Stability of solitons in $\mathcal{P}\mathcal{T}$-symmetric nonlinear potentials

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Abstract – We report on detailed investigation of the stability of localized modes in the nonlinear Schrödinger equations with a nonlinear parity-time (alias $\mathcal{P}\mathcal{T}$) symmetric potential. We are particularly focusing on the case where the spatially dependent nonlinearity is purely imaginary. Results of the Evans function analysis predict that for sufficiently small dissipation localized modes become stable when the propagation constant exceeds certain threshold value. This is the case for periodic and tanh-shaped complex potentials where the modes having widths comparable with or smaller than the characteristic width of the complex potential are stable, while broad modes are unstable. In contrast, in complex potentials that change linearly with transverse coordinate all modes are stable, which suggests that the relation between width of the modes and spatial size of the complex potential defines the stability in the general case. These results were confirmed using the direct propagation of the solutions for the mentioned examples.

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Introduction. – Since the introduction of the concept of parity-time- ($\mathcal{P}\mathcal{T}$-) symmetric potentials [1], non-Hermitian Hamiltonians possessing a purely real spectrum have received considerable attention [2] due to their relevance to quantum mechanics and optics. In the context of optical applications, it was natural that the concept was generalized to nonlinear systems [3], where the existence of localized modes was shown to be possible (we notice that experimental observation of the $\mathcal{P}\mathcal{T}$ symmetry in linear optics has been recently reported [4]). As a natural extension of this activity, the existence of stable localized nonlinear modes in nonlinear $\mathcal{P}\mathcal{T}$ symmetric lattices has been recently demonstrated in [5]. Nonlinear gain and loss compensating each other were also addressed recently within the framework of the nonlinear dimer model [6]. A general important property of nonlinear $\mathcal{P}\mathcal{T}$-symmetric systems is that they admit continuous families of localized modes parameterized by the propagation constant, similarly to what happens for the nonlinear Hamiltonian systems. This is in spite of the need to satisfy the balance between dissipation and gain, as this happens in dissipative systems of general type where localized modes appear as attractors, rather than elements of a family of solutions. In this sense, the models with $\mathcal{P}\mathcal{T}$-symmetric potentials occupy a special “place” between the Hamiltonian and dissipative systems.

A striking effect related to the existence and stability of the localized modes in nonlinear $\mathcal{P}\mathcal{T}$ lattices is that they can be stable even in the absence of modulation of the conservative part of nonlinearity [5]. The stability, however depends on the relation between the mode width and the period of the potential. More specifically only sufficiently narrow modes were found to be stable. Now we observe, that in the limit when the width of a mode goes to zero and the potential is a smooth function, the imaginary part of the potential can be approximated by the linear function (recall, that the imaginary part is an odd function). For instance, the purely imaginary nonlinear $\mathcal{P}\mathcal{T}$ potential $i\sin(2\eta)$ considered in [5] can be approximated as $i\sin(2\eta) \sim 2i\eta$ for narrow modes localized about $\eta = 0$. This leads to the natural question: is it possible to obtain stable localized modes in a nonlinear...
\(\mathcal{PT}\)-symmetric potential of a general type, provided the widths of the modes are small enough? Here we give an affirmative answer to this question and illustrate stable modes for three qualitatively different case examples: of a periodically varying dissipation and gain, i.e. the lattice case, of dissipation and gain tending to constants at the infinity (see (6) below), and to the linearly increasing gain and losses (the model (23)).

The aim of the present paper is the analytic study of the linear stability and dynamics of solitons in nonlinear \(\mathcal{PT}\)-symmetric potentials. The model we are interested in is the complex nonlinear Schrödinger equation [5]:

\[
i\partial q = -\frac{1}{2} \partial_{\eta}^2 q - [1 + V(\eta) + iW(\eta)] |q|^2 q.
\] (1)

In the optical applications \(q\) is the dimensionless electric field propagating along the \(\xi\)-direction \((\xi > 0)\) with \(\eta \in \mathbb{R}\) being the transverse coordinate. The physical mechanisms for both gain and loss are well known. The former one refers to the standard two-photon absorption, which becomes the dominating mechanism of losses in semiconductors at sufficiently high intensities, while nonlinear amplification can be realized in electrically pumped semiconductor optical amplifiers (see, e.g., [7]). We are interested in \(\mathcal{PT}\)-symmetric nonlinear potential where the \(V(\eta)\) and \(W(\eta)\) are both real and obeying the relations

\[
V(\eta) = V(-\eta) \quad \text{and} \quad W(\eta) = W(-\eta).
\] (2)

Localized modes. – We look for stationary localized solutions of eqs. (1), (2), which can be searched in the form \(q(\xi,\eta) = w(\eta)e^{i\xi}\) subject to the boundary conditions \(\lim_{\eta \to \pm \infty} q(\xi,\eta) = 0\). Bearing in mind optical applications, we refer to \(b\) as to the propagation constant. The stationary wave function \(w(\eta)\) obeys the equation

\[
\frac{1}{2} w'' - bw + [1 + V(\eta) + iW(\eta)] |w|^2 w = 0.
\] (3)

Let us calculate how many parameters one has to introduce in order to unambiguously define a localized mode \(w(\eta)\). To this end, let us first agree that we identify the modes \(w(\eta)\) and \(w(\eta)e^{i\varphi}\), \(\varphi \in \mathbb{R}\), which are not distinguishable from the physical point of view. Then a specific symmetry of eq. (3) induced by relations (2) suggests that without loss of generality the localized mode \(w(\eta) = w_r(\eta) + i w_i(\eta)\) can be chosen to be \(\mathcal{PT}\)-symmetric, i.e. having even real part and odd imaginary one:

\[
w_r(\eta) = w_r(-\eta), \quad w_i(\eta) = -w_i(-\eta).
\] (4)

Let \(\tilde{w}(\eta)\) be some solution of eq. (3) vanishing as \(\eta \to +\infty\), but not necessarily vanishing as \(\eta \to -\infty\). Then for \(\eta \to +\infty\) the nonlinear term in eq. (3) is negligible and in the corresponding limit \(\tilde{w}(\eta)\) is described by the linear equation \(\frac{1}{2} \tilde{w}'' - b \tilde{w} = 0\). Thus for \(\eta \to +\infty\) the solution \(\tilde{w}(\eta)\) behaves as \(\tilde{w}(\eta) = Ce^{i\varphi}[e^{-\sqrt{2b}\eta} + o(1)]\), where \(C\) and \(\varphi\) are real constants. Let us temporarily restrict ourselves to the case \(\varphi = 0\). Equations (4) dictate that if the solution \(\tilde{w}(\eta)\) represents a localized mode, then \(\tilde{w}(\eta)\) must obey \(\partial_{\eta}\tilde{w}(0) = 0\) and \(\tilde{w}(1) = 0\). Now, let us admit nonzero \(\varphi\) in the asymptotics for \(\tilde{w}(\eta)\). Obviously, this leads just to multiplication of \(\tilde{w}(\eta)\) by the factor \(e^{i\varphi}\). Therefore, we can formulate a weaker condition for \(\tilde{w}(\eta)\) to represent a localized mode: there must exist such \(\varphi\) that \(\tilde{w}(1)\cos \varphi + \tilde{w}(0)\sin \varphi = 0\) and \(\partial_{\eta}\tilde{w}(1)\cos \varphi - \partial_{\eta}\tilde{w}(0)\sin \varphi = 0\). These equations are compatible if and only if

\[
\tilde{w}_r(0)\partial_{\eta}\tilde{w}_r(0) + \tilde{w}_i(0)\partial_{\eta}\tilde{w}_i(0) = 0.
\] (5)

Thus any localized mode \(w(\eta)\) can be identified with a solution of eq. (5) which contains two real unknowns: \(C\) and \(b\). If we fix one of them, typically this is the propagation constant \(b\), then eq. (5) results in one or several solutions for the parameter \(C\) which indicates that eq. (3) admits continuous families of localized modes for fixed \(V(\eta)\) and \(W(\eta)\). This feature is typical for \(\mathcal{PT}\)-symmetric potentials and constitutes significant difference compared to the conventional dissipative systems.

In fig. 1(a) we show two families of localized modes on the plane \((b, U)\), where \(U = \int |q|^2 d\eta\) is the energy flow (the integration limits are omitted wherever the integration is over whole real axis), obtained for the nonlinear potential

\[
V(\eta) = 0, \quad \text{and} \quad W(\eta) = \sigma \tanh(2\eta).
\] (6)
with $\sigma = 0.7$. The modes exist only if $b$ exceeds certain threshold value. The modes of the lower (upper) curve in fig. 1(a) can be referred to as \textit{fundamental} (\textit{higher}) modes. A typical profile of a fundamental mode is shown in fig. 1(b). Below we focus on the fundamental modes.

On the other hand, the localized modes can be considered as bifurcating from the limit $\sigma = 0$, where eq. (1) reduces to the conventional nonlinear Schrödinger equation. In fig. 1(c) we show a \textit{branch} of fundamental modes found for a fixed $b$. The continuation from the limit $\sigma = 0$ will be used below for investigation of stability of the fundamental modes.

\textbf{Linear stability analysis.} – Substituting the perturbed solution $q(\xi, \eta) = e^{i\xi}w(\eta) + e^{i\xi}p_+ + e^{i\xi}p_-$, where $|p_+| \ll |w|$, and the overline stands for the complex conjugation, in eq. (1) and linearizing it around $w(\eta)$ one arrives at the eigenvalue problem $Lp = \lambda p$, where $p = (p_+ + p_- i(p_+ - p_-))^T$ (hereafter the superscript $T$ stands for matrix transposition) and $L$ is given by

$$L = \left( \begin{array}{cc} N_{11} & -\frac{1}{2} \partial^2_w + N_{21} \\ \frac{1}{2} \partial^2_w + N_{22} & N_{12} \end{array} \right),$$

where

$$N_{11} = -2[1 + V(\eta)]w_w - W(\eta)[3w^2 + w^2_0],$$  

$$N_{22} = 2[1 + V(\eta)]w_w - W(\eta)[w^2 + 3w^2_0],$$  

$$N_{12} = b[1 + V(\eta)]w^2 + 3w^2_0 - 2W(\eta)w_w,$$

$$N_{21} = -b[1 + V(\eta)][3w^2 + w^2_0] - 2W(\eta)w_w.$$  

The mode $w(\eta)$ is unstable if and only if there exists an eigenvalue $\lambda$ with positive real part.

For $V(\eta) = W(\eta) \equiv 0$ the localized mode $w(\eta)$ is a standard NLS soliton: $w_0(\eta) = w(0) = \sqrt{2b} \text{sech}(\sqrt{2b} \eta)$ and $w_0(\eta) = w(0) \equiv 0$. Designating the operator $L$ in this case by $L^{(0)}$, we recall that the spectrum of $L^{(0)}$ is well known [8]. In particular, the point spectrum of $L^{(0)}$ consists of the only eigenvalue $\lambda_0 = 0$ which is isolated and has algebraic multiplicity (a.m.) equal to 4 and geometric multiplicity (g.m.) equal to 2. The eigenfunctions corresponding to $\lambda_0$ read

$$\psi^{(0)}_{11} = (\partial_w w^{(0)}_r, 0)^T, \quad \psi^{(0)}_{12} = (0, w^{(0)}_r)^T.$$  

There also exist two generalized eigenfunctions, namely

$$\psi^{(0)}_{21} = (0, -iw^{(0)}_r)^T \quad \text{and} \quad \psi^{(0)}_{22} = (\partial_w w^{(0)}_r, 0)^T,$$

such that $L\psi^{(0)}_{2j} = \psi^{(0)}_{1j}$, $j = 1, 2$. Here $\partial_w w_r$ is obtained by means of differentiation of $w^{(0)}_r$ with respect to $b$.

Let us now assume that $V(\eta)$ and $W(\eta)$ are not equal to zero but satisfy $\mathcal{PT}$-symmetry relations (2). If at the same time $V(\eta)$ and $W(\eta)$ are small enough, then they can be considered as a perturbation to the operator $L^{(0)}$. Behavior of the spectrum of $L^{(0)}$ subject to the perturbation determines linear stability of the localized mode. In particular, a generic perturbation of the operator $E^{(0)}$ leads to splitting of the multiple eigenvalue $\lambda = 0$ into several simple eigenvalues. As a result, unstable eigenvalues can arise around $\lambda = 0$.

The multiplicity of the eigenvalue $\lambda = 0$ is related to rotational (\textit{i.e.} phase) and translational symmetries of the model. The dissipative perturbation introduced by $V(\eta)$ and $W(\eta)$ breaks the translational symmetry and preserves the rotational one. Due to the last fact $\lambda = 0$ remains an eigenvalue for $L$. However $\psi^{(0)}_{11}$ ceases to be the eigenfunction corresponding to $\lambda = 0$ and the only eigenfunction for $\lambda = 0$ is given by $\psi_{12} = (w_w, w)^T$. Disregarding parity of the functions $w(\eta)$ we observe that if $\lambda$ is an eigenvalue of the operator $L$ then $\bar{\lambda}$ is also an eigenvalue. Then, recalling eqs. (4) we find that $-\lambda$ is an eigenvalue, as well. Therefore exactly two simple eigenvalues arise in the vicinity of $\lambda = 0$ and those eigenvalues are either purely real or purely imaginary at the same time and have opposite signs.

\textbf{Evans function for $\mathcal{PT}$-symmetric potentials.} – The operator $L$ can be associated with the Evans function $E(\lambda)$ [9,10], which is an analytic function defined on the whole complex plane except for the points of the continuous spectrum of $L$. An important property of the Evans function is that it has a zero at some $\lambda$ if and only if $\lambda$ is an eigenvalue of $L$. In addition, the order of that zero is equal to the a.m. of the eigenvalue $\lambda$.

Before proceeding with explicit definition of the Evans function for the case at hand, we recall that perturbation of the operator $L$ leads to that the eigenvalue $\lambda_0$ has a.m. = 2. Hence the Evans function corresponding to the perturbed operator $L$ has a zero of the second order at $\lambda = 0$: $E(0) = \partial_{\lambda}E = 0$, $\partial^2_{\lambda}E \neq 0$ (hereafter $\partial^2_{\lambda}$ stays for $j$-th partial derivative with respect to $\lambda$ evaluated at $\lambda = 0$). Without loss of generality one can assume that $E(\lambda) < 0$ for all $\lambda \gg 1$. Then the stability of the stationary mode is determined by the sign of $\partial^2_{\lambda}E$: if $\partial^2_{\lambda}E > 0$ then $E(\lambda)$ necessarily has exactly one positive and one negative zero which corresponds to instability; vice versa, if $\partial^2_{\lambda}E < 0$ then both roots of the Evans function lie on the imaginary axis, and the solution is stable. These considerations are illustrated in fig. 2.
In order to define the Evans function for the operator \( L \) given by eq. (7) we rewrite the eigenvalue problem \((L - \lambda)p = 0\) in the form of four first-order ODEs \( \partial_\eta Y = MY \), where \( Y = (pr, p_r, \partial_\eta pr, \partial_\eta p_r)^T \) and
\[
M = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-2N_{21} & 2(\lambda - N_{22}) & 0 & 0 \\
2(N_{11} - \lambda) & 2N_{12} & 0 & 0
\end{pmatrix}
\]
(11)
The coefficients \( N_{jk} \) are given by eqs. (8). Parity of the functions \( w_r(\eta) \) and \( w_i(\eta) \) imposed by eq. (4) implies that \( N_{12} \) and \( N_{21} \) are even while \( N_{11} \) and \( N_{22} \) are odd as functions of \( \eta \).

For the sake of simplicity we suppose that \( W(\eta) = \sigma W_0(\eta) \), where \( \sigma < 1 \) is the small parameter while \( V(\eta) = O(\sigma^3) \). Then the coefficients \( N_{jk} \) become also dependent on \( \sigma \). Obviously, now the substitution \( \sigma \to -\sigma \) equivalent to the substitution \( \eta \to -\eta \) which in turn is equivalent to \( w_r(\eta) \to w_r(\eta) \) and \( w_i(\eta) \to -w_i(\eta) \). As a result, \( N_{12} \) and \( N_{21} \) are even while \( N_{11} \) and \( N_{22} \) are odd in \( \sigma \).

Now we choose four solutions \( Y_j(\eta; \sigma; \lambda), j = 1, \ldots, 4 \), such that \( Y_1 \) and \( Y_2 \) are linearly independent in \( \eta \) and vanish as \( \eta \to -\infty \), while \( Y_3 \) and \( Y_4 \) are linearly independent and vanish as \( \eta \to +\infty \). Then the Evans function is given as a \( 4 \times 4 \eta \)-independent determinant \( E(\sigma; \lambda) = \det[Y_1, Y_2, Y_3, Y_4] \). In order for \( E(\sigma; \lambda) \) to be unambiguously defined and to depend analytically on \( \sigma \) and \( \lambda \), we fix the choice of the solutions \( Y_j \) setting
\[
Y_{1,4}(\eta; 0; 0) = \begin{pmatrix}
\psi_{11}(0) \\
\partial_\eta \psi_{11}(0) \\
0 \\
0
\end{pmatrix},
\]
\[
Y_{2,4}(\eta; 0; 0) = \begin{pmatrix}
\psi_{12}(0) \\
\partial_\eta \psi_{12}(0) \\
0 \\
0
\end{pmatrix}
\]
(12a)
(recall that \( \psi_{11}(0) \) and \( \psi_{12}(0) \) are given by eqs. (9)). Requiring this choice to be consistent with parity of the coefficients \( N_{jk} \) we find that
\[
Y_3(\eta; \pm \sigma; \mp \lambda) = Y_1(-\eta; \sigma; \lambda) * J_\pm,
\]
\[
Y_4(\eta; \pm \sigma; \mp \lambda) = \pm Y_2(-\eta; \sigma; \lambda) * J_\pm
\]
(12b)
where \( J_+ = (-1, 1, 1, -1)^T \), \( J_- = (-1, -1, 1, 1)^T \), and \( * \) stands for the element-wise multiplication of the matrices. Equations (12) imply that the Evans function is even with respect to both its arguments: \( E(\sigma; -\lambda) = E(\sigma; \lambda) \). Respectively, the second derivative of the Evans function at \( \lambda = \lambda_0 \) can be searched in the form of the following expansion:
\[
\partial^2_\sigma E(\sigma; \lambda_0) = \sigma^{2k} E_{2k} + o(\sigma^{2k}), \quad E_{2k} \neq 0,
\]
(13)
where \( k = 1, 2, \ldots \) is to be determined.

Subject to the perturbation, i.e. for \( \sigma \neq 0 \), the eigenfunction \( \psi_{11} \) disappears; the solutions \( Y_{1,4}(\eta; \sigma; 0) \) become unbounded and no longer correspond to any eigenfunction of the operator \( L \). However, the eigenfunction \( \psi_{12} \) defined above persists and depends smoothly on \( \sigma \). As a result, for all \( \sigma \) the following equalities hold:
\[
Y_{2,4}(\eta; \sigma; 0) = \begin{pmatrix}
\psi_{12} \\
\partial_\eta \psi_{12}
\end{pmatrix} = (w_r, -w_r, -\partial_\eta w_r, \partial_\eta w_r)^T.
\]

Since \( \lambda_0 \) has a.m. = 2 in the spectrum of the perturbed operator, Lemma 3.3 from [10] can be applied. It states that
\[
\partial_\eta Y_3(\eta; \sigma; 0) = \partial_\eta \psi_{12}(\eta; \sigma; 0)
\]
for all \( \eta \) and \( \sigma \). Bearing in mind these facts and differentiating straightforwardly the Evans function one arrives at the following expression for the second derivative of the Evans function at \( \lambda = 0 \):
\[
\partial^2_\lambda E(\sigma; 0) = \det[Y_1, Y_3, \partial^2_\lambda Y_2 - Y_4, Y_2].
\]

Let us now recall that \( E(\sigma; \lambda) \) is an even function of \( \sigma \). It means that \( \partial_\sigma \partial^2_\lambda E(0) = 0 \) where \( \partial_\sigma \psi_{12} \) stands for the jth partial derivative with respect to \( \sigma \) evaluated at \( \sigma_0 = 0 \). Calculating the second derivative with respect to \( \sigma \) and evaluating it at \( \sigma_0 = 0 \) one arrives at
\[
\partial^2_\sigma \partial^2_\lambda E(0; 0) = \det[\partial_\sigma \psi_{12}(Y_1 - Y_4), Y_1, \partial_\sigma \partial^2_\lambda Y_2 - Y_4, Y_2] + \det[\partial_\sigma \psi_{12}(Y_1 - Y_3), Y_1, \partial_\sigma \partial^2_\lambda Y_2 - Y_4, Y_2].
\]
(15)
Using parity of the coefficients \( N_{jk} \) and the symmetries of the solutions \( Y_j \) given by eqs. (12) one can recognize that r.h.s. of eq. (15) generically is not equal to zero, i.e. we can set \( k = 1 \) in eqs. (13) and obtain
\[
\partial_\sigma \partial^2_\lambda E(0; 0) = 0, \quad \partial_\sigma \partial^2_\lambda E(0; 0) \neq 0.
\]

We failed to obtain expressions for the mixed derivatives \( \partial^2_\sigma \partial^2_\lambda (Y_1 - Y_3) \) and \( \partial_\sigma \partial_\lambda \partial^2_\lambda (Y_2 - Y_4) \) which would allow for efficient analytical or numerical computation of \( \partial^2_\sigma \partial^2_\lambda E(0; 0) \). However, using the information given by (16) one can use another representation for derivatives of the Evans function [10] which fits better for analytical and numerical investigation. To this end, we recall that apart from the eigenfunction \( \psi_{12} \) there exists a generalized eigenfunction \( \psi_{22} \) such that \( L \psi_{22} = \psi_{12} \). The adjoint operator for \( L \) reads \( L = L^T \). There also exists the adjoint eigenfunction \( \psi \) such that \( L^T \psi = 0 \). It is found in [10] that for any \( \sigma \neq 0 \) the second derivative of the Evans function at \( \lambda = 0 \) can be found as \( \partial^2_\sigma E(\sigma; 0) = 2(\psi_{22}, \psi) \), where \( \langle \cdot, \cdot \rangle \) represents standard \( L^2 \) inner product of vector-valued functions. Taking into account eqs. (16) one can construct an expansion for \( \psi_{22}, \psi \) with respect to the small parameter \( \sigma \), \( 0 < |\sigma| < 1 \). To this end, we firstly write down an expansion for the stationary mode itself. Equation (3) dictates that this expansion acquires the form:
\[
\begin{align}
L w_r^{(2)} &= w_r^{(0)} + 2w_r^{(2)} + o(\sigma^2), \\
L w_i^{(2)} &= w_i^{(0)} + o(\sigma^2),
\end{align}
\]
where \( w_r^{(2)} \) and \( w_i^{(2)} \) solve the equations
\[
L^+ w_r^{(2)} = w_r^{(0)} w_i^{(1)} - W_0(\eta) w_r^{(0)}, \quad L^+ w_i^{(2)} = -W_0(\eta) w_r^{(0)} w_i^{(0)},
\]
(17a)
(17b)
and \( L^\pm = \frac{1}{2} \partial^2_\eta - b + (2 \pm 1)(w_r^{(0)})^2 \). Notice that \( w_r^{(2)} \) is an even function of \( \eta \) while \( w_i^{(1)} \) is odd. Next, using eqs. (10)
and the definition of $\psi_{22}$ (i.e. the equation $L\psi_{22} = \psi_{12}$) we obtain an expansion for the generalized eigenfunction:

$$\psi_{22} = \left( \partial_\eta w_r^{(0)} \right) + \sigma \left( 0 \chi^{(1)} \right) + o(\sigma), \quad (18)$$

where the coefficient $\chi^{(1)}$ is an odd function of $\eta$ solving the equation

$$L^- \chi^{(1)} = -w_r^{(0)} \partial_\eta w_r^{(0)} \left[ 3W_0(\eta)w_r^{(0)} + 2w_i^{(1)} \right] + w_i^{(1)}. \quad (19)$$

Using the definition of the adjoint eigenfunction $v$ and requiring the derivative $\partial_\eta E(\sigma;0)$ to satisfy constraints (16) we observe that expansion for $v$ must have the form

$$v = \sigma \left( \partial_\eta w_r^{(0)} \right) + \sigma^2 \left( \rho^{(1)} \right) + \sigma^3 \left( 0 \phi^{(2)} \right) + o(\sigma^3). \quad (20)$$

Then $\rho^{(1)}$ is given as $\rho^{(1)} = \beta w_r^{(0)} + f$, where $f$ is a particular solution of the equation

$$L^- f = w_r^{(0)} \partial_\eta w_r^{(0)} \left[ 2w_i^{(1)} - W_0(\eta)w_r^{(0)} \right]. \quad (21)$$

The solvability condition for eq. (21) (i.e. orthogonality of its r.h.s. to $w_r^{(0)}$ is automatically provided as long as eq. (17b) holds. The coefficient $\beta$ should be chosen to satisfy the solvability condition of the equation for $\phi^{(2)}$:

$$L^+ \phi^{(2)} = 2w_i^{(1)} + 3W_0(\eta)w_r^{(0)}|w_r^{(0)}\rho^{(1)} - \partial_\eta w_r^{(0)} [6w_r^{(0)}w_i^{(2)} + (w_i^{(1)})^2 - 2W_0(\eta)w_r^{(0)}w_i^{(1)}]. \quad (22)$$

Respectively, we require r.h.s. of (22) to be orthogonal to $\partial_\eta w_r^{(0)}$ what yields $\beta = I_1/I_2$, where

$$I_1 = \int d\eta \left\{ -f w_r^{(0)} \partial_\eta w_r^{(0)} \left[ 3W_0(\eta)w_r^{(0)} + 2w_i^{(1)} \right] + \left( \partial_\eta w_r^{(0)} \right)^2 [6w_r^{(0)}w_i^{(2)} + (w_i^{(1)})^2 - 2W_0(\eta)w_r^{(0)}w_i^{(1)}] \right\},$$

and $I_2 = -\int \partial_\eta W_0(\eta)(w_r^{(0)})^4 d\eta$. Finally, using eqs. (18) and (20), we can rewrite eq. (13) in the form:

$$\partial_\eta E(\sigma;0) = 2(\psi_{21}, v) = \sigma^2 E_2 + o(\sigma^3),$$

where the coefficient $E_2$ is given as $E_2 = 2 \int (\partial_\eta w_r^{(0)} \chi^{(1)} + \partial_\eta w_r^{(0)} \rho^{(1)}) d\eta$. Functions $\chi^{(1)}$ and $\rho^{(1)}$ can be computed numerically from the linear equations (19) and (21), what gives an algorithm for obtaining the coefficient $E_2$. For given $b$ positive value of $E_2$ corresponds to the situation when the modes are unstable for small $\sigma$. Vice versa, negative $E_2$ implies stability of the modes for small $\sigma$.

Discussion of the results and conclusion. – Let us now turn to the results of the stability analysis of the particular examples (see fig. 3). We start by recalling the results for the case $V(\eta) = 0$ and $W(\eta) = \sigma \sin(2\eta)$ reported earlier in [5]. Physically, this case corresponds to the pure dissipative nonlinear lattice where domains with the nonlinear gain alternate with the nonlinear dissipation. This case was already discussed in [5], and in particular it was shown that for sufficiently small $\sigma$ the nonlinear modes become stable if the propagation constant $b$ exceeds a threshold value $b^{cr}(\sigma)$. The approach developed here based on the analysis of the Evans function allows us to compute numerically $b^{cr}(0)$, as this is shown in fig. 3(d) (the black curve 1) from which one observes that the coefficient $E_2$ changes its sign at $b^{cr}(0) \approx 1.05$ which corroborates results reported in the panel (a). This critical value corresponds to narrow modes, whose widths are of order of the period of the dissipative lattice.

Next we consider potential (6) (see also fig. 1) which is characterized by only one domain with nonlinear gain and one domain with nonlinear dissipation. From fig. 3(b) we observe that the domains of existence and stability are similar to those obtained for the sin-shaped potential: in particular, there exists the threshold for the stability of the modes. Quantitatively, now the stable modes can be broader than in the case of periodic potential: threshold value of propagation constant is remarkably lower than in the previous case. More specifically, using the approach based on the Evans function we obtain (the blue curve in panel (d)): $b^{cr}(0) \approx 0.49$. This result agrees with fig. 3(b) and with fig. 1(d) where the dependence of perturbation growth rate $\Re \lambda$ on $\sigma$ is shown.

From the results for sin- and tanh-shaped potentials we can conjecture that i) the most stable modes are localized on the scale where the dissipative potential can be approximated by the linear function, i.e. where $W(\eta) \sim 2\sigma \eta$ on the width of the mode; and ii) properly introduced
A gain and dissipation from both sides of the mode enhances its stability. These arguments readily lead to the model

\[ i \partial_t \tilde{q} = -\frac{1}{2} \partial_{\tilde{q}^2} \tilde{q} - |\tilde{q}|^2 \tilde{q} - 2i \hat{\sigma} q |\tilde{q}|^2 \tilde{q}, \tag{23} \]

where all modes in the limit \( \sigma \to 0 \) should be stable. This is indeed what happens, as one can see from fig. 3 (c): the instability threshold \( b^{\ast}(0) \) disappears completely. This is also confirmed by analysis based on the Evans function (the red curve 3 in the panel (d)).

Now from panel (c) we observe that both lines, i.e., the upper border of the existence domain \( \sigma_{app} \) (black curve) and the upper border of stability domain \( \sigma_{cr} \) (red curve), follow the parabolic scaling law \( b \sim \sigma^2 \). This law can be understood from the simple scaling arguments as follows. If \( q(\xi, \eta) \) is a solution of eq. (23), then \( \tilde{q}(\tilde{\xi}, \tilde{\eta}) = \frac{1}{\sigma} q(\xi, \eta) \) with \( \tilde{\xi} = \sigma^2 \xi \) and \( \tilde{\eta} = \sigma \eta \) is a solution of the complex NLS equation without any parameter: \( i \partial_t \tilde{q} = -\frac{1}{2} \partial_{\tilde{q}^2} \tilde{q} - |\tilde{q}|^2 \tilde{q} - 2i \hat{\sigma} \tilde{q} |\tilde{q}|^2 \tilde{q} \). This observation as well as the fact that a narrow mode “feels” only the local dissipative term, allows one to make further conclusions about the behavior of the curves in fig. 3. Since the mode widths tend to zero at \( b \to \infty \) (and \( \sigma \) bounded), and all the examples of \( W(\eta) \) were chosen to have the same slope at \( \eta = 0: W(\eta) \sim 2 \sigma \eta \), the existence curves \( \sigma_{app} \) in panels (a) and (b) tend at \( b \to \infty \) to the parabola \( \sigma_{app} \) shown in panel (c). This conjecture was supported by numerical simulations up to \( b = 20 \) where for the linear, tanh- and sin-shaped potentials we found \( \sigma_{app} \approx 3.18, 3.20, \) and 3.19, respectively.

In fig. 4 we present examples of evolution of unstable (panel (a)) and stable (panel (b)) modes. These results, obtained by direct integration of eq. (23) (i.e. of the particular case of the model (1) which is mostly “exposed” to eventual nonlinear instabilities) confirm the linear stability results. Thus, the modes belonging to the stability domains propagate undistorted over indefinitely long distances, even if they are strongly perturbed initially.

To conclude, we have investigated fundamental modes in imaginary \( PT \)-symmetric nonlinear potentials. Such potentials allow for the existence of localized modes which are stable at least in the limit when the mode is narrow enough. The stability was established both as the linear stability, on the basis of the Evans function analysis, and using direct numerical study of the mode evolution (notice that while the direct propagation ensures also the nonlinear stability of the modes in a finite domain, the nonlinear nature of the perturbation may introduce new features of the nonlinear stability of the solutions on the whole real axis). Although our analysis was performed for nonlinear potentials, the approach can be applied also for linear \( PT \) potentials, as well as to the cases where both linear and nonlinear \( PT \)-symmetric potentials are present. In the latter case stability of the modes may change dramatically. For example, in presence of a periodic linear \( PT \) potential broad small-amplitude modes are expected to be stable as long as imaginary part of the linear potential is below a critical value. This situation is in contrast to the one shown in fig. 3(a) where broad modes are unstable. As another interesting question, we would like to mention the exploration of asymmetric nonlinear modes similar to the ones reported in [6], although their existence may require more sophisticated nonlinearity landscapes.

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