Dynamically defined measures and equilibrium states

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Abstract

A technique of dynamically defined measures is developed and its relation to the theory of equilibrium states is shown. The technique uses Carathéodory’s method and the outer measure introduced in [27]. As an application, equilibrium states for contractive Markov systems [26] are obtained.

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1 Dynamically defined measures

Let $E$ be a finite set and $\Sigma := \{(..., \sigma_{-1}, \sigma_0, \sigma_1,...) : \sigma_i \in E \ \forall i \in \mathbb{Z}\}$ equipped with the product topology of the discrete topologies. Let $S: \Sigma \rightarrow \Sigma$ be the left shift map on $\Sigma$, i.e. $(S\sigma)_i = \sigma_{i+1}$ for all $i \in \mathbb{Z}$. We call the set $m[e_m,...,e_n] := \{\sigma \in \Sigma : \sigma_i = e_i \text{ for all } m \leq i \leq n\}$ for $m \leq n \in \mathbb{Z}$, a cylinder. Let $A$ denote the $\sigma$-algebra generated by the zero time partition $\{0[e] : e \in E\}$ of $\Sigma$, and define, for each integer $m \leq 1$,

$$\mathcal{A}_m := \bigvee_{i=m}^{+\infty} S^{-i}A,$$

which is the smallest $\sigma$-algebra containing all $\sigma$-algebras $\bigvee_{i=m}^{n} S^{-i}A$, $n \geq m$, where the latter consists of finite unions of cylinders $m[e_m,...,e_n]$.

Let $\phi_0$ be a finite non-negative measure on $A_0$. Let $\phi_m$ denote the measure on $\mathcal{A}_m$ given by

$$\phi_m := \phi_0 \circ S^m$$

for all $m \leq 0$. 
In [27] the following dynamically defined outer measure on $\Sigma$ was introduced. It arose very naturally from the need of measuring some subsets of $\Sigma$ which depend on the whole past under circumstances where they can be covered only with infinitely many sets $A_m \in \mathcal{A}_m$ with known values $\phi_m(A_m)$, $m \leq 0$.

**Definition 1** For $Q \subset \Sigma$, set

$$
\mathcal{C}(Q) := \left\{ (A_m)_{m \leq 0} : A_m \in \mathcal{A}_m \forall m \text{ and } Q \subset \bigcup_{m \leq 0} A_m \right\}
$$

and

$$
\Phi(Q) := \inf \left\{ \sum_{m \leq 0} \phi_m(A_m) : (A_m)_{m \leq 0} \in \mathcal{C}(Q) \right\}.
$$

It is quite straightforward to check that $\Phi$ defines an outer measure on $\Sigma$ (e.g. see [27]). Moreover, it is not difficult to verify that $\Phi$ is exactly the outer measure used for the Carathéodory’s construction of a measure if $\phi_m$’s satisfy Kolmogorov’s consistency condition (see Proposition 1). In the following, it will be shown, in particular, that our generalization of Carathéodory’s outer measure is a way for obtaining measures on product spaces from measures on subspaces which do not satisfy Kolmogorov’s consistency condition, but are consecutive images of each other under the shift map.

Another observation which can be made at this point is that the definition is possible only by the Axiom of Choice. However, this will not repel a scientifically minded reader because it will be shown below (Lemma 5) that a family of covering sets $(A_m)_{m \leq 0}$ of a Borel set can be chosen in such a way that only finitely many of them are non-empty. Therefore, the definition of $\Phi$ can be legitimately called a construction.

This is important because it will be shown in Section 2 that $\Phi$ allows to obtain certain equilibrium states. The construction of equilibrium states for dynamical systems is one of the tasks of mathematical physics. It became a subject of a rigorous mathematical analysis under the name of *Gibbs measures* [16] [21] [22] [24] since the seminal works of Bogolubov and Hacet [7], Dobrushin [12], Ruelle [20] and Sinai [23], generalizing the *Boltzmann distribution*, which has been used in physics as a distribution which minimizes the *free energy* of certain systems. The main motivation for these efforts was the construction of physically meaningful invariant measures for dynamical systems, which these systems asymptotically seek to achieve starting from measures about which we have some partial information. As far as the author is aware, the existence of equilibrium states is known only for upper semicontinuous energy functions [16], but there are no universal way for a construction of them so far even for continuous functions (usually some stronger continuity conditions, such as *summability of variation*, are required).
It will be shown in Section 2.1 that the construction of $\Phi$ allows to obtain equilibrium states for some random dynamical systems introduced in [26] as contractive Markov systems. They are a unifying generalization of les chaînes à liaisons complètes [11], $g$-measures [17] and iterated function systems with place dependent probabilities [1] which allows to formulate the results on stability of them in a language which is more systematic and natural for this kind of questions. They are known to posses sometimes several stationary states [9], [5]. Taking into the account the wide spectrum of applications of them in the modern science (e.g. [3], [2], [10], [14], [25], [4]) leaves no doubts that the questions on stability of them are no less important than curious. It was shown in [28] that the energy functions associated with such systems are not even upper semicontinuous in general (even in the case of the usual regularity conditions on the probability functions). Nevertheless, the existence of equilibrium states for them has been proved [28]. However, the task of the construction of equilibrium states for such energy functions from non-invariant initial measures remained open. All author’s attempts to use the well know technics of Gibbs measures failed. Here we obtain them using the construction of $\Phi$ even under conditions which are much weaker than those assumed in [28]. The definition of $\Phi$ resembles some of the constructions of Gibbs measures [23] and connects it with the general way in which measures are constructed in the Measure Theory [6]. Moreover, it is not difficult to see that the definition of $\Phi$ can be easily extended to more general dynamical systems.

Every theory is based just on a few examples. In almost every text book, Gibbs measures are illustrated with the help of finite Markov chains, which are just trivial cases of contractive Markov systems. If an example does not fit into the theory, the latter needs to be modified. Every working example is guiding in this situation and therefore should not be ignored.

Now, we will refine the definition to make it more easily applicable in some situations. Also, this will make the previous claims more obvious.

For each $m \leq 0$, let $\Delta_m$ denote the algebra consisting of all finite unions of cylinders of the form $\{e_m, e_{m+1}, \ldots, e_n\}, \ e_i \in E, \ m \leq i \leq n$. Set

$$\Xi(Q) := \left\{(C_m)_{m \leq 0} : C_m \in \Delta_m \text{ for all } m \leq 0 \text{ and } Q \subset \bigcup_{m \leq 0} C_m \right\}$$

for all $Q \in \Sigma$.

Since each $\Delta_m$ generates the $\sigma$-algebra $A_m$, the following lemma shows that the definition of $\Phi$ is a generalization of the usual definition of an outer measure from the standard Measure Theory, which is used for the Caratéodory’s method for obtaining a measure, as mentioned above.
Lemma 1 Let \( Q \subset \Sigma \). Then

\[
\Phi(Q) = \inf \left\{ \sum_{m \leq 0} \phi_m(C_m) : (C_m)_{m \leq 0} \in \Xi(Q) \right\}.
\]

Proof. Clearly, the left hand side of the claimed equation can not exceed it’s right hand side.

On the other hand, by the definition of \( \Phi(Q) \), there exists a sequence \(((A^n_m)_{m \leq 0})_{n \in \mathbb{N}} \subset \mathcal{C}(B)\) such that

\[
\sum_{m \leq 0} \phi_m(A^n_m) \downarrow \Phi(Q) \text{ as } n \to \infty.
\]

By the standard approximation theory, for every \( n \in \mathbb{N} \) and \( m \leq 0 \) there exists \( C^n_m \in \Delta_m \) such that \( A^n_m \subset C^n_m \) and

\[
\phi_m(C^n_m \setminus A^n_m) < \frac{2^m}{n}.
\]

Therefore,

\[
\left| \sum_{m \leq 0} \phi_m(C_m) - \Phi(Q) \right| \leq \frac{1}{n} + \left| \sum_{m \leq 0} \phi_m(A^n_m) - \Phi(Q) \right|.
\]

This implies the claim. \( \square \)

The class of the covering sets in the construction of \( \Phi \) can be restricted even further. Set

\[
\hat{\Xi}(Q) := \left\{ (B_m)_{m \leq 0} : B_m \in \Delta_m, B_m \cap B_n = \emptyset \ \forall m \neq n \text{ and } Q \subset \bigcup_{m \leq 0} B_m \right\}
\]

for all \( Q \subset \Sigma \).

Lemma 2 Let \( Q \subset \Sigma \). Then

\[
\Phi(Q) = \inf \left\{ \sum_{m \leq 0} \phi_m(B_m) : (B_m)_{m \leq 0} \in \hat{\Xi}(Q) \right\}.
\]

Proof. Let \( \hat{\Phi}(Q) \) denote the right hand side of the equation to be proved. Then obviously

\[
\hat{\Phi}(Q) \geq \Phi(Q).
\]

Now, let \((C_m)_{m \leq 0} \in \Xi(Q)\). Set

\[
B_m := C_m \setminus (C_{m+1} \cup \ldots \cup C_0) \text{ for all } m \leq 0.
\]
Then \((B_m)_{m \leq 0} \in \mathcal{B}(Q)\) and \(B_m \subset C_m\) for all \(m \leq 0\). Hence
\[
\sum_{m \leq 0} \phi_m(C_m) \geq \sum_{m \leq 0} \phi_m(B_m) \geq \Phi(Q).
\]
Therefore,
\[
\Phi(Q) \geq \hat{\Phi}(Q).
\]
This concludes the proof. 

Now, we give a formal proof that \(\Phi\) is a natural dynamical extension of the Caratéodory’s outer measure which contains the latter as the case with consistent measures \(\phi_m\)’s.

**Proposition 1** Suppose that \(\phi_{-1}\) is consistent with \(\phi_0\), i.e. \(\phi_{-1}(C) = \phi_0(C)\) for all cylinder sets \(C \in \mathcal{A}_0\). Then \(\Phi\) coincides with Caratéodory’s outer measure which extends the set function \(\phi\) defined for every cylinder set \(C \in \mathcal{A}_m, m \leq 0\), by \(\phi(C) := \phi_m(C)\).

**Proof.** Let \(M\) denote Caratéodory’s outer measure extending the set function \(\phi\). Let \(Q \subset \Sigma\) and \((A_m)_{m \leq 0} \in \mathcal{X}(Q)\). Since each \(A_m\) can be written as a finite disjoint union of some cylinders \(C_1,m, ..., C_{n,m}\) from \(A_m\),
\[
\sum_{m \leq 0} \phi_m(A_m) = \sum_{m \leq 0} \sum_{k=1}^{n_m} \phi(C_{km}) \geq M(Q).
\]
Hence,
\[
\Phi(Q) \geq M(Q).
\]
Now, let \((C_n)_{n \in \mathbb{N}}\) be a sequence of cylinders such that \(Q \subset \bigcup_{n \in \mathbb{N}} C_n\). Let \(m_1\) be the largest non-positive integer such that \(C_1 \in \mathcal{A}_{m_1}\). Set \(B_{m_1} := C_1\) and \(B_m = \emptyset\) for all \(m_1 < m \leq 0\). Recursively, assuming that all \(B_m\)’s are defined for all \(m \leq m \leq 0\), choose \(m_{n+1} < m_n\) such that \(C_{n+1} \in \mathcal{A}_{m_{n+1}}\) and set \(B_{m_{n+1}} := C_{n+1}\) and \(B_m = \emptyset\) for all \(m_{n+1} < m < m_n\). Then \((B_m)_{m \leq 0} \in \mathcal{X}(Q)\). Therefore,
\[
\sum_{n \in \mathbb{N}} \phi(C_n) = \sum_{m \leq 0} \phi(B_m) = \sum_{m \leq 0} \phi_m(B_m) \geq \Phi(Q).
\]
Thus,
\[
M(Q) \geq \Phi(Q).
\]
This completes the proof. 

The following Lemma is useful for verifying that \(\Phi(\Sigma) > 0\).

**Lemma 3** Suppose \(\phi_0\) is a measure on \(\mathcal{A}_0\) which is absolutely continuous with respect to \(\phi_0\). Let \(\Phi'\) denote the outer measure obtained from \(\phi_0\) as in Definition 1. Then for every \(\epsilon > 0\) there exists \(\delta > 0\) such that, for every \(Q \subset \Sigma\), \(\Phi'(Q) < \epsilon\) whenever \(\Phi(Q) < \delta\).
Proof. See the proof of Lemma 2 (ii) in [27].

Next, we are going to investigate what happens to $\Phi$ under the action of the shift map. The action can be considered into two ways.

**Definition 2** For $m \leq 0$, let $\Phi_m$ denote the outer measure $\Phi$ where $\phi_m$ is taken as the initial measure on $A_0$, instead of $\phi_0$.

**Lemma 4** Let $Q \subset \Sigma$. Then
$$\Phi(Q) \leq \Phi(S^{-1}Q) \leq \Phi_{(1)}(Q).$$

**Proof.** Let $(A_m)_{m \leq 0} \in \Xi(Q)$. Then $...,SA_{-1},SA_0,0) \in \Xi(SQ)$. Therefore,
$$\Phi(SQ) \leq \sum_{m \leq 0} \phi_m(SA_m).$$

Since $\phi_{m-1}(SA_m) = \phi_m(A_m)$ for all $m \leq 0$,
$$\Phi(SQ) \leq \sum_{m \leq 0} \phi_m(A_m).$$

This implies
$$\Phi(SQ) \leq \Phi(Q),$$
which is equivalent to the first inequality.

On the other hand, since $(S^{-1}A_m)_{m \leq 0} \in \Xi(S^{-1}Q)$,
$$\Phi(S^{-1}Q) \leq \sum_{m \leq 0} \phi_m(S^{-1}A_m).$$

This gives
$$\Phi(S^{-1}Q) \leq \Phi_{(1)}(Q).$$

Obviously $\Phi$ is finite since $\phi_0$ is finite. Therefore, by Lemma 4, we can make the following definition.

**Definition 3** For $Q \subset \Sigma$, set
$$\Phi^*(Q) := \lim_{m \to -\infty} \Phi_m(Q).$$

Obviously, $\Phi^*$ is an outer measure. By Lemma 4, $\Phi \leq \Phi^*$.

Next theorem bears the fruits of the construction.

**Theorem 1** The restrictions of $\Phi$ and $\Phi^*$ on the Borel $\sigma$-algebra are shift invariant measures.
Proof. We start with $\Phi$. By Lemma 4, we can define

$$\bar{\Phi}(Q) := \lim_{m \to -\infty} \Phi(S^m Q)$$

for all $Q \subset \Sigma$. It is easy to check that $\bar{\Phi}$ is an outer measure on $\Sigma$. Also, it is clear that it is shift invariant. First, we show that the restriction of $\bar{\Phi}$ on the Borel $\sigma$-algebra on $\Sigma$ is a measure. We will demonstrate it using Caratéodory’s method, i.e by showing that the $\sigma$-algebra of all $\bar{\Phi}$-measurable sets contains all cylinder sets. That is we need to show that

$$\bar{\Phi}(Q) \geq \bar{\Phi}(Q \cap C) + \bar{\Phi}(Q \setminus C)$$

for all cylinder sets $C$ and all $Q \subset \Sigma$.

So, let $C$ be a cylinder set and $Q$ a subset of $\Sigma$. Then there exists $n \geq 0$ such that $S^{-i}C \in \Delta_0$ for all $i \geq n$ and hence $S^{-i}C \in \Delta_m$ for all $i \geq n$ and $m \leq 0$. Let $i \geq n$ and $(A_m)_{m \leq 0} \in C(S^{-i}Q)$. Then

$$\sum_{m \leq 0} \phi_m(A_m) = \sum_{m \leq 0} \phi_m(A_m \cap S^{-i}C) + \sum_{m \leq 0} \phi_m(A_m \setminus S^{-i}C).$$

Since $(A_m \cap S^{-i}C)_{m \leq 0} \in C(S^{-i}(Q \cap C))$ and $(A_m \setminus S^{-i}C)_{m \leq 0} \in C(S^{-i}(Q \setminus C))$,

$$\sum_{m \leq 0} \phi_m(A_m) \geq \Phi(S^{-i}(Q \cap C)) + \Phi(S^{-i}(Q \setminus C)).$$

Hence

$$\Phi(S^{-i}Q) \geq \Phi(S^{-i}(Q \cap C)) + \Phi(S^{-i}(Q \setminus C)).$$

Taking the limit gives (1).

Let $B \subset \Sigma$ be a Borel set. The first part of the claim will follow, if we show that $\Phi(B) = \bar{\Phi}(B)$. By Lemma 4,

$$\Phi(B) \leq \bar{\Phi}(B).$$

Let $B' \subset \Sigma$ be a Borel set. Since $\bar{\Phi}$ is a Borel measure and $\Phi$ is an outer measure with $\Phi(\Sigma) = \bar{\Phi}(\Sigma)$, by Lemma 4,

$$\bar{\Phi}(\Sigma \setminus B') = \bar{\Phi}(\Sigma) - \bar{\Phi}(B') \leq \Phi(\Sigma) - \Phi(B') \leq \Phi(\Sigma \setminus B').$$

So, for $B' := \Sigma \setminus B$, we get

$$\Phi(B) \geq \bar{\Phi}(B),$$

as desired.

It is possible to deduce the second part of the claim from the first. Here we give a proof of it which is independent from the proof of the first. To show that $\Phi^*$ is a shift invariant Borel measure, observe that, by Lemma 4,

$$\Phi_{(-i)}(Q) \leq \Phi_{(-i)}(S^{-1}Q) \leq \Phi_{(-i-1)}(Q)$$
for all \( i \in \mathbb{N} \). Taking the limit gives

\[
\Phi^*(S^{-i}Q) = \Phi^*(Q).
\]

Now, we show in the same way that \( \Phi^* \) is a Borel measure. Let \((A_m)_{m \leq 0} \in \mathcal{C}(Q)\).

Observer that \((S^{-i}(A_m \cap C))_{m \leq 0} \in \mathcal{C}(S^{-i}(Q \cap C)) \text{ and } (S^{-i}(A_m \setminus C))_{m \leq 0} \in \mathcal{C}(S^{-i}(Q \setminus C))\) for all \( i \geq n \). Therefore,

\[
\sum_{m \leq 0} \phi_{m-2i}(A_m) = \sum_{m \leq 0} \phi_{m-i}(S^{-i}A_m)
\]

\[
= \sum_{m \leq 0} \phi_{m-i}(S^{-i}(A_m \cap C)) + \sum_{m \leq 0} \phi_{m-i}(S^{-i}(A_m \setminus C))
\]

\[
\geq \Phi_{(-i)}(S^{-i}(Q \cap C)) + \Phi_{(-i)}(S^{-i}(Q \setminus C))
\]

\[
\geq \Phi_{(-i)}(Q \cap C) + \Phi_{(-i)}(Q \setminus C)
\]

for all \( i \geq n \). Hence,

\[
\Phi_{(-2i)}(Q) \geq \Phi_{(-i)}(Q \cap C) + \Phi_{(-i)}(Q \setminus C).
\]

Taking the limit gives

\[
\Phi^*(Q) \geq \Phi^*(Q \cap C) + \Phi^*(Q \setminus C).
\]

This completes the proof. \(\square\)

In the following, we will use the same notation for measures obtained in Theorem 1 as for outer measures \( \Phi \) and \( \Phi^* \) if no confusion is possible.

At this point, it is easy to see that the family of the covering sets of Borel sets in the definition of \( \Phi \) can be restricted further. Set

\[
\hat{\zeta}(B) := \{(A_m)_{m \leq 0} \in \hat{\Xi}(B) : \text{ at most finitely many } A_m \neq \emptyset\}
\]

for all \( B \subset \Sigma \). Choosing elements from finitely many \( \sigma \)-algebras does not require the Axiom of Choice. The next lemma brings \( \Phi \) back to the constructive world.

**Lemma 5** Let \( B \subset \Sigma \) be Borel. Then

\[
\Phi(B) = \inf_{(A_m)_{m \leq 0} \in \hat{\zeta}(B)} \sum_{m \leq 0} \phi_m(A_m).
\]

**Proof.** First let \( C \subset \Sigma \) be compact. Obviously,

\[
\Phi(C) \leq \inf_{(A_m)_{m \leq 0} \in \hat{\zeta}(C)} \sum_{m \leq 0} \phi_m(A_m).
\]

On the other hand, by the compactness of \( C \), for every \((A_m)_{m \leq 0} \in \hat{\zeta}(C)\) there exists \((A'_m)_{m \leq 0} \in \hat{\zeta}(C)\) such that

\[
\sum_{m \leq 0} \phi_m(A_m) \geq \sum_{m \leq 0} \phi_m(A'_m) \geq \inf_{(A'_m)_{m \leq 0} \in \hat{\zeta}(C)} \sum_{m \leq 0} \phi_m(A'_m).
\]
Hence
\[ \Phi(C) \geq \inf_{(A_m)_{m \leq 0} \in \xi(C)} \sum_{m \leq 0} \phi_m(A_m). \]

Since the class of compact subsets contains the family of all cylinder sets, the claim follows by Theorem 1. \qed

It has been shown in Theorem 1 that the definition of \( \Phi \) lets to obtain a nice shift invariant measure on \( \Sigma \). The next example demonstrates that it works not always.

**Example 1** Let \( \Sigma := \{0, 1\}^\mathbb{Z} \) and \( \sigma' \in \Sigma \) given by
\[ \sigma'_i = \begin{cases} 0 & \text{if } i \text{ is even} \\ 1 & \text{otherwise} \end{cases} \]
for all \( i \in \mathbb{Z} \). Let \( \phi_0 \) be the probability measure on \( A_0 \) given by
\[ \phi_0(A) = \begin{cases} 1 & \text{if } \sigma' \in A \\ 0 & \text{otherwise} \end{cases} \]
for all \( A \in A_0 \). Then for \( (A_m)_{m \leq 0} \in \hat{\xi}(\Sigma) \) given by \( A_0 := 0[1], A_{-1} := 0[0] \) and \( A_m := \emptyset \) for all \( m \leq -2 \), \( \sum_{m \leq 0} \phi_m(A_m) = 0 \). Hence \( \Phi(\Sigma) = 0 \). Similarly, by swapping \( 0[0] \) and \( 0[1] \) if necessary, one sees that each \( \Phi(-n)(\Sigma) = 0 \). Thus \( \Phi(\Sigma) = 0 \) also.

However, from [27] we know examples where \( \Phi \) gives useful non-zero measures. It is natural to hope that, in case \( \Phi(\Sigma) > 0 \), it is a probability measure if \( \phi_0 \) is one. The following proposition shows in particular that it is not true in general.

**Proposition 2** Suppose \( \phi_0 \) is a probability measure. Then
(i) \( \Phi(\Sigma) \leq 1 \),
(ii) \( |\phi_m(A) - \phi_0(A)| \leq 1 - \Phi(\Sigma) \) for all \( A \in A_0 \) and \( m \leq 0 \).
(iii) The following are equivalent:
\begin{enumerate}
\item \( \Phi(\Sigma) = 1 \),
\item \( \phi_0(S^{-1}A) = \phi_0(A) \) for all \( A \in A_0 \),
\item \( \Phi \) uniquely extends \( (\phi_m)_{m \leq 0} \) on the Borel \( \sigma \)-algebra.
\end{enumerate}

**Proof.** (i) It is clear that \( \Phi(\Sigma) \leq 1 \), since \((\ldots, \emptyset, \emptyset, \Sigma) \in \mathcal{C}(\Sigma) \) and \( \phi_0 \) is a probability measure.

(ii) By the definition of \( \Phi \) and Theorem 1, for every \( A \in A_0 \) and \( m \leq 0 \),
\[ 1 - \phi_m(A) = \phi_m(\Sigma \setminus A) \geq \Phi(\Sigma \setminus A) = \Phi(\Sigma) - \Phi(A) \geq \Phi(\Sigma) - \phi_0(A). \]
Hence
\[ \phi_m(A) - \phi_0(A) \leq 1 - \Phi(\Sigma). \]
Applying this inequality to $\Sigma \setminus A$ gives (ii).

(iii) The implication from a) to b) follows immediately from (ii). The implication from b) to c) follows from Proposition 1 and Kolmogorov’s Consistency Theorem. The implication from c) to a) is obvious.

By Proposition 2, a normalization of $\Phi$ is in general necessary if $\Phi(\Sigma) > 0$, which is a typical situation in statistical mechanics. Non-typical in this approach is that we don’t need to require some convergence in so-called ‘thermodynamic limit’ to obtain a measure on the Borel $\sigma$-algebra.

We conclude this section by giving some sufficient conditions on $\phi_0$ for the positivity of $\Phi(\Sigma)$.

**Proposition 3** (i) Let $\nu$ be a positive Borel measure on $\Sigma$. Let $\phi'_m$ denote the absolutely continuous part of the Lebesgue decomposition of $\phi_m$ with respect to $\nu$. Suppose there exists a Borel-measurable function $f$ such that $d\phi'_m / d\nu \geq f$ for all $m \leq 0$ and $\int f \, d\nu > 0$. Then $\Phi(\Sigma) > 0$.

(ii) Suppose there exists a positive shift invariant measure on $A_0$ which is absolutely continuous with respect to $\phi_0$. Then $\Phi(\Sigma) > 0$.

**Proof.** (i) Observe that

$$\sum_{m \leq 0} \phi_m(A_m) \geq \sum_{m \leq 0} \int f \, d\nu \geq \int f \, d\nu > 0$$

for all $(A_m)_{m \leq 0} \in C(\Sigma)$. Therefore, the claim follows.

(ii) The claim follows by Lemma 3 and Proposition 2 (iii).

Next section provides in particular some physically meaningful examples where $\Phi(\Sigma) > 0$.

## 2 Equilibrium states

In this section, we intend to show that construction of $\Phi$ allows to obtain equilibrium states for some energy function $u : \Sigma \rightarrow [-\infty, 0]$.

**Definition 4** Let $P_S(\Sigma)$ denote the space of all shift invariant Borel probability measures on $\Sigma$ and $h_\Lambda(S)$ be the Shannon-Kolmogorov-Sinai entropy of $S$ with respect to $\Lambda \in P_S(\Sigma)$. $\Lambda_0 \in P_S(\Sigma)$ is said to be an equilibrium state for $u$ iff

$$h_{\Lambda_0}(S) + \int ud\Lambda_0 = \sup_{\Lambda \in P_S(\Sigma)} \left\{ h_\Lambda(S) + \int ud\Lambda \right\}.$$ 

The physical interpretation of it is that the equilibrium state is a state which minimizes the free energy of the system.
To proceed towards our goal, we need to split each \( \sigma \)-algebra \( \mathcal{A}_m, m \leq 0 \), into two pieces, the one which depends on the past and the other which depends on the future.

**Definition 5** Let \( G \) denote the \( \sigma \)-algebra on \( \Sigma \) which is generated by cylinders of the form \( 1_{[e_1, \ldots, e_n]}, e_i \in E, 1 \leq i \leq n, n \in \mathbb{N} \). For \( m \leq 0 \), let \( \mathcal{F}_m \) denote the finite \( \sigma \)-algebra on \( \Sigma \) generated by cylinders of the form \( m_{[e_m, \ldots, e_0]}, e_i \in E, m \leq i \leq 0 \). Finally, let \( \mathcal{F} \) denote the \( \sigma \)-algebra generated by \( \bigcup_{m \leq 0} \mathcal{F}_m \).

Recall that by Kolmogorov-Sinai Theorem,

\[
h_\Lambda(S) = -\sum_{e \in E} \int \mathbb{E}_\Lambda \left(1_{1_{[e]} | \mathcal{F}}\right) \log \mathbb{E}_\Lambda \left(1_{1_{[e]} | \mathcal{F}}\right) d\Lambda,
\]

where \( \mathbb{E}_\Lambda \left(1_{1_{[e]} | \mathcal{F}}\right) \) denotes the conditional expectation of the indicator function \( 1_{1_{[e]}} \) conditioned on \( \mathcal{F} \) with respect to \( \Lambda \). \( h_\Lambda(S) \) is interpreted as a measure of uncertainty of observing the next symbol of process \( \Lambda \) given its past. This is the reason for the split on the past and the future.

From (2) follows the desire to be able to compute the conditional expectations with respect to \( \Phi \), which might help to verify the equilibrium state property. This seems to require an extension of the standard theory of Martingales to adapted random variables with different underlying probability measures. Such an extension is done in the next theorem, which is a simple consequence of the construction of \( \Phi \). This can be compared to the result in [18], where \( \phi_m \)'s are assumed to be defined on the limiting \( \sigma \)-algebra and converge in some sense.

**Theorem 2** Let \( f : \Sigma \to \mathbb{R} \) be \( G \)-measurable and bounded. Suppose \( \Phi(\Sigma) > 0 \). Then

\[
\mathbb{E}_{\phi_m} (f|\mathcal{F}_m) \to \mathbb{E}_\Phi (f|\mathcal{F}) \quad \Phi\text{-a.e.}
\]

**Proof.** Though the theorem appears to be a generalization of Doob’s Martingale Theorem, the proof of it reduces to the truth of latter (just as the result in [18] does).

Let \( m \leq 0 \). Since \( \Phi|_{\mathcal{F}_m} \leq \phi_m|_{\mathcal{F}_m} \), by Radon-Nikodym Theorem, there exists \( \xi_m : \Sigma \to \mathbb{R} \mathcal{F}_m \)-measurable such that \( \Phi(A) = \int_A \xi_m d\phi_m \) for all \( A \in \mathcal{F}_m \). Therefore, by the pool-out-property of the conditional expectation,

\[
\int_A \mathbb{E}_{\phi_m} (f|\mathcal{F}_m) d\Phi = \int_A \xi_m f d\phi_m = \int_A f d\Phi
\]

for all \( A \in \mathcal{F}_m \). Thus

\[
\mathbb{E}_{\phi_m} (f|\mathcal{F}_m) = \mathbb{E}_\Phi (f|\mathcal{F}_m) \quad \Phi\text{-a.e.}
\]

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for all $m \leq 0$. Since the normalization of $\Phi$ does not alter $E_{\Phi}(f|F_m)$, by Doob’s Martingale Theorem we conclude that

$$E_{\Phi,m}(f|F_m) \to E_{\Phi}(f|F) \text{ \Phi-a.e..}$$

\[\square\]

### 2.1 Equilibrium states for random dynamical systems

Now, we are going to apply the theory developed in this paper to some random dynamical systems introduced in [26] as Markov systems. It is known that the asymptotic behavior of contractive Markov systems is similar [26], to some extent, to the trivial case of finite Markov chains. However, the question on the necessary and sufficient condition for the uniqueness of the stationary state for such systems remains open already for more than seventy years. A reason for that might be the lack of suitable mathematical tools. This paper provides some new tools. They allow us to obtain new useful results about Markov system. On the other hand, this gives an example of non-zero measures $\Phi$ obtained from measures on sub-$\sigma$-algebras which do not satisfy Kolmogorov’s consistency condition.

It was pointed out by an anonymous referee that it might be appropriate to cite in this paper the work by A. Lasota and J. Yorke [19] where a general method, called the lower bound technique, useful in providing criteria for a stability of such systems is presented.

Let $(K_{i,e}, w_e, p_e)_{e \in E}$ be a Markov system [26], i.e. $K_1, ..., K_N$ is a partition of a complete metric space $(K, d)$ into non-empty Borel subsets, $i : E \to \{1, ..., N\}$ surjective and $t : E \to \{1, ..., N\}$ such that, for every $e \in E$, $w_e : K_{i(e)} \to K_{i(e)}$ and $p_e : K_{i(e)} \to (0, 1]$ such that $\sum_{e \in E, i(e) = i} p_e(x) = 1$ for all $x \in K_i$, $i \in \{1, ..., N\}$. We can consider each $p_e$ to be extended on $K$ by zero and each $w_e$ to be extended on $K$ arbitrarily. We assume that each $p_e|_{K_{i(e)}}$ and $w_e|_{K_{i(e)}}$ is uniformly continuous, where notation $|_A$ means the restriction on a set $A$.

A Markov system is called contractive iff there exists $0 < a < 1$ such that

$$\sum_{e \in E} p_e(x)d(w_e x, w_e y) < ad(x, y) \text{ for all } x, y \in K_i, i = 1, ..., N. \quad (4)$$

Let $B(K)$ denote the Borel $\sigma$-algebra and $P(K)$ the set of all Borel probability measures on $K$. Let $U$ be the Markov operator acting on real-valued functions $f$ on $K$ by

$$Uf = \sum_{e \in E} p_e f \circ w_e$$

and $U^*$ be its adjoint operator acting on Borel probability measures $\nu$ on $K$ by $U^*\nu(B) = \int U(1_B)d\nu$ for all $B \in B(K)$.
Let $x \in K$. For each integer $m \leq 1$, let $P_x^m$ be the probability measure on the \( \sigma \)-algebra \( A_m \) given by

\[
P_x^m([m[e_1, \ldots, e_n]]) = p_{e_1}(x)p_{e_{m+1}}(w_{e_1}(x)) \cdots p_{e_n}(w_{e_{m-1}} \circ \cdots \circ w_{e_m}(x))
\]

for all cylinder sets \( [m[e_1, \ldots, e_n]], n \geq m \). It has been shown in [27] Lemma 1 that the function \( x \mapsto P_x^m(A) \) is Borel measurable for all \( A \in A_m \) (for that only the Borel measurability of \( p_e \)'s and \( w_e \)'s is needed).

**Definition 6** For \( \nu \in P(K) \), let \( \phi_m(\nu) \) be the probability measure on \( A_m \) given by

\[
\phi_m(\nu)(A) := \int P_x^m(A) d\nu(x) \quad \text{for all } A \in A_m.
\]

Observer that, for each \( i \geq 0 \), \( \phi_m(U^i\nu) \) and \( \phi_m(\nu) \circ S^{-i} \) also define measures on \( A_m \) for all \( m \leq 0 \). The following lemma states their relations to \( \phi_m(\nu) \).

**Lemma 6** Let \( m \leq 0 \) and \( \nu \in P(K) \). Then

(i) \( \phi_{m-1}(\nu)(Q) = \phi_m(U^i\nu)(Q) = \phi_m(\nu)(S^{-i}Q) \) for all \( Q \in A_m \),

(ii) \( \phi_{m-1}(\nu)(Q) = \phi_m(\nu)(S^{-i}Q) \) for all \( Q \in A_{m-1} \).

**Proof.** (i) We show only the second equation, the proof of the first is the same.
It is sufficient to check that the measures agree on all cylinders generating \( A_m \).

\[
\phi_m(\nu)(S^{-i}m[e_1, \ldots, e_n]) = \phi_m(\nu)(m[e_1, \ldots, e_n])
\]

\[
= \int \sum_{e \in E} p_e(x) P_x^m(m[e_1, \ldots, e_n]) d\nu(x)
\]

\[
= \int P_x^m(m[e_1, \ldots, e_n]) dU^i\nu(x)
\]

\[
= \phi_m(U^i\nu)(m[e_1, \ldots, e_n]).
\]

For (ii), observe that

\[
\phi_m(\nu)(S^{-i}m-1[e_1, \ldots, e_n]) = \phi_m(\nu)(m[e_1, \ldots, e_n]) = \phi_{m-1}(\nu)(m-1[e_1, \ldots, e_n]).
\]

\[\square\]

**Definition 7** For \( \nu \in P(K) \), let \( \Phi(\nu) \) and \( \Phi^*(\nu) \) be the outer measures and measures as defined in (1) and (3) with \( \phi_m(\nu) \)'s standing for \( \phi_m \)'s.

**Remark 1** Observe that, by Lemma 6 (i), measures \( \phi_m(\nu) \) satisfy Kolmogorov’s consistency condition if \( \nu \) is an invariant initial distribution of the Markov system. In this case, outer measure \( \Phi(\nu) \) is the usual outer measure used for Caratéodory’s construction of a measure. In general, \( \phi_m(\nu) \)'s have exactly the properties of \( \phi_m \)'s from Section 1. In particular, outer measure \( \Phi_{(-i)}(\nu) \) is nothing else but \( \Phi(U^i\nu) \) for all \( i \geq 0 \).
The following simple lemma states some properties of the map $\nu \mapsto \Phi(\nu)$.

**Lemma 7** Let $\nu_1, \nu_2 \in P(K)$. Then

(i) $\Phi(\nu_1) \ll \Phi(\nu_2)$ if $\nu_1 \ll \nu_2$, where $\ll$ denotes the absolute continuity relation.

(ii) For $0 \leq \alpha \leq 1$,

$$\Phi(\alpha \nu_1 + (1 - \alpha) \nu_2) \geq \alpha \Phi(\nu_1) + (1 - \alpha) \Phi(\nu_2).$$

**Proof.** (i) Observe that $\phi_0(\nu_1) \ll \phi_0(\nu_2)$. Therefore, the claim follows by Lemma 3. (ii) is a direct consequence of the supperadditivity of the infimum. \(\square\)

We will use the following initial distributions for the Markov system.

**Definition 8** Fix $x_i \in K_i$ for all $i = 1, \ldots, N$ and set

$$\nu_0 := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i},$$

where $\delta_{x_i}$ denotes the Dirac probability measure concentrated at $x_i$. Let $\emptyset \neq S \subset \{1, \ldots, N\}$ and set

$$\nu_0' := \frac{1}{|S|} \sum_{i \in S} \delta_{x_i},$$

where $|S|$ denotes the size of $S$.

**Proposition 4** (i) For $\nu \in P(K)$, the following are equivalent:

a) $\Phi(\nu)(\Sigma) = 1$,

b) $U^* \nu = \nu$,

c) $\Phi(\nu)$ uniquely extends $\phi_m(\nu)$’s on the Borel $\sigma$-algebra.

(ii) Suppose there exists $\mu \in P(K)$ such that $U^* \mu = \mu$ and $P^0_x << P^0_y$ for all $x, y \in K_i$ and $i \in S := \{j: \mu(K_j) > 0\}$. Then $\Phi(\nu_0')(\Sigma) > 0$.

**Proof.** (i) The claim follows from Lemma 6 and Proposition 2.

(ii) Clearly, it follows from the hypothesis that $P^0_x << \phi_0(\nu_0')$ for all $x \in K_i$ and $i \in S$. Therefore, $\phi_0(\mu) << \phi_0(\nu_0')$. Hence, by Lemma 7(i), $\Phi(\mu) << \Phi(\nu_0')$. Since, by (i), $\Phi(\mu)(\Sigma) > 0$, the claim follows. \(\square\)

**Example 2** It has been shown in [15] that there exists an invariant initial Borel probability distribution $\mu$ for a contractive Markov system on a Polish space if $(K_i)_{i=1}^{N}$ form an open partition of $K$. The condition of equivalence of measures $P^0_x$ and $P^0_y$ for all $x, y \in K_i$, $i = 1, \ldots, N$, is satisfied e.g. if probability functions $p_{(x)}(K_{(e)}, e \in \mathcal{E}$, of a contractive Markov system have a square summable variation and are bounded away from zero [29]. By Proposition 4 (ii), $\Phi(\nu_0)(\Sigma) > 0$ for such systems. As far as the author is aware, the first result on the equivalence of measures $P^0_x$ and $P^0_y$ was obtained by J. H. Elton in [13], in the case when the partition has the single atom and the probability functions have a summable variation (Dini continuous) and are bounded away from zero.
Definition 9 Set

\[ \Sigma_G := \{ \sigma \in \Sigma : i(\sigma_{n+1}) = t(\sigma_n) \text{ for all } n \in \mathbb{Z} \} \]

(subshift of finite type associated with the Markov system) and

\[ D := \left\{ \sigma \in \Sigma_G : \lim_{m \to -\infty} w_{\sigma_0} \circ \ldots \circ w_{\sigma_m} (x_{i(\sigma_m)}) \text{ exists} \right\} . \]

\( D \) is a \( \mathcal{F} \)-measurable, e.g. one can see it quickly as follows. Let's abbreviate \( w_m^0(\sigma) := w_{\sigma_0} \circ \ldots \circ w_{\sigma_m}(x_{i(\sigma_m)}) \) for all \( \sigma \in \Sigma \). Clearly, \( D = \{ \sigma \in \Sigma_G : (w_m^0(\sigma))_{m \leq 0} \text{ is Cauchy} \} \), by completeness of \( K \). For each \( m \leq 0 \) and \( n > 0 \) set \( Q_{mn} := \bigcap_{k \leq 0} \{ \sigma \in \Sigma_G : d(w_m^0(\sigma), w_m^0(k)(\sigma)) < 1/n \} \), which are obviously \( \mathcal{F} \)-measurable sets. Then \( D = \bigcap_{n \in \mathbb{N}} \bigcup_{m \leq 0} Q_{mn} \) and therefore it is \( \mathcal{F} \)-measurable.

Proposition 5 Suppose the Markov system is contractive. Then

(i) \( \Phi(\nu_0')| (\Sigma \setminus D) = 0 \) and

(ii) \( D \subset S^{-1}D \).

Proof. (i) It was demonstrated in [27] that

\[ \Phi(\nu_0') (\Sigma \setminus D) = 0, \]

which is a direct consequence of the contractiveness condition (4) (no continuity of \( p_e|_{K_{i(e)}} \)'s or \( w_e|_{K_{i(e)}} \)'s is required for that). Since \( \nu_0' \ll \nu_0 \), the claim follows by Lemma 7 (i).

(ii) Let \( \bar{K}_{i(e)} \) denote the closure of \( K_{i(e)} \), \( e \in E \). Since each \( w_e|_{K_{i(e)}} \) is uniformly continuous, there exists a unique continuous extension of \( w_e \) on \( \bar{K}_{i(e)} \) (e.g. Theorem 2, p. 190 in [8]), which we will denote with \( \bar{w}_e \). Let \( \sigma \in D \), then there exists \( y \in \bar{K}_{i(\sigma_0)} \) such that \( y = \lim_{m \to -\infty} w_{\sigma_0} \circ \ldots \circ w_{\sigma_m}(x_{i(\sigma_m)}) \). Hence, \( w_{\sigma_1}(y) = \lim_{m \to -\infty} w_{\sigma_1} \circ \ldots \circ w_{\sigma_m}(x_{i(\sigma_m)}) \). Therefore \( S(\sigma) \in D \). That is \( \sigma \in S^{-1}D \).

Definition 10 Set

\[ F: \Sigma \to K \]

by

\[ F(\sigma) := \begin{cases} \lim_{m \to -\infty} w_{\sigma_0} \circ \ldots \circ w_{\sigma_m}(x_{i(\sigma_m)}) & \text{if } \sigma \in D \\ x_{i(\sigma_0)} & \text{otherwise}, \end{cases} \]

We call \( F \) the coding map of the Markov system. It is obviously \( \mathcal{F} \)-Borel-measurable, as it is the pointwise limit of \( \mathcal{F} \)-Borel-measurable functions \( F_m \) given by \( F_m(\sigma) := w_m^0(\sigma) \) if \( \sigma \in D \) and \( F_m(\sigma) = x_{i(\sigma_0)} \) otherwise for all \( m \leq 0 \).

Note that \( F \) might be meaningful not only for contractive Markov systems. For example, if normalized \( \Phi(\nu_0') \) is ergodic, then \( \Phi(\nu_0')(D) = 1 \) or \( \Phi(\nu_0')(D) = 0 \) by the shift invariance of \( D \) (Proposition 5 (ii)).
Lemma 8 Suppose the Markov system is contractive. Then \( F(\Phi(\nu_0')) \) is a Radon measure.

Proof. Obviously, \( F(\Phi(\nu_0')) \) is a Borel measure, since \( F \) is in particular Borel-Borel-measurable. Furthermore, \( F(\Phi(\nu_0')) \) is regular, since \( K \) is a metric space (e.g. Theorem 7.1.7, p.70, Vol. II in [6]). So, the claim will follow if we show that \( F(\Phi(\nu_0')) \) is tight, since an intersection of a compact and a closed set in \( K \) is compact.

Let \( \epsilon > 0 \). By Lemma 3 (iii) in [27], there exist a compact set \( Q \subset \Sigma \) with \( \Phi(\nu_0)(\Sigma \setminus Q) < \epsilon \) such that \( F|_Q \) is continuous. Set \( C := F|_Q(Q) \). Then \( C \) is compact and

\[
F(\Phi(\nu_0))(K \setminus C) = \Phi(\nu_0)(\Sigma \setminus F^{-1}(C)) \\
\leq \Phi(\nu_0)(\Sigma \setminus (F|_Q)^{-1}(C)) \\
\leq \Phi(\nu_0)(\Sigma \setminus Q) \\
< \epsilon.
\]

Hence \( F(\Phi(\nu_0)) \) is tight. By Lemma 7 (i), \( F(\Phi(\nu_0')) \) is absolutely continuous with respect to \( F(\Phi(\nu_0)) \), therefore \( F(\Phi(\nu_0')) \) is also tight. \( \square \)

Definition 11 Set

\[
u(\sigma) := \begin{cases} 
\log p_{\sigma_1} \circ F(\sigma) & \text{if } \sigma \in D \\
-\infty & \text{otherwise}
\end{cases}
\]

with the definition \( \log(0) = -\infty \). We call \( \nu \) the energy function for the Markov system. Recall that \( \nu \) is not upper-semicontinuous in general [28] (even for contractive Markov systems with an open partition), therefore the existing theory of thermodynamic formalism is useless in our situation.

Observe, that with the above notation each measure \( \phi_m(F(\Lambda)) \) has the following form

\[
\phi_m(F(\Lambda))(m[e_1, ..., e_n]) = \int_{m[e_1, ..., e_n]} \exp \sum_{i=0}^{n-1} u \circ S^i d\Lambda,
\]

if \( \Lambda(D) = 1 \) and each \( K_i \) is open in \( K \), which is similar to the Sinai’s starting point for a construction of a Gibbs measure [23].

Definition 12 Let \( \nu \in P(K) \) and \( m \leq 0 \). Set

\[
\bar{\phi}_m(\nu)(A \times Q) := \int_A P_{x_A}^m(Q) d\nu(x)
\]
for all $A \in \mathcal{B}(K)$ and $Q \in \mathcal{A}_m$. It is not difficult to see that $\tilde{\phi}_m(\nu)$ extends uniquely to a probability measure on the product $\sigma$-algebra $\mathcal{B}(K) \otimes \mathcal{A}_m$ with

$$\tilde{\phi}_m(\nu)(\Omega) = \int P_x^m(\{\sigma \in \Sigma : (x, \sigma) \in \Omega\}) d\nu(x)$$

for all $\Omega \in \mathcal{B}(K) \otimes \mathcal{A}_m$ and

$$\int \psi d\tilde{\phi}_m(\nu) = \int \int \psi(x, \sigma) dP_x^m(\sigma) d\nu(x)$$

for all $\mathcal{B}(K) \otimes \mathcal{A}_m$-measurable and $\tilde{\phi}_m(\nu)$-integrable functions $\psi$ on $K \times \Sigma$ (see [28]).

Analogously to Lemma 6 one readily checks that

$$\tilde{\phi}_{m-1}(\nu)(K \times Q) = \tilde{\phi}_m(U^\ast \nu)(K \times Q)$$  \hspace{1cm} (5)

for all $Q \in \mathcal{A}_m$, $m \leq 0$.

**Proposition 6** Let $e \in E$ and $m \leq 0$. Then the following is true.

(i) \hspace{1cm} $E_{\tilde{\phi}_m(\nu')}(1_{[e]} | \mathcal{F}_m)(\sigma) = p_e \circ w_{\sigma_0} \circ \ldots \circ w_{\sigma_m}(x_{i(\sigma_m)})$  \hspace{1cm} (6)

for $\phi_m(\nu')$-a.e $\sigma \in \Sigma$.

(ii) Set $p_{cm}(x, \sigma) := p_e \circ w_{\sigma_0} \circ \ldots \circ w_{\sigma_m}(x)$ for $x \in K$ and $\sigma \in \Sigma$. Let $\nu \in \mathcal{P}(K)$. Then

$$E_{\tilde{\phi}_{m-1}(\nu)}(p_{cm-1} | K \otimes \mathcal{F}_m) = p_{cm}$$  \hspace{1cm} (7)

$\tilde{\phi}_{m-1}(\nu)$-a.e., where $K \otimes \mathcal{F}_m$ denotes the product $\sigma$-algebra of the trivial $\sigma$-algebra on $K$ and $\mathcal{F}_m$.

**Proof.** Let $m[e_m, \ldots, e_0]$ be a cylinder from $\mathcal{F}_m$. Then

$$\int_{m[e_m, \ldots, e_0]} 1_{[e]} d\phi_m(\nu'_0) = \int P_x^m(m[e_m, \ldots, e_0, e]) d\nu'_0(x)$$

$$= \int P_x^m(m[e_m, \ldots, e_0]) p_e \circ w_{\sigma_0} \circ \ldots \circ w_{\sigma_m}(x) d\nu'_0(x)$$

$$= \int \int_{m[e_m, \ldots, e_0]} p_e \circ w_{\sigma_0} \circ \ldots \circ w_{\sigma_m}(x_{i(\sigma_m)}) dP_x^m(\sigma) d\nu'_0(x)$$

$$= \int \int_{m[e_m, \ldots, e_0]} p_e \circ w_{\sigma_0} \circ \ldots \circ w_{\sigma_m}(x_{i(\sigma_m)}) d\phi_m(\nu'_0)(\sigma).$$

Therefore (i) is true.
For (ii), observe that, by (5),

\[
\int_{K \times [\sigma_{m-1}, \sigma_0]} p_e \circ w_{\sigma_0} \circ \cdots \circ w_{\sigma_{m-1}}(x) d\tilde{\phi}_{m-1}(\nu)(x, \sigma)
\]

\[
= \int \sum_{e_{m-1}} P^{m-1}_{x} (e_{m-1}, \sigma_0) p_e \circ w_{e_0} \circ \cdots \circ w_{e_{m-1}}(x) d\nu(x)
\]

\[
= \int \int_{\sigma_{m-1}} P^{m}_{x} (e_{m}, \sigma_0) p_e \circ w_{e_0} \circ \cdots \circ w_{e_m}(x) d\nu(x)
\]

\[
= \int \int_{\sigma_{m-1}} p_e \circ w_{\sigma_0} \circ \cdots \circ w_{\sigma_m}(x) d\tilde{\phi}_{m-1}(\nu)(x, \sigma),
\]

as it is claimed. \[\square\]

**Remark 2** Observe that equation (7) indicates some Martingale like behavior of functions \((p_{e_m})_{m \leq 0}\). In fact, by Kolmogorov’s Consistency Theorem, it is a martingale equation if \(\nu\) is an invariant initial distribution for the Markov system (see Proposition 4 (i)). This seems to indicate the truth of a more general Martingale Theorem where an adapted sequence of functions satisfies the recursive averaging of a martingale with respect to some measures on non-decreasing \(\sigma\)-algebras which do not satisfy the Kolmogorov’s consistency condition. Such a result has been proved in Theorem 2. According to it, the left hand side of (6) converge to the conditional expectation on the limiting \(\sigma\)-algebra almost everywhere with respect to measure \(\Phi(\nu'_0)\).

**Lemma 9** Let \(e \in E\). Suppose \(\Phi(\nu'_0)(\Sigma) > 0\) and \(\Phi(\nu'_0)(\Sigma \setminus D) = 0\). Then

\[E_{\Phi(\nu'_0)} (1_{1[e]} | \mathcal{F}) = \bar{p}_e \circ F_{\Phi(\nu'_0)} \text{ - a.e.},\]

where \(\bar{p}_e\) denotes the continuous extension of \(p_e|_{\mathcal{K}_{i(e)}}\) on the closure of \(\mathcal{K}_{i(e)}\) (e.g. Theorem 2, p. 190 in [8]).

**Proof.** By Proposition 6 (i) and Theorem 2,

\[E_{\Phi(\nu'_0)} (1_{1[e]} | \mathcal{F}) (\sigma) = \lim_{m \to -\infty} p_e \circ w_{\sigma_0} \circ \cdots \circ w_{\sigma_m}(x_i(\sigma_m))
\]

\[= \bar{p}_e \left( \lim_{m \to -\infty} w_{\sigma_0} \circ \cdots \circ w_{\sigma_m}(x_i(\sigma_m)) \right)
\]

\[= \bar{p}_e \circ F(\sigma)
\]

for \(\Phi(\nu'_0)\)-a.e. \(\sigma \in D\). The claim follows. \[\square\]
**Theorem 3** Let \( e \in E \). Suppose \( \Phi(\nu'_0)(\Sigma) > 0 \) and at least one of the following conditions holds true:

(i) the Markov system is contractive,
(ii) \( K \) is separable and \( \Phi(\nu'_0)(\Sigma \setminus D) = 0 \).

Then

\[
E_{\Phi(\nu'_0)} (1_{\{e\}}|\mathcal{F}) = p_e \circ F \quad \Phi(\nu'_0)\text{-a.e.} \quad (9)
\]

**Proof.** By Proposition 5(i) and Lemma 9,

\[
E_{\Phi(\nu'_0)} (1_{\{e\}}|\mathcal{F}) = \bar{p}_e \circ F \quad \Phi(\nu'_0)\text{-a.e.}
\]

in both cases. Suppose there exists \( e_0 \in E \) such that \( F(\Phi(\nu'_0))(\{\bar{p}_{e_0} > p_{e_0}\}) > 0 \). Observe that, by the hypothesis, \( F(\Phi(\nu'_0)) \) is a Radon measure, in case of (i) it holds true by Lemma 8, in case of (ii) it is a well known fact (e.g. Theorem 7.1.7, p. 70, Vol. II in [6]). Furthermore, the restriction of a Radon measure on a Borel set in a metric space is a Radon measure also. Therefore, by Lusin’s Theorem (e.g. Theorem 7.1.13, p.72, Vol. II in [6]), there exists a compact set \( C \subset \{\bar{p}_{e_0} > p_{e_0}\} \) such that \( F(\Phi(\nu'_0))(C) > 0 \) and the function \( \sum_{e \in E} \bar{p}_e|C \) is continuous. Then

\[
\sum_{e \in E} \bar{p}_e(x) > \sum_{e \in E} p_e(x) = 1
\]

for all \( x \in C \). However, by the above,

\[
\begin{align*}
1 & \quad < \frac{1}{F(\Phi(\nu'_0))(C)} \int_{F^{-1}(C)} \sum_{e \in E} \bar{p}_e(x) \circ F \ d\Phi(\nu'_0) \\
& \quad = \frac{1}{F(\Phi(\nu'_0))(C)} \int_{F^{-1}(C)} \sum_{e \in E} E_{\Phi(\nu'_0)} (1_{\{e\}}|\mathcal{F}) \ d\Phi(\nu'_0) \\
& \quad = 1,
\end{align*}
\]

which is a contradiction. Therefore, the claim is true. \( \square \)

**Corollary 1** Suppose \( \Phi(\nu'_0)(\Sigma) > 0 \) and it is normalized and at least one of the following conditions holds true:

(i) the Markov system is contractive,
(ii) \( K \) is separable and \( \Phi(\nu'_0)(\Sigma \setminus D) = 0 \).

Then \( \Phi(\nu'_0) \) is an equilibrium states for \( u \) and

\[
h_{\Phi(\nu'_0)}(S) = -\int u \ d\Phi(\nu'_0).
\]
Proof. The claim follows from Lemma 5 in [28]. For completeness, we prove it here in our case. By Theorem 3,

\[ h_\Phi(\nu'_0)(S) = - \sum_{e \in E} \int \phi \left( 1_{\{e\}} | \mathcal{F} \right) \log \left( \mathbb{E}_\Phi(\nu'_0) \left( 1_{\{e\}} | \mathcal{F} \right) \right) d\Phi(\nu'_0) \]

\[ = - \sum_{e \in E} \int \log p_e \circ F \ d\Phi(\nu'_0) \]

\[ = - \int \log \left( p_\sigma \circ F(\sigma) \right) d\Phi(\nu'_0)(\sigma). \]

That is

\[ h_\Phi(\nu'_0)(S) + \int u \ d\Phi(\nu'_0) = 0. \tag{10} \]

To complete the proof it remains to show that

\[ h_\Lambda(S) + \int u \ d\Lambda \leq 0 \tag{11} \]

for all $S$-invariant Borel probability measures $\Lambda$. For set

\[ g_e := E_\Lambda \left( 1_{\{e\}} | \mathcal{F} \right) \]

for each $e \in E$. Then as above

\[ h_\Lambda(S) = - \sum_{e \in E} \int \log g_e \ d\Lambda. \]

If $\Lambda(\{u = -\infty\}) > 0$, then $h_\Lambda(S) + \int u \ d\Lambda = -\infty < 0$. Otherwise, observe that

\[ h_\Lambda(S) + \int u \ d\Lambda = \sum_{e \in E} \int \log \frac{p_e \circ F}{g_e} \ d\Lambda \leq \sum_{e \in E} \int \left( \frac{p_e \circ F}{g_e} - 1 \right) \ d\Lambda. \]

It is not difficult to check that $1_{\{e\}}(p_e \circ F/g_e - 1) \in L^1(\Lambda)$ for all $e \in E$ (e.g. Lemma 2 in [28]). Therefore, by the pull-out property of the conditional expectation,

\[ \sum_{e \in E} \int \left( \frac{p_e \circ F}{g_e} - 1 \right) \ d\Lambda \]

\[ = \sum_{e \in E} \int g_e \left( \frac{p_e \circ F}{g_e} - 1 \right) \ d\Lambda \]

\[ = \sum_{e \in E} \int (p_e \circ F - g_e) \ d\Lambda \]

\[ = 1 - 1 \]

\[ = 0. \]

This implies (11), as desired. \qed
Corollary 2 Suppose $\Phi(\nu'_0)(\Sigma) > 0$ and at least one of the following conditions holds true:

(i) the Markov system is contractive,  
(ii) $\mathbb{K}$ is separable and $\Phi(\nu'_0)(\Sigma \setminus D) = 0$.

Then

$$\Phi(\nu'_0)|_{\mathcal{A}_0} = \phi_0(F(\Phi(\nu'_0))). \quad (12)$$

Proof. The claim is a straightforward consequence of Theorem 3. Observe that $\mathcal{E}_\Phi(\nu'_0) \left(1_{[e]}|\mathcal{F}\right) \circ S^{-n}$ is $\mathcal{F}$-measurable, for all $n \in \mathbb{N}$, since $S(\mathcal{F}) \subset \mathcal{F}$. Therefore, by the shift invariance of $\Phi(\nu'_0)$ and the pull-out property of the conditional expectation,

$$\phi_0(F(\Phi(\nu'_0))) (0[e_1, \ldots, e_n])$$

$$= \int P_{\mathcal{F}(\sigma)}^1 (1[e_1, \ldots, e_n]) \ d\Phi(\nu'_0)(\sigma)$$

$$= \int E_{\Phi(\nu_0)} \left(1_{[e_1]}|\mathcal{F}\right) E_{\Phi(\nu_0)} \left(1_{[e_2]}|\mathcal{F}\right) \circ S \ldots E_{\Phi(\nu_0)} \left(1_{[e_n]}|\mathcal{F}\right) \circ S^{n-1} \ d\Phi(\nu'_0)$$

$$= \int E_{\Phi(\nu_0)} \left(1_{[e_1]}|\mathcal{F}\right) \circ S^{-n+1} E_{\Phi(\nu_0)} \left(1_{[e_2]}|\mathcal{F}\right) \circ S^{-n+2} \ldots E_{\Phi(\nu_0)} \left(1_{[e_n]}|\mathcal{F}\right) \ d\Phi(\nu'_0)$$

$$= \int E_{\Phi(\nu_0)} \left(1_{[e_1]}|\mathcal{F}\right) \circ S^{-n+1} E_{\Phi(\nu_0)} \left(1_{[e_2]}|\mathcal{F}\right) \circ S^{-n+2} \ldots 1_{[e_n]} \ d\Phi(\nu'_0)$$

for all cylinders $0[e_1, \ldots, e_n] \in \mathcal{A}_0$. The claim follows. \qed

Proposition 7 Suppose $\Phi(\nu'_0)(\Sigma) > 0$ and $\Phi(\nu'_0)(\Sigma \setminus D) = 0$. Then

$$U^* F(\Phi(\nu'_0)) = F(\Phi(\nu'_0)).$$

Proof. Let $f$ be a real-valued, Borel-measurable and bounded function on $\mathbb{K}$. Let $\bar{w}_e$ denote the continuous extension of $w_e|_{K_{i(e)}}$ on the closure of $K_{i(e)}$ for
all $e \in E$. Then, by Theorem 3 and the shift invariance of $\Phi(\nu'_0)$,

$$U^* F (\Phi(\nu'_0)) (f) = \int \sum_{e \in E} p_e f \circ w_e \, dF (\Phi(\nu'_0))$$

$$= \int \sum_{e \in E} p_e f \circ \bar{w}_e \, dF (\Phi(\nu'_0))$$

$$= \int \sum_{e \in E} p_e \circ F f \circ \bar{w}_e \circ F \, d\Phi(\nu'_0)$$

$$\leq \int \sum_{e \in E} \bar{p}_e \circ F f \circ \bar{w}_e \circ F \, d\Phi(\nu'_0)$$

$$= \sum_{e \in E} \int 1_{[e]} f \circ \bar{w}_e \circ F \, d\Phi(\nu'_0)$$

$$= \sum_{e \in E} \int f \circ F \circ \bar{S} \, d\Phi(\nu'_0)$$

$$= \int f \, dF (\Phi(\nu'_0)).$$

Hence,

$$U^* F (\Phi(\nu'_0)) (f) \leq F (\Phi(\nu'_0)) (f).$$

Since $f$ was arbitrary and both measures have the same norm, they must be equal. $\square$

We conclude the paper with a list of questions which it opens.

1) Are Borel measures $\Phi$ and $\Phi^*$ equal?
2) Is $\Phi(\nu'_0)(\Sigma) > 0$ for some $\nu'_0$ for every contractive Markov system with the Feller property?
3) Can every equilibrium state for $u$ be obtained as a normalized $\Phi(\nu'_0)$ (observe that $\Lambda(D) = 1$ if $\Lambda$ is an equilibrium state for $u$)?

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Erratum: Dynamically defined measures and equilibrium states

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(Dated: 20 June 2012)

The purpose of this note is to correct some errors which the author found in the proofs of Theorem 2 and Lemma 9 in\textsuperscript{3}. The error in the proof of Theorem 2 seems to be fatal. In any case, the theorem is not that simple consequence of the construction of $\Phi$. However, Lemma 9 and all the results which follow from it still can be proved under apparently stronger assumptions. These assumptions are still strictly weaker than the usual assumptions for constructions of Gibbs measures, such as summability of variation of the energy function (see Example 2 in \textsuperscript{3} and Example 3 in\textsuperscript{2}). In particular, the assertion of the paper that the proposed measure-theoretic technique allows to construct equilibrium states in the thermodynamic sense, which minimise the free energy of the dynamical system, for some energy functions which are not upper semi-continuous is still correct.

Lemma 9 and Theorem 3 in\textsuperscript{3} can be replaced with the following lemma.

**Lemma 1** Let $e \in E$. Suppose $\mathcal{M}$ is contractive with $P^0_x \ll P^0_y$ for all $x,y \in K_i$ and $1 \leq i \leq N$, and there exists $\mu \in P(K)$ such that $U^*\mu = \mu$. Suppose each $K_i$ is open. Let $S := \{i | \mu(K_i) > 0\}$ and $\nu_0 := 1/|S| \sum_{i \in S} \delta_{x_i}$. Then

$$E_{\Phi} (\nu_0) (1_{[e]}|\mathcal{F}) = p_e \circ F \Phi(\nu_0) \text{-a.e..} \quad (1)$$

**Proof.** By the hypothesis and Proposition 4 (ii) in\textsuperscript{3}, $\Phi(\nu_0)(S) > 0$. Let $A \in \mathcal{A}_0$ be such that $\phi_0(\mu)(A) = 0$. Then $P^0_x(A) = 0$ for $\mu$-a.e. $x \in K_i$ with $i \in S$. Hence, by the hypothesis, $P^0_x(A) = 0$ for all $x \in S$. Therefore, $\phi_0(\nu_0)(A) = 0$. Thus $\phi_0(\nu_0) \ll \phi_0(\mu)$. Hence, by Lemma 2 (ii) in\textsuperscript{1}, $\Phi(\nu_0) \ll \Phi(\mu)$. It is a well know fact that for any positive shift-invariant Borel measures $\Lambda$ and $M$, absolute continuity relation $\Lambda \ll M$ implies that $E_M(1_{[e]}|\mathcal{F}) = E_\Lambda(1_{[e]}|\mathcal{F})$ $\Lambda$-a.e., (as the shift-invariance of the measures implies that $d\Lambda/dM \circ S = d\Lambda/dM$ $M$-a.e., where $d\Lambda/dM$ is the Radon-Nikodym derivative, which in turn implies that $E_M(d\Lambda/dM|\mathcal{F}) \circ S = E_M(d\Lambda/dM|\mathcal{F})$ $M$-a.e., and this implies the fact). Therefore,

$$E_{\Phi} (\nu_0) (1_{[e]}|\mathcal{F}) = E_{\Phi(\mu)} (1_{[e]}|\mathcal{F}) \Phi(\nu_0) \text{-a.e..}$$

Furthermore, by Lemma 6 in\textsuperscript{2} or Theorem 1 (iii) in\textsuperscript{4},

$$E_{\Phi(\mu)} (1_{[e]}|\mathcal{F}) = p_e \circ F \Phi(\mu) \text{-a.e.}$$

The error in the proof of Theorem 2 seems to be fatal. In any case, the theorem is not that simple consequence of the construction of $\Phi$. However, Lemma 9 and all the results which follow from it still can be proved under apparently stronger assumptions. These assumptions are still strictly weaker than the usual assumptions for constructions of Gibbs measures, such as summability of variation of the energy function (see Example 2 in\textsuperscript{3} and Example 3 in\textsuperscript{2}). In particular, the assertion of the paper that the proposed measure-theoretic technique allows to construct equilibrium states in the thermodynamic sense, which minimise the free energy of the dynamical system, for some energy functions which are not upper semi-continuous is still correct.

Lemma 9 and Theorem 3 in\textsuperscript{3} can be replaced with the following lemma.

**Proof.** The proof is the same as that of Corollary 1 in\textsuperscript{3} with the only difference that instead of referring to Theorem 3 in\textsuperscript{3} we refer to Lemma 1 above.

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