QUASI-STATIONARY STATES OF DUST FLOWS UNDER POYNTING-ROBERTSON DRAG: NEW ANALYTICAL AND NUMERICAL SOLUTIONS

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Received 1997 March 21; accepted 1997 May 22

ABSTRACT

The effect of solar or stellar radiation on dust particles’ trajectories (the Poynting-Robertson drag) has been studied by a number of authors and applied to interplanetary dust dynamics in numerical computations. Meanwhile, some important features of dust flows can be studied analytically by implementing our novel hydrodynamical approach to use the continuity equation written in the particle’s orbital elements as coordinates (Gor’kavyi, Ozernoy, & Mather). By employing this approach and integrating the continuity equation, we are able to find two integrals of motion when the Poynting-Robertson drag dominates the dissipative forces in the dust flow. These integrals of motion enable us to explore basic characteristics of dust flows from any sources in the solar system (such as asteroids, comets, Kuiper belt, etc.) or in another planetary system. In particular, we have reproduced the classical solution $n(r) \propto r^{-1}$ that approximately represents the overall distribution of dust in the solar system. We have also investigated various factors that could be responsible for the deviations of the power-law index in $n(r) \propto r^{-1}$ from $\delta = -1$, including the influences of the orbital characteristics of dust sources, the evaporation of dust particles, as well as mixtures of dust particles of both asteroidal and cometary origin. We have calculated the masses and number densities of asteroidal and cometary components of the zodiacal cloud at different distances from the Sun.

Subject headings: dust, extinction — interplanetary medium — methods: numerical — radiative transfer

1. INTRODUCTION

The Earth orbit is immersed in a multicomponent cloud of interplanetary dust and particles of various sizes. The structure and dynamics of this cloud are very important for both space missions and interpretations of numerous astronomical data. Small dust particle scattering (the zodiacal light) and emission contribute substantially to infrared data at $\lambda$1–100 $\mu$m (Hauser 1996).

The dynamics and evolution of interplanetary particles are determined by several basic effects that include (i) the Poynting-Robertson (P-R) drag; (ii) resonance effects associated chiefly with Jupiter, Mars, Earth, and Venus; (iii) gravitational encounters with these planets, which occur in the form of elastic gravitational scattering of the particles by the planets; (iv) mutual collisions of the particles; and (v) evaporation of dust particles. As we have shown earlier (Gor’kavyi, Ozernoy, & Mather 1996, 1997), the dust flow evolution governed by those processes can be described conveniently by the continuity equation written in the space of orbital coordinates:

$$\frac{\partial n(x_i, t)}{\partial t} + \frac{\partial}{\partial x_i}(n v_i) = N^+(x_i, t) - N^-(x_i, t),$$

where $i = 1 \ldots 6$, and $v_i$ is velocity along the $i$th axis in the phase space. In this equation, the “div” term $\partial(n v_i)/\partial x_i$ describes slow processes of change of particle orbital elements such as the particle transport due to both the P-R drag and the resonance effects. The terms $N^+(x_i, t)$ and $N^-(x_i, t)$ are responsible for fast processes such as the gravitational scattering of particles by the planets and the contact impacts of particles with each other and with the planets.

Let us address the evolution of interplanetary particles governed by only two processes: (1) the P-R drag that continuously changes the particle’s orbit and (2) the gravitational scattering of particles by the planets, which changes the particle’s orbit like a jump (for details, see Gor’kavyi et al. 1997, hereafter GOM). We introduce a small parameter $\epsilon = \delta n_{sc}/\delta n_{PR} \ll 1$, where $\delta n_{sc}$ and $\delta n_{PR}$ are the changes, per unit time, due to the appropriate processes in the number density of dust particles at the typical point of the $(a, e)$-space, a being semimajor axis and $e$ being eccentricity. This small parameter makes it possible to derive, by the Chapman-Enskog approach, a set of equations to get the first two consecutive approximations for particle density of the flow:

$$\frac{\partial n_{PR}(x_i, t)}{\partial t} + \frac{\partial}{\partial x_i}(n_{PR} v_i) = 0,$$

$$\frac{\partial n_{sc}(x_i, t)}{\partial t} = N^+(n_{PR}, x_i, t) - N^-(n_{PR}, x_i, t).$$

Equation (2) gives a first approximation to particle density associated with P-R drag, which is then used to get, with help of equation (3), a correcting term associated with the (second-order) process of gravitational scattering. Equation (3), applied to the dynamics of particles governed by gravitational scattering, was analyzed in our previous paper (GOM). In the present paper, we emphasize equation (2) by exploring the Poynting-Robertson drag as the leading effect for the dynamics of small particles. The paper is organized as follows. In § 1, we derive the integrals of motion of dust
particles, assuming the P-R drag as the dominating effect. In § 2, we find solutions of the continuity equation for the dust flow from a point source of dust (such as a comet or an asteroid). We also consider here the spatial distribution (in the coordinate space) of dust particles from such a source. In § 3, we generalize those analytical solutions for evaporating dust particles, whose radii decrease with time. Section 4 deals with dust flows produced by numerous real sources (217 comets and 5000 asteroids). In § 5, we summarize our results and discuss the underlying assumptions as well as further work.

2. INTEGRALS OF MOTION

The rates $v_i$ (i.e., $da/dt$, $de/dt$, ...) for the P-R drag have been given in a number of papers (e.g., Burns, Lamy, & Soter 1979; Liou, Zook, & Jackson 1995b):

$$ \frac{da}{dt} = \frac{2 + 3e^2}{a(1 - e^2)^{3/2}} \ , \ (4) $$

$$ \frac{de}{dt} = \frac{5e}{2a^2(1 - e^2)^{1/2}} \ , \ (5) $$

where $\alpha = -\beta(GM/c)$, $M$ is the mass of the Sun, and $\beta$ is the ratio of the solar radiation pressure force to the gravitation force:

$$ \beta = \frac{3LQ}{16\pi G Mcpl} \ , \ (6) $$

where $L$ is the solar luminosity, $G$ is the gravitational constant, $c$ is the speed of light, and $l$, $\rho$ are the particle's radius and density, respectively. The coefficient $Q$ depends on the optical properties of the particle, and the ratio of its size to the wavelength of light, for a perfect particle $Q = 1$. For an asteroidal dust particle, we adopt $2l = 12 \mu$m and $\rho = 2.7$ g cm$^{-3}$, so that $\beta = 0.037$ (e.g., Dermott et al. 1994).

Along the particle's trajectory, one gets

$$ \frac{de}{da} = \frac{5e(1 - e^2)}{2a(2 + 3e^2)} \ . \ (7) $$

By integrating this equation, the particle's trajectory in $(a, e)$-space is easily obtained to be

$$ a(1 - e^2)^{1/5} = \text{const} \equiv C_1 \ , \ (8) $$

in agreement with Wyatt & Whipple (1950). This integral of motion implies a continuity of the particle's trajectory in $(a, e)$-space. The trajectory is shown in Figure 1.

The flow of particles along evolutionary trajectories shown in Figure 1 is governed by an equation (derived in the appendix) that depends only on the variables $a$ and $e$ ($l =$ const):

$$ f_1(a, e) \frac{\partial n}{\partial a} + f_2(e) \frac{\partial n}{\partial e} = f_3(n, e) \ , \ (9) $$

where $f_1 = 2a(2 + 3e^2)$, $f_2 = 5e(1 - e^2)$, and $f_3 = n(6e^2 - 1)$. The value of $f_3$ is given here for the case when the particle size does not depend on the orbital parameters $a$ and $e$ (for a more general case, see the appendix). The characteristics of the partial differential equation (9) (the integrals of motion) are $C_1$ found above (eq. [8]) and $C_2$ given by

$$ C_2 = ne^{1/5} \sqrt{1 - e^2} \ . \ (10) $$

The number density $n$ of dust particles along the particle trajectory in $(a, e)$-space is shown as a function of semimajor axis $a$ in Figure 2 for various $C_2$. The physical meaning of the integrals $C_1$ and $C_2$ is the conservation of the particle's flux along the flow under the P-R effect (note that the flux of the particles is conserved in conditions when the energy is not). As the integrals of motion for the particle's flow under the P-R drag, $C_1$ and $C_2$ qualitatively play the same role as does the Tisserand criterion in the process of gravitational scattering (see GOM). As is shown in GOM,
gravitational scattering results in “jumps” in the particle's coordinates in \((a, e)\)-space. However, those jumps obey the
Tisserand criterion, implying the conservation of the particle’s energy in the planet’s coordinate system under the
gravitational scattering, which is an elastic process. For a
particle moving under the P-R drag alone, the energy is not
conserved, and the Tisserand criterion obviously breaks
down, but the role of the integrals of motion is taken by \(C_1\)
and \(C_2\). Meanwhile, for the particles that experience, along
with the P-R drift, the (in fact, instantaneous) gravitational
scattering by a planet, the Tisserand criterion holds as well.
Indeed, during a short time interval of a gravitational
scattering, the P-R drag practically does not change the particle’s
energy.

The integrals \(C_1\) (eq. [8]) and \(C_2\) (eq. [10]) fully describe
a stationary flow of dust particles produced by an arbitrary
source of dust. As for the dust distribution in the coordinate
space, it can be transformed from that in \((a, e)\)-space with
the help of the double integral. This numerical procedure is
rather complicated, but if we constrain ourselves to point
sources in \((a, e)\)-space (such as comets and asteroids with known \(a\) and \(e\)), the dust flows from such sources can be analyzed by means of a
much more simple analytical technique to be presented in
the next section.

3. PARTICLE NUMBER DENSITY DISTRIBUTION
   (ANALYTICAL SOLUTIONS FOR A POINT SOURCE)

For a point source in \((a, e)\)-space, two simple, one-
dimensional continuity equations apply:
\[
\frac{\partial}{\partial a} [n(a) v_a] = 0 ,
\]
\[
\frac{\partial}{\partial e} [n(e) v_e] = 0 .
\]
In orbital coordinates, one-dimensional distributions of the
particle number density along the \(a\)-axis and \(e\)-axis are given, in the stationary state, respectively, by
\[
n(a) = C \frac{a(1 - e^2)^{3/2}}{a^2 + 3e^2} ,
\]
\[
n(e) = C C^2 \left( \frac{2e^{3/5}}{5a(1 - e^2)^{3/2}} \right) .
\]
Here \(C\) is a constant to be found from boundary conditions,
and \(e\) in the equation for \(n(a)\) can be eliminated with the help
of the integral of motion \(C_1\) that describes, for a point
source, the trajectory of motion of the entire dust flow.
Transformation from the space of orbital elements \((a, e)\)
to \(r\)-space can be done using known analytical expressions
(Sykes 1990). In \(r\)-space, the particles having orbital elements
\((a, e, i)\) form a rather sophisticated torus-like cloud, whose
number density is described by (Sykes 1990)
\[
P(r, \phi) = S(r) \Theta(\phi) ,
\]
where
\[
S(r) = \frac{C_1}{a_r \sqrt{e^2 - (r/a - 1)^2} ,
\]
\[
\Theta(\phi) = \frac{1}{2\pi^2 \sqrt{\cos^2 \phi - \cos^2 i} ,
\]
with the limits \(a(1 - e) \leq r \leq a(1 + e)\) and \(-i \leq \phi \leq i\). For
a volume density near the ecliptic plane, \(\phi = 0\) and
\(P(r, \phi) \propto (\sin i)^{-1}\). It is convenient to introduce the surface
density
\[
\sigma(r) = \int_{-h_0}^{h_0} P(r, \phi) dh ,
\]
where \(dh = rd\phi\), and therefore
\[
\sigma(r) = C_a S(r) r ,
\]
with
\[
C_a = \int_{-1}^{1} \Theta(\phi) d\phi .
\]
It is worth mentioning that functions (16) and (18) diverge
at the pericenter and apocenter points. The total mass of
dust particles having the same orbital parameters \((a, e, i)\) in
a thin spherical region between the radii \((r_1, r_2)\) is given by
\[
\Delta m = \pi (r_1 + r_2) \int_{r_1}^{r_2} \sigma(r) dr .
\]
As a result of integration, one gets
\[
\Delta m(a, e) = C_a \frac{\pi (r_1 + r_2)}{a} \left[ \arcsin \left( \frac{r_2}{a} - 1 \right) - \arcsin \left( \frac{r_1}{a} - 1 \right) \right] ,
\]
where \(a, 1 - e_2 \leq r_2, a, 1 + e_1 \geq r_1\). If the region under
consideration, \((r_1, r_2)\), includes the pericenter or apocenter
of the dust particle orbit, the integration should be done as
far as the pericenter or apocenter, accordingly. Equation
(22), as opposed to equation (16) or equation (18), does not
involve divergences and is much more convenient to use.

The distribution of the integrated surface density in the
dust particle torus with different \((a, e)\) is shown in Figure 3.
For a dust plume produced by any given source of dust, one
can find, by using the solution (14) for \(n(e)\) of the continuity
equation (12), the total dust mass in any region \((r_1, r_2)\) of
the plume (taking into account the comment on the integration

![Figure 3](image-url)

**FIG. 3.** The surface density, \(\sigma(r)\), integrated over small \((r_2 - r_1 = 0.01\)
AU) regions as a function of radius \(r\) for three different points A, B, and C
shown in Fig. 1. The initial \((a_0, e_0)\)-data for those points are, respectively,
\((2.5, 0.29), (1.3, 0.14),\) and \((0.6, 0.054)\).
The result is given by
\[
C_m \pi (r_1 + r_2) \int_{e_1}^{e_2} \frac{1}{x \sqrt{1 - e^2}} \left\{ \arcsin \frac{1}{e} \left[ \frac{r_2(1 - e^2)}{C_1 e^{4/5}} - 1 \right] - \arcsin \frac{1}{e} \left[ \frac{r_1(1 - e^2)}{C_1 e^{4/5}} - 1 \right] \right\} \, de .
\] (24)

The integration limits are easily found by comparing the location of the apocenter or pericenter of particle orbit relative to the boundaries of the region under consideration, \((r_1, r_2)\). Thus, the problem of transformation from the space of orbital elements to the phase space is solved by reducing it to a comparatively simple, single integration to be done numerically.

The above technique can be applied to both stationary and nonstationary flows. The source of dust can appear owing to an instantaneous ejection, e.g., it can be associated with a collision of asteroids (Sykes 1990; Mann, Grun, & Wilk 1996). The resulting drift of dust particles (assumed to have the same size and located initially as a point source at \(a_0, e_0\)) is shown, for three instants of time, in Figure 3.

Let us address a situation where the source of dust has recently appeared and continues to operate. This happens, for example, when a new comet appears in a given point of the \((a, e)\)-space as a result of a "jump" due to the strong gravitational scattering of that comet by a planet. Let us suppose that the first dust particles ejected from such a source have reached the semimajor axis \(a_{\text{min}}\). In order to derive the density distribution in the formed dust cloud, it is sufficient to integrate equations (23) and (24) down to \(e_{\text{min}}\) and not through the entire dust plume. The value of \(e_{\text{min}}\) can be found from equation (8) at the known \(a_{\text{min}}\). The appearance of this dust plume is shown, as a function of time, in Figure 4. Figure 5 demonstrates a steady state plume formed from various sources of dust particles. As can be seen, small eccentricities result in the surface density \(\sigma(r) \approx \text{const}\), which implies that the volume density (i.e., accounting for inclination \(i\)) is \(n(r) \propto r^{-1}\). This result reproduces the well-known characteristic density run of the interplanetary dust.

If the source of dust has an eccentric orbit (such as the Encke comet), the density run is much steeper. The boundary between the plateau and the cutoff of all three curves on Figure 5 is located at the pericenter of the source of dust, \(q = a(1 - e)\), which is due to the fact that the pericenter of a particle experiencing the P-R drag undergoes the least shift compared with its semimajor axis and apocenter.

4. INTEGRALS OF MOTION FOR EVAPORATING PARTICLES

Here we deal with a situation, important in the dynamics of the interplanetary particles, where one needs to take into account an evaporative effect of the solar wind upon a particle (see, e.g., Johnson 1991). We describe the mass loss of a particle of mass \(m\) by
\[
\frac{dm}{dt} = -m_s \sum N_i Y_i ,
\] (25)

where \(m_s = \mu_s m_p\) is the mass of molecules sputtered from the dust particle's surface, \(\mu_s\) is the molecular weight, \(m_p\) is the proton mass, \(N_i\) is the number of particles of the \(i\)th kind in the solar wind colliding with the dust particle, and \(Y_i\) is the sputtering coefficient, i.e., the number of molecules sputtered from the dust particle by one wind particle. If the dust particles that drift to the Sun because of the P-R drag have small eccentricities, the particle's size, \(l\), changes according to a simple law \(l \propto a^\lambda\) where the power-law index \(\lambda\) is given by (Shestakova 1994)
\[
\lambda = \frac{J_0}{6(1 + J_0)} \frac{\mu_s}{\mu_w} Y_s .
\] (26)

Here \(\mu_w\) is an average molecular weight of the solar wind, \(J_0 = Mc^3/L = 0.3\) is the ratio of the corpuscular luminosity of the Sun to the radiative one \(L\), and it is assumed that \(Q = 1\). If one only accounts for the contribution of protons and \(x\)-particles to the dust particle's mass changes, one finds (Shestakova 1994)
\[
\lambda = 0.032 \mu_s Y_s ,
\] (27)

which yields \(\lambda = 0.025\) for silicates and \(\lambda = 0.56\) for ice \((H_2O)\). Bearing in mind that both \(\beta\) and \(\alpha \propto 1/l \propto a^{-\lambda}\),
comets with several distinct families et al. as well as 217 (Zapalla 1995), 5000 numbered asteroids which comprise (ITA 1994).

The positions of these sources on the diagram (sin\(i\),\(\sin e\))-space can be seen that the asteroid pericenters tend to concentrate in two regions 1.7-2.15 AU and 2.6-2.9 AU. The volume density \(n(r) = \sigma/(r \sin i)\), where \(r \sin i\) is the thickness of the dust plume. Therefore, the lower the position of an asteroidal group in Figure 7 (small \(\sin i\)), the larger is its contribution to \(n(r)\) in the ecliptic plane. As for comets, their pericenters are distributed more or less uniformly at 0.5 \(\leq q \leq 2.5\) AU.

In order to compute the contributions of asteroidal and comet sources to the volume density of the interplanetary dust, we adopt the simplest assumptions: (i) all 5000 asteroids produce dust particles with the same rate, and (ii) comets produce dust particles whenever they pass pericenter, and so their contribution into the zodiacal cloud is proportional to 1/(\(T^2\)), where \(T\) is the comet orbital period and typically \(\gamma = 2\). We neglect the comets (Sun grazers) with \(a(1-e) < 0.01\), assuming that the dust produced so close to the Sun evaporates very soon.

To visualize easily any deviations of \(n(r)\) from the “standard” solution \(n(r) = r^{-1}\) while making comparison with the data on the volume density distribution of interplanetary/zodiacal dust, it is convenient to deal with \(n(r) = \sigma/\sin i\). The expected contribution of the above 5000 asteroidal sources to \(n(r)\) is shown in Figure 9. It can be seen that the cutoff at \(r \approx 1.7-2.0\) AU, associated with the asteroidal distribution in \(q\), is mainly due to the asteroids belonging to the inner and middle groups. We have also computed \(n(r)\) due to the 1000 largest asteroids with a result that differs insignificantly from the entire sample. Therefore, our neglecting the asteroidal distribution in sizes is not critical for our conclusions.

5. SOURCES THAT FEED THE ZODIACAL CLOUD

As the sources of dust, we have incorporated the first 5000 numbered asteroids (ITA 1994), which comprise several distinct families (Zapalla et al. 1995), as well as 217 comets with \(e < 1\) taken from Marsden & Williams (1992).

The positions of these sources on the diagram (sin\(i\),\(q\)) are presented in Figure 7 (asteroids) and Figure 8 (comets). One can see that the asteroid pericenters tend to concentrate in the two regions 1.7-2.15 AU and 2.6-2.9 AU. The volume density \(n(r) = \sigma/(r \sin i)\), where \(r \sin i\) is the thickness of the dust plume. Therefore, the lower the position of an asteroidal group in Figure 7 (small \(\sin i\)), the larger is its contribution to \(n(r)\) in the ecliptic plane. As for comets, their pericenters are distributed more or less uniformly at 0.5 \(\leq q \leq 2.5\) AU.

\[
2a(2 + 3e^2) \frac{\partial n}{\partial a} + 5e(1 - e^2) \frac{\partial n}{\partial e} = n[6e^2(1 + \lambda) + \lambda - 1].
\]

\[
C_2 = ne^{1/5 - (4/5)\lambda}(1 - e^2)^{\lambda+1/2}.
\]
The expected contribution of the above 217 comet sources to \( n(r) \) is shown in Figure 10. It can be seen that the density run does not have kinks at \( r > 0.8 \) AU and is much steeper compared with what is due to the asteroidal sources (see Fig. 9).

Let us consider the combined cometary and asteroidal contributions into the interplanetary dust density. The total value of \( n(r) \) for a mixture of 25% (in number density) cometary and 75% asteroidal components is shown in Figure 11.

The density run has been approximated by \( n(r) \propto r^\delta \). The power-law indices \( \delta \) at different \( r \) are given in Table 1 for a particular \( \gamma = 2 \).

We have found that the results for \( \gamma = 1 \) and \( \gamma = 3 \) are qualitatively similar: the ratio of the volume densities (asteroids/comets) has been found to be (65%/35%) at \( \gamma = 1 \) and (80%/20%) at \( \gamma = 3 \). Accounting for evaporation of dust particles changes the results in a more substantial way: if one takes \( \lambda_a = 0.025 \) for asteroidal dust and \( \lambda_c = 0.56 \) for cometary dust (Shestakova 1994), then one gets the power-law index \( \delta = -1.3 \) in the region \( 1.0 \leq r \leq 1.7 \) AU when the ratio of the volume densities (asteroids/comets) equals 45%/55%.

The asteroidal/cometary mass ratios, which provide the observed \( \delta = -1.3 \) for different possible values of \( \gamma, \lambda_a \), and \( \lambda_c \), are given in Table 2.

The mass ratios presented in Table 2 for spherical layers at various \( r \) differ from the ratio of volume densities in the ecliptic plane (the fourth line of Table 2) because of different inclinations of the cometary and asteroidal sources of dust and, therefore, of the different thickness of the dust cloud. The last column of Table 2 is fairly consistent with the estimations for asteroidal/cometary ratios obtained from the IRAS observations while modeling the shape of the zodiacal cloud (Liou, Dermott, & Xu 1995a). It is worth noting that the evaporative power-law index \( \lambda_a \) influences

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**Table 1**

| Region (AU) | Asteroidal Dust | Cometary Dust | 75% Asteroidal Dust + 25% Cometary Dust |
|-------------|-----------------|---------------|----------------------------------------|
| \( r = 0.5 - 1.7 \)  | \(-1.03 \pm 0.02\) | \(-2.17 \pm 0.38\) | \(-1.29 \pm 0.06\) |
| \( r = 1.0 - 1.7 \)  | \(-1.04 \pm 0.02\) | \(-2.36 \pm 0.29\) | \(-1.29 \pm 0.03\) |
| \( r \approx 2.0 \)   | \(-2.6\)           | \(-2.8\)        | \(-2.6\)                          |
| \( r \approx 2.5 \)   | \(-4.0\)           | \(-2.9\)        | \(-3.8\)                          |
| \( r \approx 3.0 \)   | \(-7.4\)           | \(-3.1\)        | \(-6.4\)                          |
6. DISCUSSION AND CONCLUSIONS

The continuity equation written in the space of orbital elements, which we have proposed to use as a tool for studying the dynamical evolution of interplanetary particles, is very effective not only for gravitational scattering (Gor’kavyi et al. 1996; GOM) but, as is shown above, for the chief factor of dynamical evolution of the interplanetary dust cloud—the Poynting-Robertson effect. By solving the continuity equation, we have found two integrals of motion governed by the P-R drag and, as a result, have derived the density run for a quasi-stationary state of the interplanetary dust cloud.

Based on our approach, we are able to compute the formation of a dust cloud from many cometary, asteroidal, etc., dust sources, either observed or assumed to exist, by adopting various reasonable assumptions about the rates of dust production. As an outcome, we get the theoretical model for dust distribution in both radius and latitude in the solar system. We call “the reference model” the model that only accounts for the solar gravitational field and the P-R effect. By accounting for the other dynamical factors, we can get corrections to the reference model.

Computations performed in the framework of the reference model yield radial density runs of the interplanetary dust cloud under various model assumptions and described above. They enable us to derive some general conclusions as follows:

1. The asteroidal component of the interplanetary dust is characterized typically by \( n(r) \propto r^{-0.85} \) up to 1.7 AU with a sharp cutoff at \( r \gtrsim 2 \) AU [e.g., \( n(r) \propto r^{-2} \) at \( r \sim 3 \) AU]. The predicted cutoff is entirely consistent with what is observed for the zodiacal cloud and the main core population of the dust in the solar system (Divine 1993).

2. The power-law index \( \delta \) in the distribution of the volume density of the zodiacal dust particles, \( n(r) \propto r^{\delta} \), depends on several factors:

   a) At radii smaller than the pericenter of the source of dust, \( (r < a(1 - e)) \), the index \( \delta \equiv \delta_1 \) differs significantly from that, \( \delta \equiv \delta_2 \), at radii larger than the pericenter of the source of dust, \( (r > a(1 - e)) \).

   b) Both \( \delta_1 \) and \( \delta_2 \) depend strongly on eccentricity of the source: as \( e \) increases from 0 to 1, \( \delta_1 \) increases from \(-1\) to \(0\), whereas the value of \( \delta_2 \) increases from \(-\infty\) to \(-2.5\).

   c) Accounting for the particle’s evaporation, both \( \delta_1 \) and \( \delta_2 \) tend to increase (see Fig. 6).

   d) For a system of dust sources distributed in \( a \) and/or \( e \), the index \( \delta \) differs substantially inside and outside the system (see Fig. 12). For such distributed sources, there is a smooth transition between (and not a jump from) \( \delta_1 \) and \( \delta_2 \) near the pericenter.

3. In the region \( r < 1.7 \) AU, observations yield \( n(r) \propto r^{-1.3} \) (Divine 1993) or \( r^{-1.25} \) (Levasseur-Regourd 1996). The observed deviation from the classical law \( r^{-1} \) may be related to the contribution of the cometary dust component. This component [which alone would result in \( n(r) \propto r^{-2.4} \)] is expected to contribute as much as about 25% in number density (or about 50% in mass) to the dust cloud near Earth.

4. The ratio of the cometary dust contribution to that of the asteroidal dust into \( n(r) \) has a minimum at \( r \approx 1.8-2 \) AU, and it is large in the inner region of the solar system (\( a < 1 \) AU) as well as in the outer region of the main asteroid belt (\( a > 3 \) AU).

5. The dust created by the Kuiper belt should fill the inner region of the solar system as a more or less homogenous layer with \( \sigma(r) = \text{const} \). At \( r > 3-4 \) AU, that dust may dominate over the comet and asteroidal dust components.

Let us turn to discussing how the adopted approximations influence our conclusions. The basic approximations of our reference model are (i) a steady state character of the model and (ii) neglect of other dynamical factors except the P-R drag.

The first approximation could be easily abandoned—we are able to compute the nonstationary processes as well. For instance, Figure 4 shows different stages of dust plume.

| \( r \) (AU) | \( n_1/n_2 = 65/35 \) | \( n_1/n_2 = 75/25 \) | \( n_1/n_2 = 80/20 \) | \( n_1/n_2 = 45/55 \) |
|-------------|----------------|----------------|----------------|----------------|
| 0.0–1.0     | \( \gamma = 1, \lambda_0 = 0, \lambda_1 = 0 \)   | 39/61           | 42/58           | 37/63           | 19/81          |
| 1.0–3.0     | \( \gamma = 2, \lambda_0 = 0, \lambda_1 = 0 \)   | 56/44           | 69/31           | 75/25           | 28/72          |
| 3.0–5.0     | \( \gamma = 3, \lambda_0 = 0, \lambda_1 = 0 \)   | 21/79           | 33/67           | 40/60           | 5/95           |
| 0.0–1.0     | \( \gamma = 2, \lambda_0 = 0, \lambda_1 = 0 \)   | 46/54           | 54/46           | 52/48           | 21/79          |
| 0.95–1.05   | \( \gamma = 2, \lambda_0 = 0, \lambda_1 = 0 \)   | 46/54           | 58/42           | 62/38           | 25/75          |

Note.—Asteroidal/cometary volume densities of dust at \( r = 1 \) AU in the ecliptic plane.
formation of a source that once appeared and continues to produce dust particles. Therefore, this approximation does not imply any restriction in our approach or a failure of our model; it only results from an incompleteness of our knowledge of whether or not the real sources of dust are stationary.

The second approximation implies that our solutions fail to describe such phenomena as the resonance circumsolar dust rings near Earth (Jackson & Zook 1989), which may contain as many as \(\sim 10\%\) of the particles of the zodiacal cloud (Dermott et al. 1994; Liou et al. 1995a). In our further work, we will fully incorporate the resonance and related effects.

Some minor approximations in our computations, such as the intensity of comets as dust sources as a function of their distances from the Sun and the law of dust particle evaporation, are also worth refining. Besides, we have ignored in the above computations the distribution of dust sources in size and chemical abundance as well as the effects of observational selection in asteroidal and cometary populations. However, all uncertainties in those effects can be easily explored by varying the parameters of the model for the zodiacal cloud formation. For instance, while computing a multicomponent model of the dust cloud, we need to incorporate the other effects such as gravitational scattering of dust particles by planets (GOM), resonance capture of dust (Jackson & Zook 1989; Dermott et al. 1994), particle multiple collisions, etc.

We conclude by emphasizing that our continuity equation (1) is well suited to the study of various large-scale properties of the interplanetary dust cloud. Our approach might be fruitful for solving a number of related problems, including the structure of protoplanetary and circumstellar disks.

**APPENDIX**

**INTEGRALS OF MOTION**

Here we derive the integrals of motion based on equation (11). At first, we address the case where the particle’s size does not depend upon the orbital parameters, and thus the coefficient \(x\) in equations (4) and (5) does not depend upon the semimajor axis \(a\) and eccentricity \(e\). By differentiating equations (7) and (8), one finds

\[
\frac{\partial v_a}{\partial a} = x \left( 2 + 3e^2 \right) \frac{(1-e^2)^{3/2}}{a^2(1-e^3)^{3/2}} - \frac{2+3e^2}{a^2(1-e^3)^{3/2}}, \tag{A1}
\]

\[
\frac{\partial v_e}{\partial e} = x \left( \frac{5}{2a^2} \right) \left( \frac{e}{(1-e^3)^{1/2}} \right) \tag{A2}
\]

By substituting equations (7), (8), (A1), and (A2) into equation (11), one gets

\[
\frac{ax(2+3e^2)}{a(1-e^3)^{3/2}} \frac{\partial n}{\partial a} - \frac{na(2+3e^2)}{a^2(1-e^3)^{3/2}} + \frac{5ae}{2a^2(1-e^3)^{3/2}} \frac{\partial n}{\partial e} + \frac{5nx}{2a^2(1-e^3)^{3/2}} = 0. \tag{A3}
\]

Having multiplied equation (A3) by \(2ax^{-1}a^2(1-e^3)^{3/2}\), one finally finds

\[
2a(2+3e^2) \frac{\partial n}{\partial a} + 5e(1-e^2) \frac{\partial n}{\partial e} = n(6e^2 - 1). \tag{A4}
\]

This partial differential equation implies the following ordinary differential equation:

\[
\frac{da}{2a(2+3e^2)} = \frac{de}{5e(1-e^2)} = \frac{dn}{n(6e^2 - 1)}. \tag{A5}
\]

Each pair of equations comprising equation (A5) is solved by separating the variables to give

\[
\int \frac{1}{a} da = \frac{2}{5} \int \frac{2+3e^2}{e(1-e^2)} de \tag{A6}
\]

and

\[
\int \frac{1}{n} dn = \frac{1}{5} \int \frac{6e^2 - 1}{e(1-e^2)} de. \tag{A7}
\]
The integrals can be easily calculated to give the integrals of motion discussed in the main text:

\[ C_1 = \frac{a(1 - e^2)}{e^{2/5}} \]  

(A8)

and

\[ C_2 = ne^{1/5} \sqrt{1 - e^2}. \]  

(A9)

Let us now address the case when the particle's size, owing to evaporation, depends on the orbital parameters \( a \) and \( e \). For simplicity, we assume that the particle's radius \( l \) changes with \( a \) according to a power law. Then the coefficient

\[ \alpha = \alpha_0 a^{-\lambda}. \]  

(A10)

In this case, instead of equation (A1), one gets

\[ \frac{\partial v_a}{\partial a} = \frac{2 + 3e^2}{(1 - e^2)^{3/2}} \frac{\partial}{\partial a} \left( \frac{\alpha}{a} \right) = -\alpha_0(1 + \lambda) \frac{2 + 3e^2}{a^{2+\lambda}(1 - e^2)^{3/2}}, \]  

(A11)

as well as the following equation, instead of equation (A2):

\[ \frac{\partial v_e}{\partial e} = \frac{5}{2a^2(1 - e^2)^{3/2}} = \frac{5\alpha_0}{2a^2 + \lambda(1 - e^2)^{3/2}}. \]  

(A12)

Having substituted equations (A10), (A11), (A12), (7), and (8) into equation (11), one gets, after some algebra,

\[ 2a(2 + 3e^2) \frac{\partial n}{\partial a} + 5e(1 - e^2) \frac{\partial n}{\partial e} = n[6e^2(1 + \lambda) + \lambda - 1]. \]  

(A13)

Since the left-hand side of equations (A13) and (A4) are equal to each other, \( C_1 \) is kept the same as in equation (A8) for a nonevaporative case. Meanwhile, for the right-hand side of equation (A13), one gets, similar to equation (A7),

\[ \int \frac{1}{n} \, d\vartheta = \frac{1}{5} \int \frac{6e^2(1 + \lambda) + \lambda - 1}{e(1 - e^2)} \, de, \]  

(A14)

which yields the second integral of motion for (evaporating) particles of a changing radius:

\[ C_2 = ne^{1/5 - (4/5)\lambda}(1 - e^2)^{1+\lambda/2}. \]  

(A15)

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