Mass Matrices in $E_6$ Unification

Masako Bando,$^{1,*}$ Taichiro Kugo$^{2,**}$ and Koichi Yoshioka$^{2,***}$

$^1$Aichi University, Aichi 470-0296, Japan
$^2$Department of Physics, Kyoto University, Kyoto 606-8502, Japan

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We study a supersymmetric $E_6$ grand unified model in which the $SU(5)$ $5^*$ components are twisted in the third generation $27$. Supplementing the adjoint Higgs field to a model analyzed previously, we calculate the mass matrices for the up and down quarks and charged leptons. Although the number of free parameters is less than that of observables, an overall fitting to the observed masses and mixing angles is shown to be possible. Most notably, we find two novel, parameter-independent relations between the lepton 2-3 mixing angle $\theta_{\mu\tau}$ and the quark masses and CKM mixing angles that are in good agreement with the large lepton mixing recently observed.

§1. Introduction

The discovery of neutrino oscillation events in Superkamiokande experiments$^{1)}$ has stimulated us to reconsider the fermion mass hierarchy problem in unified theories. One of the most mysterious problems is the remarkable contrast of the mixing structure of leptons to that of quarks. Most of us believe in the possible unification of the existing matter content observed in the low energy region and that the realization of a grand unified theory (GUT) is one of the most challenging problems in particle physics. In this sense, the experimental data$^{1)}$-$^5)$ indicating a large mixing of the muon neutrino provides us with an important clue in pursuing unification and many authors have investigated neutrino physics along this line.$^6)$

The existence of right-handed neutrinos suggests a left-right symmetric gauge group which is beyond the $SU(5)$ group,$^7)$ in which all fermions of one generation are combined into a single representation $16$. In this sense, $SO(10)^8)$ would be an attractive candidate for the unified gauge group. However, the characteristic feature of the neutrino mixing structure seems to suggest that such $SO(10)$ unifications are unfeasible since they would give rise to a complete parallelism between quarks and leptons.

Among the possible simple gauge groups, $E_6^9)$ is essentially the unique candidate for the unified gauge group. Indeed, if one requires the following three conditions for the unified group, only $SO(10)$ and $E_6$ remain: (i) all the fermions, including the right-handed neutrinos of one family, belong to a single irreducible representation; (ii) the group is automatically anomaly free; and (iii) the group allows complex representations that contain our low-energy chiral fermions but not their mirror
fermions. If we further add the condition that (iv) there exists freedom in the fermion content to avoid the parallelism between the quark and lepton mass structures, then we are left with only the $E_6$ gauge group. Most remarkably, one of characteristic features of the $E_6$ group is that we have a freedom in choosing the down quark and charged lepton components in the fundamental representation $27$.

Our final goal is to unify all the fermions of three generations, but before realizing that, we should understand the difference between the mixing structures of the quark and lepton sectors. This is closely connected with the origin of the hierarchical structure of fermion masses. There have been various approaches to these problems, and to this time, most of them either employ ad hoc assumptions for hierarchical Yukawa couplings or assume certain family symmetries. Among the latter approaches, which we expect to open a gate for understanding the origin of generation, the simplest one may be to use a flavor $U(1)$ symmetry. In that case, the smallness of the Yukawa couplings are attributed to the higher-dimensional interaction terms suppressed by some fundamental scale $M_P$.

In a previous paper, the authors constructed a supersymmetric $E_6$ unified model with an extra $U(1)$ symmetry. There we showed that E-twisting family structure can reproduce all the characteristic features of the fermion mass matrices, not only the quark/lepton Dirac masses but also the neutrino Majorana masses. Despite the fact that a common $U(1)$ charge is assigned to all members in a $27$ of each generation, the model explained the qualitative features of the different mass hierarchies among generations for up and down quark sectors, as well as the mixing angles for quarks and for leptons.

In this paper, we study this model more quantitatively and aim to check whether the resultant predictions are consistent with the present experimental results. To complete our scenario, we must introduce a new Higgs field in order to allow for a difference between the quark and lepton masses and mixings. A minimal choice is to introduce a Higgs field of the adjoint representation $78$. This Higgs field is actually necessary also in order to break the $E_6$ gauge symmetry down to the standard gauge group $SU(3) \times SU(2) \times U(1)$. This is because there is no component in the fundamental representation $(27)$ that is non-singlet under the $SU(5)$ symmetry but singlet under the standard gauge group. In fact, with a particular assumption for the Higgs potential, the adjoint representation Higgs can cause the desired symmetry breaking even with the doublet-triplet splitting. The newly introduced Higgs field also induces effective Yukawa couplings which come from the higher-dimensional interaction terms via the Froggatt-Nielsen mechanism. It is found that such (non-leading) contributions can actually lead to differences between the quarks and leptons as well as between the up and down quarks. We analyze the masses and mixing angles by taking account of this additional Higgs field and find several interesting parameter-independent relations among the experimental observables, including a large lepton mixing angle.

This paper is organized as follows. In §2, we first present the field content and the charge assignments in our model with E-twisting family structure by adding a Higgs field $78$. We also explicitly list the induced higher-dimensional terms involving the $78$ field. The structure of the additional couplings is analyzed in §3. We show
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how this contribution affects the quark and lepton Yukawa couplings in the following two sections. In §6 we carry out both the analytic and numerical analyses to compare the results with the present experimental data. In spite of the complicated forms of the Yukawa matrices, we can find interesting relations between the quark and lepton mixing angles. In §7 we give summarizing discussion and further comments. The Appendix is devoted to describing the diagonalization of the $3 \times 3$ mass matrices of the type appearing in the text.

§2. Model

The supersymmetric $E_6$ unification model that was considered in Ref. 11) contains, in addition to an $E_6$ gauge vector multiplet, chiral matter multiplets corresponding to the three generation fermions $[\Psi_i \ (i = 1, 2, 3)]$ and two pairs of Higgs fields $(H, \bar{H})$ and $(\Phi, \bar{\Phi})$. The former Higgs, $H$, is for the electroweak symmetry breaking (also for the fermion masses) and the other, $\Phi$, is responsible for realizing the E-twisting (generation flipping) structure. In this paper, we additionally introduce a chiral Higgs multiplet $\phi(78)$, which is necessary to break the GUT to the standard gauge group. In Table I, we summarize all the fields we need in this paper. The $E_6$ singlet field $\Theta$ with $U(1)$ charge $-1$ plays the important role that its suitable powers compensate for the mismatch of the $U(1)$ charge in the superpotential interaction terms. The $U(1)$ flavor symmetry discriminates generations and induces a hierarchy among them. It should be noted that all the quarks and leptons in one generation have a common $U(1)$ quantum numbers.

Since we are interested in the mass terms of the ordinary fermions as well as superheavy fermions of the GUT scale, we list all the superpotentials which are invariant under $R$ parity, $U(1)$ and $E_6$ and give masses of matter superfields $\Psi_i(27)$. The Yukawa interactions are given by

$$W_Y(H) = y_{ij} \Psi_i(27)\Psi_j(27)H(27) \left( \frac{\Theta}{M_P} \right)^{f_i + f_j},$$

$$W_Y(\Phi) = y'_{ij} \Psi_i(27)\Psi_j(27)\Phi(27) \left( \frac{\Theta}{M_P} \right)^{f_i + f_j - 4},$$

where $f_i$ denotes the $U(1)$ charge of the $i$-th generation. The coupling constants $y$ and $y'$ are assumed to be order 1. However, the effective Yukawa coupling constants become multiplied by additional powers of $\lambda = \langle \Theta \rangle / M_P$, coming from the powers of $\Theta$ required by $U(1)$ quantum number matching (the Froggatt-Nielsen mechanism).

We also suppose that only the $SU(2)$ doublet components of $H$ can

| $\Psi_1$ | $\Psi_2$ | $\Psi_3$ | $H$ | $\bar{H}$ | $\bar{H}$ | $\Phi$ | $\bar{\Phi}$ | $\phi$ | $\Theta$ |
|--------|--------|--------|-----|--------|--------|--------|----------|--------|--------|
| $E_6$  | 27     | 27     | 27  | 27     | 27     | 27     | 27       | 78     | 1      |
| $U(1)$ charge | 3     | 2     | 0    | 0     | -4     | 4      | -2       | -1      |
| $R$ parity | -     | -     | -    | +     | +      | +      | +        | +       |
have the electroweak scale vacuum expectation value (VEV).

With the adjoint Higgs field $\phi$ having $U(1)$ charge $-2$, there is the following higher-dimensional operator which eventually give small mass terms:

$$W_\phi = z_{ij} M_P^{-1} \phi(78) \psi_i(27) \psi_j(27) H(27) \left( \frac{\Theta}{M_P} \right)^{f_i + f_j - 2}. \quad (2.2)$$

Here, $z_{ij}$ are order 1 couplings. Precisely speaking, there are two ways for contracting the $E_6$ group indices in this superpotential, which we will discuss in detail in the next section. There also exist the following higher-dimensional operators, which give rise to the right-handed neutrino Majorana masses:

$$W_R = x_{ij} M_P^{-1} \psi_i(27) \psi_j(27) X_k(27) X_l(27) \left( \frac{\Theta}{M_P} \right)^{f_i + f_j + x_k + x_l}. \quad (2.3)$$

Here $X_i$ represents $\Phi(27)$ or $H(27)$ with $U(1)$ charge $x_i$, and the couplings $x_{ij}$ are of order 1. For the resulting mass texture of the right-handed neutrinos, see Ref. 11).

For later convenience, we name the component fields of $\psi(27)$ as follows. The representation $27$ is decomposed under $SO(10) \subset E_6$ as

$$27 = 16 + 10 + 1, \quad (2.4)$$

which are further decomposed under $SU(5) \subset SO(10)$ as

$$16 = \nu^c, \begin{pmatrix} u_i \\ d_i \end{pmatrix}, e^c$$

$$10 = 5^* + 1, \quad 5^* = (D_i, E^c, -N^c), \quad 1 = (D^{ci}, E, -N). \quad (2.5)$$

An interesting fact is that $5^*$ appears twice in each $27$, i.e., $5^*$ of $16$ [(16, $5^*$)] and $5^*$ of $10$ [(10, $5^*$)], which we refer to as the ‘E-parity’ doublet. It is due to this doubling that we have the freedom to choose the low-energy $5^*$ candidates. This actually implies that the embedding of $SO(10)$ into $E_6$, such that $SU(5)_{GG} \subset SO(10) \subset E_6$ with Georgi-Glashow $SU(5)_{GG}$, possesses a freedom of rotation of $SU(2)_R$. The doubling of $5^*$ in each $27$ also provides the low-energy surviving down-type Higgs field with the freedom of a mixing parameter between the two $5^*$ representations in $H(27)$:

$$H(5^*) = H(10, 5^*) \cos \theta + H(16, 5^*) \sin \theta. \quad (2.6)$$

Let us pick up the low-energy matter fields among the three $\psi_i(27)$ of the above. The up-quark sector is unique, since $10$ and $5$ of $SU(5)$ appear only once in each $27$. As for the three families of (right-handed) down quarks, there is a freedom in choosing three from the six $5^*$ representations in the three $\psi_i(27)$. We have classified possible typical scenarios in Ref. 11): (i) parallel family structure; (ii) non-parallel
family structure; and (iii) E-twisted structure. Among these three possibilities, we here investigate the simplest and probably most attractive option, namely the E-twisted structure:

\[(5^1_1, 5^2_2, 5^3_3) = (\Psi_1(16, 5^*_1), \Psi_2(16, 5^*_2), \Psi_3(10, 5^*_3))\]  

This structure implies that the third family 5 falls into 10 of SO(10), which is E-twisted from that of the other two families. This twisting is realized by the suitable VEVs of the Higgs field \(\Phi\) and the usual Higgs \(H\), as shown in Ref. 11 and explained briefly below.

In order to get the actual forms of mass matrices, we should know the couplings containing the neutral components of Higgs fields, \(H(1, 16, 1), H(16, 1, 5^*), H(10, 5^*)\) and \(H(16, 5)\), which correspond to the components \(S, \nu, -\nu, -N\) and \((-N^c)\), respectively, of \(\Psi(27)\) in Eq. (2.5). In addition to this, we need also write down the form of the coupling (2.2) at the component level. The full effective mass matrices for the quarks and leptons are the sums of the contributions from direct Yukawa couplings (2.1) and the induced ones from the higher-dimensional couplings (2.2). Since the explicit forms of the tree-level Yukawa couplings have been presented in Ref. 11, we here present those of Eq. (2.2) in the next section.

§3. Coupling of \(\phi(78)\)

In this section we present the precise form of the higher-dimensional interaction (2.2) containing the adjoint Higgs \(\phi(78)\) and give the explicit component expressions for the effective Yukawa couplings induced by the VEVs of \(\phi(78)\).

The \(E_6\) adjoint \(\phi(78)\) is decomposed as follows under the subgroup \(SU(3)_L \times SU(3)_R \times SU(3)_C \subset E_6\):

\[78 = 8_L + 8_R + 8_C + (3, 3, 3) + (3^*, 3^*, 3^*).\]  

Then, \(8_{R(L)}\) is further decomposed under \(SU(2)_{R(L)} \subset SU(3)_{R(L)}\) into

\[8_{R(L)} = 3_{R(L)} + 2_{R(L)} + 2^*_{R(L)} + 1_{R(L)},\]  

among which the components of \(8_R\) and the \(SU(2)_L\) singlet part of \(8_L\) are neutral under the standard gauge symmetry. If \(SU(2)_R\) is broken by the VEV of the neutral components of \(3_R, 2_R\) and/or \(2_R^*\), this can create differences between the up and down quark sectors. However, it turns out that \(2_R\) and \(2^*_R\) do not yield sufficiently large differences. Thus we here assume that only the third component \(\phi_{3_R}\) of

\[\phi(3_R) = \sum_{a=1}^3 \phi_{3_R} \frac{\tau^a}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_{R3} & \phi_{R1} - i\phi_{R2} \\ \phi_{R1} + i\phi_{R2} & -\phi_{R3} \end{pmatrix}\]  

develops a non-vanishing VEV \(\omega\), normalized by \(\lambda^2 M_P\) for later convenience [see (3.4)]. Since we also wish to have differences between the down quarks and charged leptons (especially in the second generation), we assume that the \(SU(2)_R\) singlet \(1_R\) in \(8_R\) and the \(SU(2)_L\) singlet \(1_L\) in \(8_L\) develop non-vanishing VEVs \(\chi_R\) and \(\chi_L\).
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(normalized by $\lambda^2 M_P$), respectively. Retaining only these three VEVs, $\chi_R$, $\chi_L$ and $\omega$, we have

$$\langle \phi(8_R) \rangle_{M_P} = \lambda^2 \begin{pmatrix} \omega + \chi_R & 0 & 0 \\ 0 & -\omega + \chi_R & 0 \\ 0 & 0 & -2\chi_R \end{pmatrix},$$

$$\langle \phi(8_L) \rangle_{M_P} = \lambda^2 \begin{pmatrix} \chi_L & 0 & 0 \\ 0 & \chi_L & 0 \\ 0 & 0 & -2\chi_L \end{pmatrix}.$$  \hspace{1cm} (3.4)

(For a proper normalization of generators, $\omega$ and $\chi$ should be replaced by $\omega/\sqrt{2}$ and $\chi/(2\sqrt{3})$.) Now we derive the effective Yukawa couplings resulting from the higher-dimensional interaction (2.2), which actually reads as the following two independent $E_6$-invariant terms:

$$W_{\phi} = \sum_{i,j} s_{ij} M_P^{-1} \Psi_i(27) \Psi_j(27) \left( \phi(78) H(27) \right)_{27} \left( \theta \right)_{M_P}^{f_i + f_j - 2}$$

$$+ \sum_{i,j} a_{ij} M_P^{-1} \left( \phi(78) \Psi_i(27) \right)_{27} \Psi_j(27) H(27) \left( \theta \right)_{M_P}^{f_i + f_j - 2}.$$  \hspace{1cm} (3.5)

The $O(1)$ coupling constant $s_{ij}$ in the first term is clearly symmetric under the exchange $i \leftrightarrow j$, by definition. The second term coupling $a_{ij}$, on the other hand, need not have definite symmetry. However, we can always redefine it to become anti-symmetric through the following procedure. Since $78$ is equivalent to the generator of the $E_6$ group, we have the identity

$$\left( \phi(78) \Psi_i(27) \right)_{27} \Psi_j(27) H(27) + \Psi_i(27) \left( \phi(78) \Psi_j(27) \right)_{27} H(27)$$

$$+ \Psi_i(27) \Psi_j(27) \left( \phi(78) H(27) \right)_{27} = 0.$$  \hspace{1cm} (3.6)

This shows that the symmetric part of the second term coupling $a_{ij}$ in Eq. (3.5) is equivalent to the first term coupling $s_{ij}$. We can therefore assume that the second coupling constant $a_{ij}$ is anti-symmetric without loss of generality by a redefinition of $s_{ij}$.

The above two types of couplings in Eq. (3.5) in fact correspond to the two independent couplings through the representations $27$ and $351$ contained in the product $78 \times 27 = 27 + 351 + 1728$, which are contracted with their complex conjugate representations in $27 \times 27 = 27 + 351 + 351'$. We can thus see that the second term in Eq. (3.5), with antisymmetric coupling constant $a_{ij}$, is equivalent to the antisymmetric coupling $\left( \Psi_i(27) \Psi_j(27) \right)_{351} H(27)$. However, writing this coupling as in Eq. (3.5) is more convenient. This is because the action of $\phi(78)$ on $F(27)$ ($F = \Psi, H$) in the product $\phi(78) F(27)$ is the same as that of the $E_6$ generators. Moreover, the components in $\phi(78)$ possessing the above non-vanishing
VEVs, $\chi_R$, $\chi_L$ and $\omega$, correspond to the Cartan generators
\[
T_{3L,R} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -2
\end{pmatrix} \in SU(3)_{L,R}, \quad T_{3R} = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix} \in SU(3)_R,
\]
respectively, so that $F(27) \to \left(\langle \phi(78) \rangle F(27)\right)_{27}$ is equivalent to the replacement of each component field in $F(27)$ by the same field multiplied by its $T_{3L,R}$ and $T_{3R}$ quantum numbers and the corresponding VEVs, $\chi_R$, $\chi_L$, $\omega$. Thus the effective Yukawa couplings that result from Eq. (3.5) via the VEV $\langle \phi(78) \rangle/M_P = \chi_R T_{3R} + \chi_L T_{3L} + \omega T_{3R}$ can be found directly from the component expression for the original direct Yukawa coupling $\Psi_i(27)\Psi_j(27)H(27)$ simply by making this replacement.

The $T_{3L,R}$ and $T_{3R}$ quantum numbers of the components of $27$ can easily be found by considering its decomposition under the subgroup $SU(3)_L \times SU(3)_R \times SU(3)_C \subset E_6$ (given explicitly in the Appendix of Ref. 11). We thus find that the replacement $\Psi(27) \to M_P^{-1}(\langle \phi(78) \rangle \Psi(27))_{27}$ explicitly reads as follow for the component fields of $\Psi(27)$:

\[
\Psi(3,1,3) = \begin{pmatrix}
u_i \\
d_i \\
D_i
\end{pmatrix} \to \begin{pmatrix}(\chi_L) \nu_i \\
(\chi_L) d_i \\
-2(\chi_L) D_i
\end{pmatrix},
\]
\[
\Psi(1,3^*,3^*) = \begin{pmatrix}
u^c_i \\
d^c_i \\
D^c_i
\end{pmatrix} \to \begin{pmatrix}(\omega - \chi_R) \nu^c_i \\
(\omega - \chi_R) d^c_i \\
+2(\chi_R) D^c_i
\end{pmatrix},
\]
\[
\Psi(3^*,3,1) = \begin{pmatrix}1_L \\
2_R \\
3_R
\end{pmatrix} \begin{pmatrix}1^c_L \\
2^c_R \\
3^c_R
\end{pmatrix} \to \begin{pmatrix}\omega + \chi_R \\
-\omega + \chi_R \\
-2\chi_R
\end{pmatrix} \begin{pmatrix}-\chi_L \\
-\chi_L \\
+2\chi_L
\end{pmatrix},
\]
\[
= \begin{pmatrix}1^c_L \\
2^c_R \\
3^c_R
\end{pmatrix} \begin{pmatrix}1_L \\
2_R \\
3_R
\end{pmatrix} \begin{pmatrix}\omega + \chi_R - \chi_L N^c \\
(-\omega + \chi_R - \chi_L)(-E) \\
(-2\chi_R - \chi_L)\nu
\end{pmatrix} \to \begin{pmatrix}\omega + \chi_R - \chi_L E^c \\
(-\omega + \chi_R - \chi_L)N \\
(-2\chi_R + \chi_L)(-\nu)
\end{pmatrix}.
\]

In the antisymmetric coupling, $\phi(78)$ acts on the Higgs field $H(27)$. The neutral Higgs components $H(1)$, $H(16,1)$, $H(16,5^*)$, $H(10,5^*)$ and $H(10,5)$ correspond to the components $S$, $\nu^c$, $-\nu$, $-N$, and $-N^c$, respectively, of $\Psi(27)$ and carry the same quantum numbers as theirs given in Eq. (3.10). Thus, for $(\langle \phi(78) \rangle H(27))_{27}$,

we have only to make the following replacement:

\[
H(1) \to (-2\chi_R + 2\chi_L)H(1),
\]
\[
H(16,1) \to (\omega + \chi_R + 2\chi_L)H(16,1),
\]
\[
H(16,5^*) \to (-2\chi_R - \chi_L)H(16,5^*),
\]
\[
H(10,5^*) \to (-\omega + \chi_R - \chi_L)H(10,5^*),
\]
\[
H(10,5) \to (\omega + \chi_R - \chi_L)H(10,5).
\]
The direct Yukawa coupling of the $y_{ij}$ term in Eq. (2.1) is given by

$$W_Y(H) = y_{ij} \left[ -H(10, 5) (u_i^* u_j + v_i^* \nu_j - S_i N_j) - H(10, 5^*) (d_i^* c_j + e_i^* e_j) ight. $$

$$
\left. + H(16, 5^*) (d_i D_j^c + e_i E_j^c) - H(16, 1) (D_i d_j^c + E_i E_j^c) \right].
$$

(3.12)

Now making in this expression the above replacements, (3.8)–(3.10) and (3.11), we obtain the following explicit expression for the effective Yukawa coupling $W_{\phi}$ resulting from Eq. (3.5):

$$W_{\phi} = -H(10, 5) \left[ \left( \frac{1}{2} (\chi R + \chi L + \omega) a_{ij} + (\omega + \chi_R - \chi_L)s_{ij} \right) u_i^* u_j ight.$$\n
$$
\left. + \left( \frac{1}{2} (3\chi R + 3\chi L - \omega) a_{ij} + (\omega + \chi_R - \chi_L)s_{ij} \right) v_i^* \nu_j ight.$$

$$
\left. + \left( \frac{1}{2} (\omega - 3\chi R + 3\chi L) a_{ij} + (\omega + \chi_R - \chi_L)s_{ij} \right) S_i N_j \right]
$$

$$
-H(10, 5^*) \left[ \left( \frac{1}{2} (\chi R + \chi L - \omega) a_{ij} + (\chi_R - \chi_L - \omega)s_{ij} \right) d_i d_j^c ight.$$\n
$$
\left. + \left( \frac{1}{2} (3\chi R + 3\chi L + \omega) a_{ij} + (\chi_R - \chi_L - \omega)s_{ij} \right) e_i^* e_j \right]
$$

$$
+ H(16, 5^*) \left[ \left(- \frac{1}{2} (2\chi R - \chi_L) a_{ij} - (2\chi_R + \chi_L)s_{ij} \right) d_i d_j^c ight.$$

$$
\left. + \left( \omega + \frac{3}{2}\chi L \right) a_{ij} - (2\chi_R + \chi_L)s_{ij} \right) e_i^* e_j \right]
$$

$$
-H(16, 1) \left[ \left( \frac{1}{2} (\chi R - 2\chi L - \omega) a_{ij} + (-\omega + \chi_R + 2\chi_L)s_{ij} \right) D_i d_j^c ight.$$\n
$$
\left. + \left( \omega + 3\chi R \right) a_{ij} + (-\omega + \chi_R + 2\chi_L)s_{ij} \right) E_i^c e_j \right].
$$

(3.13)

In the expressions (3.12) and (3.13), and also henceforth in this paper, the coupling constants $y_{ij}$, $s_{ij}$ and $a_{ij}$ are no longer the original ones (which are all of order 1), but should be understood as representing the following coupling constants suppressed by the powers of $\lambda = \langle \Theta \rangle / M_P$:

$$
\left( g_{ij} \right) = \left( g_{ij}^{\text{org}, \lambda f_i f_j} \right) = \left( \begin{array}{ccc}
g^{\text{org}, \lambda^6}_{11} & g^{\text{org}, \lambda^5}_{12} & g^{\text{org}, \lambda^4}_{13} \\
g^{\text{org}, \lambda^5}_{21} & g^{\text{org}, \lambda^4}_{22} & g^{\text{org}, \lambda^3}_{23} \\
g^{\text{org}, \lambda^3}_{31} & g^{\text{org}, \lambda^2}_{32} & g^{\text{org}, \lambda^1}_{33} \\
\end{array} \right)
$$

for $g = y, s, a.$

(3.14)

It is therefore important to realize that the coupling constants $y_{ij}$, $s_{ij}$ and $a_{ij}$, as well as their linear combinations, like $Y_{ij}$ and $Y_{ij}'$ defined below, are quantities of order $\lambda^{f_i f_j}$ ($f_1 = 3$, $f_2 = 2$, $f_3 = 0$).

§4. Effective Yukawa couplings

As we mentioned above, since the $SU(5)$ 10 component appears only once in $\Psi(27)$, the identification of the three-up-type quarks is unique. Thus the effective
Yukawa texture for up-type quarks, induced after the VEVs of $\Theta$ and $\phi(78)$ are developed, can immediately be written down in terms of $y_{ij}$, $s_{ij}$ and $a_{ij}$ as

$$
\begin{pmatrix}
u_1 \\ u_1 \\ u_2 \\ u_3
\end{pmatrix} =
\begin{pmatrix}
y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33}
\end{pmatrix},
$$

(4.1)

where

$$Y_{ij} = y_{ij} + (\chi_R - \chi_L + \omega)s_{ij} + \frac{1}{2}(\chi_R + \chi_L + \omega)a_{ij}.$$  

(4.2)

(The quantity $a_{ii}$ is 0 by definition.) Note that since we have assumed that the $U(1)$ charge of the $\phi(78)$ Higgs field is $-2$, the additional Yukawa couplings, $s_{ij}$ and $a_{ij}$, coming from $\phi(78)$ do not contribute to the 3-3 element. Note also that this texture is not necessarily symmetric, because of the asymmetric coupling from $\phi(78)$.

As for the down-quark and charged-lepton sectors, the situation is more complicated, since we now utilize the E-twisted structures (2.7). As explained in Ref. 11), the E-twisted structure is realized when $\Phi(1)$ and $H(16, 1)$ develop the following VEVs:

$$\langle \Phi(1) \rangle \simeq M, \quad \langle H(16, 1) \rangle \simeq M'.$$ 

(4.3)

$\langle \Phi(1) \rangle$ gives mass terms for $\Psi_i(10, 5^*)$-$\Psi_j(10, 5)$ and $\langle H(16, 1) \rangle$ does so for $\Psi_i(16, 5^*)$-$\Psi_j(10, 5)$. Noting that $\Phi$ and $H$ have $U(1)$ charges $-4$ and 0, respectively, the superheavy mass terms are formed in the pattern depicted in Fig. 1. Now assume that $M \gg M'$, implying that the breaking scale of $E_6$ down to $SO(10)$ is higher than the breaking scale of $SO(10)$ to $SU(5)$. With this natural assumption, the components $\Psi_{1, 2}(10, 5^*) = (D_{1, 2} c, E_{1, 2})$ decouple from the light sector by forming

![Fig. 1. The E-twisted structure in which the 5* components enclosed by circles are the dominant components that remain light. The solid and dotted lines indicate heavy mass terms given by $\langle \Phi(1) \rangle$ and $\langle H(16, 1) \rangle$, respectively. The $U(1)$-suppressed terms with higher powers of $\langle \Theta \rangle/M_P$ are indicated by the thinner lines.](https://academic.oup.com/ptp/article-abstract/104/1/211/1882456)
superheavy mass terms with $\Psi_{1,2}(10,5) = (D_{1,2}, E_{1,2})$ of order $M$. Thus $D_3^c$ and $E_3$ in $\Psi_3(10,5^*)$ remain very light (massless at this stage), and we can identify them as the third generation down quark and charged lepton. The first and second generations of $5^*$, on the other hand, are not exactly $\Psi_{1,2}(16,5^*) = (d_{1,2}^c, e_{1,2})$, but are slightly mixed with the third generational one, $\Psi_3(16,5^*)$, owing to the mass mixing with $\Psi_3(10,5) = (D_3, E_3^c)$. $\Psi_3(10,5)$ forms the following mass terms by taking account also of the contributions from the $\phi(78)$ Higgs:

\[
M' D_3 \left( Y_{31} - 2\omega \tilde{s}_{31} + 3\chi_L \tilde{s}_{31} \right) d_1^c + (Y_{32} - 2\omega \tilde{s}_{32} + 3\chi_L \tilde{s}_{23}) d_2^c + y_{33} d_3^c \\
+ M'E_3 \left( Y_{31} - 2\omega \tilde{s}_{31} + 3\chi_L \tilde{s}_{31} \right) e_1 + (Y_{32} - 2\omega \tilde{s}_{32} + 3\chi_L \tilde{s}_{23}) e_2 + y_{33} e_3, \tag{4.4}
\]

where $\tilde{s}_{ij} \equiv s_{ij} + \frac{1}{2} a_{ij}$ and

\[
Y'_{ij} = y_{ij} + (\chi_R - \chi_L + \omega) s_{ij} + \frac{3}{2} (\chi_R + \chi_L + \omega) a_{ij} \\
= Y_{ij} + (\chi_R + \chi_L + \omega) a_{ij}. \tag{4.5}
\]

Thus the superheavy mass partners of $D_3$ and $E_3^c$ are not exactly $d_3^c$ and $e_3$, but the linear combinations appearing in Eq. (4.4). The light down quarks and charged leptons for the first two generations should be orthogonal to these and are, therefore, given by

\[
d_1^c = d_1^c - \frac{1}{y_{33}} (Y_{31} - 2\omega \tilde{s}_{31} + 3\chi_L \tilde{s}_{31}) d_3^c, \\
d_2^c = d_2^c - \frac{1}{y_{33}} (Y_{32} - 2\omega \tilde{s}_{32} + 3\chi_L \tilde{s}_{23}) d_3^c, \tag{4.6}
\]

\[
e_1' = e_1 - \frac{1}{y_{33}} (Y'_{31} - 2\omega \tilde{s}_{31} + 3\chi_L \tilde{s}_{31}) e_3, \\
e_2' = e_2 - \frac{1}{y_{33}} (Y'_{32} - 2\omega \tilde{s}_{32} + 3\chi_L \tilde{s}_{23}) e_3, \tag{4.7}
\]

up to unimportant minor mixings with heavier components. Using these sets of the light fields of three generations, we can obtain the final forms of the texture for the down-quark and charged-lepton Yukawa couplings.

The down quarks $d_i$–$d_3^c$ have the Yukawa couplings

\[
\begin{pmatrix}
  d_1 \\
  d_2 \\
  d_3
\end{pmatrix}
\begin{pmatrix}
  (Y_{11} - 2\omega \tilde{s}_{11}) & (Y_{12} - 2\omega \tilde{s}_{12}) & (Y_{13} - 2\omega \tilde{s}_{13}) \\
  (Y_{21} - 2\omega \tilde{s}_{21}) & (Y_{22} - 2\omega \tilde{s}_{22}) & (Y_{23} - 2\omega \tilde{s}_{23}) \\
  (Y_{31} - 2\omega \tilde{s}_{31}) & (Y_{32} - 2\omega \tilde{s}_{32}) & y_{33}
\end{pmatrix} \tag{4.8}
\]

from $H(10,5^*)$, and $d_i$–$D_3^c$ have

\[
\begin{pmatrix}
  D_1^c \\
  D_2^c \\
  D_3^c
\end{pmatrix}
\begin{pmatrix}
  Y_{11} - (\omega + 3\chi_R) \tilde{s}_{11} & Y_{12} - (\omega + 3\chi_R) \tilde{s}_{12} & Y_{13} - (\omega + 3\chi_R) \tilde{s}_{13} \\
  Y_{21} - (\omega + 3\chi_R) \tilde{s}_{21} & Y_{22} - (\omega + 3\chi_R) \tilde{s}_{22} & Y_{23} - (\omega + 3\chi_R) \tilde{s}_{23} \\
  Y_{31} - (\omega + 3\chi_R) \tilde{s}_{31} & Y_{32} - (\omega + 3\chi_R) \tilde{s}_{32} & y_{33}
\end{pmatrix} \tag{4.9}
\]
from $H(16, 5^*)$. Now, taking account of the mixing (4.6) between the $d_i^c$ and the fact that the light Higgs doublet $H(5^*)$ in the low-energy region is given by the linear combination $H(10, 5^*) \cos \theta + H(16, 5^*) \sin \theta$, the Yukawa matrix for the light down quarks is found to be

$$
\begin{pmatrix}
    d_i^c \\
    d_2 \\
    d_3
\end{pmatrix}
\begin{pmatrix}
    Y_{11}^d \cos \theta & Y_{12}^d \cos \theta & (Y_{13}^d - (\omega + 3\chi_R)s_{13}) \sin \theta \\
    Y_{21}^d \cos \theta & Y_{22}^d \cos \theta & (Y_{23}^d - (\omega + 3\chi_R)s_{23}) \sin \theta \\
    -3\chi_L s_{13} \cos \theta & -3\chi_L s_{23} \cos \theta & y_{33} \sin \theta
\end{pmatrix},
$$

where

$$
Y_{ij}^d = Y_{ij} - 2\omega s_{ij} - \frac{1}{y_{33}} (Y_{i3} - 2\omega s_{i3})(Y_{3j} - 2\omega s_{3j} + 3\chi_L s_{j3}).
$$

This is the final form of the down-quark Yukawa coupling in our model. The charged-lepton matrix is obtained in the same way. However, comparing all the Yukawa couplings for the down-quark and charged-lepton sectors, we obtain the rule that all the charged-lepton couplings can be derived by making the replacement $Y_{ij} \rightarrow Y_{ij}'$ from the corresponding down-quark couplings. Applying this rule, we immediately obtain the final form of Yukawa coupling for the three light generations of charged-leptons:

$$
\begin{pmatrix}
    e_i^c \\
    e_2 \\
    e_3
\end{pmatrix}
\begin{pmatrix}
    Y_{11}^e \cos \theta & Y_{12}^e \cos \theta & (Y_{13}^e - (\omega + 3\chi_R)s_{13}) \sin \theta \\
    Y_{21}^e \cos \theta & Y_{22}^e \cos \theta & (Y_{23}^e - (\omega + 3\chi_R)s_{23}) \sin \theta \\
    -3\chi_L s_{13} \cos \theta & -3\chi_L s_{23} \cos \theta & y_{33} \sin \theta
\end{pmatrix},
$$

where $Y^e$ is defined as

$$
Y_{ij}^e = Y_{ij}' - 2\omega s_{ij} - \frac{1}{y_{33}} (Y_{i3}' - 2\omega s_{i3})(Y_{3j}' - 2\omega s_{3j} + 3\chi_L s_{j3}).
$$

§5. Mass eigenvalues and mixings

In this section, using the above sets of Yukawa couplings for the three light generations, we derive the total $3 \times 3$ mass textures for the up-quark, down-quark and charged-lepton, $M_u$, $M_d$ and $M_e$, and obtain the mass eigenvalues and mixing angles. First we summarize the mass matrices obtained in the previous section:

mass matrix for $u$

$$
M_u = \begin{pmatrix}
    u_1^* & u_2^* & u_3^* \\
    Y_{11} & Y_{12} & Y_{13} \\
    Y_{31} & Y_{32} & 1
\end{pmatrix} y_{33} v \sin \beta,
$$

where $y_{33}$ is the VEV of the doublet $H(16, 5^*)$. The mass matrix for the down quarks is obtained in a similar way.
mass matrix for $d$

$$M_d = \begin{pmatrix}
  d_{11}^c & d_{12}^c & D_1^c \\
  Y_{d11}^d \cos \theta & Y_{d12}^d \cos \theta & (Y_{d13} - (\omega + 3\chi R) \tilde{s}_{13}) \sin \theta \\
  Y_{d21}^d \cos \theta & Y_{d22}^d \cos \theta & (Y_{d23} - (\omega + 3\chi R) \tilde{s}_{23}) \sin \theta \\
  -3\chi_L \tilde{s}_{13} \cos \theta & -3\chi_L \tilde{s}_{23} \cos \theta & y_{33} v \cos \beta
\end{pmatrix}$$

mass matrix for $e$

$$M_e^T = \begin{pmatrix}
e_{11}^e & e_{12}^e & E_3 \\
e_{21}^e & e_{22}^e & (Y_{e13} - (\omega + 3\chi R) \tilde{s}_{13}) \sin \theta \\
e_{31}^e & e_{32}^e & (Y_{e23} - (\omega + 3\chi R) \tilde{s}_{23}) \sin \theta \\
-3\chi_L \tilde{s}_{13} \cos \theta & -3\chi_L \tilde{s}_{23} \cos \theta & y_{33} v \cos \beta.
\end{pmatrix}$$

In the above, $\tan \beta$ is the mixing angle of two light Higgs doublets and $v$ is the vacuum expectation value of the standard model Higgs field. Here we have redefined all the Yukawa couplings to be normalized by $y_{33}$, but we have used the same notation $Y_{ij}$ as above, in order to avoid an overabundance of parameters. As a convention for the mass matrix $M$, we have assumed a fermion mass term given in the form

$$\mathcal{L}_{\text{mass}} = \bar{\psi} \lambda M_i \psi + \text{h.c.},$$

which explains why we have applied the transpose $T$ to the charged-lepton matrix $M_e$.

These mass matrices are diagonalized as

$$(M_u)_{\text{diag}} = U_u M_u V_u^\dagger, \quad (M_d)_{\text{diag}} = U_d M_d V_d^\dagger, \quad (M_e)_{\text{diag}} = U_e M_e V_e^\dagger,$$

and then the Cabibbo-Kobayashi-Maskawa (CKM) and Maki-Nakagawa-Sakata (MNS) mixing matrices are given by

$$V_{\text{CKM}} = U_u U_d^\dagger, \quad V_{\text{MNS}} = U_e U_\nu^\dagger,$$

where $U_\nu$ is such that it makes the matrix $U_\nu^\dagger M_\nu U_\nu$ diagonal. The matrix $M_\nu$ is the Majorana mass matrix of the light left-handed neutrinos, which we do not explicitly discuss in this paper. However, we here only note that it typically takes a hierarchical form like

$$M_\nu \propto \begin{pmatrix}
  \lambda^4 & \lambda^3 & \lambda^2 \\
  \lambda^3 & \lambda^2 & \lambda^1 \\
  \lambda^2 & \lambda^1 & 1
\end{pmatrix}.$$

Thus the mixing matrix on the neutrino side is almost unity, $U_\nu \sim 1 + O(\lambda)$, and hence the mixing matrix in the charged-lepton side $U_e$ is equal to the MNS matrix $V_{\text{MNS}}$, up to $O(\lambda)$ corrections.

It is straightforward to obtain the mass eigenvalues and mixing matrix for each mass matrix. However, we should take some care in treating the matrices for the down quarks and charged leptons, since there occurs a cancellation of the terms of leading order in $\lambda$. We present in the Appendix explicit formulas for eigenvalues and mixing matrices in such a case. Applying the formulas there to Eqs. (5.1)–(5.3), we
find the mass eigenvalues
\[ m_t = y_{33} v \sin \beta, \]
\[ m_c = y_{33} T_{22}^{u} v \sin \beta, \]
\[ m_u = y_{33} \left( T_{11}^{u} - \frac{T_{12}^{u} T_{21}^{u}}{T_{22}^{u}} \right) v \sin \beta, \]
\[ m_b = y_{33} S \sin \theta v \cos \beta, \]
\[ m_s = y_{33} \frac{T_{22}^{d}}{S} \cos \theta v \cos \beta, \]
\[ m_d = y_{33} \left( T_{11}^{d} - \frac{T_{12}^{d} T_{21}^{d}}{T_{22}^{d}} \right) \cos \theta v \cos \beta, \]
\[ m_{\tau} = y_{33} S \sin \theta v \cos \beta, \quad (= m_b) \]
\[ m_{\mu} = y_{33} \frac{T_{22}^{e}}{S} \cos \theta v \cos \beta, \]
\[ m_{e} = y_{33} \left( T_{11}^{e} - \frac{T_{12}^{e} T_{21}^{e}}{T_{22}^{e}} \right) \cos \theta v \cos \beta, \quad (5.7) \]

and the CKM and MNS matrix elements
\[ V_{us} = \frac{T_{12}^{d}}{T_{22}^{d}} - \frac{T_{12}^{u}}{T_{22}^{u}}, \]
\[ V_{ub} = -\frac{1}{S^2 T_{22}^{u}} \left[ (3 \chi R + \omega + 18 \omega \chi L \tilde{s}_{23}^2 \cot^2 \theta) (\tilde{s}_{13} T_{22}^{u} - \tilde{s}_{23} T_{12}^{u}) \right. \]
\[ \quad \left. + 6 \omega \chi L \tilde{s}_{23} \cot^2 \theta \left( f_{12} T_{22}^{u} - f_{22} T_{12}^{u} \right) \right] , \]
\[ V_{cb} = -\frac{\tilde{s}_{23}}{S^2} \left[ 3 \chi R + w + 3 \chi L \cot^2 \theta \left( T_{22}^{d} + 3 \chi L (3 \chi R + \omega) \tilde{s}_{23}^2 \right) \right] , \]
\[ V_{c2} = -\frac{T_{21}^{d}}{T_{22}^{d}} + (U_{\nu}^{1})_{12}, \]
\[ V_{c3} = \frac{3 \chi L \cot \theta}{T_{22}^{e}} (\tilde{s}_{13} T_{22}^{e} - \tilde{s}_{23} T_{21}^{e}) , \]
\[ V_{c3} = \frac{3 \chi L \tilde{s}_{23} \cot \theta}{S}. \quad (5.8) \]

Here \( S, T_{ij}^{a,d,e} \), and \( f_{ij} \) are given by the following combinations of the coupling constants:
\[ T_{ij}^{a} \equiv Y_{ij} - Y_{i3} Y_{3j}, \quad (5.9) \]
\[ T_{ij}^{d} \equiv Y_{ij} - 2 \omega \tilde{s}_{ij} - (Y_{i3} - 2 \omega \tilde{s}_{i3})(Y_{3j} - 2 \omega \tilde{s}_{3j}) - 3 \chi L (3 \chi R + \omega) \tilde{s}_{i3} \tilde{s}_{3j}, \quad (5.10) \]
\[ T_{ij}^{e} \equiv Y_{ij} - 2 \omega \tilde{s}_{ij} - (Y_{i3} - 2 \omega \tilde{s}_{i3})(Y_{3j} - 2 \omega \tilde{s}_{3j}) - 3 \chi L (3 \chi R - \omega) \tilde{s}_{i3} \tilde{s}_{3j}, \quad (5.11) \]
\[ S \equiv \left( 1 + 9 \chi L \tilde{s}_{23}^2 \cot^2 \theta \right)^{1/2}, \quad (5.12) \]
\[ f_{ij} \equiv \tilde{s}_{ij} - \tilde{s}_{i3} (Y_{3j} - \omega \tilde{s}_{3j}) - \tilde{s}_{3j} (Y_{i3} - \omega \tilde{s}_{i3}). \quad (5.13) \]

Note that in these equations, all the Yukawa couplings are normalized by \( y_{33} \), and also that \( T_{ij}^{a,d,e} \) and \( f_{ij} \) are \( O(\chi^{1+f_{ij}}) \) quantities. It is interesting that these eigenvalues
and mixing angles depend on the Yukawa couplings only through their particular combinations, like $T^{u,d,e}_{ij}$ and $f_{ij}$. Therefore, although there are many independent Yukawa couplings in the present model, the actual number of free parameters is greatly reduced. Indeed, as seen in the next section, the number of parameters is less than the number of experimentally measured quantities, and hence we can obtain a kind of sum rule for the observables. The lepton 1-2 mixing $V_{e2}$ cannot be predicted, because in the present model, the 1-2 mixing from the neutrino side is comparable to that from the charged-lepton side, and we are not specifying the contributions of higher-dimensional operators in the neutrino sector. In the following, we examine whether the low-energy experimental values can be properly reproduced with this small number of free parameters.

§ 6. Predictions

We assume in this paper that below the GUT scale, the model is described by the minimal supersymmetric standard model down to the low-energy supersymmetry breaking scale. In order to compare the predictions of our model with the experimental data, it is convenient to analyze the mass matrices at the GUT scale. We here present the numerical values of the running quark and lepton masses and mixing angles at a scale $\approx 2 \times 10^{16}$ GeV:

\[
\begin{align*}
    m_u &\sim 1.04^{+0.19}_{-0.20} \text{ MeV}, \\
    m_d &\sim 1.33^{+0.17}_{-0.19} \text{ MeV}, \\
    m_e &\sim 0.325 \text{ MeV}, \\
    m_c &\sim 302^{+25}_{-27} \text{ MeV}, \\
    m_s &\sim 26.5^{+3.3}_{-3.7} \text{ MeV}, \\
    m_\mu &\sim 68.6 \text{ MeV}, \\
    m_t &\sim 129^{+196}_{-40} \text{ GeV}, \\
    m_b &\sim 1.00 \pm 0.04 \text{ GeV}, \\
    m_\tau &\sim 1.17 \text{ GeV},
\end{align*}
\]

(6.1)

\[
\begin{align*}
    V_{us} &= 0.217 - 0.224, \\
    V_{cb} &= 0.031 - 0.037, \\
    V_{ub} &= 0.002 - 0.005.
\end{align*}
\]

(6.2)

6.1. Second and third generations

First let us concentrate on the second and third generations. As is obvious from Eqs. (5.2) and (5.3), the 3-1 and 3-2 matrix elements of $M_d$ and $M^T_e$ vanish unless $\chi_L \neq 0$, and therefore even for the simplest option we need a nonzero value of $\chi_L$. We first consider the simplest case $\chi_R = \omega = 0$. The number of free parameters is then greatly reduced. The mass eigenvalues and mixing angles can be written in the following simple forms:

\[
\begin{align*}
    m_t &= y_{33} v \sin \beta, \\
    m_c &= y_{33} T_{22} v \sin \beta, \\
    m_b &= y_{33} S v \sin \theta \cos \beta, \\
    m_s &= y_{33} \frac{T_{22}}{S} v \cos \theta \cos \beta, \\
    m_\tau &= y_{33} S v \sin \theta \cos \beta, \\
    m_\mu &= y_{33} \frac{T_{22} + 2(\chi_L a_{23})^2}{S} v \cos \theta \cos \beta, \\
    V_{cb} &= -\frac{3 \chi_L \delta_{23} T_{22}}{S^2} \cot^2 \theta,
\end{align*}
\]

(6.3)

(6.4)
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$$V_{\mu 3} = \frac{3\chi_L \tilde{s}_{23} \cot \theta}{S}. \quad (6.5)$$

Here we have denoted $T_{22}^u = T_{22}^d$ by $T_{22}$. In this case, there are 7 independent free parameters in all in the second and third generation part, $y_{33}, T_{22}, \tilde{s}_{23}, a_{23}, \theta, \tan \beta$ and $\chi_L$. However, since the VEV $\chi_L$ always appears multiplied by the Yukawa coupling $\tilde{s}_{23}$ or $a_{23}$, the number of parameters is in essence 6, with which we express all the relevant data of masses and mixings. We can ‘solve’ the above relations (6.3)–(6.5) and inversely express the parameters in terms of the observable quantities:

$$y_{33} = \frac{m_t}{v} \frac{1}{\sin \beta}, \quad (6.6)$$
$$T_{22} = \frac{m_c}{m_t}, \quad (6.7)$$
$$\tilde{s}_{23} \chi_L = -\frac{1}{3} \frac{m_c V_{cb}}{m_t V_{cb}^2 + (m_s/m_b)^2}. \quad (6.8)$$
$$(a_{23} \chi_L)^2 = \frac{1}{2} \frac{m_c}{m_t} \left( \frac{m_\mu}{m_s} - 1 \right), \quad (6.9)$$
$$\tan \theta = \frac{m_c m_s}{m_t m_b} \left( V_{cb}^2 + \left( \frac{m_s}{m_b} \right)^2 \right)^{1/2}, \quad (6.10)$$
$$\tan \beta = \frac{m_t}{m_s} \sqrt{V_{cb}^2 + \left( \frac{m_s}{m_b} \right)^2} \sin \theta. \quad (6.11)$$

Now there are 8 observables, $m_t, m_c, m_b, m_s, m_\tau, m_\mu, V_{cb}$ and $V_{\mu 3}$, and so there exist two relations between the observable quantities, which give our parameter-independent predictions. One of them is not surprising: $m_b = m_\tau$. This is the well-known $SU(5)$ GUT relation. The other is a novel relation connecting the quark and lepton mixing angles,

$$\sin^2 2\theta_{\mu \tau} = \frac{4 |V_{cb}|^2 \left( \frac{m_s}{m_b} \right)^2}{\left[ V_{cb}^2 + \left( \frac{m_s}{m_b} \right)^2 \right]^2}. \quad (6.12)$$

where we have used $\tan \theta_{\mu \tau} = V_{\mu 3}/V_{\tau 3}$. It should be kept in mind that precisely speaking the left-hand side $\theta_{\mu \tau}$ is the charged-lepton mixing angle, which may deviate from the MNS mixing angle by a neutrino-side contribution of $\lesssim O(\lambda)$. Also the masses $m_b$ and $m_s$ and the CKM element $V_{cb}$ on the right-hand side are the quantities now discussed up to $O(\lambda^2)$ corrections. Aside from these uncertainties, we can predict the $\nu_\mu$–$\nu_\tau$ mixing angle from only the quark part information. In Fig. 2, we give a comparison of the relation with the experimental data. From the figure, we can see
Fig. 2. The prediction of the lepton 2-3 mixing angle \( \sin^2 2\theta_{\mu\tau} \) from our relation (6.12). This square parameter range represents the experimental uncertainties of \( m_s/m_b \) and \( V_{cb} \). In almost the entire region, the relation is consistent with the observations.

that the relation (6.12) is quite consistent with the experimental values.

Note that the success of this relation is a characteristic feature of the twisting 5* structure in this model. To see this, let us consider the mass matrices

\[
M_u \propto \begin{pmatrix} 10_2 & x \\ 10_3 & 1 \end{pmatrix}, \quad M_d \propto \begin{pmatrix} 10_2 & 5_2^* \\ 10_3 & 5_3^* \end{pmatrix}, \quad M_e^T \propto \begin{pmatrix} 5_2^* & 5_3^* \\ 10_2 & 10_3 \end{pmatrix}.
\]

(6.13)

We have assumed the hierarchical form of the up-type Yukawa matrix. The element \( y \) is common to \( M_d \) and \( M_e \), due to the \( SU(5) \) GUT symmetry, and \( z \) is related to the \( m_s \) mass. The blank entries are irrelevant to the discussion here. The relation (6.12) itself results if the condition (i) \( x = x' \) holds. However, in order for it to predict a large lepton mixing angle \( \theta_{\mu\tau} \) [or equivalently, a ‘large’ right-hand side of (6.12)], we need the additional condition (ii) \( y \sim O(1) \). The first condition is satisfied in the present model due to the fact that \( 10_3 \) and \( 5_3^* \) come from a common single
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multiplet. This implies that we need a unification group $SO(10)$ or larger. The second condition (ii) is satisfied thanks to the $5^*$ twisting in the present model. As seen in the Appendix, the lepton mixing angle is determined solely by the parameter $y$. In the down-quark matrix side also, an $O(1)$ $y$ affects $V_{cb}$ as well as $m_s/m_b$. This successful relation is a very interesting and common feature of the generation ($5^*$) twisting mechanism, and it is valid also in the $SO(10)$ model considered by Nomura and Yanagida.\[17]\) We would like to emphasize, however, that in the present $E_6$ model, $y \sim O(1)$ also explains the top-bottom hierarchy. This is due to the fact that we have twisted not the second generation, $5^*_2$, but the third one, $5^*_3$.

We also have the inequality

$$\frac{m_\mu}{m_s} = 1 + \frac{2(a_{23} \chi_L)^2}{T_{22}} > 1. \quad (6.14)$$

(The sign of $T_{22}$ is positive since it gives $m_c/m_t$ by the relation Eq. (6.7).) This relation indicates that $m_\mu$ is always larger than $m_s$ around the GUT scale. This is indeed required to reproduce their correct low-energy mass eigenvalues when we take into account the $SU(3)$ gauge contributions to $m_s$ running in the renormalization-group evolution down to the electroweak scale. The enhancement of the $m_\mu$ mass at the GUT scale is known and is built into the Georgi-Jarlskog-type texture.\[18]\)

To see the dependence on the other parameters, we display the two Higgs mixing angles ($\tan \theta, \tan \beta$) plot in Fig. 3. Each dot satisfies the experimentally observed mass eigenvalues and mixing angles of the quarks and leptons. From this, we see that the Higgs mixing angle $\theta$ is of order $\lambda^2$, as expected from the large lepton mixing. Note that the model also predicts a small value of $\tan \beta \sim O(1)$, because the bottom Yukawa coupling is accompanied by $\sin \theta$. This may be a beautiful explanation of the reason that $m_b$ is $\lambda^2$ times smaller than $m_t$. We also find that the other parameters are naturally consistent with the low-energy experimental values at this stage.

![Fig. 3. The allowed regions of the two Higgs mixing angles $\tan \theta$ and $\tan \beta$.](image)
6.2. Including the first generation

In this section, we include the first generation effects in the two generation analysis of the previous section. Generally speaking, making analyses of such tiny parameters as the masses of the first generation might only result in unnecessary details, because, for example, other (unknown) higher-dimensional operators might become relevant. Moreover, for the lepton mixing \( V_{e3} \), a naive order of magnitude estimation predicts an order \( \lambda^1 \) value which is near the experimental upper limit, according to the CHOOZ data. Nevertheless, it would be interesting to do this analysis in the present model defined by the interaction terms (2.1)–(2.3) alone.

In order to make a full three generation analysis, we must turn on the parameters other than \( \chi_L \), because the nonzero contribution from \( \chi_L \) alone cannot create any difference between the first generation mixings of the up and down quarks (see §5). Thus we are lead to the next minimum choice to include a nonzero \( \chi_R \), while keeping \( \omega \) set to zero. In this case, \( T_u \) and \( T_d \) are no longer equal, and there are the following relations among the Yukawa couplings:

\[
T_u^{ij} - T_u^{ji} = 9 \chi_L \chi_R \tilde{s}_{ij3}, \tag{6.15}
\]

\[
T_u^{ij} - T_u^{ji} = T_d^{ij} - T_d^{ji}, \tag{6.16}
\]

\[
T_e^{ij} - T_e^{ji} = 3 \left( T_d^{ij} - T_d^{ji} \right), \tag{6.17}
\]

\[
T_e^{ij} + T_e^{ji} = T_d^{ij} + T_d^{ji} + 4(\chi_L + \chi_R)^2 a_{i3} a_{j3}. \tag{6.18}
\]

It can be easily checked that there are only 9 independent Yukawa couplings due to these relations. In the following, we take \( y_{33}, T_{ij}^{ij}, \tilde{s}_{ij3} \) and \( a_{i3} \) (\( i = 1, 2 \)) as independent free parameters. All the expressions for mass eigenvalues and mixing have been presented in the previous section. When introducing a nonzero \( \chi_R \), it can be represented by the observable quantities as before:

\[
\tilde{s}_{23} \chi_R = - \frac{1}{3} \left( \frac{m_s}{m_b} V_{ub} + V_{cb} \right). \tag{6.19}
\]

By setting \( \chi_R = 0 \) in the above equation, the relation (6.12) is reproduced. As seen in Fig. 2, the relation with \( \chi_R = 0 \) holds to very good accuracy. This fact implies that the vacuum expectation value \( \chi_R \) is significantly smaller than \( \chi_L \). In Fig. 4, we display the allowed region for \( \chi_L \) and \( \chi_R \) from the masses and mixing angles for the second and third generations alone. From this figure, one can see that \( \chi_R \) should certainly be smaller than \( \chi_L \) and may even be zero. When we use the central values of the masses and mixing angles, we have \( \tilde{s}_{23} \chi_L \sim \lambda^3 \) and \( \tilde{s}_{23} \chi_R \sim \lambda^4 \). These two scales are interesting because they imply, by Eq. (3.4) and \( \tilde{s}_{23} \sim O(\lambda f^2 + f^3 = \lambda^2) \), that the actual VEVs of the \( \chi_L \) and \( \chi_R \) components of \( \phi(78) \) are \( \langle \phi(\chi_L) \rangle \sim \lambda^3 M_P \) and \( \langle \phi(\chi_R) \rangle \sim \lambda^4 M_P \), which are just values around the GUT scale \( 10^{16} \) GeV. This is natural, since we expect that the SU(5) symmetry breaking is caused by the Higgs \( \phi(78) \).

Since \( \chi_R \) is responsible for the mixing angles of the first generation (see §5), in what follows we suppose that \( \chi_R \) generally has a nonzero value. Interestingly enough, even in that case we can find another novel relation. Among the 9 independent
Fig. 4. Typical allowed region for the adjoint Higgs VEVs. $\chi_R$ generally takes a smaller value than $\chi_L$. The central experimental values correspond to $\tilde{s}_{23}\chi_L \simeq -1.3 \times 10^{-2}$ and $\tilde{s}_{23}\chi_R \simeq -0.24 \times 10^{-2}$.

Yukawa couplings, we have already fixed 4 of them, as well as the Higgs mixing angles, from the information of the second and third generations. The remaining 5 free parameters are $T_{1u}, T_{1d}, T_{2u}, \tilde{s}_{13},$ and $a_{13}$. We still have 7 observables to be reproduced, $m_u, m_d, m_e, V_{us}, V_{ub}, V_{c2}$ and $V_{c3}$. Among them, we cannot have a precise prediction for $V_{c2}$, but we can give only an order estimation. This is because the (right-handed) neutrino sector, which we do not explicitly discuss in this paper, may contain more free parameters, i.e. higher-dimensional couplings. We thus expect one relation between the observables. It is found from the analytic solutions discussed in §5 that we now obtain another novel relation:

$$\frac{V_{us}}{V_{ub}} = \frac{V_{\mu3}}{V_{\tau3}} \frac{m_b}{m_s}. \tag{6.20}$$

This indicates that the left-hand side, which involves the first generation, can be written only in terms of the second and third generation parameters. Let us compare this formula (6.20) with the experimental data. Figure 5 shows the numerical result for Eq. (6.20), regarded as a relation between the quark and lepton mixing angles. One can see from Fig. 5 that it is also in good agreement with the experimental data. We note that in the case $\chi_R = 0$, the relation (6.20) reduces to an interesting prediction between the quark mixing angles alone,

$$V_{cb}V_{ub} = -\left(\frac{m_u}{m_b}\right)^2 V_{us}. \tag{6.21}$$

For the mass eigenvalues and mixings of the first generation, we now have the same number of parameters and observables, aside from the two quantities $V_{c2}$ and $V_{us}/V_{ub}$ mentioned above. Thus, we can inversely write down the parameters in
Fig. 5. The prediction of the lepton 2-3 mixing angle $\sin^2 2\theta_{\mu\tau}$ from the relation (6.20). This square parameter range represents the experimental uncertainties of $V_{us}$ and $V_{ub}$. In this figure, we fix $m_s/m_b = 1/40$. The predictions of the relation agree fairly well with the observations.

terms of the observables.

\[
\tilde{s}_{13} = -\tilde{s}_{23} \left( V - V_{e3} \frac{V_{\tau3}}{V_{\mu3}} \right), \tag{6.22}
\]

\[
T_{12}^u = \frac{m_c}{m_t} \left( \tilde{s}_{13} \tilde{s}_{23} + \frac{V_{ub}}{3\chi_R \tilde{s}_{23} V_{\tau3}^2} \right), \tag{6.23}
\]

\[
T_{21}^u = \frac{m_c}{m_t} \tilde{s}_{13} \tilde{s}_{23} + \frac{m_b}{m_s} \frac{V_{\tau3}}{V_{\mu3}} \left( \frac{m_u}{m_t} \frac{1}{m_b} \frac{V_{\tau3}}{V_{\mu3}} \tan \theta \right), \tag{6.24}
\]

\[
T_{11}^u = \frac{m_u}{m_t} - \frac{m_c}{m_t} \left( \frac{1}{\tilde{s}_{23}} + \frac{V_{ub}}{3\chi_R \tilde{s}_{23} V_{\tau3}^2} \right)^2, \tag{6.25}
\]

\[
2(\chi_L + \chi_R)^2 a_{13}^2 = \frac{m_\mu}{m_b} \frac{1}{V_{\tau3}} \tan \theta \left( \frac{m_c}{m_\mu} + \frac{1}{V_{\tau3}} V^2 \right) + 3V \left( T_{12}^u - T_{21}^u \right) - T_{11}^u
\]

\[
- \frac{V_{\mu3}}{V_{\tau3}} \tan \theta \left( \frac{m_s}{m_b} \frac{V_{\mu3}}{V_{\tau3}} + V_{cb} \right) \left( V - V_{e3} \frac{V_{\tau3}}{V_{\mu3}} \right)^2. \tag{6.26}
\]

On the right-hand sides of the equations, we use $\tilde{s}_{23}\chi_{L,R}$ and $\tan \theta$, which are determined by the second and third generation data. The parameter $V$ is defined by
\[ V = V_{e2} - (U^\dagger)_{12} \]

and is determined by solving the equation

\[ V^2 + aV + b = 0, \quad (6.27) \]

where

\[ a = 2(2\alpha + \beta + \gamma), \]

\[ b = \frac{m_t}{m_c} \left( \frac{m_e}{m_b} \frac{1}{V_{\tau 3}} \tan \theta + 9\chi_L \chi_R s_{23}^2 \alpha^2 \right) + (\alpha + \beta)(\alpha + \gamma) - \frac{m_u}{m_c} - \frac{1}{2(\chi_L + \chi_R)^2} a_{23}^2 m_c \left( \frac{1}{V_{\tau 3}} \frac{m_s}{m_b} \alpha \tan \theta + \frac{m_e}{m_t} (\beta - 2\gamma) \right)^2, \quad (6.28) \]

with

\[ \alpha \equiv -\frac{V_{e3} V_{\tau 3}}{V_{\mu 3}}, \]

\[ \beta \equiv \frac{-V_{ub}}{3\chi_R s_{23} V_{\tau 3}}, \]

\[ \gamma \equiv -\frac{m_t}{m_c} \frac{m_b}{m_s} V_{\tau 3} \left( \frac{m_u}{m_t} - \frac{m_d}{m_b} \frac{1}{V_{\tau 3}} \tan \theta \right). \quad (6.29) \]

As stated above, there may be subtle problems in treating such small quantities as the masses for the first generation. Here, therefore, we only give a typical result for the parameters which can actually reproduce the correct values of the observables. For the set of input parameters

\[ T^u_{11} \sim \lambda^{5.1}, \quad T^u_{12} \sim \lambda^{4.8}, \quad T^u_{21} \sim \lambda^{3.9}, \quad T^u_{22} \sim \lambda^{3.6}, \]

\[ \tilde{s}_{23} \chi_L \sim \lambda^{2.5}, \quad a_{23} \chi_L \sim \lambda^{2.1}, \quad \tilde{s}_{13} \chi_L \sim \lambda^{3.4}, \quad a_{13} \chi_L \sim \lambda^{2.2}, \]

\[ \chi_R / \chi_L \sim 0.3, \quad y_{33} \sim 0.58, \quad \tan \theta \sim \lambda^{1.8}, \quad \tan \beta \sim 9.4, \quad (6.30) \]

for instance, we get the following mass eigenvalues and mixings at the GUT scale:

\[ m_u \sim 1 \text{ MeV}, \quad m_d \sim 1 \text{ MeV}, \quad m_e \sim 0.3 \text{ MeV}, \]

\[ m_c \sim 0.4 \text{ GeV}, \quad m_s \sim 0.02 \text{ GeV}, \quad m_\mu \sim 0.07 \text{ GeV}, \]

\[ m_t \sim 100 \text{ GeV}, \quad m_b \sim 1.0 \text{ GeV}, \quad m_\tau \sim 1.0 \text{ GeV}, \]

\[ V_{us} \sim 0.2, \quad V_{cb} \sim 0.04, \quad V_{ub} \sim 0.004, \]

\[ V_{e2} \sim 0.24 + (U^\dagger)_{12}, \quad V_{e3} \sim 0.1, \quad V_{\mu 3} \sim 0.7. \quad (6.31) \]

Note that the parameters chosen in Eq. (6.30) are all of orders that are consistent with our prediction \( \sim \lambda^{l_f f_3} \). Only \( T^u_{11} \) and \( T^u_{21} \) seem larger by a factor \( \lambda^1 \) than our naive expectation. This would, however, not be a large problem, since small enhancements could occur in such combined quantities like the \( T^u_{ij} \). Among the results in Eq. (6.31), we comment on the lepton mixing angles. First, the 1-2 mixing cannot be determined in the present model unless we fix the (higher-dimensional) couplings of neutrinos. The 2-3 mixing angle is large (\( \sin^2 2\theta_{\mu 3} \simeq 1 \)), as we expect from the generation twisting structure. In addition, the 1-3 mixing \( V_{e3} \sim 0.1 \) is consistent with the CHOOZ experimental result.
§7. Summary

In this paper we have examined the $E_6$ grand unified model with E-twisted generation structure. This means that in the second and third generations we have taken a different choice of $5^*$ for the low-energy right-handed down quarks and the left-handed charged leptons. Such a twisting is possible, because the fundamental representation $27$ in $E_6$ contains two $5^*$ representations of $SU(5)$. That is, in $E_6$ GUT models, we naturally have the possibility of generation twisting without introducing extra matter fields. We have constructed such a model, supplementing an adjoint representation Higgs field, which is responsible for creating differences between the quark and lepton masses and mixings (and possibly inducing the $E_6$ gauge symmetry breaking).

Given a set of the VEVs of the adjoint Higgs components, we have investigated the structures of fermion mass matrices which are induced by the effective Yukawa couplings allowed by the flavor $U(1)$ symmetry. Because of the large gauge symmetry of $E_6$, we have found that only the specific and combined Yukawa couplings can appear in the quark and lepton mass matrices. As a consequence, we have found several novel relations between the observables. The relations indicate, notably, that the large lepton 2-3 mixing is related to the precisely measured data of the quark sector alone. This is one of the most interesting features in our $E_6$ model. As for the first generation, the approaches utilizing flavor $U(1)$ symmetries seem to have some disadvantages. That is, the predictability is somewhat weakened in such case, since the relevant quantities are small, which implies that the additional higher-dimensional operators could become relevant unless they are forbidden by some kind of symmetries. In this paper, we have only presented an example which can correctly reproduce the low-energy mass eigenvalues and mixings angles.

We have shown that the generation twisting structure is naturally incorporated in the grand unified models. In order to see whether the model can be really viable, much more work clearly needs to be done, for instance, on the analyses of the Higgs potential that causes the GUT symmetry breaking and of the Majorana mass matrix structure of the light neutrinos. The results in this paper is encouraging enough to motivate such efforts.

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In this appendix, we present general formulas for diagonalizing the $3 \times 3$ mass matrices. This is not only for symmetric-type matrices but also for non-symmetric type ones, in which some components are of the same order. Such a mass matrix appears in recent studies on the down-quarks and lepton mass matrices which induce large lepton mixing angles, as shown in this paper. In such a case, a cancellation between some elements occurs and should be taken care of. For this reason, we present formulas up to the next-to-leading order of the expansion parameter $\lambda$. We assume that all the matrix elements are real for simplicity.

First, consider a general symmetric matrix,

\[
M = \begin{pmatrix}
A \lambda^6 & B \lambda^5 & C \lambda^3 \\
B \lambda^5 & D \lambda^4 & E \lambda^2 \\
C \lambda^3 & E \lambda^2 & F
\end{pmatrix}.
\] (A.1)

It is found that this matrix has the eigenvalues

\[
m_1 = \frac{A}{F} \left( \lambda^6 - \frac{B^2}{D^2} \lambda^8 \right),
\] (A.2)

\[
m_2 = \frac{D}{F} \left( \lambda^4 + \frac{B^2}{D^2} \lambda^6 \right),
\] (A.3)

\[
m_3 = F + \frac{E^2}{F} \lambda^4,
\] (A.4)

with

\[
A = AF - C^2 - \frac{B^2}{D},
\] (A.5)

\[
B = BF - CE,
\] (A.6)

\[
C = BE - CD,
\] (A.7)

\[
D = DF - E^2.
\] (A.8)

The diagonalization $UMU^\dagger = \text{diag}(m_1, m_2, m_3)$ is implemented by a unitary matrix $U$:

\[
U = \begin{pmatrix}
\frac{B}{D} \lambda + \frac{AB}{D} \lambda^3 & -\frac{B}{D} \lambda^3 & \frac{C}{D} \lambda^3 + \frac{ABE}{DF} \lambda^5 \\
\frac{C}{F} \lambda^3 + \frac{BE}{F} \lambda^6 & E \lambda^2 + \frac{DE}{F} \lambda^6 & 1 \\
1 & 0 & 0
\end{pmatrix} \times \mathcal{N},
\] (A.9)

\[
\mathcal{N} = \begin{pmatrix}
1 - \frac{1}{2} \frac{B^2}{D^2} \lambda^2 & 0 & 0 \\
0 & 1 - \frac{1}{2} \frac{B^2}{D^2} \lambda^2 & 0 \\
0 & 0 & 1 - \frac{1}{2} \frac{E^2}{F^2} \lambda^4
\end{pmatrix}.
\] (A.10)
At leading order we have

\[ U = \begin{pmatrix}
1 & -\frac{B}{D} & \frac{C}{D} \\
\frac{B}{D} & 1 & -\frac{E}{F} \\
\frac{C}{D} & \frac{E}{F} & 1
\end{pmatrix}. \] (A.11)

With this general diagonalizing formula, we can calculate the eigenvalues and mixings for the specific types of mass matrices. The first example is the up-type mass matrix, which is not symmetric, although assumed to have hierarchical structure in any case:

\[ M_u = \begin{pmatrix}
a\lambda^6 & b\lambda^5 & c\lambda^3 \\
d\lambda^5 & e\lambda^4 & f\lambda^2 \\
g\lambda^3 & h\lambda^2 & 1
\end{pmatrix}. \] (A.12)

We have normalized the matrix elements so that the 3-3 element becomes 1 for simplicity. We apply the above formula to the symmetric matrix

\[ M = M_u M_u^\dagger. \]

It is useful to define the following three vectors proportional to the three rows of the matrix:

\[ c \equiv \begin{pmatrix} a\lambda^3 \\ b\lambda^2 \\ c \end{pmatrix}, \quad f \equiv \begin{pmatrix} d\lambda^3 \\ e\lambda^2 \\ f \end{pmatrix}, \quad i \equiv \begin{pmatrix} g\lambda^3 \\ h\lambda^2 \\ 1 \end{pmatrix}. \] (A.13)

We then obtain for the matrix \( M = M_u M_u^\dagger \),

\[ A = \frac{(c \times i) \times (f \times i)}{(f \times i)^2} = \lambda^6, \] (A.14)
\[ B = (f \times i) \cdot (c \times i) = (e - fh)(b - ch)\lambda^4, \] (A.15)
\[ C = (f \times i) \cdot (e \times f) = (e - fh)(bf - ce)\lambda^4, \] (A.16)
\[ D = (f \times i)^2 = (e - fh)^2\lambda^4. \] (A.17)

At leading order, with \( F = 1 \) in this case, the eigenvalues and mixings can be written

\[ m_1^2 = A\lambda^6, \quad m_2^2 = D\lambda^4, \quad m_3^2 = 1, \] (A.18)

and

\[ U_u = \begin{pmatrix}
1 & -\frac{b - ch}{e - fh} & \frac{bf - ce\lambda^3}{e - fh} \\
\frac{b - ch}{e - fh} & 1 & -f\lambda^2 \\
\frac{bf - ce\lambda^3}{e - fh} & \frac{-f\lambda^2}{e - fh} & 1
\end{pmatrix}, \] (A.19)

where \( U_u(M_u M_u^\dagger) U_u^\dagger = \text{diag}(m_1^2, m_2^2, m_3^2) \).

The next example is an asymmetric-type matrix, such as the down-quark (and the lepton) mass matrix in this paper:

\[ M_d = \begin{pmatrix}
a\lambda^4 & b\lambda^3 & c\lambda^3 \\
da\lambda^3 & e\lambda^2 & f\lambda^2 \\
g\lambda & h & 1
\end{pmatrix}. \] (A.20)
To consider the symmetric matrix $M = M_d M_d^\dagger$ as before, we define, in this case,

$$c \equiv \begin{pmatrix} a \lambda \\ b \\ c \end{pmatrix}, \quad f \equiv \begin{pmatrix} d \lambda \\ e \\ f \end{pmatrix}, \quad i \equiv \begin{pmatrix} g \lambda \\ h \\ 1 \end{pmatrix}. \quad (A.21)$$

Then we have the same expressions for $A, B, C$ and $D$ in terms of $c, f$ and $i$. Of course, the expressions are different when written with $a, \cdots, h$, but the differences appear only in the powers of $\lambda$. Noting that now $F = 1 + h^2$ at the leading order, we can immediately write down the eigenvalues and the mixing matrix, $U_d(M_d M_d^\dagger) U_d^\dagger = \text{diag}(m_1^2, m_2^2, m_3^2)$, from the general formula

$$m_1^2 = \left[ a - c g - \frac{(d - f g)(b - c h)}{e - f h} \right]^2 \lambda^8, \quad (A.22)$$

$$m_2^2 = \frac{(e - f h)^2}{1 + h^2} \lambda^4, \quad (A.23)$$

$$m_3^2 = 1 + h^2, \quad (A.24)$$

and

$$U_d = \begin{pmatrix} 1 & -\frac{b - c h}{e - f h} & \frac{bf - ce}{e - f h} \\ \frac{b - ch}{e - fh} & \frac{1}{e - fh} & -\frac{f + ch}{1 + h^2 \lambda^2} \\ \frac{c + bh}{1 + h^2 \lambda^3} & \frac{f + ch}{1 + h^2 \lambda^2} & 1 \end{pmatrix}. \quad (A.25)$$

Finally, we consider the charged-lepton mass matrix, the transpose of which takes the same form as the down-quark one:

$$M_e^\dagger = \begin{pmatrix} a \lambda^4 & b \lambda^3 & c \lambda^3 \\ d \lambda^3 & e \lambda^2 & f \lambda^2 \\ g \lambda & h & 1 \end{pmatrix} = M_d. \quad (A.26)$$

Then, clearly the eigenvalues $m_1, m_2, m_3 (> 0)$ are the same as those for the down-quark matrix $M_d$. The mixing matrix realizing $U_e(M_e M_e^\dagger) U_e^\dagger = \text{diag}(m_1^2, m_2^2, m_3^2)$ is found from the relation $U_d M_d U_d^\dagger = D_m \equiv \text{diag}(m_1, m_2, m_3)$ as

$$U_e = D_m^{-1} U_d M_e^\dagger = \begin{pmatrix} 1 & -\frac{d - f g}{e - f h} & -\frac{e g - d h}{e - f h} \\ \frac{d (1 + h^2) - g(f + ch)}{\sqrt{1 + h^2} \lambda} & \frac{1}{\sqrt{1 + h^2} \lambda} & -\frac{h}{\sqrt{1 + h^2} \lambda} \\ \frac{g}{\sqrt{1 + h^2} \lambda} & \frac{h}{\sqrt{1 + h^2} \lambda} & \frac{1}{\sqrt{1 + h^2} \lambda} \end{pmatrix}. \quad (A.27)$$

From the form of this mixing matrix, we can see that it depends only on the parameter $h$ whether the large mixing angle between the second and third generations is realized or not.
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