Abstract. The peristaltic motion of an incompressible fluid in two-dimensional channel is investigated. Instead of fixing the law of wall’s coordinate variation, the law of pressure variation on the wall is fixed and the border’s coordinate changes to provide the law of pressure variation on the wall. In case of small amplitude of pressure-variation on the wall $A$, expansion wave propagates along the length of channel and the wave results in the peristaltic transport of fluid. In the case of large $A$, the channel divides into two parts. The small pulsating part in the end of the tube creates the flow as a human heart, while the other big part loses this function. The solution of problem for the first peristaltic mode is stable, while the solution for the second “heart” mode is unstable and depends heavily on boundary conditions.

1. Introduction
When expansion wave propagates along the length of the tube, fluid contained in the tube is transported in the direction of the wave propagation. This kind of transport, called peristaltic, is used by many organs such as urethra, male reproductive system and gastro-intestinal tract.

Shapiro, Jaffrine and Weinberg [1] investigated the linearized problem of the peristaltic transport in two-dimensional channel and tube contained incompressible fluid. They considered also the wave length to be infinite. They found backward (reflux) time-mean flow near the tube periphery and the phenomenon of trapping: under certain conditions a bolus of fluid lying about the axis is transported with the wave speed. Jaffrine [2] continued this investigation and took into account effects of Reynolds number and wave number, when these are small but not negligible. The experiments of Weinberg, Eckstein and Shapiro [3] showed that the theory of Jaffrin [2] is valid up to Reynolds number of about 10. Takabatake and Ayukawa [4] investigated numerically the problem of peristaltic in case of two-dimensional channel. They found that validity of the perturbation solutions by Jaffrin is restricted within a narrower range than that which he had predicted. They also found the reflux near the central axis at large Reynolds number. Then Takabatake, Ayukawa and Mori [5] studied numerically the peristaltic transport and its efficiency in cylindrical tubes.

The interaction of peristalsis with elastic properties of the wall was investigated by Mittra and Prasad [6]. They used dynamic boundary conditions considering Newtonian fluid and solved this problem under the approximation of small amplitude ratio [7]. Muthu et al. [8] investigated this problem for micropolar fluids and found critical values of the parameters involving wall characteristics, which cause reversal mean flow. Muthu et al.[8], Mittra and Prasad [6] considered wall to have uniform cross section. Sankad and Radhakrishnamacharya [7] extended the analysis of Muthu et al. [8] to two-dimensional channel of non-uniform cross section.
used viscoelastic active body equations with activation parameter which was specified in the form of a traveling wave, as a boundary condition [10]. Since the hallow organ’s walls move under force produced by the muscle, the kinematic boundary conditions are changed to dynamic.

In the present study, the law of wall’s coordinate variation isn’t determined a priori. It is found from the initially definite law of pressure-variation on the wall. This way is based on the fact that some hollow organs change diameter under the signals of baroreceptors (sensors that detects the pressure). The effects of various parameters on flow rate and stability of solution have been studied.

2. The governing equations and method of solution
Consider a two-dimensional flow of incompressible Newtonian fluid with kinematic viscosity \( \nu \) and density \( \rho \) in a channel of uniform thickness \( 2h \), with a sinusoidal wave of pressure traveling along the walls of the channel with speed \( c \), amplitude \( a \) and wave length \( \lambda \). We use the following non-dimensional variables, quantities and parameters

\[
x' = \frac{x}{h}, y' = \frac{y}{\lambda}, t' = \frac{tc}{\lambda}, \xi_1' = \frac{\xi_1}{h}, \xi_2' = \frac{\xi_2}{h}, u = \frac{u'}{c}, v = \frac{v'}{c}, p = \frac{p'h^2}{\nu c \rho h},
\]

where \( x \) and \( y \) are directed normal and parallel to the channel’s wall respectively, \(-1 - \xi_1 \) is position of the first wall, \( 1 + \xi_2 \) is position of the second wall, \( u \) and \( v \) are the velocity components in the \( y \) and \( x \) directions respectively, \( p \) is pressure, Re is the Reynolds number, \( \delta \) is the wave number, \( A \) is the amplitude of pressure-variation on the wall, the non-dimensional equation are

\[
\delta^3 \left( \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + u \frac{\partial v}{\partial y} \right) + \frac{1}{Re} \frac{\partial p}{\partial x} - \frac{\delta^2}{Re} \left( \frac{\partial^2 v}{\partial x^2} + \delta^2 \frac{\partial^2 v}{\partial y^2} \right) = 0, \tag{3}
\]

\[
\delta \left( \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} \right) + \frac{1}{Re} \frac{\partial p}{\partial y} - \frac{1}{Re} \left( \frac{\partial^2 u}{\partial x^2} + \delta^2 \frac{\partial^2 u}{\partial y^2} \right) = 0, \tag{4}
\]

\[
\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0. \tag{5}
\]

The non-dimensional boundary conditions are:

\[
v = -\frac{\partial \xi_1}{\partial t}, u = 0, p = \delta \text{Re} \cos(t-y) \text{ for } x = -1 - \xi_1, \tag{6}
\]

\[
v = \frac{\partial \xi_2}{\partial t}, u = 0, p = \delta \text{Re} \cos(t-y) \text{ for } x = 1 + \xi_2. \tag{7}
\]

When the law of wall’s coordinate-variation is fixed, the governing equations aren’t changed, but boundary conditions should be written as

\[
v = -\Xi \sin(t-y), u = 0 \text{ for } x = -1 - \xi_1, \tag{8}
\]

\[
v = \Xi \sin(t-y), u = 0 \text{ for } x = 1 + \xi_2, \tag{9}
\]

where \( \Xi \) is the amplitude of wall’s coordinate-variation.

The perturbation solution of equation (3)-(7) in the term of wave number \( \delta \) (\( \delta << 1 \)) is:

\[
u = \frac{\text{ARe}\delta}{2} \sin(t-y)(x^2-1) + \frac{\text{ARe}^2\delta^2}{24} \left[ \cos(t-y)(x^2-1)(x^2-5) + 8\text{A}\sin^2(t-y) \right] + O(\delta^3), \tag{10}
\]
\[ v = \frac{A R e \delta}{6} x (x^2 - 3) \cos(t - y) - \frac{A R e^2 \delta^2}{120} [(x^2 - 5)^2 - 80 A \cos(t - y)] x \sin(t - y) + O(\delta^3), \] (11)

\[ \xi_{L,R} = \xi = -\frac{A R e}{3} \sin(t - y) + \frac{A R e^2 \delta^2}{30} [5A - 2 \cos(t - y)(5A \cos(t - y) - 2)] + O(\delta^3), \] (12)

\[ p = A R e \delta \cos(t - y) + \frac{A R e \delta^3}{2} (x^2 - 1) \cos(t - y) + O(\delta^4). \] (13)

These series are divergent not only for large value of wave number \( \delta \) but also for large value of \( R e \) and \( A \). For this reason the finite element method is used for the solution of equations (3)-(7). In the case of series the channel is infinite in the direction, but the finite element method doesn’t allow simulation of infinite channel. In this case the channel’s length is considered to be \( 8 \pi \) and the solution should satisfy the additional soft boundary conditions at the ends of channel. These conditions are

\[ \frac{\partial p}{\partial x} \bigg|_{y=4\pi} = \frac{\partial p}{\partial x} \bigg|_{y=-4\pi} = 0, \quad \frac{\partial v}{\partial x} \bigg|_{y=4\pi} = \frac{\partial v}{\partial x} \bigg|_{y=-4\pi} = 0, \quad \frac{\partial u}{\partial x} \bigg|_{y=4\pi} = \frac{\partial u}{\partial x} \bigg|_{y=-4\pi} = 0. \]

3. Flow rate

From the perturbation solution (10)-(13) flow rate is

\[ Q = \frac{1}{2\pi} \int_0^{2\pi} \int_{-1}^{1+\xi} u \, dx \, dt = \frac{A^2 R e^2 \delta^2}{3} + \frac{A^2 R e^2 \delta^4}{9} \left[ \left( \frac{65}{24} A^2 - \frac{31}{63} \right) R e^2 - \frac{22}{5} \right] + O(\delta^6). \] (14)

The series (12) for \( \xi \) allows to write the amplitude of wall’s coordinate variation as

\[ \Xi = \frac{A R e \delta}{3} + \frac{A^2 R e^2 \delta^2}{6} + O(\delta^3). \] (15)

After finding \( A \) from (15):

\[ A = \frac{3\Xi}{\delta R e} \left[ 1 - \frac{3\Xi}{2} \right] + O(\delta^2), \]

and substituting it in (14), the flow rate can be rewritten as

\[ Q = 3\Xi^2 \left[ 1 - 3\Xi \right] + O(\delta^4). \] (16)

In the case when the law of wall’s coordinate-variation is fixed (equations (3)-(5) and boundary conditions (8),(9)), from the perturbation solution we find flow rate

\[ Q = 3\Xi^2 + \left[ \frac{\Xi^4}{5\delta^2} + \frac{9\Xi^4}{4\delta^4} - \frac{\Xi^2 R e^2}{1575\delta^4} \right] \delta^4 + O(\delta^5). \] (17)

The comparison of (16) and (17) shows that both problems give the similar flow rate for small value of \( R e, \delta \) and \( A \). This conclusion is the result of the fact that one problem can be transformed into the other problem, when the solutions are stable. The law of wall’s coordinate-variation can be found from the equations (3)-(5) with boundary conditions (6) and (7). Then this law can be fixed instead of cosinusoidal law in boundary conditions (8), (9) and as a consequence both problems give identical flows.

Finally, expansion wave propagates along the length of the channel and the peristaltic transport is observed for small value of \( R e, \delta \) and \( A \). Note that the expansion wave has the similar period \( 2\pi \) as the law of pressure-variation on the wall.
The difference between the cases of different boundary conditions appears for large value of \( \text{Re}, \delta \) or \( A \). When the law of pressure-variation on the wall is fixed and the parameter is large, the width of the channel decreases (figure 5a, \( H(A) \)). The middle positions of walls after the relaxation time fall to an optimal value and stop to change (figure 5b). The optimal channel’s middle width decreases with growth of the parameters. The figures 1, 2 and 3 show how flow rate depends on \( \text{Re}, \delta \) and \( A \). When the parameter is small, the flow rate increases with growth of the parameter. When the parameter is large, the flow rate and the amplitude of coordinate-variation (figure 5a, \( \Xi(A) \)) in the central part of walls decrease with growth of the parameter. The flow in this case is created by small pulsating part of walls at the channel’s end. The
amplitude of coordinate-variation grows in this part of the wall in opposite to its behavior in the central part of channel.

The pump’s efficiency is determined by the power of walls which can be written as

\[ P = -2 \int_{|x|=1+\xi} p \frac{\partial \xi}{\partial t} dy = -2ARe \delta \int \frac{\partial \xi}{\partial t} \cos(t - y) dy. \] (18)

When the parameter is small, the expression for power (18) in term of \( \delta \) is

\[ P = \frac{2 \pi A^2 Re^2 \delta^2}{3} \left[ 1 - \delta^2 \left( \frac{17}{105} Re + \frac{4}{5} - \frac{7A^2 Re^2}{12} \right) \right] + O(\delta^4). \] (19)

The energy lost to pump fluid in term of \( \delta \) can be written as

\[ \Lambda = \frac{P}{Q} = 2\pi - \frac{\delta^2 \pi}{3} \left[ \frac{23A^2 Re^2}{12} - \frac{4Re^2}{315} - 4 \right] + O(\delta^4). \] (20)

The series (20) shows that \( \Lambda \) tends to \( 2\pi \) , when the parameters vanishes. Moreover, it shows that \( \Lambda \) decreases with growth of \( A \). When the parameters are large, the numerical simulation shows that energy lost \( \Lambda \) increases with growth of \( A \) in opposite to its behavior for small parameters.

4. Solution stability

When the parameters are large, the chaotic behavior is observed in the system (figure 4). Note that the flow doesn’t become turbulent and its structure doesn’t significantly change but the behavior of walls changes (figure 7 and 6). The instability leads to the transition from one law of wall’s coordinate-variation (figure 7) to the other (figure 6). Figure 6 shows, that channel divides into two parts. The small pulsating part at the end of the channel creates the flow as a human heart, while the other big part loses this function. When the solution becomes unstable, the peristaltic mechanism of fluid transport loses importance. The transition from stability to instability leads to decrease of the efficiency of fluid pumping and the energy lost becomes increasing with growth of \( A \).
5. Conclusion

The transition from fluid’s motion, which is created by peristaltic mechanism, to the motion, which is created by ”heart”, has been found. This transition is caused by the lost of solution’s stability and the rise of dependence on soft boundary conditions. The channel with heart is a bad pump, because the transition leads to increase of energy lost and decrease of flow rate.

When the parameters $Re$, $\delta$ or $A$ are small, the flow rate increases with growth of the parameters. If the parameters $Re$, $\delta$ or $A$ are large, the flow rate decreases with growth of the parameters. The change in dependents is caused by the lost of solution’s stability.

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References

[1] Shapiro A H, Jaffrine M Y and Weinberg S L 1969 J.Fluid Mech 37 799–825
[2] Jaffrine M Y 1971 Int. J. Eng. Sci. 11 681–99
[3] Weinberg S L, Eckstein E C and Shapiro A H 1971 J.Fluid Mech 49 461–79
[4] Takabatake S and Ayukawa K 1982 J.Fluid Mech 122 439–65
[5] Takabatake S, Ayukawa K and Mori A 1988 J.Fluid Mech 193 267–83
[6] Mittra T K and Prasad S N 1973 J. Biomech. 6 681–93
[7] Sankad G C and Radhakrishnamacharya G 2010 IJAMM 6 94–107
[8] Muthu P, Rathish K and Chandra P 2003 ANZIAM J. 45 245–60
[9] Carew E and Pedley T 1997 J. Biomech. 110 66–76
[10] Bykova A A and Regirer S A 2005 Fluid Dynamics 40 3–23