The resource theory of steering

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We present an operational framework for Einstein-Podolsky-Rosen steering as a physical resource. To begin with, we characterize the set of \textit{steering non-increasing operations} (SNIOs) –i.e., those that do not create steering– on arbitrary-dimensional bipartite systems composed of a quantum subsystem and a black-box device. Next, we introduce the notion of \textit{convex steering monotones} as the fundamental axiomatic quantifiers of steering. As a convenient example thereof, we present the \textit{relative entropy of steering}. In addition, we prove that two previously proposed quantifiers, the steerable weight and the robustness of steering, are also convex steering monotones. To end up with, for minimal-dimensional systems, we establish, on the one hand, necessary and sufficient conditions for pure-state steering conversions under stochastic SNIOs and prove, on the other hand, the non-existence of \textit{steering bits}, i.e., measure-independent maximally steerable states from which all states can be obtained by means of the free operations. Our findings reveal unexpected aspects of steering and lay foundations for further resource-theory approaches, with potential implications in Bell non-locality.

\textbf{Introduction.}– \textit{Steering}, as Schrödinger named it [1], is an exotic quantum effect by which ensembles of quantum states can be remotely prepared by performing local measurements at a distant lab. It allows [2, 3] to certify the presence of entanglement between a user with an untrusted measurement apparatus, Alice, and another with a trusted quantum-measurement device, Bob. Thus, it constitutes a fundamental notion between quantum entanglement [4], whose certification requires quantum measurements on both sides, and Bell non-locality [5], where both users possess untrusted black-box devices. Steering can be detected through simple tests analogous to Bell inequalities [6], and has been verified in a variety of remarkable experiments [7], including steering without Bell non-locality [8] and a fully loop-hole free steering demonstration [9]. Apart from its fundamental relevance, steering has been identified as a resource for one-sided device-independent quantum key-distribution (QKD), where only one of the parts has an untrusted apparatus while the other ones possess trusted devices [10, 11]. There, the experimental requirements for unconditionally secure keys are less stringent than in fully (both-sided) device-independent QKD [12].

The formal treatment of a physical property as a resource is given by a \textit{resource theory}. The basic component of this is the characterization of the physical operations under which the set of states without that property is invariant. For instance, the fundamental necessary condition for a function to be a measure of the resource is that it is monotonous –non-increasing– under the operations. That is, the resource is quantified in such a way that the operations that do not increase it on the resourceless states do not increase it on all other states either. The operations are then typically referred to as the non-increasing operations for the resource in question. Entanglement theory [4] is the most popular and best understood example of a resource theory [13, 14], with the corresponding non-increasing operations being the local operations assisted by classical communication (LOCCs) [15]. Nevertheless, resource theories have been formulated also for states out of thermal equilibrium [16], asymmetry [17], reference frames [18], and quantum coherence [19, 20], for instance.

In steering theory systems are described by an ensemble of quantum states, on Bob’s side, each one associated to the conditional probability of a measurement outcome (output) given a measurement setting (input), on Alice’s. Such conditional ensembles are sometimes called \textit{assemblages} [21–23]. The steering non-increasing operations (SNIOs) describe the physical transformations that map all unsteerable assemblages, i.e., those without steering, into some unsteerable assemblage. Curiously, up to now, no attempt for an operational framework of steering as a resource has been reported.

In this work, we derive the explicit expression of the most general SNIO, for arbitrarily many inputs and outputs for Alice’s black box and arbitrary dimension for Bob’s quantum system. With this, we provide a formal definition of steering monotones. As an example thereof, we present the relative entropy of steering, for which we also introduce, on the way, the notion of relative entropy between assemblages. In addition, we prove SNIO monotonicity for two other recently proposed steering measures, the steerable weight [22] and the robustness of steering [23], and convexity for all three measures. To end up with, we prove two theorems on steering conversion under stochastic SNIOs for the lowest-dimensional case, i.e., qubits on Bob’s side and 2 inputs \( \times 2 \) outputs on Alice’s. In the first one, we show that it is impossible to transform via SNIOs, not even probabilistically, an assemblage composed of pairs of pure orthogonal states into another assemblage composed also of pairs of pure orthogonal states but with a different pair overlap, unless the latter is unsteerable. This yields infinitely many inequivalent classes of steering already for systems of the lowest dimension. In the second one, we show that there exists no assemblage composed of pairs of pure states that can be transformed into any assemblage by stochastic SNIOs. It follows that, in striking contrast to entanglement theory, there exists no operationally well defined, measure-independent maximally steerable assemblage of minimal dimension.

\textbf{Scenario.}– We consider two distant parties, Alice and Bob, who have each a half of a bipartite system. Alice holds a so-called black-box device, which, given a classical input \( x \in [s] \), generates a classical output \( a \in [r] \), where \( s \) and \( r \) are natural numbers and the notation \( [n] := \{0, \ldots, n - 1 \} \), for \( n \in \mathbb{N} \), is introduced. Bob holds a quantum system of dimension \( d \) (\textit{qudit}), whose state he can perfectly characterize tomographically via trusted quantum measurements. The joint state of
their system is thus fully specified by an assemblage
\[ \rho_{A|X} := \{ P_{A|X}(a, x), \varrho(a, x) \}_{a,x}, \quad (1) \]
of normalized quantum states \( \varrho(a, x) \in \mathcal{L}(\mathcal{H}_B), \) with \( \mathcal{L}(\mathcal{H}_B) \) the set of linear operators on Bob’s subsystem’s Hilbert space \( \mathcal{H}_B, \) each one associated to a conditional probability \( P_{A|X}(a, x) \) of Alice getting an output \( a \) given an input \( x. \) We denote by \( P_{A|X} \) the corresponding conditional probability distribution.

Equivalently, each pair \( \{ P_{A|X}(a, x), \varrho(a, x) \} \) can be univocally represented by the unnormalized quantum state
\[ \varrho_{A|X}(a, x) := P_{A|X}(a, x) \times \varrho(a, x). \quad (2) \]
In turn, an alternative representation of the assemblage \( \rho_{A|X} \) is given by the set \( \hat{\rho}_{A|X} := \{ \hat{\rho}_{A|X}(x) \} \) of quantum states
\[ \hat{\rho}_{A|X}(x) := \sum_a |a\rangle \langle a| \otimes \varrho_{A|X}(a, x) \in \mathcal{L}(\mathcal{H}_E \otimes \mathcal{H}_B), \quad (3) \]
where \( \{ |a\rangle \} \) is an orthonormal basis of an auxiliary extension Hilbert space \( \mathcal{H}_E \) of dimension \( r. \) The states \( \{ |a\rangle \} \) do not describe the system inside Alice’s box, they are just abstract flag states to represent its outcomes with a convenient bra-ket notation. Expression (3) gives the counterpart for assemblages of the so-called extended Hilbert space representation used for ensembles of quantum states [24]. We refer to \( \hat{\rho}_{A|X} \) for short as the quantum representation of \( \rho_{A|X} \) and use either notation upon convenience.

We restrict throughout to no-signaling assemblages, i.e., those for which Bob’s reduced state \( \varrho_B \in \mathcal{L}(\mathcal{H}_B) \) does not depend on Alice’s input choice \( x. \)
\[ \varrho_B := \sum_a \varrho_{A|X}(a, x) = \sum_a \varrho_{A|X}(a, x') \quad \forall \ x, x'. \quad (4) \]
The assemblages fulfilling the no-signaling condition (4) are the ones that possess a quantum realization. That is, they can be obtained from local quantum measurements by Alice on a joint quantum state \( \varrho_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B) \) shared with Bob, where \( \mathcal{H}_A \) is the Hilbert space of the system inside Alice’s box. For any no-signaling assemblage \( \rho_{A|X}, \) we refer as the trace of the assemblage to the \( x- \)independent quantity
\[ \text{Tr}[\rho_{A|X}] := \text{Tr}_{EB}[\hat{\rho}_{A|X}] = \text{Tr}[\varrho_B] = \sum_a P_{A|X}(a, x), \quad (5) \]
and say that the assemblage is normalized if \( \text{Tr}[\rho_{A|X}] = 1 \) and unnormalized if \( \text{Tr}[\rho_{A|X}] \leq 1. \)

An assemblage \( \sigma_{A|X} := \{ \sigma_{A|X}(a, x) \}_{a,x}, \) being \( \sigma_{A|X}(a, x) \in \mathcal{L}(\mathcal{H}_B) \) unnormalised states, is called unsteerable if there exist a probability distribution \( P_A, \) a conditional probability distribution \( P_{A|XA}, \) and normalized states \( \xi(\lambda) \in \mathcal{L}(\mathcal{H}_B) \) such that
\[ \sigma_{A|X}(a, x) = \sum_\lambda P_A(\lambda) P_{A|XA}(a, x, \lambda) \xi(\lambda) \quad \forall \ x, a. \quad (6) \]

Such assemblages can be obtained by sending a shared classical random variable \( \lambda \) to Alice, correlated with the state \( \xi(\lambda) \) sent to Bob, and letting Alice classically post-process her random variable according to \( P_{A|X \lambda}, \) with \( P_{X\lambda} = P_X \times P_A \) so that condition (4) holds. The variable \( \lambda \) is called a local-hidden variable and the decomposition (6) is accordingly referred to as a local-hidden state (LHS) model. We refer to the set of all unsteerable assemblages as LHS. Any assemblage that does not admit a LHS model as in Eq. (6) is called steerable. An assemblage is compatible with classical correlations if, and only if, it is unsteerable.

The operational framework. Let us now characterise the SNIOs. We consider the general scenario of stochastic SNIOs, i.e., SNIOs that do not necessarily occur with certainty. Any assemblage transformation can be decomposed in terms of local operations assisted by communication (see Fig. 1). Bob can apply an arbitrary quantum operation on his subsystem, represented by a generalised incomplete measurement described by a completely-positive non trace-preserving map \( \mathcal{E} : \mathcal{L}(\mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_{Bf}) \) defined by
\[ \mathcal{E}(\cdot) := \sum_\omega \mathcal{E}_\omega(\cdot), \text{ with } \mathcal{E}_\omega(\cdot) := K_\omega \cdot K_\omega^\dagger, \quad (7a) \]
\[ \text{such that } \sum_\omega K_\omega K_\omega^\dagger \leq 1, \quad (7b) \]
where \( \mathcal{H}_{Bf} \) is the final Hilbert space, of dimension \( d_f, \) and \( K_\omega : \mathcal{H}_B \to \mathcal{H}_{Bf} \) is the measurement operator correspond-
As with quantum operations, the trace (12) of $\mathcal{M}(\hat{\rho}_{A|X})$ represents the probability that the physical transformation $\hat{\rho}_{A|X} \rightarrow \mathcal{M}(\hat{\rho}_{A|X})/\text{Tr}[\mathcal{M}(\hat{\rho}_{A|X})]$ takes place. Analogously, the map $\mathcal{M}_\omega$ describes the assemblage transformation that takes place when Bob post-selects the $\omega$-th outcome, which occurs with probability $\text{Tr}[\mathcal{M}_\omega(\hat{\rho}_{A|X})] = P_\Omega(\omega)$.

Steering monotonicity.— As the natural next step, we introduce an axiomatic approach to define steering measures, i.e., a set of reasonable postulates that a bona fide quantifier of the steering of a given assemblage should fulfil.

Definition 1 (SNIO-monotonicity and convexity). A function $\mathcal{S}$, from the space of assemblages into $\mathbb{R}_{\geq 0}$, is a steering monotone if it fulfills the following two axioms:

i) $\mathcal{S}(\hat{\rho}_{A|X}) = 0$ for all $\hat{\rho}_{A|X} \in \text{LHS}$.

ii) $\mathcal{S}$ does not increase, on average, under deterministic SNIOs, i.e.,

$$\sum_\omega P_\Omega(\omega) \mathcal{S} \left( \frac{\mathcal{M}_\omega(\hat{\rho}_{A|X})}{\text{Tr}[\mathcal{M}_\omega(\hat{\rho}_{A|X})]} \right) \leq \mathcal{S}(\hat{\rho}_{A|X}) \quad (13)$$

for all $\hat{\rho}_{A|X}$, with $P_\Omega(\omega) = \text{Tr}[\mathcal{M}_\omega(\hat{\rho}_{A|X})]$ and $\sum_\omega P_\Omega = 1$.

Besides, $\mathcal{S}$ is a convex steering monotone if it additionally satisfies the property:

iii) Given any real number $0 \leq \mu \leq 1$, and assemblages $\hat{\rho}_{A|X}$ and $\hat{\rho}'_{A|X}$, then

$$\mathcal{S} \left( \mu \hat{\rho}_{A|X} + (1 - \mu)\hat{\rho}'_{A|X} \right) \leq \mu \mathcal{S}(\hat{\rho}_{A|X}) + (1 - \mu)\mathcal{S}(\hat{\rho}'_{A|X}) \quad (14)$$

Condition i) reflects the basic fact that unsteerable assemblages should have zero steering. Condition ii) formalizes the intuition that, analogously to entanglement, steering should not increase —on average— under SNIOs, even if the flag information $\omega$ produced in the transformation is available. Finally, condition iii) states the desired property that steering should not increase by probabilistically mixing assemblages.

The first two conditions are taken as mandatory necessary conditions, the third one only as a convenient property. Importantly, there exists a less demanding definition of monotonicity. There, the left-hand side of Eq. (13) is replaced by $\mathcal{S}(\hat{\rho}_{A|X})/\text{Tr}[\mathcal{M}(\hat{\rho}_{A|X})]$. That is, $\mathcal{S}'$ is defined only on a map that could be generated from —say— shared random bits and describes the post-selection of the $\omega$-th output.

Condition $\mathcal{S}'$ is in many cases (including the present work) easier to prove and, together with condition iii), implies monotonicity $\mathcal{S}$.

Hence, we focus throughout on monotonicity as defined by Eq. (13) and refer to it simply as SNIO monotonicity. All three known quantifiers of steering, the two ones introduced in Refs. [22, 23] as well as the one we introduce next, turn out to be convex steering monotones in the sense of Definition 1.
The relative entropy of steering.— The first step is to introduce the notion of relative entropy between assemblages. To this end, for any two density operators \( \rho \) and \( \rho' \), we first recall the quantum von-Neumann relative entropy

\[
S_Q(\rho \| \rho') := \text{Tr}[\rho (\log \rho - \log \rho')]
\]

of \( \rho \) with respect to \( \rho' \) and, for any two probability distributions \( P_X \) and \( P'_X \), the classical relative entropy, or Kullback-Leibler divergence,

\[
S_C(P_X \| P'_X) := \sum_x P_X(x) \log P_X(x) - \log P'_X(x)
\]

of \( P_X \) with respect to \( P'_X \). The quantum and classical relative entropies (15) and (16) measure the distinguishability of states and distributions, respectively. To find an equivalent measure for assemblages, we note, for \( \hat{\rho}_{A|X}(x) \) given by Eq. (3) and \( \hat{\rho}'_{A|X}(x) := \sum_a P'_{A|X}(a|x) a \otimes \hat{\rho}'(a,x) \), that

\[
S_Q(\hat{\rho}_{A|X}(x) \| \hat{\rho}'_{A|X}(x)) = S_C(P_{A|X}(\cdot | x) \| P'_{A|X}(\cdot | x)) + \sum_a P_{A|X}(a|x) S_Q(\hat{\rho}(a,x) \| \hat{\rho}'(a,x))
\]

where \( P_{A|X}(\cdot | x) \) and \( P'_{A|X}(\cdot | x) \) are respectively the distributions over \( a \) obtained from the conditional distributions \( P_{A|X} \) and \( P'_{A|X} \) for a fixed \( x \). That is, the distinguishability between the states \( \hat{\rho}_{A|X}(x) \) and \( \hat{\rho}'_{A|X}(x) \) is \( \mathcal{L}(H_E \otimes H_B) \) equals the sum of the distinguishabilities between \( P_{A|X}(x) \) and \( P'_{A|X}(x) \) and between \( \hat{\rho}(a,x) \) and \( \hat{\rho}'(a,x) \in \mathcal{L}(H_B) \), weighted by \( P_{A|X}(a|x) \) and averaged over \( a \).

The entropy (17), which depends on \( x \), does not measure the distinguishability between the assemblages \( \rho_{A|X} \) and \( \rho'_{A|X} \). Since the latter are conditional objects, i.e., with inputs, a general strategy to distinguish them must allow for Alice choosing the input for which the assemblages’ outputs are optimally distinguishable. Furthermore, Bob can first apply a generalised measurement on his subsystem and communicate the outcome \( \gamma \) to her, which she can then use for her input choice. This is the most general procedure within the allowed SNIOs. Hence, a generic distinguishing strategy under SNIOs involves probabilistically chosen inputs that depend on \( \gamma \). Note, in addition, that the statistics of \( \gamma \) generated, described by distributions \( P_T \) or \( P_T' \), encode differences between \( \rho_{A|X} \) and \( \rho'_{A|X} \) and must therefore also be accounted for by a distinguishability measure. The following definition incorporates all these considerations.

**Definition 2** (Relative entropy between assemblages). Given any two assemblages \( \rho_{A|X} \) and \( \rho'_{A|X} \), we define the assemblage relative entropy of \( \rho_{A|X} \) with respect to \( \rho'_{A|X} \) as

\[
S_A(\rho_{A|X} \| \rho'_{A|X}) := \max_{P_X \in \{E_\gamma\}} \left[ S_C(P_T \| P'_T) + \sum_{\gamma,x} P(x|\gamma) S_Q\left( \frac{\mathbb{1} \otimes E_\gamma \rho_{A|X}(x) \mathbb{1} \otimes E_\gamma^\dagger}{P_T(\gamma)} \| \frac{\mathbb{1} \otimes E_\gamma \rho'_{A|X}(x) \mathbb{1} \otimes E_\gamma^\dagger}{P'_T(\gamma)} \right) \right],
\]

where \( E_\gamma : H_B \rightarrow H_B \) are generalised-measurement operators such that \( \sum_\gamma E_\gamma^\dagger E_\gamma = \mathbb{1} \). \( P_X \in \{E_\gamma\} \) is a conditional probability distribution of \( x \) given \( \gamma \), the short-hand notation \( P(x|\gamma) := P_{X|\Gamma}(x,\gamma) \) has been used, and

\[
\begin{align*}
P_T(\gamma) & := \text{Tr}[\mathbb{1} \otimes E_\gamma \hat{\rho}_{A|X}(x) \mathbb{1} \otimes E_\gamma^\dagger] = \text{Tr}_B[E_\gamma \hat{\rho}_{A|X}(x) E_\gamma^\dagger], \\
P'_T(\gamma) & := \text{Tr}[\mathbb{1} \otimes E_\gamma \hat{\rho}'_{A|X}(x) \mathbb{1} \otimes E_\gamma^\dagger] = \text{Tr}_B[E_\gamma \hat{\rho}'_{A|X}(x) E_\gamma^\dagger],
\end{align*}
\]

where \( \hat{\rho}'_B \) is Bob’s reduced state for the assemblage \( \rho'_{A|X} \).

In App. B, we show that \( S_A \) does not increase—an average—under deterministic SNIOs and, as its quantum counterpart \( S_Q \), is jointly convex. Hence, \( S_A \) is a proper measure of distinguishability between assemblages under SNIOs [26]. The first term inside the maximisation in Eq. (18) accounts for the distinguishability between the distributions of measurement outcomes \( \gamma \) and the second one for that between the distributions of Alice’s outputs and Bob’s states resulting from each \( \gamma \), averaged over all inputs and measurement outcomes. In turn, the maximisation over \( \{E_\gamma\} \) and \( P_X \in \{E_\gamma\} \) ensures that these output distributions and states are distinguished using the optimal SNIO-compatible strategy.

We are now in a good position to introduce a convex steering monotone. We do it with a theorem.

**Theorem 2** (SNIO-monotonicity and convexity of \( \mathcal{R}_R \)). The relative entropy of steering \( \mathcal{R}_R \) defined for an assemblage \( \rho_{A|X} \) as

\[
\mathcal{R}_R(\rho_{A|X}) := \min_{\sigma_{A|X} \in \text{LHS}} S_A(\rho_{A|X} \| \sigma_{A|X}),
\]

is a convex steering monotone.

The theorem is proven in App. B. Apart from \( \mathcal{R}_R \), recently, two other quantifiers of steering have been proposed, the steerable weight [22] and the robustness of steering [23]. In App. C we show that these are also convex steering monotones.

**Assemblage conversions and no steering bits.**— We say that
\(\Psi_{A|X}\) and \(\Psi'_{A|X}\) are pure assemblages if they are of the form

\[\Psi_{A|X} := \{P_{A|X}(a,x), |\psi(a,x)\rangle\langle\psi(a,x)|\}_{a,x}, \quad (21a)\]
\[\Psi'_{A|X} := \{P'_{A|X}(a,x), |\psi'(a,x)\rangle\langle\psi'(a,x)|\}_{a,x}, \quad (21b)\]

where \(|\psi(a,x)\rangle\) and \(|\psi'(a,x)\rangle\in \mathcal{H}_B\) and pure orthogonal assemblages if, in addition, \(\langle\psi(a,x)|\psi'(a,x)\rangle = \delta_{a,a} = \langle\psi'(a,x)|\psi(a,x)\rangle\) for all \(x\). Note that pure orthogonal assemblages are the ones obtained when Alice and Bob share a pure maximally entangled state and Alice performs a von-Neumann measurement on her share. We present two theorems about assemblage conversions under SNIOs.

The first one, proven in App. D, establishes necessary and sufficient conditions for stochastic-SNIO conversions between pure orthogonal assemblages, therefore playing a similar role here to the one played in entanglement theory by Vidal’s theorem [27] for stochastic-LOCC pure-state conversions.

**Theorem 3** (Criterion for stochastic-SNIO conversion). Let \(\Psi_{A|X}\) and \(\Psi'_{A|X}\) be any two pure orthogonal assemblages with \(d = s = r = 2\). Then, \(\Psi_{A|X}\) can be transformed into \(\Psi'_{A|X}\) by a stochastic SNIO iff: either \(\Psi_{A|X}\in \text{LHS or } P'_{A|X} = P_{A|X}\) and

\[|\langle\psi'(a,0)|\psi'(a,1)\rangle| = |\langle\psi(a,0)|\psi(a,1)\rangle| \\forall a. \quad (22)\]

In other words, no pure orthogonal assemblage of minimal dimension can be obtained via a SNIO, not even probabilistically, from a pure orthogonal assemblage of minimal dimension with a different state-basis overlap unless the former is unsteerable. Hence, each state-basis overlap defines an inequivalent class of steering, there being infinitely many of them. This is in a way reminiscent to the inequivalent classes of entanglement in multipartite [28] or infinite-dimensional bipartite [29] systems, but here the phenomenon is found already for bipartite systems of minimal dimension.

The second theorem, proven in App. E, rules out the possibility of there being a (non-orthogonal) minimal-dimension pure assemblage from which all assemblages can be obtained.

**Theorem 4** (Non-existence of steering bits). There exists no pure assemblage with \(d = s = r = 2\) that can be transformed into any assemblage by stochastic SNIOs.

Hence, among the minimal-dimension assemblages there is no operationally well defined unit of steering, or steering bit, i.e., an assemblage from which all assemblages can be obtained for free and can therefore be taken as a measure-independent maximally steerable assemblage. This is again in striking contrast to entanglement theory, where pure maximally entangled states can be defined without the need of entanglement quantifiers and each one can be transformed into any state by deterministic LOCCs [27, 30].

**Discussion and outlook.** Our analysis shows that the SNIOs are a combination of the operations that do not increase entanglement for quantum states, stochastic-LOCCs, and the ones that do not increase Bell non-locality for correlations, local wirings assisted by shared randomness. Regarding the latter, a resource-theory approach to Bell non-locality is only partially developed [31, 32]. Hence, our findings are potentially useful in Bell non-locality, for instance for axiomatic quantifiers of Bell non-locality. In addition, our work offers a number of challenges for future research. Namely, for example, the inexistence of steering bits of minimal dimension can be seen as an impossibility of steering dilution of minimal-dimension assemblages in the single-copy regime. We leave as open questions what the rules for steering dilution and distillation are for higher-dimensional systems, mixed-state assemblages, or in asymptotic multi-copy regimes, and what the steering classes are for mixed-state assemblages. Moreover, other fascinating questions are whether one can formulate a notion of bound steering and an analogue to the positive-partial-transpose criterion for assemblages.

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Appendix A: Stochastic steering non-increasing operations

In this appendix we prove Theorem 1.

Proof. Any transformation of a bipartite system can be decomposed into local operations on the subsystems assisted by generic (quantum) communication between them. Our proof strategy exploits this fact and consists of two parts. First we identify specific the types of local operations and communication that can increase steering and therefore forbid them as components of a generic stochastic SNIO. Then, we show that the most general assemblage map that excludes these forbidden components maps every unsteerable assemblage into an unsteerable assemblage and is of the form given in Eqs. (9) and (10).

Alice’s subsystem is a black-box measurement device with \( r \) classical inputs and \( s \) classical outputs, whereas Bob has full access to a qudit of dimension \( d \). Therefore, only classical information processing—wirings—is allowed on Alice’s side but arbitrary quantum operations on Bob’s, including possible coupling to local ancillas. Regarding the type of communication, since Alice has access just to classical bits, only classical communication is allowed between the parties. In addition, SNIOs cannot involve communication from Alice to Bob, as it can be used to create steering, as explained in the main text. However, one-way classical communication (1WCC) from Bob to Alice is not ruled out by these arguments and is therefore allowed. Furthermore, prior shared randomness can always be recast as 1WCC from Bob to Alice and is therefore also allowed. Finally, Alice’s local wirings must be deterministic, i.e., she is not allowed to post-select her outputs with stochastic filtering wirings. This is due, again, to the fact that post-selection can be used to create steering, as explained in the main text [25]. Nevertheless, stochastic local quantum operations on Bob’s side are not forbidden, as we see next.

In summary, we are left with the following three constituent components for stochastic SNIOs: stochastic local quantum operations on Bob’s side, 1WCC from Bob to Alice, and deterministic local wirings on Alice’s side. We denote by \( \mathcal{M} \) an assemblage map that can be decomposed into these three components. \( \mathcal{M} \) maps an arbitrary normalised initial assemblage \( \rho_{A|X} \), with \( r \) classical inputs, \( s \) classical outputs, and quantum dimension \( d \), into a final assemblage \( \rho_{A|X} = \mathcal{M}(\rho_{A|X}) \), with \( r_f \) classical inputs, \( s_f \) classical outputs, and quantum dimension \( d_f \). Without loss of generality, \( \mathcal{M} \) can always be decomposed into the following sequence (see Fig. 1).

1. Bob applies an arbitrary stochastic (incomplete) generalised measurement, described by a completely-positive non-trace-preserving map \( \mathcal{E} \), defined by Eqs. (7), to his quantum subsystem before Alice introduces an input to her device. Note that, since the non-signalling condition (4) is fulfilled, Bob has a well defined reduced quantum state \( \rho_B \), given by Eq. (4), independently of Alice still not having chosen her measurement input \( x \). Therefore, Bob’s measurement gives the outcome \( \omega \) with the \( x \)-independent probability \( P_B(\omega) \) given by Eq. (8).
2. Bob sends the outcome $\omega$ to Alice. Alice applies a local wiring, described by the normalised conditional probability distribution $P_{X|X_\omega}$, to the input $x_f$ of the final device and to $\omega$, and uses the output of this wiring as the input $x$ of her initial device. For a given $x$, her initial device outputs $a$ with a probability determined by the conditional distribution $P_{A|X}$ of the initial assemblage. In that case, Bob’s normalized state is given by the $x_f$-independent density operator

$$
\rho(a, x, \omega, x_f) := \frac{K_\omega \rho(a, x) K_\omega^\dagger}{\operatorname{Tr}[K_\omega \rho(a, x) K_\omega^\dagger]},
$$

(A1)

3. Alice applies a local wiring, described by the normalised conditional probability distribution $P_{A_f|A,X,\Omega,X_f}$, to all the previously generated classical bits, $a$, $x$, $\omega$, and $x_f$, and uses the output of this wiring as the output $a_f$ of her final device. This final processing of the bit $a_f$ does not affect Bob’s state. Thus, Bob’s system ends up in the state $\rho(a, x, \omega, x_f, a_f) := \rho(a, x, \omega, x_f)$.

We denote by $P_{i|\Omega,A,X}$ the conditional distribution of $\Omega$ given $A$ and $X$, for $X$ chosen independently of $\Omega$ (in contrast to Step 2 above), with elements $P_{i|\Omega,A,X}(\omega, a, x) := \operatorname{Tr}[K_\omega \rho(a, x) K_\omega^\dagger]$. With this, the components $\rho_{A_f|X_f}(a_f, x_f)$ of the final assemblage $\rho_{A_f|X_f}$ are explicitly given by:

$$
\rho_{A_f|X_f}(a_f, x_f) := P_{A_f|X_f}(a_f, x_f) \times \rho_f(a_f, x_f) = \sum_{a, x, \omega} P_{A_f,A,X,\Omega,X_f}(a_f, a, x, \omega, x_f) \times \rho(a_f, a, x, \omega, x_f)
$$

(A2)

$$
= \sum_{a, x, \omega} P_{A_f,A,X,\Omega,X_f}(a_f, a, x, \omega, x_f) \times \frac{K_\omega \rho(a, x) K_\omega^\dagger}{\operatorname{Tr}[K_\omega \rho(a, x) K_\omega^\dagger]}
$$

(A3)

$$
= \sum_{a, x, \omega} P_{\Omega}(\omega) P_{X|X_f,\Omega}(x, x_f, \omega) P_{A|X,\Omega}(a, x) P_{A_f|A,X,\Omega,X_f}(a_f, a, x, \omega, x_f) \times \frac{K_\omega \rho(a, x) K_\omega^\dagger}{P_{\Omega,A,X}(\omega, a, x)}
$$

(A4)

Eq. (A2) follows from basic properties of probability distributions and ensembles of states. Eq. (A3) follows from the definition of $\rho(a, x, \omega, x_f, a_f)$. Eq. (A4) follows from Bayes’ theorem together with the facts that $P_{A|X,\Omega,X_f} = P_{A|X,\Omega}$ (the output of Alice’s initial device only depends on the input $x$ and the measurement outcome $\omega$) and $P_{\Omega,X_f} = P_{\Omega}$ (the measurement outcome $\omega$ is independent of the input of Alice’s final device), and from the definition of $P_{i|\Omega,A,X}$. Next, note that, since the statistics of $A$ is fully determined by $X$ and $\Omega$ regardless of whether $X$ and $\Omega$ are independent or not, it holds that

$$
P_{A|X,\Omega} = P_{i|\Omega,A,X} = \frac{P_{A,X,\Omega}}{P_{\Omega}} = \frac{P_{i|\Omega,A,X} P_{A,X}}{P_{\Omega,X} P_{\Omega}} = \frac{P_{i|\Omega,A,X}}{P_{\Omega}} = \frac{P_{i|\Omega,A,X}}{P_{\Omega}} = \frac{P_{i|\Omega,A,X}}{P_{\Omega}},
$$

(A5)

where we have used Bayes’ theorem, that $P_{i|\Omega} = P_{\Omega}$, and that, by definition, $P_{i|X|\Omega} = P_{X}$. Inserting Eq. (A5) into Eq. (A4), we obtain

$$
\rho_{A_f|X_f}(a_f, x_f) = \sum_{a, x, \omega} P_{X|X_f,\Omega}(x, x_f, \omega) P_{A(X)(a, x)} P_{A_f|A,X,\Omega,X_f}(a_f, a, x, \omega, x_f) \times K_\omega \rho(a, x) K_\omega^\dagger
$$

$$
= \sum_{a, x, \omega} P_{X|X_f,\Omega}(x, x_f, \omega) P_{A_f|A,X,\Omega,X_f}(a_f, a, x, \omega, x_f) \times K_\omega \rho(a, x) K_\omega^\dagger, \forall (a, x_f)
$$

(A6)

where (A6) follows from the fact that $P_{i|\Omega,A,X} = P_{i|\Omega,A,X}$ and the definition of $\rho_{A|X}$. The right-hand side of Eq. (A6) gives the most general expression of the components $\rho_{A_f|X_f}(a_f, x_f)$ of $\mathcal{M}(\rho_{A|X})$ explicitly as a function of the components $\rho_{A|X}$ of $\rho_{A|X}$. The reader can straightforwardly verify that the quantum representation $\mathcal{M}(\rho_{A|X})$ of the obtained final assemblage $\mathcal{M}(\rho_{A|X})$ is given by the right-hand side of Eq. (9).

We now show that if $\rho_{A|X} \in \text{LHS}$ then $\mathcal{M}(\rho_{A|X}) \in \text{LHS}$. Replacing $\rho_{A|X}$ in Eq. (A6) by the right-hand side of Eq. (6), we write

$$
\rho_{A_f|X_f} = \sum_{a, x, \omega, \lambda} P_{\Lambda}(\lambda) P_{A|X,\Lambda}(a, x, \lambda) P_{X|X_f,\Omega}(x, x_f, \omega) P_{A_f|A,X,\Omega,X_f}(a_f, a, x, \omega, x_f) \times K_\omega \xi(\lambda) K_\omega^\dagger
$$

$$
= \sum_{a, x, \omega, \lambda} P_{\Lambda}(\lambda) P_{\Omega|\Lambda}(\omega, \lambda) P_{A|X,\Lambda}(a, x, \lambda) P_{X|X_f,\Omega}(x, x_f, \omega) P_{A_f|A,X,\Omega,X_f}(a_f, a, x, \omega, x_f) \times \xi(\lambda, \omega),
$$

(A7)
where the conditional probability $P_{\Omega | \Lambda}(\omega, \lambda) := \text{Tr}[K_{\omega} \xi(\lambda) K_{\lambda}^+]$ and the normalized state $\xi(\lambda, \omega) := \frac{K_{\omega} \xi(\lambda) K_{\lambda}^+}{P_{\Omega | \Lambda}(\omega, \lambda)}$ have been introduced. Using that $a_f$ does not explicitly depend on $\lambda$, we see that

$$P_{A | \Lambda, X, \Omega, X_f} = P_{A | \Lambda, X, \Omega, X_f}. \tag{A8}$$

In turn, using the facts that $x$ is independent of $\lambda$ and $a$ depends only on $x$ and $\lambda$, and Bayes’ theorem, we see that

$$P_{A | X, \Lambda} P_{X | X_f, \Omega} = P_{A | X, \Omega, X_f} P_{X | \Omega, X_f} = P_{A | X, \Omega, X_f}. \tag{A9}$$

Substituting into Eq. (A7) yields

$$\rho_{A | X_f} = \sum_{\omega, x, \omega, \lambda} P_{\lambda}(\omega) P_{\Omega | \Lambda}(\omega, \lambda) P_{A | X, \Omega, X_f}(a, x, \omega, \lambda, x_f) P_{A | \Lambda, X, \Omega, X_f}(a_f, a, x, \omega, \lambda, x_f) \times \xi(\lambda, \omega)
= \sum_{\omega, \lambda} P_{\Omega, \Lambda}(\omega, \lambda) P_{A | \Omega, \Lambda, X_f}(a_f, \omega, \lambda, x_f) \times \xi(\lambda, \omega)
= \sum_{\lambda} P_{\lambda} \left( \tilde{\lambda} \right) P_{A | X_f, \Lambda}(a_f, x_f, \tilde{\lambda}) \sigma \left( \tilde{\lambda} \right), \tag{A10}\tag{A11}$$

where Eq. (A10) follows from Bayes’ theorem and summing over $x$ and $a$, and Eq. (A11) follows from defining the hidden variable $\tilde{\lambda} := (\omega, \lambda)$ governed by the normalized probability distribution $P_{\lambda} := P_{\Omega, \Lambda}$. Eq. (A11) manifestly shows that $\rho_{A | X_f} \in \text{LHS}$. \qed

\textbf{Appendix B: The relative entropy of steering}

In this appendix we prove Theorem 2. The proof strategy is similar to that of the proof that the relative entropy of entanglement for quantum states is a convex entanglement monotone [33]. It relies on two Lemmas, which we state next but whose proofs we leave for App. F.

\textbf{Lemma 1.} The assemblage relative entropy $S_{A}$, defined by Eq. (18), does not increase, on average, under deterministic SNIOs. That is, for any map $\mathcal{M}$ of the form given by Eqs. (9) and (10) but with $\sum_{\omega} K_{\omega} K_{\omega}^+ = 1$ and any two assemblages $\rho_{A | X}$ and $\rho'_{A | X}$, $S_{A}$ satisfies the inequality

$$\sum_{\omega} P_{\Omega}(\omega) S_{A} \left( \frac{\mathcal{M}_{\omega}(\rho_{A | X})}{\text{Tr} [\mathcal{M}_{\omega}(\rho_{A | X})]} \right) \leq S_{A} \left( \rho_{A | X} | \rho'_{A | X} \right), \tag{B1}$$

where $\mathcal{M}_{\omega}$ is the stochastic map defined in Eq. (11), $P_{\Omega} = \text{Tr} \left[ \mathcal{M}_{\omega}(\rho_{A | X}) \right]$, and $P'_{\Omega} = \text{Tr} \left[ \mathcal{M}_{\omega}(\rho'_{A | X}) \right]$, with $\sum_{\omega} P_{\omega} = 1 = \sum_{\omega} P'_{\omega}$.

\textbf{Lemma 2.} The assemblage relative entropy $S_{A}$, defined by Eq. (18), is jointly convex. That is, given two sets $\{ \rho^{(j)}_{A | X} \}_{j=1,\ldots,n}$ and $\{ \rho'^{(j)}_{A | X} \}_{j=1,\ldots,n}$ of $n$ arbitrary assemblages each and $n$ positive real numbers $\{ \mu^{(j)} \}_{j=1,\ldots,n}$ such that $\sum_{j} \mu^{(j)} = 1$, with $n \in \mathbb{N}$, $S_{A}$ satisfies the inequality

$$S_{A} \left( \sum_{j} \mu^{(j)} \rho^{(j)}_{A | X} \| \sum_{j} \mu^{(j)} \rho'^{(j)}_{A | X} \right) \leq \sum_{j} \mu^{(j)} S_{A} \left( \rho^{(j)}_{A | X} \| \rho'^{(j)}_{A | X} \right). \tag{B2}$$

We are now in a good position to prove the theorem.

\textbf{Proof of Theorem 2.} That the relative entropy of steering $\mathcal{S}_{R}$, defined in Eq. (20), satisfies condition \textit{i}) follows immediately from its definition and the positivity of the von-Neumann relative entropy for quantum states. Conditions \textit{ii)} and \textit{iii)}, SNIO monotonicity and convexity of $\mathcal{S}_{R}$, can be proven in analogous fashion to LOCC monotonicity and convexity of the relative entropy of entanglement, respectively. We include their proofs for completeness.

To prove condition \textit{ii)}, we denote by $\sigma^{*}$ an unsteerable assemblage for which the minimisation in Eq. (20) is attained, i.e., such that

$$S_{A}(\rho_{A | X} \| \sigma^{*}) := \mathcal{S}_{R}(\rho_{A | X}) \tag{B3}$$
and by $\sigma^*_{\mu}$ an unsteerable assemblage such that

$$
S_\Lambda \left( \frac{\mathcal{K}_\omega(\rho_{A|X})}{\operatorname{Tr}[\mathcal{K}_\omega(\rho_{A|X})]} \right) := \mathcal{K}_\Lambda \left( \frac{\mathcal{K}_\omega(\rho_{A|X})}{\operatorname{Tr}[\mathcal{K}_\omega(\rho_{A|X})]} \right).
$$

(\text{B4})

Then, we write

$$
\sum_\omega P_\omega(\omega) \mathcal{K}_\Lambda \left( \frac{\mathcal{K}_\omega(\rho_{A|X})}{\operatorname{Tr}[\mathcal{K}_\omega(\rho_{A|X})]} \right) = \sum_\omega P_\omega(\omega) S_\Lambda \left( \frac{\mathcal{K}_\omega(\rho_{A|X})}{\operatorname{Tr}[\mathcal{K}_\omega(\rho_{A|X})]} \right) \left( \frac{\mathcal{K}_\omega(\sigma^*)}{\operatorname{Tr}[\mathcal{K}_\omega(\sigma^*)]} \right)
$$

\leq \sum_\omega P_\omega(\omega) S_\Lambda \left( \frac{\mathcal{K}_\omega(\rho_{A|X})}{\operatorname{Tr}[\mathcal{K}_\omega(\rho_{A|X})]} \right) \left( \frac{\mathcal{K}_\omega(\sigma^*)}{\operatorname{Tr}[\mathcal{K}_\omega(\sigma^*)]} \right)
$$

\leq S_\Lambda (\rho_{A|X} \| \sigma^*)
$$

= \mathcal{K}_\Lambda (\rho_{A|X})
$$

(\text{B5})

where Eq. (\text{B5}) follows because $\sigma^*_{\mu}$ minimises the assemblage relative entropy in each $\omega$-th term in the sum and $\frac{\mathcal{K}_\omega(\sigma^*)}{\operatorname{Tr}[\mathcal{K}_\omega(\sigma^*)]} \in$ LHS, Eq. (\text{B6}) due to Lemma 1, and Eq. (\text{B7}) due to the definition of $\sigma^*$.

To prove condition condition $iii$, we further introduce unsteerable assemblages $\sigma^*$ and $\sigma^*_\text{mix}$ such that

$$
S_\Lambda (\rho'_{A|X} \| \sigma^*) = \mathcal{K}_\Lambda (\rho'_{A|X})
$$

(\text{B8})

and

$$
S_\Lambda (\mu \rho_{A|X} + (1 - \mu) \rho'_{A|X} \| \sigma^*_\text{mix}) = \mathcal{K}_\Lambda (\mu \rho_{A|X} + (1 - \mu) \rho'_{A|X})
$$

(\text{B9})

Then, we write

$$
\mu \mathcal{K}_\Lambda (\rho_{A|X}) + (1 - \mu) \mathcal{K}_\Lambda (\rho'_{A|X}) = \mu S_\Lambda (\rho_{A|X} \| \sigma^*) + (1 - \mu) S_\Lambda (\rho'_{A|X} \| \sigma^*)
$$

\geq S_\Lambda (\mu \rho_{A|X} + (1 - \mu) \rho'_{A|X} \| \sigma^* + (1 - \mu) \sigma^*)
$$

(\text{B10})

\geq S_\Lambda (\mu \rho_{A|X} + (1 - \mu) \rho'_{A|X} \| \sigma^*_\text{mix})
$$

(\text{B11})

\geq \mathcal{K}_\Lambda (\mu \rho_{A|X} + (1 - \mu) \rho'_{A|X})
$$

(\text{B12})

where Eq. (\text{B10}) holds due to Lemma 2, Eq. (\text{B11}) because $\sigma^*_\text{mix}$ minimizes the corresponding assemblage relative entropy and $\mu \sigma^* + (1 - \mu) \sigma^* \in$ LHS, and Eq. (\text{B12}) by the definition of $\sigma^*_\text{mix}$.

\hfill \Box

\section*{Appendix C: Other convex steering monotones}

In this appendix, we show that the recently introduced steering measures steerable weight [22] and robustness of steering [23] are also convex steering monotones.

\textbf{Definition 3 (Steerable weight [22])}. The steerable weight $\mathcal{K}_w(\rho_{A|X})$ of an assemblage $\rho_{A|X}$ is the minimum $\nu \in \mathbb{R}_{\geq 0}$ such that

$$
\rho_{A|X} = \nu \hat{\rho}_{A|X} + (1 - \nu) \sigma_{A|X},
$$

\text{(C1)}

with $\hat{\rho}_{A|X}$ an arbitrary assemblage and $\sigma_{A|X} \in$ LHS.

\textbf{Definition 4 (Robustness of steering [23])}. The robustness of steering $\mathcal{K}_\text{rob}(\rho_{A|X})$ of an assemblage $\rho_{A|X}$ is the minimum $\nu \in \mathbb{R}_{\geq 0}$ such that

$$
\sigma_{A|X} := \frac{1}{1 + \nu} \rho_{A|X} + \frac{\nu}{1 + \nu} \hat{\rho}_{A|X}
$$

\text{(C2)}

belongs to LHS, with $\hat{\rho}_{A|X}$ an arbitrary assemblage.
Theorem 5 (SNIO-monotonicity and convexity of $\mathcal{S}_W$ and $\mathcal{S}_R^{\text{Rob}}$). Both $\mathcal{S}_W$ and $\mathcal{S}_R^{\text{Rob}}$ are convex steering monotones.

Proof. Let us first prove the theorem’s statement concerning the steerable weight. That $\mathcal{S}_W$ satisfies condition $i)$ of Definition 1 follows immediately from its definition. To prove that it fulfills condition $ii)$, first, we apply the map $\mathcal{M}_\omega$ to both sides of Eq. (C1) and renormalize, which yields

$$\frac{\mathcal{M}_\omega (\rho_{A|X})}{\text{Tr} [\mathcal{M}_\omega (\rho_{A|X})]} = \nu \frac{\mathcal{M}_\omega (\tilde{\rho}_{A|X})}{\text{Tr} [\mathcal{M}_\omega (\rho_{A|X})]} + (1 - \nu) \frac{\mathcal{M}_\omega (\sigma _{A|X})}{\text{Tr} [\mathcal{M}_\omega (\rho_{A|X})]}, \quad (C3)$$

Denoting by $\nu^*_\omega$ the minimum $\nu \in \mathbb{R}_{\geq 0}$ such that a decomposition of the form of Eq. (C1) allows also for one as in Eq. (C3). Furthermore, taking into account that since $\sigma _{A|X} \in \text{LHS}$ it holds that $\frac{\mathcal{M}_\omega (\sigma _{A|X})}{\text{Tr} [\mathcal{M}_\omega (\rho_{A|X})]} \in \text{LHS}$, it is also clear that

$$\mathcal{S}_W \left( \frac{\mathcal{M}_\omega (\rho_{A|X})}{\text{Tr} [\mathcal{M}_\omega (\rho_{A|X})]} \right) \leq \nu^*_\omega. \quad (C5)$$

Hence, we obtain

$$\sum_\omega P_\omega (\omega) \mathcal{S}_W \left( \frac{\mathcal{M}_\omega (\rho_{A|X})}{\text{Tr} [\mathcal{M}_\omega (\rho_{A|X})]} \right) \leq \sum_\omega P_\omega (\omega) \nu^*_\omega \leq \sum_\omega P_\omega (\omega) \mathcal{S}_W (\rho_{A|X}) \leq \mathcal{S}_W (\rho_{A|X}), \quad (C6)$$

where the last inequality is due to the fact that $\sum_\omega P_\omega \leq 1$, with $P_\omega = \text{Tr} [\mathcal{M}_\omega (\rho_{A|X})]$.

To prove the validity of condition $iii)$ for $\mathcal{S}_W$, we first write

$$\mu \rho_{A|X} + (1 - \mu) \rho'_{A|X} = \mu \left[ \mathcal{S}_W (\rho_{A|X}) \tilde{\rho}_{A|X} + (1 - \mathcal{S}_W (\rho_{A|X})) \sigma _{A|X} \right] + (1 - \mu) \left[ \mathcal{S}_W (\rho'_{A|X}) \tilde{\rho}'_{A|X} + (1 - \mathcal{S}_W (\rho_{A|X})) \sigma '_{A|X} \right] \quad (C7)$$

$$= \nu^{(\mu)} \tilde{\rho}^{(\mu)}_{A|X} + (1 - \nu^{(\mu)}) \sigma ^{(\mu)}_{A|X}, \quad (C8)$$

where Eq. (C7) holds due to the definition of $\mathcal{S}_W$ and, in Eq. (C8), we have introduced the positive real

$$\nu^{(\mu)} := \mu \mathcal{S}_W (\rho_{A|X}) + (1 - \mu) \mathcal{S}_W (\rho'_{A|X}), \quad (C9)$$

the normalized assemblage

$$\tilde{\rho}^{(\mu)}_{A|X} := \frac{1}{\nu^{(\mu)}} \left[ \mu \mathcal{S}_W (\rho_{A|X}) \tilde{\rho}_{A|X} + (1 - \mu) \mathcal{S}_W (\rho'_{A|X}) \tilde{\rho}'_{A|X} \right] \quad (C10)$$

and the normalized unsteerable assemblage

$$\sigma ^{(\mu)}_{A|X} := \frac{1}{1 - \nu^{(\mu)}} \left[ \mu (1 - \mathcal{S}_W (\rho_{A|X})) \sigma _{A|X} + (1 - \mu) (1 - \mathcal{S}_W (\rho_{A|X})) \sigma '_{A|X} \right]. \quad (C11)$$

Thus, the expression (C8) gives a decomposition of the mixture $\mu \rho_{A|X} + (1 - \mu) \rho'_{A|X}$ of the form of Eq. (C1). However, it is not necessarily the optimal one. Hence, we get

$$\mathcal{S}_W (\mu \rho_{A|X} + (1 - \mu) \rho'_{A|X}) \leq \nu^{(\mu)}, \quad (C12)$$

which, together with Eq. (C9), finishes the proof of convexity of $\mathcal{S}_W$.

Similar arguments can be employed to prove the theorem’s statement concerning the robustness of steering. That $\mathcal{S}_R^{\text{Rob}}$ satisfies condition $i)$ of Definition 1 also follows immediately by definition. Condition $ii)$ can be proven with a similar strategy to that for $\mathcal{S}_W$ and using that

$$(1 - \mu) \rho_{A|X} + \mu \rho'_{A|X} \in \text{LHS} \Rightarrow \frac{\mathcal{M}_\omega (1 - \mu) \rho_{A|X} + \mu \rho'_{A|X})}{\text{Tr} [\mathcal{M}_\omega (\rho_{A|X})]} \in \text{LHS}. \quad (C13)$$
Condition \(iii)\) can be proven by noting that Definition 4 implies that

\[
\begin{align*}
\rho_{A|X} &= [1 + \mathcal{R}_{\text{Rob}}(\rho_{A|X})]\sigma_{A|X} - \mathcal{R}_{\text{Rob}}(\rho_{A|X})\tilde{\rho}_{A|X} \\
\rho'_{A|X} &= [1 + \mathcal{R}_{\text{Rob}}(\rho'_{A|X})]\sigma'_{A|X} - \mathcal{R}_{\text{Rob}}(\rho'_{A|X})\tilde{\rho}_{A|X},
\end{align*}
\]

where the unsteerable assemblage \(\sigma'_{A|X}\) and the arbitrary assemblage \(\tilde{\rho}_{A|X}\) play respectively the same roles for \(\rho'_{A|X}\) to the ones played by \(\sigma_{A|X}\) and \(\tilde{\rho}_{A|X}\) for \(\rho_{A|X}\) in Definition 4. Then, one can introduce the positive real

\[
\nu^{(\mu)} = \mu \mathcal{R}_{\text{Rob}}(\rho_{A|X}) + (1 - \mu)\mathcal{R}_{\text{Rob}}(\rho'_{A|X}),
\]

the normalized assemblage

\[
\tilde{\rho}^{(\mu)}_{A|X} = \frac{\mu \mathcal{R}_{\text{Rob}}(\rho_{A|X})}{\nu^{(\mu)}} \tilde{\rho}_{A|X} + (1 - \mu) \mathcal{R}_{\text{Rob}}(\rho'_{A|X})\tilde{\rho}_{A|X},
\]

and the normalized unsteerable assemblage

\[
\sigma^{(\mu)}_{A|X} = \frac{(1 + \mathcal{R}_{\text{Rob}}(\rho_{A|X}))}{1 + \nu^{(\mu)}} \sigma_{A|X} + (1 - \mu) \left(1 + \mathcal{R}_{\text{Rob}}(\rho'_{A|X})\right)\sigma'_{A|X},
\]

such that

\[
\mu \rho_{A|X} + (1 - \mu)\rho'_{A|X} = \left(1 + \nu^{(\mu)}\right) \sigma^{(\mu)}_{A|X} - \nu^{(\mu)}\tilde{\rho}^{(\mu)}_{A|X},
\]

and proceed with \(\mathcal{R}_{\text{Rob}}\) analogously as with \(\mathcal{R}_W\) in Eq. (C12) above.

To end up with, it is important to mention that both \(\mathcal{R}_W\) and \(\mathcal{R}_{\text{Rob}}\) satisfy a stronger form of monotonicity than condition \(ii)\) of Definition 1. It is clear from the proof of Theorem 5 that the two measures do not increase on average not only under deterministic SNIOs, as in Eq. (13), but also under stochastic ones. Indeed, for the steerable weight, Eqs. (C4) and (C5) make it explicitly evident that the post-selected assemblage \(\frac{\mathcal{M}_{\omega}(\rho_{A|X})}{\mathcal{V}[\mathcal{M}_{\omega}(\rho_{A|X})]}\) resulting from the \(\omega\)-th measurement outcome has itself (even before averaging over \(\omega\)) smaller or equal steering than \(\rho_{A|X}\). The same fact holds for the robustness of steering, as the reader may straightforwardly verify. This property is inherited from the quantum counterparts for entanglement of these two steering measures, the best-separable approximation entanglement measure [34, 35] and the robustness of entanglement [36], both of which also satisfy this highly restrictive form of monotonicity. The relative entropy of steering introduced here is not subject to this restriction.

Appendix D: Pure-orthogonal-assemblage SNIO conversion rules

In this appendix we prove Theorem 3.

**Proof of Theorem 3.** One of the implications is trivial to prove. If \(P'_{A|X} = P_{A|X}\) and Eq. (22) holds, there exists a unitary operator \(U\) such that \(\ket{\psi'(a, x)} = U\ket{\psi(a, x)}\) for all \(a\) and \(x\). Then, \(\Psi_{A|X}\) can be transformed into \(\Psi'_{A|X}\) by means of a deterministic SNIO: namely, the one consisting of Bob applying \(U\) to his subsystem and Alice doing nothing. Likewise, if \(\Psi'_{A|X} \in \text{LHS}\), then \(\Psi_{A|X}\) can trivially be transformed into \(\Psi'_{A|X}\) by SNIOs, as any unsteerable assemblage can be created by stochastic SNIOs by definition (see discussion after Eq. (6)).

Let us then prove the converse implication. That is, assuming that \(\Psi_{A|X}\) and \(\Psi'_{A|X}\) are pure orthogonal assemblages and that the latter can be obtained from the former by a stochastic SNIO, we prove that either \(\Psi'_{A|X} \in \text{LHS}\) or \(P'_{A|X} = P_{A|X}\) and Eq. (22) is true. To this end, we first note that the no-signaling condition (4) restricts minimal-dimension pure orthogonal assemblages to a rather specific form. Namely, the fact that \(\Psi_{A|X}\) is no-signaling implies that

\begin{itemize}
  \item[i)] either \(P_{A|X}(\cdot, x)\) is a deterministic distribution for all \(x\),
  \item[ii)] or \(P_{A|X}(\cdot, x)\) is the uniform distribution for all \(x\).
\end{itemize}

If case i) holds, \(\Psi_{A|X} \in \text{LHS}\). Then, since, by assumption, \(\Psi'_{A|X}\) can be obtained via a stochastic SNIO from \(\Psi_{A|X}\), one automatically obtains that \(\Psi'_{A|X} \in \text{LHS}\).

To analyze case ii), we use that \(\Psi'_{A|X}\) is also subject to the no-signaling condition (4):

\begin{itemize}
  \item[i')] either \(P'_{A|X}(\cdot, x)\) is a deterministic distribution for all \(x\),
  \item[ii')] or \(P'_{A|X}(\cdot, x)\) is the uniform distribution for all \(x\).
\end{itemize}


That case \(i'\) is possible if case \(ii\) holds is clear, as \(i'\) corresponds to \(\Psi'_{A|X} \in \text{LHS}\). So, it only remains to show that if cases \(ii\) and \(ii'\) hold, then either \(\Psi'_{A|X} \in \text{LHS}\) or Eq. (22) holds. We show it in what follows.

Assuming that \(ii\) and \(ii'\) hold and that there is a stochastic SNIO \(M\) that maps \(\Psi_{A|X}\) into \(\Psi'_{A|X}\), i.e., such that \(M(\Psi_{A|X}) \propto \Psi'_{A|X}\), where “\(\propto\)” stands for “is proportional to”, we use Eq. (A6) to obtain

\[
\sum_{a,x,\omega} P_{X|X_f,\omega}(x,x_f,\omega) P_{A_f|A,X,\Omega,X_f}(a_f,a,x,\omega,x_f) K_\omega |\psi(a,x)\rangle \langle \psi(a,x)| K^\dagger_\omega \propto |\psi'(a_f,x_f)\rangle \langle \psi'(a_f,x_f)| \forall (a_f,x_f). \tag{D1}
\]

Since the right-hand side of Eq. (D1) is composed of a rank-one projector onto a pure state, each term of the sum in the left-hand side must be either zero or proportional to \(|\psi'(a_f,x_f)\rangle \langle \psi'(a_f,x_f)|\). In particular, this must also hold for each \(\omega\)-th term. That is, for all \(\omega\), it must hold that

\[
\sum_{a,x} P_{X|X_f,\omega}(x,x_f,\omega) P_{A_f|A,X,\Omega,X_f}(a_f,a,x,\omega,x_f) K_\omega |\psi(a,x)\rangle \langle \psi(a,x)| K^\dagger_\omega \sim |\psi'(a_f,x_f)\rangle \langle \psi'(a_f,x_f)| \forall (a_f,x_f), \tag{D2}
\]

where the symbol “\(\sim\)” is used to signify “is either equal to zero or proportional to”. Indeed, using that \(K_\omega \neq 0\) and that \(P_{X|X_f,\omega}\) and \(P_{A_f|A,X,\Omega,X_f}\) are normalised distributions, one can see by case analysis that there are always at least two different pairs \((a_f,x_f)\) for which the left-hand side of Eq. (D2) is not zero and, therefore, proportional to \(|\psi'(a_f,x_f)\rangle \langle \psi'(a_f,x_f)|\).

Let us then consider first the case

\[
K_\omega |\psi(a,x)\rangle \neq 0 \forall (a,x). \tag{D3}
\]

The other case will be considered at the end. The first step is to note that Eqs. (D2) and (D3) imply that, unless \(\Psi'_{A|X} \in \text{LHS}\),

\[
P_{X|X_f,\omega}(x|x_f,\omega) = \delta_{x_f,x \oplus f(\omega)}, \tag{D4a}
\]

\[
P_{A_f|A,X,\Omega,X_f}(a_f,a,x,\omega,x_f) \in \{0,1\} \forall (a_f,a,x), \tag{D4b}
\]

where \(f(\omega) \in \{0,1\}\). That is, for any \(\omega\) for which Eq. (D3) holds, unless \(\Psi'_{A|X} \in \text{LHS}\), the variables \(X\) and \(X_f\) must be either fully correlated or fully anticorrelated and \(P_{A_f|A,X,\Omega,X_f}(\cdot,a_f,x,\omega,x \oplus f(\omega))\) must be a deterministic distribution for all \((a_f,x)\).

To prove Eq. (D4a), suppose that it does not hold. Then there must exist \(\tilde{x}\) such that \(P_{X|X_f,\omega}(\tilde{x}|x_f,\omega) \neq 0\) for all \(x_f\). This, due to Eq. (D2), implies that

\[
\sum_a P_{A_f|A,X,\Omega,X_f}(a_f,a,\tilde{x},\omega,0) K_\omega |\psi(a,\tilde{x})\rangle \langle \psi(a,\tilde{x})| K^\dagger_\omega \sim |\psi'(a_f,0)\rangle \langle \psi'(a_f,0)|, \tag{D5a}
\]

\[
\sum_a P_{A_f|A,X,\Omega,X_f}(a_f,a,\tilde{x},\omega,1) K_\omega |\psi(a,\tilde{x})\rangle \langle \psi(a,\tilde{x})| K^\dagger_\omega \sim |\psi'(a_f,1)\rangle \langle \psi'(a_f,1)|. \tag{D5b}
\]

In turn, choosing \(\tilde{a}_f\) and \(\tilde{\pi}_f\) such that \(P_{A_f|A,X,\Omega,X_f}(\tilde{a}_f,a,\tilde{x},\omega,0) > 0\) and \(P_{A_f|A,X,\Omega,X_f}(\tilde{a}_f,a,\tilde{x},\omega,1) > 0\), which is always possible due to \(P_{A_f|A,X,\Omega,X_f}\) being a normalised distribution and does not require any extra assumption, Eqs. (D3) and (D5) imply that

\[
K_\omega |\psi(a,\tilde{x})\rangle \propto |\psi'(\tilde{a}_f,0)\rangle, \tag{D6a}
\]

\[
K_\omega |\psi(a,\tilde{x})\rangle \propto |\psi'(\tilde{a}_f,1)\rangle. \tag{D6b}
\]

This finally leads to \(|\psi'(\tilde{a}_f,0)\rangle = |\psi'(\tilde{a}_f,1)\rangle\), which is true only if \(\Psi'_{A|X} \in \text{LHS}\).

To prove Eq. (D4b) we use a similar argument. If one assumes that Eq. (D4b) is false, then there must exist a pair \((\tilde{a},\tilde{x})\) such that \(P_{A_f|A,X,\Omega,X_f}(\tilde{a}_f,\tilde{a},\tilde{x},\omega,\tilde{x} \oplus f(\omega)) > 0\) for all \(a_f\). Using this and Eqs. (D2), (D3), and (D4a), one arrives at

\[
K_\omega |\psi(\tilde{a},\tilde{x})\rangle \propto |\psi'(0,\tilde{x} \oplus f(\omega))\rangle, \tag{D7a}
\]

\[
K_\omega |\psi(\tilde{a},\tilde{x})\rangle \propto |\psi'(1,\tilde{x} \oplus f(\omega))\rangle, \tag{D7b}
\]

which, since \(|\psi'(0,\tilde{x} \oplus f(\omega))\rangle\) and \(|\psi'(1,\tilde{x} \oplus f(\omega))\rangle\) are orthogonal, yields a contradiction.

The second step is to note that Eqs. (D3) and (D4) impose restrictions on which \(a\)'s and \(x\)'s can contribute to each \(a_f\) and \(x_f\) in Eq. (D2). More precisely, one can see by case analysis that, up to relabelings of \(a_f\) and \(x_f\), only three different types of assignments are possible:
\begin{align}
K_{\omega}|\psi(0,0)\rangle &\propto |\psi'(0,0)\rangle & K_{\omega}|\psi(0,0)\rangle &\propto |\psi'(0,0)\rangle & K_{\omega}|\psi(0,0)\rangle &\propto |\psi'(0,0)\rangle \\
K_{\omega}|\psi(1,0)\rangle &\propto |\psi'(1,0)\rangle & K_{\omega}|\psi(1,0)\rangle &\propto |\psi'(1,0)\rangle & K_{\omega}|\psi(1,0)\rangle &\propto |\psi'(0,0)\rangle \\
K_{\omega}|\psi(0,1)\rangle &\propto |\psi'(0,1)\rangle & K_{\omega}|\psi(0,1)\rangle &\propto |\psi'(0,1)\rangle & K_{\omega}|\psi(0,1)\rangle &\propto |\psi'(0,1)\rangle \\
K_{\omega}|\psi(1,1)\rangle &\propto |\psi'(1,1)\rangle & K_{\omega}|\psi(1,1)\rangle &\propto |\psi'(0,1)\rangle & K_{\omega}|\psi(1,1)\rangle &\propto |\psi'(0,1)\rangle \\
\end{align}

The third step is to show that all three cases a-c are possible only if either $\Psi'_{A|X} \in \text{LHS}$ or Eq. (22) holds. Note that it is enough to show this for the case where all the eight vectors $\{|\psi(a, x)\rangle, |\psi'(a_f, x_f)\rangle\}_{a,x,a_f,x_f}$ lie on a same plane of the Bloch sphere. This is due to the fact that, since $\Psi_{A|X}$ and $\Psi'_{A|X}$ are both pure no-signaling assemblages of minimal dimension, $\{|\psi(a, x)\rangle\}_{a_f,x_f}$ and $\{|\psi'(a_f, x_f)\rangle\}_{a_f,x_f}$ are each one already contained in two planes of the Bloch sphere, as one can straightforwardly see using Eq. (4). These two planes can always be rotated so as to coincide by a unitary operation, which can in turn be absorbed in the definition of the Kraus operator $K_{\omega}$. Hence, without loss of generality, we take

$$
|\psi(0,0)\rangle = |0\rangle, \ |\psi(1,0)\rangle = |1\rangle, \ (D8a)
$$

$$
|\psi(0,1)\rangle = \cos(\varphi)|0\rangle + \sin(\varphi)|1\rangle, \ |\psi(1,1)\rangle = -\sin(\varphi)|0\rangle + \cos(\varphi)|1\rangle, \ (D8b)
$$

$$
|\psi'(0,0)\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle, \ |\psi'(1,0)\rangle = -\sin(\theta)|0\rangle + \cos(\theta)|1\rangle, \ (D8c)
$$

$$
|\psi'(0,1)\rangle = \cos(\phi)|0\rangle + \sin(\phi)|1\rangle, \ |\psi'(1,1)\rangle = -\sin(\phi)|0\rangle + \cos(\phi)|1\rangle, \ (D8d)
$$

for arbitrary $\varphi$, $\theta$, and $\phi \in [0, \pi/2]$, and where $|0\rangle$ and $|1\rangle$ represent the computational-basis states. We analyse first the case a). Dividing both vector components (in the computational basis) of the first equation of this case, one obtains that

$$
\frac{[K_{\omega}]_{00}}{[K_{\omega}]_{10}} = \frac{\cos(\theta)}{\sin(\theta)}, \text{where } [K_{\omega}]_{ij} := \langle i | K_{\omega} | j \rangle. \text{ Analogously, dividing both vector components of the second equation yields}
$$

$$
\frac{[K_{\omega}]_{01}}{[K_{\omega}]_{11}} = -\frac{\sin(\theta)}{\cos(\theta)}. \text{ Hence, introducing proportionality constants $\kappa_1 > 0$ and $\kappa_2 > 0$, the Kraus operator can be matrix-represented in the computational basis as}
$$

$$
K_{\omega} = \begin{pmatrix}
\kappa_1 \cos(\theta) & -\kappa_2 \sin(\theta) \\
\kappa_1 \sin(\theta) & \kappa_2 \cos(\theta)
\end{pmatrix}.
$$

Using Eq. (D9), the third equation of case a) implies that

$$
\kappa_1 \cos(\varphi) \sin(\theta - \phi) = \kappa_2 \sin(\varphi) \cos(\theta - \phi).
$$

Finally, the fourth equation leads to

$$
\kappa_2 \cos(\varphi) \sin(\theta - \phi) = \kappa_1 \sin(\varphi) \cos(\theta - \phi).
$$

Eqs. (D10) and (D11) can be simultaneously satisfied only if $\theta - \phi = \varphi$ or $(\theta - \phi) \times \varphi = 0$. The former option yields Eq. (22). The latter one implies that $\Psi'_{A|X} \in \text{LHS}$. In a similar fashion, for case b), the first three equations lead to Eq. (D10) and the fourth one to

$$
-\kappa_1 \sin(\varphi) \sin(\theta - \phi) = \kappa_2 \cos(\varphi) \cos(\theta - \phi).
$$

This cannot be satisfied unless $(\theta - \phi) \times \varphi = 0$, which means that $\Psi'_{A|X} \in \text{LHS}$. With a similar argument the reader can straightforwardly verify that the same thing happens for case c). This finishes the proof of the theorem for the $\omega$’s for which Eq. (D3) holds.

As the fourth and final step, it remains to treat the case where, for a certain $\omega$, there exists a pair $(\tilde{a}, \tilde{x})$ for which $K_{\omega}|\psi(\tilde{a}, \tilde{x})\rangle = 0$. Since $K_{\omega} \neq 0$, the latter is true only if the support of $K_{\omega}$ is given by the span of $|\psi(\tilde{a} \oplus 1, \tilde{x})\rangle$. Using this and the fact that there are always at least two different pairs $(a_f, x_f)$ for which the left-hand side of Eq. (D2) is not zero, one obtains that $K_{\omega} \propto |\psi'(a_f, x_f)\rangle \langle \psi'(\tilde{a} \oplus 1, \tilde{x})|$ for two different pairs $(a_f, x_f)$, which, unless $\Psi'_{A|X} \in \text{LHS}$, is a contradiction.

\section*{Appendix E: Non-existence of minimal-dimension steering bits}

In this appendix we prove Theorem 4. This section bears many similarities with App. D.
Proof of Theorem 4. We proceed by reductio ad absurdum. That is, we show that if one supposes that there exists a pure normalised assemblage $\Psi_{A|X} := \{P_{A|X}(a,x), \vert \psi(a,x) \rangle \}_{a,x}$, with $d = s = r = 2$, from which all assemblages can be obtained via stochastic SNIOs, one obtains a contradiction.

Without loss of generality, we can choose the computational basis $\{ \vert 0 \rangle, \vert 1 \rangle \}$ so that its first element coincides with $\vert \psi(0,0) \rangle$ and the element $\vert \psi(1,0) \rangle$ is in the plane that contains the vectors $\vert 0 \rangle$ and $\frac{1}{\sqrt{2}}(\vert 0 \rangle + \vert 1 \rangle)$. What is more, clearly, $\Psi_{A|X}$ cannot have a LHS model, otherwise $\Psi_{A|X}$ could not be mapped into all assemblages by stochastic SNIOs. Thus, we can safely assume that

$$\Psi_{A|X} \notin \text{LHS}. \quad (E1)$$

Hence, we take

$$\vert \psi(a,x) \rangle = \begin{cases} \vert 0 \rangle & \text{if } (a,x) = (0,0), \\ \cos(\varphi_{ax})\vert 0 \rangle + \sin(\varphi_{ax})\vert 1 \rangle & \text{if } (a,x) = (1,0), \\ \cos(\varphi_{ax})\vert 0 \rangle + e^{i\alpha_{ax}} \sin(\varphi_{ax})\vert 1 \rangle & \text{if } (a,x) \notin \{(0,0),(1,0)\} \end{cases} \quad (E2)$$

with

$$\varphi_{10} \in [0,\pi[$$
$$\alpha_{ax} \in [0,2\pi], \forall (a,x) \notin \{(0,0),(1,0)\} \quad (E3a)$$

and

$$\varphi_{a1}, \alpha_{a1} \neq \varphi_{a'1}, \alpha_{a'1} \forall a \neq a' \quad (E4)$$

Equations (E3) and (E4) hold due to the fact that $\Psi_{A|X} \notin \text{LHS}$ and the no-signalling condition (4). More precisely, if $\varphi_{10} = \{0,\pi\}$, $\vert \psi(1,0) \rangle = 0$, which implies that Bob’s reduced state is $\varrho_B = \vert 0 \rangle \langle 0 \vert$. Then, the no-signalling condition (4) implies that $\vert \psi(0,1) \rangle = \vert 0 \rangle = \vert \psi(1,1) \rangle$. Such assemblage clearly has a LHS model, which contradicts the assumption (E1). The same argument applies to (E4). Furthermore, $\Psi_{A|X} \notin \text{LHS}$ and the no-signalling principle imply also that $P_{A|X}(a,x) \neq 0$ for all $(a,x)$. To see the latter, suppose that there is a pair $(a,x)$ for which $P_{A|X}(a,x) = 0$. Then, clearly, $P_{A|X}(a \oplus 1,x) = 1$. This, together with Eq. (4), implies that there is an $\tilde{a}$ for which $P_{A|X}(\tilde{a}, x \oplus 1) = 1$, which in turn leads to $\Psi_{A|X} \notin \text{LHS}$.

Let us now consider pure orthogonal assemblages $\{\Psi_{A|X}^{\theta} \}_{\theta}$ with $d = s = r = 2$ of the form $\Psi_{A|X}^{\theta} := \{\frac{1}{2}, \vert \psi^{\theta}(a,x) \rangle \}_{a,x}$, where

$$\vert \psi^{\theta}(0,0) \rangle = \vert 0 \rangle, \quad \vert \psi^{\theta}(1,0) \rangle = \vert 1 \rangle,$$

$$\vert \psi^{\theta}(0,1) \rangle = \cos(\theta)\vert 0 \rangle + \sin(\theta)\vert 1 \rangle, \quad \vert \psi^{\theta}(1,1) \rangle = -\sin(\theta)\vert 0 \rangle + \cos(\theta)\vert 1 \rangle \quad (E5a)$$

We restrict to $0 < \theta < \pi/2$ to ensure that $\Psi_{A|X}^{\theta} \notin \text{LHS}$. If all assemblages can be obtained via stochastic SNIOs from $\Psi_{A|X}$, there must be a stochastic SNIO $M^\theta$ such $M^\theta(\Psi_{A|X}) \propto \Psi_{A|X}^{\theta}$, where “$\propto$” stands for “is proportional to”. Then, as in App. D, Eq. (A6) implies that, for all $\omega$, it must hold that

$$\sum_{a,x} P_{X|f,\Omega}(x,f,x,\omega)P_{A|X,A,X,\Omega,X_f}(a_f,a,x,\omega,x)P_{A|X}(a,x)K_{\omega}^{\theta} \langle \psi(a,x) \vert K_{\omega}^{\theta} \sim \langle \psi^{\theta}(a_f, x_f) \vert \psi^{\theta}(a_f, x_f) \rangle \quad \forall (a_f,x_f), \quad (E6)$$

where the symbol “$\sim$” is used to signify “is either equal to zero or proportional to”. However, we note again that, since $K_{\omega} \neq 0$ and $P_{X|f,\Omega}$ and $P_{A|X,A,X,\Omega,X_f}$ are normalised distributions, there are always at least two different pairs $(a_f,x_f)$ for which the left-hand side of Eq. (E6) is not zero and, therefore, proportional to $\langle \psi^{\theta}(a_f, x_f) \vert \psi^{\theta}(a_f, x_f) \rangle$, as can be seen by direct case analysis.

Let us then consider the case

$$K_{\omega}^{\theta} \langle \psi(a,x) \rangle \neq 0 \forall (a,x), \quad (E7)$$

The other case will be considered later. The first step is to note that Eqs. (E6) and (E7), together with the fact that $\Psi_{A|X} \notin \text{LHS}$, imply that

$$P_{X|f,\Omega}(x,f,x,\omega) = \delta_{x_f, x \oplus f^{\theta}(\omega)},$$

$$P_{A|X,A,X,\Omega,X_f}(a_f,a,x,\omega,x \oplus f^{\theta}(\omega)) \in \{0,1\} \forall (a_f,a,x), \quad (E8a)$$

$$P_{A|X,A,X,\Omega,X_f}(a_f,a,x,\omega,x \oplus f^{\theta}(\omega)) \in \{0,1\} \forall (a_f,a,x), \quad (E8b)$$
where \( f^\theta(\omega) \in \{0, 1\} \). That is, for any \( \omega \) for which Eq. (E7) holds, \( X \) and \( X_f \) must be either fully correlated or fully anticorrelated and \( P^\theta_{A,X|a,f,x,\omega} \) must be a deterministic distribution for all \((a, f, x)\). The proofs of Eqs. (E8) are almost identical to the proofs of Eqs. (D4) in App. D, with the only difference that, here, \( \Psi_{A|X} \notin \text{LHS} \) and \( \Psi^\theta_{A|X} \notin \text{LHS} \) are true by assumption. We therefore do not repeat the argument.

The second step is to note that Eqs. (E7) and (E8) impose restrictions on which \( \alpha \)'s and \( x \)’s can contribute to each \( a_f \) and \( x_f \) in Eq. (E6). More precisely, one can see by case analyses that, up to relabelings of \( a_f \) or \( x_f \), only one type of assignment is possible:

\[
\begin{align*}
K^\theta_{a_f} |\psi(0, 0)\rangle \propto |\psi^\theta(0, 0)\rangle, \\
K^\theta_{a_f} |\psi(1, 0)\rangle \propto |\psi^\theta(1, 0)\rangle, \\
K^\theta_{a_f} |\psi(0, 1)\rangle \propto |\psi^\theta(0, 1)\rangle, \\
K^\theta_{a_f} |\psi(1, 1)\rangle \propto |\psi^\theta(1, 1)\rangle.
\end{align*}
\]

(E9a)

(E9b)

(E9c)

(E9d)

The third step is to show that Eqs. (E9) lead to a contradiction. To this end, together with Eqs. (E2) and (E5a), Eqs. (E9a) and (E9b) respectively imply that \( [K^\theta_\omega]_{10} = 0 \) and \( [K^\theta_\omega]_{01} = -\tan(\varphi_{10}) \), where \( [K^\theta_\omega]_{ij} := \langle i | K^\theta_\omega | j \rangle \). In turn, dividing both vector components in each one of Eqs. (E9c) and (E9d), one obtains, using Eqs. (E2) and (E5b), that

\[
\begin{align*}
\frac{[K^\theta_\omega]_{11}}{[K^\theta_\omega]_{01}} &= \tan(\theta) \left( -\frac{\tan(\varphi_{10})}{\tan(\varphi_{01})e^{i\alpha_{01}}} + 1 \right), \\
\frac{[K^\theta_\omega]_{11}}{[K^\theta_\omega]_{01}} &= -1 \left( -\frac{\tan(\varphi_{10})}{\tan(\varphi_{11})e^{i\alpha_{11}}} + 1 \right).
\end{align*}
\]

(E10a)

(E10b)

Equating the right-hand sides of Eqs. (E10a) and (E10b) gives, after straightforward algebraic manipulation,

\[
\frac{1}{\tan(\varphi_{10})} = \frac{\sin^2(\theta)}{\tan(\varphi_{11})e^{i\alpha_{11}}} + \frac{\cos^2 \theta}{\tan(\varphi_{01})e^{i\alpha_{01}}}. \tag{E11}
\]

Since the last condition is independent of \( K^\theta_\omega \) and \( \Psi_{A|X} \) should be transformed by stochastic SNIOs into any member of the family \( \{ \Psi^\theta_{A|X} \}_0 \), the same condition should be fulfilled for any \( 0 < \theta < \pi/2 \). It actually suffices to choose just two assemblages \( \Psi^\theta_{A|X} \) and \( \Psi^\theta_{A|X} \), for any \( 0 < \theta_1, \theta_2 < \pi/2 \) with \( \theta_1 \neq \theta_2 \), to arrive at a contradiction. Indeed, since the angles \( \varphi_{10}, \varphi_{01}, \varphi_{11}, \alpha_{01} \) and \( \alpha_{11} \) are fixed, the only way to satisfy Eq. (E11) for both \( \theta_1 \) and \( \theta_2 \) is that

\[
\tan(\varphi_{10}) = \tan(\varphi_{01})e^{i\alpha_{01}} = \tan(\varphi_{11})e^{i\alpha_{11}}. \tag{E12}
\]

This, in turn, can happen only if \( \alpha_{01} = 0 = \alpha_{11} \) and \( \varphi_{10} = \varphi_{01} = \varphi_{11} \), which is clearly incompatible with (E4).

It remains to treat the case where, for a certain \( \omega \), there exists a pair \((\tilde{a}, \tilde{x})\) for which Eq. (E7) does not hold. By relabeling \( a \) or \( x \), we can always choose \((\tilde{a}, \tilde{x}) = (0, 0)\). Hence, we consider

\[
K^\theta_\omega |\psi(0, 0)\rangle = K^\theta_\omega |0\rangle = 0. \tag{E13}
\]

Since \( K^\theta_\omega \neq 0 \), the latter is true only if the support of \( K^\theta_\omega \) is given by the span of \( |1\rangle \). Using this and the fact that there are always at least two different pairs \((a_f, x_f)\) for which the left-hand side of Eq. (E6) is not zero, one arrives at a contradiction of the type \( K^\theta_\omega \propto |\psi^\theta(a_f, x_f)\rangle |1\rangle \) for two different pairs \((a_f, x_f)\). This finishes the proof for pure assemblages.

We finish the appendix with a remark on a difficulty to generalise Theorem 4 to the case of mixed-state assemblages, i.e., to rule out the existence of steering bits also among mixed-state assemblages. Since any mixed-state assemblage can be decomposed as a convex combination of pure assemblages and \( \mathcal{M} \) is a linear transformation, one would be tempted to trivially extend the proof above to mixed-state assemblages by using similar reasonings to those presented just above with each pure assemblage in the convex combination. However, such straightforward extension unfortunately fails. The reason for this is that each pure assemblage in the pure-assemblage decomposition of a mixed-state assemblage is, as far as we can see, not necessarily no-signalling. We emphasise that all our formalism deals only with no-signalling objects. Hence, while we strongly believe that minimal-dimension steering bits do not exist in general, i.e., even among the mixed-state assemblages, we leave the proof of this statement as an open question.
Appendix F: Proofs of Lemmas 1 and 2

Before proceeding with the proofs of the lemmas we recall some known mathematical facts necessary for the proofs. The von-Neumman relative entropy $S_Q$, defined by Eq. (15), fulfils the following properties [37]:

- Given two sets $\{q_j\}_{j=1,...,n}$ and $\{q'_j\}_{j=1,...,n}$ of $n$ arbitrary positive-semidefinite (not necessarily normalized) operators each and $n$ positive real numbers $\{\mu_j\}_{j=1,...,n}$ such that $\sum_j \mu_j = 1$, with $n \in \mathbb{N}$, $S_Q$ satisfies the joint convexity property
  \[
  S_Q \left( \sum_j \mu_j q_j \| \sum_j \mu_j q'_j \right) \leq \sum_j \mu_j S_Q \left( q_j \| q'_j \right),
  \]
  \[(F1)\]

- Given any completely-positive trace-preserving (CPTP) map $E$ and any two density operators $\varrho$ and $\varrho'$, $S_Q$ satisfies the CPTP-map contraction property
  \[
  S_Q \left( E(\varrho) \| E(\varrho') \right) \leq S_Q(\varrho \| \varrho').
  \]
  \[(F2)\]

In turn, the Kullback-Leibler divergence $S_C$ defined in Eq. (16) fulfils the following property.

- Given any two joint probability distributions $P_{X,Y}$ and $P'_{X,Y}$ over classical bits $x$ and $y$, $S_C$ satisfies the inequality
  \[
  \sum_x P_X(x) S_C \left( P_{Y|X}(\cdot,x) \| P'_{Y|X}(\cdot,x) \right) = \sum_x P_X(x) \left( \frac{\log P_{Y|X}(x)}{\log P'_{Y|X}(x)} - \frac{\log P_X(x)}{\log P'_X(x)} \right) 
  = \sum_{x,y} P_{X,Y}(x,y) \left( \frac{\log P_{X,Y}(x,y)}{\log P'_{X,Y}(x,y)} - \frac{\log P_X(x)}{\log P'_X(x)} \right) 
  = S_C \left( P_{X,Y} \| P'_{X,Y} \right) - S_C \left( P_X \| P'_X \right) 
  \leq S_C \left( P_{X,Y} \| P'_{X,Y} \right).
  \]
  \[(F3)\]

We are now in a good position to prove the lemmas.

1. Proof of Lemma 1

We begin by Lemma 1.

**Proof of Lemma 1.** First, using the definition of $\mathcal{K}$ in Eq. (18), we write the left-hand side of Eq. (B1) explicitly as

\[
\sum_{\omega} P_{\Omega}(\omega) S_A \left( \mathcal{M}_\omega \left( \hat{\rho}_{A|X} \right) \| \mathcal{M}_\omega \left( \hat{\rho}'_{A|X} \right) \right) = \sum_{\omega} P_{\Omega}(\omega) \max_{F_{X'|Y}} \{ E_\gamma \} \left[ S_C \left( P_{Y|\Omega}(\cdot,\cdot) \| P'_{Y|\Omega}(\cdot,\cdot) \right) + \sum_{\gamma,x} P_{X'|Y}(x|\gamma) P_{Y|\Omega}(\gamma,\omega) S_Q \left( \mathcal{M}_\omega \left( \hat{\rho}_{A|X} \right) \| \mathcal{M}_\omega \left( \hat{\rho}'_{A|X} \right) \right) \right],
\]

\[(F4)\]

where we have used that $P_{\Omega}(\omega) = \text{Tr}[\mathcal{M}_\omega \left( \hat{\rho}_{A|X} \right)]$ and $P'_{\Omega}(\omega) = \text{Tr}[\mathcal{M}_\omega \left( \hat{\rho}'_{A|X} \right)]$, and that $P_{Y|\Omega}(\gamma,\omega) = P_{Y|\Omega}(\gamma,\omega) P_{\Omega}(\omega)$ and $P'_{Y|\Omega}(\gamma,\omega) = P'_{Y|\Omega}(\gamma,\omega) P'_{\Omega}(\omega)$, with

\[
P_{Y|\Omega}(\gamma,\omega) := \text{Tr} \left[ \mathcal{M}_\omega \left( \hat{\rho}_{A|X} \right) \| \mathcal{M}_\omega \left( \hat{\rho}'_{A|X} \right) \right],
\]

\[(F5a)\]

and

\[
P'_{Y|\Omega}(\gamma,\omega) := \text{Tr} \left[ \mathcal{M}_\omega \left( \hat{\rho}'_{A|X} \right) \| \mathcal{M}_\omega \left( \hat{\rho}_{A|X} \right) \right],
\]

\[(F5b)\]
both of which are independent of \( x_f \) and \( a_f \). Now, since \( X_f \) and \( \Omega \) are independent variables, we can replace \( P_{X_f|\Omega} \) with \( P_{X_f|\Gamma,\Omega} \) and exchange the order of the maximisation over \( P_{X_f|\Gamma,\Omega} \) and the summation over \( \omega \) in Eq. (F4). Furthermore, the optimal measurement operators for which the maximisation over \( \{ E_\gamma \} \) is attained for each \( \omega \) depend, of course, on \( \omega \). Hence, we can also exchange the order of the summation over \( \omega \) and the maximisation over the measurement operators if we make this dependence explicit by replacing, in Eqs. (F4) and (F5), \( \{ E_\gamma \} \) with \( \{ E_{\gamma,\omega} \} \). With this, we write Eq. (F4) as

\[
\sum_\omega P_\Omega(\omega) S_Q \left( \frac{\mathcal{M}_\omega \left( \hat{\rho}_{A|X} \right)}{\text{Tr} \left[ \mathcal{M}_\omega(\hat{\rho}_{A|X}) \right]} \left| \frac{\mathcal{M}_\omega \left( \hat{\rho}'_{A|X} \right)}{\text{Tr} \left[ \mathcal{M}_\omega(\hat{\rho}'_{A|X}) \right]} \right| = \max_{P_{\Gamma|\Omega}(\cdot, \{ E_\gamma \})} \left\{ \sum_\omega P_\Omega(\omega) \left[ S \left( P_{\Gamma|\Omega}(\cdot, \{ E_\gamma \}) \| P'_{\Gamma|\Omega}(\cdot, \{ E_\gamma \}) \right) \right] \right\}
\]

Next, using Eqs. (3), (10) and (11), we write

\[
S_Q \left( \frac{1 \otimes E_{\gamma,\omega} \left[ \mathcal{M}_\omega \left( \hat{\rho}_{A|X} \right) \right] (x_f) \mathbb{1} \otimes E_{\gamma,\omega}^\dagger}{P_{\Gamma,\Omega}(\gamma, \omega)} \right) = \sum_x P_{X|\Gamma,\Omega}(x, x_f, \omega) S_Q \left( \frac{\sum_{a_f, a, x} P_{X|A,X,\Omega}(x, x_f, \omega) P_{A|A,X,\Omega,X_f}(a_f, a, x, \omega, x_f) |a_f \rangle \langle a_f| \otimes E_{\gamma,\omega} \mathbb{1} \otimes E_{\gamma,\omega}^\dagger}{P_{\Gamma,\Omega}(\gamma, \omega)} \right) \leq \sum_x P_{X|\Gamma,\Omega}(x, x_f, \omega) S_Q \left( \frac{\sum_{a_f, a, x} P_{A|A,X,\Omega,X_f}(a_f, a, x, \omega, x_f) |a_f \rangle \langle a_f| \otimes E_{\gamma,\omega} \mathbb{1} \otimes E_{\gamma,\omega}^\dagger}{P_{\Gamma,\Omega}(\gamma, \omega)} \right)
\]

where the inequality is due to Eq. (F1). On the other hand, we note that there always exists a completely positive trace-preserving map \( \mathcal{R}_{x,\omega,x_f} : \mathcal{L}(\mathcal{H}_E) \rightarrow \mathcal{L}(\mathcal{H}_E) \) such that

\[
\mathcal{R}_{x,\omega,x_f}(|a\rangle \langle a|) = \sum_{a_f} |a_f\rangle \langle a_f| P_{A|A,X,\Omega,X_f}(a_f, a, x, \omega, x_f).
\]

Hence, we can apply Eq. (F2) to the von-Neumann relative entropy in the right-hand side of Eq. (F7) with the map (F8), to get

\[
S_Q \left( \frac{\sum_{a_f, a} P_{A|A,X,\Omega,X_f}(a_f, a, x, \omega, x_f) |a_f \rangle \langle a_f| \otimes E_{\gamma,\omega} \mathbb{1} \otimes E_{\gamma,\omega}^\dagger}{P_{\Gamma,\Omega}(\gamma, \omega)} \right) \leq S_Q \left( \frac{\sum_a |a\rangle \langle a| \otimes E_{\gamma,\omega} \mathbb{1} \otimes E_{\gamma,\omega}^\dagger}{P_{\Gamma,\Omega}(\gamma, \omega)} \right)
\]
where the quantum representation (3) has been invoked again. Then, using Eqs. (F6), (F7), and (F9), we obtain

\[ \sum_{\omega} P_{1\omega}(\omega) S_A \left( \frac{\mathcal{M}_\omega(\hat{\rho}_{A|X})}{\text{Tr}[\mathcal{M}_\omega(\rho_{A|X})]} \right) \leq \max_{P_{X|\Gamma}(x,\omega)} \left\{ \sum_{\omega} P_{1\omega}(\omega) \left[ S_C(P_{\Gamma|\Omega}(\gamma,\omega)\|P'_{\Gamma|\Omega}(\omega)) + \sum_{\gamma,xf} P_{X,\Gamma}(x,\gamma,\omega) P_{X|\Gamma}(x,\gamma,\omega) \times S_Q \left( \frac{1 \otimes E_{\gamma,\omega} \hat{\rho}_{A|X}(x) K_\omega^\dagger K_\omega^\dagger E_{\gamma,\omega}^\dagger \otimes 1}{P_{\Gamma,\Omega}(\gamma,\omega)} \right) \right] \right\} \]  

The inequality (F10) follows from Eq. (F3) and from replacing \( P_{X|\Gamma}(x,\gamma,\omega) \) with \( P_{X|\Gamma}(x,\gamma,\omega) \), which cannot decrease the value of the resulting maximum.

Finally, using that, due to Bayes’ theorem, it holds that

\[ \sum_{\omega} P_{1\omega}(\omega) P_{X|\Gamma}(x,\gamma,\omega) P_{X|\Gamma}(x,\gamma,\omega) = P_{X,\Gamma}(x,\gamma,\omega), \]  

and introducing the joint variable \( \Xi := (\Gamma, \Omega) \), with values \( \xi := (\gamma, \omega) \), and the joint Kraus operators \( T_{\xi} := E_{\gamma,\omega} K_\omega \), which satisfy the normalisation condition \( \sum_{\xi} T_{\xi}^\dagger T_{\xi} = \sum_{\gamma,\omega} E_{\gamma,\omega}^\dagger E_{\gamma,\omega} K_\omega = 1 \), we write the inequality (F10) as

\[ \max_{P_{X|\Xi}(x,\xi)} \left\{ \frac{S_C(P_{\Xi}||P_{\Xi}')} + \sum_{x,\xi} P_{X,\Xi}(x,\xi) \times S_Q \left( \frac{1 \otimes T_{\xi} \hat{\rho}_{A|X}(x) T_{\xi}^\dagger \otimes 1}{P_{\Xi}(\xi)} \right) \right\} \leq \max_{P_{X|\Gamma}(x,\gamma)} \left\{ \sum_{\omega} P_{1\omega}(\omega) S_A \left( \frac{\mathcal{M}_\omega(\hat{\rho}_{A|X})}{\text{Tr}[\mathcal{M}_\omega(\rho_{A|X})]} \right) \right\}. \]  

By Definition 2, the right-hand side of Eq. (F12) coincides with the right-hand side of Eq. (B1).

\[ \square \]

2. Proof of Lemma 2

For the proof of this lemma, it is useful to re-express Eq. (18) in terms of abstract flag states representing the outcomes of Bob’s generalized quantum measurements. Introducing an auxiliary extension Hilbert space \( \mathcal{H}_{E_F} \) and an orthonormal basis of it \( \{ |\gamma\rangle \} \), where each basis member encodes the value \( \gamma \) of the measurement outcomes, and using that \( \sum_{\gamma} P_{X|\Gamma}(x,\gamma) = 1 \) for all \( \gamma \), we write

\[ S_A(\rho_{A|X}||\rho'_{A|X}) = \max_{P_{X|\Gamma}(x,\gamma)} \left\{ \sum_{x} P_{X|\Gamma}(x,\gamma) \left[ S_C(P_{\Gamma}||P'_{\Gamma}) + \sum_{\gamma} P_{\Gamma}(\gamma) S_Q \left( \frac{1 \otimes E_{\gamma} \hat{\rho}_{A|X}(x) E_{\gamma}^\dagger \otimes 1}{P_{\Gamma}(\gamma)} \right) \right] \right\} \]

\[ = \max_{P_{X|\Gamma}(x,\gamma)} \left\{ \sum_{x} P_{X|\Gamma}(x,\gamma) S_Q \left( \sum_{\gamma} |\gamma\rangle \otimes E_{\gamma} \hat{\rho}_{A|X}(x) E_{\gamma}^\dagger \otimes 1 \right) \right\}. \]  

We can now prove the lemma.
Proof of Lemma 2. Using Eq. (F13), we write the left-hand side of Eq. (B2) as

\[
S_A \left( \sum_j \mu_j^{(j)} \rho_{A|X}^{(j)} \right) \leq \max_{P_{X|\Gamma, \{E_{\gamma}\}}} \left[ \sum_x P_{X|\Gamma}(x, \gamma) S_Q \left( \sum_j \mu_j^{(j)} \sum_\gamma |\gamma\rangle \langle \gamma| \otimes E_{\gamma} \rho_{A|X}^{(j)}(x) \otimes E_\gamma \right) \right]
\]

\[
\leq \max_{P_{X|\Gamma, \{E_{\gamma}\}}} \left[ \sum_{x,j} \mu_j^{(j)} P_{X|\Gamma}(x, \gamma, j) S_Q \left( \sum_\gamma |\gamma\rangle \langle \gamma| \otimes E_{\gamma} \rho_{A|X}^{(j)}(x) \otimes E_\gamma \right) \right]
\]

where (F14) follows from Eq. (F1) and, in Eq. (F15), we exchanged the order of the maximization and the summation over \( j \) by respectively replacing \( \{E_\gamma\} \) and \( P_{X|\Gamma} \) with \( \{E_{\gamma,j}\} \) and \( P_{X|\Gamma, J} \), of elements \( P_{X|\Gamma, J}(x, \gamma, j) \). Using again Eq. (F13), one sees that, by Definition 2, the right-hand side of Eq. (F15) coincides with the right-hand side of Eq. (B2). \( \square \)

[33] V. Vedral and M. B. Plenio, Phys. Rev. A 57, 1619 (1998).
[34] M. Lewenstein and A. Sanpera, Phys. Rev. Lett. 80, 2261 (1998).
[35] S. Karnas and M. Lewenstein, quant-ph/0011066 (2000).
[36] G. Vidal and R. Tarrach, Phys. Rev. A 59, 141 (1999).
[37] E. H. Lieb, Adv. Math. 11, 267 (1973).