Topological Tverberg Theorem:  
the proofs and the counterexamples

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Abstract. The history of the Topological Tverberg Theorem (TTT) is described. Some important constructions are presented, and their properties are discussed. In particular, there is a detailed description of the cell structure of the classifying space $K(S_r,1)$, where $S_r$ is the permutation group.

Bibliography: 9 titles.

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In this paper the history of the proof of the Topological Tverberg Theorem is described and some priority questions mentioned in the survey [7] in this issue of the journal are clarified. The paper was written at the request of the editorial board of Uspekhi Matematicheskikh Nauk.

1. My initiation to TTT

My introduction to the Topological Tverberg Theorem took place in the Spring of 1980, in the student canteen at Moscow State University (in Zone B). During a lunch there, Imre Bárány told me about his success with the topological version of the Radon Theorem (TRT). The two-dimensional (for simplicity) version of TRT...
claims the following. Let $\Delta_3$ be a 3D simplex. A pair of faces $\Delta', \Delta'' \subset \partial \Delta_3$ is said to be *complementary* if
\[ \Delta' \cap \Delta'' = \emptyset \quad \text{and} \quad \dim \Delta' + \dim \Delta'' = 2. \]

For example, $\Delta'$ can be a 2D face and $\Delta''$ the opposite vertex; the other possibility is that $\Delta', \Delta''$ is a pair of skew edges.

**Theorem 1** (TRT). Let $f: \partial \Delta_3 \to \mathbb{R}^2$ be a continuous map. Then for some pair $\Delta', \Delta''$ of complementary faces of $\Delta_3$
\[ f(\Delta') \cap f(\Delta'') \neq \emptyset. \]

My immediate reaction to this statement was that it should follow from the Borsuk–Ulam Theorem (BUT) about antipodal points: BUT claims that if $g: S^2 \to \mathbb{R}^2$ is a continuous map, then for some pair $x', x'' \in S^2$ of antipodal points we have $g(x') = g(x'')$. Indeed, the two statements look similar; the only step needed, in order to relate the two, is to find a ‘universal’ map $F: S^2 \to \partial \Delta_3$ such that the images of each pair of antipodal points belong to complementary faces of $\partial \Delta_3$.

To find such a map is not hard, and here it is. First, let us replace the sphere by the cuboctahedron $\mathcal{C}$ (see Figure 1).

![Figure 1](image)

Note that $\mathcal{C}$ has 8 triangles $\delta_1, \ldots, \delta_8$ and 6 squares, that is, 14 faces in total, while $\partial \Delta_3$ has 4 triangles $d_1, \ldots, d_4$, 4 vertices $v_1, \ldots, v_4$, and 6 edges, which also makes 14 items. So let us select 4 triangles out of the 8 in $\mathcal{C}$ in such a way that any two of them are disjoint—say, $\delta_1, \ldots, \delta_4$—and define $F$ on these triangles to be linear isometries onto $d_1, \ldots, d_4$, respectively. Supposing that the faces $\delta_5, \ldots, \delta_8$ are opposite to the respective faces $\delta_1, \ldots, \delta_4$ and that the vertices $v_1, \ldots, v_4$ are opposite to the respective faces $d_1, \ldots, d_4$, we define
\[ F(\delta_k) \equiv v_{k-4} \quad \text{for} \quad k = 5, \ldots, 8. \]

It remains to extend $F$ to the six squares. But $F$ is already defined on their boundaries, sending each boundary onto a corresponding edge. So we extend $F$ inside each square in such a way that it takes the whole square into that same edge.

The existence of $F$ thus proves TRT (in 2D).
Imre also told me about his project to generalize the Linear Tverberg Theorem (LTT) to the Topological Tverberg Theorem, and we decided to join our efforts. Shortly thereafter, Andras Szücs joined our team. Our goal was to prove the following.

**Conjecture 2** (TTT). Let \( f : \partial \Delta_{(d+1)(r-1)} \to \mathbb{R}^d \) be continuous. Then for some collection \( \Delta^1, \ldots, \Delta^r \subset \partial \Delta_{(d+1)(r-1)} \) of pairwise disjoint faces we have

\[
f(\Delta^1) \cap \cdots \cap f(\Delta^r) \neq \emptyset.
\]

(LTT is the same statement, but for a linear \( f \). It was proved by Tverberg in [8].)

Our plan was somewhat similar to the above. For a simplex \( \Delta_N \) we introduced the complex \( Y_{N,r} \) of \( r \)-tuples \((y_1, \ldots, y_r) \subset \Delta_N \), where the points \( y_1, \ldots, y_r \) lie in disjoint faces of \( \Delta_N \). \( Y_{N,r} \) was going to play the role of the cuboctahedron \( \mathcal{C} \). That turned out to be a very nice CW complex; see below. It has an evident free action of the cyclic group \( \mathbb{Z}_r \). We were using the cyclic action of \( \mathbb{Z}_r \) on \( \mathbb{R}^{nr} \) which sends \((w_1, \ldots, w_r) \) to \((w_2, \ldots, w_r, w_1)\), \( w_i \in \mathbb{R}^n \). We needed it to be a free action away from the diagonal \( \mathbb{R}^n \subset \mathbb{R}^{nr} \) consisting of the vectors \((w, \ldots, w) \in \mathbb{R}^{nr} \) with \( w \in \mathbb{R}^n \) (the antipodal map above corresponds to the diagonal \( \mathbb{R}^2 \subset \mathbb{R}^4 \)). But unfortunately, that happens only when \( r \) is prime. This restricted our proof of TTT in [2] to the case of prime \( r \), but I thought at the time that the primality condition would be only a temporary obstacle.

## 2. The CW complex \( Y_{N,r} \)

The complex \( Y_{N,r} \) has several nice properties. First, not only the cyclic group \( \mathbb{Z}_r \), but also the whole permutation group \( S_r \) acts freely on it. Next, all the homotopy groups \( \pi_k(Y_{N,r}) \) are 0 for \( k \leq N-r \); see [2]. The first non-trivial homotopy group is \( \pi_{N-r+1}(Y_{N,r}) \), while the dimension \( \dim(Y_{N,r}) \) is also \( N-r+1 \). This means that \( Y_{N,r} \) is homotopy equivalent to the \((N-r+1)\)-dimensional skeleton of the universal cover of the classifying complex \( K(S_r,1) \). In other words,

\[
[K(S_r,1)]_{N-r+1} = Y_{N,r}/S_r.
\]

This is the most economical model of the skeleton of \( K(S_r,1) \).

For example, for \( r = 2 \) we have \( K(S_r,1) = K(\mathbb{Z}_2,1) = \mathbb{RP}^\infty \), and so the complex \( Y_{N,2} \) has the homotopy type of the sphere \( S^{N-1} \). For \( N = 3 \) we get \( S^2 \), with the cell structure of the cuboctahedron \( \mathcal{C} \). When \( N = 2 \) the complex \( Y_{2,2} \sim S^1 \) is a hexagon, with edges

\[
(0)(1,2), \ (1)(2,0), \ (2)(0,1), \ (0,1)(2), \ (1,2)(0) \text{ and } (2,0)(1).
\]

The CW complexes \( Y_{N,r} \) come with some extra structure. Namely, every cell \( c \subset Y_{N,r} \) is equipped with the structure of a product,

\[
c = \Delta_{d_1} \times \cdots \times \Delta_{d_r},
\]

where \( \Delta_d \) denotes the \( d \)-dimensional simplex, \( d \geq 0 \), with \( d_1 + \cdots + d_r = \dim c \). (In particular, every sphere gets such a cell structure.) We will call such a CW complex a *prism* complex.
Together with Oleg Ogievetsky we spent many happy hours attempting to prove TTT for all values of \( r \). That was a very exiting experience; we considered many beautiful questions related to TTT and we invented several nice constructions. I will present one such invention: the concept of \( \theta \)-orientation.

Before explaining it, I will talk about classic orientability.

One way of saying the prism complex \( Y \) is orientable is to demand that

- every cell \( c \) of dimension \( \dim Y - 1 \) belongs to the boundaries of at most two cells \( c' \) and \( c'' \) of dimension \( \dim Y \);
- every cell of dimension \( \dim Y \) can be assigned an orientation in such a way that each cell \( c \) of dimension \( \dim Y - 1 \) inherits the opposite orientations from its two parent cells \( c' \) and \( c'' \).

For example, the complexes \( Y_{N,2} = S^{N-1} \) are orientable.

The complexes \( Y_{N,r} \) are not orientable in the above sense once \( r > 2 \). The reason is simple: its faces of codimension 1 belong to \( r > 2 \) faces of full dimension. Yet, for our experiments we needed some canonical way of choosing the orientations of its \((N - r + 1)\)-dimensional faces of top dimension. The natural way of doing it is to use the \( \theta \)-orientability of \( Y_{N,r} \).

**Definition 3.** Let \( Y \) be a CW prism complex of dimension \( d \). It is said to be \( \theta \)-orientable if there is a choice of orientations of all its \( d \)-cells such that each cell of \( Y \) of codimension 1 inherits the same orientation from all \( d \)-dimensional cells to which it is incident.

Unlike orientability, \( \theta \)-orientability depends not only on the homotopy type of \( Y \), but also on the prism cell structure. Some triangulations of the sphere \( S^2 \) are not \( \theta \)-orientable. On the other hand, some triangulations of the Mobius strip are \( \theta \)-orientable.

**Theorem 4.** The complex \( Y_{N,r} \), with the prism structure described, is \( \theta \)-orientable.

**Proof.** Let us consider a face (cell) \( F = F(V_1, \ldots, V_r) \) of \( Y_{N,r} \) having full dimension. It corresponds to the ordered partition \( V_1, \ldots, V_r \) of the vertex set \( \{v_0, v_1, \ldots, v_j, \ldots, v_N\} \) of the simplex \( \Delta_N \) into \( r \) disjoint non-empty subsets. In that case \( F \) is the product of simplices,

\[
F = \Delta(V_1) \times \cdots \times \Delta(V_r),
\]

so to fix the orientation of \( F \) is the same as to fix the orientations on each simplex \( \Delta(V_i) \) and to take them in the order given. In case some \( V \) is a singleton, this orientation turns into a sign, \( \pm 1 \). The orientation of the simplex \( \Delta(v_{i_0}, \ldots, v_{i_k}) \) is specified by fixing some total order

\[
\overrightarrow{v_{i_0}, \ldots, v_{i_k}}
\]

on the set \( v_{i_0}, \ldots, v_{i_k} \) of its vertices. For a zero-dimensional oriented simplex corresponding to a vertex \( v \) of \( \Delta_N \) we will use the notation \( \Delta(\overleftarrow{v}) \) or \( \Delta(\overrightarrow{v}) \).

Likewise, a face \( G \) of codimension 1 of \( Y_{N,r} \) which belongs to the boundary of the face \( F \) corresponds to the ordered partition \( V'_1, \ldots, V'_r \) of the vertex set \( \{v_0, v_1, \ldots, \overleftarrow{v_j}, \ldots, v_N\} \) into \( r \) disjoint non-empty subsets, where the operation \( \overleftarrow{} \)
means the removal of the corresponding term. So all the elements of the partition \( V'_1, \ldots, V'_r \) except for one, say, \( V'_k \), coincide with the subsets \( V_1, \ldots, V_r \), while \( V'_k = V_k \setminus G \), where \( G \in \{v_0, v_1, \ldots, v_j, \ldots, v_N\} \). Therefore,

\[
G = \Delta(V_1) \times \cdots \times \Delta(V_{k-1}) \times \Delta(V'_k) \times \cdots \times \Delta(V_r),
\]

and \( \Delta(V'_k) \in \partial \Delta(V_k) \).

Now we remind the reader of the standard definition of the inherited orientation. The orientation of the simplex \( \Delta(v_{i_0}, \ldots, v_{i_s}) \) induces the orientations of all its faces of codimension 1 according to the formula

\[
\partial \Delta(V_k) \equiv \partial \Delta(v_{i_0}, \ldots, v_{i_s}) = \sum_{j=0}^{s} (-1)^j \Delta(v_{i_0}, v_{i_1}, \ldots, \hat{v}_{i_j}, \ldots, v_{i_s}). \tag{1}
\]

The meaning is that if the removed vertex \( v^G \) happens to be at an odd position in the ordered string \( \overrightarrow{v_{i_0}, \ldots, v_{i_s}} \) defining the (oriented) simplex \( \Delta(V_k) \) (that is, \( j = 0, 2, 4, \ldots \)), then it just has to be removed from it, while keeping the order of the remaining vertices for defining the orientation of the simplex \( \Delta(V'_k) \); otherwise, the removal of \( v^G \) has to be supplemented by the transposition of, say, the first two of the remaining vertices defining \( \Delta(V'_k) \). If \( V'_k \) happens to be a singleton, then the orientation of \( \Delta(V'_k) \) is the sign it gets in (1).

In case of a product of simplices one has to use the Leibniz rule:

\[
\partial[\Delta(V_1) \times \cdots \times \Delta(V_r)] = \sum_{k=1}^{r} (-1)^{\text{dim}(\Delta(V_1) \times \cdots \times \Delta(V_{k-1}))} \\
\times \Delta(V_1) \times \cdots \times \Delta(V_{k-1}) \times \partial \Delta(V_k) \times \cdots \times \Delta(V_r).
\]

Now we are going to make the choice of orientation for every face \( F \). First we take the face

\[
F_0 = F(\{v_0\}, \{v_1\}, \ldots, \{v_{r-2}\}, \{v_{r-1}, \ldots, v_N\})
\]

which corresponds to the partition into \( r - 1 \) singletons and the remaining set of \( N - r + 2 \) points; this face is an \((N - r + 1)\)-dimensional simplex. We choose its orientation according to the order \( v_{r-1} < \cdots < v_N \), while all the singletons get the +1 orientation:

\[
F_0 = F(\{\overrightarrow{v_0}, \{v_1\}, \ldots, \{v_{r-2}\}, \overrightarrow{v_{r-1}, \ldots, v_N} \}).
\]

Now let

\[
F = F(\{\overrightarrow{v_{1,1}, \ldots, v_{1,i_1}}, \overrightarrow{v_{2,1}, \ldots, v_{2,i_2}}, \ldots, \overrightarrow{v_{r,1}, \ldots, v_{r,i_r}}\})
\]

be an arbitrary face, with orientation corresponding to the orders

\[
v_{1,1} < \cdots < v_{1,i_1}, \quad v_{2,1} < \cdots < v_{2,i_2}, \quad \ldots, \quad v_{r,1} < \cdots < v_{r,i_r},
\]

on its multipoint factors, and with (+)-signs on the singleton factors. We will formulate the condition which this orientation of \( F \) has to satisfy in order to define
a global $\mathcal{O}$-orientation of $Y^{N,r}$. We assign to this orientation a word $w(F)$ in $N + r$ symbols which are the letters $\{v_0, \ldots, v_N\} \cup \{s_1, \ldots, s_{r-1}\}$:

$$w(F) = v_{1,1}, \ldots, v_{1,i_1}, s_1, v_{2,1}, \ldots, v_{2,i_2}, s_2, \ldots, s_{r-1}, v_{r,1}, \ldots, v_{r,i_r}.$$ 

The only condition the orientation on $F$ has to satisfy, is that the parity of the permutation $\pi(F) \in S_{N+r}$ taking $w(F)$ into the word

$$w(F_0) = v_0, s_1, v_1, s_2, \ldots, s_{r-1}, v_{r-1}, v_r, \ldots, v_N$$

is even.

Let us check that the orientations of faces thus prescribed form an $\mathcal{O}$-orientation.

We start with the cuboctahedron $\mathcal{C}$. Of course, $\mathcal{C}$ is orientable, but its orientation is not an $\mathcal{O}$-orientation. Its $\mathcal{O}$-orientation is easy to guess: it consists in orienting all the squares one way, and all the triangles the opposite way. We check that the $\mathcal{O}$-recipe above gives the same result.

Let us consider the 2D complex $Y^{3,2}$ constructed from the simplex

$$\Delta_3 = \{0, 1, 2, 3\}.$$ 

We take the (triangular) face $F_0 = (\{0\}, \{1, 2, 3\})$ and one of its boundary edges, say, the segment $G = (\{0\}, \{1, 3\})$. The vertex $v^G$ is the point $\{2\}$ missing in $G$. The segment $G$ is adjacent to another 2D face, the square face $F_1 = (\{0, 2\}, \{1, 3\})$.

The orientation of $F_0$ is determined by the $(+1)$-orientation of the point $\{0\}$ and the orientation of the simplex $\Delta(1, 2, 3)$ corresponding to the order $1 < 2 < 3$:

$$F_0 = (\{0\}, \{1, 2, 3\}).$$

Let us see which orientation the face $F_1$ gets via our recipe above: $(\{\overline{0, 2}\}, \{\overline{1, 3}\})$ or $(\{\overline{0, 2}\}, \{3, 1\})$? To find it we look at the parity of, say, the first permutation:

$$\begin{pmatrix} 0 & s_1 & 1 & 2 & 3 \\ 0 & 2 & s_1 & 1 & 3 \end{pmatrix}.$$ 

Since it is even, we take for the $\mathcal{O}$-orientation of the face $F_1$ the one defined by the ordering $(\{\overline{0, 2}\}, \{\overline{1, 3}\})$. The orientation that the edge $G = (\{0\}, \{1, 3\})$ inherits from the oriented face $F_0 = (\{0\}, \{1, 2, 3\})$ corresponds to the ordering $G = (\{\overline{0}\}, \{3, 1\})$, because the index $j$ in (1) has value 1; the corresponding term there is $(-1)\Delta(1, 2, 3)$. The orientation which the edge $G = (\{0\}, \{1, 3\})$ inherits from the oriented face $F_1 = (\{0, 2\}, \{1, 3\})$, corresponds to the ordering $G = (\{\overline{0}\}, \{1, 3\})$, with the $-1$ sign for the vertex $\{0\}$, since $\partial\Delta(0, 2) = +2 - 0$.

So the two orientations of $G$ do agree. The check for the other edges is done in the same way.

The general case is obtained by induction. Theorem 4 is proved.
3. Beyond the prime case

The start of the next chapter in the history of TTT came with the proof of TTT for $r$ a prime power. It was done in [6], a well-known paper left unpublished so far. Therefore, I learned about this fact much later, from the independent paper [9] by Volovikov.

Thus, the general case of TTT seemed to be within reach. Yet, nobody was able to make the final step. Some people even expressed doubts. The one case I know of was in 2011, when I was discussing TTT with David Kazhdan. But I was not convinced. Bad luck!—because four years later Florian Frick came with counterexamples to TTT, in [4]. He showed that for any $r$ distinct from a prime power the TTT does not hold...

That was the end of one chapter of TTT. Meanwhile, many other open questions around TTT have emerged; for example, see [1]—but this is another story.

4. Credits

In the report [7], §1.1, one reads: ”For these counterexamples the papers ⟨...⟩ (by M. Özaydin; M. Gromov; F. Frick; P. Blagojević, F. Frick, and G. Ziegler; and I. Mabillard and U. Wagner) are important”. This is a correct statement, since [4] appears in the list. But it is an understatement. It is the truth, but not the whole truth: without [4], there would be no counterexample. In general, it doesn’t happen that a proof of a theorem is contained in several papers by several authors. Usually, there is a pivotal paper such that the proof in question did not exist before it and does exist after it. In the case of a counterexample to TTT such a paper was written by Florian Frick, building on earlier results of Mabillard and Wagner.

The result of Frick is based on the constraint method, developed in 2015 in [3]. Remark 1.10 in [7] asserts that this method is contained in [5], an outstanding 110-page geometric paper Gromov wrote in 2010. The section discussing various generalizations contains a half-page sketch which can be interpreted as relating to the indicated method. However, this sketch is incomplete and contains inaccuracies, so much so that an attempt to correct them is made in [7]. Out of respect to Mikhail Gromov it would have been better not to initiate this discussion.

The mathematical part of A. Skopenkov’s survey is nicely written, and the reader can find there a good introduction to the subject of the Topological Tverberg Theorem.

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Bibliography

[1] I. Bárány, P. V. M. Blagojević, and G. M. Ziegler, “Tverberg’s theorem at 50: extensions and counterexamples”, Notices Amer. Math. Soc. 63:7 (2016), 732–739.
[2] I. Bárány, S. B. Shlosman, and A. Szűcs, “On a topological generalization of a theorem of Tverberg”, J. London Math. Soc. (2) 23:1 (1981), 158–164.

[3] P. V. M. Blagojević, F. Frick, and G. M. Ziegler, Barycenters of polytope skeleta and counterexamples to the topological Tverberg conjecture, via constraints, 2015, 6 pp., arXiv:1510.07984.

[4] F. Frick, “Counterexamples to the topological Tverberg conjecture”, Oberwolfach Rep. 12 (2015), 318–322.

[5] M. Gromov, “Singularities, expanders and topology of maps. Part 2: From combinatorics to topology via algebraic isoperimetry”, Geom. Funct. Anal. 20:2 (2010), 416–526.

[6] M. Özaydin, Equivariant maps for the symmetric group, preprint, Univ. of Wisconsin-Madison, 1987, 17 pp., http://digital.library.wisc.edu/1793/63829.

[7] А.Б. Скопенков, “Топологическая гипотеза Тверберга”, УМН 73:2(440) (2018), 141–174; English transl., A. B. Skopenkov, “A user’s guide to the topological Tverberg conjecture”, Russian Math. Surveys 73:2 (2018), 323–353.

[8] H. Tverberg, “A generalization of Radon’s theorem”, J. London Math. Soc. 41 (1966), 123–128.

[9] А.Ю. Воловиков, “К топологическому обобщению теоремы Тверберга”, Матем. заметки 59:3 (1996), 454–456; English transl., A. Yu. Volovikov, “On a topological generalization of the Tverberg theorem”, Math. Notes 59:3 (1996), 324–326.

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