On compression of non-classically correlated bit strings

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We show that the outcomes of measurements on correlated quantum systems that are spatially separated can be compressed much more efficiently than their classical counterparts. We show this on an example of bit strings generated by singlet correlations that we compress using Huffman coding. We then draw general conclusions on compressibility of quantumly correlated strings using Kolmogorov complexity.

Introduction.— Understanding correlations present in physical systems is crucial. This starts at a level of two particles with the phenomenon of quantum entanglement [1], quantum teleportation [2], super-dense coding [3] to name a few and ends with various fundamental phenomena in complex systems consisting of a large (usually Avogadro) number of particles such as phase transitions, quantum phase transitions [4], super conductivity and many others.

Since realisation that information is physical [5], correlations carried by physical objects have been utilised to perform computational tasks. In this context, two qubit correlations have been deployed, amongst many other applications, to perform device independent quantum key distribution [6] and private randomness amplification [7], and multi-qubit correlations are used to perform quantum computation [8].

Compression of information plays a pivotal role in information processing tasks as shown by Shannon in his groundbreaking paper [9]. The achievements of our digital era are significantly based on the fact that classical information can be efficiently compressed and stored. Quantum information, which could be the future of computational tasks as shown by Shannon in his quantum computation [8].

We now ask a question: how well Alice and Bob can loslessly compress their pairs of measurement outcomes? Since these strings originate from measurements on a singlet state, they are locally fully random and hence are un-compressible when considered separately. However, Alice can compress her data provided that she has some information about Bob’s data (and vice versa).
This compression is possible because of correlations between their bit strings.

In principle, the ultimate compression rate Alice and Bob can achieve is given by the Shannon entropy $S(x_i | y_j)$, however this rate can only be achieved in the limit of infinitely long bit strings or if the probabilities of finite strings are powers of two [12]. In all other cases one has to use some more practical compression algorithms. Since the work of Shannon many compression algorithms that loslessly compress finite data were invented. In this work we apply the Huffman prefix coding algorithm [12].

The compression procedure goes as follows. First, Alice produces the bit string $z_{ij} = x_i \oplus y_j$, i.e., she adds Bob’s bit string $y_j$ to her bit string $x_i$ (modulo two). Next, she uses the Huffman code to compress $z_{ij}$ to some shorter string $\tilde{z}_{ij}$. The string $y_j$ is not compressed. This is the losless compression, since the Huffman coding can be reversed $\tilde{z}_{ij} \rightarrow z_{ij}$ and $z_{ij} \oplus y_j = x_i$.

Before we proceed, let us give the reader an idea on how the Huffman coding algorithm is applied in our scenario by considering a specific example. Imagine that Alice generated the following bit string with the help of Bob’s data:

$$z_{ij} = 000010010000001100.$$

The length of $z_{ij}$ is $n = 18$. Alice divides her bit string into $m$ parts, each consisting of $k = n/m$ bits. In this example we choose $k = 2$, however different choices of $k$ lead to different compression efficiencies

$$00/00/10/01/00/00/00/11/00.$$

Next, Alice counts the frequency of each 2-bit sequence: $\#00 = 6$, $\#01 = 1$, $\#10 = 1$, $\#11 = 1$. She makes a table in which 2-bit sequences are ordered from the most frequent to the less frequent

$$\{(00, 6), (01, 1), (10, 1), (11, 1)\}.$$

The first term in the bracket corresponds to the sequence, whereas the second to its frequency. She follows the recursive algorithm — she starts at the end of the table and combines the two last elements into a new element whose frequency is the sum of frequencies of these elements. Next, she produces a new table in which the last two elements are superceded by the new one. Note, that the new element can take some higher position in the table because its frequency may not be the smallest. In our case the new table is

$$\{(00, 6), (10, 11, 2), (01, 1)\},$$

where $[10, 11]$ denotes that the new element is composed of 10 and 11. Alice repeats the whole procedure to obtain

$$\{(00, 6), ([10, 11], 01), 3\},$$

and after one more round she obtains

$$\{([[10, 11], 01], 00), 9\}.$$

The element $[[[10, 11], 01], 00]$ is used to produce the Huffman tree from which we obtain the Huffman prefix-free coding (for details see Fig. 1): $00 \rightarrow 0$, $01 \rightarrow 10$, $10 \rightarrow 110$ and $11 \rightarrow 111$. Therefore, after compression we get

$$\tilde{z}_{ij} = 00110100001110.$$

The length of $\tilde{z}_{ij}$ is $\tilde{n} = 14$ and the compression rate is $r_{ij} = \tilde{n}/n = 7/9 = 0.778$. It is also instructive to estimate the Shannon entropy of the string $z_{ij}$. This can be done under an assumption that each bit in $z_{ij}$ was generated independently and with respect to the same probability distribution. In this case we estimate the probabilities of 0 and 1 from frequencies, i.e., $p(0) = 7/9$ and $p(1) = 2/9$. These probabilities give the Shannon entropy $S(z_{ij}) \approx 0.764$, which shows that our Huffman compression is reasonably efficient.

The next step is to evaluate the expected Huffman compression rates for bit strings $z_{ij}$ that are generated from quantum correlations. Firstly, we note that the subsequent pairs of bits in $x_i$ and $y_j$ are generated by independent pairs of qubits in the singlet state. The probability that Alice’s measurement along direction $\vec{a}$ gives an outcome $x$ and that Bob’s measurements along $\vec{b}$ gives $y$ ($x, y \in \{0, 1\}$) is given by

$$p(x, y|\vec{a}, \vec{b}) = \frac{1}{4} \left(1 - (-1)^{x+y} \vec{a} \cdot \vec{b}\right). \quad (1)$$

Therefore, the probabilities of 0 and 1 in the bit string $z_{ij}$ are given by

$$p(0|z_{ij}) = \frac{1}{2} \left(1 - \vec{a} \cdot \vec{b}\right), \quad p(1|z_{ij}) = \frac{1}{2} \left(1 + \vec{a} \cdot \vec{b}\right). \quad (2)$$

Following the previous example, after $n$ rounds of measurements Alice produces $z_{ij}$ from her string of outcomes $x_i$ and Bob’s string of outcomes $y_j$. She divides this string into $m$ substrings of size $k = n/m$. Each substring belongs to the set of all $2^k$ possible bit strings. Now
we ask, what is the expected compression rate? The expected frequency for the substring containing #0 = l and #1 = k − l is $p(0|z_{ij})p(1|z_{ij})^{k−l}$. We can plug these frequencies into a similar table we used before and construct the Huffman code that will give us an expected compression rate.

Note that the algorithm does not depend on $m$. The only important parameters are $k$ and the probabilities $p(0|z_{ij})$ and $p(1|z_{ij})$. Interestingly, even in the case of extremal probabilities $p(0|z_{ij}) = 0$ and $p(1|z_{ij}) = 1$, or vice versa, the compression rate depends on $k$. In this case every k-bit sequence can be coded as one bit and the length of $z_{ij}$ is $n/k$. Thus, the compression rate $r_{ij} = 1/k$.

We finish our discussion of the compression algorithm with yet another example. The expected compression rate for measurements for which $\bar{a} \cdot \bar{b} = 1/\sqrt{2}$ and for $k = 2$ is $r_{ij} = 0.709$. This rate is better for larger $k$. For $k = 4$ it is $\approx 0.611$ and for $k = 8$ it is $\approx 0.605$. The corresponding Shannon entropy is $S(z_{ij}) \approx 0.601$. Note, up to now we have not exploited the non-classicality of quantum correlations. We do this in the next section.

**Compression of non-classically correlated bit strings.**—In this section we present our main result. We derive an inequality for compression rates of classically correlated quantum correlations. We do this in the next section.

The basic building block of our construction is the fact that every distance metric obeys the triangle inequality. NCD obeys the triangle inequality up to a factor that depends on the length of the uncompressed bit string $x$. We start with the triangle inequality between the bit strings $x_1, y_N, y_1$ (first two strings appear on the left-hand side of the inequality, convention we use throughout the paper)

$$NCD(x_1, y_N) \leq NCD(x_1, y_1) + NCD(y_1, y_N) + O \left( \frac{\log n}{n} \right).$$

Next, consider another triangle inequality between the strings $y_1, y_N, x_2$. Combining these two triangle inequalities we get a "rectangle" inequality

$$NCD(x_1, y_N) \leq NCD(x_1, y_1) + NCD(x_2, y_1)$$

$$+ NCD(x_2, y_N) + O \left( \frac{2 \log n}{n} \right).$$

We follow analogical steps until we are left with terms $NCD(x_i, y_i)$ or $NCD(x_{i+1}, y_i)$. Finally

$$NCD(x_1, y_N) \leq \sum_{i=1}^{N} NCD(x_i, y_i) + \sum_{i=1}^{N-1} NCD(x_{i+1}, y_i)$$

$$+ O \left( \frac{N \log n}{n} \right).$$

From now on we assume sufficiently long bit strings $N/n \ll 1$ such that the last term can be omitted. Since classically correlated bit strings obey all triangle inequalities used in this derivation, they also have to obey (7).

However, we now show that bit strings generated by the singlet state correlations violate the inequality (7). Let us consider the measurement settings $a_i$ and $b_j$

$$a_i = (\sin(i−1)\theta, \cos(i−1)\theta),$$

$$b_j = (\sin(j−1/2)\theta, \cos(j−1/2)\theta),$$

$y_1$ are uniformly complex, i.e., any n-bit subset of those strings is equally compressible. This assumption can be verified experimentally. (ii) Because of the spatial separation of Alice and Bob and the finite speed of light compression rate of the string $x_1$ alone does not depend on which measurement was chosen by Bob. If this was not the case Bob could send superluminal signals to Alice, which is not compatible with the special relativity theory (iii) Compression rate of strings $x_i$ and $y_j$ is the same as of the strings $x'_i$ and $y_j$. Here $x'_i$ denotes a string generated for $i$th setting of Alice but not at the same time as the string $y_j$. This is a counterfactual statement that cannot be tested experimentally although it has been extensively used in the literature on EPR paradox and Bell inequalities [18]. We need one more assumption, which is also of counterfactual nature (iv) The triangle inequality for any distance measure must be obeyed even for the strings that cannot be simultaneously generated [19]. For instance, $NCD(y_1, y_N)$ cannot be determined experimentally because the strings $y_1$ and $y_N$ come from measurements of incompatible observables.
where \( \theta = \pi/(2N - 1) \). This choice of directions yields \( \vec{a}_i \cdot \vec{b}_i = \vec{a}_i \cdot \vec{b}_{i+1} = \cos \frac{\pi}{2N - 2} \) and \( \vec{a}_i \cdot \vec{b}_N = 0 \), which implies \( r_{i,i} = r_{i+1,i} = r \) and \( r_{1,N} = 1 \). The uncompressibility of \( x_1 y_N \) follows from the lack of correlations between \( x_1 \) and \( y_N \) because of the orthogonality of the corresponding Bloch vectors.

For the singlet state correlations the inequality (7) simplifies to

\[
\frac{1}{2N - 1} \leq r. \tag{9}
\]

It is violated for \( N \geq 3 \). In particular, for \( N = 3 \) and \( k = 9 \) we get \( r \approx 0.199 \). For \( k = 10 \) we get \( r \approx 0.192 \). For comparison, the Shannon limit in this case is \( S(z) \approx 0.166 \). The reason why this time our compression rate is not as close to Shannon rate as in the previous examples is that the Huffman coding is not optimal if a probability of some \( k \)-bit sequence is close to 1. In such cases one needs to choose higher values of \( k \).

We now show that the violation of the inequality (7) implies violation of the inequality

\[
Z(x_1, y_N) \leq \sum_{i=1}^{N} Z(x_i, y_i) + \sum_{i=1}^{N-1} Z(x_{i+1}, y_i) + O \left( \frac{N \log n}{n} \right). \tag{10}
\]

\( Z \) is Zurek’s distance measure \([14]\) defined for two bit strings of length \( n \) as \( Z(a, b) = 2K(a, b) - K(a) - K(b) \), where \( K(a) \) is KC of the string \( a \) etc. This inequality can be derived in exactly the same way as (7). Its violation stems from 1) \( K(x_i) = K(y_j) = n \) because individual strings by Alice and Bob are purely random 2) \( Z(x_i, y_j) \leq C(x_i, y_j) \), i.e., Kolmogorov complexity is optimal by definition 3) The strings \( x_1 \) and \( y_N \) are not correlated and as such cannot be efficiently compressed, i.e., \( K(x_1 \oplus y_N) = n \). To prove it, we observe that the left-hand side of the inequality (10) equals one whereas the right-hand side is bounded from above by the right-hand side of the inequality (7), which ends the proof.

Discussion. — We have shown that bit strings generated by entangled qubits violate information-theoretic inequalities that are obeyed by classically correlated random variables. The violation occurs because the assumptions used to derive the inequalities (7) and (10) are not satisfied by quantum mechanical correlations in the presence of entanglement. A natural question is to identify the unfulfilled assumptions from the set (i) - (iv). It is clear that if one rejects a possibility of infinite speed of information propagation in the universe then it must be the assumption (iii) or (iv) that is not obeyed in quantum theory. We would like to point out that all the assumptions (i)-(iv) are satisfied by classically correlated random variables.

The interesting conclusion based on the violation of the inequalities (7) and (10) is that quantum mechanics (equivalently, violation of the assumptions (iii) and (iv)) allows for more efficient compression than classical correlations. We can also conclude that Kolmogorov complexity is not a proper description of all possible correlations occurring in nature, thus giving a possibility to distinguish between quantum and classical correlations on the level much more fundamental that the probabilistic one.

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