Study of Solutions for a quasilinear Elliptic Problem With negative exponents

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Abstract: The authors of this paper deal with the existence and regularities of weak solutions to the homogenous Dirichlet boundary value problem for the equation $-\text{div}(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = \frac{f(x)}{u^\alpha}$, $x \in \Omega$, $u = 0$, $x \in \partial \Omega$. The authors apply the method of regularization and Leray-Schauder fixed point theorem as well as a necessary compactness argument to prove the existence of solutions and then obtain some maximum norm estimates by constructing three suitable iterative sequences. Furthermore, we find that the critical exponent of $m$ in $\|f\|_{L^m(\Omega)}$. That is, when $m$ lies in different intervals, the solutions of the problem mentioned belongs to different Sobolev spaces. Besides, we prove that the solution of this problem is not in $W^{1,p}_0(\Omega)$ when $\alpha > 2$, while the solution of this problem is in $W^{1,p}_0(\Omega)$ when $1 < \alpha < 2$.

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1 Introduction

In this paper, we study existence and regularity of solutions for the following quasi-linear elliptic problem

\begin{equation}
\begin{cases}
-\text{div}(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = \frac{f(x)}{u^\alpha}, & x \in \Omega, \\
u = 0, & x \in \partial \Omega,
\end{cases}
\end{equation}

where $\Omega$ is a bounded open domain in $\mathbb{R}^N(N \geq 1)$ with smooth boundary $\partial \Omega$. $f \geq 0$, $f \not\equiv 0$, $f \in L^1(\Omega)$, $p > 1$, $\alpha > 0$.

Model (1.1) may describe many physical phenomena such as non-Newtonian flows in porous media, chemical heterogeneous catalysts, nonlinear heat equations etc. [1] [2] [3] [4]. When $p = 2$, many authors have studied this problem. In 1991, Lazer and Mckenna [5] dealt with the case when $f$ was a continuous function. They proved that the solution belonged to $W^{1,2}_0(\Omega)$ if and only if $\alpha < 3$, while it was not in $C^2(\Omega)$ if $\alpha > 1$. Later, Lair, Shaker, Zhang, and Cheng generalized this results [6] [7] [8]. Moreover, Boccardo and Orsina in [9] discussed how the summability of $f$ and the values of $\alpha$ affected the existence, regularity and nonexistence of solutions. For more results, the interested readers may refer to [10] [11]. In the case when $p \neq 2$, Giacomoni, Schindler and Takáč in [12] applied upper-lower solution method and a mountain pass theorem to prove that this problem had multiple weak solutions. And then, the authors in [13] not only improved the results in [12] but also obtained that the solution was not in $W^{1,p}_0(\Omega)$ if $\alpha > \frac{2p-1}{p-1}$. For more properties of solutions, we may refer to [14] [15]. We point out that the first eigenfunction of the $p – \text{Laplacian}$ operator with homogenous boundary value problem

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plays a major role in all of the papers mentioned above. Besides, Boccardo and Orsina claimed
that the results such as Lemma 3.3, 4.3 and 5.5 of [9] may be generalized to the case when
the linear differential operator was replaced by a monotone differential operator, for example,
$p - $ Laplacian operator. We find that the singular problem involving $p - $ Laplacian operator is
more complicated. Especially, whether or not the problem has a solution in $W_0^{1,p}(\Omega)$ depends
on the relations of $\alpha, p, N$ as well as the summability of $f$. In this paper, we discuss separately
the properties of solutions to the problem when $\alpha = 1, > 1$ and $0 < \alpha < 1$. First, we apply
the method of regularization and Leray-Schauder fixed point theorem as well as a necessary
compactness argument to overcome some difficulties arising from the nonlinearity of the differ-
ential operator and the singularity of the nonlinear terms and then obtain existence of solutions.
Moreover, some maximum norm estimates are obtained by constructing three suitable iterative
sequences. Secondly, we consider the case $0 < \alpha < 1$. By means of the maximum principle, Hopf
Lemma and a partition of unity argument, we prove that the solution of the following problem
\begin{align*}
\begin{cases}
-\text{div}(|\nabla u_1|^{p-2}\nabla u_1) + |u_1|^{p-2}u_1 = \min\{f(x), 1\}, & x \in \Omega, \\
u_1 = 0, & x \in \partial \Omega,
\end{cases}
\end{align*}
(1.2)
satisfies $\int_{\Omega} |u_1|^{-r}dx < \infty, \ \forall \ r < 1$. And then applying this result, we find an interesting
phenomenon:

1) If $f(x) \in L^m(\Omega)$ for $m > m^* = \frac{Np}{Np - (1 - \alpha)(N - p)} > 1$, then Problem (1.1) has a
solution in $W_0^{1,p}(\Omega)$;

2) If $f(x) \in L^m(\Omega)$ for $1 < m < m^*$, then Problem (1.1) has a solution in $W_0^{1,q}(\Omega)$ $q < p$.

In other words, the value of $m$ in $L^m(\Omega)$ norm of $f(x)$ determines whether or not Problem
(1.1) has a solution in $W_0^{1,p}(\Omega)$. In the case of $\alpha > 1$, we prove that this problem does not
have a solution in $W_0^{1,p}(\Omega)$ when $\alpha > 2$, where the function $f(x)$ is permitted not to be strictly
positive on $\Omega$, our result is more general than that of [13]. Furthermore, we apply the result
$\int_{\Omega} |u_1|^{-r}dx < \infty, \ \forall \ r < 1$ to prove that Problem (1.1) has a unique solution in $W_0^{1,p}(\Omega)$ in the
case when $1 < \alpha < 2$.

2 The case when $\alpha = 1$

In this section, we apply the method of regularization and Leray-Schauder fixed point the-
orem to prove the existence of solutions. Besides, we obtain $L^\infty$ norm estimates by Morse
iteration technique.

In order to prove the main results of this section, we consider the following auxiliary problem
\begin{align*}
\begin{cases}
-\text{div}(|\nabla u_n|^{p-2}\nabla u_n) + |u_n|^{p-2}u_n = \frac{f_n(x)}{(|u_n| + \frac{1}{4})^\alpha}, & x \in \Omega, \\
u_n = 0, & x \in \partial \Omega,
\end{cases}
\end{align*}
(2.1)
where $f_n = \min\{f(x), n\}$.

**Lemma 2.1.** Problem (2.1) has a unique nonnegative solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ for
any fixed $n \in N^*$. 
Proof. Let $n \in \mathbb{N}$ be fixed. For any $w \in L^p(\Omega)$, $\sigma \in [0,1]$, we get that the following problem has a unique solution $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ by applying a variational method

\[
\begin{cases}
-\text{div}(|\nabla v|^{p-2}\nabla v) + |v|^{p-2}v = \frac{\sigma f_n(x)}{(|w|+\frac{1}{n})^\alpha}, & x \in \Omega, \\
v = 0, & x \in \partial\Omega.
\end{cases}
\]  
\tag{2.2}

So, for any $w \in L^p(\Omega)$, $\sigma \in [0,1]$, we may define the map $\Gamma : [0,1] \times L^p(\Omega) \to L^p(\Omega)$ as $\Gamma(\sigma, w) = v$. It is easy to see that $\Gamma(0,w) = 0$. For all $v \in L^p(\Omega)$ satisfying $\Gamma(\sigma,v) = v$, we have $\|v\|_{L^p(\Omega)} \leq C$, where the positive constant $C$ depends on $n$. In fact, multiplying the first identity in (2.2) by $v$, and integrating over $\Omega$, we have

\[
\int_\Omega |\nabla v|^p \,dx + \int_\Omega |v|^p \,dx = \int_\Omega \frac{\sigma f_n}{(|w|+\frac{1}{n})^\alpha} v \,dx \leq n^{\alpha+1} \int_\Omega |v| \,dx.
\]

Applying the embedding theorem $W^{1,p}(\Omega) \hookrightarrow L^1(\Omega)$, we obtain

\[
\|v\|_{W^{1,p}} \leq Cn^{\alpha+1} \|v\|_{W^{1,p}},
\]

which implies

\[
\|v\|_{W^{1,p}} \leq Cn^{\frac{\alpha+1}{p}}.
\]

Due to the embedding $W^{1,p}(\Omega) \overset{\text{compact}}{\hookrightarrow} L^p(\Omega)$, we get that the map $\Gamma$ is a compact operator and $\|v\|_{L^p(\Omega)} \leq C(n)$. Then by Leray-Schauder’s fixed point theorem, we know that there exists a $u_n \in W_0^{1,p}(\Omega)$ such that $u_n = \Gamma(1,u_n)$, i.e. Problem (2.1) has a solution. Noting that

\[
\frac{f_n}{(|u_n|+\frac{1}{n})^\alpha} \geq 0, \quad \text{the maximum principle in [16, 17] shows that } u_n \geq 0, \quad u_n \in L^\infty(\Omega).
\]

Lemma 2.2. The sequence $u_n$ is increasing with respect to $n$. $u_n > 0$ in $\Omega'$ for any $\Omega' \subset \subset \Omega$, and there exists a positive constant $C_{\Omega'}$ (independent of $n$) such that for all $n \in \mathbb{N}^*$

\[
u_n \geq C_{\Omega'} > 0, \quad \text{for every } x \in \Omega'.
\]  
\tag{2.3}

Proof. Choosing $(u_n - u_{n+1})_+ = \max\{u_n - u_{n+1}, 0\}$ as a test function, observing that

\[
|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_{n+1}|^{p-2}\nabla u_{n+1})(\nabla u_n - \nabla u_{n+1})_+ \geq 0,
\]

\[
[(u_{n+1} + \frac{1}{n+1})^\alpha - (u_n + \frac{1}{n})^\alpha](u_n - u_{n+1})_+ \leq 0, \quad \text{for every } \alpha > 0,
\]

\[
0 \leq f_n \leq f_{n+1},
\]

we get

\[
0 \leq \int_\Omega |\nabla (u_n - u_{n+1})_+|^p \,dx \leq 0.
\]

This inequality yields $(u_n - u_{n+1})_+ = 0$ a.e. in $\Omega$, that is $u_n \leq u_{n+1}$ for every $n \in \mathbb{N}^*$. Since the sequence $u_n$ is increasing with respect to $n$, we only need to prove that $u_1$ satisfies Inequality (2.3). According to Lemma 2.1, we know that there exists a positive constant $C$ (depending only on $|\Omega|, N, p$) such that $\|u_1\|_{L^\infty(\Omega)} \leq C\|f_1\|_{L^\infty(\Omega)} \leq C$, then

\[-\text{div}(|\nabla u_1|^{p-2}\nabla u_1) + |u_1|^{p-2}u_1 = \frac{f_1}{(u_1 + 1)\alpha} \geq \frac{f_1}{(C+1)^\alpha}.\]

Noting that $\frac{f_1}{(C+1)^\alpha} > 0$, $\frac{f_1}{(C+1)^\alpha} \neq 0$, the strong maximum principle implies that $u_1 > 0$ in $\Omega$, i.e. Inequality (2.3) holds. \qed
Our main results are the following

**Theorem 2.1.** Let $f$ be a nonnegative function in $L^1(\Omega)$ ($f \neq 0$). Then Problem (1.1) has a solution $u \in W_{0}^{1,p}(\Omega)$ satisfying

$$
\int_{\Omega} |\nabla u|^{p-2}\nabla u \nabla \varphi dx + \int_{\Omega} |u|^{p-2}u \varphi dx = \int_{\Omega} \frac{f \varphi}{u} dx, \quad \forall \varphi \in C_{0}^{\infty}(\Omega).
$$

Moreover, suppose that $f \in L^m(\Omega)$ ($m \geq 1$), then the solution $u$ of Problem (1.1) satisfies the following properties

(i) If $m > \frac{N}{p}$ and $2 - \frac{2}{m} < p < N$, then $u \in L^\infty(\Omega)$.

(ii) If $1 \leq m < \frac{N}{p}$, then $u \in L^s(\Omega)$, for $1 < s \leq \frac{pNm}{N - pm}$.

**Proof.** Part1 (Existence). Multiplying the first identity in Problem (2.1) by $u_n$ and integrating over $\Omega$, we get

$$
\int_{\Omega} |\nabla u_n|^p dx + \int_{\Omega} |u_n|^p dx = \int_{\Omega} \frac{f_n u_n}{u_n + \frac{1}{n}} dx \leq \int_{\Omega} |f_n| dx \leq \int_{\Omega} |f| dx,
$$

i.e. $\|u_n\|_{W_{0}^{1,p}(\Omega)} \leq \|f\|_{L^1(\Omega)}^\frac{1}{p}$. Then we know that there exist $\tilde{u} \in W^{1,p}(\Omega)$ and $\tilde{V} \in L^\frac{p}{p-1}(\Omega, \mathbb{R}^N)$ such that

$$
\begin{cases}
  u_n \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega) \\
  u_n \to u \text{ a.e. in } \Omega \\
  |\nabla u_n|^{p-2}\nabla u_n \rightharpoonup \tilde{V} \text{ weakly in } L^\frac{p}{p-1}(\Omega, \mathbb{R}^N).
\end{cases}
$$

For every $\varphi \in C_{0}^{\infty}(\Omega)$, we get from Inequality (2.3) that

$$
0 \leq \left| \frac{f_n \varphi}{u_n + \frac{1}{n}} \right| \leq \frac{\|\varphi\|_{L^\infty(\Omega')}}{C_{\Omega'}} \frac{f_n}{x},
$$

where $\Omega' = \{x : \varphi \neq 0\}$. Then applying Lebesgue Dominated Convergence Theorem, one has that

$$
\lim_{n \to +\infty} \int_{\Omega} \frac{f_n \varphi}{u_n + \frac{1}{n}} dx = \int_{\Omega} \frac{f \varphi}{u} dx.
$$

Since $u_n$ satisfies the following identity

$$
\int_{\Omega} |\nabla u_n|^{p-2}\nabla u_n \nabla \varphi dx + \int_{\Omega} |u_n|^{p-2}u_n \varphi dx = \int_{\Omega} \frac{f_n \varphi}{u_n + \frac{1}{n}} dx, \quad \forall \varphi \in C_{0}^{\infty}(\Omega).
$$

Combining (2.4) – (2.6), we have

$$
\int_{\Omega} \tilde{V} \nabla \varphi dx + \int_{\Omega} |u|^{p-2}u \varphi dx = \int_{\Omega} \frac{f \varphi}{u} dx, \quad \forall \varphi \in C_{0}^{\infty}(\Omega).
$$

Next, we shall prove $\tilde{V} = |\nabla u|^p \nabla u$, a.e. in $\Omega$. Since $C_{0}^{\infty}(\Omega)$ is dense in $W^{1,p}_{0}(\Omega)$, we may choose $\xi$ with $\xi \in C_{0}^{\infty}(\Omega)$ as a test function to get

$$
\int_{\Omega} |\nabla u_n|^{p-2}\nabla u_n \nabla (u_n - \xi) dx + \int_{\Omega} |u_n|^{p-2}u_n (u_n - \xi) dx = \int_{\Omega} \frac{f_n (u_n - \xi)}{u_n + \frac{1}{n}} dx.
$$
Noting that \(|\nabla u_n|^p - 2
abla u_n - |\nabla \xi|^p - 2 \nabla \xi| (\nabla u_n - \nabla \xi) \geq 0\), we obtain that
\[
\int_{\Omega} |\nabla \xi|^p - 2 \nabla \xi \nabla (u_n - \xi) dx + \int_{\Omega} |u_n|^p - 2 u_n (u_n - \xi) dx \leq \int_{\Omega} f_n (u_n - \xi) u_n + \frac{1}{n} dx. \tag{2.9}
\]

Letting \(n \to \infty\) in (2.9) and using Identity (2.7), one get
\[
\int_{\Omega} (|\nabla \xi|^p - 2 \nabla \xi - \tilde{V}) \nabla (u - \xi) dx \leq 0, \quad \forall \xi \in C_0^\infty (\Omega). \tag{2.10}
\]

Choosing \(\xi = u \pm \varepsilon \varphi\) with \(\varphi \in C_0^\infty (\Omega)\) and letting \(\varepsilon \to 0^+\), we have \(\tilde{V} = |\nabla u|^p - 2 \nabla u\), a.e. in \(\Omega\). This proves that \(u\) is a weak solution of Problem (1.1).

**Part 2** (Regularity). (i) choosing \(\beta_k^m u_n (\beta_k^m \geq m')\) as a test function in (2.6), we obtain
\[
(\frac{\beta_k^m}{m'} + 1) \int_{\Omega} |\nabla u_n|^p \beta_k^m \frac{\partial}{\partial m'} dx + \int_{\Omega} \frac{\beta_k^m}{m'} dx = \int_{\Omega} f_n (x) \frac{\partial}{\partial m'} dx \\
\leq \int_{\Omega} f (x) u_n \frac{\partial}{\partial m'} \leq \|f\|_{L^m (\Omega)} \|u_n\|_{L^{\beta_k^m}} \tag{2.11}
\]
Define two sequences
\[
\beta_{k+1} = \frac{\beta_k^m}{m'} + 1, \quad \beta_k^* = \beta_k + pm', \quad \beta_1 = pm', \quad p^* = \frac{N_p}{N - p}, \quad m' = \frac{m}{m - 1}.
\]
According to the condition \(m > \frac{N_p}{N - p}, \quad 2 - \frac{2}{m} < p \Rightarrow 2 < \beta_1 < p^*\) and applying Sobolev embedding theorem \(W_0^1, p (\Omega) \hookrightarrow L^q (\Omega)\) with \(1 \leq q \leq p^*\), we get
\[
\|u_n\|_{L^1 (\Omega)} = \|u_n\|_{L^{pm'} (\Omega)} \leq C\|u_n\|_{W_0^1, p (\Omega)} \leq C\|f\|_{L^1 (\Omega)} \tag{2.12}
\]
Once again, applying Sobolev embedding theorem \(W_0^1, p (\Omega) \hookrightarrow L^{p^*} (\Omega)\), we get
\[
\mu^{-p} \left( \frac{pm'}{\beta_k^*} \right)^p \|u_n\|_{L^{p^*} (\Omega)} \leq \left( \frac{pm'}{\beta_k^*} \right)^p \|\nabla u_n\|_{L^{p} (\Omega)} \leq \int_{\Omega} |\nabla u_n|^p \beta_k^m dx. \tag{2.13}
\]
Combining (2.13) with (2.11) and using the definition of \(\beta_k\), we have
\[
\|u_n\|_{L^{p^*} (\Omega)} \leq \mu^p \left( \frac{pm'}{\beta_k^*} \right)^p \|f\|_{L^m} \|u_n\|_{L^{\beta_k^m}} \tag{2.14}
\]
with \(\|u_n\|_{L^p (\Omega)} = \|u_n\|_{L^\sigma (\Omega)}\). In order to prove that \(\|u_n\|_\infty\) is bounded with a bound independent of \(n\), we use a trick in [1]. Let \(F_k = \beta_k \ln \|u_n\|_{\beta_k}\). By means of Inequality (2.14), we get
\[
F_{k+1} = \beta_{k+1} \ln \|u_n\|_{\beta_{k+1}} \\
\leq \beta_{k+1} m' \left[ p \ln \mu + p \ln \beta_k^m + \ln \|f\|_{L^m} + \beta_k \ln \|u_n\|_{\beta_k} \right] \\
\leq p^* \left[ \ln \mu + \ln \frac{\beta_k^*}{pm'} \right] + p^* \ln \|f\|_{L^m} + \frac{p^*}{m'} \cdot F_k \\
\leq \lambda_k + \delta F_k, \tag{2.15}
\]
with \( \lambda_k = p^* \ln(\mu \beta_k^* \|f\|_m) \), \( \delta = \frac{\mu^*}{pm} > 1 \). Since \( \beta_k = \frac{(N+p)m-2N}{(m-1)(mp-N)} \delta^{k-1} - \frac{Npm}{mp-N} \), we obtain that

\[
\lambda_k = p^* \ln \left[ \mu \|f\|_m \left( \frac{(N+p)m-2N}{(m-1)(mp-N)} \delta^{k-1} - \frac{Npm}{mp-N} \right) \right]
\leq p^* \ln \left[ \mu \|f\|_m \right] + p^* \ln \delta^{k-1} + p^* \ln \left[ \frac{(N+p)m-2N}{(m-1)(mp-N)} \right]
\leq b + p^*(k-1) \ln \delta,
\]

where \( b = p^* \ln \left[ \mu \|f\|_m \left( \frac{(N+p)m-2N}{(m-1)(mp-N)} \right) \right] \). Using Inequalities (2.15) and \( \beta_k = \frac{(N+p)m-2N}{(m-1)(mp-N)} \delta^{k-1} - \frac{Npm}{mp-N} \), we have

\[
F_k \leq \delta^{k-1} F_1 + \lambda_{k-1} + \delta \lambda_{k-2} + \cdots + \delta^{k-2} \lambda_1
\leq \delta^{k-1} \ln(C \|f\|_i^+) + \frac{(b + p^* \ln \delta) \delta}{(\delta - 1)^2} (\delta^{k-1} - 1) + \frac{b}{1-\delta};
\]

\[
\frac{F_k}{\beta_k} \leq \frac{\delta^{k-1} \ln(C \|f\|_i^+) + \frac{(b + p^* \ln \delta) \delta}{(\delta - 1)^2} (\delta^{k-1} - 1) + \frac{b}{1-\delta}}{(N+p)m-2N} \delta^{k-1} - \frac{Npm}{mp-N}
\to \frac{(m-1)(mp-N) \left[ (\delta - 1)^2 \ln(C \|f\|_i^+) + \delta p^* \ln \delta + \delta b \right]}{(\delta - 1)^2 (Nm + pm - 2N)} := d_0 \text{ as } k \to +\infty.
\]

So

\[
\|u_n\|_{\infty} \leq \lim_{k \to \infty} \sup \|u_n\|_{\beta_k} \leq \lim_{k \to \infty} \sup \exp \left( \frac{F_k}{\beta_k} \right) = e^{d_0}.
\]

Applying Fatou’s lemma, we get

\[
\|u\|_{\infty} \leq \lim_{n \to \infty} \inf \|u_n\|_{\infty} \leq e^{d_0}.
\]

(ii). Define \( \delta = \frac{m(N-p)}{N-mp} \). According to the condition 1 < \( m < \frac{N}{p} \), we get \( \delta > 1 \). Choosing \( u_n^{p(\delta-1)+1} \) as a test function, and applying Hölder’s inequality, we arrive at

\[
(p(\delta-1) + 1) \int_{\Omega} |\nabla u_n|^p u_n^{p(\delta-1)} dx + \int_{\Omega} u_n^p dx = \int_{\Omega} \frac{f_n}{u_n + \delta} u_n^{p(\delta-1)+1} dx dx
\leq \int_{\Omega} f(x) u_n^{p(\delta-1)} dx
\leq \left( \int_{\Omega} u_n^{p(\delta-1)-\frac{m}{m-1}} dx \right)^{\frac{m-1}{m}} \|f\|_{L^m(\Omega)}, \tag{2.16}
\]

Moreover, applying Sobolev embedding theorem and using Inequality (2.16), we get

\[
\left( \int_{\Omega} |u_n|^{\frac{Np}{N-p}} dx \right)^{\frac{N-p}{N}} \leq C \int_{\Omega} |\nabla u_n|^p dx \leq C \delta^p \int_{\Omega} u_n^{p(\delta-1)} |\nabla u_n|^p dx
\leq \left( \int_{\Omega} u_n^{p(\delta-1)-\frac{m}{m-1}} dx \right)^{\frac{m-1}{m}} \|f\|_{L^m(\Omega)}.
\tag{2.17}
\]

Then for any \( 1 \leq s \leq \frac{Npm}{N-mp} \), we have

\[
\|u_n\|_{L^s(\Omega)} \leq C \|f\|_{L^m(\Omega)}^{\frac{1}{p}}.
\]
3 The case when $0 < \alpha < 1$

In this section, consider the case $0 < \alpha < 1$. We find that Problem (1.1) has a solution in $W^1_0(\Omega)$ when $f \in L^m(\Omega)$ for $1 < m^* \leq m$ with $m^* = \frac{Np}{Np-(1-\alpha)p}$, but it is not clear whether Problem (1.1) has a solution in $W^1_0(\Omega)$ when $f \in L^m(\Omega)$ for $1 < m < m^*$. Fortunately, we prove that Problem (1.1) exists a solution in $W^1_0(\Omega)$ for $1 < q < p$. Our main results are the following

**Lemma 3.1.** Let $u_n$ be the solution of Problem (2.1) and suppose that $f \in L^m(\Omega)$, with $m \geq m^*$. Then $\|u_n\|_{W^1_0(\Omega)} \leq \|f\|_{L^m(\Omega)}^{1/q}$.

**Proof.** Multiplying the first identity in Problem (2.1) by $u_n$ and integrating over $\Omega$, we get

$$\int_\Omega |\nabla u_n|^p dx + \int_\Omega |u_n|^p dx = \int_\Omega \frac{f_n u_n}{(u_n + \frac{1}{n})^\alpha} dx \leq \int_\Omega |f_n| u_n^{1-\alpha} dx \leq \|f\|_{L^m(\Omega)} \|u_n^{(1-\alpha)}\|_{L^m(\Omega)},$$

and note that

$$\left( \int_\Omega |u_n|^{p^*} dx \right)^{\frac{p}{p^*}} \leq C \int_\Omega |\nabla u_n|^p dx,$$

i.e. $\|u_n\|_{L^{p^*}(\Omega)} \leq C \|f\|_{L^m(\Omega)}^{\frac{1}{p^*}}$. Furthermore, we get $\|u_n\|_{W^1_0(\Omega)} \leq C \|f\|_{L^m(\Omega)}^{\frac{1-\alpha}{p^*}}$. \hfill $\Box$

Next, we give the first result of this section:

**Theorem 3.1.** Let $f$ be a nonnegative function in $L^m(\Omega)$ ($f \neq 0$) with $m \geq m^* > 1$. Then Problem (1.1) has a solution $u \in W^1_0(\Omega)$ satisfying

$$\int_\Omega |\nabla u|^p \nabla u \varphi dx + \int_\Omega |u|^{p-2} u \varphi dx = \int_\Omega \frac{f \varphi}{u} dx, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Moreover, the solution $u$ of Problem (1.1) satisfies the following properties

(i) If $m > \frac{N}{p}$ and $2 - \frac{2}{m} < p < N$, then $u \in L^\infty(\Omega)$.

(ii) If $1 < m^* \leq m < \frac{N}{p}$, then $u \in L^s(\Omega)$, with $1 < s \leq \frac{(p + \alpha - 1)Nm}{N - pm}$.

**Proof.** **Part 1** (Existence). Following the lines of the proof of Theorem 1.1, we obtain the existence of the solution of Problem (1.1).

**Part 2** (Regularity). (i) Using a trick of Theorem 1.1, it is easy to see that the first conclusion holds.

(ii). Define $\delta = \frac{m(p + \alpha - 1)(N - p)}{p(N - mp)}$. According to the condition $1 < m^* \leq m < \frac{N}{p}$, we get $\delta > 1$. Choosing $u_n^{(p(\delta - 1)+1}$ as a test function, and applying Hölder’s inequality, we arrive at the following relations

$$(p(\delta - 1) + 1) \int |\nabla u_n|^p u_n^{(p(\delta - 1)} dx + \int u_n^{p(\delta - 1)} dx = \int \frac{f_n}{u_n + \frac{1}{n}} u_n^{p(\delta - 1)+1} dx dx$$

$$\leq \int f(x) u_n^{p(\delta - 1)+1-\alpha} dx$$

$$\leq \left( \int u_n^{p(\delta - 1)+1-\alpha} \frac{m}{m-1} dx \right)^{\frac{m-1}{m}} \|f\|_{L^m(\Omega)}.$$
Moreover, applying Sobolev embedding theorem and using Inequality (2.11), we get

\[
\left( \int_{\Omega} |u_n|^{\frac{Np}{N-p}} dx \right)^{\frac{N-p}{N}} \leq C \int_{\Omega} |\nabla u_n|^p dx \leq C \int_{\Omega} \theta^{\frac{p\theta-1}{m-1}} \|f\|_{L^m(\Omega)} \leq \left( \int_{\Omega} u_n^{p(\theta-1) + 1 - \frac{m}{m-1}} dx \right)^{\frac{m-1}{m}} \|f\|_{L^m(\Omega)}. \tag{3.2}
\]

Using the definition of \( \delta \), we get \( \delta p^* = (p(\delta - 1) + 1 - \alpha) \frac{m}{m-1} \), then for all \( 1 \leq s \leq \frac{Nm(\rho + \alpha - 1)}{N-m(p-\alpha)} \), we have

\[
\|u_n\|_{L^s(\Omega)} \leq C \|f\|_{L^{s^*}(\Omega)}. \tag{3.3}
\]

If \( 1 \leq m < m^* \), it is not clear whether there exists a solution \( u \in W_0^{1,p}(\Omega) \) of Problem (1.1). But, we may find that Problem (1.1) has a solution \( u \in W_0^{1,q}(\Omega) \) with \( q \leq q^* = \frac{Nm(\rho + \alpha - 1)}{N-m(1-\alpha)} \). Since \( 1 \leq m < m^* \), we get that

\[
q^* - p = \frac{m[Np - (1-\alpha)(N-p)] - Np}{N - m(1-\alpha)} = \frac{Np(m-m^* - \alpha)}{m^*(N-m(1-\alpha))} < 0 \Rightarrow W^{1,p}(\Omega) \subset W_0^{1,q}(\Omega),
\]

which implies that Problem (1.1) has a solution in a larger space \( W_0^{1,q}(\Omega) \) rather than \( W^{1,p}(\Omega) \).

In this case, our main result is

**Theorem 3.2.** Suppose that \( f \in L^m(\Omega) \) with \( 1 \leq m < m^* \), then, for any \( \frac{Np}{N-p} < q \leq q^* \), Problem (1.1) has a weak solution \( u \in W_0^{1,q}(\Omega) \) satisfying

\[
\int_{\Omega} |\nabla u|^p - 2u \nabla u \phi dx + \int_{\Omega} |u|^{p-2} u \phi dx = \int_{\Omega} \frac{f \phi}{u^a} dx, \quad \forall \phi \in C_0^\infty(\Omega),
\]

if the following assumptions hold

\[
(H_1) 1 < p < \sqrt{N}, \quad 0 < \alpha < \frac{N - p^2}{N(p + 1)}, \quad \text{and} \quad \frac{Np}{N(p + \alpha - 1) + p^2} < m < m^* \quad \text{or} \quad \sqrt{N} \leq p < N, \quad 0 < \alpha < 1 - \frac{1}{m}, \quad \text{and} \quad 1 < m < m^*.
\]

In order to prove Theorem 3.2, we need the following lemmas

**Lemma 3.2.** The solution \( u_n \) of Problem (2.1) satisfies the estimates

\[
\|u_n\|_{W^{1,q}(\Omega)} \leq C := C(\|f\|_{L^m}, p, N, \alpha, |\Omega|), \quad 1 < q \leq q^*. \tag{3.4}
\]

**Proof.** Noting that \( 1 \leq m < m^* \Rightarrow \frac{p+\alpha-1}{p} < \delta = \frac{m(p(\rho+\alpha-1)(N-p))}{p(N-m^p)} < 1 \), we can’t choose \( u_n^{p(\theta-1)+1} \) as a test-function because the gradient of the function \( u_n (p(\theta-1)+1) \) is singular at \( u_n = 0 \). In order to overcome this difficulty, for any \( 0 < \theta < \frac{1}{m} \), we may replace \( u_n^{p(\theta-1)+1} \) by \( (u_n+\theta)^{p(\theta-1)+1} - \theta p(\theta-1)+1 \) in (2.1), and we get

\[
(p\delta - p + 1) \int_{\Omega} |\nabla u_n|^p (u_n + \theta)^{p(\delta-1)} dx + \int_{\Omega} u_n^{p-1} [(u_n + \theta)^{p(\delta-1)+1} - \theta p(\delta-1)+1] dx = \int_{\Omega} \frac{f_n}{(u_n + \frac{1}{m})^{\alpha}} [(u_n + \theta)^{p(\delta-1)+1} - \theta p(\delta-1)+1] dx. \tag{3.5}
\]
Then, applying Hölder’s inequality, we have
\[
\frac{(p\delta - p + 1)}{\delta p} \int_{\Omega} |\nabla[(u_n + \theta)^\delta - \theta^\delta]|^p dx + \int_{\Omega} u_n^{p-1}(u_n + \theta)^{p(\delta-1)+1} - \theta^{p(\delta-1)+1} dx
\]
\[
= \int_{\Omega} \frac{f_n}{(u_n + \theta)^\alpha}[(u_n + \theta)^{p(\delta-1)+1} - \theta^{p(\delta-1)+1}] dx
\]
\[
\leq \int_{\Omega} f(x)(u_n + \theta)^{p(\delta-1)+1} dx
\]
\[
\leq \|f\|_{L^m(\Omega)} \left( \int_{\Omega} (u_n + \theta)^{p(\delta-1)+1-\alpha} \frac{m-1}{m} dx \right)^{\frac{m-1}{m}}.
\]
Combining the above inequalities with Sobolev embedding theorem $W^{1,p}_0(\Omega) \hookrightarrow L^p(\Omega)$, we have
\[
\left( \int_{\Omega} [(u_n + \theta)^\delta - \theta^\delta]^p dx \right)^{\frac{p}{mp^*}} \leq \|f\|_{L^m(\Omega)} \left( \int_{\Omega} (u_n + \theta)^{p(\delta-1)+1-\alpha} \frac{m-1}{m} dx \right)^{\frac{m-1}{m}}.
\]
(3.5)

Letting $\theta \to 0^+$ in (3.5) and using the definition of $\delta$, we obtain the following relations
\[
\int_{\Omega} |u_n|^s dx \leq \|f\|_{L^m(\Omega)}^{\frac{mp^*}{mp-p(m-1)}},
\]
which implies
\[
\int_{\Omega} |u_n + \theta|^s dx \leq C(\|f\|_{L^m(\Omega)}^{\frac{mp^*}{mp-p(m-1)}} + \theta^{p^*}|\Omega|).
\]
(3.6)

(3.4) and (3.6) yield that
\[
\int_{\Omega} |\nabla u_n|^p(u_n + \theta)^{p(\delta-1)} dx \leq C(\|f\|_{L^m(\Omega)}^{\frac{mp^*}{mp-p(m-1)}} + \|f\|_{L^m(\Omega)}^{\frac{(m-1)p^*}{m}} |\Omega|^{\frac{m-1}{m}}).
\]

Then for $1 < q \leq q_*$, we have
\[
\int_{\Omega} |\nabla u_n|^q dx \leq \int_{\Omega} |\nabla u_n|^q(u_n + \theta)^{(\delta-1)q}(u_n + \theta)^{(1-\delta)q} dx
\]
\[
\leq \left( \int_{\Omega} |\nabla u_n|^p(u_n + \theta)^{(\delta-1)p} dx \right)^{\frac{2}{p}} \left( \int_{\Omega} (u_n + \theta)^{(1-\delta)q} dx \right)^{1-\frac{2}{p}}
\]
\[
\leq C.
\]

The following lemma plays a role in proving that Problem (1.1) has a nonnegative solution $u \in W^{1,q}_0(\Omega)$.

**Lemma 3.3.** The solution $u_1$ to Problem (2.1) with $n = 1$ satisfies
\[
\int_{\Omega} u_1^{-r} dx < \infty, \quad \forall \ r < 1.
\]
(3.7)

**Proof.** By $\frac{\min\{f(x),1\}}{(u_1+1)^r} \leq 1$, and Lemma 2.2 in [17], we know that there exists a $0 < \beta < 1$ such that $u_1 \in C^{1,\beta}(\Omega)$ and $\|u_1\|_{C^{1,\beta}(\overline{\Omega})} \leq C$, which implies that the gradient of $u_1$ exists everywhere,
then Hopf Lemma in [18] shows that $\frac{\partial u_1(x)}{\partial \nu} > 0$, in $\overline{\Omega}$, where $\nu$ is the outward unit normal vector of $\partial \Omega$ at $x$. Moreover, following the lines of the proof of Lemma in [5], we get

$$\int_{\Omega} u^r \, dx < \infty, \text{ if and only if } r > -1.$$ 

\[\Box\]

**Proof of Theorem 3.2.** According to Lemma 3.2, we know that there exists a $u \in W^{1,q}_0(\Omega)$ such that

$$\begin{aligned}
&\left\{ \begin{array}{l}
 u_n \to u \text{ weakly in } W^{1,q}_0(\Omega), \\
 u_n \to u \text{ strongly in } L^p(\Omega), \\
 u_n \to u \text{ a.e. in } \Omega.
\end{array} \right.
\end{aligned} \tag{3.8}$$

But the above convergence does not permit to pass to limit in the following identity

$$\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi \, dx + \int_{\Omega} |u_n|^{p-2} u_n \varphi \, dx = \int_{\Omega} \frac{f_n \varphi}{(u_n + \frac{1}{n})^\alpha} \, dx, \text{ for } \varphi \in C_0^\infty(\Omega). \tag{3.9}$$

We need to prove that $\nabla u_n$ converges to $\nabla u$ a.e. in $\Omega$ or $\{\nabla u_n\}$ is a Cauchy sequence in measure or in $L^1(\Omega)$.

Case 1: $\sqrt{N} \leq p < N$, $0 < \alpha < 1 - \frac{1}{m}$, and $1 < m < m^*$. Let $u_n$ and $u_k$ be the solution Problem (2.1) with $f_n$ and $f_k$, respectively. Define

$$E_{n,k,\varepsilon} = \{x \in \Omega : |u_n - u_k| < \varepsilon\}, \quad T_\varepsilon(\tau) = \max\{\min\{\tau, \varepsilon\}, -\varepsilon\},$$

then, we choose $\varphi(x) = T_\varepsilon(u_n - u_k)$ as a test function in (3.9) to get

$$\int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n - |\nabla u_k|^{p-2} \nabla u_k| \nabla T_\varepsilon(u_n - u_k) \, dx + \int_{\Omega} (|u_n|^{p-2} u_n - |u_k|^{p-2} u_k) T_\varepsilon(u_n - u_k) \, dx
\begin{aligned}
&= \int_{\Omega} \left( \frac{f_n}{(u^n + \frac{1}{n})^\alpha} - \frac{f_k}{(u^k + \frac{1}{k})^\alpha} \right) T_\varepsilon(u_n - u_k) \, dx.
\end{aligned}$$

Furthermore, we obtain

$$\int_{E_{n,k,\varepsilon}} |\nabla T_\varepsilon(u_n - u_k)|^p \, dx = -\int_{\Omega} (|u_n|^{p-2} u_n - |u_k|^{p-2} u_k) T_\varepsilon(u_n - u_k) \, dx + \int_{\Omega} \left( \frac{f_n}{(u^n + \frac{1}{n})^\alpha} - \frac{f_k}{(u^k + \frac{1}{k})^\alpha} \right) T_\varepsilon(u_n - u_k) \, dx := I_1 + I_2. \tag{3.10}$$

Noting that $u_n$ is increasing with respect to $n$ and applying the Sobolev embedding theorem $W^{1,q}(\Omega) \hookrightarrow L^p(\Omega)$, Lemma 3.3 and the condition $0 < \alpha < 1 - \frac{1}{m}$, we obtain

$$|I_1| \leq \varepsilon \int_{\Omega} (|u_n|^{p-1} + |u_n|^{p-1}) \, dx \leq C \varepsilon; \tag{3.11}$$

$$|I_2| \leq \varepsilon \int_{\Omega} \left( \frac{f(x)}{u_1^n} + \frac{f(x)}{u_1^n} \right) \, dx \leq 2 \varepsilon \|f\|_{L^m} \|u_1^{-\alpha}\|_{L^m} \leq C \varepsilon. \tag{3.12}$$

By (3.9 – 3.12), we get

$$\int_{E_{n,k,\varepsilon}} |\nabla T_\varepsilon(u_n - u_k)|^p \, dx \leq C \varepsilon. \tag{3.13}$$
Since \( u_n \to u \) strongly in \( L^p(\Omega) \), we get

\[
\text{measure}\{x \in \Omega : |u_n - u_k| > \varepsilon\} < \varepsilon. \tag{3.14}
\]

Combining (3.12) – (3.14) with Lemma 3.2, we get

\[
\int_{\Omega} |\nabla (u_n - u_k)| dx = \int_{E_{n,k,\varepsilon}} |\nabla (u_n - u_k)| + \int_{\Omega \setminus E_{n,k,\varepsilon}} |\nabla (u_n - u_k)| \\
\leq C \varepsilon^{\frac{1}{p}} + C(\text{measure}\{x \in \Omega : |u_n - u_k| > \varepsilon\})^{1 - \frac{1}{p}} \tag{3.15}
\]

which implies that \( \{\nabla u_n\} \) is a Cauchy sequence in \( L^1(\Omega) \).

By (3.8), (3.9) and (3.15), we know that Problem (1.1) has a nonnegative solution \( u \in W_0^{1,q}(\Omega) \).

Case 2: \( 1 < p < \sqrt{N}, \ 0 < \alpha < \frac{N-p^2}{N(p+1)}, \) and \( \frac{Np}{N(p+\alpha-1)+p^2} < m < m^* \). Similarly as Case 1, we may prove the conclusion holds when \( p, \alpha, N, m \) satisfy the second case.

4 The case when \( (\alpha > 1) \)

In this section, we discuss how the value of \( \alpha > 1 \) and the summability of \( f \) affect the existence and regularities of solutions. First, we study the case when \( f(x) \in L^1(\Omega) \). Our main results are as follows

**Lemma 4.1.** Let \( u_n \) be the solution of Problem (2.1) and suppose that \( f \in L^1(\Omega) \). Then \( \|u_n\|_{W_0^{1,p}(\Omega)} \leq \|f\|_{L^1(\Omega)}^{\frac{1}{p}} \) and \( \|u_n\|_{W_0^{1,p}(\Omega)} \leq C\|f\|_{L^1(\Omega)} \).

**Proof.** Our proof is similar as that in [9]. We give a brief proof. Multiplying the first identity in Problem (2.1) by \( u_n^\alpha \) and integrating over \( \Omega \), we get

\[
\alpha \int_{\Omega} |\nabla u_n|^{p-1}dx + \int_{\Omega} |u_n|^{p+\alpha-1}dx = \int_{\Omega} \frac{f_n u_n^\alpha}{(u_n + \frac{1}{n})^\alpha}dx \leq \int_{\Omega} |f_n|dx \leq \int_{\Omega} |f|dx,
\]

and note that

\[
\alpha \left(\frac{p}{p + \alpha - 1}\right) \int_{\Omega} |\nabla u_n|^{\frac{p+\alpha-1}{p}}dx = \alpha \int_{\Omega} |\nabla u_n|^{p}u_n^{\alpha-1}dx,
\]

i.e.

\[
\|u_n\|_{W_0^{1,p}} \leq C\|f\|_{L^1(\Omega)}^{\frac{1}{p}}. \tag{4.1}
\]

In order to prove that \( u_n \) is bounded in \( W_0^{1,p}(\Omega) \), we may choose \( \varphi \in C_0^\infty(\Omega), \ \Omega' = \{x \in \Omega, \ \varphi(x) \neq 0\} \). Multiplying the first identity in Problem (2.1) by \( u_n|\varphi(x)|^p \) and integrating over \( \Omega \), we get

\[
\int_{\Omega} |\nabla u_n|^{p}\varphi dx + \int_{\Omega} u_n|\varphi(x)|^{p-2}\varphi(x)|\nabla u_n|^{p-2}\nabla u_n \nabla \varphi dx = \int_{\Omega} \frac{f_n u_n|\varphi(x)|^p}{(u_n + \frac{1}{n})^\alpha}dx \\
\leq \int_{\Omega} \frac{f}{C_{\Omega'}^{\alpha-1}}|\varphi|^p dx \leq \frac{\|\varphi\|_{L^\infty(\Omega)}^p}{C_{\Omega'}^{\alpha-1}} \int_{\Omega} |f|dx. \tag{4.2}
\]
And applying Young’s inequality with $\epsilon$, one has
\[
|p \int_{\Omega} u_n |\varphi(x)|^{p-2} \varphi(x)|\nabla u_n|^{p-2} \nabla u_n \nabla \varphi|dx| \leq p \int_{\Omega} u_n |\varphi(x)|^{p-1}|\nabla u_n|^{p-1}|\nabla \varphi|dx
\leq \frac{1}{2} \int_{\Omega} |\nabla u_n|^p |\varphi|^p dx + \frac{p^2}{2} |u_n|^p |\nabla \varphi|^p dx.
\] (4.3)

Combining Inequality (4.2) with Inequality (4.3), we have
\[
\int_{\Omega} |\nabla u_n|^p |\varphi|^p dx \leq \frac{||\varphi||_{L^p(\Omega)}^p}{C_{\Omega}^{\alpha-1}} \int_{\Omega} |f|^p dx + \frac{p^2}{2} |u_n|^p |\nabla \varphi|^p dx
\leq \frac{||\varphi||_{L^p(\Omega)}^p}{C_{\Omega}^{\alpha-1}} |f|_{L^p(\Omega)}^p + \frac{p^2 ||\nabla \varphi||_{L^p(\Omega)}^p}{2} \int_{\Omega} |u_n|^p dx.
\] (4.4)

Then, by (4.1) – (4.4), we get that $u_n$ is bounded in $W^{1,p}_{loc}(\Omega)$ \hfill \Box

**Theorem 4.1.** Let $f$ be a nonnegative function in $L^1(\Omega)(f \neq 0)$. Then Problem (1.1) has a solution $u$ with $u^{\frac{p+\alpha-1}{p}} \in W^{1,p}_0(\Omega)$ satisfying
\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi|dx + \int_{\Omega} |u|^{p-2} u \varphi|dx = \int_{\Omega} \frac{f \varphi}{u_n} dx, \quad \forall \varphi \in C_0^\infty(\Omega).
\] (4.5)

Moreover, suppose that $f \in L^m(m \geq 1)$. Then the solution $u$ of Problem (1.1) has the properties

(i) If $m > \frac{N}{p}$ and $2 - \frac{2}{m} < p < N$, then $u \in L^\infty(\Omega)$.

(ii) If $1 \leq m < \frac{N}{p}$, then $u \in L^s(\Omega)$, with $1 < s \leq \frac{Nm(p+\alpha-1)}{N - pm}$.

**Proof.** \textbf{Part 1} (Existence). Following the lines of the proof of Theorem 2.1, we obtain the existence of the solution of Problem (1.1).

\textbf{Part 2} (Regularity). (i) Using a trick of Theorem 2.1, it is easy to see that the first conclusion holds.

(ii) Define $\delta = \frac{m(p+\alpha-1)(N-p)}{p(N-mp)} > \frac{p+\alpha-1}{p} > 1$. Choosing $u_n^{(p-1)+1}$ as a test function, and applying Hölder’s inequality, we arrive at
\[
(p(\delta-1)+1) \int_{\Omega} |\nabla u_n|^{p(\delta-1)} dx + \int_{\Omega} u_n^{\delta} dx = \int_{\Omega} \frac{f_n}{(u_n + \frac{1}{n})^\alpha} u_n^{p(\delta-1)+1} dx dx
\leq \int_{\Omega} f(x) u_n^{p(\delta-1)+1-\alpha} dx
\leq \left( \int_{\Omega} u_n^{(p(\delta-1)+1-\alpha) \frac{m}{m-1}} dx \right)^{m-1} \frac{1}{m} \|f\|_{L^m(\Omega)}.
\] (4.6)

Moreover, applying Sobolev embedding theorem and using Inequality (2.11), we get
\[
\left( \int_{\Omega} |u_n|^{\frac{\delta Np}{N-p}} dx \right)^{\frac{N-p}{N}} \leq C \int_{\Omega} |\nabla u_n|^p dx \leq C \delta^p \int_{\Omega} u_n^{p(\delta-1)} |\nabla u_n|^p dx
\leq \left( \int_{\Omega} u_n^{p(\delta-1)+1-\alpha} dx \right)^{m-1} \frac{m}{m} \|f\|_{L^m(\Omega)},
\] (4.7)
Using the definition of $\delta$, we get $\delta p^* = (p(\delta - 1) + 1 - \alpha) \frac{m}{m-1}$, then for all $1 \leq s < \frac{N m (p + \alpha - 1)}{N - m p}$, we have

$$\|u_n\|_{L^s(\Omega)} \leq C\|f\|_{L^m(\Omega)}^{\frac{1}{1}}.$$

Second, we study the existence and nonexistence of positive solutions in the case $f(x) \in L^\infty(\Omega)$. Our main results are as follows

**Theorem 4.2.** If $1 < \alpha < 2$, $f(x) \in L^\infty(\Omega)$, then Problem (1.1) has a unique positive solution in $W_0^{1,p}(\Omega)$.

**Proof.** Multiplying the first identity in Problem (2.1) by $u_n$, integrating over $\Omega$, and applying Lemma 2.2 and Lemma 3.3, we get

$$\int_{\Omega} |\nabla u_n|^p dx + \int_{\Omega} |u_n|^p dx = \int_{\Omega} \frac{f_n u_n}{(u_n + \frac{1}{n})^\alpha} dx \leq \int_{\Omega} |f_n|^p (u_n^{1-\alpha} dx \leq \|f\|_{L^\infty(\Omega)} \|u_n^{1-\alpha}\|_{L^1(\Omega)},$$

ie.

$$\|u_n\|_{W_0^{1,p}(\Omega)} \leq (\|f\|_{L^\infty(\Omega)} \|u_n^{1-\alpha}\|_{L^1(\Omega)})^{\frac{1}{p}} < \infty.$$

Once $W_0^{1,p}(\Omega)$ estimates are obtained, we may prove the existence of solutions with similar methods of the proof of Theorem 2.1.

Next, we consider $\alpha > 2$. Similarly as the proof of Lemma 3.3 or according to Lemma 3.2 in [13], we have

**Lemma 4.2.** Let $\Phi \in C(\overline{\Omega})$ be an eigenfunction corresponding the first eigenvalue $\lambda$ of the following problem

$$\begin{cases}
-\Delta p \Phi = \lambda \Phi^{p-1}, & x \in \Omega; \\
\Phi > 0, & x \in \Omega; \\
\Phi = 0, & x \in \partial \Omega,
\end{cases}$$

then, $\int_\Omega \Phi^r dx < \infty \iff r > -1$.

**Theorem 4.3.** If $\alpha > 2$, $f(x) \in L^\infty(\Omega)$, and $f(x) \geq \Phi_\Omega^{1-\alpha}$, in $\overline{\Omega}$, then the solution of Problem (1.1) is not in $W_0^{1,p}(\Omega)$.

**Proof.** With similar method in [13], we know that there exist $b > 0$ and $\sigma = \frac{p}{p-1+\alpha}$ such that $0 < u(x) \leq b \Phi_\Omega(x)$ in $\Omega$. Next, we argue by contradiction. Suppose that the solution $u$ of Problem (1.1) belongs to $W_0^{1,p}(\Omega)$. Since $C_\infty^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, we may choose a sequence $\{Z_n\} \subset C_0^\infty(\Omega)$ satisfying

$$Z_n \to u, \text{ in } W_0^{1,p}(\Omega), \text{ as } n \to \infty.$$

By [2] [16], we have $Z_n^+ = \max\{Z_n, 0\} \in W_0^{1,p}(\Omega), n = 1, 2, 3, \ldots$. By means of $u \geq 0$, it is easy to prove that

$$Z_n^+ \to u, \text{ in } W_0^{1,p}(\Omega), \text{ as } n \to \infty.$$
Furthermore, According to F.Riesz theorem in [19], we may assume that $Z_n^+$ converges to $u$ almost everywhere. Choosing $Z_n^+$ as a test-function in (4.5), we have

$$
\int_{\Omega} |\nabla u|^{p-2}\nabla u \nabla Z_n^+ dx + a \int_{\Omega} |u|^{p-2}u Z_n^+ dx = \int_{\Omega} \frac{f(x)}{u^\alpha} Z_n^+ dx. \tag{4.8}
$$

By the definition of weak convergence, Poincaré’s inequality and Fatou’s lemma, we get

$$
\int_{\Omega} |\nabla u|^p dx \geq \frac{1}{\Omega} \left( \int_{\Omega} (|\nabla u|^p + |u|^p) dx \right) = \lim_{n \to \infty} \int_{\Omega} (|\nabla u|^{p-2}\nabla u \nabla Z_n^+ + |u|^{p-2}u Z_n^+) dx; \tag{4.9}
$$

$$
\int_{\Omega} f(x) u^{1-\alpha} dx \leq \lim_{n \to \infty} \int_{\Omega} f(x) Z_n^+ u^{-\alpha} dx. \tag{4.10}
$$

Again by $f(x) \geq \Phi_{\frac{p-1}{p-\alpha}}$, in $\overline{\Omega}$ and $0 < u(x) \leq b \Phi_{\sigma}(x)$, in $\Omega$, we have

$$
\int_{\Omega} f(x) u^{1-\alpha} dx \geq C(M, \alpha, p) \int_{\Omega} \Phi^{\sigma(1-\alpha)+\frac{\sigma}{p}} dx = +\infty \iff \alpha > 2. \tag{4.11}
$$

By (4.8) – (4.11), we have

$$
\int_{\Omega} |\nabla u|^p dx = +\infty,
$$

which contradicts the assumption that $u \in W^{1,p}_0(\Omega)$.

**Remark 4.1.** Due to technical reasons, the authors have to give a technical condition to $f(x)$. It is not clear whether the problem still has no a solution in $W^{1,p}_0(\Omega)$ if the restriction that $f(x) \geq \Phi_{\frac{p-1}{p-\alpha}}$ is removed. Besides, what happens to the solution in the case when $\alpha = 2$? whether is $\alpha = 2$ a critical exponent or not? Up to now, this is still an open problem.

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