POSITIVE SCALAR CURVATURE
ON Pin\(^\pm\)- AND Spin\(^c\)-MANIFOLDS
AND MANIFOLDS WITH SINGULARITIES

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Abstract. It is well-known that spin structures and Dirac operators play a crucial role in the study of positive scalar curvature metrics (psc-metrics) on compact manifolds. Here we consider a class of non-spin manifolds with “almost spin” structure, namely those with spin\(^c\) or Pin\(^\pm\)-structures. It turns out that in those cases (under natural assumptions on such a manifold \(M\)), the index of a relevant Dirac operator completely controls existence of a psc-metric which is \(S^1\)- or \(C^2\)-invariant near a “special submanifold” \(B\) of \(M\). This submanifold \(B \subset M\) is dual to the complex (respectively, real) line bundle \(L\) which determines the spin\(^c\) or pin\(^\pm\) structure on \(M\). We also show that these manifold pairs \((M, B)\) can be interpreted as “manifolds with fibered singularities” equipped with “well-adapted psc-metrics”. This survey is based on our recent work as well as on our joint work with Paolo Piazza.

1. Introduction

1.1. Motivation. It is well-know that from the view-point of differential geometry, and especially problems involving scalar curvature, there is a dramatic difference between spin manifolds and non-spin manifolds. It is easy to check this condition: a smooth compact oriented manifold \(M\) admits a spin structure if and only if its second Stiefel-Whitney class \(w_2(M) = 0\). In this paper we will be interested in the question of when a manifold or pseudomanifold admits a Riemannian metric of positive scalar curvature, which we abbreviate for convenience to “psc-metric.” Now we recall:

**Theorem 1.1** ([15]). Let \(M\) be a compact non-spin simply-connected manifold with \(\dim M = n \geq 5\). Then \(M\) admits a psc-metric.

The situation for spin manifolds is completely different, and in the simply connected case can be expressed as follows:

**Theorem 1.2** ([15], [26]). Let \(M\) be a compact spin simply-connected manifold with \(\dim M = n \geq 5\). Then \(M\) admits a psc-metric if and only if \(\alpha(M) = 0\) in \(KO_n\).

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Here $KO_n$ is the $n$-the coefficient group for real $K$-theory, which by the Bott periodicity theorem is given by $\mathbb{Z}$ for $n$ divisible by 4, by $\mathbb{Z}/2$ if $n \equiv 1$ or 2 mod 8, and is 0 otherwise. The invariant $\alpha(M)$ is Atiyah’s $\alpha$-invariant, which can be identified with the $KO_n$-valued index of the $\mathcal{C}^\ell_n$-linear Dirac operator defined by the spin structure, and the map $\alpha$ passes to a surjective homomorphism $\Omega_n^{\text{spin}} \to KO_n$. Thus there are plenty of simply connected spin manifolds that do not admit a psc-metric.

We consider two classes of manifolds which are non-spin, but in some sense are very close to be spin, namely, pin$^\pm$-manifolds and spin$^c$-manifolds which are not spin. These conditions are easy to verify: if $M$ is not orientable but its orientable double cover has a spin structure, then $M$ has a pin$^\pm$-structure. If $M$ is orientable and $w_2(M) \neq 0$, then $M$ has a spin$^c$-structure exactly when there is a class $c \in H^2(M; \mathbb{Z})$ which maps to $w_2(M)$ under the mod-2 reduction map $H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{Z}/2)$.

The class $c$ gives a map $c: M \to \mathbb{C}P^\infty$ and, consequently, a complex line bundle $L \to M$. We use the notation $(M, L)$ for a manifold with a choice of spin$^c$-structure.

Let $(M, L)$ be spin$^c$-manifold. Since $M$ is finite-dimensional, the image of the map $c: M \to \mathbb{C}P^\infty$ is contained in $\mathbb{C}P^k \subset \mathbb{C}P^\infty$ for some $k$. Suppose the map $c$ is transverse to $\mathbb{C}P^{k-1}$, and let $B = c^{-1}(\mathbb{C}P^{k-1})$:

In particular, $L|_B \to B$ is the normal bundle of the inclusion $B \hookrightarrow M$. The submanifold $B$ is dual to $L \to M$.

We denote by $N(B)$ a tubular neighborhood of $B \hookrightarrow M$. Then $M$ is decomposed as

$$M = X \cup_{\partial X} N(B),$$

where $X$ is the closure of $M \setminus N(B)$. Here $X$ is a spin manifold with boundary $\partial X = \partial N(B)$ and we have a principal $S^1$-bundle $\partial X = \partial N(B) \to B$. In particular, the boundary $\partial N(B)$ has a natural free $S^1$-action, and $N(B)$ can be identified with the unit disk bundle of $L$.

**Definition 1.3.** A Riemannian metric $g$ on a spin$^c$-manifold $(M, L)$ is called well-adapted if the restriction $g|_{N(B)}$ is $S^1$-invariant (where $M = X \cup_{\partial X} N(B)$) and if the the metric on $X$ is a product metric in a collar neighborhood of $\partial X$.

The case of manifolds $M$ which are not spin but which have a double cover which is spin is also closely analogous. In this case, the double cover is classified by a map $c: M \to \mathbb{R}P^\infty$ which we can take to land in some
\[\mathbb{R}P^k \subset \mathbb{R}P^\infty\] and to be transverse to \(\mathbb{R}P^{k-1}\), and \(c\) defines a real line bundle \(L\) over \(M\). This time there is a submanifold \(B\) of \(M\) of codimension 1 and a decomposition of \(M\) as \(M = X \cup_{\partial X} N(B)\), and the submanifold \(B\) is dual to the real line bundle \(L\). In this case we have a principal \(C_2\)-bundle \(\partial X = \partial N(B) \rightarrow B\), with \(C_2 = \{\pm 1\}\), and \(N(B)\) is again the unit disk bundle of \(L\) (except that the “disks” are one-dimensional, and can be identified with \([-1, 1]\)). The parallel to Definition 1.3 is:

**Definition 1.4.** A Riemannian metric \(g\) on a nonspin manifold \(M\) with spin double cover and associated real line bundle \(L\) as above is well-adapted if the restriction \(g|_{N(B)}\) is \(C_2\)-invariant (where \(M = X \cup_{\partial X} N(B)\)) and if the metric on \(X\) is a product metric in a collar neighborhood of \(\partial X\).

Here is the question we are interested in:

**Question 1.5.** Suppose we have \((M, L)\), where \(L\) is a complex or real line bundle on \(M\), a nonspin manifold which is spin in the complex case or has a spin double cover in the real case. Under what conditions does there exist a well-adapted psc-metric \(g\) on \((M, L)\)?

We will study this under the simplifying assumptions that \(X\) and \(\partial X\) are simply connected.

1.2. **Plan.** The plan for this survey is fairly straightforward. In Section 2 we discuss the case of Question 1.5 when \(M\) has a spin double cover. This actually involves two separate questions. The first is more basic: if \(M\) is closed and nonspin but has a simply connected spin double cover, what is the necessary and sufficient condition for \(M\) to admit a Riemannian metric of positive scalar curvature? We answer this question, some special cases of which had previously been treated in [5, 14], completely in dimensions 5 and up. Then we go on to ask the more refined question of when such a psc-metric on \(M\) can be taken to be well-adapted. The main result on this, which answers the question completely, is Theorem 2.4.

In Section 3 we discuss the spin\(^c\) case of Question 1.5. This involves several new considerations: a new application of spin\(^c\) index theory, an analysis of a twisted version of positive scalar curvature, where the scalar curvature is perturbed by the curvature of a line bundle, and a study of new transfer map involving \(\mathbb{C}P^2\)-bundles. This latter map had been studied in part before by Führing [?], but with a different application in mind. The most interesting results of this section are Theorem 3.9 and Theorem 3.11.

In the final section, Section 4, we discuss a more general framework of “manifolds with singularities” (really, compact pseudomanifolds with two strata), and when they admit well-adapted psc-metrics. This section is based on recent joint work of the authors with Paolo Piazza.

2. **\(C_2\)-Bundles and \(\text{Pin}^\pm\)-Manifolds**

We begin with the case of \(C_2\)-bundles, which is slightly less complicated than the case of \(S^1\)-bundles. In this section we will be interested in the
study of manifolds $M^n$ which are not spin, but which have a spin double cover $\tilde{M}$, which we will assume for simplicity to be simply connected.

The orthogonal group $O(n)$ has two connected components, the connected component of the identity being $SO(n)$. The double cover of $SO(n)$ (which is also the universal cover when $n \geq 3$) is $Spin(n)$. But there are two ways to complete the commuting diagram of group extensions

$$
\begin{array}{c}
1 \longrightarrow \mathbb{Z}/2 \longrightarrow Spin(n) \longrightarrow SO(n) \longrightarrow 1 \\
1 \longrightarrow \mathbb{Z}/2 \longrightarrow G \longrightarrow O(n) \longrightarrow 1.
\end{array}
$$

(This can be seen, for example, using the inflation-restriction sequence in cohomology of the split group extension

$$
\begin{array}{c}
1 \longrightarrow \mathbb{Z}/2 \longrightarrow SO(n) \longrightarrow O(n) \longrightarrow 1
\end{array}
$$

with Borel cochains for the coefficient group $\mathbb{Z}/2$ [23, Chapter I].) The two possibilities for $G$ are called $Pin^+(n)$ and $Pin^-(n)$ (they are described explicitly in terms of Clifford algebras in [21, Chapter I, §2]), and a lifting of the orthogonal frame bundle to a principal bundle for $Pin^\pm$ is called a pin$^\pm$ structure on a manifold [22]. Such structures can exist even when a manifold is non-orientable, and they still make it possible to do some aspects of spinor geometry without an orientation.

Suppose $M^n$ is not spin and has a simply connected double cover $\tilde{M}$. Then up to homotopy, we have a fibration $\tilde{M} \xrightarrow{p} M \xrightarrow{c} \mathbb{R}P^\infty$, where $c$ is the classifying map of the covering map $p$. Since $\tilde{M}$ is simply connected, the Serre spectral sequence (if $\pi_1(\mathbb{R}P^\infty) \cong C_2$ acts trivially on the low dimensional cohomology of $\tilde{M}$) gives an exact sequence

$$
0 \rightarrow H^2(\mathbb{R}P^\infty, \mathbb{Z}/2) \xrightarrow{c^*} H^2(M, \mathbb{Z}/2) \xrightarrow{p^*} H^2(\tilde{M}, \mathbb{Z}/2) \xrightarrow{d_3} H^3(\mathbb{R}P^\infty, \mathbb{Z}/2)
$$

**Remark 2.1.** There are now various subcases to consider:

1. $M$ is orientable, so that $w_1(M) = 0$. Since we are assuming that $M$ is not spin, we have $w_2(M) \neq 0$, while $p^*w_2(M) = w_2(\tilde{M}) = 0$. Here $c$ is the classifying map for the universal cover of $M$, and by (1), $w_2(M)$ is pulled back from the generator of $H^2(\mathbb{R}P^\infty, \mathbb{Z}/2)$ under $c^*$.

2. $M$ is not orientable, so that $w_1(M) \neq 0$ and the generator of $H^1(\mathbb{R}P^\infty, \mathbb{Z}/2)$ pulls back under $c^*$ to $w_1(M)$. Since $p^*w_2(M) = w_2(\tilde{M}) = 0$, either $w_2(M) = 0$, in which case $M$ admits a pin$^+$-structure, or else $w_2(M) \neq 0$ but $w_2(M)$ comes from $M \xrightarrow{\gamma} \mathbb{R}P^\infty$, i.e., $w_2(M) = w_1(M)^2$, in which case $M$ admits a pin$^-$-structure [20, 19, 22]. pin$^-$ and pin$^+$-structures are also known (cf. [11, 13]) as Pin and Pin$'$ structures, respectively.
The various cases are illustrated by real projective spaces $\mathbb{R}P^n$. These are orientable for $n$ odd and non-orientable for $n$ even. $\mathbb{R}P^n$ is spin when $n \equiv 3 \mod 4$, orientable but non-spin when $n \equiv 1 \mod 4$, pin$^+$ when $n \equiv 0 \mod 4$, pin$^-$ when $n \equiv 2 \mod 4$.

The first step is to study obstructions to psc-metrics in these various cases. Here is the basic result. This fact was conjectured before [17, Conjecture 2.3.1] but we don’t know of a complete proof in the literature, although one special case (the pin$^-$-case with $n \equiv 2 \mod 4$) is in [14, Theorem 6.3].

**Theorem 2.2.** Let $M^n$ be a closed pin$^+$ or pin$^-$ manifold with fundamental group $\mathbb{Z}/2$ and spin universal cover $\tilde{M}$. Fix a Riemannian metric on $M$ and let $L$ be the determinant line bundle of the orthogonal frame bundle. $\text{So } w_1(L) = w_1(M)$.

In the pin$^-$ case, $TM \oplus L$ admits a spin structure, and we obtain an associated $\mathbb{C}^{\ell_n+1}$-linear Dirac operator $\partial / -$ with index $\alpha^-(M)$ in KO$^n+1$. In the pin$^+$ case, $TM \oplus L \oplus L \oplus L$ admits a spin structure, and we obtain an associated $\mathbb{C}^{\ell_n+3}$-linear Dirac operator $\partial / +$ with index $\alpha^+(M)$ in KO$^n+3$. These indices are obstructions to existence of a psc-metric on $M$.

Assume that $n \geq 5$, that $\alpha(\tilde{M}) = 0$ in KO$_n$ and that $\alpha^-(M) = 0$ in KO$_{n+1}$ in the pin$^-$ case, and that $\alpha^+(M) = 0$ in KO$_{n+3}$ in the pin$^+$ case. Then $M$ admits a metric of positive scalar curvature.

**Proof.** Since $L$ has a canonical flat connection, the Lichnerowicz identity $\partial^2_\pm = \nabla^*\nabla + \frac{k}{4}$, where $k$ is the scalar curvature, holds just as for the Dirac operator on a spin manifold, and so if $k > 0$, the spectrum of $\partial_\pm$ is bounded away from 0 and the indices $\alpha_\pm(M)$ must vanish. Similarly, if $M$ admits a psc-metric, then so does $\tilde{M}$, and so $\alpha(\tilde{M}) = 0$. This takes care of the necessity.

For sufficiency, we use the Bordism Theorem [17, Theorem 2.1.1], which in this case boils down to the statement that it is sufficient to check that every class in $\Omega^n_{\text{Pin}}$, $n \geq 5$, for which the necessary conditions hold has a representative of positive scalar curvature. The relevant bordism groups might as well be localized at 2 since by [19, Corollary 4 to Theorem 3], the group $\Omega^n_{\text{Pin}}$ are 2-primary torsion, and divided by the $\Omega^*_{\text{Spin}}$-submodules generated by even real projective spaces (which are obviously represented by manifolds of positive scalar curvature) are $\mathbb{F}_2$-vector spaces.

We begin with the case of $\Omega^n_{\text{Pin}} \cong \tilde{\Omega}^n_{\text{Spin}}(\mathbb{R}P^\infty)$, see [1]. Localized at 2, there is an additive splitting $M\text{Spin} \simeq \text{ko} \lor (\text{image } T)$, where $T$ is the transfer $M\text{Spin} \lor \Sigma^8B\text{PSp}(3)_+ \to M\text{Spin}$

defined by the $\mathbb{H}P^2$-bundle construction, see [26, 27]. Classes in $\Omega^n_{\text{Pin}}$ coming from the homotopy of $(\text{image } T) \lor \mathbb{R}P^\infty$ are linear combinations of classes represented geometrically by dualizing a real line bundle $L$ on a manifold with a bundle structure $\mathbb{H}P^2 \to P^{n+1} \to N^{n-7}$, where the structure group of the bundle is the isometry group of $\mathbb{H}P^2$. The line bundle $L$ has to come
from $N$ since $\mathbb{HP}^2$ is simply connected, so the resulting pin$^-$ $n$-manifold has to be an $\mathbb{HP}^2$-bundle over a pin$^-(n-8)$-manifold obtained by dualizing a real line bundle over $N$, and thus has a psc-metric. So we are reduced to looking at $\pi_\ast(ko \wedge \mathbb{RP}^\infty)$. These groups were computed in [1, Theorem 5.1] and in [19, Theorem 1]. There are summands of $\mathbb{Z}/2$ in dimensions 0 and 1 mod 8 that are detected by the index invariant $\alpha^-$, so these groups are not represented by manifolds of positive scalar curvature. The rest of the homotopy groups of $ko \wedge \mathbb{RP}^\infty$ correspond to cyclic summands generated by real projective spaces of dimension 2 mod 4, and the result is obviously true for these. So that completes the proof for the pin$^-$ case. The pin$^+$ case is quite similar using the equivalence $\text{MPin}^+ \simeq \text{MSpin} \wedge M(3)$ of [19, Theorem 1]. Everything is the same except for replacement of $M(1)$ by $M(3)$. The homotopy groups of the summand $(\text{image } T) \wedge M(3)$ are again represented by $\mathbb{HP}^2$-bundles over lower-dimensional pin$^+$ manifolds, while the homotopy groups $\pi_\ast(ko \wedge M(3))$ include cyclic summands generated by real projective spaces of dimension 0 mod 4, as well as copies of $\mathbb{Z}/2$ in dimensions 2 and 3 mod 8 that are detected by the index invariant $\alpha^+$. So again the theorem is true. □

We now want to answer question 1.5 in the case where $\tilde{M}$, $X$, and $\partial X$ are simply connected. Since we have a double covering $p: \partial X \to B$, $\pi_1(B) \cong \pi_1 N(B) \cong \pi_1(M)$. Now existence of a well-adapted psc-metric on $M$ implies existence of a psc-metric on $B$. There are a number of distinct cases:

**Remark 2.3.** Let $B$ have a simply connected spin double cover.

1. If $B$ is spin, then since the Gromov-Lawson-Rosenberg conjecture holds for the fundamental group $\mathbb{Z}/2$ [25, 6], if $B$ admits a psc-metric, then $\beta \circ c_\ast ([B]) = 0$ in $KO_{n-1}(\mathbb{R}[C_2]) = KO_{n-1} \oplus KO_{n-1}$, and the converse holds if $n \geq 6$ (so that dim $B \geq 5$). Here $\beta$ is the $KO$-assembly map discussed in [25].

2. If $B$ is oriented but non-spin, then $B$ admits a Dirac operator twisted by a direct sum of two copies the real line bundle defined by $c$ (this bundle has nontrivial $w_2$), but not an untwisted Dirac operator. The index of this twisted Dirac operator is an obstruction to psc in $KO_{n-1}$, and a variant of Stolz’s Theorem [26] shows that vanishing of this obstruction is sufficient for $B$ to admit a psc-metric if $n-1 \geq 5$ or $n \geq 6$ [5].

3. If $B$ is not orientable but has a pin$^\pm$ structure, then the obstructions to a psc-metric on $B$ are covered by Theorem 2.2 above.

Now we are ready for the main theorem of this section.

**Theorem 2.4.** Let $M^n$ be a non-spin closed $n$-manifold with a simply connected spin double cover $\tilde{M}$. Write $M = X \cup_{\partial X} N(B)$ as above, with $X$ and $\partial X$ simply connected. If $M$ admits a well-adapted psc-metric, then
\( \alpha(\tilde{M}) = 0 \) in \( KO_n \), there are additional index obstructions to a psc-metric on \( M \) enumerated in Remark 2.3 above, and \( B \) admits a psc-metric. We thus also have to have the vanishing of an index obstruction for \( B \), which depends on what subcase of Remark 2.1 applies to \( B \). The converse, i.e., the statement that the vanishing of all these index obstructions (for both \( M \) and \( B \)) implies the existence of a well-adapted psc-metric on \( M \), holds if \( n \geq 6 \).

Proof. Let \( \beta M = B \) be the “Bockstein,” the quotient of \( \partial X \) by the free \( C_2 \)-action. (The reason for the name “Bockstein” will appear later.) First suppose that \( \beta M \) is spin. Since \( N(\beta M) \) is the disk bundle of the flat line bundle \( L \) defined by the spin double cover of \( M \), restricted to \( \beta M \) where the nontriviality of the bundle is concentrated, the tangent bundle of \( M \), restricted to \( N(\beta M) \), is \( p^*T(\beta M) \oplus L \), and since the tangent bundle of \( \beta M \) is spin, we have \( w_2(M) = 0 \) (since \( X \) is also spin) and \( w_1(M) = w_1(L) \neq 0 \). So \( M \) is a pin\(^+\)-manifold. (A case to keep in mind is \( X = D^n, \partial X = S^{n-1}, \beta M = \mathbb{R}P^{n-1}, M = \mathbb{R}P^n \), in the case where \( n \equiv 0 \mod 4 \).) So we have four obstructions to a well-adapted psc-metric on \( M \): \( \alpha(\tilde{M}) \in KO_n, \alpha^+(M) \in KO_{n+3} \) from Theorem 2.2, and the two \( KO_{n-1} \)-valued obstructions to a psc-metric on \( \beta M \). (The last two are the indices of the untwisted and twisted Dirac operators on \( \beta M \).) Except for \( \alpha(\tilde{M}) \) when \( n \equiv 0 \mod 4 \), all of these obstructions are \( \mathbb{Z}/2 \)-valued. We have the following long exact sequence:

\[
\cdots \xrightarrow{\delta} \Omega^\text{Spin}_n \xrightarrow{f} \Omega^\text{Pin}^+ \xrightarrow{\beta} \Omega^\text{Spin}_{n-1}(BC_2) \xrightarrow{\delta} \Omega^\text{Spin}_{n-1} \xrightarrow{\cdots}.
\]

Here \( f \) is the forgetful map that forgets that a spin manifold is oriented and considers it as a pin\(^+\) manifold via the natural map of Lie groups \( \text{Spin}(n) \hookrightarrow \text{Pin}^+(n) \). The “Bockstein” map \( \beta \) dualizes the line bundle on a pin\(^+\) manifold defined by \( w_1 \), and produces a spin manifold one dimension lower. We call \( \beta \) the Bockstein since it is a dimension-shifting connecting map in this exact sequence, just like the classical Bockstein map for homology. The transfer map \( \delta \) takes a spin manifold with a map to \( BC_2 \) to the associated double cover of the manifold. The exact sequence \( (2) \) may be derived from a related exact sequence

\[
\cdots \xrightarrow{} \pi_n(M\text{Spin} \wedge \mathbb{Z}/2) \xrightarrow{} \Omega^\text{Pin}^+ \xrightarrow{} \Omega^\text{Pin}^- \xrightarrow{} \pi_{n-1}(M\text{Spin} \wedge \mathbb{Z}/2) \xrightarrow{} \cdots
\]

in \([19]\) Lemma 7), since \( \Omega^\text{Spin}_{n-1}(BC_2) \) splits as \( \Omega^\text{Pin}^- \oplus \Omega^\text{Spin}_{n-1} \), and \( \delta \) restricted to the second factor is just multiplication by 2 from \( \Omega^\text{Spin}_{n-1} \) to itself, which gives rise to a cofiber of \( \pi_{n-1}(M\text{Spin} \wedge \mathbb{Z}/2) \).

Now suppose that \( n \geq 6 \) and that all of the index obstructions vanish. We will use \( (2) \) to show that \( M \) admits a well-adapted psc-metric. By the Gromov-Lawson-Rosenberg conjecture for the group \( \mathbb{Z}/2 \), \( \beta M \) admits a psc-metric. Lift this to a local product metric on \( N(\beta M) \) (which is locally the product of \( \beta M \) with a flat real line bundle); this gives a \( C_2 \)-invariant psc-metric on \( \partial X = \partial(N(\beta M)) \) which is a product metric on the boundary. The double \( P \) of \( N(\beta M) \) along \( \partial X \) is a pin\(^+\)-manifold admitting a psc-metric.
which agrees with $M$ on $N(\beta M)$. By the sequence (2), $M$ is pin$^+$-bordant to the disjoint union of $P$ and a closed spin manifold $M'$. By the additivity of the index invariants, the index invariants for $M'$ vanish, so $M'$ admits a psc-metric. Thus $M' \sqcup P$ is a manifold with a psc-metric in the same pin$^+$-bordism class as $M$. We can carry the psc-metric across the bordism, and since the necessary surgery can be done on the interior of $X$, away from $\partial X$, without changing $N(\beta M)$, the resulting psc-metric on $M$ is well-adapted. This takes care of the case where $\beta M$ is a spin manifold.

If $\beta M$ is oriented but not spin (case (2) of Remark 2.3), the case studied in [5], like the case of $X = D^n$, $\partial X = S^{n-1}$, $\beta M = \mathbb{RP}^{n-1}$, $M = \mathbb{RP}^n$, in the case where $n \equiv 2 \mod 4$, things are quite similar except that $M$ is now a pin$^-$-manifold instead of a pin$^+$-manifold. The replacement for (2) in this case is the following:

\[ \cdots \rightarrow \Omega_n^{\text{Spin}} \xrightarrow{f} \Omega_n^{\text{Pin}^+} \xrightarrow{\beta} \Omega_n^{\text{Spin}^+} \xrightarrow{\delta} \Omega_{n-1}^{\text{Spin}} \rightarrow \cdots. \]

Here $\Omega_n^{\text{Spin}^+}$ is the bordism group in dimension $n-1$ for oriented manifold with a spin double cover for a given element of $H^1(-, \mathbb{Z}/2)$, $\delta$ again corresponds to the double cover, and $f$ again is the forgetful map, this time corresponding to the inclusion $\text{Spin}(n) \hookrightarrow \text{Pin}^-(n)$. We have a Bockstein map $\beta$ as before. In this case $N(\beta M)$, and hence $M$, has a pin$^-$ structure since $w_2$ is pulled back from $\mathbb{RP}^\infty$ via the map associated to $w_1$, or in other words, $w_2(M) = w_1(M)^2$. With the substitution of (3) for (2), the proof works just as before.

The final case involves a related long exact sequence of [13, Theorem 3.1], relevant to the case where $M$ is an oriented non-spin manifold and $\beta M$ is a pin$^+$ manifold. This situation arises when $X = D^n$, $\partial X = S^{n-1}$, $\beta M = \mathbb{RP}^{n-1}$, $M = \mathbb{RP}^n$, and $n \equiv 1 \mod 4$.

After renaming the maps from what they are called in [13], this sequence is as follows:

\[ \cdots \rightarrow \Omega_n^{\text{Spin}} \xrightarrow{\text{enh}} \Lambda_n \xrightarrow{\beta} \Omega_n^{\text{Pin}^+} \xrightarrow{\delta} \Omega_{n-1}^{\text{Spin}^+} \rightarrow \cdots. \]

Here $\Lambda_n$, defined in [12], is a bordism group of pairs $(M, L)$ where $M$ is an oriented manifold, $L$ is a real line bundle, and $w_2(M) = w_1(L)^2$. Again $\beta$ is the Bockstein map, sending the class of $(M, L)$ to the class of $\beta M$, dual to the line bundle $L$, $\delta$ is a transfer map, taking the class of a pin$^+$-manifold to the class of its spin double cover, and enh is an enhancement map, sending the class of a spin manifold to the class of the same manifold paired with the trivial line bundle.

We use (4) as follows. Suppose $M = X \sqcup_{\partial X} N(\beta M)$ is oriented but non-spin, with a spin double cover. There is a real line bundle $L$ associated to the spin double cover, and $w_2(M) = w_1(L)^2$, so $(M, L)$ gives a class in $\Lambda_n$. Now suppose $n \geq 6$ and all the index invariants of $M$ and $\beta M$ vanish. Applying Theorem 2.2 to $\beta M$, we see that it admits a psc-metric. Lift this to a local product metric on $N(\beta M)$. As before, $M$ is bordant (in the sense of the theory $\Lambda$) to $(M' \sqcup P, L)$, where $P$ is the double of $N(\beta M)$, $L$ lives on
$P$ and is trivial on $M'$, and $M'$ is a closed spin manifold. Again, additivity of the index invariants implies that $M'$ admits a psc-metric. Then we use the bordism method to transfer the psc-metric from $M' \sqcup P$ to $M$, doing the surgery away from $N(\beta M)$, so that the metric we get is well-adapted. That concludes the proof. □

3. $S^1$-Bundles and Spin$^c$-manifolds

3.1. Preliminary observations and examples. Let $(M, L)$ be a non-spin spin$^c$-manifold. We choose a submanifold $B \subset M$ dual to the bundle $L$; in particular, we identify the restriction $L|_B$ with the normal bundle of the embedding $B \hookrightarrow M$. Let $N(B)$ be a tubular neighborhood of $B$; we denote by $X$ the closure of $M \setminus N(B)$. We obtain a decomposition $M = X \cup \partial X$. Here $X$ is a spin manifold, whose boundary $\partial X$ is equipped with free $S^1$-action since $\partial X$ is the total space of the circle bundle $\partial X \rightarrow B$. This action is consistent with a natural $S^1$-action on the tubular neighborhood $N(B)$ since $N(B)$ is the disk bundle of the restriction $L|_B$.

Remark 3.1. Let $M$ be a non-spin simply connected spin$^c$ manifold, i.e., with $w_2(M) \neq 0$. The projective space $\mathbb{CP}^n$ is an example of such $M$ if $n$ is even. Then a spin$^c$-structure is given by a complex line bundle $L$ on $M$ such that $c_1(L)$ reduces mod 2 to $w_2(M)$. Then, as we discussed above, $B$ is dual to $L$; by construction, $c_1(L)$, and thus also $w_2(X)$, is trivial on the complement $X$ of a tubular neighborhood $N(B)$ of $B$. Thus $X$ is a spin manifold with boundary $\partial X$, which is a circle bundle over $B$. We notice that the manifold $B$ is spin, since

$$w_2(B) + (c_1(L) \mod 2) = w_2(TM|_B) = \iota^*w_2(M) = (c_1(L) \mod 2),$$

which says that $w_2(B) = 0$.

Since $B \subset M$ is a spin manifold, we have $\alpha(B) \in KO_{n-2}$ which evaluates the index of the Dirac operator $\partial_B$. Let $\Omega_n^{spin^c}$ be the spin$^c$-bordism group. We also have a natural homomorphism

$$\alpha^{spin^c}: \Omega_n^{spin^c} \rightarrow KU_n$$

which evaluates the index of the spin$^c$ Dirac operator

$$\alpha^{spin^c}: [(M, L)] \mapsto [\partial_{(M,L)}] \in KU_n.$$  

Recall that a well-adapted metric $g$ on $M$ is such a Riemannian metric that the restriction $g|_{N(B)}$ is $S^1$-invariant and $g$ is a product-metric near $\partial X = -\partial N(B)$.

The following geometrical result gives a necessary condition for existence of a well-adapted psc-metric:

Theorem 3.2 (3 Theorem C]). Let $Z$ be a compact manifold with free $S^1$-action. Then $Z$ admits an $S^1$-invariant psc-metric if and only if the quotient manifold $B = Z/S^1$ admits a psc-metric.
Example 3.3. (i) Let $B$ be a $K3$-surface (which is a simply connected spin $4$-manifold with nonzero $\hat{A}$-genus). Then $B$ does not admit a psc-metric, but there is a circle bundle $p: Y \to B$ with simply connected total space $Y$. To construct such a bundle, we choose a primitive element $c \in H^2(B, \mathbb{Z}) \cong \mathbb{Z}^{22}$ and find a complex line bundle $L(c)$ with $c_1(L(c)) = c$. Then the bundle $p: Y \to B$ is the circle bundle of the line bundle $L(c)$.

The manifold $Y$ is necessarily spin, since $T_Y \cong p^*T_B \oplus V$, where $V$ is the real tangent line bundle along the circle fibers, which is trivial, and thus $w_2(T_Y) = p^*w_2(T_B) = 0$. Furthermore, $Y$ is a spin boundary, since $\Omega_5^{\text{spin}} = 0$. Thus there is a spin $6$-manifold $X$ with $\partial X = Y$, and we can do surgery on $X$ away from the boundary to ensure that $X$ is simply connected and the pair $(X, Y)$ is $2$-connected. In particular, we obtain that $Y$ has a psc-metric $g_Y$. However, Theorem 3.2 implies that any such psc-metric $g_Y$ cannot be $S^1$-invariant since otherwise it would give a psc-metric on $B = K3$, which is not possible.

To construct a relevant spin$^c$-manifold, we glue together $X$ and the disk bundle $N(B)$ of $L(c)$ over $B$. With a little bit of work one can show that non-spin spin$^c$-manifold $M := X \cup_{\partial X} -N(B)$ comes together with a line bundle $L \to M$ dual to $B$ (and trivial over $X$), so that $L|_B$ coincides with the bundle $L(c)$ we started with. Then the manifold $M$, being non-spin and simply-connected, admits a psc-metric $g_M$. Thus we conclude that any psc-metric $g_M$ on $(M, L)$ cannot be well-adapted, since otherwise we would obtain an $S^1$-invariant psc-metric on $Y$, and, consequently, a psc-metric on $B = K3$ by Theorem 3.2.

(ii) Let $\Sigma^{10}$ be a homotopy $10$-sphere with nonzero $\alpha$-invariant (i.e., representing the generator of $KO_{10} = \mathbb{Z}_2$). We consider the spin-manifold $B = \Sigma^{10} \# \mathbb{CP}^3$. Since the $\alpha$-invariant is additive on connected sums, the manifold $B$ does not admit a psc-metric. Notice that $B$ is a fake complex projective space, so it admits a principal $S^1$-bundle $Y \to B$ for which the total space $Y$ is a homotopy $11$-sphere. There being no torsion in $\Omega_{11}^{\text{spin}}$, the exotic sphere $Y$ is a spin boundary and we can choose a spin $12$-manifold $X$ bounding $Y$, such that $(X, Y)$ is $2$-connected. Just as in the example (i), we construct a non-spin spin$^c$-manifold $(M, L)$ with $M = X \cup_{\partial X} -N(B)$ (and $L$ dual to $B$) which admits a psc metric $g_M$, but no such psc-metric is well-adapted, since otherwise it would produce a psc-metric on $B$ (again, via Theorem 3.2). These examples show that existence of a well-adapted psc-metric on a spin$^c$-manifold $(M, L)$ implies that the manifold $B$ dual to $L$ has to admit a psc-metric. Since $B$ is spin, we obtain the first obstruction $\alpha(B) \in KO_{n-2}$ for existence of a well-adapted psc-metric on $(M, L)$. In the case when the manifold $B$ is simply-connected (and $n-2 \geq 5$), this is the only obstruction for existence of a psc-metric on $B$.

Next, we choose a psc-metric $g_B$ on $B$. Then it gives us an $S^1$-invariant psc-metric on $N(B)$ and, in particular, a psc-metric $g_Y$ on $Y = \partial N(B)$. 
Let $g_M$ be some well-adapted metric on $(M, L)$ (which is not necessarily a psc-metric outside of $N(B)$) extending the above metric on $N(B)$. Then to construct a well-adapted psc-metric on $M$, it is enough to extend $g_Y$ to a psc-metric $g_X$ such that $g_X$ is a product-metric near $Y$: we keep in mind that $M = X \cup_{\partial X = Y} -N(B)$.

Consider the spin$^c$ Dirac operator $\partial_{(M, L)}$ on $(M, L)$. The operator $\partial_{(M, L)}$ depends on a choice of a connection $A_L$ on the line bundle $L$; however, since the restriction $L|_X$ is trivial, we can choose the connection $A_L$ to be flat on $L|_X$. Thus we can take the restriction $\partial_{(M, L)}|_X$ to be the usual spin Dirac operator. Assuming that $g_M$ restricts to a psc-metric $g_Y$, we obtain a proper APS$^1$-boundary problem for the Dirac operator on $(X, Y, g_Y)$. We denote by $\partial_{(X, Y, g_Y)}$ the resulting Dirac operator.

We obtain the next obstruction for existence a well-adapted psc-metric $g_M$ on $(M, L)$, given by the relative index $\alpha^{rel}(X, Y, g_Y) \in KO_n$ of the operator $\partial_{(X, Y, g_Y)}$. Notice that a priori the index $\alpha^{rel}(X, Y, g_Y)$ depends on a choice of $g_Y$ and, consequently, on a choice of a psc-metric $g_B$.

As before we fix a connection $A_L$ on the line bundle $L$ which is flat on $L|_X$, and consider again the Dirac operator $\partial_{(M, L)}$ on $(M, L)$. We have the Lichnerowicz formula:

\begin{equation}
\partial^2_{(M, L)} = \nabla^* \nabla + \frac{1}{4} s_{g_M} + \mathcal{R}_{A_L},
\end{equation}

where $g_M$ is a well-adapted psc-metric and $\mathcal{R}_{A_L}$ is a corresponding curvature form. Then we can homotope the metric on the fibers of $N(B) \to B$ to make it equal to that on a round hemisphere $S^2 \subset S^2(r)$ with the hemispherical fibers having small diameter $r$ and thus big curvature. That implies we can make $s_{g_M}$ highly positive without changing the curvature term $\mathcal{R}_{A_L}$. This allows us to bound the square of the Dirac operator $\partial^2_{(M, L)}$ away from 0.

Thus $\alpha^{spin^c}(M, L) = 0$, where $\alpha^{spin^c} : \Omega^{spin^c}_n \to KU_n$ is the index map.

We conclude: a priori there are three obstructions for existence of a well-adapted psc-metric on a spin$^c$-manifold $(M, L)$, where $M = X \cup_{\partial X = Y} -N(B)$ and $B$ is dual to $L$ as above:

(i) $\alpha(B) \in KO_{n-2}$;
(ii) $\alpha^{rel}(X, Y, g_Y) \in KO_n$;
(iii) $\alpha^{spin^c}(M, L) \in KU_n$.

We emphasize that the obstructions $\alpha(B)$ and $\alpha^{spin^c}(M, L) \in KU_n$ are primary obstructions, and the obstruction $\alpha^{rel}(X, Y, g_Y)$ is secondary one: it depends on a choice of a psc-metric $g_B$ on $B$.

A priori, it is not clear why vanishing of these three obstructions should imply existence of a well-adapted psc-metric, especially when it comes to the secondary obstruction; at least we do not know how to use this information directly to construct such a metric. First, we would like to describe geometrical meaning of the obstruction $\alpha^{spin^c}(M, L) \in KU_n$.

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Before we get to this, however, we should point out that contrary to what it might seem from the above, the obstructions (i) and (iii) on our list are actually not independent. By [10, Theorem 3], proved independently in [30, 31], \(\alpha^{\text{spin}}(M, L)\) determines \(\alpha(B)\), in the following sense: when both are integers (which happens when \(n \equiv 2 \mod 4\), so that \(KO_{n-2} \cong \mathbb{Z}\)), they are equal, and when \(\alpha^{\text{spin}}(M, L)\) is an integer but \(\alpha(B)\) is an integer mod 2 (which happens when \(n \equiv 4 \mod 8\)), then \(\alpha(B)\) is the mod 2 reduction of \(\alpha^{\text{spin}}(M, L)\). The three proofs of these facts (one by Ochanine and Fast and two by Weiping Zhang) are very interesting exercises either in index theory or in bordism theory, but would take us away from our main theme here. However, let us point out an interesting application to Example 1.12.

In the first part of this example, we constructed a case when \(M\) is a spin\(^c\) 6-manifold and \(B\) is a spin manifold with non-zero \(\hat{A}\)-genus. Thus the theorem says that whatever choice we take for \(M\), it has to satisfy \(\alpha^{c}(M) = \hat{A}(B)\).

In the second part of this example, we constructed a spin\(^c\) 12-manifold \(M\), where \(B\) is a homotopy \(\mathbb{CP}^5\) with nonzero \(\alpha\)-invariant in \(KO_{10} \cong \mathbb{Z}/2\). In this case, the theorem says that whatever choice we take for \(M\), \(\alpha^{c}(M)\) has to be odd.

### 3.2. Geometry of the index \(\alpha^{\text{spin}}\)

Let \((M, L)\) be a non-spin \(\text{spin}^c\)-manifold. We choose a metric \(g\) on \(M\), a hermitian metric \(h\) on \(L\), and a (unitary) connection \(A_L\) on \(L\). These data give us the \(\text{spin}^c\) Dirac operator \(\partial /\partial_t (M, L)\). We have the Lichnerowicz formula

\[
\partial^2 /\partial_t^2 (M, L) = \nabla^* \nabla + \frac{1}{4} \kappa_g + \mathcal{R}_L
\]

where the term \(\mathcal{R}_L\) has the following form:

\[
\mathcal{R}_L = \frac{1}{2} \sum_{j<k} F_L(e_j, e_k) \cdot e_j \cdot e_k
\]

where one sums over an orthonormal frame and \(F_L\) is the curvature of the connection \(A_L\) on the line bundle \(L\). We denote

\[
\kappa^L_g := \kappa_g + 4\mathcal{R}_L,
\]

and we say that \(\kappa^L_g\) is the \(L\)-twisted scalar curvature. Notice that \(\kappa^L_g\) depends on a choice of the hermitian metric \(h\) on \(L\) and the connection \(A_L\).

We need to consider coupling between the Riemannian curvature and the curvature of the line bundle \(L\) (which is just given by an ordinary 2-form \(\omega\), which after dividing by \(2\pi i\), has integral de Rham class representing \(c_1(L)\)). Now recall [21, Lemma D.13], which says that any 2-form \(\omega\) with \(\frac{\omega}{2\pi i}\) in the de Rham class of \(c_1(L)\) can be realized as the curvature of some unitary connection on \(L\). We call such an \(L\) a \(\text{spin}^c\) line bundle. Now we define what we mean by \(\text{spin}^c\) surgery.

**Definition 3.4.** Let \((M, L)\) be a closed \(\text{spin}^c\) manifold (i.e., \(M\) is a closed oriented manifold and \(L\) is a complex line bundle on \(M\) with \(c_1(L)\) reducing
mod 2 to $w_2(M)$). We say that $(M', L')$ be obtained from $(M, L)$ by spin$^c$ surgery in codimension $k$ if there is a sphere $S^{n-k}$ embedded in $M$ with trivial normal bundle, $M'$ is the result of gluing in $D^{n-k+1} \times S^{k-1}$ in place of $S^{n-k} \times D^k$, and there is a spin$^c$ line bundle $\mathcal{L}$ on the trace of the surgery, a bordism $(W, \mathcal{L})$: $(M, L) \rightsquigarrow (M'/L')$, such that $\mathcal{L}$ restricts to $L$ on $M$ and to $-L'$ on $M'$ respectively.

**Theorem 3.5** (Spin$^c$ surgery theorem, [7 Theorem 4.2]). Let $(M, L)$ be a closed $n$-dimensional spin$^c$ manifold. Assume that $M$ admits a Riemannian metric $g$ and $L$ admits a hermitian bundle metric $h$ and a unitary connection $A$ such that $\kappa^L_g > 0$. Let $(M', L')$ be obtained from $(M, L)$ by spin$^c$ surgery in codimension $k \geq 3$. Then there is a metric $g'$ on $M'$, and $L'$ admits a hermitian bundle metric $h'$ and a unitary connection $A'$, such that $\kappa^L_{g'} > 0$.

This leads to the following spin$^c$ bordism theorem:

**Theorem 3.6** (Spin$^c$ bordism theorem, [7 Theorem 4.3]). Let $(M, L)$ be a connected closed $n$-dimensional spin$^c$ manifold which is not spin. Assume that $M$ is simply connected and that $n \geq 5$. Also assume that there exists a pair $(M', L')$ in the same bordism class in $\Omega^\text{spin}^c_n$ with a metric $g'$ on $M'$, a hermitian metric $h'$ and a unitary connection $A'$ on $L'$ such that $\kappa^L_{g'} > 0$. Then $M$ admits a Riemannian metric $g$ and $L$ admits a hermitian bundle metric $h$ and a unitary connection $A$ such that $\kappa^L_g > 0$.

**Remark 3.7.** We emphasize that the condition that $(M, L)$ is spin$^c$, but not spin, is essential: one should not assume that if $(M, L)$ is spin$^c$ and $\alpha^c(M, L) = 0$, then one can choose a metric on $M$ and a hermitian metric and connection $A$ on $L$ so that $\kappa^L_g > 0$, for this is false. Indeed, suppose $M$ is actually spin and dim $M$ is 1 or 2 mod 8 with $\alpha(M) \neq 0$, so there is no psc-metric $g$ on $M$. Adding in the term $\mathcal{R}_L$ in this case only makes things worse, because in suitable coordinates, $\mathcal{R}_L$ has the form $\begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$, where the operator $\omega$ is constructed from the curvature of $L$, which can be any exact 2-form on $M$, so $\kappa^L_g$ cannot be strictly positive in this case unless the scalar curvature $\kappa_g$ is strictly positive, which is impossible.

Even though vanishing of the index $\alpha^c(M, L)$ does not guarantee that $\kappa^L_g > 0$ for given spin$^c$-manifold $(M, L)$, we prove that there is some representative in the same bordism class which has psc-metric $g$ with $\kappa^L_g > 0$ for an appropriate choice of bundle data.

**Theorem 3.8** ([7 Corollary 5.2]). Let $(M, L)$ be a simply connected spin$^c$ manifold with $\alpha^\text{spin}^c(M, L) = 0$ in $KU_n$. Then after changing $(M, L)$ up to spin$^c$ cobordism, we can assume that $M$ admits a Riemannian psc-metric $g$ and the line bundle $L$ over $M$ defining the spin$^c$ structure admits a hermitian metric $h$ and a connection $A$ such that $\kappa^L_g > 0$.

We notice that we do not have a dimensional restriction here; this is because “changing $(M, L)$ up to spin$^c$ cobordism” makes the problem of
Finding an appropriate psc metric and bundle data very flexible. On the other hand, the following more elegant result holds for non-spin spin\(^c\) manifolds:

**Theorem 3.9.** Let \((M, L)\) be a simply connected non spin spin\(^c\) manifold with \(\alpha^{\text{spin}^c}(M, L) = 0\) in \(KU_n\) with \(n = \dim M \geq 5\). Then \(M\) admits a Riemannian psc-metric \(g\), a hermitian metric \(h\) and a connection \(A_L\) such that \(\kappa_L^g > 0\).

Proofs of Theorems 3.8 and 3.9 are based on the above bordism Theorem and studying the kernel of the index homomorphism \(\alpha^{\text{spin}^c} : \Omega^{\text{spin}^c}_n \to KU_n\).

To explain the idea, we first recall some basic facts about spin\(^c\) bordism, see [28, Chapter XI], [24, §8], and [18]. In particular, we have an isomorphism of bordism groups

\[\Omega^{\text{spin}^c}_n \cong \tilde{\Omega}^{\text{spin}}_{n+2}(\mathbb{C}P^\infty),\]

where \(\tilde{\Omega}^{\text{spin}}_{n+2}(\mathbb{C}P^\infty)\) is the reduced bordism group. Next, classes in spin\(^c\) bordism are detected by their Stiefel Whitney numbers (which are constrained just by the Wu relations and the vanishing of \(w_1\) and \(w_3\)) and integral cohomology characteristic numbers (where in addition to the Pontryagin classes, one can use powers of \(c_1\) of the line bundle defining the spin\(^c\) structure) [28, Theorem, p. 337]. We do not need to state all these results, however, we need a few examples.

We notice that the bordism class can change, depending on the choice of spin\(^c\) structure. Thus, for example, \(\Omega^2_{\text{spin}^c} \cong \mathbb{Z}\), with all classes represented by \((\mathbb{C}P^1, L)\), \(L\) a complex line bundle with \(c_1(L)\) even, and the isomorphism to \(\mathbb{Z}\) is given by

\[(\mathbb{C}P^1, L) \mapsto \frac{1}{2}(c_1(L), [\mathbb{C}P^1]).\]

Similarly, \(\Omega^4_{\text{spin}^c} \cong \mathbb{Z}^2\), with one generator given by \((\mathbb{C}P^1, O(2))^2\), with \(\alpha^{\text{spin}^c}(\mathbb{C}P^1, O(2))^2 = 1\), and the other generator given by \((\mathbb{C}P^2, O(1))\), where \(c_1\) of the anticanonical bundle \(O(1)\) is the standard generator \(x\) of \(H^2(\mathbb{C}P^2; \mathbb{Z})\), on which \(\alpha^{\text{spin}^c}\) takes the value 0. The calculation of \(\alpha^{\text{spin}^c}\) on this generator is worked out by Hattori [16]

\[\alpha^c(\mathbb{C}P^2, O(1)) = \text{ind} \partial_{\mathbb{C}P^2, O(1)} = \langle \tilde{\mathcal{A}}(\mathbb{C}P^2) e^{x/2}, [\mathbb{C}P^2] \rangle = (1 - \frac{1}{8}x^2)(1 + \frac{1}{2}x + \frac{1}{2}x^2)\]

by the Atiyah-Singer Theorem [21, Theorem D.15, p. 399].

This last example turns out to be crucial, because there is a sense in which \(\mathbb{C}P^2\) with the bundle \(O(1)\), the dual of the tautological bundle, generates the kernel of \(\alpha^c\). In more detail, we use \((\mathbb{C}P^2, O(1))\) to construct a transfer map

\[T^{\text{spin}^c} : \Omega^{\text{spin}^c}_n(BG) \to \Omega^{\text{spin}^c}_{n+4},\]

where \(G\) is the Lie group \(SU(3)\), as follows. The group \(SU(3)\) acts transitively on \(\mathbb{C}P^2 \cong G/H\), where \(H = S(U(2) \times U(1))\), preserving the class of the
bundle $O(1)$. In particular, we obtain a fiber bundle $p : BH \to BG$ with a fiber $\mathbb{CP}^2$ and the structure group $SU(3)$.

In fact, the bundle $p : BH \to BG$ is a universal geometrical $\mathbb{CP}^2$-bundle for all $\mathbb{CP}^2$-bundles with the structure group $SU(3)$. Thus given a spin$^c$ manifold $(M, L)$ and a map $f : M \to BG$, we can form the associated $\mathbb{CP}^2$ bundle $\hat{p} : E \to M$ as a pull-back:

$$
\begin{array}{ccc}
E & \xrightarrow{j} & BH \\
\downarrow{\hat{p}} & & \downarrow{p} \\
M & \xrightarrow{f} & BG
\end{array}
$$

where $E = M \times f \mathbb{CP}^2$ has dimension $n + 4$ and has a spin$^c$ structure inherited from the spin$^c$ structure on $M$ defined by $L$ and the spin$^c$ structure on $\mathbb{CP}^2$ defined by the bundle $O(1)$.

There is another transfer map introduced by Stolz in [26] and [27]. This is defined similarly, but with $G = SU(3)$ replaced by $PSp(3)$, $H = S(U(2) \times U(1))$ replaced by $P(Sp(2) \times Sp(1))$, and $\mathbb{CP}^2$ replaced by $\mathbb{HP}^2$. One obtains a transfer map

$$
T^{\text{spin}} : \Omega^{\text{spin}}_{n-4}(BPSp(3)) \to \Omega^{\text{spin}}_n.
$$

Here is a main technical result we need:

**Theorem 3.10.** The above transfer maps

$$
T^{\text{spin}} : \Omega^{\text{spin}}_n(BSU(3)) \to \Omega^{\text{spin}}_n \quad \text{and} \quad T^{\text{spin}} : \Omega^{\text{spin}}_n(BSp(3)) \to \Omega^{\text{spin}}_n
$$

are such that $\langle \text{Im}(T^{\text{spin}}) \cup \text{Im}(T^{\text{spin}}) \rangle = \text{Ker} \alpha^{\text{spin}}$ as abelian groups.

The proof of Theorem 3.10 requires some computations with spin$^c$-bordism groups, see [2, section 5]. Then Theorem 3.10 implies that a spin$^c$-manifold $(M, L)$ spin$^c$-bordant to a union of total spaces $E$ of geometric $\mathbb{CP}^2$- and $\mathbb{HP}^2$-bundles $E \to B$.

These cases are slightly different. Consider geometric $\mathbb{CP}^2$-bundles first. We start with a trivial bundle, the spin$^c$-manifold $(\mathbb{CP}^2, O(1))$. Here we use the Fubini-Study metric $g_{FS}$ along with the usual connection on the dual of the tautological bundle. Then if $\omega$ is the Kähler form, this is also the curvature of the bundle $O(1)$ and the Ricci tensor is 6 times the metric. It is well-known that $\frac{1}{2} \kappa_{FS} = 6$, and the minimal eigenvalue of $\mathcal{R}$ is $-2$, so $\frac{1}{2} \kappa_{O(1)} = \frac{1}{2} \kappa_{FS} + \mathcal{R} \geq 6 - 2 > 0$.

Recall that $SU(3)$ acts by isometries of the standard Fubini-Study metric. Then a total space $E$ of a geometric $\mathbb{CP}^2$-bundles $E \to B$ has a canonical line bundle $L \to E$ which restricts to the bundle $O(1)$ on the fibers. Then by choosing the metric and connection so that on each fiber, we have a very small multiple of the Fubini-Study metric $g_{FS}$ and the curvature of the line bundle is the Kähler form, the curvature of the fibers will dominate everything else.
Let \( E \to B \) be a geometric \( \mathbb{HP}^2 \)-bundle over a spin\(^c\)-manifold \( B \), where the structure group \( \text{PSp}(3) \) is an isometry group of the standard metric \( g_{\mathbb{HP}^2} \) (of positive curvature). Then we can rescale the metric \( g_{\mathbb{HP}^2} \) so that its scalar curvature will dominate everything else. These arguments prove Theorem 3.8. To prove Theorem 3.9 we have to analyze the image of \( \Omega_n^{\text{spin}} \) in the group \( \Omega_n^{\text{spin}^c} \).

### 3.3. Finding a well-adapted psc-metric on a simply-connected spin\(^c\)-manifold.

Let \( (M, L) \) be a spin\(^c\)-manifold, where \( M = X \cup_{\partial X = Y} N(B) \) and \( B \) is dual to \( L \). We have identified two primary obstructions

\[
\alpha^{\text{spin}^c}(M, L) \in KU_n \quad \text{and} \quad \alpha(B) \in KO_{n-2},
\]

and a secondary obstruction \( \alpha^{\text{rel}}(X, Y, g_Y) \in KO_n \) for existence of a well-adapted psc-metric on spin\(^c\)-manifold \( (M, L) \). The secondary obstruction \( \alpha^{\text{rel}}(X, Y, g_Y) \) depends on a choice of a psc-metric \( g_Y \) on \( Y \) which is determined by a choice of a psc-metric \( g_B \) on \( B \). Then when we say that \( \alpha^{\text{rel}}(X, Y, g_Y) \) vanishes, we mean that there exists a psc-metric \( g_B \) which determines the metric \( g_Y \) so that \( \alpha^{\text{rel}}(X, Y, g_Y) = 0 \) in \( KO_n \).

We emphasize that if \( B \) is simply-connected with \( \dim B \geq 5 \), then if \( \alpha(B) \in KO_{n-2} \) vanishes, then there exists a psc-metric \( g_B \) on \( B \).

**Theorem 3.11.** Let \( (M, L) \) be a non-spin spin\(^c\)-manifold, where \( M = X \cup_{\partial X = Y} N(B) \) and \( B \) is dual to the line bundle \( L \), where \( M \) and \( B \) are simply-connected and \( \dim M = n \geq 7 \). Assume that the primary obstructions \( \alpha^{\text{spin}^c}(M, L) \in KU_n \) and \( \alpha(B) \in KO_{n-2} \) vanish and the secondary obstruction \( \alpha^{\text{rel}}(X, Y, g_Y) = 0 \) for some choice of a metric \( g_B \) (and, consequently of \( g_Y \)). Then \( (M, L) \) admits a well-adapted psc-metric.

**Proof.** Since \( \alpha^{\text{spin}^c}(M, L) \) vanishes in \( KU_n \), Theorem 3.9 says that \( M \) admits a psc-metric \( g \) so that \( \kappa^L_g \) is positive definite. This by itself is not good enough, since this metric may not be well-adapted with respect to \( B \).

However an analysis of relevant bordism groups shows that \( (M, L) \) is spin\(^c\)-bordant to a disjoint union \( M'' \sqcup (M', L') \) in the following sense: the manifold \( M'' \) is a closed spin manifold, and \( (M', L') \) is a spin\(^c\) pair, such that \( L' \) is trivial away from another closed spin manifold \( B' \), and

\[
(B, L|_B) \sim (B', L'|_{B'}) \quad \text{in} \quad \Omega^{\text{spin}}_{n-2}(\mathbb{CP}^{\infty})
\]

while \( M \sim M'' \sqcup M' \) in \( \Omega^{\text{spin}^c}_n \). We can also take \( M' \) and \( M'' \) to be simply connected. Then, since the \( \alpha \) invariants only depend on spin/spin\(^c\) bordism classes and are linear on the bordism groups, \( \alpha^{\text{spin}^c}(M', L') = 0 \) and \( \alpha(M'') = 0 \). Then we can construct \( M' \) so that it has a well-adapted psc-metric, by a slight refinement of Theorem 3.9. Also, \( M'' \) has a psc-metric by Stolz's Theorem. Putting everything together, we can push the well-adapted psc-metric on \( M'' \sqcup M' \) through the bordism to get a well-adapted psc-metric on \( M \), using the Gromov-Lawson surgery technique. First, we have to do codimension 3 surgeries on \( B' \) to convert it to \( B \), and use these surgeries
to push the metric on the tubular neighborhood. Then do surgeries on the interior to push the psc-metric from $X'$ to $X$. □

4. More General Manifolds with Singularities

The situations discussed in sections 2 and 3, as well as the paper [4], lead to a more general subject, the classification of “manifolds with singularities” or singular spaces admitting a psc-metric. Here by “manifolds with singularities” we mean compact Hausdorff spaces with a stratification, where the strata are locally closed subspaces (hence locally compact) which are themselves smooth manifolds, and usually one adds a condition on the local structure of a neighborhood of each stratum in a larger one. Various categories of manifolds with singularities are discussed in some detail in the book [29]. The prototypes of singular spaces, which are good examples to keep in mind, are either projective varieties over $\mathbb{R}$ or $\mathbb{C}$ which are not necessarily smooth, or else quotient spaces $M/G$ of a smooth manifold by a smooth action of a Lie group $G$, in the case where there can be more than one orbit type. Here we will simplify the discussion by restricting attention to the case of only two strata. Thus if $M$ is such a singular space, $M$ has a dense open subset $\tilde{M}$ which is a smooth manifold, and $M \setminus \tilde{M} = \beta M$ is a closed manifold of smaller dimension (possibly disconnected). The reasons for the local neighborhood condition are:

1. Such conditions hold in the two kinds of prototypes: algebraic varieties and quotients of smooth actions.

2. Without such a condition one can have very wild examples. For example, take a smooth manifold $M$ and collapse some closed subset $X$ to a point. The quotient space $M/X$ is the union of the open manifold $\tilde{M} = M \setminus X$ and a point, but if $X$ is pathological, the neighborhood of the singular point can be very complicated.

In the rest of this section we will consider singular manifolds (more precisely, “pseudomanifolds”) with exactly two strata, $\tilde{M}$ and $M \setminus \tilde{M} = \beta M$, such that $\beta M$ has a tubular neighborhood in $M$ homeomorphic to a fiber bundle $c(L) \to N(\beta M) \to \beta M$, where the fibers are the cone $c(L) = L \times [0, 1]/(L \times \{0\})$ on a fixed closed Riemannian manifold $(L, g_L)$, called the “link,” and the structure group of the bundle is contained in the isometry group of $(L, g_L)$ (extended to act on $c(L)$ so as to preserve the distance to the cone point). The fiber bundle has a natural section embedding $\beta M$ in $N(\beta M)$ as the union of the “vertex points” of the cone fibers. Unless $L$ is a sphere, which was the case in sections 2 and 3 where we had $L = S^0$ in section 2 and $L = S^1$ in section 3, such a pseudomanifold is generally not even homeomorphic to a topological manifold (let alone to a smooth manifold), so it certainly doesn’t admit Riemannian metrics in the usual sense. So if $L$ is not a sphere, what do we mean by a psc-metric on $M$? Extrapolating from the cases we have discussed in sections 2 and 3 we restrict attention
to “well-adapted” metrics with respect to the neighborhood structure near the singular stratum. That means we impose the following conditions.

**Definition 4.1.** Let $M^n$ be a stratified (compact) pseudomanifold with two strata $M$ and $M \setminus M = \beta M$ as above, with $\beta M$ having a tubular neighborhood $N(\beta M)$ which is a $c(L)$-bundle over $\beta M$, for a fixed link $L$. Recall that $M$ is a smooth $n$-manifold and that $\beta M$ is a closed manifold of smaller dimension. A *well-adapted metric* on $M$ will mean the following:

1. A choice of a Riemannian metric $g$ on $\hat{M}$ and of a Riemannian metric $g_\beta$ on $\beta M$, such that:
2. the restriction of $g$ to $X = M \setminus \text{int} N(\beta M)$ is a product metric $g_{\partial X} + dt^2$ in a collar neighborhood of $\partial X$,
3. the map $p: (\partial X, g_{\partial X}) \to (\beta M, g_\beta)$ is a Riemannian submersion with the given metric on $L$ as the “vertical metric” on the fibers, and
4. in a slightly smaller neighborhood of $\beta M$, with fibers $L \times [0, 1 - \varepsilon]/(L \times \{0\})$, $g$ has the local form $dr^2 + r^2 g_L + p^* g_\beta$, where $r$ is the distance to the vertex of the cone in $c(L)$.

One can easily see that Definition 4.1 specializes to Definition 1.4 when $L = S^0$.

**Definition 4.2.** If $M^n$ is a stratified pseudomanifold with two strata as in Definition 4.1, we say that a *well-adapted psc-metric* on $M$ is a well-adapted metric on $M$ in the sense of Definition 4.1, such that $g$ and $g_\beta$ are psc-metrics on $\hat{M}$ and $\beta M$, respectively. Again this agrees with our earlier terminology.

The basic question we want to study is this:

**Question 4.3.** Suppose $M^n$ is a (compact) pseudomanifold with two strata as above. Clearly, if $M$ admits a well-adapted psc-metric, then the closed manifold $\beta M$ admits a psc-metric. What additional conditions are needed to ensure the converse, at least if $n$ is sufficiently large?

We have studied this question in [7, 8, 9]. Basically, in order to get any good results on this question, we want the link manifold $L$ to be a homogeneous space for a compact Lie group $G$, with the metric on $L$ to be $G$-invariant. Such homogeneous spaces always have nonpositive curvature, and there are two rather different cases to consider: the case where $G$ is a torus, in which case $L$ itself is necessarily a torus and we might as well take $G = L$ with a flat invariant metric, or the case where $G$ is compact semisimple, in which case $L$ is a manifold of $G$-invariant positive sectional curvature. The prototype of the first case is the case where $G = L = S^1$, treated in section 8 above and in more detail in [7]. In this case, since the cone on a circle is a disk, our pseudomanifold $M$ is actually a smooth manifold, and a well-adapted psc-metric on $M$ is in particular a psc-metric on $M$ in the usual sense for smooth manifolds. If $M$ has a spin structure then $\alpha(M)$ is an obstruction to such a metric, in addition to whatever obstruction there might be to a psc-metric on $\beta M$. However, if we decompose $M$ as
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X \cup_{\partial X} N(\beta M) as above, and if \partial X and \beta M are both simply connected, then from the Gysin sequence and long exact homotopy sequence of the circle bundle \( S^1 \to \partial X \to \beta M \), one can see that \( c_1 \) of the circle bundle (which one can identify with \( c_1 \) of a complex line bundle for which \( N(\beta) \) is the unit disk bundle) has to be non-zero mod 2, and thus \( \beta M \) and \( M \) cannot be spin simultaneously. But if can happen, for example, that \( M \) is spin and \( \beta M \) is spin. An example given in [7, Remark 3.6] is the case where \( M = \mathbb{CP}^5 \# \Sigma^{10} \), where \( \Sigma^{10} \) is a homotopy sphere with \( \alpha(\Sigma^{10}) \neq 0 \) in \( KO_{10} \cong \mathbb{Z}/2 \), and \( \beta M = \mathbb{CP}^4 \), which obviously admits a psc-metric. In this case, the \( \alpha \)-invariant shows that \( M \) does not admit any psc-metric, let alone a well-adapted one.

A case not discussed in [7] is the case where \( L \) is a higher-dimensional torus. In this case, the “singular manifold” is genuinely singular, since the cone on \( L \) is not locally Euclidean. In this case, well-adapted psc-metrics never exist, because of the following calculation:

**Lemma 4.4** ([8, Lemma 3.1]). The scalar curvature function on the cone \( c(L) \) on \( L \) with the conical metric \( dr^2 + r^2 g_L \) (with the vertex \( r = 0 \) of the cone deleted) is \( (\kappa_L - \kappa_\ell) r^{-2} \), where \( \kappa_L \) is the scalar curvature of \( L \) and \( \kappa_\ell \) is the scalar curvature of a standard round sphere \( S^\ell(1) \) of radius 1, \( \ell = \dim L \).

**Corollary 4.5.** If \( L \) is flat and \( \dim L > 1 \), then the scalar curvature of \( c(L) \) tends to \(-\infty\) as \( r \to 0 \), and thus a well-adapted psc-metric is impossible.

**Proof.** Just take \( \kappa_L = 0 \) and \( \kappa_\ell = \ell(\ell - 1) \) in the above formula. For the application to well-adapted metrics, observe that such a metric is locally a Riemannian product of \( c(L) \) and \( \beta M \) up to small corrections coming from the curvature of the fiber bundle, and so there is no way to overcome the hugely negative scalar curvature on the cone. \( \square \)

The results in [8, 9] have to do with the other case where \( L \) is a homogeneous space of a compact semisimple Lie group \( G \), and the bundle \( p: \partial X \to \beta M \) is an associated bundle \( P \times_G L \) to a principal \( G \)-bundle \( G \to P \to \beta M \). Because of Lemma 4.4 we take the scalar curvature of \( L \) to be equal to \( \kappa_\ell = \ell(\ell - 1) \) (if you like, this is a normalization of the cone angle) so that the cone \( c(L) \) is scalar-flat. This ensures that if \( \beta M \) has a psc-metric, we can lift this metric to a well-adapted psc-metric on the tubular neighborhood \( N(\beta M) \). This follows from an application of the O’Neill formulas for the curvature of a Riemannian submersion.

**Proposition 4.6** ([8, Theorem 3.5]). Let \( L = G/H \) be a homogeneous space, \( \dim L = \ell \), where \( G \) is a connected compact semisimple Lie group, and \( g_L \) be a \( G \)-invariant Riemannian metric on \( L \) of constant scalar curvature equal to \( \kappa_\ell = \ell(\ell - 1) \). Let \( M = X \cup_{\partial X} N(\beta M) \) be a compact pseudomanifold of dimension \( n \) with two strata, \( \tilde{M} \cong \text{int} X \) and \( \beta M \), where \( \partial X = P \times_G L \), a principal \( G \)-bundle over \( \beta M \). If \( M \) admits a well-adapted metric of positive scalar curvature, then \( \beta M \) admits a psc-metric.
When $X$ and $\beta M$ are both spin manifolds, this gives us two obstructions to a well-adapted psc-metric, the $\alpha$-invariant of $\beta M$ (or its “higher” generalization if $\beta M$ is not simply connected), and the secondary invariant $\alpha^{rel}(X, \partial X, g_{\partial X})$, which depends on the choice of a psc-metric on $\beta M$ (or equivalently, on the Riemannian submersion metric on $\partial X$).

In the papers [8, 9], we were able to show that the vanishing of these obstructions is sometimes sufficient for $M$ to admit a well-adapted psc-metric. Not only that, but in some cases we are able to obtain information on the topology of space of well-adapted psc-metrics on $M$. For details, we refer the reader to those other papers, but here we just give an indication of some of the key techniques.

The main method for proving existence of well-adapted psc-metrics is to introduce a suitable notion of (spin) bordism for pseudomanifolds in the appropriate category, giving rise to an exact sequence of bordism groups of the form

$$
\cdots \rightarrow \Omega_{n}^{\text{spin}} \xrightarrow{i} \Omega_{n}^{\text{spin},(L,G)-\text{fb}} \xrightarrow{\beta} \Omega_{n-\ell-1}^{\text{spin}}(BG) \xrightarrow{t} \Omega_{n-1}^{\text{spin}} \rightarrow \cdots.
$$

Here $\Omega_{n}^{\text{spin},(L,G)-\text{fb}}$ is the bordism group of $n$-dimensional spin pseudomanifolds with $(L,G)$-fibered singularities, i.e., with the structure we have been talking about, where the fibration $\partial X \rightarrow \beta M$ comes from a principal $G$-bundle over $\beta M$. The “inclusion” map $i$ comes from viewing a closed spin manifold as such a pseudomanifold with empty singularities, and the “Bockstein” map $\beta$ sends $(M, X, \partial X \rightarrow \beta M)$ to the bordism class of $\beta M$ with its map to $BG$ classifying the principal $G$-bundle that defines the $(L,G)$-fibered singularity structure. The “transfer map” $t$ sends the bordism class of $\beta M \rightarrow BG$ to the total space of the associated $L$-bundle. The exact sequence (7), along with its generalization to the case where the manifolds are not simply connected, along with the surgery method of Gromov and Lawson, is the main technical tool for proving positive results.

Here is the statement of the main theorem of [8]:

**Theorem 4.7 (Existence Theorem [8, Theorem 1.2]).** Let $M = X \cup_{\partial X} N(\beta M)$ be an $n$-dimensional compact pseudomanifold with $X$ and $\beta M$ spin and simply connected. Assume that the fiber bundle $\phi: \partial M \rightarrow \beta M$ has fiber $L = G/H$ and is the associated bundle to a principal $G$-bundle over $\beta M$, where $G$ is a simply connected compact Lie group. Furthermore, assume $n \geq \ell + 6$, where $\ell = \dim L$, and that one of the following condition holds:

1. either $L$ is a spin $G$-boundary of a manifold $\bar{L}$ equipped with a psc-metric $g_{L}$, which is a product near the boundary and satisfies $g_{L}|L = g_{L}$;

2. or $\partial M = \beta M \times L$, where $L$ is an even quaternionic projective space.

Then the vanishing of the primary and secondary obstruction invariants $\alpha(\beta M) \in KO_{n-\ell-1}$ and $\alpha^{rel}(X, \partial X, g_{\partial X}) \in KO_{n}$ implies that $M$ admits an adapted psc-metric.
Case (i) of this theorem holds if $L = G$ or if $L$ is a sphere or an even complex projective space. In these cases, the idea of the proof is to show that $M$ can be obtained via surgery from the disjoint union of a psc spin manifold and the result of replacing $X$ by a $L$-bundle over $\beta M$, which clearly has a well-adapted psc-metric. Case (ii) uses a different idea; in this case, since even quaternionic projective spaces are not zero-divisors in $\Omega_{\text{spin}}^{n}$, the exactness of $[7]$ forces $[\beta M \to BG]$ to vanish in $\Omega_{\text{spin}}^{n-\ell-1}(BG)$, and this can be used with the surgery technique to show that $M$ admits a well-adapted psc-metric.

The sequel paper [9] contains a generalization of Theorem 4.7 to the non-simply connected case, as well as some results of the topology of the space of well-adapted psc-metrics when this space is non-empty. One interesting result along these lines is the following, which is nontrivial even in the simply connected case.

**Theorem 4.8** ([9 Theorem 1.5]). Let $M = X \cup_{\partial X} N(\beta M)$ be an $n$-dimensional compact pseudomanifold. Assume that the fiber bundle $\phi: \partial M \to \beta M$ has fiber $L = G/H$ and is the associated bundle to a principal $G$-bundle over $\beta M$, and assume $M$ admits a well-adapted psc-metric. Then the homotopy groups of the space of psc-metrics on $\beta M$ inject into the homotopy groups of the space of well-adapted psc-metrics on $M$.

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