THE FIELD OF MODULI OF A QUADRANGULAR RIEMANN SURFACE

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ABSTRACT. It is well known that every closed Riemann surface of genus 0 or 1 can be defined over its field of moduli and that such fields are easily computable (this being \( \mathbb{Q} \) for the genus zero case). For genus at least two, both, the computation of the field of moduli and to determine if it is a field of definition, are in general very difficult tasks. It was proved by Wolfart that if \( S \) is Riemann surface \( S \) of genus \( g \geq 2 \) for which \( S/\text{Aut}(S) \) has signature \((0; a, b, c)\), then it can be defined over its field of moduli. In this paper we note that if there is some \( H < \text{Aut}(S) \) with \( S/H \) of signature \((0; a, b, c, d)\) with \( d \notin \{a, b, c\} \), then \( S \) can also be defined over its field of moduli.

1. Introduction

As a direct consequence of the Implicit Function Theorem and the Riemann-Roch Theorem \[7\], there is a natural one-to-one correspondence between birational isomorphism classes of non-singular irreducible projective complex algebraic curves and conformal classes of closed Riemann surfaces. It is this equivalence which allows us to work at the analytical and at the algebraic settings in a parallel way. If \( C_1 \) and \( C_2 \) are non-singular complex irreducible projective algebraic curves, we denote by the symbol \( C_1 \cong C_2 \) to indicate that they are birationally equivalent (that is, the corresponding closed Riemann surfaces are conformally equivalent) and we say that \( C_1 \) and \( C_2 \) are isomorphic.

A field of definition of a closed Riemann surface \( S \) is a subfield \( K \) of \( \mathbb{C} \) for which it is possible to find a non-singular irreducible projective complex algebraic curve \( C \), whose Riemann surface structure is conformally equivalent to \( S \), defined by homogeneous polynomials with coefficients in \( K \); it is said that \( C \) is definable over \( K \). The field of moduli of \( S \), denoted by \( \mathcal{M}(S) \), may be defined, by the results in \[17\], as the intersection of all of the fields of definition of \( S \) (see Section \[2\]).

If the genus of \( S \) is zero, then \( S \) is conformally equivalent to the Riemann sphere \( \hat{\mathbb{C}} \); so it can be defined over its field of moduli \( \mathbb{Q} \). If \( S \) has genus one, then it can be described by a curve in its Legendre normal form \( E_\lambda : y^2 = x(x-1)(x-\lambda) \), where \( \lambda \in \mathbb{C} - \{0, 1\} \). As any two conformal automorphisms of order two with fixed points (necessarily four fixed points) of \( E_\lambda \) are conjugate in the group of conformal automorphisms of \( E_\lambda \), it follows that \( \mathcal{M}(E_\lambda) = \mathbb{Q}(j(\lambda)) \), where \( j \) is the classical \( j \)-function. It is also known that \( E_\lambda \) can be defined over \( \mathbb{Q}(j(\lambda)) \) \[21\] Chapter III, Prop. 1.4.

Let us assume, from now on, that \( S \) has genus at least two. If \( \text{Aut}(S) \) denotes the full group of conformal automorphisms of \( S \), then \( |\text{Aut}(S)| \leq 84(g-1) \) (Hurwitz’s bound) \[15\]. In this case, \( \mathcal{M}(S) \) is not in general a field of definition of \( S \), as it is shown by explicit examples provided by Earle \[9\] and Shimura \[20\] in the case of hyperelliptic curves and by the author \[12\] in the non-hyperelliptic curves category. Sufficient conditions for \( S \) to be definable over its field of moduli are given by Weil’s Galois descent theorem \[24\] (see Section \[2\]). If \( \text{Aut}(S) \) is trivial, then (a direct consequence of Weil’s Galois descent theorem) \( S \) can be defined over its field of moduli. Unfortunately, Weil’s conditions are in general very difficult to check if \( \text{Aut}(S) \) is non-trivial and, in general, the computation of the field of moduli of a Riemann surface \( S \), admitting non-trivial automorphisms, is a difficult task. Moreover, if we have computed explicitly the field of moduli, to determine if \( S \) can be defined over it is also a difficult problem (except for some simple cases). Even, if we already have explicitly the field of moduli and we know that the surface can be defined over it, it is a very hard problem to compute an algebraic curve defined over it representing it.

If \( S/\text{Aut}(S) \) has signature of the form \((0; a, b, c)\) (one says that \( S \) is quasiplatonic), then Wolfart \[26\] proved that \( S \) can be defined over their field of moduli (which is known to be a number field by Belyi’s theorem \[11\]).

1991 Mathematics Subject Classification. 30F10, 14H37, 14H45, 14G99.
Partially supported by project Fondecyt 1110001 and UTFSM 12.11.01.
A closed Riemann surface $S$ is called \textit{quadrangular} if there is some $H < \text{Aut}(S)$ so that $S/H$ has genus zero and exactly 4 cone points. If moreover one of the four cone points of $S/H$ has order different from the orders of the others three cone points, then we say that $S$ is \textit{quadrangular quasiplatonic}. We prove that quadrangular quasiplatonic Riemann surfaces are definable over their fields of moduli.

\textbf{Theorem 1.} Let $S$ be a quadrangular closed Riemann surface of genus at least two. Let $H < \text{Aut}(S)$ be so that $S/H$ has signature of the form $(0; a, b, c, d)$.

1. If $d \notin \{a, b, c\}$, that is, if $S$ is quadrangular quasiplatonic, then $S$ can be defined over its field of moduli.
2. If $a = c$, $b = d$, $a \neq b$, then $S$ can either be defined over its field of moduli or over a degree two extension of it.
3. If $a = b = c = d$, then $S$ can either be defined over its field of moduli or over an extension of degree at most four of it.

Part (1) of Theorem 1 states that each quadrangular quasiplatonic Riemann surface can be defined over its field of moduli. A particular case of a quadrangular quasiplatonic Riemann surface is a closed Riemann surface $S$ with $S/\text{Aut}(S)$ of signature of the form $(0; a, b, c, d)$, where $d \notin \{a, b, c\}$.

\textbf{Corollary 2.} Let $S$ be a closed Riemann surface with $S/\text{Aut}(S)$ of signature of the form $(0; a, b, c, d)$, where $d \notin \{a, b, c\}$. We may assume the the cone points of $S/\text{Aut}(S) = \mathbb{C}$ are $\infty$, $0$, $1$ and $\lambda$, where $\lambda$ has the order $d$. Then $M(S)$ is an finite extension of $\mathbb{Q}(j(\lambda))$, where $j$ is the elliptic $j$-function $j(\lambda) = (1 - \lambda + \lambda^2)^3/\lambda^2(1 - \lambda)^2$.

In [23] Fuertes-Streit studied the families of genus three Riemann surfaces $S$ admitting a group of conformal automorphisms $H$ with quotient being the sphere with 4 branch values. In each of these cases the quotient $S/H$ is one of the followings: $(0; 2, 2, 2, 3)$, $(0; 2, 2, 2, 4)$, $(0; 2, 2, 2, 6)$ and $(0; 2, 3, 3, 6)$ and, by Theorem 1 each of them can be defined over its field of moduli. In the same paper, even the computation of the corresponding fields of moduli are not explicitly given, the main results in order to obtain them are provided. As a matter of example, we provide all computations in order to obtain the field of moduli for one of these cases (see Section 3 in [9]). We also provide some information at the level of Fuchsian uniformizations. The same arguments can be done for the others cases in order to compute the corresponding fields of moduli.

In [23] Swinarski has computed explicit algebraic curves for those closed Riemann surfaces $S$ of genus four admitting a group $H < \text{Aut}(S)$ so that $S/H$ has quadrangular signature. These quadrangular signatures are given by $(0; 2, 2, 2, 3)$, $(0; 2, 2, 2, 4)$, $(0; 2, 2, 2, 5)$, $(0; 2, 2, 2, 8)$ and $(0; 2, 2, 3, 3)$. In this way, with the exception of the last signature, $S$ can be defined over its field of moduli. In the left case, that is, when $S/H$ has signature $(0; 2, 2, 3, 3)$, $S$ can be defined over its field of moduli or over an extension of degree two of it. The computations of the corresponding field of moduli may be done with the help of the provided curves in that paper.

Next we consider a particular class of quadrangular surfaces obtained by using Abelian groups. Let us now assume that there is an Abelian group $H < \text{Aut}(S)$ so that $S/H$ has signature of the form $(0; a, b, c, d)$. We say that $S$ is a homology cover of the Riemann orbifold $S/H$ if there is no other closed Riemann surface $R$ admitting an Abelian group $K < \text{Aut}(R)$ with $R/K = S/H$ and containing a non-trivial subgroup $L < K$ (acting freely on $R$) so that $R/L = S$. In other words, $S$ is a homology cover if it is a higher Abelian regular branched cover over $S/H$. An equivalent definition, using Fuchsian groups, is as follows. If $\Gamma$ is a Fuchsian group uniformizing $S/H$ and $S$ is uniformized by the derived subgroup $\Gamma'$, then we say that $S$ is the homology cover of $S/H$.

\textbf{Corollary 3.} A homology cover of a Riemann orbifold with signature of type $(0; a, b, c, d)$ can be defined over its field of moduli.
In [8] it is obtained that a homology cover $S$ of an orbifold with signature of the form $(0; a, a, a, a)$, with $a \geq 3$, admits a unique group $H < \text{Aut}(S)$ so that $H \cong \mathbb{Z}_a^3$ and $S/H$ an orbifold with signature $(0; a, a, a, a)$. This uniqueness property permits to compute explicitly the field of moduli (notice the familiarity with the case of elliptic curves). If we assume the cone points of $S/H$ to be $\infty$, 0, 1 and $\lambda \in \{0, 1\}$, then $S$ can be represented by the following projective curve.

$$C_\lambda = \left\{ \begin{array}{l} x_1^3 + x_2^3 + x_3^3 = 0 \\ \lambda x_1^3 + x_2^3 + x_3^3 = 0 \end{array} \right\} \subset \mathbb{P}_C^3.$$

In this model, the group $H$ is generated by $a_1$, $a_2$ and $a_3$, where $a_j$ is multiplication by $e^{2\pi i / a_j}$ at the coordinate $x_j$.

**Theorem 4.** Let $S$ be a closed Riemann surface which is the homology cover of an orbifold $O$ with signature $(0; a, a, a, a)$. If the conical points of $O$ are given by, up to conformal equivalence, by $\infty$, 0, 1 and $\lambda \in \mathbb{C} - \{0, 1\}$, then $\mathcal{M}(S) = \mathbb{Q}(j(\lambda))$, where $j$ is the classical $j$-function $j(\lambda) = (1 - \lambda + \lambda^2)^3 / \lambda^2 (\lambda - 1)^2$.

**Conjecture 5.** Let $S$ be a closed Riemann surface admitting an Abelian group $H < \text{Aut}(S)$ so that $S/H$ has signature of the form $(0; a, b, c, d)$. Then $S$ can be defined over its field of moduli.

Finally, another interesting class of Riemann surfaces are the hyperelliptic ones. In the case of quadrangular hyperelliptic Riemann surfaces we have the following result as a consequence of Theorem 4 and the results of Cardona-Quer in [5] and Huggins in [14].

**Corollary 6.** Every quadrangular hyperelliptic Riemann surface can be defined over its field of moduli.

2. **Preliminaries**

2.1. **Riemann orbifolds.** A Riemann orbifold of signature $(g; k_1, \ldots, k_n)$, where $n \geq 2$, can be though of a pair

$$O = (S, \{(p_1, k_1), \ldots, (p_n, k_n)\}),$$

where $S$ is a closed Riemann surface of genus $g$ (called the Riemann surface structure of the orbifold), $p_1, \ldots, p_n \in S$ are pairwise different points (called the cone points of the orbifold), and each $k_j \geq 2$ is an integer (called the cone order of the cone point $p_j$). Moreover, if $n = 2$ and $g = 0$, then we also assume that $k_1 = k_2$. With the above restriction, these Riemann orbifolds are also called good orbifolds in Thurston’s terminology. A biholomorphism between Riemann orbifolds is a biholomorphism between the underlying Riemann surfaces with the property that it takes the cone points onto the cone points preserving the cone orders. If both orbifolds are the same, we talk about conformal automorphisms of orbifolds. We denote by $\text{Aut}_{orb}(O)$ the group of conformal automorphisms of a Riemann orbifold $O$.

Riemann orbifolds are obtained as the quotient space $X/\Gamma$, where $X \in \{\mathbb{C}, \mathbb{C} \setminus \mathbb{R}\}$ and $\Gamma$ is a discontinuous group of conformal automorphisms of $X$. In this case, the cone points are the equivalence classes of those points in $X$ with non-trivial $\Gamma$-stabilizer; the cone orders being the order of such stabilizer cyclic groups.

Let the Riemann orbifold $O$ to have signature $(0; k_1, \ldots, k_n)$ and let $\Gamma$ be so that $X/\Gamma = O$. A presentation of $\Gamma$ is given as

$$\Gamma = \langle x_1, \ldots, x_n : x_1^{k_1} = \ldots = x_n^{k_n} = x_1 x_2 \ldots x_{n-1} x_n = 1 \rangle.$$

2.2. **Fields of moduli.** We denote by $\text{Gal}(\mathbb{C}/\mathbb{Q})$ the group of field automorphisms of $\mathbb{C}$ and, if $\mathbb{K}$ is a subfield of $\mathbb{C}$, then $\text{Gal}(\mathbb{C}/\mathbb{K})$ denotes the subgroup of $\text{Gal}(\mathbb{C}/\mathbb{Q})$ formed by those elements acting as the identity on $\mathbb{K}$. It is known that the fixed subfield of $\text{Gal}(\mathbb{C}/\mathbb{K})$ still $\mathbb{K}$ (since $\mathbb{C}$ is algebraically closed of characteristic zero). If $P \in \mathbb{C}[x_0, \ldots, x_n]$ is any polynomial and if $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$, then $P^\sigma \in \mathbb{C}[x_0, \ldots, x_n]$ will denote the polynomial obtained by applying $\sigma$ to each of the coefficients of $P$. 

Let $C \subset \mathbb{P}^n$ be a non-singular irreducible projective complex algebraic curve defined by the homogeneous polynomials $P_1, \ldots, P_r \in \mathbb{C}[x_0, \ldots, x_n]$. If $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$, then the polynomials $P_1^\sigma, \ldots, P_r^\sigma$ define a new non-singular irreducible projective complex algebraic curve $C^\sigma$. The corresponding Riemann surfaces of $C$ and $C^\sigma$ are topologically equivalent, but they may not be conformally equivalent. The field of moduli of $C$, denoted by $\mathcal{M}(C)$, is the fixed subfield of the group $G_C = \{ \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}) : C^\sigma \cong C \}$. Notice from the definition that if $C \cong \hat{C}$ and $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$ then $C^\sigma \cong \hat{C}^\sigma$; in particular $G_C = \text{Gal}(\hat{C}/\mathbb{C})$. In this way, if $S$ is a closed Riemann surface and $C$ is any non-singular irreducible projective algebraic curve defining $S$, then we may set $G_S = G_C$ and define the field of moduli of $S$ as $\mathcal{M}(S) = \mathcal{M}(C)$.

The above definition also permits to see that if $S$ and $R$ are conformally equivalent closed Riemann surfaces, $C_S$ and $C_R$ are non-singular irreducible projective algebraic curves defining $S$ and $R$, respectively, and $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$, then $C^\sigma_S \cong C^\sigma_R$. It follows that if there is a natural action of the group $\text{Gal}(\mathbb{C}/\mathbb{Q})$ on the moduli space $\mathfrak{M}_g$ (the complex analytic space parametrizing conformal classes of closed Riemann surfaces of genus $g$). The stabilizer of such an action at the conformal class of $S$ is given by the subgroup $G_S$.

If the closed Riemann surface $S$ is definable over $\mathbb{K}$, say by the algebraic curve $C$, and $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{K})$, then $C^\sigma = C$; so it follows that $\mathcal{M}(S) < \mathbb{K}$, that is, every field of definition of $S$ contains its field of moduli. By results of Koizumi [17] it is also known that $\mathcal{M}(S)$ is equal to the intersection of all the fields of definition of $S$. By results of Hammer-Herrlich [11], $S$ is always definable over a finite extension of its field of moduli.

As a consequence of Hammer-Herrlich’s result, one may see that $G_S = \text{Gal}(\mathbb{C}/\mathcal{M}(S))$. It is clear the contention $G_S < \text{Gal}(\mathbb{C}/\mathcal{M}(S))$. To get the other direction, let $\mathcal{M}(S) < \mathbb{K} < \mathbb{C}$ be so that $\mathcal{M}(S) < \mathbb{K}$ is a finite Galois extension and that $S$ can be defined over $\mathbb{K}$ (by Hammer-Herrlich’s result). Now, every $\tau \in \text{Gal}(\mathbb{C}/\mathcal{M}(S))$ restricts to an element $\tau^{(\mathbb{K})} \in \text{Gal}(\mathbb{K}/\mathcal{M}(S))$. In particular, we may consider the corresponding restriction of $G_S$ to a subgroup $G_S^{(\mathbb{K})}$ of $\text{Gal}(\mathbb{K}/\mathcal{M}(S))$. By the definition of $G_S$, it follows that $\text{Fix}(G_S^{(\mathbb{K})}) = \mathcal{M}(S)$ and, by the Galois fundamental theorem, that $G_S^{(\mathbb{K})} = \text{Gal}(\mathbb{K}/\mathcal{M}(S))$. Now, let $\sigma \in \text{Gal}(\mathbb{C}/\mathcal{M}(S))$ and consider the element $\sigma^{(\mathbb{K})} \in G_S^{(\mathbb{K})}$. Choose $\eta \in G_S$ so that $\eta^{(\mathbb{K})} = \sigma^{(\mathbb{K})}$, then $S^\sigma = S^{\sigma^{(\mathbb{K})}} = S^{\eta^{(\mathbb{K})}} = S^\eta \cong E$; so it follows that $\sigma \in G_S$ as desired.

2.3. **Weil’s Galois descent theorem.** Next, we recall Weil’s Galois descent theorem which provides sufficient conditions for a algebraic curve defined over any finite Galois extension $L$ of a field $\mathbb{K}$ to be defined over $\mathbb{K}$.

**Theorem 7** (Weil’s Galois descent theorem [24]). Let $C$ be a non-singular projective algebraic curve defined over a finite Galois extension $L$ of a field $\mathbb{K}$. If for every $\sigma \in \text{Gal}(L/\mathbb{K})$ there is an isomorphism $f_\sigma : C \to C^\sigma$ defined over $L$ such that for all $\sigma, \tau \in \text{Gal}(L/\mathbb{K})$ the compatibility condition $f_\sigma \tau f_\sigma = f_\tau^\sigma \circ f_\sigma$ holds, then there exists a non-singular projective algebraic curve $E$ defined over $\mathbb{K}$ and there exists an isomorphism $R : C \to E$, defined over $L$, such that $R^\sigma \circ f_\sigma = R$.

As previously noticed, a non-singular complex projective algebraic curve can be defined over a finite Galois extension of its field of moduli. So, Weil’s Galois descent theorem may be written, in our situation as follows.

**Corollary 8.** Let $C$ be a non-singular complex irreducible projective algebraic curve and let $\mathbb{K}$ be its field of moduli. If for every $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{K})$ there is a biholomorphism $f_\sigma : C \to C^\sigma$ such that for all $\sigma, \tau \in \text{Gal}(\mathbb{C}/\mathbb{K})$ the compatibility condition $f_\sigma \tau f_\sigma = f_\tau^\sigma \circ f_\sigma$ holds, then there exists a non-singular complex projective algebraic curve $E$ defined over $\mathbb{K}$ and there exists a biholomorphism $R : C \to E$ such that $R^\sigma \circ f_\sigma = R$.

2.4. **Dèbes-Emsalem’s theorem.** Next, we recall a consequence of Weil’s Galois descent theorem (due to Dèbes-Emsalem [3]) which will be useful to us. We also provide a proof as a matter of completeness. First, let us start with some notations and definitions. Let us consider non-singular complex projective algebraic curves $C$ and $D$ (so they are closed Riemann surfaces) and let $f : C \to D$ be a (branched) holomorphic covering. Assume that $\mathbb{K}$ is the field of moduli of $C$ and that $D$ is defined over a subfield $\mathbb{K}$ of the field $\mathbb{C}$. For each $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{K})$ we may consider the (branched) holomorphic covering $f^\sigma : C^\sigma \to D^\sigma = D$. We say that
they are equivalent, noted as \( \{ f^\sigma : C^\sigma \to D \} \equiv \{ f : C \to D \} \), if there is a biholomorphism \( \phi_\sigma : C \to C^\sigma \) so that \( f^\sigma \circ \phi_\sigma = f \). The field of moduli of the holomorphic covering \( f : C \to D \), denoted by \( \mathcal{M}(f : C \to D) \), is the fixed field of the subgroup
\[
\{ \sigma \in \text{Gal}(\mathbb{C}/\mathbb{K}) : \{ f^\sigma : C^\sigma \to D \} \equiv \{ f : C \to D \} \} < \text{Gal}(\mathbb{C}/\mathbb{K}).
\]

**Theorem 9** (Débes-Emsalem [3]). Let \( C \) be a non-singular complex irreducible projective algebraic curve of genus \( g \geq 2 \) whose field of moduli is \( \mathbb{K} \). Then there exists a non-singular projective algebraic curve \( B \), defined over \( \mathbb{K} \), and there exists a Galois holomorphic covering \( f : S \to B \), with \( \text{Aut}(S) \) as its deck group, so that \( \mathcal{M}(f : S \to B) = \mathbb{K} \). Moreover, if \( B \) contains at least one \( \mathbb{K} \)-rational point outside the branch locus of \( f \), then \( \mathbb{K} \) is also a field of definition of \( S/\text{Aut}(S) \).

A simple proof of Theorem 9 is as follows. Let \( \mathcal{O} = C/\text{Aut}(C) \), let \( P : C \to \mathcal{O} \) be a Galois cover with \( \text{Aut}(C) \) as its group of deck transformations and let \( \mathcal{O}_P \) the set of branch values of \( P \). Let \( G_C = \{ \sigma \in \text{Aut}(\mathbb{C}/\mathbb{K}) : C^\sigma \cong C \} \). By the definition, for each \( \sigma \in G_C \), there is a biholomorphism \( f_\sigma : C \to C^\sigma \). It follows the existence of an automorphism of the orbifold \( \mathcal{O} \), say \( M_\sigma \), uniquely determined by \( \sigma \), so that \( P^\sigma \circ f_\sigma = M_\sigma \circ P \). The uniqueness of the automorphisms \( M_\sigma \) ensures that the collection \( \{ M_\sigma : \sigma \in G_C \} \) satisfies the conditions in Weil’s Galois descent theorem, as stated in Corollary 2, for the underlying Riemann surface structure of \( \mathcal{O} \), say \( X \). It then follows the existence of an irreducible projective algebraic curve \( B \) defined over \( \mathbb{K} \) and a biholomorphism \( R : X \to B \) so that \( R = R^\sigma \circ M_\sigma \). In this way, \( f = R \circ P : C \to B \) satisfies that \( \mathcal{M}(f : C \to B) = \mathbb{K} \). Let us now assume that there is a point \( q \in B \) which is not a branch value of \( f \) and \( \mathbb{K} \)-rational point. Let us fix some point \( p \in C \) so that \( f(p) = q \). One may check that it is possible, for each \( \sigma \in G_C \), to choose \( f_\sigma : C \to C^\sigma \) so that \( f_\sigma(p) = \sigma(q) = q \). In this way, there is some \( h_\sigma \in \text{Aut}(C^\sigma) \) so that \( h_\sigma(f_\sigma(p)) = \sigma(p) \) and we may replace \( f_\sigma \) by \( h_\sigma \circ f_\sigma \) in order to have the required property.

**3. PROOF OF THEOREM 9 AND COROLLARY 2**

Let \( S \) be a closed Riemann surface of genus \( g \geq 2 \) and let \( H < \text{Aut}(S) \) be so that \( S/H \) has signature of the form \( (0; a, b, c, d) \). Let us set \( J = \langle A(z) = 1/z, B(z) = z^2 - 1 \rangle \cong \mathfrak{S}_3 \), the group of those Möbius transformations keeping invariant the set \( \{ \infty, 0, 1 \} \). We should proceed following the same arguments as for Débes-Emsalem’s theorem proof provided in the previous section.

**3.1. Proof of Theorem 9 for \( H = \text{Aut}(S) \).** Let \( P : S \to \mathbb{C} \) be a regular branched cover with \( \text{Aut}(S) \) as Deck group. We may assume that the branch values of \( P \) are given by the points \( \infty, 0, 1 \) and \( \lambda \in \mathbb{C} - \{0, 1\} \), so that \( a \) is the order of \( \infty \), \( b \) is the order of \( 0 \), \( c \) is the order of \( 1 \) and \( d \) is the order of \( \lambda \). In this case, for each \( \sigma \in G_S \), we have an isomorphism \( f_\sigma : S \to S^\sigma \) and a Möbius transformation \( M_\sigma \) so that \( P^\sigma \circ f_\sigma = M_\sigma \circ P \).

Now, the collection \( \{ M_\sigma : \sigma \in G_S \} \) may be seen to satisfy the Weil’s Galois descent conditions, so there is an isomorphism \( R : \mathbb{C} \to B \), where \( B \) is some algebraic curve defined over the field of moduli \( \mathcal{M}(S) \) of \( S \), so that the field of moduli of \( R \circ P : S \to B \) is \( \mathcal{M}(S) \).

Case (1). Let us assume that \( d \notin \{ a, b, c \} \). If \( \sigma \in G_S \), then \( M_\sigma(\lambda) = \sigma(\lambda) \) since \( \sigma(\lambda) \) is the branch value of \( P^\sigma \) with order \( d \) and \( M_\sigma \) sends branch values to branch values and preserves the order of them.

Since \( \sigma(R(\lambda)) = R^\sigma(\sigma(\lambda)) = R(M_\sigma^{-1}(\sigma(\lambda))) = R(\lambda) \), it follows that \( R(\lambda) \) is a \( \mathcal{M}(S) \)-rational point. As \( B \) is defined over \( \mathcal{M}(S) \) and it has genus zero, it follows that there is a \( \mathcal{M}(S) \)-isomorphism \( \psi : B \to \mathbb{P}^1_{\mathcal{M}(S)} \). As \( \mathbb{P}^1_{\mathcal{M}(S)} \) has infinitely many \( \mathcal{M}(S) \)-rational points, it follows that the same happens with \( B \), in particular, that \( B \) admits one of these \( \mathcal{M}(S) \)-rational points outside the branch locus of \( R \circ P \). It now follows from Débes-Emsalem’s theorem that \( S \) can be defined over \( \mathcal{M}(S) \).

Case (2). Let us assume that \( a = c, b = d \) and \( a \neq b \). Then for \( \sigma \in G_S \) it happens that \( \{ M_\sigma(0), M_\sigma(\lambda) \} = \{ 0, \sigma(\lambda) \} \). In this way, if we consider the subgroup \( L_S = \{ \sigma \in G_S : M_\sigma(\lambda) = \sigma(\lambda) \} \), then either \( L_S = G_S \) or \( L_S \) has index two in \( G_S \). In the first situation, we may proceed as in Case (1) to obtain that \( S \) can be
H is unique in the sense that if $H = H_{k,k,k,k}(0; Aut(S))$, we denote by $\hat{H}$ the unique normal subgroup of $\Gamma$ that uniformizes $S$. If $S$ has genus at least two, necessarily $H = \hat{H}$ has signature $(0; 2, k)$, with $k \geq 3$. The result follows from part (1) of Theorem 1.

3.2. Proof of Theorem 1 for $H \neq Aut(S)$. If we are in case (1) of the Theorem, then we must have that $S/\hat{Aut}(S)$ is quasiplatonic, as consequence of Singerman’s lists in [22], and the result will follow from the fact that quasiplatonic curves can be defined over their fields of moduli. If we are as in cases (2) or (3) of the Theorem, then necessarily $S/\hat{Aut}(S)$ will be either quasiplatonic or it will have a signature as already considered in Cases (1), (2) or (3) in [3, 4].

3.3. Proof of Corollary 2. Let us assume that $H = Aut(S)$ and that $d \notin \{a, b, c\}$. Next, we proceed to see that $\mathcal{M}(S)$ is an extension of $\mathbb{Q}(j(\lambda))$. If $\sigma \in G_S$, then (as previously observed) the Möbius transformation $M_\sigma$ belongs to $J$. In this way, the map $\Theta : G_S \to J$, defined by $\Theta(\sigma) = M_\sigma^{-1}$, turns out to be a homomorphism. As for $\sigma \in G_S$ we must have that $\sigma(\lambda) = M_\sigma(\lambda)$, it follows that $\sigma(\lambda)$ belongs to the $J$-orbit of $\lambda$, that is, $j(\lambda) = j(\sigma(\lambda)) = \sigma(j(\lambda))$, where $j$ is the elliptic $j$-function; in particular, $\mathbb{Q}(j(\lambda)) < \mathcal{M}(S)$.

Now, for $\sigma \in Gal(\mathbb{C}/\mathbb{Q}(j(\lambda)))$ the value $\sigma(\lambda)$ belongs to the (finite) $J$-orbit of $\lambda$. In this way, we only need to consider a finite number of orbifolds $O_T$ (whose underlying Riemann surface is $\hat{C}$ and cone points are $\infty$, 0, 1 and $T(\lambda)$, where $T \in J$, the order of $\infty$ is $a$, the order of 0 is $b$, the order of 1 is $c$ and the order of $T(\lambda)$ is $d$).

Let $\Gamma$ be a Fuchsian group with signature $(0; a, b, c, d)$ uniformizing the orbifold $O_T = S/H$ and let $\Gamma_T$ be a normal subgroup of $\Gamma$ that uniformizes $S$ with $H = \Gamma/\Gamma_T$. For each $T \in J$, we consider a Fuchsian group $\Gamma_T$ with signature $(0; a, b, c, d)$ uniformizing the orbifold $O_T$.

If $\sigma \in Gal(\mathbb{C}/\mathbb{Q}(j(\lambda)))$, then there is $T \in J$ with $M_\sigma = T$. We may lift $T$ to a conformal automorphism $N$ of the hyperbolic plane $\mathbb{H}^2$ which conjugates $\Gamma$ into $\Gamma_T$ and $\Gamma_T$ into a normal subgroup of it of the same index as $\Gamma_1$ in $\Gamma$. As $\Gamma_T$ only contains, up to conjugation, finitely many subgroups of a given finite index, it follows that the collection of those $\sigma$ with the property that $N\Gamma_1 N^{-1}$ uniformizes $S^\sigma$ has finite index in $Gal(\mathbb{C}/\mathbb{Q}(j(\lambda)))$. This provides the fact that $\mathcal{M}(S)$ is a finite degree extension of $\mathbb{Q}(j(\lambda))$.

4. Proof of Corollary 3

Let $S$ be a homology cover of a Riemann orbifold $O$ with signature $(0; a, b, c, d)$. If the genus of $S$ is at most one, then $S$ is known to be defined over its field of moduli. Let us now assume that $S$ has genus at least two. If $S/\hat{Aut}(S)$ is an orbifold with signature of type $(0; k_1, k_2, k_3)$, then $S$ is a quasiplatonic curve, so it can be defined over its field of moduli [20]. So, from now on, we also assume that $S$ is not quasiplatonic. We denote by $H < Aut(S)$ an Abelian group so that $O = S/H$ has signature $(0; a, b, c, d)$.

4.1. Case 1. Assume $a = b = c = d = k$. In this case, $S/H$ is an orbifold with signature of the form $(0; k, k, k, k)$. As we are assuming the genus of $S$ at least two, necessarily $k \geq 3$. By results in [8], the group $H$ is unique in the sense that if $\hat{H} < Aut(S)$ is an Abelian group with $S/\hat{H}$ of signature $(0; k, k, k, k)$, then $\hat{H} = H$. This implies that $H$ is a normal subgroup of $Aut(S)$, that $S/\hat{Aut}(S) = O/\hat{Aut}(O)$ and that $Aut(O)/H = Aut_{orb}(O)$. Notice that $Aut_{orb}(O)$ contains as subgroup the group

$$\hat{J} = \left\{ I(z) = z, A(z) = \frac{\lambda}{z}, B(z) = \frac{z - \lambda}{z - 1}, C(z) = \frac{\lambda(1 - z)}{\lambda - z} \right\} \cong \mathbb{Z}_2^2.$$ 

It follows the existence of a subgroup $K < Aut(S)$ so that $\hat{H} < K$ and $K/H = \hat{J}$. So $S/K = O/\hat{J}$ has signature $(0; 2, 2, 2, k)$, with $k \geq 3$. Then the result follows from part (1) of Theorem 4.
4.2. Case 2. Assume not all cone orders are equal, say \( a \neq d \). As we are assuming \( S \) not to be quasipatonic, this ensures that \( S/\text{Aut}(S) \) must have signature \((0; k_1, k_2, k_3, k_4)\). It follows from Singerman’s list of maximal Fuchsian groups that either (i) \( \text{Aut}(S) = H \) or (ii) \( H \lhd \text{Aut}(S) \) of index two. In case (ii) we may assume, without loss of generality, that \( a = b \) and \( c = d \), so \( S/\text{Aut}(S) \) has signature \((0; 2, 2, a, a)\) and, as \( a \neq d \), the result follows from Theorem 1. In case (i), as \( \text{Aut}(S) = H \), it follows that \( S/\text{Aut}(S) = \mathcal{O} \) and that \( \text{Aut}_{\text{orb}}(\mathcal{O}) \) is trivial. This last fact ensures that one of the values in \( \{a, b, c, d\} \) is different from the others three. Again, the result follows from Theorem 1.

5. Proof of Theorem 1

Let \( S \) be a homology cover of an orbifold with signature \((0; a, a, a, a)\).

5.1. If \( a = 2 \), then \( S \) is a Riemann surface of genus \( q = 1 \), \( H \cong \mathbb{Z}_2^3 \) and there is Galois cover \( \pi : S \to \hat{\mathbb{C}} \), with \( H \) as Deck group. We may assume the branch values of \( \pi \) are given by the points \( \infty, 0, 1 \) and \( \lambda \), each one with branching order 2. Let us consider the elliptic curve

\[ E_\lambda = \{y^2z = x(x - z)(x - \lambda z)\} \subset \mathbb{P}^2 \]

and the conformal involution of \( E_\lambda \) given by \( b_1(x : y : z) = [x : -y : z] \), which has exactly 4 fixed points. The degree two branched covering \( P : E_\lambda \to \hat{\mathbb{C}} \), where \( P : ([x : y : z]) = x/z \), is branched exactly at the values \( \infty, 0, 1 \) and \( \lambda \). This is enough to ensure that \( E_\lambda \) and \( S \) are conformally equivalent. As \( \mathcal{M}(C^{(2)}_\lambda) = \mathcal{M}(E_\lambda) = \mathbb{Q}(j(\lambda)) \), we are done in this case.

Another way to see that \( S \) and \( E_\lambda \) are conformally equivalent, is as follows (using the uniqueness of the homology cover). The Deck group of \( P \) is \( \langle b_1 \rangle \). The group \( G_\lambda = \langle A(w) = \lambda w, B(w) = (w(w-1)/(w-\lambda)) \rangle \cong \mathbb{Z}_2^2 \) keeps invariant these 4 branch values. The holomorphic map

\[ Q_\lambda(w) = \frac{\lambda(w^2 - \lambda)^2 - 4\lambda w(w-1)(w-\lambda)}{(w^2 - \lambda)^2 - 4\lambda w(w-1)(w-\lambda)} \]

is a regular branched covering with \( G_\lambda \) as Deck group and whose branched values are \( \infty, 0, 1 \) and \( \lambda \). One sees that \( Q_\lambda(\infty) = Q_\lambda(0) = Q_\lambda(1) = Q_\lambda(\lambda) = \lambda \). The group \( G_\lambda \) lifts to a group \( H_\lambda \) of conformal automorphisms of \( E_\lambda \) isomorphic to \( \mathbb{Z}_2^3 \), this being generated by the transformations \( b_1, b_2 \) and \( b_3 \), where

\[ b_2(x : y : z) = [\lambda xz : \lambda yz : x^2] \]

\[ b_3(x : y : z) = [\lambda(x - z)(x - \lambda z) : \lambda(\lambda - 1)y : (x - \lambda z)^2] \]

Clearly, \( Q_\lambda \circ P : E_\lambda \to \hat{\mathbb{C}} \) is a regular branched covering with \( H_\lambda \) as Deck group. It follows that \( E_\lambda \) is also a homology cover of the orbifold \( E_\lambda/\langle b_1 \rangle = C^{(2)}_\lambda/H \), in particular, \( E_\lambda \) is holomorphically equivalent to \( C^{(2)}_\lambda \).

5.2. Let us now assume \( a \geq 3 \). If the cone points of \( \mathcal{O} \) are \( \infty, 0, 1 \) and \( \lambda \), then for every \( T \in K \) we have, as \( T \) preserves the set \( \{\infty, 0, 1\} \), that \( S \) is also a homology cover of an orbifold of signature \((0; a, a, a, a)\) whose cone points are the points \( \infty, 0, 1 \) and \( T(\lambda) \). If \( \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}) \), then the Riemann surface \( S^\sigma \) is a homology cover of an orbifold of signature \((0; a, a, a, a)\) whose cone points are given by \( \infty, 0, 1 \) and \( \sigma(\lambda) \). Then, as a consequence of the previous observation and the results in \( [8] \), \( S^\sigma \cong S \) if and only if \( \sigma(\lambda) \in \text{Orb}_K(\lambda) \), where

\[ \text{Orb}_K(\lambda) = \left\{ \frac{1}{\lambda}, \frac{\lambda}{\lambda - 1}, \frac{\lambda - 1}{\lambda}, \frac{1}{1 - \lambda}, \frac{1}{1 - \lambda} \right\} \]

If \( \sigma \in G_S \), then \( \sigma(x) \in \text{Orb}_K(\lambda) \) for every \( x \in \text{Orb}_K(\lambda) \); in particular, \( \sigma(j(\lambda)) = j(\lambda) \). Reciprocally, if \( \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}) \) satisfies that \( \sigma(j(\lambda)) = j(\lambda) \) and \( \sigma(\lambda) = \mu \), then \( j(\mu) = j(\sigma(\lambda)) = j(\sigma(j(\lambda))) = j(\lambda) \); in particular, \( \mu \in \text{Orb}_K(\lambda) \). This asserts that \( \sigma \in G_S \) if and only if \( \sigma(j(\lambda)) = j(\lambda) \). In particular, \( \mathcal{M}(S) = \mathbb{Q}(j(\lambda)) \).
6. Proof of Corollary [\text{5}]

Let \( S \) be a hyperelliptic Riemann surface of genus \( g \geq 2 \) with hyperelliptic involution \( \iota : S \to S \). If there is some \( H < \text{Aut}(S) \) so that \( S/H \) has signature of type \((0; a, b, c, d)\), then \( S/\text{Aut}(S) \) has either a signature of type \((0; r, s, t)\) or of type \((0; r, s, t, u)\). In the first case \( S \) is quasiplatonic and it can be defined on its field of moduli [\text{26}]. We assume now on that \( S/\text{Aut}(S) \) has a signature of the form \((0; r, s, t, u)\). If some of the orders \( r \) or \( s \) or \( t \) or \( u \) is different from the others three, then the result will follow from Theorem [\text{1}] We still to work out the left cases. We will proceed in a direct way. As \( S/\langle \iota \rangle \) is an orbifold with signature \((0; 2, 2g+2, 2)\) and \( 2g + 2 > 4 \), we have necessarily that \( \text{Aut}(S) \neq \langle \iota \rangle \). In particular, if \( g = 2 \), then \( S \) can be defined over its field of moduli [\text{5}]. Let us assume now on that \( g \geq 3 \). As in the case that \( \text{Aut}(S)/\langle \iota \rangle \) is not cyclic \( \rangle \) can be defined over its field of moduli [\text{14}], we may also assume now on that \( \text{Aut}(S)/\langle \iota \rangle \cong \mathbb{Z}_n \), for a suitable \( n \geq 2 \). Let \( P : S \to \mathbb{C} \) be a regular two-fold branched cover. Up to composition of \( P \) at the left by a suitable Möbius transformation, we may assume that \( \text{Aut}(S) \) induces on \( \mathbb{C} \) the cyclic group of order \( n \) generated by the Möbius transformation \( A(z) = e^{2\pi i/n}z \). As we are assuming that \( S/\text{Aut}(S) \) has a signature of type \((0; r, s, t, u)\) and the branch values of \( P \) should be kept invariant under the action of \( A \), it follows that the branch values of \( P \) only can be of one of the following forms (up to composition of \( P \) at the left by a Möbius transformation of the form \( T(z) = qz \) or \( T(z) = q/z \) for a suitable \( q \neq 0 \))

(1) \( \{ \omega^k, \mu \omega^k : k = 1, ..., n \} \), where \( \mu \in \mathbb{C} \) with \( \mu^n \neq 1 \); or
(2) \( \{ \infty \} \cup \{ \omega^k, \mu \omega^k : k = 1, ..., n \} \), where \( \mu \in \mathbb{C} \) with \( \mu^n \neq 1 \); or
(3) \( \{ \infty, 0 \} \cup \{ \omega^k, \mu \omega^k : k = 1, ..., n \} \), where \( \mu \in \mathbb{C} \) with \( \mu^n \neq 1 \), where \( \omega = e^{2\pi i/n} \).

As the number of branch values of \( P \) is even, (2) is not possible. In cases (1) and (3) we have that the Möbius transformation \( B(z) = \mu z \) keeps invariant the branch values of \( P \). It follows that \( \text{Aut}(S)/\langle \iota \rangle \) contains the dihedral group \( D_n = \langle A, B \rangle \), a contradiction to our previous assumption.

7. An Example: The KFT Family

In this section we consider the family of closed Riemann surfaces of genus three admitting the symmetric group \( \mathfrak{S}_3 \) as a group of conformal automorphisms. As a consequence of Theorem [\text{1}] these surfaces can be defined over their fields of moduli. We compute the field of moduli of these surfaces and provide explicit equations in these fields. Most of all computations, in a different approach, has been carry out also in the paper [\text{9}]. We also provide Fuchsian uniformizations of these surfaces.

7.1. The hyperelliptic case. As the hyperelliptic involution is in the center of the group of conformal automorphisms of a hyperelliptic Riemann surface, it is not difficult to see that there is exactly one, up to biholomorphisms, hyperelliptic Riemann surface \( S_0 \) of genus 3 with a group of conformal automorphisms isomorphic to \( \mathfrak{S}_3 \). If we quotient \( S_0 \) by the hyperelliptic involution, then we obtain that the 8 cone points of order 2 should be invariant under the action of a group of Möbius transformations isomorphic to \( \mathfrak{S}_4 \). This permits to see that \( \text{Aut}(S_0) = \mathfrak{S}_4 \oplus \mathbb{Z}/2\mathbb{Z} \) and that \( S_0/\text{Aut}(S_0) \) has signature \((0; 2, 4, 6)\). Using the above information, one can see that \( S_0 \) can be represented by the algebraic curve \( C : y^2 = x^8 + 14x^4 + 1 \), that is, \( S_0 \) can be defined over \( \mathbb{Q} \).

7.2. The non-hyperelliptic case. A well known fact is the topological rigidity property on the action of the group \( \mathfrak{S}_4 \) as group of conformal automorphisms of closed non-hyperelliptic Riemann surfaces of genus \( g = 3 \).

\textbf{Theorem 10} (Broughton [\text{2}]). If \((S, H)\) and \((R, K)\) are so that \( S \) and \( R \) are non-hyperelliptic Riemann surfaces of genus 3 and \( H \cong K \cong \mathfrak{S}_4 \) are respective group of conformal automorphisms, then there is an orientation preserving homeomorphism between the surfaces conjugating the groups.

In Section [\text{7.3}] we provide a simple proof, based on Fuchsian groups, of Theorem [\text{10}] as a matter of completeness.
Remark 11. The hyperelliptic Riemann surface \( C : y^2 = x^8 + 14x^4 + 1 \) admits two different subgroups \( H_1 \) and \( H_2 \) inside \( \text{Aut}(C) \) with \( H_j \cong \mathcal{S}_4 \) so that \( C/H_1 \) has signature \( (0;2,4,6) \) and \( C/H_2 \) has signature \( (0;2,2,2,3) \). If \( (S,H) \) is so that \( S \) is non-hyperelliptic Riemann surface and \( \mathcal{S}_4 \cong H < \text{Aut}(S) \), then there is an orientation preserving homeomorphism \( f : S \to C \) so that \( H_2 = fHf^{-1} \). A description of these Riemann surfaces, from the point of view of Schottky uniformizations, can be found in [13].

Let \( S \) be a non-hyperelliptic closed Riemann surface of genus \( g = 3 \) and let \( \mathcal{S}_4 \cong H < \text{Aut}(S) \). As a consequence of the Riemann-Hurwitz formula [7], the orbifold \( S/H \) has signature \( (0;2,2,2,3) \). It follows that the locus, in the moduli space of genus three Riemann surfaces, of the classes of Riemann surfaces admitting the non-hyperelliptic action of \( \mathcal{S}_4 \) is one-complex dimensional.

As a consequence of Singerman’s list [22] of maximal Fuchsian groups, one has that either \( \text{Aut}(S) = H \) or \( S/\text{Aut}(S) \) has signature of the form \((0;a,b,c)\). If \( S/\text{Aut}(S) \) has signature of the form \((0;a,b,c)\), then \( S \) is quasiplatonic and so it can be defined over its field of moduli [26]. The set of these quasiplatonic surfaces form a finite subset up to conformal equivalence. Apart from the hyperelliptic case, there are only other two such Riemann surfaces; Fermat’s curve \( F : x^4 + y^4 + z^4 = 0 \) and Klein’s curve \( K : x^3y + y^3z + z^3x = 0 \). It is well known that \(|\text{Aut}(F)| = 96\) and that \(|\text{Aut}(K)| = 168\) and that \( F/\text{Aut}(F) \) has signature \((0;2,4,8)\) and that \( K/\text{Aut}(K) \) has signature \((0;2,3,7)\). All of these quasiplatonic surfaces are defined over their field of moduli, that is, over \( \mathbb{Q} \).

If \( \text{Aut}(S) = H \), the generic situation, then, as a consequence of Theorem 11, \( S \) can be defined over its field of moduli. Let \( \pi : S \to \hat{\mathbb{C}} \) be a regular branched covering with \( H \) as Deck group of cover transformations. We may assume that the branch values of \( \pi \) of order 2 are given as \( \infty, 0, 1 \) and the one of order 3 is \( \mu \in \mathbb{C} - \{0, 1\} \).

In the next sections we proceed to compute the field of moduli of the previous curves.

### 7.2.1. The KFT family

It is well known that the canonical embedding of a non-hyperelliptic Riemann surface of genus 3 is a non-singular projective algebraic curves of degree 4 (a quartic) in the complex projective plane \( \mathbb{P}^2 \). A description of such quartics for the family of non-hyperelliptic Riemann surfaces of genus 3 admitting \( \mathcal{S}_4 \) as group of conformal automorphisms has been done in [19]; called the KFT family. A study of such a family from the point of view of idempotents has been done in [10]. This family has also been studied in [41, 18, 25]. The quartics in the KFT family are of the form [19]

\[
C_\lambda = \{ x^4 + y^4 + z^4 + \lambda(x^2y^2 + y^2z^2 + z^2x^2) = 0 \} \subset \mathbb{P}^2,
\]

where \( \lambda \in \mathcal{P} = \mathbb{C} - \{-2, -1, 2\} \). For \( \lambda \in \{\pm 2, 1\} \) the curve \( C_\lambda \) is a singular quartic. The group \( H \cong \mathcal{S}_4 \), for each member of of the KFT family, is generated by the transformations

\[
A([x : y : z]) = [y : -x : -z], \quad B([x : y : z]) = [x : z : y].
\]

As a consequence of Theorem 10, every non-hyperelliptic Riemann surface of genus 3 admitting a group of conformal automorphisms isomorphic to \( \mathcal{S}_4 \) is represented by one of the curves in the KFT family. Conversely, every curve \( C_\lambda \), with \( \lambda \in \mathcal{P} \), is a closed Riemann surface of genus 3 admitting the group \( H \) as a group of conformal automorphisms.

Remark 12. The quartic \( C_0 \) corresponds to Fermat’s curve \( x^4 + y^4 + z^4 = 0 \), for which \(|\text{Aut}(C_0)| = 96\) and \( C_0/\text{Aut}(C_0) \) has signature \((0;2,3,8)\). The quartics \( C_{3\alpha} \cong C_{3\beta} \), where \( \alpha = (-1 + i\sqrt{7})/2 \), correspond to Klein’s quartic and \( C_{3\alpha}/\text{Aut}(C_{3\alpha}) \) has signature \((0;2,3,7)\). An extra automorphism of order 7 of this quartic is given by \( C([x : y : z]) = [-x + y + \alpha z : \alpha(x + y) : -x + y - \alpha z] \).

As for each \( \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}) \) one has that \( C_\lambda^\sigma = C_{\sigma(\lambda)} \), it follows that the orbits under the action of \( \text{Gal}(\mathbb{C}/\mathbb{Q}) \) of such a family are given as:

1. the orbit of \( C_\sigma \) (containing exactly all curves of the form \( C_\lambda \), where \( \lambda \) is transcendental); and
2. the orbits of the curves \( C_{\lambda_1}, \ldots, C_{\lambda_n}, \ldots \), where \( \lambda_1, \ldots, \lambda_n, \ldots \in \mathbb{P} \) is a maximal collection of algebraic numbers non-equivalent under the absolute Galois group \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \).
7.2.2. **Equivalence of curves.** In order to find explicitly the field of moduli $\mathcal{M}(C_\lambda)$, we first need to provide conditions on $\lambda_1$ and $\lambda_2$ for $C_{\lambda_1}$ and $C_{\lambda_2}$ to be conformally equivalent.

Let $G = \langle \eta(z) = z/(z - 1) \rangle \cong \mathbb{Z}_2$ and let $F(z) = z^2/(z - 1)$. The map $F$ is a regular branched cover with $G$ as Deck group.

As $\lambda \in \mathcal{P}$, we have that $\lambda^2 - \lambda - 2 \neq 0$. In this way, for each simply-connected subset $D$ of $\mathcal{P}$ we may choose one of the branches of $\sqrt{\lambda^2 - \lambda - 2}$ to get an analytic map $f(\lambda) = \sqrt{\lambda^2 - \lambda - 2}$ defined over $D$.

Let us fix $\lambda \in \mathcal{P}$ and let us consider the map $Q : \mathcal{C}_\lambda \to \hat{\mathbb{C}}$ defined as

$$Q([x : y : z]) = \frac{x^2 y^2 z^2}{(x^2 + y^2 + z^2)^3}.$$

As the polynomials $x^2 y^2 z^2$ and $x^2 + y^2 + z^2$ are invariant under $A$ and $B$, we obtain that $Q \circ A = Q = Q \circ B$. It follows from Bezout’s theorem that $Q$ has degree 24. In particular, $Q : \mathcal{C}_\lambda \to \hat{\mathbb{C}}$ is a regular branched cover with $H$ as Deck group of cover transformations.

If we fix one of the two values of $\sqrt{\lambda^2 - \lambda - 2}$, then the branch values of $Q$ are given by the points $\infty$ (of order 3), 0 (of order 2) and the following two points (each of order 2)

$$l_1(\lambda) = \frac{(2 + \lambda) (\lambda + \sqrt{\lambda^2 - \lambda - 2})^2}{(2 - \lambda - 2 \sqrt{\lambda^2 - \lambda - 2})^3}$$

$$l_2(\lambda) = \frac{(2 + \lambda) (\lambda - \sqrt{\lambda^2 - \lambda - 2})^2}{(2 - \lambda + 2 \sqrt{\lambda^2 - \lambda - 2})^3}$$

Notice that the branch value 0 is the projection under $Q$ of the fixed points of the conjugates of $A^2$ and that the branch values $l_1(\lambda)$ and $l_2(\lambda)$ are the projections of the fixed points of those elements of order two conjugate to $B$.

Let us consider the Möbius transformation $T_\lambda$ so that $T_\lambda(0) = 0$, $T_\lambda(l_1(\lambda)) = \infty$ and $T_\lambda(l_2(\lambda)) = 1$, that is,

$$T_\lambda(z) = \left( \frac{l_2(\lambda) - l_1(\lambda)}{l_2(\lambda)} \right) \frac{z}{z - l_1(\lambda)}.$$

The map $\pi_\lambda = T_\lambda \circ Q$ defines a regular branched cover with $H$ as Deck group of cover transformations whose branch values of order 2 are $\infty$, 0 and 1 and the one of order 3 is

$$\mu(\lambda) = \frac{4(2 + \lambda)^2 (\sqrt{\lambda^2 - \lambda - 2})^3}{(\lambda - \sqrt{\lambda^2 - \lambda - 2})^2 (\lambda - 2 - 2 \sqrt{\lambda^2 - \lambda - 2})^3} \in \mathbb{C} - \{0, 1\}.$$

**Remark 13.** If we change to the other value of $\sqrt{\lambda^2 - \lambda - 2}$ the roles of $l_1(\lambda)$ and $l_2(\lambda)$ get interchanged. In this case, the Möbius transformation that fixes 0 and sends $l_1(\lambda)$ to $\infty$ and $l_2(\lambda)$ to 1 is given by

$$\tilde{T}(z) = \eta(T_\lambda(z)) = \left( \frac{l_1(\lambda) - l_2(\lambda)}{l_1(\lambda)} \right) \frac{z}{z - l_2(\lambda)}$$

and the branch values of the cover map $\tilde{T} \circ Q : \mathcal{C}_\lambda \to \hat{\mathbb{C}}$ are $\infty$, 0 and 1 (of order 2) and the one of order 3 is $\eta(\mu(\lambda)) = \mu(\lambda)/(\mu(\lambda) - 1)$. In this way, to each $\lambda \in \mathcal{P}$ we have associated the two values $\mu(\lambda)$ and $\eta(\mu(\lambda))$ depending on the choice of $\sqrt{\lambda^2 - \lambda - 2}$. Let us also notice that

$$\mu(\lambda) + \eta(\mu(\lambda)) = \mu(\lambda)^2/(\mu(\lambda) - 1) = F(\mu(\lambda)) = G(\lambda) = \frac{16(1 + \lambda)^2}{27(2 + \lambda)}$$

is well defined over all $\mathcal{P}$.

Let $\mathcal{P}_0$ be the subset of $\mathcal{P}$ consisting of those values for which $\text{Aut}(C_\lambda) \neq H$. Then $\mathcal{P}_0$ is a set of isolated points. Notice that if $C_{\lambda_1} \cong C_{\lambda_2}$, then $\lambda_1 \in \mathcal{P}_0$ if and only if $\lambda_2 \in \mathcal{P}_0$. The following result was also obtained in [9] but we provide another computations as a matter of interest.
Theorem 14.  

(1) If $\lambda_1, \lambda_2 \in \mathcal{P} - \mathcal{P}_0$, then $C_{\lambda_1}$ and $C_{\lambda_2}$ are conformally equivalent if and only if $\lambda_1 = \lambda_2$. 
(2) If $\lambda_1, \lambda_2 \in \mathcal{P} - \mathcal{P}_0$, then $C_{\lambda_1}$ and $C_{\lambda_2}$ are anticonformally equivalent if and only if $\lambda_1 = \overline{\lambda_2}$. 
(3) $\mathcal{P}_0 = \{0, 3(-1 \pm i\sqrt{7})/2\}$. The curves $C_{3(-1-i\sqrt{7})/2}$ and $C_{3(-1+i\sqrt{7})/2}$ are equivalent to Klein’s curve and the curve $C_0$ is Fermat’s curve.

Proof. Given any two points $\lambda_1, \lambda_2 \in \mathcal{P}$, we may consider a simply connected domain $D \subset \mathcal{P}$ containing the points $\lambda_1$ and $\lambda_2$. Once this is done, we make a choice for a analytic branch of $\sqrt{\lambda^2 - \lambda - 2}$ in $D$. Using such a choice, we have fixed the choices of $\pi_\lambda(z)$ and of $\mu(\lambda)$ for $\lambda \in D$ (both are analytic on the parameter $\lambda \in D$).

Case (1). We assume $\lambda_1, \lambda_2 \in D - \mathcal{P}_0$. If $C_{\lambda_1} \cong C_{\lambda_2}$, then there is a conformal homeomorphism $f : C_{\lambda_1} \rightarrow C_{\lambda_2}$. As $\text{Aut}(C_{\lambda_1}) = H = \text{Aut}(C_{\lambda_2})$, it follows that there is a Möbius transformation $M$ so that $\pi_{\lambda_2} \circ f = M \circ \pi_{\lambda_1}$. Moreover, $M(0) = 0$, $M(\mu(\lambda_1)) = \mu(\lambda_2)$ and $M\{1, \infty\} = \{1, \infty\}$. It follows that either $M(z) = z$ (in which case $\mu(\lambda_2) = \mu(\lambda_1)$) or $M(z) = \eta(z)$ (in which case $\mu(\lambda_2) = \eta(\mu(\lambda_1))$). We have obtained that necessarily $G(\lambda_1) = G(\lambda_2)$. In this way, we obtain that

$$
\lambda_2 \in \left\{ \lambda_1, -\left(\frac{6 + 5\lambda_1 + \lambda_1^2 - (1 + \lambda_1)\sqrt{\lambda_1^2 - 4}}{2(2 + \lambda_1)}\right), -\left(\frac{6 + 5\lambda_1 + \lambda_1^2 + (1 + \lambda_1)\sqrt{\lambda_1^2 - 4}}{2(2 + \lambda_1)}\right) \right\}
$$

Notice that if $\lambda_1 = -5/2$, then $\lambda_2 = \lambda_1$ as $\lambda_2 \neq -1$. We assume now on that $\lambda_1 \neq -5/2$.

Let us consider the Riemann orbifolds $O_j = C_{\lambda_j}/\langle AB \rangle$ which has signature $(1; 3, 3)$. It was obtained in [19] that the $j$-invariant of the underlying Riemann surface structure $T_j$ of $O_j$ is

$$
j_3(\lambda) = \frac{(16\lambda^2 + 48\lambda + 33)^3}{108(1 + \lambda)(2 + \lambda)}
$$

Similarly, we may consider the orbifolds (all of them of genus one) obtained by quotient $C_{\lambda_j}$ by the cyclic groups $\langle A \rangle, \langle B \rangle$ and $\langle A^2 \rangle$. The corresponding $j$-invariants are

$$
j_4(\lambda) = \frac{(\lambda^2 + 18\lambda + 33)^3}{108(1 + \lambda)^3(2 + \lambda)}
$$

$$
j_2(\lambda) = \frac{-(\lambda^2 - 12\lambda - 12)^3}{108(1 + \lambda)(2 + \lambda)^4}
$$

$$
j_{2,2}(\lambda) = \frac{4(\lambda^2 + 3\lambda + 3)^3}{27(1 + \lambda)^2(2 + \lambda)^2}
$$

Next we make a comparison of $j_2, j_3, j_4, j_{2,2}$ for the three above possible values for $\lambda_2$ and those for $\lambda_1$ (this can be done with any computational software) and we obtain that the only possibility is $\lambda_2 = \lambda_1$ (as $\lambda_1 \neq -5/2$)

Case (2). The anticonformal situation is worked in a similar fashion as the previous case.

Case (3). Now we assume that $\lambda_1, \lambda_2 \in D \cap \mathcal{P}_0$. We know that $C_{\lambda_j}$ is conformally equivalent to either Klein’s curve (which is given by $\lambda = 3(-1 + i\sqrt{7})/2$ [19]) or Fermat’s curve (given with $\lambda = 0$). As each of them can be defined over $\mathbb{R}$, each of them is conformally equivalent to their conjugate, that is, $C_{\lambda_j} \cong C_{\overline{\lambda_j}}$, for $j = 1, 2$.

If $C_{\lambda_1} \cong C_{\lambda_2}$, then there exist sequences $\lambda_{1,n}, \lambda_{2,n} \in D - \mathcal{P}_0$ so that $\lambda_{1,n} \rightarrow \lambda_1$ and $\lambda_{2,n} \rightarrow \lambda_2$ as $n \rightarrow +\infty$ and so that $C_{\lambda_{1,n}} \cong C_{\lambda_{2,n}}$. By the previous case, we have that $\lambda_{2,n} = \lambda_{1,n}$. It follows that $\lambda_2 = \lambda_1$. □

It follows from Theorem 14 that the locus in $\mathcal{M}_3$ (the moduli space of genus 3) of the classes of non-hyperelliptic Riemann surfaces admitting $\mathfrak{S}_4$ as a group of conformal automorphisms is given by the set $\mathcal{P}$ after identification of the points $3(-1 - i\sqrt{7})/2$ with its conjugate $3(-1 + i\sqrt{7})/2$. 


Corollary 15. The normalization of the locus in moduli space of genus 3 consisting of classes of non-hyperelliptic Riemann surfaces admitting the symmetric group $\mathfrak{S}_4$ as group of conformal automorphisms is isomorphic to the $P = \mathbb{C} - \{-2, -1, 2\}$. The puncture corresponding to the point $-2$ corresponds to the hyperelliptic curve admitting $\mathfrak{S}_4$ as a group of conformal automorphisms.

7.2.3. Fields of moduli. The following result states that, except for Klein curve, the KFT family provides equations on the corresponding field of moduli.

Corollary 16.

(1) If $\lambda \in P - P_0$, then $M(C_\lambda) = \mathbb{Q}(\lambda)$.

(2) If $\lambda \in P_0$, then $M(C_\lambda) = \mathbb{Q}$.

Proof. If $\lambda \in P_0$, then $C_\lambda$ is either Klein’s curve or Fermat’s curve, both of them can be defined over $\mathbb{Q}$.

Let $\lambda \in P - P_0$. If $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q})$, then, as $C_\sigma^* = C_{\sigma(\lambda)}$, it follows from Theorem 14 that $C_\sigma^* \cong C_\lambda$ if and only if $\sigma(\lambda) = \lambda$.

7.3. Fuchsian uniformization of the KFT family: A proof of Theorem 10. For each $\lambda_1 \in P$ there are three values for $\lambda \in P$ so that $G(\lambda) = G(\lambda_1)$; one of them being clearly $\lambda_1$ and the others two are given by

$\lambda_2 = -\left(\frac{6 + 5\lambda_1 + \lambda_1^2 - (1 + \lambda_1)\sqrt{\lambda_1^2 - 4}}{2(2 + \lambda_1)}\right)$

$\lambda_3 = -\left(\frac{6 + 5\lambda_1 + \lambda_1^2 + (1 + \lambda_1)\sqrt{\lambda_1^2 - 4}}{2(2 + \lambda_1)}\right)$.

We have that in $P$ there is no solution for $\lambda_1 = \lambda_3$ or for $\lambda_2 = \lambda_3$ and there is exactly one solution for $\lambda_1 = \lambda_2$, this being for $\lambda_1 = -5/2$. Notice that in this case $\lambda_3 = 2 \notin P$ corresponds to the hyperelliptic curve admitting $\mathfrak{S}_4$ as a group of conformal automorphisms.

The curves $C_{\lambda_1}, C_{\lambda_2}$ and $C_{\lambda_3}$ can also be seen as follows from a Fuchsian uniformization’s point of view. Let us consider the orbifold of signature $\langle 0; 2, 2, 3 \rangle$ whose cone points of order 2 are given by $\infty, 0, 1$ and the cone point of order 3 is $\mu(\lambda_1) = \mu_1$ (once we have fixed a value for $\sqrt{\lambda_1^2 - \lambda - 2}$). Let us consider a Fuchsian group

$\Gamma = \langle x_1, x_2, x_3 : x_1^2 = x_2^2 = x_3^2 = (x_1x_2x_3)^3 = 1 \rangle$

acting on the unit disc $\mathbb{D}$ and a universal branched cover $P : \mathbb{D} \to \hat{\mathbb{C}}$ with $\Gamma$ as Deck group of covering transformations so that the fixed point of $x_3$ projects by $P$ to 0, the fixed point of $x_1$ projects to $\infty$ and the fixed point of $x_2$ projects to 1. The fixed point of $x_1x_2x_3$ projects to $\mu_1$.

As a consequence of results due to L. Keen [10] there is a fundamental domain for $\Gamma$ given by a suitable hyperbolic triangle $\Delta_1$, say with sides $s_{11}, s_{12}$ and $s_{13}$ counted in counterclockwise order, so that $x_j$ is an involution with fixed point at the middle side of $s_{1j}$.

In order to find the torsion free normal subgroups $F$ of $\Gamma$ so that $\Gamma/F \cong \mathfrak{S}_4$, up to inner conjugation in $\Gamma$, we only need to find all possible different surjective homomorphisms $\Theta : \Gamma \to \mathfrak{S}_4$ with torsion free kernel up to post-composition with automorphisms of $\mathfrak{S}_4$.

Since $\mathfrak{S}_4 = \langle A, B : A^4 = B^2 = (BA)^3 = 1 \rangle$, up to post-composition by a suitable automorphism of $\mathfrak{S}_4$, we may assume that $\Theta(x_1x_2x_3) = (BA)^{-1}$ and $\Theta(x_4) = A^2$. As there is no non-trivial automorphism of $\mathfrak{S}_4$ that fixes $A^2$ and $BA$, we have that we cannot post-compose with other non-trivial automorphisms of $\mathfrak{S}_4$ without destroying the above choices.

Now, in order for the kernel of $\Theta$ to be torsion free, we need to have that $\Theta(x_1)$ and $\Theta(x_2)$ are order two elements of $\mathfrak{S}_4$ so that $\Theta(x_1)\Theta(x_2) = A^{-1}BA^2$.

By direct inspection one obtains that the only possible choices are given by:

(1) $\Theta(x_1) = B, \Theta(x_2) = ABA^{-1}$;

(2) $\Theta(x_1) = ABA^{-1}, \Theta(x_2) = (BA)B(BA)^{-1}$.
(3) $\Theta(x_1) = (BA)B(AB)^{-1}$, $\Theta(x_2) = B$.

Each of the above three choices provides a Fuchsian group $F$ as desired (they can be computed with GAP). These three Fuchsian groups provide the uniformization of $C_{\lambda_1}$, $C_{\lambda_2}$ and $C_{\lambda_3}$.

If we consider the elements $y_1 = x_2$, $y_2 = (x_1x_2x_3)^{-1}x_1(x_1x_2x_3)$ and $y_3 = x_3$, then we have the relations $y_1^2 = y_2^2 = y_3^2 = (y_1y_2y_3)^3 = 1$ and that $\Gamma = \langle y_1, y_2, y_3 \rangle$.

Again, by the results in [16], there is another fundamental domain for $\Gamma$ given by a suitable hyperbolic triangle $\Delta_s$. These three Fuchsian groups provide the uniformization of $\Gamma$ given by a suitable hyperbolic triangle $\Delta_2$, say with sides $s_{21}$, $s_{22}$ and $s_{23}$ counted in counterclockwise order, so that $y_j$ is an involution with fixed point at the middle side of $s_{2j}$. This permits to see that there is an orientation preserving self-homeomorphism $h_1 : D \to D$ that $h_1 \circ x_j \circ h_1^{-1} = y_j$, for each $j = 1, 2, 3$; in particular, $h_1$ self-conjugates $\Gamma$ into itself. Next we observe that if $\Theta(x_1) = B$, $\Theta(x_2) = ABA^{-1}$ and $\Theta(x_3) = A^2$, then $\Theta(y_1) = ABA^{-1}$, $\Theta(y_2) = (BA)B(AB)^{-1}$ and $\Theta(y_3) = A^2$. In this way, both Fuchsian groups obtained in (1) and (2) are conjugated by the orientation preserving homeomorphism $h_2$.

All of the above also provides a proof of Theorem 10.

**Remark 17.** Let us denote the internal angles of the triangle $\Delta_1$ by $\theta_1$, $\theta_2$ and $\theta_3$, so that $\theta_1$ is the angle between the sides $s_{11}$ and $s_{12}$, $\theta_2$ is the angle between $s_{12}$ and $s_{13}$ and $\theta_3$ is the angle between $s_{13}$ and $s_{11}$. Clearly, $\theta_1 + \theta_2 + \theta_3 = 2\pi/3$. In the particular case when $\theta_1/2 = \theta_2 = \theta_3 = \pi/6$, there is a conformal automorphism $U : D \to D$ of order 4 with the same fixed points as for $x_3$ (so $U^2 = x_3$). The image under $U$ of the triangle $\Delta_1$ is a new triangle, say $\Delta_4$, whose sides, counted in counterclockwise order are $s_{41} = U(s_{11})$, $s_{42} = U(s_{12})$ and $s_{43} = s_{13}$. Let $w_1 = x_2$, $w_2 = x_3x_1x_3$ and $w_3 = x_3$. Then $w_j$ is an involution with a fixed point in the middle of the side $s_{4j}$.

If

$$\Theta(x_1), \Theta(x_2), \Theta(x_3) = (B, ABA^{-1}, A^2),$$

then

$$\Theta(w_1), \Theta(w_2), \Theta(w_3) = (ABA^{-1}, A^2BA^2, A^2),$$

so

$$A^{-1}\Theta(w_1)A, A^{-1}\Theta(w_2)A, A^{-1}\Theta(w_3)A = (B, ABA^{-1}, A^2).$$

If

$$\Theta(x_1), \Theta(x_2), \Theta(x_3) = (ABA^{-1}, (BA)B(AB)^{-1}, A^2),$$

then

$$\Theta(w_1), \Theta(w_2), \Theta(w_3) = ((BA)B(AB)^{-1}, A^{-1}BA, A^2),$$

so

$$A^2\Theta(w_1)A^2, A^2\Theta(w_2)A^2, A^2\Theta(w_3)A^2 = ((BA)B(AB)^{-1}, A^{-1}BA, A^2).$$

As a consequence, this is the case corresponding to $G(\lambda) = 4$. The curve $C_{\lambda/2}$ is uniformized by any of the two the Fuchsian groups appearing in (2) and (3) and the hyperelliptic one is uniformized by the one appearing in (1).

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