OPERATOR INTEGRALS AND SESQUILINEAR FORMS

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ABSTRACT. We consider various systematic ways of defining unbounded operator valued integrals of complex functions with respect to (mostly) positive operator measures and positive sesquilinear form measures, and investigate their relationships to each other in view of the extension theory of symmetric operators. We demonstrate the associated mathematical subtleties with a physically relevant example involving moment operators of the momentum observable of a particle confined to move on a bounded interval.

Keywords: vector measure; operator measure; operator integral; sesquilinear form; quantum observable

1. Introduction

Selfadjoint operators represent observables in the traditional (von Neumann) description of quantum mechanics when a quantum system is associated with a Hilbert space $\mathcal{H}$. By the spectral theorem, selfadjoint operators $A$ in $\mathcal{H}$ are in a bijective correspondence with spectral measures (normalized projection valued measures) $E : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$ where $\mathcal{B}(\mathbb{R})$ is the Borel $\sigma$-algebra of the real line $\mathbb{R}$ and $L(\mathcal{H})$ is the space of bounded operators on $\mathcal{H}$. The correspondence in the spectral theorem can be written as an operator integral, in the form $A = \int x \, dE(x)$. More specifically, if $E : \mathcal{B}(\mathbb{R}) \to L(\mathcal{H})$ is a normalized projection valued measure, and $f : \mathbb{R} \to \mathbb{R}$ a Borel measurable (possibly unbounded) function, there exists a unique operator, denoted $\int f \, dE$, such that its domain

$$
\text{Dom} \left( \int f \, dE \right) = \left\{ \varphi \in \mathcal{H} \bigg| \int |f(x)|^2 \, dE_{\varphi,\varphi}(x) < \infty \right\}
$$

is dense, and, for all $\psi, \varphi \in \mathcal{H}$, $\varphi \in \text{Dom}(\int f \, dE)$,

$$
\left\langle \psi \right| \left( \int f \, dE \right) \varphi \right\rangle = \int f(x) \, dE_{\psi,\varphi}(x)
$$

where $E_{\psi,\varphi}(X) := \langle \psi | E(X) \varphi \rangle$, $X \in \mathcal{B}(\mathbb{R})$. This operator is selfadjoint and its spectral measure is $X \mapsto E(f^{-1}(X))$. In addition, $\|f \, dE\|^2 = \int |f(x)|^2 \, dE_{\varphi,\varphi}(x)$, consistent with the feature that the domain consists of exactly those vectors for which the integral of the square of $f$ is finite.

However, from the operational point of view of quantum measurement theory, this definition is often considered too restrictive: in standard modern quantum theory (in particular, quantum information theory), a generalization to (normalized) positive operator (valued) measures is used instead. A physical consequence is that a positive operator measure (POM) which is not a projection valued measure (PVM) will, in particular, allow some imperfections of measurement.

Going from projection valued measures to general positive operator measures, some useful features of the theory are lost, most notably the spectral theorem and functional calculus. However, some ideas of spectral theory may be retained: According to Naimark’s dilation theory, as given e.g. in [16] or [1], for any densely defined symmetric operator $A$ in $\mathcal{H}$ there
exists a normalized POVM \( \mathbf{E} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H}) \), having the properties
\[
\langle \psi | A \varphi \rangle = \int x \, d\mathbf{E}_{\psi, \varphi}(x), \quad \psi \in \mathcal{H}, \quad \varphi \in \text{Dom}(A),
\]
and
\[
\| A \varphi \|^2 = \int x^2 \, d\mathbf{E}_{\varphi, \varphi}(x), \quad \varphi \in \text{Dom}(A).
\]
However, unlike the case of spectral measures, the domain of \( A \) need not coincide with the set of vectors for which the integral in (3) is finite. Moreover, the correspondence does not work the other way: not every POVM \( \mathbf{E} \) satisfies (3) and (4) with respect to some symmetric operator \( A \). This has been noted in the above references, along with the fact that the integral in the right hand side of (4) may well be infinite for any nonzero vector \( \varphi \). Moreover, a normalized POVM corresponding to a symmetric operator \( A \) as above is unique only if \( A \) is maximally symmetric (i.e. has no proper symmetric extension).

For these reasons, going from a POVM to a symmetric operator is not straightforward and choosing a reasonable definition for the operator integral \( \int f \, d\mathbf{E} \) (including its domain) is problematic – except when \( f \) is bounded, in which case the domain is all of \( \mathcal{H} \).

In fact, the difficulties in choosing the domain have led the authors in [1], p. 132] to consider \( \int x \, d\mathbf{E}(x) \) in a symbolic sense only, as a shorthand for the equations (3) and (4), provided they hold for the given POVM. As pointed out by Werner [18], however, even the general operator integral \( \int f \, d\mathbf{E} \) can be uniquely defined as a symmetric operator on the domain (1), so that (2) holds, in contrast to what appears to be intended in [1], p. 132]. (See the above paper by Werner, and also [8].) The reason why this does not contradict the observation that not every POVM satisfies (3) and (4) for some symmetric operator, is simply that (4) does not hold for \( A = \int x \, d\mathbf{E}(x) \), in general.

When (1) holds, with (1) dense, the POVM is called variance free [19]. For a general POVM it may be the case that only the inequality
\[
\left\| \left( \int f \, d\mathbf{E} \right) \varphi \right\|^2 \leq \int |f(x)|^2 \, d\mathbf{E}_{\varphi, \varphi}
\]
holds. The domain of (1) has a physical meaning as the set of those vector states for which the measurement distribution has finite variance. For this reason, this set is a natural domain for the variance form
\[
(\psi, \varphi) \mapsto \int x^2 \, d\mathbf{E}_{\psi, \varphi} - \langle \tilde{\mathbf{E}}[1] \psi | \tilde{\mathbf{E}}[1] \varphi \rangle \in \mathbb{C}
\]
where \( \tilde{\mathbf{E}}[1] = \int x \, d\mathbf{E} \) is the first moment operator of \( \mathbf{E} \) (see Section 5). This definition for the domain of the operator integral appears most frequently in the literature, see e.g. [18, 17, 1].

One might think that above the definition would settle the question of defining the operator integral. However, after losing the equality in (5), it is no longer clear whether the finiteness of the integral in the right hand side is actually needed to define the operator integral. Loosely speaking, the reason for the square of \( f \) appearing in the definition of the domain is connected to the multiplicativity of the projection valued measure, which is no longer true for POVMs. In fact, the square integrability domain (1) is not necessarily the largest possible one where (2) defines an operator. This is easy to see: for example, consider the POVM \( X \mapsto \mathbf{E}(X) := \mu(X)I \), where \( \mu \) is a probability measure and \( I \) the identity operator on any Hilbert space. If \( f \) is a \( \mu \)-integrable function, the integrals \( \int f \, d\mathbf{E}_{\psi, \varphi} = \langle \psi | (\int f \, d\mu) \varphi \rangle \) determine a well-defined operator with domain all of \( \mathcal{H} \), even if \( \int |f|^2 \, d\mu = \infty \), collapsing the domain (1) to \( \{0\} \). Hence, the natural definition of an operator integral needs closer mathematical examination.

A different definition has been used in [8, 9]; we call this the strong operator integral. As we will see, even this choice is not the largest reasonable, and we will also define weak operator
integrals which have still larger domains than the strong one. These are more operationally motivated, as they are constructed from the scalar measures $X \mapsto \langle \psi | E(X) | \varphi \rangle$.

The structure of the paper is as follows. We begin by considering strong operator integrals in the setting of general Banach spaces. When specializing to Hilbert spaces and positive operator measures, the role of the square integrability domain is explained. Subsequently, we proceed to introduce weak operator integrals, and investigate their connection to operators defined via quadratic forms. A physically motivated example concludes the paper.

2. Preliminaries and notations

We begin with a fairly general setting: let $E$ and $F$ be Banach spaces and $L(E, F)$ the space of bounded linear operators $T : E \to F$. (We use complex scalars as our main applications deal with complex Hilbert spaces.) Consider a measurable space $(\Omega, A)$ (where by definition $A$ is a $\sigma$-algebra of subsets of $\Omega$). A map $M : A \to L(E, F)$ is called an operator measure if it is strong operator (or briefly, strongly) $\sigma$-additive. This means that for each $x \in E$ the map $X \mapsto M_x(X) := M(X)x$ is a vector measure, i.e. $\sigma$-additive with respect to the norm in $F$. By the Orlicz-Pettis theorem it is equivalent to require that for any $x \in E$ and $y' \in F'$ (the topological dual of $F$), the function $X \mapsto M_{y', x}(X) := \langle y', M(X)x \rangle$ on $A$ is a complex measure. The following definition agrees with the usage in [4]. (We only integrate $A$-measurable functions, though this restriction could be relaxed somewhat, see e.g. [20].)

**Definition 1.** Let $\mu : A \to F$ be a vector measure and $f : \Omega \to \mathbb{C}$ an $A$-measurable function. The function $f$ is $\mu$-integrable if there is a sequence $(f_n)$ of simple functions converging to $f$ pointwise and such that $\lim_{n \to \infty} \int_X f_n \, d\mu$ exists for all $X \in A$. Then $\int_\Omega f \, d\mu := \lim_{n \to \infty} \int_\Omega f_n \, d\mu$ is called the integral of $f$ with respect to $\mu$.

**Remark 1.** It turns out to be equivalent to the above definition to require that $f$ is integrable with respect to the complex measure $\mu_{y'} := y' \circ \mu$ for every $y' \in F'$ and for each $X \in A$ one has $\nu(X) \in F$ (clearly unique) such that $\langle y', \nu(X) \rangle = \int_X f \, d\mu_{y'}$ for all $X \in A$. $y' \in F'$. (See [10], and [20] for another proof.) If $f$ is integrable with respect to every $\mu_{y'}$, it follows from the dominated convergence theorem and the uniform boundedness principle (as in e.g. [5, p. 328]) that for each $X \in A$ there is some $\nu(X) \in F''$ satisfying $\langle y', \nu(X) \rangle = \int_X f \, d\mu_{y'}$ for each $y' \in F'$, and so in case $F$ is reflexive, we can conclude that $f$ is actually $\mu$-integrable. We use this observation especially when $F$ is a Hilbert space.

Let $\mathcal{H}$ be a (complex) Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the space of bounded operators on $\mathcal{H}$. We do not have to assume that $\mathcal{H}$ is separable, except in some examples where this is clearly indicated. The identity operator of $\mathcal{H}$ is denoted by $I_\mathcal{H}$ or simply by $I$. For $\psi, \varphi \in \mathcal{H}$, we use the symbol $|\psi \rangle \langle \varphi|$ to denote the rank one operator $\eta \mapsto \langle \varphi | \eta \rangle \psi$. For a (linear) operator $A$ in $\mathcal{H}$, we let $\text{Dom}(A)$ denote the domain of $A$, i.e. the (linear) subspace of $\mathcal{H}$ on which $A$ is defined. As before, $(\Omega, A)$ is a measurable space. We let $\mathcal{B}(\Omega)$ denote the Borel $\sigma$-algebra of any topological space $\Omega$. We follow the convention $\mathbb{N} = \{0, 1, 2, \ldots\}$, and let $\chi_X$ be the characteristic function of the set $X \in A$.

**Definition 2.** Let $E : A \to \mathcal{L}(\mathcal{H})$ be a function.

(a) $E$ is a positive operator (valued) measure, or POVM for short, if $E$ is an operator measure and $E(X) \geq 0$ for all $X \in A$.

(b) A POVM $E$ is normalized if $E(\Omega) = I$.

(c) A projection valued POVM (PVM for short) which is normalized is a spectral measure.

For a POVM $E : A \to \mathcal{L}(\mathcal{H})$ and $\psi, \varphi \in \mathcal{H}$, we let $E_{\psi, \varphi}$ denote the complex measure $X \mapsto \langle \psi | E(X) | \varphi \rangle$ and $E_\varphi$ denote the $\mathcal{H}$-valued vector measure $X \mapsto E(X) \varphi$. 
Lemma 1. Appearing in the construction of the integral, defines a bounded operator. It depends on the fact that the range of any POVM is norm bounded, and the resulting integral of the integrals of simple functions forming a uniformly convergent sequence. Ultimately, this holds for these simple functions to bounded operators. The following lemma is straightforward to prove by using the usual approximation techniques appearing in the construction of the integral.

Lemma 1. Let \((\mathcal{K}, F, V)\) be a Naimark dilation of \(E\). Then for every bounded \(\mathcal{A}\)-measurable function \(f : \Omega \to \mathbb{C}\), we have \(\int f dE = V^* \left( \int f dF \right) V\).

For unbounded functions, even defining a domain for the operator valued integral needs attention. We study this question next.

3. Strong operator integrals

Let \((\Omega, \mathcal{A})\) be a measurable space. We first consider general Banach spaces \(E\) and \(F\).

Definition 3. Let \(M : \mathcal{A} \to L(E, F)\) be an operator measure and \(f : \Omega \to \mathbb{C}\) an \(\mathcal{A}\)-measurable function. We let \(D(f, M)\) denote the subset of \(E\) consisting of those \(x \in E\) for which \(f\) is integrable with respect to the vector measure \(X \mapsto M_x(X) = M(X)x\). If \(x \in D(f, M)\), we denote by \(L(f, M)x\) the integral \(\int_{\Omega} f dM_x\).

Proposition 1. If \(f : \Omega \to \mathbb{C}\) is an \(\mathcal{A}\)-measurable function, the set \(D(f, M)\), the domain of \(L(f, M)\), is a vector subspace of \(E\), and \(L(f, M) : D(f, M) \to F\) is a linear map.

Proof. See e.g. [20], Corollary 3.7.

The following proposition is an immediate consequence of Remark 1.

Proposition 2. Assume that the Banach space \(F\) is reflexive. For \(x \in E\) the following conditions are equivalent:

(i) \(x \in D(f, M)\);

(ii) \(f\) is \(M_{y', x}\) integrable for all \(y' \in F'\).

We mainly apply the above results in the case where \(F = \mathcal{H}\), a Hilbert space.

Definition 4. We say that a vector measure \(\mu : \mathcal{A} \to \mathcal{H}\) is orthogonally scattered if

\[\langle \mu(X)|\mu(Y) \rangle = 0\]

whenever the sets \(X, Y \in \mathcal{A}\) are disjoint.

Orthogonally scattered vector measures have a highly developed theory, see e.g. [12]. A basic observation is that if \(\mu : \mathcal{A} \to \mathcal{H}\) is an orthogonally scattered vector measure, by denoting \(\lambda(X) = \lambda_\mu(X) := \|\mu(X)\|^2\), we get a finite positive measure \(\lambda\) on \(\mathcal{A}\). The following result is well known and we only give a brief indication of proof.
Proposition 3. Let $\mu : A \to \mathcal{H}$ be an orthogonally scattered vector measure and $\lambda = \lambda_\mu$ the positive measure defined above. An $\mathcal{A}$-measurable function $f : \Omega \to \mathbb{C}$ is $\mu$-integrable if and only if $|f|^2$ is $\lambda$-integrable, in which case $\left\| \int_\Omega f \, d\mu \right\|^2 = \int_\Omega |f|^2 \, d\lambda$.

Proof. In one direction, one may use the argument in the proof of Lemma A.2 (b) in [3]. In the other direction a technique from the proof of Proposition 4 below may be adapted. \hfill \Box

Remark 2. (a) It follows from the above proposition that if $E$ is a Banach space and $M : A \to L(E, \mathcal{H})$ is an operator measure such that for each $x \in E$ the vector measure $M_x : A \to \mathcal{H}$ is orthogonally scattered, then the domain $D(f, M)$ of the strong operator integral $L(f, M)$ consists of precisely those vectors $x \in E$ for which $|f|^2$ is integrable with respect to the measure $X \mapsto \|M_x(X)^2\|$ on $A$.

(b) If $E : A \to \mathcal{L}(\mathcal{H})$ is a PVM, then for each $\varphi \in \mathcal{H}$ the vector measure $E_\varphi$ is orthogonally scattered and $\|E_\varphi(X)\|^2 = \langle \varphi | E(X) | \varphi \rangle$ whenever $X \in A$.

(c) Consider the Hilbert space $\ell^2 = \ell^2(\mathbb{N})$. Let $\mathcal{A}$ be the power set of $\mathbb{N}$. Let $g : \mathbb{N} \to \mathbb{C}$ be a bounded function and define $M : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ by the formula $M(X)\varphi = g\varphi \chi_X$ for all $X \in \mathcal{A}$, $\varphi \in \ell^2$. Then $M$ satisfies the assumption in (a), so that $D(f, M)$ consists of those $\varphi \in \ell^2$ for which $fg\varphi \in \ell^2$. Note that $M$ need not be a PVM, nor even a POVM. This example can be easily extended for more general measure spaces.

We have seen (the well-known fact) that for a PVM $E : A \to \mathcal{L}(\mathcal{H})$, a vector $\varphi \in \mathcal{H}$ belongs to $D(f, E)$ if and only if $|f|^2$ is integrable with respect to the measure $E_{\varphi, \varphi}$. More generally, for any POVM $E : A \to \mathcal{L}(\mathcal{H})$ we call the set $\tilde{D}(f, E) := \{ \varphi \in \mathcal{H} \mid |f|^2 \text{ is } E_{\varphi, \varphi}\text{-integrable}\}$ the square integrability domain for the integral $\int_\Omega f \, dE$. This makes sense, as it is known that $\tilde{D}(f, E)$ is a linear subspace of $\mathcal{H}$ contained in $D(f, E)$. In [3] this was given a direct elementary proof. The authors of [3] were unaware that this result essentially had already appeared in [18], where the proof is based on Naimark’s dilation theorem. (For completeness, we give a proof below reproducing the idea in [18].) The fact that $\tilde{D}(f, E)$ is a linear subspace is implied by the following easy consequence of the Cauchy-Schwarz inequality. We state it explicitly as it will also have some later use. (The terminology will be recalled at the beginning of Section 4)

Lemma 2. Let $V$ be a vector space, and $q : V \times V \to \mathbb{C}$ a positive sesquilinear form. Then $q(\varphi + \psi, \varphi + \psi) \leq 2q(\varphi, \varphi) + 2q(\psi, \psi), \quad \varphi, \psi \in V$.

Proposition 4. The vector valued integral $\int f \, dE_{\varphi}$ exists for each $\varphi \in \tilde{D}(f, E)$.

Proof. Let $(K, F, V)$ be a Naimark dilation of $E$ and $(f_n)$ a sequence of simple functions converging pointwise to $f$ with $|f_n| \leq |f|$. Then the bounded operator $\int f_n \, dE$ is defined for each $n$ according to the definition in the preceding section. Fix $\varphi \in \tilde{D}(f, E)$. Using Lemma 1 and the multiplicativity of the spectral measure $F$, we have for each $X \in \mathcal{A}$, that

$$\left\| \int_X (f_n - f_m) \, dE_{\varphi} \right\|^2 = \left\| V^* \int_X (f_n - f_m) \, dFV \varphi \right\|^2 \leq \left\| V^* \right\|^2 \left\| \int_X (f_n - f_m) \, dFV \varphi \right\|^2 = \left\| V^* \right\|^2 \int_X |f_n - f_m|^2 \, dF_{V\varphi, V\varphi} = \left\| V^* \right\|^2 \int_X |f_n - f_m|^2 \, dE_{\varphi, \varphi}.$$ 

Since $|f|^2$ is integrable, it thus follows from the dominated convergence theorem that the sequence $(\int_X f_n \, dE_{\varphi})$ of vectors is a Cauchy sequence, and thus converges. This proves the existence of the integral $\int f \, dE_{\varphi}$ of $f$ with respect to the vector valued measure $E_{\varphi}$. \hfill \Box

According to this result, we can define a linear operator $\tilde{L}(f, E) : \tilde{D}(f, E) \to \mathcal{H}, \quad \tilde{L}(f, E)\varphi := \int f \, dE_{\varphi}$. 


Since $f$ is integrable with respect to each scalar measures $E_{\psi,\varphi}$, $\psi \in \mathcal{H}$, if is integrable with respect to $E_{\varphi}$ (see e.g. [11]), it follows that $\langle \psi | \tilde{L}(f,E) | \varphi \rangle = \int f dE_{\psi,\varphi}$ for all $\varphi \in \tilde{D}(f,E)$, $\psi \in \mathcal{H}$. Since $\tilde{D}(f,E) = \{ \varphi \in \mathcal{H} \mid V \varphi \in \tilde{D}(f,F) \}$, where $\tilde{D}(f,F)$ is the domain of the selfadjoint operator $\tilde{L}(f,F)$, it now follows that

$$\tilde{L}(f,E) = V^* \tilde{L}(f,F)V.$$ 

(see also [9].)

Summarizing, for a POVM $E$ and a measurable function $f$, we have $\tilde{D}(f,E) \subset D(f,E)$ and

$$D(f,E) = \{ \varphi \in \mathcal{H} \mid \int |f| d|E_{\psi,\varphi}| < \infty \text{ for all } \psi \in \mathcal{H} \},$$

for the total variation $|E_{\psi,\varphi}|$ of $E_{\psi,\varphi}$.

In definition [3] we used the notation $L(f,M)$ but did not give it a name. From now on, we call it the strong operator integral of $f$ or the maximal strong operator integral of $f$ with respect to the operator measure $M$. If $D$ is a linear subspace of $D(f,M)$, we may call the restriction of $L(f,M)$ to $D$ a strong operator integral. Thus for a POVM $E$, the operator $\tilde{L}(f,E)$ is a strong operator integral. In this Hilbert space setting the key to our terminology is the possibility to use the whole of $\mathcal{H}$ as a “test space”: for any $\varphi$ in the appropriate domain, the integral of $f$ with respect to $E_{\psi,\varphi}$ for every $\psi \in \mathcal{H}$ exists.

**Example 1.** Let $A$ be an unbounded selfadjoint operator in $\mathcal{H}$ and $E : B(\mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H})$ its spectral measure. Then $A = L(f,E)$ and $D(f,E) = \{ \varphi \in \mathcal{H} \mid f \text{ is } E_{\psi,\varphi}-\text{integrable for all } \psi \in \mathcal{H} \}$ where $f : \mathbb{R} \rightarrow \mathbb{R}$ is the identity map. Since $E_{\psi,\varphi}(X) = \overline{E_{\varphi,\psi}(X)}$, we may also observe that if $\psi \in \mathcal{H}$, then $f$ is $E_{\varphi,\psi}$-integrable for all $\varphi$ in the dense subspace $D(f,E)$ of $\mathcal{H}$. But this does not imply that $D(f,E) = \mathcal{H}$. In particular, we see that in Proposition [2] it is not enough to assume the $M'_{y',x}$-integrability of $f$ for all $y'$ in a dense subspace of $F'$.

The above example may serve as a motivation for considering integration with respect to operator measures where the requirement for the test space described before the example is relaxed. This leads us to a host of possibilities for so-called weak operator integrals whose analysis will be our main concern in the sequel.

4. WEAK OPERATOR INTEGRALS

Often in physical applications one is led to consider the scalar measures $X \mapsto E_{\psi,\varphi}(X) = \langle \psi | E(X) | \varphi \rangle$ related to a Hilbert operator measure $E$ instead of the vector measures $E_{\varphi}$. In this section we set up a very general framework for this. For any vector spaces $V_1$, $V_2$, a map $S : V_1 \times V_2 \rightarrow C$ is said to be a sesquilinear form, or just sesquilinear, if it is linear in the second and antilinear (i.e. conjugate linear) in the first argument. Such an $S$ is positive if $V_1 = V_2$ and $S(\varphi, \varphi) \geq 0$ for all $\varphi \in V_1$. Then $S$ satisfies $S(\psi, \varphi^*) S(-\varphi, \psi^*)$ for all $\psi, \varphi \in V_1$.

Any vector space $\mathcal{V}$ may be regarded as a dense linear subspace of a Hilbert space $\mathcal{H}$: take $\mathcal{H} = \ell_K^2$ where $K$ is a Hamel basis of $\mathcal{V}$. In the context of sesquilinear forms there is, however, often a postulated way the vector space is embedded as a dense subspace of a Hilbert space. When this is the case, it is clear from the context so that, for example, there is a given norm and hence a topology on $\mathcal{V}$.

Let $\mathcal{V} \subseteq \mathcal{H}$ be a dense (linear) subspace of $\mathcal{H}$ and $S(\mathcal{V})$ the vector space of sesquilinear forms $S : \mathcal{V} \times \mathcal{V} \rightarrow C$. Assume that $E : \mathcal{A} \rightarrow S(\mathcal{V})$ is a positive sesquilinear form valued measure, i.e.

(a) $E_{\psi,\varphi} : \mathcal{A} \rightarrow \mathbb{C}$, $X \mapsto E_{\psi,\varphi}(X) := [E(X)](\psi, \varphi)$, is a complex measure for all $\psi, \varphi \in \mathcal{V}$,
We refer the reader to [31, 32] for a detailed study of such measures. Note that any POVM $E' : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ defines a unique positive sesquilinear form valued measure $E : \mathcal{A} \to \mathcal{S}(\mathcal{H})$ by setting $[E(X)](\psi, \varphi) := \langle \psi | E(X) | \varphi \rangle$ (thus, in the case of POVMs, we may put $\mathcal{V} = \mathcal{H}$ below). We always identify $E'$ with $E$ and by an abuse of notation simply write $E' = E$. Throughout this section, $f : \mathcal{A} \to \mathbb{C}$ is an $\mathcal{A}$-measurable function.

4.1. Definition. We begin with the maximal set of pairs $(\psi, \varphi)$ for which $\int f \, dE_{\psi,\varphi}$ makes sense:

$$\mathcal{W}(f, E) := \{(\psi, \varphi) \in \mathcal{V} \times \mathcal{V} | f \text{ is } E_{\psi,\varphi}\text{-integrable}\}.$$

Note that $E_{\psi,\varphi}(X) \equiv E_{\varphi,\psi}(X)$ by positivity so that $|E_{\psi,\varphi}| = |E_{\varphi,\psi}|$ and, hence, $\mathcal{W}(f, E) \subseteq \mathcal{V} \times \mathcal{V}$ is symmetric, i.e. $(\psi, \varphi) \in \mathcal{W}(f, E)$ implies $(\varphi, \psi) \in \mathcal{W}(f, E)$. We then put

$$(9) \quad \mathcal{W}_\varphi(f, E) := \{\psi \in \mathcal{V} | (\psi, \varphi) \in \mathcal{W}(f, E)\}$$

for each $\varphi \in \mathcal{V}$. Since $E_{\alpha\psi_1 + \beta\psi_2,\varphi} = \alpha E_{\psi_1,\varphi} + \beta E_{\psi_2,\varphi}$, $\alpha, \beta \in \mathbb{C}$, $\psi_1, \psi_2 \in \mathcal{V}$, it follows that each $\mathcal{W}_\varphi(f, E) \subseteq \mathcal{V} \subseteq \mathcal{H}$ is a linear subspace, and the functional $\psi \mapsto \int f \, dE_{\psi,\varphi}$ is linear on that subspace. A similar argument shows that

$$(10) \quad \mathcal{W}_{\varphi_1}(f, E) \cap \mathcal{W}_{\varphi_2}(f, E) \subseteq \mathcal{W}_{\alpha\varphi_1 + \beta\varphi_2}(f, E)$$

for any $\varphi_1, \varphi_2 \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{C}$.

We are now interested in (linear) operators $T : \text{Dom}(T) \to \mathcal{H}$ determined by these integrals through $\langle \psi | T \varphi \rangle = \int f \, dE_{\psi,\varphi}$. Accordingly, such an operator should have the property that for each $\varphi \in \text{Dom}(T)$: $\langle \psi | T \varphi \rangle = \int f \, dE_{\psi,\varphi}$, where $\psi$ runs through some subset $\mathcal{S}_\varphi$ of $\mathcal{W}_\varphi(f, E)$ which separates the points of $\mathcal{H}$ in the usual sense of self-duality of $\mathcal{H}$. We make this separation requirement to always guarantee that the vector $T \varphi$ is uniquely determined by the integrals $\int f \, dE_{\psi,\varphi}$ via the inner products just mentioned. Note that here we really want to determine $T \varphi$, and the vector $\psi$ is just in an auxiliary role. Since each $\mathcal{W}_\varphi(f, E)$ is a linear subspace, the necessarily dense linear span $\mathcal{D}_\varphi$ of such a separating subset $\mathcal{S}_\varphi$ is also included in $\mathcal{W}_\varphi(f, E)$, and by linearity, $\langle \psi | T \varphi \rangle = \int f \, dE_{\psi,\varphi}$ for all $\psi \in \mathcal{D}_\varphi$. Hence, we can take the separating subsets to be dense subspaces without restricting generality.

The above requirements imply, in particular, that $\text{Dom}(T)$ must be a subset of

$$\Gamma(f, E) := \{\varphi \in \mathcal{V} | \mathcal{W}_\varphi(f, E) \text{ is dense in } \mathcal{H}\}.$$

The requirement of choosing the separating subspaces can now be formulated as follows: Let $\mathcal{C}(f, E)$ denote the family of maps

$$\Phi : \Gamma(f, E) \to \{D \subseteq \mathcal{H} | D \text{ is a dense subspace}\},$$

$$\Phi(\varphi) \subseteq \mathcal{W}_\varphi(f, E) \text{ for all } \varphi \in \Gamma(f, E).$$

Note that $\mathcal{C}(f, E) \neq \emptyset$, because an obvious choice is $\Phi(\varphi) = \mathcal{W}_\varphi(f, E)$ for all $\varphi \in \Gamma(f, E)$. We can now state the definition of a weak operator integral.

Definition 5. We say that a linear operator $T : \text{Dom}(T) \to \mathcal{H}$ is a weak operator integral of $f$ with respect to $E$, if $\text{Dom}(T) \subseteq \Gamma(f, E)$, and there exists a map $\Phi \in \mathcal{C}(f, E)$, such that

$$\langle \psi | T \varphi \rangle = \int f \, dE_{\psi,\varphi}, \quad \text{for all } \varphi \in \text{Dom}(T), \psi \in \Phi(\varphi).$$

1If we would be interested in sesquilinear forms rather than operators, then we should consider $\psi$ and $\varphi$ in an equal footing. However, here we want to consider operator integrals, so the given requirement is clearly the most natural one.

2Note that the orthogonal complement of a separating subset $\mathcal{S}$ is $\mathcal{H}$, so $\mathcal{S}$ generates a dense subspace.
We then also say that the weak operator integral $T$ is associated with the map $\Phi$. For each $\Phi \in \mathcal{C}(f, \mathcal{E})$, we let $\mathcal{L}_W(f, \mathcal{E}, \Phi)$ denote the set of weak operator integrals associated with $\Phi$.

Note that $\Gamma(f, \mathcal{E})$ always contains at least the trivial subspace $\mathcal{D}_0 = \{0\}$, so for every choice of $\Phi$ there corresponds at least a trivial weak operator integral.

The choice of the function $\Phi$ is crucial; different choices may correspond to different operators $T$, because on the one hand, dense subspaces can even have trivial intersection, see Section 6 for an example, and on the other hand, different choices can lead to the same operator. In particular, we have the following result.

**Proposition 5.** Let $\mathcal{E} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ be a POVM. Each strong operator integral is also a weak operator integral associated with every $\Phi \in \mathcal{C}(f, \mathcal{E})$.

**Proof.** According to (7), the domain of the maximal strong operator integral is given by

$$D(f, \mathcal{E}) = \{ \varphi \in \mathcal{H} \mid \mathcal{W}_\varphi(f, \mathcal{E}) = \mathcal{H} \},$$

so $D(f, \mathcal{E}) \subseteq \Gamma(f, \mathcal{E})$. Given any $\Phi \in \mathcal{C}(f, \mathcal{E})$, equation (11) holds because of (7).

Now, given a map $\Phi \in \mathcal{C}(f, \mathcal{E})$, we set

$$\Gamma_c(f, \mathcal{E}, \Phi) := \left\{ \varphi \in \Gamma(f, \mathcal{E}) \mid \Phi(\varphi) \ni \psi \mapsto \int f d\psi_{\psi, \varphi} \text{ is continuous} \right\},$$

and use the Frechet-Riesz theorem to define a unique map

$$G(f, \mathcal{E}, \Phi) : \Gamma_c(f, \mathcal{E}, \Phi) \to \mathcal{H}, \quad \langle \psi | G(f, \mathcal{E}, \Phi) \varphi \rangle = \int f d\psi_{\psi, \varphi} \text{ for all } \psi \in \Phi(\varphi).$$

Clearly, the domain of any weak operator integral associated with the map $\Phi$ is included in $\Gamma_c(f, \mathcal{E}, \Phi)$. This observation immediately gives the following characterization.

**Proposition 6.** Fix a $\Phi \in \mathcal{C}(f, \mathcal{E})$. Given any subspace $\mathcal{D}_0$ of $\mathcal{H}$, which is included in $\Gamma_c(f, \mathcal{E}, \Phi)$, the restriction $G(f, \mathcal{E}, \Phi)|_{\mathcal{D}_0}$ is a weak operator integral (with domain $\mathcal{D}_0$) associated to $\Phi$. Conversely, every element of $\mathcal{L}_W(f, \mathcal{E}, \Phi)$ is obtained this way.

Since the intersection of two dense subspaces does not have to be dense (it can even be $\{0\}$), it is clear that $\Gamma(f, \mathcal{E})$, and therefore also $\Gamma_c(f, \mathcal{E}, \Phi)$ are not themselves linear subspaces, in general. Hence, there is no canonical choice for a maximal weak operator integral associated with a given map $\Phi$. However, it follows immediately from the above proposition that given two operators $T, T'$, such that $T' \subseteq T$ (that is, $\text{Dom}(T') \subseteq \text{Dom}(T)$ and $T' \varphi = T \varphi, \varphi \in \text{Dom}(T)$), and $T \in \mathcal{L}_W(f, \mathcal{E}, \Phi)$, it follows that $T' \in \mathcal{L}_W(f, \mathcal{E}, \Phi)$. In particular, the (nonempty) set $\mathcal{L}_W(f, \mathcal{E}, \Phi)$ is partially ordered via the usual operator ordering, or, equivalently, the inclusion of domains. Moreover, every (nonempty) totally ordered subset of $\mathcal{L}_W(f, \mathcal{E}, \Phi)$ has an upper bound in $\mathcal{L}_W(f, \mathcal{E}, \Phi)$ (the upper bound is obtained by taking the union of the domains of the operators in the chain). Hence, by Zorn’s lemma, there exists at least one maximal element in $\mathcal{L}_W(f, \mathcal{E}, \Phi)$. We call such an element a maximal weak operator integral associated to $\Phi$.

**Example 2.** For a POVM $\mathcal{E}$ and a bounded function $f$, we have $\Gamma_c(f, \mathcal{E}, \Phi) = \mathcal{H}$ regardless of the choice of $\Phi$, so every weak operator integral is a restriction of the bounded operator $\int f d\mathcal{E}$ to some subspace.

**Example 3.** Let $\mathcal{E}(X) := \mu(X)I$, where $\mu$ is a probability measure, and let $f$ be a $\mu$–integrable function. Then $\mathcal{W}(f, \mathcal{E}) = \mathcal{H} \times \mathcal{H}$, and $\Gamma_c(f, \mathcal{E}, \Phi) = \mathcal{H}$, regardless of the choice of $\Phi$, so that weak operator integrals are simply restrictions of $\varphi \mapsto (\int f d\mu) \varphi$ to some subspaces of $\mathcal{H}$. If $f$ is not $\mu$–integrable, then $\mathcal{W}(f, \mathcal{E}) = \{ (\psi, \varphi) \in \mathcal{H} \times \mathcal{H} \mid \langle \psi | \varphi \rangle = 0 \}$, and $\mathcal{W}_\varphi(f, \mathcal{E})$ is the orthogonal complement of $\{ \varphi \}$. This is dense only for $\varphi = 0$, so $\Gamma(f, \mathcal{E}) = \{0\}$. Hence, there exists only one weak operator integral, which is the zero operator defined on $\{0\}$. 

4.2. Weak operator integrals determined by a fixed separating subspace. We now look at the class $L_W(f, E, \Phi)$ with particular choices of $\Phi$. The canonical choice would be to take, for each $\varphi \in \Gamma(f, E)$, the separating subspace to be the maximal one, i.e. $\Phi(\varphi) = \mathcal{W}_\varphi(f, E)$ for each $\varphi$. However, in practice, it often happens that a fixed dense subspace (of e.g. smooth functions) is fixed. For example, this can be a linear space spanned by some physically relevant orthonormal basis of $H$ (e.g. the photon number basis of a single mode optical field).

Accordingly, we now investigate the case where a fixed dense subspace $D_s$ is given ($s$ stands for separating). For $\varphi \in \Gamma(f, E)$ we define $\Phi_{D_s}(\varphi) = D_s$ if $D_s \subseteq \mathcal{W}_\varphi(f, E)$ and $\Phi_{D_s}(\varphi) = \mathcal{W}_\varphi(f, E)$ otherwise. Then we have

**Proposition 7.** The set

$$\hat{D}_{D_s}(f, E) := \left\{ \varphi \in \mathcal{H} \mid D_s \subseteq \mathcal{W}_\varphi(f, E), \forall \psi \mapsto \int f dE_{\psi, \varphi} \in \mathbb{C} \text{ is continuous} \right\}$$

is the domain of a (clearly unique) element $\hat{L}_{D_s}(f, E) \in L_W(f, E, \Phi_{D_s})$. In the case where $E$ is a POVM, this operator is an extension of the maximal strong operator integral $L(f, E)$.

**Proof.** Clearly, $\hat{D}_{D_s}(f, E) = \left\{ \varphi \in \Gamma_c(f, E, \Phi_{D_s}) \mid D_s \subseteq \mathcal{W}_\varphi(f, E) \right\}$; in particular, $\hat{D}_{D_s}(f, E)$ is a subset of $\Gamma_c(f, E, \Phi_{D_s})$. We have to show that it is a linear space. Let $\varphi_1, \varphi_2 \in \hat{D}_{D_s}(f, E)$, and $\alpha, \beta \in \mathbb{C}$. Now $D_s \subseteq \mathcal{W}_{\varphi_1}(f, E) \cap \mathcal{W}_{\varphi_2}(f, E) \subseteq \mathcal{W}_{\alpha\varphi_1 + \beta\varphi_2}(f, E)$ (see [10]); in particular, the latter is dense, so $\alpha\varphi_1 + \beta\varphi_2 \in \Gamma(f, E)$, and $\Phi(\alpha\varphi_1 + \beta\varphi_2) = D_s$. Since $\psi \mapsto \int f dE_{\psi, \varphi}$ is continuous on $D_s$, it is continuous on $D_s$. Hence, $\alpha\varphi_1 + \beta\varphi_2 \in \Gamma_c(f, E, \Phi_{D_s})$.

We have shown that $\hat{D}_{D_s}(f, E)$ is a linear space. By Proposition 6, the restriction of $G(f, E, \Phi_{D_s})$ to $\hat{D}_{D_s}(f, E)$ is an element of $L_W(f, E, \Phi_{D_s})$. It remains to prove that in the case where $E$ is a POVM, the domain of the maximal strong operator integral is included in $\hat{D}_{D_s}(f, E)$. But this is clear because for $\varphi \in D(f, E)$, we have $D_s \subseteq \mathcal{H} = \mathcal{W}_\varphi(f, E)$, regardless of $D_s$. □

Since $L(f, E) \subseteq \hat{L}_{D_s}(f, E)$ for any POVM $E$, one can ask when these two operators are the same. Since $\|\eta\| = \sup \{\|\eta\| \mid \psi \in D_s, \|\psi\| \leq 1\}$ (as $D_s$ is dense), the following result is a direct consequence of [20, Theorem 3.5] (see also [10]).

**Proposition 8.** Suppose $E$ is a POVM, and let $D_s \subseteq \mathcal{H}$ be a dense subspace. Then $L(f, E) = \hat{L}_{D_s}(f, E)$ if and only if for each $\varphi \in \hat{D}_{D_s}(f, E)$, we have

$$\lim_{n \to \infty} \sup_{\psi \in D_s, \|\psi\| \leq 1} \int_{X_n} |f| dE_{\psi, \varphi} = 0$$

whenever the sets $X_n \in \mathcal{A}$ satisfy $X_{n+1} \subseteq X_n$, $n \in \mathbb{N}$, and $\cap_n X_n = \emptyset$.

4.3. Symmetric weak operator integrals. Since the integrals $\int f dE_{\psi, \varphi}$ are symmetric in the sense that $(\psi, \varphi) \in \mathcal{W}(f, E)$ implies $(\varphi, \psi) \in \mathcal{W}(f, E)$, and $\int f dE_{\psi, \varphi} = \int f dE_{\varphi, \psi}$, it is natural to ask when a weak operator integral is a symmetric operator. We will not look at the most general case, but concentrate on the elements of $L_W(f, E, \Phi_{D_s})$, with the fixed separating subspace $D_s \subseteq \mathcal{V}$. Since continuity properties of the integral $\int f dE_{\psi, \varphi}$ with respect to the vectors $\varphi, \psi$ are rather weak (even in the case where $E$ is POVM), knowing that

$$\langle \psi | \hat{L}_{D_s}(f, E) | \varphi \rangle = \int f dE_{\psi, \varphi} = \overline{\int f dE_{\varphi, \psi}}$$

for all $\psi \in D_s$, $\varphi \in \hat{D}_{D_s}(f, E)$, is not obviously enough to connect this to the case where $\varphi \in D_s$ and $\psi \in \hat{D}_{D_s}(f, E)$. Therefore, we now assume that the dense subspace $D_s$ satisfies the equivalent conditions of the following trivial lemma.

**Lemma 3.** Let $D_s \subseteq \mathcal{V}$ be a subspace. Then $D_s \subseteq \{ \varphi \in \mathcal{H} \mid D_s \subseteq \mathcal{W}_\varphi(f, E) \}$ if and only if $D_s \times D_s \subseteq \mathcal{W}(f, E)$. 
Proposition 9. Suppose that $D_s \subseteq V$ is a dense subspace satisfying $D_s \times D_s \subseteq W(f, E)$. We define a (clearly unique) operator $L'_{D_s}(f, E)$ whose domain and action are given by

$$D'_{D_s}(f, E) := \left\{ \varphi \in D_s \mid D_s \ni \psi \mapsto \int f dE_{\psi, \varphi} \in \mathbb{C} \text{ is continuous} \right\},$$

$$\langle \psi \vert L'_{D_s}(f, E) \varphi \rangle = \int f dE_{\psi, \varphi}, \quad \psi \in D_s, \; \varphi \in D'_{D_s}(f, E).$$

Then $L'_{D_s}(f, E) \subseteq \hat{L}_{D_s}(f, E)$. In particular, $L'_{D_s}(f, E)$ is a weak operator integral, with $L'_{D_s}(f, E) \in L_W(f, E, \Phi_{D_s})$. Moreover, if $f$ is real-valued, then $L'_{D_s}(f, E)$ is a symmetric operator.

Proof. It is clear that $L'_{D_s}(f, E)$ is a well-defined operator on the given domain $D'_{D_s}(f, E)$. (Note that the condition $D_s \times D_s \subseteq W(f, E)$ ensures that the integral is defined.) We now show that $D'_{D_s}(f, E) \subseteq \hat{D}_{D_s}(f, E)$, which by Proposition 8 implies that $L'_{D_s}(f, E) \in L_W(f, E, \Phi_{D_s})$ and $L'_{D_s}(f, E) \subseteq \hat{L}_{D_s}(f, E)$. Accordingly, let $\varphi \in D'_{D_s}(f, E)$. In particular, $\varphi \in D_s$. Since $D_s \times D_s \subseteq W(f, E)$, we have $D_s \subseteq W(\varphi, f, E)$. But $\psi \mapsto \int f dE_{\psi, \varphi}$ is continuous on $D_s$, so $\varphi \in \hat{D}_{D_s}(f, E)$. It remains to show that $L'_{D_s}(f, E)$ is symmetric if $f$ is real-valued. For that, let $\psi, \varphi \in D'_{D_s}(f, E)$. Then both of them are also in $D_s$. Hence,

$$\langle \psi \vert L'_{D_s}(f, E) \varphi \rangle = \int f dE_{\psi, \varphi} = \int f dE_{\varphi, \psi} = \langle \varphi \mid L'_{D_s}(f, E) \psi \rangle = \langle L'_{D_s}(f, E) \psi \mid \varphi \rangle.$$ 

\[ \square \]

We call an operator $L'_{D_s}(f, E)$ symmetric weak operator integral determined by $D_s$. (Even in the case where $f$ is not real valued.)

Example 4. Suppose that $H$ is separable, let $\{\varphi_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of $H$, and put $V := \text{lin}\{\varphi_n \mid n \in \mathbb{N}\}$, and $E : \mathcal{A} \to S(V)$ a positive sesquilinear form measure. Let $f : \mathcal{A} \to \mathbb{C}$ be such that

$$\sum_{n \in \mathbb{N}} \left| \int f dE_{\varphi_n, \varphi_m} \right|^2 < \infty \text{ for all } m \in \mathbb{N}. \tag{12}$$

In particular, $\int f dE_{\varphi_n, \varphi_m}$ exists for all $n, m \in \mathbb{N}$, that is, $\langle \varphi_n, \varphi_m \rangle \in W(f, E)$ for each $n$. By sesquilinearity, it follows that $\langle \psi, \varphi \rangle \in W(f, E)$ for all $\psi, \varphi \in V$, i.e. $W(f, E) = V \times V$. Hence, $V$ itself satisfies the conditions of Proposition 9 and we have the symmetric weak operator integral $L'_V(f, E)$. It now follows from (12) that for each $m \in \mathbb{N}$,

$$V \ni \psi \mapsto \int f dE_{\varphi_n, \varphi_m} = \sum_{n \in \mathbb{N}} \langle \psi \vert \varphi_n \rangle \int f dE_{\varphi_n, \varphi_m} \in \mathbb{C}$$

is continuous. Since each $\varphi \in V$ is a (finite) linear combination of the vectors $\varphi_m$, the continuity holds for each $\varphi \in V$. Hence, the domain of the symmetric weak operator integral $L'_V(f, E)$ is the whole of $V$, and its action is determined by

$$L'_V(f, E) \varphi_m = \sum_{n \in \mathbb{N}} \left( \int f dE_{\varphi_n, \varphi_m} \right) \varphi_n, \text{ for all } m \in \mathbb{N}.$$ 

Of course, an operator defined via this same formula may have a larger domain; for example, if $E_{\varphi_n, \varphi_m}(X) = \delta_{nm} \mu_n(X), \quad n, m \in N, \; X \in \mathcal{A},$

where $\delta_{nm}$ is the Kronecker delta and $\{\mu_n\}$ is a sequence of bounded positive measures on $\mathcal{A} \subseteq 2^\Omega$ then $\int f dE_{\varphi_n, \varphi_m} = \delta_{nm} \int f d\mu_m$, where $f_m := \int f d\mu_m$, and the largest possible domain of an extension of the weak operator integral $L'_V(f, E)$ is $\{ \varphi \in H \mid \sum_m |f_m(\varphi_m | \varphi)|^2 < \infty \}$. Note that

\[ ^3 \text{Obviously, } E \text{ defines a POVM if and only if } \sup_{n \in \mathbb{N}} \mu_n(\Omega) < \infty. \]
this extension is bounded if and only if \( \sup_{m \in \mathbb{N}} |f_m| < \infty \). However, the extension is not a weak operator integral, because its domain is larger than the form domain \( \mathcal{V} \) of the sesquilinear form valued measure \( \mathcal{E} \).

We immediately notice the following

**Proposition 10.** Suppose \( \mathcal{E} \) is a POVM, and the strong operator integral \( L(f, \mathcal{E}) \) is densely defined. Set \( \mathcal{D}_s = D(f, \mathcal{E}) \). Then \( L(f, \mathcal{E}) = L'_{\mathcal{D}_s}(f, \mathcal{E}) \), i.e. the strong operator integral is the symmetric weak operator integral determined by its domain.

**Proof.** If \( \varphi \in \mathcal{D}_s \) then \( \int f d\mathcal{E}_{\psi, \varphi} \) exists for all \( \psi \in \mathcal{H} \), so \( \mathcal{D}_s \times \mathcal{D}_s \subseteq \mathcal{H} \times \mathcal{D}_s \subseteq \mathcal{W}(f, \mathcal{E}) \). Hence, \( L'_{\mathcal{D}_s}(f, \mathcal{E}) \) is defined. Moreover, if \( \varphi \in \mathcal{D}_s \) then \( \psi \mapsto \int f d\mathcal{E}_{\psi, \varphi} \) is continuous on the whole \( \mathcal{H} \), and hence also on the subspace \( \mathcal{D}_s \). Thus \( \mathcal{D}_s \subseteq \mathcal{D}'_{\mathcal{D}_s}(f, \mathcal{E}) \subseteq \mathcal{D}_s \), and the proof is complete. \( \Box \)

Hence, the domain of the strong operator integral, when dense, is one choice for a separating subspace \( \mathcal{D}_s \) of a weak operator integral when \( \mathcal{E} \) is a POVM. It is easy to see that even in the general case there is a maximal choice for this subspace, which can be explicitly written down:

**Proposition 11.** The set

\[
\mathcal{D}_F(f, \mathcal{E}) := \left\{ \varphi \in \mathcal{H} \mid \int |f| d\mathcal{E}_{\psi, \varphi} < \infty \right\} = \{ \varphi \in \mathcal{H} \mid \varphi \in \mathcal{W}_\psi(f, \mathcal{E}) \}
\]

is the largest subspace \( \mathcal{D} \subseteq \mathcal{H} \) such that \( \mathcal{D} \times \mathcal{D} \subseteq \mathcal{W}(f, \mathcal{E}) \) (in the sense that any subspace \( \mathcal{D} \) with this property, is included in \( \mathcal{D}_F(f, \mathcal{E}) \)).

**Proof.** The fact that the set \( \mathcal{D}_F(f, \mathcal{E}) \) is a linear subspace of \( \mathcal{H} \) follows immediately from Lemma 2. Next we note that given \( \varphi, \psi \in \mathcal{H} \), the measure \( \mathcal{E}_{\psi, \varphi} \) is a linear combination of four measures of the form \( \mathcal{E}_{\psi_i + k \varphi, \psi_i + k \varphi} \), \( k = 0, 1, 2, 3 \). If \( (\psi, \psi) \in \mathcal{W}(f, \mathcal{E}) \) and \( (\varphi, \varphi) \in \mathcal{W}(f, \mathcal{E}) \) then \( f \) is integrable with respect to each of the four measures, since \( \mathcal{D}_F(f, \mathcal{E}) \) is a linear subspace. Hence, \( f \) is also integrable with respect to \( \mathcal{E}_{\psi, \psi} \), that is, \( (\psi, \varphi) \in \mathcal{W}(f, \mathcal{E}) \). Thus, \( \mathcal{D}_F(f, \mathcal{E}) \times \mathcal{D}_F(f, \mathcal{E}) \subseteq \mathcal{W}(f, \mathcal{E}) \). On the other hand, if \( \mathcal{D} \subseteq \mathcal{H} \) is any subspace with \( \mathcal{D} \times \mathcal{D} \subseteq \mathcal{W}(f, \mathcal{E}) \), then \( (\varphi, \varphi) \in \mathcal{W}(f, \mathcal{E}) \) for all \( \varphi \in \mathcal{D} \), so that \( \mathcal{D} \subseteq \mathcal{D}_F(f, \mathcal{E}) \). Thus, \( \mathcal{D}_F(f, \mathcal{E}) \) is the largest of such subspaces \( \mathcal{D} \). \( \Box \)

Assuming that \( \mathcal{D}_F(f, \mathcal{E}) \) is dense, we denote

\[
L'(f, \mathcal{E}) := L'_{\mathcal{D}_F(f, \mathcal{E})}(f, \mathcal{E}),
\]

and call this the largest symmetric weak operator integral determined by \( f \) and \( \mathcal{E} \). All other symmetric operator integrals are restrictions of this one. In particular, if \( \mathcal{E} \) is a POVM, Proposition 11 gives

\[
L(f, \mathcal{E}) \subseteq L'(f, \mathcal{E}).
\]

Note that this inclusion holds even in the case where \( L(f, \mathcal{E}) \) is not dense (which can easily happen even if \( \mathcal{D}_F(f, \mathcal{E}) \) is dense), because if \( \int f d\mathcal{E}_{\psi, \varphi} \) exists for all \( \psi \) then \( \int |f| d\mathcal{E}_{\psi, \varphi} < \infty \).

The following result deals with the case of spectral measures.

**Proposition 12.** Suppose that \( \mathcal{E} \) is projection valued. Then

\[
\hat{L}(f, \mathcal{E}) = L(f, \mathcal{E}) = L'(f, \mathcal{E}).
\]

**Proof.** Since \( \hat{L}(f, \mathcal{E}) = L(f, \mathcal{E}) \) is densely defined (the usual spectral integral), the weak operator integral \( L'(f, \mathcal{E}) \) exists, and is an extension of \( L(f, \mathcal{E}) \). Hence, we only need to show that \( \text{Dom}(L'(f, \mathcal{E})) \subseteq \hat{D}(f, \mathcal{E}) \). Define \( g : \Omega \to \mathbb{C} \) by \( g = \sqrt{|f|} \), and \( h : \Omega \to \mathbb{C} \) by setting \( h(x) = f(x)/(|f(x)|) \) if \( f(x) \neq 0 \), and \( h(x) = 0 \) otherwise. Then \( h \) and \( g \) are measurable, \( h \) is bounded, \( g \geq 0 \), and \( f = g^2 h \). Now

\[
L(g, \mathcal{E}) L(g h, \mathcal{E}) \subseteq L(g^2 h, \mathcal{E}) = L(f, \mathcal{E}),
\]
Indeed, if \( D \) one and in the general case, we approximate \( g \) by the usual rules of spectral calculus of unbounded functions. Now

\[
\mathcal{D}_F(f, E) = \{ \varphi \in \mathcal{H} \mid \int |f| \, dE_{\varphi, \varphi} < \infty \} = \text{Dom}(L(g, E)) = \text{Dom}(L(gh, E)).
\]

According to what has been concluded earlier by using polarization, \( f \) is \( E_{\varphi, \varphi} \)-integrable whenever both \( \psi \) and \( \varphi \) belong to \( \mathcal{D}_F(f, E) \). Since \( E \) is a spectral measure, we have

\[
\int f \, dE_{\psi, \varphi} = \int g(gh) \, dE_{\psi, \varphi} = \langle L(g, E)\psi|L(gh, E)\varphi \rangle, \quad \varphi, \psi \in \mathcal{D}_F(f, E).
\]

Indeed, if \( g \) is bounded, then this follows from the multiplicativity of the spectral measure, and in the general case, we approximate \( g \) with the sequence \( (g_n) \), where \( g_n(x) = g(x) \) if \( g(x) \leq n \), and \( g_n(x) = 0 \) otherwise, and conclude that on the one hand, \( L(g_n h, E)\varphi \to L(gh, E), L(g_n, E)\varphi \to L(g, E) \) strongly, and on the other hand, \( \int g_n^2 h \, dE_{\varphi, \varphi} \to \int g^2 h \, dE_{\varphi, \varphi} \) by dominated convergence (since \( |f| = g^2 \) is \( E_{\varphi, \varphi} \)-integrable).

Now if \( \varphi \in \text{Dom}(L(f, E)) \) then by definition, \( \psi \mapsto \int f \, dE_{\psi, \varphi} \) is continuous in \( \mathcal{D}_F(f, E) = \text{Dom}(L(g, E)) \). By the formula obtained, this implies that \( L(gh, E)\varphi \) belongs to \( \text{Dom}(L(g, E)^* \text{Dom}(L(f, E)) \). The proof is complete.

5. Sesquilinear form valued integral

5.1. The sesquilinear form valued integral of a sesquilinear form valued measure and a measurable function. Since \( E \) is a sesquilinear form valued measure, it is natural to consider the sesquilinear form valued integral of a measurable function with respect to \( E \). In this section, we first define this integral, and then consider its connection to weak operator integrals.

We start by defining a function

\[
F_{f, E} : \mathcal{W}(f, E) \to \mathbb{C}, \quad (\psi, \varphi) \mapsto F_{f, E}(\psi, \varphi) = \int f \, dE_{\psi, \varphi}.
\]

This function satisfies e.g. \( (\alpha \psi_1 + \beta \psi_2, \varphi) \in \mathcal{W}(f, E) \) and \( F_{f, E}(\alpha \psi_1 + \beta \psi_2, \varphi) = \overline{\alpha} F_{f, E}(\psi_1, \varphi) + F_{f, E}(\psi_2, \varphi) \), for any \( (\psi_1, \varphi), (\psi_2, \varphi) \in \mathcal{W}(f, E) \). In addition, \( (\psi, \varphi) \in \mathcal{W}(f, E) \) if and only if \( (\varphi, \psi) \in \mathcal{W}(f, E) \), and

\[
F_{f, E}(\overline{\psi}, \overline{\varphi}) = F_{f, E}(\overline{\varphi}, \overline{\psi}), \quad (\psi, \varphi) \in \mathcal{W}(f, E).
\]

In order to consider \( F_{f, E} \) as a sesquilinear form, we have to restrict its domain of definition to a set of the form \( \mathcal{D} \times \mathcal{D} \subseteq \mathcal{W}(f, E) \), where \( \mathcal{D} \subseteq \mathcal{V} \) is a subspace. (Clearly, any such restriction is sesquilinear.) According to Proposition 11 there is a canonical choice for \( \mathcal{D} \), namely the largest one \( \mathcal{D}_F(f, E) \). We denote the restriction of \( F_{f, E} \) to \( \mathcal{D}_F(f, E) \) by the same symbol. We say that

\[
F_{f, E} : \mathcal{D}_F(f, E) \times \mathcal{D}_F(f, E) \to \mathbb{C}
\]

is the form integral of \( f \) with respect to \( E \). The subspace \( \mathcal{D}_F(f, E) \) is the form domain.

It follows from (14) that \( F_{f, E} \) is symmetric if \( f \) is real valued. It is clearly positive if \( f \) is a positive function.

Remark 3. The form domain should not be confused with the square integrability domain, which is the form domain of the form integral of \(|f|^2\) with respect to \( E \). In the case of \( f(x) = x \) on \( \mathbb{R} \), the latter is called variance form; see Introduction.

In order to consider the connection between the (unique) form integral of \( f \) with respect to \( E \), and the various weak operator integrals, we need some preliminaries on the standard extension theory of quadratic forms.
5.2. Preliminaries on quadratic forms. We start with some basic preliminaries on the
theory of quadratic forms (see e.g. [13, 14]). A quadratic form is a sesquilinear form \( q : D \times D \to \mathbb{C} \), where \( D \subseteq H \) is a dense subspace, called the form domain. If \( q(\psi, \varphi) = \overline{q(\varphi, \psi)} \), for all \( \psi, \varphi \in D \), then \( q \) is called symmetric, and if \( q(\varphi, \varphi) \geq 0 \) for all \( \varphi \in D \), it is called positive.

The adjoint form \( q^* \) of \( q \) is defined on the same domain \( D \), via

\[
q^*(\varphi, \psi) = \overline{q(\psi, \varphi)}, \quad \varphi, \psi \in D.
\]

Inclusion \( q' \subseteq q \) between two quadratic forms is defined via the corresponding inclusion of the form domains. A linear combination of two quadratic forms is defined in the obvious way, with the domain being the intersection of the form domains. In particular, the real and imaginary parts of a quadratic form \( q \) are defined by

\[
\Re q := \frac{1}{2}(q + q^*), \quad \Im q := \frac{1}{2i}(q - q^*).
\]

A positive quadratic form \( q : D \times D \to \mathbb{C} \) is said to be closed if \( \varphi_n \in D, \varphi_n \to \varphi \in H \), and

\[
\lim_{n,m \to \infty} q(\varphi_n - \varphi_m, \varphi_n - \varphi_m) = 0
\]

imply \( \varphi \in D \) and

\[
\lim_{n \to \infty} q(\varphi_n - \varphi, \varphi_n - \varphi) = 0.
\]

It follows that \( q \) is closed if and only if \( \Re q \) is closed (see [14, p. 313]).

There is a canonical way of associating a positive selfadjoint operator to a positive closed quadratic form. It is given by the above theorem (see [14, 15]).

**Theorem 1.** Let \( q \) be a closed symmetric positive quadratic form with dense form domain \( D \). Then there exists a positive selfadjoint operator \( T \) such that \( \text{Dom}(\sqrt{T}) = D \), and

\[
q(\psi, \varphi) = \langle \sqrt{T}\psi, \sqrt{T}\varphi \rangle, \quad \text{for all } \psi, \varphi \in D.
\]

We say that \( T \) given by the above theorem is the operator associated to the quadratic form \( q \).

We will make use of the following simple corollary; it also shows that \( T \) is uniquely determined, hence the definite article.

**Proposition 13.** Let \( q \) be a closed symmetric positive quadratic form with dense form domain \( D \subseteq H \) and \( T \) a positive selfadjoint operator associated to it as in Theorem 1. Then

\[
\text{Dom}(T) = \{ \varphi \in D \mid D \ni \psi \mapsto q(\psi, \varphi) \in \mathbb{C} \text{ is continuous} \},
\]

(15)

\[
q(\psi, \varphi) = \langle \psi | T \varphi \rangle, \quad \text{for all } \psi \in D, \varphi \in \text{Dom}(T).
\]

If there is a Hilbert space \( K \), and an operator \( A : D \to K \), such that

\[
q(\psi, \varphi) = \langle A\psi | A\varphi \rangle, \quad \text{for all } \psi, \varphi \in D,
\]

then \( A^* A = T \).

**Proof.** If \( A \) is as in the lemma, we have, by the definition of the adjoint, that

\[
\text{Dom}(A^* A) = \{ \varphi \in D \mid A\varphi \in \text{Dom}(A^*) \}
\]

\[
= \{ \varphi \in D \mid D \ni \psi \mapsto q(\psi, A\varphi) \in \mathbb{C} \text{ is continuous} \}
\]

\[
= \{ \varphi \in D \mid D \ni \psi \mapsto q(\psi, \varphi) \in \mathbb{C} \text{ is continuous} \}.
\]

In particular, this holds for \( K = H \) and \( A = \sqrt{T} \), which gives (15), because \( T = (\sqrt{T})^* \sqrt{T} \). It follows that \( \text{Dom}(A^* A) = \text{Dom}(T) \), and if \( \varphi \in \text{Dom}(T), \psi \in D \), we have

\[
\langle \psi | A^* A \varphi \rangle = \langle A\psi | A\varphi \rangle = q(\psi, \varphi) = \langle \psi | T \varphi \rangle.
\]

As \( D \) is dense, this implies that \( A^* A = T \). \( \square \)


Remark 4. Note that it is nontrivial that the domain of $A^*A$ in the above proposition is actually dense. This fact follows from the above theorem. Note also that the operator $A$ is automatically closed, because the form $q$ was assumed to be closed. A special case of this result is the well-known theorem of von Neumann (see e.g. [14, p. 180]), which says that $A^*A$ is selfadjoint if $A$ is a closed densely defined operator.

5.3. Connection between form integral and weak operator integral. In the case where $D_F(f, E)$ is dense, the sesquilinear form $F_{f, E}$ is a quadratic form. In general, the adjoint form is given by

$$F_{f, E}^*(\psi, \varphi) = \int f dE_{\psi, \varphi}, \psi, \varphi \in D_F(f, E).$$

Moreover,

$$\Re(F_{f, E}) \subseteq F_{\Re(f), E}, \quad \Im(F_{f, E}) \subseteq F_{\Im(f), E},$$

where $\Re(f)$ and $\Im(f)$ are the real and imaginary parts of the function $f$, respectively. We can further decompose these into positive and negative parts, so that

$$f = f_1 - f_2 + i(f_3 - f_4),$$

$$f_i \geq 0, \quad |\Re(f)| = f_1 + f_2, \quad |\Im(f)| = f_3 + f_4.$$  

Then clearly $D_F(f, E) = \cap_{i=1}^4 D_F(f_i, E)$, so the form integral decomposes naturally as the linear combination of the corresponding positive forms:

$$F_{f, E} = F_{f_1, E} - F_{f_2, E} + iF_{f_3, E} - iF_{f_4, E}.$$  

Unfortunately, the situation is not so simple in case of the weak operator integrals. However, the following result holds:

**Proposition 14.** Suppose that $D_F(f, E)$ is dense. Then

$$L'(f, E) \supseteq L'(f_1, E) - L'(f_2, E) + iL'(f_3, E) - iL'(f_4, E) \in L_W(f, E, \Phi_{D_4}),$$

where $D_4 = D_F(f, E)$, and the inclusion can be interpreted as the ordering relation in the class $L_W(f, E, \Phi_{D_4})$ of weak operator integrals.

**Proof.** First note that since $D_F(f, E)$ is dense, so is each $D_F(f_i, E)$; hence, the weak operator integrals $L'(f_i, E)$ are defined. Denote $A := L'(f_1, E) - L'(f_2, E) + iL'(f_3, E) - iL'(f_4, E)$. By definition,

$$\text{Dom}(A) = \cap_{i=1}^4 \text{Dom}(L'(f_i, E)),$$

so that $\text{Dom}(A) \subseteq D_F(f, E)$. If $\varphi \in \text{Dom}(A)$, each functional $\psi \mapsto \int f dE_{\psi, \varphi}$ is continuous on $D_F(f, E) = \cap_{i=1}^4 D_F(f_i, E)$, and coincides with $\psi \mapsto \langle \psi \rangle L'(f, E)\varphi$ there. This implies that $\varphi \in \text{Dom}(L'(f, E))$. Hence, $A \subseteq L'(f, E)$. Since $L'(f, E) \in L_W(f, E, \Phi_{D_4})$, it follows from Proposition 8 that also $A \in L_W(f, E, \Phi_{D_4})$. This completes the proof.

We now consider the relationship between $F_{f, E}$ and $L'(f, E)$ in the case of a positive function $f$, and a POVM $E$.

**Proposition 15.** Let $E$ be a POVM and $f : \Omega \to \mathbb{C}$ a positive measurable function, such that $D_F(f, E)$ is dense. Then the quadratic form $F_{f, E}$ is symmetric, positive and closed. The associated positive selfadjoint operator $T$ (see Theorem 8), is given by

$$T = (L(\sqrt{f}, F)V)^* L(\sqrt{f}, F)V,$$

where $(K, F, V)$ is any Naimark dilation of $E$. Moreover,

$$T = L'(f, E),$$

i.e., $T$ is the largest symmetric weak operator integral determined by $f$ and $E$. In particular, $L'(f, E)$ is selfadjoint.
Proof. Clearly, $F_{f,E}(\varphi,\varphi) \geq 0$ for all $\varphi \in F_{f,E}$. Let $(K, F, V)$ be a Naimark dilation of $E$, so that $E(X) = V^*F(X)V$ and $F$ is projection valued. Now

$$D_F(f,E) = \{ \varphi \in H \mid V\varphi \in D(\sqrt{f}, F) \},$$

\begin{equation}
F_{f,E}(\varphi,\varphi) = \|L(\sqrt{f}, F)\varphi\|^2, \text{ for all } \varphi \in D_F(f,E).
\end{equation}

Now if $\varphi_n \in D_F(f,E)$, such that $\varphi_n \to \varphi \in H$, and $\lim_{n,m \to \infty} F_{f,E}(\varphi_n - \varphi_m, \varphi_n - \varphi_m) = 0$, it follows that $V\varphi_n \to V\varphi$, and $(L(\sqrt{f}, F)V\varphi_n)_n$ converges in $K$. Since $F$ is projection valued, $L(\sqrt{f}, F)$ is a closed operator on its domain, so $V\varphi \in D(\sqrt{f}, F)$, and $\lim_{n,m \to \infty} L(\sqrt{f}, F)\varphi_n = L(\sqrt{f}, F)\varphi$. But this implies that $\varphi \in D_F(f,E)$, and $\lim_{n,m \to \infty} F_{f,E}(\varphi_n - \varphi, \varphi_n - \varphi) = 0$. Hence the form $F_{f,E}$ is closed. From (16) it now follows by polarization and Proposition 13 that $(L(\sqrt{f}, F)V)^*L(\sqrt{f}, F)V$ is the selfadjoint operator associated to the form $F_{f,E}$. From Proposition 13 we immediately see that $T = L'(f,E)$. This completes the proof. \hfill \Box

**Remark 5.** Notice that in the above proposition, $V^*L(\sqrt{f}, F)V \subseteq T = L'(f,E)$, because

\begin{equation}
V^*L(\sqrt{f}, F) \subseteq (L(\sqrt{f}, F)V)^*.
\end{equation}

From (10), we know that $V^*L(f,F)V = \tilde{L}(f,E)$. Hence, in this case, the difference between the strong operator integral on the square integrability domain and the maximal symmetric weak operator integral, is in the operator inclusion (17), which can be proper because continuity of the functional $\psi \mapsto \langle L(\sqrt{f}, F)\psi | \varphi \rangle$ on $V(H) \cap \text{Dom}(L(\sqrt{f}, F))$ does not necessarily imply its continuity on the full domain $\text{Dom}(L(\sqrt{f}, F))$.

6. Application: moment operators of a POVM

Consider the operator valued moments (or moment operators) of a normalized POVM $E : B(\mathbb{R}) \to L(H)$. They are simply defined as operator integrals $\int x^k dE$ of real functions $x \mapsto x^k$ where $k \in \mathbb{N}$. Hence, we have three natural ways to defined them: $\tilde{E}[k] := \tilde{L}(x^k,E)$, $E[k] := L(x^k,E)$, and $E'[k] := L'(x^k,E)$. Recall that $\tilde{E}[k] \subseteq E[k] \subseteq E'[k]$ and their domains are

\begin{align*}
\text{Dom}(\tilde{E}[k]) &= \{ \varphi \in H \mid \int x^{2k} dE_{\varphi,\varphi}(x) < \infty \}, \\
\text{Dom}(E[k]) &= \{ \varphi \in H \mid \int |x|^k dE_{\psi,\psi} < \infty \text{ for all } \psi \in H \}, \\
\text{Dom}(E'[k]) &= \{ \varphi \in D_F(x^k,E) \mid D_F(x^k,E) \ni \psi \mapsto \int x^k dE_{\psi,\psi} \in \mathbb{C} \text{ is continuous} \}
\end{align*}

and

\begin{align*}
\text{if } D_F(x^k,E) &= \{ \varphi \in H \mid \int |x|^k dE_{\varphi,\varphi} < \infty \} \text{ is dense in } H.
\end{align*}

By comparison, the sesquilinear form valued moments are given by the form integral

\begin{equation}
F_{x^k,E}(\psi,\varphi) = \int x^k dE_{\varphi,\varphi}, \quad \psi, \varphi \in D_F(x^k,E).
\end{equation}

Since $\text{Dom}(\tilde{E}[k]) = D_F(x^{2k},E)$, we can define the variance form on this form domain as

\begin{equation}
(\psi,\varphi) \ni F_{x^{2k},E}(\psi,\varphi) - (\tilde{E}[k]|\psi)\tilde{E}[k]|\varphi).
\end{equation}

If this form is identically zero, the POVM is called variance-free [19].

Let $(K, F, V)$ be a Naimark dilation of $E$. From Proposition 12 one sees that $\tilde{F}[k] = F[k] = F'[k]$ for any $k \in \mathbb{N}$. Moreover, $\text{Dom}(F[k]V) = \text{Dom}(\tilde{E}[k])$ and

\begin{equation}
\tilde{E}[k] = V^*F[k]V.
\end{equation}

\footnote{For simplicity, we will use the symbol $x^k$ to denote the function $x \mapsto x^k$.}
(This was also proved in [9].) If \( k \) is even, i.e. \( k = 2j, j \in \mathbb{N} \), and \( \mathcal{D}_F(x^{2j}, \mathcal{E}) = \text{Dom}(\tilde{\mathcal{E}}[j]) \) is dense \((\text{which is assumed below})\), then it follows from Proposition [15] that

\[
E'[2j] = (F[j]V)^*(F[j]V)
\]

is positive and selfadjoint. Now \( \text{Dom}(E'[2j]) \) consists of exactly those vectors \( \varphi \in \text{Dom}(\tilde{\mathcal{E}}[j]) \) for which \( F[j]V \varphi \in \text{Dom}((F[j]V)^*) \). Note that \( F[j]V \) is a map \( \text{Dom}(\tilde{\mathcal{E}}[j]) \rightarrow \mathcal{K} \), so the adjoint \((F[j]V)^*\) maps from a subspace of \( \mathcal{K} \) to \( \mathcal{H} \). It is clear that

\[
V^*F[j] \subseteq (F[j]V)^*
\]

because if \( \varphi \in \text{Dom}(F[j]) \), then \( \psi \rightarrow \langle \varphi | F[j]V \psi \rangle = \langle V^*F[j] \varphi | \psi \rangle \) is continuous in \( \text{Dom}(F[j]V) \).

**Proposition 16.**

(a) Suppose that \( V^*F[j] = (F[j]V)^* \). Then \( \tilde{\mathcal{E}}[2j] = E'[2j] \).

(b) Suppose that \( F[j] \left( \text{Dom}(F[j]) \cap V(\mathcal{H}) \right) \subseteq V(\mathcal{H}) \). Then \( E'[2j] = \tilde{\mathcal{E}}[j]^* \tilde{\mathcal{E}}[j] \).

**Proof.** We have already proved (a); see [18], [19], and use \( F[2j] = F[j]F[j] \). To prove (b), note that the assumption implies \( F[j]V = VV^*F[j]V = V\tilde{\mathcal{E}}[j] \). Now a vector \( \varphi \in \mathcal{H} \) satisfies \( V\varphi \in \text{Dom}((V\tilde{\mathcal{E}}[j])^*) \) if and only if \( \psi \rightarrow \langle V\tilde{\mathcal{E}}[j]\psi | V\varphi \rangle = \langle \tilde{\mathcal{E}}[j]\psi | \varphi \rangle \) is continuous on \( \text{Dom}(\tilde{\mathcal{E}}[j]) \), which happens exactly when \( \varphi \in \text{Dom}(\tilde{\mathcal{E}}[j]^*) \). Hence, \( (F[j]V)^*V = (V\tilde{\mathcal{E}}[j])^*V = \tilde{\mathcal{E}}[j]^* \) and \( E'[2j] = (F[j]V)^*(F[j]V) = (V\tilde{\mathcal{E}}[j])^*(V\tilde{\mathcal{E}}[j]) = \{(V\tilde{\mathcal{E}}[j])^* \tilde{\mathcal{E}}[j] = \tilde{\mathcal{E}}[j]^* \tilde{\mathcal{E}}[j] \}. \)

6.1. Momentum for a bounded interval. Consider first a free (nonrelativistic) particle of mass \( m \) moving along a line which can be chosen to be \( \mathbb{R} \) without restricting generality. We use units where \( h = 1 \). Then the Hilbert space of the system is \( L^2(\mathbb{R}) \) and the (sharp) position observable is \( Q_R : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(L^2(\mathbb{R})) \),

\[
(Q_R(X)\psi)(x) := \chi_X(x)\psi(x), \quad X \in \mathcal{B}(\mathbb{R}), \; \psi \in L^2(\mathbb{R}), \; x \in \mathbb{R}.
\]

The (sharp) momentum observable is \( P_R : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(L^2(\mathbb{R})) \),

\[
P_R(Y) := \mathcal{F}^*Q_R(Y)\mathcal{F}, \quad Y \in \mathcal{B}(\mathbb{R}),
\]

where \( \mathcal{F} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \) is the Fourier-Plancherel (unitary) operator determined by

\[
(\mathcal{F}\psi)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixt}\psi(t) \, dt, \quad \psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}), \; x \in \mathbb{R}.
\]

Since \( Q_R \) and \( P_R \) are spectral measures, there is no ambiguity in defining their moment operators, see Proposition [12]. For example, \( Q_R[1] \) and \( P_R[1] \) are the usual selfadjoint position and momentum operators,

\[
(Q_R[1]\psi)(x) = x\psi(x), \quad (P_R[1]\psi)(x) = -i\psi'(x)
\]

where \( \psi'(x) := d\psi(x)/dx \) (and similarly \( \psi''(x) := d^2\psi(x)/dx^2 \)). Recall that, e.g., \( \text{Dom}(P_R[1]) \) consists of those absolutely continuous functions \( \psi \in L^2(\mathbb{R}) \) for which \( \psi' \in L^2(\mathbb{R}) \). Now the energy operator is \((2m)^{-1}P_R[1]^2 = (2m)^{-1}P_R[2] \) whose spectrum is continuous, consisting of nonnegative numbers.

Suppose then that the particle is confined to move on a (fixed) bounded interval taken to be \( I = [0, \ell] \) where \( \ell > 0 \) is the length of the interval. Note that we do not assume that the endpoints 0 and \( \ell \) can be identified so that the system is not periodic with periodic boundary conditions (indeed, in the periodic case, the position space is a circle instead of an interval).

Since the particle is strictly confined to the interval \( I \), the Hilbert space of the system is \( L^2(I) \) and the position observable is now the (restricted) spectral measure \( Q : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{L}(L^2(I)) \),

\[
(Q(X)\varphi)(x) := \chi_X(x)\varphi(x), \quad X \in \mathcal{B}(\mathbb{R}), \; \varphi \in L^2(I), \; x \in I.
\]
Indeed, let $U : L^2(\mathcal{I}) \to L^2(\mathbb{R})$ be the isometry $(U \varphi)(x) = \varphi(x)$ for $x \in \mathcal{I}$, and $(U \varphi)(x) = 0$, $x \notin \mathcal{I}$. Then $U^* : L^2(\mathbb{R}) \to L^2(\mathcal{I})$ simply acts as $(U^* \psi)(x) = \psi(x)$, $x \in \mathcal{I}$, and

$$Q(X) = U^* Q(X) U, \quad X \in B(\mathbb{R}).$$

Again, there is no ambiguity in calculating the moments of $Q$. However, the situation is totally different for the momentum POVM $P : B(\mathbb{R}) \to L(L^2(\mathcal{I}))$ which is defined similarly to $Q$:

$$P(Y) := U^* P_\mathbb{R}(Y) U, \quad Y \in B(\mathbb{R}).$$

Note that $(L^2(\mathbb{R}), P_\mathbb{R}, U)$ is a Naimark dilation of $P$.

The following questions now arise: What is the correct definition for the second moment operator of $P$? Is the second moment of $P$ (times $(2m)^{-1}$) the energy operator in this case?

The operators $P_\mathbb{R}[1]U$, $(P_\mathbb{R}[1]U)^*$, and $P[1]$ can now be explicitly determined, but a certain care has to be exercised. Namely, the domain of $P_\mathbb{R}[1]U$ consists of exactly those functions $\varphi \in L^2(\mathcal{I})$ for which $U \varphi$ is absolutely continuous, with $(U \varphi)' \in L^2(\mathbb{R})$. Now $U \varphi$ is absolutely continuous exactly when $\varphi$ is absolutely continuous in the interval $\mathcal{I} = [0, \ell]$, and vanishes at the endpoints. (If it did not vanish, then there would be a discontinuity.) The set of absolutely continuous functions $\varphi \in L^2(\mathcal{I})$ with $\varphi' \in L^2(\mathcal{I})$ and $\varphi(0) = \varphi(\ell) = 0$ is denoted by $\text{Dom}(P_0)$, and the corresponding version of the differential operator $-id/dx$ by $P_0$, acting in $L^2(\mathcal{I})$. Hence, $P_\mathbb{R}[1]U = U P_0$. This implies $P[1] = U^* P_\mathbb{R}[1]U = U^* U P_0 = P_0$, see (13). The operator $P_0$ is well known to be densely defined and closed (see e.g. (13)).

Now the adjoint of $P_\mathbb{R}[1]U = U P_0$ is a map from a subspace of $L^2(\mathbb{R})$ to $L^2(\mathcal{I})$. A vector $\psi \in L^2(\mathbb{R})$ belongs to its domain exactly when $\varphi \mapsto \langle \psi | U P_0 \varphi \rangle = \langle U^* \psi | P_0 \varphi \rangle$ is continuous in $\text{Dom}(P_0) \subseteq L^2(\mathcal{I})$. But this happens exactly when $U^* \psi = \psi|_{[0, \ell]} \in \text{Dom}(P_0)$. Now $\text{Dom}(P_0)$ consists of those vectors $\varphi \in L^2(\mathcal{I})$ which are absolutely continuous, with $\varphi' \in L^2(\mathcal{I})$, and no other restriction. Hence,

$$\text{Dom} \left( (P_\mathbb{R}[1]U)^* \right) = \left\{ \psi \in L^2(\mathbb{R}) \mid \psi|_{[0, \ell]} \text{ is absolutely continuous and } \psi|_{[0, \ell]}' \in L^2(\mathcal{I}) \right\}.$$

Obviously, this contains $\text{Dom}(P_\mathbb{R}[1])$, as required by the general inclusion $U^* \text{Dom}(P_\mathbb{R}[1]) \subseteq (P_\mathbb{R}[1]U)^*$, see (20). Now it is clear that $U^* P_\mathbb{R}[1] \neq (P_\mathbb{R}[1]U)^*$, since $\psi \in \text{Dom}(P_\mathbb{R}[1]U)^*$ does not even have to be continuous outside $[0, \ell]$. Instead, we have $P_\mathbb{R}[1](\text{Dom}(P_\mathbb{R}[1]) \cap U(L^2(\mathcal{I}))) \subseteq U(L^2(\mathcal{I}))$, because if $\varphi \in L^2(\mathbb{R})$ vanishes outside $[0, \ell]$, then $(P_\mathbb{R}[1] \varphi)(x) = 0$ for $x \notin [0, \ell]$.

Hence, we know from Proposition 16 (b) that $P'[2] = P[1] P[1] = P_0 P_0$. The domain of this operator is characterized by the boundary condition $\psi(0) = \psi(\ell) = 0$, and the requirements that $\psi$ be continuously differentiable and $\psi'' \in L^2(\mathcal{I})$. Note that this operator is selfadjoint by Proposition 15 the operator $(2m)^{-1} P'[2]$ is the Hamiltonian operator for the particle of mass $m$ confined to move in the interval $\mathcal{I}$ ("particle in a box"). The spectrum of this operator is discrete and has the complete orthonormal system of eigenvectors $\psi_n$,

$$\psi_n(x) = \sqrt{\frac{2}{\ell}} \sin(n \pi x / \ell), \quad n \in \mathbb{Z}, \quad 0 \leq x \leq \ell,$$

associated with eigenvalues $\lambda_n = n^2 \pi^2 / (2\ell^2)$, that is,

$$P'[2] \psi = \sum_{n \in \mathbb{Z}} \lambda_n \langle \psi_n | \psi \rangle \psi_n, \quad \psi \in \text{Dom}(P'[2]) = \left\{ \psi \in L^2(\mathcal{I}) \mid \sum_{n \in \mathbb{Z}} \lambda_n^2 | \langle \psi_n | \psi \rangle |^2 < \infty \right\}.$$

We will show in the Appendix that $P[2] = P[2] = P_0^2$. As required, this a restriction of $P'[2] = P_0^2 P_0$, and the difference is exactly in the additional boundary condition $\psi'(0) = \psi'(\ell) = 0$ for any $\psi \in \text{Dom}(P_0^2) \subset \text{Dom}(P_0^2 P_0)$.

To conclude, the physically reasonable definition for the second moment operator of the POVM $E$ is the symmetric weak operator integral $P'[2]$ rather than the strong operator integral.
\( \hat{P}[2] = P[2] \). With this choice, the POVM \( E \) satisfies analogous “quantization rules” as the full line momentum, up to second moments.

**APPENDIX**

The notation \( \varphi^{(k)} = d^k \varphi/dx^k \), \( \varphi^{(0)} = \varphi \), will be used. Define, for all \( n = 1, 2, \ldots \), the Sobolev-Hilbert spaces

\[
H^n(I) = \{ \varphi \in C^{n-1}(I) \mid \varphi^{(n-1)} \text{ is absolutely continuous and } \varphi^{(n)} \in L^2(I) \}
\]

where \( C^k(I) \) is the space of \( k \)-times continuously differentiable complex functions on \( I \) (and \( C^0(I) \) stands for continuous functions). For \( n = 1 \) we write \( H^1(I) = H(I) \).

We start with the definition for the moment operators that usually appears in the literature, namely \( \hat{P}[n], n \in \mathbb{N} \). The following result was briefly mentioned by Werner [19]. We give a proof here in order to emphasize that care has to be taken on absolute continuity. That care is needed can also be deduced from the fact that the only difference between the integrals that one has as a tool is their domains.

**Proposition 17.** \( \hat{P}[n] = P^0_n \), and \( P \) is variance-free.

**Proof.** According to the definition, the square integrability domain is

\[
\text{Dom}(\hat{P}[n]) := \left\{ \varphi \in L^2(I) \middle| \int x^{2n} dP_{\varphi,\varphi}(x) < \infty \right\}.
\]

Since \( \langle \varphi | P(X) \varphi \rangle = \langle U \varphi | P_R(X) U \varphi \rangle \) for \( \varphi \in L^2(I) \), it follows immediately from the usual spectral theory that \( U \text{Dom}(\hat{P}[n]) = \text{Dom}(P_R[n]) \cap U(L^2(I)) \). Each function \( \varphi : \mathbb{R} \to \mathbb{C} \) belonging to \( \text{Dom}(P_R[1]) \) is absolutely continuous, so it follows that

\[
\text{Dom}(\hat{P}[n]) = \left\{ \varphi \in H^n(I) \mid \frac{d^k}{dx^k} \varphi(0) = \frac{d^k}{dx^k} \varphi(\ell) = 0 \text{ for } k = 0, 1, \ldots, n - 1 \right\} = \text{Dom}(P^0_n).
\]

Then given a \( \varphi \in \text{Dom}(P^0_n) \), \( \hat{P}[n] \varphi(X) = \langle U \psi | P_R(X) U \varphi \rangle \) for all \( \psi \in L^2(I) \), so by the spectral theorem, \( \langle \psi | \hat{P}[n] \varphi \rangle = \langle U \psi | P_R[n] U \varphi \rangle \). The important point now is that the range of \( U \) is stable under \( P_R[n] \), i.e. \( P_R[n](\text{Dom}(P_R[n]) \cap \text{ran} U) \subseteq \text{ran} U \), since \( P_R[n] \) is a derivative and the functions in the range of \( U \) vanish outside the interval \( I \). It follows that \( P_R[n] U \varphi \) is orthogonal to \( (\text{ran} U)^\perp \), which implies that \( \langle \Psi | \hat{P}[n] \varphi \rangle = \langle \Psi | P_R[n] U \varphi \rangle \) for any \( \Psi \in L^2(\mathbb{R}) \), and so \( \hat{P}[n] \varphi = P_R[n] U \varphi \). Since \( P_R[n] \) acts as the differential operator, this clearly implies that \( \hat{P}[n] \) does the same. Hence, \( \hat{P}[n] = P^0_n \). The fact that \( P \) is variance-free follows from the relation \( U \hat{P}[n] \varphi = P_R[n] U \varphi \) (see [19]). As the proof is very short, we give it here:

\[
\| \hat{P}[1] \varphi \|^2 = \| U \hat{P}[1] \varphi \|^2 = \| P_R[1] U \varphi \|^2 = \int x^{2n} d[P_R]_{\varphi,\varphi}(x) = \int x^{2n} dP_{\varphi,\varphi}(x)
\]

for each \( \varphi \in \text{Dom}(\hat{P}[1]) \). \hfill \Box

We now proceed to the other two definitions \( P[n] \) and \( P'[n] \). The first thing to note is that both the strong operator integral \( P[n] \) and the weak one \( P'[n] \) are symmetric extensions of \( \hat{P}[n] = P^0_n \). Hence, it follows that \( \text{Dom}(P[n]) \subseteq \text{Dom}(P'[n]) \subseteq H^n(I) \), and these operators just act as \( (-i)^n d^m/dx^m \) on their respective domains.

We will first show that \( P[n] = P^0_n \), for all \( n = 1, 2, \ldots \) (see Proposition [18] below.) The following two lemmas are needed. Let \( F : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) be the Fourier-Plancherel operator\(^5\).

\(^5\)Here we want to apply \( F \) to functions in \( L^2(I) \). In our notation this would be written as \( F U \); in order to simplify the notations, we will write \( F \) instead.
Hence,

\[ (F \varphi)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\ell e^{-ixt} \varphi(t) \, dt, \quad x \in \mathbb{R}, \ \varphi \in L^2(\mathcal{I}). \]

(Note that every element of \( L^2(\mathcal{I}) \) is integrable by the Cauchy-Schwarz inequality.) The function \( F \varphi : \mathbb{R} \to \mathbb{C} \) is continuous, bounded, and belongs to \( L^2(\mathbb{R}) \).

**Lemma 4.**

(a) For any \( \varphi \in H(\mathcal{I}) \), we have

\[ [F \varphi'](x) = ix[F \varphi](x) + \frac{1}{\sqrt{2\pi}}[\varphi(\ell)e^{-ix\ell} - \varphi(0)], \quad x \in \mathbb{R} \]

(b) For any \( \varphi \in H^n(\mathcal{I}) \cap \text{Dom}(P_0^{n-1}) \), we have

\[ x^n[F \varphi](x) = (i)^n[F \varphi^{(n)}](x) + \frac{i^n}{\sqrt{2\pi}}[\varphi^{(n-1)}(\ell)e^{-ix\ell} - \varphi^{(n-1)}(0)], \quad x \in \mathbb{R}. \]

**Proof.** Straightforward application of absolute continuity and integration-by-parts. \( \square \)

**Lemma 5.** Let \( a, b \in \mathbb{C} \). Then \( x \mapsto F_\psi(x) := [\overline{\psi}(x)][ae^{-ix} - b] \) is Lebesgue-integrable over \( \mathbb{R} \) for all \( \psi \in L^2(\mathcal{I}) \), if and only if \( a = b = 0 \).

**Proof.** We only need to consider functions \( \psi_\theta \in L^2(\mathcal{I}) \), where \( \psi_\theta(t) := e^{-i\theta t} \), with \( \theta \in \mathbb{R} \). Then

\[ F_{\psi_\theta}(x) = \frac{[e^{i(x+\theta)\ell} - 1][e^{-ix\ell}a - b]}{i(x+\theta)\sqrt{2\pi}}. \]

If \( |a| \neq |b| \), then \( |F_{\psi_\theta}(x)| \geq 2|\sin(x\ell/2)|/|x|\sqrt{2\pi} \), with \( \alpha = ||a|-|b|| \), so \( F_{\psi_\theta} \) is not integrable.

If \( |a| = |b| \), take \( \theta \) so that \( a = -e^{-i\theta}b \). Then \( F_{\psi_\theta}(x) = -2b \sin((x+\theta)\ell)/\sqrt{2\pi(x+\theta)} \), which is again not integrable. The only remaining possibility is \( a = b = 0 \), and then \( F_{\psi_\theta} = 0 \) is trivially integrable. \( \square \)

**Proposition 18.** \( P[n] = \tilde{P}[n] = P_0^n \) for all \( n = 1, 2, \ldots \).

**Proof.** We have already noted that \( P[n] \) is a restriction of \( (-i)^n d^n/dx^n : H^n(\mathcal{I}) \to L^2(\mathcal{I}) \). In particular, \( \text{Dom}(P[n]) \subseteq H^n(\mathcal{I}) \). Thus, we only need to show that the vectors in \( \text{Dom}(P[n]) \) are exactly those elements of \( H^n(\mathcal{I}) \) which satisfy the boundary conditions defining \( \text{Dom}(P_0^n) \). Proceeding by induction, we first consider the case \( n = 1 \). Using Lemma 4, we get

\[ x[F \psi](x)[F \varphi](x) = -i[F \psi](x)[F \varphi'](x) + \frac{i}{\sqrt{2\pi}}[F \psi](x)[\varphi(\ell)e^{-ix\ell} - \varphi(0)], \quad x \in \mathbb{R}. \]

for \( \psi \in L^2(\mathcal{I}) \) and \( \varphi \in H(\mathcal{I}) \). By definition, \( \varphi \in \text{Dom}(P[1]) \) if and only if \( x \mapsto [F \psi](x)[F \varphi](x) \) is integrable over \( \mathbb{R} \) for all \( \psi \in L^2(\mathcal{I}) \). Since \( \varphi' \in L^2(\mathcal{I}) \), so that both \( F \psi \) and \( F \varphi' \) are in \( L^2(\mathbb{R}) \), the first term in the right hand side of (21) is integrable in any case. Hence \( \varphi \in \text{Dom}(P[1]) \) if and only if the second term is integrable for all \( \psi \in L^2(\mathcal{I}) \). But by Lemma 5, this happens exactly when \( \varphi(0) = \varphi(\ell) = 0 \), i.e. \( \varphi \in \text{Dom}(P_0) \). Thus, \( P[1] = P_0 \). Now we assume inductively that \( P[n-1] = P_0^{n-1} \). Since \( |x^{n-1}| \leq 1 + |x^n| \) for all \( x \in \mathbb{R} \), and the relevant complex measures are finite, it follows that \( P[n] \subseteq P[n-1] = P_0^{n-1} \), where the last equality follows from the induction assumption. Hence, \( \text{Dom}(P[n]) \subseteq H^n(\mathcal{I}) \cap \text{Dom}(P_0^{n-1}) \). Letting \( \varphi \in H^n(\mathcal{I}) \cap \text{Dom}(P_0^{n-1}) \) we get from Lemma 4(b) that

\[ x^n[F \psi](x)[F \varphi](x) = (i)^n[F \psi](x)[F \varphi^{(n)}](x) + \frac{i^n}{\sqrt{2\pi}}[F \psi](x)[\varphi^{(n-1)}(\ell)e^{-ix\ell} - \varphi^{(n-1)}(0)] \]

for all \( \psi \in L^2(\mathcal{I}) \). Since now \( \varphi^{(n)} \in L^2(\mathcal{I}) \) (because \( \varphi \in H^n(\mathcal{I}) \)), we can again use the same argument as before to conclude by Lemma 5 that \( \varphi \in \text{Dom}(P[n]) \) if and only if \( \varphi^{(n-1)}(\ell) = \varphi^{(n-1)}(0) = 0 \), i.e. \( \varphi \in \text{Dom}(P_0^n) \). The proof is complete. \( \square \)
For the weak operator integral $\mathcal{P}[n]$, the following result holds.

**Proposition 19.** $\mathcal{P}[1] = P_0$ and $\mathcal{P}[2n] = (P_0^n)^*P_0^n$.

**Proof.** The last statement follows from Propositions [16] (b) and [17] because the derivative of a function with support in $\mathcal{I}$ also has support in $\mathcal{I}$ (this was already mentioned in the proof of Proposition [17]).

To prove that $\mathcal{P}[1] = P_0$, recall first that $\mathcal{P}[1]$ is a symmetric extension of $\tilde{P}[1]$, and hence coincides with one of the selfadjoint extension $\mathcal{P}^{(\theta)}$ of $P_0$, or $P_0$ itself. We show that $\text{Dom}(\mathcal{P}[1])$ is a proper subspace of $\text{Dom}(\mathcal{P}^{(\theta)})$, so that $\mathcal{P}[1] = P_0$ must hold. For any $a, b \in \mathbb{C}$, define $\varphi_{a,b} : \mathbb{R} \to \mathbb{C}$ via $\varphi_{a,b}^{(b)}(t) := (b-a)t/\ell + a$. This is obviously infinitely differentiable, and satisfies the boundary conditions $\varphi_{a,b}(0) = a$ and $\varphi_{a,b}(\ell) = b$, so for a suitable choice of the two constants, the vector $\varphi_{a,b}$ will belong to the domain of a given $\mathcal{P}^{(\theta)}$. We will show that it does not belong to the form domain $D_0(x, P)$ (which is even larger than the domain of $\mathcal{P}[1]$), unless $a = b = 0$. In order to prove this, it suffices to show that $xG(x)$ is not integrable over $[1, \infty)$, where $G : \mathbb{R} \to \mathbb{C}$ is the density of the measure $P_{\varphi_{a,b}}$, i.e., $G(x) := (\mathcal{F}\varphi_{a,b})(x)^2 = (2\pi)^{-1}\int_0^\infty e^{-ixt}\varphi_{a,b}(t)\,dt$. Now in case $a = b \neq 0$, we have simply $xG(x) = 2\pi a^2(1 - \cos(x\ell))/\ell(2\pi x)$, which is not integrable. In case $a \neq b$, we put $a' := (b-a)\ell^{-1} \neq 0$, $b' := a/a'$; then we get $2\pi a^{-2}xG(x) = h(x) + x^{-2}(f(x) + x^{-1}g(x))$, where $h(x) := x^{-1}\ell + b' - b'e^{i\ell^2}$, and $f$ and $g$ are bounded real functions. Now $\int xG(x)\,dx = \infty$ is equivalent to $\int_{\ell}^{\infty} h(x)\,dx = \infty$, which is true because $h(x) \geq \frac{1}{2}(\ell + |b|)^2\ell^{-1}$ in case $|\ell + b| \neq |b|$, while $h(x) = 2|b|^2x^{-1}[1 - \cos(x\ell + \beta)]$ for some $\beta \in [0, 2\pi)$ in case $|\ell + b| = |b|$. The proof is complete. \hfill $\square$

**Remark 6.** It is interesting to compare the domains of the differential operators $\mathcal{P}[2n] = P[2n]$ and $\mathcal{P}[2n]$, both acting as restrictions of the maximal operator $(-1)^nd2n/dx^{2n}$, and thereby differing only by boundary conditions. Explicitly, we have

\[
\text{Dom}(\mathcal{P}[2n]) = \text{Dom}(P_0^{2n}) = \left\{ \varphi \in H^{2n}(\mathcal{I}) \mid \varphi^{(k)}(0) = \varphi^{(k)}(\ell) = 0, \ k = 0, 1, \ldots, 2n - 1 \right\};
\]

\[
\text{Dom}(\mathcal{P}'[2n]) = \text{Dom}(P_0^{n}*P_0^{n}) = \left\{ \varphi \in H^{2n}(\mathcal{I}) \mid \varphi^{(k)}(0) = \varphi^{(k)}(\ell) = 0, \ k = 0, \ldots, n - 1 \right\}.
\]

To obtain the last equality, recall that $\text{Dom}(P_0^{n}*P_0^{n}) = H^n(\mathcal{I})$. Hence, in the case of even index, the weak moment operator integral differs from the strong one in that half of the boundary conditions are removed. Note also that $\mathcal{P}[2n]$ is selfadjoint, because $P_0^{n}$ is closed. However, as the example $\mathcal{P}[1] = P_0$ shows, odd moments need not be.

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