A Central Limit Theorem For
Linear Random Fields

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1 Introduction

Consider a two-dimensional, linear random field, say
\[ X_{j,k} = \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} a_{r,s} \xi_{j-r,k-s} = \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} a_{j+r,k+s} \xi_{r-s}, \]
where \( a_{r,s}, r, s \in \mathbb{Z}, \) are square summable, \( \xi_{r,s}, r, s \in \mathbb{Z}, \) are i.i.d. with mean 0 and unit variance, and \( \mathbb{Z} \) denotes the integers. It is convenient to regard the array \( a = (a_{r,s} : r, s \in \mathbb{Z}) \) as an element of \( \ell^2(\mathbb{Z}^2). \) Let \( F \) denote the common distribution function of the \( \xi_{r,s} \) and \( (\Omega, \mathcal{A}, P) \) the probability space on which they are defined. If \( \Gamma \) is a finite subset of \( \mathbb{Z}^2, \) let
\[ S = S(a, \Gamma) = \sum_{(j,k) \in \Gamma} X_{j,k} \]
and
\[ \sigma^2 = \sigma^2(a, \Gamma) = E(S^2), \]
and suppose that \( \sigma^2 > 0. \) (Of course, \( S \) depends on \( \omega \in \Omega \) too, but this dependence is suppressed. As indicated dependence on \( a \) and \( \Gamma \) will only be displayed when needed for clarity.) Then
\[ S = \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} b_{r,s} \xi_{r-s}, \]
where
\[ b_{r,s} = b_{r,s}(a, \Gamma) = \sum_{(j,k) \in \Gamma} a_{j+r,k+s}, \]
and
\[ \sigma^2 = \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} b_{r,s}^2, \]
assumed to be positive. Let $\Phi$ denote the standard normal distribution and

$$G(z) = G(z; a, \Gamma, F) = P \left[ \frac{S}{\sigma} \leq z \right], \; z \in \mathbb{R}.$$ 

Sufficient conditions for $G$ to be close to $\Phi$ are developed.

There has been some recent work on the Central Limit Theorem for sums of linear processes with values in a Hilbert space [5]. Other recent work on the Central Limit Theorem for linear random fields has emphasized the Beveridge Nelson decomposition, developed by Phillips and Solo [7] for processes. This approach leads to functional versions of the CLT, [4], [6], and their references. Our approach follows that of Ibragimov [3]. For the case in which $\Gamma$ is a rectangle, it requires no additional conditions on the coefficients, but does require the innovations to be independent and does not deliver a functional version.

## 2 Generalities

Let

$$\rho = \rho(a, \Gamma) = \max_{r,s \in \mathbb{Z}} \frac{|b_{r,s}|}{\sigma}. \quad (1)$$

Interest in $\rho$ stems from the following:

**Proposition 1** Let $\mathcal{H}$ denote a class of distribution functions for which

$$\int_{\mathbb{R}} xH\{dx\} = 0, \quad \int_{\mathbb{R}} x^2H\{dx\} = 1, \text{ for all } H \in \mathcal{H}, \quad (2)$$

$$\lim_{c \to \infty} \sup_{H \in \mathcal{H}} \int_{|x| > c} x^2H\{dx\} = 0; \quad (2')$$

Then $\forall \epsilon > 0, \exists \delta = \delta_{\epsilon, \mathcal{H}},$ depending only on $\epsilon$ and $\mathcal{H}$ for which for which

$$d(G, \Phi) := \sup_{z} |G(z) - \Phi(z)| \leq \epsilon \quad (3)$$

for all all $F \in \mathcal{H}$ for all arrays $a$ and regions $\Gamma \subset \mathbb{Z}^2$ for which $\rho \leq \delta$.

**Proof.** Let $\hat{\cdot}$ denote Fourier transform (characteristic function), so that $\hat{F}(t) = \int_{\mathbb{R}} e^{itx} F\{dx\}$, and

$$L(\eta) = \frac{1}{\sigma^2} \sum_{r,s \in \mathbb{Z}} \int_{|b_{r,s}x| > \eta \sigma} |b_{r,s}x|^2 F\{dx\}.$$
for \( \eta > 0 \). Then, for any \( \eta > 0 \), \(|\hat{G}(t) - \hat{\Phi}(t)| \leq \eta|t|^3/6 + t^2L(\eta) \) for all \( t \in \mathbb{R} \) from the proof the Central Limit Theorem for independent summand (\[1\], pp. 359 - 361), and

\[
\sup_z |G(z) - \Phi(z)| \leq \frac{1}{\pi} \int_T^{-T} \left| \frac{\hat{G}(t) - \hat{\Phi}(t)}{t} \right| \, dt + \frac{24}{\pi \sqrt{2\pi T}} \tag{4}
\]

for any \( T > 0 \) by the smoothing inequality (\[2\], pp. 510 - 512). Given \( \epsilon \), let \( T_\epsilon = \frac{96}{\epsilon^2} \) and \( \eta_\epsilon = 4\epsilon/T_\epsilon^3 \). Then the left side of (4) is at most \( \frac{1}{2}T^2L(\eta_\epsilon) + \frac{1}{2}\epsilon \). Next, let \( J(c) = \sup_{H \in \mathcal{H}} \int_{|x|>c} x^2H \{ dx \} \) for \( c > 0 \), so that \( J(c) \to 0 \) as \( c \to \infty \) by (2); and let \( J^\#(z) = \inf\{c > 0 : J(c) \leq z\} \) for \( z > 0 \). Then

\[
L(\eta) \leq \sum_{r \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} b_{r,s}^2 \int_{|x| > \eta/\rho} x^2 \{ dx \} \leq J\left( \frac{\eta}{\rho} \right),
\]

and \( \delta = \eta_\epsilon/J^\#(T_\epsilon^{-2}\epsilon) \) has the desired properties. ♦

Next, let

\[
\|a\|_p = \left[ \sum_{r,s \in \mathbb{Z}} |a_{r,s}|^p \right]^{1/p} \leq \infty
\]

for \( 1 \leq p \leq 2 \). Thus, \( \|a\|_2 \) is assumed to be finite and \( \|a\|_p \) may be finite for some value of \( p < 2 \). In terms of \( \|a\|_p \) there is a simple bound on \( \rho \),

\[
\rho \leq \frac{\|a\|_p \Gamma^{1/2}}{\sigma}, \tag{5}
\]

where \( q \) denotes the conjugate, \( 1/p + 1/q = 1 \) and \( \#\Gamma \) denotes the cardinality of \( \Gamma \). In particular, \( \rho \leq \|a\|_1/\sigma \). This leads to:

**Corollary 1** Let \( \mathcal{H} \) be as in Proposition 1. If \( \|a\|_1 < \infty \), then \( \forall \epsilon > 0 \), \( \exists \kappa = \kappa(\epsilon, \mathcal{H}) > 0 \) for which (3) holds whenever \( \sigma \geq \kappa\|a\|_1 \) and \( F \in \mathcal{H} \).

**Proof.** For \( \delta \) as per in Proposition 1, pick \( \kappa \) such that \( \kappa \delta \leq 1 \). The result is then an easy consequence of the proposition and the fact that \( \rho \leq \|a\|_1/\sigma \). ♦

### 3 Rectangles

To bound \( \rho \), suppose that the maximum occurs when \( r = r_0 \) and \( s = s_0 \), say

\[
|b_{r_0,s_0}| = \max_{r,s} |b_{r,s}|,
\]
and let $\Delta b_{u,v} = b_{u,v} - b_{u,v-1} - b_{u-1,v} + b_{u-1,v-1}$ for $(u, v) \in \mathbb{Z}^2$. Then

$$b_{r_0+r,s_0+s} - b_{r_0,s_0+s} - b_{r_0+r,s_0} + b_{r_0,s_0} = \sum_{r=r_0+1}^{r_0+r} \sum_{s=s_0+1}^{s_0+s} \Delta b_{u,v}$$ \hspace{1cm} (6)

for $r, s \geq 1$. Let

$$Q_{m,n} = \sum_{r=1}^{m} \sum_{n=1}^{n} \sum_{u=r_0+1}^{r_0+r} \sum_{v=s_0+1}^{s_0+s} |\Delta b_{u,v}| = \sum_{r=r_0+1}^{r_0+m} \sum_{s=s_0+1}^{s_0+n} (r - r_0)(s - s_0)|\Delta b_{r,s}|$$ \hspace{1cm} (7)

for $m, n \geq 1$. Then $|b_{r_0,s_0}| \leq |b_{r_0+s_0}| + |b_{r_0+r,s_0}| = \sum_{r=r_0+1}^{r_0+r} \sum_{s=s_0+1}^{s_0+s} |\Delta b_{u,v}|$ for all $r, s \geq 1$ and, therefore,

$$mn|b_{r_0,s_0}| \leq \sum_{r=1}^{m} \sum_{s=1}^{n} M_{r_0+s_0} = Q_{m,n}$$

for all $m, n \geq 1$. Here

$$\sum_{r=1}^{m} \sum_{s=1}^{n} |b_{r_0+r,s_0+s}| \leq \sqrt{mn} \sqrt{\sum_{r=1}^{m} \sum_{s=1}^{n} b_{r_0+r,s_0+s}^2} \leq \sqrt{mn}\sigma,$$

and similarly, $\sum_{r=1}^{m} \sum_{s=1}^{n} |b_{r_0,s_0+s}| \leq m\sqrt{n}\sigma$ and $\sum_{r=1}^{m} \sum_{s=1}^{n} |b_{r_0+r,s_0}| \leq \sqrt{mn}\sigma$. So, $mn|b_{r_0,s_0}| \leq \sqrt{mn}\sigma + m\sqrt{n}\sigma + \sqrt{mn}\sigma + Q_{m,n}$. That is,

$$\rho = \frac{|b_{r_0,s_0}|}{\sigma} \leq (\frac{2}{\sqrt{m}} + \frac{2}{\sqrt{n}}) + \frac{Q_{m,n}}{mn\sigma}$$ \hspace{1cm} (8)

for any $m, n \geq 1$. The first two terms may be made small by taking $m$ and $n$ large. Thus, the issue is $Q_{m,n}$. Suppose now that $\Gamma$ can be written as the union of $\ell$ non-empty pairwise mutually exclusive rectangles,

$$\Gamma = \bigcup_{i=1}^{\ell} \{(j, k) : M_i \leq j \leq M_i, N_i \leq k \leq N_i\}.$$

**Proposition 2** If $\Gamma$ is of the form (3), then where $M, N \geq 1$, then

$$\rho \leq 12 \left(\frac{\sqrt{\ell}||a||_2}{\sigma}\right)^\frac{1}{\ell} + \frac{8\sqrt{\ell}||a||_2}{\sigma}.$$ \hspace{1cm} (10)

**Proof.** In this case $b_{r,s} = \sum_{i=1}^{\ell} b_{r,s}^{(i)}$, where $b_{r,s}^{(i)} = \sum_{j=M_i}^{M_i} \sum_{k=N_i}^{N_i} a_{j+r,k+s}$, $\Delta b_{r,s} = \sum_{i=1}^{\ell} \Delta b_{r,s}^{(i)}$, and $\Delta b_{r,s}^{(i)} = a_{r+M_i,s+N_i} - a_{r+M_i,s+N_i} - a_{r+M_i,s+N_i} + a_{r+M_i,s+N_i}$. So,

$$Q_{m,n} \leq mn \sum_{r=1}^{m} \sum_{s=1}^{n} \sum_{i=1}^{\ell} (|a_{r+M_i,s+N_i} - a_{r+M_i,s+N_i} - a_{r+M_i,s+N_i} + a_{r+M_i,s+N_i}|) \leq 4(mn)^2 \sqrt{\ell}||a||_2,$$
by Schwartz’ Inequality, and

$$\rho \leq \left(\frac{2}{\sqrt{m}} + \frac{2}{\sqrt{n}}\right) + \frac{4mn\ell\|a\|_2}{\sigma}$$

for any $m, n \geq 1$. Letting $m = n = \lceil(\sigma/\sqrt{\ell\|a\|_2})^{\frac{1}{2}}\rceil$, the least integer that exceeds $(\sigma/\sqrt{\ell\|a\|_2})^{\frac{1}{2}}$ then leads to (10).

When specialize to (intersections of) rectangles (with $\mathbb{Z}^2$), the proposition provides a complete analogue of Ibragimov’s theorem [3] with lots of uniformity.

**Corollary 2.** Let $\mathcal{H}$ be as in Proposition 1 and let $\mathcal{R}_\kappa$ be the collections of pairs $(a, \Gamma)$ for which $\|a\|_2 > 0$, $\Gamma$ is the $a$ rectangle, and $\sigma(a, \Gamma) \geq \kappa\|a\|_2$. Then, as $\kappa \to \infty$, the distributions of $S/\sigma$ converge to the $\Phi$ uniformly with respect to $(a, \Gamma) \in \mathcal{R}_\kappa$ and $F \in \mathcal{H}$

**References**

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