Energy and Momentum Distributions of the Magnetic Solution to \((2+1)\) Einstein-Maxwell Gravity

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Abstract

We use Møller’s energy-momentum complex in order to explicitly evaluate the energy and momentum density distributions associated with the three-dimensional magnetic solution to the Einstein-Maxwell equations. The magnetic spacetime under consideration is a one-parametric solution describing the distribution of a radial magnetic field in a three-dimensional AdS background, and representing the superposition of the magnetic field with a 2+1 Einstein static gravitational field.

Keywords: Energy-Momentum Complex, Einstein-Maxwell gravity, AdS\(_3\) spacetime.

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Introduction

One of the oldest problems in gravitation which still lacks of a definite answer is the localization of energy and momentum. Much attention has been devoted for this problematic issue. Einstein was the first to construct a locally conserved energy-momentum complex [1]. Consequently, a plethora of different energy-momentum complexes were proposed [2]-[7]. These expressions were restricted to evaluate energy distribution in quasi-Cartesian coordinates. Møller [8] proposed a new expression for an energy-momentum complex which could be utilized to any coordinate system. However, the idea of the energy-momentum complex was severely criticized for a number of reasons. Firstly, although a symmetric and locally conserved object, its nature is nontensorial and thus its physical interpretation seemed obscure [9]. Secondly, different energy-momentum complexes could yield different energy distributions for the same gravitational background [10, 11]. Thirdly, energy-momentum complexes were local objects while there was commonly believed that the proper energy-momentum of the gravitational field was only total, i.e. it cannot be localized [12]. For a long time, attempts to deal with this problematic issue were made only by proposers of quasi-local approach [13, 14].

In 1990 Virbhadra revived the interest in this approach [15]. At the same time Bondi [16] sustained that a nonlocalizable form of energy is not admissible in relativity so its location can in principle be found. Since then, numerous works on evaluating the energy distribution of several gravitational backgrounds have been completed employing the abandoned for a long time approach of energy-momentum complexes [17].

In 1996 Aguirregabiria, Chamorro and Virbhadra [18] showed that five different\(^2\) energy-momentum complexes yield the same energy distribution for any Kerr-Schild class metric. Additionally, their results were identical with the results of Penrose [20] and Tod [21] using the notion of quasi-local mass.

Later attempts to deal with this problematic issue were made (as already mentioned) by proposers of quasi-local approach. The determination as well as the computation of the quasilocal energy and quasilocal angular momentum of a (2 + 1)-dimensional gravitational background were first presented by Brown, Creighton and Mann [22]. Many attempts since then have been performed to give new definitions of quasilocal energy in General Relativity [23]. Considerable efforts have also been performed in constructing superenergy tensors [24].

In 1999 Chang, Nester and Chen [25] proved that every energy-momentum complex is

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\(^2\)Later on Virbhadra [19] came to know that Tolman’s and Einstein’s complexes which had been used in [18] were exactly the same (see footnote 1 in [19]).
associated with a Hamiltonian boundary term. Thus, the energy-momentum complexes are quasi-local and acceptable.

In this work the approach of energy-momentum complexes is implemented. The gravitational background under investigation is the (2 + 1)-dimensional rotating magnetic spacetime [26] of the Einstein-Maxwell gravity. We evaluate the energy confined in a “one-sphere” ($S^1$) of radius $r_0$ associated with the aforesaid background. Specifically, we are implementing the prescription of Møller. The specific (2 + 1)-dimensional spacetime background is described by one self-consistent integration constant, $\tilde{q}_m$. When $\tilde{q}_m = 0$, the Anti-de Sitter space is obtained. Additionally, the corresponding metric is horizonless and has no curvature singularity at the origin. The rest of the paper is organized as follows. In the first two sections we consider the concept of energy-momentum complexes in the context of General Theory of Relativity and give Møller’s prescription for the energy-momentum complex. In Section 3 we briefly present the (2 + 1)-dimensional BTZ and AdS black holes, and we give the magnetic solution to the 2+1 Einstein-Maxwell gravity, while in Section 4, using Møller’s energy-momentum complexes, we explicitly compute the energy and momentum distributions contained in a “one-sphere” of fixed radius $r_0$, as well as the effective gravitational mass of the spacetime under study. Additionally, the energy of AdS$_3$ spacetime is evaluated using Møller’s complex and the result is identical with that obtained by setting $\tilde{q}_m = 0$ in the expression for the energy associated with the magnetic solution to the 2+1 Einstein-Maxwell gravity. Finally, in Section 5 a summary of the obtained results and some concluding remarks are presented.

1 Energy-Momentum Complexes

The conservation laws of energy and momentum for an isolated, i.e., no external force acting on the system, physical system in the Special Theory of Relativity are expressed by a set of differential equations. Defining $T^\mu_\nu$ as the symmetric energy-momentum tensor of matter and of all non-gravitational fields, the conservation laws are given by

$$ T^\mu_\nu,\mu \equiv \frac{\partial T^\mu_\nu}{\partial x^\mu} = 0 $$

(1)

where

$$ \rho = T^t_t, \quad j^i = T^i_t, \quad p_i = -T^i_t $$

(2)

are the energy density, the energy current density, and the momentum density, respectively, and Greek indices run over the spacetime labels while Latin indices run over the
spatial coordinate values. Making the transition from Special to General Theory of Relativity, one adopts a simplicity principle which is called principle of minimal gravitational coupling. As a result of this, the conservation equation is now written as

$$T_{\nu,\mu} \equiv \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} T^\mu_\nu \right) - \Gamma^\kappa_{\nu\lambda} T^\lambda_\kappa = 0$$

(3)

where $g$ is the determinant of the metric tensor $g_{\mu\nu}(x)$. The conservation equation may also be written as

$$\frac{\partial}{\partial x^\mu} \left( \sqrt{-g} T^\mu_\nu \right) = \xi^\nu$$

(4)

where

$$\xi^\nu = \sqrt{-g} \Gamma^\kappa_{\nu\lambda} T^\lambda_\kappa$$

(5)

is a non-tensorial object. For $\nu = t$ this means that the matter energy is not a conserved quantity for the physical system\(^3\). From a physical point of view, this lack of energy conservation can be understood as the possibility of transforming matter energy into gravitational energy and vice versa. However, this remains an open problem and it is widely believed that in order to solve it one has to take into account the gravitational energy.

By a well-known procedure, the non-tensorial object $\xi^\nu$ can be written as

$$\xi^\nu = -\frac{\partial}{\partial x^\mu} \left( \sqrt{-g} \vartheta^\mu_\nu \right)$$

(6)

where $\vartheta^\mu_\nu$ are certain functions of the metric tensor and its first order derivatives. Therefore, the energy-momentum tensor of matter $T^\mu_\nu$ is replaced by the expression

$$\theta^\mu_\nu = \sqrt{-g} \left( T^\mu_\nu + \vartheta^\mu_\nu \right)$$

(7)

which is called energy-momentum complex since it is a combination of the tensor $T^\mu_\nu$ and a pseudotensor $\vartheta^\mu_\nu$ describing the energy and momentum of the gravitational field. The energy-momentum complex satisfies a conservation law in the ordinary sense, i.e.,

$$\theta^\mu_{\nu,\mu} = 0$$

(8)

and it can be written as

$$\theta^\mu_\nu = \chi^\mu_\nu \lambda$$

(9)

where $\chi^\mu_\nu \lambda$ are called superpotentials and are functions of the metric tensor and its first order derivatives.

\(^3\)It is possible to restore the conservation law by introducing a local inertial system for which at a specific spacetime point $\xi_\nu = 0$ but this equality by no means holds in general.
It is evident that the energy-momentum complex is not uniquely determined by the condition that its usual divergence is zero since a quantity with an identically vanishing divergence can always be added to the energy-momentum complex.

2 Møller’s Prescription

The energy-momentum complex of Møller in a four-dimensional background is given as

\[ J_{\mu}^{\nu} = \frac{1}{8\pi} \xi_{\nu, \lambda}^{\mu} \]  

where the Møller’s superpotential \( \xi_{\nu}^{\mu\lambda} \) is of the form

\[ \xi_{\nu}^{\mu\lambda} = \sqrt{-g} \left( \frac{\partial g_{\nu\sigma}}{\partial x^\kappa} - \frac{\partial g_{\nu\kappa}}{\partial x^\sigma} \right) g^{\mu\kappa} g^{\lambda\sigma} \]  

with the antisymmetric property

\[ \xi_{\nu}^{\mu\lambda} = -\xi_{\nu}^{\lambda\mu} . \]  

It is easily seen that the Møller’s energy-momentum complex satisfies the local conservation equation

\[ \frac{\partial J_{\nu}^{\mu}}{\partial x^\mu} = 0 \]  

where \( J_{0}^{0} \) is the energy density and \( J_{i}^{0} \) are the momentum density components.

Thus, in Møller’s prescription the energy and momentum for a four-dimensional background are given by

\[ P_{\mu} = \int \int \int J_{\mu}^{0} dx^1 dx^2 dx^3 \]  

and specifically the energy of the physical system in a four-dimensional background is

\[ E = \int \int \int J_{0}^{0} dx^1 dx^2 dx^3 . \]  

It should be noted that the calculations are not anymore restricted to quasi-Cartesian coordinates but can be utilized in any coordinate system.

3 AdS\(_3\) Black Holes and the Magnetic Solution to the (2 + 1) Einstein-Maxwell Gravity

In 1992 Bañados, Teitelboim, and Zanelli discovered a black hole solution (known as BTZ black hole) in (2 + 1) dimensions [27]. Till that time it was believed that no black hole
solution exists in three-dimensional spacetimes [28]. Bañados, Teitelboim, and Zanelli found a vacuum solution to Einstein gravity with a negative cosmological constant.

The starting point was the action in a three-dimensional theory of gravity

$$ S = \int d^3x \sqrt{-g} \left( R + \frac{2}{l^2} \right) $$

where the radius of curvature $l$ is related to the cosmological constant by $\Lambda = -l^{-2}$.

It is straightforward to check that Einstein’s field equations

$$ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \left( R + \frac{2}{l^2} \right) = 0 $$

are solved by the metric

$$ ds^2 = N^2(r) dt^2 - \frac{dr^2}{N^2(r)} - r^2 \left( N^\phi(r) dt + d\phi \right)^2, $$

where the squared lapse $N^2(r)$ and the angular shift $N^\phi(r)$ are given by

$$ N^2(r) = \frac{r^2}{l^2} - M + \frac{J^2}{4r^2}, \quad N^\phi(r) = -\frac{J}{2r^2} $$

with $-\infty < t < +\infty$, $0 < r < +\infty$, and $0 \leq \phi \leq 2\pi$. Since the metric satisfies Einstein’s field equations with a negative cosmological constant (see (17)), the metric is locally Anti-de Sitter

$$ ds^2 = \left( 1 + \frac{r^2}{l^2} \right) dt^2 - \frac{dr^2}{\left( 1 + \frac{r^2}{l^2} \right)} - r^2 d\phi^2 $$

and it can only differ from Anti-de Sitter space in its global properties. The two constants $M$ and $J$ are the conserved quantities mass and angular momentum, respectively. The lapse function $N(r)$ vanishes for two values of the radial coordinate $r$ given by

$$ r^2 = \frac{l^2}{2} \left( |M| - \sqrt{M^2 - \left( \frac{J}{l} \right)^2} \right) ^{1/2}. $$

\footnote{The form of the BTZ metric in quasi-Cartesian coordinates can be obtained by making the transformations

$$ x = r \cos(\phi) $$

$$ y = r \sin(\phi) $$}
The largest root, \( r_+ \), gives the black hole horizon. It is evident that in order for the horizon to exist one must have

\[
M > 0 \quad , \quad |J| \leq ML .
\]  

Therefore, negative black hole masses are excluded from the physical spectrum. There is, however, an important exceptional case. When one sets \( M = -1 \) and \( J = 0 \), the singularity, i.e., \( r = 0 \), disappears. There is neither a horizon nor a singularity to hide. The configuration is again that of Anti-de Sitter space. Thus, Anti-de Sitter emerges as a “bound state”, separated from the continuous black hole spectrum by a mass gap of one unit. For the specific case of spinless \((J = 0)\) BTZ black hole, the line element (18) takes the simple form

\[
ds^2 = \left( r^2 - M \right) dt^2 - \frac{dr^2}{r^2 - M} - r^2 d\phi^2 .\]

As it is stated in the Introduction, metric (18) of the rotating \((2 + 1)\)-dimensional BTZ black hole is not asymptotically (that is as \( r \to \infty \)) flat

\[
ds^2 = dt^2 - dr^2 - d\phi^2 .\]

The \((2+1)\)-dimensional magnetic solution to the Einstein-Maxwell field equations has been given first by Clement [29], followed by Peldán [30], Hirschmann and Welch [31] and Cataldo and Salgado [32]. In the present work, we use the generalisation to the rotating case as formulated by Dias and Lemos [33], whereby the line element is in the form

\[
ds^2 = \left( \frac{r^2}{l^2} - M \right) dt^2 - \frac{r^2}{\left( \frac{r^2}{l^2} - M \right) \left( r^2 + Q_m^2 \ln \left| \frac{r^2}{l^2} - M \right| \right)} dr^2
\]

\[
- \left( r^2 + Q_m^2 \ln \left| \frac{r^2}{l^2} - M \right| \right) d\phi^2
\]

with \( l = -1/\sqrt{\Lambda} \), the radius of a pseudo-sphere, \( \Lambda \) the cosmological constant, and \( Q_m, M \) self-consistent integration constants of the Einstein-Maxwell field equations. In the case \( Q_m = 0 \), the metric (25) reduces to the spinless three-dimensional BTZ-black hole (23). However, Cataldo et al [26] have shown recently, that the field parameter related to the
mass of the solution \( \text{(25)} \) is a pure gauge and can be rescaled to -1. Thus, the magnetic metric describing a distribution of a radial magnetic field in an AdS\(_3\) spacetime is given by the line element

\[
    ds^2 = \left( \frac{r'^2}{l^2} + 1 \right) dt'^2 - \frac{r'^2}{\left( \frac{r'^2}{l^2} + 1 \right) F'^2(r')} dr'^2 - F'^2(r') d\phi'^2
\]

\( \text{(26)} \)

with

\[
    t'^2 = \frac{\tilde{r}^2 - M l^2}{l^2} t^2
\]

\[
    r'^2 = \frac{l^2}{\tilde{r}^2 - M l^2} x^2
\]

\( \text{(27)} \)

\[
    \phi'^2 = \frac{\tilde{r}^2 - M l^2}{l^2} \phi^2
\]

where

\[
    x^2 = r^2 - \tilde{r}^2
\]

(\text{the physical spacetime holds for } r \geq \tilde{r} \text{ and } x \in [0, \infty])

and the value \( r = \tilde{r} \), for which \( g_{\phi\phi} = 0 \), satisfies the constraint

\[
    \sqrt{M} \tilde{r} < l \sqrt{M + 1}
\]

\( \text{(28)} \)

Also,

\[
    F'^2(r') = r'^2 + \tilde{q}_m^2 \ln \left( \frac{r'^2}{l^2} + 1 \right)
\]

\( \text{(29)} \)

\[
    \tilde{q}_m^2 = Q_m^2 e^{\tilde{r}^2/Q_m^2}
\]

\( \text{(30)} \)

With \( \tilde{q}_m = 0 \), the metric \( \text{(26)} \) yields the AdS\(_3\) background. However, this metric does not describe a magnetically charged three-dimensional black hole, it is horizonless, it has no curvature singularities, and it does not exhibit any signature change. According to the interpretation of Cataldo et al \[26\], a two-dimensional solenoid carrying a steady current located at spatial infinity can be considered as the source of the magnetic field given by

\[
    B(r) \sim \frac{1}{\sqrt{\frac{r^2}{l^2} + 1}}.
\]

\( \text{(31)} \)
4 Energy and Momentum Density Distributions

The aim of this section is to evaluate the effective gravitational mass of the radial magnetic field in a 2+1 Anti-de Sitter spacetime \( \text{(26)} \) using Møller’s energy-momentum complex. We first have to evaluate the superpotentials in the context of Møller’s prescription. There are four nonzero independent superpotentials

\[
\begin{align*}
\xi_{1}^{12} &= \frac{2}{l^2} \left( r^2 + \tilde{q}_m^2 \ln \left| 1 + \frac{r^2}{l^2} \right| \right) \\
\xi_{1}^{21} &= -\xi_{1}^{12} = -2 \frac{l^2}{l^2} \left( r^2 + \tilde{q}_m^2 \ln \left| 1 + \frac{r^2}{l^2} \right| \right) \\
\xi_{3}^{23} &= -2 \left( 1 + \frac{\tilde{q}_m^2}{l^2} + \frac{r^2}{l^2} \right) \\
\xi_{3}^{32} &= -\xi_{3}^{23} = 2 \left( 1 + \frac{\tilde{q}_m^2}{l^2} + \frac{r^2}{l^2} \right).
\end{align*}
\]  

(32)

By substituting the Møller’s superpotentials, as given by \( \text{(32)} \), into equation \( \text{(10)} \), one gets the energy density distribution

\[
\mathcal{J}_0^0 = \frac{r \left( 1 + \frac{\tilde{q}_m^2}{l^2} + \frac{r^2}{l^2} \right)}{2\pi l^2 \left( 1 + \frac{r^2}{l^2} \right)}
\]

while the momentum density distributions take the form

\[
\begin{align*}
\mathcal{J}_1^0 &= 0 \\
\mathcal{J}_2^0 &= 0. \tag{34} \tag{35}
\end{align*}
\]

Therefore, if we substitute equation \( \text{(33)} \) into equation \( \text{(15)} \), we get the energy of the radial magnetic field in the 2+1 Anti-de Sitter spacetime that is contained in a “sphere” of radius \( r_0 \)

\[
E(r_0) = \frac{1}{4\pi l^2} \left( \tilde{q}_m^2 + \frac{r_0^2}{l^2} \ln \left| 1 + \frac{r_0^2}{l^2} \right| \right). \tag{36}
\]

This result is the effective gravitational mass \( (E = M_{eff}) \) of the spacetime under study. Furthermore, if we independently evaluate the energy density distribution for the AdS\(_3\) metric using the Møller’s energy-momentum complex we get

\[
\mathcal{J}_0^0 = \frac{r}{2\pi l^2} \tag{37}
\]

and by integration we get the energy of the AdS\(_3\) spacetime in a “sphere” of radius \( r_0 \)

\[
E(r_0) = \frac{1}{4\pi l^2} r_0^2. \tag{38}
\]
It is evident that by setting $\tilde{q}_m = 0$ in equation (36) we get the energy of the 2+1 Anti-de Sitter spacetime (see (38)) which was independently evaluated using Møller’s prescription.

5 Conclusions

In this work, we have explicitly calculated the energy and momentum densities associated with the magnetic solution to the (2 + 1) Einstein-Maxwell gravity. The specific gravitational background describes the radial magnetic field in an AdS$_3$ spacetime which is horizonless and it does not have any curvature singularities. The magnetic solution depends on a “charge” $\tilde{q}_m$. By setting this “charge” to zero the magnetic solution to the (2 + 1) Einstein-Maxwell gravity becomes the pure AdS$_3$ spacetime. We employed Møller’s prescription in order to compute the effective gravitational mass, i.e. the total energy, contained in a “sphere” of radius $r_0$ in the aforementioned gravitational background. It should be stressed that the concept of effective gravitational mass is related to the repulsive effects of gravitation. Additionally, the corresponding momenta are zero due to the vanishing momentum density distributions. Furthermore, we have independently computed the energy of the pure AdS$_3$ spacetime using again Møller’s prescription. This result is identical to that obtained when setting $\tilde{q}_m = 0$ in the expression for the energy of the magnetic solution to the (2 + 1) Einstein-Maxwell gravity.

However, it should be pointed out that in the last years due to the AdS/CFT correspondence there has been much progress in obtaining finite stress energy tensors of asymptotically AdS spacetimes$^5$. The gravitational stress energy tensor is in general infinite due to the infinite volume of the spacetime. In order to find a meaningful definition of gravitational energy one should subtract the divergences. The proposed prescriptions so far were ad hoc in the sense that one has to embed the boundary in some reference spacetime. The important drawback of this method is that it is not always possible to find the suitable reference spacetime. Skenderis and collaborators$^6$ [35,39–41], and also Balasubramanian and Kraus [42], described and implemented a new method which provides an intrinsic definition of the gravitational stress energy tensor. The computations are universal in the sense that they apply to all asymptotically AdS spacetimes. Therefore, it is nowadays right to state that the issue of the gravitational stress energy tensor for any

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$^5$For a short review see [34].

$^6$Right after the first work of Henningson and Skenderis [35], Nojiri and Odintsov [36] calculated a finite gravitational stress energy tensor for an asymptotically AdS spacetime where the dual conformal field theory is dilaton coupled. Furthermore, Nojiri and Odintson [37], and Ogushi [38] found well-defined gravitational stress energy tensors for asymptotically AdS spacetimes in the framework of higher derivative gravity and of gauged supergravity with single dilaton respectively.
asymptotically AdS spacetime has been thoroughly understood. Finally, our results presented here provide evidence in support of Lessner’s statement [43] for the significance of Møller’s prescription.

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