LIMITING FREE ENERGY PER PARTICLE FOR ISING MODEL BY APPROXIMATING ITS FUNCTIONAL INTEGRAL

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ABSTRACT. There have been a lot of methods intended to study the limiting free energy per particle (LFEPP) for 3D Ising model in absence of an external magnetic field. These methods are elegant, but most of them are complicated and often require specialized knowledge and special skills. Sometimes, a simple method may also lead to interesting results. Here we approximate the LFEPP for Ising model from its corresponding functional integral, in which only the knowledge of mathematical analysis, linear algebra and asymptotic computation is used. The Ising LFEPP resulted, which expressed implicitly in terms of the integral, is exact formally and asymptotically. Unfortunately, this LFEPP includes a difficult integral function. Once this integral function and its 1-2 order derivative functions are determined, the final Ising LFEPP can be calculated. In the cases of: (1). sufficiently high temperature; (2). sufficiently weak exchange interaction between spins; (3). small fluctuating magnetization per site; (4). "(1)+(2)+(3)", the LFEPPs calculated for 1D and 2D Ising model are consistent with those well-known. Based on these, we further infer the LFEPP for 3D model in these special cases.

Keywords: 3D Ising model, Free energy, Functional integral, Thermodynamic limit

1. INTRODUCTION

The famous Ising model looks very simple, which consists of a lattice with a binary magnetic polarity (or "spin") assigned to each point. The nearest-neighbor Ising model without an external magnetic field in $D$-dimensions ($D = 1, 2, 3, ...$) is defined in terms of the following Hamiltonian (eg., Huang, 1987),

\begin{equation}
\mathcal{H} = -\frac{1}{2} \sum_{i,j=1}^{N} K_{ij} s_i s_j
\end{equation}

where, $i$ and $j$ are the sites $r_i$ and $r_j$ of a $D$-dimensional hyper cubic lattice with $N$ sites, respectively. $s_i = \pm 1$ are the two possible states of the $z$-components of spins localized at the lattice sites. $K_{ij}$ denotes the exchange interaction between spins localized at $r_i$ and $r_j$,

\begin{equation}
K_{ij} = \begin{cases} 
z & \text{if } i \text{ and } j \text{ are the nearest neighbors} \\
0 & \text{otherwise}
\end{cases}
\end{equation}

where $z = \beta \epsilon = \frac{\epsilon}{kT}$, $\epsilon$ the interaction energy, $T$ the temperature, and $k$ Boltzmann constant.
Such a simple model has played an important role in the theory of ferromagnetism and phase transitions, and it has been found new applications in many areas of science, eg., neuroscience, sea ice and even voter models (Viswanathan et al., 2022).

Most thermodynamic functions of this model depend on the evaluation of its LFEPP. For 1D Ising model with periodic boundary condition, the exact LFEPP, \(-\frac{\psi}{kT}\), is,

\[
-\frac{\psi}{kT} = \frac{1}{2} \int_0^{2\pi} \log (\cosh 2z - \sinh 2z \cos \omega_1) d\omega_1
\]

where we use the LFEPP in terms of integral from Berlin and Kac (1952) rather than that from Kramers and Wannier (1941) for comparison later.

For 2D Ising model imposed on periodic boundary condition, the LFEPP was evaluated by Onsager (1944) as the following,

\[
-\frac{\psi}{kT} = \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} \log \left[ \cosh^2 2z - \sinh 2z (\cos \omega_1 + \cos \omega_2) \right] d\omega_1 d\omega_2
\]

where for comparison later, we also use the LFEPP in terms of integral from Berlin and Kac (1952) rather than that from Onsager (1944).

However, there is still no an accepted LFEPP for 3D Ising model, although for this purpose there have been lots of methods or approaches which are valid for 1D, 2D and/or high-dimensional \((D > 3)\) model. The most famous ones are Mean-Field Theory (eg., Bragg and William, 1934, 1935; Williams, 1935; Landau, 1937), Transfer Matrix Method (Onsager, 1944), \(\varphi^4\) Theory (Ginzburg and Landau, 1950), Variational Calculation (Thompson, 1965). Especially the Transfer Matrix Method and the similar ones are popular in the later studies (eg., Zhang, 2007), although their results are questionable (eg., Wu et al., 2008a; Wu et al., 2008b; Fisher and Perk, 2016; Perk, 2013). Transfer Matrix Method requires specialized and abstract knowledge of spinor algebra or operator algebra, which is unfamiliar to most non-professionals. In 2015, Kocharovsky and Kocharovsky (2015) presented a method of the recurrence equations for partial contractions for 3D model, and they said that "Towards an exact solution for the three-dimensional Ising model". However, it can also be found that their method is too complicated and specialized.

Here we study the LFEPP of Ising Model from its corresponding functional integral. To our finite knowledge, we dare not say that this method is the easiest, but we think it is simpler and more easily understood by most people than those such as transfer matrix method. Although three difficult integrals are involved, the final LFEPP is formally and asymptotically exact. Based on this formal LFEPP in implicitly integral representation, we infer the LFEPP of 3D model for special cases, such as sufficiently high \(T\), sufficiently weak \(\epsilon\) between spins, small fluctuating magnetization per site, and so on. It should be pointed out that we have only obtained formally the LFEPP of Ising model, or in other words, we only provide another possible way to obtain the LFEPP of Ising model, but we have not completely solved this problem.
2. Functional integral and the LFEPP of Ising model

The partition function normalized to unity for the Ising model \( Z \) is (eg., Berlin and Kac 1952),

\[
Z = 2^{-N} \sum_{\{s_i = \pm 1\}} \exp \left( \frac{1}{2} \sum_{ij} K_{ij} s_i s_j \right)
\]

Applying the following Hubbard-Stratonovich transformation (See details in section 5.1 also in Ginzburg and Landau (1950), Amit et al. (2005) and Kopietz et al (2010)),

\[
\exp \left( \frac{1}{2} \sum_{ij} K_{ij} s_i s_j \right) = \left[ \frac{\det K}{(2\pi)^N} \right]^{1/2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{k=1}^{N} d\phi_k \\
\times \exp \left( -\frac{1}{2} \sum_{ij} \phi_i K_{ij} \phi_j + \sum_{ij} s_i K_{ij} \phi_j \right)
\]

Noted that \( K \) must be positive defined.

The partition function now is,

\[
Z = 2^{-N} \sum_{\{s_i = \pm 1\}} \exp \left( \frac{1}{2} \sum_{ij} K_{ij} s_i s_j \right)
\]

\[
= 2^{-N} \sum_{\{s_i = \pm 1\}} \left[ \frac{\det K}{(2\pi)^N} \right]^{1/2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{k=1}^{N} d\phi_k \\
\times \exp \left( -\frac{1}{2} \sum_{ij} \phi_i K_{ij} \phi_j + \sum_{ij} s_i K_{ij} \phi_j \right)
\]

\[
= 2^{-N} \left[ \frac{\det K}{(\frac{1}{2\pi})^N} \right]^{1/2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{k=1}^{N} d\phi_k \\
\exp \left\{ -\frac{1}{2} \sum_{ij} \phi_i K_{ij} \phi_j + \sum_{i} \log \left[ \cosh \left( \sum_{j} K_{ij} \phi_j \right) \right] \right\}
\]

where \( K_{ij} \) is a circulant matrix when an appropriate periodicity is used on the boundary of Ising model, and it is assumed to be positive defined when \( N \to \infty \). Details including eigenvalues and eigenvectors of \( K_{ij} \) can be found in Appendix A \( \text{(A.1,A.3,A.8)} \).
In Eq. (6), the third row is from the fact that only \( \sum_{ij} s_i K_{ij} \phi_j \) in the exponential of the second row has a dependence on the \( s_i \) (See details in Appendix B). The third row expresses the partition function of the Ising model in terms of an \( N \)-dimensional integral over variables \( \phi_i \), and the infinite-dimensional integral obtained from it when \( N \to \infty \) is called the functional integral of the Ising model. The physical meaning of the \( \phi_i \) is the fluctuating magnetization per site, for its expectation value is simply the magnetization per site (e.g., Ginzburg and Landau, 1950; Amit et al., 2005; Kopietz et al., 2010).

Functional integral of the Ising model is an exact transcription of the original model. However, it is very complicated in terms of infinite-dimensional integral. If the infinite-dimensional integral can be reduced into the product of a series of 1D integrals, it is possible to calculate this functional integral.

Similar to Eq. (B.2), but we calculate \( \sum_{\{s_i=\pm 1\}} \exp \left( \sum_{ij} s_i K_{ij} \phi_j \right) \) from another procedure as follows,

\[
2^{-N} \sum_{\{s_i=\pm 1\}} \exp \left( \sum_{ij} s_i K_{ij} \phi_j \right) = 2^{-N} \sum_{\{s_i=\pm 1\}} \exp \left( \sum_{p} s_i V_{ip} V_{pj} K_{ij} V_{jp} V_{pj} \phi_j \right) \\
= 2^{-N} \sum_{\{s_i=\pm 1\}} \exp \left[ \sum_{p} \left( \sum_{i} s_i V_{ip} \right) \lambda_p y_p \right] \\
= 2^{-N} \sum_{\{s_i=\pm 1\}} \prod_{p} \exp \left[ \left( \sum_{i} s_i V_{ip} \right) \lambda_p y_p \right] \\
= 2^{-N} \sum_{\{s_i=\pm 1\}} \prod_{p} \prod_{i} \exp \left[ s_i \left( V_{ip} \lambda_p y_p \right) \right] \\
= \prod_{p} 2^{-N} \sum_{\{s_i=\pm 1\}} \prod_{i} \exp \left[ s_i \left( V_{ip} \lambda_p y_p \right) \right] \\
= \prod_{p} 2^{-N} \prod_{i} \left[ \exp \left( V_{ip} \lambda_p y_p \right) + \exp \left( -V_{ip} \lambda_p y_p \right) \right] \\
= \prod_{p} 2^{-N} \left[ \prod_{i} 2 \cosh \left( V_{ip} \lambda_p y_p \right) \right] \\
= \prod_{p} 2^{-N} \left[ \exp \left\{ \sum_{i} \log \left[ 2 \cosh \left( V_{ip} \lambda_p y_p \right) \right] \right\} \right]
\]

where \( V_{ij} \) are the real, orthogonal, unity eigenvectors corresponding to eigenvalues \( \lambda_p \) of \( K_{ij} \) (A.8); \( 2^{-N} \) is a normalized factor (to unity) for \( \sum_{ij} \) counting a given \( i, j(p) \) twice (or \( 2^{-N} \) and \( \sum_{\{s_i=\pm 1\}} \) are one); Einstein summation convention is used in the first row of Eq. (7), for example, \( y_p = V_{pj} \phi_j = \sum_{j} V_{pj} \phi_j \).
Insert Eq. (7) into Eq. (6) with \( \sum \phi_i K_{ij} \phi_j = \sum_p \phi_i V_{ip} K_{ij} V_{pj} \phi_j = \sum_p \lambda_p y_p^2 \) and the Jacobian unity, now the partition function (6) is,

\[
Z = \left[ \frac{\det K}{(2\pi)^N} \right]^{1/2} \prod_{p=1}^{N} \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \lambda_p y_p^2 \right) \exp \left\{ \sum_i \log \left[ 2 \cosh \left( V_{ip} \lambda_p y_p \right) \right] \right\} dy_p
\]

(8)

Eq. (8) means that we can change the infinite-dimensional integral in Eq. (6) into a product of 1D integrals, i.e.,

\[
Z = \left[ \frac{\det K}{(2\pi)^N} \right]^{1/2} \prod_{p=1}^{N} \left( 2^{-N} \right) \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \lambda_p y_p^2 \right) \exp \left\{ \sum_i \log \left[ 2 \cosh \left( V_{ip} \lambda_p y_p \right) \right] \right\} dy_p
\]

\[
= \left[ \frac{\det K}{(2\pi)^N} \right]^{1/2} \prod_{p=1}^{N} 2^{-N} I_p
\]

(9)

Eq. (9) is equivalent to Eq. (6), and we call it also the functional integral of the Ising model. Once \( I_p \) is calculated, the LFEPP, \( -\frac{\psi}{kT} \), is,

\[
-\frac{\psi}{kT} = \lim_{N \to \infty} \frac{\log Z}{N}
\]

(10)

3. Calculation of \( I_p \) and the Formal LFEPP of Ising Model

In this section, we focus on the following integral,

\[
I_p = \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \lambda_p y_p^2 \right) \exp \left\{ \sum_i \log \left[ 2 \cosh \left( V_{ip} \lambda_p y_p \right) \right] \right\} dy_p
\]

(11)

Since

\[
V_{ip} = N^{-1/2} \left[ \cos \frac{2\pi}{N} (i - 1) (p - 1) + \sin \frac{2\pi}{N} (i - 1) (p - 1) \right]
\]

(12)

We can obtain,
\( I_p = \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \lambda_p y_p^2 \right) \left\{ \exp \left[ \sum_i \log \left( 2 \cosh \left( \frac{V_{ip} \lambda_p y_p}{N} \right) \right) \right] \right\} dy_p \)

\[
y_p = yN^{-1/2}
\]

\[
= \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \lambda_p \frac{y^2}{N} \right) \left\{ \exp \left[ \sum_i \log \left( 2 \cosh \left( \frac{V_{ip} \lambda_p y}{N} \right) \right) \right] \right\} \mathrm{d} \frac{y}{N^{1/2}}
\]

\[
= \frac{1}{N^{1/2}} \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \lambda_p \frac{y^2}{N} \right) \times \exp \left[ \sum_i \log \left( 2 \cosh \left[ \cos \frac{2\pi}{N} (i - 1) (p - 1) + \sin \frac{2\pi}{N} (i - 1) (p - 1) \lambda_p \frac{y}{N} \right] \right) \right] \mathrm{d} y
\]

\[
= \frac{1}{N^{1/2}} \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \lambda_p \frac{y^2}{N} \right) 2 \cosh \left( \lambda_p \frac{y}{N} \right) \times \exp \left[ \sum_{i=2} \log \left( 2 \cosh \left[ \cos \frac{2\pi}{N} (i - 1) (p - 1) + \sin \frac{2\pi}{N} (i - 1) (p - 1) \lambda_p \frac{y}{N} \right] \right) \right] \mathrm{d} y
\]

\[
y = tN
\]

\[
= N^{1/2} \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \lambda_p N t^2 \right) 2 \cosh \left( \lambda_p t \right) \times \exp \left[ \sum_{i=2} \log \left( 2 \cosh \left[ \cos \frac{2\pi}{N} (i - 1) (p - 1) + \sin \frac{2\pi}{N} (i - 1) (p - 1) \lambda_p t \right] \right) \right] \mathrm{d} t
\]

Next the following sum is taken into account,

\( J = \sum_{i=2} \log \left( 2 \cosh \left[ \cos \frac{2\pi}{N} (i - 1) (p - 1) + \sin \frac{2\pi}{N} (i - 1) (p - 1) \lambda_p t \right] \right) \)

It can be found that,
\( J = N \sum_{i=2}^{\infty} \log \left( \frac{2 \cosh \left( \frac{2\pi}{N} (i-1) (p-1) + \sin \frac{2\pi}{N} (i-1) (p-1) \right) \lambda_p t}{N} \right) \)

Subdivide the interval \( 0 - 2\pi \) into \( N \) equal intervals of length \( 2\pi/N = \Delta \theta \), and let \( \theta = (i-1)\Delta \theta \). When \( \Delta \theta \to 0 \), we have,

\[
\frac{J}{N} = \frac{1}{2\pi} \sum_{\theta=\Delta \theta}^{2\pi-\Delta \theta} \log \left( 2 \cosh \left( \lambda_p t [\cos (p-1) \theta + \sin (p-1) \theta] \right) \right) \Delta \theta \\
\to \frac{1}{2\pi} \int_{0}^{2\pi} \log \left( 2 \cosh \left( \lambda_p t [\cos (p-1) \theta + \sin (p-1) \theta] \right) \right) d\theta \\
\downarrow \\
\frac{J}{N} = \frac{1}{2\pi (p-1)} \int_{0}^{2(p-1)\pi} \log \left( 2 \cosh \left( \lambda_p t [\cos \xi + \sin \xi] \right) \right) d\xi \\
= \frac{1}{2\pi (p-1)} \left\{ \int_{0}^{2\pi} \log \left( 2 \cosh \left( \lambda_p t [\cos \xi + \sin \xi] \right) \right) d\xi + \int_{2\pi}^{4\pi} \log \left( 2 \cosh \left( \lambda_p t [\cos \xi + \sin \xi] \right) \right) d\xi + \cdots \right\} \\
= \frac{1}{2\pi} \int_{0}^{2\pi} \log \left( 2 \cosh \left( \lambda_p t [\cos \xi + \sin \xi] \right) \right) d\xi \\
= \frac{1}{\pi} \int_{0}^{\pi} \log \left( 2 \cosh \left( \lambda_p t [\cos \xi + \sin \xi] \right) \right) d\xi \\
= \frac{1}{\pi} \int_{0}^{\pi} \log \left( 2 \cosh \left( \sqrt{2} \lambda_p t \sin \xi \right) \right) d\xi \\
= \frac{1}{\pi} \int_{0}^{\pi/4} \log \left( \sqrt{2} \lambda_p t \sin \xi \right) d\xi \\

\text{Therefore insert } J \text{ into Eq. (13), and we obtain,}
I_p = N^{1/2} \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \lambda_p N t^2 \right) \{2 \cosh(\lambda_p t) \} \left\{ \frac{N}{\pi} \int_{\pi/4}^{\pi+\pi/4} \log \left( 2 \cosh \left( \sqrt{2} \lambda_p t \sin \xi \right) \right) d\xi \right\} dt

= N^{1/2} \int_{-\infty}^{+\infty} \{2 \cosh(\lambda_p t) \} \left\{ -\frac{1}{2} \lambda_p N t^2 + \frac{N}{\pi} \int_{\pi/4}^{\pi+\pi/4} \log \left( 2 \cosh \left( \sqrt{2} \lambda_p t \sin \xi \right) \right) d\xi \right\} dt

\text{where } f(\lambda_p, t) \text{ is,}

f(\lambda_p, t) = -\frac{1}{2} \lambda_p t^2 + \frac{N}{\pi} \int_{\pi/4}^{\pi+\pi/4} \log \left( 2 \cosh \left( \sqrt{2} \lambda_p t \sin \xi \right) \right) d\xi = -\frac{1}{2} \lambda_p t^2 + I_1

I_p \text{ can be solved by Laplace’s asymptotic method because it includes a large parameter } N. \text{ The required } df/dt \text{ and } d^2 f/dt^2 \text{ are as follows,}

\frac{df}{dt} = f_t = -\lambda_p t + \sqrt{2} \lambda_p \frac{\pi}{4} \tanh \left( \sqrt{2} \lambda_p t \sin \xi \right) \sin \xi d\xi = -\lambda_p t + \sqrt{2} \lambda_p I_2

\frac{d^2 f}{dt^2} = f_{tt} = -\lambda_p + \frac{2 \lambda_p^2}{\pi} \int_{\pi/4}^{\pi+\pi/4} \sec^2 \left( \sqrt{2} \lambda_p t \sin \xi \right) \sin^2 \xi d\xi = -\lambda_p + 2 \lambda_p^2 I_3

Therefore,

I_p = I_p(\lambda_p, N) = N^{1/2} \int_{-\infty}^{+\infty} 2 \cosh(\lambda_p t) \exp \left[ N f(\lambda_p, t) \right] dt

\sim \left\{ \begin{array}{ll}
4 \cosh(\lambda_p t_s) \exp \left[ N f(\lambda_p, t_s) \right] \sqrt{\frac{2\pi}{\left[ f_{tt}(\lambda_p t_s) \right]}} & t_s \in (0, \infty) \\
4 \cosh(\lambda_p t_s) \exp \left[ N f(\lambda_p, t_s) \right] \sqrt{\frac{\pi}{2\left[ f_{tt}(\lambda_p t_s) \right]}} & t_s = 0 \text{ or } t_s = \infty
\end{array} \right.

where t_s \text{ is from } f_t = 0 \text{ (Eq. (19)). } I_p = I_p(\lambda_p) \text{ or } I_p = I_p(\lambda_p, N) \text{ depend on } f(\lambda_p t_s), \text{ and generally } I_p = I_p(\lambda_p). \text{ So does the following } I_1.

Inserting the } I_p = I_p(\lambda_p, N) \text{ (i.e., Eq. (21)) into Eq. (9), we solve formally the functional integral of the Ising model, or its equivalence Eq. (6). It should be pointed out that in Eq.}
\[ I_p = 2, 3, ..., \text{ and the final } I_p \text{ should be multiplied by } I_1 \text{ (Notice that } i = 2, 3, .. \text{ in Eq. (13))}, \]

and see details in Eq. (45), and,

\begin{align*}
I_1 &= N^{1/2} \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \lambda_1 N t^2 \right) 2 \cosh(\lambda_1 t) \exp \left[ \frac{1}{2} \log(2 \cosh(\lambda_1)) \right] dt \\
&= N^{1/2} \int_{-\infty}^{+\infty} \exp \left[ N \left( -\frac{1}{2} \lambda_1 t^2 + \log(2 \cosh(\lambda_1)) \right) \right] dt \\
&= N^{1/2} 2 \int_{0}^{+\infty} \exp \left[ N \left( -\frac{1}{2} \lambda_1 t^2 + \log(2 \cosh(\lambda_1)) \right) \right] dt \\
& \sim \left\{ \begin{array}{ll}
2 \exp \left[ N \left( -\frac{1}{2} \lambda_1 t_{s_1}^2 + \log(2 \cosh(\lambda_1 t_{s_1})) \right) \right] & t_{s_1} \in (0, \infty) \\
2 \exp \left( N \ln 2 \right) & t_{s_1} = 0
\end{array} \right.
\end{align*}

where \( t_{s_1} \) is from equation of \( t = \tanh(\lambda_1 t) \).

Finally, we obtain the formal partition function of \( N \)-sites \( (N \rightarrow \infty) \) Ising model in the following,

\begin{equation}
Z = \left[ \det \frac{K}{(2\pi)^N} \right]^{1/2} \times \left( 2^{-N} I_1 \right) \times \prod_{p=2}^{\infty} \left( 2^{-N} I_p \right) \times N^{1/2} \int_{-\infty}^{+\infty} \left\{ 2 \cosh(\lambda_p t) \right\} dt \\
\times \exp \left[ -\frac{1}{2} \lambda_p N t^2 + \frac{N}{\pi} \int_{\pi/4}^{\pi/4} \log \left( 2 \cosh \left( \sqrt{2} \lambda_p t \sin \xi \right) \right) d\xi \right]
\end{equation}

\begin{equation}
\sim \left[ \det \frac{K}{(2\pi)^N} \right]^{1/2} \times \left( 2^{-N} I_1 \left( \frac{\lambda_1}{2} \right) \right) \times \prod_{p=2}^{\infty} \left( 2^{-N} I_p \left( \frac{\lambda_p}{2} \right) \right)
\end{equation}

The final \( Z \) in Eq. (23) is only the function of \( \lambda_p/2 \), ie., \( Z = Z(\lambda_p/2) \). Because \( Z \) is expressed in the form of \( \prod_{p=1}^{\infty} I_p(\lambda_p/2) \), we can easily get the LFEPP from \( \lim_{N \rightarrow \infty} \frac{\log Z(\lambda_p/2)}{N} \) according to Appendix C. It should be pointed out that \( Z \) and the LFEPP up to here are applicable to any dimensional Ising models.

However, the following are only applicable to 1-3D models, because the proofs in Appendix C are only for them. According to (C.1), (C.2) and (C.3), the LFEPP of 1-3D Ising model can be expressed implicitly in terms of integral (In the following "\( \simeq \)" means "main part")

For 1D Ising model,

\begin{equation}
-\frac{\psi}{kT} = \lim_{N \rightarrow \infty} \frac{\log Z(\lambda_p/2)}{N} \simeq \frac{1}{2\pi} \int_{0}^{2\pi} \log Z(z \cos \omega_1) d\omega_1
\end{equation}

For 2D Ising model,
(25) \[-\frac{\psi}{kT} \simeq \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_0^{2\pi} \log Z(z\cos \omega_1 + z\cos \omega_2) \, d\omega_1 \, d\omega_2\]

For 3D Ising model,

(26) \[-\frac{\psi}{kT} \simeq \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \log Z(z\cos \omega_1 + z\cos \omega_2 + z\cos \omega_3) \, d\omega_1 \, d\omega_2 \, d\omega_3\]

Again it should be pointed out that Eq. (24), Eq. (25) and Eq. (26) include three integrals, i.e., \(I_1, I_2\) and \(I_3\). All of these integrals are difficult to be calculated.

4. LFEPP OF ISING MODEL IN SOME SPECIAL CASES

To our knowledge, we find that it is difficult to solve three integrals in Eq. (18), Eq. (19) and Eq. (20) (i.e., \(I_1, I_2, I_3\)), resulting in that we can not obtain the \(t_s\) and further evaluate \(f(\lambda_p, t_s)\) and \(f_{tt}(\lambda_p, t_s)\).

To make progress, let us assume that the \(\sqrt{2}\lambda_p t\) is in some sense small, so that we may expand \(\log(\cosh(x))\) in powers of the \(x\), truncating the expansion at some order. That \(\sqrt{2}\lambda_p t\) is small in the cases of:

(1) \(t\) is small. Since

\[
(27) \quad t = \frac{y}{N} = \frac{y_p}{N^{3/2}} = \frac{\sum_j V_{pj} \phi_j}{N^{3/2}} = \frac{\sum_j \bar{V}_{pj} \phi_j}{N^2}
\]

As mentioned above, physically \(\phi_j\) represents the fluctuating magnetization, therefore, Eq. (27) is some kind of weighted mean for the fluctuating magnetization of all sites on a \(N \times N\) lattice.

(2) Another two cases result in small \(\sqrt{2}\lambda_p t\) are from \(\lambda_p\). Taking 1D Ising model as an example, \(\lambda_p = 2z\cos \frac{2\pi}{N} (p-1) = 2\frac{\lambda_p}{kT} \cos \frac{2\pi}{N} (p-1)\). A small \(\lambda_p\) means that the temperature is sufficiently high, or that the exchange interaction \(\epsilon\) between spins is sufficiently weak.

(3) Case (1)+Case (2).

When \(\sqrt{2}\lambda_p t\) is in some sense small and since

\[
\log 2 \cosh(x) \approx \log 2 + \frac{x^2}{2} - \frac{x^4}{12}
\]

\[
f(\lambda_p, t) = -\frac{1}{2} \lambda_p t^2 + I_1
\]

(28) \[
= -\frac{1}{2} \lambda_p t^2 + \frac{1}{\pi} \int_{\pi/4}^{\pi + \pi/4} \log \left(2 \cosh \left(\sqrt{2}\lambda_p t \sin \xi\right)\right) \, d\xi
\]

\[
\approx -\frac{1}{2} \lambda_p t^2 + \log 2 + \frac{(\lambda_p t)^2}{2} - \frac{(\lambda_p t)^4}{8}
\]

and,
LIMITING FREE ENERGY PER PARTICLE FOR ISING MODEL BY APPROXIMATING ITS FUNCTIONAL INTEGRAL

\[
\begin{align*}
  f_t &= -\lambda_p t + \lambda_p^2 t - \frac{\lambda_p^4}{2} t^2 \\
  f_{tt} &= -\lambda_p + \lambda_p^2 - \frac{3\lambda_p^4}{2} t^2
\end{align*}
\]

From Eq. (29) we know that \( t_s = 0 \), and \( f_{tt}(0) = -\lambda_p + \lambda_p^2 < 0 \) since \( 0 < \lambda_p < 1 \). With Laplace’s method and from Eq. (23), we obtain,

\[
Z \sim \left[ \frac{\det K}{(2\pi)^N} \right]^{1/2} \times \left( 2^{-N} I_1(\lambda_p) \times \prod_{p=2}^N (2^{-N}) I_p(\lambda_p) \right)
\]

\[
= \left[ \frac{\det K}{(2\pi)^N} \right]^{1/2} \left( 2^{-N} \right) 2 \exp(N \log 2) \sqrt{\frac{\pi}{2(\lambda_1 - \lambda_2^2)}}
\]

\[
\times \prod_{p=2}^N (2^{-N}) 4 \exp(N \log 2) \sqrt{\frac{\pi}{2(\lambda_p - \lambda_p^2)}}
\]

\[
= 2^{-N/2} \left( 2 \left[ 2 \left( 1 - \lambda_1 \right) \right]^{-1/2} \right) \prod_{p=2}^N 4 \left[ 2 \left( 1 - \frac{\lambda_p}{2} \right) \right]^{-1/2}
\]

where \( \det K = \prod_{p=1}^N \lambda_p \)

Finally, we obtain the LFEPP of Ising model according to Appendix C. Namely, For 1D Ising model,

\[
-\psi = \lim_{N \to \infty} \frac{\log Z(\lambda_p/2)}{N} = \log 2 - \frac{1}{2} \int_0^{2\pi} \log \left[ 1 - 2 \left( z \cos \omega \right) \right] d\omega
\]

For 2D Ising model,

\[
-\frac{\psi}{kT} = \log 2 - \frac{1}{2(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \log \left[ 1 - 2z \left( \cos \omega_1 + \cos \omega_2 \right) \right] d\omega_1 d\omega_2
\]

Using \( \cosh 2z \to 1 \) and \( \sinh 2z \to 2z \) when \( z \to 0 \), we can transform Eq. (3) and Eq. (4) into the following,

\[
-\frac{\psi}{kT} = \frac{1}{2(2\pi)^2} \int_0^{2\pi} \log \left( 1 - 2z \cos \omega_1 \right) d\omega_1
\]

\[
-\frac{\psi}{kT} = \frac{1}{2(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \log \left[ 1 - 2z \left( \cos \omega_1 + \cos \omega_2 \right) \right] d\omega_1 d\omega_2
\]

It can be found that Eq. (33) and Eq. (31) are almost the same, except the minus sign before the integral representation which will be discussed later. So do Eq. (34) and Eq. (32). Therefore, we can infer the LFEPP for 3D Ising model as the following,
$$- \frac{\psi}{kT} = \log 2 - \frac{1}{2} \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \log [1 - 2z (\cos \omega_1 + \cos \omega_2 + \cos \omega_3)] d\omega_1 d\omega_2 d\omega_3$$

5. Discussions

5.1. The LFEP of Ising model in an non-zero external magnetic field. The
method here can be easily generalized to the Ising model with a non-zero external magnetic
field. For this purpose, Einstein summation convention will be mainly adopted in this
section unless otherwise specified. For this case, the nearest-neighbor Ising model in $D$
dimensions ($D = 1, 2, 3, ...$) is defined in terms of the following Hamiltonian (eg., Huang,
1987),

$$\mathcal{H} = -\frac{1}{2} \sum_{i,j=1}^{N} K_{ij} s_i s_j - \sum_{i} s_i \tilde{h}_i$$

(36)

$$\mathcal{H} = -\frac{1}{2} \sum_{i,j=1}^{N} K_{ij} s_i s_j - \sum_{i} s_i \tilde{h}_i$$

where, $\tilde{h} = \beta h$ and $h$ is the Zeeman energy associated with an external magnetic field in
the $z$-direction. $\tilde{h}_i$ is an $N$-dimensional column vectors with $[\tilde{h}_i]_i = \tilde{h}$.

The corresponding partition function now is,

$$\mathcal{Z} = \sum_{\{s_i = \pm 1\}} \exp \left[ \frac{1}{2} s_i K_{ij} s_j + s_i \tilde{h}_i \right]$$

(37)

Next using the following $N$-dimensional Gaussian integrals for a positive definite matrix
$A$ (namely Hubbard-Stratonovich transformation in section 2),

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{N} \frac{d\omega_i}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} x_i A_{ij} x_j + x_i s_i \right) = [\det A]^{-1/2} \exp \left( \frac{1}{2} s_i A_{ij}^{-1} s_j \right)$$

(38)

we write the Ising model here in the following form (Let $K_{ij} = A_{ij}^{-1}$),
\[ \exp \left( \frac{1}{2} s_j K_{ij} s_j \right) = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{N} \frac{dx_i}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} x_i K_{ij}^{-1} x_j + x_i s_i \right) }{[\det A]^{-1/2}} \] 

(39)

Then the partition function is,

\[ Z = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{N} \frac{dx_i}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} x_i K_{ij}^{-1} x_j \right) \sum_{\{s_i\}} \exp \left( s_i \left( x_i + \tilde{h}_i \right) \right) }{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{N} \frac{dx_i}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} x_i K_{ij}^{-1} x_j \right) } \]

\[ \hat{x}_i = x_i + \tilde{h}_i \]

\[ x_i = \hat{x}_i \]

\[ \phi_i = K_{ij}^{-1} x_j \]

\[ \downarrow \]

\[ = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{N} \frac{dx_i}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( x_i - \tilde{h}_i \right) K_{ij}^{-1} \left( x_j - \tilde{h}_j \right) \right] \sum_{\{s_i\}} \exp (s_i x_i) }{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{N} \frac{dx_i}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} x_i K_{ij}^{-1} x_j \right) } \]

\[ \phi_i = K_{ij}^{-1} x_j \]

\[ \downarrow \]

\[ = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{N} \frac{d\phi_i}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \phi_i K_{ij} \phi_j - 2\tilde{h}_i \phi_i + h_i K_{ij}^{-1} \tilde{h}_j \right) \right] \sum_{\{s_i\}} \exp (s_i K_{ij} \phi_j) }{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{N} \frac{d\phi_i}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \phi_i K_{ij} \phi_j - 2\tilde{h}_i \phi_i + h_i K_{ij}^{-1} \tilde{h}_j \right) \right] } \]
\[
\int_{\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{N} \frac{d\phi_i}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} \phi_i K_{ij} \phi_j + \tilde{h}_i \phi_i \right) \sum_{\{s_i\}} \exp \left(s_i K_{ij} \phi_j \right) \]

(40)

\[
\frac{\det K}{(2\pi)^N} \left[ \exp \left(-\frac{1}{2} \tilde{h}_i K_{ij}^{-1} \tilde{h}_j \right) \right]^{1/2} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{i=1}^{N} d\phi_i \exp \left(-\frac{1}{2} \phi_i K_{ij} \phi_j + \tilde{h}_i \phi_i \right) \times \sum_{\{s_i\}} \exp \left(s_i K_{ij} \phi_j \right) \]

where \(\tilde{x} = x_i + \tilde{h}_i\) means that firstly \(x_i = x_i + \tilde{h}_i\), and then \(x_i = \tilde{x}_i\) without causing confusion.

\[
\sum_{\{s_i\}} = \sum_{\{s_i = \pm 1\}}. \]

According to Eq. (7), we know the item

\[
\sum_{\{s_i\}} \exp \left(s_i K_{ij} \phi_j \right) \]

in Eq. (40) can be evaluated as the follows,

\[
2^{-N} \sum_{\{s_i\}} \exp \left(s_i K_{ij} \phi_j \right) = 2^{-N} \sum_{\{s_i = \pm 1\}} \exp \left( \sum_p s_i V_{ip} V_{pj} K_{ij} V_{pj} V_{pj} \phi_j \right) \]

(41)

\[
= \prod_p 2^{-N} \sum_{\{s_i = \pm 1\}} \prod_i \exp \left[ s_i \left( V_{ip} \lambda_p y_p \right) \right] \]

With

\[
\tilde{h}_i \phi_i = \tilde{h}_i V_{ip} V_{pi} \phi_i = \tilde{h}_i V_{ip} y_p \]

and Eq. (41), we have,
\[ Z \propto \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{p=1}^{N} \exp \left( -\frac{1}{2} \lambda_p y_p^2 + \sum_i \tilde{h}_i V_{ip} y_p \right) \]
\[
\times (2^{-N}) \left[ \exp \left\{ \sum_i \log [2 \cosh (V_{ip} \lambda_p y_p)] \right\} \right] dy_p
\]
\[
= \prod_{p=1}^{N} (2^{-N}) \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \lambda_p y_p^2 + \sum_i \tilde{h}_i V_{ip} y_p \right) \]
\[
\times \left\{ \exp \left[ \sum_i \log (2 \cosh (V_{ip} \lambda_p y_p)) \right] \right\} dy_p
\]
\[
= \prod_{p=1}^{N} (2^{-N}) \, I_p
\]

And \( I_p \) now is,
\[ I_p = \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \lambda_p y_p^2 + \sum_i \tilde{h}_i V_{ip} y_p \right) \left\{ \exp \left[ \sum_i \log (2 \cosh (V_{ip} \lambda_p y_p)) \right] \right\} dy_p \]

\[ y_p = y N^{-1/2} \]

\[ \Downarrow \]

\[ = \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \lambda_p \frac{y^2}{N} \right) \exp \left( \sum_i \tilde{h}_i V_{ip} \frac{y}{N} \right) \]
\[
\times \left\{ \exp \left[ \sum_i \log \left( 2 \cosh \left( \frac{V_{ip} \lambda_p y}{N} \right) \right) \right] \right\} dy \]
\[ = \frac{1}{N^{1/2}} \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \lambda_p \frac{y^2}{N} \right) \exp \left( \sum_i \tilde{h}_i V_{ip} \frac{y}{N} \right) \]
\[
\times \left\{ \exp \left[ \sum_i \log \left( 2 \cosh \left( \frac{V_{ip} \lambda_p y}{N} \right) \right) \right] \right\} dy \]
\[ y = tN \]

\[
\downarrow
\]

\[
= N^{1/2} \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \lambda_p Nt^2 \right) \exp \left( \sum_i \tilde{h}_i V_{ip} t \right) \times \left\{ \exp \left[ \sum_i \log \left( 2 \cosh (V_{ip} \lambda_p t) \right) \right] \right\} dt
\]

\[
= N^{1/2} \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \lambda_p Nt^2 \right) \exp \left( \tilde{h}_1 t \right) \times \exp \left\{ \sum_{i=2} \tilde{h}_i \left[ \cos \frac{2\pi}{N} (i-1) (p-1) + \sin \frac{2\pi}{N} (i-1) (p-1) \right] t \right\} \\
\times 2 \cosh (\lambda_p t) \\
\times \exp \left[ \sum_{i=2} \log \left( 2 \cosh \left( \left[ \cos \frac{2\pi}{N} (i-1) (p-1) + \sin \frac{2\pi}{N} (i-1) (p-1) \right] \lambda_p t \right) \right) \right] dt
\]

\[
= N^{1/2} \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \lambda_p Nt^2 \right) \exp \left( \tilde{h}_1 t \right) \exp \left( Q_1 \right) \exp \left( Q_2 \right)
\]

When \( p = 1 \),

\[
Q_1 = \sum_{i=2} \tilde{h}_i \left[ \cos \frac{2\pi}{N} (i-1) (p-1) + \sin \frac{2\pi}{N} (i-1) (p-1) \right] t
\]

(44) \[
= \sum_{i=2} \tilde{h}_i t \\
= (N-1) \tilde{h}_i t
\]

\[
Q_2 = \sum_{i=2} \log \left( 2 \cosh \left( \left[ \cos \frac{2\pi}{N} (i-1) (p-1) + \sin \frac{2\pi}{N} (i-1) (p-1) \right] \lambda_p t \right) \right) \\
= (N-1) \log \left[ 2 \cosh (\lambda_1 t) \right]
\]

(45)
\[ I_1 = N^{1/2} \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \lambda_p N t^2 \right) \exp(\tilde{h}_1 t) \exp \left[ (N - 1)\tilde{h}_1 t \right] \times 2 \cosh(\lambda_1 t) \exp\{(N - 1) \log[2 \cosh(\lambda_1 t)]\} dt \]

\[ = N^{1/2} \int_{-\infty}^{+\infty} \exp \left\{ N \left[ -\frac{1}{2} \lambda_p t^2 + \tilde{h}_1 t + \log(2 \cosh(\lambda_1 t)) \right] \right\} dt \]

When \( p > 1 \), \( Q_1 \) can be obtained with the same procedure to Eq. (16) as the follows (as \( N \to \infty \)),

\[ Q_1 = \sum_{i=2} \tilde{h}_i \left[ \cos \frac{2\pi}{N} (i - 1) (p - 1) + \sin \frac{2\pi}{N} (i - 1) (p - 1) \right] t \]

\[ = N \sum_{i=2} \tilde{h}_i \left[ \cos \frac{2\pi}{N} (i - 1) (p - 1) + \sin \frac{2\pi}{N} (i - 1) (p - 1) \right] t \]

\[ \to \frac{N\tilde{h}_1}{2\pi} \int_0^{2\pi} |\cos (p - 1) \zeta + \sin (p - 1) \zeta| d\zeta \]

\[ = \frac{N\tilde{h}_1}{2\pi} \int_0^{2\pi} |\cos \vartheta + \sin \vartheta| d\vartheta \]

\[ = \frac{N\tilde{h}_1}{2\pi} \int_0^{2\pi} |\cos \vartheta + \sin \vartheta| d\vartheta \]

\[ = 0 \]

Since \( Q_2 \) is the same to Eq. (16),

\[ I_p = N^{1/2} \int_{-\infty}^{+\infty} \exp \left( -\frac{1}{2} \lambda_p N t^2 \right) \exp(\tilde{h}_1 t) \times 2 \cosh(\lambda_p t) \left\{ \frac{N}{\pi} \int_{\pi/4}^{\pi+\pi/4} \log \left[ 2 \cosh \left( \sqrt{2} \lambda_p t \sin \xi \right) \right] d\xi \right\} dt \]

\[ = N^{1/2} \int_{-\infty}^{+\infty} \exp(\tilde{h}_1 t) \times 2 \cosh(\lambda_p t) \left[ N \left( -\frac{1}{2} \lambda_p t^2 + I_1 \right) \right] dt \]

Therefore the partition function now is,
\[
Z = \left[ \frac{\det K}{(2\pi)^N} \right]^{1/2} \exp \left( -\frac{1}{2} \tilde{h}_i K_{ij}^{-1} \tilde{h}_j \right) (2^{-N} I_1) \prod_{p=2} (2^{-N}) I_p \\
= \left[ \frac{\det K}{(2\pi)^N} \right]^{1/2} \exp \left( -\frac{1}{2} \sum_{p} \tilde{h}_i^2 \right) (2^{-N} I_1) \prod_{p=2} (2^{-N}) N^{1/2} \int_{-\infty}^{+\infty} \exp \left( \tilde{h} t \right) \times 2 \cosh(\lambda_p t) \exp \left[ N \left( -\frac{1}{2} \lambda_p t^2 + I_1 \right) \right] dt
\]  

where \( \exp \left( -\frac{1}{2} \tilde{h}_i K_{ij}^{-1} \tilde{h}_j \right) = \exp \left( -\frac{1}{2} \sum_{p} \tilde{h}_i^2 / \lambda_p \right) \). The reason is,

\[
\tilde{h}_i K_{ij}^{-1} \tilde{h}_j = \tilde{h}_i V_{ip} K_{ij}^{-1} \tilde{h}_j V_{pj} = \sum_{p} \left( \tilde{h}_i V_{ip} \right) \frac{1}{\lambda_p} \left( V_{pj} \tilde{h}_j \right) \\
= \sum_{p} \frac{1}{\lambda_p} \left( \tilde{h}_i V_{ip} \right) \left( V_{pj} \tilde{h}_j \right) \\
= \sum_{p} \frac{1}{\lambda_p} \left( \tilde{h}_i V_{ip} \right) \left( V_{pj} \tilde{h}_j \right) \\
= \sum_{p} \frac{1}{\lambda_p} \left( \tilde{h}_i V_{ip} \right) \left( V_{pj} \tilde{h}_j \right) \\
= \sum_{p} \frac{\tilde{h}_i^2}{\lambda_p}
\]

Comparing Eq. (49) and Eq. (23), it can be found that Eq. (49) has only two more items than Eq. (23) has, namely, \( \exp \left( -\frac{1}{2} \sum_{p} \tilde{h}_i^2 / \lambda_p \right) \) and \( \exp \left( \tilde{h} t \right) \) which are related to the non-zero external magnetic field \( h \). The rest of calculation is similar to Section 2 and 3 and need not be described here. This shows that the method here can be easily generalized to the Ising model with a non-zero external magnetic field.

However, the \( \exp \left( -\frac{1}{2} \sum_{p} \tilde{h}_i^2 / \lambda_p \right) \) shows that \( \lambda_p \) will be in the denominator, which results in a non-regular \( f(\lambda_p / 2) \) (See details in Appendix C). To avoid this and according to (A.4), an extra parameter \( \alpha \) should be added to the partition function Eq. (6).
\[
Z = \sum_{\{s_i = \pm 1\}} \exp \left( \frac{1}{2} \sum_{ij} K_{ij} s_i s_j \right) \exp (2N\alpha) \exp (-2N\alpha)
\]

\[
\sum_{\{s_i = \pm 1\}} \exp (-2N\alpha) \left[ \frac{\det \tilde{K}}{(2\pi)^N} \right]^{1/2} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} d\phi_k 
\times \exp \left( -\frac{1}{2} \sum_{ij} \phi_i \tilde{K}_{ij} \phi_j + \sum_{ij} s_i \tilde{K}_{ij} \phi_j \right)
\]

Consequently and according to (A.5), (A.6) and (A.7), Eq. (31), (32) and Eq. (35) are now as follows,

For 1D Ising model,

\[
-\psi \frac{kT}{kT} = \log 2 - 2\alpha - \frac{1}{2} \frac{1}{2\pi} \int_0^{2\pi} \log \left[ 1 - 2(\alpha + z \cos \omega_1) \right] d\omega_1
\]

For 2D Ising model,

\[
-\psi \frac{kT}{kT} = \log 2 - 2\alpha - \frac{1}{2} \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \log \left\{ 1 - 2 \left[ \alpha + z (\cos \omega_1 + \cos \omega_2) \right] \right\} d\omega_1 d\omega_2
\]

For 3D Ising model,

\[
-\psi \frac{kT}{kT} = \log 2 - 2\alpha - \frac{1}{2} \frac{1}{8\pi^3} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \log \left\{ 1 - 2 \left[ \alpha + z (\cos \omega_1 + \cos \omega_2 + \cos \omega_3) \right] \right\} 
\times d\omega_1 d\omega_2 d\omega_3
\]

Therefore according to Appendix C, for the special cases \( t_s = 0 \) in section 4, the LFEPP for 1D, 2D and 3D Ising model with a non-zero external magnetic field \( h \) can be obtained by simply adding the following Eq. (55) to Eq. (52), Eq. (56) to Eq. (53) and Eq. (57) to Eq. (54), respectively.

\[
-\frac{1}{2} \frac{1}{2\pi} \int_0^{2\pi} \frac{h^2}{2(\alpha + z \cos \omega_1)} d\omega_1
\]

\[
-\frac{1}{2} \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{h^2}{2\alpha + 2z (\cos \omega_1 + \cos \omega_2)} d\omega_1 d\omega_2
\]

\[
-\frac{1}{2} \frac{1}{8\pi^3} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{h^2}{2\alpha + 2z (\cos \omega_1 + \cos \omega_2 + \cos \omega_3)} d\omega_1 d\omega_2 d\omega_3
\]
5.2. **On the method here and further work.** With a relatively simple method, we obtain an implicit integral representation of the LFEPP of N-D Ising model in absence of external magnetic field. The final LFEPP depends mainly on an integral of $I_1$, its 1-order derivative of $I_2$ and 2-order derivative of $I_3$. If $I_1$, $I_2$ and $I_3$ can be solved (probably using special functions), then the LFEPP can be obtained (asymptotically) exactly. This method, using only the knowledge of mathematical analysis, linear algebra and asymptotic computation, should be more easily understood than Transfer matrix method for 2D and 3D Ising model. To 1D model, however, it seems that this method is more difficult to be operated than matrix method.

Furthermore, we generalize this method to some extent to Ising model with an external magnetic field. It can be found that only two more items, which are related to this non-zero external magnetic field, are included in the partition function additionally. However, we have to introduce another parameter $2\alpha$ to make the $f(\lambda_p/2)$ be regular in Appendix C. Expect that $\alpha > \lambda_{\text{max}}$ (where $\lambda_p$ is the eigenvalues of $K_{ij}$), we still have no idea on what $2\alpha$ should be, although it seems and should have no effect on the final LFEPP.

On the other hand, if only the region close to the critical point for Ising model is of interest, there have been many elegant and effective methods, for example, the Conformal Bootstraps, the Renormalization Group, and so on. Since only the LFEPP in special cases are obtained with the method here, we can not calculate and discuss the related problems.

Even for the spacial cases mentioned above, our LFEPPs have a minus sign before the integral representation which is the same to the results in Feynman (2018), whereas those in section 4 have a plus sign. Although this minus sign can be converted into a plus sign by an appropriate transformation, we still have doubts about this.

Except the constant $\log 2$, our LFEPPs for these special cases is the same to those from Gaussian model. Gaussian model assumes that the probability of finding a give spin $s_i$ between $s_i$ and $s_i + ds_i$ is given by $(2\pi)^{-1/2} \exp(-s_i^2/2)ds_i$ (eg., Berlin and Kac, 1952). These two consistent results show that Ising model is Gaussian with $\langle s_i \rangle = 0$ and $\langle s_i^2 \rangle = 1$ when $T$ is sufficiently high or $\epsilon$ is sufficiently weak, especially when the fluctuating magnetization per site $t$ (or $y_p$, or $y$) is small.

Solving the LFEPPs of 3D Ising model is difficult, and lots of scientists have gone before and after to try. For us, we are trapped on the other side of the river by $I_1$, $I_2$ and $I_3$. Here we only have solved this integral in the case of $\sqrt{2}\lambda_p t \to 0$, whereas the LFEPPs of Ising model require that $\sqrt{2}\lambda_p t \in [0, +\infty]$. How to evaluate $I_1$, then $I_2$ and $I_3$, is the work in future.
APPENDIX A. Matrix $K_{ij}$

For 1D Ising model, let $K = A$, and $A$ is an $N \times N$ matrix (eg., Dixon et al., 2001). $A$ is a circulant matrix when an appropriate periodicity is used on the boundary of the Ising model.

\[
A = z \times \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 1 \\
1 & 0 & 1 & 0 & & \\
0 & 1 & 0 & 1 & & \\
0 & 0 & 1 & 0 & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
1 & 0 & 0 & 1 & 0 & \\
\end{bmatrix}
\]

with the $p$-th ($p = 1, 2, \cdots$) eigenvalue is (eg., Berlin and Kac, 1952),

\[
\lambda_p = 2z \cos \frac{2\pi}{N} (p - 1)
\]

For 2D Ising model, let $K = B$, and $B$ can be constructed from $A$ like the following (eg., Dixon et al., 2001),

\[
B = \begin{bmatrix}
A & I & 0 & 0 & \cdots & I \\
I & A & I & 0 & \cdots & 0 \\
0 & I & A & I & \cdots & I \\
0 & 0 & I & \cdots & \ddots & I \\
I & 0 & \cdots & I & A & \\
\end{bmatrix}
\]

Generally, 2D Ising model is on a rectangular lattice with $n_1$ sites in a row, $n_2$ rows, so that $N = n_1n_2$. $B$ is a circulant matrix when an appropriate periodicity is used on the boundary with the $p$-th ($p = 1, 2, \cdots$) eigenvalue is (eg., Berlin and Kac, 1952),

\[
\lambda_p = 2z \cos \frac{2\pi}{N} (p - 1) + 2z \cos \frac{2\pi n_1}{N} (p - 1)
\]

For 3D Ising model, let $K = C$, and $C$ can be constructed from $B$ like the following (eg., Dixon et al., 2001),

\[
C = \begin{bmatrix}
B & I & O & \cdots & \cdots & I \\
I & B & I & \cdots & \cdots & O \\
O & I & B & \cdots & \cdots & O \\
O & O & I & \cdots & \cdots & I \\
I & O & \cdots & \cdots & O & B \\
\end{bmatrix}
\]

Generally, 3D Ising model is on a lattice with $n_1$ sites in a row, $n_2$ rows in a plane, and $n_3$ planes so that the total number of sites $N = n_1n_2n_3$. $C$ is a circulant matrix when an
appropriate periodicity is used on the boundary with the $p$-th ($p = 1, 2, \cdots$) eigenvalue is (eg., Berlin and Kac, 1952),

\begin{equation}
\lambda_p = 2z \cos \frac{2\pi}{N} (p - 1) + 2z \cos \frac{2\pi n_1}{N} (p - 1) + 2z \cos \frac{2\pi n_1 n_2}{N} (p - 1)
\end{equation}

Generally, we assume that $K$ is positive defined (when $N \to \infty$), as the previous scientists did (eg., Ginzburg and Landau, 1950; Amit and Martin-Mayor, 2005). However, from the eigenvalues of 1D-3D Ising model above, it can be found that the eigenvalues may be zero or negative when $N$ is finite. If this happens, to make $K$ in the Hubbard-Stratonovich transformation be positive, we may write,

\begin{equation}
\sum_{ij} K_{ij} s_i s_j = N2\alpha - N2\alpha + \sum_{ij} K_{ij} s_i s_j
\end{equation}

\begin{equation}
= -N2\alpha + 2\alpha \sum_i s_i^2 + \sum_{ij} K_{ij} s_i s_j
\end{equation}

\begin{equation}
= -N2\alpha + \sum_{ij} \tilde{K}_{ij} s_i s_j
\end{equation}

where $\alpha$ is a real positive to make the eigenvalues of $\tilde{K}$ are positive. $\tilde{K}$ is not a circulant matrix now but a Toeplitz one, whose diagonal elements are $2\alpha$ (The original is zero). Since $N$ will be very large, $\tilde{K}$ is asymptotically equivalent to a circulant matrix according to Szegö theorem. $\tilde{K}$ has eigenvalues as follows,

For 1D Ising model,

\begin{equation}
\lambda_p = 2\alpha + 2z \cos \frac{2\pi}{N} (p - 1)
\end{equation}

For 2D Ising model,

\begin{equation}
\lambda_p = 2\alpha + 2z \cos \frac{2\pi}{N} (p - 1) + 2z \cos \frac{2\pi n_1}{N} (p - 1)
\end{equation}

For 3D Ising model,

\begin{equation}
\lambda_p = 2\alpha + 2z \cos \frac{2\pi}{N} (p - 1) + 2z \cos \frac{2\pi n_1}{N} (p - 1) + 2z \cos \frac{2\pi n_1 n_2}{N} (p - 1)
\end{equation}

And for any cases above, the corresponding real, orthogonal eigenvectors of $K_{ij}$, normalized to unity, are given by,

\begin{equation}
V_{ij} = N^{-1/2} \left[ \cos \frac{2\pi}{N} (i - 1) (j - 1) + \sin \frac{2\pi}{N} (i - 1) (j - 1) \right]
\end{equation}
Appendix B. Functional integral of Ising model

This section is from Ginzburg and Landau (1950), Amit et al. (2005), and Kopietz et al. (2010).

After Hubbard-Stratonovich transformation, the partition function of Ising model is,

\[ Z = \sum_{\{s_i=\pm1\}} \exp \left( \frac{1}{2} \sum_{ij} K_{ij} s_i s_j \right) \]

\[ = \sum_{\{s_i=\pm1\}} \left[ \frac{\det K}{(2\pi)^N} \right]^{1/2} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{k=1}^{N} d\phi_k \exp \left( -\frac{1}{2} \sum_{ij} \phi_i K_{ij} \phi_j + \sum_{ij} s_i K_{ij} \phi_j \right) \]

Notice the fact that only \( \sum_{ij} s_i K_{ij} \phi_j \) in the exponential of the second row of (B.1) has a dependence on the \( s_i \), then,

\[ \sum_{\{s_i=\pm1\}} \exp \left( \sum_{ij} s_i K_{ij} \phi_j \right) = \sum_{\{s_i=\pm1\}} \prod_i \exp \left( s_i \sum_j K_{ij} \phi_j \right) \]

\[ = \prod_i \left[ \exp \left( \sum_j K_{ij} \phi_j \right) + \exp \left( -\sum_j K_{ij} \phi_j \right) \right] \]

\[ = 2^N \exp \left\{ \sum_i \log \left[ \cosh \left( \sum_j K_{ij} \phi_j \right) \right] \right\} \]

Insert (B.2) into (B.1), we obtain,

\[ Z = \sum_{\{s_i=\pm1\}} \exp \left( \frac{1}{2} \sum_{ij} K_{ij} s_i s_j \right) \]

\[ = \left[ \frac{\det K}{(2\pi)^N} \right]^{1/2} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{k=1}^{N} d\phi_k \times \exp \left\{ -\frac{1}{2} \sum_{ij} \phi_i K_{ij} \phi_j + \sum_i \log \left[ \cosh \left( \sum_j K_{ij} \phi_j \right) \right] \right\} \]
Appendix C. On the $S(f) = N^{-1} \sum_{p=1}^{N} f \left( \frac{\lambda_p}{2} \right)$ when $N \to \infty$

It should be pointed out firstly that we mainly prove $S(f)$ for 3D Ising model. Those for 1D and 2D Ising model are from Berlin and Kac (1952) with minor modification.

The following sum is taken into account,

$$S(f) = \lim_{N \to \infty} N^{-1} \sum_{p=1}^{N} f \left( \frac{\lambda_p}{2} \right)$$

The largest eigenvalue occurs for $p = 1$, and we assume that $f(t)$ is regular when $t > \frac{\lambda_1}{2}$. Consequently,

$$N^{-1} \sum_{p=2}^{N} f \left( \frac{\lambda_p}{2} \right) = N^{-1} f \left( \frac{\lambda_1}{2} \right) + N^{-1} \sum_{p=2}^{N} f \left( \frac{\lambda_p}{2} \right)$$

If $f(\frac{\lambda_1}{2})$ is finite,

$$S(f) = \lim_{N \to \infty} N^{-1} \sum_{p=2}^{N} f \left( \frac{\lambda_p}{2} \right)$$

For 1D Ising model, we have,

$$\frac{\lambda_p}{2} = z \cos \frac{2\pi}{N} (p - 1)$$

where we assume $\alpha = 0$ for simplicity, and $p = 2, 3, \ldots$

Subdivide the interval $0 - 2\pi$ into $N$ equal intervals of length $\frac{2\pi}{N} = \Delta \omega_1$, and let $\omega_1 = (p - 1) \Delta \omega_1$. When $\Delta \omega \to 0$, we have,

$$N^{-1} \sum_{p=2}^{N} f \left( \frac{\lambda_p}{2} \right) = \frac{1}{2\pi} \sum_{\omega_1 = \Delta \omega_1}^{2\pi - \Delta \omega_1} f(z \cos \omega_1) \Delta \omega_1$$

$$S(f) = \lim_{\Delta \omega_1 \to 0} \frac{1}{2\pi} \sum_{\omega_1 = \Delta \omega_1}^{2\pi - \Delta \omega_1} f(z \cos \omega_1) \Delta \omega_1$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} f(z \cos \omega_1) \, d\omega_1 \quad \text{(C.1)}$$

For 2D Ising model,

$$\frac{\lambda_p}{2} = z \cos \frac{2\pi}{N} (p - 1) + z \cos \frac{2\pi n_1}{N} (p - 1)$$

where $N = n_1 n_2$.

Let $p - 1 = p_2 + p_1 n_2$, $p_2 = 0, 1, \ldots n_1 - 1$ and $p_2 = 0, 1, 2, \ldots n_2 - 1$.

Then,
\[
\frac{2 \pi}{N} (p - 1) = \frac{2 \pi}{n_1 n_2} p_2 + \frac{2 \pi}{n_1} p_1
\]
\[
\frac{2 \pi n_1}{N} (p - 1) = \frac{2 \pi}{n_2} p_2 + 2 \pi p_1
\]

And,
\[
\frac{\lambda_p}{2} = z \cos \left( \frac{2 \pi}{n_1 n_2} p_2 + \frac{2 \pi}{n_1} p_1 \right) + z \cos \left( \frac{2 \pi}{n_2} p_2 \right)
\]

Hence,
\[
N^{-1} \sum_{p=2}^{N} f \left( \frac{\lambda_p}{2} \right) = N^{-1} \sum_{p_2=1}^{n_1-1} \sum_{p_1=0}^{n_2-1} f \left( \frac{\lambda_{p_1+1}}{2} \right)
\]
\[
+ N^{-1} \sum_{p_1=1}^{n_1-1} \sum_{p_2=0}^{n_2-1} f \left( \frac{\lambda_{p_1+1}+1}{2} \right)
\]

Let \( \Delta \omega_2 = 2 \pi/n_2 \), \( \omega_2 = (2 \pi/n_2) p_2 \),
\[
\frac{\lambda_{p_2+p_1 n_2+1}}{2} = z \cos \left( \frac{\omega_2}{n_1} + \frac{2 \pi}{n_1} p_1 \right) + z \cos \omega_2
\]

Since \( \omega_2 \) always ranges between 0 and \( 2 \pi \), \( \omega_1/n_1 \) vanishes in the limit \( n_1 \), we have, as \( n_2 \to \infty \),
\[
\sum_{p_2=0}^{n_2-1} f \left( \frac{\lambda_{p_2+p_1 n_2+1}}{2} \right) \to \frac{n_2}{2 \pi} \int_{0}^{2 \pi} f \left( z \cos \left( \frac{2 \pi}{n_1} p_1 \right) + z \cos \omega_2 \right) d\omega_2
\]
and,
\[
N^{-1} \sum_{p=2}^{N} f \left( \frac{\lambda_p}{2} \right) \to \frac{1}{2 \pi n_1} \int_{0}^{2 \pi} d\omega_2 f \left( z + z \cos \omega_2 \right)
\]
\[
+ \frac{n_2}{2 \pi N} \sum_{p_1=1}^{n_1-1} \int_{0}^{2 \pi} f \left( z \cos \left( \frac{2 \pi}{n_1} p_1 \right) + z \cos \omega_2 \right) d\omega_2
\]

The summation over \( p_1 \) leads to a second independent integral. Let \( \Delta \omega_1 = 2 \pi/n_1 \), \( \omega_1 = (2 \pi/n_1) p_1 \), then,
\[
(C.2) \quad S (f) = \frac{1}{(2 \pi)^2} \int_{0}^{2 \pi} \int_{0}^{2 \pi} f \left( z \cos \omega_1 + z \cos \omega_2 \right) d\omega_1 d\omega_2
\]

For 3D Ising model,
\[
\frac{\lambda_p}{2} = z \cos \frac{2\pi}{N} (p - 1) + z \cos \frac{2\pi n_1}{N} (p - 1) + z \cos \frac{2\pi n_1 n_2}{N} (p - 1)
\]

where \(N = n_1 n_2 n_3\).

Let \(p - 1 = rn_3 + q, r = 0, 1, \ldots, n_1 n_2 - 1\), and \(q = 0, 1, \ldots, n_3 - 1\), then,

\[
\sum_{p=2}^{N} = \sum_{r=0}^{n_1 n_2 - 1} \sum_{q=1}^{n_3 - 1}
\]

and

\[
\frac{\lambda_p}{2} = z \cos \left( \frac{2\pi r}{n_1 n_2} + \frac{2\pi q}{n_1 n_2 n_3} \right) + z \cos \left( \frac{2\pi r}{n_2} + \frac{2\pi q}{n_1 n_2 n_3} \right) + z \cos \left( \frac{2\pi r + 2\pi q}{n_3} \right)
\]

Let \(\Delta \omega_3 = 2\pi/n_3\), \(\omega_3 = 2\pi q/n_3\), Since \(r\) is an integer, \(\cos (2\pi r + 2\pi q/n_3) = \cos \omega_3\). Also as \(n_1, n_2, n_3\) become large,

\[
\frac{\lambda_p}{2} = z \cos \left( \frac{2\pi r}{n_1 n_2} \right) + z \cos \left( \frac{2\pi r}{n_2} \right) + z \cos \omega_3
\]

Furthermore,

\[
\sum_{q=1}^{n_3 - 1} = \frac{n_3}{2\pi} \sum_{2\pi/n_3}^{2\pi/n_3} \Delta \omega_3 \to \frac{n_3}{2\pi} \int_{0}^{2\pi} d\omega_3
\]

Hence,

\[
N^{-1} \sum_{p=2}^{N} f \left( \frac{\lambda_p}{2} \right) \to n_3 \frac{2\pi - 2\pi/n_3}{2\pi N} \int_{0}^{2\pi} d\omega_3 \sum_{r=0}^{n_1 n_2 - 1} f \left( \frac{\lambda_p}{2} \right)
\]

Let \(r = tn_2 + s, t = 0, 1, \ldots, n_1 - 1, s = 0, 1, \ldots, n_2 - 1\),

\[
\frac{\lambda_p}{2} = z \cos \left( \frac{2\pi t}{n_1} + \frac{2\pi s}{n_1 n_2} \right) + z \cos \left( \frac{2\pi t + 2\pi s}{n_2} \right) + z \cos \omega_3
\]

Let \(\Delta \omega_2 = 2\pi/n_2, \omega_2 = 2\pi s/n_2\),

\[
\sum_{r=0}^{n_1 n_2 - 1} = \sum_{t=0}^{n_1 - 1} \sum_{s=0}^{n_2 - 1}
\]

\[
= \sum_{t=0}^{n_1 - 1} \sum_{s=0}^{n_2 - 2\pi/n_2} \Delta \omega_2 \to \sum_{t=0}^{n_1 - 1} \sum_{s=0}^{n_2 - 2\pi/n_2} \int_{0}^{2\pi} d\omega_2
\]

Then \(\cos (2\pi t + 2\pi s/n_2) = \cos \omega_2\), and,

\[
\frac{\lambda_p}{2} = z \cos \left( \frac{2\pi t}{n_1} \right) + z \cos \omega_2 + z \cos \omega_3
\]
\[ N^{-1} \sum_{p=2}^{N} f \left( \frac{\lambda_p}{2} \right) \rightarrow \frac{n_2 n_3}{(2\pi)^2} N \int_0^{2\pi} d\omega_2 d\omega_3 \sum_{t=0}^{n_1-1} f \left( \frac{\lambda_p}{2} \right) \]

Now set \( \Delta \omega_1 = 2\pi/n_1, \, \omega_1 = 2\pi t/n_1, \)

\[ N^{-1} \sum_{p=2}^{N} f \left( \frac{\lambda_p}{2} \right) \rightarrow \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} f \left( z \cos \omega_1 + z \cos \omega_2 + z \cos \omega_3 \right) d\omega_1 d\omega_2 d\omega_3 \]

And finally,

\[ S(f) = \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} f \left( z \cos \omega_1 + z \cos \omega_2 + z \cos \omega_3 \right) d\omega_1 d\omega_2 d\omega_3 \]

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