EQUIVARIANT RESOLUTION OF SINGULARITIES IN CHARACTERISTIC 0

DAN ABRAMOVICH AND JIANHUA WANG

0. Introduction

We work over an algebraically closed field \( k \) of characteristic 0.

0.1. Statement. In this paper, we use techniques of toric geometry to reprove the following theorem:

**Theorem 0.1.** Let \( X \) be a projective variety of finite type over \( k \), and let \( Z \subset X \) be a proper closed subset. Let \( G \subset \text{Aut}_k(Z \subset X) \) be a finite group. Then there is a \( G \)-equivariant modification \( r : X_1 \to X \) such that \( X_1 \) is nonsingular projective variety, and \( r^{-1}(Z_{\text{red}}) \) is a \( G \)-strict divisor of normal crossings.

This theorem is a weak version of the equivariant case of Hironaka’s well known theorem on resolution of singularities. It was announced by Hironaka, but a complete proof was not easily accessible for a long time. The situation was remedied by E. Bierstone and P. Milman [B-M2], and by O. Villamayor [V]. They gave constructions of completely canonical resolution of singularities. These constructions are based on a thorough understanding of the effect of blowing up - one carefully build up an invariant pointing to the next blowup.

The proof we give in this paper takes a completely different approach. It uses two ingredients: first, we assume that we know the existence of resolution of singularities without group actions. The method of resolution is not important: any of [H], [B-M1], [V] [K-dJ] or [B-P] would do. Second, we use equivariant toroidal resolution of singularities. Unfortunately, in [KKMS] the authors do not treat the equivariant case. But proving this turns out to be straightforward given the methods of [KKMS]. (For a similar argument in the toric case see [B].)

To this end, section 2 of this paper is devoted to proving the following:

**Theorem 0.2.** Let \( U \subset X \) be a strict toroidal embedding, and let \( G \subset \text{Aut}(U \subset X) \) be a finite group acting toroidally. Then there is a \( G \)-equivariant toroidal ideal sheaf \( I \) such that the normalized blowup of \( X \) along \( I \) is a nonsingular \( G \)-strict toroidal embedding.
1. Preliminaries

First recall some definitions. We restrict ourselves to the case of varieties over \( k \). A large portion of the terminology is borrowed from [8-dJ].

A modification is a proper birational morphism of irreducible varieties.

Let a finite group \( G \) act on a (possibly reducible) variety \( Z \). Let \( Z = \cup Z_i \) be the decomposition of \( Z \) into irreducible components. We say that \( Z \) is \( G \)-strict if the union of translates \( \cup_{g \in G} g(Z_i) \) of each component \( Z_i \) is a normal variety. We simply say that \( Z \) is strict if it is \( G \)-strict for the trivial group, namely every \( Z_i \) is normal.

A divisor \( D \subset X \) is called a divisor of normal crossings if étale locally at every point it is the zero set of \( u_1 \cdots u_k \) where \( u_1, \ldots, u_k \) is part of a regular system of parameters. Thus, in a strict divisor of normal crossings \( D \), all components of \( D \) are nonsingular.

An open embedding \( U \hookrightarrow X \) is called a toroidal embedding if locally in the étale topology it is isomorphic to a torus embedding \( T \hookrightarrow V \). (see [KKMS], II §1). One may replace “étale locally” by “complex analytically” in case \( k = \mathbb{C} \), or “formally”, obtaining the same class of embeddings. Let \( E_i, i \in I \) be the irreducible components of \( X \setminus U \). A finite group action \( G \subset \text{Aut}(U \hookrightarrow X) \) is said to be toroidal if the stabilizer of every point can be identified on the appropriate neighborhood with a subgroup of the torus \( T \). We say that a toroidal action is \( G \)-strict if \( X \setminus U \) is \( G \)-strict. In particular the toroidal embedding itself is said to be strict if \( X \setminus U \) is strict. This is the same as the notion of toroidal embedding without self-intersections in [KKMS]. For any subset \( J \) of \( I \), the components of the sets \( \cap_{i \in J} E_i - \cup_{i \notin J} E_i \) define a stratification of \( X \). Each component is called a stratum.

Recall that in [KKMS], p. 69-70 one defines the notion of a conical polyhedral complex with integral structure. As in [KKMS], p. 71, to every strict toroidal embedding \( U \subset X \) one canonically associates a conical polyhedral complex with integral structure. In the sequel, when we refer to a conical polyhedral complex, it is understood that it is endowed with an integral structure.

In [KKMS], p. 86 (Definition 2) one defines a rational finite partial polyhedral decomposition \( \Delta' \) of a conical polyhedral complex \( \Delta \). We will restrict attention to the case where \( |\Delta'| = |\Delta| \), and we will call this simply a polyhedral decomposition or subdivision.

The utility of polyhedral decompositions is given in Theorem 6* of [KKMS] (page 90), which establishes a correspondence between allowable modifications of a given strict toroidal embedding (which in our terminology are proper), and polyhedral decompositions of the associated conical polyhedral complex.

In order to guarantee that a modification is projective, one needs a bit more. Following [KKMS], p. 91, a function \( \text{ord} : \Delta \to \mathbb{R} \) defined on a conical polyhedral complex with integral structure is called an order function if:
(1) \( \text{ord}(\lambda x) = \lambda \cdot \text{ord}(x), \lambda \in \mathbb{R}^+ \)  

(2) \( \text{ord} \) is continuous, piecewise-linear  

(*) \( \text{ord}(N^Y \cap \sigma^Y) \subset \mathbb{Z} \) for all strata \( Y \).  

(4) \( \text{ord} \) is convex on each cone \( \sigma \subset \Delta \)  

For an order function on the conical polyhedral complex corresponding to \( X \), we can define canonically a coherent sheaf of fractional ideals on \( X \), and vice versa (see [KKMS], I §2). The order function is positive if and only if the corresponding sheaf is a genuine ideal sheaf. We have the following important theorem [KKMS]:

**Theorem 1.1.** Let \( F \) be a coherent sheaf of ideals corresponding to a positive order function \( \text{ord}_F \), and let \( B_F(X) \) be the normalized blowup of \( X \) along \( F \). Then \( B_F(X) \to X \) is an allowable modification of \( X \), described by the decomposition of \( |\Delta| \) obtained by subdividing the cones into the biggest subcones on which \( \text{ord}_F \) is linear.

A polyhedral decomposition is said to be **projective** if it is obtained in such a way from an order function. The corresponding modification is indeed a projective morphism.

Given a cone \( \sigma \) and a rational ray \( \tau \subset \sigma \), it is natural to define the decomposition of \( \sigma \) centered at \( \tau \), whose cones are of the form \( \sigma' + \tau \), where \( \sigma' \) runs over faces of \( \sigma \) disjoint from \( \tau \). Given a polyhedral complex \( \Delta \) and a rational ray \( \tau \), we can take the subdivision of all cones containing \( \tau \) centered at \( \tau \), and again call the resulting decomposition of \( \Delta \), the subdivision centered at \( \tau \).

From [KKMS] I §2, lemmas 1-3, p. 33-35 it follows that the subdivision centered at \( \tau \) is projective.

One can also obtain the barycentric subdivision inductively the other way: the barycentric subdivision of an \( m \)-dimensional cone \( \delta \) is formed by first taking the barycentric subdivision of all its faces, and for each one of the resulting cones \( \sigma \), including also the cone \( \sigma + b(\delta) \). This way it is clear that \( B(\Delta) \) is a simplicial subdivision.

2. **Equivariant toroidal modifications**

**Lemma 2.1.** Let \( U \subset X \) be a strict toroidal embedding, \( G \subset \text{Aut}(U \subset X) \) a finite group action. Then
(1) The group $G$ acts linearly on $\Delta(X)$.
(2) Assume that the action of $G$ is strict toroidal. Let $g \in G$, and let 
$\delta \subset \Delta(X)$ be a cone, such that $g(\delta) = \delta$. Then $g_\delta = \text{id}$.

Proof. 

(1) Clearly, $G$ acts on the stratification of $U \subset X$. Note that, from Defi-
nition 3 of [KKMS], page 59, $\Delta(X)$ is built up from the groups $M^Y$ of 
Cartier divisors on $\text{Star}(Y)$ supported on $\text{Star}(Y) \setminus U$, as $Y$ runs through 
the strata. As $g \in G$ canonically transforms $M^Y$ to $M^{g^{-1}Y}$ linearly, our 
claim follows.

(2) Assume $g : \delta \to \delta$, and $g_\delta \neq \text{id}$, then there exists an edge $e_1 \in \delta$, s.t 
$g(e_1) \neq e_1$. Denote $g(e_1) = e_2$. Assume $e_1$ corresponds to a divisor 
$E_1$, and $e_2$ corresponds to a divisor $E_2$. Since $g(e_1) = e_2$ we have 
g$(E_1) = E_2$. As $e_1, e_2$ are both edges of $\delta$, $E_1 \cap E_2 \neq \phi$. So $\cup g(E_1)$ 
can not be normal since it has two intersecting components. This is a 
contradiction to the fact that $G$ acts strictly on $X$. \hfill \Box

Lemma 2.2. Let $G \subset \text{Aut}(U \subset X)$ act toroidally. Let $\Delta_1$ be a $G$-equivariant 
subdivision of $\Delta$, with corresponding modification $X_1 \to X$. Then $G$ acts 
toroidally on $X_1$. Moreover, if $G$ acts strictly on $X$, it also acts strictly on 
$X_1$.

Proof. The fact that $G$ acts on $X_1$ follows from the canonical manner in which 
$X_1$ is costructed from the decomposition $\Delta_1$, see Theorems 6* and 7* of [KKMS], 
II §2, p. 90.

Now for any point $a \in X_1$ and $g \in \text{Stab}_a$, we have $g \circ f(a) = f \circ g(a) = f(a)$ 
hence $g \in \text{Stab}_f(a)$. Thus $\text{Stab}_a$ is a subgroup of $\text{Stab}_f(a)$, which is identified 
with a subgroup of the torus in a neighbourhood of $f(a)$. This proved that $\text{Stab}_a$ 
is identified with a subgroup of the torus.

We are left with showing that if $G$ acts strictly on $X$, then it acts strictly on 
$X_1$. Assume it is not the case. There exist two edges $\tau_1, \tau_2$ in $\Delta_1$, which are both 
edges of a cone $\delta'$, and $g(\tau_1) = \tau_2$. We choose the cone $\delta'$ of minimal dimension. 
Clearly, $\tau_1$ and $\tau_2$ cannot be both edges in $\Delta$, since $G$ acts strictly on $X$. Let 
us assume $\tau_2$ is not an edge in $\Delta$. So $\tau_2$ must be in the interior of a cone $\delta$ in 
$\Delta$, which contains $\delta'$. Now since $\delta' \cap g(\delta') \supset \tau_2 \subset \text{interior of } \delta$, we conclude: 
interior of $\delta \cap g(\delta) : \neq \phi$, which means that $g(\delta) = \delta$. From the previous lemma, 
$g_\delta = \text{id}$, so $g_\delta = \text{id}$ too, contradiction. \hfill \Box

Proposition 2.3.

(1) There is a one to one correspondence between edges $\tau_i$ in the barycentric 
subdivision $B(\Delta)$ and positive dimensional cones $\delta_i$ in $\Delta$. We denote 
this by $\tau \mapsto \hat{\delta}_\tau$.

(2) Let $\tau_i \neq \tau_j$ be edges of a cone $\hat{\delta} \in B(\Delta)$. Then $\dim \delta_{\tau_i} \neq \delta_{\tau_j}$. 


(3) If $G$ is a finite group acting toroidally on a strict toroidal embedding $U \subset X$ with corresponding polyhedral complex $\Delta$, then the action of $G$ on $X_{B(\Delta)}$ is strict.

Remark. Using this proposition, the argument at the end of [8-dJ] can be significantly simplified: there is no need to show $G$-strictness of the toroidal embedding obtained there, since the barycentric subdivision automatically gives a $G$ strict modification.

Proof.  
1. Define a map $b : \text{positive dimensional cones in } \Delta \rightarrow \text{edges in } B(\Delta)$ by  
   \[ b(\delta) = \text{the barycenter of } (\delta) \]
and define $\delta : \text{edges in } B(\Delta) \rightarrow \text{cones in } \Delta$ by  
   \[ \delta_\tau = \text{the unique cone whose interior contains } \tau \]
then it is easy to see that $b$ and $\delta$ are inverses of each other.
2. We proceed by induction on $\text{dim } \Delta$. The cone $\delta$ spanned by $\tau_i$ and $\tau_j$ must lie in some cone of $\Delta$, say $\delta^*$, which we may take of minimal dimension. We follow the second construction of the barycentric subdivision described in the preliminaries. Either $\text{dim } \delta^* \leq m - 1$, so $\delta$ is in the barycentric subdivision of the $m - 1$-skeleton of $\Delta$, in which case the statement follows by the inductive assumption, or $\text{dim } \delta^* = m$, in which case only one of $\tau_1$ and $\tau_2$ can be its barycenter, and the other is again a barycenter of a cone in the $m - 1$ skeleton.
3. From lemma 2.2, since the decomposition $B(\Delta)$ of $\Delta$ is equivariant, $G$ acts toroidally on $X_{B(\Delta)}$. Let $E_1, E_2 = g(E_1) \subset X_{B(\Delta)} \setminus U$ be divisors corresponding to edges $e_1, e_2$ in $B(\Delta)$. If $E_1 \cap E_2 \neq \emptyset$, there is a cone in $B(\Delta)$ containing $e_1, e_2$ as edges. From part (2), $\text{dim } \delta_{e_1} \neq \text{dim } \delta_{e_2}$, so $g(e_1)$ can not equal to $e_2$. This contradicts the fact that the morphism is equivariant and $g(E_1) = E_2$.

Proposition 2.4. There is a positive $G$-equivariant order function on $B(\Delta)$ such that the associated ideal $I$ induces a blowing up $B_I X_{B(\Delta)}$, which is a non-singular $G$-strict toroidal embedding, on which $G$ acts toroidally.

Proof. By the previous proposition, we know that $G$ acts toroidally and strictly on $X_{B(\Delta)}$. It follows from Lemma 2.1 that the quotient $B(\Delta)/G$ is a conical polyhedral complex, since no cone has two distinct edges in $B(\Delta)$ which are identified in the quotient. We can use the argument of [KKMS], I §2, lemmas 1-3, to get an order function ord : $B(\Delta)/G \rightarrow R$ which induces a simplicial subdivision with every cell of index 1. Denote by $\pi : B(\Delta) \rightarrow B(\Delta)/G$ the quotient map. Then $ord \circ \pi$ is an order function subdividing $B(\Delta)$ into simplicial cones of index 1. Let $I$ be the corresponding ideal sheaf. The blow up of $X_{B(\Delta)}$ along $I$ is a nonsingular strict toroidal embedding $U \subset B_I X_{B(\Delta)}$. By lemma 2.2, $G$ acts on $B_I X_{B(\Delta)}$ strictly and toroidally.
Proof of Theorem 0.2. Let $G \subset \text{Aut}(U \subset X)$ be as in the theorem. The morphism $X_{B(\Delta)} \to X$ is projective, and by the last two propositions there is a projective, toroidal $G$-equivariant morphism $Y \to X$ where $Y$ is nonsingular and such that $G$ acts strictly and toroidally on $Y$. \hfill \Box

Remark. With a little more work we can obtain a canonical choice of a toroidal equivariant resolution of singularities. One observes that the cones in the barycentric subdivision have canonically ordered coordinates, which agree on intersecting cones: for a cone $\delta$ choose the unit coordinate vectors $e_i$ to be primitive lattice vectors generating the edges $\tau$, where $i = \text{dim} \delta$, the dimension of the cone of which $\tau$ is a barycenter. Recall that in order to resolve singularities, one successively takes the subdivisions centered at lattice points $w_j$ which are not integrally generated by the vectors $e_i$. These $w_j$ are partially ordered according to the lexicographic ordering of their canonical coordinates, in such a way that if $w_j \neq w_k$ have the same coordinates (e.g. if $g(w_1) = w_2$), they do not lie in a the same cone, and therefore we can take the centered subdivision simultaneously.

We conclude this section with a simple proposition about quotients:

Proposition 2.5. Let $U \subset X$ be a strict toroidal embedding, and let $G \subset \text{Aut}(U \subset X)$ be a finite group acting strictly and toroidally. Then $(X/G, U/G)$ is a strict toroidal embedding.

Proof. Since the quotient of a toric variety by a finite subgroup of the torus is toric, we conclude that $X/G$ is still a toroidal embedding, by the definition of toroidal embedding. We need to show that it is strict. Let $q : X \to X/G$ be the quotient map. Let $Z \subset X \setminus U$ be an irreducible component. Then $q(Z) = q(\cup g(Z))$. Since the action is strict, we have $q(\cup g(Z)) \simeq Z/\text{Stab}(Z)$, which is normal.

3. Proof of theorem 0.1

Given $Z, X$ with $G$ action, $G$ finite, we may blow up $Z$ and therefore we might as well assume that $Z$ is a divisor. Let $Y = X/G, Z/G$ be the quotient, $B$ the branch locus. Define $W = B \cup Z/G$. Let $(Y', W') \to (Y, W)$ be a resolution of singularities of $Y$ with $W'$ a strict divisor of normal crossings. Let $X'$ be the normalization of $Y'$ in $K(X)$, and $Z'$ the inverse image of $W'$. Let $U = X' \setminus Z'$. By Abhyankar’s lemma, clearly $U \subset X'$ is a strict toroidal embedding, on which $G$ acts toroidally (moreover, it is $G$-strict). Applying theorem 0.2 we obtain a nonsingular strict toroidal embedding $U \subset X_1 \to X'$ as required. \hfill \Box

Acknowledgements

Thanks are due to A. J. de Jong, who was a source of inspiration for this paper. Thanks also to S. Katz and T. Pantev, and the referee, for helpful discussions and comments relevant to this paper. Special thanks to S. Kleiman, who made this collaboration possible.
References

[N-dJ] D. Abramovich and A. J. de Jong, Smoothness, semistability, and toroidal geometry, J. Alg. Geom., to appear.

[B] J.-L. Brylinski, Décomposition simpliciale d’un réseau, invariante par un groupe fini d’automorphismes, C. R. Acad. Sci. Paris Sér. A-B 288 (1979), A137–A139.

[B-M1] E. Bierstone and P. Milman, A simple constructive proof of canonical resolution of singularities, Effective methods in algebraic geometry (Castiglioncello, 1990), 11–30, Progr. Math., 94, Birkhauser Boston, Boston, MA, 1991.

[B-M2] ______, Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant, Invent. Math., to appear.

[B-P] F. Bogomolov and T. Pantev, Weak Hironaka theorem, Math. Res. Lett. 3 (1996), 299–307.

[H] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero: I, II, Ann. of Math. (2) 79 (1964), 109–326.

[KKMS] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, Toroidal embeddings I, Lecture Notes in Math, 339, Springer-Verlag, Berlin-New York, 1973.

[V] O. Villamayor, Constructiveness of Hironaka’s resolution. Ann. Sci. École Norm. Sup. (4) 22 (1989), 1–32.

Department of Mathematics, Boston University, 111 Cummington, Boston, MA 02215, USA
E-mail address: abrmovic@math.bu.edu

Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA
E-mail address: wjh@math.mit.edu