A Remark on Evolution Equation of Stochastic Logical Dynamic Systems

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Abstract—Modelling is an essential procedure in analyzing and controlling a given logical dynamic system (LDS). It has been proved that deterministic LDS can be modeled as a linear-like system using algebraic state space representation. However, due to the inherently non-linear, it is difficult to obtain the algebraic expression of a stochastic LDS. This paper provides a unified framework for transition analysis of LDSs with deterministic and stochastic dynamics. First, modelling of LDS with deterministic dynamics is reviewed. Then modeling of LDS with stochastic dynamics is considered, and non-equivalence between subsystems and global system is proposed. Next, the reason for the non-equivalence is provided. Finally, consistency condition is presented for independent model and conditional independent model.

Index Terms—Logical dynamic system, Algebraic state space representation, Conditional independence, Consistency condition, Semi-tensor product of matrices.

I. INTRODUCTION

WITH the development of networked systems, logical dynamic system (LDS) has attracted a lot of attentions due to its wide background in biological system, economical system, social system etc [1], [2], [3]. Two representative application fields of LDS are Boolean networks [1] and networked evolutionary games [4]. LDSs investigate the logic evolutionary dynamics on graphs [5]. The appeal of LDS is that it merges the interactions between nodes and the evolutionary dynamics of nodes together, which makes it a theoretically valued model [6].

As one of most fundamental problems for dynamic systems, modeling is the first step before analyzing and controlling a given LDS [7]. From a mathematical point of view, LDS can be divided into two types: (i) LDS with deterministic dynamics: the evolutionary rule for each node is deterministic; (ii) LDS with stochastic dynamics: the evolutionary rule for each node is stochastic. According to the type of LDS, there are different modeling methods [8]. Recently, a efficient mathematic tool, called semi-tensor product (STP) of matrices, is proposed to model LDS [9].

STP is a generalization of traditional matrix product, and it has been successfully applied to Boolean networks [9], finite games [13] and finite automaton [11]. Recently, STP has been applied to LDS. A control Lyapunov function approach to feedback stabilization of logical control networks is proposed in [10]. Event-triggered control of LDS is studied in [12], [14]. Controllability and observability of LDSs are considered in [15]. Optimal control of LDS is investigated using STP in [1].

However, most of existing results focused on LDS with deterministic dynamics. The reason comes from two aspects: (i) deterministic system is easily analyzed; (ii) deterministic LDS can be modelled as a linear-like system by algebraic state space method using STP. From a technical point of view, there are three key points in the process of modeling deterministic LDS into its algebraic formulation: projection, swap, and descending power, which are implemented by projection operator, swap operator, and power-reduced operator, respectively. Inspired by the successful application of algebraic formulation to deterministic LDSs, some researchers attempt to apply it to stochastic LDSs [16]-[19]. However, when we shift our attention to LDS with stochastic dynamics, power-reduced operator is not applicable any more. Therefore, it is difficult to obtain the algebraic formulation of LDS with stochastic dynamics. The intrinsic cause of the difference in the modeling of deterministic LDS and stochastic LDS is that the latter is inherently non-linear.

This paper aims at providing a unified framework for modeling LDSs with deterministic and stochastic dynamics. First, modeling of LDS with deterministic dynamics is reviewed. Then modeling of LDS with stochastic dynamics is considered, and non-equivalence between subsystems and global system is proposed. Next, the reason for the non-equivalence is revealed. Finally, consistency condition is presented for independent model and conditional independent model.

Contributions: The contributions of this paper are threefold: (i) The non-equivalence between subsystems and global system for stochastic LDS is proposed. (ii) Next, the reason for the non-equivalence is provided. We find that stochastic LDS can be modelled as a non-homogeneous Markov chain under independent condition and as a homogeneous Markov chain under conditional independent condition. (iii) Consistency condition is presented for independent model and conditional independent model.
Notations: $\mathbb{R}^n$ is denoted as the Euclidean space of all real $n$-vectors. $\mathcal{M}_{m \times n}$ is the set of $m \times n$ real matrices and $D_k := \{1, 2, \ldots, k\}$. $1_m$ is an $m$-dimensional vector with identity entries and $I_n$ is an $n \times n$ identity matrix. Row$_r(L)$ and Col$_i(L)$ represent the $r$-th row and the $i$-th column of matrix $L$, respectively. Col$(L)$ signifies the set of columns of $L$. Let $\Delta_n := \text{Col}(I_n)$, $\delta_{ir}^n := \text{Col}_i(I_n)$. $L = [\delta_{ir}^n, \delta_{ir}^n, \ldots, \delta_{ir}^n]$ is called an $n \times r$-dimensional logical matrix, which is abbreviated as $L = \delta_{ir}[i_1, i_2, \ldots, i_r]$. Denote by $L_{s \times r}$ the set of $s \times r$ logical matrices. $\Upsilon_n$ is the set of all $n$-dimensional probability vector and $\Upsilon_{p \times q}$ is the set of all $p \times q$-dimensional column stochastic matrix.

The rest of this paper is organized as follows: Section II provides some preliminaries on semi-tensor product of matrices and LDSs. Section III considers modeling of LDS with deterministic dynamics. Section IV considers the non-equivalence between subsystems and global system for stochastic LDSs. Section V and Section VI investigate the reason for the non-equivalence and consistency condition, respectively. A brief conclusion is given in Section VII.

II. PRELIMINARIES

The basic mathematical tool used in this paper is STP of matrices. Please refer to [20] for more details.

Definition 2.1: [20] Suppose $A \in \mathcal{M}_{m \times n}$, $B \in \mathcal{M}_{p \times q}$ and $l$ be the least common multiple of $n$ and $p$. The STP of $A$ and $B$ is defined as

$$A \Join B := (A \otimes I_{l/n}) (B \otimes I_{l/p}) \in \mathcal{M}_{ml/n \times ql/p},$$

where $\otimes$ is the Kronecker product of matrices.

STP has the pseudo commutativity, which is shown as follows.

Proposition 2.2: [20] STP has the following commutativity.

(i) Let $z \in \mathbb{R}^l$, $A \in \mathcal{M}_{m \times n}$, then

$$z \Join A = (I_l \Join A) \Join z.$$

(ii) Let $x \in \Delta_m$, $y \in \Delta_n$ and define a matrix $W_{[m,n]} \in \mathcal{M}_{mn \times mn}$, where

$$\text{Col}_{i(j-1)m+j}(W_{[m,n]}) = \delta_{mn}^{i+j(j-1)m}, \quad i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n,$$

$W_{[m,n]}$ is called $(m, n)$-th dimensional swap matrix. Then

$$x \Join y = W_{[m,n]}yx.$$

Proposition 2.3: Let $x \in \Delta_k$ and define a matrix, called power-reduced matrix

$$R_k = \text{diag}(\delta_1^k, \delta_1^k, \ldots, \delta_k^k).$$

Then

$$x \Join x = R_kx.$$

Assume $i \in D_k$. By identifying $i \sim \delta_i^k$, we call $\delta_i^k$ the vector form of integer $i$. A function $f : \prod_{i=1}^{\infty} D_k \to D_k$ is called a mix-valued logical function.

Proposition 2.4: [20] Let $f : \prod_{i=1}^{n} D_k, \to D_k$ be a mix-valued logical function. Then there exists a unique matrix $M_f \in M_{k \times k}$, such that

$$f(x_1, \ldots, x_n) = M_f \varphi_{i=1}^{n} x_i.$$

$M_f$ is called the structure matrix of $f$, and $k = \prod_{i=1}^{n} k_i$.

Definition 2.5: Let $A \in \mathcal{M}_{p \times n}$ and $B \in \mathcal{M}_{q \times n}$. Then the Khatri-Rao Product of $A$ and $B$ is

$$A \Join B = [\text{Col}_1(A) \Join \text{Col}_1(B), \ldots, \text{Col}_n(A) \Join \text{Col}_n(B)] \in \mathcal{M}_{pq \times n}.

An LDS consists of two aspects: (i) an undirected graph $(N, E)$ with node set $N = \{1, 2, \ldots, n\}$ and edge set $E \subset N \times N$; (ii) evolutionary rules for each node. Let $x_i(t) \in D_{k_i}$ be the state of node $i$ at time $t > 0$. Let $N_i$ be the neighbours of node $i$. If each node updates his state at time $t+1$ according to the state $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ at time $t$, then the evolutionary dynamics can be described as follows

$$\begin{align*}
  x_1(t+1) &= f_1(\{x_i(t)\}_{i \in N_1}) \\
  x_2(t+1) &= f_2(\{x_i(t)\}_{i \in N_2}) \\
  &\vdots \\
  x_n(t+1) &= f_n(\{x_i(t)\}_{i \in N_n})
\end{align*}.$$

If $f_i : \prod_{i \in N_i} D_{k_i} \to D_{k_i}$, then system (3) is called deterministic. Otherwise, if $f_i : \prod_{i \in N_i} \Upsilon_{k_i} \to \Upsilon_{k_i}$, then system (3) is called stochastic.

III. ALGEBRAIC EXPRESSION OF LDS WITH DETERMINISTIC DYNAMICS

Suppose the evolutionary rule for each node is deterministic. Consider a subset nodes $U \subseteq N$. Define a projection matrices $\Phi_U$ as follows

$$\Phi_U := \otimes_{j=1}^{\infty} \text{I}_{\gamma_j},$$

where

$$\gamma_j := \begin{cases} I_{k_j}, & j \in U \\
1_{k_j}, & j \notin U.\end{cases}$$

By virtue of vector expression to $x_i$ and $x$, the subsystems [3] can be described as follows

$$\begin{align*}
  x_1(t+1) &= M_1 \varphi_{i \in N_1} x_i(t) = M_1 \Phi_{N_1} x(t) \\
  x_2(t+1) &= M_2 \varphi_{i \in N_2} x_i(t) = M_2 \Phi_{N_2} x(t) \\
  &\vdots \\
  x_n(t+1) &= M_n \varphi_{i \in N_n} x_i(t) = M_n \Phi_{N_n} x(t),
\end{align*}$$

where $x_i(t) \in \Delta_{k_i}$, $x(t) = \varphi_{i=1}^{n} x_i(t)$, and $M_i \in L_{k_i \times k_i}$ is the structure matrix of $f_i$ with $k = \prod_{i=1}^{n} k_i$. Denote by $M_i = M_i \Phi_{N_i}$. Therefore, the global evolutionary equation can be obtained as

$$x(t+1) = Mx(t),$$

where

$$M = M_1 \Join M_2 \Join \cdots \Join M_n.$$
According to (6), global evolutionary equation (5) can be obtained from subsystems (4). On the other hand, if the global evolutionary equation (5) is given, can we obtain the subsystems (4)? The following result reveals that the subsystems (4) can be obtained from global evolutionary equation (5).

**Proposition 3.1:** Consider LDS (3) with its global evolutionary equation (5). Then $M_i$ of subsystems (4) can be obtained as follows

$$
M_i = \Phi_i M, \quad i = 1, 2, \ldots, n.
$$

**Proof:** For any $x(t) \in \Delta_k$, we have

$$
\Phi_i M x(t) = \Phi_i x(t + 1) = (\otimes_{j=1}^n \gamma_j)(\otimes_{i=1}^n x_i(t + 1)) = x_i(t + 1).
$$

The last equality comes from the following property of Kronecker product

$$(A \otimes B)(C \otimes D) = AC \otimes BD.
$$

Proposition 3.1 reveals that global evolutionary equation (5) and subsystems (4) are equivalent for deterministic LDS.

**IV. ALGEBRAIC EXPRESSION OF LDS WITH STOCHASTIC DYNAMICS**

Suppose node $\ell$ takes values from set $D_k$, according to some probabilities or the evolutionary rule for each node is stochastic. Then each evolutionary rule $f_\ell$ for node $\ell$ can be described by a transition matrix $Q_\ell = (q_{i,j})_{k \times k}$ with $k = \prod_{i=1}^n k_i$, where

$$
q_{i,j} = \Pr(x_{i}(t + 1) = i|x(t) = j), \quad \ell = 1, 2, \ldots, n.
$$

Let $p_\ell(t) = \Pr(x_{\ell}(t) = j)$ be the probability of taking $j$ of node $\ell$ at time $t$. And the probability vector of node $\ell$ at time $t$ is denoted by

$$
p_\ell(t) = [p_1(t), p_2(t), \ldots, p_k(t)]^T \in \mathcal{Y}_{k_i}.
$$

The state probability vector $p(t)$ is denoted by

$$
p(t) = [\Pr(x(t) = 1), \Pr(x(t) = 2), \ldots, \Pr(x(t) = k)]^T.
$$

Then the stochastic evolutionary equation can be described as follows

$$
\begin{align*}
p_1(t + 1) &= Q_1 p_N_1(t) = Q_1 \Phi_N_1 p(t) \\
p_2(t + 1) &= Q_2 p_N_2(t) = Q_2 \Phi_N_2 p(t) \\
&\vdots \\
p_n(t + 1) &= Q_n p_N_n(t) = Q_n \Phi_N_n p(t),
\end{align*}
$$

where $p_N_i(t) \in \mathcal{Y}_{k_N_i}$ is the state probability vector of players in $N_i$ and $k_{N_i} = \prod_{j \in N_i} k_j$.

Denote by $Q_i = Q_1 \Phi_N_1$. Construct an evolutionary equation, which is similar to (4), as follows

$$
\begin{align*}
p_1(t + 1) &= \hat{Q}_1 \kappa_{i=1}^n p_i(t) \\
p_2(t + 1) &= \hat{Q}_2 \kappa_{i=1}^n p_i(t) \\
&\vdots \\
p_n(t + 1) &= \hat{Q}_n \kappa_{i=1}^n p_i(t).
\end{align*}
$$

Construct a global evolutionary equation, which is similar to (5), as follows

$$
p(t + 1) = Q p(t),
$$

where

$$
Q = \hat{Q}_1 \ast \hat{Q}_2 \ast \ldots \ast \hat{Q}_n \in \mathcal{Y}_{k \times k}.
$$

A natural question is: whether $\kappa_{i=1}^n p_i(t) = p(t)$? What are the relationships between (6), (9) and (10)? The following example provides some interesting results.

**Example 4.1:** Consider a two-node stochastic LDS with transition matrix for node 1 and node 2 are as follows respectively

$$
\hat{Q}_1 = \begin{bmatrix}
0.3 & 0.5 & 1 & 0.2 \\
0.7 & 0.5 & 0 & 0.8
\end{bmatrix}, \quad \hat{Q}_2 = \begin{bmatrix}
0.4 & 0.2 & 0.5 & 0.7 \\
0.6 & 0.8 & 0.5 & 0.3
\end{bmatrix}
$$

According to (9), we have

$$
\begin{align*}
p_1(t + 1) &= \hat{Q}_1 \hat{p}(t), \\
p_2(t + 1) &= \hat{Q}_2 \hat{p}(t),
\end{align*}
$$

where $\hat{p}(t) = p_1(t) \ast p_2(t)$.

According to (10), we have

$$
p(t + 1) = Q \hat{p}(t).
$$

Suppose $p_1(0) = [0.4, 0.6]^T$, $p_2(0) = [0.5, 0.5]^T$, and $p(0) = p_1(0) \ast p_2(0) = [0.2, 0.2, 0.3, 0.3]^T$.

Then according to (11) and (12), it is easy to calculate that

$$
\begin{align*}
p_1(1) &= \hat{Q}_1 p(0) = [0.52, 0.48]^T, \\
p_2(1) &= \hat{Q}_2 p(0) = [0.48, 0.52]^T, \\
\hat{p}(1) &= [0.2496, 0.2704, 0.2304, 0.2496]^T.
\end{align*}
$$

Fig. 1. Evolution of state in deterministic LDS

**Example 4.1:** Consider a two-node stochastic LDS with transition matrix for node 1 and node 2 are as follows respectively

$$
\begin{align*}
\hat{Q}_1 &= \begin{bmatrix}
0.3 & 0.5 & 1 & 0.2 \\
0.7 & 0.5 & 0 & 0.8
\end{bmatrix}, \\
\hat{Q}_2 &= \begin{bmatrix}
0.4 & 0.2 & 0.5 & 0.7 \\
0.6 & 0.8 & 0.5 & 0.3
\end{bmatrix}
\end{align*}
$$

According to (9), we have

$$
\begin{align*}
p_1(t + 1) &= \hat{Q}_1 \hat{p}(t), \\
p_2(t + 1) &= \hat{Q}_2 \hat{p}(t),
\end{align*}
$$

where $\hat{p}(t) = p_1(t) \ast p_2(t)$.

According to (10), we have

$$
p(t + 1) = Q \hat{p}(t).
$$

Suppose $p_1(0) = [0.4, 0.6]^T$, $p_2(0) = [0.5, 0.5]^T$, and $p(0) = p_1(0) \ast p_2(0) = [0.2, 0.2, 0.3, 0.3]^T$.

Then according to (11) and (12), it is easy to calculate that

$$
\begin{align*}
p_1(1) &= \hat{Q}_1 p(0) = [0.52, 0.48]^T, \\
p_2(1) &= \hat{Q}_2 p(0) = [0.48, 0.52]^T, \\
\hat{p}(1) &= [0.2496, 0.2704, 0.2304, 0.2496]^T.
\end{align*}
$$
According to above analysis, it is easy to conclude that
\[ \hat{p}(t) \neq p(t), \ \forall t > 0. \]

Above Example implies that subsystems (9) and global evolutionary equation (10) with stochastic dynamics are not equivalent. In the following section, we investigate the reasons for the non-equivalence.

Remark 4.2: The difference between (9) and (10) is imperceptible, which leads to the cognitive mistake on stochastic LDS for many works. This is also the main motivation of this paper.

V. EXPLANATION FOR THE NON-EQUIVALENCE

Before investigating the non-equivalence, the following definitions are necessary.

Definition 5.1: Consider three random variables \( X, Y, Z \).

(i) Random variables \( X \) and \( Y \) are independent if
\[
\Pr(X = x, Y = y) = \Pr(X = x) \Pr(Y = y), \quad \forall x \in X, y \in Y.
\]

(ii) Random variables \( X \) and \( Y \) are conditionally independent on \( Z \) if
\[
\Pr(X = x, Y = y|Z = z) = \Pr(X = x|Z = z) \Pr(Y = y|Z = z), \quad \forall x \in X, y \in Y, z \in Z.
\]

Remark 5.2: It should be pointed out that neither independent implies to conditional independent nor conditional independent implies to independent.

Assumption 5.3: (Independence Assumption) The random variables \( x_1(t), x_2(t), \ldots, x_n(t) \) are independent for any \( t > 0 \).

Suppose Assumption 5.3 is satisfied, then the probability of \( x(t + 1) = j \) is
\[
\Pr(x(t + 1) = j) = \frac{\sum_{i=1}^{k} \Pr(x(t) = i, x(t + 1) = j)}{\sum_{i=1}^{k} \Pr(x(t) = i, x(t + 1) = j)}.
\]

According to (13), the state probability vector can be calculated as
\[
\begin{bmatrix}
\Pr(x(t + 1) = 1) \\
\Pr(x(t + 1) = 2) \\
\vdots \\
\Pr(x(t + 1) = k)
\end{bmatrix}
= \begin{bmatrix}
\prod_{i=1}^{n} \Pr(\hat{Q}_i)p(t) \\
\prod_{i=1}^{n} \Pr(\hat{Q}_i)p(t) \\
\vdots \\
\prod_{i=1}^{n} \Pr(\hat{Q}_i)p(t)
\end{bmatrix}
\]

Recall equation (9), it is easy to know that (14) is equivalent to (9) under Assumption 5.3 (independence condition).

Assumption 5.4: (Conditional Independence) The random variables \( x_1(t + 1), x_2(t + 1), \ldots, x_n(t + 1) \) are independent conditional on \( x(t) \) for any \( t > 0 \).

Suppose Assumption 5.4 is satisfied, then the probability of \( x(t + 1) = j \) is
\[
\Pr(x(t + 1) = j) = \frac{\sum_{i=1}^{k} \Pr(x(t) = i, x(t + 1) = j|x(t) = i)}{\sum_{i=1}^{k} \Pr(x(t) = i, x(t + 1) = j|x(t) = i)}.
\]

According to (15), the state probability vector can be calculated as
\[
\begin{bmatrix}
\Pr(x(t + 1) = 1) \\
\Pr(x(t + 1) = 2) \\
\vdots \\
\Pr(x(t + 1) = k)
\end{bmatrix}
= \begin{bmatrix}
\hat{Q}_1 \cdot \hat{Q}_2 \cdot \ldots \cdot \hat{Q}_n \\
\Pr(x(t) = 1) \\
\vdots \\
\Pr(x(t) = k)
\end{bmatrix}
\]

(16)
Recall equation (10), it is easy to know that (10) is equivalent to (10) under Assumption 5.4 (conditional independence condition).

**Proposition 5.5:** Consider LDS with global evolutionary equation (10). Suppose

\[ Q = Q_1 \ast Q_2 \ast \cdots \ast Q_n \in \mathcal{K}_{k \times k} \]

Then \( \hat{Q}_i \) can be obtained as follows

\[ \hat{Q}_i = \Phi_i Q, \quad i = 1, 2, \ldots, n. \]

**Proof:**

\[ \begin{align*}
\Phi_i Q &= \Phi_i [\hat{Q}_1 \ast \hat{Q}_2 \ast \cdots \ast \hat{Q}_n] \\
&= \Phi_i \otimes_{i=1}^n \text{Col}_1(\hat{Q}_i) \cdots \otimes_{i=1}^n \text{Col}_n(\hat{Q}_i) \\
&= \left[ \Phi_i \otimes_{i=1}^n \text{Col}_1(\hat{Q}_i) \cdots \otimes_{i=1}^n \text{Col}_n(\hat{Q}_i) \right] \\
&= [\text{Col}_1(\hat{Q}_i), \text{Col}_2(\hat{Q}_i), \ldots, \text{Col}_n(\hat{Q}_i)] \\
&= \hat{Q}_i.
\end{align*} \]

**Proposition 5.6:** Consider a stochastic LDS. Then

(i) (14) is equivalent to (9) under independence condition.

(ii) (16) is equivalent to (10) under conditional independence condition.

(iii) (9) is not equivalent to (10).

**Remark 5.7:** From a mathematical point of view, (14) is a non-homogeneous Markov chain, and (16) is a homogeneous Markov chain.

**VI. Consistency Condition Exploration**

According to above analysis, we know that (9) and (10) are not equivalent. Under what conditions will (9) and (10) be equivalent?

**Proposition 6.1:** (Consistency Condition) System (9) and (10) are equivalent, if and only if, for any \( p \in \mathcal{K} \) the following equation is satisfied

\[ HR_k^{-1} p = H \times p^n, \tag{17} \]

where \( R_k = \text{diag}\{\delta_1^k, \delta_2^k, \ldots, \delta_k^k\} \) and

\[ H = \times_{i=1}^n (I_{k^i} \otimes \hat{Q}_i). \]

**Proof:** Firstly, we prove that

\[ \hat{Q}_1 \ast \hat{Q}_2 \ast \cdots \ast \hat{Q}_n = H \times R_k^{-1}. \tag{18} \]

For any \( x = \delta_k^i \in \Delta_k \)

\[ \text{Col}_i(\hat{Q}_1 \ast \hat{Q}_2 \ast \cdots \ast \hat{Q}_n) = (\hat{Q}_1 \ast \hat{Q}_2 \ast \cdots \ast \hat{Q}_n) \times x \]

\[ = (\hat{Q}_1 \times x) \times (\hat{Q}_2 \times x) \times \cdots \times (\hat{Q}_n \times x) \]

\[ = \hat{Q}_1 (I_k \ast \hat{Q}_2) x^2 \cdots \times \hat{Q}_n x \]

\[ = H x^n \]

\[ = H R_k^{-1} x. \]

According to the arbitrariness of \( x \in \Delta_k \), it follows that

\[ \hat{Q}_1 \ast \hat{Q}_2 \ast \cdots \ast \hat{Q}_n = H \times R_k^{-1}. \]

Then we prove the following statement

\[ (\hat{Q}_1 p) (\hat{Q}_2 p) \cdots (\hat{Q}_n p) = H \times p^n, \quad \forall p \in \mathcal{K}. \tag{19} \]

For any \( p \in \mathcal{K}_k \)

\[ (\hat{Q}_1 p) \times (\hat{Q}_2 p) \times \cdots \times (\hat{Q}_n p) \]

\[ = \hat{Q}_1 (I_k \ast \hat{Q}_2) p^2 \times \cdots \times (\hat{Q}_n p) \]

\[ = H p^n. \]

Therefore, (9) and (10) are equivalent, if and only if

\[ HR_k^{-1} p = H p^n, \quad \forall p \in \mathcal{K}. \tag{20} \]

According to Proposition 6.1, the following corollary is obvious.

**Corollary 6.2:** System (9) and (10) are equivalent if the following equation is satisfied

\[ HR_k^{-1} = H \times p^{n-1}, \quad \forall p \in \mathcal{K}. \]

However, (20) is only a sufficient condition, not a necessary condition. The following is a counterexample.
Example 6.3: Consider a LDS with state-strategy transition matrix as follows
\[ \hat{Q}_1 = \begin{bmatrix} 0.3 & 0.4 & 0.4 & 0.3 \\ 0.7 & 0.6 & 0.6 & 0.3 \end{bmatrix}, \quad \hat{Q}_2 = \begin{bmatrix} 0.2 & 0.3 & 0.3 & 0.3 \\ 0.8 & 0.7 & 0.7 & 0.7 \end{bmatrix}. \]

Let
\[ p = [0, 0.5, 0, 0.5]^T. \]

It is easy to calculate that
\[ HR_4 \preceq p = \hat{Q}_1 \times (I_2 \otimes \hat{Q}_2) \times p = [0, 0.35, 0.65], \]
\[ H \preceq p^2 = \hat{Q}_1 \times (I_2 \otimes \hat{Q}_2) \times p^2 = [0, 0.35, 0.65], \]
therefore
\[ HR_4 \preceq p = H \preceq p^2. \]

On the other hand,
\[ HR_4 - H p = \begin{bmatrix} 0.03 & 0 & -0.02 & 0 \\ 0 & 0.02 & 0 & -0.02 \\ -0.03 & 0 & 0.02 & 0 \\ 0 & -0.02 & 0 & 0.02 \end{bmatrix}, \]
therefore
\[ HR_4 \not\preceq H \preceq p. \]

Proposition 6.4: System (9) and (10) are equivalent, if there are \( n - 1 \) matrices \( \hat{Q}_1, \hat{Q}_2, \ldots, \hat{Q}_{n-1} \), each of which has the same column. In other words, there exists \( v_i \in \Upsilon_k \), satisfying
\[ \hat{Q}_{ij} = I_k^T \otimes v_i, \quad j = 1, 2, \ldots, n - 1. \quad (21) \]

Proof: Without loss of generality, let \( \hat{Q}_1, \hat{Q}_2, \ldots, \hat{Q}_{n-1} \) be the \( n - 1 \) matrices satisfying (21).

According to (18),
\[ H \preceq R_{k-1}^{n-1} \preceq p \]
\[ = (\hat{Q}_1 \ast \hat{Q}_2 \ast \ldots \ast \hat{Q}_n)p \]
\[ = [\vee_{i=1}^n \mathrm{Col}_i(\hat{Q}_1), \ldots, \vee_{i=1}^n \mathrm{Col}_i(\hat{Q}_n)]p \]
\[ = [\vee_{i=1}^n v_i \ast \mathrm{Col}_i(\hat{Q}_1), \ldots, \vee_{i=1}^n v_i \ast \mathrm{Col}_i(\hat{Q}_n)]p \]

According to (19),
\[ H \preceq p^n \]
\[ = (\hat{Q}_1 p)(\hat{Q}_2 p) \cdots (\hat{Q}_n p) \]
\[ = [\vee_{i=1}^{n-1} v_i \ast \hat{Q}_n \ast \ldots \ast \hat{Q}_1]p \]
\[ = [\vee_{i=1}^{n-1} v_i \ast \mathrm{Col}_i(\hat{Q}_1), \ldots, \vee_{i=1}^{n-1} v_i \ast \mathrm{Col}_i(\hat{Q}_n)]p \]

Therefore,
\[ HR_{k-1}^{n-1} \preceq p = H \preceq p^n. \]

According to Proposition 6.1, system (9) and (10) are equivalent.

Example 6.5: Consider a two-node stochastic LDS with transition matrix for node 1 and node 2 are as follows respectively
\[ \hat{Q}_1 = \begin{bmatrix} 0.3 & 0.3 & 0.3 & 0.3 \\ 0.7 & 0.7 & 0.7 & 0.7 \end{bmatrix}, \quad \hat{Q}_2 = \begin{bmatrix} 0.2 & 0.6 & 0.1 & 0.4 \\ 0.8 & 0.4 & 0.9 & 0.6 \end{bmatrix}. \]

Let
\[ p = [a, b, c, 1 - a - b - c]^T \in \Upsilon_4. \]

It is easy to calculate that
\[ HR_4 \preceq p = \hat{Q}_1 \times (I_2 \otimes \hat{Q}_2) \times p \]
\[ = \begin{bmatrix} \frac{3}{10} a + \frac{2}{10} c \\ \frac{7}{10} a - \frac{7}{10} c \end{bmatrix}, \]
\[ H \preceq p^2 = \hat{Q}_1 \times (I_2 \otimes \hat{Q}_2) \times p^2 \]
\[ = HR_4 \preceq p, \quad \forall p \in \Upsilon_4. \]

The relationships between (8), (9), (10), (14) and (16) can be described as follows.

Fig. 4. Relationships between different stochastic models

VII. CONCLUSION

As one of the fundamental problems, modeling is an essential procedure before controlling a given LDS. This paper provided a unified framework for modeling LDSs with deterministic and stochastic dynamics. We first reviewed the modeling of deterministic LDS. Then non-equivalence between subsystem and global system for stochastic LDS is considered, and non-equivalence between subsystem and global system is proposed. And the reasons for the non-equivalence was provided. Finally, consistency condition was presented for independent model and conditional independent model.

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