Integer sequence of Dyck-path single changes

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Abstract

The extended Dyck words of length \( n = 2k + 1 \) \((0 < k \in \mathbb{Z})\), obtained by prefixing a 0-bit to each Dyck word of length \( 2k \), represent both the cyclic vertex classes of the odd graph \( O_k \) and the dihedral vertex classes of the middle-levels graph \( M_k \). They are also bijectively representable as signatures of their Dyck paths. A sequence is obtained by registering a single change in each such signature in the natural order of Dyck paths imposed by restricted growth strings, accompanied by the locations of those changes.

1 Introduction

In odd graphs \( O_k \) \((0 < k \in \mathbb{Z})\), uniform 2-factors \([9]\) yielding Hamilton-cycles \([8]\) are analyzed \([5]\) from the point of view of edge-supplementary arc-colorings, via restricted growth strings (or RGS’s) \([1, pg. 325]\). In this and the next sections, a slight modification of the arguments of \([5]\) shows that such RGS’s control the Dyck paths of length \( n = 2k + 1 \), that represent bijectively the cyclic classes of vertices of \( O_k \), viewed as Dyck nests (see Remark \( 5 \)). Such control is simplified by reducing the nested reconcatenations of Theorem 1, below, to a single change between the signature (see Remark \( 7\)-Theorem \( 11 \)) of each non-root Dyck nest and that of its parent nest in an associated RGS-tree. A sequence of all such changes, in Theorem 27, is our main contribution, accompanied by that of their locations, in Corollary 29.

Let \( C_k = \frac{(2k)!}{k!(k+1)!} \) be the \( k \)-th Catalan number \([10]\) \textsc{A239903}. Let \( S \) be the sequence \([10]\) \textsc{A239903} of RGS’s. It was shown in [3, 4] that the first \( C_k \) terms of \( S \) represent both the Dyck words of length \( 2k \) and the extended Dyck words of length \( n \), obtained by prefixing a 0-bit to each Dyck word, thus yielding a sole corresponding Dyck path (Remark 3).

\( S \) starts as \( S = (\beta(0), \ldots, \beta(17), \ldots) = (0, 1, 10, 11, 12, 100, 101, 110, 111, 112, 120, 121, 122, 123, 1000, 1001, 1010, 1011, \ldots) \), with the lengths of any two contiguous terms \( \beta(m-1), \beta(m) \) \((1 \leq m \in \mathbb{Z})\) constant unless \( m = C_k \) for some \( k > 1 \), in which case \( \beta(m-1) = \beta(C_k-1) = 12 \cdots k \) has length \( k \), and \( \beta(m) = \beta(C_k) = 10^k = 10 \cdots 0 \) has length \( k + 1 \).

A \( k \)-germ \((k > 1)\) is a \((k - 1)\)-string \( \alpha = a_{k-1}a_{k-2}\cdots a_2a_1 \) such that:

(a) the leftmost position of \( \alpha \), namely position \( k - 1 \), contains the entry \( a_{k-1} \in \{0, 1\} \);

(b) given \( 1 < i < k \), the entry \( a_{i-1} \) at position \( i - 1 \) satisfies \( 0 \leq a_{i-1} \leq a_i + 1 \).

Each RGS \( \beta = \beta(m) \), where \( 0 \leq m \in \mathbb{Z} \), is transformed, for every \( k \in \mathbb{Z} \) such that \( k \geq \text{length}(\beta) \), into a \( k \)-germ \( \alpha = \alpha(\beta, k) = \alpha(\beta(m), k) \) by prefixing \( k - \text{length}(\beta) \) zeros to \( \beta \).

Every \( k \)-germ \( a_{k-1}a_{k-2}\cdots a_2a_1 \) yields the \((k + 1)\)-germ \( 0a_{k-1}a_{k-2}\cdots a_2a_1 \) of its zeros to the left of its leftmost position containing a 1. We denote such an RGS again by \( \alpha \), say that the
null RGS $\alpha = 0$ represents all null $k$-germs $\alpha$ ($0 < k \in \mathbb{Z}$), and use $\alpha = \alpha(m)$, or $\beta = \beta(m)$, both for a $k$-germ and for its corresponding RGS. In fact, $\alpha = \alpha(m)$, or $\beta = \beta(m)$, will be considered to be the RGS representing all the $k$-germs $\alpha = \alpha(m)$, or $\beta = \beta(m)$, respectively, ($0 < k \in \mathbb{Z}$) leading to $\alpha$, or $\beta$, as an RGS, by stripping their zeros as indicated.

If $a, b \in \mathbb{Z}$, then let: (1) $[a, b] = \{j \in \mathbb{Z}| a \leq j \leq b\};$ (2) $[1, b] = \{j \in \mathbb{Z}| a \leq j < b\};$ (3) $]a, b] = \{j \in \mathbb{Z}| a < j \leq b\};$ (4) $]a, b[ = \{j \in \mathbb{Z}| a < j < b\}.$

Given two $k$-germs $\alpha = a_{k-1} \cdots a_2a_1$ and $\beta = b_{k-1} \cdots b_2b_1$, where $\alpha \neq \beta$, we say that $\alpha$ precedes $\beta$, written $\alpha < \beta$, whenever either

(i) $a_{k-1} < b_{k-1}$ or

(ii) $\exists i \in [1, k]$ such that $a_i < b_i$, while $a_j = b_j, \forall j \in [i, k]$.

The resulting $k$-germ order yields a bijection from $[0, C_k]$ onto the set of $k$-germs that assigns each $m \in [0, C_k]$ to a corresponding $k$-germ $\alpha = \alpha(m)$. In fact, there are exactly $C_k$ $k$-germs $\alpha = \alpha(m) < 10^k$, $\forall k > 0$. Moreover, we have the following trees $T_k$ and correspondences $F$.

**Theorem 1.** Let $k > 1$. (A) The $k$-germs are the nodes of an ordered tree $T_k$ rooted at $0^{k-1}$ such that each $k$-germ $\alpha = a_{k-1} \cdots a_2a_1 \neq 0^{k-1}$ with rightmost nonzero entry $a_i$ ($i = i(\alpha) \in [1, k]$) has parent $\beta = \beta(\alpha) = b_{k-1} \cdots b_1 < \alpha$ in $T_k$ with $b_i = a_i - 1$ and $a_j = b_j$, for every $j \neq i$ in $[1, k]$. (B) To each $k$-germ $\alpha = a_{k-1} \cdots a_1$ corresponds an $n$-string $F(\alpha)$ with initial entry 0 and having each $j \in [1, k]$ as an entry exactly twice. Moreover, $F(0^{k-1}) = 012 \cdots (k-2)(k-1)kk(k-1) \cdots 21$, and if $\alpha \neq 0^{k-1}$ and $W^i$ and $Z^i$ are the leftmost and rightmost substrings, respectively, of lengths $i = i(\alpha)$ in $F(\beta)$ and $\gamma > 0$ is the leftmost entry of $F(\beta) \setminus (W^i \cup Z^i) = X[Y$, with $Y$ starting at the leftmost entry $\gamma + 1$ of $F(\beta)$, then $F(\alpha) = W^i|Y|X|Z^i$. Furthermore, if a leftmost entry $c \in [0, k]$ is next on its right in $F(\alpha)$ to a leftmost entry $b \in [0, k]$, then $c > b$. On the other hand, $W^i$ is an ascending $i$-substring, $Z^i$ is a descending $i$-substring, and $kk$ is a substring of $F(\alpha)$.

**Proof.** The proof here is a slight modification of that of [3, Theorems 3.1–3.2], or [4, Theorems 1–2], where the rightmost appearances of each integer of $[1, k]$ in every $F(\alpha)$ as in the statement were given as asterisks, *, or equal signs, =, in the version of [5, Theorem 2]. □

## 2 Dyck words and their associated germs

A $k$-bitstring is a sequence of length $k$ whose terms are the digits 0, called 0-bits, and/or 1, called 1-bits, respectively. The weight of a $k$-bitstring is its number of 1-bits.

In this work, a Dick word of length $2k$ is a $2k$-bitstring of weight $k$ such that in every prefix the number of 0-bits is at least the number of 1-bits (which differs from the Dyck words of [8] in which the number of 1-bits is at least equal to the number of 0-bits).

The concept of empty Dyck word, denoted $\epsilon$, and whose weight is 0, also makes sense in this context. We will present each Dyck word as its associated extended Dyck word, obtained by prefixing a 0-bit. In particular, $\epsilon$ is represented by the extended Dyck word 0.

**Remark 2.** For each $k$-germ $\alpha$, where $k > 1$, we define the bitstring form $f(\alpha)$ of $F(\alpha)$ by replacing each first appearance of an integer as an entry in $F(\alpha)$ by a 0-bit and its second appearance by a 1-bit (where 0-bits and 1-bits correspond respectively to the 1-bits and
Figure 1: List of $k$-germs $\alpha$, $n$-nests $F(\alpha)$, signatures and change entries, for $k = 2, 3, 4, 5$. 

| $\alpha$ | $F(\alpha)$ | $B(\alpha)$ | $A(\alpha)$ | $i(\alpha)$ | $o(\alpha)$ | $A(\alpha)$ | $\alpha$ | $F(\alpha)$ | $B(\alpha)$ | $A(\alpha)$ | $i(\alpha)$ | $o(\alpha)$ | $A(\alpha)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0  | 01221 | 1 | / | 0 | 000 | 012344321 | 123 | / | 0 |
| 1  | 02211 | 1 | 0 | 1110 | 1 | 0 | 001 | 02344321 | 123 | 120 | 1130 | 1 | 0 |
| 10 | 0332211 | 12 | 10 | 1120 | 1 | 0 | 101 | 03443221 | 123 | 103 | 2120 | 2 | 0 |
| 11 | 0213321 | 12 | 02 | 2110 | 2 | 0 | 011 | 02213443 | 103 | 102 | 1132 | 3 | k-2 |
| 12 | 0332211 | 01 | 00 | 1210 | 4 | 0 | 012 | 03443221 | 102 | 100 | 1220 | 4 | 0 |
| 00 | 0123321 | 12 | / | 0 | 100 | 012443321 | 123 | 023 | 3110 | 5 | 0 |
| 01 | 0233221 | 12 | 10 | 1120 | 1 | 0 | 101 | 02443321 | 023 | 020 | 1130 | 6 | 0 |
| 10 | 0133221 | 12 | 02 | 2110 | 2 | 0 | 110 | 013324421 | 023 | 013 | 2121 | 7 | k-3 |
| 11 | 02213321 | 02 | 01 | 1121 | 3 | k-2 | 111 | 02214433 | 013 | 011 | 1131 | 8 | 1 |
| 12 | 0332211 | 01 | 00 | 1210 | 4 | 0 | 112 | 03324421 | 011 | 010 | 1210 | 9 | 0 |
| 10 | 0133221 | 12 | 02 | 2110 | 2 | 0 | 101 | 014433221 | 013 | 003 | 2210 | 10 | 0 |
| 11 | 02213331 | 02 | 01 | 1121 | 3 | k-2 | 112 | 02214433 | 003 | 002 | 1132 | 11 | k-2 |
| 12 | 0332211 | 01 | 00 | 1210 | 4 | 0 | 113 | 03324421 | 002 | 001 | 1221 | 12 | k-3 |
| 13 | 04433221 | 001 | 000 | 1310 | 13 | 0 | 114 | 04433221 | 001 | 000 | 1310 | 13 | 0 |
| 14 | 055443321 | 001 | 000 | 1410 | 41 | 0 | 115 | 055443321 | 001 | 000 | 1410 | 41 | 0 |

Figure 1: List of $k$-germs $\alpha$, $n$-nests $F(\alpha)$, signatures and change entries, for $k = 2, 3, 4, 5$. 

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Figure 2: List of \( k \)-germs \( \alpha \), \( n \)-nests \( F(\alpha) \), signatures and change entries, for \( k = 6 \).
0-bits of [8]). Such \( f(\alpha) \) is an \( n \)-bitstring of weight \( k \) whose support \( \text{supp}(f(\alpha)) \) is a vertex of \( O_k \) [2]. Note that \( f(\alpha) \) is an extended Dyck word of length \( n \). In fact, the extended Dyck words of length \( n \) represent both the cyclic vertex classes of \( O_k \) and the dihedral vertex classes of the middle-levels graph \( M_k \), which is the double covering graph of \( O_k \) and has Hamilton cycles lifted from those in \( O_k \) [5, 8], or independently, via ordered trees, in [3, 4, 6, 7].

**Remark 3.** Each extended Dyck word \( f(\alpha) \) yields a **Dyck path** [5] obtained as a curve \( \rho(\alpha) \) that grows from \((0,0)\) in the Cartesian plane \( \Pi \) via the successive replacement of the 0- and 1-bits of \( f(\alpha) \), from left to right, by **up-steps** and **down-steps**, namely segments \((x,y)(x+1,y+1)\) and \((x,y)(x+1,y-1)\), respectively. We assign the integers of \([0,k]\) in decreasing order (from \( k \) to 0) to the up-steps of \( \rho(\alpha) \), from the top unit layer of \( \rho(\alpha) \) in \( \Pi \) to the bottom one and from left to right at each unit layer between contiguous lines \( y, y+1 \in \mathbb{Z} \), where \( 0 \leq y \in \mathbb{Z} \). These assigned integers of \([0,k]\) correspond to their leftmost appearances as entries of \( F(\alpha) \). Each leftmost appearance \( j' \) of an integer \( j \in [1,k] \) in \( F(\alpha) \) corresponds to the starting entry of a Dyck word \( 0u1v \) in \( f(\alpha) \), where \( u, v \) are Dyck words (possibly \( \epsilon \)). Dyck word \( 0u1v \) corresponds in \( F(\alpha) \) to \( j'U_\alpha j''V \), where \( U, V \) correspond to \( u, v \) and \( j'' = j' \).

**Theorem 4.** Each extended Dyck word of length \( n \) is obtained as the bitstring form \( f(\alpha) \) of the \( n \)-string \( F(\alpha) \) associated to a unique \( k \)-germ \( \alpha \), via the procedure of Theorem 1.

**Proof.** The Lexical Procedure [3, Section 7], [4, Section 7] restores the positive integer entries of \( F(\alpha) \) corresponding to the \( k \) non-initial 0-bits of \( f(\alpha) \). These are the first appearances \( j' \) of each integer \( j \in [1,k] \) in \( F(\alpha) \). By forming the Dyck subword \( 0u1v \) of \( f(\alpha) \), the second appearance \( j'' \) of \( j \) is found by replacing its corresponding 1-bit of \( f(\alpha) \) by \( j = j'' \) in \( F(\alpha) \).

**Remark 5.** Suggested by the sets of nested intervals formed by the projections on the \( x \)-axis of the two appearances \( j', j'' \) of each integer \( j \in [1,k] \) as numbers assigned to the steps of each Dyck path \( \rho(\alpha) \), the \( n \)-strings \( F(\alpha) \) of Theorem 1 will be said to be **Dyck nests** of length \( n \), or **n-nests**. We take the tree \( T_k \) whose nodes were originally denoted via the \( k \)-germs \( \alpha \), and denote them, further, via the n-nests \( F(\alpha) \) in representation of the corresponding extended Dyck words \( f(\alpha) \). With this nest notation, \( T_k \) will be now said to be a **tree of Dyck nests**.

**Corollary 6.** The set of n-nests is in one-to-one correspondence with the set of extended Dyck words of length \( n \).

**Remark 7.** Each n-nest \( F(\alpha) \) is encoded by its **signature** \( \sigma(\alpha) \), namely the vector of floors of halfway-distances between the first, say \( j' \), and second, say \( j'' \), appearances of each integer \( j \) assigned to steps in the path \( \rho(\alpha) \). For example, if \( j'k'k''_j'' \), or \( j'(k-1)_j'k''(k-1)_j'' \), is a substring of \( F(\alpha) \), or \( F(\gamma) \), respectively, then the halfway-distance floor of \( j \) is \( \lfloor d(j', j'') \rfloor = \lfloor 3/2 \rfloor = 1 \), or \( \lfloor d(j', j'') \rfloor = \lfloor 5/2 \rfloor = 2 \), participating as the \( j \)-th entries of \( \sigma(\alpha) \), or \( \sigma(\gamma) \).

**Claim 8.** The construction of the tree \( T_k \) of Dyck nests via Theorems 1 and 4 and Corollary 6 can be simplified by changing just one single entry of \( \sigma(\beta) \) at each stage, instead of using the whole procedure in Theorem 1 in order to determine \( \sigma(\alpha) \).

**Example 9.** We exemplify Claim 8 for \( k = 2, 3, 4, 5, 6 \) in Figures 1–2. In them, the first column for each such \( k \) shows the \( k \)-germs \( \alpha = a_k \cdot \cdots a_1 \) in depth-first order of the node set
of $T_k$, in black except for $a_{i(\alpha)}$, which is in red. The second column shows the corresponding $n$-nests $F(\alpha)$ initialized on top as $F(0^{k-1}) = 012 \cdots (k-2)(k-1)kk(k-1)(k-2) \cdots 21$ (also expressible, according to Remark 3, as $01'2' \cdots (k-2)'(k-1)'kk''(k-1)'(k-2)'' \cdots 2''$) and continued from the second row on as $F(\alpha) = W^i|X|Y^i|Z^i$ (as in Theorem 1), with $W^i$, $Z^i$ in black, $Y$ in red and $X$ in green, with the parent $\beta$ of $\alpha$ in $T_k$ having $F(\beta) = W^i|X|Y^i|Z^i$. This second column has the red-green numbers underlined. The third and fourth columns have their rows as the signatures $B(\alpha) = B_{k-1}B_{k-2} \cdots B_2 B_1$ of $\beta$ (this starting at the second row) and $A(\alpha) = A_{k-1}A_{k-2} \cdots A_2 A_1$ of $\alpha$, specified by having $B_j = B_j(\alpha)$ and $A_j = A_j(\alpha)$, for each $j \in [1,k]$, as the numbers of pairs formed by the two appearances of participating integers inside the substrings of $F(\beta)$ and $F(\alpha)$, respectively, delimited between the two appearances of $j$. These third and fourth columns are determined via the black-red-green second column at each row. The fifth column, starting at the second row, is formed by four single-digit columns:

1. the value $i = i(\alpha)$ in the current application of Theorem 1; $(i$ in red if and only $i > 1)$;
2. the corresponding value of $a_i = a_i(\alpha)$ in $\alpha = a_{k-1}a_{k-2} \cdots a_2 a_1$;
3. the corresponding value of $B_i(\alpha) = B_i(\alpha)$ in the third column;
4. the value of $A_i(\alpha) = A_i(\alpha)$ in the fourth column, with $A_i$ in red if and if $A_i > 0$.

The sixth column is the depth-first order $o(\alpha)$ of $\alpha$ in $T_k$. The seventh column repeats item (4) of the fifth column. In terms of $k$, this forms an infinite integer sequence, namely the one that is the object of study in this paper. For all columns but for the second one, those rows which in the first column have $k$-germ $\alpha = a_{k-1} \cdots a_1$ with $a_1$ a local maximum (so that the following $k$-germ, say $\gamma = c_{k-1} \cdots c_1$, in the first column, if any, has $c_1 = 0$) appear underlined. All rows of the second column but for the first row has the substring $kk$ (that is, $k'k''$, in terms of the appearances $k'$ and $k''$ of $k$) either in red or in green. After the initial black $F(\alpha) = F(0^{k-1})$, the substring $kk$ is red in the two subsequent rows and becomes green in the fourth row. This corresponds to the red value $k - 2$ in the seventh column.

Let $g$ be the correspondence that assigns the values $A_i(\alpha)(\alpha)$ (as in the seventh column of Figures 1-2) to the orders $o(\alpha)$ (as in the sixth column).

**Theorem 10.** For each $k$-germ $\alpha \neq 0^{k-1}$, the signatures $B(\alpha)$ and $A(\alpha)$ of the parent $\beta$ (of $\alpha$ in $T_k$) and $\alpha$, respectively, differ solely at the $i$-th entry, that is: $B_i(\alpha)(\alpha) \neq A_i(\alpha)(\alpha)$, while $B_j(\alpha) = A_j(\alpha)$, for every $j \neq i(\alpha)$.

*Proof.* There is a single difference between the parent $\beta = b_{k-1} \cdots b_1$ of $\alpha = a_{k-1} \cdots a_1$ and the $k$-germ $\alpha$ itself, occurring at the $i$-th position, whose entry is increased in one unit from $\beta$ to $\alpha$, that is $a_i = b_i + 1$. The effect of this on $F(\alpha)$, namely the reconcatenation of the inner strings $Y$ and $Z$ of $F(\beta) = X^i|Y|Z^i|W^i$ in $F(\alpha) = X^i|Z|Y^i|W^i$, modifies just one of the halfway-distance floors $A_j = \lfloor d(j', j'')/2 \rfloor$ between the first appearance $j'$ of a $j \in [0,k]$ in $F(\alpha)$ and its second appearance, $j''$, namely $A_i = \lfloor d(i', i'')/2 \rfloor$, where $i = i(\alpha)$. Clearly, this implies the statement.

**Theorem 11.** The correspondence that assigns each $n$-nest to its signature is a a bijection.

*Proof.* Let $\alpha = a_{k-1} \cdots a_1$ be a $k$-germ. The rightmost entry of the $n$-nest $F(\alpha) = c_0c_1 \cdots c_{2k}$ is $c_{2k} = 1''$, so $A_1(\alpha)$ determines the position of $i'$. For example, if $A_1(\alpha) = 0$, then $c_{2k-1} = 1'$, so $a_1$ is a local maximum (indicated in Figures 1-2 by having $\alpha, B(\alpha), A(\alpha), \cdots, o(\alpha), A_{i(\alpha)}(\alpha)$
underlined). To obtain \( F(\alpha) \) from \( A(\alpha) \), we initialize \( F(\alpha) \) as the \( n \)-string \( F^0 = 00 \cdots 0 \). Setting the positions of \( 1^\prime, 1^\prime\prime, 2^\prime, 2^\prime\prime, \ldots, (k-1)^\prime, (k-1)^\prime\prime \) successively in place of the zeros of \( F^0 \) in their places from right to left according to the indications \( A_1(\alpha), A_2(\alpha), \ldots, A_{k-1}(\alpha) \), is done in stages: first setting the pairs \( i', i'' \) as outermost pairs from right to left; when reaching the initial 0, we restart if necessary on the right again with the replacement of the remaining zeros by the remaining pairs \( i', i'' \) in ascending fashion from right to left. For example for \( k = 6 \) and \( A(11111) = 01122, \) (or \( A(12122) = 00201 \)), we go from \( F^0 \) to

\[
\begin{align*}
&0200002100001 \quad \text{or} \quad 0300003221001 \quad \text{to} \quad 0240042130031 \\
&0236642135531 
\end{align*}
\]

the last 2 (or 1) yielding 4 (or 2) zeros separating the two 1’s, etc. Thus, given \( A(\alpha) \), we recover \( F(\alpha) \), showing that effectively the correspondence in the statement is a bijection. □

**Theorem 12.** The correspondence \( g \) between Example 9 and Theorem 10 extends uniquely for each \( k > 1 \) and each \( k \)-germ \( \alpha \), so that in terms of \( \alpha \) considered as an RGS, \( g(o(\alpha)) = A_i(\alpha) \) is expressible either as \( \ell \) or as \( k - \ell \), for a fixed \( \ell \in \mathbb{Z} \) such that \( 0 \leq \ell < k \).

**Proof.** The disjunctive options at the end of the statement depend on whether the substring \( k'k'' \) lies to the left or to the right of \( 1' \). In the first case, \( g(o(\alpha)) \) is of the form \( \ell \); otherwise, it is of the form \( k - \ell \), for if \( k \) is increased to \( k + 1 \), then the substring \( (k+1')(k+1)'' \) must separate \( k' \) and \( k'' \), thus adding one unit to \( g(o(\alpha)) \), so that \( k - \ell \) becomes \( (k + 1) - \ell \). □

**Example 13.** The nonzero values \( g(k) \) are initially as follows: \( g(3) = k - 2, g(7) = k - 3, k(8) = 1, g(11) = k - 2, g(12) = k - 3, g(17) = k - 2, g(19) = k - 4, g(21) = 1, g(22) = k - 3, \)
\( g(25) = k - 2, g(26) = 1, g(30) = k - 3, g(31) = 1, g(33) = k - 4, g(34) = 2, g(35) = 1, g(38) = k - 2, g(39) = k - 3, g(40) = k - 4, \) etc.

**Corollary 14.** (A) The leftmost entry in the substring \( W^i \) of \( F(\alpha) = X^i|Z|Y|W^i \) is \( i'' \).

(B) If the substring \( k'k'' \) of \( F(\alpha) \) appears to the left of \( i' \) in \( F(\alpha) \), then \( g(o(\alpha)) \) equals the number of pairs \( (j', j'') \) in the interval \( [i', i''] \), for all pertaining \( j \in [1, k] \). In particular, \( F(\alpha) \) ends at the substring \( 1'1'' \) if and only if \( g(o(\alpha)) = 0 \).

(C) If \( k'k'' \) lies in \( [i', i''] \) then \( k'k'' \) is green, as in Figures 1–2, and \( g(o(\alpha)) = k - j \).

**Proof.** (A)–(B) follow from Theorems 11–12. For (C): the value of \( j = j(\alpha) \) is determined as follows. If \( \beta = 1 + i(\alpha) \), where \( \beta = \beta(\alpha) \) is the parent of \( \alpha \), then \( j \) is the sum of \( g(o(\beta)) \) (which is as in (B)) plus the leftmost red number of \( F(\alpha) \); let this \( \alpha \) be denoted \( \alpha_0 \); iteratively for \( m = 0, 1, \ldots \), we determine successive new values of \( g = k - j \) as follows; if \( \alpha_m \) has a child \( \alpha_{m+1} \) with \( i(\alpha_m) = i(\alpha_{m+1}) \) and \( k'k'' \) is green in \( F(\alpha_{m+1}) \), then \( j(\alpha_{m+1}) = 1 + j(\alpha_m) \). □

**Example 15.** Let \( k = 5 \). Then, \( g(21) = g(o(1110)) = 1 \) because \( [i', i''] = [2', 2''\] contains just the pair \( (4', 4'') \), accounting for just 1 pair, concluded from Corollary 14(B). Also, in \( \alpha = 1111 \), \( k'k'' \) is green and \( g(22) = g(o(1111)) = g(o(\alpha)) = k - 3 \), where \( j = 3 \) is the sum of \( g(o(\beta)) = g(o(1110)) = g(21) = 1 \) and the leftmost red number of \( F(\alpha) \), namely 2. Also, \( g(28) = g(o(1200)) = 0 \) has child \( \alpha = 1210 \) with \( g(o(\alpha)) = g(30) = k - 3 \), because the leftmost red entry of \( F(\alpha) \) is 3. The child \( \alpha' = 1220 \) of \( \alpha \) has \( g(o(\alpha')) = 33 = k - (3 + 1) = k - 4 \). However, the child \( \alpha'' = 1230 \) of \( \alpha' \) has \( g(o(\alpha'')) = 37 \). Now, the child 121 of \( \alpha \) has \( g(o(1211)) = 1 \) because \( 1' \) is the leftmost number of \( W^1 \) and there is only 1 pair of appearances of a member of \([1, k - 1] = [1, 4]\), namely \( 3'3'' \), between \( 1' \) and \( 1'' \).
3 Sequencing single changes in nest signatures

We introduce strings $A^j_i$, for all pairs $(i, j) \in \mathbb{Z}^2$ such that $1 < i \leq j$. The entries of such strings are pairs of integers $(\iota, \ell) = \iota_{\ell}$, starting with $B^1_1 = 1_1$, initial case of notation $B^j_1 = 1_j$, where $j > 1$. The strings $A^j_i$ are as shown in Table 1. For us, the components $\iota$ in the entries $\iota_{\ell}$ represent the indices $i = i(\alpha)$ in Theorem 1 in their order of appearance in $S$, and $\ell$ is an indicator to distinguish different entries $\iota_{\ell}$ with $\iota$ locally constant; later on, $\ell$ is dropped.

TABLE 1

| $A^2_2$ | $2_1 | B^2_1$ | $B^2_1 = 2_1 | 1_1 | 1_2$; |
| $A^3_2$ | $2_2 | B^3_1$ | $B^3_1 = 2_2 | 1_1 | 1_2 | 1_3$; |
| $A^4_2$ | $2_3 | B^4_1$ | $B^4_1 = 2_3 | 1_1 | 1_2 | 1_3 | 1_4$; |
| $A^5_2$ | $2_4 | B^5_1$ | $B^5_1 = 2_4 | 1_1 | 1_2 | 1_3 | 1_4 | 1_5$; |
| $A^6_2$ | $3_1 | B^6_1 | A^2_3$ | $3_1 | 1_1 | 2_1 | 1_2 | 1_3 | 1_4$; |
| $A^7_2$ | $3_2 | B^7_1 | A^2_3$ | $A^3_3 = 3_2 | 1_1 | 2_1 | 1_2 | 1_3 | 2_3 | 1_1 | 1_2 | 1_3 | 1_4$; |
| $A^8_2$ | $3_3 | B^8_1 | A^2_3$ | $A^3_3 = 3_3 | 1_1 | 2_1 | 1_2 | 1_3 | 1_4 | 2_4 | 1_1 | 1_2 | 1_3 | 1_4 | 1_5$; |
| $A^9_2$ | $4_1 | B^9_1 | A^2_3 | A^3_3$ | $4_1 | 1_1 | 2_1 | 1_2 | 2_2 | 1_1 | 1_2 | 1_3 | 3_2 | 1_1 | 1_2 | 1_3 | 2_3 | 1_1 | 1_2 | 1_3 | 1_4$; |
| $A^{10}_2$ | $4_2 | B^2_3 | A^3_3$ | $A^4_3 = 4_2 | 1_1 | 2_1 | 1_2 | 2_2 | 1_1 | 1_2 | 1_3 | 1_4 | 2_4 | 1_1 | 1_2 | 1_3 | 1_4 | 1_5$; |
| $A^{11}_2$ | $5_1 | B^2_3 | A^2_3 | A^4_3$ | $A^5_3$; |
| $A^{12}_2$ | $5_2 | B^2_3 | A^2_3 | A^4_3 | A^5_3$; |
| $A^{13}_2$ | $6_1 | B^2_3 | A^2_3 | A^4_3 | A^5_3 | A^6_3$; |
| $A^{14}_2$ | $i+j | B^i_1 | A^j_2 | \cdots | A^{i+j-1}_2 | A^i_2 | \cdots | A^1_2 | A^0_2 | \cdots$ |

Recalling the conditions (B) and (C) of Corollary 14, let

$$h(\alpha) = \begin{cases} 
g(o(\alpha)), & \text{if } g(o(\alpha)) \text{ is as in (B)}; \\ 
g(o(\alpha)) - k, & \text{if } g(o(\alpha)) \text{ is as in (C)}. 
\end{cases}$$

(1)

Now, consider the infinite sequence $A$ of pairs of integers $k_{\ell}$ formed as the concatenation

$$A = A^1_1|A^2_2|\cdots|A^j_i|\cdots = *1_1|A^2_2|\cdots|A^j_i|\cdots,$$

(2)

where $A^1_1 = *|1_1 = *1_1$ represents the first two lines of tables as in Figures 1–2; (standing for the root of $T_\alpha$, * represents the first such line, while $A^1_1$ represents the second line).

Table 2, exemplifying display (2), has its double-line heading formed by subsequent terms of the sequence $A$. The third heading line represents the root * of the trees $T_\alpha$ and all subsequent parameters $i = i(\alpha) > 1$, as in Theorem 1. The fourth line contains the corresponding values $h(\alpha)$ for RGS’s $\alpha$ representing $k$-germs $\alpha$ with parameter $i = i(\alpha) > 1$. In every column of Table 2, the subsequent values below that fourth line are the values $h(\alpha)$ for RGS’s $\alpha$ representing $k$-germs $\alpha$ with parameter $i = i(\alpha) = 1$. Thus, below the third heading line, the subsequent values of each column represent the changes $h(\alpha)$ corresponding to all the maximal paths of trees $T_\alpha$ that, after its first node $\alpha$, has all other nodes $\alpha$ with $i = i(\alpha) = 1$. Note that $A^1_1$ is represented in table 2 in column notation as $[^{[B^1_1]}]$. In the same way, we use notation $A^1_1 = [^{[B^1_1]}]$.

Each prefix of $A$ corresponds to all $k$-germs representing a specific RGS $\alpha$, for increasing values of $k > 1$. Such prefix is assigned $h(\alpha)$ as integer of change, in accordance to Corollary 14, but for the initial position, that is assigned asterisk * to represent all the roots of the
trees $T_k$, for all $k > 1$. More specifically, all prefixes of the sequence $A$ with Catalan-number lengths $C_k$ are the strings of locations $i = i(\alpha)$ in the natural order of the corresponding trees $T_k$, while the values $h(\alpha)$ of the participating RGS’s $\alpha$ occupy the subsequent positions down below the heading lines.

### TABLE 2

| $A_1^1$ | $A_2^2$ | $A_3^3$ | $A_4^4$ |
|---------|---------|---------|---------|
| $[s]_{B_1}$ | $A_1^2$ | $[s]_{B_1}$ | $A_2^2$ | $A_3^3$ | $A_4^4$ |
| *       | 2       | 3       | 2       | 4       | 2       | 3       | 2       | 2       |
| *       | 0       | 0       | -3      | 0       | 0       | -4      | 1       | 0       | 0       | -3      | -4      | 0       |
| 0       | -2      | 0       | 1       | -2      | 0       | -2      | 0       | -3      | -2      | 0       | 1       | 2       | -2      |
| 0       | 0       | 0       | -3      | 0       | 0       | 1       | 0       | 0       | 1       | -3      | 0       | 0       | -4      |

We extend Table 2 in Figure 3 for $k = 2, 3, 4, 5, 6, 7$, where the heading line of the top layer extends the third heading line of Table 2, leading effectively all the resulting columns of values $h(\alpha)$ in the table, for $k < 7$. These values $h(\alpha)$ are also viewable in Table 3, which is a left-to-right list representation of $T_6$ whose nodes are pairs $(i(\alpha), h(\alpha))$ for the subsequent RGS’s $\alpha$ in $S$, and if some $h(\alpha)$ equals a negative integer $-\eta < 0$, then it is shown as $\bar{\eta}$. With such notation, the leftmost column of Table 3 shows the children of the root ($*, *$) of $T_6$. The adequately indented subsequent columns show the remaining descendant nodes at increasing distances from the root ($*, *$). Also in Table 3, horizontal lines separate the node sets of $T_3 - (*, *), T_4 - T_3, T_5 - T_4$ and $T_6 - T_5$.

By reading and then writing down the entries in the successive columns of Table 2, and more extensively of Figure 3, from top to bottom and then from left to right, we obtain the integer sequence $h(S)$ formed by the values $h(\alpha)$ associated to the RGS’s $\alpha$ of $S$. For example, starting with Table 2, we have that $h(S) = (h(0), \ldots, h(41)) =$

$(*, 0, 0, -2, 0, 0, 0, -3, 1, 0, 0, -2, -3, 0, 0, 0, -2, 0, -4, 0, 1, -3, 0, 0, -2, 1, 0, 0, 0, -3, 1, 0, -4, 2, 1, 0, 0, -2, -3, -4, 0, \ldots)$.

**Remark 16.** The numbers in Italics in Table 2 initiate the subsequence $h(\Phi_1)$ of $h$-values of a subsequence $\Phi_1$ of $S$. Such $h$-values will allow to present the continuation of the sequence of changes of the Dyck-path signatures. These numbers in Italics are extended with yellow background in Figure 3. Expressing $h(\Phi_1)$ with its initial terms as in Table 2, we may write $h(\Phi_1) =$

$h(j)_{j=1,2,4,6,7,11,19,21,22,26,30,33,34,35,40,\ldots} = (0, 0, -2, 0, -3, 1, -3, 0, -3, 1, 0, -4, 2, 1, 0, 0, -2, -3, -4, 0, \ldots)$.

In order to use $\Phi_1$, we recur to Catalan’s reversed triangle $\Delta'$, whose initial lines, for $k = 0, 1, \cdots, 7$, are shown on the middle left enclosure of Figure 3. In fact, $\Delta'$ is obtained from Catalan’s triangle $\Delta [3]$ by reversing its constituent lines, so that with the notation of [3], the portion of $\Delta'$ shown in its enclosure in Figure 3 may be written as in Table 4.
Figure 3: Extension of Table 2, partial view of $\Delta'$ and members of $\Phi_1$, for $k = 2, 3, 4, 5, 6, 7$
|   |   |   |   |   |
|---|---|---|---|---|
| (1, 0) |   |   |   |   |
| (2, 0) | (1, 2) | (1, 0) |   |   |
| (3, 0) | (1, 0) |   |   |   |
|   | (2, 3) | (1, 1) | (1, 0) |   |
|   | (2, 0) | (1, 2) | (1, 3) | (1, 0) |
| (4, 0) | (1, 0) |   |   |   |
|   | (2, 0) | (1, 2) | (1, 0) |   |
|   | (3, 4) | (1, 0) |   |   |
|   | (2, 1) | (1, 3) | (1, 0) |   |
|   | (2, 0) | (1, 2) | (1, 1) | (1, 0) |
| (5, 0) | (1, 0) |   |   |   |
|   | (2, 0) | (1, 2) | (1, 0) |   |
|   | (3, 0) | (1, 0) |   |   |
|   | (2, 3) | (1, 1) | (1, 0) |   |
|   | (2, 0) | (1, 2) | (1, 3) | (1, 0) |
| (4, 5) | (1, 0) |   |   |   |
|   | (2, 0) | (1, 2) | (1, 0) |   |
|   | (3, 1) | (1, 0) |   |   |
|   | (2, 4) | (1, 2) | (1, 0) |   |
|   | (2, 0) | (1, 2) | (1, 4) | (1, 0) |
|   | (3, 0) | (1, 0) |   |   |
|   | (2, 3) | (1, 1) | (1, 0) |   |
|   | (2, 1) | (1, 3) | (1, 4) | (1, 0) |
|   | (2, 0) | (1, 2) | (1, 3) | (1, 1) | (1, 0) |
| (4, 0) | (1, 0) |   |   |   |
|   | (2, 0) | (1, 2) | (1, 0) |   |
|   | (3, 4) | (1, 0) |   |   |
|   | (2, 1) | (1, 3) | (1, 0) |   |
|   | (2, 0) | (1, 2) | (1, 1) | (1, 0) |
| (3, 5) | (1, 0) |   |   |   |
|   | (2, 2) | (1, 4) | (1, 0) |   |
|   | (2, 1) | (1, 3) | (1, 1) | (1, 0) |
|   | (2, 0) | (1, 2) | (1, 2) | (1, 1) | (1, 0) |
|   | (3, 0) | (1, 0) |   |   |
|   | (2, 3) | (1, 1) | (1, 0) |   |
|   | (2, 4) | (1, 2) | (1, 1) | (1, 0) |
|   | (2, 5) | (1, 3) | (1, 2) | (1, 1) | (1, 0) |
|   | (2, 0) | (1, 2) | (1, 3) | (1, 4) | (1, 5) | (1, 0) |   |
Remark 17. Both in Table 2 and at the top layer of Figure 3, we have the representations (to be called formations) of: (1) \((A_1^1)\), namely the leftmost column, with a total of two entries; (2) \((A_1^1|A_2^2)\), namely the two leftmost columns, with a total of 5 entries; (3) \((A_1^1|A_2^2|A_3^3)\), namely the five leftmost columns, with a total of 14 entries; (4) \((A_1^1|A_2^2|A_3^3|A_4^4)\), namely the 14 columns in Table 2 or the 14 leftmost columns in Figure 3, with a total of 42 entries; and in addition Figure 3 shows in its whole top layer the formation of: (5) \((A_1^1|A_2^2|A_3^3|A_4^4|A_5^5)\), namely the top 42 columns in Figure 3, with a total of 132 entries. These five formations correspond respectively to the trees \(T_2\), \(T_3\), \(T_4\), \(T_5\) and \(T_6\).

The number of columns of values \(h(\alpha)\) in these formations are respectively: (1) \(\tau_1^1 = 1\), (2) \(\tau_2^2 = \tau_2^3 = 2\); (3) \(\tau_3^3 = \tau_3^4 = 5\); (4) \(\tau_4^4 = \tau_4^5 = 14\); and (5) \(\tau_5^5 = \tau_5^6 = 42\). We subdivide the sets of respective columns according to the corresponding lines of \(\Delta'\) considered as integer partitions \(\Delta'_k\), namely: \(\Delta'_0 = (1)\), \(\Delta_1 = (1, 1)\), \(\Delta_2 = (2, 2, 1)\), \(\Delta'_3 = (5, 5, 3, 1)\), \(\Delta'_4 = (14, 14, 9, 4, 1)\) following with \(\Delta'_5\) to be discussed subsequently.

Figure 3 contains the continuation of the formations above for \(k = 7\), namely \((A_1^1|A_2^2|A_3^3|A_4^4|A_5^5)\), extending the mentioned top layer of \(\tau_5^5 = 42\) columns by a second and third layers (with \(\tau_5^5 = 42\) and \(\tau_3^3 = 28\) columns, respectively) and then by two additional parts in the fourth layer (given in reversed order, with \(\tau_4^4 = 14\) and \(\tau_2^2 + \tau_3^1 = 5\) columns, 12 columns)
and as in Figure 3 end up with null values 

\[
\text{Remark 18. The sequence } \Phi_1 \text{ of } S, \text{ a member of a family of subsequences } \{\Phi_j|1 \leq j \in \mathbb{Z}\} \text{ satisfying rules 1–3 below, is such that } i(\Phi_1) \text{ is the subsequence of } i(S) \text{ formed by all indices } i(\alpha) \text{ larger than 1, namely the continuation of the heading line of Figure 3. Now, the rules that the subsequences } \Phi_j \text{ must satisfy, for every } 1 < j \in \mathbb{Z}, \text{ are:}
\]

1. the first term of } \Phi_j \text{ is}

\[
\phi_1 = \begin{cases} 
\text{the RGS 1} & \text{, if } j = 1; \\
\text{the smallest RGS with suffix } (j - 1)(j - 1) & \text{, if } j > 1;
\end{cases}
\]

2. if } \alpha = a_{k-1}\cdots a_2a_1 \in \Phi_j \text{ and either } a_1 = 0 \text{ or } a_{k-1}\cdots a_2a'_1 \notin \Phi_j \text{ for every } a'_1 < a_1, \text{ then } \alpha|j \in \Phi_j \text{ for } j \in [0, a_1]; \text{ in that case, if } a_{j'} \notin \Phi_j \text{ with } a_{j'} = a_{k-1}\cdots a_2(a_1 + j'), \text{ for } 1 \leq j' \in \mathbb{Z}, \text{ then } \alpha_{j'}|j \in \Phi_j, \text{ for } j \in [0, a_1];

3. for each maximal subsequence } S = (\iota, 2, \ldots, 2) \text{ of } i(S) (\iota > 2), \text{ if there are } z > 0 \text{ penultimate terms } i = 2 \text{ of } S \text{ heading each a maximal vertical prefix of length } y > 0 \text{ in } h(\Phi_j) \text{ and ending at } h(\alpha_j) = h(a_{k-1}\cdots a_3(y+j)) \text{ } (j \in [0, z]), \text{ then } \alpha_{j'} = a_{k-1}\cdots a_3(y+z)j' \in \Phi_j, \text{ for } j' \in [y, y+z], \text{ yielding vertical suffix } \{h(\alpha_{j'}); j' \in [y, y+z]\}.

The columns in the formations of Remark 17 and as in Figure 3 end up with null values 

\[h(\alpha) = 0,\] 
which correspond to the terminal nodes } \alpha \text{ of maximal paths that after their initial nodes } \beta \text{ with } i(\beta) > 1, \text{ have the remaining nodes } \beta' \text{ with } i = i(\beta') = 1. \text{ Clearly, the associated nodes } \alpha \text{ have degree 1 in the pertaining trees } \mathcal{T}_k.
Theorem 19. Let \( 1 \leq k \in \mathbb{Z} \) and let \( \alpha \) be a node of \( T_k \). Then,

1. if \( \alpha \) is a terminal node of a maximal path of \( T_k \) whose initial node \( \beta \) has \( i(\beta) > 1 \) and whose remaining nodes \( \gamma \) have \( i(\gamma) = 1 \), then \( g(\alpha) = 0 \);

2. if \( \alpha = a_k \cdots a_1 \) with \( a_k = 1 \) and \( a_j = 0 \) for \( j = 1, \ldots, k-2 \), then \( g(\alpha) = 0 \).

Proof. Item 1 in the statement arises because of the presence of the substring \( 11'' \) in \( F(\alpha) \). Item 2 arises because of the presence of the substring \( j'j'' \) in \( F(\alpha) \).

\( \square \)

| \( \tau_1 \) | \( \tau_2 \) | \( \tau_3 \) | \( \tau_4 \) | \( \tau_5 \) | \( \tau_6 \) | \( \tau_7 \) |
| --- | --- | --- | --- | --- | --- | --- |
| 429 | 297 | 165 | 90 | 48 | 60 | 75 |
| 429 | 297 | 165 | 90 | 48 | 60 | 75 |
| 429 | 297 | 165 | 90 | 48 | 60 | 75 |
| 429 | 297 | 165 | 90 | 48 | 60 | 75 |
| 429 | 297 | 165 | 90 | 48 | 60 | 75 |
| 429 | 297 | 165 | 90 | 48 | 60 | 75 |
| 429 | 297 | 165 | 90 | 48 | 60 | 75 |

Let \( 1 \leq k \in \mathbb{Z} \) and let \( \alpha \) be a node of \( T_k \). To employ subsequently, we define:

\[
h'(\alpha) = \begin{cases} 
  h(\alpha) & \text{if } h(\alpha) \geq 0; \\
  k - h(\alpha) & \text{if } h(\alpha) < 0.
\end{cases}
\]

Theorem 20. Let \( 1 \leq k \in \mathbb{Z} \). Let \( \alpha_1 \) be a node of \( T_k \). Then, \( \alpha'_1 = 1|\alpha_1 \) is a node of \( T_{k+1} \) and

1. if \( h(\alpha_1) \in \Phi_1 \), then \( h(\alpha'_1) \in \Phi_1 \) and \( h(\alpha'_1) = h'(\alpha_1) \);

2. if \( h(\alpha_1) \notin \Phi_1 \), then \( h(\alpha'_1) \notin \Phi_1 \) and \( h(\alpha'_1) = h(\alpha_1) \).

Proof. Item 1 in the statement occurs exactly when the substring \( k'k'' \) in \( F(\alpha) \) changes position from one side of \( 1' \) to the opposite side in the procedure of Theorem 1 starting at the parent \( \beta \) of \( \alpha \) and ending at \( \alpha \). Item 2 occurs exactly when that is not the case.

\( \square \)

Example 21. Since \( \alpha_1 = 1 \) is a node of \( T_2 \) as in Theorem 19 item 1, then \( \alpha'_1 = 1|\alpha_1 = 11 \) is a node of \( T_3 \) with \( h(\alpha'_1) = h(11) = h'(1) = h(1) - k = 0 - 2 = -2 \in \Phi_1 \), by Theorem 20 item 1. This is indicated by \( h(1) = 0 \) in the upper leftmost yellow box in Figure 3 and its accompanying \( h(11) = -2 \) as the upper leftmost red integer in the figure. Note that this pattern is continued by associating each yellow box in Figure 3 to a corresponding red integer for all \( k < 7 \). We can annotate this via the successive pairs \((\alpha_1, h(\alpha_1))\) taken by reading the data in Figure 3 from left to right and from top downward:

\( (1(0),11(−2)),(10(0),110(−3)),11(−2),111(1)),(100(0),1100(−4)),(110(−3),1110(1)),(111(1),1111(−3)),(122(−3),1122(1)) \).

The last pair here arises from \( h(122) = -3 \), which follows from Corollary 23, below.
Theorem 22. Let $1 < j \leq k \in \mathbb{Z}$. Let $\alpha_j = 1 \cdots (j-1)(j-1)a_{k-j-1} \cdots a_1$ be a node of $T_k$.
Then, $\alpha'_j = 12 \cdots (j-1)ja_{k-j-1} \cdots a_1$ is a node of $T_k$ and

1. if $h(\alpha_j) \in \Phi_j$ then $h(\alpha'_j) = h'(\alpha_j)$;
2. if $h(\alpha_j) \notin \Phi_j$ then $h(\alpha'_j) = h(\alpha_j)$.

Proof. Similar to the proof of Theorem 20. □

| $\alpha$ | $h(\alpha)$ | $\alpha_1$ | $h(\alpha_1)$ | $\alpha_2$ | $h(\alpha_2)$ | $\alpha_3$ | $h(\alpha_3)$ | $\alpha_4$ | $h(\alpha_4)$ | $\alpha_5$ | $h(\alpha_5)$ |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 0 * 00 | 0 | 0 | 0 | 10 | 0 | 0 | 0 | 0 | 12 | 0 | 0 |
| 1 0 01 | 0 | 0 | 0 | 11 | -2 | 12 | 0 | 0 | 11 | -2 | 12 | 0 |
| 0 * 000 | 0 | 0 | 0 | 100 | 0 | 120 | 0 | 0 | 110 | -3 | 120 | 0 |
| 1 0 001 | 0 | 0 | 0 | 101 | 0 | 121 | 0 | 0 | 111 | 1 | 121 | -2 |
| 10 0 010 | 0 | 0 | 0 | 110 | -3 | 120 | 0 | 0 | 111 | 1 | 121 | -2 |
| 11 -2 011 | -2 | 0 | 0 | 111 | 1 | 121 | 0 | 0 | 112 | 0 | 123 | 0 |
| 12 0 012 | 0 | 0 | 0 | 112 | 0 | 122 | 1 | 123 | 0 | 123 | 0 | 0 |
| 0 * 0000 | 0 | 0 | 0 | 1000 | 0 | 1200 | 0 | 0 | 1100 | -4 | 1200 | 0 |
| 1 0 0001 | 0 | 0 | 0 | 1001 | 0 | 1201 | 0 | 0 | 1101 | 0 | 1201 | 0 |
| 10 0 0010 | 0 | 0 | 0 | 1010 | 0 | 1210 | -3 | 0 | 1110 | 1 | 1210 | -3 |
| 11 -2 0011 | -2 | 0 | 0 | 1011 | -2 | 1211 | 1 | 0 | 1111 | -3 | 1211 | 1 |
| 12 0 0012 | 0 | 0 | 0 | 1012 | 0 | 1212 | 0 | 0 | 1112 | 0 | 1212 | 0 |
| 100 0 0100 | 0 | 0 | 0 | 1100 | -4 | 1200 | 0 | 0 | 1100 | 1200 | 0 |
| 101 0 0101 | 0 | 0 | 0 | 1101 | 0 | 1201 | 0 | 0 | 1101 | 0 | 1201 | 0 |
| 110 -3 0110 | -3 | 0 | 0 | 1110 | 1 | 1210 | 0 | 0 | 1110 | 1 | 1210 | -3 |
| 111 1 0111 | 1 | 0 | 0 | 1111 | -3 | 1211 | 1 | 0 | 1111 | -3 | 1211 | 1 |
| 112 0 0112 | 0 | 0 | 0 | 1112 | 0 | 1212 | 0 | 0 | 1112 | 0 | 1212 | 0 |
| 120 0 0120 | 0 | 0 | 0 | 1120 | 0 | 1220 | -4 | 1230 | 0 | 1230 | 0 |
| 121 -2 0121 | -2 | 0 | 0 | 1121 | -2 | 1221 | 2 | 1231 | -2 | 1231 | -2 |
| 122 -3 0122 | -3 | 0 | 0 | 1122 | 1 | 1222 | 1 | 1232 | -3 | 1232 | -3 |
| 123 0 0123 | 0 | 0 | 0 | 1123 | 0 | 1223 | -4 | 1233 | 0 | 1233 | 0 |

Corollary 23. Let $1 \leq k \in \mathbb{Z}$. Let $\alpha_2 = 11a_{k-3} \cdots a_1$ be a node in $T_k$. Then, $\alpha'_2 = 12a_{k-3} \cdots a_1$ is a node of $T_k$ and

1. if $h(\alpha_2) \in \Phi_2$ then $h(\alpha'_2) = h'(\alpha_2)$;
2. if $h(\alpha_2) \notin \Phi_2$ then $h(\alpha'_2) = h(\alpha_2)$.

Example 24. Applying Corollary 23 to $\alpha_2 = 11, 110, 111, 112$, with respective $h(\alpha_2) = -2, -3, 1, 0 \in \Phi_2$ yields $\alpha'_2 = 12, 120, 121, 122$ with respective $h(\alpha'_2) = 0, 0, -2, -3$. In Figure 4, the RGS’s $\alpha_2$ are shown in light-blue boxes while the corresponding RGS’s $\alpha'_2$ are shown in yellow boxes. Moreover, Figure 4 extends this coloring for $k \leq 7$.

Corollary 25. Let $1 \leq k \in \mathbb{Z}$. Let $\alpha_3 = 122a_{k-4} \cdots a_1$ be a node of $T_k$. Then, $\alpha'_3 = 123a_{k-4} \cdots a_1$ is a node of $T_k$ and

1. if $h(\alpha_3) \in \Phi_3$ then $h(\alpha'_3) = h'(\alpha_3)$;
2. if $h(\alpha_3) \notin \Phi_3$ then $h(\alpha'_3) = h(\alpha_3)$.
Example 26. Applying Corollary 25 to $\alpha_3 = 122, 1220, 1221, 1222, 1223$ with respective $h(\alpha_3) = -3, -4, 2, 1, 0 \in \Phi_3$ yields $\alpha'_3 = 123, 1230, 1231, 1232, 1233$ with respective $h(\alpha'_3) = 0, 0, -2, -3, -4$. In Figure 4, the RGS’s $\alpha_3$ are shown in thick black while the corresponding RGS’s $\alpha'_3$ are shown in thick red. Moreover, Figure 4 extends this font treatment for $k \leq 7$. For $k = 7$, numbers in Italics in Figure 4 corresponds to members of $\Psi_4$.

Both the integer-valued functions $i = i(\alpha)$ of Theorem 1 and $h = h(\alpha)$ of display (1) have the same domain, $S \setminus \beta(0)$.

Theorem 27. **(A)** The set of nodes of $T_{k+1}$ is represented by the string $A_k^k = A^1_k | A^2_k | \cdots | A^{k-1}_k | A^{k-1}_{k-1}$ with partition $\{A^1_k, A^2_k, \ldots, A^{k-1}_k, A^k_{k-1}\}$, each $A^i_k$ taken as a set of columns (Table 2, Figures 3-4) further refined by splitting the last column $A^k_{k-2}$ of $A^k_{k-1}$ into the set $B^k_{k-2}$ of its first $k - 1$ entries and the set $C^k_{k-2}$ formed by its last entry, $a_{k-1} a_{k-2} \cdots a_1 = 12 \cdots (k - 1)$. The sizes $|A^1_k|, |A^2_k|, \ldots, |A^{k-1}_k|, |B^k_{k-2}|, |C^k_{k-2}|$ form the line $\Delta^k_{k-1}$ of $\Delta'$. **(B)** The sequence $h(S \setminus \beta(0))$ is generated by stepwise consideration of the trees $T_{k+1}$ ($1 \leq k \in \mathbb{Z}$). In the $k$-th step, the determinations expressed in Theorems 20-22 are performed in the natural order of the $(k + 1)$-germs $\alpha_j$. More specifically, the $k$-step performs those determinations, namely $(\alpha_j, h(\alpha_j)) \rightarrow (\alpha'_j, h(\alpha'_j))$, for the lines of $\Delta'$ corresponding to the sets $A^j_k$ ($j = 1, \ldots, k - 1$), ending up with the determinations $(\alpha_k, h(\alpha_k)) \rightarrow (\alpha'_k, h(\alpha'_k))$ in the line corresponding to $B^k_{k-2}$ and $(\alpha_{k+1}, h(\alpha_{k+1})) \rightarrow (\alpha'_{k+1}, h(\alpha'_{k+1}))$ in the final line, corresponding to $C^k_{k-2}$.

Proof. Item (A) allows to represent the set of nodes of $T_{k+1}$ via $A_k^k$ and $\Delta'_{k-1}$. This is used in item (B) to express the stepwise nature of the generation of the sequence $h(S \setminus \beta(0))$. In fact, the methodology in the statement is obtained by integrating steps applying Theorems 20, 22 in the way prescribed, that yields the correspondence with the lines of $\Delta'$.

Example 28. Theorem 27 is exemplified via Table 5, where the lists corresponding to $T_2$, $T_3$ and $T_4$ are represented according to the respective pairs $(\alpha, h(\alpha))$ indicating column pairs $(\alpha, h(\alpha))$ and $(\alpha_j, h(\alpha_j))$, for $j = 1, 2, 3, 4, 5$, as shown in the heading line of the table.

The first pair, $(\alpha, h(\alpha))$ shows RGS’s $\alpha$ in each case and their corresponding $h(\alpha)$. The following pair, $(\alpha_1, h(\alpha_1))$, shows the $k$-germs $\alpha_1$ corresponding to the RGS’s $\alpha$ of the first column and $h(\alpha_1) = h(\alpha)$ but in bold trace if corresponding to a yellow box as in Figure 3; in that case, the subsequent determinations $(\alpha_1, h(\alpha_1)) \rightarrow (\alpha'_1, h(\alpha'_1))$ have the corresponding $h(\alpha'_1)$ in Italics. This is the case of $h(01) = 0$ in bold trace and $h(11) = -2$ in Italics, that we indicate “$h(01) = 0 \rightarrow (11) = -2$”. If a determination $(\alpha_2, h(\alpha_2)) \rightarrow (\alpha'_2, h(\alpha'_2))$ happens, then the numbers in Italics are assigned on their right to numbers in bold trace, again. The cases with bold trace and Italics in Table 5 can then be summarized as follows:

$h(01)=0 \rightarrow 1, h(11)=-2 \rightarrow 2, h(12)=0, \quad h(011)=0 \rightarrow 1, h(111)=1 \rightarrow 2, h(121)=2, \quad h(011)=0 \rightarrow 1, h(111)=-2 \rightarrow 2, h(121)=-3$.

Corollary 29. The sequence of pairs $(i(S \setminus \beta(0)), h(S \setminus \beta(0)))$ allows to retrieve any odd-graph vertex $v$, by locating its oriented $n$-cycle in its uniform 2-factor [5, 9], or its cyclic class $\mod n$, and then by locating $v$ in that cycle, or class, from its determining extended Dyck.
word; it also allows to enlist all such vertices by ordering their cycles, or classes, including all vertices in each such cycle, or class, starting with the corresponding extended Dyck word.

Proof. The function \( i(S \setminus \beta(0)) \) defined in Theorem 1 yields the required change locations while the function \( h(S \setminus \beta(0))) \) yields the specific changes, as specified in Theorem 27. This produces the corresponding signatures. Then, Theorem 10 allows to recover the original Dyck words from their signatures and thus the odd-graph vertices, by local translation in their containing \( n \)-cycles of uniform 2-factors, or cyclic classes mod \( n \).

\[ \square \]

Remark 30. Theorem 11 establishes a form of counting the Catalan’s numbers [11, ex. 6.19, pg. 219–229], as in the characterization of RGS’s in [3, 4], which differs from a similar characterization in [11, item (u), pg. 224]. In fact, our form here may be compared to that of [11, item (u), pg 224] (apart from the comparison of this with the RGS’s of [3]) in that it accompanies the Catalan’s-numbers counting list of RGS’s in apparently similar reversed order. Moreover, the first RGS, namely the null root RGS 0, correspond to the signatures \( 12 \cdots k \), for all \( 0 < k \in \mathbb{Z} \), while the last RGS of any fixed length \( k \) corresponds to the signatures \( 0^k \). These initial and terminal assignments coincide with those of the mentioned item (u). However, these counting lists with same initial and terminal terms differ in general.

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