An algorithm to compute the canonical basis of an irreducible $U_q(\mathfrak{g})$-module

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Abstract
We describe an algorithm to compute the canonical basis of an irreducible module over a quantized enveloping algebra of a finite-dimensional semisimple Lie algebra. The algorithm works for modules that are constructed as a submodule of a tensor product of modules with known canonical bases.

1 Introduction
In this paper I consider the problem of constructing the canonical basis (cf. [17]) of an irreducible module over a quantized enveloping algebra. There are several possible ways to approach this problem, and they may depend on how the module is constructed. In [4] an algorithm is described that works for any module, provided that we have a method for computing the action of elements of the algebra. In [11], [12] the irreducible module is first constructed as a submodule of a tensor product of other modules. Then using the known canonical bases of these other modules, an algorithm is described for constructing the canonical basis of the submodule.

Since constructing irreducible modules as submodules of tensor products can be rather efficient (cf. [5]), it is worthwhile to have an algorithm that is tailored to this situation. Therefore, in this paper I take the second approach above. In fact, I will describe an algorithm that is very similar to the ones in [11], [12]. The main difference is that I do not assume that the root system is of a certain type. The algorithm given here works for all types, assuming that somehow we know the canonical bases of the fundamental modules. These can, for instance, be constructed using the algorithm of [4].

This paper is organised as follows. In Section 2 the theoretical concepts, and notation that we use are introduced. Then in Section 3 a result is described concerning the form of the elements of the canonical basis of a tensor product. Then in Section 4 this is used, along with the description of a monomial basis of an irreducible module (from [10]), to give an algorithm for constructing the canonical basis. Finally in Section 5 this algorithm is compared to the algorithm from [11] in the $A_n$-case. It is shown that in that case both algorithms are very similar (but not exactly the same).
2 Preliminaries

In this section we briefly sketch the concepts and notation that we will be using. Our main reference is [3].

Let \( \mathfrak{g} \) be a semisimple Lie algebra over \( \mathbb{C} \). By \( \Phi \) we denote the root system of \( \mathfrak{g} \), and \( \Delta = \{ \alpha_1, \ldots, \alpha_l \} \) will be a fixed set of simple roots of \( \Phi \). Let \( W \) denote the Weyl group of \( \Phi \), which is generated by the simple reflections \( s_i = s_{\alpha_i} \) for \( 1 \leq i \leq l \). Let \( \mathbb{R}\Phi \) be the vector space over \( \mathbb{R} \) spanned by \( \Phi \). On \( \mathbb{R}\Phi \) we fix a \( W \)-invariant inner product \( (\ , \) ) such that \( (\alpha, \alpha) = 2 \) for short roots \( \alpha \). This means that \( (\alpha, \alpha) = 2, 4, 6 \) for \( \alpha \in \Phi \).

We work over the field \( \mathbb{Q}(q) \). For \( \alpha \in \Phi \) set \( q_\alpha = q^{(\alpha, \alpha)/2} \). For \( n \in \mathbb{Z} \) we set \( [n]_\alpha = q_\alpha^{-n+1} + q_\alpha^{-n+3} + \cdots + q_\alpha^{-n} \). Also \( [n]_\alpha! = [n]_\alpha [n-1]_\alpha \cdots [1]_\alpha \) and
\[
\begin{pmatrix} n \\ k \end{pmatrix}_\alpha = \frac{[n]_\alpha!}{[k]_\alpha ![n-k]_\alpha!}.
\]

Let \( \Delta = \{ \alpha_1, \ldots, \alpha_l \} \) be a simple system of \( \Phi \). Then the quantized enveloping algebra \( U_q = U_q(\mathfrak{g}) \) is the associative algebra (with one) over \( \mathbb{Q}(q) \) generated by \( F_\alpha, K_\alpha, K_\alpha^{-1}, E_\alpha \) for \( \alpha \in \Delta \), subject to the following relations
\[
\begin{align*}
K_\alpha K_\alpha^{-1} &= K_\alpha^{-1} K_\alpha = 1, \quad K_\alpha K_\beta = K_\beta K_\alpha \\
E_\beta K_\alpha &= q^{-\langle \alpha, \beta \rangle} K_\alpha E_\beta \\
K_\alpha F_\beta &= q^{-\langle \alpha, \beta \rangle} F_\beta K_\alpha \\
E_\alpha F_\beta &= F_\beta E_\alpha + \delta_{\alpha, \beta} \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} \\
\sum_{k=0}^{1-\langle \beta, \alpha \rangle} (-1)^k \frac{1-\langle \beta, \alpha \rangle}{k} \begin{pmatrix} 1-\langle \beta, \alpha \rangle \end{pmatrix}_\alpha E_\alpha^{1-\langle \beta, \alpha \rangle} - k E_\beta E_\alpha^k &= 0 \\
\sum_{k=0}^{1-\langle \beta, \alpha \rangle} (-1)^k \frac{1-\langle \beta, \alpha \rangle}{k} \begin{pmatrix} 1-\langle \beta, \alpha \rangle \end{pmatrix}_\alpha F_\alpha^{1-\langle \beta, \alpha \rangle} - k F_\beta F_\alpha^k &= 0,
\end{align*}
\]

where the last two relations are for all \( \alpha \neq \beta \).

Let \( U^- , U^0, U^+ \) be the subalgebras of \( U_q \) generated by respectively, \( F_\alpha \) for \( \alpha \in \Delta \), \( K_\alpha \), \( K_\alpha^{-1} \) for \( \alpha \in \Delta \), and \( E_\alpha \) for \( \alpha \in \Delta \). Then as a vector space \( U_q \cong U^- \otimes U^0 \otimes U^+ \) ([6], Theorem 4.21). Let \( \nu = \sum_k a_k \alpha_k \) with \( a_k \in \mathbb{Z}_{\geq 0} \). Then we let \( U^+ (U^-) \) be the subspace of \( U^+ (U^-) \) spanned by all \( E_{\alpha_{i_1}} \cdots E_{\alpha_{i_r}} (F_{\alpha_{i_1}} \cdots F_{\alpha_{i_r}}) \) such that \( \alpha_{i_1} + \cdots + \alpha_{i_r} = \nu \).

We let \( \lambda_1, \ldots, \lambda_l \) denote the fundamental weights, and \( P = \mathbb{Z} \lambda_1 + \cdots + \mathbb{Z} \lambda_l \) is the weight lattice. Also \( P^+ = \mathbb{Z}_{\geq 0} \lambda_1 + \cdots + \mathbb{Z}_{\geq 0} \lambda_l \) is the set of dominant weights. Now for every dominant \( \lambda \in P^+ \) there is an irreducible \( U_q \)-module \( V(\lambda) \). We have that \( V(\lambda) \) is spanned by vectors \( v_\mu \) for \( \mu \in P \), with \( K_\alpha \cdot v_\mu = q^{(\mu, \alpha)} v_\mu \). These \( v_\mu \) are called weight-vectors of weight \( \mu \). Among them there is the vector \( v_\lambda \) (which is unique up to scalar multiples), with \( U^- \cdot v_\lambda = 0 \). This \( v_\lambda \) is called the highest-weight vector. We have that \( V(\lambda) = U^- \cdot v_\lambda \). Furthermore, every finite-dimensional irreducible \( U_q \)-module is isomorphic to a \( V(\lambda) \) ([3], Theorem 5.10).
Let $M$ be a finite-dimensional $U_q$-module. Then $M$ has a crystal base $(\mathcal{M}, \mathcal{B})$ as defined in \cite{12}, 9.4. Here $\mathcal{M}$ is an $A$-submodule of $M$, where $A$ is the subring of $\mathbb{Q}(q)$ consisting of rational functions without pole at 0. And $\mathcal{B}$ is a basis of $\mathcal{M}/q\mathcal{M}$. For $\alpha \in \Delta$, we have the Kashiwara operators $\widetilde{F}_\alpha, \widetilde{E}_\alpha : \mathcal{M} \to \mathcal{M}$, and the induced operators $\tilde{F}_\alpha, \tilde{E}_\alpha : \mathcal{B} \to \mathcal{B} \cup \{0\}$ (\cite{12}, 9.2, 9.4).

There is a $\mathbb{Q}$-algebra isomorphism $\varpi : U_q \to U_q$ with $\varpi = q^{-1}$, $\overline{E}_\alpha = E_\alpha$, $\overline{F}_\alpha = F_\alpha$, and $\overline{K}_\alpha = K_\alpha^{-1}$ (\cite{12}, Proposition 11.9). If $V(\lambda)$ is an irreducible $U_q$-module with highest weight $\lambda$, and fixed highest-weight vector $v_\lambda$, then we have an induced map $\varpi : V(\lambda) \to V(\lambda)$ by $\overline{u} \cdot v_\lambda = \varpi \cdot v_\lambda$. (This is well defined by \cite{12}, Proposition 11.9.) The fixed choice for $v_\lambda$ leads to a fixed crystal base $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ of $V(\lambda)$, where $\mathcal{L}(\lambda)$ is spanned by all $\tilde{F}_{\alpha_{i_1}} \cdots \tilde{F}_{\alpha_{i_k}}(v_\lambda)$, for $r \geq 0$. Now by, e.g., \cite{12} Theorem 1.8, \cite{12} Theorem 11.10, there is a unique basis \[ \{ G_\lambda(b) \mid b \in \mathcal{B}(\lambda) \} \] of $\mathcal{L}(\lambda)$, such that

1. $G_\lambda(b) = b \mod q\mathcal{L}(\lambda)$,
2. $\overline{G}_\lambda(b) = G_\lambda(b)$.

This basis is called the canonical basis of $V(\lambda)$.

In the sequel, when we write the crystal base or the canonical basis of $V(\lambda)$ we always assume that a fixed highest-weight vector $v_\lambda$ has been chosen, which makes the choice of crystal base, canonical basis unique.

The crystal graph $\Gamma_\lambda$ of the module $V(\lambda)$ is defined as follows. The points of $\Gamma_\lambda$ are the elements of $\mathcal{B}(\lambda)$, and there is an edge $b_1 \overset{\alpha}{\to} b_2$ if $\tilde{F}_\alpha(b_1) = b_2$. There is a very elegant method to compute the crystal graph, using Littelmann’s path method. Let $\mathbb{R}P$ be the vector space over $\mathbb{R}$ spanned by the weights. Let $\Pi$ be the set of piecewise linear paths $\pi : [0, 1] \to \mathbb{R}P$, such that $\pi(0) = 0$. For $\alpha \in \Delta$ Littelmann defined operators $e_\alpha, f_\alpha : \Pi \to \Pi \cup \{0\}$ (cf. \cite{12}, \cite{12}), with the following property. Let $\lambda \in P^+$ be a dominant weight, and let $\pi_\lambda$ be the path given by $\pi_\lambda(t) = \lambda t$ (i.e., a straight line from the origin to $\lambda$). let $\Pi_\lambda$ be the set of all $f_{\alpha_{i_1}} \cdots f_{\alpha_{i_k}}(\pi_\lambda)$. Then all paths in $\Pi_\lambda$ end in an element of $P$. Furthermore, the number of paths ending in $\mu \in P$ is equal to the dimension of the weight space with weight $\mu$ in the irreducible $U_q$-module $V(\lambda)$. Now we consider the directed labeled graph with point set $\Pi_\lambda$, and edges $\pi_1 \overset{\alpha}{\to} \pi_2$ if $f_\alpha(\pi_1) = \pi_2$. This graph is isomorphic to the crystal graph of $V(\lambda)$ (\cite{12}).

Let $M_1, M_2$ be $U_q$-modules, then $M_1 \otimes M_2$ is a $U_q$-module via the comultiplication of $U_q$. There are many possible ways to define this, and the comultiplication $\Delta : U_q \to U_q \otimes U_q$ that we use is given by

\[ \Delta(E_\alpha) = E_\alpha \otimes K_\alpha^{-1} + 1 \otimes E_\alpha \]
\[ \Delta(F_\alpha) = F_\alpha \otimes 1 + K_\alpha \otimes F_\alpha \]
\[ \Delta(K_\alpha) = K_\alpha \otimes K_\alpha \]

(see \cite{12}, 9.13).
3 Canonical bases of tensor products

Here we give a description of the canonical basis of a tensor product, following [4] Chapter 9, [17] 27.3.

Let $V(\mu)$, $V(\mu')$ be two irreducible $U_q$-modules, with highest weights $\mu$, $\mu'$. Let $C = \{v_1, \ldots, v_m\}$ and $C' = \{v'_1, \ldots, v'_n\}$ be fixed canonical bases of $V(\mu), V(\mu')$. Denote the weight of $v_i$, $v'_j$ by $\nu_i$, $\nu'_j$ respectively. Then $\nu_i = \mu - \sum_k a_{k,i} \alpha_k$ with $a_{k,i} \in \mathbb{Z}_{\geq 0}$, and we say that $\sum_k a_{k,i}$ is the height of $v_i$. The height of $\nu'_j$ is defined similarly. We assume that the bases $C, C'$ are ordered according to increasing height. So $v_1 = v_{\mu}$, $v'_1 = v'_{\mu'}$ are the highest-weight vectors.

Let $(\mathcal{L}, \mathcal{B})$ and $(\mathcal{L}', \mathcal{B}')$ be crystal bases of $V(\mu)$ and $V(\mu')$ respectively. Here $\mathcal{L}, \mathcal{L}'$ are spanned by $C$ and $C'$ respectively. Furthermore, $\mathcal{B}, \mathcal{B}'$ consist of the cosets $v_1 \bmod q\mathcal{L}$, $v'_1 \bmod q\mathcal{L}'$. Now by [4], Theorem 9.17, $(\mathcal{L} \otimes \mathcal{L}', \mathcal{B} \otimes \mathcal{B}')$ is a crystal base of $V(\mu) \otimes V(\mu')$.

We let $\Theta$ be the element from [17], 4.1, and $P : U_q \otimes U_q \rightarrow U_q \otimes U_q$ is the algebra homomorphism defined by $P(a \otimes b) = b \otimes a$. We set $\Theta^0 = P(\Theta)$; then $\Theta^0 = \sum_{\eta \geq 0} \Theta^0_\eta$, where the sum runs over all $\eta = \sum_k b_k \alpha_k$ with $b_k \in \mathbb{Z}_{\geq 0}$. Furthermore, $\Theta^0_\eta \in U_+^q \otimes U_-^q$, and $\Theta^0_0 = 1 \otimes 1$. Now $\Psi_0 : V(\mu) \otimes V(\mu') \rightarrow V(\mu) \otimes V(\mu')$ is the map defined by $\Psi_0(v \otimes v') = \Theta^0(\nu \otimes \nu')$.

**Lemma 1** We have $\Psi_0(u \cdot v \otimes v') = \overline{u} \cdot \Psi_0(v \otimes v')$ for all $u \in U^-$. Furthermore, $\Psi_0^2(v \otimes v') = v \otimes v'$ for all $v \in V(\mu)$, $v' \in V(\mu')$.

**Proof.** This is the same as the corresponding results in [17], 27.3.1. The difference is that we use a different comultiplication. Denoting the comultiplication used in [17] by $\Delta_L$, we have $\Delta_L(F_\alpha) = F_\alpha \otimes K_\alpha^{-1} + 1 \otimes F_\alpha$. This means that for $u \in U^-$, we have $\Delta(u) = P(\Delta_L(u))$, where $\Delta_L(u) = \overline{\Delta_L(u)}$. The property $\Delta_L(u)\Theta = \Theta\overline{\Delta_L(u)}$ ([17], Theorem 4.1.2) now translates to $\Delta(u)\Theta^0 = \Theta^0\overline{\Delta(u)}$, where $\overline{\Delta}$ is defined similarly to $\Delta_L$. Form this the first statement follows. The second follows from $\Theta^0\overline{\Theta^0} = 1 \otimes 1$ ([17], Corollary 4.1.3).

We define a partial order on the $v_i \otimes v'_j$. We set $v_i \otimes v'_j < v_k \otimes v'_l$ if and only if $i < k$, $j > l$, and $\nu_i + v'_j = \nu_k + v'_l$.

**Proposition 2** There are unique elements $w_{ij} \in V(\mu) \otimes V(\mu')$ such that

1. $\Psi_0(w_{ij}) = w_{ij}$,
2. $w_{ij} = v_i \otimes v'_j + \sum_k \zeta_k v_{i_k} \otimes v'_{j_k}$, with $\zeta_k \in \mathbb{Q}[q]$, and $\nu_i + v'_j = \nu_i + v'_j$.

Also, $v_{i_k} \otimes v'_j < v_i \otimes v'_j$ for all $k$. The elements $w_{ij}$ form a basis of $V(\mu) \otimes V(\mu')$.

**Proof.** This goes in the same way as [17], Theorem 27.3.2. Note that

$$\Psi_0(v_i \otimes v'_j) = v_i \otimes v'_j + \sum_k \xi_k v_{i_k} \otimes v'_{j_k}, \quad (1)$$

with $\xi_k \in \mathbb{Z}[q, q^{-1}]$. From $\Theta^0_\eta \in U_+^q \otimes U_-^q$ and the assumption on the ordering of $C, C'$ it follows that $v_{i_k} \otimes v'_{j_k} < v_i \otimes v'_j$ for all $k$. Let $X$ be the set of all $(i, j)$ with $\nu_i + v'_j = \nu,$
for a certain $\nu$. Order the elements of $X$ in such a way that $v_i \otimes v'_j < v_k \otimes v'_l$ implies that $(i,j)$ appears before $(k,l)$. Let $(i,j)$ be the smallest element of $X$. Then by (1), we see that $\Psi_0(v_i \otimes v'_j) = v_i \otimes v'_j$. So in this case we set $w_{ij} = v_i \otimes v'_j$. Now choose a $(k,l) \in X$, and suppose that $w_{r,s}$ exist for all $(r,s) \in X$ appearing before $(k,l)$. Then using (2) and the triangular form of the $w_{r,s}$ we can write $\Psi_0(v_k \otimes v'_l) - v_k \otimes v'_l = \sum_{r,s} \delta_{r,s} w_{r,s}$, where $v_r \otimes v'_s < v_k \otimes v'_l$. After taking images under $\Psi_0$, and using the fact that $\Psi_0$ is an involution, we see that the $\delta_{r,s} \in \mathbb{Z}[q, q^{-1}]$ satisfy $\delta_{r,s} = -\delta_{r,s}$. This implies that there are unique $\delta_{r,s} \in q\mathbb{Z}[q]$ with $\delta_{r,s} = \delta_{r,s} - \delta_{r,s}$. Now set $w_{k,l} = v_k \otimes v'_l + \sum_{r,s} \delta_{r,s} w_{r,s}$. For the uniqueness suppose that there are $w'_{ij} \in V(\mu) \otimes V(\mu')$ satisfying 1., 2. Then we write $w'_{ij}$ as a linear combination of $w_{ij}$. By 2. the coefficients are in $\mathbb{Z}[q]$. Then 1. implies that they are in $\mathbb{Z}$. Finally, from 2. we see that one coefficient is 1, and the others are 0. 

Let $V(\lambda)$ denote the $U_q$-submodule of $V(\mu) \otimes V(\mu')$ generated by $v_\mu \otimes v'_\mu = v_1 \otimes v'_1$. So $V(\lambda)$ is the irreducible $U_q$-module with highest weight $\lambda = \mu + \mu'$. Set $L(\lambda) = (L \otimes L') \cap V(\lambda)$, and $B(\lambda) = (B \otimes B') \cap L(\lambda)/qL(\lambda)$. Then by (6), Proposition 9.10, $(L(\lambda), B(\lambda))$ is a crystal base of $V(\lambda)$ (the hypotheses of this proposition are satisfied by (4), Proposition 9.23, Lemma 9.26).

**Theorem 3** The elements of the canonical basis of $V(\lambda)$ have the form $v_i \otimes v'_j + \sum_k \zeta_k v_{i_k} \otimes v'_{j_k}$, with $\zeta_k \in q\mathbb{Z}[q]$, and $v_{i_k} \otimes v'_{j_k} < v_i \otimes v'_j$ for all $k$.

**Proof.** We have that $\Psi_0(v_i \otimes v'_j) = v_i \otimes v'_j$. So by Lemma 1, $\Psi_0$ coincides with $-$ on $V(\lambda)$ (where $u \cdot v_i \otimes v'_j = \overline{v}_i \otimes v'_j$). Hence the elements of the canonical basis of $V(\lambda)$ are invariant under $\Psi_0$. Also, since the elements of the canonical basis lie in $L(\lambda)$ and are equal to a $v_i \otimes v'_j \mod qL(\lambda)$, they must be of the form $v_i \otimes v_j + \sum_k \zeta_k v_{i_k} \otimes v'_{j_k}$ with $\zeta_k \in q\mathbb{Z}[q]$. Now Proposition 2 finishes the proof. 

Let $V(\mu_1), \ldots, V(\mu_r)$ be irreducible $U_q$-modules with canonical bases $C_i = \{v_{i,1}^1, \ldots, v_{i,m_i}^1\}$, ordered according to increasing height. We consider the tensor product $V = V(\mu_1) \otimes \cdots \otimes V(\mu_r)$. We write $v_{i_1}^1 \otimes \cdots \otimes v_{i_r}^r \in \overline{\text{span}} v_{j_1}^1 \otimes \cdots \otimes v_{j_r}^r$ if there is a $k$ with $i_1 = j_1, \ldots, i_k = j_k$ and $i_{k+1} < j_{k+1}$. Set $\lambda = \mu_1 + \cdots + \mu_r$ and let $V(\lambda)$ be the $U_q$-submodule of $V$ generated by $v_{i_1}^1 \otimes \cdots \otimes v_{i_r}^r$.

**Corollary 4** The elements of the canonical basis of $V(\lambda)$ have the form $\sum_k \zeta_k x_k$ where $\zeta_k \in q\mathbb{Z}[q]$, $x_k \in C_1 \otimes \cdots \otimes C_r$, and $x_k \leq \text{lex} v_{i_1}^1 \otimes \cdots \otimes v_{i_r}^r$.

**Proof.** The case $r = 2$ is covered by Theorem 3, so suppose $r > 2$. Let $W$ be the $U_q$-submodule of $V(\mu_2) \otimes \cdots \otimes V(\mu_r)$ generated by $v_2^1 \otimes \cdots \otimes v_r^r$. Then $W$ is the irreducible $U_q$-module with highest weight $\mu_2 + \cdots + \mu_r$. Let $\{w_1, \ldots, w_s\}$ be the canonical basis of $W$. Then by Theorem 3 the elements of the canonical basis of $V(\lambda)$ have the form

$$v_{i_1}^1 \otimes w_{j_1} + \sum_{k \geq 2} \zeta_k v_{i_k}^1 \otimes w_{j_k},$$

with $i_k < i_1$ for all $k \geq 2$, and $\zeta_k \in q\mathbb{Z}[q]$. We get the result by writing all $w_{j_k}$ for $k \geq 1$ as linear combinations of elements of $C_2 \otimes \cdots \otimes C_r$ and use induction. 

\[\Box\]
4 A monomial basis of $V(\lambda)$

In this section we first describe a basis of $V(\lambda)$, following \[14\]. Then using this we derive an algorithm for constructing the canonical basis of $V(\lambda)$, when $V(\lambda)$ is viewed as a submodule of a tensor product.

Let $\pi \in \Pi_\lambda$. Then the first direction of $\pi$ is $w(\lambda)$ for some $w \in W/W_\lambda$ (\[13\], 5.2), where $W_\lambda$ is the stabilizer of $\lambda$. Set $\phi(\pi) = w$. Let $s_{i_1} \cdots s_{i_r}$ be the reduced expression for $\phi(\pi)$, which is lexicographically the smallest. (Here $s_{i_1} \cdots s_{i_r}$ is lexicographically smaller than $s_{j_1} \cdots s_{j_r}$ if there is a $k > 0$ such that $i_1 = j_1, \ldots, i_{k-1} = j_{k-1}$ and $i_k < j_k$.) Then we define integers $n_1, \ldots, n_r$, and paths $\pi_0, \pi_1, \ldots, \pi_r$ in the following way. First, $\pi_0 = \pi$. We let $n_k$ be maximal such that $e_{\alpha_{i_k}}^{n_k} \pi_{k-1} \neq 0$, and we set $\pi_k = e_{\alpha_{i_k}}^{n_k} \pi_{k-1}$. Set $\eta_\pi = (n_1, \ldots, n_r)$, and $F_\pi = F_{\alpha_{i_1}}^{(n_1)} \cdots F_{\alpha_{i_r}}^{(n_r)}$. Let $b_\lambda \in B(\lambda)$ denote the unique element of weight $\lambda$ (it is the coset of $v_\lambda$ modulo $qL(\lambda)$). Set $b_\pi = \tilde{F}_{\alpha_{i_1}}^{n_1} \cdots \tilde{F}_{\alpha_{i_r}}^{n_r}(b_\lambda)$; then $B(\lambda) = \{b_\pi \mid \pi \in \Pi_\lambda\}$ (this follows from \[3\]).

In the sequel we let $<_B$ denote the Bruhat order on the Weyl group $W$. The lexicographical order on sequences of length $r$ is defined by $(m_1, \ldots, m_r) <_{\text{lex}} (n_1, \ldots, n_r)$ if there is a $k$ such that $m_1 = n_1, \ldots, m_{k-1} = n_{k-1}$ and $m_k < n_k$. We now define a partial order on $\Pi_\lambda$ as follows. First of all, $\pi < \sigma$ if $\phi(\pi) <_B \phi(\sigma)$. Secondly, if $\phi(\pi) = \phi(\sigma)$, then $\pi < \sigma$ if $\eta_\pi >_{\text{lex}} \eta_\sigma$. For the proof of the following theorem we refer to \[10\].

**Theorem 5**

$$F_\pi \cdot v_\lambda = G_\lambda(b_\pi) + \sum_{\sigma < \pi} \zeta_{\pi,\sigma} G_\lambda(b_\sigma),$$

where $\zeta_{\pi,\sigma} \in \mathbb{Z}[q, q^{-1}]$.

**Corollary 6** The set $\{F_\pi \cdot v_\lambda \mid \pi \in \Pi_\lambda\}$ is a basis of $V(\lambda)$.

Let $\pi \in \Pi_\lambda$, and $F_\pi = F_{\alpha_{i_1}}^{(n_1)} \cdots F_{\alpha_{i_r}}^{(n_r)}$. Then we say that $\pi$ is of weight $\nu = \sum_k n_k \alpha_{i_k}$. We note that this means that $F_\pi \cdot v_\lambda$ is a weight vector in $V(\lambda)$ of weight $\lambda - \nu$. By $\Pi_{\lambda,\nu}$ we denote the set of all $\pi \in \Pi_\lambda$ of weight $\nu$.

Suppose that $\lambda = \mu_1 + \cdots + \mu_r$, where the $\mu_i$ are dominant weights. Also suppose that we are given the modules $V(\mu_i)$ with canonical bases $C_i = \{v_{i_1}^1, \ldots, v_{i_m}^i\}$, ordered according to increasing height. We identify $V(\lambda)$ with the $U_q$-submodule of $V(\mu_1) \otimes \cdots \otimes V(\mu_r)$ generated by $v_\lambda = v_{i_1}^1 \otimes \cdots \otimes v_{i_r}^r$. Set $C = C_1 \otimes \cdots \otimes C_r$, which is a basis of $V(\mu_1) \otimes \cdots \otimes V(\mu_r)$ ordered with respect to $<_\text{lex}$ (see the previous section).

**Theorem 7** leads to the following algorithm for computing the $G_\lambda(b_\pi)$, for $\pi \in \Pi_{\lambda,\nu}$. Let $\sigma_1, \ldots, \sigma_r$ be the elements from $\Pi_{\lambda,\nu}$ that are smaller than $\pi$. We assume that the $G_\lambda(b_\sigma)$ already have been computed. Write $G_\lambda(b_{\sigma_i}) = y_i + \sum_k \zeta_k y_{i,k}$, where $y_i, y_{i,k} \in C$ and $y_i >_{\text{lex}} y_{i,k}$ for all $k$. We assume that $y_i <_{\text{lex}} y_{j}$ implies that $i > j$. Then we do the following:

1. Write $X = F_\pi \cdot v_\lambda$ as a linear combination of elements of $C$. 


2. For $i = 1, \ldots, r$ we do the following. Let $\xi_i$ be the coefficient of $y_i$ in $X$. Let $\xi_i$ be the unique element of $\mathbb{Z}[q, q^{-1}]$ such that $\xi_i = \xi_i$ and $\xi_i + \xi_i \in q\mathbb{Z}[q]$. Set $X := X + \xi_i G_\lambda(b_{\sigma_i})$.

**Proposition 7** When the loop in Step 2 terminates we have that $X = G_\lambda(b_{\pi})$.

**Proof.** Note that by Theorem 3 there are coefficients $\xi_i$ such that $G_\lambda(b_{\pi}) = F_\pi \cdot v_\lambda + \sum_{i=1}^r \xi_i G_\lambda(b_{\sigma_i})$. This implies that $\xi_i = \xi_i$. Also, by Corollary 4, we have that $G_\lambda(b_{\pi})$ is of the form $x + \sum_k \omega_k x_k$, where $x, x_k \in C$ and $\omega_k \in q\mathbb{Z}[q]$. Note that by Corollary 4 $y_1$ does not occur in any $G_\lambda(b_{\sigma_i})$, except $G_\lambda(b_{\sigma_1})$. Therefore, $\xi_1$ is uniquely determined by the requirements that it should be invariant under $\pi$, and $\xi_1 + \xi_1 \in q\mathbb{Z}[q]$. Then in the same way we see that $\xi_2$ is uniquely determined, and so on. \qed

**Example 8** Let $\Phi$ be the root system of type $G_2$. We denote the simple roots of $\Phi$ by $\alpha, \beta$, where $\beta$ is long. The fundamental module $V(\lambda_1)$ is 7-dimensional, and the canonical basis is $C_1 = \{v_1, \ldots, v_7\}$; these are weight vectors of weights $(1,0), (-1,1), (2,-1), (0,0), (-2,1), (1,-1), (-1,0)$. Here we abbreviate a weight $m\lambda_1 + n\lambda_2$ as $(m,n)$. The fundamental module $V(\lambda_2)$ is 14-dimensional and has canonical basis $C_2 = \{v_1, \ldots, v_{14}\}$. The $v_i$ are weight vectors of weights $(0,1), (3,-1), (1,0), (-1,1), (-3,2), (2,-1), (0,0), (0,0), (3,-2), (-2,1), (1,-1), (-1,0), (-3,1), (0,-1)$. A description of the action of the generators of $U_q$ on $V(\lambda_1)$ can for instance be found in [4], and the action of $U_q$ on $V(\lambda_2)$ is described in [4], 5A.4. Alternatively, these modules can be constructed using the GAP4 package QuaGroup ([4], [3]). This package has been used to perform many of the calculations in the rest of this example. Now we set $\lambda = 2\lambda_1 + \lambda_2$. Then $V(\lambda)$ is the submodule of $W = V(\lambda_1) \otimes V(\lambda_1) \otimes V(\lambda_2)$ generated by $v_1 \otimes v_1 \otimes w_1$. We construct the elements of the canonical basis of $V(\lambda)$ that are of weight $\mu = (−2, 2)$. We use the following elements of weight $\mu$:

\[
\begin{align*}
x_1 &= v_1 \otimes v_2 \otimes w_{10}, & x_2 &= v_1 \otimes v_4 \otimes w_5, & x_3 &= v_1 \otimes v_5 \otimes w_4, & x_4 &= v_2 \otimes v_1 \otimes w_{10}, \\
x_5 &= v_2 \otimes v_2 \otimes w_7, & x_6 &= v_2 \otimes v_2 \otimes w_8, & x_7 &= v_2 \otimes v_3 \otimes w_5, & x_8 &= v_2 \otimes v_4 \otimes w_4, \\
x_9 &= v_2 \otimes v_5 \otimes w_3, & x_{10} &= v_2 \otimes v_7 \otimes w_1, & x_{11} &= v_3 \otimes v_2 \otimes w_5, & x_{12} &= v_4 \otimes v_1 \otimes w_5, \\
x_{13} &= v_4 \otimes v_2 \otimes w_4, & x_{14} &= v_4 \otimes v_5 \otimes w_1, & x_{15} &= v_5 \otimes v_1 \otimes w_4, & x_{16} &= v_5 \otimes v_2 \otimes w_3, \\
x_{17} &= v_5 \otimes v_4 \otimes w_1, & x_{18} &= v_7 \otimes v_2 \otimes w_1. 
\end{align*}
\]

They are listed in lexicographical order, i.e., $x_1 <_{\text{lex}} x_2 <_{\text{lex}} \ldots <_{\text{lex}} x_{18}$. The weight space of weight $\mu$ in $V(\lambda)$ is 5-dimensional. So we get 5 paths $\pi_i$ in the crystal graph. The corresponding words in the Weyl group are $\phi(\pi_1) = s_\alpha s_\beta s_\alpha$, $\phi(\pi_2) = s_\beta s_\alpha s_\beta$, $\phi(\pi_3) = s_\alpha s_\beta s_\alpha$, $\phi(\pi_4) = s_\alpha s_\beta s_\alpha s_\beta$, $\phi(\pi_5) = s_\alpha s_\beta s_\alpha s_\beta$. Setting $\eta_i = \eta_{\pi_i}$ we have $\eta_1 = (4, 2, 1)$, $\eta_2 = (1, 5, 1)$, $\eta_3 = (3, 2, 2)$, $\eta_4 = (3, 1, 2, 1)$, $\eta_5 = (2, 1, 2, 1, 1)$. So we see that $\pi_1 < \pi_3 < \pi_4 < \pi_5$ and $\pi_2 < \pi_4$. Therefore we have

\[
\begin{align*}
G_\lambda(b_{\pi_1}) &= F_{\pi_1} v_\lambda = x_{16} + q^2 x_{15} + q^3 x_{13} + q^6 x_{12} + q^8 x_{11} + qx_9 + q^3 x_8 + q^7 x_7 + q^5 x_3 + q^8 x_2 \\
G_\lambda(b_{\pi_2}) &= F_{\pi_2} v_\lambda = x_{11} + q^3 x_7 + q^6 x_6.
\end{align*}
\]
Also
\[ F_{\pi_3}v_\lambda = x_{17} + q^2x_{16} + q^2x_{14} + q^3x_{13} + q^5x_{11} + q^3x_9 + q^5x_8 + q^9x_7. \]

All coefficients, except the first one, are in \( q\mathbb{Z}[q] \). Hence \( G_\lambda(b_{\pi_3}) = F_{\pi_3}v_\lambda \). Now
\[ F_{\pi_4}v_\lambda = (q+q^{-1})x_{16} + (q+q^3)x_{15} + (1+q^2+q^4)x_{13} + (q^3+q^5+q^7)x_{12} + (q^3+q^5+q^7)q^{11} + (1+q^2)x_9 + q^2x_8 + (q^4+q^6)x_7 + q^4x_5 + q^6x_4 + q^5x_2 + q^7x_1. \]
The coefficient of \( x_{16} \) is not in \( q\mathbb{Z}[q] \). Following the algorithm we see that \( G_\lambda(b_{\pi_4}) = F_{\pi_4}v_\lambda - (q+q^{-1})G_\lambda(b_{\pi_1}) \); we get
\[ G_\lambda(b_{\pi_4}) = x_{13} + q^2x_{12} + (q^3+q^5)x_{11} + q^2x_8 + (q^4+q^6)x_7 + q^4x_5 + q^6x_4 + q^5x_2 + q^7x_1. \]
Finally,
\[ F_{\pi_5}v_\lambda = (2q+q^2)x_{18} + (2q^3+2q+q^2)x_{17} + (2q^3+2q+q^2)x_{16} + (2q^3)x_{15} + (2q^4+3q^2+1)x_{13} + (q^3+q^5+q^7)x_{12} + (q+q^2) + (3q^5+q^7)x_{11} + q^3x_{10} + (1+2q^2+q^4)x_9 + (2q^2+3q^4+q^6)x_8 + (2q^4+3q^6+3q^8+q^{10})x_7(q^4+q^6)x_5 + q^6x_4 + (q^4+q^6)x_3 + (q^5+q^7+q^9)x_2 + q^7x_1. \]
We see that the highest basis vector not having a coefficient in \( q\mathbb{Z}[q] \) (apart from \( x_{18} \)) is \( x_{17} \). So we look at \( F_{\pi_5}v_\lambda - (q+q^{-1})G_\lambda(b_{\pi_3}) = x_{18} + (q^3+q+q^{-1})x_{16} + (q^4+2q^2+1)x_{13} + (q^4+q^2+1) + \cdots \) (here all coefficients not written lie in \( q\mathbb{Z}[q] \)). Now \( x_{16} \) does not have a coefficient in \( q\mathbb{Z}[q] \), so we look at \( F_{\pi_5}v_\lambda - (q+q^{-1})G_\lambda(b_{\pi_3}) - (q+q^{-1})G_\lambda(b_{\pi_1}) = x_{18} + (q^2+1)x_{13} + \cdots \). We see that \( G_\lambda(b_{\pi_5}) = F_{\pi_5}v_\lambda - (q+q^{-1})G_\lambda(b_{\pi_3}) - (q+q^{-1})G_\lambda(b_{\pi_1}) - G_\lambda(b_{\pi_4}) \).

**Remark.** Let \( \pi \in \Pi_\Lambda \), and let \( \phi(\pi) = s_{i_1} \cdots s_{i_r} \) be the reduced expression which is the smallest in the lexicographical order. Let \( F_\pi = F_{\alpha_1}^{(n_1)} \cdots F_{\alpha_r}^{(n_r)} \). Write \( \alpha = \alpha_{i_1} \). If \( n_1 > 1 \), then by [13], Lemma 5.3(b), \( \phi(e_{\alpha} \pi) = \phi(\pi) \), and hence \( F_{e_{\alpha} \pi} = F_{\alpha_1}^{(n_1-1)} \cdots F_{\alpha_r}^{(n_r)} \). On the other hand, if \( n_1 = 1 \), then by [13], Lemma 5.3(a) we see that \( s_\alpha \phi(e_{\alpha} \pi) \not\preceq_B \phi(e_{\alpha} \pi) \). So by [13], Lemma 5.3(b), \( \phi(\pi) = s_\alpha \phi(e_{\alpha} \pi) \). Therefore \( \phi(e_{\alpha} \pi) = s_{i_2} \cdots s_{i_r} \) which is the smallest (in the lexicographical order) reduced expression for \( \phi(e_{\alpha} \pi) \). Hence \( F_{e_{\alpha} \pi} = F_{\alpha_2}^{(n_2)} \cdots F_{\alpha_r}^{(n_r)} \). The conclusion is that
\[ F_\pi \cdot v_\lambda = \frac{1}{|n_1|} F_\alpha \cdot (F_{e_{\alpha} \pi} \cdot v_\lambda). \]
So in order to compute \( F_\pi \cdot v_\lambda \), we only have to act with \( F_\alpha \) on a vector that we already computed.

**Remark.** Instead of the algorithm described here for getting the monomials \( F_\pi \), one can also follow the procedure outlined in [13] for constructing so-called adapted strings. Instead of \( \phi(\pi) \) this procedure uses the longest element in the Weyl group. However, the monomials one gets in that case are in general different from the ones we get. Moreover, in general they do not have the nice property described in the previous remark.
5 The $A_n$-case

In this section we assume that the root system $\Phi$ is of type $A_n$. We use results from [9] to show that in this case our algorithm is very much like the algorithm described in [11].

The simple roots are $\alpha_1, \ldots, \alpha_n$, where we use the usual ordering of the nodes of the Dynkin diagram (cf. [1]).

Since the fundamental weights are all minuscule the corresponding irreducible $U_q$-modules are easy to construct (cf. [1], Chapter 5A). For $V(\lambda_k)$ we consider the set of sequences $S = \{(i_1, \ldots, i_k) \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n + 1\}$. Let $V$ be the vectorspace over $\mathbb{Q}(q)$ spanned by $v_s$ for $s \in S$. Let $s \in S$. If $i$ occurs in $s$, but $i + 1$ does not, then let $s_i^-$ be the sequence obtained from $s$ by replacing $i$ by $i + 1$, and we set $v_s^i = v_{s^-}$. Otherwise $v_s^i = 0$. Also, if $i + 1$ occurs in $s$, but $i$ does not, then let $s_i^+$ be the sequence obtained from $s$ by replacing $i + 1$ by $i$, and we set $v_s^i = v_{s^+}$. Otherwise $v_s^i = 0$. Now a $U_q$-action on $V$ is defined by $F_{\alpha_i} \cdot v_s = v_s^i$, $E_{\alpha_i} \cdot v_s = v_s^i$, and

$$K_{\alpha_i} \cdot v_s = \begin{cases} qv_s & \text{if } i \in S \text{ and } i + 1 \not\in S, \\ q^{-1}v_s & \text{if } i \not\in S \text{ and } i + 1 \in S, \\ v_s & \text{otherwise.} \end{cases}$$

Then the $U_q$-module $V$ is isomorphic to $V(\lambda_k)$. To see this we note that $v_s$ is a weight vector of weight $\mu_s = a_1\lambda_1 + \cdots + a_n\lambda_n$, where $a_i = 1$ if $i \in S$, $i + 1 \not\in S$; $a_i = -1$ if $i \not\in S$, $i + 1 \in S$, and $a_i = 0$ otherwise. Set $s_{\lambda_k} = (1, 2, \ldots, k)$, then $\mu_{s_{\lambda_k}} = \lambda_k$. If $i \in S$ and $i + 1 \not\in S$ then $s_{\alpha_i}(\mu_s) = \mu_{s^-}$. Since all elements of $S$ can be obtained from $s_{\lambda_k}$ by a sequence of "moves" $s \to s_i^-$ ([1], Proposition 3.3.1), we have that $\{\mu_s \mid s \in S\} = W \cdot \lambda_k$. Finally we compare with [1], 5A.1.

Let $L(\lambda_k)$ be the $A$-submodule of $V$ spanned by the $v_s$, and let $B(\lambda_k)$ be the set of all $v_s \mod qL(\lambda_k)$. Then $(L(\lambda_k), B(\lambda_k))$ is a crystal base of $V$ ([1], Lemma 9.6). Furthermore, $C_k = \{v_s \mid s \in S\}$ is the canonical basis of $V$. (Indeed, the $v_s$ are certainly invariant under $-$ because they are of the form $F_{\alpha_i} \cdots F_{\alpha_1} \cdot v_{s_{\lambda_k}}$. Secondly $\{v_s \mod qL(\lambda_k) \mid s \in S\} = B(\lambda_k)$.)

So the elements of $B(\lambda_k)$ are labeled by the elements of $S$. From [1] we get the action of the Kashiwara operators as follows: $F_{\alpha_i}(v_s) = v_s^{i^-} \mod qL(\lambda_k)$, and $E_{\alpha_i}(v_s) = v_s^{i^+} \mod qL(\lambda_k)$.

We write a sequence $s = (i_1, \ldots, i_k)$ as a diagram with one column of length $k$ and the elements $i_1, \ldots, i_k$ from top to bottom. For example, the sequence $(1, 4, 5)$ is

$$
\begin{array}{ccc}
1 & 1 & 4 \\
2 & 3 & \\
3 & & \\
\end{array}
= 4 \otimes 1 \otimes \frac{1}{3}.\]$$
Then the highest-weight vector $v_\lambda$ of weight $\lambda$ in $W$ is labeled by the tableau $T_\lambda$ with $i$-s in the $i$-th row. Let $V(\lambda)$ denote the submodule of $W$ generated by $v_\lambda$. Let $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ be the crystal base of $V(\lambda)$. Then by \cite{9}, the elements of $\mathcal{B}(\lambda)$ are labeled by tableaux with non-decreasing rows. In particular, these tableaux label the points in the crystal graph. From \cite{9} we get the following algorithm for computing $\tilde{F}_{\alpha_i}(T), \tilde{E}_{\alpha_i}(T)$, where $T$ is such a tableau.

1. Write the numbers in the tableau as a sequence, starting from the top right, and going along columns from right to left, top to bottom. Below each number write a $+$ if it is equal to $i$, a $-$ if it is $i + 1$ and a blank otherwise.

2. If there is a $+$ followed by a $-$ (maybe separated by blanks), then replace them by blanks. Continue until this operation is no longer possible.

3. (a) If there is no $+$ left, then $\tilde{F}_{\alpha_i}(T) = 0$. Otherwise change the $i$ corresponding to the leftmost $+$ to a $i + 1$. Rebuild the tableau, and the result is $\tilde{F}_{\alpha_i}(T)$.

   (b) If there is no $-$ left, then $\tilde{E}_{\alpha_i}(T) = 0$. Otherwise change the $i + 1$ corresponding to the rightmost $-$ into $i$. Rebuild the tableau, the result is $\tilde{E}_{\alpha_i}(T)$.

Example 9 Let the root system be of type $A_3$, and set $T = \begin{array}{ccc} 1 & 1 & 3 \\ 2 & 2 & 3 \\ \end{array}$. Then the sequence we get is $3, 1, 2, 1, 2, 3$. If $i = 2$, this corresponds to $-o + o + o$ (where we represent a blank by $o$). After the operation of step 2 this becomes $-o + ooo$. We see that

$$\tilde{F}_{\alpha_2}(T) = \begin{array}{ccc} 1 & 1 & 3 \\ 2 & 3 \\ \end{array}, \quad \tilde{E}_{\alpha_1}(T) = \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 \\ \end{array}.$$

The algorithm described in \cite{11} for computing the canonical basis of $V(\lambda)$ has the same steps as our algorithm: First for every tableau $T$ a monomial $F_T = F_{\alpha_{i_1}}^{(n_1)} \cdots F_{\alpha_{i_t}}^{(r_t)}$ is computed. Secondly, from the vectors $F_T \cdot v_\lambda$ the canonical basis is computed using a similar triangular algorithm to the one we use. Therefore, the main difference between the algorithms lies in the first step. We investigate this step a little further.

In \cite{11}, 4.1 the authors describe the following algorithm for obtaining a monomial $F_T$ from a tableau $T$. Let $i_1$ be the smallest index such that $i_1 + 1$ occurs in an $m$-th row of $T$ with $m \leq i_1$. Furthermore, $r_1$ is the number of occurrences of $i_1 + 1$ on an $m$-th row with $m \leq i$. Then $T_1$ is obtained from $T$ by replacing these $r_1$ occurrences of $i_1 + 1$ by $i_1$. Continuing with $T_1$ instead of $T$ we eventually arrive at the tableau $T_\lambda$, at which point the algorithm stops. We have obtained sequences $i_1, \ldots, i_t, r_1, \ldots, r_t$ and the monomial is $F_T = F_{\alpha_{i_1}}^{(n_1)} \cdots F_{\alpha_{i_t}}^{(r_t)}$.

Note that applying $\tilde{E}_{\alpha_i}$ amounts to replacing an $i + 1$ by $i$. Since this $i + 1$ was put there by a series of applications of $\tilde{F}_{\alpha_{i_k}}$, starting with $T_\lambda$ we see that this $i + 1$ must occur on the $m$-th row with $m \leq i$. By induction on the number of columns of $T$ it can be
shown that if $i_1$ is minimal such that $i_1 + 1$ occurs in an $m$-th row of $T$ with $m \leq i_1$, then $\tilde{E}_{\alpha_{i_1}}(T) \neq 0$. However in our algorithm we follow the lexicographically smallest reduced expression of a word in the Weyl group in order to get the sequence $i_1, \ldots$. This means that sometimes we obtain a different monomial than with the algorithm from [11], as the following example shows.

**Example 10** Set $T = \begin{array}{ccc} 1 & 1 & 4 \\ 2 & 3 \\ 3 \end{array}$. Then the monomial obtained by the algorithm of [11] is $F_{\alpha_2}F_{\alpha_3}F_{\alpha_2}F_{\alpha_1}$. Let $\pi$ be the corresponding path, then $\phi(\pi) = s_{\alpha_3}s_{\alpha_2}s_{\alpha_1}$. This means that the monomial that we obtain is $F_{\alpha_3}F_{\alpha_2}^{(2)}F_{\alpha_1}$.

We conclude that in the $A_n$-case our algorithm is very similar to, but not the same as, the algorithm described in [11].

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