Subordination Properties of Certain Subclass of p-Valent Meromorphic Functions Associated with Linear Operator

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Abstract

In this paper, subordination results are studied for certain subclass of p-valent meromorphic functions in the punctured unit disc having a pole of order p at the origin. The subclass under investigation is defined by using certain new linear operator. Moreover, we also introduced an interesting particular cases of these results in several corollaries.

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1. Introduction

Let $\Sigma_p$ denote the class of functions of the form

$$ f(z) = z^{-p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \quad (1.1) $$

which are analytic in the punctured unit disc $U^* = U \setminus \{0\}$; $U = \{z \in \mathbb{C} : |z| < 1\}$.

For two functions $f(z)$ and $g(z)$, analytic in $U$, we say that $f(z)$ is subordinate to $g(z)$ in $U$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $\omega(z)$ which (by definition) is analytic in $U$, satisfying the following conditions (see [12], [13]):

$$ \omega(0) = 0 \text{ and } |\omega(z)| < 1; \quad (z \in U) $$
such that

\[ f(z) = g(\omega(z)); \ (z \in U), \]

Indeed it is known that

\[ f(z) \prec g(z) \ (z \in U) \implies f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U). \]

In particular, if the function \( g(z) \) is univalent in \( U \), we have the following equivalence (see also [4]):

\[ f(z) \prec g(z) \ (z \in U) \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U). \]

Following the recent work of El-Ashwah [7], for a function \( f(z) \) in the class \( \Sigma_p \), given by (1.1), the operator \( L_{\lambda, \ell}^m \) is defined as following:

\[
L_{\lambda, \ell}^m f(z) = \begin{cases} 
  f(z); & m = 0 \\
  \ell z^{-p-\frac{\ell}{\lambda}} \int_0^z \left( \frac{t}{x} \right)^{p-1} L_{\lambda, \ell}^{m-1} f(t) dt; & m = 1, 2, \ldots.
\end{cases}
\]  

(1.2)

Also, following the recent work of El-Ashwah and Hassan [9] (see also [21]-[24]), for a function \( f(z) \in \Sigma_p \), given by (1.1), also, for \( \mu > 0, a, c \in \mathbb{C} \) and \( \text{Re}(c-a) \geq 0 \), the integral operator

\[ J_{p, c}^{a, \mu} : \Sigma_p \longrightarrow \Sigma_p \]

is defined for \( \text{Re}(c-a) > 0 \) as follows:

\[
J_{p, c}^{a, \mu} f(z) = \frac{\Gamma(c-p\mu)}{\Gamma(a-p\mu)\Gamma(c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} f(zt^\mu) dt,
\]

(1.3)

and for \( a = c \) as follows:

\[ J_{p, c}^{a, c} f(z) = f(z). \]

(1.4)

By iterations of the linear operators \( L_{\lambda, \ell}^m \) defined by (1.2) and \( J_{p, c}^{a, \mu} \) defined by (1.3) and (1.4), the operator

\[ I_{\lambda, \ell}^{a, \mu} (a, c, \mu) : \Sigma_p \longrightarrow \Sigma_p \]

is defined for the purpose of this paper by:

\[
I_{\lambda, \ell}^{a, \mu} (a, c, \mu) f(z) = L_{\lambda, \ell}^m (J_{p, c}^{a, \mu} f(z)) \quad \text{and} \quad J_{p, c}^{a, \mu} (L_{\lambda, \ell}^m f(z)).
\]

(1.5)

Now, it is easily to see that the generalized operator \( I_{\lambda, \ell}^{p, m} (a, c, \mu) \) can be expressed as following:

\[
I_{\lambda, \ell}^{p, m} (a, c, \mu) f(z) = z^{-p} + \frac{\Gamma(c-p\mu)}{\Gamma(a-p\mu)} \sum_{k=1-p}^{\infty} \frac{\Gamma(a+\mu k)}{\Gamma(c+\mu k)} \left[ \frac{\ell}{\ell + \lambda (k+p)} \right]^m a_k z^k,
\]

(1.6)
In view of (1.2), (1.4) and (1.5), it is clear that:

\[ I_{\lambda,\ell}^{m}(a, a, \mu) f(z) = L_{p}(\lambda) f(z). \]  

The operator \( I_{\lambda,\ell}^{m}(a, a, \mu) \) defined by (1.7) has been extensively studied by many authors with suitable restrictions on the parameters. For examples, see the following:

(i) \( \lambda, \ell \in \mathbb{N}_{0} = \mathbb{N} \cup \{0\}; p \in \mathbb{N} \) (see El-Ashwah [6]);

(ii) \( I_{\lambda,\ell}^{m}(a, a, \mu) f(z) = L_{p}(\lambda) f(z) (m \in \mathbb{N}_{0}; \lambda, \ell, \nu > 0; p \in \mathbb{N}) \) (see El-Ashwah and Aouf [8]);

(iii) \( I_{\nu,\lambda}^{m}(a + 1, a, \mu) f(z) \) (see Raina and Sharma [16]);

(iv) \( I_{\nu,\lambda}^{m}(a + 1, a, \mu) f(z) = \mathbb{Z}^{m}_{\nu,\lambda}(a, c) f(z) (\lambda, \nu > 0; a \in \mathbb{C}; c \in \mathbb{C} \setminus \mathbb{Z}_{0}^{-}; m \in \mathbb{N}_{0}) \) (see Liu and Srivastava [11] and Srivastava and Patel [18]);

(v) \( I_{\nu,\lambda}^{m}(a + 1, a, \mu) f(z) = \mathbb{Z}^{m}_{\nu,\lambda}(a, c) f(z) (\lambda, \nu > 0; a \in \mathbb{R} ; c \in \mathbb{R} \setminus \mathbb{Z}_{0}^{-}; \mathbb{Z}_{0}^{-} = \{0, 1, 2, \ldots\}; p \in \mathbb{N}) \) (see Liu and Srivastava [11] and Srivastava and Patel [18]);

(vi) \( I_{\nu,\lambda}^{m}(a + 1, a, \mu) f(z) \) (see Piejko and Sokół [15]);

(vii) \( I_{\nu,\lambda}^{m}(a + 1, a, \mu) f(z) \) (see Cho et al. [5]);

(viii) \( I_{\nu,\lambda}^{m}(a + 1, a, \mu) f(z) \) (see Yuan et al. [20]);

(ix) \( I_{\nu,\lambda}^{m}(a + 1, a, \mu) f(z) \) (see Aqlan et al. [3]);

(x) \( I_{\nu,\lambda}^{m}(a + 1, a, \mu) f(z) \) (see Lashin [10]).

### 2. Preliminaries

To establish our main results, we shall need the following lemmas:

**Lemma 1.** Using (1.6), we can obtain the following recurrence relations of the operator \( I_{\lambda,\ell}^{m}(a, a, \mu) \):

\[ z \left( I_{\lambda,\ell}^{m}(a, a, \mu) f(z) \right) = \frac{a - p \mu}{\mu} I_{\lambda,\ell}^{m}(a + 1, a, \mu) f(z) - \frac{a}{\mu} I_{\lambda,\ell}^{m}(a, a, \mu) f(z). \]  

and

\[ z \left( I_{\lambda,\ell}^{m}(a, a, \mu) f(z) \right) = \frac{c - p \mu}{\mu} I_{\lambda,\ell}^{m}(a, a, \mu) f(z) - \frac{c}{\mu} I_{\lambda,\ell}^{m}(a, a, \mu) f(z). \]  

Also,

\[ z \left( I_{\lambda,\ell}^{m+1}(a, a, \mu) f(z) \right) = \frac{\ell + \lambda p}{\lambda} I_{\lambda,\ell}^{m+1}(a, a, \mu) f(z). \]
Lemma 2 [13]. Let the function \( q(z) \) be univalent in the unit disc \( U \) and let \( \theta \) and \( \varphi \) are analytic in a domain \( D \) containing \( q(U) \) with \( q(w) \neq 0 \) for all \( w \in q(U) \). Set \( Q(z) = zz'q(z)\varphi(q(z)) \) and \( h(z) = \theta(q(z)) + Q(z) \). Suppose that
(i) \( Q(z) \) is starlike and univalent in \( U \);
(ii) \( \Re \left\{ \frac{zh'(z)}{Q(z)} \right\} > 0 \) for \( z \in U \). If \( p \) is analytic with \( p(0) = q(0) \), \( p(U) \subseteq D \) and
\[
\theta(p(z)) + zp'(z)\varphi(p(z)) < \theta(q(z)) + zq'(z)\varphi(q(z)),
\]
then
\[
p(z) < q(z) \quad (z \in U),
\]
and \( q(z) \) is the best dominant.

Lemma 3 [17]. Let \( q \) be a convex univalent function in \( U \) and let \( \delta \in \mathbb{C}, \gamma \in \mathbb{C}^* = \mathbb{C}\{0\} \) with
\[
\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max \left\{ 0, -\Re \left\{ \frac{\delta}{\gamma} \right\} \right\}.
\]
If \( p(z) \) is analytic in \( U \) with \( p(0) = q(0) \) and
\[
\delta p(z) + \gamma zp'(z) < \delta q(z) + \gamma zq'(z),
\]
then
\[
p(z) < q(z) \quad (z \in U),
\]
and \( q(z) \) is the best dominant.

In this paper, we find several sufficient conditions under which some subordination results hold for the function \( f \in \Sigma_p \) and for suitable univalent function \( q \) in \( U \). We also introduced an interesting particular cases of these results in several corollaries.

3. Subordination results

Unless otherwise mentioned, we assume throughout the remainder of the paper that
\(-1 \leq B < A \leq 1, 0 \leq \alpha < p, \lambda > 0, \ell > 0, \mu > 0, a, c \in \mathbb{C}, \Re \{a\} > p\mu, \Re \{c-a\} \geq 0, p \in \mathbb{N}, m \in \mathbb{N}_0, z \in U \) and the powers are principal.

We begin with investigating some sharp subordination results regarding the operator \( I_{p,m}^{\lambda,\ell}(a, c, \mu) f(z) \).

Theorem 1. Let \( \xi \in \mathbb{C}^* = \mathbb{C}\{0\} \). Let the function \( f \in \Sigma_p \) and the function \( q \) be univalent and convex in \( U \) with \( q(0) = 1 \). Suppose \( f \) and \( q \) satisfy any one of the following pairs of conditions:
\[
\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max \left\{ 0, -\frac{p}{\mu} \Re \left\{ \frac{a - p\mu}{\xi} \right\} \right\},
\]
(3.1)
\[ \frac{\xi}{p} \left( z^p I_{p,m}^p(a+1, c, \mu) f(z) \right) + \frac{p-\xi}{p} \left( z^p I_{p,m}^p(a, c, \mu) f(z) \right) \prec q(z) + \frac{\mu \xi}{p(a-p\mu)} zq'(z), \quad (3.2) \]

or

\[ \Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max \left\{ 0, -\frac{p}{\mu} \Re \left\{ \frac{c - p\mu - 1}{\xi} \right\} \right\}, \quad (3.3) \]

\[ \frac{\xi}{p} \left( z^p I_{\lambda,\ell}^{p,m}(a, c-1, \mu) f(z) \right) + \frac{p-\xi}{p} \left( z^p I_{\lambda,\ell}^{p,m}(a, c, \mu) f(z) \right) \prec q(z) + \frac{\mu \xi}{p(c-p\mu-1)} zq'(z), \quad (3.4) \]

or

\[ \Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max \left\{ 0, -\frac{p\ell}{\lambda} \Re \left\{ \frac{1}{\xi} \right\} \right\}, \quad (3.5) \]

\[ \frac{\xi}{p} \left( z^p I_{\lambda,\ell}^{p,m-1}(a, c, \mu) f(z) \right) + \frac{p-\xi}{p} \left( z^p I_{\lambda,\ell}^{p,m}(a, c, \mu) f(z) \right) \prec q(z) + \frac{\lambda \xi}{\ell_p} zq'(z). \quad (3.6) \]

Then

\[ z^p I_{p,m}^p(a, c, \mu) f(z) \prec q(z), \quad (3.7) \]

and \( q(z) \) is the best dominant of (3.7).

**Proof.** Let

\[ k(z) = z^p I_{p,m}^p(a, c, \mu) f(z), \quad (3.8) \]

then it is easily to show that \( k(z) \) is analytic in \( U \) and \( k(0) = 1 \). Differentiating both sides of (3.8) with respect to \( z \), followed by applications of the identities (2.1), (2.2) and (2.3) yield respectively

\[ z^p I_{\lambda,\ell}^{p,m}(a+1, c, \mu) f(z) = k(z) + \frac{\mu}{a-p\mu} z k'(z), \quad (3.9) \]

\[ z^p I_{\lambda,\ell}^{p,m}(a, c-1, \mu) f(z) = k(z) + \frac{\mu}{c-p\mu - 1} z k'(z), \quad (3.10) \]

and

\[ z^p I_{\lambda,\ell}^{p,m-1}(a, c, \mu) f(z) = k(z) + \frac{\lambda}{\ell} z k'(z). \quad (3.11) \]

Now, the subordination conditions (3.2), (3.4), and (3.6) are respectively equivalent to

\[ k(z) + \frac{\mu \xi}{p(a-p\mu)} z k'(z) \prec q(z) + \frac{\mu \xi}{p(a-p\mu)} z q'(z), \quad (3.12) \]

\[ k(z) + \frac{\mu \xi}{p(c-p\mu - 1)} z k'(z) \prec q(z) + \frac{\mu \xi}{p(c-p\mu - 1)} z q'(z), \quad (3.13) \]
and
\[ k(z) + \frac{\xi \lambda}{p \ell} z k'(z) < q(z) + \frac{\xi \lambda}{p \ell} z q'(z). \]  
(3.14)

Therefore, applying Lemma 3 to each of the subordination conditions (3.12), (3.13) and (3.14) with appropriate choices of \( \delta \) and \( \gamma \) we get the assertion (3.7) of Theorem 1. Then the proof of Theorem 1 is completed.

Putting \( q(z) = \frac{1 + Az}{1 + Bz} \) in Theorem 1, we obtain the following corollary:

**Corollary 1.** Let \( \xi \in \mathbb{C}^* \). Let the function \( f \in \Sigma_p \). Suppose any one of the following pairs of conditions is satisfied:

\[ \frac{|B| - 1}{|B| + 1} < \frac{p}{\mu} \Re \left\{ \frac{a - \mu}{\xi} \right\}, \]  
(3.15)

or

\[ \frac{|B| - 1}{|B| + 1} < \frac{p}{\mu} \Re \left\{ \frac{c - \mu - 1}{\xi} \right\}, \]  
(3.17)

or

\[ \frac{|B| - 1}{|B| + 1} < \frac{p \ell}{\lambda} \Re \left\{ \frac{1}{\xi} \right\}, \]  
(3.19)

Then

\[ z^p I_{\lambda, \ell}^{p,m}(a, c, \mu) f(z) < \frac{1 + Az}{1 + Bz}, \]  
(3.21)

and \( \frac{1 + Az}{1 + Bz} \) is the best dominant of (3.21).

**Proof.** Upon setting \( q(z) = \frac{1 + Az}{1 + Bz} \), we see that

\[ 1 + \frac{zq''(z)}{q'(z)} = \frac{1 - Bz}{1 + Bz}, \]

then, we get

\[ \Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \frac{1}{1 + |B|} \] (\( z \in U \)).

Consequently, the hypotheses (3.15), (3.17) and (3.19) imply the conditions (3.1), (3.3), and (3.5) respectively of Theorem 1. Therefore, the assertion (3.21) follows from Theorem 1. The proof of Corollary 1 is completed.
Taking \( p = A = 1 \) and \( B = -1 \) in Corollary 1, we obtain the following corollary:

**Corollary 2.** Let \( \xi \in \mathbb{C}^* \). Let the function \( f \in \Sigma \). Suppose any one of the following pairs of conditions is satisfied:

\[
\Re \left\{ \frac{a - \mu}{\xi} \right\} > 0, \tag{3.22}
\]

\[
\xi \left( z I_{m, \ell}^m (a+1, c, \mu) f(z) \right) + (1-\xi) \left( z I_{m, \ell}^m (a, c, \mu) f(z) \right) \prec \frac{1 + z}{1 - z} + \frac{\mu \xi}{a - \mu} \frac{2 z}{(1 - z)^2}, \tag{3.23}
\]

or

\[
\Re \left\{ \frac{c - \mu - 1}{\xi} \right\} > 0, \tag{3.24}
\]

\[
\xi \left( z I_{m, \ell}^m (a, c-1, \mu) f(z) \right) + (1-\xi) \left( z I_{m, \ell}^m (a, c, \mu) f(z) \right) \prec \frac{1 + z}{1 - z} + \frac{\mu \xi}{c - \mu - 1} \frac{2 z}{(1 - z)^2}, \tag{3.25}
\]

or

\[
\Re \left\{ \frac{1}{\xi} \right\} > 0, \tag{3.26}
\]

\[
\xi \left( z I_{m, \ell}^m (a, c, \mu) f(z) \right) + (1-\xi) \left( z I_{m, \ell}^m (a, c, \mu) f(z) \right) \prec \frac{1 + z}{1 - z} + \frac{\lambda \xi}{\ell} \frac{2 z}{(1 - z)^2}, \tag{3.27}
\]

Then

\[
z I_{m, \ell}^m (a, c, \mu) f(z) \prec \frac{1 + z}{1 - z}, \tag{3.28}
\]

and \( \frac{1 + z}{1 - z} \) is the best dominant of (3.27).

Taking \( a = c \) and \( m = 0 \) in Corollary 2, we obtain the following corollary:

**Corollary 3.** Let \( \xi \in \mathbb{C}^* \). Let the function \( f \in \Sigma \). Suppose any one of the following pairs of conditions is satisfied:

\[
\Re \left\{ \frac{a - \mu}{\xi} \right\} > 0, \tag{3.29}
\]

\[
\frac{\mu \xi}{a - \mu} z (zf(z))' + zf(z) \prec \frac{1 + z}{1 - z} + \frac{\mu \xi}{a - \mu} \frac{2 z}{(1 - z)^2}, \tag{3.30}
\]

or

\[
\Re \left\{ \frac{c - \mu - 1}{\xi} \right\} > 0, \tag{3.31}
\]

\[
\frac{\mu \xi}{c - \mu - 1} z (zf(z))' + zf(z) \prec \frac{1 + z}{1 - z} + \frac{\mu \xi}{c - \mu - 1} \frac{2 z}{(1 - z)^2}, \tag{3.32}
\]
or
\[
\Re \left\{ \frac{1}{\xi} \right\} > 0, \tag{3.33}
\]
\[
\frac{\lambda \xi}{\ell} z (zf(z))' + zf(z) < \frac{1 + z + \lambda \xi}{1 - z} \frac{2z}{(1 - z)^2}. \tag{3.34}
\]
Then
\[
zf(z) \prec \frac{1 + z}{1 - z}, \tag{3.35}
\]
and \( \frac{1 + z}{1 - z} \) is the best dominant of (3.35).

Also, we introduce another subordination theorem as follows:

**Theorem 2.** Let \( q(z) \) be a non zero univalent function in \( U \) with \( q(0) = 1 \). Let \( \eta \in \mathbb{C}^* \) and \( \tau, \kappa \in \mathbb{C} \) with \( \tau + \kappa \neq 0 \). Let \( f \in \Sigma_p \) and suppose that \( f \) and \( q \) satisfy the conditions:
\[
\frac{\tau z^p I_{\lambda,\ell}^{p,m}(a+1, c, \mu) f(z) + \kappa z^p I_{\lambda,\ell}^{p,m}(a, c, \mu) f(z)}{\tau + \kappa} \neq 0 \quad (z \in U),
\]
and
\[
\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\} > 0 \quad (z \in U). \tag{3.36}
\]
If
\[
\eta \left[ p + \frac{\tau z (I_{\lambda,\ell}^{p,m}(a+1, c, \mu) f(z))' + \kappa z (I_{\lambda,\ell}^{p,m}(a, c, \mu) f(z))'}{\tau I_{\lambda,\ell}^{p,m}(a+1, c, \mu) f(z) + \kappa I_{\lambda,\ell}^{p,m}(a, c, \mu) f(z)} \right] < \frac{zq'(z)}{q(z)}, \tag{3.37}
\]
then
\[
\left[ \frac{\tau z^p I_{\lambda,\ell}^{p,m}(a+1, c, \mu) f(z) + \kappa z^p I_{\lambda,\ell}^{p,m}(a, c, \mu) f(z)}{\tau + \kappa} \right] ^\eta < q(z), \tag{3.38}
\]
and \( q(z) \) is the best dominant of (3.38).

**Proof.** In view of Lemma 2, we set
\[
\theta(w) = 0 \quad \text{and} \quad \varphi(w) = \frac{1}{w}.
\]
thus
\[
Q(z) = zq'(z)\varphi(q(z)) = \frac{zq'(z)}{q(z)} \quad \text{and} \quad h(z) = Q(z).
\]
By hypothesis (3.36), we note that \( Q(z) \) is univalent, moreover
\[
\Re \left\{ \frac{zQ'(z)}{Q(z)} \right\} = \Re \left\{ \frac{z \left( \frac{zq'(z)}{q(z)} \right)'}{\frac{zq'(z)}{q(z)}} \right\} = \Re \left\{ 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\} > 0 \quad (z \in U),
\]

8
then function \( Q(z) \) is also starlike in \( U \). We furthermore get that
\[
\text{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} > 0 \quad (z \in U).
\]

Next, let the function \( p \) be defined by
\[
p(z) = \frac{\tau z P^p_{\lambda,\ell} (a+1, c, \mu) f(z) + \kappa z P^p_{\lambda,\ell} (a, c, \mu) f(z)}{\tau + \kappa} \quad (z \in U).
\]

Then \( p \) is analytic in \( U \), \( p(0) = q(0) = 1 \) and
\[
\frac{zp'(z)}{p(z)} = \eta \left[ p + \frac{\tau z (P^p_{\lambda,\ell} (a+1, c, \mu) f(z))^\prime + \kappa z (P^p_{\lambda,\ell} (a, c, \mu) f(z))^\prime}{\tau P^p_{\lambda,\ell} (a+1, c, \mu) f(z) + \kappa P^p_{\lambda,\ell} (a, c, \mu) f(z)} \right].
\]

Using (3.40) in (3.37), we have
\[
\frac{zp'(z)}{p(z)} < \frac{zq'(z)}{q(z)},
\]
which is also equivalent to
\[
zp'(z) \varphi(p(z)) < zq'(z) \varphi(q(z)),
\]
or
\[
\theta(p(z)) + zp'(z) \varphi(p(z)) < \theta(q(z)) + zq'(z) \varphi(q(z)).
\]

Therefore, by Lemma 2, we have
\[
p(z) \prec q(z),
\]
and \( q(z) \) is the best dominant. This is precisely the assertion in (3.38). The proof of Theorem 2 is completed.

Taking \( \tau = 0 \), \( \kappa = 1 \) and \( q(z) = \frac{1 + A z}{1 + B z} \) in Theorem 2, we obtain the following corollary.

**Corollary 4.** Let \( \eta \in \mathbb{C}^* \). Let \( f \in \Sigma_p \) and suppose that \( f \) satisfies the conditions:
\[
z^p P^p_{\lambda,\ell} (a, c, \mu) f(z) \neq 0 \quad (z \in U),
\]
if
\[
\eta \left[ p + \frac{z (P^p_{\lambda,\ell} (a, c, \mu) f(z))^\prime}{P^p_{\lambda,\ell} (a, c, \mu) f(z)} \right] \prec \frac{(A - B) z}{(1 + A z) (1 + B z)},
\]
then
\[
[z^p P^p_{\lambda,\ell} (a, c, \mu) f(z)]^\eta \prec \frac{1 + A z}{1 + B z},
\]
9
and $\frac{1+Az}{1+Bz}$ is the best dominant of (3.42).

Taking $p = A = 1$ and $B = -1$ in Corollary 4, we obtain the following corollary:

**Corollary 5.** Let $\eta \in \mathbb{C}^*$. Let $f \in \Sigma$ and suppose that $f$ satisfies the conditions:

$$zI^m_{\lambda,\ell}(a, c, \mu)f(z) \neq 0 \quad (z \in U),$$

if

$$\eta \left[ 1 + \frac{z (I^m_{\lambda,\ell}(a, c, \mu)f(z))'}{I^m_{\lambda,\ell}(a, c, \mu)f(z)} \right] < \frac{2z}{(1 - z^2)},$$

then

$$[zI^m_{\lambda,\ell}(a, c, \mu)f(z)]^\eta < \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant of (3.44).

Taking $a = c$, $\eta = 1$ and $m = 0$ in Corollary 5, we obtain the following corollary:

**Corollary 6.** Let $f \in \Sigma$ and suppose that $f$ satisfies the conditions:

$$zf(z) \neq 0 \quad (z \in U),$$

if

$$1 + \frac{zf'(z)}{f(z)} < \frac{2z}{(1 - z^2)},$$

then

$$zf(z) < \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant of (3.46).

Taking $\tau = 1$, $\kappa = 0$ and $q(z) = \frac{1+Az}{1+Bz}$ in Theorem 2, we obtain the following corollary.

**Corollary 7.** Let $\eta \in \mathbb{C}^*$. Let $f \in \Sigma_p$ and suppose that $f$ satisfies the conditions:

$$z^pI^{p,m}_{\lambda,\ell}(a+1, c, \mu)f(z) \neq 0 \quad (z \in U),$$

if

$$\eta \left[ p + \frac{z (I^{p,m}_{\lambda,\ell}(a+1, c, \mu)f(z))'}{I^{p,m}_{\lambda,\ell}(a+1, c, \mu)f(z)} \right] < \frac{(A-B) z}{(1+Az)(1+Bz)},$$

then

$$[z^pI^{p,m}_{\lambda,\ell}(a+1, c, \mu)f(z)]^\eta < \frac{1+Az}{1+Bz},$$
and $\frac{1+Bz}{1+Az}$ is the best dominant of (3.48).

Taking $A = p = 1$ and $B = -1$ in Corollary 7, we obtain the following corollary.

**Corollary 8.** Let $\eta \in \mathbb{C}^*$. Let $f \in \Sigma$ and suppose that $f$ satisfies the conditions:

$$zI_{\lambda,\ell}^m(a+1, c, \mu)f(z) \neq 0 \quad (z \in U),$$

if

$$\eta \left[ 1 + z \frac{I_{\lambda,\ell}^m(a+1, c, \mu)f(z)}{I_{\lambda,\ell}^m(a+1, c, \mu)f(z)} \right] \prec \frac{2z}{1 - z^2},$$

(3.49)

then

$$\left[ zI_{\lambda,\ell}^m(a+1, c, \mu)f(z) \right]^\eta \prec \frac{1+z}{1-z},$$

(3.50)

and $\frac{1+z}{1-z}$ is the best dominant of (3.50).

Taking $a = c$, $\eta = 1$ and $m = 0$ in Corollary 8, we obtain the following corollary:

**Corollary 9.** Let $f \in \Sigma$ and suppose that $f$ satisfies the conditions:

$$z^2f'(z) + \frac{a}{\mu}zf(z) \neq 0 \quad (z \in U),$$

if

$$1 + z \left( z^2f'(z) + \frac{a}{\mu}zf(z) \right) \prec \frac{2z}{1 - z^2},$$

(3.51)

then

$$\frac{\mu}{a - \mu} \left( z^2f'(z) + \frac{a}{\mu}zf(z) \right) \prec \frac{1+z}{1-z},$$

(3.52)

and $\frac{1+z}{1-z}$ is the best dominant of (3.52).

Another theorem is introduced as follows:

**Theorem 3.** Let $\eta \in \mathbb{C}^*$ and $\zeta, \tau, \kappa \in \mathbb{C}$ with $\tau + \kappa \neq 0$. Let $q(z)$ be a univalent function in $U$ with $q(0) = 1$ and

$$\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max \{0, -\Re \{\zeta\} \} \quad (z \in U).$$

(3.53)

Let $f \in \Sigma_p$ and suppose that $f$ satisfies the condition

$$\frac{\tau z^pI_{\lambda,\ell}^{p,m}(a+1, c, \mu)f(z) + \kappa z^pI_{\lambda,\ell}^{p,m}(a, c, \mu)f(z)}{\tau + \kappa} \neq 0 \quad (z \in U),$$
Set
\[\Omega(z) = \left[\tau z^p I_{\lambda,\ell}^{\mu,m}(a+1,c,\mu)f(z) + \kappa z^p I_{\lambda,\ell}^{\mu,m}(a,c,\mu)f(z)\right]^\eta \cdot \left[\tau \eta + \eta \left(\frac{\tau z^p I_{\lambda,\ell}^{\mu,m}(a+1,c,\mu)f(z) + \kappa z^p I_{\lambda,\ell}^{\mu,m}(a,c,\mu)f(z)}{\tau z^p I_{\lambda,\ell}^{\mu,m}(a+1,c,\mu)f(z) + \kappa z^p I_{\lambda,\ell}^{\mu,m}(a,c,\mu)f(z)} + p\right)\right].\] (3.54)

If
\[\Omega(z) \prec \zeta q(z) + zq'(z),\] (3.55)
then
\[\left[\tau z^p I_{\lambda,\ell}^{\mu,m}(a+1,c,\mu)f(z) + \kappa z^p I_{\lambda,\ell}^{\mu,m}(a,c,\mu)f(z)\right]^\eta \prec q(z),\] (3.56)
and \(q(z)\) is the best dominant of (3.56).

Proof. In view of Lemma 2, we set
\[\theta(w) = \zeta w \text{ and } \varphi(w) = 1 \quad (w \in \mathbb{C}),\]
thus
\[Q(z) = zq'(z)\varphi(q(z)) = zq'(z) \quad \text{and} \quad h(z) = \zeta q(z) + zq'(z).\]

Then, we note that \(Q(z)\) is univalent. Moreover, using (3.53), we find that
\[\text{Re}\left\{\frac{zQ'(z)}{Q(z)}\right\} = \text{Re}\left\{\frac{z(zq'(z))'}{zq'(z)}\right\} = \text{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > 0 \quad (z \in U),\]
then function \(Q(z)\) is also starlike in \(U\). Also, using (3.53), we get that
\[\text{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \text{Re}\left\{1 + \zeta + \frac{zq''(z)}{q'(z)}\right\} > 0 \quad (z \in U).\]

Furthermore, by using the expression of \(p(z)\) defined by (3.39) and the expression of \(zp'(z)\) defined by (3.40) we have
\[
\theta(p(z)) + zp'(z)\varphi(p(z)) = \zeta p(z) + zp'(z)
= \left[\tau z^p I_{\lambda,\ell}^{\mu,m}(a+1,c,\mu)f(z) + \kappa z^p I_{\lambda,\ell}^{\mu,m}(a,c,\mu)f(z)\right]^\eta \cdot \left[\tau \eta + \eta \left(\frac{\tau z^p I_{\lambda,\ell}^{\mu,m}(a+1,c,\mu)f(z) + \kappa z^p I_{\lambda,\ell}^{\mu,m}(a,c,\mu)f(z)}{\tau z^p I_{\lambda,\ell}^{\mu,m}(a+1,c,\mu)f(z) + \kappa z^p I_{\lambda,\ell}^{\mu,m}(a,c,\mu)f(z)} + p\right)\right]
= \Omega(z).
\]

The hypothesis (3.55) is now equivalent to
\[\zeta p(z) + zp'(z) \prec \zeta q(z) + zq'(z),\]
or
\[ \theta(p(z)) + zp'(z) \varphi(p(z)) < \theta(q(z)) + zq'(z) \varphi(q(z)). \]

Finally, an application of Lemma 2 yields
\[ p(z) \prec q(z) \]
and \( q(z) \) is the best dominant. This is precisely the assertion in (3.56). The proof of Theorem 3 is completed.

Taking \( \tau=0, \kappa=1 \) and \( q(z) = \frac{1+Az}{1+Bz} \) in Theorem 3, we obtain the following corollary.

**Corollary 10.** Let \( \eta \in \mathbb{C}^* \) and \( \zeta = \frac{|B|-1}{|B|+1} \). Let \( f \in \Sigma_p \) and suppose that \( f \) satisfies the conditions
\[ \left[ z p_{\lambda,\ell}^{,m}(a,c,\mu) f(z) \right] \eta \cdot \left[ \zeta + \eta \left( p + \frac{z f'(z) f(z)}{p_{\lambda,\ell}^{,m}(a,c,\mu) f(z)} \right) \right] \prec \zeta \frac{1+Az}{1+Bz} + \frac{(A-B)z}{(1+Bz)^2}, \]
then
\[ \left[ z p_{\lambda,\ell}^{,m}(a,c,\mu) f(z) \right] \eta \prec \frac{1+Az}{1+Bz}, \]
and \( \frac{1+Az}{1+Bz} \) is the best dominant of (3.58).

Taking \( p = A = 1, B = -1 \) and \( a = c \) in Corollary 10, we obtain the following corollary.

**Corollary 11.** Let \( \eta \in \mathbb{C}^* \). Let \( f \in \Sigma \) and suppose that \( f \) satisfies the conditions
\[ z f(z) \neq 0 \quad (z \in U), \]
and
\[ \left[ z f(z) \right] \eta \cdot \left[ \eta \left( 1 + \frac{z f'(z)}{f(z)} \right) \right] \prec \frac{2z}{(1-z)^2}, \]
then
\[ \left[ z f(z) \right] \eta \prec \frac{1+2z}{1-z}, \]
and \( \frac{1+z}{1-z} \) is the best dominant of (3.60).

**Remark 1.** The result obtained in Corollary 11 coincides with the recent result due to Mishra et al. [14, Corollary 4.9].

Taking \( \eta = 1 \) in Corollary 11, we obtain the following corollary.
**Corollary 12.** Let $f \in \Sigma$ and suppose that $f$ satisfies the conditions

$$zf(z) \neq 0 \ (z \in U), \quad (3.61)$$

and

$$zf(z) + z^2 f'(z) \prec \frac{2z}{(1-z)^2}, \quad (3.62)$$

then

$$zf(z) \prec \frac{1+z}{1-z}, \quad (3.63)$$

and $\frac{1+z}{1-z}$ is the best dominant of (3.63).

Taking $r=1$, $\kappa=0$ and $q(z)=\frac{1+Az}{1+Bz}$ in Theorem 3, we obtain the following corollary.

**Corollary 13.** Let $\eta \in \mathbb{C}^*$ and $\zeta = \frac{|B|-1}{|B|+1}$. Let $f \in \Sigma_\mu$ and suppose that $f$ satisfies the conditions

$$z^p I_{\lambda,\ell}^p (a+1, c, \mu) f(z) \neq 0 \ (z \in U), \quad (3.64)$$

and

$$\left[z^p I_{\lambda,\ell}^p (a+1, c, \mu) f(z)\right]_\eta \cdot \left[\zeta + \eta \left(\frac{z^p I_{\lambda,\ell}^p (a+1, c, \mu) f(z)}{I_{\lambda,\ell}^p (a+1, c, \mu) f(z)} + p\right)\right] \prec \zeta \frac{1+Az}{1+Bz} + \frac{(A-B)z}{(1+Bz)\zeta}, \quad (3.65)$$

then

$$\left[z^p I_{\lambda,\ell}^p (a+1, c, \mu) f(z)\right]_\eta \prec \frac{1+Az}{1+Bz}, \quad (3.66)$$

and $q(z)$ is the best dominant of (3.66).

Taking $p = A = \eta = 1$, $B = -1$ and $a = c$ in Corollary 13, we obtain the following corollary.

**Corollary 14.** Let $f \in \Sigma$ and suppose that $f$ satisfies the conditions

$$z^2 f'(z) + \frac{a}{\mu} zf(z) \neq 0 \ (z \in U), \quad (3.67)$$

and

$$\frac{\mu z}{a-\mu} \left(z \left[z f'(z) + \frac{a}{\mu} f(z)\right]\right)' \prec \frac{2z}{(1-z)^2}, \quad (3.68)$$

then

$$\frac{\mu}{a-\mu} \left(z^2 f'(z) + \frac{a}{\mu} zf(z)\right) \prec \frac{1+z}{1-z}, \quad (3.69)$$

and $\frac{1+z}{1-z}$ is the best dominant of (3.69).

**Remark 2.** Specializing the parameters in Theorems 1, 2 and 3 as mentioned before, we can obtain the corresponding subordination properties of Liu-Srivastava operator [11], Cho-Kwon-Srivastava operator [5], Yuan-Liu-Srivastava operator [20], Uralegaddi-Somanatha operator [19], and others.
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