ARITHMETIC PROGRESSIONS IN MIDDLE $\frac{1}{N}$ th CANTOR SETS

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First to fix some notation. Let $X \subset [0,1]$ be the middle $\frac{1}{N}$ th Cantor set. That is $X = \cap_{k=1}^N C_k$ where $C_0 = [0,1]$ and $C_{k+1}$ is obtained by removing the middle $\frac{1}{N}$ th from each connected component of $C_k$. Notice $C_k$ consists of $2^k$ intervals of size $(\frac{N-1}{2N})^k$. The gaps between these intervals have size at least $\frac{1}{N}(\frac{N-1}{2N})^{k-1}$. Let $a_1, ..., a_r$ be numbers and $X + a_r$ be considered modulo 1. For $\delta > 0$ let $X_\delta \supset X$ be the set obtained by deleting the middle $N^{th}$ of size at least $\delta$. This is a finite union of intervals.

**Theorem 1.** For any $a_1, ..., a_r$ we have that $\cap_{i=1}^N \frac{X + a_i}{N^{k+1}} \neq 0$.

That is, the middle $\frac{1}{N}$ th Cantor set contains arithmetic progressions and in fact more general configurations of length proportional to $\frac{N}{\log(N)}$.

Broderick, Fishman and Simmons have subsequently proved this statement using variants of Schmidt’s game [Theorem 2.1].

**Definition 2.** We say an interval $J$ of length $\frac{1}{N}$ is $k$-good if \[ J \cap \frac{X + a_i}{N^{k+1}} \neq \emptyset \]

contains $\frac{N}{k+1}$ disjoint intervals of size $\frac{1}{N}$.

We prove the Theorem by induction using the following Proposition:

**Proposition 3.** If $J$ is $k$-good then it contains a subinterval $J'$ which is $k+1$-good.

Notice that by compactness if $J$ is a closed interval and \[ J \cap_{i=1}^N \frac{X + a_i}{N^{k+1}} \neq 0 \]

for all $k$ then \[ J \cap_{i=1}^N \frac{X + a_i}{N^{k+1}} \neq 0. \]

**Lemma 4.** Let $L > k$. If $J$ is an interval of size $(\frac{N-1}{2N})^k$ and $I_1, ..., I_{2L-1}$ be the intervals removed from $C_{L-1}$ to obtain $C_L$. Then $|\{r: I_r \cap J \neq \emptyset\}| \leq 2^{L-k-1}$.

**Proof.** This is maximized if $J$ is a subinterval of $X_{\frac{1}{N}}(\frac{N-1}{2N})^{k-1}$. The estimate is achieved for those. To see that it is maximized for subintervals of $X_{\frac{1}{N}}(\frac{N-1}{2N})^{k-1}$ let us consider a $J$ with $|J| = (\frac{N-1}{2N})^k$ so that the intersections with $I_1, ..., I_{2L-1}$ are not contained in one subinterval of $X_{\frac{1}{N}}(\frac{N-1}{2N})^{k-1}$. So $J$ is contained in $U \cup G \cup V$ where $U$ and $V$ are subintervals of $X_{\frac{1}{N}}(\frac{N-1}{2N})^{k-1}$ and $G \subset (0,1] \setminus X_{\frac{1}{N}}(\frac{N-1}{2N})^{k-1}$ is the gap of size at least $\frac{1}{N}(\frac{N-1}{2N})^{k-1}$ between them. We assume $U$ is on the left of $V$. First notice no $I_r$ is contained in $G$. Now if $I_r \cap J \cap V \neq \emptyset$ then $J = U + c$ where $c - |G| \geq c - \frac{1}{N}(\frac{N-1}{2N})^{k-1} \geq d(I_r, q)$ where $q$ is the left endpoint of $V$. Let $p
be the left endpoint of $U$. There exist $I_L$ with $d(I_L, p) = d(I_r, q)$. Since $|I_L| < |G|$ it follows that $I_L \cap (U + c) = I_L \cap J = \emptyset$. So by sliding $U$ any new intersection with an $I_j$ occurs only after a previous intersection with some $I_r$ has been lost. □

**Corollary 5.** If $J$ is any interval of size $\frac{1}{N^k}$, and $I_1, \ldots, I_r$ are the intervals of length exactly $\frac{1}{N^{k+1}}\delta$ deleted to form $X_{s+1}$ then

$$|\{j : I_j \cap J \neq \emptyset\}| \leq 3 \cdot 2^{\log \frac{2N}{N^k} \left(\frac{1}{2}\right)}.$$

*Proof.* Let $p = \lceil \log \frac{2N}{N^k} \rceil$. $J$ contains at most parts of 3 subintervals of size $\left(\frac{N^k+1}{2N}\right)^p$. Since there are at most $\lceil \frac{1}{2} \rceil$ steps in the inductive process to form $X$ between deleting intervals of size $\frac{1}{N^k}$ and $\frac{\delta}{N^k}$, The corollary follows by applying the lemma. □

*Proof of Proposition.* Consider the subintervals of $J \cap \bigcup_{i=1}^{\frac{N}{\log 2(N)}} X_{\frac{1}{N^k} + a_i}$. By the assumption that $J$ is $k$-good we have at most $\frac{N^k}{N^2}$ disjoint intervals organized into $\frac{N^2}{N^k}$ blocks of $N$ consecutive intervals. (We may have other intervals too.) From $X_{\frac{1}{N^k} + a_i}$ to $X_{\frac{1}{N^k} + a_i + 1}$ we can delete portions of at most

$$3 \log \frac{2N}{N^k+1} \left(\frac{N}{N^2}\right) + 3N \log \frac{2N}{N^k+1} N \leq 3 \cdot 2^{\log \frac{2N}{N^k+1} \left(\frac{N}{N^2}\right) + \log_2 N} + 3N \log_2 N \leq 9N \log_2 N$$

of them. This estimate follows because $k$ intervals of total measure $c$ can intersect at most $2k + \delta^{-1}c$ disjoint intervals of size $\delta$. There are at most $\log \frac{2N}{N^k+1} \left(\frac{N}{N^2}\right)$ steps, and at each step we remove at most $3 \cdot 2^{\log \frac{2N}{N^k+1} \left(\frac{N}{N^2}\right)}$ intervals with total measure at most $\frac{1}{N^k}$.

We do this for each $X + a_i$ and can delete portions of at most $\frac{N^2}{N^k}$ intervals of size $\frac{1}{N^k}$. So by the pigeon hole principle one of the $\frac{N^2}{N^k}$ blocks has at least half of its intervals. This is a $k + 1$-good subinterval of $J$. □

**Remark 6.** The techniques of this note are a little robust and imply the existence of configurations for bilipschitz images of the middle $\frac{1}{N}$ cantor set where the bilipschitz constant is not too large depending on $N$. It is natural to ask if there exists $N$ so that the image of the middle $\frac{1}{N}$ cantor set under any bilipschitz map contains $3$ term arithmetic progressions.

**Question 1.** Is the bound found in this note on the order of the correct one? Is it possible to find arithmetic progressions say of order $N$?

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**References**

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