THE ERROR TERM IN THE SATO-TATE CONJECTURE

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Abstract. Let \( f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz} \in S_k^\text{new}(\Gamma_0(N)) \) be a newform of even weight \( k \geq 2 \) that does not have complex multiplication. Then \( a(n) \in \mathbb{R} \) for all \( n \), so for any prime \( p \), there exists \( \theta_p \in [0, \pi] \) such that \( a(p) = 2p^{(k-1)/2} \cos(\theta_p) \). Let \( \pi(x) = \#\{p \leq x\} \). For a given subinterval \( I \subset [0, \pi] \), the now-proven Sato-Tate Conjecture tells us that as \( x \to \infty \),

\[
\#\{p \leq x : \theta_p \in I\} \sim \mu_{ST}(I) \pi(x), \quad \mu_{ST}(I) = \frac{2}{\pi} \int_I \sin^2(\theta) \, d\theta.
\]

Let \( \epsilon > 0 \). Assuming that the symmetric power \( L \)-functions of \( f \) are automorphic and satisfy Langlands functoriality, we prove that as \( x \to \infty \),

\[
\#\{p \leq x : \theta_p \in I\} = \mu_{ST}(I) \pi(x) + O \left( \frac{x}{(\log x)^{9/8-\epsilon}} \right),
\]

where the implied constant is effectively computable and depends only on \( k, N, \) and \( \epsilon \).

1. Introduction and Statement of Results

Let

\[
f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k^\text{new}(\Gamma_0(N)), \quad q = e^{2\pi iz}
\]

be a newform of even weight \( k \geq 2 \) with trivial character. Then \( a(n) \in \mathbb{R} \) for all \( n \), and as a consequence of Deligne’s proof of the Weil conjectures, for each prime \( p \) there exists an angle \( \theta_p \in [0, \pi] \) such that

\[
a(p) = 2p^{(k-1)/2} \cos(\theta_p).
\]

For a newform associated to an elliptic curve \( E/\mathbb{Q} \) (in which case \( k = 2 \)), Sato and Tate independently conjectured the distribution of the sequence \( \{\theta_p\} \) as \( p \) varies through the primes; the following generalization of the conjecture for \( k \geq 2 \) was proven by Barnett-Lamb, Geraghty, Harris, and Taylor [1].

Theorem 1.1 (The Sato-Tate Conjecture). Suppose that \( f \in S_k^\text{new}(\Gamma_0(N), \chi) \) does not have complex multiplication, and let \( F : [0, \pi] \to \mathbb{C} \) be a Riemann-integrable function. Then

\[
\lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leq x} F(\theta_p) = \int_0^\pi F(\theta) \, d\mu_{ST},
\]

where \( d\mu_{ST} \) is the Sato-Tate measure \( \frac{2}{\pi} \sin^2(\theta) d\theta \).

Since Riemann-integrable functions can be uniformly approximated by step functions, if suffices for us to consider the function

\[
\pi_{f,I}(x) := \#\{p \leq x : \theta_p \in I\},
\]

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in which case Theorem 1.4 tells us that if \( I = [\alpha, \beta] \subset [0, \pi] \) is fixed, then

\[
\pi_{f,I}(x) \sim \mu_{ST}(I)\pi(x).
\]

The Sato-Tate Conjecture governs much of the statistical behavior of the Fourier coefficients of \( f \). It is known [14] that the Sato-Tate Conjecture follows from the analytic properties of the symmetric power \( L \)-functions associated to \( f \) that are predicted by Langlands functoriality. In order to bound the error in (1.4), one must assume that all symmetric power \( L \)-functions of \( f \) have these conjectured analytic properties.

There have been a number of estimates for the error in (1.4) under the additional assumption that the symmetric power \( L \)-functions of \( f \) satisfy the Generalized Riemann Hypothesis for symmetric power \( L \)-functions (GRH). Under this additional assumption, building on the work of Murty [11], Bucur and Kedlaya [2] proved that if \( f \) is the newform associated to an elliptic curve \( E/\mathbb{Q} \) without complex multiplication, then

\[
\pi_{f,I}(x) = \mu_{ST}(I)\pi(x) + O\left(\frac{x^{3/4}\sqrt{\log(Nx)}}{\log(Nx)}\right),
\]

where \( N \) is the conductor of \( E \). When \( f \in S_{k}^{\text{new}}(\Gamma_0(N)) \) is a newform of even weight \( k \geq 2 \) with squarefree level \( N \) (such a newform necessarily does not have complex multiplication), Rouse and the author [12] proved a completely explicit version of the Sato-Tate Conjecture with a slight improvement in Murty’s error term; this can be briefly stated as

\[
\pi_{f,I}(x) = \mu_{ST}(I)\pi(x) + O\left(\frac{x^{3/4}\log(Nkx)}{\log(x)}\right).
\]

It is important to understand the error term in the Sato-Tate Conjecture without the assumption of GRH. The goal of this note is to prove the following result, providing such an error term.

**Theorem 1.2.** Let \( f \in S_{k}^{\text{new}}(\Gamma_0(N)) \) be a newform of even weight \( k \geq 2 \) and trivial character without complex multiplication. Suppose that all of the symmetric power \( L \)-functions of \( f \) are automorphic and satisfy Langlands functoriality. If \([\alpha, \beta] \subset [0, \pi] \) is fixed, then for any \( \epsilon > 0 \),

\[
\pi_{f,I}(x) = \mu_{ST}(I)\pi(x) + O\left(\frac{x^{3/4}\log(Nkx)}{\log(x)}\right),
\]

where the implied constant is effectively computable and depends only on \( k, N, \) and \( \epsilon \).

2. **Symmetric Power \( L \)-Functions**

We will adopt the notation \( F \ll_a G \), or equivalently \( F = O_a(G) \), to indicate that \( \limsup_{x \to \infty} |F(x)/G(x)| < \infty \), where the limit superior may depend on \( a \). If there is no subscript for \( \ll \) or \( O(\cdot) \), then the implied constant is absolute. We take \( F \sim G \) to mean that \( \lim_{x \to \infty} F(x)/G(x) = 1 \).

In this section we discuss the relevant background on the symmetric power \( L \)-functions of \( f \). First, we discuss the assumption that the symmetric power \( L \)-functions of \( f \) are automorphic. We then estimate the analytic conductor of \( L(\text{Sym}^n f, s) \), a quantity that will be useful in determining dependence of important quantities on the level \( N \), the weight \( k \), and the symmetric power \( n \).
2.1. Automorphy and functoriality. Let \( f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k^\new(G_0(N)) \) be a newform of even weight \( k \geq 2 \) without complex multiplication. For each prime \( p \), define \( \theta_p \in [0, \pi] \) to be the angle for which \( a(p) = 2p^{(k-1)/2} \cos(\theta_p) \). The newform \( f \) has an associated \( L \)-function

\[
L(f, s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^{s+(k-1)/2}} = \prod_p \left( \prod_{j=0}^{1} (1 - \alpha_p^j \beta_p^{1-j} p^{-s})^{-1} \right).
\]

It is known that \( L(f, s) \) can be analytically continued to an entire function that satisfies a functional equation. By Deligne’s proof of the Weil conjectures, we know that \( |\alpha_p| = |\beta_p| = 1 \) when \( p \nmid N \) and \( |\alpha_p|, |\beta_p| \leq 1 \) when \( p \mid N \). Because \( f \) has trivial character, we have \( \alpha_p = e^{i\theta_p} \) and \( \beta_p = e^{-i\theta_p} \) for all primes \( p \nmid N \).

For each \( n \geq 0 \), the \( n \)-th symmetric power \( L \)-function of \( f \) is the degree \( n + 1 \) \( L \)-function given by the Euler product

\[
L(\text{Sym}^n f, s) = \prod_p \prod_{j=0}^{n} (1 - \alpha_p^j \beta_p^{n-j} p^{-s})^{-1}.
\]

When \( n = 0 \), \( L(\text{Sym}^n f, s) \) reduces to the Riemann zeta function \( \zeta(s) \); when \( n = 1 \), we obtain \( L(f, s) \). Conjecturally, there exists a functoriality lifting map on global automorphic functions that commutes with the local Langlands correspondence. This would imply that for all \( n \geq 1 \), \( L(\text{Sym}^n f, s) \) is an automorphic \( L \)-function. As a result, \( L(\text{Sym}^n f, s) \) would have an analytic continuation to an entire function on \( \mathbb{C} \), and this analytic continuation would satisfy a functional equation of the usual type. Specifically, there would exist a positive integer \( q_{\text{Sym}^n f} \) (the conductor), a complex number \( \epsilon_{\text{Sym}^n f} \) of modulus 1 (the root number), and a function \( \gamma(\text{Sym}^n f, s) \) (the gamma factor) so that the function

\[
\Lambda(\text{Sym}^n f, s) = q_{\text{Sym}^n f}^{s/2} \gamma(\text{Sym}^n f, s) L(\text{Sym}^n f, s)
\]

is an entire function of order 1 and satisfies the functional equation

\[
\Lambda(\text{Sym}^n f, s) = \epsilon_{\text{Sym}^n f} \Lambda(\text{Sym}^n f, 1 - s).
\]

Let \( \Gamma(s) \) be the usual Gamma function, and let

\[
\Gamma_\mathbb{R}(s) = \pi^{-s/2} \Gamma(s/2), \quad \Gamma_\mathbb{C}(s) = \Gamma_\mathbb{R}(s) \Gamma_\mathbb{R}(s + 1).
\]

It is known \([3,10]\) that under our working assumptions, we have

\[
\gamma(\text{Sym}^n f, s) = \begin{cases} \prod_{j=0}^{(n-1)/2} \Gamma_\mathbb{C}(s + (j + 1/2)(k - 1)) & \text{if } n \text{ is odd}, \\ \Gamma_\mathbb{R}(s + r) \prod_{j=1}^{n/2} \Gamma_\mathbb{C}(s + j(k - 1)) & \text{if } n \text{ is even}, \end{cases}
\]

where \( r = 1 \) if \( n/2 \) is odd and \( r = 0 \) if \( n/2 \) is even. Using the definitions of \( \Gamma_\mathbb{R}(s) \) and \( \Gamma_\mathbb{C}(s) \), we can express the \( \gamma(\text{Sym}^n f, s) \) as a constant multiple of

\[
\pi^{-(n+1)s/2} \prod_{j=1}^{n+1} \Gamma \left( \frac{s + \kappa_j \text{Sym}^n f}{2} \right)
\]

for some appropriate numbers \( \kappa_j \text{Sym}^n f \in \mathbb{C} \) with \( 1 \leq j \leq n + 1 \). The numbers \( \kappa_j \text{Sym}^n f \) satisfy the inequality \( |\kappa_j \text{Sym}^n f| \leq (n + 1) \max_j |\kappa_j \text{Sym}^n f| \). For the rest of the paper, we will assume that \( L(\text{Sym}^n f, s) \) is automorphic for all \( n \geq 1 \), though this hypothesis is only known to be
true unconditionally for \( n = 1, 2, 3, 4 \) by the work of Gelbart, Jacquet, Kim, and Shahidi [4,7–9].

Define the numbers \( \Lambda_{\text{Sym}^n f}(j) \) by

\[
-L'_L(\text{Sym}^n f, s) = \sum_{j=1}^{\infty} \Lambda_{\text{Sym}^n f}(j) j^s.
\]

A straightforward computation shows that

\[
-L'_L(\text{Sym}^n f, s) = \sum_{p} \sum_{m=1}^{\infty} \left( \sum_{j=0}^{n} (\alpha_p^j \beta_p^{n-j} m) \right) \log(p) p^{-ms}.
\]

Since \(|\alpha_p|, |\beta_p| \leq 1\) for all primes \( p \) (including the ramified ones, under our assumption of functoriality), it follows that for any positive integer \( j \), we have

\[
|\Lambda_{\text{Sym}^n f}(j)| \leq (n+1)\Lambda(j).
\]

where \( \Lambda(j) \) is the classical von Mangoldt function. Furthermore, if \( \gcd(j, N) = 1 \), then

\[
\Lambda_{\text{Sym}^n f}(j) = \begin{cases} 
U_n(\cos(m \theta_p)) \log(p) & \text{if } j = p^m, m > 0, \\
0 & \text{otherwise,}
\end{cases}
\]

where \( U_n(x) \) is the \( n \)-th Chebyshev polynomial of the second type.

2.2. The analytic conductor. We want to estimate the analytic conductor

\[
q_{\text{Sym}^n f}(s) = q_{\text{Sym}^n f} \prod_{j=1}^{n+1} (|s + \kappa_{j, \text{Sym}^n f}| + 3).
\]

An estimate of the analytic conductor will allow us to easily make estimates for \( L(\text{Sym}^n f, s) \) that are uniform as we change the symmetric power \( n \), the weight \( k \), and the level \( N \) of \( f \). Most importantly, we want an estimate of \( q_{\text{Sym}^n f} \) as \( n \to \infty \). The quality of the error term in the Sato-Tate Conjecture depends on how well one can estimate \( q_{\text{Sym}^n f} \) as a function of \( n \). We begin with an estimate for \( q_{\text{Sym}^n f} \) given by Lemma 2.1 of [13].

Lemma 2.1. As \( n \to \infty \), we have \( \log(q_{\text{Sym}^n f}) \ll_N n^3 \).

Remark 2.2. Under our assumptions of automorphy and functoriality, Cogdell and Michel prove [3] that if \( N \) is squarefree, then \( \log(q_{\text{Sym}^n f}) = n \log(N) \). With this improvement, the assumption of a squarefree level \( N \) provides considerable improvement over Lemma 2.1 when GRH is assumed. However, it will not provide any improvement without GRH because of the specific dependence of our zero-free region for \( L(\text{Sym}^n f, s) \) on \( n \).

From Lemma 2.1 and the shape of the numbers \( \kappa_{j, \text{Sym}^n f} \), we may conclude the following.

Lemma 2.3. As \( n \to \infty \), we have

\[
\log(q_{\text{Sym}^n f}(0)) \ll_{k, N} n^3, \quad \log(q_{\text{Sym}^n f}(iT)) \ll_{k, N} n^3 + n \log(T).
\]

Lemma 2.3 also allows us to determine the distribution of nontrivial zeros in the critical strip by measuring the quantity

\[
N(T, \text{Sym}^n f) = \# \{ \rho = \beta + i\gamma : 0 \leq \beta \leq 1, |\gamma| \leq T, L(\text{Sym}^n f, \rho) = 0 \}.
\]
By Theorem 5.8 of [6], we have
\[ N(T, \text{Sym}^n f) = \frac{T}{\pi} \log \left( \frac{n! T^{n+1}}{(2\pi e)^{n+1}} \right) + O(\log(n! T)). \]

Using Lemma 2.3 to give us a complete description of the dependence of \( N(T, \text{Sym}^n f) \) on \( n \), we obtain the following result, which is part of the proof of Lemma 3.4 in [13].

**Lemma 2.4.** As \( T \to \infty \), we have
\[ N(T, \text{Sym}^n f) = N(T + 1, \text{Sym}^n f) - N(T, \text{Sym}^n f) \ll_{k,N} n^2 + n \log(T). \]

### 3. Preliminary Setup

If \( \chi_I \) is the indicator function of the interval \( I = [\alpha, \beta] \), then we have
\[ \pi_{f,I}(x) = \sum_{p \leq x} \chi_I(\theta_p). \]

We approximate \( \chi_I \) with a differentiable function using the following construction.

**Lemma 3.1** (Lemma 12 of [13]). Let \( R \) be a positive integer, and let \( a, b, \delta \in \mathbb{R} \) satisfy
\[ 0 < \delta < 1/2, \quad \delta \leq b - a \leq 1 - \delta. \]

Then there exists an even periodic function \( g(y) \) with period 1 satisfying

1. \( g(y) = 1 \) when \( y \in [a + \delta, b - \delta] \),
2. \( g(y) = 0 \) when \( y \in [b + \frac{1}{2}\delta, 1 + a - \frac{1}{2}\delta] \),
3. \( 0 \leq g(y) \leq 1 \) when \( y \) is in the rest of the interval \( [a - \delta, 1 + a - \delta] \), and
4. \( g(y) \) has the Fourier expansion
\[
g(y) = b - a + \sum_{n=1}^{\infty} (a_n \cos(2\pi nx) + b_m \sin(2\pi nx)),
\]

where for all \( n \geq 1 \),
\[
|a_n|, |b_n| \leq \min \left\{ \frac{2(b - a)}{n\pi}, \frac{2}{n\pi} \left( \frac{R}{\pi n\delta} \right)^R \right\}.
\]

Let \( g(\theta) \) be defined as in Lemma 3.1 where \( a = \frac{\alpha}{2\pi} - \frac{\delta}{2} \), and \( b = \frac{\beta}{2\pi} + \frac{\delta}{2} \). We will choose \( \delta \) to be a function of \( x \) that tends to zero as \( x \) tends to infinity, and we will choose \( R \) to ensure the absolute convergence of the Fourier series. Define \( g^+(\theta; I, \delta) = g\left(\frac{\theta}{2\pi}\right) + g\left(-\frac{\theta}{2\pi}\right) \), which equals 1 for \( \theta \in I \), equals 0 for \( \theta \in [0, \alpha - 2\pi\delta] \cup [\beta + 2\pi\delta, \pi] \), and is between 0 and 1 elsewhere in the interval \( [0, \pi] \). Thus \( g^+(\theta; I, \delta) \) a pointwise upper bound for \( \chi_I(\theta) \). By repeating this construction with \( a = \frac{\alpha}{2\pi} + \frac{\delta}{2} \), and \( b = \frac{\beta}{2\pi} - \frac{\delta}{2} \), we can obtain a lower bound for \( \chi_I(\theta) \), say \( g^-(\theta; I, \delta) \). To ensure that \( g^-(\theta; I, \delta) \) is in fact a lower bound for \( \chi_I(\theta) \), we require that \( \beta - \alpha > 2\pi\delta \), which is ensured when \( x \) is sufficiently large because \( I \) is fixed.

We can express \( g^\pm(\theta; I, \delta) \) with respect to the basis of Chebyshev polynomials of the second kind \( \{U_n(\cos(\theta))\}_{n=0}^{\infty} \), which is an orthonormal basis for \( L^2([0, \pi], \mu_{ST}) \). Specifically,
\[
g^\pm(\theta; I, \delta) = a_0^\pm(I, \delta) - a_2^\pm(I, \delta) + \sum_{n=1}^{\infty} (a_n^\pm(I, \delta) - a_{n+2}^\pm(I, \delta)) U_n(\cos(\theta)),
\]

(3.2)
where \(a_n^\pm(I, \delta)\) is the \(n\)-th Fourier coefficient in the cosine expansion of \(g^\pm(\theta; I, \delta)\). From Lemma 3.1, we have

\[
|a_0(I, \pm \delta) - a_2(I, \pm \delta) - \mu_{ST}(I)| \ll \delta, \tag{3.3}
\]

\[
|a_n(I, \pm \delta) - a_{n+2}(I, \pm \delta)| \leq \frac{4}{n\pi} \left( \frac{R}{\pi n \delta} \right)^R \text{ for } n \geq 1.
\]

When summing \(g^\pm(\theta_p; I, \delta)\) over primes \(p \leq x\), we may switch the order of summation because we choose \(R\) to ensure absolute convergence. Using (3.1), (3.2), (3.3), and the prime number theorem, we have that if

\[
\Phi_{\text{Sym}^nf}(x) = \sum_{p \leq x} U_n(\cos(\theta_p)), \tag{3.4}
\]

then

\[
\pi_{f,I}(x) = \mu_{ST}(I)\pi(x) + O \left( \frac{\delta x}{\log(x)} + \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{R}{n \delta} \right)^R |\Phi_{\text{Sym}^nf}(x)| \right).
\]

Theorem 1.2 will now follow from an estimate of \(\Phi_{\text{Sym}^nf}(x)\) and choosing \(\delta\) and \(R\) optimally.

The goal of the next section is to estimate \(\Phi_{\text{Sym}^nf}(x)\), which proceeds very much like the classical prime number theorem.

4. Estimating \(\Phi_{\text{Sym}^nf}(x)\)

By our assumption of functoriality, \(|\alpha_p|, |\beta_p| \leq 1\) for all primes \(p\), and \(|\alpha_p|, |\beta_p| = 1\) for all \(p \nmid N\). Thus \(L(\text{Sym}^nf, s)\) satisfies the Ramanujan-Petersson Conjecture for all \(n \geq 1\). As such, we may use Equation 5.53 from Chapter 5 of [6] to estimate the summatory von-Mangoldt function for \(L(\text{Sym}^nf, s)\) given by

\[
\psi_{\text{Sym}^nf}(x) = \sum_{j \leq x} \Lambda_{\text{Sym}^nf}(j).
\]

Lemma 4.1. If \(n \geq 1\), then

\[
\psi_{\text{Sym}^nf}(x) = -\sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + O \left( \frac{x}{T} \log(x) \log(x^{n+1}q_{\text{Sym}^nf}(0)) \right),
\]

where \(\rho = \beta + i\gamma\) runs over the zeros of \(L(\text{Sym}^nf, s)\) in the critical strip of height up to \(T\), with any \(1 \leq T \leq x\), and the implied constant is absolute.

By our assumption of Langlands functoriality, there will be no exceptional real zeros close to \(s = 1\) (see Section 4 of [5]). As such, it remains to estimate the sum over nontrivial zeros. We invoke the zero-free region given in Theorem 5.10 of [6], which is currently the best zero-free region for a generic automorphic \(L\)-function.

Lemma 4.2. There exists an absolute constant \(c > 0\) such that \(L(\text{Sym}^nf, s)\) has no zeros in the region

\[
s = \sigma + it, \quad \sigma \geq 1 - \frac{c}{(n+1)^4 \log(q_{\text{Sym}^nf}(0)(|t| + 3))}.
\]

Using Lemmata 4.1 and 4.2 we estimate \(|\Phi_{\text{Sym}^nf}(x)|\).
Lemma 4.3. Assume the above notation, and let \( n \geq 1 \). For some constant \( 0 < c_2 < c \) (depending only on \( N \) and \( k \)), we have

\[
|\Phi_{\text{Sym}^n f}(x)| \ll_{k,N} n^3 x \exp \left( -\frac{c_2 \log(x)}{n^4(\sqrt{\log(x)} + n^3)} \right).
\]

Proof. We begin by estimating the sum over zeros in Lemma 4.1. By Lemma 4.2, if \( \rho = \beta + i\gamma \) is a nontrivial zero of \( L(\text{Sym}^n f, s) \), then

\[
|x^\rho| \leq |x^\beta| \leq x \exp \left( -\frac{c \log(x)}{(n+1)^4 \log(q_{\text{Sym}^n f}(0)(|t|+3))} \right).
\]

Thus

\[
\sum_{|\gamma| \leq T} \left| \frac{x^\rho}{\rho} \right| \ll x \exp \left( -\frac{c \log(x)}{(n+1)^4 \log(q_{\text{Sym}^n f}(0)(|T|+3))} \right) \sum_{j \leq T} \mathcal{N}(j, \text{Sym}^n f).
\]

Now, Lemma 2.4 tells us that the sum over zeros is

\[
\ll_{k,N} n^3 (\log T)^2 x \exp \left( -\frac{c \log(x)}{(n+1)^4 \log(q_{\text{Sym}^n f}(0)(|T|+3))} \right)
\]

To address the error term in Lemma 4.1, we use Lemma 2.3 to obtain

\[
x \frac{\log(x)}{T} \log(x) \log(x^{n+1} q_{\text{Sym}^n f}(0)) \ll_{k,N} n^3 \frac{x}{T} (\log x)^2.
\]

To balance our estimate for the sum over nontrivial zeros with the error term in Lemma 4.1, we choose \( T = \exp(\sqrt{\log x}) \) to obtain

\[
\psi_{\text{Sym}^n f}(x) \ll_{k,N} n^3 x \exp \left( -\frac{c_1 \log(x)}{n^4(\sqrt{\log(x)} + n^3)} \right)
\]

for some \( 0 < c_1 < c \). By a standard application of Abel summation, one has

\[
\Psi_{\text{Sym}^n f}(x) := \sum_{j \leq x} \frac{\Lambda_{\text{Sym}^n f}(j)}{\log(j)} = \psi_{\text{Sym}^n f}(x) \frac{\log(x)}{\log(x)} + \int_2^x \psi_{\text{Sym}^n f}(t) \frac{1}{t \log(t)^2} \, dt.
\]

Applying (4.1), we have that for some constant \( 0 < c_2 < c_1 \) (depending only on \( N \) and \( k \)),

\[
\Psi_{\text{Sym}^n f}(x) \ll_{k,N} n^3 x \exp \left( -\frac{c_2 \log(x)}{n^4(\sqrt{\log(x)} + n^3)} \right).
\]

We now show that \( |\Phi_{\text{Sym}^n f}(x) - \Psi_{\text{Sym}^n f}(x)| \) is small. By (2.9), if \( p \nmid N \) is prime, then

\[
\frac{\Lambda_{\text{Sym}^n f}(p)}{\log(p)} = U_n(\cos(\theta_p)).
\]

At all other prime powers \( j = p^m \), it follows from (2.8) that

\[
\left| \frac{\Lambda_{\text{Sym}^n f}(j)}{\log(j)} \right| \leq n + 1.
\]
Finally, we have $|U_n(\cos(\theta_p))| \leq n + 1$ for all $p$ by basic properties of Chebyshev polynomials. Therefore,

$$|\Phi_{\text{Sym}^n f}(x) - \Psi_{\text{Sym}^n f}(x)| \leq (n + 1) \left( \sum_{\substack{m \geq 2 \\text{prime} \leq x}} 1 + \sum_{p|N} 1 \right) \ll_N n \sqrt{x}.$$ 

This error is negligible, so we have proven the desired result. □

5. Proof of Theorem 1.2

To prove Theorem 1.2, it remains to choose $\delta$ and $R$ so that the error term in (3.5) is minimized. The factor of $n^3$ in Lemma 4.3 tells us that we must take $R$ to be at least 4 in (3.5) to ensure absolute convergence of the sum in the error term. It follows from Lemma 4.3 that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{R}{n \delta} \right)^R |\Phi_{\text{Sym}^n f}(x)| \ll_{k,N} \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{R}{n \delta} \right)^R n^3 x \exp \left( - \frac{c_2 \log(x)}{n^4 (\sqrt{\log(x)} + n^3)} \right)$$

$$\ll_{k,N} \frac{R^R}{\delta^R} \int_1^{\infty} \frac{1}{t^{R-2}} \exp \left( - \frac{c_2 \sqrt{\log(x)}}{t^4} \right) \, dt$$

$$\ll_{k,N} \frac{R^{2R}}{\delta^R} \frac{x}{(\log(x))^{\frac{R-3}{8}}}.$$ 

We have thus reduced (3.5) to

$$\pi_{f,1}(x) = \mu_{ST}(I) \pi(x) + O_{k,N} \left( \frac{\delta x}{\log(x)} + \frac{R^{5R}}{\delta^R} \frac{x}{(\log(x))^{\frac{R-3}{8}}} \right).$$

Choosing

$$\delta = R^{\frac{5}{3}} \log(x)^{\frac{3}{2R} - \frac{1}{3}},$$

we balance the error term in (5.1), which is now of order

$$\ll_{k,N} \frac{\delta x}{\log(x)} \ll_{k,N} \frac{R^{5R}}{\delta^R} \frac{x}{(\log(x))^{\frac{R-3}{8}}}.$$ 

Since we can choose $R$ to be a finite, arbitrarily large integer, we obtain the bound claimed in Theorem 1.2.

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References

[1] Tom Barnet-Lamb, David Geraghty, Michael Harris, and Richard Taylor, *A family of Calabi-Yau varieties and potential automorphy II*, Publ. Res. Inst. Math. Sci. 47 (2011), no. 1, 29–98. MR2827723 (2012m:11069)

[2] A. Bucur and K. Kedlaya, *An Application of the Effective Sato-Tate Conjecture*. Preprint.

[3] J. Cogdell and P. Michel, *On the complex moments of symmetric power $L$-functions at $s = 1$*, Int. Math. Res. Not. 31 (2004), 1561–1617. MR2035301 (2005f:11094)
Stephen Gelbart and Hervé Jacquet, *A relation between automorphic representations of GL(2) and GL(3)*, Ann. Sci. École Norm. Sup. (4) 11 (1978), no. 4, 471–542. MR533066 (81e:10025)

Jeffrey Hoffstein and Dinakar Ramakrishnan, *Siegel zeros and cusp forms*, Internat. Math. Res. Notices 6 (1995), 279–308. MR1344349 (96h:11040)

H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, vol. 53, American Mathematical Society, Providence, RI, 2004. MR2061214 (2005h:11005)

Henry H. Kim, *Functoriality for the exterior square of $GL_4$ and the symmetric fourth of $GL_2$*, J. Amer. Math. Soc. 16 (2003), no. 1, 139–183. With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak. MR1937203 (2003k:11083)

Henry H. Kim and Freydoon Shahidi, *Cuspidality of symmetric powers with applications*, Duke Math. J. 112 (2002), no. 1, 177–197. MR1890650 (2003a:11057)

C. J. Moreno and F. Shahidi, *The $L$-functions $L(s, \text{Sym}^m(r, \pi)$*, Canad. Math. Bull. 28 (1985), no. 4, 405–410. MR812115 (87a:11051)

V. Kumar Murty, *Explicit formulae and the Lang-Trotter conjecture*, Rocky Mountain J. Math. 15 (1985), no. 2, 535–551. Number theory (Winnipeg, Man., 1983). MR823264 (87h:11051)

J. Rouse and J. Thorner, *The explicit Sato-Tate conjecture and densities pertaining to Lehmer-type questions*, preprint.

Jeremy Rouse, *Atkin-Serre type conjectures for automorphic representations on GL(2)*, Math. Res. Lett. 14 (2007), no. 2, 189–204. MR2318618 (2009e:11080)

Freydoon Shahidi, *Symmetric power $L$-functions for GL(2)*, Elliptic curves and related topics, 1994, pp. 159–182. MR1269661 (95c:11066)

I. M. Vinogradov, *The method of trigonometrical sums in the theory of numbers*, Dover Publications Inc., Mineola, NY, 2004. Translated from the Russian, revised and annotated by K. F. Roth and Anne Davenport, Reprint of the 1954 translation. MR2104806 (2005f:11172)

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