Affine Invariant Maps for Log-Concave Functions

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Abstract
Affine invariant points and maps for sets were introduced by Grünbaum to study the symmetry structure of convex sets. We extend these notions to a functional setting. The role of symmetry of the set is now taken by evenness of the function. We show that among the examples for affine invariant points are the classical center of gravity of a log-concave function and its Santaló point. We also show that the recently introduced floating functions and the John- and Löwner functions are examples of affine invariant maps. Their centers provide new examples of affine invariant points for log-concave functions.

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1 Introduction

Affine invariant quantities are central in affine differential geometry and convex geometry and they and their associated inequalities have far reaching consequences for many

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other areas of mathematics. So, not surprisingly, the recent surge in the study of new affine invariants has contributed greatly to recent progress in understanding structural properties of convex bodies and resulted in numerous applications, from approximation of convex bodies by polytopes [1–4], to statistics [5], to information theory [6–12] and even quantum information theory [13–15]. Examples of such new invariants are the $L_p$-affine surface areas of the $L_p$-Brunn Minkowski theory, initiated by Lutwak in his groundbreaking paper [16], see also [17–23], the Orlicz Brunn Minkowski theory [24–26], the theory of valuations [27–31] and the theory of Fourier transformation (see e.g., Koldobsky’s book [32]).

Affine invariant quantities are intimately related to a choice of position of a convex body. The right choice of position is important for the study the related isoperimetric inequalities. These positions include the isotropic position, which arose from classical mechanics of the 19th century and which is related to a famous open problem in convex geometry, the hyperplane conjecture (see, e.g., the survey [33]). For a very long time the best results available there were due to Bourgain [34] and Klartag [35]. Recent progress has been made by Chen [36]. Other positions are the John position, also called maximal volume ellipsoid position and the Löwner position, also called minimal volume ellipsoid position. John and Löwner position are related to the Brascamp–Lieb inequality and its reverse [37, 38], to K. Ball’s sharp reverse isoperimetric inequality [39], to the notion of volume ratio [40, 41], which is defined as the $n$-th root of the volume of a convex body divided by the volume of its John ellipsoid and which finds applications in functional analysis and Banach space theory [41–44]. John and Löwner position are even relevant in quantum information theory [13, 14, 45].

A key structural property of convex bodies is that of symmetry. It is relevant in many problems. We only mention the celebrated Blaschke Santaló inequality and its reverse, the Mahler conjecture, about the the minimal volume product of polar reciprocal convex bodies. Mahler’s conjecture is still open in dimensions 4 and higher. See e.g., [42, 46–50] for partial results. A systematic study of symmetry was initiated by Grünbaum in his seminal paper [51]. The symmetry structure of convex bodies is closely related to the affine structure of the bodies. Indeed, the crucial notion in Grünbaums’s work is an affine notion, that of affine invariant point. The centroid and the Santaló point of convex bodies (with respect to which the volume of the polar body attains a minimum) are classical examples of affine invariant points. It is this notion that allows to analyze the symmetry situation. In a nutshell: the more affine invariant points, the fewer symmetries. Grünbaums’s work has been further developed recently in [52–54].

Probabilistic methods have become extremely useful in convex geometry. In this context, log-concave functions arise naturally from the uniform measure on convex bodies. Extensive research has been devoted within the last ten years to extend the concepts and inequalities from convex bodies to the setting of functions. In fact, it was observed early that the Prékopa–Leindler inequality (see, e.g., [55–58]) is the functional analog of the Brunn–Minkowski inequality (see, e.g., [59]) for convex bodies. Much progress has been made since and functional analogs of many other geometric notions and inequalities were established. Among them are the functional Blaschke–Santaló inequality [60–63] and its reverse [64], a functional affine isoperimetric inequality for log-concave functions which can be viewed as an inverse...
log-Sobolev inequality for entropy [65, 66], Alexandrov–Fenchel type inequalities [67, 68], functional analogs of the floating body [69], John ellipsoids [70, 71] and L"owner ellipsoids [72, 73] and a theory of valuations, an important concept for convex bodies (e.g., [19, 27–31]), is currently being developed in the functional setting, e.g., [74–76]. More examples can be found in e.g., [59, 77–84]).

In this paper, we extend the notion of affine invariant point and affine invariant map to the functional setting. We start out by laying the groundwork and provide the needed tools. We then put forward the definitions of affine contravariant points and affine covariant mappings for log-concave functions and establish some of their basic properties. For instance, the role of symmetry in the setting of convex bodies is now taken by the notion of evenness in the functional setting. We show that the centroid and the Santaló point of a log-concave function are examples of the affine contravariant points and that the newly developed notions of floating function [69], John function [70] and L"owner function [73] are examples of affine covariant mappings. This leads naturally to new affine contravariant points.

2 Notation and Preliminaries

Throughout the paper we will use the following notations. The set of all non-singular affine transformations on $\mathbb{R}^n$ is written as $\mathcal{A}$,

$$\mathcal{A} = \{A = T + a : T \in GL(n), a \in \mathbb{R}^n\}.$$ 

The action of an affine transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ on a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as $Af(x) = f(Ax)$.

For $z \in \mathbb{R}^n$, let $S_z$ be a translation of a function by $z$, that is, for a function $f$,

$$(S_z f)(x) = f(x + z)$$

For $s \in \mathbb{R}$ and a function $f : \mathbb{R}^n \to \mathbb{R}$, we denote by

$$G_f(s) = \{x \in \mathbb{R}^n : f(x) \geq s\}$$

the super-level sets of $f$ and by $\text{epi}(f)$ the epigraph of the function $f$,

$$\text{epi}(f) = \{(x, y) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n, y \geq f(x)\}.$$ 

Let $K$ be a convex body in $\mathbb{R}^n$, i.e., a convex compact subset $K$ of $\mathbb{R}^n$ with nonempty interior, $\text{int}(K)$. We denote by $\text{vol}_n(K)$, or simply $|K|$, the volume of $K$ and by $\mu_K$ the usual surface measure on $\partial K$, the boundary of $K$. It is the restriction of the $n - 1$ dimensional Hausdorff measure to $\partial K$. For convex bodies $K$ and $L$, their Hausdorff distance is

$$d_H(K, L) = \min\{\varepsilon : K \subseteq L + \varepsilon B^n_2, L \subseteq K + \varepsilon B^n_2\},$$

where $B^n_2$ is the $n$-dimensional unit ball.
where $B^n_2$ is the Euclidean unit ball. $B^n_2(a, r)$ is the Euclidean ball centered at $a$ with radius $r$. We write $B^n_2(r) = B^n_2(0, r)$. By $\| \cdot \|$ we denote the Euclidean norm on $\mathbb{R}^n$. For a linear operator $T : \mathbb{R}^k \to \mathbb{R}^n$ the operator norm is given by

$$\|T\|_{op} = \sup_{\|x\| \leq 1} \|Tx\|. \quad (4)$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be log-concave if it is of the form $f(x) = e^{-\psi(x)}$ where $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is a convex function. We always consider in this paper log-concave functions that are upper semi continuous, integrable and non-degenerate, i.e., the interior of the support of $f$ is non-empty, $\text{int}(\text{supp} f) \neq \emptyset$. This then implies that $0 < \int_{\mathbb{R}^n} f dx = \|f\|_1 < \infty$. Without loss of generality we may assume that $0 \in \text{int}(\text{supp} f)$. Since $\text{int}(\text{supp} f) \neq \emptyset$ the function $\psi$ is proper, i.e. $\psi(x) < \infty$ for at least one $x$.

We will denote by $LC(\mathbb{R}^n)$ or, in short, by $LC$, the set of non-degenerate, upper semi continuous, integrable, log-concave functions $f$, such that $\psi$ is proper, equipped with the $L_1$-norm,

$$LC = \{ f = e^{-\psi} : \mathbb{R}^n \to \mathbb{R}, \ 0 < \|f\|_1 < \infty \}. \quad (5)$$

We will also need the Legendre transform which we recall now. Let $z \in \mathbb{R}^n$ and let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a convex function. Then

$$L_z\psi(y) = \sup_{x \in \mathbb{R}^n} \{ (x - z, y - z) - \psi(x) \} \quad (6)$$

is the Legendre transform of $\psi$ with respect to $z$ [60, 62]. If $f = e^{-\psi}$ is log-concave, then

$$f^z(y) = \inf_{x \in \text{supp}(f)} \frac{e^{-\langle x - z, y - z \rangle}}{f(x)} = e^{-L_z\psi(y)} \quad (7)$$

is called the dual or polar function of $f$ with respect to $z$. In particular, when $z = 0$,

$$f^0(y) = \inf_{x \in \text{supp}(f)} \frac{e^{-\langle x, y \rangle}}{f(x)} = e^{-L_0\psi(y)},$$

where $L_0$, also denoted by $L$ for simplicity, is the standard Legendre transform.

In the next lemma we collect several well known properties of the generalized Legendre transform. They can be found in e.g., [60] and [62].

**Lemma 1** Let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a convex function. Let $S_z$ be as in (1). Then

(i) $L$ and $L_z$ are involutions, that is, $L(L\psi) = \psi$ and $L_z(L_z\psi) = \psi$.

(ii) $L_z = S_{-z} \circ L \circ S_z$.

(iii) $L(S_z\psi)(y) = L\psi - \langle z, y \rangle$.

(iv) Legendre transform reverses the order relation, i.e., if $\psi_1 \leq \psi_2$, then $L\psi_1 \geq L\psi_2$.
We now list some basic well-known facts on log-concave functions which will be used throughout the paper. More on log-concave functions can be found in e.g., [85].

**Lemma 2** [73] If $f$ is a non-degenerate integrable log-concave function, then $G_f(t)$ is convex and compact and has affine dimension $n$, for $0 < t < \|f\|_\infty$.

A proof of Lemma 2 can be found for instance in [73].

The following fact is a direct corollary of the functional Blaschke–Santaló inequality [60, 61] and the functional reverse Santaló inequality [64, 86].

**Lemma 3** Let $f = e^{-\psi}$ be a non-degenerate, integrable, log-concave function such that $0$ is in the interior of the support of $f$. Then $f^\circ$ is again a non-degenerate, integrable log-concave function, i.e., $0 < \int_{\mathbb{R}^n} f^\circ(x)dx < \infty$, provided that $z$ is in the interior of $\text{supp}(f)$.

A proof of the following two lemmas can be found in [73]. There, and elsewhere, we denote for a function $f$ by $\|f\|_{p,1} \leq p \leq \infty$, its $L_p$-norm.

**Lemma 4** [73] Let $(f_m)_{m \in \mathbb{N}}$ be a sequence of integrable, log-concave functions that converges pointwise to the integrable log-concave function $f$. Then, for every $s$ with $0 < s < \|f\|_\infty$ the sequence of super-level sets $(G_{f_m}(s))_{m=1}^\infty$ converges in Hausdorff metric to the super-level set $G_f(s)$.

**Lemma 5** [73] Let $(f_m)_{m \in \mathbb{N}}$ and $f$ be integrable log-concave functions such that $f_m \to f$ pointwise on $\mathbb{R}^n \setminus \partial \text{supp}(f)$. Then, $\|f_m\|_\infty \to \|f\|_\infty$.

The next lemma is Exercise 10, p. 187 of Folland [87].

**Lemma 6** [87] Suppose $1 \leq p < \infty$. If $f_n, f \in L^p$ and $f_n \to f$ a.e., then $\|f_n - f\|_p \to 0$ iff $\|f_n\|_p \to \|f\|_p$.

**Lemma 7** [85] Let $\psi$ be a convex function on $\mathbb{R}^n$. Then $\psi$ is continuous on the interior of its domain.

The following lemma is a consequence of known facts on convergence of convex and log-concave functions (see [85]) and Lemma 3.2 of [60].

**Lemma 8** Let $(f_m)^\infty_{m=1}$ and $f$ be non-degenerate integrable log-concave functions. Let $(x_m)_{m \in \mathbb{N}}$ and $x$ be in the interior of $\text{supp}(f)$. Then, we have

(i) The sequence $(f_m)^\infty_{m=1}$ converges in $L_1$ to $f$ if and only if $(f_m)^\infty_{m=1}$ converges pointwise to $f$ on $\mathbb{R}^n \setminus \partial \text{supp}(f)$.

(ii) If the sequence $(f_m)^\infty_{m=1}$ converges pointwise to $f$ on $\mathbb{R}^n \setminus \partial \text{supp}(f)$, then $(f_m)^\infty_{m=1}$ converges uniformly on the compact subsets of $\text{supp}(f)$ to $f$.

(iii) If the sequence $(f_m)^\infty_{m=1}$ converges pointwise on $\mathbb{R}^n \setminus \partial \text{supp}(f)$ to $f$ and if the sequence $(x_m)^\infty_{m=1}$ converges in $\mathbb{R}^n$ to $x$, then the sequence $(f_m^x)^\infty_{m=1}$ converges pointwise on $\mathbb{R}^n \setminus \partial \text{supp}(f)$ to $f^x$. In particular, the sequence $(f_m^x)^\infty_{m=1}$ converges in $L_1$ to $f^\circ$. 
3 Affine Contravariant Points and Covariant Mappings for Log-Concave Functions

3.1 The Definitions

Grünbaum [51] (see also Meyer et al. [52]) gave definitions of affine invariant points and affine invariant maps for convex bodies. We now extend those definitions to functions. While they can be defined for any function, we will concentrate in this section on log-concave functions and thus restrict the definition to this class. Note also that formally affine invariant points are maps.

We start with the definition of affine contravariant points for log-concave functions.

**Definition 1** A map $p : LC \to \mathbb{R}^n$ is called an affine contravariant point, if $p$ is continuous and if for every nonsingular affine map $A : \mathbb{R}^n \to \mathbb{R}^n$ one has

$$p(Af) = A^{-1}(p(f)).$$

(8)

Continuity in this definition means that $p(f_m) \to p(f)$ whenever $f_m, f \in LC$ and the sequence $(f_m)_{m \in \mathbb{N}}$ converges to $f$ in the $L_1$-norm.

We put $\mathcal{P}$ be the set of affine contravariant points on $LC$,

$$\mathcal{P} = \{ p : LC \to \mathbb{R}^n \mid p \text{ is an affine contravariant point} \},$$

(9)

and for a fixed function $f \in LC$,

$$\mathcal{P}(f) = \{ p(f) : p \in \mathcal{P} \}. $$

(10)

**Remark 1** (i) The notion of affine invariant point for log-concave functions is an extension of the concept of affine invariant points for convex bodies given in [51, 52]. Indeed, let

$$f(x) = \mathbb{1}_K(x) = e^{-I_K(x)}, \quad \text{where} \quad I_K(x) = \begin{cases} \infty & x \notin K \\ 0 & x \in K \end{cases}$$

be the characteristic function of a convex set $K \subset \mathbb{R}^n$. By Definition 1 we get for every affine map $A : \mathbb{R}^n \to \mathbb{R}^n$ and every affine contravariant point $p$,

$$A^{-1} (p(\mathbb{1}_K)) = p(A \cdot \mathbb{1}_K) = p(\mathbb{1}_{A^{-1}K}).$$

(ii) Note that if $Af = f$ for some affine map $A : \mathbb{R}^n \to \mathbb{R}^n$ and $f \in LC$, then for every $p \in \mathcal{P}$, one has

$$p(f) = p(Af) = A^{-1}(p(f)).$$

It follows that if $f$ is even, i.e., $f(x) = f(-x)$ for all $x$, then we get with $A : x \to -x$ that $p(f) = -p(f)$ for every $p \in \mathcal{P}$ and hence $\mathcal{P}(f) = \{0\}$. 
Thus even functions only have one affine contravariant point, and therefore, within the class of functions, play the role that symmetric convex bodies have in the class of convex bodies, as for those the center of symmetry is the only affine invariant point.

Next, we introduce the notion of affine covariant mappings for functions. There, continuity of a map $P : LC \to LC$ means that $P_{f_m}$ converges to $P_f$ in $L_1$-norm whenever $f, f_m, m \in \mathbb{N}$, are functions in $LC$ such that $f_m \to f$ in $L_1$-norm.

Again, affine covariant mappings can be defined for any function, but we will concentrate on log-concave functions.

**Definition 2** A map $P : LC \to LC$ is called an affine covariant mapping (for functions), if $P$ is continuous and if for every nonsingular affine map $A$ of $\mathbb{R}^n$, one has

$$P(Af) = A(P(f)) \quad (11)$$

We denote by $\mathfrak{A}$ the set of affine covariant function mappings,

$$\mathfrak{A} = \{ P : LC \to LC | P \text{ is affine covariant} \}. \quad (12)$$

**Remark 2** (i) It is easy to see that if $\lambda \in \mathbb{R}$, $p, q \in \mathfrak{P}$ and $P \in \mathfrak{A}$, then $p \circ P \in \mathfrak{P}$ and $(1 - \lambda)p + \lambda q \in \mathfrak{P}$. Thus, $\mathfrak{P}$ is an affine space and for every $f \in LC$, $\mathfrak{P}(f)$ is an affine subspace of $\mathbb{R}^n$. Moreover, for $P, Q \in \mathfrak{A}$, the maps

$$f \to (P \circ Q)(f), \quad (1 - \lambda)p + \lambda q \quad \text{and} \quad \sup[P, Q](f) = \sup[P(f), Q(f)]$$

are affine covariant mappings for functions. In that way, we can obtain many more examples of affine contravariant points and affine covariant mappings.

(ii) Properties (8) and (11) imply in particular that for every translation $S_{x_0}$ by a fixed vector $x_0$, $S_{x_0}(x) = x + x_0$, and for every $f \in LC$,

$$p(S_{x_0}f) = S_{x_0}^{-1} p(f) = p(f) - x_0, \quad \text{for every } p \in \mathfrak{P} \quad (13)$$

and

$$P(f(x + x_0)) = P(T_{x_0} f(x)) = T_{x_0}(P(f))(x) = (P(f))(x + x_0), \quad \text{for every } P \in \mathfrak{A}. \quad (14)$$

### 3.2 Centroid and Santaló Point

**Lemma 9** Let $f \in LC$ and let $(f_m)_{m=1}^\infty$ be a sequence in $LC$ that converges in $L_1$ to $f$. Then there are $t \in \mathbb{R}$ and $\rho > 0$ such that for all $m \in \mathbb{N}$ and all $x \in \mathbb{R}^n$

$$f_m(x) \leq \exp \left( -\frac{\|x\|}{\rho} + t \right). \quad (15)$$
Proof By Lemma 4 the sequence of sets
\[\{x | \psi_m(x) \leq \psi(0) + 2\}\]
converges for \(m \to \infty\) in the Hausdorff metric to
\[\{x | \psi(x) \leq \psi(0) + 2\}\].
As 0 is in the interior of the domain of \(\psi\), there is \(\rho > 0\) and \(m_0\) such that for all \(m \geq m_0\)
\[\{x | \psi_m(x) \leq \psi(0) + 2\} \subseteq B^2_2\left(\frac{\rho}{2}\right)\]  \hspace{1cm} (16)
and, using Lemma 8,
\[|\psi_m(0) - \psi(0)| < \frac{1}{4}.\]  \hspace{1cm} (17)
We show that for all \(x\) with \(\|x\| > \rho\) and all \(m \geq m_0\),
\[\psi_m(x) \geq \frac{\|x\|}{\rho} + \psi(0),\]  \hspace{1cm} (18)
which then means that we have established (15) for all \(\|x\| > \rho\).
Suppose that \(\psi_m(x) < \frac{\|x\|}{\rho} + \psi(0)\) for some \(x\) with \(\|x\| > \rho\). Then by convexity
\[\psi_m\left(\frac{\rho}{\|x\|}x\right) \leq \frac{\rho}{\|x\|} \psi_m(x) + \left(1 - \frac{\rho}{\|x\|}\right) \psi_m(0).\]  \hspace{1cm} (19)
Since \(\|\frac{\rho}{\|x\|}x\| = \rho\), it follows by (16) that
\[\frac{\rho}{\|x\|}x \notin \{x | \psi_m(x) \leq \psi(0) + 2\}\].
Therefore,
\[\psi(0) + 2 < \psi_m\left(\frac{\rho}{\|x\|}x\right).\]  \hspace{1cm} (20)
Hence, by (19) and (20)
\[\psi(0) + 2 < \psi_m\left(\frac{\rho}{\|x\|}x\right) \leq \frac{\rho}{\|x\|} \psi_m(x) + \left(1 - \frac{\rho}{\|x\|}\right) \psi_m(0)\]
By the assumption $\psi_m(x) < \frac{\|x\|}{\rho} + \psi(0)$

$$\psi(0) + 2 \leq 1 + \frac{\rho}{\|x\|} \psi(0) + \left(1 - \frac{\rho}{\|x\|}\right) \psi_m(0)$$

and by (17)

$$\psi(0) + 2 \leq 1 + \frac{\rho}{\|x\|} \psi(0) + \left(1 - \frac{\rho}{\|x\|}\right) \left(\psi(0) + \frac{1}{4}\right) = \frac{5}{4} + \psi(0).$$

This is a contradiction. Thus (18) holds.

Now, we have to consider what happens for $x$ with $\|x\| \leq \rho$. Since the sequence $(f_m)_{m=1}^\infty$ converges to $f$ in $L_1$, by Lemmas 8, 1 and 5, the sequence $(\|f_m\|_\infty)_{m=1}^\infty$ converges to $\|f\|_\infty$. Therefore, there is $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$,

$$\max_{\|x\| \leq \rho} |f_m(x)| \leq 1 + \max_{\|x\| \leq \rho} |f(x)|.$$

It follows for all $x$ with $\|x\| \leq \rho$

$$f_m(x) \leq \max_{\|x\| \leq \rho} |f_m(x)| \leq \left(1 + \max_{\|x\| \leq \rho} |f(x)|\right) \exp\left(-\frac{\|x\|}{\rho} + 1\right)$$

$$= \exp\left(-\frac{\|x\|}{\rho} + 1 + \ln\left(1 + \max_{\|x\| \leq \rho} |f(x)|\right)\right)$$

$$\leq \exp\left(-\frac{\|x\|}{\rho} + 1 + \max_{\|x\| \leq \rho} |f(x)|\right).$$

Thus we have established (15). \qed

We now present some classical examples of affine contravariant points for functions. We recall the definition of the centroid $g(f)$ of a function $f$. Provided it exists, it is defined as

$$g(f) = \frac{\int x f(x) dx}{\int f(x) dx}.$$ (21)

For log concave functions the centroid is well defined. We also recall the definition of the Santaló point $s(f)$ of a function $f \in LC [60, 62]$. It is the unique point for which

$$\min_z \int f^z(y) dy$$

is attained. Note that Santaló point must be attained in the interior of supp$(f)$ because otherwise the integral will be $\infty$.

We shall show that the centroid and the Santaló point are affine contra-variant points for log-concave functions.
Proposition 1 Let $f \in LC$.

(i) The centroid $g(f)$ is an affine contravariant point.

(ii) The Santaló point $s(f)$ is an affine contravariant point.

Proof (i) As noted above, for $f \in LC$, $g(f)$ exists. Moreover, it is easy to see that $g(Af) = A^{-1}(g(f))$ for all affine transformations $A$.

Let now $f$ be a log concave function and let $(f_m)_{m=1}^{\infty}$ be a sequence of log concave functions that converges to $f$ in the $L_1$-norm. Thus, for $\varepsilon > 0$ given, $\|f\|_1 - \varepsilon \leq \|f_m\|_1 \leq \|f\|_1 + \varepsilon$, for $m$ large enough. Then,

$$\|g(f) - g(f_m)\| \leq \|\int x f dx\| \left( |f - f_m| + 1 \right) dx + \|f\|_1 \int x (f(x) - f_m(x)) dx \leq \|f\|_1 \left( \|f\|_1 - \varepsilon \right)$$

By Lemma 9, there is $t, m_0$ and $\rho > 0$ such that for all $m \geq m_0$ and for all $x$, we have

$$\|x(f(x) - f_m(x))\| \leq 2\|x\| \exp \left( -\frac{\|x\|}{\rho} + t \right).$$

The function on the right side is integrable. Therefore we can apply the Dominated Convergence Theorem to the sequence on the left side. For almost all $x$ we have

$$\lim_{m \to \infty} \|x(f(x) - f_m(x))\| = 0.$$

(ii) First we shall show that for any non degenerate affine transform $A = T + a$ where $T \in GL(n), a \in \mathbb{R}^n$, we have that $s(Af) = A^{-1}s(f)$.

Let $z_0 = s(Af), z_1 = s(f)$. We put $u = Ax = Tx + a$, i.e., $x = T^{-1}(u - a)$, and obtain

$$\int (Af)^{(y)}(y) dy = \int \inf_{x \in \text{supp}(Af)} e^{-(x-z_0,y-z_0)} f(Ax) dy = \int \inf_{u \in \text{supp}(f)} e^{-(T^{-1}(u-a)-z_0,y-z_0)} f(u) dy$$

$$= \int \inf_{u \in \text{supp}(f)} e^{-(T^{-1}u,z_0,y-z_0)} f(u) e^{(T^{-1}a,y-z_0)} dy$$

$$= \int \inf_{u \in \text{supp}(f)} e^{-(u-Tz_0,z_0,y-z_0)} f(u) e^{(T^{-1}a,y-z_0)} dy$$

Now we introduce $w \in \mathbb{R}^n$ so that

$$(T^{-1})^t(y-z_0) = w - Tz_0 - a.$$ 

So $(T^{-1})^t y = w - Tz_0 - a + (T^{-1})^t z_0$. Hence $y = T^t w - T^t Tz_0 - T^t a + z_0$ and $dy = |\det T^t| dw = |\det T| dw$. With that change of variable, we continue the
calculation above as follows,

\[
\int \inf_{u \in \text{supp}(f)} e^{-\langle u - Tz_0, w - Tz_0 - a \rangle} e^{\langle T^{-1}a, T'w - T'Tz_0 - T'a \rangle} \left| \det T \right| dw
\]

\[
= \left| \det T \right| \int \inf_{u \in \text{supp}(f)} e^{-\langle u - Tz_0 - a, w - Tz_0 - a \rangle} \frac{f(u)}{f(u)} e^{\langle a, w - Tz_0 - a \rangle} dw
\]

\[
= \left| \det T \right| \int \inf_{u \in \text{supp}(f)} e^{-\langle u - Az_0, w - Az_0 \rangle} \frac{f(u)}{f(u)} e^{\langle a, w - Tz_0 - a \rangle} dw
\]

\[
= \left| \det T \right| \int f^{Az_0}(w) dw \geq \left| \det T \right| \int f^{z_1}(w) dw.
\]

Altogether we have

\[
\int (Af)^{z_0}(y) dy \geq \left| \det T \right| \int f^{z_1}(w) dw. \tag{23}
\]

Next we look at \( \int f^{z_1}(w) dw \) more closely. By definition,

\[
\int f^{z_1}(w) dw = \int \inf_{x \in \text{supp}(f)} e^{-\langle x - z_1, w - z_1 \rangle} f(x) dw.
\]

We put \( x = A\xi = T\xi + a \). Then the above integral equals

\[
\int \inf_{\xi \in \text{supp}(Af)} \frac{e^{-\langle T(\xi + T^{-1}(a - z_1)), w - z_1 \rangle}}{Af(\xi)} dw = \int \inf_{\xi \in \text{supp}(Af)} \frac{e^{-\langle \xi + T^{-1}(a - z_1), T'(w - z_1) \rangle}}{Af(\xi)} dw.
\]

Now let \( z_2 = T^{-1}(z_1 - a) \), that is, \( z_1 = Tz_2 + a \). Furthermore, we let

\[
v = T'(w - z_1) + z_2 = T'(w - Tz_2 - a) + z_2.
\]

Therefore \( dv = |\det T| dw \) and the latter integral equals

\[
\frac{1}{|\det T|} \int \inf_{\xi \in \text{supp}(Af)} \frac{e^{-\langle \xi - z_2, v - z_2 \rangle}}{Af(\xi)} dv = \frac{1}{|\det T|} \int (Af)^{z_2}(v) dv.
\]

Consequently, with (23)

\[
\int (Af)^{z_0}(y) dy \geq |\det T| \int f^{z_1}(w) dw = \int (Af)^{z_2}(v) dv. \tag{24}
\]
On the other hand, it’s trivially true that \( \int (Af)^{z_0} \leq \int (Af)^{z_2} \) by the definition of the Santaló point. Therefore,

\[
\int (Af)^{z_0} (y) dy = \int (Af)^{z_2} (v) dv
\]

and it follows from the uniqueness of the Santaló point that \( z_0 = z_2 \). Consequently,

\[
s(Af) = z_0 = z_2 = T^{-1}(z_1 - a) = T^{-1}(s(f)) - T^{-1}a = A^{-1}(s(f)).
\]

Now, we shall prove the continuity of the Santaló point. Let \((f_m)_{m \in \mathbb{N}}\) be a sequence of log-concave functions that converges to \( f \) in \( L_1 \). We assume that the sequence \((s(f_m))_{m \in \mathbb{N}}\) does not converge to \( s(f) \). Then, there are two cases. The first case is that

\[
\lim_{m \to \infty} \|s(f_m)\| = \infty.
\]

By the definition of the Santaló point, we have for all \( m \in \mathbb{N} \),

\[
\int f_s(f_m)(x) dx \leq \int f_s(f)(x) dx
\]

and thus by Lemma 8

\[
\lim_{m \to \infty} \int f_s(f_m)(x) dx \leq \lim_{m \to \infty} \int f_s(f)(x) dx = \int f_s(f) < \infty.
\]

It follows from the definition of the Legendre transform (6) that,

\[
\mathcal{L}_z \psi(y) = \sup_{x \in \mathbb{R}^n} [(x - z, y - z) - \psi(x)] = -(z, y - z) + \mathcal{L}_0 \psi(y - z),
\]

and

\[
f_m^{s(f_m)}(y) = \inf_{x \in \text{supp}(f_m)} \frac{e^{-(x - s(f_m), y - s(f_m))}}{f_m(x)} = e^{s(f_m), y - s(f_m)} \inf_{x \in \text{supp}(f_m)} \frac{e^{-(x, y - s(f_m))}}{f_m(x)}
\]

\[
= f_m^o(y - s(f_m))e^{s(f_m), y - s(f_m)}.
\]

Therefore,

\[
\int f_m^{s(f_m)}(y) dy = \int f_m^o(w) e^{s(f_m), w} dw.
\]

We can assume without loss of generality that \( 0 \in \text{int}(\text{supp}(f)) \). We choose \( \rho > 0 \) such that the closed ball \( B_2^n(\rho) \subset \text{int}(\text{supp}(f)) \). Since the integrands in the above
integrals are positive,

\[ \int f_m^\circ(w) e^{\langle s(f_m), w \rangle} \, dw \geq \int f_m^\circ(w) e^{\langle s(f_m), w \rangle} \mathbb{1}_{\{w \in B_2^n(\rho); \|s(f_m)\|, \frac{w}{\rho} > \frac{1}{2}\}}(w) \, dw. \]

By Lemma 8, the sequence \((f_m^\circ)_{m=1}^\infty\) converges uniformly to \(f^\circ\) on the closed ball \(B_2^n(\rho)\). Hence there exists \(m_0 \in \mathbb{N}\) such that for all \(m \geq m_0\) and all \(w \in B_2^n(\rho)\)

\[ f_m^\circ(w) \geq \frac{1}{2} \min \{ f^\circ(v) : v \in B_2^n(\rho) \}. \]

Moreover, for \(n \geq 2\), for all \(\theta\) with \(\|\theta\| = 1\),

\[
\begin{align*}
\operatorname{vol}_n \left( \left\{ w \in B_2^n(\rho) : \left( \theta, \frac{w}{\|w\|} \right) \geq \frac{1}{2} \right\} \right) & \\
& \geq \operatorname{vol}_n \left( \left\{ w \in B_2^n(\rho) : \left( \theta, \frac{w}{\|w\|} \right) \geq \frac{1}{\sqrt{2}} \text{ and } \|w\| \geq \frac{\rho}{\sqrt{2}} \right\} \right) \\
& \geq \frac{1}{n} \operatorname{vol}_{n-1}(B_2^{n-1}) \left( \frac{\rho}{\sqrt{2}} \right)^n \left( 1 - \left( \frac{1}{\sqrt{2}} \right)^n \right) \\
& \geq \frac{1}{2n} \operatorname{vol}_{n-1}(B_2^{n-1}) \left( \frac{\rho}{\sqrt{2}} \right)^n.
\end{align*}
\]

Therefore

\[
\begin{align*}
\int f_m^\circ(w) e^{\langle s(f_m), w \rangle} \mathbb{1}_{\{w \in B_2^n(\rho); \|s(f_m)\|, \frac{w}{\rho} > \frac{1}{2}\}}(w) \, dw & \\
& \geq e^{\frac{\rho}{\sqrt{2}} \|s(f_m)\|} \left( \frac{\rho}{\sqrt{2}} \right)^n \min \{ f^\circ(v) : v \in B_2^n(\rho) \} \frac{1}{4n} \operatorname{vol}_{n-1}(B_2^{n-1}).
\end{align*}
\]

If \(m\) tends to infinity the right hand side goes to infinity by assumption (25). This in turn implies that

\[ \lim_{m \to \infty} \int f_m^s(f_m)(w) \, dw = \infty, \]

contradicting (26).

The second case is that there is a converging subsequence \((s(f_{m_j}))_{j \in \mathbb{N}}\) such that

\[ \lim_{j \to \infty} s(f_{m_j}) = s_0 \neq s(f). \quad (27) \]
First, we observe that \( s_0 \in \text{int}(\text{supp}(f)) \). Otherwise, as by Lemma 8, \( \int f^{s(f)} = \lim_{m_j \to \infty} \int f_m^{s(f)} \), we have, again using Lemma 8,

\[
\infty > \int f^{s(f)}(x)dx = \lim_{j \to \infty} \int f_{m_j}^{s(f)}(x)dx \geq \lim_{j \to \infty} \int f_{m_j}^{s(f_m)}(x)dx
\]

\[
= \int f^{s_0}(x)dx = \infty,
\]

which leads to a contradiction. We show next that

\[
\lim_{j \to \infty} \int f^{s(f_m)}(x)dx = \int f^{s(f)}(x)dx.
\]

Then, by Lemma 8

\[
\int f^{s_0}(x)dx = \lim_{j \to \infty} \int f^{s(f_m)}(x)dx = \int f^{s(f)}(x)dx,
\]

which contradicts the uniqueness of the Santaló point. Thus it is enough to show (28). By Lemma 8,

\[
\lim_{j \to \infty} \int f^{s(f)}(x)dx = \int f^{s(f)}(x)dx.
\]

By the definition of the Santaló point we have for all \( j \in \mathbb{N} \)

\[
\int f_{m_j}^{s(f)}(x)dx \geq \int f_{m_j}^{s(f_m)}(x)dx
\]

and therefore, by Lemma 8

\[
\lim_{j \to \infty} \int f_{m_j}^{s(f)}(x)dx \geq \lim_{j \to \infty} \int f_{m_j}^{s(f_m)}(x)dx.
\]

Again by Lemma 8

\[
\int f^{s(f)}(x)dx = \lim_{j \to \infty} \int f_{m_j}^{s(f)}(x)dx \geq \lim_{j \to \infty} \int f_{m_j}^{s(f_m)}(x)dx
\]

\[
= \int f^{s_0}(x)dx \geq \int f^{s(f)}(x)dx.
\]

This shows that \( \int f^{s(f)}(x)dx = \int f^{s_0}(x)dx \). Thus by uniqueness of the Santaló point, we get that \( s_0 = s(f) \), contradicting (27).

In the next sections we study the Löwner function [73], the John function [70] and the floating function [69] of a log-concave function. The importance of the Löwner- and John ellipsoids in the context of convex bodies was already outlined in the introduction.
Convex floating bodies were introduced independently by Bárány and Larman [88] and Schütz and Werner [89]. They provide a way to extend the important notion of affine surface area (see e.g., [16, 90]) to all convex bodies [89]. By now floating bodies are widely used, e.g., in differential geometry [91–93] approximation theory [88, 94, 95], data science [5, 96, 97] and even economics [98]. Löwner- and John functions and floating functions serve a similar purpose within the functional setting [69, 70, 73].

4 The Floating Function of a Log-Concave Function

We start by giving the definition of the floating function for a log-concave function, which was introduced in [69]. First we recall the definition of floating set, which was also introduced in [69]. $H$ is a hyperplane and $H^+$ and $H^-$ are the two half-spaces determined by this hyperplane.

**Definition 3** [69] Let $C$ be a closed convex subset of $\mathbb{R}^n$ with non-empty interior. For $\delta \geq 0$ and a finite measure $m$ on $C$, the floating set $C_\delta$ is defined by

$$C_\delta = \bigcap \{ H^+ : \text{vol}_n (H^- \cap C) \leq \delta m(C) \}.$$  

The floating set is used to define the floating function of a convex function and a log-concave function.

**Definition 4** [69] Let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{ \infty \}$ be a convex function and $f(x) = \exp(-\psi(x))$ be an integrable log-concave function. Let $\text{epi}(\psi)$ be its epigraph and $\delta \geq 0$.

(i) The floating function of $\psi$ is defined to be this function $\psi_\delta$ such that

$$\text{epi}(\psi_\delta) = (\text{epi}(\psi))_\delta = \bigcap \left\{ H^+ : \text{vol}_{n+1} (H^- \cap \text{epi}(\psi)) \leq \delta \int_{\mathbb{R}^n} e^{-\psi(x)} \, dx \right\}.$$  

(ii) The floating function $f_\delta$ of $f$ is defined as

$$f_\delta(x) = \exp(-\psi_\delta(x)).$$  

The floating function is again a log-concave function. Denote by dom the domain of $\psi$. For all $x \notin \partial \text{dom}(\psi)$ we have $\psi(x) \leq \psi_\delta(x)$. If $f = e^{-\psi}$ is integrable, then $f_\delta$ is also integrable as

$$\int_{\mathbb{R}^n} f_\delta(x) \, dx = \int_{\mathbb{R}^n} e^{-\psi_\delta(x)} \, dx \leq \int_{\mathbb{R}^n} e^{-\psi(x)} \, dx = \int_{\mathbb{R}^n} f(x) \, dx < \infty.$$  

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Lemma 10 Let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a convex function. Then, we have for all $x_0$ in the interior of the domain of $\psi$,

$$\psi_\delta(x_0) = \sup_{(u, \alpha) \in \mathbb{R}^n \times \mathbb{R}} \alpha - \langle u, x_0 \rangle \tag{32}$$

where the supremum is taken over all $(u, \alpha(u))$ such that

$$\int_{\mathbb{R}^n} \max\{0, \alpha - \langle x, u \rangle - \psi(x)\} \, dx = \delta \int_{\mathbb{R}^n} e^{-\psi(x)} \, dx, \tag{33}$$

where $\max\{0, \alpha - \langle x, u \rangle - \psi(x)\} = 0$ if $\psi(x) = \infty$.

Proof For $(u, u_{n+1}) \in \mathbb{R}^{n+1}$ with $\|(u, u_{n+1})\| = 1$ and $\beta \in \mathbb{R}$ there is a hyperplane $H = \{x | \langle x, u \rangle + u_{n+1}x_{n+1} = \beta\}$. We may assume without loss of generality that $u_{n+1} > 0$. Then

$$\text{epi}(\psi_\delta) = \bigcap \left\{ H^+ : \text{vol}_{n+1} \left( H^- \cap \text{epi}(\psi) \right) \leq \delta \int_{\mathbb{R}^n} e^{-\psi(x)} \, dx \right\}$$

where

$$H^- = \left\{ (x, x_{n+1}) : \langle u, x \rangle + u_{n+1}x_{n+1} \leq \beta \right\} = \left\{ (x, x_{n+1}) : x_{n+1} \leq \frac{\beta}{u_{n+1}} - \left\langle \frac{u}{u_{n+1}}, x \right\rangle \right\}. \tag{34}$$

Renaming $\alpha = \frac{\beta}{u_{n+1}}$ and $v = \frac{u}{u_{n+1}}$

$$H^- = \left\{ (x, x_{n+1}) : x_{n+1} \leq \alpha - \langle v, x \rangle \right\}. \tag{35}$$

We have

$$\text{vol}_{n+1} \left( H^- \cap \text{epi}(\psi) \right) = \int_{\mathbb{R}^n} \max\{0, \alpha - \langle x, v \rangle - \psi(x)\} \, dx. \tag{36}$$

It follows that

$$\text{epi}(\psi_\delta) = \bigcap_{(\alpha, v)} \left\{ (x, x_{n+1}) : x_{n+1} \geq \alpha - \langle v, x \rangle \right\} \quad \text{and} \quad \int_{\mathbb{R}^n} \max\{0, \alpha - \langle x, v \rangle - \psi(x)\} \, dx \leq \delta \int_{\mathbb{R}^n} e^{-\psi(x)} \, dx. \tag{37}$$

Since $\text{epi}(\psi_\delta) = \{(x, x_{n+1}) : x_{n+1} \geq \psi_\delta(x)\}$

$$\psi_\delta(x) = \sup_{(\alpha, v)} \alpha - \langle v, x \rangle$$
where
\[ \int_{\mathbb{R}^n} \max\{0, \alpha - \langle v, x \rangle - \psi(x)\} \, dx \leq \delta \int_{\mathbb{R}^n} e^{-\psi(x)} \, dx. \]

We show now that it is enough to consider those \((\alpha, v)\) with equality in the latter inequality. Let us observe that if there is \(\alpha_0\) such that
\[ 0 < \int_{\mathbb{R}^n} \max\{0, \alpha_0 - \langle x, v \rangle - \psi(x)\} \, dx \leq \delta \int_{\mathbb{R}^n} e^{-\psi} \]
then there is \(\alpha_1\) with
\[ \int_{\mathbb{R}^n} \max\{0, \alpha_1 - \langle x, v \rangle - \psi(x)\} \, dx = \delta \int_{\mathbb{R}^n} e^{-\psi}. \]

We verify this. The convexity of \(\psi\) implies that by (34) the integral
\[ \int_{\mathbb{R}^n} \max\{0, \alpha - \langle x, v \rangle - \psi(x)\} \, dx \]
is finite for all \(\alpha \geq \alpha_0\). Moreover, again by the convexity of \(\psi\) the integral (35) is continuous w.r.t. \(\alpha\) for \(\alpha \geq \alpha_0\).

Consider \(x_0 \in \text{int}(\text{dom}(\ )\ )\) and suppose that \(\psi(x_0) < \psi_\delta(x_0)\). Then there is \((\alpha, v)\) satisfying (34) and we can conclude that there is \((\alpha, v)\) satisfying (35).

If \(\psi_\delta(x_0) = \psi(x_0)\) then by the theorem of Hahn–Banach there is \((\alpha, v)\) such that
\[ \alpha - \langle v, x_0 \rangle = \psi(x_0) \]
and for all \(x \in \mathbb{R}^n\) we have \(\alpha - \langle x, v \rangle \leq \psi(x)\).

\[ \square \]

**Lemma 11** For all \(x_0\) in the interior of the domain of \(\psi\) there are \(u_0\) and \(\alpha(u_0)\) such that (33) holds and
\[ \psi_\delta(x_0) = \alpha(u_0) - \langle u_0, x_0 \rangle. \]

\[ \psi_\delta(x_0) = \alpha(u_0) - \langle u_0, x_0 \rangle. \]

**Proof** By Lemma 10 there are sequences \((u_k)_{k=1}^\infty\) and \((\alpha_k)_{k=1}^\infty\) such that
\[ \psi_\delta(x_0) \geq \alpha_k - \langle x_0, u_k \rangle \geq \psi_\delta(x_0) - \frac{1}{k} \]
and for all \(k \in \mathbb{N}\)
\[ \int_{\mathbb{R}^n} \max\{0, \alpha_k - \langle x, u_k \rangle - \psi(x)\} \, dx = \delta \int_{\mathbb{R}^n} e^{-\psi} \, dx. \]

We show that the sequences \((\|u_k(x_0)\|)_{k=1}^\infty\) and \((\alpha_k(x_0))_{k=1}^\infty\) are bounded. Then, by compactness our lemma follows. Since \(x_0\) is an interior point of the domain of \(\psi\) there
is \( \rho > 0 \) such that \( B_2^n(x_0, \rho) \) is contained in the domain of \( \psi \) and \( \psi(x) \leq \psi(x_0) + 1 \) for \( x \in B_2^n(x_0, \rho) \). We have

\[
\delta \int_{\mathbb{R}^n} e^{-\psi} \, dx = \int_{\mathbb{R}^n} \max\{0, \alpha_k - \langle x, u_k \rangle - \psi(x)\} \, dx \\
\geq \int_{B_2^n(x_0, \rho)} \max\{0, \alpha_k - \langle x, u_k \rangle - \psi(x)\} \, dx \\
= \int_{B_2^n(0, \rho)} \max\{0, \alpha_k - \langle x_0 + x, u_k \rangle - \psi(x_0 + x)\} \, dx \\
\geq \int_{B_2^n(0, \rho)} \max\{0, \alpha_k - \langle x_0, u_k \rangle - \langle x, u_k \rangle - \psi(x_0) - 1\} \, dx
\]

By (38) the latter integral is bigger than

\[
\int_{B_2^n(0, \rho)} \max\{0, -\langle x, u_k \rangle + \psi_\delta(x_0) - \psi(x_0) - 2\} \, dx
\]

Since \( \psi_\delta(x_0) \geq \psi(x_0) \) the latter integral is bigger than

\[
\int_{B_2^n(0, \rho) \cap \{x : \langle x, u_k \rangle \leq 0\}} \max\{0, -\langle x, u_k \rangle - 2\} \, dx \\
\geq \int_{B_2^n(0, \rho) \cap \{x : \langle x, u_k \rangle \leq 0\}} -\langle x, u_k \rangle - 2 \, dx.
\]

The latter integral is getting arbitrarily large if the sequence \((\|u_k\|)^\infty_{k=1}\) is not bounded. This cannot be since all the integrals are bounded by \( \delta \int_{\mathbb{R}^n} e^{-\psi} \, dx \).

By (38)

\[
\alpha_k \leq \psi_\delta(x_0) + \langle x_0, u_k \rangle \leq \psi_\delta(x_0) + \|x_0\| \|u_k\|
\]

Since the sequence \((\|u_k\|)^\infty_{k=1}\) is bounded it follows that the sequence \((\alpha_k)^\infty_{k=1}\) is bounded from above. In the same way we show that the sequence is also bounded from below.

**Theorem 1** Let \( f = \exp(-\psi) \) be a function in LC and let \( \delta \geq 0 \). Then the floating operator \( F : LC \to LC \) with \( F(f) = f_\delta \) is an affine covariant mapping.

The next corollary follows immediately from the theorem, together with Remark 2.

**Corollary 1** Let \( f = \exp(-\psi) \) be a function in LC and let \( \delta \geq 0 \). Then, for all \( \lambda \in \mathbb{R} \),

\[
g(f_\delta), \quad s(f_\delta), \quad \lambda g(f_\delta) + (1 - \lambda) s(f_\delta)
\]

are affine contravariant points.
We show first the affine invariance property. Recall the super-level sets \( G_f(t) = \{ x \in \mathbb{R}^n : f(x) \geq t \} \) of a function \( f \), introduced in (2). Now we also need the sub-level sets \( E_\psi(t) \) for a convex function \( \psi : \mathbb{R}^n \to \mathbb{R} \cup \{ \infty \} \). For \( t \in \mathbb{R} \) they are defined as

\[
E_\psi(t) = \{ x \in \mathbb{R}^n : \psi(x) \leq t \}.
\]  

(39)

It’s clear that for the log-concave function \( f = e^{-\psi(x)} \) the following identity holds,

\[
G_f(t) = E_\psi(-\log t).
\]  

(40)

Lemma 12 Let \( \psi : \mathbb{R}^n \to \mathbb{R} \cup \{ \infty \} \) be a convex function and let \( f = e^{-\psi} \) be integrable and non-degenerate. Let \( \delta \geq 0 \). Then we have for any \( A \in A \),

\[
A(\psi_\delta) = (A\psi)_\delta \quad \text{and} \quad A(f_\delta) = (Af)_\delta.
\]

Proof Observe first that for a convex but not necessarily bounded set \( C \in \mathbb{R}^n \) with finite measure \( m(C) \) and \( A \in A \) one has

\[
A(C_\delta) = (AC)_\delta.
\]  

(41)

Then note that it is enough to show that \( A\psi_\delta = (A\psi)_\delta \). The statement about \( f = e^{-\psi} \) then follows easily. To prove the assertion \( A(\psi_\delta) = (A\psi)_\delta \), we show that for all \( t \in \mathbb{R} \) their sub-level sets coincide, namely \( E_A(\psi_\delta)(t) = E_{(A\psi)_\delta}(t) \). With (40) we then deduce that

\[
G_A(f_\delta)(t) = E_A(\psi_\delta)(-\log t) = E_{(A\psi)_\delta}(-\log t) = G_{(Af)_\delta}(t),
\]

which implies that \( A(f_\delta) = (Af)_\delta \). Let \( t \in \mathbb{R}^n \). We show now that \( E_A(\psi_\delta)(t) = E_{(A\psi)_\delta}(t) \). On the one hand

\[
E_A(\psi_\delta)(t) = \{ x \in \mathbb{R}^n : A\psi_\delta(x) \leq t \} = \{ x \in \mathbb{R}^n : \psi_\delta(Ax) \leq t \} = A^{-1}\{ y \in \mathbb{R}^n : \psi_\delta(y) \leq t \} = A^{-1}E_\psi(t).
\]

On the other hand, we show that \( E_{(A\psi)_\delta}(t) = A^{-1}E_\psi(t) \). For \( z = (x, y) \in \mathbb{R}^n \times \mathbb{R} \), we denote by \( \tilde{A} \) the map

\[
\tilde{A}z = \tilde{A}(x, y) = (Ax, y).
\]

Then \( \tilde{A}^{-1}z = \tilde{A}^{-1}(x, y) = (A^{-1}x, y) \) and it is clear that

\[
\text{epi}(A\psi) = \tilde{A}^{-1}(\text{epi}(\psi)).
\]

Thus by the definition of the floating set and (41),

\[
\text{epi}((A\psi)_\delta) = (\text{epi}(A\psi))_\delta = (\tilde{A}^{-1}(\text{epi}(\psi)))_\delta = \tilde{A}^{-1}((\text{epi}(\psi))_\delta) = \tilde{A}^{-1}(\text{epi}(\psi_\delta)).
\]
It follows that for all $t \in \mathbb{R}$,
\[ (E_{(A\psi)_\delta}(t), t) = \text{epi}((A\psi)_\delta) \cap \{ x \in \mathbb{R}^{n+1} : x_{n+1} = t \} = \tilde{A}^{-1}(\text{epi}(\psi)) \cap \{ x \in \mathbb{R}^{n+1} : x_{n+1} = t \} = \tilde{A}^{-1}(\text{epi}(\psi)) \cap \tilde{A}^{-1}(\{ x \in \mathbb{R}^{n+1} : x_{n+1} = t \}) = \tilde{A}^{-1}(\text{epi}(\psi) \cap \{ x \in \mathbb{R}^{n+1} : x_{n+1} = t \}) = \tilde{A}^{-1}(E_{\psi}(t), t) = (A^{-1}E_{\psi}(t), t). \]

\[ \square \]

Next we show the continuity of the floating operator.

**Proposition 2** Let $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{ \infty \}$ be a convex function and let $\psi_m : \mathbb{R}^n \to \mathbb{R} \cup \{ \infty \}$, $m \in \mathbb{N}$, be a sequence of convex functions such that the sequence $(f_m)_{m=1}^\infty = (e^{-\psi_m})_{m=1}^\infty$ converges to $f = e^{-\psi}$ in $L_1$. Then, for every $\delta > 0$, the sequence $((f_m)_\delta)_{m=1}^\infty$ converges to $f_\delta$ in $L_1$.

**Proof** By (31), $(f_m)_\delta \in L_1$ for all $m \in \mathbb{N}$ and $f_\delta \in L_1$. It suffices to show that the sequence $((\psi_m)_\delta)_{m=1}^\infty$ converges pointwise a.e. to $\psi_\delta$. Indeed, suppose this is true. Then the sequence $((f_m)_\delta)_{m=1}^\infty = (e^{-\psi_m})_{m=1}^\infty$ converges to $f_\delta = e^{-\psi_\delta}$ pointwise a.e.. The assumption that the sequence $(f_m)_{m=1}^\infty$ converges to $f$ in $L_1$ implies $\lim_{m \to \infty} \int f_m(x) \, dx = \int f(x) \, dx$ and implies by Lemma 8 that the sequence $(f_m)_{m=1}^\infty$ converges to $f$ pointwise a.e.. Moreover, we have for all $m \in \mathbb{N}$
\[ (f_m)_\delta = e^{-(\psi_m)_\delta} \leq e^{-\psi_m} = f_m. \]

The generalized Dominated Convergence Theorem (e.g., [87] p. 59, exercise 20) then yields
\[ \lim_{m \to \infty} \int_{\mathbb{R}^n} (f_m)_\delta(x) \, dx = \int_{\mathbb{R}^n} f_\delta(x) \, dx \]
and Lemma 6 that the sequence $((f_m)_\delta)_{m=1}^\infty$ converges to $f_\delta$ in $L_1$.

Since the sequence $(f_m)_{m=1}^\infty$ converges in $L_1$ to $f$, the sequence also converges pointwise to $f$ on the interior of the support of $f$. Therefore the sequence $((\psi_m))_{m=1}^\infty$ converges pointwise to $\psi$ on the interior of the domain of $\psi$.

We show that for all $x_0$ in $\mathbb{R}^n \setminus \partial \text{dom } \psi$,
\[ \lim_{m \to \infty} (\psi_m)_\delta(x_0) = \psi_\delta(x_0). \quad (42) \]

The case $x_0 \in \mathbb{R}^n \setminus \text{dom } \psi$ is easy: $\psi(x_0) = \infty$ and $\lim_{m \to \infty} \psi_m(x_0) = \psi(x_0)$. Since $\psi(x_0) \leq \psi_\delta(x_0)$ and $\psi_m(x_0) \leq (\psi_m)_\delta(x_0)$ we get
\[ \psi_\delta(x_0) = \infty = \lim_{m \to \infty} (\psi_m)_\delta(x_0). \]
Now, the case \( x_0 \in \text{int}(\text{dom } \psi) \). We show first that for all \( x_0 \) in the interior of the domain of \( \psi \),

\[
\lim_{m \to \infty} \inf (\psi_m)_\delta(x_0) \geq \psi_\delta(x_0). \tag{43}
\]

If \( \psi_\delta(x_0) = \psi(x_0) \) then

\[
\psi_\delta(x_0) = \psi(x_0) = \lim_{m \to \infty} \psi_m(x_0) \leq \lim_{m \to \infty} \inf (\psi_m)_\delta(x_0).
\]

Therefore, we can now assume that for some \( \epsilon > 0 \)

\[
\psi(x_0) + \epsilon \leq \psi_\delta(x_0).
\tag{44}
\]

Let \( \alpha \) be defined by (33) for the function \( \psi \) and for \( m \in \mathbb{N} \) let \( \alpha_m \) be defined by (33) for the function \( \psi_m \). By Lemma 11, there is \((u_0, \alpha(u_0))\) such that \( \psi_\delta(x_0) = \alpha(u_0) - \langle u_0, x_0 \rangle \). Therefore,

\[
(\psi_m)_\delta(x_0) = \sup_{u \in \mathbb{R}^n} \alpha_m(u) - \langle u, x_0 \rangle \geq \alpha_m(u_0) - \langle u_0, x_0 \rangle.
\]

In order to show (43) it is enough to show

\[
\lim_{m \to \infty} \alpha_m(u_0) = \alpha(u_0). \tag{45}
\]

We do this. By definition (33) of \( \alpha_m \) we get for all \( m \in \mathbb{N} \)

\[
\delta \int_{\mathbb{R}^n} e^{-\psi_m} \, dx = \int_{\mathbb{R}^n} \max\{0, \alpha_m(u_0) - \langle x, u_0 \rangle - \psi_m(x)\} \, dx. \tag{46}
\]

Since \((e^{-\psi_m})_{m \in \mathbb{N}}\) converges in \( L_1 \) to \( e^{-\psi} \)

\[
\delta \int_{\mathbb{R}^n} e^{-\psi} \, dx = \lim_{m \to \infty} \delta \int_{\mathbb{R}^n} e^{-\psi_m} \, dx.
\]

By (46)

\[
\delta \int_{\mathbb{R}^n} e^{-\psi} \, dx = \lim_{m \to \infty} \int_{\mathbb{R}^n} \max\{0, \alpha_m(u_0) - \langle x, u_0 \rangle - \psi_m(x)\} \, dx. \tag{47}
\]

We justify that we can interchange limit and integral. At this point we know that \( \lim_{m \to \infty} \psi_m(x) = \psi(x) \), but we do not know that \( \lim_{m \to \infty} \alpha_m(x_0) \) exists. We want to apply the Dominated Convergence Theorem. We prove now that there is a dominating, integrable function. For this, it is enough to show that there exists \( R > 0 \) and \( c > 0 \) such that for all \( m \in \mathbb{N} \) and for all \( x \in \mathbb{R}^n \)

\[
\max\{0, \alpha_m(u_0) - \langle x, u_0 \rangle - \psi_m(x)\} \leq c \, \mathbb{1}_{RB_2^n}. \tag{48}
\]
The first step towards that goal is to show that there is $R > 0$ such that for all $y \in \mathbb{R}^n$ and all $m \in \mathbb{N}$ with $\alpha_m(u_0) - \langle y, u_0 \rangle \geq \psi_m(y)$ we have that $\|y\| \leq R$. Suppose that is not the case, i.e., for every $\ell \in \mathbb{N}$ there are $m_\ell$ and $y_{m_\ell}$ such that $\|y_{m_\ell}\| \geq \ell$ and

$$\alpha_{m_\ell}(u_0) - \langle y_{m_\ell}, u_0 \rangle \geq \psi_{m_\ell}(y_{m_\ell}).$$  \hfill (49)

In fact, we may assume that

$$\lim_{\ell \to \infty} \|y_{m_\ell}\| = \infty \quad \hfill (50)$$

and that the sequence $\|y_{m_\ell}\|$, $\ell \in \mathbb{N}$, is monotonely increasing. First consider the case:

There is a subsequence $m_{i_\ell}$, $i \in \mathbb{N}$, such that for all $i \in \mathbb{N}$

$$\psi_\delta(x_0) \leq \alpha_{m_{i_\ell}}(u_0) - \langle x_0, u_0 \rangle.$$  \hfill (51)

To keep notation simple we denote this subsequence of a subsequence again by $m_i$, $i \in \mathbb{N}$. There are $\rho > 0$ and $M_0$ such that for all $m_i$, $i \in \mathbb{N}$, with $m_i \geq M_0$

$$B_2^{n+1}\left(\left(x_0, \psi(x_0) + \frac{\epsilon}{2}\right), \frac{\rho}{\|u_0\|}\right) \subseteq \text{epi} \psi_{m_i} \cap \{(x, s)| s \leq \alpha_{m_i}(x_0) - \langle x, u_0 \rangle\} = \{(x, s)| \psi_{m_i}(x) \leq s \leq \alpha_{m_i}(x_0) - \langle x, u_0 \rangle\}.$$

We may assume that $\max\{\rho, \frac{\rho}{\|u_0\|}\} < \frac{\epsilon}{4}$ where $\epsilon$ is given by (44). We prove (52). Since $x_0 \in \text{int}(\text{dom}(\psi))$ we can choose $\rho > 0$ so small that $B_2^n(x_0, \frac{\rho}{\|u_0\|})$ is a compact subset of $\text{int}(\text{dom}(\psi))$ and, by continuity of $\psi$, for all $x \in B_2^n(x_0, \frac{\rho}{\|u_0\|})$

$$|\psi(x_0) - \psi(x)| < \frac{\epsilon}{10}.$$  \hfill (53)

Moreover, by Lemma 8 the sequence $(\psi_{m})_{m \in \mathbb{N}}$ converges uniformly on $B_2^n(x_0, \frac{\rho}{\|u_0\|})$ to $\psi$. Therefore there is $M_1$ such that for all $m \geq M_1$ and all $x \in B_2^n(x_0, \frac{\rho}{\|u_0\|})$

$$|\psi(x) - \psi_{m}(x)| < \frac{\epsilon}{10},$$  \hfill (54)

where $\epsilon$ is given by (44). We show that for all $i \in \mathbb{N}$ with $m_i \geq M_1$

$$B_2^{n+1}\left(\left(x_0, \psi(x_0) + \frac{\epsilon}{2}\right), \frac{\epsilon}{4}\right) \subseteq \text{epi} \psi_{m_i}.$$

Indeed, let $(x, s) \in B_2^{n+1}\left(\left(x_0, \psi(x_0) + \frac{\epsilon}{2}\right), \frac{\epsilon}{4}\right)$. Then

$$\|\left(x, s\right) - \left(x_0, \psi(x_0) + \frac{\epsilon}{2}\right)\| \leq \frac{\epsilon}{4}$$
which implies
\[
|s - \psi(x_0) - \frac{\epsilon}{2}| \leq \frac{\epsilon}{4}. \tag{56}
\]
Therefore
\[
\psi(x_0) + \frac{\epsilon}{4} \leq s \leq \psi(x_0) + \frac{3}{4}\epsilon. \tag{57}
\]
By (54), (53) and (57) we have for all \(i \in \mathbb{N}\) with \(m_i \geq M_1\) and \((x, s) \in B_2^{n+1}\left((x_0, \psi(x_0) + \frac{\epsilon}{2}), \frac{\epsilon}{4}\right)\)
\[
\psi_{m_i}(x) - \frac{\epsilon}{10} < \psi(x) \leq \frac{\epsilon}{10} < s - \frac{3}{20}\epsilon
\]
and thus \(\psi_{m_i}(x) < s\) which means that \((x, s) \in \text{epi} \psi_{m_i}\) and we have shown (55). On the other hand, by (44) and (51) we have for all \(i \in \mathbb{N}\)
\[
\psi(x_0) + \epsilon \leq \alpha_{m_i}(u_0) - \langle x_0, u_0 \rangle.
\]
By (57) it follows for all \(i \in \mathbb{N}\) and \((x, s) \in B_2^{n+1}\left((x_0, \psi(x_0) + \frac{\epsilon}{2}), \frac{\epsilon}{4}\right)\)
\[
s + \frac{\epsilon}{4} \leq \alpha_{m_i}(u_0) - \langle x_0, u_0 \rangle \leq \alpha_{m_i}(u_0) - \langle x, u_0 \rangle + \langle x - x_0, u_0 \rangle
\]
\[
\leq \alpha_{m_i}(u_0) - \langle x, u_0 \rangle + \|u_0\|\|x - x_0\|.
\]
Since \(\rho < \frac{\epsilon}{4}\) and \(\|u_0\|\|x - x_0\| < \rho\) we have for all \(i \in \mathbb{N}\) and \((x, s) \in B_2^{n+1}\left((x_0, \psi(x_0) + \frac{\epsilon}{2}), \frac{\epsilon}{4}\right)\)
\[
s \leq \alpha_{m_i}(u_0) - \langle x, u_0 \rangle.
\]
Thus we have established (52). Now we observe that for all \(i \in \mathbb{N}\) with \(m_i \geq M_1\)
\[
(y_{m_i}, \psi_{m_i}(y_{m_i})) \in \text{epi} \psi_{m_i} \cap \{(x, s) | s \leq \alpha_{m_i}(u_0) - \langle x, u_0 \rangle\}. \tag{58}
\]
Indeed, for all \(i \in \mathbb{N}\) we have \((y_{m_i}, \psi_{m_i}(y_{m_i})) \in \text{epi} \psi_{m_i}\) and by (49)
\[
\alpha_{m_i}(u_0) - \langle y_{m_i}, u_0 \rangle \geq \psi_{m_i}(y_{m_i}).
\]
Therefore, by convexity, (52) and (58) we have for all \(i \in \mathbb{N}\)
\[
\left[(y_{m_i}, \psi_{m_i}(y_{m_i})), B_2^{n+1}\left((x_0, \psi(x_0) + \frac{\epsilon}{2}), \frac{\rho}{\|u_0\|}\right)\right] \subseteq \text{epi} \psi_{m_i} \cap \{(x, s) | s \leq \alpha_{m_i}(u_0) - \langle x, u_0 \rangle\} = \{(x, s) | \psi_{m_i}(x) \leq s \leq \alpha_{m_i}(u_0) - \langle x, u_0 \rangle\}.
\]
Consequently
\[
\delta \int_{\mathbb{R}^n} e^{-\psi_{m_i}} \, dx = \text{vol}_{n+1} \left( \{(x, s) | \psi_{m_i}(x) \leq s \leq \alpha_{m_i}(u_0) - \langle x, u_0 \rangle \} \right) \\
\geq \frac{\rho^n \text{vol}_n(B_2^n)}{\|u_0\|^n} \left\| (y_{m_i}, \psi_{m_i}(y_{m_i})) - \left( x_0, \psi(x_0) + \frac{\epsilon}{2} \right) \right\| \\
\geq \frac{\rho^n \text{vol}_n(B_2^n)}{\|u_0\|^n} \| y_{m_i} - x_0 \|.
\]

Since the sequence \( \| y_{m_i} \|, i \in \mathbb{N} \), is unbounded we arrive at a contradiction. Thus we have settled the case (51).

We assume now that (51) does not hold, i.e. we suppose that for all \( \ell \in \mathbb{N} \), except for finitely many,
\[
\alpha_{m_\ell}(u_0) - \langle x_0, u_0 \rangle \leq \psi_{\delta}(x_0) = \alpha(u_0) - \langle x_0, u_0 \rangle.
\]
In particular, for all \( \ell \in \mathbb{N} \), except for finitely many,
\[
\alpha_{m_\ell}(u_0) \leq \alpha(u_0).
\]
Let \( r \) be any positive number. By assumption (50) there is \( M_0 \) such that for all \( \ell \) with \( m_\ell \geq M_0 \) we have \( \| y_{m_\ell} \| > r \). We consider for all \( \ell \) with \( m_\ell \geq M_0 \)
\[
z_{m_\ell} = \frac{r}{\| y_{m_\ell} \|} y_{m_\ell} + \left( 1 - \frac{r}{\| y_{m_\ell} \|} \right)x_0.
\]
Then, for all \( \ell \) with \( m_\ell \geq M_0 \)
\[
\| z_{m_\ell} \| \leq r + \| x_0 \|.
\]
Therefore, by compactness, there is a subsequence \((z_{m_{\ell_i}})_{i \in \mathbb{N}}\) that converges
\[
z_0 = \lim_{i \to \infty} z_{m_{\ell_i}}
\]
and
\[
\| z_0 \| \leq r + \| x_0 \|.
\]
For ease of notation we denote the subsequence \((z_{m_{\ell_i}})_{i \in \mathbb{N}}\) by \((z_{m_i})_{i \in \mathbb{N}}\). There is \( \rho \) with
\[
0 < \rho < \frac{\epsilon}{4\|u_0\|}
\]
such that
\[
B_{2}^{n+1} \left( \left( x_0, \psi(x_0) + \frac{\epsilon}{2}, \frac{\rho}{\|u_0\|} \right) \right) \\
\subseteq \text{epi} \psi \cap \{(x, s) | s \leq \alpha(u_0) - \langle x, u_0 \rangle \} = \{(x, s) | \psi(x) \leq s \leq \alpha(u_0) - \langle x, u_0 \rangle \}.
\]
This is shown in the same way as (52). Moreover, let $x_{m_i}, i \in \mathbb{N}$, be given by

$$z_0 = \frac{r}{\| y_{m_i} \|} y_{m_i} + \left( 1 - \frac{r}{\| y_{m_i} \|} \right) x_{m_i}. \quad (63)$$

Then,

$$z_0 - z_{m_i} = \left( 1 - \frac{r}{\| y_{m_i} \|} \right) (x_{m_i} - x_0).$$

Since $z_0 = \lim_{i \to \infty} z_{m_i}$, it follows $x_0 = \lim_{i \to \infty} x_{m_i}$. Since $x_0$ is in the interior of the domain of $\psi$ there is $\alpha > 0$ such that $B^n_2(x_0, \alpha)$ is a compact subset of the interior of the domain of $\psi$. The sequence $(\psi_{m})_{m \in \mathbb{N}}$ converges uniformly to $\psi$ on $B^n_2(x_0, \alpha)$. Therefore, for every $\epsilon > 0$ there is $M_2$ so that for all $m \geq M_2$ and all $x \in B^n_2(x_0, \alpha)$

$$|\psi(x) - \psi_{m}(x)| < \frac{\epsilon}{4}.$$

Since $\psi$ is continuous at $x_0$ there is $\eta > 0$ such that for all $x \in B^n_2(x_0, \eta)$

$$|\psi(x_0) - \psi(x)| < \frac{\epsilon}{4}.$$

We may assume that $\eta < \alpha$. Therefore, for all $x \in B^n_2(x_0, \eta)$ and all $m \geq M_2$

$$|\psi(x_0) - \psi_{m}(x)| < \frac{\epsilon}{2}.$$

It follows that there is $M_3$ such that for all $i \geq M_3$

$$|\psi(x_0) - \psi_{m_i}(x_{m_i})| < \frac{\epsilon}{2}. \quad (64)$$

By (49) and (60)

$$\alpha(u_0) - \langle u_0, y_{m_i} \rangle \geq \psi_{m_i}(y_{m_i}). \quad (65)$$

Moreover, by (44) and (64)

$$\alpha(u_0) - \langle u_0, x_{m_i} \rangle = \psi_{\delta}(x_0) \geq \psi(x_0) + \epsilon \geq \psi_{m_i}(x_{m_i}) + \frac{\epsilon}{2}.$$

There is $M_4$ such that for all $i$ with $m_i \geq M_4$ we have $\|u_0\| \|x_0 - x_{m_i}\| < \frac{\epsilon}{4}$. Therefore, for all $i$ with $m_i \geq M_4$

$$\alpha(u_0) - \langle u_0, x_{m_i} \rangle = \alpha(u_0) - \langle u_0, x_0 \rangle + \langle u_0, x_0 - x_{m_i} \rangle$$

$$\geq \alpha(u_0) - \langle u_0, x_0 \rangle - \|u_0\| \|x_0 - x_{m_i}\| \geq \psi_{m_i}(x_{m_i}) + \frac{\epsilon}{4}. \quad (66)$$
By (63)

\[ \alpha(u_0) - \langle z_0, u_0 \rangle = \frac{r}{\|y_m\|} (\alpha(u_0) - \langle u_0, y_m \rangle) + \left(1 - \frac{r}{\|y_m\|}\right) \left(\alpha(u_0) - \langle u_0, x_m \rangle \right). \]

By (65), (66) and the convexity of \( \psi \) there is \( M_5 \) such that for all \( i \) with \( m_i \geq M_5 \)

\[ \alpha(u_0) - \langle z_0, u_0 \rangle \geq r \|y_m\| \psi_{m_i}(z_0) + \left(1 - \frac{r}{\|y_m\|}\right) \left(\psi_{m_i}(x_m) + \epsilon \right) \geq \psi_{m_i}(z_0). \]

By this and (62)

\[ \left[ (z_0, \psi_{m_i}(z_0)), B^{n+1}_2 \left(x_0, \psi(x_0) + \frac{\epsilon}{2}, \frac{\rho}{\|u_0\|}\right) \right] \subseteq \text{epi} \psi_{m_i} \cap \{ (x, s) | \alpha(u_0) - \langle u_0, z_0 \rangle \geq s \}. \]

This implies

\[ \delta \int_{\mathbb{R}^n} e^{-\psi_{m_i}} \, dx = \text{vol}_{n+1} \{ (x, s) | \psi_{m_i}(x) \leq s \leq \alpha(u_0) - \langle u_0, z_0 \rangle \} \]

\[ \geq \text{vol}_{n+1} \left[ (z_0, \psi_{m_i}(z_0)), B^{n+1}_2 \left(x_0, \psi(x_0) + \frac{\epsilon}{2}, \frac{\rho}{\|u_0\|}\right) \right] \]

\[ \geq \frac{\rho^n}{\|u_0\|^n} \frac{\text{vol}_n(B^{2n}_2)}{n+1} \|z_0 - x_0\|. \]

By (61) we have \( \|z_0 - x_0\| = r \). Since \( r \) was arbitrary this cannot be. Thus we have shown that there is \( R > 0 \) such that for all \( m \in \mathbb{N} \) and all \( x \) with \( \|x\| > R \)

\[ \max\{0, \alpha_m(u_0) - \langle x, u_0 \rangle - \psi_m(x)\} = 0. \]

Thus, we have shown part of (48): The support of this function is contained in \( RB^n_2 \).

We show now that there are constants \( \gamma_1 \) and \( \gamma_2 \) such that for all \( m \in \mathbb{N} \) we have

\[ \gamma_1 \leq \alpha_m(u_0) \leq \gamma_2. \tag{67} \]

We show the left side inequality first. Assume it does not hold. By Lemma 9 there is \( c_1 \in \mathbb{R} \) such that for all \( m \in \mathbb{N} \) and all \( x \in \mathbb{R}^n \),

\[ c_1 \leq \psi_m(x). \tag{68} \]

Therefore, for all \( x \in B^n_2(R) \) and all \( m \in \mathbb{N} \)

\[ \max\{0, \alpha_m(u_0) - \langle x, u_0 \rangle - \psi_m(x)\} \leq \max\{0, \alpha_m(u_0) + \|x\|\|u_0\| - c_1\} \]

\[ \leq \max\{0, \alpha_m(u_0) + R\|u_0\| - c_1\} \]

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Since we assume that the left side inequality does not hold there is $m$ such that for all $x \in \mathbb{R}^n$

$$\max\{0, \alpha_m(u_0) - \langle x, u_0 \rangle - \psi_m(x)\} = 0.$$ 

This implies $\int_{\mathbb{R}^n} e^{-\psi_m} \, dx = 0$ and this contradicts (46). Now, we show the right side inequality of (67). Assume it does not hold. Consider $x_0 \in \text{int}(\text{dom}(\psi))$. There are $\rho > 0$ and $s_0$ such that there is $M_5$ so that for all $m \geq M_5$

$$B_2^{n+1}(x_0, s_0) \subseteq epi \psi_m.$$ 

For sufficiently big $m$ we have

$$B_2^{n+1}(x_0, s_0) \subseteq epi \psi_m \cap \{(x, s) | s \leq \alpha_m(u_0) - \langle u_0, x \rangle\}.$$ 

Therefore, for sufficiently big $\alpha_m(u_0)$

$$\left[(x_0, \alpha_m(u_0) - \langle x_0, u_0 \rangle), B_2^{n+1}(x_0, s_0) \right] \subseteq epi \psi_m \cap \{(x, s) | s \leq \alpha_m(u_0) - \langle u_0, x \rangle\}.$$ 

This implies

$$\rho^n \text{vol}_n(B_2^n|\alpha_m(u_0) - (x_0, u_0) - s_0| \leq epi \psi_m \cap \{(x, s) | s \leq \alpha_m(u_0) - \langle u_0, x \rangle\}$$

$$= \delta \int_{\mathbb{R}^n} e^{-\psi_m} \, dx.$$ 

Since the sequence $(\alpha_m(u_0))_{m \in \mathbb{N}}$ is not bounded from above this cannot be true. We have shown (48).

We show that $\lim_{m \to \infty} \alpha_m(u_0)$ exists. Suppose that there are two subsequences $(\alpha_{m_j}(u_0))_{j=1}^{\infty}$ and $(\alpha_{\ell_j}(u_0))_{j=1}^{\infty}$ with

$$\lim_{j \to \infty} \alpha_{m_j}(u_0) = a < b = \lim_{j \to \infty} \alpha_{\ell_j}(u_0).$$ 

We apply the Dominated Convergence Theorem to the sequence $\max\{0, \alpha_{m_j}(u_0) - \langle x, u_0 \rangle - \psi_{m_j}(x)\}, i \in \mathbb{N}$. We have $\lim_{m \to \infty} \psi_m(x) = \psi(x)$ a.e. and by (48) a dominating function. Therefore

$$\lim_{j \to \infty} \int_{\mathbb{R}^n} \max\{0, \alpha_{m_j}(u_0) - \langle x, u_0 \rangle - \psi_{m_j}(x)\} \, dx$$

$$= \int_{\mathbb{R}^n} \max\{0, a - \langle x, u_0 \rangle - \psi(x)\} \, dx.$$
and

\[
\lim_{j \to \infty} \int_{\mathbb{R}^n} \max\{0, \alpha \ell_j(u_0) - \langle x, u_0 \rangle - \psi \ell_j(x)\} \, dx = \int_{\mathbb{R}^n} \max\{0, b - \langle x, u_0 \rangle - \psi(x)\} \, dx.
\]

By (46)

\[
\delta \int_{\mathbb{R}^n} e^{-\psi} \, dx = \int_{\mathbb{R}^n} \max\{0, a - \langle x, u_0 \rangle - \psi(x)\} \, dx = \int_{a \geq \langle x, u_0 \rangle - \psi(x)} a - \langle x, u_0 \rangle - \psi(x) \, dx \leq \delta \int_{\mathbb{R}^n} e^{-\psi} \, dx.
\]

This is a contradiction. Therefore \( a = b \) and the sequence \((\alpha_m(u_0))_{m=1}^{\infty}\) converges. By (48) we can apply the Dominated Convergence Theorem

\[
\delta \int_{\mathbb{R}^n} e^{-\psi} \, dx = \int_{\mathbb{R}^n} \max\left\{0, \lim_{m \to \infty} \alpha_m(u_0) - \langle x, u_0 \rangle - \psi(x)\right\} \, dx.
\]

It follows that \(\lim_{m \to \infty} \alpha_m(u_0) = \alpha(u_0)\) and we have shown (45) and consequently (43).

Now we show that for all \(x_0\) in the interior of the domain of \(\psi\),

\[
\limsup_{m \to \infty} (\psi_m)_{\delta}(x_0) \leq \psi_{\delta}(x_0).
\]

If

\[
\limsup_{m \to \infty} (\psi_m)_{\delta}(x_0) \leq \psi(x_0) \tag{69}
\]

then

\[
\limsup_{m \to \infty} (\psi_m)_{\delta}(x_0) \leq \psi(x_0) \leq \psi_{\delta}(x_0).
\]

Therefore we may assume that (69) does not hold, i.e. there is \(\epsilon > 0\) such that

\[
\limsup_{m \to \infty} (\psi_m)_{\delta}(x_0) \geq \epsilon + \psi(x_0). \tag{70}
\]

By Lemma 11, there are \(u_m\) and \(\alpha_m(u_m)\) such that \((\psi_m)_{\delta}(x_0) = \alpha_m(u_m) - \langle u_m, x_0 \rangle\). We show that the sequences \(\alpha_m(u_m), m \in \mathbb{N}\), and \(\|u_m\|, m \in \mathbb{N}\), are bounded. As a first step we show that \(\alpha_m(u_m) - \langle x_0, u_m \rangle, m \in \mathbb{N}\), is a bounded sequence. Suppose this is not true. Since \(x_0\) is an interior point of the domain of \(\psi\) there is \(\rho > 0\) such
that \( B^n_2(x_0, \rho) \) is compact and is contained in the interior of the domain of \( \psi \). Then the sequence \( \psi_m, m \in \mathbb{N}, \) converges uniformly on \( B^n_2(x_0, \rho) \) to \( \psi \). Therefore there is \( M_0 \) such that for all \( m \geq M_0 \) and all \( x \in B^n_2(x_0, \rho) \)

\[
|\psi(x) - \psi_m(x)| < \frac{\epsilon}{4}
\]

and by continuity of \( \psi \) in \( x_0 \)

\[
|\psi(x_0) - \psi(x)| < \frac{\epsilon}{4}.
\]

Therefore for all \( m \geq M_0 \) and all \( x \in B^n_2(x_0, \rho) \)

\[
|\psi(x_0) - \psi_m(x)| < \frac{\epsilon}{2}.
\]

Therefore, for all \( m \) with \( m \geq M_0 \)

\[
\text{epi } \psi_m \cap \{(x, s) | s \leq \alpha_m(u_m) - (u_m, x)\}
\]

\[
= \{(x, s) | \psi_m(x) \leq s \leq \alpha_m(u_0) - (u_m, x)\}
\]

\[
\sup \{(x, s) | \|x - x_0\| \leq \rho \text{ and } \psi(x_0) + \epsilon \leq s \leq \alpha_m(u_m) - (x, u_m)\}. \quad (71)
\]

We obtain for all \( m \) such that \( m \geq M_0 \) and such that \( \epsilon + \psi(x_0) \leq \alpha_m(u_m) - (x_0, u_m) \)

\[
\text{epi } \psi_m \cap \{(x, s) | s \leq \alpha_m(u_m) - (u_m, x)\}
\]

\[
\sup \{(x, \alpha_m(u_m) - (x_0, u_m)), (x, \psi(x_0) + \epsilon) | \|x - x_0\| \leq \rho, (x - x_0, u_m) \leq 0\}.
\]

Indeed, this follows by adding the inequalities \( \epsilon + \psi(x_0) \leq \alpha_m(u_m) - (x_0, u_m) \) and \( (x - x_0, u_m) \leq 0 \). The set

\[
\{(x, \psi(x_0) + \epsilon) | \|x - x_0\| \leq \rho \text{ and } (x - x_0, u_m) \leq 0\}
\]

is half of an \( n \)-dimensional Euclidean ball. Therefore for all \( m \) such that \( m \geq M_0 \) and such that \( \epsilon + \psi(x_0) \leq \alpha_m(u_m) - (x_0, u_m) \)

\[
\delta \int_{\mathbb{R}^n} e^{-\psi_m} dx = \text{vol}_{n+1} (\text{epi } \psi_m \cap \{(x, s) | s \leq \alpha_m(u_m) - (u_m, x)\})
\]

\[
\geq \text{vol}_{n+1} \{(x, s) | \|x - x_0\| \leq \rho \text{ and } \psi(x_0) + \epsilon \leq s \leq \alpha_m(u_m) - (x, u_m)\}
\]

\[
\geq \rho^n \text{vol}_{n} (B^n_2) |\alpha_m(u_m) - (x_0, u_m) - \psi_m(x_0)|.
\]

Therefore the sequence \( \alpha_m(u_m) - (x_0, u_m), m \in \mathbb{N}, \) is bounded.

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We show that the sequence $\|u_m\|, m \in \mathbb{N}$, is bounded. By (71) there is $M_0$ such that for all $m \geq M_0$

$$\operatorname{epi} \psi_m \cap \{(x, s) | s \leq \alpha_m(u_m) - \langle x, u_m \rangle\}$$

$$\supseteq \{(x, s) | \|x - x_0\| \leq \rho \text{ and } \psi(x_0) + \epsilon \leq s \leq \alpha_m(u_m) - \langle x, u_m \rangle\}.$$  

We consider the point

$$x = x_0 - \frac{\rho \|u_m\|}{u_m}.$$

We have

$$\alpha_m(u_m) - \left(x_0 - \frac{\rho \|u_m\|}{u_m}, u_m\right) = \alpha_m(u_m) - \langle x_0, u_m \rangle + \rho \|u_m\|.$$

Therefore, for all $m$ with $m \geq M_0$ and with $\alpha_m(u_m) - \langle x_0, u_m \rangle + \rho \|u_m\| \geq \psi(x_0) + \epsilon$

$$\operatorname{epi} \psi_m \cap \{(x, s) | \alpha_{m_j}(u_{m_j}) - \langle x, u_m \rangle \geq s\}$$

$$\supseteq \left[\left(x_0 - \frac{\rho}{\|u_m\|} u_m, \alpha_m(u_m) - \langle x_0, u_m \rangle + \rho \|u_m\|\right), \{x, \psi(x_0) + \epsilon\} | \|x - x_0\| \leq \rho \text{ and } \langle x - x_0, u_m \rangle \leq 0\right].$$

It follows for all $m$ with $m \geq M_0$ and with $\alpha_m(u_m) - \langle x_0, u_m \rangle + \rho \|u_m\| \geq \psi(x_0) + \epsilon$

$$\delta \int_{\mathbb{R}^n} e^{-\psi_m} dx = \operatorname{vol}_{n+1}(\operatorname{epi} \psi_m \cap \{(x, s) | \alpha_m(u_m) - \langle x, u_m \rangle \geq s\})$$

$$\geq \frac{\rho^n}{2(n + 1)} \operatorname{vol}_n(B^2_{\delta})(\alpha_m(u_m) - \langle x_0, u_m \rangle + \rho \|u_m\| - \psi(x_0) - \epsilon)$$

$$\geq \frac{\rho^{n+1} \|u_m\|}{2(n + 1)} \operatorname{vol}_n(B^2_{\delta}).$$

Therefore the sequence $\|u_m\|, m \in \mathbb{N}$, is bounded.

Therefore, by passing to a subsequence we may assume

$$\lim_{j \to \infty} \alpha_{m_j}(u_{m_j}) - \langle x_0, u_{m_j} \rangle = \limsup_{m \to \infty} \alpha_m(u_m) - \langle x_0, u_m \rangle$$

and

$$\lim_{j \to \infty} u_{m_j} = v_0, \text{ and } \lim_{j \to \infty} \alpha_{m_j}(u_{m_j}) = \beta.$$  

Then

$$\delta \int_{\mathbb{R}^n} e^{-\psi_{m_j}} dx = \int_{\mathbb{R}^n} \max\{0, \alpha_{m_j}(u_{m_j}) - \langle x, u_{m_j} \rangle - \psi_{m_j}(x)\} dx.$$  

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Since the sequence \( f_m, m \in \mathbb{N} \), converges in \( L_1 \) to \( f \) we have \( \lim_{j \to \infty} \int_{\mathbb{R}^n} e^{-\psi_{m_j}} = \int_{\mathbb{R}^n} e^{-\psi} \). Thus by Fatou’s lemma,

\[
\delta \int_{\mathbb{R}^n} e^{-\psi} \, dx = \liminf_j \int_{\mathbb{R}^n} \max\{0, \alpha_{m_j}(u_{m_j}) - \langle x, u_{m_j} \rangle - \psi_{m_j}(x)\} \, dx \\
\geq \int_{\mathbb{R}^n} \liminf_j \left( \max\{0, \alpha_{m_j}(u_{m_j}) - \langle x, u_{m_j} \rangle - \psi_{m_j}(x)\} \right) \, dx \\
= \int_{\mathbb{R}^n} \max\{0, \beta - \langle x, v_0 \rangle - \psi(x)\} \, dx.
\]

This means

\[
\psi_\delta(x_0) \geq \beta - \langle x_0, v_0 \rangle = \lim_{j \to \infty} \alpha_{m_j}(u_{m_j}) - \langle u_{m_j}, x_0 \rangle \\
= \limsup_{m \to \infty} \alpha_m(u_m) - \langle x_0, u_m \rangle = \limsup_{m \to \infty} (\psi_m)_\delta(x_0).
\]

Hence

\[
\psi_\delta(x_0) \geq \beta - \langle x_0, v_0 \rangle = \lim_{j \to \infty} \left( \alpha_{m_j}(u_{m_j}) - \langle x_0, u_{m_j} \rangle \right) = \limsup_{j \to \infty} (\psi_{m_j})_\delta(x_0).
\]

\[\square\]

### 5 The Löwner Function of a Log-Concave Function

The Löwner function of a log-concave function was introduced in [73]. It was also shown there that this is an extension of the notion of Löwner ellipsoid for convex bodies. The Löwner function is defined as follows.

**Definition 5** [73] Let \( f : \mathbb{R}^n \to \mathbb{R}^+ \), \( f(x) = e^{-\psi(x)} \) be a nondegenerate, integrable log-concave function. Then the Löwner function \( L(f) \) of \( f \) is defined as

\[
L(f)(x) = e^{-\|A_0x\| + t_0},
\]

where \((A_0, t_0) = (T_0 + b_0, t_0)\) is the solution to the minimization problem

\[
\min_{(A,t)} \int_{\mathbb{R}^n} e^{-\|Ax\| + t} \, dx = n! \, \text{vol}(B_2^n) \min_{(A,t)} \frac{e^t}{|\det T|}
\]

subject to

\[
\|Ax\| - t \leq \psi(x), \quad \text{for all } x \in \mathbb{R}^n,
\]

where the minimum is taken over all nonsingular affine maps \( A = T + b \in \mathcal{A} \) and all \( t \in \mathbb{R} \).
It was shown in [73] that the minimization problem (73) subject to the constraint condition (74) has a solution \((A_0, t_0)\) where the number \(t_0\) is unique and the affine map \(A_0\) is unique up to left orthogonal transformations. Thus the Löwner function is well defined.

A different definition of Löwner function was put forward in [72]. However, this Löwner function is not an affine covariant mapping. It is not translation invariant.

**Theorem 2** Let \(f = \exp(-\psi)\) be a function in LC. Then the Löwner operator \(L : LC \to LC\), mapping \(f\) to its Löwner function \(L(f)\) (72), is an affine covariant mapping.

The next corollary follows immediately from the theorem, together with Remark 2.

**Corollary 2** Let \(f = \exp(-\psi)\) be a function in LC. Then for all \(\lambda \in \mathbb{R}\),

\[
g(Lf), \quad s(Lf), \quad \lambda g(Lf) + (1 - \lambda)s(Lf)
\]

are affine contravariant points.

We remark that the centroid \(g(Lf)\) of the Löwner function of \(f\) was called the Löwner point of \(l(f)\) of \(f\) in [73].

**Lemma 13** Let \((f_m)_{m=1}^{\infty}\) be a sequence in LC that converges in \(L_1\) to the log-concave function \(f \in LC\). Let \((T_m, b_m, t_m), m \in \mathbb{N}\), be the minimizers for \(f_m\), \(m \in \mathbb{N}\), and let \((T, b, t)\) be the minimizer for \(f\). Then the sequences \((\|T_m\|_{\text{Op}})_{m=1}^{\infty}, (\|b_m\|)_{m=1}^{\infty}\), and \((t_m)_{m=1}^{\infty}\) are bounded.

**Proof of Lemma 13** By assumption, \(0 \in \text{int}(\text{supp}(f))\). Thus \(\psi(0) < \infty\). We consider the convex set

\[\{x | \psi(x) \leq \psi(0) + 2\}.\]

Since \(e^{-\psi}\) is integrable, \(\{x | \psi(x) \leq \psi(0) + 2\}\) is bounded. By Lemma 9 there is \(t \in \mathbb{R}\), \(\rho > 0\) and \(m_0 \in \mathbb{N}\) such that for all \(m \geq m_0\) and all \(x \in \mathbb{R}^n\)

\[f_m(x) \leq \exp\left(-\frac{\|x\|}{\rho} + t\right). \tag{75}\]

We first show that the sequence \((t_m)_{m=1}^{\infty}\) is bounded. Since 0 is an interior point of the support of \(f\) there are \(\alpha_0\) and \(\delta > 0\) such that \(B^n_2(0, \delta)\) is contained in the interior of the support of \(f\) and such that for all \(x \in B^n_2(0, \delta)\)

\[\psi(x) \leq \left(\psi(0) + \frac{1}{2}\right)1_{B^n_2(0, \delta)}.\]

By Lemma 8 the sequence \((\psi_m)_{m=1}^{\infty}\) converges uniformly to \(\psi\) on all compact subsets of the interior of the domain of \(\psi\). Therefore we can choose \(m\) so big that \(|\psi_m(x) - \psi(x)| \leq \frac{1}{4}\|x\|_{\text{Op}}\) for all \(x \in B^n_2(0, \delta)\). We have for all \(x \in B^n_2(0, \delta)\)

\[-t_m \leq \|T_m(x + b_m)\| - t_m \leq \psi_m(x) \leq \psi(0) + \frac{3}{4}. \tag{76}\]
From this inequality it follows immediately that the sequence \((t_m)_{m=1}^{\infty}\) is bounded from below. Moreover, it follows that for all \(y \in B_2^n(0, \frac{1}{4})\) we have

\[
\left\| T_m \left( y + \frac{b_m}{4} \right) \right\| - \frac{t_m}{4} \leq \frac{1}{4} \left( \psi(0) + \frac{3}{4} \right).
\]

Therefore for all \(x \in B_2^n\left(-\frac{3}{4}b_m, \frac{\delta}{4}\right)\)

\[
\| T_m(x + b_m) \| - t_m \leq -\frac{3}{4}t_m + \frac{\psi(0)}{4} + \frac{3}{16}
\]

and

\[
\exp\left( \frac{3}{4}t_m - \frac{\psi(0)}{4} - \frac{3}{16} \right) \leq \exp(-\| T_m(x + b_m) \| + t_m).
\]

It follows

\[
\int_{\mathbb{R}^n} \exp(-\| T_m(x + b_m) \| + t_m) \, dx \\
\geq \int_{B_2^n\left(-\frac{3}{4}b_m, \frac{\delta}{4}\right)} \exp\left( \frac{3}{4}t_m - \frac{\psi(0)}{4} - \frac{3}{16} \right) \, dx \\
\geq \exp\left( \frac{3}{4}t_m - \frac{\psi(0)}{4} - \frac{3}{16} \right) \text{vol}_n\left( B_2^n\left(-\frac{3}{4}b_m, \frac{\delta}{4}\right) \right) \\
= \exp\left( \frac{3}{4}t_m - \frac{\psi(0)}{4} - \frac{3}{16} \right) \left( \frac{\delta}{4} \right)^n \text{vol}_n\left( B_2^n \right).
\]

Let \(I_n\) be the \(n \times n\) identity matrix. (75) implies that \((\frac{1}{\rho} I_n, 0, t)\) satisfies (74) for \(f_m = e^{-\psi_m}, m \in \mathbb{N}\). Since \((T_m, b_m, t_m)\) is the minimizer for \(f_m = e^{-\psi_m}\)

\[
\int_{\mathbb{R}^n} \exp(-\| T_m(x + b_m) \| + t_m) \, dx \leq \int_{\mathbb{R}^n} \exp\left( -\frac{1}{\rho} \| x \| + t \right) \, dx.
\]

Therefore

\[
\exp\left( \frac{3}{4}t_m - \frac{\psi(0)}{4} - \frac{3}{16} \right) \left( \frac{\delta}{4} \right)^n \text{vol}_n\left( B_2^n \right) \leq \int_{\mathbb{R}^n} \exp\left( -\frac{1}{\rho} \| x \| + t \right) \, dx.
\]

It follows that the sequence \((t_m)_{m=1}^{\infty}\) is bounded from above. Since we know already that the sequence \((t_m)_{m=1}^{\infty}\) is bounded from below it is bounded.

Now we show that the sequence \((b_m)_{m=1}^{\infty}\) is bounded. By (76) we have for all \(x \in B_2^n(0, \delta)\)

\[
\| T_m(x + b_m) \| - t_m \leq \psi_m(x) \leq \psi(0) + \frac{3}{4}.
\]

\[\square\]
Since the sequence \((t_m)_{m=1}^{\infty}\) is bounded from above there is a constant \(c > 0\) such that for all \(x \in B_2^n(0, \delta)\) and all \(m \in \mathbb{N}\)

\[
\|T_m(x + b_m)\| \leq c. \tag{79}
\]

Therefore, for all \(\lambda \in [0, 1]\), for all \(x \in B_2^n(0, \delta)\) and all \(m \in \mathbb{N}\)

\[
\|T_m(\lambda(x + b_m))\| \leq c.
\]

It follows that for all \(m \in \mathbb{N}\) and all \(z \in \text{co}[0, B_2^n(b_m, \delta)]\)

\[
\|T_m(z)\| \leq c.
\]

By this and (77) there is a constant \(c' > 0\) such that for all \(m \in \mathbb{N}\)

\[
\int_{\mathbb{R}^n} \exp \left( -\frac{1}{\rho} \|x\| + t \right) \, dx 
\geq \int_{\mathbb{R}^n} \exp(-\|T_m(x + b_m)\| + t) dx
\geq e^{tm} \int_{\mathbb{R}^n} \exp(-\|T_m(y)\|) dy 
\geq e^{tm} \int_{\text{co}[0, B_2^n(b_m, \delta)]} \exp(-c) dx
\geq \exp(-c + c') \text{vol}_n(\text{co}[0, B_2^n(b_m, \delta)]).
\]

We have

\[
\text{vol}_n(\text{co}[0, B_2^n(b_m, \delta)]) \geq \frac{1}{n} \|b_m\| \delta^{n-1} \text{vol}_{n-1}(B_2^{n-1})
\]

and consequently

\[
\int_{\mathbb{R}^n} \exp \left( -\frac{1}{\rho} \|x\| + t \right) \, dx \geq \frac{\exp(-c + c')}{{n} \|b_m\| \delta^{n-1} \text{vol}_{n-1}(B_2^{n-1})}.
\]

Therefore the sequence \((\|b_m\|)_{m=1}^{\infty}\) is bounded.

Now we show that the sequence \((\|T_m\|_{\text{op}})_{m=1}^{\infty}\) is bounded. By (79) there is \(c > 0\) such that for all \(x \in B_2^n(\delta)\)

\[
\|T_m(x + b_m)\| \leq c.
\]

In particular for \(x = 0\),

\[
\|T_m(b_m)\| \leq c.
\]

By triangle inequality, for all \(x \in B_2^n(\delta)\)

\[
\|T_m(x)\| \leq c + \|T_m(b_m)\| \leq 2c.
\]
Therefore
\[ \| T_m \|_{\text{Op}} \leq \frac{2c}{\delta}. \] (80)

Altogether we have shown that the sequences \((t_m)_{m=1}^{\infty}\), \((\|b_m\|)_{m=1}^{\infty}\) and \((\|T_m\|_{\text{Op}})_{m=1}^{\infty}\) are bounded.

\[ \square \]

Lemma 14 Let \( f \in \text{LC} \) with minimizer \((T, b, t)\). Let \( C_k, k \in \mathbb{N} \), be compact subsets of \( \text{int}(\text{supp}(f)) \) such that \( C_k \subseteq C_{k+1} \) for \( k \in \mathbb{N} \) and

\[ \text{int}(\text{supp}(f)) = \bigcup_{k \in \mathbb{N}} C_k. \]

For all \( k \in \mathbb{N} \) the functions \( f \cdot 1_{C_k} \) are in \( \text{LC} \). Let \((T_k, b_k, t_k)\) be the minimizer for \( f \cdot 1_{C_k} = e^{-\psi} 1_{C_k} \). The sequence \((L(f \cdot 1_{C_k}))_{k=1}^{\infty}\) converges in \( L_1 \) to \( L(f) \).

Proof By Lemma 13 the sequences \((T_k)_{k=1}^{\infty}\), \((b_k)_{k=1}^{\infty}\) and \((t_k)_{k=1}^{\infty}\) are bounded. We show that

\[ \lim_{k \to \infty} (T_k, b_k, t_k) = (T, b, t). \]

Suppose this is not the case. Then there are two convergent subsequences that converge to different limits. We show that all convergent subsequences \((T_{k_j}, b_{k_j}, t_{k_j})\) converge to the same limit, the minimizer \((T, b, t)\) of \( f = e^{-\psi} \).

We have
\[ f(x) 1_{C_k}(x) \leq f(x) \leq \exp(-\|T(x + b)\| + t). \]

Therefore, for all \( k \in \mathbb{N} \)
\[ \frac{e^{tk}}{|\det T_k|} \leq \frac{e^t}{|\det T|}. \]

This implies
\[ \lim_{j \to \infty} \frac{e^{tk_j}}{|\det T_{k_j}|} \leq \frac{e^t}{|\det T|}. \]

On the other hand,
\[ f(x) \leq \lim_{j \to \infty} \exp(-\|T_{k_j}(x + b_{k_j})\| + t_{k_j}). \]

This implies
\[ \lim_{j \to \infty} \frac{e^{tk_j}}{|\det T_{k_j}|} = \frac{e^t}{|\det T|}. \] (81)
By the uniqueness of the minimizer of $f$ we get

$$\lim_{j \to \infty} (T_{k_j}, b_{k_j}, t_{k_j}) = (T, b, t).$$

This implies that $L(f \cdot 1_{C_{k_j}}(x)) = e^{-\|T_{k_j}(x+b_{k_j})\| + t_{k_j}} \to L(f)(x) = e^{-\|T(x+b)\| + t}$ pointwise and hence in $L_1$ by Lemma 8.

Lemma 15 Let $\psi_m : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}, m \in \mathbb{N}$, be a sequence of convex functions that converges pointwise to a convex function $\psi$. Moreover, let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an affine map and $t \in \mathbb{R}$ such that for all $x \in \mathbb{R}^n$

$$\psi(x) \geq \|Ax\| - t.$$ 

Then for every $\epsilon > 0$ and every $h \in \mathbb{R}$, $h > \min_{x \in \mathbb{R}^n} \psi(x) + \epsilon$ there is $m_0 \in \mathbb{N}$ such that for all $m$ with $m \geq m_0$ and all $x$ with $\|Ax\| - t \leq h$

$$\psi_m(x) \geq \|Ax\| - t - \epsilon. \quad (82)$$

The minimum $\min_{x \in \mathbb{R}^n} \psi(x)$ exists since $e^{-\psi}$ is integrable.

Proof Let $\epsilon > 0$ and $h \in \mathbb{R}$ with $h > \min_{x \in \mathbb{R}^n} \psi(x) + \epsilon$. Let $x$ be such that

$$\|Ax\| - t \leq h.$$ 

For this fixed $\epsilon$, there is positive $\eta > 0$ such that

$$\{y : \|Ay\| - t \leq h\} \supset \{y : \|Ay\| - t \leq h - \epsilon\} + \eta B_2^n.$$

By Lemma 6 we have that

$$E_{\psi_m}(s) \to E_\psi(s)$$

in Hausdorff metric for all $s > \min_x \psi$. Since $h - \epsilon > \min_x \psi(x)$ there exists $m_1 = m_1(h, \eta) \in \mathbb{N}$ such that for all $m > m_1$,

$$\{y : \psi_m(y) \leq h - \epsilon\} \subset \{y : \psi(y) \leq h - \epsilon\} + \eta B_2^n$$

$$\subset \{y : \|Ay\| - t \leq h - \epsilon\} + \eta B_2^n \subset \{y : \|Ay\| - t \leq h\}.$$ 

Let $x$ be such that $\|Ax\| - t \leq h$. If also $x$ is such that

$$x \notin \{x : \psi(x) \leq h - \epsilon\} + \eta B_2^n,$$

we then have that for all $m > m_1$

$$x \in \{y : \psi(y) \leq h - \epsilon\} + \eta B_2^n \subset \{y : \psi_m(y) \leq h - \epsilon\}.$$ 

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That is, $\psi_m(x) > h - \epsilon$ for all $m > m_1$. Hence

$$\psi_m(x) > h - \epsilon > \|Ax\| - t - \epsilon$$

for all $m > m_1$.

Otherwise assume that $x$ is such that

$$x \in \{y : \psi(y) \leq h - \epsilon\} + \eta B^a_2.$$

Since $\psi$ is lower semi continuous and since $e^{-\psi}$ is integrable the set $\{x : \psi(x) \leq h - \epsilon\}$ is a compact subset of dom($\psi$) and so is the set $\{x : \psi(x) \leq h - \epsilon\} + \eta B^a_2$, as $B^a_2$ is the closed unit ball. Thus by Lemma 8 (ii) we have that $\{\psi_m\}_{m=1}^{\infty}$ converges uniformly to $\psi$ on $\{x : \psi(x) \leq h - \epsilon\} + \eta B^a_2$. Hence for the same $\epsilon$ there exists $m_2$ such that whenever $m > m_2$, $\psi_m(x) > \psi(x) - \epsilon$ for all $x \in \{x : \psi(x) \leq h - \epsilon\} + \eta B^a_2$.

$$\psi_m(x) > \psi(x) - \epsilon$$

for all $x \in \{x : \psi(x) \leq h - \epsilon\} + \eta B^a_2$. Since $\psi(x) \geq \|Ax\| - t$ for all $x \in \mathbb{R}$, we have that on $\{x : \psi(x) \leq h - \epsilon\} + \eta B^a_2$, whenever $m > m_2$,

$$\psi_m(x) > \psi(x) - \epsilon \geq \|Ax\| - t - \epsilon.$$

Finally, let $m_0 = \max\{m_1, m_2\}$, we have (82).

**Lemma 16** Let $\psi_m : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$, $m \in \mathbb{N}$, be a sequence of convex functions that converges pointwise to a convex function $\psi$. Suppose that for all $x \in \mathbb{R}^n$

$$\psi(x) \geq \|T(x + b)\| - t. \quad (83)$$

Then for every $\epsilon > 0$, there is $m_0 \in \mathbb{N}$ such that for all $m$ with $m \geq m_0$ and all $x \in \mathbb{R}^n$

$$\psi_m(x) \geq (1 - \epsilon)\|T(x + b)\| - t - \epsilon. \quad (84)$$

**Proof** By Lemma 15, for all $\epsilon > 0$ and all $h$ with $h > \min_{x \in \mathbb{R}^n} \psi(x) + \epsilon$ there is $m_0$ such that for all $m \geq m_0$ and all $x \in \mathbb{R}^n$ with $\|T(x + b)\| - t \leq h$

$$\psi_m(x) \geq \|T(x + b)\| - t - \epsilon. \quad (85)$$

We may assume that $\psi(-b) < \infty$. We then choose $h$ so that

$$h \geq \max \left\{ 1 + |t|, \frac{\psi(-b)}{\epsilon} + 1 - t - \epsilon \right\}. \quad (86)$$

We consider now those $x \in \mathbb{R}^n$ with $\|T(x + b)\| - t - \epsilon = h$. The point $(-b, \psi(-b) + \epsilon)$ is an element of all the epigraphs of $\psi_m$ for $m \geq m_0$ by the pointwise convergence of $\{\psi_m\}_{m=1}^{\infty}$ to $\psi$. 

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In addition, claim that there is $m_1$ such that for all $m \geq m_1$ all points $(x, h)$ with $\|T(x + b)\| - t - \varepsilon = h$ are not an elements of the epigraphs of $\psi_m$. By Lemma 6 we have that

$$\{x : \psi_m(x) \leq h\} \rightarrow \{x : \psi(x) \leq h\}$$

in Hausdorff metric as $m \rightarrow \infty$. Thus for every $\eta > 0$ there exists $m_1$ such that when $m \geq m_1$

$$\{x : \psi_m(x) \leq h\} \subseteq \{x : \psi(x) \leq h\} + \eta B^n_2 \subseteq \{x : \|T(x + b)\| - t \leq h\} + \eta B^n_2.$$  

We can choose $\eta > 0$ small enough so that

$$\{x : \psi_m(x) \leq h\} \subset \{x : \|T(x + b)\| - t \leq h + \frac{\varepsilon}{2}\}.$$

Therefore, for all $(x, h)$ with $\|T(x + b)\| - t - \varepsilon = h$ and all $m \geq m_1$ we have $\psi_m(x) > h$. Hence $(x, h)$ with $\|T(x + b)\| - t - \varepsilon = h$ is not an element of the epigraphs of $\psi_m$ for $m \geq m_1$.

By convexity, no element of a ray emanating from $(-b, \psi(-b) + \varepsilon)$ through $x$ beyond $x$ is an element of any of the epigraphs of $\psi_m$ for $m > \max\{m_0, m_1\}$. Let $C(\psi)$ be the cone with apex $(-b, \psi(-b) + \varepsilon)$ and generated by the set of all $x$ with $\|T(x + b)\| - t - \varepsilon = h$. Then

$$\text{epi}(\psi_m) \cap \{x \in \mathbb{R}^{n+1} : x_{n+1} \geq h\} \subset C(\psi).$$

The boundary of the cone $C(\psi)$ is the graph of the map

$$\left(1 - \frac{\psi(-b) + \varepsilon}{h + t + \varepsilon}\right) \|T(x + b)\| + \psi(-b) + \varepsilon.$$

Indeed, this expression takes the value $\psi(-b) + \varepsilon$ for $x = -b$ and for all $x$ with $\|T(x + b)\| - t - \varepsilon = h$ we get

$$\left(1 - \frac{\psi(-b) + \varepsilon}{h + t + \varepsilon}\right) \|T(x + b)\| + \psi(-b) + \varepsilon = h + t + \varepsilon.$$

Therefore, for all $x$ with $\|T(x + b)\| - t - \varepsilon \geq h$ and all $m$ with $m \geq m_1$

$$\psi_m(x) \geq \left(1 - \frac{\psi(-b) + \varepsilon}{h + t + \varepsilon}\right) \|T(x + b)\| + \psi(-b) + \varepsilon.$$

By (83) we have $\psi(-b) \geq -t$. Therefore for all $x$ with $\|T(x + b)\| - t - \varepsilon \geq h$ and all $m$ with $m \geq m_1$

$$\psi_m(x) \geq \left(1 - \frac{\psi(-b) + \varepsilon}{h + t + \varepsilon}\right) \|T(x + b)\| - t$$
and by (85) for all \( m \geq m_0 \) and all \( x \in \mathbb{R}^n \) with \( \|T(x + b)\| - t \leq h \)

\[ \psi_m(x) \geq \|T(x + b)\| - t - \varepsilon. \]

Altogether we get for all \( x \in \mathbb{R}^n \) and all \( m > \max\{m_0, m_1\} \)

\[ \psi_m(x) \geq \left( 1 - \frac{\psi(-b) + \varepsilon}{h + t + \varepsilon} \right) \|T(x + b)\| - t - \varepsilon. \]

If \( \psi(-b) \leq 0 \) then

\[ \psi_m(x) \geq \left( 1 - \frac{\varepsilon}{h + t + \varepsilon} \right) \|T(x + b)\| - t - \varepsilon. \]

By (86) we have \( h \geq 1 + |t| \) and we get (84). If \( \psi(-b) \geq 0 \) we use \( h \geq \frac{\psi(-b)}{\varepsilon} + 1 - t - \varepsilon \) and obtain (84).

**Proof of Theorem 2** By definition \((Lf)(x) = e^{-\|A_0x\|+t_0}\), where

\[
\int_{\mathbb{R}^n} e^{-\|A_0x\|+t_0} = \min \left\{ \int_{\mathbb{R}^n} e^{-\|Ax\|+t} : (A, t) \in \mathcal{A} \times \mathbb{R}, \|Ax\| - t \leq \psi(x) \right\}.
\] (87)

We show that for every affine map \( B \) we have \( B(Lf) = L(Bf) \).

\[ B(Lf)(x) = e^{-\|A_0Bx\|+t_0}. \] (88)

On the other hand, \( L(Bf) \) arises from the solution to the following minimization problem,

\[
\min \left\{ \int_{\mathbb{R}^n} e^{-\|Ax\|+t} : (A, t) \in \mathcal{A} \times \mathbb{R}, \|Ax\| - t \leq \psi(Bx) \right\}
\]

\[
= \min \left\{ \frac{1}{|\det B|} \int_{\mathbb{R}^n} e^{-\|AB^{-1}y\|+t} dy : (A, t) \in \mathcal{A} \times \mathbb{R}, \|AB^{-1}y\| - t \leq \psi(y) \right\}
\]

\[
= \frac{1}{|\det B|} \int_{\mathbb{R}^n} e^{-\|A_0y\|+t_0} dy = \int_{\mathbb{R}^n} e^{-\|A_0Bx\|+t_0} dx.
\]

The second last equality holds by (87). This means that

\[ L(Bf)(x) = e^{-\|A_0Bx\|+t_0} = B(Lf)(x), \]

where we have used (88) in the last identity.

Now, we show the continuity of \( L \). Let \((T_m, b_m, t_m)_{m=1}^\infty\) be the minimizers of \( f_m \) and \((T_0, b_0, t_0)\) be the minimizer of \( f \). By Lemma 13 there are subsequences \((t_m_j)_{j=1}^\infty\), \((b_m_j)_{j=1}^\infty\) and \((T_m_j)_{j=1}^\infty\) that converge to some \( \bar{t}_0, \bar{b}_0 \) and \( \bar{T}_0 \). We want to argue now that
\( t_0 = t_0, \overline{b}_0 = b_0 \) and \( T_0 = T_0 \). For the ease of notation we rename the subsequence \((t_{m_j})_{j=1}^{\infty}, (b_{m_j})_{j=1}^{\infty}\) and \((T_{m_j})_{j=1}^{\infty}\) by \((t_m)_{m=1}^{\infty}, (b_m)_{m=1}^{\infty}\) and \((T_m)_{m=1}^{\infty}\).

Let \( C_k, k \in \mathbb{N}, \) be compact subsets of \( \text{int}(\text{supp}(f)) \) such that \( C_k \subseteq C_{k+1} \) for \( k \in \mathbb{N} \) and \( \text{int}(\text{supp}(f)) = \bigcup_{k \in \mathbb{N}} C_k. \)

For all \( k \in \mathbb{N} \) the functions \( f \cdot \mathbb{1}_{C_k} \) are log-concave and upper semi continuous. Let \((T_0, k, b_0, k, t_0, k)\) be the minimizer for \( f \cdot \mathbb{1}_{C_k} = e^{-\psi} \mathbb{1}_{C_k} \). The sequence \((f \cdot \mathbb{1}_{C_k})_{k=1}^{\infty}\) converges in \( L_1 \) to \( f \). By Lemma 13 the sequences \((T_{0,k})_{k=1}^{\infty}, (b_{0,k})_{k=1}^{\infty}\) and \((t_{0,k})_{k=1}^{\infty}\) are bounded. Therefore, there are convergent subsequences \((T_{0,k_j}, b_{0,k_j}, t_{0,k_j})\). We show that \((T_{0,k_j}, b_{0,k_j}, t_{0,k_j})\) converges to the minimizer \((T_0, b_0, t_0)\) of \( f = e^{-\psi} \).

We have

\[
-\frac{e^{t_0,k}}{|\det T_{0,k}|} \leq -\frac{e^{t_0}}{|\det T_0|}.
\]

This implies

\[
\lim_{j \to \infty} \frac{e^{t_0,k_j}}{|\det T_{0,k_j}|} \leq \frac{e^{t_0}}{|\det T_0|}.
\]

On the other hand,

\[
f(x) \leq \lim_{j \to \infty} \exp(-\|T_{0,k_j}(x + b_{0,k_j})\|_2 + t_{0,k_j}).
\]

This implies

\[
\lim_{j \to \infty} \frac{e^{t_0,k_j}}{|\det T_{0,k_j}|} = \frac{e^{t_0}}{|\det T_0|}.
\]

By the uniqueness of the minimizer of \( f \) we get

\[
\lim_{j \to \infty} (T_{0,k_j}, b_{0,k_j}, t_{0,k_j}) = (T_0, b_0, t_0).
\]

We consider now \( f_m = e^{-\psi_m} \) with their minimizers \((T_m, b_m, t_m)\) and the functions \( f_m \cdot \mathbb{1}_{C_k} \) with their minimizers \((T_{m,k}, b_{m,k}, t_{m,k})\). Since \( C_k \) is a compact subset of the interior of the support and \( f \) is by Lemma 7 continuous on the interior of its support

\[
0 < \min_{x \in C_k} f(x).
\]
By Lemma 8 the sequence \((f_m)_{m=1}^{\infty}\) converges uniformly on all compact subsets of the interior of the support of \(f\). For \(k\) we choose \(m_k\) so big that
\[
\| (f_{m_k} - f) 1_{C_k} \|_{\infty} \leq \left( \min_{x \in C_k} f(x) \right) \frac{1}{2^k}.
\]
It follows
\[
f_{m_k}(x) 1_{C_k}(x) \leq f(x) 1_{C_k}(x) + \left( \min_{x \in C_k} f(x) \right) \frac{1}{2^k} \leq \left( 1 + \frac{1}{2^k} \right) f(x) 1_{C_k}(x)
\]
and
\[
f_{m_k}(x) 1_{C_k}(x) \geq f(x) 1_{C_k}(x) - \left( \min_{x \in C_k} f(x) \right) \frac{1}{2^k} \geq \left( 1 - \frac{1}{2^k} \right) f(x) 1_{C_k}(x).
\]
With \(1 + t \leq e^t\)
\[
f_{m_k}(x) 1_{C_k}(x) \leq \left( 1 + \frac{1}{2^k} \right) f(x) 1_{C_k}(x) \leq \exp \left( -\|T_{0,k}(x + b_{0,k})\| + t_{0,k} + \frac{1}{2^k} \right)
\]
and thus
\[
f_{m_k}(x) 1_{C_k}(x) \leq \exp \left( -\|T_{0,k}(x + b_{0,k})\| + t_{0,k} + \frac{1}{2^k} \right). \quad (90)
\]
Moreover
\[
\left( 1 - \frac{1}{2^k} \right) f(x) 1_{C_k}(x) \leq f_{m_k}(x) 1_{C_k}(x) \leq \exp \left( -\|T_{m_k,k}(x + b_{m_k,k})\| + t_{m_k,k} \right).
\]
With \((1 - \frac{1}{2^k})^{-1} \leq 1 + \frac{1}{2^{k-1}}\)
\[
f(x) 1_{C_k}(x) \leq \exp \left( -\|T_{m_k,k}(x + b_{m_k,k})\| + t_{m_k,k} + \frac{1}{2^{k-1}} \right). \quad (91)
\]
By (90) and (91)
\[
\frac{e^{t_{0,k}}}{| \det T_{m_k,k} |} \leq \frac{e^{t_{0,k} + \frac{1}{2^k}}}{| \det T_{0,k} |} \quad \text{and} \quad \frac{e^{t_{0,k}}}{| \det T_{m_k,k} |} \leq \frac{e^{t_{0,k} + \frac{1}{2^{k-1}}}}{| \det T_{0,k} |}.
\]
Therefore
\[
\frac{e^{-\frac{1}{2^{k-1}}}}{| \det T_{0,k} |} \frac{e^{t_{0,k}}}{| \det T_{m_k,k} |} \leq \frac{e^{t_{0,k}}}{| \det T_{m_k,k} |} \leq \frac{e^{\frac{1}{2^k}}}{| \det T_{0,k} |} e^{t_{0,k}}
\]
and by (81) of Lemma 14
\[
\frac{e^{t_0}}{|\det T_0|} = \lim_{k \to \infty} \frac{e^{t_{m,k}^0}}{|\det T_{m,k}^0|}.
\]

Since \( f_{m,k} \geq f_{m,k} \cdot 1_{C_k} \) we have
\[
\frac{e^{t_{m,k}^0}}{|\det T_{m,k}^0|} \geq \frac{e^{t_{m,k}^0}}{|\det T_{m,k}^0|}
\]

and consequently
\[
\frac{e^{t_0}}{|\det T_0|} \leq \liminf_{k \to \infty} \frac{e^{t_{m,k}^0}}{|\det T_{m,k}^0|}.
\]

(92)

By Lemma 16 for every \( \epsilon > 0 \) we can choose \( m_0 \) big enough so that for all \( m \geq m_0 \)
\[
\psi_m(x) \geq (1 - \epsilon)\|T_0(x + b)\| - t_0 - \epsilon.
\]

Therefore
\[
\frac{e^{t_m}}{|\det T_m|} \leq \frac{e^{t_0 + \epsilon}}{(1 - \epsilon)^n |\det T_0|}
\]

and
\[
\limsup_{m \to \infty} \frac{e^{t_m}}{|\det T_m|} \leq \frac{e^{t_0}}{|\det T_0|}.
\]

(93)

By (92) and (93) we get
\[
\limsup_{m \to \infty} \frac{e^{t_m}}{|\det T_m|} = \frac{e^{t_0}}{|\det T_0|}.
\]

By the uniqueness of the minimizer of \( f \) we get
\[
\lim_{j \to \infty} (T_m, b_m, t_m) = (T_0, b_0, t_0).
\]

This implies that \( L(f_m)(x) = e^{-\|T_m(x+b_m)\| + t_m} \to L(f)(x) = e^{-\|T(x+b)\| + t} \) point-wise and hence in \( L_1 \) by Lemma 8. \( \square \)

### 6 The John Function of a Log-Concave Function

The John function of a log-concave function was first introduced in [70]. It is also recovered in [73]. The definition is as follows.
Definition 6 [70] Let \( f : \mathbb{R}^n \to \mathbb{R}^+ \), \( f(x) = e^{-\psi(x)} \) be a nondegenerate, integrable log-concave function. Then the John function \( J(f) \) of \( f \) is defined as

\[
J(f)(x) = t_0 \mathbb{1}_{A_0 B^n_2} = t_0 \mathbb{1}_{E_f}
\]

where \((t_0, A_0) \in \mathbb{R} \times \mathcal{A}\) is the solution to the maximization problem

\[
\max \{ t \mid \det A : t \leq \| f \|_{\infty}, A \in \mathcal{A} \} \quad \text{subject to} \quad t \mathbb{1}_{A B^n_2} \leq f. \tag{94}
\]

It was shown in [70], and again in [73], that the maximization problem (94) has a solution \((t_0, A_0)\) where the number \( t_0 \) is unique and the affine map \( A_0 \) is unique up to right orthogonal transformations. Thus the John function is well defined.

Remark A different definition of John function was put forward in [71] which is also an affine covariant mapping. We concentrate on the one given above. For the one in [71], it can be shown similarly.

The following theorem is the main theorem of this section.

Theorem 3 Let \( f = \exp(-\psi) \) be a function in \( \text{LC} \). Then the John operator \( J : \text{LC} \to \text{LC} \) mapping \( f \) to its John function \( J(f) \) is an affine covariant mapping.

The next corollary is again an immediate consequence of the theorem, together with Remark 2.

Corollary 3 Let \( f = \exp(-\psi) \) be a function in \( \text{LC} \). Then for all \( \lambda \in \mathbb{R} \),

\[
g(Jf), \quad s(Jf), \quad \lambda g(Jf) + (1 - \lambda)s(Jf)
\]

are affine contravariant points.

The affine covariance property of the John function operator was established in Lemma 2.3 of [70].

Proposition 3 ([70]) Let \( A \in \mathcal{A} \) be a nonsingular affine map. Then \( J(Af) = A(Jf) \).

It remains to prove the continuity of the John function operator on the set of log-concave functions \( \text{LC} \). Before we do that, we introduce some notation. Let \((f_m)_{m=1}^\infty\) and \( f \) be integrable log-concave functions satisfying \( f_m \to f \) in \( L_1 \). By Definition 6 there are sequences \((T_m)_{m=1}^\infty \) in \( GL(n) \), \((b_m)_{m=1}^\infty \) in \( \mathbb{R}^n \), \((t_m)_{m=1}^\infty \) in \( \mathbb{R} \) and \( T_0 \in GL(n) \), \( b_0 \in \mathbb{R}^n \), \( t_0 \in \mathbb{R} \) such that

\[
J(f_m)(x) = t_m \mathbb{1}_{T_mB^n_2+b_m}(x) \quad \text{and} \quad J(f)(x) = t_0 \mathbb{1}_{T_0B^n_2+b_0}(x).
\]

We introduce notations \( J_f(b), J_f \)

\[
J_f = t_0 | \det T_0 | = \max \left\{ t | \det T : T \in GL(n), b \in \mathbb{R}^n, t \in \mathbb{R}, t \mathbb{1}_{T B^n_2+b} \leq f \right\}
\]

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while for fixed \( b \in \mathbb{R}^n \),

\[
J_f(b) = \max \left\{ t | \det T : T \in GL(n), t \in \mathbb{R}, t_1 T B^2_n + b \leq f \right\}.
\]

It’s clear that \( J_f = \max \{ J_f(b) : b \in \mathbb{R}^n \} \). With these notations, the above assumptions read \( J_f = J_f(b_0) \) and \( J_{f_m} = J_{f_m}(b_m) \), for all \( m \). It was shown in the proof of Theorem 1 of [73] that

\[
J_f(b) = n! \text{vol}(B^n_2) \left( J_{f(b)} \right)^{-1} = n! \text{vol}_n(B^n_2) \left( J_{f(b)} \right)^{-1},
\]

where \( f_b(x) = f(x - b) \) and \( f^b \) is the polar function of \( f \) with respect to \( b \).

Note also that if \( J(f) = t_0 \mathbb{1}_{T_0B^2_n + b_0} \), the ellipsoid \( T_0B^2_n + b_0 \) centered at \( b_0 \) must be the John ellipsoid of the convex body \( G_f(t_0) \). Thus the most crucial step towards proving Theorem 3 is to show that \( t_m \to t_0 \).

**Proof of Theorem 3** By definition of the John function of \( f \) resp. \( f_m \) we have that \( t_0 \mathbb{1}_{E_f} \leq f \) resp. \( t_m \mathbb{1}_{E_{f_m}} \leq f_m \). Let \( \delta > 0 \). We can assume that \( 0 \in \text{int}(\text{supp}(f)) \). Then

\[
(1 - \delta) E_f \subseteq \text{int}(\text{supp}(f)). \tag{95}
\]

Moreover, \( (1 - \delta) E_f \) is a compact subset of the interior of the support of \( f \). By Lemma 8, the sequence \( (f_m)_{m=1}^{\infty} \) converges uniformly on \( (1 - \delta) E_f \) to \( f \). Hence for all \( \eta > 0 \), all \( \delta > 0 \) there exists \( m_0 \) such that for all \( m \geq m_0 \), for all \( x \in (1 - \delta) E_f \),

\[
(t_0 - \eta) \mathbb{1}_{(1-\delta)E_f}(x) \leq f_m(x).
\]

This and the definition of the John function imply that for all \( m \geq m_0 \)

\[
(t_0 - \eta) \text{vol}_n \left( (1 - \delta) E_f \right) \leq t_m \text{vol}_n \left( E_{f_m} \right). \tag{96}
\]

It follows

\[
t_0 \text{vol}_n \left( E_f \right) \leq \liminf_{m \to \infty} t_m \text{vol}_n \left( E_{f_m} \right). \tag{97}
\]

Therefore

\[
0 < t_0 \text{vol}_n \left( E_f \right) \leq \liminf_{m \to \infty} t_m \text{vol}_n \left( E_{f_m} \right) \leq \limsup_{m \to \infty} t_m \text{vol}_n \left( E_{f_m} \right) \leq \limsup_{m \to \infty} \| f_m \|_{L^1} \leq \| f \|_{L^1}.
\]

We put \( 2\alpha = t_0 \text{vol}_n \left( E_f \right) \). As \( \| f \|_{L^1} > 0 \), \( \alpha > 0 \). We can choose \( \eta > 0 \) and \( \delta > 0 \) small enough so that

\[
\alpha \leq (t_0 - \eta) \text{vol}_n \left( (1 - \delta) E_f \right).
\]
Therefore, for all \( m \geq m_0, \alpha \leq t_m \text{vol}_n(\mathcal{E}_{f_m}) \). There exists \( R > 0 \) such that for all \( m \in \mathbb{N} \),

\[
\mathcal{E}_{f_m} \subseteq RB^n_2.
\] (98)

Suppose not. Then for all \( R > 0 \) there is \( m \in \mathbb{N} \) such that \( \mathcal{E}_{f_m} \not\subseteq RB^n_2 \). There is \( \rho > 0 \) such that \( 0 \leq \int_{(B^n_2(\rho))^c} f \, dx < \frac{\alpha}{10} \), where \( (B^n_2(\rho))^c \) is the complement of \( B^n_2(\rho) \) in \( \mathbb{R}^n \).

On the other hand, for \( m \geq m_0, \alpha \leq t_m \text{vol}_n(\mathcal{E}_{f_m}) \).

\[
0 < \alpha \leq \int_{\mathbb{R}^n} t_m \mathbb{1}_{\mathcal{E}_{f_m}} \, dx \leq \int_{\mathbb{R}^n} f_m \, dx
\]

and consequently

\[
0 < \alpha \leq \frac{\alpha}{2} \leq \int_{B^n_2(\rho)} t_m \mathbb{1}_{\mathcal{E}_{f_m}} \, dx \leq \int_{B^n_2(\rho)} f_m \, dx.
\]

Then

\[
\int_{\mathbb{R}^n} |f - f_m| \geq \int_{B^n_2(\rho)} |f - f_m| \geq \int_{B^n_2(\rho)} f_m - \int_{B^n_2(\rho)} f \geq \frac{\alpha}{2} - \frac{\alpha}{10} = \frac{2\alpha}{5}.
\]

This contradicts the fact that the sequence \( (f_m)_{m=1}^{\infty} \) converges in \( L_1 \) to \( f \). Therefore (98) holds.

We assume now that the sequence \( (t_m \mathbb{1}_{\mathcal{E}_{f_m}})_{m=1}^{\infty} \) does not converge to \( t_0 \mathbb{1}_{\mathcal{E}_f} \) in \( L_1 \).

Then the sequence \( (t_m)_{m=1}^{\infty} \) does not converge to \( t_0 \) in \( \mathbb{R} \) or the sequence \( (\mathbb{1}_{\mathcal{E}_{f_m}})_{m=1}^{\infty} \) does not converge to \( \mathbb{1}_{\mathcal{E}_f} \) in \( L_1 \).

If the sequence \( (t_m)_{m=1}^{\infty} \) does not converge to \( t_0 \) in \( \mathbb{R} \) then there is a subsequence \( (t_{m_j})_{j=1}^{\infty} \) with

\[
\lim_{j \to \infty} t_{m_j} = \tilde{t}_0 \neq t_0.
\] (99)

Indeed, since \( 0 \leq t_m \leq \|f_m\|_\infty \) and the sequence \( (\|f_m\|_\infty)_{m=1}^{\infty} \) converges to \( \|f\|_\infty \)

the sequence \( (t_m)_{m=1}^{\infty} \) is a bounded sequence.

If the sequence \( (\mathbb{1}_{\mathcal{E}_{f_m}})_{m=1}^{\infty} \) does not converge to \( \mathbb{1}_{\mathcal{E}_f} \) in \( L_1 \) then there is \( \eta > 0 \) and a subsequence \( (\mathbb{1}_{\mathcal{E}_{f_{m_j}}})_{j=1}^{\infty} \) with

\[
\eta < \int_{\mathbb{R}^n} |\mathbb{1}_{\mathcal{E}_f} - \mathbb{1}_{\mathcal{E}_{f_{m_j}}}| \, dx = \text{vol}_n(\mathcal{E}_{f_{m_j}} \triangle \mathcal{E}_f).
\]

It follows that there is \( \tilde{\eta} > 0 \) such that for all \( j \in \mathbb{N} \)

\[
\tilde{\eta} < d_H(\mathcal{E}_{f_{m_j}}, \mathcal{E}_f).
\]
By (98) and by Blaschke’s Selection Principle there is a subsequence $\mathcal{E}_{f_m}$ that converges in the Hausdorff metric

$$\lim_{j \to \infty} \mathcal{E}_{f_m} = \overline{\mathcal{E}} \neq \overline{\mathcal{E}}_f$$

(100)

and $\overline{\mathcal{E}}$ is an ellipsoid. Altogether, there is a subsequence $(t_{m_j} \mathbb{1}_{\mathcal{E}_{f_{m_j}}})_{j=1}^{\infty}$ such that

$$\lim_{j \to \infty} t_{m_j} \mathbb{1}_{\mathcal{E}_{f_{m_j}}} = \overline{t_0} \mathbb{1}_{\overline{\mathcal{E}}}$$

(101)

pointwise where $\overline{t_0} \neq t_0$ or $\overline{\mathcal{E}} \neq \overline{\mathcal{E}}_f$. Since $t_m \mathbb{1}_{\mathcal{E}_f} \leq f_m$, it follows for all $x \in \mathbb{R}^n \setminus \partial \text{supp}(\mathcal{f})$

$$\overline{t_0} \mathbb{1}_{\overline{\mathcal{E}}} = \lim_{j \to \infty} t_{m_j} \mathbb{1}_{\mathcal{E}_{f_{m_j}}}(x) \leq \lim_{j \to \infty} f_{m_j}(x) = f(x).$$

Consider $x \in \partial \text{supp}(\mathcal{f})$. If $x \notin \overline{\mathcal{E}}$ then

$$\overline{t_0} \mathbb{1}_{\overline{\mathcal{E}}}(x) = 0 \leq f(x).$$

If $x \in \overline{\mathcal{E}}$ then there is a sequence $(x_n)_{n=1}^{\infty} \subseteq \text{int}(\overline{\mathcal{E}})$ with

$$\lim_{n \to \infty} x_n = x$$

Then, by the upper semi continuity of $f$

$$\overline{t_0} = \lim_{n \to \infty} \overline{t_0 \mathbb{1}_{\overline{\mathcal{E}}}}(x_n) \leq \lim_{n \to \infty} f(x_n) \leq f(x).$$

It follows

$$\overline{t_0} \mathbb{1}_{\overline{\mathcal{E}}} \leq f.$$  

(102)

By the definition of the John function

$$\overline{t_0} \text{vol}_n(\overline{\mathcal{E}}) \leq t_0 \text{vol}_n(\mathcal{E}_f).$$

(103)

With (97) we thus get,

$$t_0 \text{vol}_n(\mathcal{E}_f) \leq \liminf_m t_m \text{vol}_n(\mathcal{E}_f) \leq \lim_{j \to \infty} t_{m_j} \text{vol}_n(\mathcal{E}_f) = \overline{t_0} \text{vol}_n(\overline{\mathcal{E}}) \leq t_0 \text{vol}_n(\mathcal{E}_f).$$

By the uniqueness we get $\overline{t_0} \mathbb{1}_{\overline{\mathcal{E}}} = t_0 \mathbb{1}_{\mathcal{E}}$.  

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