ON THE PROPERTIES OF SPECIAL FUNCTIONS ON THE LINEAR-TYPE LATTICES

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Abstract. We present a general theory for studying the difference analogues of special functions of hypergeometric type on the linear-type lattices, i.e., the solutions of the second order linear difference equation of hypergeometric type on a special kind of lattices: the linear type lattices. In particular, using the integral representation of the solutions we obtain several difference-recurrence relations for such functions. Finally, applications to $q$-classical polynomials are given.

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1. Introduction

The study of the so-called $q$-special functions has known an increasing interest in the last years due its connection with several problems in mathematics and mathematical-physics (see e.g. [3, 6, 8, 13, 17]). A systematic study starting from the second order linear difference equation that such functions satisfy was started by Nikiforov and Uvarov in 1983 and further developed by Atakishiyev and Suslov (for a very nice reviews see e.g. [7, 13, 16]). Of particular interest is the so-called $q$-classical polynomials (see e.g. [4]) introduced by Hahn in 1949 which are polynomials on the lattice $q^s$.

Our main aim in this paper is to present a constructive approach for generating recurrence relations and ladder-type operators for the difference analogues of special functions of hypergeometric type on the linear-type lattices. Here we will focus our attention on functions defined on the $q$-linear lattice (for the linear lattice $x(s) = s$ see [4] and references therein, and for the continuous case see e.g. [18]). Therefore we will complete the work started in [16] where few recurrence relations where obtained. In fact we will prove, by using the $q$-analogue of the technique introduced in [4] for the discrete case (uniform lattice), that the solutions (not only the polynomial ones) of the difference equation on the $q$-linear lattice $x(s) = c_1 q^s + c_2$ satisfy a very general recurrent-difference relation from where several well known relations (such as the three-term recurrence relation and the ladder-type relations) follow.

The structure of the paper is as follows: In section 2 the needed results and notations from the $q$-special function theory are introduced. In sections 3 and 4 the general theorems for obtaining recurrences relations are presented. In section 5 the special case of classical $q$-polynomials are considered in details and some examples are worked out in details.

2. Some Preliminary Results

Here we collect the basic background [11, 13, 16] on $q$-hypergeometric functions needed in the rest of the work.

The hypergeometric functions on the non-uniform lattice $x(s)$ are the solutions of the second order linear difference equation of hypergeometric type on non-uniform lattices

\[
\sigma(s) \frac{\Delta}{\Delta x(s - \frac{1}{2})} \left[ \frac{\nabla y(s)}{\nabla x(s)} \right] + \tau(s) \frac{\Delta y(s)}{\Delta x(s)} + \lambda y(s) = 0, \\
\sigma(s) = \tilde{\sigma}(x(s)) - \frac{1}{2} \tilde{\tau}(x(s)) \Delta x \left( s - \frac{1}{2} \right), \quad \tau(s) = \tilde{\tau}(x(s)),
\]

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where $\Delta y(s) := y(s + 1) - y(s)$, $\nabla y(s) := y(s) - y(s - 1)$, are the forward and backward difference operators, respectively; $\delta(x(s))$ and $\bar{\tau}(x(s))$ are polynomials in $x(s)$ of degree at most 2 and 1, respectively, and $\lambda$ is a constant. Here we will deal with the linear and $q$-linear lattices, i.e., lattices of the form

$$x(s) = c_1 s + c_2 \quad \text{or} \quad x(s) = c_1(q)s^q + c_2,$$

respectively, with $c_1 \neq 0$ and $c_1(q) \neq 0$.

We will define the $k$-order difference derivative of a solution $y(s)$ of (1) by

$$y^{(k)}(s) := \Delta^{(k)}[y(s)] = \frac{\Delta}{\Delta x_{k-1}(s)} \frac{\Delta}{\Delta x_{k-2}(s)} \cdots \frac{\Delta}{\Delta x(s)}[y(s)],$$

where $x_{\nu}(s) = x(s + \frac{s}{\nu})$. It is known [13] that $y^{(k)}(s)$ also satisfy a difference equation of the same type. Moreover, for the solutions of the difference equation (1) the following theorem holds

**Theorem 2.1.** [12, 16] The difference equation (1) has a particular solution of the form

$$y_\nu(z) = \frac{C_\nu}{\rho(z)} \sum_{s=0}^{b-1} \rho_\nu(s) \nabla x_{\nu+1}(s) \frac{\sigma(s) \rho_\nu(s) \nabla x_{\nu+1}(s)}{|x_{\nu-1}(s) - x_{\nu-1}(s+1)|^{(\nu+1)}},$$

if the condition

$$\left. \left| \frac{\sigma(s) \rho_\nu(s) \nabla x_{\nu+1}(s)}{|x_{\nu-1}(s) - x_{\nu-1}(s+1)|^{(\nu+1)}} \right| \right|_a^b = 0,$$

is satisfied, and of the form

$$y_\nu(z) = \frac{C_\nu}{\rho(z)} \int_C \frac{\rho_\nu(s) \nabla x_{\nu+1}(s)}{|x_{\nu}(s) - x_{\nu}(s+1)|^{(\nu+1)}} ds,$$

if the condition

$$\int_C \left[ \frac{\sigma(s) \rho_\nu(s) \nabla x_{\nu+1}(s)}{|x_{\nu-1}(s) - x_{\nu-1}(s+1)|^{(\nu+1)}} \right] = 0,$$

is satisfied. Here $C$ is a contour in the complex plane, $C_\nu$ is a constant, $\rho(s)$ and $\rho_\nu(s)$ are the solution of the Pearson-type equations

$$\frac{\rho(s+1)}{\rho(s)} = \frac{\sigma(s) + \tau(s) \Delta x(s - \frac{1}{2})}{\sigma(s+1)} = \frac{\phi(s)}{\sigma(s+1)},$$

$$\frac{\rho_\nu(s+1)}{\rho_\nu(s)} = \frac{\sigma(s) + \tau_\nu(s) \Delta x_{\nu}(s - \frac{1}{2})}{\sigma(s+1)} = \frac{\phi_\nu(s)}{\sigma(s+1)},$$

where

$$\tau_\nu(s) = \frac{\sigma(s + \nu) - \sigma(s) + \tau(s + \nu) \Delta x(s + \nu - \frac{1}{2})}{\Delta x_{\nu}(s)},$$

$\nu$ is the root of the equation

$$\lambda_\nu + [\nu]_q \left\{ \alpha_q(\nu - 1) \bar{\gamma}^q + [\nu - 1] \frac{\bar{\gamma}^q}{2} \right\} = 0,$$

and $[\nu]_q$ and $\alpha_q(\nu)$ are the $q$-numbers

$$[\nu]_q = \frac{q^{\nu/2} - q^{-\nu/2}}{q^{1/2} - q^{-1/2}}, \quad \alpha_q(\nu) = \frac{q^{\nu/2} + q^{-\nu/2}}{2}, \quad \forall \nu \in \mathbb{C},$$

respectively. The generalized powers $[x_k(s) - x_k(z)]^{(\nu)}$ are defined by

$$[x_k(s) - x_k(z)]^{(\nu)} = (q - 1)^\nu c_1 q^{\nu(k-\nu+1)/2} \frac{\Gamma_q(s - z + \nu)}{\Gamma_q(s - z)}, \quad \nu \in \mathbb{R},$$

for the $q$-linear (exponential) lattice $x(s) = c_1(q)s^q + c_2$ and

$$[x_k(s) - x_k(z)]^{(\nu)} = c_1^q \frac{\Gamma(s - z + \mu)}{\Gamma(s - z)}, \quad \nu \in \mathbb{R},$$
for the linear lattice \( x(s) = c_1 s + c_2 \), respectively. For the definitions of the Gamma and the \( q \)-Gamma functions see, for instance, [6].

**Remark 2.2.** For the special case when \( \nu \in \mathbb{N} \), the generalized powers become
\[
[x_k(s) - x_k(z)]^{(n)} = (-1)^n c_1^n q^{-n(n-1)/2} q^{n(z+k)/2} (q^{s-z}; q)_n,
\]
for \( q \)-linear and linear lattices, respectively.

We will need the following straightforward proposition which proof we omit here (see e.g. [11 10])

**Proposition 2.3.** Let \( \mu \) and \( \nu \) be complex numbers and \( m \) and \( k \) be positive integers with \( m \geq k \). For the \( q \)-linear lattice \( x(s) = c_1 q^s + c_2 \) we have
\[
\begin{align*}
(1) \quad & \frac{[x_\mu(s) - x_\mu(z)]^{(m)}}{[x_\nu(s) - x_\nu(z)]^{(m)}} = q^{\frac{m(\mu-\nu)}{2}}, \\
(2) \quad & \frac{[x_\mu(s) - x_\mu(z)]^{(m)}}{[x_\mu(s) - x_\mu(z)]^{(k)}} = [x_\mu(s) - x_\mu(z-k)]^{(m-k)}, \\
(3) \quad & \frac{[x_\mu(s) - x_\mu(z)]^{(m)}}{[x_\nu(s) - x_\nu(z)]^{(k)}} = q^{\frac{m(\mu-\nu)}{2}} [x_\mu(s) - x_\mu(z-k)]^{(m-k)}, \\
(4) \quad & \frac{[x_\mu(s) - x_\mu(z)]^{(m+1)}}{[x_\mu(s) - x_\mu(z)]^{(m+1)}} = x_\mu(s) - x_\mu(z), \\
(5) \quad & \frac{[x_\mu(s) - x_\mu(z)]^{(m+1)}}{[x_\mu(s) - x_\mu(z)]^{(m+1)}} = x_\mu(s) + m - x_\mu(z).
\end{align*}
\]

To obtain the result for the linear lattice one only has to put in the above formulas \( q = 1 \).

3. The general recurrence relation in the linear-type lattices

In this section we will obtain several recurrence relations for the solutions (3) and (4) of the difference equation (11) in the linear-type lattices (2). Since the equation (11) is linear we can restrict ourselves to the canonical cases \( x(s) = q^s \) and \( x(s) = s \).

Let us define the function\(^4\)
\[
\Phi_{\nu, \mu}(z) = \sum_{s=0}^{b-1} \frac{\rho_\nu(s) \nabla x_{\nu+1}(s)}{[x_\nu(s) - x_\nu(z)]^{(\nu+1)}} \quad (11)
\]
and
\[
\Phi_{\nu, \mu}(z) = \int_C \frac{\rho_\nu(s) \nabla x_{\nu+1}(s)}{[x_\nu(s) - x_\nu(z)]^{(\nu+1)}} ds. \quad (12)
\]

Notice that the functions \( y_\nu \) and the functions \( \Phi_{\nu, \mu} \) are related by the formula
\[
y_\nu(z) = \frac{C_\nu}{\rho(z)} \Phi_{\nu, \nu}(z), \quad (13)
\]

**Lemma 3.1.** For the functions \( \Phi_{\nu, \mu}(z) \) the following relation holds
\[
\nabla_z \Phi_{\nu, \mu}(z) = [\mu + 1] \Phi_{\nu, \mu+1}(z), \quad (14)
\]
where \([t]_q\) denotes the symmetric \( q \)-numbers \(^4\).

\(^4\) Obviously the functions (3) correspond to the functions (11), whereas the functions \( y_\nu \) given by (4) correspond to those of (12).
Using the identity proof for the case of functions of the form (11), the other case is completely equivalent. Differences \( i = 1 \) holds

\[ \int \frac{1}{x_\nu(s) - x_\nu(z)} \left( \frac{1}{x_\nu(s) - x_\nu(z - 1)} - \frac{1}{x_\nu(s) - x_\nu(z - 1 - \mu)} \right) ds \]

Since \( x(s) - x(s - t) \) we then have

\[
\nabla_z \Phi_{\nu,\mu}(z) = \sum_{s=a}^{b-1} \frac{\rho_\nu(s) \nabla x_{\nu+1}(s)}{[x_\nu(s) - x_\nu(z)]^{(\mu+1)}}
\]

which is (14).

\[ \nabla_z \Phi_{\nu,\mu}(z) = [\mu + 1] q \nabla x_{\nu-\mu}(z) \Phi_{\nu,\mu+1}(z) \]

\[ \nabla_z \Phi_{\nu,\mu}(z) = [\mu + 1] q \nabla x_{\nu-\mu}(z) \Phi_{\nu,\mu+1}(z) \]

From (14) follows that

\[ \Delta_z \Phi_{\nu,\mu}(z) = [\mu + 1] q \Delta x_{\nu-\mu}(z) \Phi_{\nu,\mu+1}(z + 1) \]

Next we prove the following lemma that is the discrete analog of the Lemma in [14] page 14.

Lemma 3.2. Let \( x(z) \) be \( x(z) = q^z \) or \( x(z) = z \). Then, any three functions \( \Phi_{\nu_i,\mu_i}(z) \), \( i = 1, 2, 3 \), are connected by a linear relation

\[ \sum_{i=1}^{3} A_i(z) \Phi_{\nu_i,\mu_i}(z) = 0, \quad (15) \]

with non-zero at the same time polynomial coefficients on \( x(z) \), \( A_i(z) \), provided that the differences \( \nu_i - \nu_j \) and \( \mu_i - \mu_j \), \( i, j = 1, 2, 3 \), are integers and that the following condition hold:\[2\]

\[ \frac{x^k(s) \sigma(s) \rho_{\nu_0}(s)}{[x_{\nu_0-1}(s) - x_{\nu_0-1}(z)]^{(\mu_0)}} \bigg|_{s=b}^{s=a} = 0, \quad k = 0, 1, 2, \ldots, \quad (16) \]

when the functions \( \Phi_{\nu_i,\mu_i} \) are given by (17) and

\[ \int_C \nabla_z \Phi_{\nu_i,\mu_i}(z) \bigg|_{s=a}^{s=b} \nabla x_{\nu_i-1}(s) ds = 0, \quad k = 0, 1, 2, \ldots, \quad (17) \]

when \( \Phi_{\nu_i,\mu_i} \) are given by (12). Here \( \nu_0 \) is the \( \nu_i \), \( i = 1, 2, 3 \), with the smallest real part and \( \mu_0 \) is the \( \mu_i \), \( i = 1, 2, 3 \), with the largest real part.

Proof. Since in (14) we have proved the case when \( x(s) = s \) (the uniform lattice) we will restrict here to the case of the q-linear lattice \( x(s) = c_1 q^s + c_2 \). Moreover, we will give the proof for the case of functions of the form (11), the other case is completely equivalent. Using the identity

\[ \nabla x_{\nu+1}(s) = q^{\frac{\mu - \nu}{2}} \nabla x_{\nu+1}(s), \]

\[ \text{In some cases this condition is equivalent to the condition } x(s)^k \sigma(s) \rho_{\nu_0}(s) |_{s=a}^{s=b} = 0, \quad k = 0, 1, 2, \ldots. \]
as well as (3) of Proposition 2.3, we have

$$\sum_{i=1}^{3} A_i(z) \Phi(z, \mu_i) = \sum_{i=1}^{3} A_i(z) \sum_{s=0}^{b-1} \frac{\rho_{\nu_i}(s) \nabla x_{\nu_i+1}(s)}{[x_{\nu_i}(s) - x_{\nu_i}(z)]^{(\mu_i+1)}}$$

$$= \sum_{s=0}^{b-1} \sum_{i=1}^{3} A_i(z) \frac{\rho_{\nu_i}(s) \nabla x_{\nu_i+1}(s)}{[x_{\nu_i}(s) - x_{\nu_i}(z)]^{(\mu_i+1)}} = \sum_{s=0}^{b-1} \frac{1}{[x_{\nu_0}(s) - x_{\nu_0}(z)]^{(\mu_0+1)}} \times$$

$$\left( \sum_{i=1}^{3} A_i(z) q^{(\mu_0 + 1 - \mu_i - 1)} \frac{[x_{\nu_0}(s) - x_{\nu_i}(z)]^{(\mu_i-1)}}{[x_{\nu_i}(s) - x_{\nu_i}(z)]^{(\mu_i-1)}} \rho_{\nu_i}(s) \nabla x_{\nu_i+1}(s) \right)$$

Using the Pearson-type equation (6) we obtain

$$\rho_{\nu_i}(s) = \phi(s + \nu_0) \phi(s + \nu_0 + 1) \ldots \phi(s + \nu_i - 1) \rho_{\nu_i}(s), \quad (18)$$

so

$$\sum_{i=1}^{3} A_i(z) \Phi(z, \mu_i) = \sum_{s=0}^{b-1} \frac{\rho_{\nu_0}(s) \nabla x_{\nu_0+1}(s)}{[x_{\nu_0}(s) - x_{\nu_0}(z)]^{(\mu_0+1)}} \Pi(s)$$

where

$$\Pi(s) = \sum_{i=1}^{3} A_i(z) q^{(\mu_0 - \mu_i)} \frac{[x_{\nu_0}(s) - x_{\nu_i}(z)]^{(\mu_i-1)}}{[x_{\nu_i}(s) - x_{\nu_i}(z)]^{(\mu_i-1)}} \times \phi(s + \nu_0) \phi(s + \nu_0 + 1) \ldots \phi(s + \nu_i - 1). \quad (19)$$

Let us show that there exists a polynomial \( Q(s) \) in \( x(s) \) (in general, \( Q \equiv Q(z, s) \) is a function of \( z \) and \( s \)) such that

$$\frac{\rho_{\nu_0}(s) \nabla x_{\nu_0+1}(s)}{[x_{\nu_0}(s) - x_{\nu_0}(z)]^{(\mu_0+1)}} \Pi(s) = \Delta \left[ \frac{\rho_{\nu_0}(s - 1)}{[x_{\nu_0-1}(s) - x_{\nu_0-1}(z)]^{(\mu_0)}} Q(s) \right]$$

$$= \Delta \left[ \frac{\sigma(s) \rho_{\nu_0}(s)}{[x_{\nu_0-1}(s) - x_{\nu_0-1}(z)]^{(\mu_0)}} Q(s) \right]. \quad (20)$$

If such polynomial exists, then, taking the sum in \( s \) from \( s = a \) to \( b - 1 \) and using the boundary conditions (16), we obtain (13).

To prove the existence of the polynomial \( Q(s) \) in the variable \( x(s) \) in (20) we write

$$\frac{\sigma(s + 1) \rho_{\nu_0}(s + 1)}{[x_{\nu_0-1}(s + 1) - x_{\nu_0-1}(z)]^{(\mu_0)}} Q(s + 1) - \frac{\sigma(s) \rho_{\nu_0}(s)}{[x_{\nu_0-1}(s) - x_{\nu_0-1}(z)]^{(\mu_0)}} Q(s) =$$

$$= \frac{\rho_{\nu_0}(s)}{[x_{\nu_0}(s) - x_{\nu_0}(z)]^{(\mu_0+1)}} \left[ \frac{\sigma(s + 1) \rho_{\nu_0}(s + 1)}{x_{\nu_0-1}(s + 1) - x_{\nu_0-1}(z)} \right]^{(\mu_0+1)} \frac{x_{\nu_0}(s) - x_{\nu_0}(z)}{x_{\nu_0-1}(s + 1) - x_{\nu_0-1}(z)} Q(s + 1) -$$

$$- \sigma(s) \frac{x_{\nu_0}(s) - x_{\nu_0}(z)}{x_{\nu_0-1}(s) - x_{\nu_0-1}(z)} \frac{x_{\nu_0}(s) - x_{\nu_0}(z)}{x_{\nu_0-1}(s - 1) - x_{\nu_0-1}(z)} Q(s).$$

From (4) and (5) of Proposition 2.3, and using (19), the above expression becomes

$$\frac{\rho_{\nu_0}(s)}{[x_{\nu_0}(s) - x_{\nu_0}(z)]^{(\mu_0+1)}} \left\{ \phi_{\nu_0}(s) \left[ x_{\nu_0-\mu_0}(s) - x_{\nu_0-\mu_0}(z) \right] Q(s + 1) - \sigma(s) \left[ x_{\nu_0-\mu_0}(s + \mu_0) - x_{\nu_0-\mu_0}(z) \right] Q(s) \right\}.$$
Thus
\[ (\sigma(s) + \tau_{\nu_0}(s) \nabla x_{\nu_0+1}(s)) [x_{\nu_0-\mu_0}(s) - x_{\nu_0-\mu_0}(z)] Q(s + 1) = \sigma(s) [x_{\nu_0-\mu_0}(s + \mu_0) - x_{\nu_0-\mu_0}(z)] Q(s) = \nabla x_{\nu_0+1}(s) \Pi(s). \] (21)

Since \( \nabla x_{\nu_0+1}(s) \) is a polynomial of degree one in \( x(s) \), \( x_k(s) \) and \( \tau_{\nu_0}(s) \) are polynomials of degree at most one in \( x(s) \), and \( \sigma(s) \) is a polynomial of degree at most two in \( x(s) \), we conclude that the degree of \( Q(s) \) is, at least, two less than the degree of \( \Pi(s) \), i.e., \( \deg Q \geq \deg \Pi - 2 \). Moreover, equating the coefficients of the powers of \( x(s) = q^s \) on the two sides of the above equation (21), we find a system of linear equations in the coefficients of \( Q(s) \) and the coefficients \( A_i(z) \) which have at least one unknown more then the number of equations. Notice that the coefficients of the unknowns are polynomials in \( q^z \), so that after one coefficient is selected the remaining coefficients are rational functions of \( q^z \), therefore after multiplying by the common denominator of the \( A_i(z) \) we obtain the linear relation with polynomial coefficients on \( x \equiv x(z) = q^z \). This completes the proof. \( \square \)

The above Lemma when \( q \to 1 \) and \( x(s) = s \) leads to the corresponding result on the uniform lattice \( x(s) \) [3].

3.1. Some representative examples. In the following examples, and for the sake of simplicity, we will use the notation
\[ \sigma(s) = a q^{2s} + bq^s + c, \quad \tau(s) = dq^s + e, \quad \phi_{\nu}(s) = \sigma(s) + \tau_{\nu-1}(s) \nabla x_{\nu}(s) = fz^{2s} + gq^s + h. \] (22)

Example 3.3. The following relation holds
\[ A_1(z) \Phi_{\nu,\nu-1}(z) + A_2(z) \Phi_{\nu,\nu} + A_3(z) \Phi_{\nu+1,\nu}(z) = 0, \]
where the coefficients \( A_1, A_2 \) and \( A_3 \) are polynomials in \( x \equiv x(z) = q^z \), given by
\[ A_1(z) = -eq^z + \frac{b + e(q^z - q^{-z})}{a + d(q^z - q^{-z})} (dq^z + a[\nu]_q) + \left( dq^s + a[2\nu]_q \right) q^{z+2}, \]
\[ A_2(z) = -c \frac{dq^z + a[2\nu]_q}{a + d(q^z - q^{-z})} + \frac{b + e(q^z - q^{-z})}{q^z - q^{-z}} \left( q^z + \frac{a}{q^z - q^{-z}} \right) q^{z+1}, \]
\[ A_3(z) = - \frac{dq^z + a[\nu]_q}{a + d(q^z - q^{-z})}, \]
where \( a, b, c, d, \) and \( e \) are the coefficients of \( \sigma \) and \( \tau \) (22).

Proof. Using the notations of Lemma 3.2 we have \( \nu_1 = \nu, \nu_2 = \nu, \nu_3 = \nu + 1, \mu_1 = \nu - 1, \mu_2 = \nu \) and \( \mu_3 = \nu \), thus \( \nu_0 = \nu \) and \( \mu_0 = \nu \). By (19)
\[ \Pi(s) = A_1(q^{s+\frac{z}{2}} - q^{-\frac{z}{2}}) + A_2 + A_3q^{-\frac{z}{2}} \left[ (a + d(q^{s+\frac{z}{2}} - q^{-\frac{z}{2}})) q^{2\nu+2s} + (b + e(q^{s+\frac{z}{2}} - q^{-\frac{z}{2}})) q^{\nu+s} + c \right]. \] (23)

On the other hand, from (21) and because \( Q(s) = k \) is a constant notice that \( \deg(\Pi) = 2 \) we have
\[ \nabla x_{\nu_0+1}(s) \Pi(s) = k \left\{ (a + d(q^{s+\frac{z}{2}} - q^{-\frac{z}{2}})) q^{2\nu+2s} + (b + e(q^{s+\frac{z}{2}} - q^{-\frac{z}{2}})) q^{\nu+s} + c \right\} (q^s - q^{-s}) \]
\[ - (aq^{2s} + bq^s + c)(q^{\nu+s} - q^{-s}) \] (24)
where \( k \) is an arbitrary constant. Introducing (23) in (21), using the identity
\[ \nabla x_{\nu_0+1}(s) = q^z \left( q^{\frac{z}{2}} - q^{-\frac{z}{2}} \right) q^s \]
and comparing the coefficients of the powers of \( x(s) = q^s \) we get a linear system of three equations with four variables \( A_1, A_2, A_3 \) and \( k \). Choosing \( k = 1 \) and solving the corresponding system we get, after some simplifications, the coefficients \( A_1, A_2 \) and \( A_3 \). \( \square \)
In the next examples, since the technique is similar to the previous one we will omit the details.

**Example 3.4.** The following relation holds

\[ A_1(z)\Phi_{\nu,\nu}(z) + A_2(z)\Phi_{\nu,\nu+1}(z) + A_3(z)\Phi_{\nu+1,\nu+1}(z) = 0, \]

where the coefficients \( A_1, A_2 \) and \( A_3 \), are polynomials in \( x \equiv x(z) = q^z \), given by

\[
A_1(z) = f \left( a - f q^{2\nu} \right) q^z + agq - f b q^{\nu+1},
\]

\[
A_2(z) = q^{-\frac{\nu}{2}} \left( a - f q^{2\nu} \right) \left( f q^{2z} + g q^{z+1} + h q^2 \right),
\]

\[
A_3(z) = \sqrt{q} \left( a q - f q^{\nu} \right),
\]

where \( a, b, c, f, g \) and \( h \), are the coefficients of \( \sigma \) and \( \phi_\nu \).

**Example 3.5.** The following relation holds

\[ A_1(z)\Phi_{\nu-1,\nu-1}(z) + A_2(z)\Phi_{\nu-1,\nu}(z) + A_3(z)\Phi_{\nu,\nu}(z) = 0, \]

where the coefficients \( A_1, A_2 \) and \( A_3 \), are polynomials in \( x \equiv x(z) = q^z \), given by

\[
A_1(z) = q^{-\frac{\nu}{2}} \left\{ \begin{array}{l}
q f^2 q^z + a^2 h q^4 + a g b q^{\nu+4} - q^{2\nu+2} (a g^2 q - 2 f a h + f b^2) \\
- f g b q^{3\nu+1} (q^2 - q - 1) + q f q^{4\nu} (q^2 (q - 1) q - f h) + q g q^{3\nu+1} (a^2 q^2 + a^2 q^{\nu+2} (g b q^2 + f h q^2 - f h) - q^{2\nu+2} (f a h + f g b + a q^2 - f q^{2a} b q^{2\nu+2} + a g b q^{\nu+5} (b q + g q - g) + f g b q^{\nu+4} (q g + b - g) - f^2 h q^{4\nu+2} - h q^{2\nu+3} (a g^2 q^2 + f g b q^2 + f g q^2 - 2 f a h q + f b q^2 - 2 f g q^2 - f g b + f q^2) \right\},
\]

\[
A_2(z) = (q^{-\frac{\nu}{2}} - q^{\nu}) \left( f q^{2z} + g q^{3\nu+1} + h q^2 \right) \left( f q^{-2x} (f q^{2x} - a q^2) + q f^{\nu+1} (g q + b - g) - a g q^3 \right),
\]

\[
A_3(z) = f (f q^{\nu} - a q) \left[ (f q^{2x} + h q^2) (f q^{2x} - a q^2) + q g q^{z+1} (f q^{\nu} (q^2 - q - 1) - a q^2) \right],
\]

where \( a, b, c, f, g \) and \( h \), are the coefficients of \( \sigma \) and \( \phi_\nu \).

**Example 3.6.** The following relation holds

\[ A_1(z)\Phi_{\nu-1,\nu-1}(z) + A_2(z)\Phi_{\nu,\nu}(z) + A_3(z)\Phi_{\nu+1,\nu+1}(z) = 0, \]

where the coefficients \( A_1, A_2 \) and \( A_3 \), are polynomials in \( x \equiv x(z) = q^z \), given by

\[
A_1(z) = q^2 h q^4 - a g b q^{\nu+3} + q^{2\nu+2} (f q^2 - 2 f a h + a g q^2) - f g b q^{3\nu+1} + f^2 h q^{4\nu},
\]

\[
A_2(z) = q^{\frac{\nu}{2}} \left( f q^{\nu} - a q^2 \right) \left( f q^{2\nu+2} - a q^{z+2} + q g q^{2\nu+1} - b q^{\nu+2} \right),
\]

\[
A_3(z) = -q^{\frac{\nu}{2}} \left( f q^{2x} - a q^2 \right) \left( q^{\nu+1} - 1 \right) (g q^{\nu+1} + f q^{\nu+2} + h q^2),
\]

where \( a, b, c, f, g \) and \( h \), are the coefficients of \( \sigma \) and \( \phi_\nu \).

**Example 3.7.** The relation

\[ A_1(z)\Phi_{\nu,\nu}(z) + A_2(z)\Phi_{\nu,\nu+1}(z) + A_3(z)\Phi_{\nu+1,\nu+1}(z) = 0, \]

is verified when the polynomial coefficients \( A_1, A_2 \) and \( A_3 \), in the variable \( x \equiv x(z) = q^z \), are given by

\[
A_1(z) = q^{\nu+1} \left( f q^{\nu} - a q \right) \left( f q^{2\nu} - a q^{\nu+1} - f (h - b) q^{2\nu+1} - a q (g q^2 - b) \right),
\]

\[
A_2(z) = q^{-\frac{\nu}{2}} \left( f q^{2\nu} - a q \right) \left( f q^{2\nu} - a q^{\nu+1} + h q^2 \right),
\]

\[
A_3(z) = q^{\nu+1} \left( f q^{\nu} - a q \right) \left( q (g q^2 - a q - b) - f q^{\nu} (q^{\nu+1} - q - 1) \right) + q^{\nu+1} \left( (h - b) q^{\nu+1} + g q^2 - h \right),
\]

where \( a, b, c, f, g \) and \( h \), are the coefficients of \( \sigma \) and \( \phi_\nu \).
4. Recurrences involving the solutions $y_\nu$

In [16] the following relevant relation was established

$$\Delta^{(k)} y_\nu(s) = \frac{C^{(k)}_\nu}{\rho_k(s)} \Phi_{\nu, \nu-k}(s), \quad (25)$$

where

$$C^{(k)}_\nu = C_\nu \prod_{m=0}^{k-1} \left[ \alpha_q(\nu + m - 1) \tilde{c}' + [\nu + m - 1] \tilde{c}'' \right].$$

This relation is valid for solutions of the form (3) and (4) of the difference equation (1).

**Theorem 4.1.** In the same conditions as in Lemma 3.2, any three functions $y^{(k_i)}_{\nu_i}(s)$, $i = 1, 2, 3$, are connected by a linear relation

$$\sum_{i=1}^{3} B_i(s) y^{(k_i)}_{\nu_i}(s) = 0, \quad (26)$$

where the $B_i(s)$, $i = 1, 2, 3$, are polynomials.

**Proof.** From Lemma 3.2 we know that there exists three polynomials $A_i(s)$, $i = 1, 2, 3$ such that

$$\sum_{i=1}^{3} A_i(s) \Phi_{\nu_i, \nu_i-k_i}(s) = 0,$$

then, using the relation (25), we find

$$\sum_{i=1}^{3} A_i(s) (C^{(k)}_\nu)^{-1} \rho_{k_i}(s) y^{(k_i)}_{\nu_i}(s) = 0.$$

Now, dividing the last expression by $\rho_{k_0}(s)$, where $k_0 = \min\{k_1, k_2, k_3\}$, and using (18) we obtain

$$\sum_{i=1}^{3} B_i(s) y^{(k_i)}_{\nu_i}(s) = 0, \quad B_i(s) = A_i(s) (C^{(k)}_\nu)^{-1} \phi(s + k_0) \cdots \phi(s + k_i - 1),$$

which completes the proof. \(\square\)

**Corollary 4.2.** In the same conditions as in Lemma 3.2 the following three-term recurrence relation holds

$$A_1(s) y_\nu(s) + A_2(s) y_{\nu+1}(s) + A_3(s) y_{\nu-1}(s) = 0,$$

with polynomial coefficients $A_i(s)$, $i = 1, 2, 3$.

**Proof.** It is sufficient to put $k_1 = k_2 = k_3 = 0$, $\nu_1 = \nu$, $\nu_2 = \nu + 1$ and $\nu_3 = \nu - 1$ in (26). \(\square\)

**Corollary 4.3.** In the same conditions as in Lemma 3.2 the following $\Delta$-ladder-type relation holds

$$B_1(s) y_\nu(s) + B_2(s) \frac{\Delta y_\nu(s)}{\Delta x(s)} + B_3(s) y_{\nu+m}(s) = 0, \quad m \in \mathbb{Z}, \quad (27)$$

with polynomial coefficients $B_i(s)$, $i = 1, 2, 3$.

**Proof.** It is sufficient to put $k_1 = k_3 = 0$, $k_2 = 1$, $\nu_1 = \nu_2 = \nu$ and $\nu_3 = \nu + m$ in (26). \(\square\)
Notice that for the case \( m = \pm 1 \) \( (27) \) becomes

\[
B_1(s)y_{\nu}(s) + B_2(s)\frac{\Delta y_{\nu}(s)}{\Delta x(s)} + B_3(s)y_{\nu+1}(s) = 0, \tag{28}
\]

\[
\tilde{B}_1(s)y_{\nu}(s) + \tilde{B}_2(s)\frac{\Delta y_{\nu}(s)}{\Delta x(s)} + \tilde{B}_3(s)y_{\nu-1}(s) = 0, \tag{29}
\]

with polynomial coefficients \( B_i(s) \) and \( \tilde{B}_i(s), i = 1, 2, 3 \). The above relations are usually called raising and lowering operators, respectively, for the functions \( y_{\nu} \).

Let us now obtain a raising and lowering operators for the functions \( y_{\nu} \) but associated to the \( \nabla/\nabla x(s) \) operators.

We start applying the operator \( \nabla/\nabla x(s) \) to \( (13) \)

\[
\frac{\nabla}{\nabla x(s)}y_{\nu}(s) = \frac{\nabla}{\nabla x(s)} \left[ \frac{C_\nu}{\rho(s)} \Phi_{\nu,\nu}(s) \right]
= \frac{1}{\nabla x(s)} \left[ C_\nu \Phi_{\nu,\nu}(s) \left( \frac{1}{\rho(s)} - \frac{1}{\rho(s-1)} \right) + \frac{C_\nu}{\rho(s-1)} \nabla \Phi_{\nu,\nu}(s) \right],
\]

or, equivalently,

\[
\frac{\nabla \Phi_{\nu,\nu}}{\nabla x(s)} = \frac{\rho(s-1)}{C_\nu} \frac{\nabla y_{\nu}}{\nabla x(s)} - \frac{\Phi_{\nu,\nu}}{\nabla x(s)} \left[ \frac{\rho(s-1)}{\rho(s)} - 1 \right].
\]

By Lemma \( \text{[3.2]} \) with \( \nu_1 = \mu_1 = \nu_2 = \nu, \mu_2 = \nu + 1 \) and \( \nu_3 = \mu_3 = \nu + m \), there exist polynomial coefficients on \( x(s), A_i(s), i = 1, 2, 3 \), such that

\[
A_1(s)\Phi_{\nu,\nu}(s) + A_2(s)\Phi_{\nu,\nu+1}(s) + A_3(s)\Phi_{\nu+m,\nu+m}(s) = 0.
\]

From \( \text{(14)} \)

\[
\Phi_{\nu,\nu+1}(s) = \frac{1}{[\nu+1]_q} \frac{\nabla \Phi_{\nu,\nu}}{\nabla x(z)} = \frac{1}{[\nu+1]_q} \frac{\nabla \Phi_{\nu,\nu}}{\nabla x(z)}.
\]

Therefore

\[
A_1(s)\Phi_{\nu,\nu} + \frac{A_2(s)}{[\nu+1]_q} \left[ \frac{\rho(s-1)}{C_\nu} \frac{\nabla y_{\nu}}{\nabla x(s)} - \frac{\Phi_{\nu,\nu}}{\nabla x(s)} \left( \frac{\rho(s-1)}{\rho(s)} - 1 \right) \right] + A_3\Phi_{\nu+m,\nu+m} = 0.
\]

Using now the Pearson equation \( (3) \) and dividing by \( \rho(s) \) we get

\[
A_1(s)y_{\nu}(s) + \frac{A_2(q)}{[\nu+1]_q} \frac{\sigma(s)}{\phi(s-1)} \frac{\nabla y_{\nu}}{\nabla x(s)} - \frac{y_{\nu}(s)}{\nabla x(s)} \left( \frac{\sigma(s)}{\phi(s-1)} - 1 \right) + A_3\frac{C_\nu}{C_\nu+1} y_{\nu+m}(s) = 0.
\]

Multiplying both sides by \( \frac{[\nu+1]_q \phi(s-1)}{1} \),

\[
A_1(s)[\nu+1]_q \phi(s-1) y_{\nu}(s) + A_2(s)\sigma(s) \frac{\nabla y_{\nu}}{\nabla x(s)} - \frac{\sigma(s)}{\phi(s-1)} y_{\nu}(s) + A_2(s)\sigma(s) \frac{\nabla y_{\nu}}{\nabla x(s)} - A_2(s)\sigma(s) \frac{\nabla y_{\nu}}{\nabla x(s)} + A_3\frac{C_\nu}{C_\nu+1} A_3\phi(s-1) y_{\nu+m}(s) = 0.
\]

Thus we have proven the following

**Theorem 4.4.** In the same conditions as in Lemma \( \text{[3.2]} \) the following \( \nabla \)-ladder-type relation holds

\[
C_1(s)y_{\nu}(s) + C_2(s)\frac{\nabla y_{\nu}(s)}{\nabla x(s)} + C_3(s)y_{\nu+m}(s) = 0, \quad m \in \mathbb{Z}, \tag{30}
\]

with polynomial coefficients \( C_i(s), i = 1, 2, 3 \).
Notice that for the case \( m = \pm 1 \) (30) becomes
\[
C_1(s)y_\nu(s) + C_2(s)\frac{\nabla y_\nu(s)}{\nabla x(s)} + C_3(s)y_{\nu+1}(s) = 0,
\]
(31)
\[
\tilde{C}_1(s)y_\nu(s) + \tilde{C}_2(s)\frac{\nabla y_\nu(s)}{\nabla x(s)}y_\nu(s) + \tilde{C}_3(s)y_{\nu-1}(s) = 0,
\]
(32)
with polynomial coefficients \( C_i(s) \) and \( \tilde{C}_i(s), i = 1, 2, 3 \). The above relation are usually called raising and lowering operators, respectively, for the functions \( y_\nu \). Eq. (31) was firstly obtained in [16, Eq. (3.4)].

To conclude this section let us point that from formula (25) and the examples 3.3, 3.5 and 3.7 follow the relations
\[
\begin{align*}
B_1(s)y_\nu^{(1)}(s) + B_2(s)y_\nu(s) + B_3(s)y_{\nu+1}(s) &= 0, \\
B_1(s)y_\nu^{(1)}(s) + B_2(s)y_{\nu-1}(s) + B_3(s)y_\nu(s) &= 0, \\
B_1(s)y_\nu^{(1)}(s) + B_2(s)y_\nu(s) + B_3(s)y_{\nu+1}(s) &= 0,
\end{align*}
\]
(33)
respectively, being the last two expressions the lowering and raising operators for the functions \( y_\nu \). Moreover, combining the explicit values of \( A_1, A_2 \) and \( A_3 \) with formula (25), one can obtain the explicit expressions for the coefficients \( B_1, B_2 \) and \( B_3 \) in (35).

5. Applications to q-classical polynomials

In this section we will apply the previous results to the q-classical orthogonal polynomials [2, 10, 11] in order to show how the method works. We first notice that these polynomials are instances of the functions \( y_\nu \) on the lattice \( x(s) = q^s \) defined in (4). In fact we have [13, 16]
\[
P_n(x(s)) = \frac{[n]! B_n}{\rho(s) 2\pi i} \int_C \frac{\rho_n(z) \nabla x_{n+1}(z)}{[x_n(z) - x_n(s)]^{n+1}} dz,
\]
(34)
where \( B_n \) is a normalizing constant, \( C \) is a closed contour surrounding the points \( x = s, s - 1, \ldots, s - n \) and it is assumed that \( \rho_n(s) = \rho(s + n) \prod_{m=1}^{n} \sigma(s + m) \) and \( \rho_n(s + 1) \) are analytic inside \( C \) (\( \rho \) is the solution of the Pearson equation (6)), i.e., the condition (5) holds.

A detailed study of the q-classical polynomials, including several characterization theorems, was done in [2, 9, 11]. In particular, a comparative analysis of the q-Hahn tableau with the q-Askey tableau [9] and Nikiforov-Uvarov tableau [15] was done in [5]. In the following we use the standard notation for the q-calculus [8]. In particular by \( (a;q)_k = \prod_{m=0}^{k-1} (1 - aq^m) \), we denote the q-analogue of the Pochhammer symbol.

Since the q-classical polynomials are defined by (34) where the contour \( C \) is closed and \( \nu \) is a non-negative integer, then the condition (17) is automatically fulfilled, so Lemma 3.2 holds for all of them. Moreover, the Theorem 4.1 holds and there exist the non vanishing polynomials \( B_1, B_2 \) and \( B_3 \) of (26).

In the following we will assume that the three term recurrence relation is known, i.e.,
\[
x(s)P_n(x(s)) + \alpha_n P_{n+1}(x(s)) + \beta_n P_n(x(s)) + \gamma_n P_{n-1}(x(s)) = 0, \quad n \geq 0
\]
(35)
\[
P_{-1}(x(s)) = 0, \quad P_0(x(s)) = 1, \quad x(s) = q^s.
\]

where the coefficients \( \alpha_n, \beta_n \) and \( \gamma_n \) can be computed using the coefficients \( \sigma, \tau \) and \( \lambda \equiv \lambda_n \) of (1), being \( \lambda_n \) given by [8] and [9] with \( \nu = n \). For more details see, e.g., [1, 11].

Since the TTRR and the differentiation formulas for the q-polynomials are very well known (see e.g., [9, 11, 16]) we will obtain here two recurrent-difference relations involving the q-differences of the polynomials and the polynomials themselves.
5.1. The first difference-recurrence relation. If we choose \( \nu_1 = n - 1, \nu_2 = n, \nu_3 = n + 1, k_1 = 1, k_2 = 1 \) and \( k_3 = 0 \), in Theorem 4.1 one gets

\[
A_1(s)\Delta^{(1)} P_{n-1}(x(s)) + A_2(s)\Delta^{(1)} P_n(x(s)) + A_3(s)P_{n+1}(x(s)) = 0.
\]

Using [1], Eq. (6.14), page 193

\[
[\sigma(s) + \tau(s)\Delta x(s - 1/2)]\Delta^{(1)} P_n(x(s)) = \tilde{\alpha}_n P_{n+1}(x(s)) + \tilde{\beta}_n P_n(x(s)) + \tilde{\gamma}_n P_{n-1}(x(s)),
\]

where

\[
\tilde{\alpha}_n = \frac{\lambda_n}{[n]_q} \left[ q^{-\frac{\tau}{2\tau_n}}\alpha_n - \frac{B_n}{\tau_n P_{n+1}} \right], \quad \tilde{\beta}_n = \frac{\lambda_n}{[n]_q} \left[ q^{-\frac{\tau}{2\tau_n}}\beta_n + \frac{\tau_n(0)}{\tau_n} - c_3(q^{-\frac{\tau}{2}} - 1) \right],
\]

\[
\tilde{\gamma}_n = \frac{\lambda_n q^{-\frac{\tau}{2\tau_n}}} {[n]_q},
\]

to compute \( \Delta^{(1)} P_n(x(s)) = \frac{\Delta P_n(x(s))}{\Delta x(s)} \) we get

\[
\left[ A_2(s)\tilde{\alpha}_n \left( q^{-\frac{\tau}{2\tau_n}}\alpha_n - \frac{B_n}{\tau_n P_{n+1}} \right) + (\sigma(s) + \tau(s)\Delta x(s - \frac{1}{2})) A_3(s) \right] P_{n+1} +
\]

\[
\left[ A_1(s)\tilde{\alpha}_n \left( q^{-\frac{\tau}{2\tau_n}}\alpha_n - \frac{B_n}{\tau_n P_{n+1}} \right) + A_2(s)\tilde{\alpha}_n \left( q^{-\frac{\tau}{2\tau_n}}\beta_n + \frac{\tau_n(0)}{\tau_n} \right) \right] P_n +
\]

\[
\left[ A_1(s)\tilde{\alpha}_n \left( q^{-\frac{\tau}{2\tau_n}}\beta_n + \frac{\tau_n(0)}{\tau_n} \right) + A_2(s)\tilde{\alpha}_n q^{-\frac{\tau}{2\tau_n}} \gamma_n \right] P_{n-1} +
\]

\[
A_1(s)\tilde{\alpha}_n \frac{n-1}{[n-1]_q} \tau_n^{-1} P_{n-2} = 0,
\]

By (35) we may write

\[
P_{n-2}(x(s)) = \frac{x(s) - \beta_{n-1}}{\gamma_{n-1}} P_{n-1}(x(s)) - \frac{\alpha_{n-1}}{\gamma_{n-1}} P_n(x(s))
\]

so the above equality becomes

\[
\left[ \frac{\lambda_n}{[n]_q} \left( q^{-\frac{\tau}{2}}\alpha_n - \frac{B_n}{\tau_n P_{n+1}} \right) A_2(s) + (\sigma(s) + \tau(s)\Delta x(s - \frac{1}{2})) A_3(s) \right] P_{n+1}(x(s)) +
\]

\[
\left[ \frac{\lambda_n}{[n]_q} \left( q^{-\frac{\tau}{2}}\beta_n + \frac{\tau_n(0)}{\tau_n} \right) \right] A_2(s) \left( q^{-\frac{\tau}{2}}\beta_n + \frac{\tau_n(0)}{\tau_n} \right) A_2(s)\right] P_n(x(s)) +
\]

\[
\left[ \frac{\lambda_n}{[n]_q} \left( \frac{\tau_n(0)}{\tau_n} + q^{-\frac{\tau}{2}} x \right) A_1(s) + \frac{\lambda_n}{[n]_q} q^{-\frac{\tau}{2}} \gamma_n A_2(s) \right] P_{n-1}(x(s)) = 0.
\]

Comparing the above equation with the TTRR (35) one can obtain the explicit values of \( A_1, A_2, \) and \( A_3 \).

5.1.1. Some examples. Since we are working in the \( q \)-linear lattice \( x(s) = q^s \), for the sake of simplicity, we will use the letter \( x \) to denote the variable of the polynomials [9, 11]. We will consider monic polynomials, i.e., those with the leading coefficient equal to 1. In the following we need the value of \( \tau_n(x) \) for each family, which can be computed using [7].

Al-Salam-Carlitz I \( q \)-polynomials. For the Al-Salam-Carlitz I monic polynomials \( U_n^{(s)}(x; q) \) we have (see [1] see table 6.5, p.208 or [11])

\[
\sigma(x) = (1 - x)(a - x), \quad \tau_n(x) = \frac{1 - q^n}{1 - q} \left( x - (1 + a) \right),
\]

\[
\tau(x) = \tau_0(x), \quad \lambda_n = -q^{\frac{\tau}{2} - (1 - q^n)},
\]

and

\[
\alpha_n = 1, \quad \beta_n = (1 + a)q^n, \quad \gamma_n = -aq^{n-1} (1 - q^n).
\]
The constant $B_n$ is given by [1] Eq. (5.57), p. 147, $B_n = q^{\frac{1}{2}n(3n-5)}(1-q)^n$. Introducing these values into the equation (38) it becomes

$$\left[q \left(q^{-\frac{q^2}{2}} - 1\right) A_2(x) + a(1-q)q^n A_3(x)\right] U_{n+1}^{(a)}(x; q) +$$

$$\left[q^{-\frac{q^2}{2}}A_1(x) + q^{1+\frac{q^2}{2}}(1+a) \left(1 - q^{\frac{q^2}{2}}\right) A_2(x)\right] U_{n}^{(a)}(x; q) +$$

$$\left([q^{\frac{n+1}{2}}(1+a) - q^{2-n}x] A_1(x) + aq^n (1-q^n) A_2(x)\right) U_{n-1}^{(a)}(x; q) = 0.$$  

Comparing with the TTRR for the Al-Salam I polynomials we obtain a linear system for getting the unknown coefficients $A_1$, $A_2$ and $A_3$

$$q \left(q^{-\frac{q^2}{2}} - 1\right) A_2(x) + a(1-q)q^n A_3(x) = 1,$$

$$q^{-\frac{q^2}{2}}A_1(x) + q^{1+\frac{q^2}{2}}(1+a) \left(1 - q^{\frac{q^2}{2}}\right) A_2(x) = (1+a)q^n - x,$$

$$\left([q^{\frac{n+1}{2}}(1+a) - q^{2-n}x] A_1(x) + aq^n (1-q^n) A_2(x)\right) = aq^{n-1} (q^n - 1).$$

The solution of the above system is

$$A_1(x) = \frac{aq^n (1+q^\frac{x}{2}) \left((1+a)-q^{-\frac{q^2}{2}}x\right)}{aq^{-\frac{q^2}{2}}(1+q^\frac{x}{2}) - q(1+a) \left(q^{\frac{n+1}{2}}(1+a) - q^{2-n}x\right)},$$

$$A_2(x) = \frac{-aq^{-\frac{q^2}{2}}(1-q^n) - \left((1+a)q^n - x\right) \left(q^{\frac{n+1}{2}}(1+a) - q^{2-n}x\right)}{(1-q^\frac{x}{2}) \left[aq^{-\frac{q^2}{2}}(1+q^\frac{x}{2}) - q(1+a) \left(q^{\frac{n+1}{2}}(1+a) - q^{2-n}x\right)\right]},$$

$$A_3(x) = \frac{a + q^{\frac{n+1}{2} - 2n}x^2 + q^{-\frac{q^2}{2}}(a-1)q^x}{(a-1)q^n \left[aq^n + q^{\frac{q^2}{2}}(a-1)q^x\right]}.$$  

Then, the Al-Salam I q-polynomials satisfy the following relation

$$A_1(x)\Delta^{(1)} U_{n-1}^{(a)}(x; q) + A_2(x)\Delta^{(1)} U_{n}^{(a)}(x; q) + A_3(x)U_{n+1}^{(a)}(x; q) = 0,$$

where the coefficients $A_1$, $A_2$ and $A_3$ are given by (37).

Notice that the coefficients $A_1$, $A_2$ and $A_3$ are rational functions on $x$. Therefore, multiplying by appropriate factor it becomes a linear relation with polynomials coefficients.

**Alternative q-Charlier polynomials.** In this case (see [1] table 6.6, p.209)

$$\sigma(x) = q^{-1}x(1-x), \quad \tau_n(x) = -q^{\frac{n+1}{1-q}} \left(1 + aq^{1+2n} x - 1\right),$$

$$\tau(x) = \tau_0(x), \quad \lambda_n = q^{\frac{1+2n}{1-q}(1+aq^n)\left(1-q^{2n}\right)}.$$  

and, for the monic case, $\alpha_n = 1$

$$\beta_n = \frac{q^{\frac{n(1+aq^n-1+aq^{n-2n})}{1+aq^{2n-1}(1+aq^{2n+1})}}, \quad \gamma_n = \frac{aq^{3n-2} (1-q^n) (1+aq^{n-1})}{(1+aq^{2n-2})(1+aq^{2n-1})^2(1+aq^{2n})}.$$  

The corresponding normalizing constant $B_n$ is given by

$$B_n = \frac{(-1)^n q^{\frac{1}{2}n(3n-1)}(1-q)^n}{(-aq^n)^n}.$$  

Following the same procedure as before we obtain the following relation for the alternative Charlier q-polynomials:

$$A_1(x)\Delta^{(1)} K_{n-1}(x; a; q) + A_2(x)\Delta^{(1)} K_n(x; a; q) + A_3(x)K_{n+1}(x; a; q) = 0,$$
with the coefficients

\[ A_1(x) = \frac{a(1 + aq^x)}{q^2(1 + aq^{2n-1})(1 + aq^{2n-1})} x \]

\[ A_2(x) = \frac{-q^{3n+1}(1 + aq^n) x + (1 + aq^{2n}) \left(q^2 + (1 + aq^{2n+1}) + aq^{2n+1}(1 + q) + q^{2n}(1 - aq^{2n}) \right) x^2}{q^{3n}(1 + aq^n)(1 + aq^{2n})} \]

\[ A_3(x) = \frac{\frac{a+1}{q} + aq^{2n} \left(q^{\frac{2}{n}} + 1 + q^{\frac{3}{n}} \right) - q^{\frac{2}{n}} \left(1 - aq^{\frac{3}{n}} \right) x}{q^{\frac{2}{n}} (1 + aq^n)} \]

Big q-Jacobi polynomials. In this case (see [I] see table 6.2, p.204) or [II]

\[ \sigma(x) = q^{-1}(x - aq)(x - cq), \lambda_n = -q^{\frac{1-n}{2}} \frac{(1 - abq^{1+n})(1 - q^n)}{(1 - q)^2}, \]

\[ \tau_n(x) = q^{\frac{1-n}{2}} \left(1 - abq^{2+2n} q^{-x} + a(b + c)q^{1+n} - (a + c) \right), \tau(x) = \tau_0(x), \]

and, for the monic case \( \alpha_n = 1 \),

\[ \beta_n = \frac{c + a^2bq^n \left(1 + b + c \right)q^{1+n} - 1} {q^{1-n} (1 - abq^{2n}) (1 - abq^{2n+2})}, \]

\[ \gamma_n = \frac{a(1 - q^n) (1 - abq^{2n}) (1 - b^q^n) (1 - c^q^n) (c - abq^n)} {q^{1-n} (1 - abq^{2n-1}) (1 - abq^{2n+1})^2 (1 - abq^{2n+1})}. \]

The corresponding normalizing constant is

\[ B_n = \frac{(1 - q^n) q^{\frac{1-n}{2}n(3n-1)}}{(abq^{1+n}; q)_n}. \]

The big q-Jacobi polynomials satisfy the following relation

\[ A_1(x)\Delta^{(1)}(x, a, b, c; q) + A_2(x)\Delta^{(1)}(x, a, b, c; q) + A_3(x)\Delta^{(1)}(x, a, b, c; q) = 0, \]

with the coefficients \( A_1, A_2 \) and \( A_3 \) given by

\[ A_1(x) = \frac{aq^{\frac{1-n}{2}n(1 - abq^{n+1})} (1 - x)(c - bx) \left( c = (c - c) + bx \right)} {1 - abq^{2n+2}} \times \]

\[ \left\{ (1 - q)q^n \left(1 - abq^{2n+2} \right) \left[ q^{1+1-n(b+c)(c+a(1+b+c))q^{2n+1} - (1+c+1+a+b+c)q^n(1+q)} \right] - x \right\} D(x) - \]

\[ (1 - q)q^n \left(1 - abq^{2n} \right) \left[ (1 - abq^{2n})(c + a(-1 + (b + c)q^{n+1})) \right] \]

\[ q^{\frac{1-n}{2}n(1 - abq^{n+1})} (1 - x)(c - bx)\left( c = c + b(1 + a + c)q^n(1 + q) \right) N(x) \right\}, \]

\[ A_2(x) = a(1 - q^n) (1 - abq^{2n})^2 \left(1 - abq^{2n+1} \right) \left(1 - x)(c - bx) \left( c = (c - c) + bx \right) \right) N(x), \]

\[ A_3(x) = (1 - abq^{n+1}) (1 - abq^{2n+2}) (1 - x)(c - bx) D(x) + \]

\[ q^{1-\frac{1}{2}n} \left(1 - q^n \right) \left(1 - abq^{1+\frac{2}{n}} \right) (1 - abq^{2n+2})^2 \left(1 - abq^{2n+2} \right) \left( c = (c - c) + bx \right) N(x), \]

where the polynomials \( N(x) \) and \( D(x) \) are given by

\[ N(x) = \frac{aq^{2(1-q^n)(1-abq^{n+1})(1-cq^n)(c-abq^n)}} {1-1-abq^{n+1}} \left[ q^{\frac{1}{2}n(1-q^n)} \left( c + a(1+b+c)q^n \right) + q^{\frac{1-n}{2}n(3n-1)} \right] \times \]

\[ \left[ c^2 + a^2bq^n \left( c + a(1+b+c)q^{n+1} + a(1+b)(-1 + q^n + q^{n+1} - b^q^n \left(1 + q - q^n \right) - x \right) \right] \]
where the coefficients $A$

\[
\Delta^{(1)} \equiv -c + a^2 \frac{q^2}{1-q^2} (-1 - q + (b+c)q^{n+1} - q^{1+\frac{2}{n}}) + a \left[-1 + (b+c)(q^{\frac{2}{n}} + q^n + q^{n+1}) - bc(q^{\frac{2}{n}} + q^{1+\frac{2}{n}} + q^{2n+1})\right] \left[(c + a)q^{1+\frac{2}{n}} - a(b+c)q^{1+\frac{2}{n}} - q^{\frac{1}{n}} (1 - abq^{2n})x\right],
\]

respectively.

5.2. The second difference-recurrence relation. If we choose $\nu_1 = n - 1$, $\nu_2 = n$, $\nu_3 = n + 1$, $k_1 = 0$, $k_2 = 0$ and $k_3 = 1$ in Theorem 4.1, and proceeding as in the previous case one gets

\[
A_1(x)P_{n-1}(x; q) + A_2(x)P_n(x; q) + A_3(x)\Delta^{(1)}P_{n+1}(x; q) = 0,
\]

(39)

where the coefficients $A_1$, $A_2$ and $A_3$, satisfy the linear relation

\[
A_3(x) \left[\left(q^{\frac{n+1}{2}} - \frac{B_{n+1}}{\alpha_{n+1}^2 B_{n+2}}\right) (x - \beta_{n+1}) + \left(q^{\frac{n+1}{2}} \beta_{n+1} + \frac{\gamma_{n+1}(0)}{\alpha_{n+1}}\right)\right] P_{n+1} +
\]

\[
A_3(x) \left[\frac{B_{n+1}}{\alpha_{n+1}^2 B_{n+2}} - \beta_{n+1} + \left(\sigma(x) + \tau(x)\Delta x (s - \frac{1}{2})\right)\left[\frac{n+1}{\lambda_{n+1}}\right] A_2(x)\right] P_n +
\]

\[
\left(\sigma(x) + \tau(x)\Delta x (s - \frac{1}{2})\right) \left[\frac{n+1}{\lambda_{n+1}}\right] A_1(x) P_{n-1} = 0.
\]

Comparing the above relation with the three-term recurrence relation (35) one can obtain the explicit expressions for the coefficients $A_1$, $A_2$ and $A_3$ in (39).

5.2.1. Some examples.

Al-Salam and Carlitz I polynomials. Using the main data for the Al-Salam and Carlitz I polynomials we obtain the relation

\[
A_1(x)U_{n-1}(x; q) + A_2(x)U_n(x; q) + A_3(x)\Delta^{(1)}U_{n+1}(x; q) = 0
\]

where

\[
A_1(x) = aq^{n-1} (1 - q^n) x, \quad A_2(x) = a \left[1 + q^{\frac{n}{2}}\right] q^n - (1 + a)q^n - x\right] x,
\]

\[
A_3(x) = -a \left[1 - q^{\frac{n}{2}} + q^{\frac{n+1}{2}}\right].
\]

Alternative $q$-Charlier polynomials. In this case, one gets

\[
A_1(x)K_{n-1}(x; a; q) + A_2(x)K_n(x; a; q) + A_3(x)\Delta^{(1)}K_{n+1}(x; a; q) = 0,
\]

\[
A_1(x) = a \left(1 - q^n\right) \left[1 + aq^{n-1}\right] \left[aq^n(1 - q^{n-1}) + q^{\frac{n+1}{2}}(1 + aq^{n+1})\right] \left[1 + aq^{n+1} - aq^{\frac{n+1}{2}}(1 + aq^{2n+2})\right] x
\]

\[
A_2(x) = -x \left[aq^n(1 - q^{n+1}) + q^{\frac{n+1}{2}}(1 + aq^{2n+1})\right] \left[1 + aq^{n+1} - aq^{\frac{n+1}{2}}(1 + aq^{2n+2})\right] x
\]

\[
A_3(x) = a \left[1 - q^{\frac{n+1}{2}}(1 + aq^{2n+1})\right] x^2.
\]

Concluding remarks. In this paper we present a constructive approach for finding recurrence relations for the hypergeometric-type functions on the linear-type lattices, i.e., the solutions of the hypergeometric difference equation (11) on the linear-type lattices. Important instances of “discret” functions are the celebrated Askey-Wilson polynomials and $q$-Racah polynomials. Such functions are defined on the non-uniform lattice of the form $x(s) = c_1(q)q^s + c_2(q)q^{-s} + c_2(q)$ with $c_1c_2 \neq 0$, i.e., a non-linear type lattice and therefore they require a more detailed study (some preliminar general results can be found in [16]).
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