Bell’s inequalities, multiphoton states and phase space distributions

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The connection between quantum optical nonclassicality and the violation of Bell’s inequalities is explored. Bell type inequalities for the electromagnetic field are formulated for general states of quantised radiation and their violation is connected to other nonclassical properties of the field. This is achieved by considering states with an arbitrary number of photons and carefully identifying the hermitian operators whose expectation values do not admit any local hidden variable description. We relate the violation of these multi-photon inequalities to properties of phase space distribution functions such as the diagonal coherent state distribution function and the Wigner function. Finally, the family of 4-mode states with Gaussian Wigner distributions is analysed, bringing out in this case the connection of violation of Bell type inequalities with the nonclassical property of squeezing.

I. INTRODUCTION

The superposition principle and the reduction of the wave function upon measurement are the two essential features of quantum mechanics, which are responsible for most counterintuitive paradoxical situations that arise when one starts to interpret the predictions of quantum mechanics. The violation of Bell’s inequalities is one of quantum theory’s most striking consequences [1]. When outcomes of individual measurements are given an objective meaning, the locality condition of “no action at a distance” imposes constraints on the possible correlations, which are expressed through Bell’s inequalities. There exist quantum mechanical states for which these inequalities are violated bringing out the fact that in quantum mechanics we either have to give up hope of an objective interpretation of individual measurements or accept “action at a distance”. The possibility of a local “classical” theory of individual measurements is thus ruled out. Though initial work, starting with Bohm [2] concerned itself with the states of two spin-\(\frac{1}{2}\) particles, it was for particular states of the quantised electromagnetic field that experiments were first performed in this direction [3] [4].

The formulation of Bell-type inequalities is possible only for a system which has two or more kinematically independent subsystems. These subsystems could each, for example, be a spin-half system with states in a two dimensional Hilbert space, a photon with fixed energy-momentum but variable polarisation state, or even a quantum mechanical system with one canonically conjugate pair of operators \(q\) and \(p\) and states in an infinite dimensional Hilbert space [5] [6] [7]. Many discussions focus on states of two photons, for two fixed propagation vectors and variable polarization states. To go beyond a single photon in each mode, in our analysis we will deal fully with all states of 4-mode fields, which include the two-photon states as a simple special case. On the other hand, the two-mode electromagnetic fields, with different directions of propagation and fixed polarizations, provide us with a situation kinematically equivalent to the one considered in the EPR paper as the subsystems are of the \(q,p\) form.

The quantum mechanical states for which Bell’s inequalities are violated thus have essential “quantum” features and defy a deeper interpretation based on realism (the objective existence of attributes independent of their measurement) and locality (no action at a distance). On the other hand, for a state which obeys a complete set of Bell’s inequalities it is in principle possible to give a local “classical” interpretation to individual measurements. An independent notion of classicality, used in quantum optics [8] [9], for the states of the quantised electromagnetic field is based on the diagonal coherent state distribution function, and here, the specific classical theory one has in mind is Maxwell’s theory. This paper explores the connections between these two different ways of classifying the quantum mechanical

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states of the electromagnetic field as “nonclassical” and “classical”. To carry out such an analysis, we obviously have to go beyond two-photon states and formulate Bell type inequalities for general four mode states with arbitrary photon number distribution.

A typical setup used to study violation of Bell’s inequalities involves four modes of the field with propagation in two different directions, and arbitrary polarisations being allowed transverse to each direction. For photons in each propagation direction a particular polarisation is selected by a variable polariser, and finally coincidence counts are recorded using photo detectors. In order to analyse an arbitrary state of the 4-mode field and derive a Bell-type inequality for it, two crucial inputs are needed: the first is the identification of hermitian operators whose measurements and correlations are directly related to the coincidence count rates; and the second is an appropriate quantum mechanical analysis of the polariser, namely its action on a general input multiphoton state. We feel that previous analyses in this direction lack proper treatment of one or the other of these aspects.

A coincidence count may be said to be registered when one or more photons are detected at each of the two detectors simultaneously or within a pre-assigned time interval, disregarding the exact numbers of photons detected. (So this is a kind of coarse grained coincidence rate). Hence such coincidence count rates differ from intensity-intensity correlations. We introduce an appropriate set of hermitian operators in the space of states of the 4-mode field so that all such coincidence counts can be calculated from the expectation values of products of these operators. By assuming a local realist hidden variable description for the outcome of various eigen values of the above mentioned operators we derive Bell type inequalities constraining the measurable coincidence count rates.

Classical states in the quantum optical sense have an underlying classical distribution function; they have been shown not to violate Bell’s inequalities, while nonclassical states are potential candidates for such violations. The quantum optical nonclassicality is invariant under passive canonical transformations which form the group $U(n)$ for a general $n$ mode system; such transformations can be experimentally implemented in a straightforward way using optically passive elements like beam splitters and mirrors. On the other hand, such transformations can alter the potential of a state to violate Bell type inequalities. Therefore, starting with a nonclassical state we should allow it to undergo arbitrary passive canonical transformations before looking for violations of Bell’s inequalities. It may turn out that a state which obeys Bell’s inequalities is related by a passive transformation to one which violates them; such a violation obviously is a consequence of the nonclassicality of the original state. This is closely connected to the fact that passive transformations can take nonclassical nonentangled states to nonclassical entangled ones. Generally, therefore, the relationship between total photon number conserving passive canonical transformations on the one hand, and entanglement or violation of Bell-type inequalities on the other, has to be carefully analysed. We show that this capacity of $U(n)$ to alter the entanglement properties of states underlies the violation of Bell type inequalities by beams originating from independent sources discussed by Yurke and Stoler. Further we show that for their scheme to work at least one of the beams has to be in a quantum optically nonclassical state. A much larger set of quantum mechanical states of the 4-mode electromagnetic field can be analysed in our formalism opening up the possibility of more easily experimentally observing the predicted violations of Bell’s inequalities.

The material in this paper is arranged as follows: In Section II we describe the basic 4-mode setup to be used for the study of Bell’s inequalities. The coincidence count rates for a general 4-mode state are then defined and used to derive Bell type inequalities with an appropriate quantum mechanical description for the polariser. Section III explores applications to various 4-mode states. Coherent states which are “classical” in more than one way ($i.e.$, they have classical diagonal coherent state distribution functions, are not entangled in any basis and lead to classically expected “photon counting”), are shown to obey Bell type inequalities. This result is then used to prove that all quantum optically classical states also obey our Bell-type inequalities. Finally we analyse in detail the family of states of the 4-mode field with a Gaussian-Wigner distribution, bringing out their violation of the Bell-type inequalities and the connection of such violation with the nonclassical property of squeezing. Section IV contains some concluding remarks.

II. BELL’S INEQUALITIES FOR MULTIPHOTON FIELDS: CHOICE AND PREPARATION OF STATES

Consider four modes of the electromagnetic field chosen so that we have two different propagation directions labeled by wave vectors $\mathbf{k}$ and $\mathbf{k}'$ and along each direction we allow arbitrary polarisations. We can choose for convenience a basis for each polarisation space and can then label these modes with annihilation operators $a_1, a_2, a_3$ and $a_4$; the modes $a_1$ and $a_2$ refer to the linear polarisations along $x$ and $y$ for the beam represented by wave vector $\mathbf{k}$ and the modes $a_3$ and $a_4$ to the linear polarisation modes along $x'$ and $y'$ for the wave vector $\mathbf{k}'$. Without any loss of generality, we can assume that both the directions of propagation are in the plane of the paper; then $x, x'$ are also chosen to be in the same plane while $y, y'$ are in the common direction pointing out of the plane of the paper. These are indicated in the left end of Figure 1. The $U(4)$ block symbolically represents the replacement of $a_1, a_2, a_3, a_4$ by complex linear
combinations of themselves determined by a matrix of $U(4)$:

$$a'_j = \sum_k U_{jk} a_k, \quad U \in U(4) \quad (2.1)$$

which is a passive total photon number conserving canonical transformation. Polarisers $P_1$ and $P_2$ can be set at any angles $\theta_1$ and $\theta_2$ with respect to $x$ and $x'$ respectively. After passing through these polarisers the beams encounter detectors $D_1$ and $D_2$ connected to each other by a coincidence counter. This scheme is very similar in spirit to most experimental and theoretical situations which have been extensively investigated.

![Diagram](image)

**FIG. 1.** Setup to study the violation of Bell type inequalities for arbitrary states of the four mode radiation field. The $U(4)$ block represents a variable passive $U(4)$ canonical transformation mixing the four modes, $P_1, P_2$ are the polarisers oriented at angles $\theta_1, \theta_2$ with respect to the axes $x$ and $x'$ respectively. $D_1$ and $D_2$ represent the detectors and the block c.c. indicates the coincidence counter. The part of the diagram to the left of the vertical line indicates symbolically the state preparation stage, that to the right the measurements done on it.

However, the difference here is that we do not wish to assume anything about the incident 4-mode state; it could have arbitrary number of photons, and could even be a mixed state. The measurements of interest here are the coincidence count rates, either for given settings of the polarisers (i.e., $\theta_1$ and $\theta_2$) or when one or both polarisers are removed. For a given $\theta_1(\theta_2)$, $D_1(D_2)$ receives and detects photons in the corresponding linear polarisation state alone.(For simplicity alone we restrict the analysis here to linear polarisation states)

**A. Interpretation of the coincidence count rates**

A coincidence is counted when both the detectors $D_1$ and $D_2$ click simultaneously('Simultaneous' here has the meaning as stated in the introduction that within a preassigned time interval $\Delta t$ each of the two detectors registers at least one photon). We will not distinguish in this analysis between coincidences of different strengths i.e., different numbers of photons (greater than one) being received by each detector. Coincidence rates defined in this specific physical manner must be represented by suitable choices of hermitian operators and their expectation values; this will be done in detail below. In our analysis, we will need the following four types of coincidence count rates:

(a) $P(\theta_1, \theta_2)$: The first polariser at $\theta_1$ and the second one at $\theta_2$ with respect to their respective $x$ axes.
(b) $P(\theta_1, )$: The first polariser at $\theta_1$ and the second one removed.
(c) $P( , \theta_2)$: The first polariser removed and the second one at $\theta_2$.
(d) $P( , )$: Both the polarisers removed from the setup.
These coincidence counts are the measurable quantities and can be calculated for any quantum mechanical state of the 4-mode field. On the other hand, we will see that these count rates have to obey certain inequalities if we demand a realist description and locality as well, so that measurements at detector $D_1$ are independent of the setting of the polariser $P_2$ and vice versa.

B. Quantum mechanical description of the polariser

Classically, the action of a polariser is straightforward. It allows a particular polarisation state (which defines the polariser axis) to pass through without any hindrance and blocks the orthogonal one completely. For a general polarisation state, the component of the electric field along the axis passes through unaffected while the orthogonal component is absorbed. We are interested in the generalisation of this action to quantum mechanical situations. For a comparable discussion for the beam splitter action see [16]. Given the classical action, we know how the polariser should act on single photons and on coherent states. For a single photon traveling in the $z$ direction, in an arbitrary state of polarisation $c_1|1\rangle_x|0\rangle_y + c_2|0\rangle_x|1\rangle_y$ ($c_1, c_2$ being arbitrary complex constants with $|c_1|^2 + |c_2|^2 = 1$) in the $x-y$ plane and with a polariser placed in the $x$ direction, the probability of transmission is $|c_1|^2$. From this we can build up the probabilities of transmission for arbitrary states as they can be expanded in terms of number states. Consider a general pure input state $\sum_{n_1, n_2} C_{n_1n_2} |n_1\rangle_x |n_2\rangle_y$ with the polariser placed along the $x$ direction. The probability of finding $n_1$ photons after passage through the polariser should be given by $\sum_{n_2} |C_{n_1n_2}|^2$. Nevertheless, this still leaves the question about the output state open-ended; what is the state of the single-mode field which emerges out of the polariser? Exactly identical probabilities can be obtained from pure as well as mixed states emerging from the polariser. Compared to the classical case, we need to give more detailed consideration to the physical process underlying the removal of photons of particular polarisation from the beam; it could either be absorption by certain other degrees of freedom (lattice) or a change of direction caused by different refractive indices for different polarisations. In either case, after passage through the polariser the information contained in that mode, though existing, is inaccessible and therefore we should trace over that mode to obtain the outgoing state of the field.

We thus arrive at the following general prescription for the action of the polariser: for a given input two-mode state with density matrix $\rho$, the two modes being two orthogonal polarisations along the same direction of propagation, and a polariser placed at an angle $\theta$ with respect to the $x$ axis, the single-mode state $\rho(\theta)$ after passage through the polariser is obtained by taking the trace over the mode orthogonal to the linear polarisation defined by $\theta$:

$$\rho(\theta) = \sum_{n=0}^{\infty} |n\rangle \rho(n) \langle n + \frac{\pi}{2}|$$  \hspace{1cm} (2.2)

Here we have chosen the number state basis for the mode orthogonal to $\theta$; we could as well have chosen any other complete set of states. More explicitly, consider the most general density matrix for the two polarisation modes along the same direction of propagation.

$$\rho = \sum_{n_1, n_2, n'_1, n'_2} C_{n_1n_2n'_1n'_2} |n_1\rangle_x |n_2\rangle_y \langle n'_1|_y \langle n'_2|_x$$  \hspace{1cm} (2.3)

If a polariser is placed at an angle $\theta$ with respect to the $x$ axis we first rotate the basis in the $x-y$ plane to $\theta$ and $\theta + \frac{\pi}{2}$ giving us

$$\rho = \sum_{n_1, n_2, n'_1, n'_2} C_{n_1n_2n'_1n'_2}(\theta) |n_1\rangle_\theta |n_2\rangle_{\theta + \frac{\pi}{2}} \langle n'_1|_{\theta + \frac{\pi}{2}} \langle n'_2|$$  \hspace{1cm} (2.4)

Each $C_{n_1n_2n'_1n'_2}(\theta)$ is a $\theta$ dependent linear expression in the $C_{\ldots}$’s, of course conserving $n_1 + n_2$ and $n'_1 + n'_2$. Now tracing over the mode orthogonal to $\theta$ yields the final state after the passage through the polariser:

$$\rho(\theta) = \sum_{n_1, n_2} \left( \sum_{n'_1} C_{n_1n_2n'_1n'_2}(\theta) \right) |n_1\rangle_\theta \langle n'_1|$$  \hspace{1cm} (2.5)

Here we have used the language appropriate to two modes and allowed the single photon to be in either of the polarisation modes; equivalently we could have talked about a single photon in a superposition of two orthogonal polarisation states.
When $\rho$ is a pure state i.e., $C_{n_1 n_2 n_1' n_2'} = C_{n_1 n_2} C_{n_1' n_2'}^*$, the rotated coefficients $C_{n_1 n_2 n_1' n_2'}(\theta)$ also factorise, and the output state

$$\rho(\theta) = \sum_{n_1, n_1'} \left( \sum_{n_2} C_{n_1 n_2}(\theta) C_{n_1' n_2'}^*(\theta) \right) |n_1\rangle_\theta \langle n_1'| \tag{2.6}$$

could in general be mixed. For the special case of coherent states $|z_1\rangle_x |z_2\rangle_y$

$$C_{n_1 n_2}(\theta) = e^{-\frac{1}{2}(|z_1|^2 + |z_2|^2)} \frac{z_1^{n_1} z_2^{n_2}}{\sqrt{n_1!} \sqrt{n_2!}}$$

$$C_{n_1 n_2}(\theta) = e^{-\frac{1}{2}(|z_1|^2 + |z_2|^2)} \frac{(z_1 \cos \theta - z_2 \sin \theta)^{n_1} (z_1 \sin \theta + z_2 \cos \theta)^{n_2}}{\sqrt{n_1!} \sqrt{n_2!}} \tag{2.7}$$

Therefore the final density matrix after the beam emerges from the polariser is given by

$$\rho_{cs}(\theta) = e^{-|z_1|^2 - |z_2|^2} \sum_{n_1, n_1'} \frac{(z_1 \cos \theta - z_2 \sin \theta)^{n_1}}{\sqrt{n_1!}} \times \frac{(z_1^* \cos \theta - z_2^* \sin \theta)^{n_1'}}{\sqrt{n_1'!}} \sum_{n_2} \frac{(|z_1 | \sin \theta + z_2 \cos \theta|)^{2n_2}}{n_2!} |n_1\rangle_\theta \langle n_1'|$$

$$= e^{-|z_1 \cos \theta - z_2 \sin \theta|^2} \sum_{n_1, n_1'} \frac{(z_1 \cos \theta - z_2 \sin \theta)^{n_1}}{\sqrt{n_1!}} \frac{(z_1^* \cos \theta - z_2^* \sin \theta)^{n_1'}}{\sqrt{n_1'!}} |n_1\rangle_\theta \langle n_1'| \tag{2.8}$$

which is a density matrix corresponding to the pure coherent state $|z_1 \cos \theta - z_2 \sin \theta\rangle_\theta$, consisting of photons all of which are polarized in the direction $\theta$. The reason for this is the subtle fact that coherent states are not entangled in any basis we may choose for the two-mode incident state, and thus tracing over one of the modes just amounts to neglecting the corresponding factor in the product state after an appropriate rotation of the basis

$$|z_1\rangle_x |z_2\rangle_y = |z_1 \cos \theta - z_2 \sin \theta\rangle_\theta |z_1 \sin \theta + z_2 \cos \theta\rangle_{x+\frac{\pi}{2}} \rightarrow |z_1 \cos \theta - z_2 \cos \theta\rangle_{\theta} \tag{2.9}$$

On the other hand, single photon states can in general be entangled states of the two-mode field and thus would lead to mixed one-mode states after passing through the polariser. For example a pure two-mode single photon state $\frac{1}{\sqrt{2}}(|1\rangle_x |0\rangle_y + |0\rangle_x |1\rangle_y)$, after passage through a polariser placed in the $x$ direction reduces to a mixed state with density matrix $\frac{1}{2}(|0\rangle_x |0\rangle_y + |1\rangle_x |1\rangle_y).

**C. Derivation of the inequalities**

In order to define and calculate the coincidence count rates, consider the following four hermitian operators, all having eigen values 0 and 1

$$\hat{A}_1 = (I_{2 \times 2} - |00\rangle\langle 00|)_k$$

$$\hat{A}_2 = (I_{2 \times 2} - |00\rangle\langle 00|)_{k'}$$

$$\hat{A}_1(\theta_1) = (I_{\theta_1} - |0\rangle_{\theta_1} \langle 0|) I_{\theta_1 + \frac{\pi}{2}}$$

$$\hat{A}_2(\theta_2) = (I_{\theta_2} - |0\rangle_{\theta_2} \langle 0|) I_{\theta_2 + \frac{\pi}{2}} \tag{2.10}$$

The subscripts $\theta_1$ and $\theta_2$ in the last two equations refer to the directions of the polarisers. Thus $\hat{A}_1$ and $\hat{A}_1(\theta_1)$ are operators belonging to the first two modes of our 4-mode system, namely propagation direction $k$ and polarisation along $x$ or $y$; $I_{2 \times 2}$ is the unit operator for these two modes while $I_{\theta_i}$ is the unit operator for the single mode propagating in direction $k$ and with polarisation $\theta_1$, and $I_{\theta_i + \frac{\pi}{2}}$ that for the orthogonal polarisation. $\hat{A}_2$ and $\hat{A}_2(\theta_2)$ are defined in a similar way for propagation direction $k'$. Expectation values of the above operators are probabilities of finding at least one photon of the appropriate kind:
\[ \langle \hat{A}_1 \rangle = \text{probability of detecting at least one photon at } D_1 \text{ with } P_1 \text{ removed}, \]
\[ \langle \hat{A}_2 \rangle = \text{probability of detecting at least one photon at } D_2 \text{ with } P_2 \text{ removed}, \]
\[ \langle \hat{A}_1(\theta_1) \rangle = \text{probability of detecting at least one photon at } D_1 \text{ with } P_1 \text{ set at } \theta_1, \]
\[ \langle \hat{A}_2(\theta_2) \rangle = \text{probability of detecting at least one photon at } D_2 \text{ with } P_2 \text{ set at } \theta_2. \]

These operators are defined so that when we calculate their expectation values in any state, the actions of the polarisers as described in the previous subsection are automatically implemented. In an individual measurement, one of the eigen values of the hermitian operator is recorded, with the probabilities for the two possible eigen values being calculable from the wavefunction or the density matrix. On the other hand if a hidden variable description is available, then given the value of the hidden variable \( \lambda \) as well as the quantum mechanical expectation vector, we can in principle predict the outcomes of individual measurements. It is well known that while keeping the hidden variable theory completely general and by imposing the locality condition (no action at a distance), general constraints on the correlations between observed quantities can be obtained; we will perform a similar analysis for our situation and derive Bell type inequalities for the correlations among the \( \hat{A} \)'s.

Assuming the probability distribution for \( \lambda \) to be \( \mathcal{P}(\lambda) \) (for simplicity the dependence of \( \mathcal{P}(\lambda) \) on the specific 4-mode quantum state is omitted) so that

\[ \int \mathcal{P}(\lambda) d\lambda = 1, \]

we now proceed to develop expressions for the average values of various observables. Let \( a_1(\lambda), a_2(\lambda), a_1(\theta_1, \lambda), \) and \( a_2(\theta_2, \lambda) \) be the actual values of the dynamical variables \( A_1, A_2, A_1(\theta_1) \) and \( A_2(\theta_2) \) for a given value of \( \lambda \) respectively: they take values 0 or 1. The averages computed from hidden variable theory are then given by

\[ \langle A_1 \rangle_{hv} = \int a_1(\lambda) \mathcal{P}(\lambda) d\lambda \]
\[ \langle A_2 \rangle_{hv} = \int a_2(\lambda) \mathcal{P}(\lambda) d\lambda \]
\[ \langle A_1(\theta_1) \rangle_{hv} = \int a_1(\theta_1, \lambda) \mathcal{P}(\lambda) d\lambda \]
\[ \langle A_2(\theta_2) \rangle_{hv} = \int a_2(\theta_2, \lambda) \mathcal{P}(\lambda) d\lambda \]

The average values of the products of these operators, corresponding to the various coincidence count rates, are then given in local hidden variable theory by

\[ P(\lambda, \lambda)_{hv} = \langle A_1 A_2 \rangle_{hv} = \int a_{12}(\lambda) \mathcal{P}(\lambda) d\lambda = \int a_1(\lambda) a_2(\lambda) \mathcal{P}(\lambda) d\lambda \]
\[ P(\theta_1, \theta_1)_{hv} = \langle A_1(\theta_1) A_2(\theta_1) \rangle_{hv} = \int a_{12}(\theta_1, \lambda) \mathcal{P}(\lambda) d\lambda = \int a_1(\theta_1, \lambda) a_2(\lambda) \mathcal{P}(\lambda) d\lambda \]
\[ P(\theta_1, \theta_2)_{hv} = \langle A_1(\theta_1) A_2(\theta_2) \rangle_{hv} = \int a_{12}(\theta_1, \theta_2, \lambda) \mathcal{P}(\lambda) d\lambda = \int a_1(\theta_1, \lambda) a_2(\theta_2, \lambda) \mathcal{P}(\lambda) d\lambda \]

The subscript “hv” is for ‘hidden variable’ and distinguishes these averages from the quantum mechanical expectation values. Here in the first step, \( a_{12} \)'s are the hidden variable theory values of corresponding dynamical variables \( A_1 A_2 \) etc. The last step in each of the above equations is very crucial and is basically the expression of the locality assumption on the hidden variable theory. We have assumed for instance that the quantity \( a_{12}(\theta_1, \theta_2, \lambda) \) is a product of the two factors \( a_1(\theta_1, \lambda) \) and \( a_2(\theta_2, \lambda) \), each depending only on one angle and not the other; more generally what we measure for the propagation direction \( k \) does not depend on what we choose to measure (or not measure) for direction \( k' \). This assumption is “reasonable” because we can arrange the situation such that the two measurement events are space like separated.
One more assumption we make is the “no enhancement assumption\footnote{This is at the level of hidden variable description. We have already seen that the possible values of \(a_1(\theta_1, \lambda)\) and \(a_1\) are 0 or 1. So if \(a_1(\lambda) = 0\) for some \(\lambda\) then \(a_1(\theta_1, \lambda) = 0\) as well for any \(\theta_1\).} i.e., the presence of the polariser can only remove photons from the beam and is incapable of adding photons or increasing the coincidence count rate:

\[
\begin{align*}
a_4(\theta_1, \lambda) & \leq a_1(\lambda) \\
a_2(\theta_2, \lambda) & \leq a_2(\lambda)
\end{align*}
\] (2.15)

We now recall the lemma due to Clauser and Horne \footnote{We now recall the lemma due to Clauser and Horne \cite{Clauser1974} which we will use to derive inequalities for the above calculated correlation functions.} which we will use to derive inequalities for the above calculated correlation functions.

**Lemma:**

If \(0 \leq x, x' \leq X\) and \(0 \leq y, y' \leq Y\) then

\[
-XY \leq xy - xy' + x'y - Xy' - Xy \leq 0
\] (2.16)

With the “no enhancement assumption” \footnote{Integrating the above inequality over \(\lambda\), with weight function \(P(\lambda)\) we get the inequality obeyed by the coincidence count rates for any choices of angles \(\theta_1, \theta_2, \theta_1', \theta_2'\) \footnote{For such a theory to be consistent with quantum mechanics, the coincidence rates computed from a quantum mechanical calculation should also obey the above inequalities. In the above derivation we have not assumed anything about the 4-mode state and therefore we can consider arbitrary states to check for any violation.}.

\[
- P(\theta_1, \theta_2) \leq P(\theta_1, \theta_2') + P(\theta_1', \theta_2) + P(\theta_1', \theta_2') - P(\theta_1, \theta_2') \leq 0
\] (2.18)

This is the inequality we will use in our future analysis. The left hand side inequality depends upon the total coincidence count rate without the polarisers whereas the one on the right hand side does not. This result is of the same form as that derived and used by all workers on the subject; what we emphasize here is that we have clearly specified the hermitian operator observables at the quantum mechanical level, for which correlations are considered, so the actual expressions for various \(P\)’s are specific to our treatment.

For such a theory to be consistent with quantum mechanics, the coincidence rates computed from a quantum mechanical calculation should also obey the above inequalities. In the above derivation we have not assumed anything about the 4-mode state and therefore we can consider arbitrary states to check for any violation.

If a given quantum mechanical state does not obey the above inequalities then it definitely has some nontrivial quantum features which cannot be accommodated within realist hidden variable theories based on locality. We now discuss various examples of 4-mode quantum states in sequence, and examine the validity or otherwise of inequalities \footnote{This is at the level of hidden variable description. We have already seen that the possible values of \(a_1(\theta_1, \lambda)\) and \(a_1\) are 0 or 1. So if \(a_1(\lambda) = 0\) for some \(\lambda\) then \(a_1(\theta_1, \lambda) = 0\) as well for any \(\theta_1\).} in each case.

### III. MULTIPHOTON STATES, VIOLATION OF BELL’S INEQUALITIES AND PHASE SPACE DISTRIBUTIONS

#### A. Two-photon states

Before undertaking the analysis of multiphoton states, we consider here the extensively studied two photon state as a warm up exercise. It so happens that for these states our formalism reduces to the usual one and we get results identical to those already available in the literature. Consider the following pure two-photon state of the 4-mode field

\[
|\psi\rangle = \frac{1}{2}( |1\rangle_1 |0\rangle_2 |1\rangle_3 |0\rangle_4 - |1\rangle_1 |0\rangle_2 |0\rangle_3 |1\rangle_4 - |0\rangle_1 |0\rangle_2 |1\rangle_3 |1\rangle_4 + |0\rangle_1 |1\rangle_2 |1\rangle_3 |0\rangle_4 \nonumber \\
= \frac{1}{2}(a_1^+ - a_1^\dagger)(a_2^+ - a_2^\dagger)|0\rangle_1 |0\rangle_2 |0\rangle_3 |0\rangle_4
\] (3.1)

The calculations of the quantum mechanical coincidence count rates become easy if we make the following observation about the operators \(\hat{A}_1(\theta_1)\) and \(\hat{A}_2(\theta_2)\):
\[
\hat{A}_1(\theta_1) = \mathcal{U}(R_1(\theta_1)) (I_1 - |0\rangle_1 \langle 0|) I_2 \mathcal{U}^{-1}(R_1(\theta_1)) \\
\hat{A}_2(\theta_2) = \mathcal{U}(R_2(\theta_2)) (I_3 - |0\rangle_3 \langle 0|) I_4 \mathcal{U}^{-1}(R_2(\theta_2)) 
\]  
(3.2)

Here \(\mathcal{U}(R_1(\theta_1))\) and \(\mathcal{U}(R_2(\theta_2))\) are the unitary operators corresponding to the transformations \(R_1(\theta_1)\) and \(R_2(\theta_2)\) which rotate the basis in the polarisation space by angles \(\theta_1\) and \(\theta_2\) for directions \(\mathbf{k}\) and \(\mathbf{k}'\) respectively.

\[
R_1(\theta_1) = \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{pmatrix}, \quad R_2(\theta_2) = \begin{pmatrix} \cos \theta_2 & \sin \theta_2 \\ -\sin \theta_2 & \cos \theta_2 \end{pmatrix} 
\]  
(3.3)

Using this form of the operators the calculation of the expectation values of various combinations of operators is straightforward and yields the quantum mechanical coincidence count rates:

\[
P(\theta_1, \theta_2)_{\text{tp}}^{\text{qm}} = \langle \psi | \hat{A}_1(\theta_1) \hat{A}_2(\theta_2) | \psi \rangle = \frac{1}{4} \sin^2(\theta_1 + \theta_2) \\
P(\theta_1, \theta_1)_{\text{tp}}^{\text{qm}} = \langle \psi | \hat{A}_1(\theta_1) \hat{A}_2(\theta_1) | \psi \rangle = \frac{1}{4} \\
P(\theta_2, \theta_2)_{\text{tp}}^{\text{qm}} = \langle \psi | \hat{A}_1(\theta_2) \hat{A}_2(\theta_2) | \psi \rangle = \frac{1}{4} \\
P(\theta_1, \theta_2)_{\text{tp}}^{\text{qm}} = \langle \psi | \hat{A}_1(\theta_1) \hat{A}_2(\theta_1) | \psi \rangle = \frac{1}{2} 
\]  
(3.4)

(The superscript ‘tp’ here refers to two-photon states). Substituting these rates in the inequality (2.18) we get the following condition, if this state is capable of a local realist hidden variable description:

\[
-1 \leq \frac{1}{2} (\sin^2(\theta_1 + \theta_2) - \sin^2(\theta_1 + \theta_2') + \sin^2(\theta_1' + \theta_2) + \sin^2(\theta_1' + \theta_2') - 2) \leq 0 
\]  
(3.5)

We find that the above condition can be violated on either side for some values of \(\theta_1, \theta_1', \theta_2\) and \(\theta_2'\). As an example let us choose \(\theta_1 = \frac{\pi}{8}, \theta_2 = \frac{\pi}{4}, \theta_1' = \frac{3\pi}{8}\) and \(\theta_2' = 0\); then the right hand side of the above inequality becomes

\[
\frac{1}{2} (\sqrt{2} - 1) < 0 
\]  
(3.6)

which is clearly violated.

Since two-photon states have been extensively studied in the literature we connect our result with existing analyses and make some useful observations. The single hermitian operator \(\hat{A} = I - |0\rangle \langle 0|\) whose expectation value gives the probability for finding one or more photons reduces effectively to \(\alpha^* \alpha\) for the appropriate mode when at most one photon is present in the beam; therefore our results agree with existing ones [17]. For two-photon states we have the relations \(P(\theta_1, \theta_1) = P(\theta_1, \theta_2) + P(\theta_1, \theta_2 + \frac{\pi}{2})\) etc. They too are a consequence of the reduction of the \(\hat{A}\) operators to the above described form and do not remain valid for the analysis of general states.

### B. The coherent states

As a first nontrivial example we consider 4-mode coherent states defined as (omitting further subscripts outside kets)

\[
|z_1\rangle |z_2\rangle |z_3\rangle |z_4\rangle = e^{-\frac{1}{2} \left(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2\right)} e^{z_1 a_1^\dagger + z_2 a_2^\dagger + z_3 a_3^\dagger + z_4 a_4^\dagger} |0000\rangle 
\]  
(3.7)

where \(z_1, z_2, z_3, z_4\) are complex numbers. The calculation of quantum mechanical coincidence count rates for this case is rather straightforward and we get the following results (the superscript ‘cs’ means coherent states):

\[
P(\theta_1, \theta_2)_{\text{cs}}^{\text{qm}} (\theta_1, \theta_2)_{\text{cs}}^{\text{qm}} = (1 - e^{-|z_1|^2})(1 - e^{-|z_3|^2}) \\
P(\theta_1)_{\text{cs}}^{\text{qm}} = (1 - e^{-|z_1|^2})(1 - e^{-|z_3|^2} + |z_4|^2) 
\]
Dividing throughout by the positive number \( P \) always obey the inequalities (2.18). Using the facts that \( 0 \leq \rho \) originates from the unentangled nature of the coherent states and will be used now to show that coherent states always obey the inequalities (2.18). Using the facts that \( 0 \leq \rho \) and the lemma (2.16), with the identification \( x = P(\theta_1, \phi_1)_{\text{cs}} \), \( x' = P(\theta_1', \phi_1')_{\text{cs}} \), \( y = P(\theta_2, \phi_2)_{\text{cs}} \), \( y' = P(\theta_2', \phi_2')_{\text{cs}} \), \( X = Y = P(\theta_1, \phi_1)_{\text{cs}} \) we arrive at the following preliminary inequality

\[
-P(\theta_1, \phi_1)_{\text{cs}}^2 \leq P(\theta_1, \phi_1)_{\text{cs}} P(\theta_2, \phi_2)_{\text{cs}} - P(\theta_1, \phi_1)_{\text{cs}} P(\theta_1', \phi_1')_{\text{cs}} + P(\theta_1', \phi_1')_{\text{cs}} P(\theta_2, \phi_2)_{\text{cs}} + P(\theta_1', \phi_1')_{\text{cs}} P(\theta_1, \phi_1)_{\text{cs}} \leq 0
\]

(3.9)

Dividing throughout by the positive number \( P \) and using the expressions of eq. (3.8) we see that the above initial inequality reduces to the Bell form:

\[
-P(\theta_1, \phi_1)_{\text{cs}} \leq P(\theta_1, \phi_1)_{\text{cs}} - P(\theta_1, \phi_1')_{\text{cs}} + P(\theta_1', \phi_1)_{\text{cs}} - P(\theta_1', \phi_1')_{\text{cs}} \leq 0
\]

(3.10)

Thus we see explicitly that coherent states do not violate the inequalities (2.18). The reduction of equation (3.9) to equation (3.10) is possible because coincidence count rates for coherent states factorize. Further, we emphasize that it is possible to write an inequality corresponding to (3.10) for any quantum state but it can be reduced to a Bell type form (3.10) only for those special states for which coincidence count rates factorize.

C. The classical states

We can express any arbitrary state of the 4-mode radiation field in terms of projections onto coherent states (4); in particular, for a state with density matrix \( \rho \), we have

\[
\rho = \frac{1}{\pi^4} \int \varphi(z_1, z_2, z_3, z_4)z_1 z_2 z_3 z_4 d^2 z_1 d^2 z_2 d^2 z_3 d^2 z_4.
\]

(3.11a)

\[
\frac{1}{\pi^4} \int \varphi(z_1, z_2, z_3, z_4) z_1^2 z_2^2 z_3^2 z_4^2 = 1
\]

(3.11b)

The diagonal coherent state distribution function \( \varphi(z_1, z_2, z_3, z_4) \) describes the state \( \rho \). In quantum optics the states of the 4-mode field are classified into classical and nonclassical types as follows: A given state is classical if the diagonal coherent state distribution function \( \varphi \) for it is nonnegative and nowhere more singular than a delta function. Otherwise the state is nonclassical. The function \( \varphi \) undergoes a point transformation when the state undergoes a unitary evolution corresponding to a passive canonical transformation given by an element of \( U(4) \):

\[
\varphi(z_1, z_2, z_3, z_4) \rightarrow \varphi'(z_1', z_2', z_3', z_4') = \varphi(z_1', z_2', z_3', z_4') \quad (z_1' z_2' z_3' z_4') = (z_1 z_2 z_3 z_4) U^T, \quad U \in U(4).
\]

(3.12)

Therefore, the classical or nonclassical nature of a state is preserved under such transformations. Coincidence count rates, and for that matter expectation values of any observable can be calculated from the diagonal coherent state distribution function. For a classical state, the inequality (2.18) becomes, by multiplication of all terms in (3.10) by \( \varphi \) followed by integration
The above result makes use of the nonnegative nature of \( \varphi \) and will not be true for a nonclassical state. Using the normalization (3.11b), and the nonnegative nonsingular nature of \( \varphi(z_1, z_2, z_3, z_4) \), we conclude that a "classical state" will not violate Bell type inequalities defined in eq. (2.18). Further, since the classical or nonclassical status of the 4-mode state is invariant under passive canonical transformations which form the group \( U(4) \), a classical state after undergoing such transformations will still not violate Bell type inequalities. On the other hand, the nonclassical states *i.e.*, the states with negative or singular diagonal coherent state distribution functions can violate these inequalities; in fact, the violation of such an inequality implies that the underlying diagonal coherent state distribution function for the state is negative or singular *i.e.*, the state is nonclassical in the quantum optical sense.

D. Application to two-mode Gaussian states

We have seen that typically a minimum of four modes are required for the analysis of Bell’s inequalities; but if we wish to study two-mode nonclassical states and their potential to violate Bell-type inequalities then we can choose the other two modes to be in the vacuum state or in general in a classical state, then perform a suitable \( U(4) \) transformation and proceed with the analysis. If we observe any violation, it can then be attributed to the initial nonclassical two-mode state. This is very similar to the detection of squeezing, where the mixing of the squeezed signal with high intensity coherent light is required in order to measure squeezing.

Consider a 4-mode state with a general centered Gaussian distribution [13]

\[
W(\xi) = \pi^{-4}(\text{Det}G)^{\frac{1}{4}} \exp(-\xi^T G \xi), \tag{3.14a}
\]

\[
\xi^T = \begin{pmatrix}
q_1 & q_2 & q_3 & q_4 & p_1 & p_2 & p_3 & p_4
\end{pmatrix}
\]

\[
G = \text{real symmetric positive definite } 8 \times 8 \text{ matrix.} \tag{3.14b}
\]

Here \( q \)'s and \( p \)'s are quadrature components corresponding to \( a' \) and \( a^\dagger \)'s (\( q_1 = \frac{1}{\sqrt{2}} (a_1^\dagger + a_1) \), \( p_1 = \frac{i}{\sqrt{2}} (a_1^\dagger - a_1) \) etc.). For \( W(\xi) \) to represent a quantum mechanical state and to be a Gaussian Wigner distribution, \( G \) has to satisfy in addition the condition

\[
G^{-1} + i\beta \geq 0
\]

\[
\beta = \begin{pmatrix}
0_{4 \times 4} & 1_{4 \times 4} \\
-1_{4 \times 4} & 0_{4 \times 4}
\end{pmatrix} \tag{3.15}
\]

This is an expression of the uncertainty relations between the canonically conjugate \( q \)'s and \( p \)'s. The matrix \( V = \frac{1}{2} G^{-1} \) is the variance or the noise matrix. For a given state, if the least eigenvalue \( \mathcal{E}_<(V) \) of this matrix is less than \( \frac{1}{2} \), then the state is squeezed and hence nonclassical [13]:

\[
\mathcal{E}_<(V) < \frac{1}{2} \Leftrightarrow \text{Squeezing} \tag{3.16}
\]

We now proceed to study the possibility of violation of the Bell type inequality (2.18) for Gaussian states and its possible correlation with squeezing. To calculate the coincidence count rates we need to express the \( U(4) \) matrix corresponding to the rotation to the basis defined by the polarisers, as a matrix of \( Sp(8, R) \); we construct it directly from (3.3)

\[
U(\theta_1, \theta_2) = \begin{pmatrix}
\cos \theta_1 & -\sin \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sin \theta_1 & \cos \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cos \theta_2 & -\sin \theta_2 & 0 & 0 & 0 & 0 \\
0 & 0 & \sin \theta_2 & \cos \theta_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cos \theta_1 & -\sin \theta_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sin \theta_1 & \cos \theta_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cos \theta_2 & -\sin \theta_2 \\
0 & 0 & 0 & 0 & 0 & 0 & \sin \theta_2 & \cos \theta_2
\end{pmatrix} \tag{3.17}
\]
In order to calculate the coincidence count rates, we first compute the overlaps of the 4-mode Gaussian state with appropriate vacuum states. The Wigner function for a single-mode vacuum state is given by

\[
W_0(q,p) = \frac{1}{\pi} \exp(-q^2 - p^2)
\]  

(3.18)

The relevant overlaps can then be written as

\[
\begin{align*}
\Tr(\rho) & = 2\pi \int W(U(\theta_1,0)\xi)W_0(q_1,p_1)d\xi \\
& = 2\sqrt{\det(U^T(\theta_1,0)G\rho(\theta_1,0) + e_{11} + e_{55})^{-1}} \\
\Tr(\rho) & = 2\pi \int W(U(0,\theta_2)\xi)W_0(q_3,p_3)d\xi \\
& = 2\sqrt{\det(U^T(0,\theta_2)G\rho(0,\theta_2) + e_{33} + e_{77})^{-1}} \\
\Tr(\rho) & = (2\pi)^2 \int W(U(\theta_1,\theta_2)\xi)W_0(q_1,p_1)W_0(q_3,p_3)d\xi \\
& = 4\sqrt{\det(U^T(\theta_1,\theta_2)G\rho(\theta_1,\theta_2) + e_{11} + e_{33} + e_{55} + e_{77})^{-1}} \\
\Tr(\rho) & = (2\pi)^3 \int W(U(\theta_1,0)\xi)W_0(q_1,p_1)W_0(q_3,p_3)W_0(q_4,p_4)d\xi \\
& = 16\sqrt{\det(G + I_{8\times8})^{-1}}
\end{align*}
\]

(3.19)

Using these overlap integrals, and the definitions (2.10) we can immediately compute the quantum mechanical coincidence count rates

\[
\begin{align*}
P(\theta_1, \theta_2)_{\text{gauss}} & = 1 - \Tr(\rho) - \Tr(\rho) + \Tr(\rho) + \Tr(\rho) \\
& = 1 - \Tr(\rho) + \Tr(\rho) \\
& = 1 - \Tr(\rho) + \Tr(\rho) \\
& = 1 - \Tr(\rho) + \Tr(\rho) \\
& = 1 - \Tr(\rho) + \Tr(\rho)
\end{align*}
\]

(3.20)

The equations (3.19), can now be used to analyse any Gaussian Wigner state to see if it violates the inequality (2.18). Having set up the general formalism for the family of centered Gaussian Wigner states, we now consider examples of G which lead to violation of the inequality (2.18). Consider

\(^1\) \(e_{ij}\) is an \(8 \times 8\) matrix with \(e_{ij})_{ij} = 1\) and all other elements zero; \(I_{8\times8}\) is eight dimensional identity matrix
\[ G = U^{-1} S^T G_0 S U \]
\[ G_0 = \kappa I_{8 \times 8}, \quad 0 \leq \kappa \leq 1. \]
\[ \kappa = \tanh \frac{\beta}{2}, \quad \beta = \frac{\hbar \omega}{kT} \tag{3.21} \]

Here \( \kappa = 1 \) implies zero temperature and \( \kappa < 1 \) corresponds to some finite temperature, \( S \) is a 4-mode squeezing symplectic transformation, which is a \( Sp(8, \mathbb{R}) \) matrix, and \( U \) is a passive symplectic \( U(4) \) transformation whose role is to produce entanglement. As an example, we start with a state in which the modes \( a_1 \) and \( a_4 \) are squeezed by equal and opposite amounts \( u \) and the modes \( a_2 \) and \( a_3 \) are squeezed by equal and opposite amounts \( v \), and the entanglement is “maximum”. This corresponds to the choices:

\[
S = \begin{pmatrix}
  e^{-u} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & e^{v} & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & e^{-v} & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & e^{u} & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & e^{u} & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & e^{-v} & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & e^{v} & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & e^{-u} & 0
\end{pmatrix},
\]
\[ U = \frac{1}{2} \begin{pmatrix}
  1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
  -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
  -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\
  1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
  0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \\
  0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \\
  0 & 0 & 0 & 0 & 1 & -1 & -1 & 1
\end{pmatrix} \tag{3.22} \]

For this particular class of centered Gaussian Wigner states the function (dependences on \( u \) and \( v \) are left implicit)
\[
f(\theta_1, \theta_2, \theta'_1, \theta'_2) = P(\theta_1, \theta_2)_{\text{gauss}} - P(\theta_1, \theta'_2)_{\text{gauss}} + P(\theta'_1, \theta_2)_{\text{gauss}} + P(\theta'_1, \theta'_2)_{\text{gauss}} - P(\theta'_1, \theta'_2)_{\text{gauss}}, \tag{3.23} \]

can be calculated.

\footnote{This \( S \) matrix is constructed from two two-mode squeezing transformations studied in detail in \[13\], the first being the one which squeezes modes 1 and 4 by equal and opposite amounts \( u \), whereas the second one squeezes the modes 2 and 3 by equal and opposite amounts \( v \).}
FIG. 2. Violation of Bell type inequality for states with centered Gaussian Wigner distributions representing a 4-mode squeezed vacuum. (a) \( v = 0 \) i.e., Modes 1 and 4 are squeezed by equal and opposite amounts \( u \) (b) \( u = 0 \) i.e., Modes 2 and 3 are squeezed by equal and opposite amounts (c) \( u = v \) i.e., all modes are squeezed. (d) \( u = -v \) i.e., all modes are squeezed.
FIG. 3. Violation of Bell type inequality for states with centered Gaussian Wigner distributions representing 4-mode squeezed thermal states with the same temperature for all the modes. The choice of parameters for figures (a), (b), (c) and (d) is the same as that for Figure 2.

Though one ought to search over all values of the angles $\theta_1, \theta_2, \theta'_1$ and $\theta'_2$ to look for possible violations of the inequality (2.18), motivated by the choice of angles for the two-photon state, we restrict ourselves to that same choice, and plot the function $f(\pi/8, \pi/4, 3\pi/8, 0)$ for various combinations of the squeeze parameters in Figures 2 and 3. In Figure 2 we have considered the case $\kappa = 1$, i.e., the state under consideration is the squeezed vacuum; whereas for Figure 3 we have $\kappa = 0.75$ which corresponds to a squeezed thermal state. For the case of the squeezed vacuum, a clear violation is demonstrated in figures 2(b) and 2(c), for a considerable range of squeeze parameters. No violation is seen for the parameters chosen in Figure 2(d), and Figure 2(a) shows very little violation. However, for these parameter values for squeezing the violation might occur for some other values of angles $\theta_1, \theta_2, \theta'_1$ and $\theta'_2$. Figure 3 has similar features as Figure 2 but we see as expected that the amount of violation has diminished when the initial state is thermal instead of vacuum; as a consequence the small amount of violation which was present in Figure 2(a) has disappeared in Figure 3(a). As a further study, in the context of Gaussian Wigner states, it will be interesting to see the effect of phase space displacement on the violation of these inequalities; this will be taken up elsewhere.
IV. CONCLUDING REMARKS

We have developed the machinery for analysing the violation of Bell type inequalities for a general state of the 4-mode radiation field in a setup of the type described in Fig 1. A classical state in the quantum optical sense always obeys these inequalities while a nonclassical state may violate them, possibly after a $U(4)$ transformation. Starting with a general nonclassical state, we subject it to a general unitary evolution corresponding to passive canonical transformations $U(4)$ before we look for the violation of Bell type inequalities. It may turn out that a given nonclassical state does not violate Bell type inequalities but some $U(4)$ variant of it does.

A pure quantum mechanical state of a composite system is said to be entangled if we are not able to express it as a product of two factors, one belonging to each subsystem. Such states have nontrivial quantum correlations and can lead to the violation of suitable Bell’s inequalities. The simplest quantum optical system for which the notion of entanglement can be introduced is the two-mode field. The group of passive canonical transformations in this case is $U(2)$; its elements, though incapable of producing or destroying nonclassicality are capable of entangling (disentangling) originally unentangled (entangled) states. As an example we choose the nonclassical state $\ket{11}$ which is not entangled; by a simple $U(2)$ transformation $e^{i\pi/4}(a_1^+a_2 + a_2^+a_1)$ it becomes the entangled state $\frac{1}{\sqrt{2}}(\ket{20} + \ket{02})$.

However, coherent states are not entanglable in this way! If we start with a two-mode coherent state $\ket{z_1, z_2}$, it is clearly not entangled i.e., the state can be written as a product with one factor belonging to one mode and the other to the other mode. Under a $U(2)$ transformation this property is maintained. Classical states are statistical mixtures of coherent states and under $U(2)$ transform again to such mixtures of coherent states. Such a mixture can definitely have correlations which are purely classical, but it cannot have truly quantum mechanical entanglement. Thus classical states are to be regarded as not entangled, and they remain so under passive $U(2)$ transformations. However this is in general not true for a nonclassical nonentangled state which may get entangled under a suitable $U(2)$ transformation. It is a straightforward matter to generalise the above statements to $n$ mode systems where the group of passive canonical transformations is $U(n)$. There are several ways to quantify entanglement; for pure states it is unambiguous: if the reduced density matrices for the subsystems involved are also pure states then the original state is not entangled. On the other hand, if in the process of partial tracing, some information is lost then the original state is entangled. The generalisation to mixed states is nontrivial but is conceptually simple; we have to separate classical correlations from the quantum mechanical ones and this may not always be easy to do. However, as we saw for the case of classical states, we can sometimes easily conclude that a given state is nonentangled.

The above conclusions have an interesting bearing on the work on violation of Bell-type inequalities with beams originating from independent sources [14] [15] [19]. These experiments take two beams from two independent sources, pass them through some passive optical elements and show that the Bell-type inequalities are violated. The first conclusion we can draw from our analysis is that it must be the quantum optical nonclassicality of one of the beams in this experiment which has been converted into entanglement by the $U(4)$ transformation and hence led to the violation. Secondly, if the original beams were quantum optically classical, no matter what one does, no violation will be seen.

In our analysis, we have not distinguished between strengths of coincidences. The coincidence counter registers a count when simultaneously each detector detects one or more photons. This is the reason why we chose the operators $A_i$’s to have eigen values 0 and 1. In this sense, the measurements involved here are not refined. It would be interesting to further generalise the analysis by considering somewhat refined measurements where to some extent coincidences are distinguished on the basis of their strengths. However, the relevant operators in this context may be unbounded; and it is well known that the formulation of Bell type inequalities for such operators, though desirable, is nontrivial.

We have compared quantum optical nonclassicality with violation of Bell’s inequalities. When a state is nonclassical in the quantum optical sense, it does not allow a classical description based on an ensemble of solutions of Maxwell’s equations, which is a very specific classical theory. On the other hand violation of a Bell type inequality rules out any possibility of describing it by any general local “classical” hidden variable theories. Therefore, it is understandable that quantum optical nonclassicality is a necessary but not a sufficient condition for the violation of Bell’s inequalities. This disparity is partially compensated for by the freedom to perform passive canonical transformations on a nonclassical state before looking for violation of Bell’s inequalities though it is not obvious whether this freedom completely removes this discrepancy. On the other hand, if a state obeys Bell’s inequalities, it may still not allow a “classical” description. Therefore, we need a complete set of Bell’s inequalities capturing the full content of the locality assumption. These and related aspects will be explored elsewhere.
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