MODIFIED ERDÖS–GINZBURG–ZIV CONSTANTS FOR $\mathbb{Z}/n\mathbb{Z}$ AND $(\mathbb{Z}/n\mathbb{Z})^2$

AARON BERGER AND DANIELLE WANG

Abstract. For an abelian group $G$ and an integer $t > 0$, the modified Erdős–Ginzburg–Ziv constant $s'_t(G)$ is the smallest integer $\ell$ such that any zero-sum sequence of length at least $\ell$ with elements in $G$ contains a zero-sum subsequence (not necessarily consecutive) of length $t$. We compute $s'_t(G)$ for $G = \mathbb{Z}/n\mathbb{Z}$ and for $t = n, G = (\mathbb{Z}/n\mathbb{Z})^2$.

Keywords: Zero-sum sequence, Zero-sum subsequence, Erdős–Ginzburg–Ziv Constant.

1. Introduction

In 1961, Erdős, Ginzburg, and Ziv proved the following classical theorem.

Theorem 1.1 (Erdős–Ginzburg–Ziv [6]). Any sequence of length $2n - 1$ in $\mathbb{Z}/n\mathbb{Z}$ contains a zero-sum subsequence of length $n$.

Here, a subsequence need not be consecutive, and a sequence is zero-sum if its elements sum to 0. This theorem has lead to many problems involving zero-sum sequences over groups.

In general, let $G$ be an abelian group, and let $G_0 \subseteq G$ be a subset. Let $\mathcal{L} \subseteq \mathbb{N}$. Then $s_{\mathcal{L}}(G_0)$ is defined to be the minimal $\ell$ such that any sequence of length $\ell$ with elements in $G_0$ contains a zero-sum subsequence whose length is in $\mathcal{L}$. When $G_0 = G$ and $\mathcal{L} = \{\exp(G)\}$, this constant is called the Erdős–Ginzburg–Ziv constant.

When $G = \mathbb{Z}$, this problem turns out to be not very interesting — if $G_0$ contains a nonzero element, then $s_{\mathcal{L}}(G_0) = \infty$. This has lead to [2] the study of the modified Erdős–Ginzburg–Ziv constant $s'_t(G_0)$, defined as the smallest $\ell$ such that any zero-sum sequence of length at least $\ell$ with elements in $G_0$ contains a zero-sum subsequence whose length is in $\mathcal{L}$. When $\mathcal{L} = \{t\}$ is a single element, we omit the set brackets for convenience. In [3], the first author determined modified EGZ constants in the infinite cyclic case. Here we treat the finite cyclic case and extensions.

Problem 1.2 ([3, Problem 2]). Compute $s'_t(G)$ for $G = \mathbb{Z}/n\mathbb{Z}$ and $(\mathbb{Z}/n\mathbb{Z})^2$.

In this paper, we answer Problem 1.2 for $G = \mathbb{Z}/n\mathbb{Z}$ and for $t = n, G = (\mathbb{Z}/n\mathbb{Z})^2$. Note that in both cases, when $n$ does not divide $t$, the quantity $s'_t(G)$ is infinite.

Theorem 1.3. The modified EGZ constant of $\mathbb{Z}/n\mathbb{Z}$ is given by $s'_{nt}(\mathbb{Z}/n\mathbb{Z}) = (t+1)n - \ell + 1$, where $\ell$ is the smallest integer such that $\ell | n$. 

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307
E-mail addresses: bergera@mit.edu, diwang@mit.edu.
Theorem 1.4. We have \( s'_n((\mathbb{Z}/n\mathbb{Z})^2) = 4n - \ell + 1 \) where \( \ell \) is the smallest integer such that \( d \geq 4 \) and \( \ell \nmid n \).

2. The cyclic case

In this section we give the proof of Theorem 1.3. As in [10], if \( J \) is a sequence of elements of \( \mathbb{Z}/n\mathbb{Z} \) or \( (\mathbb{Z}/n\mathbb{Z})^2 \), we use \( (k \mid J) \) to denote the number of zero-sum subsequences of \( J \) of size \( k \).

Proposition 2.1. If \( d \mid n \) and \( J \) is a zero-sum sequence in \( \mathbb{Z}/n\mathbb{Z} \) of length \( 2n - d \), then \( (n \mid J) > 0 \).

Proof. By Theorem 1.1, we can break off subsequences of \( J \) of size \( d \) with sum \( 0 \) (mod \( d \)) until we have fewer than \( 2d - 1 \) remaining. In fact, since \( d \mid n \), we will have exactly \( d \) remaining. But since the sum was zero-sum to begin with, the last \( d \) must also sum to zero, so we have \( 2(n/d) - 1 \) blocks of size \( d \) with sums \( dx_1, \ldots, dx_{2(n/d)−1} \) for some \( x_i \). By Theorem 1.1, some \( n/d \) of these must sum to \( 0 \) in \( \mathbb{Z}/(n/d)\mathbb{Z} \), so the union of these blocks gives a subsequence of length \( n \) whose sum is zero in \( \mathbb{Z}/n\mathbb{Z} \).

Corollary 2.2. Let \( \ell \) be the smallest positive integer such that \( \ell \nmid n \), and let \( t \geq 1 \). If \( J \) is a zero-sum sequence in \( \mathbb{Z}/n\mathbb{Z} \) of length at least \((t+1)n−\ell+1\), then \((nt \mid J) > 0 \).

Proof. We induct on \( t \). The case \( t = 1 \) follows from Proposition 2.1 since \( \ell−1, \ldots, 1 \) all divide \( n \). Suppose the result is true for positive integers less than \( t > 1 \). Then \( J \) contains a zero-sum subsequence of length \( (t−1)n \). Remove these elements from \( J \). We are left with a zero-sum sequence of length \( 2n−\ell+1 \). This is the \( t = 1 \) case, so we can find another zero-sum subsequence of length \( n \). Combine this with the \( (t−1)n \) to get the desired subsequence of length \( nt \).

Proposition 2.3. Suppose \( \ell \nmid n \) and \( t \geq 1 \). Then there exists a zero-sum subsequence in \( \mathbb{Z}/n\mathbb{Z} \) of length \((1+t)n−\ell\) which contains no zero-sum subsequence of length \( nt \).

Proof. Consider a sequence of 0’s and 1’s with multiplicities \( a \leq tn − 1, b \leq n − 1 \) respectively where \( a + b = (t + 1)n − \ell \). Such a sequence will have no zero-sum subsequence of length \( nt \). It suffices to find \( a, b \) such that \( g = \gcd(n, \ell) \mid b \), because then we can add some constant to every term of the sequence to make it zero-sum. Note that adding a constant to every term does not introduce any new zero-sum subsequences. It suffices to take \( b = tn − g \) and \( a = n − \ell + g \leq n − \ell/2 \leq n − 1 \).

Corollary 2.2 and Proposition 2.3 together imply Theorem 1.3.

3. The case \((\mathbb{Z}/n\mathbb{Z})^2\)

In this section we prove Theorem 1.4. We first prove some preliminary lemmas.

The following results from [10] are key.

Lemma 3.1 ([10, Corollary 2.4]). Let \( p \) be a prime, and let \( J \) be a sequence of elements in \((\mathbb{Z}/p\mathbb{Z})^2\). If \( |J| = 3p - 2 \) or \( |J| = 3p - 1 \), then \((p \mid J) = 0 \) implies \((2p \mid J) \equiv -1 \) (mod \( p \)).

Lemma 3.2 ([10, Corollary 2.5]). Let \( p \) and \( J \) be as in Lemma 3.1. If \( |J| \) is a zero-sum sequence with exactly \( 3p \) elements, then \((p \mid J) > 0 \).
**Theorem 3.3** ([10, Theorem 3.2]). If $J$ is a sequence of length $4n - 3$ in $(\mathbb{Z}/n\mathbb{Z})^2$ then $(n \mid J) > 0$.

We generalize Lemma 3.2 to non-prime $n$.

**Lemma 3.4.** If $J$ is a zero-sum sequence of length $3n$ in $(\mathbb{Z}/n\mathbb{Z})^2$, then $(n \mid J) > 0$.

**Proof.** We induct on $n$. The base case $n = 1$ is clear. Assume the the lemma is true for all positive integers less than $n$. Let $n = pm$ with $p$ prime and $m < n$.

Since $3n > 4m - 3$, we can find some $m$ elements of $J$ whose sum is 0 (mod $m$). Say their sum is $mx_1$ and remove these $m$ elements. We can continue doing this until there remain only $3m$ elements. But since $J$ was a zero-sum sequence, the remaining $3m$ elements must sum to 0 (mod $m$), so by the induction hypothesis, we can remove another $m$ with sum a multiple of $m$. This gives us $3p - 2$ blocks of size $m$ whose sums are $mx_1, \ldots, mx_{3p-2}$ for some $x_i$.

If some $p$ of the $x_i$ sum to 0 (mod $p$), then combining the blocks would give us $n$ elements whose sum is 0 (mod $n$), as desired. If not, by Lemma 3.1, we must have some $2p$ of the $x_i$ summing to 0 (mod $p$), so we have $2n$ elements whose sum is 0 (mod $n$). But since $J$ itself is zero-sum and has size $3n$, the complement is zero-sum as well and has size $n$. \hfill \Box

**Proposition 3.5.** If $d \mid n$, and $J$ is a zero-sum sequence in $(\mathbb{Z}/n\mathbb{Z})^2$ of length $4n - d$, then $(n \mid J) > 0$.

**Proof.** Note that $4n - d \geq 3m$. By Theorem 3.3, we can break off subsequences of size $d$ with sum 0 (mod $d$) until we have only $3d$ elements remaining. Then by Lemma 3.4 we can break off another $d$ elements, to obtain $4(n/d) - 3$ blocks of size $d$, with sums $dx_1, \ldots, dx_{4(n/d)-3}$ for some $x_i$. By Theorem 3.3, some $n/d$ of the $x_i$ must sum to 0 in $(\mathbb{Z}/(n/d)\mathbb{Z})^2$. Combining the corresponding blocks gives a subsequence of length $n$ whose sum is zero in $(\mathbb{Z}/n\mathbb{Z})^2$. \hfill \Box

The following corollary is clear from Proposition 3.5 and Theorem 3.3.

**Corollary 3.6.** Let $\ell$ be the smallest integer greater than or equal to 4 such that $\ell \nmid n$. If $J$ is a zero-sum sequence in $(\mathbb{Z}/n\mathbb{Z})^2$ of length at least $4n - \ell + 1$, then $(n \mid J) > 0$.

**Proposition 3.7.** Suppose $4 \leq \ell \nmid n$. There exists a zero-sum sequence in $(\mathbb{Z}/n\mathbb{Z})^2$ of length $4n - \ell$ which contains no zero-sum subsequences of length $n$.

**Proof.** First, consider a sequence of the form

\[
\begin{align*}
(0, 0) & \quad a \leq n - 1 \\
(0, 1) & \quad b \leq n - 1 \\
(1, 0) & \quad c \leq n - 1 \\
(1, 1) & \quad d \leq n - 1,
\end{align*}
\]

where $a$ denotes the number of $(0, 0)$’s, etc., and $a + b + c + d = 4n - \ell$. It is easy to check that this sequence contains no zero-sum subsequence of length $n$. Now, we claim that there exists $(r, s) \in (\mathbb{Z}/n\mathbb{Z})^2$ such that adding $(r, s)$ to each term of the above sequence will result in a zero-sum sequence. Note that adding $(r, s)$ to each term does not change the fact that there is no zero-sum subsequence of length $n$.

In fact, all we need is

\[
g := \gcd(n, \ell) \mid c + d, b + d.
\]
We claim that the following \( a, b, c, d \) work.

\[
\begin{align*}
    a &= n - \ell + g + 1 \quad (\text{or } n - \ell + 2g + 1 \text{ if } g = 1) \\
    b &= n - 1 \\
    c &= n - 1 \\
    d &= n - g + 1 \quad (\text{or } n - 2g + 1 \text{ if } g = 1).
\end{align*}
\]

Note that \( g \leq \ell/2 \) because \( \ell \nmid n \), so \( a \leq n - \ell/2 + 1 \leq n - 1 \) if \( g \neq 1 \), and \( a = n - \ell + 3 \leq n - 1 \) if \( g = 1 \). It is easy to show that we always have \( a, d \geq 0 \) and \( d \leq n - 1 \), and that these \( a, b, c, d \) satisfy the divisibility relation. \( \square \)

Now, Corollary 3.6 and Proposition 3.7 imply Theorem 1.4.

### 4. Open problems

Harborth [8] first considered the problem of computing \( s_n((\mathbb{Z}/n\mathbb{Z})^d) \) for higher dimensions. He proved the following bounds.

**Theorem 4.1** (Harborth [8]). We have

\[
(n - 1)2^d + 1 \leq s_n((\mathbb{Z}/n\mathbb{Z})^d) \leq (n - 1)n^d + 1.
\]

For \( d > 2 \) the precise value of \( s_n((\mathbb{Z}/n\mathbb{Z})^d) \) is not known. See [4, 5] for some better lower bounds and [1, 9] for some better upper bounds. In general the lower bound in Theorem 4.1 is not tight, but Harborth showed that it is an equality for \( n = 2^k \) a power of 2.

**Conjecture 4.2.** If \( n = 2^k \) and \( d \geq 1 \), we have

\[
s'_n((\mathbb{Z}/n\mathbb{Z})^d) = 2^d n - \ell + 1,
\]

where \( \ell \) is the smallest integer such that \( \ell \geq 2^d \) and \( \ell \nmid n \).

By an argument similar to the \((\mathbb{Z}/n\mathbb{Z})^2\) case, we can reduce this conjecture to the case \( n = 2^d \), in which case \( \ell = 2^d + 1 \). We also have not determined the modified EGZ constants for \((\mathbb{Z}/n\mathbb{Z})^2\) for subsequences of length greater than \( n \).

**Problem 4.3.** Compute \( s'_nt((\mathbb{Z}/n\mathbb{Z})^2) \) for \( t > 1 \).

The constant \( s_n(\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}) \) is known to be \( 2m + 2n - 3 \) for \( m \mid n \) [7, Theorem 5.8.3].

**Problem 4.4.** Compute \( s'_nt(\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}) \) for \( t \geq 1 \) and \( m \mid n \).

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