THE GRADIENT RECOVERY FOR FINITE VOLUME ELEMENT METHOD ON QUADRILATERAL MESHES

YINGWEI SONG AND TIE ZHANG

ABSTRACT. We consider the finite volume element method for elliptic problems using isoparametric bilinear elements on quadrilateral meshes. A gradient recovery method is presented by using the patch interpolation technique. Based on some superclose estimates, we prove that the recovered gradient $\hat{\nabla} u_h$ possesses the superconvergence: $\| \nabla u - \hat{\nabla} u_h \| = O(h^2)\|u\|_3$. Finally, some numerical examples are provided to illustrate our theoretical analysis.

1. Introduction

The derivative (gradient) recovery techniques are postprocess techniques that reconstruct the derivative from the discrete solution to achieve better derivative approximation, for example, to obtain the superconvergent result. In the 1960s, the derivative recovery technique had been used to compute the derivative (stress) via the $C^0$-finite elements [7]. In particular, the simple averaging or weighted averaging methods were employed by engineers for linear finite elements [16]. Subsequently, new derivative recovery techniques have been developed, for example, the $L_2$-projection post-processing technique [6, 14], the well known Zienkiewicz-Zhu’s patch recovery technique (SPR) [22], the interpolation postprocess technique [10], the polynomial preserving recovery technique (PPR) [13] and the derivative patch interpolation recovery technique [19], and so on. However, all these recovery techniques were presented for the finite element methods on triangle or rectangular meshes.

Finite volume element (FVE) method is a discrete technique for solving partial differential equations. In general, it represents the conservation of an interest quantity, such as mass, momentum, or energy in fluid mechanics, so that it can be expected to simulate corresponding physical phenomena more...
effectively. Readers are referred to the monograph [8] for general presentation of the FVE method and to [1, 2, 3, 4, 9, 11, 12, 15, 17, 18, 20] and the references therein for details.

Compared with FVE methods on triangle and rectangular meshes, less works can be found for FVE methods on quadrilateral meshes. We know that finite element methods on quadrilateral meshes usually have better accuracy than that on triangle meshes, and quadrilateral meshes are more flexible than rectangular meshes in handling complicated domain geometries. So quadrilateral meshes are also used frequently in practical applications. For isoparametric bilinear FVE method solving elliptic problems on quadrilateral meshes, Schmidt, Li et al. [9, 15] first give the optimal $H^1$-error estimate; Then, Lv and Li [11] further obtain the optimal $L_2$-error estimate if the quadrilateral mesh is $h^2$-uniform; Recently, Lv and Li [12] also derive a superconvergence result in the average gradient form.

In this paper, we study the isoparametric bilinear FVE method to solve the following elliptic problem on quadrilateral meshes,

\[
\begin{aligned}
&-\text{div}(A\nabla u) + cu = f, \quad \text{in } \Omega, \\
u = 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1.1)

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain with boundary $\partial \Omega$, coefficient matrix $A = (a_{ij})_{2 \times 2}$. Our main goal is to present a gradient recovery method for the FVE solution $u_h$ on quadrilateral meshes by using the patch interpolation technique [19]. This recovery method can provide a better approximation to the gradient of the exact solution. In fact, based on some superclose estimates, we prove that the recovered gradient $Q(\nabla u_h)$ possesses the superconvergence:

\[
\|\nabla u - Q(\nabla u_h)\| = O(h^2)\|u\|_3.
\]

(1.2)

This paper is organized as follows. In Section 2, we introduce the FVE scheme and some related results. Section 3 is devoted to deriving some superclose estimates for the interpolation function. In Section 4, we present the gradient recovery method and give its superconvergence analysis. Finally, in Section 5, numerical experiments are provided to illustrate our theoretical analysis.

We shall use the standard notation for the Sobolev space $W^{m,p}(D)$ equipped with the norm $\|\cdot\|_{m,p,D}$ and the semi-norm $|\cdot|_{m,p,D}$. In order to simplify the notations, we set $W^{m,2}(D) = H^m(D)$, $\|\cdot\|_{m,2,D} = \|\cdot\|_{m,D}$, and when $D = \Omega$ we skip the index $D$. Furthermore, notations $(\cdot, \cdot)$ and $\|\cdot\|$ denote the inner product and norm in space $L_2(\Omega)$, respectively. We use letter $C$ to represent a generic positive constant, independent of the mesh size $h$. 

2. Finite volume element method on quadrilateral meshes

2.1. Partition and isoparametric bilinear transformation

Let \( T_h = \bigcup \{ K \} \) be a convex quadrilateral mesh partition of domain \( \Omega \) so that \( \Omega = \bigcup_{K \in T_h} \{ K \} \), where \( h = \max h_K, h_K \) is the diameter of element \( K \). We assume that partition \( T_h \) is regular, that is, all the inner angles of any element in \( T_h \) are uniformly bounded away from 0 and \( \pi \), and there exists a positive constant \( \gamma > 0 \) such that

\[
(2.1) \quad h_K / \rho_K \leq \gamma, \forall K \in T_h,
\]

where \( \rho_K \) denotes the diameter of the biggest ball included in \( K \).

The following strong regular and \( h^2 \)-uniform mesh conditions will be used in our analysis.

**Definition 2.1.** A regular quadrilateral partition \( T_h \) is called strongly regular if for any element \( K = \square P_1 P_2 P_3 P_4 \) in \( T_h \), it holds (see Fig. 1)

\[
(2.2) \quad |\overrightarrow{P_1 P_2} + \overrightarrow{P_3 P_4}| \leq Ch^2_K.
\]

Furthermore, a strongly regular quadrilateral partition \( T_h \) is called \( h^2 \)-uniform, if for any two adjacent elements \( K \) and \( K' \), it holds (see Fig. 1)

\[
(2.3) \quad |\overrightarrow{P_1 P_2} + \overrightarrow{P_1 P_6}| \leq Ch^2_{K'}.
\]

**Figure 1.** Two adjacent elements \( K \) and \( K' \).

Let \( l = |P_1 P_4| \) be the length of the common edge of \( K \) and \( K' \) (see Fig. 1). Since

\[
h_K = \frac{h_K \rho_K}{\rho_K} \frac{l}{h_K}, h_K' \leq \gamma h_K',
\]

then \( h_K \) in (2.3) may be replaced by \( h_K' \).

It is well known that a FVE method usually concerns two mesh partitions: a basic partition \( T_h \) and its dual partition \( T_h^* \). We here form the dual partition \( T_h^* \) in the following way. For each element \( K \in T_h \), we connect the center of \( K \) to the midpoints of its edges by straight lines. Then, for each nodal point
P in $T_h$, there exists a polygonal region $K^*_P = \Box G_1G_2G_3G_4$ surrounding $P$, $K^*_P$ is called the dual element at point $P$, and $T^*_h$ is the union of all such dual elements, see Fig. 2.

![Figure 2. The dual element $K^*_P$ surrounding point $P$.](image)

Let $\hat{K} = [0,1] \times [0,1]$ be the reference element in the $\bar{x} = (\xi, \eta)$ plane. Then, for each convex quadrilateral $K = \Box P_1P_2P_3P_4$, there exists an invertible isoparametric bilinear mapping $F_k : (\xi, \eta) \in \hat{K} \rightarrow x = (x, y) \in K$ such that (see Fig. 3)

$$x = P_1 + (P_2 - P_1)\xi + (P_4 - P_1)\eta + (P_1 - P_2 + P_3 - P_4)\xi\eta.$$  

![Figure 3. The isoparametric bilinear transformation.](image)

Let $P_i = (x_i, y_i)$. Transformation (2.4) also can be expressed as follows.

$$x = x_1 + a_1\xi + a_2\eta + a_3\xi\eta,$$

$$y = y_1 + b_1\xi + b_2\eta + b_3\xi\eta,$$

where

$$a_1 = x_2 - x_1, \quad a_2 = x_4 - x_1, \quad a_3 = x_1 - x_2 + x_3 - x_4,$$

$$b_1 = y_2 - y_1, \quad b_2 = y_4 - y_1, \quad b_3 = y_1 - y_2 + y_3 - y_4.$$
Denote the Jacobi matrix of mapping $F_K$ by $J_K$ and the determinant of $J_K$ by $\det J_K$, then we have

$$J_K = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} a_1 + a_3 \eta & a_2 + a_3 \xi \\ b_1 + b_3 \eta & b_2 + b_3 \xi \end{pmatrix}.$$ 

Furthermore, by the differentiation law of inverse function we have

$$(2.7) \quad \det J_K^{-1} = \frac{1}{J_K} \begin{pmatrix} b_2 + b_3 \xi & -(a_2 + a_3) \xi \\ -(b_1 + b_3 \eta) & a_1 + a_3 \eta \end{pmatrix}.$$ 

Under mesh conditions (2.2) and (2.3), by straightforward computation, one can derive the following results, see [9, 11].

**Lemma 2.1.** Assume that partition $T_h$ is strongly regular, then

$$(2.8) \quad |J_K|_{0,\infty} = O(h_K^2), \quad |J_K|_{0,\infty} = O(h_K), \quad |J_K^{-1}|_{0,\infty} = O(h_K^{-1}),$$

$$(2.9) \quad |J_K^{-1}|_{m,\infty} \leq C h_K^{-1}, \quad m = 1, 2, 3,$$

$$(2.10) \quad |J_K^{-1}(\xi, \eta) - J_K^{-1}(\xi', \eta')|_{0,\infty} \leq C, \quad \forall (\xi, \eta), (\xi', \eta') \in \hat{K}.$$ 

Furthermore, if partition $T_h$ is $h^2$-uniform, then for any two adjacent elements $K$ and $K'$,

$$(2.11) \quad |J_K^{-1}(\xi, \eta) - J_{K'}^{-1}(\xi', \eta')|_{0,\infty} \leq C, \quad \forall (\xi, \eta), (\xi', \eta') \in \hat{K}.$$ 

Let function $\hat{u}(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta)) = u \circ F_K(\xi, \eta)$, where $u \circ F_K$ denotes the compound function of $u(x, y)$ and the mapping $F_K(\xi, \eta)$. From Lemma 2.1 and noting that

$$\hat{\nabla} \hat{u} = J_K^T \nabla u, \quad \nabla u = J_K^{-T} \hat{\nabla} \hat{u},$$

we can derive the following estimates

$$(2.12) \quad |\hat{u}|_{m,p,K} \leq C h_K^{m-\frac{2}{p}} \|u\|_{m,p,K}, \quad m = 0, 1, 2, 3, \quad 1 \leq p \leq \infty,$$

$$(2.13) \quad |u|_{m,p,K} \leq C h_K^{-m+\frac{2}{p}} \|\hat{u}\|_{m,p,K}, \quad m = 0, 1, 2, 3, \quad 1 \leq p \leq \infty.$$ 

### 2.2. Finite volume element scheme

Consider problem (1.1). As usual, we assume that there exist positive constants $C_1$ and $C_2$ such that

$$(2.14) \quad C_1 z^T A(z, y) z \leq C_2 z^T A(z, y) z, \quad \forall z \in \mathbb{R}^2, \quad (x, y) \in \Omega,$$

We further assume that $A \in [W^{1,\infty}(\Omega)]^{2 \times 2}$, $c \in L_\infty(\Omega)$ and $c \geq 0$.

Associated with partition $T_h$ and $T_h^*$, we introduce the trial function space $U_h$ and test function space $V_h$, respectively,

$$U_h = \{ u_h \in C^0(\hat{\Omega}) : u_h|_K = P_K \circ F_K^{-1}, \quad P_K \in Q_{11}(\hat{K}), \forall K \in T_h, \quad u_h|_{\partial \Omega} = 0 \},$$

$$V_h = \{ v_h \in L_2(\Omega) : v_h|_{K^*_P} = \text{constant}, \forall P \in N_h, \quad v_h|_{K^*_P} = 0, \forall P \in \partial \Omega \},$$

where $Q_{11}(\hat{K})$ is the set of all bilinear polynomials on $\hat{K}$ and $N_h$ is the set of all mesh points of $T_h$. 
Let $u$ be the solution of problem (1.1). Using the Green formula, we obtain

$$\tag{2.15} - \int_{\partial K^*_h} n \cdot (A \nabla u) v ds + \int_{K^*_h} cu v = \int_{K^*_h} fv, \quad K^*_h \in T^*_h, \quad v \in V_h,$$

where $n$ is the outward unit normal vector on the boundary concerned. According to weak formula (2.15), we set the bilinear form $a_h(u, v)$ by

$$\tag{2.16} a_h(u, v) = \sum_{K^*_h \in T^*_h} \left( - \int_{\partial K^*_h} n \cdot (A \nabla u) v ds + \int_{K^*_h} cu v \right), \quad u \in H^2(\Omega), \quad v \in V_h,$$

and define the FVE approximation to problem (1.1): Find $u_h \in U_h$ such that

$$\tag{2.17} a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h.$$

Let $\Pi^*_h : U_h \to V_h$ be the interpolation operator defined by

$$\Pi^*_h v_h = \sum_{P \in \mathcal{N}_h} v_h(P) \chi_P, \quad \forall v_h \in U_h,$$

where $\chi_P$ is the characteristic function of the dual element $K^*_h$. Obviously, $\Pi^*_h$ is a one to one mapping from $U_h$ onto $V_h$. Then, we obtain the equivalent scheme of (2.17): Find $u_h \in U_h$ such that

$$\tag{2.18} a_h(u_h, \Pi^*_h v_h) = (f, \Pi^*_h v_h), \quad \forall v_h \in U_h,$$

which is the FVE scheme actually used in our argument. From (2.15) and (2.18), we can derive the error equation

$$\tag{2.19} a_h(u - u_h, \Pi^*_h v_h) = 0, \quad \forall v_h \in U_h.$$

Let $\Pi_h u \in U_h$ be the usual isoparametric bilinear interpolation of continuous function $u$. In our analysis, the following approximation property and trace inequality will be used frequently, see [5]. For $1 < p \leq \infty$, we have

$$\tag{2.20} \|u - \Pi_h u\|_{m,p,K} \leq Ch^{2-m} \|u\|_{2,p,K}, \quad m = 0, 1, 2,$$

$$\tag{2.21} \|u\|_{0,p,\partial K} \leq Ch^{\frac{1}{2}} (\|u\|_{0,p,K} + hK \|\nabla u\|_{0,p,K}), \quad u \in W^{1,p}(K).$$

Furthermore, the following two lemmas hold.

**Lemma 2.2** ([8]). Let $\hat{\Pi}_h^* \hat{v}_h = \hat{\Pi}_h^* \hat{v}_h$. Then for $v_h \in U_h$, we have

$$\tag{2.22} \int_{K} (\hat{v}_h - \hat{\Pi}_h^* \hat{v}_h) d\hat{K} = 0, \quad \int_{\hat{\partial} K} (\hat{v}_h - \hat{\Pi}_h^* \hat{v}_h) ds = 0,$$

$$\tag{2.23} \|v_h - \Pi_h^* v_h\|_{0,q,K} \leq Ch_K \|v_h\|_{1,q,K}, \quad 1 \leq q \leq \infty,$$

where $\hat{\partial} K \subset \partial \hat{K}$ be any one edge of the reference element $\hat{K}$.

**Lemma 2.3** ([9, 15]). Let partition $T_h$ be strongly regular, $u$ and $u_h$ be the solutions of problems (1.1) and (2.18), respectively. Then, we have

$$a_h(v_h, \Pi_h^* v_h) \geq C \|v_h\|_2^2, \quad \forall v_h \in U_h,$$

$$\|u - u_h\|_1 \leq Ch \|u\|_2.$$
3. Superclose estimate for the interpolation approximation

The superclose estimate of interpolation function usually provides a useful analysis tool in the study of superconvergence of finite element method [10, 21]. In this section, we establish some superclose estimates for the finite volume element method.

Let \( w^e \) be the piecewise constant approximation of function \( w \) on \( T_h \),

\[
(3.1) \quad w^e|_K = \frac{1}{|K|} \int_K w, \quad K \in T_h; \quad \|w - w^e\|_{a,p,K} \leq Ch_K |w|_{1,p,K}, \quad 1 \leq p \leq \infty.
\]

Lemma 3.1. Let partition \( T_h \) be strongly regular and \( a^e \) be a constant vector, \( u \in W^{3,p}(\Omega) \). Then, for \( v \in U_h \), we have

\[
(3.2) \quad |(a^e \cdot \nabla(u - \Pi_h u), \Pi_h^* v - v)_K| + |(a^e \cdot \nabla(u - \Pi_h u)_y, \Pi_h^* v - v)_K| \\
\leq Ch_K^2 \|v\|_{3,p,K} \|v_h\|_{1,q,K}, \quad 2 \leq p, q \leq \infty, \quad 1/p + 1/q = 1.
\]

Proof. Let \( w = u - \Pi_h u \) and \( e_h = \Pi_h^* v - v \). Using the isoparametric bilinear transformation, we have

\[
(3.3) \quad (a^e \cdot \nabla(u - \Pi_h u)_x, \Pi_h^* v - v)_K = \int_{\mathcal{R}} a^e \cdot J^{-T}_K \nabla (\hat{w}_\xi \xi_x + \hat{w}_\eta \eta_x) e_h J_K d\mathcal{R}
\]

\[
= \int_{\mathcal{R}} a^e \cdot J^{-T}_K (\hat{\nabla} \hat{w}_\xi \xi_x + \hat{\nabla} \hat{w}_\eta \eta_x) e_h J_K \\
+ \int_{\mathcal{R}} a^e \cdot J^{-T}_K (\hat{\nabla} \hat{w}_\xi \xi_x + \hat{\nabla} \hat{w}_\eta \eta_x) e_h J_K = E_1 + E_2.
\]

From (2.7) we obtain

\[
(3.4) \quad E_1 = \int_{\mathcal{R}} a^e \cdot J^{-T}_K (\hat{\nabla} \hat{w}_\xi (b_2 + b_3 \xi) - \hat{\nabla} \hat{w}_\eta (b_1 + b_3 \eta)) e_h \\
= \int_{\mathcal{R}} a^e \cdot (J^{-T}_K - J^{-T}_K (0, 0)) (\hat{\nabla} \hat{w}_\xi (b_2 + b_3 \xi) - \hat{\nabla} \hat{w}_\eta (b_1 + b_3 \eta)) e_h \\
+ \int_{\mathcal{R}} a^e \cdot J^{-T}_K (0, 0) (\hat{\nabla} \hat{w}_\xi b_2 - \hat{\nabla} \hat{w}_\eta b_1) e_h \\
+ \int_{\mathcal{R}} a^e \cdot J^{-T}_K (0, 0) (\hat{\nabla} \hat{w}_\xi b_3 \xi - \hat{\nabla} \hat{w}_\eta b_3 \eta) e_h = F_1 + F_2 + F_3.
\]

It follows from Lemma 2.1 and condition (2.2) that

\[
|J^{-T}_K|_\infty \leq Ch_K^{-1}, \quad |J^{-T}_K - J^{-T}_K (0, 0)|_\infty \leq C, \quad b_1 = b_2 = O(h_K), \quad b_3 = O(h_K^2),
\]

therefore, we have from (2.12), (2.20) and (2.23) that

\[
F_1 + F_3 \leq Ch_K |\hat{w}|_{2,p,\mathcal{R}} \|e_h\|_{0,q,\mathcal{R}} \leq Ch_K h_K^{2 - \frac{2}{p}} \|w\|_{2,p,K} h_K^{-\frac{2}{p}} \|e_h\|_{0,q,K} \leq Ch_K^2 \|w\|_{2,p,K} \|v\|_{1,q,K}.
\]
For $F_2$, noting that $\hat{\nabla}((\hat{\Pi}_h \hat{u})_\xi)$ and $\hat{\nabla}((\hat{\Pi}_h \hat{u})_\eta)$ are constants, then from (2.22), (3.1) and (2.12), we have

$$F_2 = \int_K \alpha^e \cdot J_K^{-T}(0,0)(\tilde{\nabla} \hat{u}_\xi b_2 - \tilde{\nabla} \hat{u}_\eta b_1) \hat{e}_h$$

$$= \int_K \alpha^e \cdot J_K^{-T}(0,0)(\tilde{\nabla} \hat{u}_\xi b_2 - (\tilde{\nabla} \hat{u}_\xi)^T b_2 - \tilde{\nabla} \hat{u}_\eta b_1 + (\tilde{\nabla} \hat{u}_\eta)^T b_1) \hat{e}_h$$

$$\leq C |J_K^{-T}|_{\infty} (|b_1| + |b_2|) |\hat{\alpha}|_{3,p,R} \|\hat{e}_h\|_{0,q,R}$$

$$\leq Ch_K^{-\frac{\xi}{q}} \|u\|_{3,p,K} h_K^{-\frac{\xi}{2}} \|e_h\|_{0,q,K} \leq Ch_K^2 \|u\|_{3,p,K} \|v\|_{1,q,K}.$$ 

Combining estimates $F_1 \sim F_3$, we obtain from (3.4) that

$$E_1 \leq Ch_K^2 \|u\|_{3,p,K} \|v\|_{1,q,K}.$$ 

Next, we estimate $E_2$. Since $|J_K^{-T}| = O(h_K^{-1})$, $|J_K| = O(h_K^2)$ and (see (2.9))

$$|\tilde{\nabla} \hat{\xi}_x| \leq |\xi_{xx} x_x + \xi_{xy} y_x| + |\xi_{xx} x_\eta + \xi_{xy} y_\eta| \leq Ch_K (|\xi_{xx}| + |\xi_{xy}|) \leq C,$$

then, we have from (2.12), (2.20) and (2.23) that

$$E_2 = \int_K \alpha^e \cdot J_K^{-T}(\tilde{\nabla} \hat{\xi}_x + \tilde{\nabla} \hat{\eta}_x) \hat{e}_h J_K \leq Ch_K |\tilde{\nabla}|_{1,p,K} \|\hat{e}_h\|_{0,q,K}$$

$$\leq Ch_K h_K^{-\frac{\xi}{q}} \|u - \Pi_h u\|_{3,p,K} h_K^{-\frac{\xi}{2}} \|e_h\|_{0,q,K} \leq Ch_K^2 \|u\|_{2,p,K} \|v\|_{1,q,K}.$$ 

The proof is completed by substituting estimates $E_1$ and $E_2$ into (3.4). $\square$

Let $\mathcal{E}_h^0$ be the union of all interior edges of elements in $T_h$.

**Lemma 3.2.** Let partition $T_h$ be $h^2$-uniform and matrix $A_M|_{\tau}$ be constant on each $\tau \in \mathcal{E}_h^0$, $u \in W^{1,p}(\Omega)$. Then, we have for $2 \leq p, q \leq \infty$, $1/p + 1/q = 1$,

$$\left| \sum_{K \in T_h} \int_{\partial K} n \cdot A_M \nabla (u - \Pi_h u)(\Pi_h^* v - v) ds \right| \leq Ch^2 \|u\|_{3,p} \|v\|_{1,q}, \quad v \in U_h.$$

**Proof.** Let $w = u - \Pi_h u$ and $e_h = \Pi_h^* v - v$. We need to estimate

$$\sum_{K \in T_h} \int_{\partial K} n \cdot A_M \nabla w e_h ds$$

$$= \sum_{K \in T_h} \sum_{\tau \in \partial K \setminus \partial \Omega} \int_{\tau} n \cdot A_M \nabla w e_h ds = \sum_{K \in T_h} \sum_{\tau \in \partial K \setminus \partial \Omega} F(\tau)$$

$$= \sum_{\tau \in \mathcal{E}_h^0} (F(\tau \cap \partial K) + F(\tau \cap \partial K')),$$

where $K$ and $K'$ be two adjacent elements with the common edge $\tau$.

Let quadrilateral element $K = \square P_1 P_2 P_3 P_4$, $\tau \in \partial K$ be an edge of $K$, for example, $\tau = P_1 P_4$, see Fig. 1. On edge $\tau (\hat{\tau} = \{ \xi = 0, 0 \leq \eta \leq 1 \})$, we have from (2.5)-(2.6) that

$$ds = \sqrt{(dx)^2 + (dy)^2} |_{\xi = 0} = \sqrt{x^2 + y^2} d\eta |_{\xi = 0} = |P_1 P_2| d\eta.$$
Therefore, we can write

\begin{equation}
F(\tau) = \int_{\tau} n \cdot A_M \nabla w e_h \, ds = \int_0^1 \left| P_1 P_4 \right| \hat{n} \cdot A_M \mathcal{J}_K^{-T}(\hat{\tau}) \hat{\nabla} \hat{w} \, \hat{e}_h \, d\eta.
\end{equation}

Let \( K' = \square P_b P_1 P_4 P_b \) be an adjacent element of \( K \) with the common edge \( \tau = \partial K \cap \partial K' = P_1 P_4 \) and \( \tau' = \tau \cap \partial K' \) (see Fig. 1). Since the outward normal vector \( n'|_{\tau'} = -n|_{\tau} \), then we have

\begin{align*}
F(\tau \cap \partial K) + F(\tau \cap \partial K') &= \int_0^1 \left| P_1 P_4 \right| \hat{n} \cdot A_M \mathcal{J}_K^{-T}(\hat{\tau}) \hat{\nabla} \hat{w}(\hat{\tau}) \hat{e}_h \, d\eta \\
&\quad - \int_0^1 \left| P_1 P_4 \right| \hat{n} \cdot A_M \mathcal{J}_{K'}^{-T}(\hat{\tau}') \hat{\nabla} \hat{w}(\hat{\tau}') \hat{e}_h \, d\eta.
\end{align*}

Set \( a_h = \left| P_1 P_4 \right| \hat{n} \cdot A_M = O(h_K) \). Noting that \( e_h \) is continuous across edge \( \tau = \tau' \) (excepting at the midpoint of \( \tau \)), we obtain

\begin{equation}
F(\tau \cap \partial K) + F(\tau \cap \partial K') = \int_0^1 \left| a_h \right| (\mathcal{J}_K^{-T}(\hat{\tau}) - \mathcal{J}_K^{-1}(0,0)) \hat{\nabla} \hat{w}(\hat{\tau}) \hat{e}_h \, d\eta \\
+ \int_0^1 \left| a_h \right| \mathcal{J}_K^{-T}(0,0)(\hat{\nabla} \hat{w}(\hat{\tau}' - \hat{\nabla} \hat{w}(\hat{\tau}')) \hat{e}_h \, d\eta \\
+ \int_0^1 \left| a_h \right| (\mathcal{J}_{K'}^{-T}(0,0) - \mathcal{J}_{K'}^{-T}(\hat{\tau}' - \hat{\nabla} \hat{w}(\hat{\tau}')) \hat{e}_h \, d\eta \\
= S_1 + S_2 + S_3.
\end{equation}

Using Lemma 2.1, trace inequality (2.21) and the finite element inverse inequality, we obtain

\begin{align*}
S_1 + S_3 &\leq Ch_K \left( ||\hat{\nabla} \hat{w}||_{0,p,K} + ||\hat{\nabla} \hat{w}||_{1,p,K} \right) ||\hat{e}_h||_{0,q,K} \\
&\quad + Ch_{K'} \left( ||\hat{\nabla} \hat{w}||_{0,p,K'} + ||\hat{\nabla} \hat{w}||_{1,p,K'} \right) ||\hat{e}_h||_{0,q,K'} \\
&\leq Ch_K \left( h_K^{1-\frac{2}{p}} ||w||_{1,p,K} + h_K^{2-\frac{2}{p}} ||w||_{2,p,K} h_K^{-\frac{2}{p}} ||e_h||_{0,q,K} \right) \\
&\quad + Ch_{K'} \left( h_{K'}^{1-\frac{2}{p}} ||w||_{1,p,K'} + h_{K'}^{2-\frac{2}{p}} ||w||_{2,p,K'} h_{K'}^{-\frac{2}{p}} ||e_h||_{0,q,K'} \right) \\
&\leq Ch^2 ||u||_{2,p,K \cup K'} ||v||_{1,q,K \cup K'}.
\end{align*}

Next, we estimate \( S_2 \). Let \( \Pi_K |_K = \Pi_K, \Pi_{K'} |_{K'} = \Pi_{K'} \). Since \( \nabla u \) is continuous across edge \( \tau = \tau' \), we have

\begin{equation}
\hat{\nabla} \hat{w}(\hat{\tau}) - \hat{\nabla} \hat{w}(\hat{\tau}') = \hat{\nabla} \hat{\Pi}_{K'} \hat{u}(\hat{\tau}) - \hat{\nabla} \hat{\Pi}_{K'} \hat{u}(\hat{\tau}').
\end{equation}

Noting that the bilinear interpolation \( \hat{\Pi}_{K'} \hat{u} \) can be written as

\( \hat{\Pi}_{K'} \hat{u} = u_1 + (u_2 - u_1) \xi + (u_4 - u_1) \eta + (u_1 - u_2 + u_3 - u_4) \xi \eta \), \( (\xi, \eta) \in \hat{K} \),
where \( u_i = u(P_i) \), and

\[
(u_1 - u_2 + u_3 - u_4) = - \int_0^1 \hat{u}_\xi(\xi,0)d\xi + \int_0^1 \hat{u}_\xi(\xi,1)d\xi
\]

\[
= \int_0^1 \int_0^1 \hat{u}_{\xi\eta}d\xi d\eta = \frac{1}{|K|} \int_K \hat{u}_{\xi\eta}d\hat{K} = (\hat{u}_{\xi\eta})^{c}(\hat{K}),
\]

then, on edge \( \hat{\tau} = \hat{\tau}' = \{\xi = 0, 0 \leq \eta \leq 1\} \), we have

\[
\hat{\nabla}\hat{\Pi}_K \hat{u}(\hat{\tau}) - \hat{\nabla}\hat{\Pi}_K \hat{u}(\hat{\tau}')
\]

\[
= (u_2 - u_1 + (\hat{u}_{\xi\eta})^{c}(\hat{K})\eta, u_4 - u_3)^T - (u_1 - u_4 + (\hat{u}_{\xi\eta})^{c}(\hat{K}')\eta, u_3 - u_2)^T
\]

\[
= (u_2 - 2u_1 + u_4, u_4 - u_1 - u_3 + u_2)^T + ((\hat{u}_{\xi\eta})^{c}(\hat{K}) - (\hat{u}_{\xi\eta})^{c}(\hat{K}'))\eta \varepsilon,
\]

where vector \( \varepsilon = (1, 0)^T \). Now, from (3.8) and (2.22) we obtain

\[
S_2 = \int_0^1 a_h \cdot \mathcal{J}^T_K(0,0)((\hat{u}_{\xi\eta})^{c}(\hat{K}) - (\hat{u}_{\xi\eta})^{c}(\hat{K}'))\eta \varepsilon d\eta
\]

\[
= \int_0^1 a_h \cdot \mathcal{J}^T_K(0,0)((\hat{u}_{\xi\eta})^{c}(\hat{K}) - \hat{u}_{\xi\eta} + \hat{u}_{\xi\eta} - (\hat{u}_{\xi\eta})^{c}(\hat{K}'))\eta \varepsilon d\eta.
\]

It follows from Lemma 2.1, trace inequality (2.21) and the finite element inverse inequality that

\[
S_2 \leq C(||\hat{u}_{\xi\eta} - (\hat{u}_{\xi\eta})^{c}||_{0,p,K} + ||\hat{u}_{\xi\eta}|_{1,p,K}|) ||\hat{\varepsilon}_h||_{0,q,K} + C(||\hat{u}_{\xi\eta} - (\hat{u}_{\xi\eta})^{c}||_{0,p,K} + ||\hat{u}_{\xi\eta}|_{1,p,K}|) ||\hat{\varepsilon}_h||_{0,q,K'}
\]

\[
\leq C||\hat{u}||_{3,p,K} ||\hat{\varepsilon}_h||_{0,q,K} + C||\hat{u}||_{3,p,K} ||\hat{\varepsilon}_h||_{0,q,K'}
\]

\[
\leq Ch^{\frac{3}{2}} ||u||_{3,p,K} h^{-\frac{3}{2}} ||\varepsilon_h||_{0,q,K} + Ch^{\frac{3}{2}} ||u||_{3,p,K} h^{-\frac{3}{2}} ||\varepsilon_h||_{0,q,K'}
\]

\[
\leq Ch^2 ||u||_{3,p,K \cup K'} ||v||_{1,q,K \cup K'}.\]

Substituting estimates \( S_1 \sim S_3 \) into (3.7), it yields

\[
(3.9) F(\tau \cap \partial K) + F(\tau \cap \partial K') \leq Ch^2 ||u||_{3,p,K \cup K'} ||v||_{1,q,K \cup K'}.
\]

The proof is completed by combining (3.9) with (3.5). \( \square \)

We known that the conventional bilinear form of finite element method for problem (1.1) reads as

\[
(3.10) a(u, v) = \int_\Omega A\nabla u \cdot \nabla v + cuv.
\]

Under strongly regular mesh condition, the following interpolation weak estimate has been established [21] for \( 2 \leq p \leq \infty, 1/p + 1/q = 1 \),

\[
(3.11) |a(u - \Pi_h u, v)| \leq Ch^2 ||u||_{3,p,K} ||v||_{1,q,K}, \forall v \in U_h.
\]

Below we will prove that estimate (3.11) also holds for the bilinear form of the FVE method \( a_h(u, \Pi_h^* v) \). In order to use the known result (3.11) in our
argument, we need to give the difference between bilinear forms $a(u,v)$ and $a_h(u,\Pi_h^*v)$.

Let $U_h + H^2(\Omega) = \{ w : w = u_h + u, u_h \in U_h, u \in H^2(\Omega) \}$ be the algebraic sum space.

**Lemma 3.3.** It holds for $w \in U_h + H^2(\Omega)$ and $v \in U_h$ that

$$ a_h(w, \Pi_h^*v) - a(w, v) = \sum_{K \in T_h} \int_{\partial K} n \cdot A\nabla w (\Pi_h^*v - v) ds $$

$$ + \sum_{K \in T_h} ( - \text{div}(A\nabla w) + cw, \Pi_h^*v - v)_K. $$

**Proof.** Using the integration by parts, we obtain

$$ \int_K A\nabla w \cdot \nabla v = - \int_K \text{div}(A\nabla w)v + \int_{\partial K} n \cdot (A\nabla w) ds, $$

and (see Fig. 2)

$$ \sum_{K \in T_h} \int_K \text{div}(A\nabla w)\Pi_h^*v = \sum_{K \in T_h} \sum_{K' \in T_h} \int_{K' \cap K} \text{div}(A\nabla w)\Pi_h^*v $$

$$ = \sum_{K \in T_h} \int_{\partial K} n \cdot (A\nabla w)\Pi_h^*v ds + \sum_{K' \in T_h} \int_{\partial K'} n \cdot (A\nabla w)\Pi_h^*v ds. $$

Then, from the definitions of $a(w,v)$ and $a_h(w,\Pi_h^*v)$ (see (2.16) and (3.10)), the desired estimate is derived.

**Theorem 3.1.** Let partition $T_h$ be $h^2$-uniform, $u \in W^{3,p}(\Omega)$. Then, we have for $2 \leq p, q \leq \infty$, $1/p + 1/q = 1$ that

$$ |a_h(u - \Pi_h u, \Pi_h^*v)| \leq Ch^2 \|u\|_{3,p} \|v\|_{1,q}, \forall v \in U_h. $$

**Proof.** Denote by $A_M$ the value of matrix $A$ at the centroid of edge $\tau \subset \partial K$.

Then, it follows from Lemma 3.3 that

$$ a_h(u - \Pi_h u, \Pi_h^*v) - a(u - \Pi_h u, v) $$

$$ = \sum_{K \in T_h} \int_{\partial K} n \cdot (A - A_M)\nabla (u - \Pi_h u)(\Pi_h^*v - v) ds $$

$$ + \sum_{K \in T_h} \int_{\partial K} n \cdot (A_M\nabla (u - \Pi_h u))(\Pi_h^*v - v) ds $$

$$ + \sum_{K \in T_h} ( - \text{div}(A\nabla (u - \Pi_h u)), \Pi_h^*v - v)_K $$

$$ + \sum_{K \in T_h} (c(u - \Pi_h u), \Pi_h^*v - v)_K $$

$$ = R_1 + R_2 + R_3 + R_4. $$
Using trace inequality (2.21) and the approximation properties, we have

\[ R_1 \leq C \sum_{K \in T_h} h_K |A|_{1,\infty} \| \nabla (u - \Pi_h u) \|_{0,\partial K} v - \Pi_h^* v \|_{0,\partial K} \]

\[ \leq Ch^2 \| u \|_{2,p} \| v \|_{1,q}, \]

\[ R_4 \leq C \sum_{K \in T_h} \| u - \Pi_h u \|_{1,p,K} v - \Pi_h^* v \|_{0,q,K} \leq Ch^2 \| u \|_{2,p} \| v \|_{1,q}. \]

Next, it follows from Lemma 3.2 that

\[ R_2 \leq Ch^2 \| u \|_{3,p} \| v \|_{1,q}. \]

Now, we need to estimate \( R_3 \). Set \( A = (a_1, a_2) \). Since

\[
\text{div}(A \nabla w) = (\text{div} a_1, \text{div} a_2) \cdot \nabla w + a_1 \cdot \nabla w_x + a_2 \cdot \nabla w_y,
\]

then we have

\[ R_3 = \sum_{K \in T_h} \left( \langle (-\text{div} a_1, \text{div} a_2) \cdot \nabla (u - \Pi_h u), \Pi_h^* v - v \rangle_K \right) \]

\[ - \sum_{K \in T_h} \langle (a_1 - a_1^c) \cdot \nabla (u - \Pi_h u)_x + (a_2 - a_2^c) \cdot \nabla (u - \Pi_h u)_y, \Pi_h^* v - v \rangle_K \]

\[ - \sum_{K \in T_h} \langle a_1^c \cdot \nabla (u - \Pi_h u)_x + a_2^c \cdot \nabla (u - \Pi_h u)_y, \Pi_h^* v - v \rangle_K. \]

Using Lemma 3.1 and the approximation properties, it yields

\[ R_3 \leq Ch^2 \| u \|_{3,p} \| v \|_{1,q}. \]

Substituting estimates \( R_1 \sim R_4 \) into (3.14), the proof is completed. \( \square \)

From Theorem 3.1, we immediately obtain the following superclose result.

\[ (3.16) \quad \| \Pi_h u - u_h \|_1 \leq Ch^2 \| u \|_3. \]

In fact, from Lemma 2.3, error equation (2.19) and the interpolation weak estimate (3.13), we obtain

\[
C \| u_h - \Pi_h u \|_2^2 \leq a_h (u_h - \Pi_h u, \Pi_h^* (u_h - \Pi_h u)) \]

\[ = a_h (u - \Pi_h u, \Pi_h^* (u_h - \Pi_h u)) \leq Ch^2 \| u \|_3 \| u_h - \Pi_h u \|_1. \]

This gives estimate (3.16)

Remark 3.1. We note that Ly and Li in [12] derived estimate (3.13) for \( p = q = 2 \) by a lengthy and complex argument. For this estimate, we here give an alternative and more accessible argument, in particular, our result holds true for \( 2 \leq p, q \leq \infty \). Such result will be useful in the study of \( W^{1,p} \)-superconvergence, see [10, 21].
4. Gradient recovery formula and superconvergence

In this section, we will construct the gradient recovery formula for the isoparametric bilinear element and make the superconvergence estimate for the recovered gradient.

Let \( P \) be an interior mesh point and elements \( K_1, K_2, K_3, K_4 \) take point \( P \) as a vertex. Denote by \( S_p = \bigcup_{i=1}^{4} K_i \) the patch recovery domain at point \( P \), see Fig. 4. Let \( F_{K_i}: \hat{K}_i \to K_i \) be the isoparametric bilinear mapping and \( \hat{S}_p = \bigcup_{i=1}^{4} \hat{K}_i \) be the reference patch recovery domain, see Fig. 4.

![Figure 4. The patch recovery domain \( S_p \) around point \( P \).](image)

We first give the gradient recovery formula on the reference domain \( \hat{S}_p \).

Let \( \hat{G}_i \) be the centroid of \( \hat{K}_i \) (\( G_i = F_{K_i}(\hat{G}_i) \) is the centroid of element \( K_i \)) and further let \( \hat{\varphi}_i \in Q_{11}(\hat{S}_p) \) be the basis function corresponding to \( \hat{G}_i \) such that \( \hat{\varphi}_i(\hat{G}_j) = \delta_{ij} \). Then, functions \( \{\hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3, \hat{\varphi}_4\} \) form the base of bilinear polynomial space \( Q_{11}(\hat{S}_p) \). Denote the piecewise smooth function space by \( W_h(\hat{S}_p) = \{\hat{w}: \hat{w}|_{\hat{K}_i} = \text{polynomial}, \hat{K}_i \in \hat{S}_p\} \).

Now, let derivative operator \( \hat{D} = \hat{D}_\xi \) or \( \hat{D}_\eta \). For \( \hat{w} \in W_h(\hat{S}_p) \), we define the derivative recovery operator \( \hat{Q}: \hat{D}\hat{w} \to \hat{Q}(\hat{D}\hat{w}) \in Q_{11}(\hat{S}_p) \) such that

\[
\hat{Q}(\hat{D}\hat{w}) = \sum_{i=1}^{4} \hat{D}\hat{w}(\hat{G}_i)\hat{\varphi}_i(\xi, \eta), \ (\xi, \eta) \in \hat{S}_p, \ \hat{w} \in W_h(\hat{S}_p).
\]

(4.1)

It is easy to see that \( \hat{Q}(\hat{D}\hat{w}) \) is the bilinear interpolation of \( \hat{D}\hat{w} \) on \( \hat{S}_p \) with the interpolation nodes \( \{\hat{G}_1, \hat{G}_2, \hat{G}_3, \hat{G}_4\} \). For \( \hat{w} \in W_h(\hat{S}_p) \), function \( \hat{D}\hat{w} \) may be discontinuous on \( \hat{S}_p \), while \( \hat{Q}(\hat{D}\hat{w}) \) is a bilinear polynomial on \( \hat{S}_p \).

From (4.1), we also obtain the gradient recovery formula on \( \hat{S}_p \):

\[
\hat{Q}(\nabla\hat{w}) = (\hat{Q}(\hat{D}_\xi\hat{w}), \hat{Q}(\hat{D}_\eta\hat{w}))^T = \sum_{i=1}^{4} \nabla\hat{w}(\hat{G}_i)\hat{\varphi}_i(\xi, \eta), \ (\xi, \eta) \in \hat{S}_p.
\]

(4.2)
Lemma 4.1. Let \( \hat{G}_0 = (\xi_0, \eta_0) \) be the centroid of \( \hat{K} \). Then
\[
\nabla (\hat{u} - \hat{\Pi}_h \hat{u})(\hat{G}_0) = 0, \ \forall \hat{u} \in P_2(\hat{K}),
\]
where \( P_2(S) \) represents the set of all quadratic polynomials on set \( S \).

Proof. Let \( (\xi_0, \eta_0) = ((\xi_1 + \xi_2)/2, (\eta_1 + \eta_2)/2) \) be the centroid of element \( \hat{K} = (\xi_1, \xi_2) \times (\eta_1, \eta_2) \). Further let \( I_\xi \) and \( I_\eta \) be the linear interpolation operators with respect to variable \( \xi \) and \( \eta \), respectively, such that \( I_\xi \hat{u}(\xi, \eta) = \hat{u}(\xi, \eta) \) and \( I_\eta \hat{u}(\xi, \eta) = \hat{u}(\xi, \eta), i = 1, 2 \). Then, the bilinear interpolation operator \( \hat{\Pi}_h \) can be represented as \( \hat{\Pi}_h = I_\eta I_\xi \). It follows from the Taylor expansion that for \( \hat{u} \in P_2(\hat{K}) \),
\[
\hat{u} - I_\xi \hat{u} = \frac{1}{2} \left( \xi - \xi_1 \right) \left( \xi - \xi_2 \right) \hat{u}_{\xi\xi}, \quad \hat{u} - I_\eta \hat{u} = \frac{1}{2} \left( \eta - \eta_1 \right) \left( \eta - \eta_2 \right) \hat{u}_{\eta\eta}.
\]
Then, we have (note that \( \hat{u}_{\xi\xi} = \text{constant} \))
\[
\hat{u} - \hat{\Pi}_h \hat{u} = \hat{u} - I_\eta \hat{u} + I_\eta \left( \hat{u} - I_\xi \hat{u} \right)
= \frac{1}{2} \left( \eta - \eta_1 \right) \left( \eta - \eta_2 \right) \hat{u}_{\eta\eta} + I_\eta \left( \frac{1}{2} \left( \xi - \xi_1 \right) \left( \xi - \xi_2 \right) \hat{u}_{\xi\xi} \right)
= \frac{1}{2} \left( \eta - \eta_1 \right) \left( \eta - \eta_2 \right) \hat{u}_{\eta\eta} + \frac{1}{2} \left( \xi - \xi_1 \right) \left( \xi - \xi_2 \right) \hat{u}_{\xi\xi}.
\]
It yields
\[
(\hat{u} - \hat{\Pi}_h \hat{u})_\xi = \left( \xi - \frac{\xi_1 + \xi_2}{2} \right) \hat{u}_{\xi\xi}, \quad (\hat{u} - \hat{\Pi}_h \hat{u})_\eta = \left( \eta - \frac{\eta_1 + \eta_2}{2} \right) \hat{u}_{\eta\eta}, \ \forall \hat{u} \in P_2(\hat{K}).
\]
The proof is completed. \( \square \)

From Lemma 4.1 we see that \( G_i = F_{K_i}(\hat{G}_i) \) is the Gauss point, that is, for any \( u = \hat{u} \circ F_{K_i}, \hat{u} \in P_2(\hat{K}_i) \), it holds
\[
\nabla (u - \Pi_h u)(G_i) = J_{K_i}^{-1} \nabla (\hat{u} - \hat{\Pi}_h \hat{u})(\hat{G}_i) = 0.
\]

Lemma 4.2. The following properties hold true for operator \( \hat{Q} \)
\[
\begin{align*}
\hat{D} \hat{u} &= \hat{Q}(\hat{D} \hat{u}), \ \forall \hat{u} \in P_2(\hat{S}_P), \\
\hat{D} \hat{u} &= \hat{Q}(\hat{D} \hat{\Pi}_h \hat{u}), \ \forall \hat{u} \in P_2(\hat{S}_P).
\end{align*}
\]

Proof. First, note that \( \hat{D} \hat{u} \in Q_{11}(\hat{S}_P) \) if \( \hat{u} \in P_2(\hat{S}_P) \). Then, equality (4.5) comes from (4.1) and the uniqueness of interpolation polynomial. Moreover, it follows from (4.3) that \( \hat{D} \hat{u}(\hat{G}_i) = \hat{D} \hat{\Pi}_h \hat{u}(\hat{G}_i) \). Thus, using (4.1) we obtain \( \hat{Q}(\hat{D} \hat{u}) = \hat{Q}(\hat{D} \hat{\Pi}_h \hat{u}) \). Together with (4.5), we complete the proof. \( \square \)

Lemma 4.3. Recovery operator \( \hat{Q} \) is bounded and
\[
\| \hat{Q}(\hat{D} \hat{u}) \|_{0, \hat{S}_P} \leq \| \hat{Q} \| \| \hat{D} \hat{u} \|_{0, \hat{S}_P}, \ \forall \hat{w} \in W_h(\hat{S}_P),
\]
where \( \| \hat{Q} \| \) represents the bound of operator \( \hat{Q} \). Furthermore, we have
\[
\| \hat{D} \hat{u} - \hat{Q}(\hat{D} \hat{\Pi}_h \hat{u}) \|_{0, \hat{S}_P} \leq C |\hat{u}|_{3, \hat{S}_P}, \ \hat{u} \in H^3(\hat{S}_P).
\]
Proof. First, it is easy to see that
\[(4.9) \quad \|\hat{\varphi}_i\|_{0,\hat{K}} \leq \left(\text{meas}(\hat{K})\right)^{\frac{1}{2}} \leq 1, \quad \hat{K} \in \hat{S}_P, \ i = 1, \ldots, 4.\]
Hence, it follows from (4.1) and the inverse inequality that
\[
\|\hat{Q}(\hat{D}\hat{w})\|_{0,\hat{S}_P} = \left(\sum_{\hat{K} \in \hat{S}_P} \|\hat{Q}(\hat{D}\hat{w})\|_{0,\hat{K}}^2\right)^{\frac{1}{2}} \\
\leq \left(\sum_{\hat{K} \in \hat{S}_P} \left(\sum_{i=1}^{4} |\hat{D}\hat{w}(\hat{G}_i)| \|\hat{\varphi}_i\|_{0,\hat{K}}\right)^2\right)^{\frac{1}{2}} \\
\leq C\left(\sum_{\hat{K} \in \hat{S}_P} \|\hat{D}\hat{w}\|_{0,\hat{K}}^2\right)^{\frac{1}{2}} = C\|\hat{D}\hat{w}\|_{0,\hat{S}_P}.
\]
Estimate (4.7) is derived. Now, for given \(\hat{v} \in L_2(\hat{S}_P)\), we introduce the linear functional
\[(4.10) \quad F(\hat{u}) = (\hat{D}\hat{u} - \hat{Q}(\hat{D}\hat{P}_h\hat{u}), \hat{v})_{\hat{S}_P}, \ \hat{u} \in H^3(\hat{S}_P).\]
Using (4.7) and the boundness of interpolation operator \(\hat{P}_h\), we obtain
\[
|F(\hat{u})| \leq \|\hat{D}\hat{u} - \hat{Q}(\hat{D}\hat{P}_h\hat{u})\|_{0,\hat{S}_P} \|\hat{v}\|_{0,\hat{S}_P} \\
\leq C\|\hat{u}\|_{3,\hat{S}_P} \|\hat{v}\|_{0,\hat{S}_P} + \|\hat{Q}\| \left(\sum_{\hat{K} \in \hat{S}_P} \|\hat{D}\hat{P}_h\hat{u}\|_{0,\hat{K}}^2\right)^{\frac{1}{2}} \|\hat{v}\|_{0,\hat{S}_P} \\
\leq C\|\hat{u}\|_{3,\hat{S}_P} \|\hat{v}\|_{0,\hat{S}_P}.
\]
Hence, \(F\) is a linear bounded functional on \(H^3(\hat{S}_P)\). Moreover, it follows from Lemma 4.2 that
\[
F(\hat{u}) = 0, \ \forall \hat{u} \in P_2(\hat{S}_P).
\]
Thus, using the Bramble-Hilbert Lemma, we derive
\[
|F(\hat{u})| \leq C\|\hat{u}\|_{3,\hat{S}_P} \|\hat{v}\|_{0,\hat{S}_P}, \ \forall \hat{v} \in L_2(\hat{S}_P),
\]
Together with (4.10), it yields
\[
\|\hat{D}\hat{u} - \hat{Q}(\hat{D}\hat{P}_h\hat{u})\|_{0,\hat{S}_P} \leq C\|\hat{u}\|_{3,\hat{S}_P}.
\]
This gives estimate (4.8). \(\square\)

Now, for \(u_h \in U_h\), we define its recovery gradient on the actual patch domain \(S_P\) by the formula:
\[(4.11) \quad Q(\nabla u_h)(x,y) = J_{K_j}^{-T} \hat{Q}(\hat{\nabla} \hat{u}_h), \ (x,y) \in K_j, \ K_j \in S_P,
\]
where \(\hat{Q}(\hat{\nabla} \hat{u}_h)\) is given by (4.2). Since
\[
Q(\nabla u_h)(G_j) = J_{K_j}^{-T} \hat{Q}(\hat{\nabla} \hat{u}_h)(\hat{G}_j) = J_{K_j}^{-T} \hat{\nabla} \hat{u}_h(\hat{G}_j) = \nabla u_h(G_j),
\]
Therefore, $Q(\nabla u_h)$ is in fact an interpolation function of $\nabla u_h$ on patch domain $S_p$ with the Gauss points $\{G_1\}$ as the interpolation nodes.

**Theorem 4.1.** Let quadrilateral partition $T_h$ be $h^2$-uniform, $u$ and $u_h$ be the solutions of problems (1.1) and (2.18), respectively, $u \in H^2(\Omega)$. Then, the recovered gradient $Q(\nabla u_h)$ satisfies the local estimate

\begin{equation}
\|\nabla u - Q(\nabla u_h)\|_{0,S_p} \leq Ch^2\|u\|_{3,S_p} + C\|\nabla(\Pi_h u - u_h)\|_{0,S_p}.
\end{equation}

**Proof.** From (4.11), Lemma 2.1 and Lemma 4.3, we obtain

\[
\|\nabla u - Q(\nabla u_h)\|_{0,S_p} = \left( \sum_{j=1}^{4} \int_{K_j} |J_{K_j}^{-1}(\hat{\nabla} \hat{u}_h - \hat{Q}(\hat{\nabla} \hat{u}_h))|^2 dK_j \right)^{\frac{1}{2}} \\
\leq C\sum_{j=1}^{4} \int_{K_j} |\hat{\nabla} \hat{u} - \hat{Q}(\hat{\nabla} \hat{u}_h)|^2 dK_j \\
= C\|\hat{\nabla} \hat{u} - \hat{Q}(\hat{\nabla} \hat{u}_h)\|_{0,S_p} \\
\leq \|\hat{\nabla} \hat{u} - \hat{Q}(\hat{\nabla} \Pi_h \hat{u})\|_{0,S_p} + \|\hat{Q}(\hat{\nabla} \Pi_h \hat{u} - \hat{\nabla} \hat{u}_h)\|_{0,S_p} \\
\leq C|\hat{u}|_{3,S_p} + C\|\nabla \Pi_h \hat{u} - \hat{\nabla} \hat{u}_h\|_{0,S_p} \\
\leq Ch^2\|u\|_{3,S_p} + C\|\nabla(\Pi_h u - u_h)\|_{0,S_p}.
\]

This gives estimate (4.12). \hfill \Box

The gradient recovery formula (4.11) is local, below we expand this formula to the whole domain $\Omega$.

We will define the global recovery formula element-wise. For an interior node $P_j$, let $S_{P_j}$ be the patch recovery domain around point $P_j$ and $Q_j = Q$ the gradient recovery operator on $S_{P_j}$ defined by (4.11). Note that for a given element $K$, there are four patch domains $\{S_{P_j}\}$ covering $K$ (or less than four if $K$ is a boundary element). In order to balance the values of $Q_j(\nabla u_h)|_{K \cap S_{P_j}}$ for different $S_{P_j}$ (or $Q_j$), we define the global recovery operator $Q_\Omega$ by the average formula,

\begin{equation}
Q_\Omega(\nabla u_h)|_K = \frac{1}{N_K} \sum_{S_{P_j} \cap K \neq \emptyset} Q_j(\nabla u_h)|_{S_{P_j} \cap K}, \quad K \in T_h,
\end{equation}

where $N_K \leq 4$ is the total number of elements in set $\{S_{P_j} : S_{P_j} \cap K \neq \emptyset\}$.

**Theorem 4.2.** Under the conditions of Theorem 4.1, the following superconvergence estimate holds

\begin{equation}
\|\nabla u - Q_\Omega(\nabla u_h)\| \leq Ch^2\|u\|_{3}.
\end{equation}

**Proof.** It follows from (4.12) that

\[
\|\nabla u - Q_\Omega(\nabla u_h)\|^2 = \sum_{K \in T_h} \|\nabla u - Q_\Omega(\nabla u_h)\|^2_{0,K}
\]
\[
\leq \sum_{K \in T_h} \frac{1}{N_K} \sum_{S \cap K \neq \emptyset} \| \nabla u - Q_j(\nabla u_h) \|_{0, S_j}^2
\leq \sum_{K \in T_h} \max_{S \cap K \neq \emptyset} \| \nabla u - Q_j(\nabla u_h) \|_{0, S_j}^2
\leq \sum_{K \in T_h} \max_{S \cap K \neq \emptyset} \left( C h^4 \| u \|_{3, S_j}^2 + C \| \nabla (\Pi_h u - u_h) \|_{0, S_j}^2 \right)
\leq C \left( h^4 \| u \|_{3}^2 + \| \nabla (\Pi_h u - u_h) \|_{2}^2 \right).
\]

We complete the proof by using estimate (3.16). \(\Box\)

### 5. Numerical example

In this section, we will present some numerical results to illustrate our theoretical analysis.

Let us consider problem (1.1) with the data:

\[
A(x, y) = \begin{pmatrix}
e^{2x} + y^3 + 1 & e^{2y} \\
e^{2x} y & e^{2y} + x^3 + 1
\end{pmatrix}, \quad c(x, y) = 1 + xy.
\]

We take \(\Omega = [0, 1]^2\) and the exact solution \(u(x, y) = 2 \sin(2\pi x) \sin(3\pi y)\).

We present numerical results using sequence of meshes \(\{T_i\}\). This quadrilateral mesh sequence is generated in the following way. We first make an original quadrilateral mesh \(T_1\) with mesh size \(h = h_1\). Then we connect the midpoints of each edge of elements in \(T_i\) \((i \geq 1)\) to obtain the refined mesh \(T_{i+1}\) which has half mesh size of \(T_i\), see Fig. 5. It is easy to see that such bisection quadrilateral meshes must be \(h^2\)-uniform. Denote by \(e(T_i)\) the error on mesh \(T_i\) in the corresponding norm, then the numerical convergence rate \(r\) is computed by the formula \(r = \ln( e(T_i)/e(T_{i+1}) ) / \ln 2\). The numerical results are given in Table 5.1. We see that an \(O(h^2)\) order of convergence rate is achieved for the recovered gradient as the theoretical prediction.

![Figure 5. Left: original mesh \(T_1\); Right: refined mesh \(T_6\) obtained by bisection partition.](image)

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Table 5.1. Convergence rate and error estimator.

| mesh size | $\|\nabla u - Q_\Omega \nabla u_h\|$ | $\|\nabla u - \nabla u_h\|$ |
|-----------|--------------------------|--------------------------|
| $h_1 = 0.373$ | 0.3621 | - | 0.6424 | - |
| $h_1/2$ | 0.9166e-1 | 1.982 | 3.2276e-1 | 0.993 |
| $h_1/4$ | 0.2314e-1 | 1.986 | 1.6194e-1 | 0.995 |
| $h_1/8$ | 0.5821e-2 | 1.991 | 0.8108e-1 | 0.998 |
| $h_1/16$ | 1.4621e-3 | 1.993 | 4.0513e-2 | 1.001 |
| $h_1/32$ | 0.3670e-3 | 1.994 | 2.0257e-2 | 1.000 |

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Yingwei Song
Department of Mathematics and the State Key Laboratory of Synthetical Automation for Process Industries
Northeastern University
Shenyang 110004, P. R. China
E-mail address: songyingwei001@163.com

Tie Zhang
Department of Mathematics
Northeastern University
Shenyang 110004, P. R. China
E-mail address: ztmath@163.com