Fault-tolerant operation of a logical qubit in a diamond quantum processor

Abobeih, M. H.; Wang, Y.; Randall, J.; Loenen, S. J. H.; Bradley, C. E.; Markham, M.; Twitchen, D. J.; Terhal, B. M.; Taminiau, T. H.

DOI
10.1038/s41586-022-04819-6

Publication date
2022

Document Version
Final published version

Published in
Nature

Citation (APA)
Abobeih, M. H., Wang, Y., Randall, J., Loenen, S. J. H., Bradley, C. E., Markham, M., Twitchen, D. J., Terhal, B. M., & Taminiau, T. H. (2022). Fault-tolerant operation of a logical qubit in a diamond quantum processor. Nature, 606(7916), 884-889. https://doi.org/10.1038/s41586-022-04819-6

Important note
To cite this publication, please use the final published version (if applicable).
Please check the document version above.
Fault-tolerant operation of a logical qubit in a diamond quantum processor

Large-scale quantum computers and quantum networks will require quantum error correction to overcome inevitable imperfections. The central idea is to encode each logical qubit of information into several physical data qubits. Non-destructive multi-qubit measurements, called stabilizer measurements, can then be used to identify and correct errors. If the error rates of all the components are below a certain threshold, it becomes possible to perform arbitrarily large quantum computations by encoding into increasingly more physical qubits. A crucial requirement is that all logical building blocks, including the error-syndrome measurement, must be implemented fault-tolerantly. At the lowest level, this implies that any single physical error should not cause a logical error.

Over the past several years, steps towards fault-tolerant quantum error correction have been made using spin qubits in silicon and in diamond, as well as in various other hardware platforms, such as superconducting qubits and trapped-ion qubits. Pioneering experiments have demonstrated codes that can detect but not correct errors, quantum error-correction protocols that can correct only one type of error, as well as non-fault-tolerant quantum error-correction protocols. A recent experiment with trapped-ion qubits has demonstrated the fault-tolerant operation of an error-correction code, albeit through destructive stabilizer measurements and post-processing.

In this work, we realize fault-tolerant encoding, gate operations and non-destructive stabilizer measurements for a logical qubit of a quantum error-correction code. Our logical qubit is based on the five-qubit code and we use a total of seven spin qubits in a diamond quantum processor (Fig. 1). Fault tolerance is made possible through the recently discovered paradigm of flag qubits. First, we demonstrate a new fault-tolerant encoding protocol based on repeated multi-qubit measurements, which herald the successful preparation of the logical state. Then, we realize the (non-universal) set of transversal single-qubit Clifford gates. Finally, we demonstrate flagged stabilizer measurements with real-time processing of the outcomes. Such measurements are a primitive for fault-tolerant quantum error correction. Although future improvements in fidelity and the number of qubits will be required to suppress logical error rates below the physical error rates, our realization of fault-tolerant protocols on the logical-qubit level is a key step towards quantum information processing based on solid-state spins.

The logical qubit

Stabilizer error-correction codes use auxiliary qubits to perform repeated stabilizer measurements that identify errors. A key requirement for fault tolerance is to prevent errors on the auxiliary qubits from spreading to the data qubits and causing logical errors. The paradigm of flag fault tolerance provides a solution with minimal qubit overhead. Auxiliary qubit errors that would...
nuclear–nuclear couplings greater than 6 Hz. b. Illustration of the main components of the experiment. We realize fault-tolerant encoding, gates and stabilizer measurements with real-time processing on a logical qubit of the five-qubit quantum error-correction code. To ensure that any single fault does not cause a logical error, an extra flag qubit is used to identify errors that would propagate to multi-qubit errors and corrupt the logical state. An illustration of such an error E is shown in red.

Fig. 2 | Non-destructive stabilizer measurements with real-time feedforward. a. Circuit diagram for the deterministic preparation of a four-qubit GHZ entangled state $\ket{\psi} = (\ket{0000} + \ket{1111})/\sqrt{2}$ using a measurement of the stabilizer $XXXX$. b. Measured expectation values of the 15 operators that define the ideal state. The obtained fidelity with the target state is 0.86(1), confirming genuine multipartite entanglement. Grey bars show the ideal expectation values. Error bars are one standard deviation.
Non-destructive stabilizer measurements

We start by demonstrating non-destructive four-qubit stabilizer measurements with real-time feedforward operations based on the measurement outcomes (Fig. 2). Despite the central role of such measurements in many error-correction codes, including the five-qubit code, the Steane code and the surface code, experimental implementations with feedforward have remained an outstanding challenge.

We benchmark the measurement by using it to deterministically create a four-qubit entangled state. We prepare the state 0000 and measure the operator XXXX. This projects the qubits into the Greenberger–Horne–Zeilinger (GHZ) state (|ψ⟩ = (0000 + 1111)/√2), with the sign determined by the measurement outcome. We process the measurement outcomes in real time using a microprocessor and apply the required correction to deterministically output the state (|ψ⟩), with a fidelity of 0.86(I). Because this result is obtained without any post-selection, it highlights that the post-measurement state is available for all measurement outcomes, satisfying one of the key requirements for error correction.

Fault-tolerant encoding

To prepare the logical qubit, we introduce a new scheme that uses repeated stabilizer measurements and a flag qubit to herald successful preparation (Fig. 3a). In contrast to the scheme introduced by Chao and Reichardt, no direct two-qubit gates between the data qubits are required (fifth section of the Supplementary Information). We demonstrate the preparation of the logical state 1− by initializing the data qubits and subsequently measuring p1, with a single-qubit Pauli error (p2), and the probabilities to obtain the opposite logical state 1+ with zero error (p3) or with a single-qubit Pauli error (p4). Note that p3, p4 are summed over all 15 possible errors. These 32 states are orthogonal and span the full five-qubit Hilbert space.

Fig. 3 | Fault-tolerant encoding of the logical qubit. a, Encoding circuit. The first stage prepares 1−, non-fault-tolerantly (‘non-FT preparation’) by starting with 0000 (an eigenstate of p1, p3) and measuring the logical operators p1 to p4. The second ‘FT verification’ stage consists of two stabilizer measurements, T1 = p1p2p3p4, T2 = p1p2p3p4, and a flag qubit measurement. Echo sequences are inserted between the measurements to decouple the qubits (not shown, see Supplementary Figs. 8 and 9). Successful preparation is heralded by satisfying a set of conditions for the measurement outcomes (see main text). Red indicates an example of an auxiliary qubit fault (an XY error in a two-qubit gate) that would propagate to a logical error but is detected by the T1 verification step. Orange indicates an example of a single fault in the verification stage that would propagate into a logical error but is detected by the flag qubit.

b,c, Probabilities to obtain the desired logical state 1− without error (p0) or with a single-qubit Pauli error (p1–p4), and the probabilities to obtain the opposite logical state 1+ without error (p5) or with a single-qubit Pauli error (p6–p10). Note that p5–p10 are summed over all 15 possible errors. These 32 states are orthogonal and span the full five-qubit Hilbert space.

Non-FT scheme

| Probability | p0 | p1 | p2 | p3 | p4 | p5 |
|-------------|----|----|----|----|----|----|
|              | 0.500 | 0.306 | 0.185 | 0.010 |

FT scheme

| Probability | p0 | p1 | p2 | p3 | p4 | p5 |
|-------------|----|----|----|----|----|----|
|              | 0.414 | 0.534 | 0.046 | 0.006 |
NV electron spin is measured to be in |0⟩). These outcomes are more reliable (Methods), increasing the fidelity of the state preparation, at the cost of a lower success probability (Supplementary Table 1).

We compare the non-FT and FT encoding schemes. We define the logical state fidelity $F_L$ as (Methods)

$$F_L = \sum_{\epsilon \in \epsilon} \text{Tr}(E - \gamma(\epsilon E \rho)),$$

in which $\rho$ is the prepared state and $\epsilon = \{I, X, Y, Z, i = 1, 2, ..., 5\}$ is the set of all single-qubit Pauli errors. The fidelity $F_L$ gives the probability that there is at most a single qubit error in the prepared state, that is, there is no logical error. We characterize the prepared state by measuring the 31 operators that define the target state (Extended Data Fig. 2 and Methods). We find that the FT encoding scheme ($F_L = 95(2)\%$) outperforms the non-FT scheme ($F_L = 81(2)\%$).

To understand this improvement, we analyse the underlying error probability distributions (Figs. 3b,c). For the five-qubit code, the $|\gamma\rangle$ state plus any number of Pauli errors is equivalent to either $|\gamma\rangle$ with at most one Pauli error (no logical error) or to $|\gamma\rangle$ with at most one Pauli error (a logical error). We calculate the overlaps between the prepared state and those states. The results show that the FT scheme suppresses logical errors, consistent with fault tolerance preventing single faults propagating to multi-qubit errors. The overall logical state fidelity $F_L$ is improved, despite the higher probability of single-qubit errors owing to the increased complexity of the sequence.

**Fault-tolerant logical gates**

The five-qubit code supports a complete set of transversal single-qubit Clifford gates, which are naturally fault tolerant (Fig. 4a). We apply four transversal logical gates $|\gamma\rangle$ (Fig. 4): $X_L = X X X X X X$, $Y_L = Y Y Y Y Y Y$, $Z_L = Z Z Z Z Z Z$, the Hadamard gate $H_L = P_0 H H H H H H$, and the phase gate $S_L = P_0 S_S S_S S_S$, in which $P_0$ is a permutation of the data qubits (Fig. 4b). These permutations are fault tolerant because we realize them by relabelling the qubits rather than by using SWAP gates (Methods). For completeness, we note that universal computation requires further non-transversal gates, constructed—for example—with auxiliary logical qubits, which are not pursued here (Methods).

Our control system performs the underlying single-qubit gates by tracking basis rotations and compiling them with subsequent gates or measurements (Fig. 4c), for example, by using the ‘worst-case’ scenario, in which the logical gates are applied physically (Fig. 4c). This includes five single-qubit gates and the corresponding extra echo sequences between the state preparation and the measurement stage. Together, the demonstrated transversal logical gates enable the fault-tolerant preparation of all six eigenstates of the logical Pauli operators.

**Fault-tolerant stabilizer measurements**

Finally, we demonstrate and characterize a flagged stabilizer measurement on the encoded state (Fig. 5a). Such measurements are a primitive for fault-tolerant quantum error-correction protocols (Methods). To ensure that the measurement is compatible with fault tolerance, the two-qubit gates are carefully ordered and a flag qubit is added to capture the auxiliary qubit errors that can propagate to logical errors (Methods).

We prepare the logical state $|\gamma\rangle$ and measure the stabilizer $s_L = X Y X Y Y$ (Methods). The resulting output consists of the post-measurement state and two classical bits of information from the measurements of the auxiliary and flag qubits (Fig. 5b). The logical state fidelity $F_L$ is given by the probability that the logical information can be correctly extracted (no logical error) when taking into account the flag measurement outcome. The interpretation of the error syndrome changes if the flag is raised (Methods). We find $F_L = 0.77(4)$ for the post-measurement state without any post-selection. Higher logical state fidelities can be obtained by post-selecting on favourable outcomes, but this is incompatible with error correction.

To illustrate the benefit of the flag qubit, we compare the logical state fidelities with and without taking the flag measurement outcome into account. Because auxiliary qubit errors that propagate to logical errors are naturally rare, no marked difference is observed (Fig. 5c). Therefore, we introduce a Pauli Y error on the auxiliary qubit (Fig. 5a). This error propagates to the two-qubit error $Y_L$. For the case without flag information, this error causes a logical flip $Z_L$ (Methods) and the logical state fidelity drops below 0.5. By contrast, with the flag qubit, this non-trivial error is detected (Fig. 5b) and remains correctable, so that the logical state fidelity is partly recovered (Fig. 5c).

**Conclusion**

In conclusion, we have demonstrated encoding, gates and non-destructive stabilizer measurements for a logical qubit of an error-correction code in a fault-tolerant way. Our results advance
stabilizer measurement as a function of the error probability $p_e$. The non-FT, Logical state fidelity $c$ flag qubit successfully detects this error.

**Fig. 5 | Fault-tolerant stabilizer measurement.**

This error will propagate to the two-qubit error compatibility with fault tolerance, we insert a error on the auxiliary qubit. Y

physical error rates. Although the demonstrated operations are of high fidelity—the experiments consist of up to 40 two-qubit gates and eight mid-circuit auxiliary qubit readouts (Fig. 5a)—improvements in both the fidelities and the number of qubits will be required.

Improved gates might be realized through tailored optimal control schemes that leverage the precise knowledge of the system and its environment. Coupling to optical cavities can further improve readout fidelities. Scaling to large code distances and several logical qubits can be realized through already-demonstrated magnetic and optical NV–NV connections that enable modular, distributed, quantum computation based on the surface code and other error-correction codes. Therefore, our demonstration of the building blocks of fault-tolerant quantum error correction is a key step towards quantum information processing based on solid-state spin qubits.

**Online content**

Any methods, additional references, Nature Research summaries, source data, extended data, supplementary information, acknowledgements, peer review information; details of author contributions and competing interests; and statements of data and code availability are available at https://doi.org/10.1038/s41586-022-04819-6.
28. Chao, R. & Reichardt, B. W. Quantum error correction with only two extra qubits. Phys. Rev. Lett. 121, 050502 (2018).
29. Chamberland, C. & Beverland, M. E. Flag fault-tolerant error correction with arbitrary distance codes. Quantum 2, 53 (2018).
30. Chao, R. & Reichardt, B. W. Flag fault-tolerant error correction for any stabilizer code. PRX Quantum 1, 010302 (2020).
31. Negnevitsky, V. et al. Repeated multi-qubit readout and feedback with a mixed-species trapped-ion register. Nature 563, 527–531 (2018).
32. Erhard, A. et al. Entangling logical qubits with lattice surgery. Nature 589, 220–224 (2021).
33. Takita, M., Cross, A. W., Córcoles, A. D., Chow, J. M. & Gambetta, J. M. Experimental demonstration of fault-tolerant state preparation with superconducting qubits. Phys. Rev. Lett. 119, 180501 (2017).
34. Gong, M. et al. Experimental exploration of five-qubit quantum error-correcting code with superconducting qubits. Nat. Sci. Rev. 9, nwab011 (2021).
35. Knill, E., Laflamme, R., Martinez, R. & Negrevergne, C. Benchmarking quantum computers: the five-qubit error correcting code. Phys. Rev. Lett. 86, 5811–5814 (2001).
36. Laflamme, R., Miquel, C., Paz, J. P. & Zurek, W. H. Perfect quantum error correction code. Phys. Rev. Lett. 77, 198–201 (1996).
37. Abobeih, M. H. et al. One-second coherence for a single electron spin coupled to a multi-qubit nuclear-spin environment. Nat. Commun. 9, 2552 (2018).
38. Abobeih, M. H. et al. Atomic-scale imaging of a 27 nuclear-spin cluster using a quantum sensor. Nature 576, 411–415 (2019).
39. Yoder, T. J., Takagi, R. & Chuang, I. L. Universal fault-tolerant gates on concatenated stabilizer codes. Phys. Rev. X 6, 031039 (2016).
40. Dolde, F. et al. High-fidelity spin entanglement using optimal control. Nat. Commun. 5, 3371 (2014).
41. Bhaskar, M. K. et al. Experimental demonstration of memory-enhanced quantum communication. Nature 580, 60–64 (2020).
42. Pompili, M. et al. Realization of a multimode quantum network of remote solid-state qubits. Science 372, 259–264 (2021).

Publisher’s note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons license, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons license and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this license, visit http://creativecommons.org/licenses/by/4.0/.

© The Author(s) 2022
**Methods**

**Sample**

We use a naturally occurring NV centre in a homo-epitaxially chemical-vapour-deposition-grown diamond with a 1.1% natural abundance of $^{13}$C and a (111) crystal orientation (grown by Element Six). A solid-immersion lens is used to enhance the photon-collection efficiency. The NV centre has been selected for the absence of $^{13}$C spins with hyperfine interactions. vapour-deposition-grown diamond with a 1.1% natural abundance of $^{13}$C and a magnetic field of about 403 G is applied using a room-temperature immersion lens is used to enhance the photon-collection efficiency.

**Qubits and coherence times**

The NV electron-spin auxiliary qubit is defined between the states $m_z = 0$ (>) and $m_z = -1$ (<<). The NV electron-spin coherence times are $T_2^* = 2.9(2)\mu s$, $T_2 = 1.182(5)$ ms and up to seconds under dynamical decoupling. The $^{14}$N nuclear-spin flag qubit is defined between the states $m_I = 0$ (>) and $m_I = -1$ (<<). The $^{13}$C nuclear-spin data qubits in this device have been characterized in detail in previous work. The NV centre has been selected for the absence of $^{13}$C spins with hyperfine component perpendicular to the applied magnetic field.

**Magnetic field**

A magnetic field of about 403 G is applied using a room-temperature permanent magnet on a XYZ translation stage. The applied field lifts the degeneracy of the $m_z = \pm 1$ states owing to the Zeeman term (first section of the Supplementary Information). We stabilize the magnetic field to <3 mG using temperature stabilization and an automatic recalibration procedure (every few hours). We align the magnetic field along the NV axis using thermal echo sequences with an uncertainty of 0.07° in the alignment.

**Single-qubit and two-qubit gates**

Single-qubit gates and echo pulses are applied using microwave pulses for the NV electron spin ($m_z = 0 \leftrightarrow m_z = -1$ transition, Hermite pulse shapes, Rabi frequency of about 15 MHz and using radio-frequency (RF) pulses for the $^{13}$C spin qubits (error function pulse shapes, typical Rabi frequency of about 2 kHz). The Hermite pulse envelopes of the microwave pulses are defined as:

$$A \left[ 1 - c \left( \frac{T - \mu}{T} \right)^2 \right] \cdot \exp \left[ - \left( \frac{T - \mu}{T} \right)^2 \right],$$

in which $c = 0.956$ for π pulses and $c = 0.667$ for π/2 pulses, $\mu = 0.5t_{\text{pulse}}$, $T = 0.1667t_{\text{pulse}}$, $t_{\text{pulse}}$ is the microwave pulse length and $A$ is the pulse amplitude, which is experimentally calibrated to achieve a π or π/2 rotation. For this work, we use $t_{\text{pulse}} = 168$ ns for π pulses and $t_{\text{pulse}} = 100$ ns for π/2 pulses. The envelope of the RF pulse is defined as:

$$f(t) = 1 - \frac{1}{2} \text{erf} \left[ \frac{2(\Delta t - t + t_0)}{\Delta t} \right] - \frac{1}{2} \text{erf} \left[ \frac{2(\Delta t + t - t_0)}{\Delta t} \right],$$

in which $\Delta t$ is the rise time, $t_0$ is the start time of the pulse, $t_{\text{pulse}}$ is the pulse length and erf(x) is the error function. We ensure that the RF pulses consisted of an integer number of periods of the RF waveform, that is, we ensure that $\omega t_{\text{pulse}} = 2n\pi$ for integer $n$. This ensures that any phase step up on the electron spin owing to the RF pulse is cancelled. Note that the $^{14}$N spin qubits (data qubits) are distinguishable in frequency owing to their hyperfine coupling to the NV electron spin.

**Compilation of gate sequences**

Our native two-qubit gates are electron-controlled nuclear-spin rotations and are equivalent to the CNOT gate up to single-qubit rotations (Supplementary Fig. 4). To implement the sequences shown in the figures, we first translate all gates into these native gates and compile the resulting sequence. Afterwards, the circuit is translated into the actual pulse sequence. At the core of this compilation process is the tracking and synchronization of the qubit phases and the corresponding pulse timings. See Supplementary Information for the details of this compilation process (Supplementary Figs. 4–9 and pseudocode 1–8).

**Echo sequences for the data qubits**

To mitigate decoherence of the data qubits owing to their spin environment, we use echo sequences that are interleaved throughout the experiments. These echo sequences ensure that the data qubits rephase each time they are operated on. Furthermore, the sequence design minimizes the time that the auxiliary electron-spin qubit is idling in superposition states, which are prone to dephasing. We use two echo stages between stabilizer measurements, as well as before and after the logical gates of Fig. 4, which provides a general and scalable solution for the timing of all gates and echoes (third section of the Supplementary Information).

An extra challenge is that, owing to the length of the sequences (up to 100 ms), we need to account for the small unwanted interactions between the nuclear-spin data qubits. The measured coupling strengths show that the strongest couplings are between qubits 3 and 2 (16.90(4) Hz) and between qubits 3 and 5 (12.96(4) Hz) (Supplementary Table 5). Such interactions can introduce correlated two-qubit errors that are not correctable in the distance-3 code considered here, which can only handle single-qubit errors in the code block.

To mitigate these qubit–qubit couplings, we decouple qubit 3 asynchronously from the other qubits (Supplementary Fig. 8). Ultimately, such local correlated errors can be suppressed entirely by larger distance codes.

**Real-time control and feedforward operations**

Real-time control and feedforward operations are implemented through a programmable microprocessor (Jaeger ADWin Pro II) operating on microsecond timescales. The microprocessor detects photon events coming from the detectors, infers the measurement outcomes and controls both the subsequent sequences in the arbitrary waveform generator (Tektronix AWG 5014c) and the lasers for the auxiliary qubit readout. The precise timing for quantum gates (1-ns precision) is based on the clock of the arbitrary waveform generator. Furthermore, the microprocessor operates various control loops that prepare the NV centre in the negative charge state, on resonance with the lasers and in the focus of the laser beam (see second and third sections of the Supplementary Information).

**Readout of the auxiliary qubit**

The electron spin (auxiliary qubit) is read out by resonantly exciting the $m_z = 0 \leftrightarrow E_c$ optical transition. For one or more photons detected, we assign the $m_z = 0$ outcome; for zero photons, we assign $m_z = \pm 1$. The
single-shot readout fidelities are $F_p = 90.5(2)$% and $F_e = 98.6(2)$% for $m_e = 0$ and $m_e = -1$, respectively (average fidelity 94.6(1)%).

Uncontrolled electron-spin flips in the excited state cause dephasing of the nuclear spins through the hyperfine interaction. To minimize such spin flips, we avoid unnecessary excitations by using weak laser pulses, so that a feedback signal can be used to rapidly turn off the laser on detection of a photon (within 2 μs). The resulting probability that the electron spin is in state $m_e = 0$ after correctly assigning $m_e = 0$ in the measurement is 0.992 (ref. 4).

For measurements that are used for heralded state preparation, that is, for which we only continue on a $m_e = 0$ outcome (see, for example, Fig. 3), we use shorter readout pulses. This improves the probability that a $m_e = 0$ outcome correctly heralds the $m_e = 0$ state, at the cost of reduced success probability (Supplementary Table 1).

**System preparation and qubit initialization**

At the start of the experiments, we first prepare the NV centre in its negative charge state and on resonance with the lasers. We then initialize the NV electron spin in the $m_e = 0$ state through a spin pumping process (fidelity > 99.7%)5. We define the electron-spin qubit between the states $m_e = 0 (|0⟩)$ and $m_e = -1 (|1⟩).$ We initialize the data qubits through SWAP sequences (Supplementary Fig. 6) into $|0⟩$ and subsequent optical reset of the auxiliary qubit (initialization fidelities 96.5–98.5%; see Supplementary Table 4). The flag qubit is initialized through a projective measurement that heralds preparation in $|0⟩$ (initialization fidelity 99.7%). Other product states are prepared by subsequent single-qubit gates.

**Final readout of the data qubits**

Measuring single-qubit and multi-qubit operators of the data qubits is performed by mapping the required correlation to the auxiliary qubit (through controlled rotations) and then reading out the auxiliary qubit4. To provide best estimates for the measurements, we correct the measured expectation values (Fig. 2 and Extended Data Figs. 1 and 2) for infidelities in the readout sequence; see Bradley et al.1 for the correction procedure.

**Fidelity of the GHZ state**

The fidelity of the prepared state $\rho$ (in Fig. 2 and Extended Data Fig. 1) with respect to the target GHZ state $\rho_0$ is obtained as

$$F = \text{Tr}(|\psi⟩⟨\psi|_0) = \frac{1}{16}(1 + |\epsilon⟩|IZZ| + |ZZI| + |IZI| + |ZIZ| + |\epsilon⟩|IIZ| + |ZZZ| + |\epsilon⟩|IYY| + |YXX|$$

$$+ |\epsilon⟩|XXX| + |YYY| + |YYX| - |YXY|$$

$$- |YYX| - |YXY| - |XXX| - |YYY| - |XYX| - |YYX|).$$

(4)

**Assessing the logical state fidelity**

The logical state fidelity $F_L$ is defined in equation (1) and gives the probability that the state is free of logical errors. Said differently, $F_L$ is the fidelity with respect to the ideal five-qubit state after a round of perfect error correction, or the probability to obtain the correct outcome in a perfect fault-tolerant logical measurement. Although fault-tolerant circuits for logical measurement exist28, we do not experimentally implement these here. Instead, we extract $F_L$ from a set of measurements, as described in the following using $-\lambda_1$ as an example.

The logical state $-\lambda_1$ is the unique simultaneous eigenstate of the five weight-3 operators $p_i$ with eigenvalue +1. Thus, we can describe the state $E|-\lambda_1$ (with $E$ a Pauli error) as the projector

$$E |-\lambda_1 = \prod_{i=1}^5 \frac{(1 + m_i p_i)}{2},$$

in which $m_i = \pm 1$ is the measurement outcome of $p_i$ and $m_i = -1$ when $E$ anticommutes with $p_i$. This projector can be expanded as a summation of 31 multi-qubit Pauli operators (including a constant), which are listed in Extended Data Fig. 2. The logical state fidelity $p_L$ in equation (1) can then be written as

$$F_L = \sum_{E} \text{Tr}(E |-\lambda_1 (-L E)p)$$

$$= \frac{1}{2} \sum_{E} \frac{1}{8} (|ZZX| + |ZIZ| + |XXZ| + |XZI| + |IJZ|)

$$+ (|YYX| + |XYI| + |IXY| + |YIX| + |ZYZ| + |ZZY|)$$

$$+ (|IYX| + |YIX| + |IYY| + |YXX| + |YYX| + |XXX|).$$

(5)

Here $\epsilon = [I, X, Y, Z, \epsilon_i, i = 1, 2, \ldots, 5]$ is the set of correctable errors for the five-qubit code. To obtain $F_L$ experimentally, we measure this set of expectation values.

**Logical state fidelity with flag**

If the flag in the circuit in Fig. 5a is not raised, then a cycle of error correction would correct any single-qubit error on a logical state. The logical state fidelity is then given by equation (1), which we now refer to as $F_{L,\text{not raised}}$. A raised flag leads to a different interpretation of the error syndrome28 (Supplementary Table 7).

For example, the $Y$ error on the auxiliary qubit in Fig. 5a leads to the output state $Y|0⟩ = -\lambda_1$ for which the eigenvalues of $s_z = XXYY, s_z = YXYX$ and $s_z = YYYX$ give the syndrome $[1, -1, -1, -1]$. Without flag, the corresponding single-qubit recovery is $Z_4$, which changes the syndrome back to all $+1$ (Supplementary Table 7). This recovery leads to the remaining state $Y|1⟩$, which is a logical $Z$ error. However, taking the flag measurement outcome into account, the syndrome is interpreted differently and the recovery is $Y|1⟩$, so that no error is left (Supplementary Table 7).

For the cases in which the flag is raised, the logical state fidelity with respect to $-\lambda_1$ is now given by

$$F_{L,\text{raised}} = \sum_{E} \text{Tr}(E |-\lambda_1 (-L E)p)$$

$$= \frac{1}{2} \sum_{E} \frac{1}{32} (5|IZX| + 6|ZZX| + 6|YIX| - 2|ZIZ|)$$

$$+ 6|XYY| + 2|XYZ| + 2|XYZZ| - 2|IZI|)$$

$$+ 2|YYXZ| + 6|XIXY| + 2|YXYY| + 2|ZZYY|$$

$$+ 2|YYXY| - 2|YYX| + 2|XXX| - 2|XXZ|),$$

(6)

with $\epsilon'$ being another set of correctable errors

$$\epsilon' = [I, X, Y, Z, \epsilon_i, i = 1, 2, \ldots, 5].$$

(7)

A detailed derivation for this set of errors and their corresponding syndromes are given in the fifth section of the Supplementary Information.

The logical state fidelity after the stabilizer measurement (Fig. 5) is calculated as the weighted sum of the fidelities conditioned on the two flag outcomes:

$$F_L = p_F \cdot F_{L,\text{raised}} + (1 - p_F) \cdot F_{L,\text{not raised}},$$

(8)

with $p_F$ being the probability that the flag is raised and $F_{L,\text{raised}}$ and $F_{L,\text{not raised}}$ are as defined above.

Finally, to construct the logical state fidelity as a function of $p_F$ (Fig. 5c), we measure $F_L$ with ($p_F = 0$) and without ($p_F = 1$) the auxiliary qubit error and calculate the outcomes for other error probabilities $p_F$, from their weighted sum:

$$F_L(p_F) = (1 - p_F) \cdot F_L(p_F = 0) + p_F \cdot F_L(p_F = 1)$$

(9)

**Error distribution in the prepared state**

The overlaps between the prepared state $\rho$ and the state $E |-\lambda_1$ with $E$ identity or a single-qubit error are written as $p_{0,\epsilon}$ and $p_{1,\epsilon}$, respectively.
These correspond to the cases that there is no logical error. The overlaps between the prepared state \( \rho \) and the state \( E_{i}^{+} \chi \), with \( E \) identity or a single-qubit error are written as \( P_{0,i} \), and \( P_{1,i} \), respectively. In these cases, there is a logical error. These overlaps are shown in Fig. 3b,c and calculated as (as \( \alpha = \omega \))

\[
P_{0,i} = \text{Tr}(\alpha_{i} \chi, \rho),
\]

\[
P_{1,i} = \sum_{E \in \{+,-,z\}} \text{Tr}(E \alpha_{i} \chi, E \rho).
\]

These overlaps can be explicitly expressed in terms of the 31 measured expectation values (see seventh section of the Supplementary Information).

**Error analysis**

The uncertainties in the measured fidelities, logical state fidelities and probabilities (\( P_{0,1,\alpha} \)) are obtained from the uncertainties in the measured expectation values using error propagation. For example, the logical state fidelity \( F_{i} \) is calculated as

\[
F_{i} = \frac{1}{2} + \frac{1}{8} \left( \sum_{\alpha} A_{\alpha} \right),
\]

in which \( A_{\alpha} \) are the 16 expectation values shown in equation (5). Assuming that the errors in the measured expectation values are independent, the standard deviation in \( F_{i} \) is:

\[
\sigma_{F_{i}} = \frac{1}{8} \left( \sum_{\alpha} \sigma_{A_{\alpha}} \right)^{\frac{1}{2}},
\]

in which \( \sigma_{A_{\alpha}} \) is the standard deviation of the expectation value \( A_{\alpha} \), and is given by a binomial distribution\(^ {42}\). Note that \( \sigma_{A_{\alpha}} \) is also corrected for the readout correction process described in Bradley et al.\(^ {1}\).

**Note added**

While finalizing this manuscript, two related preprints appeared that demonstrate destructive stabilizer measurements with a flag qubit\(^ {48} \) and flag fault-tolerant quantum error correction\(^ {49} \) with trapped-ion qubits. Furthermore, during the revision process, three related preprints appeared that demonstrate quantum error correction on a surface code using superconducting qubits\(^ {50,51} \) and realize a flag-based universal fault-tolerant gate set using trapped ions\(^ {52} \).

**Data availability**

The underlying data and software code for generating the plots presented in the main text and Supplementary Information are available at Zenodo https://doi.org/10.5281/zenodo.6461872.
Extended Data Fig. 1 | Non-destructive stabilizer measurements with a flag and real-time feedforward. a, Circuit diagram for the deterministic preparation of a four-qubit GHZ entangled state ($\psi = (0000 + 1111)/2$) using a flagged measurement of the stabilizer XXXX. b, Measured expectation values of the 15 operators that define the ideal state. The average obtained fidelity is 0.79(1)%.

c, Data post-selected on the flag not being raised. The obtained fidelity with the target state is 0.82(1)%.

d, When the flag is raised, the obtained fidelity is 0.47(5)%. Grey bars show the ideal expectation values. Note that we perform this measurement as a test of the circuit, but that the flag information in this case does not carry any specific significance.
Extended Data Fig. 2 | Measured expectation values for the encoded state. Measured expectation values of the 31 operators that define the encoded state (for the circuit in Fig. 3). Grey bars show the ideal expectation values.