Effective theory for the cosmological generation of structure

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The current understanding of structure formation in the early universe is mainly built on a magnification of quantum fluctuations in an initial vacuum state during an early phase of accelerated universe expansion. One usually describes this process by solving equations for a quantum state of matter on a given expanding background space-time, followed by decoherence arguments for the emergence of classical inhomogeneities from the quantum fluctuations. Here, we formulate the coupling of quantum matter fields to a dynamical gravitational background in an effective framework which allows the inclusion of back-reaction effects. It is shown how quantum fluctuations couple to classical inhomogeneities and can thus manage to generate cosmic structure in an evolving background. Several specific effects follow from a qualitative analysis of the back-reaction, including a likely reduction of the overall amplitude of power in the cosmic microwave background, the occurrence of small non-Gaussianities, and a possible suppression of power for odd modes on large scales without parity violation.

Keywords: cosmology, structure formation, effective equations

I. INTRODUCTION

Modern physics aims to explain complex phenomena by tracing them back to simpler basic principles. This is also true for cosmology, where the currently quite inhomogeneous universe is thought to have arisen from a simple, nearly homogeneous initial state very briefly after the big bang. An important ingredient for realizations of this scenario is inflation, an early universe phase in which the expansion was accelerated due to the presence of a postulated inflaton field whose negative pressure would drive the universe apart. In inflationary structure formation, initial perturbations are provided by small quantum fluctuations of matter field modes, including the inflaton field, which are then enlarged during the phase of accelerated expansion. Out of these initial seeds the current structure such as galaxies grows by gravitational attraction.

In this way, classical structure may arise from an unstructured vacuum state. There must thus be a transition from quantum fluctuations to classical perturbations, which is one of the fascinating aspects of this scenario, but also one of the least understood. The success of inflationary models in comparison with recent observations of structure in the cosmic microwave background indicates that the remarkably direct identification \( \phi_k = \langle \hat{\phi}_k^2 \rangle^{1/2} \) between the amplitude \( \phi_k \) of classical perturbations at wave number \( k \) and quantum fluctuations of the inflaton mode \( \hat{\phi}_k \) describes this process well. It is an interesting test for the understanding of cosmology as well as quantum physics to derive this relation, or provide a different version with the same observational success.

Details of such a relation may well be important observationally. In fact, the predictions of inflationary cosmology agree with observations but not in a completely clean way. While the result of the general form of a scale-invariant spectrum of inhomogeneities after inflation is successful, the total amplitude of predicted cosmic microwave background anisotropies is too high by several orders of magnitude compared to observations. One may achieve a lower amplitude by tuning the inflationary model, but this would eliminate a considerable part of the appeal of inflation. If only some fraction of the total quantum fluctuations are transferred to classical inhomogeneities, on the other hand, a reduction of total power would result.

Also conceptually, it is worthwhile to study the process of structure formation because there are deep issues related to the measurement problem of quantum physics. With the usual interpretation of quantum mechanics several questions immediately come to mind: What causes the wave function of the inflaton field to collapse, and why is it not the expectation value \( \langle \hat{\phi}_k \rangle \) of the perturbation operator (which would usually be used but would give zero in an initial vacuum state) but the quantum fluctuation that is identified with the classical perturbation? Such questions have been studied by several groups, justifying the outcome by a combination of different processes to model the quantum-to-classical transition. First, a matter state evolving in an inflating background becomes highly squeezed. Formally, such a state has fluctuations close to those of a classical distribution, which can then emerge after decoherence. As always, decoherence is based on the interaction of quantum degrees of freedom with an en-
environment whose properties are not measured, and thus presents a coarse-graining process of the total physical system. Not surprisingly, due to the complexity, explicit decoherence models require simplifying assumptions in particular in cosmology. A complete description is lacking: one rather studies a state on a background, without coupling it to metric inhomogeneities, followed by a decoherence phase treated separately. While this does show the right behavior of fluctuations, it is not clear that these are in fact the precise inhomogeneities coupling dynamically to metric modes.

Such issues indicate that not all crucial physical ingredients of the situation may have been included yet. One is using the inflaton as a quantum field in a dynamical universe where its coupling to the space-time metric and the gravitational field is essential, but not considered in the process above. In such a context, standard quantum field theory techniques of fields on dynamical backgrounds or even quantum gravity to describe the coupling to the metric become very complicated. Fortunately, though, the set-up of the situation, based on quantum fluctuations and classical inhomogeneities, indicates that no strong quantum gravity properties nor technical details of quantized fields are required for the process. Such situations can usually be dealt with very powerfully by effective descriptions which, as we will see in this paper, are in fact quite suitable.

In this way we will provide a description of properties of the quantum system through classical equations which are amended by quantum correction terms or by the inclusion of quantum degrees of freedom. Such a description is well known from low energy effective actions used in particle physics to describe perturbative excitations out of the vacuum of the theory. Related techniques are used in condensed matter physics in a variety of different forms. For cosmological purposes, however, this has to be generalized because the coupling to metric modes is important, and thus specifying the vacuum state would involve gravitational degrees of freedom, too, and require quantum gravity. The techniques we will use here are general enough to include also metric perturbations in addition to fluctuations of the inflaton, while reducing to the low energy effective action in the standard context. Thus, what we will be using is a proper extension of effective action schemes to a cosmological context with a dynamical metric.

In this paper our aim is to provide effective equations for a quantum state of matter on an evolving space-time, including quantum fluctuations and their back-reaction on the space-time. We focus on a discussion of the meaning and form of effective equations rather than the technical derivations or analyses, for which we refer to existing papers or future work. In a qualitative analysis we will here highlight several possible effects which show the potential behind this new type of effective equations for cosmology.

II. EFFECTIVE EQUATIONS

The basic observation of the required generalization can easily be illustrated by quantum mechanics of a system with a single degree of freedom. Instead of using a representation of states in a Hilbert space, taking the usual analytical point of view, one can treat quantum mechanics more algebraically. Akin to algebraic quantum field theory, one views the algebra of operators and their dynamics as primary and directly extracts observable information without specifying states or a quantum representation.

Dynamical information of a quantum system is contained in the expectation values \( \langle q \rangle \) and \( \langle p \rangle \) but, in contrast to a classical system, also in infinitely many additional variables. The latter represent the remaining information of a wave function given by all its moments, which we parameterize in terms of the quantum variables

\[
G^{a,n} := \langle (\hat{q} - \langle \hat{q} \rangle)^{a} (\hat{p} - \langle \hat{p} \rangle)^{n} \rangle_{\text{Weyl}} \tag{1}
\]

(i.e. Weyl ordered operators) for \( a = 0, \ldots, n \) and integer \( n \geq 2 \). At \( n = 2 \), for instance, quantum fluctuations \( G^{0,2} = \langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2 \) and \( G^{2,2} = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2 \) as well as the covariance \( G^{1,2} = \langle \hat{q} \hat{p} + \hat{p} \hat{q} \rangle - \langle \hat{q} \rangle \langle \hat{p} \rangle \) are among the quantum variables. This set of variables must be subject to the uncertainty relation

\[
G^{0,2} G^{2,2} \geq \frac{\hbar^2}{4} + (G^{1,2})^2. \tag{2}
\]

Independently of whether a Heisenberg picture for operators such as \( \hat{q} \) and \( \hat{p} \) or a Schrödinger picture for wave functions is used, equations of motion for expectation values take the form \( \dot{\langle q \rangle} = \langle [\hat{q}, H] \rangle / i\hbar \) and \( \dot{\langle p \rangle} = \langle [\hat{p}, H] \rangle / i\hbar \) whose right hand sides can, for a given Hamiltonian \( \hat{H} \), be expressed as a function of expectation values of \( \hat{q} \) and \( \hat{p} \) as well as their quantum variables. For an anharmonic oscillator with Hamiltonian

\[
\hat{H} = \frac{1}{2m} \hat{p}^2 + V(\hat{q}) = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} m \omega^2 \hat{q}^2 + \frac{1}{3} \lambda \hat{q}^3
\]

we have, for instance, equations of motion

\[
\frac{d}{dt} \langle \hat{q} \rangle = \frac{1}{m} \langle \hat{p} \rangle \tag{3}
\]

\[
\frac{d}{dt} \langle \hat{p} \rangle = -V'(\langle \hat{q} \rangle) - \lambda G^{0,2} \tag{4}
\]

correcting the classical force \( -V'(q) = -m \omega^2 q - \lambda q^2 \) by a fluctuation term which is itself dynamical, i.e. changes in time.

Equations of motion for quantum variables are thus necessary for a closed system of equations, but cannot be derived directly from commutators since they are not expectation values of operators but also involve products of expectation values. Nevertheless, using the Leibniz rule one can easily compute equations of motion such as

\[
\dot{G}^{0,2} = \frac{d}{dt} \langle (\hat{q}^2) - \langle \hat{q} \rangle^2 \rangle = \frac{\langle [\hat{q}^2, \hat{H}] \rangle}{i\hbar} - 2q \frac{\langle [\hat{q}, \hat{H}] \rangle}{i\hbar}. \tag{5}
\]
dealing with the quadratic Hamiltonian of a harmonic oscillator, such as the harmonic oscillator  
\[ H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2q^2 \]  
does have quantum variables appearing in its quantum Hamiltonian  
\[ H_Q = \langle \hat{H} \rangle = \frac{1}{2m}(p^2 + G^{2,2}) + \frac{1}{2}m\omega^2(q^2 + G^{0,2}) \]  
but these only provide the zero point energy and do not couple to expectation values. For an anharmonic oscillator, such as one with the inclusion of a cubic interaction  
\[ \frac{1}{3}!\lambda\hat{q}^3, \]  
on the other hand, we do obtain such coupling terms because  
\[ \frac{1}{3}!\lambda\langle \hat{q}^3 \rangle = \frac{1}{3}!\lambda\hat{q}^3 + !\lambda\langle \hat{q} \rangle G^{0,2} + \frac{1}{3}!\lambda G^{0,3} \]  
involves a product of fluctuations and expectation values, contributing the term \( !\lambda G^{0,2} \) to \( H \), in addition to the moment \( G^{0,3} \) of third order.

Intuitively, these coupling terms describe the motion of the peak of a wave packet, taking into account the back-reaction of spread and other deformations on the peak position. In the language of quantum field theory, one describes the coupled dynamics of \( n \)-point functions directly. The situation here is, however, more general since there is no distinguished state to be used, such as the vacuum state for \( n \)-point functions of perturbative low energy quantum field theory.

In general, one has to use a suitable class of semiclassical states for a given regime. They may be difficult to write as explicit states, but their properties can be seen from analyzing effective equations as well. Typically not all moments of a semiclassical state are required at once, and so one can derive the important ones by solving their equations of motion such as Fig. I along with those of expectation values. A simple example is again the harmonic oscillator, whose equations of motion for moments of second order, i.e. fluctuations and the covariance, decouple from the rest. It is straightforward to derive these equations and see that there is a unique solution with constant quantum variables saturating the uncertainty relation.

Such equations are in some cases easier to solve directly, rather than computing fluctuations from a complete state. The information gained is illustrated in Fig. II.

Moreover, one can express the canonical structure of the full quantum phase space through Poisson relations derived from commutators, such as  
\begin{align*}
\{G^{0,2}, G^{1,2}\} &= 2G^{0,2} \\
\{G^{0,2}, G^{2,2}\} &= 4G^{1,2} \quad (6) \\
\{G^{1,2}, G^{2,2}\} &= 2G^{2,2}.
\end{align*}

Similar relations exist between all the \( G^{a,n} \). Alternatively to deriving the equations of motion directly from expectation values of commutators one can then formulate dynamics through a quantum Hamiltonian \( H_Q = \langle \hat{H} \rangle \). It determines the usual Hamilton equations of motion not just for expectation values, \( \hat{q} = \{q, H_Q\} \) and \( \hat{p} = \{p, H_Q\} \), but also for quantum variables: \( \hat{G}^{a,n} = \{G^{a,n}, H_Q\} \). In general, i.e. unless one is dealing with the quadratic Hamiltonian of a harmonic oscillator or free particle, \( H_Q = \langle \hat{H} \rangle \) will involve quantum variables coupling to the classical ones, which gives rise to effective corrections to classical equations. A
of which now infinitely many ones exist, gives rise to an infinite number of quantum degrees of freedom. The expectation value of the Hamiltonian operator will then play the role of the Hamiltonian generating evolution of all the n-point functions.

We are here primarily interested in the situation of an inflaton field on an evolving cosmological background. The Hamiltonian operator of the inflaton field will thus provides the quantum matter Hamiltonian as the source in effective cosmological perturbation equations. One can use the quantum mechanical results described before by performing a mode decomposition of the field and by quantizing each mode individually. (A more rigorous treatment of defining quantum variables for field theories is possible.) We denote the resulting quantum variables as $\hat{G}_{a,n}^{k_1\ldots k_n}$ to indicate the wave vectors $k_i$ of each mode operator entering the quantum variable, starting with $\phi$-modes. Quantum fluctuations then automatically occur in the generated effective equations, and they do couple to metric modes since any matter Hamiltonian contains matter as well as geometrical fields. Through the evolution equations, it is then determined precisely how these quantum variables couple to classical ones and lead to classical perturbations in the course of cosmological expansion.

For a scalar field $\phi$, the inflaton, with momentum $p_\phi$ and potential $V(\phi) = \frac{1}{2} m^2 \phi^2$, we have the matter Hamiltonian

$$H_{\text{class}} = \int d^3x N \left( \frac{1}{2} q^{-3/2} p_\phi^2 + \frac{1}{2} q^{1/2} \nabla \phi \cdot \nabla \phi + q^{3/2} V(\phi) \right) + \int d^3x p_\phi \mathbf{M} \cdot \nabla \phi.$$  

This Hamiltonian is composed of the kinetic energy $\frac{1}{2} p_\phi^2/q^{3/2}$ with gradient term $\frac{1}{2} \sqrt{q} \phi \nabla \phi$, the potential energy depending on $V(\phi)$ as well as the momentum flux $p_\phi \nabla \phi$. The function $N$ and the vector $\mathbf{M}$ occur in the Hamiltonian because they determine what is considered the spatial integration over a slice of space-time in this general relativistic situation. Depending on the boosting of this slice energy flux occurs in the Hamiltonian if $\mathbf{M} \neq 0$.

For the quadratic potential, as typically used in inflation, all terms in the Hamiltonian are quadratic in the field variables. The theory is thus free when viewed on a fixed background space-time with metric components $q$, $N$ and the vector $\mathbf{M}$ which we used in the special form

$$d s^2 = -N^2 dt^2 + q(x,t)(d x + M d t) \cdot (d x + M d t) \quad (8)$$

suitable for the type of scalar perturbations. (This is relevant for structure formation in longitudinal gauge; we thus use only spatial metrics $q_{ij} = q \delta_{ij}$ which are diagonal.)

If the metric is itself dynamical rather than a fixed background, however, the Hamiltonian is not free. Its terms are no longer quadratic, and now matter couples to gravity classically but also quantum mechanically through fluctuations and higher moments.

To proceed and to show this explicitly, it is useful to write the functions $\phi$, $p_\phi$, $q$, $N$ and all components of $\mathbf{M}$ in a mode decomposition such as $\phi(t, x) = \bar{\phi}(t) + \sum_{k \neq a} \phi_k(t) e^{i k x}$. The choice of background variables, denoted by a bar, depends on the space-time gauge which, for the metric modes, one usually sets to $q = a^2$, $N = a$ (in terms of the scale factor $a$ of a Friedmann–Robertson–Walker universe) $\mathbf{M} = 0$ and identifies $- q_k / 2 a^2 = N_k / a =: \psi_k$ in final equations. The modes are, however, kept independent to compute energy-momentum components: energy density modes $\rho_k$ are derived as $a^3 \delta H / \delta N_{-k}$ and energy flux modes $V_k$ from $\delta H / \delta M_{-k}$. Pressure modes $p_k$, which are also important for cosmology, are obtained from $\delta H / \delta q_{-k}$ in a relation which follows from the definition of pressure as the negative derivative of energy by volume.

In a canonical scheme, which is essential for the general theory of effective systems as described above, irrespective of details of quantum gravity only the spatial components $q_{ij}$ of the metric will be quantized while the components $N$ and $\mathbf{M}$ remain as free functions (playing the role of Lagrange multipliers of constraints). The quantum Hamiltonian is thus

$$H_Q = \int d^3 x N \left( \frac{1}{2} q^{-3/2} p_\phi^2 + \frac{1}{2} q^{1/2} \nabla \phi \cdot \nabla \phi + q^{3/2} V(\phi) \right) + \int d^3 x \mathbf{M} \cdot (\bar{p}_\phi \nabla \bar{\phi})$$

clearly showing the non-quadratic nature of the problem.

It is to be expanded as a series of terms coupling the classical variables to each other and to quantum variables. Due to the nature of the problem of interest, i.e. relating inflaton fluctuations to classical inhomogeneities, we consider only quantum fluctuations of matter and not of the metric. We also ignore correlations between the metric and matter in this paper.

We have (with $N = a$)

$$H_Q = H_{\text{class}} + \frac{3}{2 a^2} \sum_{k} \sum_{k'} G_{k,k'}^{2,2} + \frac{a^2}{2} \sum_{k} \left( m^2 a^2 + k^2 \right) C_{0,0}^{0,2}$$

$$- \frac{3}{4 a^4} \sum_{k,k'} q_{-k-k'} G_{k,k'}^{2,2} + \frac{1}{2 a^3} \sum_{k,k'} N_{-k-k} G_{k,k'}^{2,2}$$

$$+ \frac{1}{4} \sum_{k,k'} q_{-k-k} \left( 3 m^2 a^2 - k \cdot k' \right) G_{k,k'}^{0,2}$$

$$+ \frac{1}{2} \sum_{k,k'} N_{-k-k} \left( m^2 a^2 - k \cdot k' \right) G_{k,k'}^{0,2}$$

$$+ i \sum_{k,k'} \mathbf{k} \cdot \mathbf{M}_{-k-k} G_{k,k'}^{1,2} + \cdots$$

where the dots indicate terms of higher order in the perturbations and those containing quantum correlations between matter and metric fields. The terms we kept correspond to semiclassical gravity, but here they are embedded in a larger scheme of the effective quantum matter
and gravity theories. In particular, we will also derive explicit equations of motion for the fluctuation terms.

Quantum variables are the key new contributions entering effective equations. In particular, fluctuations will give non-zero contributions to $\rho_k$, $P_k$ and $V_k$ as such

$$\frac{1}{a^3} \delta H_0 = \frac{1}{a^2} \sum_{k'} \left( \frac{1}{a^2} G^{2,2}_{k-k'k'} + \frac{m^2 a^2 + k^2 - k \cdot k'}{a^2} G^{0,2}_{k-k'k'} \right).$$

(This is one of the places where quantum field theoretical infinities can arise. In the effective treatment used here, this can be dealt with but will not play a role for the equations used below.) Moreover, using the Poisson brackets between quantum variables, such as

$$\{ G^{0,2}_{p_1, p_2}, G^{1,2}_{k_1, k_2} \} = \frac{1}{2} \left( \delta_{p_1, p_2} G^{0,2}_{k_1, k_2} + \delta_{p_1, k_2} G^{0,2}_{p_2, k_1} + \delta_{p_2, k_1} G^{0,2}_{p_1, k_2} + \delta_{p_2, k_2} G^{0,2}_{p_1, k_1} \right)$$

in analogy to (1), we obtain their equations of motion, coupled to $\psi_k$.

New terms thus result in all equations of motion which are asymptotic series containing the quantum variables. Including quantum corrections from fluctuations, we have

$$-k^2 \psi_k - 3 \frac{\dot{a}}{a} \psi_k - 3 \frac{\dot{a}^2}{a^2} \psi_k = \rho_k^{\text{class}}$$

$$+ \frac{1}{2} \sum_{k'} \left( a^{-6} G^{2,2}_{k-k'k'} + (m^2 - k' \cdot (k - k'))/a^2 \right) G^{0,2}_{k-k'k'}$$

as the constraint equation whose Lagrange multiplier is $N_k$,

$$-\dot{\psi}_k - 3 \frac{\dot{a}}{a} \psi_k - 2 \left( \frac{\dot{a}}{a} \right) \psi_k - 3 \frac{\dot{a}^2}{a^2} \psi_k = P_k^{\text{class}}$$

$$+ \frac{1}{2} \sum_{k'} \left( a^{-6} G^{2,2}_{k-k'k'} - (m^2 - k' \cdot (k - k'))/3a^2 \right) G^{0,2}_{k-k'k'}$$

for the equation of motion of $\dot{q}_k$ and

$$i k_j (\dot{\psi}_k + \frac{\dot{a}}{a} \psi_k) = V^{\text{class}}_{j, k} + i \sum_{k'} (k - k') \cdot G^{1,2}_{k-k'k'} \psi_{k'}$$

for the constraint equation whose Lagrange multiplier is the $k$-mode of the component $M_j$. The canonical derivation of these equations follows that developed for a different source of quantum corrections.

The metric modes are thus explicitly sourced by quantum correlations between different modes as well as fluctuations in (11) and (12) where $G^{0,2}_{k, 2/3} k$ and $G^{2,2}_{k, 2/3} k$ contribute. Even if matter inhomogeneities and thus the density and pressure modes $\rho^{\text{class}}, P^{\text{class}}$ and $V^{\text{class}}$ vanish in an initial state, quantum fluctuations which must always be present source and generate perturbations of the classical metric field $\psi_k$.

In this way, quantum fluctuations are the source for classical perturbations $\psi_k$. Moreover, they are themselves dynamical and change according to the metric perturbations they generate: we have

$$\dot{G}^{0,2}_{p_1, p_2} = 2 a^{-2} G^{1,2}_{p_1, p_2}$$

$$+ 4 \frac{\dot{a}^2}{a^2} \sum_k \left( \psi_{p_1, k} G^{1,2}_{p_2, k} + \psi_{p_2, k} G^{1,2}_{p_1, k} \right)$$

as the constraint equation whose Lagrange multiplier is $\dot{N}_k$.

$$\dot{\psi}_{p_1, p_2} = a^{-2} G^{2,2}_{p_1, p_2}$$

$$+ 2 \frac{\dot{a}^2}{a^2} \sum_k \left( \psi_{p_1, k} G^{2,2}_{p_2, k} + \psi_{p_2, k} G^{2,2}_{p_1, k} \right)$$

$$- \frac{1}{2} \frac{\dot{a}^2}{a^2} \left( p_1^2 + p_2^2 + 2m^2 a^2 \right) G^{0,2}_{p_1, p_2}$$

$$+ m^2 a^2 \sum_k \left( \psi_{p_1, k} G^{0,2}_{p_2, k} + \psi_{p_2, k} G^{0,2}_{p_1, k} \right)$$

$$- \frac{1}{2} \frac{\dot{a}^2}{a^2} \left( p_1^2 + p_2^2 + 2m^2 a^2 \right) G^{1,2}_{p_1, p_2}$$

$$+ 2m^2 a^2 \sum_k \left( \psi_{p_1, k} G^{1,2}_{p_2, k} + \psi_{p_2, k} G^{1,2}_{p_1, k} \right)$$

for quantum fluctuations of matter on a dynamical space-time.

With infinitely many variables, this is a complicated system of coupled equations. But one can make several observations already from the structure of this set. First, through quantum variables, different metric modes of different wave numbers couple even at the level of linear metric perturbations with a total quadratic Hamiltonian used here. In this way, correlations between the modes and thus non-Gaussianities arise, which are not included in other equations available so far but will play an increasing role for upcoming observations.

For a second observation, we now look only at terms containing $G_{k, 0}$, which are of interest because they derive from operators linear in the field modes and measure correlations between the background and inhomogeneities. They also appear on the right hand side of metric perturbation equations, e.g. $G_{k, 0}^{2,2}$ as an addition to the flux in (11) with similar contributions to $\rho_k$ and $P_k$. Such a term appears to correspond to the traditional identification between inhomogeneities and fluctuations provided that $G_{k, 0}^{1,2} = \langle \phi_k \hat{P}_0 \rangle$ can be written as $\hat{P}_0 \langle \phi_k^2 \rangle^{1/2}$. If such a relation would hold, the source term of (11), for instance, could be written as

$$V^{\text{class}}_k + i k G_{k, 0}^{1,2} = \hat{P}_0 \left( \phi_k + \sqrt{\langle \phi_k^2 \rangle} \right)$$

where the fluctuation could directly take over the role of a classical perturbation mode $\phi_k$. This form taken exactly, however, violates the assumption of a Gaussian state (which would have vanishing correlations) as well as uncertainty relations. Thus, we prove that the effective theory of a matter field on an expanding space-time must result in corrections to the usual direct identification between matter fluctuations and metric modes.

For other effects, not all the quantum variables are expected to be equally important, and suitable simplifying truncations are possible. For instance, we can restrict the
set to fluctuations only, i.e. only variables of the form $G_{k,\pm k}$ ignoring correlations. Then, $\psi_k$ together with $G_{k/2,\pm k/2}^{0,2}$, $G_{k/2,\pm k/2}^{1,2}$ and $G_{k/2,\pm k/2}^{2,2}$ forms a closed set of equations. This is particularly relevant because fluctuations are restricted by uncertainty relations (derived from the pairs $e^{i\theta_1}\hat{p}_\phi\pm e^{-i\theta_2}\hat{p}_\phi\pm e^{i\theta_2}\hat{\phi}_{-k}$ and $e^{i\theta_2}\hat{\phi}_k\pm e^{-i\theta_2}\hat{\phi}_{-k}$ of conjugate operators)

\[
(\pm 2G_{k,-k}^{0,2} + e^{2i\theta_1}G_{k,k}^{0,2} + e^{-2i\theta_2}G_{k,-k}^{0,2})(\mp 2G_{k,-k}^{2,2} + e^{2i\theta_1}G_{k,k}^{2,2} + e^{-2i\theta_2}G_{k,-k}^{2,2}) - \hbar^2 \cos^2(\theta_1 - \theta_2) \geq 0
\]

for all real $\theta_1$ and $\theta_2$ which follow as in [2], noting that $\hat{\phi}_{-k} = \hat{a}_{k}^\dagger$. Thus, all $G_{k,\pm k}$ can be zero. Nevertheless, $\psi_k = 0$ with $G_{k,k'} = 0$ for $k \neq -k'$ is a consistent solution, showing that deviations from initial Gaussian states, or quantum gravitational fluctuations, are required for structure generation. The vacuum state of a quantum field on a classical space-time, which would be Gaussian, cannot generate metric inhomogeneities.

If this is to be used for inflationary structure formation, there must be additional source terms in the equations of motion. (Other scenarios have been proposed[25]) The final source not yet included comes from contributions involving quantum variables of the metric modes, which would require quantum gravity for their derivation. While such a calculation would be challenging, it has the promise of completing the picture of structure formation in the early universe. In fact, a purely homogeneous space is not consistent with what one often expects from quantum gravity: a discrete structure of space-time. Implications of this Planck-scale discrete-ness also enter the effective equations through quantum variables involving the metric operators. They must then appear as additional source terms which would make exactly homogeneous solutions, where all $\psi_k$ vanish, inconsistent. An implementation, which is beyond the scope of this paper but would follow the same lines, would not allow tests of specific candidates of quantum gravity by the source terms of structure they provide.

\[\psi_k \neq 0 \text{ with } k \neq -k' \]

Instead of conceptual problems, we are faced with a computational problem of analyzing coupled differential equations. Since a priori infinitely many variables are present, due to the field theoretical nature but also due to the number of quantum variables describing a wave function, special solution techniques are required. As in other examples of effective equations, this usually involves the truncation of the equations to finitely many ones, based on assumptions for the magnitude of quantum variables such as the size of fluctuations compared to that of higher moments. If such a truncation has been performed, the scheme of solving coupled ordinary differential equations for expectation values and moments has strong numerical advantages over solving a partial differential equation for a state first and then integrating a possibly highly oscillatory semiclassical state to obtain moments. But also analytically this scheme is highly economical since it can show several intuitive properties as one is used to from effective equations. We have presented two examples for such considerations of only a truncated subset of quantum variables, although we did not support this here by a precise estimate of the ignored terms.

While the system of coupled equations is large and its analysis still incomplete, several qualitative effects are visible: (i) Correlations build up during evolution in an indirect process starting from quantum fluctuations, and then feed classical inhomogeneities. This results in a smaller amplitude of inhomogeneities compared to the traditional identification and could explain the observed discrepancy for the total power of inhomogeneities. (ii) States, described by quantum variables, evolve in complicated ways with all variables coupled to each other. For instance, correlations between different modes will arise, implying non-Gaussianity, also at a small level, even from linear metric perturbations. (iii) Although all quantum variables contribute as sources of $\psi_k$, there is only one fluctuation term $G_{k/2,\pm k/2}$ in its equation of motion. For this term to exist, the wave number $k/2$ must occur in the wave vector lattice for a given spatial topology. This may not be the case if we have a compact space for which the $k$-space is a lattice. For a toroidal space, for instance, only $\psi_k$ for even $k$ are generated directly while odd modes are suppressed, most strongly so for small $k$ for which it is less likely to find an existing wave number close to $k/2$. This provides a possible mechanism for the observed suppression of odd modes on large scales[26]. Since any of the components of $k$ must be even for $k/2$ to be guaranteed to lie on the wave vector lattice, the relevant parity is mirror rather than point symmetry for which indeed odd modes are suppressed. With a lattice depending on spatial topology, a precise comparison with data can reveal topological properties of space. The mechanism does not require parity violation[26] but is, in fact, a consequence of a parity invariant matter Hamiltonian combined with the fact that its main terms are quadratic in the matter field.

Regarding interpretational issues, we do not have a sharp transition from quantum to classical behavior;
instead, fluctuations always remain coupled to classical variables. In some regimes they can be ignored to an excellent approximation, in which case we obtain a classical description (akin to “decoherence without decoherence”). This must happen in cosmology: initially, quantum fluctuations are the only inhomogeneities and seed classical metric modes. After some time, metric modes grow larger and fluctuations become less and less relevant in comparison. Their increasing irrelevance for further evolution is perceived as a quantum to classical transition. This qualitative behavior still is to be seen precisely from a detailed analysis, which can shed light on the role of quantum mechanics in the problem at hand.

At a technical level, those subdominant quantum variables can then be ignored for an analysis of the equations of motion. Mathematically this provides an approximation scheme, but from the physical perspective eliminating some of the quantum variables from further considerations implies the occurrence of mixed states due to coarse-graining the degrees of freedom. In fact, not every set of quantum variables corresponds to a pure state; setting some of them to zero means that the corresponding state they determine can become mixed.

The system of differential equations is deterministic and thus appears to contradict basic quantum mechanics. However, it contains infinitely many quantum variables and thus requires infinitely many initial conditions for further evolution is perceived as a quantum to classical transition. This qualitative behavior still is to be seen precisely from a detailed analysis, which can shed light on the role of quantum mechanics in the problem at hand.

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equations of motion given by the quantum Hamiltonian $H_Q = \langle \hat{H} \rangle$. Indeed, if we choose the basis states $\psi_j$ to be given by eigenstates of $\hat{H}$ we have $H_Q(c_j) = \sum_j E_j |c_j|^2$, giving equations of motion $\frac{d}{dt} \text{Re}c_j = \{\text{Re}c_j, H_Q\} = \frac{E_j}{\hbar} \text{Im}c_j$ and $\frac{d}{dt} \text{Im}c_j = \{\text{Im}c_j, H_Q\} = -\frac{E_j}{\hbar} \text{Re}c_j$ which implies $\dot{c}_j = -i\hbar^{-1} E_j c_j$ as it follows from the Schrödinger equation. This phase space formulation brings quantum mechanics formally closer to classical mechanics, as far as dynamics is concerned, making it sometimes more straightforward to connect classical to quantum equations through effective ones. There are, however, differences in what are considered observables, an issue we will briefly come back to in the end.