A Stability Paradox in Fracture Processes: Can Local Stress Enhancement Induce Stability?

Jonas T. Kjellstadli, Eivind Bering, Martin Hendrick, Srutarshi Pradhan and Alex Hansen

PoreLab, Department of Physics, Norwegian University of Science and Technology, N-7491 Trondheim, Norway
(Dated: December 20, 2018)

By comparing the evolution of the local and equal load sharing fiber bundle models, we point out the paradoxical result that stresses may make the local load sharing model more stable than the equal load sharing model. We resolve the paradox by demonstrating that it originates from a statistical effect. Even though we use the fiber bundle model to demonstrate the paradox, we argue that it is a general feature of fracture processes.

PACS numbers:

The stability of materials against fracture is essential for our civilization. We need to be able to trust that buildings, bridges, airplanes, ships,... do not collapse. In order to prevent the collapse of structures, one needs to understand the processes that constitute fracture. Fracture has been part of engineering and materials science for a very long time. Only over the last thirty years, it has also become part of physics. Within the physics approach to fracture, there has been an emphasis on the role of disorder and fluctuations.

We may summarize the physics of fracture in a heterogeneous brittle materials as follows: The material heterogeneity implies that both the local strength of the material and the stress field it is experiencing are themselves heterogeneous. Fractures may occur and develop as a result of either the material being locally weak or locally under high stress. Applying a sufficiently large load to a material, the fracture process will start by the material failing where it is weakest. The ensuing microcracks will induce high stresses at the crack tips. If these are sufficiently high, the microcracks will grow. Hence, a competition between stress enhancement due to developing microcracks and local material weakness breaks out. At some point, the stress intensity at the crack tips has become so large that the local material weakness is no longer able to compete and catastrophic failure sets in: a macroscopic crack develops.

Essential in this summary is the opposite roles played by the heterogeneity and the stress enhancement: the heterogeneity stabilizes the fracture process whereas the stress enhancement destablizes it. In this letter we demonstrate that stress enhancement may seemingly have the opposite effect, i.e., it stabilizes the fracture process. This is a paradoxical situation which essentially turns upside down common wisdom within the physics community on how fracture processes proceed.

It turns out, however, that this stability paradox is only an apparent effect caused by the fluctuations that occur during the fracture process. We use in the following the fiber bundle model to demonstrate the paradox and its resolution. We consider two variants of the model: the equal load sharing (ELS) model where there is local heterogeneity but no local stress enhancement, and the local load sharing (LLS) model where there is a competition between local stress enhancement and local heterogeneity. Even though we use the fiber bundle model as a tool to demonstrate the stability paradox, the effect is more general. The lesson to be learned is the following: even though the average stress vs. strain curve may have a positive slope, seemingly indicating stability, the positive slope is not necessarily caused by stability, but by the evolution of the fluctuations biasing the average in such a way that makes the slope positive.

There are two main sources of fluctuations in dynamical systems such as materials failing under stress: one comes from statistical fluctuations of the probability distributions that define intrinsic properties of the system elements. Another type of fluctuations arises as a result of the system dynamics depending on the spatial structures. The first type of fluctuation has a direct relation with the system size and it normally disappears as the system size diverges due to self averaging. One can minimize the effect of these fluctuations either by making the system size larger or by increasing the number of samples. On the other hand, the dynamics-dependent fluctuations do not disappear with increasing size. It is therefore crucial to know the nature of this second type of fluctuations and its role during the entire evolution dynamics. It is this second type of fluctuations that is the cause of the stability paradox.

A fiber bundle consists of N fibers placed between two clamps. The fibers act as Hookean springs with identical spring constants κ up to an extension xi, individual for each fiber i, where they fail and cannot carry a load any more. Hence the connection between the extension x of a fiber i and the force fi it carries is

\[ f_i = \begin{cases} \kappa x & \text{if } x < x_i, \\ 0 & \text{if } x \geq x_i. \end{cases} \]

The critical extensions xi — thresholds — are drawn from a probability density p(x), with corresponding cumulative probability \[ P(x) = \int_0^x p(u)\,du. \]

In the ELS model an externally applied force F is distributed equally on all the intact fibers. This implies that...
the intact fiber with the smallest threshold breaks under the smallest force \( F \). The force per fiber \( \sigma = F/N \) required to give the bundle an extension \( x \) is on average

\[
\sigma = \kappa (1 - P(x)) \, x. \tag{2}
\]

Equivalently,

\[
\sigma = \kappa \left(1 - \frac{k}{N}\right) P^{-1} \left( \frac{k}{N} \right), \tag{3}
\]
as \( P(x) \) is the fraction of broken fibers \( k/N \) at extension \( x \) \cite{13}. The fluctuations around this average are of the first type, and disappear as \( N^{-1/2} \) as \( N \to \infty \) \cite{8}.

The load curve is the minimum force per fiber \( \sigma \) required to break the next fiber. Hence, we plot either this minimum \( \sigma \) as a function of the extension \( x \) or the fraction of broken fibers \( k/N \), see Fig. 1. Equations (2) and (3) give the average load curve for ELS. We will use the terminology that a fiber bundle is locally stable if the load \( \sigma \) must be increased to continue breaking more fibers, i.e. if the load curve is increasing. From equation (2) we determine the extension \( x_c \) at which the ELS model becomes unstable by setting \( d\sigma/dx|_{x_c} = 0 \). For an exponential threshold distribution \( P(x) = 1 - \exp(1-x) \) where \( x \geq 1 \), this gives \( x_c = 1 \). This means that the ELS model is unstable from the beginning of the failure process.

In the LLS model, the load carried by the failed fibers are distributed onto their nearest intact neighbors. Hence, there is a spatially dependent stress field. A hole is defined as a cluster (in the percolation sense) of \( h \) failed fibers. The perimeter of a hole is the set of \( p \) intact fibers that are nearest neighbors of the hole. With these definitions the force acting on an intact fiber \( i \) with the LLS model is given by

\[
f_i = \sigma \left(1 + \sum_j \frac{h_j}{p_j}\right), \tag{4}\]

where \( j \) runs over the set of neighboring holes of the fiber. The first term is the force applied to every fiber, while the second is the redistribution of the forces due to the failed fibers. Equation (4) is independent of lattice type and dimensionality.

To determine which fiber breaks next under an external load we define the effective threshold \( x_{\text{eff},i} \) of fiber \( i \) as

\[
x_{\text{eff},i} = \frac{x_i}{1 + \sum_j \frac{p_j}{p_i}}, \tag{5}\]

The breaking criterion of a fiber is then \( \sigma = \kappa x_{\text{eff},i} \), and the fiber with the smallest effective threshold will fail under the smallest applied load \( \sigma \).

The LLS model contains stress enhancement in that the fibers belonging to the perimeters of holes carry extra load. The ELS model has no stress enhancement mechanism. It is then a surprise to find in Fig. 1 how the density \( \rho \) of fluctuations around the averaged load curve for the LLS model based on the threshold distribution \( P(x) = 1 - \exp(1-x) \) where \( x \geq 1 \). The ELS model curve is based on equation (3) while the LLS model curve is based on simulations using a square lattice \( (N = 100^2) \). The background is a color map that shows the density \( \rho \) of LLS model load curves over the \( 10^5 \) samples that the average is based on. The color bar is capped at \( \rho = 0.025 \) to highlight the fluctuations with the smallest values of \( \sigma \).

![FIG. 1: (Color online) Sample averaged load curves for ELS and LLS models with threshold distribution \( P(x) = 1 - \exp(1-x) \) where \( x \geq 1 \). The ELS model curve is based on equation (3) while the LLS model curve is based on simulations using a square lattice \( (N = 100^2) \). The background is a color map that shows the density \( \rho \) of LLS model load curves over the \( 10^5 \) samples that the average is based on. The color bar is capped at \( \rho = 0.025 \) to highlight the fluctuations with the smallest values of \( \sigma \).](image)
threshold, the bias in the fluctuations begins to shift rapidly from small values of $\sigma$ to the upper bounding curve, and this shift is enough to make the average load curve increase even though the bounding curve is decreasing.

We now consider the LLS model on a square lattice but with a uniform threshold distribution on $[x_0, 1)$, i.e. $P(x) = (x - x_0)/(1 - x_0)$. The ELS model with this distribution is unstable from the beginning of the breaking process if $x_0 = 1/2$. Hence, we choose this value. The average load curves for LLS and ELS are shown in Fig. 3 together with the fluctuations around the LLS load curve. Also here there is a shift in the fluctuations around the percolation threshold, corroborated by the standard deviation in Fig. 4 when the bias of the fluctuations changes rapidly. However, in this case the fluctuations are not biased to begin with, but distributed almost uniformly around the average, but with a slight bias towards larger values. This — and the fact that the fluctuations span a smaller range of forces $\sigma$, as demonstrated by a standard deviation an order of magnitude smaller in Fig. 4 than in Fig. 2 — makes the change in the bias of the fluctuations smaller than for the exponential threshold distribution, and it is not enough to make the averaged load curve increase. Hence, the average load curve does not show any apparent stability.

The reason for the bias in the fluctuations before the percolation threshold can be explained by examining the hole structure of the LLS fiber bundle as the damage $k/N$ increases. Very early in the breaking process, when only a few fibers have broken, a single hole starts expanding and keeps growing until the entire fiber bundle has broken [10], making the growth process effectively an invasion percolation process. Hence, all fibers that break after this localization sets in are in the perimeter of this growing hole, and the force $\sigma$ required to break a fiber is a measure of the smallest threshold found along the perimeter of the hole. This process is illustrated in Fig. 5 for the exponential threshold distribution and Fig. 6 for the uniform threshold distribution. With the ex-
FIG. 5: Hole structure of a square lattice \((N = 128^2)\) LLS fiber bundle with the threshold distribution \(P(x) = 1 - \exp(1 - x)\) at three different damages: \(k/N = 0.1\) (left), \(k/N = 0.3\) (middle) and \(k/N = 0.59\) (right). Intact fibers are black, the largest hole is gray, and other broken fibers are white. From very early on in the breaking process a single hole is growing continually.

FIG. 6: Hole structure of a square lattice \((N = 128^2)\) LLS fiber bundle with a uniform threshold distribution on \([0, 1)\) at three different damages: \(k/N = 0.1\) (left), \(k/N = 0.3\) (middle) and \(k/N = 0.59\) (right). Intact fibers are black, the largest hole is gray, and other broken fibers are white. From very early on in the breaking process a single hole is growing continually.

ponential threshold distribution most of the fibers have thresholds slightly larger than 1. This causes the threshold of the failing fibers in the perimeter of the growing hole typically to be less than the average. This makes the fluctuations biased, unlike with a uniform threshold distribution.

The reason why the weaker fluctuations disappears rapidly around the percolation threshold is that the large, growing hole has permeated most of the lattice and therefore has few new areas to expand into, as shown in Figs. 5 and 6. As a result of this, there are few new neighborhoods to expand into to find new neighbors with small thresholds. This argument does not hinge on the lattice being square. Rather, it is a general effect. We therefore expect that the exponential threshold distribution will give similar results for the LLS model on other lattices, with the averaged load curve increasing around the site percolation threshold due to the shift in bias as the weaker fluctuations disappear. We show averaged LLS load curves for four lattices in 2D, 3D and 4D for the exponential threshold distribution in Fig. 7. The figure shows positive slope of load curve for all four lattices in a region around the corresponding percolation threshold, in excellent accordance with the above argument.

Even though we have limited our discussion to the fiber bundle model, the paradox presented here and its resolution should be a general feature of brittle fracture processes in disordered materials. To summarize, we have demonstrated a general mechanism resulting in the average force not being a reliable indicator of stability during fracture process due to bias in the fluctuations around the average. We find that for certain threshold distributions in the fiber bundle model this gives a pseudo-stability, the illusion of local stability due to an increasing average force even though individual systems are not locally stable. This pseudo-stability occurs around the site percolation threshold of the lattice for the systems we have studied in two to four dimensions.

The authors thank Santanu Sinha for interesting discussions. This work was partly supported by the Research Council of Norway through project number 250158 and its Centers of Excellence funding scheme, project number 262644. M. H. thanks the Swiss National Science Foundation for an early postdoc mobility grant, number 171982.

* Electronic address: jonas.t.kjellstadli@ntnu.no
† Electronic address: eivind.bering@ntnu.no
‡ Electronic address: martin.hendrick@ntnu.no
§ Electronic address: srutarshi.pradhan@ntnu.no
¶ Electronic address: alex.hansen@ntnu.no

[1] B. Lawn, *Fracture of brittle solids*, 2. Ed. (Cambridge Univ. Press, Cambridge, 1993).
[2] H. J. Herrmann and S. Roux, *Statistical models for the*
fracture of disordered media, (North-Holland, Amsterdam, 1990).

[3] S. Roux and A. Hansen, EPL 11, 37 (1990).
[4] B. K. Chakrabarti and L. G. Benguigui, Statistical physics of fracture and breakdown in disordered solids (Oxford University Press, Oxford, 1997).
[5] D. Bonamy and E. Bouchaud, Phys. Rep. 498, 1 (2011).
[6] H. E. Daniels, Proc. Roy. Soc. Ser. A 183 243 (1945).
[7] S. Pradhan, A. Hansen and B. K. Chakrabarti, Rev. Mod. Phys. 82, 499 (2010).
[8] A. Hansen, P. C. Hemmer and S. Pradhan, The fiber bundle model (Wiley-VCH, Berlin, 2015).
[9] F. T. Peirce, J. Text Ind., 17, 355 (1926).
[10] D. G. Harlow and S. L. Phoenix, J. Comp. Mat. 12, 195 (1978).
[11] M. I. Freidlin and A. D. Wentzell, Random perturbation of dynamical systems (Springer, Berlin, 2012).
[12] S. G. Abaimov, Statistical physics of non-thermal phase transitions (Springer, Berlin, 2015).
[13] E. J. Gumbel, Statistics of extremes (Dover, Mineola, 2004).
[14] S. Sinha, J. T. Kjellstadli and A. Hansen, Phys. Rev. E 92, 020401 (2015).
[15] D. Stauffer and A. Aharony, Introduction to percolation theory (Taylor & Francis, London, 1992).
[16] E. Bering, Masters thesis, NTNU, 2016.