$\Delta I = \frac{1}{2}$ Enhancement and the Glashow-Schnitzer-Weinberg Sum Rule

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Abstract

In 1967 Glashow, Schnitzer and Weinberg derived a sum rule in the soft pion and soft kaon limit relating the $\Delta I = \frac{1}{2}$ non-leptonic $K \to 2\pi$ amplitude to integrals over strange and non-strange spectral functions. Using the recent ALEPH data from $\tau$-decay, we show that the sum rule, slightly modified to reduce contributions near the cut, yields the correct magnitude decay amplitude corresponding to the $\Delta I = \frac{1}{2}$ rule.
The $\Delta I = \frac{1}{2}$ rule for kaons has been a challenge to theoreticians for more than four decades, see [1] for a review. The current-current weak non-leptonic Hamiltonian of the Standard Model leads naively to the expectation of roughly equal $\Delta I = 1/2$ and $\Delta I = 3/2$ amplitudes. Experimentally, however, the $\Delta I = 1/2$ amplitudes are enhanced by about a factor 20. QCD corrections may be computed with the help of the operator product expansion yielding a $\Delta I = 1/2$ enhancement, but its magnitude turns out too small. The situation may be improved by the use of chiral perturbation theory combined with model input, see [2] for a recent calculation.

In this note we will return to the roots and reanalyse an old current algebra calculation of the $K^0_s \to 2\pi$ matrix element by Glashow, Schnitzer and Weinberg (GSW) [3]. In this classic paper the $K^0_s \to 2\pi$ amplitude is related to integrals over spectral functions which were in turn evaluated by using the crude approximation of saturation by narrow resonances. As the integrals involve differences of large numbers, it is not astonishing that these authors do not find the right answer. Using the $K^0_s \to 2\pi$ lifetime as an input they end up with a prediction of 8 GeV for the W mass. Since precise data on the relevant spectral functions have recently been obtained by the ALEPH collaboration [5] it seemed prudent to us to reanalyse the GSW formula. It should be pointed out, however, that the ALEPH data do not saturate chiral sum rules even at $s$ as large as 3GeV$^2$ if substituted directly. In fact, it was shown [4] that if modified chiral sum rules, based on linear combinations of spectral function that vanish at the end of the integration region, are used the saturation is quite spectacular.

We will repeat briefly the basic steps of the GSW calculation. In the language of the standard model, the non-leptonic $\Delta S = 1$ weak Hamiltonian is given by

$$H_W(0) = f^2 \int dx D_{\mu\nu}(x, M^2_W) T\{j^{ud}_\mu(x) j^{su}_\nu(0) + h.c\} \quad (1)$$

with $f^2 \equiv \frac{G_F M_W^2 V_{ud} V_{us}^*}{\sqrt{2}}$ where $M_W$ is the W boson mass, $D_{\mu\nu}(x, M^2_W)$ its propagator, $G_F$ the Fermi coupling constant ($1.166 \times 10^{-5} GeV^{-2}$), $V_{ud}, V_{us}^*$ are matrix elements of the CKM matrix and

$$j^{ud}_\mu = \bar{d} \gamma_\mu (1 - \gamma_5) d, \quad j^{su}_\mu = \bar{u} \gamma_\mu (1 - \gamma_5) u. \quad (2)$$
are the strangeness conserving and strangeness changing weak currents, respectively. We consider the decay $K^0_s \rightarrow \pi^+\pi^-$ which is described by the matrix element

$$\mathcal{M} = \langle \pi^+\pi^- | H_W | K^0_s \rangle .$$

(3)

Using the conventional "soft pion" technique and the $SU(2) \times SU(2)$ commutation relations yields

$$\mathcal{M} = \frac{1}{4} f_{\pi}^2 \langle 0 | H_W | K^0_s \rangle$$

(4)

It should be remembered that the soft pion approach (or $SU(2) \times SU(2)$ chiral perturbation theory) gives all non-leptonic K-decay rates in terms of $\mathcal{M}$, and in accord with the $\Delta I = \frac{1}{2}$ rule.

The next step in the derivation is slightly controversial, namely, GSW evaluate the matrix element $\langle 0 | H_W | K^0_s \rangle$ in the soft kaon limit using the $SU(3) \times SU(3)$ algebra of currents. This step is not equivalent to evaluating the original matrix element in the $SU(3) \times SU(3)$ chiral limit where the pions and the kaon would have to be treated on the same footing from the beginning and where the amplitude would vanish. Substituting the spectral representation for the vector and axial vector correlators and exchanging the order of integration GSW obtain finally

$$\mathcal{M} = \frac{3 f_{\pi}^2}{64 \pi^2 f_{\pi}^2 f_K} \int ds \frac{s}{2\pi^2} \left[ (v(s) + a(s))^{ud} - (v(s) + a(s))^{us} \right] \left\{ \ln\left(\frac{s}{M_W^2}\right) \right\} \left\{ 1 - \frac{s}{M_W^2} \right\}$$

(5)

where $v(s)$ and $a(s)$ are the vector and axial vector spectral functions normalized according to $v(s) = a(s) = \frac{1}{2}(1 + \alpha_s s + \ldots)$ in perturbative QCD. In the derivation of Eq.5 the first and second Weinberg sum rules, which are valid in QCD, have been used.

To evaluate the integral in Eq.5, which extends from threshold to $\infty$, we split the integration range into two two parts. From threshold to $3 GeV^2$ we substitute ALEPH data, and from $3 GeV^2$ to $\infty$ we use QCD [6]

$$[(v(s) + a(s))^{ud} - (v(s) + a(s))^{us}]_{QCD} = \frac{4352}{727} \alpha_s \langle \bar{u}u \rangle \langle \bar{s}s \rangle \frac{1}{s^3}$$

(6)
where factorization has been used for the four-quark condensate and $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle$ was put. As the QCD part can only serve as an order of magnitude estimate, we should make sure that the integral is saturated by the low energy contribution accessible to experiment. Unfortunately, we know from a detailed study [4] of Weinberg’s and other chiral sum rules that this is not the case. If, however, modified sum rules build up of suitable linear combinations of spectral function sum rules are used which involve integrands that vanish at the end of the given finite integration range, precocious saturation is observed to a surprising extent. This result is understandable from the point of view of global duality in QCD. Analyticity properties of the two-point function relate the integral over a spectral function, multiplied by polynomials, to an integral over a circle in the complex plane. Perturbative QCD is expected to break down close to the time-like real axis, so it is desirable to choose the polynomial so as to vanish at the end of the region of integration over the cut.

In the present application we modify the integral Eq.5 by adding a term that vanishes by Weinberg’s sum rule,

$$\mathcal{M} = \frac{3f^2}{64\pi^2f_{\pi}^2f_K}\left[\int_{R}^{\infty} ds \frac{s^2}{2\pi^2} \left[ (v(s) + a(s))^u - (v(s) + a(s))^u \right] \ln(s/M_W^2) - s/M_W^2 \ln(R/M_W^2) \right]$$

(7)

If the integral is saturated precociously the integral from $R$ to $\infty$ should be negligible. We will assume this to be the case for the time being. In Fig. 1 we plot the result of the first integral over the experimental spectral functions as a function of the upper limit of integration $R$. As the individual contributions of the non-strange and strange spectral functions are large, we are faced with the difference of two large numbers, and it is the more amazing that the integral appears saturated for $R \geq 1.8 GeV^2$. The small oscillations are typical and will level out for larger $R$. If the high energy QCD contribution is neglected our prediction is therefore in agreement with the experimental amplitude, $|\mathcal{M}_{exp}| = 7.78 \times 10^{-7} m_K$, to within the errors expected from the soft meson approximation.

To show that the integral from $R$ to $\infty$ is indeed small we use the estimates

$$\alpha_s \langle \bar{u}u \rangle^2 = 1.9 \times 10^{-4} GeV^6, \quad \langle \bar{s}s \rangle = 0.5 \times \langle \bar{u}u \rangle$$

(8)

to find that the high energy QCD contribution is less than 1% for the
lower limit of integration $R$ between 2 to 3 GeV$^2$, in a direction bringing the prediction closer to the experimental value.

In Fig. 2 we plot the same integral using the original sum rule Eq. $7$. It is seen that the sum rule converges poorly, but the result is still consistent with the experimental value of the decay amplitude. This is in agreement with the analysis of the non-strange Weinberg sum rules where also only suitable linear combinations of sum rules that vanish at the radius $R$ converge precociously.

In Fig.1 and Fig.2 all experimental errors on the spectral functions are suppressed. This is because the statistical part of the errors is washed out completely upon integration and, hopefully, a great part of the (unknown) systematic errors cancel in the differences of spectral functions. We think that this attitude is justified a posteriori by the results plotted in the figures which show the expected shape. We also made an estimate of the small error made by neglecting the contribution of the charm quark.

Using the oscillation of the integral with $s \leq R$ in Eq. $7$ as an estimate of the total error, we arrive at a prediction for the $K^0 \rightarrow \pi^+\pi^-$ matrix element

$$|\mathcal{M}| = (9.3 \pm .9) \times 10^{-7} \quad (9)$$
Figure 2: The decay matrix element (in units of $10^{-7}m_K$) as calculated from the GSW sum rule (Equ. 5) as a function of the upper limit of integration $R$. The solid line is the experimental value

as compared to the experimental value

$$|\mathcal{M}|_{\text{exp}} = 7.78 \times 10^{-7}$$

(10)

It is remarkable that the simple formula of GSW combined with modern spectral function data should lead to the right prediction of the $\Delta I = \frac{1}{2}$ enhancement. The central value prediction is high by 15%, i.e. of the order of magnitude expected from errors of the soft kaon approximation and of the neglect of the high energy tail. Had the modern data been available to GSW they would have predicted in 1967 the existence of a charged intermediate vector boson of a mass of about 90 GeV (instead of the 8 GeV following from their assumption of narrow resonance dominance). To quote GSW ”we would lose most of our scruples about this calculation if such an intermediate boson is found”.

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