the liquid helium II specimen and the cylindrical vessel where it is contained.

As it is well known, superfluidity phenomena \[14\] in a liquid helium II specimen are associated with violating the initial $U(1)$ gauge symmetry of the Bogolubov Hamiltonian $\hat{H}$. This second-order phase transition occurs in the fixed Curie point $T_c \to 0$.

To explain in this case rotary effects (coming to appearance of thread vortices) in the liquid helium II turning model \[13\] (as well as in the liquid helium II at rest theory \[15\]), it is worth to assume that violating the initial $U(1)$ gauge symmetry in the liquid helium II theory in such a wise that the $U(1) \simeq S^1$ group space turns into the discrete quite separated set \[32\].

More exactly, it may be assumed the spontaneous breakdown

$$U(1) \simeq U_0 \otimes \mathbb{Z} \rightarrow \mathbb{Z} \simeq \tilde{U}(1); \quad \pi_0 U_0 = \pi_1 U_0 = 0. \quad (3.31)$$

Geometrically, the discrete (quite separated) set $\tilde{U}(1)$ may be seen as the circle $S^1$ cut off by its topological domains (vice verse, to get the continuous $U(1)$ group space, these topological domains would be again pasted together).

On the other hand, in the liquid helium II turning model \[13\], the appropriate group $O(2)$ of rigid rotations remains exact.

This implies the (spontaneous) breakdown

$$O(2) \otimes U(1) \simeq O(2) \otimes U_0 \otimes \mathbb{Z} \rightarrow O(2) \otimes \mathbb{Z} \simeq O(2) \otimes \tilde{U}(1) \equiv H \quad (3.32)$$

of the initial symmetry group $O(2) \otimes U(1)$ inherent in the liquid helium II turning model \[13\].

Upon some mathematical manipulations, resembling somehow ones \[29\] led to Eq. \[3.39\] in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac, the degeneration space in the liquid helium II turning model \[13\] may be founded. It proves to be

$$[\mathbb{Z} \otimes O(2)] \otimes U_0 \equiv R, \quad (3.33)$$

and it is topologically equivalent to $S^1$.

The latter fact may be explaining by reasoning that the group $O(2)$ of $2 \times 2$ orthogonal matrices with determinants $\pm 1$ always may be represented as

$$SO(2) \otimes \mathbb{Z}_2,$$

with $SO(2) \simeq U(1)$ being the group of orthogonal matrices with determinants $+1$, while $\mathbb{Z}_2 \subset \mathbb{Z}$.
It is easy to see that $\pi_1 R = \mathbb{Z}$.

In detail, this may be argued by the above ascertained topological equivalence between the liquid helium II turning degeneration space $R$ and the circle $S^1$.

On the other hand, also $\pi_0 H = \mathbb{Z}$ due to the manifest presence of the discrete multiplier $\tilde{U}(1)$ in Eq. (3.32).

Thus the general isomorphism (2.1) [9] is satisfied in the case of the liquid helium II turning model [13].

In this concrete case it implies the presence of (rectilinear) thread topological defects (vortices) in the quested model.

These rectilinear vortices contribute with the item $\Delta E$ [13], (3.30), additional to the Bogolubov helium (diagonalized) Hamiltonian $\hat{H}$ [35].

Just this increasing $\Delta E$ the helium energy (in comparison with that given by $\hat{H}$ and referring to the helium specimen at rest) is the sign of the first order phase transition occurring.

Of course, there are walls between different topological domains inside the discrete liquid helium II turning degeneration space $R$, (3.33).

On the other hand, there are no point topological defects inside $R$ since $\pi_2 S^1 = 0$.

The case when a liquid helium II is at rest is somewhat simpler than the case [13] of the liquid helium II turning together with the cylindrical vessel where it is contained.

In the former case, to justify the spontaneous appearance of rectilinear vortices [13], it may be presumed that the initial $U(1)$ rigid symmetry group of the Bogolubov helium Hamiltonian $\hat{H}$ [35] is violated in the

$$ U(1) \rightarrow \tilde{U}(1) \quad (3.34) $$

wise.

As in the liquid helium II turning case [13], $\pi_0 \tilde{U}(1) = \mathbb{Z}$.

On the other hand, it may be demonstrated (applying the manipulations similar to those necessary in the liquid helium II turning case [13] to ascertain the look (3.33) for the appropriate degeneration space $R$) that the degeneration space $\tilde{R}$ in the liquid helium II at rest theory coincides with the residual gauge symmetry group

$$ \tilde{U}(1) \simeq U_0 \otimes \mathbb{Z}. $$

As it was explained in [15] (see also [3]), the appearance of rectilinear vortices in a liquid helium II specimen is set by Eq.

$$ n = \frac{m}{2\pi\hbar} \oint_{\Gamma} \mathbf{v}^{(n)} \, d\mathbf{l}; \quad n \in \mathbb{Z}. $$

This Eq. implicates the helium mass $m$ and the tangential velocity $\mathbf{v}^{(n)}$ of a rectilinear vortex.

In this case, trivial topologies $n = 0$ correspond to disappearing the cyclic integral on the right-hand side of latter Eq. It is just the case of superfluid potential motions in the given liquid helium II specimen with rot $\mathbf{v}^{(0)} = 0$.

In Ref. [3], the explanation of vortices [15] in a (rested) liquid helium II specimen as a particular case of the Josephson effect [17] (coming to circular persistent motions of material points) was given.
Thus, formally,
\[ \pi_1 \tilde{R} = \pi_0 \tilde{U}(1) = \mathbb{Z}, \]
and this just implies the existence of thread topological defects (rectilinear vortices) in the liquid helium II at rest theory.

The alone fact \cite{3,15} referring vortices in a liquid helium II at rest to nontrivial topologies \( n \neq 0 \), while superfluid potential motions are referred to trivial topologies \( n = 0 \), is very remarkable.

Thus the connection between the superfluid and rotary effects in the liquid helium II model and the topological degeneration of appropriate data may be observed.

Finally, it may be argued, as in the liquid helium II turning model \cite{13}, that the first-order phase transition occurs also in the liquid helium II at rest model \cite{15}.

It is interesting to compare our results about the liquid helium II turning model \cite{13} and those gotten in the monograph \cite{39}.

More exactly, in the monograph \cite{39}, the group of rigid space rotations \( SO(3)_L \) (involving the orbital momentum \( L \)) for a \( ^4 \text{He} \) specimen (instead of \( O(2) \) in our case) was considered. As the author \cite{39} asserts, the appearance of vortices course violating there the initial \( U(1) \) symmetry involves also violating \( SO(3)_L \) since the direction of the (one) vortex line appears as the axis of spontaneous anisotropy. This anisotropy line can be chosen as the axis \( z \) of the (fixed) reference frame.

In our case, with the (rigid) \( O(2) \) symmetry group, is a somewhat different situation since we already have two-dimensional (rigid) rotations about the fixed axis, which, in a natural way, can be chosen as the axis \( z \). Thus it is now an already ready anisotropy (and simultaneously symmetry) line, and there are no need for violating \( O(2) \) in this case, and the above 'topological' calculations (3.32) remain legitimate.

The author \cite{39} considers a complex scalar \( \Psi = |\Psi| \exp(i\Phi) \) as the order parameter for the superfluid \( ^4 \text{He} \). Note that the similar look for the order parameter (coinciding with the helium Bose condensate wave function) was proposed in the monograph \cite{13}:
\[
\Xi(t, r) = \sqrt{N(t, r)} \, e^{i\Phi(t, r)}, \tag{3.35}
\]
with \( N(t, r) \) being the number of particles in the ground energy state \( \epsilon = 0 \) (we shall continue the analysis of the helium Bose condensate wave function in Section 4).

The different from zero order parameter \( \Psi \) implies the complete breakdown of the \( U(1)_N \) symmetry.

For a vortex with the winding number \( n_1 \), it was set \cite{39} \( \Phi(t, r) \equiv n_1 \phi \).

It is natural to guess \cite{39} that the symmetry-breaking scheme in the presence of \( SO(3)_L \) is
\[
G' = U(1)_N \otimes SO(3)_L \to H' = U(1)_Q \quad \tag{3.36}
\]
Here the remaining symmetry group \( U(1)_Q \) is the symmetry of the order parameter in \( \Xi(t, r) \). It is the rotation by the angle \( \theta \) that transforms \( \phi \to \phi + \theta \), accompanied by
the global phase rotation $\Phi \rightarrow \Phi + \alpha$, with $\alpha = -n_1 \theta$. The generator of such $U(1)_Q$ transformations is
\[ Q = L_z - n_1 N. \] (3.37)
Thus, in the absence of the orbital momentum $L$ (i.e. the group $SO(3)_L$) in the theory [39], as well as in the absence of the group $O(2)$ of rigid rotations about the axis $z$ in our case, we come, in fact, to the same discrete topology of the degeneration space (it is obviously that the $n_1 N$ item of the above generator $Q$ set the same discrete structure as (3.34), i.e. the discrete degeneration space $\tilde{U}$).

This is a very important conclusion, since the $^4$He model (turning with the vessel where it is contained and at rest) was developed in this study even when the author did know about the research stated in [39].

Returning again to the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [4], note the very transparent analogy between the rotary item $\Delta E$, (3.30), in the liquid helium II turning model [13] and the free rotator action functional $W_N$, (1.11), in the former case.

The important distinction between the both theories is, however, in the logarithmic divergence of $\Delta E$, (3.30), as $R/a \rightarrow \infty$, while the free rotator action functional $W_N$, (1.11), is suppressed actually by the value of $\epsilon(\infty) \rightarrow 0$.

The just performed brief analysis of the liquid helium II turning model [13] (as well as of the liquid helium II at rest model) suggests, by analogy, the idea to link the various rotary properties [3, 5, 7, 8, 12] of the Minkowskian Higgs model to vacuum BPS monopoles quantized by Dirac (described correctly by the free rotator action functional (1.11)) with the discrete vacuum geometry [39], including thread topological defects inside the appropriate vacuum manifold $R_{YM}$ (in the similar way, has been outlined above, in which such topological defects induce rotary effects in the liquid helium II theory).

This will be the topic of the next section, where also some arguments in favour of the first-order phase transition occurring in the Minkowskian Higgs model with vacuum BPS monopole solutions quantized by Dirac and similar to that taking place in the liquid helium II theory (these arguments are connected with the "electric" and "magnetic" contributions to the total vacuum energy).

4 How topologically nontrivial threads arise inside the discrete non-Abelian vacuum manifold.

The outlines of this topic were us projected in Introduction.

It should be begun from the fact that the Minkowskian Higgs model [9, 10, 11] with vacuum BPS monopoles is the specific model where the duality in specifying the sign of the vacuum "magnetic" field $B$ takes place. This duality is just set by the Bogonol’nyi equation (1.1) [11].
This duality implies automatically the duality in specifying the signs of magnetic charges \(m\), given by the general formula [9]

\[
m = \frac{1}{4\pi} \int_{\Gamma} dS \cdot B - \frac{1}{8\pi} \int d^3x \partial_i \{ \epsilon^{ijk} < F_{jk}^b, \Phi_b > a^{-1} \}.
\]  

(4.1)

Note that latter Eq. is irrelatively correct, actually, to the concrete choice of the vacuum geometry: either "continuous" or "discrete" one, in the quested Minkowskian Higgs model.

Really, the definition (4.1) of magnetic charges \(m\) may be recast formally to the look

\[
m = \frac{1}{4\pi} \int_{\Gamma} dS \cdot B \sim \frac{1}{4\pi} \int_{V} \text{div} \, B \, dV,
\]  

(4.2)

and this integral does not depend on the choice of the space-like surface in the (Minkowski) space. On the other hand, as it follows from this equation, if

\[
<B> \equiv \int_{V} d^3 x B = 0,
\]

this does not implies \(m = 0\). It is an important conclusion we widely utilize in the present study.

However, the actual divergence of the "magnetic" tension \(B\) at the origin of coordinates in all the Minkowskian Higgs models (for instance, [10, 26, 27]) implies that the latter integral is different from zero as the origin of coordinates lies inside the chosen space-like surface \(\Gamma\).

Furthermore, the above duality [11] in specifying the sign of the vacuum "magnetic" field \(\Phi\) affects also the generator \(h(\Phi)\) [9] of the residual \(U(1)\) gauge group in the studied Minkowskian Higgs model.

It is so since \(h(\Phi)\) may be read easily from Eq. (4.1).

Now let us return to the simpler case of "continuous" \((\sim S^2)\) vacuum geometries in Minkowskian Higgs models.

In this case the appropriate degeneration space: say, \(R\), is one-connected:

\[
\pi_0(H) = \pi_1(R) = 0,
\]

and one can always choice, in the unique way, a continuous branch of the function \(h(\Phi)\) on \(R\).

Another situation is when the degeneration space \(R\) is multi-connected ("discrete"), i.e. when

\[
\pi_0(H) = \pi_1(R) \neq 0,
\]

and therefore there exist thread topological defects in the gauge theory in question (in particular, in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [4]).

On the one hand (as we have seen this above), the nontrivial isomorphism (2.1) [9] implies the existence of thread topological defects inside the discrete vacuum manifold \(R\).
On the other hand, it turns out (see §Φ13 in [9]) that the duality in the choose of the sign of \( h(\phi) \) (and therefore also the duality [11] in the choose of the direction of \( B \)) may become unavoidable.

This occurs, in particular, when one can invert the function \( h(\Phi) \) via the gauge transformations

\[
\tilde{h}(\Phi) = \alpha h(\Phi) \alpha^{-1} = -h(\Phi),
\]

with

\[
\alpha = \exp(2\pi M)
\]

and the element \( M \) belonging to the Lee algebra of the group \( G_1 \) of global transformations compensating changes in the vacuum (YM-Higgs) configuration \( (\Phi^a, A^a_{\mu}) \) at rotations around the axis \( z \) of the chosen reference frame.

The general look of an element \( g_\theta \in G_1 \) may be set by Eq. (2.5) [9].

Herewith our above discussion about thread topological defects inside degeneration spaces in the liquid helium II models (turning [13] and at rest [15]) prompts us the idea to associate the rectilinear threads (as infinite thin tubes of infinite lengths) with the above axial symmetry.

In the monograph [9] (in §Φ12) the strict proof was given that the unitary group \( G_1 \) is a possible symmetry of alike threads (by analogy with the group \( O(2) \) of orthogonal \( 2 \times 2 \) matrices with determinants \( \pm 1 \), one utilizes in the liquid helium II turning model [13]).

Omitting details, note that the proof of this statement comes to the proof that in each topological class rectilinear threads exist giving a finite contribution to the appropriate energy integrals (due to first-order phase transitions occurred) and satisfying the appropriate equations of motion [9].

The topological type of a thread defect is determined by the group \( G_1 \): more exactly by its generator \( M \).

To prove this statement, we betake to the argument of the holonomies group [7, 9].

Let us rewrite an element (3.28) of the holonomies group \( H \simeq U(1) \) in the shape

\[
\alpha = (P \exp(- \oint A_{\mu} dx^\mu))^{-1}.
\]  

(4.5)

Due to the standard Pontryagin degree of a map theory, elements \( \alpha \) of the holonomies group \( H \) depend on integers \( n \): \( \alpha \equiv \alpha(n) \ (n \in \mathbb{Z}) \).

Passing then to the cylindrical coordinates, we come to Eq. (2.10) [9], implicating rectilinear (topologically nontrivial) threads \( A_\theta \), may be represented as (2.4).

Alternatively, \( A_\theta \) may be represented also as [9]

\[
A_\theta(\rho) = M + \beta(\rho),
\]

(4.6)

where the function \( \beta(\rho) \) approaches zero as \( \rho \to \infty \).

This also ensures [9] a finite energies densities in Minkowskian Higgs models involving (topologically nontrivial) threads.

Thus at the spatial infinity elements \( \alpha(n) \in H \) acquire the look

\[
\alpha(n) = \exp(2\pi M).
\]  

(4.7)
We see that the dependence of gauge fields $A$ on the angle $\theta$ disappears at the spatial infinity.

It is equivalent to damping, at the spatial infinity, various rotary effects, associated in Minkowskian non-Abelian theories with thread topological defects.

In particular, it is correctly for the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [4] and involving the “discrete” vacuum geometry (3.2).

On the other hand, the function $\beta(\rho)$ should be different from zero effectively only in the $\epsilon$-neighbourhood of the origin of coordinates to ensure in this infinitesimal region the above pointed collective rotary phenomena in the appropriate Minkowskian YM vacuum via the nontrivial dependence of $A_{\theta}(\rho, \theta, z)$ on $\theta$.

Indeed, above described (vacuum) YM modes $A_{\theta}(\rho, \theta, z)$ would be different from vacuum YM BPS monopoles (for instance, in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac).

It is associated with the obvious divergence as $r^{-2}$ (c.f. (3.29) [29]) of the vacuum ”magnetic” field $B$ at the origin of coordinates in Minkowskian non-Abelian models involving vacuum BPS monopole solutions.

In this case there is impossible to link YM fields $A_{\theta}(\rho, \theta, z)$, representing topologically nontrivial (rectilinear) threads in the Minkowskian Higgs model to vacuum BPS monopoles quantized by Dirac, with vacuum YM BPS monopole solutions (with same topological numbers), at least in a smooth wise (that is connected with the $\sim \partial_{\mu}A_{\nu}$ items always entering the ”magnetic” field $B$ in YM theories).

On the other hand, the value $< B^2 >$, the vacuum expectation value (VEV) of the ”magnetic” field $B$ squared, can serve [8] as the order parameter in Minkowskian Higgs models with YM fields.

Herewith its nonzero value, $< B^2 > \neq 0$, is the sign violating the initial $SU(2)$ gauge symmetry group down to its $U(1)$ subgroup, and it is the second-order phase transition [6, 30].

The natural question arise wherein. Why $< B^2 >$ but not the VEV of the squared Higgs field (in the shape of a Higgs BPS monopole mode) serves in our case as the order parameter? The explanation is the following [8]. According to Eq. (3.14) above, the Higgs (effective) mass $m/\sqrt{\lambda}$ (where $m$ and $\lambda$ are the Higgs mass and self-interacting constant, respectively) goes to infinity in the limit $V \to \infty$ at assuming that $< B^2 >$ is finite in this limit. In this case [8] the scalar (Higgs) field acquires an infinitely large mass and disappears from the spectrum of physical excitations. Thus the role of the order parameter of the physical BPS monopole vacuum is ”fixed” for $< B^2 >$ in this infinite volume limit.

If the Minkowskian Higgs model with YM fields implies the continuous $\sim S^2$ vacuum geometry, the possible (as in the BPS or ’t Hooft-Polyakov models) divergence of $B$ at the origin of coordinates testifies [6] in favour of point hedgehog topological defects located just at the origin of coordinates.

The origin of such topological defects is in the nontrivial isomorphism (2.2).
This singularity of (vacuum) "magnetic" fields can be removed by regularising appropriate energy integrals.

Also there are no contributions from point hedgehog topological defects into these energies integrals unlike the case of first-order phase transitions when such contributions appear (as, for instance, $\Delta E$ \[13\], (3.30), in the case of a liquid helium II specimen turning together with the cylindrical vessel where it is contained).

Unlike the former case, gaps in the directions of $B$ and $B_1$ in the Minkowskian YM-Higgs model with vacuum BPS monopoles quantized by Dirac and involving the "discrete" vacuum geometry (3.9) according to our assumption are evidences in favour of the first-order phase transition occurring in this model.

This is in a good agreement with the general theory of first-order phase transitions (discussed, for instance, in Ref. 30, in §3.1; see also [8]).

The discontinuous behaviour of plots for order parameters is just the sign of first-order phase transitions occurring in physical theories.

Indeed, to prove that the first-order phase transition takes place in the, we discuss now, Minkowskian YM-Higgs model with vacuum BPS monopoles quantized by Dirac, a large job is necessary. In particular, it is necessary to analyze the first derivatives of the thermodynamics potentials in the both mentioned thermodynamics phases: the "rotary" and "superfluid" inside the YMH vacuum quantized by Dirac. In the present study we do not set ourselves as the goal the such analysis, but nevertheless we would like make the one important notice already here.

A some insight in the analysis of first-order phase transitions prompts us that it should be

$$< B_1^2 > = 0,$$

i.e. one can presume that the "rotary" component of the YMH vacuum quantized by Dirac can be interpreted as a "false" vacuum. We emphasize that this is a preliminary assumption, need a further research: checking that at $< B_1^2 > = 0$ the free energy of the YMH vacuum quantized by Dirac reaches a minimum and that this minimum is not the absolute one, but this is beyond the scope of this article.

Due to Eq. (2.6), the "false" vacuum condition (4.8) comes (roughly) to the condition

$$\partial_\rho A_\theta (\rho, \theta, z) = 0.$$

The above discussed gaps do not influence however the "right" of gauge matrices $\alpha(n)$, (2.10) (taking account of the "discrete" vacuum geometry (3.9), involving [topologically nontrivial] threads), to represent the holonomies group $H$ coinciding with the residual gauge symmetry group $U(1)$ in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [4].

This can be explained by the natural isomorphism between exponential multipliers $\alpha(n)$, referring to (topologically nontrivial) threads arising because of the "discrete" vacuum geometry (3.9), and Gribov topological multipliers $v^{(n)}(x)$, referring to superfluid potential motions in the above model.
An interesting consequence of Eq. (4.8) is the following. The "step voltage" \( \phi_2(n_1 + 1) - \phi_1(n_1) \) between "neighboring" topological sectors results, due to this equation, \( < B_{n+1} - B_n > = 0 \). This can be considered as a kind of boundary conditions (at the spatial infinity) imposed onto domain walls between different topologies.

Generally speaking, the said in the present study about the "step voltage" between neighboring topological domains inside the vacuum manifold \( R_{YM} \) implies theirs infinitesimal (disappearing) contribution in the complete vacuum Hamiltonian of the order \( O(r^\alpha) \) (with \( \alpha > 0 \)) as \( r \to 0 \). Respectively, in terms of the typical domain weight \( \epsilon = \sqrt{\lambda/m} \neq 0 \), such contribution is of the order \( \epsilon^\alpha \). Thus such contribution can be neglected.

In this is the specific of the domain walls taking place between different topologies inside the vacuum manifold \( R_{YM} \) in the Minkowskian YNH model with BPS vacuum monopole solutions quantized by Dirac. The existence of such domain walls is necessary to justify the (topologically nontrivial) dynamics [3] inherent in this model (i.e. the existence of thread topological defects); but simultaneously, as we have discussed above, the vanishing contribution in the complete vacuum Hamiltonian.

Note that the dependence of the theory considered on nontrivial topologies \( n \) also disappears at the spatial infinity, i.e. in the infrared region of transferred momenta.

It is the next in turn evidence in favour of the infrared topological confinement (in the spirit [1, 23]) occurring in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac.

This "confluence" of different topologies \( n \) in the infrared region of the momentum space is closely related with the tendency of domain walls between different topological sectors of the vacuum manifold \( R_{YM} \) to become more "thin" as the spatial volume \( V \) approaches infinity.

In this limit, the typical width of a domain wall, \( \epsilon (V) \), goes, indeed, to zero, as it was discussed above (see (3.14)).

The above discussed examples of liquid helium II at rest [15] and turning together with the cylindrical vessel where it is contained [13] give us pattern of physical systems in which first-order phase transitions occur.

The "center of gravity" of the enumerated models is in the appearance of (rectilinear) threads generated appropriate vortices, i.e. thread topological defects.

On the other hand, it was demonstrated above (repeating the arguments [13]), in the case of a liquid helium II specimen turning together with the cylindrical vessel where it is contained, the "rotary" contribution \( \Delta E \), (3.30), supplements the typical (diagonalized) Bogolubov Hamiltonian \( \hat{H} [35] \).

It is the sign of the first-order phase transition taking place in the liquid helium II turning model [13].

Herewith according to the modern terminology [6], this "rotary" contribution \( \Delta E \), (3.30), is the latent heat, while the thermodynamic phase in whose framework collective solid potential rotations [13, 15] in a liquid helium II specimen turning together with
the cylindrical vessel where it is contained coexist with superfluid motions setting by the Hamiltonian \( \hat{H} \) is characterized by the supercooling phenomenon.

In this case superfluid motions inside a liquid helium II specimen represent the stable state corresponding to the second-order phase transition occurred therein and coming to violating the \( U(1) \) gauge symmetry of the helium Hamiltonian \( \hat{H} \) \(^{14}\) (in the \( 3.32 \) wise in the liquid helium II turning case \[13\]).

In turn, collective solid potential rotations in this specimen are referred to the metastable state (in the terminology \[30\]).

There are concrete computations, given in Ref. \[35\] with the example of a rested liquid helium II specimen, demonstrating discontinuities in the plot of the vacuum expectation value \( \langle |\Xi|^2(t, r) \rangle \) for the helium Bose condensate wave function \( \Xi \), \( 3.35 \), near locations of vortices cores.

Herewith the value \( \langle |\Xi|^2(t, r) \rangle \) can serve quite naturally as the order parameter in the Landau-Bogolubov helium theory \[14, 36\].

So \( \langle |\Xi|^2(t, r) \rangle = 0 \) for the helium I modification \[40\], corresponding to the initial \( U(1) \) gauge symmetry of the helium Hamiltonian \( \hat{H} \) \[35\], while \( \langle |\Xi|^2(t, r) \rangle \neq 0 \) upon this symmetry is violated in the \( 3.31 \) wise, that corresponds to the helium II modification.

Then, utilizing Eq. \( 3.35 \) \[6, 13\], the correlation (Green) function

\[
G(r) \equiv \langle \Xi(t, r), \Xi^*(t, 0) \rangle
\]

(it is different from zero for the liquid helium II modification) may be recast to the look \[35\]

\[
G(r) = \langle \Xi(t, r)\Xi^*(t, 0) \rangle
= \Xi_0^2 \exp \left[ -\frac{1}{2}\langle [\Phi(t, r) - \Phi(t, 0)]^2 \rangle \right].
\]

To a first approximation, we can neglect fluctuations in the amplitude of \( \Xi \), setting it to be a \( \Xi_0 \). On the other hand, the above discussed expression \[15\] for topological indices \( n \in \mathbb{Z} \) via circular velocities \( v^{(n)} \) is mathematically equivalent to \[35\]

\[
\frac{m}{2\pi\hbar} \oint G \Phi \, dl = \int_{\Omega} d^2 r \, n_{v}(r),
\]}

\(^{14}\)The transparent evidence in favour of the second-order phase transition occurring in a helium: actually, it is the transition between the liquid helium I high-temperature and liquid helium II modifications (the latter one possesses manifest superfluid properties described in the Landau-Bogolubov theory \[14, 35\]) is the discontinuity \[40\] in the plot of the helium heat capacity in the Curie point \( T_c \to 0 \).

For instance \[34\], in the case of superfluid helium films, the experimental data result

\[
C_p(T) \equiv T \left( \frac{\partial S}{\partial T} \right) \approx c_1 + c_2 \exp[c_3/(|T - T_c|)^{1/2}]
\]

for the helium heat capacity \( C_p(T) \), with \( S \) being the appropriate entropy; \( c_i \) (\( i = 1, 2, 3 \)) are constants.
with the vortex ”charge density” given as

\[ n_v(r) = \sum_{\alpha=1}^{N} n_\alpha \delta(r - r_\alpha) \quad (4.11) \]

for a collection \( N \) of vortices located at positions \( \{r_\alpha\} \) with integer charges \( \{n_\alpha\} \).

Without loss of generality, it may be set \( z = 0 \) in all the calculations following.

In (4.10), Eq. (1.3) \([6, 13]\), with appropriate replacing \( v_0 \rightarrow v^{(n)} \), was utilized.

Applying then the Stokes formula to (4.10), one find upon integrating:

\[ \epsilon_{ij} \partial_i \partial_j \Phi(t, r) = \partial_x \partial_y \Phi - \partial_y \partial_x \Phi \approx n_v(r). \quad (4.12) \]

To cast this Eq. in a more familiar form, one can introduce \([35]\) the value \( \tilde{\Phi}(t, r) \) dual to \( \Phi(t, r) \):

\[ \partial_i \Phi(t, r) = \epsilon_{ij} \partial_j \tilde{\Phi}(t, r). \quad (4.13) \]

Then

\[ \nabla^2 \tilde{\Phi}(t, r) = n_v(r). \quad (4.14) \]

In particular,

\[ \nabla^2 \tilde{\Phi}(t, r) = 0 \quad (4.15) \]

in the case when vortices are absent, i.e. in the case of purely superfluid motions inside a liquid helium II specimen.

Eq. (4.14) is the particular case of the Poisson equation. Thus it permits its

\[ \tilde{\Phi}(t, r) = \sum_{\alpha} n_\alpha G(r, r_\alpha) \quad (4.16) \]

solution, where the Green function \( G(r, r_\alpha) \) (defined as in the left-hand side of Eq. (4.9)) satisfies

\[ \nabla^2 G(r, r_\alpha) = \delta(r - r_\alpha). \quad (4.17) \]

For \( |r - r_\alpha| \) large enough and both points far from any boundaries, the Green function \( G(r, r_\alpha) \) has the look (see §11.8 in \([41]\))

\[ G(r, r_j) \approx \frac{1}{2\pi} \ln \left( \frac{|r - r_j|}{\xi_0} \right) + C \quad (4.18) \]

on the plane \( z = 0 \), where \( C \) is a constant which contributes to the vortex core energy.

The origin of the parameter \( \xi_0 \) entering Eq. (4.18) is following \([35]\).

It is the characteristic length related to the coefficients of the first two terms of Eq.

\[ \frac{F}{T} = \int d^2 r \left[ \frac{1}{2} A |\nabla \Xi|^2 + \frac{1}{2} a |\Xi|^2 + b |\Xi|^4 + \cdots \right] \quad (4.19) \]
for the helium free energy $F$.

In this case $\xi_0$ may be defined as \[35\]

\[\xi_0 = \sqrt{A/|a|}, \tag{4.20}\]

with $a$ given as $a = a'(T - T_c)$.

As it follows from (4.18), at setting $C = 0$, the Green function $G(\mathbf{r}, \mathbf{r}_j)$ diverges logarithmically when $|\mathbf{r} - \mathbf{r}_j| \to \infty$, while it can approach zero when

\[|\mathbf{r} - \mathbf{r}_j| \to \xi_0.\]

Additionally, $G(\mathbf{r}, \mathbf{r}_j)$ diverges logarithmically also at $|\mathbf{r} - \mathbf{r}_j| \to 0$. The latter property of $G(\mathbf{r}, \mathbf{r}_j)$, as we shall see soon, is very important.

The just performed analysis of $G(\mathbf{r}, \mathbf{r}_j)$ allows to draw the series important conclusions about the phase $\tilde{\Phi}(t, \mathbf{r})$ of the helium wave function $\Xi$ (taking over the complete collection $N$ of topologies: including the trivial one $s_\alpha = 0$, corresponding to superfluid motions \[14, 15\], in a liquid helium II specimen at rest).

So, due to (4.9),

\[<(|\Xi|^2(t, \mathbf{r}))> \equiv <\Xi(t, \mathbf{r}), \Xi^*(t, \mathbf{r})> \approx G(0) = 1/N \sum_\alpha <|\Xi|^2|_\alpha(t, \mathbf{r})>, \tag{4.21}\]

serving the order parameter in the helium theory, diverges logarithmically at each index $\alpha(n)$ ($n \in \mathbb{Z}$) in the same (4.18) \[35\] sense that the Green function $G(\mathbf{r}_\alpha, \mathbf{r}_\alpha) \approx G(0)$, i.e. in the points $\{\mathbf{r}_\alpha\}$ where quantum vortices are located in a liquid helium II specimen at rest \[15\].

This is associated immediately with (rectilinear) quantum vortices arising spontaneously \[15\] in a liquid helium II specimen at rest.

Herewith it is not important that also the phase $\Phi(t, \mathbf{r})$ (or $\hat{\Phi}(t, \mathbf{r})$) of the helium wave function $\Xi$ possesses the same behaviour that the Green function $G(\mathbf{r}, \mathbf{r}_j)$.

The said is correctly for nontrivial vortices topologies $n \neq 0$ due to Eq. (4.16). For $n = 0$, i.e. in the ”superfluid case”, $\Phi(t, \mathbf{r})$ becomes an uncertain value when $|\mathbf{r} - \mathbf{r}_j| \to \infty$ (or when $|\mathbf{r} - \mathbf{r}_j| \to 0$).

This implies \[35\] zero’s of the helium Bose condensate wave function $\Xi(t, \mathbf{r})$ in the points $\{\mathbf{r}_\alpha\}$ in which (rectilinear) quantum vortices \[15\] are located in a rested liquid helium II specimen.

On the other hand, the phase $\Phi(t, \mathbf{r})$ (or $\hat{\Phi}(t, \mathbf{r})$) of the helium wave function $\Xi$, \[3.35\], disappears from Eq. (4.21).

Summarizing, now one can assert that in the presence of (rectilinear) quantum vortices \[15\] in a liquid helium II specimen, the order parameter $<|\Xi|^2(t, \mathbf{r})>$ in the helium 15In (4.21) the Green function $G(0)$ is different from that given via Eq. (4.9) and involving the constant amplitude $\Xi_0$.

Now the amplitude of $\Xi(t, \mathbf{r})$ wouldn’t be constant, and this implies that $G(0)$ has the nontrivial look $G(\mathbf{r}_\alpha, \mathbf{r}_\alpha) \approx G(0)$.
model suffers discontinuities of the logarithmic nature in the points \( \{ r_\alpha \} \) in which these (rectilinear) quantum vortices are located (the long distances logarithmic singularities of \( G(0) \) are, probably, not associated with quantum vortices).

The discovered discontinuity (of the logarithmic nature) in the plot of the vacuum expectation value \( < |\Xi|^2(t, r) > \) for the helium Bose condensate wave function \( \Xi \), serving the order parameter in the helium theory, just testifies in favour of the first-order phase transition occurring in a liquid helium II rested specimen with arising therein quantum vortices.

There are, indeed, lot of distinctions between the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac \[4\] and the liquid helium II at rest theory \[15\] (the same concerns also the liquid helium II turning model \[13\], involving rigid \( O(2) \) rotations).

But, in spite of these distinctions, the discontinuities in the appropriate order parameters: \( < |\Xi|^2(t, r) > \) in the helium theory and \( < B^2 > \) in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac, can be considered as the common trait of the both models associated with rotary effects (including the "discrete" vacuum geometry and thread topological defects).

Moreover, drawing a parallel between the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac and the liquid helium II at rest theory \[15\], one can refer collective solid rotations inside the appropriate BPS monopole vacuum to the metastable thermodynamic phase, characterised by the latent heat equal to the "electric" energy \( \sim E^2 \). This "electric" energy is described correctly by the free rotator action functional \( (1.11) \).

Simultaneously, superfluid potential motions inside this vacuum, involving the "magnetic" energy \( \sim B^2 \), may be referred to the stable thermodynamic phase.

It is again the supecooling situation.

As it was discussed in Section 2, on the face of it, there is a contradiction between the rotary effects in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac (these effects, including collective vacuum rotations, are determined by the action functional \( (1.11) \)) and the manifest potential nature of the Minkowskian BPS monopole vacuum.

More exactly, this potential nature of the Minkowskian BPS monopole vacuum (suffered the Dirac fundamental quantization \[4\]) comes, in particular, to the colinearity \( (2.9) \) \[16\] of vectors \( \mathbf{B} \) and \( \mathbf{E} \) (vacuum "magnetic" and "electric" fields, respectively).

This is associated with the explicit look \( (1.13) \), \( (1.15) \) \[5, 7, 8, 12\] for vacuum "electric" monopoles \( \mathbf{E} \), can be expressed through vacuum Higgs BPS monopole modes \( \Phi_{(0)}(\mathbf{x}) \).

Also the vacuum "magnetic" field can be expressed through vacuum Higgs BPS monopole modes \( \Phi_{(0)}(\mathbf{x}) \), now through the Bogomol’nyi equation \( (1.1) \).

There is however a way to solve the above contradiction.

It turns out that, side by side with (topologically nontrivial) threads \( A_\theta \), one can construct also specific vacuum Higgs \( z \)-invariant solutions \[9\], having the look \( (2.7) \) in the
cylindrical coordinates, such that the vacuum Higgs field $\Phi^{(n)}_a$ specified by (2.7) merges with the appropriate Higgs vacuum BPS monopole solution in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [4], herewith belonging to the same topology $n$.

This may be achieved by the appropriate choice of the function $\phi(\rho)$ in (2.7).

Additionally, we should also claim that the covariant derivative $D\Phi$ of any vacuum Higgs field $\Phi^{(n)}_a$ specified via (2.7) merges with the covariant derivative of such a vacuum Higgs BPS monopole solution.

In other words, vacuum Higgs solutions (2.7), corresponding to the above described axial symmetry $G_1$, would merge with appropriate Higgs vacuum BPS monopole solutions actually in a smooth wise.

By this the goal is pursued to link $D\Phi^{(0)}_a$ to vacuum ”electric” monopoles (1.13) in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac.

Indeed, this confluence of vacuum Higgs solutions would occur already at the notably lesser scale of distances: at the hadronic distances scales, $\sim 1$ fm., to ensure the correct infrared behaviour [23] of quark Green functions in the Minkowskian (constraint-shell) QCD.

Actually, the following picture takes place.

As it was noted in Ref. [20], Higgs BPS monopole solutions disappear at the origin of coordinates (in its infinitesimal neighbourhood of the effective radius $\epsilon$).

In this situation the continuous and smooth prolongation of Higgs vacuum BPS monopole solutions with appropriate Higgs modes $\Phi^{(n)}_a$ [9], (2.7), seems to be quite natural. It is controlled by the proper choice of $\phi(\rho)$.

It is just the way solving the above discussed problem how to co-ordinate the ”colinearity condition” (2.9) [16] (reflecting the superfluid nature of the Minkowskian BPS monopole vacuum suffered the Dirac fundamental quantization [4]) with collective solid rotations inside this vacuum.

Really, Higgs (vacuum) modes $\Phi^{(n)}_a$, (2.7), refer to the rigid axial symmetries group $G_1$ [9].

This group, in turn, compensates local rotations of the vacuum (YM-Higgs) configuration $(\Phi^{(n)}_a, A^a_\mu)$ around the axis $z$.

On the other hand, this vacuum (YM-Higgs) configuration can be linked, in a continuous and smooth wise in its ”Higgs” part, to Higgs vacuum BPS monopoles, disappearing at the origin of coordinates [20].

By that two things are ensured.

Firstly, the continuity (and smoothness) achieved now for Higgs vacuum solutions, ”legalizes” collective solid rotations inside the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [4] (as a sign of thread topological defects inside the appropriate vacuum manifold $R_{YM}$) within the $\epsilon$-neighbourhood of the origin of coordinates. These rotations are described correctly by the action functional (1.11) [5].

Secondly, simultaneously, the Gauss law constraint (1.8) and the ”colinearity condition” (2.9) are satisfied outward of this infinitesimal region with smooth vacuum ”electric”
monopoles (1.13).

When we view very attentively Eq. (1.12) [5] for the ”rotary momentum” $I$ of the BPS monopole vacuum, we can note that the integration there is carried out over the infinite spatial volume $V \to \infty$.

At neglecting the infinitesimally small interval $[0, \epsilon(\infty)]$ in this integral, one get nevertheless the same result.

More exactly, the integral (1.12) may be evaluated in the following way [5]:

$$
I \simeq \frac{4\pi^2}{\alpha_s} \int_\epsilon^R \frac{d}{dr} \left( r^2 \frac{d}{dr} f_0^{BPS}(r) \right); \quad R \to \infty
$$

This involves the ”Higgs” BPS ansatz [7, 8]

$$
f_0^{BPS}(r) = \left[ \frac{1}{\epsilon \tanh(r/\epsilon)} - \frac{1}{r} \right],
$$

disappearing [20] at the origin of coordinates.

In the paper [5] it was proposed the following approximation for $f_0^{BPS}(r)$:

$$
f_0^{BPS}(r) \sim 1 - \frac{\epsilon}{r}
$$

with the asymptotics $f_0^{BPS}(\epsilon) \to f_0^{BPS}(0) \to 0$, while $f_0^{BPS}(\infty) \to 1$ (since $\epsilon$ also disappears in the infinite volume limit [5]). It should be taken $\epsilon(\infty) \to 0$ for $f_0^{BPS}(\epsilon)$ always in the above calculations. This result is very important and we will need it later.

We see that the extrapolation of Higgs vacuum BPS monopoles inward the interval $[0, \epsilon]$, where the potential superfluid nature of the Minkowskian BPS monopole vacuum suffered the Dirac fundamental quantization [4] is violated, does not affects the integrals (1.12), (4.22) so long as the integration limits in Eq. (4.22) are $[\epsilon, \infty]$.

This is just the region where the potential superfluid nature of that vacuum takes place.

It is determined, in the Dirac fundamental quantization scheme [4], by the Bogomol’nyi equation (1.1), YM Gauss law constraint (1.8) and Gribov ambiguity equation (1.2) as the logical consequence [2, 7, 8] of the Bogomol’nyi equation (1.1).

As to the interval $[0, \epsilon]$ (here, indeed, one must take $\epsilon = \epsilon(\infty) \to 0$), integrating inside this interval comes to integrating over the base of the infinitely narrow cylinder along the axis $z$ (in the chosen rest reference frame), so long as the Higgs BPS ansatz $f_0^{BPS}(r)$ is a function of the distance $r$ only. Inserting $f_0^{BPS}(r)$ in the approximation (4.24) in (4.22) and changing interval ibid to $[0, \epsilon]$, we see that the interval $[0, \epsilon]$ results a vestigial contribution to the ”rotary momentum” $I$, (1.12), and one can neglect it actually.

Indeed, due to required continuous stitching together Higgs vacuum BPS monopoles and Higgs ”rotary” modes (2.7) (and thus their covariant derivatives $D^2$), it is possible to write out explicitly the condition at which this stitching together should occur.
As we have already mentioned in Introduction, this comes to evaluating the quantity $\delta$ entering Eq. (2.3) for $\phi(\rho)$.

Really, utilizing the look (4.23) for the ansatz $f^{BPS}_0(r)$, it is easy to get the following transcendental equation

$$\frac{1}{r^2} - \frac{1}{\epsilon^2 \sinh^2(\frac{r}{\epsilon})} \sim r^{-2-2\delta}. \quad (4.25)$$

It is not a simple equation, and probably some numerical methods are needed for solving it. It is, however, beyond the scope of our work.

On hand thus a delimitation between superfluid potential motions and collective solid rotations inside the Minkowskian BPS monopole vacuum suffered the Dirac fundamental quantization [4].

Superfluid potential motions therein (controlled by the Bogomol’nyi and Gribov ambiguity equations) are referred actually to the spatial interval $r \in [\epsilon(\infty); \infty]$, while collective solid rotations to the spatial interval $r \in [0, \epsilon(\infty)]$.

This delimitation is the next in turn evidence in favour of the first-order phase transition occurring in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [4] and coming to the coexistence of two thermodynamic phases inside the Minkowskian BPS monopole vacuum suffered the Dirac fundamental quantization.

The thermodynamic phases of superfluid potential motions and collective solid rotations are just such phases.

To this stage of investigations about the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac, we cannot conclude regarding the concrete nature of the boundary between the mentioned thermodynamic phases as well as regarding various surface effects.

At the same time, we should like note that the transparent way to verify that the first-order phase transition occurs in any physical model is to compute the appropriate surface tension integral [13]

$$\alpha_{12} = \int_{-\infty}^{\infty} (f_1 - f_2) \, dx, \quad (4.26)$$

where $f_1$ and $f_2$ are the free energies densities referring to two thermodynamic phases coexisting at the first-order phase transition occurring in the given physical model.

Meanwhile, already now some conclusions may be drawn concerning to the domain structure of the Minkowskian vacuum manifold $R_{YM}$, (3.9). It is just associated with the $\epsilon(V) \sim V^{-1}$ dependence [7, 8].

So, intimately to the axis $z$ of the chosen rest reference frame domain walls acquire finite widths (of the order $\epsilon(0) \neq 0$, as it was explained in Section 2).

One can assume herewith that in the mentioned spatial region, topological domains inside $R_{YM}$ (at neglecting their thickness) possess the geometry of planes (perpendicular to the axis $z$.
Vice verse, at the spatial infinity topological domains merge (as it was noted above), while walls between them become infinitely thin (of the typical wide $\epsilon(\infty) \to 0$) and this promotes the infrared topological confinement \cite{23} of "large" multipliers $v^{(n)}(x)$ in the (quark, gluonic) Green functions in all the orders of the perturbation theory.

Actually, since the confluence of topological domains would occurs already at the hadronic distances scales, $\sim 1$ fm., one can speak that only in the spatial region intimately near the axis $z$ of the chosen rest reference frame the "discrete" vacuum geometry $R_{YM}$, (3.9), in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac \cite{4} is different from the continuous $\sim S^2$ vacuum geometry, (3.1), proper to the "classical" Minkowskian Higgs model with BPS monopoles \cite{9,10,11}, involving the "heuristic" FP quantization scheme \cite{28}, associated with fixing the $A_0 = 0$ gauge \cite{6}.

Nevertheless, this distinction in vacuum geometries plays the crucial role, as we see from the current discussion, for understanding the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac.

The one more important lesson we have learned from our above discussion is that the Higgs field in here studied model loses its role as an order parameter due to our claim that "the covariant derivative $D\Phi$ of any vacuum Higgs field $\Phi_a^{(n)}$ specified via (2.7) merges with the covariant derivative of vacuum Higgs BPS monopole solution". If we wish to construct (or if we presume) the gauge theory involving a first-order phase transition, such behavior of Higgs modes in no case the necessary behavior of order parameter, which must undergo a discontinuity. And, on the contrary, the "magnetic" field squared, which has the intermittent shape

$$< B >^2 = \begin{cases} < B_1 >^2 = 0, & r \to 0; \\ < B_2 >^2 \neq 0 & r \to \infty \end{cases}$$  \hspace{1cm} (4.27)

(where $r^2 = x^2 + y^2$), is a suitable "candidate" to be an order parameter for the first-order phase transition anticipated in our YMH model with vacuum monopole solutions quantized by Dirac.

But it should be remembered that in the spatial region $r \neq 0 (r \to \infty)$ the "magnetic" field $B_1$ is completely determined by the Bogomol'nyi equation (1.1), i.e. again via the Higgs scalar $\Phi$: more exactly, via its covariant derivative $D$. \cite{10}

Herewith the alone Higgs vacuum BPS monopole modes in the $x \to \infty$ limit acquire infinitely large masses limit and disappear from the spectrum of physical excitations \cite{8}. This can be seen from the estimate \cite{8}

$$\frac{1}{\epsilon} = \frac{gm}{\sqrt{\lambda}} = \frac{g^2 < B_2^2 > V}{4\pi}.$$ \hspace{1cm} (4.28)

for the natural and unique nonzero mass scale in the here discussed model. It is obvious from here that in the infinite volume limit $V \to \infty$ and when the coupling YM constant $g$ remains fixed (in the confinement region $x \to \infty$).

\footnote{Since $< B_2 >^2$ is a gauge invariant, then also $< \Phi >^2$ is gauge invariant.}
Thus a specific transmutation occurs in the discussed model when the role of the order parameter is delegated from the Higgs VEV $<\Phi>^2$ to the VEV of the "magnetic" field squared $<B_2>^2$ determined by the Bogomol'nyi equation (1.1): of course with its rotary "counterpart" $<B_1>^2 = 0$.

This curious effect is the next in turn "smile" of the "Cheshire Cat", Higgs vacuum BPS monopole modes: the alone Higgs vacuum BPS monopole modes disappear from the spectrum of physical excitations at the spatial infinity, but its covariant derivative $D$ plays the important role in the discussed theory.

As a striking proof of the said we can cite here the estimate for the "magnetic" field squared via the mass $\Delta m_\eta$ of the $\eta'$ meson given in the paper [12] and based on the here discussed two phase gauge model with YM and Higgs BPS monopole and rotary modes:

$$<B_2^2> = D\Phi^2 = \frac{2\pi^3 F_\pi^2 \Delta m_\eta^2}{N_f \alpha_s^2} = \frac{0.06 GeV^4}{\alpha_s^2}.$$  

In this equation $F_\pi \approx 130.41$ Mev is the pionic decay constant and $N_f$ is the number of flavours. We send our readers for details of this calculation to Section 5.10 of the paper [12].

The reasonable question may be asked in this context. Does it happen the above discussed transmutation in another gauge models involving Higgs and YM (vacuum) BPS monopole modes (i.e. those satisfying the Bogomol'nyi equation)? A good example of such models is the case of the $SU(5)$ spontaneous break-down in GUT to the product $SU(3) \otimes SU(2) \otimes U(1)$ of symmetries group which occurs at the temperatures in the interval $10^{14} - 10^{11}$ Gev. On the author opinion, such a high temperature scale is an important factor in answering correctly this question.

The important consequence of the presence of topologically nontrivial threads in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [4] is annihilating [9] two equal magnetic charges $m_1 = m_2 = m(n) \neq 0 (n \in \mathbb{Z})$ colliding at crossing a rectilinear topologically nontrivial thread $A_\theta(n)$: (2.4), (4.6).

This effect is the particular case of changes in the relative sign of two magnetic charges as the integration region $\Gamma$ in (4.1) intersects a (rectilinear) thread.

Suppose that the magnetic charge $|m_1| = m_1$ is concentrated to the left of the thread $A_\theta(n)$, while the magnetic charge $|m_2| = m_2$ is concentrated to the right of this thread. Herewith we concentrate these magnetic charges in points.

Concentrating magnetic charges in points does not change the matter, despite magnetic charges are distributed volume actually due to (4.1).

Due to (4.3), the both magnetic charges are specified to within their signs.

Due to (4.3), the both magnetic charges are specified to within their signs.

One can draw a closed curve connected the both magnetic charges (see Fig. 15 in [4]). It consists of two lines, $\gamma_1$ and $\gamma_2$, between the charges, one rounds in the opposite directions.

This implies that the vacuum "magnetic" tension $B$ (specifying via the Bogomol'nyi equation (1.1)) can change its sign depending on the "direction": either $\gamma_i$ or $\gamma_j$, in which one crosses the thread.
Really, we should always consider the relative sign of the charges $m_1$ and $m_2$: more precisely, our interest is the difference $m_1 - m_2$ as we move in the "direction" $\vec{\gamma}_1$; contrariwise, our interest is the difference $m_2 - m_1$ as we move in the opposite "direction" $\vec{\gamma}_2$.

Taking account of the duality (4.3) in the definition of a magnetic charge, we encounter few "alternatives" for the values of the relative magnetic charge in both the cases.

The most interesting case is when $m_1 = m_2$.

In this case some two vacuum BPS monopoles belonging to the one fixed topological class $n \in \mathbb{Z}$ annihilated mutually at approaching each other crossing the topologically nontrivial thread $A_\theta$ lying in the Minkowskian YM vacuum manifold $R_{YM}$, (3.9) (the same is correct also for excitations over the BPS monopole vacuum).

The above reasoning demonstrates that magnetic charges are not preserved in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac.

Magnetic charges cease then to be motion integrals; at the same time they are still topological invariants: continuous deformations do not change the difference $m_2 - m_1$.

In the monograph [9], in §Φ13, there was proposed the following, more general, definition of a magnetic charge, taking account of rectilinear (topologically nontrivial) threads in appropriate non-Abelian gauge models.

One considers a membrane pulling on the rectilinear thread $A_\theta$ lying along the axis $z$ of the chosen coordinate system. Then one removes this membrane from the Minkowskian space. This results a one-connected set, on which one can choose a fixed continuous branch of the magnetic tension $B$.

In fact, this "trick" comes to the restriction of the integration region in Eq. (4.1) for a magnetic charge by the "left" (respectively, "right") semi-spaces of the Minkowskian space with respect to the axis $z$. This ensures that magnetic charges are preserved in each of these semi-spaces taken separately.

In the present study we don’t have for our object deriving the concrete contribution from the above discussed annihilating effect [9] for equal topologically nontrivial magnetic charges colliding at crossing a thread $A_\theta$ in the total action functional (Hamiltonian) of the quested Minkowskian Higgs model with vacuum BPS monopoles, although such contribution exists undoubtedly.

Nevertheless, now we should like make some remarks which may be helpful for finding such contribution.

For instance, in the paper [42] possible interactions between (YM) magnetic monopoles and antimonopoles were studied involving forming so-called monopole molecules.

Herewith only two types of monopole molecules are possible [42], reflecting ordinary Feynman rules for gauge non-Abelian fields (see e.g. §9.1 in [43]).

There are, firstly, confined colourless magnetic monopoles $M$ and $\tilde{M}$ (with the symbol $M$ referring to a monopole with the magnetic charge $m$ and the symbol $\tilde{M}$ referring to an antimonopole with the magnetic charge $-m$, respectively). These monopole molecules were also referred to as monopole-antimonopole mesons in Ref. [42]. The well-known mechanism for the monopole-antimonopole annihilation is via Abrikosov threads of the
"magnetic" field \[9, 29, 44\] (see also §6.2 in [30]).

Secondly \[12, 13\], confined colour triplet and antitriplet of monopoles are also possible in the nature.

Herewith MMM (respectively, \(\tilde{M}M\tilde{M}\)) states were referred to as baryons in [42].

The presence of topologically nontrivial threads in a gauge non-Abelian theory (for instance, rectilinear threads \(2.4\) [9]), implying the annihilation of monopole pairs with equal magnetic charges colliding at crossing such a (rectilinear) topologically nontrivial thread, changes substantially the matter in comparison with the case of the Abrikosov mechanism \([29, 30, 42]\) of the monopole-antimonopole confinement.

The former case is rather similar to the case of confined colour triplet and antitriplet of monopoles (where the role of the one field in such triplets is played by the rectilinear thread \(A_\theta\)).

On the other hand, the computations about the annihilation of monopole pairs with equal magnetic charges (at least at the level of Feynman diagrams) generalizes somewhat the computations \([43, 46]\) about electron-positron pairs in QED.

Herewith unlike QED, where the creation-annihilation processes are forbidden with an only one photon taking part in such processes (by the reason of maintaining the four-momentum in a Feynman diagram involving two fermions and one photon on-shell: see e.g. §25.1 in [46]), in an YM model involving only massless gauge fields, processes with three YM fields are quite permissible.

In particular, it is correctly for the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac, where YM fields, transverse topological Dirac variables \(A_D^i\) \(i = 1, 2\), are specified [2] over the light cone \(p^2 = 0\) due to the removal \((3.11)\) [5] of temporal YM components.

The just discussed annihilation mechanism [9] for (equal) magnetic charges in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac, can cause disappearing, in a definite time interval \(\tau\), topologically nontrivial magnetic charges \(m \neq 0\).

Let us suppose that this has occurred indeed.

In this case, as it is well known, the Dirac quantization \([21, 29, 43]\) of magnetic and electric charges, taking the general look [22]

\[e_1m_2 - e_2m_1 = \frac{1}{2}n; \quad n \in \mathbb{Z};\]  

(4.29)

for two interacting (BPS, ’t Hooft-Polyakov and so on) monopoles in a Minkowskian Higgs model, implies for Higgs monopoles to have arbitrary electric charges \(e\) associated with the topological number \(n = 0\) (and thus with magnetic charges \(m = 0\)).

In the review [22] (and in other modern physical literature) such situation in a \(SU(n)\) non-Abelian gauge theory (involving the spontaneous breakdown of the initial \(SU(n)\) gauge symmetry down to its \(U(1)^{n-1}\) subgroup) when gauge fields \(A\) possess zero magnetic charges, that implies arbitrary electric charges for Higgs modes \(\Phi\) due to the Dirac quantization [21] of electric and magnetic charges, was referred to as the so-called Higgs phase.
In the notion “the Higgs phase”, one implies (for instance, in Ref. [22]) however that all magnetic charges are confined by (infinitely narrow) Meissner flux tubes, similar to ones in a superconductor [13, 13].

In turn, this involves the linearly increasing “Mandelstam” potential $O(Kr)$ (with $K$ being the string tension) between YM monopole and antimonopole.

As to Higgs vacuum BPS monopole solutions, they carry free electric charges in the Higgs phase [22].

Herewith in the concrete case of the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac, the topologically trivial Higgs BPS monopole modes contribute to the “electric” energy (1.11) [5] of the appropriate BPS monopole vacuum, induced by its topological rotations.

There is an interesting and somewhat curious occurs with topologically trivial Higgs vacuum BPS monopoles in the Higgs phase.

On the one hand, they induce vacuum ”electric” monopoles (1.13). Herewith the term ”electric” is highly relative in this context since this term is associated with temporal vacuum YM components $Z_a$ [16], (1.9), solutions to the Gauss law constraint (1.8), having a definite (rather mathematical) analogy with temporal components, $F_{0\mu}$, of Maxwell tensors, i.e. with electric strengths $E$.

On the other hand, in the Higgs phase [22], topologically trivial Higgs vacuum BPS monopoles $\Phi^a_0$ carry free electric charges, that are told immediately to vacuum ”electric” monopoles (1.13).

Thus vacuum ”electric” monopoles (1.13) become indeed electric fields in the Higgs phase.

It should be distinguished indeed between the conception of the Higgs phase understood customary (e.g. in Ref. [22]) and that in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [4].

First of all, the conception of the Higgs phase, as it is understood in [22], is based on fixing the maximum Abelian gauge (MAG) and on final violating (in the simple YM case) the initial $SU(2)$ gauge symmetry in the

$$SU(2) \rightarrow U(1) \rightarrow \mathbb{Z}$$

wise, involving the ”discrete multiplier” $\mathbb{Z}$.

Now let us discuss briefly both these features of the Higgs phase (as it is understood in modern physics).

Fixing MAG comes to singling out the maximum Abelian ($\sim U(1)$) subgroup in $SU(2)$. The simple way to do this is associated with the diagonal Pauli matrix $\tau_3$.

Then, $\tau_3$ would be considered as the generator of the maximum Abelian subgroup in $SU(2)$.

This implies maximizing the integral

$$\int d^4x \left[ (A_{\mu}^1)^2 + (A_{\mu}^2)^2 \right]; \quad A_{\mu} = A_{\mu}^a \tau_a,$$

(4.31)
or, that is equivalent (see e.g. Ref. [47]) of maximization of the quantity

$$\sum_x \sum_{\mu=1}^4 \text{Tr}[\tau_3 U_\mu(x) \tau_3 U_\mu^\dagger(x)].$$

(4.32)

On the other hand, the said is equivalent to decomposing an YM field into off-diagonal and diagonal components [48]:

$$A_\mu = A_\mu^a \tau_a + A_\mu \tau_3; \quad a = 1, 2.$$ 

Similarly, one find

$$F_{\mu\nu} = F_{\mu\nu}^a \tau_a + F_{\mu\nu} \tau_3$$

for the field (YM) strength with off-diagonal and diagonal parts given by

$$F_{\mu\nu}^a = D_{\mu}^{ab} A_{\nu}^b - D_{\nu}^{ab} A_{\mu}^b$$

and

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu},$$

respectively.

For the YM action (in the Euclidean space $E_4$) one get finally

$$S_{YM} = \int d^4x (F_{\mu\nu}^a F_{\mu\nu}^a) + F_{\mu\nu} F_{\mu\nu}.$$ 

MAG may be got [48] by imposing the Lorentz gauge

$$D_{\mu}^{ab} A_{\mu}^b = 0.$$ 

MAG in YM theories can be generalized easy for the case of the initial $SU(3)_{\text{col}}$ gauge symmetry group in ”realistic” QCD models, violated then in the

$$SU(3)_{\text{col}} \rightarrow U(1) \otimes U(1)$$

wise.

In this case MAG can be fixed by singling out the Gell-Mann matrices $\lambda_3$ and $\lambda_8$ from the total set of the Gell-Mann matrices $\lambda$.

By analogy with (4.31), now MAG fixing is equivalent to maximizing the integral

$$\int d^4x [(A_\mu^1)^2 + (A_\mu^2)^2 + (A_\mu^4)^2 + (A_\mu^5)^2 + (A_\mu^6)^2 + (A_\mu^7)^2]; \quad A_\mu = A_\mu^a \lambda_a.$$  

(4.34)

However to achieve the quark confinement in the YM theory and especially in ”realistic” QCD models, involving the initial $SU(3)_{\text{col}}$ gauge symmetry group (violated then in the

$$SU(3)_{\text{col}} \rightarrow U(1)^{n-1} (n = 2, 3)$$

wise), the appropriate Abelian $U(1)^{n-1} (n = 2, 3)$ subgroups would be also broken down further.
For the initial $SU(2)$ gauge symmetry group this occurs in the (4.30) wise, while for the initial $SU(3)_{\text{col}}$ gauge symmetry group, it is
\[ SU(3)_{\text{col}} \rightarrow U(1) \otimes U(1) \rightarrow \mathbb{Z}. \]  
(4.35)

The said can be grounded by the reasoning to provide the needs of the quark confinement in the above theories.

The sure sign of the quark confinement taking place is, at is well known, the area law [43] satisfied by Wilson loops, taking the typical shape
\[ W(C) \sim \exp\{-KA(C)\}, \]  
(4.36)
implicating the areas $A(C)$ of surfaces enclosed by appropriate paths $C$.

Just in order implementing the area law (4.36), characterizing the confinement of quarks and gluons, the phenomenological picture in which quark $q$ ($\bar{q}$) and gluons $A$ are confined by (infinitely thin) Meissner flux tubes is quite fit.

This phenomenological picture is similar to that proper to superconductors.

In particular, in superconductors, the magnetic field $B$ decays exponentially at penetrating into the superconducting region [49]:
\[ B(x) = B_0 e^{-x/\lambda_L}, \]  
(4.37)
with
\[ \lambda_L = \frac{1}{M_\gamma} \]  
(4.38)
being the penetration or London depth, depending on the "photon mass"
\[ M_\gamma = \sqrt{2} ea. \]  
(4.39)
The parameter $a$ appears in latter Eq.

It plays the role of the order parameter in the Ginzburg-Landau superconductivity model [29, 49, 50]: $a = 0$ for the normal phase, while $a \neq 0$ for the superconducting phase.

The magnetic field $B$, (4.37), decaying exponentially at penetrating into the superconducting region, is the solution to the Maxwell-London equation [29, 49]
\[ \text{rot } B = \text{rot } \text{rot } A = j_s = 2e^2a^2A, \]  
(4.40)
implicating the preserved Noether current [13, 29]
\[ j_s = -i\frac{e}{m}(\phi^*\nabla \phi - \phi \nabla \phi^*) - \frac{\epsilon^2}{m} |\phi|^2 A, \]  
(4.41)
with $\phi$ being the superconductor wave function.

Eq. (4.37) serves as the mathematical expression for the Meissner effect. The essence of this effect is in expulsion of the magnetic field from the superconducting region.
Similar locating the "magnetic" flux in the shape of infinitely narrow tubes explains the confinement effect in QCD.

The *confinement phase* can be treated [22] as dual to the Higgs phase, discussed above. In this phase gluons and quarks, that are purely electric objects, are confined by infinitely narrow "magnetic" flux tubes (strings). Herewith Higgs fields become purely magnetic objects.

The occurred in the confinement phase requires undoubtedly the spontaneous breakdown of the initial \((SU(3)_{\text{col}} \text{ or } SU(2))\) gauge symmetries groups with the appearance of Higgs modes.

In this case also MAG fixing seems to be quite natural.

In the modern literature such (phenomenological) picture of confinement based on MAG fixing and the *dual-superconductor picture* (similar to that one can observe in the Ginzburg-Landau model [50] and implicating [22] the Higgs and confinement phases) is referred to as the *Abelian dominance* [51].

But, indeed, it is not sufficient to violate the initial \(SU(n)\) \((n = 2, 3)\) gauge symmetry group in a non-Abelian model up to its \(U(1)^{n-1}\) subgroup in order to ensure the quark confinement in that model.

The thing is that the appearance of (infinitely thin) "magnetic" flux tubes (strings) confining quarks in QCD can be explained good in the framework of the well-known Nielsen-Olesen model [52].

This model serves as a highly correct representation for the Abelian Higgs theory.

With the example of the initial \(U(1)\) gauge symmetry group afterwards violated, in the original paper [52] it was demonstrated the existence of specific solutions to the equations of motions, *Nielsen-Olesen vortices* (see e.g. [29, 49]).

The way grounding the existence of Nielsen-Olesen (NO) vortices in the Abelian Higgs theory comes to violating the initial \(U(1)\) gauge symmetry group proper to that theory (this involves the nonzero value \(|\phi| \neq 0\) for the Higgs field \(\phi\) in the "asymmetric" phase, entering explicitly [29, 49] the appropriate equations of motions) and, from the topological standpoint, to the presence of thread topological defects (just interpreted as NO vortices) inside the appropriate vacuum manifold.

In the light of the said, the task to ascertain the look of this vacuum manifold becomes very important.

The investigations about the NO model [52] are not the immediate goal of the present study, but some outlines and conjectures concerning this model will be helpful for us in understanding of processes taking place in QCD if stand on the Abelian dominance viewpoint [51].

From our previous discussion about the liquid helium II theory, some analogies suggest concerning the origin of NO vortices in the Abelian Higgs model [52].

First of all, from the topological point of view, NO vortices, as a particular case of thread topological defects, would be associated with the "discrete" vacuum geometry should be assumed for the appropriate degeneration space.
Such degeneration space (vacuum manifold) can be obtained in the way similar to
that in which the discrete vacuum manifold $\tilde{R}$ has been got in the rested liquid helium II
specimen case.

However there are some principal distinctions between the latter case and the Abelian
Higgs model [52] involving NO vortices.

So, for instance, the first-order phase transition occurs primary in that model (see the
arguments [30], repeated also in Ref. [6]). It comes to a discontinuity in the plot of the
appropriate order parameter (as a such order parameter, the vacuum expectation value
$<\phi^2>$ for the Higgs field $\phi$ can be chosen). This implies the coexistence of the metastable
(symmetric) and stable (asymmetric) thermodynamic phases in the Abelian Higgs model
[52]. It is just the supercooling case (according to the argument similar to those [30] we
have encountered above at the analysis of vortices in a liquid helium II specimen).

Once thus, the way (3.31) violating the $U(1)$ gauge symmetry (fit for the second-order
phase transition [40] in a helium specimen) is not quite suitable in the Abelian Higgs
model [52] with NO vortices implying the first-order phase transition.

In the first case, the symmetric thermodynamic phase disappears entirely upon $U(1)$
violating.

This makes quite correct Eq. (3.31) for the quite separate set $\tilde{U}$, coinciding with
the degeneration space (vacuum manifold) $\tilde{R}$.

The look of this discrete space (in which all the topological domains refer to the asym-
metric thermodynamic phase) just represents the above (primary) second-order phase
transition in the helium theory.

Unlike the former case, in the Abelian Higgs model [52], in which the first-order phase
transition occurs involving coexisting [6, 30] the stable and metastable thermodynamic
phases, it may be assumed $U(1)$ violating in such a wise that a part of topological domains
in the $U(1)$ group space remains "glued together", representing herewith the 'splinter' of
the 'symmetric' (metastable) thermodynamic phase, while the rest of topological domains
become a discrete (quite separate) set, representing the asymmetric (stable) thermody-
namic phase.

Indeed, this "coexistence" of the metastable and stable thermodynamic phases in the
Abelian Higgs model [52] with NO vortices cannot last infinitely long time.

After all (generally, at lowering the temperature below the Curie point $T_c$), the Higgs
field $\phi$ in the Abelian model [52], leaving the "false" vacuum $\phi = 0$, rolls into its "true"
minimum (say, $\phi_0 \neq 0$), and the metastable (symmetric) thermodynamic phase (corre-
sponding to the above described continuous set and the "false" vacuum $\phi = 0$) disappears
herewith entirely.

From the "geometrical" standpoint, the gradual destruction of the continuous
$U(1) \simeq S^1$ group space (with forming the "bubble" [30] of the new, stable and "dis-
crete", thermodynamic phase inside the metastable, "continuous", one) occurs.

At this disappearing the metastable (symmetric) thermodynamic phase, the accumu-
lated latent heat is released; it is just the reheating situation [55].

It may be guessed that rolling the Higgs field $\phi$ in the "true" minimum $\phi_0$ corresponds
to the dual-superconducting picture of confinement in QCD based upon MAG fixing and
the Abelian Higgs model \cite{52} with NO vortices.

In connection with our discussion about the Abelian Higgs model \cite{52} (implicating NO vortices), it will be relevant to mention the very important analogy (in the sphere of the topology and phenomenology) between this model and the case \cite{13, 49, 50} of superconductors.

For instance, in Type II superconductors, the second-order phase transition occurs with definite conditions in the presence of an background magnetic field $H$.

First of all, it would be therein \cite{13, 49, 50} $\kappa > 1/\sqrt{2}$ for the Ginzburg-Landau parameter

$$\kappa = \frac{\lambda_L}{\xi},$$

with $\lambda_L$ given in (4.38) \cite{49} and $\xi$ being the coherence length which physical sense is \cite{13} the correlations radius of fluctuations in the order parameter $<\phi^2>$.

Generally speaking \cite{13}, the Ginzburg-Landau parameter $\kappa$ is a function of the temperature $T$: $\kappa \equiv \kappa(T)$.

As it was demonstrated in the original paper \cite{50} (see also \cite{13}), the inequality $\kappa > 1/\sqrt{2}$, true for Type II superconductors, is associated with the negative surface tension $\alpha_{ns} < 0$ (which look is (4.26)) between the normal (n) and superconductor (s) phases.

In the latter case it becomes advantageous, from the thermodynamic standpoint, to compensate increasing the volume energy by this negative surface energy $\alpha_{ns} < 0$.

This implies arising germs of the n-phase inside the s one at the values $H$ of the background magnetic field $H$ exceeding a field $H_{c1}$, called \cite{13} the lower critical field.

Vice verse, germs of the s-phase inside the n one arise at the values $H$ don’t exceeding a field $H_{c2}$, called the upper critical field.

Thus in the interval $H_{c1} < H < H_{c2}$, one can observe \cite{13} the mixed state in Type II superconductors (there are definite alloyed metals and combinations of metals). In this case a Type II superconductor is simultaneously in the n and s states.

At $H \leq H_{c1}$, a specimen is purely in the s-state, while it is purely in the n-state at $H \geq H_{c2}$.

It turns out that the second-order phase transition takes place in this situation, and now we would like to clarify this.

First of all, it can be argued (see, for example, the work \cite{53}) that in type II superconductors vortex lines of the n-phase repel each other; thus the vortices tend to form a vortex lattice in the type II region. In fact, due to the periodic lattice boundary conditions, even one vortex forms a square lattice with its periodic counterparts.

The main argument in favour of one or another kind of phase transitions occurring in a medium is the analysis of second derivatives of thermodynamical potentials by the temperature and pressure (for instance, the heat capacity and compressibility), which change abruptly if the second-order phase transition takes place. In the same time, the first derivatives (such as the energy and volume of the medium) do not change.

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The second case, of the free energy plot, has been analyzed, with the example of the type II superconductor, in the monograph [54]. In this monograph the behaviour of the magnetization curve \( B(H) \) in the vicinities of the critical points \( H_{c1} \) and \( H_{c2} \). This in fact the same as the analysis of the Gibbs free energy \( G \) in the external magnetic field \( H \) switched-on, which contains always the item \((BH)/4\pi\).

Such behaviour of the magnetization curve \( B(H) \) is closely related to the picture of vortices in the studied superconductor.

So, three regimes [54] can be distinguish in the \([H_{c1}, H_{c2}]\) interval.
1. Very near \( H_{c1}, \Phi_0/B \gg \lambda_L \) for the elementary flux \( \Phi_0 = \pi \hbar c/|e| \). Then the vortices are separated by distances more than \( \lambda_L \). In this case only a few neighbors are important.
2. For moderate values of \( B \), such that \( \xi^2 \ll \Phi_0/B \ll \lambda^2 \), many vortices appear within interaction range of any given one; this generate the \( \sum_{i>j} F_{ij} \) contribution into the Gibbs energy \( G \). However, it is still a good approximation to neglect details of the core.
3. Near \( H_{c2}, \xi^2 \approx \Phi_0/B \), so that the cores are almost overlapping.

Just at the careful analysis of the "regime one" one can make sure that the second order phase transition occurs in an (infinitesimal) neighbourhood of \( H_{c1} \). The (long enough) calculations [54] give the following look of the magnetic field \( B \) in a neighbourhood of \( H_{c1} \):

\[
B = \frac{2\Phi_0}{\sqrt{3\lambda^2_L}} \left\{ \ln \left[ \frac{3\Phi_0}{4\pi \lambda^2_L (H - H_{c1})} \right] \right\}^{-2}.
\]  (4.42)

\( B \) is continuous at \( H_{c1} \), corresponding to a second order phase transition (since \( B \) enters the expresiion for the free energy expression of the model studied via the \((BH)/4\pi\) item).

The similar arguments in favour of the second order phase transition occurring are applied to the regime near \( H_{c2} \). More exactly, in [54] the magnetization curves \( 4\pi M = f(H) \) for different values of \( \kappa \) were considered (see Fig. 5.2. in [54]). This analysis of curves shows that only at \( \kappa > 1/\sqrt{2} \) a second order phase transition takes place.

The attraction of vortices in type one superconductors leads [53] to forming the cylindrical hole (the "broken" s-phase) inside the magnetic field configuration (the symmetric n-phase).

For this case, the discontinuity \( \Delta \frac{\partial G/V}{\partial y} \) of the canonical free energy \( G(x, y, H/e_3^3) \), with \( y \) and \( x \) being two dimensionless ratios

\[
y = \frac{m_3^2(e_3^3)}{e_4^3}, \quad x = \frac{\lambda_3}{e_3^2} \]  (4.43)

(involving the electron mass \( m_3 \) and the Cooper pair interaction constant \( \lambda_3 \); \( e_3 \) is a scale introducing in the model by the authors [53].

Indeed,

\[
\Delta \frac{\partial G/V}{\partial y} = e_4^3 \Delta \phi^* \phi; \quad \Delta \frac{\partial G/V}{\partial H/e_3^3} = -e_3^3 \Delta B, \]  (4.44)

with \( \phi \) being the macroscopic instantaneous wave function associated with the Cooper pair.
For fixed $x$ the latent heat $L$ of the transition is defined as the discontinuity in the "energy" variable $E$ obtained from $G$ by a Legendre transformation with respect to $y$, $H/e^3_3$:

$$L = \Delta E = -y\Delta \frac{\partial G}{\partial y} - H_c \Delta \frac{\partial G}{\partial H} = V[ -ye^4_3 \Delta < \phi^* \phi > + H_c \Delta B].$$

(4.45)

For fixed $x$, the identity $\Delta G(x,y,H_c) = 0$ leads to the Clausius-Clapeyron equation, relating the different discontinuities:

$$\Delta \frac{\partial G}{\partial y} = -\frac{\partial H_c}{\partial y} \frac{\partial G}{\partial H} \Leftrightarrow e_3^4 \Delta < \phi^* \phi > = \frac{\partial H_c}{\partial y} \Delta B.$$

(4.46)

Thus it is enough to measure one of the discontinuities, and the curve $H_c(y)$.

From the topological (geometrical) point of view, the said suggests two ways for breakdown the $U(1)$ gauge symmetry inherent to the superconductivity model.

In Type II superconductors, in which the second-order phase transition takes place, the appropriate $U(1) \simeq S^1$ group manifold collapses instantly in the way when domain walls arise simultaneously between all the topological sectors of the circle $S^1$.

In Type I superconductors, in which the first-order phase transition takes place, collapsing the $U(1) \simeq S^1$ group manifold occurs gradually. At first, domain walls arise between definite topological sectors of the circle $S^1$ while another topological sectors remain unaffected (these sectors form a 'splinter' of the circle $S^1$). A distinctive feature of the first-order phase transition taking place in Type I superconductors thus the coexistence of the mentioned geometrical (topological) structures (at the temperature $T \to 0$).

In spite of the said, it is obvious that in the Curie point $T_c$, $H_{c1} \to H_{c2} \to 0$. The said allows, for all that, to construct the plots $H(T)$ for the s, n and mixed states in Type II superconductors (see Fig. 7 in [13]).

It is enough manifest that the case of Type II superconductors may be treated as a nonrelativistic limit of the Abelian Higgs NO model [52] [17].

Indeed, it is a similarity between the helium at rest theory [15] and the Abelian Higgs NO model [52] (including the Type II superconductivity [49, 50] as its particular case), namely that in both the cases the $U(1) \simeq S^1$ group space is destroyed instantly entirely (in the (3.34) wise) down to the discrete (quite separate) set $\tilde{U}(1)$ involving domain walls between topologies (the only thing that one must keep here in mind is that the $U(1)$ symmetry is rigid in the helium at rest case while it is gauge in the Abelian Higgs NO model as well as in the Type II superconductors).

It is just the sign of the second-order phase transition occurring in the Abelian Higgs NO model [52].

The outlined above parallel between the Type II superconductivity and the Abelian Higgs NO model [52] promotes also the comprehension of the dual-superconductor picture [22, 49] of the confinement of quarks and monopoles.

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17 It is so at identifying [24, 49] the density of superconducting a Cooper pair with the Higgs field squared $|\phi|$ in the NO model [52].
One can imagine [49], in his thoughts, placing some north and south magnetic monopole inside a type II superconductor in such a wise that they are separated by the distance \( d \). Thinking that the magnetic field is concentrated in cores of vortices and will not extend into the superconducting region, the field energy of this system becomes [49]

\[
V = \frac{1}{2} \int d^3x \mathbf{B}^2 \propto \frac{4\pi d}{e^2\lambda_L^2}.
\]

(4.47)

Thus, the interaction energy of magnetic monopoles grows linearly with their separation, as it would be in the dual-superconductor picture [22] of confinement.

The Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac [4] results the approach to QCD (including the confinement picture) rather another that one gets fixing MAG and assuming the dual-superconductor picture [22] of confinement (implicating the Mandelstam linearly increasing potential).

Firstly, as it was discussed in Refs. [5, 12] (see also [1]), at constructing Minkowskian Gauss-shell QCD (generalizing the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac and involving the initial \( SU(3)_{\text{col}} \) gauge symmetry group), the specific gauge for the "intermediate" \( SU(2)_{\text{col}} \) symmetry group in the "breakdown chain"

\[
SU(3)_{\text{col}} \rightarrow SU(2)_{\text{col}} \rightarrow U(1)
\]

can be fixed.

It comes [1, 5, 12] to the choice of the antisymmetric Gell-Mann matrices \( \lambda_2, \lambda_5, \lambda_7 \) as generators for \( SU(2)_{\text{col}} \).

At further assuming the discrete geometry (3.3) for \( SU(2)_{\text{col}} \), this involves lot of consequences for Gauss-shell QCD (described briefly in Ref. [1]).

Among such consequences fixing the \( \lambda_2, \lambda_5, \lambda_7 \) ("antisymmetric") "gauge", one can mention fermionic rotary (axial) degrees of freedom \( \mathbf{v}_1 = \mathbf{r} \times \mathbf{K} \) (with \( \mathbf{K} \) being the polar colour vector, \( SU(2)_{\text{col}} \) triplet).

Repeating the arguments [17], the source of the above fermionic (quark) rotary degrees of freedom may be found in the interaction item \( \sim Z^a j_{Ia(0)} \) in the Gauss-shell reduced Hamiltonian of constraint-shell QCD.

This item implicates the zero-mode solution \( Z^a \), (1.9), to the Gauss law constraint (1.6) and the fermionic charge

\[
j^{Ia(0)} \sim g \bar{\psi}^I(\lambda^a \gamma^0) \psi^I.
\]

Here it would be set \( a = 2, 5, 7 \) and \( \gamma^0 \) is the Dirac matrix; \( \bar{\psi}^I \) and \( \psi^I \) [20] are fermionic Dirac variable involving "small" Gribov multipliers \( \psi^{(0)}(\mathbf{x}) \).

Thus the existence of fermionic (quark) rotary degrees of freedom \( \mathbf{v}_1 \) (similar somewhat to rotary terms in molecules) can be associated lawfully with the "discrete vacuum geometry" assumed for Minkowskian constraint-shell QCD in the Dirac fundamental scheme [4].
Secondly, the Mandelstam linearly increasing potential $\sim Kr$\[^{43}\]$ in Minkowskian constraint-shell QCD "draws back in a background".

More exactly, the Dirac fundamental scheme\[^{4}\] for Minkowskian constraint-shell QCD implicates a linear combination of well-known instantaneous ("four-fermionic") interaction potentials: Coulomb and "golden section" ones\[^{1, 5, 12, 20}\](with coefficients depending somehow on the temperature $T$ and flavours mass scale $m$) as the source of hadronization and confinement in that model (see e.g. the papers\[^{12, 56, 57}\]).

Such linear combination may be got at finding the Green function of the Gauss law constraint equation in the BPS (Wu-Yang) monopole background\[^{18}\].

The important objection restricted the role of the Mandelstam linearly increasing potential $\sim Kr$ in QCD was given also in Ref.\[^{58}\].

It turns out that for heavy quarconia, only the Coulomb potential $\sim 1/r$ contributes in QCD.

Then the problem arises, to link (continuously and smoothly) the Mandelstam linearly increasing and Coulomb potentials in order to satisfy the heavy quarconia limit\[^{58}\].

As to the Mandelstam linearly increasing potential $\sim Kr$\[^{43}\], the hypothesis can be suggested about including this potential at (finite) temperatures $T \neq 0$ in the linear combination with the Coulomb and "golden section" potentials. This can be done via a selection of the proper coefficient for the Mandelstam potential.

Among consequences for Minkowskian constraint-shell QCD of such including one can point out the "ordinary" annihilation mechanism\[^{42}\] for monopole-antimonopole pairs via Abrikosov threads\[^{9, 44}\] of the "magnetic" field (this mechanism was us discussed above).

This is correlated closely with the dual-superconducting picture of confinement.

More exactly, at placing a monopole-antimonopole pair into the hot YM plasma, the typical length of the Abrikosov thread ("magnetic" tube) between this pair becomes\[^{30, 42}\] $\Delta l \sim (g^2 T)^{-1}$. This implicates the temperature-depended Mandelstam potential in the shape $K(g^2 T)^{-1}$.

On the other hand, this including the temperature-depended Mandelstam potential in Minkowskian constraint-shell QCD can give rise to series of problems and questions.

In particular, the problem is to check satisfying the area law (as the confinement criterion) in the situation of specific "Z-dominance" taking place actually in Minkowskian constraint-shell QCD and induced by $Z$-vortices (thread topological defects) inside the appropriate vacuum manifold, assumed to be discrete and given through\[^{39}\].

The insurmountable duality\[^{9}\] in specifying the sign of the vacuum "magnetic" field $B$ in the Bogomol’nyi equation\[^{1, 1}\] taking place at assuming the discrete geometry\[^{3, 9}\] for the vacuum manifold $R_{YM}$ in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac was us discussed above.

\[^{18}\]Wu-Yang monopoles\[^{34}\], as solutions to the equations of motions in the "purely YM" model, can be considered as spatial asymptotes for YM BPS monopoles.
Remember herewith that the crucial point for this "insurmountable duality" is the existence of gauge transformations (2.11) [9] (depending manifestly on thread solutions \(A_\theta\) via Eq. (2.10)).

Besides this duality, there is also the manifest symmetry of the "electric" action functional (1.11), squared by the topological momentum \(P_N\) (given via Eq. (1.14)), with respect to the changes in the sign of \(P_N(k) \ (k \in \mathbb{Z})\).

This becomes more obvious at recasting (1.11) to the look

\[
W_N = \int dt \frac{P_N^2(t)}{2I}.
\] (4.48)

Actually the said is equivalent to identifying each two topologies with "opposite signs": \(k\) and \(-k\).

Thus, with taking account of the "discrete" factorisation (3.3) for the residual \(U(1)\) gauge symmetry, the "total" residual gauge symmetry group of the free rotator action (1.11) amounts

\[
U_1 = U_0 \otimes \mathbb{Z}/\mathbb{Z}_2.
\] (4.49)

The same conclusion about identifying each two topologies with "opposite signs" can be drawn about the vacuum "magnetic" energy [5]

\[
\frac{1}{2} \int_\epsilon^\infty d^3x [B_\alpha^a(\Phi_k)]^2 \equiv \frac{1}{2} V < B^2 > = \frac{1}{2\alpha_s} \int_\epsilon^\infty \frac{dr}{r^2} \sim \frac{1}{2\alpha_s \epsilon} = 2\pi \frac{gm}{g^2\sqrt{\lambda}} = \frac{2\pi}{g^2\epsilon},
\] (4.50)
squared by the vacuum "magnetic" tension \(B\), set, in turn, by the Bogomol'nyi equation (1.1) and responsible for the superfluid properties of the BPS monopole vacuum (suffered the Dirac fundamental quantization [4]).

On the other hand, the Bogomol'nyi equation (1.1) [11], associated the vacuum "magnetic" field \(B\) to the Higgs BPS monopole isomultiplet \(\Phi^a\), specifies, indeed to within a sign, this vacuum "magnetic" field \(B\) and thus involves the invariance of the vacuum "magnetic" energy (4.50) with respect to changes in signs of magnetic, that is equivalent of topological, charges.

The said implies the modification of the "discrete" geometry (3.9) of the vacuum manifold \(R_{YM}\).

Now it may be written down as

\[
R'_{YM} = (\mathbb{Z}/\mathbb{Z}_2) \otimes G_0/U_0,
\] (4.51)

while the "modified" residual gauge symmetry group in the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac is \(U_1\), given by Eq. (4.49).

This implies also the "modified" discrete representation

\[
G_M = G_0 \otimes \mathbb{Z}/\mathbb{Z}_2
\] (4.52)

for the initial gauge symmetry group in this model.
It is easy to see that the topological content of the Minkowskian Higgs model with vacuum BPS monopoles quantized by Dirac does not change at the modifications (4.49), (4.51), (4.52).

Thus all the kinds of topological defects: thread and point hedgehog ones and domain walls between different topological sectors maintain in that model in spite of the modifications (4.49), (4.51), (4.52).

References

[1] L. D. Lantsman, Dirac Fundamental Quantization of Gauge Theories is the Natural Way of Reference Frames in Modern Physics, Fizika B18, 99 (2009), arXiv:hep-th/0604004.

[2] L. D. Lantsman, Superfluidity of Minkowskian Higgs Vacuum with BPS Monopoles Quantized by Dirac May Be Described as Cauchy Problem to Gribov Ambiguity Equation., arXiv:hep-th/0607079.

[3] L. D. Lantsman, Nontrivial Topological Dynamics in Minkowskian Higgs Model Quantized by Dirac., arXiv:hep-th/0610217.

[4] P. A. M. Dirac, Proc. Roy. Soc. A 114, 243 (1927); Can. J. Phys. 33, 650 (1955).

[5] D. Blaschke, V. N. Pervushin, G. Röpke, Topological Gauge Invariant Variables in QCD, MPG-VT-UR 191/99, in Proceeding of Workshop: Physical Variables in Gauge Theories, JINR, Dubna, 21-24 Sept., 1999, arXiv:hep-th/9909133.

[6] L. D. Lantsman, Superfluid Properties of BPS Monopoles arXiv:hep-th/0605074.

[7] L. D. Lantsman, V. N. Pervushin, The Higgs Field as The Cheshire Cat and his Yang-Mills "Smiles", In Proceeding of 6 International Baldin Seminar on High Energy Physics Problems (ISHEPP), June 10-15, 2002, Dubna, Russia, arXiv:hep-th/0205252; L. D. Lantsman, Minkowskian Yang-Mills Vacuum, arXiv:math-ph/0411080.

[8] L. D. Lantsman, V. N. Pervushin, JINR P2-2002-119, Yad. Fiz. 66, 1416 (2003) [Physics of Atomic Nuclei 66, 1384 (2003)], arXiv:hep-th/0407195.

[9] A. S. Schwarz, Kvantovaja Teorija Polja i Topologija, 1st ed., Moscow: Nauka, 1989 [A. S. Schwartz, Quantum Field Theory and Topology, Springer, 1993].

[10] M. K. Prasad, C. M. Sommerfeld, Phys. Rev. Lett. 35, 760 (1975); E. B. Bogomol’nyi, Yad. Fiz. 24, 449 (1976).

[11] R. Akhoury, Ju- Hw. Jung, A. S. Goldhaber, Phys. Rev. 21, 454 (1980).
[12] V. N. Pervushin, Dirac Variables in Gauge Theories, Lecture Notes in DAAD Summer School on Dense Matter in Particle and Astrophysics, JINR, Dubna, Russia, August 20-31, 2001, Phys. Part. Nucl. 34, 348 (2003) [Fiz. Elem. Chast. Atom. Yadra 34, 679 (2003)], [arXiv:hep-th/0109218].

[13] L. D. Landau, E. M. Lifschitz, Lehrbuch der Theoretischen Physik (Statistische Physik, Band 5, teil 2), in German, 1st edn. edited by H. Eserig and P. Ziesche, Berlin: Akademie-Verlag, 1980.

[14] L. D. Landau, JETP 11, 592 (1941); DAN USSR 61, 253 (1948).

[15] I. M. Khalatnikov, Teorija Sverxtekhesti, 1st edn., Moscow: Nauka, 1971.

[16] V. N. Pervushin, Teor. Mat. Fiz. 45, 394 (1980) [Theor. Math. Phys. 45, 1100 (1981)].

[17] V. N. Pervushin, Riv. Nuovo Cim. 8, N 10, 1 (1985).

[18] L. D. Faddeev, in Proc. of 4th Int. Symp. on Nonlocal Quantum Field Theory, Dubna, USSR, 1976, JINR D1-9768, p. 267.

[19] G. 't Hooft, Nucl. Phys. B 138, 1 (1978).

[20] D. Blaschke, V. N. Pervushin, G. Röpke, in Proceeding of the Int. Seminar Physical variables in Gauge Theories, Dubna, September 21-24, 1999, edited by A. M. Khvedelidze, M. Lavelle, D. McMullan and V. Pervushin (E2-2000-172, Dubna, 2000), p. 49, [arXiv:hep-th/0006249].

[21] P. A. M. Dirac, Proc. Roy. Soc. A 133, 69 (1931).

[22] F. Bruckmann, G. 't Hooft, Phys. Rep. 142, 357 (1986); [arXiv:hep-th/0010225].

[23] P. I. Azimov, V. N. Pervushin, Teor. Mat. Fiz. 67, 349 (1986) [Theor. Math. Phys. 67, (1987)].

[24] M. Engelhardt, K. Langfeld, M. Quandt, H. Reinhardt, A. Schäfke, Magnetic Monopoles, Center Vortices, Confinement and Topology of Gauge Fields, [arXiv, hep-th/9911145].

[25] M. N. Chernodub, Phys. Lett. B 637, 128 (2006), ITEP-LAT/2005-09, [arXiv:hep-th/0506107]; JETP Lett. 83, 268 (2006), ITEP-LAT/2005-13, [arXiv:hep-th/0507221].

[26] G. 't Hooft, Nucl. Phys. B 79, 276 (1974).

[27] A. M. Polyakov, Pisma JETP 20, 247 (1974) [Sov. Phys. JETP Lett. 20, 194 (1974)]; Sov. Phys. JETP Lett. 41, 988 (1975).
[28] L. D. Faddeev, V. N. Popov, Phys. Lett. B 25, 29 (1967).

[29] L. H. Ryder, Quantum Field Theory, 1st ed., Cambridge: Cambridge University Press, 1984.

[30] A. D. Linde, Elementary Particle Physics and Inflationary Cosmology, 1st edn. Moscow: Nauka, 1990, [arXiv: hep-th/0503203].

[31] L. D. Landau, E. M. Lifschitz, Theoretical Physics, v. 2. The Field Theory, edited by L. P. Pitaevskii, 7th edn. Moscow: Nauka, 1988.

[32] R. Engelking, General Topology, 2nd edn. Warszawa: Państwowe Wydawnictwo Naukowe, 1977.

[33] L. Lantsman, BPS Ansatzes as Electric Form-factors, [arXiv:0812.5080].

[34] T. T. Wu, C. N. Yang, Phys. Rev. D 12, 3845 (1975).

[35] D. R. Nelson, Defects in Superfluids, Superconductors and Membranes, Lectures presented at the 1994 Les Houches Summer School "Fluctuating Geometries in Statistical Mechanics and Field Theory.", [arXiv:cond-mat/9502114].

[36] N. N. Bogoliubov, J. Phys. 9, 23 (1947);
N. N. Bogoliubov, V. V. Tolmachev , D. V. Shirkov, Novij Metod v Teorii Sverchprovodimosti, 1st edn. (Izd-vo AN SSSR 1958: p. p. 5-9).

[37] V. G. Levich, Yu. A. Vdovin, V. A. Mjamlin, Kurs Teoreticheskoj Fiziki, v. 2, 2nd edn. Moscow: Nauka, 1971.

[38] V. N. Pervushin, V. I. Smirichinski, J. Phys. A: Math. Gen. 32, 6191 (1999), [arXiv:hep-th/9902013].

[39] G. E. Volovik, The Universe in a Helium Droplet, Oxford University Press, 2009.

[40] V. G. Levich, Yu. A. Vdovin, V. A. Mjamlin, Kurs Teoreticheskoj Fiziki, v. 1, 2nd edn. Moscow: Nauka, 1969.

[41] V. S. Vladimirov, Yravnenija Matematicheskoj Fiziki, 5th edn. Moscow: Nauka, 1988.

[42] A. D. Linde, Phys. Lett B 96, 289 (1980).

[43] T. P. Cheng, L.- F. Li, Gauge Theory of Elementary Particle Physics, 3rd edn., Oxford: Clarendon Press, 1988.

[44] A. A. Abrikosov, JETP 32, 1442 (1957).
[45] L. D. Landau, E. M. Lifshitz, Theoretical Physics, v. 4. Quantum Electrodynamics (V. B. Berestetskii, E. M. Lifshitz, L. P. Pitaevskii), edited by L. P. Pitaevskii, 3rd edn. Moscow: Nauka, 1989.

[46] A. I. Achieser, V. B. Berestetskii, Quantum Electrodynamics, 3rd edn., Moscow: Nauka, 1969.

[47] L. Del Debbio, M. Faber, J. Greensite, S. Olejnik, Phys.Rev. D 55, 2298 (1997), arXiv:hep-lat/9610005.

[48] M. A. L. Capri, D. Dudal, J. A. Gracey, V. E. R. Lemes, R. F. Sobreiro, S. P. Sorella, R. Thibes, H. Verschelde, The Infrared Behaviour of the Gluon and Ghost Propagators in SU(2) Yang-Mills Theory in the Maximal Abelian Gauge, Talk given by S.P. Sorella at the "I Latin American Workshop on High Energy Phenomenology (I LAWHEP)", December 1-3 2005, Instituto de Fisica, UFRGS, Porto Alegre, Rio Grande Do Sul, Brasil, arXiv:hep-th/0603167.

[49] F. Lenz, Topological Concepts in Gauge Theories, Lectures given at the Autumn School "Topology and Geometry in Physics", of the Graduiertenkolleg "Physical systems with many degrees of freedom", University of Heidelberg, Rot an der Rot, September 24-28, 2001, FAU-TP3-04/3, arXiv:hep-th/0403286.

[50] V. L. Ginzburg, L. D. Landau, JETP 20, 1064 (1950).

[51] A. Kronfeld, M. Laursen, G. Schrierholz, U.- J. Wiese, Phys. Lett. B 198, 516 (1987);
T. Suzuki, I. Yotsuyanagi, Phys. Rev. D 42, 4257 (1990);
S. Hioki et al., Phys. Lett. B 272,326 (1991).

[52] H. B. Nielsen, P. Olesen, Nucl. Phys. B 61, 45 (1973).

[53] K. Kajantie, M. Laine, T. Neuhaus, A. Rajantie, K. Rummukainen, Nucl. Phys. B 559, 395 (1999), arXiv:hep-th/9906028.

[54] Tinkham M. Introduction to superconductivity (2nd ed.), Dover Books (2004).

[55] P. Coles, F. Lucchin, Cosmology, the Origin and Evolution of Cosmic Structure, 2nd edn., Baffins Lane: John Wiley and Sons, LTD, 2002.

[56] Yu. L. Kalinovsky, W. Kallies, V. N. Pervushin, N. A. Sarikov, Fortschr. Phys. 38, 333 (1990).

[57] A. A. Bogolubskaya, Yu. L. Kalinovsky, W. Kallies, V. N. Pervushin, Acta Phys. Polonica 21, 139 (1990).

[58] A. A. Bykov, I. M. Dremin, A. V. Leonidov, Usp. Fiz. Nauk 143, 3 (1984) [Sov. Phys.Usp. 27, 321 (1984)].