Bipartite Producer-Consumer Networks and the Size Distribution of Firms

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Abstract

A bipartite producer-consumer network is constructed to describe the industrial structure. The edges from consumer to producer represent the choices of the consumer for the final products and the degree of producer can represent its market share. So the size distribution of firms can be characterized by producer’s degree distribution. The probability for a producer receiving a new consumption is determined by its competency described by initial attractiveness and the self-reinforcing mechanism in the competition described by preferential attachment. The cases with constant total consumption and with growing market are studied. The following results are obtained: 1, Without market growth and a uniform initial attractiveness $a$, the final distribution of firm sizes is Gamma distribution for $a > 1$ and is exponential for $a = 1$. If $a < 1$, the distribution is power in small size and exponential in upper tail; 2, For a growing market, the size distribution of firms obeys the power law. The exponent is affected by the market growth and the initial attractiveness of the firms.

Key words: size distribution of firms, bipartite networks, complex networks, econophysics

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1 Introduction

Industrial structure is an important issue both in macroeconomic and physical investigations. It is closely related to the dynamics of firms and market. From the empirical studies, it has been found that there are several “stylized facts” related to the processes of industrial evolution. One of them is the skewed distribution of firm size which is usually measured by the sales, the number of employees, the capital employed and the total assets. Such skewed distribution has usually been described by lognormal distribution since Gibrat[1] and its upper tail has been described by Pareto distribution or Zipf’s law[2,3]. In terms of cumulative distribution $P_x(x)$ for firm size $x$, this states that $P_x(x) \propto x^\mu$ for larger $x$, where $\mu$ is the exponent called Pareto index. Recently, there are more empirical researches investigating the properties of firm size distribution in detail[4,5]. Axtell reveals that the U.S. firm size is precisely described by power law distribution[6]. For the cumulative distribution of firm sizes by employees, the index is 1.059, and for the cumulative distribution of firm sizes by receipts in dollars the index is 0.994. Some scholars in Italy and Japan, by exploring the size distribution of European and Japanese firms in detail, have found more evidence for the power law distribution of firm size in upper tail including the Zipf law in firms bankruptcy or extinction[7,8]. All the indexes for cumulative distributions are ranging around 1 (from 0.7 to 1.2).

Various kinds of power-law behaviors have been observed in a wide range of systems, including the wealth distribution of individuals[9,10,11] and the price-returns in stock markets[12,13]. Pareto-Zipf law in firm size provides another interesting example which exhibits some universal characteristics similar to those observed in physical systems with a large number of interacting units. Hence the growth dynamics and size distribution of business firms have become a subject of interest among economists and physicists, especially those who working in econophysics[12]. Together with the works in macroeconomics[14,15,16], many efforts have been done from the perspectives of physics in accordance with these empirical facts. Takayasu advanced an aggregation-annihilation reaction model to study firm size dynamics[17]. Axtell has argued that complexity approach should be used to deal with the problem, and agent based modelling together with evolutionary dynamics should be helpful to understand the formation of power law[18]. Amaral et al. have studied firm growth dynamics since 1997[19]. They have developed a stochastic model based on interactions between different units of a complex system. Each unit has a complex internal structure comprising many subunits and its growth dynamics is dependent on the interactions between them. This model’s prediction goes well with the empirical result[20,21]. Some other models have also been presented, which are based on the competition dynamics[22], the information transition, herd behaviors[23], and the proportional growth for the firms’ sizes and the number of independent constituent units of the firms.[24].
The development of the research on complex networks\cite{25,26,27} has given us a new perspective to speculate the power law distribution of firm size. First, it provides us a universal tool for the research of complex systems\cite{28}. Actually, any complex systems made up by the interactive components can be described by networks, in which the components are represented by the vertices, and the interactions by the edges. Second, the empirical results demonstrate that many large networks in the real world are scale free, like the World Wide Web, the internet, the network of movie-actor collaborations, the network of citations of scientific papers, the network of scientific collaborations and so on. They all have a scale-free degree distribution with tails that decay as a power law (see \cite{27,28} as reviews). So the complex networks in the nature give us examples of power law behavior. Barabási and Albert have argued that the scale-free nature of real networks is rooted in two generic mechanisms, i.e. the growth and preferential attachment\cite{29}. We hope the mechanism responsible for the emergence of scale-free networks would give clues to understand the power law distribution of firm size. Actually the network approach has been already applied to economic analysis. Souma et al. have done some empirical studies on business networks. The results reveal the possibility that business networks will fall into the scale-free category\cite{30}. Garlaschelli and Loffredo have argued that the outcome of wealth dynamics depends strongly on the topological properties of the underlying transaction network\cite{31}. The topology and economic cycle synchronization on the world trade web have also been studied\cite{32}.

Actually, the dynamics of firms and market could also be precisely described by the network approach. We can consider that producers and consumers are the two kinds of vertices and they are related with each other by links. So they could be represented as a bipartite network, which includes two kinds of vertices and whose edges only connect vertices of different kinds. The edges between producers and consumers can stand for the consumers’ choices for their consumptions. To explore the size distribution of firms, we assume that every edge stands for one unit of consumption and the degree of a producer describes its sales. Then the size distribution of firms is corresponding to the degree distribution of producers. As the results of market competition, consumers can change their choices of the consumption, which refers to the switches of links between producers. The mechanism of preferential attachment is just a good description for the rich-getting-richer effect or the self-reinforcing mechanism in the market competition. So it is a natural way to study the formation of size distribution of firms by investigating the evolution of the network.

Bipartite network is an important kind of networks in real world and the collaboration networks of movie actors\cite{25,33} and scientists\cite{34} are the typical ones. The bipartite producer-consumer network we discuss here has different properties compared with the above collaboration networks. The links between collaborators and acts in collaboration networks are fixed but it can
also be rewired with the evolving of the network in the producer-consumer network model. So the study of producer-consumer network is also valuable to understand the properties of this kind of bipartite networks.

The presentation is organized as following. In section 2, the model A with the constant total consumption is discussed, in which producers compete in a constant market. The results reveal that there is no power law distribution of firm sizes in upper tail. In section 3 we investigate the more realistic case with growing markets. In model B, the number of producers and the market, which is described by total consumptions, both grow with the time. Led by the mechanism of preferential attachment, the size distribution of firms obeys the power law, and the exponent is affected by the growth and the initial attractiveness of the firms. In section 4, we summarize our results and give concluding remarks.

2 Model A: network evolving with constant total consumption

In the industrial structure, the scale effect of the firms determined by their technological levels is one of the factors that influence the firm size. Another one is the self-reinforcing effect in market competition. Assume that there are \( N \) producers and \( K \) consumers in the market. They form a bipartite network and the edges connect the consumers to producers. For simplicity and without losing any generality, we assume one consumer has only one edge which represents one unit of consumption. The degree of producer describes its size by means of market share, so the size distribution of firms is characterized by the degree distribution of producers. In model A, we consider the situation with constant total consumption \( K \). The total number of edges will not been changed, but as the results of competition, the consumer could switch between producers, which means some edges will be rewired. Then we concentrate on the final steady degree distribution of the model. Let \( N_k(t) \) denotes the number of vertices with degree \( k \) at time step \( t \). From any given initial distribution, the model evolves as following two steps:

1. Cutting one edge randomly at each time step. Let \( n_k(t) \) indicate the number of vertices with degree \( k \) after cutting one edge at randomly. It is determined by

\[
n_k(t) = \frac{k + 1}{K}N_{k+1}(t-1) + (1 - \frac{k}{K})N_k(t-1)
\]  

2. Connecting the edge to the producer with preferential attachment mechanism. The probability of connecting the edge to one producer with degree \( k \) is:

\[
\frac{k+a}{K-1+a},
\]

where \( a \) is a parameter called the initial attractiveness of the node.
That is related to the intrinsic competence of the firm in our discussion. It is including the technology, the distinctions of the product, and other initial features of the firm. Due to the diversity of the demand, we assume any firm has the same initial attractiveness without losing any generality. Hence, the number of vertices with degree $k$ after rewired is:

$$N_k(t) = \frac{k - 1 + a}{K - 1 + Na}n_{k-1}(t) + (1 - \frac{k + a}{K - 1 + Na})n_k(t)$$  \hspace{1cm} (2)

The eqs.(1) and (2) give the dynamics of network evolution. The boundary conditions are

$$n_0(t) = \frac{N_1(t - 1)}{K} + N_0(t - 1)$$
$$N_0(t) = (1 - \frac{a}{K - 1 + Na})n_0(t)$$
$$n_K(t) = 0$$
$$N_K(t) = \frac{K - 1 + a}{K - 1 + Na}n_{K-1}(t)$$

The eqs.(1) and (2) with the above boundary conditions give:

$$\sum_{k=K}^{K} n_k(t) = \sum_{k=K}^{K} N_k(t) = \sum_{k=K}^{K} N_k(t - 1) = N$$
$$\sum_{k=K}^{K} kn_k(t) = \sum_{k=K}^{K} kN_k(t - 1) - 1 = K - 1$$
$$\sum_{k=K}^{K} kN_k(t) = \sum_{k=K}^{K} kn_k(t) + 1 = K$$

These results approve that we have indeed cut one edge in eq.(1) and rewired it in eq.(2), while the total number of producers and consumers are all constant.

Now we can obtain the stable distribution of firm sizes by numerical solutions of the rate equations (1) and (2) with boundary conditions and we have obtained the numerical solutions for the system with total $N = 500$ producers and $K = 5000$ consumers. From any given initial distributions, the system’s stable distribution is discovered to be related with the parameters. The simulation results are shown in Figure 1. When $a = 0$, all the producers have no initial attractiveness, i.e. their advantages in competition are all from the self-reinforcing mechanism. The results indicate that many producers fail in the competition. Almost all consumers connect with few producers. In the case of $a = 1$ and for the upper tail in the case of $a < 1$, the firm size obeys the
Fig. 1. The stationary probability distribution of firm size without market growth. (a) The results for \( a = 0 \). Many producers fail in the competition. Almost all consumers connect with few producers. (b) The results for \( a \neq 0 \). In the case of \( a = 1 \) and for the upper tail in the case of \( a < 1 \), the firm size obeys the exponential distribution. When \( a > 1 \), the firm size obeys Gamma distribution. The upper inset indicates that the lower tail of firm size distribution obeys the power law in the case of \( a < 1 \). (c) The firm size distribution in the case of \( a = 1 \), \( a > 1 \) in the linear coordinates.

These results are similar with money distribution in ref [9,10] gained by transferring model. The above results indicate that we can not get power law distribution by preferential attachment in a constant market. We will discuss the case of the growing market in the next section.
3 Model B: network evolving with growing market

Model A describes the firm size distribution in a constant market whose growth rate is zero. Due to the technical progress and the enlargement of inputs, averagely, the total demand and supply always grow with the time. So we set up another model to depicts them as following. At every time step, one new producer and \( l \) new consumers enter the system. The new consumers connect existing producers with preferential probability. Meanwhile, one old consumer in the system could still switch between different producers. In contrast to model A, at each evolution step, the number of producers will increase by one and the number of consumers by \( l \).

Supposing that we have \( \bar{K} \) consumers randomly distributed in \( \bar{N} \) producers in the initial. There are \( l \) consumers and one producer enter the system at every time step from \( t = 0 \). So at time \( t \), we have \( K = \bar{K} + lt \) edges and \( N = \bar{N} + t \) producers. Let \( N_k(t) \) denotes the number of vertices with degree \( k \) at time step \( t \). The network evolves as following:

1, Cutting one edge randomly at each time step. Then the number of vertices with degree \( k \) \((n_k(t))\) is given by

\[
n_k(t) = \frac{k + 1}{K} N_{k+1}(t-1) + (1 - \frac{k}{K})N_k(t-1) \tag{3}
\]

2, Connecting the one old edge and \( l \) new links to producer with preferential attachment mechanism. We get

\[
N_k(t) = n_k(t) + \frac{(l + 1)(k - 1 + a)n_{k-1}(t)}{K - 1 + Na} - \frac{(l + 1)(k + a)n_k(t)}{K - 1 + Na} \tag{4}
\]

3, Adding one new producer to the market. So the boundary conditions are

\[
n_0(t) = N_0(t-1) + \frac{1}{K}N_1(t-1)
\]

\[
N_0(t) = n_0(t) - \frac{(l + 1)an_0(t)}{K - 1 + Na} + 1
\]

Combine eq.(3), eq.(4) and the boundary conditions, we have:

\[
\sum_{k=0} n_k(t) = \sum_{k=0} N_k(t-1)
\]

\[
\sum_{k=0} N_k(t) = \sum_{k=0} n_k(t) + 1 = \sum_{k=0} N_k(t-1) + 1
\]
Fig. 2. The stationary probability distribution of firm size with market growth. The distribution obeys power law. (a) The effects of initial attractiveness on the firm size distribution with constant $l = 30$. The exponents range from 1.85 to 2.47. (b) The effects of $l$ on the firm size distribution. The initial attractiveness $a = 10$. With the increase of $l$, the exponent changes from 2.80 to 2.01.

$$\sum_{k=0}^{\infty} kn_k(t) = \sum_{k=0}^{\infty} kN_k(t - 1) - 1$$

$$\sum_{k=0}^{\infty} kN_k(t) = \sum_{k=0}^{\infty} kn_k(t) + l + 1$$

These equations give the time evolution of the system of $l$ consumers with one producer entering the system at every time step.

We have investigated the properties of this model by numerical solutions. With $\tilde{K} = 100$ consumers randomly distributed among the $\tilde{N} = 100$ producers in the initial, the final distributions are got by the numerical solutions. If $a = 0$, the results are qualitatively the same as the case of constant market. That is many producers fail in the competition. Almost all consumers connect with few producers. The frequency distributions for $a \neq 0$ are shown in Figure 2, which indicate that the size distribution of firms obeys the power law and the exponent is related with the market growth $l$ and the initial attractiveness $a$. Larger $l$ leads to less steep slope and bigger initial attractiveness leads to steeper slope. The exponents range from 1.85 to 2.80.

From our numerical solutions of the model, we have found that the exponential tails in the numerical results are due to the limited runs of the model. If we simulate the model for longer time steps, the exponential tail will be moved to the upper end (as shown in Figure 3). So we believe that when time goes to infinity, the model with preferential choice of consumption will result in the power law distribution in the upper tail. The results of the model are
Fig. 3. The numerical solutions for different time steps. The exponential tail moves to upper end with the increase of time steps. $a = 10$, $l = 10$.

Fig. 4. The comparison between the numerical solution of the system and the simulation result for the case $a = 1$, and $l = 10$. They consist well. consistent with that of the empirical studies especially when $a$ is small.

We have done a series of computer simulations for the model $B$ with the same initial conditions. The simulation results are consistent well with those of numerical solutions of the system. We show one case in figure 3, in which $a = 10$, $l = 10$ and the simulation steps are 1000000.
4 Summary

We proposed a bipartite producer-consumer network to investigate the firm size distribution in this paper. The market dynamics is described by the evolution of the network and the firm size is characterized by its market share which is represented by the degree of producer. The instinct of competition of the firms and the self-reinforcing mechanism are introduced into the probability of connection and the consumers switch their links between the producers as for competition. The results indicate that the economic growth is an important condition for the power law distribution of firm size distribution. The growth rate and initial attractiveness of the firms will affect the exponents of the firm size distribution. Without economic growth, our results indicate that the initial attractiveness of firms $a$ is an important parameter to determine the final distribution. The final distribution is Gamma distribution when $a > 1$, is exponential when $a = 1$. When $a < 1$, the upper tail is exponential and the lower end is power. If $a = 0$, which means only the self-reinforcing mechanism works in the market competition, there would be only fewer producers surviving in the market. All these results provide understandings to the mechanism of power law distribution of firm sizes and they maybe valuable for investigating the properties of bipartite networks.

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