Competing Orders in Coupled Luttinger Liquids

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We consider the problem of two coupled Luttinger liquids both at half filling and at low doping levels, to investigate the problem of competing orders in quasi-one-dimensional strongly correlated systems. We use bosonization and renormalization group equations to investigate the phase diagrams, to determine the allowed phases and to establish approximate boundaries among them. Because of the chiral translation and reflection symmetry in the charge mode away from half filling, orders of charge density wave (CDW) and spin-Peierls (SP), diagonal current (DC) and d-density wave (DDW) form two doublets and thus can be at most quasi-long range ordered. At half-filling, Umklapp terms break this symmetry down to a discrete group and thus Ising-type ordered phases appear as a result of spontaneous breaking of the residual symmetries. Quantum disordered Haldane phases are also found, with finite amplitudes of pairing orders and triplet counterparts of CDW, SP, DC and DDW. Relations with recent numerical results and implications to similar problems in two dimensions are discussed.

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I. INTRODUCTION

The problem of the nature of the phase diagram of the cuprate superconductors remains at the center of research in the physics of strongly correlated electron systems. Recent work has focused on the possible competing orders responsible for the known features of the phase diagram as well as to the unusual physical properties of the pseudo-gap regime. A number of candidate competing orders have been considered, including antiferromagnetism (AF), d-wave pairing (DSC), incommensurate charge ordered states and other liquid crystal-like phases, and d-density wave states (DDW) (also known as staggered flux states (SF) or orbital antiferromagnetism (OAF)), among others.

SO(5) theory focuses on the competition between antiferromagnetism and d-wave superconductivity. In this theory the natural SU(2) \times U(1) symmetry of the spin and charge degrees of freedom is regarded as the result of an explicit symmetry breaking of a larger symmetry, characterized by a global SO(5) group. In this picture, this larger symmetry is not apparent except close to a quantum critical point whose quantum fluctuations suppress both antiferromagnetism and d-wave superconductivity, thus leading to a pseudo-gap regime controlled by this fixed point.

In contrast, in the stripe mechanism, the ground state of the doped Mott insulator is an inhomogeneous charge ordered state resembling a liquid crystal phase, which breaks both rotational invariance and (partially) translation invariance, i.e., it is a quantum smectic. In this picture the pseudo-gap is the spin gap which develop in these quasi-one-dimensional states, and it is not a signature of some sort of long range order. In this mechanism, macroscopic phase coherence and d-wave superconductivity result from inter-stripe Josephson couplings.

In the d-density wave state, and similarly in the physically equivalent staggered flux and orbital antiferromagnetic states, there is a hidden order which has the same d_{x^2-y^2} symmetry as a d-wave superconductor. In this phase the ground state has an ordered pattern of staggered orbital currents, and this is the order which competes with d-wave superconductivity.

However, in spite of a continued effort during the past decade or so, and largely due to the lack of systematic non-perturbative methods in two dimensions, it has been quite difficult to establish the phase diagram of reasonable two dimensional strongly correlated systems based on the Hubbard model. Much of the work done is based on mean field-type approximations which favor one type of order over others or privileges the competition among a particular pair of order parameters. While it is quite possible that these studies reveal different aspects of possible phase diagrams of some generic, possibly short range models, it is not possible at present to determine reliably the phase diagram of many of these models except sometimes at extreme regimes of some parameter. Thus different approaches, including large-N methods (and their relatives), have been used to construct spin-liquid states. Hartree-Fock, large-d and large N methods have been used to study phase separation and striped states. Similarly Hartree-Fock methods have also been used to study the competition between superconductivity and DDW order.

There is also an extensive literature on numerical simulations which work either at moderate to high temperatures (as in Quantum Monte Carlo simulations due to the fermion sign problem) or exact diagonalizations of systems which are usually too small to resolve these issues.

It is largely for these reasons, as well as for the need...
of non-perturbative results, that some of these questions have been considered in the framework of quasi-one-dimensional systems such as Hubbard-type models (in a loose sense) on chains and ladders. Many of these issues, but not all, can be studied in quasi-one-dimensional systems. However, not all of these questions can be addressed in one dimension as the physics may be quite different. For instance the two-dimensional spin liquid states in two dimensions have very specific features with no counterpart in one dimension (not even in ladders) \[23, 24, 25\]. Likewise, the description of a doped one-dimensional Mott insulator at weak coupling is a Luttinger liquid while at strong coupling is an incommensurate soliton crystal which is also a Luttinger liquid, albeit with strongly renormalized parameters. In contrast in two dimensions at weak coupling one may expect to find Fermi liquid pockets while at stronger couplings there is a host of possible liquid crystal like phases going from a solid to a stripe (or smectic) to a nematic whose behavior is markedly different from their one-dimensional counterparts (when they exist). Nevertheless, and in spite of these caveats, studies of quasi-one-dimensional systems have yielded a wealth of information on the physics of strongly correlated systems.

The simplest quasi-one-dimensional system for the study of some of the competing orders described above (and others) are ladder systems. Away from half filling Hubbard-type models on ladder systems can be reduced to the problem of two coupled Luttinger liquids. There is by now a rather extensive literature on the properties of coupled Luttinger liquids. These systems have been studied both analytically \[26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36\] and numerically \[37, 38, 39, 40, 41, 42\] partly for their theoretical simplicity as well as a laboratory to test ideas intended to work possibly in two dimensions, and for their relevance to ladder compounds \[43\]. As it turns out, systems of two coupled Luttinger liquids can support almost all of the local orders proposed for two-dimensional systems and thus shed some light on them. It is thus interesting to investigate this setting the competition between different sorts of possible ordered states, to investigate their phase diagrams systematically and to compare with numerical results.

In this paper we investigate the phase diagrams of two weakly coupled Luttinger liquids both at low doping levels and at half-filling, using bosonization and renormalization group (RG) methods. A number of authors have considered before many aspects of this problem (see in particular Refs. \[29, 31, 32, 33, 34, 35, 36\]). Although many of the phases that will discuss here have been discussed before, we also find a number of new and interesting phases as well as a number of new symmetry relations between some of these phases.

One of the motivations of this paper was the recent suggestion that the Ising-like order parameter of the \(\mathbb{Z}_2\) symmetry of the DDW phase could be observed separately from the incommensuration associated with varying the doping level \[7, 8, 22\]. If this was true it may be possible to have a stable phase on a ladder with spontaneously broken \(\mathbb{Z}_2\). Unfortunately, and in agreement with recent results by Fjærestad and Marston \[81\], we find that while the DDW order parameter does contain an Ising-like piece (as it should) it always involves the charge degree of freedom which leads to incommensurate behavior. On a ladder this leads to correlation function which decays like a power of the distance. Although our results were derived at weak coupling we expect that this behavior should extend to strong coupling as well (with the usual large but finite renormalizations of velocities and exponents.) However, in two dimensions this implies at least two (and possibly more) possible and distinct phases: a Fermi liquid like DDW phase with pockets \[5, 8\], and a smectic (or stripe) phase with DDW order. We also find that it is quite hard to reach this phase in a ladder system, at least within a naive derivation of the effective bosonized theory from Hubbard-like microscopic models, which we summarize in the Appendix A. Recent, unpublished, numerical simulations by Troyer, Chakravarty and Schollwöck \[42\] have reached similar conclusions although in a regime where the couplings are larger. These authors find exponentially decaying correlations and hence only short range DDW order, which means that the simulations reflect a quantum disordered phase (of the type described below). (See also the recent work of Stanescu and Phillips \[44\].)

The inter-twinning of charge order with some other sort of order (with a discrete symmetry group) is obviously not peculiar to DDW order. This is a rather generic situation which leads to interesting phases. It also happens for instance, and this is well known, to the Spin-Peierls or dimerized phase which upon doping in two dimensions it also becomes either a Fermi liquid driven by Fermi surface pockets at weak coupling, or a liquid crystal phase, such as a stripe state, at intermediate and strong coupling. One such example is a bond-centered stripe state which was considered at some length by Vojta, Zhang and Sachdev \[21\], or a site centered stripe of the type considered by Granath and co-workers \[7\] which has a rich phase diagram. In a ladder system these phases are Luttinger liquid which cannot be qualitatively distinguished from their weak coupling counterparts.

We also find a number of interesting symmetries relating pairs of these phases. We find that, away from half filling, the charge density wave phase (CDW) with the spin-Peierls phase (SP) (or bond-density wave (BW)), and a new diagonal current phase (DC) (described below) with the commensurate DDW phase, form two doublets under the continuous symmetry of sliding the charge profile, represented by the uniform chiral shift of the charge Luttinger field \(\phi_{c+}: \phi_{c+} \rightarrow \phi_{c+} + \alpha \text{ (mod } 4\sqrt{\pi})\), \(\phi_{c\pm} \rightarrow -\phi_{c\pm}\) (where the real number \(\alpha\) is an arbitrary phase), \(i.e.\) a chiral translation on a circle and a reflection. This continuous symmetry group is non-Abelian and it may be denoted by \(C_{\infty v}\), in Schoenflies’ symbols. Since in one dimensional quantum systems continuous symmetries cannot be broken
spontaneously, they can only exhibit at most quasi long range fluctuating order and power-law correlations. However, at half-filling Umklapp terms break the continuous symmetry \( \mathcal{C}_{4v} \) down to the finite group \( C_{4v} \), i.e., \( \phi_{c+} \to \phi_{c+} + n_\pi \phi_\perp \) (mod \( 4\sqrt{\pi} \)) and \( \phi_{s+} \to -\phi_{s+} \). Hence at half filling these symmetries can be broken spontaneously leading to true long range ordered (LRO) Ising type phases. In addition we also find four quantum disordered Haldane-like phases whose low energy physics can be described by a suitable \( O(3) \) non-linear \( \sigma \) model. In these phases there is a spin gap which remains present away from half filling. In this regime these phases are Luther-Emery liquids. There is considerable numerical and analytic evidence for these spin-gap phases which in agreement with our conclusions [37, 38, 39, 43]. We also discuss in detail the nature of the quantum phase transitions found at half-filling.

This paper is organized as follows. In Section III we present the effective Hamiltonians and the order parameters used below to characterize the different phases in their bosonized form. In Section IV we use a renormalization group analysis and the known strong coupling behaviors of the effective theory at low doping level to construct a phase diagram. In Section V we do the same type of analysis in Section III but at half filling. In Section IV we present our conclusions. In Appendix A we relate the parameters of the effective bosonized theory with those of the extended Hubbard model on the ladder, and in Appendix B we give explicit expressions for the order parameters of interest in terms of the bosonic fields.

II. MODEL HAMILTONIANS AND ORDER PARAMETERS

We begin with two coupled one-dimensional chains. To a large extent we will follow the approach used by Schulz in Ref. [29]. We consider first the non-interacting limit, and diagonalize the kinetic part in terms of “bonding” and “anti-bonding” bands (denoted by 1 and 2 respectively), i.e. symmetric and antisymmetric under the exchange of the chain labels. Including nearest neighbor (NN) hopping, the non-interacting dispersion relations are just

\[
\epsilon_{\sigma}(k) = -2t \cos k \mp t_\perp (i = 1, 2),
\]

where \( t_\perp \) is inter-chain hopping integral. This approach makes sense if \( t_\perp \) is large compared to any of the dynamically generated gaps of the system, i.e. in the weakly interacting limit.

To first order in the doping level \( \delta \), the Fermi wavevectors of two bands are, respectively, \( k_{f1,2} = \pi (1 - \delta)/2 \pm \sin^{-1}(t_\perp/2t) \), and the corresponding bare Fermi velocities are \( v_{f1,2} = \sqrt{4t^2 - t_\perp^2} \pm t_\perp \pi/2 \), where \( \pi \) is the lattice constant which will serve as the short distance cutoff in the bosonized theory. We will consider the regimes of both low doping and half-filling (discussed in Section III and Section IV respectively) and we will assume that \( t_\perp \) is not necessarily small. At half filling where the Umklapp processes dominate, the system has the particle-hole symmetry

\[
v_{f1} = v_{f2} \quad \text{and} \quad k_{f1} + k_{f2} = \pi.
\]

Away from half-filling, we will assume that the doping level \( \delta \) is large enough to suppress the effects of all Umklapp processes (See section III). However, if \( \delta \) is relatively small, the relation Eq. (2.1) still holds approximately. In this regime the difference in their Fermi velocities does not play a very important role (see however the discussion in Ref. [15]). However as the filling factor of one of the bands approaches zero, the respective Fermi velocity becomes very small and the physics is somewhat changed. In this limit there is an enhancement of the processes leading to the formation of a spin gap [21, 32]. Since we will also find spin gap phases we will ignore here this special regime since it leads to the same physics (albeit with very different parameters).

The effective theory consists then of two coupled Luttinger liquids, for the bonding and anti-bonding bands, and a set of perturbations, which we describe below, each associated with a particular coupling constant. In Appendix A we will relate these coupling constants with the interaction parameters of an extended Hubbard model on a ladder with hopping amplitudes \( t \) and \( t_\perp \), on-site Hubbard repulsion \( U \), and Coulomb interactions \( V_f \) (on the chains), \( V_L \) (on the rungs) and \( V_d \) (along the diagonals of the elementary plaquette), as well as the exchange Heisenberg interactions \( J_f \) (on the chains) and \( J_L \) (on the rungs).

We bosonize the effective theory by introducing a charge boson field and a spin boson field for both the bonding and anti-bonding Fermi fields, \( \phi_{\sigma,i} \), where \( i = 1, 2 \) and \( \sigma = c, s \), where \( c \) and \( s \) label charge and spin modes respectively. These fields are mixed under the effects of various interactions, in particular the backscattering coupling of the respective charge and spin currents and densities. The bosonized theory is diagonalized in terms of the even and odd combinations of boson fields from each band: \( \phi_{\sigma,\pm} = (\phi_{\sigma,1} \pm \phi_{\sigma,2})/\sqrt{2} \), \( \theta_{\sigma,\pm} = (\theta_{\sigma,1} \pm \theta_{\sigma,2})/\sqrt{2} \), \( \nu = c, s \).

The quadratic parts of the Hamiltonian density has the standard “universal” form:

\[
\mathcal{H}_{c,\pm} = \frac{v_{c,\pm}}{2} \left[ K_{c,\pm} \Pi_{c,\pm}^2 + \frac{1}{K_{c,\pm}} (\partial_x \phi_{c,\pm})^2 \right], \\
\mathcal{H}_{s,\pm} = \frac{v_{s,\pm}}{2} \left[ K_{s,\pm} \Pi_{s,\pm}^2 + \frac{1}{K_{s,\pm}} (\partial_x \phi_{s,\pm})^2 \right]
\]

where \( \Pi_{\sigma,\pm} \) are the momenta canonically conjugate to the bosonic fields \( \phi_{\sigma,\pm} \). The effective Luttinger parameters and velocities \( v_{c,\pm} \) and \( v_{s,\pm} \) are given by

\[
K_{c,\pm} = \sqrt{\frac{2\pi v_f - g_{c,\pm}}{2\pi v_f + g_{c,\pm}}}, \quad K_{s,\pm} = \sqrt{\frac{2\pi v_f - g_{s,\pm}}{2\pi v_f + g_{s,\pm}}}, \\
v_{c,\pm} = \sqrt{v_f^2 - \left( \frac{g_{c,\pm}}{2\pi} \right)^2}, \quad v_{s,\pm} = \sqrt{v_f^2 - \left( \frac{g_{s,\pm}}{2\pi} \right)^2},
\]

where

\[
\delta_{\perp} \to -\delta_{\perp} \quad \text{and} \quad \delta_{\pi} \to \delta_{\pi}.
\]
At intermediate and strong couplings there are also finite values of the Luttinger parameters. Also notice that these Fermi velocities, and modify the naively determined values of the charge and spin currents, and are already taken into account in the quadratic terms. Here we have ignored the effects of straightforward effects of forward-scattering chiral couplings, since they only renormalize Fermi velocities, and modify the naively determined values of the Luttinger parameters. Also notice that these expressions can only be taken seriously at weak coupling. At intermediate and strong couplings there are also finite but significant renormalization of both the Luttinger parameters and of the velocities.

Let us now discuss the non-quadratic, interaction terms. Throughout we will use Majorana Klein factors obeying the convention \( \eta_1^\dagger \eta_2^\dagger \eta_1 \eta_2 = 1 \). The backscattering and pair tunneling terms yield the bosonized expressions

\[
\mathcal{H}_{\text{int}} = \frac{\cos \sqrt{4\pi \phi_{\pm}}}{2(a^2)} (g_1 \cos \sqrt{4\pi \phi_{s-}} - g_2 \cos \sqrt{4\pi \phi_{s-}}) + \frac{\cos \sqrt{4\pi \theta_{\pm}}}{2(a^2)} (g_3 \cos \sqrt{4\pi \phi_{s-}} + g_4 \cos \sqrt{4\pi \phi_{s-}}) + g_5 \cos \sqrt{4\pi \phi_{s+}},
\]

(2.4)

where \( \phi_{s,\pm} \) and \( \theta_{s,\pm} \) are the dual fields of the charge bosons \( \phi_{s,\pm} \) and spin bosons \( \phi_{s,\pm} \) respectively. Terms labeled by the effective coupling constants \( g_1 \) and \( g_2 \) originate from the intra-band and inter-band back-scattering interactions \(-g_1(i \bar{J}^{x,y}_{1R} J^{x,y}_{1L} + 1 \rightarrow 2) - g_2(i \bar{J}^{x,y}_{2R} J^{x,y}_{2L} + 1 \rightarrow 2)\). The terms labeled by the couplings \( g_3, g_4 \) and \( g_5 \) represent singlet and triplet pair-tunneling processes \( \lambda_1 (\Delta_2 + h.c.) \) and \( \lambda_2 (\Delta_2 + h.c.) \) with \( g_3 = 2\lambda_1, g_4 = \lambda_s + \lambda_i \) and \( g_5 = \lambda_s - \lambda_i \). Three conditions, required by the \( SU(2) \) spin rotation invariance, relate the spin current and triplet tunneling couplings: \( g_{s,\pm} = (g_1 \pm g_2) / 2 \) (see also Eq. (2.3)) and \( g_5 = g_4 - g_3 \).

Near half-filling, the following additional Umklapp terms appear as

\[
\mathcal{H}_{\text{um}} = \frac{\cos(\sqrt{4\pi \phi_{\pm}} - \delta \pi x)}{2(a^2)} (g_{uc} \cos \sqrt{4\pi \theta_{\pm}} - g_{u3} \cos \sqrt{4\pi \phi_{s-}} - g_{u5} \cos \sqrt{4\pi \phi_{s+}})
\]

(2.5)

The term with coupling constant \( g_{uc} \) is the so-called “\( \eta \) pair” tunneling processes, i.e. tunneling of Cooper pairs with momentum \( 2k_F \), which has the form \( m_{11}^R m_{12}^L (1 \rightarrow 2) + h.c. \), where \( m_{11}^R = \psi R \psi L \). The terms with coupling constants \( g_{u3}, g_{u4} \) and \( g_{u5} \) represent the couplings between the respective CDW and spin density wave (SDW) couplings on each chain: \( \lambda_{sdw} (N^\dagger (1) N^\dagger (2) + h.c.) \), \( \lambda_{sdw} (N^\dagger (1) h.c.) \). Here \( N(i) \) is the \( 2k_F \) CDW order parameter of chain \( i = 1, 2 \), and \( \tilde{N}(i) \) is the \( 2k_F \) SDW order parameter of chain \( i = 1, 2 \). The coupling constants are \( g_{u3} = -\lambda_{sdw}, g_{u4,5} = (2\lambda_{sdw} + \lambda_{sdw}/2) \). Due to the \( SU(2) \) spin symmetry the condition \( g_{u5} = g_{u4} - g_{u3} \) also holds.

For the two-leg ladder, we only consider where repulsive interactions dominate, which implies that the bare values of the effective Luttinger parameters are in the regime \( K_{\pm}(0) \ll 1, K_{\mp}(0), K_{\pm}(0) \sim 1 \). Compared with Ref. 29, \( K_{\pm}(0), K_{\mp}(0) \) are not necessarily 1, for here they are determined by off-site interactions (see Appendix A).

Bosonic expressions for various order parameters are given in Appendix B. In the particle-hole (p-h) channel, the possible singlet fermionic bilinear forms which break the translational symmetry are the order parameters for the CDW and SP, DC and DDW operators as shown in Fig. 1. The CDW and SP order parameters are proportional to the real and imaginary parts of the symmetric bilinear \( \psi^\dagger L \psi^\dagger R + \psi^\dagger L \psi^\dagger R \), whereas the DC and DDW order parameters are the real and imaginary parts of the anti-symmetric version of this bilinear.

From their bosonic representations, we find that all four order parameters transform non-trivially under the symmetries broken in their associated phases (or ground states). Thus, for instance the SP and DDW order parameters are odd under the \( Z_2 \) symmetries broken spontaneously by the SP and DDW phases. However, in all four cases, these order parameters also involve a phase factor (or vertex operator) of the charge boson \( \phi_{c,+} \). Hence these order parameters also transform non-trivially under shifts of the charge boson \( \phi_{c,+} \); e. c. uniform displacements of the charge profile. This dependence means that the discrete symmetries, broken spontaneously in these phases with long range order, are intertwined with the continuous symmetry of the incommensurate doped state. Consequently, these order parameters do not truly acquire an expectation value but instead only display power law correlations. Also, while it is possible to write down bosonic expressions for operators which transform only under the discrete symmetries broken by these phases, their fermionic versions are...
We also find that these order parameters also form two doublets of the $C_{4\pi}$ group. Similarly, their triplet counterparts $SDW, SP, DC$ are called called a recent paper Ref. [46], the phases that we label as at $\sqrt{g_{1}}$ and $\sqrt{g_{2}}$ do not expect that they can change the stable RG fixed points of the renormalization group (RG) flows for the these phases associated with these order parameters. Some of the order parameters discussed above have been investigated before in Ref. [24, 30] although under different names. For example, our $CDW, DDW, SDW, SSC$ and $DSC$ order parameters are called $CDW$, $OAF$, $SDW$, $SC^{s}$ and $SC^{d}$ there. We note that in a recent paper Ref. [46], the phases that we label as $DDW, SP$ and $DC$ are called called $SF$, $P$-density wave ($PDW$), $F$-density wave ($FDW$) respectively.

Finally, in Eq. (2.4.1) we ignored the effects of the following terms

\[ \Delta H_{c} = (\Delta v_{f} + \frac{\Delta g_{c}}{\pi}) \partial_{x} \phi_{c+} \partial_{x} \phi_{c-} + (\Delta v_{f} - \frac{\Delta g_{f}}{\pi}) \Pi_{c+} \Pi_{c-}, \]

\[ \Delta H_{s} = (\Delta v_{f} - \frac{\Delta g_{s}}{\pi}) \partial_{x} \phi_{s+} \partial_{x} \phi_{s-} + (\Delta v_{f} + \frac{\Delta g_{f}}{\pi}) \Pi_{s+} \Pi_{s-}, \]

\[ \Delta H_{um} = \frac{\sin(\sqrt{4\pi} \phi_{c+} - 2\delta \pi x)}{2(\pi a)^{2}} (\Delta g_{uc} \cos \sqrt{4\pi} \theta_{c-}) \]

\[ \quad - \frac{\sin(\sqrt{4\pi} \phi_{c+} - 2\delta \pi x)}{2(\pi a)^{2}} (\Delta g_{uc} \cos \sqrt{4\pi} \theta_{c-}) \]

where $\Delta v_{f} = \delta \pi \Gamma_{1}/2$ and all other residual coupling constants vanish linearly with doping near half-filling as given in Appendix A. The quadratic residual terms in Eq. (2.4.5) and Eq. (2.4.7) are marginal perturbations, and they slightly change the scaling dimensions of various operators in Eq. (2.4.5). Because they are small, we do not expect that they can change the stable RG fixed points associated with various phases qualitatively. For the term of $\Delta g_{s}$ in Eq. (2.4.7), $\theta_{s+}$ is fixed around 0 or $\sqrt{\pi}/2$ at all the stable fixed points (see Table II and III below). The residual Umklapp terms in Eq. (2.4.8) are irrelevant away from half-filling. At half-filling, $\theta_{s+}$ is fixed at $\sqrt{\pi}/2$ (see Table III). Thus we conclude that all the non-quadratic operators are irrelevant at all the stable fixed points. Balents and Fisher [32] used a perturbative RG of the fermionic theory and found that a spin gap phase develops near half-filling, which is consistent with the argument given above. On the other hand, the continuous $C_{4\pi}$ symmetry is preserved away from half-filling where the Umklapp terms are irrelevant. Thus the conclusion that the CDW and SP, DDW and DC order parameters are incommensurate and thus exhibit quasi long range order is not affected by these terms. However, these residual terms do affect the boundaries among phases.

III. PHASE DIAGRAM IN THE INCOMMENSURATE REGIME

We will now investigate the phase diagram in the incommensurate regime, but only at low doping. In this regime, the Umklapp processes are cut off at a high energy scale of $2\pi v f \delta / a$, and can only yield renormalization of the parameters, such as the velocities, coupling constants and Luttinger parameters of the low energy effective theory. The contributions from the Umklapp terms in the RG equations away from half-filling [47] are given in terms of Bessel functions, which oscillate when a energy scale lower than that of the Umklapp process is reached. At this scale, the effects of these terms can be neglected. Below, we begin directly at the low energy scale with all the coupling constants and Luttinger parameters already renormalized by the Umklapp terms.

We will investigate the role of the remaining interactions by means of a one-loop renormalization group (RG) analysis combined with semiclassical arguments. In this regime, the charge boson $\phi_{c+}$ essentially decouples and remains gapless. Thus, to one loop order, the Luttinger parameter $K_{c+}$ does not flow. (This argument is not completely correct: there are always irrelevant couplings which do lead to finite renormalizations of $K_{c+}$; these effects do not show up at one-loop order.)

The one-loop RG equations for the coupling constants $g_{1}$ through $g_{5}$ and Luttinger parameters $K_{c-}$ and $K_{s,\pm}$ are

\[ \frac{dK_{c-}}{dl} = \frac{1}{8\pi^{2}} \frac{\left(g_{3}^{2} + g_{4}^{2} + g_{5}^{2}\right)}{g_{1} g_{2}}, \]

\[ \frac{dK_{s+}}{dl} = -\frac{K_{s+}^{2}}{8\pi^{2}} \frac{\left(g_{3}^{2} + g_{4}^{2} + g_{5}^{2}\right)}{g_{1} g_{2}}, \]

\[ \frac{dK_{s-}}{dl} = -\frac{K_{s-}^{2}}{8\pi^{2}} \frac{\left(g_{3}^{2} + g_{4}^{2} + g_{5}^{2}\right)}{g_{1} g_{2}}, \]

\[ \frac{d g_{1}}{d l} = \left(2 - K_{s+} - K_{s-}\right) g_{1} - \frac{g_{4} g_{5}}{2\pi}, \]

\[ \frac{d g_{2}}{d l} = \left(2 - K_{s+} - K_{s-}\right) g_{2} + \frac{g_{4} g_{5}}{2\pi}, \]

\[ \frac{d g_{3}}{d l} = \left(2 - \frac{1}{K_{c-}} - \frac{1}{K_{s-}}\right) g_{3} + \frac{g_{4} g_{5}}{2\pi}, \]

\[ \frac{d g_{4}}{d l} = \left(2 - \frac{1}{K_{c-}} - K_{s-}\right) g_{4} - \frac{g_{4} g_{5}}{2\pi}, \]

\[ \frac{d g_{5}}{d l} = \left(2 - \frac{1}{K_{c-}} - K_{s-}\right) g_{5} - \frac{g_{4} g_{5}}{2\pi} + \frac{g_{4} g_{5}}{2\pi}, \]

where $l = \ln(L/a)$ with the length scale $L$.

Along the $SU(2)$-invariant manifold for the spin current and pair tunneling terms, the RG equations can be
simplified to
\[
\frac{dK_{-}}{dl} = \frac{1}{8\pi^2} (g_3^2 + g_4^2 + (g_3 - g_4)^2),
\]
\[
\frac{dg_{++}}{dl} = -\frac{1}{2\pi} (g_{++}^2 + g_{--}^2) - \frac{(g_3 - g_4)^2}{4\pi},
\]
\[
\frac{dg_{+-}}{dl} = -\frac{1}{\pi} g_{+} + \frac{g_3^2}{4\pi} - \frac{g_4^2}{4\pi},
\]
\[
\frac{dg_3}{dl} = (1 - \frac{1}{K_{-}}) - \frac{g_{++} + 2g_{--}}{2\pi} g_3 + \frac{(g_{++} + g_{--} - g_4)g_4}{2\pi},
\]
\[
\frac{dg_4}{dl} = (1 - \frac{1}{K_{-}}) - \frac{g_{++} - 2g_{--}}{2\pi} g_4 + \frac{(g_{++} + g_{--} - g_4)g_4}{2\pi},
\]
with
\[
dl (g_3 - g_4 + g_5) = (1 - \frac{1}{K_{-}})(g_3 - g_4 + g_5) = 0 \tag{3.3}
\]

These equations are invariant under transformations \((g_1, g_2, g_3, g_4) \to (g_1, g_2, g_3, -g_4) \to (g_2, g_1, g_3, g_4)\). This means phase boundaries must also have such symmetries.

| \(g_1, g_2\) | \(g_3, g_4, g_5\) | \(\phi_{++}\) | \(\theta_{++}\) | \(\theta_{--}\) | Order | Dimension |
|---------------|---------------------|-----------------|-----------------|-----------------|-------|----------|
| 1 | 0, -\infty | +\infty, 0 | -\infty | / | CDW+SP+ | K_{c,+}/4 |
| 2 | 0, -\infty | -\infty, 0 | +\infty | / | DDC+DDW | K_{c,+}/4 |
| 3 | -\infty, 0 | +\infty, +\infty | / | / | DSC | 1/(4K_{c,+}) |
| 4 | -\infty, 0 | -\infty, -\infty | 0 | 0 | SSC | 1/(4K_{c,+}) |

| \(g_{++}, g_{--}\) | \(g_1, g_2, g_3\) | \(\phi_{++}\) | \(\phi_{--}\) | Transition |
|-----------------|---------------------|-----------------|-----------------|------------|
| 1 | 0.0 | +\infty, +\infty, 0 | unfixed | CDW+SP ↔ DSC |
| 2 | 0.0 | -\infty, -\infty, 0 | unfixed | DDC+DC ↔ DSC |
| 3 | -\infty, 0 | -\infty, +\infty, +\infty | &lt;\sqrt{g}/2 | DDC+DC ↔ DSC |
| 4 | -\infty, 0 | +\infty, -\infty, -\infty | 0 | CDW+SP ↔ DSC |

From now on we will use the set \((g_1, g_2, g_3, g_4)\) to represent the stable fixed points of Eq. 3.2, which are summarized in Table I. At the fixed points \((0, -\infty, +\infty, 0)\), the inter-band back-scattering coupling constant \(g_2\) is relevant, while the intra-band back-scattering coupling constant \(g_3\) is irrelevant. Both \(\lambda_s\) and \(\lambda_t\) are relevant and satisfy the relation \(\lambda_s = -\lambda_t\). By direct inspection of their scaling dimensions, we find that \(\lambda_s\) and \(\lambda_t\) are more relevant than \(g_2\). The resulting phase depends on where the RG flows go. When \(g_3 \to -\infty\), the expectation value of \(\phi_{++}\) and \(\theta_{--}\) asymptotically take the values \(\langle \phi_{++} \rangle = \sqrt{\pi}/2\) and \(\langle \theta_{--} \rangle = 0\). This is the stable fixed point for either the DDW phase or the DC phase. However, this is true only quasi-long range order (QLRO) due to the strong fluctuations of the gapless charge boson \(\phi_{++}\). In this phase these order parameters have scaling dimension \(K_{c,+}/4\). Conversely, when \(g_3 \to +\infty\), \(\langle \phi_{++} \rangle = 0\), and \(\langle \theta_{--} \rangle = \sqrt{\pi}/2\). Hence, at this fixed point we will have (naively) either a CDW phase or a spin-Peierls (SP) (or dimerized) phase. Here too there is only QLRO and the associated order parameters also have scaling dimension \(K_{c,+}/4\).

We conclude, in agreement with the recent results of Ref. 8, that the chiral translation symmetry in the field \(\phi_{c,+}\), in other terms due to the charge incommensurability, there is no true long range order of the DDW order but only (incommensurate) power law correlations. We further can see that the DDW and DC phases (and also the CDW and SP phases) form doublet representation under the \(C_{4\pi}\) group and are thus degenerate. Equivalently, the DDW and DC order parameters can be regarded as the real and imaginary parts of a single complex order parameter which can thus be rotated continuously into each other. The same relationship holds for the CDW and spin-Peierls order parameters. Thus both stable phases, CDW+SP and DDW+DC, have a continuous \(U(1)\) symmetry. Naturally, since the ladder is a one-dimensional system, this symmetry is no truly spontaneously broken as there are only power-law correlations for these order parameters. However, we will see in Section IV that at half filling, the Umklapp terms break this symmetry explicitly from \(U(1)\) down to \(Z_2\) leading to additional Ising-like phase transitions.

Similarly, we also find that \(\lambda_s\) is more relevant than \(g_1\) at \((-\infty, 0, 0, +\infty)\) while \(g_2\) and \(\lambda_t\) are irrelevant. When \(g_1 \to +\infty\), \(\langle \phi_{++} \rangle\) and \(\langle \phi_{--} \rangle\) are fixed at \(\sqrt{\pi}/2\). Thus, d-wave superconductivity (DSC) is the leading QLRO and its order parameter has scaling dimension \(1/(4K_{c,+})\). Conversely, when \(g_1 \to -\infty\), \(\langle \phi_{++} \rangle\) and \(\langle \phi_{--} \rangle\) are fixed at 0, s-wave superconductivity (SSC) is the leading QLRO and its order parameter also has scaling dimension \(1/(4K_{c,+})\).

Let us consider now the phase boundaries and the nature of the phase transitions between these possible states, at \(g_1(0) = 0\). In this regime it is more natural to represent instead the unstable fixed points with \((g_{++}, g_{--}, g_3, g_4)\). The RG flows starting with \(g_{++}(0) > 0, g_{--}(0) = 0\) and \(g_3(0) = g_4(0) = g > 0\) evolve towards...
vanishing expectation values, and the residual interactions have scaling dimension 1 operators. This system is invariant under the duality transformation $s \leftrightarrow g$. This model has been studied extensively in the literature [2, 18]. It is equivalent to a theory of two Ising models. If the coupling constant in front of both operators is the same, as it is the case in Eq. (3.3), one of the Ising models is at its critical point. Equivalently, it can be regarded as a theory of two Majorana fermions, one of which is massive. Hence this fixed point is in the universality class of the two-dimensional classical Ising model. The Ising order and disorder operators are given by $s \sqrt{\theta s}$ and $g \sqrt{\pi \theta s}$ respectively. At this fixed point, both operators have scaling dimension $1/8$, as they should at an Ising transition. A small perturbation making $g \gtrsim g_4$ or $g \lesssim g_4$ causes a flow towards the CDW+SP or DSC fixed points respectively. Thus $g_3 = g_4 > 0$ is the phase boundary between the phase CDW+SP and a d-wave superconductor at $g_3(0) = 0$ and $g_4(0) > 0$.

However, if the RG flows begin with $g_3(0) < 0$ along this direction, then the field $\phi_+ s$ is no longer critical. According to Eq. (3.2) in this regime $g_4$ is marginally relevant and $g_3 \to -\infty$, with $g_3 = g_4 > 0$ and $g_1 = g_2 < 0$. At this fixed point, the fields $\theta_s$ and $\phi_+ s$ acquire non-vanishing expectation values, and the residual interactions at this fixed point reduce to

$$\mathcal{H}_{res}^1 = \frac{g^*}{2(\pi a)^2} \left[ \cos 4\pi \theta_s - \cos 4\pi \theta_s + \cos 4\pi \phi_+ s \right],$$

(3.4)

where $g^*$ means the renormalized value of $g$. At this fixed point $K_s \to 1$, and both perturbations are dimension 1 operators. This system is invariant under the duality transformation $\phi_+ s \leftrightarrow \theta_s$. This model has been studied extensively in the literature [2, 18]. It is equivalent to a theory of two Ising models. If the coupling constant in front of both operators is the same, as it is the case in Eq. (3.3), one of the Ising models is at its critical point. Equivalently, it can be regarded as a theory of two Majorana fermions, one of which is massive. Hence this fixed point is in the universality class of the two-dimensional classical Ising model. The Ising order and disorder operators are given by $s \sqrt{\theta s}$ and $g \sqrt{\pi \theta s}$ respectively. At this fixed point, both operators have scaling dimension $1/8$, as they should at an Ising transition. A small perturbation making $g \gtrsim g_4$ or $g \lesssim g_4$ causes a flow towards the CDW+SP or DSC fixed points respectively. Thus $g_3 = g_4 > 0$ is the phase boundary between the phase CDW+SP and a d-wave superconductor at $g_3(0) = 0$ and $g_4(0) > 0$.

However, if the RG flows begin with $g_3(0) < 0$ along this direction, then the field $\phi_+ s$ is no longer critical. According to Eq. (3.2) in this regime $g_4$ is marginally relevant and $g_3 \to -\infty$, with $g_3 = g_4 > 0$ and $g_1 = g_2 < 0$. At this fixed point, the fields $\theta_s$ and $\phi_+ s$ acquire non-vanishing expectation values, and the residual interactions at this fixed point reduce to

$$\mathcal{H}_{res}^2 = \frac{\cos 4\pi \phi_s}{2(\pi a)^2} \left[ g_3^* \langle \cos 4\pi \theta_s^{-} \rangle + g_4^* \langle \cos 4\pi \phi_+ \rangle \right]$$

$$+ \frac{\cos 4\pi \phi_s}{2(\pi a)^2} \left[ g_3^* \langle \cos 4\pi \theta_s^{-} \rangle - g_2^* \langle \cos 4\pi \phi_+ \rangle \right].$$

(3.5)

At this stage of RG, the renormalized couplings satisfy $g_3^* = g_4^*$ and $g_1^* = g_2^*$. Once again we can take $\langle \cos (4\pi \theta_s^{-}) \rangle = 1$ (the renormalization of its amplitude can be absorbed in a redefined coupling constant). This effective theory has the same form as Eq. (3.3). Hence this is also a theory of two Ising models. However, unlike Eq. (3.3), the amplitudes of the two dimension one operators are not equal. Hence, generically both Ising models are off-criticality (or equivalently both species of Majorana fermions are massive). This corresponds to a finite correlation length and finite energy gap at the phase boundary. Hence, in general this is a first order transition. If $\langle \phi_+ \rangle = 0$, then the term of $\theta_s$ wins over that of $\phi_+ s$ and $g_3 \to +\infty$ in the next step RG transformation. Conversely if $\langle \phi_+ \rangle = \sqrt{g}/2$, then the term of $\phi_+ s$ wins over that of $\theta_s$ and $g_3 \to -\infty$ in the next step RG transformation. Finally, RG flows evolve to the CDW+SP fixed point in the former case while towards the DSC fixed point in the latter. Thus for $g_3 = g_4 > 0$ and $g_3(0) < 0$, the phase transition at the boundary of CDW+SP $\leftrightarrow$ DSC becomes first order as the correlation length is now finite. However, a second order transition is also possible here too. If the spin boson $\phi_+ s$ is quantum disordered, then $\langle \cos (4\pi \phi_+ \rangle) = 0$ and once again we get an Ising critical point of the same kind discussed above. Hence the general conclusion is that this phase

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**FIG. 2:** Phase boundaries with positive initial value of $g_3$ ($g_3(0) = 0.2$) and different initial values of $g_4$ with dashed line ($g_4(0) = 0$), solid points ($g_4(0) = 0.1$) and triangles ($g_4(0) = -0.1$). Phase boundaries of CDW+SP $\leftrightarrow$ DSC, DDW+DC $\leftrightarrow$ SSC become of first order for $g_3(0) < 0$.

**FIG. 3:** RG flows in the 3D parameter space with $g_3(0) > 0$. The dashed lines mark the critical surface. $|g_4|$ wins over $|g_4|$ on the left of the surface, and $|g_4|$ wins over $|g_4|$ on the right. On the critical surface, the RG trajectories flow to the line $|g_4| = |g_4|$. 

boundary may be at a second order transition (with Ising criticality) or at a first order transition, with an Ising-like tricritical point in between. Similarly, $g_s = g_4 < 0$ at $g_s = 0$ is the boundary of DDW+DC ↔ SSC, which is critical and leads to the fixed point at $(0, 0, -\infty, -\infty)$ or to a first order when $g_s > 0$ or $g_s < 0$ respectively.

Another pair of fixed points $(-\infty, 0, +\infty, \pm\infty)$ control the phase boundaries of the DDW+DC ↔ DSC transition at $g_3 = -g_4 < 0$, where $\langle \phi_{s+} \rangle = 0$, and at the phase boundary of the CDW+SP ↔ SSC transition at $g_3 = g_4 > 0$, where $\langle \phi_{s+} \rangle = \sqrt{\pi}/2$; $g_s \rightarrow -\infty$ no matter what its initial value is. The residual interaction for the $s$-sector is still described by Eq. (3.2), but now with $g_s^2 = -g_4^2$. Thus, the amplitudes of $\cos \sqrt{4\pi}\phi_{s+}$ and $\cos \sqrt{4\pi}\theta_{s+}$ are kept equal and this phase boundary is also in the universality class of the Ising critical point. The Ising order and disorder operators can be determined accordingly. The critical phase boundaries are summarized in Table I.

The initial value $g_{s-}(0)$ has important effects on phase boundaries. In Fig. 2 we present the result of a numerical integration of Eq. (3.2) for $g_{s+}(0) > 0$; $g_{s-} > 0$ favors the growth of $g_s$, but disfavors that of $|g_4|$, and conversely $g_{s-} < 0$ favors the growth of $g_s$, but disfavors that of $|g_4|$. Let us begin with the case $g_{s-}(0) > 0$. For $g_{s-}(0) \lesssim g_{s-}(0)|, at first $g_{s-}$ decreases, then it reaches a positive minimum and finally it increases. Thus $g_s$ increases faster than $|g_4|$ and eventually it wins over it. However, if $g_{s-}(0) \leq g_{s-}(0)|, g_{s-}$ decreases monotonically to negative values and $|g_4|$ still wins over $g_s$. As a result, both regions of the phase diagram with DDW+DC order and CDW+SP order expand beyond the line $g_3 = -g_4$, and the correspondingly areas of d-wave and s-wave superconductivity correspondingly shrink. Due to the symmetry of Eq. (3.2), the situation is reversed for $g_{s-}(0) < 0$. For an initial point located on one of these phase boundaries, its RG trajectory flows to the corresponding unstable fixed point, as shown in Fig. 3. For $g_{s-}(0) < 0$, the effect of $g_{s-}(0)$ is similar, but the phase boundaries CDW+SP ↔ DSC and DDW+DC ↔ SSC are now first order transitions and there are no accessible critical points.

We conclude this section with some comments on the DDW phase which has attracted considerable interest recently. Until now there is no solid numerical evidence away from half-filling. For the two-leg ladder, we find (see Appendix A) that the DDW phase may exist but it is necessarily incommensurate. We also find $V_L$ large and positive reduces $g_1$ and enhances $g_{s-}$, which is favorable for the DDW phase to exist. However, a negative $g_3$ with magnitude comparable to $|g_4|$ is also needed. Thus we suggest to look for it in the regimes $V_L \gg V_0 \gg V_0^* > 0$, which has only repulsive interactions, or for $V_L > 0 > V_0^*$, which has some attractive interactions (and thus is less physically relevant). These arguments agrees with the results of a recent two-dimensional mean-field calculation [44] that the Hubbard $U$ alone can not stabilize the DDW phase and that negative nearest-neighbor interactions are needed. However, $V_1, V_1^* < 0$ together favor d-wave superconductivity over the DDW state.

IV. THE PHASE DIAGRAM AT HALF FILLING

Let us now discuss the phase diagram at half-filling. The main change is the presence of Umklapp terms. Compared to the incommensurate case discussed in Section III, the main difference is that at half-filling the $Z_2$ symmetries behind the two-fold degeneracies found in away from half-filling are now can be spontaneously broken, with possible phase transitions between the CDW and the spin-Peierls phases, and between the DDW and the DC phases. Since much of the analysis is rather similar, here we will only sketch the main differences.

The set of RG equations are now more complicated:

\[
\begin{align*}
\frac{dK_{c+}}{dl} &= \frac{-K_s^2}{8\pi^2}(g_{uc}^2 + g_{u_3}^2 + g_{u_4}^2 + g_{u_5}^2) \\
\frac{dK_{c-}}{dl} &= \frac{1}{8\pi^2}(g_3^2 + g_4^2 + g_5^2) \\
\frac{dK_{s+}}{dl} &= \frac{-K_s^2}{8\pi^2}(g_1^2 + g_2^2 + g_5^2 + g_{u_5}^2) \\
\frac{dK_{s-}}{dl} &= \frac{-K_s^2}{8\pi^2}(g_1^2 + g_4^2 + g_{u_4}^2 + \frac{1}{8\pi^2}(g_2^2 + g_3^2 + g_{u_3}^2) \\
\frac{dg_1}{dl} &= (2 - K_{s+} - K_{s-})g_1 - \frac{g_1g_5}{2\pi} - \frac{g_4g_{u_5}}{2\pi} \\
\frac{dg_2}{dl} &= (2 - K_{s+} - K_{s-})g_2 + \frac{g_2g_{u_5}}{2\pi} + \frac{g_3g_{u_3}}{2\pi} \\
\frac{dg_3}{dl} &= (2 - \frac{1}{K_{c+}} - \frac{1}{K_{c-}})g_3 + \frac{g_3g_{u_3}}{2\pi} + \frac{g_4g_{u_4}}{2\pi} \\
\frac{dg_4}{dl} &= (2 - \frac{1}{K_{c+}} - \frac{1}{K_{c-}})g_4 - \frac{g_4g_{u_4}}{2\pi} - \frac{g_5g_{u_5}}{2\pi} \\
\frac{dg_5}{dl} &= (2 - \frac{1}{K_{c+}} - \frac{1}{K_{c-}})g_5 - \frac{g_5g_{u_5}}{2\pi} + \frac{g_3g_{u_3}}{2\pi} + \frac{g_4g_{u_4}}{2\pi} \\
\frac{dg_{uc}}{dl} &= (2 - K_{c+} - \frac{1}{K_{c-}})g_{uc} + \frac{g_3g_{u_3}}{2\pi} + \frac{g_4g_{u_4}}{2\pi} + \frac{g_5g_{u_5}}{2\pi} \\
\frac{dg_{u_3}}{dl} &= (2 - K_{c+} - K_{s-})g_{u_3} - \frac{g_1g_{u_3}}{2\pi} + \frac{g_2g_{u_4}}{2\pi} + \frac{g_5g_{u_5}}{2\pi} \\
\frac{dg_{u_4}}{dl} &= (2 - K_{c+} - K_{s-})g_{u_4} - \frac{g_1g_{u_4}}{2\pi} + \frac{g_2g_{u_5}}{2\pi} + \frac{g_5g_{u_5}}{2\pi} \\
\frac{dg_{u_5}}{dl} &= (2 - K_{c+} - K_{s+})g_{u_5} - \frac{g_1g_{u_5}}{2\pi} + \frac{g_2g_{u_4}}{2\pi} + \frac{g_5g_{u_4}}{2\pi}
\end{align*}
\]

We will not be interested here in solving these RG equations in their full glory, but only in the regime where $K_{c+} \ll 1$ and $K_{c-} \sim 1$. For this range of parameters there are a number of useful hierarchies of scales which considerably simplify the analysis.

Contrary to what happens away from half filling, the field $\phi_{c+}$ no longer decouples due to the effects of the Umklapp terms of Eq. (2.3). Clearly, $\phi_{c+}$ plays a role quite similar to that of $\theta_{c-}$. Indeed, in this regime, $g_{uc}$
is the most relevant coupling and it is associated to an operator with scaling dimension $K_{c+} + 1/K_{c-}$. This operator takes the RG flow close to a fixed point at which the field $\phi_{c+}$ acquires a gap approximately of the form $m_{c+} \approx -a^{-1}(g_{uc})^{1/(1-K_{c-}(0))}$. In this regime the field $\theta_{\infty}$ behaves roughly in the same way as in Eq. (3.2).

Here too the coupling constant $g_{\infty}$ flows to strong coupling, $1/K_{c-} \to 0$ and a gap $m_{c-}$ develops in this sector as it does away from half filling. We will set $\langle \phi_{c+} \rangle = \sqrt{\pi}/2$, correspondingly $\langle \theta_{\infty} \rangle = 0$ or $\sqrt{\pi}$ when $g_{uc}(0) > 0$ or $< 0$ respectively, so that $(\cos \sqrt{4\pi\phi_{c+}}) \approx -g_{c+}(0)K_{c+}(0)$ and $(\cos \sqrt{4\pi\theta_{\infty}}) \approx \text{sgn}(g_{uc})(a m_{c-})^{1/K_{c-}(0)}$.

Once the fields $\phi_{c+}$ and $\theta_{\infty}$ become pinned close to their classical values, the effective residual interactions among the remaining fluctuating degrees of freedom have an effective Hamiltonian of the form

$$\mathcal{H}_{\text{eff}} = \frac{\cos \sqrt{4\pi\phi_{c+}}}{2(\pi a)^2}(g_1 \cos \sqrt{4\pi\phi_{c-}} - g_2 \cos \sqrt{4\pi\theta_{c-}}) + \frac{1}{2(\pi a)^2}(g_3^* \cos \sqrt{4\pi\phi_{c-}} + g_4 \cos \sqrt{4\pi\phi_{c-}} + g_5 \cos \sqrt{4\pi\phi_{c-}}), \quad (4.2)$$

where $g_3, g_4(0) = g_{3,4}(0)(\cos \sqrt{4\pi\theta_{\infty}}) - g_{u,u}(0)(\cos \sqrt{4\pi\phi_{c+}})$ and $g_5(0) = g_5^*(0) - g_3^*(0)$. If $g_{uc}(0)$ is not small compared with the initial (or bare) values of the other coupling constants, this first step of the renormalization group flow is rather quick. In this step the marginal coupling constants cannot change very much and thus $|\langle \cos \sqrt{4\pi\phi_{c+}} \rangle| \gg |\langle \cos \sqrt{4\pi\theta_{\infty}} \rangle|$ is a good approximation. Hence the renormalized residual couplings are approximately $(g_3^*, g_4^*, g_5^*) \sim (g_{u3}(0), g_{u4}(0), g_{u5}(0))$.

| $g_{uc}$ | $g_1$ | $g_2$ | $g_3^*$, $g_4^*$, $g_5^*$ | $\theta_{\infty}$, $\phi_{c+}$, $\langle \phi_{c+} - \theta_{\infty} \rangle$ | Phase |
|---|---|---|---|---|---|
| $+\infty$ | $0$, $+\infty$, $0$, $-\infty$ | $0$ | $(0, \infty)$ | DDW |
| $+\infty$ | $0$, $-\infty$, $0$, $+\infty$ | $0$ | $(0, \infty)$ | DSC+SDW |
| $\infty$ | $0$, $+\infty$, $0$, $-\infty$ | $0$ | $(0, \infty)$ | SSC+DC |
| $\infty$ | $0$, $-\infty$, $0$, $+\infty$ | $0$ | $(0, \infty)$ | SSC+DC |
| $+\infty$ | $0$, $+\infty$, $0$, $-\infty$ | $0$ | $(0, \infty)$ | DSC+DDW |
| $+\infty$ | $0$, $-\infty$, $0$, $+\infty$ | $0$ | $(0, \infty)$ | DSC+DDW |

TABLE IV: Fixed points at half-filling: Unstable fixed points which have the common fixed value $g_{uc} = 0$. Here too $\phi_{c+} = \sqrt{\pi}/2$, and the $SU(2)$ condition requires $g_5^* = g_4^* - g_3^*$. The column on the right indicates which transition is controlled by each unstable fixed point.

The corresponding stable fixed points and values of pinned fields are summarized in Table III. The critical (or unstable) fixed points are given in Table IV. Umklapp terms break the symmetry group to $C_4$ and thus remove the degeneracy between CDW and SP phases, and between the DDW and DC phases. Hence, all four states become distinct phases with true long range order, which break the residual $Z_2$ symmetry spontaneously. At the quantum phase transitions between CDW and SP, and between DDW and DC, the symmetry is $U(1)$.

Perturbative RG studies of Refs. [11] and [30] have described the CDW and DDW fixed points with the property that the coupling constants (written in our notation) satisfy

$$-g_2 = \pm g_3 = \pm g_5 = -g_{u3} = g_{u5} = g_{uc} \to +\infty$$

$$g_1 = g_4 = g_{u4} = 0. \quad (4.3)$$

where the upper (lower) sign holds for the CDW (DDW) phase. It turns out that a model with this particular choice of coupling constants was been proposed by Scalapino, Zhang and Hanke (SZH) [34] as a ladder model of the $SO(5)$ theory. However, Lin, Balents and Fisher found that, at least to one loop order in a perturbative RG [31], the symmetry is enlarged actually to $SO(8)$. Moreover these authors found, also within a perturbative RG, that the $SO(8)$ manifold is at least locally stable, i.e. small deviations from this trajectory converge to this trajectory under the RG flow. Interestingly, the $SO(8)$ manifold is an integrable fermionic system for which a number of exact properties have been calculated using the Bethe Ansatz [30]. $SO(8)$ is clearly a dynamical symmetry which is possible because the operators that are involved (back in the fermionic representation) are all dimension 2, they are superficially marginal but become marginally relevant due to fluctuations leading to the development of a gap.

However, for more generic values of the coupling constants this dynamical symmetry does not necessarily
arise. It is not know how large the basin of attraction of the SO(8) manifold actually is. In fact using bosonization methods we find that far away from the SO(8) manifold the scaling dimensions of these operators begin to differ significantly from each other and thus evolve differently under the RG (see Eq. (4.3)). In particular, by checking their scaling dimensions, we find that the renormalized couplings can renormalize so differently from each other as 

\[ |g_2| \ll |g_3| = |g_5| \ll |g_{u3}| = |g_{u4}| \ll |g_c| \to \infty \]

\[ g_1, g_4, g_{u4} \to 0 \tag{4.4} \]

in all the four phases of CDW, DDW, SP and DC (Recall that the signs of the coupling constants change in some of the phases). Nevertheless, what is clear is that the spectrum found in these more anisotropic (and more generic) regimes is smoothly connected to the multiplets found in the SO(8) limit. In other words there is no phase transition separating these regimes, but the spectrum is organized differently.

Let us now discuss the phase transitions between the CDW and SP phases, and between the DDW and DC phases, and to the associated critical fixed points. As we noted before, these phase transitions are driven by the Umklapp terms, the most relevant of which is controlled by the coupling constant \( g_{uc} \). Hence, at the critical point separating the SP and CDW phases, and between the DDW and DC phases, the Umklapp terms are tuned to zero. The critical fixed points coincide with the stable fixed points of the incommensurate CDW+SP phase and DDW+DC phase respectively. In both cases the transition is controlled by the sign of \( g_{uc} \). We also note that the renormalized coupling constant \( g_5 \) has different signs on both sides of this phase transition. This is because, close to the transition \( g_5 \approx g_3 \cos(\sqrt{4\pi}\theta_{s-}) \), and \( (\theta_{s-}) = 0 \) in the SP phase while \( (\theta_{s-}) = \sqrt{\pi}/2 \) in the CDW phase. The same is true for the phase transition between the DDW and DC phases.

It can be shown that, if only charge interactions are considered \(30\), then \( g_{uc} = g_{u4} \) at the bare level. In this regime the CDW and SP phases are more easily accessible than the DDW and DC phases. There is strong numerical evidence for a commensurate DDW phase at half-filling in a t-J-Hubbard ladder \(41\) who included Heisenberg-like exchange interactions at the microscopic level. It is easy to see that although the inclusion of microscopic exchange interactions does not lead to a different low energy theory, it changes the strengths of the different effective couplings. In particular it makes the DDW phase more accessible. For simplicity, we discuss the conditions of the commensurate DDW phase on the SZH ladder which only includes nonzero interactions \( V_{\perp}, V_{\parallel}, J_\perp \). The coupling constants are given in the weak interaction limit in Appendix A. Let us suppose that \( V_{\perp} \) and \( J_\perp > 0 \). First of all, we need positive \( g_{uc} \) to set up the overall repulsive interaction i.e., \( U + 2V_{\perp} > 0 \). A large \( J_\perp \) helps to make \( g_{uc} > 0 \) and \( g_{u3} < 0 \) simultaneously i.e., \( \frac{1}{4} J_\perp > U - V_{\perp} > -\frac{3}{4} J_\perp \). But \( J_\perp \) can not be too large, otherwise negative \( g_{uc} \) suppresses the DDW phase. For \( |g_{u3}| > |g_{u4}| \), which can be achieved with \( U < 0 \), this phase is stabilized. But \( |U| \) can not be too large, otherwise \( g_{uc} \) would become negative. The region where the commensurate DDW was found in Ref. \(41\) agrees with this analysis. Again, we need to keep in mind that this naive analysis only makes sense in the weak coupling limit, which also neglects effects from many irrelevant operators. Thus, we do not expect this analysis to give a precise location of the phase boundary.

Now we discuss the remaining phases and phase transitions. Upon a careful study of which fields become pinned and what are their allowed expectation values, we conclude that the remaining four phases are actually quantum disordered Haldane-like states. For example, there is a phase in which d-wave superconductivity and the SDW order parameters (DSC+SDW) are quantum disordered. The order parameter for DSC is very sensitive to fluctuations in the c+ sector since \( O_{DSC} \propto e^{ip_{\perp}} \). Similarly, the x, y and z components of the SDW order parameter are controlled by fluctuations in the s± sector since \( O_{SDW} \propto (\sin(\sqrt{\pi} \theta_{s-}), \sin(\sqrt{\pi} \theta_{s+}), \cos(\sqrt{\pi} \theta_{s+})) \).

At this fixed point the fields \( \theta_{s-} \) and \( \theta_{s+} \) are not pinned and fluctuate wildly. Nevertheless, the remaining fields in the expressions for these order parameters do provide for a finite amplitude even though the fluctuations of both phase and orientation are so strong that the system is quantum disordered. The analysis of other three phases, s-wave superconductivity and triplet DC (SSC+DC\(^t\)), d-wave superconductor and triplet spin-Peiersl (DSC+SP\(^t\)), and s-wave superconductor and triplet d-density wave (SSC+DDW\(^t\)), is similar. Because of large charge gaps, the low energy physics of their spin sector, may be described by the corresponding O(3) nonlinear \( \sigma \) model without a Berry phase term, which is quantum disordered.

The phase transition between the DSC+SDW phase and the DSC+SP\(^t\) phase (see Fig. 4 and Fig. 5) is the commensurate limit of the d-wave superconductor found away from half filling. A similar relation holds for the phase transition between the SSC+DC\(^t\) phase, the SSC+DDW\(^t\) phase and the s-wave superconductor.

Finally, let us discuss the unstable fixed points with \( |g_2| = |g_4| \to \infty, g_{uc} = 0 \), summarized in Table I. The RG flows starting from the phase boundaries with \( g_{uc}(0) < 0 \) evolve towards these fixed points. At these phase boundaries, the order parameters for CDW, SP, DC and DDW have power-law correlations and have scaling dimension 3/8 at the fixed points denoted by 1, 2, 5, and 6, and scaling dimension 1/8 at the fixed points denoted by 3, 4, 7 and 8 (see Figs. 4 and 5). On these phase boundaries the d-wave and s-wave superconducting order parameters are quantum disordered. Similarly, the SDW, SP\(^t\), DC\(^t\) and DDW\(^t\) order parameters have power-law correlations and their scaling dimension is 3/8 at the points 1, 2, 5, 6 but are quantum disordered at points 3, 4, 7, 8. For \( g_{uc}(0) = 0 \), at these phase boundaries the renormalized couplings satisfy \( |g_2^*| = |g_4^*| \) as...
both low doping and at half-filling. We used (Abelian) bosonization and RG methods to study the phase diagrams of these ladders both at half filling and at low doping. Stable and unstable fixed points of the RG flows with the corresponding phases and phase boundaries were investigated in detail. First order transitions when \( g_{+}(0) < 0 \) are found and the effects of \( g_{-}(0) \) to phase boundaries are discussed. The \( C_{\infty v} \) symmetry makes CDW and spin-Peierls, DC and DDW degenerate. In the absence of Umklapp terms there is incommensurate quasi long range order. These degeneracies are removed at half-filling where true long range order appears. Power law fluctuating \( d \)-wave and \( s \)-wave superconducting phases at low doping levels become quantum disordered at half-filling, with finite amplitudes among DSC, SSC and SDW, DC\(^t\), SP\(^t\), DDW\(^t\) respectively. Suggestions on how to best find these phases in numerical simulations were given.

After this paper was submitted for publication, we became aware of the work by Tsuchiizu and Furusaki on a very similar model (at half-filling) \[46\]. In this work these authors also obtained the same eight insulating phases we found here at half-filling. Also after this work was submitted, we learned of the numerical work by Schollwock \textit{et. al.} \[51\] on a DMRG study of a similar ladder model away from the half filling. At low doping these authors found that their results are consistent with an inhomogeneous picture of the doped state in which the system is locally commensurate. It is our understanding that at long length scales the system is actually incommensurate with discommensurations (or kinks) separating the locally commensurate regions. On length scales long compared to the distance between kinks, this state behaves like an effective “elastie solid” which in one-dimension has the same quantum critical behavior as a Luttinger liquid. Thus, this state is qualitatively equivalent to our weak coupling picture, albeit with substantially renormalized parameters.

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APPENDIX A: FERMIONIC HAMILTONIAN

We considered an extended Hubbard model on a ladder with a Hamiltonian of the following form

\[
H = -t \sum_{\langle i,j \rangle} \{c_{i,j}^\dagger c_{i+1,j} + h.c.\} - t_\perp \sum_{\langle i \sigma \rangle} \{c_{i,0,\sigma}^\dagger c_{i,1,\sigma} + h.c.\} + U \sum_{i,j} n_{i,j}^\dagger n_{i,j} + V_l \sum_{i,j} n_{i,j} n_{i+1,j} + V_r \sum_{i} n_{i,1} n_{i,2} + V_d \sum_{i} (n_{i,1} n_{i+1,2} + n_{i,2} n_{i+1,1}) + J_\perp \sum_{i} \vec{S}_{i,1} \cdot \vec{S}_{i,2} + J_\parallel \sum_{ij} \vec{S}_{ij} \cdot \vec{S}_{i+1,j}
\]  

(A1)

Here \( i \) labels the sites along legs and \( j \) labels the legs (or rungs); the coupling constants \( U, V_r, V_d \) represent the on-site Hubbard interaction and various nearest and next-nearest neighbor Coulomb interactions, and \( J_\perp \) and \( J_\parallel \) are the Heisenberg interaction along the rungs and chains respectively.

After diagonalizing the kinetic part, we can rewrite the above Hamiltonian with the right and left movers of the bonding and anti-bonding bands represented by the operators \( \psi_{R1}, \psi_{L1}, \psi_{R2}, \psi_{L2} \) as below, where \( \psi_{1,2}(x) = (c_{1} \pm c_{2})/\sqrt{2a} \). In the low energy limit, the free part of the continuum Hamiltonian density can be written as

\[
\mathcal{H}_0 = v_f \left\{ \frac{\pi}{2} (J_{L1} J_{L1} + J_{R1} J_{R1}) + \frac{2}{3} \pi (\vec{J}_{L1} \cdot \vec{J}_{L1} + \vec{J}_{R1} \cdot \vec{J}_{R1}) \right\} + v_f^2 \left\{ \frac{\pi}{2} (J_{L2} J_{L2} + J_{R2} J_{R2}) + \frac{2}{3} \pi (\vec{J}_{L2} \cdot \vec{J}_{L2} + \vec{J}_{R2} \cdot \vec{J}_{R2}) \right\},
\]

(A2)

where \( J_{r,l} = \psi_{r,l}^\dagger \psi_{r,l}, \psi_{r,l}^{\dagger} \) are the right and left moving components of the charge and spin current density for the bonding \( (i = 1) \) and anti-bonding \( (i = 2) \) fermions.

The interaction part of the Hamiltonian splits into several terms. First, we have a set of terms involving only the charge currents:

\[
\mathcal{H}_{\text{int,c}} = \left\{ \frac{U}{8} + \frac{1}{2} (V_L + V_d) + \frac{V_r}{4} \right\} (J_{1R} J_{1R} + J_{1L} J_{1L} + J_{2R} J_{2R} + J_{2L} J_{2L}) + \left\{ \frac{U}{4} + (V_L + V_d)(1 - \cos 2k_{f1}) + \frac{V_r}{4} - \frac{3}{8} \cos 2k_{f1} J_{\perp} - \frac{3}{16} J_{\perp} \right\} J_{1R} J_{1L} + \left\{ \frac{U}{4} + (V_L + V_d)(1 - \cos 2k_{f2}) + \frac{V_r}{4} - \frac{3}{8} \cos 2k_{f2} J_{\perp} - \frac{3}{16} J_{\perp} \right\} J_{2R} J_{2L} + \left\{ \frac{U}{4} + V_L(1 - \frac{1}{2} \cos k_{\perp}) + V_d(1 + \frac{1}{2} \cos k_{\perp}) + \frac{V_r}{4} - \frac{3}{8} J_{\perp} \cos k_{\perp} + \frac{3}{16} J_{\perp} \right\} (J_{1R} J_{2R} + J_{1L} J_{2L}) + \left\{ \frac{U}{4} + V_L(1 - \frac{1}{2} \cos k_{\perp}) + V_d(1 + \frac{1}{2} \cos k_{\perp}) + \frac{V_r}{4} - \frac{3}{8} J_{\perp} \cos k_{\perp} + \frac{3}{16} J_{\perp} \right\} (J_{1R} J_{2L} + J_{1L} J_{2R}),
\]

(A3)

where \( k_+ = k_{f1} + k_{f2} = \pi(1 - \delta), \ k_- = k_{f1} - k_{f2} = 2 \sin^{-1} \left[ t_{\perp}/(2 \cos \frac{\pi \delta}{2}) \right] \).

Next we have the couplings involving the spin currents:

\[
\mathcal{H}_{\text{int,s}} = \left\{ \frac{-U}{6} + \frac{J_{\perp}}{4} \right\} (\vec{J}_{1R} \cdot \vec{J}_{1R} + \vec{J}_{1L} \cdot \vec{J}_{1L} + \vec{J}_{2R} \cdot \vec{J}_{2R} + \vec{J}_{2L} \cdot \vec{J}_{2L}) - \left\{ U + 2(V_L + V_d) \cos 2k_{f1} + \frac{V_r}{4} \right\} (\vec{J}_{1R} \cdot \vec{J}_{1L}) - \left\{ U + 2(V_L + V_d) \cos 2k_{f2} + \frac{V_r}{4} \right\} (\vec{J}_{2R} \cdot \vec{J}_{2L}) - \left\{ U + 2(V_L - V_d) \cos k_{\perp} + V_d(1 + \frac{1}{2} \cos k_{\perp}) J_{\perp} - \frac{3}{4} J_{\perp} \right\} (\vec{J}_{1R} \cdot \vec{J}_{2R} + \vec{J}_{1L} \cdot \vec{J}_{2L}) - \left\{ U + 2(V_L - V_d) \cos k_{\perp} - V_d(1 + \frac{1}{2} \cos k_{\perp}) J_{\perp} - \frac{3}{4} J_{\perp} \right\} (\vec{J}_{1R} \cdot \vec{J}_{2L} + \vec{J}_{1L} \cdot \vec{J}_{2R}).
\]

(A4)

Next we have the low energy couplings associated with singlet-pair and triplet-pair tunneling:

\[
\mathcal{H}_{\text{int,pt}} = \left\{ U + 2(V_L - V_d) \frac{3}{2} J_{\parallel} \cos k_{f1} + \frac{V_r}{4} \right\} (\Delta_1 \Delta_2 + h.c.) + \left\{ 2(V_L - V_d) + \frac{J_{\perp}}{4} \right\} \sin k_{f1} \sin k_{f2} (\Delta_1 \Delta_2 + h.c.)
\]

(A5)

where \( \Delta = (\psi_{R1} \psi_{L1} - \psi_{R2} \psi_{L2})/\sqrt{2} \) is the singlet pair operator on a given chain and \( \Delta \) is its triplet counterpart. Notice that 1 and 2 stand here for the chain label.
Finally, the low energy Umklapp scattering terms are

$$\mathcal{H}_{um} = e^{2i\delta \pi} \left\{ \frac{U}{4} + e^{i\delta \pi} \left[ V_\perp \left( \frac{1}{2} - \cos k_- \right) - V_\parallel \left( \frac{1}{2} + \cos k_- \right) + \frac{3}{8} J_\parallel \right] + \frac{3}{4} V_\perp + \frac{3}{16} J_\perp \right\} N_i^\dagger N_i^\dagger $$

$$ - \left\{ U + e^{i\delta \pi} \left( -2(V_\parallel - V_\perp) + \left( \frac{1}{2} + \cos k_- \right) J_\parallel \right) - V_\parallel - \frac{J_\parallel}{4} \right\} \tilde{N}_i^\dagger \tilde{N}_i^\dagger $$

$$ + \left\{ \frac{U}{2} - (V_\parallel - V_\perp - \frac{3}{4} J_\parallel)e^{i\delta \pi} - \frac{V_\parallel}{2} + \frac{3}{8} J_\perp \right\} \left( m_{1R} m_{2L} + m_{1R} m_{1L} \right) + h.c., \quad (A6)$$

Here $N_i^\dagger = \psi_{R_i}^\dagger \psi_{L_i}$ and $\tilde{N}_i^\dagger = \psi_{R_i}^\dagger (\delta/2) \psi_{L_i}$ are CDW and SDW (Néel) order parameters, respectively. $m$ is the pairing order with $2k_F$ momentum, for example, $m_R = \psi_{R_i}^\dagger \psi_{R_i}$. Following the standard Bosonization procedure with the assumption of Eq. [2.3] we arrive at the bosonized Hamiltonian density in the section II. The bare values of the weak coupling constants are given as

$$g_{c+} = U + V_\parallel \left[ 1 + \cos \pi \delta \left( 1 + \cos k_f \right) \right] + 2 V_\perp + V_\parallel \left[ 4 - \cos \pi \delta \left( 1 - \cos k_f \right) \right] + \frac{3}{4} J_\parallel \cos \pi \delta \left( 1 + \cos k_- \right),$$

$$g_{c-} = -(V_\parallel + \frac{3}{4} J_\parallel) \cos \pi \delta \left( 1 - \cos k_f \right) - V_\perp + \cos \pi \delta \left( 1 + \cos k_f \right) V_\parallel - \frac{3}{4} J_\parallel,$$

$$g_{s+} = U - V_\parallel \cos \pi \delta \left( 1 + \cos k_f \right) + V_\perp \cos \pi \delta \left( 1 - \cos k_f \right) - \frac{J_\parallel}{2} \left[ 1 - \frac{1}{2} \cos \pi \delta \right] - \frac{J_\parallel}{4},$$

$$g_{s-} = V_\parallel \cos \pi \delta \left( 1 - \cos k_f \right) + V_\perp \left[ 4 - \cos \pi \delta \left( 1 + \cos k_f \right) \right] + \frac{J_\parallel}{4} \left[ 1 - \frac{1}{2} \cos \pi \delta \right] - \frac{1}{4} J_\parallel,$$

$$g_3 = 2(V_\parallel - V_\perp + \frac{J_\parallel}{4}) \cos \pi \delta,$$

$$g_4 = U + 2(V_\parallel - V_\perp) \cos k_- - V_\perp + J_\parallel \left( \frac{1}{2} \cos k_- + \frac{3}{4} J_\parallel \right),$$

$$g_{uc} = U - 2 \cos \pi \delta \left( V_\parallel - V_\perp - \frac{3}{4} J_\parallel \right) - V_\perp + \frac{3}{4} J_\parallel,$$

$$g_{u3} = U - 2 \cos \pi \delta \left[ V_\parallel - V_\perp - J_\parallel \left( \frac{1}{4} + \frac{1}{2} \cos k_- \right) \right] - V_\perp - \frac{J_\parallel}{4},$$

$$g_{u4} = U - 2 \cos \pi \delta \left[ (V_\parallel + V_\perp) \cos k_- - J_\parallel \left( 1 + \frac{1}{2} \cos k_- \right) \right] + V_\perp + \frac{J_\parallel}{4},$$

$$g_1 = g_{s+} + g_{s-}, \quad g_2 = g_{s+} - g_{s-}, \quad g_5 = g_4 - g_3, \quad g_{u5} = g_{u4} - g_{u3}. \quad (A7)$$

where $\cos k_- = 1 - t_0^2 / (2t^2)$. Up to the first order, these coupling constants are independent of the doping $\delta$.

When away from the half-filling, the particle-hole symmetry Eq. [2.3] only holds approximately at small doping $\delta$ as $k_- = \pi / \delta \pi, \Delta k_F / a = \delta t \pi$. Taking these into account, there are some small residue terms as appearing in Eq. [2.6, 2.7, 2.8] they vanish linearly with doping. The corresponding coupling constants are

$$\Delta g_c = \frac{1}{2} (V_\parallel + V_\perp) \sin \pi \delta \cos k_f,$$

$$\Delta g_{uc} = -2 \sin \pi \delta \left( V_\parallel - V_\perp - \frac{3}{4} J_\parallel \right),$$

$$\Delta g_{u3} = -2 \sin \pi \delta \left[ V_\parallel - V_\perp - J_\parallel \left( \frac{1}{4} + \frac{1}{2} \cos k_- \right) \right] + V_\perp + \frac{J_\parallel}{4},$$

$$\Delta g_{u5} = \Delta g_{u3} - \Delta g_{u4}. \quad (A8)$$

**APPENDIX B: Bosonic Representation of the Order Parameters**

The difference of the charge density between two legs reads $\sum_{\sigma} (-)^{i+j+1} c_{j\sigma}^\dagger (i)c_{j\sigma} (i) = \sum_{\sigma} \psi_{1\sigma}^\dagger (x) \psi_{2\sigma} (x) + \psi_{2\sigma}^\dagger (x) \psi_{1\sigma} (x)$. After expressed by the right and left movers, it contains the staggered part, i.e. $O_{SDW}$. Similar situation happens to its triplet counterpart $O_{SDW,z,x,y}$. Using the bosonization identities: $\psi_{R,L} (x) = 1/\sqrt{2\pi} \exp \{ \pm i/\sqrt{\pi} \phi (x) \pm
\[ O_{CDW}(x) = \left\{ (-)^x e^{-i\delta \pi x} \right\} \psi_{1L\sigma}^\dagger(x) \psi_{2R\sigma}(x) + \psi_{1L\sigma}^\dagger(x) \psi_{1R\sigma}(x) + h.c. \]

\[ \hat{O}_{SDW; x,y}(x) \}

\[ \propto 2 e^{-i\delta \pi x} \{ \begin{array}{l} 2 \cos \sqrt{\pi \theta_c} - \cos \sqrt{\pi \phi_{s+}} \sin \sqrt{\pi \theta_s} \\ \cos \sqrt{\pi \phi_{s+}} + \cos \sqrt{\pi \theta_s} \\ - \sin \sqrt{\pi \phi_{s+}} + \sin \sqrt{\pi \phi_s} \\ \end{array} \}

\[ + \sin(\sqrt{\pi \phi_{c+} - \delta \pi x}) \begin{array}{l} -2 \sin \sqrt{\pi \theta_c} - \sqrt{\pi \phi_{s+}} \cos \sqrt{\pi \theta_s} \\ \sin \sqrt{\pi \theta_c} - \cos \sqrt{\pi \phi_{s+}} \sin \sqrt{\pi \phi_s} \\ \cos \sqrt{\pi \phi_{s+}} \cos \sqrt{\pi \phi_s} \\ \end{array} \].

\[ (B1) \]

where \( \Gamma \) equals \( i\eta(1)\eta(2) \) for the singlet and \( z \)-component of the triplet order parameters, and \( i\eta(1)\eta(2) \) for \( x, y \)-components of the triplet order parameters, and the same as below.

The difference of the bond strength between two legs is \( \sum_{\nu} \psi_{1\nu}(x) \psi_{2\nu}(x + a) + \psi_{2\nu}(x) \psi_{1\nu}(x + a) + h.c. \), similar to its triplet analog. Their staggered parts, \( O_{SP} \) and \( \hat{O}_{SP} \)

\[ O_{SP}(x) \] = \( (-)^x 2 \sin(k_f + \frac{\pi}{2}) \{ e^{-i\delta \pi x - i\delta \pi x} \begin{array}{l} 2 \sin \sqrt{\pi \theta_c} - \sin \sqrt{\pi \phi_{s+}} \cos \sqrt{\pi \theta_s} \\ \sin \sqrt{\pi \theta_c} + \sin \sqrt{\pi \phi_{s+}} \sin \sqrt{\pi \phi_s} \\ \cos \sqrt{\pi \phi_{s+}} + \cos \sqrt{\pi \phi_s} \\ \end{array} \}

\[ \propto \begin{array}{l} 2 \cos \sqrt{\pi \theta_c} - \cos \sqrt{\pi \phi_{s+}} \cos \sqrt{\pi \theta_s} \\ \sin \sqrt{\pi \theta_c} \cos \sqrt{\pi \phi_{s+}} \sin \sqrt{\pi \phi_s} \\ \cos \sqrt{\pi \phi_{s+}} \cos \sqrt{\pi \phi_s} \\ \end{array} \].

\[ (B2) \]

It is clear that \( O_{CDW} \) and \( O_{SP} \) are real and imaginary part of \( \psi_{1L\sigma}^\dagger \psi_{2R\sigma} + \psi_{1L\sigma}^\dagger \psi_{1R\sigma} \) respectively.

Next we present the staggered part of the diagonal current density, \( i \sum_{f} (-)^{j + 1} c_{j\sigma}^\dagger(i) c_{j+1\sigma}(i + 1) - h.c. \), and its triplet analog as below

\[ O_{DCD}(x) \]

\[ O_{DC; x,y}(x) \] = \( (-)^x \sin(k_f + \frac{\pi}{2}) \{ e^{-i\delta \pi x - i\delta \pi x} \begin{array}{l} 2 \cos \sqrt{\pi \theta_c} - \sin \sqrt{\pi \phi_{s+}} \cos \sqrt{\pi \theta_s} \\ \sin \sqrt{\pi \theta_c} + \sin \sqrt{\pi \phi_{s+}} \sin \sqrt{\pi \phi_s} \\ \cos \sqrt{\pi \phi_{s+}} + \cos \sqrt{\pi \phi_s} \\ \end{array} \}

\[ \propto \begin{array}{l} 2 \cos \sqrt{\pi \theta_c} - \cos \sqrt{\pi \phi_{s+}} \sin \sqrt{\pi \theta_s} \\ \cos \sqrt{\pi \phi_{s+}} \cos \sqrt{\pi \phi_s} \\ \cos \sqrt{\pi \phi_{s+}} \cos \sqrt{\pi \phi_s} \\ \end{array} \].

\[ (B3) \]

The difference of the current density along the legs is \( i \sum_{f} (-)^{j + 1} (c_{j\sigma}^\dagger(i) c_{j\sigma}(i + 1) - h.c. \). Its staggered part is

\[ (-)^x 2 \cos(k_f + \frac{\delta \pi}{2})(-i) \sum_{\sigma} \{ e^{-i\delta \pi x - i\delta \pi x \psi_{1\sigma L}(x) \psi_{2\sigma R}(x) - \psi_{1\sigma L}(x) \psi_{1\sigma R}(x) - h.c. \}. \]
Similarly, the staggered current along the rung \(i(c_{\sigma L}^\dagger(i)c_{\sigma R}(i) - h.c.)\) is:

\[
(-)^{\epsilon \eta} \frac{i}{\hbar} \sum_{\delta} e^{-i\delta\vec{O}} \left( \psi_{1\sigma L}^\dagger \psi_{2\sigma R} - \psi_{2\sigma L}^\dagger \psi_{1\sigma R} - h.c. \right)
\]

It can be shown that they satisfy the continuous relation \(\delta \vec{O}_{DDW} = \frac{\pi}{\alpha} \), so does its triplet counterpart, \(\vec{O}_{DDW'}\) staggered currents along legs and rungs have the d-wave feature. We use currents along the rung as order parameters. Their bosonized forms are

\[
O_{DDW}(x) = \frac{2\Gamma}{\pi a} \left\{ \cos(\sqrt{\pi} \phi_{c+} - \pi \delta x) \right\}
\]

\[
\left\{ 2 \sin \sqrt{\pi} \theta_{c-} \cos \sqrt{\pi} \phi_{s+} \sin \sqrt{\pi} \theta_{s-} - \cos \sqrt{\pi} \phi_{s+} \cos \sqrt{\pi} \phi_{s-} - \sin \sqrt{\pi} \theta_{s+} \cos \sqrt{\pi} \phi_{s-} \right\}
\]

\[
+ \sin(\sqrt{\pi} \phi_{c+} - \pi \delta x) \left\{ 2 \sin \sqrt{\pi} \phi_{s+} \cos \sqrt{\pi} \theta_{s-} - \sin \sqrt{\pi} \phi_{s+} \sin \sqrt{\pi} \phi_{s-} \right\}
\]

\[
\Delta_d = (\psi_{1\sigma L}^\dagger \psi_{1\sigma R} - \psi_{1\sigma L} \psi_{1\sigma R}^\dagger) - (\psi_{2\sigma L}^\dagger \psi_{2\sigma R} - \psi_{2\sigma L} \psi_{2\sigma R}^\dagger)
\]

\[
= 2\eta(1) \eta(1) \frac{e^{i\sqrt{\pi} \theta_{c+}}}{\pi a} \left( - \cos \sqrt{\pi} \phi_{c+} \sin \sqrt{\pi} \phi_{s+} \sin \sqrt{\pi} \phi_{s-} + i \sin \sqrt{\pi} \theta_{c-} \cos \sqrt{\pi} \phi_{s+} \cos \sqrt{\pi} \phi_{s-} \right)
\]

\[
\Delta_s = (\psi_{1\sigma L}^\dagger \psi_{1\sigma R} - \psi_{1\sigma L} \psi_{1\sigma R}^\dagger) + (\psi_{2\sigma L}^\dagger \psi_{2\sigma R} - \psi_{2\sigma L} \psi_{2\sigma R}^\dagger)
\]

\[
= 2\eta(1) \eta(1) \frac{e^{i\sqrt{\pi} \theta_{c+}}}{\pi a} \left( \cos \sqrt{\pi} \theta_{c-} \cos \sqrt{\pi} \phi_{s+} \cos \sqrt{\pi} \phi_{s-} + i \sin \sqrt{\pi} \theta_{c-} \sin \sqrt{\pi} \phi_{s+} \sin \sqrt{\pi} \phi_{s-} \right)
\]
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