Isoperimetric Bounds for Lower Order Eigenvalues

Fuquan Fang and Changyu Xia

Abstract

New isoperimetric inequalities for lower order eigenvalues of the Laplacian on closed hypersurfaces, of the biharmonic Steklov problems and of the Wentzell-Laplace on bounded domains in a Euclidean space are proven. Some open questions for further study are also proposed.

1 Introduction and the main results

Let $(M, g)$ be a closed Riemannian manifold of dimension $\geq 2$. The spectrum of the Laplace operator on $M$ provides a sequence of global Riemannian invariants

$$0 = \lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \leq \cdots \rightarrow \infty.$$ 

We adopt the convention that each eigenvalue is repeated according to its multiplicity. An important issue in spectral geometry is to obtain good estimates for these and other eigenvalues in terms of the geometric data of the manifold $M$ such as the volume, the diameter, the curvature, the isoperimetric constants, etc. See [1], [2], [10], [13], [31] for references.

On the other hand, after the seminal works of Bleecker-Weiner [4] and Reilly [30], the following approach is developed: the manifold $(M, g)$ is immersed isometrically into another Riemannian manifold. One then gets good estimates for $\lambda_k(M)$, mostly for $\lambda_1(M)$, in terms of the extrinsic geometric quantities of $M$. See for example [4], [15], [16], [23], [24], [35], [37]. Especially relevant for us is the quoted work of Reilly [30], where he obtained the following remarkable isoperimetric inequality for the first positive eigenvalue $\lambda_1(M)$ in the case that $M$ is embedded as a hypersurface bounding a domain $\Omega$ in $\mathbb{R}^n$:

$$\lambda_1(M) \leq \frac{n-1}{n^2} \cdot \frac{|M|^2}{|\Omega|^2}. \quad (1.1)$$

Here $|M|$ and $|\Omega|$ denote the Riemannian $(n-1)$-volume of $M$ and the Riemannian $n$-volume of $\Omega$, respectively. Moreover, equality holds in (1.1) if and only if $M$ is a round sphere. Our first result improves (1.1) to the sum of the first $n$ non-zero eigenvalues of the Laplace operator on $M$.

2000 Mathematics Subject Classification: 35P15; 53C40; 58C40, 53C42.

Key words and phrases: Isoperimetric inequalities, eigenvalues, Laplacian, biharmonic Steklov problems, Wentzell-Laplace operator.
Theorem 1.1 Let $M$ be a closed embedded hypersurface bounding a domain $\Omega$ in $\mathbb{R}^n$.
Then the first $n$ non-zero eigenvalues of the Laplacian on $M$ satisfy
\[
\sum_{i=1}^{n} \lambda_i \leq \frac{n-1}{n} \cdot \frac{|M|^2}{|\Omega|^2} \tag{1.2}
\]
and
\[
\sum_{i=1}^{n} \lambda_i \leq \frac{(n-1)\sqrt{|M|}}{|\Omega|} \left( \int_M H^2 \right)^{1/2}, \tag{1.3}
\]
where $H$ stands for the mean curvature of $M$. Moreover, equality holds in either of (1.2) and (1.3) if and only if $M$ is a sphere.

In the second part of this paper we study eigenvalues of fourth order Steklov problems. Let $\Omega$ be an $n$-dimensional compact Riemannian manifold with boundary and $\Delta$ and $\Delta$ be the Laplace operators on $\Omega$ and $\partial\Omega$, respectively. Consider the eigenvalue problem
\[
\begin{cases}
\Delta^2 u = 0 \text{ in } \Omega, \\
\partial_{\nu} u = \partial_{\nu}(\Delta u) + \xi u = 0 \text{ on } \partial\Omega,
\end{cases}
\tag{1.4}
\]
where $\partial_{\nu}$ denotes the outward unit normal derivative. This problem was first discussed by J. R. Kuttler and V. G. Sigillito [28] in the case where $\Omega$ is a bounded domain in $\mathbb{R}^n$. The eigenvalue problem (1.4) is important in biharmonic analysis and elastic mechanics. In the two dimensional case, it describes the deformation $u$ of the linear elastic supported plate $\Omega$ under the action of the transversal exterior force $f(x) = 0$, $x \in \Omega$ with Neumann boundary condition $\partial_{\nu} u|_{\partial \Omega} = 0$ (see, [33], [39]). In addition, the first nonzero eigenvalue $\xi_1$ arises as an optimal constant in an a priori inequality (see [28]). The eigenvalues of the problem (1.4) form a discrete and increasing sequence (counted with multiplicity):
\[
0 = \xi_0 < \xi_1 \leq \xi_2 \leq \cdots + \infty. \tag{1.5}
\]
Let $D_k$ be the space of harmonic homogeneous polynomials in $\mathbb{R}^n$ of degree $k$ and denote by $\mu_k$ the dimension of $D_k$, $k = 0, 1, \cdots$. For the $n$-dimensional Euclidean ball with radius $R$, the eigenvalues of (1.4) are $\xi_k = k^2(n + 2k)/R^2$, $k = 0, 1, 2, \cdots$, and the multiplicity of $\xi_k$ is $\mu_k$ (see [39], Theorem 1.5). When $\Omega$ has nonnegative Ricci curvature with strictly convex boundary, a lower bound for $\xi_1(\Omega)$ has been given in [35]. On the other hand, an isoperimetric upper bound for $\xi_1(\Omega)$ has been proven for the case where $\Omega$ is a bounded domain in $\mathbb{R}^n$ (see [39], Theorem 1.6). We have an isoperimetric inequality for the sum of the reciprocals of the first $n$ nonzero eigenvalues of the problem (1.4) on bounded domains in $\mathbb{R}^n$.

Theorem 1.2 Let $\Omega$ be a bounded domain with smooth boundary in $\mathbb{R}^n$. Then the first $n$ nonzero eigenvalues of the problem (1.4) satisfy
\[
\sum_{i=1}^{n} \frac{1}{\xi_i} \geq \frac{n^2|\Omega|}{(n+2)|\partial\Omega|} \left( \frac{|\Omega|}{\omega_n} \right)^{2/n}, \tag{1.6}
\]
with equality holding if and only if $\Omega$ is a ball, where $\omega_n$ denotes the volume of the unit ball in $\mathbb{R}^n$. 
Now we come to another Steklov problem for the bi-harmonic operator. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $\tau$ a positive constant. Denote by $\nabla^2$ and $\nabla$ the Hessian on $\mathbb{R}^n$ and the gradient operator on $\Omega$, respectively. Consider the following Steklov problem of fourth order

$$
\begin{cases}
\Delta^2 u - \tau \Delta u = 0, & \text{in } \Omega, \\
\frac{\partial^2 u}{\partial \nu^2} = 0, & \text{on } \partial \Omega, \\
\tau \frac{\partial u}{\partial \nu} - \text{div} \, M(\nabla^2 u(\nu)) - \frac{\partial \Delta u}{\partial \nu} = \lambda u, & \text{on } \partial \Omega.
\end{cases}
$$

(1.7)

This problem has a discrete spectrum which can be listed as

$$0 = \lambda_{0,\tau} < \lambda_{1,\tau} \leq \cdots \leq \lambda_{k,\tau} \nearrow +\infty.$$ 

The eigenvalue 0 is simple and the corresponding eigenfunctions are constants. Let $u_0, u_1, \ldots, u_k, \ldots$ be the eigenfunctions of problem (1.7) corresponding to the eigenvalues $0 = \lambda_{0,\tau}, \lambda_{1,\tau}, \cdots, \lambda_{k,\tau}, \cdots$. For each $k = 1, \ldots$, we have the following variational characterization

$$\lambda_{k,\tau} = \min \left\{ \frac{\int_\Omega (|\nabla^2 u|^2 + \tau |\nabla u|^2)}{\int_{\partial \Omega} u^2} \middle| u \in H^2(\Omega), u \neq 0, \int_{\partial \Omega} uu_j = 0, j = 0, \ldots, k-1 \right\}. \quad (1.8)$$

The eigenvalues and eigenfunctions on the ball in $\mathbb{R}^n$ have been determined by Buoso-Provenzano in [9]. In particular, if $B^n_R$ is the ball of radius $R$ centered at the origin in $\mathbb{R}^n$, then

$$\lambda_{1,\tau}(B^n_R) = \lambda_{2,\tau}(B^n_R) = \cdots = \lambda_{n,\tau}(B^n_R) = \frac{\tau}{R} \quad (1.9)$$

and the corresponding eigenspace is generated by $\{x_1, \ldots, x_n\}$. Buoso and Provenzano [9] also proved the following isoperimetric inequality for the sums of the reciprocals of the first $n$ non-zero eigenvalues:

$$\sum_{i=1}^n \frac{1}{\lambda_{i,\tau}(\Omega)} \geq \frac{n}{\tau} \left( \frac{|\Omega|}{\omega_n} \right)^{1/n} \quad (1.10)$$

with equality holding if and only if $\Omega$ is a ball. Further study for the eigenvalues of the problem (1.7) has been made in [8], [15], [39], etc. Our next result is an isoperimetric inequality for the sum of the first $n$ non-zero eigenvalues of the problem (1.7).

**Theorem 1.3** Let $\Omega$ be a bounded domain with smooth boundary $\partial \Omega$ in $\mathbb{R}^n$. Denoting by $\lambda_{i,\tau}$ the $i$-th eigenvalue of the (1.7), we have

$$\sum_{j=1}^n \lambda_{j,\tau} \leq \frac{\tau |\partial \Omega|}{|\Omega|}. \quad (1.11)$$

Equality holds in (1.11) if and only if $\Omega$ is a ball.

The final part of the present paper concerns the eigenvalue problem with Wentzell boundary conditions:

$$
\begin{cases}
\Delta u = 0 & \text{in } \Omega, \\
-\beta \Delta u + \partial_{\nu} u = \lambda u & \text{on } \partial \Omega,
\end{cases}
$$

(1.12)
where $\beta$ is a nonnegative constant, $\Omega$ is a compact Riemannian manifold of dimension $n \geq 2$ with non-empty boundary, $\Delta$ and $\overline{\Delta}$ denote the the Laplacian on $\Omega$ and $\partial \Omega$, respectively. When $\beta = 0$, (1.12) becomes the Steklov problem:

\[
\begin{aligned}
\begin{cases}
\Delta u = 0 & \text{in } \Omega, \\
\partial_n u = pu & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\]

which has been studied extensively (see [6], [7], [11], [18]-[22], [27], [28], [32], [36], [39], [40]).

The spectrum of the problem (1.12) consists in an increasing sequence

$\lambda_{0,\beta} = 0 < \lambda_{1,\beta} \leq \lambda_{2,\beta} \leq \cdots \leq \infty$,

with corresponding real orthonormal (in $L^2(\partial \Omega)$ sense) eigenfunctions $u_0, u_1, u_2, \cdots$.

Consider the Hilbert space

\[
H(\Omega) = \{ u \in H^1(\Omega), \text{Tr}_{\partial \Omega}(u) \in H^1(\partial \Omega) \},
\]

where Tr$\partial \Omega$ is the trace operator. We define on $H(\Omega)$ the two bilinear forms

\[
A_\beta(u, v) = \int_\Omega \nabla u \cdot \nabla v + \beta \int_{\partial \Omega} \nabla u \cdot \nabla v, \\
B(u, v) = \int_{\partial \Omega} uv,
\]

where, $\nabla$ and $\overline{\nabla}$ are the gradient operators on $\Omega$ and $\partial \Omega$, respectively. Since we assume that $\beta$ is nonnegative, the two bilinear forms are positive and the variational characterization for the $k$-th eigenvalue is

\[
\lambda_{k,\beta} = \min \left\{ \frac{A_\beta(u, u)}{B(u, u)}, u \in H(\Omega), u \neq 0, \int_{\partial \Omega} uu_i = 0, i = 0, \cdots, k - 1 \right\}.
\]

When $k = 1$, the minimum is taken over the functions orthogonal to the eigenfunctions associated to $\lambda_{0,\beta} = 0$, i.e., constant functions.

If $\Omega = B^n_R$, then [12]

$\lambda_{1,\beta} = \lambda_{2,\beta} = \cdots = \lambda_{n,\beta} = \frac{(n - 1)\beta + R}{R^2}$

and the corresponding eigenspace is generated by $\{x_i, i = 1, \cdots, n\}$. For the Steklov problem (1.13), Brock [6] showed that if $\Omega$ is a bounded domain with smooth boundary in $\mathbb{R}^n$, then the first $n$ nonzero eigenvalues of $\Omega$ satisfy

\[
\sum_{i=1}^n \frac{1}{p_i(\Omega)} \geq n \left( \frac{|\Omega|}{\omega_n} \right)^{\frac{1}{n}},
\]

with equality holding if and only if $\Omega$ is a ball. Brock’s theorem has been generalized to the eigenvalues of the problem (1.12) in [15]. We prove

**Theorem 1.4** Let $\beta \geq 0$ and $\Omega$ be a bounded domain with smooth boundary $\partial \Omega$ in $\mathbb{R}^n$. Denote by $\lambda_1, \beta \leq \lambda_2, \beta \leq \cdots \leq \lambda_n, \beta$ the first $n$ non-zero eigenvalues of the following problem with the Wentzell boundary condition.

\[
\begin{aligned}
\begin{cases}
\Delta u = 0 & \text{in } \Omega, \\
-\beta \Delta u + \partial_n u = \lambda u & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\]

\[
\begin{aligned}
\begin{cases}
\Delta u = 0 & \text{in } \Omega, \\
-\beta \Delta u + \partial_n u = \lambda u & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
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\[
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\]

\[
\begin{aligned}
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\]

\[
\begin{aligned}
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\]

\[
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\]

\[
\begin{aligned}
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\end{cases}
\end{aligned}
\]

\[
\begin{aligned}
\begin{cases}
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\[
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\end{cases}
\end{aligned}
\]

\[
\begin{aligned}
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-\beta \Delta u + \partial_n u = \lambda u & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\]

\[
\begin{aligned}
\begin{cases}
\Delta u = 0 & \text{in } \Omega, \\
-\beta \Delta u + \partial_n u = \lambda u & \text{on } \partial \Omega,
\end{cases}
\end{aligned}
\]
Then we have
\[ \sum_{i=1}^{n} \lambda_{i,\beta} \leq \frac{|\partial \Omega|}{|\Omega|} + \frac{(n-1)\beta}{n} \cdot \frac{|\partial \Omega|^2}{|\Omega|^2}. \] (1.19)

Furthermore, equality holds in (1.19) if and only if \( \Omega \) is a ball.

Taking \( \beta = 0 \) in (1.19), we have a new isoperimetric inequality for the first \( n \) nonzero Steklov eigenvalues of a bounded domain \( \Omega \subset \mathbb{R}^n \):
\[ \sum_{i=1}^{n} p_i(\Omega) \leq \frac{|\partial \Omega|}{|\Omega|} \] (1.20)
with quality holding if and only if \( \Omega \) is a ball.

It has been conjectured by Henrot [25] that the first \( n \) nonzero Steklov eigenvalues of a bounded domain \( \Omega \subset \mathbb{R}^n \) satisfy
\[ \prod_{i=1}^{n} p_i(\Omega) \leq \frac{\omega_n}{|\Omega|} \] (1.21)
which is stronger than Brock’s inequality (1.17). If \( n = 2 \), or \( n \geq 3 \) and \( \Omega \) is convex, then (1.21) is true (see [27], [26]). This result can be extended to eigenvalues of the problem (1.12). Namely, we have

**Theorem 1.5** Let the notation be as in Theorem 1.4 and when \( n \geq 3 \), assume further that \( \Omega \) is convex. Then
\[ \prod_{i=1}^{n} \lambda_{i,\beta} \leq \left( 1 + \frac{(n-1)\beta|\partial \Omega|}{n|\Omega|} \right)^n \frac{\omega_n}{|\Omega|} \] (1.22)
with quality holding if and only if \( \Omega \) is a ball.

## 2 A Proof of Theorem 1.1

In this section, we give a

**Proof of Theorem 1.1** Let \( \Delta \) and \( \Delta \) be the Laplace operators on \( \mathbb{R}^n \) and \( M \), respectively, and let \( \{u_i\}_{i=0}^{+\infty} \) be an orthonormal system of eigenfunctions corresponding to the eigenvalues
\[ 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty \] (2.1)
of the Laplacian of \( M \), that is,
\[ \overline{\Delta}u_i = -\lambda_i u_i, \quad \int_M u_i u_j = \delta_{ij}. \] (2.2)
We have \( u_0 = 1/\sqrt{|M|} \) and for each \( i = 1, \cdots \), the Rayleigh-Ritz characterization for \( \lambda_i \) is given by
\[ \lambda_i = \min_{u \neq 0, \int_M u_i u_j = 0, j = 0, \cdots, i-1} \frac{\int_M |\nabla u|^2}{\int_M u^2}. \] (2.3)
being $\nabla$ the gradient operator on $M$.

In order to obtain good upper bound for $\lambda_i$, we need to choose nice trial functions $\phi_i$ for each of the eigenfunctions $u_i$ and insure that these are orthogonal to the preceding eigenfunctions $u_0, \ldots, u_{i-1}$. We note that the coordinate functions are eigenfunctions corresponding to the first eigenvalue of the hypersphere in $\mathbb{R}^n$. For the $n$ trial functions $\phi_1, \phi_2, \ldots, \phi_n$, we simply choose the $n$ coordinate functions:

$$\phi_i = x_i, \quad \text{for } i = 1, \ldots, n,$$

(2.4)

but before we can use these we need to make adjustments so that $\phi_i \perp \text{span}\{u_0, \ldots, u_{i-1}\}$ in $L^2(\partial\Omega)$. By translating the origin appropriately we can assume that

$$\int_M x_i = 0, \quad i = 1, \ldots, n,$$

(2.5)

that is, $x_i \perp u_0$.

Nextly we show that a rotation of the axes can be made so that

$$\int_M \phi_j u_i = \int_M x_j u_i = 0,$$

(2.6)

for $j = 2, 3, \ldots, n$ and $i = 1, \ldots, j - 1$. In fact, let us define an $n \times n$ matrix $P = (p_{ji})$, where $p_{ji} = \int_M x_j u_i$, for $i, j = 1, 2, \ldots, n$. Using the orthogonalization of Gram and Schmidt (QR-factorization theorem), one can find an upper triangle matrix $T = (T_{ji})$ and an orthogonal matrix $U = (a_{ji})$ such that $T = UP$, that is,

$$T_{ji} = \sum_{k=1}^{n} a_{jk} p_{ki} = \int_M \sum_{k=1}^{n} a_{jk} x_k u_i = 0, \quad 1 \leq i < j \leq n.$$

Letting $y_j = \sum_{k=1}^{n} a_{jk} x_k$, we have

$$\int_M y_j u_i = \int_M \sum_{k=1}^{n} a_{jk} x_k u_i = 0, \quad 1 \leq i < j \leq n.$$

(2.7)

Since $U$ is an orthogonal matrix, $y_1, y_2, \ldots, y_n$ are also coordinate functions on $\mathbb{R}^n$. Thus, denoting these coordinate functions still by $x_1, x_2, \ldots, x_n$, one arrives at the condition (2.6). It follows from (2.3) that

$$\lambda_i \int_M x_i^2 \leq \int_M |\nabla x_i|^2, \quad i = 1, \ldots, n,$$

(2.8)

with equality holding if and only if

$$\Delta x_i = -\lambda_i x_i.$$

(2.9)

Integrating the equality

$$\frac{1}{2} \Delta x_i^2 = 1$$

(2.10)

on $\Omega$ and using the divergence theorem, one gets

$$|\Omega| = \int_M x_i \partial_i x_i, \quad i = 1, \ldots, n,$$

(2.11)
where \( \nu \) denotes the outward unit normal of \( \partial \Omega = M \). Taking the square of (2.11) and using the Hölder inequality, we infer
\[
|\Omega|^2 \leq \left( \int_M x_i^2 \right) \left( \int_M (\partial_\nu x_i)^2 \right), \quad i = 1, \ldots, n. \tag{2.12}
\]
Multiplying (2.8) by \( \int_M (\partial_\nu x_i)^2 \) and using (2.12), we have
\[
|\Omega|^2 \lambda_i \leq \left( \int_M |\nabla x_i|^2 \right) \left( \int_M (\partial_\nu x_i)^2 \right), \quad i = 1, \ldots, n. \tag{2.13}
\]
Observing that on \( M \)
\[
1 = |\nabla x_i|^2 = |\nabla x_i|^2 + (\partial_\nu x_i)^2, \tag{2.14}
\]
one deduces from (2.13) that
\[
|\Omega|^2 \lambda_i \leq \left( |M| - \int_M (\partial_\nu x_i)^2 \right) \left( \int_M (\partial_\nu x_i)^2 \right), \quad i = 1, \ldots, n. \tag{2.15}
\]
Summing over \( i \) and using Cauchy-Schwarz inequality, we infer
\[
|\Omega|^2 \sum_{i=1}^{n} \lambda_i \leq |M|^2 - \frac{1}{n} \left( \sum_{i=1}^{n} \int_M (\partial_\nu x_i)^2 \right)^2 = \frac{(n-1)|M|^2}{n}. \tag{2.20}
\]
This proves (1.2).

To prove (1.3), we use divergence theorem and Hölder inequality to get
\[
\int_M |\nabla x_i|^2 = - \int_M x_i \Delta x_i \leq \left( \int_M x_i^2 \right)^{1/2} \left( \int_M (\Delta x_i)^2 \right)^{1/2}, \tag{2.17}
\]
which, combining with (2.8), gives
\[
\lambda_i \left( \int_M x_i^2 \right)^{1/2} \leq \left( \int_M (\Delta x_i)^2 \right)^{1/2}. \tag{2.18}
\]
Multiplying (2.18) by \( \left( \int_M (\partial_\nu x_i)^2 \right)^{1/2} \) and using (2.12), we have
\[
\lambda_i |\Omega| \leq \left( \int_M (\Delta x_i)^2 \right)^{1/2} \left( \int_M (\partial_\nu x_i)^2 \right)^{1/2}, \quad i = 1, \ldots, n. \tag{2.19}
\]
Summing over \( i \) and using Cauchy-Schwarz inequaty, we infer
\[
|\Omega| \sum_{i=1}^{n} \lambda_i \leq \left( \sum_{i=1}^{n} \int_M (\Delta x_i)^2 \right)^{1/2} \left( \sum_{i=1}^{n} \int_M (\partial_\nu x_i)^2 \right)^{1/2} \tag{2.20}
\]
\[
= (n-1) \left( \int_M H^2 \right)^{1/2} |M|^{1/2},
\]
where, in the last equality, we have used the fact that
\[ \Delta x \equiv (\Delta x_1, \ldots, \Delta x_n) = (n-1)H, \] (2.21)
being \( H \) the mean curvature vector of \( M \) in \( \mathbb{R}^n \). Hence, (1.3) holds.

If the equality holds in (1.2), then the inequalities (2.8), (2.12), (2.15) and (2.16) must take equality sign. It then follows that (2.9) holds,
\[ \int_M (\partial_\nu x_1)^2 = \int_M (\partial_\nu x_2)^2 = \cdots = \int_M (\partial_\nu x_n)^2, \] (2.22)
and so
\[ \lambda_1 = \lambda_2 = \cdots = \lambda_n. \] (2.23)

Thus, the position vector \( x = (x_1, \ldots, x_n) \) when restricted on \( \partial \Omega \) satisfies
\[ \Delta x = -\lambda_1(x_1, \ldots, x_n). \] (2.24)

Combining (2.24) and (2.21), we have
\[ x = -\frac{(n-1)}{\lambda_1}H, \text{ on } M. \] (2.25)

Consider the function \( h = |x|^2 : M \to \mathbb{R} \). It is easy to see from (2.25) that
\[ Zh = 2(Z,x) = 0, \quad \forall Z \in \mathfrak{X}(M). \]
Thus \( g \) is a constant function and so \( M \) is a hypersphere. If the equality holds in (1.3), one can use similar arguments to deduce that \( M \) is a hypersphere in \( \mathbb{R}^n \).

**Remark 2.1** Noting that the Reilly inequality (1.1) has been strengthened to [35]
\[ \lambda_1 \leq \frac{(n-1)|M|}{\omega_n |\Omega|} \left( \frac{\omega_n}{|\Omega|} \right)^{1/n} \] (2.26)
with equality holding if and only \( M \) is a hypersphere, we believe that a stronger form of (1.2) is valid.

**Conjecture 2.1** If the conditions are as in Theorem 1.1, then
\[ \sum_{i=1}^{n} \lambda_i \leq \frac{(n-1)|M|}{\omega_n |\Omega|} \left( \frac{\omega_n}{|\Omega|} \right)^{1/n}. \] (2.27)

Moreover, the equality holds in (2.27) if and only \( M \) is a round sphere.

**Remark 2.2** It is easy to see from (2.15) that
\[ \lambda_n \leq \frac{|\partial \Omega|^2}{4|\Omega|^2} \] (2.28)
which is also new. It would be interesting to know the best possible upper bound for \( \lambda_n \).
3 Proofs of Theorems 1.2 and 1.3

In this section, we shall prove Theorems 1.2 and 1.3. Before doing this, let us recall some known facts. Let \( \{ \phi_i \}_{i=0}^\infty \) be orthonormal eigenfunctions corresponding to the eigenvalues \( \{ \xi_i \}_{i=0}^\infty \) of the problem (1.4). That is,

\[
\begin{cases}
\Delta^2 \phi_i = 0 \text{ in } \Omega, \\
\partial_{\nu} \phi_i = \partial_{\nu}(\Delta \phi_i) + \xi_i \phi_i = 0 \text{ on } \partial \Omega \\
\int_{\partial \Omega} \phi_i \phi_j = \delta_{ij}.
\end{cases}
\] (3.1)

For each \( k = 1, \cdots, \) the variational characterization for \( \xi_k \) is given by

\[
\xi_k = \inf_{\phi \in H^2(\Omega), \partial_{\nu} \phi |_{\partial \Omega} = 0, \phi |_{\partial \Omega} \neq 0} \frac{\int_{\Omega} (\Delta \phi)^2}{\int_{\partial \Omega} \phi^2}.
\] (3.2)

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) and \( \Omega^* \) the ball centered at the origin in \( \mathbb{R}^n \) such that \( |\Omega^*| = |\Omega| \). The moments of inertia of \( \Omega \) with respect to the hyperplanes \( x_k = 0 \), are defined as

\[
J_k(\Omega) = \int_{\Omega} x_k^2 \text{ for all } k \in \{1, \cdots, n\}.
\] (3.3)

By summation over \( k \), we obtain the polar moment of inertia of \( \Omega \) with respect to the origin denoted by

\[
J_0(\Omega) = \sum_{k=1}^n \int_{\Omega} x_k^2.
\] (3.4)

Note that \( J_0(\Omega) \) depends on the position of the origin. In fact, \( J_0(\Omega) \) is smallest when the origin coincides with the center of mass of \( \Omega \), i.e. when we have

\[
\int_{\Omega} x_k = 0, \ k = 1, \cdots, n.
\] (3.5)

We need the following well known isoperimetric property [5], [27]:

**Theorem 3.1** Among all domains \( \Omega \) of prescribed \( n \)-volume, the ball \( \Omega^* \) centered at the origin has the smallest polar moment of inertia, that is,

\[
J_0(\Omega) \geq J_0(\Omega^*), \ \Omega \in \mathcal{O},
\] (3.6)

for all bounded domain \( \Omega \) of prescribed \( n \)-volume \( |\Omega| \), with equality if and only if \( \Omega \) coincides with \( \Omega^* \).

By multiplication over \( k \) in (3.3), we obtain a quantity denoted by \( J(\Omega) \),

\[
J(\Omega) = \prod_{k=1}^n J_k(\Omega)
\] (3.7)

which satisfies the following isoperimetric inequality [3], [26]:

\[
J(\Omega) \geq J(\Omega^*) = \frac{|\Omega|^{n+2}}{(n+2)^n \omega_n^2}
\] (3.8)
with equality if and only if $\Omega$ is an ellipsoid symmetric with respect to the hyperplanes $x_k = 0, k = 1, \cdots, n$.

**Proof of Theorem 1.2.** By a translation of the origin in $\mathbb{R}^n$, we can assume that
\[
\int_{\Omega} x_i = 0, \ i = 1, \cdots, n. \tag{3.9}
\]
For each $i \in \{1, \cdots, n\}$, let $g_i$ be the solution of the problem
\[
\begin{cases}
\Delta g_i = x_i & \text{in } \Omega, \\
\partial_{\nu} g_i |_{\partial \Omega} = 0, \\
\int_{\partial \Omega} g_i = 0.
\end{cases} \tag{3.10}
\]
We claim that if the coordinate functions $x_1, \cdots, x_n$ are chosen properly, then
\[
g_i \perp \text{span}\{\phi_0, \cdots, \phi_{i-1}\}, \ i = 1, \cdots, n. \tag{3.11}
\]
To see this, let us fix a set of coordinate functions $x_1, \cdots, x_n$ and the solutions $g_1, \cdots, g_n$ as above. Consider the $n \times n$ matrix $H = (h_{ji})$ with $h_{ji} = \int_M g_j \phi_i$, for $i, j = 1, 2, \cdots, n$. One can find an upper triangle matrix $S = (s_{ji})$ and an orthogonal matrix $T = (t_{ji})$ such that $S = TH$, that is,
\[
s_{ji} = \sum_{k=1}^n t_{jk} h_{ki} = \int_M \sum_{k=1}^n t_{jk} g_k \phi_i = 0, \ 1 \leq i < j \leq n. \tag{3.12}
\]
Letting $y_j = \sum_{k=1}^n t_{jk} x_k$, $\tilde{g}_j = \sum_{k=1}^n t_{jk} g_k$, we have from (3.10) and (3.12) that
\[
\begin{cases}
\Delta \tilde{g}_j = y_j & \text{in } \Omega, \\
\partial_{\nu} \tilde{g}_j |_{\partial \Omega} = 0, \\
\int_{\partial \Omega} \tilde{g}_j = 0
\end{cases} \tag{3.13}
\]
and
\[
\tilde{g}_i \perp \text{span}\{\phi_0, \cdots, \phi_{i-1}\}, \ i = 1, \cdots, n. \tag{3.14}
\]
Since $T = (t_{ji})$ is an orthogonal matrix, $y_1, \cdots, y_n$ are also coordinate functions of $\mathbb{R}^n$. Thus, our claim is true. Denoting these coordinate functions and the solutions of (3.13) still by $x_1, x_2, \cdots, x_n$, and $g_1, \cdots, g_n$, respectively, we conclude from (3.9) that
\[
\xi_i \leq \frac{\int_{\Omega} x_i^2}{\int_{\partial \Omega} g_i^2}, \ i = 1, \cdots, n. \tag{3.15}
\]
From divergence theorem we know that
\[
\int_{\Omega} x_i^2 = \int_{\Omega} x_i \Delta g_i = - \int_{\Omega} (\nabla x_i, \nabla g_i) = - \int_{\partial \Omega} g_i \partial_{\nu} x_i, \tag{3.16}
\]
which gives
\[
\left( \int_{\Omega} x_i^2 \right)^2 \leq \int_{\partial \Omega} (\partial_{\nu} x_i)^2 \int_{\partial \Omega} g_i^2, \ i = 1, \cdots, n. \tag{3.17}
\]
Combining (3.15) into (3.17), we infer
\[ \xi_i \int_{\Omega} x_i^2 \leq \int_{\partial \Omega} (\partial \nu x_i)^2, \quad i = 1, \cdots, n, \] (3.18)
which implies that
\[ \sum_{i=1}^{n} \frac{1}{\xi_i} \geq \sum_{i=1}^{n} \frac{\int_{\partial \Omega} x_i^2}{\int_{\partial \Omega} (\partial \nu x_i)^2}. \] (3.19)
Using the arithmetic-geometric mean inequality and the isoperimetric inequality (3.8), we have
\[ \sum_{i=1}^{n} \frac{1}{\xi_i} \int_{\Omega} x_i^2 \int_{\partial \Omega} (\partial \nu x_i)^2 \geq \frac{n}{(n+2) \omega_n^{2/n}} \left( \prod_{j=1}^{n} \int_{\partial \Omega} (\partial \nu x_i)^2 \right) \] (3.20)
with equality holding if and only if
\[ \frac{\int_{\Omega} x_i^2}{\int_{\partial \Omega} (\partial \nu x_i)^2} = \cdots = \frac{\int_{\Omega} x_n^2}{\int_{\partial \Omega} (\partial \nu x_n)^2}, \] (3.21)
\[ \prod_{j=1}^{n} \int_{\Omega} x_i^2 = \frac{|\Omega|^{n+2}}{(n+2)^n \omega_n^2} \] (3.22)
and
\[ \int_{\partial \Omega} (\partial \nu x_1)^2 = \cdots = \int_{\partial \Omega} (\partial \nu x_n)^2. \] (3.23)
Combining (3.19) and (3.20), one gets (1.6). If the equality holds in (1.6), then (3.15), (3.17), (3.18), (3.19) and (3.20) should take equality. It follows that
\[ \int_{\Omega} x_1^2 = \int_{\Omega} x_2^2 = \cdots = \int_{\Omega} x_n^2 = \frac{|\Omega| n+2}{(n+2)^n \omega_n^2} \left( \frac{|\Omega|}{\omega_n} \right)^{2/n} \] (3.24)
and so
\[ \int_{\Omega} \sum_{i=1}^{n} x_i^2 = \frac{n|\Omega|}{n+2} \left( \frac{|\Omega|}{\omega_n} \right)^{2/n}. \] (3.25)
Consequently, we conclude from Theorem 3.1 that $\Omega$ is a ball. On the other hand, if $\Omega$ is a ball of radius $R$ in $\mathbb{R}^n$, then

$$\sum_{i=1}^n \frac{1}{\xi_i} = \frac{n}{\xi_1} = \frac{n}{\frac{n-1}{R^2}} = \frac{n^2|\Omega|}{(n+2)|\partial\Omega|}. \quad (3.26)$$

This completes the proof of Theorem 1.2.

**Remark 3.1.** Consider a more general eigenvalue problem:

$$\begin{cases}
\Delta^2 u = 0 & \text{in } \Omega, \\
\partial_\nu u = \partial_\nu (\Delta u) + \zeta \rho u = 0 & \text{on } \partial\Omega,
\end{cases} \quad (3.27)$$

where $\rho$ is a continuous positive function on $\partial\Omega$. The eigenvalues of this problem can be arranged as (counted with multiplicity):

$$0 = \zeta_0 < \zeta_1 \leq \zeta_2 \leq \cdots \leq +\infty. \quad (3.28)$$

When $\Omega$ is a bounded domain with smooth boundary in $\mathbb{R}^n$, one can use similar arguments as in the proof of (3.29) to show that the first $n$ nonzero eigenvalues of the problem (3.27) satisfy

$$\prod_{j=1}^n \zeta_j \leq \left( \frac{\omega_n}{|\Omega|} \right)^2 \cdot \left( \frac{(n+2)}{n} \int_{\partial\Omega} \frac{1}{\rho} \right)^n, \quad (3.29)$$

with equality holding implies that $\Omega$ is an ellipsoid. To see this, let us take an orthonormal set of eigenfunctions $\{\psi_i\}_{i=0}^\infty$ corresponding to the eigenvalues $\{\zeta_i\}_{i=0}^\infty$, that is,

$$\begin{cases}
\Delta^2 \psi_i = 0 & \text{in } \Omega, \\
\partial_\nu \psi_i = \partial_\nu (\Delta \psi_i) + \zeta_i \rho \psi_i = 0 & \text{on } \partial\Omega, \\
\int_{\partial\Omega} \rho \psi_i \psi_j = \delta_{ij}.
\end{cases} \quad (3.30)$$

The variational characterization for $\zeta_k$ is given by

$$\zeta_k = \inf_{\psi \in H^2(\Omega), \partial_\nu \psi|_{\partial\Omega} = 0, \psi|_{\partial\Omega} \neq 0} \frac{\int_\Omega (\Delta \psi)^2}{\int_{\partial\Omega} \rho \psi^2}, \quad k = 1, \ldots. \quad (3.31)$$

We choose the origin in $\mathbb{R}^n$ so that (3.9) holds. For each $i \in \{1, \ldots, n\}$, let $h_i$ be the solution of the problem

$$\begin{cases}
\Delta h_i = x_i & \text{in } \Omega, \\
\partial_\nu h_i|_{\partial\Omega} = 0, \\
\int_{\partial\Omega} \rho h_i = 0.
\end{cases} \quad (3.32)$$

As in the proof of Theorem 1.2 we can assume that

$$\int_{\partial\Omega} \rho x_i \psi_j = 0, \quad i = 1, 2, \ldots, n, \quad j < i. \quad (3.33)$$

It follows from (3.31) that

$$\zeta_i \int_{\partial\Omega} \rho h_i^2 \leq \int_{\Omega} x_i^2, \quad i = 1, \ldots, n. \quad (3.34)$$
Since
\[ \int_{\Omega} x_i^2 = \int_{\Omega} x_i \Delta h_i = - \int_{\partial \Omega} h_i \partial _\nu x_i \leq \left( \int_{\partial \Omega} \rho h_i^2 \right)^{1/2} \left( \int_{\partial \Omega} \left( \frac{\partial _\nu x_i}{\rho} \right)^2 \right)^{1/2}, \]
we infer from (3.34) that
\[ \zeta_i \int_{\Omega} x_i^2 \leq \int_{\partial \Omega} \left( \frac{\partial _\nu x_i}{\rho} \right)^2, \quad i = 1, \ldots, n. \]
(3.35)
By multiplication over \( i \), one gets
\[ \prod_{j=1}^n \zeta_j \prod_{i=1}^n \int_{\Omega} x_i^2 \leq \prod_{i=1}^n \int_{\partial \Omega} \left( \frac{\partial _\nu x_i}{\rho} \right)^2 \leq \left( \frac{1}{n} \sum_{i=1}^n \int_{\partial \Omega} \left( \frac{\partial _\nu x_i}{\rho} \right)^2 \right)^n = \left( \frac{1}{n} \int_{\partial \Omega} \frac{1}{\rho} \right)^n, \]
(3.37)
which, combining with (3.8), yields (3.29). Also, when the equality holds (3.29), we must have the equality in (3.8) and so \( \Omega \) is an ellipsoid.

**Proof of Theorem 1.3** Let \( u_0, u_1, u_2, \ldots \), be orthonormal eigenfunctions corresponding to the eigenvalues \( 0, \lambda_{1, \tau}, \lambda_{1, \tau}, \ldots \), that is,
\[ \begin{align*}
\Delta^2 u_i - \tau \Delta u_i &= 0, & \text{in} \ \Omega, \\
\frac{\partial ^2 u_i}{\partial \nu^2} &= 0, & \text{on} \ \partial \Omega, \\
\tau \frac{\partial u_i}{\partial \nu} - \text{div}_{\partial \Omega} \left( \nabla^2 u_i (\nu) \right) - \frac{\partial \Delta u_i}{\partial \nu} &= -\lambda_{i, \tau} u_i, & \text{on} \ \partial \Omega, \\
\int_{\partial \Omega} u_i u_j &= \delta_{ij}.
\end{align*} \]
Note that \( u_0 = 1/\sqrt{|\partial \Omega|} \). Using the same discussions as in the proof of Theorem 1.1, we can assume that
\[ \int_{\partial \Omega} x_i u_j = 0, \quad i = 1, \ldots, n, \quad j = 0, \ldots, i - 1. \]
(3.38)
Thus, we have from (1.8) that
\[ \lambda_{i, \tau} \int_{\partial \Omega} x_i^2 \leq \int_{\Omega} \left( |\nabla^2 x_i|^2 + \tau |\nabla x_i|^2 \right) = \tau |\Omega|, \quad i = 1, \ldots, n. \]
(3.39)
As in the proof of Theorem 1.1, we have for each \( i \in \{1, \ldots, n\} \) that
\[ |\Omega|^2 \leq \left( \int_{\partial \Omega} x_i \partial _\nu x_i \right)^2 \leq \left( \int_{\partial \Omega} x_i^2 \right) \left( \int_{\partial \Omega} \left( \partial _\nu x_i \right)^2 \right), \]
with equality holding if and only \( \partial _\nu x_i = \eta_i x_i \) for some constant \( \eta_i \neq 0 \).
Multiplying (3.39) by \( \int_{\partial \Omega} (\partial_{\nu} x_i)^2 \) and using (3.40), we get
\[
\lambda_{i,\tau} |\Omega|^2 \leq \tau |\Omega| \int_{\partial \Omega} (\partial_{\nu} x_i)^2, \quad 1, \cdots, n.
\] (3.41)
Dividing by \(|\Omega|^2\) and summing over \(i\), one gets
\[
\sum_{j=1}^{n} \lambda_{j,\tau} \leq \frac{\tau}{|\Omega|} \int_{\partial \Omega} \sum_{i=1}^{n} (\partial_{\nu} x_i)^2 = \frac{\tau |\partial \Omega|}{|\Omega|}.
\] (3.42)
This proves (1.11). Moreover, if equality holds in (1.11), then \( \partial_{\nu} x_i = \eta_i x_i \), for some nonzero constants \( \eta_i, i = 1, \cdots, n \). It follows that
\[
\sum_{i=1}^{n} \eta_i^2 x_i^2 = 1 \text{ on } \partial \Omega.
\] (3.43)
If \( z = \sum_{i=1}^{n} \eta_i^2 x_i^2 \), then the outward unit normal of \( \partial \Omega \) is given by
\[
\nu = \frac{\nabla z}{|\nabla z|}.
\] (3.44)
Note that
\[
\nu = (\partial_{\nu} x_1, \cdots, \partial_{\nu} x_n) = (\eta_1 x_1, \cdots, \eta_n x_n).
\] (3.45)
Comparing (3.44) and (3.45), we infer \( \eta_1 = \eta_2 = \cdots = \eta_n \), which shows that \( \partial \Omega \) is a hypersphere and so \( \Omega \) is a ball. On the other hand, we have
\[
\lambda_{1,\tau}(B^n_R) = \cdots \lambda_{n,\tau}(B^n_R) = \frac{\tau |\partial B^n_R|}{n |B^n_R|},
\] (3.46)
that is, the equality holds for balls in (1.11).

**Conjecture 3.2** Under the same assumptions of Theorem 1.3, we have
\[
\prod_{j=1}^{n} \lambda_{j,\tau} \leq \frac{\tau^n \omega_n}{|\Omega|}
\] (3.47)
with equality holding if and only if \( \Omega \) is a ball.

It should be mentioned that if \( \Omega \) is convex, then the above conjecture is true. To see this, it is enough to take the products of the \( n \) inequalities in (3.39) and use Lemma 1.1.

**4 Proofs of Theorems 1.4 and 1.5**

In this section, we prove Theorems 1.4 and 1.5. We shall need the following result [27].
Lemma 4.1 Let $\Omega$ be a bounded convex domain in $\mathbb{R}^n$. Assume that origin coincides with the center of mass of $\partial \Omega$, that is,

$$\int_{\partial \Omega} x_i \, ds = 0, \ i = 1, \cdots, n. \quad (4.1)$$

Then we have

$$\prod_{i=1}^n \int_{\partial \Omega} x_i^2 \, ds \geq \prod_{i=1}^n \int_{\partial \Omega^*} x_i^2 \, ds = \frac{|\Omega|^{n+1}}{\omega_n}, \quad (4.2)$$

with equality if and only if $\Omega = \Omega^*$.

We prove the following result from which Theorem 1.4 follows.

Theorem 4.2 Let $\beta \geq 0$ and $\Omega$ be a bounded domain with smooth boundary $\partial \Omega$ in $\mathbb{R}^n$. Let $\rho$ be a positive continuous function on $\partial \Omega$ and denote by $\eta_1, \beta \leq \eta_2, \beta \leq \cdots \leq \eta_n, \beta \leq \cdots$ the eigenvalues of the problem :

$$\begin{cases} 
\Delta u = 0 & \text{in } \Omega, \\
-\beta \Delta u + \partial_\nu u = \eta \rho u & \text{on } \partial \Omega,
\end{cases} \quad (4.3)$$

Then we have

$$\sum_{i=1}^n \eta_{i,\beta} \leq \frac{1}{|\Omega|^2} \left( |\Omega| + \beta |\partial \Omega| \right) \left( \int_{\partial \Omega} \rho^{\frac{1}{2}} \right)^2 \left( \int_{\partial \Omega} \frac{1}{\sqrt{\rho}} \right)^2. \quad (4.4)$$

Furthermore, if $\rho$ is constant, the equality holds in (4.4) if and only if $\Omega$ is a ball.

Proof of Theorem 4.2. Let $u_0, u_1, u_2, \cdots$ be orthonormal eigenfunctions corresponding to the eigenvalues $0 = \eta_{0,\beta} < \eta_{1,\beta} \leq \eta_{2,\beta} \leq \cdots$, of the problem (4.3), that is,

$$\begin{cases} 
\Delta u_i = 0 & \text{in } \Omega, \\
-\beta \Delta u_i + \partial_\nu u_i = \eta_i \rho u_i & \text{on } \partial \Omega, \\
\int_{\partial \Omega} \rho u_i u_j = \delta_{ij}.
\end{cases} \quad (4.5)$$

Note that $u_0$ is a constant function $1/(\int_{\partial \Omega} \rho)^{1/2}$. The eigenvalues $\eta_{i,\beta}, i = 1, 2, \cdots$, are characterized by

$$\eta_{i,\beta} = \min_{u \in H(\Omega) \setminus \{0\}, \ \int_{\partial \Omega} \rho u^2 = 0, \ j = 0, 1, \cdots , i-1} \frac{\int_{\Omega} |\nabla u|^2 + \beta \int_{\partial \Omega} |\nabla u|^2}{\int_{\partial \Omega} \rho u^2}. \quad (4.6)$$

As in the proof of Theorem 1.1, we can choose the coordinate functions $x_1, \cdots, x_n$ of $\mathbb{R}^n$ so that

$$\int_{\partial \Omega} \rho x_i u_j = 0, \ j < i, \ i = 1, \cdots, n. \quad (4.7)$$
Hence
\[
\eta_{i,\beta} \int_{\partial \Omega} \rho x_i^2 \leq \int_{\Omega} |\nabla x_i|^2 + \beta \int_{\partial \Omega} |\nabla x_i|^2
\]
\[
= |\Omega| + \beta \int_{\partial \Omega} |\nabla x_i|^2
\]
\[
= |\Omega| + \beta \left(|\partial \Omega| - \int_{\partial \Omega} (\partial_i x_i)^2\right), \quad i = 1, \cdots, n. \tag{4.8}
\]

We have from (3.40) that
\[
|\Omega|^2 \leq \left( \int_{\partial \Omega} \rho x_i^2 \right) \left( \int_{\partial \Omega} \rho^{-1}(\partial_i x_i)^2 \right). \tag{4.10}
\]

Multiplying (4.9) by \(\int_{\partial \Omega} \rho^{-1}(\partial_i x_i)^2\) and using (4.10), we get
\[
\eta_{i,\beta} |\Omega|^2 \leq (|\Omega| + \beta |\partial \Omega|) \int_{\partial \Omega} \rho^{-1}(\partial_i x_i)^2 - \beta \left( \int_{\partial \Omega} (\partial_i x_i)^2 \right) \left( \int_{\partial \Omega} \rho^{-1}(\partial_i x_i)^2 \right)
\]
\[
\leq (|\Omega| + \beta |\partial \Omega|) \int_{\partial \Omega} \rho^{-1}(\partial_i x_i)^2 - \beta \left( \int_{\partial \Omega} \frac{1}{\sqrt{\rho}}(\partial_i x_i)^2 \right)^2. \tag{4.11}
\]

Summing over \(i\) and using Cauchy-Schwarz inequality, one has
\[
|\Omega|^2 \sum_{i=1}^{n} \eta_{i,\beta} \leq (|\Omega| + \beta |\partial \Omega|) \int_{\partial \Omega} \rho^{-1} - \beta \left( \sum_{i=1}^{n} \int_{\partial \Omega} \frac{1}{\sqrt{\rho}}(\partial_i x_i)^2 \right)^2
\]
\[
= (|\Omega| + \beta |\partial \Omega|) \int_{\partial \Omega} \rho^{-1} - \beta \left( \int_{\partial \Omega} \frac{1}{\sqrt{\rho}} \right)^2. \tag{4.12}
\]

Dividing by \(|\Omega|^2\), we get (1.19). Moreover, when \(\rho\) is constant, the equality holds in (1.19) if and only if \(\Omega\) is a ball.

**Proof of Theorem 1.5.** Let us choose the origin in \(\mathbb{R}^n\) as the center of mass of \(\partial \Omega\). Taking \(\rho = 1\) and using the same arguments as in the proof of Theorem 1.4 we can get
\[
\lambda_{i,\beta} \int_{\partial \Omega} x_i^2 \leq |\Omega| + \beta \left(|\partial \Omega| - \int_{\partial \Omega} (\partial_i x_i)^2\right), \quad i = 1, \cdots, n. \tag{4.13}
\]

By multiplication of these inequalities, one infers
\[
\prod_{i=1}^{n} \lambda_{i,\beta} \prod_{j=1}^{n} \int_{\partial \Omega} x_j^2
\]
\[
\leq \prod_{i=1}^{n} \left(|\Omega| + \beta |\partial \Omega| - \beta \left( \int_{\partial \Omega} (\partial_i x_i)^2 \right) \right)^n
\]
\[
\leq \left( \frac{1}{n} \sum_{i=1}^{n} \left(|\Omega| + \beta |\partial \Omega| - \beta \left( \int_{\partial \Omega} (\partial_i x_i)^2 \right) \right)^2 \right)^n
\]
\[
= \left( |\Omega| + \frac{(n-1)\beta}{n} |\partial \Omega| \right)^n. \tag{4.14}
\]
Substituting (4.2) into (4.14), we obtain (1.22). It is clear from the proof that equality holds in (1.22) if and only if $\Omega$ is a ball. $\square$

**Remark 4.1** We believe that the convexity assumption in Theorem 1.5 is unnecessary.

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