Cylindrical Graph Construction
(definition and basic properties)

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Abstract

In this article we introduce the cylindrical construction for graphs and investigate its basic properties. We state a main result claiming a weak tensor-like duality for this construction. Details of our motivations and applications of the construction will appear elsewhere.

1 Introduction

This article is about a tensor-hom like duality for a specific graph construction that will be called the cylindrical construction. Strictly speaking, the construction can be described as replacement of edges of a graph by some other graph(s) (here called cylinders), where many different variants of this construction can be found within the vast literature of graph theory (e.g. Pultr templates [7, 22] and replacement [10, 14]). As our primary motivation for this construction was related to introducing a cobordism theory for graphs and was initiated through algorithmic problems, we present a general form of this edge-replacement construction in which there also may be some twists at each terminal end (here called bases) of these cylinders. In Section 3 we show how this general construction and its dual almost cover most well-known graph constructions. It is also interesting to note that the role of twists in this construction is fundamental and may give rise to new constructs and results (e.g. see Sections 3 for k-lifts as cylindrical constructions by identity cylinders and note a recent breakthrough of A.W.Marcus, D.A.Spielman and N.Srivastava [18] and references therein for the background).

In Section 2 we go through the basic definitions and notations we will be needing hereafter. An important part of this framework is going through the definitions of different categories of graphs and the pushout construction in them, where we discuss some subtleties related to the existence of pushouts in different setups in the Appendix of this article for completeness.

In Sections 3 we introduce the main construction and its dual, in which we go through the details of expressing many different well-known graph constructions as a cylindrical product

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or its dual. In this section we have tried to avoid being formal and have described the constructions in detail from the scratch using a variant of examples for clarification.

In Sections 4 and 5 we state the main duality result and some categorical perspective showing the naturality of definitions. There are related results to reductions for the graph homomorphism problem that will be considered in Section 6. Other applications of this construction will appear elsewhere.

2 Basic definitions

In this section we go through some basic definitions and will set the notations for what will appear in the rest of the article. Throughout the paper, sets are denoted by capital Roman characters as $X, Y, \ldots$ while graphs are denoted by capital RightRoman ones as $G$ or $H$. Ordered lists are denoted as $(x_1, x_2, \ldots, x_n)$, and in a concise form by Bold small characters as $x$. The notation $|\cdot|$ is used for the size of a set as in $|X|$ or for the size of a list as in $|x|$. Also, the greatest common divisor of two integers $x$ and $y$ is denoted as $\gcd(x, y)$.

We use $\{1, \ldots, k\}$ to refer to the set $\{1, 2, \ldots, k\}$, and the symbol $S_k$ is used to refer to the group of all $k$ permutations of this set. The symbol $id$ stands for the identity of the group $S_k$, and the symbol $[a_1, a_2, \ldots, a_n]$ stands for the cycle $(a_1, a_2, \ldots, a_n)$ in this group. Also, $id \simeq S_1 \leq S_k$ is the trivial subgroup containing only the identity element.

For the ordered list $x = (x_1, x_2, \ldots, x_k)$ and $\gamma \in S_k$, the right action of $\gamma$ on indices of $x$ is denoted as $x^\gamma \overset{\text{def}}{=} (x_{\gamma(1)}, x_{\gamma(2)}, \ldots, x_{\gamma(k)})$, where for any function $f$,

$$fx \overset{\text{def}}{=} (f(x_1), f(x_2), \ldots, f(x_k)).$$

We use the notation $\{x\}$ for the set $\{x_1, x_2, \ldots, x_k\}$.

2.1 Graphs and their categories

In this paper, a graph $G = (V(G), E(G), \iota_G : E(G) \to V(G), \tau_G : E(G) \to V(G))$ consists of a finite set of vertices $V(G)$, a finite set of edges $E(G)$ and two maps $\iota_G$ and $\tau_G$ called initial and terminal maps, respectively. Hence, our graphs are finite but are generally directed, and may contain loops or multiple edges. For any edge $e \in E(G)$, the vertex $u = \iota_G(e)$ is called the initial vertex of $e$ and may be denoted by $e^\to$. Similarly, the vertex $v = \tau_G(e)$ is called the terminal vertex of $e$ and may be denoted by $e^\leftarrow$. In this setting, when there is no ambiguity, the edge $e$ is sometimes referred to by the notation $uv$. Also, we say that an edge $e$ intersects a vertex $u$ if $u = e^\to$ or $u = e^\leftarrow$. Hereafter, we may freely refer to an edge $uv$ when the corresponding data is clear from the context.

The reduced part of a graph $G$, denoted by $\text{red}(G)$, is the graph obtained by excluding all isolated vertices of $G$. A graph is said to be reduced if it is isomorphic to its reduced part. Given a graph $G$ and a subset $X \subseteq V(G)$, the (vertex) induced subgraph on the subset $X$ is denoted by $G[X]$.

A (graph) homomorphism $(\sigma_V, \sigma_E)$ from a graph $G = (V(G), E(G), \iota_G, \tau_G)$ to a graph $H = (V(H), E(H), \iota_H, \tau_H)$ is a pair of maps

$$\sigma_V : V(G) \to V(H) \quad \text{and} \quad \sigma_E : E(G) \to E(H)$$

which are compatible with the structure maps (see Figure 4), i.e.,

$$\iota_H \circ \sigma_E = \sigma_V \circ \iota_G \quad \text{and} \quad \tau_H \circ \sigma_E = \sigma_V \circ \tau_G. \tag{1}$$

Note that from a categorical point of view the natural framework to describe such a construction is the category of cospans, however, we have focused on the current literature and nomenclature of graph theory to avoid unnecessary complexities.
Note that when there is no multiple edge, one may think of \( E(G) \) as a subset of \( V(G) \times V(G) \) and one may talk about the homomorphism \( \sigma \), when the compatibility condition is equivalent to the following,

\[
uv \in E(G) \quad \Rightarrow \quad \sigma(u)\sigma(v) \in E(H).
\]

The set of homomorphisms from the graph \( G \) to the graph \( H \) is denoted by \( \text{Hom}(G, H) \), and \( \text{Grph} \) stands for the category of (directed) graphs and their homomorphisms. Sometimes we may write \( G \to H \) for \( \text{Hom}(G, H) \neq \emptyset \).

The partial ordering on the set of all graphs for which \( G \leq H \) if and only if \( \text{Hom}(G, H) \neq \emptyset \), is denoted by \( \text{Grph} \leq \). Two graphs \( G \) and \( H \) are said to be homomorphically equivalent, denoted by \( G \approx H \), if both sets \( \text{Hom}(G, H) \) and \( \text{Hom}(H, G) \) are non-empty i.e., if \( G \leq H \) and \( H \leq G \). The interested reader is encouraged to refer to [14] for more on graphs and their homomorphisms.

In this paper we will be working in different categories of graphs and their homomorphisms, that will be defined in the sequel.

![Figure 1: Commutative diagrams for a graph homomorphism.](image)

**Definition 1. Marked graphs**

Let \( X = \{x_1, x_2, \ldots, x_k\} \) and \( G \) be a set and a graph, respectively, and also, consider a one-to-one map \( \varrho : X \hookrightarrow V(G) \). Evidently, one can consider \( \varrho \) as a graph monomorphism from the empty graph \( X \) on the vertex set \( X \) to the graph \( G \), where in this setting we interpret the situation as marking some vertices of \( G \) by the elements of \( X \). The data introduced by the pair \( (G, \varrho) \) is called a marked graph \( G \) marked by the set \( X \) through the map \( \varrho \). Note that (by abuse of language) we may introduce the corresponding marked graph as \( G(x_1, x_2, \ldots, x_k) \) when the definition of \( \varrho \) (especially its range) is clear from the context (for examples see Figures 2(a) and 2(b)). Also, (by abuse of language) we may refer to the vertex \( x_i \) as the vertex \( \varrho(x_i) \in V(G) \). This is most natural when \( X \subseteq V(G) \) and vertices in \( X \) are marked by the corresponding elements in \( V(G) \) through the identity mapping.

If \( \varsigma : X \to Y \) is a (not necessarily one-to-one) map, then one can obtain a new marked graph \( (H, \tau : Y \to V(H)) \) by considering the pushout of the diagram

\[
\xymatrix{ Y \ar[r]^\varsigma & X \ar[r]^\varrho & G }
\]

in the category of graphs. It is well-known that in \( \text{Grph} \) the pushout exists and is a monomorphism (e.g. Figure 2(c) presents such a pushout (i.e. amalgam) of marked graphs 2(a) and 2(b)). Also, it is easy to see that the new marked graph \( (H, \tau) \) can be obtained from \( (G, \varrho) \) by identifying the vertices in each inverse-image of \( \varsigma \). Hence, again we may denote \( (H, \tau) \) as \( G(\varsigma(x_1), \varsigma(x_2), \ldots, \varsigma(x_k)) \), where we allow repetition in the list appearing in the brackets. Note that, with this notation, one may interpret \( x_i \)'s as a set of variables in the graph structure \( G(x_1, x_2, \ldots, x_k) \), such that when one assigns other (new and not necessarily distinct) values to these variables one can obtain some other graphs (by identification of vertices) (e.g. see Figure 2(d)).
Figure 2: (a) A marked graph $G(x, y, z)$, (b) A marked graph $H(x, y, w)$, (c) The amalgamation $G(x, y, z) + H(x, y, w)$, (d) The marked graph $G(x, x, z)$, (e) A labeled graph, (f) A labeled marked graph.

On the other hand, given two marked graphs $(G, \varrho)$ and $(H, \tau)$ with $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_l\}$, one can construct their amalgam

$$(G, \varrho) +_Y (H, \tau)$$

by forming the pushout of the following diagram,

$$H \xleftarrow{\tilde{\tau}} S \xrightarrow{\tilde{\varrho}} G,$$

in which $S$ is a graph structure induced on $X \cap Y$ indicating the way edges should be identified in this amalgam such that one can define the extensions $\tilde{\tau} \overset{\text{def}}{=} \tau|_{X \cap Y}$ and $\tilde{\varrho} \overset{\text{def}}{=} \varrho|_{X \cap Y}$ as graph homomorphisms (note that pushouts exist in the category of graphs and can be constructed as pushouts of the vertex set and the edge set naturally. For more details see the Appendix). Following our previous notations we may denote the new structure by

$$G(x_1, x_2, \ldots, x_k) + H(y_1, y_2, \ldots, y_l).$$

If there is no confusion about the definition of mappings (in what follows the maps $\tilde{\tau}$ and $\tilde{\varrho}$ are embeddings of a subgraph on $X \cap Y$). Note that when $X \cap Y$ is the empty set, then the amalgam is the disjoint union of the two marked graphs. Also, by the universal property of the pushout diagram, the amalgam can be considered as marked graphs marked by $X$, $Y$, $X \cup Y$ or $X \cap Y$.

Sometimes it is preferred to partition the list of variables in a graph structure as,

$$G(x_1, x_2, \ldots, x_k; y_1, y_2, \ldots, y_l; z_1, z_2, \ldots, z_m).$$

In these cases we may either use bold symbols for an ordered list of variables and write this graph structure as $G(\mathbf{x}; y; z)$ (if there is no confusion about the size of the lists), or we may simply write

$$G(x_1, x_2, \ldots, x_k; y_1, y_2, \ldots, y_l; z_1, z_2, \ldots, z_m).$$

It is understood that a repeated appearance of a graph structure in an expression as $G(v) + G(u, w)$ is always considered as different isomorphic copies of the structure marked properly by the indicated labels (e.g. $G(v) + G(v, w)$ is an amalgam constructed by two different isomorphic copies of $G$ identified on the vertex $v$ where the vertex $w$ in one of these copies is marked).
By $K_k(v_1, v_2, \ldots, v_k)$ we mean a $k$-clique on $\{v_1, v_2, \ldots, v_k\}$ marked by its own set of vertices. Specially, a single edge is denoted by $\varepsilon(u_1, v_2)$ (i.e., $\varepsilon(v_1, v_2) \equiv K_2(v_1, v_2)$). We use the notation $G[\{x\}]$ for the induced subgraph on vertices of $G$ labeled by $x$.

**Definition 2. Labeled graphs**

An $\ell_G$-graph $G$ labeled by the set $L_G$ consists of the data $(G, \ell_G : E(G) \rightarrow L_G)$, where $G$ is a graph and $\ell_G$ is the labeling map (e.g. see Figure 2(c)). A (labeled graph) homomorphism $(\sigma_v, \sigma_E, \sigma_{\varepsilon})$ from an $\ell_G$-graph $G = (V(G), E(G), \ell_G)$ to an $\ell_H$-graph $H = (V(H), E(H), \ell_H)$ is a triple

$$\sigma_v : V(G) \rightarrow V(H), \quad \sigma_E : E(G) \rightarrow E(H) \quad \text{and} \quad \sigma_{\varepsilon} : L_G \rightarrow L_H,$$

where all maps are compatible with the structure maps, i.e., conditions of Equation 1 are satisfied as well as,

$$\ell_H \circ \sigma_E = \sigma_{\varepsilon} \circ \ell_G. \quad (4)$$

The set of homomorphisms from the $\ell_G$-graph $G$ to the $\ell_H$-graph $H$ is denoted by $\text{Hom}_\ell(G, H)$, and $\text{LGrph}$ stands for the category of labeled graphs and their homomorphisms.

It is obvious that the space of all labeled graphs, $\text{LGrph}$, is a triple

$$(G, \varrho : X \hookrightarrow V(G), \ell : E(G) \rightarrow L),$$

where $(G, \varrho)$ is a marked $\ell$-graph equipped with the labeling $\ell : E(G) \rightarrow L$ of its edges (e.g. see Figure 2(f)). Again, given two labeled marked graphs

$$(G, \varrho_G : X \hookrightarrow V(G), \ell_G : E(G) \rightarrow L_G) \quad \text{and} \quad (H, \varrho_H : Y \hookrightarrow V(H), \ell_H : E(H) \rightarrow L_H),$$

one may consider these graphs as $L_G \cup L_H$-labeled marked graphs and consider the pushout of Diagram 3 in $\text{LGrph}$, which exists when, intuitively, the $L_G \cup L_H$-labeled marked graph obtained by identifying vertices in $X \cap Y$ can be constructed by assigning the common labels of the overlapping edges in the intersection (i.e. the labels of overlapping edges must be compatible). This graph amalgam will be denoted in a concise form (when there is no confusion) by

$$G(\varrho_G(X)) + H(\varrho_H(Y)).$$

**Definition 3.** A labeled graph $G = (V, E, \ell_G, \tau_G, \ell_G)$ is said to be symmetric if there is a one to one correspondence between the labeled edges $uv$ and $vu$ that preserves the labeling (see Figure 3(a)). If for any two vertices $u$ and $v$ there only exist two edges $uv$ and $vu$ with the same label then the graph is said to be simply symmetric (see Figure 3(b)). Clearly, the data available in the structure of a simply symmetric graph can be encoded in a structure on the same set of vertices for which an edge is a subset of $V$ of size two (note that for loops we correspond two directed loops to a simple loop), while in this case the new structure may be called the corresponding generalized simple graph (see Figure 3(c)). Since usually in graph theory simple graphs do not have loops, we just freely use the word symmetric graph to refer to the directed or the generalized simple graph itself. Hereafter, $\text{SymGrph}$ stands for the category of symmetric graphs as a subcategory of $\text{LGrph}$ where we may also denote a pair of dual directed edges by a simple edge as it is usual in graph theory. ▽
2.1.1 Category of cylinders and category of \((\Gamma, m)\)-graphs

**Definition 4. Cylinders**

A cylinder \(C(y, z, \epsilon)\) of thickness \((t, k)\) (or a \((t, k)\)-cylinder for short), with the initial base \(B^−\) and the terminal base \(B^+\) (see Figure 4(a)), is a labeled marked graph with the following data,

- A labeled marked graph \(C(y, z)\).
- An ordered list \(y = (y_1, y_2, \ldots, y_k)\) with \(0 < k\) such that \(\{y_1, y_2, \ldots, y_k\} \subseteq V(C)\), and \(B^− \overset{\text{def}}{=} C[y_1, y_2, \ldots, y_k]\).
- An ordered list \(z = (z_1, z_2, \ldots, z_k)\) with \(0 < k\) such that \(\{z_1, z_2, \ldots, z_k\} \subseteq V(C)\), and \(B^+ \overset{\text{def}}{=} C[z_1, z_2, \ldots, z_k]\).
- A one to one partial function \(\epsilon: \overset{\text{1...}}{\overset{\text{k}}{\longrightarrow}} \overset{\text{1...}}{\overset{\text{k}}{\longrightarrow}}\), with \(|\text{domain}(\epsilon)| = |\text{range}(\epsilon)| = t\), indicating the equality of \(t\) elements of \(y\) and \(z\), in the sense that \(\epsilon(i) = j \iff y_i = z_j\).
- The mapping \(y_i \mapsto z_i\) (\(1 \leq i \leq k\)) determines an isomorphism of the bases \(B^−\) and \(B^+\) as labeled graphs.

Hereafter, we always refer to \(\epsilon\) as a relation, i.e. \(\epsilon = \{(i, \epsilon(i)) : i \in \text{domain}(\epsilon)\}\). Also, note that in general a cylinder is a directed graph and also has a direction itself (say from \(B^−\) to \(B^+\)). A cylinder \(C(y, z, \epsilon)\) is said to be symmetric if there exists an automorphism of \(C\) as \((\sigma_V, \sigma_E): C \rightarrow C\) that maps \(B^−\) to \(B^+\) isomorphically as labeled graphs, such that,

- \(\forall i \in \overset{\text{1...}}{\overset{\text{k}}{\longrightarrow}}\) \(\sigma(y_i) = z_i, \sigma(z_i) = y_i\),
- \(\sigma(B^−) = B^+, \sigma(B^+) = B^−\) (as labeled graphs),
- \(\epsilon \subseteq \{(j, j) : j \in \overset{\text{1...}}{\overset{\text{k}}{\longrightarrow}}\}\).

**Notation.** In what follows, we may exclude variables in our notations when they are set to their default (i.e. trivial) values. As an example, note that \(C(y, z)\) is used when \(\epsilon\) is empty (i.e. \(t = 0\)) and we may refer to it as a \(k\)-cylinder. Also, we partition the vertex set of a \((t, k)\) cylinder into the base vertices that appear in \(V(B^−) \cup V(B^+)\), and the inner vertices that do not appear on either base. A cylinder whose bases are empty graphs and has no inner vertex is called a plain cylinder.
Figure 4: (a) A general form of a cylinder, (b) A cylinder with a twist $(\lambda, \gamma)$, (c) A generalized loop cylinder.

Figure 5: (a) The identity cylinder $I_k(y, z)$ (see Examples 1 and 6), (b) The directed identity cylinder $\vec{I}_k(y, z)$.

**Example 1.** In this example we illustrate a couple of cylinders to clarify the definition.

- **The identity cylinder, $I_k(y, z)$**

  Define the *symmetric identity cylinder* on $2k$ vertices as $I_k(y, z)$, where both bases are empty graphs on $k$ vertices and for all $i \in 1...k$ the only edges are $y_i z_i$ (see Figure 5(a)). Clearly, we may define $I(y, z) \overset{\text{def}}{=} I_1(y, z)$ standing for an edge. Also, we use the notation $\vec{I}_k(y, z)$ for the directed version, where edges are directed from $B^-$ to $B^+$ (see Figure 5(b)). When the definition is clear from the context we use the same notation for both directed and symmetric forms of identity cylinder.

- **The $\sqcap$-cylinder, $\sqcap(y, z)$**

  This is a $(0, 2)$-cylinder (or a 2-cylinder for short) that is depicted in Figure 6(a), for which $\epsilon = \emptyset$. This cylinder will be used in the sequel to construct the Petersen and generalized Petersen graphs (see Example 6).

Figure 6: (a) The $\sqcap$-cylinder, (b) The path cylinder $P_n$, (c) The directed path cylinder $\vec{P}_2$ of length two, (d) The looped line-graph cylinder.

- **The path cylinder, $P_n(y, z)$**
For $n > 0$ this is 1-cylinder with $n+1$ vertices that is depicted in Figure 6(b), forming a path of length $n$. This cylinder will be used in the sequel to construct subdivision and fractional powers of a graph (see Examples 4 and 5). The cylinder $P_o$ is a $(1, 1)$ cylinder with just one vertex $y = z$ and no edge for which both bases are identified through $\epsilon = \{(1, 1)\}$ and have just one element. This cylinder will be used for contraction operation in graphs (see Example 5). Analogously, one may define directed path cylinders $P_n(y, z)$ (see Figure 6(c)).

- **The looped line-graph cylinder, $\wedge(y, z)$**

  This is a $(1, 2)$-cylinder on 3 vertices, depicted in Figure 6(d), for which $\epsilon = \{(1, 1)\}$, i.e. $y_i = z_i$. This cylinder will be used in the sequel to construct looped line-graphs (see Example 4).

- **Generalized loop cylinder,**

  This is a $(k, k)$-cylinder with $\epsilon = \{(1, 1), (2, 2), \ldots, (k, k)\}$. Any graph can be considered as a generalized loop cylinder by fixing an arbitrary subgraph of it with $k$ vertices as $B^{-} = B^{+}$ (see Figure 4(c)).

- **The deletion cylinder, $d_k(y, z)$**

  This is a $(0, k)$-cylinder with $2k$ vertices and without any edges. This cylinder will be used for deletion operation in graphs. We define $d(y, z) \overset{\text{def}}{=} d_k(y, z)$.

- **The contraction cylinder,**

  This is a particular case of generalized loop cylinder that will be used for the contraction operation in graphs. The contraction cylinder is an empty $(k, k)$ cylinder without any inner vertex. For example $P_o$ is a contraction cylinder.

- **Fiber gadget**

  A graph $M$ containing two copies $W^\alpha$ and $W^\beta$ of an indexed set $W^*$, is an fiber gadget. If we set $C = M$, $B^{-} = W^\alpha$ and $B^{+} = W^\beta$, we can see that the fibre gadget is a cylinder, where we denote it by $C(M)$.

- **Pultr templates**

  A Pultr template is a quadruple $\tau = (P, Q, \eta_1, \eta_2)$, where $P, Q$ are digraphs and $\eta_1, \eta_2 : P \rightarrow Q$ are graph homomorphisms and $Q$ admits an automorphism $\zeta$ such that $\zeta \circ \eta_1 = \eta_2$ and $\zeta \circ \eta_2 = \eta_1$. Note that if $\eta_1, \eta_2$ are one to one graph embeddings (i.e. $\eta_1(P) \simeq \eta_2(P) \simeq P$), then $Q$ has the structure of a $(t, k)$-cylinder with $k = |P|$ while $\epsilon$ and $t$ are determined by the overlaps in the range of the maps $\eta_1, \eta_2$.

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**Notation.** Given two $(t, k)$-cylinders $C(y, z)$ and $D(y, z)$, the $(t, k)$-cylinder $C(y, z) + D(y, z)$ is the amalgam obtained by identifying bases. Also, the $(2t, 2k)$-cylinder $C \bowtie D$ is the cylinder obtained by naturally taking the disjoint union of these structures (e.g. Figure 7).

**Definition 5. Twist of a cylinder**

If $|y| = k$, then $(\gamma, \lambda) \in S_k \times S_k$ is said to be a twist of a cylinder $C(y, z, \epsilon)$, if $\gamma$ and $\lambda$ are graph automorphisms of bases $B^{-}$ or $B^{+}$ (see Figure 7(b) and note that the bases $B^{-}$ and $B^{+}$ are isomorphic by definition).

A twisted cylinder $C(y', z', (\gamma, \lambda)\epsilon)$, as the result of action of a twist $(\gamma, \lambda)$ on $C(y, z, \epsilon)$, is a cylinder for which we have $y' = y\gamma$, $z' = z\lambda$ and $(\gamma, \lambda)\epsilon \overset{\text{def}}{=} \{(\gamma(i), \lambda(j)) : (i, j) \in \epsilon\}$.

Note that by definition, a twist is an element of $\text{Aut}(B^{-}) \times \text{Aut}(B^{+})$ and vice versa. The group $\text{Aut}(B^{-}) \times \text{Aut}(B^{-})$ is called the twist group of the cylinder $C$. 

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Figure 7: (a) The cylinder $C \oplus D$, (b) The cylinder $C + D$.

Figure 8: Twists of the $\cap$-cylinder (see Example 2).

**Notation.** We say that a twist $(\gamma, \delta)$ is symmetric, if $\gamma = \delta$.

**Example 2. Twists of $\cap(y, z)$**

Setting $\pi_2 \overset{\text{def}}{=} [2, 1]$, each element of $S_2 \times S_2 \simeq \{(id, id), (id, \pi_2), (\pi_2, id), (\pi_2, \pi_2)\}$ satisfies the conditions of Definition 5 and consequently, is a twist of the $\cap$-cylinder. In what follows we present the action of twists of this type on $\cap(y, z)$ as depicted in Figure 6(a).

- **Figure 8(a):** The twist $(id, id)$ on $\cap$-cylinder,
- **Figure 8(b):** The twist $(\pi_2, id)$ on $\cap$-cylinder,
- **Figure 8(c):** The twist $(id, \pi_2)$ on $\cap$-cylinder,
- **Figure 8(d):** The twist $(\pi_2, \pi_2)$ on $\cap$-cylinder.

**Definition 6. A $\Gamma$-coherent set of cylinders**

Given integers $t$ and $k$ and a subgroup $\Gamma \leq S_k$, then a set of $(t, k)$-cylinders

$$C = \{C^j(y^j, z^j, \epsilon^j) \mid j \in \hat{1}^{\hat{m}}\}$$

is said to be $\Gamma$-coherent, if

- For all $j, r \in \hat{1}^{\hat{m}}$, the mapping $y^j_i \mapsto y^r_i$ ($i \in \hat{1}^{\hat{k}}$) from $B^-(C^j)$ to $B^-(C^r)$ induces an isomorphism of these bases as labeled graphs,
- $\Gamma \leq \text{Aut}(B^-(C^j))$.

A $\Gamma$-coherent set of cylinders is said to be symmetric if each one of its cylinders is a symmetric cylinder.
Notation. When \( m = 1 \), we use the notation to \( \Gamma \)-cylinder for \( C^1(y^1, z^1, \epsilon^1) \).

**Example 3.** In this example we present two coherent sets of cylinders.

- Every set of 1-cylinders is \( S_1 \)-coherent. For instance, every set of path cylinders is \( S_1 \)-coherent.
- The \( \cap \)-cylinder and the cylinder shown in Figure 9(a) form a \( S_2 \)-coherent set of cylinders.

**Remark:** If a set of \((t,k)\)-cylinders is a \( S_k \)-coherent set, then either all bases are isomorphic to the complete graph \( K_k \) or they are isomorphic to the empty graph \( K_k \).

**Definition 7. The category of \((\Gamma, A)\)-graphs**

Given a subgroup \( \Gamma \leq S_k \) and a finite set \( A \), a \((\Gamma, A)\)-graph is a graph labeled by the set

\[
L_{\Gamma, A} = \{(\gamma^-, a, \gamma^+) \mid a \in A, \gamma^-, \gamma^+ \in \Gamma\}.
\]

When \( A \) is a set of \( m \) elements, we usually assume that \( A = \{1, \ldots, m\} \) unless it is stated otherwise, where in this case we talk about \((\Gamma, m)\)-graphs.

We say that a \((\Gamma, m)\)-graph has a symmetric labeling if all of its twists are symmetric.

Consider an edge \( e \in E(G) \) in a \((\Gamma, m)\)-graph \( G \) with \( e^- = u \) and \( e^+ = v \). Then if \( \ell_G \) is the labeling map, and \( \ell_G(e) = (\gamma^-, i, \gamma^+) \), we use the following notations

\[
\ell^\ast_G(e) \overset{\text{def}}{=} i, \quad \ell^-_G(e) \overset{\text{def}}{=} \gamma^-, \quad \ell^+_G(e) \overset{\text{def}}{=} \gamma^+.
\]

Also, if \( w \) is any one of the two ends of \( e \) (i.e. \( w = u \) or \( w = v \)) and there is not any ambiguity, then we can use the notations

\[
\ell^w_G(e) \overset{\text{def}}{=} \ell^-_G(e) = \gamma^-, \quad \ell^e_G(e) \overset{\text{def}}{=} \ell^+_G(e) = \gamma^+.
\]

Suppose that \( e_1, e_2 \in E(G) \) are incident edges, where \( e^-_1 = e^+_2 = w \) (e.g. \( e^-_1 = e^+_2 \)) for \( a, b \in \{-, +\} \), then we define a label difference function \( \Delta^w_{\Gamma}(e_1, e_2) \) as

\[
\Delta^w_{\Gamma}(e_1, e_2) \overset{\text{def}}{=} [\ell^w_{\Gamma}(e_1)]^{-1}[\ell^w_{\Gamma}(e_2)].
\]

For simple graphs, we can use the notation \( \Delta^w_{\Gamma}(e_1, e_2) \). Of course when there is not any ambiguity we may omit the subscripts or superscripts.
Let $G$ and $H$ be two $(\Gamma, m)$-graphs labeled by the maps $\ell_G$ and $\ell_H$, respectively. Then a
$(\Gamma, m)$-homomorphism from $G$ to $H$ is a graph homomorphism $(\sigma_V, \sigma_E): G \rightarrow H$ such that
$$\ell^*_G = \ell^*_H \circ \sigma_E,$$
and for any pair of two edges $e_1, e_2 \in E(G)$ that intersect in $e^a_1 = e^b_2$ (for $a, b \in \{-, +\}$),
$$\Delta^{ab}_{e_1}(e_1, e_2) = \Delta^{ab}_{H}(\sigma_E(e_1), \sigma_E(e_2)),$$
or equivalently,
$$[\ell^a_G(e_1)]^{-1}[\ell^b_H(e_2)] = [\ell^a_G(\sigma_E(e_1))]^{-1}[\ell^b_H(\sigma_E(e_2))].$$
The set of homomorphisms from the $(\Gamma, m)$-graph $G$ to the $(\Gamma, m)$-graph $H$ is denoted by
$\text{Hom}_{(\Gamma, m)}(G, H)$, and $\text{LGrph}(\Gamma, m)$ stands for the category of $(\Gamma, m)$-graphs and their
homomorphisms.

Note that when $m = 1$ and $\Gamma = \text{id}$ then $\text{Grph} \simeq \text{LGrph}(S_1, 1)$.

**Proposition 1** Let $G$ and $H$ be two $(\Gamma, m)$-graphs labeled by the maps $\ell_G$ and $\ell_H$, respectively. Then for a labeled marked graph homomorphism $(\sigma_V, \sigma_E) \in \text{Hom}_{(\Gamma, m)}(G, H)$ the following conditions are equivalent.

a) The pair $(\sigma_V, \sigma_E)$ is a $(\Gamma, m)$-homomorphism. In other words, for any pair of two
edges $e_1, e_2 \in E(G)$ that intersect at a vertex $e^a_1 = e^b_2$ (for $a, b \in \{-, +\}$),
$$\Delta^{ab}_{e_1}(e_1, e_2) = \Delta^{ab}_{H}(\sigma_E(e_1), \sigma_E(e_2)).$$

b) For any vertex $u \in V(G)$, there exists a unique constant $\alpha_u \in \Gamma$ such that for any edge
$e \in E(G)$ that intersects in $e^a = u$ (for $a \in \{-, +\}$), we have,
$$\ell^a_G(e) = \alpha_u[\ell^a_H(\sigma_E(e))].$$

**Proof.** (a $\Rightarrow$ b) Fix a vertex $u$ and an edge $e_0$ that $e^+_0 = u$ (one can do the rest with the
assumption of $e^+_0 = u$ in a similar way) and define,
$$\alpha_u \overset{\text{def}}{=} \ell^+_G(e_0)[\ell^+_H(\sigma_E(e_0))]^{-1}.$$Then for any other edge $e$ that $e^a = u$ for $a \in \{+, -\}$ we have,
$$\ell^a_G(e) = \ell^+_G(e_0)[\ell^+_H(\sigma_E(e))]^{-1}[\ell^a_G(e)]$$
$$= \ell^+_G(e_0)[\ell^+_H(\sigma_E(e))]^{-1}[\ell^a_H(\sigma_E(e))]$$
$$= \alpha_u[\ell^a_H(\sigma_E(e))].$$

(b $\Rightarrow$ a) is clear by definition of the constant $\alpha_u$. 

**Definition 8.** Consider a graph $G$ labeled by the set $L_{(\Gamma, m)}$ and the map $\ell_G$, along with a
vector $\alpha \in \Gamma^{V(G)}$. Then we define the labeled graph $G_\alpha$ on the graph $G$ with the labeling
$$\ell_{G_\alpha}(uv) = (\alpha_u\ell^-_G(uv), \ell^+_G(uv), \alpha_v\ell^-_G(uv)).$$
Remark: If \( C = \{ C_j^{y,j} | j \in \tilde{1} \ldots m \} \) is a \( \Gamma \)-coherent set of cylinders labeled by \( L_{\Gamma,m} \), then
\[
C_\alpha \overset{\text{def}}{=} \{ C_j^{y,j} | j \in \tilde{1} \ldots m \}
\]
is also a \( \Gamma \)-coherent set of cylinders.

The following results is a direct consequence of Proposition 1.

Corollary 1. Let \( G \) and \( H \) be two graphs labeled by the set \( L_{\Gamma,m} \) and the maps \( \ell_G \) and \( \ell_H \), respectively. Then, considering these graphs as objects of the category of labeled graphs \( \text{LGrph} \), and also as objects of the category of \((\Gamma,m)\)-graphs \( \text{LGrph}(\Gamma,m) \),
\[
\exists \alpha \in \Gamma^{V(G)} \quad \text{Hom}_{\Gamma,m}(G,H) \neq \emptyset \iff \text{Hom}_{\alpha}(G,H) \neq \emptyset \iff \text{Hom}_{\alpha^{-1}}(G,H) \neq \emptyset.
\]

Lemma 1. Let \( \sigma \in \text{Hom}_{\Gamma,m}(G,H) \), and also \( \ell_G \), \( \ell_H \) be the \((\Gamma,m)\)-labelings of \( G \) and \( H \) respectively.

1. If \( e_1,e_2 \in E(G) \), \( e_1^- = e_2^-, e_1^+ = e_2^+ \) and \( \ell_H(\sigma_E(e_1)) = \ell_H(\sigma_E(e_2)) \) then \( \ell_G(e_1) = \ell_G(e_2) \).

2. Let \( G \) be symmetrically labeled. If \( \alpha_u \) (as defined in Proposition 1) is a constant function on vertices of \( G \), then the image of \( \sigma \) is a symmetrically labeled subgraph of \( H \).

Proof.

1. Let \( e_1 = e_2 = uv \), then by Proposition 1 for some \( \alpha_u \) we have
\[
\ell_G(e_1) = \alpha_u \ell_H(\sigma_E(e_1)) = \alpha_u \ell_H(\sigma_E(e_2)) = \ell_G(e_2).
\]
The proof of \( \ell_G(e_1) = \ell_G(e_2) \) is similar.

2. Let \( e = uv \in E(G) \). By the hypothesis, there exist \( \lambda \in \Gamma \) and \( i \in \tilde{1} \ldots m \) such that \( \ell_G(e) = (\lambda,i,\lambda) \). Consequently, since \( \alpha_u = \alpha_v \), we have \( \ell_H(\sigma_E(e)) = (\alpha_u \lambda,i,\alpha_u \lambda) \).

3 The cylindrical graph construction

3.1 The exponential construction

Given a \( \Gamma \)-coherent set of \((t,k)\)-cylinders containing \( m \geq 1 \) cylinders, there is a canonical way of constructing an exponential graph that will be defined in the next definition.

Definition 9. The exponential graph \([C,H]_\Gamma\).

For any given labeled graph \( H \) and a \( \Gamma \)-coherent set of \((t,k)\)-cylinders
\[
C \overset{\text{def}}{=} \{ C_j^{y,j} | j \in \tilde{1} \ldots m \},
\]
the \emph{exponential graph} \([C,H]_\Gamma\) is defined (up to isomorphism) as a \((\Gamma,m)\)-graph in the following way,

i) Consider the right action of \( \Gamma \) on \( V(H)^k \) as
\[
\nu \gamma = (v_1,v_2,\ldots,v_k) \overset{\text{def}}{=} (v_{\gamma(1)},v_{\gamma(2)},\ldots,v_{\gamma(k)}),
\]
and the corresponding equivalence relation whose equivalence classes determine the orbits of this action, i.e.

\[ u \sim \gamma \Leftrightarrow \exists \gamma \in \Gamma \quad u\gamma = v. \]

Also, fix a set of representatives \( U \) of these equivalence classes and let

\[ V_U([C,H]_\Gamma) \overset{\text{def}}{=} U = \{u_1, u_2, \ldots, u_d\}. \]

ii) There is an edge \( e \overset{\text{def}}{=} u_i u_k \in E_U([C,H]_\Gamma) \) with the label

\[ \ell(e) = (\gamma^{u_i}, j, \gamma^{u_k}) \]

such that,

\[ \sigma_v(y^{u_i}) = u_i \quad \text{and} \quad \sigma_v(z^{u_k}) = u_k. \]

iii) Exclude all isolated vertices.

Using Proposition 1, one may verify that for two different sets of representatives \( U \) and \( W \) we have

\[ \forall i \in 1 \ldots d \quad \exists \gamma_i \in \Gamma \quad u_i \gamma_i = w_i, \]

and that the map \( \sigma = (\sigma_v, \sigma_E) \) defined as

\[ \forall i \in 1 \ldots d \quad \sigma_v(u_i) \overset{\text{def}}{=} w_i, \]

and for \( e = u_i u_k \) with \( \ell(e) = (\gamma^{u_i}, j, \gamma^{u_k}) \),

\[ \sigma_E(e) \overset{\text{def}}{=} w_i w_k, \quad \text{with} \quad \ell(\sigma_E(e)) = (\gamma_i, j, \gamma_k \gamma^{u_k}), \]

is an isomorphism of \( (\Gamma,m) \)-graphs, and consequently, hereafter, we assume that every exponential graph is constructed with respect to a fixed class of representatives \( U \), and we omit the corresponding subscript since we are just dealing with such graphs up to an isomorphism.

Also, note that in general an exponential graph is a directed graph, however, if one dealing with a \( \Gamma \)-coherent set of symmetric \((t,k)\)-cylinders then by the analogy between symmetric graphs and their directed counterparts, where each simple edge is replaced by a pair of directed edges in different directions, for any given symmetric labeled graph \( H \) one may talk about a symmetric exponential graph, \([C,H]_{s\Gamma}\), in which all edges are simple since by the above definition and the definition of a symmetric cylinder,

\[ uv \in E([C,H]_\Gamma) \Leftrightarrow vu \in E([C,H]_\Gamma). \]

Note that there may exist a directed cylinder \( C \) and a directed graph \( H \), such that the symmetric exponential graph \([C,H]_{s\Gamma}\) is meaningful.

**Notation.** The representative \( u_i \) of the equivalence class \([u]_\sim \Gamma \) (where \( u_i = u\gamma \) for some \( \gamma \in \Gamma \)) is denoted by \( \langle u_i \rangle_\Gamma \). Also, as before we may exclude some variables when they are set to their default values, as the concept of a \( \Gamma \)-graph which stands for a \( (\Gamma,1) \)-graph or \([C,H]_{id}\) which stands for \([C,H]_{id}\).

Moreover, to clarify the figures, we may use the name of the cylinders, instead of their index (e.g. see Figures 10 and 14).

**Example 4.** Here we consider a couple of examples for the exponential graph construction.
The role of identity
It is clear that the identity cylinder satisfies the identity property \([I, H] = H\) in both directed and symmetric cases (see Example 1).

The indicator construction (e.g. see [14] and references therein)
The exponential graph \([C, H]\) for a 1-cylinder \(C(y, z)\) is the standard indicator construction, where the symmetric case \([C, H]^s\) is again the standard construction through the corresponding automorphism (see Definition 4).

\[\begin{align*}
\text{(a)} & \quad \text{A base graph } G \text{ and the Cylinder } P_3, \\
\text{(b)} & \quad \text{The graph } G^3 = [P_3, G]_{\text{id}}.
\end{align*}\]

The \(n\)th power of a graph \(G\)
It is easy to check that the \(n\)th power \(G^n\) of a simple graph \(G\) with \(|V(G)|\), which is defined to be the graph on the vertex set \(V(G^n) \overset{\text{def}}{=} V(G)\), obtained by adding an edge \(uv\) if there exists a walk of length \(n\) in \(G\) starting at \(u\) and ending at \(v\), can be described as follows,
\[G^n = [P_n, G].\]

Note that \(G^n\) contains loops when \(n\) is even. An example of the power graph \(G^3\) is depicted in Figure 10 where Figure 10(a) shows the base graph \(G\) and a homomorphism of \(P_3\) to it, while Figure 10(b) shows the graph \(G^3\) in which the corresponding homomorphism is bolded out.

One can construct the directed version of graph power construction (see Figure 15) by
\[\overrightarrow{G^n} = [\overrightarrow{P_n}, \overrightarrow{G}].\]

Pultr right adjoint construction [6,7]
Given a Pultr template \(\tau = (P, Q, \eta_1, \eta_2)\) (see Example 1) the central Pultr functor \(\Gamma_\tau\) is a digraph functor which send any digraph \(H\) to a digraph \(\Gamma_\tau(H)\), where its vertices are the homomorphisms \(g : P \to H\), and the arcs of \(\Gamma_\tau(H)\) are pairs \((g_1, g_2)\) for which there exists a homomorphism \(h : Q \to H\) such that
\[g_1 = h \circ \eta_1, \quad g_2 = h \circ \eta_2.\]

Considering the corresponding cylinder \(C\) whose bases are isomorphic to \(P\) (see Example 1), one may verify that for one-to-one graph homomorphisms \(\eta_1\) and \(\eta_2\), we have \(\Gamma_\tau(G) = [C, G]\).

\[\text{Note that this is different from the other definition for the } n\text{th power of a graph, where an edge is added when there exists a walk of length less than or equal to } n.\]
The looped line-graph (19), also see Proposition 8,

It can be verified that the looped line-graph of a given graph H can be described as the exponential graph \([\land, H]_{S_2}\). Note that in this case, there is an extra loop on each vertex of the line-graph, since for each such vertex \(v\), there exists a homomorphism from the cylinder \(\land\) to H that maps \(y\) and \(z\) to \(v\).

Let us go through the details of such a construction for the graph H depicted in Figure 11.

1. First we present the set of vertices corresponding to the equivalence classes (see Figure 12(a)).
2. Fixed a set of representatives. Exclude isolated vertices i.e. delete any pair \(XY\) for which we can not find a graph homomorphism \(K_2 = B^{-} \rightarrow H([X,Y])\) (see Figure 12(b)).
3. To specify the edges,
   - Consider a graph homomorphism for which \(x_1 \mapsto v_1\), \(x_2 \mapsto v_2\), and \(y_1 \mapsto v_3\). Note that this homomorphism corresponds to the edge \(\langle v_1 v_3 \rangle \rightarrow \langle v_1 v_2 \rangle\) as depicted in Figure 13(a).
   - Again, consider a graph homomorphism for which \(x_1 \mapsto v_4\), \(x_2 \mapsto v_3\), and \(y_1 \mapsto v_1\), and consequently, we have the edge \(\langle v_1 v_3 \rangle \rightarrow \langle v_1 v_2 \rangle\), however, we have to use the twist \(\pi = [1,2]\) on both bases to find the corresponding representatives as \(\pi(v_1 v_3) = v_3 v_4\) and \(\pi(v_1 v_2) = v_3 v_4\) (see Figure 13(b)).
   - By continuing this procedure, we obtain the loop-lined graph as depicted in Figure 14.
• A directed example

Figure 15 shows a directed graph $T$ and the exponential graph $[^{\land},H]_{\mathbb{S}_2}$ (here note that we have excluded all default values of the parameters involved in our notation).

3.2 The cylindrical construction

Let $C \overset{\text{def}}{=} \{C'(y', z', \epsilon') \mid j \in \overline{1...m}\}$ be a set of $\Gamma$-coherent $(t, k)$-cylinders, where each $C'$ is an $\ell_{ij}$-graph. Also, let $G$ be a $(\Gamma, m)$-graph labeled by the map $\ell_{ij}$. Then the cylindrical product of $G$ and $C$ is an amalgam that, intuitively, can be described as the graph constructed by replacing each edge of $G$ whose label is $(\gamma^-, j, \gamma^+)$, by a copy of the cylinder $C'$ twisted by $(\gamma^-, \gamma^+)$, while we identify (the vertices of the) bases of the cylinders that intersect at
the position of each vertex of \( G \). Formally, this construction denoted by \( G \boxtimes_r C \) can be defined as,

\[
G \boxtimes_r C \overset{\text{def}}{=} \sum_{uv \in E(G) : \ell_{G}(uv) = (\gamma^u, \gamma^v)} C'(u^j \gamma^u, v^j \gamma^v, (\gamma^u, \gamma^v) \epsilon^j).
\]

(10)

To be more descriptive, let us describe the construction in an algorithmic way as follows.

1. **Blowing up vertices:** (This step is just for clarification and may be omitted.)

   Consider the empty graph \( G_0 \) with \( V(G_0) = \{ v_1, v_2, \ldots, v_n \} \) and \( E(G_0) = \emptyset \).

2. **Choosing the cylinders:** For every edge \( v_p v_q \in E(G) \) with the label \( \ell_{G}(v_p v_q) = (\gamma^-, j, \gamma^+) \), choose a twisted cylinder \( C'(y^j \gamma^-, z^j \gamma^+) \) in which the vertices of the bases are not identified according to \( \epsilon^j \)'s to \( G \boxtimes_r C \) along with all its vertices and edges yet. Re-mark the vertices in each base and construct the cylinder \( C'(v^j_p, v^j_q) \) for which

   \begin{itemize}
   \item Let \( v^p = y^j \gamma^- \),
   \item Let \( v^q = z^j \gamma^+ \).
   \end{itemize}

   Consider the set \( G^* \) of disjoint union of all these cylinders constructed for each edge.

3. **Identification:** Consider the amalgam constructed on \( G_0 \cup G^* \) as a marked graph and identify all vertices first with respect to markings and then with respect to \( \epsilon^j \)'s. In the end, identify all multiple edges that have the same label.

Note that given any edge \( e \in E(G \boxtimes_r C) \), there exists an index \( j \in \Gamma \) such that \( e \in E(C') \), and one may assign a well defined labeling

\[
\ell_{G \boxtimes_r C}(e) \overset{\text{def}}{=} \ell_{j}(e).
\]

Consequently, \( G \boxtimes_r C \) is an \( \ell_{G \boxtimes_r C} \)-graph.

Also, as in Definition [9] if \( C \) is a set of \( \Gamma \)-coherent symmetric \((t, k)\)-cylinders, then for any symmetric graph \( G \), one may talk about the symmetric cylindrical construct \( G \boxtimes_r C \). Note that if \( C \) is directed, then the symmetric construct \( G \boxtimes_r C \) is also a directed graph although \( G \) is symmetric.

**Notation.** If \( V(G) = \{ v_1, v_2, \ldots, v_n \} \), hereafter the vertex set of the cylindrical construct \( G \boxtimes_r C \) is assumed to be \( V(G \boxtimes_r C) = \bigcup_{j=1}^{m} \{ v^j_1, v^j_2, \ldots, v^j_r \} \) where the superscript refers to the index of the corresponding vertex in \( V(G) \) and, moreover, we may refer to the list \( v^j \overset{\text{def}}{=} \{ v^j_1, v^j_2, \ldots, v^j_r \} \). Also, we assume that in each cylinder, the set of vertices that do not appear in the vertex sets of the bases are enumerated (say from 1 to \( r \)), and for each fixed edge \( e \in E(G) \), we may refer to the list \( u^e \overset{\text{def}}{=} \{ u^e_1, u^e_2, \ldots, u^e_r \} \) consisting of non-base vertices that appear in \( G \boxtimes_r C \) when the edge \( e \) is replaced by the corresponding cylinder.

**Example 5.** In this example we go through some basic and well-known special cases.
• The identity relation
It is clear that the identity cylinder satisfies the identity property \( G \boxtimes I = G \) in both directed and symmetric cases (see Example 1). Note that, in general, cylindrical products using the identity cylinder gives rise to lifts of graphs (e.g. see [18] and references therein for the details, applications and background. Also see Figure 18).

• The replacement operation (e.g. see [14] and references therein)
The cylindrical construction \( G \boxtimes C \) for a 1-cylinder \( C = (y, z) \) is the standard replacement operation.

• The fractional power
It is easy to see that \( G \boxtimes P_n \) gives rise to an \( n \)-subdivision of \( G \) which can be described as subdividing every edge of \( G \) by \( n - 1 \) vertices (see Figure 16 for an example of this case). Also, note that one may describe the fractional power \( G^\frac{m}{n} \) as follows (see [11] for the definition and more details),

\[
G^\frac{m}{n} \overset{\text{def}}{=} [P_m, G \boxtimes P_n].
\]

• Fiber construction [20,21],
Let \( M \) be a fiber gadget (see Example 1) with \( W^a \) and \( W^b \) (as bases), and let \( G \) be a graph. Also, for each vertex \( v \) of \( G \), let \( W^v \) be a copy of \( W \), and for any edge \( uv \) of \( G \), let \( M_{uv} \) be a copy of \( M \). Then, identifying \( W^u \) and \( W^v \) with the copies of \( W^a \) and \( W^b \), respectively, in \( M \), we get a graph that is the fiber construction \( M(G) \), and can be represented as

\[
M(G) = G \boxtimes C(M).
\]

• Pultr left adjoint construction [6,7],
Given a Pultr template \( \tau = (P, Q, \eta_1, \eta_2) \) (see Example 1), the left Pultr functor \( \Lambda_\tau \) is a digraph functor which send any digraph \( G \) to a digraph \( \Lambda_\tau(G) \), whose vertices are the copies \( P_u \) for any vertex \( u \in V(G) \), and for the arcs, \( \Lambda_\tau(G) \) contains a copy \( Q_{uv} \) of \( Q \) for any arc \( uv \in E(G) \), where \( \eta_1[P] \) with \( P_u \) and \( \eta_2[P] \) with \( P_v \). Considering the corresponding cylinder \( C \) whose bases are isomorphic to \( P \), one may verify that \( \Lambda_\tau(G) = G \boxtimes C \).

Note that the fiber gadget is a special instance of Pultr template, and fiber construction is a special instance of Pultr left adjoint construction.
• **The join operation**

The join operation $G \vee H$ is a graph which is the union of two graph $G$ and $H$ with additional edges between any vertex of $G$ and any vertex of $H$.

Let $G, H$ be two graphs. Then one can show that if $H_{\epsilon}$ be a cylinder (see Figure 17(a) and 17(b)) with $B^{-} = H_{\epsilon}y, B^{+} = H_{\epsilon}z, \epsilon = \{(h, h)|h \in V(H)\}$ and we have an extra edge $yz$, then

$$G \boxtimes H_{\epsilon} \simeq G \vee H.$$  

![Figure 17: (a) A base graph H, (b) The join cylinder H_{\epsilon}, (c) The cylinder \Delta.](image)

• **The universal vertex construction**

One may verify that the result of the cylindrical construction $G \boxtimes \Delta$ (see Figure 17(c)) can be described as adding a universal vertex to the graph $G$ (i.e. a new vertex that is adjacent to each vertex in $V(G)$).

• **Deletion and contraction operations**

Let $d(y, z)$ and $c = P_{o}$ be deletion and contraction cylinders respectively, and $G$ be a graph labeled by $\{d, c, I\}$ and define $C \overset{def}{=} \{d(y, z), P_{o}, I(y, z)\}$. Then $G \boxtimes C$ is a graph obtained from $G$ in which edges labeled by $c$ are contracted and edges labeled by $d(y, z)$ are deleted. Hence, any minor is a cylindrical construct.

• **Generalized loop cylinders**

Let $C$ be a $\Gamma$-cylinder, and also let $L$ be a loop on a vertex $v$ with the label $(\lambda, \gamma) \in \Gamma \times \Gamma$. Then we define

$$Gl_{(\lambda, \gamma)}(C) \overset{def}{=} L \boxtimes_{\Gamma} C$$

which is a generalized loop cylinder. Note that by definition, any generalized loop cylinder can be constructed as mentioned above.

• **The role of twists in cylindrical construction**

As depicted in Figure 18 different labelings of $K_{3}$ in $K_{3} \boxtimes S_{2}$ $I_{2}$ can lead to different cylindrical constructions. This example shows that the choice of labeling can even affect the connectivity of the product graph.

• **One can verify that,**

**Proposition 2** *For any $\Gamma$-graph $G$, we have*

a) *For $(t, k)$-cylinders $A$ and $B$,*

$$G \boxtimes_{\Gamma} (A + B) \simeq G \boxtimes_{\Gamma} A + G \boxtimes_{\Gamma} B.$$
b) For \((t, k)\)-cylinder \(A\) and \((t', k')\)-cylinder \(B\), \(\Gamma_1 \leq S_{k}, \Gamma_2 \leq S_{k'}\) and \(\Gamma \overset{\text{def}}{=} \Gamma_1 \sqcup \Gamma_2\), then,

\[
G \boxtimes_r (A \uplus B) \simeq (G \boxtimes r_2 A) \sqcup (G \boxtimes r_2 B),
\]

where \(\sqcup\) is the disjoint union of two graphs.

Figure 18: The role of twists in cylindrical construction.

Example 6. In this example we elaborate on different descriptions of the Petersen graph using cylindrical constructions, where in this regard we will also describe any voltage graph construction as a special case of a cylindrical construction. This specially shows that a graph may have many different descriptions as a cylindrical product.

- **Petersen graph and the \(\sqcap\)-cylinder**

Consider the complete graph \(\hat{K}_5\) of Figure 19(a) as a \((S_2, 1)\)-graph, and note that the cylindrical construction \((\hat{K}_5 \boxtimes S_2 \sqcap)\) gives rise to the Petersen graph as depicted in Figure 19(b).

Figure 19: (a) The base graph \(\hat{K}_5\); (b) The Petersen graph as \(\hat{K}_5 \boxtimes S_2 \sqcap\).

- **Petersen graph and the voltage graph construction**

Let \(A\) be a set of \(k\) elements and \(\Omega\) be a group of \(n\) elements that acts on \(A\) from the
right. Then it is clear that one may identify the set $A$ as $A = \{a_1, a_2, \ldots, a_k\}$ and $\Omega$ as a permutation group acting on the index set $1..k$.

An action voltage graph is a (directed) labeled graph $(G, \ell_G : E(G) \rightarrow \Omega)$ whose edges are labeled by the elements of the group $\Omega$. The derived graph of $G$, denoted by $\tilde{G}$, is a graph on the vertex set $V(\tilde{G}) = V(G) \times A$, where there is an edge between $(u, a_i)$ and $(v, a_j)$ if $e = uv \in E(G)$ and $a_j = a_i \ell_G(e)$.

When $\Omega$ acts on itself on the right, then $\hat{A} = \Omega$ and the action voltage graph is simply called the voltage graph, when we usually refer to the corresponding derived graph as the voltage graph construct in this case. It is also easy to verify that Cayley graphs are among the well-known examples of voltage graphs (e.g. see [14] for more on voltage graphs and related topics).

Now, we show that action voltage graphs and their derived graphs are cylindrical constructs. For this, given an action voltage graph $(G, \ell_G : E(G) \rightarrow \Omega)$, we define the corresponding $(\Omega, 1)$-graph $\hat{G}$ as a graph with the same vertex and edge sets as $G$, and the labeling

$$\forall e \in E(\hat{G}) \quad \ell_{\hat{G}}(e) = (id, I_k, \rho) \iff \ell_G(e) = \rho,$$

where $I_k$ is the identity cylinder on $k$ vertices (see Figure 5). Then it is easy to see that $\hat{G} = \tilde{G} \boxtimes I_k$. Figure 20 shows the voltage graph and the corresponding $(\mathbb{Z}_5, 1)$-graph to construct the Petersen graph using this construction.

Figure 21: (a) The graph $G$, (b) The cylinder $C_\times(G)$, (c) The cylinder $C_\Box(G)$.

Example 7. NEPS of simple graphs
In this example we claim that any NEPS of simple graphs can be described as a cylindrical construction (e.g. see [14] for the definition and more details). We concentrate on the more classical cases of Cartesian product $G \square H = \text{NEPS}\{(0,1),(1,0)\}(G,H)$ and the categorical product $G \times H = \text{NEPS}\{(1,1)\}(G,H)$. Note that, the basic idea of generalizing the whole construction to arbitrary components or arbitrary NEPS Boolean sets is straightforward by noting the fact that the adjacency matrix of a NEPS of graphs is essentially constructible through suitable tensor products, using identity matrix and the adjacency matrices of the components.
For this consider a graph $G$ with $V(G) = \{v_1, \ldots, v_\nu\}$. Then define cylinders $C_\Box(G)(y, z)$ and $C_\times(G)(y, z)$ where the base of $C_\Box(G)(y, z)$ is isomorphic to $G$, base of $C_\times(G)(y, z)$ is an empty graph on $V(G)$, and the edges are described as follows (see Figure 21),

$$
\begin{align*}
y_i z_j &\in E(C_\Box(G)(y, z)) \iff i = j, \\
y_i z_j &\in E(C_\times(G)(y, z)) \iff v_i v_j \in E(G).
\end{align*}
$$

Then one may verify that $G \Box H = H \boxtimes C_\Box(G)$ and $G \times H = H \times C_\times(G)$.

The following is a classical result in the literature (see [14]) for the graph $H_G$ defined as,

$$
V(H_G) = \{f : V(G) \to V(H)\},
$$

$$
fg \in E(H_G) \iff \forall vw \in E(G) \ f(v)g(w) \in E(H).
$$

**Proposition 3.** For any pair of simple graphs $H$ and $G$, we have $H^G \simeq [C_\times(G), H]$.

**Proof.** Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $|V(H)| = m$. There is a canonical graph isomorphism between two constructions,

$$
\theta : H^G \to [C_\times(G), H],
$$

where for any $f \in V(H_G)$ we define

$$
\theta(f) = (f(v_1), f(v_2), \ldots, f(v_n)) \in V([C_\times(G), H]).
$$

Now, if $fg \in V(H_G)$, then by definition $\forall vw \in E(G), \ f(v)g(w) \in E(H)$, hence in the $H$ we can find a cylinder $C_\times(G)$ which $B^- = (f(v_1), f(v_2), \ldots, f(v_n))$, and $B^+ = (g(v_1), g(v_2), \ldots, g(v_n))$. Thus we have an edge $\theta(f)\theta(g) \in E([C_\times(G), H])$. So $\Theta$ is a graph homomorphism and it is easy to check that it is an graph isomorphism.  

\[
\begin{array}{c}
\text{Figure 22: The Cartesian product of two 2-paths and the corresponding cylindrical construction.}
\end{array}
\]

Note that similarly one may show that the strong product and the lexicographical product can be described by cylinders $C_\Box(G)$ and $C_\times(G)$, respectively. Their bases are isomorphic to $G$, and their edges are described as follows (see [9] and Figure 24),

$$
i = j \text{ or } v_i v_j \in E(G) \iff y_i z_j \in E(C_\times(G)(y, z)),
$$

$$
i, j \in 1^{k} \iff y_i z_j \in E(C_\Box(G)(y, z)).
$$

22
Figure 23: The categorical product of two 2-paths and the corresponding cylindrical construction.

\[ \begin{array}{ccc}
\times & = & \times \\
\begin{array}{c}
\times \\
\times
\end{array} & = & \\
\begin{array}{c}
\times \\
\times
\end{array}
\end{array} \]

Figure 24: (a) The graph \( G \), (b) The lexicographical product cylinder \( C \ast (G) \), (c) The strong product cylinder \( C \bowtie (G) \).

**Example 8. The zig-zag product of symmetric graphs**

In this example we show that the zig-zag product of graphs can also be presented as a cylindrical construct. We introduce two different presentations of this construction in what follows (e.g., see [14] for more details and the background).

Let \((G, \ell_G : E(G) \rightarrow D^2)\) be a symmetric \(d\)-regular labeled graph on the vertex set \(V(G)\), whose edges are labeled by the set \(D^2\) where \(D \overset{\text{def}}{=} \hat{1} \ldots d\) in a way that

\[ \forall \ v \in V(G), \ \{\ell_G(e) \mid e^- = v\} = D. \]

Then, any such graph can be represented by a rotation map as

\[ \text{Rot}_G : V(G) \times D \rightarrow V(G) \times D, \]

for which

\[ \text{Rot}_G(u, i) = (v, j) \iff \exists e = uv \in E(G) \ \ell_G(e) = (i, j). \]

Let \( G \) be a symmetric \( d\)-regular graph on the vertex set \( V(G) \), and also, let \( H \) be a symmetric \( k\)-regular graph on the vertex set \( D \), respectively, given by rotation maps \( \text{Rot}_G \) and \( \text{Rot}_H \). Then the zig-zag product of \( G \) and \( H \), denoted by \( G \otimes_z H \) is a symmetric \( k^2\)-regular graph defined on the vertex set \( V(G) \times D \) for which

\[ \text{Rot}_{G \otimes_z H}((v, h), (i, j)) = ((u, l), (j', i')) \]

if and only if

\[ \text{Rot}_H(h, i) = (h', i'), \ \text{Rot}_G(v, h') = (u, l'), \ \text{and} \ \text{Rot}_H(l', j) = (l, j'). \]

A *vertex transitive zig-zag product* \( G \otimes_z H \) is such a product when \( H \) is a vertex transitive graph.

- **The vertex transitive zig-zag product as a mixed construct**

Let \( H \) be a vertex transitive symmetric graph on the vertex set \( V(H) = \{v_1, v_2, \ldots, v_d\} \).
given by the rotation map $\text{Rot}_H$ and for each $i \in \ldots d$ fix an automorphism of $H$ as $\sigma_i$ such that $\sigma_i(v_1) = v_i$ with $\sigma_1$ equal to the identity automorphism. Let $\gamma_i \in S_d$ be the permutation induced on the index set through $\sigma_i$ and define $\Gamma \overset{\text{def}}{=} \langle \gamma_1, \gamma_2, \ldots, \gamma_d \rangle$ as the subgroup of $S_d$ generated by the permutations $\gamma_i$’s. Also, define the cylinder $C^R_H(y, z)$ (see Figure 26) labeled by $\{0, 1\}$ whose bases are two isomorphic copies of $H$ marked by the vertices $\{y_1, y_2, \ldots, y_d\}$ and $\{z_1, z_2, \ldots, z_d\}$ and labeled by 0, respectively, where the only other edge is the simple edge $y_1z_1$ labeled by 1. Moreover, let the graph $G$ be defined through the rotation map

$$\text{Rot}_G : V(G) \times D \longrightarrow V(G) \times D,$$

and define the $(\Gamma, 1)$-graph $\hat{G}$ as a graph with the same vertex and edge sets as $G$ and with the following labeling,

$$\ell_{\hat{G}}(uv) = (\gamma_i, C^R_H(y, z), \gamma_j) \iff \text{Rot}_G(u, i) = (v, j).$$

Now, the graph $\hat{G} \boxtimes C^R_H(y, z)$ is usually called the replacement product of the graphs $G$ and $H$. On the other hand, it is not hard to verify that the exponential graph $[Z, \hat{G} \boxtimes C^R_H(y, z)]$ is actually the zig-zag product $G \otimes_z H$, where $Z$ is the zig-zag cylinder depicted in Figure 26.

The vertex transitive zig-zag product as a cylindrical product

It is not hard to see that the above construction can also be described as a pure cylindrical product. For this, using the above setup, define the cylinder $C^Z_H(y, z)$ by adding edges to $C^R_H(y, z)$ such that

$$y_i, z_j \in E(C^Z_H) \iff \text{Rot}_H(y_i, l) = (y_i, l') \text{ and } \text{Rot}_H(z_j, h) = (z_j, h').$$

Then one may verify that the cylindrical product $\hat{G} \boxtimes C^Z_H$ is essentially the same as the zig-zag product $G \otimes_z H$.

Also, as a direct consequences of the definitions we have

**Proposition 4** For any pair of reduced simple graphs $G$ and $H$, there exists a vertex-surjective homomorphism $\sigma : G \rightarrow H$ if and only if there exists a coherent set of plain cylinders $C = \{C_i\}$, such that $G \cong \text{red}(H_\sigma \boxtimes C)$, where $H_\sigma$ is $H$ properly labeled according to the label of cylinders.
Figure 26: (a) $G_1$ : zig-zag base graph in cylindrical product case, (b) The zig-zag base graph in cylindrical quotient case, (c) Resulting zig-zag graph. The bold lines are representing a copy of zig-zag product cylinder placed on the edge $(v_1, v_3)$ with appropriate label. The dashed line represents a homomorphism from zig-zag quotient cylinder to the zig-zag base graph.

**Proof.** Consider the quotient graph and note that the graph induced on any two inverse images of $\sigma$ is a plain cylinder. 

This observation shows that the cylindrical construction generalizes the concept of a covering or a homomorphic pre-image in a fairly broad sense. Hence, it is important to find out about the relationships between the construction and the homomorphism problem, which is the subject of the next section.

### 4 Main duality theorem

The tensor home dualities, although not observed widely, are classic in graph theory and at least goes back to the following result of Pultr\(^3\) (see Examples 4 and 5).

**Theorem A.** \(^7,22\) For any pultr template $\tau$, functors $\Lambda_\tau$ and $\Gamma_\tau$ are left and right adjoints in $\text{Grph}_\prec$, i.e. for any two graphs $G, K$ there exists a homomorphism $\Lambda_\tau(G) \to K$ if and only if there exists a homomorphism $G \to \Gamma_\tau(K)$.

In this section we state the fundamental theorem stating a duality between the exponential and cylindrical constructions, which shows that even in the presence of twists and cylinders with mixed bases one may also deduce a correspondence.

**Theorem 1. The fundamental duality**

Let $C = \{C_j(y^j, z^j, \epsilon^j) \mid j \in \hat{1} \ldots m\}$ be a $\Gamma$-coherent set of $(t,k)$-cylinders, and also let $G$ be a $(\Gamma,m)$-graph. Then for any labeled graph $H$,

a) $\text{Hom}_\ell(G \boxtimes_{\tau} C, H) \neq \emptyset$ $\iff$ $\text{Hom}_{\Gamma,m}(G, [C, H]_\Gamma) \neq \emptyset$.

b) There exist a retraction

$$r_{G,H} : \text{Hom}_\ell(G \boxtimes_{\tau} C, H) \to \text{Hom}_{\Gamma,m}(G, [C, H]_\Gamma)$$

and a section

$$s_{G,H} : \text{Hom}_{\Gamma,m}(G, [C, H]_\Gamma) \to \text{Hom}_\ell(G \boxtimes_{\tau} C, H),$$

such that $r_{G,H} \circ s_{G,H} = 1_{\text{Hom}_\ell(G \boxtimes_{\tau} C, H)}$, where $1_{\text{Hom}_\ell(G \boxtimes_{\tau} C, H)}$ is the identity mapping.

\(^3\)Here one must be careful about the choice of categories (also see Section 6 and [6]).
Proof. \( (\Rightarrow) \) If \( \text{Hom}_r(G \boxtimes_r C, H) \neq \emptyset \), then there exists a homomorphism \( \sigma = (\sigma_v, \sigma_e) \in \text{Hom}_r(G \boxtimes_r C, H) \).

Assume that \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and

\[
\forall i \in \hat{1} \ldots n \quad \sigma_v(v^i) \overset{\text{def}}{=} u^i,
\]

where \( v^i = (v^i_1, v^i_2, \ldots, v^i_k) \) is the set of vertices that blow up at \( v_i \), and

\[
u^i \overset{\text{def}}{=} (\sigma_v(v^i_1), \sigma_v(v^i_2), \ldots, \sigma_v(v^i_k)).
\]

Now, define \( r_{G,H}((\sigma_v, \sigma_e)) = (\sigma'_v, \sigma'_e) \), where \( \sigma'_v : V(G) \to V([C, H]_r) \) and

\[
\forall i \in \hat{1} \ldots n \quad \sigma'_v(v^i) \overset{\text{def}}{=} (u^i)_r.
\]

Note that by the definition of the exponential graph construction, for each \( i \in \hat{1} \ldots n \), there exists a unique \( \alpha_i \in \Gamma \) such that \( (u^i)_r = u^i \alpha_i \), hence we have

\[
\forall i \in \hat{1} \ldots n \quad \sigma'_v(v^i) \overset{\text{def}}{=} u^i \alpha_i.
\]

Again, for each \( e = v_i v_j \in E(G) \) with \( \iota(e) = v_i \), \( \tau(e) = v_j \) and \( \ell_{\alpha_i}(e) = (\gamma^-, t, \gamma^+) \), define,

\[
\sigma'_v(e) \overset{\text{def}}{=} e' = (u^i)(u^j),
\]

such that

\[
\iota(e') = (u^i)_r = u^i \alpha_i, \quad \tau(e') = (u^j)_r = u^j \alpha_j, \quad \ell_{\langle C, H \rangle}(e') = (\alpha_i \gamma^-, t, \alpha_j \gamma^+).
\]

Now, we show that the mapping \( \sigma' = (\sigma'_v, \sigma'_e) \) is well defined and is a \((\Gamma, m)\)-homomorphism in \( \text{Hom}_{\gamma,m}(G, [C, H]_r) \). First, note that since \( \ell_{\alpha_i}(e) = (\gamma^-, t, \gamma^+) \), by cylindrical construction, the restriction of \( \sigma' \) to the cylinder on this edge \( e = v_i v_j \), gives a homomorphism

\[
\sigma' = (\sigma'_v, \sigma'_e) \in \text{Hom}_r(C(\gamma^-, \gamma^+, t), [C, H]_r),
\]

such that

\[
\sigma'_v(y^+ \gamma^-) = u^i \quad \text{and} \quad \sigma'_v(z^+ \gamma^+) = u^j.
\]

Hence, by applying the twists \((\alpha_i, \alpha_j)\) we obtain a new homomorphism

\[
\tilde{\sigma} = (\tilde{\sigma}_v, \tilde{\sigma}_e) \in \text{Hom}_r(C(y^+ \gamma^-, z^+ \gamma^+, (\gamma^-, \gamma^+)\epsilon^i, [C, H]_r),
\]

such that

\[
\tilde{\sigma}_v(y^+ \alpha_i, \gamma^-) = u^i \alpha_i = \langle u^i \rangle_r, \quad \tilde{\sigma}_v(z^+ \alpha_j, \gamma^+) = u^j \alpha_j = \langle u^j \rangle_r,
\]

and by the definition of the exponential graph, this shows that there is an edge \( e' \overset{\text{def}}{=} (u^i)_r (u^j)_r \in E([C, H]_r) \) with the label \( \ell_{\langle C, H \rangle}(e) = (\alpha_i \gamma^-, t, \alpha_j \gamma^+) \). This shows that \( \sigma' \) is a graph homomorphism.

Also, by Proposition \([\square]\) it is clear that

\[
\sigma' = (\sigma'_v, \sigma'_e) \in \text{Hom}_{\gamma,m}(G, [C, H]_r).
\]

\( (\Leftarrow) \) Fix a homomorphism \( \sigma' = (\sigma'_v, \sigma'_e) \in \text{Hom}_{\gamma,m}(G, [C, H]_r) \) and assume that

\[
\forall i \in \hat{1} \ldots n \quad \sigma'_v(v^i) = (u^i)_r \in V([C, H]_r).
\]
Also, by Proposition 1 we know that for each \( i \in \{1, \ldots, n\} \) there exists \( \alpha_i \in \Gamma \) such that for any edge \( e \) intersecting \( v_i \) we have,

\[
\ell^e_{G}(e) = \alpha_i [\ell^e_{C,H\Gamma} (\sigma_e^i (e))] .
\]

Again, fix an edge \( e = v_i v_j \in E(G) \) with the label \( \ell^e_{G}(e) = (\gamma^-, t, \gamma^+) \) and note that since \( \sigma^e \) is a homomorphism, then

\[
\sigma^e_e : E(C, H) \rightarrow E(C, H) \quad \text{and} \quad \ell^e_{C,H\Gamma} (e) = (\alpha, \gamma^-, t, \alpha_j \gamma^+) .
\]

Therefore, by the definition of the exponential graph, we have a homomorphism

\[
\tilde{\sigma}^e = (\tilde{\sigma}^e_v, \tilde{\sigma}^e_e) \in \text{Hom}_t(C, C) \quad \text{where} \quad \tilde{\sigma}^e_v = \sigma^e_v, \tilde{\sigma}^e_e = \sigma^e_e .
\]

Therefore, by the definition of the exponential graph, we have a homomorphism

\[
\sigma^e = (\sigma^e_v, \sigma^e_e) \in \text{Hom}_t(C, C) \quad \text{such that} \quad \sigma^e_v(y^\gamma^-) = u^\alpha_i \quad \text{and} \quad \sigma^e_v(z^\gamma^+), \quad \sigma^e_e = u^\alpha_j .
\]

Now, by Proposition 1 these homomorphisms are compatible at each vertex and we may define a global homomorphism

\[
s_{G,H}((\sigma^e_v, \sigma^e_e)) = (\sigma_v, \sigma_e) \triangleq \bigcup_{e \in E(G)} \sigma^e \in \text{Hom}_t(G \boxtimes_t C, H).
\]

It remains to prove that

\[
\forall \sigma^e \in \text{Hom}_t, m(G, [C, H]_t) \quad r_{G,H}(s_{G,H}(\sigma^e)) = \sigma^e ,
\]

but by definitions,

\[
r_{G,H}(s_{G,H}(\sigma^e))(e) = r_{G,H}(\sigma^e)(e) = \sigma^e(e) ,
\]

since \( \langle u^\alpha_i \rangle_t = \langle u^\alpha_j \rangle_t \). Also, for any \( e \in E(G) \) whose label is \( (\gamma^-, t, \gamma^+) \), by definitions we know that

\[
r_{G,H}(s_{G,H}(\sigma^e_e))(e) = e^\prime ,
\]

where

\[
\nu(e') = \langle u^\prime \rangle_t = u^\prime \alpha_i, \quad \tau(e') = \langle u^\prime \rangle_t = u^\prime \alpha_j, \quad \ell_{C,H\Gamma}(e') = (\alpha_i \gamma^-, t, \alpha_j \gamma^+) ,
\]

and by Proposition 1 it is clear that

\[
r_{G,H}(s_{G,H}(\sigma^e_e))(e) = \sigma^e_e(e)
\]

and consequently,

\[
r_{G,H}(s_{G,H}(\sigma^e_e)) = \sigma^e .
\]

Similarly, one may prove the following version of this theorem for symmetric graphs and their homomorphisms.

**Theorem 2.** Let \( C = \{ C_j (y^j, z^j, \epsilon^i) \mid j \in \{1, \ldots, m\} \} \) be a set of \( \Gamma \)-coherent symmetric \((t,k)\)-cylinders, and also, let \( G \) be a symmetric \((\Gamma,m)\)-graph. Then for any labeled symmetric graph \( H \),
a) Hom*_{\ell}(G \boxtimes C, H) \neq \emptyset \iff \text{Hom}_{r,m}^*(G,[C,H]_r) \neq \emptyset.

b) There exist a retraction

\[ r_{G,H} : \text{Hom}^*_\ell(G \boxtimes C, H) \to \text{Hom}_{r,m}^*(G,[C,H]_r) \]

and a section

\[ s_{G,H} : \text{Hom}_{r,m}^*(G,[C,H]_r) \to \text{Hom}^*_\ell(G \boxtimes C, H) \]

such that \( r_{G,H} \circ s_{G,H} = 1 \), where 1 is the identity mapping.

Here we go through some basic observations.

**Proposition 5.** For any graph \( G \) and positive integers \( r, s, t \) and \( k \), we have

1) \([P_{2k-1}, C_{2k+1}] \cong K_{2k+1} \).

2) \((G^*)^s \cong G^{rs} \).

3) For any two \((t,k)\)-cylinders \( A \) and \( B \) and any graph \( H \), we have

\[ [A + B,H] \to [A,H] \times [B,H], \]

where \( \times \) is the categorical product of two graphs and amalgamation is according to the standard labeling of cylinders.

**Proof.**

1. Note that for any two fixed vertices \( x \) and \( y \) in \( C_{2k+1} \), there exists a path of odd length \( d \leq 2k - 1 \) between \( x \) and \( y \) in \( C_{2k+1} \).

2. Note that \((G^*)^s \approx [P_s,[P_r,G]]\) and \((G \boxtimes P_s) \boxtimes P_r \approx (G \boxtimes P_r) \boxtimes P_s \approx G \boxtimes P_{rs} \). Hence, by Theorem [1] for any graph \( H \) we have,

\[ \text{Hom}(H,(G^*)^s) \neq \emptyset \iff \text{Hom}(H,[P_s,[P_r,G]]) \neq \emptyset \iff \text{Hom}(H \boxtimes P_s,[P_r,G]) \neq \emptyset \]
\[ \iff \text{Hom}(H \boxtimes P_r,G) \neq \emptyset \iff \text{Hom}(H \boxtimes P_{rs},G) \neq \emptyset \]
\[ \iff \text{Hom}(H,[P_{rs},G]) \neq \emptyset \iff \text{Hom}(H,(G^*)^s) \neq \emptyset. \]

Now use these equivalences once for \( H = (G^*)^s \) to get \( \text{Hom}((G^*)^s,G^{rs}) \neq \emptyset \) and once for \( H = G^{rs} \), to get \( \text{Hom}(G^{rs},(G^*)^s) \neq \emptyset. \)
3. We use Theorem \[1\] and Proposition \[2\] as follows,

\[
G \rightarrow [A + B, H] \quad \text{Theorem} \[1\] \quad G \boxtimes (A + B) \rightarrow H
\]

\[
\text{Proposition} \[2\] \quad G \boxtimes A + G \boxtimes B \rightarrow H
\]

\[
P \text{ush properties} \quad G \boxtimes B \rightarrow H \text{ and } G \boxtimes A \rightarrow H
\]

\[
\text{Theorem} \[1\] \quad G \rightarrow [A, H] \text{ and } G \rightarrow [B, H]
\]

\[
\text{Categorical product properties} \quad G \rightarrow [A, H] \times [B, H].
\]

Now, set \( G \equiv [A + B, H] \), and consequently, \([A + B, H] \rightarrow [A, H] \times [B, H]\).

\[\blacksquare\]

Definitely one may consider a category of cylinders and maps between them and consider this category and its basic properties. We will not delve into the details of this in this article.

## 5 A categorical perspective

Let \( C = \{C', \{y^j, y', c\} \mid j \in 1 \ldots m\} \) be a \( \Gamma \)-coherent set of \((t, k)\)-cylinders. We show that the cylindrical construction \(- \boxtimes C : \text{LRgr}(\Gamma, m) \rightarrow \text{LRgr}\) and the exponential graph construction \([C, -] : \text{LRgr} \rightarrow \text{LRgr}(\Gamma, m)\) introduce well-defined functors. To see this, for two objects \( G \) and \( G' \) in \( \text{Obj}(\text{LRgr}(\Gamma, m)) \), where

\[
V(G) = \{v_1, v_2, \ldots, v_n\}, \quad V(G') = \{u_1, u_2, \ldots, u_n\}
\]

consider the following definition of the cylindrical construction functor,

\[
\forall G \in \text{Obj}(\text{LRgr}(\Gamma, m)) \quad (- \boxtimes C)(G) \equiv G \boxtimes C,
\]

\[
\forall f \in \text{Hom}_{\Gamma, m}(G, G') \quad (- \boxtimes C)(f) \equiv f \boxtimes C \in \text{Hom}_{\Gamma}(G \boxtimes C, G' \boxtimes C),
\]

defined as follows, where \{\(\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \Gamma\), is provided by Proposition \[1\] with respect to \( f \),

\[
(f \boxtimes C)(x) \equiv \begin{cases} u'_{i, \alpha} & x = v^j, \quad f(v^j) = u^j \quad (\text{blow up vertices for } v^j) \\ w'_{i, \epsilon} & = x = w^e, \quad f(e) = c' \quad (\text{Internal vertices of cylinders}) \end{cases}
\]

where as mentioned before \( w^e \equiv (w^1, w^2, \ldots, w^n) \) consisting of internal vertices that appear in \( G \boxtimes C \) when the edge \( e \) is replaced by the corresponding cylinder. Also (see Figure \[28\]) we define

\[
(f \boxtimes C)(e) \equiv \begin{cases} u^j_{\alpha_i (i)} u^j_{\alpha_i (i')} & e = v^i v^j, \quad f(v^j) = u^j_i, \quad (\text{Entire edge is in base of a cylinder}) \\ u^j_{\alpha_i (i)} u^j_{\alpha_i (i')} & e = v^i v^j, \quad f(v^j) = u^j_i, \quad (\text{边缘s which adjacent to both bases of a cylinder}) \\ w^i u^j_{\alpha_i (i)} & e = w^i, \quad f(v^i) = u^j_i, \quad (\text{An internal vertex and a vertex in the initial base } B^-) \\ w^i u^j_{\alpha_i (i')} & e = w^i, \quad f(v^i) = u^j_i, \quad (\text{An internal vertex and a vertex in the terminal base } B^+) \\ w^i \in E(C^j, v^j) & e = w^i, \quad f(v^j) = u^j_i, \quad (\text{Edges which is not adjacent to any base vertices}) \end{cases}
\]

Strictly speaking, this describes \( f \boxtimes C \) as a mapping that acts as \( f \) but identically on cylinders as generalized edges.
**Proposition 6** The map $- \boxtimes C : \text{LGrph}(\Gamma, m) \rightarrow \text{LGrph}$ is a well defined functor.

**Proof.** First, we should verify that for every $f \in \text{Hom}_{\text{LGrph}}(G, G')$, the map $f \boxtimes C$ is actually a homomorphism in $\text{Hom}_{\text{LGrph}}(G \boxtimes C, G' \boxtimes C)$, but this is clear since by definition not only edges but cylinders are mapped to cylinders of the same type and moreover, if $e \in E(G \boxtimes C)$, then $(f \boxtimes C)_e(e)$ is an edge of $G' \boxtimes C$. To see this, we consider the first two cases as follows. The rest of the cases can be verified similarly.

- $e = v_i v_i'$, $f(v_i) = u_j$.
  In this case we know that
  $$(f \boxtimes C)_e(e) = u_j^{a_i} u_j^{a_i'}(e')$$
  where $\ell_G^{a_i}(e) = \gamma^-$ and $\ell_G^{a_i'}(f(e)) = \alpha, \gamma^-$ for some edge $\tilde{e} \in E(G)$. But if $l = \gamma^-(l_0)$ then $\alpha_i(l) = \alpha, \gamma^-(l_0)$ which shows that both edges $e$ and $u_j^{a_i} u_j^{a_i'}(e')$ correspond to the same edge in the cylinder $C^{\ell_G^{a_i}(e)}$.

- $e = v_i v_i'$, $f(v_i v_i') = u_i u_i'$.
  In this case we know that
  $$(f \boxtimes C)_e(e) = u_j^{a_i} u_j^{a_i'}(e')$$
  where $\ell_G(v_i v_i') = (\gamma^-, t, \gamma^+)$ and $\ell_G(u_i u_i') = (\alpha, \gamma^-, t, \alpha, \gamma^+)$. But if we assume that $l = \gamma^-(l_0)$ and $l' = \gamma^+(l'_0)$ then
  $$\alpha_i(l) = \alpha_i, \gamma^-(l_0) \quad \text{and} \quad \alpha_i(l') = \alpha_i, \gamma^+(l'_0),$$
  which shows that both edges $e$ and $u_j^{a_i} u_j^{a_i'}(e')$ correspond to the same edge in the cylinder $C^{\ell_G(v_i v_i')}$. 

Also, one may easily verify that $1_G \boxtimes C = 1_{G \boxtimes C}$ and that
$$(f \circ g) \boxtimes C = (f \boxtimes C) \circ (g \boxtimes C),$$
for every compatible pair of homomorphisms $f$ and $g$ (just attach two square like Figure 28 side by side), which shows that $- \boxtimes C$ is a well-defined (covariant) functor. ■
On the other hand, for the exponential graph construction we have,

\[ \forall H \in \text{Obj}(\text{LGrph}) \quad (\mathbb{C}, -)_{\Gamma} : \text{LGrph} \to \text{LGrph}(\Gamma, m) \text{ is a well defined functor.} \]

**Proof.** First, we should verify that for every \( f \in \text{Hom}_\Gamma(H, H') \) the map \([C, f]_{\Gamma}\) is actually a homomorphism in \( \text{Hom}_{\Gamma, m}(\mathbb{C}, H, H') \).

For this, first note that by the definition of exponential graph construction and composition of homomorphisms, if \( e = \gamma v \) with \( \ell_{(\mathbb{C}, H, H')} (e) = (\gamma, t, \gamma') \), then \( (f(v))_{\Gamma}, (f(w))_{\Gamma} \) is actually edge of \([C, H']_{\Gamma}\) with the label \((\alpha, \gamma', t, \alpha_\omega \gamma')\). Also, clearly by its definition and Proposition 1 this map is a homomorphism in \( \text{Hom}_{\Gamma, m}(\mathbb{C}, H, H') \).

Moreover, one may verify that \([C, H], 1_{\mathbb{C}})_{\Gamma} = 1_{\mathbb{C}, H'}\), and that

\[ [C, f \circ g]_{\Gamma} = [C, f]_{\Gamma} \circ [C, g]_{\Gamma}, \]

for every compatible pair of homomorphisms \( f \) and \( g \), which shows that \([C, -]\) is a well-defined (covariant) functor.

In what follows we prove that both maps \( r_{\mathbb{C}, H} \) and \( s_{\mathbb{C}, H} \) are natural with respect to indices \( G \) and \( H \). This shows that the pair \((r_{\mathbb{C}, H}, s_{\mathbb{C}, H})\) is a weak version of an adjunct pair.

![Figure 29: Naturality of \( r_{\mathbb{C}, H} \) with respect to \( H \) (see Theorem 3).](image)

**Theorem 3.** Let \( \mathbb{C} = \{C^j(\mathbb{Y}^j, \mathbb{Y}'^j, \mathbb{E}^j) \mid j \in \{1, \ldots, m\}\} \) be a \( \Gamma \)-coherent set of \((t, k)\)-cylinders. Then the retraction \( r_{\mathbb{C}, H} \) and the section \( s_{\mathbb{C}, H} \) introduced in Theorem 1 are both natural with respect to \( G \) and \( H \).

**Proof.** In what follows we prove that \( r_{\mathbb{C}, H} \) is natural with respect to its second index \( H \), and that \( s_{\mathbb{C}, H} \) is natural with respect to its first index \( G \). The other two cases can be verified similarly.

For the first claim, assume that \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and fix homomorphisms

\[ f \in \text{Hom}_\Gamma(H, H') \quad \text{and} \quad \sigma = (\sigma_\mathbb{V}, \sigma_\mathbb{E}) \in \text{Hom}_\Gamma(G \mathbb{E}, C, H), \]

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and let
\[ r_{G,H}(\sigma) = \sigma' \in \text{Hom}_{\Gamma,m}(G, [C, H]_{\Gamma}), \quad \text{and} \quad \sigma'' \overset{\text{def}}{=} \text{Hom}_{\Gamma,m}(G, [C, f]_{\Gamma})(\sigma'), \]
(see the proof of Theorem 1 and the diagram depicted in Figure 29). Then, by definitions we have
\[ \forall i \in \overline{1...n} \quad \sigma_v(v^i) = u^i \quad \Rightarrow \quad \forall i \in \overline{1...n} \quad \sigma'_v(v^i) = (u^i)_v, \]
\[ \Rightarrow \quad \forall i \in \overline{1...n} \quad \sigma''_v(v^i) = (f(u^i))_v. \]
On the other hand, let
\[ \tilde{\sigma} \overset{\text{def}}{=} \text{Hom}_{\Gamma}(G \boxtimes \Gamma, C, f)(\sigma) \quad \text{and} \quad \tilde{\tilde{\sigma}} \overset{\text{def}}{=} r_{G,H}(\tilde{\sigma}). \]
Then, again by definitions,
\[ \forall i \in \overline{1...n} \quad \sigma_v(v^i) = u^i \quad \Rightarrow \quad \forall i \in \overline{1...n} \quad \tilde{\sigma}_v(v^i) = f(u^i), \]
\[ \Rightarrow \quad \forall i \in \overline{1...n} \quad \tilde{\tilde{\sigma}}_v(v^i) = (f(u^i))_v, \]
which clearly shows that \( \sigma''_v = \tilde{\tilde{\sigma}}_v \). The equality for the edge-maps also follows easily in a similar way, and consequently, the diagram of Figure 29 is commutative and \( r_{G,H} \) is natural with respect to its second index \( H \).

Figure 30: Naturality of \( s_{G,H} \) with respect to \( G \) (see Theorem 3).

For the second claim, assume that \( V(G') = \{ w_1, w_2, \ldots, w_m \} \), fix homomorphisms
\[ g \in \text{Hom}_{\Gamma,m}(G', G), \quad \text{and} \quad \sigma' \in \text{Hom}_{\Gamma,m}(G', [C, H]_{\Gamma}), \]
with the mapping \( \beta \) such that
\[ \forall i \in \overline{1...m} \quad g(w_i) \overset{\text{def}}{=} v^i_{\beta(i)}, \]
and let \( \{ \alpha_1, \alpha_2, \ldots, \alpha_m \} \subseteq \Gamma \) be obtained by Proposition 1. Also, define,
\[ \sigma \overset{\text{def}}{=} s_{G,H}(\sigma') \in \text{Hom}_{\Gamma}(G \boxtimes \Gamma, C, H), \quad \text{and} \quad \sigma'' \overset{\text{def}}{=} \text{Hom}_{\Gamma}(g \boxtimes \Gamma, C, H)(\sigma). \]
(see the proof of Theorem 1 and the diagram depicted in Figure 30). Then, by definitions for all \( i \in \overline{1...n} \) we may choose \( u^i \) such that for all \( j \in \overline{1...m} \),
\[ \sigma'_v(v^i) = (u^i)_v = u^i \alpha^i_i \quad \Rightarrow \quad \sigma_v(v^i) = u^i \]
\[ \Rightarrow \quad \sigma''_v(w^j) = \sigma_v(g(w^j)) = \sigma_v(v^\beta(j)) = u^\beta(j). \]
On the other hand, let
\[ \tilde{\sigma} \overset{\text{def}}{=} \text{Hom}_{\Gamma,m}(g, [C, H]_{\Gamma})(\sigma') \quad \text{and} \quad \tilde{\tilde{\sigma}} \overset{\text{def}}{=} s_{G',H}(\tilde{\sigma}). \]
Then, for all \( i \in \overline{1..n} \) and \( j \in \overline{1..m} \),
\[
\sigma'_{v_i}(v_i) = \langle u_i \rangle_{\Gamma} = u_i^{\alpha_i} \quad \Rightarrow \quad \tilde{\sigma}'_{v_i}(w_j) = \sigma'_{v_i}(g(w_j)) = \sigma'_{v_i}(\beta_{\beta(j)}) = \langle u^{\beta(j)} \rangle_{\Gamma} = u^{\beta(j)}_{\alpha_{\beta(j)}}
\]
\[
\Rightarrow \quad \tilde{\sigma}'_{v_i}(w') = s_{G',L}(\tilde{\sigma}'_{v_i})(w') = u^{\beta(j)},
\]
which shows that \( \tilde{\sigma}'_{v_i} = \sigma'_{v_i} \). The equality for the edge-maps also follows easily in a similar way, and consequently, the diagram of Figure 30 is commutative and \( s_{G,L} \) is natural with respect to its first index \( G \).

**Proposition 8.** The standard line-graph construction is not cylindrical, i.e. if \( L(H) \) is the standard line-graph of the fixed graph \( H \), then there is no \( \Gamma \)-coherent set of cylinders \( C \) for which the following holds,
\[
\forall H' \approx H, \quad [C,H']_{\Gamma} \approx L(H').
\]

**Proof.** Assume that there exists \( C \) such that
\[
\forall H' \approx H, \quad [C,H']_{\Gamma} \approx L(H').
\]

Also, let \( \chi(L(H)) = n \), and let \( H' \defeq \overline{H(v) + St_{n+1}} \) be the graph obtained by identifying the central vertex \( v \) of the star graph on \( n + 1 \) vertices, \( St_{n+1} \), with an arbitrary vertex of \( H \). Clearly, \( K_{n+1} \to L(H') \).

Note that \( H \approx H' \) and \( L(H') \to L(H) \), i.e. \( [C,H']_{\Gamma} \to [C,H]_{\Gamma} \), that contradicts the functoriality of the exponential construction.

**Proposition 9** Let \( G \) be a \( \Gamma \)-graph and \( C \) be a \( \Gamma \)-coherent cylinder, also let \( H \) be a labeled graph, then,

1. If \( \sigma \in \text{Hom}_i(G \boxtimes_{\Gamma} C, H) \) is vertex-injective, then \( r_{G,H}(\sigma) \in \text{Hom}_i(G, [C,H]_{\Gamma}) \) is also vertex-injective.

2. The homomorphism \( r_{G,H}(\sigma) \in \text{Hom}_i(G, [C,H]_{\Gamma}) \) is not necessarily vertex-surjective, (even if \( \sigma \in \text{Hom}_i(G \boxtimes_{\Gamma} C, H) \) is an isomorphism).

**Proof.**

1. Let \( \sigma \in \text{Hom}_i(G \boxtimes_{\Gamma} C, H) \) be vertex-injective and \( r_{G,H}(\sigma) \in \text{Hom}_i(G, [C,H]_{\Gamma}) \). If \( r_{G,H}(\sigma)(v_i) =< \sigma(v_i) >_{\Gamma} \), where \( i \in \{1,2\} \), then \( \sigma(v_i) \) and \( \sigma(v_2) \) have not any vertex in common in \( H \), so \( < \sigma(v_1) >_{\Gamma} \neq < \sigma(v_2) >_{\Gamma} \), which means that \( r_{G,H}(\sigma)(v_i) \neq r_{G,H}(\sigma)(v_2) \).

2. Just set \( G = C_r, \ C = P_t \) and \( H = C_{rt} \), where \( r, t > 1 \), then it is easy to check that \( id \in \text{Hom}_i(G \boxtimes_{\Gamma} C, H) = \text{Hom}_i(C_r, C_{rt}) \) is an isomorphism,

Now \( [P_t, C_{rt}] \) has order \( rt \) and the order of \( C_r \) is \( r \). Hence, there is not any surjective homomorphism in \( \text{Hom}(G, [C,H]) = \text{Hom}(C_r, [P_t,C_{rt}]) \).

### 6 Adjunctions and reductions

Note that Theorem 4 and naturalities proved in previous section imply that we have an adjunction \( - \boxtimes_i C \dashv [C,-]_{\Gamma} \) between \( \text{LGrph}_{\Gamma} \) and \( \text{LGrph}_{\Gamma} \), however, this correspondence is not necessarily an adjunction between \( \text{LGrph}(\Gamma,m) \) and \( \text{LGrph} \).
In this section we consider conditions under which a class of cylinders reflects an
adjunction between LGrph(Γ, m) and LGrph, that will be called tightness. We will also consider
weaker conditions called upper and lower closedness and will consider some examples and
consequences related to computational complexity of graph homomorphism problem (e.g. see [14]
for the background and some special cases).

Lemma 2. Let C = \{C′(y, y′, e) | j \in \Gamma \ldots m\} be a set of Γ-coherent (t, k)-cylinders, and
H be a labeled graph. Then the following statements are equivalent,
a) For any (Γ, m)-graph G the retraction \( r_{G,H} \) is a one-to-one map (i.e. is invertible with
\( r_{G,H}^{-1} = s_{G,H} \)).
b) For any (Γ, m)-graph G we have
\[ |\text{Hom}_r(G \boxtimes_r C, H)| = |\text{Hom}_{r,m}(G, [C, H]_r)|. \]
c) For any (Γ, m)-graph G and any \( \sigma \in \text{Hom}_{r,m}(G, [C, H]_r) \) there is a unique \( \rho \in \text{Hom}_r(G \boxtimes_r C, H) \) such that \([C, \rho]_r \circ \eta_G = \sigma\).

Proof. (a ⇔ b): Considering the retraction
\[ r_{G,H} : \text{Hom}_r(G \boxtimes_r C, H) \rightarrow \text{Hom}_{r,m}(G, [C, H]_r) \]
and the section
\[ s_{G,H} : \text{Hom}_{r,m}(G, [C, H]_r) \rightarrow \text{Hom}_r(G \boxtimes_r C, H). \]
It is easy to see that the retraction is one-to-one if and only if the section is so and this is
equivalent to
\[ |\text{Hom}_r(G \boxtimes_r C, H)| = |\text{Hom}_{r,m}(G, [C, H]_r)|. \]
(a ⇔ c): Let \( \rho \overset{\text{def}}{=} r_{G,H}(\sigma) \). Then by definitions
\[ \forall v_1 \in V(G) \; \rho_r(v_1) = u^1 \Rightarrow \forall v_1 \in V(G) \; [C, \rho]_r([v_1]_r) = ([u^1]_r \]
\[ \Rightarrow \forall v_1 \in V(G) \; [C, \rho]_r \circ \eta_G(v_1) = ([u^1]_r, \]
which means that \([C, \rho]_r \circ \eta_G \) and \( \sigma \) act the same on vertices, and consequently, since they
are homomorphisms, they also have the same action on edges.
Now, if the retraction is one-to-one, then there is a unique map \( \rho \overset{\text{def}}{=} r_{G,H}(\sigma) \) for which
\([C, \rho]_r \circ \eta_G = \sigma \) and vice versa.

Note that for any (Γ, m)-graph, G, we have
\[ \text{Hom}_{r,m}(G, [C, G \boxtimes_r C]_r) \neq \emptyset, \]
since
\[ \eta_G \overset{\text{def}}{=} r_{G,G\boxtimes_r C}(1_{G\boxtimes_r C}) \in \text{Hom}_{r,m}(G, [C, G \boxtimes_r C]_r), \]
and moreover, it is easy to see that \( \eta : 1_{LGrph(\Gamma, m)} \Rightarrow [C, -]_r \circ (- \boxtimes_1 C) \), is a natural transformation.

Similarly, for any labeled graph H we have \( \text{Hom}_r([C, H]_r \boxtimes_r C, H) \neq \emptyset, \)
since
\[ \delta_H \overset{\text{def}}{=} s_{[C, H]_r,H}(1_{[C, H]_r}) \in \text{Hom}_r([C, H]_r \boxtimes_r C, H), \]
and moreover, it is easy to see that \( \delta : 1_{LGrph} \Rightarrow (- \boxtimes_r C) \circ [C, -]_r \), is a natural transformation.
Definition 10. Tightness
A Γ-coherent set of \((t,k)\)-cylinders \(C = \{C_j'(y^j, y^j, \epsilon^j) \mid j \in \hat{1}...m\}\) is said to be tight with respect to a labeled graph \(H\), if it satisfies any one of the equivalent conditions of Lemma 2.

A Γ-coherent set of \((t,k)\)-cylinders \(C\) is said to be lower-closed with respect to a \((\Gamma,m)\)-graph \(H\) if

\[\text{Hom}_{t,m}([C, H \boxtimes_r C], H) \neq \emptyset.\]

Also, \(C\) is said to be upper-closed with respect to a labeled graph \(G\) if

\[\text{Hom}_{t,r}(G, [C, G] \boxtimes_r C) \neq \emptyset.\]

\[\square\]

It is clear that a Γ-coherent set of \((t,k)\)-cylinders \(C\) is lower-closed with respect to a \((\Gamma,m)\)-graph \(H\) if and only if \([C, H \boxtimes_r C], \approx H\), and similarly, \(C\) is upper-closed with respect to a labeled graph \(G\) if and only if \([C, G] \boxtimes_r C \approx G\). Also, it should be noted that if a cylinder is strong in the sense of \cite{14}, i.e.

- A \((t,k)\)-cylinder \(C\) is strong if for any irreflexive \((\Gamma,m)\)-graph \(H\), and any homomorphism \(\sigma : C \to H \boxtimes_r C\), the homomorphic image \(\sigma(C)\) is contained in some cylinder \(C\).

then any strong cylinder is lower-closed with respect to all irreflexive \((\Gamma,m)\)-graphs.

Example 9.

1. \(\square\) is lower-closed

If \(|E(G)| = m\), then \([\square, G \boxtimes \square] = G \cup I_m\), where the cylinder \(\square\) is depicted in Figure 31(a). Therefore, \(\square\) is lower-closed with respect to any graph.

2. Any non-empty bipartite cylinder \(C\) is upper-closed with respect to any bipartite graph.

Let \(C\) and \(G\) be non-empty and bipartite. Therefore, \(C \to G\), and \([C, G]\) and consequently \([C, G] \boxtimes_r C\) is non-empty. Thus, \(G\) is bipartite \(G \to [C, G] \boxtimes_r C\).

3. \(C_x(G)\) is tight with respect to any graph \(H\).

Since \(|\text{Hom}(G \times F, H)| = |\text{Hom}(F, H^G)|\) (e.g. see \cite{14}), we have

\[|\text{Hom}(H \boxtimes C_x(G), F)| = |\text{Hom}(H, [C_x(G), H])|,\]

and consequently \(C_x(G)\) is tight.

\(C_x(G)\) is upper-closed with respect to \(H\), if and only if \(G \to H\).

Since \(\text{Hom}(G, G^H) \neq \emptyset\), we have

\[\text{Hom}(H, [C_x(G), H] \boxtimes C_x(G)) \neq \emptyset \iff \text{Hom}(H, H^G \times H) \neq \emptyset.\]
But \( |\text{Hom}(H, H^G \times G)| = |\text{Hom}(H, H^G)| \cdot |\text{Hom}(H, G)| \) (e.g. see [14]), and by assuming \( \text{Hom}(H, G) \neq \emptyset \), we have \( \text{Hom}(H, H^G \times G) \neq \emptyset \). It is easy to check the reverse implication.

- \( C_x(G) \) is not lower-closed with respect to a simple graph \( H \), if \( G \to H \).
  If \( G \to H \) then \( G \to G \times H \). Also, \( G \cong GL(C_x(G)) \), and consequently \( GL(C_x(G)) \to G \times H \). Thus,
  \[
  (G \times H)^G \cong [C_x(G), H \boxtimes C_x(G)] = [C_x(G), G \times H]
  \]
  has a loop (consider the fact that if \( GL(C) \to H \), then \( [C, H] \) has a loop), and since \( H \) does not have any loop,
  \[
  \text{Hom}([C_x(G), H \boxtimes C_x(G)], H) = \emptyset.
  \]

The cylinder \( C_x(G) \) shows that a tight cylinder may neither be lower nor upper-closed.

\[\blacksquare\]

**Lemma 3** Let \( C \) be a \( \Gamma \)-coherent set of \((t, k)\)-cylinders. Then,

a) If \( C \) is lower-closed with respect to a \((\Gamma, m)\)-graph, \( H \), then
  \[
  \text{Hom}_{\Gamma}(G \boxtimes_r C, H \boxtimes_r C) \neq \emptyset \iff \text{Hom}_{\Gamma,m}(G, H) \neq \emptyset.
  \]

b) If \( C \) is upper-closed with respect to a labeled graph \( G \), then
  \[
  \text{Hom}_{\Gamma,m}([C, G]_{\Gamma}, [C, H]_{\Gamma}) \neq \emptyset \iff \text{Hom}_{\Gamma}(G, H) \neq \emptyset.
  \]

**Proof.** To prove (a) by Theorem [1] and the definition we have,

\[
\text{Hom}_{\Gamma}(G \boxtimes_r C, H \boxtimes_r C) \neq \emptyset \iff \text{Hom}_{\Gamma,m}(G, [C, H]_{\Gamma}) \neq \emptyset \iff \text{Hom}_{\Gamma,m}(G, H) \neq \emptyset.
\]

The inverse implication is clear by Proposition [6] Similarly, to prove (b) we have,

\[
\text{Hom}_{\Gamma,m}([C, G]_{\Gamma}, [C, H]_{\Gamma}) \neq \emptyset \iff \text{Hom}_{\Gamma}(G \boxtimes_r C, H) \neq \emptyset \iff \text{Hom}_{\Gamma,m}(G, H) \neq \emptyset.
\]

The inverse implication is clear by Proposition [7].

**Proposition 10.** Suppose that \( C \) is a connected cylinder, then \( C \) is not tight with respect to a graph \( H \) if and only if it is not tight with respect to at least one of its connected components.

**Proof.** If we decompose \( H \) to the connected components

\[ H = H_1 \cup H_2 \cup \ldots \cup H_s, \]

we can see that

\[
|\text{Hom}(G \boxtimes C, H)| = \sum_{i=1}^{s} |\text{Hom}(G \boxtimes C, H_i)| \geq \sum_{i=1}^{s} |\text{Hom}(G, [C, H_i])| = |\text{Hom}(G, \cup_{i=1}^{s} [C, H_i])| = |\text{Hom}(G, [C, H])|.\]
Therefore, since for all \(i \in \{1, 2, \ldots, s\}\) we have \(|\text{Hom}(G \boxtimes C, H_i)| \geq |\text{Hom}(G, [C, H_i])|\), the strict inequality
\[ |\text{Hom}(G \boxtimes C, H)| > |\text{Hom}(G, [C, H])| \]
holds for \(H\) if and only if the strict inequality
\[ |\text{Hom}(G \boxtimes C, H_i)| > |\text{Hom}(G, [C, H_i])| \]
holds for one of the indices \(i \in \{1, 2, \ldots, s\}\). ■

**Example 10. Tightness of paths.**

1- **Odd paths**

(1) *Let \(k > 1\) and \(H\) be a simple graph in which \(C^1, C^2, \ldots, C^s\) are \(s\) distinct odd cycles of lengths \(c_1, c_2, \ldots, c_s\) with \(c_i \leq 2k + 1\) for all \(1 \leq i \leq s\). If \(2 \sum_{i=1}^{s} c_i > |V(H)|\), then \(P_{2k+1}^2\) is not tight with respect to \(H\).*

Note that if \(C_r\) and \(C_s\), are two odd cycles with \(s \leq r\), then there exist at least \(2s\) distinct homomorphisms from \(C_r\) to \(C_s\), (mark vertices of \(C_s\) and \(C_r\) in circular order and for any \(i \in \{1, 2, \ldots, s\}\), consider the homomorphisms \(1 \mapsto i, 2 \mapsto i+1, 3 \mapsto i+2, \ldots\) and \(1 \mapsto i, 2 \mapsto i-1, 3 \mapsto i-2, \ldots\).

Now, consider two distinct odd cycles \(C^1, C^2\) of lengths \(c_1, c_2 \leq 2k + 1\) in \(H\), and note that since any homomorphism to an odd cycle is a vertex-surjective map, for odd \(r\)
\[ |\text{Hom}(C^r, C^1) \cap \text{Hom}(C^r, C^2)| = \emptyset. \]

Therefore, if \(L\) is a loop on one vertex, then
\[ |\text{Hom}(L \boxtimes P_{2k+1}, H)| = |\text{Hom}(C^1_{2k+1}, H)| \geq 2 \sum_{i=1}^{s} c_i. \]

On the other hand, since \([P_{2k+1}, H]\) has at most \(|V(H)|\) loops we have,
\[ |\text{Hom}(L, [P_{2k+1}, H])| \leq |V(H)|, \]
and consequently,
\[ |\text{Hom}(L \boxtimes P_{2k+1}, H)| \geq 2 \sum_{i=1}^{s} c_i > |V(H)| \geq |\text{Hom}(L, [P_{2k+1}, H])|. \]

(2) \(P_{2k+1}\) is lower-closed with respect to any simple graph \(G\).

By [11] we have \(\text{Hom}(G, P_{2k+1}^2, G) \neq \emptyset\). On the other hand, we have
\[ \text{Hom}([P_{2k+1}, G \boxtimes P_{2k+1}], G) = \text{Hom}(G_{2k+1}^{2k+1}, G), \]
and consequently, \(P_{2k+1}\) is lower-closed with respect to \(G\).

(3) \(P_{2k+1}\) is not upper-closed with respect to \(G\), if \(G\) has odd girth less than or equal to \(2k + 1\).

We have
\[ \text{Hom}(G, [P_{2k+1}, G] \boxtimes P_{2k+1}) = \text{Hom}(G, (G_{2k+1}^{2k+1})_{2k+1}), \]
but the smallest odd cycle of \((G_{2k+1}^{2k+1})_{2k+1}\) is \(2k + 1\), and consequently, the odd girth of \(G\) is less than the odd girth of \((G_{2k+1}^{2k+1})_{2k+1}\). Hence, there is not any homomorphism from \(G\) to \((G_{2k+1}^{2k+1})_{2k+1}\).
2- Even paths

(1) The cylinder $P_{2k}$ is tight with respect to $H$, if and only if $H$ is a matching.

By Proposition 10 we may assume that $H$ is connected. Also, if $H$ is a connected graph that has a vertex of degree 2, then we have $2|E(H)| > |V(H)|$.

If $L$ is a loop on a single vertex, since there exist at least two homomorphisms from an even cycle to an edge, we have

$$|\text{Hom}(L \boxtimes P_{2k}, H)| = |\text{Hom}(C_{2k}, H)| \geq 2|E(H)|.$$  

Also, $|\text{Hom}(L, [P_{2k}, H])| \leq |V(H)|$, because the worst case is that $[P_{2k}, H]$ has a loop on each vertex. Therefore, when $2|E(H)| > |V(H)|$,

$$|\text{Hom}(L \boxtimes P_{2k}, H)| \geq 2|E(H)| > |V(H)| \geq |\text{Hom}(L, [P_{2k}, H])|.$$  

If $H$ does not have any vertex of degree 2, it is a matching and by Proposition 10 it is enough to assume that $H = K_2$.

Note that if $F$ is a bipartite graph, then $|\text{Hom}(F, K_2)| = 2^c$, where $c$ is the number of connected components of $F$. For any graph $G$ with $c$ connected components, $G \boxtimes P_{2k}$ is bipartite with $c$ connected components. Now, since $[P_{2k}, K_2] = L \cup L$, there exists $2^c$ homomorphisms from $G$ to $[P_{2k}, K_2]$, and consequently,

$$\text{Hom}(G, [P_{2k}, K_2]) = 2^c = \text{Hom}(G \boxtimes P_{2k}, K_2).$$

(2) $P_{2k}$ is not upper-closed with respect to any non-bipartite graph.

Note that for any non-bipartite graph $G$, the construction $[P_{2k}, G] \boxtimes P_{2k}$ is bipartite, and consequently,

$$\text{Hom}(G, [P_{2k}, G] \boxtimes P_{2k}) = \emptyset.$$  

(3) $P_{2k}$ is not lower-closed with respect to any loop free graph.

Note that for any non-empty graph $G$, $[P_{2k}, G \boxtimes P_{2k}]$ has a loop, and consequently, for a loop free graph $G$, we have

$$\text{Hom}(G, [P_{2k}, G \boxtimes P_{2k}]) = \emptyset.$$  

Corollary 2. $P_{2k+1}$ is not tight with respect to any graph with the following spanning subgraphs,

- $C_{2r+1}$ for $r \leq k$.
- A graph $H$, in which at least half of its vertices lie in an odd cycle of length less than $2k+1$.

Applications of indicator/replacement constructions to study the computational complexity of the homomorphism problem was initiated in [15] and now is well-known in the related literature (see [3-5,12-16,21]). In what follows we will consider some basic implications for the cylindrical/exponential construction as a generalization.

Problem 1. The $H$-homomorphism problem (H-Hom)

Given a labeled graph $G$, does there exist a homomorphism $\sigma : G \rightarrow H$? (i.e. find out whether $\text{Hom}_\ell(G, H) \neq \emptyset$.)
Problem 2. The \#H-homomorphism problem (\#H-Hom)

Given a labeled graph $G$, compute the number of homomorphisms $\sigma : G \rightarrow H$ (i.e. find out the number $|\text{Hom}_1(G, H)|$.)

To begin, note that as an immediate consequence of Theorem 1 we have,

**Corollary 3.** If $\leq^p_m$ stands for the many-to-one polynomial-time reduction, then

$$[C, H]_{1\text{-Hom}} \leq^p_m H\text{-Hom}.$$  

**Proof.** The reduction is exactly the cylindrical construction which is clearly a polynomial-time computable mapping of graphs. \(\blacksquare\)

The following corollary is a direct consequence of Lemma 2.

**Corollary 4.** If $\leq^f_p$ stands for the parsimonious polynomial-time reduction, and $C$ is tight with respect to $H$, then

$$\#[C, H]_{1\text{-Hom}} \leq^f_p \#H\text{-Hom}.$$  

**Corollary 5.** Let $n \geq 3$ be the odd girth of the simple graph $G$. If there exists an odd path of length less than $n$ between each pair of vertices in $G$, then $G - \text{Hom}$ is NP-complete.

**Proof.** By the assumptions, $[P_{n-2}, G] \simeq K_{|V(G)|}$ and by Corollary 3 we have

$$[P_{n-2}, G]\text{-Hom} \simeq K_{|V(G)|}\text{-Hom} \leq^p_m G\text{-Hom}.$$  

But, since $|V(G)| \geq 3$, $G - \text{Hom}$ is NP-complete. \(\blacksquare\)

**Corollary 6.** For following graphs, $G\text{-Hom}$ is NP-complete.

- $G = C_n$ for odd $n$.
- $G =$ the Petersen graph.
- $G =$ Coxeter graph.
- $G = C_n^r$ for odd $n$ and $r|n - 2$.

**Proof.** The first three cases follow from Corollary 3. For the last one, note that $C_n\text{-Hom}$ is NP-complete for odd $n$. Also by Theorem 1 we have

$$G \rightarrow [P_{n-2}, C_n] \simeq K_n$$

$$\leq^p_m G \boxtimes P_{n-2} \rightarrow C_n$$

$$\leq^p_m G \boxtimes P_{n-2} \boxtimes P_r \rightarrow C_n$$

$$\leq^p_m G \boxtimes P_{n-2} \rightarrow [P_r, C_n]$$

Let $\mathcal{H}$ be a class of graphs. Then the cylindrical closure of $\mathcal{H}$, denoted by $\Psi(\mathcal{H})$, is defined as

$$\Psi(\mathcal{H}) \overset{\text{def}}{=} \{ G : \exists C \ [C, G] \in \mathcal{H} \}.$$  

Considering the identity cylinder one gets $\mathcal{H} \subseteq \Psi(\mathcal{H})$, and moreover, the universal closure of $\mathcal{H}$ is defined as

$$\overline{\mathcal{H}} \overset{\text{def}}{=} \bigcup_{n=0}^{\infty} \Psi^n(\mathcal{H}) = \lim_{n \rightarrow \infty} \Psi^n(\mathcal{H}),$$
where $\Psi_0(\mathcal{H}) \overset{\text{def}}{=} \mathcal{H}$. Clearly, by Corollary 3, if the H-homomorphism problem is NP-complete for all $H \in \mathcal{H}$ then this problem is also NP-complete for all $H \in \overline{\mathcal{H}}$. To prove a Dichotomy Conjecture in a fixed category of graphs it is necessary and sufficient to find a class $\mathcal{H}$ such that the H-homomorphism problem is NP-complete for all $H \in \mathcal{H}$ and it is polynomial time solvable for all $H \not\in \mathcal{H}$ (e.g. see [3,5,12,10,21] for the background and results in this regard for simple and directed graphs as well as CSP’s). It is interesting to consider possible generalizations of indicator/replacement/fiber construction techniques to the case of cylindrical constructions or consider dichotomy and no-homomorphism results in the category of labeled graphs.

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Appendix: On the existence of pushouts

This section contains subtleties about the existence of pushouts in different categories of graphs. The contents are classic and have been included to clarify the details and avoid misunderstandings.

Definition 11. Serre-Stallings category of Graphs, StGrph

This is a category of graphs usually used in algebraic topology and dates back at least to the contributions of Serre and Stallings [23,24] where the existence of pushouts in this category was first addressed in [23].

The category is denoted by StGrph and a graph \((F, V, E)\) is an object of this category with two sets \(V\) (containing vertices) and \(E\) (containing edges), and two functions \(\iota_{V}: E \rightarrow V\) and \(s_{V}: E \rightarrow V\). For each edge \(e \in E\), there is an inverse edge \(s(e) = \overline{e} \in E\), and an initial vertex \(\iota(e) \in V\) where the terminal vertex defined as \(\tau(e) = \iota(s(e))\) satisfying \(\overline{\overline{e}} = e\) and \(\overline{e} \neq e\).

A StGrph-homomorphism \((\sigma_{V}, \sigma_{E})\) between two graphs \((G, V, E)\) and \((H, V', E')\) is a pair of maps that preserve structures i.e. for two maps \(\sigma_{V}: V \rightarrow V'\) and \(\sigma_{E}: E \rightarrow E'\), we have

\[
\iota_{H} \circ \sigma_{E} = \sigma_{V} \circ \iota_{G},
\]

\[
s_{H} \circ \sigma_{E} = \sigma_{E} \circ s_{G}.
\]
If one fixes orientations for graphs \( G \) and \( H \) (i.e. choosing just one edge from any pair \( \{e, \overline{e}\} \)) and call the corresponding directed graphs \( O(G) \) and \( O(H) \), respectively, then assuming the existence of an ordinary directed graph homomorphism from \( O(G) \) to \( O(H) \), then clearly, one can canonically construct a graph homomorphism from \( G \) to \( H \) in \( \text{StGrph} \).

Hence, if there exists orientations for \( \text{StGraph} \)-graphs \( G \) and \( H \) we should have \( \text{symmetry} \) of the pushout diagram we should have \( Q \) vice versa.

Hence, by commutativity of the pushout diagram we should have \( Q \) preserve such orientations.

Therefore, considering two pushout objects \((P, i^1 : H_1 \to P, i^2 : H_2 \to P)\) and \((Q, j^1 : H_1 \to Q, j^2 : H_2 \to Q)\) for which the image of \( i^1 \) and \( i^2 \) is a single loop which we denote by \( m \) and \( n \) respectively.

Hence, we have two homomorphisms \( g, f : Q \to P \), where

\[ h_E = \{(l, m), (l, m)\}, \quad f_E = \{(l, m), (l, m)\}. \]
Both \( h \) and \( f \) satisfying the pushout diagram commutativity condition which is a contradiction.

Striktly speaking, in \( \text{SymGrph} \) there exist only a unique homomorphism between a pair of loops but in \( \text{SGraph} \), there exists more than one homomorphism between a pair of loops.

**Proposition 11.** Let \( F : \text{SGraph} \rightarrow \text{SymGrph} \) be the forgetful functor, that sends each (directed) graph to its corresponding simple graph. Then \( F \) is bijective on objects, but is not faithful on objects endowed with loops or multiple edges.

Figure 32: Push out diagram in \( \text{SGraph} \), where there exist two homomorphisms between two object which commute the diagram.