What internal set theory knows about standard sets

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Abstract

We characterize those models of ZFC which are embeddable, as the class of all standard sets, in a model of internal set theory IST.

Keywords: internal set theory, standard sets, extensions of ZFC.

Introduction

In the early 60s Abraham Robinson demonstrated that nonstandard models of natural and real numbers could be used to interpret the basic notions of analysis in the spirit of mathematics of the 17-th and 18-th century, i.e. including infinitesimal and infinitely large quantities.

Nonstandard analysis, the field of mathematics which has been initiated by Robinson’s idea, develops in two different versions.

The model theoretic version, following the original approach, interprets “nonstandard” notions via nonstandard models in the ZFC universe.

On the other hand, the axiomatic version more radically postulates that the whole universe of sets (including all mathematical objects) is arranged in a “nonstandard” way, so that it contains both the objects of conventional,

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“standard” mathematics, called standard, and objects of different nature, called nonstandard. The latter type includes infinitesimal and infinitely large numbers, among other rather unusual objects.

Each of the two versions has its collective of adherents who use it as a working tool to develop nonstandard mathematics.

The most of those who follow the axiomatic version use internal set theory IST of Nelson as the basic set theory. This is a theory in the language \( L_{\in, \text{st}} \) (that is the language containing the membership \( \in \) and the unary predicate of standardness \( \text{st} \) as the only atomic predicates) which includes all axioms of ZFC in the \( \in \)-language together with three principles that govern the interactions between standard (i.e. those sets \( x \) which satisfy \( \text{st} x \)) and nonstandard objects in the set universe. (See below.)

It is known that IST is an equiconsistent extension of ZFC. Moreover, IST is a conservative extension of ZFC, so that an \( \in \)-sentence \( \varphi \) is a theorem of ZFC iff \( \varphi^{\text{st}} \) is a theorem of IST, where \( \varphi^{\text{st}} \) is the formal relativization of \( \varphi \) to the class \( S = \{ x : \text{st} x \} \) of all standard sets. This result, due to Nelson, is sometimes considered as a reason to view IST as a syntactical tool of getting ZFC theorems often in a more convenient way than traditional tools of ZFC (= the “standard” mathematics) allow.

However working with IST one should be interested to know whether its axioms reflect some sort of mathematical reality. One could expect that the relations between ZFC and IST are similar to those between the real line and the complex plane, so that each model of ZFC could be embedded, as the class of all standard sets, in a model of IST. However this is not the case: we demonstrated in \([3]\) that the least \( \in \)-model of ZFC is not embeddable in a model of IST. This observation leads us to the question:

- which “standard” models (i.e. transitive \( \in \)-models) of ZFC can be embedded, as the class of all standard sets, in a model of IST?

Let ZFGC (ZF plus Global Choice) be the theory, in the language \( L_{\in, <} \) with the binary predicates \( \in \) and \( < \) as the only atomic predicates, containing all of ZFC (with the schemata of Separation and Collection, or Replacement, in \( L_{\in, <} \)), together with the axiom saying that \( < \) wellorders the universe in such a way that each initial segment is a set.

Suppose that \( M \) is a transitive set, ordered by a relation \( < \) so that \( \langle M; \in, < \rangle \) models ZFGC. A set \( T \subseteq M \) will be called innocuous for
If, for any sets $y \subseteq x \in M$ such that $y$ is definable in the structure $\langle M; \in, < \rangle$, we have $y \in M$. (Thus it is required that $T$ does not destroy Separation in $\langle M; \in, < \rangle$ — but it can destroy Collection.)

Note that every $\mathcal{L}_{\in,<}$-formula having sets in $M$ as parameters can be naturally considered as an element of $M$. Let $\text{Truth}^M_{\in,<}$ denote the set of all closed $\mathcal{L}_{\in,<}$-formulas true in $\langle M; \in, < \rangle$.

**Theorem 1** Let $M$ be a transitive $\in$-model of ZFC. Then the existence of a wellordering $<$ of $M$, such that $\langle M; \in, < \rangle$ models ZFGC and $\text{Truth}^M_{\in,<}$ is innocuous for $\langle M; \in, < \rangle$, is necessary and sufficient for $M$ to be embeddable, as the class of all standard sets, in a model of IST. [1]

This is the main result of the paper.

The proof of the sufficiency is a modification of the original construction of an IST model by Nelson [6]. The necessity is more interesting: it is somewhat surprising that IST “knows” that the standard universe is a model of ZFGC. On the other hand the involvement of the truth relation could be expected in view of the fact that IST provides a uniform truth definition for $\in$-formulas, see Theorem 2 below.

**What IST knows about standard sets?**

The theorem answers the question in the title as follows:

- IST “knows” about the standard universe that it can be wellordered by a relation $<$ which respects the ZFC schemata of Separation and Collection, and moreover, the truth relation for the universe endowed by $<$ does not destroy Separation.

This observation could perhaps lead to new insights in the philosophy of nonstandard mathematics.

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1. By formulas of $\mathcal{L}_{\in,<}$ (with parameters in $M$) plus $T$ as an extra predicate.
2. The IST-embeddable transitive models of ZFC can be characterized in different terms. Suppose that $\langle M; \in, < \rangle$ is a model of ZFGC. Let $X$ be a collection of subsets of $M$. Say that $X$ is innocuous for $\langle M; \in, < \rangle$ if we have $y \in M$ whenever $y \subseteq x \in M$ and $y$ is definable in the second order structure $\langle \langle M; \in, < \rangle; X \rangle$. Then, a transitive model $M \models \text{ZFC}$ is embeddable, as the class of all standard sets, in a model of IST, iff there is a wellordering $<$ of $M$ such that $\langle M; \in, < \rangle$ models ZFGC and the family $X$ of all sets $X \subseteq M$, definable in $\langle M; \in, < \rangle$, is innocuous for $\langle M; \in, < \rangle$.

The equivalence of this characterization and the one given by the theorem can be easily verified directly without a reference to the IST-embeddability.
It would be interesting to get similar results for other known nonstandard set theories, including those of Hrbacek [1, 4] and Kawaï [5].

1 Internal set theory

Internal set theory \textbf{IST} is a theory in the language $L_{\in, \text{st}}$ containing all axioms of \textbf{ZFC} (in the $\in$-language) and the following “principles”:

\textbf{Transfer}:
\[ \exists x \Phi(x) \implies \exists^{\text{st}} x \Phi(x) \]
— for any $\in$-formula $\Phi(x)$ with standard parameters;

\textbf{Idealization}:
\[ \forall^{\text{st fin}} A \exists x \forall a \in A \Phi(a, x) \iff \exists x \forall^{\text{st}} a \Phi(a, x) \]
— for any $\in$-formula $\Phi(x)$ with arbitrary parameters;

\textbf{Standardization}:
\[ \forall^{\text{st}} X \exists^{\text{st}} Y \forall^{\text{st}} x (x \in Y \iff x \in X \& \Phi(x)) \]
— for any \text{st-$\in$-formula} $\Phi(x)$ with arbitrary parameters.

The quantifiers $\exists^{\text{st}} x$ and $\forall^{\text{st}} x$ have the obvious meaning (there exists a standard set $x$ ...).

We shall systematically refer to different results in \textbf{IST} from [3, 2, 3]. In particular we shall use the following theorem of [4].

\textbf{Theorem 2} There is a $\text{st-$\in$-formula}$ $\tau(x)$ such that, for any $\in$-formula $\varphi(x_1, ..., x_n)$, it is a theorem of \textbf{IST} that
\[ \forall^{\text{st}} x_1 ... \forall^{\text{st}} x_n (\varphi^{\text{st}}(x_1, ..., x_n) \iff \tau(\overline{\varphi}(x_1, ..., x_n))) . \]

Here $\overline{\varphi}$ is the formula $\varphi$ considered as a finite sequence of (coded) symbols of the $\in$-language and sets which occur in $\varphi$ as parameters. Thus the $\in$-truth in $S$ can be expressed by a single \text{st-$\in$-formula} in \textbf{IST}.

2 The necessity

Let us fix a transitive model $S$ of \textbf{ZFC} which is the standard part of a model $I$ of \textbf{IST}. To set things precisely, both $S$ and $I$ are sets in the \textbf{ZFC} universe, $S$ is a transitive $\in$-model of \textbf{ZFC}, so that $\in_S = \in | S$, $S \subseteq I$, $I$ is a model of \textbf{IST}, $\in_S = \in_I | S$, but of course $\in_I \neq \in | I$.

Our aim is to prove that there is an ordering $<$ of $S$ such that $\langle S; \in, < \rangle$ models \textbf{ZFGC} and $\text{Truth}^S_{\in, <}$ is innocuous for $\langle S; \in, < \rangle$. 

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2.1 The forcing and generic structures

Let \( \Sigma \) be the class of all structures of the form \( \sigma = \langle X; < \rangle \), where \( X \in \mathcal{S} \) is transitive and has the form \( X = \mathcal{S}_\alpha = V_\alpha \cap \mathcal{S} \) for some ordinal \( \alpha \in \mathcal{S} \), and \( < \in \mathcal{S} \) is a wellordering of \( X \).

We say that \( \sigma' = \langle X'; <' \rangle \) extends \( \sigma = \langle X; < \rangle \), symbolically \( \sigma \preceq \sigma' \), if \( X \subseteq X' \) and \( <' \) is an end-extension of \( < \).

Define a relation \( \sigma \text{ forc } \Phi(x_1, \ldots, x_n) \), where \( \sigma = \langle X; < \rangle \in \Sigma \) while \( \Phi \) is a \( \mathcal{L}_{\in, <} \)-formula and \( x_1, \ldots, x_n \in X \), by induction on the complexity of \( \Phi \).

1. If \( \Phi \) is an elementary formula of \( \mathcal{L}_{\in, <} \), i.e. \( x < y, x = y, \) or \( x \in y \), then \( \sigma \text{ forc } \Phi \) iff \( \Phi \) is true in \( \sigma \).
2. \( \sigma \text{ forc } (\Phi \& \Psi) \) iff \( \sigma \text{ forc } \Phi \) and \( \sigma \text{ forc } \Psi \).
3. \( \sigma \text{ forc } (\neg \Phi) \) iff there does not exist \( \sigma' \in \Sigma \) extending \( \sigma \) such that \( \sigma' \text{ forc } \Phi \).
4. \( \sigma \text{ forc } \exists x \Phi(x) \) iff there is \( x \in X \) such that \( \sigma \text{ forc } \Phi(x) \).

For a \( \mathcal{L}_{\in, <} \)-formula \( \Phi \), a structure \( \sigma = \langle X; < \rangle \in \Sigma \) is called \( \Phi \)-complete iff, for any subformula \( \Psi(x_1, \ldots, x_n) \) of \( \Phi \) and all \( x_1, \ldots, x_n \in X \), we have \( \sigma \text{ forc } \Psi(x_1, \ldots, x_n) \) or \( \sigma \text{ forc } \neg \Psi(x_1, \ldots, x_n) \).

**Theorem 3 (IST)** If \( \Phi \) is a closed \( \mathcal{L}_{\in, <} \)-formula with sets in \( X \) as parameters, and \( \sigma = \langle X; < \rangle \in \Sigma \) is \( \Phi \)-complete, then \( \sigma \text{ forc } \Phi \) iff \( \sigma \models \Phi \).

**Proof.** By metamathematical induction on the complexity of \( \Phi \). \( \square \)

2.2 Increasing sequence of structures

We shall define an increasing sequence of structures \( \sigma_\gamma = \langle X_\gamma; <_\gamma \rangle \in \Sigma \), \( \gamma < \lambda \), such that \( \mathcal{S} = \bigcup_{\gamma < \lambda} X_\gamma \), hence the relation \( < = \bigcup_{\gamma < \lambda} <_\gamma \) wellorders \( \mathcal{S} \). The structures \( \sigma_\gamma \) will be rather “complete” (in the sense above); then \( < \) will not destroy Replacement by an elementary chain argument.

We face, however, a problem at limit steps: how to guarantee that the unions of \( <_\gamma \) still belong to \( \mathcal{S} \). Now \( \Pi \) enters the reasoning. It occurs that

\[ V_\alpha \text{ is the } \alpha \text{-th level of the von Neumann set hierarchy.} \]

\[ \text{Here and sometimes below any } \sigma = \langle X; < \rangle \in \Sigma \text{ is understood as } \langle X; \in, < \rangle \text{.} \]
the construction can be maintained in \( I \), so that, by the IST axiom of Standardization, the unions at limit steps will be still in \( S \! \).  

Fix sets \( D \) and \( \triangleleft \) in \( I \) such that the following holds in \( I \): \( S \subseteq D \) and \( \triangleleft \) is a (strict) wellordering of \( D \). Then \( \triangleleft \) may not be a wellordering of \( D \) from the point of view of the ZFC universe \( V \), but still \( \triangleleft \) wellorders any set \( S \in \mathcal{S} \) in \( V \) by Standardization and the fact that \( S \) is a transitive set.  

We say that a structure \( \sigma \in \Sigma \) is totally complete if it is \( \Phi \)-complete for any formula \( \Phi \) of \( L_{\in,<} \). The construction depends on the frequency of totally complete structures in \( \Sigma \).  

**Case 1**: each \( \sigma \in \Sigma \) can be extended to a totally complete \( \sigma' \in \Sigma \).  

Define a sequence of structures \( \sigma_\gamma = \langle X_\gamma; <_\gamma \rangle \in \Sigma \ (\gamma < \lambda) \) such that \( X_\delta = \bigcup_{\gamma < \delta} X_\gamma \) and \( <_\delta = \bigcup_{\gamma < \delta} <_\gamma \) for all limit ordinals \( \delta < \lambda \), and \( \sigma_{\gamma+1} \) is the \( \triangleleft \)-least totally complete structure in \( \Sigma \) which properly extends \( \sigma_\gamma \).  

Let \( \lambda \) be the largest ordinal such that \( \sigma_\gamma \) is defined (and belongs to \( \Sigma \), hence to \( \mathcal{S} \)) for all \( \gamma < \lambda \); clearly \( \lambda \leq \text{"the least ordinal not in } S \text{"} \).  

**Case 2**: otherwise.  

Fix a recursive enumeration \( \{ \Phi_n : n \in \omega \} \) of all formulas of \( L_{\in,<} \). A structure \( \sigma \in \Sigma \) will be called \( n \)-complete if it is \( \Phi_k \)-complete for any \( k \leq n \).  

We set \( \lambda = \omega \) in this case, pick a structure \( \sigma_0 \in \Sigma \) not extendable to a totally complete structure, and define a sequence of structures \( \sigma_n = \langle X_n; <_n \rangle \in \Sigma \) such that, for any \( n \in \omega \), \( \sigma_{n+1} \) is the \( \triangleleft \)-least \( n \)-complete structure in \( \Sigma \) which properly extends \( \sigma_n \).  

In each of the two cases \( \langle \sigma_\gamma : \gamma < \lambda \rangle \) is a sequence of elements of \( \mathcal{S} \). It can hardly be expected that the sequence is \( \in \)-definable in \( \mathcal{S} \) as the construction refers to notions which involve the \( \in \)-truth relation for \( \mathcal{S} \). But the following holds:

**Proposition 4** The sequence \( \langle \sigma_\gamma : \gamma < \lambda \rangle \) is st-\( \in \)-definable in \( I \).

**Proof.** Apply Theorem 2. \( \square \)

### 2.3 The order

First of all we prove

**Lemma 5** \( \lambda \) is a limit ordinal and \( \bigcup_{\gamma < \lambda} X_\gamma = \mathcal{S} \).
Proof. Recall that $\lambda = \omega$ in Case 2. If $\lambda = \gamma + 1$ in Case 1 then, by the assumption of Case 1, we would be able to define $\sigma_\lambda$. Hence $\lambda$ is a limit ordinal and the relation $< = \bigcup_{\gamma < \lambda} <_\gamma$ is a wellordering of $X$.

Suppose that $X = \bigcup_{\gamma < \lambda} X_\gamma \neq S$. Then $X \in S$ as any of $X_\gamma$ has the form $S_\alpha = V_\alpha \cap S$ for some $\alpha$. Now $<$ belongs to $S$ by Standardization, being $st$-$\in$-definable in $I$ by Proposition \[.\] It follows that $\sigma = \langle X; < \rangle \in \Sigma$. Moreover $\sigma$ is totally complete. (As the limit of an increasing sequence of totally complete structures in Case 1, and by similar reasons in Case 2.) This immediately contradicts the choice of $\sigma_0$ in Case 2, while, in Case 1, adds an extra term to the sequence, which contradicts the choice of $\lambda$. \[ \square \]

It follows that $< = \bigcup_{\gamma < \lambda} <_\gamma$ is a wellordering of $S$.

Corollary 6 $\langle S; \in, < \rangle$ is a model of ZFGC.

Proof. To see that $S$ satisfies Separation in the language $L_{\in, <}$ note that $<$ is $st$-$\in$-definable in $I$ by Proposition \[ and apply Standardization in $I$. Now consider Collection. Suppose that $p, X \in S$ and $\Phi(x,y,p)$ is a $L_{\in, <}$-formula. We have to find $Y \in S$ such that the following is true in $S$:

$$\forall x \in X \left[ \exists y \Phi(x,y,p) \implies \exists y \in Y \Phi(x,y,p) \right].$$

In both Case 1 and Case 2, there is $\gamma < \lambda$ such that $p, X \in X_\gamma$ and $\sigma_\gamma$ is $(\exists y \Phi(x,y,p))$-complete. Prove that $Y = X_\gamma$ is as required.

Consider $x \in X$, hence $\in X_\gamma$. Suppose that there is $y \in S$ such that $\Phi(x,y,p)$ holds in $S$, and prove that such a set $y$ exists in $X_\gamma$.

It follows from Lemma \[ that $S$ is the union of an increasing chain of $(\exists y \Phi(x,y,p))$-complete structures. Therefore, by Theorem \[ and an ordinary model-theoretic argument, $\langle S; \in, < \rangle$ is an elementary extension of $\langle X_\gamma; \in, <_\gamma \rangle$ with respect to the formula $\exists y \Phi(x,y,p)$ and all its subformulas. This proves the existence of $y$ in $X_\gamma$. \[ \square \]

2.4 The set of true formulas is innocuous

Let $T = \text{Truth}^S_{L_{\in, <}}$ be the set of all closed $L_{\in, <}$-formulas (with sets in $S$ as parameters) true in the model $\langle S; \in, < \rangle$. The next lemma completes the proof of the necessity part in Theorem \[.\]
Lemma 7  $T$ is innocuous for $\langle S; \in, < \rangle$.

Proof. It suffices to check that $T$ is $\text{st}-\in$-definable in $\mathbb{I}$. (Then the result follows by Standardization in $\mathbb{I}$ as above.)

Let $\Phi(p_1, ..., p_k)$ be a closed $L_{\in, <}$-formula with parameters $p_1, ..., p_k \in S$. Let $n$ be the number of $\Phi(x_1, ..., x_k)$ (see Case 2 in Subsection 2.2). Take the least $\gamma < \lambda$ such that $p_1, ..., p_k \in X_\gamma$ and, in Case 2, $\gamma \geq n$. Arguing as in the proof of Corollary 4, we conclude that $\sigma_\gamma$ is an elementary substructure of $\langle S; \in, < \rangle$ with respect to $\Phi$, in particular $\Phi(p_1, ..., p_k)$ is either true or false simultaneously in both $\sigma_\gamma$ and $\langle S; \in, < \rangle$. It remains to recall that the sequence of structures $\sigma_\gamma$ is $\text{st}-\in$-definable in $\mathbb{I}$ by Proposition 4. $\blacksquare$

3  The sufficiency

This section proves the sufficiency part in Theorem 1. We start with a transitive set $S$ and a wellordering $<$ of $S$ such that $\langle S; \in, < \rangle \models \text{ZFC}$, and suppose that the set $T = \text{Truth}^S_{\in, <}$ of all closed $L_{\in, <}$-formulas (with parameters in $S$), true in $\langle S; \in, < \rangle$, is innocuous for $\langle S; \in, < \rangle$. The aim is to embed $S$, as the class of all standard sets, in a model $\mathbb{I}$ of $\text{IST}$.

3.1  The ultrafilter

To obtain $\mathbb{I}$ we shall use the construction of an adequate ultrapower of Nelson [3], modified by Kanovei [4].

Let $\text{Def}_{\in, <}(S)$ denote the collection of all sets $X \subseteq S$ definable in $\langle S; \in, < \rangle$ by a formula of $L_{\in, <}$ containing sets in $S$ as parameters.

Let $I = \mathcal{P}_\text{fin}(S) = \{ i \subseteq S : i \text{ is finite} \}$. This is a proper class in $S$. Let $\mathcal{A}$ be the algebra of all sets $X \subseteq I$ which belong to $\text{Def}_{\in, <}(S)$.

Proposition 8  There exists an ultrafilter $U \subseteq \mathcal{A}$ satisfying

(A) if $a \in S$ then the set $\{ i \in I : a \in i \}$ belongs to $U$;

(B) if $P \subseteq S \times I$, $P \in \text{Def}_{\in, <}(S)$, then the following set is in $\text{Def}_{\in, <}(S)$:

$$\{ x \in S : \text{the cross-section } P_x = \{ i : \langle x, i \rangle \in P \} \text{ belongs to } U \}$$.
(C) there is a set \( U \subseteq S \), definable in the structure \( \langle S; \in, <, T \rangle \), such that \( U = \{ U_x : x \in S \} \), where \( U_x = \{ i \in I : \langle x, i \rangle \in U \} \) for all \( x \).

**Proof.** Step 0. Let \( U_0 \) be the collection of all sets of the form

\[ I_{a_1,...,a_m} = \{ i \in I : a_1,...,a_m \in i \} \text{, where } a_1,...,a_m \in S. \]

The family \( U_0 \) obviously satisfies FIP (the finite intersection property).

Step \( n + 1 \). Suppose that a FIP family \( U_n \) of subsets of \( I \) has been constructed. Denote by \( \chi_n(x,i) \) the \( n \)-th formula in a recursive enumeration, fixed beforehand, of all \( L_{\in,<} \)-formulas with exactly two free variables.

We define \( U_{n+1} = U_n \cup \{ B_x : x \in S \} \), where \( B_x \) is equal to the set \( A_x = \{ i \in I : \langle S; \in, < \rangle \models \chi_n(x,i) \} \) whenever the family \( U_n \cup \{ B_y : y < x \} \cup A_x \) still satisfies FIP, and \( B_x = I \setminus A_x \) otherwise.

Clearly \( U = \bigcup U_n \) as is required. \([C]\) follows from the fact that the whole construction can be carried out in \( \langle S; \in, <, T \rangle \).

Let us fix such an ultrafilter \( U \subseteq A \).

Let \( U \models \Phi(i) \) mean: “the set \( \{ i \in I : \langle S; \in, < \rangle \models \Phi(i) \} \) belongs to \( U \).” (The quantifier: there exist \( U \)-many.) Then, by the choice of \( U \), we have \( U \models (a \in i) \) for any \( a \in S \), and, given a relation \( P(i,...) \) in \( \text{Def}_{\in,<}(S) \), the relation \( U \models P(i,...) \) belongs to \( \text{Def}_{\in,<}(S) \) as well.

### 3.2 The model

For \( r \geq 1 \), we let \( I^r = I \times \ldots \times I \) (\( r \) times \( I \)), and

\[ F_r = \{ f \in \text{Def}_{\in,<}(S) : f \text{ maps } I^r \text{ to } S \}. \]

Let separately \( I^0 = \{ 0 \} \) and \( F_0 = \{ \{ \langle 0, x \rangle \} : x \in S \} \).

We finally put \( F_\infty = \bigcup_{r \in \omega} F_r \), and, for \( f \in F_\infty \), let \( r(f) \) be the only \( r \) such that \( f \in F_r \).

Suppose that \( f \in F_\infty \), \( q \geq r = r(f) \), and \( i = \langle i_1,...,i_r,...,i_q \rangle \in I^q \). Then we set \( f[i] = f(\langle i_1,...,i_r \rangle) \). In particular \( f[i] = f(i) \) whenever \( r = q \).

Separately we put \( f[i] = x \) for any \( i \) whenever \( f = \{ \langle 0, x \rangle \} \in S_0 \).

Let \( f, g \in F_\infty \) and \( r = \max\{ r(f), r(g) \} \). Define

\[ f^* = g \iff U i_r U i_{r-1} \ldots U i_1 (f[i] = g[i]), \]

and
where $\mathbf{i}$ denotes $\langle i_1, \ldots, i_r \rangle$, and define $f \ast g$ similarly. (Note the order of quantifiers.) The following is a routine statement.

**Proposition 9** $\ast = \mathcal{R}$ is an equivalence relation on $F_\infty$. The relation $\ast \in$ on $F_\infty$ is $\ast -$invariant in each of the two arguments. $\Box$

Define $[f] = \{ g \in F_\infty : f \ast g \}$, let $\ast \mathcal{S} = \{ [f] : f \in F_\infty \}$ (the quotient). For $[f], [g] \in \ast \mathcal{S}$, define $[f] \ast [g]$ iff $f \ast g$. (This is independent of the choice of representatives by the proposition.)

For any $x \in \mathcal{S}$, define $\ast x = \{ \langle 0, x \rangle \}$, the image of $x$ in $\ast \mathcal{S}$.

We finally define $\mathbf{st}[f]$ iff $[f] = \ast x$ for some $x \in \mathcal{S}$.

**Theorem 10** $\langle \ast \mathcal{S}; \ast \in, \mathbf{st} \rangle$ is a model of IST. The map $x \mapsto \ast x$ is a 1-1 $\in$-embedding of $\mathcal{S}$ onto the class of all standard elements of $\ast \mathcal{S}$.

The theorem immediately implies the sufficiency part in Theorem 4.

**Proof.** We begin with an appropriate formalism. Let $\Phi(f_1, \ldots, f_m)$ be an $\in$-formula with functions $f_1, \ldots, f_m \in F$ as parameters. Put $r(\Phi) = \max\{r(f_1), \ldots, r(f_m)\}$. If $r \le q$ and $\mathbf{i} \in I^q$ then let $\Phi[\mathbf{i}]$ denote the formula $\Phi(f_1[\mathbf{i}], \ldots, f_m[\mathbf{i}])$ (an $\in$-formula with parameters in $\mathcal{S}$). Let finally $[\Phi]$ denote $\Phi(f_1, \ldots, f_m)$, which is an $\in$-formula with parameters in $\ast \mathcal{S}$.

**Proposition 11** (Loš) Let $\Phi = \Phi(f_1, \ldots, f_m)$ be an $\in$-formula with functions $f_1, \ldots, f_m \in F$ as parameters, and $r = r(\Phi)$. Then

$$[\Phi] \text{ holds in } \ast \mathcal{S} \iff U i_r U i_{r-1} \ldots U i_1 (\Phi[\mathbf{i}] \text{ holds in } \mathcal{S}).$$

**Proof.** ( $\mathbf{i}$ denotes $\langle i_1, \ldots, i_r \rangle$ in the displayed line.) The only detail one needs to note is that, since the index set $I$ is a proper class in $\mathcal{S}$, we need the global choice to carry out the ordinary argument. This is why $\mathcal{S}$ needs to be a model of ZFGC, not merely ZFC. $\Box$ (Proposition 7)

Using functions in $F_0$, we immediately conclude that the map $x \mapsto \ast x$ is an $\in$-elementary 1–1 embedding of $\mathcal{S}$ onto the class of all standard sets in $\ast \mathcal{S}$, which implies both Transfer and all of ZFC axioms in $\langle \ast \mathcal{S}; \ast \in, \mathbf{st} \rangle$. It remains to check Idealization and Standardization.
Idealization. Let $\Phi(a,x)$ be an $\in$-formula with two free variables, $a$ and $x$, and some functions in $F$ as parameters. We have to demonstrate

$$\forall^{\text{stfin}} A \exists x \forall a \in A [\Phi](a,x) \implies \exists x \forall^{\text{st}} a [\Phi](a,x)$$

in $^*S$. (It is known that the implication $\iff$ here is a corollary of other axioms of IST.) The left-hand side of (†) implies, by Proposition 11,

$$\forall_{\text{finite}} A \subseteq S U i_r U i_{r_1} ... U i_1 \exists x \forall a \in A \Phi[\langle i_1, ..., i_r \rangle](a,x)$$

in $S$, where $r = r(\Phi)$. To simplify the formula note that the leftmost quantifier is a quantifier over $I$ and define a function $\alpha \in F_{r+1}$ by $\alpha(i_1, ..., i_r, i) = i$. The last displayed formula takes the form

$$\forall i \in I U i_r U i_{r_1} ... U i_1 \exists x \forall a \in \alpha \Phi[\langle i_1, ..., i_r, i \rangle](a,x),$$

which implies $\exists x \forall a \in [\alpha] [\Phi](a,x)$ in $^*S$ by Proposition 11. Now, by the definition of the predicate $\text{st}$ in $^*S$, it suffices to check that $^*x \in [\alpha]$ in $^*S$ for any $x \in S$. This is equivalent to $U i U i_r ... U i_1 (x \in i)$, which holds by the choice of $U$.

Standardization. Recall that $U$ is definable in the structure $\langle S; \in, <, T \rangle$ by $[\text{C}]$ of Proposition 8. Therefore the model $\langle ^*S; ^*\in, \text{st} \rangle$ is definable in $\langle S; \in, <, T \rangle$ as well. Thus we have only to check that, given $x \in S$, any set $y \subseteq x$, which is definable in $\langle S; \in, <, T \rangle$, belongs to $S$. But this follows from the fact that $T$ is innocuous for $\langle S; \in, < \rangle$.

$\square$ (Theorems 10 and 1)

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