Nonlinear Electro-Hydrodynamics of Liquid Crystals

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Abstract—We present nonlinear dynamic equations for nematic and smectic A liquid crystals in the presence of an alternating electric field and explain their derivation in detail. The local electric field acting in any liquid-crystalline system is expressed as a sum of external electric field, the fields originating from feedback of liquid crystal order parameter, and a field, created by charged impurities. The system tends to decrease the total electric field, because it lowers the energy density. This basically nonlinear problem is not a pure academic interest. In the realm of liquid crystals and their applications, utilized nowadays modern experimental techniques have progressed to the point where even small deviations from the linear behavior can be observed and measured with a high accuracy. We follow hydrodynamic approach which is the macroscopic description of condensed matter systems in the low frequency and long wavelength limit. Nonlinear hydrodynamic equations are well established to describe simple fluids. Similar approaches (with degrees of freedom related to the broken orientational or translational symmetry included) have been used also for liquid crystals. However to study behavior of strongly perturbed (well above the thresholds of various electro-hydrodynamic instabilities) liquid crystals, the nonlinear equations should include soft electromagnetic degrees of freedom as well. There are many examples of such instabilities, e.g., classical Carr-Helfrich instability triggered by the competitive electric and viscous torques, flexoelectric instability, and so one. Therefore the self-consistent derivation of the complete set of the nonlinear electro-hydrodynamic equations for liquid crystals became an actual task. The aim of our work is to present these equations, which is a mandatory step to handle any nonlinear phenomenon in liquid crystals.

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1. PRELIMINARIES

Nonlinear phenomena in condensed matter, and, particularly, in liquid crystals, became one of the hottest topics in physics. Progress in experimental techniques and their accuracy led to a number of new and exciting results still waiting for the theoretical analysis (or at least rationalization). To name a few such results, it is worth to noting three breakthrough publications [1–3] (see also triggered by these works publications of other groups, where similar phenomena have been observed in different kinds of liquid crystals and under different experimental conditions [4–9]). In these works, the authors presented the evidence of localized and propagating excitations in liquid crystalline materials in an external alternating electric field. For these self-trapped wave excitations, exhibiting particle-like features system nonlinearity plays a key role, balancing excitations broadening (see, e.g., [10, 11] for Hamiltonian systems and [12–14] for driven and damped (dissipative) non-equilibrium systems).

Motivated by these works we focus in our manuscript on uniaxially symmetric types of liquid crystals, namely nematics (N) and smectic A (SmA) liquid crystals, which are ideal representative systems to study nonlinear phenomena. Nematics (nematic liquid crystals) [15] are anisotropic fluids with a long-range orientational order defined by the unit headless vector, director $\mathbf{n}$. In smectics A this uniaxial orientational anisotropy is supplemented by broken in one dimension translational symmetry. The starting point to investigate theoretically dynamic phenomena in liquid crystals is to derive the nonlinear dynamic equations. The complete set of the dynamic equations for any system should include all soft (hydrodynamic) variables related to the general conservation laws and to symmetry breaking. For sufficiently low system conductivity $\sigma$, what is usually the case for liquid crystals, soft (slow for small $\sigma$) electromagnetic degrees of freedom also should be included in consideration. We did not find in the literature consistent derivation of such general nonlinear electro-hydrodynamic equations for liquid...
crystals. However it is worth to noting two important works [16, 17], where electromagnetic degrees of freedom, were included into consideration to study dynamics of magnetic systems [16], and for quantum gravity problems [17]. The aid of our work is to present these equations which is a mandatory step to handle any nonlinear phenomenon in liquid crystals (see examples of problems, from about 50 years ago [18] and until very recent time, e.g., [19]).

For the derivation of the equations we employ the universal Poisson brackets method [20–22] permitting us to write out directly nonlinear non-dissipative dynamic equations automatically satisfying the conservation laws. Then the dissipative terms can be added based on the general thermodynamic relations providing positive entropy production, including besides orientational elastic energy, hydrodynamic motion, electric conductivity (nematics are typically weak electrolytes) and flexoelectric polarization. The obtained set of equations is not only of its own intellectual interest, but also of relevance to study many nonlinear phenomena in liquid crystals. To avoid too cumbersome expressions with too many unknown phenomenological parameters in this publication we restrict ourselves to only nematic and smectic $A$ liquid crystals. Thanks to their high symmetry (with an infinite order rotation axis) the equations can be represented in a relatively compact form. However if needed the equations can be formulated for less symmetric types of liquid crystals (e.g., for biaxial nematics or tilted smectic C liquid crystals).

Here we derive the general hydrodynamic equations for a nematic or a smectic-$A$ sample in an external electromagnetic field. The nonlinear dynamic equations of nematics and smectics-$A$ are known (see, e.g., [20, 21, 23]). The linear theory of the low-frequency electro-hydrodynamic instability also was developed long ago [24]. However, a consistent nonlinear hydrodynamic theory with both reactive and dissipative (irreversible) terms, that includes soft degrees of freedom of the electromagnetic field, is absent. This lacuna in the theory description exists in spite of many relevant works in the field [25–30]. The matter is that in these publications the authors derived nonlinear dynamic equations for nematics or smectics-$A$ by writing out all terms of a given order allowed by the symmetry, and then performing selection of the terms satisfying the conservation laws. The problem with such method is that at each step taking into account next-order nonlinear terms, one has to repeat again the procedure and to check again that all conservation laws are satisfied.

The structure of the paper is as follows. In Section 2 we illustrate our approach, deriving well known equations of the electromagnetic field in an isotropic continuous medium. In Section 3 we derive non-linear equations of the nematics in the presence of an alternating electromagnetic field. In Section 4 we derive non-linear equations of the smectic $A$ in the presence of an alternating electromagnetic field. In Section 5 we sum up the results of our investigation. In Appendix A we derive the expressions of the Poisson brackets for vector fields. In Appendix B we examine the role of the finite velocity of the medium in the electromagnetic effects.

2. ELECTROMAGNETIC FIELD

The main goal of our article is to derive a complete set of nonlinear dynamic equations of nematics in the external alternative electromagnetic field. To illustrate main features of our approach, we give the derivation of the well-known equations of the electromagnetic field in an isotropic continuous medium. We use the approximation of weak conductor, where the polarizability of the medium can compete with the conductivity at frequencies interested for us. Thus both, polarizability of the medium and its conductivity are taken into account. The polarizability can be introduced within thermodynamic approach, whereas the conductivity is a kinetic phenomenon.

The electromagnetic field in a continuous medium is governed by Maxwell equations

$$\partial_t \mathbf{D} = c \nabla \times \mathbf{H},$$  \hspace{1cm} (1)

$$\partial_t \mathbf{B} = -c \nabla \times \mathbf{E}.$$  \hspace{1cm} (2)

Here $\mathbf{E}$ is electric field, $\mathbf{B}$ is magnetic induction field, $\mathbf{D}$ is displacement field, $\mathbf{H}$ is magnetic field, and $c$ is speed of light. The electric field $\mathbf{E}$ and the magnetic induction field $\mathbf{B}$ are expressed via the vector potential of the electromagnetic field. Equation (1) reflects internal electromagnetic processes in the medium.

Equations (1), (2) have to be supplemented by conditions for divergencies of $\mathbf{D}$, and $\mathbf{B}$. We assume

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \cdot \mathbf{B} = 0.$$  \hspace{1cm} (3)

The relation $\nabla \cdot \mathbf{B} = 0$ is exact (valid for any electromagnetic field) and the relation $\nabla \cdot \mathbf{D} = 0$ implies electro-neutrality of the medium, that is the charge density in the liquid crystal is assumed to be equal to zero. Both conditions (3) are, obviously, consistent with the Maxwell Eqs. (1), (2).

To close the system of Eqs. (1), (2) one should relate $\mathbf{D}$, $\mathbf{H}$ to $\mathbf{E}$, $\mathbf{B}$. This deceptively simple exercise required some care.

First of all, we consider an ideal dielectric, where the dissipative effects (like electrical current) can be neglected. Such dielectrics is a subject of thermodynamic approach. The thermodynamic identity for the dielectric is written in terms of the density of the internal energy $U$ [31]:

$$dU = \frac{1}{4\pi} \mathbf{E} \cdot \mathbf{D}_0 + \frac{1}{4\pi} \mathbf{H}_0 \cdot \mathbf{B} + \ldots,$$  \hspace{1cm} (4)
where $\mathbf{D}_0$ is displacement field and $\mathbf{H}_0$ is magnetic field taken solely with account of the electric and magnetic polarizability of the medium. The dots in Eq. (4) stay for additional terms related to other thermodynamic quantities like mass density $\rho$. In the quadratic approximation the relation (4) leads to

$$\mathbf{D}_0 = e\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}_0.$$  

(5)

where $e$ is the electric permittivity coefficient and $\mu$ is the magnetic permeability coefficient.

The dynamic equation for the ideal dielectric can be found by the Poisson brackets method. In terms of the Poisson brackets method dynamic equation for a mechanical variable $\phi$ is [32]

$$\frac{\partial \phi}{\partial t} = \{\mathcal{H}, \phi\},$$  

(6)

where $\mathcal{H}$ stands for Hamiltonian (energy of the system), and $\{\ldots\}$ designate the Poisson brackets. Unlike classical mechanics, variables, describing the continuous medium, are fields, depending on both, time and space coordinates. The equation for the fields has the same form (6). However, Hamiltonian $\mathcal{H}$ of the medium is the integral over space of the energy density $U$ dependent of the fields:

$$\mathcal{H} = \int dV U(\mathbf{D}_0, \mathbf{B}, \ldots),$$  

(7)

where dots stay for other thermodynamic variables. To derive the dynamic equations, one should know expressions for the Poisson brackets for the fields, see [20, 21].

Here we consider the dynamic equations for the electromagnetic field. The simplest way to find the Poisson brackets for $\mathbf{D}_0$, $\mathbf{B}$ is to exploit the expressions of $\mathbf{E}$, $\mathbf{B}$ via the vector potential $\mathbf{A}$. Then with general expressions for the Poisson brackets from Appendix A, one calculates the needed for us Poisson bracket, including electric displacement and magnetic field vectors. The Poisson bracket is

$$\{B_j(\mathbf{r}), D_{0k}(\mathbf{x})\} = 4\pi e_{jmk} \partial_m \delta(\mathbf{r} - \mathbf{x}),$$  

(8)

see Appendix A. Here $e_{jmk}$ is Levi-Civita antisymmetric tensor and derivative $\partial_m \delta(\mathbf{r} - \mathbf{x})$ means $\delta(\mathbf{r} - \mathbf{x})/\partial r_m$. It is important to note that the expression (8) derived for Poisson bracket $\{B_j(\mathbf{r}), D_{0k}(\mathbf{x})\}$ coincides with the expression presented by Volovik in [17]. The Poisson brackets of $\mathbf{B}$, $\mathbf{D}_0$ with other hydrodynamic variables are zero.

Exploiting the expression (8), one immediately finds

$$\partial_v \mathbf{B} = \{\mathcal{H}, \mathbf{B}\} = -e \nabla \times \mathbf{E},$$  

$$\partial_v \mathbf{D}_0 = \{\mathcal{H}, \mathbf{D}_0\} = e \nabla \times \mathbf{H}_0,$$  

(9)

where we used the expression (7) and the thermodynamic identity (4). The same result is obtained in Appendix A, see Eq. (A.27). Equation (9) is Eq. (1), where $\mathbf{D}$, $\mathbf{H}$ are substituted by $\mathbf{D}_0$, $\mathbf{H}_0$, as it should be in an ideal dielectric [31].

Now one should include into consideration the dissipative terms. Equation (2) is exact, and so no dissipative terms can be added to the equation. We include the kinetic term, proportional to $\mathbf{E} = 4\pi \delta \mathcal{H}/\delta \mathbf{D}_0$ to the equation for $\mathbf{D}_0$:

$$\partial_v \mathbf{D}_0 = e \nabla \times \mathbf{H} - 4\pi \sigma \mathbf{E}.$$  

(10)

The kinetic coefficient $\sigma$ is conductivity and the last term in Eq. (10) is current of free charges (with sign minus). Here we ignored the difference between $\mathbf{H}$ and $\mathbf{H}_0$ neglecting the magnetic dissipation. Comparing Eqs. (10) and (1), we conclude

$$\partial_v \mathbf{D} = \partial_v \mathbf{D}_0 + 4\pi \sigma \mathbf{E}.$$  

(11)

It is the standard expression for the displacement field including the polarization term and the current of free charges.

Further we present a generalization of the scheme for liquid crystals, nematics and smectics $A$. Magnetic effects in the liquid crystals are weak. Therefore we take the magnetic permeability coefficient to be equal to unity, $\mu = 1$, and ignore effects related to magnetic dissipation. In this case $\mathbf{H} = \mathbf{B}$, the equality will be implied below. In contrast, the electric polarizability and the electric conductivity play a crucial role in the liquid crystals. Both phenomena should be taken into account to adequately describe the liquid crystal.

3. DYNAMICS OF NEMATICS

We start with the thermodynamic description of nematics based on thermodynamical identities. Considering the nematic in the first approximation as an ideal dielectric, one introduces the density of energy $U$, depending on density of mass $\rho$, density of entropy $S$, density of momentum $\mathbf{J}$, and director $\mathbf{n}$, as well as on the displacement field $\mathbf{D}_0$ and the magnetic induction field $\mathbf{B}$ [31]. The notation $\mathbf{D}_0$ implies that one takes into account solely the contribution to the displacement field, related to polarization of the liquid crystal, as it should be in an ideal dielectric. The contribution to $\mathbf{D}$ related to conductivity (i.e., dissipation) has to be introduced separately, as a kinetic term.

The thermodynamic identity for the nematic in the presence of the electric and magnetic fields is a direct generalization of the thermodynamic identity for a solid dielectric, see [31], it is

$$dU = \frac{1}{4\pi} Ed\mathbf{D}_0 + \frac{1}{4\pi} \mathbf{H}_0 d\mathbf{B} + T dS + \mu d\rho + v d\mathbf{J}$$  

$$+ \frac{\partial U}{\partial \mathbf{n}} d\mathbf{n} + \frac{\partial U}{\partial (\partial \mathbf{n})} d(\partial \mathbf{n}).$$

(12)

Here as above $\mathbf{E}$ is the electric field, $\mathbf{H}_0$ is the magnetic field, $T$ is temperature, $v$ is velocity, and $\mu$ is chemical potential. As above for $\mathbf{D}_0$, the symbol $\mathbf{H}_0$ means that we consider solely the contributions to the
magnetic field related to the magnetic polarization, ignoring the kinetic processes. Note that according to Galilean invariance \( J = \rho v \). The pressure of the system \( p \) is related to \( U \) via the Legendre transform

\[
p = \frac{1}{4\pi} \rho D_\text{e}E + \frac{1}{4\pi} \rho B_\text{H} + \rho \mu + ST + Jv - U. \tag{13}
\]

One obtains from Eq. (13) the following thermodynamic identity for the pressure

\[
\frac{\partial U}{\partial n} - \frac{\partial U}{\partial (\partial n)} d(\partial n, n),
\]

equivalent to the thermodynamic identity (12).

### 3.1. Non-Dissipative Equations

The non-dissipative dynamic equations of any physical system can be constructed using the Poisson brackets method, see [20, 21]. In the framework of the method, the dynamic equation for any field is written as (6) where Hamiltonian \( \mathcal{H} \) is the integral over space (7) of the energy density \( U \) dependent of the fields. For the nematics

\[
\mathcal{H} = \int dVU(D_\text{e}, B, J, S, \rho, n, \partial n).
\tag{15}
\]

The dynamic equations for the fields \( D_\text{e}, B, J, S, \rho, n \) can be written in the form (6) by exploiting the proper expressions for their Poisson brackets.

Following methods of [20–22, 33] we derive the full set of the Poisson brackets needed to study nemato-dynamics in the presence of the electromagnetic field (including the electromagnetic field itself). Since the macroscopic dynamic equations are local, the Poisson brackets for any pairs of hydrodynamic variables have to be proportional to delta-function or its space derivative. The Poisson brackets for the hydrodynamic variables \( \rho, S, J \) are known [20, 21], non-zero Poisson brackets are

\[
\{J_i(r), \rho(x)\} = \rho(r) \partial_i \delta(r - x),
\tag{16}
\]

\[
\{J_i(r), S(x)\} = S(r) \partial_i \delta(r - x),
\tag{17}
\]

\[
\{J_i(r), J_k(x)\} = J_i(r) \partial_k \delta(r - x) + \partial_i \delta(r - x) J_k(x). \tag{18}
\]

Here as above derivatives like \( \partial_i \delta(r - x) \) mean \( \partial \delta(r - x)/\partial r_i \).

For director, we exploit the following Poisson bracket

\[
\{J_i(r), n_k(x)\} = -\partial_i n_k \delta(r - x)
- \partial_i \delta(r - x) \delta^k_{ik} n_j(x), \tag{19}
\]

where \( \delta^k_{ik} = \delta_{ik} - n_i n_k \). The expression is derived for a normalized vector field in Appendix A, see formula (A.17). The Poisson bracket (19) coincides with ones found by Volovik for ideal rods frozen into the fluid flow [34]. Corrections to the model can be introduced in terms of kinetic terms (see Section 3.2).

The Poisson bracket for the electromagnetic field is determined by Eq. (8). The Poisson brackets of \( B, D_\text{e} \) with other variables are zero. It implies, particularly, that we do not include the electromagnetic contribution to the momentum density \( J \).

With the Poisson brackets in hands we obtain the full set of non-dissipative, nonlinear dynamic equations for nematics

\[
\partial_i J_i = \partial_x \{\mathcal{H}, J_i\} = -\partial_i \partial_x v_i - \partial_i p + 1 \frac{1}{4\pi} \partial_i \left( D_{ik} B_k + B_k H_{ik} \right)
- \frac{1}{4\pi c} \partial_i [D_\text{e} \times B],
\tag{20}
\]

\[
\partial_i n_i = \partial_x \{\mathcal{H}, n_i\} = -\nu \partial_i v_i + n_i \delta^k_{ik} \partial_j v_k.
\tag{21}
\]

Remind that \( v = J/\rho \).

Equations (20), (21) have the same form as in absence of the electromagnetic field. The equation for the momentum density can be written as

\[
\partial_t J_i = \partial_x \{\mathcal{H}, J_i\} = -\partial_i \partial_x v_i - \partial_i p + 1 \frac{1}{4\pi} \partial_i \left( D_{ik} B_k + B_k H_{ik} \right)
- \frac{1}{4\pi c} \partial_i [D_\text{e} \times B],
\tag{22}
\]

where

\[
\Xi_i = \partial_x \{\mathcal{H}, n_i\} = \partial_x \{\mathcal{H}, \rho\} - \partial_x \{\mathcal{H}, \mu\} - \partial_x \{\mathcal{H}, TS + Jv\} - \partial_x \{\mathcal{H}, \mu\}.
\tag{23}
\]

it is (up to sign) the variational derivative of the energy over \( n \).

The last term in Eq. (22), derived by using Eqs. (2), (3), (9), is merely (with the sign minus) time derivative of the momentum density of the electromagnetic field (see Appendix A). Equations (20)–(22) lead to the energy conservation law

\[
\partial_t U = -\nabla Q^* = - \frac{c}{4\pi} \nabla [E \times H]
- \nabla \left[ \frac{\partial U}{\partial (\nabla n_i)} (\nu \partial_i v_i - n_k \delta^k_{ik} \partial_j v_j) \right]
- \partial_i \{\rho \mu + TS + Jv \} v_i
- \partial_i \{\mathcal{H}, n_i \delta^k_{ik} \partial_x v_k \},
\tag{24}
\]

where the energy flow density is designated by \( Q^* \).

We consider relatively small frequencies and non-relativistic hydrodynamic motions. In this situation the induced magnetic field is weak, one can say, that \( H \) is small over parameter \( v/c \). However, the magnetic field cannot be excluded from the equations. The reason is that there is the contribution to the equation for \( U \) (24) related to the electromagnetic energy flow, its density is determined by Poynting vector

\[
S = \frac{c}{4\pi} E \times H. \tag{25}
\]
Thus though the magnetic field has the smallness $v/c$, in the expression for electromagnetic energy flow (25) it is compensated by the factor $c$.

3.2. Kinetic Terms

We have derived the dynamic equations using the Poisson brackets method. In accordance with the general ideology, the equations should be supplemented by the kinetic terms. The scheme of constructing the terms based on the fact that these terms are proportional to the variational derivatives of the energy over the dynamic variables (or their derivatives) with some coefficients (kinetic coefficients). The kinetic coefficients have to satisfy Onsager symmetry and to lead to positive entropy production. Following this approach for our case (nematic liquid crystals) it is necessary to add kinetic terms to the obtained above dynamic Eqs. (9), (20)–(22). The phenomenological coefficients in front of these contributions are known as kinetic coefficients.

Presented below kinetic contributions into electrohydrodynamic equations of nematic liquid crystals are introduced separately in the literature. For the director $n$ and hydrodynamic degrees of freedom they are presented in [15, 21, 25]. Contributions for flexoelectric effect and finite conductivity but without hydrodynamic equations of nematic liquid crystals are written in [27, 28, 35], and for the linearized electro-hydrodynamic equations in [36]. We collect all the kinetic terms together.

The equations including dissipative terms can be found by adding of the following dissipative contributions to Eqs. (9), (19), (20)

$$\partial_t n_i = \{H, n_i\} + \frac{1}{\gamma} \frac{1}{2} \nabla \cdot \boldsymbol{v} + \left(1 - \frac{\lambda}{2}\right) \left(\sigma^{\perp}_{ik} \delta_{ik}^t + \sigma^{\perp}_{ij} \delta_{i}^t n_j\right) \partial_t \nu_j,$$

$$\partial_t D_{ik} = c e_{iknm} \partial_k H_n - 4 \pi \sigma_{ik} n_m E_k,$$

$$\partial_t J_i = \{H, J_i\} + \partial_k \left(\eta^{\perp}_{iknm} \partial^t_n \nu_m\right) + \partial_k \left[\frac{1 - \lambda}{2} n_{ik} \delta_{ik}^t + n_{ik} \delta_{i}^t \sigma^{\perp}_{ik}\right] \nu_j,$$

$$\partial_t S = \{H, S\} + \partial_t \left(\frac{\kappa_{ik}}{\gamma} \partial_k T\right) + \frac{R}{\gamma},$$

$$\partial_t U = -\nabla \cdot \boldsymbol{Q}^{(c)} + \partial_t \left(\frac{\sigma_{ik}}{\gamma} \partial_k T\right) + \frac{1}{\gamma} \nabla \cdot \boldsymbol{v} + \partial_t \left(\frac{v \eta^{\perp}_{iknm} \partial^t_n \nu_m\right)$$

$$-\nabla \cdot \left[\frac{\partial_t U}{\nabla^2} \right] + \partial_t \left[\frac{1 - \lambda}{2} n_{ik} \delta_{ik}^t + n_{ik} \delta_{i}^t \sigma^{\perp}_{ik}\right] \nu_j,$$

where

$$R = \frac{\kappa_{ik}}{T} \partial_i T \partial_k T \frac{1}{\gamma} \frac{1}{2} \eta^{\perp}_{iknm} \partial^t_n \nu_m + \sigma_{ik} E_k, \quad (31)$$

Here $\sigma_{ik}$ is the electric conductivity tensor, $\kappa_{ik}$ is the thermal conductivity tensor, $\eta^{\perp}_{iknm}$ is the viscosity tensor, $\gamma$ is the coefficient of the rotational viscosity and $\lambda$ is some kinetic coefficient.

Onsager symmetry leads to the conclusions, that the tensors $\sigma_{ik}$ and $\kappa_{ik}$ are symmetric and that $\eta^{\perp}_{iknm} = \eta^{\perp}_{nmik}$. The kinetic contribution to the stress tensor should be symmetric. The term with the kinetic coefficient $\lambda$ is explicitly symmetric. The symmetry of the viscous contribution implies $\eta^{\perp}_{iknm} = \eta^{\perp}_{nmik}$. Since $R/\gamma$ is the entropy production rate, $R$ should be positive. It imposes some inequalities for the components of the tensors $\sigma_{ik}$, $\kappa_{ik}$, $\eta^{\perp}_{iknm}$ and leads to $\gamma > 0$.

The important point here is that the terms with the kinetic coefficient $\lambda$ do not contribute to the entropy production rate, such kinetic terms are called reactive. The term with $\lambda$ describes deviations from the model of the nematic liquid crystal as ideal rods frozen into the fluid flow, which corresponds to $\lambda = 1$. Note that for the similar model for frozen disc-like molecules, $\lambda = -1$ [34]. For such value of $\lambda$ the sum of the reactive terms and of the terms obtained via the Poisson bracket (19), is equivalent to using the alternative expression for the Poisson bracket $\{J_i, n_k\}$ that can be obtained from Eq. (A.18).

Equation (27) has to be compared with the Maxwell Eq. (1). We conclude that

$$\partial_t D_i = \partial_t D_{0i} + 4 \pi \sigma_{ik} E_k, \quad (32)$$

it is a generalization of Eq. (11). The last term in Eq. (32) is a generalization of the usual contribution to the displacement field, related to conductivity [31], for the case of the anisotropic medium (nematic). Above we ignored a dissipative contribution to the magnetic field $H$. If the magnetic induction field $B$ is much larger than the electric field $E$, corrections related to the nematic velocity could be taken into account. The case is analyzed in Appendix B.

In the nematics the kinetic tensors $\sigma_{ik}$, $\kappa_{ik}$, $\eta^{\perp}_{iknm}$ are “uniaxial.” Say, the electric conductivity tensor and the heat conductivity tensor are written as

$$\sigma_{ik} = \sigma_{ik} n_k n_i + \sigma_{ik} (\delta_{ik} - n_i n_k), \quad (33)$$

$$\kappa_{ik} = \kappa_{ik} n_k n_i + \kappa_{ik} (\delta_{ik} - n_i n_k). \quad (34)$$

To determine the fourth rank viscosity tensor $\eta^{\perp}_{iknm}$, five coefficients $\eta_{1i} - \eta_{5i}$ are needed [21, 37–39]:

$$\eta^{\perp}_{ikm} A_{kl} = 2 \eta_{1i} A_{ij} + 2 (\eta_2 - \eta_3) (A_{ik} n_j n_i + A_{jk} n_i n_k)$$

$$+ (\eta_4 - \eta_3) \delta_{ij} A_{kl} + 2 (\eta_5 + \eta_3 - 2 \eta_1) n_i n_j n_k n_l A_{kl}$$

$$+ (\eta_5 - \eta_4 + \eta_3) (\delta_{i} n_k n_l A_{kl} + n_i n_j A_{kl}),$$
where \( A_q = \partial_i v_j + \partial_j v_i \). The quantities \( \eta_1 - \eta_5 \) are called dynamic viscosity coefficients.

Positiveness of the entropy production, that is the condition \( R > 0 \) (see Eq. (31)) leads to the inequalities

\[
y > 0, \quad \sigma_i > 0, \quad \kappa_i > 0, \quad \kappa_i > 0,
\]

imposed on the coefficients in the expressions (33), (34) for the electric conductivity tensor and the heat conductivity tensor. The coefficients \( \eta_1 - \eta_5 \), entering the viscosity tensor \( \eta_{ikmm} \) see Eq. (35), should satisfy the following conditions

\[
\eta_4(2\eta_1 + \eta_2) > (\eta_3 - \eta_4)^2; \quad \eta_2, \eta_3, \eta_4 > 0; \quad 2(\eta_1 + \eta_5) - \eta_4 + \eta_2 > 0,
\]

guarantying positiveness of the viscous entropy production.

### 3.3. Minimal Model for Nematics

In nematics the displacement field \( \mathbf{D}_0 \) in the main approximation can be written as

\[
\mathbf{D}_0 = \hat{\varepsilon} \mathbf{E} + 4\pi \mathbf{P}_{p}.
\]

(35)

Here \( \mathbf{E} \) is electric field, the matrix \( \hat{\varepsilon} \) is the permittivity matrix of the nematic and \( \mathbf{P}_{p} \) represents the flexoelectric contribution to the polarization vector [24, 40] related to a non-homogeneity of the director \( \mathbf{n} \). The permittivity matrix and \( \mathbf{P}_{p} \) are

\[
\hat{\varepsilon}_{ik} = \varepsilon_i (\delta_{ik} - \eta_i n_k) + \varepsilon_{ik} n_k, \quad \mathbf{P}_{p} = \zeta_1 \mathbf{n}(\nabla \mathbf{n}) + \zeta_2 (n \nabla) \mathbf{n},
\]

(36)

(37)

where \( \varepsilon_i, \varepsilon_{ik}, \zeta_1, \zeta_2 \) are phenomenological coefficients, characterizing dielectric susceptibility and flexoelectric response.

Further we ignore effects related to magnetic polarization of the nematics, assuming \( \mu = 1 \) and discarding the magnetic dissipative effects. However there is another magnetic effect coming from a finite conductivity of the nematics. It is known that the electric current density of a moving conductor is equal to

\[
j_i = \sigma_{ik} \left( E_k + \varepsilon_{ik\alpha} \frac{V_k}{c} B_{\alpha} \right),
\]

see [31]. Of course in practice velocities of the nematics are much smaller than \( c \). However, the contribution to the electric current density related to the magnetic field can be relevant if \( B \gg E \). The case is analyzed in Appendix B. Below we neglect the magnetic contribution to the current density.

The internal energy \( U \) in the same approximation is written as

\[
U = \frac{1}{8\pi} \mathbf{D}_0 \hat{\varepsilon}^{-1} \mathbf{D}_0 - \mathbf{P}_{p} \hat{\varepsilon}^{-1} \mathbf{D}_0 + \frac{1}{2\rho} J^2 + F_F + U_0(S, \rho).
\]

(38)

Then the identity \( \mathbf{E} = 4\pi \partial U/\partial \mathbf{D}_0 \) (see Eq. (12)) reproduces Eq. (35). The term \( F_F \) in Eq. (38) is Frank energy:

\[
F_F = \frac{K_1}{2} (\nabla n)^2 + \frac{K_2}{2} |n(\nabla \times n)|^2 + \frac{K_3}{2} |n \times (\nabla \times n)|^2,
\]

(39)

where \( K_1, K_2, K_3 \) are splay, twist and bend Frank modiues.

For any realistic flows, the liquid crystal (nematic or smectic \( A \)) can be treated as incompressible, that is mass density \( \rho \) is constant and \( \nabla \mathbf{v} = 0 \). Furthermore, in the case of large thermal conductivity, temperature \( T \) is homogeneous. In the opposite case of small thermal conductivity the specific entropy \( S/\rho \) is homogeneous. Both limit cases enable one to exclude temperature (entropy) from the consideration. The set of equations can be made more compact assuming a single constant approximations for Frank elastic energy, and keeping only a single flexoelectric coefficient (all the more that in the main approximation it is also the case for smectic \( A \) liquid crystals, see Section 4).

Our model correctly identifies the important excitation modes, and their characteristic time and space scales. If necessary, these assumptions of the simplified model can be easily relaxed at the cost of more cumbersome set of equations. As a note of caution it is important not to overplay with such simplifications. Special care should be taken assuming a single (isotropic) viscosity coefficient. For example, for various types of electro-hydrodynamic instabilities in nematics, just the interplay between different viscosity coefficients determines the threshold of the instability. Another striking example is provided by a lyotropic smectic \( A \) liquid crystal. In this case [41] a simple shear within water layers is determined by the very low \( 10^{-2} \) Poise water viscosity, while all other hydrodynamic motions include several hundred times larger membrane viscosity.

### 4. SMECTICS

To span a wide range of possibilities to apply our results, in this section we consider the nonlinear elecro-hydrodynamics of a smectic-\( A \) Instead of the director \( \mathbf{n} \) smectic \( A \) is characterized by the displacement \( u \) of the smectic layers in \( z \)-direction, where \( z \)-axis is perpendicular to the equilibrium positions of the smectic layers. It is convenient to formulate the equations in terms of the variable \( W = z - u \), what allows to formulate the equations in the invariant under rotations form (see e.g., [15, 21]). Then we deal with the thermodynamic identity

\[
dU = \frac{1}{4\pi} \mathbf{E} \cdot \mathbf{D}_0 + \frac{1}{4\pi} \mathbf{H}_d \mathbf{B} + TdS + \mu d\rho + \nu d\mathbf{J}
\]

\[
+ \frac{\partial U}{\partial (\partial_i W)} \frac{d(\partial_i W)}{dW} + M_{ik} d(\partial_i \partial_k W),
\]

(40)

instead of Eq. (12).
In the main approximation the smectic elastic energy contributions into $U$ can be written as

$$U_{sm} = \frac{B}{8} [(\nabla W)^2 - 1]^2 + \frac{K}{2} (\nabla^2 W)^2, \quad (41)$$

instead of Eq. (39). Therefore

$$U = \frac{1}{8\pi \epsilon_0} D_0 \hat{e}^{-1} D_0 - \mathbf{P}_f \hat{e}^{-1} D_0 + \frac{1}{2\rho} J^2 + U_{sm} + U_0(S, \rho). \quad (42)$$

The permittivity matrix $\hat{\epsilon}$ and the flexoelectric contribution to the polarization vector $\mathbf{P}_f$ of the smectic are written as

$$\epsilon_{ik} = \epsilon_{zz}(\delta_{ik} - l/1_k) + \epsilon_{ij}/l_k, \quad (43)$$

$$\mathbf{P}_f = \omega_{il} (\nabla \Theta) + \omega_{i} (\mathbf{I}), \quad (44)$$

instead of Eqs. (36), (37). In the expressions (42), (43) $\mathbf{I}$ is the unit vector perpendicular to the smectic layers: $\mathbf{I} = \nabla W/|\nabla W|$. It is worth to note, that pure flexoelectric instability in smectic $A$ liquid crystals (as far as we know) was not observed experimentally. However there is no doubts about the very existence of the flexoelectric effect in smectics $A$ (see e.g., [42], where the coefficient $\omega_{i}$ has been measured).

To formulate the non-dissipative dynamic equations of the smectics we use the same Poisson bracket method, as for the nematics. The Poisson bracket for the variable $W$ has form

$$\{\mathbf{J}(\mathbf{r}), W(x)\} = -\nabla W \delta(\mathbf{r} - \mathbf{x}), \quad (44)$$

instead of Eq. (19). The other expressions (16)–(18) for the Poisson brackets remain unchanged. Then the equation for the variable $W$ is

$$\partial_t W = \{\hat{\epsilon}, W\} = -\nabla W, \quad \partial_t \mu = v_z - \nabla u, \quad (45)$$

where we used the expressions (40), (44). The equation for the displacement $u$ is obtained after the substitution $W = z - u$.

The equation for the momentum density of the smectic is

$$\partial_t J_i = -\partial_k(v_k J_i) - \partial_k(M_{ik} \partial_t W) - \partial_k(M_{ik} \partial_\mu \partial_t J)W - \partial_i p + \frac{1}{4\pi} \partial_k(D_{ik} E_i + B_k H_0) - \frac{1}{4\pi c} \partial_k |\mathbf{D}_0 \times \mathbf{B}|, \quad (46)$$

instead of Eq. (22). Here

$$M_i = \frac{\partial U}{\partial (\partial_i W)} - \partial_k M_{ik}, \quad (47)$$

and

$$M_{ik} = \frac{\partial U}{\partial (\partial_i \partial_k W)}. \quad (48)$$

Pressure $p$ is determined by the same relation (13).

Next step is to add dissipative terms to the written above reactive equations for the smectics. The dissipative terms in the equations for electric displacement field and momentum density are analogous to the dissipative terms in Eqs. (27), (28) where one should substitute $\mathbf{n} \rightarrow \mathbf{l}$. The dissipative contributions to the equations for $W$ and energy density $U$ are

$$\partial_t W = -\nabla W + \Theta, \quad (49)$$

$$\Theta = -\varepsilon_1 |\nabla W|^2 h - \frac{\varepsilon_2}{T} \nabla W \nabla T, \quad (50)$$

$$\partial_t U = -\nabla Q_{sm}^{(s)} + \partial_i(\kappa_{ik} \partial_k T) + \partial_i(v_k \eta_{iknn} \partial_n v_m) + \nabla \left( \frac{\partial W}{\partial (\nabla W)} \Theta \right) + \partial_i(M_{ik} \partial_k \Theta) - \partial_i(\partial_\mu M_{ik} \Theta), \quad (51)$$

where

$$h = -\nabla \frac{\partial U}{\partial (\nabla W)} + \partial_i \partial_k M_{ik}, \quad (52)$$

is the variational derivative of the energy over $W$, in turn non-dissipative energy flow density for smectics is

$$Q_{sm}^{(s)} = \frac{c}{4\pi} [\mathbf{E} \times \mathbf{H}_{0}]_l + M_i (v_k \partial_k W) + M_{ik} \partial_k (v_k \partial_i W) + (\rho \mu + TS + Jw) v_i. \quad (53)$$

In Eq. (47) $\xi_1$, $\xi_2$ are so-called permeation coefficients [15]. The equation for the entropy density is

$$\partial_t S = \{\mathbf{J}, S\} + \partial_i \left( \frac{\kappa_{ik}}{T} \partial_k T + \frac{\xi_2}{T} \partial_i W h + R \right), \quad (54)$$

$$R = \kappa_{ik} \partial_i T \partial_k T + \xi_1 (\nabla W)^2 \tilde{h}^2 + 2 \frac{\xi_2}{T} h (\nabla W) \nabla T + \eta_{iknm} \partial_i v_k \partial_n v_m + \sigma_{ik} E_i E_k. \quad (55)$$

The positive entropy production implies $\xi_1 > 0$, $\kappa_{ik} > \xi_2^2/T$, in addition to the conditions, analogous to ones formulated for nematics.

5. CONCLUSIONS

To conclude we would like to stress that liquid crystals are far from being exhausted as a topic of research. Since typically liquid crystals are soft (easily excited and deformed) systems, nonlinear physics is one of the prominent direction to study liquid crystals. In this work we have shown how to include electromagnetic field in the description of nonlinear dynamic phenomena in nematic and smectic $A$ liquid crystals in external alternating electric field. Our main result (the formulated set of dynamic equations for nematic and smectic $A$ liquid crystals) does allow an entry point to study different nonlinear dynamic phenomena in liquid crystals. To name a few

- electrically driven dynamic three-dimensional localized and moving excitations (directrons);
- classical driven by electric field coarsening kink (separating stable and unstable director configurations) dynamics. Several interesting questions, requiring the complete set of nonlinear dynamic equations, are naturally arose here;
dynamic transitions between isotropic and nematic liquid crystal phases in nonequilibrium systems driven by an external alternating electric field (see, e.g., [43]);

- anisotropic viscous flow effects created in liquid crystals by rotating colloidal particles (see said above about assumption on a single isotropic shear viscosity coefficients, and recent publication [44]);

- a role of defects (disclinations in director field and hydrodynamic vortices in a driven nematic liquid crystal cell, see [45]).

In all cases enlisted above the instability is triggered either by external electric field (which influences the system via dielectric anisotropy, flexoelectric coupling, or, in smectics, via so-called electroclinic effect [46, 47]) or by externally imposed flow. Then well above the instability thresholds the system behavior is controlled by nonlinear phenomena.

To analyze quantitatively some problems, using the derived in our work equations, three further steps should be performed: (1) to linearize and to solve the linear equations; (2) to include non-linear terms relevant for a specific problem under consideration; (3) to solve the corresponding nonlinear set of equations. Unfortunately, it is almost impossible to combine all three steps in one manuscript. Besides it requires rather heavy and bulky numerics, since to describe more or less realistic situation, one has to solve the set of coupled Floquet equations in a multi-parametric phase space. That is why we postpone these steps for the further works which are in progress now.

APPENDIX A
DERIVATION OF POISSON BRACKETS

Here we derive some expressions for Poisson brackets needed to construct the nonlinear equations presented in the paper. Note that the expressions are universal [20, 21] that is they are independent of the concrete form of Hamiltonian (energy of the system). We need extended Poisson brackets to include into consideration electromagnetic degrees of freedom.

The derived in this Appendix expressions are also useful to formulate nonlinear reactive equation describing nematic director dynamics.

Vector Fields

Let us consider Hamiltonian dynamics for a system described by canonically conjugated vector fields \( \mathbf{p}, \mathbf{q} \). Hamiltonian of the system is written as

\[
\mathcal{H} = \int dV H(\mathbf{p}, \mathbf{q}, \mathbf{p}, \mathbf{q}). \tag{A.1}
\]

Canonical equations for the fields \( \mathbf{p}, \mathbf{q} \) are written as

\[
\dot{\mathbf{p}} = -\frac{\delta \mathcal{H}}{\delta \mathbf{q}} = -\frac{\partial H}{\partial \mathbf{q}} + \mathbf{i} \frac{\partial H}{\partial (\partial_i \mathbf{q})}, \tag{A.2}
\]

\[
\frac{\partial \mathbf{q}}{\partial \mathbf{p}} = \frac{\delta \mathcal{H}}{\delta \mathbf{p}} = \frac{\partial H}{\partial \mathbf{p}}. \tag{A.3}
\]

The equations can be rewritten as

\[
\partial_j \mathbf{p} = \{\mathcal{H}, \mathbf{p}\}, \quad \partial_j \mathbf{q} = \{\mathcal{H}, \mathbf{q}\}, \tag{A.4}
\]

where \{..., ...\} designate Poisson brackets. Non-zero Poisson brackets for the canonically conjugated fields \( \mathbf{p}, \mathbf{q} \) are

\[
\{p_i(r), q_k(x)\} = \delta_{ik} \delta(x - r). \tag{A.5}
\]

Being substituted into Eq. (A.4), the expression (A.5) leads to Eqs. (A.2), (A.3).

Equations (A.2), (A.3) lead to the following equation for the canonical momentum density

\[
\partial_j (-\mathbf{p} \partial_i \mathbf{q}) + \partial_k \Pi_{ik} = 0, \tag{A.6}
\]

\[
\Pi_{ik} = \left( \frac{\partial^2 H}{\partial \mathbf{p} \partial \mathbf{p}} \right)_{\delta \mathbf{p}, \delta \mathbf{q}} \delta_{ik} + \frac{\partial H}{\partial (\partial_i \mathbf{q})} \partial_j \mathbf{q}. \tag{A.7}
\]

In general case the stress tensor (A.7) is not symmetric. Therefore the conservation of the angular momentum law based on the canonical momentum density is not granted. To overcome the difficulty we exploit rotational invariance of \( H \)

\[
\epsilon_{iak} \Pi_{kn} = -\epsilon_{iak} \epsilon_{njl} (p_j q_n) - \epsilon_{iak} \epsilon_{njl} \left[ q_j \frac{\partial H}{\partial (\partial_j q_n)} \right] = 0, \tag{A.8}
\]

Using (A.8) and the standard relation \( \epsilon_{iak} \epsilon_{njl} = \delta_{ik} \delta_{ij} - \delta_{ik} \delta_{kj} \). Eq. (A.6) can be rewritten as

\[
\partial_j (-\mathbf{p} \partial_i \mathbf{q}) + \frac{1}{2} \partial_k (\Pi_{ik} + \Pi_{ki}) + \frac{1}{2} \partial_k (\Pi_{ik} - \Pi_{ki}) = 0, \tag{A.9}
\]

where

\[
\partial_k (\Pi_{ik} - \Pi_{ki}) = \partial_k (\epsilon_{iak} \epsilon_{njl} \Pi_{ji}) = -\partial_j \partial_k (p_j q_k) - \epsilon_{ikl} \epsilon_{njl} \partial_k \partial_m \left[ q_j \frac{\partial H}{\partial (\partial_m q_l)} \right]. \tag{A.10}
\]

The last term in Eq. (A.10) can always be rewritten as a derivative of a symmetric tensor, using the identity

\[
\partial_k \partial_j M_{kij} = \partial_k \partial_j \left[ M_{kij} - \frac{1}{2} \epsilon_{kij} (\epsilon_{pqk} M_{pqk} - \epsilon_{qik} \epsilon_{qkM_{ijkl}}) \right]. \tag{A.11}
\]

One can easily check, that the combination in the square brackets in Eq. (A.11) is invariant under the permutation \( i \leftrightarrow k \). Therefore Eq. (A.9) can be rewritten as

\[
\partial_j [-\mathbf{p} \partial_i \mathbf{q} + \partial_k (p_k q_i)] - \partial_k T_{ik} = 0, \tag{A.12}
\]

where the stress tensor \( T_{ik} \) is symmetric.
Thus we conclude that for vector fields the correct form for the momentum density is

$$J_i = -p_i \partial_i \mathbf{q} + p_i \partial_i (p_i \mathbf{q}).$$ \hspace{1cm} (A.13)

Besides the canonical contribution, the expression (A.13) contains the additional term characteristic of the vector fields. Using the relation (A.5), one can derive from Eq. (A.13) the expression

$$\{J_i (\mathbf{r}), J_j (\mathbf{x})\} = J_i (\mathbf{r}) \partial_j (\mathbf{r} - \mathbf{x}) + \partial_j (\mathbf{r} - \mathbf{x}) J_j (\mathbf{x}),$$

coinciding with Eq. (18), thus confirming its universality.

Starting with the relation (A.5), one derives from the expression (A.13) the following expressions for the Poisson brackets

$$\{J_i (\mathbf{r}), p_k (\mathbf{x})\} = p_k (\mathbf{r}) \partial_i (\mathbf{r} - \mathbf{x}) + \partial_i (\mathbf{r} - \mathbf{x}) p_k (\mathbf{q}),$$ \hspace{1cm} (A.14)

$$\{J_i (\mathbf{r}), q_k (\mathbf{x})\} = -\partial_i q_k (\mathbf{r} - \mathbf{x}) + \partial_i (\mathbf{r} - \mathbf{x}) q_k (\mathbf{q}),$$ \hspace{1cm} (A.15)

$$\{J_i (\mathbf{r}), b_k (\mathbf{x})\} = b_k (\mathbf{r}) \partial_i (\mathbf{r} - \mathbf{x}) - \partial_i (\mathbf{r} - \mathbf{x}) b_k (\mathbf{q}),$$ \hspace{1cm} (A.16)

where $\mathbf{b} = \nabla \times \mathbf{q}$. One derives from Eqs. (A.14), (A.15) for unit vectors

$$\{J_i (\mathbf{r}), p_k (\mathbf{x})\} = -\left( \partial_i \frac{p_k}{p} \right) \delta (\mathbf{r} - \mathbf{x})$$

$$- \partial_i \delta (\mathbf{r} - \mathbf{x}) \left[ \frac{p_k}{p} \partial_i \left( \frac{p_k}{p} \right) \right].$$ \hspace{1cm} (A.17)

$$\{J_i (\mathbf{r}), q_k (\mathbf{x})\} = -\left( \partial_i \frac{q_k}{q} \right) \delta (\mathbf{r} - \mathbf{x})$$

$$- \partial_i \delta (\mathbf{r} - \mathbf{x}) \left[ \frac{q_k}{q} \partial_i \left( \frac{q_k}{q} \right) \right].$$ \hspace{1cm} (A.18)

where the expressions in square brackets are functions of $\mathbf{x}$. Note that the expressions (A.17) and (A.18), derived for the vector fields, coincide with ones found by Volovik for ideal rods and discs frozen into the liquid crystal flow [34].

Electromagnetic Field

We begin with non-dissipative dynamic equations for the electromagnetic field in Lagrangian formulation. It is convenient to exploit Weyl gauge where both the electric field $\mathbf{E}$ and the magnetic induction field $\mathbf{B}$ are expressed in terms of the vector potential $\mathbf{A}$ as

$$\mathbf{E} = -\frac{1}{c} \partial_t \mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}.$$ \hspace{1cm} (A.19)

Thus Hamiltonian (energy of the system) is written as

$$\mathcal{H} = \int dV U (\mathbf{D}_0, \mathbf{B}, ...),$$ \hspace{1cm} (A.21)

$$dU = \frac{1}{4\pi} \mathbf{E} d\mathbf{D}_0 + \frac{1}{4\pi} \mathbf{H}_0 d\mathbf{B} + ..., \hspace{1cm} (A.22)

where dots stand for other variables.

One obtains from Eqs. (A.20), (A.22)

$$dL = \frac{1}{4\pi} \mathbf{D}_0 d\mathbf{E} - \frac{1}{4\pi} \mathbf{H}_0 d\mathbf{B} + ...$$ \hspace{1cm} (A.23)

Therefore the variable, canonically conjugated to $\mathbf{A}$, is

$$\frac{\partial L}{\partial (\partial_t \mathbf{A})} = -\frac{\mathbf{D}_0}{4\pi c}.$$ \hspace{1cm} (A.24)

Thus, we arrive at the following expression for the Poisson bracket

$$\{A_i (\mathbf{r}), D_{0i} (\mathbf{x})\} = 4\pi c \delta_{\mathbf{r} \mathbf{x}} \delta (\mathbf{r} - \mathbf{x}).$$ \hspace{1cm} (A.25)

Taking curl of the relation (A.25), we find

$$\{B_i (\mathbf{r}), D_{0i} (\mathbf{x})\} = 4\pi c \epsilon_{ijk} \delta_{\mathbf{r} \mathbf{x}} \delta (\mathbf{r} - \mathbf{x}),$$ \hspace{1cm} (A.26)

that is at Eq. (8). It is important to note that the expression (A.25) derived for Poisson bracket $\{A_i (\mathbf{r}), D_{0i} (\mathbf{x})\}$ coincides with the expression presented by Volovik in [16], moreover both obtained expressions (A.25) and (A.26) coincides with ones given by Volovik in [17].

Now, we can derive the canonical equations for the electromagnetic field. Using the thermodynamic identity (A.20) and the expression (A.26), we end up with Maxwell equations

$$\partial_t \mathbf{D}_0 = \{\mathcal{H}, \mathbf{D}_0\} = c \nabla \times \mathbf{H}_0,$$ \hspace{1cm} (A.27)

$$\partial_t \mathbf{B} = \{\mathcal{H}, \mathbf{B}\} = -c \nabla \times \mathbf{E}.$$ \hspace{1cm} (A.28)

Note that Eqs. (A.28) is a direct consequence of the relations (A.19). One can find the contribution to the momentum density related to the electromagnetic field. Substituting to the expression (A.13) $p = -(4\pi c)^{-1} \mathbf{D}_0$, $\mathbf{q} = \mathbf{A}$, one finds

$$J^m = \frac{1}{4\pi c} \mathbf{D}_0 \times \mathbf{B},$$ \hspace{1cm} (A.29)

where we assumed $\nabla \mathbf{D}_0 = 0$. The condition means zero density of free charges.

**CONDUCTIVITY AT FINITE VELOCITY**

We do not know publications with derivation of the conductivity for moving liquid crystals (in spite of the fact that it is well known how to perform such calculations). Thus we have presented the results of such calculations in this Appendix. One of the byproduct of our calculation is that in a sufficiently strong magnetic
field, moving liquid crystal electric conductivity depends on magnetic field even in the limit \( v \ll c \).

Here we pass to the thermodynamic potential.

\[
\tilde{U} = U - \frac{1}{4\pi} \mathbf{B} \mathbf{H}_0, \quad (B.1)
\]

\[
d\tilde{U} = \frac{1}{4\pi} \mathbf{E} \mathbf{dD}_0 - \frac{1}{4\pi} \mathbf{B} \mathbf{dH}_0 + \ldots, \quad (B.2)
\]

Thus, the time derivative of the energy density is

\[
\partial_t \tilde{U} = \frac{1}{4\pi} \mathbf{D} \mathbf{E}_0 - \frac{1}{4\pi} \mathbf{B} \mathbf{D}_0 - \frac{1}{4\pi} \mathbf{B} \mathbf{D}_0 + \ldots, \quad (B.3)
\]

where we present the electromagnetic terms. We are interested in the dissipative contribution to \( \partial_t \tilde{U} \).

If the velocity \( v \) of a system is nonzero then the relation between the current density of the free charges and the electric field \( \mathbf{E} \) is

\[
j_i = \sigma_i \left( E_k + \epsilon_{ken} \frac{v}{c} B_m \right). \quad (B.4)
\]

It is a consequence of the fact, that the vectors \( \mathbf{E}, \mathbf{B} \) and the vectors \( \mathbf{D}, \mathbf{H} \) can be presented as the components of four-dimensional antisymmetric tensors of second order (with the condition that the charge density is zero) [31]. The arguments behind Eq. (B.4) are based on Lorentz invariance, and go back to Minkowski (1907). Quantity (B.4) is the kinetic (dissipative) contribution to \( -\mathbf{D} \mathbf{dD}_0/(4\pi) \). It is interesting to note that the second contribution in Eq. (B.4) can be relevant even at \( v/c \ll 1 \) if \( \mathbf{B} \gg \mathbf{E} \). Similar arguments lead to the conclusion that the dissipative contribution to the magnetization derivative \( \partial_t \mathbf{H}_0/(4\pi) \) is

\[
\epsilon_{ij} \frac{v}{c} \sigma_{jk} \left( E_k + \epsilon_{kmn} \frac{v}{c} B_m \right). \quad (B.5)
\]

We find from Eqs. (B.2), (B.4), (B.5) that

\[
R = \frac{1}{4\pi} \sigma_{ij} \left( E_j + \epsilon_{iak} \frac{v}{c} B_k \right) \left( E_j + \epsilon_{ilm} \frac{v}{c} B_l \right) + \ldots, \quad (B.6)
\]

where \( R/T \) is the entropy production rate. Thus, the only difference in comparison with the scheme presented in the main text is in the substitution \( \mathbf{E} \rightarrow \mathbf{E}' \) where

\[
\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \quad (B.7)
\]

compare with [31], Eq. (63.1). Let us stress that the relations derived in the section, are correct at an arbitrary velocity \( \mathbf{v} \).

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**CONFLICT OF INTEREST**

The authors declare that they have no conflicts of interest.

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