LOCAL BOUNDARY CONDITIONS FOR THE DIRAC OPERATOR
AND ONE-LOOP QUANTUM COSMOLOGY

Peter D. D’Eath\textsuperscript{(a)} and Giampiero Esposito\textsuperscript{(a,b)}

\textsuperscript{(a)} Department of Applied Mathematics and Theoretical Physics
Silver Street, Cambridge CB3 9EW, U. K.

\textsuperscript{(b)} St. John’s College, Cambridge CB2 1TP, U. K.

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Abstract. This paper studies local boundary conditions for fermionic fields in quantum

cosmology, originally introduced by Breitenlohner, Freedman and Hawking for gauged su-

pergravity theories in anti-de Sitter space. For a spin-$\frac{1}{2}$ field the conditions involve the

normal to the boundary and the undifferentiated field. A first-order differential operator for

this Euclidean boundary-value problem exists which is symmetric and has self-adjoint ex-

tensions. The resulting eigenvalue equation in the case of a flat Euclidean background with

a three-sphere boundary of radius $a$ is found to be: $F(E) = |J_{n+1}(Ea)|^2 - |J_{n+2}(Ea)|^2 = 0, \forall n \geq 0$. Using the theory of canonical products, this function $F$ may be expanded
in terms of squared eigenvalues, in a way which has been used in other recent one-loop calculations involving eigenvalues of second-order operators. One can then study the generalized Riemann \( \zeta \)-function formed from these squared eigenvalues. The value of \( \zeta(0) \) determines the scaling of the one-loop prefactor in the Hartle-Hawking amplitude in quantum cosmology. Suitable contour formulae, and the uniform asymptotic expansions of the Bessel functions \( J_m \) and their first derivatives \( J'_m \), yield for a massless Majorana field:

\[
\zeta(0) = \frac{11}{360}.
\]

Combining this with \( \zeta(0) \) values for other spins, one can then check whether the one-loop divergences in quantum cosmology cancel in a supersymmetric theory.

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I. INTRODUCTION

In the last few years, a number of authors have investigated the one-loop approximation in quantum cosmology and the boundary terms in the asymptotic expansion of the heat kernel for fields of various spins.\textsuperscript{1–11} For bosonic fields the most natural boundary conditions are local: either Dirichlet or Neumann, or perhaps a combination of the two.\textsuperscript{11,12} In the case of fermionic fields one has a choice of local or non-local boundary conditions.\textsuperscript{8}

The possibility of non-local boundary conditions arises because of the first-order nature of the fermionic operators (a precise mathematical treatment of the Dirac operator can be found in Ref. 13). For example, take a quantum cosmological model in which the Dirac field is regarded as a perturbation around a Friedmann background gravitational model containing a family of three-spheres of radius $a(t)$.\textsuperscript{14} Using two-component spinor notation, the unprimed spin-$\frac{1}{2}$ field $\psi_A$ on a given three-sphere may be split into a sum $\psi_A^{(+)} + \psi_A^{(-)}$, where $\psi_A^{(+)}$ is a sum of harmonics having positive eigenvalues for the three-dimensional Dirac operator $\epsilon n_{AA'}e^{BA'j(3)}D_j$ on the $S^3$, and $\psi_A^{(-)}$ is a sum of harmonics having negative eigenvalues. Here $\epsilon n_{AA'}$ is the spinor version of the unit Euclidean normal\textsuperscript{14} to the three-sphere, $e^{BA'j}$ is the spinor version of the orthonormal spatial triad on the three-sphere, and $(3)D_j$ is the three-dimensional covariant derivative ($j = 1, 2, 3$).\textsuperscript{14} A similar decomposition may be applied to the primed field $\tilde{\psi}_{A'}$, which is taken to be independent of $\psi_A$, not related by any conjugation operation. Boundary conditions suitable for investigating the Hartle-Hawking quantum state\textsuperscript{15,16} may be found by studying the classical version of the
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Hartle-Hawking path integral, i.e. by asking for data on a three-sphere bounding a compact region with a Riemannian metric, such that the classical Dirac equation is well-posed.

For a massless field, if one uses spectral boundary conditions, one is forced to specify $\psi_A^{(+)i}$ and $\tilde{\psi}_{A'}^{(+)}$ on the boundary, and not $\psi_A^{(-)}$ and $\tilde{\psi}_{A'}^{(-)}$. Heat-kernel calculations relevant to the one-loop quantum amplitude with such non-local boundary conditions are described in a subsequent paper.\(^{17}\)

Alternatively, one can examine possible local boundary conditions for fermionic fields (particularly for spin $\frac{1}{2}$). For a spin-$\frac{1}{2}$ field $\left(\psi_A, \tilde{\psi}_{A'}\right)$ in Riemannian space, referred to here as a Majorana spin-$\frac{1}{2}$ field,\(^{18}\) these conditions are

$$\sqrt{2} e_{lA}^{A'} \psi^A = \epsilon \tilde{\psi}^{A'},$$  \hspace{1cm} (1.1)

on the bounding surface. We use the word Majorana loosely: quantum amplitudes computed formally in Lorentzian geometries, using Lorentzian Majorana spinors $\left(\psi_A, \tilde{\psi}_{A'}\right)$, where a bar denotes complex conjugation, can be continued analytically to the Euclidean regime by replacing $\tilde{\psi}_{A'}$ by $\tilde{\psi}_{A'}$, which is freed from being the conjugate of $\psi_A$. Recently, a different definition of Euclidean Majorana spinors has been discussed in Ref. 19; our use of the word Majorana is different from that in Ref. 19.

In the boundary conditions (1.1) $e_{lA}^{A'}$ is again the Euclidean normal, and $\epsilon$ will be taken to be either $+1$ or $-1$. Boundary conditions of the kind (1.1) have also been studied in the present context in Ref. 12, where (though using a different formalism) the more general possibility $\epsilon = e^{i\theta}$ has been considered, with $\theta$ taken to be a real function of position on the boundary. However, the results of Sec. II on self-adjointness for the Dirac problem
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only hold in the case of real $\epsilon$, and attention will be restricted to this case. The special case $\epsilon = \pm 1$ is part of a set of boundary conditions introduced by Breitenlohner, Freedman and Hawking$^{20,21}$ for gauged supergravity theories in anti-de Sitter (hereafter referred to as ADS) space. The conditions (1.1) are generalized to higher spins in an obvious way by requiring for spin 1 that

$$2 \epsilon n_A^{A'} \epsilon n_B^{B'} \psi^{AB} = \epsilon \tilde{\psi}^{A'B'} , \quad (1.2)$$

where $\left( \psi_{AB}, \tilde{\psi}_{A'B'} \right)$ is the Maxwell field strength. Similar boundary conditions may be written down for the spin-$\frac{3}{2}$ symmetric field strength $\left( \psi_{ABC}, \tilde{\psi}_{A'B'C'} \right)$ and for the Weyl spinor $\left( \psi_{ABCD}, \tilde{\psi}_{A'B'C'D'} \right)$ in the case of spin 2. For a complex scalar field $\phi$, the conditions require the vanishing of $Re \phi$ and $\frac{\partial}{\partial n} Im \phi$, or of $Im \phi$ and $\frac{\partial}{\partial n} Re \phi$. As with the spectral boundary conditions above, the classical massless Dirac equation for $\left( \psi_A, \tilde{\psi}_{A'} \right)$ is well-posed in a compact Riemannian region bounded by a surface, on which the quantity $\left( \sqrt{2} \epsilon n_A^{A'} \psi^{A'} - \epsilon \tilde{\psi}^{A'} \right)$ appearing in Eq. (1.1) is specified (at least in the case of a spherically symmetric geometry). Similar results hold for other spins.

The relevance to gauged supergravity of the set of local boundary conditions including (1.1,2) is as follows. ADS can be seen as the maximally supersymmetric solution of the $O(N)$ gauged supergravity theories. It has topology $S^1 \times R^3$, and its closed timelike curves can be removed by considering its covering space CADS. CADS is conformally flat, conformally embedded in the Einstein static universe, and its boundary is the product of the time axis of the Einstein universe and the two-sphere. The solutions of the twistor equation,$^{22}$
subject to a suitable boundary condition,\textsuperscript{21} generate the rigid supersymmetry transformations between massless linearized fields of different spins on an ADS background. In Ref. 20 the rigid supersymmetry transformations map classical solutions of the linearized field equations, subject to boundary conditions of the type (1.1,2) at infinity, to classical solutions for an adjacent spin, again obeying the boundary conditions at infinity. The set of boundary conditions including (1.1,2) is thus in a certain sense supersymmetric.

Motivated by quantum cosmology, we examine fields on a flat Euclidean background bounded by a three-sphere of radius $a$. There are again\textsuperscript{23} solutions of the twistor equation, subject to a certain boundary condition, which generate rigid supersymmetry transformations among classical solutions obeying boundary conditions of the type (1.1,2) on the bounding $S^3$. However, these rigid transformations do not map eigenfunctions of the spin-$s$ wave operators to eigenfunctions for adjacent spin $s \pm \frac{1}{2}$ with the same eigenvalues. Hence no cancellation can be expected \textit{a priori} between adjacent spins in a one-loop calculation of the functional determinant about flat space (with $S^3$ boundary) for a supersymmetric theory.

A different type of supersymmetric boundary condition is suggested by the work of Ref. 24. In simple supergravity the spatial tetrad $e^{AA'}_i$ (from which the intrinsic spatial metric $h_{ij}$ is constructed) and the projection $\left(\epsilon \bar{\psi}_i^{A'} - \sqrt{2} \, e_{A'} \, \psi_i^{A} \right)$ formed from the spatial components $\left(\psi_i^{A}, \bar{\psi}_i^{A'}\right)$ of the spin-$\frac{3}{2}$ potential, transform into each other under half of the local supersymmetry transformations at the boundary. Further, the supergravity action, suitably modified by a boundary term, is invariant under this class of local supersymmetry transformations. One is thus led to specify $e^{AA'}_i$ and $\left(\epsilon \bar{\psi}_i^{A'} - \sqrt{2} \, e_{A'} \, \psi_i^{A}\right)$ on the boundary.
in computing the quantum amplitude. One could further check, along the lines of Ref. 25, whether any local one-loop surface counterterms are permitted by this remaining local supersymmetry, i.e. whether there is any cancellation between the one-loop determinants for spin 2 and spin $\frac{3}{2}$. Correspondingly, in studying the path integral, one arrives at the consideration of these local boundary conditions$^{11}$ by requiring that transition amplitudes are invariant under BRST transformations (so that results do not depend on the gauge-fixing term) and that supersymmetry is respected. Extending this to $O(N)$ supergravity, one finds, following the usual supersymmetry transformation laws, that for spin-$\frac{1}{2}$ fields
\[
(\bar{\psi}^A - \sqrt{2} \epsilon n^A \psi^A) \quad \text{should again be specified on the boundary, as in Eq. (1.1). However, for spins higher than $\frac{1}{2}$ these boundary conditions are typically different from those of Breitenlohner, Freedman and Hawking, involving projections of potentials rather than field strengths.}
\]

Sec. II demonstrates self-adjointness of the spin-$\frac{1}{2}$ local boundary-value problem. Sec. III examines the boundary conditions (1.1) for a massless Majorana spin-$\frac{1}{2}$ field on a flat Euclidean background, bounded by a three-sphere of radius $a$. The eigenvalue equation arising in the evaluation of the one-loop functional determinant is derived. We then turn to the generalized Riemann $\zeta$-function formed from the squared eigenvalues. The value of $\zeta(0)$ measures the one-loop divergence of the quantum amplitude prescribed by the given boundary conditions. It also determines the scaling or $a$-dependence of the one-loop amplitude, which is proportional to $a^{\zeta(0)}$ for a bosonic field and $a^{-\zeta(0)}$ for a fermionic field (in the case of a scale-independent measure). In Sec. IV the general structure of the $\zeta(0)$ calculation for spin $\frac{1}{2}$ is described. This involves generalizing previous work for bosonic...
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fields, and the calculation depends on the theory of canonical products. The value of \( \zeta(0) \) is then computed in terms of its various contributions in Secs. V-VIII. The result \( \zeta(0) = \frac{11}{360} \) for a massless Majorana spin-\( \frac{1}{2} \) field disagrees with the result \( \frac{17}{180} \) of a recent calculation, which uses a quite different approach. A discussion of this problem is given in Sec. IX, together with other concluding remarks.

II. SELF-ADJOINTNESS OF THE BOUNDARY-VALUE PROBLEM

We study the Hartle-Hawking path integral

\[
K_{HH} = \int e^{-I_E} D\psi^A D\tilde{\psi}^{A'} ,
\]

taken over the class of spin-\( \frac{1}{2} \) fields \((\psi^A, \tilde{\psi}^{A'})\) which obey Eq. (1.1) on the bounding \( S^3 \). Here, with our conventions (see below),

\[
I_E = \frac{i}{2} \int d^4x \sqrt{g} \left[ \tilde{\psi}^{A'} (\nabla_{AA'} \psi^A) - (\nabla_{AA'} \tilde{\psi}^{A'}) \psi^A \right] ,
\]

is the Euclidean action for a massless spin-\( \frac{1}{2} \) field \((\psi^A, \tilde{\psi}^{A'})\). The fields are defined on the ball in Euclidean four-space bounded by a three-sphere of radius \( a \), subject to the local boundary conditions (1.1). The unprimed and primed spinors are taken to transform under independent groups \( SU(2) \) and \( \widetilde{SU}(2) \), appropriate to Euclidean space. The fermionic fields are taken to be anti-commuting, and Berezin integration is being used.
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As with a bosonic one-loop path integral, one would like to be able to express the path integral (2.1) in terms of a suitable product of eigenvalues. The eigenvalue equations naturally arising from variation of the action (2.2) are

\[
\nabla A A' \psi^A_n = \lambda_n \tilde{\psi}^A_{nA'}, \quad (2.3)
\]

\[
\nabla A A' \tilde{\psi}^{A'}_n = \lambda_n \psi_n A, \quad (2.4)
\]

subject again to the boundary conditions (1.1). One would expect to expand a typical pair \((\psi^A, \tilde{\psi}^{A'})\) in a complete set of eigenfunctions (assuming provisionally that such a complete set exists); one also needs the cross-terms in \(I_E\) to vanish. The gaussian fermionic path integral (2.1) is then formally proportional to the product of the eigenvalues \(\prod_n \left( \frac{\lambda_n}{\tilde{\mu}} \right)\), where the constant \(\tilde{\mu}\) with dimensions of mass has been introduced in order to make the product dimensionless.\textsuperscript{26,27} In fact the eigenvalues \(\lambda_n\) for this problem are purely imaginary in the case of a general Riemannian four-manifold with boundary. Further, in the particular example of Euclidean four-space bounded by a three-sphere, the eigenvalues occur in equal and opposite pairs \(\pm \lambda_n\), so that the formal product \(\prod_n \left( \frac{\lambda_n}{\tilde{\mu}} \right)\) can instead be written as \(\prod_n \left( \frac{|\lambda_n|}{\mu} \right)\), a product of positive real numbers. This formal expression for the path integral (2.1) must then be regularized using zeta-function methods.

The typical cross-term \(\Sigma\) appearing in \(I_E\) is

\[
\Sigma = \frac{i}{2} \int d^4 x \sqrt{g} \left[ (\lambda_n + \lambda_m) \tilde{\psi}^A_m \tilde{\psi}^{A'}_{mA'} + (\lambda_n + \lambda_m) \psi^A_n \psi_{mA} \right]. \quad (2.5)
\]
Using the eigenvalue equations (2.3,4), the square bracket in Eq. (2.5) may be rewritten as

\[ a \nabla A A' \left( \psi_m \overline{\psi}_m \right) + b \nabla A A' \left( \psi_n \overline{\psi}_n \right), \]

with \( a = -b = \frac{\lambda_n + \lambda_m}{\lambda_n - \lambda_m} \), provided \( \lambda_n \neq \lambda_m \). Thus \( \Sigma \) becomes

\[
\Sigma = \frac{i}{2} \frac{\lambda_n + \lambda_m}{\lambda_n - \lambda_m} \left[ \int_{\partial M} d^3 x \sqrt{h} e n_{AA'} \psi_m \overline{\psi}_n - \int_{\partial M} d^3 x \sqrt{h} e n_{AA'} \psi_n \overline{\psi}_m \right] = 0 ,
\]

by virtue of the local boundary conditions (1.1), provided \( \lambda_m \neq \lambda_n \). In the degenerate case where the eigenvalues are equal, a linear transformation within the degenerate eigenspace can be found such that the cross-terms again vanish.

The property that \( I_E \) can be written as a diagonal expression in terms of a sum over eigenfunctions suggests that the Dirac action used here, subject to local boundary conditions, can be expressed in terms of a self-adjoint differential operator acting on fields \( \left( \psi^A, \overline{\psi}^{A'} \right) \). We shall see that this is indeed the case, and that the eigenvalues \( \lambda_n \) are all purely imaginary. The proof will be described in the case of a flat background geometry, but can readily be generalized to the case of curved space.

Consider the space of spinor fields such as

\[ w \equiv \left( \psi^A, \overline{\psi}^{A'} \right) , \quad z \equiv \left( \phi^A, \overline{\phi}^{A'} \right) , \]

defined on the ball of radius \( a \) in Euclidean four-space, subject to the boundary conditions (1.1) (and to suitable differentiability conditions, to be specified later). Considering the action (2.2) and eigenvalue equations (2.3,4), one is led to study the map

\[
C : \left( \psi^A, \overline{\psi}^{A'} \right) \rightarrow \left( \nabla^A B, \overline{\psi}^{B'}, \nabla^A B, \psi^B \right) .
\]
Since the notion of self-adjointness involves the idea of reality, we also need to introduce a conjugation operation on Euclidean spinors, the dagger operation
\[ (\psi^A)^+ \equiv \epsilon^{AB} \delta_{BA'} \overline{\psi}^{A'} , \quad (\overline{\psi}^{A'})^+ \equiv \epsilon^{A'B'} \delta_{B'A} \psi^A . \] (2.9)

Here \( \delta_{BA'} \) is an identity matrix preserved by \( SU(2) \) transformations, and the alternating spinor \( \epsilon^{AB} \) realizes the isomorphism between spin space and its dual, raising and lowering indices according to the rules: \( \lambda^A = \epsilon^{AB} \lambda_B, \ \lambda_A = \lambda^B \epsilon_{BA} \) (and similarly for \( \epsilon^{A'B'} \)).

Moreover, the bar symbol \( \overline{\psi}^A = \overline{\psi}^{A'} \) denotes the usual complex conjugation of \( SL(2, C) \) spinors. Note that the dagger operation has the property
\[ \left( (\psi^A)^+ \right)^+ = \epsilon^{AC} \delta_{CB'} (\overline{\psi}^{B'} + ) = \epsilon^{AC} \delta_{CB'} \epsilon^{B'D'} \delta_{D'F} \psi^F = -\psi^A , \] (2.10)
and hence is anti-involutory.

From now on in this Section, despite the requirement that spinors in the path integral be anti-commuting Grassmann quantities, we study commuting spinors, for simplicity of exposition of the self-adjointness. It may easily be checked that the dagger operation has the following properties:

\[ (\psi_A + \lambda \phi_A)^+ = \psi_A^+ + \lambda^* \phi_A^+ , \] (2.11)

\[ \epsilon_{AB}^+ = \epsilon_{AB} , \quad (\psi_A \phi_B)^+ = \psi_A^+ \phi_B^+ , \] (2.12)

\[ (\psi_A)^+ \psi^A > 0 , \ \forall \psi_A \neq 0 \] , (2.13)
where the symbol \( \ast \) denotes complex conjugation of scalars. We can now define the scalar product

\[
(w, z) \equiv \int_M \left[ \psi_A^+ \phi^A + \bar{\psi}_{\bar{A}}^+ \bar{\phi}^{A'} \right] \sqrt{g} d^4 x .
\]  

(2.14)

This is indeed a scalar product, because it satisfies the following properties, for all vectors \( u, v, w \) and \( \forall \lambda \in C \),

\[
(u, u) > 0 \quad \forall u \neq 0 ,
\]

(2.15)

\[
(u, v + w) = (u, v) + (u, w) ,
\]

(2.16)

\[
(u, \lambda v) = \lambda (u, v) , \quad (\lambda u, v) = \lambda^* (u, v) ,
\]

(2.17)

\[
(v, u) = (u, v)^* .
\]

(2.18)

It will turn out that the operator \( iC \) is symmetric using this scalar product, i.e. that 

\[
(iCz, w) = (z, iCw) \quad \forall z, w.
\]

This result will be used in the course of proving further that the operator \( iC \) has self-adjoint extensions.

Let us now compute \( (Cz, w) \) and \( (z, Cw) \) for typical vectors \( w \) and \( z \) given by Eq. (2.7). From the definitions,

\[
(Cz, w) = \int_M (\nabla_{AB'} \phi^A) + \bar{\psi}_{B'}^B \sqrt{g} d^4 x + \int_M \left( \nabla_{BA'} \bar{\phi}^{A'} \right)^+ \psi_B^B \sqrt{g} d^4 x .
\]

(2.19)

Similarly, but integrating by parts, one finds

\[
(z, Cw) = \int_M (\nabla_{AB'} \phi^A) \bar{\psi}_{B'}^B \sqrt{\bar{g}} d^4 x + \int_M \left( \nabla_{BA'} \left( \bar{\phi}^{A'} \right)^+ \right) \psi_B^B \sqrt{g} d^4 x
\]

\[
- \int_{\partial M} (e^{n_{AB'}} \phi^A) \bar{\psi}_{B'}^B \sqrt{h} d^3 x - \int_{\partial M} (e^{n_{BA'}} \left( \bar{\phi}^{A'} \right)^+ \psi_B^B \sqrt{h} d^3 x .
\]

(2.20)
This may now be simplified using Eq. (2.9), the identity
\[
\left( e^{n}A^A' \phi_A \right)^+ = \epsilon^{A'B'} \delta_{B'C} e^{n}D_{D'} \phi_D = -\epsilon^{A'B'} \delta_{B'C} \left( e^{n}D' \right) \phi_{D'} ,
\]
and the boundary conditions on $S^3 : \sqrt{2} e^{n}C' \psi_C = \tilde{\psi}^{B'} , \sqrt{2} e^{n}A^A' \phi_A = \tilde{\phi}^{A'}$. The sum of the boundary terms in Eq. (2.20) vanishes. Therefore, in verifying that $(iCz, w) = (z, iCw)$, it is sufficient to check that the corresponding volume integrands are equal. This involves detailed use of the Infeld-van der Waerden connection symbols $\sigma^{AA'}_{AA'}$ and $\sigma^{AA'}_a$, where in particular we take $\sigma_0 = -\frac{i}{\sqrt{2}} I , \sigma_i = \frac{\Sigma_i}{\sqrt{2}} (i = 1, 2, 3)$, where $\Sigma_i$ are the Pauli matrices. In simplifying Eqs. (2.19,20) one uses, for example, the flat-space relations
\[
\left( \nabla_{BA'} \tilde{\phi}^{A'} \right)^+ = \delta_{BF'} \tilde{\phi}^{F'}_C \partial_a \left( e^{C} \phi \right) \quad \text{and} \quad \left( \nabla_{BA'} \left( \tilde{\phi}^{A'} \right)^+ \right) = -\delta_{CF'} \tilde{\phi}^{F'}_B \partial_a \left( e^{C} \phi \right).
\]
A straightforward but tedious calculation leads to the desired relation $(iCz, w) = (z, iCw)$, $\forall w, z$.

Every symmetric operator has a closure, and the operator and its closure have the same closed extensions. Moreover, a closed symmetric operator on a Hilbert space is self-adjoint if and only if its spectrum is a subset of the real axis (see Ref. 29, p 136). To prove self-adjointness for our boundary-value problem, we recall a result due to Von Neumann (see Ref. 29, p 143) : given a symmetric operator $A$ with domain $D(A)$, and a map $F : D(A) \to D(A)$ such that
\[
F(\alpha w + \beta z) = \alpha^* F(w) + \beta^* F(z) , \quad (2.22)
\]
\[
(w, w) = (Fw, Fw) , \quad (2.23)
\]
\[
F^2 = \pm I , \quad (2.24)
\]
13
then $A$ has self-adjoint extensions. In the present case (see Eq. (2.9)) let $D$ denote the operator

$$D : (\psi^A, \overline{\psi}^{A'}) \to \left(\psi^{A+}, \left(\overline{\psi}^{A'}\right)^+\right),$$

(2.26)

and let $F = iD$ and $A = iC$. It can be checked that the operator $F$ maps $D(A)$ to $D(A)$. Denoting by $k$ an integer $\geq 2$ and defining

$$D(A) \equiv \left\{ (\psi^A, \overline{\psi}^{A'}) : \psi^A \text{ and } \overline{\psi}^{A'} \text{ are } C^k, \text{ and } \sqrt{2} \varepsilon n^{AA'} \psi_A = \epsilon \overline{\psi}^{A'} \text{ on } S^3 \right\},$$

(2.27)

one finds that $F$ maps $\left(\psi^A, \overline{\psi}^{A'}\right)$ to $\left(\beta^A, \overline{\beta}^{A'}\right) = \left(i\left(\psi^A\right)^+, i\left(\overline{\psi}^{A'}\right)^+\right)$ such that:

$$\sqrt{2} \varepsilon n^{AA'} \beta_A = \gamma \overline{\beta}^{A'} \text{ on } S^3,$$

(2.28)

where $\gamma = \epsilon^*$. The boundary condition (2.28) is clearly of the type which occurs in Eq. (2.27) provided $\epsilon$ is real, and the differentiability of $\left(\beta^A, \overline{\beta}^{A'}\right)$ is not affected by the action of $F$. In deriving Eq. (2.28), we have used Eq. (2.21). The requirement of self-adjointness enforces the choice of a real function $\epsilon$, which for simplicity we have taken in Sec. I to be a constant. Moreover, in view of Eq. (2.10), Eqs. (2.22,24) hold when $F = iD$, provided we write Eq. (2.24) as $F^2 = -I$. Since, in Ref. 29, condition (2.24) is written as $F^2 = I$, and examples are later given (see p 144 therein) where $F$ is complex conjugation, some explanation is in order. Our problem is in the Euclidean regime, where the only possible conjugation is the dagger operation, which is anti-involutory on spinors with an odd number of indices. Thus we are using a slight generalization of Von Neumann’s theorem. Note
here that if \( F : D(A) \to D(A) \) satisfies Eqs. (2.22)-(2.25), then the same is clearly true of \( \tilde{F} = -iD = -F \). Hence

\[
-F \, D(A) \subseteq D(A) \quad ,
\]

(2.29)

\[
F \, D(A) \subseteq D(A) \quad .
\]

(2.30)

Acting with \( F \) on both sides of Eq. (2.29), we find

\[
D(A) \subseteq F \, D(A) \quad ,
\]

(2.31)

using the property \( F^2 = -I \). Eqs. (2.30,31) imply that \( F \, D(A) = D(A) \), so that \( F \) takes \( D(A) \) onto \( D(A) \) in our case. Comparison with the proof on p 144 of Ref. 29 shows that this is sufficient to generalize Von Neumann’s theorem to the Dirac operator with the boundary conditions (1.1).

It remains to verify conditions (2.23,25). First, note that

\[
(Fw, Fw) = (iDw, iDw)
\]

\[
= \int_M (i\psi_A^+) + i\psi^A+ \sqrt{g} d^4x + \int_M (i\tilde{\psi}_{A'}^+) + i\tilde{\psi}^{A'}\sqrt{g} d^4x
\]

\[
= (w, w) \quad ,
\]

(2.32)

using Eqs. (2.10,14), for commuting spinors. Second,

\[
FAw = (iD) (iC) w = i \left[ i \left( \nabla^A_{B'} \tilde{\psi}^{B'} , \nabla^A_{B'} \psi^{B} \right) \right] + = \left( \nabla^A_{B'} \tilde{\psi}^{B'} , \nabla^A_{B'} \psi^{B} \right) + \quad ,
\]

(2.33)

\[
AFw = (iC) (iD) w = iCi \left( \psi^{A+} , \left( \tilde{\psi}^{A'} \right)^+ \right) = - \left( \nabla^A_{B'} \left( \tilde{\psi}^{B'} \right)^+ , \nabla^A_{B'} \psi^{B+} \right) \quad ,
\]

(2.34)
Local Boundary Conditions for the Dirac Operator ... yielding Eq. (2.25), with the help of Eq. (2.9) and of the fact that the complex conjugate of $\sigma^A_{A'}a^a$ is equal to $\sigma^A_{A'}a^a$, $\forall a = 0, 1, 2, 3$. To sum up, it has been shown that the operator $iC$ arising in this boundary-value problem is symmetric and has self-adjoint extensions. In particular, the eigenvalues $\lambda_n$ of $C$ are purely imaginary. The same result follows for a curved four-dimensional Riemannian space with boundary, by a straightforward generalization.

Throughout this Section, we have considered only the first-order operator $iC :$

$$\left(\psi^A, \bar{\psi}^A\right) \rightarrow \left(i\nabla^A_{B'}\bar{\psi}^{B'}, i\nabla^A_{B'}\psi^B\right)$$

which appears naturally in varying the action (2.2), subject to the local boundary conditions (1.1), and whose eigenvalues $i\lambda_n$ appear in the formal product expression $\prod_n \left(\frac{|\lambda_n|}{\mu}\right)$ for the path integral (2.1). An alternative procedure is, of course, to square up the Dirac operator and study the second-order operator $C^+C$, which in our flat background is just minus the Laplacian acting on spinors. This approach has been taken, for example, in Refs. 9, 11, 12. The path integral (2.1) can instead be evaluated in terms of eigenvalues of $C^+C$, but an additional boundary condition, which can be written in the form

$$\sqrt{2} \epsilon n^A_{A'} \nabla^A_{B'} \bar{\psi}^{B'} = \epsilon \nabla^A_{B'} \psi^B , \quad (2.35)$$

must be imposed together with Eq. (1.1). [In Refs. 9, 11, this condition is written as

$$\left(\epsilon n^{BB'} \nabla_{BB'} + \frac{1}{2} trK\right)\left(\sqrt{2} \epsilon n^A_{A'} \psi^A + \epsilon \bar{\psi}^{A'}\right) = 0 ,$$

where $K_{ij}$ is the second fundamental form of the boundary]. This extra condition is automatically obeyed by the eigenfunctions $\left(\psi^A_n, \bar{\psi}^A_n\right)$ of the first-order problem. The
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eigenfunctions of the first- and second-order problems coincide, and the eigenvalues of the second-order problem are of course $|\lambda_n|^2$. The formal product expressions for the path integral (2.1) agree. The extra condition (2.35) involves not just $\psi^A$ and $\tilde{\psi}^{A'}$ on the bounding surface, but also their normal derivatives. When viewed in terms of the path integral subject to the boundary conditions (1.1), condition (2.35) seems extraneous. We have therefore preferred to use only the boundary conditions (1.1) and to study the associated first-order Dirac operator.

The product is regularized by defining the zeta-function

$$\zeta(s) \equiv \sum_{n,k} d_k(n) |\lambda_{n,k}|^{-2s}.$$  \hfill (2.36)

Here we modify the notation in anticipation of Secs. III, IV: the eigenvalues $\lambda_{n,k}$ in the example studied there are labelled by two integers $n$ and $k$, and the degeneracy $d_k(n)$ depends only on $n$. Because the $|\lambda_{n,k}|^2$ are the eigenvalues for a second-order self-adjoint problem, $\zeta(s)$ converges for $Re(s) > 2$ and can be analytically continued to a meromorphic function regular at the origin, with poles only at $s = \frac{1}{2}, 1, \frac{3}{2}, 2$ (see Eqs. (4.9,11) of Ref. 30). The formal expression $\log \prod_{n,k} \left( \frac{|\lambda_{n,k}|}{\mu} \right)$ is then evaluated, following the standard procedure,\textsuperscript{27,30} as $-\frac{1}{2}\zeta'(0) - \zeta(0) \log \tilde{\mu}$. 

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III. LOCAL BOUNDARY CONDITIONS AND EIGENVALUE EQUATION

In this Section we consider the eigenvalue equation for a massless Majorana field subject to the boundary conditions (1.1) on a three-sphere of radius \( a \), bounding a region of Euclidean four-space centred on the origin. The field \( (\psi^A, \tilde{\psi}^{A'}) \) may be expanded in terms of harmonics on the family of spheres centred on the origin,\(^{14} \) as

\[
\psi^A = \frac{\tau^{-3/2}}{2\pi} \sum_{npq} \alpha_{np}^{pq} \left( m_{np}(\tau) \rho^{nqA} + \tilde{r}_{np}(\tau) \tilde{\sigma}^{nqA} \right), \quad (3.1)
\]

\[
\tilde{\psi}^{A'} = \frac{\tau^{-3/2}}{2\pi} \sum_{npq} \alpha_{np}^{pq} \left( \tilde{m}_{np}(\tau) \tilde{\rho}^{nqA'} + r_{np}(\tau) \sigma^{nqA'} \right). \quad (3.2)
\]

Here \( \tau \) is the radius of a three-sphere. In the summation, \( n \) runs from 0 to \( \infty \), \( p \) and \( q \) from 1 to \((n+1)(n+2)\). The \( \alpha_{np}^{pq} \) are a collection of matrices introduced for convenience, where, for each \( n \), \( \alpha_{np}^{pq} \) is block-diagonal in the indices \( pq \), with blocks \( \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \). The harmonics \( \rho^{nqA} \) and \( \sigma^{nqA'} \) have positive eigenvalues \( \frac{1}{2}(n + \frac{3}{2}) \) of the intrinsic three-dimensional Dirac operator on \( S^3 \), while the harmonics \( \tilde{\sigma}^{nqA} \) and \( \tilde{\rho}^{nqA'} \) have negative eigenvalues \( -\frac{1}{2}(n + \frac{3}{2}) \).

The \textit{tilde} symbol does not denote any conjugation operation: pairs such as \( m_{np}, \tilde{m}_{np} \) are independent functions of \( \tau \). The harmonics \( \tilde{\rho}^{nqA'} \) and \( \tilde{\sigma}^{nqA} \) may be re-expressed in terms of the harmonics \( n_A^{A'} \rho^{npA} \) and \( n_A^{A'} \sigma^{npA'} \), where \( n^{AA'} = i \epsilon^{AA'} \) is the Lorentzian normal,\(^{14} \) using the relations

\[
\tilde{\rho}^{nqA'} = 2n_A^{A'} \sum_d \rho^{ndA}(A_n^{-1} H_n)^{dq}, \quad (3.3)
\]
Local Boundary Conditions for the Dirac Operator ...

\[ \bar{\sigma}^{nqA} = 2n^{-A} \sum_d \sigma^{ndA'} (A^{-1}_n H_n)^{dq} \quad . \] (3.4)

Here the matrix \( A^{-1}_n H_n \) of Ref. 14 is again block-diagonal in the indices \( dq \), for each \( n \), with blocks \( \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \).

For simplicity, consider first that part of the boundary condition (1.1) which involves the \( \rho \) harmonics. The relations (3.1)-(3.3) yield for each \( n \):

\[ -i \sum_{pq} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{pq} m_{np}(a) \rho^{nqA} = \epsilon \sum_{pq} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{pq} \tilde{m}_{np}(a) \sum_d \rho^{ndA} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{dq} . \] (3.5)

In the typical case of the indices \( p, q, d = 1, 2 \), this implies

\[ -im_{n1}(a) (\rho^{n1A} + \rho^{n2A}) - im_{n2}(a) (\rho^{n1A} - \rho^{n2A}) = \epsilon \tilde{m}_{n1}(a) (\rho^{n2A} - \rho^{n1A}) \]
\[ + \epsilon \tilde{m}_{n2}(a) (\rho^{n2A} + \rho^{n1A}) . \] (3.6)

Similar equations hold for adjacent indices \( p, q, d = 2k + 1, 2k + 2 \). Since the harmonics \( \rho^{n1A} \) and \( \rho^{n2A} \) on the bounding three-sphere are linearly independent, we have

\[ -i \left[ m_{n1}(a) + m_{n2}(a) \right] = \epsilon \left[ \tilde{m}_{n2}(a) - \tilde{m}_{n1}(a) \right] , \] (3.7)
\[ -i \left[ m_{n1}(a) - m_{n2}(a) \right] = \epsilon \left[ \tilde{m}_{n2}(a) + \tilde{m}_{n1}(a) \right] , \] (3.8)

whose solution is

\[ -im_{n1}(a) = \epsilon \tilde{m}_{n2}(a) , \] (3.9)
\[ im_{n2}(a) = \epsilon \tilde{m}_{n1}(a) . \] (3.10)
In the same way, the part of Eq. (1.1) involving the $\sigma$ harmonics leads to

$$
\epsilon \sum_{pq} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right)^{pq} r_{np}(a) \sigma^{nqA'} = i \sum_{pq} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right)^{pq} \tilde{r}_{np}(a) \sum_{d} \sigma^{ndA'} \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)^{dq},
$$

(3.11)

which implies, for example

$$
\epsilon \left[ r_{n1}(a) + r_{n2}(a) \right] \sigma^{n1A'} + \epsilon \left[ r_{n1}(a) - r_{n2}(a) \right] \sigma^{n2A'} = -i \left[ \tilde{r}_{n1}(a) - \tilde{r}_{n2}(a) \right] \sigma^{n1A'}
$$

$$
+ i \left[ \tilde{r}_{n1}(a) + \tilde{r}_{n2}(a) \right] \sigma^{n2A'},
$$

(3.12)

leading to

$$
-i \tilde{r}_{n1}(a) = \epsilon r_{n2}(a),
$$

(3.13)

$$
i \tilde{r}_{n2}(a) = \epsilon r_{n1}(a),
$$

(3.14)

and similar equations for other adjacent indices $p, q, d$. Thus, defining

$$
x \equiv m_{n1}, X \equiv m_{n2}, \tilde{x} \equiv \tilde{m}_{n1}, \tilde{X} \equiv \tilde{m}_{n2}, y \equiv r_{n1}, Y \equiv r_{n2}, \tilde{y} \equiv \tilde{r}_{n1}, \tilde{Y} \equiv \tilde{r}_{n2},
$$

(3.15)

we may cast Eqs. (3.9,10,13,14) in the form

$$
-i x(a) = \epsilon \tilde{X}(a), \quad iX(a) = \epsilon \tilde{x}(a),
$$

(3.16)

$$
-i \tilde{y}(a) = \epsilon Y(a), \quad iY(a) = \epsilon y(a).
$$

(3.17)

Again, similar equations hold relating $m_{np}, \tilde{m}_{np}, r_{np}$ and $\tilde{r}_{np}$ at the boundary for adjacent indices $p = 2k + 1, 2k + 2$ ($k = 0, 1, ..., \frac{n}{2}(n + 3)$).

Now we turn to the eigenvalue equations (2.3,4), described with the help of the preceding decomposition. Studying the case of coupled modes with adjacent indices $p, q =$
Local Boundary Conditions for the Dirac Operator ...

2k + 1, 2k + 2 for a given value of n, as above, we write \( l = n + \frac{3}{2} \) and \( E = i\lambda_n = -Im(\lambda_n) \).

Introducing \( \forall n \geq 0 \) the operators

\[
L_n \equiv \frac{d}{d\tau} - \frac{l}{\tau}, \quad M_n \equiv \frac{d}{d\tau} + \frac{l}{\tau},
\]

the eigenvalue equations are found to be

\[
L_n x = E\tilde{x} \quad , \quad M_n \tilde{x} = -Ex \quad ,
\]

\[
L_n y = E\tilde{y} \quad , \quad M_n \tilde{y} = -Ey \quad ,
\]

\[
L_n X = E\tilde{X} \quad , \quad M_n \tilde{X} = -EX \quad ,
\]

\[
L_n Y = E\tilde{Y} \quad , \quad M_n \tilde{Y} = -EY \quad .
\]

We now define \( \forall n \geq 0 \) the differential operators

\[
P_n \equiv \frac{d^2}{d\tau^2} + \left[ E^2 - \frac{((n + 2)^2 - \frac{1}{4})}{\tau^2} \right] , \quad (3.23)
\]

\[
Q_n \equiv \frac{d^2}{d\tau^2} + \left[ E^2 - \frac{((n + 1)^2 - \frac{1}{4})}{\tau^2} \right] . \quad (3.24)
\]

Eqs. (3.19)-(3.22) lead straightforwardly to the following second-order equations:

\[
P_n \tilde{x} = P_n \tilde{X} = P_n \tilde{y} = P_n \tilde{Y} = 0 \quad , \quad (3.25)
\]

\[
Q_n y = Q_n Y = Q_n x = Q_n X = 0 \quad . \quad (3.26)
\]

The solutions of Eqs. (3.25,26) which are regular at the origin are

\[
\tilde{x} = C_1 \sqrt{\tau} J_{n+2}(E\tau) \quad , \quad \tilde{X} = C_2 \sqrt{\tau} J_{n+2}(E\tau) \quad , \quad (3.27)
\]
Local Boundary Conditions for the Dirac Operator ...

\[ x = C_3 \sqrt{\tau} J_{n+1}(E \tau) \quad , \quad X = C_4 \sqrt{\tau} J_{n+1}(E \tau) \quad , \quad (3.28) \]

\[ \tilde{y} = C_5 \sqrt{\tau} J_{n+2}(E \tau) \quad , \quad \tilde{Y} = C_6 \sqrt{\tau} J_{n+2}(E \tau) \quad , \quad (3.29) \]

\[ y = C_7 \sqrt{\tau} J_{n+1}(E \tau) \quad , \quad Y = C_8 \sqrt{\tau} J_{n+1}(E \tau) \quad . \quad (3.30) \]

In order to find the equation obeyed by \( E \), we must now insert Eqs. (3.27)-(3.30) into the boundary conditions (3.16,17), taking into account also the first-order system given by Eqs. (3.19)-(3.22). This gives the following eight equations :

\[ -iC_3 J_{n+1}(Ea) = \epsilon C_2 J_{n+2}(Ea) \quad , \quad (3.31) \]

\[ iC_4 J_{n+1}(Ea) = \epsilon C_1 J_{n+2}(Ea) \quad , \quad (3.32) \]

\[ -iC_5 J_{n+2}(Ea) = \epsilon C_8 J_{n+1}(Ea) \quad , \quad (3.33) \]

\[ iC_6 J_{n+2}(Ea) = \epsilon C_7 J_{n+1}(Ea) \quad , \quad (3.34) \]

\[ C_1 = - \frac{EC_3 J_{n+1}(Ea)}{E J_{n+2}(Ea) + (n + 2) J_{n+2}(Ea)} \quad , \quad (3.35) \]

\[ C_2 = - \frac{EC_4 J_{n+1}(Ea)}{E J_{n+2}(Ea) + (n + 2) J_{n+2}(Ea)} \quad , \quad (3.36) \]

\[ C_7 = \frac{EC_5 J_{n+2}(Ea)}{E J_{n+1}(Ea) - (n + 1) J_{n+1}(Ea)} \quad , \quad (3.37) \]

\[ C_8 = \frac{EC_6 J_{n+2}(Ea)}{E J_{n+1}(Ea) - (n + 1) J_{n+1}(Ea)} \quad . \quad (3.38) \]
Note that these give separate relations among the constants $C_1, C_2, C_3, C_4$ and among $C_5, C_6, C_7, C_8$. For example, eliminating $C_1, C_2, C_3, C_4$, using Eqs. (3.31,32,35,36) and the useful identities

$$EaJ_{n+1}(Ea) - (n + 1)J_{n+1}(Ea) = -EaJ_{n+2}(Ea) \quad ,$$
$$EaJ_{n+2}(Ea) + (n + 2)J_{n+2}(Ea) = EaJ_{n+1}(Ea) \quad ,$$

one finds

$$-i\epsilon \frac{J_{n+1}(Ea)}{J_{n+2}(Ea)} = \epsilon^2 \frac{C_2}{C_3} = \epsilon^2 \frac{C_4}{C_1} = -i\epsilon^3 \frac{J_{n+2}(Ea)}{J_{n+1}(Ea)} \quad ,$$

which implies (since $\epsilon = \pm 1$):

$$J_{n+1}(Ea) = \pm J_{n+2}(Ea) \forall n \geq 0 \quad ,$$

which is the desired set of eigenvalue conditions. Exactly the same set of eigenvalue conditions would have arisen from eliminating $C_5, C_6, C_7, C_8$.

Note that, since $\frac{J_{n+2}(z)}{J_{n+1}(z)}$ is an odd function of $z$, eigenvalues occur in equal and opposite pairs $\pm E$. The degeneracy of a particular eigenvalue $E$ corresponding to a given $n$ is $(n + 1)(n + 2)$, since this is the number of indices $p$ in Eqs. (3.1,2) for a given $n$.

The limiting behaviour of the eigenvalues can be found under certain approximations. For example, if $n$ is fixed and $|z| \to \infty$, one has the standard asymptotic expansion:

$$J_n(z) \sim \sqrt{\frac{2}{\pi z}} \cos \left( z - \frac{n\pi}{2} - \frac{\pi}{4} \right) + O\left(z^{-\frac{3}{2}}\right) \quad .$$

\[3.43\]
Thus, writing Eq. (3.42) in the form $J_{n+1}(E) = \kappa J_{n+2}(E)$, where $\kappa = \pm 1$ and we set $a = 1$ for simplicity, two asymptotic sets of eigenvalues result:

$$E^+ \sim \pi \left( \frac{n}{2} + L \right) \text{ if } \kappa = 1$$

$$E^- \sim \pi \left( \frac{n}{2} + M + \frac{1}{2} \right) \text{ if } \kappa = -1$$

where $L$ and $M$ are large integers (both positive and negative). One can also obtain an estimate of the smallest eigenvalues, for a given large $n$. The asymptotic expansions in Sec. 9.3 of Ref. 32 show that these eigenvalues have the asymptotic form

$$|E| \sim \left[ n + o(n) \right]$$

In general it is very difficult to solve Eq. (3.42) numerically, because the recurrence relations which enable one to compute Bessel functions starting from $J_0$ and $J_1$ are a source of large errors when the argument is comparable with the order. However, a remarkable numerical study has been carried out in Ref. 33. In that paper, the authors study eigenvalues of the Dirac operator with local boundary conditions, in the case of neutrino billiards. This corresponds to massless spin-$\frac{1}{2}$ particles moving under the action of a potential describing a hard wall bounding a finite domain. The authors end up with an eigenvalue equation of the kind $J_l(k_{nl}) = J_{l+1}(k_{nl})$, and compute the lowest 2600 positive eigenvalues $k_{nl}$.
IV. GENERAL STRUCTURE OF THE $\zeta(0)$ CALCULATION FOR THE 
SPIN-$\frac{1}{2}$ FIELD SUBJECT TO LOCAL BOUNDARY CONDITIONS ON $S^3$

In Sec. III we derived the eigenvalue equation (3.42) which, setting $a = 1$ for simplicity, can be written in the non-linear form

$$F(E) = [J_{n+1}(E)]^2 - [J_{n+2}(E)]^2 = 0 \quad \forall n \geq 0 \quad .$$

(4.1)

The function $F$ is the product of the entire functions (functions analytic in the whole complex plane) $F_1 = J_{n+1} - J_{n+2}$ and $F_2 = J_{n+1} + J_{n+2}$, which can be written in the form

$$F_1(z) = J_{n+1}(z) - J_{n+2}(z) = \gamma_1 z^{(n+1)} e^{g_1(z)} \prod_{i=1}^{\infty} \left(1 - \frac{z}{\mu_i}\right) e^{\frac{z}{\mu_i}} \quad ,$$

(4.2)

$$F_2(z) = J_{n+1}(z) + J_{n+2}(z) = \gamma_2 z^{(n+1)} e^{g_2(z)} \prod_{i=1}^{\infty} \left(1 - \frac{z}{\nu_i}\right) e^{\frac{z}{\nu_i}} \quad .$$

(4.3)

In Eqs. (4.2,3), $\gamma_1$ and $\gamma_2$ are constants, $g_1$ and $g_2$ are entire functions, the $\mu_i$ are the (real) zeros of $F_1$ and the $\nu_i$ are the (real) zeros of $F_2$. In fact, using the terminology in Ref. 34 (see pp 194-195 therein), $F_1$ and $F_2$ are entire functions whose canonical product has genus 1. Namely, in light of the asymptotic behaviour of the eigenvalues (see Eqs. (3.44,45)), we know that $\sum_{i=1}^{\infty} \frac{1}{|\mu_i|} = \infty$ and $\sum_{i=1}^{\infty} \frac{1}{|\nu_i|} = \infty$, whereas $\sum_{i=1}^{\infty} \frac{1}{|\mu_i|^2}$ and $\sum_{i=1}^{\infty} \frac{1}{|\nu_i|^2}$ are convergent. This is why $e^{\frac{z}{\mu_i}}$ and $e^{\frac{z}{\nu_i}}$ must appear in Eqs. (4.2,3), which are called the canonical-product representations of $F_1$ and $F_2$. The genus of the canonical product for $F_1$ is the minimum integer $h$ such that $\sum_{i=1}^{\infty} \frac{1}{|\mu_i|^{h+1}}$ converges, and similarly for $F_2$, replacing $\mu_i$ with $\nu_i$. If the genus is equal to 1, this ensures that no higher powers of $\frac{z}{\mu_i}$ and $\frac{z}{\nu_i}$
are needed in the argument of the exponential. However, there is a very simple relation between \( \mu_i \) and \( \nu_i \). As remarked already in Sec. III, the zeros of \( F_1(z) \) are minus the zeros of \( F_2(z) : \mu_i = -\nu_i, \forall i \). Hence

\[
F(z) = \tilde{\gamma} z^{2(n+1)} e^{(g_1 + g_2)(z)} \prod_{i=1}^{\infty} \left(1 - \frac{z^2}{\mu_i^2}\right), \quad (4.4a)
\]

where \( \tilde{\gamma} = \gamma_1 \gamma_2 \), and \( \mu_i^2 \) are the positive zeros of \( F(z) \).

It turns out that the function \( (g_1 + g_2) \) in Eq. (4.4a) is actually a constant, so that we can write

\[
F(z) = F(-z) = \gamma z^{2(n+1)} \prod_{i=1}^{\infty} \left(1 - \frac{z^2}{\mu_i^2}\right). \quad (4.4b)
\]

In fact, the following theorem holds (see Ref. 35, pp 250-251, and in particular Ref. 36, pp 12-17).

**Theorem 4.1** Let \( f \) be an entire function. If \( \forall \varepsilon > 0 \ \exists A_\varepsilon \) such that:

\[
\log \max \left\{ 1, |f(z)| \right\} \leq A_\varepsilon |z|^{1+\varepsilon}, \quad (4.5)
\]

then \( f \) can be expressed in terms of its zeros as

\[
f(z) = e^{A+Bz} \prod_{i=1}^{\infty} \left(1 - \frac{z}{\nu_i}\right) e^{\nu_i}. \quad (4.6)
\]

If we now apply theorem 4.1 to the functions \( F_1(z)z^{-(n+1)} \) and \( F_2(z)z^{-(n+1)} \) (see Eqs. (4.2,3)), we discover that the well-known formula (see Ref. 32, relation 9.1.21, p 360):

\[
J_n(z) = \frac{i^{-n}}{\pi} \int_0^{\pi} e^{i z \cos \theta} \cos(n\theta) \, d\theta, \quad (4.7)
\]
Local Boundary Conditions for the Dirac Operator ...

leads to the fulfillment of Eq. (4.5) for $F_1(z)z^{-(n+1)}$ and $F_2(z)z^{-(n+1)}$. Thus these functions will satisfy Eq. (4.6) with constants $A_1$ and $B_1$ for $F_1(z)z^{-(n+1)}$, and constants $A_2$ and $B_2 = -B_1$ for $F_2(z)z^{-(n+1)}$. The fact that $B_2 = -B_1$ is well-understood if we look again at Eqs. (4.2,3). We are most indebted to Dr. R. Pinch for providing this argument.

The property that $F(z)$ admits the canonical-product expansion (4.4b) in terms of eigenvalues permits us to evaluate $\zeta(0)$, where $\zeta(s)$ has been defined in Eq. (2.36), following a method described in Ref. 4. The heat kernel $G(t)$, defined by

$$G(t) \equiv \sum_{n,k} e^{-|\lambda_{n,k}|^2t}, \quad (4.8)$$

has the familiar asymptotic expansion

$$G(t) \sim \sum_{n=0}^{\infty} B_n t^{\frac{n}{2} - 2}, \quad (4.9)$$

valid as $t \to 0^+$, where in particular $B_4 = \zeta(0)$. Defining the generalized zeta-function

$$\zeta(s, x^2) \equiv \sum_{n,k} \left( |\lambda_{n,k}|^2 + x^2 \right)^{-s}, \quad (4.10)$$

one then has

$$\Gamma(3)\zeta(3, x^2) = \int_{0}^{\infty} t^2 e^{-x^2 t} G(t) \, dt$$

$$\sim \sum_{n=0}^{\infty} B_n \Gamma \left( 1 + \frac{n}{2} \right) x^{-2-n}, \quad (4.11)$$

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where the asymptotic expansion holds as $x \to \infty$. On the other hand, defining $m \equiv n + 2$, one also has the identity

$$
\Gamma(3)\zeta(3, x^2) = \sum_{m=2}^{\infty} N_m \left( \frac{1}{2x} \frac{d}{dx} \right)^3 \log \left[ (ix)^{-2(m-1)} (J_{m-1}^2(ix) - J_m^2(ix)) \right], \quad (4.12)
$$

following from the canonical-product expansion (4.4b), where $N_m$ is the degeneracy of the eigenvalues, given by $N_m = (m^2 - m)$. Since the functions $(J_{m-1}^2(ix) - J_m^2(ix))$ admit a uniform Debye expansion at large $x$, one can obtain an asymptotic expansion of the right-hand side of Eq. (4.12) valid as $x \to \infty$, which then yields an alternative asymptotic expansion

$$
\Gamma(3)\zeta(3, x^2) \sim \sum_{n=0}^{\infty} q_n x^{-2-n}, \quad (4.13)
$$

valid as $x \to \infty$. By comparison of Eqs. (4.11) and (4.13) one finds the coefficients $B_n$ and in particular $\zeta(0) = B_4 = \frac{4\pi}{2}$.

The relevant uniform asymptotic expansions of $J_n(ix)$ and $J'_n(ix)$, valid for $x \to \infty$ uniformly in the order $n$, are given in Appendix A. The form of these expansions makes it necessary to re-express the function $F(z)$ of Eq. (4.1) in terms of Bessel functions and their derivatives of the same order. Using the identity

$$
J'_l(x) = J_{l-1}(x) - \frac{l}{x} J_l(x), \quad (4.14)
$$

we find

$$
J_{m-1}^2(x) - J_m^2(x) = \left( J'_m + \frac{m}{x} J_m - J_m \right) \left( J'_m + \frac{m}{x} J_m + J_m \right) \\
= J'_m^2 + \left( \frac{m^2}{x^2} - 1 \right) J_m^2 + 2 \frac{m}{x} J_m J'_m. \quad (4.15)
$$
Thus, making the analytic continuation $x \rightarrow ix$, defining $\alpha \equiv \sqrt{m^2 + x^2}$ and using the notation of Eqs. (A1,2), we obtain:

$$J_{m-1}^2(ix) - J_m^2(ix) \sim \frac{(ix)^{2(m-1)}}{2\pi} \alpha e^{2\alpha} e^{-2m \log(m+\alpha)} \left[ \Sigma_1^2 + \Sigma_2^2 + 2\frac{m}{\alpha} \Sigma_1 \Sigma_2 \right], \quad (4.16)$$

where the functions $\Sigma_1$ and $\Sigma_2$ have asymptotic series as described in Appendix A:

$$\Sigma_1 \sim \sum_{k=0}^{\infty} \frac{u_k(m/\alpha)}{m^k}, \quad (4.17)$$

$$\Sigma_2 \sim \sum_{k=0}^{\infty} \frac{v_k(m/\alpha)}{m^k}, \quad (4.18)$$

and the functions $u_k$ and $v_k$ are polynomials given in Refs. 32, 37. The asymptotic series on the right-hand sides of Eqs. (4.17,18) can be re-expressed, defining

$$t \equiv \frac{m}{\alpha}, \quad (4.19)$$

as

$$\sum_{k=0}^{\infty} \frac{u_k(m/\alpha)}{m^k} \sim 1 + \frac{a_1(t)}{\alpha} + \frac{a_2(t)}{\alpha^2} + \frac{a_3(t)}{\alpha^3} + \ldots, \quad (4.20)$$

$$\sum_{k=0}^{\infty} \frac{v_k(m/\alpha)}{m^k} \sim 1 + \frac{b_1(t)}{\alpha} + \frac{b_2(t)}{\alpha^2} + \frac{b_3(t)}{\alpha^3} + \ldots, \quad (4.21)$$

where

$$a_i(t) = \frac{u_i(t)}{t^i}, \quad b_i(t) = \frac{v_i(t)}{t^i}, \quad \forall i \geq 0 \quad (4.22)$$
Local Boundary Conditions for the Dirac Operator ...

Following Eqs. (4.1,12,16), we define

\[ \tilde{\Sigma} \equiv \Sigma_1^2 + \Sigma_2^2 + 2t\Sigma_1\Sigma_2 \quad , \]  

(4.23)

and study the asymptotic expansion of \( \log(\tilde{\Sigma}) \) in the relation

\[ \log \left[ (ix)^{-2(m-1)} (J_{m-1}^2 - J_m^2)(ix) \right] \sim -\log(2\pi) + \log(\alpha) + 2\alpha - 2m\log(m + \alpha) + \log(\tilde{\Sigma}) . \]  

(4.24)

From the relations (4.17)-(4.19) and (4.23), \( \tilde{\Sigma} \) has the asymptotic expansion

\[ \tilde{\Sigma} \sim c_0 + \frac{c_1}{\alpha} + \frac{c_2}{\alpha^2} + \frac{c_3}{\alpha^3} + ... \quad , \]  

(4.25)

where

\[ c_0 = 2(1 + t) \quad , \quad c_1 = 2(1 + t)(a_1 + b_1) \quad , \]  

(4.26, 27)

\[ c_2 = a_1^2 + b_1^2 + 2(1 + t)(a_2 + b_2) + 2ta_1b_1 \quad , \]  

(4.28)

\[ c_3 = 2(1 + t)(a_3 + b_3) + 2(a_1a_2 + b_1b_2) + 2t(a_1b_2 + a_2b_1) \quad . \]  

(4.29)

Higher-order terms have not been computed in Eq. (4.25) because they do not affect the result for \( \zeta(0) \), as we will show in detail in Sec. VII. Defining

\[ \Sigma \equiv \frac{\tilde{\Sigma}}{c_0} \quad , \]  

(4.30)

and making the usual expansion

\[ \log(1 + \omega) = \omega - \frac{\omega^2}{2} + \frac{\omega^3}{3} - \frac{\omega^4}{4} + \frac{\omega^5}{5} + ... \quad , \]  

(4.31)
valid as $\omega \to 0$, we find

$$
\log(\tilde{\Sigma}) = \log(c_0) + \log(\Sigma)
$$

$$
\sim \log(c_0) + \frac{A_1}{\alpha} + \frac{A_2}{\alpha^2} + \frac{A_3}{\alpha^3} + \ldots ,
$$

where

$$
A_1 = \left( \frac{c_1}{c_0} \right) , \quad A_2 = \left( \frac{c_2}{c_0} \right) - \frac{\left( \frac{c_1}{c_0} \right)^2}{2} ,
$$

$$
A_3 = \left( \frac{c_3}{c_0} \right) - \left( \frac{c_1}{c_0} \right) \left( \frac{c_2}{c_0} \right) + \frac{\left( \frac{c_1}{c_0} \right)^3}{3} .
$$

Using Eqs. (2.13)-(2.23) of Ref. 37 and our Eqs. (4.22), (4.26)-(4.29), (4.33)-(4.35), we find after a lengthy calculation:

$$
A_1 = \sum_{r=0}^{2} k_1 r^r , \quad A_2 = \sum_{r=0}^{4} k_2 r^r , \quad A_3 = \sum_{r=0}^{6} k_3 r^r ,
$$

where

$$
k_{10} = -\frac{1}{4} , \quad k_{11} = 0 , \quad k_{12} = \frac{1}{12} ,
$$

$$
k_{20} = 0 , \quad k_{21} = -\frac{1}{8} , \quad k_{22} = k_{23} = \frac{1}{8} , \quad k_{24} = -\frac{1}{8} ,
$$

$$
k_{30} = \frac{5}{192} , \quad k_{31} = -\frac{1}{8} , \quad k_{32} = \frac{9}{320} , \quad k_{33} = \frac{1}{2} ,
$$

$$
k_{34} = -\frac{23}{64} , \quad k_{35} = -\frac{3}{8} , \quad k_{36} = \frac{179}{576} .
$$

Note that, interestingly, all factors of $(1 + t)$ in the denominators of $A_1, A_2$ and $A_3$ in Eqs. (4.33)-(4.35) have cancelled against factors in the numerators.
Local Boundary Conditions for the Dirac Operator ...

The result of these calculations, following from Eqs. (4.24) and (4.32)-(4.39), may be summarized in the expression

$$\log \left[ (ix)^{-2(m-1)} \left( J_{m-1}^2 - J_m^2 \right) (ix) \right] \sim \sum_{i=1}^{5} S_i(m, \alpha(x)) + \text{higher-order terms}, \quad (4.40)$$

where

$$S_1 \equiv -\log(\pi) + 2\alpha, \quad (4.41)$$

$$S_2 \equiv -(2m - 1) \log(m + \alpha), \quad (4.42)$$

$$S_3 \equiv \sum_{r=0}^{2} k_{1r} m^r \alpha^{-r-1}, \quad (4.43)$$

$$S_4 \equiv \sum_{r=0}^{4} k_{2r} m^r \alpha^{-r-2}, \quad (4.44)$$

$$S_5 \equiv \sum_{r=0}^{6} k_{3r} m^r \alpha^{-r-3}. \quad (4.45)$$

The asymptotic series (4.40) will be sufficient for computing $\zeta(0)$. The infinite sum over $m$ in the expression (4.12) can be evaluated following Eq. (4.40) with the help of the formulae derived using contour integration:

$$\sum_{m=0}^{\infty} m^{2k} \alpha^{-2k-l} = \frac{\Gamma \left( k + \frac{1}{2} \right) \Gamma \left( \frac{l}{2} - \frac{1}{2} \right)}{2\Gamma \left( k + \frac{l}{2} \right)} x^{1-l} \quad , \quad k = 1, 2, 3, \ldots \quad , \quad (4.46)$$

$$\sum_{m=0}^{\infty} m \alpha^{-1-l} \sim \frac{x^{1-l}}{\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{2^r}{r!} \tilde{B}_r x^{-r} \frac{\Gamma \left( \frac{r}{2} + \frac{1}{2} \right) \Gamma \left( \frac{l}{2} - \frac{1}{2} + \frac{r}{2} \right)}{2\Gamma \left( \frac{1}{2} + \frac{l}{2} \right)} \cos \left( \frac{r\pi}{2} \right). \quad (4.47)$$
Local Boundary Conditions for the Dirac Operator ...

Here \( l \) is a real number obeying \( l > 1 \) and \( \tilde{B}_0 = 1, \tilde{B}_2 = \frac{1}{6}, \tilde{B}_4 = -\frac{1}{30} \) etc. are Bernoulli numbers. Thus, defining the functions \( W_\infty \) and \( W_\infty^1, ..., W_\infty^5 \) of \( x \) by

\[
W_\infty \equiv \sum_{m=0}^{\infty} (m^2 - m) \left( \frac{d}{2x \, dx} \right)^3 \left[ \sum_{i=1}^{5} S_i(m, \alpha(x)) \right] = \sum_{i=1}^{5} W_\infty^i ,
\]

we find following Eq. (4.12) and the discussion of Sec. VII that

\[
\Gamma(3) \zeta(3, x^2) \sim W_\infty + \sum_{n=5}^{\infty} \hat{q}_n x^{-2-n} .
\]

Since, in Eq. (4.11), \( \zeta(0) = B_4 \) is found from the coefficient of \( x^{-6} \), we see that \( \zeta(0) \) can be computed from the asymptotic series for \( W_\infty \) as \( x \to \infty \).

**V. CONTRIBUTION OF \( W_\infty^1 \) AND \( W_\infty^2 \)**

The term \( W_\infty^1 \) does not contribute to \( \zeta(0) \). In fact, using Eqs. (4.41) and (4.46)-(4.48)

we find

\[
W_\infty^1 = \frac{3}{4} \sum_{m=0}^{\infty} (m^2 - m) \alpha^{-5}
\]

\[
\sim \frac{x^{-2}}{4} - \frac{3}{4} \frac{x^{-3}}{\Gamma(\frac{1}{2})} \sum_{r=0}^{\infty} \frac{2^r r!}{r} \tilde{B}_r x^{-r} \frac{\Gamma\left(\frac{r}{2} + \frac{1}{2}\right) \Gamma\left(\frac{r}{2} + \frac{3}{2}\right)}{2 \Gamma\left(\frac{r}{2}\right)} \cos\left(\frac{r\pi}{2}\right) ,
\]

which implies that \( x^{-6} \) (whose coefficient contributes to \( \zeta(0) \)) does not appear.

The contribution \( \zeta^2(0) \) of \( W_\infty^2 \) to \( \zeta(0) \) is obtained using the identities

\[
\left( \frac{1}{2x \, dx} \right)^3 \log\left( \frac{1}{m + \alpha} \right) = (m + \alpha)^{-3} \left[ -\alpha^{-3} - \frac{9}{8} m \alpha^{-4} - \frac{3}{8} m^2 \alpha^{-5} \right] ,
\]

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\( (m + \alpha)^{-3} = \frac{(\alpha - m)^3}{x^6} \), \hspace{1cm} (5.3)

so that Eq. (5.2) becomes
\[
\left( \frac{1}{2x} \frac{d}{dx} \right)^3 \log \left( \frac{1}{m + \alpha} \right) = -x^{-6} + mx^{-6} \alpha^{-1} + \frac{m}{2} x^{-4} \alpha^{-3} + \frac{3}{8} mx^{-2} \alpha^{-5} . \hspace{1cm} (5.4)
\]

Now the series for \( W_\infty^2 \) is convergent, as may be checked using Eqs. (4.42,48) and the form (5.2). However, when the sum over \( m \) is rewritten using the splitting (5.4), the individual pieces become divergent. These ‘fictitious’ divergences may be regularized using the device of Ref. 4: dividing by \( \alpha^{2s} \), summing using Eqs. (4.46,47) and then taking the limit \( s \to 0 \).

With this understanding, and using the sums \( \rho_i \) defined in Eqs. (B1)-(B12), we find:
\[
W_\infty^2 = -2x^{-6} \rho_1 + 2x^{-6} \rho_2 + x^{-4} \rho_3 + \frac{3}{4} x^{-2} \rho_4 + 3x^{-6} \rho_5 - 3x^{-6} \rho_6
\]
\[
- \frac{3}{2} x^{-4} \rho_7 - \frac{9}{8} x^{-2} \rho_8 - x^{-6} \rho_9 + x^{-6} \rho_{10} + \frac{x^{-4}}{2} \rho_{11} + \frac{3}{8} x^{-2} \rho_{12} . \hspace{1cm} (5.5)
\]

When odd powers of \( m \) greater than 1 occur, we can still apply Eq. (4.47) after re-expressing \( m^2 \) as \( \alpha^2 - x^2 \). Thus, applying again the contour formulae (4.46,47), only \( \rho_1 \) and \( \rho_9 \) are found to contribute to \( \zeta^2(0) \), leading to
\[
\zeta^2(0) = -\frac{1}{120} + \frac{1}{24} = \frac{1}{30} . \hspace{1cm} (5.6)
\]

**VI. EFFECT OF \( W_\infty^3 \), \( W_\infty^4 \) AND \( W_\infty^5 \)**

The term \( W_\infty^3 \) does not contribute to \( \zeta(0) \). In fact, using the relations (4.43) and (4.48) we find
\[
W_\infty^3 = -\frac{1}{8} \sum_{r=0}^{2} k_1 r (r + 1)(r + 3)(r + 5) \left[ \sum_{m=0}^{\infty} (m^2 - m) m^r \alpha^{-r-7} \right] . \hspace{1cm} (6.1)
\]
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In light of Eqs. (4.46,47), $x^{-6}$ does not appear in the asymptotic expansion of Eq. (6.1) at large $x$.

For the term $W_\infty^4$ a remarkable cancellation occurs. In fact, using Eqs. (4.44,48) we find

$$W_\infty^4 = -\frac{1}{8} \sum_{r=0}^{4} k_{2r}(r+2)(r+4)(r+6) \left[ \sum_{m=0}^{\infty} (m^2 - m) m^r \alpha^{-r-8} \right].$$  \hspace{1cm} (6.2)

The application of Eqs. (4.38,46,47) leads to

$$\zeta^4(0) = \frac{1}{2} \sum_{r=0}^{4} k_{2r} = 0.$$ \hspace{1cm} (6.3)

Finally, using Eqs. (4.45,48) we find

$$W_\infty^5 = -\frac{1}{8} \sum_{r=0}^{6} k_{3r}(r+3)(r+5)(r+7) \left[ \sum_{m=0}^{\infty} (m^2 - m) m^r \alpha^{-r-9} \right].$$  \hspace{1cm} (6.4)

Again, the contour formulae (4.46,47) lead to

$$\zeta^5(0) = -\frac{1}{2} \sum_{r=0}^{6} k_{3r} = -\frac{1}{360},$$ \hspace{1cm} (6.5)

in light of Eq. (4.39).
VII. VANISHING EFFECT OF HIGHER-ORDER TERMS

We now prove the statement made after Eq. (4.29), namely that there is no need to compute the explicit form of $c_k$ in Eq. (4.25), $\forall k > 3$. In fact, the formulae (4.33)-(4.35) can be completed by

$$A_4 = \left(\frac{c_4}{c_0}\right) - \frac{\left(\frac{c_4}{c_0}\right)^2}{2} - \left(\frac{c_1}{c_0}\right) \left(\frac{c_4}{c_0}\right) + \left(\frac{c_1}{c_0}\right)^2 \left(\frac{c_2}{c_0}\right) - \frac{\left(\frac{c_1}{c_0}\right)^4}{4}, \quad (7.1)$$

plus infinitely many others, and the general term has the structure

$$A_n = \sum_{p=1}^{l} h_{np} (1 + t)^{−p} + \sum_{r=0}^{2n} k_{nr} t^r, \quad \forall n \geq 1, \quad (7.2)$$

where $l < n$, the $h_{np}$ are constants, and $r$ assumes both odd and even values. We have indeed proved that $h_{11} = h_{21} = h_{31} = 0$, but the calculation of $h_{np}$ for all values of $n$ is not obviously feasible. However, we will show that the exact value of $h_{np}$ does not affect the $\zeta(0)$ value. Thus, $\forall n > 3$, we must study

$$H_n^\infty \equiv \sum_{m=0}^{\infty} (m^2 - m) \left(\frac{1}{2x} \frac{d}{dx}\right)^3 \left[\frac{A_n}{\alpha^n}\right] = H_n^{A,\infty} + H_n^{B,\infty}, \quad (7.3)$$

where, defining

$$a_{np} \equiv (p - n)(p - n - 2)(p - n - 4), \quad b_{np} \equiv 3 \left(-p^3 + (3 + 2n)p^2 - (n^2 + 3n + 1)p\right), \quad (7.4a)$$

$$c_{np} \equiv 3 \left(p^3 - (p^2 + p)n - p\right), \quad d_{np} \equiv -p(p + 1)(p + 2), \quad (7.4b)$$

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one has

\[ H_{n,A}^{\infty} \equiv \sum_{p=1}^{l} h_{np} \sum_{m=0}^{\infty} (m^2 - m) \left( \frac{1}{2x} \frac{d}{dx} \right)^3 [\alpha^{p-n}(m + \alpha)^{-p}] \]

\[ = \sum_{p=1}^{l} h_{np} \sum_{m=0}^{\infty} (m^2 - m) \left[ a_{np}\alpha^{p-n-6}(m + \alpha)^{-p} + b_{np}\alpha^{p-n-5}(m + \alpha)^{-p-1} \right. \]

\[ + c_{np}\alpha^{p-n-4}(m + \alpha)^{-p-2} + d_{np}\alpha^{p-n-3}(m + \alpha)^{-p-3} \right] , \]  

(7.5)

\[ H_{n,B}^{\infty} \equiv -\frac{1}{8} \sum_{r=0}^{2n} k_{nr}(r + n)(r + n + 2)(r + n + 4) \left[ \sum_{m=0}^{\infty} (m^2 - m) m^r \alpha^{-r-n-6} \right] . \]  

(7.6)

Because we are only interested in understanding the behaviour of Eq. (7.5) as a function of \( x \), the application of the Euler-Maclaurin formula \(^{38}\) described in Appendix C is more useful than the splitting (5.3). In so doing, we find that the part of the Euler-Maclaurin formula involving the integral on the left-hand side of Eq. (C2), when \( n = \infty \), contains the least negative power of \( x \). Thus, if we prove that the conversion of Eq. (7.5) into an integral only contains \( x^{-l} \) with \( l > 6 \), \( \forall n > 3 \), we have proved that \( H_{n,A}^{\infty} \) does not contribute to \( \zeta(0) \), \( \forall n > 3 \). This is indeed the case, because in so doing we deal with the integrals defined in Eqs. (C4)-(C11), where \( I_1^{np}, I_3^{np}, I_5^{np} \) and \( I_7^{np} \) are proportional to \( x^{-3-n} \), and \( I_2^{np}, I_4^{np}, I_6^{np} \) and \( I_8^{np} \) are proportional to \( x^{-4-n} \), where \( n > 3 \).

Finally, in Eq. (7.6) we must study the case when \( r \) is even and the case when \( r \) is odd. In so doing, defining

\[ \Sigma(I) \equiv \sum_{m=0}^{\infty} m^{2+r} \alpha^{-r-n-6} \quad , \quad \Sigma(II) \equiv \sum_{m=0}^{\infty} m^{1+r} \alpha^{-r-n-6} \]  

(7.7)
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we find for \( r = 2k > 0 \) (\( k \) integer)

\[
\Sigma_{(I)} = \frac{x^{-3-n}}{2} \frac{\Gamma \left( k + \frac{3}{2} \right) \Gamma \left( \frac{n}{2} + \frac{3}{2} \right)}{\Gamma \left( 3 + k + \frac{n}{2} \right)},
\] (7.8)

and for \( r = 2k + 1 \) (\( k \) integer \( \geq 0 \))

\[
\Sigma_{(I)} \sim \frac{x^{-3-n}}{\Gamma \left( \frac{1}{2} \right)} \sum_{l=0}^{\infty} \left\{ \frac{2^l \tilde{B}_l}{l!} x^{-l} \Gamma \left( \frac{l}{2} + 1 \right) \cos \left( \frac{l\pi}{2} \right) \right. \\
\left. \left[ \frac{\Gamma \left( \frac{n}{2} + \frac{3}{2} + \frac{l}{2} \right)}{\Gamma \left( \frac{n}{2} + \frac{3}{2} \right)} + \ldots + (-1)^{(1+k)} \frac{\Gamma \left( \frac{n}{2} + \frac{5}{2} + k + \frac{l}{2} \right)}{\Gamma \left( \frac{n}{2} + \frac{5}{2} + k \right)} \right] \right\}. \] (7.9)

Moreover, we find for \( r = 2k > 0 \)

\[
\Sigma_{(II)} \sim \frac{x^{-4-n}}{\Gamma \left( \frac{1}{2} \right)} \sum_{l=0}^{\infty} \left\{ \frac{2^l \tilde{B}_l}{l!} x^{-l} \Gamma \left( \frac{l}{2} + 1 \right) \cos \left( \frac{l\pi}{2} \right) \right. \\
\left. \left[ \frac{\Gamma \left( \frac{n}{2} + 2 + \frac{l}{2} \right)}{\Gamma \left( \frac{n}{2} + 3 \right)} + \ldots + (-1)^k \frac{\Gamma \left( \frac{n}{2} + 2 + k + \frac{l}{2} \right)}{\Gamma \left( \frac{n}{2} + 3 + k \right)} \right] \right\}, \] (7.10)

and for \( r = 2k + 1 \) (\( k \geq 0 \))

\[
\Sigma_{(II)} = \frac{x^{-4-n}}{2} \frac{\Gamma \left( k + \frac{3}{2} \right) \Gamma \left( \frac{n}{2} + 2 \right)}{\Gamma \left( \frac{n}{2} + k + \frac{n}{2} \right)}. \] (7.11)

Once more, in deriving Eqs. (7.8,11) we used Eq. (4.46), and in deriving Eqs. (7.9,10) we used Eq. (4.47). Thus also \( H_{\infty}^{n,B} \) does not contribute to \( \zeta(0) \), \( \forall n > 3 \), and our proof is completed.
In light of Eqs. (5.6, 6.3, 6.5), and using the result proved in Sec. VII, we conclude that for the complete Majorana field \( (\psi^A, \bar{\psi}^{A'}) \):

\[
\zeta(0) = \frac{11}{360}.
\]  

(8.1)

This value, found by a direct calculation, disagrees with the result \( \zeta(0) = \frac{17}{180} \) in Ref. 11, which was computed using a different, indirect, approach. If there are \( N \) massless Majorana fields, the full \( \zeta(0) \) in Eq. (8.1) should be multiplied by \( N \). Following the remarks in the Introduction, this \( \zeta(0) \) value can now be combined with the \( \zeta(0) \) values for other spins, in order to check whether the one-loop divergences in quantum cosmology cancel in higher-\( N \) supergravity theories (in the case of our simple background geometry). At present it can only be said that existing results are inconclusive on this question (see Sec. IX).

IX. CONCLUDING REMARKS

Note, following the comments at the end of Sec. II, that the zeta-function studied in Ref. 11, formed from the eigenvalues of the squared-up Dirac operator, is the same as the zeta-function studied here, formed from squared eigenvalues, since the boundary conditions agree. Unfortunately it is hard to understand the discrepancy between the spin-\( \frac{1}{2} \) result of Ref. 11 and the present result, since the approach of Refs. 9, 11 involves...
finding general expressions for $\zeta(0)$ for various spins and types of boundary condition, on a general Riemannian manifold with boundary. Only at the end of the calculation in Ref. 11 does one restrict attention to a flat background with spherical boundary; degeneracies and a specific eigenvalue equation play no role.

Turning to the question of $\zeta(0)$ in extended supergravity models with boundaries present, recall that two sets of possible ‘supersymmetric’ boundary conditions were suggested in the Introduction. The first set, Breitenlohner-Freedman-Hawking boundary conditions,\textsuperscript{20,21} involve field strengths at the boundary for gauge fields (spins $s = 1, \frac{3}{2}, 2$), being either of magnetic or electric type, depending on the sign of the quantity $\epsilon$ introduced in Sec. I. For bosonic fields, the magnetic conditions involve fixing the magnetic field $B_i$ on the surface ($s = 1$) or fixing the magnetic part $B_{ij}$ of the Weyl tensor ($s = 2$), which involves the normal derivative of the metric. Similarly the electric conditions fix the electric field $E_i$ ($s = 1$) or the electric part $E_{ij}$ of the Weyl tensor ($s = 2$). This points to a difficulty with such boundary conditions, since $E_{ij}$ involves second derivatives of the metric in the normal direction, so that electric boundary conditions cannot be formulated in terms of canonical variables (using Hamiltonian language) for $s = 2$. This in turn affects the quantization of such a system, since in the case when $E_{ij}$ is fixed one cannot find a Euclidean action $I$ for linearized gravity with these boundary conditions, such that $\delta I = 0$ gives the linearized $s = 2$ field equations subject to specified $E_{ij}$ at the boundary. As a result, one cannot formulate a Hartle-Hawking path integral in the usual way. A similar problem arises for $s = \frac{3}{2}$, in both the magnetic and electric cases : the boundary conditions again cannot be written in terms of canonical variables, so that again one cannot
write down the quantum amplitude as a path integral. Thus the Breitenlohner-Freedman-Hawking boundary conditions are problematic for spins $s = \frac{3}{2}$ and 2 in our example.

One is therefore led to study the alternative *locally supersymmetric* boundary conditions suggested in the Introduction, following from Ref. 24, and used in Ref. 11. In the case of a spin-$\frac{1}{2}$ field, these again take the form (1.1), and for a complex scalar field they again involve Dirichlet and Neumann conditions on the real and imaginary parts (or vice-versa). For spin 1, they can be regarded as magnetic, and for spins $\frac{3}{2}$ and 2 they involve fixing the projection $\left(\epsilon \tilde{\psi}^A_i - 2\epsilon n_A^A \psi^i_A\right)$ of the spin-$\frac{3}{2}$ potential and the spatial tetrad $e^{AA'}_i$ (and hence intrinsic metric $h_{ij}$) up to gauge. Refs. 9, 11 give detailed calculations of $\zeta(0)$ in a general background geometry for these boundary conditions, including the contribution of gauge-fixing and ghost terms. They find no cancellations when summing over spins for any higher-$N$ supergravity theory, allowing also for the topological contribution of anti-symmetric tensor fields where appropriate.\textsuperscript{9,11} In the case of flat Euclidean four-space bounded by a three-sphere, the resulting $\zeta(0)$ value for $s = \frac{1}{2}$ has already been quoted. For $s = 0$ the result $\zeta(0) = \frac{7}{45}$ can be confirmed by a *direct* calculation based on finding the eigenvalue conditions and using the methods of Ref. 4. The values found in Ref. 11 for $s = 1, \frac{3}{2}$ and 2 in this case are $\zeta(0) = -\frac{38}{45}, \frac{107}{180}$ and $-\frac{803}{45}$. However, the discrepancy found in this paper for $s = \frac{1}{2}$ makes it particularly necessary to check also the result for $s = \frac{3}{2}$; both fermionic cases $s = \frac{1}{2}$ and $\frac{3}{2}$ involve mixed boundary conditions in a crucial way, and are somewhat similar, although the extra technical difficulties of spin $\frac{3}{2}$ may render a *direct* calculation of $\zeta(0)$ along the lines of this paper impossible. Given
these uncertainties, one cannot say that the question of the one-loop finiteness of extended supergravity, in the presence of boundaries, has been settled.

A further interesting question, related to recent considerations in the literature, is raised by this work in the case of gauge fields. The value of $\zeta(0)$ depends on whether one first restricts the classical theory to a set of physical degrees of freedom, by choice of gauge, and then quantizes, or whether one quantizes the full theory in BRST-invariant fashion with gauge-averaging and ghost terms included. For example, the original quantum gravity ($s = 2$) calculation of Schleich\textsuperscript{1} worked with physical degrees of freedom, using a transverse-traceless gauge and Dirichlet boundary conditions on the three-sphere to obtain $\zeta(0) = -\frac{278}{45}$, which differs from the result $\zeta(0) = -\frac{803}{45}$ of Ref. 11. Similarly the $s = 1$ calculation of Louko,\textsuperscript{2} working in a transverse gauge with Dirichlet (magnetic) boundary conditions on the three-sphere, gave $\zeta(0) = -\frac{77}{180}$, as compared to the result $\zeta(0) = -\frac{38}{45}$ of Ref. 11. [Electric boundary conditions give $\zeta(0) = \frac{13}{180}$ with physical degrees of freedom,\textsuperscript{8} while again $\zeta(0) = -\frac{38}{45}$ in Ref. 11]. The present authors have also done a calculation for spin $\frac{3}{2}$, working with physical degrees of freedom, in the gauge $e_{AA'}^j \psi^A_j = 0$, $e_{AA'}^j \tilde{\psi}^A_j = 0$, subject to the boundary conditions of the previous paragraph on the three-sphere, giving $\zeta(0) = -\frac{289}{360}$, as compared to the result $\zeta(0) = \frac{197}{180}$ of Ref. 11. In the BRST method, the local boundary conditions are chosen such that transition amplitudes are invariant under BRST transformations, and hence, by the Fradkin-Vilkovisky theorem,\textsuperscript{41} the amplitudes do not depend on the gauge-fixing term. In a related example,\textsuperscript{42} the method involving physical degrees of freedom has been shown to be formally equivalent to a method involving Faddeev-Popov ghosts (and hence to a BRST method). But apparently in the present
example of quantum fields inside a three-sphere, the relation between the two approaches is only formal. One expects, for example, that the $\zeta(0)$ value for a physical-degrees-of-freedom calculation might depend on the choice of gauge conditions. In this situation, since gauge invariance is the underlying physical principle, one is forced to choose the gauge-invariant BRST method.

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APPENDIX A

In Sec. IV, we use uniform asymptotic expansions of regular Bessel functions \( J_n \) and their first derivatives \( J'_n \), relying on the relations 9.3.7 and 9.3.11 on p 366 of Ref. 32. In those formulae, the argument of \( J_n \) and \( J'_n \) is \( \frac{n}{\cosh \gamma} \), where \( \gamma \) is fixed and positive and \( n \) is large and positive. If we put \( \frac{n}{\cosh \gamma} = x \), we find that

\[
\begin{align*}
e^\gamma &= \frac{n}{x} \pm \sqrt{\frac{n^2}{x^2} - 1}, \quad \tanh \gamma = \pm \frac{1}{n} \sqrt{n^2 - x^2}.
\end{align*}
\]

Thus, making the analytic continuation \( x \rightarrow ix \) and then defining \( \alpha = \sqrt{n^2 + x^2} \), we may write

\[
\begin{align*}
J_n(ix) &\sim \frac{(ix)^n}{\sqrt{2\pi}} \alpha^{-\frac{1}{2}} e^\alpha e^{-n \log(n+\alpha) \Sigma_1}, \quad (A1) \\
J'_n(ix) &\sim \frac{(ix)^{n-1}}{\sqrt{2\pi}} \alpha^{-\frac{1}{2}} e^\alpha e^{-n \log(n+\alpha) \Sigma_2}, \quad (A2)
\end{align*}
\]

where the functions \( \Sigma_1 \) and \( \Sigma_2 \) admit the asymptotic expansions : 

\[
\Sigma_1 \sim \sum_{k=0}^{\infty} u_k \left( \frac{\alpha}{n^k} \right),
\]

\[
\Sigma_2 \sim \sum_{k=0}^{\infty} v_k \left( \frac{\alpha}{n^k} \right),
\]

valid uniformly in the order \( n \) as \( |x| \rightarrow \infty \). The functions \( u_k \) and \( v_k \) are polynomials, given by the relations (9.3.9) and (9.3.13) on p 366 of Ref. 32.
We now write the infinite sums used for the $\zeta(0)$ calculation of Secs. IV-VIII. In Eq. (5.5) we have

\begin{align*}
\rho_1 &\equiv \sum_{m=0}^{\infty} m^3, \quad \rho_2 \equiv \sum_{m=0}^{\infty} m^4 \alpha^{-1}, \quad \rho_3 \equiv \sum_{m=0}^{\infty} m^4 \alpha^{-3}, \quad (B1, 2, 3) \\
\rho_4 &\equiv \sum_{m=0}^{\infty} m^4 \alpha^{-5}, \quad \rho_5 \equiv \sum_{m=0}^{\infty} m^2, \quad \rho_6 \equiv \sum_{m=0}^{\infty} m^3 \alpha^{-1}, \quad (B4, 5, 6) \\
\rho_7 &\equiv \sum_{m=0}^{\infty} m^3 \alpha^{-3}, \quad \rho_8 \equiv \sum_{m=0}^{\infty} m^3 \alpha^{-5}, \quad \rho_9 \equiv \sum_{m=0}^{\infty} m, \quad (B7, 8, 9) \\
\rho_{10} &\equiv \sum_{m=0}^{\infty} m^2 \alpha^{-1}, \quad \rho_{11} \equiv \sum_{m=0}^{\infty} m^2 \alpha^{-3}, \quad \rho_{12} \equiv \sum_{m=0}^{\infty} m^2 \alpha^{-5}. \quad (B10, 11, 12)
\end{align*}

Of course, the sums (B1)-(B12) do not by themselves make sense as they are written. But, as discussed in Sec. V, the sums appear in convergent linear combinations and their values are regularized as explained following Eq. (5.4).
If $f$ is a function which obeys suitable differentiability and integrability conditions, the Euler-Maclaurin formula is a very useful tool which can be used to estimate the sum $\sum_{i=0}^{n} f(i)$. Denoting by $\widetilde{B}_s$ the Bernoulli numbers, defined by the expansion

$$
\frac{t}{e^t - 1} = \sum_{s=0}^{\infty} \widetilde{B}_s \frac{t^s}{s!} , \quad |t| < 2\pi ,
$$

(C1)

the following theorem holds.$^{38}$

**Theorem C.1** Let $f$ be a real- or complex-valued function defined on $0 \leq t < \infty$. If $f^{(2m)}(t)$ is absolutely integrable on $(0, \infty)$ then, for $n = 1, 2, \ldots,$

$$
\sum_{i=0}^{n} f(i) - \int_{0}^{n} f(x) \, dx = \frac{1}{2} \left[ f(0) + f(n) \right]
$$

$$
+ \sum_{s=1}^{m-1} \frac{\widetilde{B}_{2s}}{(2s)!} \left[ f^{(2s-1)}(n) - f^{(2s-1)}(0) \right] + R_m(n) ,
$$

(C2)

where the remainder $R_m(n)$ satisfies

$$
| R_m(n) | \leq \left( 2 - 2^{1-m} \right) \frac{|\widetilde{B}_{2m}|}{(2m)!} \int_{0}^{n} |f^{(2m)}(x)| \, dx .
$$

(C3)

Eq. (C2) can be used to evaluate infinite sums, setting $n = \infty$, if the corresponding derivatives, and the integrals in Eqs. (C2,3) are well-defined. Typically one considers $\bar{n}$ terms in the sum involving Bernoulli numbers in Eq. (C2), and neglects terms starting with some value $s = \bar{n} + 1 \leq (m - 1)$. Eq. (C3) can be used to show that the remainder
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$R_m(n)$ is bounded in absolute value by a constant times the first neglected term in the sum in Eq. (C2), provided $f^{(2m)}$ obeys suitable conditions specified on p 38 of Ref. 38.

The integrals which arise in Sec. VII in taking $n = \infty$ on the left-hand side of Eq. (C2) are

\[ I_1^{np} \equiv \int_0^\infty y^2 \left( y + \sqrt{x^2 + y^2} \right)^{-p} \left( x^2 + y^2 \right)^{\frac{p}{2} - \frac{2}{3} - 3} \, dy \quad , \quad (C4) \]

\[ I_2^{np} \equiv \int_0^\infty y \left( y + \sqrt{x^2 + y^2} \right)^{-p} \left( x^2 + y^2 \right)^{\frac{p}{2} - \frac{2}{3} - 3} \, dy \quad , \quad (C5) \]

\[ I_3^{np} \equiv \int_0^\infty y^2 \left( y + \sqrt{x^2 + y^2} \right)^{-p-1} \left( x^2 + y^2 \right)^{\frac{p}{2} - \frac{2}{3} - \frac{2}{3}} \, dy \quad , \quad (C6) \]

\[ I_4^{np} \equiv \int_0^\infty y \left( y + \sqrt{x^2 + y^2} \right)^{-p-1} \left( x^2 + y^2 \right)^{\frac{p}{2} - \frac{2}{3} - \frac{2}{3}} \, dy \quad , \quad (C7) \]

\[ I_5^{np} \equiv \int_0^\infty y^2 \left( y + \sqrt{x^2 + y^2} \right)^{-p-2} \left( x^2 + y^2 \right)^{\frac{p}{2} - \frac{2}{3} - 2} \, dy \quad , \quad (C8) \]

\[ I_6^{np} \equiv \int_0^\infty y \left( y + \sqrt{x^2 + y^2} \right)^{-p-2} \left( x^2 + y^2 \right)^{\frac{p}{2} - \frac{2}{3} - 2} \, dy \quad , \quad (C9) \]

\[ I_7^{np} \equiv \int_0^\infty y^2 \left( y + \sqrt{x^2 + y^2} \right)^{-p-3} \left( x^2 + y^2 \right)^{\frac{p}{2} - \frac{2}{3} - \frac{3}{3}} \, dy \quad , \quad (C10) \]

\[ I_8^{np} \equiv \int_0^\infty y \left( y + \sqrt{x^2 + y^2} \right)^{-p-3} \left( x^2 + y^2 \right)^{\frac{p}{2} - \frac{2}{3} - \frac{3}{3}} \, dy \quad . \quad (C11) \]
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