TESTING QCD PREDICTIONS FOR MULTIPLICITY DISTRIBUTIONS AT HERA

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Abstract

On the basis of a recently introduced generalization of the negative binomial distribution the influence of higher-order perturbative QCD effects on multiplicity fluctuations are studied for deep inelastic $e^+p$ scattering at HERA energies. It is found that the multiplicity distributions measured by the H1 Collaboration indicate violation of infinite divisibility in agreement with pQCD calculations. Attention is called to future experimental analysis of combinants whose nontrivial sign-changing oscillations are predicted using the generalized negative binomial law.

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Over the past 15 years of experimental and theoretical study of multiplicity distributions (MDs) in a variety of collision processes a universal law emerged known as the negative binomial (NB) regularity [1]. The negative binomial distribution (NBD) proved to be very successful in describing the observed shape of MDs not only in full phase-space but also in restricted subdomains. At the same time the NBD naturally arises in branching processes which share many common features with the parton cascades of QCD. Thus the theoretical basis of the NB regularity is well founded, at least qualitatively.

The NBD has two parameters, the shape parameter $k$ and the mean value $\langle n \rangle$. Its analytic form can be written as

$$P_n = \frac{1}{n!} \frac{\Gamma(k+n)}{\Gamma(k)} \left( \frac{\theta}{1+\theta} \right)^n P_0$$

with

$$P_0 = (1+\theta)^{-k} \quad \text{and} \quad \theta = \frac{\langle n \rangle}{k}.$$  

The probability generating function $G(z) = \sum_{n=0}^{\infty} P_n z^n$ has the form

$$G(z) = (1 + (1 - z) \theta)^{-k}.$$  

A fundamental property of the distribution is that asymptotically $P_n$ exhibits a nice scaling behaviour [2]

$$\lim_{n \to \infty, \langle n \rangle \to \infty, \text{ } n/\langle n \rangle \text{ fixed}} P_n = \frac{1}{\langle n \rangle} \psi \left( \frac{n}{\langle n \rangle} \right)$$

which is the famous KNO scaling law [3]. The preasymptotic MDs can be reconstructed from the asymptotic scaling function $\psi(z = n/\bar{n})$ via Poisson transform:

$$P_n = \int_0^{\infty} \psi(z) \frac{(\bar{n} z)^n}{n!} e^{-\bar{n} z} dz.$$  

For the NBD $\psi(z)$ turns out to be a gamma distribution. In fact Eq. (5) is a special superposition of Poisson distributions, known as Poisson mixing in the mathematical literature [4]. The superposition is such that the mean values $\langle n \rangle$ of the Poisson components vary according to $\psi(z)$ which is called the mixing distribution. According to the theorem of Ottestad and Consael [4] the resulting discrete distribution $P_n$ has probability generating
function equivalent to the characteristic function of \( \psi(z) \) and, consequently, the factorial moments (cumulants) of \( P_n \) are equivalent to the ordinary moments (cumulants) of \( \psi(z) \).

During the last few years discrepancies have been found between the NBD and the MDs measured at the highest available energies. This is not surprising; it is very unlikely that a two-parameter discrete distribution can describe all the details of MDs in \( ll, lh \) and \( hh \) collisions. Besides the experimentally found deviations, the perturbative QCD calculations also indicate that the NB regularity is not general enough to account higher-order pQCD effects \([6]\). These manifest in the violation of infinite divisibility of the MDs in sharp contradiction to the NBD being infinitely divisible. The analysis of experimental data confirmed the pQCD calculations \([7,8]\).

In a recent paper we have generalized the NBD by incorporating some pQCD-based characteristics of MDs \([9]\). The model involves an additional parameter, \( \mu \), with the following physical meaning. Let us preserve the asymptotic scaling form of the NBD (i.e. the gamma distribution) but not simply in the scaling variable \( z \), rather, in a certain power of it, say in \( z^\mu \) with \( \mu > 0 \). Then the asymptotic scaling function can be written as

\[
\psi(z) = \frac{\mu}{\Gamma(k)} \lambda^k z^{\mu k - 1} \exp(-[\lambda z]^\mu)
\]

which is the generalized gamma distribution \([5]\). The ordinary gamma distribution is the \( \mu = 1 \) special case. Obviously, the \( q \)th moment of the modified \( \psi(z) \) involves the fractional rank \( q/\mu \). Due to the theorem of Ottestad and Consael quoted before, the same rescaled rank \( q/\mu \) appears in the \( q \)th factorial moment of \( P_n \) defined by the Poisson transform of the modified \( \psi(z) \). Thus one can reproduce possible enhancement (\( \mu < 1 \)) or suppression (\( \mu > 1 \)) of multiplicity fluctuations with respect to the NBD by fitting the Poisson transform of the generalized gamma distribution to the observed \( P_n \).

The Poisson transform of Eq. (6) can be expressed in terms of Fox’s generalized hypergeometric function \([9]\). Without going into the details we recapitulate the final result:

\[
P_n = \frac{1}{n! \Gamma(k)} H^{1,1}_{1,1} \left[ \frac{1}{\theta} \begin{array}{c} (1 - n, 1) \\ (k, 1/\mu) \end{array} \right], \quad 0 < \mu < 1
\]

and

\[
P_n = \frac{1}{n! \Gamma(k)} H^{1,1}_{1,1} \left[ \theta \begin{array}{c} (1 - k, 1/\mu) \\ (n, 1) \end{array} \right], \quad \mu > 1
\]
where
\[
\theta = \frac{\langle n \rangle \Gamma(k)}{\Gamma(k + 1/\mu)}
\] 
and \( H(\cdot) \) denotes the Fox-function. The probability generating function of \( P_n \) is given by
\[
G(z) = \frac{1}{\Gamma(k)} H_{1,1}^{1,1} \left[ 1 \left| \begin{array}{c} 1, 1 \\ t \end{array} \right| \begin{array}{c} (1, 1) \\ (k, 1/\mu) \end{array} \right], \quad 0 < \mu < 1
\] 
and
\[
G(z) = \frac{1}{\Gamma(k)} H_{1,1}^{1,1} \left[ t \left| \begin{array}{c} 1 - k, 1/\mu \\ 0, 1 \end{array} \right| \right], \quad \mu > 1
\] 
with \( t = (1 - z) \theta \). For \( \mu = 1 \) the negative binomial distribution is recovered. The necessity of two separate expressions for \( P_n \) and \( G(z) \) follows from the existence conditions of \( H(x) \) discussed in [9].

The splitted parameter space in \( \mu \) reflects an important difference between the two expressions for \( P_n \) given by Eqs. (7-8). According to the theorem of Bondesson [10] the generalized gamma density (6) is infinitely divisible only for \( 0 < \mu \leq 1 \). For \( \mu > 1 \) its characteristic function is entire analytic function of finite order and hence it must have complex zeroes which is not permitted for infinitely divisible entire characteristic functions [10]. Let us recall Maceda’s theorem [4] which states that a discrete distribution \( P_n \) defined by Poisson mixing (5) is infinitely divisible if and only if the mixing distribution \( \psi(z) \) with \( z \in (0, \infty) \) is infinitely divisible. Accordingly, in case of Eq. (7) \( P_n \) satisfies the requirements of infinite divisibility, just as the NBD for \( \mu = 1 \), whereas the \( \mu > 1 \) case given by Eq. (8) violates this feature. Thus our simple generalization of the NBD is capable of reproducing a basic prediction of pQCD calculations. The additional parameter \( \mu \) measures the degree of violation of infinite divisibility for \( \mu > 1 \).

To see how Eqs. (7-8) work in practice we carried out fits to the recent multiplicity data of the H1 Collaboration measured in deep inelastic \( e^+p \) scattering at HERA over a large kinematic region [11]. The MDs have been studied in pseudorapidity \( \eta^* \) domains of varying size in the current fragmentation region of the hadronic centre-of-mass frame. Comparison of the MDs to the NB and lognormal (LN) distributions showed that both models give an acceptable description of the uncorrected data only in the smallest pseudorapidity domains. For widening \( \eta^* \)-intervals the quality of fits becomes progressively worse [11].
In our fitting procedure numerical evaluation of the integral defining the Poisson transform (5) has been carried out using 96-point gaussian quadrature. It is turned out that keeping the shape parameter fixed at \( k = 1 \) and fitting only \( \langle n \rangle \) and \( \mu \) produces reasonable description of the H1 data for \( P_n \) tabulated in ref. [11]. Thus our theoretical \( P_n \) is the Poisson transform of the Weibull distribution [5]. The scale parameter of \( \psi(z) \) is restricted to \( \lambda = \Gamma(1/\mu)/\mu \) by the normalization condition \( \int_0^\infty z \psi(z) dz = 1 \) [9]. The results of fits are collected in Table 1. As is seen the values of the \( \chi^2 \) are satisfactory, with the only exception of the fit corresponding to \( 1 < \eta^* < 3 \) and \( W = 80 \div 115 \) GeV. In this case the NBD provides a better account of the data. Let us observe that \( \mu > 1 \) and increases for widening \( \eta^* \)-intervals signalling increasingly dominant violation of infinite divisibility. In our opinion this is the reason why the NB and LN distributions (both being infinitely divisible) produce progressively worse fits for large pseudorapidity domains. The quality of our fits is illustrated in Fig. 1 for \( 1 < \eta^* < 4 \).

In the remaining part of this Letter we consider the question of infinite divisibility of MDs on the basis of combinants. According to Lévy’s theorem [4] a discrete distribution is infinitely divisible if and only if its probability generating function can be written in the form

\[
G(z) = \exp(\lambda(g(z) - 1)) \tag{12}
\]

where \( \lambda > 0 \) and \( g(z) \) is another probability generating function. The discrete distributions having \( G(z) \) of the form of Eq. (12) are known also as compound Poisson distributions. These can be regarded as convolutions of Poisson singlet, Poisson doublet, Poisson triplet, etc. distributions since Eq. (12) can be rewritten as

\[
G(z) = \prod_{q=1}^{\infty} \exp(C_q(z^q - 1)) \tag{13}
\]

i.e. as the product of the generating functions of the Poisson components having mean values \( C_q \). The \( C_q \) are proportional to the probabilities given by \( g(z) \) with constant of proportionality \( \lambda \) hence they can not be negative for infinitely divisible distributions [4].

After Kauffmann and Gyulassy [12] the quantities \( C_q \) acquired the name combinants in multiparticle phenomenology. They can be expressed in terms
of the probability ratios $P_q = P_q / P_0$ according to

$$C_q = P_q - \frac{1}{q} \sum_{i=1}^{q-1} i C_i P_{q-i}$$  \hspace{1cm} (14)$$

see \cite{1,12,13}. Although the requirement $P_0 > 0$ can be a drawback for full phase-space analysis of MDs, in restricted subdomains the combinants have a number of advantages. First of all, as Eq. (14) shows, the knowledge of $C_q$ requires only the finite number of probabilities $P_{n \leq q}$. This is extremely useful since testing the violation of infinite divisibility of $P_n$ can be realized without the possible influence of truncating the high-multiplicity tail which can mimic pQCD effects \cite{1,14}. Furthermore, one need not know the probabilities $P_n$ themselves; the combinants can be determined directly from the unnormalized topological cross-sections $\sigma_n$ because they involve only ratios of $P_n$. Thus the statistical and systematic uncertainties of $\sigma_{\text{tot}}$ are filtered out by the combinants. They also share some common features with the factorial cumulants, e.g. $C_1 = \langle n \rangle$ and $C_{q \geq 2} = 0$ for the Poisson distribution, further, both quantities should be nonnegative for infinitely divisible distributions.

Since the generalization of the NBD obtained by the Poisson transform of Eq. (6) can violate infinite divisibility it is of interest to study the behaviour of its combinants. For $\mu \leq 1$ one expects positive $C_q$ in all orders but for $\mu > 1$ sign-changing oscillations may appear in the $q$-dependence of $C_q$ similarly to the factorial cumulant-to-moment ratios $H_q$ of the distribution \cite{9}. We have calculated the combinants of the generalized NBD with the help of Eq. (14) using the same numerical integration procedure for the evaluation of $P_n$ as described earlier. A typical result for the behaviour of $\log |C_q|$ over the $\mu$-$q$ plane is shown in Fig. 2a for $k = 3/2$ and $\langle n \rangle = 10$ (with the same $k$ the behaviour of $H_q$ is studied in \cite{9}). The peculiar structure for $\mu > 1$ is due to sign-changing oscillations of $C_q$. At a fixed $q$ the neighbouring bumps are $C_q$ intervals of opposite sign in $\mu$. For $\mu \leq 1$ $C_q$ is always positive as expected. In Fig. 2b slices are shown with $q = 5, 20$ (left) and with $\mu = 1, 4$ (right). It is seen that the pattern of oscillations is nontrivial, i.e. not alternating in sign as $q$ takes even/odd values. Qualitatively similar sign-changing oscillations of $C_q$ occur for a different choice of $\langle n \rangle$ and $k$.

Finally we provide a further example for the usefulness of combinants in the analysis of MDs. It is often stated that the factorial cumulant-to-moments raios $H_q$ are very sensitive to tiny details of the high-multiplicity
tail of $P_n$ (and thus to truncation effects as well). Let us consider this widespread opinion somewhat closer for the NBD. The unnormalized factorial moments $\xi_q$ of the NBD are given by

$$\xi_q = \frac{\Gamma(k+q)}{\Gamma(k)} \theta^q$$  \hspace{1cm} (15)$$

where $\theta$ is given in Eq. (2). The unnormalized factorial cumulants $f_q$ take the form

$$f_q = k \Gamma(q) \theta^q$$  \hspace{1cm} (16)$$
hence for $H_q = f_q/\xi_q$ one obtains [6]

$$H_q = k \frac{\Gamma(k) \Gamma(q)}{\Gamma(k+q)} = k B(k,q)$$  \hspace{1cm} (17)$$

with $B(\cdot)$ denoting the Euler beta-function. The combinants of the NBD are given by [15,16]

$$C_q = \frac{k}{q} \left( \frac{\theta}{1+\theta} \right)^q.$$  \hspace{1cm} (18)$$

Comparing Eqs. (1) and (15) we get

$$q! P_q = \frac{\Gamma(k+q)}{\Gamma(k)} \left( \frac{\theta}{1+\theta} \right)^q = \frac{\xi_q}{(1+\theta)^q}$$  \hspace{1cm} (19)$$

whereas the comparison of Eqs. (18) and (16) yields

$$q! C_q = k \Gamma(q) \left( \frac{\theta}{1+\theta} \right)^q = \frac{f_q}{(1+\theta)^q}.$$  \hspace{1cm} (20)$$

Thus one arrives at the rather unexpected result

$$H_q = \frac{C_q}{P_q} = 1 - \frac{1}{q} \sum_{i=1}^{q-1} i C_i \frac{P_q-i}{P_q}$$  \hspace{1cm} (21)$$

which shows that the factorial cumulant-to-moment ratios $H_q$ of the NBD are completely insensitive to the $P_{n>q}$ tail of the distribution and carry essentially the same information as the combinants. This is a very special property of the NBD which does not hold in general for infinitely divisible distributions.
In summary, we have investigated the violation of infinite divisibility of MDs. This particular feature plays a central role in recent studies of multiplicity fluctuations since the higher-order pQCD calculations predict its appearance for MDs measured in full phase-space and in restricted subdomains. As a consequence, departures arise from the NB regularity.

In a previous paper we have generalized the NBD by extending the validity of its asymptotic scaling form to a certain power of the scaling variable. By this modification one can reproduce possible suppression/enhancement of multiplicity fluctuations with respect to the NBD as well as the violation of infinite divisibility of MDs. Fitting the generalized NBD to the recent experimental data for $P_n$ measured by the H1 Collaboration in deep inelastic $e^+p$ scattering at HERA we have obtained good agreement. According to our results the violation of infinite divisibility of $P_n$ is increasingly dominant in widening pseudorapidity intervals.

Investigating the combinants of the generalized NBD we have found non-trivial sign-changing oscillations of these quantities for the subdomain of parameter space violating infinite divisibility. In our view the higher-order pQCD effects on multiplicity fluctuations can be studied most effectively by measuring the combinants in restricted phase-space volumes. The constraints of conservation laws are less pronounced, hence the dynamical effects are more visible, and the sign-changing oscillations of the combinants are not corrupted by finite statistics effects. The pQCD prediction for the $q$-dependence of $C_q$ (location of the first minimum, etc.) would be very important to identify properly the dynamics underlying the oscillations.

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### Table 1.

Results of fits to the H1 data for $P_n$ with the Poisson transform of the generalized gamma distribution (6) for fixed shape parameter $k = 1$ (Weibull case). The errors are only statistical.

| $\eta^*\text{-interval}$ | $W$ (GeV) | $\mu$       | $\langle n \rangle$ | $\chi^2$/d.o.f. |
|--------------------------|-----------|-------------|---------------------|-----------------|
| $1 < \eta^* < 2$        |           |             |                     |                 |
| $80 \div 115$          | 1.96 ± 0.08 | 2.44 ± 0.04 | 18/12               |
| $115 \div 150$         | 1.82 ± 0.07 | 2.50 ± 0.04 | 5/13                |
| $150 \div 185$         | 1.87 ± 0.08 | 2.61 ± 0.05 | 12/13               |
| $185 \div 220$         | 1.84 ± 0.09 | 2.64 ± 0.06 | 7/13                |
| $1 < \eta^* < 3$        |           |             |                     |                 |
| $80 \div 115$          | 2.46 ± 0.07 | 4.88 ± 0.06 | 44/16               |
| $115 \div 150$         | 2.09 ± 0.06 | 5.05 ± 0.07 | 25/17               |
| $150 \div 185$         | 2.07 ± 0.07 | 5.29 ± 0.08 | 9/19                |
| $185 \div 220$         | 2.10 ± 0.08 | 5.33 ± 0.09 | 7/20                |
| $1 < \eta^* < 4$        |           |             |                     |                 |
| $80 \div 115$          | 3.70 ± 0.14 | 6.42 ± 0.06 | 17/17               |
| $115 \div 150$         | 3.18 ± 0.11 | 7.02 ± 0.07 | 18/19               |
| $150 \div 185$         | 2.92 ± 0.09 | 7.49 ± 0.08 | 16/21               |
| $185 \div 220$         | 2.75 ± 0.10 | 7.67 ± 0.09 | 12/21               |
| $1 < \eta^* < 5$        |           |             |                     |                 |
| $80 \div 115$          | 4.66 ± 0.23 | 6.87 ± 0.06 | 12/17               |
| $115 \div 150$         | 4.44 ± 0.18 | 7.70 ± 0.06 | 26/20               |
| $150 \div 185$         | 4.16 ± 0.16 | 8.39 ± 0.08 | 19/21               |
| $185 \div 220$         | 4.08 ± 0.18 | 8.79 ± 0.08 | 16/22               |

Fig. 1: The best-fit theoretical distributions for $1 < \eta^* < 4$, see the text for details. The corresponding parameters are collected in Table 1. The displayed errors are only statistical.

Fig. 2a: Sign-changing oscillations of the combinants $C_q$ of the Poisson transformed generalized gamma distribution. The parameters kept fixed are $k = 3/2$ and $\langle n \rangle = 10$. For clarity only the odd-rank combinants are displayed.

Fig. 2b: Slices through Fig. 2a with $q = 5, 20$ (left) and with $\mu = 1, 4$ (right). The $\mu = 1$ case (smooth curve) corresponds to the NBD.
Figure 1
Figure 2a

Figure 2b