Derived McKay correspondence via pure-sheaf transforms

Timothy Logvinenko

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Abstract

In most cases where it has been shown to exist the derived McKay correspondence $D(Y) \sim \rightarrow D^G(C^n)$ can be written as a Fourier-Mukai transform which sends point sheaves of the crepant resolution $Y$ to pure sheaves in $D^G(C^n)$. We give a sufficient condition for $E \in D^G(Y \times C^n)$ to be the defining object of such a transform. We use it to construct the first example of the derived McKay correspondence for a non-projective crepant resolution of $C^3/G$. Along the way we extract more geometrical meaning out of the Intersection Theorem and learn to compute $\theta$-stable families of $G$-constellations and their direct transforms.

1 Introduction

It was observed by McKay in [McK80] that the representation graph (better known now as the McKay quiver) of a finite subgroup $G$ of $SL_2(C)$ is the Coxeter graph of one of the affine Lie algebras of type ADE, while the configuration of irreducible exceptional divisors on the minimal resolution $Y$ of $C^2/G$ is dual to the Coxeter graph of the finite-dimensional Lie algebra of the same type. It followed that the subgraph of nontrivial irreducible representations coincided with the graph of irreducible exceptional divisors. This led Gonzalez-Sprinberg and Verdier in [GSV83] to construct an isomorphism of the $G$-equivariant $K$-theory of $C^2$ to the $K$-theory of $Y$, which induced naturally a choice of such bijection. This became known as the (classical) McKay correspondence.

In [Rei97] M.Reid proposed that the $K$-theory isomorphism might lift to the level of derived categories. It became known as the derived McKay correspondence conjecture:

**Conjecture 1.** Let $G$ be a finite subgroup of $SL_n(C)$ and let $Y$ be a crepant resolution of $C^n/G$, if one exists. Then

$$D(Y) \sim \rightarrow D^G(C^n)$$

where $D(Y)$ and $D^G(C^n)$ are bounded derived categories of coherent sheaves on $Y$ and of $G$-equivariant coherent sheaves on $C^n$, respectively.

To date and to the extent of our knowledge this conjecture has been settled for the following situations:

1. $G \subset SL_2,3(C)$; $Y$ the distinguished crepant resolution $G$-Hilb; ([KV98], Theorem 1.4; [BKR01], Theorem 1.1).
2. $G \subset \text{SL}_3(\mathbb{C})$ abelian; $Y$ any projective crepant resolution; 
   ([CI04], Theorem 1.1).

3. $G \subset \text{SL}_n(\mathbb{C})$ abelian; $Y$ any projective crepant resolution; 
   ([Kaw05], special case of Theorem 4.2).

4. $G \subset \text{Sp}_{2n}(\mathbb{C})$; $Y$ any symplectic (crepant) resolution; 
   ([BK04], Theorem 1.1).

In the case 3 the construction is not direct and it isn’t clear what form does the equivalence (1.1) take, but in each of the cases 1, 2 and 4, the equivalence (1.1) is constructed directly and we observe that the constructed functor sends point sheaves $O_y$ of $Y$ to pure sheaves (i.e. complexes with cohomologies concentrated in degree zero) in $D^G(\mathbb{C}^n)$. Another property (cf. though [Orl97], Theorem 2.18) that these functors share is that each can be written as a Fourier-Mukai transform $\Phi_E(\otimes \rho_0)$ (see Def. 3) for some object $E \in D^G(Y \times \mathbb{C}^n)$.

A straightforward application (Prop. 3) of the established machinery of Fourier-Mukai transforms shows that if an equivalence (1.1) is a Fourier-Mukai transform $\Phi_E(\otimes \rho_0)$ which sends point sheaves to pure sheaves, then its defining object $E$ is itself a pure sheaf. Moreover, the fibers of $E$ over $Y$ have to be simple ($G$-$\text{End}_{\mathbb{C}^n}(E|_y) = \mathbb{C}$ for all $y \in Y$), orthogonal in all degrees ($G$-$\text{Ext}^i_{\mathbb{C}^n}(E|_{y_1}, E|_{y_2}) = 0$ if $y_1 \neq y_2$) and the Kodaira-Spencer maps have to be isomorphisms.

Let $Y$ now be any irreducible separated scheme of finite type over $\mathbb{C}$. A gnat-family $\mathcal{F}$ on $Y$ is a coherent $G$-sheaf on $Y \times \mathbb{C}^n$, flat over $Y$, such that for any $y \in Y$ the fiber $\mathcal{F}|_y$ of $\mathcal{F}$ is a $G$-constellation supported on a single $G$-orbit. That is, $\mathcal{F}|_y$ is a finite length coherent $G$-sheaf on $\mathbb{C}^n$ whose support is a single $G$-orbit and whose global sections have $G$-representation structure of the regular representation. Such family $\mathcal{F}$ has a well-defined Hilbert-Chow morphism $\pi_\mathcal{F} : Y \to \mathbb{C}^n/G$, it sends any $y \in Y$ to the $G$-orbit that $\mathcal{F}|_y$ is supported on (Prop. 2). Let $Y$ and $\mathcal{F}$ be any such $\pi_\mathcal{F}$ is birational and proper. In this paper we give a sufficient condition for the functor $\Phi_\mathcal{F}(\otimes \rho_0)$ to be an equivalence (1.1). Notable, in the view of Prop. 3, is that this condition only asks for the non-orthogonality locus of $\mathcal{F}$ to be of high enough codimension. The simplicity of $\mathcal{F}$ and the Kodaira-Spencer maps being isomorphisms follow automatically:

**Theorem 1.** Let $G$ be a finite subgroup of $\text{SL}_n(\mathbb{C})$. Let $Y$ be an irreducible separated scheme of finite type over $\mathbb{C}$ and $\mathcal{F}$ be a gnat-family on $Y$. Assume $Y$ and $\mathcal{F}$ such that the Hilbert-Chow morphism $\pi_\mathcal{F}$ is birational and proper.

If for every $0 \leq k < (n + 1)/2$, the codimension of the subset

$$N_k = \{ (y_1, y_2) \in Y \times Y \setminus \Delta \mid G$-$\text{Ext}^k_{\mathbb{C}^n}(\mathcal{F}|_{y_1}, \mathcal{F}|_{y_2}) \neq 0 \}$$

(1.2)

in $Y \times Y$ is at least $n + 1 - 2k$, then the functor $\Phi_\mathcal{F}(\otimes \rho_0)$ is an equivalence of categories $D(Y) \sim D^G(\mathbb{C}^n)$.

Once $\Phi_\mathcal{F}(\otimes \rho_0)$ is known to be an equivalence usual methods ([Rob98], Theorem 6.2.2 and [BKR01], Lemma 3.1) apply to show that $Y$ is non-singular and $\pi_\mathcal{F}$ is crepant. The set $N_k$ in (1.2) can be thought of as the locus of the degree $k$ non-orthogonality in $\mathcal{F}$.

Our proof of Theorem 1 is based on the ideas introduced in [BO95] and [BKR01], particularly on the Intersection Theorem trick introduced in the latter. However, not wishing to restrict ourselves
to just quasi-projective schemes necessitates more work in applying the Intersection Theorem. This
is done in Section 2, which is a self-contained piece of abstract derived category theory for a locally
noetherian scheme $X$. There we propose a generalisation of the concept of the homological dimension
of $E \in D^b_{coh}(X)$ which we call Tor-amplitude, and use it to show that the inequality

$$\text{hom. dim. } E \geq \text{codim}_X \text{ Supp } E$$

of [BM02], Corollary 5.5 refines to

$$\text{Tor-amp } E \geq \text{codim}_X \text{ Supp } E + \text{coh-amp } E.$$  

Other notable points of our proof of Theorem 1 are a different approach to Grothendieck duality
when constructing the left adjoint to $\Phi_F(- \otimes \rho_0)$ and an application of [Log06], Prop. 1.5 which states
that outside the exceptional set of $Y$ any gnat-family has to be locally isomorphic to the universal
family of $G$-clusters. The latter is everywhere simple and its Kodaira-Spencer maps are isomorphisms.
Then the locus of points of $Y$ where objects of $\mathcal{F}$ are not simple or the Kodaira-Spencer map isn’t an
isomorphism turns out to have too high a codimension to exist at all.

The question of an existence of a derived McKay correspondence which sends point sheaves to
pure sheaves is thus reduced to that of an existence of a gnat-family satisfying the non-orthogonality
condition of Theorem 1. This is particularly relevant whenever $G$ is abelian, for then all the gnat-
families on a given resolution $Y \to \mathbb{C}^n/G$ had been classified and their number was shown to be finite
and non-zero ([Log06], Theorem 4.1).

When $n = 3$, Theorem 1 reduces to:

**Corollary 1.** Let $G$ be a finite subgroup of $\text{SL}_3(\mathbb{C})$. Let $Y$, $\mathcal{F}$ and $\pi_\mathcal{F}$ be as in Theorem 1. Let $E_1, \ldots, E_k$ be the irreducible exceptional surfaces of $\pi_\mathcal{F}$. Then if general points of any surface $E_i$
are orthogonal in degree 0 in $\mathcal{F}$ to general points of any surface $E_j$ (including case $j = i$) and of any
curve $E_i \cap E_m$, then $\Phi_\mathcal{F}(- \otimes \rho_0)$ is an equivalence of categories.

By a general point of an intersection of $k$ exceptional surfaces we mean a point that doesn’t lie on
an intersection of any $k + 1$ exceptional surfaces.

In Section 4 we show how to compute the degree 0 non-orthogonality locus of a gnat-family. We use this in Section 5 to give following application of Corollary 1: for $G$ the abelian subgroup of $\text{SL}_3(\mathbb{C})$ known as $\frac{1}{2}(1, 1, 4) \oplus \frac{1}{2}(1, 0, 1)$ (see Section 5.1) and for $Y$ a certain non-projective crepant
resolution of $\mathbb{C}^3/G$ (see Section 5.2) we construct a gnat-family $\mathcal{F}$ on $Y$ which satisfies the condition
in Corollary 1. This gives the first example of the derived McKay correspondence for a non-projective
crepant resolution of $\mathbb{C}^3/G$.

It also leads to an important observation: the properties that $\mathcal{F}$ must then possess in view of
Proposition 3 imply that $Y$ is a fine moduli space of $G$-constellations, representing the functor of all
gnat-families whose members (fibres over closed points) are isomorphic to members of $\mathcal{F}$. At present
the only moduli functors known for $G$-constellations come from the notion of $\theta$-stability. Their fine
moduli spaces $M$ (cf. [CI04]) are constructed via the method introduced by King in [Kin94]. However,
$Y$ can’t be one of $M$ as these are all, due to the GIT nature of their construction in [Kin94], projective
over $\mathbb{C}^n/G$. This raises the question as to whether there could exist a more general notion of ‘stability’,
related perhaps to Bridgeland-Douglas stability [Bri02], which would allow for functors with non-
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2 Cohomological and Tor amplitudes

We clarify terminology and introduce notation. By a point of a scheme we mean both a closed and non-closed point unless specifically mentioned otherwise. Given a point $x$ on a scheme $X$ we write $(\mathcal{O}_x, m_x)$ for the local ring of $x$, $k(x)$ for the residue field $\mathcal{O}_x/m_x$ and $\iota_x$ for the point-scheme inclusion $\text{Spec } k(x) \hookrightarrow X$. Given an irreducible closed set $C \subset X$, we write $x_C$ for the generic point of $C$ and we sometimes write simply $(\mathcal{O}_C, m_C)$ for the local ring of $x_C$. All complexes are cochain complexes. Given a right (resp. left) exact functor $F$ between two abelian categories $\mathcal{A}$ and $\mathcal{B}$, we denote by $\mathbf{L} F$ (resp. $\mathbf{R} F$) the left (resp. right) derived functor between the appropriate derived categories, if it exists, and by $\mathbf{L}^i F(\bullet)$ (resp. $\mathbf{R}^i F(\bullet)$) the $-i$-th cohomology of $\mathbf{L} F(\bullet)$ (resp. the $i$-th cohomology of $\mathbf{R} F(\bullet)$).

For $X$ a smooth variety the results of Lemmas 1 and 2 below have appeared in the proof of Proposition 1.5 in [BO95]. We show them to hold in a more general setting of a locally noetherian scheme.

Lemma 1. Let $X$ be a locally noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on $X$ and $C$ be an irreducible component of $\text{Supp}_X \mathcal{F}$. Then for every point $x \in C$

$$\mathbf{L}^i \iota^*_x \mathcal{F} \neq 0 \quad \text{for } 0 \leq i \leq \text{codim}_X(C).$$

(2.1)

Proof. Recall (cf. [Mat86], §19) that if a minimal free resolution $L_\bullet$ of a finitely generated module $M$ for a local ring $(R, \mathfrak{m}, k)$ exists, then

$$\dim_k \text{Tor}^i(M, k) = \text{rk} L_i$$

Since $X$ is locally noetherian minimal free resolutions of $\mathcal{F}$ exist in all local rings. Write $F_C$ for the localisation of $\mathcal{F}$ to the local ring $\mathcal{O}_C$ of $x_C$. As $\mathbf{L}^i \iota^*_x \mathcal{F} = \text{Tor}^i_{\mathcal{O}_C}(F_C, k(x))$ it suffices to prove that the length of the minimal free resolution of $F_C$ is at least $\text{codim}_X(C)$.

Consider the standard filtration ([Ser00], I, §7, Theorem 1) of $F_C$ by submodules $0 = M_0 \subset \cdots \subset M_n = F_C$ with each $M_i/M_{i-1}$ isomorphic to $\mathcal{O}_C/p$ for some $p \in \text{Supp}_{\mathcal{O}_C}(F_C)$. As the defining ideal of $C$ is minimal in $\text{Supp}_X(\mathcal{F})$, $\text{Supp}_{\mathcal{O}_C}(F_C)$ consists of just $m_C$. So each $M_i/M_{i-1}$ is isomorphic to $k_C$ and hence $F$ is a finite-length $\mathcal{O}_C$-module. Then by the New Intersection Theorem (e.g. [Rob98], Theorem 6.2.2) the length of the minimal resolution of $F_C$ is at least $\dim \mathcal{O}_C$. As $\dim \mathcal{O}_C = \text{codim}_X(C)$ the claim follows. \hfill $\square$

Lemma 2. Let $X$ be a locally noetherian scheme. Let $\mathcal{F}$ be a coherent sheaf on $X$ of finite Tor-dimension. For any $p \in \mathbb{Z}$ define

$$D_p = \{ x \in X \mid \mathbf{L}^i \iota^*_x \mathcal{F} \neq 0 \text{ for some } i \geq p \}. $$

(2.2)

Then each $D_p$ is closed and $\text{codim}_X(D_p) \geq p$. 

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Proof. It suffices to prove both claims for the case $X = \text{Spec } R$ with $R$ noetherian. Write $F$ for $\Gamma(F)$. As $L^p \tau_x^* F = \text{Tor}^R_p(F, k(x))$ the first claim follows from the upper semicontinuity theorem ([GD63], Théorème 7.6.9).

For the second claim let $C$ be any irreducible component of $D_p$ and let $F_C$ be the localisation of $F$ to the local ring $O_C$. Then $\text{Tor}^{O_C}_p(F_C, k(x_C)) \neq 0$ by the defining property of $D_p$. We have ([Mat86], §19, Lemma 1) \[
\text{proj dim}_{O_C} F_C = \sup \{ i \in \mathbb{Z} \mid \text{Tor}^{O_C}_i(F_C, k(x_C)) \}
\]
hence $\text{proj dim}_{O_C} F_C \geq p$. By the Auslander-Buchsbaum equality we have \[
\text{proj dim}_{O_C} F_C = \text{proj dim}_{O_C} F_C + \text{depth}_{O_C} F_C
\]
and thus $\text{codim}_X C = \text{dim } O_C \geq \text{depth}_{O_C} O_C \geq p$ as required. \hfill \Box

The main idea behind the proof of the following proposition we owe to Bondal and Orlov in [BO95], Proposition 1.5.

**Proposition 1.** Let $X$ be a locally noetherian scheme and $F \in D^{b}_{\text{coh}}(X)$ an object of finite $\text{Tor}$-dimension. Denote by $H^i$ the $i$-th cohomology sheaf of $F$. Then for any point $x \in X$ we have \[
- \sup \{ i \in \mathbb{Z} \mid x \in \text{Supp } H^i \} = \inf \{ j \in \mathbb{Z} \mid L^j \tau_x^* F \neq 0 \}. \tag{2.3}
\]

Let $C$ be an irreducible component of $\text{Supp } H^l$ for some $l$ such that also $C \nsubseteq \text{Supp } H^m$ for any $m < l$. Then \[
\text{codim}_X C - \inf \{ i \in \mathbb{Z} \mid C \subseteq \text{Supp } H^i \} = \sup \{ j \in \mathbb{Z} \mid L^j \tau_x^* F \neq 0 \}. \tag{2.4}
\]

**Proof.** Fix a point $x \in X$. The main ingredient of the proof is the standard spectral sequence (eg. [GM03], Proposition III.7.10) associated to the filtration of $L \tau_x^* F$ by the rows of the Cartan-Eilenberg resolution of $F$: \[
E^{-p,q}_2 = L^p \tau_x^*(H^q) \Rightarrow E_{\infty}^{q-p} = L^{q-p} \tau_x^*(F). \tag{2.5}
\]

Denote by $h$ the highest non-zero row of $E_2$. As all rows above row $h$ and all columns to the right of column 0 in $E_2$ consist entirely of zeroes
we conclude by inspection of the complex that $0 = E^0_\infty$ for all $n > h$ and $\mathcal{H}^h|_x = E^0_2 = E^h_\infty = L^{-h}(s_*(F))$. This gives (2.3).

To obtain (2.4) set $x$ to be the generic point of $C$ and define $E^{\bullet\bullet}_x$ as above. For any $m < l$ we have $C \not\subseteq \text{Supp } \mathcal{H}^m$ and hence $L^{-h}_x \mathcal{H}^m = 0$. So all the rows of $E^{\bullet\bullet}_x$ below $l$ consist of zeroes. On the other hand, $C$ is an irreducible component of $\mathcal{H}^l$ and by Lemma 2 the set of points $y \in X$, such that there is a non-zero $L^i_x(s_*(\mathcal{H}^l))$ with $i > d$, is closed and of codimension at least $d + 1$. Then this set can not contain $x$ for the closure of $x$ is $C$ whose codimension is $d$. Hence all columns to the left of column $-d$ in $E^{\bullet\bullet}_2$ consist entirely of zeroes. We conclude that $E^0_\infty = 0$ for all $n > l - d$ and $L^d_x(s_*(\mathcal{H}^l)) = E^{d-d}_2 = E^{l-d}_\infty = L^{d-l}_x(s_*(F))$. Thus, as $L^d_x(s_*(\mathcal{H}^l)) \neq 0$ by Lemma 1, we obtain (2.4). 

**Definition 1.** Let $A$ be an abelian category and $E^{\bullet}$ be a cochain complex of objects of $A$. Define its cohomological amplitude, denoted by coh-amp $E$, to be the length of the minimal interval in $\mathbb{Z}$ containing the set

$$\{i \in \mathbb{Z} \mid H^i(E) \neq 0\}. \quad (2.6)$$

If no such interval exists we say that coh-amp $E = \infty$.

Trivially coh-amp $E$ is the minimal length of a bounded complex quasi-isomorphic to $E$, if any exist, and infinity, if none do.

**Definition 2.** Let $R$ be a ring or a sheaf of rings and $E^{\bullet}$ be a cochain complex of objects of $\text{Mod} - R$. Define its Tor-amplitude, denoted by Tor-amp$_R E^{\bullet}$, to be the length of the minimal interval in $\mathbb{Z}$ containing the set

$$\{i \in \mathbb{Z} \mid \exists A \in \text{Mod} - R \text{ such that } \text{Tor}^i_R(E^{\bullet}, A) \neq 0\}. \quad (2.7)$$

If no such interval exists we say that Tor-amp$_R E = \infty$.

Def. 2 can be seen to be equivalent to [Kuz05], Def. 2.20.

Let now $X$ be any scheme. It follows from [Har66], Prop 4.2, that an object of $D^b(\text{Mod} - X)$ has finite Tor-amplitude if and only if it is of finite Tor-dimension, i.e. quasi-isomorphic to a bounded complex of flat sheaves.

**Lemma 3.** Let $X$ be a locally noetherian scheme and $E \in D^b_{\text{coh}}(X)$ an object of finite Tor-dimension. Denote by $l$ the length of the shortest complex of flat sheaves quasi-isomorphic to $E$, and by $k$ the length of the smallest interval in $\mathbb{Z}$ containing the set

$$\{i \in \mathbb{Z} \mid \exists x \in X \text{ such that } L^i_x(s_*(E)) \neq 0\}. \quad (2.8)$$

Then $l = \text{Tor-amp}_{\mathcal{O}_X} E = k$.

**Proof.** Implications $l \geq \text{Tor-amp}_{\mathcal{O}_X} E$ and $\text{Tor-amp}_{\mathcal{O}_X} E \geq k$ are trivial. We claim that $k \geq l$.

Let $n, k \in \mathbb{Z}$ be such that the interval $[-n - k, -n]$ contains the set (2.8). Then (2.3) and (2.4) of Proposition 1 show that $H^i(E) = 0$ unless $i \in [n, n + k]$. Since resolutions by flat modules exist on $X$, there exists a complex $F^{\bullet}$ of flat sheaves quasi-isomorphic to $E$ and with $F_i = 0$ for all $i > n + k$. 

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We claim that we can truncate $F^\bullet$ at degree $n$ and keep it flat, i.e. the sheaf $F^n/\text{Im } F^{n-1}$ is flat. But as $\mathcal{H}_i(F^\bullet) = 0$ for $i < n$, the complex

$$
\cdots \to F^{n-2} \to F^{n-1} \to F^n \to 0 \to \cdots
$$

is a flat resolution of $F^n/\text{Im } F^{n-1}$. Hence $L^1 i_x^* (F^n/\text{Im } F^{n-1}) = L^{n+1} i_x^*(E)$ and so vanishes for all $x \in X$ by assumption. Thus we obtain a length $k$ complex of flat-sheaves quasi-isomorphic to $E$, i.e. $k \geq l$.

Whenever $X$ is a quasi-projective scheme, or any other scheme where there exist resolutions by locally-free sheaves, replacing the word ‘flat’ by the word ‘locally-free’ throughout Lemma 3 and its proof shows that for any $E \in D_{\text{coh}}^b(X)$ its Tor-amplitude is the length of the shortest complex of locally-free sheaves quasi-isomorphic to $E$. In other words, Tor-amp$_{\mathcal{O}_X} E$ is the homological dimension of $E$ introduced in [BM02]. The following can thus be compared to the inequality hom.dim.$E \geq \text{codim } C$ of [BM02]:

**Theorem 2.** Let $X$ be a locally noetherian scheme and $E \in D_{\text{coh}}^b(X)$ an object of finite Tor-dimension. Then

$$
\text{Tor-amp}_{\mathcal{O}_X} E \geq \text{codim } \text{Supp } E + \text{coh-amp } E
$$

and for any irreducible component $C$ of $\text{Supp } E$ we have

$$
\text{Tor-amp}_{\mathcal{O}_C} E_C = \text{codim } C + \text{coh-amp}_{\mathcal{O}_C} E_C.
$$

**Remark:** To see that the inequality (2.9) can be strict, consider $X = \mathbb{A}^1$ and $E = \mathcal{O}_X \oplus \mathcal{O}_x$ for some closed point $x \in X$.

**Proof.** Denote by $\mathcal{H}^i$ the $i$th cohomology sheaf of $E$. Set

$$
n = \inf_{x \in \text{Supp } E} \{ i \in \mathbb{Z} | x \in \text{Supp } \mathcal{H}^i \},
$$

$$
m = \sup_{x \in \text{Supp } E} \{ i \in \mathbb{Z} | x \in \text{Supp } \mathcal{H}^i \},
$$

$$
l = \inf_{x \in \text{Supp } E} \{ i \in \mathbb{Z} | L^i i_x^* E \neq 0 \},
$$

$$
h = \sup_{x \in \text{Supp } E} \{ i \in \mathbb{Z} | L^i i_x^* E \neq 0 \},
$$

and observe that $m - n = \text{coh-amp } E$ and $h - l = \text{Tor-amp}_{\mathcal{O}_X} E$ (Lemma 3).

By (2.3) of Proposition 1 we have

$$
-m = l.
$$

Let $D$ be any irreducible component of $\text{Supp } \mathcal{H}^n$. We then have

$$
\text{codim } \text{Supp } E - n \leq \text{codim } D - n = \sup \{ i \in \mathbb{Z} | L^i i_D^* E \neq 0 \} \leq h
$$

with the middle equality due to (2.4) of Proposition 1 applied to $D$. Subtracting (2.11) from (2.12) we obtain $(m - n) + \text{codim } \text{Supp } E \leq (h - l)$. This shows (2.9).

To obtain (2.10) we observe that on $\text{Spec } \mathcal{O}_C$ the support of the localisation $E_C$ consists of a single point $x_C$. Therefore applying the above argument to $X' = \text{Spec } \mathcal{O}_C$ and $E' = E_C$ we have $D = x_C = \text{Supp } E'$ which makes both the inequalities in (2.12) into equalities. □
3 Derived McKay correspondence

Given a scheme $S$ denote by $D_{qc}(S)$ (resp. $D(S)$) the full subcategory of the derived category of $\mathcal{O}_S\text{-Mod}$ consisting of complexes with quasi-coherent (resp. bounded and coherent) cohomology. For $S$ a scheme of finite type over $\mathbb{C}$ and $H$ a finite group acting on $S$ on the left by automorphisms an $H$-sheaf is a sheaf $E$ of $\mathcal{O}_S$-modules equipped with a lift of the $H$-action to $E$. For the technical details see [BKR01], Section 4. Denote by $\mathcal{O}_S\text{-Mod}^H$ (resp. $\mathbf{QCoh}^H_S$, $\text{Coh}^H_S$) the abelian category of $H$-sheaves (resp. quasi-coherent, coherent $H$-sheaves) on $S$ and by $D^H_{qc}(S)$ (resp. $D^H(S)$) the full subcategory of the derived category of $\mathcal{O}_S\text{-Mod}^H$ consisting of complexes with quasi-coherent (resp. bounded and coherent) cohomology.

3.1 Integral transforms

Let $N$ and $M$ be schemes of finite type over $\mathbb{C}$. Denote by $\pi_N$ and $\pi_M$ the projections $N \times M \to N$ and $N \times M \to M$.

**Definition 3.** Let $E$ be an object of $D_{qc}(N \times M)$ of finite Tor-dimension. An integral transform $\Phi_E$ is a functor $D_{qc}(N) \to D_{qc}(M)$ defined by

$$\Phi_E(-) = R\pi_M^*(E \otimes \pi_N^*(-)). \quad (3.1)$$

The object $E$ is called the kernel of the transform. If $\Phi_E$ is an equivalence of categories it is further called a Fourier-Mukai transform.

If a group $G$ acts on $N$ and $M$ then, for any $E \in D^G_{qc}(N \times M)$ of finite Tor-dimension, (3.1) defines an integral transform $D^G_{qc}(N) \to D^G_{qc}(M)$. If the group action on $N$ is trivial denote by $(- \otimes \rho_0)$ the functor $D_{qc}(N) \to D_{qc}(N)$ which gives a sheaf the trivial $G$-equivariant structure. It is exact and has an exact left and right adjoint $(-)^G$, the functor of taking the $G$-invariant part ([BKR01], Section 4.2). We also use the terms integral and Fourier-Mukai transform for the functors $D_{qc}(N) \to D^G_{qc}(M)$ of the form $\Phi_E(- \otimes \rho_0)$ where $\Phi_E$ is some integral transform $D^G_{qc}(N) \to D^G_{qc}(M)$.

When $N$ and $M$ are smooth and proper varieties it is well known that $\Phi_E$ has a left adjoint $\Phi_{E^\vee \otimes \pi_M^*(\omega_M)[\dim M]}$ ([BO95], Lemma 1.2). The lemma below allows to generalise this to certain integral transforms between non-proper schemes. We use methods of Verdier-Deligne as per the exposition in [Del66] to which we refer the reader for all the necessary definitions.

**Lemma 4.** Let $N$ and $M$ be separable schemes of finite type over $\mathbb{C}$ with $M$ smooth of dimension $n$. Let $E \in D(N \times M)$ be of finite homological dimension with $\text{Supp}(E)$ proper over $N$. Then the functor

$$\pi_N^*(-) \otimes E : D(N) \to D(N \times M)$$

has a left adjoint

$$R\pi_{N*}(- \otimes E^\vee \otimes \pi_M^*(\omega_M))[n] : D(N \times M) \to D(N). \quad (3.2)$$
Proof. First we compactify $M$: choose an open immersion $M \hookrightarrow \bar{M}$ with $\bar{M}$ smooth and proper [Nag62]. Then $\pi_N$ decomposes as an open immersion $\iota : N \times M \hookrightarrow N \times \bar{M}$ followed by the projection $\pi_N : N \times M \rightarrow N$. As $\pi_N$ is smooth and proper Grothendieck-Serre duality for smooth and proper morphisms (e.g. [Har66], VII4.3) implies that $\pi_N^* : D(N) \rightarrow D(N \times M)$ has a left adjoint

$$R \pi_N^*(-) \otimes \pi_M^* \omega_{\bar{M}}[n]$$

where $\pi_M : N \times \bar{M} \rightarrow \bar{M}$ is the projection onto the second component.

By the duality for open immersions ([Del66], Prop. 4) the left adjoint to the (exact) functor $\iota^*(-)$ exists as an (exact) functor $\iota_!$ from $\text{Coh}(N \times M)$ to the category pro-$\text{Coh}(N \times \bar{M})$. For the definition of pro-$\text{Coh}(N \times \bar{M})$ and the generalities on pro-objects see [Del66], n° 1. The functor $\iota_!$ may be calculated as follows: given $\mathcal{A} \in \text{Coh}(N \times M)$ take any $\mathcal{A} \in \text{Coh}(N \times \bar{M})$ which restricts to $\mathcal{A}$ on $N \times M$. Then

$$\iota_!(\mathcal{A}) = \varinjlim \text{Hom}(\mathcal{I}^n \mathcal{A}, -)$$

(3.3)

where $\mathcal{I}$ is the ideal sheaf defining the complement $N \times (\bar{M} \setminus M)$.

Finally, as $E$ is of finite homological dimension, the left adjoint of $(-) \otimes^L E$ is $(-) \otimes^L E^!$ where $E^!$ is $R \text{Hom}(E, \mathcal{O}_{N \times \bar{M}})$. Therefore the left adjoint of $\pi_N^* (-) \otimes^L E$ exists as the functor

$$R \pi_N^*(\iota_!(-) \otimes^L E^! \otimes \pi_M^*(\omega_M))[n]$$

(3.4)

from pro-$D(N \times M)$ to pro-$D(N)$. To finish the proof it suffices now to show that $\iota_!(\pi^M \otimes E^!)$ is $\iota_* (\pi^M \otimes E^!)$. Then applying the projection formula to $\iota_* (\pi^M \otimes E^!)$ in (3.4) and observing that $\iota \circ \pi_M = \pi_M$ and $\iota \circ \pi_N = \pi_N$ yields (3.2).

We have $\text{Id} = \iota^* \iota_!$ on $\text{QCoh}(N \times M)$ ([GD60], Prop. 9.4.2). It induces by the adjunction of [Del66], Prop. 4 natural transformations $\Upsilon : \iota_! \rightarrow \iota_*$ of functors $\text{Coh}(N \times M) \rightarrow \text{pro-}\text{Coh}(N \times \bar{M})$ and $\Upsilon' : \iota_!(\pi^M \otimes E^!) \rightarrow \iota_*(\pi^M \otimes E^!)$ of functors $D(N \times M) \rightarrow \text{pro-}D(N \times \bar{M})$. By [Del66], Prop. 3 and the exactness of $\iota_!$ and $\iota_*$, to show $\Upsilon'$ to be an isomorphism of functors it suffices to show that $\Upsilon$ is an isomorphism on the cohomology sheaves of $\pi^M \otimes E^!$. The support of these is proper over $N$ by the assumption on $E$. For any $\mathcal{A} \in \text{Coh}(N \times M)$ we have

$$\text{Hom}(\iota_!(\mathcal{A}), \iota_*(\mathcal{A})) = \lim \text{Hom}_{N \times \bar{M}}(\mathcal{I}^k \mathcal{A}, \iota_*(\mathcal{A}))$$

(3.5)

using the notation of (3.3). From the construction of the adjunction in [Del66], Prop. 4 it is immediate that $\Upsilon(\mathcal{A})$ is the unique element of RHS of (3.5) which restricts to $N \times M$ as $\text{Id} \in \text{Hom}_{N \times M}(\mathcal{A}, \mathcal{A})$. If $\text{Supp}(\mathcal{A})$ is proper over $N$, we can take $\mathcal{A} = \iota_* \mathcal{A}$ in (3.3). Moreover, $\mathcal{I}^k \iota_!(\mathcal{A}) = \iota_*(\mathcal{A})$ for all $k$. Therefore (3.3) yields $\iota_!(\mathcal{A}) = \iota_*(\mathcal{A})$ and moreover the RHS of (3.5) is just $\text{Hom}(\iota_! \mathcal{A}, \iota_! \mathcal{A})$. It is then clear that $\Upsilon(\mathcal{A}) = \text{Id}$, as required. \hfill \Box
3.2 $G$-constellations and $gnt$-families

**Definition 4.** Let $G$ be a finite subgroup of $\text{GL}_n(\mathbb{C})$. A $G$-constellation is a coherent $G$-sheaf $V$ on $\mathbb{C}^n$ whose global sections $\Gamma(V)$ have the $G$-representation structure of the regular representation $V_{\text{reg}}$.

Two $G$-constellations $V, W$ are orthogonal in degree $k$ if $G\text{-Ext}_{\mathbb{C}^n}^k(V, W) = G\text{-Ext}_{\mathbb{C}^n}^k(W, V) = 0$.

Let now $Y$ be a scheme of finite type over $\mathbb{C}$. We endow $Y$ with the trivial $G$-action, thus we can speak of $G$-sheaves on $Y$ and on $\mathbb{C}^n$.

**Definition 5.** A $gnt$-family on $Y$ (short for $G$-natural or geometrically natural) is an object $F$ of $\text{CoC}^G(Y \times \mathbb{C}^n)$, flat over $Y$, such that for every closed $y \in Y$ the fiber $F_y$ is a $G$-constellation supported on a single $G$-orbit. The Hilbert-Chow map $\pi_F$ of $F$ is the map $Y \to \mathbb{C}^n/G$ defined by $y \mapsto \text{Supp}_{\mathbb{C}^n} F_y$. A $gnt$-family on a fixed scheme $Y \to \mathbb{C}^n/G$ is a $gnt$-family on $Y$ whose Hilbert-Chow map coincides with $\pi$.

Two subsets $C$ and $C'$ of $Y$ are orthogonal in degree $k$ in $F$ if for every $y \in C$ and $y' \in C'$ the fibers $F_y$ and $F_{y'}$ are orthogonal in degree $k$. The family $F$ is orthogonal in degree $k$ if $Y$ is in $G$-degree $k$ in $F$.

**Proposition 2.** For any $gnt$-family $F$ its Hilbert-Chow map $\pi_F$ is a morphism.

**Proof.** Denote by $R$ the ring $\mathbb{C}[x_1, \ldots, x_n]$. For any $G$-constellation $V$, the action of $R$ on $H^0(V)$ restricts to the action of $R^G$ on $H^0(V)^G$. Clearly

$$(\text{Ann}_R H^0(V))^G \subseteq \text{Ann}_{R^G} H^0(V)^G.$$ (3.6)

The LHS of (3.6) is the image of $\text{Supp}_{\mathbb{C}^n} V$ in $\mathbb{C}^n/G$. If this support is a single $G$-orbit, then $(\text{Ann}_R H^0(V))^G$ is maximal in $R^G$ and (3.6) is an equality. Therefore it suffices to construct a morphism $Y \to \mathbb{C}^n/G$ which sends each $y \in Y$ to $\text{Ann}_{R^G} H^0(F_y)^G$. We construct it thus: the invariant part of $\pi_{Y*}(F)$ is a line bundle on $Y$, which has a $R^G$-module structure induced from $F$. This structure defines a homomorphism $R^G \to O_Y$. The corresponding morphism $Y \to \mathbb{C}^n/G$ is easily seen to send each $y \in Y$ to $\text{Ann}_{R^G} H^0(F_y)^G$. $\square$

**Lemma 5.** If $F$ is a $gnt$-family on $Y$ and $\pi_F : Y \to \mathbb{C}^n/G$ is proper, then $F$ is of finite homological dimension in $D^G(Y \times \mathbb{C}^n)$ and the integral transform $\Phi_F : D^G_{\text{qc}}(Y) \to D^G_{\text{qc}}(\mathbb{C}^n)$ restricts to $D^G(Y) \to D^G(\mathbb{C}^n)$.

**Proof.** Let $\iota$ be the open immersion $Y \times \mathbb{C}^n \to Y \times \mathbb{P}^n$. As $\text{Supp} \ F$ is proper over $Y$, $\iota_* F$ is coherent. Quite generally, given any coherent sheaf $A$ on $Y \times \mathbb{P}^n$ flat over $Y$, consider the adjunction co-unit $\xi : \pi_Y^* \pi_{Y*} A \to A$. As $\pi_Y$ is proper and $A$ is flat over $Y$, $\pi_Y^* \pi_{Y*} A$ is lfr (locally free of finite rank). Twisting by some power of $\pi_{Y*} O(1)$ we can make $\xi$ surjective. But then $\ker \xi$ is again coherent and flat over $Y$. We set initially $A = \iota_* F$ and repeat this construction until $\ker \xi$ becomes lfr. This has to happen eventually as $\iota_* F$ is flat over $Y$ and $\mathbb{P}^n$ is smooth. Thus we obtain an lfr resolution of $\iota_* F$ of finite length. Restricting it to $Y \times \mathbb{C}^n$ demonstrates the first claim.

For the second claim: since $\pi_Y$ is flat, the pullback $\pi_Y^*(O(- \otimes \rho_0)$ is exact and takes $D(Y)$ to $D^G(Y \times \mathbb{C}^n)$. Since $F$ is of finite homological dimension, $F \otimes -$ takes $D^G(Y \times \mathbb{C}^n)$ to $D^G(Y \times \mathbb{C}^n)$. $\square$
Moreover the image $\text{Im}(\mathcal{F} \otimes -)$ lies in the full subcategory of $D^G(Y \times \mathbb{C}^n)$ consisting of the objects with support in $\text{Supp} \mathcal{F}$. Finally, $\pi_\mathcal{F}$ being proper implies that $\text{Supp} \mathcal{F}$ is proper over $\mathbb{C}^n$, hence $R\pi_{\mathcal{C}^n*}$ takes $\text{Im}(\mathcal{F} \otimes -)$ to $D^G(\mathbb{C}^n)$ ([GD61], Corollaire 3.2.4).

The following demonstrates a certain relevance of gnat-families:

**Proposition 3.** Let $G$ be a finite subgroup of $\text{SL}_n(\mathbb{C})$, $Y$ a variety and $E \in D^G(Y \times \mathbb{C}^n)$ an object such that $\Phi_E(- \otimes \rho_0)$ is an equivalence $D(Y) \overset{\sim}{\rightarrow} D^G(\mathbb{C}^n)$ which sends point sheaves on $Y$ to pure sheaves. Then $E$ is a gnat-family over $Y$ and its Hilbert-Chow map $\pi_E$ is a crepant resolution of $\mathbb{C}^n/G$. Moreover

$$G\text{-Ext}^1(E_{|y_1}, E_{|y_2}) = \begin{cases} \mathbb{C} & \text{if } y_1 = y_2, \ i = 0 \\ 0 & \text{if } y_1 \neq y_2 \end{cases} \quad (3.7)$$

and for any $y \in Y$ the (Kodaira-Spencer) map $\text{Ext}^1(\mathcal{O}_y, \mathcal{O}_y) \rightarrow G\text{-Ext}^1(E_{|y}, E_{|y})$ is an isomorphism.

**Proof.** By [Huy06], Example 5.1(vi), $E_{|y} = \Phi_E(\mathcal{O}_y \otimes \rho_0)$, whence the assertion (3.7) and the Kodaira-Spencer maps being isomorphisms. By [Bri99], Lemma 4.3, it follows that $E$ is a pure sheaf flat over $Y$. Then by Lemma 4 the inverse of $\Phi_E(- \otimes \rho_0)$ is $\Phi_{E^\vee[n]}(-)^G$. It maps $\mathcal{O}_{\mathbb{C}^n}$ to $(\pi_Y^* E^\vee[n])^G$, so the cohomology sheaves of $(\pi_Y^* E^\vee[n])^G$ are coherent $\mathcal{O}_Y$-modules. Since $\pi_Y^*$ is affine, the support of $E^\vee[n]$ is finite over $Y$. As $\text{Supp}(E^\vee[n]) = \text{Supp} E$, we conclude that for each $y \in Y$ the support of $E_{|y}$ is a finite union of $G$-orbits. The simplicity of $E_{|y}$ further implies that it has to be a single $G$-orbit. To show that $\Gamma(E_{|y})$ has $G$-representation structure of $V_{\text{reg}}$ it suffices, by flatness of $E$, to show it for any single $y \in Y$. As the set $\{E_{|y}\}_{y \in Y}$ is an image of a spanning class of $D(Y)$ under $\Phi(- \otimes \rho_0)$, it is a spanning class for $D^G(\mathbb{C}^n)$. Hence for every $G$-orbit $Z$ in $\mathbb{C}^n$ there exists $y \in Y$ such that $E_{|y}$ is supported at $Z$. Choose $Z$ to be any free orbit. The only simple $G$-sheaf supported on a free orbit is its structure sheaf, therefore $\Gamma(E_{|y}) \cong V_{\text{reg}}$. We conclude that $E$ is a gnat-family and that $\pi_E$ is surjective and an isomorphism outside the singularities of $\mathbb{C}^n/G$. By [Rob98], Theorem 6.2.2 and [BKR01], Lemma 3.1, $Y$ is smooth and $\pi_E$ is crepant. It remains to show that $\pi_E$ is proper, which is equivalent to $\text{Supp}_{\mathcal{C}^n \times \mathbb{C}^n} E$ being proper over $\mathbb{C}^n$ and that follows, e.g., from $\pi_{\mathcal{C}^n*} E$ having to be coherent, as it is a cohomology sheaf of the complex $\Phi_E(\mathcal{O}_Y \otimes \rho_0)$. \hfill \Box

### 3.3 Main results

We now give the proof of Theorem 1. Its general framework follows those of [BO95], Theorem 1.1 and of [BKR01], Theorem 1.1. We note two principal differences: [BO95] works with smooth varieties, while we assume $Y$ to be a not necessarily smooth scheme (whence the content of Section 2); [BKR01] adopts a two-step strategy to establish the left adjoint of $\Phi_{\mathcal{F}}(- \otimes \rho_0)$, whereas our Lemma 4 achieves this directly.

**Proof of Theorem 1.** We divide the proof into five steps:

**Step 1:** We claim that $\Phi_{\mathcal{F}}(- \otimes \rho_0)$ has a left adjoint $(\Psi_{\mathcal{F}})^G$, where $\Psi_{\mathcal{F}}$ is a certain integral transform $D^G(\mathbb{C}^n) \rightarrow D^G(Y)$.
Recall that $\Phi _F = R \pi _{C^n*}(\mathcal{F} \otimes \pi _Y^*(-))$. The issue here is the left adjoint of $\pi _Y^*(-)$ as $\pi _Y$, though smooth, is manifestly non-proper. But the support of $\mathcal{F}$ is proper, so by Lemma 4 the functor $R \pi _{Y*}((- \otimes \mathcal{F}^\vee |n])$ is the left adjoint to $\pi _Y^*(-) \otimes \mathcal{F}$. The claim now follows, for $\pi _{C^n*}$ is the left adjoint to $R \pi _{C^n*}$ and $(-)^G$ is the left (right) adjoint of $- \otimes \rho _0$.

Step 2: We claim that the composition $(\Psi _F)^G \circ \Phi _F (- \otimes \rho _0)$ is an integral transform $\Phi _Q$ for some $Q \in D(Y \times Y)$ and that for any closed point $(y_1, y_2)$ in $Y \times Y$ and any $k \in \mathbb{Z}$ we have

$$L^k \iota _{y_1, y_2}^* Q = G^* \text{Ext}^k(\mathcal{F}_{|y_1}, \mathcal{F}_{|y_2})^\vee. \quad (3.8)$$

The first assertion is a standard result due to Mukai in [Muk81], Proposition 1.3. The second assertion follows from the formula (5) of [BKR01], Sec. 6, Step 2 by the adjunction of $L \iota _{y_1, y_2}^*$ and $\iota _{y_1, y_2}^*$.\hfill

Step 3: We claim that $Q$ is a pure sheaf and that its support lies within the diagonal $Y \to Y \times Y$.

First note that since $Y \times Y$ is of finite type over $\mathbb{C}$, it is certainly Jacobson (see [GD66], §10.3) and so any closed set of $Y \times Y$ is uniquely identified by its set of closed points. We implicitly use this property at several points of the argument below.

Recall the closed set $N_k$ of (1.2). As the support of any $G$-constellation is proper and as $\omega _{C^n} = \mathcal{O}_{C^n} \otimes \rho _0$ as a $G$-sheaf since $G \subseteq \text{SL}_n(\mathbb{C})$, Serre duality applies to yield

$$G^* \text{Ext}^k_{C^n}(\mathcal{F}_{|y_1}, \mathcal{F}_{|y_2}) = G^* \text{Ext}^n_{C^n}(\mathcal{F}_{|y_2}, \mathcal{F}_{|y_1})^\vee.$$  

It follows that $\text{codim } N_k = \text{codim } N_{n-k}$ for all $k$.

Let $C$ be an irreducible component of $\text{Supp }Q$. Denote by $y_C$ its generic point, by $\mathcal{O}_C$ the local ring of $y_C$ and by $Q_C$ the localisation of $Q$ to $\mathcal{O}_C$. For any $k$ denote by $M_k$ the set $\{ y \in Y \times Y \mid L^k \iota _y^* Q \neq 0 \}$ and let $l$ and $m$ be the infimum and the supremum of the set $\{ k \in \mathbb{Z} \mid y_C \in M_k \}$, hence $\text{Tor}^{\text{amp}}_{\mathcal{O}_C} Q_C = m - l$ (Lemma 3). By (3.8) the closure of $M_l \setminus \Delta$ is $N_l$, so $y_C \in M_l$ implies $y_C \in \Delta$ or $y_C \in N_l$. Similarly for $N_m$. Thus either $y_C \in \Delta$ or $y_C \in N_l \cap N_m$. The latter would imply that

$$\text{codim } C \geq \text{codim } N_l \geq n - 2l + 1$$

$$\text{codim } C \geq \text{codim } N_m = \text{codim } N_{n-m} \geq 2m - n + 1$$

and therefore that $\text{codim } C \geq m - l + 1$. But then $\text{codim } C$ would be strictly greater than $\text{Tor}^{\text{amp}}_{\mathcal{O}_C} Q_C$, which contradicts Theorem 2. Thus $y_C$ lies within $\Delta$ and, since $Y$ is separated, so does all of $C$.

We have now shown that $\text{Supp } Q \subseteq \Delta$, so $\text{codim } \text{Supp } Q \geq n$. But as $\mathbb{C}^n$ is smooth and $n$-dimensional, (3.8) implies

$$L^k \iota _y^* Q = 0 \quad \forall y \in Y, \ k \neq 0, \ldots, n \quad (3.9)$$

so $\text{Tor}^{\text{amp}} Q \leq n$. By Theorem 2 $\text{Tor}^{\text{amp}} Q = n$ and $\text{coh}^{\text{amp}} Q = 0$. Together with (3.9) this implies that $Q$ is a pure sheaf.

Step 4: We claim that $Q$ is the structure sheaf $\mathcal{O}_\Delta$ of the diagonal $\Delta$ and therefore $\Phi _F (- \otimes \rho _0)$ is fully faithful.
The adjunction co-unit $\Phi_Q \to \Id_{D(Y)}$ induces a surjective $\mathcal{O}_{Y \times Y}$-module morphism $Q \xhookrightarrow{\sim} \mathcal{O}_\Delta$. Let $K$ be its kernel, we then have a short exact sequence
\[ 0 \to K \to Q \xhookrightarrow{\sim} \mathcal{O}_\Delta \to 0. \tag{3.10} \]
Choosing some closed point $(y, y) \in \Delta$ and applying functor $L i_{y,y}^* (\_)$ to (3.10) we obtain a long exact sequence of $\mathbb{C}$-modules
\[ \cdots \to G^* \text{Ext}^n_{\mathcal{O}_Y} (\mathcal{F}_y, \mathcal{F}_y)^* \xrightarrow{\alpha_y} \Omega^1_{Y,y} \to K_{y,y} \to G^* \text{End}_{\mathbb{C}^n} (\mathcal{F}_y)^* \xrightarrow{\epsilon_y} \mathbb{C} \to 0 \to \cdots. \]

The map $\epsilon_y$ is surjective due to any $G$-constellation having automorphisms consisting of scalar multiplication. It is an isomorphism whenever $\mathcal{F}_y$ is simple, i.e. when the scalar multiplication automorphisms are all we get. The map $\alpha_y$ is the dual of the Kodaira-Spencer map of $\mathcal{F}$ at $y \in Y$, which takes a tangent vector at $y$ to the infinitesimal deformation in that direction in the family $\mathcal{F}$. Hence for any $y \in Y$, such that $\mathcal{F}_y$ is simple and such that the Kodaira-Spencer map of $\mathcal{F}$ is injective at $y$, the long exact sequence above shows that $K|_{y,y} = 0$.

Having proved that $\text{Supp} Q \subseteq \Delta$ we have proved by (3.8) that any two $G$-constellations in $\mathcal{F}$ are orthogonal. Denoting by $q$ the quotient map $\mathbb{C}^n \to \mathbb{C}^n / G$ we claim that for any closed point $x \in \mathbb{C}^n / G$, such that $q^{-1}(x)$ is a free orbit of $G$, the fiber $\pi_{x}^{-1}(x)$ consists of at most a single point. This is because, by definition of $\pi_x$, all the $G$-constellations parametrised by $\pi_{x}^{-1}(x)$ are supported on $q^{-1}(x)$ - and any two $G$-constellations supported at the same free orbit are easily seen to be isomorphic.

Thus $\pi_x$ is an isomorphism on the smooth locus $X_0$ of $\mathbb{C}^n / G$. By [Log06], Proposition 1.5 the family $\mathcal{F}$ on $X_0$ (identified with an open subset of $Y$ via $\pi_x$) is locally isomorphic to the canonical $G$-cluster family $q_* \mathcal{O}_{\mathbb{C}^n}|_{X_0}$. As any $G$-cluster is simple and as the Kodaira-Spencer map of $q_* \mathcal{O}_{\mathbb{C}^n}|_{X_0}$ is trivially injective $K|_{y,y} = 0$ for any $y \in X_0$. Therefore $\text{codim}_{Y \times Y} \text{Supp} K \geq n + 1$, as $X_0$ is open in $\Delta$.

On the other hand, since $\text{Tor-amp} Q = \text{Tor-amp} \mathcal{O}_\Delta = n$, the short exact sequence (3.10) implies that $\text{Tor-amp} K \leq n$. As that is smaller than the codimension of its support, $K = 0$ by Theorem 2. Thus $Q \simeq \mathcal{O}_\Delta$, the adjunction co-unit is an isomorphism and $\Phi_{\mathcal{F}} (\_ \otimes \rho_0)$ is fully faithful.

Step 5: We claim that $\Phi_{\mathcal{F}} (\_ \otimes \rho_0)$ is an equivalence of categories.

As $D(Y)$ is fully faithfully embedded in $D^G (\mathbb{C}^n)$ the trivial Serre functor of the latter induces a trivial Serre functor on the former. Therefore the left adjoint to $\Phi_{\mathcal{F}} (\_ \otimes \rho_0)$ is also its right adjoint. Then $\Phi_{\mathcal{F}} (\_ \otimes \rho_0)$ is an equivalence of categories by [Bri99], Theorem 3.3.

\[ \square \]

Proof of Corollary 1. It suffices to demonstrate that $\mathcal{F}$ satisfies the condition of Theorem 1. Thus we have to show that $\text{codim} N_0 \geq 4$ and $\text{codim} N_1 \geq 2$. But, as seen in the proof of Theorem 1, $N_k$ lies within the fibre product $Y \times_{\mathbb{C}^n / G} Y$ for all $k$. As $\pi_x$ is birational its fibres are at most divisors and so the codimension of $Y \times_{\mathbb{C}^n / G} Y$ is at least 2.

It remains to show that $N_0 \geq 4$. The assumptions of the Corollary ensure that $N_0$ is contained in the union of all sets of form $(E_i \cap E_j) \times (E_k \cap E_l) \text{ or } E_i \times (E_i \cap E_j \cap E_k)$, and the codimension of each of these sets is 4. \[ \square \]

4 Orthogonality in degree zero

Throughout this section we denote by $G$ a finite abelian subgroup of $\text{SL}_n (\mathbb{C})$, by $Y$ a smooth scheme of finite type over $\mathbb{C}$ and by $\mathcal{F}$ a gnat-family on $Y$. We assume that the Hilbert-Chow morphism
\[ \pi_F \] associated to \( F \) is birational and proper. The main purpose of this section is to show how, given any pair of closed points of \( Y \), one checks whether the corresponding pair of \( G \)-constellations are orthogonal in degree 0.

We denote by \( V_{\text{gv}} \) the representation of \( G \) given by its inclusion into \( \text{SL}_n(\mathbb{C}) \). The (left) action of \( G \) on \( V_{\text{gv}} \) induces a right action of \( G \) on \( V_{\text{gv}}^\vee \) which we make into a left action by setting:

\[ g \cdot f(v) = f(g^{-1} \cdot v) \quad \text{for all } v \in V_{\text{gv}}, \ f \in V_{\text{gv}}^\vee, \ g \in G. \quad (4.1) \]

We denote by \( x_1, \ldots, x_n \) the common eigenvectors of the action of \( G \) on \( V_{\text{gv}}^\vee \). We denote by \( R \) the symmetric algebra \( S(V_{\text{gv}}^\vee) \) with the induced left action of \( G \). Then \( R = \mathbb{C}[x_1, \ldots, x_n] \) and as an affine \( G \)-scheme \( \mathbb{C}^n \) is Spec \( R \). We denote by \( G^\vee \) the character group \( \text{Hom}(G, \mathbb{C}^*) \) of \( G \). A rational function \( f \in K(\mathbb{C}^n) \) is said to be \( G \)-homogeneous of weight \( \chi \in G^\vee \) if we have \( f(g \cdot v) = \chi(g) \cdot f(v) \) for all \( v \in \mathbb{C}^n \) where \( f \) is defined. We denote by \( \rho(f) \) the weight \( \chi \) of such \( f \). It follows from (4.1) that \( G \) acts on \( f \) by \( \rho(f)^{-1} \).

From here on we employ freely the terminology and the results of [Log06].

### 4.1 The McKay quiver of \( G \)

By a **quiver** we mean a vertex set \( Q_0 \), an arrow set \( Q_1 \) and a pair of maps \( h : Q_1 \rightarrow Q_0 \) and \( t : Q_1 \rightarrow Q_0 \) giving the head \( hq \in Q_0 \) and the tail \( tq \in Q_0 \) of each arrow \( q \in Q_1 \). By a **representation of a quiver** we mean a graded vector space \( \bigoplus_{i \in Q_0} V_i \) and a collection of linear maps \( \{\alpha_q : V_{\text{tg}} \rightarrow V_{\text{hq}}\}_{q \in Q_1} \).

**Definition 6.** The **McKay quiver of \( G \)** is the quiver whose vertex set \( Q_0 \) are the irreducible representations \( \rho \) of \( G \) and whose arrow set \( Q_1 \) has dim \( \text{Hom}_G(\rho_i, \rho_j \otimes V_{\text{gv}}) \) arrows going from the vertex \( \rho_i \) to the vertex \( \rho_j \).

We have \( V_{\text{gv}}^\vee = \bigoplus \mathbb{C} x_i \) as \( G \)-representations. Denote by \( U_\chi \) the 1-dimensional representation on which \( G \) acts by \( \chi \in G^\vee \). By Schur’s lemma

\[ G \cdot \text{Hom}(U_\chi \otimes V_{\text{gv}}^\vee, U_\chi) = \begin{cases} \mathbb{C} & \text{if } \chi_j = \chi_i \rho(x_k)^{-1} \quad k \in \{1, \ldots, n\} \\ 0 & \text{otherwise} \end{cases}. \]

Thus each vertex \( \chi \) of the McKay quiver of \( G \) has \( n \) arrows emerging from it and going to vertices \( \chi \rho(x_k)^{-1} \) for \( k = 1, \ldots, n \). We denote the arrow from \( \chi \) to \( \chi \rho(x_k)^{-1} \) by \( (\chi, x_k) \). Let now \( A \) be a \( G \)-constellation viewed as an \( R \times G \)-module ([Log06], Section 1.1) and let \( \bigoplus A_\chi \) be its decomposition into irreducible representations of \( G \). Then the \( R \times G \)-module structure on \( A \) defines a representation of the McKay quiver into the graded vector space \( \bigoplus A_\chi \), where the map \( \alpha_{\chi, x_k} \) is just the multiplication by \( x_k \), i.e.

\[ \alpha_{\chi, x_k} : A_\chi \rightarrow A_{\chi \rho(x_k)^{-1}}, \ v \mapsto x_k \cdot v. \quad (4.2) \]

### 4.2 Degree 0 orthogonality of \( G \)-constellations

Let \( A \) and \( A' \) be two \( G \)-constellations and \( \phi \) be an \( R \times G \)-module morphism \( A \rightarrow A' \). Let \( \bigoplus A_\chi \) and \( \bigoplus A'_\chi \) be decompositions of \( A \) and \( A' \) into one-dimensional representations of \( G \). By \( G \)-equivariance \( \phi \) decomposes into linear maps \( \phi_\chi : A_\chi \rightarrow A'_\chi \).
Let \( \{ \alpha_q \} \) and \( \{ \alpha'_q \} \) be the corresponding representations of the McKay quiver into graded vector spaces \( \bigoplus A^\chi \) and \( \bigoplus A'^\chi \), as per (4.2). Each \( \alpha_q \) is a linear map between one-dimensional vector spaces \( A_{tq} \) and \( A_{hq} \) and so is either a zero-map or an isomorphism, similarly for the maps \( \alpha'_q \). So for each arrow of the McKay quiver we distinguish the following four possibilities:

**Definition 7.** Let \( q \) be an arrow of McKay quiver of \( G \). With the notation above we say that with respect to an ordered pair \( (A, A') \) of \( G \)-constellations the arrow \( q \) is:

1. a type \([1, 1]\) arrow, if both \( \alpha_q \) and \( \alpha'_q \) are isomorphisms.
2. a type \([1, 0]\) arrow, if \( \alpha_q \) is an isomorphism and \( \alpha'_q \) is a zero map.
3. a type \([0, 1]\) arrow, if \( \alpha_q \) is a zero map and \( \alpha'_q \) is an isomorphism.
4. a type \([0, 0]\) arrow, if both \( \alpha_q \) and \( \alpha'_q \) are zero maps.

**Proposition 4.** Let \( q \) and \( (A, A') \) be as in Definition 7 and let \( \phi \) be any \( R \rtimes G \)-module morphism \( A \to A' \). Then:

1. If \( q \) is a \([1, 0]\) arrow, then \( A_{hq} \subseteq \ker \phi \).
2. If \( q \) is a \([0, 1]\) arrow, then \( A_{tq} \subseteq \ker \phi \).
3. If \( q \) is a \([1, 1]\) arrow, \( A_{tq} \) and \( A_{hq} \) either both lie in \( \ker \phi \) or both don’t.

**Proof.** Write \( q = (\chi, i) \) where \( \chi \in G^\vee \) and \( i \in \{1, \ldots, n\} \). Recall that \( \alpha_q \) is the map \( A_{tq} \to A_{hq} \) corresponding to the action of \( x_i \) on \( A_{tq} \). Then \( R \)-equivariance of the morphism \( \phi \) implies a commutative square

\[
\begin{array}{ccc}
A_{hq} & \xrightarrow{\phi_{hq}} & A'_{hq} \\
\alpha_q \downarrow & & \alpha'_q \downarrow \\
A_{tq} & \xrightarrow{\phi_{tq}} & A'_{tq}
\end{array}
\]

from which all three claims immediately follow. \( \square \)

**Corollary 2.** Let \( (A, A') \) be an ordered pair of \( G \)-constellations. If every component of the McKay quiver path-connected by \([1, 1]\)-arrows has either a \([0, 1]\)-arrow emerging from it or a \([1, 0]\)-arrow entering it, then

\[ \text{Hom}_{R \rtimes G}(A, A') = 0. \]

If, also, every component has either a \([0, 1]\)-arrow entering it or a \([1, 0]\)-arrow emerging from it, then we further have

\[ \text{Hom}_{R \rtimes G}(A', A) = 0 \]

and therefore \( A \) and \( A' \) are orthogonal in degree 0.
### 4.3 Divisors of zeroes

The Hilbert-Chow morphism $\pi_F : Y \to \mathbb{C}^n/G$ is birational, thus it defines a notion of $G$-Cartier and $G$-Weil divisors on $Y$ ([Log06], Section 2). The family $\mathcal{F}$, in a sense of a sheaf of $\mathcal{O}_Y \otimes (R \times G)$-modules on $Y$, can be written as $\bigoplus_{\chi \in G^t} \mathcal{L}(-D_\chi)$, where $D_\chi$ are $G$-Weil divisors. For any other such expression $\bigoplus \mathcal{L}(-D'_\chi)$ of $\mathcal{F}$ there exist $f \in K(Y)$ such that $D'_\chi = D_\chi + (f)$ for all $\chi \in G^t$ ([Log06], Section 3.1).

**Definition 8.** Let $q = (\chi, x_k)$ be an arrow in the McKay quiver of $G$. We define the divisor of zeroes $B_q$ of $q$ in $\mathcal{F}$ to be the Weil divisor

$$D_{\chi^{-1}} + (x_k) - D_{\chi^{-1}\rho(x_k)}.$$  \hspace{0.5cm} (4.3)

Note that $B_q$ is always an ordinary, integral Weil divisor on $Y$.

**Proposition 5.** Let $(\chi, x_k)$ be an arrow in the McKay quiver of $G$ and $B_{\chi,x_k}$ be its divisor of zeroes in $\mathcal{F}$. Let $y$ be a closed point of $Y$ and $A$ be the $G$-constellation $\mathcal{F}_y$. Then in the corresponding representation $\{\alpha_q\}_{q \in Q_0}$ of the McKay quiver the map $\alpha_{\chi,x_k}$ is a zero map if and only if $y \in B_{\chi,x_k}$.

**Proof.** The map $\alpha_{\chi,x_k} : A_\chi \to A_{\chi\rho(x_k)^{-1}}$ is the action of $x_k$ on $A_\chi$. This map is the restriction to the point $y$ of the global section $\beta$ of the $\mathcal{O}_Y$-module

$$\text{Hom}_{G,\mathcal{O}_Y}(\mathcal{O}_Y x_k \otimes \mathcal{F}_y, \mathcal{F}_{\chi\rho^{-1}(x_k)})$$ \hspace{0.5cm} (4.4)

defined by $x_k \otimes s \mapsto x_k \cdot s$ for any section $s$ of the $\chi$-eigensheaf $\mathcal{F}_\chi$.

As $G$ acts on a monomial of weight $\chi$ by $\chi^{-1}$ the $\chi$-eigensheaf of $\mathcal{F}$ is $\mathcal{L}(-D_{\chi^{-1}})$. Hence (4.4) is canonically isomorphic to the following sub-$\mathcal{O}_Y$-module of $K(\mathbb{C}^n)$:

$$\mathcal{L}(D_{\chi^{-1}} + (x_k) - D_{\chi^{-1}\rho(x_k)})$$ \hspace{0.5cm} (4.5)

and the isomorphism maps $\beta$ to the global section $1 \in K(\mathbb{C}^n)$ of (4.5). Which vanishes precisely on the Weil divisor $B_{\chi,x_k} = D_{\chi^{-1}} + (x_k) - D_{\chi^{-1}\rho(x_k)}$. \hfill \Box

Proposition 5 together with Corollary 2 show that the data of the divisors of zeroes of $\mathcal{F}$ is all that is necessary to determine whether any given pair of closed points of $Y$ are orthogonal in degree 0 in $\mathcal{F}$.

### 4.4 Direct transforms

Let $Y'$ and $Y''$ be two crepant resolutions of $\mathbb{C}^n/G$ isomorphic outside of a closed set of codimension $\geq 2$. E.g. the case $n = 3$ where all crepant resolutions are related by a chain of flops ([Kol89]). We fix a birational isomorphism and use it to identify $Y'$ and $Y''$ along the isomorphism locus $U$. Since the complement of $U$ is of codimension $\geq 2$ in $Y'$ (resp. $Y''$) any line bundle or divisor on $U$ extends uniquely to a line bundle or a divisor on $Y'$ (resp. $Y''$). The same is true of a family of $G$-constellations as for $G$ abelian any such family is a direct sum of line bundles. For any family $\mathcal{V}'$ of $G$-constellations on $Y'$ we define its direct transform $\mathcal{V}''$ to $Y''$ to be the unique extension to $Y''$ of the restriction of $\mathcal{V}'$ to $U$. Observe that if $\mathcal{V}'$ is of form $\bigoplus \mathcal{L}(-D'_\chi)$ for some $G$-Weil divisors $D'_\chi$ on $Y'$ then $\mathcal{V}''$ is the family $\bigoplus \mathcal{L}(-D''_{\chi})$ where each $D''_{\chi}$ is the direct transform of $D'_\chi$.  

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If $\mathcal{F}$ can be shown to be a direct transform of some everywhere orthogonal in degree 0 family $\mathcal{F}'$ on some $Y'$, it greatly reduces the number of calculations necessary to determine the degree 0 non-orthogonality locus of $\mathcal{F}$. Let $U$ be as above. As $\mathcal{F}$ is the direct transform of $\mathcal{F}'$, the restriction of $\mathcal{F}$ to $U \subset Y$ is isomorphic to the restriction of $\mathcal{F}'$ to $U \subset Y'$. So the calculations only have to be carried out for points in $Y \times Y \setminus U \times U$.

### 4.5 Theta stability and gnat-families

We recall basic facts about $\theta$-stability for $G$-constellations, cf. [CI04], Section 2.1. Let $\mathbb{Z}(G) = \bigoplus_{\chi \in G^\vee} \mathbb{Z}\chi$ be the representation ring of $G$ and set

$$\Theta = \{\theta \in \text{Hom}_G(\mathbb{Z}(G), \mathbb{Q}) \mid \theta(\text{reg}) = 0\}$$

For any $\theta \in \Theta$, a $G$-constellation $A$ is $\theta$-stable (resp. $\theta$-semistable) if for every sub-$R \rtimes G$-module $B$ of $A$ we have $\theta(B) > 0$ (resp. $\theta(B) \geq 0$). We say that $\theta$ is generic if every $\theta$-semistable $G$-constellation is $\theta$-stable. This is equivalent to $\theta$ being non-zero on any proper subrepresentation of $\text{reg}$.

Let $\pi$ be any proper birational morphism $Y \to \mathbb{C}^n/G$. A gnat-family $\mathcal{V}$ on $Y \xrightarrow{\pi} \mathbb{C}^n/G$ is normalized if $\mathcal{V}^G \simeq \mathcal{O}_Y$. Such $\mathcal{V}$ can be written uniquely as $\bigoplus_{\chi \in G^\vee} \mathcal{L}(-D_\chi)$ for some $G$-Weil divisors $D_\chi$ with $D_{\chi_0} = 0$ ([Log06], Cor. 3.5). Denote by $\mathfrak{E}$ the set of all prime Weil divisors on $Y$ whose image in $\mathbb{C}^n/G$ is either a point or a coordinate hyperplane $x_i^{[G]} = 0$. As $G$ is abelian, any branch divisor of $\mathbb{C}^n \to \mathbb{C}^n/G$, if it exists, is one of the hyperplanes $x_i^{[G]} = 0$. Hence, by [Log06], Prop. 3.14 and 3.15, each $D_\chi$ is of form $\sum_{E \in \mathfrak{E}} q_{\chi,E}$. Denote by $U$ the open subset of $Y$ consisting of points lying on at most one divisor in $\mathfrak{E}$.

**Definition 9.** Let $\theta$ be an element of $\Theta$. We define a map

$$w_\theta : \left\{\text{normalized gnat-families on } Y \xrightarrow{\pi} \mathbb{C}^n/G\right\} \to \mathbb{Q}$$

by

$$w_\theta(\mathcal{V}) = \sum_{E \in \mathfrak{E}} \sum_{\chi \in G^\vee} \theta(\chi) q_{\chi,E}. \quad (4.6)$$

The domain of definition of $w_\theta$ is finite ([Log06], Corollary 3.16), so for any $\theta \in \Theta$ there is at least one normalized gnat-family maximizing $w_\theta$.

**Proposition 6.** Let $\mathcal{M}$ be any family which maximizes $w_\theta(\mathcal{M})$. Then for any point $y \in U$ the fiber of $\mathcal{M}$ at $y$ is a $\theta$-semistable $G$-constellation. If, moreover, $\theta$ is generic, then such family $\mathcal{M}$ is unique.

**Proof.** Write $\mathcal{M}$ as $\bigoplus \mathcal{L}(-M_\chi)$. Suppose that the fiber of $\mathcal{M}$ is not $\theta$-semistable at some $y \in U$. Denote this fiber by $A$, its decomposition into irreducible representations by $\bigoplus_{\chi \in G^\vee} A_\chi$ and the corresponding representation of the McKay quiver by $\{\alpha_q\}$. As $A$ isn’t $\theta$-semistable there exists a non-empty proper subset $I$ of $G^\vee$ such that $A' = \bigoplus_{\chi \in I} A_\chi$ is a sub-$R \rtimes G$-module of $A$ and $\theta(A') < 0$. Denote by $J$ the complement $G^\vee \setminus I$. Denote by $Q_{I \to J}$ the subset $\{q \in Q_1 \mid tq \in I,hq \in J\}$ of the arrow set $Q_1$ of the McKay quiver and similarly for $Q_{J \to I}, Q_{J \to J}$. Then $A'$ being closed
under the action of $R$ implies that for any $q \in Q_{I\rightarrow J}$ the map $\alpha_q$ is a zero map. Which by Proposition 5 implies $y \in B_q$.

The support of each $M_\chi$ consists only of the prime divisors in $\mathfrak{E}$ ([Log06], Prop. 3.14 and 3.15). The same is true of the principal divisors $(x_i)$ for their images in $\mathbb{C}^n/G$ are the coordinate hyperplanes $x_i^{[G]} = 0$. Therefore, by their defining equation (4.3), the support of each of the divisors of zeroes $B_q$ of $\mathcal{M}$ consists also only of the prime divisors in $\mathfrak{E}$. As $y$ lies on all $B_q$ with $q \in Q_{I\rightarrow J}$, $y$ must lie on at least one divisor in $\mathfrak{E}$. But, as $y \in U$, $y$ also lies on at most one divisor in $\mathfrak{E}$. Denote this unique divisor by $E$, then

$$q \in Q_{I\rightarrow J} \Rightarrow E \subset B_q.$$ (4.7)

Define a new $G$-Weil divisor set $\{M'_\chi\}$ by setting $M'_\chi$ to be $M_\chi$ if $\chi \in I$ and $M_\chi + E$ if $\chi \in J$. Then divisors $\{B'_q\}$ defined from $\{M'_\chi\}$ by equations (4.3) can be expressed as

$$B'_q = \begin{cases} B_q & \text{if } q \in Q_{I\rightarrow I}, Q_{J\rightarrow J} \\ B_q + E & \text{if } q \in Q_{J\rightarrow I} \\ B_q - E & \text{if } q \in Q_{I\rightarrow J} \end{cases}. \quad (4.8)$$

Since $\{B_q\}$ are all effective (4.8) and (4.7) imply that $\{B'_q\}$ are also all effective. Therefore $\bigoplus \mathcal{L}(-M'_\chi)$ is a normalized gnat-family. But

$$w_\theta(\mathcal{M}') = w_\theta(\mathcal{M}) + \sum_{\chi \in J} \theta(\chi) \quad (4.9)$$

which contradicts the maximality of $w_\theta(\mathcal{M})$ since $\sum_{\chi \in J} \theta(\chi) = -\theta(A') > 0$.

For the second claim let $\mathcal{N} = \bigoplus \mathcal{L}(-N_\chi)$ be another normalized family $\theta$-semistable over $U$. Let $B'_q$ be divisors of zeroes of $\mathcal{N}$. Then

$$B_q - B'_q = (M_q - N_q) - (M_q - N_q). \quad (4.10)$$

Take any $E' \in \mathfrak{E}$ such that the sets $\{m_{\chi,E'}\}$ and $\{n_{\chi,E'}\}$ of the coefficients of $E'$ in $\{M_\chi\}$ and $\{N_\chi\}$ are distinct. Then $J' = \{\chi \in G' \mid n_{\chi,E'} > m_{\chi,E'}\}$ is a non-empty proper subset of $G'$. Denote by $J'$ its complement. For any $q \in Q_{J'\rightarrow J'}$ the coefficient of $E'$ in the RHS of (4.10) is strictly positive. As $B'_q$ is effective we conclude that $q \in Q_{J'\rightarrow J'}$ implies $E' \subset B_q$. So for any $y \in E'$ the restriction $(\bigoplus_{\chi \in J'} \mathcal{L}(M_\chi)|_y$ is a sub-$R \times G$-module of $\mathcal{M}_y$. But as $\mathcal{M}$ is $\theta$-semistable on $U$ and as $U \cap E' \neq \emptyset$ we must have $\sum_{\chi \in J'} \theta(\chi) \geq 0$. Similarly if $q \in Q_{J'\rightarrow J'}$, then the RHS of (4.10) is strictly negative, so $E' \subset B'_q$ and $\theta$-semistability of $\mathcal{N}$ implies $\sum_{\chi \in J'} \theta(\chi) = -\sum_{\chi \in J} \theta(\chi) \geq 0$. Therefore $\sum_{\chi \in J'} \theta(\chi) = 0$ and $\theta$ is not generic. \[\Box\]

The fine moduli space $M_{\theta}$ of $\theta$-stable $G$-constellations can be constructed via GIT theory, together with the universal family $\mathcal{M}_\theta$. The Hilbert-Chow morphism $\pi_\theta$ of $\mathcal{M}_\theta$ is projective. As the universal family is defined up to an equivalence of families, that is up to a twist by a line bundle, we can assume $\mathcal{M}_\theta$ to be normalised.

Assume for the rest of this section that $n = 3$. If $\theta$ is generic, then $M_{\theta}$ is a projective crepant resolution of $\mathbb{C}^3/G$ and $\mathcal{M}_\theta$ is everywhere orthogonal in all degrees. As any two crepant resolutions
of a canonical treefold are connected by a chain of flops, $M_\theta$ and $Y$ are isomorphic outside of a codimension 2 subset. The maps $Y \xrightarrow{\pi} \mathbb{C}^3/G$ and $M_\theta \xrightarrow{\pi_\theta} \mathbb{C}^3/G$ fix a choice of a birational isomorphism between $Y$ and $M_\theta$. This, as described in Section 4.4, defines a notion of direct transforms between $Y$ and $M_\theta$.

**Corollary 3.** Let $\theta \in \Theta$ be generic. Let $\mathcal{M}$ be the unique normalized gnat-family on $Y$ which maximizes the map $w_\theta$. Then $\mathcal{M}$ is isomorphic to the direct transform of $M_\theta$ from $M_\theta$ to $Y$.

**Proof.** By the first claim of Proposition 6, $\mathcal{M}$ is $\theta$-stable on $U$. So, by its definition, is the direct transform of $M_\theta$ to $Y$. Hence, by the second claim of Proposition 6, $\mathcal{M}$ and the direct transform of $M_\theta$ must be isomorphic. \qed

## 5 Non-projective example

In this section we give an application of the Theorem 1 whereby we construct explicitly a derived McKay correspondence for a choice of an abelian $G \subset \text{SL}_3(\mathbb{C})$ and of a non-projective crepant resolution $Y$ of $\mathbb{C}^3/G$.

### 5.1 The group

We set the group $G$ to be $\frac{1}{6}(1,1,4) \oplus \frac{1}{2}(1,0,1)$. That is, the image in $\text{SL}_3(\mathbb{C})$ of the product $\mu_6 \times \mu_2$ of groups of 6th and 2nd roots of unity, respectively, under the embedding:

\[
(\xi_1, \xi_2) \mapsto \begin{pmatrix}
\xi_1 \\
\xi_2 \\
\xi_1^3 \xi_2
\end{pmatrix}.
\] (5.1)

We denote by $\chi_{i,j}$ the character of $G$ induced by $(\xi_1, \xi_2) \mapsto \xi_1^i \xi_2^j$. Calculating the McKay quiver of $G$ (cf. Section 4.1), we obtain:

![McKay Quiver](image.png)
The way we’ve chosen to depict the McKay quiver reflects the fact that it has a universal cover quiver naturally embedded into \( \mathbb{R}^2 \). This point of view will not be essential for our argument but a curious reader should consult [CI04], Section 10.2 and [Log04], Section 6.4.

### 5.2 The resolution

We define the crepant resolution \( Y \) of \( \mathbb{C}^3/G \) using methods of toric geometry. For the specifics related to \( G \)-constellations see [Log03], Section 3.

We define the relevant notation. The embedding (5.1) defines a surjection of torii

\[
0 \longrightarrow G \longrightarrow (\mathbb{C}^*)^3 \longrightarrow T \longrightarrow 0.
\]

(5.2)

Applying \( \text{Hom}(\bullet, \mathbb{C}^*) \) to (5.2) we obtain the character lattices of the torii:

\[
0 \longrightarrow M \longrightarrow \mathbb{Z}^3 \overset{\rho}{\longrightarrow} G^\vee \longrightarrow 0.
\]

(5.3)

Given any character \( m = (k_1,k_2,k_3) \in \mathbb{Z}^3 \) of \( (\mathbb{C}^*)^3 \) we denote by \( x^m \) the Laurent monomial \( x_1^{k_1}x_2^{k_2}x_3^{k_3} \) in \( R \). Applying \( \text{Hom}(\bullet, \mathbb{Z}) \) to (5.3) we obtain the dual lattices

\[
0 \longrightarrow (\mathbb{Z}^3)^\vee \longrightarrow N \longrightarrow \text{Ext}^1(G^\vee, \mathbb{Z}) \longrightarrow 0.
\]

Let \( e_1, e_2, e_3 \) be the basis of \( (\mathbb{Z}^3)^\vee \) dual to \( x_1, x_2, x_3 \). The dual lattice \( N \) is generated over \( (\mathbb{Z}^3)^\vee \) by \( \frac{1}{6}(1,1,4) \) and \( \frac{1}{2}(1,0,1) \). The quotient space \( \mathbb{C}^3/G \) is the toric variety given by a single cone \( \sigma_{\geq 0} = \sum \mathbb{R}_{\geq 0} e_i \) in \( N \). Let \( Y \) be the toric variety whose fan \( \mathcal{F} \) in \( N \) is the subdivision of \( \sigma_{\geq 0} \) which triangulates the junior simplex \( \Delta = \{ (k_1,k_2,k_3) \in \sigma_{\geq 0} \mid \sum k_i = 1 \} \) as depicted below

![Toric Variety Diagram](image)

where by \( e_i \) we denote the following elements of \( N \)

\[
\begin{align*}
    e_1 &= (1,0,0) & e_2 &= (0,1,0) & e_3 &= (0,0,1) \\
    e_4 &= \frac{1}{6}(1,1,4) & e_5 &= \frac{1}{3}(1,1,1) & e_6 &= \frac{1}{2}(1,1,0) \\
    e_7 &= \frac{1}{6}(1,4,1) & e_8 &= \frac{1}{2}(1,0,1) & e_9 &= \frac{1}{6}(4,1,1) \\
    e_{10} &= \frac{1}{2}(0,1,1).
\end{align*}
\]
Denote by \( \pi \) the map \( Y \to \mathbb{C}^3/G \) defined by the inclusion of \( \mathfrak{g} \) into \( \sigma \geq 0 \). All the maximal cones of \( \mathfrak{g} \) are basic in \( N \), so \( Y \) is smooth. The generators \( e_i \) of the rays of \( \mathfrak{g} \) lie in \( \Delta \), so the map \( \pi \) is crepant([Rei87], Prop. 4.8). Finally, the argument of [KKMSD73], Chapter III, §2E, Example 2 shows that \( \pi \) is non-projective.

The quotient torus \( T \) acts on \( Y \) and to each \( k \)-dimensional cone \( \sigma \) in \( \mathfrak{g} \) corresponds a \((3-k)\)-dimensional orbit of \( T \). We denote it by \( S_\sigma \) and denote by \( E_\sigma \) the closure of \( S_\sigma \), it is the union of all orbits \( S_{\sigma'} \) with \( \sigma \subseteq \sigma' \). For each cone \( \langle e_i \rangle \) in the fan \( \mathfrak{g} \), we denote by \( S_i \) the codimension 1 orbit \( S_{\langle e_i \rangle} \) and by \( E_i \) the divisor \( E_\langle e_i \rangle \). Similarly we use \( S_{i,j} \) and \( E_{i,j} \) for the codimension 2 orbit \( S_{\langle e_i, e_j \rangle} \) and the surface \( E_{\langle e_i, e_j \rangle} \) and we use \( E_{i,j,k} \) for the toric fixed point \( E_{\langle e_i, e_j, e_k \rangle} \).

### 5.3 The family

The map \( Y \to \mathbb{C}^3/G \) defines the notion of \( G \)-Weil divisors on \( Y \). Any normalized gnat-family on \( Y \to \mathbb{C}^3/G \) is of the form \( \bigoplus_{X \in G^\vee} L(-D_X) \) for some \( G \)-Weil divisors \( D_X \) with \( D_{X,0,0} = 0 \). Moreover, as explained in [Log06], Section 3.5, there exists the maximal shift family \( \bigoplus L(-M_\chi) \) such that for any other normalized gnat-family \( \bigoplus L(-D_X) \) we have

\[
M_\chi \geq D_X
\]

for all \( \chi \in G^\vee \). We denote this family by \( \mathcal{F} \) and shall prove it to satisfy the assumptions of Corollary 1.

In the notation of Section 5.2 each divisor \( M_\chi \) is of form \( \sum q_{\chi,i} E_i \). The coefficients \( q_{\chi,i} \) can be calculated via formula

\[
q_{\chi,i} = \inf \{ e_i(m) \mid m \in \sigma_\chi^{-1} \cap \rho^{-1}(\chi) \}.
\]

A detailed example of such calculation can be seen in [Log03], Example 4.21. In our case, we obtain \( q_{\chi,i} \) to be:

| \( \chi \backslash i \) | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----------------|---|---|---|---|---|---|---|
| \( \chi_{0,0} \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \chi_{4,0} \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \chi_{1,0} \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \chi_{3,1} \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \chi_{0,1} \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \( \chi_{5,0} \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

The principal \( G \)-Weil divisors \( (x_k) \) can be calculated with a formula

\[
(x_i) = \frac{1}{12} \sum_{j=1}^{10} e_j (x_i^2) E_j,
\]

(5.8)
Substituting the data of (5.9) and (5.7) into the formula (4.3) we calculate for every arrow of the McKay
quiver its divisor of zeroes in $\mathcal{F}$:

$$
(x_1) = E_1 + \frac{1}{6} E_4 + \frac{1}{3} E_5 + \frac{1}{2} E_6 + \frac{1}{6} E_7 + \frac{1}{2} E_8 + \frac{4}{6} E_9
$$

$$
(x_2) = E_2 + \frac{1}{6} E_4 + \frac{1}{3} E_5 + \frac{1}{2} E_6 + \frac{4}{6} E_7 + \frac{1}{2} E_9 + \frac{1}{2} E_{10}
$$

$$
(x_3) = E_3 + \frac{4}{6} E_4 + \frac{1}{3} E_5 + \frac{1}{2} E_7 + \frac{1}{2} E_8 + \frac{1}{6} E_9 + \frac{1}{2} E_{10}
$$

Substituting the data of (5.9) and (5.7) into the formula (4.3) we calculate for every arrow of the McKay
quiver its divisor of zeroes in $\mathcal{F}$:

$B_{x_{0,0},1} = E_1$
$B_{x_{0,0},2} = E_2$
$B_{x_{0,0},3} = E_3$
$B_{x_{4,0},1} = E_1$
$B_{x_{4,0},2} = E_2$
$B_{x_{4,0},3} = E_3$
$B_{x_{x_{1},0},1} = E_1 + E_5 + E_9$
$B_{x_{x_{1},0},2} = E_2 + E_5 + E_7$
$B_{x_{x_{1},0},3} = E_3 + E_5 + E_9$
$B_{x_{x_{1},1},1} = E_1 + E_6 + E_8 + E_9$
$B_{x_{x_{1},1},2} = E_2 + E_6 + E_7 + E_9$
$B_{x_{x_{1},1},3} = E_3 + E_6 + E_7 + E_9$
$B_{x_{x_{1},2},1} = E_1 + E_6$
$B_{x_{x_{1},2},2} = E_2 + E_6 + E_7 + E_10$
$B_{x_{x_{1},2},3} = E_3 + E_6 + E_7 + E_10$
$B_{x_{x_{1},3},1} = E_1 + E_8$
$B_{x_{x_{1},3},2} = E_2 + E_8 + E_7 + E_10$
$B_{x_{x_{1},3},3} = E_3 + E_8 + E_7 + E_{10}$

$B_{x_{0,1},1} = E_1 + E_4 + E_5 + E_6 + E_7 + E_8 + E_9$
$B_{x_{0,1},2} = E_2 + E_4 + E_5 + E_6 + E_7 + E_9 + E_{10}$

5.4 A sample calculation

Corollary 2 together with the table (5.10) are all that we need to check any two $G$-constellations in $\mathcal{F}$
for the degree 0 orthogonality. Below we give an example of a calculation which verifies that any
point on the torus orbit $S_8$ and any point on the torus orbit $S_{1,7}$ are orthogonal in degree 0 in $\mathcal{F}$.

Let $a$ be any point of $S_8$. Then $a$ lies on no divisor $E_i$ other than $E_8$. Hence $a \in B_q$ if and
only if $E_8 \subset B_q$. Let $A$ be the fiber of $\mathcal{F}$ at $a$ and $\{\alpha_q\}$ be the corresponding representation of the
McKay quiver. By Proposition 5 for any arrow $q$ the map $\alpha_q$ is a zero map if and only if $E_8 \in B_q$.
On Figure 4 we use the table (5.10) and mark all the zero-maps in $\{\alpha_q\}$ by drawing a line through the
horizontal arrow of the McKay quiver. Similarly if $b$ is a point of $S_{1,7}$ then $b$ lies on no $E_i$ other
than $E_1$ and $E_7$. Let $B$ be the fiber of $\mathcal{F}$ at $b$ and $\{\beta_q\}$ be the corresponding representation. As above
$\beta_q$ is a zero-map if and only if either $E_1$ or $E_7$ belongs to $B_q$. On Figure 5 we mark all the zero-maps
$\{\beta_q\}$.
On Figure 6 we combine the markings of Figures 4 and 5. The arrows left unmarked are the arrows of type \([1,1]\) with respect to the pair \(A,B\) (Def. 7). It is clear that the components path-connected by \([1,1]\)-arrows are: \(\{\chi_{0,0}, \chi_{2,1}, \chi_{5,0}, \chi_{1,1}\}\), \(\{\chi_{5,1}, \chi_{4,1}, \chi_{2,0}\}\), \(\{\chi_{1,0}, \chi_{4,0}, \chi_{3,0}\}\). Now, with Cor. 2 in mind, we search the borders of these four regions for the \([1,0]\) and \([0,1]\)-arrows. The \([1,0]\)-arrows are the ones unmarked on Figure 4 but marked on Figure 5 and vice versa for \([0,1]\)-arrows. On Figure 7 we’ve marked on the border of each region an incoming and an outgoing \([0,1]\)-arrow. By Cor. 2 we see that \(A\) and \(B\) are orthogonal in degree 0.

### 5.5 Final calculations

We now claim that \(\mathcal{F}\) is the direct transform of the universal family of \(G\)-clusters on \(G\)-Hilb\((\mathbb{C}^3)\). In the notation of Section 4.5 define \(\theta_+ \in \Theta\) by \(\theta_+(\chi_{0,0}) = 1 - |G|\) and \(\theta_+(\chi) = 1\) for \(\chi \neq \chi_{0,0}\). Evidently \(\theta_+\) is generic. It follows from the original observation by Ito and Nakajima in [IN00], §3, that \(G\)-clusters can be identified with \(\theta_+\)-stable \(G\)-constellations, thus identifying \(G\)-Hilb\((\mathbb{C}^3)\) with
the fine moduli space $\mathcal{M}_{\theta_+}$. On the other hand, inequalities (5.5) imply that $\mathcal{F}$ maximizes $\omega_{\theta_+}$ on $Y \xrightarrow{\pi} \mathbb{C}^3/G$. Hence, by Corollary 3, $\mathcal{F}$ is the direct transform of $\mathcal{M}_{\theta_+}$ from $G\text{-Hilb}(\mathbb{C}^3)$ to $Y$.

For a detailed description of an algorithm which allows one to calculate the toric fan of $G\text{-Hilb}(\mathbb{C}^3)$ see in [CR02]. For our group $G$ we obtain:

The general points of an exceptional surface $E_i$, as per the statement of Corollary 1, are precisely the codimension 1 torus orbit $S_i$. Similarly, the general points of an exceptional curve $E_i \cap E_j$ are precisely the codimension 2 torus orbit $S_{i,j}$. Comparing Figure 8 with the fan of $Y$ on Figure 3 we see that the only codimension 1 or 2 torus orbits in $Y$ whose corresponding cones aren’t also contained in the fan of $G\text{-Hilb}(\mathbb{C}^3)$ are $S_{1,7}$, $S_{2,4}$ and $S_{3,9}$. The argument in Section 4.4 reduces verifying that $\mathcal{F}$ satisfies the conditions of Corollary 1, to checking that each of these three orbits is orthogonal in degree 0 in $\mathcal{F}$ to every codimension 1 orbit $S_i$.

We claim that, in fact, it suffices to check it for just one of these orbits. Let $\phi$ be the rotation of the fan of $Y$ around the ray $e_5$ which rotates Figure 2 clockwise by $2\pi/3$. Let $\psi$ be the rotation of the plane containing the McKay quiver on the Figure 3 anti-clockwise by $2\pi/3$ with center at $\chi_{0,0}$. Observe that the permutation of the divisors $E_i$ defined by $\phi$ and the permutation of the arrows of the McKay quiver defined by $\psi$ leave the numerical data (5.10) of divisors of zeroes of $\mathcal{F}$ invariant\(^1\). It follows that the orthogonality calculation of Section 5.4 for any pair of torus orbits $S, S'$ and the same calculation for $\phi(S), \phi(S')$ differ on Figures 4-7 only by a rotation by $\psi$. The claim now follows as the cones of $S_{1,7}$, $S_{2,4}$ and $S_{3,9}$ are permuted by $\phi$.

We choose to treat $S_{1,7}$. We repeat the calculation of Section 5.4 for $S_{1,7}$ and every other orbit $S_i$ and list below the analogues of Figure 7. From them, as elaborated in Section 5.4, the reader could readily ascertain the orthogonality in $\mathcal{F}$ of the torus orbits involved.

We conclude, by Corollary 1, that the integral transform $\Phi_{\mathcal{F}}(-\otimes \rho_0)$ is an equivalence of categories $D(Y) \to D^G(\mathbb{C}^3)$ and that a posteriori the family $\mathcal{F}$ is everywhere orthogonal in all degrees.

\(^1\)This invariance is a consequence of the fan of $Y$ being symmetric and of $\mathcal{F}$ being intrinsically defined as the maximal shift family.
\[(S_1, S_{1,7}) \text{ and } (S_7, S_{1,7})\]
\[(S_2, S_{1,7})\]
\[(S_3, S_{1,7})\]
\[(S_4, S_{1,7})\]
\[(S_5, S_{1,7})\]
\[(S_6, S_{1,7})\]
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