The Locally Gaussian Partial Correlation

Håkon Otneim
Dag Tjøstheim

Abstract

It is well known that the dependence structure for jointly Gaussian variables can be fully captured using correlations, and that the conditional dependence structure in the same way can be described using partial correlations. The partial correlation does not, however, characterize conditional dependence in many non-Gaussian populations. This paper introduces the local Gaussian partial correlation (LGPC), a new measure of conditional dependence. It is a local version of the partial correlation coefficient that characterizes conditional dependence in a large class of populations. It has some useful and novel properties besides: The LGPC reduces to the ordinary partial correlation for jointly normal variables, and it distinguishes between positive and negative conditional dependence. Furthermore, the LGPC can be used to study departures from conditional independence in specific parts of the distribution. We provide several examples of this, both simulated and real, and derive estimation theory under a local likelihood framework. Finally, we indicate how the LGPC can be used to construct a powerful test for conditional independence, which, again, can be used to detect Granger causality in time series.

1 Introduction

Estimation of conditional dependence and testing for conditional independence are extremely important topics in classical as well as modern statistics. In the last two decades, for instance, there has been a very intense development using conditional dependence in probabilistic network theory. This comes in addition to conditional multivariate time series analysis and copula analysis.

For jointly Gaussian variables, conditional dependence is measured by the partial correlation coefficient. Given three jointly Gaussian stochastic variables $X_1$, $X_2$ and $X_3$, $X_1$ and $X_2$ are conditionally independent given $X_3$ if and only if the partial correlation between $X_1$ and $X_2$ given $X_3$ is equal to zero. There is a rich literature on applications of the partial correlation coefficient to Gaussian networks, path analysis and causality. The Gaussian assumption is strict, however, and it is easy to find non-Gaussian examples where the partial correlation function completely fails in describing conditional dependence. We will give some explicit examples of that in Section 4.3.

The purpose of this paper is to introduce a new concept for measuring conditional dependence. This concept retains all the properties of the ordinary partial correlation in the Gaussian case, but seeks to avoid the weaknesses of this measure in the non-Gaussian case. We do this by fitting a family of Gaussian distributions to a given continuous multivariate distribution and exploiting the simple conditioning rules of the Gaussian distribution locally. This approach produces a new measure of conditional dependence, the local Gaussian partial correlation (LGPC) which, being directly related to the ordinary partial correlation, is easy to interpret, and reduces to the very same ordinary partial correlation in the Gaussian case. Moreover, it distinguishes between positive and negative conditional dependence whereas competing non-linear measures report only the strength of the conditional dependence on some non-negative scale.

The local view gives much more flexibility. It allows the conditional dependence to be stronger or weaker in certain regions of a multivariate distribution than in others. This is of particular interest in finance, where description of tail behavior is important. The local aspect also makes it possible to focus tests of conditional independence to selected parts of the distribution, which may potentially increase the power of the test.

The local approach has been shown to be advantageous in other areas of statistics, such as the measurement of nonlinear dependence, density estimation and spectral analysis, see for instance Tjøstheim and Hufthammer

*Corresponding author. Department of Business and Management Science, NHH Norwegian School of Economics. Helleveien 30, 5045 Bergen, Norway. hakon.otneim@nhh.no
†University of Bergen
The present paper, then, represents the first attempt to model conditional dependence locally. There are several other approaches to modeling conditional dependence and testing for conditional independence in the non-Gaussian case. For the most part, they revolve around test statistics in conditional independence tests. Such statistics are usually based on measures of distance between some property of the sample and the corresponding property of the population under the null hypothesis of conditional independence. As a consequence, the conditional dependence measures are not always easy to interpret, and can, as mentioned above, not distinguish between negative and positive conditional dependence. We present a comprehensive set of references to this literature in Section 5.

A quick summary of the paper is as follows: Partial correlation in the global case can be defined in several ways, which all coincide in the Gaussian case. We will examine this in the following section and make a choice of a measure that is most convenient for localization. The LGPC is defined in Section 3 by using a distributional local Gaussian approach, where it will be seen that the LGPC can be introduced at several levels of computational and theoretical complexity. Visualization is important when studying local conditional dependence, which involves at least three scalar variables. Several simulated and a real data example are given, in particular cases where the ordinary global partial correlation fails completely. Estimation theory comes next in Section 4, whereas testing is treated in Section 5, including a fairly comprehensive comparison with existing tests.

2 Conditional and partial correlation

Let \( X = (X_1, \ldots, X_p) \) be a random vector. We will in this paper denote by \( (X^{(1)}, X^{(2)}) \) a partition of \( X \), and in our treatment below, \( X^{(1)} \) will consist of the first and second component in \( X \) so that \( X^{(1)} = (X_1, X_2) \), and \( X^{(2)} \) will contain the remaining components: \( X^{(2)} = (X_3, \ldots, X_p) \). In any case, we will assume that the mean vector \( \mu \) and covariance matrix \( \Sigma \) of \( X \) exist, and we will partition them correspondingly, writing

\[
\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},
\]

where \( \Sigma_{11} \) and \( \Sigma_{22} \) are the covariance matrices of \( X^{(1)} \) and \( X^{(2)} \) respectively. There are two main concepts of correlation when \( X^{(2)} \) is given, the partial and the conditional correlation. They coincide in several joint distributions, among them the Gaussian. We will in this section provide some details about this distinction and explain our preference for using the partial correlation as a starting point when defining the LGPC.

The partial variance-covariance matrix of \( X^{(1)} \) given \( X^{(2)} \) is

\[
\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}.
\]

Note that in the Gaussian case \( \Sigma_{11.2} \) is the covariance matrix in the conditional (Gaussian) distribution of \( X^{(1)} \) given \( X^{(2)} \). Similarly, if \( X^{(1)} = (X_1, X_2) \), the partial correlation between \( X_1 \) and \( X_2 \) is then defined as

\[
\rho_{11.2} = \frac{\sigma_{11.2}^{(1,2)}}{\sqrt{\sigma_{11.1}^{(1,1)} \sigma_{11.2}^{(2,2)}}},
\]

where \( \sigma_{11.2}^{(i,j)} \) refers to the element in position \((i, j)\) in the partial covariance matrix \( \Sigma_{11.2} \). This, in turn, in the Gaussian case, can be identified with the correlation matrix in the conditional distribution of \( X^{(1)} \) given \( X^{(2)} \), and this fact, in particular equations (2) and (3) will serve as the starting point for our definition of local partial correlation.

The partial variance or covariance of \( X^{(1)} \) given \( X^{(2)} \) can also be considered as the variance or covariance between residuals of projections of \( X_1 \) and \( X_2 \) on the linear space spanned by \( X^{(2)} \).
\[ \sigma_{11,2} = \text{Cov} \left( X_i - \tilde{X}_i(X^{(2)}), X_j - \tilde{X}_j(X^{(2)}) \right), \quad i, j = 1, 2, \]

where \( \tilde{X}_i(X^{(2)}) = E(X_i) + \Sigma_{X_i, X^{(2)}} \Sigma_{X^{(2)}}^{-1} (X^{(2)} - E(X^{(2)})) \) is the projection of \( X_i \) on \( X^{(2)} \). The expression in terms of residuals is often taken as the starting point when defining the global partial correlation, see e.g. Lawrance (1976). Moreover, using eq. (3), it can be shown, that all of the partial correlations between two variables \( X_i \) and \( X_j \) in a set of variables \( V \) having correlation matrix \( \Omega = \{ \omega_{ij} \} = \{ \rho_{X_i, X_j} \} \) and given the other variables in \( V \) (i.e. given \( V \setminus \{ X_i, X_j \} \)) is obtained as

\[ \rho_{X_i, X_j \setminus \{ X_i, X_j \}} = - \frac{p_{ij}}{\sqrt{p_{ii} p_{jj}}} \]

where \( p_{ij} \) is element \( (i, j) \) in the precision matrix \( \Sigma^{-1} \).

The conditional covariance of \( X_1 \) and \( X_2 \) given \( X^{(2)} \) is defined by

\[ \text{Cov}(X_1, X_2 | X^{(2)}) = E \left( (X_1 - E(X_1 | X^{(2)}))(X_2 - E(X_2 | X^{(2)})) | X^{(2)} \right), \]

and this is the covariance between \( X_1 \) and \( X_2 \) in the conditional distribution of \( (X_1, X_2) \) given \( X^{(2)} \). The conditional covariance matrix can be written

\[ \Sigma_{11|2} = \begin{pmatrix} \sigma_{11|X^{(2)}} & \sigma_{12|X^{(2)}} \\ \sigma_{21|X^{(2)}} & \sigma_{22|X^{(2)}} \end{pmatrix} \]

resulting in the conditional correlation

\[ \rho_{12|X^{(2)}} = \frac{\sigma_{12|X^{(2)}}}{\sqrt{\sigma_{11|X^{(2)}}} \sqrt{\sigma_{22|X^{(2)}}}}, \]

which is the correlation between \( X_1 \) and \( X_2 \) in the conditional distribution of \( (X_1, X_2) \) given \( X^{(2)} \), thus coinciding with \( \rho_{11,2} \) of eq. (3) in the Gaussian case.

Baba, Shibata, and Sibuya (2004) give the following result for the relationship between conditional and partial quantities in a more general situation.

**Corollary 1.** For any random vectors \( X^{(1)} = (X_1, X_2) \) and \( X^{(2)} = (X_3, \ldots, X_p) \) the following two conditions are equivalent:

(i) \( E(X^{(1)} | X^{(2)}) = \alpha + BX^{(2)} \) for a vector \( \alpha \) and a matrix \( B \), and \( \Sigma_{11|2} \) independent of \( X^{(2)} \).

(ii) \( \Sigma_{11|2} = \Sigma_{11|2} \) almost surely.

Either of the conditions in Corollary 1 is, according to Baba, Shibata, and Sibuya (2004), valid not only for the multivariate normal, but also for elliptical, multivariate hypergeometric, multinomial and Dirichlet distributions. Moreover, for the multivariate Gaussian, zero partial correlation (or conditional correlation) is equivalent to conditional independence between \( X_1 \) and \( X_2 \) given \( X^{(2)} \). We will now use such results to construct a generalized partial correlation function, which characterizes conditional dependence in a broader class of distributions.

### 3 The local Gaussian partial correlation

#### 3.1 Definition

In all of the following it is assumed that the \( X \)-variables are continuous and have a density function. In order to localize the partial correlation we need to look at the concept of local Gaussian approximations. Let \( f \)
be a multivariate density function. Given a point \( x \), we can approximate \( f \) in a neighborhood of \( x \) by a multivariate Gaussian density

\[
\psi(x, v) = \frac{1}{(2\pi)^{p/2} |\Sigma(x)|^{1/2}} \exp \left\{ -\frac{1}{2} (v - \mu(x))^T \Sigma^{-1}(x) (v - \mu(x)) \right\},
\]

where \( x = (x_1, \ldots, x_p) \), \( \mu(x) = \{\mu_i(x)\} \) and \( \Sigma(x) = \{\sigma_{ij}(x)\} \) for \( i, j = 1, \ldots, p \). Moving to another point \( y \), there is another (generally different) Gaussian approximation \( \psi(y, v) \). In this way we approximate \( f \) by a family of multivariate Gaussian densities defined by a set of smooth parameter functions \( \{\mu(x), \Sigma(x)\} \), and if \( f \) is itself a Gaussian density, then the parameter functions collapse to constants corresponding to the true parameter values, and \( \psi(x) \equiv f(x) \). Hjort and Jones (1996) provide the general framework for estimating such parameter functions non-parametrically from a given data set using a local likelihood procedure, and the basic idea in the following treatment is to replace the components in the partial covariance matrix (4) on the marginally normal scale by considering the marginals separately, and by fixing \( \mu_i(z) = 0 \) and \( \sigma_i(z) = 1 \) for \( i = 1, \ldots, p \), which leaves a matrix \( \mathbf{R}(z) = \{\rho_{ij}(z)\}_{i,j} \) of local correlations to be estimated. An alternative, used by Jordanger and Tjøstheim (2017) is to allow \( \mu_i(z) \) and \( \sigma_i(z) \) to depend on \( z \).

In the following, we will not always distinguish between \( Z_i \) and \( Z_{i,n} \). In fact, we conclude our asymptotic analysis in Section 4.2 by showing that under certain technical conditions, the error made by estimating \( \mathbf{R}(z) \) using the empirically transformed variables \( Z_{i,n} \), instead of \( Z_i \), is smaller in the limit than the estimation error made when estimating the local correlations themselves.

Define the random vector \( \mathbf{Z} \) by this transformation of \( \mathbf{X} = (X^{(1)}, X^{(2)}) = (X_1, X_2, \ldots, X_p) \) to marginal standard normality:

\[
\mathbf{Z} = \left( \Phi^{-1}(F_{X_1}(X_1)), \Phi^{-1}(F_{X_2}(X_2)), \ldots, \Phi^{-1}(F_{X_p}(X_p)) \right).
\]

Assume then that the probability density function of \( \mathbf{Z} \) can be written on the local simplified Gaussian form (cf. Otneim and Tjøstheim 2017, 2018)

\[
f_{\mathbf{Z}}(\mathbf{z}) = \psi(\mathbf{z}, \mathbf{R}(\mathbf{z})) = \frac{1}{[2\pi |\mathbf{R}(\mathbf{z})|]^{1/2}} \exp \left\{ -\frac{1}{2} \mathbf{z}^T \mathbf{R}^{-1}(\mathbf{z}) \mathbf{z} \right\},
\]

where \( \mathbf{R}(\mathbf{z}) \) is the local correlation matrix with \( \mathbf{R}(\mathbf{z}) = \{\rho_{ij}(z_i, z_j)\}_{i,j} \) and \( \rho_{ii}(z_i) = 1 \), and where the means and standard deviations have been set and fixed to 0 and 1 correspondingly, as indicated above. This means that we can use the terms local correlation and local covariance interchangeably within this family of distributions. We will in this paper also refer to \( \mathbf{X} \) and its probability density function \( f_{\mathbf{X}} \) as being on the \( x \)-scale, and to \( \mathbf{Z} \) and its probability density function \( f_{\mathbf{Z}} \) as being on the \( z \)-scale. For further discussion of the simplified \( z \)-representation we refer to Otneim and Tjøstheim 2017.
Denote by \((Z^{(1)}, Z^{(2)})\) the partitioning of \(Z\) corresponding to the partitioning \((X^{(1)}, X^{(2)})\) of \(X\). A natural definition of the local partial covariance matrix of \(Z^{(1)}|Z^{(2)}\) is the local version of eq. 2:

\[
\Sigma_{11,2}(z) = R_{11}(z^{(1)}) - R_{12}(z)R_{22}^{-1}(z^{(2)})R_{21}(z),
\]

which is a 2 \times 2 matrix, and we define the local Gaussian partial correlation \(\alpha(z)\) of \(Z^{(1)}\) given \(Z^{(2)}\) in accordance with the ordinary (global) partial correlation provided by eq. 3:

\[
\alpha(z) = \frac{\{\Sigma_{11,2}(z)\}_{12}}{\{\Sigma_{11,2}(z)\}_{11}^{1/2} \{\Sigma_{11,2}(z)\}_{22}^{1/2}}.
\]

or, if \(Z^{(2)} = Z_3\) is scalar,

\[
\alpha(z) = \rho_{12|3}(z_1, z_2|z_3) = \frac{\rho_{12}(z_1, z_2) - \rho_{13}(z_1, z_3)\rho_{23}(z_2, z_3)}{\sqrt{1 - \rho_{13}^2(z_1, z_3)}\sqrt{1 - \rho_{23}^2(z_2, z_3)}}.
\]

The monotone relation between the \(x\)-scale and the \(z\)-scale is given in eq. 5. The \(z\)-representation is in a sense analogous to a copula representation which is based directly on uniform variables \(U_i = F_{X_i}^{-1}(X_i)\) (sometimes referred to as the \(u\)-scale), but avoids the problems of unbounded densities on bounded support that often occur in common copula models. It is of course possible to introduce an LGPC \(\alpha(x)\) directly on the \(x\)-scale, but this representation is in many ways harder to handle both computationally and asymptotically. For a multivariate Gaussian distribution, we have that \(\alpha_X(x) = \alpha_Z(z) = \alpha\). In the remainder of this paper, we will mainly write in terms of the \(z\)-representation using the LGPC \(\alpha(z) = \alpha_Z(z)\), but when we write the local partial correlation between \(X_1\) and \(X_2\) given \(X_3 = x_3\) at the point \((x_1, x_2, x_3)\), this is simply \(\alpha(z)\) inserted with \(z_i = \Phi^{-1}(F_{X_i}(x_i)), i = 1, \ldots, p\).

### 3.2 Properties

The local Gaussian partial correlation is clearly closely related to the partial correlation between jointly normally distributed variables. We see this also from the following properties that we will establish next:

1. The LGPC \(\alpha(z)\) satisfies \(-1 \leq \alpha(z) \leq 1\).
2. If \(X\) is jointly normally distributed, then the LGPC coincides with the ordinary (global) partial, and thus conditional correlation.
3. For stochastic vectors having joint density function on \(z\)-scale of the form 6, the LGPC \(\alpha(z)\) is identically equal to zero if and only if \(X_1\) and \(X_2\) are conditionally independent given \(X^{(2)}\). Note that \(\alpha(z) \equiv 0\) if and only if \(\alpha(x) \equiv 0\).
4. The LGPC is invariant with regard to a set of monotone transformations \(Y = h(X) = (h_1(X_1), \ldots, h_p(X_p))\).

Property 1 is trivially true if \(R(z)\) is a valid correlation matrix. By removing the \(z\)-dependence in the local correlations, it follows immediately from eq. 6 and from the results referred to in Section 2 that property 2 holds.

To see that conditional independence between \(X_1\) and \(X_2\) given \(X^{(2)}\) is equivalent to \(\alpha(z) \equiv 0\), or equivalently \(\alpha(x) \equiv 0\), we need to follow a few simple steps that would also work in the global Gaussian case. Working on the standard normal \(z\)-scale and assuming that \((z_1, z_2, z_3)\) is jointly locally normally distributed having density function \(f_Z(z)\) as given by 6, it follows from Otneim and Tjøstheim (2018) that we can calculate conditional distributions in the same way as in the global Gaussian case. This is indeed an important advantage of using the Gaussian family as local approximant as compared to other families of parametric distributions. For example, the conditional density of \(z_1|z_3\) at the point \(z_1\) is given by
where
\[ \mu_{1|3}(z_1) = R_{12}(z_1, z_3)R_{22}(z_3)^{-1}z_3 \quad \text{and} \quad \sigma_{1|3}(z_1) = \Sigma z_1|z_3, \]
where the latter expression is defined in the same way as (7). The conditional density \( f_{Z_1|Z_3}(z_1|z_3) \) is defined in the same way. If \( Z_1 \) and \( Z_2 \) are conditionally independent given \( Z_3 \), then
\[
f_{Z_1, Z_2|Z_3}(z_1, z_2|z_3) = f_{Z_1|Z_3}(z_1|z_3) f_{Z_2|Z_3}(z_2|z_3)
\]
and we at once identify the conditional density of \( (Z_1, Z_2)|Z_3 \) as another Gaussian distribution, but without any cross-term involving \( z_1z_2 \), which then implies that the off-diagonal element in the partial covariance matrix (7) is identically equal to zero. We see immediately from our definition (8) that the LGPC is also identically equal to zero.

The converse statement also follows by looking at the conditional density of \( (Z_1, Z_2)|Z_3 \). Let \( f_z(z) \) be on the locally Gaussian form (6). Again, we have from Otneim and Tjøstheim (2018) that the conditional density of \( (Z_1, Z_2|Z_3) \) is locally associated with a Gaussian distribution, with local conditional mean vector and local conditional covariance matrix given by expressions corresponding to (10). The off-diagonal in the local conditional covariance matrix is zero if \( \alpha(z) \equiv 0 \), or equivalently \( \alpha(x) \equiv 0 \), in which case one may factorize the joint conditional density of \( (Z_1, Z_2|Z_3) \) into two factors, one depending only on \( z_1 \), and one depending only on \( z_2 \). Hence, \( Z_1 \) and \( Z_2 \) are conditionally independent given \( Z_3 \).

Let the transformation \( Y = h(X) \) define the \( y \)-scale in the same way as the transformation (5) defines the \( z \)-scale. We define the LGPC in terms of the marginally standard normal \( Z \)-variables, which for the \( Y \)-variables can be calculated as
\[
Z = \left( \Phi^{-1}(F_{Y_1}(Y_1)), \ldots, \Phi^{-1}(F_{Y_p}(Y_p)) \right)
\]
but if \( h_j(\cdot) \) is monotone for all \( j = 1, \ldots, p \),
\[
F_{Y_j}(y_j) = P(h_j(X_j) \leq y_j) = P(X_j \leq h_j^{-1}(y_j)) = P(X_j \leq x_j) = F_{X_j}(x_j),
\]
where \( x_j \) is the point on the \( x \)-scale corresponding to the point \( y_j \) on the \( y \)-scale. The \( Z \)-variables are the same for the stochastic variables \( X \) and \( Y = h(X) \). Their LGPC-function \( \alpha(z) \) must therefore be the same as well according to our definition in the preceding section.

## 4 Estimating conditional dependence via the LGPC

### 4.1 Estimation by local likelihood

We see from the defining equations (7) and (8) that the basic building blocks for the local Gaussian partial correlation are the local Gaussian correlation functions, that populate the local correlation matrix \( R(z) \) in
the density function $f_Z$ in (6). Consider the random vector $X = (X_1, X_2, \ldots, X_p)$ having joint probability density function (pdf) $f(x)$, and its transformed counterpart $Z = (\Phi^{-1}(F_1(X_1)), \ldots, \Phi^{-1}(F_p(X_p)))$, on the marginally standard normal $z$-scale. The relation between the pdf of $X$ and the pdf of $Z$ is given by

$$f_X(x) = f_Z\left(\Phi^{-1}(F_1(x_1)), \ldots, \Phi^{-1}(F_p(x_p))\right) \prod_{i=1}^{p} \frac{f_i(x_i)}{\phi(\Phi^{-1}(F_i(x_i)))},$$  \hspace{1cm} (11)

where $f_i(\cdot)$, $i = 1, \ldots, p$ are the marginal density functions of $X$, and $\phi(\cdot)$ is the standard normal density function. Otneim and Tjøstheim (2017) provide more details of this construction in the context of multivariate density estimation. Let $X_1, \ldots, X_n$ be a random sample identically distributed as $X$ and construct the pseudo standard normal observations $\bar{Z}_1, \ldots, \bar{Z}_n$ as

$$\bar{Z}_i = \left(\Phi^{-1}\left(\widehat{F}_1(X_{i1})\right), \ldots, \Phi^{-1}\left(\widehat{F}_p(X_{ip})\right)\right),$$  \hspace{1cm} (12)

where $X_i = (X_{i1}, X_{i2}, \ldots, X_{ip})$, and $\widehat{F}_j(\cdot)$ is an estimate of the marginal distribution function of $X_j$, for example the empirical distribution function. Next, we must produce an estimate $\hat{R}(z)$ of the local correlation matrix, and we will consider two variations of this task in the following sub-sections. It is then natural to estimate the local Gaussian partial covariance matrix as

$$\hat{\Sigma}_{1.2}(z) = \hat{R}_{1.1}(z) - \hat{R}_{1.2}(z)\hat{R}_{2.2}^{-1}(z)\hat{R}_{2.1}(z),$$  \hspace{1cm} (13)

and if we assume that $X_1$ and $X_2$ constitute the two first elements in $X$, we estimate the local Gaussian partial correlation between $X_1$ and $X_2$ given $X^{(2)}$ on the $z$-scale as

$$\hat{\rho}(z) = \left\{ \hat{\Sigma}_{1.2}(z) \right\}_{12}^{11} \left\{ \hat{\Sigma}_{1.2}(z) \right\}_{22}^{-1},$$  \hspace{1cm} (14)

A corresponding value of $\hat{\rho}(x)$ at the point $x = \widehat{F}^{-1}(\Phi(z))$ is obtained by inserting $z = \Phi^{-1}(\widehat{F}(x))$ in (14). The curse of dimensionality dictates that nonparametric estimates of multivariate regression and density functions quickly deteriorates as the number of variables increases. This is also true for local correlation surfaces, but it can be alleviated somewhat by introducing heavier smoothing towards the global Gaussian model.

Otneim and Tjøstheim (2017) propose a different strategy. By estimating each local correlation $\rho_{ij}(z)$, $1 \leq i < j \leq p$, sequentially in the bivariate problem involving only the variables $Z_i$ and $Z_j$, they effectively impose a pairwise dependence structure in order to circumvent the curse of dimensionality. This is indeed possible when calculating the LGPC as well, and it can be done for an arbitrary dimension $p$. It is described in more detail in Section 4.1.2.

In the case that $p = 3$, and all variables involved are scalars, we do provide in this paper the empirical option and corresponding estimation theory to fit a full trivariate normal distribution locally to $(Z^{(1)}, Z^{(2)}) = (Z_1, Z_2, Z_3)$ without any structural simplifications. This constitutes a novelty in the local Gaussian literature, and we present the procedure in the following sub-section.

4.1.1 Estimation of $R(z)$ when $p = 3$ and $X^{(2)}$ is a scalar

If $\dim(X) = 3$, then $\dim(X^{(2)}) = 1$, and $R(z)$ is a $3 \times 3$ symmetric matrix of local correlation functions having arguments $z = (z_1, z_2, z_3)$:

$$R(z) = \begin{pmatrix} 1 & \rho_{12}(z) & \rho_{13}(z) \\ \rho_{12}(z) & 1 & \rho_{13}(z) \\ \rho_{13}(z) & \rho_{23}(z) & 1 \end{pmatrix},$$
containing three parameter functions that must be estimated from data. Hjort and Jones (1996) provide the local likelihood framework that we need to perform this task. Based on the sample (12), we estimate $R(z)$ by fitting the parametric family $\psi(z, R)$, defined in (6), locally to the density $f_Z(z)$ by maximizing the local likelihood function

$$\hat{R}(z) = \arg \max_R n^{-1} \sum_{i=1}^{n} K_b(\hat{Z}_i - \hat{z}) \log \psi(\hat{Z}_i, R) - \int K_b(y - \hat{z}) \psi(y, R) \, dy$$

(15)

in each point $z$, where $K_b(x) = |b|^{-1/2} K(b^{-1/2} x)$, $K$ is a non-negative and radially symmetric kernel function that satisfies $\int K(x) \, dx = 1$, and $b$ is a diagonal $3 \times 3$ matrix of bandwidths that serve as smoothing parameters. The estimate $\hat{f}_Z(z) = \psi(z, \hat{R}(z))$ aims at minimizing its locally weighted Kullback-Leibler distance to $f_Z(z)$, and we refer to Hjort and Jones (1996) and Tjøstheim and Hufthammer (2013) for much more details about this construction, and to Section 4.2 for a review of relevant estimation theory.

Finally, we obtain the estimated LGPC $\hat{\alpha}(z)$ by plugging the estimated local correlations $\hat{R}(z)$ into equations (13) and (14) and a corresponding value of $\hat{\alpha}(x)$ at the point $x = \hat{F}^{-1}(\hat{\Phi}(z))$ by inserting $z = \hat{F}^{-1}(\hat{F}(x))$.

4.1.2 Estimation of $R(z)$ when $X^{(2)}$ is a vector

As mentioned above, the complexity of the estimation problem (15) increases sharply with the number of variables involved, much in the same way as other nonparametric estimation methods suffer under the curse of dimensionality. Otneim and Tjøstheim (2017) suggest a way to circumvent this issue when estimating local correlation matrices. Their basic idea is that local correlations in $R(z)$ are modeled as functions of their corresponding pair of variables only:

$$R(z) = \{\rho_{jk}(z_j, z_k)\}_{j < k}, \quad j, k = 1, 2, \ldots, p,$$

(16)

which reduces the estimation of the $p$-variate correlation functions $\rho_{jk}(z)$ to a series of bivariate estimation problems. We suggest to use this approach also when modelling partial dependence when the number of variables in $X$ is bigger than three. This allows nonlinear dependence between variables to be approximated only by the pairwise structure (16), while still being computationally tractable. An analogue to the pairwise approximation is the additive approximation in nonparametric regression. See Otneim and Tjøstheim (2017) for a brief discussion.

We turn again to Hjort and Jones (1996) for the means to estimate the simplified version of $R(z)$. We transform the observations to the z-scale as in (12), but now estimate the individual components in $R(z)$ sequentially, taking only into consideration the pair of variables in question:

$$\hat{\rho}_{jk}(\hat{z}_j, \hat{z}_k) = \arg \max_{\rho_{jk}} n^{-1} \sum_{i=1}^{n} K_b(\hat{Z}_i - \hat{z}) \log(\hat{Z}_i, \rho_{jk}) - \int K_b(y - \hat{z}) \psi(y, \rho_{jk}) \, dy,$$

(17)

where all running variables and samples are bivariate subsets corresponding to the indices $(j, k)$, $\psi(\cdot, \rho)$ is the bivariate version of (6), and $b$ now is a $2 \times 2$ diagonal matrix of bandwidths. After estimating all local correlations in this way, we proceed to calculate the LGPC using equations (13) and (14) as above.

4.2 Asymptotic theory

Equations (7), (8) and their empirical counterparts (13) and (14) demonstrate clearly that the LGPC is nothing more than a deterministic function of the local correlation matrix, in the same way as the ordinary partial correlation is a function of the ordinary correlation matrix. The asymptotic behavior of $\hat{\alpha}(z)$ can thus be derived directly from the asymptotic behavior of $R(z)$, which, for the most part, has been established.
Then, by writing

where

Let

Theorem 1.

Consider first the full trivariate fit described in Section 4.1.1 with \( z = (z_1, z_2, z_3) \). We estimate the local correlation matrix \( R(z) \) by maximizing the local likelihood function in (15), which we denote by \( L_n(R(z), z) \), with the only exception that we, for now, assume the true transformation between the \( x_- \) and the \( z \)-scale to be known. For a fixed matrix of bandwidths \( b \), denote by \( R_b(z) \) the local correlation that satisfies, as \( n \to \infty \),

\[
\nabla L_n(R_b, z) \to \int K_b(y - z)u(y, R_b(z))\{f_Z(y) - \psi(y, R_b)\}\,dy = 0, \tag{18}
\]

where \( f_Z(z) = f_{Z_1, Z_2, Z_3}(z_1, z_2, z_3) \) is the density function of \( Z \), and \( u(\cdot, \rho_b) \) is the column vector of local score functions \( \nabla \log \psi(\cdot, \rho_b) \), where the gradient is taken with respect to the parameters \( \rho_{12}(z), \rho_{13}(z) \) and \( \rho_{23}(z) \). Denote by \( \rho = \rho(z) = (\rho_{12}(z), \rho_{13}(z), \rho_{23}(z)) \) the vector of local correlations. The joint limiting distribution of \( \hat{\rho} \) (for convenience only stated on the \( z \)-scale below) is given by the following result as \( n \to \infty \) and \( b \to 0 \):

**Theorem 1.** Let \( \{Z\}_n \) be observations on the marginally standard normally distributed random vector \( Z \) having joint density \( f_Z(z) \). Assume that the following conditions hold:

1. For any sequence of bandwidth matrices \( b_n \) tending to zero element-wise, there exists for the trivariate marginally standard Gaussian vector \( Z \) a unique set of local correlations \( \rho_b(z) \) that satisfies (18), and there exists a \( \rho_0(z) \) such that \( \rho_b(z) \to \rho_0(z) \).
2. \( \{Z\}_n \) is \( \alpha \)-mixing with mixing coefficients satisfying \( \sum_{m \geq 1} m^\alpha(m)^{1 - 2/\delta} < \infty \) for some \( \lambda > 1 - 2/\delta \) and \( \delta > 2 \).
3. \( n \to \infty \), and each bandwidth \( b \) tends to zero such that \( nb^{\lambda+2-2/\delta} = O(n^{\epsilon_0}) \) for some constant \( \epsilon_0 > 0 \).
4. In a given point \( z \), the parameter space \( \Theta \) for each local correlation \( \rho(z) \) is a compact subset of \((-1, 1)\).
5. The kernel function \( K(\cdot) \) satisfies \( \sup_z |K(z)| < \infty \), \( \int |K(y)|\,dy < \infty \), \( \partial/\partial z_i K(z) < \infty \), and \( \lim_{z_i \to \infty} |z_i K(z)| = 0 \) for \( i = 1, 2, 3 \).

Then, by writing \( b_{n1} = b_{n2} = b_{n3} = b \to 0 \) assuming they all converge to zero at the same rate,

\[
\sqrt{nb^2}J_bM_b^{-1/2}(\hat{\rho}_n - \rho_0) \xrightarrow{D} N(0, I), \tag{19}
\]

where \( I \) is the \( 3 \times 3 \) identity matrix,

\[
J_b = \int K_b(y - z)u(y, \rho_b(z))u^T(y, \rho_b(z))\psi(y, \rho_b(z))\,dy
\]

\[
- \int K_b(y - z)\nabla u(y, \rho_b(z))\{f_Z(y) - \psi(y, \rho_b(z))\}\,dy,
\]

and

\[
M_b = b^3 \int K_b^2(y - z)u(y, \rho_b(z))u^T(y, \rho_b(z))f_Z(y)\,dy
\]

\[
- b^3 \int K_b(y - z)u(y, \rho_b(z))f_Z(y)\,dy \int K_b(y - z)u^T(y, \rho_b(z))f_Z(y)\,dy.
\]
Theorem 1 and its assumptions above are slight variations of the assumptions and the results presented by Hjort and Jones (1996), and in particular Theorem 3 by Tjøstheim and Hufthammer (2013). The latter reference contains a detailed argument concerning the convergence rate of local likelihood estimates in multi-parameter problems, such as ours. Their analysis of the asymptotic behavior of \( J_b \) and \( M_b \) reveals that the local parameter estimates converge more slowly than, for example, the trivariate kernel density estimator (that converges as \( 1/(nb^3) \)), and it is a straightforward exercise to modify their proof to suit our particular situation, which reveals that the three local correlations converge jointly in distribution at the rate \( 1/(nb^5) \). Furthermore, Tjøstheim and Hufthammer (2013) demonstrate that an analytic expression of the leading term in the covariance matrix \( J_b^{-1}M_b(J_b^{-1})^T \) is not practically available. In applications we may circumvent this problem by using the bootstrap, see also Lacal and Tjøstheim (2018), or by approximating \( J_b \) and \( M_b \) using empirical moments corresponding to the integrals defining them. We refer to Appendix 7.1 for some details regarding the proof of Theorem 1.

Moving on to the higher dimensional case, we will see below that the pairwise estimation structure that we describe in Section 4.1.2 simplifies the asymptotic analysis of \( \hat{R}(z) \) considerably. This result is also available in the literature already, so we modify the notation slightly, and reproduce Theorem 3 in Otneim and Tjøstheim (2018). For a pair of variables (\( X_j, X_k \)) and its corresponding pair of transformed variables (\( Z_j, Z_k \)) (the relation between the \( x \)- and \( z \)-scale is again assumed to be known at this stage). We then estimate the single local correlation \( \rho_{jk}(x_j, x_k) = \rho_{jk}(z_j, z_k) \) by maximizing the local likelihood function in (17), denoted by \( L_n(\rho_{ij}, z) \). For a fixed matrix of bandwidths \( b \), now a diagonal \( 2 \times 2 \)-matrix, define \( \rho_b \) in a similar way as above as the solution to

\[
\frac{\partial L_n(\rho_b, z)}{\partial \rho} \to \int K_b(y - z)u(y, \rho_b)\left\{f_{jk}(y) - \psi(y, \rho_b)\right\} \, dy = 0 \tag{20}
\]

as \( n \to \infty \), but now \( f_{jk}(z) \) is the joint density of \( (Z_j, Z_k) \), \( u(\cdot, \rho_b) \) is the local score function \( \partial \log \psi(\cdot, \rho_b)/\partial \rho \), and \( K(\cdot) \) and \( \psi(\cdot) \) are bivariate versions of the kernel function and the simplified Gaussian density associated with \( \Theta \) respectively. Denote by \( \rho = (\rho_{jk})_{j \neq k} \) the vector of all local correlations defined between all pairs of variables in \( X \). The joint limiting distribution of \( \hat{\rho}(z) \) is then given by the following result (Otneim and Tjøstheim 2017):

**Theorem 2.** Let \( \{Z\}_n \) be observations on the marginally standard normally distributed random vector \( Z \), and for each pair of variables \( (Z_j, Z_k) \), \( j < k \leq p \), assume that the following conditions hold:

1. For any sequence of bandwidth matrices \( b = b_n \) tending to zero element-wise, there exists for the bivariate marginally standard Gaussian vector \( (Z_j, Z_k) \) a unique \( \rho_b(z) \) that satisfies (20), and there exists a \( \rho_0(z) \) such that \( \rho_b(z) \to \rho_0(z) \).

2. For each pair \( (j, k) \) of variables, with \( 1 \leq j \leq p, 1 \leq k \leq p, j \neq k \), \( \{(Z_j, Z_k)\}_n \) is \( \alpha \)-mixing with mixing coefficients satisfying \( \sum_{m \geq 1} m^{-\lambda} \alpha(m)^{1-2/\delta} < \infty \) for some \( \lambda > 1 - 2/\delta \) and \( \delta > 2 \).

3. \( n \to \infty \), and each bandwidth \( b \) tends to zero such that \( nb^4 = O(n^{\epsilon_0}) \) for some constant \( \epsilon_0 > 0 \).

4. For a given location \( z \), the parameter space \( \Theta \) for each local correlation \( \rho(z) \) is a compact subset of \((-1, 1)\).

5. The kernel function \( K(\cdot) \) satisfies \( \sup_\Omega |K(z)| < \infty \), \( \int |K(y)| \, dy < \infty \), \( \partial/\partial z_i K(z) \to \infty \), and \( \lim_{z_i \to \infty} |z_i K(z)| = 0 \) for \( i = 1, 2 \).

Then, by letting \( b_n^2 \) (assuming identical convergence rates) mean the product \( b_{nj} b_{nk} \) corresponding to the pairs of variables defining the components in \( \rho \),

\[
\sqrt{nb_n^2}(\hat{\rho}(z) - \rho_0(z)) \overset{d}{\to} N(0, \Omega),
\]

where \( \Omega \) is a diagonal matrix with components

\[
\Omega^{(\ell, \ell)} = \frac{f_{\ell}(z_\ell)}{u^2(z_\ell, \rho_0, \ell(z_\ell))} \psi^2(z_\ell, \rho_0, \ell(z_\ell)), \tag{21}
\]
where $\ell = 1, \ldots, p(p-1)/2$ runs over all pairs of variables, and $f_\ell(\cdot)$ is the bivariate marginal density function of the pair $Z_\ell$.

For a proof, we refer to Otneim and Tjøstheim (2018). There are two important differences between Theorems 1 and 2. First of all, we notice in Theorem 2 that the convergence rate is equal to the usual nonparametric rate of $1/nb^2$: this is because we estimate the local correlations sequentially based on pairs of variables. The second difference is that the leading term of the asymptotic covariance matrix in Theorem 2 is easy to write up analytically, as we have done in equation (21).

We can now calculate the limiting distribution of the LGPC by means of the delta method. Denote by $g : \mathbb{R}^{(p-1)/2} \to \mathbb{R}$ the translation from the vector of local correlations between the components in $Z$ (or, equivalently, $X$) to the local partial correlation $\alpha(z)$ between $Z_1$ and $Z_2$ given $Z_3, \ldots, Z_p$. From the expression of our estimate (14) we have that $\hat{\alpha}(z) = g(\hat{\rho}(z))$, so we use the delta method to see that

$$\sqrt{nb_m} (\hat{\alpha}(z) - \alpha(z)) \overset{d}{\to} \mathcal{N}(0, \nabla g(\rho)^T \Lambda \nabla g(\rho)),$$

where $m$ is equal to 2 or 5, and $\Lambda$ is either equal to the leading term of $J_b^{-1} M_b (J_b^{-1})^T$, or $\Omega$, depending on whether we use the full trivariate locally Gaussian fit described in Section 4.1.1 or the sequentially pairwise simplification described in Section 4.1.2. In any case, it is only a matter of basic differentiation to work out an expression for $\nabla g(\cdot)$, and this task has been deferred to the Appendix, section 7.2.

Finally, we present the limiting distribution of the LGPC between $Z_1$ and $Z_2$ given $Z_3, \ldots, Z_p$ when the components in $Z = (Z_1, \ldots, Z_p)$ are replaced by marginally standard normal pseudo-observations as defined in equation (12). The following result ensures that Theorems 1 and 2, and thus eq. (22), still holds.

**Theorem 3.** Under the conditions in Theorems 1 and 2, assume further that the kernel function $K(\cdot)$ has bounded support. Replacing the marginally standard normal vector $Z$ with the approximately marginally standard normal vector $\tilde{Z}$ as defined in eq (22), does not change the conclusions in Theorems 1 and 2.

**Proof.** See the Appendix, Section 7.3.

The practicality of the other conditions of Theorems 1 and 2 have been discussed in more detail by Otneim and Tjøstheim (2018).

### 4.3 Examples

#### 4.3.1 Simulated examples

We will in this section provide some demonstrations on how the estimated LGPC can be used to reveal nonlinear departures from conditional independence. In the first and second example we use the pairwise simplification of Section 4.1.2. In the third example, we employ the fully trivariate model of Section 4.1.1 and with exception of the third simulated example, very similar results are obtained for both estimation methods. All plots are presented on the $x$-scale.

First, we provide some reference pictures from the simplest situation imaginable. Let $X_1$, $X_2$ and $X_3$ be independent standard (and thus also jointly) normal variables. Since $X_1$ is conditionally independent from $X_2$ given $X_3$ (written $X_1 \perp X_2 | X_3$) in this particular case, we also know from property 3 in Section 3.2 that the LGPC between $X_1$ and $X_2$ is zero everywhere.

In Figure 1 we see the estimated LGPC between $X_1$ and $X_2$ given $X_3 = 0$ for three samples, having sample size 100, 500 and 2000 respectively, mapped out on a grid, where blue colors indicate negative local partial correlation, and red colors indicate positive local partial correlation. These dependence maps are inspired by Berentsen, Kleppe, and Tjøstheim (2014), and all computations in this paper may be reproduced by following instructions in the online appendix. In this and all other examples in this paper we select bandwidths based
on a simple plug-in formula that follows naturally from classical
asymptotic arguments, \( b = cn^{-1/9} \) for the full trivariate
fit, and \( b = cn^{-1/6} \) for the bivariate simplification,
where the constant \( c \) controls the amount of
smoothing and must be chosen appropriately based on
the task at hand. Håkon Otneim (2016) argues that
\( c = 1.75 \) is a good choice within the realm of
density estimation, and we will in the next section
see that this choice also gives good power when
testing for conditional independence in many
instances, but also that even smaller values may
be beneficial in others. For the visual display
of conditional dependence maps, however,
we tend to prefer more smoothing, and in Figure 2
we have used \( c = 4 \). We will typically calculate the LGPC
at different levels of smoothing during the
initial exploration of data. Otneim and Jøsøeh (2017)
also suggest a cross validation algorithm for
bandwidth selection in the density estimation
context.

As expected, \( \hat{\alpha}(x) \) is close to zero in all three
plots in Figure 1.

We have noted before that the ordinary partial
 correlation coefficient characterizes conditional
dependence between jointly normal variables.
It is not hard, on the other hand, to construct
elements in which strong nonlinear relationships
remain completely undetected by the partial correlation.
Consider, for example, the
following structural equation:

\[
X_2 = X_1^2 + X_3,
\]

where we observe all components in \((X_1, X_2, X_3)\). There is, obviously, a strong dependence between
\( X_1 \) and \( X_2 \), and furthermore, this dependence is
deterministic when conditioning on \( X_3 = x_3 \). It is well known,
however, that \( X_2 \) and \( X_1^2 + X_3 \) are uncorrelated if \( E(X_1) = E(X_1^2) = 0 \). The LGPC easily
reveals conditional dependence between \( X_1 \) and \( X_2 \) in this case. Let us generate \( n = 500 \) independent
observations each from \( X_1 \sim N(0, 1) \) and \( X_3 \sim N(0, 1) \) and calculate \( X_2 \) by (23). The sample partial
correlation between \( X_1 \) and \( X_2 \) is -0.037 in this case, but the LGPC, on the other hand, which in this case is calculated using the simplified
pairwise structure defined in Section 4.1.2 is displayed in Figure 2a along with the observations. The LGPC
indicates strong departures from conditional independence
on the plane defined by \( X_3 = 0 \). Indeed, we identify a region of negative conditional relationship on the left
hand side of the plot, which makes good sense because
small values of \( X_2 \) is typically observed together with
large values of \( X_1 \) and vice versa. The opposite
phenomenon is clearly visible in the right hand half of
the plot. In Figure 2b we have plotted the estimated LGPC along the curve \( X_2 = X_1^2 \) (that is indicated as a dashed line in Figure 2a), along with a 95% confidence band, calculated
using the limiting distribution defined in Theorem 2. The
full trivariate fit presented in Section 4.1.1 gives a very similar picture.

This points to an important feature of the LGPC: It is able to distinguish between positive and negative conditional relationships, which, to our knowledge, has until now not been possible beyond the linear and jointly Gaussian setting using the ordinary partial correlation. Our approach also allows exploration of conditionally
different dependence patterns across different levels
of the conditioning variable. Let us demonstrate how
this works in another constructed example: Generate
\( X_3 = \rho \sim U(-1, 1) \), and then generate \((X_1, X_2)\) from the

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bivariate Gaussian distribution having standard normal marginals, and correlation coefficient equal to \( \rho \). We observe \( X = (X_1, X_2, \rho) \) and seek to visualize the dependence between \( X_1 \) and \( X_2 \) conditional on \( \rho \) using the LGPC.

We see the results from this exercise in Figure 3. In panel (a), we generate \( n = 1000 \) simulated pairs \((X_1, X_2)\) from this model. In panel (b)-(d) we see the estimated LGPC plotted over suitable grids, and where the conditioning variable \( X_3 = \rho \) has been fixed at the respective values \(-0.9, 0\), and \(0.9\), and we see clearly how the dependence between \( X_1 \) and \( X_2 \) changes in these cases. In this particular example it is straightforward to see that \( X_1 \) and \( X_2 \) are uncorrelated and that the ordinary partial correlation between \( X_1 \) and \( X_2 \) given \( X_3 \) is equal to zero. Furthermore, we note that the pairwise simplification defined in Section 4.1.2 would also not be able to measure the conditional relationship between \( X_1 \) and \( X_2 \) given \( X_3 \) in this case, as we see clearly from eq. (9), because \( X_1 \) and \( X_2 \) are both marginally independent from \( X_3 \). This means that the two pairwise correlations \( \rho_{13}(z_1, z_3) \) and \( \rho_{23}(z_2, z_3) \) are equal to zero. This example also shows that the form of the LGPC can depend very strongly on the value of the conditioning variable.

4.3.2 Empirical example: Granger causality

Let us now, before we bring this section to its conclusion, briefly demonstrate a practical implementation of the LGPC in a situation involving real data. In particular, we will look at Granger causality, which, it goes without saying, has been a central concept in economics and econometrics ever since its inception by C. W. Granger (1969). In layman’s terms, the time series \( \{X_t\} \) Granger causes \( \{Y_t\} \) if past and present values of \( \{X_t\} \) are helpful when predicting future values of \( \{Y_t\} \). Formally, \( \{X_t\} \) Granger causes \( \{Y_t\} \) if (C. W. Granger 1980)

\[
Y_t \not\perp \mathcal{I}^*(t-1) \mid \mathcal{I}^-_X(t-1),
\]

where \( \perp \) denotes independence, \( \not\perp \) denotes dependence, and where \( \mathcal{I}^*(t-1) \) is all information available at time \( t-1 \) and \( \mathcal{I}^-_X(t-1) \) is the same information, with the exception of the values of \( \{X_t\} \) up to, but not including, time \( t \). Of course, the hypothesis \( \text{(24)} \) can not, in practice, be tested in its full generality. By taking only effects up to the first lag into account, we may formulate a sufficient (but not necessary) condition for \( \text{(24)} \): \( Y_t \not\perp X_{t-1} \mid Y_{t-1} \), the converse of which constitutes a testable null hypothesis:
(a) Observed pairs $(X_1, X_2)$ from the conditionally Gaussian model.

(b) The estimated LGPC between $X_1$ and $X_2$ given $\rho = -0.9$.

(c) The estimated LGPC between $X_1$ and $X_2$ given $\rho = 0$.

(d) The estimated LGPC between $X_1$ and $X_2$ given $\rho = 0.9$.

Figure 3: The conditionally Gaussian model
Figure 4: Local partial dependence maps for the S&P 500 data
There are many ways to carry out this test in practice. The simplest is based on the further restriction of linear relationships between \( \{X_t\} \) and \( \{Y_t\} \) and is thus a test for conditional uncorrelatedness rather than conditional independence. Nonparametric tests that have power against many nonlinear types of conditional dependence have also been developed, and we refer to Section 4 for further references to this literature and a new test for conditional independence based on the LGPC.

One fairly recent method for testing (25) is based on the maximal conditional correlation and was introduced by Huang (2010) and later extended to the time series case by Cheng and Huang (2012). The latter authors use this test to examine whether trading volume Granger caused index value, or vice versa, on the S&P 500 stock index during the first decade of this century. Using the daily price series \( \{P_t\} \) and volume \( \{V_t\} \) they define the log-differenced series

\[
R_t = 100 \log \left( \frac{P_t}{P_{t-1}} \right) \quad \text{and} \quad V_t^* = \log \left( \frac{V_t}{V_{t-1}} \right).
\]

Further, they denote by \( R_{t-1} \not \perp V_t^* \) the Granger non-causality from \( \{R_t\} \) to \( \{V_t^*\} \), which they analyze by putting the following null hypothesis to the test:

\[
H_0 : V_t^* \perp R_{t-1} \mid Y_{t-1}. \tag{25}
\]

Granger non-causality in the opposite direction, \( V_{t-1}^* \not \perp R_t \) is formulated and tested for in the obvious way.

A linear test rejects \( R_{t-1} \not \perp V_t^* \), but not \( V_{t-1}^* \not \perp R_t \). The nonparametric test by Cheng and Huang (2012) rejects both, leading to the natural conclusion that there are nonlinear relationships in the latter situation that remain unseen by linear models. What is it though, in this particular data set, that leads to such results? Estimates of the local partial Gaussian correlation provide some clues towards answering that question.

We obtain daily observations on price and trading volume on the S&P 500 index from January 1st 2000 until December 31st 2009 on the S&P 500 index from Yahoo Finance (2018), and plot the estimated LGPC between \( V_t^* \) and \( R_{t-1} \) as defined in Section 4.1.1 on the plane defined by \( V_{t-1}^* = 0 \) in Figure 4a. Departures from \( \alpha(V_t^*, R_{t-1}) \equiv 0 \) provide evidence against \( R_{t-1} \not \perp V_t^* \). Similarly, we plot the estimated LGPC between \( R_t \) and \( V_{t-1}^* \) on the plane \( R_{t-1} = 0 \) in Figure 4b in which departures from \( \alpha(R_t, V_{t-1}^*) \equiv 0 \) provide evidence against \( V_{t-1}^* \not \perp R_t \). We have used a proportionality constant of \( c = 3.5 \) to calculate the estimates (see discussion on bandwidth selection in Section 4.3.1). The observations can be seen in the background of both plots.

The differences between Figures 4a and 4b are subtle, but important. In Figure 4a, \( \tilde{\alpha}(V_t^*, R_{t-1}) \) is mostly negative, especially in the data rich portions of the sample space (other values of the conditioning variable \( V_{t-1}^* \) than zero give similar pictures). Indeed, the global partial correlation in this situation is \( \tilde{\alpha}_{\text{glob}} = -0.086 \), which is significantly different from zero. The global partial correlation in the second situation is very small in absolute value though: \( \tilde{\alpha}_{\text{glob}} = -0.0018 \), but Figure 4b still reveals departures from conditional independence of similar magnitudes as in Figure 4a. The difference is that the estimated LGPC is positive (primarily in the second and fourth quadrants) as well as negative (in the first and third quadrants), but this pattern collapses to the constant global value \( \alpha(R_t, V_{t-1}^*) = \alpha_{\text{glob}} \equiv 0 \) as the bandwidths tend to infinity. In Figures 4c and 4d we can explore the conditional dependence between \( R_t \) and \( V_{t-1} \) at high and low levels of \( R_{t-1} \), respectively, and we see even more clear differences in this dimension, especially in the first and second quadrants.

The \( p \)-values for tests of the hypotheses (25) and its opposite counterpart, using our new test for conditional independence as presented in the next section, are both equal to 0.
5 Testing for conditional independence

5.1 The recent fauna of nonparametric tests

Property 3 in Section 3.2 states that the LGPC characterizes conditional dependence within a large class of distributions: \( X_1 \) and \( X_2 \) are independent given \( X_3 \) if and only if the locally Gaussian partial correlation between them is identically equal to zero everywhere on the sample space of \((X_1, X_2, X_3)\). The road is therefore short to a test for conditional independence that may have power against a great deal of nonlinear alternatives compared to a test based on the ordinary partial correlation coefficient. One can, however, no longer claim that «few methods exist for testing for conditional independence between \( X_1 \) and \( X_2 \) given \( X_3 \) in a general nonparametric setting» as some of the earlier references mentioned below do. In fact, the last decade or so has seen the publication of many new tests for conditional independence, some of which are presented along with rigorous derivation of asymptotic properties and thorough simulations.

Su and White have published a series of such tests: Su and White (2007) is based on detecting differences between estimated characteristic functions (which is also the method used by Wang and Hong (2017)), Su and White (2008) is based on estimating the Hellinger distance between conditional density estimates, Su and White (2012) use local polynomial quantile regression to test for conditional independence, and Su and White (2014) present a test based on empirical likelihood. Huang (2010) introduces the maximal nonlinear conditional correlation which is used in a test for conditional independence, in turn extended to dependent data by Cheng and Huang (2012). Song (2009) constructs a test via Rosenblatt transformations, while Bergsma (2010) and Bouezmarni, Rombouts, and Taamouti (2012) present new tests for conditional independence based on copula constructions. Bouezmarni and Taamouti (2014) test for conditional independence by measuring the \( L_2 \) distance between estimated conditional distribution functions, and Patra, Sen, and Székely (2015) use empirical transformations to translate conditional independence to joint independence, tests for which exist in abundance in a large body of literature stretching decades back in time. Wang et al. (2015) introduce the conditional distance correlation which they use to construct a test for conditional independence. Most of the papers mentioned here refer to a discussion paper by Linton and Gozalo (1997), who formulate conditional independence in terms of probability statements which then forms the basis of a test. A version of this work has been published later as Linton and Gozalo (2014). Also, Delgado and Manteiga (2001) develop a test for conditional independence in a nonparametric regression framework. Finally, we mention that there is a small literature on testing by way of characterizing conditional independence via reproducing kernel Hilbert spaces (RKHS), see, for example, Zhang et al. (2012) and references therein.

Apart from some notational similarities with Zhang et al. (2012), a test based on the LGPC is quite different from the methods quoted above. First of all it is semi-parametric and does not rely on traditional kernel estimates of density-, distribution-, or characteristic functions. This opens up the possibility for better power properties. Furthermore, due to our transformation of the data to marginal standard normality, we do not necessarily have to specify weight functions in our test functional to lessen the impact of outliers - unless of course we wish to test for conditional independence in a specific portion of the sample space.

On the other hand, we have to pay a price when imposing structure on the dependence. Certain types of conditional dependence will remain invisible to our test statistic, but simulation experiments show that our test performs on par with, and sometimes better than existing, fully nonparametric tests.

5.2 A test for conditional independence based on the LGPC

In accordance with many central references mentioned above, and indeed with a larger literature on general independence testing based on local measures of dependence, we construct a test statistic for testing

\[ H_0 : X_1 \perp X_2 | X_3 \]

or, equivalently in terms of the marginally Gaussian pseudo observations
by aggregating our local measure of dependence over the sample space of $X$ (or $Z$). A natural test statistic on the z-scale is

$$T_{n,b} = \int_S h(\tilde{\alpha}_b(z)) \, dF_n(z), \quad (27)$$

where $h(\cdot)$ is a real-valued function that is typically even and non-negative for most standard applications, but not necessarily so, and $S \subseteq \mathbb{R}^p$ is an integration area that can be altered in order to focus the test to specific portions of the sample space. Standard laws of large numbers ensure that, under regularity conditions, $T_{n,b}$ converges in probability towards its population value

$$T = \int_S h(\alpha(z)) \, dF(z).$$

This way, departures from many types of conditional independence lead to large values of $T_{n,b}$ that, if larger than a critical value depending on the chosen significance level, leads to the rejection of (26). One might expect that approximate $p$-values for our test can be readily extracted from the limiting distribution of the test statistic $T_{n,b}$, which can be derived along the same lines as Lacal and Tjøstheim (2018). Several authors, for example Teräsvirta, Tjøstheim, and Granger (2010), have noted, however, that asymptotic analysis of nonparametric test statistics on the form (27) tend to be too crude for finite sample applications. We have therefore chosen to approximate the null distribution of $T_{n,b}$ using the bootstrap. The validity of the bootstrap in such contexts has been examined by Lacal and Tjøstheim (2018).

In accordance with our treatment so far, let $\{Z_{1,i}, Z_{2,i}, Z_{3,i}\}, i = 1, \ldots, n$ be observations on the $p$-variate stochastic vector $Z$, $p \geq 3$, with $Z_1$ and $Z_2$ being scalar and $Z_3$ being $p - 2$-variate. In order to calculate the statistic $T_{n,b}$ for testing the null hypothesis $Z_1 \perp Z_2 \mid Z_3$, we must estimate the joint local correlation matrix $\Sigma(z)$ of $Z = (Z_1, Z_2, Z_3)$, which we may also use to estimate the conditional probability density functions of $Z_1 \mid Z_3$ and $Z_3 \mid Z_3$ according to Otneim and Tjøstheim (2018). We exploit this in the following algorithm designed to produce approximate resampled versions of $T_{n,b}$:

1. Use the local correlation estimates to estimate the conditional densities $f_{Z_1 \mid Z_3}(\cdot)$ and $f_{Z_2 \mid Z_3}(\cdot)$ by means of the method by Otneim and Tjøstheim (2018). In practice, in order to reduce the computational load, we evaluate $\hat{f}_{Z_1 \mid Z_3}$ and $\hat{f}_{Z_2 \mid Z_3}$ on a fine grid on their support, over which a continuous representation of the estimates are produced using cubic splines.

2. Using the accept-reject algorithm, generate $B$ samples, each of size $n$, from $\hat{f}_{Z_1 \mid Z_3}$ and $\hat{f}_{Z_2 \mid Z_3}$, leading to $B$ replicates

$$Z^*_m = \{Z_{1,i}^*, Z_{2,i}^*, Z_{3,i}^*\}_i, \quad i = 1, \ldots, n, \quad m = 1, \ldots, B,$$

of $Z$ under the null hypothesis.

3. Calculate $(T_{n,b})^*_m$, $m = 1, \ldots, B$ for the replicated data sets and obtain the approximate $p$-value for the conditional independence test.

### 5.3 Comparing with other tests

Su and White (2008) formulate a simulation experiment for evaluating their nonparametric test for conditional independence by generating data from 10 different data generating processes (DGP$s$), and then check the power and level for their test. Many of the later works that were discussed in Section 5.1 contain very similar experiments, and some of them even present simulation results for tests published prior to Su and White (2008). This allows us to present comprehensive comparisons between the new test presented above, and many competitors.
The null hypotheses of conditional independence between various types of linear and nonlinear time series dependence: distance test that Su and White (2008) present, at different levels of smoothing. Their constant causality test, CM and KS are tests by Linton and Gozalo (1997) that use statistics of the two versions of corresponding results from the literature, and they are presented in tables 1 and 2.

| Test | DGP | Level | Power |
|------|-----|-------|-------|
|      |     | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 9     | 10    |
| CHF  |     | 0.034 | 0.058 | -     | -     | 0.780 | 0.792 | 0.520 | 0.780 | 0.728 | 0.580 |
| CM   |     | 0.054 | 0.058 | 0.060 | 0.048 | 0.920 | 0.548 | 0.504 | 0.412 | 0.384 | 0.188 |
| HEL, c=1 |     | 0.096 | 0.060 | 0.048 | 0.072 | 0.668 | 0.756 | 0.388 | 0.860 | 0.828 | 0.680 |
| HEL, c=1.5 |     | 0.068 | 0.056 | 0.052 | 0.056 | 0.888 | 0.940 | 0.512 | 0.924 | 0.952 | 0.812 |
| HEL, c=2 |     | 0.072 | 0.036 | 0.072 | 0.048 | 0.952 | 0.944 | 0.576 | 0.940 | 0.988 | 0.912 |
| KS   |     | 0.042 | 0.056 | 0.056 | 0.040 | 0.780 | 0.404 | 0.380 | 0.288 | 0.292 | 0.156 |
| LGPC, c=1.0 |     | 0.054 | 0.048 | 0.046 | 0.046 | 0.910 | 0.722 | 0.559 | 0.990 | 0.968 | 0.866 |
| LGPC, c=1.4 |     | 0.047 | 0.043 | 0.046 | 0.047 | 0.971 | 0.855 | 0.727 | 0.969 | 0.916 | 0.765 |
| MCC, c=1 |     | 0.046 | 0.050 | 0.050 | 0.047 | 0.746 | 0.717 | 0.400 | 0.873 | 0.566 | 0.320 |
| MCC, c=1.5 |     | 0.040 | 0.052 | 0.056 | 0.055 | 0.814 | 0.779 | 0.329 | 0.889 | 0.618 | 0.341 |
| MCC, c=2 |     | 0.041 | 0.050 | 0.053 | 0.062 | 0.852 | 0.793 | 0.218 | 0.860 | 0.631 | 0.348 |
| SCM  |     | 0.076 | 0.060 | 0.084 | 0.064 | 0.924 | 0.464 | 0.352 | 0.500 | 0.224 | 0.196 |
| SEL  |     | 0.054 | 0.038 | -     | -     | 0.840 | 0.856 | 0.760 | 0.904 | 0.716 | 0.556 |
| SKS  |     | 0.064 | 0.056 | 0.088 | 0.068 | 0.728 | 0.236 | 0.288 | 0.340 | 0.120 | 0.112 |

Table 1: Level and power, n = 100

Let \((\epsilon_{1,t}, \epsilon_{2,t}, \epsilon_{3,t})\) be IID observations from a \(N(0, I_3)\)-distribution, where \(I_3\) is the 3 × 3 identity matrix. We will test \(H_0: X_{1,t} \perp X_{2,t} | X_{3,t}\) in the following 10 cases taken from Su and White (2008), which cover various types of linear and nonlinear time series dependence:

1. \((X_{1,t}, X_{2,t}, X_{3,t}) = (\epsilon_{1,t}, \epsilon_{2,t}, \epsilon_{3,t})\).
2. \(X_{1,t} = 0.5X_{1,t-1} + \epsilon_{1,t}, X_{2,t} = 0.5X_{2,t-1} + \epsilon_{2,t}, X_{3,t} = X_{1,t-1}\).
3. \(X_{1,t} = \epsilon_{1,t} \sqrt{0.01 + 0.5X_{1,t-1}^2}, X_{2,t} = 0.5X_{2,t-1} + \epsilon_{2,t}, X_{3,t} = X_{1,t-1}\).
4. \(X_{1,t} = \epsilon_{1,t} \sqrt{h_{1,t}}, X_{2,t} = \epsilon_{2,t} \sqrt{h_{2,t}}, X_{3,t} = X_{1,t-1}, h_{1,t} = 0.01 + 0.9h_{1,t-1} + 0.05X_{1,t-1}^2, h_{2,t} = 0.01 + 0.9h_{2,t-1} + 0.05X_{2,t-1}^2\).
5. \(X_{1,t} = 0.5X_{1,t-1} + 0.5X_{2,t} + \epsilon_{1,t}, X_{2,t} = 0.5X_{2,t-1} + \epsilon_{2,t}, X_{3,t} = X_{1,t-1}\).
6. \(X_{1,t} = 0.5X_{1,t-1} + 0.5X_{2,t} + \epsilon_{1,t}, X_{2,t} = 0.5X_{2,t-1} + \epsilon_{2,t}, X_{3,t} = X_{1,t-1}\).
7. \(X_{1,t} = 0.5X_{1,t-1} + 0.5X_{2,t} + \epsilon_{1,t}, X_{2,t} = 0.5X_{2,t-1} + \epsilon_{2,t}, X_{3,t} = X_{1,t-1}\).
8. \(X_{1,t} = 0.5X_{1,t-1} + 0.5X_{2,t} + \epsilon_{1,t}, X_{2,t} = 0.5X_{2,t-1} + \epsilon_{2,t}, X_{3,t} = X_{1,t-1}\).
9. \(X_{1,t} = \epsilon_{1,t} \sqrt{0.01 + 0.5X_{1,t-1}^2 + 0.25X_{2,t}^2}, X_{2,t} = X_{2,t-1} + \epsilon_{2,t}, X_{3,t} = X_{1,t-1}\).
10. \(X_{1,t} = \epsilon_{1,t} \sqrt{h_{1,t}}, X_{2,t} = \epsilon_{2,t} \sqrt{h_{2,t}}, X_{3,t} = X_{1,t-1}, h_{1,t} = 0.01 + 0.1h_{1,t-1} + 0.4X_{1,t-1}^2 + 0.5X_{2,t}^2, h_{2,t} = 0.01 + 0.2h_{2,t-1} + 0.5Y_{2,t}^2\).

The null hypotheses of conditional independence between \(X_{1,t}\) and \(X_{2,t}\) given \(X_{3,t}\) is true for DGP 1–4, and these will be used to check the level of the test, while for DGP 5–10 we measure the power. By evaluating our test using the sample sizes \(n = 100\) and \(n = 200\) at the 5% level, we can harvest a great number of corresponding results from the literature, and they are presented in tables 1 and 2.

The first set of simulation results are collected from Su and White (2008): LIN is a standard linear Granger causality test, CM and KS are tests by Linton and Gozalo (1997) that use statistics of the two versions of the nonparametric test developed by Delgado and Manteiga (2001). Finally, HEL refers to the Helliger distance test that Su and White (2008) present, at different levels of smoothing. Their constant \(c\) has the same meaning as our smoothing constant discussed in Section 4.3.

Next, we move to Su and White (2014), who provide simulations for their test for conditional independence based on the empirical likelihood (SEL), as well as the test by Su and White (2007) that is based on properties
of the conditional characteristic function (CHF). Cheng and Huang (2012) provide simulations for their tests based on the maximal conditional correlation (MCC) at various levels of smoothing.

Finally, we include results from simulations using our new test based on the LGPC, using the trivariate specification defined in Section 4.1.1, for two different levels of smoothing, and include them in Tables 1 and 2. We highlight the new results in grey to indicate that they, as opposed to all the other numbers, have not appeared in the literature before. Also, to the best of our knowledge, these results have not been compared simultaneously before.

We see in Table 1 that our test has the correct level and is quite powerful against all alternative specifications of conditional dependence, in the additive models 5 and 6, as well as the remaining examples 7 to 10, that are more multiplicative in nature.

When \( n = 200 \) we can include simulation results reported by Bouezmarni, Rombouts, and Taamouti (2012) for their conditional independence test based on measuring the Hellinger distance between copula density estimates (shortened BRT from the names of the authors) as well as the results reported by Bouezmarni and Taamouti (2014) on their test based on \( L_2 \) distances between estimated conditional distribution functions, which is abbreviated by BT. We include in Table 2 all results from these papers, which amounts to three levels of smoothing for each method. We see that the test based on the LGPC, in particular the case with \( c = 1.4 \), exhibits the best over-all performance among the examples listed in the table, which, again to the best of our knowledge, include all such simulation results that have been published to date. It is seen that the standard linear Granger causality test (LIN) has a miserable performance in these examples.

Su and White (2014) then define two extensions to a subset of the data generating processes defined above.

| ↓ Test | DGP → | Level | Power |
|--------|--------|-------|-------|
|        |        | 1     | 2     | 3     | 4     | 5     | 6     | 7     | 8     | 9     | 10    |
| BRT, \( c=1 \) | 0.037 | 0.025 | 0.028 | 0.032 | 0.995 | 0.996 | 0.979 | 0.989 | 0.904 | 0.785 |
| BRT, \( c=1.5 \) | 0.044 | 0.025 | 0.025 | 0.037 | 0.971 | 0.993 | 0.931 | 0.997 | 0.931 | 0.759 |
| BRT, \( c=2 \) | 0.064 | 0.023 | 0.023 | 0.052 | 0.943 | 0.979 | 0.873 | 0.997 | 0.912 | 0.728 |
| BT, \( c_1=1, c_2=1 \) | 0.047 | 0.051 | 0.041 | 0.053 | 0.996 | 0.812 | 0.852 | 1.000 | 0.936 | -     |
| BT, \( c_1=0.85, c_2=0.7 \) | 0.048 | 0.044 | 0.064 | 0.056 | 0.988 | 0.728 | 0.792 | 1.000 | 0.908 | -     |
| BT, \( c_1=0.75, c_2=0.6 \) | 0.036 | 0.048 | 0.052 | 0.052 | 0.976 | 0.719 | 0.808 | 1.000 | 0.896 | -     |
| CHF   | 0.046 | 0.042 | -     | -     | 0.976 | 0.988 | 0.820 | 0.952 | 0.944 | 0.864 |
| CM    | 0.044 | 0.056 | 0.060 | 0.048 | 0.992 | 0.740 | 0.788 | 0.680 | 0.476 | 0.360 |
| HEL, \( c=1 \) | 0.064 | 0.052 | 0.080 | 0.080 | 0.900 | 0.960 | 0.596 | 0.992 | 0.968 | 0.880 |
| HEL, \( c=1.5 \) | 0.064 | 0.056 | 0.048 | 0.036 | 0.980 | 1.000 | 0.808 | 0.992 | 0.972 | 0.972 |
| HEL, \( c=2 \) | 0.044 | 0.060 | 0.056 | 0.048 | 1.000 | 1.000 | 0.864 | 1.000 | 1.000 | 0.996 |
| KS    | 0.068 | 0.053 | 0.048 | 0.084 | 0.952 | 0.552 | 0.660 | 0.532 | 0.336 | 0.284 |
| LGPC, \( c=1.0 \) | 0.039 | 0.052 | 0.054 | 0.054 | 0.995 | 0.948 | 0.818 | 1.000 | 1.000 | 0.985 |
| LGPC, \( c=1.4 \) | 0.042 | 0.057 | 0.058 | 0.042 | 1.000 | 0.993 | 0.956 | 1.000 | 1.000 | 0.958 |
| LIN   | 0.043 | 0.053 | 0.042 | 0.050 | 1.000 | 0.354 | 0.250 | 0.113 | 0.172 | 0.143 |
| MCC, \( c=1 \) | 0.049 | 0.051 | 0.057 | 0.054 | 0.982 | 0.983 | 0.831 | 1.000 | 0.947 | 0.679 |
| MCC, \( c=1.5 \) | 0.046 | 0.048 | 0.049 | 0.053 | 0.995 | 0.989 | 0.872 | 1.000 | 0.968 | 0.738 |
| MCC, \( c=2 \) | 0.045 | 0.045 | 0.047 | 0.057 | 0.997 | 0.995 | 0.735 | 1.000 | 0.971 | 0.745 |
| SCM   | 0.048 | 0.060 | 0.064 | 0.068 | 0.980 | 0.648 | 0.620 | 0.720 | 0.352 | 0.280 |
| SEL   | 0.052 | 0.033 | -     | -     | 0.992 | 1.000 | 0.972 | 1.000 | 0.884 | 0.864 |
| SKS   | 0.056 | 0.028 | 0.064 | 0.072 | 0.964 | 0.324 | 0.512 | 0.552 | 0.148 | 0.136 |

Table 2: Level and power, \( n = 200 \)
We report the level and power results obtained by Su and White (2014) for the methods CHF, CM, KS and the new test based on the multivariate simplification of the LGPC as defined in Section 4.1.2. The second extension introduced by Su and White (2014) increases the dimension of the conditioning variable but where \( \epsilon_{3,t} \sim \mathcal{N}(0, I_3) \).

The first extension turns the conditioning variable in DGP1-DGP2 and DGP5-DGP10 into a bivariate vector \( X_{3,t} \), where we define DGP1’ in the same way as above,

1’. \( (X_{1,t}, X_{2,t}, X_{3,t}) = (\epsilon_{1,t}, \epsilon_{2,t}, \epsilon_{3,t}) \),

but where \( \epsilon_{3,t} \sim \mathcal{N}(0, I_2) \), and where we define DGP2’ and DGP5’-DGP10’ by setting \( X_{3,t} = (X_{1,t-1}, X_{1,t-2}) \), keeping \( X_{2,t} \) as above, and,

2’. \( X_{1,t} = 0.5X_{1,t-1} + 0.25X_{1,t-2} + \epsilon_{1,t}, \)

3’. \( X_{1,t} = 0.5X_{1,t-1} + 0.25X_{1,t-2} + 0.5X_{2,t} + \epsilon_{1,t}, \)

4’. \( X_{1,t} = 0.5X_{1,t-1} + 0.25X_{1,t-2} + 0.5X_{2,t}^2 + \epsilon_{1,t}, \)

5’. \( X_{1,t} = \sqrt{n} \epsilon_{1,t}, \) \( h_t = 0.01 + 0.5X_{2,t-1}^2 + 0.25X_{1,t-2}^2 + 0.25X_{2,t}^2 \), and

10’. Same as DGP10 above, except for the new definition of \( X_{3,t}. \)

We report the level and power results obtained by Su and White (2014) for the methods CHF, CM, KS and SEL on these data by testing the null hypothesis

\[ H_0 : X_{t,1} \perp X_{t,2} \mid X_{t,3}, \]

and include results from our new test based on the multivariate simplification of the LGPC as defined in Section 4.1.2. The results are quite acceptable and compares well with other nonparametric methods.

The second extension introduced by Su and White (2014) increases the dimension of the conditioning variable \( X_{3,t} \) once more, so that \( \epsilon_{3,t} \sim \mathcal{N}(0, I_3) \), and where DGP2’ and DGP5’-DGP10’ are defined by by setting \( X_{3,t} = (X_{1,t-1}, X_{1,t-2}, X_{1,t-3}) \), and, 

| Sample size | Test | DGP → | Level | Power |
|-------------|------|--------|-------|-------|
|             |      |        | 1’    | 2’    | 5’    | 6’    | 7’    | 8’    | 9’    | 10’   |
| n = 100     | CHF  |        | 0.028 | 0.042 | 0.720 | 0.704 | 0.412 | 0.564 | 0.460 | 0.556 |
|             | CM   |        | 0.028 | 0.016 | 0.656 | 0.360 | 0.108 | 0.512 | 0.164 | 0.208 |
|             | KS   |        | 0.040 | 0.020 | 0.400 | 0.284 | 0.056 | 0.380 | 0.124 | 0.176 |
|             | LGPC, c=1.75 |         | 0.048 | 0.033 | 0.951 | 0.668 | 0.294 | 0.517 | 0.406 | 0.538 |
|             | SEL  |        | 0.052 | 0.040 | 0.844 | 0.828 | 0.620 | 0.568 | 0.440 | 0.528 |
| n = 200     | CHF  |        | 0.030 | 0.040 | 0.948 | 0.944 | 0.748 | 0.828 | 0.724 | 0.860 |
|             | CM   |        | 0.050 | 0.032 | 0.940 | 0.588 | 0.304 | 0.792 | 0.304 | 0.364 |
|             | KS   |        | 0.046 | 0.024 | 0.776 | 0.432 | 0.168 | 0.696 | 0.216 | 0.284 |
|             | LGPC, c=1.75 |         | 0.059 | 0.035 | 0.998 | 0.923 | 0.415 | 0.769 | 0.698 | 0.833 |
|             | SEL  |        | 0.058 | 0.026 | 0.972 | 0.988 | 0.932 | 0.832 | 0.684 | 0.832 |
| n = 400     | CHF  |        | 0.040 | 0.036 | 1.000 | 0.984 | 0.972 | 0.996 | 0.920 | 0.984 |
|             | CM   |        | 0.056 | 0.024 | 1.000 | 0.884 | 0.552 | 0.980 | 0.556 | 0.668 |
|             | KS   |        | 0.060 | 0.024 | 1.000 | 0.732 | 0.324 | 0.952 | 0.384 | 0.524 |
|             | LGPC, c=1.75 |         | 0.048 | 0.018 | 1.000 | 0.996 | 0.584 | 0.960 | 0.928 | 0.978 |
|             | SEL  |        | 0.040 | 0.030 | 1.000 | 1.000 | 1.000 | 1.000 | 0.836 | 0.884 |

Table 3: Level and power, 4-dimensional data
Again, we observe simulation results in Table 4. Contrary to Su and White (2014), we have also run our test based on the LGPC performs mostly on par with other non-parametric tests in this setting as well.

The full potential of the LGPC-test may not have been reached in these experiments. If one compares with the recent econometric literature on this topic. The results are promising, and suggest that the newly developed test provide powerful improvements to existing methods.

The purpose of this paper is two-fold. First we define and develop the LGPC, a local measure of conditional dependence that has useful properties and is easy to interpret. We then explore the possibility of using the LGPC in estimation of conditional dependence and in testing for conditional independence, which contributes to the recent econometric literature on this topic. The results are promising, and suggest that the newly developed test provide powerful improvements to existing methods.

For 3 scalar variables $X_1, X_2, X_3$ a full trivariate analysis depending on all 3 coordinates can be undertaken as in Section 4.1.1. For a higher dimensional conditioning variable $X_3$ a pairwise approximation and simplification

| Sample size | Test | DGP | Level | Power |
|-------------|------|-----|-------|-------|
| $n = 100$   | LGPC, $c=1.75$ | DGP → | 0.068 | 0.911 |
| $n = 200$   | CHF  | 0.028 | 0.964 | 0.852 |
|             | CM   | 0.050 | 0.756 | 0.192 |
|             | KS   | 0.048 | 0.500 | 0.096 |
|             | LGPC, $c=1.75$ | 0.062 | 0.993 | 0.363 |
|             | SEL  | 0.056 | 0.996 | 0.860 |
| $n = 400$   | CHF  | 0.032 | 1.000 | 0.792 |
|             | CM   | 0.050 | 0.992 | 0.400 |
|             | KS   | 0.044 | 0.840 | 0.220 |
|             | LGPC, $c=1.75$ | 0.048 | 0.938 | 0.538 |
|             | SEL  | 0.056 | 1.000 | 0.888 |

Table 4: Level and power, 5-dimensional data.

$2'$. $X_{1,t} = 0.5X_{1,t-1} + 0.25X_{1,t-2} + 0.125X_{1,t-3} + \epsilon_{1,t}$.
$5'$. $X_{1,t} = 0.5X_{1,t-1} + 0.25X_{1,t-2} + 0.125X_{1,t-3} + 0.5X_{2,t} + \epsilon_{1,t}$.
$6'$. $X_{1,t} = 0.5X_{1,t-1} + 0.25X_{1,t-2} + 0.125X_{1,t-3} + 0.5X_{2,t} + \epsilon_{1,t}$.
$7'$. $X_{1,t} = 0.5X_{1,t-1}X_{2,t} + 0.25X_{1,t-2} + 0.125X_{1,t-3} + \epsilon_{1,t}$.
$8'$. $X_{1,t} = 0.5X_{1,t-1} + 0.25X_{1,t-2} + 0.125X_{1,t-3} + 0.5X_{2,t}\epsilon_{1,t}$.
$9'$. $X_{1,t} = \sqrt{h_t}\epsilon_{1,t}$, $h_t = 0.01 + 0.5X_{1,t-1}^2 + 0.25X_{1,t-2}^2 + 0.125X_{1,t-3}^2 + 0.25X_{2,t}^2$, and $10'$. Same as DGP10 and DGP10' above, except for the new definition of $X_{3,t}$.

Again, we observe simulation results in Table 4. Contrary to Su and White (2014), we have also run our test for $n = 100$ which reveals that we can obtain some power also in that case. All in all, the simplified test on the LGPC performs mostly on par with other non-parametric tests in this setting as well.

The full potential of the LGPC-test may not have been reached in these experiments. If one compares with independence testing using the local Gaussian correlation a considerable increase in power was obtained by focusing the tests appropriately, see Berentsen and Tjøstheim (2014). This was done by exploiting the typical local dependence pattern for financial variables. To do this here, one must ascertain whether such patterns exist for local conditional dependence. Finally, the simplified test based on pairwise relations may be better able to fight the curse of dimensionality as the dimension $p$ of $X^{(2)} = (X_3, \ldots, X_p)$ increases, cf. Otneim and Tjøstheim (2018).

6 Conclusion and outlook

The purpose of this paper is two-fold. First we define and develop the LGPC, a local measure of conditional dependence that has useful properties and is easy to interpret. We then explore the possibility of using the LGPC in estimation of conditional dependence and in testing for conditional independence, which contributes to the recent econometric literature on this topic. The results are promising, and suggest that the newly developed test provide powerful improvements to existing methods.

For 3 scalar variables $X_1, X_2, X_3$ a full trivariate analysis depending on all 3 coordinates can be undertaken as in Section 4.1.1. For a higher dimensional conditioning variable $X_3$ a pairwise approximation and simplification
has been described in Section 4.1.2. This can be likened to additive approximation in nonparametric regression. Both the full trivariate approach and the pairwise simplification have been mainly analyzed at the 2-scale using the transformed variables $Z = \Phi^{-1}(F(X))$. The corresponding LGPC $\alpha(z)$ is invariant to monotone transformations to the marginals. The LGPC $\alpha(x)$ on the 2-scale can be obtained from $\alpha(z)$ by taking $x = F^{-1}(\Phi(z))$, and it is in general different from $\alpha(z)$ and is not invariant to monotone transformations of the marginals. An analogue is the different transformation properties of the Pearson correlation and the copula structure in describing joint dependence. For multivariate Gaussian variables, $\alpha(x) \equiv \alpha(z) \equiv \alpha$, which is the ordinary global partial correlation.

There is a potential for much further work. The $X_1$ and $X_2$ variables can be made into vectors leading to a local dependence between groups of variables. A next natural step may be structural equations as used in network analysis and Pearls-type causality (Pearl 2000), and ultimately further exploring the relationship between that type of causality and Granger causality.

All methods presented in this paper have been implemented in the R programming language, and is available in the package Ig (Håkon Otneim 2019). In addition, we provide code and data for easy replication of the results presented here in an online appendix.

7 Appendix

7.1 Some details regarding the proof of Theorem 1

For fixed bandwidths we have from standard arguments, such as those provided by Hjort and Jones (1996), that

$$\sqrt{n}b^3(\hat{\rho}_n - \rho_0) \xrightarrow{a.s.} \mathcal{N}(0, J_b^{-1}M_b(J_b^{-1})^T),$$

which corresponds to the usual rate for nonparametric density estimation. As $b \to 0$, however, we must take extra care when considering the asymptotic covariance matrix. Tjostheim and Hufthammer (2013), Section 4, write the Taylor expansions of $M_b$ and $J_b$, being functionals of three variables $(x_1, x_2, x_3)$ or $(z_1, z_2, z_3)$, as

$$M_b = I_M + II_M + o(b^3) \quad \text{and} \quad J_b = I_J + II_J + o(b^2),$$

and consider each term. For example, we write their first term in the expansion of $M_b$ in the three-variate case as follows:

$$I_M \sim \int K^2(w_1, w_2, w_3)Ab_w b_w^T A f(z + b_1 w_1 + b_2 w_2 + b_3 w_3) \, dw_1 \, dw_2 \, dw_3,$$

where $K$ is a product kernel, and, in our case, $b_w$ is the vector defined by $b_w^T = \begin{pmatrix} 1 & b_1 & b_2 & b_3 \end{pmatrix}$, and $A$ is the $3 \times 4$ matrix $A = \begin{pmatrix} v & v_{z_1} & v_{z_2} & v_{z_3} \end{pmatrix}$ with $v^T = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$, $v_i = \partial L_n/\partial \rho_i$, where the index $i = 1, 2, 3$ represents the three local correlations, and $v_{z_i} = \partial v/\partial z_i$. In the next step, they compute the matrix $\int K^2(w_1, w_2, w_3)b_w b_w^T \, dw_1 \, dw_2 \, dw_3$, which, by omitting all constant factors and exploiting that $\int w_k K^2(w) \, dw = 0$ for $k = 1, 3$, in our case becomes the diagonal matrix

$$H = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & b_1^2 w_1^2 & 0 & 0 \\
0 & 0 & b_2^2 w_2^2 & 0 \\
0 & 0 & 0 & b_3^2 w_3^2
\end{pmatrix}.$$

The second term defining $M_b$ has smaller order than $I_M$ as $b \to 0$, while the first term of $J_b$ can be treated exactly as $I_M$ above resulting in a matrix of order $b^2$. 23
The second term in $J_b$ can similarly be written as $H(J_b) = B(\psi(z, \theta_b(z)) - f(z)) + o(b^2)$, where $B = \nabla v(z, \theta_b(z))$ are second derivatives of the local likelihood function with respect to the local correlations, and $\psi(z, \theta_b(z)) - f(z) = O(b^2)$ according to Hjort and Jones [1996]. Hence, it follows that the leading term of $J_b^{-1} M_b [J_b^{-1}]^T$ is $O(b^{-2})$, which again means that the convergence rate in eq. (28) must be modified accordingly in order to balance the convergence in eq. (19). Extracting an analytic expression for the leading term of the asymptotic covariance matrix is, as mentioned by Tjøstheim and Hufthammer [2013], possible by means of symbolic manipulation software, but the expression is very complicated and of little practical use. See also the corresponding proof of Theorem 3 in Tjøstheim and Hufthammer [2013].

7.2 The asymptotic variance of the LGPC

The limiting distribution of the LGPC is given in eq. (22). In this section, we will present the basic steps needed to work out the value of $\nabla g(\rho)$. For the sake of this particular argument, we may simplify notation quite a bit, leaving out what we do not need: We drop the $z$-dependence, and simply write $\Sigma = \Sigma_{12}(z)$ for the $2 \times 2$ local partial covariance matrix between $Z_1$ and $Z_2$ given $(Z_3, \ldots, Z_p)$. The vector of local correlations $\rho$ is indexed by $k$, which in turn means that $k$ corresponds to a specific pair $(i, j)$: $\rho_k = \rho_{ij}$, with $\rho_{ij} = \rho_{ji}$. Let $\Sigma(k)$ be the element-wise matrix of partial derivatives

$$\Sigma(k) = \frac{\partial \Sigma}{\partial \rho_{ij}} = \frac{\partial \Sigma}{\partial \rho_{kj}},$$

which means that element $k$ in the gradient $\nabla g(\rho)$ is given by

$$\langle \nabla g(\rho) \rangle_k = \frac{\Sigma_{111} \Sigma_{222} \Sigma_{12}^{(k)} - \Sigma_{12} \left( \Sigma_{111}^{(k)} \Sigma_{222} + \Sigma_{111} \Sigma_{222}^{(k)} \right)}{(\Sigma_{111} \Sigma_{222})^2},$$

(29)

where the double subscript to $\Sigma$ and $\Sigma(k)$ here means matrix elements. In order to avoid confusion with the double subscripts defining matrix partitions in (7), we re-label the matrix blocks $R_{11}, R_{12}$ and $R_{22}$ to $R_1, R_2$ and $R_3$ respectively, with $R_{21} = R_2^T$. Following basic differentiation rules for matrices (see e.g. Van den Bos [2007]), we have that, element-wise,

$$\frac{\partial (R_2 R_3^{-1} R_2^T)}{\partial \rho_{jk}} = \frac{\partial R_2}{\partial \rho_{jk}} R_3^{-1} R_2^T - R_2 R_3^{-1} \frac{\partial R_3}{\partial \rho_{jk}} R_3^{-1} R_2^T + R_2 R_3^{-1} \frac{\partial R_2^T}{\partial \rho_{jk}} R_2^{-1} R_3^{-1} R_2^T.$$

(30)

The value of the gradient \( \langle \nabla g(\rho) \rangle_k \) depends on $k$, and when differentiating (7) with respect to $\rho_k$ there are three different cases we must consider:

1. If $\rho_k = \rho_{12}$ the second term in (7) vanishes, because $\rho_{12}$ is only present in $R_1$. Indeed,

$$R_{11} = \begin{pmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{pmatrix},$$

which means that

$$\Sigma(k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

and we easily see from (29) that in this particular case, \( \langle \nabla g(\rho) \rangle_k \) = 1.

2. If $\rho_k = \rho_{ij}$ where $i \in \{1, 2\}$ and $j \in \{3, \ldots, p\}$, then $\rho_{ij}$ is an element in $R_2$, but not $R_1$ or $R_3$. For that reason, we are left with the second term in (7), and furthermore, the middle term in the derivative (30) above will also vanish. The $2 \times (p - 2)$ matrix $\partial R_{12}/\partial \rho_{ij}$ will contain only zeros except for a 1 in the position of $\rho_{ij}$; and $\partial R_{21}/\partial \rho_{ij}$ is its transposed. Write $C = R_{22} R_{21}$, and we see that $\Sigma(k)$ in this case is a simple linear combination of the elements in $C$:

$$\Sigma(k) = \frac{\partial R_{12}}{\partial \rho_{ij}} C + C^T \frac{\partial R_{21}}{\partial \rho_{ij}}.$$
from which we select the elements needed to calculate \( \{ \nabla g(\rho) \}_k \) through (29).

3. In this case, we look at the derivative of \( g(\rho) \) with respect to the local correlations located in \( R_3 \), that is, \( \rho_{ij} \) for \( i,j \in \{3,\ldots,p\} \). We only need to keep the middle term in (30). We have that

\[
\Sigma^{(k)} = -C^T \frac{\partial R_3}{\partial \rho_k} C,
\]

where \( \frac{\partial R_3}{\partial \rho_k} \) is a symmetric \((p-2) \times (p-2)\) matrix of zeros everywhere, except for 1s in the positions corresponding to \( \rho_k \) with respect to which we perform the differentiation. Finally, equation (30) provides the final expression for \( \{ \nabla g(\rho) \}_k \).

### 7.3 Proof of Theorem 3

Theorem 3 states that using marginally standard normal pseudo-observations instead of exactly standard normally distributed observations for estimating the LGPC does not change the conclusions in Theorems 1 and 2. The result may be expected, as the empirical distribution functions \( \hat{F}_i, i = 1, \ldots, p \) converge at a rate faster than the local correlations \( n^{-\frac{1}{2}} \) vs. \((nb^2)^{-\frac{1}{2}}\) or \((nb^2)^{-\frac{1}{2}}\) depending on whether the rate in Theorem 1 or 2 is used), but we must attend to some details nevertheless.

Proving joint asymptotic normality of the local Gaussian correlations with marginally normally distributed variables \((Z_1, \ldots, Z_p) = (\Phi^{-1}(F_1(X_1)), \ldots, \Phi^{-1}(F_p(X_p)))\), where \( F_i \) is the cdf of \( X_i \) for \( i = 1, \ldots, p \), relies on proving asymptotic normality for the variables

\[
Y_n(z) = \frac{1}{n} \sum_{i=1}^n K_b(Z_i - z) v(Z_i, \rho_0),
\]

where \( v(\cdot) = \partial L(\cdot)/\partial \rho \) in the same way as in Section 7.1. This has been done by Otneim and Tjøstheim (2017) in the iid case using arguments from Schervish (2012), and by Otneim and Tjøstheim (2018) in the \( \alpha \)-mixing case, using arguments from Fan and Yao (2008). We need here to prove asymptotic normality of the variables

\[
\hat{Y}_n(z) = \frac{1}{n} \sum_{i=1}^n K_b(\hat{Z}_i - z) v(\hat{Z}_i, \rho_0),
\]

with \( \hat{Z}_i = \Phi^{-1}(F_n(X_i)) \), and \( F_n(X_i) \) is the application of the marginal empirical distribution function to the corresponding components of the vector \( X_i \).

In order to ease notation we state the proof of Theorem 3 in the simplified case only, where the local correlations are estimated using pairs of variables, which is the case that is treated in Theorem 2. It can be extended to the trivariate case (that we treat in Theorem 1) by adding the appropriate terms in the derivations below, without changing the final conclusion. Write \( \hat{Y}_n(z) = Y_n(z) - (Y_n(z) - \hat{Y}_n(z)) \) and do a Taylor expansion of \( Y_n(z) \) around \( F_n(X_i) \). Since we use product kernels, we then need derivatives of (assuming without loss of generality that \( K(x) = K(-x) \))

\[
H_{x,b} = K \left( \frac{z_1 - \Phi^{-1}(y_1)}{b} \right) K \left( \frac{z_2 - \Phi^{-1}(y_2)}{b} \right) v(\Phi^{-1}(y)),
\]

where \( y = F(x), y_i = F(x_i) \), and we write \( K(x_1, x_2) = K(x_1)K(x_2) \). Writing \( k(z) = K'(z) \) and \( v_{y_i} = \partial v_i/\partial y_i \),
\[
\frac{\partial H_{z,b}}{\partial y_1} = k \left( \frac{z_1 - \Phi^{-1}(y_1)}{b} \right) K \left( \frac{z_2 - \Phi^{-1}(y_2)}{b} \right) \frac{v(\Phi^{-1}(y))}{b\phi(\Phi^{-1}(y_1))} + K \left( \frac{z_1 - \Phi^{-1}(y_1)}{b} \right) K \left( \frac{z_2 - \Phi^{-1}(y_2)}{b} \right) \frac{v_1(\Phi^{-1}(y))}{\phi(\Phi^{-1}(y_1))},
\]

\[
\frac{\partial H_{z,b}}{\partial y_2} = K \left( \frac{z_1 - \Phi^{-1}(y_1)}{b} \right) K \left( \frac{z_2 - \Phi^{-1}(y_2)}{b} \right) \frac{v(\Phi^{-1}(y))}{b\phi(\Phi^{-1}(y_2))} + K \left( \frac{z_1 - \Phi^{-1}(y_1)}{b} \right) K \left( \frac{z_2 - \Phi^{-1}(y_2)}{b} \right) \frac{v_2(\Phi^{-1}(y))}{\phi(\Phi^{-1}(y_1))}.
\]

A typical term in the Taylor expansion of \( Y_n(z) \) takes the following form:

\[
K_b \left( \Phi^{-1}(F(X_i)) - z \right) v \left( \Phi^{-1}(F(X_i)), \rho_0 \right) = K_b \left( \Phi^{-1}(F_n(X_i)) - z \right) v \left( \Phi^{-1}(F_n(X_i)), \rho_0 \right) + k \left( \frac{z_1 - \Phi^{-1}(F_n(X_{i,1}))}{b} \right) K \left( \frac{z_2 - \Phi^{-1}(F_n(X_{i,2}))}{b} \right) \frac{v(\Phi^{-1}(F_n(X_i)))}{b\phi(\Phi^{-1}(F_n(X_{i,1})))} (F(X_{i,1}) - F_n(X_{i,1})) + K \left( \frac{z_1 - \Phi^{-1}(F_n(X_{i,1}))}{b} \right) K \left( \frac{z_2 - \Phi^{-1}(F_n(X_{i,2}))}{b} \right) \frac{v_1(\Phi^{-1}(F_n(X_i)))}{\phi(\Phi^{-1}(F_n(X_{i,1})))} (F(X_{i,1}) - F_n(X_{i,1})) + \text{analogous terms for the second (and possibly third) index involving } (F(X_{i,2}) - F_n(X_{i,2})),
\]

where \( F_n^* \) comes from the mean value theorem. The challenge in proving the desired result is to control the behavior of quantities on the form \( 1/\phi(\Phi^{-1}(F(x))) \), because this fraction tends to infinity as \( |x| \to \infty \). The key to this problem is the assumption that the kernel function \( K(\cdot) \), and thus its derivative \( k \), have compact support. This means, that in the expressions above, \( K = k \equiv 0 \) if for some \( M > 0 \)

\[
\frac{|z_1 - \Phi^{-1}(F_n^*(X_{i,1}))|}{b} \geq M,
\]

or

\[
|z_1 - \Phi^{-1}(F_n^*(X_{i,1}))| \geq Mb,
\]

or, by removing the absolute value signs,

\[
\Phi^{-1}(F_n^*(X_{i,1})) \leq z_1 - Mb \quad \text{or} \quad \Phi^{-1}(F_n^*(X_{i,1})) \geq z_1 + Mb,
\]

where \( M \) may be large. The same reasoning applies of course to the second index as well. Letting \( n \to \infty \) and using consistency of the empirical distribution function, it follows that the kernel function term is zero if

\[
X_{i,1} \leq F^{-1}(\Phi(z_1 - Mb)) \quad \text{or} \quad X_{i,1} \geq F^{-1}(\Phi(z_1 + Mb)).
\]

The same reasoning applies to the derivative of the kernel function, where we consider the function \( k/b \) instead of \( K \). All of this implies that the challenge of controlling the magnitude of the Taylor terms above as \( x \to \infty \) disappears, since, taking boundedness of the other terms into account,

\[
k \left( \frac{z_1 - \Phi^{-1}(F_n^*(X_{i,1}))}{b} \right) K \left( \frac{z_2 - \Phi^{-1}(F_n^*(X_{i,2}))}{b} \right) \frac{v(\Phi^{-1}(F_n^*(X_i)))}{b\phi(\Phi^{-1}(F_n^*(X_{i,1})))} \quad \text{and} \quad K \left( \frac{z_1 - \Phi^{-1}(F_n^*(X_{i,1}))}{b} \right) K \left( \frac{z_2 - \Phi^{-1}(F_n^*(X_{i,2}))}{b} \right) \frac{v_1(\Phi^{-1}(F_n^*(X_i)))}{\phi(\Phi^{-1}(F_n^*(X_{i,1})))}
\]

are bounded almost surely as \( n \to \infty \) and \( b \to 0 \).
All this means that if we can prove that

\[ \frac{1}{n} \sum_{i=1}^{n} (F_n(X_i) - F(X_i)) \overset{P}{\to} 0 \]

at an appropriate rate, then we are done by Slutsky’s Theorem.

We need a bound on the supremum of \( |F_n(X_i) - F(X_i)| \) and \( |F_n(x) - F(x)| \) as \( n \to \infty \). According to Arcones [1995], the best law of the iterated logarithm has been obtained by Rio [1995]. One statement of this is as follows:

Let \( f(x) \in L_p, 2 < p \leq \infty, \sum \alpha k2^{(p-2)} < \infty \), then the sequence \( f(X_i) \) satisfies the compact LIL (law of the iterated logarithms), i.e.

\[
\left\{ (n2\log \log n)^{-\frac{1}{2}} \sum_{i=1}^{n} (f(X_i) - E[f(X_i)]) \right\}_{n=1}^{\infty}
\]

is relatively compact and its limit set is \([-\sigma, \sigma]\) where

\[ \sigma^2 = \text{Var}(f(X_i)) + 2 \sum_{j=1}^{\infty} \text{Cov}(f(X_i), f(X_j)). \]

For a fixed \( x \), take \( f(X_i) = I(X_i \leq x) \). Then \( E(f(X_i)) = F(x) \) and \( f(x) \in L_p \) for all \( p \), and

\[
\left\{ (n2\log \log n)^{-\frac{1}{2}} \sum_{j=1}^{n} (I(X_j \leq x) - F(x)) \right\}
\]

has its limit in \([-\sigma, \sigma]\). Here

\[ \text{Var}(I(X_i \leq x)) = F(x)(1 - F(x)) \leq \frac{1}{4}, \]

and according to Davydov’s Lemma (see e.g. Fan and Yao [2008] p. 278),

\[ |\text{Cov}(I(X_i \leq x), I(X_j \leq x))| \leq 8\|I(X_i \leq x)\|_p\|I(X_j \leq x)\|_q \alpha(|j - i|)^{1-p^{-1}-q^{-1}}, \quad p, q > 1, \quad \frac{1}{p} + \frac{1}{q} < 1, \]

with \( \|I(X_i \leq x)\|_p = (E|I(X_i \leq x)|^p)^{1/p} \). We may for example take \( p = q = 4 \), leading to \( \|I(X_i \leq x)\|_4 = F(x)^{1/4} \leq 1 \).

Since we have assumed exponential mixing (much weaker conditions suffice), then Rio’s condition \( \sum \alpha k2^{(p-2)} < \infty \) is satisfied, and the bounds of \(-\sigma\) and \( \sigma \) can be made independent of \( x \).

Having

\[ F_n(x) = \frac{1}{n} \sum_{j=1}^{n} I(X_j \leq x) \]

means that

\[ n^{\frac{1}{2}} [2\log \log n]^{-\frac{1}{2}} (F_n(x) - F(x)) \]

has its limits in \([-\sigma, \sigma]\) or that the supremum of \( |F_n(x) - F(x)| \) is almost surely bounded by \( n^{-1/2} (2\log \log n)^{1/2} \).

This immediately implies that

\[ \frac{1}{n} \sum_{i=1}^{n} (F_n(X_i) - F(X_i)) \]

is almost surely (and consequentially in probability) bounded by \( n^{-\frac{1}{2}} (2\log \log n)^{\frac{1}{2}} \) which for all conceivable bandwidths is faster than the convergence rates of the CLT for \( Y_n(z) \), which is \( (nb^2)^{-\frac{1}{2}} \).
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