The weak separation in higher dimensions

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Abstract. For an odd integer \( r > 0 \) and an integer \( n > r \), we introduce a notion of weakly \( r \)-separated subsets of \([n] = \{1, 2, \ldots, n\}\). When \( r = 1 \), this corresponds to the concept of weak separation introduced and studied by Leclerc and Zelevinsky. In this paper, extending results due to Leclerc-Zelevinsky, we develop a geometric approach to establish a number of nice combinatorial properties of maximal weakly \( r \)-separated collections.

Keywords: weakly separated sets, cyclic zonotope, fine zonotopal tiling, higher Bruhat order

MSC Subject Classification 05E10, 05B45

1 Introduction

Let \( n \) be a positive integer and let \([n]\) denote the set \(\{1, 2, \ldots, n\}\). For subsets \(X, Y \subseteq [n]\), we write \(X < Y\) if the maximal element \(\max(X)\) of \(X\) is smaller than the minimal element \(\min(Y)\) of \(Y\), letting \(\max(\emptyset) := 0\) and \(\min(\emptyset) := n + 1\) if \(Z = \emptyset\). An interval in \([n]\) is a nonempty subset \(\{a, a+1, \ldots, b\}\) in it, denoted as \([a, b]\) (so \([n]\) = \([1, n]\)).

The well-known concept of strongly separated sets introduced by Leclerc and Zelevinsky is extended as follows.

Definition. For \( r \in \mathbb{Z}_{\geq 0}\), sets \(A, B \subseteq [n]\) are called (strongly) \(r\)-separated if there is no sequence \(i_1 < i_2 < \cdots < i_{r+2}\) of elements of \([n]\) such that the elements with odd indices (namely, \(i_1, i_3, \ldots\)) belong to one of \(A - B\) and \(B - A\), while the elements with even indices (\(i_2, i_4, \ldots\)) belong to the other of these two sets (where \(A' - B'\) denotes the set difference \(i: A' \ni i \notin B'\)). Accordingly, a set-system \(S \subseteq 2^{[n]}\) of subsets of \([n]\) is called \(r\)-separated if any two members of \(S\) are such.

Equivalently, \(A, B \subseteq [n]\) are \(r\)-separated if there are intervals \(I_1 < I_2 < \cdots < I_{r'}\) in \([n]\) with \(0 \leq r' \leq r + 1\) such that one of \(A - B\) and \(B - A\) is included in \(I_1 \cup I_3 \cup \ldots\),
and the other in \( I_2 \cup I_4 \cup \ldots \). If, in addition, \( r' + |I_1| + \cdots + |I_{r'}| \) is as small as possible, we say that \((I_1, \ldots, I_{r'})\) is the \textit{interval cortege} associated with \( A, B \).

In particular, \( A, B \) are 0-separated if \( A \subseteq B \) or \( B \subseteq A \), and 1-separated if either \( \max(A - B) < \min(B - A) \) or \( \max(B - A) < \min(A - B) \). The 1-separation relation is just what is called the strong separation one in [8]. The case \( r = 2 \) was studied by Galashin [6] (who used the term “chord separated” for 2-separated sets). A study for a general \( r \) is conducted in Galashin and Postnikov [7].

When \( A, B \) are \( r \)-separated but not \((r - 1)\)-separated, they are called \((r + 1)\)-\textit{interlaced}. In other words, the interval cortege associated with such \( A, B \) consists of \( r + 1 \) intervals. For example, \( A = \{1, 2, 5, 6, 7, 10\} \) and \( B = \{2, 3, 6, 9\} \) have the interval cortege \( (\{1\}, \{3\}, \{5, 7\}, \{9\}, \{10\}) \), and therefore they are 5-interlaced.

Another sort of set separation introduced by Leclerc and Zelevinsky is known under the name of \textit{weak separation} (which appeared in [5] in connection with the problem of characterizing quasi-commuting flag minors of a quantum matrix; for a discussion on this and wider relations between the weak separation and quantum minors, see also [10, Sect. 8]). We generalize that notion to “higher dimensions” in the following way (where the term “higher dimensions” is justified by appealing to a geometric interpretation, defined later). When \( A, B \subseteq [n] \) are such that \( \min(A - B) < \min(B - A) \) and \( \max(A - B) > \max(B - A) \), we say that \( A \) \textit{surrounds} \( B \).

\textbf{Definition.} Let \( r \) be a positive \textit{odd} integer. Sets \( A, B \subseteq [n] \) are called \textit{weakly \( r \)-separated} if they are \( r'\)-interlaced with \( r' \leq r + 2 \), and if \( r' = r + 2 \) takes place, then either (a) \( A \) surrounds \( B \) and \( |A| \leq |B| \), or (b) \( B \) surrounds \( A \) and \( |B| \leq |A| \). Accordingly, a set-system \( W \subseteq 2^{|n|} \) is called \textit{weakly \( r \)-separated} if any two members of \( W \) are such.

In other words, \( A \) and \( B \) are weakly \( r \)-separated if they are either (strongly) \( r \)-separated or \((r + 2)\)-interlaced, and in the latter case, for the interval cortege \((I_1, \ldots, I_{r+2})\) associated with \( A, B \), if the cardinalities of \( A \) and \( B \) are different, say, \( |A| < |B| \), then \( I_1 \cup I_2 \cup \ldots \cup I_{r+2} \) contains \( A - B \) (and \( I_2 \cup I_4 \cup \ldots \cup I_{r+1} \) contains \( B - A \)). For example, \( \{1, 2, 6\} \) and \( \{2, 3, 4, 5\} \) are 1-separated, whereas \( \{1, 2, 5, 6, 7\} \) and \( \{1, 3, 4, 5\} \) are 3-interlaced (having the interval cortege \((\{2\}, \{3, 4\}, [6, 7]\)) but not weakly 1-separated.

When \( r = 1 \), the notion of weak 1-separation turns into the weak separation of [8].

In this paper we generalize, to an arbitrary odd \( r \geq 1 \), two results on weakly separated collections obtained in [8]. One of those says that

\begin{equation}
(1.1) \text{the maximal possible sizes (numbers of members) of strongly and weakly separated collections in } 2^{|n|} \text{ are the same and equal to } \frac{1}{2} n(n+1) + 1 = \binom{n}{2} + \binom{n}{1} + \binom{n}{0}. \end{equation}

To formulate a generalization of (1.1), let \( r < n \) and denote the maximal possible size \( |S| \) of an \( r \)-separated collection \( S \subseteq 2^{|n|} \) by \( s_{n,r} \). Also when \( r \) is odd, denote the maximal possible size of a weakly \( r \)-separated collection \( W \subseteq 2^{|n|} \) by \( w_{n,r} \). Extending results in [8] (for \( r = 1 \)) and [6] (for \( r = 2 \), Galashin and Postnikov [7] showed that

\begin{equation}
(1.2) \ s_{n,r} = \left( \left\lfloor \frac{n}{2} \right\rfloor \right) + \left( \binom{n}{r} \right) + \cdots + \left( \binom{n}{0} \right). \end{equation}

We prove the following
Theorem 1.1  Let $r$ be odd. Then $w_{n,r} = s_{n,r}$.

Another impressive result in [8] says that a weakly separated collection can be transformed into another one by making a flip (a sort of mutations) “in the presence of four witnesses”. This relies on the following property (Theorem 7.1 in [8]):

(1.3) let $\mathcal{W} \subset 2^{[n]}$ be weakly separated, and suppose that there are elements $i < j < k$ of $[n]$ and a set $X \subseteq [n] - \{i,j,k\}$ such that $\mathcal{W}$ contains four sets (“witnesses”) $Xi, Xk, Xij, Xjk$ and a set $U \subseteq \{Xj, Xik\}$; then the collection obtained from $\mathcal{W}$ by replacing $U$ by the other member of $\{Xj, Xik\}$ is again weakly separated.

Hereinafter, for disjoint subsets $A$ and $\{a,\ldots,b\}$ of $[n]$, we use the abbreviated notation $Aa\ldots b$ for $A \cup \{a,\ldots,b\}$.

We generalize (1.3) as follows.

Theorem 1.2  For an odd $r$ and $r' := (r + 1)/2$, let $P = \{p_1,\ldots, p_{r'}\}$ and $Q = \{q_0,\ldots, q_r\}$ consist of elements of $[n]$ such that $q_0 < p_1 < q_1 < p_2 < \ldots < p_{r'} < q_{r'}$, and let $X \subseteq [n] - (P \cup Q)$. Define the set of neighbors (or “witnesses”) of $P,Q$ to be

$$ N = N(P,Q) := \{S \subseteq P \cup Q : S \neq P,Q, r' \leq |S| \leq r' + 1\}. $$

Suppose that a weakly $r$-separated collection $\mathcal{W} \subset 2^{[n]}$ contains a set $U \in \{X \cup P, X \cup Q\}$. If, in addition, $\mathcal{W}$ contains the sets of the form $X \cup S$ for all $S \in N$, then the collection obtained from $\mathcal{W}$ by replacing $U$ by the other member $U'$ of $\{X \cup P, X \cup Q\}$ is weakly $r'$-separated as well.

(Obviously, $P$ and $Q$ are not weakly $r$-separated, and $|P \cup Q| = r + 2$ easily implies that any two sets in $N \cup \{P,Q\}$ except for $P,Q$ are weakly $r$-separated.) In general, for two weakly $r$-separated collections $\mathcal{W}$ and $\mathcal{W}'$, if there are $P,Q,X$ as above such that $\mathcal{W}' = (\mathcal{W} - \{X \cup P\}) \cup \{X \cup Q\}$ and $\mathcal{W} = (\mathcal{W}' - \{X \cup Q\}) \cup \{X \cup P\}$, then we say that $\mathcal{W}'$ is obtained from $\mathcal{W}$ by a raising (combinatorial) flip, while $\mathcal{W}$ is obtained from $\mathcal{W}'$ by a lowering flip.

Our method of proof of the above theorems (and more) appeals to a geometric approach and uses some facts on fine zonotopal tilings, or cubillages, on a cyclic zonotope in a space $\mathbb{R}^d$. One of them is that the maximal by size (strongly) $(d - 1)$-separated collections $S$ in $2^{[n]}$ one-to-one correspond to the cubillages $Q$ in a cyclic zonotope $Z(n,d)$ generated by (a cyclic configuration of) $n$ vectors in $\mathbb{R}^d$; one may say that the vertex set of $Q$ “encodes” $S$. (When $d = 2$, a cubillage becomes a rhombus tiling on a planar $n$-zonogon, and a bijection between these tilings and the maximal strongly separated collections in $2^{[n]}$ is well-known. For $d = 3$, a bijection between the corresponding cubillages and maximal 2-separated sets was originally established in [9]. For a general $d$, the corresponding bijection was recently shown by Galashin and Postnikov [7].)

Another useful fact, inspired by a result in the classical work due to Manin and Schechtman [9] on higher Bruhat orders, is that any cubillage on $Z(n,d-1)$ can be lifted as a certain $(d-1)$-dimensional subcomplex, that we call an $s$-membrane, in some cubillage on $Z(n,d)$. For more details and other relevant facts, see [3][10].

We further develop the theory of cubillages by constructing a certain fragmentation $Q^c$ of a cubillage $Q$ on $Z(n,d)$, introducing a class of $(d-1)$-dimensional subcomplexes
in $Q^n$, called \textit{w-membranes}, and showing (in Theorem 6.4) that when $d$ is odd, the vertex set of any w-membrane forms a maximal by size weakly $(d-2)$-separated collection in $2^{[n]}$. It turns out that the collections of this sort (over all cubillages on $Z(n,d)$) constitute a poset with a unique minimal element and a unique maximal element and where neighboring collections are linked by flips; this is obtained as a consequence of Theorems 6.4 and 1.2.

In light of this, given an odd $r$ and $n > r$, we can specify three classes $W_{n,r}$, $W_{n,r}^=$, and $W_{n,r}^*$ of weakly $r$-separated collections $W$ in $2^{[n]}$, in which $W$ is maximal by inclusion, maximal by size, and representable, respectively. Here we call $W$ \textit{representable} if it can be represented as the vertex set of a w-membrane in a cubillage on $Z(n,r+2)$ (in particular, $W$ is maximal by size). Then $W_{n,r} \supseteq W_{n,r}^= \supseteq W_{n,r}^*$.

This paper is organized as follows. Sect. 2 contains basic definitions and reviews some useful facts on cyclic zonotopes and cubillages. Sect. 3 recalls the construction of $s$-membranes in cubillages and describes their properties needed to us. Here we also introduce the so-called \textit{bead-thread} relation on vertices of a cubillage, which is used in the proof of Theorem 1.1. Sect. 4 is devoted to proving Theorem 1.1, and Sect. 5 proves a sharper version of Theorem 1.2 (given in Proposition 5.1). Also we explain that the results of Sect. 5 can be strengthened in the following way: instead of the whole set $N(P,Q)$ of neighbors of $P,Q$, it suffices to take into account only those neighbors that are at distance $\leq 2$ from $P$ or from $Q$ (Proposition 5.2 and Corollary 5.3).

Sect. 6 introduces w-membranes in the fragmentation of a cubillage and proves the above-mentioned results on w-membranes in a cubillage on $Z(n,d)$ and representable $(d-2)$-separated collection in $2^{[n]}$, and on the poset of such collections (Theorem 6.4 and Corollary 6.5). Sect. 7 demonstrates an example of “non-pure” weakly $r$-separated collections $W$ in $2^{[n]}$, in the sense that $W \in W_{n,r}$ but $|W| < w_{n,r}$, thus showing that the inclusion $W_{n,r} \supseteq W_{n,r}^= \supseteq W_{n,r}^*$ can be strict (in contrast to the well-known purity result for $r = 1$, saying that $W_{n,1} = W_{n,1}^=$). Also we raise two conjectures on weakly $r$-separated set-systems (in Sects. 6 and 7). The paper finishes with the Appendix that gives proofs of two propositions stated in Sect. 6.

2 Preliminaries

This section contains additional definitions, notation and conventions that will be needed later on. Also we recall some known properties of cubillages.

- Let $n, d$ be positive integers with $n \geq d > 1$. By a \textit{cyclic configuration} of size $n$ in $\mathbb{R}^d$ we mean an ordered set $\Xi$ of $n$ vectors $\xi_i = (\xi_i(1), \ldots, \xi_i(d)) \in \mathbb{R}^d$, $i = 1, \ldots, n$, satisfying:

\begin{equation}
\begin{aligned}
(a) \quad & \xi_i(1) = 1 \text{ for each } i, \\
(b) \quad & \text{for the } d \times n \text{ matrix } A \text{ formed by } \xi_1, \ldots, \xi_n \text{ as columns (in this order), any flag minor of } A \text{ is positive.}
\end{aligned}
\end{equation}

A typical (and commonly used) sample of such configurations $\Xi$ is generated by the Veronese curve; namely, take reals $t_1 < t_2 < \cdots < t_n$ and assign $\xi_i := \xi(t_i)$, where $\xi(t) = (1, t, t^2, \ldots, t^{d-1})$. 

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The zonotope $Z = Z(\Xi)$ generated by $\Xi$ is the Minkowski sum of line segments $[0, \xi_i]$, $i = 1, \ldots, n$. A fine zonotopal tiling is a subdivision $Q$ of $Z$ into $d$-dimensional parallelotopes such that: any two intersecting ones share a common face, and each face of the boundary of $Z$ is contained in some of these parallelotopes. For brevity, we liberally refer to these parallelotopes as cubes, and to $Q$ as a cubillage.

- When $n, d$ are fixed, the choice of one or another cyclic configuration $\Xi$ (subject to \eqref{2.1}) does not matter in essence, and for this reason, we unify notation $Z(n, d)$ for $Z(\Xi)$, referring to it as the cyclic zonotope for $(n, d)$.

- Let $\pi$ denote the projection $\mathbb{R}^d \to \mathbb{R}^{d-1}$ given by $(x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_{d-1})$. Due to \eqref{2.1}, the vectors $\pi(\xi_1), \ldots, \pi(\xi_n)$ form a cyclic configuration as well, and we may say that $\pi$ projects $Z(n, d)$ to the zonotope $Z(n, d - 1)$.

- Each subset $X \subseteq [n]$ naturally corresponds to the point $\sum_{i \in X} \xi_i$ in $Z(n, d)$, and the cardinality $|X|$ is called the height, or level of this subset/point. (W.l.o.g., we usually assume that all combinations of vectors $\xi_i$ with coefficients 0,1 are different.)

- Depending on the context, we may think of a cubillage $Q$ on $Z(n, d)$ in two ways: either as a set of $d$-dimensional cubes (and write $C \in Q$ for a cube $C$ in $Q$) or as the corresponding polyhedral complex. The $0$- and $1$-dimensional cells (or faces) of $Q$ are called vertices and edges, respectively. A simple fact is that, by the subset-to-point correspondence, each vertex is identified with a subset of $[n]$. In turn, each edge $e$ is a parallel translation of some segment $[0, \xi_i]$; we say that $e$ has color $i$, or is an $i$-edge. When needed, $e$ is regarded as a directed edge (according to the direction of $\xi_i$).

- Let $V(Q)$ denote the set of vertices of a cubillage $Q$. Galashin and Postnikov \cite{7} showed the following important correspondence between cubillages and separated set-systems:

\begin{equation}
\label{2.2}
\text{for any cubillage } Q \text{ on } Z(n, d), \text{ the set } V(Q) \text{ of its vertices (regarded as subsets of } [n]) \text{ constitutes a maximal by size } (d - 1)\text{-separated collection in } 2^{[n]}; \text{ conversely, for any maximal by size } (d - 1)\text{-separated collection } S \subseteq 2^{[n]}, \text{ there exists a cubillage } Q \text{ on } Z(n, d) \text{ with } V(Q) = S.
\end{equation}

- When a cell (face) $C$ of $Q$ has the lowest point $X \subseteq [n]$ and when $T \subseteq [n]$ is the set of colors of edges in $C$, we say that $C$ has the root $X$ and type $T$, and may write $C = (X \mid T)$. One easily shows that $X \cap T = \emptyset$. Another well-known fact is that for any cubillage $Q$, the types of all $(d$-dimensional) cubes in it are different and form the set $\binom{n}{d}$ of $d$-element subsets of $[n]$ (so $Q$ has exactly $\binom{n}{d}$ cubes).

- For a closed subset $U$ of points in $Z = Z(n, d)$, let $U^\text{fr}$ ($U^\text{rear}$) be the part of $U$ “seen” in the direction of the last, $d$-th, coordinate vector $e_d$ (resp. $-e_d$), i.e., the set formed by the points $x \in \pi^{-1}(x') \cap U$ with $x_d$ minimum (resp. maximum) for all $x' \in \pi(U)$. It is called the front (resp. rear) side of $U$.

In particular, $Z^\text{fr}$ and $Z^\text{rear}$ denote the front and rear sides, respectively, of (the boundary of) the zonotope $Z$. We call $Z^\text{fr} \cap Z^\text{rear}$ the rim of $Z$ and denote it as $Z^\text{rim}$.

- When a set $X \subseteq [n]$ is the union of $k$ intervals and $k$ is as small as possible, we say that $X$ is a $k$-interval. Note that for such an $X$, its complementary set $[n] - X$ is a $k'$-interval with $k' \in \{k - 1, k, k + 1\}$. In the next section we will use the following
known characterization of the sets of vertices in the front and rear sides of a zonotope in an odd dimension (cf., e.g., [3]).

(2.3) Let $d$ be odd. Then for $Z = Z(n, d)$,

(i) $V(Z^{\text{fr}})$ is formed by all $k$-intervals of $[n]$ with $k \leq (d - 1)/2$;

(ii) $V(Z^{\text{rear}})$ is formed by the subsets of $[n]$ complementary to those in $V(Z^{\text{fr}})$; specifically, it consists of all $k$-intervals with $k < (d - 1)/2$, all $(d - 1)/2$-intervals containing at least one of the elements 1 and $n$ and all $(d + 1)/2$-intervals containing both 1 and $n$.

This implies that $V(Z^{\text{rim}})$ consists of the $k$-intervals with $k < (d - 1)/2$ and $(d - 1)/2$-intervals containing at least one of 1 and $n$; the set of inner vertices in $Z^{\text{fr}}$, i.e., $V(Z^{\text{fr}}) - V(Z^{\text{rim}})$ consists of the $(d - 1)/2$-intervals containing none of 1 and $n$, whereas $V(Z^{\text{rear}}) - V(Z^{\text{rim}})$ consists of the $(d + 1)/2$-intervals containing both 1 and $n$.

- Consider a cube $C = (X | T)$ and let $T = (p(1) < p(2) < \cdots < p(d))$. This cube has $2d$ facets $F_1, \ldots, F_d, G_1, \ldots, G_d$, where $F_i = F_i(C)$ is viewed as $(X | T - p(i))$, and $G_j = G_j(C)$ as $(X p(j) | T - p(j))$.

(For a set $A$ and an element $a \in A$, we abbreviate $A - \{a\}$ to $A - a$.)

3 S-membranes and bead-threads

In this section we recall the definition of s-membranes, associate with a cubillage a certain path structure, and review some basic properties.

**Definition.** Let $Q$ be a cubillage on $Z(n, d)$. An s-membrane in $Q$ is a subcomplex $M$ of $Q$ such that $M$ (regarded as a subset of $\mathbb{R}^d$) is bijectively projected by $\pi$ to $Z(n, d - 1)$.

Then each facet of $M$ is projected to a cube of dimension $d - 1$ in $Z(n, d - 1)$ and these cubes constitute a cubillage on $Z(n, d - 1)$, denoted as $\pi(M)$. In view of (2.2) and (1.2) (applied to $\pi(Q)$), we obtain that

(3.1) all s-membranes $M$ in a cubillage $Q$ on $Z(n, d)$ have the same number of vertices, which is equal to $s_{n,d-2}$, and the vertex set of $M$ (regarded as a collection in $2^{[n]}$) is $(d - 2)$-separated.

Two s-membranes are of an especial interest. These are the front side $Z^{\text{fr}}$ and the rear side $Z^{\text{rear}}$ of $Z = Z(n, d)$ (in these cases the choice of a cubillage on $Z$ is not important.) Following terminology in [4 [5], their projections $\pi(Z^{\text{fr}})$ and $\pi(Z^{\text{rear}})$ are called the standard and anti-standard cubillages on $Z(n, d - 1)$, respectively.

Next we distinguish certain vertices in cubes. When $n = d$, the zonotope turns into the cube $C = (\emptyset|[d])$, and there holds:

(3.2) the front side $C^{\text{fr}}$ (rear side $C^{\text{rear}}$) of $C = (\emptyset|[d])$ has a unique inner vertex, namely, $t_C := \{i \in [n]: d - i \text{ odd}\}$ (resp. $h_C := \{i \in [n]: d - i \text{ even}\}$.
(In fact, we will use (3.2) when \( d \) is odd, in which case it can be obtained from (2.3). A direct proof of (3.2) for an arbitrary \( d \) as follows (a sketch). The facets of \( C \) are \( F_i = F_i(C) := (\emptyset)[d] - i \) and \( G_i = G_i(C) := ([i])[d] - i \), \( i = 1, \ldots, d \) (cf. (2.1)). A facet \( F_i \) is contained in \( C^{fr} \) (\( C^{rear} \)) if, when looking at the direction \( e_d \), \( C \) lies “behind” (resp. “before”) the hyperplane containing \( F_i \), or, equivalently, \( \det(A_i) > 0 \) (resp. \( \det(A_i) < 0 \)), cf. (2.1)(b), where \( A_i \) is the matrix with the columns \( \xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_d, \xi_i \) (in this order). It follows that \( F_i \subset C^{fr} \) if and only if \( d - i \) is even. By “central symmetry”, \( G_i \subset C^{fr} \) if and only if \( d - i \) is odd.

Now consider a vertex \( X \subseteq [d] \) of \( C \). If \( X \) (resp. \([d] - X \)) has consecutive elements \( i - 1, i \), then \( X \in G_{i-1} \) and simultaneously \( X \in G_i \) (resp. \( X \in F_{i+1} \) and \( X \in F_i \)). This implies that \( X \) is in both \( C^{fr} \) and \( C^{rear} \), i.e., \( X \in C^{rim} \). The remaining vertices of \( C \) are just \( t_C \) and \( h_C \) as in (3.2); one can see that the former (latter) is contained in all facets \( F_j \) and \( G_i \) with \( d - j \) even and \( d - i \) odd (resp. \( d - j \) odd and \( d - i \) even). So \( t_C \) lies in \( C^{fr} \), and \( h_C \) in \( C^{rear} \); moreover, both are not in \( C^{rim} \) (since \( C \) is full-dimensional).)

When \( n \) is arbitrary and \( Q \) is a cubillage on \( Z = Z(n, d) \), we distinguish vertices \( t_C \) and \( h_C \) of a cube \( C \in Q \) in a similar way; namely (cf. (3.2),

\[
(3.3) \text{ if } C = (X \mid T) \text{ and } T = (p_1 < \ldots < p_d), \text{ then } t_C = X \cup \{p_i : d - i \text{ odd}\} \text{ and } h_C = X \cup \{p_i : d - i \text{ even}\}.
\]

Also for each vertex \( v \) of \( Q \), unless \( v \) is in \( Z^{rear} \), there is a unique cube \( C \in Q \) such that \( t_C = v \), and symmetrically, unless \( v \) is in \( Z^{fr} \), there is a unique cube \( C \in Q \) such that \( h_C = v \) (to see this, consider the line going through \( v \) and parallel to \( e_d \)).

Therefore, by drawing for each cube \( C \in Q \), the edge-arrow from \( t_C \) to \( h_C \), we obtain a directed graph whose connectivity components are directed paths beginning at \( Z^{fr} - Z^{rim} \) and ending at \( Z^{rear} - Z^{rim} \). We call these paths bead-threads in \( Q \). It is convenient to add to this graph the elements of \( V(Z^{rim}) \) as isolated vertices, forming degenerate bead-threads, each going from a vertex to itself. Let \( B_Q \) be the resulting directed graph. Then

\[
(3.4) \text{ } B_Q \text{ contains all vertices of } Q, \text{ and each component of } B_Q \text{ is a bead-thread going from } Z^{fr} \text{ to } Z^{rear}.
\]

Note that the heights \( |X| \) of vertices \( X \) along a bead-thread are monotone increasing when \( d \) is odd, and constant when \( d \) is even.

4 Proof of Theorem 1.1

Let \( r \) be odd and \( n > r \). We have to show that

\[
(4.1) \text{ if } W \text{ is a weakly } r\text{-separated collection in } 2^{[n]}, \text{ then } |W| \leq \left( \begin{array}{c} n \\ \leq r+1 \end{array} \right).
\]

This is valid when \( r = 1 \) (cf. (1.1)) and is trivial when \( n = r+1 \). So one may assume that \( 3 \leq r \leq n - 2 \). We prove (4.1) by induction, assuming that the corresponding inequality holds for \( W', n', r' \) when \( n' \leq n, r' \leq r, \) and \( (n', r') \neq (n, r) \).
Define the following subcollections in \( \mathcal{W} \):

\[
\mathcal{W}^- := \{ A \subseteq [n - 1] : \{ A, A_n \} \cap \mathcal{W} \neq \emptyset \}, \quad \text{and} \\
\mathcal{T} := \{ A \subseteq [n - 1] : \{ A, A_n \} \subseteq \mathcal{W} \},
\]

(referring to elements \( A, A_n \) in \( \mathcal{W} \) as \textit{twins}). Observe that

\[
(4.2) \quad \text{any } A, B \in \mathcal{W}^- \text{ are weakly } r\text{-separated.}
\]

Indeed, this is trivial when \( A, B \in \mathcal{W} \) or \( A_n, B_n \in \mathcal{W} \). Assume that \( A \in \mathcal{W} \), \( B' := Bn \in \mathcal{W} \) and that \( A, B' \) are \((r+2)\)-interlaced (for if \( A, B' \) are \( r' \)-interlaced with \( r' \leq r+1 \), then so is for \( A, B \), and we are done). Since \( \max(B' - A) = n > \max(A - B') \) and \( r+2 \) is odd, \( B' \) surrounds \( A \). Therefore, \( \min(B' - A) < \min(A - B') \) and \( |B'| \leq |A| \). Then \( |B| < |A| \) and \( \min(B - A) = \min(B' - A) < \min(A - B) \), implying that \( A, B \) are weakly \( r \)-separated, as required.

By induction, \( |\mathcal{W}^-| \leq \binom{n-1}{\frac{r-1}{2}} \). Also one can observe that \( |\mathcal{W}| = |\mathcal{W}^-| + |\mathcal{T}| \). Therefore, using the identity \( \binom{n}{j} = \binom{n}{j-1} + \binom{n}{j} \) for any \( j \leq n-1 \), in order to obtain the inequality in \((4.1)\), it suffices to show that

\[
|\mathcal{T}| \leq \binom{n-1}{\frac{r-1}{2}}. \tag{4.3}
\]

For \( i = 0, 1, \ldots, n - 1 \), define \( \mathcal{T}^i := \{ A \in \mathcal{T} : |A| = i \} \). We will rely on two claims.

Claim 1 For each \( i \), the collection \( \mathcal{T}^i \) is \((r-1)\)-separated; moreover, \( \mathcal{T}^i \) is weakly \((r-2)\)-separated.

Proof Let \( A, B \in \mathcal{T}^i \). Take the interval cortege \( (I_1, \ldots, I_{r'}) \) for \( A, B \), and let for definiteness \( I_r \subseteq A - B \). Then \( (I_1, \ldots, I_r, I_{r+1} := \{ n \}) \) is the interval cortege for \( A \) and \( B' := Bn \). Since \( |A| < |B'| \) and \( \max(A - B') < \max(B' - A) = n \), the fact that \( A, B' \) are weakly \( r \)-separated implies that \( r' + 1 \) is strictly less than \( r + 2 \). Then \( r' \leq r \), which means that \( A, B \) are \((r-1)\)-separated. Since \( |A| = |B| \) and \( r \) is odd, we also can conclude that \( A, B \) are weakly \((r-2)\)-separated. \( \blacksquare \)

Now consider the zonotope \( Z = Z(n-1, r) \). For \( j = 0, 1, \ldots, n - 1 \), define \( \mathcal{S}^j (\mathcal{A}^j) \) to be the set of vertices \( X \) of \( Z^{fr} \) (resp. \( Z^{re} \)) with \( |X| = j \). We extend each collection \( \mathcal{T}^i \) to \( \mathcal{D}^i \), defined as

\[
\mathcal{D}^i := \mathcal{T}^i \cup (\mathcal{S}^{i+1} \cup \ldots \cup \mathcal{S}^{n-1}) \cup (\mathcal{A}^0 \cup \mathcal{A}^1 \cup \ldots \cup \mathcal{A}^{i-1}). \tag{4.4}
\]

Claim 2 \( \mathcal{D}^i \) is weakly \((r-2)\)-separated.

Proof The vertex sets of \( Z^{fr} \) and \( \pi(Z^{fr}) \) are essentially the same (regarding a vertex as a subset of \([n-1]\)), and similarly for \( Z^{re} \) and \( \pi(Z^{re}) \). Since \( \pi(Z^{fr}) \) and \( \pi(Z^{re}) \) are cubillages on \( Z(n-1, r-1) \) (the so-called “standard” and “anti-standard” ones), \((3.2)\) implies that both collections \( V(Z^{fr}) = \mathcal{S}^0 \cup \ldots \cup \mathcal{S}^{n-1} \) and \( V(Z^{re}) = \mathcal{A}^0 \cup \ldots \cup \mathcal{A}^{i-1} \) are \((r-2)\)-separated, and therefore, they are weakly \((r-2)\)-separated as well.

Next, by \((2.3)(i)\), each vertex \( X \) of \( Z^{fr} \) is a \( k \)-interval with \( k \leq (r-1)/2 \). Such an \( X \) and any subset \( Y \subseteq [n-1] \) are \( k' \)-interlaced with \( k' \leq 2k + 1 \). Then \( k' \leq r \)
and this holds with equality when $X$ and $Y$ are $r$-interlaced and $Y$ surrounds $X$. It follows that $X$ is weakly $(r - 2)$-separated from any $Y \subseteq [n - 1]$ with $|Y| \leq |X|$ (in particular, if $X \in S^j$ and $j \geq 1$, then $X$ is weakly $(r - 2)$-separated from each member of $T^i \cup A^0 \cup \ldots \cup A^{i-1}$).

Symmetrically, by (2.3)(ii), each vertex $X$ of $Z_{\text{rear}}$ is the complement (to $[n - 1]$) of a $k$-interval with $k \leq (r - 1)/2$. We can conclude that such an $X$ is weakly $(r - 2)$-separated from any $Y \subseteq [n - 1]$ with $|Y| \geq |X|$.

Now the result is provided by the inequalities $|X| > |A| > |X'|$ for any $X \in S^{i+1} \cup \ldots \cup S^{n-1}$, $A \in T^i$, and $X' \in A^0 \cup \ldots \cup A^{i-1}$.

By induction, $|D^i| \leq \binom{n-1}{r-1}$. Then, using (1.2) and (3.1) (relative to $n - 1$ and $r - 2$), we have

$$|D^i| \leq \binom{n-1}{r-1} = s_{n-1,r-2} = |V(Z^r)|. \quad (4.5)$$

Let $S' := S^0 \cup S^1 \cup \ldots \cup S^i$ and $A' := A^0 \cup A^1 \cup \ldots \cup A^{i-1}$. Since $S^{i+1} \cup \ldots \cup S^{n-1} = V(Z^r) - S'$, we obtain from (4.4) and (4.5) that

$$|T^i| = |D^i| - (|V(Z^r) - S'|) - |A'| \leq |S'| - |A'|. \quad (4.6)$$

We now finish the proof by using a bead-thread techniques (as in Sect. 3). Fix an arbitrary cubillage $Q$ in $Z = Z(n - 1, r)$. Let $R^i$ be the set of vertices $X$ of $Q$ with $|X| = i$, and let $B$ be the set of paths (bead-threads) in the graph $B_Q$ beginning at $Z^r$ and ending at $Z_{\text{rear}}$. Since $r$ is odd, each edge $(X,Y)$ of $B_Q$ is “ascending” (satisfies $|Y| > |X|$). This implies that each path $P \in \mathcal{P}$ beginning at $S'$ must meet either $R^i$ or $A'$, and conversely, each path meeting $R^i \cup A'$ begins at $S'$. This and (4.6) imply

$$|T^i| \leq |R^i|. \quad \square$$

Summing up these inequalities for $i = 0, 1, \ldots, n - 1$, we have

$$|T| = \sum_i |T^i| \leq \sum_i |R^i| = |V_Q| = s_{n-1,r-1} = \binom{n-1}{r-1},$$

yielding (1.3) and completing the proof of Theorem 1.1. \quad \square

5 Proof of Theorem 1.2

Let $r, r', P = \{p_1, \ldots, p_r\}, Q = \{q_0, \ldots, q_r\}$ and $X$ be as in the hypotheses of Theorem 1.2 (where $r$ is odd and $r' = (r + 1)/2$). We will use the following notation and terminology.

Let $A, B \subset [n]$. The interval cortege for $A, B$ is denoted by $\mathcal{I}(A, B)$, and when it is not confusing, we refer to the intervals in it concerning $A - B$ ($B - A$) as $A$-bricks (resp. $B$-bricks). When $A \cap B = \emptyset$, we may abbreviate $A \cup B$ as $AB$. When $A, B$ are not weakly $r$-separated, we say that the pair $\{A, B\}$ is bad.

Note that for $P, Q, X$ as above and for the set $\mathcal{N} = \mathcal{N}(P, Q)$ of neighbors of $P, Q$ (defined in (1.4)), we have: $PX$ and $XQ$ are $(r + 2)$-interlaced, $XQ$ surrounds $XP$, $|XQ| > |XP|$, and $\{XP, XQ\}$ is the unique bad pair in the collection $\{XS: S \in \mathcal{N} \cup \{P, Q\}\}$. We are going to prove a slightly sharper version of Theorem 1.2.
Proposition 5.1 Let $Y \subset [n]$ be different from $XP$ and $XQ$. Let $U \in \{XP,XQ\}$. Suppose that the pair $\{Y,U\}$ is bad. Then there exists a neighbor $S \in \mathcal{N}(P,Q)$ such that $\{Y, XS\}$ is bad either.

(To obtain Theorem 1.2 consider $\mathcal{W}, U$ as in this theorem and let $U'$ be the set in $\{XP, XQ\}$ different from $U$. Suppose that $\{Y, U'\}$ is bad for some $Y \in \mathcal{W} - \{U\}$. By Proposition 5.1 applied to $Y, U'$, there exists $S \in \mathcal{N}(P,Q)$ such that $\{Y, XS\}$ is bad. But $\mathcal{W}$ is weakly $r$-separated and contains both $Y$ and $XS$.)

Proof W.l.o.g., one may assume that $Y \cap X = \emptyset$. We first prove the assertion for the case $U = XP$ (obtaining the result for $U = XQ$ as a consequence, as we explain in the end of the proof). We will write $\mathcal{N}$ for $\mathcal{N}(P,Q)$, and $\mathcal{I}$ for $\mathcal{I}(Y, XP)$.

Suppose, for a contradiction, that no pair $\{Y, XS\}$ with $S \in \mathcal{N}$ is bad. This will impose restrictions on $Y$ (as exposed in the claims below) and eventually will lead us to the conclusion that no $Y$ is possible at all.

Claim 1. Suppose that $q \in Q$ and $q \notin Y$. Then $q$ belongs to no $Y$-brick in $\mathcal{I}$.

Proof If $q$ belongs to a $Y$-brick $A$, then $q \notin Y$ implies that $\min(A) < q < \max(A)$ (in view of $\min(A), \max(A) \in Y = XP$). Then taking the neighbor $S := Pq \in \mathcal{N}$, we observe that the $Y$-brick $A$ of $\mathcal{I}$ is replaced in $\mathcal{I}(Y, XS)$ by three bricks, namely, by two $Y$-bricks (one containing $\min(A)$ and the other containing $\max(A)$) and the $XS$-brick $\{q\}$, while the other bricks of $\mathcal{I}$ preserve. Then $|\mathcal{I}(Y, XS)| = |\mathcal{I}| + 2$, whence $\{Y, XS\}$ is bad.

Claim 2. Suppose that $Y \cap P \neq \emptyset$. Then each $q \in Q$ forms the single-element $Y$-brick $\{q\}$ in $\mathcal{I}$.

Proof Fix $p \in Y \cap P$ and consider $q \in Q$. Define $S' := P - p$ and $S'' := S'q$. Then $S''$ (but not $S'$) is in $\mathcal{N}$, and $|XS''| = |XP|$.

Compare $\mathcal{I}$ with the interval cortege $\mathcal{I}' := \mathcal{I}(Y, XS')$. Since $p \in Y \cap P$, we have $Y - XS' = (Y - XP) \cup p$ and $XS' - Y = XP - Y$. Three cases are possible: (a) $p$ occurs in an $Y$-brick $A$ of $\mathcal{I}$; or (b) $p$ occurs in an $XP$-brick $B$ of $\mathcal{I}$; or (c) no brick of $\mathcal{I}$ contains $p$. In case (a), $A$ preserves in $\mathcal{I'}$. In case (b), $B$ is replaced by three bricks in $\mathcal{I'}$, namely, by two $XS'$-bricks (containing $\min(B)$ and $\max(B)$) and the $Y$-brick $\{p\}$. And in case (c), either $\{p\}$ becomes a new $Y$-brick (which is the first or last brick in $\mathcal{I'}$), or $p$ extends some $Y$-brick of $\mathcal{I}$. It follows that $|\mathcal{I'}| \geq |\mathcal{I}|$ (with equality in case (a) and, possibly, in case (c)). Also each $Y$-brick $A$ of $\mathcal{I}$ induces an $Y$-brick $A'$ such that $A \subseteq A'$.

Now compare the interval cortege $\mathcal{I}' := \mathcal{I}(Y, XS'')$. Two cases are possible. (a) Let $q \notin Y$. Then $q \in XS'' - Y$ and Claim 1 implies that either $q$ forms a new $X$-$\text{-brick}$ (which is the first or last brick in $\mathcal{I}'$), yielding $|\mathcal{I}'| > |\mathcal{I}|$, or $q$ extends some $XS''$-brick, yielding $|\mathcal{I}'| = |\mathcal{I}|$. (b) Let $q \in Y$. Since $q \in Y - XP$, $q$ belongs to some $Y$-brick $A$ of $\mathcal{I}$. If $A = \{q\}$, we are done. So suppose that $|A| \geq 2$. This $A$ indices an $Y$-brick $A'$ of $\mathcal{I'}$ with $A \subseteq A'$. Then $|A'| \geq 2$, and therefore the removal of $q$ from $Y$ (in view of $q \in XS''$) transforms $A'$ into a nonempty $Y$-brick in $\mathcal{I}'$, yielding $|\mathcal{I}'| = |\mathcal{I}|$.

Thus, in all cases (unless $\{q\}$ is a $Y$-brick in $\mathcal{I}$), we have $|\mathcal{I}'| \geq |\mathcal{I}'| \geq |\mathcal{I}| = r + 2$. So $Y$ and $XS''$ are $r''$-interlaced with $r'' \geq r + 2$. Moreover, using $|XS''| = |XP|$, one
can conclude that the badness of \( \{ Y, XP \} \) implies that \( \{ Y, XS'' \} \) is bad either (even if \( r'' = r + 2 \)); a contradiction.

Suppose that \( Y \cap P \neq \emptyset \). By Claim 2, \( \mathcal{I} \) contains \( Y \)-bricks \( \{ q_0 \}, \ldots, \{ q_r \} \). Assume that \( p_i \in Y \) and consider the \( XP \)-brick \( B \) of \( \mathcal{I} \) such that \( q_{i-1} < B < q_i \). Since \( q_{i-1} < B < q_i \) and \( p_i \neq \min(B), \max(B) \) (in view of \( \min(B), \max(B) \in XP - Y \)), at least one takes place: \( \min(B) < p_i < q_i \) or \( q_{i-1} < p_i < \max(B) \). Assume the former (the latter case is treated similarly). Taking \( S := (P - p_i) \cup q_i \in N \) and \( \mathcal{T}' := \mathcal{I}(Y, XS) \), we observe that the \( Y \)-brick \( \{ q_i \} \) of \( \mathcal{I} \) is replaced by the \( Y \)-brick \( \{ p_i \} \) in \( \mathcal{T}' \), and the \( XP \)-brick \( B \) is replaced by the nonempty \( XS \)-brick containing the element \( \min(B) \). Also \( |XS| = |XP| \). As a result, we can conclude that \( |\mathcal{T}'| \geq |\mathcal{T}| \) and that the badness of \( \{ Y, XP \} \) implies that of \( \{ Y, XS \} \). This contradiction implies that

\[
Y \cap P = \emptyset \quad \text{(whence } Y \cap XP = \emptyset). \tag{5.1}
\]

**Claim 3.** Each \( XP \)-brick of \( \mathcal{I} \) contains at most one element of \( P \).

**Proof** Suppose that there is an \( XP \)-brick \( B \) of \( \mathcal{I} \) such that \( |B \cap P| \geq 2 \). Then \( B \) contains elements \( p_i, p_{i+1} \) for some \( i \). Since \( p_i < q_i < p_{i+1} \), \( B \) contains \( q_i \) as well. Then \( q_i \notin Y \). Form the neighbor \( S := (P - p_i) \cup q_i \) and compare \( \mathcal{I} \) and \( \mathcal{T}' := \mathcal{I}(Y, XS) \). The \( XP \)-brick \( B \) of \( \mathcal{I} \) is replaced by the \( XS \)-brick of \( \mathcal{T}' \) (containing \( q_i, p_{i+1} \)). We obtain \( |\mathcal{T}'| = |\mathcal{T}| \), and the badness of \( \{ Y, XP \} \) implies that of \( \{ Y, XS \} \) (taking into account that \( |XS| = |XP| ) \); a contradiction.

By this claim and (5.1), \( \mathcal{I} \) has at least \( r' \) \( XP \)-bricks, and the elements of \( P \) are distributed among \( r' \) different \( XP \)-bricks.

**Claim 4.** Suppose that \( \mathcal{I} \) has an \( XP \)-brick \( B \) such that \( |B| \geq 2 \) and \( B \) contains an element \( p \in P \). Then each \( q \in Q \) forms an \( Y \)-brick in \( \mathcal{I} \).

**Proof** Suppose that there is \( q \in Q \) such that \( \{ q \} \) is not a brick of \( \mathcal{I} \). Then, in view of Claim 1, either (a) \( q \notin Y \) and \( q \) belongs to no \( Y \)-brick in \( \mathcal{I} \), or (b) \( q \in Y \) and \( q \) belongs to an \( Y \)-brick \( A \) with \( |A| \geq 2 \) in \( \mathcal{I} \). Form the neighbor \( S := (P - p) \cup q \) and compare \( \mathcal{I} \) and \( \mathcal{T}' := \mathcal{I}(Y, XS) \).

In case (a), \( q \) either forms a new (single-element) \( X.. \)-brick of \( \mathcal{T}' \) or is included in an \( X.. \)-brick of \( \mathcal{T} \) which extends (or is inserted in) an \( XP \)-brick of \( \mathcal{I} \). And in case (b), in view of \( |A| \geq 2 \), \( A \) is replaced by a nonempty \( Y \)-brick in \( \mathcal{T}' \). In both cases, in view of \( |B| \geq 2 \), the removal of \( p \) turns \( B \) into a nonempty \( X.. \)-brick in \( \mathcal{T}' \). As a result, \( |\mathcal{T}'| \geq |\mathcal{T}| \), and the badness of \( \{ Y, XP \} \) implies that of \( \{ Y, XS \} \); a contradiction.

Using the above claims, we now examine three possible cases for \( Y \) and \( U = XP \).

I. Let \( |\mathcal{I}| = 2r' + 1 \) (= \( r + 2 \)) and let \( Y \) surround \( XP \). Then \( |Y| > |XP| \) and the bricks of \( \mathcal{I} \) are viewed as \( A_0 < B_1 < A_1 < \cdots < B_{r'} < A_{r'} \), where each \( A_i \) (\( B_j \)) concerns \( Y \) (resp. \( XP \)). By (5.1) and Claim 3, each \( B_i \) contains \( p_i \), \( i = 1, \ldots, r' \). Moreover, Claim 4 implies \( B_i = \{ p_i \} \) for each \( i \) (for \( |B_i| \geq 2 \) would imply \( A_{r'} = \{ q_j \} \), \( j = 0, \ldots, r' \); but then \( |Y| = |Q| = r' + 1 = |P| + 1 < |XP| \)). Thus, \( X = \emptyset \).

Suppose that \( q \notin Y \) for some \( q \in Q \). Then, by Claim 1, \( q \notin A_0 \cup \cdots \cup A_{r'} \). Hence either (a) \( q < A_0 \), or (b) \( q > A_{r'} \). On the other hand, there is \( i \) such that (c) \( p_i < q < A_i \) or (d)
$A_i < q < p_{i+1}$. In cases (a) and (b), taking $S := Pq$ and $T' := \mathcal{I}(Y,XS)$, we have $|T'| > |I| = r + 2$ (since $\{q\}$ forms a new brick in $T'$). And in case (c) (resp. (d), if we take as $S$ the neighbor $(P - p_i) \cup q$ (resp. $(P - p_{i+1}) \cup q$), then the brick $\{p_i\}$ (resp. $\{p_{i+1}\}$) is replaced by $\{q\}$. This implies $|T'(Y,S)| = |I|, |S| = |P| < |Y|$, and $Y$ surrounds $S$. So in all cases, $\{Y,S\}$ is bad.

Therefore, we may assume that $Q \subseteq Y$. Then $q_0 < p_1 < q_1 < \cdots < p_r < q_r$ implies $q_i \in A_i$ for $i = 0, \ldots, r'$. If $|A_i| \geq 2$ for some $i$, then considering a neighbor $S$ of the form either $Pq_i$ (with $i \in \{0,r'\}$) or $(P - p_j) \cup q_i$ for some $j \in \{i,i + 1\}$ and arguing as above, we obtain that $\{Y,S\}$ is bad.

The remaining case is: $A_i = \{q_i\}$ for all $i = 0, \ldots, r'$. But then $Y = Q$, contrary to the hypotheses of the proposition (when $X = \emptyset$).

II. Let $|I| = 2r' + 1$ and let $XP$ surround $Y$. Then $|XP| > |Y|$ and the intervals in $\mathcal{I}$ are viewed as $B_0 < A_1 < B_1 < \cdots < A_r < B_r$, where each $A_i$ ($B_i$) concerns $Y$ (resp. $XP$). By Claim 3, each $B_i$ contains at most one element of $P$. By Claim 4, if there is $i$ such that $|B_i| \geq 2$ and $B_i \cap P \neq \emptyset$, then each element of $Q$ forms an $Y$-brick in $\mathcal{I}$; this is impossible since $|Q| = r' + 1$ but the number of $Y$-bricks in $\mathcal{I}$ equals $r'$. Therefore, all $XP$-bricks of $\mathcal{I}$, except for one brick, say, $B_i$, are singletons $\{p_1\}, \ldots, \{p_r\}$. Assume that $i > 1$ (the case $i < r'$ is treated similarly). Then $B_0 = \{p_1\}$. Since $q_0 < p_1$ and $p_1 < A_1$, $q_0$ belongs to no $Y$-brick, and therefore $q_0 \notin Y$. Taking $S := Pq_0$, we have: $|\mathcal{I}(Y,XS)| = |I| = r + 2, |XS| > |Y|$, and $XS$ surrounds $Y$, whence $\{Y,XS\}$ is bad.

III. Let $|I| > 2r' + 1 (= r + 2)$. If there is $q \in Q$ such that either $q \notin Y$, or $q \in Y$ and the $Y$-brick in $\mathcal{I}$ containing $q$ has size $\geq 2$, then $S := Pq$ satisfies $|\mathcal{I}(Y,XS)| \geq |I|$, and therefore $\{Y,XS\}$ is bad. So we may assume that each $q \in Q$ forms the $Y$-brick $\{q\}$ in $\mathcal{I}$. Let $a$ and $b$ be the numbers of $Y$- and $XP$-bricks in $\mathcal{I}$, respectively; then $a + b > 2r' + 1$, $a \geq r' + 1$ and $|a - b| \leq 1$. Consider $S' := Pq_0$ and $S'' := Pq_r$. We assert that at least one of $\{Y,XS'\}$ and $\{Y,XS''\}$ is bad.

To see this, let $A, A'$ (resp. $B, B'$) be the first and last $Y$-bricks (resp. $XP$-bricks) in $\mathcal{I}$, respectively. Denote the numbers of $Y$-bricks and $XS'$-bricks in $T' := \mathcal{I}(Y,XS')$ by $\tilde{a}$ and $\tilde{b}$. Then $\tilde{a} = a - 1$ (since the transformation $XP \mapsto XP_{q_0}$ removes the $Y$-brick $\{q_0\}$, while preserving the other $Y$-bricks), and either $\tilde{b} = b$ (when $q_0 < B$) or $\tilde{b} = b - 1$ (when $B < q_0$); for in this case the two $X$-bricks which one precedes and the other succeeds $\{q\}$ merge). Hence $|I'| = \tilde{a} + \tilde{b} \geq a + b - 2$. Consider three possibilities for $a$.

Let $a \geq r' + 3$. Then $b \geq r' + 2$ (since $b \geq a - 1$). We have $\tilde{a} + \tilde{b} \geq a + b - 2 \geq 2r' + 3$, implying that $\{Y,XS'\}$ is bad.

Let $a = r' + 2$. If $b \geq r' + 2$, then $\tilde{a} + \tilde{b} \geq a + b - 2 \geq 2r' + 2$, whence $\{Y,XS'\}$ is bad.

And if $b = r' + 1$, then $Y$ surrounds $XP$ and at least one holds: $A = \{q_0\}$ or $A' = \{q_r\}$ (since all but one $Y$-bricks in $\mathcal{I}$ include elements of $Q$, in view of $|Q| = r' + 1 = a - 1$). One may assume that $A = \{q_0\}$ (the case $A' = \{q_r\}$ is symmetric). Then $q_0 < B$, and therefore $\tilde{b} = b$. So $\tilde{a} + \tilde{b} = a + b - 1 = 2r' + 2$, and $\{Y,XS'\}$ is bad.

Now let $a = r' + 1$. Then $Y = Q$. Three cases for $b$ are possible: (a) $b = r'$, (b) $b = r' + 1$, and (c) $b = r' + 2$. In case (a), $|I| = 2r' + 1, Y$ surrounds $XP$, and the badness of $\{Y,XP\}$ implies $|XP| < |Y| = r' + 1$. Since $|P| = r' \geq |XP|$, we have $X = \emptyset$. But then $Y = Q = XQ$, contradicting the hypotheses in the proposition. In
case (b), \(a = b\) implies that either \(q_0 < B\) or \(q_0 > B'\). Assume the former (the latter is symmetric). Then \(\tilde{b} = b = r' + 1\) and \(\tilde{a} = a - 1 = r'\). Hence \(XS'\) surrounds \(Y\) and \(|T'| = 2r' + 1\). Also \(|Y - XS'| = |Q - q_0| = r'\) and \(|XS' - Y| = |XP| \geq b = r' + 1\). So \(\{Y, XS'\}\) is bad. Finally, in case (c), we have \(\tilde{a} = a - 1 = r'\) and \(\tilde{b} = b - 1 = r' + 1\). Then \(|T'| = 2r' + 1\), \(XS'\) surrounds \(Y\), and \(|XS' - Y| \geq b = r' + 1 > r' = \tilde{a} = |Y - XS'|\), and \(\{Y, XS'\}\) is bad again.

Thus, the proposition is valid when \(U = XP\).

It remains to consider the case \(U = XQ\). We reduce it to the previous case, using the following observation. For \(A \subseteq [n]\), let \(\overline{A}\) denote the complementary set \([n] - A\). One can see that \(I(A, B) = I(\overline{A}, \overline{B})\) and that the bricks for \(A - B\) coincide with those for \(\overline{B} - \overline{A}\). It follows that if \(A, B\) are \((r + 2)\)-interlaced (where \(r\) is odd, as before) and \(A\) surrounds \(B\), then \(\overline{A}, \overline{B}\) are \((r + 2)\)-interlaced, \(\overline{B}\) surrounds \(\overline{A}\), and \(|A| - |B| = |\overline{B} - \overline{A}|\). Therefore, if \(A, B\) are weakly \(r\)-separated then so are \(\overline{A}, \overline{B}\).

Now for \(Y, P, Q, X\) and \(U = XQ\) as in the proposition, consider \(Y' := Y, X' = XPQ\) (= \([n] - (X \cup P \cup Q)\) and \(U' := XQ\). Suppose that \(\{Y, U\}\) is bad. Then \(\{Y', U'\}\) is bad as well. Note also that \(U' = X'P\). By the proposition applied to \(Y', P, Q, X'\) and \(U',\) there exists \(S' \in \mathcal{N}(P, Q)\) such that \(\{Y', X'S'\}\) is bad. Then \(\{Y, X'S'\}\) is bad as well. Take \(S := (P \cup Q) - S'\). One can see that \(S \in \mathcal{N}(P, Q)\) and \(X'S' = XS\). Therefore, \(\{Y, XS\}\) is bad, as required.

This completes the proof of the proposition and implies Theorem 1.2.

Analyzing the above prove, we observe that when handling a bad pair \(\{Y, XP\}\), we construct every time a bad pair \(\{Y, XS\}\) such that the neighbor \(S \in \mathcal{N}(P, Q)\) is of the form either \(Pq\) or \((P - p) \cup q\) for some \(q \in Q\) and \(p \in P\). This leads to the following strengthening of Corollary 5.3 (where \(A \Delta B\) denotes the symmetric difference \((A - B) \cup (B - A)\) of sets \(A, B\)).

**Proposition 5.2** Let \(r, n, P, Q, X\) be as above, and define \(\mathcal{N}_{P}^{2} (\mathcal{N}_{Q}^{2})\) to be the set of members \(S \in \mathcal{N}(P, Q)\) such that \(|P \Delta S| \leq 2\) (resp. \(|Q \Delta S| \leq 2\).) Let \(Y \subset [n]\) be different from \(XP\) and \(XQ\). Then:

(i) if \(\{Y, XP\}\) is bad, then there exists \(S \in \mathcal{N}_{P}^{2}\) such that \(\{Y, XS\}\) is bad; and
(ii) if \(\{Y, XQ\}\) is bad, then there exists \(S \in \mathcal{N}_{Q}^{2}\) such that \(\{Y, XS\}\) is bad.

This implies a sharper version of the theorem on combinatorial flips.

**Corollary 5.3** For \(r, n, P, Q, X\) as in Theorem 1.2, if a weakly \(r\)-separated collection \(\mathcal{W} \subset 2^{[n]}\) contains the set \(XP\) (\(XQ\)) and the sets \(XS\) for all \(S \in \mathcal{N}_{P}^{2}\) (resp. \(S \in \mathcal{N}_{Q}^{2}\), then the collection obtained from \(\mathcal{W}\) by replacing \(XP\) by \(XQ\) (resp. \(XQ\) by \(XP\)) is weakly \(r\)-separated as well.

6 Weakly \(r\)-separated collections generated by cubillages

In Sects. 2.3 we outlined an interrelation between (strongly) \(*\)-separated collections from one side, and cubillages and s-membranes from the other side (see 2.2 and 3.1).
This section is devoted to geometric aspects of the weak \( r \)-separation (assuming that \( r \) is odd). Being motivated by geometric constructions for maximal weakly 1-separated collections elaborated in [3, 4], we explain how to construct maximal by size weakly \( r \)-separated collections by use of the so-called \( w \)-membranes; these are analogs of \( s \)-membranes in certain fragmentations of cubillages.

In subsections below we introduce the notions of fragmentation and \( w \)-membrane, demonstrate their properties (extending results from [4, Sect. 6]) and finish with a theorem saying that the vertex set of any \((r + 1)\)-dimensional \( w \)-membrane gives rise to a maximal by size weakly \( r \)-separated collection (for corresponding \( n \)). Note that in Sects. 6.1–6.3 the dimension \( d \) of a zonotope/cubillage in question is assumed to be arbitrary (not necessarily odd).

### 6.1 Fragmentation

Let \( Q \) be a cubillage on \( Z(n, d) \). For \( \ell = 0, 1, \ldots, n \), we denote the “horizontal” hyperplane at “height” \( \ell \) in \( \mathbb{R}^d \) by \( H_\ell \), i.e., \( H_\ell := \{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 = \ell \} \). The fragmentation of \( Q \) is meant to be the complex \( Q^\equiv \) obtained by cutting \( Q \) by all \( H_1, \ldots, H_{n-1} \).

Such hyperplanes subdivide each cube \( C = (X | T) \) of \( Q \) into \( d \) pieces \( C_1^\equiv, \ldots, C_d^\equiv \), where \( C_h^\equiv \) is the (closed) portion of \( C \) between \( H_{|X|+h-1} \) and \( H_{|X|+h} \). We say that \( C_h^\equiv \) is \( h \)-th fragment of \( C \) and, depending on the context, may also think of \( Q^\equiv \) as the set of fragments over all cubes. Let \( S_h(C) \) denote \( h \)-th horizontal section \( C \cap H_{|X|+h} \) of \( C \) (where \( 0 \leq h \leq d \)); this is the convex hull of the set of vertices

\[
(X \mid (T_h))^\equiv = \{ x \cup A : A \subset T, \ |A| = h \}.
\]

(Using terminology of [9], \( S_h(C) \) is said to be a hyper-simplex. It turns into a usual simplex when \( h = 1 \) or \( d - 1 \).) Observe that for \( h = 1, \ldots, d \),

\[
(6.2) \quad h \text{-th fragment } C_h^\equiv \text{ of } C \text{ is the convex hull of the set of vertices } (X \mid (T_h)) \text{ and } (X \mid (T_h)) \text{; it has two } "\text{horizontal}" \text{ facets, namely, } S_{h-1}(C) \text{ and } S_h(C), \text{ and } 2d \text{ other facets (conditionally called } "\text{vertical}" \text{ ones), namely, the portions of } F_i(C) \text{ and } G_i(C) \text{ between } H_{|X|+h-1} \text{ and } H_{|X|+h} \text{ for } i = 1, \ldots, d, \text{ denoted as } F_{h,i}(C) \text{ and } G_{h,i}(C), \text{ respectively.}
\]

Here \( F_i(C) \) and \( G_i(C) \) are the facets of \( C = (X | T) \) defined in [2, 4], letting \( T = (p(1) < p(2) < \cdots < p(d)) \). We call \( S_{h-1}(C) \) and \( S_h(C) \) the lower and upper facets of the fragment \( C_h^\equiv \), respectively. Note that \( S_0(C) \) and \( S_d(C) \) degenerate to the single points \( X \) and \( XT \), respectively. The vertical facets \( F_{d,i}(C) \) and \( G_{1,i}(C) \) (for all \( i \)) degenerate as well.

The horizontal facets are “not fully seen” under the projection \( \pi \). To visualize all facets of fragments of \( Q^\equiv \), it is convenient to look at them as though “from the front and slightly from below”, i.e., by use of the projection \( \pi^\ell : \mathbb{R}^d \to \mathbb{R}^{d-1} \) defined by

\[
x = (x_1, \ldots, x_d) \mapsto (x_1 - \epsilon x_d, x_2, \ldots, x_{d-1}) =: \pi^\ell(x)
\]

for a sufficiently small \( \epsilon > 0 \). (Compare \( \pi^\ell \) with \( \pi \).) Figure 1 illustrates the case \( d = 3 \); here the fragments of a cube \( C = (X | T) \) with \( T = (i < j < k) \) are drawn.
Using this projection, we obtain slightly slanted front and rear sides of objects in $Q^\oplus$. More precisely, for a closed set $U$ of points in $Z = Z(n,d)$, let $U^{\epsilon,\text{fr}}$ ($U^{\epsilon,\text{rear}}$) be the subset of $U$ “seen” in the direction $e_d + \epsilon e_1$ (resp. $-e_d - \epsilon e_1$), where $e_i$ is $i$-th coordinate vector, i.e., formed by the points $x \in (\pi^\epsilon)^{-1}(x') \cap U$ with $x_d$ minimum (resp. maximum) for all $x' \in \pi^\epsilon(U)$. We call it the $\epsilon$-front (resp. $\epsilon$-rear) side of $U$.

Obviously, $Z^{\epsilon,\text{fr}} = Z^{\text{fr}}$ and $Z^{\epsilon,\text{rear}} = Z^{\text{rear}}$. Also for a cube $C = (X \mid T)$ in $Z$, $C^{\epsilon,\text{fr}} = C^{\text{fr}}$ and $C^{\epsilon,\text{rear}} = C^{\text{rear}}$. As to fragments of $C$, their $\epsilon$-front and $\epsilon$-rear sides are viewed as follows:

\[(6.4) \quad \text{for } h = 1, \ldots, d, \quad C_h^{\epsilon,\text{fr}} \text{ is the union of } C_h^{\text{fr}} \text{ and the lower facet } S_{h-1}(C) \text{ (degenerating to the point } X \text{ when } h = 1; \text{ in its turn, } C_h^{\epsilon,\text{rear}} \text{ is formed by union of } C_h^{\text{rear}} \text{ and the upper facet } S_h(C) \text{ (degenerating to the point } X \cup T \text{ when } h = d).\]

So $C_h^{\epsilon,\text{fr}} \cup C_h^{\epsilon,\text{rear}}$ is just the boundary of $C^\oplus$.

**6.2 W-membranes** Membranes of this sort represent certain $(d-1)$-dimensional subcomplexes of $Q^\oplus$. To introduce them, we consider small deformations of cyclic zonotopes in $\mathbb{R}^{d-1}$ using the projection $\pi^\epsilon$. More precisely, given a cyclic configuration $\Xi = (\xi_1, \ldots, \xi_n)$ as in (2.1), define

$$\psi_i := \pi(\xi_i) \quad \text{and} \quad \psi_i^\epsilon := \pi^\epsilon(\xi_i), \quad i = 1, \ldots, n.$$ 

Then $\Psi = (\psi_1, \ldots, \psi_n)$ obeys (2.1) (with $d-1$ instead of $d$), and when $\epsilon$ is small enough, $\Psi^\epsilon = (\psi_1^\epsilon, \ldots, \psi_n^\epsilon)$ obeys the condition (2.1)(b), though slightly violates (2.1)(a); yet we keep the term “cyclic configuration” for $\Psi^\epsilon$ as well. Consider the zonotope in $\mathbb{R}^{d-1}$ generated by $\Psi^\epsilon$, denoted as $Z^\epsilon(n,d-1)$ (when it is not confusing).

**Definition.** A w-membrane of a cubillage $Q$ on $Z(n,d)$ is a subcomplex $M$ of the fragmentation $Q^\oplus$ such that $M$ (regarded as a subset of $\mathbb{R}^d$) is bijectively projected by $\pi^\epsilon$ to $Z^\epsilon(n,d-1)$.

A w-membrane $M$ has facets (of dimension $d-1$) of two sorts, called H-tiles and V-tiles. Each H-tile is a horizontal facet of some fragment (viz. the section $S_h(C)$ of a cube $C$ in $Q$ at height $h \in [d-1]$). And V-tiles are vertical facets of some fragments $C^\oplus_h$ (see (5.2)).

**6.3 Acyclicity and the lattice structure of w-membranes** Let $C(n,d)$ denote the set of all cubes in $Z(n,d)$ (occurring in all cubillages there). For $C, C' \in C(n,d)$,
we say that \( C \) immediately precedes \( C' \) if \( (C')^{\text{rear}} \) and \( (C')^{\text{fr}} \) have a common facet. As a far generalization of the known acyclicity property for cubes in a cubillage, one can show the following

**Proposition 6.1** The directed graph \( \Gamma_{n,d} \) whose vertices are the cubes in \( C(n,d) \) and whose edges are the pairs \( (C,C') \) such that \( C \) immediately precedes \( C' \) is acyclic.

(As a consequence, the transitive closure of this “immediately preceding” relation forms a partial order on \( C(n,d) \).) This proposition enables us to construct a partial order on the set of fragments for a cubillage \( Q \), which in turn is used to show that the set of w-membranes in \( Q^\equiv \) forms a distributive lattice.

More precisely, given a cubillage \( Q \) on \( Z(n,d) \), consider fragments \( \Delta = C_i^\equiv \) and \( \Delta' = (C')_j^\equiv \) of \( Q^\equiv \). Let us say that \( \Delta \) immediately precedes \( \Delta' \) if the \( c \)-rear side of \( \Delta \) and the \( e \)-front side of \( \Delta' \) share a facet. In other words, either \( C \neq C' \) and \( \Delta^{\text{rear}} \cap (\Delta')^{\text{fr}} \) is a V-tile, or \( C = C' \) and \( j = i + 1 \). The following is important for us.

**Proposition 6.2** The directed graph \( \Gamma_Q^\equiv \) whose vertices are the fragments in \( Q^\equiv \) and whose edges are the pairs \( (\Delta,\Delta') \) such that \( \Delta \) immediately precedes \( \Delta' \) is acyclic.

Proofs of Propositions 6.1 and 6.2 will be given in the Appendix.

From Proposition 6.2 it follows that the transitive closure of the above relation on the fragments of \( Q^\equiv \) forms a partial order; denote it as \( (Q^\equiv, \prec) \).

Next we associate with a w-membrane \( M \) of \( Q \) the set \( Q^\equiv(M) \) of fragments in \( Q^\equiv \) lying in the region of \( Z(n,d) \) between \( Z^{\text{fr}} \) and \( M \). The constructions of \( \pi^\equiv \) and \( M \) lead to the following property: for fragments \( \Delta, \Delta' \) of \( Q^\equiv \), if \( \Delta \) immediately precedes \( \Delta' \) and if \( \Delta' \in Q^\equiv(M) \), then \( \Delta \in Q^\equiv(M) \) as well. This implies a similar property for fragments \( \Delta, \Delta' \) with \( \Delta \prec \Delta' \). So \( Q^\equiv(M) \) is an ideal of \( (Q^\equiv, \prec) \). One can check that a converse property is also true: any ideal of \( (Q^\equiv, \prec) \) is expressed as \( Q^\equiv(M) \) for some w-membrane \( M \) of \( Q \). Therefore,

(6.5) the set \( \mathcal{M}^w(Q) \) of w-membranes of a cubillage \( Q \) on \( Z(n,d) \) is a distributive lattice in which for \( M, M' \in \mathcal{M}^w(Q) \), the w-membranes \( M \land M' \) and \( M \lor M' \) satisfy \( Q^\equiv(M \land M') = Q^\equiv(M) \cap Q^\equiv(M') \) and \( Q^\equiv(M \lor M') = Q^\equiv(M) \cup Q^\equiv(M') \); the minimal and maximal elements of this lattice are \( Z^{\text{fr}} \) and \( Z^{\text{rear}} \), respectively.

Suppose that \( M \in \mathcal{M}^w(Q) \) is different from \( Z^{\text{fr}} \); then \( Q^\equiv(M) \neq \emptyset \). Take a maximal (relative to the order \( \prec \) in \( Q^\equiv \)) fragment \( \Delta \) in \( Q^\equiv(M) \). Then \( \Delta^{\text{c,\text{rear}}} \) is entirely contained in \( M \). Indeed, if a facet \( F \in \Delta^{\text{c,\text{rear}}} \) lies in \( Z^{\text{rear}} \), then \( F \) is automatically in \( M \). And if \( F \) is not in \( Z^{\text{rear}} \), then \( F \) is shared by \( \Delta^{\text{c,\text{rear}}} \) and \( (\Delta')^{\text{c,\text{fr}}} \) for another fragment \( \Delta' \). Hence \( \Delta \) immediately precedes \( \Delta' \), implying that \( \Delta' \) lies in the region between \( M \) and \( Z^{\text{rear}} \). Then \( F \) is in \( M \), as required.

For \( \Delta \) as above, the set \( Q^\equiv(M) - \{ \Delta \} \) is again an ideal of \( (Q^\equiv, \prec) \), and therefore it is expressed as \( Q^\equiv(M') \) for a w-membrane \( M' \). Moreover, \( M' \) is obtained from \( M \) by replacing the disk \( \Delta^{\text{c,\text{rear}}} \) by \( \Delta^{\text{c,\text{fr}}} \). We call the transformation \( M \mapsto M' \) the lowering flip in \( M \) using \( \Delta \), and call the reverse transformation \( M' \mapsto M \) the raising flip in \( M' \) using \( \Delta \). As a result, we obtain the following nice property.
Corollary 6.3 Let $M$ be a $w$-membrane of a cubillage $Q$. Then there exists a sequence of $w$-membranes $M_0, M_1, \ldots, M_k \in \mathcal{M}^w(Q)$ such that $M_0 = Z^r$, $M_k = M$ and for $i = 1, \ldots, k$, $M_i$ is obtained from $M_{i-1}$ by the raising flip using some fragment of $Q$.

6.4 Weakly $r$-separated collections via $w$-membranes

Now we throughout assume that $r$ is odd and $d = r + 2$. Consider a cubillage $Q$ on $Z = Z(n, d)$. Based on Theorem 1.2 and Corollary 6.3, we establish the main result of Sect. 6.

Theorem 6.4 For any $w$-membrane $M$ of a cubillage $Q$ on $Z(n, d)$, the set $V(M)$ of vertices of $M$ (regarded as subsets of $[n]$) constitutes a maximal by size weakly $r$-separated collection in $2^{[n]}$ (where, as before, $r$ is odd and $d = r + 2$). In particular, all $w$-membranes in $Q$ have the same number of vertices, namely, $w_{n,d-2}$ ($= s_{n,d-2}$).

Proof Let $M \in \mathcal{M}^w(Q)$ and consider a sequence $Z^r = M_0, M_1, \ldots, M_k = M$ as in Corollary 6.3. Let $\Delta_1, \ldots, \Delta_k$ be the fragments of $Q$ such that $M_i$ is obtained from $M_{i-1}$ by the raising flip using $\Delta_i$. The collection $V(Z^r)$ is weakly $r$-separated (as it is strongly $r$-separated, cf. (3.3)), and our aim is to show that if $V(M_{i-1})$ is weakly $r$-separated, then so is $V(M_i)$.

To show this, consider $w$-membranes $M, M'$ of $Q$ such that $M'$ is obtained from $M$ by the raising flip using a fragment $\Delta \in Q^e$. Let $\Delta = C^r_h$ for a cube $C = (X \mid T = (p(1) < \ldots < p(d)))$ and $h \in [d]$. By explanations in Sect. 3, $C^r$ and $C^{r\text{rear}}$ differ by exactly two vertices; namely, $V(C^r) = V(C^{r\text{rim}}) \cup \{t_C\}$ and $V(C^{r\text{rear}}) = V(C^{r\text{rim}}) \cup \{h_C\}$, where $t_C = Xp(2)p(4)\ldots p(d-1)$ and $h_C = Xp(1)p(3)\ldots p(d)$ (cf. (3.3)). Define $R$ to be the set of vertices of $C^{r\text{rim}}$ occurring in $\Delta$, and let $r' := (d-1)/2$. We consider three cases.

Case 1: $h \leq r'$. Since the vertices of $\Delta$ are formed by the sections $S_{h-1}(C)$ and $S_h(C)$,

$$V(\Delta) = (X \mid (h_{h-1})) \cup (X \mid (h_h)) \quad \text{and} \quad R \subseteq V(\Delta^r) \cup V(\Delta^{r\text{rear}})$$

(cf. (3.1)). Also $V(\Delta^r) \subseteq V(\Delta^{r\text{fr}})$ and $V(\Delta^{r\text{rear}}) \subseteq V(\Delta^{r\text{fr}})$. If $h < r'$, then all vertices of $\Delta$ belong to $C^{r\text{rim}}$; this implies $V(\Delta^{r\text{fr}}) = R = V(\Delta^{r\text{rear}})$. And if $h = r'$, then the only vertex of $\Delta$ not in $R$ is $t_C$. Since $t_C \in V(C^r)$, $t_C$ belongs to $\Delta^{r\text{fr}}$. But $t_C$ also lies in the upper facet $S_{r'}(C)$ (in view of $|p(2)p(4)\ldots p(d-1)| = r'$), and this facet is included in $\Delta^{r\text{fr}}$. Hence $t_C \in \Delta^{r\text{fr}} \cap \Delta^{r\text{rear}}$, implying $V(\Delta^{r\text{fr}}) = V(\Delta^{r\text{rear}})$.

Case 2: $h \geq r' + 2$. This is “symmetric” to the previous case. If $h > r' + 2$, then all vertices of $\Delta$ belong to $C^{r\text{rim}}$, implying $V(\Delta^{r\text{fr}}) = R = V(\Delta^{r\text{rear}})$. And if $h = r' + 2$, then $\Delta^{r\text{fr}}$ includes the lower facet $S_{r'+1}(C)$, which in turn contains the vertex $h_C$ (since $|p(1)p(3)\ldots p(d)| = r' + 1$). Also $h_C \in V(C^{r\text{rear}})$ implies $h_C \in V(\Delta^{r\text{rear}})$, and we again obtain $V(\Delta^{r\text{fr}}) = V(\Delta^{r\text{rear}})$.

Thus, in both cases the raising flip $M \mapsto M'$ using $\Delta$ does not change the vertex set of the $w$-membrane in question.

Case 3: $h = r' + 1$. This case is most important. Now the lower facet $S_{h-1}(C)$ of $\Delta$ contains $t_C$, while the upper facet $S_{h-1}(C)$ contains $h_C$. Hence $t_C \in V(\Delta^{r\text{fr}})$ and $h_C \in V(\Delta^{r\text{rear}})$. On the other hand, neither $t_C$ belongs to $\Delta^{r\text{fr}}$ ($= \Delta^{r\text{fr}} \cup S_{r'}(C)$), nor $h_C$ belongs to $\Delta^{r\text{fr}}$ ($= \Delta^{r\text{fr}} \cup S_{r'}(C)$).
It follows that \( V(\Delta, \text{rear}) = (V(\Delta, \text{fr}) - \{t_C\}) \cup \{h_C\} \), and therefore the raising flip \( M \mapsto M' \) using \( \Delta \) replaces \( t_C \) by \( h_C \), while preserving the other vertices of the w-membrane. Note also that the vertices of \( \Delta \) different from \( t_C, h_C \) just form the collection of sets \( XS \) such that \( S \) runs over \( \mathcal{N}(P, \bar{Q}) \), the set of neighbors of \( P := p(2)p(4) \ldots p(d-1) \) and \( \bar{Q} := p(1)p(3) \ldots p(d) \).

Now applying Theorem \( \mathcal{L}2 \) to \( W := V(M) \), \( X, \bar{P}, \bar{Q} \) and \( U := X\bar{P} \), we conclude that \( W(M') \) is weakly \( r \)-separated, as required.

This completes the proof of the theorem.

It should be noted that any w-membrane in a cubillage on \( Z(n, 3) \) can be expressed as a quasi-combined tiling in the planar zonogon \( Z(n, 2) \), and in this particular case, the statement of Theorem \( 6.4 \) with \( r = 1 \) is equivalent to Theorem 3.4 in [3].

Also Theorem \( 6.4 \) together with (6.5) implies the following property of the set \( W_{n,r} * \) of representable maximal by size weakly \( r \)-separated collections in \( 2^{|n|} \) (defined in the Introduction).

**Corollary 6.5** \( W_{n,r} * \) is a poset with the unique minimal element \( V(Z^\text{fr}(n, r+2)) \) and the unique maximal element \( V(Z^\text{rear}(n, r+2)) \) in which any two neighboring elements are linked by a (raising or lowering) combinatorial flip.

A natural question is whether any two members of the set \( W_{n,r} = \) (including \( W_{n,r} * \)) can be connected by a sequence of flips. This is strengthened in the following

**Conjecture 1** Let \( r \) be odd and \( n > r + 1 \). Then any maximal by size weakly \( r \)-separated collection \( W \subseteq 2^{|n|} \) is realizable, i.e., there exists a cubillage \( Q \) on \( Z(n, r+2) \) and a w-membrane \( M \) in (the fragmentation of) \( Q \) such that \( V(M) = W \).

This together with Theorem \( 6.4 \) would imply \( W_{n,r} * = W_{n,r} = \). The above assertion has been proved for \( r = 1 \); see Theorem 3.5 in [3].

**Remark 1.** When \( d \) is even, we also can argue as in Sects. \( 6.1 \) \( 6.3 \) and consider the set (lattice) \( M^w(Q) \) of w-membranes \( M \) in (the fragmentation \( Q^\text{fr} \) of) a cubillage \( Q \) on \( Z = Z(n, d) \) (since \( 6.5 \) remains valid). However, the size of \( V(M) \) can now exceed the value \( s_{n,d-2} \). For example, let \( n = d \), i.e., \( Z \) is the cube \( C = (\emptyset | [d]) \). All vertices of \( C \) except for \( t_C \) and \( h_C \) are in \( C^\text{rim} \), and we have \( s_{d,d-2} = |V(C^\text{fr})| = |V(C^\text{rear})| = |V(C^\text{rim})| + 1 \). On the other hand, the evenness of \( d \) implies that both \( t_C \) and \( h_C \) belong to the section \( S = S_{d/2}(C) \) of \( C \). Take the w-membrane \( M \) containing \( S \) (which exists and is unique). Then \( V(M) \) contains all vertices of \( C \), and we obtain \( |V(M)| = |V(C^\text{rim}) + 2| = s_{d,d-2} + 1 \).

7 The non-purity phenomenon for the weak \( r \)-separation

Suppose that \( R \) is a symmetric binary relation on elements of a set \( N \) and let \( G \) be the graph whose vertices are the elements of \( N \) and whose edges are the pairs \( \{u, v\} \) of distinct vertices subject to \( uRv \). Let \( C \) be the set of cliques in \( G \) (where a clique is meant to be an inclusion-wise maximal subset of vertices of which any two are connected by edge). Then \( C \) is said to be pure if all cliques of \( G \) have the same size.
Recall that for an odd \( r \) and \( n > r \), \( W_{n,r} \) denotes the set of all maximal by inclusion weakly \( r \)-separated collections in \( 2^{[n]} \). It was shown in [2] that \( W_{n,1} \) is pure for any \( n \) (which affirmatively answers Leclerc-Zelevinsky’s conjecture in [8] on maximal weakly separated set-systems). In other words, \( W_{n,1} = W_{n,1}^= \).

In this section we show that \( W_{n,r} \) need not be pure when \( n = 6 \) and \( r = 3 \). In fact, we borrow a construction from [5, Sect. 3] where it is used to demonstrate the non-purity behavior for strongly 3-separated set-systems. (Note that by a general result due to Galashin and Postnikov [7],

\[
(7.1) \text{ the set } S_{n,r'} \text{ of all inclusion-wise maximal strongly } r'-\text{separated collections in } 2^{[n]}
\]

is pure if and only if \( \min\{r', n - r'\} \leq 2 \), among all integers \( r', n \);

so \( (n, r') = (6, 3) \) is the smallest case when the non-purity of \( S_{n,r'} \) happens.

To construct a non-pure set-system of our interest, consider the zonotope \( Z = Z(6,4) \). Note that the set \( V(Z) \) of vertices of (the boundary of) \( Z \) consists of all intervals and all 2-interval corteges containing 1 or 6. (This relies on two observations: (a) any \( A \subseteq [6] \) is a vertex of some cubillage in \( Z \), and therefore \( V(Z) \cup \{A\} \) is (strongly) 3-separated, by [2,2]; and (b) the intervals and the 2-intervals containing 1 or 6 are exactly those subsets of \([6]\) that are \( r' \)-interlaced with any subset of \([6]\), where \( r' \leq 4 \).)

A direct enumeration shows that \( |V(Z)| = 52 \). Therefore, \( 2^6 - 52 = 12 \) subsets of the set \([6]\) are not in \( V(Z) \); these are:

\[
(7.2) \quad 24, 245, 25, 235, 35, 135, 1356, 136, 1346, 146, 1246, 246.
\]

(Recall that \( a \cdots b \) stands for \( \{a, \ldots, b\} \).) Let \( A_i \) denote \( i \)-th member in this sequence (so \( A_1 = 24 \) and \( A_{12} = 246 \)). Form the collection

\[
\mathcal{A} := \text{Sp}(Z) \cup \{A_1, A_5, A_9\}.
\]

It consists of 52 + 3 = 55 sets, whereas the number \( s_{6,3} = w_{6,3} \) is equal to \( \binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} = 57 \). Now the non-purity of \( W_{6,3} \) is implied by the following

**Lemma 7.1** \( \mathcal{A} \) is a maximal weakly 3-separated collection in \( 2^{[6]} \).

**Proof** As mentioned above, any two \( X \in V(Z) \) and \( Y \in \mathcal{A} \) are 3-separated, and therefore they are weakly 3-separated. Observe that \( |A_{i-1} \triangle A_i| = 1 \) for any \( 1 \leq i \leq 12 \) (where \( A_0 := A_{12} \) and \( A \triangle B \) stands for \( (A - B) \cup (B - A) \)). Then any \( A, A' \in \{A_1, A_5, A_9\} \) satisfy \( |A \triangle A'| \leq 4 \). This implies that \( A \) and \( A' \) are 3-separated. Therefore, the collection \( \mathcal{A} \) is weakly 3-separated.

The maximality of \( \mathcal{A} \) follows from the observation that adding to \( \mathcal{A} \) any member of \( \{A_i : 1 \leq i \leq 12, i \neq 1, 5, 9\} \) would violate the weak 3-separation. Indeed, a routine verification shows that \( A_1 \) is not weakly 3-separated from any of \( A_6, A_7, A_8 \), and similarly for \( A_5 \) and \( \{A_{10}, A_{11}, A_{12}\} \), and for \( A_9 \) and \( \{A_2, A_3, A_4\} \).

**Remark 2.** To visualize a verification in the above proof, one can use the circular diagram in Fig. 2 where the sets from the sequence in (7.2) are disposed in the cyclic order. Here the sets \( A_1, A_5, A_9 \) are drawn in boxes and connected by lines with those sets where the weak 3-separation is violated.
In conclusion note that in light of the complete characterization for the strong separation case in \((7.1)\), it is tempting to characterize all pairs \((n, r)\) (with \(r\) odd) for which \(W_{n,r}\) is pure. In particular, this is so when \(r = 1\) (by [2]), and it is not difficult to check that \(W_{n,r}\) is pure if \(n - r \leq 2\). We conjecture that the remaining cases of \((n, r)\) give the non-purity (similarly to \((7.1)\)).

**Conjecture 2** For an odd \(r\) and \(n > r\), \(W_{n,r}\) is pure if and only if \(\min\{r, n-r\} \leq 2\).

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Appendix: Proofs of two propositions on acyclicity

Proof of Proposition 6.1 Let $C$ immediately precede $C'$, and let the cubes $C, C'$ and the facet $F := C^{-} \cap (C')^{+}$ be of the form $(X | T), (X' | T')$ and $(\tilde{X} | \tilde{T})$, respectively. Then $T = \tilde{T} \alpha$ and $T' = \tilde{T} \beta$ for some $\alpha, \beta \in [n]$. Four cases are possible (as illustrated in Fig. 3):

(i) $X = X' = \tilde{X}$;
(ii) $X, X', \tilde{X}$ are different (then $\tilde{X} = X \alpha = X' \beta$);
(iii) $X \neq X' = \tilde{X}$ (then $\tilde{X} = X \alpha$);
(iv) $X' \neq X = \tilde{X}$ (then $\tilde{X} = X' \beta$).

Let us associate with a cube $C'' = (X'' | T'')$ a label $\omega(C'') \in \{0, 1, 2\}$ by the following rule:

(*) $\omega(C'') = 0$ if $n \neq X'', T''$; $\omega(C'') = 1$ if $n \in T''$; $\omega(C'') = 2$ if $n \in X''$.

The following observation is the key.

Claim For $C, C'$ as above, $\omega(C) \leq \omega(C')$.

Proof of the Claim We may assume that $\omega(C) \neq \omega(C')$. Then $n$ belongs to neither $\tilde{T}$ nor $X \cap X'$. This implies that either $\alpha = n$ or $\beta = n$ (in view of $\tilde{T} = T - \alpha = T' - \beta$). Note that, as explained in Sect. [3] (in the proof of [32]),

(A.1) for a cube $C$, a facet $F_i(C) (G_i(C))$ is in $C^{+}$ if and only if $d - i$ is even (resp. odd).

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Using this for $C$ and $F$ as above and considering the inclusion $F \subset C^{\text{rear}}$, one can conclude that if $\alpha = n$, then the root $X$ of $F$ and the root $X$ of $C$ are different (taking into account that $n$ is the maximal element in $T$). In its turn, $F \subset (C')^{\text{fr}}$ implies that if $\beta = n$, then $X = X'$. This leads to the following:

(A.2) $\alpha = n$ is possible only in cases (ii) and (iii), whereas $\beta = n$ is possible only in cases (i) and (iii).

In particular, case (iv) is impossible at all (when $\omega(C) \neq \omega(C')$). As to the other three cases, we obtain from (A.2) that

(a) in case (i), $\omega(C) = 0$ and $\omega(C') = 1$ (since $n = \beta \in T'$);

(b) in case (ii), $\omega(C) = 1$ (since $n = \alpha \in T$) and $\omega(C') = 2$ (since $\tilde{X} = X\alpha = X'\beta$ implies $\alpha \in X'$);

(c) in case (iii), if $\alpha = n$ then $\omega(C) = 1$ and $\omega(C') = 2$ (since $X' = X\alpha$), and if

$\beta = n$ then $\omega(C) = 0$ and $\omega(C') = 1$.

Thus, $\omega(C) \leq \omega(C')$ holds in all cases, as required. \[\Box\]

Now we finish the proof of the proposition by induction on $n$. This is trivial when $n = d$, so assume that $n > d$ and that the assertion is valid for $(n', d')$ with $n' < n$.

Suppose, for a contradiction, that $\Gamma_{n,d}$ has a directed cycle $C = (C_0, C_1, \ldots, C_k = C_0)$ (where each $C_i$ immediately precedes $C_{i+1}$). Then the Claim implies that $\omega(C_i)$ is the same number $q$ for all $i$. Consider three cases (where $C_i = (X_i | T_i)$).

Case 1: $q = 0$. Then $C$ is a directed cycle in $\Gamma_{n-1,d}$, contrary to the inductive assumption.

Case 2: $q = 2$. Define $X'_i := X_i - n$ and $C'_i := (X'_i | T_i)$, $i = 0, \ldots, k$. Then each $C'_i$ is a cube in $Z(n-1,d)$, and the sequence $C'_0, C'_1, \ldots, C'_k$ forms a directed cycle in $\Gamma_{n-1,d}$; a contradiction.

Case 3: $q = 1$. Define $T'_i := T_i - n$ and $C'_i := (X'_i | T'_i)$, $i = 0, \ldots, k$. Then each $C'_i$ can be regarded as a cube in $Z(n-1,d-1)$ (in view of $|T'_i| = d-1$). Considering (A.2) and using the fact that $n$ is the maximal element in $T_i$, one can conclude that if $(Y | U)$ is a facet in $C'_i^{\text{fr}}$, then $(Y \cap [n-1 | U \cap [n-1])$ is a facet in $(C'_i)^{\text{rear}}$, and similarly for facets in $C'_i^{\text{fr}}$ and $(C'_i)^{\text{fr}}$. Then the fact that $C_i^{\text{rear}} \cap C_{i+1}^{\text{fr}}$ is a facet implies that $(C'_i)^{\text{fr}} \cap (C'_{i+1})^{\text{rear}}$ is a facet as well. This means that $C'_{i+1}$ immediately precedes $C'_i$. Therefore, the sequence $C'_k, C'_{k-1}, \ldots, C'_1, C'_0$ forms a directed cycle in $\Gamma_{n-1,d-1}$; a contradiction.

This completes the proof of the proposition. \[\Box\]

**Proof of Proposition 6.2** For a fragment $\Delta = C_\infty^\text{fr}$ of a cube $C = (X | T)$, denote $|X| + h - 1/2$ by $\ell(\Delta)$, called the **height** of $\Delta$.

Suppose that there exist fragments $\Delta_0, \Delta_1, \ldots, \Delta_k = \Delta_0$ forming a cycle in $\Gamma_Q$. Consider two consecutive fragments $\Delta = \Delta_{i-1}$ and $\Delta' = \Delta_i$. Then the sides $\Delta_i^{\text{rear}}$ and $\Delta_i^{\text{fr}}$ share a facet $F$, and either (a) $F$ is a vertical facet of both (in terminology of (6.2)),

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or (b) $F$ is the upper facet of $\Delta$ and the lower facet of $\Delta'$. Obviously, $\ell(\Delta') = \ell(\Delta)$ in case (a), and $\ell(\Delta') = \ell(\Delta) + 1$ in case (b). This implies

$$\ell(\Delta_0) \leq \ell(\Delta_1) \leq \cdots \leq \ell(\Delta_{k-1}) \leq \ell(\Delta_0).$$

Then all fragments $\Delta_i$ have the same height, and therefore each pair of consecutive fragments shares a vertical facet. But this means that the sequence of cubes containing these fragments forms a cycle in the graph $\Gamma_{n,d}$, contrary to Proposition 6.1. \qed