B-SEQUENCES AND APPROXIMATIONS OF GENERALIZED
COHEN-MACULAY IDEALS

YUKIHIDE TAKAYAMA

Abstract. We introduce the notion of b-sequence for finitely generated modules over Noetherian rings, which characterizes long Bourbaki sequences. Our main concern is an application of this notion to generalized Cohen-Macaulay approximation, which we introduced in [6]. We will show how we can construct long Bourbaki sequences of non-trivial type characterizing generalized Cohen-Macaulay rings by finding suitable b-sequences.

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Introduction

Relation between Bourbaki sequences and local cohomomogies has been studied several times, for example, by Evans-Griffith [5] and Auslander-Buchweitz [2]. In [6], we studied approximations of generalized Cohen-Macaulay modules by (non-CM) maximal generalized Cohen-Macaulay modules.

Let \((R, \mathfrak{m})\) be a Gorenstein local ring and consider a generalized Cohen-Macaulay ideal \(I \subset R\) of codimension \(r(\geq 2)\), which is an ideal such that the local cohomology is \(H^i_{\mathfrak{m}}(R/I) \cong M_i, i = 0, \ldots, n - r - 1\), for some finite length \(R\)-modules \(M_i\). Then there exists a maximal generalized Cohen-Macaulay module \(M\) fitting into a length \(r\) long Bourbaki sequence

\[
(1) \quad 0 \rightarrow F_{r-1} \rightarrow \cdots \rightarrow F_1 \rightarrow M \rightarrow I \rightarrow 0 \quad (\text{exact})
\]

where \(F_i\) are \(R\)-free modules, such that \(H^i_{\mathfrak{m}}(M) \cong H^i_{\mathfrak{m}}(I)\) for \(i \leq n - r\) and \(H^{n-r+1}_{\mathfrak{m}}(M) = 0\). If we restrict ourselves to consider \(M\) satisfying the additional homological condition

\[
(2) \quad H^i_{\mathfrak{m}}(M) = 0 \quad (n - r + 2 \leq i \leq n - 1),
\]

then \(M\) is unique up to direct \(R\)-free summands (See Proposition 5). We will call the Bourbaki sequence \(M\) an approximation sequence and \(M\) an approximation module of \(I\). The proof of this fact is carried out by constructing, with a homological method, approximation modules \(M\) that always satisfy the condition (2).

In this paper, we are interested in approximation sequences that do not satisfy the condition (2). We do not know a systematic method to construct such sequences, particularly in the case of \(r \geq 3\). Recall that length 2 Bourbaki sequence can be constructed by finding basic elements ([4] Chapter VII §4). In section 1 we introduce the notion of b-sequences for Bourbaki sequences of arbitrary length, which plays a similar role to basic elements. Then we give a characterization of long Bourbaki sequences in terms of b-sequences (Theorem 3). Section 2 gives a characterization
of (non-trivial) approximation sequences that do not satisfy the condition (2) in a
typical case in terms of b-sequences (Theorem 12). Some examples in the case of
$r = 3$ are considered in section 3, where we focus on the special case of approximation
modules $M$ such that $H^t_{m+1}(M) = H^t_{m-1}(M) = K$ (field) and $H^t_m(M) = 0$ otherwise
for $i < n$.

For a set $S$, we will denote by $\langle S \rangle$ the module generated by $S$. Also, for a module
$M$ over a ring $R$, the $i$th syzygy module will be denoted by $\Omega_i(M)$.

1. b-sequences for Modules
1.1. b-sequences and long Bourbaki sequences. Recall that length 2 Bourbaki
sequences over a normal domain

$$0 \to F \to M \to I \to 0,$$

where $F$ is a $R$-free module, $M$ is a finitely generated torsion-free $R$ module and
$I \subset R$ is an ideal, can be constructed by finding basic elements in $M$ ([1] Chapter VII
§4). In this section, we introduce the notion of b-sequence, which is a counterpart of
basic elements for long Bourbaki sequences

$$0 \to F_{r-1} \to \cdots \to F_1 \to M \to I,$$

in particular for $r \geq 3$. We first prove

**Lemma 1.** Let $R$ be a Noetherian ring and $M$ be a finitely generated $R$-module
with a presentation $0 \to \text{Ker} \varepsilon \to U \xrightarrow{\varepsilon} M \to 0$, with a finitely generated free
$R$-module $U$. Also let $f : F \to G$ be a monomorphism of $R$-modules where $G$ is
free of rank $G = q$. Then, following are equivalent.

(i) We have an exact sequence

$$0 \to F \xrightarrow{f} G \xrightarrow{\beta} M \xrightarrow{I} 0 \quad (\text{exact})$$

for an ideal $I \subset R$.

(ii) We have $\beta_1, \ldots, \beta_q \in U \setminus \text{Ker} \varepsilon$ and $\varphi \in \text{Hom}_R(U, R)$ such that
(a) $\text{Ker}(\varphi) = \langle \beta_1, \ldots, \beta_q \rangle + \text{Ker} \varepsilon$, and
(b) we have the following commutative diagram

\[
\begin{array}{cccccc}
0 & \to & F & \xrightarrow{f} & G & \xrightarrow{\beta} M & \xrightarrow{I} 0 \\
| & & | & & | & & |
0 & \to & \text{Ker} \beta \circ f & \to & \text{Ker} \beta & \to & 0 \\
| & & | & & | & & |
0 & \to & F & \xrightarrow{f} & G & \xrightarrow{\beta} 0 \\
| & & | & & | & & |
0 & \to & \langle \beta_1, \ldots, \beta_q \rangle \cap \text{Ker} \varepsilon & \to & \langle \beta_1, \ldots, \beta_q \rangle & \xrightarrow{I} 0 \\
| & & | & & | & & |
0 & \to & 0 & & 0 & & 0
\end{array}
\]
where $\beta(m_i) = \beta_i$ for all $i$ with $\{m_1, \ldots, m_q\}$ a free basis of $G$.
In this case we have $I = \text{Im} \varphi$.

Proof. We first prove $(ii)$ to $(i)$. We set $I = \varphi(U)$. Then by $(a)$ we have the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \langle \beta_1, \ldots, \beta_q \rangle + \text{Ker} \varepsilon & \longrightarrow & U & \overset{\varphi}{\longrightarrow} & I & \longrightarrow & 0 \\
& & \varepsilon \downarrow & & \varepsilon \downarrow & & \varepsilon \downarrow & & \varepsilon \downarrow \\
0 & \longrightarrow & \langle \varepsilon(\beta_1), \ldots, \varepsilon(\beta_q) \rangle & \longrightarrow & M & & & & 0
\end{array}
$$

Then we can define a well-defined map $\psi : M \rightarrow I$ by $\psi(\epsilon(a)) = \varphi(a)$ for all $a \in U$, and we have $\text{Ker} \psi = \langle \varepsilon(\beta_1), \ldots, \varepsilon(\beta_q) \rangle$. Also we define $g : G \rightarrow M$ by $g = \varepsilon \circ \beta$. Then, from the above diagram, we have an exact sequence

$$
G \xrightarrow{g} M \xrightarrow{\psi} I \longrightarrow 0.
$$

On the other hand, we have

$$
\text{Ker} \ g = \left\{ \sum_{i=1}^{q} h_i m_i \mid \varepsilon(\sum_{i=1}^{q} h_i \beta_i) = 0, h_i \in R \right\}
$$

$$
= \left\{ \sum_{i=1}^{q} h_i m_i \mid \sum_{i=1}^{q} h_i \beta_i \in \text{Ker} \varepsilon, h_i \in R \right\}
$$

$$
= \left\{ \sum_{i=1}^{q} h_i m_i \mid \beta(\sum_{i=1}^{q} h_i m_i) \in \langle \beta_1, \ldots, \beta_q \rangle \cap \text{Ker} \varepsilon \right\}
$$

$$
= \left\{ u \in G \mid \beta(u) \in \langle \beta_1, \ldots, \beta_q \rangle \cap \text{Ker} \varepsilon \right\}
$$

Now let $u \in G$ be such that $\beta(u) \in \langle \beta_1, \ldots, \beta_q \rangle \cap \text{Ker} \varepsilon$. Then $u$ must be in $f(F)$. In fact, by $(b)$ we can choose $v \in F$ such that $(\beta \circ f)(v) = \beta(u)$. Thus $u - f(v) \in \text{Ker} \beta \cong \text{Ker}(\beta \circ f) \subseteq F$, and we have $u \in f(v) + \text{Ker} \beta \subseteq f(F)$ as required. Thus we have $\text{Ker} g \subseteq \text{Im} f$ and the converse inclusion is clear by $(b)$. Consequently, we have a desired exact sequence.

Next we prove $(i)$ to $(ii)$. Given an exact sequence

$$
0 \longrightarrow F \overset{f}{\longrightarrow} G \overset{g}{\longrightarrow} M \overset{\psi}{\longrightarrow} I \longrightarrow 0
$$

with an ideal $I \subseteq R$. Then $\psi \in \text{Hom}_R(M,R)$ and we have $\text{Ker} \varepsilon + N = \varepsilon^{-1}(\text{Ker} \psi)(\subseteq U)$ for some submodule $N(\neq 0)$ of $U$. Then we can choose a finite set of generators $\{\beta_i\}$ of $N$ such that $\varepsilon(\beta_i) = g(m_i)$ $(\forall i)$ where $\{m_i\}$ is a $R$-free basis of $G$. Then
we have the following diagram:

\[
\begin{array}{ccccccc}
0 & \rightarrow & G & \xrightarrow{g} & M & \xrightarrow{\psi} & I & \rightarrow & 0 \\
\uparrow & & \downarrow{\beta} & & \downarrow{\varepsilon} & & \downarrow & \\
0 & \rightarrow & \langle \beta_1, \ldots, \beta_q \rangle & \rightarrow & U & \rightarrow & 0 \\
\end{array}
\]

where we define \( \beta(m_i) = \beta_i \) (\( \forall i \)). Then by defining \( \varphi = \psi \circ \varepsilon \), we have \( \{\beta_i\}_i \) and \( \varphi \in \text{Hom}_R(U, R) \) satisfying the condition (ii)(a). Now we prove (ii)(b). Since \( \text{Ker} g = \text{Ker}(\varepsilon \circ \beta) = \text{Im} f \cong F \) we readily have the following diagram:

\[
\begin{array}{ccccccc}
0 & \rightarrow & 0 & & 0 & & 0 & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & \text{Ker} \beta \circ f & \rightarrow & \text{Ker} \beta & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & F & \xrightarrow{f} & G & \xrightarrow{g} & M \\
\downarrow{\beta} & & \downarrow & & \downarrow{\varepsilon|_{\text{Im} \beta}} & & \downarrow & \\
0 & \rightarrow & \langle \beta_1, \ldots, \beta_q \rangle \cap \text{Ker} \varepsilon & \rightarrow & \langle \beta_1, \ldots, \beta_q \rangle & \xrightarrow{\varepsilon|_{\text{Im} \beta}} & M \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & & & & & & 0 & \\
\end{array}
\]

Notice that since \( \text{Ker} \beta \subset \text{Ker}(\varepsilon \circ \beta) = \text{Ker} g = \text{Im} f \) we have the exactness of the first row. Since \( \text{Im}(\beta \circ f) = \beta(\text{Ker}(\varepsilon \circ \beta)) = \text{Ker}(\varepsilon|_{\text{Im} \beta}) = \langle \beta_1, \ldots, \beta_q \rangle \cap \text{Ker} \varepsilon \), we have a well-defined surjection

\[
\beta \circ f : F \longrightarrow \langle \beta_1, \ldots, \beta_q \rangle \cap \text{Ker} \varepsilon
\]
as required. \( \square \)

Now we introduce the notion of b-sequence.

**Definition 2.** Let \( R \) be a Noetherian ring. For a finitely generated \( R \)-module \( M \) with a presentation \( 0 \rightarrow \text{Ker} \varepsilon \rightarrow U \xrightarrow{\varepsilon} M \rightarrow 0 \) and a \( R \)-module monomorphism \( F \rightarrow G \) where \( G \) is \( R \)-free, the sequence \( \beta_1, \ldots, \beta_q \in U \setminus \text{Ker} \varepsilon \) together with \( \varphi \in \text{Hom}_R(U, R) \) satisfying the condition (ii) in Lemma 1 is called a b-sequence for the pair \( (f : F \rightarrow G, M) \).

From Lemma \( \square \) we immediately have a characterization of long Bourbaki sequences.
Theorem 3. Let $r \in \mathbb{Z}$ be $r \geq 2$. Let $R$ be a Noetherian ring and $M$ be a finitely generated $R$-module. Consider a $R$-module homomorphism $f_1 : F_1 \rightarrow M$ from a $R$-free module $F_1$. Then, following are equivalent.

(i) We have a long Bourbaki sequence of length $r$

$$0 \rightarrow F_{r-1} \xrightarrow{f_{r-1}} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} M \xrightarrow{\psi} I \rightarrow 0$$

where $I \subset R$ is an ideal and $F_i$ are $R$-free modules.

(ii) There exists a $b$-sequence $\{\beta_i\}$, $\varphi$ for $(\text{Ker} f_1 \hookrightarrow F_1, M)$ such that

$$0 \rightarrow F_{r-1} \xrightarrow{f_{r-1}} \cdots \xrightarrow{f_2} F_2 \xrightarrow{f_2} \text{Ker} f_1 \rightarrow 0 \quad (\text{exact})$$

Remark 4. Notice that a $b$-sequence in the case of length 2 Bourbaki sequence is not the same as a sequence of basic elements in the sense of $\mathbb{H}$. If we choose a suitable $b$-sequence $\{\beta_i\}$ under a suitable condition, $\{\varepsilon(\beta_i)\}$ can be a sequence of basic elements.

1.2. Syzygies of Artinian Gorenstein rings. A $b$-sequence has slightly more explicit description for some class of modules over Gorenstein local rings. Let $(R, \mathfrak{m})$ be a Gorenstein local ring of dimension $n$. We will denote the dual $\text{Hom}_R(-, R)$ by $(-)^*$. Let $J_i \subset R$ ($i = 0, \ldots, d$) ($d \leq n - r - 1$) be Gorenstein ideals of grade $n$ and set $M_i = R/J_i$. Let $(F^{(i)}_*, \partial^{(i)}_*)$ be a minimal $R$-free resolution of $M_i$. By self-duality of the resolution we immediately have

Lemma 5. For all $i$, $\Omega_i(M_i) \cong \Omega_{n-i+1}(M_i)^* \cong \partial^{*}_{n-i+1}(\Omega_{n-i+1}(M)^*)$

Now consider the module $M = \bigoplus_{i = 0}^d \Omega_i(M_i)$. By Lemma 5 we have

Proposition 6. Let $(\{\beta_i\}, \varphi)$ be a $b$-sequence for $M$. Then $\varphi = \bigoplus_{i = 0}^d a_i \circ \partial^{(i)}_i$ where $a_i \in \partial^{*}_i(\Omega_i(M_i))^*$.

This range of $a_i$ has more explicit description if $J_i = \mathfrak{m} = (x_1, \ldots, x_n)$ and $R = S = K[x_1, \ldots, x_n]$. For $I = \{i_1, \ldots, i_u\} \subset \{1, \ldots, n\} = [n]$, we denote by $e_I$ a base $e_{i_1} \wedge \cdots \wedge e_{i_u}$ of the Koszul complex $K_*$ over $S$ of sequences $x_1, \ldots, x_n$. A dual base to $e_I$ is denoted by $e_i^*$. For $J, K \subset [n]$ with $J \cap K = \emptyset$ we define $\sigma(J, K) = (-1)^i$ where $i = |\{(j, k) \in J \times K \mid j > k\}|$. Then we have $x_J \wedge x_K = \sigma(J, K)x_{J \cup K}$.

Corollary 7. Let $M_i = K(= R/\mathfrak{m})$ for all $i$. Then $a_i$ in Proposition 6 is an element from

$$\left\langle \sum_{k=1}^i (-1)^{k+1} \sigma(J \setminus \{j_k\}, [n] - (J \setminus \{j_k\})) x_{j_k} e_{[n] - (J \setminus \{j_k\})}^* : J = \{j_1, \ldots, j_i\} \subset [n] \right\rangle$$

2. Approximation of generalized Cohen-Macaulay ideals

2.1. Approximation modules. Let $(R, \mathfrak{m})$ be a Gorenstein local ring and consider a generalized Cohen-Macaulay ideal $I \subset R$ of codimension $r$ ($r \geq 2$) such that $H^i_m(R/I) = M_i$ for $i = 0, \ldots, n - r - 1$ where $M_i$ are finite length $R$-modules. Then we have the following result, which is an immediate consequence from Lemma 1.3 [1] and Theorem 1.1 [2].
Proposition 8. For a generalized Cohen-Macaulay ideal $I \subset R$ of codimension $r (\geq 2)$ there exists a maximal generalized Cohen-Macaulay module $M$ fitting into a Bourbaki sequence

$$0 \rightarrow F_{r-1} \rightarrow \cdots \rightarrow F_1 \rightarrow M \rightarrow I \rightarrow I$$

such that $H^i_m(M) \cong H^i_m(I)$ for $i \leq n - r$, $H^{n - r + 1}_m(M) = 0$. Moreover, if we assume the homological condition (2), then $M$ is unique up to $R$-free direct summands.

Notice that in Proposition 8, the ideal $I$ is approximated by the module $M$ in a similar sense to Auslander-Buchweitz (see [6] for detail). We will call the maximal Cohen-Macaulay module $M$ (or long Bourbaki sequence) an approximation module (or approximation sequence).

More specific result can be obtained when we consider a special class of ideals.

Proposition 9. Let $(R, m)$ be a regular local ring and let $I \subset R$ be an ideal of codimension $r (\geq 2)$. Assume that we have an approximation sequence

$$0 \rightarrow F_{r-1} \rightarrow \cdots \rightarrow F_1 \rightarrow \Omega_{t+1}(N) \oplus H \rightarrow I \rightarrow 0$$

for some $R$-free modules $H, F_1, \ldots, F_{r-1}$ and a finite length $R$-module $N$. Then, we have

$$(3) \quad H^i_m(R/I) = \begin{cases} N & i = t \\ 0 & i < n - r, i \neq t \end{cases}$$

Also the converse holds if

(i) $r = 2$, or

(ii) $r \geq 3$ and we assume the homological condition (2) for the approximation module $M$ of $I \subset R$.

Proof. The initial part of the proposition is clear. For the converse, the case $r = 2$ is Proposition 3.1 [6]. Now assume that $I \subset R$ satisfies (3) and the condition (ii). Then by Proposition 8 we have an approximation sequence

$$0 \rightarrow F_{r-1} \rightarrow \cdots \rightarrow F_1 \rightarrow X \rightarrow I \rightarrow 0$$

with

$$H^i_m(X) \cong H^i_m(I) \quad (i \leq n - r), \quad H^i_m(X) = 0 \quad (n - r + 1 \leq i \leq n - 1),$$

and it remains to show that if $X$ is a $R$-module of maximal dimension with the property that for $s := t + 1$ with $0 < s < n - r + 1$ one has

$$H^i_m(X) \cong \begin{cases} 0 & \text{for } i < n \text{ and } i \neq s \\ N & \text{for } i = s \end{cases},$$

then $X \cong \Omega_s(N) \oplus H$ with some $R$-free module $H$. But this fact is already proved in the proof of Proposition 3.1 [6].

As proved in Proposition 8 and Proposition 9, the homological condition (2) assures the uniqueness of approximation modules $M$. If we do not assume this condition, we have a large varieties of $M$ even in cohomologically very simple cases. For example,
Proposition 10. Let $r \geq 3$ and $0 \leq t \leq n-r-1$ be integers. Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field $K$ and $m = (x_1, \ldots, x_n)$. Let $M$ be a maximal generalized CM module over $S$ with depth $M = t + 1$. Consider a minimal $S$-free resolution of $M$:

$$F_* : 0 \rightarrow F_{n-t-1} \xrightarrow{\varphi_{n-t-1}} F_{n-t-2} \xrightarrow{\varphi_{n-t-2}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \rightarrow 0.$$  

Also let $N$ be a non-zero finite length module over $S$. Then the following are equivalent.

(i) For any $l \in \mathbb{Z}$ such that $n-r+2 \leq l \leq n-1$, we have

$$H^i_m(M) = \begin{cases} 
K & \text{if } i = t + 1 \\
N & \text{if } i = l \\
0 & \text{if } i < n, i \neq t + 1, l
\end{cases}$$

(ii) $\Omega_{n-l}(M) \cong F_{n-l}/E_{n+t+2-l}$, and $N^\vee \cong \Omega_{n-l}(M)^* / \text{Im } \varphi^*_n$.

where we denote $\Omega_s(K)$ simply by $E_s$, and we define $(-)^* = \text{Hom}_S(-, S(-n))$ and $(-)^\vee = \text{Hom}_S(-, K)$.

Proof. We first prove (i) to (ii). By taking the dual of $F_*$, we have

$$0 \rightarrow F_0^* \xrightarrow{\varphi_0^*} F_1^* \xrightarrow{\varphi_1^*} F_2^* \xrightarrow{\varphi_2^*} \cdots \xrightarrow{\varphi_{n-t-2}^*} F_{n-t-2}^* \xrightarrow{\varphi_{n-t-1}^*} F_{n-t-1}^* \rightarrow 0.$$  

Then by local duality the $j$th cohomology of this complex is

$$\text{Ext}_S^j(M, S(-n)) \cong H^{n-j}_m(M)^\vee = \begin{cases} 
K & \text{if } j = n - t - 1 \\
N^\vee & \text{if } j = n - l \\
0 & \text{if } j \neq n - t - 1, n - l
\end{cases}$$

for $j \geq 1$. Thus

$$0 \rightarrow \text{Im } \varphi^*_{n+1-l} \rightarrow F^*_{n+1-l} \rightarrow \cdots \rightarrow F^*_{n-t-1} \rightarrow K \rightarrow 0$$

is exact and $F^*_{n+1-l}$ to $F^*_{n-t-1}$ part is a begining of a minimal free resolution of $K$, which is isomorphic to the corresponding begining of the Koszul complex $(K_*, \partial_*)$ of the sequence $x_1, \ldots, x_n$. Namely,

$$F^*_{n-t-1} \cong K_0, \ldots, F^*_{n+1-l} \cong K_{t-2} \text{ and } \text{Im } \varphi^*_{n+1-l} \cong E_{t-1}.$$  

On the other hand, we have $N^\vee \cong \text{Ker } \varphi^*_{n+1-l} / \text{Im } \varphi^*_{n-l}$ and $E_{t-1} \cong \text{Im } \varphi^*_{n+1-l} \cong F^*_{n-l} / \text{Ker } \varphi^*_{n+1-l}$. Now set $U := \text{Coker } \varphi^*_{n-l} = F^*_{n-l} / \text{Im } \varphi^*_{n-l}$. Then

$$U / N^\vee \cong (F^*_{n-l} / \text{Im } \varphi^*_{n-l}) / (\text{Ker } \varphi^*_{n+1-l} / \text{Im } \varphi^*_{n-l}) \cong F^*_{n-l} / \text{Ker } \varphi^*_{n+1-l} = E_{t-1}.$$  

Thus we have

$$0 \rightarrow N^\vee \rightarrow U \rightarrow E_{t-1} \rightarrow 0$$

Taking the dual, we have

$$0 \rightarrow E_{t-1}^* \rightarrow U^* \rightarrow (N^\vee)^*.$$  

Since $N$ has finite length, $N^\vee$ has also finite length by Matlis duality, so that $(N^\vee)^* = 0$. Also $E_{t-1}^* \cong E_{n+t+2-l}$ by selfduality of Koszul complex. Thus we have $U^* \cong E_{n+t+2-l}$. Then by dualizing the exact sequence

$$F^*_{n-1-l} \xrightarrow{\varphi^*_{n-l}} F^*_{n-l} \rightarrow U \rightarrow 0$$  

(4)
we have

\[ 0 \rightarrow E_{n+l+2-l} \rightarrow F_{n-l} \xrightarrow{\varphi_{n-l}} F_{n-1-l} \rightarrow \Omega_{n-1-l}(M) \rightarrow 0. \]

This proves the first condition of \((ii)\). Now from the short exact sequence

\[ 0 \rightarrow \Omega_{n-l}(M) \rightarrow F_{n-1-l} \xrightarrow{\varphi_{n-1-l}} \Omega_{n-1-l}(M) \rightarrow 0 \]

we have the long exact sequence

\[ 0 \rightarrow \Omega_{n-1-l}(M) \xrightarrow{\varphi_{n-1-l}} F_{n-1-l}^* \rightarrow \Omega_{n-l}(M) \rightarrow N^\vee \rightarrow 0 \]

since we have \(\text{Ext}_{S}^{n-l}(M, S(-n)) \cong H_{m}^{I}(M)^\vee = N^\vee\) by local duality. Notice that we have \(\Omega_{n-l}(M)^* \cong \text{Ker} \varphi_{n-1-l}\) from the short exact sequence \(F_{n+1-l} \xrightarrow{\varphi_{n+1-l}} F_{n-l} \rightarrow \Omega_{n-l}(M) \rightarrow 0\). This proves the second condition in \((ii)\).

Next we prove \((ii)\) to \((i)\). By \((ii)(a)\) we have a \(S\)-free resolution of \(M\):

\[ 0 \rightarrow K_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_{n+t+3-l}} K_{n+t+2-l} \xrightarrow{\partial_{n+t+2-l}} F_{n-l} \xrightarrow{\varphi_{n-l}} \cdots \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \rightarrow 0 \]

where \(F_i\) are \(S\)-free modules. By taking the dual, we have the complex

\[ 0 \rightarrow M^* \xrightarrow{\varphi_0^*} F_0^* \xrightarrow{\varphi_1^*} \cdots \xrightarrow{\varphi_1^*} F_{n-l}^* \xrightarrow{\partial_{n+t+2-l}^*} K_{n+t+2-l}^* \xrightarrow{\partial_{n+t+3-l}^*} \cdots \xrightarrow{\partial_{n+t+3-l}^*} K_n^* \rightarrow 0 \]

Then by local duality and selfduality of Koszul complex we compute

\[ H_{m}^{I}(M) \cong \text{Ext}_{S}^{t-i}(M, S(-n))^\vee = \begin{cases} K & \text{if } i = t + 1 \\ 0 & \text{if } i \leq l - 1, i \neq t + 1 \end{cases} \]

Now by dualizing the exact sequence

\[ K_{n+t+2-l} \xrightarrow{\partial_{n+t+2-l}} F_{n-l} \xrightarrow{\varphi_{n-l}} \text{Ker} \varphi_{n-1-l} \rightarrow 0 \]

we have

\[ 0 \rightarrow (\text{Ker} \varphi_{n-1-l})^* \rightarrow F_{n-l}^* \xrightarrow{\partial_{n+t+2-l}^*} K_{n+t+2-l}^* \]

so that we have \(\Omega_{n-l}(M)^* = (\text{Ker} \varphi_{n-1-l})^* \cong \text{Ker}(\partial_{n+t+2-l}^*)\). Then with the second condition in \((ii)\) we compute

\[ H_{m}^{I}(M)^\vee \cong \text{Ext}_{S}^{n-i}(M, S(-n)) = \text{Ker} \partial_{n+t+2-l}^*/\text{Im} \varphi_{n-l}^* \cong \Omega_{n-l}(M)^*/\text{Im} \varphi_{n-l}^* = N^\vee \]

as required. \(\Box\)

A typical class of the modules that do not satisfy the homological condition \((2)\) is \(\bigoplus_{i=0}^{n-r-1} \Omega_{i+1}(M_i) \oplus \bigoplus_{i=n-r+2}^{n-1} \Omega_{i}(N_i)\) for finite length modules \(M_i\) and \(N_i\), which we will consider in the next subsection.
2.2. Approximation sequences of non-trivial type. In this subsection we assume \((R, m)\) to be regular local. In the proof of Proposition 8 we construct approximation modules \(M\) in a homological method, which always entails the homological condition (2). See [6] and [1]. Now we are interested in the following problem: how can we construct approximation sequences as in Proposition 8 that do not satisfy the homological condition (2)? The simplest answer to this question is to make the direct sum of an approximation sequence as in Proposition 8 and the following exact sequences:

\[
0 \rightarrow G^{(i)}_n \rightarrow \cdots \rightarrow G^{(i)}_i \rightarrow \Omega_i(N_i) \rightarrow 0 \quad (i = n - r + 2, \ldots, n - 1)
\]

where \(N_i\) are any finite length \(R\)-modules and \(G^{(i)}_\bullet\) are minimal \(R\)-free resolutions of \(N_i\). Then we have a Bourbaki sequence with the approximation module \(M' = M \oplus \bigoplus_{i=n-r+2}^{n-1} \Omega_i(N_i)\) and the map from \(M'\) to the ideal \(I\) is trivial on \(\bigoplus_{i=n-r+2}^{n-1} \Omega_i(N_i)\) part. We will call this an approximation sequence of trivial type.

Now we will consider approximation sequences of non-trivial type. Let \(r \in \mathbb{Z}\) be \(r \geq 2\) and \(n \geq r + 1\). Consider a long Bourbaki sequence of length \(r\)

\[
0 \rightarrow F_{r-1} \rightarrow \cdots \rightarrow F_2 \rightarrow F_1 \xrightarrow{g} M \oplus N \xrightarrow{\phi} I \rightarrow 0
\]

where \(I \subset R\) is a generalized Cohen-Macaulay ideal of codimension \(r\), \(F_i\) are \(R\)-free modules, and \(M = \bigoplus_{i=0}^{n-r-1} \Omega_{i+1}(M_i)\) and \(N = \bigoplus_{i=n-r+2}^{n-1} \Omega_i(N_i)\). From this sequence, we construct the following diagram, where \(U_\bullet \xrightarrow{\varepsilon} M\) and \(V_\bullet \xrightarrow{\eta} N\) are minimal free resolutions of \(M\) and \(N\), and the third row is the mapping cone \(C(\alpha_\bullet)\) of a chain map \(\alpha_\bullet\), which is a \(R\)-free resolution of \(I\).

\[
\begin{array}{cccccc}
\cdots & \rightarrow & F_2 & \rightarrow & F_1 & \xrightarrow{g} \text{Ker } \phi & \rightarrow 0 \\
\downarrow & & \alpha_2 & \downarrow & \alpha_1 & \downarrow & \\
\cdots & \rightarrow & U_1 \oplus V_1 & \rightarrow & U_0 \oplus V_0 & \xrightarrow{\varepsilon \oplus \eta} M \oplus N & \rightarrow 0 \\
\downarrow & & & & \downarrow & \phi & \\
\cdots & \rightarrow & U_1 \oplus V_1 \oplus F_1 & \rightarrow & U_0 \oplus V_0 & \rightarrow & I & \rightarrow 0 \\
& & & & \downarrow & & \\
& & & & 0
\end{array}
\]

Let \(p_1 : U_0 \oplus V_0 \rightarrow U_0\) and \(p_2 : U_0 \oplus V_0 \rightarrow V_0\) be the first and the second projections. From the diagram (6) we know that \(\text{Ker } \phi = \text{Im } g = ((\varepsilon \oplus \eta) \circ \alpha_1)(F_1)\) and then by considering the ranks of the modules in the short exact sequence

\[
0 \rightarrow \text{Ker } \phi \rightarrow M \oplus N \xrightarrow{\phi} I \rightarrow 0
\]

we have

\[
\text{rank}(\text{Ker } \varphi) = \text{rank}(M) + \text{rank}(N) - 1.
\]

On the other hand, we have

\[
(\varepsilon \circ p_1 \circ \alpha_1)(F_1) \oplus (\eta \circ p_2 \circ \alpha_1)(F_1) \supset ((\varepsilon \oplus \eta) \circ \alpha_1)(F_1) = \text{Ker } \phi.
\]
Thus we have

\[(10) \quad \text{rank} \mathcal{I}_{U_0} + \text{rank} \mathcal{I}_V \geq \text{rank}(M) + \text{rank}(N) - 1\]
\[(11) \quad \text{rank}(M) \geq \text{rank} \mathcal{I}_{U_0}\]
\[(12) \quad \text{rank}(N) \geq \text{rank} \mathcal{I}_V\]

where \(\mathcal{I}_{U_0} := (\varepsilon \circ p_1 \circ \alpha_1)(F_i)(\subseteq M)\) and \(\mathcal{I}_V := (\eta \circ p_2 \circ \alpha_1)(F_i)(\subseteq N)\). From this we know that \((\text{rank} \mathcal{I}_{U_0}, \text{rank} \mathcal{I}_V) = (\text{rank}(M), \text{rank}(N)), (\text{rank}(M) - 1, \text{rank}(N))\), or \((\text{rank}(M), \text{rank}(N) - 1)\). Under this situation, we have

**Lemma 11.** Following are equivalent.

(i) the approximation sequence \([9]\) is of non-trivial type.

(ii) For any free basis \(\{m_i\}_i\) of \(F_1\) there exists an index \(i\) such that \(\alpha_1(m_i) \notin U_0\) and \(\alpha_1(m_i) \notin V_0\).

Proof. We will prove (i) to (ii). We assume that for all \(i\) we have either \(\alpha_1(m_i) \in U_0\) or \(\alpha_1(m_i) \in V_0\), and will deduce a contradiction. First of all, we have equality in \([9]\), and then from \([10]\) we have

\[\text{rank}(M) + \text{rank}(N) - 1 = \text{rank} \mathcal{I}_{U_0} + \text{rank} \mathcal{I}_V.\]

Thus we have \((\text{rank} \mathcal{I}_{U_0}, \text{rank} \mathcal{I}_V) = (\text{rank}(M) - 1, \text{rank}(N))\) or \((\text{rank}(M), \text{rank}(N) - 1)\). Also, since \(\text{Ker} \phi = \mathcal{I}_{U_0} \oplus \mathcal{I}_V\), we have by \([17]\)

\[(13) \quad \mathcal{I} \cong (M/\mathcal{I}_{U_0}) \oplus (N/\mathcal{I}_V)\]

**case** \((\text{rank} \mathcal{I}_{U_0}, \text{rank} \mathcal{I}_V) = (\text{rank}(M) - 1, \text{rank}(N))\): Since we have \(\text{rank} N/\mathcal{I}_V = \text{rank}(N) - \text{rank} \mathcal{I}_V = 0\), \(N/\mathcal{I}_V\) is 0 or a torsion-module. But since \(I\) is torsion-free, we must have \(N = \mathcal{I}_V\) by \([13]\). Thus \(\text{Ker} \phi = \mathcal{I}_{U_0} \oplus N\) and then the Bourbaki sequence \([5]\) must be of trivial-type

\[\cdots \to F'_1 \oplus V_1 \to F'_0 \oplus V_0 \to M \oplus N \xrightarrow{\phi} I \to 0\]

where \(F'_i\), a \(S\)-free resolution of \(I_{U_0}\), a contradiction.

**case** \((\text{rank} \mathcal{I}_{U_0}, \text{rank} \mathcal{I}_V) = (\text{rank}(M), \text{rank}(N) - 1)\): In this case we have \(\text{rank} M/\mathcal{I}_{U_0} = 0\). Since \(M/\mathcal{I}_{U_0} \subset I\) by \([13]\) and \(I\) is torsion-free, we must have \(M/\mathcal{I}_{U_0} = 0\). Thus \(\text{Ker} \phi = M \oplus \mathcal{I}_V\) and the Bourbaki sequence \([5]\) is obtained by combining

\[0 \to M \oplus \mathcal{I}_V \to M \oplus N \xrightarrow{\phi} I \to 0\]

with the minimal \(R\)-free resolution \(U_*\) of \(M\) and a minimal \(R\)-free resolution of \(\mathcal{I}_V\).

Now since \(M\) must satisfy \(H^i_m(M) = H^i_m(I) \cong H^{i-1}_{m-1}(R/I)\) for \(i = 1, \ldots, n - r\) and depth \(R/I \leq n - r - 1\), we must have depth \(M \leq n - r\). Thus by Auslander-Buchsbaums formula, the length of \(U_*\) must be \(\geq r\), which exceeds the length of our Bourbaki sequence, a contradiction.

Now we show (ii) to (i). Assume that \([5]\) is of trivial type. Then we must have \(\alpha_1(F_1) = p_1(\alpha_1(F_1)) \oplus p_2(\alpha_1(F_1))\). From this we immediately obtain the required result. \(\square\)
From Lemma 11 we immediately have

**Theorem 12.** The approximation sequence (3) is of non-trivial type if and only if

(i) there exists a b-sequence \( \{ \beta_i \}_i \subset U_0 \oplus V_0 \) and

(ii) the submodule \( N := \{ \{ \beta_i \}_i \} \) of \( U_0 \oplus V_0 \) cannot be decomposed in the form of \( N = A \oplus B \) for some \((0 \neq) A \subset U_0 \) and \((0 \neq) B \subset V_0\)

### 3. Some Applications in Codimension 3

#### 3.1. b-sequences for \( E_{t+1} \) and \( E_{t+1} \oplus E_{n-1}(d) \) \((d \in \mathbb{Z})\).

As an application of our theory, we will consider a special case. Let \( S = K[x_1, \ldots, x_n] \) and \( \mathfrak{m} = (x_1, \ldots, x_n) \).

We consider the standard grading with \( \deg(x_i) = 1 \) for all \( i \). Also, in the following, the dual \((-)^* \) always denotes \( \text{Hom}_S(-, S(-n)) \).

We now consider the graded approximation module \( M = E_{t+1} \) and \( E_{t+1} \oplus E_{n-1}(d) \), for arbitrarily \( d \in \mathbb{Z} \).

First of all, by Lemma 8 and Corollary 7 we have the following.

**Corollary 13.** Following are equivalent.

(i) We have a length 3 Bourbaki sequence

\[
0 \rightarrow F \xrightarrow{f} G \xrightarrow{g} E_{t+1} \oplus E_{n-1}(d) \rightarrow I(c) \rightarrow 0 \quad \text{(exact)}
\]

where \( I \subset S \) is a graded ideal and \( F \) and \( G \) are finitely generated \( S \)-free modules.

(ii) \( \text{rank } F = \text{rank } G - n + 2 - \binom{n-1}{t} \) and we have a b-sequence \( \{ \{ \beta_i \}_i, \varphi \} \) for \( (f, E_{t+1} \oplus E_{n-1}(d)) \) where \( \beta_i \subset K_{t+1} \oplus K_{n-1}(d) \| E_{t+2} \oplus E_n(d) \) and \( \varphi = (a, b) \in A \times B \), with

\[
A = \left\{ \sum_{j=1}^{n-t} (-1)^{j+1} \sigma(L\{i_j\}, ([n]\{L\} \cup \{i_j\})x_j e_{([n]\{L\} \cup \{i_j\})) \mid L = \{i_1, \ldots, i_{n-t}\} \subset [n] \right\}
\]

\[
B = \{ (-1)^j x_j e_{[n]\{i\}} - (-1)^j x_i e_{[n]\{j\}} \mid 1 \leq i < j \leq n \},
\]

and thus \( \varphi : K_{t+1} \oplus K_{n-1}(d) \rightarrow S(-n) \) is a degree \( n + c \) homomorphism.

In this case, we have \( I = \varphi(K_{t+1} \oplus K_{n-1}(d))(c) \)

We also consider the case of \( M = E_{t+1} \).

**Corollary 14.** Following are equivalent.

(i) We have a length \( r \geq 3 \) Bourbaki sequence

\[
0 \rightarrow F_{r-1} \rightarrow \cdots \rightarrow F_2 \rightarrow F_1 \xrightarrow{f_1} E_{t+1} \rightarrow I(c) \rightarrow 0 \quad \text{(exact)}
\]

where \( I \subset S \) is a graded ideal, and \( F_i \) are finitely generated \( S \)-free modules.

(ii) We have \( \text{rank Ker } f_1 = \text{rank } F_1 + 1 - \binom{n-1}{t} \) and a b-sequence \( \{ \{ \beta_i \}_i \subset K_{t+1} \| E_{t+2}, \varphi \in A \} \) for \((Ker f_1 \rightarrow F_1, E_{t+1})\) where

\[
A = \left\{ \sum_{j=1}^{n-t} (-1)^{j+1} \sigma(L\{i_j\}, ([n]\{L\} \cup \{i_j\})x_j e_{([n]\{L\} \cup \{i_j\})) \mid L = \{i_1, \ldots, i_{n-t}\} \subset [n] \right\}
\]

and thus \( \varphi : K_{t+1} \rightarrow S(-n) \) defines a degree \( n + c \) homomorphism. In this case, we have \( I = \varphi(K_{t+1})(n-c) \).
A small application of this explicit formula is

**Corollary 15.** There is no graded ideal $I \subset S$ of codimension $r \geq 2$ of depth$(S/I) = 0$ such that local cohomology is trivial except $H^0_m(S/I) = K(c)$ (for some $c \in \mathbb{Z}$) and having a length $r$ approximation sequence with approximation module $E_1$.

**Proof.** Assume that there exists an ideal $I \subset S$ such that $H^0_m(S/I) = K(c)$ and $H^1_m(S/I) = 0$ $(0 < i < n - r)$ having the following approximation sequence:

$$0 \rightarrow F_{r-1} \rightarrow \cdots \rightarrow F_1 \rightarrow E_1 \xrightarrow{\varphi} I(c) \rightarrow 0.$$ 

Then by Corollary 14 there exists a b-sequence $(\{\beta_i\}, \varphi)$ with $\beta_i \in K_1 \setminus E_2$ and $(0 \neq) \varphi \in \mathcal{A}$. Then $(\{\beta_i\}) + E_2 = \text{Ker}(\varphi : K_1 \rightarrow S(-n))$. Since $\varphi$ is a non-zero element from $\mathcal{A} = \langle x_1e_1^+ + \cdots + x_ne_n^* \rangle$ we must have $\{\beta_i\} \subset E_2(\neq E_1)$, a contradiction.

**Remark 16.** By Proposition 9 it is assured that an ideal as in Corollary 15 has a length $r$ approximation sequence with approximation module $E_1 \oplus H$, with non-trivial $S$-free module $H$. We will show later that there exists an approximation sequence with approximation module $E_1 \oplus E_{n-1}$, due to Corollary 14. See Example 20.

### 3.2. Numerical condition for codimension 3.

Now we consider in particular the case of Corollary 15. The existence of approximation sequences as in Corollary 13 only implies that codim $I \leq 3$. To assure that the codimension is exactly 3, we need additional condition. We have

**Proposition 17.** Let $n \geq 4$ and $t \leq n - 4$. Assume that we have the following long Bourbaki sequence

$$0 \rightarrow \bigoplus_{i=1}^p S(-a_i) \rightarrow \bigoplus_{i=1}^q S(-b_i) \rightarrow E_{t+1} \oplus E_{n-1}(d) \rightarrow I(c) \rightarrow 0$$

with $I \subset S$ a graded ideal and $c \in \mathbb{Z}$. Then we have codim $I \leq 3$ and the equality holds if and only if

1. $q = p + \binom{n-1}{t} + n - 2$;
2. $\sum_{i=1}^q b_i - \sum_{i=1}^p a_i = n^2 - (2 + d)n + c + d + \binom{n-2}{t-1} + \binom{n-1}{t}$;
3. $\sum_{i=1}^q b_i^2 - \sum_{i=1}^p a_i^2 = n^3 - (3 + 2d)n^2 + (d^2 + 4d + 1)n - c^2 - d^2$

$$+ \binom{n-1}{t}(t+1)^2 - \binom{n-2}{t}(2t+1) - 2\binom{n-3}{t-1}.$$

**Proof.** Now from the sequence (13), we construct the mapping cone $C(a_\bullet)$ in a similar way to (10). The cone gives a $S$-free resolution $F_\bullet$ of the residue ring $S/I$.

$$F_\bullet : 0 \rightarrow F_{n-t} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow S/I \rightarrow 0$$
where

\[
F_0 = S \\
F_1 = K_{t+1}(-c) \oplus K_{n-1}(d-c) = S(-t-1-c)^{\beta_1} \oplus S(-n+1+d-c)^n \\
F_2 = K_{t+2}(-c) \oplus K_n(d-c) \oplus G(-c) \\
\quad = S(-t-2-c)^{\beta_2} \oplus S(-n+d-c) \oplus \bigoplus_{i=1}^q S(-b_i - c) \\
F_3 = K_{t+3}(-c) \oplus F(-c) = S(-t-3-c)^{\beta_3} \oplus \bigoplus_{i=1}^p S(-a_i - c) \\
F_i = K_{t+i}(-c) = S(-t-i-c)^{\beta_i} \quad (4 \leq i \leq n-t) \\
\text{with} \quad \beta_i = \binom{n}{t+i} \quad i = 1, \ldots, n-t.
\]

Now we compute the Hilbert series \( \text{Hilb}(S/I, \lambda) \) of \( S/I \). We have

\[
(15) \quad \text{Hilb}(S/I, \lambda) = \frac{Q(\lambda)}{(1-\lambda)^n}
\]

with

\[
Q(\lambda) = \sum_{i,j} (-1)^i \beta_{i,j} \lambda^j
\]

\[
= 1 - n\lambda^{n-1+c-d} + \lambda^{n+c-d} + \sum_{i=1}^q \lambda^{b_i+c} - \sum_{i=1}^p \lambda^{a_i+c} + (-1)^t \lambda^c \sum_{i=t+1}^n \binom{n}{i} \lambda^i
\]

where \( \beta_{i,j} \) are as in \( F_i = \bigoplus_j S(-j)^{\beta_{i,j}} \), \( i = 0, \ldots, n-t \). (see Lemma 4.1.13). Since we have \( H_m^i(S/I) = H_{m+1}^i(M) \) for \( 0 \leq i \leq n-4 \), we know that \( \dim S/I \geq n-3 \), i.e., \( \text{codim} I \leq 3 \). To assure that \( \text{codim} I \geq 3 \) we must have \( Q(1) = Q'(1) = Q''(1) = 0 \) (see Corollary 4.1.14(a)).

Now by straightforward computations, we have

(1) \( Q(1) = 0 \) holds for all \( n, t, c \) and \( p \)
(2) \( Q'(1) = 0 \) holds if and only if

\[
\sum_{i=1}^q b_i - \sum_{i=1}^p a_i = n^2 - (2 + d)n + c + d + \binom{n-2}{t-1} + \binom{n-1}{t} t.
\]

(3) \( Q''(1) = 0 \) holds if and only if

\[
\sum_{i=1}^q b_i^2 - \sum_{i=1}^p a_i^2 = n^3 - (3 + 2d)n^2 + (d^2 + 4d + 1)n - c^2 - d^2 + \binom{n-1}{t} (t+1) - \binom{n-2}{t} (2t+1) - 2 \binom{n-3}{t-1}
\]

as required.
3.3. Examples. Now we give a few concrete examples in codimension 3.

Example 18 (approximation module $E_{t+1}$ with $t = \text{depth } S/I = 1$). We first give an application of Corollary 13. Namely, a codimension 3 ideal $I$ with approximation module $E_{t+1}$. Let $t = 1$ and $n = 6$. Then $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5, A_6\} = \partial_1^2(E_2) \subset K_2^*$ where

\[
\begin{align*}
A_1 &= x_1e_{16}^* + x_2e_{26}^* + x_3e_{36}^* + x_4e_{46}^* + x_5e_{56}^*, \\
A_2 &= -x_1e_{15}^* - x_2e_{25}^* - x_3e_{35}^* - x_4e_{45}^* + x_6e_{66}^*, \\
A_3 &= x_1e_{14}^* + x_2e_{24}^* + x_3e_{34}^* - x_5e_{45}^* - x_6e_{66}^*, \\
A_4 &= -x_1e_{13}^* - x_2e_{23}^* + x_4e_{43}^* + x_5e_{53}^* + x_6e_{66}^*, \\
A_5 &= x_1e_{12}^* - x_3e_{23}^* - x_4e_{42}^* - x_5e_{52}^* - x_6e_{66}^*, \\
A_6 &= x_2e_{12}^* + x_3e_{13}^* + x_4e_{24}^* + x_5e_{15}^* + x_6e_{66}^*.
\end{align*}
\]

We choose a $b$-sequence $\{(\beta_i)_t, a\}$ with $a \in \mathcal{A}$ and $\beta_i \in K_2 \setminus E_3$ as follows:

\[
a = x_6A_1 - x_5A_2 + x_4A_3 = x_1x_4e_{14}^* + x_1x_5e_{15}^* + x_1x_6e_{16}^* + x_2x_4e_{24}^* + x_2x_5e_{25}^* + x_2x_6e_{26}^* + x_3x_4e_{34}^* + x_3x_5e_{35}^* + x_3x_6e_{36}^*.
\]

\[
\beta_1 = e_{12}, \quad \beta_2 = e_{13}, \quad \beta_3 = e_{23}, \quad \beta_4 = e_{45}, \quad \beta_5 = e_{46}, \quad \beta_6 = e_{56}
\]

Then we obtain the long Bourbaki sequence

\[
0 \longrightarrow S^2(-3) \xrightarrow{f} S^6(-2) \xrightarrow{g} E_2 \xrightarrow{\varphi} I \longrightarrow 0.
\]

We can check that all the conditions in Proposition 17 are satisfied so that we must have codim $I = 3$. If $\{n_i\}$ and $\{m_j\}$ are free bases of $S^2(-3)$ and $S^6(-2)$, we have

\[
f : \quad S^2(-3) = Sn_1 \oplus Sn_2 \quad \longrightarrow \quad S^6(-2) = Sm_1 \oplus \cdots \oplus Sm_6
\]

\[
g : \quad S^6(-2) = Sm_1 \oplus \cdots \oplus Sm_6 \quad \longrightarrow \quad \partial_2(\beta_i) \quad (i = 1, \ldots, 6)
\]

and

\[
\varphi : \quad E_2 \xrightarrow{\partial_2(e_{ij})} I \quad \text{for} \quad i,j = (1,2), (1,3), (2,3), (4,5), (4,6), (5,6)
\]

\[
\partial_2(e_{ij}) \xrightarrow{\partial_2(e_{ij})} 0 \quad \text{for} \quad i,j \neq (1,2), (1,3), (2,3), (4,5), (4,6), (5,6)
\]

and we obtain $I = (x_1x_4, x_1x_5, x_1x_6, x_2x_4, x_2x_5, x_2x_6, x_3x_4, x_3x_5, x_3x_6) = (x_1, x_2, x_3)(x_4, x_5, x_6)$.

Example 19 (approximation module $E_{t+1} \oplus E_{n-1}$ with $t = \text{depth } S/I = 1$). We continue to consider the situation in Example 18. As an application of Corollary 15, we can see that the same ideal fits into a long Bourbaki sequence with approximation module $E_{t+1} \oplus E_{n-1} = E_1 \oplus E_5$. In this case, we must also consider $\mathcal{B} = \{B_{ij} \mid 1 \leq
where \( \{m_i\}_i \) is a free basis of \( L \). Notice that \( \{\beta_i\}_i \leq 1 \) satisfies the condition of Proposition \[13\].

Then we have an approximation sequence of non-trivial type

\[
0 \to S^2(-3) \oplus S(-6) \xrightarrow{f} S^6(-2) \oplus S^6(-5) \xrightarrow{g} E_2 \oplus E_5 \xrightarrow{\varphi} I \to 0
\]

where

\[
f : S^2(-3) \oplus S(-6) = \bigoplus_{i=1}^3 S n_i \quad \mapsto \quad S^6(-2) \oplus S^6(-5) = \bigoplus_{i=1}^{12} S m_i
\]

and

\[
g : S^6(-2) \oplus S^6(-5) = \bigoplus_{i=1}^{12} S m_i \quad \mapsto \quad E_2 \oplus E_5
\]

and

\[
\varphi
\]

\[
\partial_1(e_{ij})
\]

\[
\partial_1(e_{ij})
\]

\[
\partial_5(e_{23456})
\]

\[
\partial_5(e_{12356})
\]

\[
\partial_5(e_{ijklm})
\]

and the ideal \( I \) is the same as that in Example \[13\]. We can also check that this sequence satisfies the numerical condition in Theorem \[16\].

\textbf{Example 20} (approximation module \( E_{t+1} \oplus E_{n-1}(d) \) with \( t = \text{depth} S/I = 0 \).

By Corollary \[13\], we do not have a long Bourbaki sequence with an approximation module \( E_1 \) and a codimension 3 generalized CM ideal \( I \). However, there are long Bourbaki sequences with approximation modules \( E_1 \oplus E_5(d) \) for \( d \in \mathbb{Z} \), which is an application of Corollary \[13\]. Let \( k = 1 \) and \( n = 6 \). Then, we choose a b-sequence
Let $\{\beta_i\}, \phi$ as follows: We set $\beta_i \in K_1 \oplus K_5(1)$ to be

$$
\begin{align*}
\beta_1 &= -x_6e_{12345} + x_5e_{12346}, \\
\beta_2 &= x_6^2e_3 - x_1^2e_{13456}, \\
\beta_3 &= x_6^3e_2 - x_1^2e_{12456}, \\
\beta_4 &= x_2^4x_5e_2 - x_1e_{12345}, \\
\beta_5 &= x_1^2x_6e_2 - x_1^2e_{12346}, \\
\beta_6 &= -x_6^4e_{12346} + x_2^2e_{12456}, \\
\beta_7 &= e_{23456}, \\
\beta_8 &= e_{12356}.
\end{align*}
$$

Also let $\phi = (a, b) \in A \times B$ be

$$
\begin{align*}
a &= x_1^3e_1^* + x_1^2x_2e_2^* + x_1^2x_3e_3^* + x_1^2x_4e_4^* + x_1^2x_5e_5^* + x_1^2x_6e_6^*, \\
b &= -x_2^5B_56 + x_5^6B_{23} \\
&= x_2^5x_6e_{12346} + x_2^5x_5e_{12345} + x_3^5x_6e_{13456} + x_2^5x_6e_{12456}
\end{align*}
$$

where

$$
\begin{align*}
A &= \langle x_1e_1^* + \cdots + x_6e_6^* \rangle \\
B &= \langle B_{ij} = (-1)^i x_i e_{[i] \cdot j} - (-1)^j x_i e_{[i] \cdot j} : 1 \leq i < j \leq 6 \rangle.
\end{align*}
$$

Then we can check that $\{\beta_i\}, \phi$ is a b-sequence for $(F' \rightarrow G, E_1 \oplus E_5(1))$ where

$$
\begin{align*}
G &= \langle m_1, \ldots, m_8 \rangle = S(-5) \oplus S^4(-6) \oplus S(-8) \oplus S^2(-4), \\
F' &= \langle x_1^3m_1 + x_6m_4 - x_5m_5, x_1^3m_3 - x_1^2m_5 + x_1^2m_6, \\
x_2^5m_1 - x_2m_2 + x_3m_3 - x_2^3m_7 + x_2^2x_4m_8 \rangle.
\end{align*}
$$

Also we know that the condition of Theorem 12 is satisfied. Thus we have a non-trivial approximation sequence

$$
0 \rightarrow F \xrightarrow{f} G \xrightarrow{g} E_1 \oplus E_5(1) \xrightarrow{\phi} I(2) \rightarrow 0
$$

where $g(m_i) = \beta_i$, $i = 1, \ldots, 8$, and $F = S(-10) \oplus S^2(-7) = \langle u, v, w \rangle$ with $f(u) = x_1^3m_3 - x_2^3m_4 + x_1^2m_6, f(v) = -x_2^5m_1 + x_6m_4 - x_5m_5$ and $f(w) = -x_1^3m_1 - x_2m_2 + x_3m_3 - x_2^3m_7 + x_2^2x_4m_8$. The map $\phi$ is as follows: $\phi(x_i) = x_i x_1^2 (i = 1, \ldots, 6), \phi(e_{12345}) = x_2^5x_5, \phi(e_{12346}) = x_2^5x_6, \phi(e_{12356}) = 0, \phi(e_{12456}) = x_2^5x_6, \phi(e_{13456}) = x_3x_6^5, \phi(e_{23456}) = 0$. The ideal is $I = \text{Im} \phi = x_1^2m + (x_5^5x_6, x_2^5x_5, x_3^5x_6, x_2x_4^5)$. Finally we can check that this approximation sequence satisfies the numerical condition of Theorem 17 so that codim $I = 3$.

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YUKIHIDE TAKAYAMA, DEPARTMENT OF MATHEMATICAL SCIENCES, RITSUMEIKAN UNIVERSITY, 1-1-1 NOJIHGASHI, KUSATSU, SHIGA 525-8577, JAPAN

E-mail address: takayama@se.ritsumei.ac.jp