The Wavefront Sets of Unipotent Supercuspidal Representations

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Abstract

Let $G(k)$ be a semisimple $p$-adic group, inner to split. In this article, we compute the algebraic and canonical unramified wavefront sets of the irreducible supercuspidal representations of $G(k)$ in Lusztig’s category of unipotent representations.

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6.1 Classical groups

6.2 Exceptional groups

1 Introduction

Let $G$ be a connected semisimple algebraic group defined over a $p$-adic field $k$ with residue field $F_q$ and let $G(k)$ be the group of $k$-rational points. Let $X$ be an irreducible admissible representation of $G(k)$ with distribution character $\Theta_X$. For each nilpotent orbit $O$ in the Lie algebra $g(k)$ of $G(k)$, let $\hat{\mu}_O$ denote the Fourier transform of the associated orbit integral. In [Har99], Harish-Chandra proved that there are complex numbers $c_0(X) \in \mathbb{C}$ such that

$$\Theta_X(\exp(\xi)) = \sum_O c_0(X)\hat{\mu}_O(\xi)$$  (1.0.1)

for $\xi \in g(k)$ a regular element in a small neighborhood of 0. The formula (2.3.1) is called the local character expansion of $X$.  

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One of the most fundamental invariants which can be extracted from the local character expansion is the so-called wavefront set. This is the set of nilpotent orbits

$$WF(X) = \max \{ \emptyset \subset g(k) \mid c_0(X) \neq 0 \}$$

It is common in the literature to consider a slightly coarser invariant called the geometric wavefront set. This is the set of nilpotent orbits over an algebraic closure $k$ of $k$ which meet some orbit in $WF(X)$. This set is denoted by $kWF(X)$. A longstanding conjecture of Mœglin and Waldspurger [MW87, Page 429] asserts that $kWF(X)$ is a singleton, for all $X$. Another closely related invariant is the canonical unramified wavefront set, which was recently introduced by the third-named author in [Oka21, Section 2.2.3]. This final invariant is a refinement of $kWF(X)$.

Lusztig in [Lus93, Section 0.3] defined the notion of a unipotent representation of $G(k)$ (see Definition 2.6.1 below) and completed a Langlands classification for this class of representations, under the assumption that $G$ is simple and adjoint. These restrictions on $G$ have since been removed as a result of various works, culminating with [Sol18].

Let $G^\vee$ denote the complex Langlands dual group. The (enhanced) Langlands parameter of an irreducible unipotent representation $X$ is the $G^\vee$-orbit of a triple $(\tau, n, \rho)$, where $\tau$ is a semisimple element in $G^\vee$, $n$ is a nilpotent element in $g^\vee$ (the Lie algebra of $G^\vee$) such that $\text{Ad}(\tau)n = qn$, and $\rho$ is an irreducible representation of a certain finite group $A^1_{\tau}$, see section 2.7. When $G$ is adjoint, $A^1_{\tau}$ is the component group of the centralizer of $\tau$ and $n$ in $G^\vee$. Let $O_X$ denote the nilpotent $G^\vee$-orbit of $n$.

It is natural to ask if and how the local character expansion is related to the Langlands parameter $(\tau, n, \rho)$ of $X$. At one extreme, we have the coefficient $c_0(X)$. It has long been known that when $X$ is tempered, $c_0(X) \neq 0$ if and only if $X$ is square integrable, and in this case, $c_0(X)$ equals, up to a sign, the ratio between the formal degrees of $X$ and the Steinberg representation. An interpretation of the formal degree in terms of the Langlands parameters was conjectured first by Reeder [Ree00]; this interpretation was verified in the case of split exceptional groups by Reeder ([Ree00] and in the remaining cases by Opdam [Opd16].

At the other extreme, we have the wavefront set of $X$. In this article, we compute the geometric and canonical unramified wavefront sets of all supercuspidal unipotent representations of $G(k)$ when $G$ is an inner twist of a split group. In all such cases, we find that $kWF(X)$ is a singleton, and is uniquely determined by the nilpotent part $n$ of the Langlands parameter. More precisely, we prove in Theorem 3.0.2

$$kWF(X) = d(O_X), \quad KWF(X) = d_A(O_X, 1). \quad (1.0.2)$$

Here $d$ and $d_A$ are the duality maps defined by Spaltenstein, also Lusztig and Barbasch-Vogan, and Achar, respectively, see section 2.2. In particular, the geometric wavefront set $kWF(X)$ determines the nilpotent orbit $O_X$ via the duality map $d$. We emphasize that the simplicity of the formulas (1.0.2) is due to the fact that $X$ is supercuspidal and therefore equal to its Aubert-Zelevinsky dual ($Aub95$). In general, one expects that the wavefront set of $X$ is obtained by duality from the nilpotent parameter associated to the AZ dual of $X$, see [CMO21]. This expression for the wavefront set is closely related to Lusztig’s formula for the Kawanaka wavefront set of an irreducible unipotent representation of a finite reductive group [Lus92, Theorem 11.2]. In fact, the finite reductive group results from loc.cit. play an important role in the construction and analysis of test functions in the local character expansion [BM97, Oka21].

We remark that the methods in this paper show that the canonical unramified wavefront set (and so the geometric wavefront set as well) is in fact a singleton for all depth-zero supercuspidal representations, see Proposition 2.5.4. The irreducibility of the geometric wavefront set for depth-zero supercuspidal representations was established independently in [AGS22], but also as a consequence of the results in [BM97, Oka21].

2 Preliminaries

Let $k$ be a nonarchimedean local field of characteristic 0 with residue field $\mathbb{F}_q$ of sufficiently large characteristic, ring of integers $\mathfrak{o} \subset k$, and valuation $\text{val}_k$. Fix an algebraic closure $\bar{k}$ of $k$ with Galois group $\Gamma$, and let $K \subset \bar{k}$ be the maximal unramified extension of $k$ in $k$. Let $\mathcal{O}$ be the ring of integers of $K$. Let $\text{Frob}$ be the geometric Frobenius element of $\text{Gal}(K/k)$, the topological generator which induces the inverse of the automorphism $x \mapsto x^q$ of $\mathbb{F}_q$. 

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Let $\mathbf{G}$ be a connected semisimple algebraic group defined over $\mathbb{Z}$, and let $\mathbf{T} \subset \mathbf{G}$ be a maximal torus. For any field $F$, we write $\mathbf{G}(F)$, $\mathbf{T}(F)$, etc. for the groups of $F$-rational points. The $\mathbb{C}$-points are denoted by $G$, $T$, etc. Let $G_{\text{ad}} = G/Z(G)$ denote the adjoint group of $\mathbf{G}$.

Write $X^*(\mathbf{T}, \bar{k})$ (resp. $X_*(\mathbf{T}, \bar{k})$) for the lattice of algebraic characters (resp. co-characters) of $\mathbf{T}(\bar{k})$, and write $\Phi(\mathbf{T}, \bar{k})$ (resp. $\Phi^\vee(\mathbf{T}, \bar{k})$) for the set of roots (resp. co-roots). Let

$$R = (X^*(\mathbf{T}, \bar{k}), \Phi(\mathbf{T}, \bar{k}), X_*(\mathbf{T}, \bar{k}), \Phi^\vee(\mathbf{T}, \bar{k}), \langle , \rangle)$$

be the root datum corresponding to $\mathbf{G}$, and let $W$ the associated (finite) Weyl group. Let $\mathbf{G}^\vee$ be the Langlands dual group of $\mathbf{G}$, i.e. the connected reductive algebraic group corresponding to the dual root datum

$$R^\vee = (X_*(\mathbf{T}, \bar{k}), \Phi^\vee(\mathbf{T}, \bar{k}), X^*(\mathbf{T}, \bar{k}), \Phi(\mathbf{T}, \bar{k}), \langle , \rangle).$$

Set $\Omega = X_*(\mathbf{T}, \bar{k})/\mathbb{Z}\Phi^\vee(\mathbf{T}, \bar{k})$. The center $Z(\mathbf{G}^\vee)$ can be naturally identified with the irreducible characters $\text{Irr}_\mathbb{C}$, and dually, $\Omega \cong X^*(Z(\mathbf{G}^\vee))$. For $\omega \in \Omega$, let $\zeta_\omega$ denote the corresponding irreducible character of $Z(\mathbf{G}^\vee)$.

For details regarding the parametrization of inner twists of a group $\mathbf{G}(k)$, see [Vog93, §2], [Kot84], [Kal16, §2], or [ABPS17, §1.3] and [FOS19, §1]. We only record here that the set of equivalence classes of inner twists of the split form of $\mathbf{G}$ are parametrized by the Galois cohomology group

$$H^1(\Gamma, G_{\text{ad}}) \cong H^1(F, G_{\text{ad}}(K)) \cong \Omega_{\text{ad}} \cong \text{Irr}(Z(\mathbf{G}^\vee)),$$

where $\mathbf{G}^\vee_\mathbb{C}$ is the Langlands dual group of $G_{\text{ad}}$, i.e., the simply connected cover of $\mathbf{G}^\vee$, and $F$ denotes the action of Frob on $\mathbf{G}(K)$. We identify $\Omega_{\text{ad}}$ with the fundamental group of $G_{\text{ad}}$. The isomorphism above is determined as follows: for a cohomology class $h$ in $H^1(F, G_{\text{ad}}(K))$, let $z$ be a representative cocycle. Let $u \in G_{\text{ad}}(K)$ be the image of $F$ under $z$, and let $\omega$ denote the image of $u$ in $\Omega_{\text{ad}}$. Set $F_\omega = \text{Ad}(u) \circ F$. The corresponding rational structure of $G_{\text{ad}}$ is given by $F_\omega$. Let $G_\omega$ be the connected semisimple group defined over $k$ such that $G(K)^{F_\omega} = G_\omega(k)$. Note that $G^1 = G$ (where we view $G$ as an algebraic group over $k$ for this equality).

If $H$ is a complex reductive group and $x$ is an element of $H$ or $\mathfrak{h}$, we write $H(x)$ for the centralizer of $x$ in $H$, and $A_H(x)$ for the group of connected components of $H(x)$. If $S$ is a subset of $H$ or $\mathfrak{h}$ (or indeed, of $H \cup \mathfrak{h}$), we can similarly define $H(S)$ and $A_H(S)$.

In this section we will recall some standard facts about the Bruhat-Tits building (all of which can be found in [Tit79]).

Fix a $\omega \in \Omega$ and let $G_\omega$ be the inner twist of $G$ corresponding $\omega$ as defined in the previous section. Let $\mathcal{B}(G_\omega, k)$ denote the (enlarged) Bruhat-Tits building for $G_\omega(k)$. Let $\mathcal{B}(G, K)$ denote the (enlarged) Bruhat-Tits for $G(K)$. For an apartment $A$ of $\mathcal{B}(G, K)$ and $\Omega \subseteq A$ we write $A(\Omega, A)$ for the smallest affine subspace of $A$ containing $\Omega$. The inner twist $G_\omega$ of $G$ gives rise to an action of the Galois group $\text{Gal}(K/k)$ on $\mathcal{B}(G, K)$ and we can (and will) identify $\mathcal{B}(G_\omega, k)$ with the fixed points of this action. We use the notation $c \subseteq \mathcal{B}(G_\omega, k)$ to indicate that $c$ is a face of $\mathcal{B}(G_\omega, k)$. Given a maximal $k$-split torus $\mathbf{T}$ of $G_\omega$, write $A(\mathbf{T}, k)$ for the corresponding apartment in $\mathcal{B}(G_\omega, k)$. Write $\Phi(\mathbf{T}, k)$ (resp. $\Psi(\mathbf{T}, k)$) for the set of roots of $G(k)$ (resp. affine roots) of $\mathbf{T}(k)$ on $G_\omega(k)$. For $\psi \in \Phi(\mathbf{T}, k)$ write $\psi' \in \Phi(T, k)$ for the gradient of $\psi$, and $W = W(\mathbf{T}, k)$ for the Weyl group of $G(k)$ with respect to $\mathbf{T}(k)$. For a face $c \subseteq \mathcal{B}(G_\omega, k)$ there is a group $P^c_{\omega}$ defined over $k$ such that $P^c_{\omega}(k)$ identifies with the stabiliser of $c$ in $G(k)$. There is an exact sequence

$$1 \to U_c(\mathfrak{o}) \to P^c_{\omega}(\mathfrak{o}) \to L^c(\mathbb{F}_q) \to 1,$$  

(2.1.1)
where $U_c(\mathfrak{o})$ is the pro-unipotent radical of $P_c^1(\mathfrak{o})$ and $L_c^1$ is the reductive quotient of the special fibre of $P_c^1$. Let $L_c$ denote the identity component of $L_c^1$, and let $P_c$ be the subgroup of $P_c^1$ defined over $\mathfrak{o}$ such that $P_c(\mathfrak{o})$ is the inverse image of $L_c(\mathbb{F}_q)$ in $P_c^1(\mathfrak{o})$. We also write $T$ for the well defined split torus scheme over $\mathfrak{o}$ with generic fibre $T$. This scheme $T$ defined over $\mathfrak{o}$ is a subgroup of $P_c$ and the special fibre of $T$, denoted $T$, is a maximal torus of $L_c$. For $c$ viewed as a face of $B(G, K)$, the stabiliser of $c$ in $G(K)$ identifies with $P_c^1(\mathcal{O})$.

It has pro-unipotent radical $U_c(\mathcal{O})$ and $P_c^1(\mathcal{O})/U_c(\mathcal{O}) = L_c(\mathbb{F}_q)$. For $c$ a face lying in $B(G^\omega, k) \subseteq B(G, K)$, $F_{\omega}$ stabilises $P_c(\mathcal{O})$ and induces a Frobenius on $L_c(\mathbb{F}_q)$. The group $L_c(\mathbb{F}_q)$ consists of the fixed points of this Frobenius. The groups $P_c(\mathfrak{o})$ obtained in this manner are called $(k)$-parahoric subgroups of $G^\omega$. When $c$ is a chamber, then we call $P_c(\mathfrak{o})$ an Iwahori subgroup of $G$.

For this paper it will be convenient to fix a maximal $k$-split torus $T$ of $G^\omega$ containing $T$ and defined over $k$. We have that $A(T, k) = A(T, 1, K)^{\text{Gal}(K/k)}$. We will also fix a $\text{Gal}(K/k)$-stable chamber $c_0$ of $A(T, 1)$. There is a natural map $\Phi_c(T, 1, K)$ is isomorphic to $\text{Gal}(k)$ and defined over $k$, $\Phi_c(T, 1, K)$ fixes an embedding $\Phi_c(T, 1, K)$ and an isomorphism between $\tilde{W}$ and $N_{G(\mathcal{O})}(T(1, K))/T(1, K)^{\times}$. Write

$$\tilde{W} \rightarrow W, \quad w \mapsto w$$

(2.1.2)

for the natural projection map. For a face $c \subseteq A$ let $W_c$ be the subgroup of $\tilde{W}$ generated by reflections in the hyperplanes through $c$. The special fibre of $T_1$ (as a scheme over $\mathcal{O}$) which we denote by $T_1$, is a split maximal torus of $L_1(\mathbb{F}_q)$. Write $\Phi_c(T_1, \mathbb{F}_q)$ for the root system of $L_c$ with respect to $T_1$. Then $\Phi_c(T_1, \mathbb{F}_q)$ naturally identifies with the set of $\psi \in \Psi(T_1, K)$ that vanish on $c$, and the Weyl group of $T_1$ in $L_c$ is isomorphic to $W_c$.

Recall that a choice of $x_0$ fixes an embedding $\Phi(T_1, K) \rightarrow \Phi(T_1, L)$. If we fix a set of simple roots $\Delta \subseteq \Phi(T_1, K)$, this embedding determines a set of extended simple roots $\tilde{\Delta} \subseteq \Psi(T_1, K)$. When $\Phi(T_1, K)$ is irreducible, $\tilde{\Delta}$ is just the set $\Delta \cup \{1 - a_0\}$ where $a_0$ is the highest root of $\Phi(T_1, K)$ with respect to $\Delta$. When $\Phi(T_1, K)$ is reducible, say $\Phi(T_1, K) = \cup_i \Phi_i$, where each $\Phi_i$ is irreducible, then $\tilde{\Delta} = \cup_i \Delta_i$, where $\Delta_i = \Phi_i \cap \Delta$. Fix $\Delta$ so that the chamber cut out by $\Delta$ is $c_0$. Let

$$P(\Delta) := \{J \subseteq \tilde{\Delta} : J \cap \Delta_i \subseteq \Delta_i, \forall i\}.$$

Each $J \in P(\Delta)$ cuts out a face of $c_0$ which we denote by $c(J)$. In particular $c(\Delta) = x_0$. Note that since $\Omega \simeq \tilde{W}/W \simeq \mathbb{Z}\Phi(T_1, K)$ (recall $G$ is semisimple), and $W \simeq \mathbb{Z}\Phi(T_1, K)$ acts simply transitively on the chambers of $A(T_1, K)$, the action of $\tilde{W}$ on $A(T_1, K)$ induces an action of $\Omega$ on the faces of $c_0$ and hence on $\tilde{\Delta}$ (and $P(\Delta)$). For $\omega \in \Omega$ let $\sigma_\omega$ denote the corresponding permutation of $\tilde{\Delta}$.

Let $P^\omega(\Delta) := \{J \in P(\Delta) \mid \sigma_\omega(J) = J\}$

and let $c_0^\omega$ be the chamber of $B(G^\omega, k)$ lying in $c_0$. The set $P^\omega(\Delta)$ is an indexing set for the faces of $c_0^\omega$. For $J \in P^\omega(\Delta)$ write $c^\omega(J)$ for the face of $c_0^\omega$ corresponding to $J$. The face $c^\omega(J)$ lies in $c(J)$. Moreover for $J, J' \in P^\omega(\Delta)$ (resp. $P(\Delta)$) we have $J \subseteq J'$ if and only if $c^\omega(J) \supseteq c^\omega(J')$ (resp. $c(J) \supseteq c(J')$).

### 2.2 Nilpotent orbits

Let $N$ be the function which takes a field $F$ to the set of nilpotent elements in $g(F)$, and let $N_o$ be the functor which takes $F$ to the set of adjoint $G(F)$-orbits on $N(F)$. When $F$ is $k$ or $K$, we view $N_o(F)$ as a partially ordered set with respect to the closure ordering in the topology induced by the topology on $F$. When $F$ is algebraically closed, we view $N_o(F)$ as a partially ordered set with respect to the closure ordering in the Zariski topology. For brevity we will write $N_o(F'/F)$ (resp. $N_o(F'/F)$) for $N(F' \rightarrow F)$ (resp. $N_o(F' \rightarrow F)$) where $F \rightarrow F'$ is a morphism of fields. For $(F, F') = (k, K)$ (resp. $(k, k)$, $(K, K)$), the map $N_o(F'/F)$ is strictly increasing (resp. strictly decreasing). We will simply write $N_o$ for $N_o(\mathcal{C})$ and $N_o$ for $N_o(\mathcal{C})$. In this case we also define $N_o(c) = \{p \in N_o \mid p \subseteq c\}$ to be the set of all pairs $(b, C)$ such that $b \in N_o$ and $C$ is a conjugacy class in the fundamental group $A(\mathcal{O})$ of $\mathcal{O}$ (resp. Lusztig’s canonical quotient $\tilde{A}(\mathcal{O})$ of $A(\mathcal{O})$, see [Som01, Section 5]). There is a natural map

$$\Omega : N_o \rightarrow N_o, \quad (\mathcal{O}, C) \mapsto (\tilde{\mathcal{O}}, \tilde{C})$$

(2.2.1)
where $\bar{C}$ is the image of $C$ in $\tilde{A}(\mathbb{O})$ under the natural homomorphism $A(\mathbb{O}) \to \tilde{A}(\mathbb{O})$. There are also projection maps $\text{pr}_1 : N_{o,c} \to N_o$, $\text{pr}_1 : N_{o,c} \to N_o$. We will typically write $N_\gamma$, $N_\gamma'$, $N_{o,c}$, and $N_{o,c}'$ for the sets $N$, $N_o$, $N_{o,c}$, and $N_{o,c}$ associated to the Langlands dual group $G^\vee$. When we wish to emphasise the group we are working with we include it as a superscript e.g. $N_{o,c}^{G^\omega}$. Note that since $G^\omega$ splits over $K$ we have that $N_{o,c}^{G^\omega}(F) = N_{o,c}(F)$ for field extensions $F$ of $K$.

### 2.2.1 Classical results and constructions

Recall the following classical results and constructions related to nilpotent orbits.

**Lemma 2.2.1** (Corollary 3.5, [Pom77] and Theorem 1.5, [Pom80]). Let $F$ be algebraically closed with good characteristic for $G$. Then there is canonical isomorphism of partially ordered sets $\Lambda^F : N_{o,c}^{G^\omega}(F) \cong N_o$.

Write

$$d : N_0 \to N_0', \quad d : N_0' \to N_0,$$

(2.2.2)

duality maps defined by Spaltenstein ([Spa82, Proposition 10.3]), Lusztig ([Lus84, §13.3], and Barbasch-Vogan ([BV85, Appendix A]). Write

$$d_S : N_{o,c} \to N_o', \quad d_S : N_o' \to N_o$$

(2.2.3)

for the duality maps defined by Sommers in [Som01, Section 6] and

$$d_A : N_{o,c} \to N_{o,c}'', \quad d_A : N_{o,c}'' \to N_{o,c}$$

(2.2.4)

for the duality maps defined by Achar in ([Ach03, Section 1]). These duality maps are compatible in the following sense. For $\mathbb{O} \in N_o$

$$d_S(\mathbb{O}, 1) = d(\mathbb{O})$$

and for $(\mathbb{O}, C) \in N_{o,c}$

$$d_A(\mathbb{O}(\mathbb{O}, C)) = (d_S(\mathbb{O}, C), \bar{C}')$$

for some $\bar{C}'$.

There is a natural pre-order $\preceq_A$ on $N_{o,c}$ defined by Achar in [Ach03, Introduction] by

$$(\mathbb{O}, C) \preceq_A (\mathbb{O}', C') \iff \mathbb{O} \preceq \mathbb{O}' \text{ and } d_S(\mathbb{O}, C) \succeq d_S(\mathbb{O}', C').$$

Write $\sim_A$ for the equivalence relation on $N_{o,c}$ induced by this pre-order, i.e.

$$(\mathbb{O}_1, C_1) \sim_A (\mathbb{O}_2, C_2) \iff (\mathbb{O}_1, C_1) \preceq_A (\mathbb{O}_2, C_2) \text{ and } (\mathbb{O}_2, C_2) \preceq_A (\mathbb{O}_1, C_1)$$

Write $[(\mathbb{O}, C)]$ for the equivalence class of $(\mathbb{O}, C) \in N_{o,c}$. The $\sim_A$-equivalence classes in $N_{o,c}$ coincide with the fibres of the projection map $\mathbb{O} : N_{o,c} \to N_{o,c}$ [Ach03, Theorem 1]. So $\preceq$ descends to a partial order on $N_{o,c}$, also denoted by $\preceq_A$. The maps $d, d_A$ are all order reversing with respect to the relevant pre/partial orders.

### 2.2.2 Structure of $N^{G^\omega}(K)$

In [Oka21, Section 2] the third-named author establishes a number of results about the structure of $N_{o,c}^{G^\omega}(K) = N_{o,c}^{G^\omega}(F)$ which we now briefly summarize.

Let $T$ be a maximal $k$-split torus of $G^\omega$, $T_1$ be a maximal $K$-split torus of $G^\omega$ defined over $k$ and containing $T$, and $x_0$ be a special point in $A(T_1, K)$. In [Oka21, Section 2.1.5] the third-named author constructs a bijection

$$\theta_{x_0, T_1} : N_{o,c}^{G^\omega}(K) \cong N_{o,c}.$$

**Theorem 2.2.2.** [Oka21, Theorem 2.20, Theorem 2.27, Proposition 2.29] The bijection

$$\theta_{x_0, T_1} : N_{o,c}^{G^\omega}(K) \cong N_{o,c}$$

is natural in $T_1$, equivariant in $x_0$, and makes the following diagram commute:

$$
\begin{array}{c}
\Lambda^{G^\omega}(K) \\
\downarrow_{\Lambda^k} \downarrow_{\text{pr}_1}
\end{array}
\begin{array}{c}
N_{o,c}(\kappa) \\
\downarrow_{\text{pr}_1}
\end{array}
\begin{array}{c}
N_{o,c} \\
\end{array}
$$

(2.2.5)
The composition
\[ d_{S, T} := d_S \circ \theta_{x_0, T}, \]
is independent of the choice of \( x_0 \) and natural in \( T_1 \) [Oka21, Proposition 2.32].

For \( \mathcal{O}_1, \mathcal{O}_2 \in \mathcal{N}_0(K) \) define \( \mathcal{O}_1 \leq_A \mathcal{O}_2 \) by
\[ \mathcal{O}_1 \leq_A \mathcal{O}_2 \iff \mathcal{N}_0(\tilde{K}/K)(\mathcal{O}_1) \leq \mathcal{N}_0(\tilde{K}/K)(\mathcal{O}_2), \quad \text{and} \quad d_{S, T_1}(\mathcal{O}_1) \geq d_{S, T_2}(\mathcal{O}_2) \]
and let \( \sim_A \) denote the equivalence classes of this pre-order. This pre-order is independent of the choice of \( T_1 \) and the map
\[ \theta_{x_0, T_1} : (\mathcal{N}_0(K), \leq_A) \to (\mathcal{N}_{0, c}, \leq_A) \]
is an isomorphism of pre-orders.

**Theorem 2.2.3.** The composition \( \mathcal{O} \circ \theta_{x_0, T_1} : \mathcal{N}_0(K) \to \mathcal{N}_{0, \bar{c}} \) descends to a (natural in \( T_1 \)) bijection
\[ \tilde{\theta}_{T_1} : \mathcal{N}_0(K) / \sim_A \to \mathcal{N}_{0, \bar{c}} \]
which does not depend on \( x_0 \).

### 2.2.3 Lifting nilpotent orbits

Define
\[ \mathcal{I}_0 = \{(c, \mathcal{O}) \mid c \subseteq B(G), \mathcal{O} \in \mathcal{N}_0^{L_r}(\mathbb{F}_q)\}. \tag{2.2.6} \]
There is a partial order on \( \mathcal{I}_0 \), defined by
\[ (c_1, \mathcal{O}_1) \leq (c_2, \mathcal{O}_2) \iff c_1 = c_2 \text{ and } \mathcal{O}_1 \leq \mathcal{O}_2 \]
In [Oka21, Section 1.1.2] the third author defines a strictly increasing surjective map
\[ \mathcal{L} : (\mathcal{I}_0, \leq) \to (\mathcal{N}_0(K), \leq). \tag{2.2.7} \]
The composition
\[ [\bullet] \circ \mathcal{L} : (\mathcal{I}_0, \leq) \to (\mathcal{N}_0(K) / \sim_A, \leq_A) \]
is also strictly increasing [Oka21, Corollary 4.7, Lemma 5.3].

Recall the groups \( G = G(\mathbb{C}), T = T(\mathbb{C}) \) from section 2. Call a pseudo-Levi subgroup \( L \) of \( G \) standard if it contains \( T \) and write \( Z_L \) for its center. Let \( \mathcal{A} = \mathcal{A}(T_1, K) \).

**Lemma 2.2.4.** [Oka21, Section 14, Corollary 2.19] There is a \( W \)-equivariant map
\[ \mathcal{L}_{x_0} : \{\text{faces of } \mathcal{A}\} \to \{(L, tZ_L^c) \mid L \text{ a standard pseudo-Levi}, \ Z_L^c(tZ_L^c) = L\} \tag{2.2.8} \]
where \( c_1, c_2 \) lie in the same fibre iff
\[ \mathcal{A}(c_1, A) + X_s(T_1, K) = \mathcal{A}(c_2, A) + X_s(T_1, K). \]
Moreover, if \( \mathcal{L}_{x_0}(c) = (L, tZ_L^c) \) then \( L \) is the complex reductive group with the same root datum as \( L_c(\mathbb{F}_q) \) and thus there is an isomorphism \( \tilde{\mathcal{N}}^{L_r} : \mathcal{N}_0^{L_r}(\mathbb{F}_q) \simeq \mathcal{N}_0^{L_r} \).

Recall from the end of section 2.1 the definitions of \( \Delta, \tilde{\Delta}, c_0, \mathcal{P}(\tilde{\Delta}) \) and \( c(J) \). Note that the definitions of \( c_0 \) and \( c(J) \) depend on a choice of \( x_0 \) and \( \Delta \). Let \( L_J \) denote the pseudo-Levi subgroup of \( G \) generated by \( T \) and the root groups corresponding to \( \vec{\alpha} \) for \( \alpha \in J \). Then \( \text{pr}_1 \circ \mathcal{L}_{x_0}(c(J)) = L_J \) (indeed by [Oka21, Lemma 2.21], this group should not depend on \( x_0 \)). Define
\[ \mathcal{I}_{x_0, \Delta} = \{(J, \mathcal{O}) \mid J \in \mathcal{P}(\tilde{\Delta}), \ \mathcal{O} \in \mathcal{N}_0^{L_{c(J)}}(\mathbb{F}_q)\}, \tag{2.2.9} \]
\[ \mathcal{K}_{\tilde{\Delta}} = \{(J, \mathcal{O}) \mid J \in \mathcal{P}(\tilde{\Delta}), \ \mathcal{O} \in \mathcal{N}_0^{L_J}(\mathbb{C})\}. \tag{2.2.10} \]
The map
\[ \iota_{x_0} : \mathcal{I}_{x_0, \Delta} \rightarrow \mathcal{K}_\Delta, \quad (J, \emptyset) \mapsto (J, \Lambda_{c,0}(\emptyset)) \]
is an isomorphism. Let
\[ L : \mathcal{K}_\Delta \rightarrow \mathcal{N}_{c,o} \quad \text{(2.2.11)} \]
be the map that sends \((J, \emptyset)\) to \((Gx, tZ^*_G(x))\) where \(x \in \emptyset\) and \((L, tZ^*_L) = \mathcal{L}(c(J))\) where \(\mathcal{L}\) is the map from \([\text{Oka21}]\) Corollary 2.19. By \([\text{CMO21}], \text{Theorem 2.1.7}\) the diagram
\[ \begin{array}{ccc}
\mathcal{I}_{x_0, \Delta} & \xrightarrow{\sim} & \mathcal{K}_\Delta \\
\downarrow \mathcal{L} & & \downarrow L \\
\mathcal{N}_c(K) & \xrightarrow{\sim} & \mathcal{N}_{c,o}
\end{array} \quad \text{(2.2.12)} \]
commutes. Define
\[ \Gamma = \mathcal{O} \circ L. \]
This map can be computed using Achar’s algorithms in \([\text{Ach03}], \text{Section 3.4}\).

### 2.3 Wavefront sets

Let \(X\) be an admissible smooth representation of \(\mathbf{G}^\omega(k)\) and let \(\Theta_X\) be the character of \(X\). Recall that for each nilpotent orbit \(\emptyset \in \Lambda_{c,0}(\mathbf{G}^\omega(k))\) there is an associated distribution \(\mu_\emptyset\) on \(C^*_p(\mathfrak{g}^\omega(k))\) called the nilpotent orbital integral of \(\emptyset\) \([\text{Ran72}]\). Write \(\hat{\mu}_\emptyset\) for the Fourier transform of this distribution. Generalizing a result of Howe \([\text{How74}]\), Harish-Chandra in \([\text{Har99}]\) showed that there are complex numbers \(c_\emptyset(X) \in \mathbb{C}\) such that
\[ \Theta_X(\exp(\xi)) = \sum_\emptyset c_\emptyset(X) \hat{\mu}_\emptyset(\xi) \quad \text{(2.3.1)} \]
for \(\xi \in \mathfrak{g}^\omega(k)\) a regular element in a small neighborhood of 0. The formula \((2.3.1)\) is called the local character expansion of \(\pi\). The \((p\text{-adic})\) wavefront set of \(X\) is
\[ \text{WF}(X) := \max\{\emptyset \mid c_\emptyset(X) \neq 0\} \subseteq \mathcal{N}_c(k). \]

The geometric wavefront set of \(X\) is
\[ \text{kWF}(X) := \max\{\mathcal{N}_c(\mathbf{k}/\emptyset) \mid c_\emptyset(X) \neq 0\} \subseteq \mathcal{N}_c(\mathbf{k}), \]
see \([\text{Wal18}, \text{p. 1108}]\) (warning: in \([\text{Wal18}]\), the invariant \(\text{kWF}(X)\) is called simply the ‘wavefront set’ of \(X\)).

In \([\text{Oka21}], \text{Section 2.2.3}\) the third author has introduced a third type of wavefront set for depth-0 representations, called the canonical unramified wavefront set. This invariant is a natural refinement of \(\text{kWF}(X)\). We will now define \(\text{WF}(X)\) and explain how to compute it.

Recall from Equation \((2.2.4)\) the lifting map \(\mathcal{L}\). For every face \(c \subseteq \mathcal{B}(\mathbf{G})\), the space of invariants \(X^{	ext{U}_c(\omega)}\) is a \((\text{finite-dimensional})\) \(\mathcal{L}_c(\mathbb{F}_q)\)-representation. Let \(\text{WF}(X^{	ext{U}_c(\omega)}) \subseteq \Lambda_{c,0}^\text{Lc}(\mathbb{F}_q)\) denote the Kawanaka wavefront set \([\text{Kaw87}]\) and let
\[ \text{WF}_c(X) := \{[\mathcal{L}(c, \emptyset)] \mid \emptyset \in \text{WF}(X^\text{U}_c(\omega))\} \subseteq \mathcal{N}_c(K). \quad \text{(2.3.2)} \]

**Definition 2.3.1.** Let \(X\) be a depth-0 representation of \(\mathbf{G}^\omega(k)\). The canonical unramified wavefront set of \(X\) is
\[ \text{WF}(X) := \max\{\text{WF}_c(X) \mid c \subseteq \mathcal{B}(\mathbf{G}^\omega, k)\} \subseteq \mathcal{N}_c(K)/\sim_A. \quad \text{(2.3.3)} \]

Fix \(T, T_1, c_0, x_0, \Delta\) as at the end of section \([2.1]\). By \([\text{Oka21}], \text{Lemma 2.36}\) we have that
\[ \text{WF}(X) = \max\{\text{WF}_{c,J}(X) \mid J \in \mathbf{P}^\omega(\hat{\Delta})\}. \quad \text{(2.3.4)} \]
We will often want to view \(\text{WF}(X)\) and \(\text{WF}_c(X)\) as subsets of \(\mathcal{N}_o, \bar{c}\) using the identification \(\tilde{\theta}_{T_1}\) from Theorem \([22.3]\). We will write
\[ \text{WF}(X, \mathbb{C}) := \tilde{\theta}_{T_1}(\text{WF}(X)), \quad \text{WF}_c(X, \mathbb{C}) := \tilde{\theta}_{T_1}(\text{WF}_c(X)). \]
We will also want to view $\tilde{\kappa}\WF(X)$, which is naturally contained in $N_\sigma(\tilde{k})$, as a subset of $N_\sigma$ and so will write

$$\tilde{\kappa}\WF(X, \mathbb{C}) := \Lambda^{\tilde{\kappa}}(\kappa\WF(X)).$$

By [Oka21, Theorem 2.37], if $\kappa\WF(X)$ is a singleton, then $\tilde{\kappa}\WF(X)$ is also a singleton and

$$\kappa\WF(X, \mathbb{C}) = (\tilde{\kappa}\WF(X, \mathbb{C}), \tilde{C}).$$ (2.3.5)

for some conjugacy class $\tilde{C}$ in $\tilde{\Lambda}(\tilde{\kappa}\WF(X, \mathbb{C}))$.

### 2.4 Isogenies

Let $f : H' \to H$ be an isogeny of connected reductive groups defined over $k$. Let $f_k : H'(k) \to H(k)$ denote the corresponding homomorphism of $k$-points. We note that $N^{H} \cong N^{H'}$ and so we can compare the canonical unramified wavefront sets of representations of the two groups. In an upcoming paper [Oka22], the third author proves the following result about the behaviour of the canonical unramified wavefront set under isogeny.

**Lemma 2.4.1.** Let $X$ be an irreducible admissible depth-0 representation of $H(k)$ and write $X'$ for the representation of $H'(k)$ obtained by pulling back along $f_k$. Then $X'$ decomposes as a finite sum of irreducible admissible representations $X' = \bigoplus_i X'_i$ and $\kappa\WF(X) = \kappa\WF(X'_i)$ for all $i$.

### 2.5 Depth-0 representations

Let $\omega \in \Omega$. Recall that a smooth irreducible representation $X$ of $G(\omega)(k)$ has depth-0 if $X^{U,\omega}(\sigma) \neq 0$ for some $c \subseteq B(G^{\omega}(k))$. Write $\Pi^0(G^{\omega}(k))$ for the subset of $\Pi(G^{\omega}(k))$ consisting of depth-0 representations. Let

$$S^\omega := \{(c, \sigma) : c \subseteq B(G^{\omega}(k)), \sigma \text{ a cuspidal representation of } L_{\kappa}(\mathbb{F}_q)\}$$

and for $(c_1, \sigma_1), (c_2, \sigma_2) \in S^\omega$ write $(c_1, \sigma_1) \sim (c_2, \sigma_2)$ if they are associate in the sense of [MP96, Section 5]. Suppose $(c_1, \sigma_1), (c_2, \sigma_2) \in S^\omega$ are such that $\sigma_1$ is a subrepresentation of $X^{U,\omega}(\sigma)$ for $i \in \{1, 2\}$. Then by [MP96, Theorem 5.2], we have that $(c_1, \sigma_1) \sim (c_2, \sigma_2)$. Thus there is a well defined map

$$\supp : \Pi^0(G^{\omega}(k)) \to S^\omega / \sim$$

which attaches to $X$ the well-defined association class of $(c, \sigma) \in S^\omega$ where $(c, \sigma)$ is such that $\sigma$ appears as a subrepresentation of $X^{U,\omega}(\sigma)$. This association class called the unrefined minimal K-type of $X$, but we write $\supp(X)$ for brevity.

Note that for $(c_1, \sigma_1), (c_2, \sigma_2) \in S^\omega$, if $(c_1, \sigma_1) \sim (c_2, \sigma_2)$ then $(c_1, \WF(\sigma_1)) \sim_K (c_2, \WF(\sigma_2))$ where $\WF$ denotes the Kawanaka wavefront set and $\sim_K$ is the equivalence relation defined in [Oka21, Section 1.1.1]. In particular, since $\mathcal{L}$ is constant on $\sim_K$-classes we have that

$$\mathcal{L}(c_1, \WF(\sigma_1)) = \mathcal{L}(c_2, \WF(\sigma_2)).$$ (2.5.1)

For $X \in \Pi^0(G^{\omega}(k))$ define $\kappa\WF_{\supp}(X)$ to be $\sim_{\mathcal{A}}$-class of the well-defined orbit in Equation 2.5.1 applied to the association class of $\supp(X)$. We will write

$$\kappa\WF_{\supp}(X, \mathbb{C}) := \bar{\theta}_{\mathcal{T}_1}(\kappa\WF_{\supp}(X)).$$

**Lemma 2.5.1.** For $X \in \Pi^0(G^{\omega}(k))$ we have that

$$\kappa\WF_{\supp}(X) \leq_{\mathcal{A}} \emptyset$$

for some $\emptyset \in \kappa\WF(X)$. In particular, when $\kappa\WF(X)$ is a singleton we have

$$\kappa\WF_{\supp}(X) \leq_{\mathcal{A}} \kappa\WF(X).$$
Proof. Let $\text{supp}(X) = [(c, \sigma)]$. Then $\sigma$ is a subrepresentation of $X_{U,c}(\sigma)$ and so $WF(\sigma) \leq \emptyset$ for some $\emptyset \in WF(X_{U,c}(\sigma))$. Thus
$$\mathcal{L}(c, WF(\sigma)) \leq_A \mathcal{L}(c, \emptyset) \in KWF_c(X).$$

The result then follows from the fact that
$$KWF(X) = \max\{KWF_c(X) : c \subseteq B(G^\omega, k)\}.$$

\hfill $\square$

Remark 2.5.2. In fact the analogous result for the unramified wavefront set (as defined in [Ok], Section 1.0.5]), holds as well.

For an Iwahori-spherical representation $X$ this inequality says nothing because $\text{supp}(X) = [(c_0, \text{triv})]$ and so $KWF_{\text{supp}}(X, \mathbb{C}) = (\{0\}, 1)$. For supercuspidal representations however, the inequality is in fact an equality. We now proceed to prove this.

Lemma 2.5.3. Let $X$ be a depth-0 supercuspidal representation. Let $c'$ be a face of $B(G^\omega, k)$ with $X_{U,c}(\sigma) \neq 0$ and suppose that $\tau$ is an irreducible constituent of $X_{U,c}(\sigma)$. Then $\tau$ is a cuspidal representation of $L_{c'}(\mathbb{F}_q)$ and in particular $[(c', \tau)] = \text{supp}(X)$.

Proof. Let $\text{supp}(X) = [(c, \sigma)]$ and $c', \tau$ be as in the statement of the lemma. Let $[(M, \tau')]$ be the cuspidal data for $\tau$ (i.e. a conjugacy class of Levi of $L_{c'}(\mathbb{F}_q)$ and cuspidal representation of said Levi). In particular, if $M$ is included into any parabolic $P$ so that $P$ has Levi decomposition $P = MU$, then $\tau'$ is a subrepresentation of $\tau^U$. Now, all of the parabolics of $L_{c'}(\mathfrak{o})$ are conjugate to a parabolic of the form $P_{c'}(\mathfrak{o})/U_{c'}(\mathfrak{o})$ where $c' \subseteq \overline{c'}$. Thus (conjugating $M$ appropriately) we can find a $c''$ such that $M$ is a Levi factor of $P_{c''}(\mathfrak{o})/U_{c''}(\mathfrak{o})$ and so $L_{c''}(\mathbb{F}_q) \simeq M$. We thus have that
$$\tau' \subseteq \tau^U \subseteq (X_{U,c}(\sigma))_{U,c}(\mathbb{F}_q) = X_{U,c''}(\mathbb{F}_q).$$

In particular $(c'', \tau')$ is an unrefined minimal $K$-type for $X$. Thus by [MP96, Theorem 5.2], we have that $(c'', \tau') \sim (c', \sigma)$. In particular $c''$ is also a minimal face and so $c'' = c'$ and $\tau = \tau'$. Thus $\tau$ is a cuspidal representation of $L_{c'}(\mathbb{F}_q)$ and $[(c', \tau)] = [(c, \sigma)]$.

\hfill $\square$

Proposition 2.5.4. Let $X$ be a depth-0 supercuspidal representation of $G^\omega(k)$. Then
$$KWF_{\text{supp}}(X) = KWF(X).$$

Proof. Recall that
$$KWF(X) = \max\{KWF_c(X) : c \subseteq B(G^\omega, k)\}.$$ 

Suppose $c \subseteq B(G^\omega, k)$ is such that $X_{U,c}(\sigma) \neq 0$. Then for any $\tau$ an irreducible constituent of $X_{U,c}(\sigma)$ we have by Lemma 2.5.3 that $[(c, \tau)] = \text{supp}(X)$. Thus we must have that $KWF_c(X) = KWF_{\text{supp}}(X)$. If $c \subseteq B(G^\omega, k)$ is such that $X_{U,c}(\sigma) = 0$ then $KWF_c(X, \mathbb{C}) = (\{0\}, 1)$. Thus
$$KWF(X) = KWF_{\text{supp}}(X).$$

\hfill $\square$

Remark 2.5.5. Although we have only stated the results for inner twists of the split group, the proposition in fact holds for all connected reductive groups (with identical proof).

Note that since $KWF_{\text{supp}}$ is a singleton by construction, this establishes Moeglin and Waldspurger’s conjecture for all depth-0 supercuspidal representations for connected reductive groups.
2.6 Unipotent supercuspidal representations

Let

\[ S_{\text{unip}}^\omega := \{(c, \sigma) \in S^\omega : \sigma \text{ is unipotent}\}. \]

Definition 2.6.1. Let \( X \) be an irreducible \( G^\omega(k) \)-representation. We say that \( X \) has unipotent cuspidal support if \( \text{supp}(X) \in S_{\text{unip}}^\omega \). Write \( \Pi^{\text{unip}}(G^\omega(k)) \) for the subset of \( \Pi(G(k)) \) consisting of all such representations.

Call a supercuspidal representation unipotent if it has cuspidal unipotent support. By \([\text{Mor96}]\), the irreducible unipotent supercuspidal representations are all obtained by compact induction \( \text{ind}_{P_{c}(\sigma)}^{G(k)}(\sigma^\dagger) \), where \( P_{c}(\sigma) \) is a maximal parahoric subgroup and \( \sigma^\dagger \) contains an irreducible Deligne-Lusztig cuspidal unipotent representation \( \sigma \) of \( L_{c}(F_q) \) upon restriction to \( L_c(F_q) \). In particular \( X \in \Pi^{\text{unip}}(G^\omega(k)) \) is supercuspidal if and only if \( \text{supp}(X) = \{(c, \sigma)\} \) where \( c \) is a minimal face of \( B(G^\omega, k) \).

2.7 Langlands classification of unipotent supercuspidal representations

Let \( W_k \) be the Weil group of \( k \) with inertia subgroup \( I_k \) and set \( W'_k = W_k \times SL(2, \mathbb{C}) \). We will think of a Langlands parameter for \( G \) as a continuous morphisms \( \varphi : W'_k \to G^\omega \) such that \( \varphi(w) \) is semisimple for each \( w \in W_k \) and the restriction of \( \varphi \) to \( SL(2, \mathbb{C}) \) is algebraic. A Langlands parameter \( \varphi \) is called unramified if \( \varphi(I_k) = \{1\} \). Let \( G^\omega(\varphi) \) denote the centralizer of \( \varphi(W'_k) \) in \( G^\omega \). Define

\[ Z^1_{G^\omega}(\varphi) = \text{preimage of } G^\omega(\varphi)/Z(G^\omega) \text{ under the projection } G^\omega \to G^\omega_{\text{ad}}, \]

and let \( A^1_\varphi \) denote the component group of \( Z^1_{G^\omega}(\varphi) \). An enhanced Langlands parameter is pair \((\varphi, \rho)\), where \( \rho \in \text{Irr}(A^1_\varphi) \). A parameter \((\varphi, \rho)\) is called \( G^\omega \)-relevant (recall that \( G^\omega \) is an inner twist of the split form, \( \omega \in \Omega_{\text{ad}} \)) if \( \rho \) acts on \( Z(G^\omega) \) by a multiple of the character \( \zeta_\omega \).

Define the elements

\[ s_\varphi = \varphi(\text{Frob}, 1), \quad u_\varphi = \varphi(1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}). \]

Following \( [\text{AMS18}] \), consider the possibly disconnected reductive group

\[ G_\varphi = Z^1_{G^\omega}(\varphi(W_k)), \]

which is defined analogously to \( Z^1_{G^\omega}(\varphi) \). Then \( u_\varphi \in G_\varphi \) and by \( [\text{AMS18}, \text{(92)}] \)

\[ A^1_\varphi \simeq G_\varphi(u_\varphi)/G_\varphi(u_\varphi)^\circ. \]

An enhanced Langlands parameter \((\varphi, \rho)\) is called discrete if \( G^\omega(\varphi) \) does not contain a nontrivial torus (this notion is independent of \( \rho \)). A discrete parameter is called cuspidal if \((u_\varphi, \rho)\) is a cuspidal pair. This means that every \( \rho^\varphi \) which occurs in the restriction of \( \rho \) to \( A^1_{G^\omega} \) defines a \( G^\omega_{\text{sc}} \)-equivariant local system on the \( G^\omega_{\text{sc}} \)-conjugacy class of \( u_\varphi \) which is cuspidal in the sense of Lusztig.

A Langlands correspondence for unipotent supercuspidal representations has been obtained by \( [\text{Mor96}] \) when \( G \) is simple and adjoint, see also \( [\text{Lus95}] \). For arbitrary reductive \( K \)-split groups, this correspondence is available by \( [\text{FOS19}, \text{FO20}] \). Let \( \text{Irr}(G^\omega(k))_{\text{cusp, unip}} \) denote the set of equivalence classes of irreducible unipotent supercuspidal \( G^\omega(k) \)-representations. Let \( \Phi(G^\omega)_{\text{cusp, nr}} \) denote the set of \( G^\omega \)-equivalence classes of unramified cuspidal enhanced Langlands parameters \((\varphi, \rho)\) which are \( G^\omega \)-relevant.

Theorem 2.7.1. For every \( \omega \in \Omega_{\text{ad}} \), there is a bijection

\[ \Phi(G^\omega)_{\text{cusp, nr}} \longrightarrow \text{Irr}(G^\omega(k))_{\text{cusp, unip}}. \]

This bijection satisfies several natural desiderata (including formal degrees, equivariance with respect to tensoring by weakly unramified characters), see \( [\text{FOS14}, \text{Theorem 2}] \).
For $X$ a unipotent supercuspidal representation of $G^\omega(k)$ let $\varphi$ denote the corresponding Langlands parameter. We will write $O^\vee_X \in \mathcal{N}_0^\vee$ for the $G^\vee$-orbit of 
\[ n_\varphi = d_\varphi(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}). \]

**Lemma 2.7.2.** Let $X, X' \in \text{Irr}(G^\omega(k))_{\text{cusp.unip}}$. If $\text{supp}(X) = \text{supp}(X')$ then $O^\vee_X = O^\vee_{X'}$. 

**Proof.** This follows by inspecting the explicit classification in [FOS19]. \[ \square \]

For $[(c, \sigma)] \in S^\omega$ with $c$ a minimal face we write $O^\vee(c, \sigma)$ for the common nilpotent parameter of all $X \in \text{Irr}(G^\omega(k))_{\text{cusp.unip}}$ with $\text{supp}(X) = [(c, \sigma)]$.

We will recall the explicit classification in the section 5.

### 3 Main result

**Proposition 3.0.1.** Suppose $G$ is simple and adjoint and let $[(c, \sigma)] \in S^\omega$ be such that $c$ is a minimal face. Then 
\[ \mathfrak{L}(c, \text{WF}(\sigma)) = d_A(O^\vee(c, \sigma), 1). \]

This proposition will be proved in Section 6.

**Theorem 3.0.2.** Let $G$ be a split semisimple group defined over $k$. Let $\omega \in \Omega$ and let $G^\omega$ denote the corresponding inner twist of $G$. Let $X$ be an irreducible supercuspidal $G^\omega(k)$-representation with unipotent cuspidal support. Then $^k\text{WF}(X), ^k\text{WF}(X)$ are singletons, and 
\[ ^k\text{WF}(X, C) = d_A(O^\vee_X, 1) \]
\[ ^k\text{WF}(X, C) = d(O^\vee_X). \]

**Proof.** Suppose first that $G$ is simple and adjoint.

Let $X$ be a unipotent supercuspidal representation of $G^\omega(k)$. By Proposition 3.0.1 we have that \[ ^k\text{WF}(X) = ^k\text{WF}_{\text{supp}}(X). \]

Write $\text{supp}(X) = [(c, \sigma)]$. By definition \[ ^k\text{WF}_{\text{supp}}(X, C) = \mathfrak{L}(c, \text{WF}(\sigma)). \]

By Proposition 3.0.1 we have that \[ \mathfrak{L}(c, \text{WF}(\sigma)) = d_A(O^\vee(c, \sigma), 1). \]

Since $O^\vee_X = O^\vee(c, \sigma)$ we get that 
\[ ^k\text{WF}(X; C) = d_A(O^\vee_X, 1) \]
as required.

Applying Lemma 2.4.1 we get that the theorem holds for all simply-connected simple groups. Since wavefront sets behave as expected with respect to products, the theorem holds for all simply-connected semisimple groups. Finally, applying Lemma 2.4.1 again we get that the theorem holds for all split semisimple groups $G$. \[ \square \]

### 4 Unipotent cuspidal representations of finite reductive groups

For the explicit results about the parametrization of unipotent representations of finite reductive groups, we refer to [Lus84, §4, §8.1] and [Car93, §13.8, §13.9]. The relevant results for the Kawanaka wavefront sets and unipotent support are in [Lus92, §10, §11]. The classification of unipotent representations is independent of the isogeny, so in this section, we may assume without loss of generality that the group $G$ is simple and adjoint.
4.1 Classical groups

4.1.1 $A_{n-1}(q)$. The group $G = PGL(n)$ does not have unipotent cuspidal representations.

4.1.2 $A_n(q^2)$. The group $G = PU(n+1)$ has unipotent representations if and if $n = \frac{r(r+1)}{2} - 1$, for some integer $r \geq 2$. The unipotent $A_n(q^2)$-representations are in one-to-one correspondence with partitions of $n + 1$, and so are the geometric nilpotent orbits of $G$. When $n = \frac{r(r+1)}{2} - 1$, the cuspidal unipotent representation $\sigma$ is unique and it is parametrized by the partition

$$(1, 2, 3, \ldots, r).$$

The Kawanaka wavefront set if $WF(\sigma) = (1, 2, 3, \ldots, r)$.

4.1.3 $B_n(q), C_n(q)$. Suppose $G$ is $SO(2n + 1)$ or $PSp(2n)$ over $\mathbb{F}_q$. The group $G(\mathbb{F}_q)$ has a unipotent cuspidal representation (and in this case the cuspidal representation is unique) if and only if $n = r^2 + r$ for a positive integer $r$. The unipotent representations of $G(\mathbb{F}_q)$ are parametrized by symbols

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_a \\ \mu_1 & \cdots & \cdots & \mu_b \end{pmatrix},$$

$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_a$, $0 \leq \mu_1 < \mu_2 < \cdots < \mu_b$, $a - b$ odd and positive, and $\lambda_1, \mu_1$ are not both zero, such that

$$n = \sum \lambda_i + \sum \mu_j - \frac{(a + b - 1)}{2}.$$

Let $d = a - b$ be the defect of the symbol. Two unipotent representations belong to the same family if their symbols have the same entries with the same multiplicities. For the unipotent cuspidal representation $\sigma$, the corresponding symbol has defect $d = 2r + 1$ and it is

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & 2r \end{pmatrix}.$$

The geometric nilpotent orbits of $SO(2n + 1)$ (resp., $PSp(2n)$) are parametrized by partitions of $2n + 1$ (resp., $2n$), where the even (resp., odd) parts occur with even multiplicity. The Kawanaka wavefront set of the unipotent cuspidal representation $\sigma$ is

$WF(\sigma) = \begin{cases} (1, 1, 3, 3, \ldots, 2r - 1, 2r - 1, 2r + 1), & G = SO(2n + 1) \\ (2, 2, 4, 4, \ldots, 2r, 2r), & G = PSp(2n) \end{cases}$ (4.1.1)

4.1.4 $D_n(q)$. Suppose $G$ is the split orthogonal group $PSO(2n)$ over $\mathbb{F}_q$. There exists a unipotent cuspidal representation (and in this case it is unique) if and only if $n = r^2$ for a positive even integer $r$. The type $D_n$-symbols are

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_a \\ \mu_1 & \cdots & \cdots & \mu_b \end{pmatrix},$$

$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_a$, $0 \leq \mu_1 < \mu_2 < \cdots < \mu_b$, $a - b$ is divisible by 4, and $\lambda_1, \mu_1$ are not both zero, such that

$$n = \sum \lambda_i + \sum \mu_j - \frac{(a + b)(a + b - 2)}{4}.$$

One symbol and the symbol if the row swapped are regarded the same. The irreducible unipotent $G(\mathbb{F}_q)$-representations are in one-to-one correspondence with the type $D_n$-symbols, except if the symbol has identical rows, then there are two nonisomorphic irreducible unipotent representations attached to it. The defect $d = a - b$ is even.

For the unipotent cuspidal representation $\sigma$, the corresponding symbol has defect $d = 2r$ and it is

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & 2r - 1 \end{pmatrix}.$$
The geometric nilpotent orbits of $PSO(2n)$ are parametrized by partitions of $2n$ with the even parts occurring with even multiplicity. The Kawanaka wavefront set of the unipotent cuspidal representation is

$$WF(\sigma) = (1, 1, 3, 3, \ldots, 2r - 1, 2r - 1).$$ (4.1.2)

4.1.5 $2D_n(q^2)$. The group $2D_n(q^2)$ admits unipotent cuspidal representations if and only if $n = r^2$, for some odd positive integer $r$, and in this case the unipotent cuspidal representation is unique. The type $2D_n$-symbols are

$$\left( \begin{array}{cccc} \lambda_1 & \lambda_2 & \cdots & \lambda_d \\ \mu_1 & & \cdots & \mu_b \end{array} \right),$$

$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_d$, $0 \leq \mu_1 < \mu_2 < \cdots < \mu_b$, $a - b \equiv 2 \mod 4$, and $\lambda_1, \mu_1$ are not both zero, such that

$$n = \sum \lambda_i + \sum \mu_j - \frac{(a + b)(a + b - 2)}{4}.$$

One symbol and the symbol if the row swapped are regarded the same. The irreducible unipotent $2D_n(q^2)$-representations are in one-to-one correspondence with the type $2D_n$-symbols.

For the unipotent cuspidal representation $\sigma$, the corresponding symbol and Kawanaka wavefront set are the same as in the split case $D_n(q)$ (except $r$ is now odd).

4.1.6 $3D_4(q^3)$. The group $3D_4(q^3)$ has eight unipotent representations: six are in the principal series, in one-to-one correspondence with the irreducible representations of the Weyl group of type $G_2$, and two unipotent cuspidal representations, denoted $3D_4[1]$ and $3D_4[-1]$.

The geometric nilpotent orbits of $3D_4(q^3)$ are parametrized by partitions of 8 with even parts occurring with even multiplicity. The unipotent cuspidal representations have Kawanaka wavefront set

$$WF(3D_4[1]) = WF(3D_4[1]) = (1, 1, 3, 3).$$ (4.1.3)

4.2 Exceptional groups

Suppose $G(\mathbb{F}_q)$ is $\mathbb{F}_q$-split. In the table below, we will list all unipotent cuspidal $G(\mathbb{F}_q)$-representations. The irreducible unipotent representations of $G(\mathbb{F}_q)$ are partitioned into families, each family being in one-to-one correspondence with the set

$$M(\Gamma) = \Gamma\text{-orbits in } \{(x, \tau) \mid x \in \Gamma, \ \tau \in \tilde{Z}(x)\},$$

for a finite group $\Gamma$. Each group $\Gamma$ is uniquely attached to a special nilpotent orbit $\mathbb{O}^\gamma$ in the dual Lie algebra, such that $\Gamma = \tilde{A}(\mathbb{O}^\gamma)$, where $\tilde{A}(\mathbb{O}^\gamma)$ is Lusztig’s canonical quotient.

In Table II for each unipotent cuspidal representation $\sigma$, we will record the corresponding Kawanaka wavefront set, the nilpotent orbit $\mathbb{O}^\gamma$ corresponding to $\sigma$ and its canonical quotient $\tilde{A}(\mathbb{O}^\gamma)$, the pair $(x, \tau) \in M(\tilde{A}(\mathbb{O}^\gamma))$ that parametrizes $\sigma$. The geometric nilpotent orbits are given in the Bala-Carter notation.

| $G(\mathbb{F}_q)$ | Cuspidal $\sigma$ | $WF(\sigma)$ | $\mathbb{O}^\gamma$ | $\tilde{A}(\mathbb{O}^\gamma)$ | $(x, \tau)$ |
|-------------------|-------------------|---------------|---------------------|------------------|-------------|
| $G_2$             | $G_2[1]$          | $G_2(a_1)$    | $G_2(a_1)$          | $S_3$            | $(1, \epsilon)$ |
|                   | $G_2[-1]$         |               |                     |                  | $(g_2, \epsilon)$ |
|                   | $G_2[\theta^l], \ l = 1, 2$ | | | | |
| $F_4$             | $F_4[1]$          | $F_4(a_3)$    | $F_4(a_3)$          | $S_4$            | $(1, \lambda^\gamma)$ |
|                   | $F_4[-1]$         |               |                     |                  | $(g_2, \epsilon)$ |
|                   | $F_4[1]$          |               |                     |                  | $(g_2, \epsilon)$ |
|                   | $F_4[\theta^l], \ l = 1, 2$ | | | | |
|                   | $F_4[\pm i]$      |               |                     |                  | $(g_3, \theta^\epsilon)$ |
| $E_6$             | $E_6[\theta^l], \ l = 1, 2$ | | | | |
|                   | $E_6(a_1)$        | $D_4(a_1)$    | $D_4(a_1)$          | $S_3$            | $(g_5, \theta^\epsilon)$ |
| $E_7$             | $E_7[\xi]$       | $A_4 + A_1$   | $A_4 + A_1$         | $\mathbb{Z}/2$   | $(g_2, 1)$ |
|                   | $E_7[\zeta]$     |               |                     |                  | $(g_2, \epsilon)$ |
5 Langlands parameters for unipotent supercuspidal representations

Recall that $X \in \Pi_{\text{Lang}}^{\text{cus}}(G^\omega(k))$ is supercuspidal if and only if $\text{supp}(X) = [(c, \sigma)]$ where $c$ is a minimal face of $B(G^\omega, k)$. Since every association class of faces of $B(G^\omega, k)$ contains a face of $\omega_0$ we may assume that $c$ is of the form $c^\omega(J)$ for some $J \in \mathbf{P}^\omega(\Delta)$. Moreover $c^\omega(J)$ is a minimal face if and only if $J$ is maximal in $\mathbf{P}^\omega(\Delta)$. In this section we list the set of possible pairs $(J, \sigma)$ (up to $\sim$) along with $\Omega^\vee(c^\omega(J), \sigma)$, where $J \in \mathbf{P}^\omega(\Delta)$ is maximal and $\sigma$ is a unipotent cuspidal representation of $L_{c^\omega(J)}(F_q)$, for $G$ split, simple and adjoint, and $\omega \in \Omega$. We use the conventions of Lus 93, Section 6.10 to specify the set $J \subseteq \Delta$. Note when $G$ is of classical type, the group $L_{c^\omega(J)}(F_q)$ is also of classical type and so if it admits a unipotent cuspidal representation, then it has exactly one unipotent cuspidal representation. Thus for the classical types we will only record the $J$ and $\Omega^\vee(c^\omega(J), \sigma)$. The explicit parameters can be found in Lus 95, Ree 00, FO20, §4.7, and FOS19.

5.1 Classical groups

5.1.1 $\text{PGL}(n)$ If $G = \text{PGL}(n)$, then $G^\prime = \text{SL}(n, \mathbb{C})$ and $Z(G^\prime) = \mathbb{Z}/n\mathbb{Z}$. Hence $\Omega = \text{Irr}(Z(G^\prime))$ can be identified with $C_n$. For $\omega \in \Omega$, the inner form $G^\omega$ admits unipotent supercuspidal representations if and only if $\omega$ has order $n$ and $J = \emptyset$. In this case $\Omega^\vee(c^\omega(J), \sigma)$ is the principal nilpotent orbit.

5.1.2 $\text{SO}(2n+1)$ If $G = \text{SO}(2n+1)$, $G^\prime = \text{Sp}(2n, \mathbb{C})$ and $Z(G^\prime) = \mathbb{Z}/2\mathbb{Z}$. The inner forms are parametrized by $\hat{Z}(G^\prime) \cong C_2 = \{1, -1\}$.

1. If $\omega = 1$, then $J$ is of the form $D_{\ell} \times B_1$, where $\ell + t = n$, $\ell = a^2$, $t = b(b + 1)$, $a, b$ nonnegative integers, $a$ even. Let

$$\delta = \begin{cases} 
    b - a & \text{if } b \geq a \\
    a - b - 1 & \text{if } a > b,
\end{cases}$$

(5.1.1)

and $\Sigma = a + b$. The nilpotent orbit $\Omega^\vee(c^\omega(J), \sigma)$ is parameterized by the partition

$$\lambda = (2, 4, \ldots, 2\delta) \cup (2, 4, \ldots, 2\Sigma).$$

(5.1.2)

2. If $\omega = -1$, then $J$ is of the form $D_{\ell} \times B_1$, where $\ell + t = n$, $\ell = a^2$, $t = b(b + 1)$, $a, b$ nonnegative integers, where $a$ is now odd. The nilpotent orbit $\Omega^\vee(c^\omega(J), \sigma)$ is defined analogously to the $\omega = 1$ case.

5.1.3 $\text{PSp}(2n)$ If $G = \text{PSp}(2n)$, then $G^\prime = \text{Spin}(2n + 1, \mathbb{C})$, and $Z(G^\prime) = \mathbb{Z}/2\mathbb{Z}$. The inner forms are parametrized by $\hat{Z}(G^\prime) \cong C_2 = \{1, -1\}$.
1. If $\omega = 1$, then $J$ is of the form $C_\ell \times C_t$, where $\ell + t = n$, $\ell = a(a + 1)$, $t = b(b + 1)$, $a, b$ nonnegative integers and $a \geq b$. Let $\delta = a - b$ and $\Sigma = a + b$. The nilpotent orbit $O^\vee(\sigma^\omega(J), \sigma)$ is parameterized by the partition
\[ \lambda = (1, 3, \ldots, 2\delta - 1) \cup (1, 3, \ldots, 2\Sigma + 1) \] (5.1.3)
where $\cup$ means union of partitions.

2. If $\omega = -1$, then $J$ is of the form $J = C_\ell \times C_t$, where $2\ell + t = n - 1$ and $t = \frac{a(a + 1)}{2} - 1$, $\ell = b(b + 1)$, $a, b$ are nonnegative integers. If $a = 0, 1$, we interpret $J$ as being $J = C_\ell \times C_t$. Let $a'$ be such that $a = 2a'$ if $a$ is even and $a = 2a' + 1$ if $a$ is odd. Let $\Sigma = b + a'$ and
\[ \delta = \begin{cases} b - a' & \text{if } 2b \geq a \\ a' - b & \text{if } 2b < a. \end{cases} \]
The nilpotent orbit $O^\vee(\sigma^\omega(J), \sigma)$ is parameterized by the partition
\[ \lambda = \begin{cases} (1, 5, \ldots, 4\delta + 1) \cup (3, 7, \ldots, 4\delta - 1) & \text{if } a \text{ is even and } 2b \geq a \\ (1, 5, \ldots, 4\delta + 1) \cup (1, 5, \ldots, 4\delta - 3) & \text{if } a \text{ is even and } 2b < a \\ (3, 7, \ldots, 4\delta + 3) \cup (1, 5, \ldots, 4\delta - 3) & \text{if } a \text{ is odd and } 2b \geq a \\ (3, 7, \ldots, 4\delta + 3) \cup (3, 7, \ldots, 4\delta - 1) & \text{if } a \text{ is odd and } 2b < a. \end{cases} \]

5.1.4 $PSO(2n)$ If $G = PSO(2n)$, then $G^\vee = Spin(2n, \mathbb{C})$, and
\[ Z(G^\vee) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})^2, & \text{if } n \text{ even,} \\ \mathbb{Z}/4\mathbb{Z}, & \text{if } n \text{ odd.} \end{cases} \]

Let $\tau$ be the standard diagram automorphism of type $D_n$. Let $\{1, -1\}$ be the kernel of the isogeny $Spin(2n, \mathbb{C}) \rightarrow SO(2n, \mathbb{C})$. Write the four characters of $Z(G^\vee)$ as $\Omega = \{\eta, \rho, \eta\rho, \rho\}$, where $\tau(\eta) = \eta$ and $\eta(-1) = 1$.

1. If $\omega = 1$, then $J$ is of the form $D_\ell \times D_t$, where $\ell + t = n$, $\ell = a^2$, $t = b^2$, $a, b$ even non-negative integers, $a \geq b$. Let $\delta = a - b, \Sigma = a + b$. The nilpotent orbit $O^\vee(\sigma^\omega(J), \sigma)$ is parameterized by the partition
\[ \lambda = (1, 3, \ldots, 2\delta - 1) \cup (1, 3, \ldots, 2\Sigma - 1). \] (5.1.4)

2. If $\omega = \eta$, then $J$ is of the form $^2D_\ell \times ^2D_t$, where $\ell + t = n$, $\ell = a^2$, $t = b^2$, and $a, b$ are odd positive integers, $a \geq b$. The nilpotent orbit $O^\vee(\sigma^\omega(J), \sigma)$ is defined analogously to the $\omega = 1$ case.

3. If $\omega = \rho, \eta\rho$, then $J$ can take one of the following two forms:
   (i) $J$ is of the form $^2A_t$, where $t = n - 1$ is even, $t = \frac{a(a + 1)}{2} - 1$, $a$ is a non-negative integer. This means that $a \equiv 0 \pmod{4}$, $a \equiv 3 \pmod{4}$. There are four ways to embed $J$ into the affine Dynkin diagram $\hat{D}_n$, two of them are $\rho$-stable, and the other two $\eta\rho$-stable. In all cases the nilpotent orbit $O^\vee(\sigma^\omega(J), \sigma)$ is parameterized by the partition
\[ \lambda = \begin{cases} (3, 3, 7, 7, \ldots, 2a - 1, 2a - 1) & a \equiv 0 \pmod{4}, \\ (1, 1, 5, 5, \ldots, 2a - 1, 2a - 1) & a \equiv 3 \pmod{4}. \end{cases} \] (5.1.5)

   (ii) $J$ is of the form $D_\ell \times ^2D_t$, where $2\ell + t = n - 1$, $t = \frac{a(a + 1)}{2} - 1$ and $\ell = b^2$, $a, b$ are non-negative integers. Let $a'$ be such that $a = 2a'$ if $a$ is even and $a = 2a' + 1$ if $a$ is odd. Let $\Sigma = b + a'$ and
\[ \delta = \begin{cases} b - a' & \text{if } 2b > a \\ a' - b & \text{if } 2b \leq a. \end{cases} \]

The nilpotent orbit $O^\vee(\sigma^\omega(J), \sigma)$ is parameterized by the partition
\[ \lambda = \begin{cases} (3, 3, \ldots, 4\delta - 1) \cup (1, 5, \ldots, 4\delta - 3) & \text{if } a \text{ is even and } 2b > a \\ (3, 3, \ldots, 4\delta - 1) \cup (3, 7, \ldots, 4\delta - 1) & \text{if } a \text{ is even and } 2b \leq a \\ (1, 5, \ldots, 4\delta + 1) \cup (3, 7, \ldots, 4\delta - 5) & \text{if } a \text{ is odd and } 2b > a \\ (1, 5, \ldots, 4\delta + 1) \cup (1, 5, \ldots, 4\delta + 1) & \text{if } a \text{ is odd and } 2b \leq a. \end{cases} \]
5.2 Exceptional groups

5.2.1 $G_2$ If $G = G_2$, then $G^\vee = G_2(\mathbb{C})$, and $Z(G^\vee) = \{1\}$. If $\omega = 1$ then $J$ is of the form $G_2$ and there are 4 choices for $\sigma$ as enumerated in Table II. In all cases

$$\mathcal{O}^\vee(c^\omega(J), \sigma) = G_2(a_1).$$

5.2.2 $F_4$ If $G = F_4$, then $G^\vee = F_4(\mathbb{C})$, and $Z(G^\vee) = \{1\}$. If $\omega = 1$ then $J$ is of the form $F_4$ and there are 7 choices for $\sigma$ as enumerated in Table II. In all cases

$$\mathcal{O}^\vee(c^\omega(J), \sigma) = F_4(a_3).$$

5.2.3 $E_6$ If $G = E_6$, then $G^\vee = E_6(\mathbb{C})$, and $Z(G^\vee) = \{1, \zeta, \zeta^2\}$.

1. If $\omega = 1$ then $J$ is of the form $E_6$ and there are 2 choices for $\sigma$ as enumerated in Table II. In both cases

$$\mathcal{O}^\vee(c^\omega(J), \sigma) = D_4(a_1).$$

2. If $\omega \in \{\zeta, \zeta^2\}$ then $J$ is of the form $^3D_4$ and $\sigma = D_4[1]$ or $D_4[-1]$. In both cases

$$\mathcal{O}^\vee(c^\omega(J), \sigma) = E_6(a_3).$$

5.2.4 $E_7$ If $G = E_7$, then $G^\vee = E_7(\mathbb{C})$, and $Z(G^\vee) = \{1, -1\}$.

1. If $\omega = 1$ then $J$ is of the form $E_7$ and there are 2 choices for $\sigma$ as enumerated in Table II. In both cases

$$\mathcal{O}^\vee(c^\omega(J), \sigma) = A_4 + A_1.$$  

2. If $\omega = -1$, then $J$ is of the form $^2E_6$. There are three cuspidal unipotent representations afforded by $J$: $^2E_6[1]$, $^2E_6[\theta]$, $^2E_6[\theta^2]$. In all cases

$$\mathcal{O}^\vee(c^\omega(J), \sigma) = E_7(a_5).$$

5.2.5 $E_8$ If $G = E_8$, then $G^\vee = E_8(\mathbb{C})$, and $Z(G^\vee) = \{1\}$. If $\omega = 1$ then $J$ is of the form $E_8$ and there are 13 choices for $\sigma$ as enumerated in Table II. In all cases

$$\mathcal{O}^\vee(c^\omega(J), \sigma) = E_8(a_7).$$

6 Proof of Proposition 3.0.1

In this section, we will prove Proposition 3.0.1.

6.1 Classical groups

In each case we show that

$$\mathfrak{l}(J, \text{WF}(\sigma)) = d_A(\lambda, 1)$$

where $\lambda$ is the partition parameterizing $\mathcal{O}^\vee(c^\omega(J), \sigma)$. We will use the machinery of [Ach03, Section 3.4] to prove this equality.

6.1.1 $\text{PGL}(n)$ Let $\omega \in \Omega \simeq C_n$ be of order $n$. Let $J = \varnothing$. Then $\sigma = \text{triv}$, $\text{WF}(\sigma) = \{0\}$, and $\mathcal{O}^\vee(c^\omega(\varnothing), \sigma)$ is the principal orbit $O^\vee_{\text{prin}}$. We need to show that

$$\mathfrak{l}(\varnothing, \{0\}) = d_A(O^\vee_{\text{prin}}, 1).$$

But both sides are equal to $(\{0\}, 1)$ and so we have equality.
6.1.2 \( SO(2n + 1) \) Consider the cases \( \omega = 1, -1 \) simultaneously. Fix integers \( a, b \) as in section 5.1.2 to fix \( J \) and hence \( \sigma \). By section 4

\[
WF(\sigma) = (1, 1, 3, 3, \ldots, 2a - 1, 2a - 1) \times (1, 1, 3, 3, \ldots, 2b - 1, 2b - 1, 2b + 1).
\]

Let \( \delta, \Sigma, \lambda \) be as in section 5.1.2. We have that

\[
\lambda' = (\delta, \delta - 1, \delta - 1, \ldots, 1, 1) \vee (\Sigma, \Sigma, \Sigma - 1, \Sigma - 1, \ldots, 1, 1)
\]

\[
= \begin{cases} 
(2b, 2b - 2, 2b - 2, \ldots, 2a, 2a - 1, 2a - 1, \ldots, 1, 1) & \text{if } b \geq a \\
(2a - 1, 2a - 1, 2a - 3, 2a - 3, \ldots, 2b + 1, 2b + 1, 2b, 2b, \ldots, 1, 1) & \text{if } a > b
\end{cases}
\]

so \( \pi(\lambda) = \emptyset \). We also have

\[
d(\lambda) = (2b + 1, 2b - 1, 2b - 1, \ldots, 1, 1) \cup (2a - 1, 2a - 1, \ldots, 1, 1).
\]

Since \( (1, 1, 3, 3, \ldots, 2a - 1, 2a - 1) \) only has parts with even multiplicity,

\[
\Xi(J, WF(\sigma)) = \langle (1, 1, 3, 3, \ldots, 2a - 1, 2a - 1) \rangle \cdot d(\lambda) = \langle \emptyset \rangle \cdot d(\lambda) = \langle \pi(\lambda) \rangle \cdot d(\lambda) = d_A(\lambda, 1)
\]

where \( \pi(\lambda) \) is the subpartition of \( \lambda' \) defined by Achar in \((\text{Ach03, Equation } 8)\).

6.1.3 \( PSp(2n) \)

1. Let \( \omega = 1 \). Fix integers \( a, b \) as in section 5.1.3 (1) to fix \( J \) and hence \( \sigma \). By section 4

\[
WF(\sigma) = (2, 2, 4, 4, \ldots, 2a, 2a) \times (2, 2, 4, 4, \ldots, 2b, 2b).
\]

Let \( \delta, \Sigma, \lambda \) be as in section 5.1.3 (1). We have that

\[
\lambda' = (\delta, \delta - 1, \delta - 1, \ldots, 1, 1) \vee (\Sigma + 1, \Sigma, \Sigma, \ldots, 1, 1)
\]

\[
= (2a + 1, 2a - 1, 2a - 1, \ldots, 2b + 1, 2b + 1, 2b, 2b, \ldots, 1, 1)
\]

so \( \pi(\lambda) = \emptyset \). We also have

\[
d(\lambda) = (2a, 2a, \ldots, 2b + 2, 2b + 2, 2b, 2b, 2b, \ldots, 2, 2, 2).
\]

Since \( (2, 2, 4, 4, \ldots, 2a, 2a) \) only has parts with even multiplicity,

\[
\Xi(J, WF(\sigma)) = \langle (2, 2, 4, 4, \ldots, 2a, 2a) \rangle \cdot d(\lambda) = \langle \emptyset \rangle \cdot d(\lambda) = \langle \pi(\lambda) \rangle \cdot d(\lambda) = d_A(\lambda, 1).
\]

2. Let \( \omega = -1 \). Fix integers \( a, b \) as in section 5.1.3 (2) to fix \( J \) and hence \( \sigma \). Then

\[
WF(\sigma) = (2, 2, 4, 4, \ldots, 2b, 2b) \times (1, 2, \ldots, a) \times (2, 2, 4, 4, \ldots, 2b, 2b).
\]

Let \( \delta, \Sigma, \lambda \) be as in section 5.1.3 (2). We have that

\[
\lambda' = \begin{cases} 
(\Sigma + 1, \Sigma^4, \ldots, 1^4) \vee (\delta^3, (\delta - 1)^4, \ldots, 1^4) & \text{if } a \text{ is even and } 2b \geq a \\
(\Sigma + 1, \Sigma^4, \ldots, 1^4) \vee (\delta, (\delta - 1)^4, \ldots, 1^4) & \text{if } a \text{ is even and } 2b < a \\
((\Sigma + 1)^3, \Sigma^4, \ldots, 1^4) \vee (\delta, (\delta - 1)^4, \ldots, 1^4) & \text{if } a \text{ is odd and } 2b \geq a \\
((\Sigma + 1)^3, \Sigma^4, \ldots, 1^4) \vee (\delta^3, (\delta - 1)^4, \ldots, 1^4) & \text{if } a \text{ is odd and } 2b < a
\end{cases}
\]

\[
= \begin{cases} 
(2b + 1, (2b)^2, \ldots, (a + 1)^2, a^4, \ldots, 1^4) & \text{if } a \text{ is even and } 2b \geq a \\
(a + 1, (a - 1)^4, \ldots, (2b + 1)^4, (2b)^2, \ldots, 1^4) & \text{if } a \text{ is even and } 2b < a \\
(2b + 1, (2b)^2, \ldots, (a + 1)^2, a^4, \ldots, 1^4) & \text{if } a \text{ is odd and } 2b \geq a \\
(a^3, (a - 2)^4, \ldots, (2b + 1)^4, (2b)^2, \ldots, 1^4) & \text{if } a \text{ is odd and } 2b < a
\end{cases}
\]

Thus \( \pi(\lambda) = \emptyset \) since all even parts of \( \lambda' \) have even multiplicity. Moreover

\[
d(\lambda) = (2, 2, 4, 4, \ldots, 2b, 2b) \cup (1, 1, 2, 2, \ldots, a, a) \cup (2, 2, 4, 4, \ldots, 2b, 2b)
\]

in all cases. Thus

\[
\Xi(J, WF(\sigma)) = \Xi(J, (2, 2, \ldots, 2b, 2b) \times (1, 1, \ldots, a, a) \cup (2, 2, \ldots, 2b, 2b))
\]

\[
= \langle (2, 2, 4, 4, \ldots, 2b, 2b) \rangle \cdot d(\lambda) = \langle \emptyset \rangle \cdot d(\lambda) = \langle \pi(\lambda) \rangle \cdot d(\lambda) = d_A(\lambda, 1)
\]

where \( J = C_1 \times C_{r+1} \).
6.1.4 \(PSO(2n)\)

1. Let \(\omega \in \{1, \eta\}\). Fix integers \(a, b\) as in section 6.1.3 (1), (2) to fix \(J\) and hence \(\sigma\). By section 6.1.3

\[
WF(\sigma) = (1, 1, 3, 3, \ldots, 2a - 1, 2a - 1) \times (1, 1, 3, 3, \ldots, 2b - 1, 2b - 1).
\]

Let \(\delta, \Sigma, \lambda\) be as in section 6.1.3 (1). We have that

\[
\lambda^t = (\delta, \delta - 1, \delta - 1, \ldots, 1, 1) \cup (\Sigma, \Sigma - 1, \Sigma - 1, \ldots, 1, 1)
\]

\[
= (2a, 2a - 2, 2a - 2, \ldots, 2b, 2b, 2b - 1, 2b - 1, 1, 1)
\]

so \(\pi(\lambda) = \emptyset\) since all odd parts have even multiplicity. We also have

\[
d(\lambda) = (2a - 1, 2a - 1, \ldots, 2b + 1, 2b + 1, 2b - 1, 2b - 1, 2b - 1, 1, 1, 1, 1, 1).
\]

Since \((1, 1, 3, 3, \ldots, 2a - 1, 2a - 1)\) only has parts with even multiplicity,

\[
\mathcal{J}(J, WF(\sigma)) = \langle \langle (1, 1, 3, 3, \ldots, 2a - 1, 2a - 1) \rangle d(\lambda) = \pi(\lambda) d(\lambda) = d_A(\lambda, 1).
\]

2. Let \(\omega \in \{\rho, \eta\}\). We will treat the cases (i) and (ii) simultaneously. Fix integers \(a, b\) as in section 6.1.3 (3) (ii) to fix \(J\) and hence \(\sigma\) (we treat (i) as the case with \(b = 0\)). By section 6.1.3

\[
WF(\sigma) = (1, 1, 3, 3, \ldots, 2b - 1, 2b - 1) \times (1, 2, \ldots, a) \times (1, 1, 3, 3, \ldots, 2b - 1, 2b - 1).
\]

Let \(\delta, \Sigma, \lambda\) be as in section 6.1.3 (3) (ii). We have that

\[
\lambda^t = \begin{cases} 
(\Sigma^3, (\Sigma - 1)^4, \ldots, \delta^3, \delta^4, \ldots, (\delta - 1)^4) & \text{if } a \text{ is even and } 2b \leq a \\
(\Sigma^3, (\Sigma - 1)^4, \ldots, (\delta - 1)^4, \delta^4) & \text{if } a \text{ is even and } 2b < a \\
(\Sigma + 1)^3, (\Sigma^4, \ldots, (\delta - 1)^4, \delta^4) & \text{if } a \text{ is odd and } 2b < a \\
(2b + 1, (2b)^2, a^2, a^4) & \text{if } a \text{ is even and } 2b \geq a \\
(a + 1, (a - 1)^2, (2b + 1)^2, (2b)^4) & \text{if } a \text{ is even and } 2b < a \\
(2b + 1, (2b)^2, a^2, a^4, 1^4) & \text{if } a \text{ is odd and } 2b \geq a \\
(a^3, (a - 2)^2, (2b + 1)^4, (2b)^4, 1^4) & \text{if } a \text{ is odd and } 2b < a.
\end{cases}
\]

(6.1.3)

(6.1.4)

Thus \(\pi(\lambda) = \emptyset\) since all even parts of \(\lambda^t\) have even multiplicity. Moreover

\[
d(\lambda) = (2, 2, 4, 4, \ldots, 2b, 2b) \cup (1, 1, 2, 2, \ldots, a, a) \cup (2, 2, 4, 4, \ldots, 2b, 2b)
\]

in all cases. Thus

\[
\mathcal{J}(J, WF(\sigma)) = \mathcal{J}(J, (1, 1, \ldots, 2b - 1, 2b - 1)) \times (1, 1, \ldots, a) \cup (1, 1, 3, 3, \ldots, 2b - 1, 2b - 1))
\]

\[
= \langle \langle (2, 2, 4, 4, \ldots, 2b, 2b) \rangle d(\lambda) = \langle \langle \emptyset \rangle d(\lambda) = \langle \pi(\lambda) \rangle d(\lambda) = d_A(\lambda, 1)
\]

where \(J = D_t \times D_{t+1+i}\).

6.2 Exceptional groups

6.2.1 Split forms Suppose that \(G\) is split, of exceptional type, and that \(\omega = 1\). As can be seen in Section 6.2.2 \(J\) is always equal to \(\Delta\). Thus,

\[
\mathcal{J}(J, WF(\sigma)) = (WF(\sigma), 1).
\]

On the other hand, the nilpotent \(\mathcal{O}^\vee := \mathcal{O}^\vee(c^\omega(J), \sigma)\) is always special. Thus,

\[
d_A(\mathcal{O}^\vee, 1) = (d(\mathcal{O}^\vee), 1)
\]

by the general properties of \(d_A\), see [Ach03, Section 3]. So for Proposition 6.2.1 it suffices to show that

\[
WF(\sigma) = d(\mathcal{O}^\vee)
\]

for all \(\sigma\). This follows by comparing the orbits in Table 4.4 and in section 5.2.
6.2.2 Non-split forms of $E_6$ Suppose $G$ is of type $E_6$ and $\omega \in \{\zeta, \zeta^2\}$. Then $J$ is of the form $^3D_4$, and $WF(\sigma) = (1,1,3,3)$ for both $\sigma = D_4[1]$ and $\sigma = D_4[-1]$. The orbit $(1,1,3,3)$ is the orbit $A_2$ in Bala-Carter notation. Thus we need to show that

$$\mathfrak{E}(J,(1,1,3,3)) = d_A(E_6(a_3),1).$$

We note that $E_6(a_3)$ is special and $d(E_6(a_3)) = A_2$ so we must show that

$$\mathfrak{E}(J,A_2) = (A_2,1).$$

Since $J \subseteq \Delta$ this follows from [Oka21, Proposition 2.30].

6.2.3 Non-split forms of $E_7$ Suppose $G$ is of type $E_7$ and $\omega = -1$. Then $J$ is of the form $^2E_6$, and $WF(\sigma) = D_4(a_1)$ for all possible $\sigma$. Thus we need to show that

$$\mathfrak{E}(J,D_4(a_1)) = A_2(E_7(a_5),1).$$

We note that $E_7(a_5)$ is special and $d(E_7(a_5)) = D_4(a_1)$ so we must show that

$$\mathfrak{E}(J,D_4(a_1)) = (D_4(a_1),1).$$

Since $J \subseteq \Delta$ this follows from [Oka21, Proposition 2.30].

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