Non-perturbative correlation masses
in the hot electroweak phase

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Abstract

The effective action describing the long range fluctuations in the high temperature phase of the electroweak standard theory is a strongly coupled SU(2)-Higgs-model in three dimensions. We outline in detail a model in which the spatial correlation scales in this phase are calculated as inverse relativistic bound state masses. Selection rules for these states are derived. The correlation masses are calculated by evaluating the bound state Green’s function. The scalar-scalar-potential and its influence on the masses is investigated. The predictions for the correlation masses agree very well with the lattice data available now.

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1 Introduction

The perturbative treatment of the electroweak standard model (SM) leads to a spectacularly good agreement with experiments. It works because the Higgs mechanism renders the non-Abelian gauge bosons massive. This remains true in the Higgs phase (broken phase) of the SM at high temperatures for small Higgs masses, as has been discussed carefully in the last years [1–6]. But due to the rising Higgs-plasma-mass at very high temperatures $T > T_c \sim 100\text{GeV}$ a new hot (unbroken) phase of the electroweak theory was argued to exist in [7] in which according to perturbative calculations the gauge bosons are massless. A first order phase transition between the two phases could have important effects, in particular the possibility of a baryogenesis at this state has been discussed extensively [8]. It now is practically excluded for the experimentally allowed range of $m_H \gtrsim 60\text{GeV}$ but variants of the standard model in particular the supersymmetric standard model are under discussion.

If one wants to discuss the phase transition in detail – critical bubbles, sphaleron transitions, etc. – one needs the effective action (or at least the effective potential) of the theory. However, for vanishing or small Higgs vacuum expectation values the infrared problem comes up like in QCD and we expect non-perturbative effects. In the case of large $m_H \gtrsim 70\text{GeV}$, where the phase transition starts to fade away, this non-perturbative regime might cover even the broken phase at the temperatures of the phase transition [10].

At high temperatures the non-zero Matsubara-modes are heavy and the IR-sensitive part of the theory is an effective three dimensional theory of the zero modes whose parameters are obtained from the original theory by matching a set of amplitudes in 4 and 3 dimensions [1,6]. The three dimensional action can be simplified further by integrating out the massive zero modes of the time component of the gauge field. In the high temperature expansion higher derivative terms are suppressed and neglecting electroweak mixing one ends up with the three dimensional SU(2)-Higgs model. An appropriate way to treat such an IR-sensitive theory where non-perturbative effects are expected is Monte Carlo simulations on the lattice like in QCD. This is simpler than in the latter case because the fermions have already been integrated out as non-zero Matsubara-modes. Such calculations have been performed by several groups [11–16]. They confirmed that there are confinement effects to be discussed in detail later on. An effective 3-dimensional gauge coupling rising in the infrared was also argued for in the framework of exact renormalization group equations [17].

In this paper we will discuss the spatial correlation scales of the physical states, i.e. of the corresponding gauge invariant operators, in the 3-dimensional SU(2)-Higgs-model. The simple-minded perturbative picture would predict long range correlations because of the massless gauge bosons, but the neglect of IR-effects is obviously wrong. Based on 1-loop gap equations it was argued that one has just another Higgs-phase [18]. The predicted vector-boson-mass is much smaller than the one calculated in the gauge invariant lattice calculations (but curiously not much different from (Landau) gauge fixed lattice results [16]). In a recent letter [19] we proposed a model for two-dimensional bound states of light constituents in the hot electroweak phase. The Green’s functions of these bound states have been evaluated following a method which was developed by Simonov [20] in 4-dimensional QCD. In this paper we elaborate the model in detail. The evaluation of the Green’s function is generalized and reorganized. The bound state correlation masses are first calculated analytically for a linear scalar-scalar-potential. Several modifications of the potential and their influences on the correlation masses are discussed. The parameters
of the potential are then fixed from lattice calculations of the Wegner-Wilson-loop. Finally we can compare our predictions for higher correlations with recent gauge invariant lattice calculations \[13,15\].

Chapter 2 introduces the effective 3-dimensional SU(2)-Higgs-model. Chapter 3 discusses the possible bound states. The corresponding Green's functions are derived in chapter 4. They are evaluated in the next chapter. In chapter 6 we calculate the correlation masses for a linear potential, discuss modifications of this potential and their influence on the masses. The parameters of our model are fixed in chapter 7. Chapter 8 gives the comparison of our results with lattice data. The intercept of the potential is investigated in chapter 9. Finally we present our conclusions.

\section{The SU(2)-Higgs-model in three dimensions}

At high temperatures the SM can effectively be described by the three dimensional SU(2)-Higgs-model \[6\] with the Lagrangian

\begin{equation}
\mathcal{L} = \frac{1}{4} F^a_{ij} F^a_{ij} + \left( D_i \phi \right)^\dagger (D_i \phi) + m_3^2 \phi^\dagger \phi + \lambda_3 (\phi^\dagger \phi)^2 , \quad D_i = \partial_i - g_3 A_i^a \tau^a_2 .
\end{equation}

Even variants of the SM, e.g the minimal supersymmetric standard model, can be described by this effective theory in a large part of the parameter space \[21\]. The squared mass \( m_3^2 \), the quartic coupling \( \lambda_3 \) and the three dimensional gauge coupling \( g_3 \) depend on the parameters of the fundamental theory and on the temperature. The quartic coupling has the mass-dimension 1, while \( g_3 \) has the mass-dimension 1/2.

The non-perturbative aspects we are interested in are dominated by the gauge boson sector. In contrast to a four dimensional theory this sector has a natural mass scale given by the gauge coupling. It is therefore natural to express dimensioned quantities in units of powers of \( g_3 \). The parameters of the model can then be represented by the two dimensionless quotients

\begin{equation}
\bar{\lambda}_3 = \frac{\lambda_3}{g_3^2} \quad \text{and} \quad \bar{m}_3^2 = \frac{m_3^2}{g_3^4} ,
\end{equation}

sometimes called \( x \) and \( y \) in the literature \[6\]. The mass \( m_3 \) depends in general on the renormalization scale. We work, however, with fixed values of \( \bar{m}_3^2 \) and hence at a fixed scale. This is similar to lattice calculations where \( \bar{m}_3^2 \) is fixed at the scale given by the lattice constant.

In order to uncover the full symmetry of the SU(2)-Higgs-model we replace the Higgs-doublet \( \phi(x) \) by a \( 2 \times 2 \)-matrix-field \( \Phi(x) \) via

\begin{equation}
\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \Phi \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{with} \quad \Phi = \Phi_0 \mathbb{1}_{2 \times 2} + i \Phi_i \tau^i ,
\end{equation}

where the \( \tau^a \) are the Pauli matrices. Due to the identity \( \text{Tr} \Phi^\dagger \Phi = 2 \phi^\dagger \phi \) the potential (eq. (1)) depends only on this trace.

Expressing the gauge field by the usual matrix notation \( A_i = A_i^a \tau^a_2 \) the action can be written as

\begin{equation}
S_3 = \int d^3 x \left\{ \frac{1}{2} \text{Tr} F_{ij} F_{ij} + \frac{1}{2} \text{Tr} (D_i \Phi)^\dagger (D_i \Phi) + V_3 \left( \frac{1}{2} \text{Tr} \Phi^\dagger \Phi \right) \right\} .
\end{equation}
It is invariant under the transformation
\[ \Phi(x) \to U(x) \Phi(x) V, \quad (5) \]
\[ D_i(x) \to U(x) D_i(x) U(x)\dagger, \quad (6) \]
\[ F_{ij}(x) \to U(x) F_{ij}(x) U(x)\dagger. \quad (7) \]

\[ U(x) \] is an element of the local gauge group SU(2)\(_G\) and depends in general on \( x \), while \( V \) is a \( x \)-independent element of the isospin group SU(2)\(_I\). Hence the first index \( \alpha \) of the matrix \( \Phi_{\alpha a} \) refers to the gauge group and the second index \( a \) to the global SU(2)\(_I\).

has a first gauge index \( \alpha \) and a second isospin index \( a \).

3 Bound states of two scalars

In the 3-dimensional SU(2)-Higgs-model like in all gauge theories in principle everything has to be expressed in terms of gauge invariant quantities. For the well known formulation of the Higgs mechanism for generating massive gauge bosons this just implies a rather trivial reformulation in terms of gauge invariant objects (unitary gauge). In the case of a confining theory such a gauge invariant description is standard and absolutely mandatory. Indeed some time ago the 4-dimensional SU(2)-Higgs model was a prominent model for describing an effective electroweak interaction of composite \( W \), Higgs, quarks and leptons. But it is not clear just from a gauge invariant formulation whether one is in the Higgs or in the confinement phase. Contrary to the case of the original Abbot-Farhi-model for compositeness in electroweak interaction \[ [23, 24] \] we argue here that in the 3-dimensional case of the hot electroweak theory one really has a non-perturbative theory with genuine composite states in 2+1 dimensions. In this we differ from reference \[ [18] \] where a Higgs-mechanism also in this phase is advocated.

Nonlocal operators  The nonlocal operators corresponding to the bound state of two fundamental scalars \( \Phi \) are either

\[ \text{Tr} \ \Phi(x)\dagger T(x, \bar{x}) \Phi(\bar{x}) \quad (8) \]
or

\[ \text{Tr} \ \Phi(x)\dagger T(x, \bar{x}) \Phi(\bar{x}) \tau^i. \quad (9) \]

The matrix \( T(x, \bar{x}) \) with gauge-indices is the link-operator (also called gauge field transporter) defined by

\[ T(x, \bar{x}) = P \exp \left( -ig_3 \int^{\bar{x}}_x A_k dz_k \right), \quad (10) \]

where \( P \) denotes the path-ordering operator along some given path between \( x \) and \( \bar{x} \). The operator of eq. (8) is an isospin-singlet, the operator of eq. (8) is an isospin-triplet; both are gauge singlets. Since they are non-local all possible angular momentum states contribute.

The local interpolating fields

\[ \text{Tr} \ \Phi(x)\dagger \Phi(x) \quad \text{and} \quad \text{Tr} \ \Phi(x)\dagger D_k \Phi(x) \tau^i \quad (11) \]

are included in these operators in the limit \( \bar{x} \to x \). The index of the vector field corresponds to the spatial direction of the link operator. Note that \( \text{Tr} \ \Phi(x)\dagger \Phi(x) \tau^i \) vanishes while \( \text{Tr} \ \Phi(x)\dagger D_k \Phi(x) \) is a total derivative.
Selection rules The local limit shows that the isospin-singlet operator overlaps with scalar states while the isospin-triplet operator overlaps with vector states. The obvious question arises if there are scalar isospin-triplets or vector isospin-singlets as well. Let us first assume that the path between \( x \) and \( \bar{x} \) is a straight line. In coordinate gauge the operator \( T(x, \bar{x}) \) is then equal to the \( \mathbb{1} \)-matrix. The gauge-invariant operators \( (8) \) and \( (9) \) can be evaluated in this gauge. One gets

\[
\text{Tr} \Phi(-r)\Phi^\dagger(r) O_I \quad \text{with} \quad O_I \in \{ \mathbb{1}_{2\times2}, \tau^j \} .
\]

(12)

Here we have changed the coordinate system; the origin is now at the center of the bound state.

The projection on parity-even and parity-odd states respectively is

\[
\frac{1}{2} \text{Tr} \left( \Phi(-r)\Phi^\dagger(r) \pm \Phi(r)\Phi^\dagger(-r) \right) O_I .
\]

(13)

Expressing \( \Phi(r) \) by real fields \( \Phi_0 \) and \( \Phi_i \) as in eq. (3) the parity-even states become

\[
\left( \Phi_0(-r)\Phi_0(r) + \Phi_i(-r)\Phi_i(r) \right) \text{Tr} \mathbb{1}_{2\times2} O_I
\]

(14)

while the parity-odd states are

\[
i \left( \Phi_0(-r)\Phi_i(r) - \Phi_i(-r)\Phi_0(r) - \Phi_k(-r)\Phi_l(r)\epsilon_{kli} \right) \text{Tr} \tau^i O_I .
\]

(15)

One sees the parity-even isospin-triplets (with \( O_I = \tau^j \)) and parity-odd isospin-singlets (with \( O_I = \mathbb{1}_{2\times2} \)) vanish. Hence in particular no scalar isospin-triplets and no vector-isospin-singlets are allowed.

If the path between \( x \) and \( \bar{x} \) is not a straight line there may be an overlap with states which do no respect this selection rule. Nevertheless, there is no local interpolating field composed of two scalar fields alone which corresponds to these states in the limit \( \bar{x} \to x \). The interpolating fields in that case would contain at least an additional gauge boson field \( F_{ij} \) and thus correspond to hybrid states in QCD. The latter ones are several hundred MeV (order of magnitude of the “gluon constituent” mass) heavier than the corresponding pure quark-antiquark states. We assume that a similar mechanism holds also in our case.

The derivation of the selection rules given above can be generalized to “blocked” operators which are used in lattice investigations.

4 Green’s functions and correlation masses

In order to get some insight into the dynamical structure of the electroweak theory above the critical temperature and to compare with QCD in the confining phase we have proposed in reference [19] a bound state model with a potential whose parameters were determined by comparing with results of lattice simulations for Wegner-Wilson-loops. Since the fundamental scalars might be very light one has to treat the problem with relativistic kinematics. For that we adopted the method of Simonov [20] to our case. We reorganize and generalize our calculations of reference [19] in this paper. In the described model the correlation masses are determined from the exponential falloff of the Green’s function.
The Green’s function of a fundamental scalar  

We start from the Green’s function $G(x, y, A)_{a\alpha b\beta}$ that transforms the matrix field $\Phi(x)_a$ into $\Phi(y)_b$. Neglecting the scalar self-interaction it has in the worldline formalism the form

$$
G(x, y, A)_{a\alpha b\beta} = \int_0^\infty ds \exp(-m_3^2 s) \int_x^y Dz \left[ P \exp \left( - \int_0^s d\tau \left( \frac{1}{4} \dot{z}_k^2(\tau) + ig_3 A_k \dot{z}_k \right) \right) \right]_{\alpha,\beta} \delta_{ab}
$$

where $m_3^2$ is the squared mass of the fundamental scalar. $s$ is the Schwinger proper time. $z_k(\tau)$ is a path which connects $x$ with $y$ and is parameterized by $\tau$; $\tau$ runs from 0 to $s$. $\dot{z}_k(\tau)$ denotes the derivative of $z_k(\tau)$ with respect to $\tau$.

The first term in the exponent alone describes the free propagation of a scalar field. The second term includes the interaction with the gauge field $A_k$; $g_3$ is the three dimensional gauge coupling. Scalar self-interactions are neglected (quenched approximation). The exponential function is ordered along the path; $P$ denotes the path-ordering operator. The gauge field in eq. is treated as a fixed background field. Later on we will quantize averaging over this field. (In principle there is one worldline action for every index pair $(\alpha, \beta)$.)

The path-ordered exponential function (eq. (17)) is matrix valued. It is identical to

$$
\exp \left( - \frac{1}{4} \int_0^s d\tau \dot{z}_k^2(\tau) \right) \left[ P \exp \left( - ig_3 \int_0^s d\tau A_k \dot{z}_k \right) \right]_{\alpha,\beta} = \exp \left( - \frac{1}{4} \int_0^s d\tau \dot{z}_k^2(\tau) \right) T(x, y)_{\alpha,\beta},
$$

where $T(x, y)$ is the link-operator from eq. (16).

Using this the Green’s function (eq. (16)) can be written in Feynman-Schwinger-representation

$$
G(x, y, A)_{a\alpha b\beta} = \int_0^\infty ds \exp(-m_3^2 s) \int_x^y Dz \exp \left( - \frac{1}{4} \int_0^s d\tau \dot{z}_k^2(\tau) \right) T(x, y)_{\alpha\beta} \delta_{ab}.
$$

The path used for the evaluation of $T(x, y)$ is the one integrated over in the path integral. All interactions of the scalar with the gauge field are expressed by this link-operator.

The Green’s function of a pair of scalars  

The Green’s function $G(x, \bar{x}, y, \bar{y})$ connecting the two isospin singlet operators $\text{Tr} \Phi(x)^\dagger T(x, \bar{x}) \Phi(\bar{x})$ and $\text{Tr} \Phi(y)^\dagger T(y, \bar{y}) \Phi(\bar{y})$ is built up by two fundamental Green’s functions (eq. (19)) and two link operators $T$

$$
G(x, \bar{x}, y, \bar{y})
$$

$$
= \left\langle G(\bar{x}, \bar{y}, A)_{a\alpha b\beta} T(y, y)_{\beta\gamma} G(y, x, A)_{\gamma\alpha \delta} T(x, \bar{x})_{\delta\alpha} \right\rangle_A
$$

$$
= 2 \int_0^\infty ds \int_0^\infty d\bar{s} \exp \left( -m_3^2 (s + \bar{s}) \right) \int \mathcal{D}z \mathcal{D}\bar{z}
$$

$$
\times \exp \left( - \frac{1}{4} \int_0^s d\tau \dot{z}_k^2(\tau) - \frac{1}{4} \int_0^\bar{s} d\tau \dot{\bar{z}}_k^2(\tau) \right) \left\langle P \exp \left( -ig_3 \int_{x, \bar{x}, \bar{y}, y} A_k \dot{z}_k \right) \right\rangle_A.
$$
The brackets \( \langle \ldots \rangle_A \) symbolize the functional integration over the gauge field \( A \). In principle this averaging contains all perturbative and non-perturbative effects. The last factor is the expectation value of the Wegner-Wilson-loop in a gauge field background. It results from four link-operators; two in the Green’s functions of the fundamental scalars (eq. (19)) and two in the link-operators of the bound states (eq. 8). The paths \( z(\tau) \) and \( \bar{z}(\bar{\tau}) \) determine the both sides of the Wegner-Wilson-loop.

The Green’s function of a triplet is correspondingly

\[
G_T(x, \bar{x}, y, \bar{y}) = \left\langle \left( G(\bar{x}, \bar{y}, A)_{a\alpha b\beta} \tau_{bc}^\alpha \tau_{\beta\gamma}^j \langle T(y, x, A)_{c\gamma d\delta} \tau_{\delta\alpha}^j \rangle_A \right) \right\rangle_A
\]

For a given \( i = j \) the Green’s function of a triplet is identical to the Green’s function of a singlet, but due to the selection rules discussed in section 3 the iso-singlet and iso-triplet states must have different angular momenta.

The correlation mass

The correlation mass of the bound states \( M \) describes the fall off of the Green’s function at large distances \( \Theta \)

\[
G(x, \bar{x}, y, \bar{y}) \propto \exp(-M\Theta) .
\]

The extension of the states has to be small as compared to their distance

\[
|x - \bar{x}|, |y - \bar{y}| \ll |x - y|, |\bar{x} - \bar{y}| \approx \Theta .
\]

An exact definition of \( M \) is given by

\[
M = - \lim_{\Theta \rightarrow \infty} \frac{1}{\Theta} \ln \left( G(x, \bar{x}, y, \bar{y}) \right) .
\]

From the form of the Green’s function \( G(x, \bar{x}, y, \bar{y}) \) it follows that \( M \) depends only on \( m_3^2 \) and on the expectation value of the Wegner-Wilson-Loop. The other variables used in eq. (20) are integrated out. All possible angular momenta contribute to \( G(x, \bar{x}, y, \bar{y}) \). They have to be isolated later.

5 The calculation of the correlation masses

As in Simonovs original work \[20\] the correlation masses are calculated from eq. (25) by simplifying the Green’s function of the bound state. The calculations are, however, reorganized and generalized. Furthermore we have in our (scalar) case no problems with chiral symmetry-breaking and spin-spin-interactions. In order to gain a manageable form we have to make two approximations.

The parameterization of the paths

In a first step the paths of both scalars are parameterized by a common parameter \( \gamma \) via

\[
\tau = \gamma s \quad \bar{\tau} = \gamma \bar{s} \quad \text{with} \quad 0 \leq \gamma \leq 1 .
\]
The kinetic terms of eq. (20) become
\[
\frac{1}{4} \int_0^s d\tau \dot{z}_k^2(\tau) + \frac{1}{4} \int_0^\bar{s} d\bar{\tau} \ddot{z}_k^2(\bar{\tau})
\]
\[
= \frac{1}{4} \int_0^1 d\gamma \left( \frac{1}{s} \left( \frac{\partial z_k(\gamma)}{\partial \gamma} \right)^2 + \frac{1}{s} \left( \frac{\partial \bar{z}_k(\gamma)}{\partial \gamma} \right)^2 \right) .
\]  
(27)

The spatial vectors \( z_k(\gamma) \) and \( \bar{z}_k(\gamma) \) are then replaced by a kind of “center of mass” and a relative coordinate
\[
R_k = \frac{s \bar{s}}{s + \bar{s}} \left( \frac{1}{s} z_k + \frac{1}{\bar{s}} \bar{z}_k \right) \quad \quad u_k = z_k - \bar{z}_k .
\]  
(28)

For \( z_k \) and \( \bar{z}_k \) one gets
\[
z_k = R_k + \frac{s}{s + \bar{s}} u_k \quad \quad \bar{z}_k = R_k - \frac{\bar{s}}{s + \bar{s}} u_k .
\]  
(29)

The path integrals \( \int \int Dz \bar{D}\bar{z} \) in eq. (20) transform into path integral over the new variables \( \int \int DR \bar{Du} \) and the kinetic terms (eq. (27)) become
\[
\frac{1}{4} \int_0^1 d\gamma \left( \frac{s + \bar{s}}{s \bar{s}} \left( \frac{\partial R_k(\gamma)}{\partial \gamma} \right)^2 + \frac{1}{s + \bar{s}} \left( \frac{\partial u_k(\gamma)}{\partial \gamma} \right)^2 \right) .
\]  
(30)

Mixed terms cancel. This parameterization is always possible and no approximation has been made.

**The classical path of the center of mass** Now as in [20] we make the crucial assumption that the classical path dominates the trajectory of the center of the bound state. This approximation is justified for heavy bound states. It is not obvious which scale has to be used as reference, therefore we can not decide here if the “large mass condition” is fulfilled; we will come back to this question in section 10.

In this approximation the path integral \( \int DR \) can be replaced by an appropriate parameterization of \( R_k(\gamma) \). We choose the \( x_3 \)-axes as the direction of the path
\[
R_1 = R_2 = 0 \quad \quad R_3 = \gamma \Theta = \vartheta ,
\]  
(31)

where \( \Theta \) is the distance between the two bound states introduced in eq. (24). The parameter \( \vartheta \) will be interpreted in connection with eq. (37).

The Green’s function of the bound state becomes
\[
G(x, \bar{x}, y, \bar{y})
\]
\[
\propto \int_0^\infty ds \int_0^\infty d\bar{s} \exp \left( -m_3^2(s + \bar{s}) \right) \int \bar{D}u
\]
\[
\times \exp \left( -\frac{1}{4} \int_0^1 d\gamma \left( \frac{s + \bar{s}}{s \bar{s}} \Theta^2 + \frac{1}{s + \bar{s}} \left( \frac{\partial u_k(\gamma)}{\partial \gamma} \right)^2 \right) \right) \left\{ \mathcal{P} \exp \left( -ig_3 \int x, \bar{x}, y, \bar{y} A_k dz_k \right) \right\} A .
\]  
(32)
The modified area law. The Wegner-Wilson-Loop is unchanged by the former transformations. Motivated by lattice results we propose the ansatz

$$\left\langle P \exp \left( -i g_3 \oint_{x,x',y,y'} A_k dz_k \right) \right \rangle \propto \exp \left( -\Theta \int_0^1 d\gamma V \left( \sqrt{u_1^2(\gamma) + u_2^2(\gamma)} \right) \right)$$

for the gauge field averaged expression. This replacement generalizes the evaluation of the Wegner-Wilson-loop in the references [20] and [19]. As we will see in section 6.2, effects important at small distances can be taken into account (cf. sect. 6.2) with this generalization.

Note that the averaging over the gauge field is included in the modified area law (eq. (33)). The non-perturbative effects originate in the gauge boson sector and are represented by this averaging. We thus assume that all non-perturbative effects on the bound state masses can be expressed by a scalar-scalar-potential $V(r)$. 

Eq. (33) evaluated for rectangular Wegner-Wilson-loops with infinite length $T$ leads to the two-dimensional potential

$$V(r) = -\lim_{T \to \infty} \frac{1}{T} \ln(W(r,T)) \quad .$$

Here $W(r,T)$ denotes the expectation value of the rectangular loop with length $T$ and width $r$. In section 6 we use this definition to determine $V(r)$ from lattice data.

Using the replacement of eq. (33) the $u_3$-path integral in the Green’s function of the bound state (eq. (32)) becomes independent of $\Theta$. Therefore it does not contribute to the correlation mass. Since the Green’s function is only needed up to a constant we disregard this contribution. The two remaining coordinates are combined to a vector $\vec{u}=(u_1,u_2)$.

Evaluation of the Green’s function. In the next step we find a Hamiltonian for a corresponding Schrödinger type equation whose Green’s function is the one of eq. (32) with the replacement of eq. (33). The latter should behave as $G \propto \exp(-M_i \Theta)$ in the channel $i$. This has two advantages: First the separation of states with different angular momentum is very simple and second the solution of the resulting differential equations is much simpler than performing the path integrals.

For this purpose it is advisable to introduce new variables. First $\gamma$ is replaced by the parameter $\vartheta$ from eq. (31). Then the Schwinger proper time variables are substituted by

$$\mu = \frac{\Theta}{2s} \quad \bar{\mu} = \frac{\Theta}{2\bar{s}} \quad \tilde{\mu} = \frac{\mu \bar{\mu}}{\mu + \bar{\mu}} = \frac{\Theta}{2(s+\bar{s})} \quad .$$

The Green’s function (eq. (32)) becomes

$$G(x,\bar{x},y,\bar{y}) \propto \int_0^\infty d\mu \int_0^\infty d\bar{\mu} \left( \frac{\Theta}{2\mu \bar{\mu}} \right)^2 \exp \left( -\frac{\Theta}{2} \left[ m_3^2 \left( \frac{1}{\mu} + \frac{1}{\bar{\mu}} \right) + (\mu + \bar{\mu}) \right] \right) \int D\vec{u} \exp(-B)$$

with

$$B = \int_0^\Theta d\vartheta \left[ \frac{\tilde{\mu}}{2} \left( \frac{\partial \vec{u}(\vartheta)}{\partial \vartheta} \right)^2 + V(|\vec{u}|) \right] \quad .$$

If we look first to the $\vec{u}$-path integral the formal analogy to a two dimensional system is obvious: a particle with mass $\tilde{\mu}$ at position $\vec{u}$ and time $\vartheta$ moves in the potential $V(|\vec{u}|)$. The Euclidean action is $B$; the time interval runs from 0 to $\Theta$. Hence, the component $\vartheta=R_3$ of the center of mass coordinate takes over the role of the quasi time.
**The Schrödinger equation of the two dimensional problem.** The time independent Schrödinger equation corresponding to the two dimensional problem is

$$ H \Psi(\vec{u}) = \epsilon \Psi(\vec{u}) \quad (38) $$

where $\epsilon$ is the energy eigenvalue and the Hamilton operator is given by

$$ H = -\frac{1}{2\bar{\mu}} \left( \frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} \right) + V(|\vec{u}|) \quad . \quad (39) $$

Due to the rotational symmetry of $H$ we can separate the radial from the angular variables and the problem simplifies to the solution of the radial equation. Using $r=|\vec{u}|$ one gets

$$ -\frac{1}{2\bar{\mu}} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{l^2}{r^2} \right) \psi_{nl}(r) + V(r) \psi_{nl}(r) = \epsilon_{nl} \psi_{nl}(r) \quad , \quad (40) $$

where $n=1,2,\ldots$ is the radial and $l=0,\pm1,\pm2,\ldots$ is the angular momentum quantum number.

If $V(r)$ is less singular than $1/r$ at $r=0$, the boundary conditions are

$$ \lim_{r \to \infty} \psi_{nl}(r) = 0 \quad \text{and} \quad \psi_{nl}(r) = r^{|l|} \quad \text{at} \quad r = 0 . \quad (41) $$

For a given potential $V(r)$ and fixed quantum numbers $n$ and $l$ it is possible to solve the radial equation and to calculate the eigenvalue $\epsilon_{nl}$. In section \[ we derive the results for some potentials of interest.

**The Green’s function of a particular state** The solution of the Schrödinger equation (38) allows us to express the $\mu$-dependent part of the Green’s function by

$$ \int D\vec{u} \exp(-B) = \sum_{n,l} \psi_{nl}(\vec{u}(\Theta)) \psi_{nl}(\vec{u}(0)) \exp(-\epsilon_{nl}\Theta) \quad (42) $$

Thus the contribution of a particular state with quantum numbers $n$ and $l$ to the $\Theta$-dependence of the Green’s function (eq. (36)) is given by

$$ G_{nl}(x,\bar{x},y,\bar{y}) \propto \int_0^\infty d\mu \int_0^\infty d\bar{\mu} \left( \frac{\Theta}{2\mu \bar{\mu}} \right)^2 \exp\left( -\frac{\Theta}{2} \left[ m_3^2 \left( \frac{1}{\mu} + \frac{1}{\bar{\mu}} \right) + (\mu + \bar{\mu}) + 2\epsilon_{nl}(2\bar{\mu}) \right] \right) . \quad (43) $$

**The saddle point method** The $\mu$- and the $\bar{\mu}$-integrals are evaluated with the saddle point method. For large values of $\Theta$ both integrals are dominated by the exponential function. In the limiting case of $\Theta \to \infty$ only the maximum of the exponent with respect to $\mu$ and $\bar{\mu}$ contributes.

The calculation may be abbreviated using the fact that the exponent is symmetric in $\mu$ and $\bar{\mu}$. This is due to the fact that eq. (21) is symmetric in $s$ and $\bar{s}$. This symmetry in turn originates from the identity of the masses of both fundamental scalars. The saddle point equations for $\mu$ and $\bar{\mu}$ are hence equivalent. The minimum of both is unique and therefore the same. It is possible to identify both parameters. Using $\mu = \bar{\mu} = 2\bar{\mu}$ (cf. eq. 35) we arrive at the Green’s function

$$ G_{nl}(x,\bar{x},y,\bar{y}) \propto \exp(-M_{nl}(\mu)\Theta) \quad (44) $$

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with the correlation mass

\[ M_{nl}(\mu) = \frac{m_3^2}{\mu} + \mu + \epsilon_{nl}(\mu) \]  

(45)

where \( \mu \) fulfills the saddle point equation

\[ \frac{\partial M_{nl}(\mu)}{\partial \mu} = -\frac{m_3^2}{\mu^2} + 1 + \frac{\partial \epsilon_{nl}(\mu)}{\partial \mu} = 0 \]  

(46)

Thus the calculation of the correlation mass from the Green’s function of the bound state (eq. (20)) is reduced to the solution of a differential equation (eq. (40)) and a minimization problem (eq. (46)).

6 The correlation masses

6.1 Linear potential

The solutions of the Schrödinger equation (eq. (38)) and the corresponding eigenvalues \( \epsilon_{nl} \) depend on the scalar-scalar-potential. In this subsection we investigate the simple and important case of a linear potential

\[ V(r) = \sigma r \]  

(47)

The radial equation Substituting \( r \) in eq. (40) by the dimensionless variable

\[ \rho = (\mu \sigma)^{1/3} r \]  

(48)

results in the eigenvalue equation

\[ -\frac{\partial^2 \psi_{nl}(\rho)}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial \psi_{nl}(\rho)}{\partial \rho} + \left( \frac{l^2}{\rho^2} + \rho \right) \psi_{nl}(\rho) = a_{nl} \psi_{nl}(\rho) \]  

(49)

The boundary conditions (eq. (41)) are unchanged.

The numerical solution of this differential equation does not cause any difficulties. The eigenfunction with the lowest quantum numbers are displayed in figure 1. The corresponding eigenvalues \( a_{nl} \) are:

| \( a_{nl} \) | \( l=0 \) | \( l=1 \) | \( l=2 \) |
|---|---|---|---|
| \( n=1 \) | 1.74 | 2.87 | 3.82 |
| \( n=2 \) | 3.67 | 4.49 | 5.26 |

The eigenvalues of interest \( \epsilon_{nl}(\mu) \) can by calculated from the dimensionless eigenvalues \( a_{nl} \) of eq. (49) via

\[ \epsilon_{nl}(\mu) = \frac{\sigma^{2/3}}{\mu^{1/3}} a_{nl} \]  

(50)

Therefore the \( \mu \)-dependence of \( \epsilon_{nl}(\mu) \) is a power law in the case of a linear potential.
Figure 1: The solutions of the radial equation (49) for the quantum numbers \( n = 1, 2 \) and \( l = 0, 1, 2 \). The Green’s function of the bound states (51) can be expressed by these solutions. The notation s-, p- and d-wave corresponds to different angular momentum quantum numbers \( l \), as in atomic physics.

**The saddle point equation** Using \( \epsilon_{nl}(\mu) \) from eq. (50) the saddle point equation (50) becomes

\[
- \frac{m_3^2}{\mu^2} + 1 - \frac{1}{3} \mu^{-4/3} \sigma^{2/3} a_{nl} = 0 .
\]  

(51)

It is solved by

\[
\mu = z(a_{nl}, m_3^2 \sigma^{1/3})^3 \sqrt{\sigma} \quad \text{with} \quad b = \frac{m_3^2}{\sigma} \quad (52)
\]

where \( z(a, b) \) is a positive and real solution of the cubic equation

\[
z^3 - \frac{1}{3} a z - b = 0 .
\]  

(53)

The solution corresponding to the minimum of \( \mathcal{M}(\mu) \) is

\[
z(a, b) = \frac{2^{1/3}}{3} a \left( 27 b + \sqrt{729 b^2 - 4 a^3} \right)^{-1/3} + \frac{1}{21/3} \left( 27 b + \sqrt{729 b^2 - 4 a^3} \right)^{1/3} .
\]  

(54)

It is unique if \( b \geq 0 \); for \( \frac{2}{21} a^{3/2} < b < 0 \) it corresponds to a local minimum of \( \mathcal{M} \). If \( b \) is even smaller there is no positive and real solution anymore. We therefore treat only the case that \( m_3^2 \) and hence \( b \) is positive.

The correlation mass (eq. (43)) evaluates to

\[
\frac{\mathcal{M}_{nl}}{\sqrt{\sigma}} = 4 z \left( a_{nl}, \frac{m_3^2}{\sigma} \right)^{3/2} - 2 z \left( a_{nl}, \frac{m_3^2}{\sigma} \right)^{-3/2} \frac{m_3^2}{\sigma} ,
\]  

(55)

with the function \( z \) given in eq. (54). It depends only on the dimensionless eigenvalue \( a_{nl} \), on the string tension \( \sigma \) and on the squared mass \( m_3^2 \) of the fundamental scalar. The
result scales with $\sigma$. In figure 2 we plot the correlation masses of the bound states with the lowest quantum numbers versus the squared mass of the fundamental scalar.

Our results show a hierarchy of bound states. The lowest state corresponds to the 1s-wave ($n=1$, $l=0$). Due to the selection rules derived in section 3 it has to be an isospin-singlet. The 1p-state ($n=1$, $l=1$) is heavier, it is a vector-isospin-triplet. A dense spectrum of higher excitations follows. Note that the string tension determines the typical scale of the bound state masses. It is a genuine non-perturbative quantity.

For the case that $m_3^2 \lesssim \sigma$ our treatment is by no means an analogue of the non-relativistic quark model. This can be seen by comparing eq. (40) with eq. (45). From (45) we would deduce a constituent mass of the elementary scalars

$$m_{\text{constit}} = \mu + \frac{m_3^2}{2\mu}$$

whereas from the kinetic terms we would deduce $m_{\text{constit}} = \mu$. For $m_3^2 \gtrsim \sigma$ there is no contradiction since $\mu$ goes to $m_3^2$ in that case. Furthermore the constituent mass shows a marked dependence on the state for $m_3^2 \lesssim \sigma$.

The mass parameter $\mu$ varies from $0.24g_3^2$ to $0.4g_3^2$ going from 1s to 2p corresponding to a quasi static constituent mass of half that value at $m_3^2 = 0$.

\footnote{Thus it is about half of the constituent mass introduced in a very recent paper by Buchm"uller and Philipsen~\cite{35}. Note that we also have a binding energy $\epsilon_{nl}$. The constituent W-boson in~\cite{35} does not enter directly the mass formula in our dynamical approach. However it corresponds to the inverse correlation length of the gauge field strength and hence determines the binding energy via the string tension (eq. (65) and appendix 3).}
6.2 A nonlinear potential

The procedure presented above can be extended in a straightforward way to a general potential \[26\]. For the problem under consideration several modifications of the linear potential are expected to occur.

As in QCD we do expect screening of the potential by spontaneous creation of a pair of two fundamental scalars. The potential would look like the full line in figure 3a. The dashed line shows the corresponding linear potential. So far lattice results do not indicate any screening and the low lying bound states should not be affected by the screening anyway. We therefore neglect this effect.

From lattice calculations the potential can only be determined up to an additional constant. In the stochastic vacuum model, outlined in appendix B, the linear rise of the potential sets in only at distances \( r \) which are larger than the correlation length of two gauge fields in the vacuum. The shape of the potential is given in figure 3b (full line). If the wave functions obtained from the Schrödinger equation are not dominated by the region of small \( r \), the potential can be approximated by a linear potential with a negative intercept

\[
V(r) = V_0 + \sigma r \quad \text{with} \quad V_0 < 0
\]  

(dotted line). The size of the intercept will be discussed in section 9.

It is easy to see that the correlation masses corresponding to this potential are

\[
M_{nl} = M_{nl}^{\text{lin}} + V_0 ,
\]

where \( M_{nl}^{\text{lin}} \) are the masses calculated from the linear potential. Due to the lowering of the potential the correlation masses of the bound states are lowered as well.

The difference between the real potential and the approximation (eq. (57)) may, if it is small, be taken into account as a perturbation (see below).

At small distances \( r \) the potential should be dominated by the perturbative gauge boson exchange. The lowest order is calculated in appendix A. This leads to the two dimensional Coulomb potential

\[
V_C(|\vec{u}|) = \frac{3}{8\pi} g_3^2 \ln(\Lambda |\vec{u}|) .
\]
The constant $\Lambda$ cannot be fixed due to the logarithmic divergence.

If the exchanged particle acquires the mass $m$ we obtain the two dimensional Yukawa potential

$$V_Y(|\vec{u}|) = -\frac{3}{8\pi} g_3^2 K_0(m |\vec{u}|) ,$$  \hspace{1cm} (60)

where $K_0$ is the modified Bessel function. Note that both potentials merge at small distances $|\vec{u}| = r \ll m^{-1}$ with a suitable choice of $\Lambda$. At these distances the mass does not make any difference.

If the perturbative contribution to the potential vanishes in the limit $r \to \infty$ as $V_Y(r)$ does, the linear potential is not changed here. This situation is sketched in figure 3c. It turns out that in this case the corrections to the correlation masses $\mathcal{M}_{nl}$ can be calculated perturbatively. One gets

$$\mathcal{M}_{nl} = \mathcal{M}_{nl}^{\text{lin}} + \delta \mathcal{M}_{nl}$$ \hspace{1cm} (61)

with

$$\delta \mathcal{M}_{nl} = 2\pi \int_0^\infty dr \ r V_Y(r) \psi_{nl}(r)^2 .$$ \hspace{1cm} (62)

Since the Yukawa potential is negative, the masses are lowered.

7 The potential on the lattice

In order to compare the correlation masses of the bound states with lattice data, as will be done in the next section, we have to determine $V(r)$. In this section we analyze lattice data of Wegner-Wilson-loops and fix the parameters of the potential using a suitable fit.

The symmetric electroweak phase of the three dimensional SU(2)-Higgs-model has been investigated by several groups on the lattice [11–16]. Nevertheless, only the authors of reference [11–13] calculated the Wegner-Wilson-loops. The data have been reanalyzed by us [26] and will be given in a form which is appropriate for the purpose of calculating the bound state masses.

The lattice data are expressed in terms of the parameters $\beta_G$, $\beta_H$ and $\beta_R$. For $\beta_G$ there is the simple relation

$$\beta_G = \frac{4}{a_L g_3^2} ,$$ \hspace{1cm} (63)

where $a_L$ is the lattice constant. The data we use are evaluated at $\beta_G = 12$ and hence correspond to $a_L = \frac{1}{3} g_3^{-2}$. The more complicated relation between the other lattice parameters and the (normalized) continuum parameters $\tilde{m}_3^2 = m_3^2/g_3^4$ and $\tilde{\lambda}_3 = \lambda_3/g_3^2$ (eq. (3)) are given in reference [27]. The data of [11] correspond to $\lambda_3 = 0.0239$ and $\tilde{m}_3^2 = 0.73$; the $\lambda_3 = 0.0957$ results [12, 13] cover the range $\tilde{m}_3^2 = -0.022$ to 0.524.

The Wegner-Wilson-loops on the lattice are rectangular and have extensions from 2 to $N = 15$ resp. 24 lattice units. The extrapolation to infinite length is very secure, since the relevant quantity $-\ln W(r,T)$ (cf. eq. (34)) rises practically linear in $T$ for $T \geq 8a$. We found that the lattice data can be reproduced by the three parameter fit of the form

$$V(r) = C - \frac{3}{8\pi} g_3^2 K_0(m r) + \sigma r$$ \hspace{1cm} (64)

The fitted constant $C$ is without any physical significance since it depends on the lattice renormalization procedure. An effective intercept $C = V_0$ like in eq. (57) due to non-perturbative effects will be discussed in section 4. In table 4 we give the results for the relevant parameters $m$ and $\sigma$ for the available values of $\lambda_3$ and $\tilde{m}_3^2$. 

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Table 1: The fitted parameter of the function (eq. 64). The values at $\tilde{\lambda}_3 = 0.0957$ and $m^2_3 = 0.073$ are interpolated. $N$ is the number of lattice points, $m$ is the effective gauge boson mass and $\sigma$ is the string tension.

| $\tilde{\lambda}_3$ | $m^2_3$ | $N$ | $m/g^2_3$ | $\sigma/g^4_3$ |
|---------------------|---------|-----|------------|----------------|
| 0.0957              | 0.5254  | 15  | 1.045      | 0.1370         |
| 0.0957              | 0.3699  | 15  | 1.105      | 0.1390         |
| 0.0957              | 0.2153  | 15  | 1.029      | 0.1345         |
| 0.0957              | 0.1229  | 15  | 1.083      | 0.1364         |
| 0.0957              | 0.0615  | 15  | 1.046      | 0.1335         |
| 0.0957              | 0.0309  | 15  | 0.999      | 0.1296         |
| 0.0957              | 0.0003  | 15  | 0.886      | 0.1208         |
| 0.0957              | -0.0058 | 24  | 0.770      | 0.1135         |
| 0.0957              | -0.0220 | 24  | 0.411      | 0.0876         |
| 0.0957              | 0.073   | 15  | 1.058      | 0.1345         |
| 0.0239              | 0.073   | 15  | 1.051      | 0.1326         |

The contribution $\frac{3}{8\pi}g_3^2K_0(mr)$ indicates that the coloured objects exchanged between the scalars have an effective mass of about $1g_3^2$. It is suggestive to identify it with some magnetic mass of the gauge boson. The mass $m$ of our fit is, however, larger than the magnetic mass obtained in lattice simulations in Landau gauge [16] and also larger than the one predicted in reference [18] from gap equations.

Comparing the potential parameters at $\tilde{\lambda}_3 = 0.0239$ with the ones at $\tilde{\lambda}_3 = 0.0957$ (interpolated) one sees practically no effect of the quartic coupling. This can be understood from the fact that the exchange of a scalar is suppressed by a factor $\tilde{\lambda}_3^2$ in comparison to the gauge boson exchange.

The shape of the scalar-scalar-potential depends at $m^2_3 \geq 0.0615$ only marginally on $\tilde{\lambda}_3$. In this mass range $V(r)$ is nearly exclusively determined by the gauge boson sector. This behavior is expected for large scalar masses, since the scalars decouple here. From the data of table 1 we conclude that the mass-limit from which on the influence of the scalars on $V(r)$ is negligible is small compared to the other mass scales of the problem. Above this limit the string tension $\sigma$ should be identical to the one of the pure SU(2)-Yang-Mills-Theory. Indeed, Teper [28] finds for the latter one roughly $\sigma \approx 0.137g_3^4$ at $\beta_G = 12$ based on the correlation of Polyakov-loops. This value is similar to the string tensions of table 1. He extrapolates his results to $\beta_G = \infty$ in order to gain a continuum value of the string tension. We stay, however, with the potential at $\beta_G = 12$, since we compare with lattice data at the same value in the next chapter.

The data point $m^2_3 = -0.022$ is at the critical temperature. The fact that the critical $m^2_3$ is negative is in agreement with other lattice investigations [14].
Table 2: The masses of the bound states up to an arbitrary additional constant $C$. They have been calculated using the potential eq. (64) with the parameters of table 1 and $C = 0$.

| $\bar{\lambda}_3$ | $\overline{m}_3^2$ | $M_{1s}$/g^3 | $M_{2s}$/g^3 | $M_{1p}$/g^3 | $M_{2p}$/g^3 | $M_{1d}$/g^3 |
|---------------------|-------------------|----------------|----------------|----------------|----------------|----------------|
| 0.0957              | 0.5254            | 1.93           | 2.48           | 2.27           | 2.71           | 2.53           |
| 0.0957              | 0.3699            | 1.73           | 2.31           | 2.09           | 2.55           | 2.36           |
| 0.0957              | 0.2153            | 1.48           | 2.08           | 1.85           | 2.33           | 2.14           |
| 0.0957              | 0.1229            | 1.31           | 1.95           | 1.70           | 2.21           | 2.00           |
| 0.0957              | 0.0615            | 1.15           | 1.82           | 1.56           | 2.09           | 1.88           |
| 0.0957              | 0.0309            | 1.05           | 1.74           | 1.47           | 2.01           | 1.79           |
| 0.0957              | 0.0003            | 0.91           | 1.61           | 1.34           | 1.88           | 1.67           |
| 0.0239              | 0.073             | 1.18           | 1.84           | 1.58           | 2.10           | 1.89           |
| 0.0239              | 0.089             | 1.22           | 1.87           | 1.62           | 2.13           | 1.92           |

8 Comparison of the correlation masses with lattice calculations

The correlation masses in our model We calculated the correlation masses $M_{nl}$ of the bound states with the method presented above. We based this calculations on the potential of eq. (64) with $m$ and $\sigma$ from table 1 and the arbitrary choice $C = 0$. The effect of the term $-3\pi g^2 K_0(mr)$ is very small. It lowers the masses by less than 2% and can safely be treated perturbatively. This is due to the large extensions of the bound states as compared to $1/m$. Our results for the correlation masses are listed in table 2.

In lattice calculations the correlations of pairs of the operators (eq. (8)) respectively (eq. (9)) are investigated. The relative orientation can be varied. Using special combinations of these pairs it is possible to extract single angular momentum states. The method of blocking operators and the diagonalization of mass matrices are used to suppress the mixing of different states. In this way one gets the correlations masses corresponding to definite spin and isospin quantum numbers.

The quantum numbers of the 1s-state are the same as those of the Higgs-particle; the quantum numbers of the 1p-state are identical with those of the $W$-boson. Therefore the operators used to investigate the 1s- and the 1p-state are the same operators which are used in the broken phase to investigate the Higgs- and $W$-boson, respectively. Therefore the 1s-mass is sometimes called Higgs-mass, the 1p-mass is called $W$-mass in the literature.

Philipsen, Teper and Wittig [15] calculated the correlation masses of the bound states at $\bar{\lambda}_3 = 0.0239$ and $\overline{m}_3^2 = 0.089$ on the lattice. They find a spectrum of higher excitations.

The masses of the low lying states are determined with high accuracy

$$
M_{1s}^{PTW} = (0.839 \pm 0.015) g_3^2 , \\
M_{1p}^{PTW} = (1.27 \pm 0.06) g_3^2 , \\
M_{2p}^{PTW} = (1.47 \pm 0.04) g_3^2 .
$$

Since the Wegner-Wilson-loops have not been measured by Philipsen et al., we have to use the potentials analyzed above to compare with our results. We do not have any data at the values of $\bar{\lambda}_3$ and $\overline{m}_3^2$ used in reference [15]. In view of the small $\overline{m}_3^2$-dependence of
Figure 4: The correlation mass of the 1s-state vs. the squared mass of the fundamental scalar. The points with error-bars are the lattice results of reference [13]. The full line connects the results of our model. Both data sets are normalized at \( m_3^2 = 0 \).

Since unfortunately the constant \( C \) of the potential cannot be deduced from the lattice data, we cannot compare the masses directly, but only the mass differences, as given in the table

|                         | \( (M_{1p} - M_{1s}) g_3^{-2} \) | \( (M_{2s} - M_{1s}) g_3^{-2} \) |
|-------------------------|----------------------------------|----------------------------------|
| our model               | 0.40                             | 0.65                             |
| Philipsen et al.        | \( 0.43 \pm 0.06 \)              | \( 0.63 \pm 0.05 \)              |

Within the errors the data predicted by our model agree very well with those from the lattice.

Gürtler et al. [13] have calculated the 1s-masses as well as an upper bound on the 1p-masses at \( \lambda_3 = 0.0957 \). There results for the 1s-masses for different scalar masses \( m_3^2 \) are plotted in figure 4 in comparison with our results. Both data sets are normalized at \( m_3^2 = 0 \). One sees that the dependence of the bound state mass on the mass of the fundamental scalar is well described by our model. The main contribution to the variation of \( M_{1s} \) is due to the explicit \( m_3^2 \)-dependence of the Green’s function (eq. (20)), while the modification of the potential with \( m_3^2 \) is not significant.

9 The intercept

The constant \( C \) in eq. (64) can not be fixed from lattice data of the Wegner-Wilson-Loop. One could choose it to give good agreement with the results of reference [13], resulting in \( C = -0.44 g_3^2 \). Similarly one could choose the constant to give the best possible agreement with the masses from reference [13]. This would lead to the slightly
larger value $C = -0.38 \, g_3^2$. The difference between these two values can in our opinion not be attributed to the different values of the quartic coupling. As we showed in section 7 the latter has only a small influence on the potential. It is rather due to a small discrepancy between the two lattice investigations, which is, however, within the statistical error.

In order to get some theoretical understanding of the origin of the intercept it seems to be necessary to investigate the vacuum structure of the theory. In the bound state model we have considered the binding forces between the fundamental scalars. The parameters were taken from lattice calculations of the Wegner-Wilson loops. The resulting spectra strongly support a picture of the effective 3-dimensional electroweak theory above the critical temperature which is very similar to QCD with light scalar quarks, the linear confinement playing an essential role. Though there is no analytic proof for confinement in (4-dimensional) QCD there exists a simple model which yields confinement for non-Abelian gauge theories in a very natural way, the model of the stochastic vacuum \[29,30\]. In appendix \[B\] we give the essential features of this model for a three-dimensional theory. The model yields an asymptotically linear potential and relates the vacuum expectation value of the gauge boson fields $\langle g_3^2 F F \rangle$ and their correlation lengths $a$ to the string tension. For small inter-quark separations the potential is quadratic and becomes linear only at a distance of a few correlation lengths $a$ and thus corresponds to the situation depicted in figure 3b. If we denote by $D(z^2)$ the (rotationally invariant) correlation function of the gluon fields in the vacuum (see eq. (79)) the string tension is given by:

$$\sigma = \frac{\pi \langle g_3^2 F F \rangle}{12 N_C} \int_0^\infty dz \, z \, D(z^2)$$

whereas the effective intercept $C = V_0$ (see figure 3b) is given by:

$$V_0 = - \frac{\langle g_3^2 F F \rangle}{6 N_C} \int_0^\infty dz \, z^2 \, D(z^2)$$

From these equations we can deduce:

$$V_0 = - K \sigma a$$

where $a$ is defined by

$$a = \int_0^\infty dz \, D(z^2)$$

and $K$ is of the order 1, the numerical value depends on the form of $D(z^2)$. Unfortunately neither the condensate $\langle g_3^2 F F \rangle$ nor the correlation length have been calculated on the lattice so far, so we apply here a simplifying argument in order to get an idea of the order of magnitude of $V_0$.

If $a$ is the correlation length between two field strengths $F$ we expect the correlation between the product of two field strengths $F^2$ to be of order $a/2$. Since the product $F^2$ interpolates a glue-ball (W-ball) we expect $2/a$ to be near the glue-ball (W-ball) mass. In QCD this seems to be fulfilled qualitatively, with $2/a \approx 1.5$ GeV. The W-ball mass has been obtained in lattice calculations \[13\] and found to be $m_G = 1.60 \pm 0.04 \, g_3^2$. Assuming the correlation length to be $a = 2/m_G$ we obtain together with the string tension of the lattice calculations (cf. tab. \[1\]) the value $V_0 = -0.17 K \, g_3^2$ which is indeed of the same order.

\footnote{Taking $a/4$ (corresponding to 4 gluons in the glueball \[35\]) would improve the agreement discussed below. We thank O. Philipsen for a discussion of this point.}
of magnitude as the constant $C$ which we need to fix the absolute values of the bound-state masses. It would be extremely interesting to determine the correlator $\langle F(x)T(x,0)F(0) \rangle$ on the lattice in order to see if considerations similar as in QCD lead to confinement in this 3-dimensional theory.

10 Discussion and conclusions

The comparison with the lattice data in the last section shows that the correlation masses of physical (gauge invariant) objects in the hot electroweak phase are described well by the proposed model. It turns out that it is similar to QCD but without the problem of chiral symmetry breaking and spin-spin-interactions. The fact that the dense spectrum of higher excitations predicted by us in reference [19] has been confirmed by lattice calculation is in our opinion a success of the model on its own. The numerical agreement of the mass-differences is much better than it could be expected in view of the approximations made. Even the $m^2_3$-dependence of the 1s-mass is explained by the model. In consideration of this fact we will discuss the assumptions of our calculations again.

The Green’s function of the fundamental scalar (eq. (16)) does not take into account the interactions with other scalars (quenched approximation). Nevertheless, the influence of the dynamical scalars is taken into account. The fact that the scalar-scalar-potential measured on the lattice is not the same for all data sets is due the dynamical scalars. The $\lambda_3$-dependence is small, as far as we can conclude this from the existing data. The modification of the potential as function of the mass $m^2_3$ is only large for very small values of this mass. The direct perturbative exchange of a scalar can be taken into account like the perturbative gauge boson contribution. It is, however, suppressed by a factor $\lambda^2_3$ compared to the latter one. The scalar contributions to the scalar self-energy is neglected by the quenched approximation as well. All these terms are suppressed by at least a factor $\lambda^2_3$ when compared to those contributions taken into account. This approximation is therefore well-founded. Only the influence of the dynamical scalars of the scalar-scalar-potential for very small squared masses $m^2_3$ is of numerical relevance; this effect is included in our calculations.

In the calculation of the correlation masses from the Green’s function (eq. (20)) in section 5 we made two approximations: the choice of the classical path for the center of mass trajectory and the modified area law.

By the restriction to a straight and steady movement of the center of mass $R(\gamma)$ fluctuations of the bound state as a unit are excluded. This approximation is well-founded for heavy bound states. As we have seen, the typical scale of the problem is the string tension. The masses of the bound states are $\mathcal{M} \gtrsim 3\sqrt{\sigma}$ (cf. fig. 2) and hence larger than this scale. It might, nevertheless, be interesting to check this approximation or to calculate the next order corrections. Note that the relative movement of both fundamental scalars is treated fully relativistically in spite of the restriction to the center of mass path.

In this context the obvious question arises why it is possible to reduce the treatment of the relative movement to a Schrödinger equation. Here it is important to note that it is only an analogous non-relativistic problem which is described by this equation. The parameters $\mu$ and $\bar{\mu}$ then have the meaning of constituent masses. They are averaged by the Schwinger-proper-time integral later on. The evaluation of this integral with the saddle point method is exact in the limit $\Theta \to \infty$ investigated by us. Neither the non-relativistic treatment of the analog problem nor the saddle point method is based on any
approximations. One may, at most, ask if the limit $\Theta$ to infinity can be extrapolated by the lattice investigations we are comparing with.

The modified area law is included in our calculations of the correlation masses. The correlation with the usual area law was already discussed in section 5. An extension to a potential which depends on all three components of $u$ does not cause any problem, is, however, not adequate since the connection with the scalar-scalar-potential calculated on the lattice would get lost.

Finally we neglected the effect of the screening. The Wegner-Wilson-loops calculated on the lattice show no screening of the potential up to distances of about $8g_3^{-2}$. In figure 5 we plotted $V(r)$ in comparison with the bound state wave functions. One sees that the wave functions corresponding to the lowest quantum numbers have extensions between $8g_3^{-2}$ and $10g_3^{-2}$. To exclude screening effects at all the linear part of the potential has to be confirmed at distances which are slightly larger than those of today's data. It is, on the other hand, not expected that the lowest excitations are influenced by screening effects.

The very good agreement of the correlation masses calculated by us with the lattice data justifies our approximations a posteriori. As long as the errors of the lattice data do not get smaller than today we see no need to go along without these assumptions.

A more interesting question is the calculation of the scalar-scalar-potential. The Bessel-function contribution (eq. (64)) seems to originate in the exchange of a colored massive particle or quasi-particle. The mass is of order $g_3^2$; the nature of the exchanged particle is not yet revealed. As explained in appendix $\mathbb{B}$ the string tension and the intercept can in principle be calculated with the model of the stochastic vacuum. The gluon correlator is an important input function for this model, which needs confirmation from the lattice.

Another interesting point, which is not fully clarified yet, is the treatment of negative values of $m_3^2$. It is not attractive to have a fundamental scalar with negative squared mass. One could try to use the perturbative mass calculated from the second derivative of the effective potential at $\varphi=0$ instead. The latter one diverges, however, due to the $\varphi^2 \ln(\varphi)$-term of the potential. This infrared-divergence shows the breakdown of the perturbation
theory in the symmetric phase. A much more speculative possibility is the generation of a scalar mass by a gluon-condensate, similar to the generation of a gauge boson mass by a scalar condensate in the broken phase. A gluon condensate due to the instability of the naive vacuum for small Higgs vev’s should also be very important for the generation of the non-perturbative part of the electroweak effective potential.

An important outcome of this investigation is that the hot electroweak phase is strongly interacting like a pure 3-dimensional Yang-Mills-theory. The spectrum of gauge invariant states is, however, totally different from the latter one. The existence of low-lying 1s- and 1p-states is only due to the fundamental scalars; the gauge invariant spin 0- and spin 1-operators of eq. (11) do not exist in pure Yang-Mills-Theories. The gauge bosons in the pure gauge theory are only defined in a definite gauge while the gauge-invariant spin 1-object is a W-ball. Indeed there is the interesting observation from lattice calculations [15, 31] that glueballs and Higgs states are practically decoupled. The string tension is universal in a given confining theory and thus the former remark fits very well together with the fact mentioned above that the string tension responsible for $\Phi$-binding is nearly the one of pure Yang-Mills theory in lattice calculations.

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A Perturbative contributions to the potential

Using the cumulant expansion [32] the expectation value of the rectangular Wegner-Wilson-loop can be expressed as

$$ W(r,T) = \text{Tr} \exp \left( -\frac{g^2}{2} \int dx \int dy \langle \langle A_i(x) A_j(y) \rangle \rangle + \text{higher cumulants} \right). $$

To lowest order in this expansion one has

$$ \ln W(r,T) = -\frac{g^2}{4} \int dx \int dy \langle \langle A^a_i(x) A^a_j(y) \rangle \rangle. $$

The perturbative contribution is now calculated from the exchange of a perturbative gauge boson between the two long sides of the loop. With $\vec{x}$ and $\vec{y}$ as two-dimensional vectors and $x_1$ and $y_1$ the corresponding components running along the long sides one finds

$$ \lim_{T \to \infty} \frac{1}{T} \ln W(r,T) = -\frac{g^2}{4} \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} dx_1 \int_{-T/2}^{T/2} dy_1 \langle \langle A^a_1(x_1, \vec{x}) A^a_1(y_1, \vec{y}) \rangle \rangle $$

$$ = \frac{g^2}{4} \int_{-\infty}^{\infty} dy_1 \langle \langle A^a_1(0, \vec{x}) A^a_1(y_1, \vec{y}) \rangle \rangle, $$

where we have used translational invariance in the last line.

The two-point-cumulant is now replaced by the perturbative propagator in Feynman-gauge via

$$ \langle \langle A_i^a A_j^b \rangle \rangle \to \delta^{ab} \delta_{ij} \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 + m^2} e^{ik(y-x)}. $$
We split the momentum coordinates in a first component \( k_1 \) and a two dimensional vector \( \vec{k} \). The perturbative part of the potential is

\[
V_{\text{pert}}(\vec{y} - \vec{x}) = \frac{g_3^2}{4} \int_{-\infty}^{\infty} dy_1 \int \frac{d^3k}{(2\pi)^3} \frac{1}{k_1^2 + k^2 + m^2} e^{i(k_1 y_1 + \vec{k}(\vec{y} - \vec{x}))}
\]

(74)

It is thus the two-dimensional Fourier-transformation of the perturbative propagator in momentum space with \( k_1 = 0 \). The derivation holds for \( m = 0 \) as well.

B The model of the stochastic vacuum in three dimensions

We introduced a model of bound states in the hot electroweak phase and showed how to calculate the correlation masses within this model. The knowledge of the scalar-scalar-potential \( V(r) \) is an important requirement of this calculation. The origin of the asymptotically linear potential itself has not been explained so far. This can be attempted by adopting the model of the stochastic vacuum, which was originally proposed by Dosch and Simonov \cite{[23], [30]} for QCD, to the three dimensional SU(2)-Higgs-model.

The basic ingredient of the model of the stochastic vacuum is the assumption that the complicated contributions of non-perturbative field configurations can be approximated by a simple stochastic process. In that way already the assumption that this process has a convergent linked cluster expansion leads very naturally to linear confinement. Making the more restrictive approximation of a Gaussian stochastic process leads to a very predictive model since now in principle the full non-perturbative contribution is approximated by a single correlator. This correlator can be used to determine observables as the static quark-antiquark potential in QCD, soft high energy cross sections and others. It can also be compared with lattice calculations and all results turned out to be very satisfactory.

In the following we shortly exhibit basic features of the model adopted to three dimensions and SU(\( N_c \)) as gauge group with \( N_c = 2 \), the case considered here. In three dimensions we can introduce besides the field tensor \( F_{ij} \) the vector

\[
\tilde{F}_k^a = \epsilon_{ijk} F_{ij}^a
\]

(76)

which in an Abelian theory satisfies the Bianchi identity

\[
\partial_k \tilde{F}_k^a = 0
\]

(77)

In order to form gauge invariant correlators of the field strength tensors we have to transport the color content of all fields at point \( x \) to a fixed reference point \( y \). This is done by the gauge field transporter \( T(x, y) \) (eq. (10)) along a straight line from \( x \) to \( y \).

\[
F_{ij}^a(x, y) \frac{\tau^a}{2} = T(x, y)^{-1} F_{ij}^a(x) \frac{\tau^a}{2} T(x, y)
\]

(78)

The correlator \( \langle g_3^2 F_{ij}^a(x, y) F_{kl}^b(x', y) \rangle_A \) depends in general on the reference point \( y \). We make the assumption that it can be approximated by an expression depending only on the difference \( z = x - x' \).

\[
\langle F_{ij}^a(x, y) F_{kl}^b(x', y) \rangle_A = \frac{\delta^{ab}}{6(N_c^2 - 1)} \langle FF \rangle \left[ \kappa (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) D(z^2) + (1 - \kappa) \left( \frac{1}{2} \frac{\partial}{\partial z_i} (z_k \delta_{jl} - z_l \delta_{jk}) + \frac{1}{2} \frac{\partial}{\partial z_j} (z_l \delta_{ik} - z_k \delta_{il}) \right) D_1(z^2) \right].
\]

(79)
Here \(\langle FF\rangle\) is the gluon condensate

\[
\langle FF\rangle = \sum_{ij} \sum_a \langle F^a_{ij}(0)F^a_{ij}(0)\rangle_A
\]  

(80)

and \(D\) and \(D_1\) are correlation functions which are supposed to fall off with a characteristic correlation length \(a\).

In an Abelian gauge (without monopoles) the Bianchi identity (eq. (77)) forces the first contribution to vanish, i.e. \(\kappa=0\), but in a non-Abelian theory we have no longer the Bianchi identity and hence there is no reason that the correlator \(D(z^2)\) vanishes. Lattice results [33] showed \(\kappa \approx 0.74\) in QCD. No lattice results are available for the 3-dimensional correlator.

From the correlator one can obtain the area law by first transforming the line integral over the potential \(A^a_i\) into a surface integral over the field strength \(F^a_{ij}\) by means of the non-Abelian Stokes theorem and then applying the cluster expansion. For subtleties due to the path ordering we refer to reference [34].

For a rectangular Wegner-Wilson-loop of extension \(T\) in \(z_0\)-direction and \(r\) in \(z_1\)-direction we obtain:

\[
\ln W(r, T) = \ln N_c - \frac{g^2\langle FF\rangle}{6N_c} \left[ \int_0^T dz_0 \int_0^r dz_1 \kappa (rT - rz_0 - Tz_1 + z_0z_1) D(z^2) \right.

+ \left. (1-\kappa) \left( \frac{1}{2}Tz_1 + \frac{1}{2}rz_0 - z_1z_0 \right) D_1(z^2) \right]
\]

(81)

For small values of \(r\) and \(T\) the non-constant term in \(\ln W\) behaves as \(-\frac{1}{24N_c}g^2\langle FF\rangle r^2T^2\), independent of the value of \(\kappa\), whereas for large values of \(r\) and \(T\) only the correlator \(D(z^2)\) gives an area law.

The static potential is obtained from \(W(r, T)\) by the limit of eq. (34). It increases quadratically in \(r\) for \(r \ll a\) and linearly for \(r \gg a\) (cf. fig. 3b). Asymptotically, i.e. for \(r \gg a\), the potential \(V(r)\) can be written in the form of eq. (57). The string tension is determined by the gluon condensate and the correlation length via eq. (65). The value of the intercept \(V_0\) depends on the value of \(\kappa\) and the form of the correlators \(D\) and \(D_1\). For \(\kappa = 1\) one obtains eq. (66).

Therefore the linear rising of the potential at large distances as well as a negative intercept can be explained within the model of the stochastic vacuum. Note that the model of the stochastic vacuum explains only the non-perturbative part of the potential. The perturbative part has to be added. In the three-dimensional case it is, however, not easy to distinguish from the non-perturbative part because the squared gauge coupling has dimension of a mass.
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