Abstract

Embedding of a Green-Schwarz superbrane into a generic curved target space in a general covariant way is considered. It is demonstrated explicitly, that the customary superbrane formulation based on finite-component spinors extends to a superspaces of restricted curving only, with the General Coordinate Transformations realized nonlinearly over its orthogonal type subgroups. Infinite-component, world, spinors and a recently constructed corresponding Dirac-like equation, enable a possibility of a manifestly covariant generic curved target space superbrane formulation.

1 Introduction

Quantum gravity, a theory based on General Relativity and Quantum Theory, is still by no means one of the most outstanding problems of the contemporary physics. There where quite a number of various attempt to tackle this problem without achieving a substantial breakthrough. Superstring theory, with its subsequent superbrane based theories, is considered to be the most promising candidate on this road. Superstring theory avoids the ultraviolet infinities that arise in attempt to quantize gravity, it unfolds a profound way to unify all fundamental interactions, and it strikes with its geometrical beauty and uniqueness.

In the conventional lagrangian formulation for superbranes, the \((p + 1)\)-dimensional curved (locally reparametrizable) brane world volume \(\mathcal{R}^{p+1}\) is
embedded in a flat $D$-dimensional Minkowski spacetime $M^{1, D-1}$ (Poincaré invariance). On the other hand, macroscopic gravity is described classically by Einstein’s theory, corresponding to a generic curved Riemannian $\mathcal{R}^4$ manifold (general covariance). Thus one is faced with an apparent difference in the manifest symmetries of these two theories. This difference is not only of a principal nature, but is crucial for numerous practical questions such as non-perturbative gravitational solutions (Schwarzshild) etc. One can certainly hope to reconstruct the full general covariance starting from the field theory of superbranes embedded in a flat space. However, difficulties encountered along this line support a more pragmatic (and in our opinion in fact the only) approach to construct an a priori fully generally-covariant target space superbrane theory. In other words it is desirable to construct, from the very beginning, a generic curved target space formulation for superbranes. Having achieved this task, one can, among other things, be in a position to study naturally the superbrane theory on manifolds with some of the dimensions compactified.

The aim of this paper is to study a generic curved target space symmetry of the superbrane action, i.e. the group theoretical constraints enforced on the action by the spinorial representations properties. As pointed out with Y. Ne’eman, by making use of the “there is no target to a target” argument, one can not embed a superstring into a generic curved target space [1]. In this paper we show explicitly, by studying the curved target space symmetries of a Green-Schwarz superbrane action, that the spinorial representations properties of the superDiff($D, R$) group determine possible target space curvings. There are two clearly distinguished cases: finite-dimensional and infinite dimensional spinorial representations of the superDiff($D, R$) group non-linearly realized over its orthogonal-type, e.g. Spin$(1, D-1)$, and $SL(D, R)$ subgroups, respectively.

## 2 Bosonic brane curved space embedding

The bosonic branes and superbranes [2] are considered in turn below in order to point out their similarities and differences as for the question of a generic curved target space embedding, as well as to fix notation and to point out transformation properties of relevant entities.

Let us start with a bosonic p-brane embedded in a flat $D$-dimensional Minkowski spacetime $M^{1, D-1}$. The Poincaré $P(1, D-1)$ group, i.e. its
homogeneous Lorentz subgroup $SO(1, D - 1)$, are the physically relevant target space symmetries, while the $(p + 1)$-dimensional brane world volume is preserved by the General Coordinate Transformation (GCT) group $SDiff(p + 1, R)$.

The flat target space $p$-brane action, that permits a straightforward transition to the supersymmetric case, is given by the following expression:

$$S = \int d^{p+1}\xi \left( -\frac{1}{2p} \sqrt{-\gamma} \gamma^{ij} \partial_i X^m \partial_j X^n \eta_{mn} + \frac{p-1}{2p} \sqrt{-\gamma} \right) + \frac{1}{(p+1)!} \epsilon_{i_1i_2\ldots i_{p+1}} \partial_{i_1} X^{m_1} \partial_{i_2} X^{m_2} \ldots \partial_{i_{p+1}} X^{m_{p+1}} + \frac{1}{(p+1)!} \epsilon_{\tilde{m}_1\tilde{m}_2\ldots \tilde{m}_{p+1}} \partial_{\tilde{i}_1} X^{\tilde{m}_1} \partial_{\tilde{i}_2} X^{\tilde{m}_2} \ldots \partial_{\tilde{i}_{p+1}} X^{\tilde{m}_{p+1}} A_{m_1 m_2 \ldots m_{p+1}}(X) \right),$$

where $i = 0, 1, \ldots, p$ labels the coordinates $\xi^i = (\tau, \sigma, \rho, \ldots)$ of the brane world volume with metric $\gamma_{ij}(\xi)$, and $\gamma = \det(\gamma_{ij})$; $m = 0, 1, \ldots, D - 1$ labels the target space coordinates $X^m(\xi^i)$ with metric $\eta_{mn}$, and $A_{m_1 m_2 \ldots m_{p+1}}$ is a $(p + 1)$-form characterizing a Wess-Zumino-like term in the action.

This action can be generalized in a straightforward manner for a generic curved target space to read in terms of the target space world variables as follows:

$$S = \int d^{p+1}\xi \left( -\frac{1}{2p} \sqrt{-\gamma} \gamma^{ij} \partial_i X^{\tilde{m}} \partial_j X^{\tilde{n}} g_{\tilde{m}\tilde{n}} + \frac{p-1}{2p} \sqrt{-\gamma} \right) + \frac{1}{(p+1)!} \epsilon_{\tilde{i}_1\tilde{i}_2\ldots \tilde{i}_{p+1}} \partial_{\tilde{i}_1} X^{\tilde{m}_1} \partial_{\tilde{i}_2} X^{\tilde{m}_2} \ldots \partial_{\tilde{i}_{p+1}} X^{\tilde{m}_{p+1}} A_{\tilde{m}_1 \tilde{m}_2 \ldots \tilde{m}_{p+1}}(X) \right),$$

where $\tilde{m} = 0, 1, \ldots, D - 1$ labels the curved target space coordinates $X^{\tilde{m}}(\xi^i)$, with riemannian metric $g_{\tilde{m}\tilde{n}}$.

The flat target space vector $X^m$ transforms w.r.t. a linearly realized $D$-dimensional vectorial representation $D_{SO(1, D - 1)}^{(v)}$ of the Lorentz group $SO(1, D - 1)$. The generic curved target space vector $X^{\tilde{m}}$ transforms w.r.t. a nonlinearly realized $D$-dimensional vectorial representation $D_{Diff(D)}^{(v)}$ of the GCT group, $Diff(D, R)$. As for the relevant physical subgroups, $X^{\tilde{m}}$ transforms w.r.t. a linearly realized $D$-dimensional vectorial representation $D_{GL(D,R)}^{(v)}$ of the maximal linear subgroup $GL(D, R)$ (i.e. $SL(D, R)$), as well as w.r.t. a linearly realized $D$-dimensional vectorial representation $D_{SO(1, D - 1)}^{(v)}$ of the Lorentz subgroup $SO(1, D - 1)$ of the $Diff(D, R)$ group. The $Diff(D, R)$ is realized nonlinearly over $GL(D, R)$, while both $Diff(D, R)$ and $GL(D, R)$ groups are realized nonlinearly over $SO(1, D - 1)$.
The off-shell tensorial structure of the target space metric $g_{\tilde{m}\tilde{n}}$ is described by a symmetric second rank irreducible representation of the $SL(D, R)$ group, while the on-shell states are characterized by the relevant little group, that is a subgroup of the $SO(1, D)$ group. The off-shell, and on-shell tensor calculus is effectively given by the $SL(D, R)$, and $SO(1, D)$ group representations, respectively, and the $Diff(D, R)$ symmetry is nonlinearly realized over its relevant subgroup in question.

It is a well known, and for this considerations an important fact, that besides the scalar representations, vector representations of the $Diff(D, R)$, $GL(D, R)$, and $SL(D, R)$ groups have the same dimensionality, $D$, as the vector representation of the $SO(1, D-1)$ group. Due to this fact, there are rectangular “$D$-bines” matrices, that connect mutually vectors of the above four groups. For instance, $X^m = e^m_n X^n$ connects mutually $Diff(D, R)$ and $SO(1, D-1)$ vectors taking into account the nonlinear realization of the $Diff(D, R)$ group over its $SO(1, D-1)$ subgroup, etc.

3 Superbrane curved space embedding

The flat target space super-$p$-brane action reads \[ S = \int d^{p+1}\xi \left( -\frac{1}{2p} \sqrt{-\gamma} \gamma^{ij} \Pi_i^m \Pi_j^n \eta_{mn} + \frac{p-1}{2p} \sqrt{-\gamma} \right) + \frac{1}{(p+1)!}\epsilon_{i_1i_2\cdots i_{p+1}} \partial_{i_1} Z^{a_1} \partial_{i_2} Z^{a_2} \cdots \partial_{i_{p+1}} Z^{a_{p+1}} B_{a_1a_2\cdots a_{p+1}} \right) \] (3)

Here, the target space is a supermanifold with superspace coordinates $Z^a(\xi^i) = (X^m(\xi^i), \Theta^\alpha(\xi^i))$, $m = 0, 1, \ldots, D-1$, $\alpha = 1, 2, \ldots, 2[D^2]$. Moreover, $\Pi_i^m = \partial_i X^m + i\bar{\Theta} \Gamma^m \partial_i \Theta$ (4)

and $\Gamma^m$ are the corresponding $D$-dimensional target space gamma-matrices. Note that $\Theta^\alpha$ transforms w.r.t. fundamental (and its contragradient) spinorial representation of the $Spin(1, D-1) \simeq SO(1, D-1)$ group, i.e. the double covering of the $SO(1, D-1)$ group.

As for the symmetry transformation properties of the action, it is essential that the second term in $\Pi_i^m$ transforms w.r.t. a target space transformations as the $\partial_i X^m$ term itself. This is guaranteed, here, by the very construction of the $D$-dimensional gamma-matrices, $(\Gamma^m)^\beta_\alpha$ as well as the $Spin(1, D-1)$
group fundamental spinorial representations properties. In particular, for Spin(1, D−1) tensor calculus, a product of a vector by a fundamental spinorial and its contragradient representation contains these spinorial representations upon reduction. Note, that this is not a generic feature of Classical Lie groups/algebras, e.g. for the SL(n, R) case, and a luck of this feature can endanger a superbrane formulation.

3.1 Finite-component spinors

Let us consider now the customary curved target space super-p-brane action. Its derivation is based on a method inherited from supergravity and used extensively in curving the target space in superstrings. The action is given by the following expression [3]

\[ S = \int d^{p+1}\xi \left( -\frac{1}{2p} \sqrt{-\gamma} \gamma^{ij}(\xi) E^\tilde{m}_i E^\tilde{n}_j g_{\tilde{m}\tilde{n}} \right) + \frac{p-1}{2p} \sqrt{-\gamma} + \frac{1}{(p+1)!} \epsilon_{i_1i_2\ldots i_{p+1}} E^\tilde{a}_{i_1} E^\tilde{a}_{i_2} \ldots E^\tilde{a}_{i_{p+1}} B_{\tilde{a}_1\tilde{a}_2\ldots\tilde{a}_{p+1}} \right). \]  

Here, the target space is a supermanifold with superspace coordinates \( Z^\tilde{a} = (X^\tilde{m}, \Theta^\alpha) \), where \( \tilde{m} = 0, 1, \ldots, D - 1 \) and \( \alpha = 1, 2, \ldots, 2[D] \). The number of supersymmetries, \( N \), is inessential for present discussion, and thus suppressed. Furthermore, \( E^a_i = (\partial_i Z^\tilde{a}) E^a_\tilde{a}(Z) \), where \( E^a_{\tilde{a}} \) is the supervielbein and \( a = (m \alpha) \) is the so called “tangent space index”. As pointed in [1], there are no flat tangents to a curved target space, which is a tangent to the brane world volume itself - there are no frames over frames. Were it not for the spinors, generic curving could have been achieved by replacing \( X^m(\xi^i) \) by \( X^\tilde{m}(\xi^i) \), a world vector carrying finite linear representation of \( SL(D, R) \) and nonlinear representation of \( Diff(D, R) \), as done in the bosonic brane case. The \( X^\tilde{m} \) transforms w.r.t. nonlinear \( Diff(D, R) \) representations, that are induced from linear representations of a formal mathematical flat target space Lorentz group \( SO(1, D - 1) \). The “tangent space index”, \( a \), labels coordinates of this formal mathematical group.

The transformation group of the bosonic part of \( E^\tilde{m}_i \) is now the full target space GCT group, that is nonlinearly realized under its \( SL(D, R) \) subgroup. One has

\[ super\ Diff(D, R) \rightarrow Diff(D, R) \rightarrow SL(D, R), \]  

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where in the first step superdiffeomorphisms are mapped to diffeomorphisms, as given by $superDiff(D, R)/\mathbb{Z}_2 \simeq Diff(D, R)$, and in the second step $Diff(D, R)$ is nonlinearly realized under $SL(D, R)$. An 'effective’ transformation group of the part of $E^{\tilde{m}}_\alpha$ that contains spinorial variables $\Theta^\alpha$ is a subgroup of $superDiff(D, R)$ that has finite-dimensional spinorial representations, and allows for gamma-matrices construction. One has here,

$$superDiff(D, R) \rightarrow Spin(1, D - 1), \quad (7)$$

with superdiffeomorphisms nonlinearly realized under its $Spin(1, D - 1)$ subgroup. The symmetry of the above superbrane action is an intersection of the symmetries of the terms given by even and odd variables, i.e. an orthogonal type subgroup of $superDiff(D, R)$. In conclusion, the mathematics of the even part of a curved superspace yields restrictions on a possible curving, allowing $X^{\tilde{m}}(\xi^i)$ to be coordinates of homogeneous coset spaces, e.g. flat, De Sitter, anti de Sitter etc., with an orthogonal type structure group.

### 3.2 Infinite-component spinors

Topology of the $Diff(D, R)$ group is given by the topology of its maximal compact subgroup $SO(D)$, and thus, for $D \geq 3$, the universal covering of the $Diff(D, R)$ group is its double covering $\overline{Diff}(D, R)$. Likewise, the universal covering group of the $SL(D, R)$ group is its double covering $\overline{SL}(D, R)$. It turns out, due to a way the $SO(D)$, i.e. $SO(1, D - 1)$, subgroup is embedded into $SL(D, R)$, that $\overline{SL}(D, R)$ and $\overline{Diff}(D, R)$ groups are given by infinite matrices [4]. There are no finite-dimensional $\overline{Diff}(D, R)$ and/or $\overline{SL}(D, R)$ spinorial representations! The unitary infinite-dimensional $\overline{SL}(D, R)$ spinorial representations are constructed for various dimensions [5]. The theory of these representations on fields is amended with a ‘demunitarizing automorphism”, that provides a correct physical interpretations of the Lorentz subgroup quantities [4].

We have constructed recently a Dirac-like equation for infinite-component spinorial $\overline{SL}(D, R)$, $D \geq 3$ fields, together with an explicit form of the $D$-vector operator, $\Gamma^{\tilde{m}}$, that generalizes Dirac’s gamma-matrices [6]. This equation is lifted up, by making use of appropriate pseudo-frames, infinite matrices related to the quotient $\overline{Diff}(D, R)/\overline{SL}(D, R)$, to a fully $\overline{Diff}(D, R)$ covariant Dirac-like wave equation for a world spinor field. Moreover, the pseudo-frames provide for a construction of generalized gamma-matrices, $\tilde{\Gamma}^{\tilde{m}}$ in a generic curved space.
Let us consider now a superbrane action that reads,

$$S = \int d^{p+1} \xi \left( -\frac{1}{2p} \sqrt{-\gamma} \gamma^{ij} \Pi_{\tilde{m}i} \Pi_{\tilde{m}j} g_{\tilde{m}\tilde{n}} + \frac{p-1}{2p} \sqrt{-\gamma} \right),$$  \hspace{1cm} (8)

where $\Pi_{\tilde{m}i}$ is given by the following expression,

$$\Pi_{\tilde{m}i} = \partial_i X_{\tilde{m}} + i \Theta_{\tilde{\alpha}} (\tilde{\Gamma}^{\tilde{m}})_{\tilde{\alpha}} \partial_i \Theta_{\tilde{\alpha}}$$ \hspace{1cm} (9)

The target space is a generic curved supermanifold with superspace coordinates $Z^{\tilde{a}}(\xi^i) = (X_{\tilde{m}}(\xi^i), \Theta^{\tilde{\alpha}}(\xi^i))$, where $\tilde{m} = 0, 1, \ldots, D - 1$ and $\tilde{\alpha} = 1, 2, \ldots, \infty$.

This action is, by construction, invariant under the full General Coordinate Transformations group, and thus describes a superbrane embedded into a generic curved target space.

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**References**

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