General Relativistic Models for the Electron

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Abstract
A model is proposed for the classical electron as a point charge with finite electromagnetic self-energy. Modifications of the Reissner-Nordstrøm (spin 0) and Kerr-Newman (spin 1/2) solutions of the Einstein-Maxwell equations are derived. It is conjectured that spacetime curvature very close to the point charge deforms the electric and magnetic fields such as to reduce the self-energy to a finite value by means of Hawking polarization of the vacuum, much like that around a black hole.

1 Introduction
In a recent paper [1], we showed how a phenomenological representation of vacuum polarization could be introduced into the formalism of classical electrodynamics. Modification of the classical theory is significant only within the classical radius of the electron \( r_0 = e^2/mc^2 \approx 2.8 \times 10^{-13} \text{ cm} \) but this makes possible a consistent model for point charges with finite electromagnetic self-energies. Further, the self-interaction of a charged particle with its own electromagnetic field was shown to be equivalent to its reaction to the vacuum polarization.

In this paper, we reformulate these results in the context of general relativity. We obtain two new solutions of the Einstein-Maxwell equations for a point charge by modification of the Reissner-Nordstrøm and Kerr-Newman metrics. In contrast to the originals, our solutions possess finite electromagnetic self-energies. The stress-energy tensors in our solutions imply the presence of finite charge distributions. These are conjectured to represent vacuum polarization produced by the ultrastrong electromagnetic fields in the vicinity of point charges, analogous to the phenomena proposed by Hawking [2] for strong gravitational fields.

It was shown in [1] that, for the electron treated as a point charge and point mass, an effective dielectric constant for the surrounding vacuum

\[
\epsilon(r) = \exp(r_0/2r)
\]

leads to a self-energy entirely electromagnetic in origin, viz

\[
W = \frac{1}{8\pi} \int \mathbf{E} \cdot \mathbf{D} \, d^3 \mathbf{r} = \frac{1}{8\pi} \int_0^\infty \frac{e^2}{\epsilon(r) r^4} 4\pi r^2 \, dr = mc^2
\]
The net charge density (polarization plus free) around an isolated electron is accordingly given by
\[ \rho(r) = \frac{er_0}{8\pi r^4} \exp(-r_0/2r) \]  
(3)

Previous applications of general relativity to the structure of the electron have usually been based on perfect fluid models involving additional non-electromagnetic forces\[^3\]-\[^5\]. A number of other workers have considered elementary particles from the viewpoint of general relativity\[^6\]-\[^10\]. In some instances negative gravitational mass was introduced in the interior of the particle in order to achieve a finite rest energy. We find such assumptions unappealing. In this paper, we propose a model for the electron based on the following premises:
1. The electron is a point charge; 2. The electromagnetic self-energy is finite and accounts for the total rest mass. We thus reject any non-electromagnetic contributions to the electron mass. This is consistent with the nearly zero rest mass of the electron’s uncharged weak isodoublet partner—the neutrino; 3. Very close to the electron, for distances \( r \ll r_0 \), the electromagnetic energy density becomes sufficiently massive to fall within the scope of general relativity. Specifically, the curvature of spacetime deforms the electromagnetic fields sufficiently to reduce the self-energy to a finite value.

### 2 Scalar Point Charge

We consider first the hypothetical electron without spin or magnetic moment, described by a static, spherically-symmetrical metric of the form
\[ ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]  
(4)

The Reissner-Nordstrøm metric\[^11\],\[^12\] has this structure with
\[ f_{\text{RN}}(r) = 1 - 2M/r + \frac{Q^2}{r^2} \]  
(5)

where \( M \) and \( Q \) are mass and charge in geometrized units:
\[ M = Gm/c^2, \quad Q = \sqrt{Ge}/c^2 \]

Solution of the Einstein-Maxwell equations
\[ R_{\lambda\mu} - \frac{1}{2} R g_{\lambda\mu} = \frac{8\pi G}{c^4} T_{\lambda\mu} \]  
(6)

gives the stress-energy tensor
\[ T_0^0 = T_1^1 = -T_2^2 = -T_3^3 = -\frac{e^2}{8\pi r^4} \]  
(7)

This represents a point charge since
\[ 4\pi T_\mu^\lambda = F^\lambda_{\mu\nu} F^\nu_{\mu\sigma} - \frac{1}{4} g^\lambda_{\mu\nu} F^\nu_{\mu\sigma} F^\mu_{\nu\sigma} \]  
(8)
implies that the only non-zero field tensor components are
\[ F_{10} = -F_{01} = E_r = e/r^2 \]  
(9)
The electromagnetic self-energy, given by
\[ W = -\int_0^\infty T^0_0 \frac{4\pi r^2 \, dr}{4\pi r^2} \]  
(10)
remains divergent. For the case \( Q > M \), which obtains here, \( g_{00} \) in the metric (4) has no real roots, indicating the absence of a horizon. The electron is not a conventional black hole, despite the singularity at \( r = 0 \).

To obtain the desired electromagnetic self energy \( mc^2 \), we do some reverse engineering on the Reissner-Nordstrøm (RN) metric, replacing the function (5) by
\[ f(r) = 1 - \frac{2M}{r} \exp(-Q^2/2Mr) \]  
(11)
This is actually a very small change, differing from the original form by less than 1 percent down to the Planck length \((\sqrt{\hbar G}/c^3) \approx 1.6 \times 10^{-33} \text{ cm})\). With removal of the singularity at \( r = 0 \), the metric now becomes Lorentz flat as \( r \to 0 \) (as well as \( r \to \infty \)). Note that
\[ Q^2/2Mr = e^2/2mc^2r = r_0/2r \]  
(12)
where \( r_0 \) is the classical electron radius. The modified metric in the Einstein-Maxwell equations can be solved for a stress-energy tensor with the nonvanishing components:
\[ T^0_0 = T^1_1 = -\frac{e^2 e^{-r_0/2r}}{8\pi r^4}, \quad T^2_2 = T^3_3 = \frac{e^2 e^{-r_0/2r}}{8\pi r^4} (1 - r_0/4r) \]  
(13)
The electromagnetic energy is now given by
\[ W = -\int_0^\infty T^0_0 \frac{4\pi r^2 \, dr}{4\pi r^2} = \int_0^\infty \frac{e^2 e^{-r_0/2r}}{8\pi r^4} 4\pi r^2 \, dr = \frac{e^2}{r_0} = mc^2 \]  
(14)
As advertised, we obtain a finite electron rest mass, entirely electromagnetic in origin, the result originally sought by Lorentz.

The Ricci scalar for our metric is nonvanishing:
\[ R = \frac{Q^4}{2Mr^5} e^{-Q^2/2Mr} \]  
(15)
impling a gaussian curvature of spacetime absent in the original RN solution, which is Ricci flat.

The original Reissner-Nordstrøm model can be interpreted as a point charge in a hypothetical “bare” vacuum. The modified metric implies some sort of finite electron charge distribution. We conjecture that this represents quantum-electrodynamic vacuum polarization. The field tensor \( G_{\lambda\mu} \) for a polarizable
medium is obtained from $F_{\lambda\mu}$ by the substitutions $E \rightarrow D$ and $B \rightarrow H$, such that (8) generalizes to Minkowski’s stress-energy tensor:

$$4\pi T^\lambda_\mu = F^{\lambda\nu} G_{\mu\nu} - \frac{1}{4} g^{\lambda\nu} F_{\nu\sigma} G_{\lambda\sigma}$$

(16)

In particular,

$$T^0_0 = -\frac{1}{8\pi} E_r D_r$$

(17)

The stress-energy tensor (13) is consistent with

$$D_r = \frac{e}{r^2}, \quad E_r = \frac{e}{r^2} e^{-r_0/2r}$$

(18)

as already anticipated in Eq (1). The electric displacement $D$ corresponds to the Reissner-Nordstrøm field from the “free” electron point charge, while the electric field $E$ takes account of all charges including those induced in the vacuum. In this way we preserve the point-charge structure of the bare electron, namely the validity of Gauss’ law

$$\int D(r) \cdot dS = \frac{e}{r^2} 4\pi r^2 = 4\pi e$$

(19)

for arbitrarily small radius $r$. The introduction of an inhomogeneous dielectric constant $\epsilon(r) = \exp(r_0/2r)$ follows a suggestion by Weisskopf. As in the case of a dense plasma, the dielectric constant increases with charge density. In this case, the “plasma” is made of virtual positron-electron pairs produced by vacuum polarization.

The nonvanishing trace of the stress-energy tensor (13) indicates contributions other than the electromagnetic-field. One might interpret the additional terms in $T^2_2$ and $T^3_3$ as the presence of a viscous fluid, with components $p_\theta$ and $p_\phi$, representing tangential pressure exerted by the virtual particles. Accordingly, the fields (18) do not represent a solution of Maxwell’s equations, except in the limit $r_0 \rightarrow 0$.

3 Electron with Spin

We extend our model to include electron spin by drawing from the theory of rotating black holes. Kerr first solved Einstein’s equations for a black hole with angular momentum. Newman generalized this result to include electrical charge. The metric in Kerr-Newman geometry, in the coordinates introduced by Boyer and Lindquist, can be written

$$ds^2 = -\frac{\Delta}{\rho^2} \left[ dt - a \sin^2 \theta d\phi \right]^2$$

$$+ \frac{\sin^2 \theta}{\rho^2} \left[ (r^2 + a^2) d\phi - a dt \right]^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$$

(20)
where
\[ \rho^2 \equiv r^2 + a^2 \cos^2 \theta \tag{21} \]
and
\[ \Delta \equiv a^2 + r^2 f(r) \]
The spin angular momentum per unit mass is represented by the parameter
\[ a \equiv S/M \]
For the original Kerr-Newman metric, \( f(r) \) is given by Eq (5), and
\[ \Delta_{\text{KN}} = r^2 - 2Mr + a^2 + Q^2 \]
Computation of the electromagnetic self energy \( W \) with this metric gives a divergent result, just as in the Reissner-Nordstrøm case.
It is therefore suggested that we try the modified functional form for \( f(r) \) in Eq (11). Following are explicit expressions for the nonvanishing elements of the metric tensor (20):
\[
\begin{align*}
g_{00} &= -1 - \frac{r^2[f(r) - 1]}{\rho^2} \\
g_{11} &= \frac{\rho^2}{a^2 + r^2 f(r)} \\
g_{22} &= \rho^2 \\
g_{33} &= (a^2 + r^2) \sin^2 \theta - \frac{a^2 r^2 [f(r) - 1]}{\rho^2} \sin^4 \theta \\
g_{30} &= g_{03} = \frac{ar^2 [f(r) - 1]}{\rho^2} \sin^2 \theta \tag{22}
\end{align*}
\]
The corresponding Jacobian is
\[ \sqrt{-g} = \rho^2 \sin \theta \tag{23} \]
independent of \( f(r) \).
The stress-energy tensor follows after a lengthy computation. We display only the \( T_0^0 \) component:
\[
T_0^0 = \frac{c^4}{16\pi \rho^6} \left[ -4a^2 r^2 - 4r^4 + 2a^2 \rho^2 + 4r^2 \rho^2 - 2\rho^4 \\
+ (4a^2 r^2 + 4r^4 - 2a^2 \rho^2 - 4r^2 \rho^2 + 2\rho^4) f(r) \\
+ (4a^2 r^3 + 4r^5 - 4a^2 r \rho^2 - 6r^3 \rho^2 + 4r \rho^4) f'(r) \\
+ (-a^2 r^2 \rho^2 - r^4 \rho^2 + r^2 \rho^4) f''(r) \right] \tag{24}
\]
with \( \rho \) defined in Eq (21).
The electromagnetic energy is given by
\[ W = - \int T_0^0 \sqrt{-g} \, dr \, d\theta \, d\phi \tag{25} \]
Integrating over angles and putting in the explicit form (11) for \( f(r) \)

\[
W = \frac{c^4}{G} \int_0^\infty \frac{Q^2 e^{-Q^2/2Mr}}{8aMr^4} \left[ (4aMr^2 + aQ^2r) \\
-(Q^2r^2 - 4a^2Mr + a^2Q^2) \arctan \left( \frac{a}{r} \right) \right] dr
\]  

(26)

This works out simply to

\[
W = \frac{c^4}{G} M = mc^2
\]  

(27)

implying again that the electron rest mass is purely electromagnetic.

For distances \( r \gg a, r_0 \), the dominant components of the electric and magnetic fields closely approximate a rotating ring of charge with radius \( a \), as previously shown in the analysis of Israel\[18\] and of Pekeris and Frankowski\[19\]:

\[
E_r \approx \frac{e}{r^2} - \frac{3ea^2 \cos^2 \theta}{r^4}, \quad E_\theta \approx -\frac{2ea^2 \cos \theta \sin \theta}{r^4}
\]

\[
B_r \approx \frac{2\mu}{r^3} \cos \theta, \quad B_\theta \approx \frac{\mu}{r^3} \sin \theta
\]  

(28)

neglecting terms of higher order in \( a/r \) and \( r_0/r \). We have used the following conversions from geometrized units: \( M = Qa = QS/M = \sqrt{G\mu/c^2} \), \( Q = \sqrt{Ge/c^2} \) and \( S = Gs/c^3 \). The magnetic moment is given by

\[
\mu = \frac{e\hbar}{2mc}
\]  

(29)

for spin angular momentum \( s = \hbar/2 \). Remarkably, this corresponds to a spin g-factor of 2, in agreement with Dirac theory, as first noted by Carter\[20\].

One might imagine, in concept, the spherically-symmetrical polarization charge distribution of Eq (3) set into rapid rotation, causing parallel current elements to attract, thus compressing the distribution into a disc and ultimately into a Kerr-Newman ring.

In summary, we have proposed a plausible resolution of the long-standing paradox for the classical electron—how it can be a point charge yet have finite electromagnetic self-energy\[21\]. The suggested mechanism involves warping of spacetime in ultrastrong fields, leading to Hawking polarization of the vacuum. We note that this further connection between general relativity and quantum mechanics might contribute to a realization of a unified theory extensively discussed by many workers, notably Wheeler\[22\] and Sachs\[23\].

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