COMBINATORIAL PSEUDO-TRIANGULATIONS

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Abstract. We prove that a planar graph is generically rigid in the plane if and only if it can be embedded as a pseudo-triangulation. This generalizes the main result of [4] which treats the minimally generically rigid case.

The proof uses the concept of combinatorial pseudo-triangulation, CPT, in the plane and has two main steps: showing that a certain “generalized Laman property” is a necessary and sufficient condition for a CPT to be “stretchable”, and showing that all generically rigid plane graphs admit a CPT assignment with that property.

Additionally, we propose the study of combinatorial pseudo-triangulations on closed surfaces.

1. Introduction

The study of pseudo-triangulations in the plane was initiated recently [1, 10], but it is rapidly becoming a standard topic in Computational Geometry. This paper continues the program established in [4, 9, 11, 12], showing that pseudo-triangulations provide a missing link between planarity and rigidity in geometric graph theory. Our main result is:

Theorem 1. For a plane graph $G$, the following conditions are equivalent:

1. $G$ is generically rigid in the plane.
2. $G$ can be stretched to become a pseudo-triangulation of its vertex set with the given topological embedding.

Recall that a plane graph is a graph together with a given (non-crossing) topological embedding in the plane. It is interesting to observe that property (1) is a property of the underlying (abstract) graph, while property (2) in principle depends on the embedding. That is, our result in particular implies that pseudo-triangulation stretchability of a plane graph is independent of the embedding. Another consequence is that:

Corollary 1. The class of planar and generically rigid (in the plane) graphs coincides with the class of graphs of pseudo-triangulations.

The implication from (2) to (1) in Theorem 1 follows from [9]. Here we prove the implication from (1) to (2) in two steps, via the concept of combinatorial pseudo-triangulation introduced in Section 3. Section 5 proves that generically rigid plane graphs can be turned into combinatorial pseudo-triangulations with a certain “generalized Laman” property, and Section 4 proves that combinatorial
pseudo-triangulations with this property can be stretched. In Section 6 we initiate the study of combinatorial pseudo-triangulations on closed surfaces, which we think is an interesting topic to be developed further.

2. PSEUDO-TRIANGULATIONS AND RIGIDITY

Let $A$ be a finite point set in the Euclidean plane, in general position. A pseudo-triangle in the plane is a simple polygon with exactly three convex angles. A pseudo-triangulation of $A$ is a geometric (i.e., with straight edges) non-crossing graph with vertex set $A$, containing the convex hull edges of $A$ and in which every bounded region is a pseudo-triangle. A vertex $v$ in a geometric graph $G$ is called pointed if all the edges of $G$ lie in a half-plane supported at $v$ or, equivalently, if one of the angles incident to $v$ is greater than $180^\circ$. A pseudo-triangulation is called pointed if all its vertices are pointed. The following numerical result has been stated several times in different forms, and is crucial to some of the nice properties of pseudo-triangulations:

**Lemma 1.** Let $G$ be a non-crossing straight-line embedding of a connected graph in the plane. Let $e$, $x$ and $y$ denote the numbers of edges, non-pointed vertices and pointed vertices in $G$. Then, $e \leq 3x + 2y - 3$, with equality if and only if the embedding is a pseudo-triangulation.

**Proof.** Let $f$ be the number of bounded faces of the embedding. By Euler’s formula, $x + y + f = e + 1$. We now double-count the number of “big” and “small” angles in the embedding (that is, angles bigger and smaller than $180$ degrees, respectively). The total number of angles equals $2e$. The number of big angles equals $y$, and the number of small angles is at least $3f$ (every bounded face has at least three corners) with equality if and only if the embedding is a pseudo-triangulation. These equations give the statement. \(\square\)

Recall that a graph is generically rigid in the plane, see [3] or [13], if any generically embedded bar and joint framework corresponding to the graph has no non-trivial infinitesimal motions. Generic rigidity is a property of the graph, and not of any particular embedding. In fact, edge-minimal generically rigid graphs on a given number $n$ of vertices are characterized by Laman’s Condition: they have exactly $2n - 3$ edges and every subset of $k$ vertices spans a subgraph with at most $2k - 3$ edges, see [5]. Generically rigid graphs with $|E| = 2n - 3$ are also known as Laman graphs.

The connection between rigidity and pseudo-triangulations was first pointed out in Streinu’s seminal paper [12] where it is proved that the graphs of pointed pseudo-triangulations are minimally generically rigid graphs, i.e. Laman graphs. In [4] it was shown that a graph $G$ has a realization as a pointed pseudo-triangulation in the plane if and only if the graph is a planar Laman graph. The following theorem in [9] extends Streinu’s result to non-minimally rigid graphs and relates the number of non-pointed vertices to the degree to which a planar rigid graph is overbraced.

**Theorem 2.** Let $G$ be the graph of a pseudo-triangulation of a planar point set in general position. Then:

1. $G$ is infinitesimally rigid, hence rigid and generically rigid.
2. Every subset of $x$ non-pointed plus $y$ pointed vertices of $G$, with $x + y \geq 2$, spans a subgraph with at most $3x + 2y - 3$ edges.
Property (2) will be crucial in our proof of Theorem 1. Observe, however, that it is not a property of the graph $G$, but a property of the specific straight-line embedding of $G$.

Another remarkable connection between planarity, rigidity, and pseudo-triangulations concerns planar rigidity circuits. These are redundantly rigid graphs such that the removal of any edge leaves a minimally rigid graph. They can, by our results, be realized as pseudo-triangulations with exactly one non-pointed vertex. Rigidity circuits, or Laman circuits, have the nice property that the number of faces equals the number of vertices and that their geometric dual (which exists and is unique) is also a Laman circuit, see [2]. In [8] we show, using techniques developed here and Maxwell’s classical theory of reciprocal diagrams [6], that if $C$ is a planar Laman circuit, then $C$ and its geometric dual $C^*$ can be realized as pseudo-triangulations with the same directions for corresponding edges.

3. Combinatorial pseudo-triangulations in the plane

We now consider a combinatorial analog of pseudo-triangulations. Let $G$ be a plane graph. We call angles of $G$ the pairs of consecutive edges in the vertex rotations corresponding to the embedding. Equivalently, an angle is a vertex-face incidence. By a labelling of angles of $G$ we mean an assignment of “big” or “small” to every angle of $G$. Such a labelling is called a combinatorial pseudo-triangulation labelling (or CPT-labelling, for short) if every bounded face has exactly three angles labelled “small”, all the angles in the unbounded face are labelled “big”, and no vertex is incident to more than one “big” angle.

The embedded graph $G$ together with a CPT-labelling of its angles is called a combinatorial pseudo-triangulation, or CPT. In figures we will indicate the large angles by an arc near the vertex between the edge pair. Figure 1 shows three graphs with large angles labelled. The one in the left is not a CPT, because the exterior face has three small angles. Figure 1b is a CPT, whose bounded faces are three “triangles” and a hexagonal “pseudo-triangle”. If possible we shall draw large angles larger than 180°, small ones as angles smaller than 180° and edges as straight non-crossing segments, but it has to be observed that this is sometimes not possible, since there are non-stretchable CPT’s, such as the one in Figure 1c.

Following the terminology of true pseudo-triangulations we say that the interior faces of a CPT are pseudo-triangles with the three small angles joined by three pseudo-edges. As in the geometric case, a vertex is called pointed if there is a big angle incident to it and the CPT is called pointed if this happens at every vertex, see

![Figure 1](image-url)
Figure 1b. The following result and its proof, a straightforward counting argument using Euler’s formula, are completely analogous to the geometric situation.

**Lemma 2.** Every combinatorial pseudo-triangulation in the plane with \(x\) non-pointed and \(y\) pointed vertices has exactly \(3x + 2y - 3\) edges.

We recall here the main result of [4]:

**Theorem 3.** Given a plane graph \(G\), the following conditions are equivalent:

(i) \(G\) is generically minimally rigid (isostatic),
(ii) \(G\) satisfies Laman’s condition,
(iii) \(G\) can be labelled as a pointed CPT,
(iv) \(G\) can be stretched to a pointed pseudo-triangulation preserving the given topological embedding.

Throughout the paper, we say that a combinatorial pseudo-triangulation \(G\) has the **generalized Laman property** or is **generalized Laman** if every subset of \(x\) non-pointed plus \(y\) pointed vertices, with \(x + y \geq 2\), induces a subgraph with at most \(3x + 2y - 3\) edges. This property is inspired by Theorem 2 and it is crucial to our proof; see Theorem 4. We call it the generalized Laman property because it restricts to the Laman condition for the pointed case.

**Lemma 3.** The generalized Laman property is equivalent to requiring that every subset of \(x'\) non-pointed plus \(y'\) pointed vertices of \(G\), with \(x' + y' \leq n - 2\) be incident to at least \(3x' + 2y'\) edges.

**Proof.** Using Lemma 2, \(x\) plus \(y\) vertices satisfy the condition in the definition of generalized Laman if and only if the cardinalities, \(x'\) and \(y'\), of the complementary sets of vertices satisfy this reformulated one. \(\Box\)

Lemma 3 implies that the generalized Laman property forbids vertices of degree 1 and that vertices of degree 2 must be pointed. Moreover, any edge cutset separating the graph into two components, each containing more than a single vertex, has cardinality at least 3.

The following is a more detailed formulation of our main result, Theorem 1.

**Theorem 4.** Given a plane graph \(G\), the following conditions are equivalent:

(i) \(G\) is generically rigid,
(ii) \(G\) contains a spanning Laman subgraph,
(iii) \(G\) can be labelled as a CPT with the generalized Laman property.
(iv) \(G\) can be stretched as a pseudo-triangulation (with the given embedding and outer face).

The equivalence of (i) and (ii) is Laman’s theorem and the fact that (i) and (iii) follow from (iv) is Theorem 2. We will prove (ii)\(\Rightarrow\)(iii) (Section 5) and (iii)\(\Rightarrow\)(iv) (Section 4).

Note that having the generalized Laman property is not superfluous in the statement, even for pointed CPT’s. Figure 2a shows a combinatorial pointed pseudo-triangulation (CPPT) which is not rigid because the innermost three link chain has a motion or, equivalently, which is not Laman because those four pointed vertices are incident to only seven edges.

It is also not true that every rigid CPT has the generalized Laman property, as Figure 2b shows, where the two non-pointed vertices are incident to only five edges.
If we do not require the generalized Laman property, it is easy to show that every rigid graph possesses a CPT labelling. One can start with a minimally rigid spanning subgraph, which has a CPPT labelling by Theorem 3, and then insert edges while only relabelling angles of the subdivided face. For details see [7]. At each step one pointed vertex must be sacrificed. But it is not obvious how to preserve the generalized Laman condition in this process, even though one starts out with a Laman graph. In Section 5 we show that this can be done.

4. Generalized Laman CPT’s can be stretched

Here we prove the implication (iii)⇒(iv) of Theorem 4. Our proof is based on a partial result contained in Section 5 of [4]. To state that result we need to introduce the concept of corners of a subgraph.

Let $G = (V, E)$ be a CPT. Since $G$ comes (at least topologically) embedded in the plane, we have an embedding of every subgraph of $G$. If $H$ is such a subgraph, every angle in $H$ is a union of one or more angles of $G$. Also, $H$ comes with a well-defined outer face, namely the region containing the outer face of $G$. We say that a vertex $v$ of $H$ incident to the outer face is a corner of $H$ if either

1. $v$ is pointed in $G$ and its big angle is contained in the outer face of $H$, or
2. $v$ is non-pointed in $G$ and it has two or more consecutive small angles contained in the outer face of $H$.

The following statement is Lemma 15 in [4]:

**Lemma 4.** Let $H \subset G$ be a subgraph of a CPT and suppose that it is connected and contains all the edges interior to its boundary cycle (that is to say, $H$ is the graph of a simply connected subcomplex of $G$).

Let $e$, $x$, $y$ and $b$ denote the numbers of edges, non-pointed vertices, pointed vertices and length of the boundary cycle in $S$, respectively. Then, the number $c_1$ of corners of the first type (big angles in the outer boundary) of $H$ equals

$$c_1 = e - 3x - 2y + 3 + b.$$ 

We say that a plane graph has non-degenerate faces if the edges incident to every face form a simple closed cycle. The following statement is part of Theorem 7 of [4]:

**Theorem 5.** For a combinatorial pseudo-triangulation $G$ with non-degenerate faces the following properties are equivalent:

(i) $G$ can be stretched to become a pseudo-triangulation with the given assignment of angles.
Every subgraph of $G$ with at least three vertices has at least three corners.

With this, in this section we only need to prove that:

**Theorem 6.** Let $G$ be a generalized Laman CPT. Then:

1. Faces of $G$ are non-degenerate.
2. Every subgraph $H$ of $G$ on at least 3 vertices has at least 3 corners.

Hence, $G$ can be stretched.

**Proof.**
1. Every face in a plane graph has a well-defined contour cycle. What we need to prove is that no edge appears twice in the cycle. For the outer face this is obvious, since all angles in it are big: a repeated edge in the cycle would produce two big angles at each of its end-points. Hence, assume that there is a repeated edge $a$ in the contour cycle of a pseudo-triangle of $G$. This implies that $G \setminus a$ has two components, “one inside the other”. Let us call $H$ the interior component. We will show that the set of vertices of $H$ violates the generalized Laman property by Lemma 3.

Indeed, let $f$ and $e$ be the number of bounded faces and edges of $H$. Let $x$ and $y$ be the numbers of non-pointed and pointed vertices in it. The number of edges incident to the component is $e + 1$ (for the edge $a$). Hence, the generalized Laman property says that:

$$e + 1 \geq 3x + 2y.$$ 

On the other hand, twice the number of edges of $H$ equals the number of angles of $G$ incident to $H$ minus one (because the removal of the edge $a$ merges two angles into one). The number of small angles is at least $3f$ and the number of big angles is exactly $y$. Hence,

$$2e + 1 \geq 3f + y.$$ 

Adding these two equalities we get $3e + 2 \geq 3f + 3y + 3x$, which violates Euler’s formula $e + 1 = f + y + x$.

2. Observe first that there is no loss of generality in assuming that $H$ is connected (if it is not, the statement applies to each connected component and the number of corners of $H$ is the sum of corners of its components) and that $H$ contains all the edges of $G$ interior to its boundary cycle (because these edges are irrelevant to the concept of corner). We claim further that there is no loss of generality in assuming that the boundary cycle of $H$ is non-degenerate. Indeed, if $H$ has an edge $a$ that appears twice in its boundary cycle, its removal creates two connected components $H_1$ and $H_2$, whose numbers of vertices we denote $v_1$ and $v_2$. We claim that each $H_i$ contributes at least $\min\{v_i, 2\}$ corners to $H$. Indeed, if $v_i$ is 1 or 2, then all vertices of $H_i$ are corners in $H$. If $v_i \geq 3$, then $H_i$ has at least three corners and all but perhaps one are corners in $H$. Hence, $H$ has at least $\min\{v_1, 2\} + \min\{v_2, 2\} \geq \min\{v_1 + v_2, 3\}$ corners, as desired.

Hence, we assume that $H$ consists of a simple closed cycle plus all the edges of $G$ interior to it.

Let $y$, $x$, $e$, and $b$ be the numbers of pointed vertices, non-pointed vertices, edges and boundary vertices of $H$, respectively. Let $V$ be the set of vertices of $H$ which are either interior to $H$ or boundary vertices, but not corners. $V$ consists of $x + y - c_1 - c_2$ vertices, where $c_1$ and $c_2$ are the corners of type 1 and 2 of $H$, respectively. Hence, Lemma 3 implies that the number of edges incident to $V$ is at least

$$2(y - c_1) + 3(x - c_2) = 3x + 2y - 2c_1 - 3c_2.$$
On the other hand, the edges incident to \( V \) are the \( e-b \) interior edges of \( H \) plus at most two edges per each boundary non-corner vertex. Hence,

\[
3x + 2y - 2c_1 - 3c_2 \leq e - b + 2(b - c_1 - c_2),
\]
or, equivalently,

\[
c_2 \geq 3x + 2y - e - b.
\]

Using Lemma 4 this gives \( c_1 + c_2 \geq 3 \).

**Corollary 2.** The following properties are equivalent for a combinatorial pseudo-triangulation \( G \):

(i) \( G \) can be stretched to become a pseudo-triangulation (with the given assignment of angles).

(ii) \( G \) has the generalized Laman property.

(iii) \( G \) has non-degenerate faces and every subgraph of \( G \) with at least three vertices has at least three corners.

**Proof.** The implications (i) \( \Rightarrow \) (ii), (ii) \( \Rightarrow \) (iii), and (iii) \( \Rightarrow \) (i), are, respectively, Theorems 2, 6 and 5.

To these three equivalences, Theorem 7 of [4] adds a fourth one: that a certain auxiliary graph constructed from \( G \) is 3-connected in a directed sense. That property was actually the key to the proof of (iii) \( \Rightarrow \) (i), in which the stretching of \( G \) is obtained using a directed version of Tutte’s Theorem saying that 3-connected planar graphs can be embedded with convex faces.

5. Obtaining Generalized Laman CPT-labellings

**Theorem 7.** The angles of a generically rigid plane graph can be labelled so that the labelling is a CPT satisfying the generalized Laman condition.

The proof of the theorem proceeds by induction on the number \( n \) of vertices. As base case \( n = 3 \) suffices. Consider a generically rigid graph \( G \) with more than three vertices. Since \( G \) is generically rigid it contains a Laman spanning subgraph \( L \). Let \( v \) be a vertex of minimal degree in \( L \). Vertices in \( L \) have minimum degree at least two and \( L \) has fewer than \( 2n \) edges (where \( n \) is the number of vertices), so \( v \) has degree two or three:

- If \( v \) has degree two in \( L \), then \( G \setminus v \) is generically rigid, because \( L \setminus v \) is a spanning Laman subgraph of it. By inductive hypothesis, \( G \setminus v \) has a generalized Laman CPT labelling. Lemma 5 shows that \( G \) has one as well.

- If \( v \) has degree three in \( L \), then \( G \setminus v \) is either generically rigid (and then we proceed as in the previous case) or it has one degree of freedom. If the latter happens there must be two neighbors \( a \) and \( b \) of \( v \) such that if we insert the edge \( e = ab \) into \( G \setminus v \) we get again a generically rigid graph. Since the plane embedding of \( G \) induces a plane embedding of \( G' := G \setminus v \cup e \), we have by inductive hypothesis a generalized Laman CPT labelling of \( G' \). Lemma 6 shows that \( a \) and \( b \) can be chosen so that there is a CPT labelling of \( G' \) extending to one of \( G \).

**Lemma 5.** If \( G \setminus v \) is generically rigid, then every generalized Laman CPT labelling of \( G \setminus v \) extends to one of \( G \).
Proof. Let $T$ be the region of $G \setminus v$ where $v$ needs to be inserted. This region will either be a pseudo-triangle (of the CPT labelling of $G$) or the exterior region of the embedding. For now, we assume that $T$ is a pseudo-triangle. At the end of the proof we mention how to proceed in the (easier) case where $T$ is the exterior region.

Let $a$, $b$ and $c$ be the three corners of $T$. There are the following cases:

(a) If there is a pseudo-edge, say $ab$, containing at least two neighbors of $v$, then let $a'$ and $b'$ be the neighbors closest to $a$ and $b$ on that pseudo-edge. We may have $a = a'$ or $b = b'$, but certainly $a' \neq b'$. We separate two subcases:

(a.1) If all the neighbors of $v$ are on the pseudo-edge $ab$, then let us label $v$ as pointed, with the big angle in the face containing $c$. The vertices which are not neighbors of $v$ keep the angles they had in $G \setminus v$. The neighbors of $v$ are labelled with both angles small, except that if $a = a'$ (respectively, $b \neq b'$) then $a'$ (resp. $b'$) gets a big angle in the face containing $a$ (resp. $b$). See Figure 3(a.1).

(a.2) If there is a neighbor of $v$ not on $ab$, let $c'$ be one of them, chosen closest to $c$ (it may be $c$ itself). We label $v$ as non-pointed, and the remaining angles as before except if $c' \neq c$, we also put a big angle at $c'$, in the face containing $c$. See Figure 3(a.2).

(b) If no pseudo-edge contains two neighbors, then $v$ has either two or three neighbors. The labelling is as in Figure 3(b), with one of the edges from $v$ removed (and hence $v$ pointed) in the case of only two neighbors. One of the corners of $T$ may coincide with the corresponding neighbor of $v$. In this case that neighbor will get two small angles.

In all cases, the CPT labelling has the property that every vertex that was non-pointed on $G \setminus v$ will remain non-pointed in $G$. This in turn has the following consequence: let $l$ be the number of neighbors of $v$ that were pointed in $G \setminus v$ and are not pointed in $G$. By the count of edges in a CPT, the degree $d$ of $v$ equals $l + 2$ if $v$ is pointed and $l + 3$ if $v$ is not pointed. This relation follows also from the above case study. The neighbors of $v$ that keep their status are precisely the points $a'$, $b'$ and (if it exists) $c'$ in each case.

This is crucial in order to prove the generalized Laman property, which we now do. Let $S$ be a subset of vertices of $G \setminus v$. Since the subgraph induced by $S$ on $G$ and $G \setminus v$ is the same, and since no vertex changed from non-pointed to pointed,
the generalized Laman property of \( S \) in \( G \setminus v \) implies the same for \( S \) in \( G \). But we also need to check the property for \( S \cup v \). For this, by Lemma 3 it will be enough if the number of neighbors of \( v \) in \( S \) that did not change from pointed to non-pointed is at most two if \( v \) is pointed and at most three if \( v \) is non-pointed. This follows from the above equations \( d = l + 2 \) and \( d = l + 3 \) respectively.

As promised, we now address the case where \( T \) is the exterior region of the embedding of \( G \setminus v \). This case can actually be considered a special case of (a.1) above, since the exterior region has only “one pseudo-edge”. And, indeed, a labelling similar to the one shown in Figure 3(a.1) works in this case, where the arc \( a'b' \) now should be understood as the segment of the boundary of the exterior region of \( G \setminus v \) that becomes interior in \( G \).

**Lemma 6.** Let \( v \) be a vertex of degree three in a Laman spanning subgraph of \( G \) and let \( e = ab \) be an edge between two neighbors of \( v \) such that \( G' := G \setminus v \cup e \) is generically rigid.

1. Every CPT labelling of \( G' \) extends to one of \( G \).
2. If \( a \) and \( b \) are consecutive neighbors of \( v \), then every generalized Laman CPT labelling of \( G' \) extends to a generalized Laman CPT labelling of \( G \).
3. If \( a \) and \( b \) are at “distance two” among neighbors of \( v \), that is, if there is a vertex \( w \) such that \( a, w \) and \( b \) are consecutive neighbors of \( v \), then either
   1. every generalized Laman CPT labelling of \( G' \) extends to a generalized Laman CPT labelling of \( G \), or
   2. there is a generically rigid subgraph \( H \) of \( G \setminus v \) containing \( w \) and with \( v \) lying inside a bounded region of \( H \).

It should be clarified what we mean by “extends” here. We mean that all the labels of angles common to \( G \) and \( G' \) have the same status. We exclude from this the angles at the end-points of the edge \( ab \), which may change status, even if these angles could in principle be considered to survive in \( G \), split into two edges \( av \) and \( vb \).

**Proof.** 1. As in the previous Lemma, we start with a generalized Laman CPT labelling of \( G' = G \setminus v \cup e \). Let \( T_1 \) and \( T_2 \) be the two pseudo-triangles containing \( e \). (As in the previous Lemma, the case where one of them, say \( T_1 \), is the exterior region can be treated as if \( T_1 \) was a pseudo-triangle with all neighbors of \( v \) in the same pseudo-edge). We call \( a_i, b_i \) and \( c_i \) the vertices of \( T_i \), in such a way that the pseudo-edge from \( a_i \) to \( b_i \) contains \( a \) and \( b \), in this order. Clearly, \( a \) coincides with at least one of \( a_1 \) and \( a_2 \), and \( b \) with one of \( b_1 \) and \( b_2 \). Figure 4 shows the four possibilities, modulo exchange of all \( a \)'s and \( b \)'s or all \( 1 \)'s and \( 2 \)'s. In all cases we have drawn the union as a pseudo-quadrilateral even if in parts (a) and (b) a pseudo-triangle would in principle be possible. But that would actually imply that \( G \setminus v \) is generically rigid (because it can be realized as a generalized Laman CPT), in which case we could use Lemma 5.

We will try to extend the labelling independently in \( T_1 \) and \( T_2 \). We concentrate on one of them, say \( T_1 \). Let \( a'_1 \) and \( b'_1 \) be the neighbors of \( v \) closest to \( a_1 \) and \( b_1 \), respectively, on the pseudo-edge \( a_1 b_1 \). Observe that \( a'_1 \) will coincide with \( a_1 \) if \( a_1 \) is a neighbor, and will coincide with \( a \) if \( a \) is the only neighbor on the path \( a a_1 \). If not all the neighbors of \( v \) in \( T_1 \) are on the pseudo-edge \( a_1 b_1 \), let \( c'_1 \) be one which is closest to \( c_1 \) (possibly \( c_1 \) itself). We assign labels as follows: All non-neighbors of \( v \) keep their labels. All the angles at \( v \) and at neighbors of \( v \) are labelled small,
with the following exceptions: If \( a' \neq a_1 \) (respectively, \( b' \neq b_1 \) or \( c' \neq c_1 \)) the angle at \( a' \) on the edge to \( a_1 \) is big (respectively, the angle at \( b' \) on the edge to \( b_1 \) or the angle at \( c' \) on the edge to \( c_1 \)). Also, if all neighbors are on the pseudo-edge \( a_1b_1 \), then the angle at \( v \) on the pseudo-triangle \( a_1b_1c_1 \) is labelled big. Figure 5 schematically shows the two cases. For future reference, observe that the neighbors of \( v \) whose status does not change are precisely \( a' \), \( b' \) and, unless \( v \) gets a big angle, \( c' \).

This clearly produces a pseudo-triangulation of \( T_1 \), and we use the same idea in \( T_2 \). Observe also that we have not put big angles where there were none before. In particular, no vertex other than perhaps \( v \) will receive two big angles, and no vertex that was not pointed in \( G' \) will be pointed in \( G \).

But \( v \) itself may actually get two big angles, one on \( T_1 \) and one on \( T_2 \). This will happen if all neighbors of \( v \) in \( T_1 \) are on the pseudo-edge \( a_1b_1 \) and all neighbors in \( T_2 \) are on the pseudo-edge \( a_2b_2 \). In this case, since \( v \) has at least three neighbors, one of \( a'_1 \), \( a'_2 \), \( b'_1 \) or \( b'_2 \) is different from \( a \) and \( b \). Without loss of generality, suppose that \( a'_2 \) is different from \( a \). In particular \( a_2 \neq a \) and then \( a_1 = a = a'_1 \) (as in parts (b), (c) or (d) of Figure 4). In this case \( a \) has received a small angle both in \( T_1 \) and in \( T_2 \), but it was originally pointed in \( G' \). We are then allowed to change the angle of \( a \) in \( T_1 \) to be big, and that of \( v \) in the same pseudo-triangle to be small. Figure 6 shows the change. This finishes the proof of part 1 of the Lemma.

Before going into the proofs of parts 2 and 3 we make the following observations about the CPT labelling that we have just constructed:

(a) The number of neighbors of \( v \) that do not change status is three if \( v \) is pointed and four if \( v \) is non-pointed. This can be proved with a case study.
using our explicit way of labelling, but it also follows from global counts of small angles in the CPT’s $G$ and $G'$.

(b) Assume now that $G'$ has the generalized Laman property, and let us try to prove the property for $G$. Every subset $S$ of vertices of $G$ not containing $v$ satisfies the generalized Laman count in this CPT labelling: indeed, the subgraph induced by $G$ on $S$ is contained in the one induced by $G'$, and no vertex changed from non-pointed to pointed. Hence, the Laman count translates from $G'$ to $G$.

(c) But if we try to prove the generalized Laman property for $S \cup v$, we encounter a problem: Suppose that $a$ and $b$ do not both belong to $S$, so that the subgraph induced by $G$ on $S$ is the same as that induced by $G'$. Suppose also that $S$ is tight in $G'$, meaning that the subgraph induced has exactly the number of edges permitted by the generalized Laman count. Then, we need to prove that if $v$ is pointed (respectively, non-pointed) at most two (respectively, three) of the neighbors of $v$ in $S$ keep their pointedness status. But, globally, we know that three (respectively, four) of the neighbors of $v$ keep their status, so we cannot finish the proof.

However, from this analysis we get very precise information on the cases where $G$ happens not to have the generalized Laman property. Namely, if $S \cup v$ fails to satisfy the generalized Laman count, $S$ has the following four properties:

- $S$ does not contain both $a$ and $b$. Otherwise, the deleted edge $ab$ allows for one extra edge to be inserted and the count is satisfied.
- $S$ is tight in $G'$, otherwise again an extra edge is allowed to be inserted. This implies that if $G'$ is embedded as a pseudo-triangulation (via Theorem 6), then the subgraph $G'|_S$ induced by $S$ is itself a pseudo-triangulation. Indeed, let $e_S$, $x_S$ and $y_S$ be the numbers of edges, non-pointed vertices and pointed vertices of $G'|_S$. Let $x$ and $y$ be the numbers of vertices of $S$ which are non-pointed and pointed in $G'$, we have

$$e_S = 3x + 2y - 3 \geq 3x_S + 2y_S - 3.$$

Hence, $G'|_S$ is a pseudo-triangulation by Lemma 1.
- In particular, $G'|_S$ is generically rigid. Since $G'|_S$ does not contain $ab$, this implies that $S$ moves rigidly in the 1-degree of freedom (1-dof) mechanism $G \setminus v$. 

**Figure 6**
• $S$ contains all the neighbors of $v$ that did not change their status when we extended the CPT labelling. In particular, at least one of $a$ or $b$ did change its status.

2. We now suppose that $a$ and $b$ are consecutive neighbors of $v$. Suppose that $T_1$ is the pseudo-triangle in which there is no other neighbor. Certainly, $T_1$ does not impose any change of status for $a$ or $b$. So, if $a$ or $b$ change their status, this must be because of what happens in the pseudo-triangle $T_2$. There are two cases:

• If our CPT-labelling of $T_2$ only changes the status of one of $a$ or $b$, we can restore its status by the same type of change that we used when $v$ got two big angles: in the pseudo-triangle $T_1$ we change the angle of $v$ from big to small, and that of the neighbor that changed status in $T_2$ (which, clearly, had a big angle on $T_2$ and hence a small angle in $T_1$), from small to big. We now have a CPT-labelling where neither $a$ nor $b$ changed status, hence the generalized Laman property holds.

• If our CPT-labelling of $T_2$ changes the status of both $a$ and $b$, let $a'$ and $b'$ be the neighbors of $v$ in the pseudo-edge $a_2b_2$ that do not change their status by the CPT-labelling of $T_2$. We can try to apply the trick of the previous case at vertex $a$ and at vertex $b$. If the first one fails, we have a CPT-labelling where $a$, $a'$ and $b'$ did not change their status and without the generalized Laman property. By our final remarks in the proof of part 1, this implies that $a$, $a'$ and $b'$ move rigidly in the 1-dof mechanism $G \setminus v$. Similarly, if the second fails, $b$, $b'$ and $a'$ move rigidly. Hence, if both fail, $a, b, a'$ and $b'$ move rigidly in $G \setminus v$, which contradicts our initial choice of the edge $e = ab$.

3. Applying the construction of part 1 to the edge $e$, it turns out that $w$ must be one of the points that does not change status (that is, one of the points $a'_i$, $b'_i$ or $c'_i$ of Figure 5). Suppose that the resulting CPT is not generalized Laman, for some generalized Laman CPT of $G'$. We want to prove that there is a rigid component of $G \setminus v$ that includes $w$ and contains $v$ in the interior of a cycle.

By the remarks at the end of the proof of part 1, if the generalized Laman property fails in $G$, then there is an induced subgraph $S$ of $G \setminus v$ that contains all the neighbors of $v$ that did not change status (in particular, contains $w$), and which is itself a pseudo-triangulation (with respect to the CPT labelling of $G'$). If this pseudo-triangulation already contains $v$ in the interior of a pseudo-triangle, then the claim is proved. But it may happen that $v$ is in the exterior of this sub-pseudo-triangulation (see a schematic picture in Figure 7).

The crucial point now is that in the construction of part 1 the vertices that do not change status cannot all lie on the same pseudo-edge of the pseudo-quadrilateral of $G \setminus v$ containing $v$. Indeed, one of them is either $a'_1$ or $a'_2$ (this is the point marked $a'$ in Figure 7), and lies on one of the two pseudo-edges “on the $a$ side”. Another one is either $b'_1$ or $b'_2$ (marked $b'$ in Figure 7), and lies on the two pseudo-edges “on the $b$-side”.

So, if $v$ is exterior to $S$, then the boundary of $S$ contains a concave chain connecting two different pseudo-edges of the pseudo-quadrilateral. Together with the opposite part of the pseudo-quadrilateral this produces a pseudo-triangle in $G \setminus v$ that contains $v$ in its interior and all the vertices that did not change status on its boundary. This pseudo-triangle, together with everything in its exterior, is a
pseudo-triangulation, hence generically rigid (here, we are assuming that \( G' \) and, in particular, \( G \setminus v = G' \setminus e \) has been stretched, via Theorem 6).

\[
\text{Proof.} \quad \text{Let } v \text{ be a vertex of degree three in a Laman spanning subgraph of } G. \text{ Then, there is an edge } e = ab \text{ between two neighbors of } v \text{ such that } G' := G \setminus v \cup e \text{ is generically rigid and every generalized Laman CPT labelling of } G' \text{ extends to a generalized Laman CPT labelling of } G.
\]

We now look at edges of the form \( e = w_{i-1}w_{i+1} \). At least one of them must restore rigidity in \( G \setminus v \cup e \): if not, let \( G_0 \) be the graph consisting of the \( 2k \) edges \( w_{i}w_{i+1} \) and \( w_{i-1}w_{i+1} \). Our hypothesis is that \( G \setminus v \cup G_0 \) is still a 1-dof mechanism. Hence, in the rest of the proof we suppose that this is not the case.

\[
\text{Let } S \text{ be the set of neighbors of } v \text{ in } G \text{ with the property that } w_{i-1}w_{i+1} \text{ restores rigidity. In the rest of the proof we show that there is a } w_i \in S \text{ such that } e = w_{i-1}w_{i+1} \text{ is as claimed in the statement. We argue by contradiction, so assume the claim is not true. Part 3 of the previous Lemma says that then for each } w_i \in S \text{ there is a generically rigid subgraph of } G \setminus v \text{, let us denote it } H_{w_i} \text{, such that } w_i \in H_{w_i} \text{ and } v \text{ lies in the interior of a bounded face of } H_{w_i}. \text{ We now claim that the same holds for every } S' \subset S: \text{ there is a generically rigid subgraph } H_{S'} \text{ of } G \setminus v \text{ such that (1) } S' \subset H_{S'} \text{ and (2) } v \text{ lies in the interior of a bounded face of } H_{S'}. \text{ Indeed, after we know this for one-element subsets we just need to show that from } H_{S_1} \text{ and } H_{S_2}, \text{ we can construct } H_{S_1 \cup S_2}. \text{ We consider first the union of the two graphs } H_{S_1} \text{ and } H_{S_2}. \text{ It clearly contains both } S_1 \text{ and } S_2, \text{ and } v \text{ lies in the interior of a face: the intersection of the faces } F_1 \text{ of } H_{S_1} \text{ and } F_2 \text{ of } H_{S_2} \text{ that contain } v. \text{ The only problem is that } H_{S_1} \cup H_{S_2} \text{ may not be generically rigid. Since } H_{S_1} \text{ and } H_{S_2} \text{ clearly intersect (the boundaries } \partial F_1 \text{ and } \partial F_2 \text{ of } F_1 \text{ and } F_2 \text{ are two cycles around } v \text{ that must intersect because they both contain neighbors of } v \text{ in } G), \text{ if their union is not generically rigid then they intersect in a single point. This point must actually be in the two cycles } \partial F_1 \text{ and } \partial F_2. \text{ That is, the cycles “are tangent and one is inside the other”. Since } S_1 \subset \partial F_1 \text{ and } S_2 \subset \partial F_2 \text{ consist only of neighbors of } v \text{ in } G, \text{ this implies that one of } S_1 \text{ and } S_2 \text{ (say } S_1) \text{ consists of a single point } w_{i_1} \text{, which is the intersection point. In particular, } S_1 \subset H_{S_2} \text{ and we can take } H_{S_1 \cup S_2} = H_{S_2}.}
\]

\(\text{Figure 7}\)
So, taking \( S' = S \), the conclusion is that all the \( w_i \)'s such that \( w_{i-1}w_{i+1} \) restores rigidity lie in a rigid subgraph of \( G \setminus v \). Our final claim is that under these conditions all neighbors of \( v \) move rigidly in the 1-dof mechanism \( G \setminus v \), which (as above) contradicts the fact that \( G \) is rigid. Indeed, assume without loss of generality that the vertex \( w_1 \) is such that \( w_{kk}w_2 \) restores rigidity. To seek a contradiction, let \( w_{i+1} \) be the first neighbor of \( v \) (in the order \( w_1, \ldots, w_k \)) which does not move rigidly with \( w_1 \) and \( w_2 \), and let \( w_{j+1} \) be the first neighbor after \( w_i \) which does not move rigidly with \( w_{i}w_{i+1} \). In particular, both \( w_{i-1}w_{i+1} \) and \( w_{j-1}w_{j+1} \) restore rigidity and, by the above conclusion, the three vertices \( w_1, w_i \) and \( w_j \) lie in a rigid subgraph. This subgraph has two vertices in common with both \( \{w_1, \ldots, w_i\} \) and \( \{w_i, \ldots, w_j\} \), which lie respectively in two rigid subgraphs by the choice of \( w_i \) and \( w_j \). Hence, all \( \{w_1, \ldots, w_j\} \) lies in a rigid subgraph, in contradiction with the choice of \( w_i \). □

6. Pseudo-triangulations on closed surfaces

This section contains a couple of observations on the concept of pseudo-triangulations (and combinatorial ones) on closed surfaces. We believe it would be interesting to develop this concept further.

Let \( G \) be a graph embedded on some closed surface of genus \( g \). Every closed surface can be realized as the quotient of the sphere, the Euclidean plane, or the hyperbolic plane, by a discrete group of isometries, so in each case there is a well-defined notion of distance and angle. Similar to the situation in the plane, a pseudo-triangulation of the surface is a graph embedding with geodesic arcs such that every face has exactly three angles smaller than \( 180^\circ \). A combinatorial pseudo-triangulation is a (topological) embedding together with an assignment of “big” and “small” to angles such that every face has exactly three small angles and every vertex has at most one big angle. One difference with the situation in the plane is that now there is no “outer” face.

**Proposition 1.** Let \( G \) be embedded on a surface \( S \) of genus \( g \). If \( G \) possesses a combinatorial pseudo-triangular assignment then the numbers \( e, x \) and \( y \) of edges, non-pointed vertices and pointed vertices satisfy \( e = 3x + 2y + 6 + 6g \) if \( S \) is orientable, and \( e = 3x + 2y - 6 + 3g \) if \( S \) is non-orientable.

**Proof:** The number of angles equals twice the number of edges. There are \( 2e - y \) small angles. The number of small angles equals three times the number of faces, \( f \), which together with Euler’s formula \( x + y - e + f = 2 - 2g \) in the orientable case, or \( x + y - e + f = 2 - g \) in the non-orientable case yields the desired relationship between \( x, y \) and \( e \). □

The tree in Figure 1a is an example of a pointed pseudo-triangulation of the sphere. A triangular prism which can realized as a pointed pseudo-triangulation in the plane, cannot be realized as a pointed pseudo-triangulation of the sphere since, by Proposition 1, any CPT labelling of the prism for the sphere must have three non-pointed vertices. The graph of a cube, which has no CPT labelling for the plane, since it has too few edges to be rigid, can be realized as a pseudo-triangulation of the sphere, with two non-pointed vertices, see Figure 8 in which the pointed vertices are placed on the equator, and the two non-pointed vertices are at the poles. In this geometric realization the large angles are exactly \( 180^\circ \), so that pseudo-triangles are actual triangles, although the complex is not a triangulation since it is not regular.
In Figure 9 we see the well known embedding of the one-skeleton of the octahedron into the torus with all square faces. In Figure 10 we have modified this construction to embed the octahedron graph as a pointed pseudo-triangulation of the torus. The octahedron graph, which is overbraced as a framework in the plane, has no pointed pseudo-triangular embedding in either the plane or the sphere.
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