Spacetime Relationalism in GR and QG

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Abstract

We provide here a philosophical basis for [4, 5], based on the notion of spacetime relationalism. We argue that the view which is more cleanly compatible with GR is that in which the spacetime manifold is a relational (or “emergent”) property of the aggregate or physical sum of gravitational field portions, field that is considered material and for which spacetime geometry (in the sense of the metric properties) is one of its material properties. All of the previous philosophical notions are defined in relation to a clear ontology of material individuals, briefly described in the appendix. Furthermore, this fits well with the phase space of GR, thing that also may inform how to proceed in the quantum realm.

1.1 Spacetime Relationalism in GR

Regarding space and time, Newton had a peculiar position. In his words: “absolute space, in its own nature, without regard to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position to bodies: and which is vulgarly taken for immovable space... Absolute motion is the translation of a body from one absolute place into another: and relative motion, the translation from one relative place into another... Absolute, true and mathematical time, of itself, and from it s own nature flows equably without regard to anything external, and by another name is called duration: relative, apparent and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, which is commonly used instead of true time ...” - Isaac Newton. That is, if all matter in the universe suddenly disappears, space and time still would exist, according to this view. In this way, we would need to introduce a second substance in our ontology, a “Spatio-Temporal Substance”, and which is, of course, different and independent from standard material substance. This point of view is called “Space-time substantivalism”. Note, however, that he gives a role to this substance: it defines absolute inertial motion.

Leibniz, among others, had quite a different view, but which is more appropriate for the notion of Reality adopted in the Appendix. Regarding space, it states that Things are presented in reality not in an arbitrary way, but by keeping separation relations and, then, we could say that, given more than one thing, then a property of this system emerges and which is just physical space. Regarding time, note that, given a Thing, it can present, at least in principle, different states. Indeed, we will postulate that all Things are Mutable; that is, they have at least two different possible physical states (e.g., states \((q, p) = (2, 5)\) and \((q, p) = (1, 2)\) in classical mechanics.) Then, we postulate that there is at least one thing in the universe that Changes: that is, for which its state effectively mutates from one to another. The new point of view is simply that time is (actually, a layered notion, consisting in) Mutability and Change (note that only Mutability is analogous

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1Proto-physically speaking, we can think of its physical meaning as resulting from the axiom that any signal from one Thing takes a finite time duration (see next footnote) to reach a second one (not superimposed and separated from the first.) Distance values, in this scheme, are related to the actual different quantitative values of these time durations.
to Separation\textsuperscript{3}. The advantage of this point of view is that, by postulating the existence of mutability, change, and separation (and, in particular, as phenomena tied intrinsically to Things) in the way we did, we then only need the set of Things $\Theta$ in our ontology, space and time being now emergent properties from Reality $R$ (in $\Theta$), i.e., no new substance is needed. The key difference is, of course, that if all the Things in the universe suddenly disappear, then so do space and time. This point of view is called “Space-time relationalism”. In Leibniz’s words: “I have said more than once that I hold space to be something merely relative, as time is; that I hold it to be an order of coexistents, as time is an order of successions” (note, however, that Newton also mentions this notion, but he says it “is vulgarly taken for immovable space”\textsuperscript{4}).

Now, we adopt as a criterion for the existence of a Physical Material Field in spacetime if there are accelerations on particles that cannot be globally eliminated by passing to another reference frame (although, it may be locally), since, if they could, in that case it would just be an innocuous artificial acceleration caused by the acceleration of the reference frame itself and nothing more than that. Electromagnetic fields obviously satisfy this criterion and they also carry energy; thus, they are material entities. But also gravitational accelerations satisfy this criterion: they can be locally eliminated but never globally, this is just Einstein’s equivalence principle. Thus, they have to be caused by a Physical Material Field, which we call the Gravitational Field. In each spacetime region (actually, infinitesimal) in which they are eliminated, the spacetime geometry is flat. But, since the elimination cannot be done globally, this means that the global metric cannot be flat. In this way, we can characterize a gravitational field by a curved spacetime geometry. We could also say that spacetime geometry is actually a property of the gravitational field, since it depends on the configuration of the latter\textsuperscript{5}. This is the actual content of the theory: spacetime geometry becomes a property\textsuperscript{6} of a material\textsuperscript{6} entity. Once he introduced this setup, Einstein realized that the motion of a massive body affected by the gravitational field alone is such that its spacetime trajectory is a geodesic of the curved spacetime geometry (i.e., taking advantage of the mentioned identification, one can describe gravitational notions in terms of metric variables.) Indeed, the geodesic equation equates the coordinate acceleration to a geometry-dependent term (and, of course, that doesn’t depend on the mass of the body) that, if the geometry is curved, cannot be transformed away globally. If the geometry is flat, it can be transformed away globally, and, in this way, the non-accelerated spacetime trajectory of a free particle in flat spacetime is also a geodesic. Thus, absolute inertial motion is related to absolute geodesity, i.e., it depends on the geometry of spacetime; but, since, in General Relativity, geometry is a property of the Gravitational Field, then, in a rather curious twist, Newton’s absolute substantial space, the one that defined inertial motion, was just the Gravitational Field! \textsuperscript{2} (note that a geodesic can be seen as a spacetime trajectory of constant and non-zero 3–velocity in one, perhaps local, inertial frame and zero in another, that is, inertial motion is not defined

\textsuperscript{2}Proto-physically, time duration is related to a strict ordering in the successive events that comprise this change, i.e., it’s yet another new layer in the concept of time (as distance is in space.) That is, the set of all points of space is a different thing and prior to the metric one puts into it. For the set of points itself, we take the emergent/relation view. In this sense, one thing is mutability and change, and other very different is the duration of this change; analogously, one thing is separation and other is the actual distance that separates two points. In both cases, mutability-change and separation are previous to, respectively, duration and distance.

\textsuperscript{3}Newton and Leibniz also had different views regarding locality. Newton considered valid the principle of locality or individuation by separation, while Leibniz didn’t (since he believed that two indistinguishable things must actually be the same; this even if they are separated, because, being a relationalist regarding the nature of space, he doesn’t consider the different positions of the particles as a property that distinguishes them, since position would not be an intrinsic and individual property of each particle, like, say, spin, but an emergent property of the system of two particles.) Not that there seem to be some similarities with the non-locality established by quantum entanglement.

\textsuperscript{4}That is to say, the geometry that was always studied, both from the physical and mathematical point of view, was always really the Gravitational Field itself. When Riemann mathematically studied geometry, we could say that, in retrospect, what he was studying was simply the mathematical structure of the Gravitational Field. And this because mathematical geometry always had its basis in physical geometry, instead of being a mere mathematical abstraction. In this way, and clarifying the fact that behind it is the Gravitational Field, we will call geometry to all properties of this field having to do with issues such as areas, volumes, durations, and related concepts, such as the Levi-Civita covariant derivative, the curvature, etc.

\textsuperscript{5}In the Ashtekar variables $[A^i_a, E^a]$ formulation of General Relativity, one can see this in an explicit way since the area of a surface or region $S$ is given by $a_S[E^a] = \int_S \sum_{j=1}^3 E_j^a m_a m_b e_S$, which is, clearly, a functional $a_S : M \rightarrow \mathbb{R}$ on the phase space $M$ of the field described by it to the real numbers, that is, an authentic property of the field.

\textsuperscript{6}The field equations predict gravitational waves, i.e., perturbations in the gravitational field, and, therefore, in spacetime geometry, that travel through space carrying energy extracted from the system which emitted them. Thus, according to this theory, the gravitational field indeed has energy and therefore it’s matter.
here in relation to movement with respect to some absolute space in absolute rest, as Newton conceived it and which is its weak point, but in terms of geodesity, which is unaffected by a change of frames, that is, it’s compatible with the absolute relativity of velocity; what connects this with Newton is that he perceived that a new substance was needed to define inertial motion and he just postulated it in the manner and the means which were accessible to him at his time, but which later resulted to be the Gravitational Field in the more precise, both mathematically-conceptually as well as philosophically, description of General Relativity, which was informed by later discoveries in experimental-theoretical physics as well as pure mathematics.) Furthermore, there’s actually a phenomenon in General Relativity, called frame-dragging, in which the geodesics are dragged along the rotation of a very massive object; in this way, the notion of inertial frame is also being dragged, hence the name, and this is possible because this notion is really dependent on the state of the Gravitational Field around the massive object (which affects this field.) This effect has been experimentally observed through gyroscopes in orbit around Earth.

In this way, by absorbing Newton’s absolute spacetime into the Gravitational Field, General Relativity seems to suggest or call for a relationist theory of spacetime. Thus, the ontological picture one gets in General Relativity is the following: the elements of $\Theta_{GR}$ are only fields (gravitational and non-gravitational), from the Reality given by $\Theta_{GR}$ emerges only fields as a relational structure, and, furthermore, this structure carries a metric which is actually a property of one of these fields (the Gravitational Field, of course.)

To make a precise theory that explains how space and time emerge from Things, and what their physical meanings are, is a task for proto-physics. And, indeed, the relational sketch we made here can actually be made more precise. We only consider all the atomic or minimal Things, that is, those Things which cannot be further decomposed as the aggregation of other Things. First, we assume Mutability, that is, the state of a Thing has more than just one state (in this way, a process is a change from one state to another, and, thus, a change in the values of the properties of the Thing.) Before continuing, note that, since the actual Universe is just a single one and fixed, the set of all Things is also one and fixed, since it represents that Universe; furthermore, the proper historyootnote{Note that there’s ontological continuity between Newton’s theory and General Relativity via this reinterpretation of the former’s absolute space. This type of continuities support the notion that there’s a Reality composed of elements and properties and which we can discover in more detail as our investigation of it advances.} of each of these Things is not a generic, variable one, it’s also one and fixed once and for all. Now, if we assume as ontological hypothesis that the states of a Thing along its proper history $h$ have a partial order $\leq$ and that this order is continuous (that is, if $h$ is divided into two subsets $h_p$ and $h_f$ such that every state in $h_p$ temporally precedes any state in $h_f$, then there exists only one state $s_0$ such that $s_1 \leq s_0 \leq s_2$, where $s_1 \in h_p$ and $s_2 \in h_f$), then it’s known from standard analysis that there’s a bijection $T : h \rightarrow \mathbb{R}$ such that $h_1 = \{s(\tau) / \tau \in \mathbb{R}\}$, $s_0 = s(0)$, and $s_1 = s(1)$, where the unit change $(s_0, s_1)$ is arbitrary. Thus, we get a metric on $h$ and this metric is the proper duration or proper time. What we have done here is to derive (and, in this way, give physical meaning) the notion of metric duration from ontological axioms about Things and properties of their set of states. Now, assume that any pair of Things interact. Suppose Thing $x$, whose history is $h(\tau_x)$, interacts with Thing $y$, whose history is $h(\tau_y)$, that this action starts from $x$ at $\tau^{x}_0$, then reaches $y$ and disturbs it, and then reflects back to $x$, which gets disturbed at $\tau^{1}_x$ when the reflex action reaches it. We postulate that for any two separated things, there always exists a minimum positive bound for the interval $(\tau^{1}_x - \tau^{0}_x)$ defined by the reflex action after considering all the possible types of interactions. We define that $\tau^{0}_y$ is simultaneous with $\tau^{1/2}_x = \frac{1}{2}((\tau^{1}_x + \tau^{0}_y)$. Of course, this means that Things $y$ and $x$ in their states at $\tau^{0}_y$ and $\tau^{1/2}_x$, respectively, are Separated. We now define a distance $d$ between $\tau^{0}_y$ and $\tau^{1/2}_x$ by $d(x, y) = \frac{\epsilon}{2} | \tau^{1}_x - \tau^{0}_y |$, where $\epsilon$ is said to be the distance between $x$ and $y$.

\footnote{Although, of course, the substantivalist could still claim that there seems to be some sort of “asymmetry” regarding the respective roles of the Gravitational Field and the other matter fields, the former being related to the metric geometry of spacetime, and even serving as some sort of “background” (more on this below) due to its special appearance in the equations of the other fields via the Levi-Civita connection of the derivative, while the other fields do not seem to be related to geometry in this way. But the relationist could counter that the Gravitational Field carries energy, and so it’s definitely a type of standard matter substance, rather than a new kind of substance, and that the Levi-Civita connection is just a part of the very complex way in which this field interacts with the other matter fields (the other part being, of course, the Einstein field equations.)}

\footnote{Recall that any state $s$, in this framework, is such that all properties of the Thing have defined and fixed (for that state) real values.}

\footnote{Which we take as unique and whose points consists in the Thing in all its allowed legal states once fixed an initial one.}
where \( c \) is some universal constant (we actually postulate that \( d \) is a distance on the equivalence class \( E \) of states simultaneous to \( r_{y}^{0} \)). Geometric space \( E_{G} \) is the completion of the metric space \((E, d)\). Thus, \( E \) is dense in \( E_{G} \), that is, it’s a plenum (which means that every open neighbourhood in \( E_{G} \) of a Thing in \( E \) contains other Things from \( E \).) Thus, since we already had a physical meaning for the time duration, this induces a physical meaning for the spatial distance \( d \). Note that the metrical time \( \tau \) and the distance \( d \) are, again, one and fixed once and for all (that is, we don’t consider a given, fixed set of Things and the possibility of having different metrics on it, since that’s just a mathematical device, which only occurs in the human mind, not in the Universe which is just one and is the thing we are trying to model with all this.) In this way, from the set of Things and their states we get a relational theory of space and time with well defined physical meanings for time durations and spatial distances, meanings which ultimately are only intrinsically dependent on the physical interpretation of the Things themselves (which is a primitive notion) and their states.

Thus, the standard notion of spacetime can be abstracted from the set of all Things and their relations of separation and mutability (note that we don’t include change, which is not modeled by, e.g., General Relativity; instead, one derives conclusions from this theory about the basic separation-mutability relations and, when living in reality, one simply eventually experiences them but with change shaping the actual way in which we do it, in particular, as a novel element of which the theory is silent, since it can talk about processes, evolutions, etc., but always studies them in the whole, the full 4-dimensional manifold, field, the full worldline, etc., the actual change is absent; this doesn’t mean that the theory says that change is “unreal”, it simply never entered into the things it describes in the first place; change may be or may be not real, but General Relativity is not the theory that will tell us which option is true; see the final paragraph of this section for more about this.)

The semantic formulation of General Relativity would be as follows: at the mathematical side we have \( \langle M, g_{ab}, F_{a_{1}...a_{n}}^{(i)} \rangle \), where \( M \) is a 4-dimensional smooth manifold (with the usual topological requirements), \( g_{ab} \) a smooth lorentzian metric, and \( F_{a_{1}...a_{n}}^{(i)} \) a set of tensor fields, where the triple in question is a fixed solution to the Einstein field equations \( G_{ab} = \kappa \sum_{i} \tau_{ab}^{(i)} \) (this in order to match the previous notion of a single and fixed Universe), while, at the physical side, the set of all Things \( \Theta \), their set of legal states \( S_{L} \), any Thing \( x \) with its history (which can be considered to be an infinitesimal portion of a physical field evolving in time and proper duration \( \tau_{x} \), and the different simultaneity spaces \( E \). The semantic map \( \varphi \) takes \( p \in M \) to \( x | s \) (that is, Thing \( x \) in state \( s \in S_{L} \); or, more precisely, just the state \( s \) itself of Thing \( x \); see next paragraph), \( \varphi(p) = x | s \), a timelike curve \( \gamma \) to the history \( h \) of Thing \( x \), \( \varphi(\gamma) = h \), the number \( \tau[\gamma] \equiv \int_{\gamma} \sqrt{-g_{ab}(\frac{dx}{d\lambda})^{a}(\frac{dx}{d\lambda})^{b}} \) d\( \lambda \) to the proper duration \( \tau_{x} \) of \( h \), \( \varphi(\tau[\gamma]) = \tau_{x} \), and the distance of the riemannian metric \( E_{ab} \) induced by \( g_{ab} \) on \( \varphi^{-1}(E) \) to \( d \). Thus, General Relativity is an exactification of our relational theory of spacetime 11.

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11 In fact, change seems to be rather tricky to define in a non-circular or tautological way, since, no matter how one frames it, it always tends to sound like something in the lines of “change means that something changes from this to that”, as if change were something that can only be experienced.

12 General Relativity is a field theory. A field as a whole is a single thing instead of several, nevertheless, we postulate, and can think of it, as the total union of infinitesimal portions of the field in different regions of spacetime and, in this way, we actually have an uncountable infinity of things in this universe. What this actually means is that the mere existence of a field of any kind gives rise, via these axioms, to the emergence of spacetime by the very nature of the field as something infinitely divisible into minor things or portions of it. The fact that actions take a while to go from one portion to another is then implemented or reflected in the hyperbolic nature of field equations in General Relativity. With this we can also see more clearly why one has to choose a single solution to build the spacetime. Consider, for simplicity, that a minimal field portion is a field portion of the whole field on a small compact region \( U \) of spacetime and build the phase space of this portion. If, instead of considering the state of this portion only for a given solution, we consider “the state of the field”, then we would need to consider the whole of that phase space, which, for example, contains an infinity of field configurations whose support is the compact region \( U \). But, of course, we want just one state per one of these small compact regions (imagine, for simplicity, that they are the points of the spacetime), since those states will be the points of the relational spacetime which have to coincide, for each solution, with the manifold to which the region \( U \) belongs (see the paragraph in this section about the “problem of time” for an additional clarification and a more precise mathematical formulation.) This and the desire to fit a single Universe is what motivates the assumption in question, that is, the Universe is not a “phase space” which encompasses all possible solutions; instead, only one of these solutions can describe the single Universe in which we live.
So, how do we build explicitly this spacetime in terms of fields, then? In general, one just takes the Gravitational Field for this. Indeed, one uses a coordinate chart to characterize the points in $M$ (when there’s an interpretation, we will do a slight notational abuse and write $M$ instead of the correct $\varphi [M]$), and then the proper times and distances of the metric $g_{ab}$ to give physical meaning to these coordinates (the proper time of a timelike curve is taken as its parameter and this parameter is taken as a time coordinate, and the same with spacelike curves, distances, and spatial coordinates; note how it’s a property of the Gravitational Field, the proper time, the thing that allows us to distinguish different states separated by the temporal partial order via the different values that this property takes on these states); but, since the metric is actually a property of the Gravitational Field (which is, of course a matter field), what one is doing with this process is to characterize the points $p$ via the values of properties of this field, i.e., via states (of portions) of the latter (and, actually, properties for which we have explicit physical interpretations.) Note that a key property of spacetime is that the solutions for all fields are fixed and that all fields superimpose with each other on the same point of spacetime for all points (since they permeate all of space), and this is modeled by the interposition operation $\times$ at the ontological level; thus, we also postulate in our set of axioms that a point $p \in M$ is actually given by certain equivalence classes $p = [s_{F(i)}]$ of field states (once spacetime emerges, the equivalence manifests itself as the standard superposition principle of fields.) This gives us liberty for choosing any of the fields to characterize space and time via their states, and this is the origin of the notion of spacetime as a manifold, i.e., a set of points which can be mapped to $\mathbb{R}^n$ in different, though relatable, ways; of course, this is reflected in the liberty for choosing different coordinate systems, which are nothing more than mathematical abstractions from the fact that a given spacetime point can be characterized, in different but equivalent ways, by the values of the properties of different fields (and how many values per point is what will determine the dimension of the spacetime; of course, we postulate that this dimension is equal to 4); this, in turns, leads us to postulate the principle of physical (this when solutions are fixed; more on this below) general covariance, which states that the equations, and physical information, of the fields cannot depend on the choice of a particular coordinate system, thing which leads us to formulate them with tensor fields and other intrinsic geometric objects. Nevertheless, note that fields do not “inhabit” spacetime: they produce it. Fields exist by themselves, in an autonomous and absolute fashion, even when, for convenience, one represents them mathematically as functions on spacetime, this is just a mathematical representation that one shouldn’t take literally and translate it to the physics in such a naive way, as if mathematics were a perfect and transparent mirror of physical reality: it’s not, since, as we saw, one can consistently build a relational proto-physical theory of spacetime starting from Things. Note that the functions of the form $f : M \rightarrow \mathbb{R}$ are properties of minimal portions of the field, since at each state $M \ni p = [s_{F(i)}] = (t, \vec{x})$ we get $f(t, \vec{x}) \in \mathbb{R}$, that is, associated to the state we get a real number which is just the values of the properties in that state (but note that $f$ encompasses properties of all the minimal field portions, since its domain includes all of space; for the portion that evolves along the curve given by $\vec{x} = \vec{x}_0 = \text{const.}$, then $f(t, \vec{x}_0)$ indeed represents the value of a property for that portion as the latter evolves); due to general covariance, the point $p \in M$ could be constructed using another field, and, therefore, the same function $f$ also represents properties of portions of this other field; thus, the algebra $C^\infty (M)$ can be seen as the algebra of properties of all the minimal field portions of any of the matter fields (bear in mind, though, that this algebra is not the algebra of functions on the phase space of the whole field, where each function there actually represents a property of the whole field and also their domain comprises all the possible solutions.)

Note that the notion of a given fixed spacetime $(M, g_{ab})$ but with a matter field $F_{a_1 \ldots a_n}$ on it, considered

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13 When we build the spacetime using a field, we are actually postulating that each minimal sub-Thing, of the infinitude whose aggregate forms the whole field, is allocated at different spacetime points, and this implies the standard notion of the field as an extended entity (and also separates points in $M$) in spacetime. Now, since all portions have analogous properties (being the same type of entities), then we can consider the same property (say, the intensity) and see how it varies when we pass from each minimal portion to another. Thus, this is why (once we choose, as just mentioned, another one of these fields and its states as a reference field for characterizing the spacetime points) we represent fields as functions on $M$ (since the two fields will generally have different values for their properties on the same spacetime point on which their states superimpose, and, therefore, if we use the values of the properties of one of the fields as coordinates for that point, the values of the properties of the other field will be represented as functions of those coordinates, since the latter also characterize the state of this second field, but by values other than the ones of its properties on that state; this is also the case for the metric, since the values of the field $g_{\mu \nu} (t, \vec{x})$ do not directly represent proper times and distances.)
as a variable for its matter field equation and for which one studies the different solutions (all over the same \((M, g_{ab})\), is not something to which we can give a physical interpretation, since it only exists in the mind (the Universe is a single one and fixed) and is just a mathematical device for studying the matter fields at different sub-regions of the whole spacetime (since, in that case, the solutions can indeed be different in each patch; these solutions would need to approximate the actual full solution, particularized to each patch, of the Einstein equations for the Universe) and when the interaction between gravity and the rest of the fields can be ignored.

In retrospective, the only thing that deserves to be called spacetime, and as a structure common to all the matter fields, is just the set \(M \ni p \ni [s_{F^{(i)}}]\) and its dimension, that is, the bare differentiable manifold. Distances, durations, temporal partial order, causality, and the distinction between mutability and separation, are simply mere properties of the Gravitational Field. Nevertheless, these properties result manifold. Distances, durations, temporal partial order, causality, and the distinction between mutability and separation, are simply mere properties of the Gravitational Field. Nevertheless, these properties result useful for characterizing \(M\), since one can easily associate them to curves, have physical interpretations which in experience are perceived in a direct and obvious way by us, and the fact that it’s possible to consider a fixed Gravitational Field as “background” to study the dynamics of the other matter fields (when the backreaction can be ignored), while the converse seems more complex since it doesn’t seem to be very clear how one is supposed to choose a solution for the other matter fields as background due to the now variable metric being part of the non-gravitational field’s equations themselves (see next paragraph for more on this point); as we mentioned before, this is actually the only “asymmetry” that remains between gravity and the other fields. Note that in our relationist theory of how spacetime emerges from Things we used the metric time to characterize space (both for the notions of separation and metric distance), but this is legit since, due to the fact that we are considering the real spacetime which is one and fixed, all the solutions and notions are fixed and then we can use one aspect from one to characterize another notion or viceversa (in this case, since we already have physical interpretations for the states in the Thing’s history, then it’s natural to introduce the partial order since, because of this, it naturally, in a transitive fashion, acquires a physical interpretation and so on.)

The case in which both the spacetime metric \(g_{ab}\) and the other matter fields are variable and unspecified is even less passible of a full physical interpretation, and the manifold \(M\) becomes just an arbitrary abstract thing (the coordinates too, since one cannot interpret them in terms of proper times and distances of the variable metric because one uses those coordinates to study the very variability of the metric via the field equations), whose only purpose is to remind us of the fact that, whatever the solution is, it must imply mutability and separation; this gives rise to the so-called “Problem of Time” in canonical quantum (but also in classical) gravity. We cannot associate this situation to the real spacetime, which is one and fixed, it’s just a (very useful) mathematical fantasy that only exists in the mind. The manifold and its coordinates remain here as residues, product of abstracting them from the real ontology but for then artificially give them an independent status from this ontology in our mathematical models. Note that we defined spacetime as \(M \ni p \ni [s_{F^{(i)}}]\), i.e., mathematically, it’s the manifold and a solution for the fields \(F^{(i)}\). The standalone, “abstract” manifold, and, in particular, on which many different field solutions are considered, is a mathematical fantasy and it doesn’t mean a “single and fixed real spacetime” on which these different solutions live (of course, assuming the case of all fields variable, not the case with a background, fixed metric.) Instead, it means that the mere mathematical structure of the real spacetime associated to each of those solutions is the same for all of the latter (in particular, that’s why one can write two different of these solutions on the same “abstract” manifold, but where each solution must have different property values for the same “abstract” manifold point, since it’s the same “abstract” manifold structure but for different field configurations and real spacetime, and therefore different physical field values are associated to a same “abstract” manifold point, otherwise they would both be just the same solution and real spacetime.) In particular, this is why, in the definition of a spacetime singularity, the points of the manifold on which the curvature diverges are not considered to be part of the spacetime, since no Gravitational Field can be defined on them, and, therefore, they would be points of a mere mathematical, “abstract” manifold. In a more precise way, consider the (kinematical) phase space

\[
X = \{ [h_{ab}, \pi^{ab}] / h_{ab}, \pi^{ab} \in C^\infty(\Sigma) \mbox{ (as fields on } \Sigma) \}
\]
of GR (where $h_{ab}$ and $\pi^{ab}$ are, respectively, a smooth riemannian metric on a Cauchy surface $\Sigma$ in a compact, boundaryless $M$ and its conjugate momentum tensor density) and the set of properties $C(X)$ of the field (note that these are properties of the field on the whole spacetime and that their domain contains points from any solution, these are phase space properties.) We start with space, since time will require more considerations. Consider now the subset $F \subset C(X)$, of properties which are of the form

$$F_f ([h, \pi]) = \int_{\Sigma} f(\epsilon(h), \forall [h, \pi] \in X,$$

where $f \in C^\infty(\Sigma)$ and $\epsilon(h)$ is the volume element of $h$, i.e., $\epsilon(h) = \sqrt{h} d^3x$. Evidently, for a field portion in a volume $V_1 \subset \Sigma$, the value of this property is

$$F_f ([h, \pi]) = \int_{V_1} f(\overline{x}) \sqrt{h} d^3x,$$

and for an infinitesimal field portion at $\overline{x}$, simply

$$f(\overline{x}) \left( \sqrt{h} d^3x \right) |_{\overline{x}}$$

(see here as a 4-form $f(\overline{x}) \epsilon(h) |_{\overline{x}}$, so that it’s actually mathematically well-defined; note, though, that there’s a 1-1 correspondence between $f(\overline{x}) \left( \sqrt{h} d^3x \right) |_{\overline{x}}$ and the actual value $f(\overline{x})$, since the functional form of the volume element is known and fixed and the only arbitrary variation is in the function $f$, and, in this way, we can consider these latter finite values, the volumetric densities at $\overline{x}$ rather than the infinitesimal quantities.) Thus, for a given phase space point $[h, \pi]$, the values of those properties for an infinitesimal field portion at $\overline{x}$ are given by the values of the functions $f(\overline{x})$; but, since all the functions $f$ cover all the possible smooth functions on $\Sigma$, which we know should indeed correspond to all the possible properties of the infinitesimal field portions for the spatial part of a solution, this means that $\overline{x}$ characterizes the values for all the possible properties of the infinitesimal field portion, that is, the generic state $s$ of this portion is identified with the generic space point $p \in \Sigma$ of coordinates $\overline{x}$. In this way, if we relationally build space via the states $s$ of infinitesimal gravitational field portions using this phase space, we get, for any solution, an identification of relational space with the initial manifold $\Sigma$ that was assumed to represent space once a solution is chosen (of course, $\Sigma$ is interpreted as the physical space only after a given solution is considered; note that the manifold is there all the time since it was assumed from the onset that space would have that type of mathematical structure and not because everything happens on a “background” space.) Furthermore, one can intuitively see that, if the phase space functions vary over all possible phase space points, and if space points are states of infinitesimal portions of the field for a given solution, then a phase space property $F_f$ whose domain of integration is restricted to all the different infinitesimal portions of the field for a given solution will give rise to a space function in the space algebra $A_{sp.} = C^\infty(\Sigma)$ for the space that relationally arises from that solution. That is, in this view, the space functions, rather than being physically independent, actually come, or are derived, and acquire physical and mathematical meaning from phase space functions (which are taken now as the fundamental objects) via certain process once space relationally arises when a solution is fixed (the value of the space function on an event is actually the value of a property for an infinitesimal field portion in the state identified with that point; of course, this may seem trivial and that we put the space functions there by hand, but the relevant thing here is the reversion in the logic: in the usual approach, we have the space functions, which exist independently from the fields, while, here, without the physical field, no space function can arise in the first place.) Now, since, of course, we can consider more general phase space functions too, for the Gravitational Field, length can also be seen in this way (that is, as a property of infinitesimal portions of the field, as claimed before), since it’s a functional of the form $l \left[ (\gamma; h_{ab}) |_{\gamma} \right]$ with the curve of the coordinate $x_1$ taken as $\gamma$, and therefore

$$\tilde{l}(\gamma) \left[ h_{ab}, \pi^{ab} \right] (\overline{x}) \doteq \sqrt{h_{11}(\overline{x})} \delta(x_2, x_3) \frac{1}{\sqrt{\epsilon(\overline{x})}}$$

is an authentic (distributional) density, whose integral is the value $l \left[ (\gamma; h_{ab}) |_{\gamma} \right]$ itself. This is why General Relativity is special and much more revolutionary than what it already seems at first: in its phase space
formulation, and unlike other non-gravitational theories, the manifold $M$ used there no longer has physical meaning as spacetime, since all fields are unspecified and therefore no a priori solution to any physical field can be used to relationally build the spacetime, even the manifold $M$ itself.

Thus, one should be very careful in making literal interpretations of the mathematics. When there are fixed solutions, they indeed relate to physical spacetime and the coordinates have physical interpretations, since they ultimately refer to the values of the properties of the matter fields (in these solutions.) But, in the variable metric and other fields too situation, one loses this, the coordinates become mere abstract mathematical parameters and the general covariancem of the gravity field equations with respect to them now looks very similar to the gauge symmetry in, e.g., electromagnetism, where $A'_a = A_a + \partial_a \chi$ (here, both $A'_a$ and $A_a$ give rise to the same $F_{ab}$, since $F_{ab} = \partial_{[a} A_{b]} = \partial_{[a} A'_{b]}$, with $\chi$ being an abstract scalar field without physical interpretation.) We refer to this second case as gauge general covariance, as opposed to physical general covariance. In the case of a fixed metric with variable non-gravitational matter fields (and no backreaction being considered), we have physical general covariance of the non-gravitational matter fields equations, but, since the coordinates have physical interpretations thanks to the fixed metric solution, the invariance of the non-gravitational matter field action under these transformations doesn’t generate constraints like in the case for the gravitational action. That is, the link (invariance of the action) $\implies$ (constraints) is broken here, due to the fact that the coordinates are not mere abstract, gauge mathematical parameters for this situation, and, actually, we can get non-trivial Hamiltonians/metric time evolutions. The mathematics of the models indeed recognizes these differences, since, in the fixed metric case, there’s no implied intrinsic variability of the metric from the part of the non-gravitational actions (the metric is just an external thing to this action), while metric variability is, of course, intrinsic in the gravitational action; the proof that this implies, respectively, the validity or not of the link (invariance of the action) $\implies$ (constraints) is shown explicitly in the actual calculations, where in both cases the actions are invariant under the coordinate changes, but only in the second case we get constraints associated to these invariances. The Problem of Time could be partially solved if one can make sense of the case with variable metric but with a fixed non-gravitational matter field solution (and no backreaction being considered), since one could study the gravitational variability by parameterizing the points of the real spacetime $M$ with the values of the properties of these other matter fields (although, of course, one cannot interpret the resulting “time” as duration, since that’s a property of the Gravitational Field, and, instead, the interpretation is as whatever the properties of these other fields are.\footnote{\footnote{\textsuperscript{14}}}{\textsuperscript{14}} But, in the case with metric and other fields taken as variable, the resulting formalism is pure mathematical fantasy, which, of course, cannot have any time in it, since that formalism doesn’t have any relation to real spacetime (although, the formalism in question is the only way we have to express the equations of gravity.) The only possible interpretation in this case is in “potential” terms, that is, if all the solutions maintain a same peculiarity (e.g., a given hypersurface is spacelike for all the solutions, or the fact that for any solution $M$ will refer to mutability and speration.) Nevertheless, all these interpretations are only potential, in the sense that, eventually, once a solution is fixed, the actual interpretation holds; the formalism itself has no true, actual physical interpretation in this sense. Thus, in the phase space of Gravity, there’s no background spacetime (both in terms of manifold and metric), and this leads to a Hamiltonian $H_{EH} = \sum_i N_{(i)} C^{(i)}$ given by the constraints $C^{(i)}$, which, in turn, leads to gauge invariant phase space properties $F$ (i.e., the ones that commute with the constraints, $\{ F, C^{(i)} \} = 0$) that lack an adequate time evolution (since $\frac{dF}{dt} = \{ F, H_{EH} \} = \sum_i N_{(i)} \{ F, C^{(i)} \} = 0$.) The only solution for this problem is to consider a solution $g^{1}_{ab}$ to the gravitational equations, to particularize the gauge invariant phase space function $F$ (whose domain comprises all possible solutions) for $g^{1}_{ab}$, and then make the change of coordinates $t \rightarrow \tau$.

\footnote{\textsuperscript{15}Although, this seems problematic, considering that the metric enters into the equations of these other fields via the Levi-Civita connection, that is, the equations of these other fields depend on the variable metric. On the other hand, since the other fields don’t enter into the pure gravitational action, the latter still interprets that there isn’t any fixed spacetime and that’s why there still are constraints (compare with the case in which the other fields are variable and the metric is fixed.) In general, the only way in which the other fields relate to gravity is via the Einstein equations, and this is done by adding the other field’s actions to the pure gravitational action; but, of course, this would imply the variability of the other fields, since one would be considering their field equations via their action. One could try to put these other fields into the Einstein equations by hand, but, since they are fixed, this would restrict the variability of the metric. Thus, all this seems to point out that this other possibility (variable metric and other fields fixed) is untenable or forbidden by the formalism itself.}
(where \( \tau \) is the proper time according to \( g^{1}_{ab} \)) in order to study the true time evolution (with respect to \( \tau \), of course) of \( F \) and of properties such as \( g^{1}_{ab}(\tau) \), the curvature, etc., and this process would need to be considered as a separate case for each different solution; alternatively, one can start with a solution \( g^{1}_{ab} \), a chart defined in terms of \( \tau \), and then build the natural gauge invariant properties, such as the curvature. In any case, the solution, and therefore the physical spacetime, is always needed (note that, as mentioned before, the field has to be the gravitational field and one cannot simply consider any other field as fixed for doing this, since, due to the special role of the gravitational field as the one that gives the spacetime metric, if this field is taken as variable, then all other fields must be taken as variable too.) Thus, to obtain an actual time, one needs to fix a solution. In phase space, the metric is the intrinsic variable, and therefore no solution is assumed, this is why the “time” evolution there is only gauge, and if one goes to the quantum theory with this picture, this gives the impression that there’s “no time” in quantum gravity. But this is just an artifact of the phase space picture, and can be solved in the classical case in the way we mentioned above. The real problem in quantum gravity is the one related to the continuum. But if we don’t have continuum, we don’t know how to pass to the spacetime picture, which in the classical case is something as trivial as just picking a given solution, and then we are stuck in the phase space picture with no time! This is the real core of the problem of time once we are in quantum gravity. To solve it, one needs an analogue of the spacetime picture, but, of course, without the continuum. Note that a fully diffeomorphism invariant property is not necessarily a gauge invariant phase space property, since the symmetries implemented in phase space by the constraints do not exhaust the full diffeomorphism group of the manifold \( M \); among those properties are, for example, the spacetime volume; it’s often argued that the “real” properties are the gauge invariant phase space ones; but, again, this is another artifact of the phase space picture: those other properties are certainly not gauge invariant phase space properties, because the phase space picture is artificially restrictive, but make perfect physical sense as properties in the spacetime picture (furthermore, one doesn’t calculate their time evolution via phase space, but, as mentioned above, via the spacetime picture: time evolution is the specific change with respect to a duration.)

In order to define a coordinate and, in this way, to characterize in a concrete (and, of course, relational) way a point of the “abstract” manifold, consider the metric area of a surface \( S \subset \Sigma \), sub-manifold of \( \Sigma \), that is,

\[
a^{S}(\{h, \pi\}) = \int_{S} \epsilon(q), \forall \{h, \pi\} \in X
\]

(where \( q \) is the metric induced in \( S \) by the metric \( h \) in \( \Sigma \).) This is done, for example, when using spherical coordinates in a metric \( h \) with spherical symmetry, where the radial coordinate of any point \( p \) in a topological sphere \( S \) is defined as\[r (p) = \sqrt{\frac{1}{4\pi} a^{S}(\{h, \pi\})}\]

Now, as we move to bigger spheres, \( r \), of course, just monotonically increases, and, in this way, this property of the field separates the points in \( \Sigma \) and gives them physical meaning, since it assigns to each of them a different value (along a curve of fixed angular coordinates, of course; note, too, that for being able to do this, a manifold structure on \( \Sigma \) is assumed in the first place.) This process is indeed the relational way we use everyday in GR to define the points of the manifold (since we use the values of properties of the Gravitational Field to characterize them), but it’s too tied to the classical structure of the theory for being of any general use.

Indeed, consider the quantization of the phase space property \( a^{S} \) in LQG. There, the differentiable manifold \( \Sigma \) is maintained, and \( a^{S} \) is promoted to an area operator, \( \hat{a}^{S} \). The eigenstates of this operator are the so-called spin network functions, the simplest of which is the Wilson loop of spin \( s \in \frac{1}{2} \mathbb{N} \bigcup \{0\} \). If this loop punctures the surface \( S \), then its area eigenvalue is proportional to \( \sqrt{s(s+1)} \). If we now consider different spheres, as in the classical case, then the area eigenvalue will still always be the same if the loop still punctures these other spheres too. But, since, in a relational approach, these values give physical meaning to the points of space, this means that all these different spheres in \( \Sigma \) are physically the same. Thus, there’s\[\text{To see it as a property of an infinitesimal field portion, i.e., as a density, proceed as in the case of the length.}^{16}\]
an obvious conflict between the physical meaning of this operator and the surfaces to which it supposedly gives their area. In the most benign case, this may indicate that this particular mathematical incarnation of the formalism is just very awkward, but in the worst case, that it's actually a wrong approach. The more sensible position seems to be the latter one, since, as was mentioned, the manifold is a direct consequence of the assumed continuous range of classical properties. Thus, one cannot just dispose one and keep the other, since the coordinate charts are just a mathematical abstraction of the values of field properties; if the values of field properties are discrete, then no classical charts and, therefore, no classical manifold. In this way, since this formalism contradicts itself from the physical point of view, one should completely reformulate the approach in a way that avoids this type of problems.

In addition, and from a relational point of view, there's also a second issue with the phase space functionals for geometrical properties, like the previous $a^S$, which is the following. In a relational theory, only after a solution $g$ is selected does the manifold $\Sigma$ (and the surface $S$) acquire physical meaning. This means that only the spacetime picture version, $a^S([h, \pi])$, of the phase space functional $a^S$ can be given the interpretation of area of $S$, since in that case $S$ has a physical interpretation as a surface in physical space, while the phase space functional cannot be given the mentioned interpretation because no solution is selected, and, therefore, the $S$ in it is just an abstract parameter. In this way, if we go to the quantum theory, the eigenvalues of $\hat{a}^S$ cannot be strictly interpreted as the possible area values for a physical surface $S$ (even when one suspects that this spectrum is on the right track.) Furthermore, if we simply eliminate the continuum from the phase space area functional, we lose all the topological and differential geometric information, which was encoded in $S$: we would be calculating the area of something which is missing! Thus, we conclude from this that a spacetime picture is needed if one pretends to study geometrical properties like areas and volumes: it's only when both metric, $q$, and surface, $S$, meet that they acquire, simultaneously, physical meanings, which are also mutually consistent with each other.

Finally, it's usually said that something exists if it exists at an instant of time in the present. In the context of our ontology, this is misleading. Reality is a single thing and a Thing in it either exists or not, independently of time. If a Thing indeed exists in Reality, then it has a state space with several states accessible to it. Now, as the Thing changes, it's actually its state the one that changes. That is, the thing that disappears, so to speak, as the Thing changes, is the previous state, not the Thing itself. These are the two levels of concepts that shouldn't be confounded (if a given Thing is annihilated, the basic ontological substance is still conserved and a new Thing of a different kind has to emerge, like photons emerge from the annihilation of an electron with a positron.) What presents itself and unfolds in time or change is the Thing in a particular state. Spacetime is the network of spatiotemporal relations between all the states of all Things in Reality. Regarding the relativity of simultaneity in special relativity, the changes that this introduces in the pre-relativistic view on space, time and change are the following. Consider the worldline $\gamma_A$ of a changing Thing $A$. As the Thing changes, we reach the event $e \in \gamma_A$ at the present moment. In pre-relativistic physics, one can establish what are all the events that are simultaneous with $e$, what events are in the past of this absolute surface of simultaneity and which ones are in its future. This exhausts all the events in spacetime. Events in the past are taken as non-existent anymore, events in the simultaneous present as currently existing, and future events as still not existing. In this way, as the Thing changes, the surface of simultaneity advances. This view is, of course, that of presentism, and is the natural one if one accepts change as something real. In special relativity, one can only establish the absolute causal past $J^-(e)$ of event $e$ and its absolute causal future $J^+(e)$, but no notion of surface of simultaneity, to which $e$ would belong, can be determined objectively, the only present for event $e$ is just the event itself. Nevertheless, one can establish that events in $J^-(e)$ already happened, and not only according to Thing $A$, but also according to any other Thing. For example, if $\gamma_B$ is the worldline of changing Thing $B$ and event $p \in \gamma_B$ is also in $J^-(e)$, then we can affirm that, according to $B$, event $p$ already happened to it (since, once $p$ is in $J^-(e)$, if $B$ emitted a light signal at $p$, this signal hits $A$ in $e$, but, then, when $A$ receives, it concludes that $B$ already left event $p$ behind.) Symmetric notions apply to the causal future. Thus, the causal past and future are objective and absolute: what happened, happened. The so-called block view of spacetime, in which all events somehow always exist, is not necessary. Indeed, it would be necessary if, say, event $e$ that, according to $A$, is present, is nevertheless in the past of, say, $B$ (assuming that the spacetime is causally well-behaved); but, if this were the case, this would be contradictory, since, then, $B$ could be receiving,
in its present, light signals that \( A \), according to its present, has not even emitted yet. This contradiction only dissolves if the present of \( B \) is neither in \( J^+(e) \) nor \( J^-(e) \), but then the block view is not necessary anymore. In this way, for each changing Thing, one can establish a present, composed of a single event in its worldline, and an absolute causal past and future, for which the events in the former are not anymore and the events in the latter still not are, and where any other changing Thing, according to its own present, will agree that those events indeed are not anymore or still not are. Thus, what we get is a picture vaguely similar to that of presentism, in the sense of becoming and absolute (causal) pasts and futures, but where the notion of surface of present is not valid anymore. In the case of a field, the changing Thing is any portion of the field localized along any timelike worldline (this is also valid for the metric tensor field itself once it’s fixed, since in that case one can take the proper time of the worldlines as coordinates, while this is not possible if the metric is considered as the variable, since the whole formalism doesn’t even correspond to the physical spacetime, as we mentioned before.)

1.2 Critique of the Theory in 1.1

While the theory indeed success in giving a purely relational proto-physical theory of spacetime, it has a particularly ad-hoc flavor. Pretty much everything is postulated and the theory, at best, shows that all these suppositions can hold together to give the desired theory, without needing to assume another substance besides matter, aspect which captures the relationalism one is looking for. Let’s recapitulate all the ad-hoc assumptions: 1) Mutability, which basically amounts to postulate as axiom the very essence of time; 2) the temporal partial ordering, which gives rise to durations; 3) the finite speed propagation of causal influences, which gives rise to distance and Separation, the latter being, again, akin to postulate as axiom the very essence of space; 4) the field’s nature as an infinitely divisible and extended Thing; 5) the (differentiable) manifold structure of the set \( M \) and its dimensionality. We will see in the next sections how the mathematical formalism of the spectral standard model can be used to make a proto-physical relational theory of spacetime in which all of the mentioned points 1),...,5) can actually be derived rather than pressuposed.

Note that the mathematical model of the spacetime as a manifold was actually already included in the proto-physical theory that was built in the previous section, and this was a consequence of the particular assumptions that were made in that theory. Thus, the distinction between \( M \) and \( \varphi [M] \) in the semantic interpretation of the mathematical formalism of General Relativity which we did is more a mere book-keeping device. The true non-trivial content of that semantic interpretation resides in the metric \( g_{ab} \), which is a particular mathematical construct, being interpreted in terms of the durations and distances of the relational proto-physical theory, since those physical notions in the latter didn’t have a precise quantitative meaning, and, therefore, what the semantic interpretation introduces is an actual way for calculating those values, and this will determine a range of consequences, proper of this particular physical theory. Note also that, in this interpretation, the functions in \( C^\infty (M) \) acquire their physical interpretation from that of \( M \) in a transitive way.

But, consider now the following situation. What if we assume the whole mathematical formalism of General Relativity and only give a physical interpretation to \( C^\infty (M) \) and as the algebra of properties of all the minimal field portions of any of the matter fields? If we do this, we can take a point \( p \) from the (abstract) set \( M \) and consider the values \( f(p) \) for some \( f \in C^\infty (M) \). But, since those are values of properties, then this implies that the points \( p \) must be the states of those minimal portions, that is, they acquire this interpretation transitively from that of \( f \) as properties. The assumption in the mathematical formalism about \( M \) being a set with different points then implies Mutability, and its manifold structure implies the other assumptions made in the original proto-physical theory. Indeed, the fields being represented as functions on \( M \) implies now the infinite subdivision property that they have. The abstract metric \( g_{ab} \) induces a partial order in the histories made from points in \( M \), which, given their interpretation as states, is a temporal order and therefore implies physical duration (and also exactified.) The hyperbolic nature of the field’s equations implies the finite speed (and a maximal one, not matter the specifics of the influence) of causal influences,
which, in turns implies the physical Separation of the points in $M$ and implies that the mathematical distance calculated from the metric is the actual physical distance (and, again, also exactified.) Thus, we recover the previous proto-physical theory of spacetime, where now the mathematical assumptions made in the mathematical formalism become the physical or ontological assumptions made in the ontological theory discussed initially.

From the semantical point of view, these two different ways are certainly equivalent, and also note that the “input” information needed is basically the same, since, what was an ontological assumption in the first approach, now must be assumed in the mathematical formalism, and, thus, there’s really no gain in terms of explanatory power by shifting from one approach to the other. The advantage of the first approach is that we can concentrate in the actual aspects concerning to physical spacetime without making any compromise with a particular exactification, and, then, this adds philosophical clarity to the discussion and also means we can use this proto-physical theory to interpret physically other physical theories as well (say, the space and time notions in classical mechanics.) On the other hand, the advantage of the second approach is that the ontological theory of spacetime is now suggested or derived from the mathematical formalism of a given physical theory, and, since science evolves and corrects itself, this seems more convenient because the ontological theory will be up-to-date with the latest developments, while the first approach risks at getting trapped in assumptions that are later discarded by more up-to-date physical theories. Thus, the adequate way of proceeding is actually the following. One should take a given physical theory and study the implied ontology, and then abstract the latter in a separate proto-physical theory in order to achieve more philosophical clarity. Evidently, the relational proto-physical theory of spacetime discussed in the previous section was abstracted from General Relativity in its current differentiable manifold mathematical formulation, in which most of the intuitive notions we associate to space and time are put in by hand into that theory, since they do seem to be fundamental, at least at the level to which this physical theory operates.

### 2.1 Spacetime Relationalism and the Spectral Standard Model of NCG: The Spectral Standard Model and NCG

Consider a compact, $n$-dimensional spin manifold $M$ and its associated Dirac operator $\mathcal{D}$ (which is the covariant derivative on sections of the spinor bundle, i.e., the spinor fields.) It’s known that such spinorial structure gives rise to a riemannian metric $g_{ab}$. Using the volume element $\varepsilon = \sqrt{\det g}$ of this metric, we can build the space $L^2(S_n)$ of square integrable spinor fields and in this way consider the canonical spectral triple, defined as $(L^2(S_n), C^\infty(M), \mathcal{D})$. One can also show there that $\text{tr}^+ (f | \mathcal{D}^{-n}) = k_n \int_M f \varepsilon$, where $f \in C^\infty(M)$ and $k_n$ is a constant. Then a general spectral triple is defined as $(\mathcal{H}, A, D)$, where $\mathcal{H}$ is a Hilbert space, $A$ is a sub $\ast$-algebra of bounded operators (not necessarily commutative), and $D$ a self-adjoint operator that mimics some properties of $\mathcal{D}$; the general or “non-commutative” integral is defined as $\int a \triangleq \frac{1}{k_n} \text{tr}^+ (a | D^{-p})$, where $a \in A$ and $p$ is certain number. The seminal reconstruction theorem of NCG states that if a general spectral triple satisfies certain regularity assumptions and $A$ is commutative, then there’s a compact, $n$-dimensional spin manifold $M$ such that $(\mathcal{H}, A, D) \cong (L^2(S_n), C^\infty(M), \mathcal{D})$ (and, of course, this is what justifies to see general spectral triples as genuine generalizations of the notion of spin manifolds to the non-commutative case and to call $\int$ a genuine generalization of the notion of metric integration to the non-commutative case) [3].

The key observation on which the spectral standard model is based is the following [3]. For a canonical spectral triple, one gets $\int f D^{-2} = k' \int_M f R \sqrt{\det g}$, where $R$ is the Ricci scalar, that is, the standard Einstein-Hilbert action. Now consider an almost commutative spectral triple based on the algebra $C^\infty(M) \otimes M_N(\mathbb{C})$ (where $M_N(\mathbb{C})$ are the $N \times N$ complex matrices), Hilbert space $L^2(S_n) \otimes M_N(\mathbb{C})$ (the latter equipped with the Hilbert-Schmidt norm), and Dirac operator $D = \mathcal{D} \otimes I$. Then the so-called spectral action given by $\text{tr}^+ (F(D/A))$ (where $F$ is a real, positive even function), which is purely geometric and generalizes to the non-commutative case an action based on a function $F(K)$ (where $K$ is a scalar of the curvature), gives rise to the standard action of gravity coupled to a Yang-Mills field. That is, we get, via non-commutative
geometry, a “purely geometric” unification of the forces. The metric is, of course, given by the Dirac operator $\mathcal{D}$, while the gauge fields correspond to “fluctuations of the metric”, and the gauge symmetries come from the group of automorphisms of the almost commutative manifold (in the commutative case, this group is just the group of diffeomorphisms of the manifold.)

The fact that the spectral action on an almost commutative space allows one to derive the standard gravitational equations as well as the existence of Yang-Mills fields and their equations, all this from a non-commutative version of a purely geometrical action means that the “asymmetry” between gravity and the rest of matter fields that the spacetime substantivalist could claim is completely erased now, and, therefore, this provides a strong argument for substantial monism in physics. Of course, this substance, rather than being “all geometry” is just “all matter”, and the previous unification of the equations means that the dynamics of all the matter fields can be expressed in “geometric” terms, which simply means that all of geometry, both the commutative part as well as the non-commutative one, is nothing more than just a substantial property of these matter fields. The Gravitational Field is only “special” in the fact that the geometry of the usual spacetime $M$ is one of its properties, while the geometrical elements which are a property of the other fields (the fluctuations of the metric) correspond to phenomena that arise due to the finite dimensional algebra part. But this is not surprising since, being geometry always a property of matter fields and being the usual spacetime commutative algebra a part of the tensor product, then its geometry will necessary have to be associated to one of the matter fields. Besides, one could say that there’s truly just one matter entity, since the spectral action is a single one and referred to the algebra of the whole almost commutative spacetime; thus, the rewriting of this as fields on $M$ (the standard commutative spacetime) and satisfying their particular field equations can be seen as a mere convenient way of expressing the total information of this single entity and that shouldn’t be taken very literally in an ontological sense.

2.2 A Relational Theory of Spacetime from the Spectral Standard Model

Here we do something which was already partially explained in section 1.2. We formulate an apparently “new” physical theory by the following axioms. At the mathematical side, assume a commutative spectral triple $(\mathcal{H}, A, D)$ that satisfies the regularity conditions of the reconstruction theorem. We now interpret physically the algebra $A$ as the algebra of properties of all the minimal Things (where the algebra elements for some different minimal Things may be the same), and that the dynamics of these Things is constrained by the spectral action. If we now apply the reconstruction theorem, we get that $(\mathcal{H}, A, D) \cong (L^2(S_n), C^\infty(M), \mathcal{D})$ and $f a = \int_M f a \mathcal{D}$.

In the usual relational construction, the states $s$ are such that the properties $f$ take a value $f(s)$ in that state. That is, the properties are functions of the state and of the state space $M \ni s$, i.e., $f : M \rightarrow \mathbb{C}$. This scheme doesn’t say how many states are, and, therefore, in order to have Mutability and Separation, one must postulate that there are at least two states in $M$. This because the set $M$ is taken as ontologically fundamental and the properties refer to it for their mathematicial definition and implementation, and this means that the structure of $M$ (including its cardinality) always will be fundamental ontological postulates. Nevertheless, even if there’s only one single state $s \in M$, the very definition of property as a function $f : M \rightarrow \mathbb{C}$ already implies the existence of an infinite number of different properties, since there are as many different of those functions as elements in $\mathbb{C}$. Note that this is not an exclusive consequence of taking $\mathbb{C}$ as the quantity-value object, since this just makes that the result is at least a possibility (since, if that set had one single element, then there would be only one single possible function): what makes the result to follow is the definition of the property as a function and the axioms (ZFC) of Set Theory (we will investigate this further in the next section.) Now, by the Gelfand duality, each $s \in M$ has asociated to it an algebraic state given by a Dirac measure $\delta_s : C(M) \rightarrow \mathbb{C}$, $\delta_s(f) = f(s)$, $\forall f \in C(M)$. Thus, $M$ is identified with the set of pure states on the algebra. Now, one could take the algebra and its structure as fundamental and the states as the ones that refer to it for their mathematicial definition and implementation. Furthermore, these algebraic states $\omega : C(M) \rightarrow \mathbb{C}$ also associate values to the algebra elements via the expectation value $\omega(f)$, which, for the previous pure states is just $f(s)$; thus they indeed generalize the notion of state that
was define only in the context of the “states-first” approach via $M$ and the functions on it as properties. In this way, the Mutability and Separation in $M$ translates to the existence of at least two of these algebraic states and these notions are now also valid for the case of a non-commutative algebra $A$, since the notion of algebraic states still makes sense in that context. The set of pure states will only be a manifold in the case of a commutative algebra. This approach is, of course, more natural in quantum theories, where the very formalism of that theory is a non-commutative probability theory which replaces the classical probability theory given by a measure on phase space. Also, since now the states are the functions of the form $\omega : A \rightarrow C$, even if the algebra has one single element, the existence of Mutability and Separation are now a consequence of $C$ having more than one element, the definition of states as functions and the axioms of Set Theory, by a reasoning identical to the previous one. Thus, the change to this algebraic view shows that $C$ having more than one element, the definition of states as functions and the axioms of Set Theory are a sufficient condition for obtaining Mutability and Separation in the implied ontological theory (in the scheme with $M$, these are necessary conditions, but not sufficient ones, since the cardinality of $M$ must still be postulated; note that we are analyzing this only at the level of the most basic structure, namely, Set Theory and functions, and, therefore, we don’t postulate a differentiable manifold structure on $M$ at this point.)

The Dirac operator allows to reconstruct the metric $g_{ab}$, while the spectral action implies the vacuum Einstein field equations for it (the metric is, of course, of riemannian signature, and, therefore, we will have to add as an axiom that what we actually recover here is the action in its Wick rotated form of an originally lorentzian theory; we must do this since we need the hyperbolic character of the equations to recover space; of course, it would be more desirable to have a theory of lorentzian spectral triples and to prove the emergence of the standard model from them.) Thus, the rest of the argument goes as in the second approach to semantic interpretations discussed in section 1.2, and we simply recover General Relativity and its implied relational ontological theory of space and time. Note that the reconstruction theorem alone is not enough, since it would only give the Mutability; we also need the spectral action and therefore the field equations in order to give physical meaning to a part of $M$ as describing physical Separation.

Note that the previous construction, since it uses the algebraic view, is such that the basic object is the one of properties, while the notions of states sits on it, and this allows us to build a theory which includes the notions of properties, states, and symmetries over the basic notion of properties. This is desirable in a relational approach to spacetime, since, given that the basic definition of material Thing is as a substantial element with properties (states are not mentioned), if only Things exist, then is natural to build the mathematical formalism which will model Things as one based on notions that are actually fundamental to Things (that is, their properties), and not derived notions, like states.

One reason why this approach is relevant is that it explains more than the usual approach because the assumptions made now are all of a same type (the ZFC and restrictions and regularity assumptions on the structure of the spectral triple, where the latter models the structure of the properties of minimal Things), while in the old approach seems like a set of disconnected and arbitrary assumptions of different kinds, thing which gives it the discussed ad-hoc flavor; thus, since, as mentioned, the assumptions on the spectral triple have nothing to do, in principle, with the standard, intuitive notions of space and time, which are ad-hoc in the usual approach, we could say then that our “new” physical theory here, based on the spectral standard model, truly explains in a non trivial/ad-hoc way the origin of these notions from Things and their properties in a relational way. It’s also relevant because in this second approach it’s made manifest a supposition, the commutativity of $A$, which in the first approach cannot be relaxed while in the second it can (but, in spite of this, the notion of metric integration still makes sense.) Thus, even if we start only from minimal Things and their properties, the relational emergence of standard, smooth manifold spacetime is only guaranteed when $A$ is commutative. This opens the possibility of having a new physical theory, with non-commutative $A$, in which minimal Things and their properties are fundamental and assumed, but where standard spacetime is forcefully absent. That is, in the usual approach, the ad-hoc assumptions given by the standard notions of space and time give the usual algebraic structure to $C^\infty(M)$, but, if we relax the former, we lose the manifold structure and therefore the algebraic structure on the latter, while in the second approach, one can start from an algebraic structure on the set of properties, which will give rise to
the manifold and the standard spacetime according to the commutativity of this algebra (of course, in the theory of previous sections, standard spacetime is also secondary and not fundamental, since it’s derived from minimal Things and their properties, which are assumed to be the fundamentals of the ontology; nevertheless, the axioms that make standard spacetime to appear relationally from matter are ad-hoc, they can be put there but also removed without any real justification more than that their assumption leads to the desired result, while, in the approach discussed here, the emergence of standard spacetime from minimal Things is forcefully forbidden if $A$ is non-commutative). 

In this way, the main learning from all of this is that standard spacetime is not only something ontologically non-fundamental because it can be relationally derived from Things (as in the theory in the previous section), but because there may be cases in which, even when there are indeed Things, standard spacetime cannot even be relationally derived from them, and, of course, much less substantively so.

That $A$ may not be commutative is suggested by the quantization of gravity. Indeed, its elements are the properties of (minimal) fields and, when all fields including gravity are quantized (and, therefore, one cannot use a classical Gravitational Field as background to build a classical, commutative spacetime), these properties form non-commutative algebras. Although, the relation between $A$ and the quantum field algebras doesn’t seem straightforward, since, in the classical case, $M$ (and, therefore, $C^\infty(M)$) is built from a single solution, while the algebras of functions on phase space (which are the ones actually related to the quantum field algebras) have as domains the phase space points coming from all solutions. Nevertheless, recall that our relationist approach here gives a way of obtaining the algebra of physical 3-space purely from Field properties (represented here by the phase space functionals), and, therefore, it can be used as a guide to obtaining the corresponding quantum counterparts, as done in [1, 2]. In any case, this still shows that it seems unlikely that the quantum formalism based on the quantum algebra of gravity could have some standard spacetime as background (even the very standard manifold $M$, and not only the metric, which is an obvious thing.)

Appendix: Elementary Philosophy of Physics (Ontology and Semantics) [1]

A.1. Ontology

Physical theories assume that Reality is made of something. In ontology, this something is called a Substance. In science, it’s assumed that Reality is made of a single type of Substance, which is called Material Substance. Therefore, science’s ontology is monistic. In ordinary language, we refer to this Substance simply as Matter. Nevertheless, the “matter” of which we are talking about is not necessarily some chunk of something with some mass, as, say, a piece of steel. The notion of Matter used in science can become quite subtle and unintuitive. In fact, the best and more up-to-date physical theories should, and must, be the ones that dictate the specifics of the ontology and the particular properties of matter. A key property of Material Substance is that two material individuals can Associate to form a third material individual. If $a$ and $b$ are these two material individuals, we will denote the association operation as $\circlearrowright$, and, in this sense, if $c$ is the third material individual mentioned before, we write (we will denote as $S$ the set which contains our material individuals):

$$c = a \circlearrowright b.$$ 

We are talking about a fully non-commutative case, not the “almost commutative” one on which the spectral standard model is based, composed of a tensor product of algebras, one which is the standard commutative algebra of a manifold.

Other metaphysical theories may introduce other types of Substances, like “Spatio-Temporal Substance”, “Immaterial Souls”, “Mental Substance” (this one is famously associated with Descartes and his dualist, in terms of Substance, solution to the, also famous, Mind-Body problem, i.e., what’s the mind and how does it interact with the physical body?), etc. But, in science, only matter interacts and associates with matter.
That is, in science we have different types of “matter”, which are each characterized by the binary association property, \(+ : S \times S \rightarrow S\) among its elements. We postulate that all Material Substance carries Energy, where this concept is defined according to the best physical theory at disposition. In general, it’s a Property (see below) of matter which is represented, in one way or another, by an additive, real function on matter (which depends on the reference frame adopted, though), i.e., an \(h(a)\) such that

\[ h(a + b) = h(a) + h(b), \]

for any two elements \(a, b\) from \(S\), and which “generates time translations” in the context of that theory (this is usually done in what is called the “Hamiltonian formulation” of a dynamical theory.) What this means is that Energy is the Property of a Thing that allows us to determine, in a physical theory, how the Properties of this Thing change in time (see below for these terms.)

Thus, if something doesn’t possess energy, then it cannot be considered matter.

Of course, the things that inhabit Reality are much more than mere undifferentiated bunches of material substance. The “fauna” of Reality is very rich, with individuals that can have very peculiar and different Substantial Properties. A Substantial Property is a property/quality that a material element possess or bears and it’s as objective as the existence of the material element itself (properties are not substantial elements themselves, though) We will call Thing to a matter element which possess substantial properties. Things also associate to produce other Things. We denote the set of all Things that exist as \(\Theta\). We mention that a very particular type of properties are the so-called Emergent Properties, that is, properties which are present in the aggregate Thing \(c = a + b\), but not in its component Things, \(a\) and \(b\), before the aggregation.

Reality is the Thing \(R \in \Theta\) which consists in the association or aggregation of all Things in \(\Theta\)\(^{20}\).

A.2. Semantics

A mathematical theory without a factual interpretation is only mathematical, not physical. A scalar field satisfying the Laplace equation may be the potential for a gravitational field or for a static electric field. It is the factual interpretation (which interprets this field as the potential for a gravitational field or for a static electric field) the thing that transforms it into a theory of physics. The physical interpretation maps mathematical objects\(^{19}\) to objects of the real world. The purely semantic map, \(\varphi\), assumes a certain kind of realism (which is an ontological hypothesis), simply the same that is needed to give meaning to the fundamental maxim of the scientific method (we will consider only classical realism here). In this way, the interpretation here is very simple: factual item \(y\) of this physical system will be represented mathematically by, say, the mathematical function \(f\) (which is a construct.) We don’t need anything else, we simply refer to the factual item because it exists, it is simply there, we simply point to it (semantically speaking, not pointing in a literal sense.) When a particular mathematical model from a physical theory is interpreted in terms of some physical notions, one says that these (often vague in the quantitative sense) notions have been exactified.

The purely semantic interpretation takes a mathematical construct and relates it to the real world. But this is not enough to give physical meaning to this construct. The purely semantic interpretation gives us, at best, the physical referents of the construct, that is, those physical entities (i.e., the Things that comprise

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19Mathematically, one can take a Boolean \(\sigma\)-algebra structure for the triple \((S, +, \Box)\) (where \(\Box\) represents the null element, or minimal element, \(R\) the maximal element, see below, and \(+\) the “disjunction” or \(\sup\), that is, the minimal upper bound between elements; one also has the \(\inf\) or “conjunction”, which is interpreted as the Interposition \(\times\) of material individuals.)

20One could say, what about time and space, are not they part of Reality too? This is a complex issue. They are intrinsically tied to Things in a certain specific sense, as we will see. Also, what about logic? We use logic to understand Reality, but we don’t take it as part of Reality (it’s not even a thing.)

21These mathematical objects are mere mental constructs made by human brains. In this way, we do not adhere to mathematical Platonism (which says they exist in their “own objective reality”) However, this does not imply a free subjectivism, since different humans can manage to understand each other through the same concepts. Then, in practice, we can pretend as if Platonism were true.
the Reality hypothesized by the physical theory in consideration) to which the physical interpretation refers. What is important to note is that, in a full theory of physics, this construct is not isolated. Indeed, there are also other mathematical constructs in the theory, and, in general, they are all inter-related with one another (in the logical sense.) These interrelations constitute what is called the mathematical Sense of the construct. Some of these other constructs are also physically interpreted in terms of purely semantic interpretations, and, thus, the mathematical sense becomes also a physical sense. In this way, given a physical theory, we will adopt the point of view in which the meaning of a construct is fully established by its total Sense and its Reference class ($\Sigma \ni \sigma$ and $\Sigma \subseteq \Theta_{\text{Universe}}$) of Things ($\sigma$), both which can only be read once the theory has been fully established in its mathematical axioms and semantic interpretations. Note that, in a completely axiomatized theory, certain basic constructs will determine the meaning of the other constructs of the theory (in general, these basic constructs will be those whose physical interpretations are made in terms of factual elements that are usually taken as factual primitives, such as the notions of length, lapses of time, facts, propensity, etc.) Space and time, in particular, are quite generic to most theories. In Scientific Ontology, one can actually make theories that give them a precise physical meaning. These theories are appropriately called “proto-physics” (to construct these theories for space and time will be the main aim here) Although, one often needs to insert them into a physical theory in order to have a more exact (in a quantitative sense) meaning.

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