Determining the type of initial anisotropy of elastic material from a series of experiments

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Abstract. An experimental program has been developed to identify the type of initial elastic anisotropy of the material. The program includes a series of compression experiments to determine the orientation of the principal axes of anisotropy of the material and additional experiments to find the orientation of the canonical axes of anisotropy. In order to distinguish isotropic and cubic materials, it is necessary to fulfill an experiment on biaxial tension-compression in the direction of two canonical axes of anisotropy, and a shear experiment in the same plane. Similar experiments make it possible to identify trigonal, tetragonal and hexagonal materials. To identify triclinic, monoclinic, and rhombic materials, three shear experiments in planes determined by the canonical axes of anisotropy are required.

1. Introduction

Many modern structural materials have anisotropy of elastic properties and exhibit nonlinear behavior even at small deformations. In [1, 2], several variants of relations were proposed that describe the nonlinear dependence of stresses on strains in isotropic and anisotropic materials of various types. To develop experimental programs for concretizing the material constants and functions included in these constitutive relations, it is necessary to know the type of initial elastic anisotropy of the material, since the number of material constants or functions is different for different types of material.

At infinitely small strains and constant temperature, the proposed constitutive relations asymptotically tend to Hooke’s law, which expresses a linear relationship between strains and stresses:

\[ \varepsilon = C \cdot S, \]

where \( \varepsilon \) is the strain tensor, \( C \) is the fourth-rank constant elastic compliance tensor, \( S \) is the stress tensor. The inverse dependence of stresses on strains has the form \( S = N \cdot \varepsilon \), where \( N \) is the material elasticity tensor. The structure of tensors \( N \) and \( C \) is the same for the same type of material.

Numerous works [3–7] are devoted to the analysis of the structure and properties of the tensor \( C \). This analysis shows that in the general case, the tensor \( C \) has internal symmetry: \( C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij} \) — and therefore has 21 independent components. Different crystallographic systems have a certain number of symmetry elements. Therefore, the tensor \( C \) has a smaller number of independent components: from 2 for isotropic and gyrotropic materials.
to 13 for monoclinic [8]. Each type of anisotropic material has its own structure of elastic compliance tensor $C$. For this reason the type of initial elastic anisotropy of the material can be determined from experiments at infinitesimal strains.

In articles [9–12] methods for determining 21 components of the elasticity tensor in a laboratory coordinate system using acoustic waves [12], methods of holographic and speckle interferometry [9], and experiments on static uniform deformation [10] are given.

In [13], the problem of determining the type of anisotropic material was solved in general terms on the basis of the decomposition of the tensor of elastic moduli known from experiments into an isotropic (constant) part and parts containing two deviators and a non or. The formulae for the decomposition of the matrix of elastic moduli of materials of all crystallographic syngonies are given. These decompositions for different types of anisotropic materials have a different form, therefore, using them, the problem of identifying anisotropic material can be completely solved.

Another approach to solving this problem is based on the construction of a complete system of polynomial invariants with respect to orthogonal transformations of the coordinate system [3].

However, in all the indicated works [9–12], to establish the type of anisotropic material, it is necessary to know the 21 components of the elasticity tensor or elastic compliance tensor in some coordinate system.

In the article [14] an approach is proposed for determining whether the elastic properties of anisotropic materials belong to different symmetry classes using the concept of the optimal symmetric approximation for the elastic constant tensor.

An alternative approach to solving the problem of identifying the type of anisotropy of a material is to develop an experimental program with which it is possible to classify materials by crystallographic systems before determining elastic constants [5] if the symmetry of the properties of some anisotropic material is a priori unknown. Such a system of experiments makes it possible to identify the type of symmetry of the properties of an anisotropic material. The development of this system of experiments is an urgent task. A preliminary determination of the type of anisotropy and, therefore, the symmetry of the material will reduce the number of experiments required to find all the elastic constants (components of tensor $N$ or $C$).

2. General approach to constructing a program of experiments to identify the type of elastic symmetry of an anisotropic material

As an object of study, we consider a cubic sample of representative size, the edges of which are directed along the axes of the laboratory coordinate system.

The base test in the program is an experiment to determine the position of the main axes of anisotropy in a material. In accordance with the definition of V. V. Novozhilov [15], the main axes of anisotropy are the principal axes of the stress tensor arising in the anisotropic material in response to a purely bulk deformation.

Pure volumetric deformation is difficult to implement in experiments. Let us show that the principal axes of the anisotropy of the material can be defined as the principal axes of the tensor of deformations arising in the anisotropic material in response to a purely bulk deformation.

At pure bulk deformation, the tensor of small strains has the form $\varepsilon = \varepsilon_0 (e^1 e^1 + e^2 e^2 + e^3 e^3)$, where $\varepsilon_0$ characterizes the relative change in volume. The stress tensor describing the response of a material to volumetric deformation generally has the form

$$S = \sqrt{3} \varepsilon_0 \left[ \left( \frac{n_{00}}{\sqrt{3}} - \frac{n_{01}}{2\sqrt{6}} \right) a^1 a^1 + \left( \frac{n_{00}}{\sqrt{3}} - \frac{n_{01}}{2\sqrt{6}} + \frac{n_{02}}{2\sqrt{2}} \right) a^2 a^2 + \left( \frac{n_{00}}{\sqrt{3}} + \frac{n_{01}}{\sqrt{6}} \right) a^3 a^3 \right], \quad (2)$$

where $n_{00}, n_{01}, n_{02}$ are material constants.
Moreover, in an arbitrary laboratory coordinate system with a basis \( \vec{e}^1, \vec{e}^2, \vec{e}^3 \) tensor (2) is a symmetric general tensor. The main axes of this tensor — vectors \( \vec{a}^1, \vec{a}^2, \vec{a}^3 \) — determine the directions of the main axes of anisotropy.

When loading with hydrostatic pressure

\[
S_0 = -\sqrt{3}\sigma \left( e^1 e^1 + e^2 e^2 + e^3 e^3 \right)
\]

deformations generally occur in the material

\[
\epsilon = -\sqrt{3}\sigma \left[ \left( \frac{c_{00}}{\sqrt{3}} - \frac{c_{01}}{2\sqrt{6}} - \frac{c_{02}}{2\sqrt{3}} \right) \vec{a}^1 \vec{a}^1 + \left( \frac{c_{00}}{\sqrt{3}} - \frac{c_{01}}{2\sqrt{6}} + \frac{c_{02}}{2\sqrt{3}} \right) \vec{a}^2 \vec{a}^2 + \left( \frac{c_{00}}{\sqrt{3}} + \frac{c_{01}}{\sqrt{6}} \right) \vec{a}^3 \vec{a}^3 \right],
\]

where \( c_{00}, c_{01}, c_{02} \) are material constants.

The principal axes of this strain tensor coincide with the principal axes of the stress tensor (2).

As a result of experiment (3), the principal values \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) and the principal vectors \( \vec{a}^1, \vec{a}^2, \vec{a}^3 \) of the strain tensor will be determined. The triple of vectors \( \vec{a}^1, \vec{a}^2, \vec{a}^3 \) sets the directions of the main axes of anisotropy of the material. Due to the symmetry of the strain tensor, three cases are possible:

1) \( \varepsilon_1 = \varepsilon_2 = \varepsilon_3, \vec{a}^1, \vec{a}^2, \vec{a}^3 \) is an arbitrary orthogonal trihedron;
2) \( \varepsilon_1 = \varepsilon_2 \neq \varepsilon_3 \), the vector \( \vec{a}^3 \) is determined, vectors \( \vec{a}^1 \) and \( \vec{a}^2 \) are orthogonal and located in a plane perpendicular to the vector \( \vec{a}^3 \);
3) \( \varepsilon_1 \neq \varepsilon_2 \neq \varepsilon_3, \vec{a}^1, \vec{a}^2, \vec{a}^3 \) are deformation ellipsoid axes.

In accordance with the representations of the elastic compliance tensors \([3, 7, 10, 12]\) in the case of isotropic and cubic materials, the strains calculated by relations (1) are volumetric:

\[
\epsilon = -c_{00}\sigma \left( \vec{a}^1 \vec{a}^1 + \vec{a}^2 \vec{a}^2 + \vec{a}^3 \vec{a}^3 \right) - \frac{c_{01}}{2\sqrt{2}} \sigma \left( 2\vec{a}^3 \vec{a}^3 - \vec{a}^1 \vec{a}^1 - \vec{a}^2 \vec{a}^2 \right)
\]

occur under hydrostatic pressure. When hydrostatic stresses are applied, materials of a rhombic, monoclinic, and triclinic syngony produce strains of the most general form

\[
\epsilon = -c_{00}\sigma \left( \vec{a}^1 \vec{a}^1 + \vec{a}^2 \vec{a}^2 + \vec{a}^3 \vec{a}^3 \right) - \frac{c_{01}}{2\sqrt{2}} \sigma \left( 2\vec{a}^3 \vec{a}^3 - \vec{a}^1 \vec{a}^1 - \vec{a}^2 \vec{a}^2 \right) - \frac{\sqrt{3}c_{02}}{2\sqrt{2}} \sigma \left( \vec{a}^2 \vec{a}^2 - \vec{a}^1 \vec{a}^1 \right).
\]

Thus, the comprehensive compression experiment allows to determine the orientation of the main axes of anisotropy in the material, as well as to assign the material to one of three groups: isotropic or cubic; tetragonal, trigonal or hexagonal; rhombic, monoclinic or triclinic. The significant disadvantages of this experimental program are:

1) the complexity of the experiment on hydrostatic loading associated with the need to measure all six strains of a cubic sample;
2) the ambiguity in determining the position of the main axes of anisotropy \( \vec{a}^1, \vec{a}^2, \vec{a}^3 \) for uniaxial crystals (tetragonal, trigonal and hexagonal materials \([4]\)) and cubic material.

To overcome these shortcomings, in the present work, as in \([16]\), it is proposed to replace the hydrostatic loading experiment with three agreed compression tests.
3. The program of experiments to determine the main and canonical axes of anisotropy of the material

Let us consider the possibility of determining the principal axes of anisotropy from three compression experiments. The stress tensor (3) can be represented as the sum of three tensors

\[ S_0 = S_1 + S_2 + S_3, \tag{4} \]

where \( S_1 = -t_1 e^1 \cdot e^1, S_2 = -t_2 e^2 \cdot e^2, S_3 = -t_3 e^3 \cdot e^3 \), each of which describes the compression along one of the axes of some fixed Cartesian coordinate system \( Ox'y'z' \) with the basis \( e^1, e^2, e^3 \), which will be called the laboratory coordinate system in the future. Each stress tensor is corresponded to the strain tensor associated with it by Hooke’s law (1):

\[ \varepsilon_1 = C \cdot S_1, \quad \varepsilon_2 = C \cdot S_2, \quad \varepsilon_3 = C \cdot S_3. \tag{5} \]

Due to the linearity of Hooke’s law, taking into account formulae (4) and (5), we obtain

\[ \varepsilon_0 = C \cdot S_0 = C \cdot (S_1 + S_2 + S_3) = \varepsilon_1 + \varepsilon_2 + \varepsilon_3. \tag{6} \]

The additivity of representation (6) allows us to replace the comprehensive compression experiment with three uniaxial compression tests, in which it is possible to measure all the components of the strain tensor of the sample. If three such uniaxial experiments are performed, then the main axes of the anisotropy of the material can be defined as the main axes of the strain tensor (6), which describes the response of the sample to comprehensive compression.

If it is not possible to experimentally measure all the components of the strain tensors \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \), then these components can be expressed in terms of the geometric parameters of the deformed samples.

The principal axes of the tensor (6) are the principal axes of the anisotropy of the material. The directions of the main axes of anisotropy of the material are determined by unit eigenvectors \( \vec{a}^i, i = 1, 2, 3 \) of the tensor \( \varepsilon_0 \), which are determined from the condition \( \varepsilon_0 \cdot \vec{a}^i = \varepsilon_i \vec{a}^i \), where \( \varepsilon_i, i = 1, 2, 3 \) are the eigenvalues of the tensor \( \varepsilon_0 \). To find the eigenvalues of the tensor \( \varepsilon_0 \), we use the characteristic equation

\[ \varepsilon^3 - I_1(\varepsilon_0)\varepsilon^2 + I_2(\varepsilon_0)\varepsilon - I_3(\varepsilon_0) = 0, \]

where \( I_1(\varepsilon_0), I_2(\varepsilon_0), I_3(\varepsilon_0) \) are the algebraic invariants of the tensor \( \varepsilon_0 \).

The principal axes and principal values of the strain tensor \( \varepsilon_0 \) (6) are determined by linear algebra methods. Finding the principal axes of this tensor means determining the orientation of the principal axes of the anisotropy of the material in space relative to the laboratory coordinate system. Finding the eigenvalues of the tensor \( \varepsilon_0 \) allows us to attribute the material to one of the three groups specified in the Section 2, according to the number of different eigenvalues of the tensor.

As indicated above, in the case of multiple eigenvalues of the tensor \( \varepsilon_0 \), the main axes of anisotropy are determined in the general case up to three parameters. However, tensors of elastic properties \( \mathbf{N} \) and \( \mathbf{C} \) have the smallest number of independent constants exclusively in the basis associated with the symmetry elements of the material properties. For crystals, such a basis is associated with crystallophysical axes \([6, ?, ?]\), and for composite materials or wood, with the preferred directions associated with their structure.

In the case when the internal structure of some anisotropic material is not known in advance, the basis of the Cartesian coordinate system associated with the symmetry elements of the properties of this material can be determined from a system of mechanical experiments. We call the axes of such a coordinate system the canonical axes of anisotropy of the material. The associated basis vectors are denoted by \( \vec{k}^1, \vec{k}^2, \vec{k}^3 \).
The canonical axes of anisotropy are the axes of the Cartesian coordinate system in which the tensors describing the properties of the material have the smallest number of nonzero independent constants. The canonical axes of anisotropy associated with the symmetry elements of the crystal structure always coincide with the main axes of anisotropy defined by V. V. Novozhilov. However, the main anisotropy axes, found in accordance with the definition of V. V. Novozhilov, are arbitrary in choosing their directions and, in the general case, may not coincide with the canonical axes. It follows from the foregoing that the definition of the main axes of anisotropy given by V. V. Novozhilov is necessary for finding the canonical axes of anisotropy in materials from mechanical macroexperiments, but is not sufficient. Next, we show how the canonical axes of anisotropy of a material can be determined from the results of mechanical macroexperiments.

We consider three possible cases of the multiplicity of the principal values of the strain tensor \( \epsilon_0 \) (6) and for each case we indicate the ways in which the position of the canonical axes of anisotropy can be determined.

If, according to the results of experiments (5), it turns out that all the eigenvalues of the tensor \( \epsilon_0 \) (6) are different \( (\epsilon_1 \neq \epsilon_2 \neq \epsilon_3) \), then its eigenvectors are uniquely determined and determine the directions of the main axes of anisotropy \( \vec{a}^1, \vec{a}^2, \vec{a}^3 \). The canonical axes of anisotropy \( \vec{k}^1, \vec{k}^2, \vec{k}^3 \) in this case coincide with the main axes of anisotropy, and the material can be classified as rhombic, monoclinic or triclinic in its properties.

If two of the three eigenvalues of the strain tensor (6) are equal \( (\epsilon_1 = \epsilon_2 \neq \epsilon_3) \), then only one main axis of anisotropy is uniquely determined — the main rotary axis \( \vec{a}^3 \). The basis vectors \( \vec{a}^1 \) and \( \vec{a}^2 \) can be arbitrarily selected, and the basis vectors \( \vec{k}^1, \vec{k}^2 \) must be associated with a lateral rotary axis. Vector \( \vec{k}^3 = \vec{a}^3 \). The bases \( \vec{a}^1, \vec{a}^2, \vec{a}^3 \) and \( \vec{k}^1, \vec{k}^2, \vec{k}^3 \) are connected by the orthogonal rotation tensor

\[
Q_3 = \cos \varphi \left( \vec{k}^1 \vec{k}^1 + \vec{k}^2 \vec{k}^2 \right) + \sin \varphi \left( \vec{k}^1 \vec{k}^2 - \vec{k}^2 \vec{k}^1 \right) + \vec{k}^3 \vec{k}^3, \tag{7}
\]

so that \( \vec{a}^i = \vec{k}^i \cdot Q_3, \ i = 1, 2, 3 \).

To find the angle \( \varphi \) in the case of trigonal material, it is necessary to conduct one experiment: biaxial tension-compression along the vectors \( \vec{a}^1, \vec{a}^2 \). In this experiment, the stress tensor has the form \( S_4 = t_4 (\vec{a}^1 \vec{a}^1 - \vec{a}^2 \vec{a}^2) \), and the measured components of the strain tensor are

\[
\varepsilon_{11} = -\varepsilon_{22} = t_4 (C_{1111} - C_{1122}), \quad \varepsilon_{33} = \varepsilon_{12} = 0,
\]

\[
\varepsilon_{13} = 2t_4 C_{1123} \sin 3\varphi, \quad \varepsilon_{23} = 2t_4 C_{1123} \cos 3\varphi. \tag{8}
\]

From expressions (8), the angle \( \varphi \) for the trigonal material is determined by the formula

\[
\varphi^{(tr)} = \frac{1}{3} \arctan \frac{\varepsilon_{13}}{\varepsilon_{23}}. \tag{9}
\]

To determine the angle \( \varphi \) in the case of tetragonal material, two experiments are required: tension-compression along the vectors \( \vec{a}^1, \vec{a}^2 \) and a shift in the plane of these vectors, which can be performed by tension-compression at an angle 45° to the directions \( \vec{a}^1, \vec{a}^2 \). In the first of these experiments, the stress tensor is \( S_4 = t_4 (\vec{a}^1 \vec{a}^1 - \vec{a}^2 \vec{a}^2) \), and the measured components of the strain tensor are

\[
\varepsilon_{11} = -\varepsilon_{22} = t_4 \left( (C_{1111} - C_{1122}) \cos^2 2\varphi + 2C_{1212} \sin^2 2\varphi \right),
\]

\[
\varepsilon_{12} = t_4 \left( 2C_{1212} - (C_{1111} - C_{1122}) \right) \sin 2\varphi \cos 2\varphi, \quad \varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0. \tag{10}
\]
In the second experiment, the stress tensor has the form \( \mathbf{S}_5 = t_5 \left( \bar{a}^1 \bar{a}^2 + \bar{a}^2 \bar{a}^4 \right) \), and the measured strains are

\[
\begin{align*}
\bar{\varepsilon}_{11} &= -\bar{\varepsilon}_{22} = t_5 \left( 2C_{1212} - (C_{1111} - C_{1122}) \right) \sin 2\varphi \cos 2\varphi, \\
\bar{\varepsilon}_{12} &= t_5 \left( (C_{1111} - C_{1122}) \sin^2 2\varphi + 2C_{1212} \cos^2 2\varphi \right), \\
\bar{\varepsilon}_{13} &= \bar{\varepsilon}_{23} = \bar{\varepsilon}_{33} = 0.
\end{align*}
\] (11)

From expressions (10), (11), the angle \( \varphi \) for the tetragonal material is determined by the formulae

\[
\varphi^{(t)} = -\frac{1}{4} \arctan \frac{2\bar{\varepsilon}_{12}t_5}{\varepsilon_{11}t_5 - \varepsilon_{12}t_4} \quad \text{or} \quad \varphi^{(t)} = -\frac{1}{4} \arctan \frac{2\bar{\varepsilon}_{11}t_4}{\varepsilon_{11}t_5 - \varepsilon_{12}t_4}.
\] (12)

If in both experiments the loads are equal \( t_4 = t_5 \), then expressions (12) for the angle \( \varphi^{(t)} \) are simplified:

\[
\varphi^{(t)} = -\frac{1}{4} \arctan \frac{2\bar{\varepsilon}_{12}}{\varepsilon_{11} - \varepsilon_{12}} \quad \text{or} \quad \varphi^{(t)} = -\frac{1}{4} \arctan \frac{2\bar{\varepsilon}_{11}}{\varepsilon_{11} - \varepsilon_{12}}.
\] (13)

The orientation of the canonical axes of anisotropy relative to the laboratory coordinate system is determined by the found angle \( \varphi \) (formulae (9) and (12) for trigonal and tetragonal materials, respectively) from the relations

\[
\bar{k}^1 = \bar{a}^1 \cos \varphi - \bar{a}^2 \sin \varphi, \quad \bar{k}^2 = \bar{a}^1 \sin \varphi + \bar{a}^2 \cos \varphi, \quad \bar{k}^3 = \bar{a}^3.
\]

The elastic properties of the hexagonal material are invariant with respect to any rotations around the vector \( \bar{k}^3 = \bar{a}^3 \) [4]. Therefore, in the hexagonal material under the action of stresses \( \mathbf{S}_4 = t_4 \left( \bar{a}^1 \bar{a}^3 - \bar{a}^2 \bar{a}^3 \right) \) the measured components of the strain tensor are

\[
\bar{\varepsilon}_{11} = -\bar{\varepsilon}_{22} = t_4(C_{1111} - C_{1122}), \quad \bar{\varepsilon}_{12} = \bar{\varepsilon}_{13} = \bar{\varepsilon}_{23} = \bar{\varepsilon}_{33} = 0,
\]

since for such material

\[
C_{1212} = \frac{1}{2}(C_{1111} - C_{1122}).
\]

Under the action of stresses \( \mathbf{S}_5 = t_5 \left( \bar{a}^1 \bar{a}^2 + \bar{a}^2 \bar{a}^4 \right) \), measured strains in the hexagonal material are

\[
\bar{\varepsilon}_{12} = t_5(C_{1111} - C_{1122}), \quad \bar{\varepsilon}_{11} = \bar{\varepsilon}_{22} = \bar{\varepsilon}_{13} = \bar{\varepsilon}_{23} = \bar{\varepsilon}_{33} = 0.
\]

If as a result of an experiment to determine the position of the principal anisotropy axes according to V. V. Novozhilov, it turns out that the principal values of the strain tensor (6) are equal \( \varepsilon_1 = \varepsilon_2 = \varepsilon_3 \), then the axes of the laboratory coordinate system in which the experiment was performed should be selected as the main axes of anisotropy \( \bar{a}^1, \bar{a}^2, \bar{a}^3 \). Such a choice of vectors allows the use of uniaxial compression experiments to determine the position of the canonical axes of anisotropy \( \bar{k}^1, \bar{k}^2, \bar{k}^3 \) in a cubic material.

The mutual orientation of the vector bases \( \bar{a}^i \) and \( \bar{k}^i \) is determined by the orthogonal rotation tensor \( \mathbf{Q} = q_{ij} \bar{k}^i \bar{k}^j: \bar{a}^i = \bar{k}^i \cdot \mathbf{Q}, i = 1, 2, 3 \). For the components of the tensor \( \mathbf{Q} \) the identities hold

\[
q_{11}^2 + q_{21}^2 + q_{31}^2 = 1, \quad q_{12}^2 + q_{22}^2 + q_{32}^2 = 1, \quad q_{13}^2 + q_{23}^2 + q_{33}^2 = 1,
\]

\[
q_{11}q_{12} + q_{21}q_{22} + q_{31}q_{32} = 0, \quad q_{11}q_{13} + q_{21}q_{23} + q_{31}q_{33} = 0, \quad q_{12}q_{13} + q_{22}q_{23} + q_{32}q_{33} = 0.
\] (14)
Let us show that in order to determine the position of the canonical axes of anisotropy \( \bar{k} \) relative to the axes of the laboratory system \( \bar{a} \), it suffices to conduct two experiments on uniaxial compression in the laboratory coordinate system. In the first experiment the stress tensor is \( S_1 = -t\bar{a}\bar{a}^\top \). The measured strains are expressed in terms of compliance constants and components of tensor \( Q \) as follows:

\[
\bar{\varepsilon}_{11} = -t \left[ Q_1 (C_{1111} - (C_{1122} + 2C_{1212})) + C_{1122} + 2C_{1212} \right], \\
\bar{\varepsilon}_{22} = -t \left[ Q_4 (C_{1111} - (C_{1122} + 2C_{1212})) + C_{1122} \right], \\
\bar{\varepsilon}_{33} = -t \left[ Q_5 (C_{1111} - (C_{1122} + 2C_{1212})) + C_{1122} \right], \\
\bar{\varepsilon}_{12} = -t Q_7 (C_{1111} - (C_{1122} + 2C_{1212})), \\
\bar{\varepsilon}_{13} = -t Q_8 (C_{1111} - (C_{1122} + 2C_{1212})), \\
\bar{\varepsilon}_{23} = -t Q_9 (C_{1111} - (C_{1122} + 2C_{1212})),
\]

where indicated

\[
Q_1 = q^4_{11} + q^4_{12} + q^4_{13}, \quad Q_4 = q^2_{11}q^2_{22} + q^2_{12}q^2_{22} + q^2_{13}q^2_{23}, \quad Q_5 = q^2_{11}q^2_{22} + q^2_{12}q^2_{22} + q^2_{13}q^2_{23}, \quad Q_7 = q_{21}q^3_{11} + q_{22}q^3_{12} + q_{23}q^3_{13}, \quad Q_8 = q_{31}q^3_{11} + q_{32}q^3_{12} + q_{33}q^3_{13}, \quad Q_9 = q_{21}q^3_{11} + q_{22}q^3_{12} + q_{23}q^3_{13}.
\]

In the second experiment the stress tensor is \( S_2 = -t\bar{a}^2\bar{a}^2 \), and the measured components of the strain tensor \( \bar{\varepsilon}_2 \) are expressed in terms of compliance constants and components of tensor \( Q \):

\[
\bar{\varepsilon}_{11} = -t \left[ Q_4 (C_{1111} - (C_{1122} + 2C_{1212})) + C_{1122} + 2C_{1212} \right], \\
\bar{\varepsilon}_{22} = -t \left[ Q_2 (C_{1111} - (C_{1122} + 2C_{1212})) + C_{1122} \right], \\
\bar{\varepsilon}_{33} = -t \left[ Q_6 (C_{1111} - (C_{1122} + 2C_{1212})) + C_{1122} \right], \\
\bar{\varepsilon}_{12} = -t Q_{10} (C_{1111} - (C_{1122} + 2C_{1212})), \\
\bar{\varepsilon}_{23} = -t Q_{11} (C_{1111} - (C_{1122} + 2C_{1212})), \\
\bar{\varepsilon}_{13} = -t Q_{12} (C_{1111} - (C_{1122} + 2C_{1212})),
\]

where indicated

\[
Q_2 = q^4_{21} + q^4_{22} + q^4_{23}, \quad Q_6 = q^2_{21}q^2_{32} + q^2_{22}q^2_{32} + q^2_{23}q^2_{33}, \quad Q_{10} = q_{11}q^3_{21} + q_{12}q^3_{22} + q_{13}q^3_{23}, \quad Q_{11} = q_{31}q^3_{21} + q_{32}q^3_{22} + q_{33}q^3_{23}, \quad Q_{12} = q_{11}q^3_{21} + q_{12}q^3_{22} + q_{13}q^3_{23}.
\]

To find the nine components of the tensor \( Q \), we use six relations (14) and four independent relations from (15), (16):

\[
-t Q_7 (C_{1111} - (C_{1122} + 2C_{1212})) = \bar{\varepsilon}_{12}, \quad -t Q_8 (C_{1111} - (C_{1122} + 2C_{1212})) = \bar{\varepsilon}_{13}, \\
-t Q_9 (C_{1111} - (C_{1122} + 2C_{1212})) = \bar{\varepsilon}_{23}, \quad -t Q_{10} (C_{1111} - (C_{1122} + 2C_{1212})) = \bar{\varepsilon}_{12}.
\]
Eliminating the factor \((C_{1111} - (C_{1122} + 2C_{1212}))\) from them, we obtain three equations for the components of the tensor \(Q\):

\[
q_{11}q_{11}^3 + q_{21}q_{21}^3 + q_{31}q_{31}^3 = \frac{\varepsilon_{11}}{\varepsilon_{12}} \left( q_{21}q_{11}^3 + q_{22}q_{12}^3 + q_{23}q_{13}^3 \right),
\]

\[
q_{21}q_{31}q_{11}^2 + q_{22}q_{32}q_{12}^2 + q_{23}q_{33}q_{13}^2 = \frac{\varepsilon_{23}}{\varepsilon_{12}} \left( q_{21}q_{11}^3 + q_{22}q_{12}^3 + q_{23}q_{13}^3 \right),
\]

\[
q_{11}q_{21}^3 + q_{12}q_{22}^3 + q_{13}q_{23}^3 = \frac{\varepsilon_{12}}{\varepsilon_{12}} \left( q_{21}q_{11}^3 + q_{22}q_{12}^3 + q_{23}q_{13}^3 \right),
\]

(17)

A numerical solution of the system of equations (14), (17) allows us to find the components of the tensor \(Q\), that is, to determine the orientation of the canonical coordinate system in the cubic material relative to the laboratory coordinate system from the strains measured in experiments.

In a particular case, one of the main axes of anisotropy, determined from compression experiments, can coincide with the canonical axis of anisotropy, for example, \(\vec{a}^3 = \vec{k}^3\). Moreover, a numerical solution of the indicated system (14), (17) cannot be obtained, since \(Q_5 = Q_6 = Q_8 = Q_9 = Q_{11} = Q_{12} = 0\).

In this case, the rotation tensor has the form (7)

\[
Q_3 = \cos \varphi \left( \vec{k}^1 \vec{k}^1 + \vec{k}^2 \vec{k}^2 \right) + \sin \varphi \left( \vec{k}^1 \vec{k}^2 - \vec{k}^2 \vec{k}^1 \right) + \vec{k}^3 \vec{k}^3,
\]

so that \(\vec{a}^i = \vec{k}^i \cdot Q_3, i = 1, 2, 3\), and the orientation of the canonical axes of anisotropy \(\vec{k}^1, \vec{k}^2\) with respect to the laboratory axes \(\vec{a}^1, \vec{a}^2\) is determined by one parameter — the angle \(\varphi\). Then, in both uniaxial compression experiments, some strain components will be equal to zero, namely: \(\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{13} = \varepsilon_{23} = 0\). When a sample is compressed along the vector \(\vec{a}^1\), nonzero deformations will appear in it:

\[
\varepsilon_{11} = -t \left[ Q_1 \left( C_{1111} - (C_{1122} + 2C_{1212}) \right) + C_{1122} + 2C_{1212} \right],
\]

\[
\varepsilon_{22} = -t \left[ Q_4 \left( C_{1111} - (C_{1122} + 2C_{1212}) \right) + C_{1122} \right],
\]

\[
\varepsilon_{33} = -t C_{1122},
\]

\[
\varepsilon_{12} = -t Q_7 \left( C_{1111} - (C_{1122} + 2C_{1212}) \right),
\]

moreover \(Q_1 = \cos^4 \varphi + \sin^4 \varphi, Q_4 = 2 \sin^2 \varphi \cos^2 \varphi, Q_7 = -\sin \varphi \cos \varphi \cos 2\varphi\). The angle \(\varphi\) is determined by the expression

\[
\varphi^{(c)} = \frac{1}{2} \arctan \frac{\varepsilon_{22} - \varepsilon_{33}}{\varepsilon_{12}}.
\]

(18)

If under compression with the same stresses along different laboratory axes, the axial as well as transverse deformations will be the same, and the shear deformations will be zero, that is \(\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{11} = \varepsilon_{33}, \varepsilon_{ij} = \varepsilon_{ij} = 0\ (i \neq j)\), then the canonical anisotropy axes \(\vec{k}^1, \vec{k}^2, \vec{k}^3\) coincide with the main anisotropy axes \(\vec{a}^1, \vec{a}^2, \vec{a}^3\), but the type of material remains unknown.

Thus, a system of mechanical experiments was developed that allows one to determine the directions of the canonical axes of anisotropy associated with symmetry elements of material properties in a homogeneous cubic sample. The position of these axes relative to the laboratory coordinate system is determined for uniaxial crystals by the relations (7), (9), (12), (13), for a cubic material — by the relations (14), (17) or (7), (18).
4. The program of experiments to determine the type of anisotropy of the material
When determining the orientation of the canonical anisotropy axes relative to the laboratory coordinate system, one can identify trigonal, tetragonal, and cubic materials by the presence of nonzero components of the strain tensor in certain experiments in accordance with expressions (8), (10), (11), (15), (16). In the case of unambiguous determination of the position of the canonical axes of anisotropy (for triclinic, monoclinic and rhombic materials) and in the case of coincidence of the canonical axes of anisotropy with the axes of the laboratory coordinate system, additional experiments are required to identify the type of material.

4.1. The program of experimental identification of isotropic and cubic materials.
Let shear deformations, for example, \( \varepsilon_{12} \), \( \bar{\varepsilon}_{12} \) be reliably determined in experiments on uniaxial compression in a laboratory coordinate system, then the material is cubic, and its canonical anisotropy axes are determined by relations (14), (17) or (7), (18). If the measured shear deformations do not exceed the measurement errors and it can be considered that \( \varepsilon_{ij} = \bar{\varepsilon}_{ij} = 0 \) \((i \neq j)\), then, as shown above, the canonical axes of anisotropy \( \vec{k}^1, \vec{k}^2, \vec{k}^3 \) coincide with the main axes \( \vec{a}^1, \vec{a}^2, \vec{a}^3 \) and axes of the laboratory coordinate system.

In this case, in order to distinguish between an isotropic material and a cubic one, it is necessary to conduct a biaxial tension-compression experiment along the directions of vectors \( \vec{a}^1, \vec{a}^2 \) with a stress tensor \( S_4 = t_4 (\vec{a}^1 \vec{a}^1 - \vec{a}^2 \vec{a}^2) \) and a shear experiment in this plane with a stress tensor \( S_5 = t_5 (\vec{a}^1 \vec{a}^2 + \vec{a}^2 \vec{a}^1) \).

In response to stresses \( S_4 \) in both isotropic and cubic materials, in accordance with the expressions of the tensor \( C \) strains
\[
\varepsilon_{4i}^{(4)} = \varepsilon_{4i} = t_4 (C_{1111} - C_{1122}) (\vec{a}^1 \vec{a}^1 - \vec{a}^2 \vec{a}^2) = (C_{1111} - C_{1122}) S_4,
\]
will arise, where \( C_{ijkl} \) are elastic compliance tensor components.

In a shear experiment at stresses \( S_5 \) in an isotropic material strains
\[
\varepsilon_{5i}^{(5)} = 2t_5 C_{1212} (\vec{a}^1 \vec{a}^2 + \vec{a}^2 \vec{a}^1) = t_5 (C_{1111} - C_{1122}) (\vec{a}^1 \vec{a}^2 + \vec{a}^2 \vec{a}^1) = (C_{1111} - C_{1122}) S_5
\]
will arise. Then
\[
\frac{\varepsilon_{11}^{(4)}}{t_4} = \frac{\varepsilon_{12}^{(4)}}{t_5} = \frac{\varepsilon_{12}^{(5)}}{t_5},
\]
(19)
since for an isotropic material [4]
\[
C_{1212} = C_{2323} = C_{1313} = \frac{1}{2} (C_{1111} - C_{1122}).
\]

In a cubic material at the shear \( S_5 \) in the plane of the vectors \( \vec{a}^1, \vec{a}^2 \), strains arise, described by the tensor
\[
\varepsilon_{5}^c = 2t_5 C_{1212} (\vec{a}^1 \vec{a}^2 + \vec{a}^2 \vec{a}^1) = 2C_{1212} S_5.
\]

For components of the elastic compliance tensor of a cubic material
\[
C_{1212} = C_{2323} = C_{1313} \neq \frac{1}{2} (C_{1111} - C_{1122}),
\]
so
\[
\varepsilon_{5}^c = 2C_{1212} S_5 \neq (C_{1111} - C_{1122}) S_5.
\]

If condition (19) is satisfied, then the material is isotropic, and if not satisfied, then cubic.

In experiments on biaxial tension-compression and shear in the plane of the vectors \( \vec{a}^1, \vec{a}^2 \) the components of the strain tensors \( \varepsilon_4, \varepsilon_5 \) are expressed in terms of the measured geometric parameters of the deformed samples.
4.2. The program of experimental identification of uniaxial crystals.

For tetragonal, trigonal, and hexagonal materials, the canonical axes of anisotropy are defined in Section 3. If the canonical axes of anisotropy $\vec{k}^1$, $\vec{k}^2$ do not coincide with the main axes of anisotropy $\vec{a}^1$, $\vec{a}^2$ then from a biaxial tension-compression experiment the trigonal material can be distinguished from tetragonal and hexagonal ones by the presence of non-zero strain components $\varepsilon_{13}$ and $\varepsilon_{23}$.

In the case of hexagonal material in a tension-compression experiment along vectors $\vec{a}^1$, $\vec{a}^2$ with a stress tensor $\mathbf{S}_4 = t_4(\vec{a}^1 \vec{a}^1 - \vec{a}^2 \vec{a}^2)$ the measured components of the strain tensor have the form

$$\varepsilon_{11} = -\varepsilon_{22} = t_4(C_{1111} - C_{1122}), \quad \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0.$$

Hexagonal material differs from tetragonal one in that in the indicated experiment for the former $\varepsilon_{12} = 0$, and for the latter $\varepsilon_{12} \neq 0$ and it is determined by the corresponding expression (10).

Let us consider the case when the principal axes of anisotropy, determined experimentally, coincide with the canonical axes of anisotropy of uniaxial crystals, i.e., the angle $\varphi = 0$. In this case, in a tensile-compression experiment along vectors $\vec{a}^1$, $\vec{a}^2$ with a stress tensor $\mathbf{S}_4 = t_4(\vec{a}^1 \vec{a}^1 - \vec{a}^2 \vec{a}^2)$ the strain tensors in accordance with the expressions for the elastic compliance tensor $\mathbf{C}$ have the form:

for the trigonal material

$$\varepsilon_t^r = t_4(C_{1111} - C_{1122})(\vec{a}^1 \vec{a}^1 - \vec{a}^2 \vec{a}^2) + 2t_4C_{1123}(\vec{a}^2 \vec{a}^3 + \vec{a}^3 \vec{a}^2),$$

(20)

for tetragonal and hexagonal materials

$$\varepsilon_t = \varepsilon_t^h = t_4(C_{1111} - C_{1122})(\vec{a}^1 \vec{a}^1 - \vec{a}^2 \vec{a}^2) = (C_{1111} - C_{1122}) \mathbf{S}_4,$$

(21)

In the shear experiment under loading $\mathbf{S}_5 = t_5(\vec{a}^1 \vec{a}^2 + \vec{a}^2 \vec{a}^1)$ the following strains will arise: in the trigonal material

$$\varepsilon_t^s = t_5(C_{1111} - C_{1122})(\vec{a}^1 \vec{a}^2 + \vec{a}^2 \vec{a}^1) + 2t_5C_{1123} (\vec{a}^1 \vec{a}^3 + \vec{a}^3 \vec{a}^1),$$

in the tetragonal material

$$\varepsilon_t = 2t_5C_{1212} (\vec{a}^1 \vec{a}^2 + \vec{a}^2 \vec{a}^1) = 2C_{1212} \mathbf{S}_5.$$

(22)

In the hexagonal material, the shear $\mathbf{S}_5$ in the plane of vectors $\vec{a}^1$, $\vec{a}^2$ gives rise to strains described by the tensor

$$\varepsilon_h = 2t_5C_{1212} (\vec{a}^1 \vec{a}^2 + \vec{a}^2 \vec{a}^1) = t_5(C_{1111} - C_{1122})(\vec{a}^1 \vec{a}^2 + \vec{a}^2 \vec{a}^1) = (C_{1111} - C_{1122}) \mathbf{S}_5,$$

(23)

since for hexagonal material [4]

$$C_{1212} = \frac{1}{2}(C_{1111} - C_{1122}).$$

If the canonical axes of anisotropy $\vec{k}^i$ coincide with the main axes of anisotropy $\vec{a}^i$, then from the biaxial tension-compression experiment, in accordance with expressions (20), (21), the trigonal material can be distinguished from tetragonal and hexagonal ones by the presence of non-zero strain components $\varepsilon_{23}$.

In accordance with expressions (21)–(23), the identification of the type of tetragonal and hexagonal materials in the canonical axes of anisotropy can be carried out similarly to the identification of the type of cubic and isotropic materials: if the reactions of the material to loading $\mathbf{S}_4$, $\mathbf{S}_5$ are the same in tension-compression and shear experiments, then the material is hexagonal, and if the reactions are different, then the material is tetragonal.
4.3. The program of experimental identification of triclinic, monoclinic and rhombic materials. In this case to identify the type of material, it is necessary to conduct three experiments on shear in planes defined by the principal axes of anisotropy. These experiments can be performed as tension-compression in the direction of pairs of vectors lying in the plane of the two main axes of anisotropy and rotated around the third axis by the angle 45°.

At the shear in the plane of the vectors $\vec{a}^1$, $\vec{a}^2$ with the stress tensor $S_5 = t_5 (\vec{a}^1 \vec{a}^3 + \vec{a}^3 \vec{a}^1)$, following strains arise:

in the triclinic material

$$\varepsilon_5^{(t)} = 2t_5 \left[ C_{1112} \vec{a}^1 \vec{a}^1 + C_{2212} \vec{a}^2 \vec{a}^2 + C_{3312} \vec{a}^3 \vec{a}^3 + \right.$$ \hspace{2cm} (24)

$$+ C_{1212} \left( \vec{a}^1 \vec{a}^2 + \vec{a}^2 \vec{a}^1 \right) + C_{2312} \left( \vec{a}^2 \vec{a}^3 + \vec{a}^3 \vec{a}^2 \right) + C_{3112} \left( \vec{a}^1 \vec{a}^3 + \vec{a}^3 \vec{a}^1 \right) \right],$$

in the monoclinic material

$$\varepsilon_5^{(m)} = 2t_5 \left[ C_{1112} \vec{a}^1 \vec{a}^1 + C_{2222} \vec{a}^2 \vec{a}^2 + C_{3322} \vec{a}^3 \vec{a}^3 + C_{1212} \left( \vec{a}^1 \vec{a}^2 + \vec{a}^2 \vec{a}^1 \right) \right],$$

in the rhombic material

$$\varepsilon_5^{(r)} = 2t_5 C_{1212} \left( \vec{a}^1 \vec{a}^2 + \vec{a}^2 \vec{a}^1 \right) = 2C_{1212} S_5.$$ (26)

At the shear in the plane of the vectors $\vec{a}^2$, $\vec{a}^3$ the stress tensor $S_6 = t_6 (\vec{a}^2 \vec{a}^3 + \vec{a}^3 \vec{a}^2)$ leads to strains:

in the triclinic material

$$\varepsilon_6^{(t)} = 2t_6 \left[ C_{1123} \vec{a}^1 \vec{a}^1 + C_{2223} \vec{a}^2 \vec{a}^2 + C_{3323} \vec{a}^3 \vec{a}^3 + \right.$$ \hspace{2cm} (27)

$$+ C_{1223} \left( \vec{a}^1 \vec{a}^2 + \vec{a}^2 \vec{a}^1 \right) + C_{2323} \left( \vec{a}^2 \vec{a}^3 + \vec{a}^3 \vec{a}^2 \right) + C_{3123} \left( \vec{a}^1 \vec{a}^3 + \vec{a}^3 \vec{a}^1 \right) \right],$$

in the monoclinic material

$$\varepsilon_6^{(m)} = 2t_6 \left[ C_{2323} \left( \vec{a}^2 \vec{a}^3 + \vec{a}^3 \vec{a}^2 \right) + C_{3123} \left( \vec{a}^1 \vec{a}^3 + \vec{a}^3 \vec{a}^1 \right) \right],$$

in the rhombic material

$$\varepsilon_6^{(r)} = 2t_6 C_{2323} \left( \vec{a}^2 \vec{a}^3 + \vec{a}^3 \vec{a}^2 \right) = 2C_{2323} S_6.$$ (29)

Stress tensor $S_7 = t_7 (\vec{a}^1 \vec{a}^3 + \vec{a}^3 \vec{a}^1)$ causes deformations:

in the triclinic material

$$\varepsilon_7^{(t)} = 2t_7 \left[ C_{1131} \vec{a}^1 \vec{a}^1 + C_{2231} \vec{a}^2 \vec{a}^2 + C_{3331} \vec{a}^3 \vec{a}^3 + \right.$$ \hspace{2cm} (30)

$$+ C_{1231} \left( \vec{a}^1 \vec{a}^2 + \vec{a}^2 \vec{a}^1 \right) + C_{2331} \left( \vec{a}^2 \vec{a}^3 + \vec{a}^3 \vec{a}^2 \right) + C_{3131} \left( \vec{a}^1 \vec{a}^3 + \vec{a}^3 \vec{a}^1 \right) \right],$$

in the monoclinic material

$$\varepsilon_7^{(m)} = 2t_7 \left[ C_{2331} \left( \vec{a}^2 \vec{a}^3 + \vec{a}^3 \vec{a}^2 \right) + C_{3131} \left( \vec{a}^1 \vec{a}^3 + \vec{a}^3 \vec{a}^1 \right) \right],$$

in the rhombic material

$$\varepsilon_7^{(r)} = 2t_7 C_{3131} \left( \vec{a}^1 \vec{a}^3 + \vec{a}^3 \vec{a}^1 \right) = 2C_{3131} S_7.$$ (32)

By comparing the measured components of the strain tensors in three experiments with the components calculated by formulae (24)–(32), it is possible to determine the type of anisotropic material. In triclinic material, in response to loading $S_5$, $S_6$, $S_7$ deformations of a general form occur, in a monoclinic material some components of the strains are equal to zero. In rhombic material, strain tensors are similar to stress tensors.
5. Conclusion
The program consisting of three experiments on uniaxial compression of cubic samples to determine the orientation of the main axes of anisotropy of the material is proposed. The program includes a series of compression experiments to determine the orientation of the principal axes of anisotropy of the material and additional experiments to find the orientation of the canonical axes of anisotropy. For all cases when the principal anisotropy axes found from the indicated experiments may not coincide with the canonical axes of the anisotropy of the material, additional experimental programs are developed to establish the position of the canonical axes of anisotropy in the material. Experimental programs are developed for all classes of elastic symmetry that make it possible to unambiguously establish the type of anisotropic material from the results of experiments in the canonical axes of anisotropy. Proposed experimental programs can provide preliminary determination of the type of material anisotropy.

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