On the Göttche Threshold

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Abstract. For a line bundle \( L \) on a smooth surface \( S \), it is now known that
the degree of the Severi variety of cogenus-\( \delta \) curves is given by a universal
polynomial in the Chern classes of \( L \) and \( S \) if \( L \) is \( \delta \)-very ample. For \( S \)
rational, we relax the latter condition substantially: it suffices that three key
loci be of codimension more than \( \delta \). As corollaries, we prove that the condition
conjectured by Göttche suffices if \( S \) is \( \mathbb{P}^2 \) or \( S \) is any Hirzebruch surface, and
that a similar condition suffices if \( S \) is any classical del Pezzo surface.

1. Introduction

Fix \( \delta \geq 0 \). Fix a smooth irreducible projective complex surface \( S \), and a line
bundle \( L \). Let \( |L| \) be the complete linear system, and \( |L|^\delta \subset |L| \) the Severi variety,
the locus of reduced curves \( C \) of cogenus \( \delta \); so \( \delta \) is the genus drop, \( \delta := p_a C - p_g C \),
or \( \delta = \chi(\tilde{O}_C) - \chi(O_C) \) where \( \tilde{C} \) is the normalization. Let \( |L|^\delta_+ \subset |L|^\delta \) be the
sublocus of \( \delta \)-nodal curves. Often enough when \( S \) is rational, \( |L|^\delta_+ \) is open and
dense in \( |L|^\delta \), so that \( \deg |L|^\delta_+ = \deg |L|^\delta \); see Prp. 2 below.

The degree \( \deg |L|^\delta_+ \) can be found recursively if \( S \) is the plane [28, Thm. 3C.1],
[7, Thm. 1.1], if \( S \) is any Hirzebruch (rational ruled) surface [35, §8], or if \( S \) is any
classical del Pezzo surface (that is, its anticanonical bundle is very ample) [35, §9].
If \( \delta \) and \( S \) are arbitrary, but \( L \) is sufficiently ample, then by [23, 24], by [34], or by
[21], there’s a universal polynomial \( G_\delta(S, L) \) in the Chern classes of \( S \) and \( L \) with

\[
\text{(+) } \deg |L|^\delta_+ = G_\delta(S, L).
\]

Further, set \( r := \dim |L| \). In those cases, \( \deg |L|^\delta_+ \) is the number of \( \delta \)-nodal
curves through \( r - \delta \) general points, and each curve is counted with multiplicity 1
by [19, Lem. (4.7)]. See [20] for a brief survey of related work and open problems.

Given \( \delta \) and \( S \), for precisely which \( L \) does \textcircled{4} hold? It is known [21, Thm. 4.1]
that \textcircled{4} holds if \( L \) is \( \delta \)-very ample, that is if, for any subscheme \( Z \subset S \) of length
\( \delta + 1 \), the restriction map \( H^0(L) \to H^0(L|_Z) \) is surjective. In particular, \textcircled{4} holds

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for \( S = \mathbb{P}^2 \) and \( \mathcal{L} = \mathcal{O}(d) \) if \( d \geq \delta \). Previously, this bound had been confirmed by F. Block \cite{6} Thm.1.3, who also coined the term Göttzsche threshold for the value of \( d \) at which \( \Box \) begins to hold. However, as conjectured by Göttzsche \cite{13} Conj. 4.1, Rmk. 4.4 and proved by Block \cite{6} Thm. 1.4 for \( \delta = 3, \ldots, 14 \), in fact the threshold appears to be \( \lceil \delta/2 \rceil + 1 \) if \( \delta \geq 3 \); whereas, it is 1 if \( \delta = 0, 1, 2 \). Göttzsche \cite{13} Rmk. 4.3, 4.4 also conjectured a value for the threshold if \( S \) is any Hirzebruch surface.

Here we prove Göttzsche’s conjectured value is at least an upper bound on the threshold if \( S \) is \( \mathbb{P}^2 \) or if \( S \) is any Hirzebruch surface, and we prove a similar bound if \( S \) is any classical del Pezzo surface; seeCors. 3, 4, 6 and Rmk. 5 stated just below. Although we cannot say exactly when the bound is tight, in Rmk. 5 we show it isn’t if \( S \) is the first Hirzebruch surface, the blowup of \( \mathbb{P}^2 \) at a point. We derive those results directly from our main results, Thm. 1 and Prp. 2 stated next.

Note that the term immersed is used here in the sense of differential geometry; specifically, we call an embedded curve \( D \subset S \) immersed if \( D \) is reduced and the tangent map \( T_D \rightarrow T_S \) is injective, where \( D \) is the normalization.

**Theorem 1.** Assume \( S \) is rational with canonical class \( K \). Let \( V \) be a closed subset of \( |\mathcal{L}| \) that contains every \( D \in |\mathcal{L}| \) such that either

1. \( D \) is nonreduced, or
2. \( D \) has a component \( D_1 \) with \( -K \cdot D_1 \leq 0 \), or
3. \( D \) has a nonimmersed component \( D_1 \) with \( -K \cdot D_1 \approx 1 \).

Then the closure of \( |\mathcal{L}|^\delta - V \) has codimension \( \delta \) at all its points (if any), and its sublocus of immersed curves is open and dense, and is smooth off \( V \). Further, if \( \text{codim} V > \delta \), then \( |\mathcal{L}|^\delta \) has codimension \( \delta \) at all its points, and \( \text{deg} |\mathcal{L}|^\delta = G_\delta(S, \mathcal{L}) \).

**Proposition 2.** Under the conditions of Thm. 1, assume \( D \in V \) also if either

4. \( D \) has a component \( D_1 \) with a point of multiplicity at least 3 and with \( -K \cdot D_1 \leq 3 \), or
5. \( D \) has two components \( D_1, D_2 \) with a common point that is double on \( D_1 \) and with \( -K \cdot D_1 = 1 \) or \( -K \cdot D_2 = 1 \), or
6. \( D \) has two components \( D_1, D_2 \) with a common point that is double on \( D_1 \) and on \( D_2 \) and with \( -K \cdot D_1 = 2 \) and \( -K \cdot D_2 = 2 \), or
7. \( D \) has two components \( D_1, D_2 \) with a common point that is double on \( D_1 \) and simple on \( D_2 \) and with \( -K \cdot D_1 = 2 \), or
8. \( D \) has three components \( D_1, D_2, D_3 \) with a common point that is simple on each and with \( -K \cdot D_1 = 1 \), or
9. \( D \) has two components \( D_1, D_2 \) with a common point that is simple on each and at which they are tangent and with \( -K \cdot D_1 = 1 \), or
10. \( D \) has a component \( D_1 \) with a nonnodal double point and with \( -K \cdot D_1 \leq 2 \).

Then in the closure of \( |\mathcal{L}|^\delta - V \), its sublocus of nodal curves is open and dense. Further, if \( \text{codim} V > \delta \), then \( |\mathcal{L}|^\delta \) is open and dense in \( |\mathcal{L}|^\delta \), and \( \Box \) holds.

**Corollary 3.** Assume \( S = \mathbb{P}^2 \) and \( \mathcal{L} = \mathcal{O}(d) \). If \( d \geq \lceil \delta/2 \rceil + 1 \), then \( \Box \) holds.

**Corollary 4.** Assume \( S \) is the Hirzebruch surface with section \( E \) of self-intersection \( -e \) with \( e \geq 0 \). Assume these subloci of \( |\mathcal{L}| \) have codimension more than \( \delta \): (1) the nonreduced curves, (2) if \( e \geq 1 \), the curves with \( E \) as a component. Then \( \Box \) holds.
Remark 5. Göttsche [13] Rmk. 4.3, 4.4 stated without proof that the codimension condition of Cor. 3 is equivalent to essentially this condition: say $\mathcal{L} = \mathcal{O}(nF + mE)$ where $F$ is a ruling, and set $p := n - em$; then either $m = 0$, $p = 1$, and $\delta = 1$ or

$$m + p \geq 1 \quad \text{and} \quad \delta \leq \begin{cases} \min(2m, p) & \text{if } e \geq 1, \\ \min(2m, 2p) & \text{if } e = 0. \end{cases} \tag{5.1}$$

In fact, more is true; the proof of this equivalence plus the main results yield the following statements. Assume $e \geq 1$ and $m \geq 2$ and $p \geq 0$. Assume the nonreduced $D \in |\mathcal{L}|$ appear in codimension more than $\delta$, or equivalently,

$$\delta \leq \min(2m, 2p + e + 1) \tag{5.2}$$

Assume $\delta \geq p + e$ too. Then there are curves in $|\mathcal{L}|^\delta$ with $E$ as a component, and they form a component of $|\mathcal{L}|^\delta$ of codimension $\delta - e + 1$; the other components are of codimension $\delta$. Lastly, if $e = 1$, then $\deg |\mathcal{L}|^\delta = G_\delta(S, \mathcal{L})$; further, $[\mathbb{F}]$ holds at least if $\delta = p + 1$ too.

Corollary 6. Assume $S$ is a classical del Pezzo surface. Assume these subloci of $|\mathcal{L}|$ have codimension more than $\delta$: (1) the nonreduced curves, (2) the curves with a $-1$-curve as a component. Then $[\mathbb{F}]$ holds.

Section 2 derives the three corollaries from the theorem and the proposition. It also proves the remark. Section 3 proves four lemmas about the Severi variety and the relative Hilbert scheme. Section 4 uses those lemmas to prove the theorem and the proposition, which are the main results.

Throughout, $\delta, S, \mathcal{L}, K$, and so forth continue to be as above. In particular, $C$ denotes a reduced member of $|\mathcal{L}|$, and $D$ an arbitrary member. In addition, $\Gamma$ denotes an arbitrary reduced curve on $S$, usually integral, but not always.

As some loci may be empty, we adopt the convention that the empty set has dimension $-1$, and so codimension 1 more than the dimension of the ambient space. Thus, in the theorem and the proposition, the hypothesis $\text{codim } V > \delta \geq 0$ implies that $\dim |\mathcal{L}| \geq 0$; in particular, $\mathcal{L}$ is nontrivial.

2. Proof of the corollaries and the remark

Before addressing the corollaries and the remark, we prove the following lemma, which we use to handle the bounds in Cor.3 and Rmk.3.

Lemma 7. Assume that $S$ is rational and that $D \in |\mathcal{L}|$. Then $H^2(S, \mathcal{L}) = 0$ and $\dim |\mathcal{L}| \geq D \cdot (D - K)/2$. Equality holds and $H^3(S, \mathcal{L}) = 0$ if this condition obtains: every component $\Gamma$ of $D$ satisfies $-K \cdot \Gamma \geq 1$, and every $\Gamma$ that is a $-1$-curve appears with multiplicity 1.

Proof. Since $S$ is integral, $H^0(S, \mathcal{O}_S) = 1$. Since $S$ is rational, $H^q(S, \mathcal{O}_S) = 0$ for $q = 1, 2$. Hence the Riemann–Roch theorem yields

$$\dim |\mathcal{L}| = D \cdot (D - K)/2 + \dim H^1(S, \mathcal{L}) - \dim H^2(S, \mathcal{L}). \tag{7.1}$$

Thus it suffices to study the vanishing of $H^1(S, \mathcal{L})$ and $H^2(S, \mathcal{L})$.

Given a component $\Gamma$ of $D$, let $m_\Gamma$ denote its multiplicity of appearance. Set $m := \sum m_\Gamma$, and proceed by induction on $m$. Suppose $m = 0$. Then $D = 0$. So $\mathcal{L} = \mathcal{O}_S$. Hence in this case, both groups vanish.

Suppose $m \geq 1$. Fix a component $\Gamma$, and set $\mathcal{L}' := \mathcal{L}(-\Gamma)$. Form the standard
sequence $0 \to \mathcal{L}' \to \mathcal{L} \to \mathcal{L}\Gamma \to 0$, and take cohomology to get this sequence:

$$H^q(S, \mathcal{L}') \to H^q(S, \mathcal{L}) \to H^q(S, \mathcal{L}\Gamma) \quad \text{for} \quad q = 1, 2.$$ 

By induction, $H^2(S, \mathcal{L}') = 0$. As $\Gamma$ is a curve, $H^2(\Gamma, \mathcal{L}\Gamma) = 0$. Thus $H^2(S, \mathcal{L}) = 0$, as desired.

Assume the stated condition obtains. Then by induction, $H^1(S, \mathcal{L}') = 0$. Thus, it suffices to show $H^1(\Gamma, \mathcal{L}\Gamma) = 0$.

Let $K\Gamma$ be the canonical class. By adjunction, $\mathcal{O}_\Gamma(K\Gamma) = \mathcal{O}_\Gamma(D - \Gamma + K\Gamma)$.

The latter group is dual to $H^1(\Gamma, \mathcal{L}\Gamma)$, and so \((\text{Thm. 1 and Prp. 2}$. Thus it remains to consider \((1)\).

Note in passing that, if $\mathcal{L}'$ is a ruling. Then every curve $\Gamma$ is equivalent to $\mathcal{L} = O_S(m\Gamma)$ where $\Gamma$ is a $-1$-curve and $m \geq 1$, then $$(\text{1.1}) \quad \text{yields $dim H^1(S, \mathcal{L}) = m(m - 1)/2$.}$$

**Proof of Cor. 3.** Note $deg K = -3$; so $-K \cdot \Gamma \geq 3$ for every integral curve $\Gamma$ on $S$, and $-K \cdot \Gamma \geq 9$ if $\Gamma$ is singular. So no $D \in |\mathcal{L}|$ satisfies any of $(2)$–$(10)$ of Thm. \[1\] and Prp. \[2\]. Thus it remains to consider $(1)$.

The nonreduced $D \in |\mathcal{L}|$ are of the form $D = A + B$ with $A, B$ effective. Set $b = deg B$. Fix $b \geq 1$. Then these $D$ form a locus of dimension $dim |A| + dim |B|$, so of codimension $b(4d - 5b + 3)/2$ owing to Lem. \[4\]. But $d \geq 2b$. So

$$b(4d - 5b + 3)/2 - (2d - 1) = (b - 1)(4d - 5b - 2)/2 \geq (b - 1)(3b - 2)/2 \geq 0.$$ 

Therefore, when $b = 1$, the codimension achieves its minimum value, namely, $2d - 1$. This value is more than $\delta$, as desired.

**Proof of Cor. 4.** For the following basic properties of Hirzebruch surfaces, see [15] Ch. V, §2. Let $F$ be a ruling. Then every curve $\Gamma$ is equivalent to $nF + mE$ with $n, m \geq 0$. Suppose $\Gamma$ is integral and $\Gamma \neq E$. Then $n > 0$ and $n - m \geq 0$. Further, $-K = (e + 2)F + 2E$. Finally, $F^2 = 0$ and $F \cdot E = 1$.

Hence $-K \cdot \Gamma = n + (n - m)e + 2m$. Suppose $-K \cdot \Gamma \leq 3$. Then either $n = 1$ and $m = 0$, or $n, m, e = 1$. In first case, $-K \cdot \Gamma = 2$; further, $\Gamma = F$, so $\Gamma$ is smooth. In the second case, $-K \cdot \Gamma = 3$; further, $\Gamma \cdot F = 1$, whence $\Gamma$ is smooth. On the other hand, $E$ is smooth, and $-K \cdot E = 2 - e$. So if $-K \cdot E \leq 1$, then $e \geq 1$.

In $|\mathcal{L}|$ consider the locus of $D$ with a component $\Gamma$ such that $-K \cdot \Gamma \leq k$. By the above, if $k = 1$, then $\Gamma = E$ and $e \geq 1$. So by hypothesis, the locus has codimension more than $\delta$. Further, if $k = 3$, then $\Gamma$ is smooth. Thus all the hypotheses of Thm \[3\] and Prp. \[2\] obtain; whence, \((\text{1.1})\) holds, as asserted.

**Proof of Rmk. 5.** Fix a section $G$ of $S$ complementary to $E$. Then $G$ is equivalent to $eF + E$, so that $\mathcal{L} = O(pF + mG)$. Let’s see that, if there’s a $D \in |\mathcal{L}|$, then $m \geq 0$; further, $p \geq 0$ if also either $e = 0$ or $e \geq 1$ and $D$ doesn’t contain $E$. Indeed, as $|F|$ has no base points, $m = D \cdot F \geq 0$. If $e = 0$, then $S = \mathbb{P}^1 \times \mathbb{P}^1$; whence by symmetry, $p \geq 0$. If $e \geq 1$, then $p = D \cdot E \geq 0$.

Note that, if the nonreduced $D \in |\mathcal{L}|$ form a locus of codimension more than
δ, then \( \dim |L| \geq 0 \); in particular, \( L \) is nontrivial. Then \( m \geq 0 \). Further, if some \( D \in |L| \) doesn’t contain \( E \), then \( p = D \cdot E \geq 0 \). In particular, if the codimension condition of Cor. 4 obtains, then \( m, p \geq 0 \). On the other hand, if \( \delta \geq 0 \), then \( m, p \geq \delta \geq 0 \). Thus to prove the remark, we may assume \( m, p \geq 0 \) and \( m + p \geq 1 \).

If \( m = 0 \) and \( p = 1 \), then \( \dim |L| = 1 \), no \( D \in |L| \) contains \( E \), and every \( D \) is reduced; whence, then the codimension condition of Cor. 4 obtains if and only if \( \delta \leq 1 \), if and only if either \( \delta = 1 \) or \( \delta \geq 1 \). If \( m = 0 \) and \( p \geq 2 \), then \( \dim |L| \geq 2 \), no \( D \in |L| \) contains \( E \), and the nonreduced \( D \) form a locus of codimension 1. Hence, then the codimension condition of Cor. 4 obtains if and only if \( \delta = 0 \), if and only if \( \delta \geq 0 \). Thus, to complete the proof, we may assume \( m \geq 1 \); further, if \( e = 0 \), then by symmetry, we may assume \( p \geq 1 \) too.

The proof of Cor. 4 yields \(-K \cdot F = 2\) and \(-K \cdot G = e + 2\). Also \( L = \mathcal{O}(pF + mG) \) and \( m, p \geq 0 \). So Lem. 4 yields this formula:

\[
\dim |L| = pm + p + m + me(1 + m)/2.
\]

The \( D \in |L| \) containing \( E \) are of the form \( D = A + E \) with \( A \) effective. Set

\[
(7.2) \quad L' := \mathcal{O}_S((p + e)F + (m - 1)G).
\]

Then \( A \in |L'| \). But we now assume \( p \geq 0 \) and \( m \geq 1 \). So Lem. 4 yields

\[
\dim |L'| = pm - 1 + m + me(1 + m)/2 \geq 1.
\]

If \( e \geq 1 \), then \( \dim |E| = 0 \) as \( E^2 = -e \) (whereas if \( e = 0 \), then \( \dim |E| = 1 \)); so the \( D \in |L| \) containing \( E \) form a nonempty locus of codimension exactly \( p = 1 \):

\[
\dim |L| - \dim |L'| = p + 1.
\]

Thus, if \( e \geq 1 \), then the \( D \in |L| \) containing \( E \) appear in codimension more than \( \delta \) if and only if \( \delta \leq p \).

By the same token, if \( e \geq 1 \) and if \( m \geq 2 \), then the \( A \in |L'| \) containing \( E \) appear in codimension \( p + e + 1 \). Conversely, if \( e \geq 1 \) and if there exists such an \( A \), then \( m - 2 = (A - E) \cdot F \geq 0 \). Thus if \( e \geq 1 \), then there exists a \( D \in |L| \) containing \( 2E \) if and only if \( m \geq 2 \); if so, then these \( D \) form a locus of codimension \( 2p + e + 2 \).

Given a nonreduced \( D \in |L| \), say \( D = A + 2B \) with \( A, B \) effective and \( B \neq 0 \). Say \( B \) is equivalent to \( aF + bG \). Then \( A \) is equivalent to \( (p - 2a)F + (m - 2b)G \). Since \( A \) and \( B \) are effective, \( m - 2b \geq 0 \) and \( b \geq 0 \). If \( e \geq 1 \), assume \( D \) does not contain \( E \). Then \( p - 2a \geq 0 \) and \( a \geq 0 \) for any \( e \). Hence, for fixed \( a \) and \( b \), these \( D \) form a locus of dimension \( \dim |A| + \dim |B| \); so Lem. 4 yields its codimension to be

\[
e(a, b) := 2pb + 2am - 5ab + a + b + (1 + 4m - 5b)be/2.
\]

The above analysis assumed given some \( D \) and \( A \) and \( B \). However, given \( a, b \geq 0 \) such that \( p - 2a \geq 0 \) and \( m - 2b \geq 0 \), set

\[
A := (p - 2a)F + (m - 2b)G, \quad B := aF + bG, \quad D := A + 2B.
\]

Then \( A \) and \( B \) are effective. Also, \( D \in |L| \), and \( D \) does not contain \( E \). Further, \( B \neq 0 \) if \( a + b \geq 1 \). So the above analysis yields a locus of nonreduced members of \( |L| \) of codimension \( e(a, b) \).

Note \( e(0, 1) = 2p + 1 + 2e(m - 1) \). But \( p \geq 2a \) and \( m \geq 2b \). So if \( b \geq 1 \), then

\[
e(a, b) - e(0, 1) = (2p + 1)(b - 1) + a(2m - 5b + 1) + (4m - 5b - 4)(b - 1)e/2 \\
\geq (3a + 1 + (3b - 4)e/2)(b - 1).
\]
The latter term is nonnegative. Further, 

\[ \epsilon(a, 0) = a(2m + 1) \geq \epsilon(1, 0) = 2m + 1. \]

Thus \( \min(\epsilon(a, b), \epsilon(0, 1)) = \epsilon(1, 0) + 2e(m - 1) \).

Suppose \( \epsilon = 0 \). Then we are assuming \( m, p \geq 1 \). Hence the nonreduced \( D \in |L| \) form a nonempty locus of codimension exactly \( \min(2m + 1, 2p + 1) \). Thus the codimension condition of Cor. 4 obtains if and only if \( (5.1) \) obtains, as asserted.

Suppose \( \epsilon \geq 1 \) and the codimension condition of Cor. 4 obtains. In this case, we assume \( m \geq 1 \) and \( p \geq 0 \). Then, as proved above, \( \delta \leq p \). So if \( p \leq 1 \), then \( \delta \leq 2m \). If \( p \geq 2 \), take \( a := 1 \) and \( b := 0 \); then the codimension condition yields \( \epsilon(1, 0) \geq \delta \). But \( \epsilon(1, 0) = 2m + 1 \). Thus \( (5.1) \) obtains, as asserted.

Conversely, suppose \( \epsilon \geq 1 \) and \( (5.1) \) obtains. Then, as proved above, the \( D \in |L| \) not containing \( E \) appear in codimension more than \( \delta \). Also, the nonreduced \( D \in |L| \) not containing \( E \) appear in codimension \( \min(2m + 1, 2p + 1 + 2e(m - 1)) \). But we assume \( m - 1 \geq 0 \). Thus the codimension condition of Cor. 4 obtains, as asserted.

Finally, assume \( \epsilon \geq 1 \) and \( m \geq 2 \). Then \( 2e(m - 1) \geq \epsilon + 1 \). Let \( W \) be the locus of all nonreduced curves. Then \( \text{codim} W = \min(2m + 1, 2p + e + 2) \). Thus \( \text{codim} W > \delta \) if and only if \( (5.2) \) obtains, as asserted. Assume \( (5.2) \) does obtain.

Assume \( \delta \geq p + e \) too. Set \( \delta' := \delta - p - e \). Then \( \delta' \leq p + 1 \) as \( \delta \leq 2p + e + 1 \); so \( \delta' \leq p + e \) as \( \epsilon \geq 1 \). Further, \( \delta \leq 2m \), so \( \delta' \leq 2m - p - e \). Hence \( \delta' \leq 2m - 2 \), except possibly if \( p = 0 \); but then, \( \delta' \leq 1 \), so after all \( \delta' \leq 2m - 2 \) as \( m \geq 2 \).

Consider the \( L' \) of \((12)\). By the above analysis, the Severi variety \( |L'|^{\delta'} \) is nonempty and everywhere of codimension \( \delta' \) in \( |L'| \), so of codimension \( \delta - e + 1 \) in \( |L| \). Further, \( |L'|^{\delta'} \) contains a dense open subset of curves \( A \) not containing \( E \). Set \( D := A + E \). Then \( D \in |L|^{\delta} \) as \( p_a D = p_a A + p_a E + A \cdot E - 1 \) and \( p_a D = p_a A + p_a E - 1 \) by general principles. Conversely, given a \( D \in |L|^{\delta} \) containing \( E \), set \( A := D - E \); then, plainly, \( A \in |L'|^{\delta'} \), and \( A \) does not contain \( E \).

Recall that \( \text{codim} W > \delta \); further, if \( \Gamma \) is an integral curve with \( -K \cdot \Gamma \leq 1 \), then \( \Gamma = E \). Let \( V \) be the union of \( W \) and the locus of \( D \in |L| \) containing \( E \). Then by Thm. 11 the closure of \( |L| - V \) has codimension \( \delta \) everywhere. Consequently, there are \( D \in |L|^{\delta} \) containing \( E \), and they form a component of \( |L|^{\delta} \). The other components of \( |L|^{\delta} \) are of codimension \( \delta \); as asserted.

Lastly, assume \( \epsilon = 1 \) in addition. Then \( -K \cdot E = 1 \) and \( E \) is immersed. Thus Thm. 11 yields \( \deg|L|^{\delta} = G_\delta(S, L) \), as asserted.

Further, by Prp. 2, the nodal curves form an open and dense subset of \( |L|^{\delta} \). Assume \( \delta = p + 1 \) also. Then \( \delta' = 0 \). So the \( D \in |L|^{\delta} \) containing \( E \) are the \( D \in |L| \) of the form \( A + E \) where \( A \in |L'| - V \). The \( A \) that meet \( E \) transversally form a dense open sublocus, because the restriction map \( H^0(S, L) \to H^0(L, E) \) is surjective as \( H^1(S, \mathcal{O}) = 0 \) by Lem. 14. Hence the nodal locus is open and dense in \( |L|^{\delta} \). Thus \((13)\) holds, as asserted.

**Proof of Cor. 6** Since \( S \) is a classical del Pezzo surface, we may regard \( S \) as embedded in a projective space with \( -K \) as the hyperplane class. Let \( \Gamma \subset S \) be an integral curve. Suppose \( -K \cdot \Gamma = 1 \). Then \( \Gamma \) is a line. So adjunction yields \( \Gamma^2 = -1 \). Hence \( \Gamma \) is a \(-1\)-curve. In \( |L| \) consider the locus of \( D \) with a component \( D_1 \) such that \( -K \cdot D_1 = 1 \); by hypothesis, this locus therefore has codimension more than \( \delta \). If \( -K \cdot \Gamma = 2 \), then \( \Gamma \) is an integral plane conic, so smooth. Finally, if \( -K \cdot \Gamma = 3 \), then \( \Gamma \) is either a twisted cubic, so smooth, or else an integral plane
cubic, so has no point of multiplicity at least 3. Thus all the hypotheses of Thm. 1 and Prp. 2 obtain; whence, \( \square \) holds, as asserted.

3. Four lemmas

We now set the stage to prove Thm. 1 and Prp. 2. First off, we recall some basic deformation theory from [8] and [14].

Fix the reduced curve \( C \in |L| \). There exist a smooth (analytic or étale) germ
\[
(\Lambda, 0) := (\text{Def}_{\text{loc}}(C), 0)
\]
and a family \( \mathcal{C}_{\Lambda}/\Lambda \) realizing a miniversal deformation of the singularities of \( C \); that is, given any family \( \mathcal{C}_B/B \) and point \( b \in B \) such that the fiber \( \mathcal{C}_b \) is a multigerm of \( C \) along its singular locus \( \Sigma \), there exists a map of germs \( (B, b) \to (\Lambda, 0) \) such that the multigerm \( (\mathcal{C}_B, \Sigma) \) is the pullback of the multigerm \( (\mathcal{C}_\Lambda, \Sigma) \). The tangent map \( T_bB \to T_0\Lambda \) is canonical. Further, there is an identification
\[
(7.3) \quad T_0\Lambda = H^0(C, \mathcal{O}_C/\mathcal{J})
\]
where \( \mathcal{J} \) is the Jacobian ideal of \( C \), the first Fitting ideal of its Kähler differentials.

Denote the cogenus of \( C \) by \( \delta(C) \) and the normalization map by
\[
n: \tilde{C} \to C.
\]
So \( \delta(C) = \dim H^0(n_*\mathcal{O}_{\tilde{C}}/\mathcal{O}_C) \). Denote the locus of \( a \in \Lambda \) with \( \delta(\mathcal{C}_a) = \delta(C) \) by \( \Lambda^{\delta(C)} \). It is called the equigeneric locus or \( \delta \)-constant stratum. Its codimension is \( \delta(C) \). Its reduced tangent cone \( (C_0\Lambda^{\delta})_{\text{red}} \) is a vector space; namely,
\[
(7.4) \quad (C_0\Lambda^{\delta})_{\text{red}} = H^0(C, \mathcal{A}/\mathcal{J})
\]
under the identification \( \mathcal{J} \). Here \( \mathcal{A} \) denotes the conductor ideal sheaf; namely,
\[
\mathcal{A} := \mathcal{H}\mathcal{om}(n_*\mathcal{O}_{\tilde{C}}, \mathcal{O}_C).
\]
The following lemma regarding \( \mathcal{A} \) is fundamental. It is more or less well known.

**Lemma 8.** Denote by \( K_{\tilde{C}} \) the canonical class of \( \tilde{C} \). Then
\[
(8.1) \quad \mathcal{A} \cdot n^*\mathcal{O}_S(C) = \mathcal{O}_C(K_{\tilde{C}} - n^*K)
\]
where, doing double duty, \( n \) also denotes the composition \( n: \tilde{C} \to C \hookrightarrow S \).

Let \( \tilde{M} \) be a line bundle on \( \tilde{C} \), and \( \tilde{C}_1, \ldots, \tilde{C}_b \) be the components of \( \tilde{C} \). Then
\[
(8.2) \quad \dim H^1(\tilde{C}, \mathcal{A} \cdot n_*\tilde{M} \otimes \mathcal{O}_S(C)) \leq \sum_{i=1}^b \max(0, 1 + \deg(\tilde{M}^{-1}(n^*K)|\tilde{C}_i)).
\]

**Proof.** By adjunction, \( \mathcal{O}_C(K_{\tilde{C}}) = \mathcal{O}_C \otimes \mathcal{O}_S(C + K) \). And relative duality yields
\[
n_*\mathcal{O}_{\tilde{C}}(K_{\tilde{C}}) = \mathcal{H}\mathcal{om}(n_*\mathcal{O}_{\tilde{C}}, \mathcal{O}_C(K_{\tilde{C}})) = \mathcal{A} \otimes \mathcal{O}_C(K_{\tilde{C}}).
\]
Hence \( \mathcal{A} \otimes \mathcal{O}_S(C) = n_*\mathcal{O}_{\tilde{C}}(K_{\tilde{C}}) \otimes \mathcal{O}_S(-K) \). But \( n \) is finite, and that equation is just the image under \( n_* \) of \( (8.1) \). Thus \( (8.1) \) holds.

By the same token, \( H^1(\tilde{C}, \mathcal{A} \cdot n_*\tilde{M} \otimes \mathcal{O}_S(C)) = H^1(\tilde{C}, \tilde{M}(K_{\tilde{C}} - n^*K)) \). By duality, the right side is just \( H^0(\tilde{C}, \tilde{M}^{-1}(n^*K))^{\vee} \); whence, \( (8.2) \) holds. \( \square \)

Since \( C \in |L| \), the tangent map \( T_{C}/|L| \to T_0\Lambda \) is just this restriction map:
\[
(8.3) \quad H^0(S, |L|)/\text{Im} H^0(S, \mathcal{O}_S) \to H^0(C, \mathcal{O}_C/\mathcal{J}).
\]
Consequently, using Lem. 8 we can prove the following results about the Severi variety and the Hilbert scheme. The results about the Severi variety are already known in various forms, see [4] (10.1), p. 845], [7] Prp. 2.21 p. 355], [32] Thm. 2.8,
LEMMA 9. Assume \( C \in |\mathcal{L}|^{\delta} \). Set \( \lambda := \dim \ker(H^1(S, \mathcal{O}_S) \to H^1(S, \mathcal{L})) \) and \( \alpha := \dim \ker(H^1(C, \mathcal{A} \cdot \mathcal{O}_C(C)) \to H^1(C, \mathcal{O}_C(C))) \). Then

\[
(9.1) \quad \delta - \alpha - \lambda \leq \dim_c |\mathcal{L}| - \dim_{\mathcal{L}} |\mathcal{L}|^{\delta} \leq \delta \quad \text{and} \\
(9.2) \quad (C_C|\mathcal{L}|^{\delta})_{\text{red}} \subset H^0(\overline{C}, \mathcal{O}_{\overline{C}}(K_{\overline{C}} - n^*K)).
\]

In addition, assume \( \lambda = 0 \) and \( \alpha = 0 \). Then

\[
(9.3) \quad (C_C|\mathcal{L}|^{\delta})_{\text{red}} = H^0(\overline{C}, \mathcal{O}_{\overline{C}}(K_{\overline{C}} - n^*K)).
\]

Finally, assume \( C \) is immersed too. Then \( |\mathcal{L}|^{\delta} \) is smooth at \( C \).

**Proof.** Plainly, \( |\mathcal{O}_S(C)|^{\delta} \) is, locally at \( C \), the preimage of the equigeneric locus \( \Lambda^{\delta} \) in \( \Lambda := \text{Def}_{\text{loc}}(C) \). As codimension cannot increase on taking a preimage from a smooth ambient target, the right-hand bound holds in (9.1).

In general, let \( f : X \to Y \) be a map of schemes, \( x \in X \) a point, \( y := f(x) \in Y \) the image. Plainly, \( f \) induces maps of tangent spaces \( T_x : T_x(X) \to T_y(Y) \) and tangent cones \( C_x(X) \to C_y(Y) \), so a map of reductions \( C_x(X)_{\text{red}} \to C_y(Y)_{\text{red}} \). Thus \( C_x(X)_{\text{red}} \subset T_f^{-1}(C_y(Y)_{\text{red}}) \). Now, take \( |\mathcal{L}|^{\delta} \to \Lambda^{\delta} \) for \( f \), and take \( C \) for \( x \). Therefore, \( (C_C|\mathcal{L}|^{\delta})_{\text{red}} \) lies in the preimage of \( (C_0\Lambda^{\delta})_{\text{red}} \) in \( T_C|\mathcal{L}|^{\delta} \). However, \( T_C|\mathcal{L}|^{\delta} \subset T_C|\mathcal{L}| \). Thus \( (C_C|\mathcal{L}|^{\delta})_{\text{red}} \) lies in the preimage of \( (C_0\Lambda^{\delta})_{\text{red}} \) in \( T_C|\mathcal{L}| \).

Further, the tangent map \( T_C|\mathcal{L}| \to T_C\Lambda \) is given by this composition:

\[
(9.4) \quad \theta : H^0(S, \mathcal{L})/\text{Im} H^0(S, \mathcal{O}_S) \xrightarrow{\simeq} H^0(C, \mathcal{O}_C(C)) \xrightarrow{\nu^*} H^0(C, \mathcal{O}_C/\mathcal{A}).
\]

Therefore, (7.4) and the injectivity of \( \eta \) yield

\[
(9.5) \quad (C_C|\mathcal{L}|^{\delta})_{\text{red}} \subset \theta^{-1} H^0(C, \mathcal{A}/\mathcal{A}) \subset \nu^{-1} H^0(C, \mathcal{A}/\mathcal{A}).
\]

Consider the following composition:

\[
(9.6) \quad \xi : H^0(C, \mathcal{O}_C(C)) \xrightarrow{\nu^*} H^0(C, \mathcal{O}_C/\mathcal{A}) \xrightarrow{\nu} H^0(C, \mathcal{O}_C/\mathcal{A}).
\]

The left-exactness of \( H^0 \) yields \( H^0(C, \mathcal{A}/\mathcal{A}) = \ker \rho \) and \( H^0(C, \mathcal{A} \cdot \mathcal{O}_C(C)) = \ker \xi \). But \( \nu^{-1} \ker \nu = \ker \xi \). Hence \( \nu^{-1} H^0(C, \mathcal{A}/\mathcal{A}) = H^0(C, \mathcal{A} \cdot \mathcal{O}_C(C)) \). But (8.1) implies \( H^0(C, \mathcal{A} \cdot \mathcal{O}_C(C)) = H^0(\overline{C}, \mathcal{O}_{\overline{C}}(K_{\overline{C}} - n^*K)) \). Thus (9.2) holds.

The above considerations also yield \( \nu^{-1} H^0(C, \mathcal{A}/\mathcal{A}) = \ker \xi \). So (9.5) yields

\[
(9.7) \quad \dim \mathcal{C} |\mathcal{L}|^{\delta} = \dim (C_C|\mathcal{L}|^{\delta})_{\text{red}} \leq \dim \ker \xi.
\]

On the other hand, the long exact cohomology sequences involving \( \eta \) and \( \xi \) yield

\[
(9.8) \quad - \dim |\mathcal{L}| + \dim H^0(C, \mathcal{O}_C(C)) - \lambda = 0
\]

\[
(9.9) \quad \dim \ker \xi - \dim H^0(C, \mathcal{O}_C(C)) + \dim H^0(C, \mathcal{O}_C/\mathcal{A}) - \alpha = 0.
\]

But \( \dim H^0(C, \mathcal{O}_C/\mathcal{A}) = \delta \). Thus, combined, (9.7) and (9.8) and (9.9) yield the left-hand bound in (9.1).

In addition, assume \( \lambda = 0 \) and \( \alpha = 0 \). To prove (9.3), let’s show both sides of (9.2) are of the same dimension. The left-hand side is of dimension \( \dim |\mathcal{L}| - \delta \) by (9.1). On the other hand, (9.8) and (9.9) yield \( \dim \ker \xi = \dim |\mathcal{L}| - \delta \), and the considerations after (9.6) show \( \ker \xi \) is equal to the right-hand side, as desired.

Finally, assume \( C \) is immersed too. Then \( \Lambda^{\delta} \) is smooth at \( C \) by Thm. 2.59(1)(c) of [14] p. 355. So \( T_0\Lambda^{\delta} = H^0(C, \mathcal{A}/\mathcal{A}) \) by (7.4). Always, \( T_C|\mathcal{L}|^{\delta} \) maps into \( T_0\Lambda^{\delta} \);
so $T_C|\mathcal{L}^\delta$ lies in the preimage $\mathbf{T}$ of $T_0A^\delta$ in $T_C|\mathcal{L}|$. But $\mathbf{T}$ is a vector space of codimension $\delta$ owing to the above analysis; indeed, $\mathbf{T} = \theta^{-1}H^0(C, A/\mathfrak{J})$, and in \[9.5\], the two extreme terms are of codimension $\delta$. But codim $T_C|\mathcal{L}^\delta \leq \delta$ by \[9.1\]. Thus dim $T_C|\mathcal{L}^\delta = \dim C|\mathcal{L}^\delta$. Thus $|\mathcal{L}|^\delta$ is smooth at $C$. \[\square\]

In the remaining two lemmas, we assume $S$ is regular; that is, $H^1(S, \mathcal{O}_S) = 0$. As a consequence, in Thm.\[1\] and Prp.\[2\] instead of assuming $S$ is rational, we may assume $S$ is regular. But the “generalization is illusory,” as noted in \[30\] (v), p.\[116\] in a similar situation. Indeed, assume dim $|\mathcal{L}| \geq 1$, else, $K \cdot C' \geq 0$, but $-K \cdot \Gamma \geq 1$ for every component $\Gamma$ of $C'$ as $C \notin V$. Since $H^1(S, \mathcal{O}_S) = 0$, Castelnuovo’s Criterion implies $S$ is rational.

The first lemma below addresses the immersedness of a general member of $|\mathcal{L}|^\delta$. The discussion involves another invariant of the reduced curve $C$, namely, the (total) multiplicity of its Jacobian ideal $\mathfrak{J}$, or what is the same, the colength of its extension $\mathfrak{J}\mathcal{O}_C$ to the normalization of $C$. This invariant was introduced by Teissier \[31\], II.67, p.\[139\] in order to generalize Plücker’s formula for the class (the degree of the dual) of a plane curve.

This invariant was denoted $\kappa(C)$ by Diaz and Harris \[8\] (3.2), p.\[441\], but they defined it by the formula

$$\kappa(C) = 2\delta(C) + m(C)$$

where $m(C)$ denotes the (total) ramification degree of $\tilde{C}/C$. The two definitions are equivalent owing to the following formula, due to Pien \[27\], p.\[261\]:

$$\mathfrak{J}\mathcal{O}_C = A \cdot \mathfrak{R}$$

where $\mathfrak{R}$ is the ramification ideal.

The invariant $\kappa(C)$ is upper semicontinuous in $C$; see \[31\] p.\[139\] or \[8\] bot., p.\[450\]. So $|\mathcal{L}|^\delta$ always contains a dense open subset $|\mathcal{L}|^\delta_{\mathfrak{K}}$ on which $\kappa(C)$ is locally constant, termed an equiclassical locus in \[8\].

By definition, $C$ is immersed if and only if $m(C) = 0$. Thus if $C \in |\mathcal{L}|^\delta$, then $\kappa(C) \geq 2\delta$, and $C$ is immersed if and only if $\kappa(C) = 2\delta$. Further, if so, then every curve $D$ in every component of $|\mathcal{L}|^\delta$ containing $C$ is immersed.

**Lemma 10.** Assume $S$ regular, and $C \in |\mathcal{L}|^\delta$. Assume $-K \cdot C_1 \geq 1$ for every component $C_1$ of $C$. If some $C_1$ is not immersed, then $-K \cdot C_1 = 1$.

**Proof.** Fix a $C_1$. Assume $C_1$ is not immersed, but $-K \cdot C_1 \geq 2$. Then there’s a point $\tilde{P}$ in the normalization of $C_1$ at which $n$ ramifies. Set $A' := A \cdot n, \mathcal{O}_{\tilde{C}}(-\tilde{P})$. Then owing to Lem.\[5\] the restriction map

$$H^0(C, \mathcal{O}_C(C)) \to H^0(C, \mathcal{O}_C/A')$$

is surjective. Since $S$ is regular, the following restriction map too is surjective:

$$H^0(S, \mathcal{L}) \to H^0(C, \mathcal{O}_C(C))$$

Set $\mathfrak{K} := n_*(\mathfrak{J}\mathcal{O}_C)$. Then $A' \supset \mathfrak{K}$ owing to Pien’s Formula \[9.10\]. But $\mathfrak{K} \supset \mathfrak{J}$. Set $\Lambda := \text{Def}_{1,\mathfrak{J}}(C)$. It follows, as in the proof of Lem.\[5\] that the image of $T_C|\mathcal{L}|$ in $T_0A$ is transverse to $A'/\mathfrak{J}$. Thus the image of $|\mathcal{L}|$ in $\Lambda$ contains a 1-parameter equigeneric family whose tangent space at 0 is transverse to $A'/\mathfrak{J}$ inside $A/\mathfrak{J}$.

Diaz and Harris \[8\] (5.5), p.\[459\] proved that $\mathfrak{K}/\mathfrak{J}$ is the reduced tangent cone
to the locus of equiclassical deformations. Thus the above 1-parameter family exits \(|\mathcal{L}|\), while remaining in \(|\mathcal{L}|^0\), contrary to the openness of \(|\mathcal{L}|^0\) in \(|\mathcal{L}|^0\).

Finally, we consider the smoothness over \(\mathbb{C}\) of the relative Hilbert scheme of a family. To be precise, given a family of curves with parameter space \(B\) and total space \(\mathcal{E}_B\), denote by \(\mathcal{E}_B^n\) the relative Hilbert scheme of \(n\) points. Further, if \(B \subset |\mathcal{L}|\), take \(\mathcal{E}_B\) to be the total space of the tautological family.

**Lemma 11.** Assume \(S\) regular, and \(-K \cdot C_1 \geq 1\) for every component \(C_1\) of \(C\). Fix \(n \geq 0\). Then the relative Hilbert scheme \(\mathcal{E}_B^n\) is smooth over \(\mathbb{C}\) along the Hilbert scheme \(C^n\) of \(C\) over \(\mathbb{C}\).

**Proof.** The proof has three steps: (1) show that \(\mathcal{E}_B^n\) is smooth over \(\mathbb{C}\) along \(C^n\); (2) show that, for any point \(z \in C^n\), the image in \(T_{0}\Lambda\) of the tangent space \(T_z\mathcal{E}_B^n\) contains \(\mathbb{A}/\mathbb{A}\) in \(T_{0}\Lambda\); and (3) show that \(\mathcal{E}_B^n\) is smooth over \(\mathbb{C}\) along \(C^n\).

The hypothesis that \(S\) is regular and \(-K \cdot C_1 \geq 1\) is not used in the first two steps.

Step (1) was done in [29] Prp. 17. Here’s the idea. First, embed \(\mathcal{E}_B^n\) in \(S^n \times \Lambda\), where \(S^n\) is the Hilbert scheme. The latter is smooth by Fogarty’s theorem. Form the tangent bundle-normal bundle sequence (constructed barhandedly as (6) in [29]); it’s the dual of the Second Exact Sequence of Kähler differentials [15] Prp. 8.12, p. 176). It shows the question is local analytic about the singularities of \(C\), as the smoothness in question is equivalent to the surjectivity of the right-hand map owing to [11] (17.12.1)). So we may replace \(C\) by an affine plane curve \(\{f = 0\}\).

Take a vector space \(V\) of polynomials containing \(f\) and also every polynomial of degree at most \(n\). Form the tautological family \(\mathcal{E}_V/V\). Its relative Hilbert scheme \(\mathcal{E}_V^n\) is smooth over \(\mathbb{C}\) along \(C^n\) owing to the analogous tangent bundle-normal bundle sequence; its right-hand map is surjective by choice of \(V\). Finally, as \(\Lambda\) is versal, there’s a map of germs \(\lambda:\ (V,0) \rightarrow (\Lambda,0)\) such that \(\mathcal{E}_V^n\) is the pullback of \(\mathcal{E}_\Lambda^n\). It’s smooth as the map on tangent spaces is surjective. Thus \(\mathcal{E}_\Lambda^n\) is smooth over \(\mathbb{C}\) along \(C^n\), as desired.

To do Step (2), we may assume that \(z\) represents a subscheme \(Z\) of \(C\) supported on its singular locus \(\Sigma\), because the map of tangent spaces (essentially the map on the left in [29] (6))) is the product of the corresponding maps at the various points \(p\) in the support of \(Z\), and these maps are clearly surjective at the \(p\) where \(C\) is smooth. Set \(\mathcal{O} := \mathcal{O}_{C,\Sigma}\), and let \(I \subset \mathcal{O}\) be the ideal of \(Z\). Then \(T_z\mathcal{E}_\Lambda^n\) is the set of first-order deformations of the inclusion map \(I \hookrightarrow \mathcal{O}\). Further, the map \(T_z\mathcal{E}_\Lambda^n \rightarrow T_{0}\Lambda\) forgets the inclusion, and just keeps the deformation of \(\mathcal{O}\).

Let \(J\) be the Jacobian ideal of \(\mathcal{O}\), the ideal of \(\Sigma\). Then [16] yields \(T_{0}\Lambda = \mathcal{O}/J\). Further, let \(A\) be the conductor ideal of \(\mathcal{O}\).

The map \(T_z\mathcal{E}_\Lambda^n \rightarrow T_{0}\Lambda\) factors through the set \(D(\mathcal{O}, I)\) of first-order deformations of the pair \((\mathcal{O}, I)\) with \(I\) viewed as an abstract \(\mathcal{O}\)-module. The map \(D(\mathcal{O}, I) \rightarrow T_{0}\Lambda\) was studied by Fantechi, Göttsche and van Straten in [10] Sec. C]; they showed that, in \(\mathcal{O}/J\), the image of this map contains \(A/J\).

It remains to show \(T_z\mathcal{E}_\Lambda^n \rightarrow D(\mathcal{O}, I)\) is surjective. So take \((\mathcal{O}', I') \in D(\mathcal{O}, I)\). As \(\mathcal{O}\) is Gorenstein, \(\text{Ext}^1_{\mathcal{O}}(I, \mathcal{O}) = 0\). Hence, since deformations are flat, the Property of Exchange [1] Thm. (1.10)] implies this natural map is bijective:

\[\text{Hom}_{\mathcal{O}}(I', \mathcal{O}') \otimes_{\mathcal{O'}} \mathcal{O} \rightarrow \mathcal{O}
\]

So the inclusion map \(I \hookrightarrow \mathcal{O}\) lifts to a map \(I' \rightarrow \mathcal{O}\). The latter is injective and its
cokernel is flat owing to the Local Criterion of Flatness, as \( O' \) is flat and \( I' \to O' \) reduces to an injection with flat cokernel, namely, \( I \to 0 \).

Finally, consider Step (3). Since \( \Lambda \) is versal, there exists a map of germs 
\( (|\mathcal{L}|, C) \to (\Lambda, 0) \) such that the germ \( (|\mathcal{L}|, z) \) is the pullback of the germ \( (C, z) \), which is smooth over \( \mathbb{C} \) by Step (1). Since \( (|\mathcal{L}|, C) \) and \( (\Lambda, 0) \) are smooth over \( \mathbb{C} \), the pullback \( (|\mathcal{L}|, z) \) is therefore smooth over \( \mathbb{C} \) by general principles, if the images in \( T_N\Lambda \) of the tangent spaces \( T_{C_{|\mathcal{L}|}} \) and \( T_{zC}\mathcal{L} \) sum to \( T_N\Lambda \).

Owing to \( \text{(8.3)} \) and to Step (2), the latter holds if this composition is surjective:
\[
\operatorname{H}^0(S, \mathcal{L}) \to \operatorname{H}^0(C, \mathcal{O}_C(C)) \to \operatorname{H}^0(C, \mathcal{O}_C/A).
\]

However, the first map is surjective as \( S \) is regular, and the second map is surjective by Lem.\([S]\) with \( M = 0c \) owing to the hypothesis \(-K \cdot C_1 \geq 1\). \( \square \)

4. Proof of the main results

Thm.\([1]\) can now be proved by revisiting the construction in \([21]\) of the universal polynomial \( G_S(S, \mathcal{L}) \) and making use of the lemmas in the preceding section.

**Proof of Thm.\([1]\):** First, \( \text{(8.1)} \) yields \( \text{codim}_C |\mathcal{L}|^\delta \leq \delta \) for all \( C \in |\mathcal{L}|^\delta \). Also, \( \operatorname{H}^1(S, \mathcal{O}_S) = 0 \) as \( S \) is rational, and if \( C \in (|\mathcal{L}|^\delta - \mathcal{V}) \), then \( \operatorname{H}^1(C, \mathcal{A} \cdot \mathcal{O}_C(C)) = 0 \) by \( \text{(8.2)} \); hence, if \( C \in (|\mathcal{L}|^\delta - \mathcal{V}) \), then \( \text{(8.1)} \) yields \( \text{codim}_C |\mathcal{L}|^\delta \geq \delta \). Therefore, if \( \text{codim} \mathcal{V} \geq \delta \), then \( \text{codim}_C |\mathcal{L}|^\delta = \delta \) for all \( C \) in the closure \( (|\mathcal{L}|^\delta - \mathcal{V}) \), and then \( (|\mathcal{L}|^\delta - \mathcal{V})^{-} = |\mathcal{L}|^\delta \).

Note that Lem.\([10]\) and the discussion before it imply that, if \( C \in (|\mathcal{L}|^\delta - \mathcal{V})^{-} \), then \( C \in |\mathcal{L}|^\delta \) if only if \( C \) is immersed, and that \( |\mathcal{L}|^\delta \) is open and dense in \( |\mathcal{L}|^\delta \). Further, the last assertion of Lem.\([8]\) now implies \( |\mathcal{L}|^\delta \) is smooth at \( C \) if \( C \notin \mathcal{V} \).

It remains to compute \( \text{deg} |\mathcal{L}|^\delta \) assuming \( \text{codim} \mathcal{V} > \delta \). Denote by \( g \) the common arithmetic genus \( p_a D \) of the \( D \in |\mathcal{L}| \). Bertini’s theorem \([15]\) Cor. 10.9, p. 274] yields a \( \delta \)-plane \( \mathbb{P} \subset |\mathcal{L}| \) avoiding \( \mathcal{V} \cup (|\mathcal{L}| - |\mathcal{L}|^\delta) \) and such that \( \mathbb{C}^{[n]} \) is smooth over \( \mathbb{C} \) for \( n \leq g \). But \( \mathbb{C}^{[n]} \) is, by \([2]\) Thm. 5, p. 5], cut out of \( \mathbb{P} \times S^{[n]} \), where \( S^{[n]} \) is the Hilbert scheme, by a transversally regular section of the rank-\( n \) bundle \( \mathcal{L}^{[n]} \) that is obtained by pulling \( \mathcal{L} \) back to the universal family and then pushing it down. Hence the topological Euler characteristic \( \chi(\mathbb{C}^{[n]}) \) can be computed by integrating polynomials in the Chern classes of \( \mathcal{L}^{[n]} \) and \( S^{[n]} \). But, as Ellingsrud, Göttsche, and Lehn \([9]\) show, such integrals admit universal polynomial expressions in the Chern classes of \( S \) and \( \mathcal{L} \).

Following \([18]\), define \( n_h(\mathbb{P}) \) by this relation:
\[
\sum_{n=0}^\infty q^n \chi(\mathbb{C}^{[n]}) = \sum_{h=-\infty}^g n_h(\mathbb{P}) q^{g-h}(1 - q)^{2h-2}.
\]
For \( D \in |\mathcal{L}| \), define \( n_D(D) \) similarly. By additivity of the Euler characteristic, these definitions are compatible: \( \chi(\mathbb{P}, n_h) = n_h(\mathbb{P}) \) where \( n_h : \mathbb{P} \to \mathbb{Z} \) is the constructible function \( b \mapsto n_h(\mathbb{E}_b) \). By \([26]\) App. B.1], if \( D \) is reduced of geometric genus \( g \), then \( n_h(D) = 0 \) for \( h < g(D) \). Thus the \( n_h(\mathbb{P}) \) admit universal polynomial expressions.

For each \( \epsilon \), Lem.\([8]\) implies \( |\mathcal{L}|^\epsilon \) is of codimension \( \epsilon \) at every \( D \in (|\mathcal{L}|^\epsilon - \mathcal{V}) \). So \( |\mathcal{L}|^\epsilon - \mathcal{V} \) is empty if \( \epsilon > \dim |\mathcal{L}| \). Further, replacing \( \mathbb{P} \) by a more general \( \delta \)-plane if necessary, we may assume \( \mathbb{P} \cap |\mathcal{L}|^\epsilon \) is empty if \( \delta < \epsilon \leq \dim |\mathcal{L}| \). Then there are only finitely many \( D \in \mathbb{P} \) of cogenus \( \delta \), and none of greater cogenus. Thus
\[
n_{g-\delta}(\mathbb{P}) = \sum_{D \in \mathbb{P} \cap |\mathcal{L}|^\epsilon} n_{g}(D).
\]
Alternatively, instead of using \([9.1]\) to bound the codim \(|\mathcal{L}|^\epsilon\), we could use \([25]\) Cor.9], which asserts that, given any family of locally planar curves whose \(n\)th relative Hilbert scheme is smooth over \(\mathcal{E}\) and any \(\epsilon \leq n\), the curves of cogenus \(\epsilon\) form a locus of codimension at least \(\epsilon\) in the base.

Finally, as each \(D \in \mathbb{P} \cap |\mathcal{L}|^\delta\) is immersed, \(n_{\delta}(D) = 1\) by \([29]\) Eqn.5] plus \([5]\) Prp.3.3. Alternatively, this statement follows from \([29]\) Thm.A], because \(|\mathcal{L}|^\delta\) is smooth at \(D\). Thus \(n_{\delta}(\mathbb{P}) = \deg |\mathcal{L}|^\delta\).

Lastly, we prove Prp.2 which provides conditions under which the nodal curves in the Severi variety \(|\mathcal{L}|^\delta\) form a dense open subset \(|\mathcal{L}|^\delta_+\). It is well known that \(|\mathcal{L}|^\delta_+\) is open and dense if \(S\) is the plane; see \([38]\) Thm.2, p.220] and \([4]\) (10.7), p.847] and \([7]\) Prp.2.2, p.355]. Similar arguments work if \(S\) is a Hirzebruch surface; see \([35]\) Prp.8.1, p.74]. The broadest statement is given in \([32]\) Thm.2.8, p.8].

However, even that statement is not broad enough to cover our needs. Moreover, our approach appears to be new in places. In addition, the appendix develops the ideas in \([32]\] further, so as to provide another proof of Prp.2 and the codimension statement in Thm.1.

**Proof of Prp. \([2]\)** Clearly, \(\deg |\mathcal{L}|_+^\delta = \deg |\mathcal{L}|_+^\delta\) if \(|\mathcal{L}|^\delta_+\) is open and dense in \(|\mathcal{L}|\). Thus Thm.\([4]\] and the first assertion of Prp.\([2]\] yield the second.

To prove the first assertion, assume \(C \in \mathcal{L}^\delta\), \(P \in C\), and consider the local Milnor number \(\mu(C,P)\), which vanishes if \(C\) is smooth at \(P\). It is, by \([14]\) Thm.6.2(2), p.114], upper semicontinuous in this sense: there is an (analytic or étale) neighborhood \(B\) of the point in \(|\mathcal{L}|^\delta\) representing \(C\) and a neighborhood \(U\) of \(P\) in the tautological total space \(\mathcal{E}_B\) such that, for each \(b \in B\),

\[
\mu(C,P) \geq \sum_{Q \in \mathcal{E}_b \cap U} \mu(\mathcal{E}_b, Q).
\]

So the total Milnor number \(\mu(C) := \sum_z \mu(C, z)\) too is upper semicontinuous in \(C\).

Therefore, \(|\mathcal{L}|^\delta_+\) always contains a dense open subset \(|\mathcal{L}|^\delta_+\mu\) on which \(\mu(C)\) is locally constant. So fix \(C \in |\mathcal{L}|^\delta_+\). Then after \(B\) is shrunk, equality holds in \((11.1)\). Therefore, there is a section \(B \to \mathcal{E}_B\) along which the family is equisingular by work of Zariski’s \([36], 37]\), of Lê and Ramamujam’s \([22]\) and of Teissier’s—see both \([14]\) Prp.2.62, p.359] and \([31]\) Thm.5.3.1, p.123], as well as the historical note \([31]\) 5.3.10, p.129].

Consider Milnor’s Formula \(\mu(C) = 2\delta - \sum_{Q \in C}(r(Q) - 1)\) where \(r(Q)\) is the number of branches of \(C\) at \(Q\); see \([14]\) Prp.3.35, p.208]. It implies \(\mu(C) \geq \delta\), with equality if and only if \(C\) is \(\delta\)-nodal. So the nodal locus \(|\mathcal{L}|^\delta_+\) is always a union of components of \(|\mathcal{L}|^\delta_\mu\). Thus to complete the proof of Prp.2 it suffices to show \(|\mathcal{L}|^\delta_+ - V\) consists of nodal curves. So assume \(C \in |\mathcal{L}|^\delta_+ - V\).

First of all, \(C\) is immersed by Lem.\([10]\]. So Lem.\([9]\] implies \(|\mathcal{L}|^\delta_\mu\) is smooth at \(C\) with tangent space equal to \(H^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(K_{\tilde{C}} - n^*K))\).

Form the composition \(B \to \mathcal{E}_B \to S\) of the above equisingular section and of the projection. Denote the preimage of \(P \in S\) by \(B'\). Evidently, \(\dim \mathcal{B} - \dim \mathcal{B}' \leq 2\).

By equisingularity, \(P\) has the same multiplicity \(m\) on every \(D \in B'\). Denote by \(S'\) the blowup of \(S\) at \(P\), by \(E\) the exceptional divisor, by \(C'\) the strict transform of \(C\). Set \(C' := \mathcal{O}_S(C')\) and \(\delta' := \delta - m(m - 1)/2\). Taking strict transforms gives a map \(B' \to |\mathcal{L}|^\delta_\mu\). It is injective as taking images gives an inverse.

Denote by \(n' : \tilde{C} \to C'\) the normalization map, by \(K'\) the canonical class of \(S'\).
Then (11.2) yields \((\mathcal{C}_C, |\mathcal{L}|^{\delta})_{\text{red}} \subset H^0(\tilde{C}, \mathcal{O}_C(K_{\tilde{C}} - n^*K'))\). Therefore,

\[
\dim H^0(\tilde{C}, \mathcal{O}_C(K_{\tilde{C}} - n^*K)) - \dim H^0(\tilde{C}, \mathcal{O}_C(K_{\tilde{C}} - n^*K')) \leq \dim_C |\mathcal{L}|^{\delta} - \dim_C |\mathcal{L}'|^{\delta'} = \dim C \ B - \dim_{\text{C}'} B' \leq 2.
\]

The groups in (11.2) belong to the long exact cohomology sequence arising from

\[
0 \to \mathcal{O}_C(K_{\tilde{C}} - n^*K') \to \mathcal{O}_C(K_{\tilde{C}} - n^*K) \to \mathcal{O}_{n^*E} \to 0.
\]

Further, \(H^1(\tilde{C}, \mathcal{O}_C(K_{\tilde{C}} - n^*K)) = 0\) by (11.2) as \(C \notin V\). Hence (11.2) is equal to

\[
\dim H^0(\mathcal{O}_{n^*E}) - \dim H^1(\tilde{C}, \mathcal{O}_C(K_{\tilde{C}} - n^*K')) = 0.
\]

But \(\deg(n^*E) = m; \) so \(\dim H^0(\mathcal{O}_{n^*E}) = m\).

Denote by \(C_1, \ldots, C_h\) the components of \(C\), by \(\tilde{C}_i\) the normalization of \(C_i\). Set

\[k_i := -K \cdot C_i \quad \text{and} \quad m_i := \text{mult}(P, C_i) = \deg(n^*E(\tilde{C}_i)) \geq 0.\]

Now, \(n^*K' = n^*K + n^*E\). Therefore, (8.1) and (8.2) yield

\[
\dim H^1(\tilde{C}, \mathcal{O}_C(K_{\tilde{C}} - n^*K')) \leq \sum_{i=1}^h \max(0, 1 - k_i + m_i).
\]

Note \(m = \sum_{i=1}^h m_i\). Consequently, (11.4) and (11.3) yield

\[
\sum_{i=1}^h s_i \leq 2 \quad \text{where} \quad s_i := m_i - \max(0, 1 - k_i + m_i).
\]

Note \(m_i \geq s_i \geq 0\) for all \(i\), as \(0 \leq \max(0, 1 - k_i + m_i) \leq m_i\) since \(k_i \geq 1\) owing to (2) of Thm.1. Also, \(s_i = 0\) if \(k_i = 1\) for any \(i\) and any \(m_i\); conversely, if \(s_i = 0\) and \(m_i \geq 1\), then \(k_i = 1\). Further, \(m_i = s_i\) if and only if \(k_i \geq m_i + 1\), as both conditions are obviously equivalent to \(\max(0, 1 - k_i + m_i) = 0\). Clearly, \(k_i \leq m_i + 1\) if and only if \(s_i = k_i - 1\).

Using (11.3), let’s now rule out \(m \geq 3\). Aiming for a contradiction, assume

\[
(11.6) \quad m_1 \geq \cdots \geq m_h \quad \text{and} \quad m_1 + \cdots + m_h = m \geq 3.
\]

Now, (11.5) yields \(s_1 \leq 2\) as \(s_1 \geq 0\) for all \(i\). So if \(m_1 \geq 3\), then \(m_1 - 2 \leq 1 - k_1 + m_1\); whence, \(k_1 \leq 3\), contrary to (4) of Prp.2. Thus (11.6) yields \(2 \geq m_1 \geq m_2 \geq 1\).

Suppose \(m_1 = 2\). Then (5) of Prp.2 rules out \(k_1 = 2\) and \(k_2 = 1\). So \(k_1 \geq 2\) and \(k_2 \geq 2\). Suppose \(k_1 = 2\). Then \(s_1 = 1\). So \(s_1 \leq 1\). If \(m_2 = 2\), then \(k_2 = 2\), contrary to (6) of Prp.2. If \(m_2 = 1\), then already \(k_1 = 2\) is contrary to (7) of Prp.2. Suppose \(k_1 \geq 3\). Then \(s_1 = 2\). So \(s_2 = 0\). So \(k_2 = 1\). But this case was already ruled out. Thus the case \(m_1 = 2\) is ruled out completely.

Lastly, suppose \(m_1 = 1\). Then (11.3) yields \(m_2 = 1\) and \(m_3 = 1\) too. So (8) of Prp.2 yields \(k_i \geq 2\) for \(i = 1, 2, 3\). Hence \(s_i = 1\) for \(i = 1, 2, 3\), contradicting (11.5). Thus \(m = 2\), as claimed.

Finally, given \(m = 2\), let’s show \(P\) is a simple node. Since \(C\) is immersed at \(P\), it is locally analytically given by an equation of the form \(y^2 = x^{2k}\) for some \(k \geq 1\). Denote by \(\tilde{P}, \tilde{Q} \in \tilde{C}\) the points above \(P\) on the branches with equations \(y = x^k\) and \(y = -x^k\). Then (8.1) and (8.2) imply

\[
\dim H^1(\tilde{C}, \mathcal{O}_C(K_{\tilde{C}} - n^*K - \tilde{P} - \tilde{Q})) = 0,
\]

because \(k_i \geq 1\) for all \(i\) owing to (2) of Thm.1 and because either \(k_i \geq 2\) for \(i = 1, 2\) if \(\tilde{P} \in \tilde{C}_1\) and \(\tilde{Q} \in \tilde{C}_2\) owing to (9) of Prp.2 or \(k_1 \geq 3\) if \(\tilde{P}, \tilde{Q} \in \tilde{C}_1\) owing to (10) of Prp.2. Hence the following restriction map is surjective:

\[
H^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(K_{\tilde{C}} - n^*K)) \to H^0(\tilde{C}, \mathcal{O}_{\tilde{P} + \tilde{Q}}).
\]

Therefore, there’s a section of \(\mathcal{O}_{\tilde{C}}(K_{\tilde{C}} - n^*K)\) that doesn’t vanish at \(\tilde{P}\), but
Appendix A. An alternative proof by Ilya Tyomkin

Our goal is to use the deformation theory of maps to provide an alternative proof of Prp.\[2\] and the codimension statement in Thm.\[1\]. The general idea goes back to Arbarello and Cornalba \[3\], but the proof contains new ingredients, most of which were introduced in \[32\].

A.1. Notation.

Let \( \delta, S, \mathcal{L}, K, \mathcal{L}^\delta, \mathcal{L}^3 \) be as in the Introduction. Again, we work over complex numbers \( \mathbb{C} \), but as is standard, we denote the residue field at a point \( p \) by \( k(p) \). Moreover, as our treatment is purely algebraic, all the statements and proofs are valid over an arbitrary algebraically closed field of characteristic 0.

Given a morphism \( f: X \to Y \), and \( p^1, \ldots, p^r \in X \) points where \( X \) is smooth, \( \text{Def}(X, f; p) \) denotes the functor of deformations of \( (X, f; p^1, \ldots, p^r) \); i.e., if \((T, 0)\) is a local Artinian \( \mathbb{C} \)-scheme, then \( \text{Def}(X, f; p)(T, 0) \) is the set of isomorphism classes of this data: \((X_T, f_T; p_T^1, \ldots, p_T^r; t)\) where \( X_T \) is \( T \)-flat, each \( p_T^i: T \to X_T \) is a section, \( f_T: X_T \to Y \times T \) is a \( T \)-morphism, and \( t \) is an isomorphism

\[ t: (X_0, f_0; p^1_0, \ldots, p^r_0) \to (X, f; p^1, \ldots, p^r). \]

Let \( \text{Def}^\delta(X, f; p) \) denote the set of first-order deformations \( \text{Def}(X, f; p)(T, 0) \) where \( T := \text{Spec}(\mathbb{C}[\epsilon]) \) and \( \mathbb{C}[\epsilon] \) is the ring of dual numbers.

If \( X \) and \( Y \) are smooth, set \( N_f := \text{Coker}(T_X \to f^* T_Y) \); it’s the normal sheaf.

A.2. Three Lemmas.

Lemma 12. Let \((C; p^1, \ldots, p^r)\) be a smooth curve with marked points, and \( f: C \to S \) a map that does not contract components of \( C \). Then there is a natural exact sequence

\[ 0 \to \bigoplus_{i=1}^r T_{p^i}(C) \to \text{Def}^\delta(C, f; p) \to H^0(C, N_f) \to 0. \]

Proof. Consider the forgetful map \( \phi: \text{Def}^\delta(C, f; p) \to \text{Def}^\delta(C, f) \). It is surjective by the infinitesimal lifting property, since \( C \) is smooth at all the \( p^i \). Its kernel is canonically isomorphic to \( \bigoplus_{i=1}^r \text{Def}^\delta(p^i \to C) \), so to \( \bigoplus_{i=1}^r T_{p^i} C \). Finally, since \( T_C \to f^* T_S \) is injective, \( \text{Def}^\delta(C, f) = \text{Ext}^1(L_{C/S}, O_C) = H^0(C, N_f) \), where \( L_{C/S} \) is the cotangent complex of \( f: C \to S \); see \[17\] (2.1.5.6), p. 138; Prp. 3.1.2, p. 203; Thm. 2.1.7, p. 192) or \[16\] pp. 374–376].

Lemma 13. Let \( C \) be a smooth curve, \( f: C \to S \) a map, \( D \subset S \) a closed curve. Set \( Z := D \times_S C \), and assume \( Z \) is reduced and zero-dimensional. Let \( g: Z \to D \) be the inclusion, and set \( T := \text{Spec}(\mathbb{C}[\epsilon]) \) and \((Z_T, g_T) := (C_T, f_T) \times_{S \times T} (D \times T)\). Then sending \((C_T, f_T)\) to \((Z_T, g_T)\) defines a map \( d\psi: \text{Def}^\delta(C, f) \to \text{Def}^\delta(Z, g) \). Furthermore, \( d\psi(H^0(C, N_f')) = 0 \).
Proof. To prove $d\psi$ is well defined, it suffices to show that $Z_T$ is $T$-flat. Let $0 \in T$ be the closed point, $q \in Z \subset Z_T$ a preimage of $0$, and $h = 0$ a local equation of $D$ at $f(q)$. Then there exists an exact sequence $0 \to \mathcal{O}_{C_T,q} \to \mathcal{O}_{C_T,q} \to \mathcal{O}_{Z_T,q} \to 0$ where the first map $m_h$ is the multiplication by $f_T^*(h)$. Also, $m_h \otimes k(0): \mathcal{O}_{C,q} \to \mathcal{O}_{C,q}$ is injective, since the locus of zeroes of $f^*(h)$ in $C$ is of codimension 1, and so $f^*(h) \in \mathcal{O}_{C,q}$ is not a zero-divisor. Thus, $\mathcal{O}_{Z_T,q}$ is flat by the local criterion of flatness [12 Cor. 5.7]. Thus $d\psi$ is well defined.

As $Z$ is reduced, $Z \cap \text{Supp}(N^\text{tor}_f) = \emptyset$. Set $U := C \setminus \text{Supp}(N^\text{tor}_f)$. Then $d\psi$ factors through $\text{Def}^1(U, f|_U) = N_f(U) = (N_f/N^\text{tor}_f)(U)$. Thus $d\psi(H^0(C, N^\text{tor}_f)) = 0$. □

Lemma 14. Let $W$ be an algebraic variety, $C_W \to W$ a flat family of reduced curves, $\tilde{C}_W \to C_W$ the normalization, and $Z_W \subset \tilde{C}_W$ a reduced closed subvariety quasi-finite over $W$. Then there exists an étale morphism $U \to W$ and sections $s_i: U \to \tilde{C}_U$ such that the following two conditions hold: (1) $C_U \to U$ is equinormalizable, i.e., $\tilde{C}_U \to U$ is flat and $\tilde{C}_U \to C_U$ is the normalization for any $u \in U$; and (2) $Z_U \to U$ is étale and $Z_U = \bigcup_{i=1}^r \eta_i(U)$.

Proof. The generic fiber $\tilde{C}_u$ is normal since normalization commutes with arbitrary localizations. Then it is geometrically normal, since the characteristic is zero; and hence $\tilde{C}_u \to \eta$ is smooth by flat descent. Then $\tilde{C}_W \to W$ is generically smooth by generic flatness theorem, i.e., there exists an open dense subset $U_0 \subset W$ such that $\tilde{C}_{U_0} \to U_0$ is smooth. In particular, $\tilde{C}_{U_0} \to U_0$ is flat and has normal fibers. But, $\tilde{C}_u \to C_u$ is finite for any $u \in U_0$, and hence the normalization. Furthermore, for any étale map $U \to U_0$, the family $C_U \to U$ is equinormalizable since normalization commutes with étale base changes.

The morphism $Z_W \to W$ is finite, and $Z_W$ is reduced. Thus, $Z_u \to \eta$ is finite and étale since the characteristic is zero. Hence, after shrinking $U_0$, we may assume that $Z_{U_0} \to U_0$ is finite and étale. Then there exists an étale morphism $U \to U_0$ such that $Z_U$ is the disjoint union of $\deg(Z_u \to \eta)$ copies of $U$ and the map $Z_U \to U$ is the natural projection. Hence $U$ is as needed. □

A.3. The results.

Proposition 15. Let $W \subseteq |L|^g$ be an irreducible subvariety, $C_W \to W$ the tautological family of curves, $f_W: C_W \to S$ the natural morphism, and $f_0: C_{W_0} \to S$ a general closed point. Assume that $C_0$ is reduced.

1. Then there exists a natural embedding $T_0(W) \hookrightarrow H^0(C_0, N_{f_0}/N^\text{tor}_{f_0})$.
2. If $-K.C \geq 1$ for any irreducible component $C \subseteq C_0$, then
   \[
   \dim(W) \leq h^0(C_0, N_{f_0}/N^\text{tor}_{f_0}) \leq -K.C_0 + p_g(C_0) - 1. \tag{15.1}
   \]
3. If (15.1) is equality and $-K.C > 1$ for an irreducible component $C$ of $C_0$, then $C$ is immersed.
4. If (15.1) is equality and $-K.C > 1$ for any irreducible component $C$ of $C_0$, then $N_{f_0}$ is invertible and $T_0(W) \to H^0(C_0, N_{f_0})$ is an isomorphism.

Proof. Pick a smooth irreducible closed curve $D \subset S$ in a very ample linear system such that $h^0(S, L(-D)) = 0$. Then $D \cap C_W$ is finite for any $w \in W$, and is reduced for almost all $w \in W$ by Bertini’s theorem. In particular, $D \cap C_0$ is reduced since $0 \in W$ is general. Hence the projection $Z_W := \tilde{C}_W \times_S D \to W$ is finite, since it is a projective morphism with finite fibers. Let $g_0: Z_0 \to D$ be
the closed immersion. Then, by Lem. 13 and Lem. 14, there exists a commutative diagram

\[
\begin{array}{ccc}
\mathbf{T}_0(W) & \to & \text{Def}^1(\widetilde{C}_0, f_0) \\
\downarrow & & \downarrow \\
\mathbf{T}_{\mathcal{Z}_0}(|\mathcal{L} \otimes \mathcal{O}_D|) & \to & \text{Def}^1(Z_0, g_0) \\
\end{array}
\]

where \( \mathbf{T}_0(W) \to \mathbf{T}_{\mathcal{Z}_0}(|\mathcal{L} \otimes \mathcal{O}_D|) \) is injective since \( W \subseteq |\mathcal{L}| \subseteq |\mathcal{L} \otimes \mathcal{O}_D| \) by the choice of \( D \); and \( \mathbf{T}_{\mathcal{Z}_0}(|\mathcal{L} \otimes \mathcal{O}_D|) \to \text{Def}^1(Z_0, g_0) \) is injective since \( \mathbf{T}_{\mathcal{Z}_0}(|\mathcal{L} \otimes \mathcal{O}_D|) \) is a subspace of the space of first-order embedded deformations of \( g_0(Z_0) \subset D \), and the latter is canonically isomorphic to \( \bigoplus_{p \in g_0(Z_0)} \mathbf{T}_p(D) = \text{Def}^1(Z_0, g_0) \). Thus, the composition \( \mathbf{T}_0(W) \to H^0(Z_0, \mathcal{N}_{g_0}) \) is injective, and hence so is \( \mathbf{T}_0(W) \to H^0(\widetilde{C}_0, \mathcal{N}_{f_0}/N_{\text{tor}}) \) as asserted by (1).

(2) The first inequality in (15.1) follows from (1). Since both sides of the second inequality in (15.1) are additive with respect to unions, we may assume that \( \mathcal{N}_0 \) is irreducible. Let \( 0 \to \mathcal{N}_{f_0}/N_{\text{tor}} \to \mathcal{F} \) be an invertible extension such that \( c_1(\mathcal{F}) = c_1(\mathcal{N}_0) \). By the assumption, \( c_1(\mathcal{F}) = c_1(\mathcal{N}_{f_0}) - 1 \). Thus, \( h^0(\widetilde{C}_0, \mathcal{F}) = c_1(\mathcal{F}) + 1 = c_1(\omega_{\widetilde{C}_0}) - K.C_0 + 1 \). By Riemann–Roch theorem, since \( h^0(\widetilde{C}_0, \mathcal{F} \otimes \omega_{\widetilde{C}_0}) = 0 \); and hence (15.1) holds.

(3) Once again, we may assume that \( \mathcal{N}_0 \) is irreducible. To prove that \( \mathcal{N}_0 \) is immersible, it is sufficient to show that \( \mathcal{N}_{\text{tor}} \neq 0 \). Assume to the contrary that \( \mathcal{N}_{\text{tor}} = 0 \). Pick an invertible extension \( 0 \to \mathcal{N}_{f_0}/N_{\text{tor}} \to \mathcal{F} \) with \( c_1(\mathcal{F}) = c_1(\mathcal{N}_0) \). By the assumption, \( c_1(\mathcal{F}) = c_1(\mathcal{N}_{f_0}) - 1 \). Thus, \( h^0(\widetilde{C}_0, \mathcal{F}) = c_1(\mathcal{F}) + 1 = c_1(\omega_{\widetilde{C}_0}) - K.C_0 + 1 \). By Riemann–Roch theorem, which is a contradiction.

(4) Note that by (3) we have: \( \mathcal{N}_{\text{tor}} = 0 \), and hence \( \mathcal{N}_{f_0} \) is invertible. Then by (2), \( \dim(\mathbf{T}_0(W)) = h^0(\widetilde{C}_0, \mathcal{N}_{f_0}/N_{\text{tor}}) = h^0(\widetilde{C}_0, \mathcal{N}_f_0) \). Thus, (1) implies that \( \mathbf{T}_0(W) \hookrightarrow H^0(\widetilde{C}_0, \mathcal{N}_f_0) = H^0(\widetilde{C}_0, \mathcal{N}_{f_0}/N_{\text{tor}}) \) is an isomorphism.

\[ \square \]

**Remark 16.** By definition, \( \delta := p_a(C_0) - p_g(C_0) \). Hence, if \( S \) is rational and \( -K.C_0 \geq 1 \), then the adjunction formula and Lem. 7 yield

\[ -K.C_0 + p_g(C_0) - 1 = C_0.(C_0 - K)/2 - \delta = \dim|\mathcal{L}| - \delta. \]

**Proposition 17.** Fix a point \( q \in S \), and a curve \( E \subset S \). Let \( W \subseteq |\mathcal{L}| \) be an irreducible subvariety, \( C_W \to W \) the tautological family of curves, \( \widetilde{C}_W \to C_W \) the normalization, \( f_W: \widetilde{C}_W \to S \) the natural morphism, and \( 0 \in W \) a general closed point. Assume that \( C_0 \) is reduced and immersed, \( \dim(W) = -K.C_0 + p_g(C_0) - 1 \), and \( -K.C \geq 1 \) for any irreducible component \( C \) of \( C_0 \).

(1) If \( -K.C > 1 \) for any irreducible component \( C \) of \( C_0 \), then \( q \notin C_0 \).

(2) Let \( q_0 \in C_0 \) be a point of multiplicity at least three, and \( p^1, p^2, p^3 \in \widetilde{C}_0 \) three distinct preimages of \( q_0 \). Then there exists an irreducible component \( C \subseteq C_0 \) such that \( -K.C \leq \left| \widetilde{C} \cap \{ p^1, p^2, p^3 \} \right| \).

(3) Let \( q_0 \in C_0 \) be a singular point with at least two tangent branches, and \( p^1, p^2 \in \widetilde{C}_0 \) the preimages of \( q_0 \) on these branches. Then there exists an irreducible component \( C \subseteq C_0 \) such that \( -K.C \leq \left| \widetilde{C} \cap \{ p^1, p^2 \} \right| \).
(4) If \(-K.C > 1\) for any irreducible component \(C\) of \(C_0\), then any branch of \(C_0\) intersects \(E\) transversally. Furthermore, if \(C_0^{\text{sing}} \cap E \neq \emptyset\), then there exists an irreducible component \(C \subseteq C_0\) such that \(C^{\text{sing}} \cap E \neq \emptyset\) and \(-K.C = 2\).

**Proof.** First, note that \(N_{f_0}\) is invertible since \(C_0\) is immersed. Thus, the embedding \(T_0(W) \hookrightarrow H^0(\tilde{C}_0, N_{f_0}/N_{f_0}^{\text{tor}}) = H^0(\tilde{C}_0, N_{f_0})\) of Prp.13 (1) is an isomorphism by Prp.13 (2). Then \(h^0(\tilde{C}_0, N_{f_0}) = -K.C_0 + p_\beta(C_0) - 1 = \chi(N_{f_0})\), and hence \(h^1(\tilde{C}_0, N_{f_0}) = 0\). Let \(A_W \subseteq C_W\) be the locus of singular points of the fibers \(C_W \rightarrow W\). Set \(Z_W := \nu^{-1}(A_W) \cup f_W^{-1}(q \cup E) \subset \tilde{C}_W\), where \(\nu: \tilde{C}_W \rightarrow C_W\) is the normalization. Then \(Z_W \subset \tilde{C}_W\) is locally closed, and \(Z_W \rightarrow W\) has finite fibers. Thus, by Lem.12 there exists an étale neighborhood \(U\) of \(0\) and disjoint sections \(s_i: U \rightarrow \tilde{C}_U\) such that \(Z_U = \cup_{i=1}^r s_i(U)\). Set \(p_i := s_i(0)\). Then the isomorphism \(T_0(W) \rightarrow H^0(\tilde{C}_0, N_{f_0}) = \text{Def}^1(\tilde{C}_0, f_0; \underline{p})\) factors through \(\text{Def}^1(\tilde{C}_0, f_0; \underline{p})\) for any \(1 \leq i_1 < \cdots < i_m \leq r\), where \(\underline{p} = (p^{i_1}, \ldots, p^{i_m})\).

Consider the exact sequence of Lem.12

\[ 0 \rightarrow \bigoplus_{j=1}^m \left(T_{\tilde{C}_0} \otimes k(p^{i_j})\right) \rightarrow \text{Def}^1(\tilde{C}_0, f_0; \underline{p}) \rightarrow H^0(\tilde{C}_0, N_{f_0}) \rightarrow 0,\]

the restriction map \(\gamma: H^0(\tilde{C}_0, N_{f_0}) \rightarrow \bigoplus_{j=1}^m (N_{f_0} \otimes k(p^{i_j}))\), and the forgetful map \(\beta: \text{Def}^1(\tilde{C}_0, f_0; \underline{p}) \rightarrow \bigoplus_{j=1}^m \text{Def}^1(p^{i_j}, f_0|_{p^{i_j}}) = \bigoplus_{j=1}^m (f_0^* T_S \otimes k(p^{i_j})).\)

Then the following diagram is commutative:

\[
\begin{array}{ccc}
T_0(W) & \xrightarrow{\beta} & \bigoplus_{j=1}^m (f_0^* T_S \otimes k(p^{i_j})) \\
\| & & \| \\
T_0(W) & \xrightarrow{\gamma} & \bigoplus_{j=1}^m (N_{f_0} \otimes k(p^{i_j})) \\
\end{array}
\]

From the long exact sequence of cohomology associated to the short exact sequence of sheaves \(0 \rightarrow N_{f_0}(\sum_{j=1}^m p^{i_j}) \rightarrow N_{f_0} \rightarrow \bigoplus_{j=1}^m N_{f_0} \otimes k(p^{i_j}) \rightarrow 0\) we obtain:

\(\text{Ker}(\gamma) = H^0(\tilde{C}_0, N_{f_0}(\sum_{j=1}^m p^{i_j})\), and \(\text{Coker}(\gamma) \subseteq H^1(\tilde{C}_0, N_{f_0}(\sum_{j=1}^m p^{i_j})).\)

Thus, the map \(\gamma\) is not surjective if and only if \(h^1(\tilde{C}_0, N_{f_0}(\sum_{j=1}^m p^{i_j})) \neq 0\), since \(h^1(\tilde{C}_0, N_{f_0}) = 0\). In particular, if \(\gamma\) is not surjective then there exists an irreducible component \(C \subseteq C_0\) such that \(c_1(N_{f_0}(\sum_{j=1}^m p^{i_j}))|_C \leq c_1(\omega_C)|_C\), or, equivalently, \(-K.C \leq |C \cap \{p^{i_j}\}|_j=1^m|_1\).

Let \(q_0 \in C_0\) be either a singular point, or a point of intersection \(C_0 \cap E\), or \(q_0 = q\). Assume that \(\nu(s_i(0)) = q_0\) for \(1 \leq i \leq m\). Since \(0 \in W\) is general, \(\nu \circ s_i = \nu \circ s_j\) for all \(1 \leq i \leq j \leq m\). Set \(i_j := j\), and consider diagram (17.1). For any \(1 \leq j \leq m\), the tensor product \(f_0^* T_S \otimes k(p^j)\) is canonically isomorphic to \(T_0(S) = \text{Def}^1(q_0 \rightarrow S)\), and \(\beta\) factors through the diagonal map \(\Delta: T_0(S) \rightarrow \bigoplus_{j=1}^m (f_0^* T_S \otimes k(p^j)).\) Hence \(\text{Im}(\gamma) \subseteq \text{Im}(\pi \circ \Delta)\).

(1) Assume to the contrary that \(q \in C_0\), and set \(q_0 := q\). Without loss of generality, \(\nu(s_i(0)) = q_0\). Set \(m := 1\), and consider diagram (17.1). Then the image of \(T_0(W)\) in \(\text{Def}^1(p^i, f_0|_{p^i})\) is trivial since \(q\) is fixed. Thus, \(\gamma\) is the zero map, and hence there exists an irreducible component \(C \subseteq C_0\) such that \(-K.C \leq 1\), which is a contradiction.

(2) Assume that \(q_0 \in C_0\) is a singular point of multiplicity at least three.
Without loss of generality, $s_4(0), s_2(0), s_3(0)$ are preimages of $q_0$. Set $m := 3$, and consider diagram (17.1). Then $\dim(\text{Im}(\gamma)) \leq \dim(\text{Im}(\pi \circ \Delta)) = 2 < 3$, and hence $\gamma$ is not surjective. Thus, there exists an irreducible component as asserted.

(3) Assume that $C_0$ has at least two tangent branches at $q_0$. Without loss of generality, $s_1(0), s_2(0)$ are the preimages of $q_0$ on the tangent branches. Set $m := 2$, and consider diagram (17.1). Then $\dim(\text{Im}(\gamma)) \leq \dim(\text{Im}(\pi \circ \Delta)) = 1 < 2$, and hence $\gamma$ is not surjective. Thus, there exists an irreducible component as asserted.

(4) Assume that $q_0 \in C_0 \cap E$. Then $q_0 \notin E^{\text{sing}}$ by (1). Without loss of generality, $s_1(0)$ is a preimage of $q_0$. Assume to the contrary that $d_0(T_{s_1(0)}(C_0)) = T_{q_0}(E)$. Set $m := 1$, and consider diagram (17.1). The image of $\gamma$ belongs to the image of $\text{Def}^1(q_0 \to E) = T_{q_0}(E) \to N_{q_0} \otimes k(p^1)$, which is zero. Thus, there exists an irreducible component $C \subset C_0$ such that $-K.C \leq 1$, which is a contradiction. Hence no branch of $C_0$ is tangent to $E$. Assume now that $q_0 \in C_0^{\text{sing}}$. Without loss of generality, $s_1(0)$ and $s_2(0)$ are preimages of $q_0$. Set $m := 2$, and consider diagram (17.1). The image of $\gamma$ belongs to the image of $T_{q_0}(E) \to \oplus_{i=1}^2 (N_{q_0} \otimes k(p^i))$, which is at most one-dimensional. Hence, $\gamma$ is not surjective, and hence there exists an irreducible component $C \subset C_0$ such that $-K.C \leq |C \cap \{p^1, p^2\}| \leq 2$. However, $-K.C \geq 2$ by the assumption. Hence $p^1, p^2 \in C$, $q_0 \in C^{\text{sing}}$, and $-K.C = 2$. □

A.4. Conclusions and final remarks.

First, let us prove the assertion about the codimension in Thm.1. The upper bound follows easily from the fact that the locus of equigeneric deformations in the space of all deformations has codimension $\delta$, as explained at the very beginning of the proof of Lem.9. The lower bound follows from Prp.15 (2) and Rmk.10 applied to every irreducible component $W \subseteq |L|^{\delta} \setminus V$.

Second, let us prove the most difficult part of Prp.2, namely the nodality of a general curve in $|L|^{\delta} \setminus V$: Pick an irreducible component $W \subseteq |L|^{\delta} \setminus V$, and let $0 \in W$ be a general closed point. Then $\dim(W) = -K.C_0 + p_\delta(C_0) - 1$ by Thm.1 and Rmk.10. Furthermore, $C_0$ is immersed by Prp.15 (3) and assumption (3) of Thm.1. By Prp.17 (2), if $C_0$ has a point of multiplicity at least three, then we get a contradiction to assumption (4), or (5), or (6), or (7), or (8) of Prp.2. Similarly, by Prp.17 (3), if $C_0$ has a singular point with at least two tangent branches, then we get a contradiction to assumption (9) or (10) of Prp.2. Thus, $C_0$ is nodal.

Third, note that Prp.15 and Prp.17 imply few previously known results about families of curves on algebraic surfaces such as [33] Thm. 2, p. 220, [3] (3.1), p. 95, [4] (10.7), p. 847], [7] Prp.2.2, p. 355, [35] Prp.8.1, p. 74], and [32] Thm. 2.8, p. 8].

Finally, let us mention that in positive characteristic Prp.15 and Prp.17 are no longer true. It was shown in [33] that to have exist $S, L, W$ as in the Propositions such that: (a) for any étale morphism $U \to W$ the family $C_U$ is not equinormalizable, (b) $\dim(W) = -K.C_0 + p_\delta(C_0) - 1$, and (c) all curves $C_w$ are non-immersed, have tangent branches, and intersect each other non-transversally. However, at least for toric surfaces $S$, it was shown that the bound $\dim(W) \leq -K.C_0 + p_\delta(C_0) - 1$ holds true in arbitrary characteristic.

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