Matter Power Spectrum in $f(R)$ Gravity with Massive Neutrinos

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Abstract

The effect of massive neutrinos on matter power spectrum is discussed in the context of $f(R)$ gravity. It is shown that the anomalous growth of density fluctuations on small scales due to the scalaron force can be compensated by free streaming of neutrinos. As a result, models which predict observable deviation of the equation-of-state parameter $w_{DE}$ from $w_{DE} = -1$ can be reconciled with observations of matter clustering if the total neutrino mass is $O(0.5 \text{ eV})$. 
\( f(R) \) gravity is a simple and nontrivial extension of General Relativity (GR) which has recently received much attention as an alternative to the \( \Lambda \)-Cold-Dark-Matter (\( \Lambda \)CDM) model that can explain the present cosmic acceleration. The basic idea is to modify the Einstein-Hilbert action by using the general function of Ricci scalar \( R \) as

\[
S = \int d^4x \sqrt{-g} f(R) + S_m, \tag{1}
\]

where \( S_m \) indicates a matter action (see Ref. [1] for a recent review and an extensive list of references). If \( f''(R) \) is not zero identically, this modified gravity contains an additional degree of freedom corresponding to a new scalar field, dubbed scalaron in Ref. [2] where the simplest, \( f(R) = R + R^2/6M^2 \), variant of such theory (with small one-loop quantum gravitational corrections) was used to construct an internally self-consistent inflationary model of the early Universe with a graceful exit from inflation to the subsequent radiation-dominated Friedmann-Robertson-Walker (FRW) stage after the gravitational creation of matter and reheating. However, \( f(R) \) gravity can be used to describe dark energy in the present Universe as well, although with the use of more complicated functions \( f(R) \) having a nontrivial structure for low values of \( R \).

To have the correct Newtonian limit for \( R \gg R_0 \equiv R(t_0) \sim H_0^2 \), where \( t_0 \) is the present moment and \( H_0 \) is the Hubble constant, as well as the standard matter-dominated FRW stage with the scale factor behaviour \( a(t) \propto t^{2/3} \) driven by cold dark matter and baryons, the following conditions should be fulfilled:

\[
|f(R) - R| \ll R, \quad |f'(R) - 1| \ll 1, \quad Rf''(R) \ll 1, \quad R \gg R_0, \tag{2}
\]

where the prime denotes a derivative with respect to the argument \( R \). The same conditions guarantee smallness of non-GR corrections to a space-time metric for more general backgrounds for which GR has to be used in full. In addition, the stability condition \( f''(R) > 0 \) has to be satisfied to guarantee that the standard matter- and radiation-dominated FRW stages remain attractors with respect to an open set of neighboring isotropic cosmological solutions in \( f(R) \) gravity. In quantum language, this condition means that the scalaron is not a tachyon. Note that the other stability condition, \( f'(R) > 0 \), which means that gravity is attractive and the graviton is not a ghost, is automatically fulfilled in this regime. Specific functional forms that satisfy these and other necessary conditions have been proposed in Refs. [3–5], and much work has been conducted on their cosmological consequences.
We can express field equations derived from action (1) in the following Einsteinian form,

\[ R_{\mu}^\nu - \frac{1}{2} \delta_{\mu}^\nu R = -8\pi G \left( T_{\mu(\text{m})}^{\nu} + T_{\nu(\text{DE})}^{\mu} \right), \quad (3) \]

where

\[ 8\pi G T_{\nu(\text{DE})}^{\mu} \equiv (F - 1) R_{\nu}^{\mu} - \frac{1}{2} (f - R) \delta_{\nu}^{\mu} + (\nabla_{\mu} \nabla_{\nu} - \delta_{\mu}^{\nu} \Box) F, \quad (4) \]

and we have also defined \( F(R) \equiv f'(R) = df/dR \). Working in the spatially flat Friedmann-Robertson-Walker (FRW) space-time with the scale factor \( a(t) \), we find

\[ 3H^2 = 8\pi G \rho - 3(F - 1)H^2 + \frac{1}{2} (FR - f) - 3H \dot{F}, \quad (5) \]

\[ 2\dot{H} = -8\pi G \rho - 2(F - 1)\dot{H} - \ddot{F} + H \dot{F}, \quad (6) \]

where \( H \) is the Hubble parameter and \( \rho \) is the energy density of the material content, which we assume to consist of nonrelativistic matter. From these expressions, the effective DE equation-of-state parameter \( w_{\text{DE}} \equiv P_{\text{DE}}/\rho_{\text{DE}} \) is given by

\[ 1 + w_{\text{DE}} = \frac{-2\dot{H} - 8\pi G \rho}{3H^2 - 8\pi G \rho} = \frac{2(1 - F)(-\ddot{a}/a + H^2) + \dddot{F} - H \dot{F}}{3(1 - F)(-\dddot{a}/a) + (R - f)/2 - 3H \dddot{F}}. \quad (7) \]

Note that the phantom crossing is naturally realized in viable \( f(R) \) theories as shown in Refs. [3, 6–8].

We specifically adopt the following functional form that satisfies all the above requirements[5],

\[ f(R) = R + \lambda R_s \left[ \left( 1 + \frac{R^2}{R_s^2} \right)^{-n} - 1 \right], \quad (8) \]

where \( n, \lambda, \) and \( R_s \) are model parameters. It is sufficient to describe the recent evolution of the Universe, although it has to be modified both for \( R < R_0 \) (including the region \( R < 0 \)) and for very large positive \( R \) to avoid problems in the early Universe cosmology (see Refs. [6, 9]).

In \( f(R) \) gravity, the evolution equation for density fluctuation with comoving wave number \( k \) in Fourier space, \( \delta_k \), in the deeply subhorizon regime is given by [10, 11]

\[ \ddot{\delta}_k + 2H \dot{\delta}_k - 4\pi G_{\text{eff}} \rho \delta_k = 0, \quad (9) \]

where

\[ G_{\text{eff}} = \frac{G}{1 + 3\frac{k^2}{a^2} \frac{F'}{F}}. \quad (10) \]
In the high-curvature regime, the differential equation (9) can be solved analytically [9], and we analyzed it in the general regime in terms of numerical calculation in the previous paper [8]. It is revealed that the effective gravitational coefficient $G_{\text{eff}}$ enhances the growth of density fluctuations compared with that in the standard $\Lambda$CDM model due to an extra force mediated by the scalaron. To suppress the deviation of the matter power spectrum from that in the $\Lambda$CDM model within, say 10%, the parameter space for $n$ and $\lambda$ should be restricted. For example, $\lambda$ should be greater than 8 in $n = 2$ [8].

In this paper, we show that the anomalous growth of density fluctuations on small scales due to the scalaron is rectified if neutrino masses are taken into consideration. That is, the free streaming of neutrinos partially erases small-scale density fluctuations, which can be harmful in Einstein gravity, to compensate their unwanted extra growth in $f(R)$ gravity.

To set up the initial condition, we calculate the evolution of density fluctuations for $z \gtrsim 10$ using CAMB [12, 13], and obtained $P(k, z = 10)$, power spectrum at $z = 10$. We assume one mass eigenstate for neutrino, and all three neutrinos have practically the same masses. Note that the density parameter of neutrinos is

$$\Omega_\nu h^2 \simeq \begin{cases} 1.685 \times 10^{-5} & (\sum m_\nu = 0) \\ \frac{\sum m_\nu}{94.1 \text{ eV}} & (\sum m_\nu \neq 0), \end{cases} \tag{11}$$

and the free streaming of massive neutrinos suppresses fluctuations below the scale [14, 15]

$$k_{fs}(z) \simeq \frac{0.35}{(1 + z)^{1/2}} \left( \frac{m_\nu}{1 \text{ eV}} \right) \left( \frac{\Omega_m}{0.27} \right)^{1/2} h \text{ Mpc}^{-1}. \tag{12}$$

Therefore, $\delta_k$ in Eq. (9) implies density fluctuations for CDM below that scale.

At $z \gtrsim 10$, the effect of modified gravity is sufficiently small. Therefore, we start to solve the evolution of density fluctuations in $f(R)$ gravity using Eq. (9) at $z = 10$. We have found that the physical wave number crossing the scalaron mass today is given by

$$\frac{k_s}{a_0} \simeq 3.2 \lambda^{1.88} H_0, \tag{13}$$

for $n = 2$[8], which is to be compared with Eq. (12). It takes $1.07 \times 10^{-3} h$ and $8.44 \times 10^{-3} h$ Mpc$^{-1}$ for $\lambda = 1$ and $3$, respectively.

The present power spectrum is constructed as

$$P(k, z = 0) = P(k, z = 10) \left( \frac{\delta(z = 0)}{\delta(z = 10)} \right)^2. \tag{14}$$
**FIG. 1:** Deviation of power spectrum for \(n = 2\), \(\lambda = 1\), 3. Captions indicate total neutrino mass, and it is assumed that three neutrinos have the same masses.

**FIG. 2:** Left: Relative deviation of linear matter power spectrum from ΛCDM model at \(k = 0.174\ h\ Mpc^{-1}\) as a function of \(\lambda\) for various values of neutrino mass. Right: The region between two lines for each \(n\) corresponds to the parameter space where the deviation of power spectrum is smaller than 10%.

Figure 1 shows cancellation of the contributions from \(f(R)\) gravity and neutrino masses. Equations (12) and (13) explain the threshold wave number. In particular, the transition occurs at a similar wave number for \(\lambda = 3\) when we take \(n = 2\). The deviation at \(k = 0.174\ h\ Mpc^{-1}\), which is the scale corresponding to \(\sigma_8\) normalization [8], is minimum when the total neutrino masses are 0.6 and 0.5 eV for \(\lambda = 1\) and 3, respectively.

In the left panel of Fig. 2, the relative deviation of the power spectrum from that of the ΛCDM model is depicted as a function of \(\lambda\) for various total neutrino masses with \(n = 2\). The right panel of Fig. 2 shows the range of total neutrino masses where the relative deviation of the power spectrum at \(k = 0.174\ h\ Mpc^{-1}\) remains smaller than 10%.

As seen in the right panel, models with smaller values of \(\lambda\), which tend to exhibit a larger deviation from the ΛCDM model, become compatible with the observation of fluc-
FIG. 3: Possible range of time variation of $w_{\text{DE}}$ (left) and growth index $\gamma$ (right) as a function of total neutrino mass. $\gamma$ is measured at the comoving wave number $k = 0.174\, h\, \text{Mpc}^{-1}$.

In other words, models with deviations of $w_{\text{DE}}$ and the growth index $\gamma(z) = \log \left( \frac{\dot{\delta}}{H\delta} \right) / \log \Omega_m$ large enough to be observable become viable thanks to the suppression of small-scale fluctuations due to finite neutrino masses. Figure 3 depicts the maximum range of the time variation of $w_{\text{DE}}$ and $\gamma$ that can be realized for each value of the total neutrino mass. That is, if we choose a minimum possible value of $\lambda$ for each total neutrino mass, $w_{\text{DE}}$ varies between the upper and lower edges of the spatula shape at the corresponding neutrino mass. Similarly, $\gamma$ evolves between the upper and lower edges of the ax-shape. Note that $\gamma$ takes an almost constant value, 0.55, in the $\Lambda$CDM model, which corresponds to the upper bound in $f(R)$ gravity.

Finally, we comment on how nonlinear effects may modify our results. So far, nonlinear effects on matter clustering in $f(R)$ gravity have been analyzed in higher order perturbation theory [16] and N-body simulations [17]. The latter has shown that nonlinear effects lower the relative deviations of the power spectrum, $(P_{\text{RG}} - P_{\Lambda\text{CDM}})/P_{\Lambda\text{CDM}}$, by about 5% in a specific case $f(R) = R - 2\Lambda - f_{R0}R_0^2/R$ with $|f_{R0}| = 10^{-4}$, where $R_0$ is the current value of the scalar curvature. It has also been shown that the nonlinear effects tend to suppress the anomalous growth of small-scale fluctuations observed in linear perturbation in $f(R)$ theory. We may therefore conclude that nonlinear effects will not change our results qualitatively.

In conclusion, neutrino rest-masses with $\sum \nu m_\nu = O(0.5\, \text{eV})$ relax the most critical constraint on $f(R)$ gravity, which follows from the anomalous $k$-dependent growth of density perturbations in the cold dark matter + baryon component at recent redshifts, making possible for $w_{\text{DE}}$ and $\gamma(k)$ deviate from those of the $\Lambda$CDM model noticeably. One can
distinguish the effects of $f(R)$ and massive neutrinos by analyzing the ratio of correlation between galaxies and curvature perturbation probed by weak lensing to that between galaxies and velocity field, namely, an estimator $E_G = \Omega_m/(F\beta)$, where $\beta$ is the growth rate[18]. According to the current observational analysis, both Einstein gravity and $f(R)$ gravity are consistent with the data. However, in the future, one can break the degeneracy in principle.

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