Compressible primitive equations: formal derivation and stability of weak solutions

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Abstract

We present a formal derivation of compressible primitive equations for atmosphere modelling. They are obtained from the 3D compressible Navier–Stokes equations with an \textit{anisotropic viscous stress tensor} depending on the density. Then, we study the stability of weak solutions to this problem by introducing an intermediate model obtained by a suitable change of variables. This intermediate model is more practical and it is simpler to achieve the main result.

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1. Introduction

Among equations of geophysical fluid dynamics (see Buntebarth [5]), classically the equations governing the motion of the atmosphere are primitive equations (PEs). In the hierarchy of geophysical fluid dynamics models, they are situated between non-hydrostatic models and shallow water models.

\textit{Derivation of the compressible PEs.} Compressible primitive equations (CPEs) are obtained from the hydrostatic approximation (see, for instance, Pedlowski [11] or Temam and Ziane [12]) of the full three-dimensional set of Navier–Stokes equations for atmosphere modelling.
Neglecting phenomena such as the evaporation and solar heating, the PEs read

\[
\begin{align*}
\frac{d}{dt} \rho + \rho \text{div} U &= 0, \\
\rho \frac{d}{dt} \mathbf{u} + \nabla_x p &= D, \\
\partial_y p &= -g \rho, \\
p(\rho) &= c^2 \rho
\end{align*}
\] (1)

where

\[
\frac{d}{dt} = \partial_t + \mathbf{u} \cdot \nabla_x + v \partial_y
\]

with \(x = (x_1, x_2)\) the horizontal and \(y\) the vertical coordinate.

\(U\) is the three-dimensional velocity vector with component \(u = (u_1, u_2)\) for the horizontal velocity and \(v\) for the vertical one. The terms \(\rho, p, g\) stand for the density, the barotropic pressure and the gravity vector \((0, 0, g)\). The constant \(c^2\) is usually set to \(RT\) where \(R\) is the specific gas constant for air and \(T\) the temperature.

In this paper, the diffusion term \(D\) reads

\[
D = 2 \text{div}_x \left( v_1(t, x, y) D_x(\mathbf{u}) \right) + \partial_z \left( v_2(t, x, y) \partial_y \mathbf{u} \right).
\]

It is obtained by introducing an anisotropic viscous tensor in the initial Navier–Stokes equations where \(\text{div}_x\) stands for \(\partial_{x_1} + \partial_{x_2}\), \(D_x = (\nabla_x + \nabla_x^t)/2\) and \(v_1(t, x, y) \neq v_2(t, x, y)\) represent the anisotropic pair of viscosity depending on the density \(\rho\).

The main difference with respect to the classical viscous term found in the literature (see for instance Temam and Ziane [12]) is that viscosities depend on the density.

Mathematical analysis of CPEs. The mathematical analysis of PEs for atmosphere modelling was first carried out by Lions et al [9]. These authors have taken into account evaporation and solar heating with constant viscosities. They produced the mathematical formulation in two- and three-dimensions based on the works of J Leray and obtained the existence of weak solutions for all time (see also Temam and Ziane [12] where the result was proved by different means).

Following Temam and Ziane [12], Ersoy and Ngom [6] showed the global weak existence for the 2D version of model (1) by a useful change of vertical coordinates.

Currently, to our knowledge, there is no way to prove an existence or stability result for model (1). One of the difficulties encountered is to obtain energy estimates. Indeed, proceeding by standard techniques, multiplying the conservation of the momentum equations of system (1) by \((\mathbf{u}, v)\), we get

\[
\frac{d}{dt} \int_\Omega (\rho |\mathbf{u}|^2 + \rho \ln \rho - \rho + 1) \, dx \, dy + \int_\Omega \left( 2v_1 |D_x(\mathbf{u})|^2 + v_2 |\partial_y \mathbf{u}|^2 \right) \, dx \, dy + \int_\Omega \rho g v \, dx \, dy
\]

where the sign of the integral \(\int_\Omega \rho g v \, dx \, dy\) is unknown. There is no way to control, \textit{prima facie}, the integral term \(\int_\Omega \rho g v \, dx \, dy\) introduced by the hydrostatic equation \(\partial_y p = -g \rho\). To overcome this problem, we change the variables and study an intermediate problem. Following Ersoy and Ngom [6], setting

\[
z = 1 - e^{-g/c^2 y} \quad \text{and} \quad w(t, x, z) = e^{-g/c^2 y} v(t, x, y)
\]

and assuming

\[
v_1(t, x, y) = \tilde{v}_1 \rho(t, x, y) \quad \text{and} \quad v_2(t, x, y) = \tilde{v}_2 \rho(t, x, y) e^{2y} \quad \text{with} \ \tilde{v}_1 > 0,
\]
we obtain the following model:

\[
\begin{aligned}
\frac{d}{dt} \xi + \xi(\text{div}_x u + \partial_z w) &= 0, \\
\rho \frac{d}{dt} u + \nabla_x p &= D_z, \\
\partial_z \xi &= 0, \\
p(\xi) &= c^2 \xi
\end{aligned}
\]  

(2)

where \(d/dt\) denotes

\[
\frac{d}{dt} = \partial_t + u \cdot \nabla_x + w \partial_z
\]

and

\[
D_z = 2\text{div}_x (v_1(t, x, z) D_x(u)) + \partial_z (v_2(t, x, z) \partial_z u).
\]  

(3)

Consequently, in the computation of the energy the integral term vanishes since the right-hand side of the equation \(\partial_z \xi\) becomes 0. Thus, we can obtain preliminary estimates.

In order to show the weak stability, the additional required estimates are provided by the BD-entropy (see, for instance, Bresch et al. [1–4]) by adding a regularizing term to equations (2). In this paper, we have added a quadratic friction source term. Combining this term to the viscous one (3) brings regularity on the density which is required to pass to the limit in the non-linear terms (e.g. for the term \(\xi u \otimes u\) where typically a strong convergence of \(\sqrt{\xi} u\) is needed). Finally, energy and BD-entropy estimates are enough to show a weak stability result for model (2) and by the reverse change of variables for model (1).

Currently, the question of the existence of weak solutions remains an open question for model (2) (and also for model (1)).

This paper is organized as follows. In section 2, starting from the 3D compressible Navier–Stokes equations with an anisotropic viscous tensor, we formally derive model (1). Then, we present the main result in section 2.2. We provide a complete proof in section 3.2. We provide a complete proof in section 3.2.

2. Formal derivation of the atmosphere model

We consider the Navier–Stokes model in a bounded three-dimensional domain with periodic boundary conditions on \(\Omega_2\) and free conditions on the rest of the boundary. More precisely, we assume that the motion of the medium occurs in a domain \(\Omega = \{(x, y); x \in \Omega_2, 0 < y < H\}\) where \(\Omega_2 = T^2\) is the bi-dimensional torus and \(H\) the characteristic scale of the altitude. The full Navier–Stokes equations are

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - \text{div}\sigma - \rho f &= 0, \\
p &= p(\rho),
\end{aligned}
\]  

(4-6)

where \(\rho\) is the density of the fluid and \(u = (u, v)^t\) stands for the fluid velocity with \(u = (u_1, u_2)^t\) the horizontal component and \(v\) the vertical one. \(\sigma\) is the total asymmetric stress tensor. The pressure law is given by the equation of state:

\[
p(\rho) = c^2 \rho
\]

(7)

for some given positive constant \(c\). The term \(f\) regroups the quadratic friction source term and the gravity strength:

\[
f = -R \sqrt{u_1^2 + u_2^2} (u_1, u_2, 0)^t - g k
\]

where \(R\) is a positive constant, \(g\) is the gravitational constant and \(k = (0, 0, 1)^t\) (where \(X^t\) stands for the transpose of tensor \(X\)).
Remark 1. As we will see later, the friction term is a mathematical remedy to ensure the stability of weak solutions of the problem.

The total stress tensor is

$$\sigma = -pI + 2 \Sigma.D(u) + \lambda \text{div}(u) I_3$$

where the term $\Sigma.D(u)$ reads

$$\begin{pmatrix}
2\mu_1 D_x(u) & \mu_2 \left( \partial_j u + \nabla_x v \right) \\
\mu_2 \left( \partial_j u + \nabla_x v \right)^t & 2\mu_3 \partial_j v
\end{pmatrix}$$

with $I_3$ the identity matrix. In the definition above, the term $\Sigma(t,x,y)$ stands for the following non-constant anisotropic viscous tensor (see, for instance, [6–8]):

$$
\begin{pmatrix}
\mu_1 & \mu_1 & \mu_2 \\
\mu_1 & \mu_1 & \mu_2 \\
\mu_3 & \mu_3 & \mu_3
\end{pmatrix}
$$

The term $D_x(u)$ is the strain tensor with respect to the horizontal variable $x$, i.e.

$$2D_x(u) = \nabla_x u + \nabla_x^t u = \left( \partial_{x_i} u_j + \partial_{x_j} u_{i} \right)_{1 \leq i,j \leq 2}.$$

The last term $\lambda \text{div}(u)$ is the classical normal stress tensor where $\lambda$ is the volumetric viscosity.

Remark 2. Let us remark that if we play with the magnitude of viscosity $\mu_i$, the matrix $\Sigma$ will be useful to set a privileged flow direction.

The Navier–Stokes system is closed with the following boundary conditions on $\partial \Omega$:

$$\begin{align*}
&v|_{y=0} = v|_{y=H} = 0, \\
&\partial_y u|_{y=0} = \partial_y u|_{y=H} = 0.
\end{align*}$$

(8)

We also assume that the distribution of the horizontal component of the velocity $u$ and the density distribution are known at the initial time $t = 0$:

$$\begin{align*}
u(0,x,y) &= u_0(x,y), \\
\rho(0,x,y) &= \xi_0(x)e^{-g/c_2 y}
\end{align*}$$

(9)

where $\xi_0$ is a bounded positive function:

$$0 \leq \xi_0(x) \leq M < +\infty.$$

Remark 3. The expression of $\rho$ at time $t = 0$ is quite natural since in the atmosphere the density is stratified, i.e. for each altitude $y$, the density has the profile of the given function $\xi_0$. Moreover, it is also mathematically justified at the end of section 2.1, more precisely see equation (13).

2.1. Formal derivation of the CPEs

Taking advantage of the shallowness of the atmosphere, we assume that the characteristic scale for the altitude $H$ is small with respect to the characteristic length $L$. In this context, we also assume that the vertical movements and variations are very small compared with the horizontal ones which justifies the following approximation: let $\varepsilon$ be a ‘small’ parameter such as

$$\varepsilon = \frac{H}{L} = \frac{V}{U}.$$
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where \( V \) and \( U \) are respectively the characteristic scale of the vertical and horizontal velocity. We introduce the characteristic time \( T \) such as \( T = L/U \) and the pressure unit \( P = \bar{p} U^2 \) where \( \bar{p} \) is a characteristic density. Finally, we note the dimensionless quantities of time, space, fluid velocity, pressure, density and viscosities:

\[
\hat{t} = \frac{t}{T}, \quad \hat{x} = \frac{x}{L}, \quad \hat{y} = \frac{y}{H}, \quad \hat{u} = \frac{u}{U}, \quad \hat{v} = \frac{v}{V},
\]

\[
\hat{\rho} = \frac{\rho}{\rho U^2}, \quad \hat{\rho} = \frac{\rho}{\bar{p}}, \quad \hat{\lambda} = \frac{\lambda}{x}, \quad \hat{\mu}_j = \frac{\mu_j}{\bar{\mu}_j}, \quad j = 1, 2, 3.
\]

With these notations, the Froude number \( Fr \), the Reynolds number associated with the viscosity \( \mu_i \), \( Re_i \), \( (i = 1, 2, 3) \), the Reynolds number associated with the viscosity \( \lambda \), \( Re_\lambda \) and the Mach number \( Ma \) are, respectively,

\[
Fr = \frac{U}{\sqrt{g H}}, \quad Re_i = \frac{\rho UL}{\mu_i}, \quad Re_\lambda = \frac{\rho UL}{\lambda}, \quad Ma = \frac{U}{c}.
\] (10)

Applying this scaling to system (4)–(7), using the definition of the dimensionless number (10) and dropping \( \hat{\cdot} \), we get the following non-dimensional system:

\[
\begin{align*}
\partial_t \rho + \nabla (\rho \vec{u}) + \partial_y (\rho \vec{v}) &= 0, \\
\partial_t (\rho \vec{u}) + \nabla (\rho \vec{u} \otimes \vec{u}) + \partial_y (\rho \vec{v} \vec{u}) + \frac{1}{M_a^2} \nabla \rho + r \rho |\vec{u}| &+ \frac{1}{\epsilon^2} \partial_y \left( \frac{1}{\epsilon^2} \partial_y \vec{u} + \nabla \vec{v} \right) \\
\partial_y \rho &= -\frac{1}{\epsilon^2} \frac{M_a^2}{F_r^2} \frac{\xi}{\rho} \\
\partial_t (\rho \vec{v}) + \nabla (\rho \vec{u} \vec{v}) + \partial_y (\rho \vec{v}^2) + \frac{1}{\epsilon^2} \partial_y \left( \frac{1}{\epsilon^2} \partial_y \vec{v} + \nabla \vec{v} \right) + \frac{2}{\epsilon^2} \partial_y \left( \frac{1}{\epsilon^2} \partial_y \vec{v} \vec{u} \right) + \frac{1}{\epsilon^2} \partial_y \left( \lambda \partial_y \vec{u} + \lambda \partial_y \vec{v} \right)
\end{align*}
\] (11)

where we have noted \( R = r/L \).

Next, assuming the following asymptotic regime:

\[
\frac{\mu_1}{Re_1} = v_1, \quad \frac{\mu_i}{Re_i} = \epsilon^2 v_i, \quad i = 2, 3 \quad \text{and} \quad \frac{\lambda}{Re_\lambda} = \epsilon^2 \gamma.
\]

and dropping all terms of order \( O(\epsilon) \), system (11) reduces to the following CPEs:

\[
\begin{align*}
\partial_t \rho + \nabla (\rho \vec{u}) + \partial_y (\rho \vec{v}) &= 0, \\
\partial_t (\rho \vec{u}) + \nabla (\rho \vec{u} \otimes \vec{u}) + \partial_y (\rho \vec{v} \vec{u}) + \frac{1}{M_a^2} \nabla \rho + r \rho |\vec{u}| &+ \frac{1}{\epsilon^2} \partial_y \left( \frac{1}{\epsilon^2} \partial_y \vec{u} + \nabla \vec{v} \right) \\
\partial_y \rho &= -\frac{M_a^2}{F_r^2} \frac{\xi}{\rho} \\
\partial_t (\rho \vec{v}) + \nabla (\rho \vec{u} \vec{v}) + \partial_y (\rho \vec{v}^2) + \frac{1}{\epsilon^2} \partial_y \left( \frac{1}{\epsilon^2} \partial_y \vec{v} + \nabla \vec{v} \right) + \frac{2}{\epsilon^2} \partial_y \left( \frac{1}{\epsilon^2} \partial_y \vec{v} \vec{u} \right) + \frac{1}{\epsilon^2} \partial_y \left( \lambda \partial_y \vec{u} + \lambda \partial_y \vec{v} \right)
\end{align*}
\] (12)

holding in the domain \( \Omega = \{(x, y); x \in \Omega_x \subset \mathbb{R}^2, 0 < y < 1\} \).

Simplifying by setting \( M_a = F_r \), the hydrostatic equation of system (12) gives

\[
\rho(t, x, y) = \xi(t, x)e^{-y}
\] (13)

for some function \( \xi = \xi(t, x) \) that we call again ‘density’. 

In what follows, we note
\[ \rho_A \]
Remark 4. This expression of the density justifies the choice of the initial data (9) for the density \( \rho \).

In what follows, we note
\[ v_1(t, x, y) = \bar{v}_1 \rho(t, x, y) \quad \text{and} \quad v_2 = \bar{v}_2 \rho(t, x, y) e^{2y}. \] (14)
for some positive constant \( \bar{v}_1 \) and \( \bar{v}_2 \).

2.2. The main result
In order to define a weak solution of the CPEs, we introduce the set of functions \( \rho \in \mathcal{P}(u, v; y, \rho_0) \) which satisfies
\[
\rho \in L^\infty(0, T; L^1(\Omega)), \quad \sqrt{\rho} u \in L^2(0, T; (L^2(\Omega))^2),
\]
\[
\sqrt{\rho} v \in L^\infty(0, T; L^2(\Omega)), \quad \sqrt{\rho} D_x(u) \in L^2(0, T; (L^2(\Omega))^{2\times 2}), \quad \sqrt{\rho} \partial_y v \in L^2(0, T; L^2(\Omega)),
\]
with \( \rho \gtrsim 0 \) and where \( (\rho, \sqrt{\rho} u, \sqrt{\rho} v) \) satisfies
\[
\begin{cases}
\partial_t \rho + \text{div}_x(\sqrt{\rho} \sqrt{\rho} u) + \partial_y(\sqrt{\rho} \sqrt{\rho} v) = 0, \\
\rho_{t=0} = \rho_0.
\end{cases}
\]
We also define the following integral operators for any smooth test function \( \varphi \) with compact support such as \( \varphi(T, x, y) = 0 \) and \( \varphi_0 = \varphi_{t=0} \):
\[
\mathcal{A}(\rho, u, v; \varphi, dy) = -\int_0^T \int_\Omega \rho u \delta_y \varphi \, dx \, dy \, dt
\]
\[
+ \int_0^T \int_\Omega (2v_1(t, x, y) \rho D_x(u) - \rho u \otimes u) : \nabla_x \varphi \, dx \, dy \, dt
\]
\[
+ \int_0^T \int_\Omega r \rho |u| \varphi \, dx \, dy \, dt - \int_0^T \int_\Omega \rho \text{div}(\varphi) \, dx \, dy \, dt
\]
\[
- \int_0^T \int_\Omega u \partial_y (v_2(t, x, y) \partial_y \varphi) \, dx \, dy \, dt - \int_0^T \int_\Omega \rho v u \partial_y \varphi \, dx \, dy \, dt
\] (15)
\[
\mathcal{B}(\rho, u, v; \varphi, dy) = \int_0^T \int_\Omega \rho v \varphi \, dx \, dy \, dt
\]
and
\[
\mathcal{C}(\rho, u; \varphi, dy) = \int_\Omega \rho \varphi_0 \, dx \, dy
\] (16)
Under these definitions, we consider weak solutions of the CPEs in sense of the distributions. More precisely, we will say that:

Definition 1. A weak solution of system (12) on \([0, T] \times \Omega\), with boundary conditions (8) and initial conditions (9), is a collection of functions \( (\rho, u, v) \) such as \( \rho \in \mathcal{P}(u, v; y, \rho_0) \) and the following equality holds for all smooth test function \( \varphi \) with compact support such as \( \varphi(T, x, y) = 0 \) and \( \varphi_0 = \varphi_{t=0} \):
\[
\mathcal{A}(\rho, u, v; \varphi, dy) + \mathcal{B}(\rho, u, v; \varphi, dy) = \mathcal{C}(\rho, u; \varphi, dy).
\]
Then, we can state the main result:

Theorem 1. Let \( (\rho_n, u_n, v_n) \) be a sequence of weak solutions of system (12), with boundary conditions (8) and initial conditions (9), satisfying entropy inequalities (23) and (40) such as
\[
\rho_n \geq 0, \quad \rho_0^n \rightarrow \rho_0 \text{ in } L^1(\Omega), \quad \rho_0^n u_0^n \rightarrow \rho_0 u_0 \text{ in } L^1(\Omega).
\]
Then, up to a subsequence,

- \( \rho_n \) converges strongly in \( C^0(0, T; L^{3/2}(\Omega)) \),
- \( \sqrt{\rho_n} \) converges strongly in \( L^2(0, T; (L^{3/2}(\Omega))^2) \),
- \( \rho u_n \) converges strongly in \( L^1(0, T; (L^1(\Omega))^2) \) for all \( T > 0 \),
- \( (\rho_n, \sqrt{\rho_n} u_n, \sqrt{\rho_n} v_n) \) converges to a weak solution of system (12),
- \( (\rho_n, u_n, v_n) \) satisfies the energy inequality (23), the entropy inequality (40) and converges to a weak solution of (12)–(8).

The proof of the main result is divided into three parts:

- in section 3.1, we perform a change of variables using \((\xi, u, w = e^{-y}v)\) as unknowns instead of \((\rho, u, v)\) and we obtain an intermediate model,
- in sections 3.2.2–3.2.6, we prove the stability of weak solutions of the model problem,
- in section 3.2.7, by the reverse change of variables, we prove the main result.

3. Stability of weak solutions for the CPEs

As pointed out in section 1, the classical techniques fail. To overpass this difficulty, following Ersoy et al we perform a useful change of variables which transform the initial problem into a more simpler and more practical for mathematical analysis.

3.1. A model problem; an intermediate model

Let us first remark that the structure of the density \( \rho \), defined as a tensorial product (see equation (13)), suggests the following change of variables:

\[
z = 1 - e^{-y}
\]

where the vertical velocity in the new coordinates becomes

\[
w(t, x, z) = e^{-y}u(t, x, y).
\]

(17)

(18)

Since the new vertical coordinate \( z \) is defined as \( (d/dy)z = e^{-y} \), multiplying by \( e^y \) system (12) and using the viscosity profile (14) and the change of variables (17)–(18) provides the following model, called model problem:

\[
\begin{align*}
\partial_t \xi + \text{div}_x (\xi u) + \partial_z (\xi w) &= 0, \\
\partial_t (\xi u) + \text{div}_x (\xi u \otimes u) + \partial_z (\xi u w) + \nabla_x \xi + r|u|u &= 2\tilde{\nu}_1 \text{div}_x (\xi D_x(u)) + \tilde{\nu}_2 \partial_z (\xi \partial_z u), \\
\partial_z \xi &= 0
\end{align*}
\]

(19)

holding in the domain \( \Omega' = \{(x, z); x \in \Omega'_x, 0 < z < 1 - e^{-1}\} \) where \( \Omega'_x = \mathbb{T}^2 \) is the bi-dimensional torus.

In the new variables, the boundary conditions (8) and the initial conditions (9) become periodic conditions on \( \Omega'_x \),

\[
\begin{align*}
w|_{z=0} = w|_{z=h} &= 0, \\
\partial_z u|_{z=0} = \partial_z u|_{z=h} &= 0
\end{align*}
\]

(20)

and

\[
\begin{align*}
u(0, x, y) &= u_0(x, z), \\
\xi(0, x) &= \xi_0(x)
\end{align*}
\]

(21)

where \( h = 1 - e^{-1} \).
3.2. Mathematical study of the model problem

In this section, we show the stability of weak solutions of system (19). To this end, we will say that:

**Definition 2.** A weak solution of system (19) on \([0, T] \times \Omega\), with boundary (20) and initial conditions (21), is a collection of functions \((\xi, u, w)\), if \(\xi \in \mathcal{P}(u, w; z, \xi_0)\) and the following equality holds for all smooth test function \(\phi\) with compact support such as \(\phi(T, x, y) = 0\) and \(\phi_0 = \phi_{t=0}\):

\[
A(\xi, u; \phi, dz) = C(\xi, u; \phi, dz)
\]

where \(A\) and \(C\) are given by (15) and (16).

We then have the following result:

**Theorem 2.** Let \((\xi_n, u_n, w_n)\) be a sequence of weak solutions of system (19), with boundary conditions (20) and initial conditions (21), satisfying entropy inequalities (23) and (40) such as

\[
\xi_n \geq 0, \quad \xi_n \to \xi_0 \quad \text{in} \quad L^1(\Omega), \quad \xi_n u_n \to \xi_0 u_0 \quad \text{in} \quad L^1(\Omega).
\]

Then, up to a subsequence,

- \(\xi_n\) converges strongly in \(C^0(0, T; L^{3/2}(\Omega'))\),
- \(\sqrt{\xi_n} u_n\) converges strongly in \(L^2(0, T; (L^{3/2}(\Omega'))^2)\),
- \(\xi_n u_n\) converges strongly in \(L^1(0, T; (L^1(\Omega'))^2)\) for all \(T > 0\),
- \((\xi_n, \sqrt{\xi_n} u_n, \sqrt{\xi_n} w_n)\) converges to a weak solution of system (19),
- \((\xi_n, u_n, w_n)\) satisfies the energy inequality (23), the entropy inequality (40) and converges to a weak solution of \((19)-(20)\).

We divide the proof of theorem 2 into three steps:

- in section 3.2.1, we obtain suitable *a priori* bounds on \((\xi, u, w)\),
- in sections 3.2.2–3.2.5, we show the compactness of sequences \((\xi_n, u_n, w_n)\) in appropriate space function,
- in section 3.2.6, we prove that we can pass to the limit in all terms of system (19) which ends the proof of theorem 2.

### 3.2.1. Energy and entropy estimates

A part of *a priori* bounds on \((\xi, u, w)\) are obtained by the physical energy inequality which is obtained in a classical way by multiplying the momentum equation by \(u\), using the mass equation and integrating by parts. We obtain the following inequality:

\[
\frac{d}{dt} \int_{\Omega} \left( \frac{u^2}{2} + (\xi \ln \xi - \xi + 1) \right) dx \, dz + \int_{\Omega} \xi (2\tilde{v}_1|D_x(u)|^2 + \tilde{v}_2|\partial_z u|^2) \, dx \, dz + \rho \int_{\Omega} |\xi u|^3 \, dx \, dz \leq 0
\]

(23)

which provides the uniform estimates:

- \(\sqrt{\xi} u\) is bounded in \(L^\infty(0, T; (L^2(\Omega'))^2)\),
- \(\xi^{1/2} u\) is bounded in \(L^2(0, T; (L^2(\Omega'))^2)\),
- \(\sqrt{\xi} \partial_z u\) is bounded in \(L^2(0, T; (L^2(\Omega'))^2)\),
- \(\sqrt{\xi} D_x(u)\) is bounded in \(L^2(0, T; (L^2(\Omega'))^2)\),
- \(\xi \ln \xi - \xi + 1\) is bounded in \(L^\infty(0, T; L^1(\Omega'))\).
The strong convergence of $\sqrt{\xi}u$ required to pass to the limit in the non-linear term $\xi u \otimes u$ is obtained by the mathematical BD-entropy. To this end, we first take the gradient of the mass equation, then we multiply by $2\bar{v}_1$ and write the term $\nabla_x \xi$ as $\xi \nabla_v \ln \xi$ to obtain
\begin{align*}
\partial_t (2\bar{v}_1 \xi \nabla_v \ln \xi) + \text{div}_v ((2\bar{v}_1 \xi \nabla_v \ln \xi \otimes u) + \partial_z (2\bar{v}_1 \xi \nabla_v \ln \xi w) + \text{div}_v (2\bar{v}_1 \xi \nabla_v^2 u) + \partial_z (2\bar{v}_1 \xi \nabla_v w) = 0. \tag{29}
\end{align*}
Next, we sum equation (29) with the momentum equation of system (19) to get the equation
\begin{align*}
\partial_t (\xi (u + 2\bar{v}_1 \nabla_v \ln \xi)) + \text{div}_v (\xi (u + 2\bar{v}_1 \nabla_v \ln \xi \otimes u) + \partial_z (\xi w u) + 2\bar{v}_1 \partial_z \nabla (\xi w) + 2\bar{v}_1 \text{div}_v (\xi A_\xi (u)) - \bar{v}_2 \partial_z (\xi \partial_z u) + r \xi |u| u + \nabla \xi \xi = 0, \tag{30}
\end{align*}
where $A_\xi (u) = (\nabla_v u - \nabla_v^T u)/2$ is the vorticity tensor. The mathematical BD-entropy inequality is then obtained by multiplying the previous equation by $u + 2\bar{v}_1 \nabla_v \ln \xi$ and by integrating by parts. To this end, multiplying equation (30) by the term $u + 2\bar{v}_1 \nabla_v \ln \xi$ and integrating over $\Omega$, we have to compute each term of the following integral:
\begin{align*}
\int_{\Omega} \partial_t (u + 2\bar{v}_1 \nabla_v \ln \xi))(u + 2\bar{v}_1 \nabla_v \ln \xi) \, dx \, dz + 
\int_{\Omega} \text{div}_v (\xi (u + 2\bar{v}_1 \nabla_v \ln \xi \otimes u))(u + 2\bar{v}_1 \nabla_v \ln \xi) \, dx \, dz + 
\int_{\Omega} \partial_z (\xi w u)(u + 2\bar{v}_1 \nabla_v \ln \xi) \, dx \, dz + 
\int_{\Omega} \partial_z \nabla (\xi w) (u + 2\bar{v}_1 \nabla_v \ln \xi) \, dx \, dz + 
\int_{\Omega} 2\bar{v}_1 \text{div}_v (\xi A_\xi (u))(u + 2\bar{v}_1 \nabla_v \ln \xi) \, dx \, dz + 
\int_{\Omega} \partial_z (\xi \partial_z u) (u + 2\bar{v}_1 \nabla_v \ln \xi) \, dx \, dz + 
\int_{\Omega} \nabla \xi \xi (u + 2\bar{v}_1 \nabla_v \ln \xi) \, dx \, dz = 0. \tag{31}
\end{align*}
The first two terms read as follows:
\begin{align*}
\int_{\Omega} \partial_t (u + 2\bar{v}_1 \nabla_v \ln \xi))(u + 2\bar{v}_1 \nabla_v \ln \xi) \, dx \, dz + 
\int_{\Omega} \text{div}_v (\xi (u + \bar{v}_1 \nabla_v \ln \xi \otimes u))(u + 2\bar{v}_1 \nabla_v \ln \xi) \, dx \, dz = 
\frac{1}{2} \int_{\Omega} \xi \partial_t |u + 2\bar{v}_1 \nabla_v \ln \xi|^2 \, dx \, dz + 
\int_{\Omega} (u + 2\bar{v}_1 \nabla_v \ln \xi) \partial_z \xi \, dx \, dz + 
\frac{1}{2} \int_{\Omega} (\xi u \cdot \nabla) |u + 2\bar{v}_1 \nabla_v \ln \xi|^2 \, dx \, dz + 
\int_{\Omega} (u + 2\bar{v}_1 \nabla_v \ln \xi) \partial_z \xi \, dx \, dz + 
\frac{1}{2} \int_{\Omega} (u + 2\bar{v}_1 \nabla_v \ln \xi) \partial_z \xi \, dx \, dz + 
\int_{\Omega} (u + 2\bar{v}_1 \nabla_v \ln \xi) \partial_z \xi \, dx \, dz
\end{align*}
which is also
\begin{align*}
\int_{\Omega} \partial_t (u + 2\bar{v}_1 \nabla_v \ln \xi))(u + 2\bar{v}_1 \nabla_v \ln \xi) \, dx \, dz + 
\int_{\Omega} \text{div}_v (\xi (u + \bar{v}_1 \nabla_v \ln \xi \otimes u))(u + 2\bar{v}_1 \nabla_v \ln \xi) \, dx \, dz = 
\frac{1}{2} \int_{\Omega} \xi \partial_t |u + 2\bar{v}_1 \nabla_v \ln \xi|^2 \, dx \, dz + 
\int_{\Omega} |\text{div}_v (\xi u) |u + 2\bar{v}_1 \nabla_v \ln \xi|^2 \, dx \, dz + 
\int_{\Omega} (u + 2\bar{v}_1 \nabla_v \ln \xi) \partial_z \xi \, dx \, dz + 
\frac{1}{2} \int_{\Omega} (u + 2\bar{v}_1 \nabla_v \ln \xi) \partial_z \xi \, dx \, dz + 
\frac{1}{2} \int_{\Omega} (u + 2\bar{v}_1 \nabla_v \ln \xi) \partial_z \xi \, dx \, dz. \tag{32}
\end{align*}
Remarking that
\begin{align*}
\partial_z (\xi w u) = \partial_z (\xi w (u + 2\bar{v}_1 \nabla_v \ln \xi)) - 2\bar{v}_1 \partial_z w \nabla_v \xi,
\end{align*}
we have
\[
\int \bar{v}_i (\xi w (u + 2\bar{v}_i \nabla_x \ln \xi)) (\xi w u) (u + 2\bar{v}_i \nabla_x \ln \xi) \, dx \, dz
\]
\[
-2\bar{v}_i \int \nabla_x \xi \partial_z w (u + 2\bar{v}_i \nabla_x \ln \xi) \, dx \, dz
\]
\[
= \frac{1}{2} \int \xi w \partial_z |u + 2\bar{v}_i \nabla_x \ln \xi|^2 \, dx \, dz + \int (u + 2\bar{v}_i \nabla_x \ln \xi)^2 \partial_z (\xi w) \, dx \, dz
\]
\[
+ 2\bar{v}_i \int w \nabla_x \xi \partial_z u \, dx \, dz
\]
which is finally
\[
\int \xi w \partial_z (\xi w (u + 2\bar{v}_i \nabla_x \ln \xi)) (u + 2\bar{v}_i \nabla_x \ln \xi) \, dx \, dz
\]
\[
-2\bar{v}_i \int \nabla_x \xi \partial_z w (u + 2\bar{v}_i \nabla_x \ln \xi) \, dx \, dz
\]
\[
= -\frac{1}{2} \int |u + 2\bar{v}_i \nabla_x \ln \xi|^2 \partial_z (\xi w) \, dx \, dz + \int (u + 2\bar{v}_i \nabla_x \ln \xi)^2 \partial_z (\xi w) \, dx \, dz
\]
\[
+ 2\bar{v}_i \int w \nabla_x \xi \partial_z u \, dx \, dz. \tag{33}
\]
Summing (32) and (33), we obtain
\[
\int \partial_t (\xi w (u + 2\bar{v}_i \nabla_x \ln \xi)) (u + 2\bar{v}_i \nabla_x \ln \xi) \, dx \, dz
\]
\[
+ \int \text{div}_x (\xi (u + 2\bar{v}_i \nabla_x \ln \xi) \otimes u) (u + 2\bar{v}_i \nabla_x \ln \xi) \, dx \, dz
\]
\[
+ \int \partial_t (\xi w u) (u + 2\bar{v}_i \nabla_x \ln \xi) \, dx \, dz
\]
\[
= \frac{1}{2} \frac{d}{dt} \int \xi |u + 2\bar{v}_i \nabla_x \ln \xi|^2 \, dx \, dz + 2\bar{v}_i \int w \nabla_x \xi \partial_z u \, dx \, dz. \tag{34}
\]
The fourth term in equation (31), i.e. 2\bar{v}_i \int \partial_t \nabla_x (\xi w) (u + 2\bar{v}_i \nabla_x \ln \xi) \, dx \, dz gives
\[
\int \partial_t \nabla_x (\xi w) (u + 2\bar{v}_i \nabla_x \ln \xi) \, dx \, dz = -\int \nabla_x (\xi w) \partial_t u \, dx \, dz
\]
\[
= \int \xi w \partial_t \text{div}_x (u) \, dx \, dz
\]
\[
= \int (w \partial_t \text{div}_x (\xi w)) - w \nabla_x \xi \partial_z u \, dx \, dz.
\]
Differentiating the equation of the conservation of the mass with respect to \(z\),
\[
\partial_t \text{div}_x (\xi u) = -\xi \partial_x^2 w,
\]
we get
\[
2\bar{v}_i \int \partial_t \nabla_x (\xi w) (u + 2\bar{v}_i \nabla_x \ln \xi) \, dx \, dz = -2\bar{v}_i \int (\xi w \partial_x^2 w - w \nabla_x \xi \partial_z u) \, dx \, dz
\]
\[
= 2\bar{v}_i \int \xi |\partial_x w|^2 \, dx \, dz - 2\bar{v}_i \int w \nabla_x \xi \partial_z u \, dx \, dz. \tag{35}
\]
In order to compute the term $-2\tilde{v}_1 \int_{\Omega} \text{div}_x (\xi A_x(u))(u + 2\tilde{v}_1 \nabla_x \ln \xi) \, dx \, dz$ in equation (31), we have just to remark that thanks to periodic conditions, we have

$$\int_{\Omega} \text{div}_x (\xi A_x(u)) \nabla_x \ln \xi \, dx \, dz = 0$$

which leads to

$$-2\tilde{v}_1 \int_{\Omega} \text{div}_x (\xi A_x(u))(u + 2\tilde{v}_1 \nabla_x \ln \xi) \, dx \, dz = 2\tilde{v}_1 \int_{\Omega} \xi |A_x(u)|^2 \, dx \, dz. \tag{36}$$

The fifth and sixth terms in equation (31) simply read

$$r \int_{\Omega} |u|^3 u \, dx \, dz = r \int_{\Omega} |u|^3 x \, dx \, dz + 2\tilde{v}_1 r \int_{\Omega} |u| u \nabla_x \ln \xi \, dx \, dz \tag{37}$$

and

$$-\tilde{v}_2 \int_{\Omega} \partial_x (\xi \partial_x u)(u + 2\tilde{v}_1 \nabla_x \ln \xi) \, dx \, dz = \tilde{v}_2 \int_{\Omega} |\partial_x u|^2 \, dx \, dz. \tag{38}$$

The last term $\int_{\Omega} \nabla_x \xi (u + 2\tilde{v}_1 \nabla_x \ln \xi) \, dx \, dz$ gives

$$\int_{\Omega} \nabla_x \xi (u + 2\tilde{v}_1 \nabla_x \ln \xi) \, dx \, dz = \frac{d}{dt} \int_{\Omega} (\xi \log \xi - \xi + 1) \, dx \, dz$$

$$+ 8\tilde{v}_1 \int_{\Omega} |\nabla_x \sqrt{\xi}|^2 \, dx \, dz. \tag{39}$$

Finally, summing the terms (34) to (39), we obtain the entropy inequality:

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( |u|^2 + 2\tilde{v}_1 \nabla_x \ln \xi |^2 + 2(\xi \log \xi - \xi + 1) \right) \, dx \, dz$$

$$+ \int_{\Omega} 2\tilde{v}_1 \xi |\partial_x u|^2 + 2\tilde{v}_1 \xi |A_x(u)|^2 + \tilde{v}_2 |\partial_x u|^2 \, dx \, dz$$

$$+ \int_{\Omega} r|u|^3 + 2\tilde{v}_1 r |u| u \nabla_x \xi + 8\tilde{v}_1 |\nabla_x \sqrt{\xi}|^2 \, dx \, dz = 0 \tag{40}$$

which gives the following estimates:

$$\nabla \sqrt{\xi} \text{ is bounded in } L^\infty(0, T; (L^2(\Omega))^3), \tag{41}$$

$$\sqrt{\xi} \partial_x u \text{ is bounded in } L^2(0, T; L^2(\Omega)), \tag{42}$$

$$\sqrt{\xi} A_x(u) \text{ is bounded in } L^2(0, T; (L^2(\Omega))^2). \tag{43}$$

This finishes the first step of the proof of theorem 2.

**Remark 5.** Estimate (41) is a straightforward consequence of estimates

$$\sqrt{\xi} (u + 2\tilde{v}_1 \nabla_x \ln \xi) \in L^\infty(0, T, (L^2(\Omega))^2) \quad \text{and} \quad \sqrt{\xi} u \in L^\infty(0, T, (L^2(\Omega))^2)$$

since

$$\sqrt{\xi} (u + 2\tilde{v}_1 \nabla_x \ln \xi) = \sqrt{\xi} u + 2\tilde{v}_1 \frac{\nabla_x \ln \xi}{\sqrt{\xi}}.$$ 

To show the compactness of sequences $(\xi_n, u_n, w_n)$ in appropriate space function we follow the work of Mellet and Vasseur [10]. To this end, we divide this second step of the proof of theorem 2 into 4 parts:

1. In section 3.2.2, we show the convergence of the sequence $\sqrt{\xi_n}$.
2. In section 3.2.3, we seek bounds of $\sqrt{\xi_n} u_n$ and $\sqrt{\xi_n} w_n$.
3. In section 3.2.4, we prove the convergence of $\xi_n u_n$.
4. In section 3.2.5, we prove the convergence of $\sqrt{\xi_n} u_n$. 


3.2.2. Convergence of $\sqrt{\xi_n}$. Let us first prove the following.

**Lemma 1.** For every $\xi_n$ satisfying the mass equation of system (19), we have

- $\sqrt{\xi_n}$ is bounded in $L^\infty(0, T, H^1(\Omega'))$,
- $\partial_t \sqrt{\xi_n}$ is bounded in $L^2(0, T, H^{-1}(\Omega'))$.

Then, up to a subsequence, the sequence $\xi_n$ converges almost everywhere and strongly in $L^2(0, T; L^2(\Omega'))$. Moreover, $\xi_n$ converges to $\xi$ in $C^0(0, T; L^{3/2}(\Omega'))$.

**Proof of lemma 1.** $\sqrt{\xi_n}$ is bounded in $L^\infty(0, T, H^1(\Omega'))$ since we have

$$||\sqrt{\xi_n}(t)||^2_{L^2(\Omega')} = ||\xi_n^0||_{L^1(\Omega')}$$

from the continuity equation and by estimate (41).

Using again the mass conservation equation, we write

$$\partial_t(\sqrt{\xi_n}) = -\frac{1}{2} \sqrt{\xi_n} \text{div}_x(u_n) - u_n \nabla_x \sqrt{\xi_n} - \sqrt{\xi_n} \partial_z w_n$$

$$= \frac{1}{2} \sqrt{\xi_n} \text{div}_x(u_n) - \text{div}_x(u_n \sqrt{\xi_n}) - \sqrt{\xi_n} \partial_z w_n.$$

Then, from estimates (27), (41), (42) and (43), we get

$$\partial_t(\sqrt{\xi_n})$$

is bounded in $L^2(0, T, H^{-1}(\Omega'))$.

We have then the compactness of $\sqrt{\xi_n}$ in $C^0(0, T, L^2(\Omega'))$ by Aubin’s lemma, i.e.

$\sqrt{\xi_n}$ converges strongly to $\sqrt{\xi}$ in $C^0(0, T, L^2(\Omega'))$.

We also have, by Sobolev embeddings, bounds of $\sqrt{\xi_n}$ in spaces $L^\infty(0, T, L^p(\Omega'))$ for all $p \in [1, 6]$. Consequently, for $p = 6$, we get bounds of $\xi_n$ in $L^\infty(0, T, L^3(\Omega'))$

and we deduce that

$$\xi_n u_n = \sqrt{\xi_n} \sqrt{\xi_n} u_n$$

is bounded in $L^\infty(0, T, L^{3/2}(\Omega'))^2$.

It follows that $\partial_t \xi_n$ is bounded in $L^\infty(0, T, W^{-1,3/2}(\Omega'))$ since

$$\partial_t \xi_n = -\text{div}_x(\xi_n u_n) - \xi_n \partial_z w_n$$

and estimate (42) holds.

To conclude, writing

$$\nabla_x \xi_n = 2\sqrt{\xi_n} \nabla_x \sqrt{\xi_n} \in L^\infty(0, T; L^{3/2}(\Omega'))^2,$$

we deduce bounds of $\xi_n$ in $L^\infty(0, T; W^{1,3/2}(\Omega'))$. Then, using again Aubin’s lemma provides compactness of $\xi_n$ in the intermediate space $L^{3/2}(\Omega')$: compactness of $\xi_n$ in $C^0(0, T; L^{3/2}(\Omega'))$. ■
3.2.3. Bounds of $\sqrt{\xi_n} u_n$ and $\sqrt{\xi_n} w_n$. To prove the convergence of the momentum, we have to control bounds of $\sqrt{\xi_n} u_n$ and $\sqrt{\xi_n} w_n$. Thus, we have to prove the following:

**Lemma 2.** We have $\sqrt{\xi_n} u_n$ bounded in $L^\infty(0, T; (L^2(\Omega'))^2)$ and $\sqrt{\xi_n} w_n$ bounded in $L^2(0, T; L^2(\Omega'))$.

**Proof of lemma 2.** We have already bounds of $\sqrt{\xi_n}$ (see estimates (24)). There is left to show bounds of $\sqrt{\xi_n} w_n$ in $L^2(0, T; L^2(\Omega'))$. As $\xi_n = \xi_n(t, x)$ and estimate (42) holds, by the Poincaré inequality, we have

$$\int_0^h |\sqrt{\xi_n} w_n|^2 \, dz \leq c \int_0^h |\partial_z(\sqrt{\xi_n} w_n)|^2 \, dz.$$

Consequently, the following inequality:

$$\int_{\Omega} \xi_n |w_n|^2 \, dx \, dz \leq c \int_{\Omega} \xi_n |\partial_z w_n|^2 \, dx \, dz$$

gives bounds of $\sqrt{\xi_n} w_n$ in $L^2(0, T; L^2(\Omega'))$.

3.2.4. Convergence of $\xi_n u_n$. As bounds of $\sqrt{\xi_n} u_n$ and $\sqrt{\xi_n} w_n$ are provided by lemma 2, we are able to show the convergence of the momentum.

**Lemma 3.** Let $m_n = \xi_n u_n$ be a sequence satisfying the momentum equation (19). Then we have

$$\xi_n u_n \to m \quad \text{in} \quad L^2(0, T; (L^p(\Omega'))^2) \quad \text{strong,} \quad \forall \, 1 \leq p < 3/2$$

and

$$\xi_n u_n \to m \quad \text{a.e.} \quad (t, x, y) \in (0, T) \times \Omega'.$$

**Proof of lemma 3.** Writing $\nabla_x(\xi_n u_n)$ as

$$\nabla_x(\xi_n u_n) = \sqrt{\xi_n} \nabla_x u_n + 2 \sqrt{\xi_n} u_n \otimes \nabla_x \sqrt{\xi_n}$$

provides

$$\nabla_x(\xi_n u_n) \text{ bounded in } L^2(0, T; (L^1(\Omega'))^2 \times \Omega'). \quad (44)$$

Next, we have

$$\partial_z(\xi_n u_n) = \sqrt{\xi_n} \partial_z(\xi_n u_n) \text{ is bounded } L^2(0, T; (L^{3/2}(\Omega'))^2). \quad (45)$$

Then, from bounds (44) and (45), we deduce

$$\xi_n u_n \text{ is bounded } L^2(0, T; (W^{1,1}(\Omega'))^2). \quad (46)$$

On the other hand, we have

$$\partial_t(\xi_n u_n) = - \mathrm{div}_x(\xi_n u_n \otimes u_n) - \partial_z(\xi_n u_n w_n) - \nabla_x \xi_n$$

$$\quad + 2 \tilde{v}_1 \mathrm{div}_x(\xi_n D_x(u_n)) + \tilde{v}_2 \partial_z(\xi_n \partial_z u_n) - r \xi_n |u_n| u_n.$$

As

$$\xi_n u_n \otimes u_n = \sqrt{\xi} u_n \otimes \sqrt{\xi} u_n,$$

we have

$$\partial_t(\xi_n u_n) = \mathrm{div}_x(\xi_n u_n \otimes u_n) - \nabla_x \xi_n$$

$$\quad + 2 \tilde{v}_1 \mathrm{div}_x(\xi_n D_x(u_n)) + \tilde{v}_2 \partial_z(\xi_n \partial_z u_n) - r \xi_n |u_n| u_n.$$
we deduce bounds of
\[ \xi_n u_n \otimes u_n \in L^\infty(0, T; (L^1(\Omega'))^{2 \times 2}). \]

Particularly, we have
\[
\text{div}_x(\xi_n u_n \otimes u_n) \text{ bounded in } L^\infty(0, T; (W^{-2,4/3}(\Omega'))^2).
\]

Similarly, as \( \xi_n u_n w_n = \sqrt{\xi_n} u_n \sqrt{\xi_n} w_n \in (L^1(\Omega'))^2 \), we also have
\[ \partial_t (\xi_n u_n w_n) \text{ bounded in } L^\infty(0, T; (W^{-2,4/3}(\Omega'))^2). \]

Moreover, as \( \sqrt{\xi_n} \partial_n u_n \in L^2(0, T; (L^{3/2}(\Omega'))^2) \) and \( \sqrt{\xi_n} \partial_n D_\xi(u_n) \in L^2(0, T; (L^{3/2}(\Omega'))^{2 \times 2}) \), we get bounds of
\[
\partial_t (\sqrt{\xi_n} \partial_n u_n), \quad \text{div}_x(\sqrt{\xi_n} \partial_n D_\xi(u_n)) \in L^2(0, T; (W^{-1,3/2}(\Omega'))^2).
\]

We also have bounds of \( \nabla_x \xi_n \in L^\infty(0, T, (W^{-1,3/2}(\Omega'))^2) \).

Using \( W^{-1,3/2}(\Omega') \subset W^{-1,4/3}(\Omega') \), we obtain
\[
\partial_t (\xi_n u_n) \text{ bounded in } L^2(0, T; (W^{-2,4/3}(\Omega'))^2). \tag{48}
\]

Using Aubin’s lemma with the bounds (46), (48) provides the compactness of
\[ \xi_n u_n \in L^2(0, T; (L^p(\Omega'))^2), \forall p \in [1, 3/2]. \]

3.2.5. Convergence of \( \sqrt{\xi_n} u_n \) and \( \xi_n w_n \). Let us note that, up to section 3.2.4, we can always define \( u = m/\xi \) on the set \( \{ \xi > 0 \} \), but we do not know, a priori, if \( m \) equals zero on the vacuum set. To this end, we need to prove the following lemma:

**Lemma 4.**

1. The sequence \( \sqrt{\xi_n} u_n \) satisfies
   - \( \sqrt{\xi_n} u_n \) converges strongly in \( L^2(0, T; (L^2(\Omega'))^2) \) to \( m/\xi \).
   - We have \( m = 0 \) almost everywhere on the set \( \{ \xi = 0 \} \) and there exists a function \( u \) such that \( m = \xi u \) and
     \[ \xi_n u_n \rightarrow \xi u \text{ strongly in } L^2(0, T; (L^p(\Omega'))^2) \text{ for all } p \in [1, 3/2], \]
     \[ \sqrt{\xi_n} u_n \rightarrow \sqrt{\xi} u \text{ strongly in } L^2(0, T; (L^2(\Omega'))^2). \]

2. The sequence \( \sqrt{\xi_n} w_n \) converges weakly in \( L^2(0, T; L^2(\Omega')) \) to \( \sqrt{\xi} w \).

To prove lemma 4, we adapt the proof of Mellet and Vasseur [10]. As already pointed out by Bresch et al [3], the presence of the term \( r\xi |u|_u \) simplifies also this proof.

**Proof of lemma 4.** To start, we set \( m_n = \xi_n u_n \).

Since \( m_n/\sqrt{\xi_n} \) is bounded in \( L^\infty(0, T; (L^2(\Omega'))^2) \) Fatou’s lemma yields
\[
\int_{\Omega} \liminf_n \frac{m_n^2}{\xi_n} < \infty.
\]

In particular, we have \( m(t, x, z) = 0 \) almost everywhere on the set \( \{ \xi(t, x) = 0 \} \). So, if we define the limit velocity \( u(t, x, z) \) by setting
\[ u(t, x, z) = \begin{cases} \frac{m(t, x, z)}{\xi(t, x)} & \text{if } \xi(t, x) \neq 0, \\ 0 & \text{if } \xi(t, x) = 0, \end{cases} \]

then, we have
\[ m(t, x, z) = \xi(t, x) u(t, x, z) \]
and 
\[ \int_{\Omega} \frac{m^2_n}{\xi_n} \, dx \, dz = \int_{\Omega} \xi u^2 \, dx \, dz < \infty. \]

Next, since \( m_n \) and \( \sqrt{\xi_n} \) converge almost everywhere, it is readily seen that on the set \( \{ \xi(t, x) \neq 0 \} \),
\[ \sqrt{\xi_n} u_n = \frac{m_n}{\sqrt{\xi_n}} \text{ converges almost everywhere to } \sqrt{\xi} u = \frac{m}{\sqrt{\xi}}. \]

Moreover, for a constant \( M > 0 \), we have
\[ \sqrt{\xi_n} u_n \mathbb{1}_{|u_n| \leq M} \to \sqrt{\xi} u \mathbb{1}_{|u| \leq M} \text{ almost everywhere.} \quad (49) \]

As a matter of fact, the convergence holds almost everywhere on the set \( \{ \xi(t, x) \neq 0 \} \cup \{ \xi(t, x) = 0 \} \) and we have
\[ \sqrt{\xi_n} u_n \mathbb{1}_{|u_n| \leq M} \leq M \sqrt{\xi}. \]

To complete the proof, we cut the \( L^2 \) norm as follows:
\[
\int_{\Omega} |\sqrt{\xi_n} u_n - \sqrt{\xi} u|^2 \, dx \, dz \leq \int_{\Omega} |\sqrt{\xi_n} u_n \mathbb{1}_{|u_n| \leq M} - \sqrt{\xi} u \mathbb{1}_{|u| \leq M}|^2 \, dx \, dz
+ 2 \int_{\Omega} |\sqrt{\xi_n} u_n \mathbb{1}_{|u_n| > M}|^2 \, dx \, dz
+ 2 \int_{\Omega} |\sqrt{\xi} u \mathbb{1}_{|u| > M}|^2 \, dx \, dz.
\]

It is obvious that \( \sqrt{\xi_n} u_n \mathbb{1}_{|u_n| \leq M} \) is uniformly bounded in \( L^\infty(0, T; (L^2(\Omega'))^2) \), then using (49) gives the convergence of the first integral:
\[ \int_{\Omega} |\sqrt{\xi_n} u_n \mathbb{1}_{|u_n| \leq M} - \sqrt{\xi} u \mathbb{1}_{|u| \leq M}|^2 \, dx \, dz \to 0. \quad (50) \]

Finally, writing
\[
\int_{\Omega} |\sqrt{\xi_n} u_n \mathbb{1}_{|u_n| > M}|^2 \, dx \, dz \leq \frac{1}{M} \int_{\Omega} \xi_n |u_n|^3 \, dx \, dz, \quad (51)
\]
\[
\int_{\Omega} |\sqrt{\xi} u \mathbb{1}_{|u| > M}|^2 \, dx \, dz \leq \frac{1}{M} \int_{\Omega} |u|^3 \, dx \, dz \quad (52)
\]
and putting together (50), (51) and (52), we deduce
\[ \lim_{n \to +\infty} \sup \int_{\Omega} |\sqrt{\xi_n} u_n - \sqrt{\xi} u|^2 \, dx \, dz \leq \frac{C}{M}, \quad \forall M > 0 \]

which ends the first point of the lemma by taking \( M \to +\infty \).

The second part of the theorem is done by weak compactness. As \( \sqrt{\xi_n} w_n \) is bounded in \( L^2(0, T; L^2(\Omega')) \), there exists, up to a subsequence, \( \sqrt{\xi_n} w_n \) which converges weakly to some limit \( l \) in \( L^2(0, T; L^2(\Omega')) \). Next, we define \( w \) as
\[
w = \begin{cases} 
\frac{l}{\sqrt{\xi}} & \text{if } \xi > 0, \\
0 & \text{a.e. if } \xi = 0
\end{cases}
\]
where the limit \( l \) is written: \( l = \sqrt{\xi} (l/\sqrt{\xi}) = \sqrt{\xi} w. \)

This finishes the second point of the proof of theorem 2.
3.2.6. Convergence step. Gathering the previous results, we show straightforwardly that we can pass to the limit in all terms of system (19) in the sense of theorem 2. To this end, let \((\xi_n, u_n, w_n)\) be a weak solution of system (19) satisfying lemma 1 to 4 and let \(\phi \in C_c^\infty ([0, T] \times \Omega')\) be a smooth function with compact support such as \(\phi(T, x, z) = 0\) and \(\phi(0, x, z) = \phi_0(x, z)\). Then, writing each term of the weak formulation of system (19), we have

- For the first integral, we have
  \[
  \int_0^T \int_\Omega \partial_t (\xi_n u_n) \phi \, dx \, dz \, dt = - \int_0^T \int_\Omega \xi_n u_n \partial_t \phi \, dx \, dz \, dt - \int_\Omega \xi_n^0 u_0^0 \phi_0 \, dx \, dz.
  \]

Using convergences (22) and lemma 3, we get

- For the following integral, we write
  \[
  \int_0^T \int_\Omega \xi_n u_n \partial_t \phi \, dx \, dz \, dt - \int_0^T \int_\Omega \xi_n^0 u_0^0 \phi_0 \, dx \, dz \rightarrow - \int_0^T \int_\Omega \xi u \partial_t \phi \, dx \, dz - \int_\Omega \xi_0 u_0 \phi_0 \, dx \, dz.
  \]

- For the following integral, we write
  \[
  \int_0^T \int_\Omega \xi_n u_n \otimes u_n \cdot \phi \, dx \, dz \, dt = - \int_0^T \int_\Omega \xi_n u_n \otimes u_n : \nabla \phi \, dx \, dz \, dt\]
  then, from equality (47) and lemma 4, we have
  \[
  - \int_0^T \int_\Omega \xi_n u_n \otimes u_n : \nabla \phi \, dx \, dz \, dt \rightarrow - \int_0^T \int_\Omega \xi u \otimes u : \nabla \phi \, dx \, dz \, dt.
  \]

- Writing
  \[
  \int_0^T \int_\Omega \partial_t (\xi_n u_n w_n) \cdot \phi \, dx \, dz \, dt = - \int_0^T \int_\Omega \xi_n u_n w_n \cdot \partial_t \phi \, dx \, dz \, dt,
  \]
  as \(\xi_n u_n w_n = \sqrt{\xi_n} u_n \sqrt{\xi_n} w_n\), by lemma 4, we get
  \[
  - \int_0^T \int_\Omega \xi_n u_n w_n \cdot \partial_t \phi \, dx \, dz \, dt \rightarrow - \int_0^T \int_\Omega \xi u w \cdot \partial_t \phi \, dx \, dz \, dt.
  \]

- For the following integral, we write
  \[
  \int_0^T \int_\Omega \nabla \xi_n \cdot \phi \, dx \, dz \, dt = - \int_0^T \int_\Omega \xi_n \nabla \phi \, dx \, dz \, dt\]
  then, lemma 1 provides
  \[
  - \int_0^T \int_\Omega \xi_n \nabla \phi \, dx \, dz \, dt \rightarrow - \int_0^T \int_\Omega \xi \nabla \phi \, dx \, dz \, dt.
  \]

- We write the integral as follows
  \[
  \int_0^T \int_\Omega \nabla \xi_n \cdot (\nabla \xi_n D_x (u_n)) \cdot \phi \, dx \, dz \, dt = - \int_0^T \int_\Omega \xi_n D_x (u_n) : \nabla \phi \, dx \, dz \, dt\]
  Since \(D_x (u_n) = \frac{1}{2}(\nabla_x u_n + \nabla_x u_n^t)\), expanding the term in the last integral gives
  \[
  - \int_0^T \int_\Omega \xi_n D_x (u_n) : \nabla \phi \, dx \, dz \, dt = \frac{1}{2} \int_0^T \int_\Omega (\xi_n u_n : \Delta_x \phi + \nabla_x \phi \nabla_x (\sqrt{\xi_n}) : \sqrt{\xi_n} u_n) \, dx \, dz \, dt
  \]
  \[
  + \frac{1}{2} \int_0^T \int_\Omega (\xi_n u_n : \nabla_x (\nabla_x \phi) + \nabla_x \sqrt{\xi_n} : \nabla_x \phi : \sqrt{\xi_n} u_n) \, dx \, dz \, dt.
  \]
From estimate (41), the sequence $\nabla_x \sqrt{\xi_n}$ weakly converges, and using lemmas 1, 3 and 4, we obtain
\[
\frac{1}{2} \int_0^T \int_\Omega (\xi_n u_n \cdot \Delta_x \phi + \nabla_x \phi \nabla_x (\sqrt{\xi_n}) \cdot \sqrt{\xi_n} u_n) \, dx \, dz \, dt \\
+ \frac{1}{2} \int_0^T \int_\Omega (\xi_n u_n \cdot \text{div}_x (\nabla_x \phi) + \nabla_x \phi u_n) \, dx \, dz \, dt \\
\to \frac{1}{2} \int_0^T \int_\Omega (\xi u \cdot \Delta_x \phi + \nabla_x \phi \nabla_x (\sqrt{\xi}) \cdot \sqrt{\xi} u) \, dx \, dz \, dt \\
+ \frac{1}{2} \int_0^T \int_\Omega (\xi u \cdot \text{div}_x (\nabla_x \phi) + \nabla_x \phi u) \, dx \, dz \, dt.
\]
Hence
\[
- \int_0^T \int_\Omega \xi \mathcal{D}_x (u_n) : \nabla_x \phi \, dx \, dz \, dt \to - \int_0^T \int_\Omega \xi \mathcal{D}_x (u) : \nabla_x \phi \, dx \, dz \, dt.
\]

- We have straightforwardly
\[
\int_0^T \int_\Omega \xi \mathcal{D}_x (u_n) : \phi \, dx \, dz \, dt \to \int_0^T \int_\Omega \xi u_n : \xi \mathcal{D}_x (\phi) \, dx \, dz \, dt.
\]

Using lemma 3 provides the following convergence:
\[
\int_0^T \int_\Omega \xi \mathcal{D}_x (u_n) : \phi \, dx \, dz \, dt \to \int_0^T \int_\Omega \xi u : \xi \mathcal{D}_x (\phi) \, dx \, dz \, dt.
\]

- The convergence of the integral
\[
\int_0^T \int_\Omega r \xi_n |u_n| \phi \, dx \, dz \, dt \to \int_0^T \int_\Omega r \xi |u| \phi \, dx \, dz \, dt
\]
is obtained by lemma 4, and finishes the proof of theorem 2.

3.2.7. Proof of theorem 1. Following Ersoy and Ngom [6], to finish the proof of theorem 1 we consider a sequence $(\xi_n, u_n, w_n)$ of weak solution of system (19). All obtained estimates in steps 3.2.2–3.2.6 hold if we replace $\xi_n$ by $\rho_n$ and $w_n$ by $v_n$, since
\[
\rho(t, x, y) = \xi(t, x)e^{-y} \quad \text{and} \quad w(t, x, z) = v(t, x, y)e^{-y}
\]
where $(d/dy)z = e^{-y}$. Moreover, by the change of variables $z = 1 - e^{-t}$ in integrals, we have the following properties:
- $\|\rho\|_{L^1(\Omega)} = \alpha \|\xi\|_{L^1(\Omega)}$,
- $\|\nabla_x \rho\|_{L^1(\Omega)} = \alpha \|\nabla_x \xi\|_{L^1(\Omega)}$,
- $\|\partial_t \rho\|_{L^1(\Omega)} = \alpha \|\xi\|_{L^1(\Omega)}$

where
\[
\alpha = \int_0^1 (1 - z) \, dz < +\infty.
\]
We deduce then
\[
\|\rho\|_{W^{1,1}(\Omega)} = \alpha \|\xi\|_{W^{1,1}(\Omega)}
\]
which provides
\[
\rho \in L^\infty(0, T; W^{1,2}(\Omega))
\]
and
\[
\partial_t \rho \in L^2(0, T; L^2(\Omega)).
\]
Again, by the change of variable in integrals, the fact that $v \in L^2(0, T; L^2(\Omega))$ is obtained from the inequality:

$$
\|v\|_{L^2(\Omega)} = \int_{\Omega} \int_0^1 |v(t, x, y)|^2 \, dy \, dx
= \int_{\Omega} \int_0^1 e^{-e^{-1}} \left( \frac{1}{1-z} \right)^3 |w(t, x, z)|^2 \, dz \, dx
< e^3 \|w\|_{L^2(\Omega')}.
$$

Finally, all estimates on $u$ remaining true, theorem 1 is proved.

4. Perspectives

In this paper, we have presented compressible primitive equations where viscosities are anisotropic and density dependent. We have established a stability result for weak solutions by introducing a useful change of variable. The question of the existence of weak solutions for these equations remains an open question. However, with the obtained estimations, it may be possible to construct an approximate sequence of solutions, as Faedo–Galerkin approach and to adapt the technique presented by Vaïgant et al [13]. Although, their models do not take into account the anisotropy and the dynamical viscosity is constant, useful additional estimates can be derived, particularly, to show that the density is bounded. The work is actually in progress.

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