Hiding Planets behind a Big Friend: Resonant Planets with an Inclined External Companion

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ABSTRACT

Recent observations suggest that a large fraction of Kepler super-Earth systems have external giant planet companions (cold Jupiters), which can shape the architecture of the inner planets, in particular their mutual inclinations. The dynamical perturbation from cold Jupiters may account for the population of misaligned planets in the Kepler data. The effectiveness of this mechanism can be hindered by a strong planet-planet coupling in the inner system. In this paper, we study how mean-motion resonances (MMRs) affect this coupling and the expected misalignment. We derive approximate analytic expressions for the mutual inclination excitations in the inner planet system induced by a inclined companion, for various period ratios and perturber properties. In most cases, the mutual inclination is proportional to a dimensionless parameter that characterizes the strength of the perturber relative to the coupling in the inner system. We show that the MMR strengthens the inner coupling, reducing the mutual inclination induced by the perturber by a factor of a few. We find that the resonance is resilient to the perturbation, and derive a criterion for the libration of the resonant angle. Our results have applications for constraining unseen planetary perturbers, and for understanding the architecture of multiplanet systems.

Key words: Planetary Systems — Celestial mechanics

1 INTRODUCTION

Among the thousands of exoplanets detected to date, the most common are super-Earths or mini-Neptunes (Lissauer et al. 2011; Fabrycky et al. 2014; Winn & Fabrycky 2015). They have masses and sizes between that of the Earth and Neptune, and orbital periods below 300 days. About 30 % of Sun-like stars host super-Earths, with each system containing an average of 3 planets (Zhu et al. 2018). These planetary systems are generally “dynamically cold”, with eccentricities $e \sim 0.02$ and mutual inclinations $I \sim 2^\circ$ (e.g., Winn & Fabrycky 2015). However, there is an observed excess of single transiting planets, that could be sign of a dynamically hot sub-population of misaligned planets (so-called Kepler dichotomy, Lissauer et al. 2011; Read et al. 2017; He et al. 2019). There is also an indication that the mutual inclination dispersion in a given system increases rapidly with decreasing number of planets (Zhu et al. 2018). In addition, a fraction of super-Earth systems have orbital periods close to mean-motion resonances (see, e.g., Winn & Fabrycky 2015; Choksi & Chiang 2020).

In recent years, long-period giant planets (i.e. “Cold Jupiters” or CJs) have been observed in an increasing number of super-Earth systems. Statistical analyses combining radial velocity and transit observations confirmed a strong correlation between the occurrence of super-Earth systems and CJs: depending on the metalicities of their host stars, 30-60 % of inner super-Earth systems have CJ companions (Zhu & Wu 2018; Bryan et al. 2019). Massive companions may shape the architecture of super-Earth systems through various dynamical processes. It has been suggested that stars with CJs or with high metallicities have smaller multiplicity of inner Super-Earths (Zhu & Wu 2018). Moreover, Masuda et al. (2020) found that these CJ companions are typically misaligned with their inner systems (mutual inclination of $I_p \sim 12$ deg on average).

The dynamical perturbations from external companions can excite the eccentricities and mutual inclinations of super-Earths, thereby influencing the observability (co-transiting geometry) and stability of the inner system. This subject has been studied extensively in recent years, in a general framework (e.g.; Boué & Fabrycky 2014; Carrera et al. 2016; Lai & Pu 2017; Huang et al. 2017; Becker & Adams 2017; Read et al. 2017; Pu & Lai 2018; Denham et al. 2019; Pu & Lai 2020) or in connection to specific systems with misalignment and obliquity measurements or detected cold companions (e.g.; Jontof-Hutter et al. 2017; Gratia & Fabrycky 2017; Weiss et al. 2020; Xuan & Wyatt 2020; Damasso et al. 2020; Pan et al. 2020). Kepler-88 for example comprises two inner planets, misaligned by $3^\circ$, and an outer giant companion (Weiss et al. 2020).

In this paper, we will consider two inner super-Earths perturbed by a misaligned CJ. The theoretical framework has been studied previously in Boué & Fabrycky (2014) and Lai & Pu (2017) in particular. They found that the outer body

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induces a differential nodal precession in the inner system, which introduces a misalignment between initially coplanar super-Earths. Nevertheless, the mutual inclination between the inner planets can remain small if their mutual coupling is strong enough to temper the perturber’s influence.

However, these analyses focused on pure secular interactions, and are no longer valid when the inner planets have commensurable periods, i.e. when they are in mean-motion resonance (MMR). While the analysis of Kepler’s period ratio distribution does not evidence a strong excess of commensurability, the proximity to mean-motion resonance has been found in systems with external CJs, such as the Kepler-88 system mentioned above, where the two inner planets are close to the 2:1 MMR. It is also unclear to what extent an external companion can affect the inner MMRs of super-Earths.

In this paper, we carry out a theoretical study to determine how the mutual inclination of two initially coplanar inner planets in MMR is influenced by an inclined external perturber. As a related issue, we will determine the effect of the perturber on the resonance. We focus on inclination dynamics, and restrict the analysis to circular orbits. Eccentricity will be taken into account in a future paper. Order two MMRs are the strongest for which the resonance is coupled to the inclination. We will focus on this type of resonances in this paper.

Section 2 describes the setup and characteristic timescales (frequencies) of the problem, and the anticipated results. In Section 3, we consider the restricted 3-body problem, where a test mass is in MMR with an inner planet, perturbed by an external body. We present the equations of motion derived from the Hamiltonian, the approximate analytical solution and a comparison with the numerical results. We show that the MMR increases the coupling between the inner planets, and thus temper the "disruptive" influence of the external perturber. We also derive a condition for the resonance to hold (i.e. the resonant angle to librate) throughout the precession cycle. Section 4 presents the case where the test mass is inner to the planet. In this case, a secular resonant feature can trigger strong misalignments for "intermediate" couplings. In Section 5, we generalize these results to the problem of two finite-mass planets in MMR. We summarize our key findings and conclude in Section 6.

2 SET-UP AND TIMESCALES

Consider two planets (masses \(m_1\) and \(m_2\)) in circular orbits (semi-major axes \(a_1\) and \(a_2\)) around a central star (mass \(M\)). The orbital angular momenta are denoted by \(L_1 = L_1\hat{I}_1\) and \(L_2 = L_2\hat{I}_2\), where \(\hat{I}_1\) and \(\hat{I}_2\) are unit vectors. The two planets are initially close to coplanarity. An external perturber (mass \(m_p\)) moves in a circular inclined orbit, with semi-major axis \(a_p\), orbital angular momentum \(L_p = L_p\hat{I}_p\), and inclination \(I_p\) with respect to the inner planets’ initial orbital plane. Throughout the paper, we assume \(L_p \gg L_1, L_2\), so that \(L_p\) is constant in time. The question we are interested in is: How does the mutual inclination \(I\) of the two inner planets evolve?

Let us define the dimensionless masses and Keplerian orbital frequencies (mean-motions) of the planets:

\[
\mu_k = \frac{m_k}{M}, \quad \nu_k = \sqrt{\frac{GM}{a_k^3}},
\]

where \(k = 1, 2, p\) labels the different bodies.

The inner planets revolve around the central star following a nearly Keplerian motion, perturbed by both their fellow inner planet and the outer companion. The dominant inclination evolution is associated with the secular precession of the longitude of the node, i.e. the slow rotation of the planet’s orbital plane. Denote \(\omega_{ik}\) the characteristic precession frequency of planet \(i\) induced by the gravitational torque from planet \(k\). The frequencies of mutual interactions in the inner two planets are

\[
\begin{align*}
\omega_{12} &\equiv \frac{1}{4} n_2 \mu_2 \sqrt{\frac{a_1}{a_2}} b^{(1)}_1 \left(\frac{a_1}{a_2}\right), \\
\omega_{21} &\equiv \frac{\omega_{12} L_1}{L_2} = \frac{1}{4} n_1 \mu_1 \left(\frac{a_1}{a_2}\right)^2 b^{(1)}_1 \left(\frac{a_1}{a_2}\right), \\
\omega_{1p} &\equiv \frac{1}{4} n_p \mu_p \sqrt{\frac{a_1}{a_p}} b^{(1)}_1 \left(\frac{a_1}{a_p}\right), \\
\omega_{2p} &\equiv \frac{1}{4} n_p \mu_p \sqrt{\frac{a_2}{a_p}} b^{(1)}_1 \left(\frac{a_2}{a_p}\right),
\end{align*}
\]

(see Eq. 7.11 in Murray & Dermott 2000). The \(b^{(j)}_k\) functions are the Laplace coefficients:

\[
b^{(j)}_k(\alpha) = \frac{1}{\pi} \int_0^{2\pi} \frac{\cos(jx)dx}{(1 - 2\alpha \cos x + \alpha^2)^{\frac{3}{2}}}.
\]

Lai & Pu (2017) considered pure secular interactions between the planets and the perturber. They showed that the inclination dynamics of the inner planets depend on the dimensionless coupling parameter

\[
\varepsilon_{12} = \frac{\omega_{2p} - \omega_{1p}}{\omega_{21} + \omega_{12}},
\]

which measures the relative strength of the perturber with respect to the coupling in the inner system. For \(\varepsilon_{12} \ll 1\), the inner planets are strongly coupled and remain "rigidly" coplanar; for \(\varepsilon_{12} \gg 1\), the planets are weakly coupled, and each undergoes free nodal precession driven by the perturber.

In this paper, we consider the situation where the two planets are near the \(j : (j - 2)\) MMR, i.e. \((j - 2)n_1 \approx jn_2\). This MMR introduces two resonant coupling frequencies between the two planets, given by (see Eqs. 8.19 and 8.20 in Murray & Dermott 2000)

\[
\begin{align*}
\omega_{12,\text{res}} &\equiv \frac{1}{4} n_2 \mu_2 \sqrt{\frac{a_1}{a_2}} b^{(j-1)}_1 \left(\frac{a_1}{a_2}\right), \\
\omega_{21,\text{res}} &\equiv \omega_{12,\text{res}} L_1 = \frac{1}{4} n_1 \mu_1 \left(\frac{a_1}{a_2}\right)^{\frac{5}{2}} b^{(j-1)}_1 \left(\frac{a_1}{a_2}\right).
\end{align*}
\]

Note that these resonant coupling frequencies have a similar order of magnitude as the secular frequencies:

\[
\begin{align*}
\omega_{12,\text{res}} &= \frac{\omega_{21,\text{res}}}{\omega_{21}} = \frac{b^{(j-1)}_1 \left(\frac{a_1}{a_2}\right)^{\frac{5}{2}}}{b^{(1)}_1 \left(\frac{a_1}{a_2}\right)^{\frac{5}{2}}} \approx \frac{5}{4} \left(\frac{a_1}{a_2}\right)^{j-2}.
\end{align*}
\]

In this new study, we will show that in the presence of
The difference from the non-resonant case (Eq. 7) is the resonant-induced precession terms. Here we present the case where planet 2 has negligible mass. In order to have a theoretical understanding of the problem, we will first simplify it by supposing that one of the planets is the test mass planet, and the other is the outer perturber. For $a_p \gg a_2$, $e_{12,\text{res}}$ is proportional to $\mu_p/a_p^3$. For the 1 : 3 MMR, it is approximately given by:

$$e_{12,\text{res}} \approx 0.1 \left( \frac{\mu_p}{100 \mu} \right) \left( \frac{10 a_2}{a_p} \right)^3,$$

where $\mu$ is the largest of $\mu_1$ and $\mu_2$.

### 3 TEST MASS-PLANET RESONANCE PERTURBED BY AN EXTERNAL BODY: OUTER TEST MASS

In order to have a theoretical understanding of the problem, we will first simplify it by supposing that one of the planets has negligible mass. Here we present the case where planet 2 is a test mass ($m_2 \ll m_1, m_p$). For convenience, we adopt the notations that the primed quantities represent the finite-mass planet and the unprimed quantities the test mass: $a_1 = a' < a_2 = a < a_p$.

Let us study the problem in the orbital plane of planet 1, which is rotating uniformly due to the influence of the outer perturber. We position the frame such that the ascending node of the outer perturber corresponds to the origin of longitude ($\Omega_p = 0$, see Fig. 1).

#### 3.1 Hamiltonian and evolution equations

In the purely secular problem, the inclination dynamics is governed by the evolution of the angular momentum axis of each planet, so that a vector approach is convenient (Lai & Pu 2017). However, near MMR, the system has an additional degree of freedom, the resonant angle, that is not accounted for in the vector approach. In the following, we use Hamiltonian mechanics, with the expression of the Hamiltonian valid for $I \ll 1$.

Let us write down the Hamiltonian per unit mass of the test mass in the orbital plane of the inner finite-mass planet, using the canonical variables

$$\lambda, \Lambda = \sqrt{GM}a,$$

$$-\Omega, Z = \sqrt{GM}a(1 - \cos I) \simeq \sqrt{GM}aI^2/2,$$

where $\lambda$ is the mean longitude and $\Omega$ the longitude of the ascending node. The Hamiltonian has several pieces. First, at order 2 in $I$, the interactions between the inner planets and with the central star give (Eq. 8.20 in Murray & Dermott 2000):

$$H_{21} = -\frac{GM}{2a} - \frac{1}{2}\omega_{\text{res}}I^2 \cos \phi + \frac{1}{2} \omega_{21}I^2,$$

where the second and third terms represent respectively the resonant ($\omega_{\text{res}} \equiv \omega_{21,\text{res}}$) and secular interactions on the test mass ($m_2$) from the planet ($m_1$). The resonant angle $\phi$ is defined as

$$\phi = j\lambda_2 - (j - 2)\lambda_1 - 2\Omega.$$

Second, in the presence of the outer perturber $m_p$, the secular interaction on $m$ from $m_p$ gives

$$H_{2p} = \frac{1}{2} \omega_{2p}\Lambda \sin^2 \theta_p,$$

where $\theta_p$ is the angle between the orbital planes of $m$ and $m_p$, which is given by

$$\cos \theta_p = \cos I \cos I_p + \cos \Omega \sin I \sin I_p.$$

Finally, the frame that we are in (Fig. 1), where $L_1$ and $L_p$ are constant, rotates at the frequency $-\omega_{1p}\cos I_p$ around $I_p$, so that a term must be added to the Hamiltonian (Tremaine & Yavetz 2014):

$$H_{1p} = (\omega_{1p}\cos I_p) L_1 - \frac{L_2}{m_2} = \omega_1\Lambda \cos I_p \sin \theta_p.$$

Note that an additional secular interaction term $H_{\text{sec},0}(a)$ that does not depend on any angle should be theoretically accounted for. However, it plays little role in the dynamics, only shifting the resonant location by a negligible amount. We neglect it in our analysis.

To proceed, it is convenient to make a canonical transformation to a new set of variables, replacing the fast varying $\lambda$ for the more slowly varying resonant angle $\phi$. The new angle-action pairs are

$$\theta_1 = \phi, \quad J_1 = \Lambda/j,$$

$$\theta_2 = \Omega, \quad J_2 = \Lambda(2j - I^2/2),$$

obtained via the generating function

$$S = [j\lambda - (j - 2)\lambda' - 2\Omega] J_1 + \Omega J_2.$$  

This adds a new term to the Hamiltonian, to take into account the time dependence of $\lambda' = n'I$:

$$H_S = \frac{\partial S}{\partial t} = -\frac{j - 2}{j} n' \Lambda.$$

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Figure 1. Coordinate system used in Section 3. The z-axis is along $\hat{l}_1$, the orbital angular momentum unit vector of $m_1$ (the finite-mass planet), and $\hat{l}_2$ and $I$ specify the orbital angular momentum axes of $m_p$ (the external perturber) and $m$ (the test mass). This frame is not inertial: it is rotating uniformly due to the influence of $m_p$. 

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**MNRAS 000, 1–16 (2020)**
Thus, the Hamiltonian (in the rotating frame) for the test mass due to the interaction with \( M, m' = m_1 \), and \( m_p \) is
\[
H = H_{21} + H_{2p} + H_{\text{rot}} + H_S. \tag{24}
\]

The Hamilton-Jacobi equations to the first orders in \( I, \mu' \) and \( \mu_p \) give
\[
\frac{d\phi}{dt} = \frac{\partial H}{\partial J_1} \bigg|_{J_2} = j \frac{\partial H}{\partial \lambda} + j \frac{J_2}{\Lambda^2} \frac{\partial H}{\partial I}, \tag{25}
\]
\[
\frac{d\Omega}{dt} = \frac{\partial H}{\partial J_1} \bigg|_{J_2} = -\frac{1}{\Lambda^2} \frac{\partial H}{\partial I}, \tag{26}
\]
\[
\frac{dJ_1}{dt} = -\frac{\partial H}{\partial \phi} \bigg|_{\Omega} = -\frac{1}{2} \omega_{\text{res}} \Lambda I^2 \sin \phi, \tag{27}
\]
\[
\frac{dJ_2}{dt} = -\frac{\partial H}{\partial \Omega} \bigg|_{\phi} = \Lambda I \sin I_p \sin \Omega (\omega_{2p} \cos \theta_p - \omega_{1p} \cos I_p). \tag{28}
\]

On the other hand, we can retrieve the evolution of \( a \) and \( I \) from the evolution of \( J_1 \) and \( J_2 \):
\[
\frac{da}{dt} = \frac{2J_1}{GM} \frac{dJ_1}{dt}, \tag{29}
\]
\[
\frac{dI}{dt} = -\frac{1}{\Lambda} \frac{dJ_2}{dt}. \tag{30}
\]

Using Eqs. (25-30), to the first orders in \( I \), we obtain a set of differential equations:
\[
\frac{da}{dt} = -j \omega_{\text{res}} a I^2 \sin \phi, \tag{31}
\]
\[
\frac{dI}{dt} = -\omega_{\text{res}} I \sin \phi + \frac{\Delta \omega_p}{2} \sin(2I_p) \sin \Omega, \tag{32}
\]
\[
\frac{d\phi}{dt} = -\Delta n - 2\omega_{\text{res}} \cos \phi - \frac{\Delta \omega_p}{T} \sin(2I_p) \cos \Omega, \tag{33}
\]
\[
\frac{d\Omega}{dt} = \omega_{\text{res}} \cos \phi - \omega_{21} - \Delta \omega_p \cos^2 I_p + \frac{\Delta \omega_p}{2T} \sin(2I_p) \cos \Omega, \tag{34}
\]

where \( \Delta \omega_p = \omega_{2p} - \omega_{1p} \) is the relative precession frequency induced by the outer perturber, and \( \Delta n \) characterizes the distance from the nominal resonant location:
\[
\Delta n \equiv (j - 2)n_1 - jn_2. \tag{35}
\]

In the absence of external perturber, \( \Delta \omega_p = 0 \), and the Hamiltonian does not depend on \( \Omega \). Thus, the conjugated momentum \( J_2 \) is constant, which is equivalent at first orders to the conservation of the quantity
\[
(a_1/a_2)(1 + j I^2/2) \equiv \alpha_0. \tag{36}
\]

In that case, Eqs. (31) and (32) are not independent. As the evolution of \( \Omega \) does not impact the overall dynamics, we are then left with a "classic" one-degree of freedom problem. However, in the presence of the perturber, \( \alpha_0 \) depends strongly on \( \Omega \), and all equations (31-34) must be taken into account.

The full system of differential equations is not solvable analytically without further approximation: the precession and resonance are strongly entangled. Of course, the equations can be integrated numerically.

### 3.2 Decoupling resonance and precession

Our interest here is to determine the amplitude and frequency of the inclination variations, due to both the resonance and the precession. Equations (31-34) can be solved numerically.
Now, let us study the decoupled problem. We assume that the resonance is at the equilibrium \( \phi = \pi \) (we will give conditions on the validity of this hypothesis below). The evolution of the system on the secular timescale is then governed by:

\[
\frac{\mathrm{d} I}{\mathrm{d} t} = \frac{\Delta \omega_p}{2} \sin(2I_p) \sin \Omega,
\]
\[
\frac{\mathrm{d} \Omega}{\mathrm{d} t} = -\omega_{\text{prec}} + \frac{\Delta \omega_p}{2I} \sin(2I_p) \cos \Omega.
\]

This problem is identical to the well-known forced eccentricity problem. The equations can be rewritten in the complex form:

\[
\frac{d}{dt} \left( I e^{i\Omega} \right) = -i\omega_{\text{prec}} I e^{i\Omega} + i \frac{1}{2} \Delta \omega_p \sin(2I_p).
\]

As \( \Delta \omega_p \sin(2I_p) \) and \( \omega_{\text{prec}} \) are constant, this differential equation is solvable, giving:

\[
I e^{i\Omega} = I_{\text{free}} e^{i(-\omega_{\text{prec}} t + \psi)} + I_{\text{forced}},
\]

where

\[
I_{\text{forced}} = \frac{1}{2} \frac{\Delta \omega_p}{\omega_{\text{prec}}} \sin(2I_p),
\]

\[
= \frac{1}{2} \varepsilon_{12, \text{res}} \sin(2I_p) \text{ if } \varepsilon_{12} \ll 1.
\]

is the forced inclination, with \( \varepsilon_{12, \text{res}} \) given by Eq. (11). The constant \( I_{\text{free}} \) ("free inclination") and the phase \( \psi \) are determined by the initial conditions.

The secular behavior is thus a libration or circulation of the longitude of the node coupled to an oscillation of the inclination around the forced inclination. When the longitude of the node circulates, i.e. when \( I_{\text{free}} > I_{\text{forced}} \), the maximum inclination amplitude is independent of the initial conditions and is given by

\[
I_{\text{max}} - I_{\text{min}} = 2I_{\text{forced}}.
\]

Figure 3 represents the Hamiltonian map of the problem for a given \( \varepsilon_{12} \). The analytical solution (Eq.42) is also shown for several different initial conditions. Figure 2 depicts the agreement between the exact numerical solution and the analytical solution for the decoupled problem.

Now let us consider the validity of the "decoupling" assumption, that the inner planets remain in resonance throughout the precession cycles. From Eq. (31), it is clear that the only possible equilibrium of the resonance is for \( \sin \phi = 0 \), similar to the non-perturbed MMR. In the latter as in our case, the equilibrium at \( \phi = 0 \) is unstable, which is revealed by a stability analysis around that point. Let us do a similar analysis for the \( \phi = \pi \) case, by studying the motion close to the precession solution, which we denote by \( I_{\text{prec}} \) and \( \Omega_{\text{prec}} \). Their expressions are given by Eq. (42) and satisfy Eqs. (39-40). Moreover, we suppose that the longitude of the node circulates with a precession frequency roughly equal to \( \omega_{\text{prec}} \). We slightly perturb the cycle as follows:

\[
a = a' \left( \frac{j - 2}{j} - \frac{2\omega_{\text{res}}}{3n'} \right)^{-\frac{3}{2}} + \delta a,
\]

\[
I = I_{\text{prec}} + \delta I,
\]

\[
\phi = \pi + \delta \phi,
\]

\[
\Omega = \Omega_{\text{prec}} + \delta \Omega.
\]

After some algebra, the differential equations Eqs. (31-34) reduce to:

\[
\frac{1}{\omega_{\text{res}}} \frac{d^2 \delta \phi}{dt^2} = \left( \frac{3}{2} \pi' j(j - 2) I_{\text{prec}}^2 \right) \omega_{\text{prec}} \frac{\Delta \omega_p}{I_{\text{prec}}} \sin(2I_p) \cos \Omega_{\text{prec}} \delta \phi
\]

\[
- \omega_{\text{res}} \frac{\Delta \omega_p}{I_{\text{prec}}} \sin(2I_p) \Omega_{\text{prec}}.
\]
\[ \sin \Omega \text{ is negligible. This gives the condition:} \]
\[ \frac{\omega_{\text{prec}}}{\omega_{\text{res}}} \sin(2I_p) \frac{\Delta \omega_p}{n'} < \frac{3}{2} j(j - 2) \frac{r_{\text{prec}}}{r_{\text{res}}} \quad \text{(stability condition).} \]
\[ \quad (51) \]

In Figure 4 we show an example where Eq. (51) is not fulfilled and the resonant angle is not librating. Thus, the secular solution Eq. (42) is no longer valid. The resonant angle circulates faster than the precession, so that it can be averaged in Eqs. (31-34), and we get back to the non-resonant problem. The inclination angle and longitude of the node then approximately follow the pure secular behavior described in Lai & Pu (2017).

If the condition (51) is not satisfied, then the stability is not guaranteed, but can nevertheless occur, depending on the entanglement between \( \phi \) and \( \Omega \). We will examine more thoroughly this possibility in the next subsection, in the particular case where the problem is close to its fixed point.

### 3.3 Equilibrium around the fixed point

In the previous subsection, we have examined the case where we could effectively decouple the precession from the libration of the resonant angle (\( \Omega \approx \omega_{\text{prec}} \)). We found that this assumption is valid for inclinations large enough to resist the perturbation from the outer perturber (Eq. 51). Here, we examine the full problem without such assumptions, but close to the only stable fixed point. It is given by:

\[ a_{\text{eq}} = a' \left( \frac{j - 2}{j} + \frac{2}{j} \left( \omega_{\text{prec}} - \omega_{\text{res}} \right) \right)^{-\frac{3}{2}}, \]
\[ I_{\text{eq}} = I_{\text{forced}}, \]
\[ \phi_{\text{eq}} = \pi, \]
\[ \Omega_{\text{eq}} = 0. \]

A stability analysis can be performed rigorously by deriving the eigenvalues of the 4 × 4 matrix corresponding to the evolution equations linearized near the fixed point (see Appendix A). It reveals that the onset of instability occurs when
\[ \varepsilon_{12, \text{res}} \sin(2I_p) \lesssim \sqrt{\mu'}. \]

This stability condition is roughly equivalent to Eq. (51) in the strong coupling regime, where \( I_{\text{prec}} \) is set to \( I_{\text{forced}} \) and \( \omega_{\text{prec}} \sim \omega_{\text{res}} \sim \mu'n' \). Fig. 6 shows the parameter space (\( \phi, I \)), describing the MMR at fixed \( \Omega \) around the stability threshold.

The stability analysis also reveals that the stability can be restored for even smaller \( \varepsilon_{12, \text{res}} \sin(2I_p) \). In that case, the precession frequency can be comparable to the resonant frequency, but the strong coupling between \( \phi \) and \( \Omega \) enables the libration to survive. An example of such evolution is shown in Fig. 5. Nevertheless, this island of stability below the stability threshold Eq. (56) exists only close to the forced inclination, and is thus easily destroyed.

### 3.4 Results for mutual inclination

We now compute the expected mutual inclination for the inner planets as a function of the coupling parameter \( \varepsilon_{12} \). This can be done by doing numerical calculations starting from appropriate initial conditions (see below) at different \( \varepsilon_{12} \) (Fig. 7), or by varying adiabatically \( \varepsilon_{12} \) (Fig. 8). We integrate Eqs. (31-34) with the odeint function of the Python module SciPy.

For the calculations depicted in Fig. 7, the integrations start at the theoretical equilibrium given by Eqs. (52-55), and are then carried on for several precession periods. We estimate the forced inclination by averaging the mutual inclination over the entire integration time. We confirm the theoretical relation (see Eq. 43) between the forced inclination and the parameter \( \varepsilon_{12, \text{res}} \). In the strong coupling regime, the inclination is proportional to \( \varepsilon_{12, \text{res}} \), whereas in the weak coupling regime, the inclination reaches \( I_p \), the perturber’s misalignment. The strong coupling regime of the purely secular case is also represented in Fig. 7, where the inclination is proportional to \( \varepsilon_{12} \). The difference between the resonant and non-resonant cases is
\[ \frac{\varepsilon_{12, \text{res}}}{\varepsilon_{12}} = \frac{\omega_{21}}{\omega_{21, \text{res}} + \omega_{21}}. \]

For the 1 : 3 MMR, this factor is approximately equal to 1/2. Thus, the coupling between the inner planets in the resonant
The external perturber is inclined by $a_1 = a_2 = a' < a_p$. We consider the problem in the orbital plane of the planet, so that Fig. 1 still applies, except $\tilde{z} = \tilde{l}_2$ and $\tilde{l} = \tilde{l}_1$.

4.1 Hamiltonian and evolution equations

As in Section 3, the Hamiltonian per unit mass is a sum of several terms. To the order 2 in $I$, the interaction with the other planet and the central star is

$$H_{12} = -\frac{GM}{2a} - \frac{1}{2} \omega_{res} I_p^2 \cos \phi + \frac{1}{2} \omega_{12} I^2.$$  \hspace{1cm} (58)

where $\omega_{res}$ now denotes $\omega_{12, res}$. The perturbation from the outer body writes

$$H_{1p} = \frac{1}{2} \omega_{1p} \Lambda \sin^2 \theta_p,$$  \hspace{1cm} (59)

where $\theta_p$ is the relative angle between the orbital planes of $m$ and $m_p$ (Eq. 18). The term associated to the rotating frame (the orbital plane of planet 2) is:

$$H_{res} = (\omega_{2p} \cos I_p) I_p \frac{\Lambda_1}{m} = \omega_{2p} \Lambda \cos I_p \cos \theta_p.$$  \hspace{1cm} (60)

Finally, when making a canonical transformation from $\lambda$, $-\Omega$, to $\phi$, $\Omega$, an additional term must be added (see Section 3):

$$H_S = \frac{j}{j - 2} n' \Lambda.$$  \hspace{1cm} (61)

Rewriting the Hamilton-Jacobi equations to the first orders in $I$, $\mu'$ and $\mu_p$ (see Section 3), we obtain:

$$\frac{d\phi}{dt} = (j - 2) \omega_{res} a I^2 \sin \phi,$$  \hspace{1cm} (62)

$$\frac{dI}{dt} = -\omega_{res} I \sin \phi - \frac{\Delta \omega_p}{2} \sin(2I_p) \sin \Omega,$$  \hspace{1cm} (63)

$$\frac{d\Omega}{dt} = \omega_{res} \cos \phi - \omega_{12} + \Delta \omega_p \cos^2 I_p = \frac{\Delta \omega_p}{2} \sin(2I_p) \cos \Omega,$$  \hspace{1cm} (64)

where $\Delta n$ characterizes the distance from the nominal location of the MMR (Eq. 35). This system of differential equations is similar to the outer test mass case, only the coefficient in front of the semi-major axis and the signs in front of the $\Delta \omega_p$ terms are different.

4.2 Forced inclination and MMR stability

From Equations (62-65), we define the "total" precession frequency $\omega_{\text{prec}}$ of the test mass in the rotating frame as:

$$\omega_{\text{prec}} \equiv |\omega_{\text{res}} + \omega_{12} - (\omega_{2p} - \omega_{1p}) \cos^2 I_p|.$$  \hspace{1cm} (66)

Following the approach of Section 3.2, we can solve Eqs. (62-65) exactly by assuming that the resonant angle $\phi$ stays close to $\pi$. The forced inclination retains the same expression as Eq. 43.
is depicted in Fig. 9. The stability condition to always be fulfilled, which is not the case for the right plot. The corresponding temporal evolution of the latter exponentially with a characteristic time \( \tau_{\text{res}} = 10 \) (Eq. 43), the dark dashed line represents the forced inclination limit without resonance. The blue shade represents the envelope if the system is in resonance (the dark blue corresponds to \( |I - I_0| < \Delta I_{\text{forced}} \), and the light blue corresponds to \( |I - I_0| < 2\Delta I_{\text{forced}} \)). The red zone corresponds to the theoretical instability of the resonance (Eq. 51). In the left plot, \( I_0 \) is high enough for the stability condition to always be fulfilled, which is not the case for the right plot. The corresponding temporal evolution of the latter is depicted in Fig. 9.

Figure 9. Time evolution of the longitude of the node \( \Omega \) and the resonant angle \( \phi \), corresponding to the right panel of Fig. 8, where \( \varepsilon_{12} \) increases exponentially. In the beginning, the inner planet pair is in resonance (the perturber has a negligible impact). Then, the perturber has enough influence to break the resonance (red zones, Eq. 51 is not satisfied). When the inclination gets forced to higher values as \( \varepsilon_{12} \) increases (Eq. 43), the stability is restored. Finally, the inclination stabilizes around \( I_p \) while the perturber keeps growing. Again, the stability condition Eq. (51) does not hold and the resonant angle circulates.

A comparison between the precession frequencies for the outer test mass case (Eq. 33, \( \omega_{\text{prec},2} \)) and inner test mass case (Eq. 66, \( \omega_{\text{prec},1} \)) is represented in Figure 10. As it appears clearly in both the plot and Eq. (66), \( \omega_{\text{prec}} \) can be null in the inner test mass case. This is an indication of a secular inclination resonance. In this case, the forced inclination normally diverges (see Eq. 43). Since our analytic theory breaks down for large inclination, we will not dwell on the study of the cases close to \( \omega_{\text{prec}} = 0 \).

The stability of the MMR, and thus the validity of the assumption \( \phi \approx \pi \), leads to the same constraints in this case as in the outer test mass case. By assuming \( \tilde{\Omega} \approx \omega_{\text{prec}} \) and performing a stability analysis around the precessing solutions, we recover the stability condition given by Eq. (51). On the other hand, a stability analysis can be performed as in Sec-

Figure 10. True precession frequency of the test mass as a function of the precession frequency induced by the fellow planet, for a perturber misalignment \( I_p = 10^\circ \) and a pair of inner planets in 1:3 MMR (\( \omega_{\text{prec},2} \) corresponds to the outer test mass case, \( \omega_{\text{prec},1} \) to the inner test mass case). For the inner test mass case (blue), the true precession frequency cancels as \( \omega_{12} + \omega_{\text{res}} = \Delta \cos^2 I_p \), leading to a significant increase in the relative inclination \( I \).
1:3 MMR, \( \ell_p = 10^{-1}, \quad \mu' = 10^{-05} \)

10 1
Iforced (deg)

\[ \text{Coupling parameter} \]

\[ 12 \]

\[ \begin{align*}
1 & \quad 10 \quad 1 \quad 100 \quad 101 \quad 102
\end{align*} \]

\[ \text{Analytic} \quad \text{Numerical} \]

\[ \text{Unstable} \]

\[ \text{Secular resonance} \]

\[ \text{Analytic} \quad \text{Numerical} \]

Figure 11. Forced mutual inclination as a function of \( \varepsilon_{12} \), for configurations identical to Fig. 7 except that \( m_1 \) is now the test mass. Each purple point represents an integration with initial conditions close to equilibrium (Eqs. 67-70), and the blue line represents the theoretical forced inclination (Eq. 43). The red line corresponds to the theoretical instability derived in the Appendix (Eq. A10) and described in Sec. 3.3 and 4.2. The dark dashed line represents the forced inclination limit assuming the divergence at resonance (dark dashed line represents the forced inclination limit assuming the divergence at resonance (Eq. A10) and described in Sec. 3.3 and 4.2. The dark dashed line represents the forced inclination limit assuming resonance (\( \phi \approx \pi \), Eq. 43), and the light gray one represents the forced inclination limit in the purely secular case. The equations are not valid for large inclinations close to the secular resonance (\( \varepsilon_{12} \sim 1 \)).

4.3 Results for mutual inclination

Again, there are different ways to compute the expected mutual inclination for this problem as a function of the coupling parameter \( \varepsilon_{12} \). Varying adiabatically \( \varepsilon_{12} \) is compromised here because of the divergence at \( \omega_{0 \text{res}} = 0 \), corresponding to \( \varepsilon_{12} = (1 + \omega_{0 \text{res}}/\omega_{12})/\cos^2(\ell_p) \approx 1 \). We instead integrate the system of equations Eqs. (62-65) for different \( \varepsilon_{12} \), close to the theoretical equilibrium Eqs. (67-70), for several precession periods, and average the inclination to derive the forced inclination. The results are shown in Fig. 11. We confirm that the forced inclination is proportional to \( \varepsilon_{12, \text{res}} \) for strong couplings as given in Eq. (43), when the stability criterion Eq. (A10) is fulfilled.

5 PLANET-PLANET RESONANCE PERTURBED BY AN EXTERNAL BODY

When both inner planets have finite masses, the inclination dynamics become more complex. In the plane of reference (to be specified below), let \( I_1 \) and \( I_2 \) be the inclinations of the two planets, and \( \Omega_1 \) and \( \Omega_2 \) the longitudes of the node. We define the orientation of the reference plane so that the external perturber has \( \Omega_p = 0 \). The mutual inclination between the two inner planets is given by:

\[ \cos I = \cos I_1 \cos I_2 + \cos(\Omega_2 - \Omega_1) \sin I_1 \sin I_2. \]  

5.1 Strong coupling limit

In the strong coupling limit (\( \varepsilon_{12} \ll 1 \)), the inner planets remain nearly coplanar with small mutual inclination. Let \( L = L_1 + L_2 \equiv \hat{L} \) be the total angular momentum of the two inner planets, with \( L \approx L_1 + L_2 \). The total torque on the inner planets from the external perturber is

\[ \omega_{1p} \cos \theta_{1p} L_1 \times \hat{I}_p + \omega_{2p} \cos \theta_{2p} L_2 \times \hat{I}_p. \]  

With \( \hat{I}_1 \) nearly parallel to \( \hat{I}_2 \), we find

\[ \frac{dL}{dt} \approx \omega_p (\hat{I}_1 \cdot \hat{I}_p) L \times \hat{I}_p, \]

where

\[ \omega_p = \frac{\omega_{1p} L_1 + \omega_{2p} L_2}{L}. \]

Thus, \( \hat{I} \) rotates around \( \hat{I}_p \) at the rate \( -\omega_p \hat{I}_1 \hat{I}_p \). This rate is approximately constant for \( \varepsilon_{12} \ll 1 \), or \( \mu_1 \) or \( \mu_2 \ll 1 \) (Appendix B). To examine the mutual inclination between the two inner planets, it is convenient to work in this rotating frame in the strong coupling regime, where we have the following relations:

\[ I = I_1 + I_2, \]

\[ L_1 \sin I_1 = L_2 \sin I_2, \]

\[ \Omega_1 = \Omega_2 + \pi. \]

Thus, a small mutual inclination \( I \) implies that both \( I_1 \) and \( I_2 \) are small.

We now write down the Hamiltonian for the two inner planets. First, to the order 2 in \( I \), the secular interactions between the inner planets and with the central star give (Murray & Dermott 2000):

\[ H_{12/21} = -\frac{GM m_1}{2a_1} - \frac{GM m_2}{2a_2} + \frac{1}{2} \omega_{12} L_1 I_1^2, \]

(note that \( \omega_{12} L_1 = \omega_{21} L_2 \)). Second, the resonant interaction is a combination of three terms, which correspond to two resonant angles as well as a combination of the two:

\[ \phi_1 = j \lambda_2 - (j - 2) \lambda_1 - 2 \Omega_1, \]

\[ \phi_2 = j \lambda_2 - (j - 2) \lambda_1 - 2 \Omega_2, \]

\[ H_{\text{res}} = \frac{1}{2} \omega_{12, \text{res}} L_1 I_1^2 \cos \phi_1 - \frac{1}{2} \omega_{21, \text{res}} L_2 I_2^2 \cos \phi_2 \]

\[ + \omega_{12, \text{res}} L_1 I_1 I_2 \cos \left( \frac{\phi_1 + \phi_2}{2} \right). \]

The secular interaction on \( m_1 \) and \( m_2 \) from \( m_p \) gives:

\[ H_{1p} = \frac{1}{2} \omega_{1p} L_1 \sin^2 \theta_{1p}, \]

\[ H_{2p} = \frac{1}{2} \omega_{2p} L_2 \sin^2 \theta_{2p}. \]
with
\[
cos \theta_1 = \cos I_1 \cos I_p + \cos \Omega_1 \sin I_1 \sin I_p, \tag{84}
\]
\[
\cos \theta_2 = \cos I_2 \cos I_p + \cos \Omega_2 \sin I_2 \sin I_p. \tag{85}
\]

Finally, due to the rotation of our reference plane, we add a "rotational" term to the Hamiltonian (Tremaine & Yavetz 2014):
\[
H_{\text{rot}} = (\omega_p \cos I_p)(L_1 + L_2)
= \omega_p \cos I_p(\cos \theta_1 L_1 + \cos \theta_2 L_2). \tag{86}
\]

The Hamiltonian of the two inner planets in the rotating frame is thus:
\[
H = H_{12/21} + H_{\text{res}} + H_{1p} + H_{2p} + H_{\text{rot}}. \tag{87}
\]

It has four angle-action pairs:
\[
\lambda_1, L_1 = m_1 \sqrt{GMa_1}, \tag{88}
\]
\[
\lambda_2, L_2 = m_2 \sqrt{GMa_2}, \tag{89}
\]
\[-\Omega_1, Z_1 \approx L_1 I_1^2/2, \tag{90}\]
\[-\Omega_2, Z_2 \approx L_2 I_2^2/2. \tag{91}\]

We can then derive the eight Hamilton-Jacobi equations (Appendix C). We notice that the Hamiltonian only depends on \(\lambda_1 \) and \(\lambda_2 \) through \(j \lambda_2 - (j - 2) \lambda_1 \). The associated conserved quantity is
\[
J = jL_1 + (j - 2)L_2, \tag{92}
\]
which implies that \(a_1 \) and \(a_2 \) are not independent. We will thus replace their evolution equations by the evolution of \(\alpha = a_1/a_2 \). The problem is now described by six differential equations.

Finally, since in our rotating frame \(I_1 \) is related to \(I_2 \), \(\Omega_1 \) is related to \(\Omega_2 \), and \(\phi_1 = \phi_2 \equiv \phi \), we have (Appendix C)
\[
\frac{d\alpha}{dt} = ((j - 2)\omega_{12,\text{res}} + j\omega_{21,\text{res}}) \alpha I^2 \sin \phi, \tag{93}
\]
\[
\frac{dI}{dt} = -(\omega_{12,\text{res}} + \omega_{21,\text{res}}) I \sin \phi - \frac{\Delta \omega_p}{2} \sin(2I_p) \sin \Omega_1, \tag{94}
\]
\[
\frac{d\phi}{dt} = -\Delta n - 2(\omega_{12,\text{res}} + \omega_{21,\text{res}}) \cos \phi + \frac{\Delta \omega_p}{T} \cos \Omega_1 \sin(2I_p), \tag{95}
\]
\[
\frac{d\Omega_1}{dt} = -(\omega_2 + \omega_1) + (\omega_{12,\text{res}} + \omega_{21,\text{res}}) \cos \phi - \frac{\Delta \omega_p}{2T} \cos \Omega_1 \sin(2I_p), \tag{96}
\]

where we still have \(\Delta \omega_p = \omega_p - \omega_1 \) and \(\Delta n = (j - 2)n_1 - jn_2 \).

These equations are essentially similar to the corresponding equations in the test mass cases studied in Sections 3 and 4 (Eqs. 31-34 and 62-65). The only difference is that the precession frequencies between the inner planets, \(\omega_{12} \) and \(\omega_{21} \), \(\omega_{12,\text{res}} \) and \(\omega_{21,\text{res}} \) are added up, strengthening their coupling. Thus, in the strong coupling limit, the forced mutual inclination and the maximum inclination amplitude are
\[
I_{\text{forced}} = \frac{1}{2} \varepsilon_{12,\text{res}} \sin(2I_p), \tag{97}
\]
\[
I_{\max} - I_{\min} = \varepsilon_{12,\text{res}} \sin(2I_p), \tag{98}
\]

where \(\varepsilon_{12,\text{res}} \) is given by Eq. (11). Equations (97-98) are valid for arbitrary \(I_p \), as long as we are in the strong coupling regime \(\varepsilon_{12} \ll 1 \).

### 5.2 Weak coupling limit

In the weak coupling limit, corresponding to \(\varepsilon_{12} \gg 1 \), the vectors \(\mathbf{I}_1 \) and \(\mathbf{I}_2 \) precess around \(\mathbf{I}_p \) independently, with constant \(\cos \theta_1 \) and \(\cos \theta_2 \). We can thus expect to have
\[
I_{\max} - I_{\min} = 2I_p. \tag{99}
\]

This is valid for arbitrary \(I_p \).

### 5.3 General coupling, small \(I_p \)

In the general case, the total angular momentum of the inner planets \(\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 \) has a complex motion because its misalignment \(I_p \) with the outer perturber varies in time (see Appendix B). In order to write down the Hamiltonian of the MMR, we have to assume that \(I_1 \) and \(I_2 \) are small. We therefore study the problem in the initial orbital plane of the inner planets, assuming they are initially coplanar or nearly coplanar – note that strict coplanarity might compromise the stability of the MMR. To ensure \(I_1, I_2 \) remain small during the evolution, we assume that the perturber inclination is not too large (\(I_p \ll 1 \)).

The Hamiltonian is then the same as Section 5.3 (Eqs. 78, 81-83), without the "rotational" term:
\[
H = H_{12/21} + H_{\text{res}} + H_{1p} + H_{2p}. \tag{100}
\]

We derive the eight associated Hamilton-Jacobi equations in Appendix C, and the reduction to six differential equations. Those equations cannot be solved analytically, but can be integrated numerically.

Figure 12 shows the maximum amplitude of the mutual inclination between the inner planets as a function of the coupling parameter \(\varepsilon_{12} \), for different mass ratios \(m_1/m_2 \), obtained numerically. The integrations start in a nearly coplanar configuration for the pair of inner planets in the 1:3 MMR. Note that to guarantee the stability of the MMR (see Eq. 51), the initial inclination \(I_{\min} \) should not be too small; we find that \(I_{\min} = 1^\circ \) ensures this for the range of coupling parameters we have tested. The integrations are then carried on for several precession periods, and the maximum mutual inclination amplitude \(I_{\max} - I_{\min} \) is then computed. In the strong coupling regime \((\varepsilon_{12} \ll 1)\), we recover our analytical results (Eq. 98) that the maximum mutual inclination is proportional to \(\varepsilon_{12,\text{res}} \); in the weak coupling regime, the maximum amplitude reaches \(2I_p \). For intermediate values of \(\varepsilon_{12} \), the behavior is very similar to the test mass cases, with a smooth change between regimes for \(m_2 < m_1 \) and a secular resonance feature for \(m_1 < m_2 \), where the mutual inclination \(I \) reaches very high values around \(\varepsilon_{12} \sim 1 \). Once again, all other things being equal, we observe that it is harder to increase the mutual inclination in a resonant configuration compared to the non-resonant case for strong coupling \(\varepsilon_{12} < 1 \).

### 6 SUMMARY AND DISCUSSION

#### 6.1 Key results

We have calculated the excitation of mutual inclination in an inner two-planet system due to an inclined external companion. Such a configuration resembles the observed super-Earth systems surrounded by cold Jupiters. Our results, summarized in Fig. 12, generalize that of previous studies
Hiding Planets behind a Big Friend: Resonant Planets with an Inclined External Companion

6.2 Caveats

Our results apply for arbitrary $I_p$ and coupling $\epsilon_{12,\text{res}}$, provided that the mutual inclination between the inner planets is small. This is due to the approximate Hamiltonian used to model mean-motion resonance. Consequently, our results do not hold if both $I_p$ and $\epsilon_{12,\text{res}}$ are large, because in this case large misalignments can be generated. In addition, the amplitude of the mutual inclination near the secular resonance, which occurs at $\epsilon_{12} \sim 1$ and $m_1 \lesssim m_2$ (see Fig. 12), can be large, and is therefore not constrained by our model.

More importantly, we have restricted this study to circular orbits. However, misalignment may be accompanied with non-zero eccentricities, either because they have the same origin or because they are coupled. The entanglement between eccentricity and misalignment is complex near mean-motion resonance. We will tackle this issue in a forthcoming paper.

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DATA AVAILABILITY

The Python algorithm used for the integration of the differential equations characterizing the problem is available on request.

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APPENDIX A: STABILITY AROUND THE TRUE EQUILIBRIUM

In the outer test mass case (Section 3), the differential equations around the true equilibrium (Eqs. 52-55) writes:

\[
\begin{align*}
\frac{\dd a}{\dd t} &= j\omega_{\text{res}} a_{\text{eq}} I_{\text{forced}}^2 \delta \phi, \\
\frac{\dd I}{\dd t} &= \omega_{\text{res}} I_{\text{forced}} \delta \phi + \frac{\Delta \omega_p}{2} \sin(2I_p) \delta \Omega, \\
\frac{\dd \delta \phi}{\dd t} &= -\frac{3}{2} n j \left( \frac{a'}{a_{\text{eq}}} \right)^2 \frac{\dd a}{\dd t} + \frac{\Delta \omega_p}{I_{\text{forced}}} \sin(2I_p) \delta I, \\
\frac{\dd \delta \Omega}{\dd t} &= -\frac{\Delta \omega_p}{2I_{\text{forced}}^2} \sin(2I_p) \delta I, \\
\end{align*}
\]

which can be rewritten as

\[
\frac{d}{dt} \begin{pmatrix} \delta a \\ \delta I \\ \delta \phi \\ \delta \Omega \end{pmatrix} = \begin{pmatrix} 0 & 0 & j a_{\text{eq}} \omega_{\text{res}} I_{\text{forced}}^2 & 0 \\ 0 & \omega_{\text{res}} I_{\text{forced}} & 0 & \Delta \omega_p \sin(2I_p)/2 \\ -\Delta \omega_p \sin(2I_p)/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta a \\ \delta I \\ \delta \phi \\ \delta \Omega \end{pmatrix},
\]

The eigenvalues have the form:

\[
\lambda_{\pm} = \pm \frac{1}{2\sqrt{2}} \sqrt{x \pm \sqrt{x^2 - y}},
\]

where, assuming \((a'/a_{\text{eq}})^2 \approx (j - 2)/j:\)

\[
x = 8 \omega_{\text{res}} \omega_{\text{prec}} - 4 \omega_{\text{prec}}^2 - \frac{3}{2} n j (j - 2) \Delta \omega_p^2 \sin^2(2I_p) \frac{\omega_{\text{res}}}{\omega_{\text{prec}}},
\]

\[
y = 24 j (j - 2) \omega_{\text{prec}}^2 \sin^2(2I_p) \omega_{\text{res}}.
\]

As we always have (for whatever \(j\)) \(\omega_{\text{prec}} > 2 \omega_{\text{res}}\), then \(x < 0\). The stability of the equilibrium then depends on \(x^2 - y\): unstable when negative, stable when positive.

\[x^2 - y < 0 \quad \text{(instability)} \]

\[
\Leftrightarrow \frac{3}{2} n j (j - 2) \frac{\omega_{\text{res}}}{\omega_{\text{prec}}} \Delta \omega_p^2 \sin^2(2I_p) - \sqrt{24n^2 j (j - 2) \omega_{\text{res}} \Delta \omega_p \sin(2I_p) - 8 \omega_{\text{res}} \omega_{\text{prec}} + 4 \omega_{\text{prec}}^2} < 0.
\]

This expression is a polynomial of degree 2 in \(\Delta \omega_p \sin(2I_p)\), so that we can derive an instability condition:

\[
\left( \frac{\omega_{\text{prec}}}{n'} \right)^2 \left( \sqrt{\frac{\omega_{\text{prec}}}{2 \omega_{\text{res}}} - 1} \right) \sqrt{\frac{16}{3(j - 2)}} < \frac{\Delta \omega_p \sin(2I_p)}{n'} < \left( \frac{\omega_{\text{prec}}}{n'} \right)^2 \left( \sqrt{\frac{\omega_{\text{prec}}}{2 \omega_{\text{res}}} + 1} \right) \sqrt{\frac{16}{3(j - 2)}},
\]

If \(\varepsilon_{12,\text{res}}\) is small, it reduces to:

\[
\frac{\omega_{\text{prec}}}{n'} \left( \sqrt{\frac{\omega_{\text{prec}}}{2 \omega_{\text{res}}} - 1} \right) \sqrt{\frac{16}{3(j - 2)}} < \epsilon_{12,\text{res}} \sin(2I_p) < \sqrt{\frac{\omega_{\text{prec}}}{n'} \left( \sqrt{\frac{\omega_{\text{prec}}}{2 \omega_{\text{res}}} + 1} \right) \sqrt{\frac{16}{3(j - 2)}}} \approx \sqrt{\mu_f}.
\]

An example of a stable evolution with \(\varepsilon_{12,\text{res}} \sin(2I_p)\) below the lower limit is shown in Fig. 5. It can be shown by studying numerically the roots of Eq. (A10) that, for reasonable mass ratios \(\mu' \lesssim 10^{-3}\), the stability is ensured for any coupling if \(\varepsilon_{12,\text{res}}\) and \(I_p\) roughly satisfy:

\[
\varepsilon_{12,\text{res}} \sin(2I_p) > \sqrt{\mu_f}.
\]

The same reasoning can be hold for the inner test mass case, leading to the same instability condition.

APPENDIX B: ROTATING FRAME

The evolution of the total angular momentum \(\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 = L \hat{\mathbf{i}}\) is governed by

\[
\frac{\dd \mathbf{L}}{\dd t} = (\omega_{1p} \cos \theta_{1p} \mathbf{L}_1 + \omega_{2p} \cos \theta_{2p} \mathbf{L}_2) \wedge \hat{\mathbf{i}}_p.
\]

If the relative inclination \(I\) between \(\mathbf{L}_1\) and \(\mathbf{L}_2\) is small, then:

\[
\mathbf{L}_1 = L_1 \hat{\mathbf{i}} + (\mathbf{L}_1 - \frac{L_1}{L} \mathbf{L}) = L_1 \hat{\mathbf{i}} + \frac{L_2 \mathbf{L}_1 - L_1 \mathbf{L}_2}{L} + O(I^2) = L_1 \hat{\mathbf{i}} + O(I^2).
\]
Moreover, \( \cos \theta_{1p} = \cos \theta_{2p} + O(I) = \cos I_p + O(I) \), where \( I_p \) is the angle between \( \mathbf{L} \) and \( \mathbf{L}_p \). Thus

\[
\frac{d\mathbf{L}}{dt} = \frac{\omega_{1p}L_1 + \omega_{2p}L_2 \cos I_p \mathbf{L} \wedge \mathbf{i}_p}{L}.
\]  

(B3)

Let us now consider the possible variation of \( I_p \). We have:

\[
L \cos I_p = L_1 \cos \theta_{1p} + L_2 \cos \theta_{2p}.
\]  

(B4)

The relative inclinations \( \theta_{1p} \) and \( \theta_{2p} \) are unchanged by the perturber, but vary as \( m_1 \) and \( m_2 \) interact. However, we have:

\[
\frac{d}{dt} L_1 \cos \theta_{1p} = -\frac{d}{dt} L_2 \cos \theta_{2p},
\]  

(B5)

so that

\[
\frac{dL \cos I_p}{dt} = 0.
\]  

(B6)

After some algebra, using \( L^2 = L_1^2 + L_2^2 + 2L_1L_2 \cos I \):

\[
\frac{dI_p}{dt} = -\frac{L_1L_2}{L} \tan I_p \frac{dI}{dt},
\]

\[
= -\frac{L_1L_2}{L} \sin I \cos(\Omega_2 - \Omega_1) \cos I_p(\omega_{2p} \cos \theta_{2p} - \omega_{1p} \cos \theta_{1p}).
\]  

(B7)

Inspiring from Section 3, we suppose that \( \cos I_p \) stays roughly constant and \( \sin I \cos(\Omega_2 - \Omega_1) = I_{\text{free}} \sin(\omega_{\text{prec}} t + \psi) \), and we get:

\[
\Delta I_p \approx \frac{L_1L_2}{L} \varepsilon_{12,\text{res}} I_{\text{free}} \cos^2 I_p.
\]  

(B8)

All in all, we can suppose \( I_p \) constant only if \( L_1 \) or \( L_2 \) is zero (test mass), or if \( \varepsilon_{12} \) is small (strong coupling).

**APPENDIX C: FULL SET OF EQUATIONS FOR THE COMPARABLE MASS CASE**

**C1 General coupling, small \( I_p \)**

From the Hamilton-Jacobi equations applied to the Hamiltonian Eq. (100), we get the evolution of the angles:

\[
\frac{d\lambda_1}{dt} = \frac{\partial H}{\partial L_1} \bigg|_{Z_1} = \frac{\partial H}{\partial L_1} \bigg|_{J_1} - \frac{I_1}{2Z_1} \frac{\partial H}{\partial T_1} \bigg|_{L_1} = n_1 + O(\mu', \mu_p),
\]  

(C1)

\[
\frac{d\lambda_2}{dt} = \frac{\partial H}{\partial L_2} \bigg|_{Z_2} = \frac{\partial H}{\partial L_2} \bigg|_{J_2} - \frac{I_1}{2Z_2} \frac{\partial H}{\partial T_2} \bigg|_{L_2} = n_2 + O(\mu', \mu_p),
\]  

(C2)

\[
\frac{d\Omega_1}{dt} = -\frac{\partial H}{\partial Z_1} \bigg|_{L_1} = -\frac{1}{L_1(I_1, T_1)} \frac{\partial H}{\partial T_1} \bigg|_{L_1}
\]

\[
= -\omega_{12} + \omega_{12, \text{res}} I_1 \cos(\Omega_2 - \Omega_1) + \omega_{12, \text{res}} \cos \phi_1 - \omega_{12, \text{res}} \frac{I_2}{I_1} \cos \left( \frac{\phi_1 + \phi_2}{2} \right) + \omega_{12} \sin \phi_1 \cos \theta_{1p} (\cos(\Omega_1 - \Omega_p) \sin I_p - I_1 \cos I_p),
\]  

(C3)

\[
\frac{d\Omega_2}{dt} = -\frac{\partial H}{\partial Z_2} \bigg|_{L_2} = -\frac{1}{L_2(I_2, T_2)} \frac{\partial H}{\partial T_2} \bigg|_{L_2}
\]

\[
= -\omega_{21} + \omega_{21, \text{res}} I_2 \cos(\Omega_2 - \Omega_1) + \omega_{21, \text{res}} \cos \phi_2 - \omega_{21, \text{res}} \frac{I_1}{I_2} \cos \left( \frac{\phi_1 + \phi_2}{2} \right) + \omega_{21} \sin \phi_2 \cos \theta_{2p} (\cos(\Omega_1 - \Omega_p) \sin I_p - I_2 \cos I_p),
\]  

(C4)

and the evolution of the momenta

\[
\frac{dL_1}{dt} = \frac{\partial H}{\partial \lambda_1} = \frac{j - 2}{2} \omega_{12, \text{res}} L \left( I_1^2 \sin \phi_1 + I_2^2 \sin \phi_2 - 2I_1I_2 \sin \left( \frac{\phi_1 + \phi_2}{2} \right) \right),
\]  

(C5)

\[
\frac{dL_2}{dt} = \frac{\partial H}{\partial \lambda_2} = \frac{j - 2}{2} \omega_{21, \text{res}} L \left( I_1^2 \sin \phi_1 + I_2^2 \sin \phi_2 - 2I_1I_2 \sin \left( \frac{\phi_1 + \phi_2}{2} \right) \right),
\]  

(C6)

\[
\frac{dZ_1}{dt} = \frac{\partial H}{\partial \Omega_1} = -\omega_{12} L_1 I_1 \sin(\Omega_2 - \Omega_1) - \omega_{12, \text{res}} L \left( I_1^2 \sin \phi_1 - I_1I_2 \sin \left( \frac{\phi_1 + \phi_2}{2} \right) \right) + \omega_{12} \cos \theta_{1p} L_1 I_1 \sin I_p \sin \Omega_1,
\]  

(C7)

\[
\frac{dZ_2}{dt} = \frac{\partial H}{\partial \Omega_2} = \omega_{21} L_2 I_2 \sin(\Omega_2 - \Omega_1) - \omega_{21, \text{res}} L \left( I_2^2 \sin \phi_2 - I_1I_2 \sin \left( \frac{\phi_1 + \phi_2}{2} \right) \right) + \omega_{21} \cos \theta_{2p} L_2 I_2 \sin I_p \sin \Omega_2.
\]  

(C8)
Deriving the semi-major axes and inclinations from the momenta, we finally get

\[
\frac{\text{d}a_1}{\text{d}t} = \frac{2L_1 - \text{d}L_1}{m_1^2GM} \frac{\text{d}t}{\text{d}t} = (j - 2)\omega_{12,\text{res}}a_1 \left(I_1^2 \sin \phi_1 + I_2^2 \sin \phi_2 - 2I_1 I_2 \sin \frac{\phi_1 + \phi_2}{2}\right),
\]  
(C9)

\[
\frac{\text{d}a_2}{\text{d}t} = \frac{2L_2}{m_2^2GM} \frac{\text{d}t}{\text{d}t} = -J\omega_{21,\text{res}}a_2 \left(I_1^2 \sin \phi_1 + I_2^2 \sin \phi_2 - 2I_1 I_2 \sin \frac{\phi_1 + \phi_2}{2}\right),
\]  
(C10)

\[
\frac{\text{d}I_1}{\text{d}t} = \frac{1}{L_1 I_1} \frac{\text{d}Z_1}{\text{d}t} - \frac{L_1 I_2}{2L_1} \frac{\text{d}L_1}{\text{d}t} = -\omega_{12}I_2 \sin(\Omega_2 - \Omega_1) - \omega_{12,\text{res}} \left(I_1 \sin \phi_1 - I_2 \sin \frac{\phi_1 + \phi_2}{2}\right) + \omega_{1p} \cos \theta_{1p} \sin I_p \sin(\Omega_1 - \Omega_p),
\]  
(C11)

\[
\frac{\text{d}I_2}{\text{d}t} = \frac{1}{L_2 I_2} \frac{\text{d}Z_2}{\text{d}t} - \frac{L_2 I_1}{2L_2} \frac{\text{d}L_2}{\text{d}t} = \omega_{21}I_1 \sin(\Omega_1 - \Omega_1) - \omega_{21,\text{res}} \left(I_2 \sin \phi_2 - I_1 \sin \frac{\phi_1 + \phi_2}{2}\right) + \omega_{2p} \cos \theta_{2p} \sin I_p \sin(\Omega_2 - \Omega_p).
\]  
(C12)

From there, we can get to six equations by replacing the evolution of \(a_1\) and \(a_2\) by \(\alpha = a_1/a_2\) and the evolution of \(\lambda_1\) and \(\lambda_2\) by \(j\lambda_2 - (j - 2)\lambda_1:\)

\[
\frac{\text{d}}{\text{d}t} \left[\frac{1}{\alpha} \frac{\text{d}a_1}{\text{d}t} - \frac{1}{\alpha} \frac{\text{d}a_2}{\text{d}t} \right] = (j - 2)\omega_{12,\text{res}} + j\omega_{21,\text{res}} \left(I_1^2 \sin \phi_1 + I_2^2 \sin \phi_2 - 2I_1 I_2 \sin \frac{\phi_1 + \phi_2}{2}\right),
\]  
(C13)

\[
\frac{\text{d}I_1}{\text{d}t} \left(\frac{1}{\alpha} \frac{\text{d}a_1}{\text{d}t} - \frac{1}{\alpha} \frac{\text{d}a_2}{\text{d}t} \right) = \omega_{1p} \cos I_p \sin I_p \sin(\Omega_1 - \Omega_p).
\]  
(C14)

The final set of equations is Eqs. (C13, C14, C11, C12, C3,C4).

### C2 Strong coupling

Assuming \(L\) is rotating around \(L_p\) at the rate \(-\omega_p \cos I_p\) with \(I_p\) constant, we can reduce the problem to four equations by choosing as the reference plane the plane perpendicular to \(L\). New terms have then to be added to take into account the rotation:

\[
\frac{\text{d}\Omega_1}{\text{d}t} = \frac{\omega_p}{I_1} \cos I_p \left(\cos(\Omega_1 - \Omega_p) \sin I_p - I_1 \cos I_p\right),
\]  
(C15)

\[
\frac{\text{d}\Omega_2}{\text{d}t} = \frac{\omega_p}{I_2} \cos I_p \left(\cos(\Omega_2 - \Omega_p) \sin I_p - I_2 \cos I_p\right),
\]  
(C16)

\[
\frac{\text{d}I}{\text{d}t} = \frac{\omega_p}{I} \sin I_p \sin(\Omega_1 - \Omega_p),
\]  
(C17)

\[
\frac{\text{d}I}{\text{d}t} = \frac{\omega_p}{I} \cos I_p \sin I_p \sin(\Omega_2 - \Omega_p).
\]  
(C18)

Moreover, our choice of frame ensures that \(I = I_1 + I_2\), \(L_1 \sin I_1 = L_2 \sin I_2\) and \(\Omega_1 = \Omega_2 + \pi\). We can thus make use of the following relations:

\[
\frac{\text{d}I}{\text{d}t} = \frac{\text{d}I_1}{\text{d}t} + \frac{\text{d}I_2}{\text{d}t},
\]  
(C19)

\[
\omega_p - \omega_{1p} = \frac{L_2}{L} \Delta \omega_p,
\]  
(C20)

\[
\omega_{2p} - \omega_p = \frac{L_1}{L} \Delta \omega_p,
\]  
(C21)

Finally, we get:

\[
\frac{\text{d}\alpha}{\text{d}t} = \alpha (j - 2)\omega_{12,\text{res}} + j\omega_{21,\text{res}} I^2 \sin \phi,
\]  
(C23)

\[
\frac{\text{d}I}{\text{d}t} = -(\omega_{12,\text{res}} + \omega_{21,\text{res}}) I \sin \phi - \frac{\Delta \omega_{1p}}{2} \sin(2I_p) \sin(\Omega_1 - \Omega_p),
\]  
(C24)

\[
\frac{\text{d}\phi}{\text{d}t} = -\Delta n - (\omega_{12,\text{res}} \cos \phi + \frac{L_2}{L} \Delta \omega_{1p} \cos^2 I_p - \frac{\Delta \omega_p}{2I} \cos(\Omega_1 - \Omega_p) \sin(2I_p),
\]  
(C25)

\[
\frac{\text{d}I_1}{\text{d}t} = -(\omega_{12} + \omega_{21}) + (\omega_{12,\text{res}} + \omega_{21,\text{res}}) \tan \phi + \frac{L_2}{L} \Delta \omega_{1p} \cos^2 I_p - \frac{\Delta \omega_p}{2I} \cos(\Omega_1 - \Omega_p) \sin(2I_p).
\]  
(C26)
In order to ensure that $\frac{d\Omega_1}{dt} = \frac{d\Omega_2}{dt}$, $\Delta \omega_p$ should be negligible with respect to $\omega_{12} + \omega_{21} + \omega_{12,\text{res}} + \omega_{21,\text{res}}$ (strong coupling limit).

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