Hecke groups, linear recurrences, and Kepler limits

Barry Brent

11h 26 February 2019

Abstract
We study the linear fractional transformations in the Hecke group $G(\Phi)$ where $\Phi$ is either root of $x^2 - x - 1$ (the larger root being the “golden ratio” $\phi = 2\cos \frac{\pi}{5}$). Let $g \in G(\Phi)$ and let $z$ be a generic element of the upper half-plane. Exploiting the fact that $\Phi^2 = \Phi - 1$, we find that $g(z)$ is a quotient of linear polynomials in $z$ such that the coefficients of $z^1$ and $z^0$ in the numerator and denominator of $g(z)$ appear themselves to be linear polynomials in $\Phi$ with coefficients that are certain multiples of Fibonacci numbers. We make somewhat less detailed observations along similar lines about the functions in $G(2\cos \frac{\pi}{k})$ for $k \geq 5$.

1 Introduction
Let $G(\lambda)$ be the Hecke group generated by the linear fractional transformations $S : z \mapsto -1/z$ and $T_\lambda : z \mapsto z + \lambda$ and let $G_k = G(2\cos \frac{\pi}{k})$. This article describes numerical experiments carried out to study Hecke groups, mainly $G_k$ for $k \geq 5$. In this article, an $n$-tuple of symbols

$\vec{w} = \{w_1, w_2, w_3, \ldots, w_n\}$

representing an ordered set of integers is called a word on $\mathbb{Z}$ and we write $n = |\vec{w}|$. A typical element of $G(\lambda)$ takes the form

$T_{\lambda}^{w_1} S T_{\lambda}^{w_2} S \ldots S T_{\lambda}^{w_n} = g_{\lambda; \vec{w}}$.

This representation is not unique. For example, a function $g \in G(\lambda)$ can be described by a word of length $n$ for arbitrarily large $n$, because any word representing $g$ can be padded with zeros and the resulting word will also represent $g$. Consequently, when studying all $g$ represented by words $\vec{w}$ with $|\vec{w}| \leq N$, we can restrict attention to the words satisfying $|\vec{w}| = N$.

Let $\phi, \phi^*$, represent the larger and smaller roots of $x^2 - x - 1$, respectively. The problem of expressing (for $g \in G_5$) $g = g_{\phi; \vec{w}}$ in terms of the $w_i$ was raised by Leo in $[23]$ and discussed by his student Sherkat in $[26]$; the first purpose of
this article is to write down conjectures addressing this question. Our calculations indicate that, for arbitrary \( \lambda, z \in \mathbb{C} \), \( g_{\lambda z}(z) \) is a rational function of \( z \) and \( \lambda \) in polynomials of \( \lambda \)-degree \( \leq k \). Here are the first few:

\[
g_{\lambda;\{w_1\}}(z) = \frac{1 \cdot z + w_1 \lambda}{0 \cdot z + 1},
\]

\[
g_{\lambda;\{w_1, w_2\}}(z) = \frac{w_1 \lambda \cdot z + w_1 w_2 \lambda^2 - 1}{1 \cdot z + w_2 \lambda},
\]

and

\[
g_{\lambda;\{w_1, w_2, w_3\}}(z) = \frac{(w_1 w_2 \lambda^2 - 1) \cdot z + w_1 w_2 w_3 \lambda^3 - (w_1 + w_3) \lambda}{w_2 \lambda \cdot z + w_2 w_3 \lambda^2 - 1}.
\]

Following [26], we simplify the above expressions for \( g_{\lambda z} \) when \( \lambda = \phi \) or \( \phi^* \) by repeatedly making the substitution \( \Phi^2 = \Phi + 1 (\Phi = \phi \) or \( \phi^* \).) The coefficients in \( g_{\phi z} \) become linear polynomials in \( \Phi \):

\[
g_{\Phi;\{w_1\}}(z) = \frac{1 \cdot z + w_1 \Phi}{0 \cdot z + 1},
\]

\[
g_{\Phi;\{w_1, w_2\}}(z) = \frac{w_1 \Phi \cdot z + w_1 w_2 \Phi + w_1 w_2 - 1}{1 \cdot z + w_2 \Phi},
\]

and

\[
g_{\Phi;\{w_1, w_2, w_3\}}(z) = \frac{(w_1 w_2 \Phi + w_1 w_2 - 1) \cdot z + (2w_1 w_2 w_3 - w_1 - w_3) \Phi + w_1 w_2 w_3}{w_2 \Phi \cdot z + w_2 w_3 \Phi + w_2 w_3 - 1}.
\]

Further calculations suggest that the coefficients of \( \Phi^1 \) and \( \Phi^0 \) in these expressions are always a linear combinations of first-degree monomials \( h \) in the \( w_i \) such that the numerical coefficient of \( h \) is \( \pm 1 \) times a Fibonacci number determined by the total degree of \( h \); details are in the next section.

It is well known, of course, that the consecutive ratios \( F_n/F_{n-1} \) of Fibonacci numbers converge to \( \phi \). More generally, the limit of the ratios of consecutive elements of a linear recurrence \( L \), when it exists, is called by Fiorenza and Vincenzi the Kepler limit of \( L \). Certain roots of other polynomials than \( x^2 - x - 1 \) are also Kepler limits [19, 20], so we are led to consider the possibility that the \( G_5 \) phenomenon generalizes; our observations tend to confirm this guess. Section 2 describes what we found out about \( G_5 \), section 3 describes less detailed observations for \( G_k, 5 \leq k \leq 33 \), and the final section provides some detail about our numerical experiments; documentation in the form of Mathematica notebooks is on our ResearchGate pages [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

We state merely empirical claims in this article. We make several conjectures,
but they, too, are based on empirical evidence, not on theoretical reasoning. When we say we have observed convergence of a sequence $s_n$ (say) to a limit $S$, we mean that our plots of 1000 values of $\log |S - s_n|$ are apparently linear, with negative slope. We relied on our eyesight in this matter: we did not fit our data to curves with a statistical package. Interested readers are invited to inspect the plots on our ResearchGate pages.

In the following section our observations were made on words in $W$ of length 20, and those in the last section tested words of length 25. This means that we have in fact tested the claims on all shorter words as well.

In our tests, we identified functions in the $G_k$ with their matrix representations: A function

$$T^{w_1}_\lambda ST^{w_2}_\lambda S...ST^{w_n}_\lambda$$

was identified with the corresponding matrix product.

More information about the Hecke groups is available, for example, in [3].

Remark 1. The book [21] by Khovanskii apparently describes another method for approximating roots of polynomials using convergent sequences of ratios of elements of numerical sequences; but these sequences are not linear recurrences. (We have not seen [21], but Khovanskii's's method is described in [24], where the book is cited.)

2 The group $G_5$

We make the following definitions.

1. The Fibonacci numbers are defined with the convention that $F_0 = 0, F_1 = F_2 = 1$, etc. It will be convenient to write $F_{-1} = 1$ as well in contexts where (see below) $\overrightarrow{S} = \emptyset$.

2. $\chi$ is the following Dirichlet character:

$$\chi(n) = \begin{cases} 
0 & \text{if } n \equiv 0 \pmod{4}, \\
1 & \text{if } n \equiv 1 \pmod{4}, \\
0 & \text{if } n \equiv 2 \pmod{4}, \\
-1 & \text{if } n \equiv 3 \pmod{4}.
\end{cases}$$

Alternatively, with $(a|b)$ representing the Kronecker symbol, $\chi(n) = (n|2)$ if $n \equiv 0, 1, 2, 3, 4 \text{ or } 6 \pmod{8}$, and $\chi(n) = -(n|2)$ otherwise.

3. $W$ is the set of words $\overrightarrow{w}$ on $\mathbb{Z}$. The empty word $\overrightarrow{\emptyset}$ verifies $\overrightarrow{\emptyset} \in W$ and $\overrightarrow{\emptyset} \subset \overrightarrow{w}$ for any $\overrightarrow{w} \in W$. 


4. We write the cardinal number of a set $\sigma$ as $|\sigma|$. We apply the same notation to words $\vec{w} \in W$. We write $|\vec{\emptyset}| = 0.$

5. (a) For $\vec{w} \in W, \vec{w} \neq \vec{\emptyset}$, let $\vec{s} = \{w_{j_1}, w_{j_2}, ..., w_{j_m}\} \subset \vec{w} = \{w_1, ..., w_n\}$. If all of the $j_m \equiv m \pmod{2}$, then we write $\vec{s} \ll_1 \vec{w}$. We also write $\vec{\emptyset} \ll_1 \vec{w}$.

(b) If $\vec{s} \ll_1 \vec{w}$ and $|\vec{s}| > 1$, we write $\vec{s} \ll_2 \vec{w}$. 

(c) Let $\vec{s}, \vec{w}$ be as in definition 5a, except that all of the $j_m$ satisfy $j_m \equiv m - 1 \pmod{2}$. Then we write $\vec{s} \ll_3 \vec{w}$. We also write $\vec{\emptyset} \ll_3 \vec{w}$.

(d) If $\vec{s} \ll_3 \vec{w}$ and $|\vec{s}| > 1$, we write $\vec{s} \ll_4 \vec{w}$.

6. (a) For $\vec{w} \in W$, the formal product

$$m_{\vec{w}} = \prod_{w_i \in \vec{w}} w_i.$$ 

We also write

$$m_{\vec{\emptyset}} = 1.$$ 

(b) $M_{\vec{w}}$ is the set of all linear combinations with coefficients in the integers of monomials $m_{\vec{w}}$ such that $\vec{s} \subset \vec{w}$.

(c) $M$ is the union of the $M_{\vec{w}}$ as $\vec{w}$ ranges over $W$.

Remark 2. In view of the identities $\Phi^2 = \Phi + 1$ for $\Phi = \phi$ or $\phi^*$, it is clear that

(i) For each $1 \leq j \leq 8$, there is a function $f_j : W \rightarrow M$ such that $f_j(\vec{w}) \in M_{\vec{w}}$ and, for all $g_{\vec{w}'} \in G(\Phi)$ and $3z > 0$,

$$g_{\Phi, \vec{w}'}(z) = \frac{(f_1(\vec{w})\Phi + f_2(\vec{w}))z + f_3(\vec{w})\Phi + f_4(\vec{w})}{(f_5(\vec{w})\Phi + f_6(\vec{w}))z + f_7(\vec{w})\Phi + f_8(\vec{w})}.$$ 

Referring to the introduction, for example:

$$f_3(w_1, w_2, w_3) = 2w_1w_2w_3 - w_1 - w_3$$
and
\[ f_6(w_1, w_2, w_3) = 0. \]
(ii) For each \( 1 \leq j \leq 8 \), there is a function \( \nu_j : W \times W \mapsto Z \) determined by the condition
\[ f_j(\vec{w}) = \sum_{\vec{s} \subset \vec{w}} \nu_j(\vec{s}, \vec{w}) m_{\vec{s}}. \]

The following observations describe the functions represented by words of length \( \leq 20 \) in \( G_k, 5 \leq k \leq 50 \).

**Observation 1.**
(a) \( \nu_1(\vec{s}, \vec{w}) = \begin{cases} \chi(|\vec{w}| - |\vec{s}|)|F_{|\vec{s}|}| & \text{if } \vec{s} \ll_1 \vec{w}, \\ 0 & \text{otherwise.} \end{cases} \)
(b) \( \nu_2(\vec{s}, \vec{w}) = \begin{cases} \chi(|\vec{w}| - |\vec{s}|)|F_{|\vec{s}|-1} & \text{if } \vec{s} \ll_2 \vec{w}, \\ 0 & \text{otherwise.} \end{cases} \)
(c) \( \nu_3(\vec{s}, \vec{w}) = \begin{cases} -\chi(|\vec{w}| - |\vec{s}| - 1)|F_{|\vec{s}|}| & \text{if } \vec{s} \ll_1 \vec{w}, \\ 0 & \text{otherwise.} \end{cases} \)
(d) \( \nu_4(\vec{s}, \vec{w}) = \begin{cases} -\chi(|\vec{w}| - |\vec{s}| - 1)|F_{|\vec{s}|-1} & \text{if } \vec{s} \ll_2 \vec{w}, \\ 0 & \text{otherwise.} \end{cases} \)
(e) \( \nu_5(\vec{s}, \vec{w}) = \begin{cases} \chi(|\vec{w}| - |\vec{s}| - 1)|F_{|\vec{s}|-1} & \text{if } \vec{s} \ll_3 \vec{w}, \\ 0 & \text{otherwise.} \end{cases} \)
(f) \( \nu_6(\vec{s}, \vec{w}) = \begin{cases} \chi(|\vec{w}| - |\vec{s}| - 1)|F_{|\vec{s}|-1} & \text{if } \vec{s} \ll_4 \vec{w}, \\ 0 & \text{otherwise.} \end{cases} \)
(g) \( \nu_7(\vec{s}, \vec{w}) = \begin{cases} \chi(|\vec{w}| - |\vec{s}|)|F_{|\vec{s}|} & \text{if } \vec{s} \ll_3 \vec{w}, \\ 0 & \text{otherwise.} \end{cases} \)
(h) \( \nu_8(\vec{s}, \vec{w}) = \begin{cases} \chi(|\vec{w}| - |\vec{s}|)|F_{|\vec{s}|-1} & \text{if } \vec{s} \ll_3 \vec{w}, \\ 0 & \text{otherwise.} \end{cases} \)

**Conjecture 1.** It appears that observation 1 may hold for words of arbitrary length and all \( k \geq 5 \).

### 3 Higher-order Hecke groups

**Definition:** let \( t(x) \) be a polynomial \( \sum_{j=0}^d a_j x^j \) and \( \gamma(t) := \gcd\{ j \text{ s.t. } a_j \neq 0 \} \). If \( \gamma(t) = 1 \), we say that \( t \) is stable. Whether or not \( t \) is stable, we associate to it the family of \( d^{th} \)-order linear recurrences \( \Lambda_t \) with kernel \( \{-a_{d-1}, -a_{d-2}, \ldots, -a_0\} \).
Let \( \lambda = 2 \cos \frac{\pi}{k} \) with minimal polynomial \( p_\lambda = p \) (say.) Under certain conditions \[19, 20\], a root \( x = \kappa_p \) of \( p(x) \) is the Kepler limit of one of the \( L_p \in \Lambda_p \). The elements of \( G(\lambda) = G_k \) have the form

\[
g_{\lambda, \pi}(z) = \frac{\sum_{j=0}^{d-1} f_{\lambda, 1,j}(\overrightarrow{w}) \lambda^j}{\sum_{j=0}^{d-1} f_{\lambda, 3,j}(\overrightarrow{w}) \lambda^j} \cdot z + \sum_{j=0}^{d-1} f_{\lambda, 4,j}(\overrightarrow{w}) \lambda^j
\]  

(1)

(Equation (1) is clear, as in the \( G_5 \) case, by substitution.)

For pragmatic reasons, we restricted our attention to \( f = f_{\lambda, 1,0} \) in the following observations.

Observation 2. For \( 5 \leq k \leq 500 \), \( \gamma(p) = 1 \) if \( k \) is odd, \( \gamma(p) = 2 \) if \( k \) is even.

Conjecture 2. For polynomials of the form \( p = p_\lambda \), the statements in the above observation hold for all \( k \geq 5 \).

Observation 3. Let \( 5 \leq k \leq 33 \).

(a) There is a function \( \nu^{(k)}: W \times W \rightarrow \mathbb{Z} \) such that

\[
f(\overrightarrow{w}) = \sum_{\overrightarrow{w} \subset \overrightarrow{w}} \nu^{(k)}(\overrightarrow{s}, \overrightarrow{w}) m_{\overrightarrow{x}}.
\]  

(2)

with

\[\overrightarrow{s_1}, \overrightarrow{s_2} \subset \overrightarrow{w}\]

and

\[
|\overrightarrow{s_1}| = |\overrightarrow{s_2}| \Rightarrow |\nu^{(k)}(\overrightarrow{s_1}, \overrightarrow{w})| = |\nu^{(k)}(\overrightarrow{s_2}, \overrightarrow{w})|
\]  

(3)

for all \( \overrightarrow{w} \in W \) s.t. \( |\overrightarrow{w}| = 25 \).

(b) If \( k \) is odd, then for some particular \( L_p \in \Lambda_p \) and all \( \overrightarrow{s} \subset \overrightarrow{w} \) s.t. \( |\overrightarrow{w}| = 25 \):

(b1) \( |\nu^{(k)}(\overrightarrow{s}, \overrightarrow{w})| \) \( \in L_p \) and (b2) \( \kappa_p = \lambda \).

(In our experiments the sum on the r.h.s. of equation (2) typically contains over \( 6 \times 10^4 \) terms, but twelve or fewer distinct values of \( |\nu^{(k)}(\overrightarrow{s}, \overrightarrow{w})| \).

(c) Suppose \( 6 \leq k \leq 32 \) is even. Then

(c1) clause (b1) still holds, but (b2) does not; in this situation, we found no \( L_p \) for which \( \kappa_p \) exists. (By design, our searches stopped with the first instance of \( L_p \) satisfying (a), so this is far from decisive.)

(c2) For \( k = 8, 10, 14, 16, 18, 22, 26, \) and \( 32 \), the ratios of consecutive elements of the \( L_p \) we found in the experiments form two convergent sub-sequences with different limits.)
For \( k = 6, 12, 20, 24, 28, \) and \( 30, \) the \( L_p \) terminate in a sequence in which alternate members are zero, so that the requisite ratios are alternately zero or undefined.

(d) Suppose \( k = 12, 14, 20, 22, 24, 28, \) or \( 30. \) After a substitution \( y = x^2, \) \( p(x) \) is transformed to a stable polynomial \( q_{\lambda^2}(y) = q(y) \) (say), and then \( \lambda^2 \) is the Kepler limit of a linear recurrence \( L_q \in \Lambda_q \) containing the \( \|x^{(k)}(\vec{s}, \vec{w})\|. \)

**Conjecture 3.** In the above observation, clause (a) holds for all \( k \geq 5, \) clause (b) holds for all odd \( k \geq 5, \) and clause (c1) holds for all even \( k \geq 6. \) One of clauses (c2) or (c3) holds for any even \( k \geq 6. \) Clause (d) holds for an unbounded set of even \( k \geq 6. \)

The conditions on polynomials under which linear recurrences with Kepler limits that are killed by them were established in \([25]\) (cited in \([19]\)).

A procedure (which can be invoked from computer algebra systems) for computing \( p = p_{\lambda} \) for \( \lambda = 2 \cos \frac{\pi}{k} \) one at a time for individual \( k \) has appeared in \([2]\); some information about the constant terms, in \([1]\); and, about the degree, in \([22]\).

### 4 Data on the linear recurrences

#### 4.1 Coefficients

This is a list of distinct coefficients of the \( m_{\vec{s}} \) appearing in our calculations for equation (2), \( 5 \leq k \leq 33: \)

5: 1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181, 10946, 28657

6: 1, 3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, 177147, 531441

7: 1, 4, 14, 47, 155, 507, 1652, 5373, 17460, 56714, 184183

8: 1, 2, 8, 28, 96, 328, 1120, 3824, 13056, 44576, 152192, 519616

9: 1, 6, 27, 109, 417, 1548, 5644, 20349, 72846, 259579

10: 1, 5, 25, 100, 375, 1375, 5000, 18125, 65625, 237500, 859375, 3109375

11: 1, 6, 27, 110, 429, 1637, 6172, 23104, 86090, 319792

12: 1, 4, 15, 56, 209, 780, 2911, 10864, 40545, 151316, 564719
13: 1, 6, 27, 110, 429, 1638, 6188, 23255, 87190, 326646
14: 1, 7, 49, 245, 1078, 4459, 17836, 69972, 271313, 1044435, 4002467
15: 1, 5, 20, 74, 265, 936, 3290, 11560, 40699, 143755, 509771
16: 1, 2, 16, 88, 416, 1820, 7616, 31008, 124032, 490312
17: 1, 8, 44, 208, 910, 3808, 15504, 62016, 245157
18: 1, 3, 18, 81, 333, 1323, 5184, 20196, 78489, 304722, 1182519
19: 1, 10, 65, 350, 1700, 7752, 33915, 144210
20: 1, 8, 45, 220, 1000, 4352, 18411, 76380, 312455
21: 1, 7, 35, 154, 636, 2534, 9877, 37962, 144571, 547239
22: 1, 11, 121, 847, 4840, 24684, 117249, 531069, 2326588
23: 1, 12, 90, 544, 2907, 14364, 67298
24: 1, 8, 44, 208, 911, 3824, 15656, 63136, 252241
25: 1, 10, 65, 350, 1700, 7752, 33915, 144210
26: 1, 13, 169, 1352, 8619, 48165, 247247, 1197196
27: 1, 18, 189, 1518
28: 1, 12, 91, 560, 3059, 15484, 74382
29: 1, 14, 119, 798, 4655, 24794
30: 1, 7, 35, 155, 650, 2653, 10676, 42635, 169555
31: 1, 16, 152, 1120, 7084
32: 1, 2, 32, 304, 2240, 14168
33: 1, 11, 77, 440, 2244, 10659, 48278, 211486

4.2 Initial segments for observations 3a - 3c

We searched for initial segments $I$ of linear recurrences $L_p(p = p_\lambda, \lambda = 2\cos\frac{\pi}{k})$ with length equal to that of the kernel of $L_p$ (so that $I$ determines $L_p$) such
that a sufficiently long initial segment of $L_p$ contains the elements listed above for corresponding $k$.

5: $\{0, 1\}$
6: $\{0, 1\}$
7: $\{0, 0, 1\}$
8: $\{0, 0, 1, 2\}$
9: $\{0, 0, 1\}$
10: $\{0, 0, 1, 5\}$
11: $\{0, 0, 0, 0, 1\}$
12: $\{0, 0, 0, 1\}$
13: $\{0, 0, 0, 0, 0, 1\}$
14: $\{-3, 1, -3, 0, -3, 0\}$
15: $\{0, 0, 0, 1\}$
16: $\{0, 0, 0, 0, 0, 0, 0, 1, 2\}$
17: $\{0, 0, 0, 0, 0, 0, 0, 0, 1\}$
18: $\{0, 0, 0, 0, 1, 3\}$
19: $\{0, 0, 0, 0, 0, 0, 0, 0, 0, 1\}$
21: $\{0, 0, 0, 0, 0, 1\}$
22: $\{-1, 1, -1, 0, -1, 0, -1, 0, -1, 0\}$
23: $\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1\}$
24: $\{0, 0, 0, 0, 0, 0, 0, 1\}$
25: $\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1\}$
26: $\{-1, -1, -1, 0, -1, 0, -1, 0, -1, 0, 1, 0\}$
27: $\{0, 0, 0, 0, 0, 0, 0, 0, 0, 1\}$
4.3 Initial segments for observation 3d

This is a list of initial segments for $L_q \in \Lambda_q, q = q\lambda^2, \lambda = 2\cos\frac{\pi}{k}, k = 12, 14, 20, 22, 24, 28, \text{ and } 30$, satisfying the conditions of observation 3d.

12: \{0, 1\}
14: \{1, 0, 0\}
20: \{0, 0, 1\}
22: \{1, 0, 0, 0, 0\}
24: \{0, 0, 0, 1\}
28: \{0, 0, 0, 0, 0, 1\}
30: \{0, 0, 0, 0, 0, 1\}

4.4 Kernels for the linear recurrences

We list these for the convenience of the reader. Below is the list of kernels for the $L_p \in \Lambda_p, (p = p\lambda, \lambda = 2\cos\frac{\pi}{5}, 5 \leq k \leq 33$.

5: \{1, 1\}
6: \{0, 3\}
7: \{1, 2, -1\}
8: \{0, 4, 0, -2\}
9: \{0, 3, 1\}
10: \{0, 5, 0, -5\}
11: \{1, 4, -3, -3, 1\}
12: \{0, 4, 0, -1\}
13: \{1, 5, -4, -6, 3, 1\}
14: \{0, 7, 0, -14, 0, 7\}
15: \{-1, 4, 4, -1\}
16: \{0, 8, 0, -20, 0, 16, 0, -2\}
17: \{1, 7, -6, -15, 10, 10, -4, -1\}
18: \{0, 6, 0, -9, 0, 3\}
19: \{1, 8, -7, -21, 15, 20, -10, -5, 1\}
20: \{0, 8, 0, -19, 0, 12, 0, -1\}
21: \{-1, 6, 6, -8, -8, -1\}
22: \{0, 11, 0, -44, 0, 77, 0, -55, 0, 11\}
23: \{1, 10, -9, -36, 28, 56, -35, -35, 15, 6, -1\}
24: \{0, 8, 0, -20, 0, 16, 0, -1\}
25: \{0, 10, 0, -35, 1, 50, -5, -25, 5, 1\}
26: \{0, 13, 0, -65, 0, 156, 0, -182, 0, 91, 0, -13\}
27: \{0, 9, 0, -27, 0, 30, 0, -9, 1\}
28: \{0, 12, 0, -53, 0, 104, 0, -86, 0, 24, 0, -1\}
29: \{1, 13, -12, -66, 55, 165, -120, -210, 126, 126, -56, -28, 7, 1\}
30: \{0, 7, 0, -14, 0, 8, 0, -1\}
31: \{1, 14, -13, -78, 66, 220, -165, -330, 210, 252, -126, -84, 28, 8, -1\}
Below is the list of kernels for the $L_q \in \Lambda_q$, $q = q_{2^2}$, $\lambda = 2 \cos \frac{\pi}{k}$, $k = 12, 14, 20, 22, 24, 28,$ and 30, satisfying the conditions of observation 3d.

12: \{4, -1\}
14: \{7, -14, 7\}
20: \{8, -19, 12, -1\}
22: \{11, -44, 77, -55, 11\}
24: \{8, -20, 16, -1\}
28: \{12, -53, 104, -86, 24, -1\}
30: \{7, -14, 8, -1\}

References

[1] Chandrashekar Adiga, Ismail Naci Cangul, and HN Ramaswamy. “On the Constant Term of The Minimal Polynomial of $\cos(2^{\frac{\pi}{n}})$ over $\mathbb{Q}$”. In: Filomat 30.4 (2016), pp. 1097–1102.

[2] Abdelmejid Bayad and Ismail Naci Cangul. “The minimal polynomial of $2 \cos(\frac{\pi}{q})$ and Dickson polynomials”. In: Applied Mathematics and Computation 218.13 (2012), pp. 7014–7022.

[3] Bruce C Berndt and Marvin Isadore Knopp. Hecke’s theory of modular forms and Dirichlet series. Vol. 5. World Scientific, 2008.

[4] B. Brent. $f_1$. https://www.researchgate.net/publication/331267286 2019.

[5] B. Brent. $f_2$. https://www.researchgate.net/publication/331267869 2019.

[6] B. Brent. $f_3$. https://www.researchgate.net/publication/331268477 2019.

[7] B. Brent. $f_4$. https://www.researchgate.net/publication/331268858 2019.

[8] B. Brent. $f_5$. https://www.researchgate.net/publication/331271499 2019.
[9] B. Brent. f6. https://www.researchgate.net/publication/331270450
2019.
[10] B. Brent. f7. https://www.researchgate.net/publication/331270400
2019.
[11] B. Brent. f8. https://www.researchgate.net/publication/331270786
2019.
[12] B. Brent. k=14. https://www.researchgate.net/publication/331314929
2019.
[13] B. Brent. k=22. https://www.researchgate.net/publication/331314667
2019.
[14] B. Brent. k=26. https://www.researchgate.net/publication/331314850
2019.
[15] B. Brent. Kepler39. https://www.researchgate.net/publication/331304911
2019.
[16] B. Brent. Kepler40. https://www.researchgate.net/publication/331314752
2019.
[17] B. Brent. Kepler41. https://www.researchgate.net/publication/331314923
2019.
[18] B. Brent. Kepler42. https://www.researchgate.net/publication/331314924
2019.
[19] Alberto Fiorenza and Giovanni Vincenzi. “From Fibonacci sequence to
the golden ratio”. In: Journal of Mathematics 2013 (2013).
[20] Alberto Fiorenza and Giovanni Vincenzi. “Limit of ratio of consecutive
terms for general order-k linear homogeneous recurrences with constant
coefficients”. In: Chaos, Solitons & Fractals 44.1-3 (2011), pp. 145–152.
[21] Alexey Nikolaevitch Khovanskii. The application of continued fractions
and their generalizations to problems in approximation theory. Translated
by Peter Wynn. Noordhoff Groningen, 1963.
[22] Derik H Lehmer. “A note on trigonometric algebraic numbers”. In: The
American Mathematical Monthly 40.3 (1933), pp. 165–166.
[23] John Garret Leo. Fourier coefficients of triangle functions, Ph.D. thesis.
http://halfaya.org/ucla/research/thesis.pdf 2008.
[24] J Mc Laughlin and B Sury. “Some Observations on Khovanskii’s Matrix
Methods for Extracting Roots of Polynomials”. In: INTEGERS: The
Electronic Journal of Combinatorial Number Theory 7.A48 (2007).
[25] Henri Poincare. “Sur les equations lineaires aux differentielles ordinaires
et aux differences finies”. In: American Journal of Mathematics (1885),
pp. 203–258.
[26] Hooman Sherkat. Investigation of the Hecke group G5 and its Eisenstein
series. http://halfaya.org/ucla/research/sherkat.pdf 2007.