g factor in a light two body atomic system:
a determination of fundamental constants
to test QED

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Abstract: Energy levels of a two-body atomic system in an external homogeneous magnetic field can be presented in terms of magnetic moments of their components, however, those magnetic moments being related to bound particles differ from their free values. Study of bound $g$ factors in simple atomic systems are now of interest because of a recent progress in experiments on medium $Z$ ions and of a new generation of muonium experiments possible with upcoming intensive muon sources. We consider bound corrections to the $g$ factors in several atomic systems, experimental data for which are available in literature: hydrogen, helium-3 ion, muonium, hydrogen-like ions with spinless nuclei with medium $Z$.

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1. Introduction

High-resolution spectroscopy of hydrogen and hydrogen-like atoms provides significant amount of data for precision QED tests and determination of basic fundamental physical constants such as the Rydberg constant $R_{\infty}$, the fine structure constant $\alpha$, etc. However, not only the spectrum of a free hydrogen-like atom can be studied, but also its interaction with an external field and in particular with a homogeneous magnetic field. From experimental and practical point of view two kinds of two-body systems are of particular interest. Those are muonium and an hydrogen-like ion with a spinless nucleus at not too high $Z$.

$g$ factor of a bound lepton (electron or muon) can be presented in the form [1]

$$g = 2 \cdot \left(1 + a + b\right), \quad (1)$$

where $I$ stands for the Dirac value, $a$ is anomalous magnetic moment of a free lepton and an additional correction $b$ is due to the binding effects. The problems for calculation of $b$ in muonium and an hydrogen-like ion are similar but they have some different significant features.

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2. Motivation: muonium

Properties of the hydrogen atom are affected by effects of the proton structure which limit accuracy of any theoretical prediction. The muonium atom is, in contrast to hydrogen, a pure leptonic system suited for an accurate QED test. A value of its hyperfine structure has been measured at LAMPF with a great accuracy \[ \nu_{hfs}(exp) = 4463.302.776(51) \text{ kHz}. \] (2)

The experimental uncertainty of \( \nu_{hfs} \) is only \( 1.1 \cdot 10^{-8} \) in fractional units. The calculation has not been able to reach this level yet. The theoretical problems are:

- It is not possible to do any exact calculations. One have to expand either in \( \alpha \) (i.e. to select diagrams with a different number of QED loops), or \( Z\alpha \) (i.e. over the binding effects) or \( m/M \) (i.e. over recoil effects). A possible contribution of presently uncalculated terms of higher order has been estimated as \( 5 \cdot 10^{-8} \) [3, 4].
- One has to take into account hadronic effects (see e.g. [3, 5]).
- It appears that a largest problem of the theoretical calculation is an experimental uncertainty. The leading contribution into the ground state hyperfine splitting (so-called Fermi energy)

\[
\nu_F = \frac{16}{3} (Z\alpha)^2 Z^2 R_\infty c m_e \frac{m_\mu}{m_\mu + m_e} \left( 1 + a_\mu \right)
\]

contains the electron-to-muon mass ratio \( m_e/m_\mu \). The most accurate determination of the mass ratio has been performed at LAMPF [2] studying Breit-Rabi levels of the muonium ground state in presence of homogenous magnetic field. For any successful interpretation of the experiment, \( g \) factors of muon and electron in muonium are needed.

Special features of the theory of \( g \) factors of electron and muon in muonium are:

- It is necessary to find a theoretical expression for the bound \( g \) factors with an uncertainty below 0.01 ppm.
- All small parameters, \( \alpha, Z\alpha \) and \( m_e/m_\mu \), are of about the same value (\( \alpha = Z\alpha \sim 1/137 \)) and hence QED, binding and recoil effects are equally important.

The theory up to the third order in either of these three small parameters has been known up to now and our target is a complete evaluation of fourth-order corrections.

3. Motivation: Hydrogen-like ions with spinless nuclei

Recent Mainz-GSI experiment [6, 7, 8] have provided precision measurement of

- ion cyclotron frequency

\[
\omega_{ion} = \frac{(Z - 1)e}{M_{ion}} B
\]

- and spin precession frequency

\[
\omega_{spin} = \frac{e}{2m_e} \frac{\hbar}{\hbar} B
\]
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with uncertainty below 1 ppb. A comparison of these two frequencies leaves us with two values: of the $g$ factor of a bound electron and of the electron-to-ion mass ratio. Accuracy of the latter is essentially the same as of determination of the electron-to-proton mass ratio in Ref. [9] and in fact the uncertainty for $m_e/M_{ion}$, derived from Ref. [9] is bigger than both the experimental uncertainty for frequencies and the theoretical uncertainty for the bound $g$ factor (see e.g. [10]). The comparison of experiment with theory provides the most accurate determination of the electron-to-proton mass ratio and a test of QED calculations. The latter is possible if the experiment on two different ions is performed and hence a ratio of their $g$ factors is free of any uncertainty due to the proton-to-electron mass ratio.

Published experimental data are related to the ion of carbon-12 [6], recent efforts has been directed to the oxygen-16 ion [8]. Precision measurements with other ions are also possible and expected.

The theory must target a level of accuracy as good as few parts in $10^{10}$. The details of theory differ from muonium: the recoil effects are pretty small and less important than the QED contributions. The most important problem is to take into account the binding effects. The latter are less important for lighter H-like ions, namely for hydrogen-like ions of helium-4 and beryllium-10.

4. Muonium: calculations

The “old” theoretical expressions for the ground state in muonium (up to the third order) are of the form (see e.g. [11])

$$g'_e \text{(up to 3rd)} = g_e^{(0)} \cdot \left\{ 1 - \frac{(Z\alpha)^2}{3} \left[ 1 - \frac{3}{2} \frac{m_e}{m_\mu} \right] + \frac{\alpha(Z\alpha)^2}{4\pi} \right\} \tag{6}$$

and

$$g'_\mu \text{(up to 3rd)} = g_\mu^{(0)} \cdot \left\{ 1 - \frac{\alpha(Z\alpha)}{3} \left[ 1 - \frac{3}{2} \frac{m_e}{m_\mu} \right] \right\} \tag{7}$$

Although $Z = 1$ for muonium we keep it to simplify classification of contributions.

We present here a complete result for the fourth order corrections

$$g'_e \text{(4th)} = g_e^{(0)} \cdot \left\{ -\frac{(Z\alpha)^2(1+Z)}{2} \left( \frac{m_e}{m_\mu} \right)^2 - \frac{5\alpha(Z\alpha)^2}{12\pi} \frac{m_e}{m_\mu} \right. \right.$$

$$\left. - \frac{(Z\alpha)^4}{12} - (0.289 \ldots) \cdot \frac{\alpha^2(Z\alpha)^2}{\pi^2} \right\} \tag{8}$$

and

$$g'_\mu \text{(4th)} = g_\mu^{(0)} \cdot \left\{ -\frac{\alpha(Z\alpha)(1+Z)}{2} \left( \frac{m_e}{m_\mu} \right)^2 + \frac{\alpha^2(Z\alpha)}{6\pi} \frac{m_e}{m_\mu} - \frac{97}{108} \alpha(Z\alpha)^3 \right\} \tag{9}$$

Most of terms in Eqs. (8), (9) are of the kinematic origin and we have systematically checked them. All of those kinematic terms can be derived by several methods applied originally for derivation of the third order corrections which are all of the kinematic origin (cf. [11]). Non-kinematic corrections include a high-order Breit contribution to $g'_e$ [12] and a screening contribution to $g'_\mu$. The latter was calculated using of the corrections to the Dirac wave function due to hyperfine interaction [13] and due to external magnetic field [14]. The result agrees with calculations for the screening tensor in Refs. [15, 16].

The results of a re-evaluation of LAMPF data [17, 2] taking into account Eqs. (8), (9) is shown in Table 1.

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Table 1. Re-evaluation of LAMPF data. The fractional corrections $\delta_{e,\mu}$ to the derived value of the ratio of the muon magnetic moment and proton magnetic moment are related to the fourth order contribution to $g_{e,\mu}$ found in this work.

| Experiment | LAMPF, 1982, [17] | LAMPF, 1999, [2] |
|------------|-------------------|-------------------|
| $\mu_e/\mu_p$ | 3.183 290 0(13) | 3.183 289 03(38) |
| $\mu_\mu/\mu_p$ | 3.183 346 1(13) | 3.183 345 14(38) |
| $\delta_e$ | $-0.4 \cdot 10^{-8}$ | $-0.3 \cdot 10^{-8}$ |
| $\delta_\mu$ | $1.2 \cdot 10^{-8}$ | $1.2 \cdot 10^{-8}$ |

5. H-like ions with spinless nuclei: calculations

We present a theoretical expression for the binding correction to the electron $g$ factor in the form

$$b = b_{\text{kin}} + b_{\text{rel}} + b_{\text{rec}} + b_{\text{rad}} + b_{\text{nucl}}.$$ (10)

The kinematic term was discussed above in the case of muonium. The relativistic corrections are known exactly thanks to Breit [12]. The nuclear corrections were considered in [18, 1, 19]. For $Z < 10$ an analytic expression [1]

$$b_{\text{nucl}} = \frac{4}{3} (Z \alpha)^4 (m R_N)^2$$ (11)

has an adequate accuracy.

The recoil effects are small enough. We have performed an evaluation based on the Groth equation and confirm here the leading term

$$b_{\text{rec}} = -\frac{1}{24} (Z \alpha)^4 \frac{m}{M} ,$$ (12)

obtained recently in Ref. [20]. There were also numerical calculations by Shabaev and Yerokhin [22] and we apply their data to obtain our final results.

The essential binding effects contribute into the radiative corrections

$$b_{\text{rad}} = b_{\text{VP}} + b_{\text{WK}} + b_{\text{SE}} + b_{\text{2loop}} .$$ (13)

The vacuum polarization effects were studied in Ref. [18] numerically and in Ref. [14] analytically. The Wichman-Kroll contribution was investigated in Ref. [1]. The light-by-light contributions are needed more study [21]. The two-loop contributions have not been yet calculated and only some conservative estimations are possible. The biggest contribution and the biggest uncertainty in $b_{\text{rad}}$ comes from the one-loop electron self energy. Fractional uncertainty of numerical data for $b_{\text{SE}}(Z)$ increases at lower $Z$ and one can improve accuracy of low $Z$ results after properly fitting data. We fit here numerical data from Ref. [18]. An essential improvement for $Z = 2$ and $Z = 4$ has been achieved. After our analysis has been completed a paper [23] with data more accurate than those in Ref. [18]. Results of our fitting are in fair agreement with new numerical calculations.

The final results for ions of interest are collected in Table 2 below. Note, that the theoretical accuracy for the hydrogen-like ions of helium-4 and beryllium-10 is higher than for carbon and oxygen and from theoretical point of view they are better suited for determination of the electron-to-proton mass ratio.

A comparison of carbon-to-oxygen and oxygen-to-oxygen leads to

$$g(^{12}\text{C}^{5+})/g(^{16}\text{O}^{7+}) = 1.000 497 273 4(9) ,$$ (14)
Table 2. The binding contributions to $g$ factors of some hydrogen-like ions.

| Ion     | $b$  |
|---------|------|
| $^4\text{He}^+$ | $-70\,948.84(3)$ |
| $^{10}\text{Be}^{3+}$ | $-283\,865.0(2)$ |
| $^{12}\text{C}^{5+}$ | $-638\,857.4(5)$ |
| $^{16}\text{O}^{7+}$ | $-1\,136\,142.5(8)$ |
| $^{18}\text{O}^{7+}$ | $-1\,136\,141.9(8)$ |

and

$$g^{(16}\text{O}^{7+}) - g^{(18}\text{O}^{7+}) = -1.2 \cdot 10^{-9}.$$  \hspace{1cm} (15)

Equation (14) can be used to test QED calculations, while Eq. (15) is rather useful to check for experimental systematical errors. The experimental result for the ratio in Eq. (14) is

$$g^{(12}\text{C}^{5+})/g^{(16}\text{O}^{7+}) = 1.000\,497\,273\,1(15),$$  \hspace{1cm} (16)

where we take into account a possible correlation between systematic errors for the carbon experiment [6] and the oxygen measurement [8]. The experimental value is in a fair agreement with the theoretical prediction (14).

6. Muon-to-electron mass ratio

Several results of determination of the muon-to-electron mass ratio are collected in Table 3. The improvement of theory for the $g$ factors of a bound electron and muon in the muonium atom is not that important for present level of accuracy. However, a number of promising projects for intensive muon sources for experiments in particle physics are now under consideration [36]. The accurate theory of the $g$ factors in muonium will be useful for upcoming experiments.

7. Proton-to-electron mass ratio

Determination of the proton-to-electron mass ratio is summarized in Table 4 below. Presently the most accurate value of the electron-to-proton mass ratio comes from an experiment related to the $g$ factor of a bound electron. Theory and experiment contribute into uncertainty at the same level. Experiments with lighter ions might reduce the theoretical uncertainty substantially.

8. The fine structure constant

The mass ratios of electron-to-muon and electron-to-proton are considered above. Both are important for precision determination of the fine structure constant $\alpha$. A precision value of the latter is strongly needed for any accurate QED test.

Comparison of theory and experiment for the muonium hyperfine structure interval allows to determine $\alpha$ if a value of the muon mass is known in a proper units, e.g. in unit of the electron mass, or the muon magnetic moment in unit of the magnetic moment of electron or proton.

A promising method to determine the fine structure constant has been developing by Chu and coworkers [45]. It is based on the photon recoil spectroscopy and offers a precision value of $h/M_{\text{atom}}$. The experiment in Ref. [45] deals with cesium atoms, but some other atoms are also used in several
Table 3. Muon-to-electron mass ratio. The results marked with † differ from originally published. The result of Beltrami et al. is corrected accordingly to Ref. [4]. The result of Casperson et al. is presented accordingly to Ref. [24]. The results marked with † are found with taking into account fourth order corrections to the bound $g$ factors of electron and muon. The CODATA result is based on some result derived from $1s\ HFS\ Mu$ with some overoptimistic estimation of theoretical uncertainty. The result from $1s\ HFS\ Mu$ quoted here is calculated with estimation of theoretical uncertainty as explained in Ref. [3] and with use of $1/\alpha = 137.035\ 999\ 58(52)$ from anomalous magnetic moment of electron [25, 26].

| Method | $m_\mu/m_e$ | Ref. |
|--------|-------------|------|
| Precession of $\mu^+$ in water | 206.768 60(29) | Crow et al., 1972, [27] |
| Precession of $\mu^+$ in Br$_2$ | 206.768 35(11) | Klempt et al., 1982, [24] |
| $(g - 2)$ of $\mu^+$ | 206.771 4(21) | Bailey et al., 1977, [28] |
| $(g - 2)$ of $\mu^+$ | 206.766 8(20) | Bailey et al., 1977, [29] |
| $(g - 2)$ of $\mu^+$ | 206.768 70(27) | Brown et al., 2001, [30, 31] |
| $3d_{5/2} - 2p_{3/2}$ transitions in muonic $^{24}\text{Mg}$ and $^{28}\text{Si}$ | 206.767 94(64)† | Beltrami et al., 1986, [32] |
| Breit-Rabi for Mu | 206.768 18(54)† | Casperson et al., 1977, [33] |
| Breit-Rabi for Mu | 206.768 22(8)† | Mariam et al., 1982, [17] |
| Breit-Rabi for Mu | 206.768 283(25)† | Liu et al., 1999, [2] |
| Breit-Rabi for Mu | 206.768 283(11)† | Liu et al., 1999, [2] |
| $1s\ HFS\ Mu$ | 206.768 283(17) | Meyer et al., 2000, [34] |
| Adjustment | 206.768 265 7(63) | CODATA, 1998, [35] |

Table 4. Proton-to-electron mass ratio. The result for $^9\text{Be}^+$ differs from the original value in Ref. [37] because we have applied a more accurate theory from Ref. [38] and a more accurate value of the beryllium-9 mass [39]. We present two theoretical evaluation of experiment [6]. The value marked with † has been obtained in this work.

| Method | $m_p/m_e$ | Ref. |
|--------|-------------|------|
| $g$ factor in $^9\text{Be}^+$ | 1 836.152 92(29)† | Wineland et al., 1983, [37] |
| $g$ factor in $^{12}\text{C}^+$ | 1 836.152 673 3(14) | Häffner et al., 2000, [6] |
| $g$ factor in $^{12}\text{C}^+$ | 1 836.152 673 6(13)† | Häffner et al., 2000, [6] |
| Precession of $\ell$ and $p$ | 1 836.152 667 0(39) | Farnham et al., 1995, [9] |
| Precession of $\ell$ and $H_\ell^2$ | 1 836.152 680(88) | Gabrielse et al., 1990, [41] |
| Gross structure in H and D | 1 836.152 667(85) | de Beauvoir et al., 2000, [42], Huber et al., 1998, [43] |
| Antiprotonic He | 1 836.152 58(24) | Hori et al., 2001, [44] |
| Adjustment | 1 836.152 667 5(39) | CODATA, 1998, [35] |
experiments by other groups. In order to obtain precise value for $\alpha$, one can combine $\hbar/M_{\text{atom}}$ with a precision value of the Rydberg constant

$$R_{\infty} = \frac{\alpha^2 m_e c}{2 \hbar},$$

(17)

the cesium-to-proton mass [46] and the electron-to-proton mass ratio.

The presently most accurate value of the fine structure constants comes from study of anomalous magnetic moment of electron [25]. It used to be quoted as a QED value. However, the major uncertainty of this value of $\alpha$ comes from not perfect understanding of a motion of a light particle (i.e. electron) in a Penning trap [26]. It is important that the value of the electron-to-proton mass ratio from the bound $g$ factor is free of problem of an electron in Penning trap in contrast to the former best value of the ratio from University of Washington [26]. That makes a value of $\alpha$ from Chu experiment really independent from those from $g-2$ of electron. The recent preliminary result by Chu has a relative uncertainty as low as $7 \cdot 10^{-9}$ and some progress is still possible [45].

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