The ΛCDM Limit of the Generalized Chaplygin Gas Scenario

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Abstract. We explicitly demonstrate that, contrary to recent claims, the dynamics of a generalized Chaplygin gas model with an equation of state $p = -C$ (where $C$ is a positive constant) is equivalent to that of a standard ΛCDM model to first order in the metric perturbations. We further argue that the analogy between the two models goes well beyond linear theory and conclude that they cannot be distinguished based on gravity alone.

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1. Introduction

Observational evidence strongly suggests that we live in a (nearly) flat Universe which has recently entered an accelerating phase \[1, 2, 3, 4\]. In the context of general relativity such a period of accelerated expansion must be induced by an exotic ‘dark energy’ component violating the strong energy condition \[5, 6, 7, 8, 9\], though this is not necessarily so in the context of more general models (see for example \[10\]). There is also strong evidence that most of the matter in the Universe, an essential ingredient for structure formation, is in a non-baryonic form.

One can therefore ask if it is possible to have some component of the energy budget of the universe which simultaneously accounts for both the dark energy and the dark matter, or if two different components are inevitable. An interesting toy model candidate for the unification of dark matter and dark energy is a perfect fluid with an exotic equation of state known as the (generalized) Chaplygin gas, for which

\[ p = -\frac{C}{\rho^\alpha}. \]

(1)

Here \( \rho \) is the density, \( p \) is the pressure, \( C \) is a positive constant and \( 0 \leq \alpha \leq 1 \). Both the best-motivated Chaplygin gas (which has \( \alpha = 1 \)) \[11, 12\] and simple generalizations thereof \[13\] have recently attracted considerable attention at this toy model level. In fact a connection between string theory and the original Chaplygin gas has also been claimed (see for example \[14\] and references therein).

In the \( \alpha = 0 \) case it is straightforward to show that the background equations for the Chaplygin gas model are identical to those of the familiar \( \Lambda \)CDM scenario. However, there has been a recent claim \[15\] that this similarity breaks down at first order in the metric perturbations.

In fact, we have recently shown \[16\] that for \( \alpha > 0 \) the computation of precise predictions in the context of this type of models does need to take into account the non-linear behavior since the background equations will cease to be valid at late times even on large cosmological scales. In broad terms, this is related to the fact that in general

\[ \langle p \rangle \equiv -\langle C/\rho^\alpha \rangle \neq -C/\langle \rho \rangle^\alpha, \]

(2)

where \( \langle \ldots \rangle \) represents a spatial average. This means that all previous predictions \[17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29\] relying on the background evolution or linear evolution of density perturbations must be re-evaluated. However, this is only a problem if \( \alpha \neq 0 \).

In this paper we explicitly show that, contrary to the claim in \[15\], the evolution of linear density perturbations in the context of a Chaplygin gas model with \( \alpha = 0 \) is identical to that of a standard \( \Lambda \)CDM model. We further conclude that these models are in fact indistinguishable as far as gravity is concerned.
2. The background dynamics

The dynamics of a flat homogeneous and isotropic Friedmann-Robertson-Walker (FRW) universe is fully described by

\[
\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho a^2}{3},
\]

\[
\left(\frac{\ddot{a}}{a}\right) - \left(\frac{\dot{a}}{a}\right)^2 = -\frac{4\pi G (\rho + 3p)a^2}{3},
\]

if the equation of state \( p = p(\rho) \) is provided. Here a dot represents a conformal time derivative \( (d/d\eta) \), \( a \) is the scale factor, \( \rho \) is the energy density and \( p \) is the pressure. It is possible to show using Eqs. (3) and (4)

\[
\dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + p) = 0.
\]

Now, let us consider two possible scenarios:

- Model I: a generalized Chaplygin gas model with a density \( \rho_I \), and pressure \( p_I = -C \).
- Model II: a ΛCDM scenario with pressureless cold dark matter with density \( \rho_m \), and a cosmological constant with density \( \rho_\Lambda \), and pressure \( p_\Lambda = -\rho_\Lambda = -C \).

where \( C \) is a positive constant. It is trivial to verify that the background equations (and hence the dynamics) for models I and II are identical when one identifies the total densities and pressures in both cases, that is

\[
\rho_I = \rho_{II} = \rho_m + \rho_\Lambda,
\]

\[
p_I = p_{II} = p_\Lambda = -C.
\]

We can therefore physically interpret \( C \) in the Chaplygin gas case as the equivalent vacuum energy density of the ΛCDM model.

3. Growth of Perturbations

For a general case of \( n \) gravitationally interacting fluids, the linear evolution of perturbations in the synchronous gauge is given by

\[
\ddot{h} + \mathcal{H}\dot{h} + 3\mathcal{H}^2 \sum_i (1 + 3v_i^2) \Omega_i \delta_i = 0
\]

\[
\dot{\delta}_i + (1 + \omega_i)(\theta_i + \dot{h}/2) + 3\mathcal{H}(v_i^2 - \omega_i)\delta_i = 0
\]

\[
\dot{\theta}_i + \mathcal{H}(1 - 3v_i^2)\theta_i + \frac{v_i^2}{1 + \omega_i} \nabla^2 \delta_i = 0
\]

where \( h \) is the trace of the perturbation to FRW metric, \( \mathcal{H} \equiv \dot{a}/a \), \( \delta_i \) is the density contrast of the \( i \)th-fluid obeying \( p_i = \omega_i\rho_i \) with an adiabatic sound speed \( v_i \) and an element velocity divergence \( \theta_i \). Note that Eqs. (9) and (10) apply for all \( i = 1, \ldots, n \).

In models I and II the sound speed is identically zero so that Eq. (10) reduces to \( \theta_i = 0 \) at all times in the initially unperturbed synchronous gauge. Using the fact that \( \omega_1/w_1 = 3\mathcal{H}(1 + w_1) \) and

\[
\dot{h} = -2\dot{\delta}_1 + 6\omega_1\mathcal{H}\delta_1 = -2 \frac{d}{d\eta} \left( \frac{\delta_1}{1 + w_1} \right),
\]

\( h, \delta, \theta, \mathcal{H} \) are all functions of \( \eta \) as defined in [10].
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it is straightforward to show that the evolution of density perturbations in model I is given by

$$\ddot{\delta}_s + \mathcal{H} \dot{\delta}_s - \frac{3}{2} \mathcal{H}^2 (1 + \omega_I) \delta_s = 0,$$

(12)

where $\delta_s = \delta_I / (1 + w_I)$. Let us now consider the evolution of matter perturbations, $\delta_m$, in the context of model II. Using equation (8) plus the fact that $\dot{h} = -2 \dot{\delta}_m$ and the relation $\Omega_m = \rho_m / (\rho_m + \rho_\Lambda) = 1 + w_{II}$ it is easy to show that

$$\ddot{\delta}_m + \mathcal{H} \dot{\delta}_m - \frac{3}{2} \mathcal{H}^2 (1 + w_{II}) \delta_m = 0.$$

(13)

It is now immediate to see that the evolution of $\delta_I$ and $\delta_{II} \equiv \delta \rho_{II} / \rho_m = \delta_m / (1 + \rho_\Lambda / \rho_m) = \delta_m (1 + w_{II})$ is identical if we identify $w_{II}$ with $w_I$. It is straightforward to show from Eq. (2) that $\delta_{II}$ has an asymptotic behavior $\propto a^{-3}$ at late times. In reference [15] the authors wrongly concluded that the Chaplygin gas model with $\alpha = 0$ and ΛCDM would differ in their first order evolution on the basis of their different evolutions in $\delta_m$ and $\delta_I$. As we have explicitly shown above, these are not the right variables to compare.

In fact, given that the evolution of the density perturbations in Fourier space for models I and II is independent of the wave-number $k$, the fact that the models are equivalent to zeroth order in the metric perturbations necessarily implies their equivalence to first order. Note that a perturbation with an infinite wavelength ($k = 0$) is uniform and can be studied using the equations for the background evolution.

4. Beyond linear theory

It is also possible to show that the models are in fact equivalent beyond first order in the metric perturbations. From the point of view of gravity, the models are equivalent if, given the same initial conditions, the evolution of the metric and the energy-momentum tensor of the perfect fluid driving the expansion of the Universe,

$$T^{\alpha \beta} = (\rho + p) u^\alpha u^\beta + pg^{\alpha \beta},$$

(14)

is identical for both models. To specify the evolution of the energy-momentum tensor one needs to know the evolution of $\rho$ and $3$ of the components of the 4-velocity $u^\alpha$. Note that the constant pressure is specified and that the 4th component of the 4-velocity can be determined from the other 3 using the condition $u^\alpha u_\alpha = -1$.

On the other hand, given initial conditions at the time $t_i$ for the components of the metric $g^{\alpha \beta}$, and their first derivatives it is possible to use the Einstein equations to compute the second derivatives of the metric with respect to time everywhere, and use this to calculate the new values of the metric components and their first derivatives at a subsequent time $t_i + dt$.

Also, the conservation of the energy-momentum tensor ($T^{\alpha \beta}{}_{;\alpha} = 0$) gives us 4 equations which relate the components of the energy-momentum tensor at the instants $t_i + dt$ and $t_i$. Given that the evolution of the energy-momentum tensor depends on 4 variables (other than the metric) it is completely determined by the 4 equations which need to be satisfied in order for the energy-momentum tensor conservation to hold.
Hence, we conclude that if the initial conditions are the same for both models (I and II) the dynamics will be identical so that the models can not be distinguished based on gravity alone.

5. Conclusions

We have shown that gravity alone cannot distinguish between a generalized Chaplygin gas model with $\alpha = 0$ and a standard $\Lambda$CDM scenario. Given similar initial conditions, their dynamics will be similar so that the cosmological predictions of both models are identical. Given that both of them are, to a certain extent, toy models with a somewhat tenuous motivation from fundamental physics, this is probably as far as they can meaningfully be compared.

In any case, our present results, together with those of [16] show that the equation of state $W$ with $0 \leq \alpha \leq 1$ basically describes a one parameter family of toy models interpolating from $\Lambda$CDM to the original Chaplygin gas model. As such they can be useful for studying cosmological constraints on unified dark matter models, though one must keep in mind that as soon as $\alpha \neq 0$ the analysis becomes quite subtle.

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