An effective potential for electron-nucleus scattering in neutrino-pair bremsstrahlung in neutron star crust

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Abstract. We derive an analytic approximation for the emissivity of neutrino-pair bremsstrahlung (NPB) due to scattering of electrons by atomic nuclei in a neutron star (NS) crust of any realistic composition. The emissivity is expressed through generalized Coulomb logarithm by introducing an effective potential of electron-nucleus scattering. In addition, we study the conditions at which NPB in the crust is affected by strong magnetic fields and outline the main effects of the fields on neutrino emission in NSs. The results can be used for modelling of many phenomena in NSs, such as cooling of young isolated NSs, thermal relaxation of accreting NSs with overheated crust in soft X-ray transients and evolution of magnetars.

1. Introduction

It is well known that studies of thermal evolution of NSs allow one to explore the physics of superdense matter in NS interiors [1]. The thermal evolution is mainly regulated by neutrino emission from various NS layers [2]. Here we focus on the important neutrino emission mechanism in the NS crust, which is the emission of neutrino pairs (of any flavors) in collisions of electrons ($e$) with atomic nuclei ($A; Z$), called also NPB,

$$e + (A, Z) \rightarrow e + (A, Z) + \nu + \bar{\nu}. \quad (1)$$

It has been extensively studied as reviewed in [2].

We employ the formalism of [3] which includes the most complete collection of physical effects important in NPB and allows one to calculate the emissivity of the process $Q \text{[erg s}^{-1} \text{cm}^{-3}]$ for any composition of the NS crust. To use these results in modeling of NS phenomena, it is convenient to have an analytic fit of $Q$. The authors of [3] fitted $Q$ for the ground-state crust. Here we calculate $Q$ for different compositions and obtain the required fit. In addition, we investigate the conditions at which the NPB is affected by magnetic fields. More extended version of this investigation is published elsewhere [4].

2. Formalism

Let us analyze the NPB in the crust of an NS. Under the crust we mean the envelope of the star which contains atomic nuclei; it extends [1] to the density $\rho \approx 1.5 \times 10^{14} \text{ g cm}^{-3}$. The nuclei are crystallized or form Coulomb liquid. At any value of $\rho$ the matter is assumed to contain spherical nuclei of one species. The nuclei are immersed in the sea of electrons, and at densities
higher than the neutron drip density \( \rho_{ND} \approx (4 - 6) \times 10^{11} \text{ g cm}^{-3} \) [2], also in the sea of free neutrons.

Following [3] we consider the case of ultrarelativistic and strongly degenerate electrons. Then \( p_F \gg m_e c \), that is \( \rho \gg 10^{6} \text{ g cm}^{-3} \), where \( p_F = \hbar (3\pi^2 n_e)^{1/3} \) is the electron Fermi momentum, \( m_e \) the electron rest-mass, and \( n_e \) is the electron number density. The electrons are degenerate at \( c p_F \gg k_B T \), \( T \) being the temperature and \( k_B \) the Boltzmann constant. The electrons form nearly ideal Fermi gas and the nuclei (ions) are fully ionized. The electric neutrality of the matter implies \( n_e = Z n_i \), where \( n_i \) is the number density of the nuclei, and \( Z \) is the nucleus charge number. Let us introduce also \( A_{tot} \), the total number of nucleons per one nucleus, and \( A = A_{nuc} \), the total number of nucleons confined in one nucleus. In the outer crust (\( \rho < \rho_{ND} \)) we have \( A_{tot} = A \), while in the inner crust (\( \rho \geq \rho_{ND} \)) one gets \( A_{tot} > A \). The mass density of the matter in the crust is \( \rho \approx m_u A_{tot} n_i \), where \( m_u \) is the atomic mass unit.

Let us introduce the dimensionless parameters, \( x = \frac{p_F}{m_e c}, \quad \Gamma = \frac{Z^2 e^2}{k_B T a}, \quad \tau = \frac{T}{T_{pl}}, \quad \xi = \frac{R_p p_F}{h} \),

\[ Q = 5.362 \times 10^{15} Z^2 (n_i/10^{34} \text{ g cm}^{-3}) (T/10^9 \text{ K})^6 LR_{NB} \text{erg cm}^{-3} \text{s}^{-1}. \]

Here \( R_{NB} = 1 + 0.00554 Z + 0.0000737 Z^2 \) is an approximate non-Born correction. Furthermore, \( L \) is the generalized Coulomb logarithm to be determined; \( L = L_{liq} \) in the liquid phase; \( L = L_{ph} + L_{sl} \) in the solid phase, where \( L_{ph} \) is the contribution of electron-phonon scattering, and \( L_{sl} \) is the contribution of Bragg diffraction of electrons on the static lattice. For liquid ions,

\[ L_{liq} = \int_{0}^{1} S_{liq}(q)|V(q)|^2 R_c(y) y^3 dy, \]

where \( hq \) is the electron momentum transfer in a reaction event and \( y = hq/(2p_F) \). The function \( S_{liq}(q) \) is the ion structure factor in the Coulomb liquid [5]; \( R_c(y) = 1 + 2y^2 (\ln y)/(1 - y^2) \) comes from the squared matrix element; \( V(q) = F(q)/(y^2 + y_0^2) \) is the Fourier transform of the Coulomb potential screened by electron polarization (included into \( y_0 \) [3]); \( F(q) \) is the nuclear form factor which takes into account proton charge distribution within the nucleus [3].

Let us consider this charge distribution as uniform. Then \( F(q) = F(u) = 3(\sin u - u \cos u)/u^3, \quad u = 2\xi y \). At \( \rho \lesssim 10^{12} \text{ g cm}^{-3} \) one typically has \( R_p \ll a \), and finite sizes of atomic nuclei are unimportant (\( \xi \to 0, \quad F(q) = 1 \)). At \( 10^{12} \lesssim \rho \lesssim 10^{13} \text{ g cm}^{-3} \) the approximation of uniformly charged proton core works well but at higher \( \rho \) it breaks down [1]. However, in this case one can take realistic proton charge distribution, calculate its root-mean-square (rms) value and use this value instead of \( R_p \) in the formulas obtained formally for the uniform charge distribution [6].

The expression for \( L_{ph} \) is

\[ L_{ph} = \int_{(4Z)^{-1/3}}^{1} S_{ph}(q)|V(q)|^2 R_c(y) y^3 dy. \]
An appropriate effective structure factor \(S_{\text{ph}}\) is obtained in [7] and takes into account multiphonon processes, \(S_{\text{ph}} = \exp(w_1y^2) - 1\) \(\exp(-wy^2)\),
\[
 w_1 = (12\pi^2)^{1/3} \frac{Z^{2/3}u_{-2}}{\Gamma} \frac{b\tau}{\sqrt{(b\tau)^2 + u_{-2}^2 \exp(-7.6\tau)}},
\]
(6)
\[
 w = (12\pi^2)^{1/3} \frac{Z^{2/3}u_{-2}}{\Gamma} \left(1 + \frac{u_{-1}}{2u_{-2}} \exp(-9.1\tau)\right),
\]
(7)
where \(b = 231\), \(u_{-1} = 2.798\), and \(u_{-2} = 12.972\). The crystal is assumed to have the body centered cubic structure, but the results are almost insensitive to the lattice type [3]. Finally,
\[
 L_{\text{fi}} = \frac{1}{12Z} \sum_{\mathbf{K} \neq 0} (1 - y^2)y^2|V(K)|^2I(t\nu, y) \exp(-wy^2),
\]
(8)
where \(K\) is a reciprocal lattice vector, \(y = \hbar|\mathbf{K}|/(2p\nu)\),
\[
 \frac{1}{t\nu} = \frac{\Gamma}{Z} \left(\frac{4}{3\pi Z}\right)^{2/3} \sqrt{1 - y^2}|V(K)| \exp(-wy^2).
\]
(9)
The function \(I(t\nu, y)\) was analyzed in [3]. It accounts for the electron band structure effects [8].

Note that the approach [3] neglects the quantum effects of ion motion in the liquid state \(S_{\text{liq}}(q)\) in (4) is classical, independent of \(\tau\). Note also that in our case \(L\) is explicitly independent of \(x\). We will calculate and fit the Coulomb logarithm which can be presented as a function of four arguments, \(L = L(Z, \Gamma, \tau, \xi)\).

### 3. Coulomb logarithm and effective potential

Our fit of \(L\) will use the concept of effective potential for the electron-nucleus scattering in NPB. Similar effective potentials have been introduced by A. Y. Potekhin to fit Coulomb logarithms [6] for electric and thermal conductivities of electrons due to electron-nucleus scattering.

In our case \(V_{\text{eff}}\) is introduced as
\[
 L = \frac{1}{0} |V_{\text{eff}}(y)|^2 \mathcal{R}_c(y)y^3 dy, \quad |V_{\text{eff}}|^2 = S_{\text{eff}}(y) \left|\frac{\mathcal{F}(2\xi y)}{y^2}\right|^2,
\]
(10)
where \(S_{\text{eff}}\) is the effective structure factor which we present in the form \(S_{\text{eff}} = \exp(w_{\text{eff}}y^2) - 1\) \(\exp(-w_{\text{eff}}y^2)\). Then \(w_{\text{eff}}\) and \(w_{\text{eff}}\) can be taken similar to (6) and (7),
\[
 w_{\text{eff}} = B \frac{br_{\ast}}{\sqrt{(b\tau_{\ast})^2 + u_{-2}^2 \exp(-7.6(b\tau_{\ast} + 18/\Gamma_{\ast}))}}
\]
(11)
\[
 w_{\ast} = B \left[1 + \frac{u_{-1}}{2u_{-2}\tau_{\ast}} \exp\left(-9.1\left\{\tau_{\ast} + \frac{126}{\Gamma_{\ast}}\right\}\right)\right],
\]
(12)
\[
 B = 3 \frac{(12\pi^2)^{1/3}u_{-2}Z^{4/5}}{5 + (\xi/Z)\Gamma_{\ast}/(200 + \Gamma_{\ast})} \left(\frac{\Gamma_{\ast}^2 + 1037}{(1 + \Gamma_{\ast}/204)^4} \frac{\Gamma_{\ast}^{(1+0.15\xi)/2}}{Z^{0.10}}\right)^{-1/2}.
\]
(13)

Here \(\tau_{\ast}\) and \(\Gamma_{\ast}\) are rescaled \(\tau\) and \(\Gamma\), respectively. They are related to original \(\Gamma\), \(\tau\), \(\xi\) and \(Z\) as
\[
 \tau_{\ast} = 0.095 \frac{2\tau}{0.095\sqrt{\tau^2 + 4}}^{1/\Lambda}, \quad \Lambda = 1 + \frac{(Z/35.5)^2}{1 + \Gamma/(223Z)},
\]
(14)
Figure 1: Density dependence of NPB emissivity $Q$ for ground-state (G) and accreted (A) crusts of neutron star at seven temperatures, $T = 10^7.75$, $10^8$, ..., $10^9.25$ K. Lines are numerical values, while symbols are our fits. At $\rho \gtrsim 10^{13}$ g cm$^{-3}$ the smooth composition model of ground-state crust (G-S) is used.

\[
\Gamma_* = \Gamma \exp \left\{ -\frac{\tau_*^{0.053} (Z/11.3)^{4/9}}{1 + \tau_* G [\Gamma/(19.3 Z^{1.7})]^H} \right\},
\]

(15)

with $G = 0.5 + 0.002 Z \xi$ and $H = [1.52 + 0.9(1 - \tau_*)]/(1 + 0.5 \xi)$.

For an analytic integration of (10) we approximate $R_c$ as $R_c(y) \approx 1 - y$ (with the maximum relative error of 0.08 at $y = 0.3$ over $0 \leq y \leq 1$) and $|F(w)|^2 \approx \exp(-\alpha u^2)$ with $\alpha = 0.23$ (maximum absolute error of 0.032 at $u = 1.6$ for any $u \geq 0$). Then

\[
L = f \left( w^* + 4\alpha \xi^2 \right) - f \left( w^* - w_r^* + 4\alpha \xi^2 \right), \quad f(x) = \frac{1}{2} \left[ \sqrt{\frac{\pi}{x}} \text{erf}(\sqrt{x}) + \ln(x) - \text{Ei}(-x) \right],
\]

(16)
erf$(x)$ and Ei$(x)$ being the error function and exponential integral, respectively; $f(x) = 0.71139 + x/6$ as $x \to 0$. Note that $L_{ph}$ roughly reproduces the total $L$ [4]; thus $L_{ph}$ is basic for our fit. Modified $w^*$ and $w_r^*$ accurately reproduce $L_{liq}$, while $\Gamma_*$ and $\tau_*$ tune the fit of $L_{liq}$.

The fit contains a number of coefficients determined by comparing with the calculations. The calculations have been done on the wide grid of the parameters: $Z = 6, \ldots, 50$ (10 points), $A = 1.9 Z, \ldots, 3 Z$ (10 points), $R_p = (0.0, \ldots, 0.2) \times a$ (10 points), $\log n_i [\text{cm}^{-3}] = 29, \ldots, 34$ (15 points), and $\log T [K] = 8, \ldots, 9$ (15 points). We have excluded some points as described in [4]. After that the fit has been done on 172550 grid points. The rms relative fit error of $L$ is about 8%, with the maximum error $\sim 50\%$ at $Z = 11$, $A = 21$, $n_i = 10^{34}$ cm$^{-3}$, $T = 10^8$ K, and $R_p = 0.022 a$, which is sufficient for applications.

Fig. 1 displays $Q$ versus $\rho$ in the NS crust at seven temperatures, $T = 10^7.75$, $10^8$, ..., $10^9.25$ K. We use two models of the crust, the ground-state and accreted ones [1, 9]. The numerical values of $Q$ for the ground-state crust are shown by the long-dashed lines, and for the accreted crust by the short-dashed lines. The fits are plotted by filled dots and triangles, respectively. The accreted crust is composed of lighter nuclei with lower $Z$ leading to weaker NPB. Slight jumps of the calculated and fitted $Q$ values at some densities are associated with the change of nuclides in dense matter with growing $\rho$ [1]. The fits reproduce calculated $Q$ values quite well.

Figure 2: Density-temperature diagram for the ground-state NS crust. The densely shaded region is expected to be weakly affected by the magnetic field $B = 10^{10} G$, while slightly shaded region by the field $B = 10^{15} G$ (see text for details).
At $\rho \gtrsim 10^{13}$ g cm$^{-3}$ the accreted crust is almost indistinguishable from the ground-state one, and we do not plot the data for the accreted crust at higher densities. Note that the NPB for the dense ground-state crust starts to depend on the proton density profiles within the nuclei (see above). To demonstrate the quality of our fits in this regime we use the smooth-composition model of the ground-state crust at $\rho \gtrsim 10^{13}$ g cm$^{-3}$ (the solid lines), calculate the rms proton core radii and use them in the fits (squares). Again, the fit quality remains good.

4. Effects of magnetic fields on NPB

Many NSs possess strong magnetic fields which can affect the NPB. Accurate consideration of NPB in a $B$-field is still not done. Here we formulate the conditions at which $B$-fields can modify the NPB (through electrons and atomic nuclei).

A field $B$ changes the motion of electrons because of Landau quantization of electron states; e.g. [1]. In our case of strongly degenerate relativistic electrons the importance of magnetic effects is mostly determined by the characteristic density $\rho_B \approx 7045(A_{\text{tot}}/Z)(B/10^{12}$ G)$^{3/2}$ g cm$^{-3}$. At $\rho \lesssim \rho_B$ the electrons occupy the ground Landau level while at $\rho \gg \rho_B$ they populate many Landau levels. Accordingly, we expect that the field $B$ strongly affects the NPB at $\rho \lesssim \rho_B$.

Strong magnetic fields modify vibration properties (phonon modes) of Coulomb crystals and, hence, influence the NPB. Phonon modes of magnetized Coulomb crystals have been studied in a number of works, e.g. [10, 11] and references therein. Contributions of various modes to physical properties of matter are affected by the fields in certain ranges of $\rho$ and $T$ (see Fig. 6 of [11], where $B$-fields modify phonon heat capacity). Magnetic field effects are mostly pronounced at $T \ll T_B = h\omega_B/k_B$, where $\omega_B = ZeB/(A_{\text{nuc}}n_Bc)$ is the ion cyclotron frequency. It is natural to expect that at $T \gtrsim T_B$ the NPB is not affected by $B$-fields through the ion motion.

Fig. 2 shows possible ranges of $\rho$ and $T$ in the NS crust where magnetic fields can influence the NPB for the smooth composition model of the ground-state matter. We expect that the densely shaded region is weakly affected by the magnetic $B = 10^{16}$ G, whereas the slightly shaded region is weakly affected by the field $B = 10^{15}$ G. Any of these two regions is bounded by the density $\rho_B$ and by the temperature $T_B$. Hence, the fields $B \lesssim 10^{15}$ G have practically no effects on the NPB in the NS crust under formulated conditions (see above), while higher fields may affect the NPB, especially at low densities. In addition, in Fig. 2 we plot the neutron drip density $\rho_{\text{ND}} \approx 4.3 \times 10^{14}$ g cm$^{-3}$, the melting temperature $T_m$ of the crystal and the ion plasma temperature $T_{\text{pi}}$ for the ground state matter neglecting the effects of magnetic fields.

5. Neutrinos and magnetic fields

Based on [2] let us outline general effects of $B$-fields on neutrino emission of NSs. There are many neutrino emission mechanisms there. In addition, the $B$-fields introduce new, specific mechanisms, like synchrotron emission of neutrino-pairs by electrons. Some effects of $B$-fields are almost unexplored (e.g., on plasmon decay into neutrino pairs). The strongest effects of $B$-fields occur in the low-density matter, first of all in the outer NS crust, while deeper layers are affected by magnetic fields (like $B \lesssim 10^{14}$ G) much weaker.

Higher $B$ influence the neutrino emission in deeper layers, in the inner crust and the NS core. To modify the strongest neutrino mechanism, the direct Urca process (which can be open at high densities in the inner NS core), one typically needs $B \sim 10^{16}$ G. At this $B$ charged particles in the core populate many Landau levels, and the main $B$-field effect will be to broaden the direct Urca threshold [12]. One needs superstrong $B \gtrsim 10^{18}$ G for the particles in the NS core to occupy one or a few Landau levels, but such fields are close to the maximum fields which NSs can contain [1]. The direct Urca process in the extreme case of only one populated Landau level is well studied [12]. If $B$ is lower and more Landau levels are occupied, the direct Urca emissivity will generally tend to the $B = 0$ case but the broadening of the direct Urca threshold
will be strong. In addition, neutrino emission in NS cores is strongly affected by superfluidity and superconductivity of NS matter; such effects are superimposed by the effects of $B$-fields which further complicates studies of neutrino emission of NSs.

6. Conclusions
Using the formalism [3], which includes rich spectrum of physical effects, we have derived a universal fit for the NPB emissivity $Q$ in an NS crust. We have expressed the Coulomb logarithm $L$ which determines $Q$ using the method of effective potential $V_{\text{eff}}$.

Our fit is valid for any realistic composition of the NS crust, in wide ranges of parameters, particularly, at densities from about $10^8$ g cm$^{-3}$ to the crust bottom and at temperatures from about a few times of $10^7$ K to a few times of $10^9$ K. It is valid for the nuclei with $6 \lesssim Z \lesssim 50$ and $1.9Z \lesssim A_{\text{nucl}} \lesssim 3Z$. The fit is convenient for using in computer codes which simulate thermal evolution of NSs. The fit assumes the presence of nuclei of one type at any values of $\rho$ and $T$. For a multicomponent plasma one can employ the approach of mean nucleus (mean ion), e.g. [1].

Our fit neglects the effects of $B$-fields but we have studied its applicability in magnetized NSs and we have also outlined general effects of magnetic fields on neutrino emission processes in NSs.

As the leading neutrino process in a NS crust, NPB is important for modelling transient phenomena in warm NSs, particularly, thermal relaxation in young (age 10–100 yr) isolated NSs [13, 6] and in accreting NSs with overheated crust in soft X-ray transients (e.g. [14, 15]). The NPB can also be important in X-ray superbursts in accreting NSs (e.g. [16]), and in thermal evolution of magnetars [17], as well as in massive cooling white dwarfs.

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