Symmetrization and Entanglement of Arbitrary States of Qubits

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Given two arbitrary pure states \(|\phi\rangle\) and \(|\psi\rangle\) of qubits or higher level states, we provide arguments in favor of states of the form \(\frac{1}{\sqrt{2}}(|\psi\rangle|\phi\rangle + i|\phi\rangle|\psi\rangle)\) instead of symmetric or anti-symmetric states, as natural candidates for optimally entangled states constructed from these states. We show that such states, firstly have on the average a high value of concurrence, secondly can be constructed by a universal unitary operator independent of the input states. We also show that these states are the only ones which can be produced with perfect fidelity, by any quantum operation designed for intertwining two pure states with a relative phase. A probabilistic method is proposed for producing any pre-determined relative phase into the combination of any two arbitrary states.

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I. INTRODUCTION

Entanglement [1, 2] is a quantum mechanical resource that can be used for many computational and communication purposes. During the past few years many experimental efforts have been reported for creating entanglement [3] along with theoretical scenarios [4, 5, 6, 7, 8, 9] for generating as much entanglement as possible. A remarkable scenario has been put forward by Buzek and Hillery in [8] which stems from the most apparent property of entangled states, that is, their symmetry property. Consider two systems \(A\) and \(B\) in a pure state \(|\Psi\rangle_{AB}\) which can not be decomposed into a product of state vectors of the two parts. Such a state is called entangled and has a value of entanglement according to various measures of entanglement, defined for measuring this property. A symmetric state of the form

\[
|\Psi\rangle_{AB} = N(|\psi\rangle_A \otimes |\phi\rangle_B + |\phi\rangle_A \otimes |\psi\rangle_B),
\]

is a prototype of a pure state having this property. The scenario of [8] is thus based on the following natural question: Given two systems \(A\) and \(B\) respectively in pure states \(|\psi\rangle\) and \(|\phi\rangle\), is it possible to construct a quantum machine \(M\) which takes these two states as input and produces with exact fidelity, a symmetric and hence entangled output state? An even simpler task for this machine is to symmetrize an unknown state \(|\psi\rangle\) of \(A\) with a fixed reference state \(|\phi\rangle\) of \(B\), that is

\[
|\psi\rangle_A \otimes |\phi\rangle_B \otimes |v_0\rangle_M \rightarrow (|\psi\rangle_A \otimes |\phi\rangle_B + |\phi\rangle_A \otimes |\psi\rangle_B) \otimes |v_\psi\rangle_M
\]

where \(|v_0\rangle\) and \(|v_\psi\rangle\) are the initial and final states of the machine. The authors of [8] show by a simple argument that linearity and unitarity of quantum mechanics do not allow such machines to exist. Remarkably however, they succeed to construct an optimal machine which produces an output mixed state \(\rho_{AB}^{\text{out}}\) which has a universal (input-independent) fidelity equal to \(9+3\sqrt{3}/4 = 0.946\), with the ideal symmetric state \(|\Psi\rangle_{AB} = N(|\psi\rangle_A \otimes |\phi\rangle_B + |\phi\rangle_A \otimes |\psi\rangle_B)\). Here \(N\) is a normalization constant. They then proceed to show that the output impure state \(\rho_{AB}^{\text{out}}\) is indeed quantum mechanically entangled or inseparable by using the Peres-Horodecki’s criterion [10, 11] and showing that for all input states the partially transposed matrix \((\rho_{AB})^{T_A}\) has one negative eigenvalue. The negativity of this eigenvalue however depends on the input state.

We should also mention another related scheme for optimal entangling, the one proposed by Alber [12] who considers anti-symmetrization of an arbitrary input state with a reference state yielding the result that the output is a maximally disordered mixture of anti-symmetric Bell states. For the case of qubits however, there is only one anti-symmetric Bell state, namely \(|\phi^-\rangle := \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\) so that the output carries no information about the input states.

In this paper we want to study another scenario for entangling states. Our scenario is based on the idea that optimal entanglement need not necessarily be obtained by symmetrization. By looking at symmetrization and entanglement...
as two different tasks we propose different methods for their production. Our scenario may not be optimal in the class of all conceivable operations, however it is simple both theoretically and experimentally. Moreover it clarifies to some extent the relation between symmetrization and entanglement. By discussing it we hope at least to raise some questions for further study.

The paper is organized as follows: In Sec. II we propose a simple method for entangling qubits based on maximization of the entanglement of formation of a mixed state for which we have a closed formula as given by Wootters [13]. In Sec. III we consider the problem of symmetrization separately and show that given two arbitrary states \(|\phi\rangle\) and \(|\psi\rangle\) no quantum operation can produce a state of the form

\[
N(|\phi\rangle \otimes |\psi\rangle + e^{i\theta} |\psi\rangle \otimes |\phi\rangle),
\]

unless \(\theta = \pm \frac{\pi}{4}\). Thus the only machine which can produce with exact fidelity, linear combinations of the above form for arbitrary states, is the one found in Sec. II for optimal entanglement. In this same section we propose a probabilistic method for producing generalized symmetric states of two arbitrary states, i.e. states of the form given in (2), where \(N\) is a normalization constant and \(e^{i\theta}\) is any predetermined phase.

II. PRODUCTION OF MAXIMAL ENTANGLEMENT

Suppose that we have two qubits \(A\) and \(B\) in arbitrary pure states \(|\psi\rangle_A\) and \(|\phi\rangle_B\). When we want to produce entangled states of the form similar to the one discussed in the introduction (i.e. symmetric or anti-symmetric states), by passing these qubits from a quantum machine \(M\), it is rather natural to think that an obstacle is that at the output the two parts are entangled with the machine itself in the form

\[
|\Psi^{\text{out}}\rangle := |\psi\rangle_A |\phi\rangle_B |X\rangle_M + |\phi\rangle_A |\psi\rangle_B |Y\rangle_M.
\]

where \(|X\rangle\) and \(|Y\rangle\) are states of the machine. It is easy to see that such an operation is linear and unitary provided that the machine states be two fixed input-independent states satisfying the following relations

\[
\langle X|X\rangle + \langle Y|Y\rangle = 1 \quad \text{and} \quad \langle X|Y\rangle + \langle Y|X\rangle = 0
\]

A convenient parametrization of these inner products is

\[
\langle X|X\rangle = \frac{1 + \xi}{2}, \quad \langle Y|Y\rangle = \frac{1 - \xi}{2}, \quad \langle X|Y\rangle = i\eta
\]

where \(\xi\) and \(\eta\) are real parameters. We now ask under what condition, the output density matrix \(\rho^{\text{out}}_{AB}\) of the two systems \(A\) and \(B\) has the maximum value of entanglement.

For two qubits in a mixed state \(\rho\), we have a well established measure of entanglement given by a closed formula. It has been introduced by Wootters and Hill in [13, 14] and is directly related to the entanglement of formation of such a mixed state. It is called concurrence and is denoted by \(C\) and is given by

\[
C = \max (0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)
\]

where \(\lambda_1\) to \(\lambda_4\) are the eigenvalues of the following matrix in decreasing order:

\[
R = \sqrt{\rho^{*} \hat{\rho} \rho^{*}}.
\]

Here

\[
\hat{\rho} = (\sigma_2 \otimes \sigma_2) \rho^{*} (\sigma_2 \otimes \sigma_2),
\]

\(\sigma_2\) is the second Pauli matrix and \(\rho^{*}\) is the complex conjugate of \(\rho\) in the computational basis. Equivalently \(\lambda_i\)'s can be taken to be the square root of eigenvalues of the matrix \(\hat{\rho}\).

Here and in what follows we designate the density matrix of the two qubits simply by \(\rho\) instead of \(\rho_{AB}\). For the calculation of eigenvalues we can take without loss of generality, \(|\psi\rangle = a|0\rangle + b|1\rangle\) and \(|\phi\rangle = |0\rangle\) to obtain

\[
|\Psi^{\text{out}}\rangle = \begin{pmatrix} a|X\rangle + a|Y\rangle \\ b|Y\rangle \\ b|X\rangle \\ 0 \end{pmatrix}
\]
from which we obtain $\rho^{\text{out}}$ by taking the trace of $|\Psi^{\text{out}}\rangle \langle \Psi^{\text{out}}|$ over the machine states. The result is

$$\rho = \begin{pmatrix}
|a|^2 & ab(-i\eta + \frac{1+\xi}{2}) & ab(i\eta + \frac{1+\xi}{2}) & 0 \\
\overline{ab}(i\eta + \frac{1+\xi}{2}) & |b|^2(\frac{1+\xi}{2}) & i|b|^2\eta & 0 \\
\overline{ab}(-i\eta + \frac{1+\xi}{2}) & -i|b|^2\eta & |b|^2(\frac{1+\xi}{2}) & 0 \\
0 & 0 & 0 & |a|^2
\end{pmatrix} \tag{10}\]

where we have used the relations in (4).

Using (5) we obtain

$$\tilde{\rho} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & |b|^2(\frac{1+\xi}{2}) & i|b|^2\eta & -a\overline{b}(i\eta + \frac{1+\xi}{2}) \\
0 & -i|b|^2\eta & |b|^2(\frac{1+\xi}{2}) & -a\overline{b}(-i\eta + \frac{1+\xi}{2}) \\
0 & -\overline{ab}(-i\eta + \frac{1+\xi}{2}) & -\overline{ab}(i\eta + \frac{1+\xi}{2}) & |a|^2
\end{pmatrix}. \tag{11}\]

Finally the matrix $\rho\tilde{\rho}$ is found to be

$$\rho\tilde{\rho} = \begin{pmatrix}
0 & a\overline{b}|b|^2m_- & a\overline{b}|b|^2m_+ & -2(a\overline{b})^2(\eta^2 + \frac{1-\xi^2}{4} - i\eta\xi) \\
0 & |b|^4(\frac{1+\xi^2}{2} + \eta) & i|b|^4(1 - \xi) & -a\overline{b}|b|^2m_+ \\
0 & -i|b|^4(1 + \xi) & |b|^4(\frac{1+\xi^2}{2}) & -a\overline{b}|b|^2m_- \\
0 & 0 & 0 & 0
\end{pmatrix}. \tag{12}\]

where $m_{\pm} := (\eta^2 + \frac{1-\xi^2}{4} \pm i\eta(1 + \xi)).$

The eigenvalues of this matrix are easily determined, it has obviously two zero eigenvalues, the other two being the eigenvalues of the central 2 by 2 sub-matrix. The square root of its eigenvalues which are the eigenvalues of the matrix $R$ are in decreasing order

$$\lambda_1 = |b|^2(\frac{\sqrt{1-\xi^2}}{2} + \eta) \quad \lambda_2 = |b|^2(\frac{\sqrt{1-\xi^2}}{2} - \eta) \quad \lambda_3 = 0 \quad \lambda_4 = 0. \tag{13}\]

where in taking the square roots we have used the Cauchy Schwartz inequality for the inner products in (5) which implies that $\eta \leq \frac{\sqrt{1-\xi^2}}{2}$. 

Putting all this together with equation (6), leads to the following value for the concurrence

$$C = 2|b|^2\eta. \tag{14}\]

To obtain maximum concurrence we have to take $\eta = \frac{1}{4}$ which according to the Cauchy Schwartz inequality forces us to choose $\xi = 0$. This then means that the vectors $|X\rangle$ and $|Y\rangle$ are the same modulo a crucial phase, that is

$$|X\rangle = \frac{1}{\sqrt{2}}|c\rangle \quad |Y\rangle = \frac{1}{\sqrt{2}}|c\rangle, \tag{15}\]

where $|c\rangle$ is a normalized state. Interestingly the output state now turns out to be disentangled from the machine, so that the machine produces a pure entangled state:

$$|\psi\rangle_A \otimes |\phi\rangle_B \rightarrow |\Psi^{\text{out}}_{AB}\rangle = \frac{1}{\sqrt{2}}(|\psi\rangle_A \otimes |\phi\rangle_B + i|\phi\rangle_A \otimes |\psi\rangle_B) \tag{16}\]

Having determined the optimal choice of $\eta$ by taking the state $|\phi\rangle$ in the basis state $|0\rangle$, we can now calculate the concurrence produced for any two arbitrary input states. Since the output state is pure, its concurrence can be calculated from an alternative formula for any two input pure states

$$C = |\langle \Psi^{\tilde{\psi}} | \rangle| \tag{17}\]

where we have abbreviated $|\Psi^{\text{out}}_{AB}\rangle$ to $|\Psi\rangle$ and

$$|\tilde{\Psi}\rangle = (\sigma_2 \otimes \sigma_2)|\Psi^*\rangle = \frac{1}{\sqrt{2}}(|\tilde{\psi}\rangle_A \otimes |\tilde{\phi}\rangle_B - i|\tilde{\phi}\rangle_A \otimes |\tilde{\psi}\rangle_B). \tag{18}\]
Inserting this in (17) and using the fact that for every qubit state $|\psi\rangle$, $\langle \psi | \tilde{\psi} \rangle = 0$, we obtain

$$ C = \langle \psi | \tilde{\phi} \rangle \langle \phi | \tilde{\psi} \rangle $$

(19)

If the two initial states are two spin states in definite directions on the Bloch sphere, that is if $|\psi\rangle = |\tilde{\mathbf{n}}\rangle$ and $|\phi\rangle = |\tilde{\mathbf{m}}\rangle$ we find after straightforward calculations that

$$ C = \frac{1}{2} (1 - \mathbf{n} \cdot \mathbf{m}). $$

(20)

The details of this calculation is given in the appendix. The average of this concurrence over all input states is $\frac{1}{2}$. Thus this transformation produces on the average, an entanglement which as measured by concurrence is $1/2$. Equation (20) also shows that maximal entanglement is produced by intertwining two anti-parallel spin states on the Bloch sphere.

Up to now we have shown the possibility of the desirable entangling transformation. In view of (10) the actual form of the transformation is given by

$$ U = \frac{1}{\sqrt{2}} (I + iP) = e^{i\frac{\pi}{4} P} = e^{i\frac{\pi}{4} \mathbf{\sigma} \cdot \mathbf{\sigma}'} $$

(21)

where $P$ is the permutation operator (i.e. $P|\alpha, \beta\rangle = |\beta, \alpha\rangle$) which for two dimensional spaces is related to the Pauli matrices as $P = \frac{1}{2} (I + \mathbf{\sigma} \cdot \mathbf{\sigma}')$. The above argument and the combination of the last two formulas also tell us how to produce states with a definite amount of entanglement or concurrence, when we have partial information about the input states. Let us fix the spin state $|\phi\rangle = |\tilde{\mathbf{m}}\rangle$ in the direction $z$. We then need only take a spin state $|\psi\rangle = |\tilde{\mathbf{n}}\rangle$ with $\mathbf{n}$ making an angle $\theta$ with the $z$ axis (this still leaves the angle $\mathbf{B}$ undetermined) and apply the spin spin interaction (21) to produce a concurrence of $C = \frac{1}{2}(1 - \cos \theta)$. In this way by just adjusting the initial value of $\theta$ we can produce entangled states with any desired value of entanglement, ranging from the minimum value of $C = 0$ for $\theta = 0$ to the maximum value of $C = 1$ for $\theta = \pi$.

Moreover the output density matrix also retain some information about the input states, since it is easily verified from (10) that

$$ \rho_{A}^{(\text{out})} = tr_B (|\Psi^{\text{out}}\rangle_{AB} \langle \Psi^{\text{out}}|) = \frac{1}{2} (|\psi\rangle_A \langle \psi| + |\phi\rangle_A \langle \phi|), $$

(22)

with an identical formula for $\rho_B^{\text{out}}$. Thus the fidelity of reduced one particle density matrices of the output with any of the states $|\psi\rangle$ or $|\phi\rangle$ is $\frac{1}{2}$.

If we are interested in determining how much the output state $|\Psi^{\text{out}}\rangle = \frac{1}{\sqrt{2}} (|\psi\rangle|\phi\rangle + i|\phi\rangle|\psi\rangle)$ is close to a symmetric state $|\Psi^{\text{sym}}\rangle = N(|\psi\rangle|\phi\rangle + |\phi\rangle|\psi\rangle)$), then we can find the overlap of these two states. Knowing that $2N^2(1+|\langle \psi | \phi \rangle|^2) = 1$, we find:

$$ |\langle \Psi^{\text{out}} | \Psi^{\text{sym}} \rangle|^2 = \frac{1}{2}(1 + |\langle \psi | \phi \rangle|^2). $$

(23)

When averaged over all the input states this will give an overlap of $\frac{3}{4} = 0.75$.

### III. SYMMETRIZATION

By separating the issue of symmetrization from that of entanglement we will have more freedom in constructing states exhibiting each of these properties. Concerning symmetrization problem, we can now ask a more general question than the one considered in $\S$.

Is it possible to have a quantum machine which takes two input states $|\psi\rangle$ and $|\phi\rangle$ and produces generalized symmetric state as follows?

$$ |\psi\rangle_A |\phi\rangle_B |v\rangle_M \rightarrow (|\psi\rangle|\phi\rangle + e^{i\theta}|\phi\rangle|\psi\rangle)_{AB} |v'\rangle_M. $$

(24)

where $\theta$ is a predetermined phase. Note that for simplicity we have suppressed all the $\otimes$ signs. Here $|v\rangle$ is the initial normalized state of the machine and $|v'\rangle$ is the state of the machine after operation and it certainly depends on the initial states $|\psi\rangle$ and $|\phi\rangle$.

We will see that such a machine exists only when $e^{i\theta} = i$. To see this we note that if such a machine exists, it has to act as follows

$$ |0\rangle|0\rangle|v\rangle \rightarrow |0\rangle|0\rangle|v_0\rangle $$

$$ |1\rangle|0\rangle|v\rangle \rightarrow (|1\rangle|0\rangle + e^{i\theta}|0\rangle|1\rangle)|v_1\rangle. $$

(25)

(26)
where $|v_0\rangle$ and $|v_1\rangle$ are two of the machine states. Unitarity then demands the following relations among these machine states

$$
\langle v_0|v_0\rangle = 1 \quad \langle v_1|v_1\rangle = \frac{1}{2}.
$$

(27)

We now consider an input state like $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle$ which according to (24) should be transformed to

$$
|\Psi\rangle^\text{id} = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|0\rangle + e^{i\theta}(|0\rangle|0\rangle + |0\rangle|1\rangle)\right)|v_2\rangle
$$

(28)

where $|v_2\rangle$ is another state of the machine. On the other hand linearity of quantum mechanics requires that in view of (25), this input state be transformed to the state

$$
|\Psi\rangle^\text{out} = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle|v_0\rangle + (|1\rangle|0\rangle + e^{i\theta}|0\rangle|1\rangle)|v_1\rangle\right).
$$

(29)

Comparing $|\Psi\rangle^\text{id}$ and $|\Psi\rangle^\text{out}$, we find $|v_0\rangle = (1+e^{i\theta})|v_2\rangle$ and $|v_2\rangle = |v_1\rangle$, which are compatible with the norm condition (24) only if $e^{i\theta} = i$. This proves the negative part of the theorem. The positive part has been demonstrated already in section 2, where we have shown that the operator $U = e^{i\frac{\pi}{4}}P$ intertwines any two arbitrary states $|\psi\rangle$ and $|\phi\rangle$ to $\frac{1}{\sqrt{2}}(|\psi\rangle|\phi\rangle + i|\phi\rangle|\psi\rangle)$.

The interesting point is that from two different starting points we have arrived at the above form of states, one from requiring maximum entanglement production in section 2 and the other from requiring universal symmetrization. Note that while the arguments and the results for entanglement production are specific to qubits, those for symmetrization are valid in any dimension. This raises the question as to whether this coincidence holds true also in other dimensions? We will touch upon this question in the conclusion of the paper.

Although we do not have a quantum machine which can put two arbitrary states $|\psi\rangle$ and $|\phi\rangle$ into a combination with a general relative phase $e^{i\theta} \neq i$, we can achieve this probabilistically by using Fredkin gates which do controlled swap operations. The circuit shown in Fig. 1 which is a generalization of the one given in [16] for producing symmetric and antisymmetric superposition of input states, performs such an operation. Note that no three body interaction is needed for implementing the Fredkin gate and such a gate can be constructed by a combination of two body operations exactly [16]. In fact such a circuit develops the state $|\psi\rangle|\phi\rangle|0\rangle$ to the state

$$
|\Psi^{out}\rangle = \frac{1}{2}(e^{-i\theta}|\phi\rangle|\psi\rangle + e^{i\theta}|\psi\rangle|\phi\rangle)_{AB}|0\rangle_M + \frac{1}{2}(e^{i\theta}|\psi\rangle|\phi\rangle - e^{-i\theta}|\phi\rangle|\psi\rangle)_{AB}|1\rangle_M
$$

(30)

Before measurement of the control qubit the output state of $A$ and $B$ is a separable and hence disentangled mixed state given by the density matrix

$$
\rho_{AB}^{(out)} = \frac{1}{2}\left(|\psi\rangle\langle\psi| \otimes |\phi\rangle\langle\phi| + |\phi\rangle\langle\phi| \otimes |\psi\rangle\langle\psi|\right).
$$

(31)

Once the control qubit (the state of the machine) is measured in the basis $|0\rangle$ and $|1\rangle$, the two systems $A$ and $B$ will be projected onto one of the normalized states

$$
|\Psi_+\rangle = \frac{1}{N^+}(|\phi\psi\rangle + e^{i\theta}|\psi\phi\rangle)
$$

(32)

$$
|\Psi_-\rangle = \frac{1}{N^-}(|\phi\psi\rangle - e^{i\theta}|\psi\phi\rangle)
$$

(33)

![FIG. 1: Quantum network for performing probabilistic generalized symmetrization.](image)
where $N_+$ and $N_-$ are normalization factors. The result of the measurement of the control bit will declare which one of these two states have been produced. These states are produced with probabilities $P_+ = \frac{N_+^2}{N}$ and $P_- = \frac{N_-^2}{N}$. These states have now quantum entanglement as measured by their concurrence given by

$$C_+ = |\langle \Psi_+ | \tilde{\Psi}_+ \rangle| = \frac{2}{N_+} \langle \phi | \tilde{\psi} \rangle \langle \psi | \tilde{\phi} \rangle$$

(34)

$$C_- = |\langle \Psi_- | \tilde{\Psi}_- \rangle| = \frac{2}{N_-} \langle \phi | \tilde{\psi} \rangle \langle \psi | \tilde{\phi} \rangle$$

(35)

The average concurrence of the states produced will be given by

$$\bar{C} = P_+ C_+ + P_- C_- = \langle \phi | \tilde{\psi} \rangle \langle \psi | \tilde{\phi} \rangle,$$

(36)

which is equal to the concurrence we obtain for the state $\frac{1}{\sqrt{2}} (|\phi\rangle |\psi\rangle + i|\psi\rangle |\phi\rangle)$. However as far as entanglement production is concerned, this machine has no advantage over the simple deterministic machine proposed in section II, since with that transformation we could exactly produce every desirable value including the maximum value of entanglement.

IV. DISCUSSION

We have shown that there is no quantum mechanical process which can change a product state $|\psi\rangle |\phi\rangle$ into a state $N(|\psi\rangle |\phi\rangle + e^{i\theta} |\phi\rangle |\psi\rangle)$ with perfect fidelity unless $e^{i\theta} = \pm 1$. This result is true in any dimension. In two dimensions where we are dealing with qubits and have a closed formula for the entanglement of formation of a mixed state [13, 14], we have shown that these states are also the ones which have the maximum value of entanglement, when $|\phi\rangle$ and $|\psi\rangle$ are two initially fixed states. An interesting question is whether this correspondence exists also in higher dimensions or not. If this correspondence holds also in higher dimension then we can conjecture that the states of the form $|\Psi\rangle = \frac{1}{\sqrt{2}} (|\psi\rangle |\phi\rangle + i|\phi\rangle |\psi\rangle)$ rather the symmetric or anti-symmetric ones are the states which have the maximum entanglement when averaged over all the input product states. A hint already comes from calculating the I-concurrence of these states as given in [14]. The I-concurrence of a pure state $|\Psi\rangle_{AB}$ of two systems $A$ and $B$ is defined to be

$$C = \sqrt{1 - 2tr(\rho_{AB})^2} = \sqrt{1 - 2tr(\rho_B)^2}$$

(37)

where $\rho_A$ and $\rho_B$ are the reduced density matrices of the subsystems $A$ and $B$ respectively. If the two systems are in a pure state as above (i.e. $|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}} (|\psi\rangle |\phi\rangle + i|\phi\rangle |\psi\rangle)$, we find that

$$\rho = \frac{1}{2} (|\phi\rangle \langle \phi | + |\psi\rangle \langle \psi | + i |\phi\rangle \langle \psi | + i |\psi\rangle \langle \phi | - i |\psi\rangle \langle \psi |)$$

(38)

where $\rho$ stands for the density matrix of any of the two subsystems. A simple calculation shows that

$$C = 1 - |\langle \phi | \psi \rangle|^2$$

(39)

This concurrence reduces to [20] for qubits calculated from Wootters formula. Moreover for orthogonal states it gives the maximum value 1 and when averaged over all product states it gives the value $\frac{1}{2}$.

V. APPENDIX

In this section we present in detail the calculation leading to equation [20]. For two qubits $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ and $|\phi\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$, we have according to [13], $C = |ad - bc|^2$. Let the qubit states $|\psi\rangle$ and $|\phi\rangle$ correspond to unit vectors $n = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ and $m = (\sin \theta' \cos \phi', \sin \theta' \sin \phi', \cos \theta')$ respectively. Then we will have

$$|\psi\rangle \equiv |n\rangle = \begin{pmatrix} \cos \frac{\theta}{2} e^{-i\phi} \\ \sin \frac{\theta}{2} e^{i\phi} \end{pmatrix} \quad |\phi\rangle \equiv |m\rangle = \begin{pmatrix} \cos \frac{\theta'}{2} e^{-i\phi'} \\ \sin \frac{\theta'}{2} e^{i\phi'} \end{pmatrix}$$

(40)
from which we find

$$C = | \sin \frac{\theta'}{2} \cos \frac{\theta}{2} e^{-i \frac{\phi' - \phi}{2}} - \sin \frac{\theta}{2} \cos \frac{\theta'}{2} e^{i \frac{\phi' - \phi}{2}} |^2. \quad (41)$$

Simplifying this expression leads to

$$C = \frac{1}{2} (1 - \cos \theta \cos \theta' - \sin \theta \sin \theta' \cos (\phi - \phi')) \quad (42)$$

which is nothing but the expression $C = \frac{1}{2} (1 - \mathbf{m} \cdot \mathbf{n})$ written in components.

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