A note on transitive union-closed families.

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Abstract
We show that the Union-Closed Conjecture holds for the union-closed family generated by the cyclic translates of any fixed set.

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1 Introduction

If $X$ is a set, a family $\mathcal{F}$ of subsets of $X$ is said to be \textit{union-closed} if the union of any two sets in $\mathcal{F}$ is also in $\mathcal{F}$. The celebrated Union-Closed Conjecture (a conjecture of Frankl [2]) states that if $X$ is a finite set and $\mathcal{F}$ is a union-closed family of subsets of $X$ (with $\mathcal{F} \neq \{\emptyset\}$), then there exists an element $x \in X$ such that $x$ is contained in at least half of the sets in $\mathcal{F}$. Despite the efforts of many researchers over the last forty-five years, and a recent Polymath project [5] aimed at resolving it, this conjecture remains wide open. It has only been proved under very strong constraints on the ground-set $X$ or the family $\mathcal{F}$; for example, Balla, Bollobás and Eccles [1] proved it in the case where $|\mathcal{F}| \geq \frac{2}{3}2^{|X|}$; more recently, Karpas [4] proved it in the case where $|\mathcal{F}| \geq \left(\frac{1}{2} - c\right)2^{|X|}$ for a small absolute constant $c > 0$; and it is also known to hold whenever $|X| \leq 12$ or $|\mathcal{F}| \leq 50$, from work of Vučković and Živković [8] and of Roberts and Simpson [7]. We note that Reimer [6] proved that the average size of a set in an arbitrary finite union-closed family $\mathcal{F}$ is at least $\frac{1}{2} \log_2(|\mathcal{F}|)$; this yields (by averaging) a good approximation to the Union-Closed

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Conjecture in the case where $\mathcal{F}$ is large, e.g. it implies that there is an element contained in at least an $\Omega(1)$-fraction of the sets in $\mathcal{F}$, in the case where $|\mathcal{F}| = 2^{\Omega(n)}$.

If $X$ is a set and $\mathcal{F}$ is a family of subsets of $X$, we say $\mathcal{F}$ is transitive if the automorphism group of $\mathcal{F}$ acts transitively on $X$. (The automorphism group of $\mathcal{F}$ is the set of all permutations of $X$ that preserve $\mathcal{F}$.) Informally, $\mathcal{F}$ is transitive if all points of $X$ ‘look the same’ with respect to $\mathcal{F}$. Even the special case of the Union-Closed Conjecture for transitive families is wide open.

In this note, we prove the conjecture in the special case where $X$ is $\mathbb{Z}_n$, the cyclic group of order $n$, and $\mathcal{F}$ is the (transitive) union-closed family consisting of all unions of cyclic translates of some fixed set. This is a question asked in the Polymath project [5].

Theorem 1. Let $n \in \mathbb{N}$, and let $R \subseteq \mathbb{Z}_n$ with $R \neq \emptyset$. Let $\mathcal{F} = \{A + R : A \subseteq \mathbb{Z}_n\}$ be the set of all unions of cyclic translates of $R$. Then the average size of a set in $\mathcal{F}$ is at least $n/2$. In particular, the Union-Closed Conjecture holds for $\mathcal{F}$.

Our proof is surprisingly short. In fact, we establish the following slightly more general result.

Theorem 2. Let $(G, +)$ be a finite Abelian group, and let $R \subseteq G$ with $R \neq \emptyset$. Let $\mathcal{F} = \{A + R : A \subseteq G\}$ be the set of all unions of translates of $R$. Then the average size of a set in $\mathcal{F}$ is at least $|G|/2$. In particular, the Union-Closed Conjecture holds for $\mathcal{F}$.

We note that the family $\mathcal{F}$ in the statement of Theorem 2 is clearly transitive and union-closed, since $x \mapsto x + x_0$ is an automorphism of $\mathcal{F}$ for any $x_0 \in G$, and $(A_1 + R) \cup (A_2 + R) = (A_1 \cup A_2) + R$ for any $A_1, A_2 \subseteq G$.

We remark that it is possible to deduce a slightly weaker form of Theorem 2 from a theorem of Johnson and Vaughan (Theorem 2.10 in [3]). In fact, the result of Johnson and Vaughan, after applying a quotienting argument, yields that there is an element of $G$ contained in at least $(|\mathcal{F}| - 1)/2$ of the sets in $\mathcal{F}$. (Since $\mathcal{F}$ may have odd size, for example when $G$ is $\mathbb{Z}_3$ and $R = \{0, 1\}$, this is not quite enough to deduce Theorem 2.)

We are indebted to Zachary Chase for bringing this paper of Johnson and Vaughan to our attention.

A short explanation of our notation and terminology is in order. As usual, if $G$ is an Abelian group, and $A, B \subseteq G$, we write $A + B = \{a + b : a \in A, b \in B\}$ for the sumset of $A$ and $B$. Similarly, if $a \in G$ and $B \subseteq G$, we define $a + B = \{a + b : b \in B\}$. For any $x \in G$, we let $-x$ denote the inverse of $x$ in $G$, and for any set $A \subseteq G$, we let $-A = \{-a : a \in A\}$. We say a subset $A \subseteq G$ is symmetric if $A = -A$. If $X$ is a finite set, we write $\mathcal{P}(X)$ for the power-set of $X$.

2 Proof of Theorem 2.

Before proving Theorem 2, we introduce some useful concepts and notation. Let $G$ be a fixed, finite Abelian group, and let $R \subseteq G$ be fixed. For any set $A \subseteq G$, we define its $R$-neighbourhood to be

$$N_R(A) := A + R,$$

where $A + R := \{a + r : a \in A, r \in R\}$.
and its $R$-interior to be
$$\text{Int}_R(A) := \{ x \in G : x + R \subseteq A \}.$$ We note that, if $R$ is symmetric and contains the identity element $0$ of $G$, then the $R$-neighbourhood of any set $A$ is precisely the graph-neighbourhood of $A$ in the Cayley graph of $G$ with generating-set $R \setminus \{0\}$, and similarly, the $R$-interior of $A$ is precisely the graph-interior of $A$ with respect to this Cayley graph.

**Proof of Theorem 2.** Let $G$ be a fixed, finite Abelian group and let $R \subseteq G$ be a fixed, nonempty subset of $G$. Let
$$\mathcal{F} = \{ A + R : A \subseteq G \}$$ be the union-closed family consisting of all unions of translates of $R$.

We define a function $f : \mathcal{P}(G) \to \mathcal{P}(G)$ by
$$f(S) = -(G \setminus \text{Int}_R(S)) \quad \text{for all } S \subseteq G.$$ It is clear that for any set $S \subseteq G$, $|\text{Int}_R(S)| \leq |S|$, since for any element $r \in R$, the function $x \mapsto x + r$ is an injection from $\text{Int}_R(S)$ into $S$. Hence,
$$|S| + |f(S)| \geq |G| \quad \text{for all } S \subseteq G. \quad (1)$$

Next, we observe that $f(S) = (-(G \setminus S)) + R$ for all $S \subseteq G$. \quad (2)

Indeed, for any $x \in G$, it holds that $x \in f(S)$ iff $-x \notin \text{Int}_R(S)$ iff $(-x + R) \cap (G \setminus S) \neq \emptyset$ iff $x \in (-(G \setminus S)) + R$. It follows that $f(\mathcal{P}(G)) \subseteq \mathcal{F}$.

Finally, we observe that the restriction $f|_\mathcal{F}$ is an injection. This might seem surprising at first glance, but it follows immediately from the fact that
$$N_R(\text{Int}_R(A + R)) = A + R \quad \text{for all } A \subseteq G. \quad (3)$$ To see (3), let $S = A + R$ and observe that $N_R(\text{Int}_R(S)) \subseteq S$ holds by definition (in fact for any set $S$). On the other hand, if $S = A + R$, then we have $A \subseteq \text{Int}_R(S)$ and therefore $S = A + R \subseteq N_R(\text{Int}_R(S))$. Hence, $S = N_R(\text{Int}_R(S))$, as required.

Putting everything together, we see that $f|_\mathcal{F}$ is a bijection from $\mathcal{F}$ to itself and satisfies
$$|S| + |f(S)| \geq |G| \quad \text{for all } S \in \mathcal{F}.$$ Therefore,
$$\frac{1}{|\mathcal{F}|} \sum_{S \in \mathcal{F}} |S| = \frac{1}{2|\mathcal{F}|} \sum_{S \in \mathcal{F}} (|S| + |f(S)|) \geq \frac{1}{2|\mathcal{F}|} \sum_{S \in \mathcal{F}} |G| = |G|/2,$$ proving the first part of the theorem. It follows that
$$\frac{1}{|G|} \sum_{x \in G} \left| \left\{ S \in \mathcal{F} : x \in S \right\} \right| = \frac{1}{|G|} \frac{1}{|\mathcal{F}|} \sum_{S \in \mathcal{F}} |S| \geq 1/2,$$ so by averaging, there exists $x \in G$ such that at least half the sets in $\mathcal{F}$ contain $x$, and so the Union-Closed Conjecture holds for $\mathcal{F}$. \qed
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