Breather wave and lump-type solutions of new (3 + 1)-dimensional Boiti–Leon–Manna–Pempinelli equation in incompressible fluid

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Under investigation is a new (3+1)-dimensional Boiti–Leon–Manna–Pempinelli equation. The main results are listed as follows: (i) lump solutions; (ii) interaction solutions between lump wave and solitary waves; (iii) interaction solutions between lump wave and periodic waves; and (iv) breather wave solutions. Furthermore, graphical representation of all solutions is studied and shown in some 3D and contour plots.

KEYWORDS
breather wave solutions, lump solutions, periodic waves, solitary waves

MSC CLASSIFICATION
35C08; 68M07; 33F10

1 INTRODUCTION

The well-known (3 + 1)-dimensional Boiti–Leon–Manna–Pempinelli (BLMP) equation can be read as

\[(u_x + u_y)_t + (u_y + u_z)_{xxx} - 3(u_y + u_z)_x = 0, \tag{1}\]

where \(u = u(x, y, z, t)\). The (3 + 1)-dimensional BLMP equation can be used to describe the wave propagation in incompressible fluid and the interaction of the Riemann wave. Many results about BLMP equation have been obtained. Darvishi et al\(^2\) presented the stair and step soliton solutions. Zuo et al\(^3\) derived the bilinear form, Lax pairs, and Bäcklund transformations. Tang and Zai\(^4\) obtained the new periodic-wave solutions. Liu et al\(^5,6\) given the new three-wave and non-traveling wave solutions. Mabrouk and Rashed\(^7\) obtained some new analytical solutions. Li and Ma\(^8\) presented the multiple-lump waves solutions. Osman and Wazwaz\(^9\) derived lump waves, breather waves, mixed waves, and multisoliton wave solutions. Peng et al\(^10\) studied the breather waves and rational solutions. Xu\(^11\) given the Painlevé analysis, lump-kink solutions, and localized excitation solutions. Wang et al\(^12\) constructed N-soliton solutions and illustrated novel nonlinear wave phenomena.

In this paper, a new (3 + 1)-dimensional BLMP equation is studied as follows:\(^13\)

\[(u_x + u_y + u_z)_t + (u_x + u_y + u_z)_{xxx} + (u_x(u_x + u_y + u_z))_x = 0. \tag{2}\]

The new BLMP equation was first proposed by Wazwaz.\(^13\) Compared with the well-known (3 + 1)-dimensional BLMP equation (1), Equation (2) adds the \(u_x\) to each of the potential derivatives \(u_y + u_z\). The integrability, compatibility conditions, multiple soliton solutions, and multiple complex soliton solutions were discussed by Painlevé test and Hirota’s direct method.
Based on the transformation

\[ u = -2[\ln \xi(x, y, z, t)]_t, \]

the new BLMP equation has the bilinear form

\[ [D^4_x + D_y D^2_x + D_z D^2_x + D_t D_x + D_t D_y + D_t D_z] \xi \cdot \xi = 0. \]

The organization of this paper is as follows. Section 2 gives the lump solutions; Section 3 obtains the interaction solutions between lump wave and solitary waves; Section 4 derives the interaction solutions between lump wave and periodic waves; and Section 5 presents the breather wave solutions. All results have been verified to be correct by using Mathematica software.14-39 Section 6 gives the conclusion.

2 | LUMP SOLUTIONS

Generally speaking, the lump solutions of nonlinear integrable equations can be assumed as follows:

\[ \xi = \gamma_3 + (\alpha_4 t + \alpha_1 x + \alpha_2 y + \alpha_3 z)^2 + (\beta_4 t + \beta_1 x + \beta_2 y + \beta_3 z)^2, \]

where \( \alpha_i (1 \leq i \leq 4) \), \( \beta_i (1 \leq i \leq 4) \) and \( \gamma_3 \) are undetermined real parameters. Substituting Equation (5) into Equation (4) and making the coefficients of \( x^2 \), \( y^2 \), \( xy \), \( xz \), and so on be zero, undetermined real parameters in Equation (5) have the following results:

1. \( \alpha_3 = -\alpha_1 - \alpha_2, \beta_2 = -\beta_1 - \beta_3 \)
2. \( \beta_1 = -\alpha_3 \alpha_4 / \beta_4, \alpha_3 = \alpha_1 \beta_3 / \beta_1, \beta_2 = -\alpha_2 \beta_3 / \alpha_1 + \alpha_2 \), \( \beta_4 = (\alpha_1 + \alpha_2) \alpha_4 / \beta_3 \)
3. \( \alpha_3 = \alpha_1 \beta_3 / \beta_1, \beta_1 = -\alpha_1 \beta_3 / \alpha_1 + \alpha_2, \beta_2 = -\alpha_2 \beta_3 / \alpha_1 + \alpha_2 \)
4. \( \alpha_3 = -\alpha_1 - \alpha_2, \beta_3 = \alpha_1^2 + \alpha_2 \alpha_4 / \beta_1, \beta_2 = -\alpha_1 (\alpha_1 + \alpha_2) / \beta_1 \)
5. \( \alpha_2 = -\alpha_1, \beta_2 = -\beta_1, \alpha_3 = \beta_3 = 0 \)
6. \( \alpha_4 = \beta_4 = 0, \beta_3 = -\alpha_1 (\alpha_1 + \alpha_2 + \alpha_3) + \beta_1 (\beta_1 + \beta_2) / \beta_1 \)
7. \( \alpha_3 = -\alpha_1 - \alpha_2, \beta_1 = -\beta_2 - \beta_4, \beta_4 = 2\alpha_1 \alpha_4 \beta_1 / (\alpha_1 - \beta_1) \)
8. \( \alpha_3 = -\alpha_1 - \alpha_2, \beta_1 = \alpha_1 \beta_2 / \alpha_2, \beta_3 = -\alpha_1 (\alpha_1 + \alpha_2) / (\alpha_1 - \beta_1) \)

Substituting Equations (6)–(13) into Equations (3) and (5), eight different types of lump solutions are derived. As an example, substituting Equation (6) into Equations (3) and (5), we have

\[ u = -[2[2\alpha_1 [\alpha_4 t + \alpha_1 x + \alpha_2 y - (\alpha_1 + \alpha_2) z] + 2\beta_1 [\beta_4 t + \beta_1 x \\
- (\beta_1 + \beta_3) y + \beta_3 z)]]/[\gamma_3 + [\alpha_4 t + \alpha_1 x + \alpha_2 y - (\alpha_1 + \alpha_2) z]^2 \]

\[ + [\beta_4 t + \beta_1 x - (\beta_1 + \beta_3) y + \beta_3 z]^2]. \]

Graphical representation of lump solution (14) is shown in Figure 1. It is obvious that there are a peak and a bottom in Figure 1, and they are symmetric. In the peak, solution (14) has a maximum \( 2\sqrt{5} \) at \( x = -1/\sqrt{5} \) and \( y = 0 \). In the bottom, solution (14) has a minimum \(-2\sqrt{5} \) at \( x = 1/\sqrt{5} \) and \( y = 0 \).
FIGURE 1  Lump solution (14) with $\alpha_1 = \beta_3 = -2, \beta_4 = -3, \alpha_4 = 2, t = z = 0, \alpha_2 = \beta_1 = \gamma_3 = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 2  Solution (17) with $\alpha_1 = \beta_3 = \gamma_1 = -2, \beta_4 = \varphi_1 = -3, \alpha_4 = \varphi_2 = 2, \gamma_2 = z = 0, \alpha_2 = \beta_1 = \gamma_3 = \varphi_4 = 1$, when (A,D) $t = -3$, (B,E) $t = 0$, and (C,F) $t = 3$ [Colour figure can be viewed at wileyonlinelibrary.com]
In this section, we will discuss the interaction phenomenon between lump wave and solitary waves. Considering the following mixed functions:

$$
\xi = \gamma_3 + (\alpha_1 t + \alpha_2 x + \alpha_3 y + \alpha_4 z)^2 + (\beta_4 t + \beta_1 x + \beta_2 y + \beta_3 z)^2 \\
+ \gamma_1 e^{\phi_1 + \phi_2 + \phi_3 + \phi_4} + \gamma_2 e^{-\phi_1 - \phi_2 - \phi_3 - \phi_4},
$$

(15)

where $\phi_i (1 \leq i \leq 4)$ and $\gamma_i (i = 1, 2)$ are undetermined real parameters. As an example, substituting Equations (15) and (6) into Equation (4) and making the coefficients of $e^{\phi_1 + \phi_2 + \phi_3 + \phi_4} - \phi_1 - \phi_2 - \phi_3 - \phi_4$ be zero, undetermined real parameters in Equation (15) have the following results:

$$
\alpha_3 = -\alpha_1 - \alpha_2, \beta_2 = -\beta_1 - \beta_3, \phi_3 = -\phi_1 - \phi_2.
$$

(16)

Substituting Equation (16) into Equations (3) and (15), the interaction solutions between lump wave and solitary waves can be read as

$$
u = -2[\gamma_2 \phi_2 \exp[-t \phi_4 - \phi_1 x - \phi_2 y + (\phi_1 + \phi_2) z] + 2\alpha_1 [\alpha_4 t + \alpha_1 x \\
+ \alpha_2 y - (\alpha_1 + \alpha_2) z] + 2\beta_3 \gamma_2 \exp[-t \phi_4 - \phi_1 x - \phi_2 y \\
+ (\phi_1 + \phi_2) z] + [\alpha_4 t + \alpha_1 x + \alpha_2 y - (\alpha_1 + \alpha_2) z]^2 + [\beta_4 t + \beta_1 x \\
- (\beta_1 + \beta_3) y + \beta_2 z]^2 + \gamma_1 e^{\phi_1 + \phi_2 + \phi_3 + \phi_4} - \phi_1 - \phi_2 - \phi_3 - \phi_4].
$$

(17)

FIGURE 3 Solution (17) with $\alpha_1 = \beta_3 = \gamma_1 = \gamma_2 = -2, \beta_4 = \phi_1 = -3, z = 0, \alpha_4 = \phi_2 = 2, \alpha_2 = \beta_1 = \gamma_3 = \phi_3 = 1$, when (A) $t = -4$, (B) $t = -2$, (C) $t = 0$, (D) $t = 2$, and (E) $t = 4$ [Colour figure can be viewed at wileyonlinelibrary.com]
Graphical representation of the interaction solutions (17) is shown in Figures 2–4. Figure 2 describes the interaction phenomenon between lump wave and one solitary wave. Figures 3 and 4 represent the interaction phenomenon between lump wave and two solitary waves.

4 | INTERACTION SOLUTIONS BETWEEN LUMP WAVE AND PERIODIC WAVES

In this section, we will investigate the interaction phenomenon between lump wave and periodic waves. Assuming the following mixed functions:

\[
\xi = \gamma_3 + (\alpha_4 t + \alpha_1 x + \alpha_2 y + \alpha_3 z)^2 + (\beta_4 t + \beta_1 x + \beta_2 y + \beta_3 z)^2 \\
+ \gamma_2 \sin (\mu_4 t + \mu_1 x + \mu_2 y + \mu_3 z) + \gamma_1 \cos (t\varphi_4 + \varphi_1 x + \varphi_2 y + \varphi_3 z),
\]

where \( \mu_i (1 \leq i \leq 4) \) is undetermined real parameter.

Substituting Equations (18) and (6) into Equation (4) and making the coefficients of \( \cos (t\varphi_4 + \varphi_1 x + \varphi_2 y + \varphi_3 z)x^2 \), \( \sin (\mu_4 t + \mu_1 x + \mu_2 y + \mu_3 z)x^2 \) and so on be zero, undetermined real parameters in Equation (18) have the following results:

\[
\alpha_3 = -\alpha_1 - \alpha_2, \quad \beta_2 = -\beta_1 - \beta_3, \quad \varphi_3 = -\varphi_1 - \varphi_2, \quad \mu_3 = -\mu_1 - \mu_2.
\]

**FIGURE 4** The corresponding contour plots of Figure 3 [Colour figure can be viewed at wileyonlinelibrary.com]
Graphical representation of the interaction solutions (20) is shown in Figures 5 and 6. Figure 5 describes the interaction phenomenon between lump wave and one periodic wave. Figure 6 represents the interaction phenomenon between lump wave and two periodic waves.

Substituting Equation (19) into Equations (3) and (18), the interaction solutions between lump wave and periodic waves can be written as

\[
 u = - \left[ 2 \alpha_4 \left[ \alpha_4 t + \alpha_1 x + \alpha_2 y - (\alpha_1 + \alpha_2) z \right] + 2 \beta_1 \left[ \beta_4 t + \beta_1 x - (\beta_1 + \beta_3) y + \beta_3 z \right] + \gamma_2 \mu_1 \cos \left[ \nu_4 t + \mu_1 x + \mu_2 y - (\mu_1 + \mu_2) z \right] - \gamma_1 \varphi_1 \sin \left[ \varphi_4 + \varphi_1 x \right] + \varphi_2 y - (\varphi_1 + \varphi_2) z \right] \] 
+ \gamma_1 \cos \left[ \nu_4 + \varphi_1 x + \varphi_2 y - (\varphi_1 + \varphi_2) z \right].
\]

Graphical representation of the interaction solutions (20) is shown in Figures 5 and 6. Figure 5 describes the interaction phenomenon between lump wave and one periodic wave. Figure 6 represents the interaction phenomenon between lump wave and two periodic waves.
5 | BREather Wave Solutions

In this section, we will study the breather wave solutions. Choosing the following mixed functions:

\[
\xi = k_1 e^{i\phi_4 + \phi_1 x + \phi_2 y + \phi_3 z} + \gamma_2 \sin (\mu_4 t + \mu_1 x + \mu_2 y + \mu_3 z) \\
+ \gamma_1 \cos (\nu_4 t + \nu_1 x + \nu_2 y + \nu_3 z) + e^{-i\phi_4 - \phi_1 x - \phi_2 y - \phi_3 z},
\]

where \(\nu_i (1 \leq i \leq 6)\) and \(k_1\) are unknown real parameters. Substituting Equation (21) into Equation (4), we have the following results:

\[
\nu_3 = -\nu_1 - \nu_2, \phi_3 = -\phi_1 - \phi_2, \mu_3 = -\mu_1 - \mu_2.
\]

Substituting Equation (22) into Equations (3) and (21), the breather wave solutions can be presented as

\[
u = -[2 - \phi_1 \exp(-t\phi_4 - \phi_1 x - \phi_2 y + (\phi_1 + \phi_2) z) + \gamma_2 \mu_1 \cos(\mu_4 t + \mu_1 x) \\
+ \mu_2 y - (\mu_1 + \mu_3) z] + k_1 \phi_1 e^{i\phi_4 + \phi_1 x + \phi_2 y - (\phi_1 + \phi_2) z} - \gamma_1 \nu_1 \sin(\nu_4 t + \nu_1 x) \\
+ \nu_2 y - (\nu_1 + \nu_2) z] / [\exp(-t\phi_4 - \phi_1 x - \phi_2 y + (\phi_1 + \phi_2) z) \\
+ k_1 e^{i\phi_4 + \phi_1 x + \phi_2 y - (\phi_1 + \phi_2) z} + \gamma_2 \sin(\mu_4 t + \mu_1 x + \mu_2 y - (\mu_1 + \mu_2) z) \\
+ \gamma_1 \cos(\nu_4 t + \nu_1 x + \nu_2 y - (\nu_1 + \nu_2) z)].
\]

Graphical representation of the breather wave solutions (23) is displayed in Figure 7.

6 | Conclusion

Recently, a new \((3 + 1)\)-dimensional BLMP equation is introduced by Wazwaz. At present, there is no literature on the lump solution of this equation. In this paper, breather wave and lump-type solutions of the new \((3 + 1)\)-dimensional BLMP equation are presented. Lump-type solutions contain the interaction solutions between lump and solitary waves, and the interaction solutions between lump and periodic waves, which have not been studied in any literature. Furthermore, graphical representation for lump solution is displayed in Figure 1. Interaction phenomenon of lump-type solutions are demonstrated in Figures 2–7. All the results have been verified to be correct by Mathematica software.

Conflict of Interest

This work does not have any conflicts of interest.
AUTHOR CONTRIBUTIONS
Jian-Guo Liu has made substantial contributions to the writing and calculation of thesis. Prof. Abdul-Majid Wazwaz has made substantial contributions to the promotion of language, acquisition of data, or analysis and interpretation of data.

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