Fermionization and bosonization of expanding 1D anyonic fluids

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The momentum distribution of an expanding cloud of one-dimensional hard-core anyons is studied by an exact numerical approach, and shown to become indistinguishable from that of a non-interacting spin-polarized Fermi gas for large enough times (dynamical fermionization). We also consider the expansion of one-dimensional anyons with strongly attractive short-range interactions suddenly released from a parabolic external potential, and find that momentum distribution approaches that of its dual system, the ideal Bose gas (dynamical bosonization). For both processes the characteristic time scales are identified, and the effect of the initial confinement is analyzed comparing the dynamics associated with both harmonic and hard-wall traps.

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The recent realization of ultracold atoms in tight waveguides has spurred extensive studies of one-dimensional (1D) systems exhibiting Bose-Fermi duality 1. The inextricable link between interactions and exchange statistics in 1D allows to relate the Bose gas with δ-interactions with spin-aligned attractive Fermi gas with inverse coupling constant. The upshot is that, in spite of the different symmetry of the many-body wavefunction, dual systems share many of its properties, including any local correlation function.

Paradigmatic examples are the 1D gas of spin-polarized fermions and impenetrable bosons (the so-called Tonks-Girardeau (TG) gas 2) which have been prepared in the laboratory 3. Similarly, the duality between non-interacting bosons and spin-polarized fermions with strongly attractive odd-wave interactions (fermionic Tonks-Girardeau (FTG) gas 4) has recently been explored. A staggering manifestation of the BF duality is the dynamical fermionization in which the momentum distribution of the TG gas approaches during a 1D free expansion that of non-interacting fermions 5, 6. The reverse phenomenon has been predicted for FTG under the same dynamics, in which asymptotically the momentum distribution of the ideal Bose gas is exhibited 7. Far from being limited to the Bose-Fermi mappings, the duality in lower dimension extends to particles with fractional statistics, whose realization with ultracold atoms has recently been proposed 8. Motivated by its application to transport in quantum Hall fluids, and the possibility of performing universal quantum computation 2, the properties of 1D anyons are under current scrutiny. Particularly, several studies have been devoted to 1D anyons with short-range delta interactions (anyonic Lieb-Liniger model 9, 10, 11, 12, 13, 14) and the limiting cases of impenetrable anyons 15, 16, 17, 18, though much less is known about its dynamics.

In this letter, we shall consider the anyonic generalization of TG and FTG gases, and show that they do undergo dynamical “dualization” during their free time evolution. Hard-core anyonic gas. Strongly δ-interacting anyons, at low enough densities may reach the so-called Tonks-Girardeau regime in which the particles become impenetrable. We assume that the gas of hard-core anyons (HCA) is initially confined in a harmonic trap of frequency ωn, whose eigenstates are denoted by φn(x).

The description in such regime is simplified introducing a dual system, a spin polarized Fermi gas, whose ground-state wavefunction is a normalized Slater determinant ΨF(x1, . . . , xN) = 1/(det[(N−1)N]) ϕn(x1) .

The wavefunction of the ATG gas is obtained from ΨF by imposing the correct symmetry under permutation of particle using the anyon-fermion (AF) mapping 19. ΨHCA(x1, . . . , xN) = Aθ|φ 1(−x1, . . . , −xN)ΨF(x1, . . . , xN), where Aθ = N j<k eδ/2[ψ(x− xj)] is the one-parameter family of unitary operators which generalizes the well-known Bose-Fermi map 2 and ε(x) = 1 (−1) if x > 0 (< 0) and ε(0) = 0. Varying the statistical parameter θ, the map smoothly extrapolates between spin-polarized fermions (θ = 0), and the well-known bosonic Tonks-Girardeau gas (θ = π) 10. Note that A−θ = Aθ = Aθ−1 and the expectation values coincide ⟨ΨHCA, ΨHCA⟩ = ⟨ΨF, ΨF⟩, which imply that both dual systems share the same density profile 15 ρθ(x, t) = N ∫ |Ψ(x, x2, . . . , xN; t)|2 dx2 . . . dxN = ∑n=0N−1 ϕn(x, t)2. However, its momentum distribution n(k) = (2π)−1 ∫ dzdy eik(x−y)ρ(x, y) is drastically different. For the spin polarized fermions, described by an Slater determinant wavefunction, the reduced single-particle density matrix (RSPDM) simply reads ρF(x, y) = N ∫ |ΨF(x, x2, . . . , xN)|2 dx2 . . . dxN = ∑n=0N−1 ϕn(x, y)2 (normalized to N), whence it follows that nF(k) = ∑n=0N−1 |ϕn(k)|2 (the tilde denoting Fourier transform). For the HCA gas, making use explicitly of the anyon-Fermi map, the orthonormality of the single-particle eigenstates, and the Laplace expansion of the determinant in the dual wavefunction, it is found...
that the RSPDM can be efficiently computed as
\[ \rho_{HCA}(x, y) = \frac{1}{(2\pi\hbar)^N} \int_{\mathcal{D}} \cdots \int_{\mathcal{D}} \prod_{n=1}^{N} \phi_n^*(x_n) \phi_n(y) \prod_{i=1}^{N} A_{i\alpha}(x_i, y) d^4x_i, \]
where \( A(x, y) = (P^{-1})^T \text{det} P \) and the elements of the matrix \( P(x, y) \) are
\[ P_{ln} = \delta_{ln} - (1 - e^{-i\theta(x-y)} \epsilon(y-x)) \int_x^y dz \phi_n^*(z) \phi_n(z), \]
a result which holds under time-evolution. This generalizes for any anyonic parameter, the result recently obtained by Pezer and Buljan \cite{20} for the Tonks-Girardeau gas \( \theta = \pi \). As shown in Fig. 1 an asymmetric momentum distribution results for any statistical parameter other than \( \theta = 0, \pi \), \cite{14, 18}. Moreover, for \( \theta = 0 \), \( P_{ln} = \delta_{ln} \) so the RSPDM becomes diagonal, and the familiar result for spin-polarized fermions is recovered. In what follows we shall be interested in the expansion dynamics after suddenly switching off the confining potential at \( t = 0 \). We find that, as the time of evolution goes by, the momentum distribution of the HCA gas approaches that of its dual system, the spin-polarized ideal Fermi gas.

Provided that \( A_0 \) is time-independent (the quantum statistics is preserved under time-evolution), the expansion dynamics of the manybody wavefunction is found by using the anyon-fermion mapping. The self-similar evolution of the single-particle states according to the time-dependent Schrödinger equation, \( i\hbar \frac{\partial \phi_n(x, t)}{\partial t} = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{\omega_0^2 x^2}{2} \right] \phi_n(x, t) + \Theta(t) \), with \( \Theta(t) \) being the Heaviside step function, can be found exploiting the scaling law, \( \phi_n(x, t) \sim \phi_n(x/b(t), 0) e^{i\pi x^2/2\hbar - iE_n \tau(t)/\hbar} \sqrt{b(t)} \) where the scaling factor \( b(t) \) is the solution of the differential equation \( \dot{b} + \omega^2(t) b = \omega_0^2 / b^3 \) satisfying \( b(0) = 1 \) and \( b(0) = 0 \), \( E_n = \hbar \omega_0(n+1/2) \), and \( \tau(t) = \int_0^t dt' / b^2(t') \), \cite{21}.

For \( \omega(t) = \omega_0 \sqrt{1 + \omega_0^2 t^2} \) it follows that \( b(t) = \sqrt{1 + \omega_0^2 t^2} \), and therefore, for all \( t \gg \omega_0^{-1} \) the expansion becomes ballistic \cite{6, 22}. During the expansion a dynamical fermionization occurs, see Fig. 2 by means of which the momentum distribution tends to that of the non-interacting spin-polarized Fermi gas, \( n_F(k) = \sum_{n=0}^{N} |\phi_n(k)|^2 \) where \( \phi_n(k) = (-1)^n (2x^2_0/\pi)^{1/4} e^{-k^2 x_0^2/2} H_n(\sqrt{2k} x_0) \), \( H_n \) are the Hermite polynomials and \( x_0 = \sqrt{\hbar / m \omega_0} \). Indeed, the stationary phase method (SPM) allows to find the asymptotic form of the momentum distribution for \( t \to \infty \), \( n_{\text{HCA}}(k) \sim |\omega_0/b| n_F(\omega_0 k/b) \sim n_F(k) \). Intuitively, as the gas expands, the particles stop to interact, and asymptotically one find the distribution of quasi-momenta, which are integrals of motion. The fermionization time scale can be approximated as \( t_F(\theta) \approx \sin(\theta/2)/N \omega_0 \), dependent on the statistical parameter \( \theta \) and decreasing with the number of particles \( N \).

Expansion from a box. The scaling law is specific of the harmonic confinement and we may rightly wonder how the subsequent time-evolution differs from the expansion from other type of traps. In what follows we shall compare the previous results with the expansion from a box-like trap \cite{24}. The main difference is that at variance with the parabolic potential the expansion from a hard-wall confinement is not self-similar, as the space-time dynamics already exhibits a rich transient structure at the single-particle level. For a hard-wall confinement of width \( L \), the well-known orthonormal eigenstates read \( \varphi_j(x) = (2/L)^{1/2} \sin(k_j x) \chi_0(L)(x) \) \( (k_j = j \pi / L, j \in \mathbb{N}, \chi_0(L)(x) \) is the characteristic function in the interval \( [0, L] \), and the corresponding energy eigenvalues are \( E_j = \hbar^2 k_j^2 / 2m \). The exact time-evolution of the single-particle eigenstates of
A hard-wall trap in a 1D free expansion $\varphi_j(x,t)$ can be found using the superposition principle $\varphi_j(x,t) = \int_{-\infty}^{\infty} dx' K_0(x,t|x',t = 0)\varphi_j(x',0)$ where the free propagator $K_0(x,t|x',t = 0) = (m/2\pi\hbar)^{1/2} \exp(\imath m(x-x')^2/2\hbar t)$. After performing the integral explicitly, $\varphi_j(x,t) = \frac{1}{\sqrt{2\pi}} \sum_{\alpha = \pm 1} \alpha \left[ e^{\imath \alpha k_j x} M(x - L, \alpha k_j, \tau) - M(x, \alpha k_j, \tau) \right]$ where $\tau = \hbar t/m$. The Moshinsky function is defined as $M(x, k, t_j) = e^{t_j^2} w\left[ -\frac{1}{2} \sqrt{\imath} (k - \frac{x}{t_j}) \right] / 2^{23}$, and the Faddeeva function is related to the complementary error function $w(z) = e^{-z^2} \text{Erfc}(-iz)$. For all $j > 1$, the density profile $|\varphi_j(x,t)|^2$ bifurcates in two main branches moving with $\pm k_j$ after a critical time $t_c = mL^2/2\pi\hbar$, reflecting the underlying bimodal momentum distribution [20].

For the free expansion of a TG gas the transition to the ballistic regime sharply takes place at $t_N$ [23]. To compare the expansion from both harmonic and hard-wall traps we impose the particle number $N$ and total energy of the AHC gas to be the same. For the many-body ground-state, the following relation must hold between the frequency $\omega_0$ and the length of the box, $L$: $\omega_0 = \hbar^2 (N + 1)/(2N + 1)/6mL^2N$. Figure 3 shows how the momentum distribution is broadened with the initial hard-wall confinement with respect to the harmonic case, due to the dispersion relation $E_n \propto n^2$ and the presence of higher momentum components. The reference case for $\theta = \pi$, the bosonic TG gas, presents a sharper peak at $k = 0$ when prepared in the later trap. Note also how asymptotically the quasi-momentum distribution is already flat (for $N = 5$) for the box confinement, while in the harmonic trap case keeps an oscillating profile, both of which are eventually mapped into the coordinate density profile in a $t_F$ time scale.

Anyonic attractive Tonks-Girardeau gas. The anyonic attractive Tonks-Girardeau (AATG) gas has recently been introduced in analogy with the Fermionic Tonks-Girardeau gas (FTG) [13]. The FTG results when a spin-polarized Fermi gas is driven through a p-wave Feschbach resonance inducing strongly attractive short range odd-wave interactions [27]. Indeed, in a given sector $x_1 < x_2$, the resulting pseudopotential consists of a hard-core wall at $x_1 - x_2 = 0$ and a square well of depth $V$ and width $a$ in the limit $a \to 0$, $V \to \infty$, keeping $V = (\hbar^2 a^2)/4m$. The dual system of the FTG is simply the ideal Bose gas, whose ground state is simply described by a Hartree product $\Psi_B(x_1, \ldots, x_N) = \prod_{n=1}^{N} \phi_0(x_n)$, $\phi_0$ being the single particle ground state of the external potential. This motivates, the study of the 1D anyons with such infinitely attractive pseudopotential - the AATG gas-, whose ground state wavefunction is given by the anyon-boson mapping $\Psi_{AATG}(x_1, \ldots, x_N) = \prod_{1 \leq j < k \leq N} \epsilon(\hat{x}_k - \hat{x}_j) e^{-\imath \theta/2} \epsilon(\hat{x}_k - \hat{x}_j) \Psi_B(x_1, \ldots, x_N)$, where the mapping holds under any unitary time evolution [13].

A one-parameter family of RSPDM for the AATG results from direct computation in the line of [27],

$$\rho_{AATG}(x,y) = N\phi_0(x)\phi_0(y)|P_{0}^\theta(x,y)|^{N-1},$$

where

$$P_0^\theta(x,y) = \int dz \epsilon(y - z) \epsilon(x - z) e^{-\imath \theta/2} \epsilon(y-z) - \epsilon(x- z)|\phi_0(z)|^2$$

$$= 1 - [1 + e^{-\imath \theta \epsilon(y-x)}] \int_x^y dz |\phi_0(z)|^2.$$

In particular, for the harmonic confinement, $P_0^\theta(x,y) = 1 - \frac{1 + e^{-\imath \theta \epsilon(y-x)}}{1 + e^{-\imath \theta \epsilon(y-x)}} |\text{Erf}(y/x_0 \sqrt{b}) - \text{Erf}(x/x_0 \sqrt{b})|$. The first term in Eq. (2), $N \phi_0^2(x)\phi_0(y)$, corresponds to the RSPDM of the dual non-interacting Bose system. Moreover, the momentum distribution $n_\theta(k)$ can be obtained by the usual double Fourier transform. Figure 4 shows $n_\theta(k)$ for a gas of AATG in a harmonic trap as a function of the statistical parameter $\theta$. The ideal gas distribution $N|\phi_0(k)|^2$ is recovered from the general expression for $\theta = \pi$, whereas the FTG gas is obtained at $\theta = 0$. However, for any other value of the statistical parameter $\theta$ the momentum distribution of the trapped system is asymmetric as a result of the anyonic permutation symmetry. We shall next describe the one dimensional expansion dynamics of a given AATG, following a sudden
switch-off of the hard-wall trap. Figure 5 shows how as the system expands in free space, the AATG gas undergoes bosonization reaching asymptotically the (symmetric) momentum distribution of the ideal Bose gas, \( n_B(k) = |\phi_0(k)|^2 \). For \( t \to \infty \) the argument of each of the complementary error functions in \( P_0^0(x, y) \) tends to vanish, and hence, \( \text{Erf}(0) = 0 \), and \( P_0^0(x, y) = 1 \). It follows that \( p_{\text{AATG}}(x, y, t) \sim N\phi_0^2(x)\phi_0(y) \), meaning that any one-particle observable, both local and non-local, becomes identical for both the AATG and ideal Bose gases. The SPM leads for the the asymptotic momentum distribution \( n_{\text{AATG}}(k) \sim \frac{k_0}{|\phi_0|} n_B(\omega_0 k/k_0) \sim n_B(k) \).

For \( N, \omega_0 t \gg 1 \), the off-diagonal correlations vary exponentially as \( p_{\text{AATG}}(x, y, t) \sim \rho_B(x, y, t) \exp[-i\theta(x-y)/2] N \cos(\theta/2) |y-x|/\omega_0 t \sqrt{\pi \sigma_0} \). Hence, the bosonization time scale becomes \( t_B(\theta) \approx N \cos(\theta/2) \omega_0^{-1} \), this is, \( t_B(\theta) \approx N^2 \tau_F (\pi - \theta) \). Should we have considered the expansion from a hard-wall trap, the characteristic time scale would be \( t_B(\theta) \approx N m L^2 \cos(\theta/2)/2\pi \hbar \).

In conclusion, we have shown that 1D hard-core anyons undergo dynamical fermionization during a free time-evolution, namely, the momentum distribution of the gas approaches that of spin-polarized non-interacting fermions. We have further shown that the complementary dynamical process can take place: for anyons with strongly attractive short range interactions (the anyonic attractive Tonks-Girardeau gas), the freely time-evolving momentum distribution tends to that of an ideal Bose gas. In both cases the momentum distribution becomes symmetric, as the role of the anyonic permutation symmetry diminishes. One may expect that the recently developed methods for the dynamics of a bosonic Lieb-Liniger gas [28], could equally be generalized for fractional statistics allowing to study the expansion dynamics for finite interactions [29].

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