IDENTIFYING CARTESIAN DECOMPOSITIONS PRESERVED BY TRANSITIVE PERMUTATION GROUPS

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1. Introduction

Intuitively, a Cartesian decomposition of a finite set $\Omega$ is a way of identifying $\Omega$ with a Cartesian product $\Gamma_1 \times \cdots \times \Gamma_\ell$ of smaller sets $\Gamma_i$. However we do not wish to distinguish between two such identifications if the second can be obtained from the first by re-naming the elements in the individual sets $\Gamma_i$. Nor do we wish to distinguish between, say, $\Gamma_1 \times \Gamma_2$ and $\Gamma_2 \times \Gamma_1$. Thus by a Cartesian decomposition we will mean an equivalence class of identifications of $\Omega$ with a Cartesian product under a certain notion of equivalence. Our formal Definition 2.1 encompasses these ideas, but at first reading this may not be apparent. We therefore develop the concept further in Section 2 before giving the formal definition.

Our aim is to describe the theory of Cartesian decompositions preserved by some member of a large family of finite transitive permutation groups called innately transitive groups. Innately transitive groups are defined in Section 3 and for such a group $G$, the Cartesian decompositions preserved by $G$ correspond to certain families of subgroups, called Cartesian systems, of a normal subgroup $M$ of $G$. Many Cartesian decompositions correspond to direct decompositions of $M$. These are called $M$-normal and are defined in Section 4. The non-normal $G$-invariant Cartesian decompositions occur for $M$ of the form $M = T^k$, where $T$ is a nonabelian simple group, and are often related to factorisations of $T$. The various types of such non-normal Cartesian decompositions are discussed in Section 5 and simple examples illustrating most of the possibilities are given in Section 6.

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2. THE CONCEPT OF Cartesian decompositions

In this section we develop the concept of a Cartesian decomposition, giving a formal definition in Definition 2.1. An identification of a finite set $\Omega$ with a Cartesian product $\Omega_1 \times \cdots \times \Omega_\ell$ is a bijection $\varphi : \Omega \to \Omega_1 \times \cdots \times \Omega_\ell$. A second bijection $\varphi' : \Omega \to \Delta_1 \times \cdots \times \Delta_\ell'$ is defined to be equivalent to $\varphi$ if $\ell = \ell'$, and there exist a permutation $\pi \in S_\ell$ and bijections $\alpha_1 : \Omega_1 \to \Delta_1$, $\ldots$, $\alpha_\ell : \Omega_\ell \to \Delta_\ell$ such that

$$\varphi'(\omega) = (\alpha_1 \times \cdots \times \alpha_\ell)_\pi(\varphi(\omega)) \quad \text{for all} \quad \omega \in \Omega,$$

where $(\alpha_1 \times \cdots \times \alpha_\ell)_\pi$ denotes the bijection $\Omega_1 \times \cdots \times \Omega_\ell \to \Delta_1 \times \cdots \times \Delta_\ell$ defined by

$$(\alpha_1 \times \cdots \times \alpha_\ell)_\pi(\omega_1, \ldots, \omega_\ell) = (\alpha_{1\pi^{-1}}(\omega_{1\pi^{-1}}), \ldots, \alpha_{\ell\pi^{-1}}(\omega_{\ell\pi^{-1}})).$$

Whether or not a second bijection is equivalent to $\varphi$ can be decided using the preimages of the natural projection maps $\sigma_i : \Omega_1 \times \cdots \times \Omega_\ell \to \Omega_i$ as follows. For $i = 1, \ldots, \ell$ set

$$\Gamma_i = \{\omega \mid \sigma_i(\varphi(\omega)) = \omega_i \} \ | \ \omega_i \in \Omega_i\}.$$

It is easy to see that each of the $\Gamma_i$ is a partition of $\Omega$, and that

$$|\gamma_1 \cap \cdots \cap \gamma_\ell| = 1 \quad \text{for all} \quad \gamma_1 \in \Gamma_1, \ldots, \gamma_\ell \in \Gamma_\ell.$$

Moreover two bijections $\varphi$ and $\varphi'$ are equivalent under the equivalence relation defined above if and only if they give rise to the same set of partitions of $\Omega$; the proof of this is left to the reader. Hence each equivalence class of Cartesian decompositions gives rise to a unique set of partitions satisfying $[\Pi]$. Conversely, if $\Gamma_1, \ldots, \Gamma_\ell$ are partitions of $\Omega$ such that $[\Pi]$ holds, then $\Omega$ can be identified with the Cartesian product $\Gamma_1 \times \cdots \times \Gamma_\ell$ as follows. Let $\omega \in \Omega$, and let $\gamma_1 \in \Gamma_1, \ldots, \gamma_\ell \in \Gamma_\ell$ such that $\omega \in \gamma_1 \cap \cdots \cap \gamma_\ell$. Such blocks $\gamma_i$ exist and are unique because each $\Gamma_i$ is a partition of $\Omega$. Define $\psi : \Omega \to \Gamma_1 \times \cdots \times \Gamma_\ell$ by $\psi(\omega) = (\gamma_1, \ldots, \gamma_\ell)$. Property $[\Pi]$ ensures that $\psi$ is a bijection.

Now the motivation behind the following definition should be clear.

**Definition 2.1.** If $\Omega$ is a finite set, then a set $\mathcal{E} = \{\Gamma_1, \ldots, \Gamma_\ell\}$ of partitions of $\Omega$ is said to be a *Cartesian decomposition* of $\Omega$ if $[\Pi]$ holds.

This definition of Cartesian decompositions enables us to study Cartesian decompositions that are invariant under the action of a permutation group. If $\mathcal{E}$ is a Cartesian decomposition of $\Omega$ and $g \in \text{Sym} \Omega$ then we say that $\mathcal{E}$ is invariant under $g$ if the partitions in $\mathcal{E}$ are permuted by $g$. The stabiliser in $\text{Sym} \Omega$ of a Cartesian decomposition $\mathcal{E}$ is obviously a subgroup of $\text{Sym} \Omega$. 
We give two simple examples of Cartesian decompositions that are invariant under the action of some transitive permutation group.

**Example 2.2.** Let $\Omega = \{1, 2\} \times \{1, 2, 3\}$. Then the identity map $(i, j) \mapsto (i, j)$ of $\Omega$ is a bijection whose corresponding Cartesian decomposition contains the two partitions given by the columns and the rows of the following grid.

| (1, 1) | (1, 2) | (1, 3) |
|-------|-------|-------|
| (2, 1) | (2, 2) | (2, 3) |

Hence the Cartesian decomposition corresponding to the identity map on $\Omega$ consists of the following two partitions, namely the rows of the grid,

$$
\Gamma_1 = \{\{(1, 1), (1, 2), (1, 3)\}, \{(2, 1), (2, 2), (2, 3)\}\}
$$

and the columns of the grid

$$
\Gamma_2 = \{\{(1, 1), (2, 1)\}, \{(1, 2), (2, 2)\}, \{(1, 3), (2, 3)\}\}.
$$

Note that the two partitions in this Cartesian decomposition have different sizes. Such a Cartesian decomposition is said to be *inhomogeneous*. The group $G = S_2 \times S_3$ in its natural action on $\Omega$ is the full stabiliser of $\{\Gamma_1, \Gamma_2\}$ in $\text{Sym} \Omega$, and is transitive on $\Omega$, but, as this Cartesian decomposition is inhomogeneous, no element of $G$ swaps $\Gamma_1$ and $\Gamma_2$. Hence $G$ is intransitive on the Cartesian decomposition $\{\Gamma_1, \Gamma_2\}$ of $\Omega$.

**Example 2.3.** Let $\Gamma$ be a finite set, let $\ell \geq 2$, and let $\Omega = \Gamma \times \cdots \times \Gamma = \Gamma^\ell$. The wreath product $W = \text{Sym} \Gamma \wr S_\ell$ is the semidirect product of its normal subgroup $N = (\text{Sym} \Gamma)^\ell$ and a subgroup $H \cong S_\ell$. The product action of $W$ on $\Omega$ is defined by

$$(\gamma_1, \ldots, \gamma_\ell)^{xh} = (\gamma_1^{xh^{-1}}, \ldots, \gamma_\ell^{xh^{-1}})$$

for all $(\gamma_1, \ldots, \gamma_\ell) \in \Omega$, $x = (x_1, \ldots, x_\ell) \in N$, and $h \in H$, where we write the image of $\gamma \in \Gamma$ under $y \in \text{Sym} \Gamma$ as $\gamma^y$. Clearly $W$ is transitive on $\Omega$. The Cartesian decomposition corresponding to the identity map on $\Omega$ is $E = \{\Gamma_1, \ldots, \Gamma_\ell\}$, where $\Gamma_i$ is the partition of $\Omega$ into disjoint subsets according to the $i$-th coordinate of a point in $\Omega$, that is to say, the parts of $\Gamma_i$ are indexed by $\Gamma$ and the $\gamma$-part is the set of all points $(\gamma_1, \ldots, \gamma_\ell)$ with $\gamma_i = \gamma$. Thus $|\Gamma_i| = |\Gamma|$ for all $i$. A Cartesian decomposition $\{\Gamma_1, \ldots, \Gamma_\ell\}$ for which the $\Gamma_i$ all have the same cardinality is said to be *homogeneous*. Thus $E$ is homogeneous. Also each element $xh \in W$ maps the partition $\Gamma_i$ to the partition $\Gamma_{ih}$. Thus $W$ preserves
the Cartesian decomposition $\mathcal{E}$. In fact $W$ is the full stabiliser of $\mathcal{E}$ in $\text{Sym}\Omega$ and $W$ permutes the partitions $\Gamma_i$ transitively.

A Cartesian decomposition $\mathcal{E}$ is called $G$-transitive if it is $G$-invariant and $G$ acts transitively on the set $\mathcal{E}$ of partitions. Thus the Cartesian decomposition of Example 2.3 is $W$-transitive but not $N$-transitive.

3. Innately Transitive Groups

The class of primitive permutation groups that preserve a Cartesian decomposition of the underlying set is well-understood, and is described in [Pra90]. The original aim of the research presented in this article was to extend this result to describe Cartesian decompositions that are preserved by a permutation group in which all minimal normal subgroups are transitive. Such a group is said to be quasiprimitive. We found that the methods used to achieve this goal give a description of Cartesian decompositions preserved by a larger class of groups, namely those that have at least one transitive minimal normal subgroup. Such a group is said to be innately transitive and a transitive minimal normal subgroup of an innately transitive group is called a plinth. In particular each primitive or quasiprimitive group is innately transitive, and the class of innately transitive groups also contains many interesting non-quasiprimitive groups. Innately transitive groups are studied in [BP]. The essential reason why they provide us with the right context for our research is the following result.

Proposition 3.1. Let $G$ be an innately transitive group acting on $\Omega$, and let $M$ be a plinth of $G$. If $\{\Gamma_1, \ldots, \Gamma_\ell\}$ is a $G$-invariant Cartesian decomposition of $\Omega$, then each $\Gamma_i$ is an $M$-invariant partition of $\Omega$.

A plinth $M$ of an innately transitive group $G$ on $\Omega$ is transitive on $\Omega$. It is well-known (see, for example, [DM Theorem 1.5A]) that, for a transitive permutation group $M$ on $\Omega$ and a fixed $\omega \in \Omega$, there is a one-to-one correspondence between the set of $M$-invariant partitions of $\Omega$ and the set of subgroups $K$ of $M$ containing the stabiliser $M_\omega$. Thus the subgroups $K_1, \ldots, K_\ell$ of $M$ corresponding to $M$-invariant partitions $\Gamma_1, \ldots, \Gamma_\ell$ of $\Omega$ are sufficient to determine the partitions $\Gamma_i$, and we can decide from certain properties of the $K_i$ whether or not the $\Gamma_i$ form a Cartesian decomposition of $\Omega$.

More precisely, let $G$ be an innately transitive group acting on $\Omega$ with plinth $M$, and let $\{\Gamma_1, \ldots, \Gamma_\ell\}$ be a $G$-invariant Cartesian decomposition of $\Omega$. By Proposition 3.1 each
of the $\Gamma_i$ is an $M$-invariant partition of $\Omega$. Fix $\omega \in \Omega$, let $\gamma_1 \in \Gamma_1, \ldots, \gamma_\ell \in \Gamma_\ell$ be such that $\{\omega\} = \gamma_1 \cap \cdots \cap \gamma_\ell$, and, for $i = 1, \ldots, \ell$, set $K_i = M_{\gamma_i}$. It is proved in [BPS, Lemma 2.2] that the set $\{K_1, \ldots, K_\ell\}$ is invariant under conjugation by $G_\omega$ and has the following two important properties:

(2) $\bigcap_{i=1}^\ell K_i = M_\omega$.

(3) $K_i \left( \prod_{j \neq i} K_j \right) = M$ for all $i \in \{1, \ldots, \ell\}$.

Definition 3.2. Let $G$ be a transitive permutation group on $\Omega$ with plinth $M$ and let $\mathcal{K} = \{K_1, \ldots, K_\ell\}$ be a $G_\omega$-invariant set of subgroups of $M$ such that (2) and (3) hold. Then $\mathcal{K}$ is said to be a Cartesian system of subgroups in $M$ with respect to $\omega$.

Conversely, using the correspondence explained above, any Cartesian system in $M$ leads to a $G$-invariant Cartesian decomposition. Thus the set of Cartesian decompositions invariant under the action of an innately transitive group can be studied via the set of Cartesian systems in the plinth.

Theorem 3.3. Let $G$ be an innately transitive group acting on a set $\Omega$ with plinth $M$ and let $\omega$ be a fixed element of $\Omega$. Then there is a one-to-one correspondence between the set of $G$-invariant Cartesian decompositions of $\Omega$ and the set of Cartesian systems in $M$ with respect to $\omega$.

Consider Examples 2.2 and 2.3 in terms of Cartesian systems.

Example 3.4. If $G$ and $\Omega$ are as in Example 2.2, then $G$ has no transitive minimal normal subgroup, so $G$ is not innately transitive. On the other hand, the group $W = NH$ in Example 2.3 is innately transitive on $\Omega$, and $W$ preserves the Cartesian decomposition $\mathcal{E} = \{\Gamma_1, \ldots, \Gamma_\ell\}$ of $\Omega$. Suppose that $|\Gamma_i| \geq 5$ so that the plinth is $M = M_1 \times \cdots \times M_\ell = (\text{Alt}\Gamma)^\ell$. Set $\omega = (\gamma, \ldots, \gamma)$. An easy computation shows that the Cartesian system corresponding to $\mathcal{E}$ with respect to $\omega$ is $\{K_1, \ldots, K_\ell\}$ where each $K_i = (M_i)_{\gamma} \times \prod_{j \neq i} M_j$.

If $G$ is innately transitive with an abelian plinth then $G$ is primitive (see [BP]) and so, as we mentioned above, all $G$-invariant Cartesian decompositions have been determined in [Pra90]. Thus for the rest of the paper we will assume that each plinth is nonabelian.
4. Normal and non-normal Cartesian systems

The Cartesian system $\mathcal{E} = \{\Gamma_1, \ldots, \Gamma_\ell\}$ presented in Example 2.3 has the property that the group $M = (\text{Alt} \Gamma)^\ell$ can be written as a direct product of $\ell$ subgroups with the action of the $i^{th}$ direct factor corresponding to the $M$-action on the $i^{th}$ partition $\Gamma_i$. This is a very useful property, and is perhaps a property possessed by most of the transitive Cartesian decompositions that might come readily to mind. We formalise this property of $G$-invariant Cartesian decompositions for innately transitive groups $G$ with nonabelian plinths.

**Definition 4.1.** Let $G$ be an innately transitive group acting on $\Omega$ with a non-abelian plinth $M$, and let $\mathcal{K} = \{K_1, \ldots, K_\ell\}$ be a Cartesian system of subgroups in $M$ with respect to some $\omega \in \Omega$. Then $\mathcal{K}$ is said to be $M$-normal if there are normal subgroups $M_1, \ldots, M_\ell$ of $M$ such that $M = M_1 \times \cdots \times M_\ell$ and each $K_i = (M_i \cap M_\omega) \times \prod_{j \neq i} M_j$. A $G$-invariant Cartesian decomposition of $\Omega$ is said to be normal if the corresponding Cartesian system is $M$-normal for some plinth $M$.

Normal Cartesian decompositions are considered natural, and they can be determined using the direct factorisations of the plinth. On the other hand, not every Cartesian decomposition is normal. The simplest non-normal Cartesian decomposition preserved by an innately transitive group is given in the following example.

**Example 4.2.** Let $G \cong \Pi\Gamma L_2(9)$ and consider the unique transitive action of $G$ on a set $\Omega$ of 36 points. The group $G$ is innately transitive on $\Omega$, because $G$ has a unique minimal normal subgroup $M \cong A_6$ and $M$ is transitive on $\Omega$. Moreover if $\omega \in \Omega$, then $M_\omega \cong D_{10}$, and it is easy to see that $M$ has subgroups $K_1, K_2$, both isomorphic to $A_5$, such that $\{K_1, K_2\}$ is a Cartesian system of subgroups in $M$ with respect to $\omega$. Hence $G$ preserves a Cartesian decomposition $\{\Gamma_1, \Gamma_2\}$ of $\Omega$, where each $\Gamma_i$ has six parts of size 6. Since $M$ is simple and is the unique minimal normal subgroup of $G$, this Cartesian decomposition cannot be normal.

5. Factorisations of finite simple groups

Let $G$ be an innately transitive group on $\Omega$ with a nonabelian plinth $M$. If $M$ is simple then it is very unusual for $G$ to preserve a nontrivial Cartesian decomposition of $\Omega$, any such decomposition is non-normal, and in fact all such possibilities have been classified.
explicitly, see [BPS, Theorem 6.1]. In this section we outline a theory of non-normal Cartesian decompositions preserved by innately transitive groups with a nonabelian plinth, in particular pointing out the role of simple group factorisations.

The group $M$ is a minimal normal subgroup of $G$, and so $M$ is a nonabelian characteristically simple group and hence is of the form $M = T_1 \times \cdots \times T_k$, where the $T_i$ are finite simple groups, each isomorphic to the same simple group $T$. Moreover, the group $G$ acts transitively by conjugation on the set $\{T_1, \ldots, T_k\}$. For $i = 1, \ldots, k$, let $\sigma_i : M \to T_i$ denote the $i$-th projection map.

By Theorem 3.3, there is a one-to-one correspondence between the set of $G$-invariant Cartesian decompositions of $\Omega$ and the set of Cartesian systems in $M$ relative to a given point $\omega$. Since in a Cartesian system $\{K_1, \ldots, K_\ell\}$ the factorisation property (3) holds, we obtain factorisations of the simple direct factors of $M$ using the natural projection maps $\sigma_i$, as follows. For all $i$, (3) gives the following factorisations of $T_i$:

\[(4) \quad T_i = \sigma_i(K_j) \left( \bigcap_{m \neq j} \sigma_i(K_m) \right), \quad \text{for all } j = 1, \ldots, \ell.\]

Many of the subgroups $\sigma_i(K_j)$ may coincide with $T_i$, so we are really interested in the following sets:

\[(5) \quad F_i = \{ \sigma_i(K_j) \mid \sigma_i(K_j) \neq T_i, \ j = 1, \ldots, \ell \}.\]

The set $F_i$ is essentially independent of $i$, in the sense that if $i_1, i_2 \in \{1, \ldots, k\}$ then there is some $g \in G$ such that $T_{i_1}^g = T_{i_2}$, and then we have $F_{i_1}^g = F_{i_2}$. In particular $|F_i|$ is independent of $i$.

Since the size of $F_i$ is an invariant of the corresponding Cartesian decomposition, one natural way of subdividing the class of Cartesian decompositions invariant under innately transitive groups is to use the number $|F_i|$. This is achieved in a forthcoming paper where Cartesian decompositions in each sub-class are described in detail. In the rest of this section we summarise the results of that paper.

Using results on factorisations of finite simple groups in [BP98] it is easy to prove that $|F_i| \leq 3$. Moreover, if $F_i = \{A, B, C\}$ then the information contained in (4) is that

\[(6) \quad T_i = A(B \cap C) = B(C \cap A) = C(A \cap B).\]

This is called a strong multiple factorisation of $T_i$. Such strong multiple factorisations of finite simple groups are classified in [BP98]. (All possibilities for $T, A, B, C$ are listed
This classification can be used to describe the $G$-invariant Cartesian decompositions for which the corresponding $F_i$ have 3 elements. If $F_i = \{A, B\}$, then the information contained in (4) is precisely that $T = AB$, a factorisation of the finite simple group $T$, and moreover each such factorisation may occur in relation to some Cartesian decomposition; see also Example 6.4 in the next section.

If $|F_i| = 1$, then the corresponding Cartesian decomposition is either $M$-normal, or the Cartesian system elements contain full diagonal subgroups isomorphic to $T$ covering exactly two of the simple direct factors of $M$. If $|F_i| = 0$, then the $K_i$ are subdirect subgroups of $M$ and the corresponding Cartesian decomposition is $M$-normal.

6. Examples of Cartesian systems

In this section we give some examples of nontrivial Cartesian systems preserved by innately transitive groups that illustrate the various sub-classes described in Section 5. In these examples $T$ is a finite simple group and $D(T \times T)$ denotes the straight diagonal subgroup $\{(t, t) \mid t \in T\}$ of $T \times T$. We recall that the sets $F_i$ assigned to a Cartesian system are defined in (5). The first example is the smallest one with $F_i = \emptyset$, and shows that every simple group $T$ can occur in this case.

Example 6.1. Let $G = T \text{wr} D_8 = M \rtimes D_8$, where $M = T^4$ and $D_8$ acts naturally on the four simple direct factors of $M$, that is to say, elements of $D_8$ either fix setwise, or interchange, the first two, and the last two, simple direct factors of $M$. Let

$$K_1 = D(T \times T) \times T \times T \quad \text{and} \quad K_2 = T \times T \times D(T \times T)$$

so that $K_1 \cap K_2 = D(T \times T) \times D(T \times T)$ is normalised by $D_8$. Then the $M$-coset action on $\Omega = [M : K_1 \cap K_2]$ can be extended to $G$ with $(K_1 \cap K_2) \rtimes D_8$ as the stabiliser of the point $\omega = K_1 \cap K_2$. According to Definition 3.2, $\{K_1, K_2\}$ is a Cartesian system of subgroups in $M$ with respect to $\omega$. Hence $G$ preserves the corresponding Cartesian decomposition of $\Omega$. Clearly in this example we have $|F_i| = 0$, and this Cartesian system is $M$-normal.

In Example 6.1 the diagonal subgroups $D(T \times T)$ involved in $K_1$ and $K_2$ are disjoint in the sense that the diagonal subgroup of $K_1$ is contained in the direct product of the first two simple direct factors of $M$, while the diagonal subgroup of $K_2$ is contained in the direct product of the last two simple direct factors. It turns out that any two diagonal
subgroups involved in a Cartesian system with $|\mathcal{F}_i| = 0$ are disjoint. This also means that each such Cartesian system is normal.

Next we present two examples with $|\mathcal{F}_i| = 1$, and show that the class of Cartesian systems with $|\mathcal{F}_i| = 1$ contains both normal and non-normal examples. Moreover, the first example shows that, for each nonabelian simple group $T$ and each of its proper subgroups $A$, there is such an example with plinth a direct power of $T$ and $\mathcal{F}_i = \{A\}$.

**Example 6.2.** Let $A$ be a proper subgroup of $T$, let $M = T \times T$, $G = M \times S_2 = T \wr S_2$, $K_1 = T \times A$, and $K_2 = A \times T$. Then the $M$-coset action on $\Omega = [M : K_1 \cap K_2]$ can be extended to $G$ with point stabiliser $(K_1 \cap K_2) \times S_2$, and $\{K_1, K_2\}$ is a Cartesian system of subgroups in $M$. Thus $G$ preserves the corresponding Cartesian decomposition of $\Omega$, $|\mathcal{F}_i| = 1$, and the Cartesian system is $M$-normal. Indeed it is not difficult to see that all Cartesian systems with $|\mathcal{F}_i| = 1$, and involving no diagonal subgroups, are normal.

**Example 6.3.** Set $T = A_6$, $A = A_5$, and $B = \text{PSL}_2(5)$. Then there exists an element $\tau \in \text{Aut}(T)$ that swaps $A$ and $B$, such that $\tau^2 = 1$. Let $G_1 = M_1 \rtimes S_2$ where $M_1 = T \times T$ and the nontrivial element $x$ of $S_2$ acts via $(t_1, t_2)^x = (t_2', t_1')$. Note that $x$ normalises the subgroup $A \times B$ of $M_1$. Let $G = G_1 \wr S_2 = (G_1 \times G_1) \rtimes S_2 = M \rtimes D_8$, with $M \cong T^4$ the unique minimal normal subgroup of $G$. Define

$$K_1 = A \times B \times D(T \times T) \quad \text{and} \quad K_2 = D(T \times T) \times A \times B.$$ 

Then the $M$-coset action on $\Omega = [M : K_1 \cap K_2]$ can be extended to $G$ with point stabiliser $(K_1 \cap K_2) \rtimes D_8$, and $\{K_1, K_2\}$ is a Cartesian system of subgroups in $M$. Hence $G$ preserves the corresponding Cartesian decomposition of $\Omega$, $|\mathcal{F}_i| = 1$, but, as the $K_i$ involve diagonal subgroups, this Cartesian system is not normal.

Finally we present two further examples, one with $|\mathcal{F}_i| = 2$, and one with $|\mathcal{F}_i| = 3$.

**Example 6.4.** Let $A$ and $B$ be two subgroups of $T$ such that $T = AB$, let $M = T^2$, $K_1 = A \times B$, $K_2 = B \times A$, and $G = M \rtimes S_2 = T \wr S_2$. Then the $M$-coset action on $\Omega = [M : K_1 \cap K_2]$ can be extended to $G$ with point stabiliser $(K_1 \cap K_2) \rtimes S_2$, and $\{K_1, K_2\}$ is a Cartesian system of subgroups in $M$. Hence $G$ preserves the corresponding Cartesian decomposition of $\Omega$, $|\mathcal{F}_i| = 2$, and the Cartesian system is not normal.

**Example 6.5.** Let $T$ be a finite simple group such that $T$, $A$, $B$, $C$ form a strong multiple factorisation of $T$, that is to say, the equations in $(\text{6})$ hold. Let $M = T^3$, $G = M \rtimes C_3 = T \wr C_3$,

$$K_1 = A \times B \times C, \quad K_2 = B \times C \times A, \quad \text{and} \quad K_3 = C \times A \times B.$$
Then the $M$-coset action on $\Omega = [M : K_1 \cap K_2 \cap K_3]$ can be extended to $G$ with point stabiliser $(K_1 \cap K_2 \cap K_3) \rtimes C_3$, and $\{K_1, K_2, K_3\}$ is a Cartesian system of subgroups in $M$. Hence $G$ preserves the corresponding Cartesian decomposition of $\Omega$, $|\mathcal{F}_i| = 3$, and the Cartesian system is not normal.

In Example 6.4, in addition to the Cartesian system given, there is also an $M$-normal Cartesian system for the same action of $G$ formed by the subgroups $(A \cap B) \times T$ and $T \times (A \cap B)$. Similarly, in Example 6.5 there is an $M$-normal Cartesian system for the same action of $G$ formed by the subgroups $(A \cap B \cap C) \times T \times T$, $T \times (A \cap B \cap C) \times T$, and $T \times T \times (A \cap B \cap C)$. This turns out to be a rather general phenomenon, and will be studied further in our forthcoming paper.

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