On Taylor series expansion of \((1 + z)^A\) for \(|z| > 1\)

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Abstract

It is well known that the Taylor series expansion of \((1 + z)^A\) does not converge for \(|z| > 1\) where \(A\) is a real number which is not equal to zero or a positive integer. A limited series expansion of this expression is obtained in this paper for \(|z| > 1\) as a product of convergent series.

Keywords:

1. Introduction

It is well known that the Taylor series expansion of \((1 + z)^A\) is given by

\[
(1 + z)^A = \sum_{n=0}^{\infty} \binom{A}{n} z^n
\]  

(1)

for \(|z| < 1\) where \(\binom{A}{n}\) is the binomial choose function. It is well known that the series expansion does not converge for \(|z| > 1\) where \(A\) is a real number which is not equal to zero or a positive integer.

We could obtain a limited series expansion for \(|z| > 1\) by writing the above expression as follows

\[
(1 + z)^A = (1 + \frac{z}{2} + \frac{z}{2})^A = (1 + \frac{z}{2})^A(1 + \frac{z}{1 + \frac{z}{2}})^A = (1 + \frac{z}{2})^A(1 + \frac{z}{z + 2})^A
\]  

(2)

The second term in the above equation has a convergent series representation, given that \(|\frac{z}{z+2}| < 1\). If \(|\frac{z}{z+2}| > 1\), we can write

\[
(1 + \frac{z}{2})^A = (1 + \frac{z}{4})^A(1 + \frac{z}{z + 4})^A
\]  

(3)

Repeating this procedure iteratively, if \(m_0\) is the minimum value for which \(|\frac{z}{2m_0}| < 1\), we can write

\[
(1 + z)^A = (1 + \frac{z}{2m_0})^A \prod_{r=1}^{m_0}(1 + \frac{z}{z + 2r})^A
\]  

(4)

Each of the terms in the above product of terms has a convergent series representation. Given that we can write the convergent series expansion for each of the terms above as \((1 + \frac{z}{2m})^A = \)
\[ \sum_{m=0}^{\infty} \binom{A}{m} \left( \frac{z}{z+2} \right)^m = \sum_{m=0}^{\infty} \binom{A}{m} \left( \frac{z}{z+2} \right)^m, \]

where \( \binom{A}{m} \) represents the Choose function \([1]\), we have the series expansion for \((1 + z)^A\) expressed as a product of convergent series, which converges for \(|z| > 1\) as follows:

\[ (1 + z)^A = \left[ \sum_{n=0}^{\infty} \binom{A}{n} \left( \frac{z}{2^{m+1}} \right)^n \prod_{r=1}^{m+1} \sum_{m=0}^{\infty} \binom{A}{m} \left( z + 2r \right)^m \right] \]

(5)

2. Section 2

Let us consider the following binomial expression

\[ \left( \frac{1}{z+2} \right)^m \]

in the above equation 5 as Taylor series around a point \( z = 0 \), we have for \( m > 0 \)

\[ \frac{1}{(z+2)^m} = \sum_{j=0}^{\infty} b(j, r, m) z^j \]

(6)

where \( b(0, r, m) = \frac{1}{z+2} \) and \( b(j, r, m) \) is given as follows for \( j = 1, 2, 3... \)

\[ b(j, r, m) = \binom{m+j-1}{j} \frac{(-1)^j}{(2r)^m+j}; \]

(7)

For \( m = 0 \), \( \frac{1}{z+2} = 1 \). Now we can write the series expansion of \((1 + z)^A\) which converges for \( 1 < |z| < 2 \), as a product of terms expanded in Taylor series as follows:

\[ (1 + z)^A = \left[ \sum_{n=0}^{\infty} \binom{A}{n} \left( \frac{z}{2} \right)^n \prod_{r=1}^{m+1} \sum_{m=0}^{\infty} \binom{A}{m} z^m \sum_{j=0}^{\infty} b(j, 1, m) z^j \right] \]

(8)

For the case of \( m_0 > 1 \) for \( |z| > 2 \), we can write as follows:

\[ (1 + z)^A = \left[ \sum_{n=0}^{\infty} \binom{A}{n} \left( \frac{z}{2} \right)^n \prod_{r=1}^{m_0} \sum_{m=0}^{\infty} \binom{A}{m} z^m \sum_{j=0}^{\infty} b(j, m_0, m) z^j \prod_{r=1}^{m_0-1} \sum_{m=0}^{\infty} \binom{A}{m} z^m [1 \left( \frac{1}{z+2r} \right)^m] \right] \]

(9)

The last term in the above equation \( \left( \frac{1}{z+2} \right)^m \) can be expressed as follows:

\[ \left( \frac{1}{z+2} \right)^m = (z+2)^{-m} = 2^{-r-m}(1 + \frac{z}{2r})^{-m} \]

(10)

The term \((1 + \frac{z}{2r})^{-m}\) can be recursively expanded using Eq.9 by substituting \( z \to \frac{z}{2r} \) and \( A \to -m \) and \( m_0 \to m_0 - r \) to obtain the series expansion of \((1 + z)^A\) which converges for \(|z| > 1\) as a product of terms expanded in Taylor series.

3. Section 3

Let us consider the following binomial expression
where $A$ is a real number which is not equal to zero or a positive integer and $z = \frac{x}{y}$ and $|z| > 1$. Writing $(x + y)^A = (1 + z)^A y^A = (1 + z)^A y^A$, we can write the series expansion of this expression using results obtained in equations 5 and 9 as follows:

$$
(x + y)^A = y^A \sum_{n=0}^{\infty} \left( \begin{array}{c} A \\ n \end{array} \right) \frac{z^n}{2^m} \prod_{r=1}^{\infty} \sum_{m=0}^{\infty} \left( \begin{array}{c} A \\ m \end{array} \right) \frac{1}{z + 2^r}^m
$$

(12)

$$
(x + y)^A = y^A \sum_{n=0}^{\infty} \left( \begin{array}{c} A \\ n \end{array} \right) \frac{z^n}{2^m} \prod_{r=1}^{\infty} \sum_{m=0}^{\infty} b(j, m_0, m) \sum_{j=0}^{m} b(j, m_0, m) \sum_{m=0}^{\infty} \left( \begin{array}{c} A \\ m \end{array} \right) \frac{1}{z + 2^r}^m
$$

(13)

4. Conclusion

It has been shown that the Taylor series expansion of $(1 + z)^A$ can be expanded as a product of convergent series, for $|z| > 1$ where $A$ is a real number which is not equal to zero or a positive integer.

5. Acknowledgements

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6. References

[1] Abramowitz, M. and Stegun, I. A. (Eds.). Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, 1972.