New bounds for numbers of primes in element orders of finite groups

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Abstract
Let \( \rho(n) \) denote the maximal number of different primes that may occur in the order of a finite solvable group \( G \), all elements of which have orders divisible by at most \( n \) distinct primes. We show that \( \rho(n) \leq 5n \) for all \( n \geq 1 \). As an application, we improve on a recent bound by Hung and Yang for arbitrary finite groups.

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element orders, finite groups, primes, solvable groups

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1 INTRODUCTION AND STATEMENT OF RESULTS

Let \( G \) be a finite solvable group, and let \( \omega(n) \) denote the number of distinct prime divisors of \( n \). We define \( \omega(G) \) as the maximum of \( \omega(\text{ord}(g)) \), over all \( g \in G \). Then, finally, we define \( \rho(n) \) as the maximum of \( \omega(|G|) \), taken over all finite solvable groups \( G \) with \( \omega(G) = n \). Work on bounding \( \rho(n) \) has been undertaken by the second author in the series of papers [8, 10, 11]; see also [15] for some related problems. He [9] showed that there is a function \( C(n) \) such that

\[
\rho(n) \leq C(n)n, \tag{1}
\]

where \( C(n) \) is bounded, and \( \lim_{n \to \infty} C(n) = 4 \). Of interest is determining the smallest constant \( K \) such that

\[
\rho(n) \leq Kn, \quad (n \geq 1).
\]

Yang [19] showed that \( K \leq 17/3 \). A weaker (but simpler) proof that \( K \leq 7 \) was given by Hung and Yang [5]. It is conjectured that \( K = 3 \).

If one gets a good value of \( K \), then, as outlined in Moretó [15], one can deploy this directly to some related problems. For example, Yang and Qian [21] examined the analogous problem for character codegrees and obtained a bound of the shape \( K + 3 \). This was improved recently by Yang [19] to \( K + 2 \). At almost the same time, Moretó [15, Corollary 1.4] reduced this to \( K + 3/2 \). Moretó notes (see the commentary before Theorem 1.3 in [15]) that this result is sharp. This shows that a lot of interest has been expended on bounds involving \( K \). We use a mix of theory and computation to prove the following.

**Theorem 1.** Let \( G \) be a finite solvable group. For all \( n \geq 1 \), we have \( \rho(n) \leq 5n \).
The same methods leading to the proof of Theorem 1 allow us to show, for example, that \( \rho(n) \leq 4.1n \) for all \( n \geq 1000 \). However, as will be made clear in the proof of Theorem 1, it is difficult to remove this restriction on \( n \).

As a result of Theorem 1, we can obtain the following improvement on the size of \( \rho_g(n) \) for arbitrary finite groups. We define \( \rho_g(n) \) in the same way as \( \rho(n) \) except that for the former we consider arbitrary finite groups \( G \), not necessarily solvable.

**Corollary 1.** Let \( G \) be a finite group. For all \( n \geq 1 \), we have \( \rho_g(n) \leq 141n^4 \).

This improves on work first examined by Zhang [22], and improved by the second author (see Remark 16.19(b) in [6, p. 213]) and then Moretó [14] to a bound of the form \( \rho_g(n) \leq An^4 \log n \) for some unspecified \( A \). Hung and Yang [5] proved a version of Corollary 1 with 210 in place of 141.

We remark that recent work [20] (Lemma 2.1) could be improved slightly with our Theorem 1 and our (12) in Section 3.

The outline of this paper is as follows. In Section 2 we collect some relevant background material. This allows us to prove Theorem 1 in Section 3. We give the proof of Corollary 1 in Section 4, and outline some potential future work stemming from Theorem 1.

## 2 | BACKGROUND

We begin by recalling the observation by the second author [9, p. 651] that

\[
\rho(n) \leq \pi(\rho(n)) + 4n, \tag{2}
\]

where \( \pi(x) \) counts the number of primes not exceeding \( x \). Since \( \pi(x) \sim x / \log x \), then, given any \( \epsilon \) we can find an \( n_0 \) for which \( n \geq n_0 \) guarantees that \( \pi(n) \leq (1 + \epsilon)n / \log n \). Hence, rearranging (2) gives

\[
\rho(n) \leq 4n \left(1 - \frac{1 + \epsilon}{\log n_0}\right)^{-1}. \tag{3}
\]

Others have, hitherto, used the simple (and elegant) bound due to Rosser and Schoenfeld [17, Corollary 1], namely, that

\[
\pi(n) \leq 1.25506n / \log n, \quad (n \geq 2). \tag{4}
\]

When \( n \) is small, say, \( n < 10^8 \), we could alternatively compute \( \pi(n) \) exactly on a computer.

For larger values of \( n \), we could use, for example, the result by Buethe [4] that \( \pi(x) \leq li(x) \) for all \( 2 \leq x \leq 10^{19} \), where \( li(x) = \int_2^x dt / \log t \). We can then bound \( li(x) \) by integrating-by-parts and truncating the error term, as in Bennett et al. [1] or in Saouter et al. [18]. This can be augmented with by another result by Buethe [3], which has been extended recently by Johnston [7], using the partial verification of the Riemann hypothesis by Platt and the third author [16]. Namely, we have \( \pi(x) - li(x) \leq 1/(8\pi) \sqrt{x \log^2 x} \) for \( x \leq 10^{25} \). Finally, for still larger values of \( n \), we could use any of the (many) results in Broadbent et al. [2].

The difficulty with the above is that we run into trouble for small \( n \). If we want a bound that holds for all \( n \), we are limited by the poor bounds on \( \rho(n) \) for small \( n \), mainly for \( n \leq 100 \).

Indeed, Yang’s bound of \( K \leq 17/3 \) comes about from the “worst” bound on \( \rho(n) \) occurring at \( n = 9 \). We make an improvement below to show that worst cases are now due to \( n = 8 \) and \( n = 9 \), which give us \( K \leq 5 \).

Having stressed above the need to examine values of \( \rho(n) \) for small \( n \), we proceed to list what is known. We have, from [8, 11] that

\[
\rho(1) = 2, \quad \rho(2) = 5, \quad \rho(3) = 8, \quad \rho(4) = 12. \tag{5}
\]

We also have the following collection of bounds (see [8, section 1.4] and [22] and [11])

\[
\rho(n + 1) \leq \rho(n) + n + 2, \quad \rho(n) \leq \frac{n(n + 3)}{2} - 2, \quad \rho(n) \geq 2n, \tag{6}
\]

where the second bound is true for \( n \geq 4 \), and the others for all \( n \geq 1 \).
The exact values of \( \rho(n) \) in (5) and the second formula in (6) give the following pairs, in which \( C(n) \) is the bound in (1):
\[
(n, C(n)) \in \{(5, 18), (6, 25), (7, 33), (8, 42), (9, 52), (10, 63)\}.
\] (7)

Yang makes a small improvement when \( n = 9 \) by using (2). If \( \rho(9) = 52 \), then, since \( \pi(52) = 15 \), we have that the right-hand side of (2) is 51, whence \( \rho(9) \leq 51 \), a contradiction. Hence \( \rho(9) \leq 51 \); this method is unable to reduce \( \rho(9) \) further, though small improvements are possible for \( n \geq 10 \). This (and some additional computation to examine some larger values of \( n \)) leads to Yang’s version of our Theorem 1 with constant \( 51/9 = 17/3 \) in place of our 5.

3 A NEW RECURSIVE ARGUMENT

For a given \( n \geq 5 \) and positive integer \( k \), suppose that \( \rho(n) = k \). Let \( G \) be a finite solvable group such that \( \omega(G) = n \) and \( \omega(|G|) = k \). We will show that there is a contradiction if \( k \) is too large. This will allow us to give an upper bound on \( k \). By [8, Lemma 1.3], we may assume that \( G \) is \( \sigma \)-reduced, that is, every nontrivial Sylow subgroup of \( G \) is isomorphic to a chief factor of \( G \). Also write \( n_i = \omega(|G^{(i)}/G^{(i+1)}|) \) for \( i \geq 0 \), where \( G^{(i)} \) is the \( i \)th term of the derived series of \( G \) for \( i \geq 0 \). As is common, we write \( G, G', G'', \) and \( G''' \) for \( G^{(i)}, i = 0, 1, 2, 3 \), respectively. Clearly \( n_1 \leq n \) for all \( i \). So we can write \( n_0 + n_1 + n_2 = 3n - \lambda \) for some integer \( \lambda \geq 0 \). (Clearly \( \lambda \) depends on \( G \).) We therefore have that \( \omega(|G'''|) = k - 3n + \lambda \). Now let \( \tau \) be the set of primes less than or equal to \( \omega(|G'''|) \), and let \( H \) be a Hall \( \tau \)-subgroup of \( G \) (so that all prime divisors of \( |H| \) are greater than \( \omega(|G'''|) \)). Again by [8, Lemma 1.3], let \( H_0 \) be a \( \sigma \)-reduced subgroup of \( H \) such that \( \omega(|H_0|) = \omega(|H|) \). (Note that all \( \tau, H, \) and \( H_0 \) depend on \( G \)). Then, by [9, Theorem 7(a)], we have that \( \omega(|H|) = \omega(|H_0|) \leq 4 \omega(H_0) \leq 4 \omega(H) \leq 4n \). Thus,
\[
k = \rho(n) = \omega(|G|) \leq \omega(|H|) + |\tau| \leq 4n + \pi(k - 3n + \lambda).
\] (8)

If the right-most inequality in (8) is less than \( k \), we have a contradiction. This means we need those values of \( \lambda \) such that
\[
\pi(k - 3n + \lambda) \leq k - (4n + 1).
\] (9)

It is clear that if (9) is satisfied at all, then it will only be satisfied if \( \lambda \leq L \) for some \( L \) (and \( L \) only depends on \( k \) and \( n \)). Hence, for all/any such \( \lambda \), we derive a contradiction to (8), showing that \( G \) cannot exist and thus eliminating \( k \) as a possible value for \( \rho(n) \).

For larger values of \( \lambda \), we use a variation of (2). By following the proof of the second author [9], we have \( \rho(n) \leq \pi(\rho(n)) + 4n - \lambda \). Since in these cases we have \( \lambda \geq L + 1 \), we obtain a contradiction (and hence can eliminate this particular value of \( k \)) if
\[
4n - L \leq k - \pi(k).
\] (10)

Note that it is easier to solve (10) if \( L \) is large, that is, if there are many \( \lambda \) for which (9) is true.

Let us illustrate this method when \( n = 9 \): We already know that \( k \leq 51 \). Suppose \( k = 51 \). Then, (9) is true for all \( 0 \leq \lambda \leq 22 \), and so, for such \( \lambda \), we can conclude that \( \rho(9) \leq 50 \). For larger \( \lambda \), we turn to (10): For \( L = 22 \), the left-hand side is 14, whereas the right-hand side is 36, hence we still can conclude that \( \rho(9) \leq 50 \).

We continue in this way and can eliminate \( \rho(9) \in [46, 50] \) but cannot rule out \( \rho(9) = 45 \). There, (9) is only satisfied for \( 0 \leq \lambda \leq 4 \), whence \( L = 4 \), but then (10) is not satisfied. Hence, the best we can do is to conclude that \( \rho(9) \leq 45 \).

This approach yields the following improved upper bounds on \( \rho(n) \):
\[
(n, C(n)) = \{(8, 40), (9, 45), (10, 49), (11, 53), (12, 57)\}.
\] (11)

We find that the worst bounds for \( \rho(n)/n \) arise when \( n = 8 \) and \( n = 9 \). The values on (5) and (11), and a verification on a standard desktop computer of the above method establishes that \( \rho(n) \leq 5n \) for all \( n \leq 265 \).

When \( n \geq 266 \), we use the bound on \( \pi(n) \) from (4) in (2), obtaining
\[
\rho(n) \leq 1.25506 \cdot \frac{\rho(n)}{\log(\rho(n))} + 4n.
\]
Now, using the third relation in (6), namely, that $\rho(n) \geq 2n$, we have

$$\rho(n) \left(1 - 1.25506 \cdot \frac{1}{\log(2n)}\right) \leq \rho(n) \left(1 - 1.25506 \cdot \frac{1}{\log(\rho(n))}\right) \leq 4n.$$ 

For $n = 266$, we have

$$4 \cdot \left(1 - 1.25506 \cdot \frac{1}{\log(2 \cdot 266)}\right)^{-1} \approx 4.9997 ... < 5.$$ 

As a result, we can conclude that $\rho(n) \leq 5n$ for $n \geq 266$.

We remark in passing that this part of the proof could be substantially improved by using better results for $\pi(n)$. Indeed, Rosser and Schoenfeld [17] give other bounds that can be rewritten into the form of (3) as required. However, as noted earlier, the difficult cases come from small values of $n$: here for $n = 8, 9$.

We note that the improvement for $\rho(9)$ allows one, via a quick inductive argument, to improve the second bound in (6) to

$$\rho(n) \leq \frac{n(n + 3)}{2} - 9, \quad (n \geq 9).$$

(12)

We also remark that we have used the bound $\rho(n) \geq 2n$, which holds for all $n$. We could obtain sharper results, for larger $n$, if we knew that even $\rho(n) \geq 3n$ for $n \geq 4$. For example, by using this bound, the 4.9997 ... above could be improved to 4.925041. Now note that by [8, Remark 1.11], we already know that $\rho(n) \geq 3n$ whenever $n$ is divisible by 4. One gets this by taking direct products of suitable groups as constructed in [8, Example 1.9]. To prove that $\rho(n) \geq 3n$ for all $n \geq 4$, all one needs to do is to construct three groups, using the same method as in [8, Example 1.9], to prove that, in the notation of [8], $\rho_3(5) = 15$, $\rho_3(6) = 18$, and $\rho_3(7) = 21$, respectively. With these groups, and taking direct products with groups as in [8, Example 1.9], all values of $n \geq 4$ can be reached, and so $\rho(n) \geq 3n$ for $n \geq 4$. Since the expected improvements to our bounds in this paper seem to be minor, we leave the construction of the three above-mentioned groups to the interested reader.

4 | AN APPLICATION TO A RELATED PROBLEM AND A PROOF OF COROLLARY 1

There is also a related problem for arbitrary finite $G$, that is, not just for the finite solvable groups we have considered hitherto. Whereas for solvable $G$, we have a linear bound of $\rho(n)$ in $n$, for arbitrary $G$, the sharpest result to date is by Hung and Yang [5], who show that $\rho_G(n) \leq 210n^4$.

We note that, toward the bottom of p. 1910 in [5], the proof gives the following bound:

$$\rho_G(n) \leq n^4 \left(28C(n) + \frac{C(n)^2}{n^2} + \frac{C(n)}{n^3}\right).$$

(13)

Hung and Yang use a weaker version of our Theorem 1 with $\rho(n) \leq 7n$ for all $n$. They also prove that $\rho_G(1) \leq 4$, and so, with $n \geq 2$, they can take the term in the parentheses of (13) to be 209.125 < 210.

We obtain values for $C(n)$ from (5), (7), and (11), and use Theorem 1. This shows that the term in the parentheses in (13) is at most 140.41, which proves Corollary 1.

We have not made any other attempt at improving the result. A close inspection of the proof in [5] shows that, at least when $n$ is large, one should be able to replace the 28 in (13) by 8. This, and the fact that $\lim_{n \to \infty} C(n) = 4$, means that one has $\rho_G(n) \leq (32 + \varepsilon)n^4$ for $n$ sufficiently large, and for any positive $\varepsilon$.

We conclude by noting that the method used in Section 3 could also be deployed in the paper by the second author [12]. Finally, since specific bounds on $\pi(x)$ are used by Moretó in the proofs of Theorems 1.1 and 1.2 in [13], it is possible that our methods may be of some use there as well.
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