PROBABILISTIC APPROACH TO QUANTUM SEPARATION EFFECT FOR FEYNMAN-KAC SEMIGROUP

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Abstract. The quantum tunnelling phenomenon allows a particle in Schrödinger mechanics to tunnel through a barrier that it classically could not overcome. Even infinite potentials do not always form impenetrable barriers. We discuss an answer to the following question: What is a critical magnitude of potential, which creates an impenetrable barrier and for which the corresponding Schrödinger evolution system separates? In addition we describe some quantitative estimates for the separating effect in terms of cut-off potentials.

1. Introduction

The main motivation for our study comes from the notion of the quantum tunnelling effect, a phenomenon which illuminates a striking difference between classical and quantum mechanics. It allows a microscopic particle to pass through the classically forbidden potential barrier, even if its height is infinite. It is easily predicted and explained by Schrödinger mechanics - the eigenstates of the Hamiltonian of the system cannot be localised. In this work we consider a quantum well and investigate the possibility that a particle trapped in a well cannot escape, that is the possibility that the barrier separates two regions. To be more precise, we consider the domain $D$ and its boundary $K = \partial D$ that separates $D$ and its complement $D^c$. Then we fix the special class of potentials $V$ singular near $K$ and consider the Hamiltonian of the system, that is the operator

$$H_V = \Delta - V$$

initially defined for a function belonging to $C^\infty_c(\mathbb{R}^d \setminus K)$. Here $\Delta$ is the positive standard Laplace operator. Then we consider the Feynman-Kac semigroup $\exp(-tH_V)$ generated by the Hamiltonian $H_V$ and we denote by $p^V_t(x,y)$ the corresponding heat kernel. We address the question: when does $\exp(-tH_V)$ separate $D$ and $D^c$, that is when is $p^V_t(x,y) = 0$ for $x \in D$ and $y \in D^c$?

When the domain $D$ has a smooth boundary, the problem considered by us has been satisfactorily resolved by Wu in [18]. In our work we generalise the results obtained by Wu. The essential difference compared to [18] is that we do not require smoothness.
of the considered domain $D$. The domains we consider are irregular fractals of some special but still general type including the Koch snowflake domain. In addition, we study quantitative estimates of the tunnelling effect in our setting. Namely, we consider cut-off potentials and we estimate the rate at which they suppress the semigroup kernel $p_t(x, y)$ when $x, y$ are separated by the boundary $K = \partial D$, see Section 5 below.

In order to deal with irregular domains, we develop a new approach, different than in Wu. For the case of the separation problem, it is still an elementary and simple probabilistic argument based on the Paley-Zygmund inequality and Blumenthal's zero-one law. It becomes more involved Brownian paths analysis for the quantitative description of the tunnelling.

The assumptions we impose on the potential $V$, are optimal within the classes we consider. The estimates which we discuss in our note are strictly connected to the boundary behaviour of the Brownian motion. We mention [3], [4] as papers studying diffusion in this direction. Our motivation for the techniques we use partially comes from the analysis in [3, 4] and [17].

We would like to add that the questions concerning separation can be posed for any semigroup of operators, even without direct relations to Schrödinger mechanics. We mention work [7] where the authors study similar phenomena for certain types of divergence form elliptic operators. The separation phenomenon for semigroups is also related to regularity theory of the solutions of Partial Differential Equations which was investigated in [8] and [16]. Interestingly in [8] and [16] sufficient and often necessary conditions for the regularity of the system (which contradicts the separation) are formulated in terms of integrability of the coefficients of the corresponding operators whereas we consider an assumption which can be essentially formulated in term of integrability of the potential $V$.

2. Preliminaries

Let $X = \{X_t\}_{t \geq 0}$ be the standard Brownian motion in $\mathbb{R}^d$. It is by now classical that $X = \{X_t\}_{t \geq 0}$ is a strong Markov process with continuous trajectories, see e.g. [2] for basic properties of Brownian motion. Let $D \subset \mathbb{R}^d$ be open, $D^c = \mathbb{R}^d \setminus D$ be its complement. Let $V \geq 0$ be locally bounded on $D \cup \text{int}(D^c) \subset \mathbb{R}^d$. Following [5], we define the Feynman-Kac functional by the formula

\begin{equation}
(1) \quad e_V(t) = \exp(-A_V(t))
\end{equation}

where

\[ A_V(t) = \int_0^t V(X_s)ds. \]

Then the one parameter family of operators $\{T_t, t \geq 0\}$

\begin{equation}
(2) \quad T_t f(x) = E^x \{e_V(t) f(X_t)\}
\end{equation}

where by $E^x$ we denote expected value over the Brownian motion starting at point $x \in \mathbb{R}^d$, is called Feynman-Kac semigroup, see [5, (26) p. 76]. It is well known that the operators $\{T_t, t \geq 0\}$ form the one parameter strongly continuous symmetric semigroup of contractions on $L^p(\mathbb{R}^d)$ for all $1 \leq p < \infty$. Moreover $C_c^\infty(D \cup \text{int}(D^c))$
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is contained in the domain of its infinitesimal generator $-H_V$, and for $\phi \in C_c^\infty(D \cup \text{int}(D^c))$ we have

$$H_V \phi(x) = (\Delta - V(x))\phi(x).$$

We say that the semigroup $\{T_t, t \geq 0\}$ separates the sets $D$ and $D^c$, if all operators $T_t$ preserve the subspace $L^2(D) \subset L^2(R^d)$, that is

$$T_t(L^2(D)) \subset L^2(D).$$

Note that the operators $\{T_t, t \geq 0\}$ are symmetric so (3) implies that the subspace $L^2(D^c)$ is also preserved.

In our approach the Paley-Zygmund inequality plays a crucial role. It bounds the probability that a positive random variable is small, in terms of its mean and variance. Let us recall the statement of this result.

**Proposition 1.** Suppose that $Z \geq 0$ is a positive random variable with finite variance and that $0 < \theta < 1$. Then

$$P(Z \geq \theta E(Z)) \geq (1 - \theta)^2 \frac{E(Z)^2}{E(Z^2)}.$$ 

*Proof.* Note that

$$E(Z) = E(Z \chi_{Z < \theta E(Z)}) + E(Z \chi_{Z \geq \theta E(Z)}).$$

Obviously, the first addend is at most $\theta E(Z)$. By the Cauchy-Schwarz inequality the second one is at most

$$E(Z^2)^{1/2} E(\chi_{Z \geq \theta E(Z)})^{1/2} = E(Z^2)^{1/2} P(Z \geq \theta E(Z))^{1/2}.$$ 

This proves the required estimate. $\square$

3. **Quantum separation for Feynman-Kac semigroups**

Consider a closed subset $K \subset R^d$ with dimension $d \geq 2$. We will impose fractal like type regularity requirements for $K$. Namely for any $\gamma > 0$ we define the $\gamma$ neighbourhood $K_\gamma$ of $K$ by the formula

$$K_\gamma = \{x \in R^d: \inf_{y \in K} |x - y| \leq \gamma\}.$$

In what follows we will always assume that there exist an exponent $0 < \alpha < d$ and a positive constant $C_1$ such that

$$|K_\gamma \cap B(x, r)| \leq C_1 r^{\alpha \gamma^{d-\alpha}}$$

for all $x \in R^d$ and $1 \geq r \geq \gamma > 0$. We also assume that there exists a positive constant $C_2$ such that

$$|K_\gamma \cap B(x, r)| \geq C_2 r^{\alpha \gamma^{d-\alpha}}$$

for all $x \in K_{r/2}$ and all $1 \geq r \geq \gamma > 0$.

The above regularity conditions are frequently considered in the literature and are motivated by the notion of Minkowski dimension (which is also called box dimension) see for example §3.1 and Proposition 3.2 of [9]. These conditions are also closely related to the notion of Ahlfors regularity, which is often used in the context of analysis on metric spaces, see for example [6] and [11]. Using the standard techniques, one can
check that these conditions are satisfied for most of the standard fractal constructions including the classical van Koch snowflake curve, again see for example [9]. If the boundary of the region $D$ is regular enough, for example if $K = \partial D$ is an immersed $C^1$ manifold then it is immediate that the estimates (4) and (5) hold with $\alpha$ equals to topological dimension of $K$.

We define the distance from $K$ by the formula $d_K(x) = \inf\{d(x, y) : y \in K\}$ and then we set

$$V_\beta = C_V d_K^{-\beta}. \tag{6}$$

The precise value of the constant is irrelevant for our analysis, so in what follows we fix $C_V = 1$.

The first result which we going to discuss can be stated in the following way

**Theorem 2.** Assume that a closed subset $K \subset \mathbb{R}^d$, for some $d \geq 2$ satisfies conditions (4) and (5) with some $d > \alpha > d - 2$. Suppose next that $\{X_s\}_{s \geq 0}$ is the Brownian motion starting at point $x$ contained in $K$ that is such that $X_0 = x \in K$. Then

$$P \left( \int_0^\delta V_\beta(X_s) \, ds = \infty, \forall \delta > 0 \right) = 1$$

for every $\beta$ such that $\beta + \alpha \geq d$.

**Remark.** It is not difficult to show that if $\{X_s\}_{s \geq 0}$ is the Brownian motion in the Euclidean space $\mathbb{R}^d$ starting at the origin then

$$P \left( \int_0^\delta |X_s|^{-2} \, ds = \infty, \forall \delta > 0 \right) = 1.$$ 

Hence there is no point to study the case $\beta \geq 2$ and we can assume that $\alpha > d - 2$.

**Proof.** Note that by taking intersection of $K$ with a closed ball $B(x, 1)$ we can assume without loss of generality that the set $K$ is compact. Next note that it follows from (4) and (5) that if one takes $a > 0$ such that $C_2/2 \geq a^{d-\alpha}C_1$ then for $C_3 = C_2/2$

$$\left| (K_\gamma \setminus K_{a\gamma}) \cap B(x, r) \right| \geq C_3 r^{\alpha} \gamma^{d-\alpha} \quad \forall x \in K_{r/2}. \tag{7}$$

Now for any $n \in \mathbb{N}$ we set

$$K'_n = K_{a^n} \setminus K_{a^{n+1}},$$

where $a$ is the constant from estimate (7). We define a sequence of random variables $Z_n$ by the following formula.

$$Z_n = \int_0^{\delta^2} \chi_{K'_n}(X_s) \, ds$$

where $\chi_{K'_n}$ is the characteristic function of the set $K'_n$ described above and $\{X_t, t \geq 0\}$ is the Brownian motion process starting at some fixed point $x \in K$.

Following the idea of [17], we shall verify assumptions of the Paley-Zygmund inequality for each random variable $Z_n$. Set $b_n = a^{n(d-\alpha)}\delta^{2-d+\alpha}$. We shall prove the following estimates for the expected values of $Z_n$ and $Z_n^2$

$$E(Z_n) \geq Cb_n \quad \text{and} \quad E(Z_n^2) \geq Cb_n^2 \tag{8}$$
(9) \[ E(Z^2_n) \leq c b_n^2 \]
valid for all \( n \in \mathbb{N} \) such that \( a^n \leq \delta^2 \). The constants \( c, C \) in (8) and (9) do not depend on \( n \). In order to prove (8) we note that if \( d > 2 \) then for positive constants \( C, C', c > 0 \)
\[ C' r^{2-d} \exp \left( -\frac{c' r^2}{\delta^2} \right) \leq \int_0^{\delta^2} t^{-d/2} \exp \left( -\frac{r^2}{4t} \right) dt \leq \int_0^{\delta^2} t^{-d/2} \exp \left( -\frac{r^2}{4t} \right) dt \leq C(1+|\log(\delta/r)|) \exp \left( -\frac{cr^2}{\delta^2} \right). \]
Whereas for \( d = 2 \)
\[ C'(1+|\log(\delta/r)|) \exp \left( -\frac{c' r^2}{\delta^2} \right) \leq \int_0^{\delta^2} t^{-d/2} \exp \left( -\frac{r^2}{4t} \right) dt \leq C(1+|\log(\delta/r)|) \exp \left( -\frac{cr^2}{\delta^2} \right). \]

Now assume that \( d > 2 \). By Fubini’s theorem, using the standard Gaussian distribution \( p_t(x,y) = (4\pi t)^{-d/2} e^{-\frac{|x-y|^2}{4t}} \) we have
\[
E(Z_n) = \int_0^{\delta^2} \int_{\mathbb{R}^d} \chi K_n(y)(4t\pi)^{-d/2} \exp \left( -\frac{|x-y|^2}{4t} \right) dy dt
= \int_{\mathbb{R}^d} \int_0^{\delta^2} \chi K_n(y)(4t\pi)^{-d/2} \exp \left( -\frac{|x-y|^2}{4t} \right) dt dy.
\]
By (10) and then by (7)
\[
E(Z_n) = \int_{\mathbb{R}^d} \int_0^{\delta^2} \chi K_n(y)(4t\pi)^{-d/2} \exp \left( -\frac{|x-y|^2}{4t} \right) dt dy
\geq C \int_{|x-y| \leq \delta} \chi K_n(y)|x-y|^{2-d} dy \geq c\delta^{(2-d)} \int_{K_n \cap B(x,\delta)} dy
\geq cC' \delta^{(2-d)} a^n (d-\alpha) \delta^\alpha = c' b_n.
\]
This proves estimate (8).

In the next step of the proof we will verify estimate (9).
\[
Z^2_n = \int_0^{\delta^2} \int_0^{\delta^2} \chi K_n(X_s(\omega)) \chi K_n(X_t(\omega)) ds dt
= \int_{s \leq t \leq \delta^2} \chi K_n(X_s(\omega)) \chi K_n(X_t(\omega)) ds dt + \int_{t \leq s \leq \delta^2} \chi K_n(X_s(\omega)) \chi K_n(X_t(\omega)) ds dt.
\]
Now for \( t > s \) it follows from the independence of \( X_s \) and \( X_t - X_s \)
\[
\int \int_{s \leq t \leq \delta^2} \chi K_n(X_s(\omega)) \chi K_n(X_t(\omega)) ds dt d\omega
= \int \int_{s \leq t \leq \delta^2} \chi K_n(X_s(\omega)) \chi K_n(X_t(\omega) - X_s(\omega) + X_s(\omega)) ds dt d\omega
= \int_0^{\delta^2} \int_0^t \int \int \chi K_n(y) \chi K_n(z+y) (4\pi)^d (s(t-s))^{d/2} \exp \left( -\frac{|x-y|^2}{4s} \right) \exp \left( -\frac{|z|^2}{4(t-s)} \right) dz dy ds dt.
\]
Next, by (10)

\[
\int_0^t \int_0^t \int_0^t \frac{\chi_K'(y)\chi_K'(z+y)}{4\pi^d|s(t-s)|^{d/2}} \exp \left( -\frac{|x-y|^2}{4s} \right) \exp \left( -\frac{|z|^2}{4(t-s)} \right) dz dy ds dt \\
\leq C \int \int \frac{\chi_K'(y)\chi_K'(z+y)}{|x-y|^{d-2}|z|^{d-2}} \exp \left( -\frac{c|x-y|^2}{\delta^2} \right) \exp \left( -\frac{c|z|^2}{\delta^2} \right) dz dy =: I.
\]

To estimate the term \( I \) we note in the same way as at the beginning of the proof that without loss of generality we can assume that \( K \subset B(x, 1) \) and for any \( m \in \mathbb{N} \) set

\[ A_m = \{ z \in \mathbb{R}^d : a^{m+1} < |z| \leq a^m \}. \]

Then

\[
\int \chi_K'(z+y)|z|^{2-d} \exp \left( -\frac{c|z|^2}{\delta^2} \right) dz \\
= \sum_{m \in \mathbb{N}} \int_{A_m} \chi_K'(z+y)|z|^{2-d} \exp \left( -\frac{c|z|^2}{\delta^2} \right) dz \\
\leq C \sum_{a^m < \delta} a^{m(2-d)} a^m a^{n(d-\alpha)} + C \sum_{a^m \geq \delta} a^{m(2-d)} a^m a^{n(d-\alpha)} \exp \left( -\frac{ca^{2m}}{\delta^2} \right) \\
\leq Ca^{n(d-\alpha)} \delta^{2-d+\alpha} + Ca^{n(d-\alpha)} \delta^{2-d+\alpha} \sum_{a^m \geq \delta} \left( \frac{a^m}{\delta} \right)^{2-d+\alpha} \exp \left( -\frac{ca^{2m}}{\delta^2} \right) \\
\leq Ca^{n(d-\alpha)} \delta^{2-d+\alpha} = Cb_n.
\]

By the above estimate

\[
\int \int \chi_K'(y)\chi_K'(z+y)|x-y|^{2-d} \exp \left( -\frac{c|x-y|^2}{\delta^2} \right) |z|^{2-d} \exp \left( -\frac{c|z|^2}{\delta^2} \right) dz dy \\
\leq Ca^{n(d-\alpha)} \delta^{2-d+\alpha} \int \chi_K'(y)|x-y|^{2-d} \exp \left( -\frac{c|x-y|^2}{\delta^2} \right) dy.
\]

Now the repetition of the same calculation applied to the remaining integral gives the required estimate

\[ I \leq Cb_n^2 = C \left( a^{n(d-\alpha)} \delta^{2-d+\alpha} \right)^2. \]

This proves estimate (9).

Next, by the Paley-Zygmund inequality with \( \theta = 1/2 \) it follows from estimates (8) and (9) that there exists a constant \( \sigma > 0 \) independent of \( n, \delta \) such that for an appropriate constant \( c \)

\[
P\left( Z_n \geq ca^{n(d-\alpha)} \delta^{2-d+\alpha} = cb_n \right) \geq P\left( Z_n \geq \frac{E(Z_n)}{2} \right) \geq \frac{E(Z)^2}{4E(Z^2)} \geq \sigma.
\]
Now consider the Feynman-Kac functional $A_V$ with $V_{\beta} = d_K^{-\beta}$ for some $\beta \geq d - \alpha$. Then for any sequence $\delta_j$ decreasing to 0 we have

$$P \left( A_V = \int_0^{\delta^2} V(X_s) ds = \infty : \forall \delta > 0 \right)$$

$$= P \left( \bigcap_j \left\{ \int_0^{\delta_j} V(X_s) ds = \infty \right\} \right) = \lim_{j \to \infty} P \left( \int_0^{\delta_j} V(X_s) ds = \infty \right)$$

$$\geq \lim_{j \to \infty} P \left( \int_0^{\delta_j} \chi_{K^n}(X_s) ds \geq c a_n^{(d-\alpha)} \delta_j^{2-d+\alpha} \text{ for infinitely many } n \right) \geq \sigma.$$ 

The last inequality follows by a variant of Borel-Cantelli Lemma since we have

$$P \left( \int_0^{\delta_j} \chi_{K^n}(X_s) ds \geq c a_n^{(d-\alpha)} \delta_j^{2-d+\alpha} \right) \geq \sigma.$$ 

Now the event

$$\Omega_V = \left\{ \int_0^{\delta^2} V(X_s) ds = \infty : \forall \delta > 0 \right\}$$

is measurable with respect to $\sigma$-field $\cap_{t>0} \mathbb{F}_t$ so by Blumenthal’s zero-one law it must have probability equal to 0 or 1. Thus $P(\Omega_V) = 1$. This ends the proof of Theorem 2 in the case $d > 2$. For $d = 2$ the proof is a simple modification of the above argument. One can use the version of estimates (10) corresponding to $d = 2$ in the calculations. We skip the details here. $\square$

We are now in position to state the main result of this section.

**Theorem 3.** Suppose that the set $D \subset \mathbb{R}^d$ is open simply connected and that its boundary $K = \partial D$ satisfies condition (4) for some $d > \alpha > 0$ and $d \geq 2$. Let $\{T_t, t \geq 0\}$ be the Feynman-Kac semigroup generated by $L_{V_{\beta}} = \Delta - V_{\beta}$, where the potential $V_{\beta}$ is defined by (6). Assume also that $\alpha + \beta \geq d$.

Then the subspace $L^2(D)$ of $L^2(\mathbb{R}^d)$ is invariant under the action $T_t$ that is

$$T_t(L^2(D)) \subset L^2(D)$$

for all $t \geq 0$.

**Proof.** Let us denote by $\Omega_x$ the set of paths starting from $x \in D$ with $\tau \leq t$, where $\tau$ denotes the first hitting time into $K = \partial D$. Let $d\mu(x, \tau)$ be the joint distribution of the vector $(X_\tau, \tau)$. By the strong Markov property and the analysis above

$$P \left( X \in \Omega_x : \text{ for all } \delta > 0 \int_{\tau}^{\tau + \delta^2} V(X_s) ds = \infty \right)$$

$$= \int P \left( X_0 = y, \text{ for all } \delta > 0 \int_0^{\delta^2} V(X_s) ds = \infty \right) d\mu(y, \tau) = \mu(\Omega_x).$$
Hence the event:

\[
\text{for all } \delta > 0 \text{ we have } \int_{\tau}^{\tau+\delta^2} V(X_s)ds = \infty
\]

holds a.s on \( \Omega_x \). The theorem follows by applying Feynman-Kac formula (2). \( \square \)

4. Singularity of \( V \) forcing separation.

In this section we prove that Theorem 3 is optimal, that is that the condition \( \alpha + \beta \geq d \) is also necessary. By \( p_t(x,y) \) we denote the standard Gaussian distribution. Recall that

\[
p_t(x,y) = (4\pi t)^{-d/2}e^{-\frac{|x-y|^2}{4t}}
\]

Then as before by \( p_t^V(x,y) \) we denote the kernel of the Feynman-Kac semigroup corresponding to the positive potential \( V \). In these terms Theorem 3 can be stated in the following way: For any \( \beta \) such that \( \alpha + \beta \geq d \)

\[
p^V_{\beta t}(x,y) = 0 \quad \forall t > 0, x \in D \quad \text{and} \quad \forall y \in D^c.
\]

Next we shall show that if \( \alpha + \beta < d \) then

\[
p^V_{\beta t}(x,y) > 0 \quad \forall t > 0, x, y \in \mathbb{R}^d,
\]

see Theorem 4 and Corollary 5 below.

First for any \( t > 0 \) we set

\[
\Gamma_t(x,y) = \int_0^t p_s(x,y)ds = \int_0^t (4\pi s)^{-d/2}e^{-\frac{|x-y|^2}{4s}}ds.
\]

**Theorem 4.** Assume that \( V \in L^1_{\text{loc}}(\mathbb{R}^d) \) is a positive locally integrable potential. Suppose also that for some fixed \( t \)

\[
V * \Gamma_t(x) + V * \Gamma_t(y) < \infty.
\]

Then \( p_t^V(x,y) > 0 \) for every \( t > 0 \).

**Proof.** Let us start with recalling the equivalent version of Feynmann-Kac formula based on the notion of the Brownian bridge. Let \( \{Y_s, t \geq s \geq 0\} \) be the Brownian bridge stochastic process connecting points \( x, y \in \mathbb{R}^d \) for a definition, see for example [14, Example 3, p. 243]. Using the above notion the heat kernel corresponding to the Feynmann-Kac semigroup can be written as

\[
p^V_t(x,y) = \int \exp \left( -\int_0^t V(Y_s)ds \right) d\mu_{x,y}(Y)
\]

where \( d\mu_{x,y} \) is the Brownian bridge measure defined on the set \( \Omega_{x,y}^t \) of continuous sample paths connecting \( x \) and \( y \) and normalised in such way that \( \int d\mu_{x,y}(\omega) = p_t(x,y) \), see [15, Theorem 6.6]. Note that in our notation \( \frac{\mu_{x,y}(\Omega_{x,y}^t)}{p_t(x,y)} \) is the standard Brownian bridge probability measure on the set \( \Omega_{x,y}^t \). We will call \( d\mu_{x,y} \) the Brownian bridge measure.
Next by Chebyshev’s inequality
\[
\mu_{x,y} \left( \left\{ \int_0^t V(Y_s) \, ds \geq A \right\} \right) \leq \frac{1}{A} \int \int \int_0^t V(Y_s)p_s(x-z)p_{t-s}(z-y) \, dz \, ds
\]
\[
\leq \frac{C}{At^{d/2}} \left( \int_0^{t/2} \int \int_0^t V(z)p_s(x-z) \, dz \, ds + \int_0^{t/2} \int \int_0^t V(z)p_s(y-z) \, dz \, ds \right)
\]
\[
\leq \frac{C}{At^{d/2}} (V \ast \Gamma_t(x) + V \ast \Gamma_t(y)).
\]

Hence for sufficiently large \(A\)
\[
\mu_{x,y} \left( \left\{ \int_0^t V(Y_s) \, ds \leq A \right\} \right) \geq p_t(x,y) - \frac{C}{At^{d/2}} (V \ast \Gamma_t(x) + V \ast \Gamma_t(y)) \geq \frac{1}{2} p_t(x,y).
\]

Thus
\[
p_t^V(x,y) = \int \exp \left( - \int_0^t V(Y_s) \, ds \right) d\mu_{x,y}(Y) \geq \frac{1}{2} e^{-A} p_t(x,y) > 0.
\]

This concludes proof of Theorem 4. □

As a direct consequence of Theorem 4 we obtain the following corollary

**Corollary 5.** Under the assumptions of Theorem 3 the condition \(\alpha + \beta \geq d\) is necessary for separation.

**Proof.** The proof is essentially repetition of the proof of estimate (8) from the proof of Theorem 2. More precisely one can notice that it we set \(t = \delta^2\) then
\[
V \ast \Gamma_t(x) \leq C \sum_{n>0} a^{-n\beta} E(Z_n) \leq C \sum_{n>0} a^{-n\beta} a^{n(d-\alpha)}.
\]

Recall that \(a < 1\) so the above sum is finite if \(\alpha + \beta < d\). Now Corollary 5 follows from Theorems 3 and 4. □

As an illustration of Theorem 3 and Corollary 5 we would like to describe the construction of the van Koch snowflake curve.

**Example 6. Van Koch snowflake.** Consider an equilateral triangle \(K_0\) with sides of unit length. Next define a curve \(K_1\) by replacing the middle of all edges of \(K_0\) by the two sides of the equilateral triangle based on the middle every segment. Next define \(K_2\) by repeating the same procedure on each of the twelfth edges of \(K_1\). Von Koch snowflake, which we denote by \(K\) is the self-similar set obtained by iteration of this procedure. Its Minkowski dimension is equal to \(\alpha = \log 4 / \log 3\) and it satisfies assumptions (4) and (5) with this \(\alpha\) and \(d = 2\).

Thus as a straightforward consequence of our results we obtain the following corollary
Corollary 7. Suppose that \( D \subset \mathbb{R}^2 \) is the region of the plane inside the Von Koch snowflake \( K \) and \( p_{t}^{V_{\beta}} \) is the heat kernel corresponding to the operator \( L + d_{K}^{\beta} \). Then for any \( \beta \geq 2 - \log 4 / \log 3 \)

\[
p_{t}^{V_{\beta}}(x, y) = 0 \quad \forall t > 0, x \in D \quad \text{and} \quad \forall y \in D^c.
\]

When \( \beta < 2 - \log 4 / \log 3 \) then

\[
p_{t}^{V_{\beta}}(x, y) > 0 \quad \forall x, y \in \mathbb{R}^2 \quad \text{and} \quad \forall t > 0.
\]

5. Estimates of the rate of separation for truncated potentials

In this section we consider the potential

\[
V_{A}^{\beta} = C_{V}A^{\beta} \chi_{d_{K}(x) \leq A^{-1}} = C_{V}A^{\beta} \chi_{K_{A^{-1}}}
\]

We fix a supercritical exponent \( \alpha + \beta > d \). Denote by \( p_{t}^{A}(x, y) \) the kernel of the Feynman-Kac semigroup generated by the operator

\[
-L_{V_{A}^{\beta}} = \Delta - V_{A}^{\beta}
\]

with the natural domain. It is well known that the functions \( p_{t}^{A}(x, y) \) are continuous in \( x, y, t \in \mathbb{R}^d \times (0, \infty) \).

It is convenient for us to introduce at this point the following definition of the uniform domain type. Our definition is a variant of Definition 3.2 of [10] but is motivated by the definition of NTA (nontangentially accessible) domains introduced by Jerison and Kenig in [12]. See also the discussion in Section 3.1.3 of [10].

Definition 8. Let \( D \subset \mathbb{R}^d \) be a connected subset of \( \mathbb{R}^d \). We say that \( D \) satisfies the inside NTA condition if there are constants \( c, C \) such that such that, for any \( x, y \in D \) in the interior of \( D \) there exists a continuous curve \( \gamma_{x,y} : [0, 1] \to D \) such that \( \gamma_{x,y}(0) = x \) and \( \gamma_{x,y}(1) = y \) and the following two properties are satisfied

1. The length \( L(\gamma_{x,y}) \) is at most \( C|x - y| \).
2. For any \( z \in \gamma_{x,y}([0, 1]) \)

\[
d_{\partial D}(z) \leq c \min(L(\gamma_{x,z}), L(\gamma_{y,z})).
\]

We say that \( D \) is an NTA domain if both \( D \) and \( D^c \) satisfies the inside NTA condition.

The following statement is the main result discussed in this section

Theorem 9. Suppose that set \( D \subset \mathbb{R}^d \) is open simply connected and that its boundary \( K = \partial D \) satisfies conditions (4) and (5) for some \( d > \alpha > 0 \) and that \( \alpha + \beta > d \geq 2 \).

Let \( D \) be an NTA domain. There exists a constant \( \sigma > 0 \) such that for any points \( x \in D \), any ball \( B_0 \) separated from \( D \), that is \( B_0 \subset D^c = \mathbb{R}^d \setminus D \), and any positive \( t > 0 \) there exists a constant \( C = C(x, t) \) such that

\[
\int_{B_0} p_{t}^{A}(x, y) dy \leq CA^{-\sigma}
\]

for all \( A > 0 \).
Remarks. 1. By the standard elliptic estimates (or by a slightly more technical variant of our argument) one can obtain a pointwise estimate $p^d_t(x,y) \leq CA^{-\sigma}$.

2. In this work we are interested in the existence of a positive $\sigma$ satisfying (12). We want to point out however that our methods can be strengthened at the cost of some further work. We do not present the details.

3. The value of $\sigma$ in our approach depends on the domain. The constant $C$ depends on the domain and the value of the multiplicative constant $C_V$. For the sake of simplicity we will consider only the case $C_V = 1$. Without loss of generality we can assume that $t = 1$. First we prove a series of technical lemmata concerning properties of the Brownian motion.

Lemma 10. There exist constants $0 < \zeta < 1$ and $\sigma_0 > 0$ such that

$$P \left( \int_0^{\delta^2} V^A(X_t) dt < \zeta E \left( \int_0^{\delta^2} V^A(X_t) dt \right) \right) \leq 1 - \sigma_0$$

for all $\delta > 0$ and $A > 0$ and every starting point $X_0 = x \in K$.

Proof. The statement is an immediate corollary to (11). We have to replace $\chi_{K^c_n}$ in the definition of $Z_n$ by $\chi_{d_k(x) \leq A^{-1}}$. Then the statement follows exactly by the calculations leading to (11) which we use in the proof of Theorem 2. To avoid repetition we omit the details.

Let $w_\lambda = \lambda(1,1,\ldots,1) \in \mathbb{R}^d$ where $\lambda \in \mathbb{R}$ will be specified later. Consider the decomposition of $\mathbb{R}^d$ into congruent cubes of side length $\delta$ obtained by the $\delta \mathbb{Z}^d$ translations of $w_\lambda + [0, \delta]^d$ and denote by $Q_1,\ldots,Q_M$ all these cubes, such that $Q_j \cap \partial D \neq \emptyset$ for $j = 1,\ldots,M$. By (4) one immediately gets $\frac{1}{4} \delta^{-\alpha} \leq M \leq C \delta^{-\alpha}$ where $C$ does not depend on $\lambda$. We fix some small $v > 0$, $\delta \approx A^{-v}$ and $H \in \mathbb{N}$ in such a way that $2H + 2 = \delta^{-2}$. For any $h \in \{0,\ldots,H\}$ we set

$$I_h = [2h\delta^2, (2h + 1)\delta^2) \quad \text{and} \quad J_h = [(2h + 1)\delta^2, (2h + 2)\delta^2)$$

Let $1 \leq H_0 \leq H$ be a fixed number. We will consider multi-indexes of the form $j_1,\ldots,j_{H_0},k_1,\ldots,k_{H_0}$ such that $1 \leq j_s \leq M$, $0 \leq k_s \leq H$, $k_1 < k_2 < \ldots < k_{H_0}$ for all $1 \leq s \leq H_0$.

Let $\Omega$ be the set of all Brownian paths $X_t$ such that $X_0 = x$ and $X_1 \in B_0$. We define the family of the subsets $\Phi_{j_1,\ldots,j_{H_0},k_1,\ldots,k_{H_0}} \subset \Omega$ of $\Omega$ by requiring the following conditions.

(1) For every $1 \leq s \leq H_0$ the path $X_t$, $t \in I_{k_s}$ crosses the boundary $K$ at least one time.

(2) For every $1 \leq s \leq H_0$ the first time of hitting the set $K$ for $t \in I_{k_s}$ is the time $t_s$ such that $X_{t_s} \in Q_{j_s}$.

(3) In addition $X_t \notin K$ for $t \in I_h$ for any $h$ not listed in the sequence $k_1,\ldots,k_{H_0}$.

Claim. There exists $\lambda \in [0, \delta)$ such that the Wiener measure of the set $E_\lambda$ of trajectories which do not uniquely determine cubes $Q_{j_1},\ldots,Q_{j_{H_0}}$ is zero.

Proof. Indeed, denote by $\tau \in I_{k_j}$ the first hitting time into $K$. Fix the $d - 1$ or lower dimensional common wall $W$ of two cubes and observe that all of its translations
$W_\lambda = W + \omega_\lambda$ are pairwise disjoint. Hence the events \{$X_\tau \in W_\lambda$\} are pairwise disjoint, so only for at most countably many of $\lambda$ will the event \{$X_\tau \in W_\lambda$\} have positive Wiener measure. The claim follows. \hfill $\square$

From now on we fix $\lambda$ given by the claim and consider the grid of cubes corresponding to $\omega_\lambda$. We subtract the set $E_\lambda$ from $\Omega$ and $\Phi_{j_1, \ldots, jH_0, k_1, \ldots, kH_0}$ and denote the new sets again by $\Omega$ and $\Phi_{j_1, \ldots, jH_0, k_1, \ldots, kH_0}$. Now it is straightforward to see that the sets $\Phi_{j_1, \ldots, jH_0, k_1, \ldots, kH_0}$ are pairwise disjoint. Moreover the set

\[ \bigcup_{j_1, \ldots, jH_0, k_1, \ldots, kH_0} \Phi_{j_1, \ldots, jH_0, k_1, \ldots, kH_0} \]

(up to a subset of measure zero) consists of $X_t \in \Omega$ such that the paths \{$X_t$\}$_{t \in I_h}$ for $h = 1, \ldots, H$ cross $K$ for exactly $H_0$ out of $H$ intervals $I_h$. We will use these facts in the sequel.

Next, denote by $q_{js}$ the center of $Q_{js}$. For any $\eta \geq 1$ and $x, y \in \mathbb{R}^d$, such that

\[ |x - q_{js}| \leq \eta \delta \sqrt{\log H}, \quad \text{and} \quad |y - q_{js}| \leq \eta \delta \sqrt{\log H} \]

we define $p(x, y)$ as the Brownian bridge measure $d\mu_{x,y}$ (see the definition in Section 4) of the set

(13) $\Psi_0 = \left\{ X_t \in \Omega_{x,y}^{s2}: \{X_t\}_{t \in [0, \delta^2]} \cap K \cap Q_{js} \neq \emptyset, |X_{\tau_{js}} - X_{\tau_{js} + \gamma_{s,H}}| \leq \frac{\eta \delta}{\log H} \right\}$

where by $\tau_{js}$ we denote the first hitting time of $\{X_t\}_{t \in [0, \delta^2]}$ into $K \cap Q_{js}$ and $\gamma_{s,H} = \delta^2(\log H)^{-3}$. Then we define $\tilde{p}(x, y)$ similarly, as the $d\mu_{x,y}$ measure of

(14) $\Psi_1 = \left\{ X_t \in \Psi_0: \int_{\tau_{js}}^{\tau_{js} + \gamma_{s,H}} V^A(X_u)du < \zeta E \left( \int_{\tau_{js}}^{\tau_{js} + \gamma_{s,H}} V^A(X_u)du \right) \right\}$

where the constant $0 < \zeta < 1$ has been defined in Lemma 10. The following elementary observation is critical for our argument

**Lemma 11.** Under the above definitions there exists a constant $\beta < 1$ such that

\[ \tilde{p}(x, y) \leq \beta p(x, y). \]

uniformly for all indices $x, y, A, j_s$ defined above and sufficiently large $H \geq H_{\min}(\eta)$.

**Proof.** Denote by $\tau$ the first hitting time of $\{X_t\}_{t \in [0, \delta^2]}$ in $K \cap Q_{js}$. By the definition of $\Psi_0$, the variable $\tau$ is well defined for paths in $\Psi_0$. Denote by $d\mu(\tau, w)$ the joint distribution of the variables $\tau, X_\tau$ for $X_t \in \Psi_0$, and zero away from $\Psi_0$. Obviously $d\mu$ is supported on $S = [0, \delta^2] \times (K \cap Q_{js})$. Next for any subset $G \subset \mathbb{R}^d$ put

\[ \nu_1(G) = P \{ (X_u)_{u \in [\tau, \tau + \gamma_{s,H}]}, X_\tau = w, X_{\tau + \gamma_{s,H}} \in w + G \} \]

\[ = P \{ X_0 = 0, X_{\gamma_{s,H}} \in G \}. \]
Note that at this point $\nu_1$ is just the standard Gaussian distribution and recall that $\gamma_{\delta,H} = \delta^2 (\log H)^{-3}$. Next we observe that
\[
p(x, y) = \mu_{x,y}(\Psi_0) = \int_S P \left( |w - X_{\tau+\gamma_{\delta,H}}| \leq \frac{\eta\delta}{\log H}, X_{2\delta^2} = y \right) d\mu(\tau, w)
\]
\[
= \int_S \int_{|z| \leq \frac{\eta\delta}{\log H}} P \left( X_{\tau+\gamma_{\delta,H}} = w - z, X_{2\delta^2} = y \right) d\nu_1(z) d\mu(\tau, w)
\]
\[
\geq \left( 1 - \frac{C\eta^2}{\sqrt{\log H}} \right) \int_S \int_{|z| \leq \frac{\eta\delta}{\log H}} p_{2\delta^2-(\tau+\gamma_{\delta,H})}(w - y) d\nu_1(z) d\mu(\tau, w)
\]
\[
\geq \left( 1 - \frac{C\eta^2}{\sqrt{\log H}} \right) \int_S \int_{|z| \leq \frac{\eta\delta}{\log H}} p_{2\delta^2-(\tau+\gamma_{\delta,H})}(w - y) p_{\gamma_{\delta,H}}(z) dz d\mu(\tau, w)
\]
\[
\geq \left( 1 - \frac{C\eta^2}{\sqrt{\log H}} \right)^2 \int_S p_{2\delta^2-(\tau+\gamma_{\delta,H})}(w - y) d\mu(\tau, w).
\]
In the above estimates we used the first and third of the elementary inequalities
\[
\left( 1 - \frac{C\eta^2}{\sqrt{\log H}} \right) p_{2\delta^2-(\tau+\gamma_{\delta,H})}(w - y) \leq p_{2\delta^2-(\tau+\gamma_{\delta,H})}(w - y + z)
\]
\[
\left( 1 - \frac{C\eta^2}{\sqrt{\log H}} \right) p_{2\delta^2-(\tau+\gamma_{\delta,H})}(w - y + z) \leq p_{2\delta^2-(\tau+\gamma_{\delta,H})}(w - y)
\]
\[
\int_{|z| \leq \frac{\eta\delta}{\log H}} p_{\gamma_{\delta,H}}(z) dz \geq 1 - \frac{C}{\log H}
\]
valid for any $w$ and $\tau$ from the domain of integration, and $z, y, \eta$ as above.

Now set
\[
\Phi_{V^A} = \left\{ X_u \in [\tau, \tau + \gamma_{\delta,H}] : \int_{\tau_j}^{\tau_j + \gamma_{\delta,H}} V^A(X_u) du < \zeta E \left( \int_{\tau_j}^{\tau_j + \gamma_{\delta,H}} V^A(X_u) du \right) \right\}
\]
and then put
\[
\nu_2(G) = P \left( \{ X_u \in [\tau, \tau + \gamma_{\delta,H}] : X_u \in \Phi_{V^A}, X_\tau = w, X_{\tau+\gamma_{\delta,H}} \in w + G \} \right).
\]
It follows from Lemma 10 that $\nu_2(\mathbb{R}^d) \leq 1 - \sigma_0$ so
\[
\bar{p}(x, y) = \mu_{x,y}(\Psi_1)
\]
\[
= \int_S \int_{|z| \leq \frac{\eta\delta}{\log H}} P \left( X_{\tau+\gamma_{\delta,H}} = w - z, X_{2\delta^2} = y \right) d\nu_2(z) d\mu(\tau, w)
\]
\[
\leq \left( 1 + \frac{C\eta^2}{\sqrt{\log H}} \right) (1 - \sigma_0) \int_S p_{2\delta^2-(\tau+\gamma_{\delta,H})}(w - y) d\mu(\tau, w).
\]
The lemma follows for $\beta = 1 - \frac{\sigma_0}{2}$ and sufficiently large $H \geq H_{\min}(\eta)$. \qed
Recall that we denote the centre of the cube $Q_j$ by $q_j$. Slightly abusing the notation we again denote by $\tau_{j}$ the first hitting time of $\{X_t\}_{t \in I_k}$ into $K \cap Q_j$. We put

$$\Lambda = \bigcup_{1 \leq H_0 \leq H} \bigcup_{j_1, \ldots, j_{H_0}, k_1, \ldots, k_{H_0}} \Phi_{j_1, \ldots, j_{H_0}, k_1, \ldots, k_{H_0}}$$

where we sum over the set of all indices $1 \leq H_0 \leq H$ and $j_1, \ldots, j_{H_0}, k_1, \ldots, k_{H_0}$ such that the system of inequalities

$$|X_{2k_j}\delta^2 - q_j| \leq \eta \delta \sqrt{\log H}, \quad |X_{(2k_j+2)\delta^2} - q_j| \leq \eta \delta \sqrt{\log H}$$

and

$$|X_{\tau_j} - X_{\tau_j + \gamma \delta, H}| \leq \frac{\eta \delta}{\log H}$$

is not satisfied for at least one of $1 \leq s \leq H_0$. Then obviously

$$\Lambda \subset \tilde{\Lambda} = \left\{ \{X_t\}_{t \in [0,1]} : \max_{h \in \{0,1,\ldots,H\}} |X_{2h}\delta^2 - X_{(2h+2)\delta^2}| > 2\eta \delta \sqrt{\log H} \right\}$$

and

$$\bigcup \left\{ \{X_t\}_{t \in [0,1]} : \sup_{t_1, t_2 \in [0,1], |t_1 - t_2| \leq \gamma \delta, H} |X_{t_1} - X_{t_2}| > \frac{\eta \delta}{\log H} \right\}.$$ 

The set $\tilde{\Lambda}$ is of small probability. In fact we have

**Lemma 12.** For any $\rho > 0$ there exists a constant $\eta$ such that the set $\tilde{\Lambda}$ defined above satisfies the estimate

$$P(\tilde{\Lambda}) \leq H^{-\rho}$$

for all sufficiently large $H$.

**Proof.** This lemma follows exactly in the same way as the proof of Hölder regularity of the Brownian motion. Directly one can easily check that

$$P \left( |X_{2h\delta^2} - X_{(2h+2)\delta^2}| > \eta \delta \sqrt{\log H / 2} \right) \leq \exp(-c\eta \log H)$$

And then sum-up the estimates. The second part can be verified in a similar way after applying the reflection principle. The lemma follows.

From now on we fix large $\rho$ and the corresponding $\eta = \eta_0$ given by Lemma 12. We assume $H \geq H_{\min}(\eta_0)$. We will need the following standard fact about NTA domains

**Lemma 13.** A) Assume $D \subset \mathbb{R}^d$ is an NTA domain, and let $B_0$, be a closed ball contained in the interior of $D$ and separated from $K = \partial D$. Then, there exists $\gamma > 0$, such that the harmonic function on $D \setminus B_0$ vanishing on $K$ and equal to 1 on $\partial B_0$ satisfies

$$h(x) \leq C(d_K(x))^{\gamma}. \quad (15)$$

The same statement is valid for $D^c$.

B) For dimension $d = 2$ estimate (15) holds both for the domain $D$ and its complement $D^c$ with exponent $\gamma = 1/2$. 
Proof. We briefly sketch the proof. We start with Part A. Since $D$ satisfies NTA condition, the Dirichlet problem for $D$ is solvable and there exists a function, harmonic in $D \setminus B_0$ such that $h(x) = 0$ for $x \in K$ and $h(y) = 1$ for $y \in \partial B_0$. Define $K_j = \{ x \in D : d_K(x) = a^j \}$ for sufficiently small fixed $a < 1$. Fix $x \in K_j$. Let $B_1 \subset D^c$ be a ball with center $y_0$ and radius $r$ such that $|x - y_0| \leq 2d_K(x)$ and, for some $c$ depending only on the domain $D$

$$2a^j = 2d_K(x) \geq r \geq cd_K(x) = ca^j.$$ Such a ball exists by the NTA conditions for $D$. Now observe that $P\{X_{r^2} \in B_1\} \geq p_0$ and $P\{\text{there exists } t \leq r^2, \text{ such that } X_t \in K_{j-1}\} \leq C_1 \exp\left(-\frac{r}{C_1 a}\right)$ where $p_0, C_1$ depends only on the NTA constants of the domain, not on $a$. Consequently, for some small enough $a > 0$, we have

$$P\{\text{diffusion starting from } K_j \text{ hits } K, \text{ not hitting } K_{j-1} \text{ before }\} \geq P\{X_{r^2} \in B_1\} - P\{\text{there exists } t \leq r^2, \text{ such that } X_t \in K_{j-1}\} \geq \frac{p_0}{2}$$ for all $j \in \mathbb{N}$. Hence, any harmonic function bounded by 1 on $K_{j-1}$ and vanishing on $K$ must be bounded by $1 - \frac{p_0}{2}$ on $K_j$. By a simple induction argument, we obtain $h(x) \leq (1 - \frac{p_0}{2})^{j-j_0}$ for $x \in K_j$, where $j_0$ is the minimal index such that $K_{j_0}$ does not intersect $B_0$. Now Part A of Lemma 15 follows by the maximum principle.

Part B of the theorem is a consequence of Beurling projection theorem, see [1, Theorem 3-6, page 43]. A probabilistic proof of Beurling projection theorem is described in [13].

□

Remark Note that in the whole paper we use the NTA conditions only to obtain estimate (15).

In the next lemma we estimate the probability of the set of paths for which the number $H_0$ is small.

Lemma 14. There exist constants $C, \theta$ such that for any $\kappa'' > \kappa' > 0$ the following estimate holds

$$P\left(\{X_t\}_{t \in [0,1]} : \#\{h : X_t \text{ hits the boundary for } t \in I_h\} < H^{\kappa'}\right) \leq CH^{\kappa''} \delta^\theta.$$

Proof. Let $\Phi_{j_1,\ldots,j_{H_0},k_1,\ldots,k_{H_0},0} = \Phi_{j_1,\ldots,j_{H_0},k_1,\ldots,k_{H_0}} \setminus A$. Note that

$$\Phi_{j_1,\ldots,j_{H_0},k_1,\ldots,k_{H_0},0} \subset \tilde{\Phi}_{j_1,\ldots,j_{H_0},k_1,\ldots,k_{H_0}}$$

where

$$\tilde{\Phi}_{j_1,\ldots,j_{H_0},k_1,\ldots,k_{H_0}} =$$

$$= \left\{ \{X_t\}_{t \in [0,1]} : X_t = \bar{X}_t \text{ for all } t \leq (2k_{H_0} + 2)\delta^2 \text{ and some } \bar{X}_t \in \Phi_{j_1,\ldots,j_{H_0},k_1,\ldots,k_{H_0},0} \right\}.$$

Next, observe that for a fixed $H_0 \geq 1$, the sets $\tilde{\Phi}_{j_1,\ldots,j_{H_0},k_1,\ldots,k_{H_0}}$ are mutually disjoint. Let $\theta = 2\gamma$, ($\gamma$ defined by (15)). We will prove the estimate

$$P\left(\Phi_{j_1,\ldots,j_{H_0},k_1,\ldots,k_{H_0},0}\right) \leq C_0 \delta^\theta P\left(\tilde{\Phi}_{j_1,\ldots,j_{H_0},k_1,\ldots,k_{H_0}}\right).$$
Next, we define the measure $\nu_3$ by the formula

$$
\nu_3(G) = P \left( X \in \tilde{\Phi}_{j_1, \ldots, j_{H_0}, k_1, \ldots, k_{H_0}} : X_{(2k_{H_0} + 2)\delta^2} \in G \right).
$$

By the Markov property of Brownian motion we have

$$
P \left( \Phi_{j_1, \ldots, j_{H_0}, k_1, \ldots, k_{H_0}, 0} \right) = \int P \left( X_0 = x, X_{1-(2k_{H_0} + 2)\delta^2} \in B_0 \text{ and } X_s \cap K = \emptyset \text{ for } 0 \leq s \leq 1 - (2k_{H_0} + 2)\delta^2 \right) d\nu_3(x)
$$

where by the definition of $\Phi_{j_1, \ldots, j_{H_0}, k_1, \ldots, k_{H_0}}$, for $x \in \text{supp}\{d\nu_3\}$ we have

$$
|x - q_{k_{H_0}}| \leq \delta^2 \sqrt{\log H}.
$$

By Lemma 13

$$
P \left( X_0 = x, X_{1-(2k_{H_0} + 2)\delta^2} \in B_0 \text{ and } X_s \cap K = \emptyset \text{ for } 0 \leq s \leq 1 - (2k_{H_0} + 2)\delta^2 \right) \leq P \left( X_0 = x, \text{ and } \{X_s\}_{1-(2k_{H_0} + 2)\delta^2} \text{ hits first time into } B_0, \text{ not into } K \right) \leq C \left( \delta^2 \sqrt{\log H} \right)^{\gamma}
$$

and consequently, since $\nu_3(\mathbb{R}^d) = P \left( \tilde{\Phi}_{j_1, \ldots, j_{H_0}, k_1, \ldots, k_{H_0}} \right)$ it follows that

$$
P \left( \tilde{\Phi}_{j_1, \ldots, j_{H_0}, k_1, \ldots, k_{H_0}, 0} \right) \leq C \left( \delta^2 \sqrt{\log H} \right)^{\gamma} P \left( \tilde{\Phi}_{j_1, \ldots, j_{H_0}, k_1, \ldots, k_{H_0}} \right).
$$

Hence, since $\tilde{\Phi}_{j_1, \ldots, j_{H_0}, k_1, \ldots, k_{H_0}}$ are mutually disjoint, we obtain

$$
\sum_{1 \leq H_0 \leq H_{\kappa'}} \sum_{j_1, \ldots, j_{H_0}, k_1, \ldots, k_{H_0}} P \left( \Phi_{j_1, \ldots, j_{H_0}, k_1, \ldots, k_{H_0}, 0} \right) \leq CH^\kappa'' \delta^\theta.
$$

Now the lemma follows from the above estimates and Lemma 12.

Now we are ready to prove Theorem 9.

**Proof of Theorem 9.** First we restrict the domain of the expectation integral in Feynman-Kac formula to the set of trajectories which hit $K$ at a time $t \in I_h$ for some $h$. If the path does not cross $K$ at any $t \in I_h$ for some $h$, then it must cross $K$ at a time $t \in J_h$ for some $h$, and we repeat the argument replacing $I_h$ by $J_h$.

Now consider the set

$$
\Phi_{\text{small}} = \bigcup_{H_0 \leq H_{\kappa}} \bigcup_{j_1, \ldots, j_{H_0}, k_1, \ldots, k_{H_0}} \Phi_{j_1, \ldots, j_{H_0}, k_1, \ldots, k_{H_0}, 0}
$$

Recall, that we set $\delta \approx A^{-v}$ so by Lemma 14

$$
\int_{\Phi_{\text{small}}} e_V(t) 1_{B_0}(X_t) \leq P(\Phi_{\text{small}}) \leq CH^\kappa'' \delta^\theta \leq CA^{-v(\gamma - \kappa'')}. \tag{16}
$$
Next consider the following sets

\[
Φ_{bad}^{j_1,...,j_{H_0},k_1,...,k_{H_0},0} = Φ_{j_1,...,j_{H_0},k_1,...,k_{H_0},0}^{bad}
\]

\[
\cap \left\{ \int_{τ_{j_s}}^{τ_{j_s}+\gamma_{δ,H}} V^A(X_s)ds \leq ζ E \left( \int_{τ_{j_s}}^{τ_{j_s}+\gamma_{δ,H}} V^A(X_s)ds \right) \right. \text{on every } I_{k_s} \right\};
\]

\[
Φ_{essential} = \bigcup_{H_0>H^α} \bigcup_{j_1,...,j_{H_0},k_1,...,k_{H_0}} Φ_{j_1,...,j_{H_0},k_1,...,k_{H_0},0};
\]

\[
Φ_{bad} = \bigcup_{H_0>H^α} \bigcup_{j_1,...,j_{H_0},k_1,...,k_{H_0}} Φ_{j_1,...,j_{H_0},k_1,...,k_{H_0},0};
\]

\[
Φ_{good} = Φ_{essential} \setminus Φ_{bad}.
\]

Recall, that we defined \( δ \approx A^{-v} \) and \( \gamma_{δ,H} = δ^2(\log H)^{-3} \). Let \( 2vd < β + α - d \). If \( \{X_t\}_{t∈[0,1]} \in Φ_{good} \) then, for at least one \( j_s \) we have

\[
\int_{τ_{j_s}}^{τ_{j_s}+\gamma_{δ,H}} V^A(X_s)ds \geq ζ E \left( \int_{τ_{j_s}}^{τ_{j_s}+\gamma_{δ,H}} V^A(X_s)ds \right) \geq CA^{β+α-d-2dv} ≥ C(A').
\]

for any \( t < β + α - d - 2dv \) and for some constant \( C > 0 \) depending only on the domain and \( γ \), but not on \( A \). Hence, for any fixed \( ρ > 0 \) and sufficiently large \( A \) we have

\[
\int_{Φ_{good}} e^V(t)1_{B_0}(X_t) ≤ \exp(-CA') ≤ C′A^{-ρ}.
\]

Next we shall prove that for any \( ρ > 0 \) we have \( P(Φ_{bad}) ≤ A^{-ρ} \). To do this we first will show that

\[
P \left( Φ_{bad}^{j_1,...,j_{H_0},k_1,...,k_{H_0}} \right) ≤ β^{H_0}P \left( Φ_{j_1,...,j_{H_0},k_1,...,k_{H_0}} \right).
\]

We set \( p^0_{k_1}(x,y) = p(x,y) \) and \( p^0_{k_2}(x,y) = \tilde{p}(x,y) \) where \( p \) and \( \tilde{p} \) are defined by (13) and (14). Moreover, for \( h \neq k_1,...,k_{H_0} \) we define \( p^0_h(x,y) \) as the Brownian bridge measure \( dμ_{x,y} \) of

\[
p^0_h(x,y) = μ_{x,y}(\{X_t\} \in Ω_{x,y}^{2δ^2}: \{X_t\}_{t∈[0,δ]} ∩ K = \emptyset,)
\]

Now if we apply the Markov property and replace each \( \tilde{p} \) by \( p \) we get the following inequality

\[
P(Φ_{bad}^{j_1,...,j_{H_0},k_1,...,k_{H_0}}) =
\]

\[
\int_{B_0} \int p^0_{k_1}(x_1)p^0_{k_1}(x_1,x_2)…p^0_{H-1}(x_{H-1},y)dx_1…dx_{H-1}dy \leq \text{ (by Lemma 11)}
\]

\[
≤ β^{H_0} \int_{B_0} \int p^0_{k_1}(x_1)p^0_{k_1}(x_1,x_2)…p^0_{H-1}(x_{H-1},y)dx_1…dx_{H-1}dy
\]

\[
= β^{H_0}P(Φ_{j_1,...,j_{H_0},k_1,...,k_{H_0}}).
\]
The above inequality proves (18). Now, for $\delta \approx A^{-v}$ and any $C > 0$ we have

$$\sum_{H_0 \geq H^\kappa} \sum_{j_1, \ldots, j_{H_0}, k_1, \ldots, k_{H_0}} P(\Phi_{j_1, \ldots, j_{H_0}, k_1, \ldots, k_{H_0}}) \leq C \beta^{H^\kappa} \leq C_v C \delta^\rho$$

for sufficiently large $H$. Combining all the above estimates we get that if $0 < \beta + \alpha - d - 2dv$ then for any $\rho < \gamma \beta + \alpha - d$ and $\kappa''$ small enough

$$\int e^V(t) 1_{B_0}(X_t) \leq C H^{\kappa''} \delta^\rho \leq C A^{-v(\gamma - \kappa'')} \leq C A^{-\rho}.$$

This ends the proof.

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