Derivations from the even parts into the odd parts for Hamiltonian superalgebras

Yuan Chang, Liangyun Chen

School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, CHINA

Abstract

Let $W_1$ and $H_0$ denote the odd parts of the general Witt modular Lie superalgebra $W$ and the even parts of the Hamiltonian Lie superalgebra $H$ over a field of characteristic $p > 3$, respectively. We give a torus of $H_0$ and the weight space decomposition of the special subalgebra of $W_1$ with respect to the torus. By means of the derivations of the weight 0 and three series of outer derivations from $H_0$ into $W_1$, the derivations from the even parts of Hamiltonian superalgebra to the odd parts of Witt superalgebra are determined.

Key words: Torus; Weight space decomposition; Derivations

1 Introduction

As the natural generalization of Lie algebras, Lie superalgebras become an efficient tool for analyzing the properties of physical systems. The theory of Lie superalgebras is closely related to many branches of mathematics. In particular, V.G.Kac classified the finite-dimensional simple Lie superalgebras over algebraically closed field of characteristic zero [1]. For modular Lie superalgebras, as we know, [2,3] should be the earliest papers. With the development of modular Lie superalgebras, Cartan-type Lie superalgebras play an important role in the category of modular Lie superalgebras. Eight families of $\mathbb{Z}$-graded Cartan-type Lie superalgebras were constructed over a field of characteristic $p > 3$ [4,7]. Determining the superderivation algebras are very important subjects in modular Lie superalgebras. The superderivation algebras were studied in one-by-one fashion for the

Corresponding author (L. Chen): chenly640@nenu.edu.cn.
Supported by NNSF of China (Nos. 11171055 and No. 11471090).
finite dimensional and simple ones. [11] used a uniform method to determine the superderivation algebras of $\mathbb{Z}$-graded Cartan-type Lie superalgebras.

For Lie superalgebras, the even parts are closely connected with Lie algebras and the odd parts are the modulars of the even parts. Determining the derivations of the even parts and the derivations from the even parts into the odd parts for Lie superalgebras are very interesting. [12, 13] respectively determine the derivations of the even parts and the derivations from the even parts into the odd parts for contact Lie superalgebras. The odd $\mathbb{Z}$-homogeneous derivations and negative $\mathbb{Z}$-homogeneous derivations from the even parts of Hamiltonian Lie superalgebras into the even parts of Witt Lie superalgebras. Moreover, there are more outer derivations for the even parts of Hamiltonian Lie superalgebras than for the even parts of Lie superalgebras.

This paper is organized as follow. In Section 2, we give the basic notations and concepts. In Section 3, we introduce the suitable generating set and the proper torus $T$ of the even parts $H$ in the finite-dimensional Hamiltonian Lie superalgebras $H$. Then we give the special subalgebra $\mathfrak{g}_1$ of $W_\mathbb{T}$ and the weight space decompositions of $\mathfrak{g}_1$ with respect of the torus $T$ that have the same weights with some generators of $H$. In Section 4, we give three series of outer derivations from the even parts $H\bar{l}_m$ into the odd parts $W_\mathbb{T}$. We characterize the derivations vanishing on the top of $H\bar{l}_m$ with the above results. In Section 5, we determine the derivation algebras from the even parts of Hamiltonian Lie superalgebras to the odd parts of Witt modular Lie superalgebras.

2 Basic

For a vector superspace $V = V_{\mathbb{T}} \oplus V_{\overline{\mathbb{T}}}$, we denote by $p(x) = \alpha$ the parity of a homogeneous element $x \in V_{\alpha}, \alpha \in \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{0, 1\}$. If $V = \oplus_{i \in \mathbb{Z}} V_i$ is a $\mathbb{Z}$-graded vector space and $x \in V$ is a $\mathbb{Z}$-homogeneous element, write $zd(x)$ for the $\mathbb{Z}$-degree of $x$. The symbol $p(x)$ (resp. $zd(x)$) always implies that $x$ is a $\mathbb{Z}_2$- (resp. $\mathbb{Z}_r$)-homogeneous element. If $L = \oplus_{-r \leq i \leq s} L_i$ is a $\mathbb{Z}$-graded Lie superalgebra, then $\oplus_{-r \leq i \leq s} L_i$ is called the top of $L$.

Now, we review the notions of modular Lie superalgebras $W$ and $H$ of Cartan-type and their grading structures. Throughout $\mathbb{F}$ is an algebraically closed field of characteristic $p > 3$. We write $\mathbb{N}$ for the set of natural numbers and $\mathbb{N}_+$ for the set of positive integers. Fix two positive integers $m$ and $n$. For $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbb{N}_+^m$, put $|\alpha| = \sum_{i=1}^m \alpha_i$. In [17], denote by $\mathcal{O}(m)$ the divided power algebra over $\mathbb{F}$ with an $\mathbb{F}$-basis $\{x^{(\alpha)} | \alpha \in \mathbb{N}_+^m\}$. For $\varepsilon_i = (\delta_{i1}, \cdots, \delta_{im})$, we abbreviate $x^{(\varepsilon_i)}$ to $x_i, i = 1, \cdots, m$. Let $\Lambda(n)$ be the exterior superalgebra over $\mathbb{F}$ in $n$ variables $x_{m+1}, \cdots, x_{m+n}$. Denote the tensor product by $\mathcal{O}(m, n) = \mathcal{O}(m) \otimes_\mathbb{F} \Lambda(n)$. Obviously, $\mathcal{O}(m, n)$ is an associative superalgebra with a $\mathbb{Z}_2$-gradation induced by the trivial $\mathbb{Z}_2$-gradation of $\mathcal{O}(m)$ and the natural $\mathbb{Z}_2$-gradation of $\Lambda(n)$. Moreover, $\mathcal{O}(m, n)$ is super-commutative. For $g \in \mathcal{O}(m), f \in \Lambda(n)$, we write $gf$.
for \( g \otimes f \). The following formulas hold in \( \mathcal{O}(m, n) \):

\[
x^{(\alpha)} x^{(\beta)} = \left( \frac{\alpha + \beta}{\alpha} \right) x^{(\alpha + \beta)} \quad \text{for } \alpha, \beta \in \mathbb{N}_+^m;
\]

\[
x_k x_l = -x_l x_k \quad \text{for } k, l = m + 1, \ldots, m + n;
\]

\[
x^{(\alpha)} x_k = x_k x^{(\alpha)} \quad \text{for } \alpha \in \mathbb{N}_0^m, k = m + 1, \ldots, m + n,
\]

where \( \left( \frac{\alpha + \beta}{\alpha} \right) := \prod_{i=1}^{m} \left( \frac{\alpha_i + \beta_i}{\alpha_i} \right) \). Put \( I_0 := \{1, 2, \ldots, m\}, I_1 := \{m + 1, \ldots, m + n\} \) and \( I := I_0 \cup I_1 \). Set

\[
\mathbb{B}_k := \{\langle i_1, i_2, \ldots, i_k \rangle | m + 1 \leq i_1 < i_2 < \cdots < i_k \leq m + n\}
\]

and \( \mathbb{B} := \mathbb{B}(n) = \bigcup_{k=0}^{n} \mathbb{B}_k \), where \( \mathbb{B}_0 = \emptyset \). For \( u = \langle i_1, i_2, \ldots, i_k \rangle \in \mathbb{B}_k \), set \( |u| := k \), \( x^u = x_{i_1} \cdots x_{i_k} \), \( |\emptyset| = 0 \) and \( x^0 = 1 \). For convenience, we use \( \langle s \rangle \) to stand for \( \langle \delta_1, s, \delta_2, \ldots, \delta_{(m+n)} \rangle \), for \( s \in I_1 \). Set \( \mathbb{B}^0 = \{u \in \mathbb{B}|u| \text{ is even}\} \). Clearly, \( \{x^{(\alpha)} x^u | \alpha \in \mathbb{N}_0^m, u \in \mathbb{B}\} \) constitutes an \( \mathbb{F}\)-basis of \( \mathcal{O}(m, n) \). Let \( \partial_1, \partial_2, \ldots, \partial_{m+n} \) be the linear transformations of \( \mathcal{O}(m, n) \) such that \( \partial_i(x^{(\alpha)}) = x^{(\alpha-\epsilon_i)} \) for \( i \in I_0 \) and \( \partial_i(x_k) = \delta_{ik} \) for \( i \in I_1 \). Obviously, \( p(\partial_i) = \mu(i) \), where \( \mu(i) := 0, i \in I_0 \), and \( \mu(i) := 1, i \in I_1 \). Then \( \partial_1, \partial_2, \ldots, \partial_{m+n} \) are superderivations of the superalgebra \( \mathcal{O}(m, n) \). Let

\[
W(m, n) := \left\{ \sum_{r \in I} f_r \partial_r | f_r \in \mathcal{O}(m, n), r \in I \right\}.
\]

Then \( W(m, n) \) is an infinite-dimensional Lie superalgebra contained in \( \text{Der}(\mathcal{O}(m, n)) \). One can verify that

\[
[fD, gE] = f D(g)E - (-1)^{p(f)p(g)} g E(f)D + (-1)^{p(f)p(g)} fg[D, E]
\]

for \( f, g \in \mathcal{O}(m, n), D, E \in \text{Der}(\mathcal{O}(m, n)) \). Specially,

\[
[f \partial_i, g \partial_j] = f \partial_i(g) \partial_j - (-1)^{p(f)p(g)} g \partial_j(f) \partial_i \quad \text{for } f, g \in \mathcal{O}(m, n), i, j \in I.
\]

Hereafter, suppose that \( m = 2r, r \in \mathbb{N}_0 \). If \( n \) is even, then we set \( n = 2s + 1 \), otherwise \( n = 2s + 1 \), where \( s \in \mathbb{N}_0 \). Define a linear mapping \( D_H : \mathcal{O}(m, n) \rightarrow W(m, n) \) by means of

\[
D_H(f) := \sum_{i \in I} \tau(i) (-1)^{\mu(i)p(f)} \partial_i(f) \partial_{i'} \quad \text{for all } f \in \mathcal{O}(m, n),
\]

where

\[
i' = \begin{cases}  
i + r, & 1 \leq i \leq r, \\
i - r, & r + 1 \leq i \leq m, \\
i + s, & m + 1 \leq i \leq m + s, \\
i - s, & m + s < i \leq m + 2s, \\
i, & \text{the other}, \end{cases} \quad \tau(i) = \begin{cases} 1, & 1 \leq i \leq r, \\
-1, & r < i \leq m, \\
1, & i \in I_1. \end{cases}
\]
The following equation holds:

$$[D_H(f), D_H(g)] = D_H(D_H(f)(g))$$

for all $f, g \in \mathcal{O}(m, n)$.

Put $H(m, n) := \text{span}_F \{ D_H(f) | f \in \mathcal{O}(m, n) \}$. Obviously, $H(m, n)$ is an infinite-dimensional $\mathbb{Z}_2$-graded subalgebra of $W(m, n)$.

Fix two $m$-tuples of positive integers:

$$\underline{t} := (t_1, t_2, \cdots, t_m) \in \mathbb{N}^m, \quad \pi := (\pi_1, \pi_2, \cdots, \pi_m),$$

where $\pi_i = p^{t_i} - 1, i = 1, 2, \cdots, m$. Let $\mathcal{A} := \mathcal{A}(m; \underline{t}) = \{ \alpha \in \mathbb{N}_0^m | \alpha_i \leq \pi_i, \ i \in I_0 \}$. Then

$$\mathcal{O}(m, n; \underline{t}) := \text{span}_F \{ x^{(\alpha)} x^u | \alpha \in \mathcal{A}, \ u \in \mathbb{B} \}$$

is a finite-dimensional subalgebra of $\mathcal{O}(m, n)$ with a $\mathbb{Z}$-grading structure:

$$\mathcal{O}(m, n; \underline{t}) = \bigoplus_{i=0}^\xi \mathcal{O}(m, n; \underline{t})_i,$$

where $\mathcal{O}(m, n; \underline{t})_i := \text{span}_F \{ x^{(\alpha)} x^u | \alpha + |u| = i \}$ and $\xi := |\pi| + n$. Set

$$W(m, n; \underline{t}) := \left\{ \sum_{i \in I} f_i \partial_i | f_i \in \mathcal{O}(m, n; \underline{t})_i, \ i \in I \right\}.$$

Then $W(m, n; \underline{t})$ is a finite-dimensional simple Lie superalgebra [4], which is called the generalized Witt Lie superalgebra [4], and is denoted by $W$. We note that $W(m, n; \underline{t})$ possesses a standard $F$-basis

$$\{ x^{(\alpha)} x^u \partial_j | \alpha \in \mathcal{A}, u \in \mathbb{B}, j \in I \}$$

and a $\mathbb{Z}$-grading structure:

$$W(m, n; \underline{t})_r = \bigoplus_{r=1}^{\xi-1} W(m, n; \underline{t})_r,$$

where $W(m, n; \underline{t})_r := \text{span}_F \{ x^{(\alpha)} x^u \partial_j | |\alpha| + |u| = r + 1, \ j \in I \}$. Set

$$H(m, n; \underline{t}) := \left\{ D_H(f) | f \in \bigoplus_{i=0}^\xi \mathcal{O}(m, n; \underline{t})_i \right\},$$

where $\xi := |\pi| + n$. Then $H(m, n; \underline{t})$ is a finite-dimensional $\mathbb{Z}$-graded simple subalgebra of $W(m, n; \underline{t})$ [4], which is called the Hamiltonian superalgebra.

In the following sections, $\mathcal{O}(m, n; \underline{t})$, $W(m, n; \underline{t})$ and $H(m, n; \underline{t})$ will be denoted by $\mathcal{O}$, $W$ and $H$. In addition, the even parts of $W$ and $H$ will respectively be denoted by $W_0$ and $H_0$, and the odd parts of $W$ will be denoted by $W_1$. 

4
3 Torus and Weight Space Decompositions

Recall
\[ H_\mathfrak{H} = \text{span}_\mathbb{F}\{D_H(x^{(\alpha)}x^u)|\alpha \in \mathbb{A}, u \in \mathbb{B}^0, (\alpha, u) \neq (\pi, \omega)\}. \]

Set
\[ J := \text{span}_\mathbb{F}\{D_H(x^{(\alpha)}x^u)|\alpha \in \mathbb{A}, u \in \mathbb{B}^0, (\alpha, u) \neq (\pi, \omega), (\alpha, u) \neq (\pi, \emptyset)\}, \]

where \( \mathbb{B}^0 = \{u \in \mathbb{B}|\|u\| is even\} \) and \( \omega = \langle m+1, m+2, \cdots, m+n \rangle \in \mathbb{B}_n \). Obviously, \( J \) is a subspace of \( H_\mathfrak{H} \) of codimension 1:
\[ H_\mathfrak{H} = J \oplus \mathbb{F}D_H(x^{(\pi)}). \]

Lemma 3.1. [16] \( J \) is a maximal idea of \( H_\mathfrak{H} \).

We review the generating sets of \( J \). Set
\[ \mathcal{M} = \{D_H(x^{(q, u)})|1 \leq q_i \leq n_i, i \in I_0\} \text{ and } \mathcal{N} = \{D_H(x_i x^u)|i \in I_0, u \in \mathbb{B}_2\}. \]

Lemma 3.2. [16] \( J \) is generated by \( \mathcal{M} \cup \mathcal{N} \cup J_0 \).

Set \( g_1 = \text{span}_\mathbb{F}\{x^u \partial_r|r \in I_1, u \in \mathbb{B}, p(x_u \partial_r) = \mathbf{1}\} \). And \( g_1 \) is a \( \mathbb{Z} \)-graded subspace of \( W_\mathfrak{T} \). Since \( C_{W_\mathfrak{T}}((W_\mathfrak{H})^{-1}) = g_1, C_{W_\mathfrak{T}}(J) \subseteq C_{W_\mathfrak{T}}((H_\mathfrak{H})^{-1}) = C_{W_\mathfrak{T}}((W_\mathfrak{H})^{-1}) = g_1 \).

Proposition 3.3. The following statements hold:

(1) If \( \omega = n \) is even, then \( C_{W_\mathfrak{T}}(J) = 0 \).

(2) If \( \omega = n \) is odd, then \( C_{W_\mathfrak{T}}(J) = \mathbb{F}D_H(x^{(\pi)}). \)

Proof. Since \( C_{W_\mathfrak{T}}(J) \) is a \( \mathbb{Z} \)-graded subalgebra of \( W_\mathfrak{T} \), we note that \( C_{W_\mathfrak{T}}(J) \subseteq g_1 \). If \( D \in C_{W_\mathfrak{T}}(J) \), we see that \( D \in g_1 \). Thus one may assume that
\[ D = \sum_{r \in I_1} f_r \partial_r, \tag{3.1} \]

where \( f_r \in \Lambda(n) \). For any \( i \in I_0 \), since \( D_H(x^{(2\alpha_i)}) \in J \), we have \([D, D_H(x^{(2\alpha_i)})] = f_i \partial_i \) and therefore \( f_i \partial_i = 0 \). This proves that \( f_i = 0 \) for all \( i \in I_0 \). By Eq.(3.1), we can obtain that
\[ D = \sum_{r \in I_1} f_r \partial_r, \text{ where } f_r \in \Lambda(n). \tag{3.2} \]

(1) When \( \omega = n \) is even, there are three cases to discuss.

Case 1. If \( 2 \leq zd(f_r) \leq n - 2 \) for \( r \in I_1 \), assume that \( f_k \neq 0 \) for some \( k \in I_1 \). We choose \( k, k' \in I_1 \). Then
\[ 0 = [D_H(x^k x^{k'}), D] = \sum_{r \in I_1} (x^{k'} \partial_r(f_r) - x^k \partial_r(f_r)) \partial_r - f_k \partial_{k'} + f_{k'} \partial_k. \tag{3.3} \]
By comparison the coefficients of $\partial_k$ in Eq. (3.3), we can find $x^k \in f_k$ and $x^{k'} \notin f_k$. For $l$, $l' \in I_1 \setminus \{k, k'\}$, we have

$$0 = [D_H(x^lx'^l), D] = \sum_{r \in I_1} (x'^l \partial_r (f_r) - x^l \partial_r (f_r)) \partial_r - f_l \partial_l' + f_l' \partial_l.$$

(3.4)

Since the coefficients of $\partial_k$ in Eq. (3.4) yield that $x'^l \partial_r (f_k) - x^l \partial_r (f_k) = 0$, we can obtain that $x^l, x'^l \in f_k$ or $x^l, x'^l \notin f_k$. So $p(f_k)$ is odd and $p(f_k \partial_k)$ is even, which is a contradiction. This proves that $D = 0$.

Case 2. If $zd(f_r) = 0$ for $r \in I_1$, then we assume that

$$D = \sum_{r \in I_1} a_r \partial_r, \quad \text{where } a_r \in F.$$

For any $k, l \in I_1$, we obtain

$$0 = [D_H(x^kx^l), \sum_{r \in I_1} a_r \partial_r] = -a_l \partial_k - a_k \partial_l.

(3.5)

By comparison the coefficients in Eq. (3.5), we obtain that $a_r = 0$ for any $r \in I_1$. It is obvious that $D = 0$.

Case 3. If $zd(f_r) = n$ for $r \in I_1$. We can assume

$$D = \sum_{r \in I_1} a_r x^r \partial_r, \quad \text{where } a_r \in F.$$

Arguing as the proof of Case 2, we can obtain $a_r = 0$ for any $r \in I_1$ and $D = 0$.

(2) We declare $|w| = n$ is odd. For any basis element $D_H(x^{(\alpha)}x^{(\mu)})$ of $J$, it is clear that $[D_H(x^{(\mu)}), D_H(x^{(\alpha)}x^{(\nu)})] = 0$. Therefore, $FD_H(x^{(\nu)}) \subseteq C_{W'}(J)$. We will show the converse inclusion. Every element $D \in C_{W'}(J)$ may be written as Eq. (3.2). We distinguish two cases to discuss.

Case 1. $0 \leq zd(f_r) \leq n - 3$ for $r \in I_1$. Arguing just as the proof of the above Case 1 and Case 2, we can prove that $D = 0$.

Case 2. $zd(f_r) = n - 1$ for $r \in I_1$. Then one may assume that

$$f_r = \sum_{s \in I_1} c_{rs} x^s, \quad \text{where } c_{rs} \in F.$$

Then we put $s := \omega - \langle s \rangle$ where for $s \in I_1$. For $k, k' \in I_1$, we have the equation that

$$0 = [D_H(x^kx^{k'}), D].$$

Then

$$0 = [x^{k'} \partial_{k'} - x^k \partial_k, \sum_{r \in I_1} (\sum_{s \in I_1} c_{rs} x^s) \partial_r] = \sum_{r \in I_1} (c_{rk} x^k - c_{r'k} x^{k'}) \partial_r + (\sum_{s \in I_1} c_{ks} x^s) \partial_k - (\sum_{s \in I_1} c_{k's} x^s) \partial_{k'}.$$

By the calculation of the coefficients of $\partial_r$ for $r \in I_1$, we obtain that

$$D = \sum_{r \in I_1} c_{rr'} x^{r'} \partial_r.$$
For any $k, l \in I_1$, we have the equation that
\[ 0 = [D_H(x^k x^l), D] = [x^k \partial^l - x^l \partial^k, \sum_{r \in I_1} c_{rr} x^r \partial_r]. \]

We obtain that $(c_{kk'} + (-1)^{k' + l - 3} c_{kl}) x^k \partial^l - (c_{ll'} + (-1)^{l' + k - 3} c_{kl}) x^l \partial^k = 0$, and therefore, $c_{k'k} = (-1)^{k' + l} c_{kl}$ for $k, l \in I_1$. Set $\mu = c_{i'j}$ where $i = m + 1$. So far, we prove that
\[ D = \mu \sum_{r \in I_1} (-1)^{r + (m+1)} x^r \partial^r = \mu D_H(x^{\omega}). \]

\[\square\]

**Theorem 3.4.** Suppose $\phi \in \text{Der}(H_\Theta, W_\pi)$ and $\phi(\mathcal{J}) = 0$. Then the following statements hold:

1. If $|\omega| = n$ is even, then $\phi = 0$.
2. If $|\omega| = n$ is odd, then $\phi(D_H(x^{(\pi)})) = \mu D_H(x^{\omega})$ for some $\mu \in F$. Conversely, any linear mapping $\phi : H_\pi \to W_\pi$ vanishing on $\mathcal{J}$ and satisfying $\phi(D_H(x^{(\pi)})) = \mu(D_H(x^{\omega}))$ for any fixed $\mu \in F$ is a derivation from $H_\pi$ into $W_\pi$.

**Proof.** By Lemma 3.1 and $[\phi(D_H(x^{(\pi)})), \mathcal{J}] = 0$, that is $\phi(D_H(x^{(\pi)})) \in C_{W_\pi}(\mathcal{J})$, we can get the consequences directly. \[\square\]

For any fixed $\mu \in F$, define the linear mapping $\Gamma_\mu : H_\pi \to W_\pi$ by means of $\Gamma_\mu(\mathcal{J}) = 0$ and $\Gamma_\mu(D_H(x^{(\pi)})) = \mu D_H(x^{\omega})$. In the case that $|\omega| = n$ is odd, by Theorem 3.4, $\Gamma_\mu$ are outer derivations from $H_\pi$ into $W_\pi$, where $\mu \in F$, and $zd(\Gamma_\mu) = |\omega| - |\pi|$.

Suppose $L$ is a Lie superalgebra and $V$ an $L$-module. Denote by $\text{Der}(L, V)$ the superderivation space and $\text{Ind}(L, V)$ the inner derivation space. Clearly, $\text{Der}(L, V)$ is an $L$-submodule of $\text{Hom}_F(L, V)$. A derivation $D : L \to V$ is called inner if there is $v \in V$ such that $D(x) = xv$ for all $x \in L$. Assume in addition that $L = \bigoplus_{r \in Z} L_r$ is $\mathbb{Z}$-graded and finite-dimensional, and $V = \bigoplus_{r \in Z} V_r$ is a $\mathbb{Z}$-graded $L$-module. Then the superderivation space inherits a $\mathbb{Z}$-graded $L$-module structure
\[ \text{Der}(L, V) = \bigoplus_{r \in Z} \text{Der}_r(L, V). \]

Let $(L, [p])$ be a restricted Lie algebra. An element $x \in L$ is $p$-semisimple provided that $x \in \Sigma_{r \in \mathbb{N}F} x^{[p]^r}$. An abelian restricted subalgebra $T$ of $L$ is called a torus if every element in $T$ is $p$-semisimple. Let $T \subseteq L_0 \cap L_{\pi}$ be a torus of $L$ with the weight space decomposition:
\[ L = \bigoplus_{\alpha \in \Theta} L_\alpha, \quad V = \bigoplus_{\beta \in \Delta} V_\beta. \]

Then there exist subsets $\Theta_i \subset \Theta$ and $\Delta_i \subset \Delta$ such that $L_i = \bigoplus_{\alpha \in \Theta_i} L_\alpha$ and $V_j = \bigoplus_{\beta \in \Delta_i} V_\beta \cap V_\beta$. Hence $L$ and $V$ have the corresponding $\mathbb{Z} \times T^*$-grading structures, respectively, where $T^*$ is the dual space of $T$. Of course $\text{Der}(L, V)$ inherits a $\mathbb{Z} \times T^*$-grading
Lemma 3.7. A superderivation $\phi \in \text{Der}(L, V)$ is called a \textit{weight-derivation} if it is $T^*$-homogeneous. Every superderivation is a sum of weight-derivations.

Set $T = \text{span}_\mathbb{F}\{D_H(x_i x_{i'})| i \in (1, \cdots, r) \cup (m + 1, \cdots, m + s)\}$. Obviously, $T$ is a torus of $H_T$. For any $D_H(x_i x_{i'}) \in T$, we have

$$D_H(x_i x_{i'})^p = D_H(x_i x_{i'})$$

$$[D_H(x_i x_{i'}), D_H(x^{(a)} x^n)] = (\alpha_i \delta_{i' \in I_0} - \alpha_i \delta_{i \in I_0} + \delta_{i' \in u} - \delta_{i \in u})D_H(x^{(a)} x^n). \quad (3.6)$$

For $\alpha \in A$ and $u \in B$, define a linear function $(\alpha + u)$ on $T$ such that

$$(\alpha + u)(D_H(x_i x_{i'})) = (\alpha_i \delta_{i' \in I_0} - \alpha_i \delta_{i \in I_0} + \delta_{i' \in u} - \delta_{i \in u}).$$

Further, $H_T$ and $W_T$ both have weight space decompositions about $T$:

$$H_T = \bigoplus_{(\alpha + u)} H_T(\alpha + u), \quad W_T = \bigoplus_{(\alpha + u)} W_T(\alpha + u).$$

\textbf{Lemma 3.5.} \cite{13} Suppose that $L$ is a $\mathbb{Z}$-graded subalgebra of $W_T$ and $L_{-1} = (W_T)_{-1}$. Let $E \in L$ and $\phi \in \text{Der}(L, W_T)$ such that $\phi((W_T)_{-1}) = 0$. Then $\phi(E) \in \mathfrak{g}_1$ if and only if $[E, (W_T)_{-1}] \subseteq \text{ker}\phi$.

\textbf{Lemma 3.6.} \cite{11} A weight-derivation $\phi \in \text{Der}(L, V)$ is inner if it is a nonzero weight-derivation. In particular, any derivation $\phi \in \text{Der}(L, V)$ is inner modulo a zero weight-derivation.

\textbf{Lemma 3.7.} Let $i, j \in I_0$ and $k, l \in I_1$. For $q_1, q_2 \in \mathbb{F}$, set $q_1 \equiv 0 \pmod{p}$, $q_2 \equiv 1 \pmod{p}$. Then the following statements hold:

1. $\mathfrak{g}_1(q_1 e_{i}) = \left(\frac{1 + (-1)^{n+1}}{2}\right) \left(\sum_{r \in I_1} \mathbb{F} x^r \partial_r\right)$.

2. $\mathfrak{g}_1(q_2 e_{i}) = \left(\frac{1 + (-1)^{n+1}}{2}\right) \left(\mathbb{F} x^{m+n} \partial_{i'}\right)$.

3. $\mathfrak{g}_1(\varepsilon_i + (k,l)) = \left(\frac{1 + (-1)^{n+1}}{2}\right) \left(\mathbb{F} x^k x^{m+n} \partial_{i'}\right)$.

4. $\mathfrak{g}_1(\varepsilon_{i'} + (k,l)) = \left(\frac{1 + (-1)^{n+1}}{2}\right) \left(\sum_{r \in I_1} \mathbb{F} x^r \partial_r\right)$.

5. $\mathfrak{g}_1(\varepsilon_{i'}, \varepsilon_{j'}) = \left(\frac{1 + (-1)^n}{2}\right) \left(\mathbb{F} x^{m+n} \partial_{i'} + \mathbb{F} x^{m+n} \partial_{j'}\right)$.

6. $\mathfrak{g}_1(\varepsilon_k e_{i'}) = \left(\frac{1 + (-1)^{n+1}}{2}\right) \left(\sum_{r \in I_1} \mathbb{F} x^r \partial_r\right)$.

7. $\mathfrak{g}_1((k,l)) = \mathbb{F} x^{m+n} \partial_{i'} + \mathbb{F} x^{m+n} \partial_{i'}$.

Where $x^l, x''$ are both in $x^{m+n}$ for $l \in \{m + 1, \cdots, m + s\}$. 

8
Proof. Compare Eq. (3.6) with the following equation for $i \in (1, \cdots, r) \cup (m + 1, \cdots, m + s)$,

$$[D_H(x_i x_i'), x^u \partial_r] = \delta_{i' \in u} - \delta_{i \in u} - \delta_{i' r} + \delta_{i r},$$

where $x^u \partial_r \in g_1$.

By calculation, we can obtain the equations (1)–(7). \hfill \Box

4 Derivation Algebras

In this section, we determine the derivations from $\mathcal{J}$ into $W_T$ which vanish on the top of $\mathcal{J}$. To this aim, one needs to investigate the action of the derivations on the generators of $\mathcal{J}$. We shall use the fact that $\mathcal{J}$ is generated by $\mathcal{M} \cup \mathcal{N} \cup \mathcal{J}_0$ (see Lemma 3.2). Recall that $g_1 = C_{W_T}((W_T)^{-1}) = \text{span}_F \{x^u \partial_r | r \in I, u \in B, p(x_u \partial_r) = 1\}$.

For simplicity, put

$$E(g_1) = \bigoplus_{r \in \mathbb{Z}} g_{1(2r)}, \quad O(g_1) = \bigoplus_{r \in \mathbb{Z}} g_{1(2r+1)},$$

where $g_{1(2r)} = \{x^u \partial_i | |u| = 2r + 1, i \in I_0\}$ and $g_{1(2r+1)} = \{x^u \partial_i | |u| = 2r, i \in I_1\}$.

4.1 Exceptional derivations

Throughout this section assume that $|\omega| = n$ is odd. In this section we shall consider three series of the so-called exceptional derivations from $H_\mathcal{T}$ into $W_T$. And we will see that these exceptional derivations are all outer.

Let us define the first series of exceptional derivations from $H_\mathcal{T}$ into $W_T$. For $i \in I_0$ and $q \in \mathbb{N}$, define

$$\Phi_i^{(q)} : H_\mathcal{T} \to W_T, \quad D_H(f) \mapsto \partial_i^{p^q}(f)D_H(x^\omega).$$

Since $\text{Ker}(D_H) = F1$, note that $\Phi_i^{(q)}$ is well defined.

**Lemma 4.1.** $\Phi_i^{(q)} \in \text{Der}(H_\mathcal{T}, W_T)$ and $\text{ad}(\Phi_i^{(q)}) = n - p^q$.

Now we define the second series of exceptional derivations. We have known that $(\text{ad}\partial_i)^{p^q}$ is a derivation of $H_\mathcal{T}$. For $i \in I_0$ and $q \in \mathbb{N}$, define

$$\Theta_i^{(q)} : H_\mathcal{T} \to W_T, \quad D_H(f) \mapsto x^\omega(\text{ad}\partial_i)^{p^q}(D_H(f)).$$

Moreover, we have the following

**Lemma 4.2.** $\Theta_i^{(q)} \in \text{Der}(H_\mathcal{T}, W_T)$ and $\text{ad}(\Theta_i^{(q)}) = n - p^q$. 

9
Let Lemma 4.4. consider the elements in M equation holds

4.2 Derivations vanishing on the top

Lemma 4.3. \( \Psi^{(i)} \in \mathcal{D}_H(\mathcal{H}^W \mathcal{T}) \) and \( \text{ad}(\Psi^{(i)}) = n - 2 \).

Proof. For \( i \in (1, \cdots, r) \cup (m + 1, \cdots, m + s) \), define

\[
\Psi^{(i)} : H_{\mathcal{T}} \to W_{\mathcal{T}}, \quad \mathcal{D}_H(f) \mapsto \partial_i \partial_r(f) \mathcal{D}_H(x^\omega) \quad \text{for } f \in \mathcal{O}(m, n; \mathcal{I}).
\]

Case 1. \( u \neq 0, v \neq 0 \). Since \( |u| \geq 2, |v| \geq 2 \) in this case, two sides of Eq. (4.1) vanish.

Case 2. \( u = 0, v \neq 0 \). The left-hand side of Eq. (4.1) is as follows

\[
\Psi^{(i)}([\mathcal{D}_H(x^{(\alpha)} x^u), \mathcal{D}_H(x^{(\beta)} x^v)]) = \Psi^{(i)}(\mathcal{D}_H(\sum_{j=1}^{m} \tau(j) \partial_j(x^{(\alpha)}) \partial_r(x^{(\beta)} x^v)))
\]

\[
= \sum_{j=1}^{m} \tau(j)(x^{(\alpha-\epsilon_j)})(x^{(\beta-\epsilon_j)}) \partial_i \partial_r(x^u) \mathcal{D}_H(x^\omega)
\]

The right-hand side of Eq. (4.1) is as follows

\[
[\Psi^{(i)}(\mathcal{D}_H(x^{(\alpha)} x^u)), \mathcal{D}_H(x^{(\beta)} x^v)] + [\mathcal{D}_H(x^{(\alpha)} x^u), \Psi^{(i)}(\mathcal{D}_H(x^{(\beta)} x^v))]
\]

\[
= 0 + [\mathcal{D}_H(x^{(\alpha)}), x^{(\beta)} \partial_i \partial_r(x^v) \mathcal{D}_H(x^\omega)]
\]

\[
= \sum_{j=1}^{m} \tau(j)(x^{(\alpha-\epsilon_j)})(x^{(\beta-\epsilon_j)}) \partial_i \partial_r(x^v) \mathcal{D}_H(x^\omega)
\]

This proves Eq. (4.1) in this case.

Case 3. \( u \neq 0, v = 0 \). The proof is analogous to Case 2.

Case 4. \( u = v = 0 \). Obviously two sides of Eq. (4.1) vanish. \( \square \)

### 4.2 Derivations vanishing on the top

Throughout this section assume that \( |\omega| = n \) is odd unless otherwise stated. Let us consider the elements in \( \mathcal{M} \).

**Lemma 4.4.** Let \( \phi \in \text{Der}(\mathcal{J}, \mathcal{W}_\mathcal{T}) \) be homogeneous such that \( \phi(\mathcal{J}_1 \oplus \mathcal{J}_0) = 0 \). Suppose that \( zd(\phi) \) is even and \( \phi(\mathcal{D}_H(x^{(b, a)})) = 0 \) for all \( b < a \leq \pi_i \), where \( a \) is a fixed positive integer and \( i \in I_0 \). Then the following statements hold:
(1) \( a \equiv 0 \pmod{p} \). If \( a \) is not any \( p \)-power, then \( \phi(D_H(x^{(aq)}) = 0 \). If \( a = p^q \) for some \( q \in \mathbb{N} \), then there is \( \lambda_i^{(q)} \in \mathbb{F} \) such that
\[
(\phi - \lambda_i^{(q)} D_i^{(q)})(D_H(x^{(aq)})) = 0.
\]

(2) \( a \equiv 1 \pmod{p} \). If \( a - 1 \) is not any \( p \)-power, then \( \phi(D_H(x^{(aq)}) = 0 \). If \( a - 1 = p^q \) for some \( q \in \mathbb{N} \), then there is \( \mu_i^{(q)} \in \mathbb{F} \) such that
\[
(\phi - \mu_i^{(q)} D_i^{(q)})(D_H(x^{(aq)})) = 0.
\]

(3) In the other conditions, \( \phi(D_H(x^{(aq)}) = 0 \).

Proof. (1) When \( a \equiv 0 \pmod{p} \), by Lemmas 3.5 and 3.7 (1), we may assume that
\[
\phi(D_H(x^{(aq)})) = \sum_{r \in I_1} \alpha_r x^r \partial_r, \quad \text{where } \alpha_r \in \mathbb{F}. \tag{4.2}
\]

If \( a \) is not any \( p \)-power, then \( a \) is written as the \( p \)-adic form \( a = \sum_{r=1}^t c_r p^r \), \( c_r \neq 0 \). Obviously, \( \binom{a}{p^r} \neq 0 \pmod{p} \). For \( i \in I_0 \), we have
\[
[D_H(x^{(p^r a)} x_{r'}), D_H(x^{(a-p^r+1)a)}}] = \tau(i') \binom{a}{p^t} D_H(x^{(aq)}). \tag{4.3}
\]

Since \( a - p^r + 1 < a \), there is the equation that \( \phi(D_H(x^{(a-p^r+1)a)}) = 0 \). On the other hand, we have
\[
[D_H(x^{2a}) x_{r'}), D_H(x^{(aq)})] = \tau(i') D_H(x^{(a-1)a} x_{r'}).\]

It follow from Eq. (4.2) that \( \phi(D_H(x^{((a-1)a})) = 0 \). Since \( r' < a - 1 \), it follows that
\[
\phi(D_H(x^{(p^r a)} x_{r'})) = \tau(i') [D_H(x^{(2a)}) , D_H(x^{(aq)})] = 0.
\]

Then \( \phi(D_H(x^{(p^r a)} x_{r'})) = 0 \). It is obtained that \( \phi(D_H(x^{(aq)}) = 0 \).

If \( a = p^q \) for some \( q \in \mathbb{N} \), then \( \text{ad} \phi + a \) is odd. Since \( [D_H(x^{k} x'), D_H(x^{(aq)})] = 0 \) for \( k, l \in I_1 \), applying \( \phi \) on the equation, we have
\[
(\alpha_r x^l \partial_{r'}(x^l) + \alpha_l x^l \partial_{r'} - (\alpha_k x^k \partial_{r'}(x^k) + \alpha_l x^l) \partial_{r'} = 0.
\]

Furthermore, \( \alpha_r x^l \partial_{r'}(x^l) + \alpha_k x^k \partial_{r'} = 0 \). Without loss of generality, assume that \( k < l' \). It follows that \( \alpha_{k'}(-1)^{k'+l-3} + \alpha_l = 0 \), that is, \( \alpha_l = (-1)^{k'+l-3} \alpha_{k'} \). Recall that \( a = p^q \). Put \( \lambda_i^{(q)} := c_{m+1} \). Then we obtain from Eq. (4.2) that
\[
\phi(D_H(x^{(aq)})) = \lambda_i^{(q)} D_H(x^{(aq)}).
\]

(2) When \( a \equiv 1 \pmod{p} \), by Lemmas 3.5 and 3.7 (2), we may assume that
\[
\phi(D_H(x^{(aq)})) = \mu_i x^{m+1} a \partial_{r'}, \tag{4.4}
\]

11
where $\mu_i \in \mathbb{F}$ and $x^l$, $x^u$ are both in $x^{m}$ for $l \in \{m + 1, \cdots, m + s\}$.

If $a - 1$ is not any $p$–power, then $a - 1$ is written as the $p$–adic form $a - 1 = \sum_{r = 1}^{t} c_r P^r$; $c_t \neq 0$. Obviously, $\binom{a}{p} \neq 0$ (modp). Just as the proof of (1), we have $\phi(D_H(x^{(a_{(i)})})) = 0$.

If $a - 1 = p^q$ for some $q \in \mathbb{N}$, then $\text{ad}(\phi) + a$ is even. For $k \in I_1 \setminus \{m + n\}$, we obtain that
\[
0 = [D_H(x^k x^{m+n}), D_H(x^{(a_{(i)})})] = -x^k \partial_{m+n}(x^\mu x^{m+n}) \partial_v.
\]
This implies that $x^\mu x^{m+n} = x^\omega$. Therefore, we can prove
\[
\phi(D_H(x^{(a_{(i)})})) = \mu_i x^\omega \partial_v.
\]

(3) It is direct to prove by Lemmas 3.5 and 3.7.

**Lemma 4.5.** Let $\phi \in \text{Der}(\mathcal{J}, W_T)$ be homogeneous such that $\phi(J_{-1} \oplus J_0) = 0$. Suppose that $zd(\phi)$ is odd and $\phi(D_H(x^{(b_{(i)})})) = 0$ for all $b < a \leq \pi_i$, where $a$ is a fixed positive integer and $i \in I_0$. Then $\phi(D_H(x^{(a_{(i)})})) = 0$.

**Proof.** We also consider the three cases of $a$. If $a \equiv 0$ (modp) and $a$ is not any $p$–power, we can prove $\phi(D_H(x^{(a_{(i)})})) = 0$ in the similar method of Lemma 4.3 (1). If $a = p^q$ for some $q \in \mathbb{N}$, then $zd(\phi) + a$ is even. So we can obtain $\phi(D_H(x^{(a_{(i)})})) \in \mathcal{E}(\mathfrak{g}_1)$, which is a contradiction with Lemma 3.7 (1). Then $\phi(D_H(x^{(a_{(i)})})) = 0$. If $a \equiv 1$ (modp) and $a - 1$ is not any $p$–power, we can prove $\phi(D_H(x^{(a_{(i)})})) = 0$ in the similar method of Lemma 4.3 (2). If $a - 1 = p^q$ for some $q \in \mathbb{N}$, then $zd(\phi) + a$ is odd. We can obtain $\phi(D_H(x^{(a_{(i)})})) \in \mathcal{O}(\mathfrak{g}_1)$, which is a contradiction with Lemma 3.7 (2). Then $\phi(D_H(x^{(a_{(i)})})) = 0$.

In the other conditions, it is direct to prove by Lemmas 3.5 and 3.7.

Put $\mathfrak{D} = \sum_{i \in I_1} x^i x^{m+n} \partial_i$. For the elements in $\mathcal{N}$, we have the following lemma.

**Lemma 4.6.** Let $\phi \in \text{Der}(\mathcal{J}, W_T)$ be homogeneous such that $\phi(J_{-1} \oplus J_0) = 0$. Then there is $\lambda \in \mathbb{F}$ such that $\lambda \text{ad}(\mathfrak{D})(\mathcal{N}) = 0$.

**Proof.** Recall that $\mathcal{N} = \{D_H(x_i x^u)| i \in I_0, u \in \mathbb{B}_2\}$. By Lemma 3.7 (3), we obtain that
\[
\phi(D_H(x_i x^u x^l)) = \alpha_{i,(k,l)} x^k x^l x^{m+n} \partial_v,
\]
where $i \in I_0$, $k, l \in I_1$ and $\alpha_{i,(k,l)} \in \mathbb{F}$. Note that $\phi(J_0) = 0$. For $s \in I_1 \setminus \{l\}$, one can get the equation that $[D_H(x^s x^s), D_H(x_i x^{(a_{(i)})})] = D_H(x_i x^s x^l)$. Applying $\phi$ to the equation, we obtain $\alpha_{i,(k,l)} = \alpha_{i,(s,l)}$. This proves that $\alpha_{i,(k,l)}$ is only dependent on the choice of $i$. We denote $\alpha_i = \alpha_{i,(k,l)}$ for $i \in I_0$. Then $\phi(D_H(x_i x^u x^l)) = \alpha_i x^u x^{m+n} \partial_v$ for all $x^u \in \mathbb{B}_2$.

For $i, j \in I_0$, we have $[D_H(x_i x_j), D_H(x_j x^u)] = \tau(j')(1 + \delta_{i,j'})D_H(x_i x^u)$. Applying $\phi$ to the equation we have $\tau(i)\alpha_i = \tau(j)\alpha_j$ for $i, j \in I_0$. Let $\lambda = \frac{1}{2}\tau(i)\alpha_i$. Then
\[
(\phi - \lambda \text{ad}(\mathfrak{D}))(D_H(x_i x^u)) = \alpha_i x^u \partial_v - 2\tau(i)\lambda x^u \partial_v = 0.
\]

The proof is complete.
Proposition 4.7. Let \( \phi \in \text{Der}(J, W_T) \) be homogeneous and \( \phi(J_{-1} \oplus J_0) = 0 \). Then there are \( \lambda, \lambda^{(s_r)} \) and \( \mu^{(s_r)} \in \mathbb{F} \), where \( r \in I_0 \) and \( 1 \leq s_r < t_r \), such that

\[
\phi = \lambda(\text{ad}\mathcal{D}) + \sum_{r \in I_0} \sum_{1 \leq s_r \leq t_r - 1} (\lambda^{(s_r)} \Phi_i^{(s_r)} + \mu^{(s_r)} \Theta_i^{(s_r)}).
\]

Proof. In view of Lemmas 4.4 and 4.5, there are \( \lambda^{(s_r)} \) and \( \mu^{(s_r)} \in \mathbb{F} \), and we can prove by induction on \( a_i \) that

\[
\left( \phi - \sum_{r \in I_0} \sum_{1 \leq s_r \leq t_r - 1} (\lambda^{(s_r)} \Phi_i^{(s_r)} + \mu^{(s_r)} \Theta_i^{(s_r)}) \right) (\text{D}_H(x^{(a_i,i)})) = 0,
\]

for all \( i \in I_0, 1 \leq a_i \leq \pi_i \). By Lemma 4.6, there is \( \lambda \in \mathbb{F} \) such that

\[
\varphi = \phi - \sum_{r \in I_0} \sum_{1 \leq s_r \leq t_r - 1} (\lambda \Phi_i^{(s_r)} + \mu \Theta_i^{(s_r)}) - \lambda(\text{ad}\mathcal{D})
\]

vanish on \( \mathcal{N} \). It is easy to see that \( \varphi \) vanish on \( \mathcal{M} \) and Lemma 3.2 ensures that \( \varphi = 0 \). The remaining is clear and the proof is complete. \( \square \)

Theorem 4.8. Let \( \phi \in \text{Der}(H_{\overline{W}}, W_T) \) be homogeneous and \( \phi((H_{\overline{W}})_{-1} \oplus (H_{\overline{W}})_0) = 0 \). If \( n \) is even, then there are \( \mu, \lambda, \lambda^{(s_r)} \) and \( \mu^{(s_r)} \in \mathbb{F} \), where \( r \in I_0 \) and \( 1 \leq s_r < t_r \), such that

\[
\phi = \Gamma_{\mu} + \lambda(\text{ad}\mathcal{D}) + \sum_{r \in I_0} \sum_{1 \leq s_r \leq t_r - 1} (\lambda^{(s_r)} \Phi_i^{(s_r)} + \mu^{(s_r)} \Theta_i^{(s_r)}).
\]

Proof. It is direct to obtain the consequence of Propositions 3.3 and 4.7. \( \square \)

Obviously, we can get the following conclusion for the even integer \( n \) by Proposition 3.3 and Lemma 3.7.

Theorem 4.9. Let \( \phi \in \text{Der}(H_{\overline{W}}, W_T) \) be homogeneous and \( \phi((H_{\overline{W}})_{-1} \oplus (H_{\overline{W}})_0) = 0 \). If \( n \) is even, then \( \phi = 0 \).

4.3 Homogeneous derivation

Lemma 4.10. \(^{[13]} \) Suppose that \( L \) is a \( \mathbb{Z} \)-graded subalgebra of \( W_T \) satisfying \( L_{-1} = (W_T)_{-1} \). Let \( \phi \in \text{Der}(L, W_T) \) with \( \text{ad}(\phi) = t \). Then there is \( E \in (W_T)_t \) such that

\[
(\phi - \text{ad}E)(L_{-1}) = 0.
\]

Lemma 4.11. Let \( \phi \in \text{Der}(J, W_T) \) be \( \mathbb{Z} \)-homogeneous with the odd degree such that \( \phi(J_{-1}) = 0 \). Then there is \( \lambda_i \in \mathbb{F} \) such that \( (\phi - \lambda_i \Psi^{(i)})(J_0) = 0 \).
Proof. Since \(zd(\phi)\) is odd, \(zd(\phi(J_0))\) is odd. We can obtain \(\phi(J_0) \in O(g_1)\). For \(i \in I_0\), by Lemmas 3.3 and 3.7 (4), we can assume that

\[
\phi(D_H(x_ix_i')) = \sum_{r \in I_1} \alpha_r x^r \partial_r, \quad \text{where } \alpha_r \in \mathbb{F}. \tag{4.5}
\]

For arbitrary \(k, l \in I_1\), with \(k \neq l'\), we have

\[
0 = [D_H(x_k x_l), \phi(D_H(x_i x_i'))] = \sum_{r \in I_1} \alpha_r (x_k \partial_{k'} - x_k \partial_r) x^r \partial_r - \alpha_l x^l \partial_{k'} + \alpha_r x^r \partial_r.
\]

Furthermore, \(\alpha_k x^k \partial_{k'}(x^l) + \alpha_k x^k = 0\). Without loss of generality, assume that \(k < l'\). It follows that \(\alpha_k (-1)^{k+l-3} + \alpha_l = 0\), that is, \(\alpha_l = (-1)^{k+l} \alpha_k\). Put \(\lambda_i := c_{m+1}\). Then we obtain from Eq. (4.5) that

\[
\phi(D_H(x_i x_i')) = \lambda_i D_H(x^{\omega}).
\]

For \(i, j \in I_0\), with \(j \neq i'\), by Lemmas 3.3 and 3.7 (5), we can assume that

\[
\phi(D_H(x_i x_j)) = \alpha_{i'} x^{\omega} x^{m+n} \partial_r + \alpha_{j'} x^{\omega} x^{m+n} \partial_r.
\]

Then there is a contradiction. So \(\phi(D_H(x_i x_j)) = 0\).

By Lemmas 3.3 3.7 (6) and (7), for \(k, l \in I_1\) with \(l \neq k'\), we can assume that

\[
\phi(D_H(x_k x_k)) = \sum_{r \in I_1} \alpha_r x^r \partial_r, \tag{4.6}
\]

\[
\phi(D_H(x_k x_l)) = \beta_k x^k \partial_{k'} + \beta_l x^l \partial_{l'}, \tag{4.7}
\]

where \(\alpha_r, \beta_{k'}, \beta_{l'} \in \mathbb{F}\) and \(x^l, x^{l'}\) are both in \(x^{\omega}\) and \(x^{\omega'}\) for \(l \in \{m+1, \ldots, m+s\}\). Since \([D_H(x_k x_k), D_H(x_s x_t)] = 0\) for \(s, t \in I_1 \setminus \{k, k'\}\), applying \(\phi\) on the equation, we have that

\[
0 = \sum_{r \in I_1} \alpha_r x^r \partial_r, x_i \partial_{s'} - x_s \partial_{s'} = \alpha_i x^i \partial_{s'} - \alpha_s x^s \partial_{s'} - \alpha_{x_i} x_i \partial_{s'}(x^i) \partial_{s'} + \alpha_{x_s} x_s \partial_{s'}(x^s) \partial_{s'}.
\]

By the calculation of the coefficients of \(\partial_r\) and \(\partial_{s'}\), we have \(\alpha_i = (-1)^{l+s} \alpha_s\) for \(s, t \in I_1 \setminus \{k, k'\}\). We will particularly discuss \(\alpha_k\) and \(\alpha_{k'}\). For \(t \in I_1 \setminus \{k, k'\}\), we have the equation that

\[
[D_H(x_k x_k), D_H(x_k x_t)] = -D_H(x_k x_t).
\]

Applying \(\phi\) on the equation, by Eq. (4.7), we have

\[
\alpha_k x^k \partial_{k'} - \alpha_k x^k \partial_{l'} + \alpha_{k'} x_k \partial_{k'}(x^k) \partial_{k'} + \alpha_{x_k} x_k \partial_{k'}(x^k) \partial_{k'} = \beta_k x^k \partial_{k'} + \beta_{l'} x^l \partial_{l'}. \]

Since \(x^l, x^{l'}\) are both in \(x^{\omega'}\) with \(l \in \{m+1, \ldots, m+s\}\), we have that \(\alpha_k = (-1)^{l+k} \alpha_s\) and \(\alpha_{k'} = (-1)^{l+k} \alpha_s\). Furthermore, we also get \(\beta_{l'} = \beta_{l'} = 0\), for \(k, t \in I_1\), with \(t \neq k'\). Then we obtain that \(\phi(D_H(x_k x_t)) = 0\), for \(k, t \in I_1\), with \(t \neq k'\). Put \(\lambda_k := c_{m+1}\). Then we obtain from Eq. (4.6) that

\[
\phi(D_H(x_k x_k)) = \lambda_k D_H(x^{\omega}).
\]

14
For $i \in (1, \cdots, r) \cup (m+1, \cdots, m+s)$, we put
\[ \varphi = \phi - \lambda_i \Psi(i), \]
where $\lambda_i \in \mathbb{F}$. By the above proof, obviously, $\varphi(\mathcal{J}_0) = 0$.

**Lemma 4.12.** Let $\phi \in \text{Der}(\mathcal{J}, W_\mathcal{T})$ be $\mathbb{Z}$-homogeneous with the even degree such that $\phi(\mathcal{J}_1) = 0$. Then $\phi(\mathcal{J}_0) = 0$.

**Proof.** Since $zd(\phi)$ is even, $zd(\phi(\mathcal{J}_0))$ is even. We can obtain $\phi(\mathcal{J}_0) \in E(g_1)$. By Lemmas 3.5 and 3.7 for $i \in I_0$, $k, l \in I_1$ we can obtain that $\phi$ vanish on $D_H(x_i x_I)$, $D_H(x_k x_l)$ and $D_H(x_k x_j)$. For $i, j \in I_0$, with $j \neq i, i'$, by Lemma 3.7 (5), we can assume that
\[ \phi(D_H(x_i x_j)) = \alpha_{i'} x^{\Psi(i')} x^{m+n} \partial_{i'} + \alpha_{j'} x^{\Psi(j')} x^{m+n} \partial_{j'}. \quad (4.8) \]
Since $[D_H(x_i x_j), D_H(x_{i'} x_{j'})] = \tau(i)D_H(x_j x_{i'}) + \tau(j)D_H(x_i x_{i'})$, applying $\phi$ on the equation, we can obtain that $\alpha_i = \alpha_j = \alpha_{i'} = \alpha_{j'} = 0$ by Eq.(4.8). Then the conclusion is proved. \[ \square \]

**Proposition 4.13.** Let $H_H$ be the even parts of $H$ and $W_\mathcal{T}$ the odd parts of $W$. $\mathcal{J}$ is an idea of $H_H$ of codimension 1. Then the following conclusion holds:
\[ \text{Der}(\mathcal{J}, W_\mathcal{T}) = \begin{cases} \text{ad}W_\mathcal{T} + \mathbb{F}\Psi(i) + \sum_{r \in I_0} \sum_{1 \leq s_r \leq t_r - 1} (\mathbb{F}\Phi_i(s_r) + \mathbb{F}\Theta_i(s_r)) & \text{n is odd}, \\ \text{ad}W_\mathcal{T} & \text{n is even}. \end{cases} \]

**Proof.** It is direct to prove by Proposition 4.7, Lemmas 4.9, 4.10 and 4.11. \[ \square \]

Obviously, we can get the following conclusion by Theorems 4.8, 4.9 and Proposition 3.3.

**Theorem 4.14.** Let $H_H$ be the even parts of $H$ and $W_\mathcal{T}$ the odd parts of $W$. Then the following conclusion holds:
\[ \text{Der}(H_H, W_\mathcal{T}) = \begin{cases} \mathbb{F}T_i + \text{ad}W_\mathcal{T} + \mathbb{F}\Psi(i) + \sum_{r \in I_0} \sum_{1 \leq s_r \leq t_r - 1} (\mathbb{F}\Phi_i(s_r) + \mathbb{F}\Theta_i(s_r)) & \text{n is odd}, \\ \text{ad}W_\mathcal{T} & \text{n is even}. \end{cases} \]

**References**

[1] V. G. Kac, Lie superalgebras, Adv. Math., 26(1977), 8–96.

[2] Yu. Kochetkov and D. Leites, Simple Lie algebras in characteristic 2 recovered from superalgebras and on the notion of a simple finite group, Contemp. Math., 131(2)(1992), 59–67.

[3] V. M. Petrogradski, Identities in the enveloping algebras for modular Lie superalgebras, J. Algebra, 145(1992), 1–21.
[4] Y. Z. Zhang, Finite-dimensional Lie superalgebras of Cartan-type over fields of prime characteristic, Chin. Sci. Bull, 42(1997), 720–724.

[5] J. Y. Fu, Q. C. Zhang and C. P. Jiang, The Cartan-type modular Lie superalgebra $KO$. Comm. Algebra, 34(1)(2006), 107–128.

[6] W. D. Liu, and Y. H. He, Finite-dimensional special odd Hamiltonian superalgebras in prime characteristic. Commun. Contemp. Math. 11(4)(2009), 523–546.

[7] W. D. Liu and J. X. Yuan, Special odd Lie superalgebras in prime characteristic, Sci. China Math., 55(3)(2012), 567–576

[8] F. M. Ma and Q. C. Zhang, Derivation algebra of modular Lie superalgebra $K$ of Cartan-type, J. Math. (Wuhan), 20(4)(2000), 431–435.

[9] Y. Wang and Y. Z. Zhang, Derivation algebra $\text{Der}(H)$ and central extensions of Lie superalgebras, Commun. Algebra, 32(2004), 4117–4131.

[10] Q. C. Zhang and Y. Z. Zhang, Derivation algebras of modular Lie superalgebras $W$ and $S$ of Cartan type, Acta. Math. Sci., 20(1)(2000), 137–144.

[11] W. Bai and W. D. Liu, Superderivations for modular graded Lie superalgebras of Cartan-type, Algebr. Represent. Theory, 17(1)(2014), 69–86.

[12] W. D. Liu and Y. Z. Zhang, Derivations for the even parts of the modular Lie superalgebras $W$ and $S$ of Cartan type, Internat. J. Algebra. Comput., 17(2007), 661–714.

[13] W. D. Liu and B. L. Guan, Derivations from the even parts into the odd parts for Lie superalgebras $W$ and $S$, J. Lie Theory 17(2007), 449–468.

[14] B. L. Guan and W. D. Liu, Derivations of the even part into the odd part for modular contact superalgebra, J. Math. (Wuhan), 32(2012), 402–414.

[15] B. L. Guan and L. Y. Chen, Derivations of the even part of contact Lie superalgebra, J. Pure Appl. Algebra, 216(6)(2012), 1454–1466.

[16] W. D. Liu, Y. C. Su and Y. Z. Zhang, Even parts of Hamiltonian superalgebras in prime characteristic. Linear Algebra Appl., 31(2010), 379–398.

[17] H. Strade, Simple Lie algebras over fields of positive characteristic, I. Structure Theory, Walter de Gruyter, Berlin and New York, 2004.