Random access codes and non-local resources

Anubhav Chaturvedi$^\dag$ and Marcin Pawlowski

Institute of Theoretical Physics and Astrophysics, National Quantum Information Centre, Faculty of Mathematics, Physics and Informatics, University of Gda´nsk, Wita Stwosza 57, 80-308 Gda´nsk, Poland

Karol Horodecki

Institute of Informatics, National Quantum Information Centre, Faculty of Mathematics, Physics and Informatics, University of Gda´nsk, Wita Stwosza 57, 80-308 Gda´nsk, Poland

(Dated: October 7, 2016)

Abstract

It is known that a PR-BOX (PR), a non-local resource and $(2 \rightarrow 1)$ random access code (RAC), a functionality wherein Alice encodes 2 bits into 1 bit message and Bob learns one of randomly chosen Alice’s inputs are equivalent under the no-signaling condition. In this work we introduce generalizations to PR and $(2 \rightarrow 1)$ RAC and study their inter-convertibility. We introduce generalizations based on the number of inputs provided to Alice, $B_n$-BOX and $(n \rightarrow 1)$ RAC. We show that a $B_n$-BOX is equivalent to a no-signaling $(n \rightarrow 1)$ RACBOX (RB). Further we introduce a signaling $(n \rightarrow 1)$ RB which cannot simulate a $B_n$-BOX. Finally to quantify the same we provide a resource inequality between $(n \rightarrow 1)$ RB and $B_n$-BOX, and show that it is saturated. As an application we prove that one requires atleast $(n - 1)$ PRs supplemented with a bit of communication to win a $(n \rightarrow 1)$ RAC.

We further introduce generalizations based on the dimension of inputs provided to Alice and the message she sends, $B_n^+(+)$-BOX, $B_n^(-)$-BOX and $(n \rightarrow 1, d)$ RAC $(d > 2)$. We show that no-signaling condition is not enough to enforce strict equivalence in the case of $d > 2$. We introduce classes of no-signaling $(n \rightarrow 1, d)$ RB, one which can simulate $B_n^+(+)$-BOX, second which can simulate $B_n^-(+)$-BOX and third which cannot simulate either. Finally to quantify the same we provide a resource inequality between $(n \rightarrow 1, d)$ RB and $B_n^+(+)$-BOX, and show that it is saturated.

I. INTRODUCTION

Popescu and Rohrlich $[1]$ have found that no-signaling condition allows for more non-locality (more violation of Bell inequalities) than what is allowed by quantum theory. They introduced PR-BOX (PR), the most non-local system that violate the Clauser-Horne-Shimony-Holt (CHSH) inequality. This paved the way to the quest for information theoretic principles that restrict the violation of Bell inequalities, in particular CHSH inequality to the Tsirelson bound. Using a PR Alice and Bob could win with certainty a $(2 \rightarrow 1)$ Random Access Code (RAC). Most basic RAC is a functionality wherein Alice has two bits and is allowed to communicate only one bit to Bob, for which we shall use the notation $(2 \rightarrow 1)$ specifying the encoding of 2 input bits into 1 bit message. Bob is not allowed to communicate with Alice and guesses one of Alice’s bits depending on a random choice. Within both classical and quantum theory it is not possible to win the RAC with certainty, however within the quantum theory the maximal probability of winning a RAC is higher than that within the classical theory.

RACs are a broad category of communication tasks which have found use in a variety of applications. For instance RAC serve as basic primitives for cryptography in classical information theory $[2, 6]$. Whereas in quantum senero they were a basis of the Wiesner’s first quantum protocols $[4, 5]$, semi-device independent cryptography $[6, 7]$ and randomness expansion $[8, 9]$. Apart of information theoretic tasks RACs have also found application in foundations of quantum mechanics $[10, 12]$. In particular the principle of information causality $[12]$ was primarily based on RACs.

The notion of inter-convertibility between resources is fundamental to any theory of resources, for instance entanglement theory $[13, 15]$, quantum communication theory $[16]$, thermodynamics $[17, 19]$ or in general no-signaling theory $[20, 21]$. The question raised in $[22]$ is whether a functionality (RAC) can be used to simulate a PR. As said a PR can simulate a RACBOX (RB), an arbitrary BOX which when supplemented with one bit of communication can win a RAC with certainty. It was shown that under the no-signaling condition a RB can simulate a PR. Thus the authors established equivalence between a no-signaling system (PR) and a functionality (no-signaling RAC). Furthermore they provided an example of a signaling RB which cannot simulate a PR. In this way a signaling resource (no signaling RB) was shown to be a weaker resource as compared to a no-signaling resource (no-signaling RB).

In this work we study in depth the relationship between dynamic resources RAC and static non-local resources. The paper is divided into two parts: In the first part we consider the case of bits. We introduce $B_n$-BOX a generalization of the PR with respect to number of bits supplied to Alice. We show that a $B_n$-BOX...
BOX is equivalent to a no-signaling \((n \rightarrow 1)\) RB. Here the notation \((n \rightarrow 1)\) implies that Alice encodes \(n\) bits into a single bit message. Further we find a bad signaling \((n \rightarrow 1)\) RB which cannot simulate the \(B^n\)-BOX. To quantify the same we provide a resource inequality and show that it is saturated. We show that \(n-2\) \((2 \rightarrow 1)\) RBs and a bit of classical communication cannot win a \((n \rightarrow 1)\) RAC. Using the equivalence we provide a protocol for winning a \((n \rightarrow 1)\) RAC using \(n-1\) PRs or equivalently \(n-1\) \((2 \rightarrow 1)\) RBs and a single bit of classical communication.

In the second part we focus on the more general case of \(d\) RBs. The case of higher dimensions is more intricate. We introduce two distinct non-local boxes \(B^d_{-\text{RB}}(+)\) BOX and \(B^d_{-\text{RB}}(-)\) BOX and show that these boxes can win a \((n \rightarrow 1, d)\) RAC. Here the notation \((n \rightarrow 1, d)\) specifies the encoding of \(n\) \(d\) bits into a single \(d\) bit message where \(d > 2\). However here no-signaling condition is not enough to enforce equivalence. We show using explicit examples that there exists no-signaling \((n \rightarrow 1, d)\) RB which cannot simulate \(B^d_{-\text{RB}}(+)\)-BOX or \(B^d_{-\text{RB}}(-)\)-BOX. Yet again to quantify the same we provide a resource inequality and show that it is saturated.

Abstract as these results may sound, they have connection to cryptography. It has been found by Wim van Dam [23] that using \(2^n-1\) PR boxes and single bit of communication one can achieve \((2^n \rightarrow 1)\) RAC. It has been further noted [24] that in case of device independent key such a protocol can be used as a hacking attack. Let Alice and Bob (the honest parties) share together \(n-1\) PR boxes with Eve (an eavesdropper), where \(n\) is the key length. Then upon certain wiring on their side, and leakage of single bit to Eve (e.g., by a Trojan-horse program), via the van Dam protocol Eve is in a favorable position that she can choose to learn the particular bit of key shared by Alice and Bob. Knowing this attack, we ask if there exist a smaller box than \(n-1\) PR boxes, which does the same task, which would make such a protocol significantly more difficult to detect. We give negative answer this question: any box which achieves the attack, is equivalent to \(n-1\) PR boxes, which amounts to certain non-negligible ”memory” inside of the Alice’s and Bob’s devices, making the attack harder to perform.

The attack invoked above can be however performed only in a world where extremal non-signaling boxes like PR box can be prepared. There are yet important reasons for such a world do not exist. One of them is the so called Information Causality principle [25]. Indeed, it disallow not only such boxes like PR to exist, but also disallows for the access in a manner of the RAC:

\[
\sum_i I(a_i : E | e=i) \leq 1
\]  

where \(a_i\) are the bits that Alice and Bob has (the key bits), and \(I\) denotes conditional mutual information (conditioned upon choice of the bit \(e=i\) by Eve). The RHS is the number of leaked bits, so that essentially the effect of RAC is suppressed: only the amount of leakage is known to Eve. One can therefore consider the secrecy extraction not only under quantum or non-signaling eavesdropping, but also under other principles such as the Information Causality, which we leave however for future work.

\[\text{FIG. 1. (a) (}n \rightarrow 1\text{) RAC. (b) }B_n\text{-BOX (c) (}n \rightarrow 1\text{) RB acts like an }n \rightarrow 1\text{ RAC, provided that }A' = A. \text{ In particular when }A \text{ is sent as a message to Bob and he inputs it into }A', \text{ then }B = a_b. \text{ (d) No-signaling }n \rightarrow 1\text{ RB satisfies }B = a_b \oplus A \oplus A'.\]

\[\text{II. THE CASE OF BITS}\]

In this section we provide generalization based on the number of input bits provided to Alice to \((2 \rightarrow 1)\) RAC and PR. We start of with defining the resources under consideration and subsequently study their relationships.

\((n \rightarrow 1)\) RAC, \((n \rightarrow 1)\) RB and \(B_n\)-BOX

Let us define a \((n \rightarrow 1)\) RAC. This is a box wherein, Alice is assigned \(n\) input bits \(a_0, a_1, \ldots, a_{n-1}\). Bob is assigned an input \(b \in \{0, 1, 2, \ldots, n-1\}\) to decide which of Alice’s bit he gets. Bob has a bit of output \(B\). For a \((n \rightarrow 1)\) RAC when \(B = a_b\) for all possible inputs [see Fig. [1]]. Consider a box that has an additional output on Alice’s side \(A\) and one more input \(A'\) on Bob’s side [see Fig. [1]]. Further suppose that it is no-signaling from Bob to Alice. Such a box we call \((n \rightarrow 1)\) RB when the following condition holds: if \(A = A'\), then it acts like a \((n \rightarrow 1)\) RAC i.e., \(B = a_b\). However when \(A \neq A'\) we do not put any restrictions. This box is described by a probability distribution \(P(A, B|a_0, a_1, \ldots, a_{n-1}, A', b)\) with a condition i.e., for all \(i \in \{0, 1, 2, \ldots, n-1\}, \)

\[
P(B = a_b | A' = A, b = i) = 1 \quad (2)
\]
Notice that we have the freedom to define the probability distribution of \((n \to 1) RB\) as long as it can be turned into a perfect \((n \to 1) RAC\). This implies we can have both signaling (only possible from Alice to Bob) and no-signaling \((n \to 1) RB\).

A no-signaling \((n \to 1) RB\) is a \((n \to 1) RB\) with an additional condition, namely when no message is sent Bob should not able gain any information about Alice’s inputs (no-signaling condition) i.e. \(P(B|a_0, a_1, \ldots, a_{n-1}, A', b) = P(B|a_0, a_1', \ldots, a_{n-1}', A', b)\) for all possible values of \(B, A', b\).

Now we characterize no-signaling \((n \to 1) RB\) by the following lemma.

**Lemma 1** A no signaling \((n \to 1) RB\) for \(A \not= A'\) acts as an anti-\((n \to 1) RAC\), i.e., it satisfies

\[
B = a_0 \oplus A \oplus A' \tag{3}
\]

Let Alice and Bob share a no-signaling \((n \to 1) RB\). Suppose Alice does not send the message and Bob chooses \(A'\) randomly,

\[
P(A = A') = P(A \not= A') = \frac{1}{2} \tag{4}
\]

Let \(P(B = a_i|b = i)\) denote the probability that Bob’s outcome is correct when no message was sent. The no-signaling condition along with the fact that Alice’s inputs are uniformly distributed implies \(P(B = a_i|b = i) = \frac{1}{2}\) see Fig. 1. Upon conditioning on the events \(A = A'\) and \(A \not= A'\) we obtain,

\[
P(B = a_i|A = A', b = i)P(A = A) + \\
P(B = a_i|A \not= A', b = i)P(A \not= A') = \frac{1}{2} \tag{5}
\]

From (4) and (1),

\[
\frac{1}{2} + P(B = a_i|A \not= A', b = i)\frac{1}{2} = \frac{1}{2} \tag{6}
\]

implies,

\[
P(B = a_i|A \not= A', b = i) = 0 \tag{7}
\]

If \(A \not= A'\), \(B = a_i \oplus 1\) or,

\[
P(B = a_i \oplus 1|A \not= A', b = i) = 1 \tag{8}
\]

Thus equations (3) and (8) lead to desired result (6). ■

We shall now present an instance of a signaling \((n \to 1) RAC\) which performs its duty regarding \((n \to 1) RAC\) when supplemented with a bit of communication but is signaling from Alice to Bob. It is a \((n \to 1) RB\) with an additional condition,

\[
P(B = a_i|A \not= A', b = i) = \frac{1}{2} \tag{9}
\]

In this case when no message is sent Bob could still gain some information about Alice’s inputs (no-signaling condition) as,

\[
P(B = a_i|b = i) = \frac{3}{4} \tag{10}
\]

Finally we define our contender from non-local resources as a generalization to the PR. A \(B_n\)-BOX is a bipartite no signaling resource (correlation) wherein, Alice’s side has \(n - 1\) input bits \(x_1, x_2, x_3, \ldots, x_{n-1}\) and an output bit \(X\). Bob’s side has \(n\) possible inputs corresponding to a input bit \(y \in \{0, 1, 2, 3, \ldots, n - 1\}\) and an output bit \(Y\). A \(B_n\)-BOX is described by the probability distribution \(P(X, Y|x_1, x_2, \ldots, x_{n-1}, y)\) such that,

\[
P(X, Y|x_1, x_2, \ldots, x_{n-1}, y) = \begin{cases} 
\frac{1}{2} & \text{for } X \oplus Y = x_y, \\
0 & \text{else.}
\end{cases} \tag{11}
\]

The condition,

\[
X \oplus Y = x_y \tag{12}
\]

will be called \(B_n\) correlations, where \(x_0 = 0\).

**Relationships**

We shall proof the following theorem which deals with equivalence between a \(B_n\)-BOX and a no-signaling \((n \to 1) RB\).

**Theorem 1** A \(B_n\)-BOX and a no-signaling \((n \to 1) RB\) are strictly equivalent.

In order to prove equivalences between two resources it is necessary and sufficient to show that each can simulate the other. Here we provide a protocol using which Alice and Bob sharing a \(B_n\)-BOX can win a \((n \to 1) RAC\).

1. Alice receives \(n\) input bits \(a_0, a_1, \ldots, a_{n-1}\) and she inputs \(x_i = a_0 \oplus a_i\) for \(i \in \{1, 2, \ldots, n - 1\}\).
2. Alice obtains an output bit \(X\) from \(B_n\)-BOX sends the message \(m = a_0 \oplus X\).
3. Bob receives \(y = b\) and obtains output bit \(Y\).
4. Bob outputs the final answer \(B = m \oplus Y = a_0 \oplus X \oplus Y\).

Now for a \(B_n\)-BOX, \(X \oplus Y = x_y\). Therefore \(B = a_0 \oplus x_y = a_i\) if \(y \in \{1, 2, \ldots, n - 1\}\) and \(B = a_0\) if \(y = 0\).

Now to complete the proof, we provide a protocol using which Alice and Bob sharing a no-signaling \((n \to 1) RB\) can simulate the statistics of \(B_n\)-BOX.

1. Alice receives \(n - 1\) input bits \(x_1, x_2, \ldots, x_{n-1}\) and she inputs \(a_i = x_i\) for \(i \in \{1, 2, \ldots, n - 1\}\) and fixes \(a_0 = 0\).
2. Alice obtains an output bit \(X = A\) from the \((n \to 1) RB\).
3. Bob receives \(y \in \{0, 1, 2, \ldots, n - 1\}\) and fixes \(A' = 0\) obtains output bit \(Y = B\).

Observe whenever \(X = A = A' = 0\), using (2) we get \(Y \oplus X \oplus Y = B = a_0 = x_i\). Further whenever \(X = A \neq 0\), using (6) we obtain \(Y \oplus X = Y \oplus 1 = B \oplus 1 = a_0 \oplus 1 \oplus 1 = \frac{1}{2}\) as desired.
We shall now provide the following resource inequality which implies that having access to any \((n \to 1)\) RB (signaling or no-signaling), one bit of communication (c-bit) and one shared random bit (sr-bit) we can simulate a \(B_n\)-BOX and additionally obtain erasure channel \(\xi\) with probability of erasure \(\epsilon = P(y \neq 0)\).

**Resource Inequality 1** between a \((n \to 1)\) RB and a \(B_n\)-BOX:

We show that the following inequality holds for any \((n \to 1)\) RB,

\[
(n \to 1) RB + 1c - bit + 1sr - bit \geq B_n - BOX + \xi
\]

(13)

where \(\xi\) is a bit erasure channel.

Since by definition \((n \to 1)\) RB plus 1 bit of communication offers a RAC we shall prove the following inequality instead,

\[
(n \to 1) RAC + 1sr - bit \geq B_n - BOX + \xi
\]

(14)

In order to reproduce a \(B_n\)-BOX in the case when \(y = 0\), one can use just shared randomness since Alice’s and Bob’s outputs must be the same i.e., \(X \oplus Y = 0\). The \((n \to 1)\) RAC is not used up and can be utilized for communication of the bit \(a_0\). But when \(y \neq 0\), Bob will need the \((n \to 1)\) RAC to reproduce \(B_n\) correlations as \(X \oplus Y = x_y\), and in this case no communication will be performed.

Let \(z\) denote the bit to be communicated. Alice puts \(a_0 = z\) and \(a_i = x_i\) where \(i \in \{1,2,..n - 1\}\), while Bob inputs \(b = y\). Alice and Bob use shared random bits for outputs. When \(y = 0\), Bob simply outputs the shared random bit and \(B_n\)-BOX is reproduced. When \(y \neq 0\), Bob performs a CNOT gate with his output \(B\) being the control bit and his shared random bit being target bit. When \(y \neq 0\) we need to have correlations when \(x_1 = 0\) and anti correlations when \(x_1 = 1\) given that Bob inputs \(b = y = i\). From the definition of a \((n \to 1)\) RAC, when \(b = y = i\) where \(i \in \{1,2,..n - 1\}\), we have \(B = x_i\). Hence, when \(x_1 = 0\) the shared random bit is bot flipped and Alice and Bob have correlations and when \(x_1 = 1\) the bit is flipped and they have anti correlations. Thus the protocol perfectly simulates a \(B_n\)-BOX.

When \(y = 0\), Bob’s output \(B = a_0 = z\), hence the message is perfectly transmitted, whereas for \(y \neq 0\) Bob’s output \(B = xy\) and the message is lost. Thus we obtain an erasure channel with probability of erasure \(\epsilon = P(y \neq 0) = \frac{n - 1}{n}\) (assuming Bob’s input are uniformly distributed).

**Tightness of resource inequality**\([2]\). This resource inequality is trivial for a no-signaling \((n \to 1)\) RB. However using the signaling \((n \to 1)\) RB defined above we can tighten the inequality through the following theorem.

**Theorem 2** Assume that \(x_1, x_2,..x_n, y\) are generated uniformly at random. Let us suppose for the signaling \((n \to 1)\) RB described above, a channel \(\Lambda\) satisfies the following inequality:

\[
(n \to 1) RB + 1c - bit \geq B_n - BOX + \Lambda
\]

(15)

Then the capacity of bit channel \(\Lambda\) is upper bounded by \(\frac{1}{n}\).

For the proof see Appendix A. The theorem shows that in order to simulate a \(B_n\)-BOX by such a no-signaling \((n \to 1)\) RB, we need, in addition at-least \(\frac{n - 1}{n}\) bit of communication. Thus, in this respect the signaling \((n \to 1)\) RB is weaker than a no-signaling \((n \to 1)\) RB.

**Lemma 2** \(n - 2\) no-signaling \((2 \to 1)\) RBs and 1 c-bit can not win a \((n \to 1)\) RAC.

(2 \to 1) RACs as building blocks for \((n \to 1)\) RAC

It is a known fact that some number of no-signaling \((2 \to 1)\) RBs and 1 c-bit can be used to construct a general \((n \to 1)\) RAC. Through the following theorem we shall show that the minimum number of no-signaling \((2 \to 1)\) RBs to win a \((n \to 1)\) RAC when Alice is allowed to communicate \(1\) c-bit is \(n - 1\).

**Theorem 3** \(n - 1\) no-signaling \((2 \to 1)\) RBs are necessary and sufficient to win a \((n \to 1)\) RAC when Alice is allowed to communicate 1 c-bit.

We shall prove this theorem in two parts. First we shall prove the following lemma,

**Lemma 2** \(n - 2\) no-signaling \((2 \to 1)\) RBs + 1 c-bit cannot win a \((n \to 1)\) RAC.

For the proof see Appendix B. To complete the proof we provide a protocol which uses \(n - 1\) no-signaling \((2 \to 1)\) RB and 1 c-bit of additional communication to win a \((n \to 1)\) RAC. The protocol we provide uses two subroutines namely, concatenation and addition (see Appendix C). For instance a \((7 \to 1)\) RAC requires 6 no-signaling \((2 \to 1)\) RBs and a c-bit of communication [see Fig. 2].

**III. FROM BITS TO DITS.**

In this section we provide further generalization based on the dimension \(d\) of inputs provided to Alice and message she sends to \((n \to 1)\) RAC and \(B_n\)-BOX. We start with defining the resources under consideration and subsequently study their relationships.

\[
(n \to 1, d) \text{ RAC}, (n \to 1, d) \text{ RB}, B_{n, d}(+)\text{-BOX} \text{ and } B_{n, d}^d(-)\text{-BOX}
\]

We start by defining a \((n \to 1, d)\) RAC. This is a box wherein, Alice is assigned \(n\) input dits \(a_0, a_1,..a_{n-1}\) where \(a_i \in \{0,1,..d - 1\}\) for \(i \in \{0,1,..n - 1\}\). Bob is assigned an input \(b \in \{0,1,2,..n - 1\}\) to decide which of Alice’s dit
he gets. Bob has a dit of output $B$. Such a box is a $(n \to 1, d)$ RAC when $B = a_b$ for all possible inputs [see Fig. 2].

Let us consider another box that has an additional output dit on Alice’s side and one more input dit $A'$ on Bob’s side [see Fig. 3]. Further suppose that it is no-signaling from Bob to Alice. Such a box we call $(n \to 1, d)$ RB when the following condition holds: if $A = A'$, then it acts like a $(n \to 1, d)$ RAC i.e., $B = a_b$. However when $A \neq A'$ we do not put any restrictions. This box is described by a probability distribution $P(A, B|a_0,a_1,...,a_{n-1},A',b)$ with a condition i.e., for all $i \in \{0,1,2,...,n-1\}$,

$$P(B = a_i|A' = A, b = i) = 1$$  \hspace{1cm} (16)

This box is designed such that when supplemented with one bit of communication, it offers a perfect $(n \to 1, d)$ RAC. A no-signaling $(n \to 1, d)$ RB is a $(n \to 1, d)$ RB with an additional condition, namely when no message is sent Bob should not able gain any information about Alice’s inputs (no-signaling condition), i.e. $P(B|a_0,a_1,...,a_{n-1},A',b) = P(B|a_0,a_1',...a_{n-1}',A',b)$ for all possible values of $B, A', b$ [see Fig. 3].

The case of $d > 2$ presents itself with intricate details. To see this let Alice and Bob share a no-signaling $(n \to 1, d)$ RB. Suppose Alice does not send the message and Bob chooses $A'$ randomly,

$$P(A = A') = \frac{1}{d}$$  \hspace{1cm} (17)

Let $P(B = a_i|b = i)$ denote the probability that Bob’s outcome is correct when no message was sent. The no-signaling condition along with the fact that Alice’s inputs are uniformly distributed implies $P(B = a_i|b = i) = \frac{1}{d}$.

Now,

$$P(B = a_i|b = i) = P(B = a_i, A = A'|b = i) + P(B = a_i, A \neq A'|b = i) = \frac{1}{d}$$  \hspace{1cm} (18)

$$P(B = a_i|b = i) = P(B = a_i|A = A', b = i)P(A = A) + P(B = a_i|A \neq A', b = i)P(A \neq A') = \frac{1}{d}$$  \hspace{1cm} (19)

From (16, 17),

$$\frac{1}{d} + P(B = a_i|A \neq A', b = i)\frac{d-1}{d} = \frac{1}{d}$$  \hspace{1cm} (20)

implies,

$$P(B = a_i|A \neq A', b = i) = 0$$  \hspace{1cm} (21)

Notice that this does not completely specify the value (or probability distribution) of $B$ except for the fact that it must not be $a_i$, as compared to the case of $d = 2$. Thus, even under no-signalling for the case $d > 2$ we have the freedom to define probability distribution of no-signaling $(n \to 1, d)$ RB as long as it can be turned into a perfect $(n \to 1, d)$ RAC. This implies we can define subclasses of no-signaling $(n \to 1, d)$ RB based on additional condition over the probability distribution. W.L.o.g when no message is sent we assume that Bob always inputs $A'$, then we have the following three different no-signaling $(n \to 1, d)$ RB,

- **no-signaling $(n \to 1, d)$ RB (+) :** This particular instance of no-signaling $(n \to 1, d)$ RB is defined by additional condition over probability distribution $P(B +_d A = a_i|A, A' = 0, b = i) = 1$ (where $+_d$ is addition modulo $d$).

- **no-signaling $(n \to 1, d)$ RB (-) :** This one is defined by additional condition over probability distribution $P(B -_d A = a_i|A, A' = 0, b = i) = 1$ (where $-_d$ is subtraction modulo $d$).

- **no-signaling $(n \to 1, d)$ RB (3) :** This particular instance of no-signaling $(n \to d)$ RB is defined by additional condition over probability distribution $P(B = j|A \neq 0, A' = 0, b = i) = \frac{1}{d}$ for $j \in \{0,1,2,...,d-1\} - a_i$. This is a bad instance in the sense that while it fulfills its duties as a $(n \to 1, d)$ RAC when $A' = A$ but it cannot simulate either $B_{n}^{d}(+)$-BOX or $B_{n}^{d}(-)$-BOX.

The case of $d > 2$ is also rich in complexity when it comes to defining a generalization to the PR (or a $B_{n}$-BOX). For instance we define two possible generalizations namely, $B_{n}^{d}(+)$-BOX and $B_{n}^{d}(-)$-BOX. A $B_{n}^{d}(+)$-BOX is a bipartite no signaling resource (correlation) wherein, Alice’s side has $n - 1$ input dits $x_1, x_2, x_3,...,x_{n-1}$ and an output dit $X$. Bob’s BOX has $n$ possible inputs corresponding to a input dit $y \in \{0,1,2,3,...,n-1\}$ and
We will prove the above theorem by giving explicit protocols. We provide a protocol using which Alice and Bob sharing a $B^d(+)\text{-BOX}$ and 1 c-dit of communication can win a $(n \to 1, d)$ RAC with certainty.

1. Alice receives $n$ input bits $a_0, a_1, \ldots, a_{n-1}$ and she inputs $x_i = a_i - a_0$ for $i \in \{1, 2, \ldots, n-1\}$.
2. Alice obtains an output bit $X$ from $B^d_n(+) \text{-BOX}$ sends the message $m = X + d a_0$.
3. Bob receives $y \in \{0, 1, 2, \ldots, n-1\}$ and obtains output bit $Y$.
4. Bob outputs the final answer $B = m + d Y = a_0 + d X + d Y$.

Now for a $B^d_n(+)\text{-BOX}$, $X + d Y = x_y$. Therefore $B = a_0 + d x_y = a_i$ if $y \in \{1, 2, \ldots, n-1\}$ and $B = a_0$ if $y = 0$.

Finally to complete to proof we provide a protocol using which Alice and Bob sharing no-signaling $(n \to 1, d)$ RB (+) can simulate a $B^d_n(+)\text{-BOX}$ perfectly.

1. Alice receives $n-1$ input bits $x_1, x_2, \ldots, x_{n-1}$ and she inputs $a_i = x_i$ for $i \in \{1, 2, \ldots, n-1\}$ and fixes $a_0 = 0$.
2. Alice obtains an output bit $X = A$ from the no-signaling $(n \to 1, d)$ RB (+).
3. Bob receives $y \in \{0, 1, 2, \ldots, n-1\}$ and fixes $A' = 0$ obtains output bit $Y = B$.

Observe whenever $X = A = A' = 0$, $y + d X = Y = B = a_0 = x_i$. Further whenever $X = A \neq 0$, $y + d X = B + d A = a_0 = x_i$.

**Theorem 5** A $B^d_n(-)\text{-BOX}$ and a no-signaling $(n \to 1, d)$ RB (-) are strictly equivalent for $d > 2$.

The proof is omitted as it is similar to that of Theorem 4.

We shall now provide the following resource inequality which implies that having access to any $(n \to 1, d)$ RB, one dit of communication (c-dit) and one shared random dit (sr-dit) we can simulate a $B^d_n(+)\text{-BOX}$ (or $B^d_n(-)\text{-BOX}$) and additionally obtain erasure dit channel $\xi_d$ with probability of erasure $\epsilon = p(y \neq 0)$:

**Resource Inequality 2** between a $(n \to 1, d)$ RAC and a $B^d_n(+)\text{-BOX}$:

We show that the following inequality holds for any $(n \to 1, d)$ RB,

\[(n \to 1, d) RB + 1c - dit + 1sr - dit \geq B^d_n(+) - BOX + \xi_d\]  

where $\xi_d$ is a dit erasure channel.

Since by definition $(n \to 1, d)$ RB plus 1 dit of communication offers a $(n \to 1, d)$ RAC we shall prove the following inequality instead,

\[(n \to 1, d) RAC + 1sr - dit \geq B^d_n(-) - BOX + \xi_d\]  

**Theorem 4** A $B^d_n(+)\text{-BOX}$ and a no-signaling $(n \to 1, d)$ RB (+) are equivalent for $d > 2$.

**Relationships**

We begin with showing through the following theorem that shows $B^d_n(+)\text{-BOX}$ is equivalent to a no-signaling $(n \to 1, d)$ RB (+).

\[a_0, a_1, \ldots, a_n \in \{0, 1, \ldots, d\}, \quad b \in \{0, 1, \ldots, n\}, \quad x_1, \ldots, x_n \in \{0, 1, \ldots, d\}, \quad y \in \{0, 1, \ldots, n\}\]

\[\begin{array}{c}
(n \to 1, d) \text{ RAC} \\
B^d_n(+) \text{ BOX (}X \leftarrow Y = x_y, a \in A\text{)} \\
B^d_n(-) \text{ BOX (}X \leftarrow Y = x_y, x \in X\text{)} \\
\end{array}\]

FIG. 3. (a) $(n \to 1, d)$ RAC. (b) $B^d_n(+)\text{-BOX}$ (c) $(n \to 1, d)$ RB acts like an $(n \to 1)$ RAC, provided that $A' = A$. In particular when $A$ is sent as a message to Bob and he inputs it into $A'$, then $B = a_b$. (d) $B^d_n(-)\text{-BOX}$.

receives output bit $Y$. A $B^d_n(+)\text{-BOX}$ is described by the probability distribution $P(X, Y|X_1, x_2, \ldots, x_{n-1}, y)$ such that,

\[P(X, Y|X_1, x_2, \ldots, x_{n-1}, y) = \begin{cases} \frac{1}{2} & \text{if } X + d Y = x_y; \\ 0 & \text{else.} \end{cases}\]

The condition,

\[X + d Y = x_y\]

will be called $B^d_n(+)\text{-BOX}$ correlations.

Similarly a $B^d_n(-)\text{-BOX}$ is described by the probability distribution $P(X, Y|X_1, x_2, \ldots, x_{n-1}, y)$ such that,

\[P(X, Y|X_1, x_2, \ldots, x_{n-1}, y) = \begin{cases} \frac{1}{2} & \text{if } X - d Y = x_y; \\ 0 & \text{else.} \end{cases}\]

The condition,

\[X - d Y = x_y\]

will be called $B^d_n(-)\text{-BOX}$ correlations.
In order to reproduce a $B^d_n(\pm)$-BOX (or $B^d_n(\mp)$-BOX) in the case when $y = 0$, one can use just shared randomness of the form $s_A + d s_B = 0$. The $(n \to 1, d)$ RAC is not used up and can be utilized for communication of the dit $a_0$. But when $y \neq 0$, Bob will need the $(n \to 1, d)$ RAC to reproduce $B^d_n(\pm)$ correlations as $X + d Y = x y$, and in this case no communication will be performed.

Let $z$ denote the dit to be communicated. Alice puts $a_0 = z$ and $a_i = x_i$ where $i \in \{1, 2, \ldots, n - 1\}$, while Bob inputs $b = y$. Alice and Bob use shared random dits for outputs. When $y = 0$, Bob simply outputs the random dit and $B^d_n(\pm)$-BOX is reproduced. When $y \neq 0$, Bob adds $B$ to her shared random bit $s_B$. From the definition of a $(n \to 1, d)$ RAC, when $b = y = 1$ where $i \in \{1, 2, \ldots, n - 1\}$, we have $B = x_i$. Hence, when $x_i = 0$ the shared random bit remains the same so that $X + d Y = s_A + d s_B = 0$ and when $x_i \neq 0$ Bob produces $Y = s_B + B = s_B + d x_i$ so that $X + d Y = s_A + d s_B + d x_i = x_i$ and in this case the message is lost. Thus we obtain a dit erasure channel with probability of erasure $\epsilon = P(y = 0) = \frac{s - 1}{n}$ (assuming Bob’s input are uniformly distributed).

Now we proceed to show that the no-signaling $(n \to 1, d)$ RB (3) cannot simulate the $B^d_n(\pm)$-BOX. We shall use the case of $n = 2, d = 3$ for simplicity.

**Tightness of resource inequality**. This resource inequality is trivial for a no-signaling $(2 \to 1, 3)$ RB (+) when trying to simulate $B^d_2(\pm)$-BOX or no-signaling $(2 \to 1, 3)$ RB (-) trying to simulate $B^d_2(\mp)$-BOX. However using the no-signaling $(2 \to 1, 3)$ RB (3) defined above we can tighten the inequality through the following theorem.

**Theorem 6** Assume that $x, y$ (inputs to the $B^d_2(\pm)$-BOX) are generated uniformly at random. Let us suppose for the no-signaling $(2 \to 1, 3)$ RB (3) described above, a channel $\lambda_3$ satisfies the following inequality:

$$ (2 \to 1, 3) RB + 1c - 3it \geq B^d_2(\pm) - BOX + \Lambda_4 $$

Then the mutual information of 3 bit channel is upper bounded by $\frac{1}{2}$.

For the proof see Appendix D. The theorem shows that in order to simulate a $B^d_2(\pm)$-BOX by such a no-signaling $(2 \to 1, 3)$ RB (3), we need, in addition at least $\frac{1}{2} 3it$ of communication. Thus, in this respect the no-signaling $(2 \to 1, 3)$ RB (3) is weaker than a no-signaling $(2 \to 1, 3)$ RB (+) or no-signaling $(2 \to 1, 3)$ RB (-). It is straightforward to generalize above theorem for arbitrary $n, d$.

**Theorem 7** Assume that $x_1, x_2, \ldots, x_n, y$ are generated uniformly at random. Let us suppose for the no-signaling $(n \to 1, d)$ RB (3) described above, a channel $\lambda_d$ satisfies the following inequality:

$$ (n \to 1, d) RB + 1c - dit \geq B^d_n(\pm) - BOX + \Lambda_d $$

Then the mutual information of dit channel is upper bounded by $\frac{1}{d}$.

The proof of the above theorem follows directly from the proof of Theorem 2 and [6].

**IV. CONCLUSIONS AND FURTHER DIRECTIONS**

In this work we introduced generalizations of a static no-signaling non-local resource PR based on number of inputs provided to Alice, namely $B^d_n$-BOX and then based on the dimension of the inputs provided to Alice, namely $B^d_n(\pm)$-BOX and $B^d_n(\mp)$-BOX.

In the former case we show that a $B^d_n$-BOX can win with a certainty a functionality ($n \to 1$) RAC. Furthermore a no-signaling ($n \to 1$) RB can simulate a $B^d_n$-BOX. Hence the two resource are shown to be equivalent. We bring up a signaling ($n \to 1$) RB and show that it cannot simulate the $B^d_n$-correlation. To quantify the above we provide a resource inequality and show it is saturated. As an application to the above we prove that under the restriction that Alice is only allowed to communicate 1 bit of communication we require at-least $(n - 1)$ no-signaling ($2 \to 1, d$) RB (or PR) in order to win a $(n \to 1)$ RAC.

In the latter case of dimension $d > 2$ we find that the no-signaling condition is not enough to enforce a strict equivalence between $B^d_1(\pm)$-BOX (or $B^d_1(\mp)$-BOX) and no-signaling $(n \to 1, d)$ RB. We introduce three classes of no-signaling ($n \to 1, d$) RB, namely $(+), (-), (3)$. We show that a $B^d_n(\pm)$-BOX (or $B^d_n(\mp)$-BOX) is strictly equivalent to no-signaling ($n \to 1, d$) RB (+) or (-). However $(n \to 1, d)$ RB (3) cannot simulate $B^d_n(\pm)$-BOX (or $B^d_n(\mp)$-BOX). Finally to quantify the same we provide a resource inequality and show that it is saturated.

We have shown that in the case of higher dimension $d > 2$ there exists no-signaling ($n \to 1, d$) RB which cannot simulate all of extremal non-local resources which can be used to win a $(n \to 1, d)$ RAC. However the question remains open that whether there exists any extremal non-local resource which cannot win a $(n \to 1, d)$ RAC.

Such equivalences can be generalized to multiparty scenario. In [26] we study equivalence between $n$-party Svetlichny BOXes and a generalization of the standard two party RAC, Controlled Random Access Codes (C-RAC).

**V. ACKNOWLEDGMENTS**

We acknowledge useful discussions with Michal Horodecki. This project was supported by grant, Harmonia 4 (Grant number: UMO-2013/08/M/ST2/00626), ERC AdG QOLAPS. AC would like to acknowledge CCNSB/CSTAR of IIT-Hyderabad and support by Prof. Indranil Chakraborty. KH acknowledges grant Sonata Bis 5 (Grant number: 2015/18/E/ST2/00327).
APPENDIX A: PROOF OF THEOREM 2

We provide the proof for resource inequality and consequently the in-equivalence of signaling \((n \rightarrow 1)\) RB and \(B_n\)-BOX. We give the proof in two parts:

1. We shall show that if the bit of communication is not used to send the output of Alice’s RB \(A\) then \(B_n\)-BOX cannot be obtained.
2. If the bit of communication is used to send \(A\) and \(B_n\) BOX is obtained, then the capacity of obtainable channel \(A\) is upper bounded by \(\frac{1}{n}\) bit.

Part I

The Part I says that if we do not input \(A\) to \(A'\) then \(B_n\)-BOX cannot be obtained.

Let us denote \(m\) for the one-bit message to be communicated to Bob. The goal is to obtain perfect \(B_n\) correlations i.e. \(Y = X \otimes x_y\) in any case \(m = 0\) or 1. Bob’s output for any give \(m\) depends on \(A\)’s settings on Bob’s side: \(Y = Y(b, A', B)\). For any fixed \(m = m_0\) there are two possible cases \(A' = A\) and \(A \neq A'\). In the first case \(B_n\) correlations are obtained using by processing a perfect RAC. But in the case \(A' \neq A\) the signaling \(n \rightarrow 1\) RB offers a random \(B\) which does not depend on the work of RAC. Hence \(Y\) can be obtained solely from processing of \(y\) i.e. \(Y = Y(y)\). Since we want to obtain perfect \(B_n\) correlations \(Y(y = 0) = X\) and \(Y(y \neq 0) = X \otimes x_y\). By adding \(Y(y = 0)\) and \(Y(y \neq 0)\) Bob can compute \(x_y\). We therefore obtain, that in the case \(A' \neq A\), the value of \(x_y\) must be known to Bob.

However signaling \(n \rightarrow 1\) RB is no-signaling from Bob to Alice. For \(y \in \{0, 1, \ldots, n-1\}\) and \(n > 2\) Alice cannot know in advance the value of \(y\) in order to send \(m = x_y\). These leaves only one option to send \(A\) which shall be dealt with in the next part.

Part II

We will show using information theoretic tools, that if the signaling \(n \rightarrow 1\) RB considered in Theorem 2 supplemented with one bit of communication is to reproduce exactly \(B_n\)-BOX and some channel, then the mutual information of the channel must be bounded by \(\frac{1}{n}\) (assuming that Alice’s output of the RB \(A\) will be inserted directly into as Bob’s second input to the RB i.e. \(A' = A\).

Assumptions: Alice is given variables \(x_1, x_2, \ldots, x_{n-1}\) and \(z\). Bob is given variable \(y \in \{0, 1, \ldots, n-1\}\). Both are given access to common variable \(s\) such that \(x_1, x_2, \ldots, x_{n-1}, z, y, s\) are mutually independent. Alice generates \(A\) from \(x_1, x_2, \ldots, x_{n-1}, z, s\) and inputs \(a_0, a_1, \ldots, a_{n-1}\) to RAC. Bob
generates b from y, s and inputs it into RAC. These strategies result in shared joint probability distribution 
P(x_1, x_2, x_{n-1}, z, y, s, b, A', A, B), where B = a_b is obtained from (n → 1) RAC on Bob's side, and Y is generated out of b, B, s, y by Bob. First we shall express the Theorem 2 in other words. Under Assumptions 1, if variables x_1, x_2, x_{n-1}, y, A, B perfectly reproduce B_n correlations, there holds:

\[ I(z : B, b, y, s) \leq \frac{1}{n} \tag{30} \]

where z is the message that Alice sends to Bob. We shall prove the above in two parts:

1. First we shall use entropies and correlation to state the fact that to simulate the B_n BOX Bob has to guess perfectly X when y = 0 and x_y \oplus X when y \neq 0.

2. Second we shall show that it is impossible to send more than 1.31t through a channel with 1.31t capacity. As in our case Alice would like to send both x_1 and z which bounds Bob’s possible information gain about z.

**Lemma 3** Under Assumptions 1, if variables (x_1, x_2, x_{n-1}, y, A, B) simulate perfectly B_n correlations, there holds:

\[ I(B : X | b, s, y = 0) = H(X | b, s, y = 0) \tag{31} \]

\[ I(B : X \oplus x_y | b, s, y \neq 0) = H(X \oplus x_y | b, s, y \neq 0) \tag{32} \]

In order to reproduce B_n-correlations given y = 0, Bob should perfectly guess X, whereas given y \neq 0 he should perfectly guess X \oplus x_y. This implies that there must be max_j [p(a = j | B = i, b = k, y = 0, s = i)] = 1. Then for y = 0 the values of variables B, b, s should determine uniquely the value of X i.e. H(X | B, b, s, y = 0) = 0. In such a case, i.e. I(X : B | b, s, y = 0) = H(X | b, s, y = 0). Analogously, we obtain I(X \oplus x_y : B | b, s, y \neq 0) = H(X \oplus x_y | b, s, y \neq 0).

One cannot send more than one bit through a single-bit wire.

Here, we prove the following theorem which provides the key argument in the proof of Theorem 2. Namely it’s shows a tradeoff between Bob’s correlations with X and X \oplus x_y (that should be high if he simulates B_n correlations) and his correlations with z.

**Theorem 8** Under aforementioned assumptions, there holds:

\[ \frac{1}{n} \left[ \sum_{i=1}^{n-1} I(X \oplus x_i : B | b, s, y = i) + I(X : B | b, s, y = 0) \right] + \frac{1}{n} I(X : B | b, s, y = 0) \leq \frac{1}{n} I(X : X \oplus x_2 : \ldots : X \oplus x_{n-1} : z | b, s) + H(B | b, s, y) \tag{33} \]

In the proof for the above theorem we use the following fact, which captures that one cannot send reliably n bits (n > 1) through a single bit wire, unless the bits are correlated.

**Lemma 4** For any random variables S_1, S_2, \ldots, S_n, T, V there holds:

\[ \sum_{i=1}^{n} I(S_i : T | V) \leq I(S_1 : S_2 : \ldots : S_n : T | V) \tag{34} \]

where I(S_1 : S_2 : \ldots : S_n : T | V) = \sum_{i=0}^{n} H(S_i | V) + H(T | V) - H(S_1, S_2, \ldots, S_n, T | V)

First we proof the above fact without conditioning. We shall use the following fact recursively. For any random variables S_i, S_j, T it follows directly from strong subadditivity:

\[ H(S_i S_j U) + H(T) \leq H(S_i T) + H(S_j T) \tag{35} \]

By expressing mutual information via Shannon entropies, we can rewrite LHS as:

\[ n H(T) + \sum_{i=1}^{n} H(S_i) - \sum_{i=1}^{n} H(S_i T) \tag{36} \]

Using (35) n-1 times we can upper bound LHS by,

\[ H(T) + \sum_{i=1}^{n} H(S_i) - H(S_1, S_2, \ldots, S_n, T) \equiv I(S_1 : S_2 : \ldots : S_n : T) \tag{37} \]

Which is the desired result without conditioning on V.

We can now fix V = v, and the thesis will hold for conditional distribution p(S_1, S_2, \ldots, S_n | T | V = v):

\[ \sum_{i=1}^{n} I(S_i : T | V = v) \leq I(S_1 : S_2 : \ldots : S_n : T | V = v) \tag{38} \]

The thesis is obtained after multiplying each side by p(V = v), and summing over range of variable V.

Moving on with the proof of Theorem 8. Let us reformulate LHS and fix s=j:

\[ \frac{1}{n} \left[ \sum_{i=1}^{n-1} I(X \oplus x_i : B | b, s, y = i) + I(X : B | b, s, y = 0) \right] + I(z : B | b, s, y) \]

By decomposing the last term into n terms, which depend on the value of y we obtain:

\[ \frac{1}{n} \left[ \sum_{i=1}^{n-1} (I(X \oplus x_i : B | b, s, y = i) + I(z : B | b, s, y = i)) + I(X : B | b, s, y = 0) \right] \tag{39} \]

We use Lemma 4 (for n = 2) pairwise to show that the above quantity is upper bounded by

\[ \frac{1}{n} \left[ \sum_{i=1}^{n-1} (I(X \oplus x_i : z | b, s, y = i) + I(X : z | b, s, y = i)) + I(B : X \oplus x_i, z : b, s, y = i) \right] + I(B : X, z : b, s, y = 0) \tag{40} \]
Observe that \((X \oplus x_i, z|s = j)\) is independent from \((y, b|s = j)\), hence there is \(I(X \oplus x_i : z|s = j, y = i) = I(X \oplus x_i : z|s = j)\) and similarly \(I(X : z|s = j, y = 0) = I(X : z|b, s = j)\). Multiplying both sides these equalities by \(p(s = i)\) and summing up over values of \(s\) we get \(I(X \oplus x_i : z|b, s = j)\) and \(I(X : z|b, s = j, y = 0) = I(X : z|b, s)\). Applying the same to (41) and using the latter equalities we obtain:

\[
\frac{1}{n} \sum_{i=1}^{n-1} \left( I(X \oplus x_i : z|b, s) + I(B : X \oplus x_i, z|b, s, y = i) \right) + I(X : z|b, s) + I(B : X, z|b, s, y = 0) \]  

(41)

So that we can use Lemma 4 to the terms \(\sum_{i=1}^{n-1} I(X \oplus x_i : z|b, s) + I(X : z|b, s)\) to obtain:

\[
\frac{1}{n} \sum_{i=1}^{n-1} \left( I(X : X \oplus x_1 \oplus x_2 : \ldots X \oplus x_{n-1} : z|b, s) + I(B : X \oplus x_i, z|b, s, y = i) + I(B : X, z|b, s, y = 0) \right) \]  

(42)

The terms \(I(B : X \oplus x_i, z|b, s, y = i)\) are bounded by \(H(B|b, s, y = i)\). Similarly the term \(I(B : X, z|b, s, y = 0)\) is bounded by \(H(B|b, s, y = 0)\) which because of the factor \(\frac{1}{n}\) give rise to \(H(B|b, s, y)\) and the assertion follows. ■

Finally we come back to the proof of Theorem 2. We now prove the main result. To this end we first observe that in fact it is sufficient to show:

\[
I(z : B|b, y, s) \leq \frac{1}{n} \]  

(43)

Indeed from the chain rule: \(I(z : B, b, s, y) = I(z : y, b, s) + I(z : B|b, y, s)\), but \(I(z : y, b, s) = 0\) by assumption. Hence we get,

\[
I(z : B, b, s, y) = I(z : B|b, y, s) \leq \frac{1}{n} \]  

(44)

which is the desired bound. To show (43), we use Theorem 8 and Lemma 3. From Theorem 8 we have:

\[
\frac{1}{n} \sum_{i=1}^{n-1} \left( I(X \oplus x_i : B|b, s, y = i) + I(X : B|b, s, y = 0) \right) \]  

\[
I(z : B|b, s, y) \leq \frac{1}{n} \sum_{i=1}^{n-1} \left( I(X \oplus x_1 \oplus x_2 : \ldots X \oplus x_{n-1} : B, s) + H(B|b, s, y) \right) \]  

(45)

Now using Lemma 3 we get:

\[
\frac{1}{n} \sum_{i=1}^{n-1} \left( H(X \oplus x_i : B|b, s, y = i) + H(X|b, s, y = 0) \right) \]  

\[
I(z : B|b, s, y) \leq \frac{1}{n} \sum_{i=1}^{n-1} \left( H(X \oplus x_i : B) + H(X|b, s) + H(z|b, s) \right) - H(X, X \oplus x_1, X \oplus x_2 : \ldots X \oplus x_{n-1} : B, s) \]  

(46)

Now because \((X|s = j)\) and \((X \oplus x_i|s = j)\) are independent from \((b, y = j)\), we have for each \(j\) that \(H(X|b, s = j, y = 0) = H(X|b, s = j)\) and \(H(X \oplus x_i|b, s = j, y = i) = H(X \oplus x_i|b, s = j)\). And because for fixed \(s = j, z\) is independent from \(b\), there is \(H(z|b, s = j) = H(z|s = j)\). Averaging these equalities over \(P(s = i)\) we obtain that the terms \(\sum_{i=1}^{n-1} H(X \oplus x_i|b, s) + H(X|b, s, y)\) of LHS and RHS cancel each other and the inequality reads:

\[
I(z : B|b, s, y) \leq \frac{1}{n} \left[ H(z|s) \right] - H(X, X \oplus x_1, X \oplus x_2 : \ldots X \oplus x_{n-1} : B, s) \]  

(47)

Since \(z\) is independent from \(s\), \(H(z|s) = H(z) = 1\). Now, \(H(X, X \oplus x_1, X \oplus x_2 : \ldots X \oplus x_{n-1} : B, s)\) equals \(H(X, x_1 : \ldots x_{n-1} : z|s)\) as we can add \(X\) to \(X \oplus x_i\) reversibly. From the data processing inequality and the independence of \(s\) from \((x, z)\), we get \(H(X, x_1 : \ldots x_{n-1} : z) \geq H(x_1 : \ldots x_{n-1} : z|s) = H(x_1 : \ldots x_{n-1} : z) = n\). Hence the term \(\frac{1}{n} H(z|s)\) is bounded from above by \(\frac{1}{n^2}\). The last term is trivially upper bounded 1, which gives desired total upper bound \(\frac{1}{n}\).

APPENDIX B: PROOF OF LEMMA 2

Here we show that one requires at least \((n - 1)\) of \((2 \rightarrow 1)\) RBs (or equivalently PRs) to win a \((n \rightarrow 1)\) RAC with certainty. In particular we show that \((n - 2)\) \((2 \rightarrow 1)\) RBs (or equivalently PRs) cannot win a \((n \rightarrow 1)\) RAC with certainty.

For the first step, we shall show that a no-signaling \((2 \rightarrow 1)\) RB (or equivalently a PR) cannot win a \((3 \rightarrow 1)\) RAC.

**Lemma 5** A no-signaling \((2 \rightarrow 1)\) RB cannot win a \((3 \rightarrow 1)\) RAC.

As a part of the task at hand, Alice is provided with three input bits \(a_0, a_1, a_2\) and Bob with a bit \(b \in \{0, 1, 2\}\). Additionally Alice is allowed to communicate 1 bit of message \(m\) to Bob. Finally Bob is required to guess \(\hat{B} = \hat{a}_b\).

Alice and Bob share a no-signaling \((2 \rightarrow 1)\) RB. Depending on \(a_0, a_1, a_2\), Alice inputs \(a_0 \equiv a_0(\hat{a}_0, \hat{a}_1, \hat{a}_2), a_1 \equiv a_1(\hat{a}_0, \hat{a}_1, \hat{a}_2)\) to the RB. She receives an output \(A\) for the RB. Alice prepares a message \(m\) \(\equiv m(\hat{a}_0, \hat{a}_1, \hat{a}_2)\) to send to Bob. Bob inputs \(b \equiv b(m, \hat{a}_0, \hat{A} = A(\hat{b}, m)\). He receives an output \(B = a_0 \oplus A \oplus A'\) from the RB. He outputs \(\hat{B} = \hat{B}(b, m, B)\).

**Observation 1** The output of Alice’s side of a no-signaling \((2 \rightarrow 1)\) RB is random and uncorrelated with her inputs.

**Notice Bob can fix her inputs** \(A', b, \text{ and } B \oplus A' = A \oplus a_0\). But under no-signaling condition Bob cannot gain any
information about $a_0$. This implies that output of Alice's side of the RB $A$ must be random and generated in a way such that it is independent of her inputs $a_0, a_1$ in order to hide any information about $a_0, a_1$. Further it follows from the fact that $a_0 \equiv a_0(a_0, a_1, a_2), a_1 \equiv a_1(a_0, a_1, a_2)$ and $A$ is independent of $a_0, a_1, a_2$.

Let us consider Bob's lab, he receives 2 bits, $m$ as message from Alice and $B$ as output from the RB given that he inputs $b = b_0$ for a particular run. There are the following possible strategies:

1. Alice sends some $m = m_0$ which depends on her inputs $a_0, a_1, a_2$ where $m_0$ is independent of $A$. Since $B = a_0 \oplus A \oplus A'$ output of the RB is some random value. So without information of $A$ the RB is of no use. Now Bob is only left with one bit of information $m$. Alice does not know in advance the value $b$. Therefore Bob can only guess one bit with certainty. The reason for this is simply that one cannot encode more than one bit through a single bit wire reliably. To see this suppose Bob wants to learn $a_0, a_1$ notice that for each value of $m$ Bob’s simplest strategy can rely on the following possibilities:

   (a) $P_g(a_0|m = 0) = 1, P_g(a_0|m = 1) = 1$
   (b) $P_g(a_1|m = 0) = 1, P_g(a_1|m = 1) = 1$
   (c) $P_g(a_0|m = 0) = 1, P_g(a_0|m = 1) = 1$
   (d) $P_g(a_1|m = 0) = 1, P_g(a_0|m = 1) = 1$

Notice that first two possibilities are simply sending $m = a_0$ and $m = a_1$ respectively. Further lets assume third possibility works, and Bob guess the value $a_0 = 0$ given $m = 0$ and $a_1 = 0$ given $m = 1$. It is easy to see that such a probability distribution $P(a_0, a_1, m)$ cannot exist as the probability $P'(a_0 = 1, a_1 = 1) = 0$. This implies the reduced distribution $P(a_0, a_1)$ is no longer randomly distributed, as it ought to be. Similarly arguments apply to the fourth case.

2. Alice sends $m = A$, and the RB works perfectly that is $B = a_0$. Hence depending on the choice of $b$ Bob can perfectly guess one bit in each run. W.l.o.g Alice can encode $a_0 = a_0$ and $a_1 \equiv a_1(a_1, a_2)$. Again as a working RB is a single bit wire given $b = 1$ and it follows from observation $A$ is uncorrelated with (has no information about) $a_1, a_2$, Bob cannot perfectly guess both $a_1$ and $a_2$ simultaneously. In this case Bob can guess two bits (one for each turn, for each assignment of $b$), which is still not good enough.

3. Alice sends $m \equiv m(a_0, a_1, a_2, A)$ (excluding the case $m = A$ or $m = A \oplus 1$). Again as its a single bit, and $a_0, a_1, a_2$ are independent from $A$, Bob cannot get the value of $A$ perfectly and hence the RB wont work perfectly. Say with some probability of guessing $P_g(A|m)$. Bob could guess the value of $A$ perfectly in that case Bob can guess two bits (one for each turn) however with probability $1 - P_g(A|m)$ Bob can only guess 1 bit. Therefore Bob can guess $P_g(A|m)(2) + (1 - P_g(A|m))(1) = P_g(A|m) + 1$ bits (one for each turn). Hence In this case Bob could guess at best two bits but the average is lower. Therefore there exist no strategy using which Alice and Bob sharing a no-signaling $(2 \rightarrow 1)$ RB and a bit of communication can win a $(3 \rightarrow 1)$ RAC. Furthermore we make the following observation.

**Observation 2** Its always better to have a working RB i.e. $m = A$ then sending a fixed message

This observation directly follows from comparison between strategies one and two above. As in the case of a fixed message only one bit can be guessed with certainty while a working RB enables Bob to guess two bits (one for each turn).

Now we proceed with proof of Lemma 2. Alice is provided with $n$ input bits $a_0, a_1, \ldots a_{n-1}$ and Bob with a nit $b \in \{0, 1, \ldots, n - 1\}$. Additionally Alice is allowed to communicate 1 cbit of message $m$ to Bob. Finally Bob is required to output $\hat{B} = \hat{a}_0$.

Alice and Bob share $(n - 2)$ no-signaling $(2 \rightarrow 1)$ RB. Depending on $a_0, a_1, \ldots a_{n-1}$ and outputs from other RB $A^i$ where $i \in \{1, 2, \ldots, n - 2\} - \{i\}$. Alice decides her inputs $a_0^i, a_1^i$ to the $i$th RB where $i \in \{1, \ldots, n-2\}$. Alice sends an output $A^i$ from the RB. Alice prepares a message $m$ to send to Bob depending on $a_0, a_1, \ldots a_{n-1}$ and $A_1, A_2, \ldots A_{n-2}$. Bob inputs $b^i, A^i \in \{0, 1\}$ to $i$th RB depending on $b$, output from other RB $B^j$ where $j \in \{1, 2, \ldots, n - 2\} - \{i\}$ and message from Alice $m$. She receives output $B^i = a^i_{b_0} \oplus A^i \oplus A^i$ from ith RB. Depending upon $B_1, B_2, \ldots B_{n-2}, m, b$ she outputs $\hat{B}$.

Consider Bob’s lab, he receives $n-1$ bits, $m$ as message from Alice and $B_i$ as output from ith RB given that he inputs $b^i = b_0^i$ for each run where $i \in \{1, 2, \ldots, n - 2\}$.

Following observation we seek a greedy strategy, in the sense that we want to activate maximum number of RB. As there is only 1 cbit message allowed, Alice simply sends output $A_{n-2}$ in order to activate the n - 2th RB. This allows Alice and Bob to transmit reliably two bits (though only one per run) $a_0^{n-2}$ or $a_1^{n-2}$. There are following possible strategies:

1. Alice uses a fixed input $a_0^{n-2} = m_0, a_1^{n-2} = m_1$ depending upon $a_0, a_1, \ldots a_{n-1}$. In this case both the inputs $m_0, m_1$ do not carry any information about $A^1, A^2, \ldots A^{n-3}$ so no other RB works, and $B^1, B^2, \ldots B^{n-3}$ are random and consequently useless. In this case as Alice is not aware of $b$ only 2 bits can be guessed perfectly by Bob (one for each turn depending on $b^0$). Similar arguments
Lemma 6 For any fully connected binary tree with a root node of degree 2 and k − 1 interior nodes with degree 3, the number of leaf nodes l equals k + 1.

For any tree the following holds,

\[ |E| = |V| - 1 \]  \hspace{1cm} (48)

where \(|E|\) is the number of edges, \(|V|\) is the number of nodes. Let \(|E| = m\), this implies,

\[ m = k + l - 1 \]  \hspace{1cm} (49)

Also for any graph the following holds,

\[ 2|E| = \sum_{v \in V} \text{deg}(v) \]  \hspace{1cm} (50)

so we have 1 root node with degree 2, \(k - 1\) interior nodes with degree 3 and inputs available to Alice form the leaf nodes with degree 1.

\[ 2m = 3(k - 1) + l + 2 \]  \hspace{1cm} (51)

substituting value of \(m\) from (49) we have,

\[ l = k + 1 \]  \hspace{1cm} (52)

Hence we have number of available inputs to Alice \(l\) is exactly \(k + 1\), where \(k\) is the number of RB available.

APPENDIX C: \((n - 1)\) NO-SIGNALING \((2 \to 1)\) RBS AS BUILDING BLOCKS FOR \((n \to 1)\) RAC.

Here we provide a protocol for winning an \((n \to 1)\) RAC using \((n - 1)\) no-signaling \((2 \to 1)\) RBs (or equivalently PR). We start with defining two subroutines CONCATENATION and ADDITION in general for use in the protocol for construction later. Further we use the simple fact that every natural number \(n\) has a binary representation to give the protocol for the construction.

**CONCATENATION**

No-signaling \((2 \to 1)\) RB can be arranged in an inverted pyramid like structure to win a \((n \to 1)\) RAC where \(n = 2^k\) for \(k \in N\) in the same way as the PR in the [12]. The trick is to supply the outputs of the first layer of \((2 \to 1)\) RB as inputs the next layer of \((2 \to 1)\) RB on Alice’s side.

For winning \((2^k \to 1)\) RAC using \((2^k - 1)\) no-signaling \((2 \to 1)\) RB: Alice has \(2^k\) inputs bits, \(a_0, a_1, a_2, \ldots, a_{(2^k-1)}\), she supplies them pairwise as input to \(2^{k-1}\) \((2 \to 1)\) RBs which form the top most layer \(r = 1\) of the inverted pyramid. Each of the no-signaling \((2 \to 1)\) RBs, \(RB(i)\) will give a bit output \(A_i\) where \(i \in \{1, 2, 3, \ldots, 2^{k-1}\}\). Supply the outputs \(A_i\) where \(i \in \{0, 1, 2, \ldots, 2^{k-1}\}\) pair-wise to \(2^{k-2}\) no-signaling \((2 \to 1)\) RBs which forms the next layer \(r = 2\). Each of these RBs, \(RB(i)\) will in-turn output \(A_i\) where \(i \in \{2^{k-1} + 1, 2^{k-1} + 2, 2^{k-1} + 3, \ldots, 2^{k-1} + 2^{k-2}\}\) and repeat the above until the layer \(r = k\) with \(2^{k-r} = 1\) \((2 \to 1)\) RB\((2^{k-1})\) and the final output forms the message \(m = \sum_{k} B_k\). Bob receives \(b \in \{0, 1, 2, 3, \ldots, 2^{k-1}\}\) or input bits, \(b_k\), which describes the index, \(b = \sum_{k} b_k 2^k\). Depending on which he reads a suitable message \(B_k\), using a box at each layer. Finally, she outputs \(B = m @ B_1 @ B_2 \ldots @ B_k = a_0\).

Cost: For \(n = 2^k\), \(a \to 1\) RAC requires \(n - 1\) no-signaling \((2 \to 1)\) RB. However when \(n \in N\) and we are only allowed to use concatenation, we first find \(k \in N\) such that, \(2^{k-1} \leq n \leq 2^k\), then construct a \((2^k \to 1)\) RAC using the protocol above.

For example of winning \((2^3 \to 1)\) RAC using \((2^3 - 1)\) \((2 \to 1)\) RB see Fig. 4.
FIG. 4. The figure demonstrates the use of CONCATENATION of $(2 \to 1)$ RB to form a $(2^3 \to 1)$ RB. RESOURCE: $2^3 - 1 = 7$ $(2 \to 1)$ RB. In the figure Bob is trying to learn $a_{101}$ or $a_{111}$.

**ADDITION**

Let us start with a no-signaling $(n \to 1)$ RB(1) and no-signaling $(m \to 1)$ RB(2). We aim at winning $(n + m \to 1)$ RAC. The protocol to achieve that is as follows,

For winning $(n + m \to 1)$ BOX using a no-signaling $(n \to 1)$ RB, no-signaling $(m \to 1)$ RB, an additional no-signaling $(2 \to 1)$ RB and 1 c-bit of communication: Alice has $n$ input bits $a_0, a_1, \ldots, a_{n-1}$ corresponding to no-signaling $(n \to 1)$ RB(1) and $m$ inputs bits $a_0, a_{n+1}, \ldots, a_{n+m-1}$ corresponding to no-signaling $(m \to 1)$ RB(2), in total $n + m$ input bits. Alice obtains a output bit $A_1$ from no-signaling $(n \to 1)$ RB(1) and $A_2$ from no-signaling $(m \to 1)$ RB(2). The cost of addition is an additional no-signaling $(2 \to 1)$ RB(3) whose input bits are $A_1$ and $A_2$ and output bit is $A_3$. Alice then sends $m = A_3$ On Bob’s end, Bob receives $b \in \{0, 1, n-1, n, n + m - 1\}$ and message $m$.

If $0 \leq b < n$ then Bob enters 0 in the no-signaling $(2 \to 1)$ RB(3) and obtains $B_3$ and enters $b$ into no-signaling $(n \to 1)$ RB(1) to get $B_1$ and outputs $B = b \oplus B_1 \oplus B_3 = a_b$. If $n \leq b \leq n + m - 1$ then Bob enters 1 in the no-signaling $(2 \to 1)$ RB(3) and obtains $B_3$ and enters $b$ into no-signaling $(m \to 1)$ RB(2) to get $B_2$ and outputs $B = b \oplus B_2 \oplus B_3 = a_b$.

For example winning $(m + n \to 1)$ RAC using no-signaling $(m \to 1)$ RB, no-signaling $(n \to 1)$ RB and a $(2 \to 1)$ RB [see Fig. 2].

Finally, protocol for winning $(n \to 1)$ RAC using only $(2 \to 1)$ RB for any $n \in N$.

**Protocol for winning $(n \to 1)$ RAC using no-signaling $(2 \to 1)$ RBs:** Any Natural number $n$ can be broken into sum of some discrete powers of 2, i.e. $n = \sum_{i=0}^{k-1} \alpha_i 2^i$ where $\alpha_i \in \{0, 1\}$, $i \in \{0, k - 2\}$ and $\alpha_{k-1} = 1$, $k \in N$ such that $2^{k-1} \leq n < 2^k$. The coefficients $(\alpha_0, \alpha_1, \ldots, \alpha_{k-1})_2$ form the binary representation. We use a variable $count \in N$ and initialize it to $count = 0$ for cost calculation given later. Alice receives $n$ input bits and for $i \in \{0, k - 1\}$ such that $\alpha_i = 1$ repeats the following steps,

1. Use CONCATENATION of $2^i - 1$ no-signaling $(2 \to 1)$ RB to construct a $(2^i \to 1)$ RAC for $i > 0$ and for $i = 0$ simply take the first input bit $a_0$. Update $count = count + 1$ and RB(LEFT) = $RB \oplus 2^i \to 1$ RB.

2. If $count = 1$ let $RB(LEFT) = RB(LEFT)$.

3. If $count > 1$: use a no-signaling $(2 \to 1)$ RB and ADDITION of $(x \to 1)$ RB(LEFT) and $(y \to 1)$ RB(LEFT) to form the updated $(x + y \to 1)$ RB(LEFT).

Alice sends the output of final (bottom most) $(2 \to 1)$ RB as the message bit $m$. Bob receives the message bit $m$ and $b \in \{0, 1, 2, \ldots, n - 1\}$ and outputs $B = a_b$, by following corresponding parts of CONCATENATION and ADDITION.

**Cost:** The variable count stores the total number of indexes $i$ such that $i = 1$. CONCATENATION repeated count times uses $(n - count)$ no-signaling $(2 \to 1)$ RB. ADDITION is repeated $(count - 1)$ each time costing 1 no-signaling $(2 \to 1)$ RB. Finally total cost is $(n - count) + (count - 1) = n - 1$ no-signaling $(2 \to 1)$ RB. For example winning $(7 \to 1)$ RAC using (6) $(2 \to 1)$ RB [see Fig. 6]. It is to be noted at this point that the protocol given above is just one of many possibilities for simulation of $(n \to 1)$ RAC using $(n - 1) (2 \to 1)$ RB. In particular any construction which forms a fully connected binary tree with no-signaling $(2 \to 1)$ RB as nodes can be used for the simulation $(n \to 1)$ RAC. In Appendix B we give the proof that $(n - 2) (2 \to 1)$ RB (or equivalently PR) cannot simulate $(n \to 1)$ RAC.
Classical and quantum winning probability of \((n \rightarrow 1)\) RAC using the above protocol and corresponding bounds on \((2 \rightarrow 1)\) RAC.

One can win \((n \rightarrow 1)\) RAC using \(n-1\) no-signaling \((2 \rightarrow 1)\) RB. Let the quantum winning probability over \((n \rightarrow 1)\) RAC be \(T_n\). As the protocol above requires no-signaling \((2 \rightarrow 1)\) RB, and therefore \(T_2 = \frac{2+\sqrt{2}}{4}\) and \(C_2 = \frac{3}{4}\) are enough to determine \(C_n\) and \(T_n\) for all \(n \in \mathbb{N}\). Let the winning probability of \((2 \rightarrow 1)\) RAC be \(0 < P_2 < 1\). As an example for \((7 \rightarrow 1)\) RAC (Figure 3) the probability that Bob guesses \(a_0\) correctly is \(P(B = a_0|b = 0) = P_2^2 + \frac{1-P^2}{2}\). \(P(B = a_0|b = 1) = \frac{1}{2}P_2^2 + \frac{1}{2}(1-P_2)^2P_2^2\). Now Bob’s inputs are uniformly distributed, hence \(P_B = P(B = a_0) = \frac{P(B=\overline{a}|b=0) + 6P(B=\overline{a}|b=1)}{7}\). So \(T = 0.08723\). Similarly for \((n \rightarrow 1)\) RAC for \(n \in \{2, \ldots, 10\}\) the protocol, \(C_n\) and \(T_n\) are provided in [see Fig. 7]. While \(C_n\) are not optimal for \(n = 2k+1\) [27], \(T_n\) are numerically close to known optimal values.

**APPENDIX D**

Here we provide the proof for resource inequality [2] and consequently the in-equivalence of no-signaling \((2 \rightarrow 1, 3)\) RB (3) and \(B_{2}^{\text{3} (+)}\). We give the proof in two parts:

1. We shall show that if the 3it of communication is not used to send the output of Alice’s RB \(A\) but \(B_{2}^{\text{3} (+)}\) is obtained, then the channel \(A_3\) is a depolarizing channel: it outputs \(z\) or a random 3it with \(\frac{1}{2}\) probability.

2. If the 3it of communication is used to send \(A\) and \(B_{2}^{\text{3} (+)}\) is obtained, then the capacity of obtainable channel \(A_3\) is upper bounded by \(\frac{1}{2}\) 3it.

**Proof part 1.**

Let \(m\) be the message 3it to be sent to Bob. The goal is to obtain \(B_{2}^{\text{3} (+)}\)-BOX i.e. \(Y = x, y, z\) \(X\) in any case \(m = 0, 1, 2\). In general for any \(m\) Bob’s output is a function of his input \(y\) along with the RB \(Y = Y(y, b, B)\).

Now there are two cases:

1. \(A = A’\): In this case \(B_{2}^{\text{3} (+)}\)-BOX is obtained by processing a perfect \((2 \rightarrow 1, 3)\) RAC.

2. \(A \neq A’\): In this case no-signaling \((2 \rightarrow 1, 3)\) RB (3) outputs \(B\) which does not depend on the work of RAC and hence is useless. Hence Bob’s outcome only depends on the input she receives i.e. \(Y = Y(y)\). Now we need for \(B_{2}^{\text{3} (+)}\)-BOX, \(Y(y = 0) = Y(y = 1) = -x\). Notice Bob can compute \(x_1\) using the fact \(Y(y = 1) = -x\). Therefore in this case Bob must know the value of \(x_1\).

Therefore \(P_g(x_1|m = m_0) = 1\) or \(P_g(A|m = m_0) = 1\) or both. Where \(P_g\) denotes Bob’s guessing probability. W.l.o.g. we assume that all three values of \(m\) occur with non-zero probability. Bob’s simplest strategy of (guessing only one variable, \(x_1\) or \(A\) for given \(m\)) can rely on 8 different cases:

1. \(P_g(A|m = 0) = 1, P_g(A|m = 1) = 1\) and \(P_g(A|m = 2) = 1\).
2. \(P_g(x_1|m = 0) = 1, P_g(x_1|m = 1) = 1\) and \(P_g(x_1|m = 2) = 1\).
3. \(P_g(A|m = 0) = 1, P_g(A|m = 1) = 1\) and \(P_g(A|m = 2) = 1\).
4. \(P_g(A|m = 0) = 1, P_g(A|m = 1) = 1\) and \(P_g(A|m = 2) = 1\).
5. \(P_g(x_1|m = 0) = 1, P_g(x_1|m = 1) = 1\) and \(P_g(x_1|m = 2) = 1\).
6. \(P_g(x_1|m = 0) = 1, P_g(x_1|m = 1) = 1\) and \(P_g(x_1|m = 2) = 1\).
7. \(P_g(x_1|m = 0) = 1, P_g(x_1|m = 1) = 1\) and \(P_g(x_1|m = 2) = 1\).
8. \(P_g(A|m = 0) = 1, P_g(x_1|m = 1) = 1\) and \(P_g(x_1|m = 2) = 1\).

In the third case (equivalently for fourth to eighth cases) Bob makes a perfect guess of \(A\) for \(m = 0, 1\) and of \(x_1\) for \(m = 2\). We shall now show by an example (others are analogous), that for the conditions in the third case such a joint probability distribution \(P(A, x_1, m)\) cannot exist. Suppose Bob makes a perfect guess, e.g. \(A = 0\) for \(m = 0\), \(A = 1\) for \(m = 1\) and \(x_1 = 0\). We find that \(P(A = 2, x_1 = 1) = 0\) and \(P(A = 2, x_1 = 2) = 0\), which implies the reduced probability distribution \(P(A, x_1)\) is no longer randomly distributed, but RB works in a way such that \(A\) and \(x_1\) are generated independently at random. From the only two possible cases we see that in the first case, \(m\) is simply used to send \(A\). This case is further dealt with in the Part 2. In the second case the message bit is used to send \(x_1\) in this \(B_{2}^{\text{3} (+)}\) BOX is obtained and the RB serves as an depolarizing 3it channel with probability \(\frac{1}{2}\).

**Proof part 2**

: We shall use information theoretic tools to show that if the no-signaling \((2 \rightarrow 1, 3)\) RB (3) supplemented with one bit of communication is able to reproduce exactly the \(B_{2}^{\text{3} (+)}\)-BOX and some 3it channel, then the mutual information of that channel must be bounded by \(\frac{1}{2}\) (assuming that Alice’s output of RB \(A\) is directly inserted into Bob input to the RB i.e. \(A = A’\)).

We shall use the following common assumptions:

**Assumptions:** Alice is supplied with two 3its \(x_1, z\), Bob
| \((n \rightarrow 1)\) RAC | Protocol | Classical Bound | Quantum Bound |
|------------------------|----------|----------------|---------------|
| \((2 \rightarrow 1)\) RAC | ![Diagram](image1) | \(C_1 = 0.75\) | \(T_2 = \frac{1 + \sqrt{2}}{2} \approx 0.85355\) |
| \((3 \rightarrow 1)\) RAC | ![Diagram](image2) | \(C_1 = \frac{2^{\frac{3}{2}T_1 - C_2} + C_2}{2} = 0.66666\) | \(T_3 = \frac{2^{\frac{3}{2}T_1 - T_2}}{2} \approx 0.78454\) |
| \((4 \rightarrow 1)\) RAC | ![Diagram](image3) | \(C_4 = C_2^2 + (1 - C_2)^2 = 0.625\) | \(T_4 = T_3^2 + (1 - T_3)^2 \approx 0.75\) |
| \((5 \rightarrow 1)\) RAC | ![Diagram](image4) | \(C_5 = \frac{4(C_2^2 + C_2 - C_2^2 - C_2)}{2} = 0.6\) | \(T_5 = \frac{4(T_2 + T_3 - T_2 T_3)}{2} \approx 0.71213\) |
| \((6 \rightarrow 1)\) RAC | ![Diagram](image5) | \(C_6 = \frac{2^{\frac{3}{2}T_1 - C_2^2} + C_2^2}{2} = 0.58333\) | \(T_6 = \frac{2^{\frac{3}{2}T_1 - T_2 T_3}}{2} \approx 0.70118\) |
| \((7 \rightarrow 1)\) RAC | ![Diagram](image6) | \(C_7 = \frac{4(C_2^2 + C_2 - C_2^2 - C_2)}{2} = 0.57142\) | \(T_7 = \frac{4(T_2 + T_3 - T_2 T_3)}{2} \approx 0.69725\) |
| \((8 \rightarrow 1)\) RAC | ![Diagram](image7) | \(C_8 = C_2 C_4 + (1 - C_2)(1 - C_4) = 0.5425\) | \(T_8 = T_2 T_4 + (1 - T_2)(1 - T_4) \approx 0.67677\) |
| \((9 \rightarrow 1)\) RAC | ![Diagram](image8) | \(C_9 = \frac{4(C_2 C_4 + C_4 - C_2 C_4 - C_4)}{2} = 0.55555\) | \(T_9 = \frac{4(T_2 T_4 + T_4 - T_2 T_4)}{2} \approx 0.65839\) |
| \((10 \rightarrow 1)\) RAC | ![Diagram](image9) | \(C_{10} = \frac{4(C_2 C_4 + C_4 - C_2 C_4 - C_4)}{2} = 0.55\) | \(T_{10} = \frac{4(T_2 T_4 + T_4 - T_2 T_4)}{2} \approx 0.65\) |

**FIG. 6.** (Color online) The figure demonstrates the use of protocol described in Appendix C and \(n - 1\) no-signaling \((2 \rightarrow 1)\) RB to win a \((n \rightarrow 1)\) RAC for \(n \in \{2, 3, \ldots, 10\}\). The C and Q bounds, \(C_n\) and \(T_n\) are derived recursively using \(C_2 = 0.75\) and \(T_2 = \frac{1 + \sqrt{2}}{4}\).

is given a bit \(y\). Both are given access to shared random 3bit \(s\) such that \(x_1, z, y, s\) are mutually independent. Alice generates \(X\) and inputs for the RB \(a_0, a_1\) from \(x_1, z, s\). Similarly Bob generates his input to the RB \(b\) from \(y, s\). These strategies result in shared joint probability distribution \(P(x_1, z, y, s, b, B, X, Y)\) such that \(B = a_0\) is obtained from the RAC on Bob’s side (as \(A\) is always inserted into \(A'\)) , and \(Y\) is generated out of \(b, B, s, y\). We shall first reformulate Theorem 6 in other words. Under the aforementioned assumptions, if variables \(x_1, y, X, Y\) perfectly reproduce the \(B_2^3(+)\)-BOX, there holds:

\[
I(z : B, b, y, s) \leq \frac{1}{2}
\]  

(53)
We shall prove this theorem in two parts:

1. First we shall use entropies and correlation to state the fact that to simulate the $B^3_{\alpha}$-BOX Bob has to guess perfectly $X$ when $y = 0$ and $x_1 - 3 X$ when $y = 1$.

2. Second we shall show that it is impossible to send more than 3 3it through a channel with 3 3it capacity. As in our case Alice would like to send both $x_1$ and $z$ which bounds Bob’s possible information gain about $z$.

**Lemma 7** Under the aforementioned assumptions, if variables $(x_1, y, X, Y)$ simulate the $B^3_{\alpha}$-BOX, there holds:

\[
I(B : x_1 - 3 \, X | b, s, y = 1) = H(x_1 - 3 \, X | b, s, y = 1)
\]

\[
I(B : X | b, s, y = 0) = H(X | b, s, y = 0)
\]

Now in order to perfectly reproduce $B^3_{\alpha}$-BOX given $y = 0$, he should perfectly guess $X$. On the other hand, given $y = 1$ he should perfectly guess $x_1 - 3 \, X$. So we must have:

\[
J(B, b, s, y = 0 \rightarrow X) = 1
\]

and

\[
J(B, b, s, y = 1 \rightarrow x_1 - 3 \, X) = 1
\]

where $J(X \rightarrow Y) = \Sigma_i P(X = i) \max_j [P(Y = j | X = i)]$. Which implies that there must be $\max_j [P(X = j | B = l, b = l, y = 0, s = i)] = 1$, and consequently $H(X | B, b, y = 0, s) = 0$ which directly leads to (54). Analogously for (55).

One cannot send more than one 3it through a single 3it wire.

Here we prove the main argument of Theorem 6. That is, the following theorem shows the tradeoff between Bob’s correlations with $X$ and $x_1 - 3 \, X$ and his correlations with $z$.

**Theorem 9** Under aforementioned assumptions, there holds:

\[
\frac{1}{2} I(x_1 - 3 \, X : B | b, s, y = 1) +
\]

\[
\frac{1}{2} I(X : B | b, s, y = 1) + I(z : B | b, s, y) \leq
\]

\[
\frac{1}{2} I(X : x_1 - 3 \, X : z | b, s) + H(B | b, s, y)
\]

To prove this theorem we use the following fact:

\[
I(S : T | V) + I(T : U | V) \leq I(S : U | V) + I(T : S | U | V)
\]

The LHS of (59) can be reformulated as, upon fixing $s = i$:

\[
\frac{1}{2} I(x_1 - 3 \, X : B | b, s, i, y = 1) +
\]

\[
\frac{1}{2} I(X : B | b, s, i, y = 1) + I(z : B | b, s, i, y = 0)
\]

\[
I(z : B | b, s, = i, y = 1)
\]

Now using (59) to first and third terms and to second and fourth terms, we find that the above quantity is upper bounded by:

\[
\frac{1}{2} I(x_1 - 3 \, X : z | b, s, i, y = 1) +
\]

\[
I(B : x_1 - 3 \, X, z | b, s, i, y = 1) +
\]

\[
I(X : z | b, s, i, y = 0) +
\]

\[
I(B : X, z | b, s, i, y = 0)]
\]

Now as $(x_1 - 3 \, X, z | s = i)$ is independent from $(y, b | s = i)$, therefore $I(x_1 - 3 \, X : z | b, s, i, y = 1) = I(x_1 - 3 \, X : z | b, s = 1)$. And since $(X, z | s = i)$ is independent from $(y, b | s = i)$, there is $I(X : z | b, s, i, y = 0) = I(X : z | b, s = i)$. Multiplying these with $P(s = i)$ and summing over values of $s$ we obtain:

\[
\frac{1}{2} I(x_1 - 3 \, X : z | b, s) +
\]

\[
I(B : x_1 - 3 \, X, z | b, s, y = 1) +
\]

\[
I(X : z | b, s) +
\]

\[
I(B : X, z | b, s, y = 0)]
\]

We can now use (59) to the first and third terms to obtain the upper bound:

\[
\frac{1}{2} I(x_1 - 3 \, X : X | b, s) +
\]

\[
I(z : X, x_1 - 3 \, X | b, s) +
\]

\[
I(B : x_1 - 3 \, X, z | b, s, y = 1) +
\]

\[
I(B : X, z | b, s, y = 0)]
\]

Finally, we shall proof the inequality (53). From the chain rule: $I(z : B, b, s, y) = I(z : y, b, s) + I(z : B | y, b, s)$ but $I(z : y, s, b) = 0$ $(b = b(y, s))$. Hence (53) can be reformulated as:

\[
I(z : B | b, s, y) \leq \frac{1}{2}
\]

To prove (64) we shall reformulate (58) using (54) and (55):

\[
\frac{1}{2} [H(X | b, s, y = 0) + H(x_1 - 3 \, X | b, s, y = 1)] +
\]

\[
I(z : B | b, s, y) \leq \frac{1}{2} [H(X | b, s) + H(x_1 - 3 \, X | b, s) +
\]

\[
H(z | b, s) - H(X, x_1 - 3 \, X, z | b, s) +
\]

\[
H(B | b, s, y) (65)
\]
Now as \((X|s = i)\) and \((x_1 - 3 X|s = i)\) are independent from \((b, y|s = i)\), we have for each \(i\) that \(H(X|b, s = i, y = 0) = H(X|b, s = i)\) and \(H(x_1 - 3 X|b, s = i, y = 1) = H(x_1 - 3 X|b, s = i)\). And because for some fixed \(s = i\), \(z\) is independent of \(b\), we have \(H(z|b, s = i) = H(z|s = i)\). Averaging these equalities over \(P(s = i)\) we obtain:

\[
I(z : B|y, b, s) \leq \frac{1}{2} [H(z|s) - H(X, x_1 - 3 X, z|b, s)] + H(B|y, s, b) \tag{66}
\]

Now \(z\) is independent of from \(s\), \(H(z|s) = H(z|s) = 1\). And \(H(X, x_1 - 3 X, z|s) = H(X, x_1, z|b, s)\) as we can add \(X\) to \(x_1 - 3 X\). Using the data processing inequality and the independence of \(s\) form \((x, z)\) we get, \(H(z, X, x_1|s) \geq H(z, x_1|s) = H(z, x_1) = 2\). Hence the first two terms are upper bounded by \(-\frac{1}{2}\). The last term trivially upper bounded by 1, which results in \(\frac{1}{2}\) proving (64).