Poly-Cauchy numbers and polynomials of the second kind with umbral calculus viewpoint

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Abstract
In this paper, we consider the poly-Cauchy polynomials and numbers of the second kind which were studied by Komatsu. We note that the poly-Cauchy polynomials of the second kind are the special generalized Bernoulli polynomials of the second kind. The purpose of this paper is to give various identities of the poly-Cauchy polynomials of the second kind which are derived from umbral calculus.

1 Introduction
As is well known, the Bernoulli polynomials of the second kind are defined by the generating function to be

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} \quad (\text{see [1, p.130]}). \quad (1)$$

When $x = 0$, $b_n = b_n(0)$ are called the Bernoulli numbers of the second kind (see [1, p.131]).

Let $\text{Lif}_k(x)$ be the polylogarithm factorial function, which is defined by

$$\text{Lif}_k(x) = \sum_{n=0}^{\infty} \frac{x^m}{n!(m+1)^k} \quad (\text{see [2–7]}). \quad (2)$$

The poly-Cauchy polynomials of the second kind $\hat{c}_n^{(k)}(x)$ ($k \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0}$) are defined by the generating function to be

$$\text{Lif}_k(-\log(1+t))(1+t)^x = \sum_{n=0}^{\infty} \hat{c}_n^{(k)}(x) \frac{t^n}{n!} \quad (\text{see [2, 3]}). \quad (3)$$

When $x = 0$, $\hat{c}_n^{(k)} = \hat{c}_n^{(k)}(0)$ are called the poly-Cauchy numbers of the second kind, defined by

$$\sum_{n=0}^{\infty} \hat{c}_n^{(k)} \frac{t^n}{n!} = \text{Lif}_k(-\log(1+t)). \quad (4)$$

In particular, if we take $k = 1$, then we have

$$\text{Lif}_1(-\log(1+t))(1+t)^x = \frac{t}{(1+t)\log(1+t)}(1+t)^x = \frac{t(1+t)^{x-1}}{\log(1+t)}. \quad (5)$$

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Thus, we note that
\[ \hat{c}_n^{(l)}(x) = b_n(x - 1) = B_n^{(l)}(x), \] (6)
where \( B_n^{(l)}(x) \) are the Bernoulli polynomials of order \( \alpha \) (see [8]) as their numbers [9, p.257 and p.259].

When \( x = 0 \), \( \hat{c}_n^{(l)} = \hat{c}_n^{(l)}(0) = b_n(-1) = B_n^{(l)}(\alpha) \) are the Bernoulli numbers of order \( \alpha \).

The falling factorial is defined by
\[ (x)_n = x(x - 1) \cdots (x - n + 1) = \sum_{l=0}^{n} S_1(n, l)x^l, \] (7)
where \( S_1(n, l) \) is the signed Stirling number of the first kind.

For \( m \in \mathbb{Z}_{\geq 0} \), it is well known that
\[ (\log(1 + t))^m = m! \sum_{l=m}^{\infty} S_1(l, m) \frac{t^l}{l!}. \]
\[ = \sum_{l=0}^{\infty} S_1(l + m, m) \frac{m!}{(l + m)!} t^{l+m} \] (see [10, p.62]). (8)

For \( \lambda \in \mathbb{C} \) with \( \lambda \neq 1 \), the Frobenius-Euler polynomials of order \( r \) are defined by the generating function to be
\[ \left( \frac{1 - \lambda}{e^t - \lambda} \right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda) \frac{t^n}{n!} \] (see [11–13]).

In this paper, we investigate the properties of the poly-Cauchy numbers and polynomials of the second kind with umbral calculus viewpoint. The purpose of this paper is to give various identities of the poly-Cauchy polynomials of the second kind which are derived from umbral calculus.

2 Umbral calculus

Let \( \mathbb{C} \) be the complex number field and let \( \mathcal{F} \) be the set of all formal power series in the variable \( t \):
\[ \mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \middle| a_k \in \mathbb{C} \right\}. \] (9)

Let \( \mathbb{P} = \mathbb{C}[x] \) and let \( \mathbb{P}^* \) be the vector space of all linear functionals on \( \mathbb{P} \). \( \langle L | p(x) \rangle \) is the action of the linear functional \( L \) on the polynomial \( p(x) \), and we recall that the vector space operations on \( \mathbb{P}^* \) are defined by \( (L + M)p(x) = \langle (L + M)p(x) \rangle, \ (cL)p(x) = \langle cLp(x) \rangle \), where \( c \) is a complex constant in \( \mathbb{C} \). For \( f(t) \in \mathcal{F} \), let us define the linear functional on \( \mathbb{P} \) by setting
\[ \langle f(t) | x^n \rangle = a_n \quad (n \geq 0). \] (10)
Then, by (9) and (10), we get

\[ \langle t^k | x^n \rangle = n! \delta_{n,k} \quad (n, k \geq 0), \]

where \( \delta_{n,k} \) is Kronecker’s symbol.

For \( f_L(t) = \sum_{k=0}^{\infty} \frac{(t | x^k)}{k!} \), we have \( \langle f_L(t) | x^n \rangle = \langle L | x^n \rangle \). That is, \( L = f_L(t) \). The map \( L \mapsto f_L(t) \) is a vector space isomorphism from \( \mathbb{P}^n \) onto \( \mathcal{F} \). Henceforth, \( \mathcal{F} \) denotes both the algebra of formal power series in \( t \) and the vector space of all linear functionals on \( \mathbb{P} \), and so an element \( f(t) \) of \( \mathcal{F} \) will be thought of as both a formal power series and a linear functional.

We call \( \mathcal{F} \) the umbral algebra and the umbral calculus is the study of umbral algebra. The order \( O(f(t)) \) of a power series \( f(t) \) (\( \neq 0 \)) is the smallest integer \( k \) for which the coefficient of \( t^k \) does not vanish. If \( O(f(t)) = 1 \), then \( f(t) \) is called a delta series; if \( O(f(t)) = 0 \), then \( f(t) \) is called an invertible series (see [10, 14, 15]). For \( f(t), g(t) \in \mathcal{F} \) with \( O(f(t)) = 1 \) and \( O(g(t)) = 0 \), there exists a unique sequence \( s_n(x) \) (deg \( s_n(x) = n \)) such that \( \langle g(t)f(t)^k | s_n(x) \rangle = n! \delta_{n,k} \) for \( n, k \geq 0 \). The sequence \( s_n(x) \) is called the Sheffer sequence for \( (g(t), f(t)) \) which is denoted by \( s_n(x) \sim (g(t), f(t)) \) (see [10, 15]).

For \( f(t), g(t) \in \mathcal{F} \) and \( p(x) \in \mathbb{P} \), we have

\[ \langle f(t)g(t) | p(x) \rangle = \langle f(t) | g(t)p(x) \rangle = \langle g(t) | f(t)p(x) \rangle, \]

and

\[ f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t) | x^k \rangle}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k | p(x) \rangle}{k!} x^k. \]

Thus, by (13), we get

\[ t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad \text{and} \quad e^y p(x) = p(x + y). \]

Let us assume that \( s_n(x) \sim (g(t), f(t)) \). Then the generating function of \( s_n(x) \) is given by

\[ \frac{1}{g(f(t))} e^{\tilde{f}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \quad \text{for all } x \in \mathbb{C}, \]

where \( \tilde{f}(t) \) is the compositional inverse of \( f(t) \) with \( \tilde{f}(f(t)) = t \) (see [10, 15]).

For \( s_n(x) \sim (g(t), f(t)) \), we have the following equation:

\[ f(t)s_n(x) = ns_{n-1}(x) \quad (n \geq 0), \]

\[ s_n(x) = \sum_{j=0}^{n} \frac{1}{j!} \langle g(\tilde{f}(t))^{-1} \tilde{f}(t)^j | x^n \rangle x^j, \]

and

\[ s_n(x + y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}(y), \]

where \( p_n(x) = g(t)s_n(x) \) (see [10, p.21]).
Let us assume that
\[ p_n(x) \sim (1, f(t)) \quad q_n(x) \sim (1, g(t)) \]
Then the transfer formula is given by
\[ q_n(x) = x \left( \frac{f(t)}{g(t)} \right)^n x^{n-1} p_n(x) \quad (n \geq 0) \]
(see [10, p.51]).

For
\[ s_n(x) \sim (g(t), f(t)) \quad r_n(x) \sim (h(t), l(t)) \]
let us assume that
\[ s_n(x) = \sum_{m=0}^{n} C_{n,m} r_n(x) \quad (n \geq 0). \]  (19)

Then we have
\[ C_{n,m} = \frac{1}{m!} \left( \frac{h(t)}{g(t)} \right)^m \left[ \frac{d^m (\bar{f}(t))}{dt^m} \right] \]
(see [10, p.132]). \hspace{1cm} (20)

### 3 Poly-Cauchy numbers and polynomials of the second kind

From (3), we note that \( \tilde{c}_n^{(k)}(x) \) is the Sheffer sequence for the pair
\[
\left( g(t) = \frac{1}{\text{Lif}_k(-t)} f(t) = e^t - 1 \right),
\]
that is,
\[
\tilde{c}_n^{(k)}(x) \sim \left( \frac{1}{\text{Lif}_k(-t)}, e^t - 1 \right). \]  (21)

Because for \( \bar{f}(t) = \log(1 + t) \), using the formula (15), we get
\[
\text{Lif}_k(-\log(1 + t))(1 + t)^x = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}
\]
which is the generating function of \( \tilde{c}_n^{(k)}(x) \) in (3).

From (21), we have
\[
\frac{1}{\text{Lif}_k(-t)} \tilde{c}_n^{(k)}(x) \sim (1, e^t - 1), \]  (22)
and
\[
(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l \sim (1, e^t - 1). \]  (23)

By (22) and (23), we get
\[
\tilde{c}_n^{(k)}(x) = \text{Lif}_k(-t)(x)_n = \sum_{m=0}^{n} S_1(n,m) \text{Lif}_k(-t)x^m
= \sum_{m=0}^{n} S_1(n,m) \sum_{a=0}^{m} \frac{(-1)^a}{a!(a+1)^k} t^a x^m
\]
\[\begin{align*}
&= \sum_{m=0}^{n} \sum_{a=0}^{m} S_1(n, m) \left(\frac{(-1)^{a(m)}}{(a+1)^k}\right) x^{m-a} \\
&= \sum_{m=0}^{n} \sum_{j=0}^{m} S_1(n, m) \left(\frac{(-1)^{m-j}(m)}{(m-j+1)^k}\right) x^{j} \\
&= \sum_{j=0}^{n} \left\{ \sum_{m=j}^{n} S_1(n, m) \left(\frac{(-1)^{m-j}(m)}{(m-j+1)^k}\right) x^{j} \right\}.
\end{align*}\]

By (17) and (21), we get

\[\hat{c}_n^{(k)}(x) = \sum_{j=0}^{n} \frac{1}{j!} \partial^j \left(\frac{1}{\text{Lif}_k(-\log(1+t)) (\log(1+t))^j} \right) x^n |x|^j.\]  

Now, we observe that

\[\begin{align*}
&\left[\partial^j \left(\frac{1}{\text{Lif}_k(-\log(1+t)) (\log(1+t))^j} \right) x^n \right] \left| x^n \right| \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)^k} \left( (\log(1+t))^m |x^n| \right) \\
&= \sum_{m=0}^{n-j} \frac{(-1)^m}{m!(m+1)^k} \sum_{l=0}^{n-m} \frac{S_1(l + m + j, m + j)}{(l + m + j)!} (m + j)^{m+j} |x|^j \\
&= \sum_{m=0}^{n-j} \frac{(-1)^m (m + j)!}{m!(m+1)^k} S_1(n, m + j). \tag{26}
\end{align*}\]

From (25) and (26), we have

\[\begin{align*}
\hat{c}_n^{(k)}(x) &= \sum_{j=0}^{n} \frac{1}{j!} \sum_{m=0}^{n-j} \frac{(-1)^m (m + j)!}{m!(m+1)^k} S_1(n, m + j) x^{j} = \sum_{j=0}^{n} \left\{ \sum_{m=0}^{n-j} \frac{(-1)^m (m+1)^{m+j}}{(m+1)^k} S_1(n, m + j) \right\} x^{j} \\
&= \sum_{j=0}^{n} \left\{ \sum_{m=j}^{n} S_1(n, m) \right\} x^{j}, \tag{27}
\end{align*}\]

which is the same as the expression in (24). From (1), we note that

\[\frac{1}{\text{Lif}_k(-t)} x^n \sim (1, e^t - 1), \quad x^n \sim (1, t).\]  

(28)

For \(n \geq 1\), by (19) and (28), we get

\[\begin{align*}
\frac{1}{\text{Lif}_k(-t)} x^n &\sim x \left(\frac{t}{e^t - 1}\right)^n x^{n-1} = x \left(\frac{t}{e^t - 1}\right)^n x^{n-1} \\
&= x B_{n-1}^{(n)}(x) = \sum_{l=0}^{n-1} \binom{n-1}{l} x^{l+1}. \tag{29}
\end{align*}\]
Thus, by (29), we see that

\[
\hat{c}_n^{(k)}(x) = \sum_{l=0}^{n-1} \binom{n-1}{l} p_{n-1-l}^{(1)} \text{Lif}_k(-t)x^{l+1}
\]

\[
= \sum_{l=0}^{n-1} \sum_{m=0}^{l+1} (-1)^m \binom{n-1}{l} \binom{l+1}{m} \frac{B_{n-1-l}^{(m)}}{(m+1)^k} x^{l+1-m}
\]

\[
= \sum_{l=0}^{n-1} \sum_{j=0}^{l} (-1)^{l+1-j} \binom{n-1}{l} \binom{l+1}{j} \frac{B_{n-1-l}^{(j)}}{(l+2-j)^k} x^j
\]

\[
= \sum_{l=0}^{n-1} (-1)^{l+1} \binom{n-1}{l} \frac{B_{n-1-l}^{(l)}}{l+2} x^l
\]

\[
+ \sum_{j=1}^{n} \left( \sum_{l=j-1}^{n-1} (-1)^{l+1-j} \binom{n-1}{l} \binom{l+1}{j} \frac{B_{n-1-l}^{(j)}}{(l+2-j)^k} \right) x^j.
\]

(30)

Therefore, by (27) and (30), we obtain the following theorem.

**Theorem 1** For \( n \geq 1, 1 \leq j \leq n \), we have

\[
\sum_{m+j}^{n} \frac{(-1)^{m+j}}{(m-j+1)^k} S_1(n, m) \binom{n}{j} \frac{B_{n-1-j}^{(m)}}{(m+1)^k} = \sum_{l=0}^{n-1} \sum_{j=0}^{l} (-1)^{l+1-j} \binom{n-1}{l} \binom{l+1}{j} \frac{B_{n-1-l}^{(j)}}{(l+2-j)^k}.
\]

In addition, for \( n \geq 1 \), we have

\[
\hat{c}_n^{(k)} = \sum_{m=0}^{n} S_1(n, m) \binom{n}{j} \frac{(-1)^m}{(m+1)^k} = \sum_{l=0}^{n-1} (-1)^{l+1} \binom{n-1}{l} \frac{B_{n-1-l}^{(l)}}{(l+2)^k}.
\]

From (18), we note that

\[
\hat{c}_n^{(k)}(x+y) = \sum_{j=0}^{n} \binom{n}{j} \hat{c}_j^{(k)}(x)p_{n-j}(y),
\]

(31)

where \( p_n(y) = \frac{1}{\text{Lif}_k(-t)} \hat{c}_n^{(k)}(y) \sim (1, e^t - 1) \).

By (22) and (23), we get

\[
(y)_n = p_n(y) \sim (1, e^t - 1).
\]

(32)

Thus, from (31) and (32), we have

\[
\hat{c}_n^{(k)}(x+y) = \sum_{j=0}^{n} \binom{n}{j} \hat{c}_j^{(k)}(x)(y)_{n-j}.
\]

(33)

By (14), (16), and (21), we get

\[
\hat{c}_n^{(k)}(x+1) - \hat{c}_n^{(k)}(x) = (e^t - 1) \hat{c}_n^{(k)}(x) = n \hat{c}_n^{(k)}(x).
\]
For \( s_n(x) \sim (g(t), f(t)) \), the recurrence formula for \( s_n(x) \) is given by

\[
s_{n+1}(x) = \left( x - \frac{g(t)}{f(t)} \right) \frac{1}{f'(t)} s_n(x) \quad \text{(see [10])}.
\]  

By (21) and (34), we get

\[
\tilde{c}_{n+1}^{(k)}(x) = \left( x - \frac{\text{Lif}_k^{(t)}(t)}{\text{Lif}_k^{(t)}} \right) e^{-\gamma_n^{(k)}(x)}
= x \tilde{c}_n^{(k)}(x - 1) - e^{-\gamma_n^{(k)}(x)} \frac{\text{Lif}_k^{(t)}(t)}{\text{Lif}_k^{(t)}} \tilde{c}_n^{(k)}(x).
\]  

We observe that

\[
\frac{\text{Lif}_k^{(t)}(t)\text{Lif}_k^{(m)}(m)}{\text{Lif}_k^{(t)}} \tilde{c}_n^{(k)}(x) = \text{Lif}_k^{(t)}(t) \frac{1}{\text{Lif}_k^{(t)}} \tilde{c}_n^{(k)}(x) = \text{Lif}_k^{(t)}(t)(x)_n
= \sum_{l=0}^{n} S_l(n, l) \text{Lif}_k^{(t)}(-t)x^l
= \sum_{l=0}^{n} S_l(n, l) \sum_{m=0}^{l} (-1)^m \binom{l}{m} x^{l-m}
= \sum_{j=0}^{n} \left( \sum_{l=j}^{n} (-1)^{l-j} \binom{l}{j} (l-j+2)^k S_l(n, l) \right) x^l.
\]  

Therefore, by (35) and (36), we obtain the following theorem.

**Theorem 2** For \( n \geq 0 \), we have

\[
\tilde{c}_{n+1}^{(k)}(x) = x \tilde{c}_n^{(k)}(x - 1) - \sum_{j=0}^{n} \left( \sum_{l=j}^{n} (-1)^{l-j} \binom{l}{j} (l-j+2)^k S_l(n, l) \right) (x - 1)^j.
\]

From (11), we note that

\[
\tilde{c}_n^{(k)}(y) = \left( \sum_{l=0}^{\infty} \tilde{c}_l^{(k)}(y) \frac{y^l}{l!} \right) x^n
= \left[ \text{Lif}_k^{(t)}(-\log(1 + t))(1 + t)^y|x^n \right]
= \left[ \text{Lif}_k^{(t)}(-\log(1 + t))(1 + t)^y|x^{n-1} \right]
= \left\langle \hat{a}_t(\text{Lif}_k^{(t)}(-\log(1 + t))(1 + t)^y)|x^{n-1} \right\rangle
= \left\langle \hat{a}_t(\text{Lif}_k^{(t)}(-\log(1 + t))(1 + t)^y)|x^{n-1} \right\rangle
= \left\langle \hat{a}_t(\text{Lif}_k^{(t)}(-\log(1 + t))(1 + t)^y)|x^{n-1} \right\rangle
\]  

where \( \hat{a}f(t) = \frac{d^t f(t)}{dt} \).

Since \( t \text{Lif}_k^{(t)}(t) = \text{Lif}_{k-1}^{(t)}(t) - \text{Lif}_k^{(t)}(t) \), we get

\[
\text{Lif}_k^{(t)}(t) = \frac{\text{Lif}_{k-1}^{(t)}(t) - \text{Lif}_k^{(t)}(t)}{t}.
\]
By (37) and (38), we see that

\[ \tilde{c}_n^{(k)}(y) = y\tilde{c}_{n-1}^{(k)}(y - 1) \]
\[ + \left\{ \frac{\text{Li}_{k-1}(- \log(1 + t)) - \text{Li}_k(- \log(1 + t))}{(1 + t) \log(1 + t)} (1 + t)^y \right\} x^{n-1} \]
\[ = y\tilde{c}_{n-1}^{(k)}(y - 1) \]
\[ + \left\{ \frac{\text{Li}_{k-1}(- \log(1 + t)) - \text{Li}_k(- \log(1 + t))}{t(1 + t)} \right\} \frac{t}{\log(1 + t)} x^{n-1} \].

(39)

From (1), (6), and (38), we note that

\[ \tilde{c}_n^{(k)}(y) = y\tilde{c}_{n-1}^{(k)}(y - 1) + \sum_{l=0}^{n-1} \frac{B_l^{(1)}}{l} (n-1)_l \]
\[ \times \left\{ \frac{\text{Li}_{k-1}(- \log(1 + t)) - \text{Li}_k(- \log(1 + t))}{t} (1 + t)^y \right\} x^{n-l} \]
\[ = y\tilde{c}_{n-1}^{(k)}(y - 1) + \sum_{l=0}^{n-1} \frac{B_l^{(1)}}{l} (n-1)_l \]
\[ \times \left\{ \frac{\text{Li}_{k-1}(- \log(1 + t)) - \text{Li}_k(- \log(1 + t))}{t} \right\} \frac{t}{n-l} \]
\[ = y\tilde{c}_{n-1}^{(k)}(y - 1) + \sum_{l=0}^{n-1} \binom{n-1}{l} B_l^{(1)} \left[ \tilde{c}_{n-l}^{(k-1)}(y - 1) - \tilde{c}_{n-l}^{(k)}(y - 1) \right] \]
\[ = y\tilde{c}_{n-1}^{(k)}(y - 1) + \frac{1}{n} \sum_{l=0}^{n-1} \binom{n}{l} B_l^{(1)} \left[ \tilde{c}_{n-l}^{(k-1)}(x - 1) - \tilde{c}_{n-l}^{(k)}(x - 1) \right]. \]

(40)

It is not difficult to show that \( \tilde{c}_0^{(k)}(y - 1) = \tilde{c}_0^{(k-1)}(y - 1) \). Since \( \tilde{c}_0^{(k)}(y - 1) = \tilde{c}_0^{(k-1)}(y - 1) \), by (40), we obtain the following theorem.

**Theorem 3** For \( n \geq 1 \), we have

\[ \tilde{c}_n^{(k)}(x) = x\tilde{c}_{n-1}^{(k)}(x - 1) + \frac{1}{n} \sum_{l=0}^{n} \binom{n}{l} B_l^{(1)} \left[ \tilde{c}_{n-l}^{(k-1)}(x - 1) - \tilde{c}_{n-l}^{(k)}(x - 1) \right]. \]

For \( n \geq m \geq 1 \), we compute

\[ \left\langle (\log(1 + t))^m \text{Li}_k(- \log(1 + t)) \right\rangle x^n \]

in two different ways.

On the one hand,

\[ \left\langle (\log(1 + t))^m \text{Li}_k(- \log(1 + t)) \right\rangle x^n = \left\langle \text{Li}_k(- \log(1 + t)) \sum_{l=0}^{m} \frac{m!}{l! (l + m)!} S_l(l + m, m) t^l x^n \right\rangle \]
On the other hand, we get

\[
\begin{align*}
\mathcal{F}(\log(1+t)) &= \sum_{l=0}^{n-m} \frac{m!}{(l+m)!} S_l(l+m,m)(n-m) \mathcal{L}(\log(1+t))|x^{n-l-m}| \\
&= \sum_{l=0}^{n-m} m! \binom{n}{l+m} S_l(l+m,m) c^{(k)}_{n-l-m}.
\end{align*}
\]  

(41)

Now, we observe that

\[
\begin{align*}
\partial_t \mathcal{F}(\log(1+t)) &= (\log(1+t))^m \mathcal{L}(\log(1+t))|x^n| \\
&= (\log(1+t))^m \mathcal{L}(\log(1+t))|xx^{n-1}| \\
&= \partial_t \mathcal{F}(\log(1+t))|x^{n-1}|.
\end{align*}
\]  

(42)

By (42) and (43), we get

\[
\begin{align*}
\mathcal{F}(\log(1+t)) &= \sum_{l=0}^{n-m} \frac{(m-1)!}{(l+m-1)!} S_l(l+m-1,m-1) \\
&\quad \times \left\{ (m-1) \mathcal{L}(\log(1+t))(1+t)^{-1} |x^{l+m-1}x^{n-1}| \\
&\quad + \mathcal{L}_m(\log(1+t))(1+t)^{-1} |x^{l+m-1}x^{n-1}| \right\} \\
&= (m-1) \sum_{l=0}^{n-m} \frac{(m-1)!}{(l+m-1)!} S_l(l+m-1,m-1)(n-1)_{l+m-1} \\
&\quad \times \left\{ \mathcal{L}(\log(1+t))(1+t)^{-1} |x^{m-l}| \\
&\quad + \sum_{l=0}^{n-m} \frac{(m-1)!}{(l+m-1)!} S_l(l+m-1,m-1)(n-1)_{l+m-1} \\
&\quad \times \mathcal{L}_m(\log(1+t))(1+t)^{-1} |x^{m-l}| \right\} \\
&= \sum_{l=0}^{n-m} \frac{(m-1)!}{(l+m-1)!} \binom{n-1}{l+m-1} S_l(l+m-1,m-1) \\
&\quad \times \left\{ (m-1)c^{(k)}_{n-l-m}(-1) + c^{(k-1)}_{n-l-m}(-1) \right\}.
\end{align*}
\]  

(44)

Therefore, by (41) and (44), we obtain the following theorem.
Theorem 4 For $n \geq m \geq 1$, we have

$$\sum_{l=0}^{n-m} m! \binom{n}{l+m} S_1(l+m,m)c_{n-l-m}^{(k)}$$

$$= \sum_{l=0}^{n-m} (m-1)! \binom{n-1}{l+m-1} S_1(l+m-1,m-1)$$

$$\times \left\{ (m-1)c_{n-l-m}^{(k)}(-1) + c_{n-l-m}^{(k-1)}(-1) \right\}.$$  

In particular, if we take $m = 1$, then we get

$$c_{n}^{(k-1)}(-1) = \sum_{l=0}^{n-1} (-1)^l l! \binom{n}{l+1} c_{n-l-1}^{(k)}.$$  

Remark For $s_n(x) \sim (g(t), f(t))$, it is known that

$$\frac{d}{dx} s_n(x) = \sum_{l=0}^{n-1} \binom{n}{l} \langle \bar{f}(t) x^{n-l} | x \rangle$$  

(see [10, p.108]). (45)

By (21) and (45), we easily show that

$$\frac{d}{dx} c_n^{(k)}(x) = (-1)^n m! \sum_{l=0}^{n-1} \frac{(-1)^{l-1}}{(n-l)! l!} c_l^{(k)}(x),$$  

which is a special case of Proposition 2 in [4].

Let us consider the following two Sheffer sequences:

$$c_n^{(k)}(x) \sim \left( \frac{1}{L_i(-t)} e^t - 1 \right),$$  

(46)

and

$$B_n^{(r)}(x) \sim \left( \left( \frac{e^t - 1}{t} \right)^r , t \right).$$

Suppose that

$$c_n^{(k)}(x) = \sum_{m=0}^{n} C_{n,m} B_m^{(r)}(x).$$  

(47)

By (20), we see that

$$C_{n,m} = \frac{1}{m!} \left\{ \frac{(\log(1+t))^r}{L_i(-\log(1+t))} \right\}$$

$$\left. \left( \log(1+t) \right)^m x^n \right|_{x^n}$$

$$= \frac{1}{m!} \left\{ L_i(-\log(1+t)) \left( \frac{t}{\log(1+t)} \right)^r \left( \log(1+t) \right)^m x^n \right|_{x^n}$$
Indeed, Let us denote

\[ a = \frac{1}{S_1(l + m, m)B_a^{(a-r+1)}(1)c^{(k)}_{n-l-m-a}}. \] (48)

Therefore, by (47) and (48), we obtain the following theorem.

**Theorem 5** For \( n \geq 0 \), we have

\[ c_n^{(k)}(x) = \sum_{m=0}^{n} \left[ \sum_{l=0}^{n-m} \sum_{a=0}^{n-l-m} \left( \frac{n-l-m}{l+m} \right) \frac{S_1(l + m, m)B_a^{(a-r+1)}(1)c^{(k)}_{n-l-m-a}}{a!} \right] B_m^{(r)}(x). \]

**Remark** The Narumi polynomials of order \( a \) are defined by the generating function to be

\[ \sum_{k=0}^{\infty} N_k^{(a)}(x) \cdot t^k = \left( \frac{t}{\log(1+t)} \right)^{-a} (1+t)^x \quad \text{see \cite{10}, p.127}. \] (49)

Indeed, \( N_k^{(a)}(x) = B_k^{(k+r+1)}(x+1), N_k^{(a)}(x) \sim \left( \frac{t-1}{t} \right)^a, a^t - 1 \).

By (48) and (49), we get

\[ C_{n,m} = \sum_{l=0}^{n-m} \sum_{a=0}^{n-l-m} \left( \frac{n-l-m}{l+m} \right) \frac{S_1(l + m, m)B_a^{(a-r+1)}(1)c^{(k)}_{n-l-m-a}}{a!} \] (50)

From (47) and (50), we have

\[ c_n^{(k)}(x) = \sum_{m=0}^{n} \left[ \sum_{l=0}^{n-m} \sum_{a=0}^{n-l-m} \left( \frac{n-l-m}{l+m} \right) \frac{S_1(l + m, m)B_a^{(a-r+1)}(1)c^{(k)}_{n-l-m-a}}{a!} \right] B_m^{(r)}(x). \] (51)
By (1), we easily show that

$$C_{n,m} = \sum_{l=0}^{n-m} \sum_{a=0}^{n-m-l} \sum_{a_1+\ldots+a_r=a} \binom{n}{l+m} \binom{n-l-m}{a} \binom{a}{a_1,\ldots,a_r} \times S_1(l+m,m) b_{n+m} \cdots b_{a} c_{n-m-l-a}^{(k)}.$$  \hspace{1cm} (52)

From (47) and (52), we can derive the following equation:

$$\hat{c}^{(k)}(x) = \sum_{m=0}^{n} \sum_{l=0}^{n-m-n-l} \sum_{a=0}^{n-m-l} \sum_{a_1+\ldots+a_r=a} \binom{n}{l+m+1} \binom{n-l-m}{a} \binom{a}{a_1,\ldots,a_r} \times S_1(l+m,m) \left( \prod_{i=1}^{r} b_i \right) \hat{c}^{(k)}_{n-m-l-a} \right] b_m(x). \hspace{1cm} (53)$$

For (20) and (24), let

$$\hat{c}^{(k)}(x) = \sum_{m=0}^{n} C_{n,m} H_m^{(r)}(x|\lambda), \hspace{1cm} (54)$$

where, by (20), we get

$$C_{n,m} = \frac{1}{m!(1-\lambda)^r} \left( \text{Lif}_k \left( - \log (1+t) \right) (1+t-\lambda)^r \left( (\log (1+t))^m x^l \right) \right)$$

$$= \frac{1}{m!(1-\lambda)^r} \sum_{l=0}^{n-m} \sum_{a=0}^{m} \frac{m!}{(l+m)!} S_1(l+m,m)(n)_l \hspace{1cm} \times \left( \text{Lif}_k \left( - \log (1+t) \right) (1+t-\lambda)^r x^{n-l-m} \right). \hspace{1cm} (55)$$

We observe that

$$\left( \text{Lif}_k \left( - \log (1+t) \right) (1+t-\lambda)^r x^{n-l-m} \right)$$

$$= \sum_{a=0}^{r} \binom{r}{a} (1-\lambda)^r a \left( \text{Lif}_k \left( - \log (1+t) \right) x^{n-l-m-a} \right)$$

$$= \sum_{a=0}^{r} \binom{r}{a} (1-\lambda)^r a (n-m-l)_a \left( \text{Lif}_k \left( - \log (1+t) \right) x^{n-l-m-a} \right)$$

$$= \sum_{a=0}^{r} \binom{r}{a} (1-\lambda)^r a (n-m-l)_a \hat{c}^{(k)}_{n-l-m-a}.$$  \hspace{1cm} (56)

Thus, by (55) and (56), we get

$$C_{n,m} = \sum_{l=0}^{n-m} \sum_{a=0}^{r} \binom{n}{l+m} \binom{r}{a} (n-m-l)_a (1-\lambda)^r a \left( \text{Lif}_k \left( - \log (1+t) \right) x^{n-l-m-a} \right) \hat{c}^{(k)}_{n-l-m-a}.$$  \hspace{1cm} (57)

Therefore, by (54) and (57), we obtain the following theorem.
Theorem 6 For $n \geq 0$, we have

$$
\hat{c}_n(x) = \sum_{m=0}^{n} \left( \sum_{l=0}^{n-m} \sum_{a=0}^{r} \binom{H}{l} (n-m-l+a) S_l(l+m,m) \sum_{\alpha=0}^{r} a \right) x^m 
$$

For $\hat{c}_n(x) \sim \left( \frac{1}{\text{Lif}_k(-t)}, e^t - 1 \right)$, and $(x)_n \sim (1,e^t-1)$, let us assume that

$$
\hat{c}_n(x) = \sum_{m=0}^{n} C_{n,m}(x)_m. \quad (58)
$$

From (20), we note that

$$
C_{n,m} = \frac{1}{m!} \left[ \text{Lif}_k(-\log(1+t)) t^m[x^n] \right] 
$$

$$
= \frac{1}{m!} \left[ \text{Lif}_k(-\log(1+t)) x^m \right] 
$$

$$
= \left( \binom{n}{m} \text{Lif}_k(-\log(1+t)) x^{n-m} \right) 
$$

$$
= \left( \binom{n}{m} \hat{c}_{n-m}^{(k)} \right). \quad (59)
$$

Therefore, by (58) and (59), we obtain the following theorem.

Theorem 7 For $n \geq 0$, we have

$$
\hat{c}_n(x) = \sum_{m=0}^{n} \binom{n}{m} \hat{c}_{n-m}^{(k)}(x)_m. 
$$

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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