Entanglement-Enabled Communication

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Abstract—We introduce and analyse a multiple-access channel with two senders and one receiver, in the presence of i.i.d. noise coming from the environment. Partial side information about the environmental states allows the senders to modulate their signals accordingly. An adversarial jammer with its own access to information on environmental states and the modulation signals can jam a fraction of the transmissions. Our results show that for many choices of the system parameters, entanglement shared between the two senders allows them to communicate at non-zero rates with the receiver, while for the same parameters the system forbids any communication without entanglement-assistance, even if the senders have access to common randomness (local correlations). We complement these results by demonstrating that there even exist model parameters for which entanglement-assisted communication is no longer possible, but a hypothetical use of nonlocal no-signalling correlations between Alice and Bob could enable them to communicate to Charlie again. While it was long-known that quantum correlations can improve the performance of quantum communication systems, our results show dramatic manifestations of quantum pennage in otherwise entirely classical systems. We believe that they open the door to realistic, disruptive and near-term applications of quantum technologies in communication.

Index Terms—Information theory, Entanglement, Quantum communication, Multiple-Access Channel, Cooperation, Arbitrarily Varying Channel

I. INTRODUCTION

What new possibilities does quantum nonlocality offer us? This question, that was posed in the 1994 publication [18] on quantum nonlocality, has not lost any of its appeal.

It is now known that quantum technology offers dramatic advantages in the areas of computing [12], secret communication [3], randomness generation [7] and metrology [4]. That is, a quantum system can outperform their classical analogues, for instance a quantum computer a classical Turing machine, a quantum communication line with quantum states and quantum detectors a classical communication system, etc. It is not known how quantum communication can interplay with existing communication systems.

In most of the aforementioned known technological implementations of quantum effects, quantum entanglement [11], [19] has been identified as the crucial enabling property providing the quantum advantage. Yet, not much can be found on the role of quantum entanglement as a plug-in resource for an otherwise classical communication system. Instead, much of the previous literature on quantum communication is concerned with complete quantum systems. This manuscript highlights the benefits of quantum correlations (entanglement) over classical correlations in a classical communication system.

The closest we can so far get to a comparison of quantum and classical correlation on equal terms is in the setting of nonlocal games, where two or more players sitting in closed labs receive queries and have to give answers without communicating. Here, the entanglement (or whatever other physical correlation considered) is thought of as enclosed in a black box, which each player operates by choosing classical settings and reading classical outcomes, while it is unimportant for the players that the origin of the correlations is that the black boxes contain quantum systems in an entangled quantum state, and that each box internally performs a measurement of a quantum mechanical observable on its quantum system.

The prototypical nonlocal game, named after Clauser, Horne, Shimony and Holt (CHSH) [6], who developed the original idea of Bell [2], has two players, each with binary inputs and binary outputs: $x, y \in \{0, 1\}$, $\alpha, \beta \in \{0, 1\}$. The players may agree on a strategy before the game starts, including a shared random variable $\lambda$; however, when play starts, they are separated, each receives their input (Alice $x$, Bob $y$), and without consulting each other, they have to respond with outputs $\alpha$ (Alice) and $\beta$ (Bob). Alice and Bob win the game if

$$xy = \alpha \oplus \beta,$$

otherwise they lose. Assuming uniform distribution on the inputs $x$ and $y$, [6, 2] showed that the maximum winning probability for classically correlated players is $3/4$, corresponding to the easily-verified fact that if $\alpha = \alpha(x)$ and $\beta = \beta(y)$ are functions of $x$ and $y$ alone, respectively, then Eq. (I) can be satisfied in only 3 out of 4 cases. Interestingly, and crucially, with a quantum strategy, where each player holds one of two quantum bits (qubits) that are prepared in a maximally entangled state, and by making suitable quantum measurements on their respective systems, they can win with probability $\cos^2 \frac{\pi}{8} = \frac{1}{2} \left(1 + \frac{1}{\sqrt{2}}\right) \approx 0.85 > 3/4$. There are generalisations of the CHSH game, with more input and outputs and different winning predicates [13, 21, 17].

In the present paper, we show how to harness this advantage in a setting where Alice and Bob wish to communicate over a joint channel to a single receiver, i.e., they face correlated noise and have to cooperate (or even coordinate) to maximise their throughput. The basic model is that of the multiple-access channel (MAC), which was introduced in the work of [3].
Shannon [20], and solved by Ahlswede and Liao [1, 15]. To obtain the strongest possible separations, however, we will add the following features to the channel model: a jammer who can influence the noisy channel in an “arbitrary” way by choosing from a finite list of options in each transmission, access to partial information regarding the channel state for two technical devices (called modulators) under the control of the sending parties, and the option to use further (non-signalling) resources between them. We call this model the modulated arbitrarily varying multiple-access channel with (partial) environmental state information (MAVMACEI).

As is the case for the ordinary multiple-access channel (MAC), a noise process randomly generates different channel realizations. The generating process is modeled by the random variable $X : Y$ where both $X$ and $Y$ are sources of perfect random bits. Similar to the CHSH game, $X$ is made available to a technical device called a “modulator” in Alice’s possession and $Y$ is made available to a similar device in Bob’s possession. Both modulators can accept a further, optional input. Both devices can modify the information transmitted by their respective owner depending on their received input. The channel states $X : Y$ and the outputs $\alpha, \beta$ of the modulators are revealed to James, who then selects an additional input (state) to the channel. To limit his otherwise overwhelming capabilities, we subject James to a power constraint $\Lambda \in [0, 1]$.

This last subtlety of our model allows us to demonstrate the differences between purely classical coding on the one hand side and entanglement-assisted encoding on the other hand side in the strongest possible sense: there exist choices for the value of the power constraint, such that the rate region of our MAVMACEI may consist of the single point $\{(0, 0)\}$ for purely classical coding but may have non-empty interior under entanglement-assisted coding. As the model consists of binary alphabets only, noise is modelled by random bit flips. A key observation then is that the jammer’s state knowledge enables her to flip bits only at positions that were not in error already, thereby effectively increasing the probability of an error from e.g. $1/4$ to $1/4 + \Lambda$. The MAVMACEI thus becomes useless for message transmission purpose as soon as $\Lambda \geq 1/4 = 0.25$.

To re-establish the possibility of communicating from Alice and Bob to Charlie, one may resort to communication between Alice and Bob, for example in the spirit of conferencing [22]. In this work, we explicitly forbid any communication between Alice and Bob. Instead, we allow for “entanglement-modulated encoding”. When this method is used, a source of entangled quantum states is available to Alice and Bob. Based on their inputs $X$ or $Y$, the modulation units then perform a measurement on a shared quantum state. Their outputs $\alpha, \beta$ modify the sending parties input and can be read by James.

We show how such an entanglement-assisted coding scheme is able to reduce the effective initial noise to $2 - \sqrt{2}$, such that the operating point at which the entanglement-assisted MAVMACEI cannot reliably transmit any messages any more is reached much later, when $\Lambda \geq \frac{1}{2\sqrt{2}} \approx 0.35$. In the region $\Lambda \in [0.25, 0.35]$ we therefore have plenty of space for channel models having an empty capacity region without entanglement-assistance and a non-empty capacity region with entanglement-assistance.

To allow for a fair comparison, we study a third situation where Alice and Bob share a so-called “local correlation” instead of an entangled state. This type of correlation can be established without resorting to a use of quantum mechanical devices. The modulation units then modify the sending parties signals using potentially correlated inputs $\alpha$ and $\beta$. Again, $\alpha$ and $\beta$ are made available to James.

To complete our analysis, we show that arbitrary non-signalling correlations can enable Alice and Bob to reduce the noise (but not the interference) to zero.

In all cases, given the input $s = s(\alpha, \beta, x, y)$ of James the channel output is

$$c = a \oplus b \oplus x \cdot y \oplus \alpha \oplus \beta \oplus s. \quad (2)$$

Let $\mathcal{R}_e$ ($e$ being the abbreviation for “entanglement”) denote the rate region of this channel when access to an entangled state is given. Let $\mathcal{R}_r$ ($r$ being the abbreviation for “random”) denote the rate region of the channel when only classical coordination resources can be used and $\mathcal{R}_d$ ($d$ being the abbreviation for “deterministic”) the rate region when no coordination resource can be used. Our result is a separation as follows:

- If Alice and Bob use independent encoding or correlated encoding but $\Lambda \geq 1/4$ then none of them can transmit at a positive rate. The respective capacity regions $\mathcal{R}_d, \mathcal{R}_r$ consist only of one trivial element: $\mathcal{R}_d = \mathcal{R}_r = \{(0, 0)\}$.
- If $\Lambda < 1/4$ then $\mathcal{R}_r \neq \{(0, 0)\}$.
- If Alice and Bob correlate their encoding using a shared entangled state then the respective capacity region $\mathcal{R}_e$ satisfies $\mathcal{R}_e \neq \{(0, 0)\}$ even when $\Lambda$ satisfies the weaker constraint $\Lambda < \frac{1}{2\sqrt{2}}$. [25]
In other words: choosing \( \Lambda = 1/4 \) proves the existence of a classical communication that has zero capacity without entanglement assistance and nonzero capacity with entanglement assistance.

Hints towards the possibility of such statement have been found earlier already in [10] for classical zero-error communication assisted by entanglement between the communicating parties. The very recent work [14] demonstrated again a benefit of using entanglement to assist the task of classical communication.

II. OUTLINE

We introduce our notation and definitions in section III after which we state our main result together with the key elements of its proof in section IV. The final section V and the appendix C are devoted to the proof of the technical details.

III. NOTATION AND DEFINITIONS

A. Notation

The basic building blocks of the systems studied in this work are the alphabet \( \{0,1\} \) and the state space \( \mathbb{C}^2 \). For two elements \( x,y \in \{0,1\} \), \( x \oplus y \) denotes addition modulo two.

Given an arbitrary finite alphabet (set) \( X \), the set of probability distributions on it is denoted \( \mathcal{P}(X) \). The corresponding state space for quantum systems on a finite dimensional Hilbert space \( \mathcal{H} \) is denoted \( \mathcal{S}(\mathcal{H}) \). All sets in this work are considered finite, and likewise all Hilbert spaces will be finite dimensional. A quantum state will typically be denoted \( \Psi \).

For an element \( x \in X \) the symbol \( \delta_x \) denotes an element of \( \mathcal{P}(X) \) with the property \( \delta_x(x') = 1 \) if and only if \( x = x' \). The symbol \( \pi \) denotes the unique distribution with the property \( \pi(x) = |X|^{-1} \) for all \( x \in X \). To save space, we may occasionally write \( p_i \) instead of \( p(i) \). The set \( \mathcal{P}(X) \) can be written as the convex hull conv of the set \( \{ \delta_x \}_{x \in X} \). Composite alphabets are defined as \( X \times Y := \{(x,y): x \in X, y \in Y\} \) and composite quantum systems are modelled on tensor products \( \mathcal{H} \otimes \mathcal{H}' \).

The n-fold composition of \( X \times \ldots \times X \) is written \( X^n \). If \( X = \{0,1\} \) and \( x^n, y^n \in X^n \) then \( x^n \oplus y^n \) is defined component-wise as \((x^n \oplus y^n)_i := x_i \oplus y_i \).

The scalar product of \( x,y \in \mathbb{C}^d \) is denoted \( \langle x,y \rangle \). A positive operator-valued measurement (POVM) on a Hilbert space \( \mathbb{C}^d \) is a collection \( (M_i)_{i=1}^I \) of non-negative (meaning that \( \langle x,M_i y \rangle \geq 0 \) for all \( i = 1,\ldots,I \)) matrices such that \( \sum_{i=1}^I M_i = \mathbb{1} \), where \( \mathbb{1} \) is the identity map on \( \mathbb{C}^d \). For pure states, we may occasionally write \( \langle \psi | \psi \rangle = |\psi \rangle \langle \psi | \). For any vector such that \( \langle \psi , \Psi \psi \rangle = 1 \). The trace of any matrix \( M \) on \( \mathbb{C}^d \) is denoted \( \text{tr}(M) \).

A classical channel \( W \) with input alphabet \( X \) and output alphabet \( Y \) is completely defined by the matrix \( (w(y|x))_{x \in X,y \in Y} \) where each matrix entry is a conditional probability \( w(y|x) \) of mapping an input \( x \in X \) to an output \( y \in Y \). We thus identify channels with their corresponding matrix of conditional probabilities and write \( W \) for both the former and the latter. A channel \( W \) is a linear map from \( \mathcal{P}(X) \) to \( \mathcal{P}(Y) \) satisfying

\[
W_p(y) = \sum_x p(x)w(y|x).
\]

The set of channels with input alphabet \( X \) and output alphabet \( Y \) is denoted \( \mathcal{C}(X,Y) \). There are five particular channels acting on binary alphabets that deserve a specific symbol: the first is the identity on \( \{0, 1\} \), denoted \( \mathbb{1} \). It holds \( I \delta_x = \delta_x \) for all \( x \in \{0,1\} \). The second is the bit-flip on \( \{0,1\} \), denoted \( \mathbb{F} \). It holds \( \mathbb{F} \delta_x = \delta_{x \oplus 1} \) for all \( x \in \{0,1\} \). The third is the binary symmetric channel with parameter \( \nu \in [0,1] \) that we denote \( \text{BSC}(\nu) \). It is defined as

\[
\text{BSC}(\nu) = \nu \mathbb{1} + (1 - \nu) \mathbb{F}.
\]

The fourth is the interference channel \( I \in \mathcal{C}(\{0,1\}^2, \{0,1\}) \) that acts as

\[
I(\delta_{x,y}) = \delta_{x \oplus y}.
\]

The fifth is \( J \in \mathcal{C}(\{0,1\}^3, \{0,1\}) \) and can, with the convention \( J_0 := \mathbb{1} \) and \( J_1 := \mathbb{F} \), be written down in the form of an arbitrarily varying channel as

\[
J = (J_0 \circ J_1)^{x,y=0}.
\]

The symbol \( W_p \) denotes the output distribution of \( W \) upon input \( p \), it holds \( W_p(y) := \sum_x p(x)w(y|x) \). The symbol \( (W,p) \) stands for the joint distribution of input- and output symbols. It holds \( (W,p)(y,x) := p(x)w(y|x) \).

The entropy of a probability distribution \( p \in \mathcal{P}(X) \) is defined as

\[
H(p) := -\sum_{x \in X} p(x) \log_2(p(x)).
\]

The mutual information of a probability distribution \( p \in \mathcal{P}(X) \) and a channel \( W \in \mathcal{C}(X,Y) \) is defined as

\[
I(p;W) := H(p) + H(Wp) - H((W,p)).
\]

The symbol \( H(p) \) is, whenever unambiguously possible, identified with the value \( p(1) \). In some cases only this value \( p(1) \) may be given, and in those cases we write \( h(p) \) or \( h(p(1)) \) to denote the entropy \( H(p) \) of the probability distribution \( p \).

The mutual information of a probability distribution \( p \in \mathcal{P}(X) \) and a channel \( W \in \mathcal{C}(X,Y) \) is defined as

\[
I(p;W) := H(p) + H(Wp) - H((W,p)).
\]

Given \( n \in \mathbb{N} \) and a number \( t \) satisfying \( 0 \leq t \leq n \), the typical set \( T^n_t \subset \{0,1\}^n \) is defined as \( T^n_t := \{ x^n : N(1|x^n) = t \} \), where we set \( N(1|x^n) := |\{i : x_i = 1\}| \) for every \( x \in \{0,1\}^n \) and \( x^n \in \{0,1\}^n \). In cases where \( n \in \mathbb{N} \) is clear from the context, the distribution \( \pi_t \in \mathcal{P} \) is defined as

\[
\pi_t(x^n) := \begin{cases} |T^n_t|^{-1}, & x^n \in T^n_t, \\ 0, & \text{else} \end{cases}
\]

For \( \delta \geq 0 \) and \( p \in \mathcal{P}(X) \), the \( \delta \)-typical set \( T^n_{p,\delta} \) is defined as the set of all \( x^n \in X^n \) such that \( |N(1|x^n) - n \cdot p| \leq n \delta \).

For a number \( \nu \in [0,1] \) we will typically use the abbreviation \( \nu' := 1 - \nu \).
B. Definitions

This subsection contains the formal definitions of our models. In the following, A and B stand for the alphabets that Alice and Bob can use to compose their code-words. C is the alphabet at the receiver, Charlie.

Definition 1 (Code). A code for block length \( n \) consists of message pairs \( \{(u, v)\}_{u,v=1}^{U,V} \) and the corresponding code words \( a^n_u, b^n_v \in A^n, B^n \) together with a collection \( (D_{u,v})_{u,v=1}^{U,V} \) of decoding sets satisfying \( D_{u,v} \subseteq C^n \) for all \( u, v \in [U], [V] \) and \( D_{u,v} \cap D_{u',v'} = \emptyset \) whenever \( (u, v) \neq (u', v') \).

Remark 1. Definition 1 covers the case where \( U = 1 \) or \( V = 1 \). In our application we will treat the case \( A = B = C \) only.

Next we define what non-signalling and what local correlations are. The study of such correlation goes back to the work of Cirel’son [5] and [13]. The following two definitions are taken from [16].

Definition 2 (Non-Signalling Correlation). Let \( A, B, X, Y \) be alphabets. A conditional probability distribution \( \{q(a, b | x, y)\}_{a \in A, b \in B, x \in X, y \in Y} \subseteq \mathcal{P}(A \times B) \) is called non-signalling if

\[
\forall a, x, y, y' : \sum_b q(a, b | x, y) = \sum_b q(a, b | x, y')
\]

\[
\forall b, x, x' : \sum_a q(a, b | x) = \sum_a q(a, b | x')
\]

An example for a non-signalling correlation is the one presented in Lemma 10.

Definition 3 (Local Correlation). Let \( A, B, X \) and \( Y \) be alphabets. A conditional probability distribution \( \{q(a, b | x, y)\}_{a \in A, b \in B, x \in X, y \in Y} \subseteq \mathcal{P}(A \times B) \) is called local if

\[
q(a, b | x, y) = \sum_c p(c) q_1(a | e, x) q_2(b | e, y)
\]

for some distribution \( p \in \mathcal{P}(E) \) on some finite alphabet \( E \).

To avoid simplified access to the tools we use to treat a special type of multiple-access channel with and without entanglement assistance we first define a special form of AVC with state knowledge at the receiver and a power constraint.

Definition 4 (A VC with environmental information at the jammer). Let \( W \in C(A \times X \times Y, C) \) and \( q \in \mathcal{P}(Y) \). Transmission over the AVC \( (W, p) \) with environmental information at the jammer (AVCIE) is a model where the success probability is given by

\[
\min_{s} \frac{1}{M} \sum_{m} \sum_{y^n, y'^n} S(s^n | y^n)p^{\otimes n}(y^n)w^{\otimes n}(D_m | s^n, y^n, a_m^n).
\]

Here \( \{D_m\}_{m=1}^{M} \) are decoding sets (it holds \( D_m \subseteq C^n \) for all \( m \) and \( D_m \cap D_{m'} = \emptyset \) if \( m \neq m' \)) and the minimization \( \min_{s} \) is understood to take place only over admissible jamming strategies. In the most general case all strategies \( S \in C(Y^n, S^n) \) are admissible.

The restriction of admissible codes treated here requires that transmission takes place under a power constraint \( \Lambda \geq 0 \) with constraint function \( I \) \( (l : S \to \mathbb{R}_+) \). Such a constraint requires every \( s^n \) that is selected by the jammer to obey the power constraint \( \sum_{i=1}^{n} l(s_i) \leq n\Lambda \).

This is equivalent to saying that, for every \( s^n \in S^n \), validity of the inequality \( \sum_{i=1}^{n} l(s_i) > n\Lambda \) implies \( S(s^n | y^n) = 0 \) for all \( y^n \in Y^n \).

Definition 5 (Rate and Capacity for AVCEI). A number \( R \geq 0 \) is called an achievable rate for the AVCEI \( (W, p) \) under power constraint \( \Lambda \geq 0 \) (with constraint function \( I \)) if there exists a sequence \( (C_n)_{n \in \mathbb{N}} \) of codes with success probability according to Definition 4 going to 1 as \( n \to \infty \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \log M_n = R.
\]

The capacity of the AVCEI is defined in the usual way as the supremum over achievable rates. It is denoted as \( C_{AVCEI} \).

Arbitrarily varying channels have so far withstood the attempt of a description solely in terms of entropic quantities. Rather, the notion of symmetrizability has proven to be a useful tool when describing their capacities. A key ingredient to the proofs of our results will be a result derived in [9]. Its use requires that we employ the notation of symmetrizability.

Definition 6 (Symmetrizability). Let \( W \in C(S \times X \times Y) \) be an AVC. If there exists a \( \sigma \in C(X, S) \) such that

\[
\sum_{s \in S} \sigma(s | x)w(y | s, x) = \sum_{s \in S} \sigma(s | x)w(y | s, x')
\]

holds for all \( x, x' \in X \) and \( y \in Y \) then \( W \) is called symmetrizable.

The model of the AVCEI can be generalized to a situation with two senders, Alice and Bob, as follows:

Definition 7 (AVMAC with environmental information at the jammer (AVMACIE)). An AVMAC \( W \in C(A \times B \times X \times Y, C) \), together with a probability distribution \( q \in \mathcal{P}(Y) \) is called AVMACIE if the following holds:

For every choice of code defined by decoding sets \( (D_{u,v})_{u,v=1}^{U,V} \) satisfying the usual condition \( D_{u,v} \cap D_{u',v'} = \emptyset \) if \( (u, v) \neq (u', v') \) and corresponding code-words \( (a^n_u, b^n_v)_{u,v=1}^{U,V} \) the probability for successful message transmission over the channel is given by

\[
\min_{S} \frac{1}{UV} \sum_{u,v} \sum_{y^n, y'^n} S(s^n | y^n)q^{\otimes n}(y^n)w^{\otimes n}(D_{u,v} | s^n, y^n, a^n_u, b^n_v).
\]

where the minimization \( \min_{S} \) is over admissible jamming strategies.

A strategy \( S \) is called admissible under a power constraint \( \Lambda \geq 0 \) with constraint function \( I \) \( (l : S \to \mathbb{R}_+) \) if, for every \( s^n \), \( \sum_{i=1}^{n} l(s_i) \leq n\Lambda \) implies \( S(s^n | y^n) = 0 \) for all \( y^n \).

Definition 8 (Rate region for AVMACIE). A pair \( (R_A, R_B) \) of non-negative number is called achievable for the AVMACIE
under power constraint $\Lambda \geq 0$ with constraint function $l$ if there exists a sequence $(C_n)_{n \in \mathbb{N}}$ of codes with success probability according to Definition \ref{def:success_probability}, going to 1 as $n \to \infty$ such that

$$
\lim_{n \to \infty} \frac{\log U_n}{n} \geq R_A \quad \text{and} \quad \lim_{n \to \infty} \frac{\log V_n}{n} \geq R_B. \quad (13)
$$

The rate region of the AVMACEI is defined as the convex closure of the set of achievable rate pairs. It is denoted as $\mathcal{R}_A(V, p)$.

The definition of the AVMACEI is open to an extension towards a model where Alice and Bob share a communication resource. This resource may or may not be of a quantum mechanical nature, and this flexibility allows us to explain the usefulness of entanglement as a plug-in communication resource within an otherwise completely classical communication system.

In the following definition, the so-called ‘environment’ selects states on an alphabet $E_A \times E_B \times Y$ i.i.d. according to a distribution $p$. From every triplet $(e_A, e_B, y)$ the part $e_A$ is fed into a so-called ‘modulator’ that is able to modify Alices signals, and $e_B$ is revealed to a corresponding modulator for Bob’s signals. The symbol $y$ is revealed to James.

**Definition 9** (Flexibly modulated AVMACEI with partial state information (MAVMACEI)). An MAVMACEI consists of a multiple-access channel (MAC) $\mathcal{W} \in \mathcal{C}(E_A \times E_B \times \hat{Y} \times M_a \times M_B \times A \times B \times S, \mathcal{C})$ together with a distribution $p \in \mathcal{P}(E_A \times E_B \times \hat{Y})$. To show the relation to the AVMACEI we will abbreviate $\hat{Y} \times M_a \times M_B$ simply as $Y$.

Given any shared resource in the form of a channel $Q \in \mathcal{C}(E_A \times E_B, M_a \times M_B)$, Alice and Bob can use this resource to create the AVMACEI $\hat{W}$ with transition probabilities $w(e \left| a, b, y \right.)$ given by

$$
\sum_{e_A, e_B} w(e \left| e_A, e_B, y, \alpha, \beta, a, b, s \right.) p'(e_A, e_B | y).
$$

and distribution $q(\alpha, \beta | e_A, e_B) \hat{p}(y)$ where $p(e_A, e_B, y) = p'(e_A, e_B | y) \hat{p}(y)$.

**Definition 10** (Code for the MAVMACEI). A code for the MAVMACEI is a shared resource $Q \in \mathcal{C}(E_A \times E_B, M_a \times M_B)$ plus a code for the AVMACEI as in Definition \ref{def:code}. The code is called deterministic if $\alpha$ is a function only of $e_A$ and $\beta$ a function only of $e_B$. It is called “jointly random modulated” if $Q$ is a local correlation. It is called entanglement-modulated if $Q(\alpha, \beta | e_A, e_B) \equiv \text{tr}(|\Psi M_{A,e_A,B} \otimes M_{B,e_B,a,\beta}|)$ for local measurements $M_{A,e_A}$ and $M_{B,e_B}$ depending explicitly on the environmental states $e_A, e_B$.

**Definition 11** (Rate region of the MAVMACEI). A pair $(R_A, R_B)$ is said to be achievable with

1) deterministically modulated codes
2) jointly random modulated codes
3) entanglement-modulated codes

under power constraint $\Lambda \geq 0$ if there exists a corresponding sequence $(C_n)_{n \in \mathbb{N}}$ of respective codes such that

$$
\begin{align*}
\lim_{n \to \infty} \hat{e}(C_n, \tau, \Lambda) &= 0 \\
\lim_{n \to \infty} \frac{1}{n} \log U_n &\geq R_A \\
\lim_{n \to \infty} \frac{1}{n} \log V_n &\geq R_B
\end{align*}
$$

for the AVMACEI as described in Definition \ref{def:code}.

The letters $\mathcal{R}_A(N, \Lambda), \mathcal{R}_r(N, \Lambda), \mathcal{R}_e(N, \Lambda)$ denote the sets $(R_A, R_B) \in \mathbb{R}^2$ of achievable rates when deterministically-, jointly random- or entanglement modulated codes are used.

**Remark 2.** The specific MAVMACEI studied in this manuscript will be denoted $N$ henceforth. It is one where every alphabet is binary, where

$$
\begin{align*}
p(e_A, e_B, y) &= \pi(e_A)\pi(e_B)\delta(y, e_A \cdot e_B) \\
\hat{w}(e | e_A, e_B, y, \alpha, \beta, a, b) &= \delta(c, 0 \pm e_A \cdot e_B \mp a \pm b \pm s).
\end{align*}
$$

This particular situation is depicted in Figure 7. When this model is used, $e_A$ and $e_B$ do not have any direct impact on the channel. Their only influence is via the modulators.

To keep notation lean we use $i$ and $j$ instead of $e_A$ and $e_B$ in what follows.

We achieve our separation result by showing the existence of values $\Lambda \in [0, 1]$ for the power constraint such that no rate pairs other than $(0, 0)$ are achievable for $N$ as described in Remark 2 under jointly random modulated coding, while for the same values of the power constraint $\Lambda$ there are entanglement-modulated codes achieving strictly more rate pairs. To demonstrate this result we first define the modulation scheme that we use:

**Definition 12** (EPR Modulation). Einstein-Podolsky-Rosen (EPR) modulation is used when the state $\Psi = |\psi\rangle \langle \psi|$ for $\psi = \frac{\sqrt{2}}{\sqrt{2}} (e_0 \otimes e_1 + e_1 \otimes e_0)$ and the measurements are defined by the unitary matrices

$$
U_\theta :=
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
$$

choosing angles

$$
\begin{align*}
\theta_0 &= 0, \quad \theta_1 := \pi/4, \quad \tau_0 := \pi/8, \quad \tau_1 := -\pi/8
\end{align*}
$$

and then setting

$$
\begin{align*}
M_{A,x,i} := U_{\theta_x} |e_i\rangle \langle e_i| U_{\theta_y}^{-1} \\
M_{B,y,j} := U_{\tau_y} |e_i\rangle \langle e_i| U_{\tau_x}^{-1}
\end{align*}
$$

for all $i, j, x, y \in \{0, 1\}$, where $\{e_0, e_1\}$ is the standard basis of $\mathbb{C}^2$.
IV. RESULTS AND PROOF OF MAIN RESULT

In the sequel, we will jump between three specific instances of the three channel models introduced earlier. We use $I(s) = s$ as cost function on $\{0, 1\}$ and $\Lambda \in [0, 1]$ as power constraint.

The first model is the MASMACE $\mathcal{N}$ from Remark 2.

The second model is the AVMACM $\mathcal{L} = (J_{xy} \circ I)^{1}_{x,y=0}$ with environmental states $(x, y)$ distributed according to $\pi \otimes \pi$.

The third is the AVCEI $\mathcal{J} = (J_s \circ J_{y_j})^{1}_{s,y=0}$ with environmental states $y$ distributed according to some distribution $\omega$.

Our main result is the following:

**Theorem 3.** The following are true:

1. $\mathcal{R}_d(\mathcal{N}, \Lambda) = \mathcal{R}_r(\mathcal{N}, \Lambda) = \{(0, 0)\}$ for $\Lambda \geq 1/4$
2. $\Lambda < 1/4 \Rightarrow \mathcal{R}_d(\mathcal{N}, \Lambda), \mathcal{R}_r(\mathcal{N}, \Lambda) \neq \{(0, 0)\}$
3. $\mathcal{R}_s(\mathcal{N}, \Lambda) \neq \{(0, 0)\}$ if $\Lambda < \frac{2}{7}$

The rate regions $\mathcal{R}$ are defined in Definition 7.

**Remark 4.** It holds $\frac{1}{4} - \frac{1}{2\sqrt{2}} \approx 0.10$.

To prove validity of this theorem, we need a couple of lemmas. The first lemma reduces the channel that Alice and Bob can create to one which is a simple concatenation of the basic interference channel $I$ with a binary symmetric channel.

**Lemma 5** (Restrictions from Classical Strategies). Let $(X, Y)$ be uniformly distributed. For every choice of $A, B \in C(\{0, 1\}, \{0, 1\})$ the MAC $\mathcal{L} = C(\{0, 1\}, \{0, 1\})$ defined by

\[ L(c(a, b)) := \sum_{x,y} \sum_{i,j} \mathbb{A}(\psi(x, y)) J_{x,y}^{i,j}(c(a \oplus b)) \]  

has the following properties: First,

\[ L \in \text{conv} \{BSC(\nu) \circ I : \frac{1}{4} \leq \nu \leq \frac{3}{4}\} \]  

Second, for every fixed choice of $b \in \{0, 1\}$ there is a $\nu \in [\frac{1}{4}, \frac{3}{4}]$ such that $BSC(\nu)$ equals to

\[ \delta_b \rightarrow L(A|b) \]  

Third, for every $E$ and $p \in \mathcal{P}(E)$ the MAC defined via

\[ L_p := \sum_{x,y} \sum_{i,j} \mathbb{A}(\psi(x, y)) J_{x,y}^{i,j} \circ E \]  

satisfies

\[ L_p \in \text{conv} \{BSC(\nu) \circ E : \frac{1}{4} \leq \nu \leq \frac{3}{4}\} \]  

**Lemma 6** suggests the jammer, given his knowledge about the states $x \cdot y \oplus i \oplus j$ of the BSC, could simply realize a number of bit flips such that the total number of bit flips in any transmission of $n$ (signal) bits over the channel, is roughly equal to $n/2$ and all bit flips are placed at random.

That such a strategy is indeed sufficient to prohibit any communication over the AVMACM is the result of the following lemma:

**Lemma 6** (Impact of jammer state knowledge on AVMACM). Let $\Lambda \geq 1/4$. The rate region $\mathcal{R}_s(\mathcal{L}, \Lambda)$ of $\mathcal{L}$ with jointly randomized modulation equals $\{(0, 0)\}$.

**Lemma 9** (An achievable rate region). The region consisting of all $(R_A, R_B)$ such that

\[ R_A \leq C^I(\mathcal{J}, \frac{1}{4}) \]  

\[ R_B \leq C^I(\mathcal{J}, \frac{1}{4}) \]  

\[ R_A + R_B \leq C^I(\mathcal{J}, \frac{1}{4}) \]  

is contained in $\mathcal{R}_d(\mathcal{N}, \Lambda)$, and $\mathcal{R}_d(\mathcal{N}, \Lambda)$ is strictly larger than $\{(0, 0)\}$ whenever $\Lambda < 1/4$.

**Remark 8.** We will use the core idea in the proof of Theorem 7 to prove Theorem 8. If James knows the output of $p$ as well as those of the modulators, then he will effectively see an AVCEI $(1, \psi)$ acting on the joint input $A \oplus B$ of Alice and Bob.

Equipped with the results outlined in Theorem 7 we can prove achievability of a nontrivial rate region for the MASMACE without any correlation between the senders by simply reducing e.g. the action of Bob to continuously transmitting just the symbol 0. In this case, the effective channel for Alice becomes a BSC and Theorem 7 gives the capacity $C$ of this effective BSC and proves achievability of the rate pair $(C, 0)$. Switching the roles of Alice and Bob proves achievability of the rate pair $(0, C)$. Application of time sharing then yields the following achievable rate region:

**Lemma 9** (An achievable rate region). The region consisting of all $(R_A, R_B)$ such that

\[ R_A \leq C^I(\mathcal{J}, \frac{1}{4}) \]  

\[ R_B \leq C^I(\mathcal{J}, \frac{1}{4}) \]  

\[ R_A + R_B \leq C^I(\mathcal{J}, \frac{1}{4}) \]  

is contained in $\mathcal{R}_d(\mathcal{N}, \Lambda)$, and $\mathcal{R}_d(\mathcal{N}, \Lambda)$ is strictly larger than $\{(0, 0)\}$ whenever $\Lambda < 1/4$.

**Lemma 10** (EPR Measurements). Let $\psi = \frac{1}{2\sqrt{2}}(e_0 \otimes e_0 + e_1 \otimes e_1) \in C^2$ and $\Psi = |\psi\rangle \langle \psi|$. Let $M_A, M_B$ be the measurements from Definition 7. Define

\[ q(\alpha, \beta|x, y) := \text{tr}(M_{A|x, \alpha} \otimes M_{B|y, \beta} \Psi) \]  

(33)

(for all $\alpha, \beta, x, y \in \{0, 1\}$). Then it holds

\[ q(|0, 0) = (\frac{1}{4} + t, \frac{1}{4} - t, \frac{1}{4} - t, \frac{1}{4} + t) \]  

\[ q(|1, 0) = q(|0, 0) \]  

\[ q(|0, 1) = q(|0, 0) \]  

\[ q(|1, 1) = (\frac{1}{4} - t, \frac{1}{4} + t, \frac{1}{4} + t, \frac{1}{4} - t) \]  

where outputs are indexed lexicographically and $t = \frac{1}{\sqrt{2}}$. 

**Lemma 6** together with Lemma 5 therefore implies statement 1) of Theorem 3. To prove statements 2) and 3) in Theorem 5 we will make use of the following result:

**Theorem 7** (Jammer state knowledge and power constraints).

\[ C^I(\mathcal{J}, \frac{1}{4}) = 0 \]  

\[ C^I(\mathcal{J}, \frac{1}{4}) = 1 - h(\frac{1}{4} + \frac{1}{4}) > 0. \]  

(29)
These measurements can be used to realize what we would like to call ‘EPR modulated encoding’. The impact of this scheme on the communication system is described in the following Lemma which describes how the use of EPR modulated encoding increases the capacity region of our MAC.

**Lemma 11** (EPR modulated encoding [6]). For every $\Lambda \in [0, 1]$ the rate region $R_v$ consisting of all $(R_A, R_B)$ such that

\[
\begin{align*}
R_A &\leq C_c(A, J, \frac{2 - \sqrt{2}}{2}) \\
R_B &\leq C_c(A, J, \frac{2 + \sqrt{2}}{2}) \\
R_A + R_B &\leq C_c(A, J, \frac{2 - \sqrt{2}}{2})
\end{align*}
\]

is a subset of $R_e(N, \Lambda)$.

**Remark 12.** It is straightforward to verify that the rate region becomes even larger by using a nonlocal correlation taking the form as in Lemma [7] but with $t = 1/4$ instead of $1/2$ - as defined in [18, Equation (7)]. A look at equation (124) in the proof of Lemma [7] verifies that for $t = 1/4$ Alice and Bob will effectively transmit over the channel

\[
\text{BSC}(0) \circ I = I.
\]

Such a scheme is thus able to cancel any noise coming from the environment and transform the channel into the binary interference channel $I$. The achievable rate region then contains the set as described in Lemma [7] but with $c = 0$.

**V. Proofs**

**Proof of Lemma 3** Let $A, B \in C(\{0, 1\}, \{0, 1\})$. First, we consider extreme channels only. Under this restriction our potential choices for $A$ are limited to the set $E := \{1, \text{BSC}, 0, 1\}$. The transition probabilities from an input $(a, b)$ by Alice and Bob to an output $c$ for Charlie read as

\[
\mathbb{P}(c|a, b) = \sum_{x, y} \delta(c, x \cdot y \oplus Ax \oplus By \oplus a \oplus b).
\]

The notion $Ax$ is an abbreviation for the symbol $t$ such that $A(t|x) = 1$, likewise $By$ is an abbreviation for the symbol $t$ such that $B(t|y) = 1$.

Let us first consider a fixed choice of $a$ and $b$ such that $a \oplus b = 0$, and choose $c = 0$ as well. We need to understand the distributions of the random variables $f_{AB}(X, Y)$ where

\[
f_{AB}(x, y) := x \cdot y \oplus Ax \oplus By.
\]

If $(X, Y)$ are uniformly distributed, the distribution of $f_{AB}(X, Y)$ is contained in $\text{conv}\{r, s\}$ for $r = \left(\frac{1}{4}, \frac{3}{4}\right)$ and $s = \left(\frac{1}{4}, \frac{1}{4}\right)$ - independent from the particular choice of $A, B \in E$.

To see this, we use the symmetry of $\pi \otimes \pi$ under exchange of $x$ with $y$ implying the statement needs only be proven for all 4 choices $AB$ of the form $f_{AA}$ plus half of the remaining choices $AB$ (since the results for e.g. $01$ equals that for $00\)$. It is also evident that the distribution of $f_{11}$ equals that of $f_{BB}$, and likewise for $f_{12}$. Since $\delta(a, b \oplus 1) = \delta(a \oplus 1, b)$ and $\mathbb{P} = y \oplus 1$ we can easily confirm that the result holds for all choices $AB \in \{1 \text{F}, 1 \text{F}, 01, 0 \text{F}\}$ (since $r = \text{F}$) if it holds for only one of them. The same reasoning applies to the set $\{10, 00, 11\}$ and $\{11, 10, 00\}$. We thus set out to prove our result for the choices $\{11, 10, 00\}$:

\[
\begin{align*}
\mathbb{P}(f_{11}(X, Y) = 0) &= \frac{1}{4} \sum_{x, y} \delta(0, xy \oplus x \oplus y) \\
&= \frac{1}{4}(\delta(0, 0) + \delta(0, 1) + \delta(0, 1) + \delta(0, 1 \oplus 1 \oplus 1)) \\
&= \frac{1}{4}.
\end{align*}
\]

\[
\begin{align*}
\mathbb{P}(f_{10}(X, Y) = 0) &= \frac{1}{4} \sum_{x, y} \delta(0, xy) \\
&= \frac{1}{4} \delta(0, 0) + \delta(0, 0) + \delta(0, 0) + \delta(0, 1) \\
&= \frac{3}{4}.
\end{align*}
\]

\[
\begin{align*}
\mathbb{P}(f_{00}(X, Y) = 0) &= \frac{1}{4} \sum_{x, y} \delta(0, xy) \\
&= \frac{1}{4} \delta(0, 0) + \delta(0, 0) + \delta(0, 0) + \delta(0, 1) \\
&= \frac{1}{4}.
\end{align*}
\]

This reasoning together with our calculation demonstrates that

\[
\delta_x \mapsto \sum_{x, y} \frac{1}{4} \delta(\cdot, x \cdot y \oplus Ax \oplus By \oplus z)
\]

equals $\text{BSC}(\nu)$ for some $1/4 \leq \nu \leq 3/4$, depending on the choice of $A, B \in E$. Since the joint input $z$ by Alice and Bob is realized as $z = I(a, b) = a \oplus b$, the claim is proven for extremal strategies $A$ and $B$. Thus the convex set $X$ of all possible effective channels that Alice and Bob can generate satisfies

\[
X \subset \text{conv}\{\text{BSC}(\frac{1}{4}) \circ I, \text{BSC}(\frac{1}{4}) \circ I\}.
\]

That in fact $X = \text{conv}\{\text{BSC}(\frac{1}{4}) \circ I, \text{BSC}(\frac{1}{4}) \circ I\}$ can be seen by choosing the input 0 for Bob. In that case, Alice’s choice 0 generates the effective channel $\text{BSC}(3/4)$ and if Alice chooses 1 she generates the channel $\text{BSC}(1/4)$. This proves the proposed first claim of Lemma 3. The second claim is a comparably simple consequence of the first. The third claim follows by noting that $X$ is a convex set, and any random choice, including jointly random choices as the ones defined in Definition 3 of elements taken from $X$, will again produce an element of $X$. Thus the third claim is proven.

For the proofs of Lemma 14 and Theorem 7 we define an explicit jamming strategy for James in Definition 14 (see the Appendix) and prove that it produces a certain distribution of identities and bit flips in the communication between the sending parties and Charlie. An identity channel is, in this picture, identified with the letter 0 and the bit flip with the letter 1. This is motivated by the identities $1(\delta_x) = \delta_{x \oplus 0}$ and $\mathbb{P}(\delta_x) = \delta_{x \oplus 1}$ that hold for every $x \in \{0, 1\}$.

The idea of the jamming strategy is as follows: Under i.i.d. noise generated by the environment, the ‘original’ channel states $\alpha^x$ will most likely have a certain minimal number $t_1$
and a certain maximal number $t_2$ of ones. We will assume for simplicity but without loss of generality that $\frac{t_2-t_1}{4t} \in [0, \frac{1}{2}]$. James will receive the products $(x_1y_1, \ldots, x_ny_n)$ of the environmental states. He will then generate a bit string of his own, which is equal to zero wherever $x_iy_i = 1$, and has a random pattern of zeroes and ones at those places where the original string is equal to zero. The task for him is to make an adequate selection of his random pattern such that the resulting sum of the original string and his string does on average look as if the convex combination of two channels $\nu \otimes \pi$ defined via

$$I(\nu \otimes \pi) := I(\nu BSC) + I(\pi BSC)$$

(55)

where $J_0 := 1$ and $J_1 := E \otimes \nu$ (compare Section III). The effective channel is generated by the inputs $(x, y)$ of the environment and the inputs $(\alpha, \beta)$ by Alice and Bob. For every two message bits $(a_i, b_i)$ that Alice and Bob transmit, the channel $I$ converts them to a bit $a_i \oplus b_i$ which then serves as an input to the channel $J_{s'}$ where

$$s' = x \cdot y \oplus \alpha \oplus \beta.$$  

(56)

The bits $x$ and $y$ are chosen at random by the environment, according to the distribution $\pi \otimes \pi$. The probability that when $x$ is detected by Alice and $y$ is detected by Bob an input $\alpha$ is chosen by Alice and an input $\beta$ is chosen by Bob is given by

$$q(\alpha, \beta|x, y) := \sum_{e \in E} p(e)A(\alpha|e, x)B(\beta|\epsilon, y),$$  

(57)

where $E$ is some finite alphabet and $p \in \mathcal{P}(E)$ (compare Definition 3).

According to Definition 9 the jammer knows $s'$ as defined in (56). Thus no matter which strategy $q$ of the form (57) Alice and Bob choose, we can deduce from Lemma 5 that there will be a resulting $\nu \in [\frac{1}{4}, \frac{3}{4}]$ such that for the purpose of analyzing the capacity region of the resulting channel, the AVCEI

$$\mathcal{J} = \{J_s \circ J_y \circ I\}_{s, y=0}$$  

(58)

is the correct model. In order to show that for all choices $\nu \in [\frac{1}{4}, \frac{3}{4}]$ the capacity region of $(\mathcal{J}, \nu)$ equals $\{(0, 0)\}$ we can apply the jammer’s strategy as defined in Definition 13. Without loss of generality, we may and will assume that $\nu \in [0, \frac{1}{2}]$. For the strategy from Definition 13 Lemma 13 and Lemma 15 show that for every $\epsilon > 0$ it yields an effective sequence $(\nu'_n)_{n \in \mathbb{N}}$ of distributions over the channel states $s \oplus y$ that satisfies $\lim_{n \to \infty} \nu'_n(1) = \pi_{\epsilon}(1)$ for $\pi_{\epsilon} \in \mathcal{P}(\{0, 1\})$ defined via $\pi_{\epsilon}(1) := \frac{1}{2} - \epsilon$ and

$$p'_n \leq \frac{1}{1 - \delta} \pi_{\epsilon}$$  

(59)

for $\delta = 2^{-nc(\epsilon)}$ where $c(\epsilon) > 0$ is some suitable constant. Thus an application of this strategy ensures that for every input strings $a^n, b^n$ and output string $c^n$ for Charlie we have

$$\sum_{\alpha^n, s^n, \beta^n} \nu \otimes n(a^n)J_n(\delta_{\alpha^n} \otimes \delta_{\beta^n})(c^n)S(s^n|\alpha^n)$$  

$$= \sum_{\alpha^n, s^n} \hat{p}_n(\alpha^n)J_n(\delta_{\alpha^n} \otimes \delta_{\beta^n})(c^n)S(s^n|\alpha^n)$$  

(60)

$$\leq \frac{1}{1 - \delta} \sum_{\alpha^n} \pi_{\epsilon} \otimes n(s^n)J_n(\delta_{\alpha^n} \otimes \delta_{\beta^n})(c^n)$$  

$$= \frac{1}{1 - \delta} BSC(\frac{1}{2} - \epsilon)(\delta_{\alpha^n} \otimes \beta^n)(c^n)S(s^n|\alpha^n).$$  

(61)

For every $\epsilon > 0$, $\delta$ goes to zero when $n$ goes to infinity. Thus $\mathcal{J}$ is effectively turned into the channel $BSC(1/2 - \epsilon) \circ I$. For $BSC's$ it holds - for all $\rho, \sigma \in [0, 1] -

$$BSC(\rho) \circ BSC(\sigma) = BSC(\rho \sigma \circ \rho' \sigma') \circ I$$  

(64)

$$= I \circ BSC(\rho) \circ BSC(\sigma).$$  

(65)

With the choice $\rho = \sigma = \frac{1}{2} - \sqrt{\epsilon/2}$ and by the data processing inequality (c.f. Lemma 3.1 in [8]) and the capacity formula for the MAC it thus follows for every pair of achievable rates $(R_A, R_B)$ and every $\epsilon > 0$ that

$$R_A + R_B \leq \max_{p_{1}, p_{2}} I(p_1 \otimes p_2; BSC(\rho) \circ I)$$  

$$\leq \max_{p_{1}, p_{2}} I(p_1 \otimes p_2; BSC(\rho) \circ BSC(\sigma) \circ I)$$  

(66)

$$= \max_{p_{1}, p_{2}} I(p_1 \otimes p_2; I \circ BSC(\rho) \circ BSC(\sigma))$$  

$$\leq 2 - 2h(\frac{1}{2} - \sqrt{\epsilon/2}).$$  

(67)

For values $\nu \in [\frac{1}{4}, \frac{3}{4}]$ the strategy from Definition 12 applies as well, the only modification is that James randomly selects his input on those channel states $y_i$ that are equal to one. Thus the rate region of $(\mathcal{J}, \nu)$ consists of the single element $\{(0, 0)\}$, as claimed.

**Proof of Theorem 7.** Our goal is to give a lower bound on the capacity $C_{\text{AV}}(\mathcal{J}, \omega)$ by using the results of [9] and let that lower bound match an upper bound derived by explicitly quantifying the impact of one particular, valid jamming strategy (compare Definition 13 in the Appendix).

To recapitulate the preliminaries, the power constraint $\Lambda \in [0, 1]$ and the BSC parameter $\omega \in [0, 1]$ are given from the statement of the Theorem. Without loss of generality we may assume that $\omega \neq \frac{1}{2}$. We set $\omega' := 1 - \omega$ and $\Lambda' := 1 - \Lambda$.

According to 2 the calculation of the capacity of an AVC with a power constraint on the jammer requires the definition of a function $p \rightarrow \Lambda(p)$ (compare equation (2.13) in [9]). This definition requires us to first fix a function $l$ and then minimize over the entire set of symmetrizers (compare Definition 9 of the AVC. We will study this function for the special class of binary AVCs that take the form

$$L_{\nu} = (BSC(\nu), BSC(1 - \nu))$$  

(70)

for some $\nu \in [0, 1]$. The question arises, whether we can explicitly write down the set of all symmetrizers for such
AVCs. It is clear that for each $\theta \in [0, 1]$ the map $q(s|x) := BSC(\theta)(s|x)$ is a symmetrizer:

$$
\sum_{s=0}^{1} q(s|0)L_{\nu,s}(\delta_1) = q(0|0)\left(\nu'\nu\right) + q(1|0)\left(\nu'\nu\right) = q(1|1)L_{\nu,1}(\delta_0) + q(0|1)L_{\nu,0}(\delta_0).
$$

Moreover, since

$$
\left\{ \nu'\nu, \nu'\nu \right\}
$$

is a linearly independent set whenever $\nu \neq 1/2$, we know that $\left\{ BSC(\theta) : \theta \in [0, 1] \right\}$ is the complete set of symmetrizers, for every AVC $L_{\nu}$, if $\nu \neq 1/2$.

We can now proceed and calculate the function $\Lambda_0 : \mathcal{P}(\{0, 1\}) \to \mathbb{R}$ for every AVC $L_{\nu}$. The specific form of our cost function $l$ which is defined via $l(s) = s$ leads to a function $\Lambda_0$ as follows:

$$
\Lambda_0(p) := \min_{0 \leq \theta \leq 1} \sum_{x=0}^{1} p(x) BSC(\theta)(s|x)l(s)
$$

$$
= \min_{0 \leq \theta \leq 1} \sum_{x=0}^{1} p(x) BSC(\theta)(1|x)
$$

$$
= \min_{0 \leq \theta \leq 1} p(0)\theta + p(1)(1 - \theta)
$$

$$
= \min\{p(0), p(1)\}.
$$

We define the following sets so that capacity formulas (c.f. [9] Theorem 3) can be written more efficiently:

$$
P^s_{\Lambda} := \left\{ q \in \mathcal{P}(\{0, 1\}) : q(1) \leq \Lambda \right\} \quad (80)
$$

$$
P^x_{\Lambda} := \left\{ p : \Lambda_0(p) > \Lambda \right\}. \quad (81)
$$

By part 1) of Theorem 3 in [9] and using the symbol $C^i_{\Lambda}(L_{\nu})$ to denote what is the equivalent to $C(1,\Lambda)$ there (if $g(x) = 1$ for all $x \in \{0, 1\}$ with $g$ being the function as used in [9] and $L_{\nu}$ as defined in [70] being the AVC) we get

$$
\max\min\{p(0), p(1)\} < \Lambda \implies C^i_{\Lambda}(L_{\nu}) = 0. \quad (82)
$$

From 2) of the same theorem we get

$$
\max\min\{p(0), p(1)\} > \Lambda \implies C^i_{\Lambda}(L_{\nu}) = \max_{p \in P^x_{\Lambda}} \min_{q \in P^s_{\Lambda}} I(p; \sum_{i=0}^{1} q(i) BSC(\nu(i))). \quad (83)
$$

Let $\Lambda < 1/2$ so that $1/2 \in P^x_{\Lambda}$. Observe that for every choice $q \in \mathcal{P}(\{0, 1\})$ the channel $\sum_{i=0}^{1} q(i) BSC(\nu(i))$ (where $\nu_0 := 1 - \nu$ and $\nu_1 := 1 - \nu$) is a BSC. Further, $P^s_{\Lambda}$ and $P^x_{\Lambda}$ are both convex sets. We can therefore calculate (83) even more explicitly:

$$
\max_{p \in P^x_{\Lambda}} \min_{q \in P^s_{\Lambda}} I(p; \sum_{i=0}^{1} q(i) BSC(\nu(i))) = \min_{q \in P^s_{\Lambda}} \max_{p \in P^x_{\Lambda}} I(p; BSC(\sum_{i} q(i) \nu(i))) = \min_{q \in P^s_{\Lambda}} I(\tau; BSC(\sum_{i} q(i) \nu(i))) = \min_{q \in P^s_{\Lambda}} (1 - h(\sum_{i} q(i) \nu(i))) = 1 - \max_{\tau \leq \Lambda} h(\tau\nu_1 + \tau'\nu_0). \quad (88)
$$

We conclude

$$
\max\min\{p(0), p(1)\} > \Lambda \implies C^i_{\Lambda}(L_{\nu}) = 1 - \max_{\tau \leq \Lambda} h(\tau). \quad (89)
$$

If $\Lambda \geq 1/2$ then by application of the jamming strategy in Definition [13] it follows $C^i_{\Lambda}(L_{\nu}) = 0$, and thus

$$
C^i_{\Lambda}(L_{\nu}) = 1 - \max_{\tau \leq \Lambda} h(\tau). \quad (90)
$$

It also becomes evident that the map $\Lambda \to C^i_{\Lambda}(L_{\nu})$ is continuous.

To recapitulate, we are now able to explicitly calculate the capacity $C^i_{\Lambda}(L_{\nu})$ for every $\Lambda \in [0, 1]$. What is missing is the relation between $C^i_{\Lambda+\delta}(L_{\nu})$ and $C^i_{\Lambda}(L_{\nu}, \omega)$. We proceed regarding this second problem. For the binary AVC $L_{\nu} = (1, 1, F)$, with state information at the jammer, the relation

$$
C^i_{\Lambda}(L_{\nu}, \omega) \geq C^i_{\Lambda+\delta}(L_{\nu}) \quad (91)
$$

holds for every $\Lambda_1, \Lambda_2, \delta > 0$ satisfying $\Lambda_1 + \Lambda_2 + \delta \leq 1$ by Lemma [13] (remember the right hand side of the inequality refers to the capacity as defined in [9]).

The inequality (91) is, roughly speaking, due to the fact that (for every $n \in \mathbb{N}$) James can always use a strategy where he first samples a string $s^n$ from the distribution $\omega$ and then, given $s^n$, decide how to use the remaining roughly $n\Lambda$ bit flips that he is allowed to use in order to prevent Alice from transmitting her messages.

From our previous discussion we know that this implies

$$
C^i_{\Lambda}(L_{\nu}, \omega) \geq 1 - \max_{\tau \leq \Lambda_1 + \Lambda_2 + \delta} h(\tau). \quad (92)
$$

Therefore by continuity of the right hand side of (92)

$$
C^i_{\Lambda}(L_{\nu}, \omega) \geq 1 - \max_{\tau \leq \Lambda_1 + \Lambda_2} h(\tau), \quad (93)
$$

so that we ultimately end up with the statement $\forall \Lambda_1, \Lambda_2 \geq 0 : \Lambda_1 + \Lambda_2 \geq 1:\n
$$
C^i_{\Lambda}(L_{\nu}, \omega) = C^i_{\Lambda}(L_{\nu}, \omega) \geq C^i_{\Lambda+\delta}(L_{\nu}). \quad (94)
$$

To show that indeed $C^i_{\Lambda}(L_{1}, \omega) = C^i_{\Lambda+\omega}(L_{1})$ we use the strategy defined in [13] in a manner similar to our application in equations (38) to (69), but with

$$
\tau(\omega, \Lambda) := \min\left(\frac{1}{2}, \omega + \Lambda\right) \quad (95)
$$
and \( p_n \to \tau(\omega, \Lambda) - \varepsilon \) instead of \( p_n \to \frac{1}{2} - \varepsilon \). Then, the effective channel for the communication from Alice to Bob is upper bounded by

\[
\frac{1}{1-\delta} BSC(\tau(\omega, \Lambda) - \varepsilon_n)^{\otimes n} \quad (96)
\]

for some sequence \((\varepsilon_n)_{n \in \mathbb{N}}\) of positive numbers converging to \(\varepsilon\). It follows that for every \(\varepsilon > 0\), a code \((C_n)_{n \in \mathbb{N}}\) for reliable message transmission can have rate \(R\) being no larger than \(1 - h(\tau(\omega, \Lambda) - \varepsilon)\). Thus

\[
\forall \omega \in [0, \frac{1}{2}) : \quad C_A^t(L, \omega) \leq 1 - \max_{|\tau| \leq \Lambda} h(\omega + \tau). \quad (97)
\]

This includes the special case where \(\omega + \Lambda > 1\) and \(C_A^t(L, \omega) = 0\). For the particular choice \(\omega = \frac{3}{4}\) and \(\Lambda = \frac{1}{4}\) we have \(\omega + \Lambda > 1\)

\[
C_A^t(L, \frac{3}{4}) = 0. \quad (98)
\]

If \(\omega = \frac{2 - \sqrt{\sqrt{2}}}{4}\) and \(\Lambda = \frac{1}{4}\) then \(\omega + \Lambda = \frac{3 - \sqrt{2}}{4}\) and

\[
\frac{3 - \sqrt{2}}{4} < \frac{3 - 1}{4} = \frac{1}{2} \quad (99)
\]

so that we get

\[
C_A^t(L, \frac{2 - \sqrt{\sqrt{2}}}{4}) = 1 - h(\frac{3 - \sqrt{2}}{4}) > 0. \quad (100)
\]

The situation \(\omega \in (\frac{1}{4}, 1]\) is dealt with by letting James apply his random disturbances to those indices \(i\) where \(x_i y_i = 1\) instead of those where \(x_i y_i = 0\). Thus

\[
\forall \omega \in [0, 1] : \quad C_A^t(L, \omega) \leq 1 - \max_{|\tau| \leq \Lambda} h(\omega + \tau). \quad (102)
\]

Proof of Lemma 70 Let \(d \in \mathbb{N}\) and \(\psi \in \mathbb{C}^d \otimes \mathbb{C}^d\) be given by

\[
\psi = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} e_i \otimes e_i.
\]

Let \(U : \mathbb{C}^d \to \mathbb{C}^d\) be any matrices. It is known that

\[(U \otimes \mathbb{I}) \psi = (\mathbb{I} \otimes U^\dagger) \psi.
\]

If \(U_\theta, U_r : \mathbb{C}^d \to \mathbb{C}^d\) are unitary matrices with only real entries, then even

\[(U_\theta \otimes U_r) \psi = (U_\theta \cdot U_r^{-1} \otimes \mathbb{I}) \psi.
\]

Let

\[
P_i = |e_i\rangle \langle e_i|, \quad P_j = |e_j\rangle \langle e_j|
\]

be the projections onto the computational basis. Then

\[
\langle (U_\theta \cdot U_r) \psi, (P_i \otimes P_j) (U_\theta \otimes U_r) \psi \rangle = \langle (U_\theta \cdot U_r^{-1}) \otimes \mathbb{I} \rangle \psi, (P_i \otimes P_j) (U_\theta \cdot U_r^{-1} \otimes \mathbb{I}) \psi \rangle \quad (103)
\]

\[
= \langle (U_\theta \cdot U_r^{-1}) \otimes \mathbb{I} \rangle e_j \otimes e_j, (P_i \otimes \mathbb{I}) (U_\theta \cdot U_r^{-1} \otimes \mathbb{I}) e_j \otimes e_j \rangle \quad (104)
\]

\[
= \langle U_\theta \cdot U_r^{-1} \rangle e_j, P_i \cdot U_\theta \cdot U_r^{-1} e_j \rangle \quad (105)
\]

\[
= \langle P_i \cdot U_\theta \cdot U_r^{-1} \otimes e_j, P_i \cdot U_\theta \cdot U_r^{-1} \rangle e_j \rangle \quad (106)
\]

\[
= \langle U_\theta \cdot U_r^{-1} \rangle_{i,j}^2 \quad (107)
\]

We now define the desired measurements as follows. Alice chooses an angle \(\theta \in \{\theta_0, \theta_1\}\), depending on whether her bit \(x\) equals 0 or 1. Bob chooses an angle \(\tau \in \{\tau_0, \tau_1\}\) depending on his input. Their measurements are then defined by

\[
M_{A,x,i} := U_{\theta_0} P_i U_{\theta_0}^{-1} \quad (108)
\]

\[
M_{B,y,j} := U_{\tau_0} P_j U_{\tau_0}^{-1} \quad (109)
\]

for all \(i, j, x, y \in \{0, 1\}\). Thus following equations (103) to (107) their probability of measuring \((i, j)\) upon input \((x, y)\) is

\[
\mathbb{P}(i, j | x, y) = \langle (U_{\theta_0} \otimes U_{\tau_0}) \psi, (P_i \otimes P_j) (U_{\theta_0} \otimes U_{\tau_0}) \psi \rangle = \langle U_{\theta_0} \cdot U_{\tau_0}^{-1} \rangle_{i,j}^2. \quad (110)
\]

We now specify \(U_\theta\) further by setting

\[
\theta_0 := 0, \quad \theta_1 := \pi/4 \quad (112)
\]

\[
\tau_0 := \pi/8, \quad \tau_1 := -\pi/8. \quad (113)
\]

Then the measurement probabilities evaluate to

\[
\mathbb{P}(|00, 11\rangle | 0, 0) = \frac{1}{2} |(U(\frac{\pi}{8} \cdot 0, 0)|^2 + |U(\frac{\pi}{8}, 1, 1)|^2 \rangle \quad (114)
\]

\[
= \cos^2(\frac{\pi}{8}) \quad (115)
\]

\[
\mathbb{P}(|00, 11\rangle | 0, 1) = \frac{1}{2} |(U(\frac{\pi}{8} \cdot 0, 0)|^2 + |U(\frac{\pi}{8}, 1, 1)|^2 \rangle \quad (116)
\]

\[
= \cos^2(\frac{\pi}{8}) \quad (117)
\]

\[
\mathbb{P}(|00, 11\rangle | 1, 0) = \mathbb{P}(|00, 11\rangle | 0, 1) \quad (118)
\]

\[
\mathbb{P}(|01, 10\rangle | 1, 1) = \frac{1}{2} |(U(\frac{3\pi}{8} \cdot 0, 0)|^2 + |U(\frac{3\pi}{8}, 1, 1)|^2 \rangle \quad (119)
\]

\[
= \sin^2(\frac{3\pi}{8}) \quad (120)
\]

It holds \(\cos^2(\frac{\pi}{8}) = (1 + \frac{\sqrt{2}}{2})/2\) and since \(\sin(3\pi/8)^2 = \cos^2(\pi/8)\) we get the desired result. ■

Proof of Lemma 77 Let \(q\) have the form as stated in the Lemma. If Alice and Bob use EPR modulated encoding the probability that Charlie receives \(c\) upon Alice sending \(a\) and Bob sending \(b\) is - if James is not considered for a moment - given by

\[
\mathbb{P}(c | a, b) = \sum_{x, y, \alpha, \beta} \sum_{x, y} q(\alpha, \beta | x, y) \delta(c, x y + \alpha \oplus \beta \oplus a + b) \quad (121)
\]

\[
= 4(2^{l^{\frac{1}{2}} + 4l^{\frac{1}{4}}}) \delta(c, a \oplus b) + 2^{l^{\frac{1}{2}} + 4l^{\frac{1}{4}}}) \delta(c, 1 \oplus a + b) \quad (122)
\]

\[
= (l^{\frac{1}{4}} + l^{\frac{1}{2}}) \cdot I \quad (123)
\]

\[
BSC(\frac{1}{4^{l^{\frac{1}{4}}}}) \circ I \quad (124)
\]

For the specific value \(t = \frac{1}{4^{l^{\frac{1}{2}}}}\) we arrive at an effective MAC

\[
BSC((1 - \frac{1}{\sqrt{2}})/2) \circ I = BSC(\frac{1}{\sqrt{2}}) \circ I. \quad (125)
\]
To specify the entire code we let Bob transmit zeroes only, such that \((J, \frac{2-s}{2})\) models the transmission for Alice. Theorem 7 then shows that pairs of the form \((R_A, 0)\) with \(R_A \leq C_{\Lambda_A}^0(J, \frac{2-s}{2})\) are achievable. Switching the roles of Alice and Bob and using time sharing proves the remaining two inequalities.

VI. APPENDIX

**Lemma 13** (Connecting Capacities). For every \(\Lambda \in [0, 1]\) and \(\Lambda_1, \Lambda_2 \geq 0, \delta > 0\) such that \(\Lambda_1 + \Lambda_2 + \delta = \Lambda\) it holds

\[
C_{\Lambda}^0((1, \mathbb{F}), \Lambda_2) \geq C_{\Lambda_1 + \Lambda_2 + \delta}^0((1, \mathbb{F}))
\]

**Proof:** Let \(R \geq 0\) and \((C_n)_{n \in \mathbb{N}}\) be any sequence of codes with rate \(R\) for \((1, \mathbb{F})\) under power constraint \(\Lambda\) and cost function \(l(i) = i\). Decompose \(\Lambda\) as \(\Lambda = \Lambda_1 + \Lambda_2 + \delta\) where \(\Lambda_1, \Lambda_2 \geq 0\) and \(\delta > 0\) can be arbitrarily small. Let the average error of the code given a jammer state \(s_n^k\) be

\[
\bar{e}(s^n) := \frac{\sum_m w_m h(D_m(s^n, s_n^k))}{M}.
\]

For every \(\varepsilon > 0\) there is \(N_1 \in \mathbb{N}\) such that for all \(n \geq N_1\)

\[
1 - \varepsilon \leq \min_{l(s^n) \leq \Lambda} \bar{e}(s^n).
\]

Let \(p \in \mathcal{P}(\{0, 1\})\) be defined by \(p(1) := \Lambda_2\). Then there is an \(N_2 \in \mathbb{N}\) such that

\[
p^\otimes n(T^n_{p, \delta}) \geq 1 - \delta.
\]

For each \(s_2^n \in T^n_{p, \delta}\) and \(s_1^n\) with the property \(l(s_1^n) \leq \Lambda_1\) it holds

\[
l(s_1^n \oplus s_2^n) \leq \Lambda_1 + \delta + \Lambda_2 \leq \Lambda.
\]

This implies (for \(\varepsilon' := 1 - \varepsilon\))

\[
\varepsilon' \leq \sum_{s_2^n \in T^n_{p, \delta}} p^\otimes n(s_1^n) \min_{l(s_2^n) \leq \Lambda_1} \bar{e}(s_1^n \oplus s_2^n)
\]

\[
\leq \sum_{s_1^n} p^\otimes n(s_1^n) \min_{l(s_2^n) \leq \Lambda_1} \bar{e}(s_1^n \oplus s_2^n) + p^\otimes n(T^n_{p, \delta}).
\]

This latter inequality in combination with inequality \([22]\) allows us to conclude that \(R\) is achievable for the AVCEI \(((1, \mathbb{F}), \Lambda_2)\) under power constraint \(\Lambda_1\). It follows

\[
C_{\Lambda_1}^0((1, \mathbb{F}), \Lambda_2) \geq C_{\Lambda_1 + \Lambda_2 + \delta}^0((1, \mathbb{F}))
\]

for every \(\delta > 0\).

Our strategy assumes James applies a certain level \(\Lambda\) of disturbance to any incoming binary sequence that has a ratio of zeroes and ones within a predefined range.

**Definition 13.** Let \(\Lambda \in [0, 1]\). Set \(\Lambda_n := \lfloor n\Lambda \rfloor\) and let \(t_1 \leq t_2\). \(K\) be natural numbers such that \(\Lambda_n - K - t_2 + t_1 \geq 0\) and \(t_1 + \Lambda_n \leq n\). For any \(k \in \{0, 1, \ldots, K\}\) we set

\[
\chi(K, t, k) := k - (t - t_1) + (\Lambda_n - K).
\]

Let \(t \in \mathbb{N}\) satisfy \(t_1 \leq t \leq t_2\). Let Given any \(\alpha^n \in T^n_t\) and an \(s^n \in T_{n-1}k\) for some \(k \in \{0, \ldots, K\}\) we define a new string \(S^n \in \{0, 1\}^n\) element-wise via

\[
S(s^n-t, \alpha^n) = \begin{cases} 0, & \text{if } \alpha_i = 1 \\ s_{N(\alpha^n)} \oplus \alpha_{N(\alpha^n)}, & \text{else}. \end{cases}
\]

Given a selection \(\lambda_1, \ldots, \lambda_K \geq 0\) of real numbers satisfying \(\sum_{k=0}^{K} \lambda_k = 1\) the jammer strategy for \(\alpha^n \in \bigcup_{t=t_1}^{t_2} T^n_t\) is defined as

\[
\hat{S}(|\alpha^n|) := \sum_{k=0}^{K} \sum_{s^n \in T^n_{\chi(K, t, k)}} \lambda_k \delta(s^n, \alpha^n). \tag{136}
\]

The complete strategy \(S\) is to apply \(\hat{S}\) whenever \(\alpha^n \in \bigcup_{t=t_1}^{t_2} T^n_t\) and apply \(S(-|\alpha^n|) := \delta_{(0, \ldots, 0)}\). else.

Observe that this strategy obeys the power constraint \(\Lambda\). In addition, the following Lemma holds:

**Lemma 14.** Let \(n, \Lambda, t_1, t_2, K\) and \(\lambda_1, \ldots, \lambda_K \geq 0\) be as in Definition 13. Then it holds for every \(t \in \mathbb{N}\) with the property \(t_1 \leq t \leq t_2\) that

\[
\sum_{k=0}^{K} \sum_{\alpha^n \in T^n_t} \sum_{s^n \in T^n_{\chi(K, t, k)}} \lambda_k \frac{\delta(s^n, \alpha^n)}{|T^n_{\chi(K, t, k)}|} \geq \sum_{k=0}^{K} \lambda_k \pi_t(K, 0, k), \tag{137}
\]

where \(\pi_t\) is defined in \([7]\).

**Proof of Lemma 14.** The idea behind James’ strategy is that for a number of potential choices of \(k \in \mathbb{N}\) he will attempt to add a number \(\hat{k}\) of ones at random positions \(i \in [n]\), but restrict himself to those where \(\alpha_i = 0\). Our interest is to quantify the distribution of the resulting sequence of identities and bit flip channels that Alice and Bob will need to transmit over. Such a sequence is in one-to-one correspondence with the corresponding state sequence \(s^n \oplus \alpha^n\).

Consider now \(k\) as fixed for the moment. If a given \(c^n \in \{0, 1\}^n\) has the property \(N(1|c^n) \neq t + k\) then James’ strategy will assign probability zero to the event \(\alpha^n \oplus s^n = c^n\). Thus, we may assume \(N(1|c^n) = t + k\). We may then consider all partitions of \(c^n\) into two parts where the first part has exactly \(t\) elements which are all equal to one. Each such decomposition corresponds to one choice of \(\alpha^n\) from \(T^n_t\). Using this decomposition we can explicitly calculate the probability
that $\alpha^n + s^n = c^n$ given that $\alpha^n \in T_t$ as

$$
\sum_{\alpha^n \in T_t} \frac{1}{|T_t|} \mathbb{P}(\alpha_t + s_t = c^n) =
$$

$$
= \frac{1}{|T_t|} \sum_{\alpha^n \in T_t} \sum_{s^n : S(s^n|\alpha^n) > 0} \frac{1}{|T^n_c|} \delta(\alpha^n + s^n, c^n)
$$

$$
= \binom{n}{t}^{-1} \binom{t+k}{t} \binom{n-t}{k}^{-1} \frac{n!}{t!(n-t)!(t+k)!} \frac{1}{n!} \frac{1}{(n-t)!}
$$

$$
= \binom{n}{t+k}^{-1}
$$

$$
= \frac{1}{|T_t+k|}.
$$

This particular strategy of James aims at producing the same distribution of $\alpha^n + s^n$, which should ideally match a desired target i.i.d. distribution, for a wide range of potential types $t$ of $\alpha^n$ - preferably at least those which are typical for $\omega$.

Given any type $t$ with the property $t_1 \leq t \leq t_2$, James samples an integer $k'$ from the set $[K]$ according to $\lambda_0, \ldots, \lambda_K$. He then applies a corresponding (random) number $\chi(K, t, k') := k - (t-t_1) + (\Lambda_n - K)$ of bit flips to the channel according to the definition of $S(s^n|\alpha^n)$. By definition of the procedure and the equations preceding (143), given any $t$, this produces - independent from $t$ - the distribution

$$
p' := \sum_{k=0}^{K} \lambda_k \pi_{t+k-(t-t_1)+(\Lambda_n-K)} = \sum_{k=0}^{K} \lambda_k \pi_{\chi(K, 0, k)}
$$

which is well defined as long as the constraints

$$
t_1 + \Lambda_n \leq n, \quad K \leq t_1 + \Lambda_n, \quad t_1 + \Lambda_n \geq K + t_2
$$

are satisfied. Further, $p'$ does not depend on $t$ in case the constraints are satisfied. ■

**Lemma 15** (Distribution of Noise). Let $\omega \in (0, \frac{1}{2})$, $\Lambda \in (0, \frac{1}{2} - \omega]$ and $0 < \varepsilon < \frac{1}{4} \min \{\Lambda, \omega\}$. For every $n \in \mathbb{N}$, set $t_1 := \lfloor (\omega - \varepsilon) n \rfloor$ and $t_2 := \lfloor (\omega + \varepsilon) n \rfloor$ and $K := \lfloor n \varepsilon / 2 \rfloor$ and $\Lambda_n := \lfloor n \Lambda \rfloor$ and $p_n := \frac{\Lambda + K + t_1}{n}$.

The sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ defined by $\varepsilon_n := \Lambda - \omega - p_n$ is non-negative and converges to $\varepsilon$.

Set $k_0 := \min \{ n, p_n - k \}$. Define for every $k \in \mathbb{N}$ a number $c_k = 1$ if $k \in \{ k_0 - \lfloor \frac{K}{2} \rfloor, k_0 + \lfloor \frac{K}{2} \rfloor \}$ and $c_k = 0$ else. Set

$$
\lambda_k := \frac{c_k}{p_n^{\otimes n} \sum_{l} c_l T_l^{\otimes n}} p_n^{\otimes n}(T_k^n).
$$

Then for every $t$ satisfying $t_1 \leq t \leq t_2$

$$
\sum_{\alpha^n \in T^n_t} \frac{S(s^n|\alpha^n)}{|T^n_t|} \delta(c^n, \alpha^n + s^n) = \tilde{p}(c^n)
$$

for $\tilde{p}$ defined via

$$
\tilde{p} := \sum_k c_k \lambda_k \pi_k.
$$

There is a positive number $c(\varepsilon)$ such that

$$
p_n \leq \frac{1}{1 - 2^{-n c(\varepsilon)}} p_n^{\otimes n}.
$$

**Proof:** The properties of $(\varepsilon_n)_{n \in \mathbb{N}}$ are obvious from its definition.

We assume for simplicity that $K$ is an even number, which can be achieved by choosing $\varepsilon$ appropriately and does not stop us from choosing arbitrarily small $\varepsilon > 0$. Then, the particular structure of $\tilde{p}(c^n)$ follows from Lemma 13.

Observe that $\sum_k c_k \delta^{\otimes k}(T_k) \geq 1 - 2^{-n c(\varepsilon)}$ (c.f. [8]) for some $c(\varepsilon) > 0$ satisfying $\lim_{n \to 0} c(\varepsilon) = 0$ and $\tilde{p}(c^n) = p_n^{\otimes n}(c^n)$ for every $c^n \in T^n_t$ with $k \in \{ k_0 - \lfloor \frac{K}{2} \rfloor, k_0 + \lfloor \frac{K}{2} \rfloor \}$. For all other $c^n$, $\tilde{p}(c^n) = 0$.

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