We discuss some classical and quantum properties of 2d gravity models involving metric and a scalar field. Different models are parametrized in terms of a scalar potential. We show that a general Liouville - type model with exponential potential and linear curvature coupling is renormalisable at the quantum level while a particular model (corresponding to $D = 2$ graviton - dilaton string effective action and having a black hole solution ) is finite. We use the condition of a “split” Weyl symmetry to suggest possible expressions for the “effective action” which includes the quantum anomaly term.

* e-mail: russo@slacvm.bitnet.
** e-mail: aat11@amtp.cam.ac.uk . On leave of absence from the Department of Theoretical Physics, P.N. Lebedev Physics Institute, Moscow 117924, Russia.
1. Classical action and solutions

A study of 2d quantum gravitational models may teach us important lessons about quantum gravity in general. For example, one would like to understand the effect of quantum corrections on the properties of classical (e.g. black hole) solutions. A particular “string-inspired” 2d model which contains black hole solutions [1] was recently discussed in ref.[2]. In this paper we shall consider some effects of quantum gravitational corrections on such type of scalar - tensor 2d models. We shall first make a field redefinition which brings the action into a canonical form (with different models being parametrised by a scalar potential). This form of the action facilitates the study of renormalisation and is useful in trying to determine the structure of the “effective action” which accounts for the general covariance of the quantum theory, i.e. is invariant under the background (“split”) Weyl symmetry in the conformal gauge [3]. We shall find that the “string-inspired” model corresponds to a Liouville model with an exponential potential and is finite within the standard loop expansion. We shall suggest two possible ansatze for the corresponding “effective action” based on two different choices of a metric in terms of which the anomaly contribution is constructed.

We shall consider the following model for a scalar field interacting with gravity in two dimensions\(^1\)

\[
S = \int d^2 x \sqrt{g} \left[ \frac{1}{2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi + F(\phi) R + U(\phi) \right].
\]

(1.1)

Here \(\phi\) is dimensionless so that \(U = \mu \bar{U}, \; [\mu] = cm^{-2}\). Eq.(1.1) is the most general local reparametrisation invariant action containing terms of dimension \(\leq 2\) (an arbitrary function \(K(\phi)\) in the kinetic term of the scalar field can be absorbed into a redefinition of \(\phi\)).

\(^1\) We shall assume that the metric has Euclidean signature. Continuation to Minkowski signature is straightforward. We have chosen the standard plus sign for the kinetic term of \(\phi\). When \(F\) is non-zero this does not guarantee that \(\phi\) is a physical field. In fact, the scalar field mixes with the conformal factor of the metric and the signature of the resulting kinetic matrix is \((+ -)\). Since the euclidean path integral is in any case unstable and should be defined by an analytic continuation the choice of the minus sign in front of (1.1) is also legitimate.
The action (1.1) is of interest as a simple tractable model of 2d gravity: the introduction of a scalar field is necessary in order to get a non-trivial local reparametrisation invariant action in two dimensions (similar models were previously discussed in refs.[4]). Related actions appear from higher dimensional Einstein action after a reduction to two dimensions.

Power counting implies that (1.1) should be renormalisable in a generalized sense (i.e. assuming that the form of the scalar functions $F$ and $U$ may change under renormalization). To study some formal properties of (1.1) it is useful to make a redefinition of $\phi$ and a Weyl rescaling of the metric which effectively replace $F$ by a linear function. Within the perturbation theory expansion we may ignore possible singularities of such field transformation. In general, one should take into account the region of definition of the scalar field and also orders of critical points of $F$ [5]. As a result, eq.(1.1) takes the form

$$S = \int d^2 x \sqrt{g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} q \phi \tilde{R} + V(\phi) \right), \quad (1.2)$$

where $q = \text{const}$, $\tilde{R} = R(\tilde{g})$ and

$$\frac{1}{2} q \phi = F(\phi), \quad g_{\mu\nu} = e^{2\rho} \tilde{g}_{\mu\nu} \quad (1.3)$$

$$\rho(\phi) = \frac{F}{q^2} - \frac{1}{4} \int^\phi d\phi' \left( \frac{dF}{d\phi'} \right)^{-1}, \quad V = U e^{2\rho}. \quad (1.4)$$

Thus the class of models (1.1) is parametrized by one arbitrary function $V$ of the scalar field.

To illustrate the transformation (1.3),(1.4) let us consider the metric – dilaton functional (string effective action) which generates the $\sigma$-model Weyl anomaly coefficients in the case of $D = 2$ target space$^2$

$$S = \frac{1}{8} \int d^2 x \sqrt{g} e^{-2\phi} \left( R + 4(\nabla \Phi)^2 + c \right). \quad (1.5)$$

$^2$ $c$ is proportional to an effective central charge, $\alpha' = 1$; a numerical factor in front of the action can be absorbed into a constant part of $\Phi$. We have chosen the plus sign in front of (1.5) in order for the transformation which puts it into the form (1.1),(1.2) to be real (e.g. not changing the sign of the metric). If we start with the action (1.5) with the minus sign we get (1.2) also with the minus sign. As we have noted, there is no $a$ priori reason for a particular choice of sign from the present point of view of two dimensional theory. However, if one interprets (1.5) as originating from a higher dimensional theory (in which “transverse” gravitons should have physical sign in the action) or demands correspondence
Eq. (1.5) can be represented in the form (1.1):

\[ S = \int d^2 x \sqrt{\tilde{g}} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{8} \phi^2 R + \frac{1}{8} c \phi^2 \right] , \quad \phi \equiv e^{-\Phi} . \]  

(1.6)

By applying the transformation (1.3), (1.4) to (1.6)

\[ q \phi = \frac{1}{4} \phi^2 , \quad \rho = \frac{1}{8 q^2} \phi^2 - \log \phi , \quad g_{\mu\nu} = \frac{e^{\phi/q}}{4 q \phi} \tilde{g}_{\mu\nu} , \]  

(1.7)

we conclude that (1.5) takes the form (\( \mu = c/8 \))

\[ S = \int d^2 x \sqrt{\tilde{g}} \left[ \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} q \phi \tilde{R} + \mu e^{\phi/q} \right] , \]  

(1.8)

or

\[ S = q^2 \int d^2 x \sqrt{\tilde{g}} \left[ \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} + \frac{1}{2} \tilde{\phi} \tilde{R} + \mu e^{\tilde{\phi}} \right] . \]

It is interesting to note that the simple Liouville structure of the potential in (1.8) is due to a particular relative coefficient of the dilaton and Einstein terms in (1.5) (if the coefficient 4 is replaced by \( \gamma \) one finds (1.2) with \( V = \mu \phi^{1 - \frac{1}{4} \gamma e^{\phi/q}} \)). This particular structure of (1.5) is also responsible for the existence of a black hole - type classical solution [1].

The classical field equations which follow from (1.2) are

\[ -\tilde{\nabla}^2 \varphi + \frac{1}{2} q \tilde{R} + V' = 0 , \]  

(1.9)

\[ q \tilde{\nabla}_\mu \tilde{\nabla}_\nu \varphi - \partial_\mu \varphi \partial_\nu \varphi + \tilde{g}_{\mu\nu} \left[ \frac{1}{2} (\partial \varphi)^2 + V - q \tilde{\nabla}^2 \varphi \right] = 0 . \]  

(1.10)

The trace of (1.10) represents the classical Weyl anomaly of (1.2):

\[ -q \tilde{\nabla}^2 \varphi + 2 V = 0 . \]  

(1.11)

with the string effective action which also includes physical scalar field not coupled to the curvature (tachyon) one should change the sign in (1.5). While this choice of overall sign is not important at the classical level it becomes relevant once the quantum corrections are included (see sect.3). Though we shall stick to the choice of the plus signs in (1.5),(1.2) it is easy to modify the results of sect.3 to the case of the minus sign: one should change the overall signs in (3.1),(3.13)-(3.15),(3.18)-(3.20) as well as the signs of the anomaly coefficients \( A, \ B \) (the sign of the anomaly term is invariant).
Eqs. (1.9) and (1.11) imply that

\[ \tilde{R} = \frac{2}{q^2} (2V - qV') . \]  

(1.12)

Since \( q \tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi - \partial_\mu \phi \partial_\nu \phi \sim \tilde{g}_{\mu\nu} \) the metric which solves (1.10) always has the Killing vector [5] \( \xi_\mu = \varepsilon_{\mu\nu} \tilde{\nabla}^\nu (e^{-\phi/q}) \). In conformal coordinates where \( \tilde{g}_{\mu\nu} = e^{2\sigma} \delta_{\mu\nu} \) eqs. (1.9) and (1.11) reduce to

\[-\partial^2 \phi - q \partial^2 \sigma + V'e^{2\sigma} = 0 , \]  

(1.13)

\[-q \partial^2 \phi + 2Ve^{2\sigma} = 0 . \]  

(1.14)

In the case of exponential potential \( V = \mu e^{\varphi/q} \) corresponding to (1.5),(1.8) one finds that eqs. (1.10),(1.13) and (1.14) have a spherically symmetric solution

\[ 2\sigma + \frac{1}{q} \varphi = k , \quad \varphi = ax^2 + b , \]  

(1.15)

\[ a = \frac{\mu}{q} e^k , \quad a, b, k = \text{const} . \]

In terms of the original metric and dilaton which appear in (1.5) this corresponds to the \( D = 2 \) “black hole” solution of ref.[1]

\[ \Phi = \Phi_0 - \frac{1}{2} \log \phi , \]

\[ g_{\mu\nu} = e^{2\rho+2\sigma} \delta_{\mu\nu} = \frac{1}{\phi^2} e^{2\rho+\frac{1}{q}\phi} \delta_{\mu\nu} = e^{2\Phi+k} \delta_{\mu\nu} = \frac{d}{\phi} \delta_{\mu\nu} , \quad d = e^k/4q , \]  

(1.16)

or in Minkowski notation

\[ ds^2 = \frac{dx^+dx^-}{b' - a' x^+ x^-} , \quad \Phi = \Phi_0 - \frac{1}{2} \log (b' - a' x^+ x^-) , \quad a' = \frac{c}{4}, \quad b' = \frac{M}{\sqrt{c}} . \]  

(1.17)

We see that (1.7) transforms the regular solution (1.15) into the singular one: the zero of \( \phi \) in (1.15) is transformed into the singularity of the original fields in (1.16).

Let us note that it is possible to find an explicit solution for arbitrary potential \( V \). This is easy to do in terms of the variables \( (g_{\mu\nu}, \Phi) \) used in (1.5). Consider the following generalization of (1.5)

\[ S = \frac{1}{8} \int d^2x \sqrt{g} e^{-2\Phi} [ R + 4(\partial \Phi)^2 + U(\Phi) ] , \]  

(1.18)
\[ U = c + c_1 e^{2\Phi} + c_2 e^{4\Phi} + \cdots = c + \bar{U} \ . \]

The corresponding equations of motion can be represented in the form

\[ R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi + F_1(\Phi)g_{\mu\nu} = 0 , \]  \hfill (1.19)

\[ -\frac{1}{2} \nabla^2 \Phi + (\partial \Phi)^2 + F_2(\Phi) = 0 , \quad \Phi - \frac{1}{4} U' , \quad F_2 \equiv - \frac{1}{4} U . \]  \hfill (1.20)

In \( D = 2 \) \( R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R \) so that \( g_{\mu\nu} \) is proportional to \( \nabla_\mu \nabla_\nu \Phi \). Hence the solution has the Killing vector \( \xi_\mu = \varepsilon_{\mu\nu} \nabla_\nu \Phi \). Fixing the coordinates so that

\[ ds^2 = f d\theta^2 + f^{-1} dx^2 , \quad f = f(x) , \quad \Phi = \Phi(x) , \]  \hfill (1.21)

with \( \theta \) parametrizing the direction along the Killing vector flow, we find that the \( (xx) \) and \( (\theta\theta) \) components of (1.19) reduce to \( (R = -f''') \)

\[ -\frac{1}{2} f^{-1} f''' + 2\Phi'' + f^{-1} f' \Phi' + f^{-1} F_1 = 0 , \quad -\frac{1}{2} f f''' + f f' \Phi' + f F_1 = 0 . \]  \hfill (1.22)

As a consequence, \( \Phi''(x) = 0 \), i.e.

\[ \Phi = \Phi_0 - bx , \]  \hfill (1.23)

so that

\[ f(x) = ae^{-2bx} - \frac{2}{b} e^{-2bx} \int_x dx' e^{2bx'} \hat{F}_2(x') , \]

\[ \hat{F}_2(x) \equiv - \frac{1}{4} U(\Phi_0 - bx) \equiv - \frac{1}{4} (c + \bar{U}(x)) , \]  \hfill (1.24)

i.e.

\[ f(x) = \frac{1}{4b^2} + ae^{-2bx} + \frac{1}{2b} e^{-2bx} \int_x dx' e^{2bx'} \bar{U}(x') . \]  \hfill (1.25)

The equivalent solution was found in [6]. For a large class of potentials it represents a black hole - type configuration (if \( \bar{U} = 0 \) the coordinate transformation \( x^+x^- = \alpha + \beta e^{2bx}, \ x^+/x^- = e^{2t} \) brings the metric into the form (1.17)). In the context of string theory \( \bar{U} \) in (1.18) represents higher loop corrections to the dilaton potential. To the one-loop order we find from (1.25) [6]

\[ f(x) = \frac{1}{4b^2} + ae^{-2bx} + hxe^{-2bx} + O(e^{-4bx}) , \quad a = M \sqrt{c}, \quad h = \frac{c_1}{2b} . \]  \hfill (1.26)

The genus 1 correction dominates over the “mass term” \( ae^{-2bx} \) in the weak coupling region \( x \to \infty \). The curvature \( R = -e^{-2bx}(4b^2 \frac{M}{\sqrt{c}} - 2c_1 + 2bc_1 x) + O(e^{-4bx}) \) has very different
behaviour as compared to the tree-level solution (in particular, it changes sign at some point).

2. Perturbative renormalization

Let us now discuss the renormalization of the model (1.2). Though there are no propagating degrees of freedom, there are nontrivial ultra-violet divergences. On dimensional grounds the counter-terms should have the structure

\[
\Delta S = \int d^2 x \sqrt{\tilde{g}} \left[ K(\varphi) \tilde{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + P(\varphi) \tilde{R} + Q(\varphi) \right], \tag{2.1}
\]

so that after a non-linear renormalization of the scalar field and the metric the renormalised action takes the same form as the original action (1.2) with some new potential \( \tilde{V} \). For a particular form of the potential the model (1.2) is not renormalizable in the usual sense unless \( \tilde{V} \) is simply proportional to \( V \). It is straightforward to compute \( \tilde{V} \) in the one-loop approximation. The “on-shell” counter-term should be gauge-independent so that we may use the simplest quantum conformal gauge\(^4\), \( \tilde{g}_{\mu\nu} = e^{2\sigma} \bar{g}_{\mu\nu} \). In this gauge the action (1.2) takes the form

\[
S = \int d^2 x \sqrt{\bar{g}} \left[ \frac{1}{2} \bar{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + q \bar{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \sigma + \frac{1}{2} q \varphi \bar{R} + V(\varphi) e^{2\sigma} \right]. \tag{2.2}
\]

Now the divergences can be computed using the background field method. It is useful to interpret (2.2) as a particular case of the \( D = 2 \) \( \sigma \)-model

\[
S = \int d^2 x \sqrt{\bar{g}} \left[ \frac{1}{2} G_{ij}(X) \bar{g}^{\mu\nu} \partial_\mu X^i \partial_\nu X^j + \frac{1}{2} \bar{R} \Psi(X) + T(X) \right], \tag{2.3}
\]

\[
X^i = (\varphi, \sigma), \quad G_{ij} = \begin{pmatrix} 1 & q \\ q & 0 \end{pmatrix}, \quad \Psi = q \varphi, \quad T = V(\varphi) e^{2\sigma}. \tag{2.4}
\]

\(^3\) Equivalently, one may use the classical field equations (1.9),(1.10) to transform (2.1) into the form \( \int d^2 x \sqrt{\bar{g}} \tilde{V}(\varphi) \).

\(^4\) The 1-loop renormalization of (1.1) in harmonic-type gauges was recently discussed in ref.[7]. A surprising result that the “on-shell” counter-term is gauge dependent was found. This is probably due to the fact that the boundary divergences (which can mix with the volume ones on the equations of motion (1.9),(1.10)) were ignored. Our conformal gauge result disagrees with the expressions found in [7].
Since the metric $G$ is flat and the dilaton $\Psi$ is linear, the only non-trivial divergences correspond to a renormalisation of the tachyon. This is formally true to all orders in the loop expansion. The only subtle point that may complicate the renormalisation of the model (1.2) as compared to the $\sigma$-model (2.3),(2.4) is that of a reparametrization invariance of a cutoff. This issue is irrelevant in the 1-loop approximation.

The Weyl anomaly coefficient corresponding to the tachyonic coupling has the following well-known structure [8]:

$$\bar{\beta}^T = -\frac{1}{4\pi} G^{ij} D_i D_j T + (G^{ij} \partial_i \Psi \partial_j T - 2T) \equiv \beta^T + \Delta \beta^T .$$

(2.5)

Here the first term $\beta^T$ corresponds to the genuine UV divergence while $\Delta \beta^T$ represent the classical Weyl anomaly. Computing (2.5) for the particular couplings in (2.4) we find

$$\bar{\beta}^T = \beta^T = \frac{1}{\pi q^2} e^{2\sigma} (V - qV') , \quad \Delta \beta^T = 0 .$$

(2.6)

Thus the condition of UV finiteness coincides with the condition of background Weyl invariance and is satisfied if $V = qV'$, i.e. if

$$V = \mu e^{\phi/q} .$$

(2.7)

It should be stressed that this conclusion holds only if both the scalar field and the metric are quantised: the scalar models (1.6),(1.8) are not finite if quantised on a fixed curved background.

Remarkably, the potential (2.7) corresponds to the “string-inspired” model (1.5) (see eq.(1.8)). It is possible to check the finiteness of the model (1.5) directly by going into the conformal gauge in (1.5) and rewriting the resulting action in $\sigma$-model form (1.20) (see (3.18)-(3.21)). The UV finiteness condition is satisfied only if a potential term is the same as in (1.5), i.e. $c e^{-2\Phi}$ . A weaker renormalizability condition is satisfied if $V - qV'$ is proportional to the potential itself, i.e. in the case of the Liouville potential

$$V = \mu e^{\gamma \phi} , \quad \gamma = \text{const} .$$

(2.8)

This, of course, is the expected conclusion since the $\sigma$-models corresponding to (1.5) and (1.8) are related by a field redefinition which leaves the on-shell finiteness condition invariant.
Then the divergence can be absorbed into a renormalization of $\mu$ (or a shift of $\phi$). We conclude that the Liouville theory is the only 2d scalar field theory which remains renormalizable after coupling to 2d quantum gravity (with the linear curvature term included as in (1.2)).

Using the conformal gauge it is easy to compute the 1-loop effective action in the theory (1.2). Expanding near the background $\bar{g}_{\mu\nu} = \bar{g}_{\mu\nu}$, $\phi = \bar{\phi}$ and accounting for the mixing between the scalar field and the conformal factor one finds

$$\Gamma_{\text{eff}}[\bar{g}, \bar{\phi}] = \frac{1}{2} \log \det \Delta_{ij} - \log \det \Delta_{gh},$$

(2.9)

where $\Delta_{gh}$ is the standard ghost operator corresponding to the conformal gauge and

$$\Delta_{ij} = \Delta_{ij}^{(0)} + M_{ij},$$

(2.10)

$$\Delta_{ij}^{(0)} = \begin{pmatrix} \bar{\Delta} & q\bar{\Delta} \\ q\bar{\Delta} & 0 \end{pmatrix}, \quad M_{ij} = \begin{pmatrix} V'' & 2V' \\ 2V' & 4V \end{pmatrix}, \quad \bar{\Delta} = -\nabla^2, \quad V' = \frac{\partial V}{\partial \bar{\phi}}.$$

The $V$-independent part of (2.9) is given by (we fine tune the coefficient of the cosmological term $\lambda \sqrt{\bar{g}}$ to zero)

$$\Gamma_{\text{eff}}^{(0)} = \frac{1}{8} A \int \bar{R} \Delta^{-1} \bar{R} + \text{const}, \quad A = \frac{(-2 + 26)}{12\pi}.$$  

(2.11)

The flat metric scalar Laplacian terms $\log \det \Delta$ cancel out (the scalar and conformal factor contributions are compensated by the two ghost contributions) so that the model has zero dynamical degrees of freedom (diagonalizing $\Delta_{ij}^{(0)}$ one finds that the signs of the eigenmodes are opposite so that the euclidean path integral should be defined by an analytic continuation as in $D = 4$ Einstein theory).

In the case of the model (1.8) $V = \mu e^{\bar{\phi}/q}$ one may compute (2.9) on the “black hole” solution (1.15), i.e. $\bar{g}_{\mu\nu} = e^{2\bar{\sigma}} \delta_{\mu\nu}$, $\bar{\phi} = ax^2 + b$, $2\bar{\sigma} + \bar{\phi}/q = k$. Using that $\sqrt{\bar{g}}V(\bar{\phi}) = \mu e^k = \bar{\mu} = \text{const}$ and that the metric is conformally flat one finds that the $\bar{\mu}$-dependent terms in $\det \Delta_{ij}$ cancel out so that there is no non-trivial correction to the anomaly term (2.11) in the effective action (in particular, there are no “extra” negative modes of $\Delta_{ij}$). If one repeats the calculation by starting with the classical action in the original parametrization (1.5) one finds again that the effective action is given simply by (2.11) but now the background metric $\bar{g}_{\mu\nu}$ is given by the classical solution for $g_{\mu\nu}$ (1.16), i.e. $\bar{g}_{\mu\nu} = d \delta_{\mu\nu}/(ax^2 + b)$ (we are assuming that starting with (1.5) one defines the quantum theory in terms of the metric $g_{\mu\nu}$). The resulting integral in (2.11) is less divergent than
in the first case. If one starts with the action in yet another parametrization (3.15) and defines the determinants in terms of the metric \( \hat{g}_{\mu\nu} \) (3.16) (which is flat on the classical solution) the effective action is given by (2.11) with \( \bar{g}_{\mu\nu} = \text{const} \delta_{\mu\nu} \), i.e. is trivial. Let us note that starting with (3.15) one can formally argue that the effective action is trivial to all orders in perturbation theory (non-perturbatively, however, there is a subtlety related to the fact that \( \psi \) should be non-negative in order to preserve a correspondence between (1.8) and (3.15)).

Computing (2.9) for the flat metric and constant scalar background one finds the following expression for the effective potential

\[
V_{\text{eff}} = \frac{1}{2\pi} \int_0^{\Lambda^2} d^2 p \ \ln \det(\delta^i_j p^2 + M^i_j),
\]

\[
M^i_j = G^{ik} M_{kj}, \quad G^{ik} = \begin{pmatrix} 0 & -1/q \\ -1/q & 1/q^2 \end{pmatrix},
\]

i.e.

\[
V_{\text{eff}} = \frac{1}{2} m_1^2 \left( \ln \frac{A^2}{m_1^2} + 1 \right) + \frac{1}{2} m_2^2 \left( \ln \frac{A^2}{m_2^2} + 1 \right) + \text{const},
\]

\[
m_{1,2}^2 = 2q^{-2} [V - qV' \pm (V^2 - 2qVV' + q^2VV'')^{1/2}].
\]

If \( V = \mu e^{\gamma \varphi} \) then \( m_1^2 = 4q^{-2} \mu e^{\gamma \varphi} (1 - \gamma q) \), \( m_2^2 = 0 \) so that the effective potential vanishes in the case of the model (1.8), i.e. when \( \gamma = 1/q \).

The usual kinetic term for the conformal factor \( \sigma \) is missing in eq.(2.2). Since such term is generated at the 1-loop level (appearing both from the ghost and scalar determinants) one may consider a resummation of the standard loop expansion by adding the Weyl anomaly term

\[
S_{\text{anom}} = \frac{1}{8} A \int \tilde{\Delta}^{-1} \tilde{R} = \frac{1}{2} A \int d^2 x \sqrt{\bar{g}} (\partial \mu \sigma \partial^\mu \sigma + \tilde{R} \sigma) + \frac{1}{8} A \int \tilde{R} \Delta^{-1} \tilde{R}, \quad (2.12)
\]

\[
A = \frac{1}{12\pi} (26 - D_{\text{eff}}),
\]

( \( D_{\text{eff}} \) is a number of effective scalar degrees of freedom) to the classical action (2.2) and quantizing the resulting theory. The corresponding \( D = 2 \sigma \)-model will be given by eq.(2.3) with the modified metric and dilaton

\[
G_{ij} = \begin{pmatrix} 1 & q \\ q & A \end{pmatrix}, \quad \Psi = q \varphi + A \sigma. \quad (2.13)
\]
Computing the $\bar{\beta}^T$-function (2.5) we find

$$\beta^T = -\frac{1}{4\pi A - q^2} (AV'' - 4qV' + 4V) ,$$

$$(2.14)$$

$$\Delta \beta^T = \frac{e^{2\sigma}}{A - q^2} (AQV' - 2q^2V - qAV' + 2AV) - 2Ve^{2\sigma} \equiv 0 .$$

$$(2.15)$$

Eq.(2.14) reproduces (2.6) in the case of $A = 0$. As it is clear from (2.14) the Liouville theory (2.8) is still renormalizable, with the zero of the $\beta$-function corresponding to the coefficient $\gamma$ in (2.8) which satisfies

$$A\gamma^2 - 4q\gamma + 4 = 0 .$$

$$(2.16)$$

We conclude that the model (1.5),(1.8) with the potential (2.7) with $\gamma = 1/q$ is no longer finite. The parameter $\mu$ in eq.(1.8),(2.7) (or $c$ in (1.5)) is running with scale unless $A = 0$.

3. Quantum effective action with anomaly

We have seen that the action (1.5) or (1.8) is special being not renormalised at the quantum level within the naive loop expansion. It would be interesting to understand its modification by finite quantum gravitational corrections and, in particular, to determine the fate of its classical black hole solution. As an attempt in this direction we shall suggest how one can find a “quantum” analog of (1.5),(1.8) using the DDK-type argument [3].

In the above discussion in sect. 2 we were ignoring the issue of maintaining the general covariance of the quantum theory. Instead of using an invariant regularisation (which is complicated in the conformal gauge) one may adopt a non-invariant cutoff adding at the same time some counterterms which are necessary in order to satisfy the reparametrisation invariance Ward identities. The resulting “effective action” should generate a theory which is invariant under the background Weyl symmetry [3], $\hat{g} \rightarrow e^{2\tau(x)} \hat{g}, \sigma(x) \rightarrow \sigma(x) - \tau(x)$. Since the metric $\hat{g} = e^{2\sigma} \hat{g}$ is left unchanged this transformation should be an exact symmetry of the theory, i.e. the $\bar{\beta}$-functions of the couplings in the “effective action” should vanish. The basic assumption (which can be justified to a certain extent in the absence of a scalar potential) is that the conformal factor dependence of the covariant quantum measure and regularisation can be represented by a local “effective action” containing only simplest lowest derivative terms [3].

Since in the conformal gauge the actions (1.5),(1.8) describe the systems of two interacting scalar fields a priori one might expect that the corresponding “effective action” is
a general $D = 2$ $\sigma$-model with the couplings being functions of the two arguments which solve the Weyl invariance conditions. It is natural to assume, however, that for a particular choice of the quantum variables only some particular structures are actually appearing. In what follows we shall discuss two (inequivalent) suggestions for the effective action which may correspond to (1.5),(1.8). They will be described by two particular $D = 2$ $\sigma$-models which generalize the actions (1.5) and (1.8) to the case when the anomaly terms are included.

While equivalent at the classical level (being related by the field redefinition (1.7)) the models (1.5) and (1.8) are not necessarily equivalent at the quantum level since the field redefinition involves a Weyl rescaling of the metric which may introduce an additional anomaly. Given that the scalar field and the conformal factor are mixed in the classical action it is not obvious which particular metric should be used in the definition of the quantum theory (i.e. in a measure and in a cutoff).  

Let us first discuss the ansatz for the effective action which is most natural if we use (1.8) as a starting point. We shall assume that the classical action (2.2) should be replaced at the quantum level by (cf. (2.2),(2.12))

$$S_{\text{eff}} = \int d^2 x \sqrt{\bar{g}} \left[ \frac{1}{2} \bar{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \frac{1}{2} A \bar{g}^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma + q \bar{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \sigma 
+ \frac{1}{2} \left( q \varphi + B \sigma \right) R + T(\varphi, \sigma) \right].$$

(3.1)

Since the two scalar fields can be linearly combined and rescaled in order to eliminate the mixing term the only a priori free parameters (in addition to $T$) are the two coefficients of the curvature couplings (the coefficient of the mixing term can be taken to be equal to $q$ without loss of generality). We shall fix $A$ at its natural value $A = \frac{25-D_{\text{eff}}}{12\pi}$, where $D_{\text{eff}} = 1 + N$ and $N$ is a number of extra matter scalar fields which may contribute to the conformal anomaly. The condition of the vanishing of the total central charge gives (see (2.3),(2.13))

$$\bar{\beta} \Psi = \frac{1}{12\pi} (D_{\text{eff}} + 1 - 26) + G^{ij} \partial_i \Psi \partial_j \Psi
= \frac{1}{12\pi} (D_{\text{eff}} - 25) + \frac{1}{A - q^2} \left[ Aq^2 - 2q^2 B + B^2 \right] = 0 .$$

(3.2)

\footnote{A “preferred” metric may be selected by coupling a gravitational system to extra matter fields.}
Eq. (3.2) has two solutions: $B = 2q^2 - A$ and $B = A$. The correspondence with the classical limit $A = 0$ selects the solution $B = A$. A solution of the condition of the vanishing of the “tachyon” Weyl anomaly coefficient (2.5) is given by $T = \mu e^{a\sigma + \gamma \varphi}$, where the constants $a$ and $\gamma$ satisfy the relation

$$-\frac{1}{2}A\gamma^2 - \frac{1}{2}a^2 + q\gamma a + 2\pi(a - 2)(A - q^2) = 0 .$$  \hfill (3.3)

Eq. (3.3) is not sufficient in order to determine both $a$ and $\gamma$. We shall make an additional assumption of correspondence between the “effective action” and (1.8),(1.5) in the classical limit. In particular, the effective equations should have asymptotically flat solutions. Let us solve the vacuum equations for arbitrary $a$ and $\gamma$. Setting $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$ and introducing the complex coordinates $z, \bar{z}$ we find that the equations of motion which follow from (3.1) become

$$\partial \bar{\partial} \varphi + q \partial \bar{\partial} \sigma = \frac{\mu}{4} e^{a\sigma + \gamma \varphi} ,$$  \hfill (3.4)
$$q \partial \bar{\partial} \varphi + A \partial \bar{\partial} \sigma = \frac{\mu}{4} a e^{a\sigma + \gamma \varphi} .$$  \hfill (3.5)

Combining eqs. (3.4),(3.5) we get

$$\sigma = -\alpha \varphi , \quad \alpha \equiv \frac{a - \gamma q}{aq - A\gamma}$$  \hfill (3.6)

(we have dropped the sum of arbitrary analytic and anti-analytic functions which may appear in $\sigma$). Inserting $\sigma$ (3.6) into eq. (3.4) we obtain

$$\partial \bar{\partial} \varphi = \bar{\mu} e^{(\gamma - a\alpha)\varphi} , \quad \bar{\mu} \equiv \frac{\mu\gamma}{4(1 - q\alpha)} .$$  \hfill (3.7)

Eq. (3.7) is the Liouville equation with the general solution

$$\varphi = \frac{1}{a\alpha - \gamma} \log \left[ \frac{\bar{\mu}(a\alpha - \gamma)(1 + f(z)f'(\bar{z}))^2}{2f'(z)f'(\bar{z})} \right] .$$  \hfill (3.8)

One can check that this solution satisfies also the constraints implied by general covariance. This solution does not approach the classical solution (1.15),(1.16) unless $a\alpha = \gamma$. In the latter case an important simplification occurs and the solution of (3.7) takes the form

$$\varphi = \frac{\mu\gamma a}{4(a - q\gamma)} z\bar{z} + \text{const} .$$  \hfill (3.9)

Using the definition of $\alpha$ in (3.6) we conclude that $a\alpha = \gamma$ implies the following relation between $\gamma$ and $a$:

$$1 - 2q\frac{\gamma}{a} + A\frac{\gamma^2}{a^2} = 0 .$$  \hfill (3.10)
Combining eq. (3.10) with eq. (3.3) we obtain
\[ a = 2, \quad \gamma = \frac{2q}{A}(1 - C), \quad C \equiv (1 - \frac{A}{q^2})^{1/2}. \] (3.11)

In the limit \( A \to 0 \) the value \( \gamma = 1/q \) is reproduced.

The transformation which connects \((\varphi, \tilde{g})\) and \((\Phi, g)\) should be such that the parameter \( q \) disappears in the effective action for \( \Phi \) and \( g \) since it was absent in the classical action (1.5) \((q \) can be eliminated by shifting \( \Phi \)). The required transformation for \( A \neq 0 \) is (cf. (1.7))
\[ a \rho = \gamma \varphi - \log(Cq\varphi/2), \quad qC\varphi = \frac{1}{4}e^{-2\Phi}, \quad \lambda = \rho + \sigma, \] (3.12)
where \( \lambda \) is the conformal factor of metric \( g_{\mu\nu} \), i.e.
\[ g_{\mu\nu} = e^{2\rho}\tilde{g}_{\mu\nu} = e^{2\lambda}\bar{g}_{\mu\nu}. \]

Inserting (3.12) into eq. (3.1) and assuming that \( a = 2 \) we find that the resulting effective action in terms of the original variables \((g, \Phi)\) of (1.5) is
\[ S_{\text{eff}} = \int d^2x \sqrt{\hat{g}} \left[ \frac{1}{8}e^{-2\Phi}(\hat{R} + 4\partial_{\mu}\Phi\partial^{\mu}\Phi - 4\partial_{\mu}\lambda\partial^{\mu}\Phi + c e^{2\lambda}) \right. \\
+ \left. A\left(\frac{1}{2}\partial_{\mu}\lambda\partial^{\mu}\lambda + \frac{1}{2}\partial_{\mu}\Phi\partial^{\mu}\Phi - \partial_{\mu}\Phi\partial^{\mu}\lambda + \frac{1}{2}(\lambda - \Phi)\bar{R}\right) \right]. \] (3.13)

By construction this action should give the vanishing Weyl anomaly coefficients. This can be checked explicitly and made more transparent by rewriting (3.13) in terms of the new variables
\[ \psi = \frac{1}{4}e^{-2\Phi}, \quad \kappa = \lambda - \Phi, \]
\[ S_{\text{eff}} = \int d^2x \sqrt{g} \left[ \frac{1}{2}(\hat{R}\psi + 2\partial_{\mu}\psi\partial^{\mu}\kappa + \frac{1}{4}c e^{2\kappa}) + \frac{1}{2}A(\partial_{\mu}\kappa\partial^{\mu}\kappa + \hat{R}\kappa) \right], \] (3.14)
i.e.
\[ S_{\text{eff}} = S + S_{\text{anom}}, \quad S = \frac{1}{2} \int d^2x \sqrt{\tilde{g}}(\hat{R}\psi + \frac{1}{4}c), \] (3.15)
\[ S_{\text{anom}} = \frac{1}{8}A\left[ \int \hat{R}\Delta^{-1}\hat{R} - \int \bar{R}\Delta^{-1}\bar{R} \right], \]
where we have introduced the metric
\[ \hat{g}_{\mu\nu} = e^{2\kappa}\tilde{g}_{\mu\nu} = e^{-2\Phi}g_{\mu\nu} = 4\psi g_{\mu\nu}. \] (3.16)

14
The classical solution of (1.5), i.e. of (3.15) with \(A = 0\) corresponds to the flat \(\hat{g}_{\mu\nu}\), i.e. \(\hat{R} = 0\). The structure of (3.15) (namely, the absence of \(\psi\) in its quantum part \(S_{\text{anom}}\)) implies that the same is true also for a non-vanishing \(A\), i.e. the extremum of (3.15) is given by \(^7\)

\[
\kappa = \lambda - \Phi = 0, \quad e^{-2\Phi} = \frac{c}{4}z\bar{z} + \frac{M}{\sqrt{c}}. \tag{3.17}
\]

This is still the classical black hole solution, i.e. the incorporation of the quantum Weyl anomaly according to (3.13),(3.15) has not altered the classical result.

As it is clear from (3.15) the classical action (1.5) has a very simple representation in terms of \(\hat{g}\) and \(\psi\). \(^8\) It appears as if \(\hat{g}\) is a “preferred” metric in terms of which the anomaly contribution is to be constructed. One could have found (3.15) by directly supplementing the “naive” conformal anomaly term \(\int \partial_{\mu}\lambda \partial^{\mu}\lambda\) by extra terms needed to satisfy the condition of the background Weyl invariance.

The action (3.15) is not, however, a unique quantum extension of (1.5) which includes the anomaly term and satisfies the condition of background Weyl invariance. To find another one let us start directly with the action (1.5)

\[
S = \frac{1}{2} \int d^2x \sqrt{g} \left[ \psi^{-1} \partial_{\mu} \psi \partial^{\mu} \psi + \psi R + c \psi \right], \quad \psi = \frac{1}{4} e^{-2\Phi}, \tag{3.18}
\]

written in the conformal gauge \(g_{\mu\nu} = e^{2\lambda} \tilde{g}_{\mu\nu}\) (cf. (3.13))

\[
S = \frac{1}{2} \int d^2x \sqrt{\tilde{g}} \left[ \psi^{-1} \partial_{\mu} \psi \partial^{\mu} \psi + 2 \partial_{\mu} \psi \partial^{\mu} \lambda + \psi \tilde{R} + c \psi e^{2\lambda} \right]. \tag{3.19}
\]

Let us assume that the anomaly term to be added to (3.19) has its standard \(\lambda\)-dependent form. The suggested ansatz for the quantum analog of (3.19) is thus the following

\[
S'_{\text{eff}} = \frac{1}{2} \int d^2x \sqrt{\tilde{g}} \left[ \psi^{-1} \partial_{\mu} \psi \partial^{\mu} \psi + 2 \partial_{\mu} \psi \partial^{\mu} \lambda + A \partial_{\mu} \lambda \partial^{\mu} \lambda + \tilde{R}(\psi + B\lambda) + T(\psi, \lambda) \right], \tag{3.20}
\]

\(^7\) This solution can be obtained also from (3.6), (3.9) by using the transformation (3.12).

\(^8\) This representation makes perturbative finiteness of the model (1.5) manifest and also simplifies the analysis of the classical solutions in the case of a non-trivial scalar potential (1.18), i.e. when

\[
S = \frac{1}{2} \int d^2x \left[ \tilde{R}\psi + \frac{1}{4} U(\psi) \right], \quad U = c + c_1 \psi^{-1} + c_2 \psi^{-2} + ... .
\]
where $B$ and $T$ are to be determined from the background Weyl invariance condition. It is straightforward to check that the $\sigma$-model which corresponds to (3.20), i.e. (2.3) with

$$X^i = (\psi, \lambda), \quad G_{ij} = \begin{pmatrix} \psi^{-1} & 1 \\ 1 & A \end{pmatrix}, \quad \Psi = \psi + B\lambda,$$

has the vanishing metric Weyl anomaly coefficient $\bar{\beta}^G$ provided $B = A$. In fact, the metric in (3.21) is actually flat (for any $A$) and the second covariant derivative of the dilaton $\Psi$ vanishes if $B = A$. The condition of the vanishing of the total central charge (3.2) is satisfied if $A = \frac{1}{12\pi} (25 - D_{\text{eff}})$. The vanishing of the tachyon Weyl anomaly coefficient $\bar{\beta}^T$ (2.5) gives

$$A\partial_\psi^2 T - 2\partial_\psi \partial_\lambda T + \psi^{-1} \partial_\lambda^2 T + \frac{A\psi^{-1}}{2(A - \psi)} (A\partial_\psi - \partial_\lambda) T$$

$$- 4\pi \psi^{-1} (A - \psi)(\partial_\lambda - 2) T = 0 .$$

(3.22)

Remarkably, there is no solution to eq.(3.22) of the “classical” type $\psi f(\lambda)$ and hence the scalar field dependence of the potential should change. There are solutions of the form $T = e^{a\lambda} W(\psi)$. However, if we want $W$ to approach $c\psi$ in the classical region $\psi \to \infty$ we may set $a = 2$ (this is similar to the $a = 2$ condition (3.11) we have discussed above in the derivation of (3.13)). We are thus assuming that only the $\psi$ dependence of the potential gets modified. Then (3.22) gives the following differential equation for the potential $W$

$$A\psi(A - \psi)W'' + \left[ \frac{1}{2} A^2 - 4\psi(A - \psi) \right] W' + (3A - 4\psi) W = 0 ,$$

(3.23)

$$T = e^{2\lambda} W(\psi) .$$

In the classical case $A = 0$, $W = c\psi$. For a general $A$ one can find a solution of (3.23) by expanding

$$W = c\psi + c_1 + c_2\psi^{-1} + O(\psi^{-2}) .$$

(3.24)

Eq.(3.23) gives

$$c_1 = -\frac{A}{4} c , \quad c_2 = -\frac{A^2}{32} c , \quad \text{etc.}$$

(3.25)

Eq.(3.24) is analogous to the expansion of a string loop corrected dilaton potential in terms of a string coupling (cf. eq.(1.18)). If eq.(3.23) were true in string theory it would give a nonperturbative expression for the dilaton potential. In sect. 2 we have seen that the presence of the “string loop corrections” in the potential (1.18) changes the classical black hole geometry (see eq.(1.26)). It is interesting to note that in the case of the effective
action (3.20) the combined effect of the anomaly term $A\partial\lambda\partial\lambda$ and the first correction $c_1$ to the potential (3.24) is such that there is no substantial modification of the black hole solution in the asymptotic weak coupling region. A solution of the equations corresponding to (3.20) with a general $T$ satisfying (3.23) is likely to be very different from the classical ($A = 0$) solution.

Comparing the two suggestions for the effective action (3.14) and (3.20) it is clear that they correspond to the two possible choices of a metric in terms of which the anomaly contribution is constructed. In (3.14) this is $\hat{g}$ (3.16) which is natural from the point of view of the “canonical” form of the action (1.8). The use of $\hat{g}$ makes it possible to keep the classical form of the potential term and of the vacuum solution. In (3.20) one employs the original metric $g$ of (1.5) but then the potential term and the vacuum solution become complicated.

In ref.[2] the effect of quantum corrections on the black hole solution in the model (1.5) was analysed in the $1/N$ approximation (i.e. the metric and $\phi$ were considered as classical). The inclusion of the conformal anomaly term due to scalar matter fields has led to a drastic modification of the classical solution. Our discussion suggests that once the quantum gravitational corrections are accounted for in a way consistent with the general covariance (so that the “effective action” in the conformal gauge satisfies the condition of the background Weyl invariance) the structure of the anomaly terms and hence of the corresponding vacuum solutions may be different. To justify further the ansatze (3.15) or (3.20) it is important to understand a relation between an approximation in which they can be considered as candidates for a quantum effective action and the standard loop and $1/N$ expansions.

A.T. would like to acknowledge a financial support of Trinity College, Cambridge. The research of J.R. is supported by INFN.
References

[1] E. Witten, Phys.Rev. D44 (1991) 314;
    S. Elitzur, A. Forge and E. Rabinovici, Nucl.Phys. B359 (1991) 581;
    G. Mandal, A. Sengupta and S. Wadia, Mod.Phys.Lett. A6 (1991) 1685.
[2] C.G. Callan, S.B. Giddings, J.A. Harvey and A. Strominger,
    Santa Barbara preprint UCSB-TH-91-54.
[3] F. David, Mod.Phys.Lett. A3 (1988) 1651; Nucl.Phys. B293 (1988) 332;
    J. Distler and H. Kawai, Nucl.Phys. B321 (1989) 509;
    S.R. Das, S. Nair and S.R. Wadia, Mod.Phys.Lett. A4 (1989) 1033;
    J. Polchinski, Nucl.Phys. B324 (1989) 123;
    T. Banks and J. Lykken, Nucl.Phys. B331 (1990) 173;
    A.A. Tseytlin, Int.J.Mod.Phys. A5 (1990) 1833;
    E. D’Hoker, Mod.Phys.Lett. A6 (1991) 745;
    A. Cooper, L. Susskind and L. Thorlacius, Nucl.Phys. B363 (1991) 132.
[4] R. Marnelius, Nucl.Phys. B211 (1983) 14;
    C. Teitelboim, Phys.Lett.Phys.Lett. B126 (1983) 41;
    T. Yoneya, Phys.Lett. B148 (1984) 111;
    R. Jackiw, in: Quantum Theory of Gravity, ed. S.Christensen (Adam
    Hilger, Bristol 1984);
    A. Chamseddine, Phys.Lett. B256 (1991) 2930.
[5] T. Banks and M. O’Loughlin, Nucl.Phys. B362 (1991) 649.
[6] M. McGuigan, C. Nappi and S. Yost, IAS preprint IASSNS-HEP-91-57.
[7] S.D. Odintsov and I.L. Shapiro, Phys.Lett. B263 (1991) 183;
    Madrid preprint FTUAM-33 (1991);
    D. Mazzitelli and N. Mohammedi, Trieste preprint IC/91/238 (1991).
[8] C.G. Callan, D. Friedan, E. Martinec and M.J. Perry,
    Nucl.Phys. B262 (1985) 593.