Estimation of quantum channels with finite resources

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We investigate the problem of determining the parameters that describe a quantum channel. It is assumed that the users of the channel have at best only partial knowledge of it and make use of a finite amount of resources to estimate it. We discuss simple protocols for the estimation of the parameters of several classes of channels that are studied in the current literature. We define two different quantitative measures of the quality of the estimation schemes, one based on the standard deviation, the other one on the fidelity. The possibility of protocols that employ entangled particles is also considered. It turns out that the use of entangled particles as a new kind of nonclassical resource enhances the estimation quality of some classes of quantum channel. Further, the investigated methods allow us to extend them to higher dimensional quantum systems.

I. INTRODUCTION

Quantum information processing has attracted a lot of interest in recent years, following Deutsch’s investigations concerning the potentiality of a quantum computer, i.e., a computer where information is stored and processed in quantum systems. Their application as quantum information carriers gives rise to outstanding possibilities, like secret communication (quantum cryptography) and the implementation of quantum networks and quantum algorithms that are more efficient than classical ones.

Many investigations concern the transmission of quantum information from one party (usually called Alice) to another (Bob) through a communication channel. In the most basic configuration the information is encoded in qubits. If the qubits are perfectly protected from environmental influence, Bob receives them in the same state prepared by Alice. In the more realistic case, however, the qubits have a nontrivial dynamics during the transmission because of their interaction with the environment. Therefore, Bob receives a set of distorted qubits because of the disturbing action of the channel.

Up to now investigations have focused mainly on two subjects: Determination of the channel capacity and reconstruction schemes for the original quantum state under the assumption that the action of the quantum channel is known. Here we focus our attention on the problem that precedes, both from a logical and a practical point of view, all those schemes: The problem of determining the properties of the quantum channel. This problem has not been investigated so far, with the exception of very recent articles. The reliable transfer of quantum information requires a well known intermediate device. The knowledge of the behaviour of a channel is also essential to construct quantum codes. In particular, we consider the case when Alice and Bob use a finite amount of qubits, as this is the realistic case. We assume that Alice and Bob have, if ever, only a partial knowledge of the properties of the quantum channel and they want to estimate the parameters that characterize it.

The article is organized as follows. In section II, we shall give the basic idea of quantum channel estimation and introduce the notation as well as the tools to quantify the quality of channel estimation protocols. We shall then continue with the problem of parametrizing quantum channels appropriately in section III. Then we are in a position to envisage the estimation protocol for the case of one parameter channels in section IV. In particular, we shall investigate the optimal estimation protocols for the depolarizing channel, the phase damping channel and the amplitude damping channel. We shall also give the estimation scheme for an arbitrary qubit channel. In section V we explore the use of entanglement as a powerful nonclassical resource in the context of quantum channel estimation. Section VI deals with higher dimensional quantum channels before we conclude in section VII.

II. A GENERAL DESCRIPTION OF CHANNEL ESTIMATION

The determination of all properties of a quantum channel is of considerable importance for any quantum communication protocol. In practice such a quantum channel can be a transmission line, the storage for a quantum system, or an uncontrolled time evolution of the underlying quantum system. The behaviour of such channels is generally not known from the beginning, so we have to find methods to gain this knowledge.

This is in an exact way only possible if one has infinite resources, which means an infinite amount of well prepared quantum systems. The influence of the channel on each member of such an ensemble can then be studied, i.e., the corresponding statistics allows us to characterize the channel. In a practical application, however, such a condition will never be fulfilled. Instead we have to come along with low numbers of available quantum systems. We therefore
cannot determine the action of a quantum channel perfectly, but only up to some accuracy. We therefore speak of channel estimation rather than channel determination, which would be the case for infinite resources.

A quantum channel describes the evolution affecting the state of a quantum system. It can describe effects like decoherence or interaction with the environment as well as controlled or uncontrolled time evolution occurring during storage or transmission. In mathematical terms a quantum channel is a completely positive linear map \( \mathcal{C} \) (CP-map) \(^9\), which transforms a density operator \( \rho \) to another density operator \( \rho' = \mathcal{C} \rho \).

Each quantum channel \( \mathcal{C} \) can be parametrized by a vector \( \vec{\lambda} \) with \( L \) components. For a specific channel we shall therefore write \( \mathcal{C}_\vec{\lambda} \) throughout the paper. Depending on the initial knowledge about the channel, the number of parameters differs. The goal of channel estimation is to specify the parameter vector \( \vec{\lambda} \).

The protocol Alice and Bob have to follow in order to estimate the properties of a quantum channel is depicted in figure 1. Alice and Bob agree on a set of \( N \) quantum states \( \rho_i, i = 1, 2, \ldots, N \), which are prepared by Alice and then sent through the quantum channel \( \mathcal{C}_\vec{\lambda} \). Therefore, Bob receives the \( N \) states \( \rho'_i = \mathcal{C}_\vec{\lambda} \rho_i \). He can now perform measurements on them. From the results he has to deduce an estimated vector \( \vec{\lambda}^{\text{est}} \) which should be as close as possible to the underlying parameter vector \( \vec{\lambda} \) of the quantum channel.

How can we quantify Bob’s estimation? To answer this we introduce two errors or cost functions which describe how good the channel is estimated.

The first obvious cost function is the statistical error

\[
c_s(N, \vec{\lambda}) \equiv \sum_{\ell=1}^{L} (\lambda_\ell - \lambda^{\text{est}}_\ell (N))^2 \tag{1}
\]

in the estimation of the parameter vector \( \vec{\lambda} \). Note that the elements of the estimated parameter vector \( \vec{\lambda}^{\text{est}} \) strongly depend on the available resources, i.e. the number \( N \) of systems prepared by Alice. We also emphasize that \( c_s \) describes the error for one single run of an estimation protocol. However, we are not interested in the single run error \( c_s \) but in the average error of a given protocol. Therefore, we sum over all possible measurement outcomes \( J \) to get the mean statistical error

\[
\bar{c}_s(N, \vec{\lambda}) \equiv \langle c_s(N, \vec{\lambda}) \rangle_J
\]

while keeping the number \( N \) of resources fixed.

Though this looks as a good benchmark to quantify the quality of an estimation protocol it has a major drawback. The cost function \( c_s \) strongly depends on the parametrization of the quantum channel. While this is not so important if one compares different protocols using the same parametrization it anyhow would be much better if we could give a cost function which is independent of any specifications. We define such a cost function with the help of the average overlap

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**FIG. 1.** Basic scheme of channel estimation. Alice sends \( N \) quantum state \( \rho_i \) to Bob. The channel maps these states onto the states \( \rho'_i = \mathcal{C}_\vec{\lambda} \rho_i \), on which Bob can perform arbitrary measurements. Note that Bob’s measurements are designed with the knowledge of the original quantum states \( \rho_i \). His final aim will be to present an estimated vector \( \vec{\lambda}^{\text{est}} \) being as close as possible to the underlying parameter vector \( \vec{\lambda} \) of the quantum channel.
\[
\mathcal{F}(C_1, C_2) \equiv \left\langle F(C_1 | \psi \rangle \langle \psi | C_2 | \psi \rangle \right\rangle_{| \psi \rangle} \tag{2}
\]

between two quantum channels \(C_1\) and \(C_2\), where we average the fidelity \([11]\)
\[
F(\rho_1, \rho_2) \equiv \text{Tr}^2 \sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}} \tag{3}
\]

between two mixed states \(\rho_1 = C_1 \rho\) and \(\rho_2 = C_2 \rho\) over all possible pure quantum states \(\rho = |\psi \rangle \langle \psi|\) \([\text{11}]\). Since the fidelity ranges from zero to one, the fidelity error is given by
\[
c_f(N, \vec{\lambda}) \equiv 1 - F(C_{\vec{\lambda}}, C_{\vec{\lambda}_{\text{est}}}(N)) \tag{4}
\]

which now is zero for identical quantum channels. Again we average over all possible measurement outcomes to get the mean fidelity error
\[
\overline{c}_f(N, \vec{\lambda}) \equiv \langle c_f(N, \vec{\lambda}) \rangle_J \tag{5}
\]

which quantifies the whole protocol and not a specific single run.

In the first part of this paper we are only dealing with qubits described by the density operator
\[
\rho \equiv \frac{1}{2} (1 + \vec{s} \cdot \vec{\sigma}) \tag{6}
\]

with Bloch vector \(\vec{s} = \text{Tr}(\rho \vec{\sigma})\) and the Pauli matrices \(\vec{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)\). The action of a channel is then completely described by the function
\[
\vec{s}' = C(\vec{s}), \tag{7}
\]

which maps the Bloch vector \(\vec{s}\) to a new Bloch vector \(\vec{s}'\). In particular, the mixed state fidelity equation (3), for two qubits with Bloch vectors \(\vec{s}_1\) and \(\vec{s}_2\), see equation (6), then simplifies to \([11]\)
\[
F(\vec{s}_1, \vec{s}_2) = \frac{1}{2} \left[ 1 + \vec{s}_1 \cdot \vec{s}_2 + \sqrt{(1 - |\vec{s}_1|^2) (1 - |\vec{s}_2|^2)} \right] \tag{8}
\]

which leads to an average channel overlap, equation (3),
\[
\mathcal{F}(C_1, C_2) \equiv \int \frac{d\Omega}{4\pi} F(C_1(\vec{n}), C_2(\vec{n})) \tag{9}
\]

for the two qubit channels \(C_1\) and \(C_2\). As emphasized above we only average over pure input states with unit Bloch vector
\[
\vec{n} = (\cos \Phi \sin \Theta, \sin \Phi \sin \Theta, \cos \Theta) \tag{10}
\]

and Bloch sphere element \(d\Omega = \sin \Theta \, d\Theta \, d\Phi\). By inserting equations (8) and (3) into equations (4) and (5) we get a cost function for the comparison of two qubit channels which is independent of the chosen parametrization.

### III. Parametrization of a Quantum Channel

As we have already mentioned in section \([1]\), the qubits that Alice sends to Bob are fully characterized by their Bloch vector. Therefore, the disturbing action of the channel modifies the initial Bloch vector \(\vec{s}\) according to equation (7). It has been shown that for completely positive operators the action of the quantum channel \(C\) is given by an affine transformation \([12]\)
\[
\vec{s}' = \mathcal{M} \vec{s} + \vec{u} \tag{11}
\]

\(^1\)Since for our estimation protocol we are only sending pure states through the quantum channel we also only average over pure states.
where $\mathcal{M}$ denotes a $3 \times 3$ invertible matrix and $\vec{v}$ is a vector. The transformation is thus described by 12 parameters, 9 for the matrix $\mathcal{M}$ and 3 for the vector $\vec{v}$. These 12 parameters have to fulfill some constraints to guarantee the complete positivity of $C$. The definition of the parameters is somewhat arbitrary. We will not use the parameters as defined in [12], but adopt a different parametrization

$$
\begin{pmatrix}
  s'_1 \\
  s'_2 \\
  s'_3
\end{pmatrix} =
\begin{pmatrix}
  2\lambda_7 - \lambda_1 - \lambda_4, & 2\lambda_{10} - \lambda_1 - \lambda_7, & \lambda_4 - \lambda_7 \\
  2\lambda_6 - \lambda_2 - \lambda_5, & 2\lambda_{11} - \lambda_2 - \lambda_6, & \lambda_5 - \lambda_6 \\
  2\lambda_9 - \lambda_3 - \lambda_6, & 2\lambda_{12} - \lambda_3 - \lambda_9, & \lambda_6 - \lambda_9
\end{pmatrix}
\begin{pmatrix}
  s_1 \\
  s_2 \\
  s_3
\end{pmatrix} +
\begin{pmatrix}
  \lambda_1 + \lambda_4 - 1 \\
  \lambda_2 + \lambda_5 - 1 \\
  \lambda_3 + \lambda_6 - 1
\end{pmatrix}
$$

(12)

in terms of the parameter vector $\vec{\lambda} = (\lambda_1, \ldots, \lambda_{12})^T$. In general, the choice of the best parametrization of quantum channels depends on the relevant features of the channel and also on which observables can be measured. The reason for the choice of the parametrization, equation (12), will become clear at the end of the section IV, in which a protocol for the general channel is described.

IV. ESTIMATION OF CHANNEL PARAMETERS

Although the characterization of the general quantum channel requires the determination of 12 parameters, only a smaller number of parameters must be determined in practice for a given class of quantum channels. Indeed, the knowledge of the properties of the physical devices used for quantum communication gives information on some known channels and allows to reduce the number of parameters to be estimated. We shall now examine in detail some known channels described by only one parameter.

A. The depolarizing channel

The first channel we consider is the depolarizing channel. The relation (11) between the Bloch vectors $\vec{s}$ and $\vec{s}'$ of Alice’s and Bob’s qubit, respectively, reduces to the simple form

$$
\vec{s}_\lambda ' = (1 - 2\lambda)\vec{s},
$$

where $0 \leq \lambda \leq 1/2$ is the only parameter that describes this quantum channel. This channel is a good model when quantum information is encoded in the photon polarization that can change along the transmission fiber via random rotations of the polarization direction. If Alice prepares the qubit in the pure state $|\psi\rangle$, the qubit Bob receives is described by the state

$$
\rho' = (1 - \lambda)|\psi\rangle\langle\psi| + \lambda|\tilde{\psi}\rangle\langle\tilde{\psi}|,
$$

(13)

where $|\tilde{\psi}\rangle$ denotes the state orthogonal to $|\psi\rangle$. We note that the depolarizing channel has no preferred basis. Therefore, its action is isotropic in the direction of the input state. This means that the action of the channel is described by equation (13) even after changing the basis of states.

Bob must estimate $\lambda$, which ranges between 0 (noiseless channel) and 1/2 (total depolarization). The estimation protocol is the following: Alice prepares $N$ qubits in the pure state

$$
|\uparrow_{\vec{n}}\rangle = e^{-i\Phi/2} \cos \frac{\Theta}{2} |\uparrow_z\rangle + e^{i\Phi/2} \sin \frac{\Theta}{2} |\downarrow_z\rangle
$$

which denotes spin up in the direction $\vec{n}$, equation (10), in terms of eigenvectors $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$ of $\sigma_z$. The state $|\uparrow_{\vec{n}}\rangle$ is sent to Bob through the channel. Bob knows the direction $\vec{n}$ and measures the spin of the qubit he receives along $\vec{n}$. The outcome probabilities are

$$
P(|\uparrow_{\vec{n}}\rangle) = 1 - \lambda, \quad P(|\downarrow_{\vec{n}}\rangle) = \lambda.
$$

Since Alice sends a finite number of qubits, Bob can only determine frequencies of measurements instead of probabilities. After Bob has measured $i \leq N$ qubits with spin down and the remaining $N - i$ with spin up, his estimate of $\lambda$ is $\lambda_{\text{est}} = i/N$. Note that in a single run of the probabilistic estimation method we can get results $\lambda_{\text{est}} > 1/2$, which is outside the range of the depolarizing channel. However, for the calculation of the average errors we do not have to take that into account. The cost function $c_{s}$, equation (11), is thus
The average error is easily obtained when one considers that Bob’s frequencies of measurements occur according to a binomial probability distribution, since each of the \( i \) measurements of spin down occurs with probability \( \lambda \) and each of the spin up measurements occurs with probability \( 1 - \lambda \). Therefore, the mean statistical error reads

\[
c_{s}(N, \lambda) = \left( \lambda - \frac{i}{N} \right)^2.
\]

This function, shown in figure 2 a), scales with the available finite resources \( N \). It vanishes when the channel faithfully preserves the polarization, \( \lambda = 0 \). The largest average error occurs for \( \lambda = 1/2 \), when the two actions of preserving and changing the polarization have the same probability to occur.

The cost function \( c_f \) is obtained from equations (4) and (9)

\[
c_f(N, \lambda) = 1 - \left[ \sqrt{\lambda \frac{i}{N}} + \sqrt{(1 - \lambda) \left(1 - \frac{i}{N}\right)} \right]^2,
\]

which leads to the fidelity mean error

\[
\bar{c}_f(N, \lambda) = 1 - \sum_{i=0}^{N} \binom{N}{i} \lambda^i (1 - \lambda)^{N-i} c_f(N, \lambda)
\]

\[
= 1 - \frac{1}{N} \sum_{i=0}^{N} \binom{N}{i} \lambda^i (1 - \lambda)^{N-i} \left( \sqrt{\lambda i} + \sqrt{(1 - \lambda)(N - i)} \right)^2,
\]

which is shown in figure 2 b). Although \( \bar{c}_f \) does not show a simple \( 1/N \)-dependency, it clearly decreases for increasing values of the number of qubits \( N \). For a given quality \( \bar{c}_s \) or \( \bar{c}_f \) one can use figure 2 and read off the number \( N \) of needed resources.

**B. The phase damping channel**

The phase-damping channel acts only on two components of the Bloch vector, leaving the third one unchanged:
\[ \vec{s}'_\lambda = ((1 - 2\lambda) s_1, (1 - 2\lambda) s_2, s_3)^T. \]

Here \( 0 \leq \lambda \leq 1/2 \) is the damping parameter. This channel, contrarily to the depolarizing channel, has a preferred basis. In terms of density matrices, it transforms the initial state

\[ \rho = \begin{pmatrix} \rho_{\downarrow\downarrow} & \rho_{\downarrow\uparrow} \\ \rho_{\uparrow\downarrow} & \rho_{\uparrow\uparrow} \end{pmatrix} \]

where \( \rho_{\downarrow\uparrow} \equiv \langle \downarrow \mid \rho \mid \uparrow \rangle \), etc., into

\[ \rho' = \begin{pmatrix} \rho_{\downarrow\downarrow} (1 - 2\lambda) \rho_{\downarrow\uparrow} \\ (1 - 2\lambda) \rho_{\uparrow\downarrow} \rho_{\uparrow\uparrow} \end{pmatrix}. \]

We note that here the parameter \( \lambda \) only appears in the off-diagonal terms. For this reason, the phase damping channel is a good model to describe decoherence \[13\]. Indeed, a repeated application of this channel leads to a vanishing of the off–diagonal terms in \( \rho' \), whereas its diagonal terms are preserved.

Since the parameter \( \lambda \) of the phase damping channel appears in the off-diagonal elements of \( \rho' \), the protocol is the following: Alice sends \( N \) qubits with the Bloch vector in the \( x - y \) plane; for instance, qubits in the state \( |\uparrow_x\rangle = (|\downarrow_z\rangle + |\uparrow_z\rangle)/\sqrt{2} \) can be used. This would correspond to a density operator whose matrix elements are all equal to 1/2, can be used. The density matrix of Bob’s qubit is then given by

\[ \rho'_x = \frac{1}{2} \begin{pmatrix} 1 & (1 - 2\lambda) \\ (1 - 2\lambda) & 1 \end{pmatrix}. \]

Now he measures the spin in the \( x \) direction. The theoretical probabilities are

\[ P(\uparrow_x) = \langle \uparrow_x \mid \rho' \mid \uparrow_x \rangle = 1 - \lambda, \]
\[ P(\downarrow_x) = \langle \downarrow_x \mid \rho' \mid \downarrow_x \rangle = \lambda, \]

and we denote the frequency of a spin–down measurement as \( i/N \), leading to \( \lambda^{est} = i/N \). The mean statistical error has again the form

\[ \bar{c}_x(N, \lambda) = \sum_{i=0}^{N} \binom{N}{i} \lambda^i (1 - \lambda)^{N-i} \left( \lambda - \frac{i}{N} \right)^2 = \frac{\lambda(1 - \lambda)}{N}, \]

as for the depolarizing channel. For the fidelity, equation (8), we obtain

\[ F(\vec{s}'_\lambda, \vec{s}'_{\lambda^{est}}) = 1 + \left[ 2\lambda \lambda^{est} - \lambda - \lambda^{est} \\ + 2\sqrt{\lambda(1 - \lambda)\lambda^{est}(1 - \lambda^{est})} \right] (s'_1 + s'_2) \]

and after averaging over the Bloch surface, according to equation (11),

\[ F(C_\lambda, C_{\lambda^{est}}) = 1 - \frac{2}{3} \lambda - \frac{2}{3} \lambda^{est} + \frac{4}{3} \lambda \lambda^{est} + \frac{4}{3} \sqrt{\lambda(1 - \lambda)\lambda^{est}(1 - \lambda^{est})} \]

Therefore the fidelity mean error \( \bar{c}_f \) reads

\[ \bar{c}_f(N, \lambda) = \sum_{i=0}^{N} \binom{N}{i} \lambda^i (1 - \lambda)^{N-i} \frac{2}{3} \left[ \lambda + (1 - 2\lambda) \frac{i}{N} - 2\sqrt{\lambda(1 - \lambda) \frac{i}{N} \left( 1 - \frac{i}{N} \right)} \right] \]
\[ = \frac{4}{3} \left[ \lambda (1 - \lambda) - \frac{1}{N} \sum_{i=0}^{N} \binom{N}{i} \lambda^i (1 - \lambda)^{N-i} \sqrt{\lambda(1 - \lambda) i (N-i)} \right], \]

which is shown in figure 3 b). This mean error is very similar to that obtained for the depolarizing channel.

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FIG. 3. a) The average statistical mean error $\bar{c}_s(N, \lambda)$, equation (17), and b) the fidelity mean error $\bar{c}_f(N, \lambda)$, equation (18), for a phase damping channel with parameter $\lambda$, are shown for different values of the number of qubits $N$.

C. The amplitude damping channel

The amplitude damping channel affects all components of the Bloch vector according to

$$\vec{s}_\lambda' = \left( \sqrt{1 - \lambda} s_1, \sqrt{1 - \lambda} s_2, (1 - \lambda) s_3 + \lambda \right)^T$$

where $0 \leq \lambda \leq 1$ is the damping parameter. The density matrix, equation (16), is transformed into

$$\rho' = \begin{pmatrix} \lambda + (1 - \lambda)\rho_{++} & \sqrt{1 - \lambda} \rho_{+\downarrow} \\ \sqrt{1 - \lambda} \rho_{\downarrow+} & (1 - \lambda)\rho_{++} \end{pmatrix}.$$ 方程式

This channel is a good model for spontaneous decay [14] from an atomic excited state $|\uparrow_z\rangle$ to the ground state $|\downarrow_z\rangle$. Repeated applications of this channel cause all elements but one of the density matrix to vanish. Now the parameter $\lambda$ appears in all the elements of $\rho'$ and the channel clearly possesses a preferred basis.

If Alice and Bob know that they are using an amplitude damping channel, Alice sends all $N$ qubits in the state $|\uparrow_z\rangle$. The density operator of the qubit received by Bob is

$$\rho' = \begin{pmatrix} \lambda & 0 \\ 0 & 1 - \lambda \end{pmatrix}.$$ 方程式

He measures the spin along the $z$ direction (the diagonal elements of $\rho'$ are the probabilities to find spin down and spin up, respectively). We denote the frequency of spin down measurements with $i/N$. The statistical cost function is again

$$\bar{c}_s(N, \lambda) = \sum_{i=0}^{N} \left( \frac{N}{i} \right) \lambda^i(1 - \lambda)^{N-i} \left( \lambda - \frac{i}{N} \right)^2 = \frac{\lambda(1 - \lambda)}{N}$$ 方程式

Using equation (19) we obtain

$$F(\vec{s}_\lambda', \vec{s}_{\lambda'}^{\text{est}}) = \frac{1}{2} \left[ 1 + \sqrt{(1 - \lambda)(1 - \lambda^{\text{est}})} \left( s_1^2 + s_2^2 + (1 - \lambda)(1 - \lambda^{\text{est}})s_3^2 \right) + \left[ \lambda(1 - \lambda^{\text{est}}) + \lambda^{\text{est}}(1 - \lambda) \right] s_3 + \lambda\lambda^{\text{est}} + (1 - s_3)^2 \sqrt{\lambda(1 - \lambda)} \sqrt{\lambda^{\text{est}}(1 - \lambda^{\text{est}})} \right]$$ 方程式

for the fidelity cost function, equation (8), and

$$F(C_\lambda, C_{\lambda^{\text{est}}}) = \frac{1}{6} \left[ 4 + 2\sqrt{(1 - \lambda)(1 - \lambda^{\text{est}})} - \lambda - \lambda^{\text{est}} + 4\lambda\lambda^{\text{est}} + 4\sqrt{\lambda(1 - \lambda)}\lambda^{\text{est}}(1 - \lambda^{\text{est}}) \right]$$ 方程式

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for the averaged fidelity (9). Thus
\[
\bar{c}_f(N, \lambda) = \sum_{i=0}^{N} \binom{N}{i} \lambda^i (1 - \lambda)^{N-i} \left[ 1 - \frac{1}{6} \left( 1 + 2 \sqrt{1 - \lambda} \left( 1 - \frac{i}{N} \right) - \lambda - \frac{i}{N} \sqrt{1 - \lambda} \left( 1 - \frac{i}{N} \right) \right) + 4 \lambda \frac{i}{N} + 4 \sqrt{\lambda (1 - \lambda) \frac{i}{N} \left( 1 - \frac{i}{N} \right)} \right] = \frac{1}{3} \left[ 1 + \lambda (1 - 2 \lambda) - \sqrt{\frac{1 - \lambda}{N}} \sum_{i=0}^{N} \binom{N}{i} \lambda^{N-i} (1 - \lambda)^i \right] - \frac{2}{N} \sqrt{\lambda (1 - \lambda)} \sum_{i=0}^{N} \binom{N}{i} \lambda^i (1 - \lambda)^{N-i} \sqrt{i (N-i)}
\]
(21)
which is illustrated by figure 4.

![Figure 4](image)  
**FIG. 4.** a) The average statistical mean error \(\bar{c}_s(N, \lambda)\), equation (20), and b) the fidelity mean error \(\bar{c}_f(N, \lambda)\), equation (21), for the amplitude damping channel with parameter \(\lambda\), are shown for different values of the number of qubits \(N\).

We see from the figures (2)–(4) that for a fixed number of resources the mean error \(\bar{c}_f\) has a local maximum for small values of the parameter \(\lambda\). For the amplitude damping channel there is also a second local maximum for large values of \(\lambda\).

### D. The general quantum channel

We now come to the problem of estimating the 12 parameters \(\vec{\lambda} = (\lambda_1, \ldots, \lambda_{12})^T\) of the general quantum channel. The protocol is summarized in table 1: Alice prepares 12 sets of qubits, divided into 4 groups. Bob measures the spin of the three sets in each group along \(x\), \(y\), and \(z\), respectively. From the measurements on each set he gets an estimate of one of the parameters \(\lambda_i\). The parametrization (12) has been chosen for this purpose.

The statistical cost function for the general channel is thus a generalization of the cost functions for one–parameter channels,
\[
\bar{c}_s(N, \vec{\lambda}) = \sum_{i_1=0}^{N/12} \cdots \sum_{i_{12}=0}^{N/12} \prod_{n=1}^{12} \binom{N/12}{i_n} \lambda_n^i (1 - \lambda_n)^{N/12-i_n} \sum_{j=1}^{12} (\lambda_j - \lambda_{j}^{\text{est}})^2
\]
\[= \frac{1}{N/12} \sum_{j=1}^{12} \lambda_j (1 - \lambda_j).
\]

The mean fidelity error can be calculated numerically. However, we do not give the expression here as it gives no particular further insight.
TABLE I. The protocol for the general channel. Alice prepares 4 sets of $N/4$ qubits each in a state corresponding to the spin pointing in a well prepared direction (left); Bob splits each set into 3 subsets of $N/12$ qubits each and performs a spin measurement along $x$, $y$ or $z$, respectively (center); the frequency of each set of spin measurements gives the estimate of one channel parameter (right).

| Alice | Bob | Estimate |
|-------|-----|----------|
| $| \downarrow_x \rangle$ | $\sigma_z$ | $\lambda_1^{st}$ |
|       | $\sigma_x$ | $\lambda_2^{st}$ |
|       | $\sigma_y$ | $\lambda_3^{st}$ |
| $| \uparrow_x \rangle$ | $\sigma_z$ | $\lambda_3^{st}$ |
|       | $\sigma_x$ | $\lambda_2^{st}$ |
|       | $\sigma_y$ | $\lambda_1^{st}$ |
| $| \uparrow_y \rangle$ | $\sigma_z$ | $\lambda_1^{st}$ |
|       | $\sigma_x$ | $\lambda_1^{st}$ |
|       | $\sigma_y$ | $\lambda_1^{st}$ |

V. THE PAULI CHANNEL FOR QUBITS

Up to now we have only considered estimation methods based on measuring single qubits sent through the quantum channel. However, we are not restricted to these estimation schemes. Instead of sending single qubits we could use entangled qubit pairs [6]. In this section we will demonstrate the superiority of such entanglement-based estimation methods for the estimation of the so called Pauli channel by comparing both schemes (for a different use of entanglement as a powerful resource when using Pauli channels, see [15]).

The Pauli channel is widely discussed in the literature, especially in the context of quantum error correction [17]. Its name originates from the error operators of the channel. These error operators are the Pauli spin matrices $\sigma_x$, $\sigma_y$ and $\sigma_z$. The operators define the quantum mechanical analogue to bit errors in a classical communication channel since $\sigma_x$ causes a bit flip ($| \downarrow \rangle$ transforms into $| \uparrow \rangle$ and vice versa), $\sigma_z$ causes a phase flip ($| \downarrow \rangle + | \uparrow \rangle$ transforms into $| \downarrow \rangle - | \uparrow \rangle$) and $\sigma_y$ results in a combined bit and phase flip. The Pauli channel is described by three probabilities $\vec{\lambda} \equiv (\lambda_1, \lambda_2, \lambda_3)^T$ for the occurrence of the three errors. If an initial quantum state $\rho$ is sent through the channel, the Pauli channel transforms $\rho$ into

$$\rho' = (1 - \lambda_1 - \lambda_2 - \lambda_3) \rho + \lambda_1 \sigma_x \rho \sigma_x + \lambda_2 \sigma_y \rho \sigma_y + \lambda_3 \rho \sigma_z.$$

Thus we see that the density operator $\rho$ remains unchanged with probability $1 - \lambda_1 - \lambda_2 - \lambda_3$, whereas with probability $\lambda_1$ the qubit undergoes a bit flip, with probability $\lambda_3$ there occurs a phase flip, and with probability $\lambda_2$ both a bit flip and a phase flip take place.

A. Estimation using single qubits

We will first describe the single qubit estimation scheme for the Pauli channel, before we switch to the entanglement-based one in the next subsection. Following the general estimation scheme presented in section II the protocol to estimate the parameters of the Pauli channel, equation (22), requires the preparation of three different quantum states with spin along three orthogonal directions. Alice sends (i) $M = N/3$ qubits in the state $| \uparrow_z \rangle$, (ii) $M$ qubits in the state $| \uparrow_x \rangle$, and (iii) $M$ qubits in the state $| \uparrow_y \rangle$. Bob measures their spins along the direction of spin-preparation, namely the $z$, $x$, and $y$ axes, respectively. The measurement probabilities of spin down along those three directions are then given by

$$P(\downarrow_z) = \lambda_1 + \lambda_2,$$
$$P(\downarrow_x) = \lambda_2 + \lambda_3,$$
\[ P(\psi^2) = \lambda_1 + \lambda_3. \]

The estimated parameter values can be calculated via

\[
\begin{align*}
\lambda_{1}^{\text{est}} &= \frac{1}{2} \left[ \frac{i_3}{M} - \frac{i_1}{M} + \frac{i_2}{M} \right] \\
\lambda_{2}^{\text{est}} &= \frac{1}{2} \left[ \frac{i_4}{M} - \frac{i_2}{M} + \frac{i_3}{M} \right] \\
\lambda_{3}^{\text{est}} &= \frac{1}{2} \left[ \frac{i_2}{M} - \frac{i_3}{M} + \frac{i_1}{M} \right]
\end{align*}
\]

where \( i_k/M, k = 1, 2, 3, \) denote the frequencies of spin down results along the directions \( x, y \) and \( z \). We note that, although the probabilities \( \lambda_k \) are positive or vanish, their estimated values \( \lambda_k^{\text{est}} \) may be negative. This occurs because in the present case the measured frequencies are not the estimates of the parameters. Nonetheless, the average cost functions can always be evaluated. For the statistical cost function we find

\[
\tilde{\mathcal{E}}_{\text{est}}(N, \tilde{\lambda}) = \sum_{i_1=0}^{N/3} \sum_{i_2=0}^{N/3} \sum_{i_3=0}^{N/3} \left( \begin{array} {c}
N/3 \\
i_1 \end{array} \right) \left( \begin{array} {c}
N/3 \\
i_2 \end{array} \right) \left( \begin{array} {c}
N/3 \\
i_3 \end{array} \right) (P(\downarrow_1))^{i_1} (1 - P(\downarrow_1))^{N/3-i_1} \times (P(\downarrow_2))^{i_2} (1 - P(\downarrow_2))^{N/3-i_2} \\
\times \left[ (\lambda_1 - \lambda_1^{\text{est}})^2 + (\lambda_2 - \lambda_2^{\text{est}})^2 + (\lambda_3 - \lambda_3^{\text{est}})^2 \right] \\
= \frac{9}{2N} [\lambda_1(1 - \lambda_1 - \lambda_2) + \lambda_2(1 - \lambda_2 - \lambda_3) + \lambda_3(1 - \lambda_3 - \lambda_1)].
\]

(23)

for the average error \( \tilde{\mathcal{E}}_{\text{est}} \) of the estimation with separable qubits. For fixed \( N \) the average error has a maximum at \( \lambda_1 = \lambda_2 = \lambda_3 = 1/4 \), in which case all acting operators occur with the same probability. On the other hand, the average error vanishes when faithful transmission or one of the errors occurs with certainty.

Instead of using the statistical cost function \( c_s \) we can also use the fidelity based cost function \( c_f \). The average error \( \tilde{\mathcal{E}}_{\text{est}} \) of the estimation via separable qubits is then given by

\[
\tilde{\mathcal{E}}_{\text{est}}(N, \tilde{\lambda}) = \sum_{i_1=0}^{N/3} \sum_{i_2=0}^{N/3} \sum_{i_3=0}^{N/3} \left( \begin{array} {c}
N/3 \\
i_1 \end{array} \right) \left( \begin{array} {c}
N/3 \\
i_2 \end{array} \right) \left( \begin{array} {c}
N/3 \\
i_3 \end{array} \right) (P(\downarrow_1))^{i_1} (1 - P(\downarrow_1))^{N/3-i_1} \times (P(\downarrow_2))^{i_2} (1 - P(\downarrow_2))^{N/3-i_2} \\
\times (P(\downarrow_3))^{i_3} (1 - P(\downarrow_3))^{N/3-i_3} \times c_f(N, \tilde{\lambda}).
\]

(24)

The cost function \( c_f \) cannot be calculated analytically. We shall compare the results (23) and (24) with the same mean errors for a different estimation scheme, where entangled pairs are used, that we are now going to illustrate.

**B. Estimation using entangled states**

All the protocols that we have considered so far are based on single qubits prepared in pure states. These qubits are sent through the channel one after another. However, one can envisage estimation schemes with different features. An interesting and powerful alternative scheme is based on the use of entangled states [6]. In this case the estimation scheme requires Alice and Bob to share entangled qubits in the \( |\psi^-\rangle \equiv \frac{1}{\sqrt{2}} (|\downarrow\rangle|\uparrow\rangle - |\uparrow\rangle|\downarrow\rangle) \) Bell state. Thus from \( N \) initial qubits Alice and Bob can prepare \( N/2 \) Bell states. Alice sends her \( N/2 \) qubits of the entangled pairs through the Pauli channel, which transforms the entangled state into the mixed state

\[
\rho' = \mathcal{C}(|\psi^-\rangle\langle\psi^-|) = \lambda_1 |\phi^-\rangle\langle\phi^-| + \lambda_2 |\phi^+\rangle\langle\phi^+| + \lambda_3 |\psi^-\rangle\langle\psi^-| + (1 - \lambda_1 - \lambda_2 - \lambda_3) |\psi^-\rangle\langle\psi^-|,
\]

where \( |\phi^\pm\rangle \equiv (|\downarrow\rangle|\downarrow\rangle \pm |\uparrow\rangle|\uparrow\rangle) / \sqrt{2} \) and \( |\psi^\pm\rangle \equiv (|\downarrow\rangle|\uparrow\rangle \pm |\uparrow\rangle|\downarrow\rangle) / \sqrt{2} \) are the four Bell states [3]. Bob performs \( N/2 \) Bell measurements, i.e., projects each of the \( N/2 \) pairs of qubits onto one of the four Bell states with probabilities
\[ P(|\phi^-\rangle) = \lambda_1, \quad P(|\phi^+\rangle) = \lambda_2, \quad P(|\psi^-\rangle) = \lambda_3, \quad P(|\psi^+\rangle) = 1 - \lambda_1 - \lambda_2 - \lambda_3. \]

Consequently, the estimated parameter values are directly given by

\[ \lambda_1^{\text{est}} = \frac{i_1}{N/2}, \quad \lambda_2^{\text{est}} = \frac{i_2}{N/2}, \quad \lambda_3^{\text{est}} = \frac{i_3}{N/2}, \]

We stress that now the estimates of the parameters \( \lambda_i \) are always nonnegative, contrarily to the estimation scheme with single qubits.

The average error \( \bar{c}_s^{\text{ent}} \) of the entangled estimation based on the statistical cost function \( c_s \) reads \([6]\)

\[
\bar{c}_s^{\text{ent}}(N, \vec{\lambda}) = \sum_{i_1+i_2+i_3+i_4=N/2} \frac{(N/2)!}{i_1!i_2!i_3!i_4!} \lambda_1^{i_1} \lambda_2^{i_2} \lambda_3^{i_3} (1 - \lambda_1 - \lambda_2 - \lambda_3)^{i_4} \sum_{k=1}^{3} (\lambda_k - \lambda_k^{\text{est}})^2 
\]

\[
= \frac{1}{N/2} [\lambda_1(1 - \lambda_1) + \lambda_2(1 - \lambda_2) + \lambda_3(1 - \lambda_3)].
\]

As we have shown already in \([6]\) \( \bar{c}_s^{\text{ent}} \) is always smaller or equal to \( \bar{c}_s^{\text{sep}} \),

\[
\Delta_s(N, \vec{\lambda}) = \bar{c}_s^{\text{sep}}(N, \vec{\lambda}) - \bar{c}_s^{\text{ent}}(N, \vec{\lambda})
\]

\[
= \frac{1}{2N} \left[ 5(1 - \lambda_1 - \lambda_2 - \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3) + \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 \right] \geq 0
\]

If we use the fidelity–based cost function \( c_f \) we can also write down the average error \( \bar{c}_f^{\text{ent}} \) for the entangled estimation scheme. It reads

\[
\bar{c}_f^{\text{ent}}(N, \vec{\lambda}) = \sum_{i_1+i_2+i_3+i_4=N/2} \frac{(N/2)!}{i_1!i_2!i_3!i_4!} \lambda_1^{i_1} \lambda_2^{i_2} \lambda_3^{i_3} (1 - \lambda_1 - \lambda_2 - \lambda_3)^{i_4} c_f(N, \vec{\lambda})
\]

\[
= \frac{1}{N/2} [\lambda_1(1 - \lambda_1) + \lambda_2(1 - \lambda_2) + \lambda_3(1 - \lambda_3)].
\]

and can be evaluated numerically. In figure \([5]\) we show the difference

\[
\Delta_f(N, \vec{\lambda}) = \bar{c}_f^{\text{sep}}(N, \vec{\lambda}) - \bar{c}_f^{\text{ent}}(N, \vec{\lambda})
\]

between the average error obtained with single qubits and entangled qubits. We compare the two average errors when the same number of qubits are used. The figure shows that the use of entangled pairs always leads to an enhanced estimation and therefore we can consider entanglement as a nonclassical resource for this application.

FIG. 5. The difference \( \Delta(N, \vec{\lambda}) \) equation \([26]\) between the fidelity mean errors \( \bar{c}_f^{\text{sep}} \), equation \([24]\), and \( \bar{c}_f^{\text{ent}} \), equation \([25]\) for the qubit Pauli channel. \( \Delta \) is plotted as a function of \( \lambda_1 \) and \( \lambda_3 \), while keeping \( \lambda_2 = 0 \) and \( N = 6 \) fixed.
Before ending this section we want to add some comments about our findings. We have found that the mean statistical error has the form

\[ \tilde{c}_s(N, \vec{\lambda}) = \frac{L}{N} \sum_{i=1}^{L} \lambda_i (1 - \lambda_i) \]  

(27)

whenever the estimates of the parameters \( \lambda_i \) are directly given by the frequencies of measurements. This occurs also for the relevant case of the entanglement–based protocol for the Pauli channel, but not for the qubit–based protocol. Although the parametrization we have used is better indicated since the \( \lambda_i \) represent the probabilities of occurrence of the logic errors, one might be tempted to use a different parametrization \( \vec{\lambda}' \) that gives again the expression (27). Indeed, this can be done and actually the errors for the \( \vec{\lambda}' \) are smaller. Nonetheless, one can show that in spite of this improvement, the scheme based on entangled pairs still gives an enhanced estimation even with the new parameters \( \vec{\lambda}' \). This is not the case of the one–parameter channels, where the use of entangled pairs does not lead to enhanced estimation.

VI. THE GENERALIZED PAULI CHANNEL

In the previous section we have proposed an entanglement–based method for estimating the parameters which define the Pauli channel. As we shall show in this section, this method can be easily extended to the case of quantum channels defined on higher dimensional Hilbert spaces.

Let us start by considering the most general possible trace preserving transformation of a quantum system described initially by a density operator \( \rho \). This transformation can be written in terms of quantum operations \( A_i \) as

\[ \rho' = C \rho = \sum_i \lambda_i A_i \rho A_i^\dagger \]  

(28)

with \( \sum_i \lambda_i A_i A_i^\dagger = 1 \). According to the Stinespring theorem \[1\] this is the most general form of a completely positive linear map. The set of operators \( A_i \) can be interpreted as error operators which characterize the action of a given quantum channel onto a quantum system and the set of parameters \( \lambda_i \) as probabilities for the action of error operators \( A_i \).

In particular, we can consider the action of the quantum channel, equation (28), onto only one, say the second, of the subsystems of a bipartite quantum system. For sake of simplicity we assume that the two subsystems are \( D \)–level systems. If only the second particle is affected by the quantum channel, equation (28), then the final state is given by

\[ \rho' = \sum_i \lambda_i (1 \otimes A_i) \rho (1 \otimes A_i)^\dagger \]  

(29)

where \( 1 \) is the identity operator. In the special case of an initially pure state, i. e. \( \rho = |\psi\rangle \langle \psi| \), the final state \( \rho' \) becomes

\[ \rho' = \sum_i \lambda_i |\psi_i\rangle \langle \psi_i| \]  

(30)

where we have defined \( |\psi_i\rangle \equiv (1 \otimes A_i) |\psi\rangle \).

Our aim is the estimation of the parameter values \( \lambda_i \) which define the quantum channel. This can be done by projecting the state of the composite quantum system onto the set of states \( |\psi_i\rangle \). As we shall show, this set can be perfectly distinguished if and only if they are mutually orthogonal. Hence, we impose the condition

\[ \langle \psi_i | \psi_j \rangle = \langle \psi | (1 \otimes A_i)^\dagger (1 \otimes A_j) | \psi \rangle = \delta_{i,j} \]  

(31)

on the initial state \( |\psi\rangle \). It means that the initial state \( |\psi\rangle \) of the bipartite quantum system should be chosen in such a way that it is mapped onto a set of mutually orthogonal states \( |\psi_i\rangle \) by the error operators \( A_i \). It is worth to remark the close analogy between this estimation strategy and quantum error correction. A non-degenerate quantum error correcting code \[8\] corresponds to a Hilbert subspace which is mapped onto mutually orthogonal subspaces under the action of the error operators. In this sense, a state \( |\psi\rangle \) satisfying the condition (31) is a one-dimensional non-degenerated error correcting code.
A necessary but not sufficient condition for the existence of a state \( |\psi\rangle \) satisfying equation (31) is given by the inequality

\[
n \leq D'
\]

(32)

where \( n \) is the total number of error operators which describe the quantum channel and \( D' \) is the dimension of the Hilbert space. This inequality shows why the use of an entangled pair enhances the estimation scheme. Although the first particle is not affected by the action of the noisy quantum channel, it enlarges the dimension of the total Hilbert space. This inequality shows why the use of an entangled pair enhances the estimation scheme. Although the action of the Pauli channel is defined by a set of orthogonal states can only be designed in principle if the number of error operators \( n \) is smaller than \( D' \). For instance, in the case of the Pauli channel for qubits considered in the previous section, condition (32) does not hold if we use single qubits for the estimation. The action of the Pauli channel is defined by a set of \( n = 4 \) error operators (including the identity) acting onto qubits \( (D' = 2) \).

Let us now apply our consideration to the case of a generalized Pauli channel. The action of this channel is given by

\[
|\psi\rangle \equiv \frac{1}{\sqrt{D}} \sum_{i=0}^{D-1} |i\rangle_1 \otimes |i\rangle_2
\]

(35)

where \( |i\rangle_1 \) and \( |i\rangle_2 \), \( j = 0, \ldots, D - 1 \), are basis states for the two particles. In fact, the action of the operators \( 1 \otimes U_{\alpha,\beta} \) onto state \( |\psi_{0,0}\rangle \) generates the basis

\[
|\psi_{\alpha,\beta}\rangle \equiv 1 \otimes U_{\alpha,\beta}|\psi_{0,0}\rangle = \frac{1}{\sqrt{D}} \sum_{k=0}^{D-1} e^{i\frac{2\pi}{D}\alpha k} |k\rangle_1 \otimes |k - \beta\rangle_2
\]

(36)

of \( D^2 \) maximally entangled states for the total Hilbert space. Thereby, the action of the generalized Pauli channel yields

\[
\rho' = \sum_{\alpha,\beta=0}^{D-1} \lambda_{\alpha,\beta} |\psi_{\alpha,\beta}\rangle \langle \psi_{\alpha,\beta}|
\]

when the initial state is \( \rho = |\psi_{0,0}\rangle \langle \psi_{0,0}| \). The particular values of the coefficients \( \lambda_{\alpha,\beta} \) can now be obtained by projecting onto the states \( |\Psi_{\alpha,\beta}\rangle \). The quality of this estimation strategy according to the statistical error is given by

\[
\bar{e}_s(N, \bar{\lambda}) = \frac{1}{N/2} \sum_{\alpha,\beta=0}^{D-1} \lambda_{\alpha,\beta}(1 - \lambda_{\alpha,\beta})(1 - \delta_{\alpha,0}\delta_{\beta,0})
\]

with \( N/2 \) the number of pairs of quantum systems used in the estimation.
VII. CONCLUSIONS

We have examined the problem of determining the parameters that describe a noisy quantum channel with finite resources. We have given simple protocols for the determination of the parameters of several classes of quantum channels. These protocols are based on measurements made on qubits that are sent through the channel. We have also introduced two cost functions that estimate the quality of the protocols. In the most simple protocols measurements are performed on each qubit. We have also shown that more complex schemes based on entangled pairs can give a better estimate of the parameters of the Pauli channel. Our investigations stress once more the usefulness of entanglement in quantum information.

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