A path integral approach to business cycle models with large number of agents

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Abstract
This paper presents an analytical treatment of economic systems with an arbitrary number of agents that keeps track of the systems’ interactions and agents’ complexity. This formalism does not seek to aggregate agents. It rather replaces the standard optimization approach by a probabilistic description of both the entire system and agents’ behaviors. This is done in two distinct steps. A first step considers an interacting system involving an arbitrary number of agents, where each agent’s utility function is subject to unpredictable shocks. In such a setting, individual optimization problems need not be resolved. Each agent is described by a time-dependent probability distribution centered around his utility optimum. The entire system of agents is thus defined by a composite probability depending on time, agents’ interactions and forward-looking behaviors. This dynamic system is described by a path integral formalism in an abstract space—the space of economic variables—and is very similar to a statistical physics or quantum mechanics system. The usual utility optimization of a representative agent is recovered as a particular case. Compared to a standard optimization, such a description eases the treatment of systems with small number of agents. It becomes however useless for a large number of agents. In a second step therefore, we show that for a large number of agents, the previous description is equivalent to a more compact description in terms of field theory. This yields an analytical though approximate treatment of the system. This field theory does not model the aggregation of a microeconomic system in the usual sense. It rather describes an environment of a large number of interacting agents. From this description, various phases or equilibria may be retrieved, along with individual agents’ behaviors and their interactions with the environment. For illustrative purposes, this paper studies a business cycle model with a large number of agents.

Keywords Path integrals · Statistical field theory · Business cycle · Budget constraint · Multi-agent model · Interacting agents

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Introduction

In many instances, representative agent models have proven unrealistic, lacking both the collective and emerging effects stemming from agents’ interactions. To remedy these pitfalls, various paths have been explored: complex systems, networks, agent-based systems or econophysics. However, agent-based and networks models rely on numerical simulations and may lack microeconomic foundations. Econophysics builds on statistical facts and empirical aggregate rules to derive macroeconomic laws. These laws are prone, like ad hoc macroeconomics, to the Lucas critique (see Lucas 1976). The gap between microeconomic foundations and multi-agent systems remains.

The present paper attempts to fill this gap by adapting statistical physics methods to describe multiple interacting agents. It is an introduction to the method developed in Gosselin et al. (2017), illustrated by a basic economic application to a business cycle model. Our setup models individual, i.e., microeconomic interactions for a large number of agents in the context of statistical field theory which allows to recover a global, macroeconomic description of the system.

In physics, the use of field theory to study dynamic models with large number of degrees of freedom in condensed matter has a long history. The field formulation provides a compact way to find the possible macroeconomic features emerging from the interactions of a large set of microeconomic structures.

In economics, once adapted to account for standard microeconomic concepts such as preferences, constraints, rationality and economic interactions, the field formalism allows an analytical treatment of a broad class of models with an arbitrary number of agents.

By combining statistical physics and economics, our field formalism provides an understanding of the transition from the individual to the collective scale. The microeconomic setup shapes the field model from which emerges a specific collective background or phase. The field description, in turn, describes the impact of this collective background on individual behaviors. This provides a back and forth interpretation between scales and accounts for the interplay between phases and the interactions at the microeconomic level.

Additionally, field theory has the particularity to distinguish microscopic interactions that fade away at large scales from those that become predominant. Applied to economic systems, the relevance or irrelevance of some microeconomic concepts during the change of scale could indirectly shed some light on the aggregation problem of economic variables.

The statistical approach of economic systems presented here is a two-step process. In a first step, the usual model of optimizing agents is replaced by a probabilistic description of the system. In an interacting system involving an arbitrary number of agents, each agent is described by an intertemporal utility function depending on an arbitrary number of variables. However each agent’s utility function is subject to unpredictable shocks. In such a setting, individual optimization problems are discarded. Each agent is described by a time-dependent probability distribution centered
around this agent’s utility optimum. Unpredictable shocks deviate each agent from his optimal action, depending on each individual shock variance. When these variances are null, standard optimization results are recovered. This blurred behavior can be justified by the inherent complexity of agents: each period, their goals and behaviors can be modified by some internal, unobservable and individual shocks.

This setup is a path integral formalism in the abstract space of agents’ economic variables, i.e., the state space. It is actually very similar to the statistical physics or quantum mechanics systems. This description is a good approximation of standard descriptions and allows to solve otherwise intractable problems. Compared to standard optimization techniques, such a description markedly eases the treatment of systems with a small number of agents. Working with a probability distribution is often easier than solving optimization equations. This approach is thus consistent and useful in itself. It provides an alternative to the standard modeling in the case of a small number of interacting agents. The average dynamics recovered is close and at times identical to the standard approach. It also allows to study the set of agents’ dynamics and its fluctuations under some external shocks.

This formalism, useful for small sets, becomes intractable for a large number of agents. It can nonetheless be conveniently modified using methods of statistical field theory developed by Kleinert (1989), into another and more efficient description directly grounded on our initial path integral formalism. In a second step, therefore, the individual agents’ description is replaced by a model of field theory that replicates the properties of the system when \( N \), the number of agents, is large. This modeling, although approximate, is compact enough to allow an analytical treatment of the system. A double transformation is thus performed with respect to the usual optimization models. The optimization problem is first replaced by a statistical system of \( N \) agents, that is then itself replaced by a specific field theory with a large number of degrees of freedom.

This field theory does not represent an aggregation of microeconomic systems in the usual sense. It rather describes an environment of an infinite number of interacting agents, from which various phases or equilibria may be retrieved, as well as the behaviors of the agents, and the way they are influenced by, or interact, with their environment. This is the so-called phase transition of field theory: the configuration of the ground state represents an equilibrium for the whole set of agents, and shapes interactions and individual dynamics. Depending on the parameters of the system, the form of the ground state may change drastically the description at the individual level. It is thus possible to compare the particular features of the macroeconomic state of a system and those of the individual level. As such, it may confirm or invalidate some aspects of the representative agent models.

To sum up, the advantages of statistical field theories are threefold. They allow, at least approximatively, to deal analytically with systems with large degrees of freedom, without reducing them to mere aggregates. They reveal features otherwise hidden in an aggregate context. Actually, they allow switching from micro- to macroeconomic description, and vice-versa, and to interpret one scale in the light of the other. Moreover, and relevantly for economic systems, these model may exhibit phase transition. Depending on the parameters of the model, the system may experience structural
changes in behaviors, at the individual and collective scale. In that, they allow to consider the question of multiple equilibria in economics.

Section 1 reviews the literature. Section 2 presents a probabilistic formalism for a system with $N$ identical economic agents, interacting through mutual constraints. Section 3 introduces and discusses the associated field formalism for a large number of agents. In Sect. 4, we present an application of this formalism to a business cycle model. Section 5 concludes.

1 Literature review

By several aspects, our work is related to the multi-agent system economic literature, notably agent-based models (see Gaffard and Napoletano 2012) and economic networks (Jackson 2010). Both rely on numerical simulation of multi-agent system, but are often concerned with different types of model. Agent-based models deal with general macroeconomics models, whereas network models rather deal with lower-scale models, such as contract theory, behavior diffusion, information sharing or learning. In both type of settings, agents are typically defined by, and follow, various set of rules. These rules allow for equilibria and dynamics that would otherwise remain inaccessible to the representative agent setup.

The agent-based approach is similar to ours in that it does not seek to aggregate all agents, but considers the interacting system in itself. It is however highly numerical, model dependent and relies on microeconomic relations, such as ad hoc reaction functions, that may be too simplistic. On the contrary, statistical field theory accounts for the transition between scales. Macroeconomic patterns do not emerge from the sole dynamics of a large set of agents, but are grounded on particular behaviors and interactions structures. Describing these structures in terms of field theory allows the study the emergence of a particular phase at the macroeconomic scale, and in turn its impact at the individual level.

Econophysics is closer to our approach (for a review, see Abergel et al. 2011a, b and references therein; or Lux 2008, 2016). It often considers the set of agents as a statistical system. Moreover, Kleinert (2009) has already used path integrals to model the stock prices’ dynamics. However, econophysics does not fully apply the potentiality of field theory to economic systems and rather focuses on empirical laws. But the absence of micro-foundations casts some doubts on the robustness of these observed empirical laws. Our approach, in contrast, keeps track of usual microeconomics concepts such as utility functions, expectations and forward looking behaviors. It includes these behaviors in the analytical treatment of multi-agent systems by translating the main characteristics of a system of optimizing agents in terms of a statistical system.

Mean field theory formalism applied to game theory and economics is based like our formalism on a system of large number of agents but the two approaches differ in many respects.

Technically speaking, the fields we are referring to differ from those used in mean field theory. While in mean field theory, the term “fields” refer to probability distributions, in our formalism they are abstract complex functions defined on the state space, analogous to second quantized wave functions in quantum theory.
Besides, mean field theory studies the time evolution of the density of agents in the state space, the space of economic variables. This evolution is modeled by a transport equation coupled to the Hamilton–Jacobi–Bellman equations of fully rational agents (for an account of mean field game theory, see for example Bensoussan et al. 2018; Lasry et al. 2010a, b; Gomes et al. 2015 for its economic applications). Interactions between agents and the population as a whole can be included through the density of population. Thus, mean field theory is an intermediate scale between the macro- and microscale. It does not seek to aggregate agents but rather replaces them by an overall probability distribution.

Our formalism on the contrary focuses on the direct interactions between agents at the microlevel, their economic structural relations and the emergence of particular phases or states induced by these specific microeconomic relations. These phases in turn impact each agent and his stochastic dynamics at the microlevel. Besides, our formalism does not need any assumption about agents’ rationality. The randomness in agents behaviors stems from their large number.

2 A probabilistic description of economic agents in interaction

This section presents a probabilistic formalism for a system with $N$ identical economic agents in interaction. Agents are described by intertemporal utility functions, but do not optimize these utilities. Instead, each agent chooses for his action a path randomly distributed around the optimal path. The agent’s behavior can be described as a weight that is an exponential of the intertemporal utility, that concentrates the probability around the optimal path. This feature models some internal uncertainty as well as non-measurable shocks. Gathering all agents, it yields a probabilistic description of the system in terms of a probabilistic weight. This weight includes utility functions and internalizes forward-looking behaviors, such as intertemporal budget constraints and interactions among agents. These interactions may for instance arise through constraints, since income flows depend on other agents demand. The probabilistic description then allows to compute the transition functions of the system, and in turn compute the probability for a system to evolve from an initial state to a final state within a given timespan. They have the form of Euclidean path integrals.

For the sake of clarity, the description in terms of probabilistic representation is first explained discarding constraints. This modelization, however suitable for simple models, would be a limitation for most economic models in which constraints are relevant. So that these constraints will be considered in a second step. In a third step, we will show that the interactions between agents are best described in terms of mutual constraints. We end the section by discussing the transition functions associated with a system for a large number of economic agents and the transition to the field formalism.

2.1 Principles

To keep track of the agents’ main microeconomic features, several conditions must be satisfied. First, optimization equations should, at least in some basic cases and
in average, be recovered. Second, this probabilistic description should account for the agents’ individual characteristics, such as constraints, interactions and forward-looking behavior.

The probabilistic description presented here involves a probability density for the state of the system at each period. In a system composed of \( N \) agents, each defined by a vector of action \( X_i(t) \):

**Notation 1** We denote \( X(t) \) the concatenation \( (X_1(t), \ldots, X_N(t)) \) of the action vectors.

**Notation 2** We denote \( X = (X(0), \ldots, X(T)) \) the concatenation of these vectors over the entire timespan, where \( T \) is the time horizon.

We define a probability density \( P(X(t)) \) for the set of actions \( X(t) \) that describes the state of the system at time \( t \). Consider first the intertemporal utility of an agent \( i \):

\[
U_{c}^{(i)}(t) = \sum_{n \geq 0} \beta^n u_{t+n}^{(i)}(X_i(t+n), X(t+n-1))
\]

where \( u_{t+n}^{(i)} \) is the instantaneous utility at time \( t + n \). The superscript “c” in \( U_{c}^{(i)}(t) \) stands for the “classical” intertemporal utility. In the optimization setup, \( X_i(t+n) \) is the agent \( i \) control variable, the variables \( (X_j(t+n-1))_{j \neq i} \) are the actions of the other agents, and \( (X_j(t+n-1)) \) are the actions of the set of all agents.

The above utility can encompass any quantity optimized, such as the production or utility functions of consumer/producer models. It may also describe the interaction of several substructures within an individual agent (see Gosselin et al. 2017, and previous formulations in Lotz 2011; Gosselin and Lotz 2012; Gosselin et al. 2013, 2015), or the motion mechanisms, such as decision and control, in the Neurosciences literature.

Rather than optimizing \( U_{t}^{(i)} \) on \( X_i(t) \), we postulate that agent \( i \) will choose an action \( X_i(t) \) and a plan updated every period \( X_i(t+n), n > 0 \), for its future actions. However, unlike the optimization setup, this plan follows a conditional probabilistic law proportional to:

\[
\exp(U_{t}^{(i)}) = \exp \left( \sum_{n \geq 0} \frac{\beta^n}{\sigma_{t+n}^2} u_{t+n}^{(i)}(X_i(t+n), X(t+n-1)) \right).
\]

This probabilistic law for \( X_i(t) \) and the plan \( (X_i(t+n))_{n \geq 0} \) is conditional to the economic variables \( X(t+n-1) \), perceived as exogenous by agent \( i \). The uncertainty at time \( t \) about agent \( i \) behavior at time \( t+n \), i.e., the variability of the agents’ actions, is measured by the factor \( \sigma_{t+n}^2 \). Usually, one can expect \( \sigma_{t+n}^2 \) to increase with \( n \). We will assume that \( \sigma_{t+n}^2 \) depends on \( n \) only, where \( n \) is the forecasting horizon. We thus write \( \sigma_{t+n}^2 \equiv \sigma_{i,n}^2 \), and define \( \sigma_{i,0}^2 \).
Due to the time dependency of the uncertainty, the function $U^{(i)}_t$ differs slightly from $U^{c(i)}_t$. When the uncertainty is constant, the functions $U^{(i)}_t$ and $U^{c(i)}_t$ are proportional.

Remark that, for a usual convex utility with a maximum, the closest the choices of the $X_i (t + n)$ to $U^{(i)}_t$ optimum, the higher the probability associated with $X_i (t + n)$. When $\sigma^2_{i,t} (t + n) \to 0$, the agent’s action is close to his optimum. Our choice of utility is therefore coherent with a probability peaked around the optimization optimum. It is different from the usual description in terms of optimal path of actions, but encompasses this approach in average.

The random description (1) is justified for several reasons. First, it is a measure of agents’ ignorance about other agents’ actions or about the environment modifications. As a consequence, each agent’s action deviates randomly from its optimum. Second, even optimizing agents may be subject to some indeterminacy in behaviors and goals. Third, in many applications, we consider that agents have the same utility function or can be divided in groups characterized by their utilities. For a large number of heterogenous agents however, goals or personal determinants are not identical. The weight translates these differences by assuming each agent departs randomly from an average utility function. The weight represents and aggregates these various uncertainties.

The meaning of (1) can be illustrated through the particular case $\sigma^2_{i,n} = \sigma^2$ and $u^{(i)}_t = u_t \forall i$. Then, for $\sigma^2 = 1$, we face a mild uncertainty: agents share some but not all information and goals. When $\sigma^2 \to \infty$, weight (1) becomes a non-normalized uniform distribution. It describes a fully heterogenous set of fully uninformed agents who largely differ in goals and behave quite randomly due to the lack of information. On the contrary when $\sigma^2 \to 0$, agents are well informed and very homogenous, and consequently rarely depart from the optimal behavior defined by the average utility function. This limit $\sigma^2 \to 0$ corresponds to the particular case of a representative agent optimizing his utility.

The setup developed in Gosselin et al. (2017) allows for strategic behaviors and difference of information between agents, but the probabilistic description is simplified for non-strategic agents with no information about others. This is the hypothesis chosen in this paper. In such a setting, agent $i$ considers the variables $X_j (t + n)$ as random noises and integrates them out. The probability for $X_i (t)$ and $X_i (t + n), n > 0$ will thus be proportional to:

$$\int \exp \left( U^{(i)}_t \right) \prod_{j \neq i} \prod_{s \geq t} \exp \left( - \frac{X^2_j (s)}{2\sigma^2_{j,i,t} (s)} \right) dX_j (s)$$

Here, $\exp \left( - \frac{X^2_j (s)}{2\sigma^2_{j,i,t} (s)} \right)$ is a subjective Gaussian weight of variance $\sigma^2_{j,i,t} (s)$ attributed by agent $i$ at time $t$ to $X_j (s)$. In general, if no information is available to agent $i$, we can assume that $\sigma^2_{j,i,t} (s) \to \infty$ and $\exp \left( - \frac{X^2_j (s)}{2\sigma^2_{j,i,t} (s)} \right) \to \delta \left( X_j (s) \right)$, where $\delta \left( X_j (s) \right)$ is the Dirac delta function, i.e., a function that is peaked on 0 , and null everywhere else.
Thus, as long as no further information is available, other agents may be considered as random perturbations: agent \( i \) set their future actions to 0 and discard them.

When there are no constraints and no inertia in \( u_i(t) \), i.e., when \( u_i(t) \) solely depends on \( X_i(t) \) and on other agents’ previous actions \( \{X_j(t-1)\}_{j \neq i} \), the time periods are independent. Actually, action \( X_i(s) \) at times \( s > t \) are independent of \( X_i(t) \). Consequently, \( \exp \left( \frac{\beta_n}{\sigma^2_{u_i(n)}} u_i(t+n) \right) \). The \( X_i(s) \) for \( s > t \) can be integrated out, and the probability associated with the action \( X_i(t) \) given the past variables becomes:

\[
\int \left( \int \exp \left( U_i(t) \right) \prod_{j \neq i} \exp \left( -\frac{X_j^2(s)}{2\sigma^2_{j,i,t}(s)} \right) dX_j(s) \right) \prod_{s > t} dX_i(s)
\]

\[\propto \exp \left( \frac{u_i(X_i(t), X(t-1))}{\sigma^2_i} \right) \]

or in terms of conditional probabilities:

\[
P(X_i(t) \mid (X(t-1))) = \frac{\exp \left( \frac{u_i(X_i(t), X(t-1))}{\sigma^2_i} \right)}{\mathcal{N}_i}
\]  \( (2) \)

where the normalization factor is defined by:

\[
\mathcal{N}_i = \int \exp \left( \frac{u_i(X_i(t), X(t-1))}{\sigma^2_i} \right) dX_i(t).
\]  \( (3) \)

In the following, the normalization factor will only be reintroduced if needed. Formula (2) shows that each agent is described by his instantaneous utility. The lack of information induces a short-sighted behavior: in the absence of any period overlap, i.e., without any constraint, the behavior of agent \( i \) is described by a random distribution peaked around the optimum of \( u_i(X_i(t), X(t-1)) \) which models exactly the optimal behavior of an agent influenced by individual random shocks.

As a consequence, gathering all \( N \) agents, the full system \( X(t) \) is described by a probability weight at each time \( t \):

\[
\prod_{i} \exp \left( \frac{u_i(X_i(t), X(t-1))}{\sigma^2_i} \right)
\]

for any \( (X_k(t-1)) \). Assuming that \( \sigma^2_i = \sigma^2 \forall i \), a particular path for the whole system \( X \) is defined by the probability, up to the normalization factors:

\[
P(X) = \prod_{i} \prod_{i} \exp \left( \frac{u_i(X_i(t), X(t-1))}{\sigma^2} \right).
\]  \( (4) \)
2.2 Introducing constraints

To introduce constraints, either in an exact way for simple cases or as first approximation in the general case, we must distinguish two types of constraints.

2.2.1 Instantaneous constraints

We define an instantaneous constraint as a dynamic identity between the control variables of the system. A standard example is the capital accumulation dynamics for a single producer/consumer:

\[ K_i (t + 1) - (1 - \delta) K_i (t) = Y_i (t) - C_i (t) + \epsilon_i (t) \]  

(5)

where \( C_i (t) \) is the consumption, \( \delta \) the depreciation rate, \( \epsilon_i (t) \) a Gaussian random shock centered around 0 and of variance \( \eta^2 \), and \( Y_i = F (K_i) \) the income. The function \( F \) may depend on other variables, such as technology, that themselves depend on the environment provided by other agents. We can generalize Eq. (5) for any constrained subset \( X_i^{(2)} (t) \) of the vector \( X_i (t) \) of economic variables:

\[ X_i^{(2)} (t + 1) - X_i^{(2)} (t) + H (X_i (t), X (t - 1)) = \epsilon_i (t) \]  

(6)

for some vector-valued function \( H \) and noise \( \epsilon_i (t) \) of the same dimension as \( X_i^{(2)} (t) \). The variance of \( \epsilon_i (t) \)'s components is assumed to be constant and equal to \( \eta^2 \). The inclusion of this constraint in our probabilistic description is straightforward. If we assume that \( \epsilon_i (t) \) is independent from any of the variables, the density of probability for the system (2) is modified by the adjunction of a Gaussian term:

\[
P (X_i (t) | (X (t - 1))) = \exp \left( \frac{u_i^{(i)} (X_i (t), X (t - 1))}{\sigma^2} \right) \exp \left( -\frac{\left( X_i^{(2)} (t + 1) - X_i^{(2)} (t) + H (X_i (t), X (t - 1)) \right)^2}{2 \eta^2} \right). \]  

(7)

Summing over agents and periods yields a statistical weight for a path \( X \) of the system:

\[
P (X) = \exp \left( \sum_{i,t} \frac{u_i^{(i)} (X_i (t), X (t - 1))}{\sigma^2} \right) \exp \left( -\sum_{i,t} \frac{\left( X_i^{(2)} (t + 1) - X_i^{(2)} (t) + H (X_i (t), X (t - 1)) \right)^2}{2 \eta^2} \right). \]  

(8)
In continuous time, an integral replaces the sum over \( t \) and formula (18) becomes:

\[
P(X) = \exp \left( \sum_i \int \frac{u^{(i)}_t(X_i(t), X(t-1))}{\sigma^2} \, dt \right) \\
\exp \left( -\frac{1}{2\eta^2} \sum_i \int_{0}^{T} \left( \frac{d}{dt} X^{(2)}_i + H(X_i(t)) \right)^2 \, dt \right) .
\]

(9)

#### 2.2.2 Intertemporal constraints

We will first consider an economic agent optimizing a quadratic utility under some budget constraint. We will then extend the result to \( N \) agents with quadratic utilities under linear arbitrary constraints. Finally, we will consider the general case of arbitrary utility.

Consider the quadratic utility of an agent whose action vector \( X_i(t) \) is his sole consumption. His utility reduces to:

\[
u^{(i)}_t(X_i(t), X(t-1)) = u(C_i(t)) .
\]

(10)

His current account intertemporal constraint is of the form:

\[
C_i(t) = B_i(t) + Y_i(t) - B_i(t+1)
\]

(11)

where \( Y_i(t) \) is first considered as an exogenous random variable, such as income in standard optimization models. The state variable \( B_i(t) \) represents the usual Treasury Bond. Both the interest rate \( r \) and the discount factor \( \beta \) are discarded here for the sake of simplicity (see Gosselin et al. 2017). We do this explicitly for the interest rate in our example in Sect. 4.

If we were to keep the state variable \( B_i(t) \) in our description, we could consider (11) as an instantaneous constraint similar to (5). However, we will rather replace the state variable \( B_i(t) \) and describe the system in terms of the usual control variable \( C_i(t) \). This is in line with the usual models where an intertemporal constraint such as:

\[
\sum_{t \geq 0} Y_i(t) - \sum_{t \geq 0} C_i(t) = 0
\]

is combined with the usual consumption smoothing induced by the Euler equation. In our formalism, both these elements will appear in a probabilistic form.

Successive periods are interconnected through the constraint. We assume that \( \sigma_i^2 = \sigma^2 \) and \( \sigma_i^2(n) = \sigma^2(n) \), \( \forall i, \forall n > 0 \). When \( C_i(t) \) is replaced by the state variable \( B_i(t) \), the intertemporal probability weight (1) becomes:
This measures the probability for a choice $C_i (t)$ and $C_i (t + n), n = 1 \ldots T$. Alternatively, it is the probability for the state variable $B_i$ to follow a path $B_i (t + n), n \geq 0$ starting from $B (t)$. The time horizon $T$ represents the expected remaining duration at time $t$ of the interaction process. It should depend decreasingly on $t$, but, for the sake of simplicity, it is assumed to follow a random exponential process. As a consequence, the mean expected duration will be a constant written $T$, irrespective of $t$. Integrating over the $B_i (t + n)$ with $n \geq 2$ yields a transition probability between $B_i (t)$ and $B_i (t + 1)$ written $P (B_i (t), B_i (t + 1))$, the probability to reach $B_i (t + 1)$ given $B_i (t)$. It is equal to:

$$
P (B_i (t), B_i (t + 1)) = \int \prod_{i=2}^{T} dB_{i+i} \exp \left( \frac{u (B_i (t) + Y_i (t) - B_i (t + 1))}{\sigma^2} + \sum_{n>0} \frac{u (B_i (t + n) + Y_i (t + n) - B_i (t + n + 1))}{\sigma^2 (n)} \right).
$$

(12)

Computing $P (B_i (t), B_i (t + 1))$ rather than the transition function for $C_i (t)$ does not change our approach. It merely requires to be applied to the state variable $B_i (t)$ rather than to the control variable $C_i (t)$. In this case, due to the overlapping nature of state variables, the probability transition $P (B_i (t), B_i (t + 1))$ now measures a probability involving two successive periods, so that the probability for the path $C_i (t + n), n \geq 0$ has to be rebuilt from the data $P (B_i (t), B_i (t + 1))$.

Consider a quadratic utility function of the form $u (C_i (t)) = -\alpha (C_i (t) - \bar{C})^2$ with objective $\bar{C}$ or, should it be non-quadratic, its second-order approximation. Rescale it for the sake of simplicity as $-\alpha (C_i (t) - \bar{C})^2 \to -\frac{(C_i (t) - \bar{C})^2}{2}$. The transition probability between two consecutive state variables thus becomes:

$$
P (B_i (t), B_i (t + 1)) = \int \prod_{n=2}^{T} dB_i (t + n)
\exp \left( -\frac{(C_i (t) - \bar{C})^2}{2\sigma^2} - \sum_{n>0} \frac{(C_i (t + n) - \bar{C})^2}{2\sigma^2 (n)} \right).
$$

(13)
The successive integrals are performed using the budget constraint (11). We find:

\[
P(B_i (t), B_i (t + 1)) = \exp \left( -\frac{(B_i (t) + Y_i (t) - B_i (t + 1) - \tilde{C})^2}{2\sigma^2} - \frac{(B_i (t + 1) + \sum_{n>0} (Y_i (t + n) - \tilde{C}))^2}{2\sum_{n=1}^T \sigma^2 (n)} \right)
\]

(14)

where the transversality condition \(B_i (t) \to 0\) as \(t \to T\) has been imposed. Recall that the number of periods \(T\) is the expected mean process duration. Online Appendix 1 shows how to recover the transition probability for \(C_i (t)\) by integrating over \((Y_i (t + n))_{n>0}, B_i (t)\) and \(B_i (t + 1)\). The integration over the variables is conditional to agent’s information set at time \(t\). The computation is performed under the condition that the income \(Y_i (t + n)\) is centered around \(\tilde{Y}\) with variance \(\theta_2^2 (t + n)\) for all agents. We will assume that both \(\tilde{Y}\) and \(\theta_2^2 (t + n)\) are unknown. We write \(\tilde{Y}_i (t)\) and \(\theta_2^2 (t, n)\) the time \(t\) estimation of these variables, and assume them to be identical for all agents, so that \(\tilde{Y}_i (t) = \tilde{Y} (t)\) and \(\theta_2^2 (t, n) = \theta_2^2 (n) \forall i\).

Online Appendix 1 shows that the statistical weight for the consumption path \(C_i\) becomes in first approximation for \(T\) large:

\[
P(\hat{C}_i (t)) = \exp \left( -\sum_{t=1}^T \frac{(C_i (t) - C_i (t + 1))^2}{2\sigma^2} - \frac{(\sum_{t=1}^T Y_i (t) - \sum_{t=1}^T C_i (t))^2}{2\theta^2} \right)
\]

(15)

where \(\sigma_e^2\) is an effective variance depending on \(\sigma^2, \sigma^2 (n), n > 0\) and \(\theta_2^2 (n)\). The variance \(\theta^2\) accounts for unexpected random deviations around the budget constraint. Equation (15) modifies (4) when agents are facing some constraints. Online Appendix 3 extends this formula for a budget constraint (11) with an exogenous interest rate. Moreover, we can generalize (15) to any system with a constraint similar to (11), by coming back to the notation \(X_i (t)\) for general economic variables. We assume that a subset \(X_i^{(1)} (t)\) of \(X_i (t)\) is constrained intertemporally and that the utility \(u_i^{(i)} (X_i (t), X (t - 1))\) decomposes as:

\[
u_i^{(i)} (X_i (t), X (t - 1)) = u_i^{(i,0)} \left( X_i \setminus X_i^{(1)} (t), X (t - 1) \right) + u_i^{(i,1)} \left( X_i^{(1)} (t), X (t - 1) \right)
\]

where \(X_i \setminus X_i^{(1)}\) denotes the complementary variables of \(X_i^{(1)}\) in \(X_i\). This last hypothesis merely simplifies the exposition.

Assuming the individual utilities \(u_i^{(i,1)} \left( X_i^{(1)} (t), X^{(1)} (t - 1) \right)\) have a quadratic approximation around some reference value \(X_0\):
where $A$ is the matrix of second derivatives of the utility $u_t^{(i)}$. We consider an intertemporal constraint of the form:

$$\sum_{0 \leq t \leq T} \hat{Y}_i(t) = \sum_{0 \leq t \leq T} X_i^{(1)}(t)$$

(16)

for some exogenous vector-valued flow variable $\hat{Y}_i(t)$ whose dimension is the same as that of $X_i^{(1)}(t)$. Redefining the variables $X_i^{(1)}(t)$ when necessary, we can set: $A = -Id/2$, where $Id$ is the identity matrix. Using (4) for the unconstrained variables $(X_i \setminus X_i^{(1)}) (t)$, the generalization of (15) for an individual path $X_i$ is:

$$P(X_i) = \exp \left( \sum_{0 \leq t \leq T} \frac{u_t^{(i,0)} \left( (X_i \setminus X_i^{(1)}) (t), X (t - 1) \right)}{\sigma^2} \right) \times \exp \left( - \sum_{0 \leq t \leq T} \frac{\left( X_i^{(1)} (t) - X_i^{(1)} (t + 1) \right)^2}{2\sigma_e^2} \right) \left( \sum_{0 \leq t \leq T} \hat{Y}_i(t) - \sum_{0 \leq t \leq T} X_i^{(1)} (t) \right)^2 \right) \right) . \tag{17}
$$

Summing over the agents inside the exponential yields the global weight for a path $X$ of the system with an intertemporal constraint:

$$P(X) = \exp \left( \sum_{i, 0 \leq t \leq T} \frac{u_t^{(i,0)} \left( (X_i \setminus X_i^{(1)}) (t), X (t - 1) \right)}{\sigma^2} \right) \times \exp \left( - \sum_{i, 0 \leq t \leq T} \frac{\left( X_i^{(1)} (t) - X_i^{(1)} (t + 1) \right)^2}{2\sigma_e^2} \right) \left( \sum_{0 \leq t \leq T} \hat{Y}_i(t) - \sum_{0 \leq t \leq T} X_i^{(1)} (t) \right)^2 \right) . \tag{18}
$$

If we consider a continuous time, an integral replaces the sum over $t$ and formula (18) becomes:
\[
\begin{align*}
P(X) &= \exp\left( \sum_i \int_0^T \frac{u_i^{(i,0)}(X_i \setminus X_i^{(1)}(t), X(t-1))}{\sigma_2^2} dt \right) \\
& \quad \times \exp\left( -\sum_i \left( \frac{1}{2\sigma_2^2} \int_0^T \left( \frac{d}{dt} X_i^{(1)}(t) \right)^2 dt \right.ight. \\
& \quad \left. \left. + \left( \int_0^T dt \hat{Y}_i(t) - \int_0^T dt X_i^{(1)}(t) \right)^2 \right) \right). 
\end{align*}
\]

Equations (18) and (19) describe the statistical weight associated with a path for a system with quadratic utility and intertemporal constraint. Gosselin et al. (2017) shows that non-quadratic corrections to the utility can be included, if needed, by adding a term \(\hat{V}_1(X_i(t))\) to the weights (18) and (19). This term may be written as a series expansion around the minimum \(X_0\). It is analog to a potential term in physics, i.e., a term without time derivatives (see online Appendix 2 for more details).

When utility functions \(u_i^{(i,0)}\) are independent from \(i\), i.e., agents share an identical average utility function \(u_i^{(0)}\), we can define:

\[
V_1(X_i(t)) = \hat{V}_1(X_i(t)) + \frac{u_i^{(0)}((X_i \setminus X_i^{(1)}(t), X(t-1))}{\sigma_2^2}.
\]

In continuous time, this yields for \(P(X)\):

\[
\begin{align*}
P(X) &= \exp\left( -\sum_i \left( \int_0^T \left( \frac{d}{dt} X_i^{(1)}(t) \right)^2 \frac{2}{2\sigma_2^2} + V_1(X_i(t)) \right) dt \right. \\
& \quad \left. \left. + \left( \int_0^T dt \hat{Y}_i(t) - \int_0^T dt X_i^{(1)}(t) \right)^2 \right) \right). 
\end{align*}
\]

Ultimately, we can directly generalize (20) when the income \(\hat{Y}_i(t)\) is itself a function of the variables of the system. This will be the case in Sect. 4, when considering a business cycle model. Assuming the form \(\hat{Y}_i(t) = F(X_i(t))\) yields:

\[
\begin{align*}
P(X) &= \exp\left( -\sum_i \left( \int_0^T \left( \frac{d}{dt} X_i^{(1)}(t) \right)^2 \frac{2}{2\sigma_2^2} + V_1(X_i(t)) \right) dt \right. \\
& \quad \left. \left. + \left( \int_0^T dt F(X_i(t)) - \int_0^T dt X_i^{(1)}(t) \right)^2 \right) \right). 
\end{align*}
\]
2.2.3 Interdependent constraints

The above computations were performed assuming that the constraint included some exogenous, i.e., totally independent from other agents, variable \( Y_i (t) \). However, for a system of \( N \) agents, constraints are more likely imposed on agents by the entire set of interacting agents. For example, the variable \( Y_i (t) \) in the constraint (11) represented the agent’s income. In a context of \( N \) interacting agents, this variable depends on others’ activity. In our simple model (10), it is on their consumption. In a system of consumer/producer, the others’ consumption generates the flow of income \( Y_i (t) \). In other word, agent \( i \) income \( Y_i (t) \) depends on other agents’ consumptions \( C_j (t) \) - or possibly \( C_j (t - 1) \) if we assume a lag between agents actions and their effect.

More generally, for a system with a large number of agents, the income \( Y_i (t) \) may depend on endogenous variables that can still be considered as exogenous in agent \( i \)’s perspective. Thus, our benchmark hypothesis in this section will be that agents are too numerous to be manipulated by a single agent.

The previous procedures developed for the constraint of a single agent remain valid and can be generalized directly. Again, we impose a constraint of the form (16) in continuous time:

\[
\int_0^T Y_i (t) \, dt = \int_0^T X_i^{(1)} (t) \, dt
\]

for each agent. When the individual agent considers \( Y_i (t) \) as exogenous, (19) applies. But, if \( Y_i (t) \) depends endogenously on other agents, (19) must be modified accordingly. Assume for example that \( Y_i (t) = \sum_j \alpha_j X_j^{(1)} (t) \) for the \( i \)th agent. The coefficients \( \alpha^j_i \) are of order \( \frac{1}{N} \), where \( N \) is the number of agents. The last term in (19) for the \( i \)th agent can be replaced by:

\[
\left( \int \alpha_i \, dY_i (t) - \int \alpha_i \, dX_i^{(1)} (t) \right)^2 = \frac{1}{2\theta^2} \int \int X_i^{(1)} (s) X_i^{(1)} (t) \, ds \, dt + \frac{1}{2\theta^2} \sum_{i, j} \int \int V_2 \left( X_i^{(1)} (s), X_j^{(1)} (t) \right) \, ds \, dt
\]

for some constant \( \nu \) depending on the system, and with

\[
V_2 \left( X_i^{(1)} (s), X_j^{(1)} (t) \right) = \left( \sum_k \alpha_j^k \alpha_i^k - \left( \alpha_j^i + \alpha_i^j \right) \right) X_i^{(1)} (s) X_j^{(1)} (t) .
\]

The term:

\[
\frac{1}{2\theta^2} \int \int X_i^{(1)} (s) X_i^{(1)} (t) \, ds \, dt
\]

depends only on the individual agent \( i \). It is irrelevant in modeling the interactions between agents. Moreover, Gosselin et al. (2017) shows that it can often be approximated by a term proportional to \( \int \left( X_i^{(1)} (t) \right)^2 \, dt \). It can thus be included in the
contribution \( V_1 (X_i (t)) \) of (20). As a result, Eq. (23) transcribes the constraints in some non-local interactions between agents. Each agent’s constraint is shaped by the environment other agents create. Equation (23) also accounts, when necessary, for some nonlinear constraints \( V \left( X_i^{(1)} (s) , X_j^{(1)} (t) \right) \), where \( V \) is an arbitrary function.

This discussion can be generalized straightforwardly to constraints involving up to \( k \) agents. In that case, any interdependent intertemporal constraint or any interaction between \( k \) agents is modeled by:

\[
\sum_{k \geq 2} \sum_{i_1, \ldots, i_k} \int_{0}^{T} \cdots \int_{0}^{T} \frac{V_k \left( X_{i_1} (s_1), \ldots, X_{i_k} (s_1) \right)}{2 \theta^2} ds_1 \ldots ds_k. \tag{25}
\]

The functions \( V_k \) depend on the particular interactions to model and are similar to an interaction potential in physics. Usually, as in (23), these terms can be expanded in products involving subsets of agents:

\[
\sum_{k \geq 2} c_k \sum_{i_1, \ldots, i_k} \int_{0}^{T} \cdots \int_{0}^{T} \frac{X_{i_1} (s_1) \ldots X_{i_k} (s_1)}{2 \theta^2} ds_1 \ldots ds_k
\]

with constant coefficients \( c_k \). Section 4 will present an example of an interaction term involving technology and capital [Eq. (49)].

### 2.2.4 Full statistical weight

The global statistical weight of the set of agents in the continuous time version includes both instantaneous constraints (9), individual intertemporal constraints (20) and intertemporal constraints (25). More precisely, consider that \( X_i (t) \) can be divided into three subsets of variables:

\[
X_i (t) = \left( \left( X_i \setminus \left( X_i^{(1)} \cup X_i^{(2)} \right) \right) (t) , X_i^{(1)} (t) , X_i^{(2)} (t) \right). \tag{26}
\]

The intertemporal constraints only involve \( X_i^{(1)} (t) \), whereas instantaneous constraints involve \( X_i^{(2)} (t) \). The global statistical weight is thus:

\[
P (X) = \exp \left( -A_1 - A_2 \right) \tag{27}
\]

with

\[
A_1 = \sum_{i} \int_{0}^{T} \left( \frac{\left( \frac{d}{dt} X_i^{(1)} (t) \right)^2}{2 \sigma_i^2} + V_1 (X_i (t)) \right) dt
+ \frac{1}{2 \eta^2} \sum_{i} \int_{0}^{T} \left( \frac{d}{dt} X_i^{(2)} (t) + H (X_i (t)) \right)^2 dt
\]
$$A_2 = \sum_{k \geq 2} \sum_{i_1, \ldots, i_k} \int_0^T \int_0^T V_k \left( X_{i_1} (s_1), \ldots, X_{i_k} (s_1) \right) \frac{d s_1 \ldots d s_k}{\theta^2}.$$ 

The contribution $A_1$ is the part of the statistical weight that considers agents but excludes their interactions. It includes agents’ utilities, along with their individual intertemporal constraints and their instantaneous constraints. The intertemporal constraint is given by (20). Assuming that $\hat{Y}_i (t)$ is a function of other agents’ decisions, the term:

$$\left( \int_0^T ds \hat{Y}_i (s) - \int_0^T ds X_i^{(1)} (s) \right)^2$$

is composed of two parts, as in (22). The first part is (24) that can be included in $V_1 \left( X_s^{(i)} \right)$. The second part is (23), an interaction term that can be included in the contribution $A_2$. Actually, the contribution $A_2$ accounts for any type of interaction between agents through a potential that depends on an arbitrary number of agents.

Finally, a slight generalization of (27) will later prove useful. Assuming the $N$ agents have different lifespan $T_1, \ldots, T_1$, we define $P_{T_1, \ldots, T_1} (X)$ the probability for a path with variable individual lifespan by:

$$P_{T_1, \ldots, T_1} (X) = \exp \left( - A'_1 - A'_2 \right) \tag{28}$$

with

$$A'_1 = \sum_i \int_0^{T_i} \left( \frac{d}{dt} X_i^{(1)} (t) \right)^2 + V_1 \left( X_i (t) \right) \right) dt$$

$$+ \frac{1}{2 \eta^2} \sum_i \int_0^{T_i} \left( \frac{d}{dt} X_i^{(2)} (t) + H \left( X_i (t) \right) \right)^2 dt$$

$$A'_2 = \sum_{k \geq 2} \sum_{i_1, \ldots, i_k} \int_0^{T_i} \int_0^{T_j} V_k \left( X_{i_1} (s_1), \ldots, X_{i_k} (s_1) \right) \frac{d s_1 \ldots d s_k}{\theta^2}.$$ 

### 2.3 Probability transition functions

Our formalism replaces the optimization problem by a probabilistic approach that allows to compute the probability transition functions (or transition functions) for the system between an initial and a final state. We first define $P \left( X, \bar{X} \right)$ as the set of paths $X$ such that $X(0) = X$ and $X(T) = \bar{X}$. In formula (27), $P \left( X \right)$ is the probability density for a given path $X$. We then define the probability of transition between an initial state $X$ and a final state $\bar{X}$ of the system, as a sum of (27) over all paths. This probability is computed as a multiple integral:

$$P_T \left( X, \bar{X} \right) = \int_{P \left( X, \bar{X} \right)} P \left( X \right) \prod_i \prod_t dX_i \left( t \right). \tag{29}$$
The integrand \( \int P(T, X) \prod_i \prod_t dX_i(t) \) can be understood as the sum over the paths \( X(t) \) between \( X = X(0) \) and \( X = X(T) \). A compact notation for this path integral is \( \prod_i \prod_t dX_i(t) = \prod_t D X_i \) (see Peskin and Schroeder (1995) or Zinn-Justin (1993) for more on path integrals). Similarly using (28) we can define 
\[
P_T(1, \ldots, N; X, X) = \int P_T(1, \ldots, N; X, X) \prod_i \prod_t dX_i(t).
\]
(30)

However, the integrals in (29) and (30) are difficult to compute, particularly when the number of agents is large. Moreover, non-local terms in (27) add another source of complexity. Techniques based on the perturbation expansion of the potential terms \( V_1(X_i(t)) \) and \( V_k(X_i(s_1), \ldots, X_i(s_k)) \) in terms of Feynman graphs exist and may be used in some case. Nevertheless, for a large number of agents, another method exists, based on statistical field theory. This formalism will consider a set of an infinite number of agents and compute the symmetrized Laplace transform of (30) for any number \( N \) of indistinguishable agents among this set, defined by:

\[
G_{\alpha}(X, X) = \sum_{\sigma_N} \int_0^T \exp(-\alpha(T_1 + \cdots + T_N)) P_{T_1, \ldots, T_N}(X, \sigma_N(X)) \ dT_1 \ldots dT_N
\]
(31)

where the sum runs over the permutations of \( N \) elements. Once (31) computed, the symmetrized version of \( P_T(X, X) \) can be recovered analytically using an inverse Laplace transform of \( G_{\alpha} \). However, this function \( G_{\alpha}(X, X) \) is of interest in itself. First, up to a normalization factor \( \alpha^N \), it computes an average transition probability for a variable lifespan between agents: the timespans \( T_i \) are assumed to follow an exponential random law with mean \( 1/\alpha \). This models a random horizon whose probability decreases quickly away from the mean.

Second, and more importantly to our field theory formalism, the function \( G_{\alpha}(X, X) \) computes a transition probability for indistinguishable agents. Actually, summing over all permutations of final points amounts to compute the transition probability between \( N \) initial points and \( N \) final point, irrespective of the attribution of these final points to particular initial states. This simplification makes sense in the context of large number of agents where we rather study the transition from one set to another, as a whole.

3 A field theoretic formulation for interactions between large number agents

When the present formalism is applied to a large number of agents, transition functions can be computed as the so-called correlation functions of a field theory (see Kleinert 1989) whose action is directly derived from individual agents’ statistical weight defined in Sect. 2. Starting from the expression (27) defining \( P(X) \) for a system, a functional of an abstract quantity, or “field,” is built, that will keep the collective
aspects of the system and allow the computation of the individual agents’ transition functions (31) defined in Sect. 2. Field theory allows to inspect the phases of the system that encapsulate the background in which individual agents evolve. Given the parameters of the system, several phases may exist: the system may experience phase transition, switching from one type of dynamic to another.

We can now explain how to associate a field representation to (27). The idea is the following (see online Appendix 4 for a detailed presentation). For a large number of agents, the system described by (27) involves a large number of variables $X(t)$ that are hardly tractable. We consider the space $\mathcal{H}$ of square-integrable complex functions defined on the space of a single agent’s actions. The space $\mathcal{H}$ describes the collective behavior of the system. Each function $\Psi$ of $\mathcal{H}$ encodes a particular state of the system. Then, to each function $\Psi$ of $\mathcal{H}$, we associate a statistical weight, i.e., a probability describing the state encoded in $\Psi$. This probability is written $\exp(-S(\Psi))$, where $S(\Psi)$ is a functional, i.e., a function of the function $\Psi$. The form of $S(\Psi)$ is derived directly from the form of (27).

This description does not represent an aggregation to a single representative agent. It keeps the information about individual agents among the whole system, which allows to compute the transition functions for several agents among the system, and to find some collective features of the system as a whole.

The method presented here is an adaptation of tools in statistical field theory described in Kleinert (1989). It relies on building a field action, starting from the probabilistic description (27) in the following successive steps.

Replacing the economic variables by a field Rather than describing agents by a set of economic variables $X(t)$, we consider a (complex valued) function $\Psi(x)$ where the vector $x$ belongs to the same space as the $X_i(t)$. For agents described by their consumption $C_i(t)$ and a stock of individual capital $K_i(t)$, the field will be a function $\Psi(x) = \Psi(c,k)$.

This function is an abstract encoding of the distribution of consumption and capital among the whole set of agents. It is not a distribution of probability for these values. Only a functional $S(\Psi)$ of this field will give some information about this distribution.

Translating the individual part of $P(X)$ in terms of field The individual part $A_1$ of (27) is the weight depending only on individual agents, excluding their mutual interactions. Recall that:

$$A_1 = \sum_i \int_0^T \left( \frac{d}{ds} X_i^{(1)}(s) \right)^2 + V_1 \left( X_s^{(i)} \right) ds + \frac{1}{2\eta^2} \sum_i \int_0^T \left( \frac{d}{dt} X_i^{(2)}(t) + H(X_i(t)) \right)^2 dt.$$

Under some conditions on $\sigma^2$ (see Gosselin et al. 2017), we can associate to $A_1$ the functional:
\[ S_0 (\Psi) = \int \left( \Psi^\dagger (x) \left( -\frac{\sigma^2}{2} \nabla_1^2 + V_1 (x) + \alpha \right) \Psi (x) \right) \, dx \\
- \int \Psi^\dagger (x) \left( \frac{\eta^2}{2} \nabla_2^2 + H (x) \cdot \nabla_2 + \frac{1}{2} (\nabla_2 \cdot H) (x) \right) \Psi (x) \]  

(32)

where \( \alpha \) is the parameter arising in the Laplace transform described in (31) (see Sect. 2), and where \( \Psi^\dagger (x) \) denotes the complex conjugate of \( \Psi (x) \).

Some notations of (32) need to be explained. First, consider \( \nabla \), the gradient operator, a vector whose \( l \)th coordinate is the first derivative \( \frac{\partial}{\partial x_l} \): \( \nabla = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_2} \right) \) and the associated Laplacian \( \nabla^2 \):

\[ \nabla^2 = \sum_l \frac{\partial^2}{\partial x_l^2} \]

where the sum runs over the coordinates \( x_l \) of the vector \( x \). Applying this to our previous example where \( \Psi (x) = \Psi (c, k) \), we get \( \nabla^1 = \frac{\partial}{\partial c} \), \( \nabla^2 = \frac{\partial^2}{\partial c^2} \) and \( \nabla_2 = \frac{\partial}{\partial k} \), \( \nabla_2^2 = \frac{\partial^2}{\partial k^2} \).

Recall also [see (6)] that \( H (x) \) is a vector-valued function with as many components as \( x_2 \). In (32), \( H (x) \cdot \nabla_2 \) is an operator acting on \( \Psi (x) \) as:

\[ H (x) \cdot \nabla_2 \Psi (x) = \sum_l H_l (x) \frac{\partial}{\partial (x_2)_l} \Psi (x). \]

Ultimately, \( (\nabla_2 \cdot H) (x) \) is the function \( \sum_l \frac{\partial}{\partial (x_2)_l} H_l (x) \). In both expressions, the sums run over the coordinates of \( x_2 \).

**Adding the interaction terms of** \( P (X) \) The last part of \( P (X) \) specifically describes the interaction between agents. We call it \( A_2 \).

\[ A_2 = \sum_{k \geq 2} \sum_{i_1, \ldots, i_k} \int_0^T \int_0^T V_k \left( X^{(i_1)}_{s_1}, \ldots, X^{(i_k)}_{s_k} \right) \frac{1}{2 \theta^2} \, ds_1 \ldots ds_k. \]

In terms of field, it is translated into the functional:

\[ S_I (\Psi) = \frac{1}{2 \theta^2} \sum_{k \geq 2} \int \Psi (x_1) \ldots \Psi (x_k) V_k (x_1 \ldots x_k) \Psi^\dagger (x_1) \ldots \Psi^\dagger (x_k) \, dx_1 \ldots dx_k. \]
Adding $S_I (\Psi)$ to $S_0 (\Psi)$ yields the field action:

$$S (\Psi) = S_0 (\Psi) + \int (\Psi^\dagger (x) \Psi^\dagger (y) (V_2 (x, y)) \Psi (x) \Psi (y)) \, dx \, dy$$

$$= \int \left( \Psi^\dagger (x) \left( -\frac{\sigma_e^2 \nabla_1^2}{2} + V_1 (x) + \alpha \right) \Psi (x) \right) \, dx$$

$$- \int \Psi^\dagger (x) \left( \frac{\eta^2}{2} \nabla_1^2 + H (x) \cdot \nabla_2 + \frac{1}{2} (\nabla_2 \cdot H) (x) \right) \Psi (x)$$

$$+ \frac{1}{2 \theta^2} \sum_{k \geq 2} \int \Psi (x_1) \ldots \Psi (x_k) V_k (x_1 \ldots x_k) \Psi^\dagger (x_1) \ldots \Psi^\dagger (x_k) \, dx_1 \ldots dx_k. \quad (33)$$

**Adding source fields to the action** The above functional $S (\Psi)$ gathers all the necessary information. But to compute the transition functions associated with the system described by (27), it has to be supplemented with so-called source fields.

Consider a complex function $J (x)$ in $H$, and add to $S (\Psi)$ the quadratic terms:

$$\int \left( J (x) \Psi^\dagger (x) + J^\dagger (x) \Psi (x) \right) \, dx. \quad (34)$$

The system can be described by an action with source:

$$S (\Psi, J) = \int \left( \Psi^\dagger (x) \left( -\frac{\sigma_e^2 \nabla_1^2}{2} + V_1 (x) + \alpha \right) \Psi (x) \right) \, dx$$

$$- \int \Psi^\dagger (x) \left( \frac{\eta^2}{2} \nabla_1^2 + H (x) \cdot \nabla_2 + \frac{1}{2} (\nabla_2 \cdot H) (x) \right) \Psi (x)$$

$$+ \frac{1}{2 \theta^2} \sum_{k \geq 2} \sum_{i_1 \ldots i_k} \Psi (x_{i_1}) \ldots \Psi (x_{i_k}) V_k (x_{i_1} \ldots x_{i_k}) \Psi^\dagger (x_{i_1}) \ldots \Psi^\dagger (x_{i_k}) \, dx_{i_1} \ldots dx_{i_k}$$

$$+ \int \left( J (x) \Psi^\dagger (x) + J^\dagger (x) \Psi (x) \right) \, dx. \quad (34)$$

**Statistical weight for the field** Once the action $S (\Psi, J)$ defined in (34) for the field is computed, we can associate a statistical weight to this action. This statistical weight has the form:

$$\exp (-S (\Psi, J)). \quad (35)$$

This weight is the density of probability for a given configuration $(\Psi, J (x))$. This statistical weight is directly linked to our initial description of the system in terms of paths. Actually, the successive derivatives of $S (\Psi, J)$ with respect to $J (x)$ and $J^\dagger (x)$ will allow to recover the transition functions, as shown below.

**Computing the transition functions via path integrals over $\Psi (x)$** Once the weight (35) is obtained, we can compute the transition functions defined in (31). To do so, we first need to introduce two notations.
Notation 3 We denote $X^k(t)$ the vector $(X_1(t), \ldots, X_k(t))$ of $k$ action vectors. Agents being identical, any set of $k$ agents among the entire set of $N$ agents is equivalent.

Notation 4 We denote $X^k$ and $\overline{X}^k$ any initial and final conditions for this set of agents.

Using these notations, a result of statistical field theory (see Kleinert 1989) states that the transition probability $G_\alpha(X^k, \overline{X}^k)$ defined in (31) for $k$ agents between the initial state $X^k$ and the final state $\overline{X}^k$ is:

$$G_\alpha(X^k, \overline{X}^k) = \left( \frac{\delta}{\delta J(x_{i_1})} \frac{\delta}{\delta J^\dagger(y_{i_1})} \right) \cdots \left( \frac{\delta}{\delta J(x_{i_N})} \frac{\delta}{\delta J^\dagger(y_{i_N})} \right) \exp \left( - S(\Psi, J) \right)$$

$$= \left[ \frac{\delta}{\delta J(x_{i_1})} \frac{\delta}{\delta J^\dagger(y_{i_1})} \right] \cdots \left( \frac{\delta}{\delta J(x_{i_N})} \frac{\delta}{\delta J^\dagger(y_{i_N})} \right) \exp \left( - \int \left( \Psi^\dagger(x) \left( -\sigma^2 \frac{\nabla^2}{2} + V_1(x) + \alpha \right) \Psi(x) \right) dx ight.$$

$$+ \int \Psi^\dagger(x) \left( \frac{\eta^2}{2} \nabla^2 + H(x) \cdot \nabla_2 + \frac{1}{2} (\nabla_2 H)(x) \right) \Psi(x)$$

$$- \frac{1}{2 \rho^2} \sum_{k \geq 2} \sum_{i_1, \ldots, i_k} \Psi(x_{i_1}) \cdots \Psi(x_{i_k}) V_k(x_{i_1} \cdots x_{i_k}) \Psi^\dagger(x_{i_1}) \cdots \Psi^\dagger(x_{i_k})$$

$$+ J(x) \Psi^\dagger(x) + J^\dagger(x) \Psi(x) \right) \mathcal{D}\Psi \mathcal{D}\Psi^\dagger \right]_{J=J^\dagger=0}. \quad (36)$$

As before, the notation $D\Psi D\Psi^\dagger$ denotes an integration over the space of functions $\Psi(x)$ and $\Psi^\dagger(x)$, that is an integral in an infinite dimensional space. Actually, these integrals are formal and solely computed in simple cases. The form of $S(\Psi)$ is often sufficient to derive good qualitative insights about the results. In terms of field theory, formula (36) means that the transition functions are the correlation functions of the field theory with action $S(\Psi)$.

The formulation (36) shows how the agents’ transitions, i.e., their dynamical and stochastic properties, take place in a surrounding. We do not seek to compute the dynamics of the whole system, but rather derive agent’s behaviors from the global properties of a substratum, i.e., the global action for the field $\Psi(x)$.

Remark that this change in formulation is related to the introduction of a variable number $k$ of agents in (36). In Sect. 2, the system was described by a fixed number of agents. Here, the focus being on the environment, we can compute the transition functions for an arbitrary number of agents in this environment.

3.1 Non-trivial vacuum, phase transition and Green function

In practice, computing $2k$ derivatives in (36) to derive the $k$ agents transition functions is not necessary. It is generally sufficient to know the “free” transition function for one
agent $G^{(0)}_\alpha \left( \overline{X}^{[1]}, \overline{X}^{[1]} \right)$, computed by setting $V_1 = V_k = 0$ in (36). From there, the
transition functions $G_\alpha \left( \overline{X}^{[k]}, \overline{X}^{[k]} \right)$ for several agents can be deduced by techniques
such as Feynman graphs.

However, the graph expansion can sometimes be avoided in first approximation. Hints about the system’s behavior can be obtained by inspecting the minima of (33): assume that there is no instantaneous constraint, so that $\nabla_1 = \nabla$ and (33) reduces to:

$$S (\Psi) = \int \left( \Psi^\dagger (x) \left( -\sigma_e^2 \nabla_e^2 + V_1 (x) + \alpha \right) \Psi (x) \right) dx$$

$$+ \frac{1}{2\theta^2} \sum_{k \geq 2} \int \Psi (x_1) \ldots \Psi (x_k) V_k (x_1 \ldots x_k) \Psi^\dagger (x_1) \ldots \Psi^\dagger (x_k) dx_1 \ldots dx_k.$$ 

First look for a field minimizing the action $S (\Psi)$, that is a field $\Psi_0$ solution of

$$\frac{\partial}{\partial \Psi} S (\Psi) = 0.$$ 

If such a non-null solution does exist, the system is said to have a non-trivial vacuum for $S (\Psi_0)$. Then, let $\Psi = \Psi_0 + \delta \Psi$ and expand $S (\Psi)$ to the second order in $\delta \Psi$. That is:

$$S (\Psi) \simeq S (\Psi_0) + \int \delta \Psi^\dagger (x) \left( -\sigma_e^2 \nabla_e^2 + V_1 (x) + \alpha \right) \delta \Psi (x) dx$$

$$+ \int \delta \Psi^\dagger (x) V (\Psi_0, x, y) \delta \Psi (y) dx dy$$

with

$$V (\Psi_0, x) = \frac{1}{2\theta^2} \sum_{k \geq 2} \sum_{l_1, l_2} \int_{x_i, i \neq l_1} \prod_{x_i, i \neq l_1} \Psi_0 (x_i) \right] V_k (x_{l_1} \ldots x_{l_2})$$

$$\left[ \prod_{x_i, i \neq l_2} \Psi_0^\dagger (x_i) \right] \delta (x - x_{l_1}) \delta (y - x_{l_2}) dx_1 \ldots dx_k$$

and $\delta (x - x_{l_1})$ is the Dirac function. It is then a classical computation to show that in first approximation the one agent transition function is determined by the quadratic part in $\delta \Psi$ and satisfies the differential equation:

$$\left( -\sigma_e^2 \nabla_e^2 + \alpha + V_1 (x) + V (\Psi_0, x, y) \right) G_\alpha (x, y) = \delta (x - y)$$

in that case, the transition function describing the system can be computed at least approximatively numerically. A consequence of this setup is the notion of phase transition. For some values of the parameters, the only vacuum of the theory may be $\Psi_0 = 0$. In that case, $V (\Psi_0, x, y) = 0$, so that the transition function is in first approximation given by the solution of:

$$\left( -\sigma_e^2 \nabla^2 + \alpha + V_1 (x) \right) G_\alpha (x, y) = \delta (x - y).$$
On the other hand, if for another range of parameters $\Psi_0 \neq 0$, then the transition function is computed by (38) and we say that the system experiences a phase transition. The qualitative properties of the system in the phase $\Psi_0 \neq 0$ differ from those in the phase $\Psi_0 = 0$. Probabilities of transition and average values of quantities may differ from one phase to another.

A simple and standard example is the following. Assume a system described by the following action for a real field (this simplification does not impair the argument) depending on a one-dimensional variable $x$:

$$S(\Psi) = \int \left( -\Psi(x) \frac{\nabla^2}{2} \Psi(x) + g(\Psi(x))^2 + (\Psi(x))^4 + \alpha(\Psi(x))^2 \right) dx.$$ (40)

The parameter $g$ describes some microeconomic properties of the system of agents. For $g + \alpha > 0$, the functional (40) is positive and the minimum is $\Psi(x) = 0$. As a consequence, the second-order expansion (37) around the minimum is:

$$\int \delta \Psi(x) \left( -\nabla^2 + 2(g + \alpha) \right) \delta \Psi(x) dx.$$ 

One can show that the two points transition function for such an action is proportional to:

$$G^{(1)}_{\alpha}(x, y) \sim \exp \left( -\sqrt{2} \frac{|g + \alpha|}{2} |x - y| \right) \sqrt{2} \frac{|g + \alpha|}{|x - y|}.$$ (41)

This $G^{(1)}_{\alpha}(x, y)$ characterizes the first phase of the system.

On the other hand for $g + \alpha < 0$, the functional (40) has two minima:

$$\Psi_0(x) = \pm \sqrt{\frac{|g + \alpha|}{2}}.$$ 

In this example, the second-order expansion (37) is the same for both solutions, so that we can choose $\Psi_0(x) = \sqrt{\frac{|g + \alpha|}{2}}$. The second-order expansion (37) around $\Psi_0$ becomes:

$$\int \delta \Psi(x) \left( -\nabla^2 + 8 |g + \alpha| \right) \delta \Psi(x) dx$$

whose two points transition function is proportional to:

$$G^{(2)}_{\alpha}(x, y) \sim \exp \left( -\sqrt{8} \frac{|g + \alpha|}{8} |x - y| \right) \sqrt{8} \frac{|g + \alpha|}{|x - y|}.$$ (42)

This $G^{(2)}_{\alpha}(x, y)$ defines the second phase of the system.

Both phases can be interpreted as described by the same Gaussian transition function:

$$G(t, x, y) = \frac{\exp \left( -\frac{(x-y)^2}{2t} \right)}{\sqrt{2\pi t}}.$$
but with a different average time horizon for the interaction. Actually, we have:

\[ G^{(i)}_{\alpha}(x, y) \sim \int \frac{\exp \left( -\frac{(x-y)^2}{2t} \right)}{\sqrt{2\pi t}} \exp (-\alpha_i t) \, dt \]

with

\[
\alpha_1 = 2(g + \alpha) \\
\alpha_2 = 8|g + \alpha|.
\]

This means that the time horizon \( \frac{1}{\alpha_1} \), i.e., the average time of interaction between agents, is phase dependent: although the time horizon of agents remains \( \frac{1}{\alpha} \), the phase will determine the ease of agents’ transitions from one state to another and impact their effective time horizon.

If for example \( \alpha_2 > \alpha_1 \), \( G^{(2)}_{\alpha}(x, y) \) defined in (42) is an exponential decreasing faster in the variable \(|x - y|\) than \( G^{(1)}_{\alpha}(x, y) \) [see (41)]. As a consequence, the probability of transition between two points \( x \) and \( y \) is lower in phase 2 than in phase 1. For reasons inherent to the structure of the model and its interactions, the second phase presents some inertia, whereas the first phase presents a higher mobility in the state space.

### 4 Application: revisiting a standard business cycle model

In this section, we present an application of our formalism to a standard Business Cycle model. The usual assumptions of the standard model are maintained (see Romer 1996), but agents now interact through technology. In such a model, we show that a non-trivial vacuum may appear. For some values of the parameters, the equilibrium may experience a discontinuous shift. The different phases of the system induce different individual behaviors. In the following, we will present the model, compute the effect of the agents’ interactions on individual dynamics for each phase and provide an interpretation of the results.

#### 4.1 Description

#### 4.1.1 The model

We consider a system with a large number of identical consumer/producer agents. Each agent \( i \) consumes at time \( t \) a quantity \( C_i(t) \) has a stock of capital \( K_i(t) \) and a technology \( A_i(t) \). The saving variable \( B_i(t) \) is equal to the stock of capital \( K_i(t) \) used in the production function, as usually assumed in standard Business Cycle models.

On the consumer side, we consider a utility function of the standard form (Romer 1996; Obstfeld and Rogoff 1996):
\[ u (C_i (t)) = \frac{C_i (t)^{1-\theta} - 1}{1-\theta} \]

where the coefficient \( \theta \) measures the relative risk aversion, i.e., the inverse of the elasticity of substitution between consumption at different dates. A quadratic approximation of \( u (C_i (t)) \) can be found by an expansion around some minimal value \( \hat{C} \) for the consumption:

\[ u (C_i (t)) = \left( C_i (t) - \hat{C} \right) - \frac{\theta}{2} \left( C_i (t) - \hat{C} \right)^2. \quad (43) \]

We can thus rewrite:

\[ u (C_i (t)) = \left( C_i (t) - \hat{C} \right) - \frac{\theta}{2} \left( C_i (t) - \hat{C} \right)^2 = -\frac{\theta}{2} \left( C_i (t) - \hat{C} \right)^2 + \frac{1}{2\theta} \quad (44) \]

with \( \tilde{C} = \hat{C} + \frac{1}{\beta} \). We assume that \( \tilde{C} \gg 1 \). As usual, this constant ensures decreasing marginal utility. The utility used in the sequel will be the quadratic approximation (44).

As consumers, agents each optimize their intertemporal utility function. Written in continuous time:

\[ U (C) = \int_0^T u (C_i (t)) \, dt. \]

Since the discount factor \( \beta \) does not impair the main arguments of this section, we set \( \beta = 1 \). Under the usual budget constraint:

\[ C_i (t) = r_i (t) B_i (t) + Y_i (t) - \frac{d}{dt} B_i (t) \quad (45) \]

where \( r_i (t) \) is the \( i \)th agent or sector interest rate. In continuous time, integrating (45) over the entire period yields the overall budget constraint:

\[ \int_0^T (Y_i(t) - C_i(t)) \exp \left( -\int r_i (t) \, dt \right) \, dt = 0. \]

On the production side, assuming some uncertainty in the capital accumulation process yields a dynamic equation for capital:

\[ \dot{K}_i (t) = Y_i (t) - C_i (t) - \delta (K_i (t)) + \varepsilon (t) \quad (46) \]

where \( \varepsilon (t) \) is a random term of variance \( \nu^2 \) and \( \delta (K_i (t)) \) describes the depreciation of \( K_i (t) \).

We endogenize the production \( Y_i(t) \) and treat it as a function of capital: \( Y_i (t) = A_i (t) F_i (K_i (t)) \). From this relation, we can deduce the form of the interest rate faced by each sector:

\[ r_i (t) = A_i (t) F'_i (K_i (t)) + r_c. \]
That includes an exogenous (or minimal) interest rate $r_c$, plus some individual determinants depending on the capital depreciation, rates of return, environment and technology of each sector. That is, $r_i$ is defined by the marginal productivity in the sector plus some collective effect $r_c$. Remark that usually, $r_c = -\delta$, but we assume that other determinants allow to consider $r_c$ as an independent variable. It is always possible to set $r_c = -\delta$ if needed.

To complete the model, the dynamics of technology should be modeled. We assume that $A_i(t)$ is a stochastic process with specific features. Its dynamics includes an intrinsic part that fluctuates around a technology growth path. Besides, we assume that technology and capital stock influence each other. Part of the technology random process will thus describe technology’s interaction with capital. Since the dynamics of $A_i(t)$ is probabilistic, we will provide its precise description in the next section.

### 4.1.2 Probabilistic description

Let us now apply the method presented in Sect. 2. Three variables describe our model. The variables $K$ and $C$ are standard control variables. As such, they must be assigned a statistical weight describing their dynamics. The third variable, technology, does not qualify as a control variable. It could be treated as an exogenous parameter. However, since our formalism aims at studying interactions between variables, and exploring the consequences of these interactions, we will treat technology as a variable of the system interacting with capital, and give it a statistical weight. So that the probability describing the system can be decomposed into several statistical weights, respectively, due to consumption, capital and technology.

**Statistical weight of consumption** The first weight corresponds to the consumption behavior of agents with utility (43) under an intertemporal budget constraint (45). Online Appendix 3 shows that the exogenous interest rate in the constraint modifies the statistical weight (15) associated with consumption under constraint in:

\[
\exp \left( -\frac{1}{2\sigma^2} \sum_t C_i(t) - \bar{C} - \frac{(C_i(t+1) - \bar{C})}{(1+r)} \right)^2 + \sum_t C_0 \right) \exp \left( -\frac{\int_0^T (Y_i(t) - C_i(t)) \exp (-\int r_i(t) \, dt) \, dt)^2}{2\theta^2} \right)
\]

where $\sigma^2 = \frac{\sigma_e^2}{\delta}$ measures the uncertainty in consumption behavior among agents. For $\sigma^2 \ll 1$, the weight is peaked around the usual optimal Euler equation in continuous time. The parameter $\sigma^2$ is the uncertainty in consumption behavior used in (15). Remark that $C_0 = \frac{1}{2\sigma_e^2}$, and that the sum $\sum_t C_0 = TC_0$ measures the agents’ relative risk aversion cumulated over the entire timespan to change consumption.
Online Appendix 5 shows that the part due to the overall intertemporal constraint:

\[- \left( \int_0^T (Y_i(t) - C_i(t)) \exp \left( - \int r_i(t) \, dt \right) \, dt \right)^2 \over 2 \theta^2 \]  

(47)

can be neglected in the first approximation. In continuous time, this leaves us with

\[
\exp \left( - \int dt \left( \dot{C}_i(t) - r \left( C_i(t) - \bar{C} \right) \right)^2 \over 2 \sigma^2 + C_0 \int dt \right). 
\]  

(48)

**Statistical weight of capital** The second weight models the capital dynamics. Equation (46) shows that \( \varepsilon_i(t) = \dot{K}_i(t) - (Y_i(t) - C_i(t) - \delta(K_i(t))) \) is a Gaussian variable with variance \( \nu^2 \). The associated statistical weight is thus Gaussian and writes:

\[
\exp \left( - \int dt \left( \dot{K}_i(t) - (AF_i(K_i(t)) - C_i(t) - \delta(K_i(t))) \right)^2 \over 2 \nu^2 \right). 
\]  

(49)

Since \( \varepsilon_i(t) \) is consumption independent, this weight will be multiplied by (48).

**Statistical weight of technology** The third weight accounts for technology. Recall that this is a particular variable in our setting, in that its dynamics can be seen as intrinsic, or resulting from capital interaction. This reflects on its weight.

**Statistical weight of intrinsic technology** We first consider the contribution inherent to the technology itself. We denote \( \langle A \rangle \) the system’s average technology, to be computed later, but that is phase dependent. Let us also denote \( A_0 \) an exogenous level of technology and \( \bar{A} \) an agent “natural” technology level within the society with \( \bar{A} = \kappa \langle A \rangle + A_0 \), and \( \kappa < 1 \). Thus, \( \bar{A} \) mixes the average technology in the system and the constant level \( A_0 \). We choose the technology contribution to be of the following form:

\[
\exp \left( - \int dt \left( \frac{(\dot{A}_i(t) - gA_i(t))^2}{2\lambda^2} + (A_i(t) - \bar{A})^2 \right) \right) \]  

(50)

The first term \( \frac{(\dot{A}_i - gA_i)^2}{2\lambda^2} \) models the agent’s technology endowment as fluctuating around a technology growth path \( g \). Actually, the distribution is centered around the paths solutions of \( (\dot{A}_i(t) - gA_i(t)) = 0 \). We will simplify the sequel by setting \( g = 0 \), so that \( A_i(t) \) can be seen as a detrended variable. However, the growth factor \( g \) could be reintroduced if needed. We consider \( \lambda^2 \gg 1 \), which means that the level of technology can adapt relatively quickly to \( \bar{A} \). The second term, \( (A_i(t) - \bar{A})^2 \), is the difference between the agent’s technology and agents’ potential level of technology in the system. This insures that in the absence of any other forces, the agent should be driven toward this optimal level of technology. As a consequence, (50) models an individual technology that is driven by individual factors as well as a collective level of technology.
**Statistical weight of capital–technology interaction** In Sect. 2, we modeled interactions by adding a potential involving several agents [see (25)]. Here, we model the impact of capital on technology by introducing an additional term in (50) such as:

\[
\exp \left( -\gamma \int \int_{t_j < t_i} \sum_j A_i (t_i) H (K_i (t_i), K_j (t_j)) K_j (t_j) \, dt_j dt_i \right). \tag{51}
\]

This term describes the value added accumulated in the different sectors by capital stocks. The function \( H \) is any positive function and represents the impact of sector \( i \) on sector \( j \). We assume reciprocal interactions and assume that \( H \) is symmetric:

\[
H (K_i (t_i), K_j (t_j)) = H (K_j (t_j), K_i (t_i)).
\]

The constant \( \gamma \) measures the magnitude of these interactions. To check that (51) represents the impact of capital stock on technology, notice that the agent’s technology \( A_i (t_i) \) is multiplied by a weighted sum of other agents’ past capital stocks. This weight thus models the interaction between technology and the global capital stock.

This interaction weight would however be incomplete since the various sectors’ technology may also in turn accelerate the dynamics of \( K_i (t) \). Considering the interaction between capital and technology is reciprocal is equivalent to adding the term:

\[
\exp \left( -\gamma \int \int_{t_j < t_i} \sum_j A_i (t_i) H (K_i (t_i), K_j (t_j)) K_j (t_j) \, dt_i dt_j \right) \tag{52}
\]

Here, inverting \( t_j < t_i \) accounts for the reversal of roles: it is the past technology that impacts capital stock. Consequently, the statistical weight for technology and its interaction with capital stock is:

\[
\exp \left( -\int dt \left( \frac{(\dot{A}_i (t) - gA_i (t))^2}{2\lambda^2} + (A_i (t) - \bar{A})^2 \right) \right.
\]

\[
\left. -\gamma \int \int \sum_j A_i (t_i) H (K_i (t_i), K_j (t_j)) K_j (t_j) \, dt_j dt_i \right) \tag{53}
\]

To better understand the second term of (53), note that it can be replaced in (53) for \( |\gamma| \ll 1 \) in first approximation by the quadratic term:

\[
+ \frac{\gamma}{\sqrt{|\gamma|}} \int \sum_i \left( A_i (t_i) - \frac{\sqrt{|\gamma|}}{2} \int \sum_j H (K_i (t_i), K_j (t_j)) K_j (t_j) \, dt_j \right)^2 dt_i. \tag{54}
\]

Actually, the expansion of (54) yields the second term of (53), plus a quadratic term in \( A_i (t_i) \) and a quadratic term in \( K_j (t_j) \). The quadratic term in \( K_j (t_j) \) is of mag-
nitude $\left(\sqrt{|\gamma|}\right)^3$ and can be neglected. The quadratic term in $A_i(t_i)$ is of magnitude $\sqrt{|\gamma|}A_i^2(t_i)$ which is negligible with respect to $(A_i(t_i) - \bar{A})^2$.

Equation (54) shows that for $\gamma < 0$, the interaction is attractive: the higher the capital stock, the higher the technology. Interactions between capital and technology increase the likelihood for paths satisfying:

$$A_i(t_i) - \frac{\sqrt{|\gamma|}}{2} \int \sum_j H\left(K_i(t_i), K_j(t_j)\right) K_j(t_j) \, dt_j = 0.$$  

On the contrary, for $\gamma > 0$, the interaction is repulsive: interactions between capital and technology increase the likelihood of paths satisfying:

$$A_i(t_i) - \frac{\sqrt{|\gamma|}}{2} \int \sum_j H\left(K_i(t_i), K_j(t_j)\right) K_j(t_j) \, dt_j \to \infty.$$  

Depending on society’s stock of capital, we can define a certain threshold $\bar{A}$ of required technology:

$$\bar{A} = \frac{\sqrt{|\gamma|}}{2} \int \sum_j H\left(K_i(t_i), K_j(t_j)\right) K_j(t_j) \, dt_j.$$  

Agents with technology endowment higher than this threshold have an advantage and are thus driven on a technology growth path. Agents below this threshold $\bar{A}$ will be evicted. We will study both cases $\gamma < 0$ and $\gamma > 0$ later on.

**Overall statistical weight** We can now gather the three contributions (48) (49) and (53). The overall statistical weight writes:

$$\exp\left(-\int dt \left(\frac{(\dot{C}_i(t) + r (C_i(t) - \bar{C}))^2}{2\sigma^2}\right) + C_0 \int dt - \sum_i \int dt \left(\frac{\left(\dot{K}_i(t) - (A_i(t) F_i(K_i(t)) - C_i(t) - \delta (K_i(t)))\right)^2 + \varsigma^2 C_i^2(t)}{2\nu^2}\right)\right)$$

$$\times \exp\left(-\sum_i \int dt_i \left(\frac{(\dot{A}_i(t_i))^2}{2\lambda^2} + (A_i(t_i) - \bar{A})^2\right)\right)$$

$$- \gamma \int \sum_{i,j} A_j(t) H\left(K_j(t_j), K_i(t_i)\right) K_i(t_i) \, dt_i \, dt_j.$$  

(55)

### 4.1.3 Field theoretic description

The model has been described in terms of probabilities. We can now transcribe it in terms of field, and apply the method presented in Sect. 3. We introduce a field $\Psi(K, C, A)$ depending on the relevant variables of the system, consumption, capital and technology. Online Appendix 6.1 presents the field theoretic formulation of the
system given the above assumptions. Choosing the usual linear depreciation function \( \delta (K) = \delta K \) and a “distance function” of the form \( H (K_2, K_1) = 1 \) yields the field formulation of the system:

\[
S (\Psi) = \frac{1}{2} \int \Psi^\dagger (K, C, A) O \Psi (K, C, A) \\
+ \frac{\gamma}{2} \int \Psi^\dagger (K_1, C_1, A_1) \Psi^\dagger (K_2, C_2, A_2) \left( A_2 K_1 + A_1 K_2 \right) \Psi (K_1, C_1, A_1) \Psi (K_2, C_2, A_2) \tag{56}
\]

where the second-order differential operator \( O \) is defined, after rescaling \( 2 (\alpha - C_0) \rightarrow \alpha - C_0 \), by:

\[
O = -\omega^2 \frac{\partial^2}{\partial C^2} - \frac{1}{\lambda^2} \frac{\partial^2}{\partial A^2} - \nu^2 \frac{\partial^2}{\partial K^2} \\
+ \left( A - \bar{A} \right)^2 - 2 \left( C - AF (K) + \delta K \right) \frac{\partial}{\partial K} \\
+ 2 \left( AF' (K) + r_c \right) \frac{\partial}{\partial C} + \varsigma^2 \left( C - \bar{C} \right)^2 \\
+ 2 \left( 2 A F' (K) + r_c - \delta \right) + \alpha - C_0 .
\]

The quadratic part of the action \( S (\Psi) \), namely \( \int \Psi^\dagger (K, C, A) O \Psi (K, C, A) \), describes the individual behavior of the agents. The quartic part of the action represents the interaction between technology and capital stocks among agents.

In the sequel, the production function \( F (K) \), is assumed to be \( F (K) = AK^\varepsilon \) with \( \varepsilon < 1 \). It can be approximated, above a minimal stock of capital \( \bar{K} \), by a Taylor expansion:

\[
AK^\varepsilon \simeq A \bar{K}^\varepsilon \left( 1 + \varepsilon \left( \frac{K - \bar{K}}{\bar{K}} \right) - \frac{\varepsilon (1 - \varepsilon)}{2} \left( \frac{K - \bar{K}}{\bar{K}} \right)^2 \right)
\]

4.2 Results

We have described the model in terms of field theory. We can now search for non-trivial phases in the system, and study their properties. These emerging phases will then allow us to compute the transition functions first without, then with interactions.

4.2.1 Phases of the system

Once the field action \( S (\Psi) \) (56) is found, the minima that define the phases of the system are found in online Appendix 6.2. The stability of these minima is studied in online Appendix 6.3. The phases correspond to different economic global equilibria for the system. The trivial phase, i.e., \( \Psi_1 (K, C, A) = 0 \), is analogous to a system linearized around its classical equilibrium. On the contrary, a non-trivial phase reveals
the emergence of another equilibrium. In each of these phases, we can compute and study the variables’ average values (online Appendix 6.2.2) and the agents’ transition probability functions (online Appendix 6.4).

The results show that a non-trivial minimum \( \Psi_1 (K, C, A) \) for \( S(\Psi) \) exists, provided some conditions on the parameters are fulfilled. The minimum \( \Psi_1 (K, C, A) \) is a product of several Gaussian functions in the variables \( K, C, A \) whose precise form is not necessary to the discussion. The precise form for \( \Psi_1 (K, C, A) \) is given in online Appendix 6.2.3.

This non-trivial minimum of \( S(\Psi) \) exists when:

\[
\gamma > 0 \quad A_0 > \left( 1 + \sqrt{2} \right) \tilde{C} \quad \lambda > 1
\]  

and:

\[
1 \gg \frac{A_0 \varepsilon}{(1 - \varepsilon) \tilde{K}^{1 - \varepsilon}} - \delta > 0.
\]  

Given that \( \varepsilon < 1 \), these conditions are jointly satisfied for \( A_0 \) and \( \tilde{K} \) relatively large, and for a specific return of capital \( \frac{A_0 \varepsilon}{(1 - \varepsilon) \tilde{K}^{1 - \varepsilon}} \) exceeding the depreciation value. This specific return of capital is computed with the technology \( \frac{A_0}{(1 - \varepsilon) \tilde{K}^{1 - \varepsilon}} \), which is the average technology reached by an agent whose minimal technology \( A_0 \) has been raised to its natural level by the agent’s interaction with the society’s technology [see the definition of \( \tilde{A} \) before (50)]. Thus, a non-trivial phase only occurs when a specific level of technology has been achieved by the society. An additional condition on the relative risk aversion \( C_0 \) exists (see details in online Appendix 6.2.3). Qualitatively, for an intermediate range of values for \( C_0 \), the non-trivial phase is possible and stable. We give below some interpretation for these conditions.

The equilibrium values of the variables in both phases and the global patterns of the system are computed using quadratic expansions of \( S(\Psi) \) around the minima \( \Psi_1 (K, C, A) = 0 \) and \( \Psi_1 (K, C, A) \neq 0 \), respectively. This expansion allows in turn to find the average values for \( K, C \) and \( A \) in each phase (see online Appendix 6.2.2). For \( \Psi_1 (K, C, A) = 0 \) and neglecting the interaction terms, the average values for the relevant variables are:

\[
\langle A \rangle_0 = \tilde{A} = \frac{A_0}{(1 - \varepsilon)}
\]

\[
\langle C \rangle_0 = \tilde{C} + \sqrt{\frac{2}{\pi}} \varepsilon \sigma
\]

\[
\langle K \rangle_0 = \frac{A_0}{1 - \varepsilon} \tilde{K}^{\varepsilon} (1 - \varepsilon) - \chi \frac{Y}{Y}
\]

with

\[Y = \delta - \frac{A_0 \varepsilon \tilde{K}^{\varepsilon - 1}}{(1 - \varepsilon)}.\]
For the phase with $\Psi_1(K, C, A) \neq 0$, one finds:

$$
\langle A \rangle_1 = \frac{A_0}{(1 - \varepsilon)} - \frac{1}{2} \frac{K^\varepsilon A_0 (1 - \varepsilon)}{Y^2 (1 - \varepsilon)^3} - \frac{\left( \tilde{C} + \sqrt{\frac{2}{\pi}} \sigma - K'_1 \right) Y (1 - \varepsilon)}{Y^2 (1 - \varepsilon)^3} \gamma \rho
$$

$$
\langle C \rangle_1 = \tilde{C} + \sqrt{\frac{2}{\pi}} \sigma - \frac{\sigma^2 \left( \frac{A_0}{1 - \varepsilon} \right)}{2 \left( \varepsilon^2 \sigma^2 + (A F' (K) + r_c)^2 \right)} \delta - \frac{A_{0f} K^\varepsilon - 1}{1 - \varepsilon} \gamma \rho
$$

$$
\langle K \rangle_1 = \frac{\tilde{A}_0}{1 - \varepsilon} \tilde{K}^\varepsilon (1 - \varepsilon) - x - \frac{1}{2} \frac{K^\varepsilon A_0 (1 - \varepsilon) - Y (1 - \varepsilon) x (\varepsilon x - K \delta (1 - \varepsilon))}{Y^4 K (1 - \varepsilon)^3} \gamma \rho
$$

$$
- \left( \frac{\tilde{K}^\varepsilon (1 - \varepsilon)(1 - Y) K'_1}{2Y^3} + \frac{\varepsilon^2 A_0}{2Y^2} + \frac{\sigma^2 \left( \frac{A_0}{1 - \varepsilon} \right)}{2 \left( \varepsilon^2 \sigma^2 + (A F' (K) + r_c)^2 \right) Y^2} \right) \gamma \rho
$$

(60)

where $\sqrt{\rho}$ is the $L^2$ norm of $\Psi_1$ and:

$$
x = \tilde{C} + \sqrt{\frac{2}{\pi}} \sigma - K'_1
$$

$$
K'_1 = -\frac{2}{\pi} \tilde{C} + \sqrt{\frac{2}{\pi}} \sigma - A \tilde{K}^\varepsilon (1 - \varepsilon) - \frac{\sqrt{2 \delta - \frac{A_{0f} K^\varepsilon - 1}{1 - \varepsilon}}}{\sqrt{\pi}} \nu
$$

In the non-trivial phase, the value of $\rho \gamma$ depends on the parameters of the system. An estimation is given in online Appendix 6.2.4. However, here it is enough here to know that $\rho \gamma \ll 1$.

In the non-trivial phase, both the average level of consumption $\langle C \rangle_1$ and the average level of technology $\langle A \rangle_1$ are lower than in the trivial phase: $\langle C \rangle_1 < \langle C \rangle_0$, $\langle A \rangle_1 < \langle A \rangle_0$. Indeed, the non-trivial phase emerges under high relative risk aversion, and hence low level of consumptions. The effect on average capital stock $\langle K \rangle_1$ is mitigated: since the technology level has decreased, a higher level of capital stock may be required to reach average levels of consumption. Equation (61) shows that when capital or/and consumption volatility, $\nu$ and $\sigma$, respectively, are high, capital stock is lower in the non-trivial phase. Uncertainty hinders accumulation. But when these volatilities are low, capital stock is higher than in the trivial phase.

We can also compute the average production level in both phases. It appears that in our range of approximations:

$$
\langle Y \rangle_1 < \langle Y \rangle_0.
$$

Let us now discuss the conditions (57) and (58) in which this non-trivial phase should appear. When $\gamma < 0$, capital and technology are mutually enhanced. This prevents the non-trivial phase to appear. A high level of consumption and capital can be reached. But when $\gamma > 0$, the interaction between capital and technology is
selective. In such a setting, the initial endowment in technology is crucial. Agents endowed with a level of technology above a certain threshold are favored, whereas for agents poorly endowed, this level acts as a ceiling. In average, our results show that overall, the society experiences lower technology, lower production and lower consumption.

Finally, the non-trivial phase corresponds to a large minimal technology and intermediate values of $C_0$. Actually, risk aversion has to be large enough to reduce capital accumulation. Still, this reduced capital accumulation must be sufficient to reach optimal consumption. To do so, technology must be relatively high and compensate for a lower stock of capital. This equilibrium must also be sustainable. Thus, a large $C_0$ is outside the limits of our model.

4.2.2 Transition functions without interaction

In the above, we had first translated the classical business model in terms of probabilistic description, then translated it into statistical field. Now that the various phases of the system of the statistical field have been described, we can examine the transition functions of the system. In other terms, having found the collective background, we can now return to the individual level that may include several agents and their interactions. The transition functions will allow us to describe the individual agents’ dynamics in each phase and deduce the equilibrium values of these dynamics, as well as the dynamic equations of their average paths. The computation of the transition functions for each phase is performed in online Appendix 6.4.1 and 6.4.2. The average path for each phase is derived in online Appendix 6.4.3.

In the following, we will add an index $\iota = 0$ or $1$ to the phase dependent parameters. The minimum for the trivial phase $\iota = 0$ is $\Psi_1 = 0$. The non-trivial phase $\iota = 1$ corresponds to $\Psi_1 \neq 0$.

Transition functions In a given a phase $\iota$ of the system, neglecting in a first step the quartic interaction term in (56), the probability of transition for an agent from a state $(C', K', A')$ to a state $(C, K, A)$ during a timespan $t$, is equal to:

$$G(C, K, A, C', K', A', t) = \frac{1}{\sqrt{(2\pi)^3 \Omega^2 t^3}} \exp \left( -\frac{(C - \bar{C}_\iota) - (C' - \bar{C}_\iota)(1 + \beta t))^2}{2\Omega^2 t} \right)$$

$$\times \exp \left( -\frac{(K - \bar{K} + \delta \bar{K} + \bar{C}_\iota)(1 - \alpha t) - (C' - \bar{C}_\iota) t + A' \bar{K}_\iota t}{2\Omega^2 t} \right)$$

$$\times \exp \left( -\frac{\lambda^2 (A - A')^2}{2t} - \frac{(\delta A / 2 - \bar{A}_\iota)^2}{2t} - t - m, t \right)$$

(62)

where $(C', K', A')$ is the initial state of the system and $(C, K, A)$ the final state for a process of duration $t$. 
Parameters Two types of parameters appear in Eq. (62). Some parameters, such as \( \Omega^2, \alpha \) and \( \beta \), do not depend on the phase of the system. They are:

\[
\Omega^2 = \frac{\sigma^2}{\lambda^2} \left( \nu^2 + \frac{2\hat{K}^2\epsilon}{\lambda^2\alpha^2} + \frac{3\sigma^2}{2(\beta^2 - \alpha^2)} \right)
\]

\[
\alpha = \delta - \frac{A + A'}{2} F' \left( \frac{K + K'}{2} \right)
\]

\[
\beta = \frac{A + A'}{2} F' \left( \frac{K + K'}{2} \right) + r_c - \delta.
\]

The parameter \( \Omega^2 \) is a global variance of the system which mixes the variances of the variables \( C, K \) and \( A \). This parameter is independent from the phase, since the phases do not affect volatilities but average values in this particular model. For \( r_c = \delta \), the parameter \( \beta \) is the average rate of return of capital for a process starting from \((C', K', A')\) and reaching \((C, K, A)\). Finally, the parameter \( \alpha \) measures the spread between marginal productivity and capital depreciation. These two variables do not depend on the phase but merely on the producer capital and technology levels.

Other parameters in Eq. (62) are phase dependent. These parameters are the average values of technology and capital in a given phase. They are, for the first phase:

\[
\bar{A}_0 = \langle A \rangle_0 = \frac{A_0}{1 - \chi} \\
\bar{C}_0 = \langle C \rangle_0 = \bar{C} + 2\sigma \\
m_0 = 0
\]

and for the second phase:

\[
\bar{A}_1 = A_0 + \chi \langle A \rangle_1, \quad \bar{C}_1 = \Gamma_1 \\
m_1 = \bar{A}_1^2 - \left( (1 - \varepsilon) \bar{A}_1 + \varepsilon \Gamma_3 \right)^2 + \frac{\left( (1 - \varepsilon) \bar{A}_1 + \varepsilon \Gamma_3 \right)^2 - 2\bar{C} \left( (1 - \varepsilon) \bar{A}_1 + \varepsilon \bar{A}_1 \right) - \bar{C}_1^2}{(1 - \varepsilon) \hat{K}^\varepsilon}
\]

where the average values \( \langle A \rangle_i \) have been defined in (59) and (60).

Because some parameters are phase dependent, the agent’s transition probability depends on the phase of the system.

The parameter \( m_i \) is a measure of the system’s inertia. It is null in the trivial phase but positive in the non-trivial phase. The fact that it should appear as an exponential term \( \exp(-m_i t) \) in (62) is relevant. Transitions are quickly dampened through the interaction process in the non-trivial phase, and transition probabilities are reduced. The strong inertia of the system keeps agents closer to their initial values than in the trivial phase. Thus, the parameter \( m_i \) characterizes the economic environment defined by the phase, so that mobility in the non-trivial phase is lower than in the trivial phase.
Note also that multi-agents interactions do not appear directly in (62), but only through parameters. These parameters encode the collective effects of the interactions and their impact at the individual level.

**Average paths and classical dynamics** Formula (62) is valid for small $t$. However, it is sufficient to find the agent’s average path in each phase. This is straightforward: for a Gaussian weight, the most likely path is found by setting the exponent in (62) to 0, which is also the equation for the average path. This yields the relations between the initial and final points in a given phase $i$:

$$0 = \left( (C - \bar{C}_i) - (C' - \bar{C}_i)(1 + (\alpha + \beta)t) \right)$$

$$0 = \left( \left( K - \bar{K} + \frac{\delta \bar{K} + \bar{C}_i}{\alpha} \right) - \left( \left( K' - \bar{K} + \frac{\delta \bar{K} + \bar{C}_i}{\alpha} \right)(1 - \alpha t) - (C' - \bar{C}_i) t + A' \bar{K}^e t \right) \right)$$

$$0 = \frac{\lambda^2 (A - A')^2}{2t} + \frac{(A + A' - \bar{A}_i)^2}{2} t. \tag{64}$$

The treatment of these equations is usual. The equilibrium values are first found by setting both initial and final values equal to the equilibrium:

$$(C', K', A') = (C, K, A) = (C_e, K_e, A_e).$$

One finds directly:

$$K_e = \frac{(1 - \varepsilon) \bar{A}_i \bar{K}^e - \bar{C}_i}{\delta - \bar{K}^e \bar{A}_i \varepsilon}$$

$$C_e = \bar{C}_i$$

$$A_e = \bar{A}_i.$$

Then, replacing these values in (64) yields directly the relations for the most likely path, or equivalently, the average path:

$$\left( C - \bar{C}_i \right) = \left( C' - \bar{C}_i \right) + \left( C' - \bar{C}_i \right)(\alpha + \beta)t$$

$$\left( K - K_e \right) = \left( K' - K_e \right) - \alpha \left( K' - K_e \right) t - (C' - \bar{C}_i)t$$

$$\lambda^2 (A - A') = -\frac{A + A' - \bar{A}_i}{2} t. \tag{65}$$

Using (65) and $\frac{K + K'}{2} \to K$ in the limit of small $t$ leads to a differential equation for the average path in phase $i$: 
\[
\frac{d}{dt} (C(t) - \bar{C}_e) = (C(t) - \bar{C}_e) (AF' (K(t)) + r_c - \delta)
\]
\[
\frac{d}{dt} (K(t) - K_e) = (AF' (K(t)) - \delta)(K(t) - K_e) - (C(t) - \bar{C}_t)
\]
\[
\lambda^2 \frac{d (A - \bar{A}_i)}{dt} = - \frac{(A - \bar{A}_i)}{2}.
\]

(66)

Actually, the above equations are those of a simplified model of capital accumulation: the standard approach is recovered in average. The first equation is the usual Euler equation with interest rate. The second and third equations describe the dynamics of capital and technology, respectively. The system’s interactions are encompassed in the equation with interest rate. The second and third equations describe the dynamics of the standard approach is recovered in average. The first equation is the usual Euler actually, the above equations are those of a simplified model of capital accumulation: the system moves along the unstable equilibrium, the linear approximation breaks down. Actually, for large values of \((K(t) - K_e)\), marginal productivity falls below the depreciation rate, and capital accumulation stops. However, once the phases of the system have been found, and the average dynamics equations have been written, interpretations are standard.

### 4.2.3 Corrections due to the interaction term

We now study the individual dynamics that has been created by the interacting system as a whole. To take into account the individual interactions and their impact on (62), we must return to the field theoretic formulation and find the modification of the transition functions due to the quartic interaction term in (56):

\[
I = \frac{\gamma}{2} \int \Psi^\dagger (K_1, C_1, A_1) \Psi^\dagger (K_2, C_2, A_2) (A_2 K_1 + A_1 K_2) \Psi (K_1, C_1, A_1) \Psi (K_2, C_2, A_2).
\]

(67)

Online Appendix 6.5.1 computes this correction. We reintroduce in the probability transitions the contribution of the interaction term (67) that was discarded while computing (62) and show that the Green function is modified at the first order in \(\gamma\) as:

\[
\tilde{G} (C, K, A, C', K', A', t) = G (C, K, A, C', K', A', t) \times \exp (-\gamma V (C, K, A, C', K', A', t))
\]

where the function \(V (C, K, A, C', K', A', t)\) depends on the initial and final states:

\[
V (C, K, A, C', K', A', t) = \\
= (2r^2 AK + \frac{r^3}{12} (A - A')(C - C') + \frac{t^3}{2} (A - A')(K - K') - \frac{1}{12} \bar{K}^e r^3 (A - A')^2) \\
+ A \left( \frac{1}{3} r^3 (C - C') + t^2 (K - K') - \frac{1}{3} \bar{K}^e r^3 (A - A') \right) + \gamma t^2 (A - A') K.
\]
Online Appendix 6.5.2 also shows that an average path with initial conditions \( (C(0), K(0), A(0), \dot{C}(0), \dot{K}(0), \dot{A}(0)) \) is modified at the first order in \( \gamma \) for small interaction timespans. Defining \( \delta C(t), \delta K(t) \) and \( \delta A(t) \) the respective consumption, capital and technology deviations from their average paths \((66)\) due to the interactions, we can write, in phase \( \iota \):

\[
\begin{align*}
\delta C(t) &= 0 \\
\delta K(t) &= \gamma \left( \frac{7ct^5}{720A^2_0} C(0) + \frac{ct^4}{48A^2_0} K(0) + \left( \frac{bt^3}{6K^\varepsilon A^2_0} + \frac{K^\varepsilon ct^5}{90} \right) A(0) \right) \\
&\quad + \gamma \left( -\frac{7ct^6}{1440A^2_0} \dot{C}(0) + \frac{ct^5}{60A^2_0} \dot{K}(0) + \left( \frac{bt^4}{24K^\varepsilon A^2_0} + \frac{3K^\varepsilon ct^6}{160A^2_0} \right) \dot{A}(0) \right) \\
\delta A(t) &= \gamma \left( \frac{ct^3}{6K^\varepsilon A^2_0} K(0) \right) + \gamma \left( \frac{ct^4}{24} \dot{K}(0) + t \dot{A}(0) \right)
\end{align*}
\]

where

\[
\begin{align*}
b &= 2 \left( \nu^2 + \frac{2K^{2\varepsilon} \alpha^2 \beta}{\lambda^2 \alpha^2} + \frac{3\sigma^2}{2(2\alpha + \beta) \beta} \right) \\
c &= \frac{2}{\lambda^2}.
\end{align*}
\]

The interpretation of \((68)\) is the following. From \((68)\), we directly find the impact of the agent’s initial state on the deviations \( \delta K(t) \) and \( \delta A(t) \):

\[
\begin{align*}
\frac{\partial (\delta K(t))}{\partial K(0)} &= \frac{ct^4}{48A^2_0} > 0 \\
\frac{\partial (\delta K(t))}{\partial A(0)} &= \frac{bt^3}{6K^\varepsilon A^2_0} + \frac{K^\varepsilon ct^5}{90} > 0 \\
\frac{\partial (\delta A(t))}{\partial K(0)} &= \frac{ct^3}{6K^\varepsilon A^2_0} > 0.
\end{align*}
\]

The interaction between individual capital and technology stocks produces a synergy effect, and both stocks increase faster. This effect is proportional to the stocks’ initial values, so that the higher these initial individual values, the faster both stocks increase. Moreover, the polynomial time dependency of the elasticities shows that the accumulation dynamics is faster than a linear process. Remark that this synergy effect is not contradictory with the eviction effect described in the non-trivial phase. Actually, the results presented here are only valid at the individual level. In other words, the individual dynamics in a given phase does not detect the collective mechanisms of interactions. The latter are only detectable when analyzing the phase and are indeed hidden in apparently exogenous parameters shaping the agents’ environment.

The accumulation dynamics is however dampened by fluctuations in technology stocks, measured by \( \tilde{A}_i^2 \lambda^2 \). The higher these fluctuations, the slower the accumulation
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...process. The initial direction of the system, given by the terms proportional to \( \dot{X}(0) \) amplify this synergy effect. Actually, we can also compute from (68) the impact of the agent’s initial momentum on the deviations \( \delta K(t) \) and \( \delta A(t) \):

\[
\begin{align*}
\frac{\partial (\delta K(t))}{\partial \dot{K}(0)} &= \gamma \frac{ct^5}{60A_i^2} > 0 \\
\frac{\partial (\delta K(t))}{\partial \dot{A}(0)} &= \gamma \frac{bt^4}{24K^eA_i^2} + \frac{3K^e ct^6}{160A_i^2} > 0 \\
\frac{\partial (\delta A(t))}{\partial \dot{A}(0)} &= \gamma t > 0 \\
\frac{\partial (\delta A(t))}{\partial \dot{K}(0)} &= \gamma \frac{ct^4}{24} > 0.
\end{align*}
\]

A system that initially started to accumulate both capital and technology stocks will accelerate faster than a system that was initially in a constant equilibrium.

The effect of the initial value of consumption is ambiguous. Improvements in technology, measured by the dynamics of \( A \), increase productivity and rates of return. It is thus optimal for agents to increase their savings and reduce their consumption. Capital stock is positively correlated to \( C(0) \), as shown by its elasticity with respect to \( C(0) \):

\[
\frac{\partial (\delta K(t))}{\partial C(0)} = \gamma \frac{7ct^5}{720A_i^2} > 0.
\]

In other word, a high level of initial consumption is an indicator of wealth. The agents’ interactions induce an accumulation process that favors capital expenditures. Since consumption elasticity with respect to the initial value consumption \( \frac{\partial (\delta C(t))}{\partial C(0)} \) is null, any increase of wealth is spent on capital stock.

Besides, any initial increase in the consumption rate impairs capital accumulation since:

\[
\frac{\partial (\delta K(t))}{\partial \dot{C}(0)} = -\gamma \frac{7ct^6}{1440A_i^2} < 0.
\]

An initial increase in consumption will be smoothed over the entire timespan and will eventually dampen the accumulation process.

**4.2.4 Two agents transition functions and interaction**

The field formalism presented here allows also the study of interaction between individual agents. Consider the simplest example of a two agents’ dynamics. Discarding interactions, the probability of transition between a two agents’ initial state:

\[
\left((K_1, C_1, A_1)_i, (K_2, C_2, A_2)_i\right)
\]

and a final state:

\[
\left((K_1, C_1, A_1)_f, (K_2, C_2, A_2)_f\right)
\]
is simply the product of the transition probabilities (62) for each agent:

\[
G \left( (K_1, C_1, A_1)_i, (K_2, C_2, A_2)_i, (K_1, C_1, A_1)_f, (K_2, C_2, A_2)_f \right) \\
\equiv G \left( (K_1, C_1, A_1)_i, (K_2, C_2, A_2)_i, t \right) G \left( (K_1, C_1, A_1)_f, (K_2, C_2, A_2)_f, t \right)
\]

since they are considered independent. The global interaction effect, i.e., the impact of the entire system on each agent, is included in the phase of the system through the parameters of the transition probabilities.

When the interaction term is included, the two agents’ transition probability has to be modified. Online Appendix 6.6 computes this correction. Consider again the initial state \( (K_1, C_1, A_1)_i, (K_2, C_2, A_2)_i \) and the final state \( (K_1, C_1, A_1)_f, (K_2, C_2, A_2)_f \). We show that the interaction term:

\[
I = \frac{\gamma t}{2} \int \langle C_1 \rangle \langle K_2 \rangle \langle K_1 \rangle \langle A_2 \rangle \Psi(K_1, C_1, A_1) \Psi(K_2, C_2, A_2)
\]

modifies the transition probabilities in the following way:

\[
G_I \left( (K_1, C_1, A_1, t)_i, (K_2, C_2, A_2, t)_i, (K_1, C_1, A_1, t)_f, (K_2, C_2, A_2, t)_f \right) \\
\equiv G \left( (K_1, C_1, A_1)_i, (K_2, C_2, A_2)_i, (K_1, C_1, A_1)_f, (K_2, C_2, A_2)_f \right) \exp(-V_I)
\]

where

\[
V_I = \gamma t^2 \left( \langle A_1 \rangle \langle K_2 \rangle + \langle K_1 \rangle \langle A_2 \rangle \right) \\
+ \frac{\gamma t^3}{24} \left( \langle A_1 \rangle \Delta C_2 + \Delta C_1 \langle A_2 \rangle - K^e \left( \langle A_1 \rangle \Delta A_2 + \Delta A_1 \langle A_2 \rangle \right) \right)
\]

and with

\[
\langle X_j \rangle = \frac{(X_j)_i + (X_j)_f}{2} \\
\Delta X_j = (X_j)_f - (X_j)_i
\]

for any variable \( X = C, K, \) or \( A \) and agent \( j = 1 \) or \( 2 \). The quantity \( \langle X_j \rangle \) computes the average value of \( X \) for agent \( j \) along the path, and \( \Delta X_j \) the variation of \( X \) along this path.

Due to the interaction, the two agents’ transition probabilities are now entangled. The average trajectory for one agent is modified by the other agent’s path. We write \( \delta X_{2 \rightarrow 1} (t) \) the correction of agent 1’s trajectory due to agent 2 for \( X = C, K, \) or \( A \), and \( \delta X_{1 \rightarrow 2} (t) \) the correction of agent 2’s trajectory due to agent 1. Online Appendix 6.6 shows that:

\[
\delta K_{2 \rightarrow 1} (t) = \gamma bt \langle A_2 \rangle \\
\delta A_{2 \rightarrow 1} (t) = \gamma \left( \frac{1}{6} c^e \frac{\Delta C_2}{2} - \frac{1}{6} c K^e \frac{\Delta A_2}{2} \right) t^2 + \gamma ct \langle K_2 \rangle
\]
\[
\delta K_{1 \rightarrow 2} (t) = \gamma bt \langle A_1 \rangle \\
\delta A_{1 \rightarrow 2} (t) = \gamma \left( \frac{1}{6} c \frac{\Delta C_1}{2} - \frac{1}{6} c K^e \frac{\Delta A_1}{2} \right) t^2 + \gamma ct \langle K_1 \rangle
\]

(70)

with

\[
\bar{X}_i = \frac{X_i (0) + X_i (t)}{2} \\
\frac{\Delta X_i}{2} = \frac{X_i (t) - X_i (0)}{2}
\]

Formula (70) allows to find the dependency of an agent behavior on other agent’s path. The elasticities are:

\[
\frac{\partial (\delta K_1 (t))}{\partial A_2 (0)} = \frac{\partial (\delta A_1 (t))}{\partial K_2 (t)} = t > 0 \text{ (and other elasticities with respect to } \bar{X}_2 \text{ are null)}
\]

which means that the average technology of agent 2 impacts positively the accumulation of capital for agent 1, and that the accumulated stock of agent 2 accelerates the technology improvement for agent 1. Agent 2 participates to the environment of agent 1, and both its capital and technology stocks influence the other agents.

These elasticities are proportional to the interaction timespan: the longer agents interact, the higher the final stocks. The elasticities with respect to the initial direction of agent’s 2 path may seem counterintuitive.

\[
\frac{\partial (\delta A_1 (t))}{\partial \Delta A_2 (0)} = - \frac{1}{6} c K^e t^2 < 0 \\
\frac{\partial (\delta A_1 (t))}{\partial C_2 (t)} = \frac{1}{6} c t^2 > 0.
\]

The technology stock is negatively correlated to other agents’ accumulation rate. This is the consequence of the acceleration of accumulation process for both agents and our choice of representation of a path as function of the average value of the path \( \bar{X}_2 \): since the dynamic follows an accelerating pattern, its representative curve is below the average \( \bar{X}_2 \) most of the time. As a consequence, the accumulated stock is below the linear approximation in \( \bar{X}_2 \). The term proportional to \( \frac{\Delta X_2}{2} \) is thus a correction to this linear approximation.

### 4.3 Synthesis and discussion

Applying our formalism to a basic business cycle model has shown the implications of introducing multiple agents interacting through technology and capital stocks. This has allowed us to inspect setups not accessible to the usual representative agents’ models. It has also allowed to detect collective effects due to large number of agents, such as the appearance of multiple phases or macroeconomic equilibria. In turn, we have seen how these phases impact the individual agents’ dynamics. These individual
dynamics are formally identical to those used in representative agents’ models. At this point however, some major differences with standard models appear:

In our formalism, the individual agents’ dynamics are derived from a collective background. They emerge from the model, but cannot be imposed as a defining point of the model. This translates into several features.

First, the individual dynamics parameters are not exogenous. As mentioned above, they depend on the global system. A change of phase in the entire system induces a structural break that actually modifies the parameters of the agent’s dynamic equations.

Second, the fact that individual behaviors emerge from the system extends to any subset of agents. Their dynamics and interactions, too, can and should be deduced from the collective background (see Sect. 4.2.4). This allows a straightforward and detailed analysis of agents’ interactions, while preserving the agents’ heterogeneity.

As a result, and third, individual agents cannot be considered as representative agents. Section (4.2.3) demonstrates that individual features do not aggregate to produce similar effects at the macrolevel. The synergy effect in Eq. (69) shows that an agent may experience a virtuous circle between his capital and technology, even in the non-trivial phase characterized at a macrolevel by an eviction effect and a lower production. These two macroeconomic features are present at the individual level, but only as a hidden externality that shapes the agent’s environment through seemingly exogenous parameters.

To conclude, the representative agent paradigm cannot detect some macroeconomic features from the description of particular agents. Some conclusions at the individual level do not aggregate.

5 Conclusion

This paper has presented an analytical treatment of economic systems with an arbitrary number of agents that keeps track of the systems’ interactions and agents’ complexity. As significant results, we have shown that a field theory formalism may reveal some emerging equilibria and studied the influence of these equilibria on the agent’s individual dynamics. This method can be applied to various economic models.

In this paper we have, for the sake of clarity, deliberately set aside some matters developed in Gosselin et al. (2017), such as strategic behaviors and heterogeneity among agents, in information, goals or actions. However, our formalism extends to such cases. Social interactions and economic networks could also be included. These subjects are under current research.

Ultimately, our formalism should shed some lights on the matter of aggregation. Indeed, even though field theory does not deal with aggregates, it should allow to recover macroeconomic quantities through averages and explore the relations between these macroeconomic quantities.

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