Lowest Landau level on a cone and zeta determinants

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Abstract
We consider the integer QH state on Riemann surfaces with conical singularities, with the main objective of detecting the effect of the gravitational anomaly directly from the form of the wave function on a singular geometry. We suggest the formula expressing the normalisation factor of the holomorphic state in terms of the regularized zeta determinant on conical surfaces and check this relation for some model geometries. We also comment on possible extensions of this result to the fractional QH states.

Keywords: quantum Hall effect, singular surfaces, zeta determinant, gravitational anomaly

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum Hall states on curved backgrounds, in particular on compact Riemann surfaces, have recently attracted a renewed attention, in part due to the derivation of the gravitational anomaly in the integer [1, 20] and fractional quantum Hall states [8, 9, 15, 18]. There are two main reason as to why considering QH states on curved backgrounds is interesting both from theoretical and experimental perspectives.

First, it provides a way to learn about the properties of the quantum Hall states, which do not transpire in planar geometry. Indeed, Riemann surfaces are naturally endowed with parameter spaces, such as the (infinite-dimensional) space of admissible Riemannian metrics and the (finite-dimensional) moduli spaces of complex structures. In addition, there is a parameter space of Aharonov–Bohm phases, corresponding to the flat gauge connections of...
the magnetic field along the surface, as well as a possibility of considering an inhomogeneous magnetic field. Considering adiabatic transport of QH states on these parameter spaces and employing the standard framework of Berry connections and curvatures, one potentially can elucidate novel adiabatic transport coefficients in quantum Hall effect.

These geometric considerations became important since the early days of theoretical studies of QHE. In the classical [25] Laughlin considered a gedanken experiment with QHE on the ribbon-shaped surface, and explained the quantization of the Hall conductance via the adiabatic transport of the lowest Landau level (LLL) wave functions when the Aharonov–Bohm phase of the magnetic field through the hole makes a full circle in the parameter space. In this argument the transport of electric charge is induced by adiabatic motion on the parameter space, which in the case of the torus and higher genus Riemann surfaces corresponds to the moduli space of flat connections, as was later explained in [3]. In the well known [4] the anomalous (Hall) viscosity, which is the adiabatic coefficient controlling the response to the shear motion of the electronic liquid, was identified with the adiabatic curvature on the complex structure moduli space of flat tori. In [30] this calculation was also generalized to the fractional QHE.

The gravitational anomaly gives rise to a novel adiabatic transport coefficient, which transpires in the adiabatic curvature of QH states on moduli spaces of Riemann surfaces of genus \( g \geq 2 \) [21], see also [6]. This is a universal finite size correction \( \frac{\alpha}{4\pi}(g - 1) \) to the Hall viscosity, where \( e_{\text{H}} \) is the central charge with \( e_{\text{H}} = 1 - 3\nu^{-1} \) for Laughlin states at the filling fraction \( \nu \). Coupling QH states to gravity can be also used as a tool to find the density of states function on the LLL [13] as well as the density-density correlation functions [8, 17].

The second reason behind the interest in QH states on curved backgrounds is the possibility of their experimental realisation. Recently synthetic Landau levels on a cone were constructed in a photon resonator [31]. There it was observed that at the cone tip the local density of states increases due to the curvature singularity and the measured excess is consistent with predictions of the theory of QHE in the curved space [32] and, in particular, for the density of states function (Bergman kernel) [13]. Graphene is a recent example where curvature of the sample occurs, giving rise to the strain-induced magnetic fields and quantum Hall effect, see e.g. [38] for review. The large curvature and high strain are observed in triangular nanoscale graphene bubbles [27], providing yet another system where the theory of QHE in curved spaces could potentially be tested.

Now we explain how the gravitational anomaly in a QH state can be computed. We are interested in the large \( N \) expansion of the so-called generating functional \( \log Z \), where \( Z \) is simply the normalization factor of the holomorphic QH wave function. For the integer QH state and for the Laughlin state \( Z \) has another interpretation as the partition function of the 2d Coulomb plasma. On a compact Riemann surface \( \Sigma \) in a constant perpendicular magnetic field with total flux \( k \) the generating functional admits the following asymptotic expansion

\[
\log Z[\delta_0, \delta] = \sum_{m=0}^{\infty} k^{2-m} c_{2-m} S_{2-m}(\delta_0, \delta),
\]

(1)

at large \( k \). Here \( \delta_0 \) is a background metric on \( \Sigma \), which can be taken, e.g., as constant scalar curvature metric, \( \delta \) is an arbitrary curved metric on \( \Sigma \) and \( S_m \) denotes certain geometric functionals, see equations (11)–(13). Since the number of particles \( N \) and the total flux of the magnetic field \( k \) are related by the Riemann-Roch formula \( N = k + 1 - g \), where \( g \) is the genus of \( \Sigma \), large \( N \) and large \( k \) expansions are essentially equivalent. Equation (1) was derived by various methods for 2d Coulomb gas in the planar domains [39], for the integer QH state on compact Riemann surfaces [20, 22] and for the fractional QH states such as the Laughlin states [8, 9, 15, 23].
The gravitational anomaly is given by the Liouville functional $S_0(g_0, g) = S_L(g_0, g)$, see equation (13) for the definition, in the order $O(1)$ term in the expansion above. Consider the regularized spectral determinant of the scalar laplacian operator $\Delta_g$, i.e. regularized product of all non-zero eigenvalues of $\Delta_g$. The latter is usually defined as $\text{det}' \Delta = e^{-\zeta(0, \Delta)}$ via the spectral zeta function of the laplacian, defined in equation (23). Then the Liouville functional represents the effect of the gravitational (conformal) anomaly

$$\frac{\text{det}' \Delta_g}{\text{det}' \Delta_{g_0}} = e^{-\frac{1}{6} S_L(g_0, g)},$$

when the metric is changed by a Weyl factor $g = e^{\sigma} g_0$, where $\sigma$ is an arbitrary smooth scalar function on $\Sigma$.

Although subleading at large $k$, the gravitational anomaly term in equation (1) becomes the most important in certain situations, particularly for singular Riemann surfaces. These include cones, cusps, and, in the case of Riemann surfaces with boundary, corners. One of the assumptions going into the derivation of (1) is the $C^\infty(\Sigma)$-smoothness of the metric $g$, when the underlying asymptotic expansion of the Bergman kernel is valid [20]. In the non-smooth case the formula (1) breaks down, and analysis has to be reconsidered. In this paper we suggest the following formula for $\log Z$ for the integer QH, in the limit of large number of particles

$$\log Z = c_2 k^2 + c_1 k + (\zeta(0, \Delta_{\text{sing}}) - \zeta(0, \Delta_{g_0})) \log(k h_{\text{sing}}) - \frac{1}{2} \log \frac{\text{det}' \Delta_{\text{sing}}}{\text{det}' \Delta_{g_0}} + c_0 + O(1/k).$$

Here $\zeta(s, \Delta_{\text{sing}})$ is the spectral zeta function of the laplacian for the metric with conical singularities on $\Sigma$ and the constants $c_m$ are related to the values of the geometric functionals. The first logarithmic term involves the length scale $l_{\text{sing}}$ induced by the singularities, which is roughly the diameter of the singular manifold. This length scale is in general bigger than the magnetic length $l_B = \sqrt{\hbar/eB}$.

We test the formula above for model geometries, where $Z$ can be computed explicitly and find simple expressions for $c_0, c_1, c_2$ as the functions of cone angles. Zeta determinant $\text{det}' \Delta_{\text{sing}}$ is a more complicated object, which in our examples is expressed through the Barnes double zeta function (22). The study of spectral geometry on manifolds with regular singularities was initiated by Cheeger [11], and the formulas for spectral zeta function on cones, relevant to the present work, were obtained in [2, 14, 33, 34]. The standard approach to the spectral zeta function on manifolds with regular singularities involves a combination of heat kernel methods and solving the spectral problem for the appropriate self-adjoint extensions of the associated laplacian [7, 11].

In this regard equation (3) suggests a novel view on the singular zeta determinants. In order to construct $Z$ on the lhs of (3) one needs to know only the following ingredients: the holomorphic wave functions on the lowest Landau level, the volume form on $\Sigma$ derived from the singular metric and the corresponding potential function for the metric, the Kähler potential. Recall that the holomorphic LLL wave functions solve the first order $\bar{\partial}$-equation on $\Sigma$ and therefore are metric-independent [13]. Remarkably, definition of $Z$ involves no knowledge of the spectrum of $\Delta_g$—it appears that the relevant information about the spectrum is encoded in the lowest energy level of the magnetic Shrödinger equation.

There is a certain degree of similarity between equation (3) and the behavior of the free energy at criticality for the system of characteristic size $l$ in presence of singularities [10]. In the singular case the first three terms in (3), were recently computed in [24] and the behavior...
of free energy was interpreted as the emergence of conformal symmetry in the presence of singularities. In another recent work on singularities in QH states [5, 19] gravitational anomaly for geometric defects on higher genus curves with \( \mathbb{Z}_n \) symmetry was studied. Landau levels on a cone have been extensively studied in the literature, see e.g. [28] and especially [29] for the comprehensive list of references. However in the present work we are only interested in the lowest Landau level, where subtleties associated with the construction of the higher Landau level wave functions do not arise.

2. Integer QH state on curved backgrounds

We begin with the definition of the integer QH state on the compact Riemann surface \( \Sigma \). Consider a smooth background metric \( g_0 \) on \( \Sigma \), which without loss of generality we choose to be a constant scalar curvature metric, and a constant perpendicular magnetic field \( B_0 \), with the total flux \( k \in \mathbb{Z}_+ \) through the surface \( k = \frac{1}{2} \int B_0 \sqrt{g_0} d^2z \). The latter is described by the holomorphic line bundle \( L_k \) of degree \( k \) on \( \Sigma \) with the Hermitian metric \( h_{k0}^0(z, \bar{z}) \) (magnetic potential), related to the magnetic field as

\[
B_0 = -g_0^{\bar{z}z} \partial_{\bar{z}} \partial_z \log h_{k0}^0(z, \bar{z}).
\]

The degeneracy \( N \) of the LLL is fixed by the Riemann-Roch theorem \( N = k + 1 - g \). The integer QH state for \( N \) particles is given by the Slater determinant of the wave functions on completely filled LLL

\[
\Psi(z_1, ..., z_N) = \frac{1}{\sqrt{N!}} \det [s_j(z_l)]_{l,j=1}^N,
\]

and its Hermitian norm is

\[
|\Psi(z_1, ..., z_N)|_{h_0}^2 = \frac{1}{N!} |\det s_j(z_l)|^2 \prod_{j=1}^N h_{k0}^0(z_j, \bar{z}_j).
\]

By construction, this state is \( L^2 \) normalized.

Suppose we now have an arbitrary curved metric of the same area, which can be conveniently parameterized by the relative Kähler potential \( \phi \),

\[
g_{zz} = g_{0zz} + \partial_z \partial_{\bar{z}} \phi.
\]

We would like to define the normalized integer QHE state in the curved metric. Setting magnetic field to constant in the new metric

\[
B = -g^{\bar{z}z} \partial_{\bar{z}} \partial_z \log h^k(z, \bar{z}) = k
\]

leads to the relation between the magnetic potentials \( h^k = h_{k0}^0 e^{-k \phi} \). The holomorphic part of the integer QHE state can be taken as in equation (5) and its Hermitian norm in the new metric is given by
\[ |\Psi(z_1, \ldots, z_N)\rangle_\hbar = \frac{1}{Z[g_0, g]} \frac{1}{N!} \det s_j(z_i)^2 \prod_{j=1}^N h_j^k(z_j, \bar{z}_j). \]

By construction the normalization factor \( Z \) of this state depends on the pair of metrics \( g_0, g \). It reads

\[ Z[g_0, g] = \frac{1}{(2\pi)^N N!} \int_{\Sigma^N} \det s_j(z_i)^2 \prod_{j=1}^N h_j^k(z_j, \bar{z}_j) e^{-k\phi(z_j, \bar{z}_j)} \sqrt{g} \, dz_j. \] (8)

We can equivalently rewrite \( Z \) as the determinant of the matrix of \( L^2 \) norms

\[ Z[g_0, g] = \det \frac{1}{2\pi} \int_{\Sigma} s_j(z) s_l(z) h_0^k(z, \bar{z}) e^{-k\phi(z)} \sqrt{g} \, dz = \det \langle s_j, s_l \rangle_g. \] (9)

In particular, this makes the boundary condition \( Z[g_0, g_0] = 1 \) obvious. If the basis of sections \( s_j \) is orthogonal (but not orthonormal) in the metric \( g \), so that \( \langle s_j, s_l \rangle_g = \delta_{jl} n_j \) for some \( n_j \), then the normalization factor reduces to the product of normalization factors \( n_j \) for one particle LLL states

\[ Z[g_0, g] = \prod_{j=1}^N n_j. \]

The logarithm of \( Z \) is called the generating functional. For the smooth metrics \( g_0, g \) the generating functional admits large \( k \) asymptotic expansion with the coefficients given by geometric functionals of the metrics. The first few terms of the expansion read [20, equation (5.2)],

\[ \log Z = -k^2 S_{AY}(g_0, \phi) + \frac{k}{2} S_M(g_0, \phi) + \frac{1}{6} S_L(g_0, \phi) - \frac{5}{192k} \left( \frac{1}{2\pi} \int_{\Sigma} R^2 \sqrt{g} \, dz - \frac{1}{2\pi} \int_{\Sigma} R_0^2 \sqrt{g_0} \, dz \right) + O(1/k^2). \] (10)

Here the subleading terms in \( 1/k \) consist of integrals of curvature invariants of the metric, i.e. given by polynomials in scalar curvature \( R \) and its derivatives. The first three terms are given by the Aubin-Yau, Mabuchi and Liouville functionals of the Kähler potential

\[ S_{AY}(g_0, \phi) = \frac{1}{2\pi} \int_{\Sigma} (\phi \partial_j \partial_l \phi + \phi \sqrt{g_0}) \, d^2 z, \]

\[ S_M(g_0, \phi) = \frac{1}{2\pi} \int_{\Sigma} \left( -\frac{1}{2} \phi R_0 \sqrt{g_0} + \sqrt{g} \log \frac{\sqrt{g}}{\sqrt{g_0}} \right) \, d^2 z, \]

\[ S_L(g_0, \phi) = \frac{1}{2\pi} \int_{\Sigma} \left( -\log \frac{\sqrt{g}}{\sqrt{g_0}} \partial_j \partial_l \log \frac{\sqrt{g}}{\sqrt{g_0}} + \frac{1}{2} R_0 \sqrt{g_0} \log \frac{\sqrt{g}}{\sqrt{g_0}} \right) \, d^2 z. \] (11)-(13)

The important assumption in the derivation of the expansion equation (10) is smoothness of the metrics. In particular, we can immediately see, that the \( \int R^2 \) term blows up when curvature has delta-function singularities.

Now we consider what happens when the metric \( g \) is singular.
3. Conical singularities and geometric functionals

Two types of singularities are of particular interest in the quantum Hall states. One is the well-known magnetic singularities, when the magnetic field has the form \( B(z) = B_0(z) - 2\pi \varphi \delta(z_0) \) with \( B_0(z) \) being a smooth function and \( \delta(z_0) \) is the Dirac delta function at the point \( z_0 \). In the integer quantum Hall effect the delta-function term with \( \varphi \in \mathbb{R} \) corresponds to the Aharonov–Bohm flux, when the surface is not simply-connected as e.g. in the Laughlin’s quantization argument [25]. In the fractional quantum Hall states the delta-function singularity creates a quasi-hole excitation [26] with the charge \( \varphi \in \mathbb{Z} \) quantized on a compact surface.

A more rigid type of singularity is the curvature singularity of the metric on the surface. In the case of conical singularities at points \( z_\alpha \) the Ricci tensor

\[
R_{\bar{z}z} = -\partial_{\bar{z}} \partial_z \log g_{\bar{z}z}
\]

for the metric \( ds^2 = 2g_{\bar{z}z}dz d\bar{z} \) on \( \Sigma \) in local complex coordinates has the form

\[
R_{\bar{z}z} = 2\lambda g_{\bar{z}z} + \pi \sum_\alpha (1 - a_\alpha) \delta^2(z_\alpha),
\]

where \( \delta^2(z_\alpha) \) is the Dirac delta-function, which in a local euclidean chart \( U_\alpha \) around \( z_\alpha \) satisfies \( \int_{U_\alpha} \delta^2(z_\alpha) d^2z = 1 \). The conical singularities are characterized locally by the opening angle \( 2\pi a_\alpha \) with \( 0 \leq a_\alpha < 1 \), where \( a_\alpha \to 0 \) limit corresponds to a cusp and \( a \to 1 \) limit to a smooth surface. In equation (15) we also implicitly assumed that it is possible to choose a metric of constant scalar curvature outside the singular points, which will be the case in specific examples. The curvature singularities can be considered both in the integer and fractional QH states, but here we focus on the integer QH state. Since we are interested in the purely gravitational contribution to the generating functional, we set the AB-fluxes to zero \( \varphi = 0 \), although we shall note that both types of singularities can be considered simultaneously [24].

Near the conical point \( z_\alpha \) the metric has the following singular behavior

\[
g_{\bar{z}z} \sim |z - z_\alpha|^{2(1-a_\alpha)} f_{\text{reg}}(z),
\]

as follows from equations (14) and (15) and the identity \( \partial_{\bar{z}} \partial_z \log |z - z_\alpha|^2 = \pi \delta^2(z_\alpha) \). Hence, for a smooth background metric \( g_0 \) the Kähler potential in (6) is smooth at the singular point. Then it is not hard to check that the Aubin-Yau and Mabuchi functionals ((11) and (12)) are finite since all the terms are integrable. In the Liouville functional only the second term is integrable and the kinetic term behaves as \( \delta^2(z_\alpha) \log |z - z_\alpha| \) hence it diverges. The higher-order terms in the expansion of \( \log Z \) also diverge since they contain integrals of products of delta-functions at conical points. Hence the smooth-case expansion (10) breaks down at the order \( O(1) \).

In the next two sections we will compute asymptotics of \( \log Z \) for two model singular geometries and check the proposed general form (3) of the expansion in the singular case.

4. Integer QH state on the spindle

The spindle (or, american football) \( \Sigma_a \), see figure 1 (left), is the sphere with two antipodal conical points with equal opening angles \( 2\pi a \), \( 0 < a \leq 1 \) with \( a = 1 \) corresponding to the round sphere. The exists a metric of constant scalar curvature outside the conical points on this geometry [37] which reads
We normalized the total area as 
\[ \int_{\Sigma_a} \sqrt{g} d^2z = 2\pi \]
and in our notations \( \sqrt{g} = 2g_{\bar{z}z} \). The Ricci curvature of this metric is given by
\[ R_{\bar{z}z} = 2ag_{\bar{z}z} + \pi(1 - a)\delta^2(0) + \pi(1 - a)\delta^2(\infty), \]
and the Euler formula holds \( \int_{\Sigma_a} R \sqrt{g} d^2z = 8\pi \) since the surface is topologically a sphere. At \( a = 1 \) the metric (16) reduces to the standard round metric on the sphere \( g_0 = \frac{1}{(1 + |z|^2)^{3/2}} \). Thus the Kähler potential (6) for the spindle relative to the round metric reads
\[ \phi_{sp} = \log \left( \frac{1 + |z|^{2a}}{1 + |z|^2} \right)^{1/a}. \]

The constant magnetic field \( B_0 = k \) on the round sphere and on the spindle \( B = k \) correspond to the following magnetic potentials
\[ h_0^k(z, \bar{z}) = \frac{1}{(1 + |z|^2)^{k}}, \quad h^k(z, \bar{z}) = h_0^k e^{-k\phi_{sp}} = \frac{1}{(1 + |z|^{2a})^{k/a}}. \]

On the round sphere, the orthonormal basis of sections with respect to the inner product (4) reads
\[ s_j(z) = \sqrt{\frac{(k + 1)!}{(k + j + 1)(j - 1)!}} z^{j-1}, \quad j = 1, \ldots, k + 1. \]

Plugging the basis (18) into the determinantal formula (9) for the generating functional and performing the integrals we obtain
\[ \log Z_{sp} = \log \det M_{ij}, \]
\[ M_{ij} = \frac{(k + 1)!}{(k - j + 1)(j - 1)!} B \left( 1 + \frac{j - 1}{a}, 1 + \frac{k + 1 - j}{a} \right) \delta_{ij}, \]
where \( B(x, y) \) is the standard Beta-function. Thus we arrive at the following formula for the normalization factor

\[ \log Z_{sp} = \log \det M_{ij}, \]

\[ M_{ij} = \frac{(k + 1)!}{(k - j + 1)(j - 1)!} B \left( 1 + \frac{j - 1}{a}, 1 + \frac{k + 1 - j}{a} \right) \delta_{ij}, \]

\[ \text{Figure 1. Spindle } \Sigma_a \text{ (‘american football’) versus simple cone } D_a. \]
\[ Z_{sp} = \left( \frac{\Gamma(2 + k)}{\Gamma(2 + \frac{k}{a})} \right)^{k+1} \left( \frac{\prod_{j=1}^{k} \Gamma(1 + \frac{j}{a})}{G(2 + k)} \right)^2, \]

(20)

where \( G(z) \) is the Barnes \( G \)-function, see e.g. [16]. As a consistency check we immediately see that the boundary condition \( Z|_{a=1} = 1 \) holds, since \( G(n) = \prod_{j=1}^{n-2} \Gamma(j+1) \) for integer \( n \).

The large \( k \) asymptotics of \( \log Z \) can be now evaluated using Euler–Maclaurin summation formula, see Appendix A for details,

\[
\log Z_{sp} = -\frac{1-a}{2a} k^2 + \frac{\log a}{2} k + \frac{1}{6} \left( a + \frac{1}{a} - 2 \right) \log k + 2 \zeta'_2(0,a,1,1) - 2 \zeta'_2(-1) + \frac{13}{12} (1-a) + \frac{3}{2} \log a + \mathcal{O}(1/k)
\]

(21)

where \( \zeta_2 \) is the Barnes double zeta function, defined by the following double series,

\[
\zeta_2(s,a,b,x) = \sum_{m,n=0} \frac{1}{(am + bn + x)^s}.
\]

(22)

For more details on this function we refer to [35] and references therein.

Note that non-local \( k^2 \log k \) and \( k \log k \) terms cancel in (21), which is an important consistency check. The \( \log k \) and \( \mathcal{O}(1) \) terms are related to the spectral zeta function for the scalar laplacian \( \Delta_s \), defined as

\[
\zeta(s,\Delta_s) = \sum_{\lambda \neq 0} \lambda^{-s},
\]

(23)

where the sum goes over all positive eigenvalues \( \lambda \) with multiplicities. The zeta determinant of laplacian on the spindle with cone angle \( 2\pi a \) was computed in [33], see Appendix B for more details. Here we quote the result in equation (B.8)

\[
\log \det' \Delta_s = -\zeta'(0,\Delta_s) = -4 \zeta'_2(0,a,1,1) + \frac{a}{2} - 2 \log a - \left( \frac{a}{6} + \frac{1}{6a} - 1 \right) \log \frac{a}{2}.
\]

At \( a = 1 \) this reduces to the value of the zeta determinant on the round sphere

\[
\log \det' \Delta_{s0} = -4 \zeta'_2(-1) + \frac{1}{2} - \frac{2}{3} \log 2,
\]

(24)

with the area normalized to \( 2\pi \), see equation (B.10). We also need the value of the zeta function at zero: \( \zeta(0,\Delta_s) = -1 + \frac{s}{2} + \frac{s^2}{12} \) and \( \zeta(0,\Delta_{s0}) = -\frac{3}{4} \). Then we can recast the formula for the generating functional (21) in the form announced in equation (3)

\[
\log Z_{sp} = c_2 k^2 + c_1 k + \left( \zeta(0,\Delta_s) - \zeta(0,\Delta_{s0}) \right) \log kl_{ang} - \frac{1}{2} \log \frac{\det' \Delta_s}{\det' \Delta_{s0}} + c_0 + \mathcal{O}(1/k).
\]

(25)

It is straightforward to check that the first two terms in (25) coincide with the values of the Aubin-Yau and Mabuchi functionals, computed for the Kähler potential \( \phi_{sp} \) and the round metric,

\[
c_2 = -S_{AY}(g_0,\phi_{sp}) = -\frac{1-a}{2a}, \quad c_1 = \frac{1}{2} S_M(g_0,\phi_{sp}) = \frac{1}{2} \log a.
\]

(26)
and thus to the order $O(k)$ the expansion if consistent with the smooth case (10). There is also a correction constant $c_0$, which has a rather simple form

$$c_0 = \frac{5}{6}(1 - a + \log a).$$

(27)

While the exact nature of this constant is not clear, the natural guess is that it is presumably related to the value of the Liouville functional, which needs to be appropriately regularized.

Finally, the length scale here is given by $l_{\text{sing}} = \sqrt{2/a}$ and it is related to the distance between the two conical point as $d(0, \infty) = \pi l_{\text{sing}}/2$.

5. IQHE on a cone

Here we make the same calculation on the simple cone $D_a$ (see figure 1, right) with the angle $2\pi a$ and flat metric outside the singularity. We parameterize $D_a$ by the complex coordinate $z$ in the unit disk $0 \leq |z| \leq 1$. The metric on the cone

$$ds^2 = \frac{2adz d\bar{z}}{|z|^{2(1-a)}}$$

(28)

and has the area $2\pi$. Then we can solve the condition (7) of the constant perpendicular magnetic field and find the magnetic potential

$$h^k(z, \bar{z}) = e^{-\frac{1}{2}|z|^2}.$$

The flux of the magnetic field, when restricted to $D_a$ equals $\int_{D_a} B \sqrt{g} d^2 z = 2\pi k$. On the flat disk $a = 1$ the orthonormal basis of holomorphic LLL states is given by

$$s_j(z) = \sqrt{k^j/(j-1)!} z^{j-1}, \quad j \leq k.$$

This basis is infinite when considered on $\mathbb{C}$, therefore we introduce the cutoff $j \leq k$, restricting to the first $k$ states supported inside the disk.

Now we use this basis to determine the normalization factor (9) of the integer QH state one the cone $D_a$.

$$Z_{\text{cone}} = \prod_{j=1}^{k} k^j \left( \frac{k}{a} \right)^{-\frac{j-1}{2}} \frac{\Gamma\left(\frac{j-1}{2} + 1\right)}{\Gamma(j)}.$$

(29)

The asymptotic expansion of $\log Z_{\text{cone}}$ at large $k$ is obtained in equation (A.11) in appendix A and explicit formulas for the spectral zeta function on the cone with Dirichlet boundary conditions are presented in equations ((B.11)–(B.13)) in appendix B. Putting these together we arrive at the relation

$$\log Z_{\text{cone}} = b_0 k^2 + b_1 k + \left( \zeta(0, \Delta_g) - \zeta(0, \Delta_{g_0}) \right) \log k l_{\text{sing}} - \frac{1}{2} \log \det' \Delta_g + b_2 + O(1/k),$$

(30)

where $\zeta(0, \Delta_g) = \frac{1}{2\pi}(a + 1/a)$ and $\zeta(0, \Delta_{g_0}) = \frac{1}{6}$. The constants in this case are given by...
The length scale $l_{\text{sing}} = \sqrt{2/a}$ in this case is the distance between the conical point and the edge of the cone, $d(0, 1) = l_{\text{sing}}$.

6. Discussion

In this note we suggest and check in simple examples the following formula for the normalization factor of the integer QH state on singular Riemann surfaces

$$Z = \left[ \frac{\det' \Delta_{g_{\text{sing}}}^\text{reg}}{\det' \Delta_{g_0}^\text{reg}} \right]^{-1/2} \cdot (kl_{\text{sing}})^{\zeta_{\Delta_{g_{\text{sing}}}}(0) - \zeta_{\Delta_{g_0}}(0)} \cdot e^{c_2 k^2 + c_1 k + c_0 + O(1/k)},$$

(31)

Here $g_0$ is a smooth background metric, $\det' \Delta_{g_{\text{sing}}}^\text{reg}$ is the zeta regularized spectral determinant of the laplacian on a singular surface, $\zeta_{\Delta_{g_{\text{sing}}}}(s)$ is the spectral zeta function, $l_{\text{sing}}$ is a certain length scale associated with singularities and $c_m$ are simple constants related to the values of geometric functionals. In this work we tested this formula for the model geometries of the flat cone and the spindle, figure 1.

Some aspects of the relation equation (31) remain to be better understood. In particular, note that in view of equation (2), the power of $\det' \Delta_{g_{\text{sing}}}^\text{reg}$ in equation (31) is half of its smooth case value (10). On the one hand this is not completely surprising, since the expansion of $\log Z$ breaks down at the order $O(1)$ where $S_L$ starts to diverge, and there is no apriori reason to expect that equation (10) is valid in the singular case. However, it is possible that a regularized version of the equation (10) continues to hold, and the value of the constant $c_0$ could be equal to the $\frac{1}{12}S_\text{reg}^L$ for the regularized Liouville functional on the singular geometry. Also the definition of $l_{\text{sing}}$ remains to be understood for more general geometries.

Another interesting aspect of equation (31) that we would like to discuss is the appearance of the Barnes double zeta function in the explicit formulas (A.10) and (A.11) for $Z$. It would be interesting to compute $O(1)$ term in the norm of the Laughlin state on singular surface characterized by the metric $g_{\text{sing}}$. In the notations of equation (8) this is given by the following integral

$$Z_\beta[g_0, g_{\text{sing}}] = \frac{1}{N_0} \int_{\mathbb{C}^{k+1}} | \det s_j(z_m)|^{2\beta} \prod_{j=1}^{k+1} h^{3k}(z_j, \bar{z}_j) \sqrt{g_0} \, dz_j,$$

where $\beta \geq 1$ and $N_0$ is a normalisation constant chosen such that in the smooth limit $g_{\text{sing}} \to g_0$ we have $Z_\beta[g_0, g_0] = 1$.

Comparing to the $\beta = 1$ case, we conjecture that the terms up to $O(k)$ in $\log Z$ should coincide with the smooth case [15, equation (1.1)], i.e. are given by the values of the geometric functionals computed on the singular geometry. Conversely, we expect the $O(1)$ term to be a non-trivial $\beta$-deformation of the formulas for zeta determinants (e.g. one obvious guess is that $\zeta'_{\Delta_{g_{\text{sing}}}}(0, a, b, c)$ will appear with values of arguments $b, c \neq 1$) and depending on $\beta$.

In [24] the large $k$ behavior of the generating functional (3) was interpreted as emergence of conformal symmetry in presence of singularities. Better understanding of the $O(1)$ term in equation (3) could help answer the question: exactly what kind of conformal theory emerges in the singular case? If one is to think of the conical singularities as insertions vertex operators...
of charge $a$, then after the subtraction of large magnetic field terms, the $O(1)$ term could be
interpreted as a correlation function in the appropriate CFT. There is an example of a CFT
where Barnes special functions, such as Barnes double gamma function (closely related to $\zeta'_2$),
show up in correlation functions,—namely, the Liouville field theory [12, 40]. Emergence of
the Liouville quantum gravity in QH states at the tip of a cone is a thrilling possibility\(^2\), which
we suggest as an avenue for further investigations.

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**Appendix A**

Here we derive the asymptotics of $\log Z$ ((21) and (30)). We begin we a collection of use-
ful asymptotic formulas, which can be found e.g. in [16]. At large $z$ the Gamma function,
Barnes $G$-function, digamma function $\psi(z)$ and polygamma function $\psi_n(z)$ admit the follow-
ing expansions

$$
\log \Gamma(1 + z) = \left(z + \frac{1}{2}\right) \log z - z + \frac{1}{2} \log 2\pi + \frac{1}{12z} + O(1/z), \quad (A.1)
$$

$$
\log G(1 + z) = \left(\frac{z^2}{2} - \frac{1}{12}\right) \log z - \frac{3}{4} \log z + \frac{z}{2} \log 2\pi + \frac{1}{12z} + O(1/z), \quad (A.2)
$$

$$
\psi(1 + z) = \frac{\Gamma'(1 + z)}{\Gamma(1 + z)} = \log z + O(1/z), \quad (A.3)
$$

$$
\psi_n(1 + z) = \frac{d^n \psi(1 + z)}{dz^n} = O(1/z), \quad n > 1 \quad (A.4)
$$

where $\zeta_R(s)$ is the Riemann zeta function and $\zeta'_R(s) = \frac{d}{ds} \zeta_R(s)$. We will need the values of the
digamma and polygamma functions at 1:

$$
\psi(1) = -\gamma, \\
\psi_n(1) = (-1)^{n+1} n! \zeta_R(n + 1), \quad n > 1, \quad (A.5)
$$

where $\gamma$ is the Euler-Mascheroni constant. More generally

$$
\psi_n(z) = (-1)^{n+1} n! \zeta_H(n + 1, z),
$$

where the Hurwitz zeta function is defined as

$$
\zeta_H(s, z) = \sum_{j=0}^{\infty} \frac{1}{(j + z)^s}.
$$

\(^2\) Relevance of the Liouville theory to the description of QH states on singular surfaces has also been advocated by
P Wiegmann.
The first derivative of the Barnes double zeta function \( \zeta'_j(s, a, b, x) = \frac{\partial}{\partial s} \zeta_j(s, a, b, x) \) admits the following asymptotic expansions \[35\] for large and small \( a \), respectively,

\[ a \gg 1, \quad \zeta'_j(0, a, 1, 1) = -\frac{1}{12} \left( 3 + a + \frac{1}{a} \right) \log a + \left( \frac{1}{12} - \zeta'_j(-1) \right) a \]

\[-\frac{1}{4} \log 2\pi + \frac{\gamma}{12} a^2 + \sum_{j=2}^{\infty} \frac{B_{2j} \zeta_j(2j-1)}{2j(2j-1)} a^{2j-1} \]

\[ a \ll 1, \quad \zeta'_j(0, a, 1, 1) = \left( \frac{1}{12} - \zeta'_j(-1) \right) \frac{1}{a} - \frac{1}{4} \log 2\pi + \frac{\gamma}{12} a + \sum_{j=2}^{\infty} \frac{B_{2j} \zeta_j(2j-1)}{2j(2j-1)} a^{2j-1}. \]

(A.6)

The following integral appears below

\[ \int_0^z \log \Gamma(1 + x) \, dx = -\frac{1}{2} z(z + 1) + \frac{1}{2} z \log 2\pi + z \log \Gamma(1 + z) - \log G(1 + z). \]  

(A.7)

The large \( k \) asymptotics of the sum of \( \Gamma \)-functions can be calculated via the Euler–Maclaurin formula

\[ \sum_{j=1}^{k} \log \Gamma(1 + \frac{j}{a}) = \int_0^{k} \log \Gamma \left( 1 + \frac{j}{a} \right) \, dx + \frac{1}{2} \log \Gamma \left( 1 + \frac{k}{a} \right) + \frac{1}{12} \left( \psi \left( 1 + \frac{k}{a} \right) - \psi(1) \right) \]

\[ + \sum_{j=2}^{\infty} \frac{B_{2j}}{(2j)!} \left( \psi_{2(j-1)} \left( 1 + \frac{k}{a} \right) \right) \bigg|_{x=0} - \psi_{2(j-1)} \left( 1 + \frac{k}{a} \right) \bigg|_{x=0} \].

Now we use equation (A.7) and formulas ((A.1)–(A.4)) to find the large \( k \) asymptotics

\[ \sum_{j=1}^{k} \log \Gamma \left( 1 + \frac{j}{a} \right) = \frac{1}{2a} k^2 \log k - \left( \frac{1}{2a} \log a + \frac{3}{4a} \right) k^2 + \left( \frac{1}{2a} + \frac{1}{2} \right) k \log k \]

\[ - \left( \frac{1}{2} \log a + \frac{1}{2a} \log a + \frac{1}{2} + \frac{1}{2a} - \frac{1}{2} \log 2\pi \right) k + \frac{1}{12} \left( 3 + a + \frac{1}{a} \right) \log k \]

\[ + \zeta'_j(0, a, 1, 1) + \frac{1}{2} \log 2\pi + O(1/k). \]

(A.8)

Another way to compute this asymptotic expansion is to use the exact formula

\[ \prod_{j=1}^{k} \Gamma \left( 1 + \frac{j}{a} \right) = k!(2\pi)^{\frac{k-1}{2}} a^{-\frac{k-1}{2} - \frac{1}{2} - 1} \left[ \Gamma_2(k + 1, 1, a) \right]^{-1}. \]

(A.9)

This follows by induction from the functional equation \[35, \text{proposition 8.6}\] for the Barnes double Gamma function \( \Gamma_2 \) (see e.g. \[35, \text{section 1}\] for the definition), taking into account \( \Gamma_2(1, 1, 1) = \sqrt{2\pi a}^{-1/2} \). Then asymptotics (A.8) follows from the asymptotic expansion of the logarithm of \( \Gamma_2 \) \[35, \text{proposition 8.11}\].

Now we use the asymptotic formula (A.8) and equations ((A.1) and (A.2)) in order to find asymptotics of the generating functionals equations ((20) and (29))

12
\[ \log Z_{\text{sp}} = \frac{a - 1}{2a} k^2 + \frac{\log a}{2} k + \frac{1}{6} \left( a + \frac{1}{a} - 2 \right) \log k + 2\zeta'_2(0, a, 1, 1) - 2\zeta'_4(-1) + \frac{13}{12} (1 - a) + \frac{3}{2} \log a + \mathcal{O}(1/k), \]  
(A.10)

\[ \log Z_{\text{cone}} = -\frac{3(1 - a)}{4a} k^2 + \left( \frac{1}{a} - 1 + \log a \right) \frac{k}{2} + \left( \frac{a}{12} + \frac{1}{12a} - \frac{1}{6} \right) \log k + \zeta'_2(0, a, 1, 1) - \zeta'_4(-1) + \frac{1}{2} \log a + \mathcal{O}(1/k). \]  
(A.11)

It is not hard to show that

\[ \zeta'_3(0, 1, 1, 1) = \zeta'_4(-1), \]  
(A.12)

hence the boundary condition \( Z|_{a=1} = 1 \) holds.

**Appendix B**

Here we collect formulas for the spectral zeta function on the conical surfaces, derived in [33–35]. In order to compare the results with equations (A.10) and (A.11), we express the answers in terms of \( \zeta'_3(0, a, 1, 1) \) and also adjust the formulas, taking into account the normalization of the area \( A = 2\pi \), which is adopted in this paper for consistency with previous work.

In [33] the spectral zeta function is computed for the metric on the spindle

\[ d^2 s = d\theta^2 + \frac{1}{a^2} \sin^2 \theta d\varphi^2. \]  
(B.1)

The cone angle here is \( 2\pi/a \) and the area of the surface in this metric equals \( 4\pi/a \). Under the coordinate change \( \tan \theta/2 = 1/r^a \) and \( z = re^{i\varphi} \) the metric above reads

\[ d^2 s = \frac{4dzd\bar{z}}{a^2 |z|^{2(1-1/a)} (1 + |z|^{2/a})^2}. \]

We start with the formula [33, theorem 4.16] for the spectral zeta function\(^3\) in the metric (B.1)

\[ \zeta'(0, \Delta s_{1/a}) = -\left( \frac{a}{3} + \frac{1}{3a} \right) \log a - 2 \log 2\pi + \frac{a}{3} + 1 + \frac{1}{2a} + \log \Gamma \left( 1 + \frac{1}{a} \right) - 2a\zeta'_4(-1) - 2a\zeta'_4 \left( -1, 1 + \frac{1}{a} \right) + 2i \int_0^\infty \log \frac{\Gamma(1 + i\bar{z}) \Gamma(1 + \frac{1}{a} + i\bar{z})}{\Gamma(1 - i\bar{z}) \Gamma(1 + \frac{1}{a} - i\bar{z})} e^{2\pi y} dy. \]  
(B.2)

Now, we use the following formula [35, proposition 5.1] for the first derivative of the Barnes double zeta function at \( s = 0 \)

\[ \zeta'_3(0, a, b, x) = \left( -\frac{1}{2} \zeta' \left( 0, \frac{x}{a} \right) + \frac{a}{b} \zeta' \left( -1, \frac{x}{a} \right) - \frac{1}{12a} \right) \log a + \frac{1}{2} \log \Gamma \left( \frac{x}{a} \right) - \frac{1}{4} \log 2\pi - \frac{a}{b} \zeta' \left( -1, \frac{x}{a} \right) - \frac{a}{b} \zeta' \left( -1, \frac{x}{a} \right) + i \int_0^\infty \log \frac{\Gamma \left( \frac{x + iby}{a} \right)}{\Gamma \left( \frac{iby}{a} \right)} e^{2\pi y} dy. \]

\(^3\) Note the corrected [36] coefficient in the fifth term, see [33, theorem 4.16].
in order to express the last integral in (B.2) in terms of \( \zeta'_2(0; a, b, x) \) as follows

\[
\zeta'(0, \Delta g_{a/x}) = 2\zeta'_2(0, a, 1, a) + 2\zeta'_2(0, a, 1, a + 1) - \frac{1}{2a} - \log 2\pi
\]

\[
= 4\zeta'_2(0, a, 1, 1) - \frac{1}{2a} + \log a. \tag{B.3}
\]

The relevant values of the Hurwitz zeta function can be found, e.g. in \([16, 35]\),

\[
\zeta_H(-1, z) = -\frac{1}{2}z^2 + \frac{1}{2}z - \frac{1}{12}, \quad \zeta_H(0, z) = \frac{1}{2} - z, \quad \zeta_H(s, 1) = \zeta_R(s).
\]

In the second line in equation (B.3) we use the formulas

\[
\zeta'_2(0, a, 1, a) = \zeta'_2(0, a, 1, 1) + \frac{1}{2} \log a
\]

\[
\zeta'_2(0, a, 1, a + 1) = \zeta'_2(0, a, 1, 1) + \frac{1}{2} \log 2\pi.
\]

which easily follow from the definition (22). The following relation

\[
\zeta'_2(0, 1/a, 1, 1) = \zeta'_2(0, a, 1, 1) + \left( \frac{1}{4} + \frac{a}{12} + \frac{1}{12a} \right) \log a
\]

also follows from (22). This relation is needed to invert the cone angle \( 1/a \to a \), in order to compare with the metric (16). We obtain

\[
\zeta'(0, \Delta g) = 4\zeta'_2(0, a, 1, 1) - \frac{a}{2} + \frac{1}{3} \left( a + \frac{1}{a} \right) \log a, \tag{B.4}
\]

for the spindle with cone angle \( a \) and the metric

\[
d^2s = \frac{4a^2dzd\bar{z}}{|z|^{2(1-a)}(1 + |z|^{2a})^2}. \tag{B.5}
\]

This metric has the area \( 4\pi a \) and differs from the metric (16), which has the area \( 2\pi \), by a factor \( 2a \). Hence the corresponding laplacians are related as \( \Delta_g = 2a \Delta g_{a/x} \), where \( \Delta_g = 2g^{zz}\partial_z\partial_{\bar{z}} \) is the laplacian in the metric (16) and \( \Delta g_{a/x} \) is the laplacian in the metric (B.5). Using the relation

\[
\zeta'(0, C\Delta_g) = \zeta'(0, \Delta_g) - \zeta(0, \Delta_g) \log C, \tag{B.6}
\]

we obtain

\[
\zeta'(0, \Delta_g) = \zeta'(0, \Delta g_{a/x}) - \zeta(0, \Delta_g) \log 2a.
\]

The value of the zeta function at zero can be read off from [33, theorem 4.15]

\[
\zeta(0, \Delta_g) = -1 + \frac{a}{6} + \frac{1}{6a}. \tag{B.7}
\]

Thus we arrive at the final formula for the logarithm of the regularized determinant of laplacian in the metric (16),

\[
\zeta'(0, \Delta_g) = 4\zeta'_2(0, a, 1, 1) - \frac{a}{2} + 2 \log a + \left( -1 + \frac{a}{6} + \frac{1}{6a} \right) \log \frac{a}{2}. \tag{B.8}
\]

In the smooth limit at \( a = 1 \) the values of zeta function equations (B.7) and (B.8) reduce to the following values
\[ \zeta(0, \Delta_{g_0}) = -\frac{2}{3}, \]  
(B.9)

\[ \zeta'(0, \Delta_{g_0}) = 4\zeta'_d(-1) - \frac{1}{2} + \frac{2}{3}\log 2, \]  
(B.10)

where we used (A.12). The standard value of \( \zeta'(0) \) on the sphere of area \( 4\pi \) is \( \zeta'(0, \Delta_{S^2}) = 4\zeta'_d(-1) - \frac{1}{2} \) and the log 2 term in (B.10) is because the area of the sphere in the metric \( g_0 \) is \( 2\pi \).

Explicit formulas for the zeta function on the cone were obtained in [34]. In the notations of [34] the cone has the area \( \pi l^2/\nu \), where \( \nu = 1/a \). Again, using the relation (B.6) we can adopt the formulas in [34] to our case

\[ \zeta(0, \Delta_{g_0}) = \zeta(0, \Delta_g) = \frac{1}{6}, \]

(B.11)

\[ \zeta'(0, \Delta_{g_0}) = 2\zeta'_d(-1) - \frac{1}{6}\log 2 + \frac{5}{12} + \frac{1}{2}\log 2\pi, \]

(B.12)

\[ \zeta'(0, \Delta_g) = 2\zeta'_d(0, a, 1, 1) + \frac{1}{12} \left( a + \frac{1}{a} \right) \log \frac{a}{2} + \frac{5a}{12} + \frac{1}{2}\log a + \frac{1}{2}\log 2\pi, \]

(B.13)

where the notations in the formulas above imply the metric \( g \) on the cone, as given in equation (28), \( g_0 \) is the metric on the disk \( a = 1 \), with both metrics of the area \( 2\pi \).

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